DYNAMICAL R-MATRICES FOR INTEGRABLE MAPS

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Abstract
The integrability of two symplectic maps, that can be considered as discrete-time analogs of the Garnier and Neumann systems is established in the framework of the \( r \)-matrix approach, starting from their Lax representation. In contrast with the continuous case, the \( r \)-matrix for such discrete systems turns out to be of dynamical type; remarkably, the induced Poisson structure appears as a linear combination of compatible “more elementary” Poisson structures. It is also shown that the Lax matrix naturally leads to define separation variables, whose discrete and continuous dynamics is investigated.
1. Introduction

In a number of recent papers, it has been shown that the well-established techniques devised to construct integrable finite-dimensional hamiltonian systems out of integrable hierarchies of nonlinear evolution equations (often denoted “soliton hierarchies”) [1] can also be applied in a discrete context, naturally leading to integrable maps [2-5]. As remarkable results, I would like to quote the construction of discrete-time versions of the Toda-lattice [6] and of the Calogero-Moser system [7].

In this context, the author has shown [5] that, starting from the so-called “Toda Hierarchy with Sources”, one readily obtains integrable maps equipped with a Lax pair. In a further paper [8], the connection of these maps with the stationary Toda flows and with the finite gap sector of the solution manifold of the Toda hierarchy has been rigorously established.

In the present paper, we investigate the algebraic structure underlying two of such integrable maps, namely the Discrete Garnier (DG) [3] and the Discrete Neumann (DN) [4] systems, in terms of the “classical $r$-matrix” formalism as generalized in a fundamental paper by Babelon and Viallet [9]. Surprisingly, we find that the associated $r$-matrices are of dynamical type, in contrast with those pertaining to the corresponding continuous systems [10,11], and moreover appear as a linear combination of “more elementary” $r$-matrices, so that the associated Poisson structure is itself a linear combination of compatible Poisson structures.

We recall that $r$-matrices of dynamical type have recently been discovered for an extremely important class of integrable finite-dimensional continuous time systems, namely the Calogero-Moser class [12].

A remarkable property of the maps under scrutiny is their separability in terms of “roots variables” naturally provided by the Lax representation, in full analogy with the continuous systems [11].

The DG and DN systems and their Lax representation are tersely reviewed in Section 2. In Section 3 we derive the associated $r$-matrices, while in Section 4 we define separation variables and investigate the corresponding dynamics.

A few (possibly) interesting open problems are mentioned in Section 5.

2. Two Integrable Maps and their Lax Representation

The DG system is given by the following Lagrangean map:

$$
<\Psi_n,\Psi_{n-1}>\Psi_{n-1}+<\Psi_n,\Psi_n>\Psi_n+<\Psi_n,\Psi_{n+1}>\Psi_{n+1} = \Lambda\Psi_n \quad (2.1)
$$

In (2.1), $n \in \mathbb{Z}$, $\Psi_n$ is an $\mathbb{R}^N$-vector of components $\Psi_n^{(j)}$, $\Lambda$ is a diagonal matrix with distinct entries $(\lambda_1, \cdots, \lambda_N)$ and the symbol $<\cdot,\cdot>$ denotes the usual euclidean inner product in
As explained in [5], the system (2.1) arises by taking $N$ replicas of the Toda-lattice spectral problem [13]:

$$a_{n-1}\Psi_{n-1}^{(j)} + b_n\Psi_n^{(j)} + a_n\Psi_{n+1}^{(j)} = \lambda_j\Psi_n^{(j)}.$$  \hspace{1cm} (2.2)

corresponding to $N$ different values of the spectral parameter $\lambda$, and by restricting the dynamical variables $(a_n, b_n)$ to the invariant manifold:

$$a_n = 2 < \Psi_n, \Psi_{n+1} >; \quad b_n = < \Psi_n, \Psi_n >$$  \hspace{1cm} (2.3)

The corresponding Lagrange function reads:

$$L(x, y) = < x, y >^2 + \frac{1}{2} < x, x > - < x, \Lambda x >$$

where $x = \Psi_n, \; y = \Psi_{n+1}$.

Canonical variables $q, p$ can be introduced in the standard way [2]:

$$q = \Psi_n; \quad p = \frac{\partial L}{\partial y}|_{(x = \Psi_{n-1}, y = \Psi_n)} = \frac{\Psi_{n-1}}{< \Psi_n, \Psi_{n-1} >}$$ \hspace{1cm} (2.4)

thus allowing to recast (2.1) in the form of the following map:

$$q' = (\Lambda < q, q > - q - p)/g$$ \hspace{1cm} (2.5a)

$$p' = gq$$ \hspace{1cm} (2.5b)

with $g^2 = < q, \Lambda q > - < q, q >^2 - < p, q >$. The map (2.5) is symplectic for the standard symplectic form $dq \wedge dp$. Exploiting the procedure outlined in [5], the map (2.5) can be represented in the following discrete Lax form:

$$L = AL'A^{-1}$$ \hspace{1cm} (2.6a)

where $L, A$ are $2 \times 2$ matrices, given by:

$$L = \begin{pmatrix} -(\lambda/2+ < p, R\lambda q >) & \sqrt{< p, q >} (1+ < p, R\lambda p > / < p, q >) \\ -\sqrt{< p, q >} (1+ < q, R\lambda q >) & \lambda/2+ < p, R\lambda q > \end{pmatrix}$$ \hspace{1cm} (2.6b)

$$A = \begin{pmatrix} (\lambda - < q, q >)/\sqrt{< p, q >} & -\sqrt{< p', q' >}/\sqrt{< p, q >} \\ 1 & 0 \end{pmatrix}$$ \hspace{1cm} (2.6c)
where
\[ R_\lambda = (\lambda I - \Lambda)^{-1} \]

The meromorphic invariant function \( \Delta(\lambda) \equiv det[L(\lambda)] \) is the generating function of the conserved quantities:
\[ \Delta(\lambda) = -\lambda^2/4 + \sum_{j=1}^{N} \frac{I_j}{\lambda - \lambda_j} \]  
(2.7a)

\[ I_j = \sum_{k \neq j} \frac{(p_j q_k - p_k q_j)^2}{\lambda_k - \lambda_j} + p_j^2 + < p, q > q_j^2 - \lambda_j p_j q_j \]  
(2.7b)

and an explicit computation shows that they are mutually in involution
\[ \{ I_j, I_k \} = 0, \]
thus entailing the complete integrability of the map (2.5).

Similar considerations hold for the DN system [4], described by the map:
\[ \frac{\Psi_{n-1}}{2 < \Psi_{n-1}, \Psi_n>} + b_n \Psi_n + \frac{\Psi_{n+1}}{2 < \Psi_{n+1}, \Psi_n>} = \Lambda \Psi_n \]  
(2.8)

where the discrete motion is constrained on the unit sphere \( S^N \): \( < \Psi_n, \Psi_n > = 1 \), and accordingly the Lagrange multiplier \( b_n \) is determined to be: \( b_n = < \Psi_n, \Lambda \Psi_n > - 1 \).

Eq.(2.8) is a Lagrangean map for the Lagrange function:
\[ \mathcal{L}(x, y) = log < x, y > - < x, \Lambda x > \]
on \( S^N \).

In terms of the hamiltonian variables \( q, p \), defined as in (2.4) by:
\[ q = \Psi_n; \quad p = \frac{\partial \mathcal{L}}{\partial y} |_{x=\Psi_{n-1}, y=\Psi_n} = \frac{\Psi_{n-1}}{2 < \Psi_n, \Psi_{n-1}>} \]
eq(2.8) becomes the map:
\[ p' = q||(\Lambda - b)q - p|| ; \quad q' = \frac{(\Lambda - b)q - p}{||(\Lambda - b)q - p||} \]  
(2.9a)

\[ ||q|| = 1, \quad < q, p > = \frac{1}{2} \]  
(2.9b)

which is symplectic for the Poisson brackets:
\[ \{ q, q \} = 0; \quad \{ q, p \} = I - q \otimes q; \quad \{ p, p \} = p \wedge q \]  
(2.9c)
It enjoys the Lax representation (2.6a), with:

\[
L = \begin{pmatrix}
\frac{1}{2} + \frac{1}{\|p\|} < p, R\lambda q > & -\frac{1}{\|p\|} < p, R\Lambda p > \\
\frac{1}{\|q\|} < q, R\Lambda q > & -\frac{1}{2} - \frac{1}{\|p\|} < p, R\lambda >
\end{pmatrix}
\]  
(2.10a)

\[
A = \begin{pmatrix}
\frac{(\lambda - p)/\|p\| - \|p\|/\|p\|}{1} & 0 \\
\end{pmatrix}
\]  
(2.10b)

Again, the invariant function \( \Delta(\lambda) \equiv det[L(\lambda)] \) is the generating function of the integrals of motion:

\[
\Delta(\lambda) = \sum_j \frac{I_j}{\lambda - \lambda_j}
\]  
(2.11a)

with

\[
I_j = \sum_{k \neq j} \frac{(p_j q_k - p_k q_j)^2}{\lambda_k - \lambda_j} - q_j p_j
\]  
(2.11b)

The conserved quantities \( I_j \) have been shown in [4] to be in involution for the Poisson brackets (2.9c).

3. r-Matrix Formulation

As it has been shown for the first time in [9], whenever a hamiltonian system is associated with a Lax matrix whose eigenvalues are in involution for a given Poisson bracket, it enjoys an r-matrix representation of the form:

\[
\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]
\]  
(3.1)

We have used the standard notations:

\[
L_1 \equiv L \otimes 1 = \sum_i L_i e^i \otimes 1
\]

\[
L_2 \equiv 1 \otimes L = \sum_i L_i 1 \otimes e^i
\]

\[
r_{12} = \sum_{i,k} r_{ik} e^i \otimes e^k
\]

\[
r_{21} = \sum_{i,k} r_{ki} e^i \otimes e^k
\]

where \( \{e^i\}_{i=1}^M \) is a basis for the matrix Lie-algebra which \( L \) belongs to, and, in general, the coefficients \( r_{ik} \) will be functions on the phase space (dynamical r-matrix).
For both the systems introduced in Sec. 2, the Lie algebra is obviously $sl(2)$, and we will use the Cartan-Weil basis:

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(3.2a)

rather than the Pauli basis:

$$\sigma^1 = \sigma^+ + \sigma^-; \quad \sigma^2 = i(\sigma^- - \sigma^+); \quad \sigma^3$$

(3.2b)

As a matter of fact, since $L$ has an additional (rational) dependence upon the spectral parameter $\lambda$, it has to be regarded as an element of the loop algebra $\mathcal{G} = sl(2) \otimes \mathbb{C}(\lambda, \lambda^{-1})$, namely as a formal Laurent series in $\lambda$ with coefficients in $sl(2)$:

$$L(\lambda) = \sum_{k=-\infty}^{p} L_k \lambda^k$$

(3.3)

Incidentally, we notice that for matrices (2.6b), (2.10a), the series (3.3) is actually uniformly convergent in any compact subset of the annulus $\text{max}_j |\lambda_j| < |\lambda| < \infty$.

The trace form on $\mathcal{G}$, given by:

$$(L) \equiv \text{res } tr L(\lambda) = tr L_{-1}$$

(3.4)

allows to identify $\mathcal{G}$ with its dual (the space of linear functions on $\mathcal{G}$), and to consider the $r$–matrix as an endomorphism $R$ on $\mathcal{G}$, rather than as an element of $\mathcal{G} \otimes \mathcal{G}$. In such a “dual” picture, eq.(3.1) induces the following Poisson bracket between two functions on $\mathcal{G}$:

$$\{f, g\}_L = (L, [df, dg]_R)$$

(3.5a)

with

$$[X, Y]_R = [X, R(Y)] + [R(Y), X] \quad (X, Y \in \mathcal{G})$$

(3.5b)

and:

$$R(X) = \sum_{j,k} r_{jk} e^{i(j)}(e^{(k)}, X)$$

(3.5c)

As is well known [9], eq.(3.1) implies involutivity of the invariants of the matrix $L$.

In the dual picture, it is convenient to look at the functions:

$$f_k = \text{res } \frac{1}{2} tr \lambda^k L^2 = \frac{1}{2} (L, \lambda^k L)$$

(3.6a)
From (3.5):
\[
\{f_k, f_j\} = (L, [df_k, df_j]_R) = (L, [R(\lambda^k L), \lambda^j L]) + (k \leftrightarrow j) = 0 \quad (3.6b)
\]

In our concrete cases, where \(L\)-matrices are given by (2.6b) and (2.10a), the functions \(f_k\) are related to the invariants \(I_j\) (2.7b), (2.11b) by the formula:

\[
f_k = \sum_j \lambda^k_j I_j \quad (3.7a)
\]

We can summarize the previous result by the following Theorem:

**Theorem 1.**
The Hamiltonian flows of the functions \(f_k\) correspond to completely integrable continuous-time Hamiltonian systems, endowed with the Lax representation:

\[
\frac{\partial L}{\partial t_k} = [L, R(\lambda^k L)]
\]

The proof of the above Theorem is straightforward. The complete integrability follows from (3.6b). As for the Lax representations, we have, by definition:

\[
\frac{\partial L}{\partial t_k} := \{L, f_k\} \quad (3.7b)
\]

On the other hand, for any basis element \(\sigma^l\), and for any integer \(j\), it holds:

\[
\{(\sigma^l, \lambda^j L), f_k\} = (\lambda^j \sigma^l, [L, R(\lambda^k L)])
\]

namely:

\[
tr \sigma^l \frac{\partial (res \lambda^j L)}{\partial t_k} = tr \sigma^l (res \lambda^j [L, R(\lambda^k L)])
\]

and thus

\[
res \lambda^j (\frac{\partial L}{\partial t_k} - [L, R(\lambda^k L)]) = 0
\]

whence the result. ⊳

It is worth to notice that there is an intimate relation between an integrable map and the Hamiltonian flows of its invariant functions; namely it holds the following Proposition:

**Proposition I**
An integrable map is a Backlund transformations for the Hamiltonian flows of its invariants.
The proof of the above assertion is trivial: let us denote by \( x \) a point in the phase space \( M = (\mathbb{R}^{2N}, \omega) \), and by \( x' = \Phi(x) \) a symplectic map possessing the \( N \) invariant functions \( F_j \), in involution with respect to the Poisson bracket induced by \( \omega \).

Let:

\[
K^{(j)} = \frac{\partial x}{\partial t_j} = [x, F_j]|_x
\]

then:

\[
\frac{\partial x'}{\partial t_j} = \dot{\Phi}(x) \cdot \frac{\partial x}{\partial t_j} = \dot{\Phi}(x) \cdot [x, F_j(x)] = \{\Phi(x), F_j(x)\};
\]

but \( F_j \) is an invariant function for \( \Phi \), and thus:

\[
\frac{\partial x'}{\partial t_j} = \{\Phi(x), F_j(\Phi(x))\}
\]

and finally, since \( \Phi \) is symplectic:

\[
\frac{\partial x'}{\partial t_j} = [x, F_j]|_{x'=\Phi(x)}
\]

i.e. \( \Phi \) maps solutions into solutions. ♦

We now proceed to the explicit evaluation of the \( r \) matrices.

1) DG system.

By direct calculation, we have:

\[
\{L_3(\lambda), L_3(\mu)\} = 0
\]

\[
\{L_3(\lambda), L_\pm(\mu)\} = \pm \frac{2}{\lambda - \mu} [L_\pm(\lambda) - L_\pm(\mu)]
\]

\[
\{L_\pm(\lambda), L_\pm(\mu)\} = -\frac{1}{\sqrt{<p,q>}} [L_\pm(\lambda) - L_\pm(\mu)]
\]

\[
\{L_\pm(\lambda), L_\mp(\mu)\} = \pm \frac{4}{\lambda - \mu} [L_3(\lambda) - L_\pm(\mu)] + \frac{1}{\sqrt{<p,q>}} [L_\pm(\lambda) - L_\mp(\mu)]
\]

(3.8)

The above formulas already show the appearance of the “dynamical term” \( \frac{1}{\sqrt{<p,q>}} \). They entail the following expression for the Poisson bracket \( \{L_1, L_2\} \):

\[
\{L_1(\lambda), L_2(\mu)\} =
\]
\[ = -\frac{1}{\lambda - \mu} \{\Pi, L_1(\lambda) + L_2(\mu)\} + \frac{1}{2\sqrt{<p, q>}} ([\sigma^3 \otimes (\sigma^- - \sigma^+), L_1(\lambda)] - [(\sigma^- - \sigma^+) \otimes \sigma^3, L_2(\mu)]) \]

(3.9)

where \( \Pi \) is the usual permutation operator:

\[ \Pi = \sum_i \sigma^i \otimes \sigma^i \]

Comparing with the general formula (3.1), we have the final expression for the \( r \)-matrix:

\[ r_{12}(\lambda, \mu) = -\frac{1}{\lambda - \mu} \Pi + \frac{1}{2\sqrt{<p, q>}} \sigma^3 \otimes (\sigma^- - \sigma^+) \]

(3.10)

To derive the endomorphism \( R(X) \) (3.5c) that takes part in the Poisson bracket (3.5a), first of all we recall [10] that the term \( \frac{1}{\lambda - \mu} \Pi \) dualizes into the difference of the projectors \( \mathcal{P}_+ \) and \( \mathcal{P}_- \) on positive and (strictly) negative \( \lambda \)-powers:

\[ \mathcal{P}_+(X) = \sum_{k \geq 0} X_k \lambda^k \]
\[ \mathcal{P}_-(X) = \sum_{k < 0} X_k \lambda^k \]

As for the dynamical term, we note that:

\[ res \lambda^{-1} L = \begin{pmatrix} 0 & \sqrt{<p, q>} \
-\sqrt{<p, q>} & 0 \end{pmatrix} \]

whence it follows:

\[ \sqrt{<p, q>} = res \frac{1}{2} \lambda^{-1} (L_+ - L_-) = +\frac{1}{2}(L, \lambda^{-1}(\sigma^- - \sigma^+)) \]

so that:

\[ R(X) = X_- - X_+ + \frac{1}{(L, \lambda^{-1}(\sigma^- - \sigma^+))} \sigma_3(\sigma^- - \sigma^+, X) \]

(3.11)

It might be of some interest looking at the role of the dynamical part of the \( r \)-matrix in the Jacobi identity, that we write down both in the tensor picture and in the dual picture:

\[ [L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}]] + [L_1, \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cyclic perm.} = 0 \]

(3.12a)

\[ (L, [X, B(Y, Z) + \{(L, Y), R(Z)\} - \{(L, Z), R(Y)\}]) + \text{cyclic perm.} = 0 \]

(3.12b)
In (3.12a) all quantities are understood to belong to $G \otimes G \otimes G$ and, as usual, the subscript denotes the space which the corresponding tensor acts on nontrivially:

$$L_1 = L \otimes 1 \otimes 1, \quad r_{12} = \sum_{jk} r_{jk} e^i \otimes e^k \otimes 1, \text{etc.}$$

In (3.12b), we have shortly denoted by $\mathcal{B}(\cdot, \cdot)$ the usual Yang-Baxter term, namely:

$$\mathcal{B}(X, Y) = [R(X), R(Y)] - R([X, Y]_R)$$

For the sake of simplicity, we focus our attention on eq.(3.12a), and write for a moment:

$$r_{12} = r^{(c)}_{12} + r^{(d)}_{12}$$

where $r^{(c)}_{12}$ stands for the constant part $-\frac{1}{\lambda - \mu} \Pi$ and $r^{(d)}_{12}$ stands for the dynamical part

$$\frac{1}{2\sqrt{\langle p, q \rangle}} \sigma^3 \otimes (\sigma^- - \sigma^+).$$

We see the following:

(i) As is well known $r^{(c)}$ satisfies the Yang-Baxter equation:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0$$

(ii) The mixed terms, containing both $r^{(c)}$ and $r^{(d)}$ vanish.

(iii) The dynamical part $r^{(d)}$ yield both quadratic and cubic terms in $\frac{1}{\sqrt{\langle p, q \rangle}}$. The quadratic terms in the first commutator of (3.12a) cancel with the quadratic terms in the second commutator, while the cubic terms, appearing just in the second commutator, cancel among themselves due to cyclic permutations.

We point out that property (ii) entails that the Jacobi identity equation (3.12a) splits into two equations, involving separately $r^{(c)}$ and $r^{(d)}$, which are both satisfied. In terms of Poisson brackets, this means that for the DG system the Poisson bracket (3.1) (or (3.5a)) is actually the sum of two compatible Poisson brackets, generated by $r^{(c)}$ and $r^{(d)}$ respectively.

(2) DN system

The Poisson brackets between the elements of the Lax matrix now read:

$$\{L_3(\lambda), L_3(\mu)\} = 0$$

$$\{L_3(\lambda), L_{\pm}(\mu)\} = \pm \frac{2}{\lambda - \mu} [L_\pm(\lambda) - L_{\pm}(\mu)] + \frac{1}{\| p \|} L_{\pm}(\mu)(L_\pm(\lambda) - L_{\mp}(\lambda));$$

$$\{L_{\pm}(\lambda), L_{\pm}(\mu)\} = \frac{1}{\| p \|} [L_{\pm}(\lambda) - L_{\pm}(\mu)] + \frac{2}{\| p \|} (L_3(\lambda)L_{\pm}(\mu) - L_3(\mu)L_{\pm}(\lambda));$$
\[ \left\{ L_{\pm}(\lambda), L_{\mp}(\mu) \right\} = \pm \frac{4}{\lambda - \mu} [L_3(\lambda) - L_3(\mu)] + \frac{1}{\|p\|} [L_{\pm}(\mu) - L_{\mp}(\lambda)] + \frac{2}{\|p\|} [L_{\mp}(\lambda)L_3(\mu) - L_{\pm}(\mu)L_3(\lambda)] \] (3.13)

Comparing with (3.7a), we notice that linear terms in \( L \) are the same, up to the substitution \( \sqrt{<p,q>} \leftrightarrow \|p\| \); consequently, the \( r \)-matrix will be of the form:

\[ r_{12} = -\frac{\Pi}{\lambda - \mu} + \frac{1}{2\|p\|} \sigma^3 \otimes (\sigma^- - \sigma^+) + r_{12}^{(d_2)} \] (3.14a)

where \( r_{12}^{(d_2)} \) takes care of quadratic terms in \( L \) in (3.13). Skipping out the computational details, we report the result:

\[ r_{12}^{(d_2)}(\lambda, \mu) = \frac{1}{2\|p\|} (\sigma^+ - \sigma^-) \otimes [\sigma^3, L(\mu)] \] (3.14b)

and correspondingly:

\[ r_{21}^{(d_2)}(\lambda, \mu) = \frac{1}{2\|p\|} [\sigma^3, L(\lambda)] \otimes (\sigma^+ + \sigma^-) \] (3.14b)

It is also possible to write the dynamical term in an invariant form. In fact we have

\[ \|p\| = -\text{res tr } L\sigma^- = -(L, \sigma^-) \]

Consequently, the \( r \)-matrix (3.14) dualizes to:

\[ R(X) = X_- - X_+ - \frac{1}{2(L, \sigma^-)} \sigma^3(\sigma^- - \sigma^+, X) - \frac{1}{2(L, \sigma^-)}(\sigma^+ + \sigma^-)([\sigma^3, L], X) \] (3.15)

The check of the Jacobi identity is now considerably more involved than in the DG case. However, the same remarkable phenomenon occurs: namely, the Jacobi identity equations decouples in three equations individually satisfied by \( r^{(c)} \), \( r^{(d_1)} \), \( r^{(d_2)} \), thus entailing that the Poisson structure characterizing DN is actually the sum of three compatible Poisson structures engendered by \( r^{(c)} \), \( r^{(d_1)} \), \( r^{(d_2)} \). It is perhaps worthwhile to point out that the dynamical nature of \( r_{12}^{(d_2)} \), namely the presence of the \( \|p\|^{-1} \) factor is indeed essential: the tensor \([\sigma^3, L(\lambda)] \otimes (\sigma^+ + \sigma^-) \) alone is not an \( r \)-matrix.

4. Separability

In this Section we will show that the integrable maps denoted as DG and DN, as well as the integrable continuous flows (3.7b) associated with them enjoy the classical property of separability. Namely, we will show that there exists a set of canonical coordinates \((\mu_j, \pi_j)_{j=1}^N\) such that:
\[
\pi_j = \frac{\partial W}{\partial \mu_j} \tag{4.1}
\]

where the function \( W_j \) depend just upon the variable \( \mu_j \) and the integrals of motion:

\[
W_j = W_j(\mu_j, \{ I_k \}_{k=1}^N)
\]

Paraphrasing, with slight modifications, the derivation presented in [11] we will introduce the variables \( \mu_j \) as the zeroes of suitable polynomials taking part in the Lax matrix.

To this aim we notice that the \( L_- \) element of the Lax matrices (2.6b), (2.10a) can be written as:

\[
L_- = \alpha \frac{P(\lambda)}{Q(\lambda)} \tag{4.2a}
\]

where

\[
\alpha = -\sqrt{< p, q >} \quad (DG); \quad \alpha = \|p\| \quad (DN) \tag{4.2b}
\]

and:

\[
Q(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_j) \tag{4.3a}
\]

\( P(\lambda) \) being a monic polynomial of degree \( N \) for the DG system, of degree \( N - 1 \) for the DN system, whose real zeroes, which are functions of the canonical variables \( q, p \), we denote by \( \mu_j \):

\[
P(\lambda) = \prod (\lambda - \mu_j) \tag{4.3b}
\]

Following [11] we set:

\[
\pi_j = L_3(\mu_j) \tag{4.4}
\]

i.e.:

\[
\pi_j = -\left( \frac{\mu_j}{2} + \sum_{k=1}^{N} \frac{p_k q_k}{\mu_j - \lambda_k} \right) \quad j = 1, \cdots, N \quad (DG)
\]

\[
\pi_j = \frac{1}{2} + \sum_{k=1}^{N} \frac{p_k q_k}{\mu_j - \lambda_k} \quad j = 1, \cdots, N - 1 \quad (DN).
\]

Then, by direct calculation, starting from formulas (3.8), (3.9), and paraphrasing again [11] we can establish the following theorem:

**Theorem 2:** The variables \( \pi_j, \mu_j \) are canonically conjugated. Moreover, they are separation variables.
The separation equations are obviously given by:

$$\pi_j^2 = \Delta(\mu_j)$$  \hspace{1cm} (4.5)

and consequently the functions $W_j$ of formula (4.1) read:

$$W_j = \int_{\mu_j}^{\mu} d\lambda \sqrt{\Delta(\lambda)}$$  \hspace{1cm} (4.6)

We shall now look at the continuous flows of the invariants $f_k$ taking part in Theorem 1. From the Lax equation:

$$\frac{\partial L}{\partial t_k} = [L, R(\lambda^k L)]$$  \hspace{1cm} (4.7)

we deduce:

$$\frac{\partial L}{\partial t_k} = 2L - M(k) - 2M(k) L_3$$

whence:

$$\frac{\partial L}{\partial t_k} |_{\lambda=\mu_r} = -2M(k)(\mu_r)\pi_r$$  \hspace{1cm} (4.8)

It is easily seen that, both in the DG and in the DN case, only the constant part of the r-matrix contributes to $M(k)(\mu_r)$. Indeed, a direct and simple calculation leads to the formula (here and in the following a superscript dot denotes differentiation: e.g. $\dot{Q}(\lambda_k) \equiv \frac{\partial Q}{\partial \lambda} |_{\lambda=\lambda_s}$):

$$M(k)(\mu_r) = 2\alpha \sum_s \frac{\lambda_s^k}{\mu_r - \lambda_s} \frac{P(\lambda_s)}{Q(\lambda_s)}$$  \hspace{1cm} (4.9)

taking into account that:

$$\frac{\partial L}{\partial t_k} |_{\lambda=\mu_r} = -\alpha \frac{\dot{P}(\mu_r)}{Q(\mu_r)} \frac{\partial \mu_r}{\partial t_k}$$

we get from (4.8):

$$\frac{\partial \mu_r}{\partial t_k} = \frac{4Q(\mu_r)}{\dot{P}(\mu_r)} \sum_s \frac{\lambda_s^k}{\mu_r - \lambda_s} \frac{P(\lambda_s)}{Q(\lambda_s)}$$  \hspace{1cm} (4.10)

Recalling the so-called Lagrange interpolation formula, which is a plane consequence of the residues theorem:

$$\sum_r \frac{\mu_r^l}{\dot{P}(\mu_r)(\mu_r - \lambda_s)} = -\frac{\lambda_l}{P(\lambda_s)} \quad (l \leq \deg(P))$$  \hspace{1cm} (4.11)

we can write:
\[
\sum_r \frac{\mu_r}{Q(\mu_r)\sqrt{\Delta(\mu_r)}} \frac{\partial \mu_r^l}{\partial t_k} = - \sum_s \frac{\lambda_s^{k+l}}{Q(\lambda_s)}
\]  

(4.12)

Eq. (4.12) can be immediately integrated, yielding to the Jacobi inversion problem (solvable in terms of Riemann \( \Theta \) functions):

\[
\sum_r \int_{\mu_r(t_k)}^{\mu_r(t_k^{(0)})} d\lambda \frac{\lambda^l}{Q(\lambda)\sqrt{\Delta(\lambda)}} = -(t_k - t_k^{(0)}) \sum_s \frac{\lambda_s^{k+l}}{Q(\lambda_s)}
\]

(4.13)

In particular, for the first nontrivial flows \((k = 0 \text{ in the DG case and } k = 1 \text{ in the DN case})\) we get:

\[
\sum_r \int_{\mu_r(t)}^{\mu_r(t^{(0)})} d\lambda \frac{\lambda^l}{Q(\lambda)\sqrt{\Delta(\lambda)}} = -(t - t^{(0)}) \times \left( \frac{\delta_{l,N-2}}{\delta_{l,N-1}} \right) (DN)
\]

(4.14)

The evaluation of the discrete-time evolution of the separation variables \( \mu_j \) is considerably more involved. The basic starting point is now the discrete Lax equation (2.6a), that entails both in the DG and in the DN case:

\[
L'_3(\mu_j) = -\pi_j
\]

(4.15a)

to be of course complemented by:

\[
L'_3(\mu'_j) = \pi'_j
\]

(4.14b)

Eq. (4.15a) allows one to express the quantities \{p'_k q'_k\} in terms of \{\mu_j\}, \{\pi_j\}; by inserting such expression into (4.15b), one gets a set of formulas relating \{\mu'_j\}, \{\pi'_j\} to \{\mu_j\}, \{\pi_j\} or, in other words, relating \{\mu'_j\} to \{\mu_j\} through the map invariants. The explicit calculation are rather cumbersome but on the other hand straightforward, relying upon repeated applications of the residues theorem. Omitting the details, we just report the final result:

\[
\mu'_j - (M - \Lambda) = \epsilon - \frac{2Q(\mu'_j)\pi'_j}{P(\mu'_j)} - 2 \sum_s \frac{1}{\mu'_j - \mu_s} \frac{Q(\mu_s)\pi_s}{P(\mu_s)}
\]

(4.16)

where:

\( \epsilon = 1 \) and \( j = 1, \cdots, N - 1 \) in the Neumann case
\( \epsilon = 0 \) and \( j = 1, \cdots, N \) in the Garnier case
\( M = \sum_k \mu_k \) ; \( \Lambda = \sum_k \lambda_k \)

By recalling that \( \pi_j = \sqrt{\Delta(\mu_j)} \) and getting rid of the square roots, one gets an algebraic equation of degree \( 2N(2(N - 1)) \) in the DG (DN) case, as the coefficients of powers \( 2N \) \( 2, 2N + 1(2N, 2N - 1) \) identically vanish. Of course, only half of the roots of that algebraic
equation will also solve (4.16), where a definite sign for the $\pi_j$ has to be chosen. Once the $\mu_j$ have been found, the frequencies $\nu_j$ of the maps can be evaluated by hyperelliptic integrals. In fact, denoting by $\theta_j$ the variable conjugated to $I_j$, given by:

$$
\theta_j = \sum_k \int^{\mu_k} \frac{\partial \pi(\mu)}{\partial I_j} d\mu = \frac{1}{2} \sum_k \int^{\mu_k} \frac{1}{(\lambda_j - \mu) \sqrt{\Delta(\mu)}} d\mu
$$

we have:

$$
\nu_j(I_1, \cdots, I_N) = \frac{1}{2} \sum_k \int^{\mu_k'} \frac{1}{(\lambda_j - \mu) \sqrt{\Delta(\mu)}} d\mu
$$

and the maps linearize to:

$$
\theta_j(n) = \nu_j n + \theta_j(0)
$$

**Concluding Remarks**

We have shown that the $r$-matrix approach to integrable systems can be successfully applied to the discrete time case, starting from the Lax representation. It remains an open question whether one can construct different Lax pairs, possibly in terms of $N \times N$ matrices with a polynomial dependence upon the spectral parameter, leading to constant $r$-matrices. Another problem worth to be looked at is the role of the $r$-matrix as far as the discrete-time dynamics is concerned; in fact, since to an integrable map one can associate a (family of) interpolating Hamiltonian flow(s), engendered by a (family of) Hamilton function(s) functionally dependent on the invariants, one has to expect that the matrix $A$ entering in the discrete Lax representation be expressible as well as a function of the matrices $M^{(k)}$ that define the compatible continuous dynamics and are constructed through the $r$-matrix. Finally, a few words about quantisation: without going here into the stimulating problem of quantising symplectic maps, we just mention two points: i) once symmetrized with respect to $p$ and $q$ the classical invariants (2.7b) (2.11b) have a simple quantum version to commuting (formally) self-adjoint operators; ii) as illustrated in [11] in the continuous case, the separation equations can be quantised to a set of decoupled one-dimensional multiparametric ordinary differential equations of Schroedinger type.

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