FALTINGS MODULAR HEIGHT AND
SELF-INTERSECTION OF DUALIZING SHEAF

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0. Introduction

Let $K$ be a number field, $O_K$ the ring of integers of $K$ and $X$ a stable curve over $O_K$ of genus $g \geq 2$. In this note, we will prove a strict inequality

$$\frac{(\hat{c}_1(\omega_{X/S}, \Phi_{can})^2)}{[K : \mathbb{Q}]} > \frac{4(g - 1)}{g} \text{Height}_{Fal}(J(X_K)),$$

where $\omega_{X/S}$ is the dualizing sheaf of $X$ over $S = \text{Spec}(O_K)$, $\Phi_{can}$ is the canonical Hermitian metric of $\omega_{X/S}$ and $\text{Height}_{Fal}(J(X_K))$ is the Faltings modular height of the Jacobian of $X_K$ (cf. Corollary 2.3). As corollary, for any constant $A$, the set of all stable curves $X$ over $O_K$ with

$$\frac{(\hat{c}_1(\omega_{X/O_K}, \Phi_{can})^2)}{[K : \mathbb{Q}]} \leq A$$

is finite under the following equivalence (cf. Theorem 3.1). For stable curves $X$ and $Y$ over $O_K$, $X$ is equivalent to $Y$ if $X \otimes_{O_K} O_{K'} \simeq Y \otimes_{O_K} O_{K'}$ for some finite extension field $K'$ of $K$.

In §1, we will consider semistability of the kernel of $H^0(C, L) \otimes O_C \rightarrow L$, which gives a generalization of [PR]. In §2, an inequality of self-intersection and height will be treated. Finally, §3 is devoted to finiteness of stable arithmetic surfaces with bounded self-intersections of dualizing sheaves.

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THIS IS A TENTATIVE VERSION.
1. Semistability of the Kernel of $H^0(C, L) \otimes \mathcal{O}_C \to L$

Throughout this section, we will fix an algebraically closed field $k$. Let $C$ be a smooth projective curve over $k$. For a non-zero torsion free sheaf $E$ on $C$, an average degree $\mu(E)$ of $E$ is defined by $\mu(E) = \deg(E)/\text{rk} E$. $E$ is said to be stable (resp. semistable) if $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$) for all non-zero proper subsheaves $F$ of $E$.

Let $L$ a line bundle on $C$. We set

$$E(L) = \ker(H^0(C, L) \otimes \mathcal{O}_C \to L) \quad \text{and} \quad M(L) = \text{Im}(H^0(C, L) \otimes \mathcal{O}_C \to L).$$

Clearly, $h^0(C, L) = h^0(C, M(L))$. If $h^0(C, L) \geq 2$, then $E(L) \neq 0$ and

$$\mu(E(L)) = -\frac{\deg(M(L))}{h^0(C, L) - 1} = -\frac{\deg(M(L))}{h^0(C, M(L)) - 1}.$$

Moreover, $\mu(E(L)) \geq -2$ if and only if $h^0(C, L) \geq \frac{1}{2} \deg(M(L)) + 1$. The main purpose of this section is to give a generalization of A. Paranjape and S. Ramanan’s result [PR].

**Theorem 1.1.** Let $C$ be a smooth projective curve of genus $g \geq 1$ over $k$ and $L$ a line bundle on $C$ such that $h^0(C, L) \geq 2$ and $\mu(E(L)) \geq -2$. Then, $E(L)$ is semistable. Moreover, we have the following.

1. If $\deg L \geq 2g + 1$, $E(L)$ is stable.
2. In the case where $\deg L = 2g$ and $C$ is not hyperelliptic, $E(L)$ is stable if and only if $h^0(C, L \otimes \omega_C^{-1}) = 0$.
3. $E(\omega_C)$ is stable if and only if $C$ is not hyperelliptic, where $\omega_C$ is the dualizing sheaf of $C$ over $k$.

**Proof.** First of all, we will prepare several lemmas.

**Lemma 1.2.** Let $X$ be a $d$-dimensional projective variety over $k$ and $E$ a vector bundle of rank $r$ on $X$. If $E$ is generated by global sections, then there is a subvector space $V$ of $H^0(X, E)$ over $k$ such that $\dim_k V = d + r$ and $V \otimes \mathcal{O}_X \to E$ is surjective.

**Proof.** First, we consider a case where $r = 1$. Let $\varphi : X \to \mathbb{P}(H^0(X, E))$ the morphism induced by the complete linear system $|E|$. Since $\dim \varphi(X) \leq d$, there is a linear subspace $T$ of $\mathbb{P}(H^0(X, E))$ such that $\text{codim} T = d + 1$ and $\varphi(X) \cap T = \emptyset$. Let $\{s_0, s_1, \ldots, s_d\}$ be a basis of $H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ and $\pi : \varphi(X) \to \mathbb{P}^d$ the morphism induced by the projection $\mathbb{P}(H^0(X, E)) \setminus T \to \mathbb{P}^d$. Here we consider a subvector space $V$ of $H^0(X, E)$ generated by

$$(\pi \circ \varphi)^*(s_0), (\pi \circ \varphi)^*(s_1), \ldots, (\pi \circ \varphi)^*(s_d).$$

Then, it is easy to see that $V$ is a desired vector subspace.

Next, we consider a general case. Let $f : Y = \mathbb{P}(E) \to X$ be the projective bundle of $E$ and $\mathcal{O}_Y(1)$ the tautological line bundle of $Y$. Since $E$ is generated by global sections, so is $\mathcal{O}_Y(1)$. Thus, by the previous observation, there is a subvector space $V$
of $H^0(Y,\mathcal{O}_Y(1))$ such that $\dim V = d + r$ and $V \otimes \mathcal{O}_Y \to \mathcal{O}_Y(1)$ is surjective. Since $H^0(Y,\mathcal{O}_Y(1)) \cong H^0(X, E)$, we can view $V$ as a subvector space of $H^0(X, E)$. Pick up a point $x \in X$. Let us consider the natural homomorphism $V \to H^0(Y_x, \mathcal{O}_{Y_x}(1))$. This is surjective because $V \otimes \mathcal{O}_{Y_x} \to \mathcal{O}_{Y_x}(1)$ is surjective and $W \otimes \mathcal{O}_{Y_x} \to \mathcal{O}_{Y_x}(1)$ is not surjective for every proper subvector space $W$ of $H^0(Y_x, \mathcal{O}_{Y_x}(1))$. Therefore, $V \to E_x$ is surjective. Thus, we get our assertion. □

**Lemma 1.3.** Let $C$ be a smooth projective curve of genus $g$ over $k$ and $W$ a non-zero vector bundle on $C$ such that $h^0(C, W) = 0$ and $W^*$ is generated by global sections. Then, we have the following.

1. If $h^1(C, (\det W)^{-1}) = 0$, then $\deg W \leq -\mathrm{rk} W - g$.
2. If $h^1(C, (\det W)^{-1}) \neq 0$, then $\deg W \leq -2 \mathrm{rk} W$. Moreover, equality holds if and only if either $W \cong E(\omega_C)$, or $C$ is hyperelliptic and $W \cong E(\mathcal{O}_C(mg_2^1))$ for some $1 \leq m \leq g - 1$, where $g_2^1$ is the hyperelliptic divisor.

**Proof.** By Lemma 1.2, there is a surjective homomorphism $\mathcal{O}_C^{rk W+1} \to W^*$. Clearly, the kernel of it is $\det W$. Thus, we get

\begin{equation}
0 \to W \to \mathcal{O}_C^{rk W+1} \to (\det W)^{-1} \to 0.
\end{equation}

Therefore, since $h^0(C, W) = 0$, we have

\begin{equation}
\mathrm{rk} W + 1 \leq h^0(C, (\det W)^{-1}).
\end{equation}

If $h^1(C, (\det W)^{-1}) = 0$, then, by Riemann-Roch theorem,

\[ h^0(C, (\det W)^{-1}) = -\deg W + 1 - g. \]

Thus, by (1.3.2), we obtain (1).

If $h^1(C, (\det W)^{-1}) \neq 0$, then $(\det W)^{-1}$ is special. Thus, by Clifford’s theorem (cf. Chapter IV, Theorem 5.4 of [Ha]),

\begin{equation}
\frac{-\deg W}{2} + 1.
\end{equation}

Therefore, by (1.3.2) and (1.3.3), we have $\deg W \leq -2 \mathrm{rk} W$.

If $\deg W = -2 \mathrm{rk} W$ holds, then we have

\[ \mathrm{rk} W + 1 = h^0(C, (\det W)^{-1}) \quad \text{and} \quad h^0(C, (\det W)^{-1}) = \frac{-\deg W}{2} + 1. \]

By equality conditions of Clifford’s theorem, we have either $\det W = \mathcal{O}_C$, $\det W = \omega_C^{-1}$, or $C$ is hyperelliptic and $\det W = \mathcal{O}_C(-mg_2^1)$ for some $1 \leq m \leq g - 1$. The case $\det W = \mathcal{O}_C$ is impossible because $\mathrm{rk} W = 0$ in this case. If $\det W = \omega_C^{-1}$, then $\mathrm{rk} W = g - 1$. Therefore, by the exact sequence (1.3.1), we have $W \cong E(\omega_C)$. By the same way, if $C$ is hyperelliptic and $\det W = \mathcal{O}_C(-mg_2^1)$ for some $1 \leq m \leq g - 1$, then $W \cong E(\mathcal{O}_C(mg_2^1))$.

Conversely, if $W \cong E(\omega_C)$, or $W \cong E(\mathcal{O}_C(mg_2^1))$, then it is easy to see that $\deg W = -2 \mathrm{rk} W$. □
Lemma 1.4. Let $C$ be a smooth projective curve of genus $g$ over $k$ and $L$ a line bundle on $C$. If $L$ is generated by global sections and $h^0(C, L) \geq 2$, then

$$h^0(C, E(L)^*) \geq h^0(C, L).$$

Moreover, equality holds if and only if

$$H^0(C, L) \otimes H^0(C, \omega_C) \to H^0(C, L \otimes \omega_C)$$

is surjective. In particular, if $g \geq 2$ and $C$ is not hyperelliptic, then $h^0(C, E(\omega_C)^*) = g$.

Proof. Let us consider the exact sequence:

$$0 \to L^{-1} \to H^0(C, L)^* \otimes \mathcal{O}_C \to E(L)^* \to 0.$$

Thus, we have

$$h^0(C, E(L)^*) \geq h^0(C, L).$$

Further, equality holds if and only if

$$H^1(C, L^{-1}) \to H^1(C, H^0(C, L)^* \otimes \mathcal{O}_C)$$

is injective. Considering Serre's duality, the injectivity is equivalent to the surjectivity of

$$H^0(C, H^0(C, L) \otimes \omega_C) \to H^0(C, L \otimes \omega_C).$$

Thus, we have our lemma because $H^0(C, H^0(C, L) \otimes \omega_C) \simeq H^0(C, L) \otimes H^0(C, \omega_C)$. □

Let us start the proof of Theorem 1.1. Let $W$ be a proper non-zero subvector bundle of $E(L)$. Then, clearly, $h^0(C, W) = 0$ because $h^0(C, E(L)) = 0$. Moreover, since $M(L)$ is locally free, $E(L)^*$ is a quotient of $H^0(C, L)^* \otimes \mathcal{O}_C$. Thus so is $W^*$. Therefore, $W^*$ is generated by global sections. So we can apply Lemma 1.3. If $h^1(C, (\det W)^{-1}) = 0$, then we have $\mu(W) < \mu(E(L))$ as follows.

$$\mu(W) \leq -1 - \frac{g}{\text{rk} W} < -1 - \frac{g}{\text{rk} E(L)} = -\frac{\deg(M(L)) - h^1(C, M(L))}{\text{rk} E(L)} \leq \mu(E(L)).$$

If $h^1(C, (\det W)^{-1}) \neq 0$, then $\mu(W) \leq -2$ by Lemma 1.3. Thus, $E(L)$ is semistable.

Next, we will consider stability of $E(L)$ for each case (1) – (3).

(1) In this case, $\mu(E(L)) > -2$. Thus, stability is trivial.

(2) First, we assume that $E(L)$ is stable. If $h^0(C, L \otimes \omega_C^{-1}) \neq 0$, then $\omega_C$ is a subsheaf of $L$. Thus, $E(\omega_C)$ is a subsheaf of $E(L)$. On the other hand, $\mu(E(L)) = \mu(E(\omega_C)) = -2$. Thus, $E(L)$ is not stable. This is a contradiction.
Next we assume that \( h^0(C, L \otimes \omega_C^{-1}) = 0 \). If \( E(L) \) is not stable, by Lemma 1.3, there is a subbundle \( W \) of \( E(L) \) such that \( W \) is isomorphic to \( E(\omega_C) \). Thus, we have an exact sequence:

\[
0 \to L \otimes \omega_C^{-1} \to E(L)^* \to E(\omega_C)^* \to 0.
\]

By Lemma 1.4, \( h^0(C, E(L)^*) \geq g + 1 \) and \( h^0(C, E(\omega_C)^*) = g \). Thus, \( h^0(C, L \otimes \omega_C^{-1}) \neq 0 \). This is a contradiction.

(3) By Lemma 1.3, it is easy to see that if \( C \) is not hyperelliptic, then \( E(\omega_C) \) is stable. Here, we assume that \( C \) is hyperelliptic. Then, \( \omega_C \simeq \mathcal{O}_C((g - 1)g_2^1) \). Thus, \( \omega_C \) has a subsheaf \( \mathcal{O}_C(g_2^1) \), which implies that \( E(\omega_C) \) has a subsheaf \( E(g_2^1) \). On the other hand, \( \mu(E(\omega_C)) = \mu(E(g_2^1)) = -2 \). Thus, \( E(\omega_C) \) is not stable. □

2. Inequality of self-intersection and height

Let \( K \) be a number field and \( \mathcal{O}_K \) the ring of integers of \( K \). Let us consider a pair \((V, h)\) of an \( \mathcal{O}_K \)-module \( V \) of finite rank and Hermitian metric \( h_\sigma \) on \( V_\sigma \) for each \( \sigma \in K_\infty \). We define \( L^2 \)-degree \( \text{deg}_{L^2}(V, h) \) of \((V, h)\) by

\[
\text{deg}_{L^2}(V, h) = \log \# \left( \frac{V}{\mathcal{O}_K x_1 + \cdots + \mathcal{O}_K x_t} \right) - \frac{1}{2} \sum_{\sigma \in K_\infty} \log \det(h_\sigma(x_i, x_j)),
\]

where \( x_1, \ldots, x_t \in V \) and \( \{x_1, \ldots, x_t\} \) is a basis of \( V \otimes K \). Using the Hasse product formula, it is easily checked that \( \text{deg}_{L^2}(V, h) \) does not depend on the choice of \( \{x_1, \ldots, x_t\} \).

The purpose of this section is to prove the following theorem, which is a variant of Theorem II in [Bo] and gives a refine result in special cases.

**Theorem 2.1.** Let \( f : X \to S = \text{Spec}(\mathcal{O}_K) \) be a regular arithmetic surface of genus \( g \geq 1 \) and \( L \) a line bundle on \( X \) such that \( L \) is \( f \)-nef and \( \text{deg}(L_Q) > 0 \). Let \( V \) be a \( \mathcal{O}_K \)-submodule of \( H^0(X, L) \) and \( h \) a Hermitian metric of \( V \) such that the natural homomorphism \( V_Q \otimes \mathcal{O}_{X_Q} \to L_Q \) is surjective. Let \( h_L \) the quotient metric of \( L \) induced by \( h \) via the surjective homomorphism \( V_\sigma \otimes \mathcal{O}_{X_\sigma} \to L_\sigma \) on each infinite fiber \( X_\sigma \). If \( \text{ker}(V_Q \otimes \mathcal{O}_{X_Q} \to L_Q) \) is semistable, then

\[
\frac{1}{2} \left( \frac{\text{deg}_{L^2}(V, h)}{\text{deg}(L_Q)} \right) > \frac{\text{deg}_{L^2}(V, h)}{\text{rk} V}.
\]

**Proof.** Let \( Q \) be the image of \( V \otimes \mathcal{O}_X \to L \) and \( S \) the kernel of \( V \otimes \mathcal{O}_X \to L \). Then, \( Q \) is a torsion free sheaf of rank 1 and \( S \) is a torsion free sheaf of rank \( \text{rk} V - 1 \). Using the natural metric \( f^*(h) \) of \( V \otimes \mathcal{O}_X \), we can give the quotient metric \( h_Q \) to \( Q \) and the submetric \( h_S \) to \( S \). Clearly, on each infinite fiber, \( h_Q \) coincides with \( h_L \) of \( L \).

Here, we calculate \( \widehat{c}_1(S, h_S) \) and \( \widehat{c}_2(S, h_S) \) in terms of \( \widehat{c}_1(V, h) \), \( \widehat{c}_1(Q, h_Q) \) and the extension class of \( 0 \to S \to V \otimes \mathcal{O}_X \to Q \to 0 \). First of all, we get

\[
\widehat{c}_1(S, h_S) = f^*(\widehat{c}_1(V, h)) - \widehat{c}_1(Q, h_Q).
\]
We set
\[(2.1.2) \quad \rho = \hat{c}_2(f^*(V, h)) - \hat{c}_2((S, h_S) \oplus (Q, h_Q)).\]

Then, by Proposition 7.3 of [Mo], we have that \(\rho \geq 0\), and \(\rho = 0\) if and only if the exact sequence:
\[(2.1.3) \quad 0 \to (S, h_S) \to f^*(V, h) \to (Q, h_Q) \to 0\]
splits orthogonally on each infinite fiber. It follows from (2.1.2) that
\[(2.1.4) \quad \hat{c}_2(S, h_S) = -(\rho + \hat{c}_2(Q, h_Q)) - \deg(L_{\mathbb{Q}}) \deg_L^2(V, h) + \hat{c}_1(Q, h_Q)^2.\]

In our situation, (2.1.3) doesn’t split on each infinite fiber because a trivial bundle doesn’t have an ample sub-line bundle as its direct summand. So we get \(\rho > 0\). Moreover, since \(\text{rk} \, Q = 1\), we obtain \(\hat{c}_2(Q, h_Q) \geq 0\). Hence, by (2.1.4),
\[(2.1.5) \quad \hat{c}_2(S, h_S) < -\deg(L_{\mathbb{Q}}) \deg_L^2(V, h) + \hat{c}_1(Q, h_Q)^2,\]

Since \(S_{\mathbb{Q}}\) is semistable vector bundle, by virtue of Corollary 8.9 in [Mo], we obtain
\[(2.1.6) \quad (\text{rk} \, V - 2)\hat{c}_1(S, h_S)^2 \leq 2(\text{rk} \, V - 1)\hat{c}_2(S, h_S).\]

Combining (2.1.1), (2.1.5) and (2.1.6), we have
\[(2.1.7) \quad \frac{1}{2} \hat{c}_1(Q, h_Q)^2 > \frac{\deg_L^2(V, h)}{\text{rk} \, V}.\]

On the other hand, since \(Q \subseteq L\) and \(Q \otimes K = L \otimes K\), there is a vertical effective 1-cycle \(Z\) on \(X\) such that
\[\hat{c}_1(Q, h_Q) = \hat{c}_1(L, h_L) - Z.\]

Therefore,
\[\hat{c}_1(Q, h_Q)^2 = \hat{c}_1(L, h_L)^2 - 2(L \cdot Z) + Z^2.\]

Hence, since \(L\) is \(f\)-nef and \(Z^2 \leq 0\), the above implies that
\[(2.1.8) \quad \hat{c}_1(Q, h_Q)^2 \leq \hat{c}_1(L, h_L)^2.\]

Thus, by (2.1.7) and (2.1.8), we finally get our inequality. \(\square\)

By Theorem 1.1 and Theorem 2.1, we have
Corollary 2.2. Let $f : X \to S = \text{Spec}(O_K)$ be a regular arithmetic surface of genus $g \geq 1$ and $L$ a line bundle on $X$ such that $L$ is $f$-nef, $\deg(L_{\mathbb{Q}}) > 0$, $L_{\mathbb{Q}}$ is generated by global sections and $\text{rk} H^0(X, L) \geq \frac{1}{2} \deg(L_{\mathbb{Q}}) + 1$. Let $h$ be a Hermitian metric of $H^0(X, L)$ and $h_L$ the quotient metric of $L$ induced by $h$ via the surjective homomorphism $H^0(X, L) \otimes \mathcal{O}_{X, \sigma} \to L_{\sigma}$ on each infinite fiber. Then we have

$$\frac{1}{2} \deg(L_{\mathbb{Q}}) \geq \frac{\deg(L_{\mathbb{Q}})}{\text{rk} H^0(X, L)}.$$ 

Let $C$ be a compact Riemann surface of genus $g \geq 1$ and $\Omega_C$ the sheaf of holomorphic 1-forms on $C$. The natural Hermitian metric $\langle \ , \ \rangle_{\text{can}}$ of $H^0(C, \Omega_C)$ is defined by

$$\langle \alpha, \beta \rangle_{\text{can}} = \frac{\sqrt{-1}}{2} \int_C \alpha \wedge \overline{\beta}.$$ 

Since $H^0(C, \Omega_C) \otimes \mathcal{O}_C \to \Omega_C$ is surjective, the Hermitian metric $\langle \ , \ \rangle_{\text{can}}$ induces the quotient Hermitian metric of $\Omega_C$. We denote this metric by $\Phi_{\text{can}}$ and call it the canonical metric of $\Omega_C$. Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(C, \Omega_C)$ with respect to $\langle \ , \ \rangle_{\text{can}}$. Then, the Kähler metric $k_{\text{can}} = \Phi_{\text{can}}^{-1}$ is given by

$$\omega_1 \otimes \overline{\omega}_1 + \cdots + \omega_g \otimes \overline{\omega}_g.$$ 

Let $K$ be a number field, $O_K$ the ring of integers of $K$ and $S = \text{Spec}(O_K)$. Let $f : X \to S$ be an arithmetic surface of the genus $g \geq 1$ with the invertible dualizing sheaf $\omega_{X/S}$. We can give the above canonical metric to $\omega_{X/S}$ on each infinite fiber. By abuse of notation, we denote this metric by $\Phi_{\text{can}}$.

Let $A$ be an abelian variety of dimension $g$ over $K$ such that $A$ has semi-stable reduction. Let $\pi : N(A) \to S$ be the Neron model of $A$ and $\varepsilon : S \to N(A)$ the identity of the group scheme $N(A)$. Set $\omega_{A/S} = \varepsilon^*(\det(\Omega_{N(A)/S}))$. For each infinite place $\sigma \in K_\infty$, we give a Hermitian metric $\langle \ , \ \rangle_{\sigma}$ of $\omega_{A/S}$ defined by

$$\langle \alpha, \beta \rangle_{\sigma} = \left(\frac{\sqrt{-1}}{2}\right)^g \int_{A_\sigma} \alpha \wedge \overline{\beta}.$$ 

The Faltings modular height $\text{Height}_{\text{Fal}}(A)$ of $A$ is given by

$$\text{Height}_{\text{Fal}}(A) = \frac{\deg_L(\omega_{A/S}, \langle \ , \ \rangle)}{[K : \mathbb{Q}]}.$$ 

The following corollary is the main result of this note which is a refinement of the inequality (4.9) in [Bo].
Corollary 2.3. Let $K$ be a number field, $O_K$ the ring of integers of $K$ and $S = \text{Spec}(O_K)$. Let $f : X \to S$ be a stable arithmetic surface of genus $g \geq 2$. Then we have

$$\frac{(\hat{c}_1(\omega_{X/S}, \Phi_{\text{can}}))^2}{[K : \mathbb{Q}]} > \frac{4(g - 1)}{g} \text{Height}_{\text{Fal}}(J(X_K)),$$

where $J(X_K)$ is the Jacobian of $X_K$.

Proof. First of all, it is well known that

$$\deg_{L^2}(H^0(X, \omega_{X/S}), (\cdot)_{\text{nat}}) = \text{Height}_{\text{Fal}}(J(X_K)).$$

Let $\mu : Y \to X$ be a minimal resolution of $X$. Then, $\omega_{Y/S}$ is $f$-nef, $H^0(Y, \omega_{Y/S}) = H^0(X, \omega_{X/S})$, and $(\hat{c}_1(\omega_{X/S}, \Phi_{\text{can}}))^2 = (\hat{c}_1(\omega_{Y/S}, \Phi_{\text{can}}))^2$. Thus, our assertion follows from Corollary 2.2 \[\square\]

3. Finiteness of stable arithmetic surfaces with bounded self-intersections of dualizing sheaves

In this section, we will consider an application of the inequality of Corollary 2.3. Let $g$ an integer with $g \geq 2$. For a scheme $T$, we set

$$\bar{M}_g^s(T) = \{X \to T \mid X \to T \text{ is a stable curve of genus } g \text{ with smooth generic fibers}\}.$$

For $X, Y \in \bar{M}_g^s(\text{Spec}(O_K))$, we define an equivalence $X \sim Y$ by the following:

$$X \sim Y \iff X \otimes_{O_K} O_{K'} \simeq Y \otimes_{O_K} O_{K'} \text{ for some finite extension field } K' \text{ of } K.$$

For a constant $A$, we denote by $B_{g}^{\text{can}}(K, A)$ a subset of $\bar{M}_g^s(\text{Spec}(O_K))/\sim$ consisting of the classes of stable curves with

$$\frac{(\hat{c}_1(\omega_{X/S}, \Phi_{\text{can}}))^2}{[K : \mathbb{Q}]} \leq A.$$

Then, we have

Theorem 3.1. If $g \geq 2$, then $B_{g}^{\text{can}}(K, A)$ is finite for any number field $K$ and any constant $A$.

Proof. By Corollary 2.3 and the finiteness property of the Faltings modular height (cf. [Fa]), it is sufficient to prove the following lemma.
Lemma 3.2. Let $K$ be a number field and $O_K$ the ring of integers of $K$. Let $X$ and $X'$ be stable curves over $O_K$ of genus $g \geq 2$. If $X_K$ is isomorphic to $X'_K$ over $K$, then this isomorphism extends to an isomorphism over $O_K$.

Proof. Since $X_K$ is isomorphic to $X'_K$ over $K$, there is a rational map $\phi : X \dashrightarrow X'$ over $O_K$. Let $\mu : Y \to X$ be a minimal succession of blowing-ups such that $\phi \cdot \mu$ induces a morphism $\mu' : Y \to X'$. There are effective divisors $Z$ and $Z'$ on $Y$ such that

$$\omega_{Y/O_K} = \mu^*(\omega_{X/O_K}) \otimes O_Y(Z) = \mu'^*(\omega_{X'/O_K}) \otimes O_Y(Z').$$

Clearly, $\text{Supp}(Z)$ is contracted by $\mu$ and $\text{Supp}(Z')$ is contracted by $\mu'$. Moreover, by the minimality of $\mu$, $Z$ and $Z'$ has no common components. Let us consider

$$(\mu^*(\omega_{X/O_K}) \otimes O_Y(Z) \cdot Z) = (\mu'^*(\omega_{X'/O_K}) \otimes O_Y(Z') \cdot Z).$$

If $Z \neq 0$, then the left hand side of the above is negative. But the right hand side is non-negative. Therefore, $Z = 0$. By the same way, $Z' = 0$. Thus, we have

$$\omega_{Y/O_K} = \mu^*(\omega_{X/O_K}) = \mu'^*(\omega_{X'/O_K}).$$

Here, we assume that $\mu$ is not an isomorphism. Then, there is a curve $C$ on $Y$ such that $C$ is contracted by $\mu$, but is not contracted by $\mu'$. Then, $(\mu^*(\omega_{X/O_K}) \cdot C) = 0$, but $(\mu'^*(\omega_{X'/O_K}) \cdot C) > 0$. This is a contradiction. Thus, the rational map $\phi : X \dashrightarrow X'$ is actually a morphism and $\phi^*(\omega_{X'/O_K}) = \omega_{X/O_K}$. Hence, $\phi$ is finite because $\omega_{X/O_K}$ and $\omega_{X'/O_K}$ are ample. Moreover, $X$ and $X'$ are normal. Therefore, by Zariski main theorem, $\phi$ is an isomorphism. \qed

Next, we give a variant of Theorem 3.1. Let $C$ be a compact Riemann surface of genus $g \geq 1$ and $\{\omega_1, \cdots, \omega_g\}$ an orthonormal basis of $H^0(C, \Omega_C)$ with respect to $\langle , \rangle_{\text{can}}$. We set the normalized Kähler form $\mu_{\text{can}}$ as follows.

$$\mu_{\text{can}} = \frac{\sqrt{-1}}{2g} \sum_{i=1}^{g} \omega_i \wedge \bar{\omega}_i.$$  

We can give another metric $\Phi_{\text{Ar}}$ of $\Omega_C$ by the following way, which is called the Arakelov metric. Let $\Delta$ be the diagonal of $C \times C$, $p : C \times C \to C$ the first projection, and $q : C \times C \to C$ the second projection. Let $h_\Delta$ be the Einstein-Hermitian metric of $O_{C \times C}(\Delta)$ with respect to $p^*\mu_{\text{can}} + q^*\mu_{\text{can}}$ such that

$$\int_{C \times C} \log(h_\Delta(1,1))(p^*\mu_{\text{can}} + q^*\mu_{\text{can}})^2 = 0,$$

where 1 is the canonical section of $O_{C \times C}(\Delta)$. Since $\Omega_C$ is canonically isomorphic to $O_{\Delta}(-\Delta)$, $(h_\Delta^{-1})|_{\Delta}$ induces the metric $\Phi_{\text{Ar}}$ on $\Omega_C$. It is well known that $c_1(\Omega_C, \Phi_{\text{Ar}}) = 2(g - 1)\mu_{\text{can}}$. Here we set

$$\nu_{\text{Ar}}(C) = \int_C \log \left( \frac{\Phi_{\text{Ar}}}{\Phi_{\text{can}}} \right) \mu_{\text{can}}.$$  

Then, we have the following lemma.
Lemma 3.3. Let $K$ be a number field, $O_K$ the ring of integers of $K$ and $S = \text{Spec}(O_K)$. Let $f : X \to S$ be a stable arithmetic surface of genus $g \geq 1$. Then,
\[
(\hat{c}_1(\omega_{X/S}, \Phi_{can})^2) = (\hat{c}_1(\omega_{X/S}, \Phi_{Ar})^2) + 2(g - 1)\nu_{Ar}(X/S),
\]
where $\nu_{Ar}(X/S) = \sum_{\sigma \in K(C)} \nu_{Ar}(X_{\sigma})$.

Proof. We set $\rho = \Phi_{Ar}/\Phi_{can}$. Then
\[
(\hat{c}_1(\omega_{X/S}, \Phi_{can})^2) = (\hat{c}_1(\omega_{X/S}, \Phi_{Ar})^2) + \sum_{\sigma \in K(C)} \int_{C_{\sigma}} \log(\rho_{\sigma}) c_1(\omega_{X_{\sigma}}, (\Phi_{Ar})_{\sigma}).
\]
Thus we get our lemma because $c_1(\omega_{X_{\sigma}}, (\Phi_{Ar})_{\sigma}) = 2(g - 1)(\mu_{can})_{\sigma}$. □

For constants $A_1$ and $A_2$, we denote by $B^Ar_g(K, A_1, A_2)$ a subset of $\bar{M}_g^s(\text{Spec}(O_K))/\sim$ consisting of the classes of stable curves with
\[
\frac{(\hat{c}_1(\omega_{X/S}, \Phi_{Ar})^2)}{[K : \mathbb{Q}]} \leq A_1 \quad \text{and} \quad \frac{\nu_{Ar}(X/S)}{[K : \mathbb{Q}]} \leq A_2.
\]
Thus, Theorem 3.1 and Lemma 3.3 imply

Corollary 3.4. For any number field $K$ and any constants $A_1$ and $A_2$, $B^Ar_g(K, A_1, A_2)$ is finite.

Remark 3.5. Optimistically, we can guess that
\[
B^Ar_g(K, A) = \left\{ X \to S \in \bar{M}_g^s(\text{Spec}(O_K)) \left| \frac{(\hat{c}_1(\omega_{X/S}, \Phi_{Ar})^2)}{[K : \mathbb{Q}]} \leq A \right\} / \sim
\]
is finite for any constant $A$. Indeed, as L. Szpiro pointed out in his letter, $B^Ar_g(K, 0)$ is finite. For, let $f : X \to S$ be a stable arithmetic surface of genus $g \geq 2$. Then, by [Zh], if $f$ is not smooth, then $(\hat{c}_1(\omega_{X/S}, \Phi_{Ar})^2) > 0$. Thus, by Shafarevich Conjecture which was proved by Faltings [Fa], $B^Ar_g(K, 0)$ is finite.

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