NEW LARGE-CARDINAL AXIOMS AND THE
ULTIMATE-L PROGRAM

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ABSTRACT. We will consider a number of new large-cardinal properties, the $\alpha$-tremendous cardinals for each limit ordinal $\alpha > 0$, the hyper-tremendous cardinals, the $\alpha$-enormous cardinals for each limit ordinal $\alpha > 0$, and the hyper-enormous cardinals. For limit ordinals $\alpha > 0$, the $\alpha$-tremendous cardinals and hyper-tremendous cardinals have consistency strength between I3 and I2. An $\omega$-enormous cardinal has consistency strength greater than I0, and also all the large-cardinal axioms discussed in the second part of Hugh Woodin's paper on suitable extender models, not known to be inconsistent with ZFC and of greater consistency strength than I0. Ralf Schindler and Victoria Gitman have developed the notion of a virtual large-cardinal property, and a clear sense can be given to the notion of "virtually $\omega$-enormous". A virtually $\omega$-enormous cardinal can be shown to dominate a Ramsey cardinal.

It can be shown that a cardinal $\kappa$ which is a critical point of an elementary embedding $j : V_{\lambda+2} \prec V_{\lambda+2}$, in a context not assuming choice, is necessarily a hyper-enormous cardinal. Building on this insight, we can obtain the result that the existence of such an elementary embedding is in fact outright inconsistent with ZF. The assertion that there is a proper class of $\alpha$-enormous cardinals for every limit ordinal $\alpha > 0$ can be shown to imply a version of the Ultimate-L Conjecture.

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To my beloved wife Mari Mnatsakanyan, without whom this work would not have been possible.
In what follows we will present a number of new large-cardinal axioms, and applications of them. Let us begin by presenting the definitions of the new large-cardinal properties to be considered.

1. Definitions of the new large-cardinal properties

Definition 1.1. Suppose that $\alpha$ is a limit ordinal such that $\alpha > 0$. We say that an uncountable regular cardinal $\kappa$ is $\alpha$-tremendous if there exists an increasing sequence of cardinals $\langle \kappa_\beta : \beta < \alpha \rangle$ such that $V_{\kappa_\beta} \prec V_\kappa$ for all $\beta < \alpha$, and if $n > 1$ and $\langle \beta_i : i < n \rangle$ is an increasing sequence of ordinals less than $\alpha$, then if $\beta_0 \neq 0$ then for all $\beta < \beta_0$ there is an elementary embedding $j : V_{\kappa_\beta n-2} \prec V_{\kappa_\beta n-1}$ with critical point $\kappa_{\beta'}$ and $j(\kappa_{\beta'}) = \kappa_{\beta_0}$ and $j(\kappa_{\beta_0}) = \kappa_{\beta_{i+1}}$ for all $i$ such that $0 \leq i < n-2$, and if $\beta_0 = 0$ then there is an elementary embedding $j : V_{\kappa_\beta n-2} \prec V_{\kappa_\beta n-1}$ with critical point $\kappa' < \kappa_0$ and $j(\kappa') = \kappa_0$ and $j(\kappa_{\beta_0}) = \kappa_{\beta_{i+1}}$ for all $i$ such that $0 \leq i < n-2$.

Definition 1.2. A cardinal $\kappa$ such that $\kappa$ is $\kappa$-tremendous is said to be hyper-tremendous.

Definition 1.3. Suppose that $\alpha$ is a limit ordinal such that $\alpha > 0$, and that $\langle \kappa_\beta : \beta < \alpha \rangle$ together with a family $\mathcal{F}$ of elementary embeddings witness that $\kappa$ is $\alpha$-tremendous, with just one embedding in the family $\mathcal{F}$ witnessing $\alpha$-tremendousness for each finite sequence of ordinals less than $\alpha$. Suppose that $n > 1$ and $\langle \beta_i : i < n \rangle$ is an increasing sequence of ordinals less than $\alpha$ and suppose $\beta_{i-1} < \beta' < \alpha$. Suppose that, given any such $n$ and any such sequence of ordinals $\langle \beta_i : i < n \rangle$ and any such ordinal $\beta'$, and given any embedding $j : V_{\kappa_{\beta_0 n-2}} \prec V_{\kappa_{\beta_0 n-1}}$ from the family $\mathcal{F}$, witnessing the $\alpha$-tremendousness of $\kappa$ for the sequence $\langle \beta_i : i < n \rangle$, there is an embedding $k : V_{\kappa_{\beta_{i-1} n-1}} \prec V_{\kappa_{\beta_{i-1}}}$ from the family $\mathcal{F}$, which witnesses the $\alpha$-tremendousness of $\kappa$ for the sequence $\langle \beta_0, \beta_1, \ldots, \beta_{i-1}, \beta' \rangle$ such that $k$ agrees with $j$ on $V_{\kappa_{\beta_0 n-2}}$. Then the cardinal $\kappa$ is said to be $\alpha$-enormous.

Definition 1.4. A cardinal $\kappa$ such that $\kappa$ is $\kappa$-enormous is said to be hyper-enormous.

We will shortly establish that the $\alpha$-tremendous cardinals and hyper-tremendous cardinals are consistent relative to $\text{I}_2$. We shall also establish that the $\alpha$-enormous cardinals and hyper-enormous cardinals have greater consistency strength than $\text{I}_0$, or any other previously considered large-cardinal axiom not known to be inconsistent with ZFC. Some of the motivations for conjecturing that these large cardinals are consistent with ZFC will become apparent in the results appearing in the
remainder of this paper, and the final section will briefly discuss the intuition which inspired the original formulation of the definitions.

Let us begin by establishing that the $\alpha$-tremendous cardinals for limit ordinals $\alpha > 0$ and the hyper-tremendous cardinals have consistency strength strictly between $I_3$ and $I_2$.

2. The consistency strength of the $\alpha$-tremendous cardinals and hyper-tremendous cardinals

**Definition 2.1.** A cardinal $\kappa$ is said to be an $I_3$ cardinal if it is the critical point of an elementary embedding $j : V_\delta \prec V$. $I_3$ is the assertion that an $I_3$ cardinal exists, and $I_3(\kappa, \delta)$ is the assertion that the first statement holds for a particular pair of ordinals $\kappa, \delta$ such that $\kappa < \delta$.

**Definition 2.2.** A cardinal $\kappa$ is said to be an $I_2$ cardinal if it is the critical point of an elementary embedding $j : V \prec M$ such that $V_\delta \subset M$ where $\delta$ is the least ordinal greater than $\kappa$ such that $j(\delta) = \delta$. $I_2$ is the assertion that an $I_2$ cardinal exists, and $I_2(\kappa, \delta)$ is the assertion that the first statement holds for a particular pair of ordinals $\kappa, \delta$ such that $\kappa < \delta$.

In this section we wish to show that the $\alpha$-tremendous cardinals and hyper-tremendous cardinals have consistency strength strictly between $I_3$ and $I_2$.

**Theorem 2.3.** Suppose that $\kappa$ is $\omega$-tremendous as witnessed by $\langle \kappa_i : i < \omega \rangle$. Then there is a normal ultrafilter $U$ on $\kappa_0$ such that the set of all $\kappa' < \kappa_0$ such that $I_3(\kappa', \delta)$ for some $\delta < \kappa_0$, is a member of $U$.

**Proof.** Suppose that $\kappa$ is $\omega$-tremendous and that $\langle \kappa_i : i \in \omega \rangle$ together with a certain family $\mathcal{F}$ of elementary embeddings witness the $\omega$-tremendousness of $\kappa$. It can be assumed without loss of generality that all the embeddings in $\mathcal{F}$ with critical point $\kappa_0$ have the same restriction to $V_{\kappa_0+1}$ and as such give rise to the same normal ultrafilter on $\kappa_0$, denoted by $U$ in what follows. We may use reflection to show the existence of a $\kappa'_0 < \kappa_0$ belonging to any fixed member of $U$, such that $\langle \kappa'_0, \kappa_0, \kappa_1, \ldots \rangle$, together with a certain family $\mathcal{F}_0$ of elementary embeddings, witness $\omega$-tremendousness of $\kappa$. Then we can repeat this procedure to find a $\kappa'_1$ belonging to the same fixed member of $U$ such that $\kappa'_0 < \kappa'_1 < \kappa_0$, such that $\langle \kappa'_0, \kappa'_1, \kappa_0, \kappa_1, \ldots \rangle$, together with a certain family $\mathcal{F}_1$ of elementary embeddings, witness $\omega$-tremendousness of $\kappa$. We can continue in this way, and we can also arrange things so that there is a sequence of embeddings $j_n : V_{\kappa_{n-1}} \prec V_{\kappa_n}$ with critical point...
\( \kappa'_0 \) for all \( n > 1 \), which can be chosen by induction, such that for each \( n > 1 \), \( j_n \) coheres with \( j_m \) for all \( m \) such that \( 1 < m < n \), and the embeddings from \( \mathcal{F}_n \) that have critical point \( \kappa'_0 \) can be chosen so as to be coherent with \( j_n \). In this way we obtain a sequence \( \langle \kappa'_n : n < \omega \rangle \) and a sequence of embeddings \( j_n \) with the previously stated properties. The existence of such a pair of sequences for any given element of \( U \) yields the claimed result.

**Theorem 2.4.** Suppose that \( \kappa \) is an I2 cardinal. Then there is a normal ultrafilter \( U \) on \( \kappa \) concentrating on the hyper-tremendous cardinals.

**Proof.** Suppose that \( \kappa \) is an I2 cardinal and let the elementary embedding \( j : V \prec M \) with critical point \( \kappa \) witness that \( \kappa \) is an I2 cardinal, the supremum of the critical sequence being \( \delta \). If we let \( U \) be the ultrafilter on \( \kappa \) arising from \( j \) we can easily show that the set of \( \kappa' < \kappa \) such that there is an elementary embedding \( k_{\kappa'} : V_{\delta} \prec V_{\delta} \), with critical sequence consisting of \( \kappa' \) followed by the critical sequence of \( j \), is a member of \( U \) (denoted by \( X \) hereafter). Then the sequence of ordinals belonging to this set, together with a family of embeddings that can be derived from the sequence of embeddings \( \langle k_{\kappa'} : \kappa' \in X \rangle \) witness that \( \kappa \) is hyper-tremendous. Since it also follows that \( \kappa \) is hyper-tremendous in \( M \), the desired result follows.

This completes the proof that the \( \alpha \)-tremendous cardinals and hyper-tremendous cardinals have consistency strength strictly between I3 and I2. In the next section we show that \( \omega \)-enormous cardinals have greater consistency strength than any previously considered extension of ZFC not known to be inconsistent.

3. **Consistency strength of \( \omega \)-enormous cardinals**

We wish to show that \( \omega \)-enormous cardinals have greater consistency strength than any previously considered large-cardinal axiom not known to be inconsistent with ZFC. We shall begin by defining some large-cardinal axioms discussed in [2].

**Definition 3.1.** We say that an ordinal \( \lambda \) satisfies Laver’s axiom if the following holds. There is a set \( N \) such that \( V_{\lambda+1} \subseteq N \not\subseteq V_{\lambda+2} \) and an elementary embedding \( j : L(N) \prec L(N) \), such that

1. \( N = L(N) \cap V_{\lambda+2} \) and \( \text{crit}(j) < \lambda \);
2. \( N^\lambda \subseteq L(N) \);
3. for all \( F : V_{\lambda+1} \to N \setminus \{\emptyset\} \) such that \( F \in L(N) \) there exists \( G : V_{\lambda+1} \to V_{\lambda+1} \) such that \( G \in N \) and such that for all \( A \in V_{\lambda+1} \), \( G(A) \in F(A) \).
Definition 3.2. We define the sequence \( \langle E^0_\alpha(V_{\lambda+1}) : \alpha < \Upsilon_{V_{\lambda+1}} \rangle \) to be the maximum sequence such that the following hold.

1. \( E^0_0(V_{\lambda+1}) = L(V_{\lambda+1}) \cap V_{\lambda+2} \) and \( E^0_1(V_{\lambda+1}) = L((V_{\lambda+1})^\#) \cap V_{\lambda+2} \).
2. Suppose \( \alpha < \Upsilon_{V_{\lambda+1}} \) and \( \alpha \) is a limit ordinal. Then \( E^0_\alpha(V_{\lambda+1}) = L(\cup \{ E^0_\beta(V_{\lambda+1}) : \beta < \alpha \}) \cap V_{\lambda+2} \).
3. Suppose \( \alpha + 1 < \Upsilon_{V_{\lambda+1}} \). Then for some \( X \in E^0_\alpha(V_{\lambda+1}) \), \( E^0_\alpha(V_{\lambda+1}) < X \), where by this we mean that there is a surjection \( \pi : V_{\lambda+1} \rightarrow E^0_\alpha(V_{\lambda+1}) \) with \( \pi \in L(X, V_{\lambda+1}) \), and \( E^0_{\alpha+1}(V_{\lambda+1}) = L(X, V_{\lambda+1}) \cap V_{\lambda+2} \), and if \( \alpha + 2 < \Upsilon_{V_{\lambda+1}} \) then \( E^0_{\alpha+2}(V_{\lambda+1}) = L((X, V_{\lambda+1})^\#) \cap V_{\lambda+2} \).
4. Suppose \( \alpha < \Upsilon_{V_{\lambda+1}} \). Then there exists \( X \subseteq V_{\lambda+1} \) such that \( E^0_\alpha(V_{\lambda+1}) \subseteq L(X, V_{\lambda+1}) \) and such that there is a proper elementary embedding \( j : L(X, V_{\lambda+1}) \hookrightarrow L(X, V_{\lambda+1}) \), where this means that \( j \) is non-trivial with critical point below \( \lambda \), and for all \( X' \in L(X, V_{\lambda+1}) \cap V_{\lambda+2} \) there exists \( Y \in L(X, V_{\lambda+1}) \cap V_{\lambda+2} \) such that \( \langle X_i : i < \omega \rangle \in L(Y, V_{\lambda+1}) \), where \( X_0 = X' \) and \( X_{i+1} = j(X_i) \) for all \( i \geq 0 \).
5. Suppose \( \alpha < \Upsilon_{V_{\lambda+1}} \), \( \alpha \) is a limit ordinal, and let \( N = E^0_\alpha(V_{\lambda+1}) \). Then either
   - (a) \( \text{cof}(\Theta^N)^{L(N)} < \lambda \), or
   - (b) \( \text{cof}(\Theta^N)^{L(N)} > \lambda \) and for some \( Z \in N, L(N) = (\text{HOD}_{V_{\lambda+1}\cup \{Z\}})^{L(N)} \).
   Here \( \Theta^N = \sup\{ \Theta^{L(X, V_{\lambda+1})} : X \in N \} \) where \( \Theta^{L(X, V_{\lambda+1})} \) is the supremum of the ordinals \( \gamma \) which can serve as the codomain of a surjection with domain \( V_{\lambda+1} \) where the surjection is an element of \( L(X, V_{\lambda+1}) \).
6. Suppose \( \alpha + 1 < \Upsilon_{V_{\lambda+1}} \), \( \alpha \) is a limit ordinal, and let \( N = E^0_\alpha(V_{\lambda+1}) \). Then either
   - (a) \( \text{cof}(\Theta^N)^{L(N)} < \lambda \), and \( E^0_{\alpha+1}(V_{\lambda+1}) = L(N^\lambda, N) \cap V_{\lambda+2} \), or
   - (b) \( \text{cof}(\Theta^N)^{L(N)} > \lambda \), and \( E^0_{\alpha+1}(V_{\lambda+1}) = L(\mathcal{E}(N), N) \cap V_{\lambda+2} \), where \( \mathcal{E}(N) \) is the set of elementary embeddings \( k : N \prec N \).

Define \( N := L(\cup \{ E^0_\alpha(V_{\lambda+1}) : \alpha < \Upsilon_{V_{\lambda+1}} \}) \cap V_{\lambda+2} \). Suppose that \( \text{cof}(\Theta^N) > \lambda \) and \( L(N) \neq (\text{HOD}_{V_{\lambda+1}\cup \{Z\}})^{L(N)} \) for all \( Z \in N \), and further there is an elementary embedding \( j : L(N) \hookrightarrow L(N) \) with \( \text{crit}(j) < \lambda \). Then we say that \( \lambda \) satisfies Woodin’s axiom.

Theorem 3.3. Suppose that \( \kappa \) is \( \omega \)-enormous as witnessed by \( \langle \kappa_i : i \in \omega \rangle \). Then \( V_{\kappa_0} \) is a model for the assertion that there is a proper class of \( \lambda \) satisfying both Laver’s axiom and Woodin’s axiom.

Proof. Suppose that \( \kappa \) is \( \omega \)-enormous and that the sequence \( \langle \kappa_i : i \in \omega \rangle \), with supremum \( \lambda \), witnesses the \( \omega \)-enormousness of \( \kappa \). From the family of embeddings witnessing \( \omega \)-enormousness we may construct an
elementary embedding $j : V_\lambda < V_\lambda$ with critical sequence $\langle \kappa_i : i \in \omega \rangle$. It is clearly sufficient to prove that $\lambda$ satisfies both Laver’s axiom and Woodin’s axiom, via embeddings extending $j$, in $V_\kappa$. The elementary embedding $j$ has a unique extension to $V_{\lambda+1}$, we will begin by showing that this extension is an elementary embedding $V_{\lambda+1} < V_{\lambda+1}$.

By making use of Skolem hull arguments, the $\omega$-enormousness of $\kappa$, and the fact that $V_{\kappa_n} < V_\kappa$ for all $n$, it is possible to show the following. Given any $X \in V_{\lambda+1}$, and any $m, n \in \omega$, there is a $\kappa' < \kappa_0$ and a $\Sigma_m$-elementary embedding $j_n : V_{\kappa_n} <_{\Sigma_m} V_\kappa$ with critical point $\kappa'$, where $j_n(\kappa') = \kappa_0$ and $j_n(\kappa_i) = \kappa_{i+1}$ for integers $i$ such that $0 \leq i < n$, such that $X \in \text{range } j_n$. Furthermore the $j_n$ may be chosen so that the functions $j_n | V_{\kappa_{n-1}}$ cohere with one another and may be glued together into a single mapping $j'$ with domain $V_\lambda$, with an obvious extension to $V_{\lambda+1}$, and one may construct a set $X'$ by gluing together all the sets of the form $(j_n)^{-1}(X \cap V_{\kappa_n})$ so that $j'(X') = X$. Since each $j_n$ is $\Sigma_m$-elementary, and also each $V_{\kappa_n} < V_\kappa$, it can be shown that the function $j'$ is a $\Sigma_m$-elementary embedding from $V_{\lambda+1}$ to $V_{\lambda+1}$. Let us elaborate on this point further.

Suppose that $\phi$ is a $\Sigma_m$ formula in the first-order language of set theory with free variables $x_1, x_2, \ldots x_n$. Suppose that the formula is true in the structure $V_{\lambda+1}$ under the interpretation which assigns $a_i$ to $x_i$ for each integer $i$ such that $1 \leq i \leq s$, where $a_1, a_2, \ldots a_s \in V_{\lambda+1}$. We can assume $a_1, a_2, \ldots a_r \in V_{\kappa_p}$ for some non-negative integer $p$ and some integer $r$ such that $0 \leq r \leq s$, and that $a_{r+1}, a_{r+2}, \ldots a_s \in V_{\lambda+1} \setminus V_\lambda$. If $r = s$, then clearly there is no difficulty in establishing that $\phi$ holds in $V_{\lambda+1}$ under the interpretation which assigns $j'(a_i)$ to each $x_i$. The case that requires discussion is when $r < s$, and clearly we can suppose without loss of generality that $r = 0$ and $s = 1$. Thus $a_1 \in V_{\lambda+1} \setminus V_\lambda$ and using $V_{\kappa_n} < V_\kappa$ for each $n$ we may obtain a sequence $\langle b_n : n \in \omega \rangle$ with $b_n \in V_{\kappa_n}$, $\langle b_n \cap V_{\kappa_{n-1}} : n \in \omega \rangle$ strictly increasing under the inclusion relation and the union of that sequence being $a_1$, and $\phi$ holds in $V_{\text{rank } b_n+1}$ under the interpretation which assigns $b_n$ to $x_1$ for each $n$. It can also be arranged that $j'(\text{rank } b_n) = \text{rank } b_{n+1}$. Then clearly for each $n$, the formula $\phi$ holds in $V_{\text{rank } b_{n+1}+1}$ under the interpretation which assigns $j'(b_n)$ to $x_1$. Since this holds for each finite ordinal $n$, the conclusion that $\phi$ holds in $V_{\lambda+1}$ under the interpretation assigning $j'(a_1)$ to $x_1$ follows. Since this discussion applied to arbitrary $\Sigma_m$ formulas $\phi$, it is now established that $j'$ is indeed a $\Sigma_m$-elementary embedding from $V_{\lambda+1}$ into $V_{\lambda+1}$.
We have outlined how to construct a $\Sigma_m$-elementary embedding $j' : V_{\lambda+1} \prec V_{\lambda+1}$ with critical sequence $\langle \kappa', \kappa_0, \kappa_1, \ldots \rangle$, and furthermore the construction may be done in such a way that $j'(j' \restriction V_\lambda) = j \restriction V_\lambda$, independently of the value of $m$. Now the stated result that the unique extension of $j$ to $V_{\lambda+1}$ is an elementary embedding from $V_{\lambda+1}$ to $V_{\lambda+1}$ also follows.

This also gives rise to an embedding, also denoted by $j$ by abuse of notation, such that $j : V_\kappa \prec M$ with $V_\lambda \subset M$. There is a well-ordering $W$ of $(V_{\lambda+1})^M$ of least possible length in the range of $j$. Note that $j^{-1}(W), j(W) \in L(W, V_{\lambda+1})$ and $L(j(W), V_{\lambda+1}) = L(W, V_{\lambda+1})$. If we now consider the restriction of $j$ to $L(W, V_{\lambda+1}) \cap V_\kappa$ we can argue from the $\omega$-enormousness of $\kappa$ in a similar way to previously, that $j$ is an elementary embedding from $L(W, V_{\lambda+1}) \cap V_\kappa$ into itself and so that $\lambda$ satisfies Laver’s axiom in $V_\kappa$ via an embedding extending $j$. Here, when we say “in a similar way to previously”, what we have in mind is that for any given $X$ in the range of the aforementioned restriction of $j$, we can obtain a series of embeddings $j_n$ with domain $V_{\kappa_n}$ with $X$ in the range of each $j_n$ and such that the sequence of $j_n$ “approximates” $j$. For the object $X$ is definable from some parameter in $V_{\lambda+1}$ together with an ordinal that is in the range of $j$, so that the same techniques as before are applicable. This fact is used to argue, using the same type of reasoning as before, that the restriction of $j$ given above is indeed an elementary embedding into the specified codomain.

Suppose that a sequence of sets $\langle E_\alpha^0(V_{\lambda+1}) : \alpha < \beta \rangle$ satisfies requirements (1)-(6) of the definition of Woodin’s axiom, relativized to $V_\kappa$, for some $\beta \leq \Upsilon_{V_{\lambda+1}}$, and define $N$ to be the unique possible candidate for $E_\beta^0$ if it exists. It can be shown by transfinite induction that $L(j(N) \cup V_{\lambda+1}) \cap V_\kappa = L(N) \cap V_\kappa$. Then, by similar reasoning to that given previously in establishing that $\lambda$ satisfies Laver’s axiom in $V_\kappa$, and considering that the action of $j$ on the elements of such an $N$ is determined by $j \restriction V_\lambda$, it can be shown that the restriction of $j$ to $L(N) \cap V_\kappa$ is an elementary embedding $L(N) \cap V_\kappa \prec L(N) \cap V_\kappa$, proper in the case where $\beta < \Upsilon_{V_{\lambda+1}}$. Since this is so for every $N$ satisfying all the previously stated requirements, we may now conclude by transfinite induction that $\lambda$ satisfies Woodin’s axiom in $V_\kappa$ via an embedding extending $j$. This completes the argument. □

This completes the demonstration that $\omega$-enormous cardinals have greater consistency strength than any previously considered extension of ZFC not known to be inconsistent.
4. Virtually ω-enormous and hyper-enormous cardinals

Ralf Schindler and Victoria Gitman in [4] have introduced the notion of virtual large-cardinal properties. Given any large-cardinal property defined with reference to a set-sized elementary embedding \( j : V_\alpha \prec V_\beta \) or family of such embeddings, the corresponding virtual large-cardinal property is defined in the same way except by means of elementary embeddings \( j : (V_\alpha)^V \prec (V_\beta)^V \) where \( j \in V[G] \) for a set generic extension of \( V \). The notion of a virtually ω-enormous cardinal is clear.

**Theorem 4.1.** If \( \kappa \) is virtually ω-enormous as witnessed by \( \langle \kappa_i : i < \omega \rangle \), then \( \kappa_0 \) is Ramsey.

**Proof.** Suppose that \( \kappa \) is virtually ω-enormous as witnessed by \( \langle \kappa_i : i < \omega \rangle \) and a family of elementary embeddings from some generic extension \( V[G] \) of \( V \), and let \( f : [\kappa_0]^\omega \rightarrow \{0, 1\} \) be a 2-coloring of the finite subsets of \( \kappa_0 \), where \( f \in V \). There exists a normal ultrafilter \( U \in V[G] \) on \( \kappa_0 \) concentrating on a sequence of cardinals cofinal in \( \kappa_0 \) witnessing virtual hyper-tremendousness of \( \kappa_0 \) in \( V \), and this can be chosen in such a way that it is homogeneous for \( f \mid [\kappa_0]^\omega \). □

**Theorem 4.2.** If \( \kappa \) is a measurable cardinal, and \( V = \text{HOD} \), then there is a sequence cofinal in \( \kappa \) witnessing the virtual hyper-enormousness of \( \kappa \).

**Proof.** Suppose that \( j : V \prec M \) with critical point \( \kappa \) witnesses the measurability of \( \kappa \). Then there is an elementary embedding \( j' : V_\kappa \prec (M \cap V_j(\kappa)) \) which appears in a generic extension of \( M \) (here using the hypothesis \( V = \text{HOD} \)). Iterating reflection yields the desired result. □

5. Inconsistency of the choiceless cardinals

In this section we wish to prove the following theorem.

**Theorem 5.1.** It is not consistent with ZF that there exists an ordinal \( \lambda \) and a non-trivial elementary embedding \( j : V_{\lambda+2} \prec V_{\lambda+2} \).

**Proof.** The same reasoning that shows that every I2 cardinal \( \kappa \) has a normal ultrafilter \( U \) concentrating on the hyper-tremendous cardinals, also shows in ZF that if \( \kappa \) is a critical point of an elementary embedding \( V_{\lambda+2} \prec V_{\lambda+2} \), then there is a normal ultrafilter \( U \) concentrating on a sequence \( \langle \kappa_\alpha : \alpha < \kappa \rangle \) which witnesses that \( \kappa \) is hyper-enormous. In [6], Gabriel Goldberg has also shown, using iterated collapse forcing, that if the existence of such an embedding is consistent with ZF then it is also consistent with \( V_\lambda \) being well-orderable (using a well-ordering
where if \(m < n\) and \(\langle \kappa'_i : i \in \omega \rangle\) is the critical sequence of \(j\), the map \(j^{n-m}\) maps the restriction of the well-ordering to \(V_{\kappa'_n+1} \setminus V_{\kappa'_m}\) to the restriction of the well-ordering to \(V_{\kappa'n+1} \setminus V_{\kappa'_n}\).

So suppose the conjunction of these two hypotheses, in ZF, and let \(\kappa\) be the critical point of the embedding and let \(S\) be the sequence \(\langle \kappa_\alpha : \alpha < \kappa \rangle\) mentioned before. For each \(\alpha < \kappa\), let \(E_\alpha\) be the equivalence relation on \([\kappa_\alpha]^{\omega}\) which holds if two sets of ordinals less than \(\kappa_\alpha\) whose elements in order constitute two sequences of countably infinite length, if and only if the two sequences in question have the same tail. There is a sequence \(\langle C_\alpha : \alpha < \kappa \rangle\) such that for each \(\alpha < \kappa\), \(C_\alpha\) is a choice set for the equivalence classes of \(E_\alpha\), and for each pair \((\alpha, \beta)\) with \(\alpha < \beta\), when one is choosing an elementary embedding \(j'\) from a fixed family of embeddings witnessing the hyper-enormousness of \(\kappa\), one can without loss of generality choose it so that \(j'(C_\alpha) = C_\beta\). Then using the embedding \(j\) one can extend this to a family of choice sets \(\langle C_\alpha : \alpha < \lambda \rangle\), such that if \(\alpha < \beta < \kappa'_n\) then an elementary embedding \(j'\) can be chosen which is part of a fixed family of embeddings witnessing the hyper-enormousness of \(\kappa'_n\), such that \(j'(C_\alpha) = C_\beta\).

This allows one to construct a choice set \(C\) for the corresponding equivalence relation \(E\) on \([\lambda]^{\omega}\). The method is as follows. Given an \(X \in [\lambda]^{\omega}\), one may find an \(X' \in V_{\kappa'_n}\) for any given \(n > 0\) such that \(X' \in [\rho]^{\omega}\) for a \(\rho\) of cofinality \(\omega\) between \(\kappa'_{n-1}\) and \(\kappa'_n\) and an embedding \(e_{X,n} : V_{\rho+1} \prec V_{\lambda+1}\) which carries a sequence of hyper-enormous cardinals cofinal in \(\rho\) to the critical sequence of \(j\) or a tail thereof, such that \(e_{X,n}(X') = X\). This can be used together with the sequence of choice sets \(\langle C_\alpha : \alpha < \lambda \rangle\) to choose a member of the equivalence class of \(X\), depending on \(n\). Using the relation mentioned earlier between the different choice sets \(C_\alpha\), one can argue that this data can be chosen in such a way that the function mapping \(n\) to the chosen member of the equivalence class of \(X\) is in fact eventually constant, and that a choice set for the equivalence relation \(E\) can be constructed in this way.

However, this gives rise to a contradiction using the method of proof of Kunen’s inconsistency theorem. And this contradiction was obtained from a set of assumptions which are provably consistent by forcing relative only to ZF plus the existence of an elementary embedding \(V_{\lambda+2} \prec V_{\lambda+2}\). Thus the existence of an elementary embedding \(V_{\lambda+2} \prec V_{\lambda+2}\) is in fact inconsistent with ZF.

\(\square\)
6. A proof of the Ultimate-L Conjecture

In this section, we will seek to give a proof of Hugh Woodin’s Ultimate-L Conjecture. The most important sources for Hugh Woodin’s Ultimate-L program are [1], [2], and [3]. We must begin by giving the statement of the axiom $V = \text{Ultimate-L}$, following Definition 7.14 of [3].

**Definition 6.1.** The axiom $V = \text{Ultimate-L}$ is defined to be the assertion that

1. There is a proper class of Woodin cardinals.
2. Given any $\Sigma_2$-sentence $\phi$ which is true in $V$, there exists a universally Baire set of reals $A$, such that, if $\Theta^{L(A,\mathbb{R})}$ is defined to be the least ordinal $\Theta$ such that there is no surjection from $\mathbb{R}$ onto $\Theta$ in $L(A,\mathbb{R})$, then the sentence $\phi$ is true in $\text{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta^{L(A,\mathbb{R})}}$.

Now let us recall a set of definitions from [3].

**Definition 6.2.** Suppose that $N$ is a transitive proper class model of $\text{ZFC}$ and that $\delta$ is a supercompact cardinal in $V$. We say that $N$ is a weak extender model for the supercompactness of $\delta$ if for all $\gamma > \delta$, there exists a normal fine $\delta$-complete measure $\mu$ such that $\mu(N \cap P_\delta(\gamma)) = 1$.

**Definition 6.3.** A sequence $N := \langle N_\alpha : \alpha \in \text{Ord} \rangle$ is weakly $\Sigma_2$-definable if there is a formula $\phi(x)$ such that

1. For all $\beta < \eta_1 < \eta_2 < \eta_3$, if $(N_\phi)^{V_{\eta_1}} \models \beta = (N_\phi)^{V_{\eta_3}} \models \beta$ then $(N_\phi)^{V_{\eta_1}} \models \beta = (N_\phi)^{V_{\eta_2}} \models \beta = (N_\phi)^{V_{\eta_3}} \models \beta$;
2. For all $\beta \in \text{Ord}$, $N \models \beta = (N_\phi)^{V_\gamma} \models \beta$ for sufficiently large $\eta$, where, for all $\gamma$, $(N_\phi)^{V_\gamma} = \{ a \in V_\gamma : V_\gamma \models \phi[a] \}$. Suppose $N \subseteq V$ is an inner model such that $N \models \text{ZF-C}$. Then $N$ is weakly $\Sigma_2$-definable if the sequence $\langle N \cap V_\alpha : \alpha \in \text{Ord} \rangle$ is weakly $\Sigma_2$-definable.

We can now state the result we plan to prove in this section.

**Theorem 6.4.** Suppose that for each limit ordinal $\alpha > 0$ there is a proper class of $\alpha$-enormous cardinals. Then the following version of the Ultimate-L conjecture, given as Conjecture 7.41 in [3], holds. Suppose that $\delta$ is an extendible cardinal (in fact one can even suppose only that $\delta$ is a supercompact cardinal). Then there is a weak extender model $N$ for the supercompactness of $\delta$ such that

1. $N$ is weakly $\Sigma_2$-definable and $N \subseteq \text{HOD}$;
2. $N \models \text{“}V = \text{Ultimate-L}\text{”}$.
3. $N \models \text{GCH}$. 


Proof of Theorem [6.4]. Let us give the long awaited definition of Ultimate-L. We claim that what follows is the correct definition of Ultimate-L, assuming that there are sufficiently many large cardinals in $V$ as outlined in the hypotheses for Theorem [6.3]. The correct way to define it when we are making weaker large-cardinal assumptions still remains to be discovered.

Suppose that $\kappa$ is $\omega$-enormous as witnessed by $\langle \kappa_n : n \in \omega \rangle$, where clearly we may assume without loss of generality that the latter sequence is in HOD, and we will do so. Then we may consider all the sets of ordinals of the form $j''\lambda$ where $\lambda := \sup \{ \kappa_n : n \in \omega \}$ for some sequence $\langle \kappa_n : n \in \omega \rangle$ with the properties previously described, and $j$ is an elementary embedding $V_{\lambda+1} \prec V_{\lambda+1}$ with critical sequence $\langle \kappa_n : n \in \omega \rangle$. Some of these sets of ordinals will be members of HOD. We define Ultimate-L to be the smallest enlargement of $L$ containing every such set of ordinals in HOD obtained in this way from some $\omega$-enormous cardinal $\kappa$. The hypothesis of $\omega$-enormousness of $\kappa$ implies that in this model, there will indeed exist at least one elementary embedding $j : V_{\lambda+1} \prec V_{\lambda+1}$ with critical sequence $\langle \kappa_n : n \in \omega \rangle$ which gives rise to sufficient data to witness the $\omega$-enormousness of $\kappa$. The necessary elementary embedding can be constructed using similar arguments to those of Section 3. Thus any $\alpha$-enormous cardinal for any limit ordinal $\alpha > 0$ will indeed remain $\alpha$-enormous in Ultimate-L, and further clearly this inner model will satisfy GCH. Further this model is easily seen to be weakly $\Sigma_2$-definable and a subclass of HOD, and a weak extender model for the supercompactness of any extendible cardinal $\delta$ given the stated large-cardinal hypothesis. In order to see the last point, it is necessary to observe that given the stated large-cardinal assumptions, any supercompact cardinal is necessarily hyper-enormous, and all necessary elementary embeddings for witnessing this do descend to Ultimate-L. We must now show that this model is indeed a model for the axiom $V=\text{Ultimate-L}$ as stated at the start of this section.

Clearly, our version of Ultimate-L is a model for the assertion that there is a proper class of Woodin cardinals. Suppose then, that some $\Sigma_2$-sentence is true in Ultimate-L, so we are required to find a universally Baire set of reals $A$ in Ultimate-L such that the $\Sigma_2$-sentence in questions holds in $(HOD)^{L(A,R)} \cap V_{L(A,R)}$. From well-known generic absoluteness results which are known to hold assuming a proper class of Woodin cardinals, it is sufficient to prove that this does obtain in some set-generic extension of Ultimate-L. So choose an ordinal $\beta$ such that $(V_\beta)^{\text{Ultimate-L}}$ is a $\Sigma_2$-elementary substructure of Ultimate-L,
and choose a $\gamma < \beta$ such that $V_\gamma$ models the $\Sigma_2$-sentence. Now consider a generic extension of Ultimate-L where $A$ is a universally Baire set chosen to contain enough data so that, in the generic extension, $\Theta^{L(A,R)} = \beta$, and $(HOD)^{L(A,R)} \cap V_\gamma$ in the generic extension is equal to the intersection of the Ultimate-L of the ground model and $V_\gamma$. In this generic extension, the desired result obtains, so the aforementioned generic absoluteness results imply that it obtains in our ground model as well. This completes the proof of Theorem 6.4.

\[\Box\]

7. Concluding Remarks

The new large cardinals were inspired by Victoria Marshall’s work on reflection principles in [5] and are plausibly the correct generalisation of the reflection principles which were demonstrated by her in that work to imply the existence of $n$-huge cardinals. The large cardinal axiom used to prove the Ultimate-L conjecture certainly has quite substantial consistency strength and some skepticism about its consistency would certainly be quite reasonable at this stage, but it may be that the further study of the inner model theory of Ultimate-L and inner models which approximate it from within will provide new insights and increased confidence in consistency. In the mean time, it may very well be that the Ultimate-L conjecture is provable from just an extendible cardinal as originally envisaged by Hugh Woodin, so in that sense much work remains to be done.

If these new large cardinals are indeed consistent then the study of them appears to be quite fruitful, and it may be that the addition to ZFC of an axiom schema asserting for each $n < \omega$ the existence of a hyper-enormous cardinal $\kappa$ such that $V_\kappa \prec_{\Sigma_n} V$, together with the axiom $V=\text{Ultimate-L}$, will eventually come to be accepted as the correct “effectively complete” theory of $V$, assuming that confidence develops over time that this theory is consistent.

References

[1] Hugh Woodin. Suitable Extender Models I. Journal of Mathematical Logic, Vol. 10, Nos. 1 & 2 (2010), pp. 101–339.
[2] Hugh Woodin. Suitable Extender Models II: Beyond $\omega$-huge. Journal of Mathematical Logic, Vol. 11, No. 2 (2011), pp. 151–436.
[3] Hugh W. Woodin. In Search Of Ultimate-L: The 19th Midrasha Mathematical Lectures. The Bulletin of Symbolic Logic, 23(1), 1-109. doi:10.1017/bsl.2016.34
[4] Victoria Gitman and Ralf Schindler. Virtual Large Cardinals, pre-print.
[5] M. Victoria Marshall R. Higher order reflection principles, Journal of Symbolic Logic, vol. 54, no. 2, 1989, pp. 474–489.
[6] Gabriel Goldberg. On the consistency strength of Reinhardt cardinals, preprint.