On some new difference sequence spaces of fractional order

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Abstract
Let $\Delta^{(\alpha)}$ denote the fractional difference operator. In this paper, we define new difference sequence spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$. Also, the $\beta-$dual of the spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ are determined and calculated their Schauder basis. Furthermore, the classes $(\mu(\Gamma, \Delta^{(\alpha)}, u) : \lambda)$ where $\mu \in \{c_0, c\}$ and $\lambda \in \{c_0, c, \ell_\infty, \ell_1\}$.

Key words: Difference operator $\Delta^{(\alpha)}$, Sequence spaces, $\beta-$dual, Matrix transformations.

1. Preliminaries, background and notation

By a sequence space, we mean any vector subspace of $\omega$, the space of all real or complex valued sequences $x = (x_k)$. The well-known sequence spaces that we shall use throughout this paper are as following:

- $\ell_\infty$: the space of all bounded sequences,
- $c$: the space of all convergent sequences,
- $c_0$: the space of all null sequences,
- $cs$: the space of all sequences which form convergent series,
- $\ell_1$: the space of all sequences which form absolutely convergent series,
- $\ell_p$: the space of all sequences which form $p$-absolutely convergent series, where $1 < p < \infty$.

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Let $X, Y$ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, we say that $A$ defines a matrix mapping from $X$ into $Y$, and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$, is in $Y$; where

$$
(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}). 
$$

(1)

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $(X : Y)$, we denote the class of all matrices $A$ such that $A : X \to Y$. Thus, $A \in (X : Y)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y$ for all $x \in X$. A sequence $x$ is said to be $A$-summable to $\alpha$ which is called as the $A$-limit of $x$.

If a normed sequence space $X$ contains a sequence $(b_n)$ with the property that for every $x \in X$ there is a unique sequence of scalars $(\alpha_n)$ such that

$$
\lim_{n \to \infty} \|x - (\alpha_0b_0 + \alpha_1b_1 + \ldots + \alpha_nb_n)\| = 0,
$$

then $(b_n)$ is called a Schauder basis (or briefly basis) for $X$. The series $\sum \alpha_kb_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$, and written as $x = \sum \alpha_kb_k$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for triangle matrices $A, B$ and a sequence $x$. Further, a triangle matrix $U$ uniquely has an inverse $U^{-1} = V$ that is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$. We write additionally $U$ for the set of all sequences $u$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$.

For a sequence space $X$, the matrix domain $X_A$ of an infinite matrix $A$ is defined by

$$
X_A = \{x = (x_k) \in \omega : Ax \in \lambda\},
$$

(2)

which is a sequence space. The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been recently employed by Wang [6], Ng and Lee [7], Aydın and Başar [17] and Altay and Başar [12].

The gamma function may be regarded as a generalization of $n!$ ($n$–factorial), where $n$ is any positive integer. The gamma function $\Gamma$ is defined for all $p$
real numbers except the negative integers and zero. It can be expressed as an improper integral as follows:

\[ \Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt. \] (3)

From the equality (3) we deduce following properties:

(i) If \( n \in \mathbb{N} \) then we have \( \Gamma(n+1) = n! \)
(ii) If \( n \in \mathbb{R} \setminus \{0, -1, -2, -3, \ldots\} \) then we have \( \Gamma(n+1) = n\Gamma(n) \).

For a proper fraction \( \alpha \), Baliarsingh and Dutta have defined a fractional difference operators \( \Delta^\alpha : w \to w \), \( \Delta^{(\alpha)} : w \to w \) and their inverse in [20] as follows:

\[ \Delta^\alpha(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha + 1)}{i!\Gamma(\alpha + 1 - i)} x_{k+i} \] (4)

\[ \Delta^{(\alpha)}(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha + 1)}{i!\Gamma(\alpha + 1 - i)} x_{k-i} \] (5)

\[ \Delta^{-\alpha}(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1 - \alpha)}{i!\Gamma(1 - \alpha - i)} x_{k+i} \] (6)

and

\[ \Delta^{(-\alpha)}(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1 - \alpha)}{i!\Gamma(1 - \alpha - i)} x_{k-i} \] (7)

where we assume throughout the series defined in (4)-(7) are convergent. In particular, for \( \alpha = \frac{1}{2} \),

\[ \Delta^{1/2}x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \ldots \]

\[ \Delta^{-1/2}x_k = x_k + \frac{1}{2}x_{k+1} + \frac{1}{8}x_{k+2} + \frac{1}{16}x_{k+3} + \frac{5}{128}x_{k+4} + \frac{7}{256}x_{k+5} + \ldots \]

\[ \Delta^{(1/2)}x_k = x_k - \frac{1}{2}x_{k-1} - \frac{1}{8}x_{k-2} - \frac{1}{16}x_{k-3} - \frac{5}{128}x_{k-4} - \frac{7}{256}x_{k-5} - \ldots \]

\[ \Delta^{(-1/2)}x_k = x_k + \frac{1}{2}x_{k-1} + \frac{1}{8}x_{k-2} + \frac{1}{16}x_{k-3} + \frac{5}{128}x_{k-4} + \frac{7}{256}x_{k-5} + \ldots \]
Baliarsingh have been defined the spaces $X(\Gamma, \Delta^\alpha, u)$ for $X \in \{\ell_\infty, c_0, c\}$ by introducing the fractional difference operator $\Delta^\alpha$ and a positive fraction $\alpha$ in [18]. In this article, Baliarsingh have been studied some topological properties of the spaces $X(\Gamma, \Delta^\alpha, u)$ and established their $\alpha-$, $\beta-$ and $\gamma-$duals.

Following [18], we introduce the sequence spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ and obtain some results related to these sequence spaces. Furthermore, we compute the $\beta-$dual of the spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ . Finally, we characterize some matrix transformations on new sequence spaces.

2. The sequence spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$

In this section, we define the sequence spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ and examine the some topological properties of this sequence spaces.

The notion of difference sequence spaces was introduced by Kizmaz [2]. It was generalized by Et and Çolak [4] as follows:

Let $m$ be a non-negative integer. Then

$$\Delta^m(X) = \{x = (x_k) : \Delta^m x_k \in X\}$$

where $\Delta^0 x = (x_k)$, $\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ for all $k \in \mathbb{N}$ and

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}. \quad (8)$$

Furthermore, Malkowsky E., et al.[9] have been introduced the spaces

$$\Delta^{(m)}_u X = \{x \in \omega : \Delta^{(m)}_u x \in X\} \quad (9)$$

where $\Delta^{(m)}_u x = u\Delta^{(m)} x$ for all $x \in \omega$ . In this study, the operator $\Delta^{(m)}_u : \omega \to \omega$ was defined as follows:

$$\Delta^{(m)}_u x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i}. \quad (10)$$

Let $\alpha$ be a proper fraction and $u \in U$. We define the sequence spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ as follows:

$$c_0(\Gamma, \Delta^{(\alpha)}, u) = \{x \in \omega : \left(\sum_{j=0}^{k} u_j \Delta^{(\alpha)} x_j \right) \in c_0\} \quad (11)$$
and
\[ c(\Gamma, \Delta^{(a)}, u) = \{ x \in \omega : (\sum_{j=0}^{k} u_j \Delta^{(a)} x_j) \in c \}. \] (12)

Now, we define the triangle matrix \( \Delta^{(a)}_u(\Gamma) = (\tau_{nk}) \),

\[
\tau_{nk} = \begin{cases} 
\sum_{i=0}^{n-k} (-1)^i \frac{\Gamma(\alpha + 1)}{i!\Gamma(\alpha + 1 - i)} u_{i+k}, & (0 \leq k \leq n) \\
0, & k > n
\end{cases}
\] (13)

for all \( k, n \in \mathbb{N} \). Further, for any sequence \( x = (x_k) \) we define the sequence \( y = (y_k) \) which will be used, as the \( \Delta^{(a)}_u(\Gamma) \)–transform of \( x \), that is

\[
y_k = \sum_{j=0}^{k} u_j \Delta^{(a)} x_j = \sum_{j=0}^{k} u_j (x_j - \alpha x_{j-1} + \frac{\alpha(\alpha - 1)}{2!} x_{j-2} + ...)
\]

\[
= \sum_{j=0}^{k} \left( \sum_{i=0}^{k-j} (-1)^i \frac{\Gamma(\alpha + 1)}{i!\Gamma(\alpha + 1 - i)} u_{i+j} \right) x_j
\] (14)

for all \( k \in \mathbb{N} \).

It is natural that the spaces \( c_0(\Gamma, \Delta^{(a)}, u) \) and \( c(\Gamma, \Delta^{(a)}, u) \) may also be defined with the notation of (2) that

\[
c_0(\Gamma, \Delta^{(a)}, u) = (c_0)_{\Delta^{(a)}_u(\Gamma)} \quad \text{and} \quad c(\Gamma, \Delta^{(a)}, u) = c_{\Delta^{(a)}_u(\Gamma)}.
\] (15)

Before the main result let us give some lemmas, which we use frequently throughout this study, with respect to \( \Delta^{(a)} \) operator.

**Lemma 1.** [20, Theorem 2.2]

\[ \Delta^{(a)} o \Delta^{(\beta)} = \Delta^{(\beta)} o \Delta^{(a)} = \Delta^{(a+\beta)}. \]

**Lemma 2.** [20, Theorem 2.3]

\[ \Delta^{(a)} o \Delta^{(-a)} = \Delta^{(-a)} o \Delta^{(a)} = Id \]

where \( Id \) the identity operator on \( \omega \).
Theorem 1. The sequence spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ are $BK-$spaces with the norm
\[
\|x\|_{c_0(\Gamma, \Delta^{(\alpha)}, u)} = \|x\|_{c(\Gamma, \Delta^{(\alpha)}, u)} = \sup_k \left| \sum_{j=0}^{k-1} u_j \Delta^{(\alpha)} x_j \right| .
\] (16)

Proof. Since (15) holds and $c_0, c$ are $BK-$spaces with respect to their natural norms (see [21, pp. 16-17]) and the matrix $\Delta^{(\alpha)}(\Gamma) = (\tau_{nk})$ is a triangle, Theorem 4.3.12 Wilansky [24, pp. 63] gives the fact that $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ are $BK-$spaces with the given norms. This completes the proof.

Now, we may give the following theorem concerning the isomorphism between the spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ respectively:

Theorem 2. The sequence spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$ are linearly isomorphic to the spaces $c_0$ and $c$, respectively, i.e, $c_0(\Gamma, \Delta^{(\alpha)}, u) \cong c_0$ and $c(\Gamma, \Delta^{(\alpha)}, u) \cong c$.

Proof. We prove the theorem for the space $c(\Gamma, \Delta^{(\alpha)}, u)$. To prove this, we should show the existence of a linear bijection between the spaces $c(\Gamma, \Delta^{(\alpha)}, u)$ and $c$. Consider the transformation $T$ defined, with the notation of (14), from $c(\Gamma, \Delta^{(\alpha)}, u)$ to $c$ by $x \mapsto y = T x = \Delta^{(\alpha)}(\Gamma) x$. The linearity of $T$ is clear. Further, it is trivial that $x = \theta$ whenever $T x = \theta$ and hence $T$ is injective. Let be $y = (y_k) \in c$ and we define a sequence $x = (x_k) \in c(\Gamma, \Delta^{(\alpha)}, u)$ by
\[
x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} \frac{y_{k-i} - y_{k-i-1}}{u_{k-i}} .
\] (17)

Then by Lemma [2] we deduce that
\[
\sum_{j=0}^{k} u_j \Delta^{(\alpha)} x_j = \sum_{j=0}^{k} u_j \Delta^{(\alpha)} \left( \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} \frac{y_{j-i} - y_{j-i-1}}{u_{j-i}} \right)
\]
\[
= \sum_{j=0}^{k} u_j \Delta^{(\alpha)} \left( \frac{y_j - y_{j-1}}{u_j} \right)
\]
\[
= \sum_{j=0}^{k} (y_j - y_{j-1}) = y_k
\] (18)
Hence, \( x \in c(\Gamma, \Delta^{(\alpha)}, u) \) so \( T \) is surjective. Furthermore one can easily show that \( T \) is norm preserving. This complete the proof.

3. The \( \beta \)-Dual of The Spaces \( c_0(\Gamma, \Delta^{(\alpha)}, u) \) and \( c(\Gamma, \Delta^{(\alpha)}, u) \)

In this section, we determine the \( \beta \)-dual of the spaces \( c_0(\Gamma, \Delta^{(\alpha)}, u) \) and \( c(\Gamma, \Delta^{(\alpha)}, u) \). For the sequence spaces \( X \) and \( Y \), define the set \( S(X, Y) \) by

\[
S(X, Y) = \{ z = (z_k) \in \omega : xz = (x_kz_k) \in Y \text{ for all } x \in X \}. \tag{19}
\]

With the notation of (19), \( \beta \)-dual of a sequence space \( X \) is defined by

\[
X^\beta = S(X, cs).
\]

**Lemma 3.** \( A \in (c_0 : c) \) if and only if

\[
\lim_{n \to \infty} a_{nk} = \alpha_k \text{ for each fixed } k \in \mathbb{N}, \tag{20}
\]

\[
\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{21}
\]

**Lemma 4.** \( A \in (c : c) \) if and only if (20) and (21) hold, and

\[
\lim_{n \to \infty} \sum_k a_{nk} \text{ exists.} \tag{22}
\]

**Lemma 5.** \( A = (a_{nk}) \in (\ell_\infty : \ell_\infty) \) if and only if

\[
\sup_n \sum_k |a_{nk}| < \infty. \tag{23}
\]

**Theorem 3.** Define the sets \( \Gamma_1, \Gamma_2 \) and a matrix \( T = (t_{nk}) \) by

\[
t_{nk} = \begin{cases} 
  t_k - t_{k+1}, & (k < n) \\
  t_n, & (k = n) \\
  0, & (k > n)
\end{cases}
\]
for all \(k, n \in \mathbb{N}\) where \(t_k = a_k \sum_{i=0}^{k} (-1)^i \frac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} u_{k-i}\)

\[
\Gamma_1 = \left\{ a = (a_n) \in w : \lim_{n \to \infty} t_{nk} = \alpha_k \text{ exists for each } k \in \mathbb{N} \right\}
\]

\[
\Gamma_2 = \left\{ a = (a_n) \in w : \sup_{n \in \mathbb{N}} \sum_k |t_{nk}| < \infty \right\}
\]

\[
\Gamma_3 = \left\{ a = (a_n) \in w : \sup_{n \in \mathbb{N}} \lim_{n \to \infty} \sum_k t_{nk} \text{ exists} \right\}.
\]

Then, \(\{c_0(\Gamma, \Delta^{(\alpha)}, u)\}^\beta = \Gamma_1 \cap \Gamma_2\) and \(\{c(\Gamma, \Delta^{(\alpha)}, u)\}^\beta = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3\).

**Proof.** We prove the theorem for the space \(c_0(\Gamma, \Delta^{(\alpha)}, u)\). Let \(a = (a_n) \in w\) and \(x = (x_k) \in c_0(\Gamma, \Delta^{(\alpha)}, u)\). Then, we obtain the equality

\[
\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[ a_k \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} \frac{1}{u_{k-i}} \right] (y_k - y_{k-i-1})
\]

\[
= \sum_{k=0}^{n} \left[ a_k \sum_{i=0}^{k} (-1)^i \frac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} \frac{1}{u_{k-i}} \right] (y_k - y_{k-i-1})
\]

\[
= \sum_{k=0}^{n-1} \left[ a_k \sum_{i=0}^{k} (-1)^i \frac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} \frac{1}{u_{k-i}} - a_{k+1} \sum_{i=0}^{k+1} (-1)^i \frac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} \frac{1}{u_{k+1-i}} \right] y_k
\]

\[
+ [ a_n \sum_{i=0}^{n} (-1)^i \frac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} \frac{1}{u_{n-i}} ] y_n
\]

\[= T_n y \]  \hspace{1cm} (24)

Then we deduce by (24) that \(ax = (a_k x_k) \in cs\) whenever \(x = (x_k) \in c_0(\Gamma, \Delta^{(\alpha)}, u)\) if and only if \(Ty \in c\) whenever \(y = (y_k) \in c_0\). This means that \(a = (a_k) \in \{c_0(\Gamma, \Delta^{(\alpha)}, u)\}^\beta\) if and only if \(T \in (c_0 : c)\). Therefore, by using Lemma 3, we obtain:

\[
\lim_{n \to \infty} t_{nk} = \alpha_k \text{ exists for each } k \in \mathbb{N}, \hspace{1cm} (25)
\]

\[
\sup_{n \in \mathbb{N}} \sum_k |t_{nk}| < \infty. \hspace{1cm} (26)
\]

Hence, we conclude that \(\{c_0(\Gamma, \Delta^{(\alpha)}, u)\}^\beta = \Gamma_1 \cap \Gamma_2\).
4. Some matrix transformations related to the sequence spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$

In this final section, we state some results which characterize various matrix mappings on the spaces $c_0(\Gamma, \Delta^{(\alpha)}, u)$ and $c(\Gamma, \Delta^{(\alpha)}, u)$.

We shall write throughout for brevity that

$$\tilde{a}_{nk} = z_{nk} - z_{n,k+1} \quad \text{and} \quad b_{nk} = \sum_{j=0}^{n} \left( \sum_{i=0}^{n-j} (-1)^i \frac{\Gamma(\alpha + 1)}{i!\Gamma(\alpha + 1 - i)} u_{i+j} \right) a_{jk}$$

for all $k, n \in \mathbb{N}$, where

$$z_{nk} = a_{nk} \sum_{i=0}^{k} (-1)^i \frac{\Gamma(1 - \alpha)}{i!\Gamma(1 - \alpha - i)} \frac{1}{u_{k-i}}.$$

Now, we may give the following theorem.

**Theorem 4.** Let $\lambda$ be any given sequence space and $\mu \in \{c_0, c\}$. Then, $A = (a_{nk}) \in (\mu(\Gamma, \Delta^{(\alpha)}, u) : \lambda)$ if and only if $C \in (\mu : \lambda)$ and

$$C^{(n)} \in (\mu : c)$$

for every fixed $n \in \mathbb{N}$, where $c_{nk} = \tilde{a}_{nk}$ and $C^{(n)} = (c^{(n)}_{mk})$ with

$$c^{(n)}_{mk} = \begin{cases} 
  z_{nk} - z_{n,k+1}, & (k < m) \\
  z_{nm}, & (k = m) \\
  0, & (k > m)
\end{cases}$$

for all $k, m \in \mathbb{N}$.

**Proof.** Let $\lambda$ be any given sequence space. Suppose that (27) holds between the entries of $A = (a_{nk})$ and $C = (c_{nk})$, and take into account that the spaces $\mu(\Gamma, \Delta^{(\alpha)}, u)$ and $\mu$ are linearly isomorphic.

Let $A = (a_{nk}) \in (\mu(\Gamma, \Delta^{(\alpha)}, u) : \lambda)$ and take any $y = (y_k) \in \mu$. Then, $C\Delta_u^{(\alpha)}(\Gamma)$ exists and $\{a_{nk}\}_{k \in \mathbb{N}} \in \mu(\Gamma, \Delta^{(\alpha)}, u)^\beta$ which yields that (28) is necessary and $\{c_{nk}\}_{k \in \mathbb{N}} \in \mu^\beta$ for each $n \in \mathbb{N}$. Hence, $Cy$ exists for each $y \in \mu$ and thus by letting $m \to \infty$ in the equality

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1} \left( z_{nk} - z_{n,k+1} \right) y_k + z_{nm}y_m; \quad (m, n \in \mathbb{N})$$

(29)
we have that \( Cy = Ax \) and so we have that \( C \in (\mu : \lambda) \).

Conversely, suppose that \( C \in (\mu : \lambda) \) and (28) hold, and take any \( x = (x_k) \in \mu(\Gamma, \Delta^{(\alpha)}, u) \). Then, we have \( \{c_{nk}\}_{k \in \mathbb{N}} \in \mu^\beta \) which gives together with (28) that \( \{a_{nk}\}_{k \in \mathbb{N}} \in \mu(\Gamma, \Delta^{(\alpha)}, u)^{\beta} \) for each \( n \in \mathbb{N} \). So, \( Ax \) exists. Therefore, we obtain from the equality

\[
\sum_{k=0}^{m} c_{nk} y_k = \sum_{k=0}^{m} \left[ \sum_{j=k}^{m} (-1)^j \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - i) \Gamma(i)} u_{i+j} \right] c_{nj} x_k \quad \text{for all } n \in \mathbb{N},
\]

as \( m \to \infty \) that \( Ax = Cy \) and this shows that \( A \in (\mu(\Gamma, \Delta^{(\alpha)}, u) : \lambda) \). This completes the proof.

**Theorem 5.** Suppose that the entries of the infinite matrices \( A = (a_{nk}) \) and \( B = (b_{nk}) \) are connected with the relation (27) and \( \lambda \) be given sequence space and \( \mu \in \{c_0, c\} \). Then \( A = (a_{nk}) \in (\lambda : \mu(\Gamma, \Delta^{(\alpha)}, u)) \) if and only if \( B = (b_{nk}) \in (\lambda : \mu) \).

**Proof.** Let \( z = (z_k) \in \lambda \) and consider the following equality with (27)

\[
\sum_{k=0}^{m} b_{nk} z_k = \sum_{j=0}^{n} \left[ \sum_{k=0}^{m} (-1)^j \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - i) \Gamma(i)} u_{i+j} \right] a_{jk} z_k \quad (m, n \in \mathbb{N}),
\]

which yields as \( m \to \infty \) that \( (Bz)_n = [\Delta^{(\alpha)} u(\Gamma)(Az)]_n \). Hence, we obtain that \( Az \in \mu(\Gamma, \Delta^{(\alpha)}, u) \) whenever \( z \in \lambda \) if and only if \( Bz \in \mu \) whenever \( z \in \lambda \).

We will have several consequences by using Theorem 4 and Theorem 5. But we must give firstly some relations which are important for consequences:

\[
\sup_{n \in \mathbb{N}} \sum_{k} |a_{nk}| < \infty \quad (31)
\]

\[
\lim_{n \to \infty} a_{nk} = \alpha_k \quad \text{exists for each fixed } k \in \mathbb{N} \quad (32)
\]

\[
\lim_{k \to \infty} a_{nk} = 0 \quad \text{for each fixed } n \in \mathbb{N} \quad (33)
\]

\[
\lim_{n \to \infty} \sum_{k} a_{nk} \quad \text{exists} \quad (34)
\]

\[
\lim_{n \to \infty} \sum_{k} a_{nk} = 0 \quad (35)
\]
\[
\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty \quad (36)
\]  
\[
\lim_{n \to \infty} \sum_k |a_{nk}| = 0 \quad (37)
\]  
\[
\sup_{n,k} |a_{nk}| < \infty \quad (38)
\]  
\[
\lim_{m \to \infty} \sum_k |a_{nk}| = \sum_k |\alpha_k| \quad (39)
\]

Now, we can give the corollaries:

**Corollary 1.** The following statements hold:

(i) \( A = (a_{nk}) \in (c_0(\Gamma, \Delta(\alpha), u)_\ell : \ell_\infty) = (c(\Gamma, \Delta(\alpha), u) : \ell_\infty) \) if and only if (31) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (28) also holds.

(ii) \( A = (a_{nk}) \in (c_0(\Gamma, \Delta(\alpha), u)_c : c) \) if and only if (31) and (32) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (28) also holds.

(iii) \( A = (a_{nk}) \in (c_0(\Gamma, \Delta(\alpha), u)_c : c_0) \) if and only if (31) and (33) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (28) also holds.

(iv) \( A = (a_{nk}) \in (c_0(\Gamma, \Delta(\alpha), u) : \ell) = (c(\Gamma, \Delta(\alpha), u) : \ell) \) if and only if (30) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (28) also holds.

(v) \( A = (a_{nk}) \in (c_0(\Gamma, \Delta(\alpha), u)_0 : \ell_\infty) = (c(\Gamma, \Delta(\alpha), u) : \ell_\infty) \) if and only if (36) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (28) also holds.

**Corollary 2.** The following statements hold:

(i) \( A = (a_{nk}) \in (\ell_\infty : c_0(\Gamma, \Delta(\alpha), u)) \) if and only if (34) hold with \( b_{nk} \) instead of \( a_{nk} \).

(ii) \( A = (a_{nk}) \in (c : c_0(\Gamma, \Delta(\alpha), u)) \) if and only if (31), (35) and (36) hold with \( b_{nk} \) instead of \( a_{nk} \).

(iii) \( A = (a_{nk}) \in (c_0 : c_0(\Gamma, \Delta(\alpha), u)) \) if and only if (31) and (33) hold with \( b_{nk} \) instead of \( a_{nk} \).

(iv) \( A = (a_{nk}) \in (c : c_0(\Gamma, \Delta(\alpha), u)) \) if and only if (33) and (38) hold with \( b_{nk} \) instead of \( a_{nk} \).

(v) \( A = (a_{nk}) \in (\ell_\infty : c(\Gamma, \Delta(\alpha), u)) \) if and only if (32) and (39) hold with \( b_{nk} \) instead of \( a_{nk} \).

(vi) \( A = (a_{nk}) \in (c : c(\Gamma, \Delta(\alpha), u)) \) if and only if (31), (32) and (34) hold with \( b_{nk} \) instead of \( a_{nk} \).
(vii) $A = (a_{nk}) \in (c_0 : c(\Gamma, \Delta^{(\alpha)}, u))$ if and only if (31) and (32) hold with $b_{nk}$ instead of $a_{nk}$.

(viii) $A = (a_{nk}) \in (\ell : c(\Gamma, \Delta^{(\alpha)}, u))$ if and only if (32) and (38) hold with $b_{nk}$ instead of $a_{nk}$.

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