Powers of doubly-affine integer square matrices
with one non-zero eigenvalue

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Abstract

When doubly-affine matrices such as Latin and magic squares with a
single non-zero eigenvalue (1EV) are powered up they become constant
matrices after a few steps. The process of compounding squares of orders
$m$ and $n$ can then be used to generate an infinite series of 1EV squares of
orders $mn$.

The Cayley-Hamilton theorem is used to understand this 1EV prop-
erty, where their characteristic polynomials: $x^n = L_n x^{n-1}$, have just two
terms, with $n$ the order of the square and $L_n$ the row-column sum.

1 Introduction

We consider a diagonal Latin square [DLS] $A = sud4a$ from Cameron, Rogers
and Loly [6] [DMPS] of four symbols (here 1, 2, 3, 4 in arithmetic progression)
which has just one non-zero eigenvalue (1EV), and then its matrix square $A^2$
and cube $A^3$:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
2 & 1 & 4 & 3
\end{bmatrix}
\begin{bmatrix}
27 & 23 & 27 & 23 \\
23 & 27 & 23 & 27 \\
23 & 27 & 23 & 27 \\
27 & 23 & 27 & 23
\end{bmatrix}
\begin{bmatrix}
250 & 250 & 250 & 250 \\
250 & 250 & 250 & 250 \\
250 & 250 & 250 & 250 \\
250 & 250 & 250 & 250
\end{bmatrix}
\] (1)

The intermediate square of two symbols still has rows ($R$), columns ($C$) and
diagonals (main $d_1$, dexter $d_2$) (RCD) with a common linesum (100), but the
final stage is a constant matrix with identical elements, with all higher powers
then constant matrices. We described $sud4a$ as a mini-Sudoku in DMPS because
of its design as a miniature version of order 2 Latin squares as subsquares in a
$4 \times 4$ frame, contrasted with the well known Sudoku solutions of a $9 \times 9$ puzzle
of a larger Latin square of 9 symbols (1, 2, ...9) with one of each in every 3-by-3
subsquare. This result in (1) characterizes the present study and the sequence
is quite dramatic when each unique element is designated by a colour, declining
from four colours, through two, to a uniform single colour.
This study was prompted by a note to an online group of magic square bloggers c. 27 September 2016 by Miguel Amela [AMA], in Powers of Associative Magic Squares [2], which renewed interest w.r.t "powering-up matrixwise" associative magic squares (also called regular or symmetric in much of the earlier literature - see Loly, Cameron, Trump and Schindel [15] [LAA], which also includes references to earlier literature). This is quite different from the issue of multimagic squares which remain DDA when all their elements are squared, cubed, etc. see Boyer [5] and Derksen, Eggermont and Van den Essen [9].

In particular Miguel Amela was concerned to show counterexamples to Proposition 6.1 by Cook, Bacon and Hillman [8] [CBH]:

"Let $M$ be any associative $p \times p$ magic square with magic constant $m$. Then $M^{2n+1}$ is magic with magic constant $m^{2n+1}$, and $M^{2n}$ is semimagic."

We will show counterexamples, in part because we find examples that become constant matrices at orders $n = 4, 5, 8, 16, ...$.

CBH stated that the proof of their proposition "involves matrix algebra and an analysis of the eigenvalues ...", but details were not given. An alternation for odd versus even powers of associative magic squares was also claimed. We will illuminate this claim with a variety of examples, while extending beyond the specific constraints of associative magic squares, i.e. those magic squares with antipodal pair sums of $1 + n^2$ (or Latins with pair sums $1 + n$).

While CBH were concerned with the "magicness" of the powered-up squares, the present work evolves from Amela’s later observation [30 September 2016] linking some counterexamples to certain order 4 associative magic squares having the property of just one non-zero eigenvalue, "1EV" - again see our LAA.

The corresponding characteristic polynomials are obtained by evaluating: $\det (A - x I_n) = 0$, where $I_n$ is the $n$-by-$n$ identity matrix. For the first three powers ($p = 1, 2, 3$) of $sud4a$ the characteristic polynomials [CharPoly] are given in turn with their corresponding eigenvalues ($\lambda_i$), singular values (SVs: $\sigma_i$), and matrix rank $r$ (equal to the number of non-zero SVs), and sum of the 4th powers of the second to last SVs, $R$. 

| $p$ | CharPoly | $\lambda_i$ | $\sigma_i$ | $r$ | $R$ |
|-----|-----------|--------------|------------|-----|-----|
| 1   | $x^4 = 10x^4$ | 10, 0, 0, 0  | 10, 4, 2, 0 | 3   | 272 |
| 2   | $x^4 = 100x^4$ | 100, 0, 0, 0 | 100, 8, 0, 0 | 2   | 4096|
| 3   | $x^4 = 1000x^4$ | 1000, 0, 0, 0 | 1000, 0, 0, 0 | 1   | 0   |

Table 1 - $sud4a$ - RCD linesum ($s_4 = 10$)$^p$. The sole eigenvalue and leading singular value are equal to RCD (and the 1EV), $\lambda_1 = (10)^p$.

**Note the vanishing R-index which is the sum of the 4th powers of all but the lead SV.**

In DMPS we introduced "singular value clans", which are characterised by a particular set of singular values, as a powerful way to classify different magic and Latin squares of the same order, and the $R$-index:

$$ R = \sum_{i=2}^{n} \sigma_i^4 $$  \hspace{1cm} (2)
which is an integer for all integer squares - see DMPS.

The rest of this paper continues with some preliminary matrix spectral issues, before completing the order 4 Latin square issues, then moving on to the parallel situation for magic squares, and after that the results of compounding squares of orders $m$ and $n$, as an extension of Chan and Loly [7] in order to generate an infinite series of 1EV squares of orders $mn$, which the present authors know [22] [RCL] will also have just 1EV.

The present approach builds on our previous 2004-7 studies of magic square spectra in [15] [LAA] which revealed several cases with 1EV and included a discussion of non-diagonable matrices as well as the insight gained by looking at Jordan Normal Forms, together with our more recent 2013 extension which also included Latin squares in DMPS. For sud4a these all have Mattingly's [16] multiplicity ($\mu$) of zero eigenvalues (here $\mu = 3$), originally for magic squares.

The number of SVs (and thus the matrix rank) decrease from 3 through 2 to 1 in this powering. We will later use the Cayley-Hamilton theorem [13] which states that the characteristic polynomial is also satisfied by the matrix itself, to understand this 1EV property. These 1EV squares provide hitherto unexpected examples which become constant on matrix powering.

Integer Latin and magic squares each have characteristic row and column sums which are therefore doubly-affine [DA]. In addition magic squares have the same sums for both major diagonals. All of our early DA squares can be scaled up by constructing larger compound (or composite) DA squares of multiplicative order following an ancient strategy extended by Chan and Loly [CL]. Circa 2008 we realized that this preserved the 1EV property. Details and more varieties (not needed here) are included in a companion paper [RCL].

2 Terminology for magic and Latin squares

Our interest in 1EV matrices began c. 2004 with magic squares as reported in a conference in 2007 [14] [IWMS] and published in 2009 [LAA], but we did not envisage this powering issue, even though we were aware of prior interest in powering up magic squares as indicated in section 3.5 of that work. However in September 2016 Miguel Amela [2] [AMA] rekindled this by drawing our attention to squaring 8 associative (or symmetric or regular) [3] order 4 magic squares which we had previously identified as having 1EV [14],[15].

We are particularly concerned to be more precise with the use of both magic and semimagic terms by grounding them in a long tradition that stretches more than a century, e.g. Andrews classic work [3], itself largely a compilation of works by several authors who published in The Monist.

We find it worth clarifying magic terminology since CBH appear to have followed Stark [24] who used "magic" when just the row and column sums are equal, usually called semimagic, thus ignoring the two main diagonals usually included in the definition of traditional magic squares. Since having the same diagonal sums is a feature of the great majority of studies on magic squares [12], we include the diagonals in our definition, and thus use the familiar term
of semi-magic squares for those lacking one or both diagonals.

Here magic is reserved for "classic" magic squares of sequential integers with the same RCD sums [3], and semimagic whenever one or both diagonals do not have the same linesum, so we leave "magic" and "semimagic" for those which have non-sequential elements. We note that semimagic and Latin squares are also doubly-affine [DA], while magic squares are diagonal doubly-affine [DDA]. We use magic squares with elements from the consecutive set 1, 2, 3, ...n^2, and our Latin squares have n symbols in each row and column whose elements run from 1, 2, ...n, which are also the most common historically. Then their linesums (a.k.a. magic constants, L_n, in the abstract) for magic and Latin squares are respectively:

$$S_n = \frac{n}{2}(1 + n^2), \quad s_n = \frac{n}{2}(1 + n)$$

(3)

e.g. S_3 = 15, S_4 = 34, ... and s_2 = 3, s_3 = 6, s_4 = 10, ..., replacing the generic L_n used in the Abstract. We note that the elements used here are in arithmetic progression, but that there is some interest in magic squares which are not, e.g. Fibonacci magic squares in CBH which we examine near the end.

All magic squares have 8 phases on rotation and reflection, and we have previously noted in LAA some variations in spectral properties as a result that turn out to be relevant here. Amongst the varieties of magic squares several types have further properties due to additional constraints, e.g. associated (or associative or regular) where all antipodal pairs have the same sum, pandiagonal where all parallel broken diagonals also have the magic linesum, ultramagic which are both associative and pandiagonal, etc. see LAA. Further there are other squares of interest, particularly with Ben Franklin's bent-diagonal squares which at least are semimagic due to their half row and column sum property - see Schindel, Rempel and Loly [23] [PRSA].

Our tests include orders 3, 4, 5, 8 and 16 for magic squares, but began with the Latin square sud4a which affords by far the simplest example. Also we do not restrict ourselves to associative magic squares, knowing already that other types in LAA exhibit 1EVs. Also our order four 1EV examples multiply up to order 16 by the basic compounding process described in 2002 by Chan and Loly [7] [CL] which preserves the 1EV property.

For the 880 order 4 magic squares we use the indexing common to the book by Benson and Jacoby [4], as well as the web pages of Harvey Heinz [12]. Note also that CBH use "regular" for sequential elements - we prefer to reserve regular as an alternate to associative as per Andrews 1917, 2004 [3].

3 Doubly-affine considerations

It is helpful now to rephrase CBH’s Proposition 1.1: "If A and B are n × n semimagic matrices with magic constants M and N, respectively, then AB and BA are semimagic with magic constant MN."

A proof was given by CBH which remains valid if we edit it to apply to doubly-affine square matrices as a new Proposition:
PROPOSITION 1: If $A$ and $B$ are $n \times n$ DA matrices with magic constants $M$ and $N$, respectively, then $AB$ and $BA$ are DA with magic constant $MN$.

3.1 Multiplying by constant square matrices

CBH also noted that: "If $A$ is magic and if some power of $A$ is constant, then since a constant matrix times a magic square is constant, it follows that all higher powers are magic."

Our Proposition 1 now takes care of multiplying by a constant square matrix. Such a constant matrix can be represented by $cE_n$ where $c$ is a multiplicative factor and $E_n$ is an $n \times n$ unit matrix consisting of all 1’s.

In Tables below we will characterize the nature of the DA integer squares by their "Type", i.e. magic or diagonal DA (DDA), and semimagic or simply DA.

Also the elements do not need to be integers for either the Latin or magic squares, although that is usually the case. Both Latin squares and magic squares are characterised by constant row and column linesums, and even without the same linesum(s) for both diagonals, are both DA, hence its use in the title. All magic squares are characterised by having the same linesum on all rows, columns and both diagonals.

4 Characteristic Polynomials and the Cayley-Hamilton Theorem

In Table 1 we saw that $sudda$ has rank 3, with characteristic polynomial:

$$x^4 - 10x^3 = 0, \text{ or } x^4 = 10x^3. \quad (4)$$

We will build more evidence later for the role of a single non-zero eigenvalue (1EV) but for now turn to looking at the characteristic polynomials for an understanding of the powering of 1EV magical squares to constancy - see [15].

For $sudda$ in (1) has solutions $x = 10$, the linesum eigenvalue, and the triply degenerate $x = 0$. So according to the Cayley-Hamilton theorem [13] one can substitute the matrix $A$ for $x$ in the characteristic equation (if $0$ is the null matrix and $I$ is the identity matrix of the same order, here $n = 4$, with $s_1 = \lambda_1$):

$$A^4 - 10A^3 = 0, \text{ or } A^4 = 10A^3. \quad (5)$$

Which means that both $A^3$ AND $A^4$ are constant matrices, and then multiplying both sides successively by $A$, for $A^5 = 10A^4$, and so on for constancy at all higher powers. We see that all powers higher than $p = 2$ are constant matrices with just one non-zero eigenvalue ($\mu = 3$).

We conclude more generally that Cayley-Hamilton applied to 1EV characteristic polynomials amounts to:

PROPOSITION 2: Doubly-affine squares with 1EV power up $A$ to constant matrices.
Which we will illustrate by examples where constancy occurs at either the cube or quartic power.

4.1 Singular Values (SVs) and Gramian matrix product

The source of the matrix singular values (SVs), the Gramian matrix product used to obtain the squares of the SVs from the product of a matrix and its transpose, either $\mathbf{A}\mathbf{A}^T$ or $\mathbf{A}^T\mathbf{A}$, is quite similar to the squared product of a matrix with itself, with the same row and column sums and for $sud4a$ is:

\[
\begin{bmatrix}
30 & 22 & 20 & 28 \\
22 & 30 & 28 & 20 \\
20 & 28 & 30 & 22 \\
28 & 20 & 22 & 30
\end{bmatrix}
\]

while differing in the information that it yields. Those SVs lead us to the entropic measures introduced by Newton and De Salvo [17] [NDS] in 2010 for their Sudoku study, which we extended to a wider set of Latin squares as well as magic squares in DMPS, and used here for comparing successive powers.

5 Measures of the diversity of matrix elements

Our results will be presented in terms of the ”Compression” entropic measure $(C)$ introduced for Sudoku solutions following NDS, which offers an insight into the increase in uniformity upon powering up through squaring, cubing, etc. through a measure based on the singular values of matrices. Specifically one takes the Shannon entropy, $H$, which is a function of the singular values normalized by their sum, $\hat{\sigma}_i$, and then used to obtain the percentage Compression $C$ in NDS:

\[
H = -\sum_i \hat{\sigma}_i \ln(\hat{\sigma}_i) , \quad C = (1 - H/\ln(n)) \times 100\%
\]

An alternative ”Spread” measure of the elements divides the difference between max and min elements of matrices by their average value:

\[
Spread = n(\text{Max}[f_i] - \text{Min}[f_i])/L(n),
\]

where $L(n)$ is the row-column linesum eigenvalue. For $sud4a$:

| $p$ | $r$ | $C\%$ | Spread  | Type                          |
|-----|-----|-------|---------|-------------------------------|
| 1   | 3   | 35.0603 | 1.2     | diagonal Latin (DLS) ($R = 272$) |
| 2   | 2   | 80.9527 | 0.16    | DDA (& associative)           |
| 3   | 1   | 100    | 0       | constant; $250E_4$            |

Table 2 - $sud4a$ - $p$ is the matrix power, $r$ the matrix rank, $C\%$ the Compression (here to 4 decimal places). The spectra listed in Table 1 are not repeated here.
In our tables $C\% = 100$ means that every element of the cubed and higher powers of $sud4a$ has all elements the same. Increasing powers have decreasing Spread, although we prefer the increasing NDS Compression measure.

6 A note on Jordan Normal Form

The 1EV squares that produce the constant matrix are non-diagonalizable [see LAA], as revealed by the algebraic and geometric multiplicities of the eigenvalues. In these situations a decomposition of the non-diagonalizable matrix $A$ is still possible via the Jordan normal form, $A = PJP^{-1}$ where $P$ is an invertible matrix and $J$ is the Jordan matrix [13],[20]. We note that this form allows powers of the matrix to be computed easily $A^n = PJ^nP^{-1}$, since we have $PP^{-1} = I$, the identity matrix.

The Jordan form of the square is useful since it is composed of $q$ blocks each of rank $r_i$, $i = 1..q$, whose sum corresponds to the rank of the square and the maximum blocksize $k$ indicates the exponent required to reduce our magic square to a constant matrix.

The Jordan normal form (often called the Jordan canonical form) is an upper triangular matrix which has each non-zero off-diagonal entry equal to 1, immediately above the main diagonal (on the superdiagonal), and with identical diagonal entries to the left and below them to form blocks, e.g. the 3-by-3, 2-by-2 and 1-by-1 blocks:

$$
\begin{bmatrix}
\lambda_1 & 1 & 0 \\
0 & \lambda_1 & 1 \\
0 & 0 & \lambda_1 \\
\end{bmatrix}
\begin{bmatrix}
\lambda_2 & 1 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_{q-1} \\
\end{bmatrix}
\begin{bmatrix}
\lambda_q & 0 \\
0 & \lambda_q \\
\end{bmatrix}.
$$

(9)

For a diagonalizable matrix the Jordan form simply has eigenvalues along the main diagonal, with 1’s above the diagonals in each block of dimension greater than one and where all non-displayed entries are zero [19].

7 Nilpotent Square Matrices with 1EV

A nilpotent matrix is a square matrix $N$ such that $N^q = 0$ for some positive integer $q$, where its smallest value is sometimes called the degree or index of $N$.

Let $Z$ be a Latin or magic square of order $n$ with magic constant linesum $l$ and $n - 1$ zero eigenvalues, then we can write $Z$ in terms of $N$, a nilpotent
matrix of order \( n \) and index \( k \leq n \) with zero line sum and the square matrix of order \( n \) of all ones \( E_n \), in the following fashion:

\[
Z = N + \left( \frac{I}{n} \right) E_n, \text{ or } N = Z - \left( \frac{I}{n} \right) E_n.
\]  

(10)

The square \( N \) will be nilpotent since it will have all zero eigenvalues as required. If we raise \( Z \) to the \( k \)th power we will end up with a polynomial in \( N \) and \( E_n \) composed of terms \( N^k, \left( \frac{I}{n} \right)^k E_n^k \), and cross-terms in \( \left( \frac{I}{n} \right)^p N^k E_n^p \), with appropriate binomial coefficients. The first term \( N^k \) vanishes since we have index \( k \), the crossterms will all vanish because they involve multiplying \( N \) by \( E_n \) which simply gives zero elements because effect of \( E_n \) is to sum the rows/columns of \( N \) that has zero linesum. Since \( E_n^k = n^{k-1} E_n \), we end up for our magic square that \( Z^k = \left( \frac{I}{n} \right) E_n \), a constant matrix.

Also for a nilpotent matrix \( N \) of index \( k \) the Jordan form, \( J_k \), has just 1's above the main diagonal, from which it follows that to reduce the Jordan form to a zero matrix we need to raise it successively to the \( k \)th power while watching the one's migrate out of the resulting products.

For the 1EV order 4 DA squares of interest here the Jordan form is:

\[
J_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \text{ with } \lambda = l, \text{ the linesum eigenvalue.}
\]  

(11)

The largest blocksize is 3 which has rank 2, so the rank of this ”4-square” is \( r_4 = 2 + 1 = 3 \). If \( L_4 \) is the DA linesum the diagonal elements for \( J_4 \) are \( \{0, 0, 0, L_4^3\} \), and all superdiagonal elements are zero. This form is characteristic of all \( n = 4 \) 1EV magic squares, and a \( J_5 \) example for \( n = 5 \) will be shown later.

8 Jordan forms for powers of sud4a (\( \mu = 3 \))

For sud4a and its powers in the Introduction the second power has all element pairs symmetric about the centre with the same sum, there 50, and can be classified as associative (or regular) using terminology from the literature of magic squares [15], with higher powers trivially so. However the initial sud4a is clearly not associative.

We now show their Jordan Forms [20],[13] in terms of blocks along the main diagonal:

\[
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 100 \\ 1000 \end{bmatrix}
\]  

(12)

Here the off-diagonal ”1’s” move out for each rise in power, with their matrix ranks first from an order 3 block of rank 2 and the diagonal ”10” for rank 8.
\[ r = 2+1 = 3, \] next an order 2 block and the diagonal "100" for rank \( r = 1+1 = 2, \) and then final rank \( r = 0+1 = 1, \) after which the higher powers are all constant matrices.

### 9 lat4a - 3 non-zero eigenvalues [3EV] (\( \mu = 1 \))

There are many singular squares of order 4 and higher with three non-zero eigenvalues, the abbreviation 3EV will be useful. Here we compare sud4a with lat4a, from which we obtained the former by moving the second row to the last.

Now the first three powers of lat4a are:

\[
lat4a = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1
\end{bmatrix}, \begin{bmatrix}
30 & 28 & 22 & 20 \\
28 & 30 & 20 & 22 \\
22 & 20 & 30 & 28 \\
20 & 22 & 28 & 30
\end{bmatrix}, \begin{bmatrix}
264 & 264 & 236 & 236 \\
264 & 264 & 236 & 236 \\
268 & 264 & 236 & 236 \\
268 & 264 & 236 & 236
\end{bmatrix}, ... \quad (13)
\]

None of these are associative, but all are antipodal about the center [or symmetric about both main diagonals], and all still four symbol Latin. The spectra of lat4a and its first four powers are:

- \( p = 1: \lambda_i = 10, -4, -2, 0; \sigma_i = 10, 4, 2, 0; \)
- \( p = 2: \lambda_i = \sigma_i = 100, 16, 4, 0; \)
- \( p = 3: \lambda_i = 10^3, -64, -8, 0; \sigma_i = 10^3, 64, 8, 0; \)
- \( p = 4: \lambda_i = \sigma_i = 10^4, 256, 16, 0; \)

all with multiplicity \( \mu = 1 \) of zero eigenvalues. Then other measures are:

| \( p \) | \( r \) | C\% | Spread | Type | \( R \) |
|---|---|---|---|---|---|
| 1 | 3 | 35.0003 | 1.2 | Latin \( d1 = 4, d2 = 16 \) | 272 |
| 2 | 3 | 61.4828 | 0.4 | DA \( d1 = 120, d2 = 80 \) | 65,792 |
| 3 | 3 | 80.5474 | 0.144 | DA \( d1 = 928, d2 = 1072 \) | 16,781,312 |
| 4 | 3 | 90.7518 | 0.0544 | DA \( d1 = 10272, d2 = 9728 \) | 4,295,032,832 |

Table 3 - lat4a

In Table 3 note how Compression gradually increases towards the 100% of uniformity, while the Spread decreases gradually towards uniformity, but neither is ever achieved. Here the average of the diagonal sums is the common RC linesum in Table 1.

The Jordan Normal form was already useful to us for non-diagonalizable sud4a, as shown in [15] for magic squares, so the diagonal form for lat4a which is diagonalizable has eigenvalues \(-4, -2, 0, 10\) down the main diagonal, \( r = 3. \)

A discussion of eigenvalue and singular value spectra for low order Latin and magic squares that is useful in the present context is found in [6].
10 The $n = 3$ associative magic square ($\mu = 0$)

Aside from rotations and reflections there is just one third order magic square, the ancient Loshu. Here $A = \text{loshu} = \{\{4, 9, 2\}, \{3, 5, 7\}, \{8, 1, 6\}\}$, which is the top-bottom reflection of $A_1$ of CBH. We show it followed by its square, cube and quartic matrix powers ($p = 1, 2, 3$):

$$\begin{bmatrix}
4 & 9 & 2 \\
3 & 5 & 7 \\
8 & 1 & 6
\end{bmatrix}, \begin{bmatrix}
59 & 83 & 83 \\
83 & 59 & 83 \\
83 & 83 & 59
\end{bmatrix}, \begin{bmatrix}
1149 & 1029 & 1197 \\
1173 & 1125 & 1027 \\
1053 & 1221 & 1101
\end{bmatrix}, ... \quad (14)
$$

Here the non-singular characteristic polynomials are respectively:

$p = 1$: $\lambda_i = 15, 2i\sqrt{6}, -2i\sqrt{6}; \sigma_i^2 = 225, 48, 12; R = 2448$

$p = 2$: $\lambda_i = 225, -24, -24; \sigma_i = 225, 24, 24; \text{note the degeneracy in both spectra.}$ Here $R = 663, 552$

$p = 3$: $\lambda_i = 3375, \sqrt{13824}I, -\sqrt{13824}I; \sigma_i = 3375, \sqrt{27648}, \sqrt{6912}; R = 812, 187, 648$. 

While the matrix elements increase rapidly, notice how the relative spread of the elements progressively decreases on powering up. These are tabled up to sixth matrix power in the following table

| $p$ | $r$  | $C\%$ | Spread | Type    |
|-----|------|-------|--------|---------|
| 1   | 3    | 14.7017 | 1.4   | magic ($R = 2448$) |
| 2   | 3    | 46.5804 | 0.32  | DA, $d1 = 177$ |
| 3   | 3    | 73.2057 | 0.1707| DDA     |
| 4   | 3    | 88.8869 | 0.0341| DA, $d1 = 51777$ |
| 5   | 3    | 95.3843 | 0.0182| DDA     |
| 6   | 3    | 98.2995 | 0.0036| DA, $d1 = 11362977$ |

Table 4 - $n = 3$ Loshu magic square - the linesum at each power is $S_3 = 15^p$, i.e. 15, 225, 3375, 50625, ...

In Table 4 the odd and even powers are in agreement with the alternation in CBH’s Propositions 2.1 for 3-by-3’s, and their general Proposition 6.1, as their “magic” alternates with “semimagic” as powers increase and the Compression monotonically tends to 100%, while the Spread tends progressively to zero.

In LAA we showed the Jordan Form of the $\text{loshu}$, which has just the eigenvalues on the main diagonal, and also discussed the 8 phases of magic squares using the Loshu as an example of changing eigenvalues from $15, 2i\sqrt{6}, -2i\sqrt{6}$ to $15, 2\sqrt{6}, -2\sqrt{6}$ on rotation by 90 degrees in the first table on page 2667 of LAA.

11 Prior applications and studies of matrix powering

The previous 3EV cases of $\text{latAa}$ and $\text{loshu}$ both illustrate the gradual increase in Compression and reduction in Spread which are more representative of larger
general (not doubly-affine) matrices where the iterative powering up of matrices towards constancy has proved useful, e.g. for the largest eigenvalue of large matrices - Pitman [21] reference to Von Mises, etc. with addition of the corresponding eigenvector evolutions.

The Compression and Spread measures in Tables 2 and 3 are an indicator of what happens in the cases of larger matrices of full rank, but singular magic and Latin squares may converge much more rapidly as shown by some examples shown below. First we use our earlier results for the spectral properties of the well known complete set of 880 fourth order magic squares where we elucidated the 1EV phenomenon [LAA], and where we noted earlier powering literature, before a brief discussion of 1EV fifth order magic squares from the complete set of some 275 million. Higher orders are impossible to survey in their entirety, but prompted by CBH we take a look at order 8 in the context of Franklin squares [PRSA].

12 n=4 magic squares - Dudeney Groups and SV clans

From Dudeney 1917 [10] we have an important classification of Frenicle’s 1699 count into the Dudeney Group classification listing of the 880 distinct order 4 magic squares. We refer to the Appendix of Benson and Jacoby [4] for a list of the 880 with their Dudeney Group designation. In 2007-9 LAA noted [in that reference’s table 3] that the 7 Dudeney groups [I,II,III,IV,V,VI-P,VI-S] were singular with at least one zero eigenvalue. Then in DMPS we gave a summary of order $n = 4$ magic squares which have 880 distinct members in 13 Dudeney groups and the then new 63 SV clans which we introduced.

Further Dudeney Groups I (pandiagonal), II (semi-pandiagonal, semi-bent) and III (associative/regular, semi-pandiagonal), each have 48 distinct members, each with the same three singular value clans, $\alpha, \beta, \gamma$, characterised by their singular values, as shown in DMPS in that reference’s Figure 2.

In the associatives (Group III) there are eight 1EV squares:

| clan | $\sigma_i$ | Frenicle indices | $R$ |
|------|------------|-----------------|-----|
| alpha | $34, 8\sqrt{5}, 2\sqrt{5}$ | 290, 360, 790, 803 | 102, 800 |
| beta | $34, 4\sqrt{17}, 2\sqrt{17}$ | 99, 377, 489, 535 | 78, 608 |
| gamma | $34, 2\sqrt{9}, 3\sqrt{13}, 4\sqrt{3}$ | e.g. 126 (of forty total) | 74,000 |

Table 5 - summary of Group III SV clans.

For comparison with CBH we begin with one which agrees with their Proposition 6.1, choosing the 1EV $f^{360}$ which is the transpose of AMA’s fourth order associative example. AMA noted that 8 Group III Associative MSs with multiplicity $m = 3$ of zero eigenvalues [now identified as those with the single linesum eigenvalue, 1EV] as described in LAA. AMA’s counterexample to CBH’s Prop. 6.1, although AMA only took this to the second power and did not note the constancy which begins at power $p = 3$. $f^{360}$ is also a column permutation of
MATLAB’s magic(4) which is (29) in LAA, p.2668: col.1->col.3; col. 2->col.1; col.3->col.4; col.4->col.2.

12.1 $f_{360}$ clan alpha associative ($\mu = 3$) 1EV

Now:

\[
\begin{bmatrix}
2 & 11 & 7 & 14 \\
13 & 8 & 12 & 1 \\
16 & 5 & 9 & 4 \\
3 & 10 & 6 & 15
\end{bmatrix},
\begin{bmatrix}
301 & 285 & 293 & 277 \\
325 & 277 & 301 & 253 \\
253 & 301 & 277 & 325 \\
277 & 293 & 285 & 301
\end{bmatrix}, 9.826 \times 10^4
\]

(15)

The eigenvalues of these matrices are respectively: 34, 0, 0, 0; 1156, 0, 0, 0; and 39304, 0, 0, 0, note how this becomes a matrix of identical elements after power 2, as did $sud4a$, and SVs - clan alpha in Table 5. The linesums are $S_4^p = 34p$, i.e. 34, 1156, 39304. Their characteristic polynomials have the form: $-S_4x^3 + x^4 = 0$, or $x^4 = S_4x^3$, where $S_4 = 34$, with eigenvalues $\lambda_i = 34, 0, 0, 0$, and similarly for the square and cube with 34 replaced by 1156 and 39304 respectively. The next table summarizes the results up to matrix power $p = 3$:

| $p$ | $r$ | $C\%$ | $Spread$ | Type | $R$ |
|-----|-----|-------|----------|------|-----|
| 1   | 3   | 37.284 | 1.7647   | magic | 102,800 |
| 2   | 2   | 82.7039 | 0.2491 | DDA  | 40,960,000 |
| 3   | 1   | 100     | 0        | constant: $9.826 \times 10^4$ | 0 |

Table 6 - $f_{360}$ Group III, clan alpha - Note the vanishing R-index.

This does not agree with CBH’s Proposition 6.1 in their alternation between ”magic” and ”senimagic”, in fact the behaviour is totally different because of the constancy and since for $p = 2$ we find our DDA instead of their ”senimagic”. Note how the rank descends from 3 through 2 to rank 1, i.e. a matrix of equal elements, and remains there for higher powers.

N.B. CBH late in their paper in Proof of their Proposition 5.1 for an unusual order 8 magic square also indicate this point, but did not link it to the 1EV property. More on this later.

12.2 $f_{299}$ clan beta associative ($\mu = 3$) 1EV

Consider also a second 1EV from Group III, but now in SV clan beta:

$f_{299} = \{\{2, 7, 13, 12\}, \{16, 9, 3, 6\}, \{11, 14, 8, 1\}, \{5, 4, 10, 15\}\}$.

Now the only differences from Table 6 are a slightly different elements on squaring, and compressions: $C\% = 31.5627$ and 75.7272, as well as a $Spread$ of 0.3460 for $p = 2$.
12.3 **Associative \((\mu = 1)\) 3EV magic square \(f175\)**

For comparison we now look at one of the forty 3EV Associative (Regular, semi-pandiagonal) Group=III in LAA’s (31) which affords an example on a par with CBH. Table 5 in LAA shows that it is a member of SV clan alpha in Table 5. \(\mathbf{A} = f175\) is shown next, followed by its matrix square and cube:

\[
\begin{bmatrix}
1 & 12 & 8 & 13 \\
14 & 7 & 11 & 2 \\
15 & 6 & 10 & 3 \\
4 & 9 & 5 & 16
\end{bmatrix}
\begin{bmatrix}
341 & 261 & 285 & 269 \\
285 & 301 & 309 & 261 \\
261 & 309 & 301 & 285 \\
269 & 285 & 261 & 341
\end{bmatrix}
\]

For \(f175\): \(\lambda_i = 34, 8, -8, 0\); \(\sigma^2 = \{1156, 320, 20, 0\}\) showing the singularity, while the results are otherwise qualitatively similar to those in Table 4 for \(n=3\) with \(C\%\) rising from 37.2284 at \(p = 1\) to 99.7369 at \(p = 6\) and \(\text{Spread}\) declining from 1.7647 to 0.0009, with the Jordan Form for \(f175\) just has the eigenvalues on the diagonal.

While it is not our goal to give a complete account of all \(n = 4\) magic squares, it does seem worthwhile to note other 1EV cases which are not associative magic squares and thus become constant under powering. These are found in Dudeney’s [10] Group I Pandiagonals and his Group II Semi-pandiagonal and Semi-bent [see Table 3 in LAA], of which we choose one of the latter next.

12.4 **\(f27\) Group II semi-pandiagonal, semi-bent rank oscillator \((1 \leftarrow \mu \rightarrow 3)\) - not associative**

A question that must be addressed is whether associativity in magic squares such as \(f360\) is necessary for the powering pattern given by CBH of alternation of "magic" and "semimagic". In Dudeney’s Group II [semi-pandiagonal, semi-bent magic squares, LAA] there are 32 magic squares with multiplicity of zero eigenvalues \(\mu = 1\), and 16 oscillating on reflection or rotation [LAA Table 3] between \(\mu = 1\) and \(\mu = 3\), e.g. from the list in Benson and Jacoby [4] we take first \(f27\) with \(\mu = 1\) with eigenvalues \(\lambda_i = \{34, -8, 8, 0\}\) which is similar to \(f175\) in behaviour under powering, with \(f27\) having a diagonal Jordan Form the same as for \(f175\) above so not shown explicitely. Here the powering details (omitted) follow CBH’s Proposition 6.1, thus extending its scope beyond just the associative magic squares in Group II.

12.4.1 **\(f27\) rotated \(\pi/2\)**

Now with a different result compared to \(f27\) but with the pattern of constancy by power \(p = 3\) as in Table 6 for \(f360\), and with the exception only of a slightly different Type for \(p = 2\) of DA with \(d2 = 1092\) instead of DDA. So Dudeney [10] Group II "semi-bent, semi-pandiagonals" have the same behaviour for the rotated phase as did the Group III associatives for all their phases. Thus 1EV magic squares do not need to be associative!
We note also that Group I Pandiagonals have another 16 phase oscillators - see Table 3 in LAA for sixteen more 1EV magic squares.

12.5 Non-singular Group IX $f_{181}$ ($\mu = 0$) - not associative

For the sake of comparison with previous singular case we choose $A = f_{181} = \{1, 12, 13, 8\}, \{16, 9, 4, 5\}, \{2, 7, 14, 11\}, \{15, 6, 3, 10\}$, taken from equation (39) in LAA. The characteristic polynomial is $(x - 34)(x + 8)(x^2 - 8x + 24) = 0$, and now a diagonal Jordan form holds the eigenvalues. Now we tabulate the properties through to the fifth power:

| $p$ | $r$ | $C\%$ | spread | Type               |
|-----|-----|--------|--------|--------------------|
| 1   | 4   | 26.767 | 1.76471| magic ($R = 92, 288$) |
| 2   | 4   | 68.2067| 0.33218| DA $d_1 = 1236, d_2 = 1124$ |
| 3   | 4   | 90.4131| 0.08060| DA $d_1 = 38728, d_2 = 39496$ |
| 4   | 4   | 97.1769| 0.02011| DA $d_1 = 1339536, d_2 = 1335056$ |
| 5   | 4   | 99.3030| 0.00441| DA $d_1 = 45397024, d_2 = 4549248$ |

Table 7 - $f_{181}$ non-singular and not associative.

Because $f_{181}$ is not an associative magic square this was not expected to agree with CBH’s Proposition 6.1 in their alternation between ”magic” and ”semimagic”.

13 $n = 5, \mu = 4$ 1EV associative magic square

Although there are no singular order 5 Latin squares [DMPS], our LAA paper with Walter Trump already provides an order 5 magic square with just 1EV (one just four cases),

$$A = \text{laa44} = \begin{bmatrix} 2 & 11 & 21 & 23 & 8 \\ 16 & 14 & 7 & 6 & 22 \\ 25 & 17 & 13 & 9 & 1 \\ 4 & 20 & 19 & 12 & 10 \\ 18 & 3 & 5 & 15 & 24 \end{bmatrix},$$

for which the characteristic polynomial for $p = 1$ gives by Cayley-Hamilton $0 = A^5 - 65A^4$ or $A^5 = 65A^4$, so that by contrast with the earlier $n = 4$ 1EV cases of $sud4a$ and $f_{27}$ the constancy occurs first at $p = 4$ instead of $p = 3$ (as found previously):

| $p$ | $r$ | $C\%$ | spread | Type       | $R$          |
|-----|-----|--------|--------|------------|--------------|
| 1   | 4   | 25.3638| 1.84615| magic      | 706,000      |
| 2   | 3   | 64.2240| 0.41894| DDA        | 82,414,854,400 |
| 3   | 2   | 92.6908| 0.10487| DDA        | (very large) |
| 4   | 1   | 100    | 0      | $3570125E_5$ | 0            |

Table 8 - A 1EV $n = 5$ magic square ($\mu = 4$) with R,C linesums of $(S_5 = 65)^p$. Note again the vanishing R-index for a 1EV, but now at $p = 4$. 
This shows its rank 4 dropping by one in each step until it is a constant matrix at \( p = 4 \), instead of constancy at \( p = 3 \) as for \( f360 \). This associative magic square differs from CBH’s Proposition 6.1. Now the powers of the Jordan Forms see the off-diagonal 1’s moving out until constancy is reached in a similar fashion to (12), i.e. a single 4-by-4 block of rank 3 for the zero eigenvalues, whose rank reduces by 1 for each increase in power, with powers of 65 left on the lower right corner. This shows why constancy is attained at \( p = 4 \) rather than the previous \( p = 3 \) examples.

13.1 \( n = 5, (\mu = 0) \) Ultramagic non-singular magic square

In LAA we examined \( laa45 \), an ultramagic square (associative and pandiagonal). Here the results are otherwise qualitatively similar to those in Table 4 for \( n = 3 \) with \( C\% \) rising \( p \) as 23.4758 to 99.4352 at \( p = 5 \) and Spread declining from 1.84615 to 0.003. This agrees with CBH’s 2010 Proposition 6.1 which predicts alternation between their ”magic”, our DDA, and their ”semimagic”, our DA.

We also note that there are no singular \( n = 5 \) Latin squares DMPS, and so no 1EVs [DMPS].

14 Are there 1EV squares in order 6 and higher?

From DMPS there are singular rank 4 order 6 Latin squares but we are not aware of any 1EV examples. Moreover there are no \( n = 6 \) associative magic squares - see Trump [25].

For \( n = 6 \) and higher there are too many magic squares to count [25], but some of these are discussed in our LAA. So special cases must be considered, one of which we address next.

14.1 \( n=8 \) Franklin bent diagonal squares - rank 3 (\( \mu = 5 \))

Here we consider questions concerning the complete set of \( n = 8 \) Franklin squares which were first counted by Schindel, Rempel and Loly [SRL] in [23] [PRSA]. These are characterised by common half row and column sums (and thus are at least semimagic) as well as a complete set of \( 4n \) bent diagonals [PRSA]. Franklin’s handful of squares also had another property that all 2-by-2 quartets have the half row-column sum, and which is considered a further constraint. SRL were also the first to find magic Franklin squares since exactly 1/3 of their definitive count were also DA in their rows and columns. Moreover in DMPS we stated on p. 2675 that all 368,640 eighth order natural pandiagonal Franklin squares have three non-zero eigenvalues and that they are singular with rank 3.

Our landmark count of the order 8 Franklin bent diagonal semimagic squares did not consider their spectral properties [PRSA], but in DMPS we reported that they are rank 3 and thus highly singular.

CBH drew attention to a particular magic square, \( BF \), which they derived from a bona fide Franklin square of Neal Abrahams on Suzanne Ale-
jandre’s web site [1] by RC permutations of rows 1,2 and then rows 3,4. For Abrahams square (not shown here): 
\[ \lambda_i = \{260, -43.7128, 11.7128, 0, 0, 0, 0, 0\}, \]
\[ \sigma^2 = \{67600, 4\{1365 + \sqrt{1,755,705}\} \text{ for } 21520.2\text{ and } 319.758\}, \]
the latter for integer \( R = 463, 223, 040 \) and \( r = 3 \), which is consistent with Table 3 in DMPS, and which is neither magic nor 1EV: \( d_1 = 228, d_2 = 292 \).

### 14.1.1 CBH’s ”Franklin” 8th order 1EV magic squares (\( \mu = 7 \))

CBH claimed to study ”BEN FRANKLIN’S 8X8 MAGIC SQUARE MATRIX, BF. Added to the puzzle in retrospect is Proposition 5.1 of CBH: ”The Alejandre Form of the Ben Franklin 8-by-8 Magic Square and all of its \( k \)th integral powers are magic, with magic constant \( (260)^k \).”

Their proof noted that for \( k > 2 \) the powered up matrix was a constant matrix (all elements equal), but this was understated and they did not appear to extend this to other magic squares, and then we found that their’s was not a bona fide Franklin bent-diagonal square! Unfortunately this is not a true Franklin square [see PRSA] because while magic (in RCD), it is not so in all the bent diagonals. CBH use a row permutation of Abrahams’ bona fide Franklin square, which is not magic in the diagonals [1], involving swapping row 1 and 2 and then row 3 and 4. While a bona fide magic square [RCD], in their proof of their Proposition 5.1 CBH state that powers of this magic square for the cube and higher are constant (magic) matrices with magic constant \( (260)^{k+1} \), \( k > 2 \). After correcting the last cell in CBH in the first row from a duplicate 10 to 19:

\[
BF = \begin{bmatrix}
14 & 3 & 62 & 51 & 46 & 35 & 30 & 19 \\
52 & 61 & 4 & 13 & 20 & 29 & 36 & 45 \\
11 & 6 & 59 & 54 & 43 & 38 & 27 & 22 \\
53 & 60 & 5 & 12 & 21 & 28 & 37 & 44 \\
55 & 58 & 7 & 10 & 23 & 26 & 39 & 42 \\
9 & 8 & 57 & 56 & 41 & 40 & 25 & 24 \\
50 & 63 & 2 & 15 & 18 & 31 & 34 & 47 \\
16 & 1 & 64 & 49 & 48 & 33 & 32 & 17
\end{bmatrix}
\] (17)

We note that the modified square has just a single non-zero eigenvalue, in contrast with three in the rank 3 Franklin n=8 set [6],[23]. It has characteristic polynomial and spectra:

\[ -260x^7 + x^8, \lambda_i = 260, 0, 0, 0, 0, 0, 0; \]
with the same SVs as above for Abrahams’ square, where we note as an aside that \( \sigma^2 \) are not always integer, and again with \( R = 463, 223, 040 \), which is identical with Abrahams square above, i.e. both are in the same SV clan [DMPS].

This all suggests to us that many more 1EV squares may be expected by applying similar row and/or column permutations to other bona fide Franklin squares. The Jordan form of this remarkable magic square is simply an enlargement to \( n = 8 \) of the \( n = 4 \) Jordan forms in (11) for \( sud4a, J_4, \) etc. by the addition of 4 more rows and columns of zeroes, with the magic eigenvalue as appropriate, now 260, and so need not take up display space! Moreover the powering
is similar to Table 6 with \( C\% = 61.4849, 97.872, 0 \); \( \text{Spread} = 1.9385, 0.0152, 0.0 \), with linesums \( (S_p)^p = 260^p \), i.e. 260, 67600, ...

\[
\begin{array}{cccc}
 p & r & C\% & \text{spread} & \text{Type} \\
 1 & 3 & 61.4849 & 1.93846 & \text{magic } (R = 463, 223, 040) \\
 2 & 2 & 97.872 & 0.0151479 & \text{DDA} \\
 3 & 1 & 100 & 0 & \text{constant } E_9 \\
\end{array}
\]

Table 9 - 1EV \( n = 8 \) magic square, \( BF \).

N.B. For "the proof" of Proposition 6.1 CBH state that it involves matrix algebra and analysis of the eigenvalues, so perhaps they found the single eigenvalue property?

Nordgren [18] recently showed that odd matrix powers of true pandiagonal Franklin squares [PRSA] are also pandiagonal Franklin squares, and confirmed the rank 3 property.

15 Compounding \textit{sud4a} to order 16 1EV DLS

Compounding is an ancient idea (more than a thousand years) for generating larger magic squares, initially used for multiplying up the non-singular \( n = 3 \) \textit{loshu} to order 9. A paper with Wayne Chan [7] [CL] on rediscovering the basic compound idea used here, included iterating a pandiagonal and most-perfect \( n = 4 \) with Euler’s 1779 pandiagonal \( n = 7 \) to the highly compounded \( n = 12, 544 = 4^27^2 \), also pandiagonal.

Now the 1EV property extends to compounded 1EV Latin and magic squares. Next we apply compounding to our 1EV Latin square, \textit{sud4a}, before considering the magic square parallels [22] [RCL]. The order of the squares increases to multiples of \( n = 4 \), i.e. \( n = 16, 64, 256, \ldots \) of which the first will be considered in detail here.

In 2004 we began work on an expansion of the compounding idea to include Latin squares, and general magic squares, which lead to a formula for the eigenvalues and singular values of compounded magic (and Latin) squares which was included in our talk at IWMS-16 in 2007 at Windsor, Ontario [see the conference PowerPoint slides - see IWMS]. It became clear to us by 2008 that compounding 1EVs would produce larger 1EVs. Due to space restrictions the compound work was not included in the LAA conference proceedings, with the idea that it would be featured in a follow-up paper [22] [RCL], which now follows the present work.

So in the present context it is obvious that the compounded 1EV \( n = 4 \) Latin and magic squares compound to \( n = 16 \) 1EV squares with \( \mu = 15 \) and rank 5, and thus worth raising to higher powers!

The idea is to place incremented versions of the basic square in positions in a larger square with the same underlying pattern, which in this case is a Latin square of 16 symbols which takes a lot of space which we abbreviate by suggesting an order four filled with \( A = \textit{sud4a}, B = \textit{sud4a} + 4\textbf{E}_4, C = \textit{sud4a} + 8\textbf{E}_4, D = \textit{sud4a} + 12\textbf{E}_4 \):
\[
\begin{bmatrix}
A & B & C & D \\
C & D & A & B \\
D & C & B & A \\
B & A & D & C
\end{bmatrix}
\] (18)

or explicitly with blank rows and columns:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 & 15 & 16 & 13 & 14 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 12 & 11 & 10 & 9 & 16 & 15 & 14 & 13 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15
\end{bmatrix}
\] (19)

As expected from the compound spectral formulae of RCL this is diagonal Latin [DL] with rank 5. The results on powering-up are similar to \textit{sud4a} in Table 2, thus agreeing with our Proposition 2, but now with C\% = 55.8491 and \textit{Spread} = 1.76471 at \(p = 1\). Both \textit{sud4a} and its compound are DLs. Now the linesums in each row are \((s_{16})^p = 136^p\), i.e. for \(p = 1, 2, 3\) these are 136, 18496, 2515456. Their characteristic polynomials are respectively: \(x^{16} = 136x^{15}; x^{16} = 18496x^{15}; x^{16} = 2,515,456x^{15}\), etc. The first of these says that the eigenvalues are 136 and fifteen zeroes, i.e. another 1EV square, with multiplicity \(\mu = 15\), and so on.

Now we find an \(n = 16\) Jordan form, of which we show just an 8-by-8 portion of the order 16 as the rest contains the only zero elements:
where two 3-by-3 rank 2 Jordan blocks along the diagonal are followed after nine rows and columns of zeros before the corner diagonal element of the linesum, 136. So we have a degree $2 + 2 - 1 = 3$ compound square, i.e. raising it to the cube power will provide us with a constant matrix of the previous form: \[ Z^k = (l \frac{p}{n}) E_n \], with the observed rank decreasing by 2 each time because we have two Jordan blocks, each contributing now a rank 1 decrease at each power. This Jordan structure explains why constancy is attained at $p = 3$, as for $sud4a$ itself, and not much higher as might have been expected by Cayley-Hamilton which gives $A_{16} = 136A_{15}$.

Continuing this compounding also gives larger Latin squares or orders $4 * 16 = 64, 4 * 64 = 256, ..., all 1EV$.

### 16 n=16 compounding of associative 1EV $f360$

Now we follow the previous compounding of $sud4a$ with the analogous approach for an $n = 4$ 1EV magic square above. An example for a different order four magic square is shown explicitly in CL. The Jordan form here differs from the compound of $sud4a$ only by the magic linesum which here is $l_{16} = S_{16} = Sum(1..256)/16 = 2056$.

For the 1EV compounding of $f360$ the powers show constancy again at $p = 3$ as for the previous $sud4a$ compounding, and the same reduction in rank as for the compounding of $sud4a$, now with $C\%$ increasing from 64.3023 and Spread decreasing from 1.98444 at $p = 1$ (compare with Table 6).

### 16.1 Larger compound magic squares

For 1EV compound magic squares combinations of the basic 1EV magic squares $n = 4, 5 & 8$ featured in this work this means that compounding will produce orders $4 * 5 = 20, 8 * 5 = 40, 16 * 5 = 90, etc., in addition to all those that we could make just from powers of $n = 4, 5 & 8$.

For $n_1 = 4, n_2 = 5$, the 1EV order 20 compound magic squares have rank 6 as expected [RCL], and power up to constancy at $p = 4$, the same as for the $n = 5$ 1EV component, as discussed earlier. This may be continued for magic squares to multiples of $n = 5$, and then products of powers of 4 and 5, i.e. $4 * 5 = 20, 16 * 5 = 80, 4 * 25 = 100, ...$
17 Products of different $n = 4$ 1EV magic squares

Squaring a matrix is a pair product, and cubing a triple product. For the associative Group III’s we now try mixing different members of the same clan.

17.1 Double and Triple products of different alpha squares

[See Table 5 for details of the alpha and beta clans.]

Consider the following pair of alpha squares as a matrix product:

$\text{pair}_1 = f_{360}.f_{790}$; $\lambda_i = \{1156, 80, 80, 0\}$, $d1 = 1316, d2 = 1156$, which is no longer 1EV, but now 3EV and DA, which indicates that even with the same clan the 1EV property is destroyed in this pair. We have tested powers of $\text{pair}_1$ to 4th order and it is always 3EV, but slowly approaching constancy.

However including either component $f_{360}$ or $f_{790}$ as a third matrix before or after $\text{pair}_1$ always renews the 1EV property:

$f_{360}.\text{pair}_1 = f_{360}.(f_{360}.f_{790})$ 1EV, constant also at $p = 2, 3, ...$

$\text{pair}_1.f_{360} = (f_{360}.f_{790}).f_{360}$ 1EV, constant also at $p = 3, 4, ...$

$f_{790}.\text{pair}_1 = f_{790}.(f_{360}.f_{790})$ 1EV, constant also at $p = 3, 4, ...$

These show that when the same square is adjacent the constancy appears in the first (squared) power of the triple product, whereas when they are separated by the other the constancy is delayed to the cube.

17.2 Pair products of alpha and beta squares

Multiplying a mixed pair of 1EV SVs, an alpha square with a beta square also produces a 3EV $m = 1$ square, e.g. try 1EV beta $f_{489}$ with alpha $f_{790}$:

$\text{pair}_2 = f_{489}.f_{790}$; $\lambda_i = \{1156, -32, 16, 0\}$, $d1 = 1140, d2 = 1204$, so no longer 1EV, but now 3EV DA, which again suggests that mixing clans destroys the 1EV property.

Again we have tested powers of $\text{pair}_2$ to 4th order and it is always 3EV, but slowly approaching constancy. However the following triples are now all 1EV, with the same pattern as for the previous cases which used a single clan:

$f_{489}.\text{pair}_2 = f_{489}.(f_{489}.f_{790})$ 1EV, constant also at $p = 2, 3, ...$

$\text{pair}_2.f_{489} = (f_{489}.f_{790}).f_{489}$ 1EV, constant also at $p = 3, 4, ...$

$\text{pair}_2.f_{790} = (f_{489}.f_{790}).f_{790}$ 1EV, constant also at $p = 2, 3, ...$

$f_{790}.\text{pair}_2 = f_{790}.(f_{489}.f_{790})$ 1EV, constant also at $p = 3, 4, ...$

17.3 Commutation

Non-commuting (matrix) operators encountered in the matrix formulation of quantum mechanics prompt us to consider this issue for the pair products above. First with the same clan pair, $\text{pair}_1 = f_{360}.f_{790}$:

$\text{comm}_1 = f_{360}.f_{790} - f_{790}.f_{360}$: $\lambda_i = \{-80, 80, 0, 0\}$, $d1 = 0, d2 = 0$, so the 3EV $\text{pair}_1$ has a 2EV commutator.
Second with the mixed clan pair, pair2 = bj489.f790:

\[ \text{comm}2 = bj489.f790 - f790.bj489 : \lambda_i = \{48, -32, -16, 0\}, \]

d1 = 0, d2 = 96, and the 3EV pair2 has a 3EV commutator.

With this insight we discontinue further exploration of these products.

18 Fibonacci magic square \((\mu = 0)\)

We include this since CBH introduced Herta Freitag’s order four magic square [11] even though it is not associative, nor are the elements in arithmetic progression. Taking sixteen elements from the Fibonacci series, and its \(p = 2\) square:

\[
A = \begin{bmatrix}
13 & 89 & 97 & 34 \\
110 & 21 & 63 & 39 \\
68 & 94 & 55 & 16 \\
42 & 29 & 18 & 144
\end{bmatrix} , A^2 = \begin{bmatrix}
17983 & 13130 & 12815 & 10361 \\
9662 & 17284 & 16160 & 11183 \\
15636 & 13660 & 15831 & 9162 \\
11008 & 10215 & 9483 & 23583
\end{bmatrix} , \ldots \quad (21)
\]

These are non-singular we do not have a 1EV square:

\(\lambda_i = \{233, 116.003, -68.1711, -47.8322\}\)

The results for the first four powers are similar to Tables 3 and 7:

| \(p\) | \(r\) | \(C\%\) | spread | Type |
|------|------|--------|--------|------|
| 1    | 4    | 13.1301| 2.24893| magic|
| 2    | 4    | 38.8696| 1.06254| DA   |
| 3    | 4    | 62.4492| 0.53231| DA   |
| 4    | 4    | 78.6555| 0.26527| DA   |

Table 10 - Freitag Fibonacci square slowly converging towards constancy.

For the record we compounded this magic square to \(n = 16\), rank 7, \(C\% = 52.2854\), \(Spread = 2.28586\).

18.1 Prime 1EV Latin squares

An example of a prime arithmetic progression is given on MathWorld [26]. From the ten terms given with a difference of 210 we take the first four for a prime Latin square modelled on \(sud4a\) in (1) which is also 1EV,

\[
A = \begin{bmatrix}
199 & 409 & 619 & 829 \\
619 & 829 & 199 & 409 \\
829 & 619 & 409 & 199 \\
409 & 199 & 829 & 619
\end{bmatrix} . \quad (22)
\]

Here \(\lambda_i = \{2056, 0, 0, 0\}\); \(\sigma_i = \{2056, 840, 420, 0\}\), \(r = 3\), \(C\% = 34.6522\), \(Spread = 1.22568\), and note the closeness of these values to those for \(sud4a\) which were \(C\% = 35.06\), \(Spread = 1.2\).
Clearly there are many more of these $n = 4$ prime 1EV squares from the other selections from the ten in MathWorld [26], in addition to more from compounding, and possibly for order 7, 8, 9, ..., Latin squares (we did not find any for orders 5 and 6 in DMPS).

19 Gerschgorin Disks for 1EV $n = 4$ magic squares

Following the discussion of the location of eigenvalues w.r.t to these disks in Ortega [20], we have not seen any previous use of these disks in the literature of magic or Latin squares, and the opportunity to look at the extreme case of just 1EV. It suffices to describe the image for $sud4a$ of four circles centered on the $x$-axis at 1, 4, 2, 3 with radii 9, 6, 8, 7 respectively (the element sums of the consecutive columns), eigenvalue 0 at the origin and eigenvalue 10 at the point where the four circles touch.

20 Conclusions

It was necessary to review previous ”magic” terminology for these powers, leading to our use of doubly-affine and diagonal doubly-affine to characterize these magical squares.

Using doubly-affine in our Proposition 1 to include both Latin and magic squares we showed that certain associative $n = 4, 5$ 1EV squares power up to constancy in a few steps. This was understood by using the Cayley-Hamilton theorem for the corresponding characteristic equations, leading to our Proposition 2.

CBH’s Proposition 6.1 had claimed that for any associative magic squares their odd powers remained ”magic”, while their even powers were just ”semimagic”, but we found exceptions. In addition we found evidence that another type did satisfy their proposition. A different variation of CBH’s alternation between magic and semimagic found in some of our examples.

We found constancy for Latin $sud4a$ at $p = 3$, as well as for its compounding to $n = 16$, and for some $n = 4$ examples, while constancy occurred at $p = 4$ for an $n = 5$ associative 1EV magic square, as well as for their compounding to multiplicative orders. It was necessary to study their Jordan forms in order to understand constancy at $p = 3$ for their constancy at $p = 3$.

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