On a Singular Limit Problem for Nonlinear Maxwell’s Equations

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Abstract: In this paper we study the following nonlinear Maxwell’s equations
\[ \varepsilon E_t + \sigma(x, |E|)E = \nabla \times H + F, \quad H_t + \nabla \times E = 0, \]
where \( \sigma(x, s) \) is a monotone graph of \( s \). It is shown that the system has a unique weak solution. Moreover, the limit of the solution as \( \varepsilon \to 0 \) converges to the solution of quasi-stationary Maxwell’s equations.

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1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ and $Q_T = \Omega \times (0, T]$ for any fixed $T > 0$. Let $E$ and $H$ be the electric and magnetic fields, respectively, in $\Omega$ (here and thereafter a bold letter represents a vector in $\mathbb{R}^3$). Let $\sigma$ be the electric conductivity in the field, which is assumed to be a function of $x$ and $|E|$. Consider the following Maxwell’s equations (see Landau-Lifschitz [13]):

\begin{align}
\varepsilon E_t + \sigma(x, |E|)E &= \nabla \times H + F, \quad (x, t) \in Q_T, \\
H_t + \nabla \times E &= 0, \quad (x, t) \in Q_T, \\
\nabla \times E &= 0, \quad (x, t) \in \partial \Omega \times (0, T], \\
E(x, 0) &= E_0(x), H(x, 0) = H_0(x), \quad x \in \Omega,
\end{align}

where $\varepsilon$ is the dielectric parameter and other physical parameters are normalized.

In some applications ([4, 11]), the electric conductivity, $\sigma$, strongly depends on the electric field $|E|$, hence the electric current density. Particularly, the electric conductivity may act like a switch-like function in some electromagnetic fields. On the other hand, for many types of micron devices and other industrial problems (such as microwave heating [11, 8, 17] etc.) the experiment shows that the displacement current, $\varepsilon E_t$, is often negligible since it is small in comparison of the eddy current, $J = \sigma E$. This motivates us to investigate the nonlinear problem (1.1)-(1.4) and the singular limit problem as $\varepsilon \to 0$. It is shown that there exists a unique global solution to (1.1)-(1.4). Moreover, the limit of the solution converges to the solution of the quasi-stationary system (i.e., the system (1.1)-(1.4) with $\varepsilon = 0$ in (1.1)). This limit solution provides new existence result for the quasi-stationary system. Indeed, when $\varepsilon = 0$, the system (1.1) becomes

\begin{equation}
\sigma(x, |E|)E = \nabla \times H + F,
\end{equation}

Thus, one can solve Eq. (1.5) for $|E|$ in terms of $|\nabla \times H|$ and known data,

\[|E| = g(x, |\nabla \times H|),\]
where \( g(x, s) \) is the inverse function of \( \sigma(x, s) s \).

It follows from (1.2) that \( \mathbf{H} \) satisfies

\[
\mathbf{H}_t + \nabla \times [\rho(x, |\nabla \times \mathbf{H}|) \nabla \times \mathbf{H}] = 0,
\]

(1.6)

where

\[
\rho(x, |\nabla \times \mathbf{H}|) = \frac{1}{\sigma(x, |\mathbf{E}|)} = \frac{1}{\sigma(x, g(x, |\nabla \times \mathbf{H}|))}
\]

represents the electric resistivity in the field.

The research on Maxwell’s equations is of great interest because of the important applications in plasma physics, semiconductor-superconductor modeling and other industrial problems ([8, 9, 13, 17] etc.). The study on the system (1.1)-(1.4) as well as the quasi-stationary form (1.6) received considerable attention recently. In [12], the authors established the well-posedness for a quasi-stationary system, where a constitutive relation between the magnetic field \( \mathbf{H} \) and the magnetic induction \( \mathbf{B} \) is assumed to be nonlinear. In [13], the author studied the regularity of weak solution to a linear system of (1.6) with minimal requirement on coefficients. There is a special interest when

\[
\rho(|\nabla \times \mathbf{H}|) = |\nabla \times \mathbf{H}|^{p-2}, \quad p > 2.
\]

On one hand, if \( \mathbf{H} \) is restricted in one direction (scalar field) then the evolution system (1.6) becomes the p-Laplacian which has been studied extensively (see [4] and the references therein). On the other hand, in a recent work [14] (also see [1, 3] for the scalar case), it is shown that the limit of the solution to (1.6) as \( p \to \infty \) is the unique solution to Bean’s critical-state model in the superconductivity theory ([2]). Thus, for large \( p \) the system (1.6) provides a good approximation to Bean’s model. More recently, the author of [10] studied the similar problem to this paper in a domain with a bounded complement in \( \mathbb{R}^3 \). The conditions on \( \sigma \) in [10] is quite different from ours here. Like many nonlinear problems, the major difficulty is how to pass the weak limit of an approximate solution for a nonlinear function \( \sigma(x, s) \). This is done by employing a monotonicity argument ([3]). The monotonicity of \( \sigma(x, s) \) in \( s \) is essential in the proof.
In §2, we use the finite element method to establish the well-posedness of the system (1.1)-(1.4) for fixed $\varepsilon > 0$. In §3, we show that the singular limit of the solution to (1.1)-(1.4) has a unique limit. Moreover, the limit solution solves the quasi-stationary Maxwell’s equations. Some examples are also discussed in this section.

2. Existence and Uniqueness for fixed $\varepsilon > 0$

Introduce some standard spaces (see [4, 7]).

\[ H(\text{div}, \Omega) = \{ U \in L^2(\Omega)^3, \text{div}U \in L^2(\Omega) \}; \]
\[ H(\text{curl}, \Omega) = \{ U \in L^2(\Omega)^3, \text{curl}U \in L^2(\Omega)^3 \}; \]
\[ H(\text{div}0, \Omega) = \{ U \in H(\text{div}, \Omega) : \text{div}U = 0 \text{ in } \Omega \}, \]
\[ H_0(\text{curl}, \Omega) = \{ U \in H(\text{curl}, \Omega) : N \times U = 0 \text{ on } \partial\Omega \}, \]

where $N$ is the exterior unit normal on $\partial\Omega$.

Note that the trace of a function in $H(\text{curl}, \Omega)$ is well defined (see [4] for example).

We shall assume the following conditions on $\sigma(x, s)$ and data $E_0(x), H_0(x)$ and $F(x, t)$.

\textbf{H}(2.1): Let $\sigma(x, s)$ be measurable in $\Omega \times [0, \infty)$ and monotone increasing in $s$. Moreover,

\[ \int_0^{s^2} \sigma(x, \sqrt{s})dx \geq a_0s^{p+2} - a_1, \text{ if } s \text{ is sufficiently large,} \]
\[ 0 \leq \sigma(x, s) \leq b_0(1 + s^p), s \in [0, \infty), \text{ for } p \geq 0, \]

where the constants $a_0 > 0, a_1 \geq 0$ and $b_0 \geq 0$.

\textbf{H}(2.2): Assume that $H_0 \in H(\text{curl}, \Omega) \cap H(\text{div}0, \Omega), F \in H^1(0, T; H^1(\Omega))$.

\textbf{Definition 2.1}: A pair of vector fields $(E(x, t), H(x, t))$ is said to be a weak solution of the problem (1.1)-(1.4), if

\[ E \in L^2(0, T; H_0(\text{curl}, \Omega)) \cap L^{p+2}(0, T; \Omega), H \in L^2(0, T; H(\text{div}0, \Omega)) \cap H(0, T, L^2(\Omega)) \]
which satisfy the following integral identities:

\[
\int \int_{Q_T} \left[ -\varepsilon \mathbf{E} \cdot \Phi_t + \sigma(x, |\mathbf{E}|) \mathbf{E} \cdot \Phi \right] dxdt
= \int \int_{Q_T} \mathbf{H} \cdot (\nabla \times \Phi) dxdt + \varepsilon \int_{\Omega} \mathbf{E}_0 \cdot \Phi(x, 0) dx,
\]

(2.1)

\[
\int \int_{Q_T} \left[ -\mathbf{H} \cdot \Psi_t + \mathbf{E} \cdot (\nabla \times \Psi) \right] dxdt = \int_{\Omega} [\mathbf{H}_0(x) \cdot \Psi(x, 0)] dx
\]

(2.2)

for all test functions \( \Phi \in H^1(0, T; H_0(curl, \Omega)) \), \( \Psi \in H^1(0, T; H(curl, \Omega)) \) with \( \Phi(x, T) = \Psi(x, T) = 0 \).

First of all, we derive some energy estimates. A special attention is paid on how various constants depend on \( \varepsilon \) since we will study the singular limit problem in section 3.

**Lemma 2.1:** Under the assumptions H(2.1)-H(2.2) there exist constants \( C_1, C_2 \) and \( C_3 \) such that

\[
\sup_{[0, T]} \int_{\Omega} \left[ \varepsilon |\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{E}|^{p+2} \right] dx + \int_0^T \int_{\Omega} \left[ \varepsilon |\mathbf{E}_t|^2 + |\mathbf{H}_t|^2 \right] dxdt
\]

\[
\leq C_1 \int_{\Omega} \left[ |\mathbf{E}_0|^2 + |\mathbf{H}_0|^2 + |\nabla \times \mathbf{H}_0|^2 \right] dx + C_2 \int \int_{Q_T} [||\mathbf{F}|^2 + |\mathbf{F}_t|^2]dxdt + C_3,
\]

where \( C_1, C_2 \) and \( C_3 \) depend only on known data.

**Proof:** Note that for any vector fields \( \mathbf{A}, \mathbf{B} \in H(curl, \Omega) \) with either \( \mathbf{A} \) or \( \mathbf{B} \) in \( H_0(curl, \Omega) \), the following identity holds:

\[
\int_{\Omega} \mathbf{A} \cdot (\nabla \times \mathbf{B}) dx = \int_{\Omega} \mathbf{B} \cdot (\nabla \times \mathbf{A}) dx.
\]

Taking inner product to the system (1.1) and (1.2) by \( \mathbf{E} \) and \( \mathbf{H} \), respectively, we add up the resulting equations to obtain

\[
\sup_{[0, T]} \int_{\Omega} \left[ \varepsilon |\mathbf{E}|^2 + |\mathbf{H}|^2 \right] dx + \int \int_{Q_T} \sigma(x, |\mathbf{E}|) |\mathbf{E}|^2 dxdt
\]

\[
\leq C \int_{\Omega} \left[ |\mathbf{E}_0|^2 + |\mathbf{H}_0|^2 \right] dx + \int \int_{Q_T} [||\mathbf{E} \cdot \mathbf{F}|^2]dxdt
\]

(2.3)

where the constant \( C \) depends only on known data, but not on \( \varepsilon \).
We first assume that $\sigma(x, s)$ is differentiable with respect to $s$. Then we formally differentiate Eq.(1.1) and Eq.(1.2) with respect to $t$ to obtain

$$
\varepsilon E_{tt} + \sigma(x, |E|) E_t + \sigma_s(x, |E|)(|E|) E = \nabla \times H_t + F_t,
$$

$$
H_{tt} + \nabla \times E_t = 0.
$$

It is clear that

$$
\int \int_{Q_T} (\nabla \times E_t) \cdot H_t dx dt = \int \int_{Q_T} (\nabla \times H_t) \cdot E_t dx dt.
$$

We take the inner product by $E_t$ for the first equation and by $H_t$ for the second equation and add up the resulting equations to obtain:

$$
\sup_{[0,T]} \int_{\Omega} \left[ \varepsilon |E_t|^2 + |H_t|^2 \right] dx dt + \int \int_{Q_T} \left[ \sigma |E_t|^2 + \sigma_s(|E|) E \cdot E_t \right] dx dt \leq C,
$$

where $C$ depends only on known data.

Note that $\sigma_s \geq 0$, we see that

$$
\int \int_{Q_T} \sigma_s(|E|) E \cdot E_t dx dt
$$

$$
= \int \int_{Q_T} \sigma_s(|E|) \frac{d}{dt} |E|^2 dx dt
$$

$$
= \int \int_{Q_T} \sigma_s |E||E_t|^2 dx dt \geq 0.
$$

It follows that

$$
\sup_{[0,T]} \int_{\Omega} \left[ \varepsilon |E_t|^2 + |H_t|^2 \right] dx dt + \int \int_{Q_T} \sigma |E_t|^2 dx dt \leq C.
$$

Since the above estimate does not depend on the differentiability of $\sigma$ with respect to $s$, therefore the above estimate holds as long as $\sigma$ is monotone increasing with respect to $s$.  

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Now we take the inner product by $E_t$ to (1.1) and by $H_t$ to (1.2) and then add up the resulting equations to obtain

$$\int \int_{Q_T} [\varepsilon |E_t|^2 + |H_t|^2] dx dt + \frac{1}{2} \int_0^T \int_\Omega |E(x,t)|^2 \sigma(x, \sqrt{s}) ds dx$$

$$\leq \int \int_{Q_T} E_t \cdot F dx dt + C \int_0^T \int_\Omega |E_0(x)|^2 \sigma(x, \sqrt{s}) ds dx + \int_\Omega |\nabla \times H_0|^2 dx + C.$$ 

Now

$$\int \int_{Q_T} E_t \cdot F dx dt$$

$$= \int_\Omega [E(x, T) \cdot F(x, T) - E_0(x) \cdot F(x, 0)] dx - \int \int_{Q_T} E \cdot F_t dx dt$$

$$\leq \int_\Omega \left[ \frac{a_0}{4} |E(x, T)|^{p+2} + \frac{16}{a_0} |F_t|^\frac{p+2}{p+1} \right] dx dt.$$ 

On the other hand, by the assumption H(2.1) we may assume that the growth condition of $\sigma(x, s)$ on $s$ holds for all $s \geq M_0$, i.e.,

$$\int_0^s \sigma(x, \sqrt{s}) dx \geq a_0 s^{p+2} - a_1, \text{ if } s \geq M_0,$$

where $M_0$ is a fixed constant.

It follows that

$$\int \int_\Omega |E(x, T)|^2 \sigma(x, s) ds dx$$

$$\geq a_0 \int_\Omega \int_{\{x: |E(x, T)| \geq M_0\}} |E(x, T)|^{p+2} dx - C. \quad (2.4)$$

Combining (2.3)-(2.4) yields

$$\sup_{0 \leq t \leq T} \int_\Omega [\varepsilon |E|^2 + |H|^2] dx + \sup_{0 \leq t \leq T} \int_\Omega |E(x, t)|^{p+2} dx + \int_0^T \int_\Omega [\varepsilon |E_t|^2 + |H_t|^2] dx dt$$

$$\leq C \int \int_{Q_T} [F]^2 + |F_t|^2 dx dt + \int \int_\Omega [|E_0|^2 + |H_0|^2 + |\nabla \times H_0|^2] dx + C.$$ 

Q.E.D.
Theorem 2.2: Under the assumptions $H(2.1)-H(2.2)$ the problem (1.1)-(1.4) has a unique weak solution. Moreover, \[
\text{curl } E \in L^2(Q_T), \ E_t \in L^2(Q_T) 
\]
and \[
H_t \in L^2(Q_T), \ \nabla \times H \in L^{\frac{p+2}{p+1}}(Q_T). 
\]

Proof: The proof is based on the finite element method (see [14] for parabolic equations). The monotonicity of $\sigma(x, s)$ on $s$ plays an important role. We shall first deal with the case where $\sigma(x, s)$ is continuous on $s$. For convenience, we rewrite the system (1.1)-(1.4) to the following form:

\[
\varepsilon W_{tt} + \sigma(x, |W_t|)W_t = \nabla \times [H_0 - \nabla \times W] + F, \quad (x, t) \in Q_T, \quad (2.5)
\]

\[
N \times (W_t) = 0, \quad x \in \partial \Omega, 0 \leq t \leq T, \quad (2.6)
\]

\[
W(x, 0) = 0, W_t(x, 0) = E_0(x), \quad x \in \Omega, \quad (2.7)
\]

where $W$ is defined as follows:

\[
W(x, t) = \int_0^t E(x, \tau)d\tau.
\]

It is clear that if $W$ is a solution of the system (2.5)-(2.7) then a pair of functions defined by

\[
E(x, t) = W_t(x, t), H(x, t) = H_0(x) - \nabla \times W(x, t)
\]

will be a weak solution of (1.1)-(1.4). Let $\{e_k\} = \{e_k^{(1)}, e_k^{(2)}, e_k^{(3)}\}$ be a smooth basis of $H_0(curl, \Omega)$ and orthonormal in $L^2(\Omega)^3$, i.e.

\[
< e_i, e_j >= \delta_{ij},
\]

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Now we expand the known data as follows:

\[
H_0(x) = \sum_{k=1}^{\infty} \text{diag}[a_k \circ e_k],
\]

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\[ E_0(x) = \sum_{k=1}^{\infty} \text{diag}[b_k \circ e_k], \]
\[ F(x, t) = \sum_{k=1}^{\infty} \text{diag}[g_k(t) \circ e_k], \]

where \( a_k, b_k \) and \( g_k \) are \( 3 \times 1 \) matrices, the symbol \( \circ \) is the matrix product and \( \text{diag}[\cdot] \) represents the diagonal vector of a matrix.

Let
\[ W_n(x, t) = \sum_{k=1}^{n} \text{diag}[c_n^{(k)} \circ e_k], \]

where \( c_n^{(k)}(t) \) is a \( 3 \times 1 \) vector which is determined by the following ordinary differential system:
\[ \varepsilon \frac{d^2}{dt^2} c_n^{(k)} + \sigma(x, |W_{nt}|) \frac{d}{dt} c_n^{(k)} = A_k(W_n^{(k)}, e_k) + B_k(t), \quad \text{(2.8)} \]
\[ c_n^{(k)}(0) = 0, \quad \text{(2.9)} \]
\[ \frac{d}{dt} c_n^{(k)}(0) = b_k, \quad \text{(2.10)} \]

where
\[ W_n^{(k)} = \text{diag}[c_n^{(k)} \circ e_k], \]
\[ A_k(W_n^{(k)}, e_k) = \int_{\Omega} \text{diag}\{[\nabla \times W_n^{(k)}] \circ [\nabla \times e_k]\} dx, \]
\[ B_k(t) = \int_{\Omega} \text{diag}\{[\nabla \times H_0 + F] \circ e_k\} dx, \quad k = 1, 2, \cdots n. \]

Now we define the approximate solution \((E_n, H_n)\) as follows:
\[ E_n(x, t) = W_{nt}(x, t), H_n(x, t) = H_{0n} - \nabla \times W_n(x, t), \]

where
\[ H_{0n}(x) = \sum_{k=1}^{n} \text{diag}[a_k \circ e_k]. \]

Equivalently, then \((E_n, H_n)\) satisfies the following system in the weak sense:
\[ \varepsilon E_{nt} + \sigma(x, |E_n|)E_n = \nabla \times H_n, \quad (x, t) \in Q_T, \quad \text{(2.11)} \]
\[ H_{nt} + \nabla \times E_n = 0, \quad (x, t) \in Q_T. \quad \text{(2.12)} \]
Similar to Lemma 2.1, one can easily derive the following energy estimates:

\[
\sup_{[0,T]} \int_{\Omega} \left[ \varepsilon |E_n|^2 + |H_n|^2 + |E_n|^{p+2} \right] \, dx + \int_0^T \int_{\Omega} \left[ \varepsilon |E_{nt}|^2 + |H_{nt}|^2 \right] \, dx \, dt \\
\leq C_1 \int_{\Omega} \left[ |E_{0n}|^2 + |H_{0n}|^2 + |\nabla \times H_{0n}|^2 \right] \, dx + C_2 \int_{Q_T} \left[ |F_n|^2 + |F_{nt}|^2 \right] \, dx \, dt + C_3,
\]

where \( C_1, C_2 \) and \( C_3 \) are independent of \( n \) and \( \varepsilon \).

By the weak compactness property, we can extract a subsequence (still denoted by \((E_n, H_n)\)) such that

- \( E_n \to E, E_{nt} \to E_t, H_{nt} \to H_t \), weakly in \( L^2(Q_T) \),
- \( H_n \to H \), weakly in \( L^2(0,T; W^{1,p+2}(\Omega)) \),
- \( H_n \to H \), a.e. in \( Q_T \).

Moreover,

\( E_n \to E \) weakly in \( L^{p+2}(Q_T) \).

Next we claim that the sequence \( E_n \) converges to \( E \) strongly in \( L^2(Q_T)^3 \). To prove the claim we only need to show that \( \{E_n\} \) is a Cauchy sequence in \( L^2(Q_T)^3 \). Let

\[
E_n^*(x,t) = E_n(x,t) - E_m(x,t), \quad H_n^*(x,t) = H_n(x,t) - H_m(x,t).
\]

By energy estimates, we see

\[
\sup_{0 \leq t \leq T} \int_{\Omega} \left[ |E_n^*|^2 + |H_n^*|^2 \right] \, dx + \\
\int_{Q_T} \left\{ [\sigma(x, |E_n|)E_n - \sigma(x, |E_m|)E_m] \cdot [E_n - E_m] \right\} \, dx \, dt \\
\leq C \int_{\Omega} \left[ |E_{0n} - E_{0m}|^2 + |H_{0n} - H_{0m}|^2 \right] \, dx + C \int_{Q_T} \left[ |F_n - F_m|^2 \right] \, dx \, dt,
\]

where \( C \) is a constant independent of \( n \) and \( m \).

Note that \( \sigma(x,s) \) is monotonic increasing in \( s \), then

\[
\frac{[\sigma(x, |E_n|)E_n - \sigma(x, |E_m|)E_m] \cdot [E_n - E_m]}{\sigma(x, |E_n|) - \sigma(x, |E_m|)} \frac{||E_n||^2 - ||E_m||^2}{2} \geq 0.
\]

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It follows that
\[
\sup_{0 \leq t \leq T} \int_\Omega \left( |E_n^*|^2 + |H_n^*|^2 \right) dx 
\leq C \int_\Omega \left( |E_{0n} - E_{0m}|^2 + |H_{0n} - H_{0m}|^2 \right) dx 
+ C \int_{Q_T} |F_n - F_m|^2 dx dt.
\]
This implies that \( E_n, H_n \) are Cauchy sequences since both \( E_{0n}, H_{0n} \) and \( F_n \) are Cauchy sequences in \( L^2(Q_T)^3 \). Hence,
\[
E_n, H_n \to E, H \text{ strongly in } L^2(Q_T).
\]
After taking a subsequence if necessary, we see that
\[
E_n \to E, \quad \text{a.e. in } Q_T.
\]

To show the existence of a weak solution to (1.1)-(1.4), we only need to show
\[
\sigma(x, |E_n|)E_n \to \sigma(x, |E|)E \quad \text{in } L^1(Q_T).
\]

As \( \sigma(x, s) \) is continuous on \( s \) and \( E_n \) converges to \( E \) almost everywhere in \( Q_T \), we know
\[
\sigma(x, |E_n|)E_n \to \sigma(x, |E|)E \quad \text{a.e. in } Q_T.
\]
We now show that \( \sigma(x, |E_n|)E_n \) is equip-integrable in \( Q_T \). We adopt a technique used for scalar elliptic and parabolic equations. Let \( A \) be any measurable subset of \( Q_T \). For any large \( m > 0 \),
\[
\int_A \int \sigma(x, |E_n|) |E_n| dx dt 
\leq \int_A \int_{|E_n| \leq m} \sigma(x, |E_n|) |E_n| dx dt 
+ \int_A \int_{|E_n| \geq m} \sigma(x, |E_n|) |E_n| dx dt 
\equiv I_1 + I_2.
\]
The assumption on \( \sigma(x, s) \) yields
\[
I_1 \leq C \int_A \int_{|E_n| \leq m} [1 + |E_n|^p] |E_n| dx dt,
\]
which can be arbitrarily small if $|A|$ is small since $E_n \in L^{p+2}(Q_T)$.

On the other hand,

$$I_2 \leq \frac{1}{m} \int \int_{A \cap \{|E_n| \geq m\}} \sigma(x, |E_n|)|E_n|^2\,dx\,dt \leq \frac{C}{m},$$

which is also small if $m$ is sufficiently large.

This concludes that $\sigma(x, |E_n|)E_n$ is equip-integrable in $Q_T$.

It follows by Vitali’s theorem that

$$\sigma(x, |E_n|)E_n \to \sigma(x, |E|)E \quad \text{in } L^1(Q_T).$$

Finally, we show that $(E, H)$ is a weak solution of (1.1)-(1.4). By multiplying Eq.(2.11) and Eq.(2.12) by test functions $\Psi$ and $\Phi$, respectively, and then taking integration over $Q_T$, after some routine calculations and taking the limit, we see that $(E, H)$ is a weak solution to the system (1.1)-(1.4).

Now we consider the case where $\sigma(x, s)$ is discontinuous on $s$ at some points. Without loss of generality, we may assume that $\sigma(x, s)$ has a jump only at one point $s = 1$. In this case $\sigma(x, s)$ is not uniquely defined at $s = 1$. We shall understand the value of $\sigma(x, 1)$ in the following sense:

$$\sigma(x, 1) \in [\sigma(x, 1-), \sigma(x, 1+)],$$

where $\sigma(x, 1\pm)$ represents the right or left limit as $s \to 1$.

By the standard approximation, we can construct a smooth approximation sequence $\sigma_m(x, s)$ such that

1. $\sigma_m(x, s)$ is monotonic increasing for all $s \geq 0$,
2. $\sigma_m(x, s) = \sigma(x, s)$, if $|s - 1| \geq \frac{1}{m}$.

Let $(E_m, H_m)$ be a solution of (1.1)-(1.4) in which $\sigma(x, s)$ is replaced by $\sigma_m(x, s)$. By the same energy estimate we see that there exists a measurable function $\beta(x, t) \in L_{p+1}^{\frac{p+2}{p+1}}(Q_T)$ such that

$$\sigma_m(x, |E_m|) \to \beta(x, t), \text{ weakly in } L_{p+1}^{\frac{p+2}{p+1}}(Q_T).$$
Define
\[
A_m = \{(x, t) : 1 - \frac{1}{m} \leq |E| \leq 1 + \frac{1}{m}\},
A = \{(x, t) : |E(x, t)| = 1\}.
\]

Since \(\sigma(x, s)\) is continuous except at \(s = 1\), we see
\[
\beta(x, t) = \sigma(x, |E|)E, \text{ if } (x, t) \in Q_T \setminus A.
\]

Now it is clear that
\[
A = \bigcap_{m=1}^{\infty} A_m.
\]

Recall that \(\sigma_m(x, s) = \sigma(x, s)\) if \(|s - 1| \geq \frac{1}{m}\). It follows that for all \((x, t) \in A_m\)
\[
\sigma(x, 1 - \frac{1}{m}) \leq \sigma_m(x, |E|) \leq \sigma(x, 1 + \frac{1}{m}).
\]

Consequently, as \(m \to 0\),
\[
\sigma(x, 1-) \leq \beta(x, t) \leq \sigma(x, 1+), (x, t) \in A.
\]

Thus, \((E, H)\) is a weak solution of (1.1)-(1.4).

Finally, we show the uniqueness. Suppose \((E, H)\) and \((E^*, H^*)\) are two solutions of (1.1)-(1.4). Let
\[
\hat{E} = E - E^*, \hat{H} = H - H^*.
\]

Similar to the calculation in deriving energy estimates, we find
\[
\sup_{0 \leq t \leq T} \int_{\Omega} [||\hat{E}|^2 + |\hat{H}|^2] dx + \int_{Q_T} [\sigma(x, |E|)E - \sigma(x, |E^*|)E^*] \cdot [E - E^*] dx dt \leq 0.
\]

The monotonicity of \(\sigma(x, s)\) implies that the second term in the above inequality is nonnegative. It follows that
\[
\sup_{0 \leq t \leq T} \int_{\Omega} [||\hat{E}|^2 + |\hat{H}|^2] dx \leq 0.
\]

Therefore, the uniqueness follows immediately.

Q.E.D.
3. Singular Limit Problem

In this section we shall show that the solution of (1.1)-(1.4) has a limit as $\varepsilon \to 0$, which solves Maxwell’s equations in quasi-stationary fields, i.e. the system (1.1)-(1.4) with $\varepsilon = 0$. A weak solution of the quasi-stationary system is defined as in Definition 2.1 with $\varepsilon = 0$.

From now on we denote by $(E_\varepsilon, H_\varepsilon)$ the weak solution of the system (1.1)-(1.4).

**Theorem 3.1:** The limit of $(E_\varepsilon, H_\varepsilon)$ as $\varepsilon \to 0$ solves the quasi-stationary system (1.1)-(1.4) with $\varepsilon = 0$ in the weak sense. Moreover, the weak solution is unique if $\sigma(x, s) > 0$ for all $(x, s) \in \Omega \times \mathbb{R}^+$. 

**Proof:** The crucial step in proving the convergence is to show $\sigma(|E_\varepsilon|)E_\varepsilon \to \sigma(|E|)E$, a.e. in $Q_T$ as $\varepsilon \to 0$.

The monotonicity of $\sigma(x, s)$ in $s$ plays a key role. First of all, from Lemma 2.1 and the weak compactness we see

$$E_\varepsilon \to E, \ H_\varepsilon \to H, \text{ weakly in } L^2(Q_T),$$

$$\nabla \times H_\varepsilon \to \nabla \times H, \text{ weakly in } L^{\frac{4+2}{p+4}}(Q_T),$$

$$\sigma(x, |E_\varepsilon|)E_\varepsilon \to J(x, t), \text{ weakly in } L^{\frac{4+2}{p+4}}(Q_T),$$

where $J(x, t) \in L^{\frac{4+2}{p+4}}(Q_T)$. Moreover, as $\text{div}H_\varepsilon(x, t) = 0$, by the decomposition property of $H^1(\Omega)$ property, after extracting a subsequence if necessary we see that $H_\varepsilon \to H$, strongly in $L^2(Q_T)$ and

$$H_\varepsilon \to H, \text{ a.e. in } Q_T.$$ 

Next we show $J(x, t) = \sigma(x, |E|)E$, a.e. in $Q_T$.

We use a monotonicity argument. As a first step, we show

$$\lim_{\varepsilon \to 0} \int \int_{Q_T} \sigma(x, |E_\varepsilon|)|E_\varepsilon|^2 dxdt = \int \int_{Q_T} J \cdot E dxdt.$$
Here we adopt an idea from [10]. Let $\lambda(t)$ be a nonnegative smooth function and

$$\lambda'(t) \leq 0, \lambda(0) = 1, \lambda(T) = 0.$$ 

Define an operator $L$ in $L^{p+2}(Q_T)^3$ as follows:

$$ L[E] = \sigma(x, |E|)E. $$

Since $\sigma(x, s)$ is monotonic increasing in $s$, then the operator $L$ is monotonic increasing, that is,

$$ < L[E_\varepsilon] - L[E], E_\varepsilon - E > \geq 0. $$

It is clear that

$$ < L[E_\varepsilon] - L[E], E_\varepsilon - E > = < L[E_\varepsilon], E_\varepsilon > - < L[E], E > - < L[E_\varepsilon], E > + < L[E], E > $$

It follows that

$$ \lim_{\varepsilon \to 0} \inf \ < L[E_\varepsilon], E_\varepsilon > \geq < J, E >. \quad \text{(3.1)} $$

On the other hand, from the system (1.1)-(1.2) we have

$$ \int_0^T \int_\Omega \sigma(E_\varepsilon)|E_\varepsilon|^2 \lambda(t) dx dt $$

$$ = - \int_0^T \int_\Omega [\varepsilon \lambda(t)E_{\varepsilon t} \cdot E_\varepsilon + \lambda H_{\varepsilon t} \cdot H_\varepsilon] dx dt + \int \int_{Q_T} F \cdot E_\varepsilon dx dt $$

$$ = - \int_0^T \int_\Omega \left\{ \frac{\partial}{\partial t} \left[ \frac{1}{2} (\varepsilon |E_\varepsilon|^2 + |H_\varepsilon|^2) \lambda(t) \right] - \lambda'(t) \left[ \frac{1}{2} (\varepsilon |E_\varepsilon|^2 + |H_\varepsilon|^2) \right] \right\} dx dt + $$

$$ \int \int_{Q_T} F \cdot E_\varepsilon dx dt $$

$$ \leq \int_0^T \int_\Omega \lambda'(t) \left[ \frac{1}{2} |H_\varepsilon|^2 \right] dx dt + \frac{1}{2} \int \int_\Omega \left[ \varepsilon |E_\varepsilon(x,0)|^2 + |H_\varepsilon(x,0)|^2 \right] dx + $$

$$ \int \int_{Q_T} F \cdot E_\varepsilon dx dt. \quad \text{(3.2)} $$
Since $\lambda(t) \leq 0$, it follows that
\[
\lim_{\varepsilon \to 0} \sup \int_0^T \int_\Omega \sigma(|E_\varepsilon|)|E_\varepsilon|^2 \lambda(t) \, dx \, dt
\leq \frac{1}{2} \int_0^T \int_\Omega \lambda(t)|H|^2 \, dx \, dt + \frac{1}{2} \int_\Omega |H_0(x)|^2 \, dx + \int \int_{Q_T} E \cdot F \, dx \, dt.
\]
(3.3)

Recall from Definition 2.1 that $(E_\varepsilon, H_\varepsilon)$ satisfies the following integral equations:
\[
\int_0^T \int_\Omega \left[ -\varepsilon E_\varepsilon \cdot \Phi_t + \sigma(x, |E_\varepsilon|) E_\varepsilon \cdot \Phi \right] \, dx \, dt
= \int_0^T \int_\Omega \left[ -H_\varepsilon \cdot (\nabla \times \Phi) \right] \, dx \, dt + \int \int_{Q_T} F \cdot \Phi \, dx \, dt + \int_\Omega \varepsilon E_0 \cdot \Phi(x, 0) \, dx,
\]
\[
\int_0^T \int_\Omega \left[ -H_\varepsilon \cdot \Psi_t + E_\varepsilon \cdot (\nabla \times \Psi) \right] \, dx \, dt = \int \Omega [H_0(x) \cdot \Psi(x, 0)] \, dx.
\]
(3.5)

Now by choosing $\Phi = \lambda(t)E$ and $\Psi = \lambda(t)H$ (note that the condition at $t = T$ is satisfied since $\lambda(T) = 0$), we obtain
\[
\int_0^T \int_\Omega [\lambda(t)J \cdot E] \, dx \, dt = \frac{1}{2} \int_0^T \int_\Omega [\lambda(t)|H|^2] \, dx \, dt + \frac{1}{2} \int_\Omega |H_0(x)|^2 \, dx + \int \int_{Q_T} E \cdot F \, dx \, dt.
\]
It follows by (3.3) that
\[
\lim_{\varepsilon \to 0} \sup \lambda(t)L[E_\varepsilon], E_\varepsilon > \leq \lambda(t)J, E >.
\]
(3.6)
Combination of (3.1) and (3.6) yields
\[
\lim_{\varepsilon \to 0} \left< \lambda(t)L[\mathbf{E}_\varepsilon], \mathbf{E}_\varepsilon \right> = \left< \lambda(t)\mathbf{J}, \mathbf{E} \right>.
\] (3.7)

Consequently, by choosing \(\lambda(t)\) properly we have
\[
\lim_{\varepsilon \to 0} \left< L[\mathbf{E}_\varepsilon], \mathbf{E}_\varepsilon \right> = \left< \mathbf{J}, \mathbf{E} \right>.
\] (3.8)

For any vector field \(\mathbf{W} \in L^2(Q_T) \cap L^{p+1}(Q_T)\), the monotonicity of \(\sigma(x, s)\) in \(s\) implies
\[
\int \int_{Q_T} \left[ \sigma(x, |\mathbf{E}_\varepsilon|)\mathbf{E}_\varepsilon - \sigma(x, |\mathbf{W}|)\mathbf{W} \right] \cdot [\mathbf{E}_n - \mathbf{W}] dxdt \geq 0. \tag{3.9}
\]

We rewrite the above inequality to the following form:
\[
\int \int_{Q_T} \left\{ \sigma(|\mathbf{E}_\varepsilon|)|\mathbf{E}_\varepsilon|^2 - \sigma(|\mathbf{E}_\varepsilon|)\mathbf{E}_\varepsilon \cdot \mathbf{W} \right\} dxdt \\
\geq \int \int_{Q_T} \left[ \sigma(x, |\mathbf{W}|)\mathbf{W} \right] \cdot [\mathbf{E}_\varepsilon - \mathbf{W}] dxdt. \tag{3.10}
\]

We take the limit as \(\varepsilon \to 0\) and use (3.8) for the first term in (3.10) to obtain
\[
\int \int_{Q_T} \left\{ \mathbf{J} \cdot \mathbf{E} - \mathbf{J} \cdot \mathbf{W} \right\} dxdt \\
\geq \int \int_{Q_T} \left[ \sigma(x, |\mathbf{W}|)\mathbf{W} \right] \cdot [\mathbf{E} - \mathbf{W}] dxdt.
\]

Equivalently,
\[
\int \int_{Q_T} \left\{ [\mathbf{J} - \sigma(x, |\mathbf{W}|)]\mathbf{W} \right\} \cdot [\mathbf{E} - \mathbf{W}] dxdt \geq 0. \tag{3.11}
\]

Set \(\mathbf{W} = \mathbf{E} + \delta \mathbf{Y}\), where \(\delta > 0\) is small parameter and \(\mathbf{Y} \in L^2(Q_T) \cap L^{p+1}(Q_T)\) is arbitrary.

With the above choice of \(\mathbf{W}\) in the equality (3.10), we obtain
\[
\int \int_{Q_T} \left\{ \mathbf{Y} \cdot [\mathbf{J} - \sigma(|\mathbf{E} + \delta \mathbf{Y}|)](\mathbf{E} + \delta \mathbf{Y}) \right\} dxdt \geq 0.
\]

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When $\sigma(x, s)$ is continuous in $s$, then we let $\delta \to 0$ to obtain
\[
J(x, t) = \sigma(x, |E|)E,
\]
since $Y(x, t)$ is arbitrary in $L^2(Q_T) \cap L^{p+1}(Q_T)$.

When $\sigma(x, s)$ has a jump at a point, say, $s = 1$. Then as in §2 we understand that the value of $\sigma(x, s)$ at $s = 1$ is
\[
\sigma(x, 1-) \leq \sigma(x, 1) \leq \sigma(x, 1+).
\]
By using the same procedure as in §2, we can derive the above inequality.

Finally, by taking limit for (2.1)-(2.2) we see that $(E, H)$ is a weak solution of the quasi-stationary system.

To prove the uniqueness, we assume that $(E, H)$ and $(E^*, H^*)$ are two weak solutions to the quasi-stationary system. Let $\hat{E} = E - E^*$ and $\hat{H} = H - H^*$. Then the energy estimate implies
\[
\sup_{0 \leq t \leq T} \int_\Omega |\hat{H}|^2 dx + \int_{Q_T} [\sigma(x, |E|)E - \sigma(x, |E^*|)E^*] \cdot [E - E^*] \, dx \, dt \leq 0.
\]
The monotonicity of $\sigma(x, s)$ in $s$ implies the second term in the above inequality is nonnegative. It follows that
\[
\hat{H} = 0, \quad a.e. \text{in } Q_T.
\]
From the definition of weak solution, we have $\hat{E} = 0$ as long as $\sigma > 0$.

To conclude this section, we consider two special classes of electric conductivity

$\sigma(x, s) = s^p$ and

$\sigma(x, s) = \begin{cases} 
a, & \text{if } |s| \leq 1; 
b, & \text{if } |s| > 1,
\end{cases}$

where $0 < a < b$.

For the first case with $F = 0$, it is easy to see that
\[
E = |\nabla \times H|^{-\frac{p}{p+1}} \nabla \times H.
\]

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It follows from Theorem 3.1 that there exists a unique weak solution to the following evolution system

\[ H_t + \nabla \times [\|\nabla \times H\|^{-\frac{1}{2}} \nabla \times H] = G, \quad (x, t) \in Q_T, \]

subject to the initial-boundary conditions:

\[ N \times (\nabla \times H) = 0, \quad (x, t) \in \partial \Omega \times [0, T], \]
\[ H(x, 0) = H_0(x), \quad x \in \Omega, \]

where \( G \) is a known exterior magnetic field. This existence result is not covered in [16]. More regularity of the weak solution can be established as in [16]. We shall not repeat it here.

For the second case, we see that

\[ E = \frac{1}{a} \nabla \times H + \frac{1}{a} F, \]

in the region \( Q_T^- = \{(x, t) : |E| < 1\} \) and

\[ E = \frac{1}{b} \nabla \times H + \frac{1}{b} F, \]

in the region \( Q_T^+ = \{(x, t) : |E| > 1\} \). Note that

\[ \text{div} H(x, t) = \text{div} H_0(x) = 0, \]

we see that

\[ \nabla \times \nabla \times H = -\Delta H. \]

It follows that \( H \) satisfies the parabolic equation:

\[ H_t - \frac{1}{a} \Delta H = -\frac{1}{a} \nabla \times F, \quad (x, t) \in Q_T^- \]

and

\[ H_t - \frac{1}{b} \Delta H = -\frac{1}{b} \nabla \times F, \quad (x, t) \in Q_T^+. \]

The regularity theory of parabolic equations implies that \( H(x, t) \) is smooth in \( Q_T^\pm \).
The interface between $Q^-_T$ and $Q^+_T$ is defined by
\[ \Gamma = \{(x, t) : |E| = 1\}, \]
which is a free boundary.

**Remark 3.1:** We may allow that $\sigma(s) = 0$ if $|E| < 1$ in the above example. In this case, one must consider the full system (1.1)-(1.2) in order to define a weak solution. However, in this case the uniqueness of the weak solution does not hold.

**Remark 3.2:** It is not clear whether or not $\Gamma$ is indeed a hypersurface in $R^3 \times (0, \infty]$. It would be of great interest to study the smoothness of the interface $\Gamma$ and to find the free boundary conditions for $H$.

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