Information-disturbance tradeoff in quantum measurements

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We present a simple information-disturbance tradeoff relation valid for any general measurement apparatus: The disturbance between input and output states is lower bounded by the information the apparatus provides in distinguishing these two states.

PACS numbers: 03.67.-a,03.65.Yz,42.50.Lc,03.65.Ta

Extraction of information from a quantum system cannot always be without feedback. This was clear since the early days of quantum mechanics: It was the spirit of the original form of the Heisenberg uncertainty “principle”, as derived from the gedanken-experiment of the Heisenberg microscope [1]. Since then, much more refined descriptions of allowed quantum measurements have been put forth [2,3], so that we now know that the Heisenberg principle can be easily circumvented [2,3], and that its correct interpretation must be carefully adjusted (see Ref. [4] for a recent review on the subject). The upshot is that there is no “unavoidable dynamical disturbance” attached to all measurements. The debate on that state is lower bounded by the amount of information the experimenter has on the system [7].

We show that at least one state must be modified by the measurement and then we give a bound on such modification. For the sake of clarity, we give proofs of a very simple case, and postpone the general derivation to the appendix.

Before attempting a derivation of an information-disturbance tradeoff, we have to appropriately define these two quantities.

Information: Intuitively, one would expect that the information extracted from a measurement should be defined as a function of the outcome statistics only, such as the entropy of the probability of the outcomes. This is easily shown to be inadequate: Think of a measurement device that returns random outcomes (according to a well defined probability) without yielding any information on the system. A “good” measurement should have outcomes in some way correlated to the initial state of the system, so to provide information on the system. Thus, a suitable expression for the information-part of our tradeoff is through the mutual information I the measurement provides on which of two equally-probable input states the system is in [4]. It supplies the fraction of a bit the measurement tells us on which one is the input state, and varies continuously between I = 0 (no knowledge) and I = 1 (complete knowledge). Alternatively, we can employ the binary entropy H_{2}(p_e) of the probability p_e of making an error when determining which state: It is a measure of the uncertainty on the determination of which state. The two quantities are simply related as I = 1 − H_{2}(p_e). Information is measured in bits. To obtain an adimensional quantity (in order to relate information and disturbance), we will consider the ratio between information I (or uncertainty H_{2}) and the maximum information (or maximum uncertainty) that can be obtained, i.e. one bit in this case.

Disturbance: A system is disturbed by a physical process when its initial and final states do not coincide. The fidelity $\text{tr}|\sqrt{\rho} \sqrt{\sigma}|^2$ [12], a simple function of the Bures distance, is the most appropriate mea-
measurement model $\{\Pi_k\}$ (see Fig. 1). The measured system interacts unitarily with an external ancillary system describing the measurement apparatus. The ancillary system then undergoes a Lüders-type projective measurement $M$, i.e. such that its POVM elements are orthogonal projectors $\{\Pi_k = |k\rangle\langle k|\}$. The system output state is then the partial trace (over the ancillary Hilbert space $A$) conditioned on obtaining the result $k$ on the ancilla, i.e. $\rho'_{(k)} = \frac{\text{Tr}_A \left[ (I_H \otimes |k\rangle\langle k|) U (\rho \otimes \sigma) U^\dagger \right]}{\text{Tr} \left[ (I_H \otimes |k\rangle\langle k|) U (\rho \otimes \sigma) U^\dagger \right]}$, where $\sigma$ is the initial state of the ancilla and $U$ is the unitary interaction that correlates the system to the apparatus, acting on $H \otimes A$. Notice that there is no assumption on the joint post-measurement state in Eq. (2), which combines the Born rule on the ancillary space $A$ with the rule to obtain the state of a subsystem from a partial trace on the joint state.

For the sake of clarity, we will start analyzing the simple case in which the input states of the system $\rho$ and of the apparatus $\sigma = |0\rangle\langle 0|$ are pure and no entanglement is generated by the unitary $U$. The general situation will be analyzed subsequently. The unitary will thus evolve two different input states $|\psi_1\rangle$ and $|\psi_2\rangle$ according to the evolution $|\psi_1'\rangle|a_1\rangle = U|\psi_1\rangle|0\rangle$ and $|\psi_2'\rangle|a_2\rangle = U|\psi_2\rangle|0\rangle$. A unitary does not change the scalar product, hence $\langle \psi_1 | \psi_2 \rangle = \langle \psi_1' | \psi_2' \rangle a_1 a_2$. We assume that the measurement is informative, i.e. the apparatus is able to correlate to the system somehow. This implies that there must exist some $|\psi_1\rangle$ and $|\psi_2\rangle$ that give rise to different states in the apparatus, i.e. $|a_1\rangle \neq |a_2\rangle$. Thus, $|a_1\rangle a_2 < 0$ so that $|\langle \psi_1 | \psi_2 \rangle|^2 < |\langle \psi_1' | \psi_2' \rangle|^2$, i.e. the output states are less distinguishable than the input: their fidelity has increased. In the general case (see the appendix), this can be formalized in the following way. For any informative measurement, there exist at least two system states $\rho_1$ and $\rho_2$ such that

$$F(\rho_1, \rho_2) < F(\rho_1', \rho_2'),$$

where $\rho_1'$, $\rho_2'$ are the output states corresponding to $\rho_1$, $\rho_2$.
when the measurement results are the same. This implies that for any measurement there exists at least one state that is modified.

Call such a state \(|\psi\rangle\). The scalar product between \(|\psi\rangle\) and its evolved counterpart \(|\psi'\rangle\) is \(|\langle \psi | \psi' \rangle| = |\langle \psi | \psi' \rangle| (a |a\rangle | + |a'\rangle |) \leq |\langle a |a\rangle | | + |\langle a' |a\rangle | |\), where \(|a\rangle\) and \(|a'\rangle\) are the apparatus states corresponding to system inputs \(|\psi\rangle\) and \(|\psi'\rangle\) respectively, and where \(|\psi''\rangle\) is the system output corresponding to input \(|\psi'\rangle\). In general the evolution \(U\) will generate entanglement between system and apparatus so that the system output state will be a mixed state (see appendix). The probability of error \(p_e\) in discriminating between two states \(|a\rangle\) and \(|a'\rangle\) can be calculated from state discrimination theory as \(p_e = (1 - \sqrt{1 - |\langle a |a\rangle |^2})/2\), whence \(|\langle a |a\rangle |^2 = 4p_e(1 - p_e)\).

The uncertainty in this discrimination is given by the Shannon entropy of the related probability distribution \(\{p_e, 1 - p_e\}\), i.e. the binary entropy \(H_2(p_e)\). It measures the bits of information one would gain by discovering which of the two states the apparatus is in after the unitary interaction. Since \(4p_e(1 - p_e) \leq H_2(p_e)\), we find that \(|\langle \psi | \psi' \rangle|^2 \leq H_2(p_e)\): the fidelity between the input and output states is upper bounded by the binary entropy related to the discrimination of the two states by the apparatus. This can be restated in the form of a tradeoff relation as

\[
1 - \frac{|\langle \psi | \psi' \rangle|^2}{1 - H_2(p_e)} \geq 1 - H_2(p_e) .
\]

In the general situation (see the appendix), this information-disturbance tradeoff takes the equivalent form

\[
1 - F(\phi, \phi') \geq 1 - H_2(p_e) : \quad (5)
\]

The disturbance \(1 - F\) between input \(\phi\) and output \(\phi'\) is lower bounded by the mutual information \(1 - H_2(p_e)\) on which of the two states \(\phi\) and \(\phi'\) is present at the input. This is the main result of the paper. By rearranging the terms of \((5)\) as \(1 - F(\phi, \phi') + H_2(p_e) \geq 1\), we can also give it a different interpretation: The disturbance \(1 - F\) between input and output plus the uncertainty \(H_2(p_e)\) in the discrimination by the apparatus of these two states cannot be made arbitrarily small. Equivalently, we can say that the mutual information on which state plus the fidelity of these two states are upper bounded by one.

Since the inequality \(4p_e(1 - p_e) \leq H_2(p_e)\) is tight only for \(p_e = 0, 1/2, 1\), the bound \((4)\) is not tight in general. It is achieved only if the apparatus cannot discriminate between \(\phi\) and \(\phi'\) at all, or if it can discriminate between them exactly.

Even though the state reduction rule is not a quantum prerogative, the tradeoff we derived is a purely quantum effect. In classical mechanics, an informative non-disturbing measurement which perfectly correlates the outcomes with the state of a system will collapse a mixed state into a pure state: The effect of such a measurement is to reduce the “volume” that the state of the system occupies in phase space (a sort of “classical state-reduction”). In classical mechanics, there is no lower bound to such volume and two pure states, which occupy zero volume, can always be distinguished without disturbance. In contrast, in quantum mechanics the “volume” of a state must occupy in phase space is lower bounded by \(\hbar/2\). On one hand two non-identical pure states may overlap and their conclusive discrimination may not be possible. On the other hand, if the post-measurement state is perfectly correlated with the outcome (Lüders or von Neumann type apparatuses) and the measure is sharp enough to sufficiently constrain the volume in one direction of the phase space, the post-measurement state must “expand” in other directions to preserve the minimum volume. For other types of apparatuses the situation is not as cut-and-dried as we have shown, at least one pure state of the system must be modified by any informative measurement. So, while in classical mechanics the system will evolve compatibly with its pre-measurement trajectory in phase space (only the “thickness” of the trajectory may be reduced), in quantum mechanics the phase-space expansion might have observable consequences and the system might not evolve compatibly with its pre-measurement trajectory.

In conclusion, we have derived an information-disturbance tradeoff which is valid for any measurement device: Any measurement modifies at least one state of the system, and the fidelity between input and output states is upper bounded by the binary entropy related to the discrimination of the two states by the apparatus. This can be restated in the form of a tradeoff relation as

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Even though the state reduction rule is not a quantum prerogative, the tradeoff we derived is a purely quantum effect. In classical mechanics, an informative non-disturbing measurement which perfectly correlates the outcomes with the state of a system will collapse a mixed state into a pure state: The effect of such a
Eq. (3) is true. Incidentally, note that the converse also partially holds: If a measurement decreases the fidelity, then all unitaries \( U \) corresponding to its indirect measurement models will transfer some information to the probe state (this does not automatically imply that the measurement is informative, since the modification of the probe state may be ignored the last stage of the apparatus, the von Neumann measure \( M \) of Fig. 1). In fact, the no-signaling property of factorized unitary maps \([22]\) implies that any non-factorized unitary \( U \) of the indirect measurement model can send a signal from the system to the probe, i.e. \( U \neq U_R \otimes U_A \) implies that there exist two states \( \varphi_1, \varphi_2 \) such that \( F(\sigma'_1, \sigma'_2) < 1 \), where \( \sigma'_i = \text{Tr}[U(\varphi_i \otimes \sigma) U^\dagger] \) is the final state of the probe, \( \sigma \) is its initial state, and \( U_R \) and \( U_A \) are arbitrary unitaries acting only on the system and on the ancillary Hilbert spaces respectively. It is possible to evaluate which states are modified by the measurement process for each outcome \( k \), by considering the map \( L_k \) as a linear operator on the operator space of the states of the system. One then immediately sees that only the eigenstates of \( L_k \) are not altered, while superpositions of eigenstates with different eigenvalues are.

Proof of Eq. (4): In general, the input states to the apparatus may be mixed. The probability of making a mistake when discriminating two mixed states \( \varphi_1 \) and \( \varphi_2 \) is given by \( p_c = 1/2 - \text{Tr}[|\varphi_1 - \varphi_2||4 \Box 23] \). By using the property \( \text{Tr}[|\varphi_1 - \varphi_2|] \leq 2 \sqrt{1 - F(\varphi_1, \varphi_2)} \) \([15]\), we can write \( p_c \geq (1 - \sqrt{1 - F(\varphi_1, \varphi_2)})/2 \), where the equality is attained for pure states \([15]\). The binary entropy \( H_2(x) \equiv -x \log_2 x - (1 - x) \log_2 (1 - x) \) for \( x \in [0, 1] \) satisfies the inequalities \( x \leq H_2(1/2 - 1/2 \sqrt{1 - x}) \) and \( x \leq H_2(1/2 + 1/2 \sqrt{1 - x}) \). Moreover, for \( x \leq 1/2 \), it is monotonically increasing so that we can write

\[
x \leq H_2 \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - x} \right) \leq H_2(y) ,
\]

for any \( y \) such that \( \frac{1}{2} - \frac{1}{2} \sqrt{1 - x} \leq y \leq \frac{1}{2} \). Choosing \( x = F(\varphi_1, \varphi_2) \) and \( y = p_c \), we obtain \( F(\varphi_1, \varphi_2) \leq H_2(p_c) \), i.e. Eq. (5) from (6), which is valid when \( p_c \leq 1/2 \). If \( p_c \geq 1/2 \) instead, we proceed analogously starting from

\[
x \leq H_2 \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - x} \right) \leq H_2(y') ,
\]

valid for \( 1/2 \leq y' \leq 1/2 + \frac{1}{2} \sqrt{1 - x} \). Choosing \( x = F(\varphi_1, \varphi_2) \) and \( y' = 1 - p_c \), we obtain Eq. (4) for \( p_c \geq 1/2 \), by recalling that \( H_2(1 - p_c) = H_2(p_c) \).

I thank Vittorio Giovannetti for very useful discussions and criticisms. Financial support comes from the Ministero Italiano dell’Università e della Ricerca (MIUR) through FIRB (bando 2001) and PRIN 2005.

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