Research Article

Maclaurin Coefficient Estimates for New Subclasses of Bi-univalent Functions Connected with a $q$-Analogue of Bessel Function

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1. Introduction, Definitions, and Preliminaries

The theory of $q$-calculus plays an important role in many areas of mathematical, physical, and engineering sciences. Jackson (see [1, 2]) was the first to have some applications of the $q$-calculus and introduced the $q$-analogue of the classical derivative and integral operators (see also [3]). Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \Delta := \{z \in \mathbb{C} : |z| < 1\},$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent functions in $\Delta$. If $g \in \mathcal{S}$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in \Delta,$$

then, the Hadamard (or convolution) product of $f$ and $g$ is defined by

$$(f \ast g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \Delta.$$

If $f$ and $F$ are analytic functions in $\Delta$, we say that $f$ is subordinate to $F$, written $f \prec F$, if there exists a Schwarz function, which is analytic in $\Delta$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \Delta$, such that $f(z) = F(w(z))$, $z \in \Delta$. Furthermore, if the function $F$ is univalent in $\Delta$, then we have the following equivalence (see [4, 5]):

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\Delta) \subset F(\Delta).$$

The Bessel function of the first kind of order $\nu$ is defined by the infinite series (see [6])

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k! \Gamma(k+\nu+1)} \quad (z \in \mathbb{C}, \nu \in \mathbb{R})$$

where $\Gamma$ stands for the Gamma function. Recently, Szász and Kupán [7] investigated the univalence of the normalized Bessel function of the first kind $k_\nu : \Delta \rightarrow \mathbb{C}$ defined by (see also [8–10])

$$k_\nu(z) := 2^\nu \Gamma(\nu+1)z^{1-\nu}J_\nu(z^2)$$

$$= z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(k+\nu)}{4^{k-1}(k-1)! \Gamma(k+\nu)} z^k, \quad (z \in \Delta, \nu \in \mathbb{R}).$$
For $0 < q < 1$, the $q$-derivative operator for $k_n$ is defined by

\[
\partial_q k_n(z) = \partial_q \left[ z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(v+1)}{4^{k-1}(k-1)! \Gamma(k+v)} z^k \right] = \frac{k_n(z) - k_n(z)}{z(q-1)} = 1 + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{\Gamma(v+1)}{4^{k-1}(k-1)! \Gamma(k+v)} [k, q] z^{k-1}, \quad z \in \Delta,
\]

where

\[
[k, q] = \frac{1-q^k}{1-q} = 1 + \sum_{k=1}^{\infty} q^k, \quad [0, q] = 0.
\]

Using the definition formula (8), we will define the next two products:

(i) For any nonnegative integer $k$, the $q$-shifted factorial is given by

\[
[k, q]! = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{j=1}^{\infty} [1, q][2, q][3, q] \cdots [k, q], & \text{if } k \in \mathbb{N}. \end{cases}
\]

(ii) For any positive number $r$, the $q$-generalized Pochhammer symbol is defined by

\[
[r, q]_k = \begin{cases} 1, & \text{if } k = 0, \\ [r, q][r+1, q] \cdots [r+k-1, q], & \text{if } k \in \mathbb{N}. \end{cases}
\]

For $0 < \lambda < -1$, and $0 < q < 1$, El-Deeb and Bulboacă [11] define the function $\mathcal{F}^q_{\lambda} : \Delta \rightarrow \mathbb{C}$ by

\[
\mathcal{F}^q_{\lambda}(z) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(v+1)}{4^{k-1}(k-1)! \Gamma(k+v)} \frac{[k, q]!}{[\lambda + 1, q]_{k-1}} z^k, \quad z \in \Delta.
\]

A simple computation shows that

\[
\mathcal{F}^q_{\lambda}(z) \ast \mathcal{M}_{q, \lambda+1}(z) = z \partial_q k_n(z), \quad z \in \Delta,
\]

where the function $\mathcal{M}_{q, \lambda+1}$ is given by

\[
\mathcal{M}_{q, \lambda+1}(z) = z + \sum_{k=2}^{\infty} \frac{[\lambda + 1, q]_{k-1}}{[k-1, q]!} z^k, \quad z \in \Delta.
\]

Using the definition of $q$-derivative along with the idea of convolutions, El-Deeb and Bulboacă [11] introduce the linear operator $\mathcal{N}^q_{\lambda} : \mathcal{A} \rightarrow \mathcal{A}$ defined by

\[
\mathcal{N}^q_{\lambda}(f)(z) = \mathcal{F}^q_{\lambda}(z) \ast f(z) = z + \sum_{k=2}^{\infty} \psi_k [k, q] z^k, \quad z \in \Delta, \quad (\nu > 0, \lambda > 1, 0 < q < 1),
\]

where

\[
\psi_k = \frac{(-1)^{k-1} \Gamma(v+1)}{4^{k-1}(k-1)! \Gamma(k+v)} \frac{[k, q]!}{[\lambda + 1, q]_{k-1}}.
\]

Remark 1. From the definition relation (14), we can easily verify that the next relations hold for all $f \in$: 

(i) $[\lambda + 1, q] \mathcal{N}^q_{\lambda}(f)(z) = [\lambda, q] \mathcal{N}^q_{\lambda+1}(f)(z) + q^\nu z \partial_q \left( \mathcal{N}^q_{\lambda+1}(f)(z) \right), \quad z \in \Delta;
\]

(ii) $\lim_{q \rightarrow 1} \mathcal{N}^q_{\lambda}(f)(z) = \mathcal{F}^q_{\lambda} \ast f(z) = \mathcal{F}^q_{\lambda}(z) = z + \sum_{k=2}^{\infty} \frac{k!}{[\lambda + 1, q]_{k-1}} \frac{(-1)^{k-1} \Gamma(v+1)}{4^{k-1}(k-1)! \Gamma(k+v)} a_k z^k, \quad z \in \Delta.
\]

The Koebe one quarter theorem (see [12]) proves that the image of $\Delta$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Therefore, every function $f \in \mathcal{S}$ has an inverse $f^{-1}$ satisfying

\[
f^{-1}(f(z)) = z (z \in \Delta),
\]

\[
f^{-1}(\omega) = \omega \left( |\omega| < r_0(f) ; r_0(f) \geq \frac{1}{4} \right),
\]

where

\[
f^{-1}(\omega) = \omega - a_2 \omega^2 + (2a_3 - a_2) \omega^3 - (5a_4 - 5a_2 a_3 + a_4) \omega^4 + \cdots.
\]

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\Delta$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\Delta$. Let $\Sigma$ denote the class of bi-univalent functions in $\Delta$ given by (1). For a brief history and interesting examples in the class $\Sigma$, see [13]. Brannan and Taha [14] (see also [15–17]) introduced certain subclasses of the bi-univalent functions class $\Sigma$ similar to the familiar subclasses $\mathcal{S}'(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha (0 < \alpha < 1)$, respectively (see [13]). Thus, following Brannan and Taha [14], a function $f \in \mathcal{A}$ is
said to be in the class $S^*_p(\alpha)$ of strongly bi-starlike functions of order $\alpha(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left| \arg \left( \frac{zf'(z)}{f(z)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ z \in \Delta) \right. \right) \tag{19}$$

$$\left| \arg \left( \frac{zg'(w)}{g(w)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ w \in \Delta), \right. \right)$$

where the function $g$ is given by

$$g(w) = w - a_1w^2 + (2a_2^2 - a_3)w^3 - (5a_3^3 - 5a_4a_3 + a_4)w^4 + \ldots, \tag{20}$$

and $g$ is the extension of $f^{-1}$ to $\Delta$. The classes $S^*_p(\alpha)$ and $K_p(\alpha)$ of bi-starlike functions of order $\alpha$ and biconvex functions of order $\alpha(0 < \alpha \leq 1)$, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^*_p(\alpha)$ and $K_p(\alpha)$, they found nonsharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [14, 17]).

The objective of the present paper is to introduce new subclasses of the function class $\Sigma$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class $\Sigma$ employing the techniques used earlier by Srivastava et al. [18].

Now, we define the subclasses of functions $Q^\beta_p(\alpha, \gamma, \lambda, \nu)$, $R^\beta_p(\beta, \gamma, \lambda, \nu)$, and $Q_p(\alpha, \gamma, \lambda, \nu)$ as follows:

**Definition 2.** Let $f(z)$ be given by (1), then $f(z)$ is said to be in the class $Q^\beta_p(\alpha, \gamma, \lambda, \nu)$ if the following conditions are satisfied:

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ z \in \Delta) \right. \right) \tag{21}$$

$$\left| \arg \left( \frac{zg'(w)}{g(w)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ w \in \Delta), \right. \right)$$

where the function $g$ is given by (2).

Putting $q \longrightarrow 1^-$, we obtain that

$$\lim_{q \longrightarrow 1^-} Q^\beta_p(\alpha, \gamma, \lambda, \nu) = Q_p(\alpha, \gamma, \lambda, \nu),$$

where

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ z \in \Delta) \right. \right) \tag{23}$$

$$\left| \arg \left( \frac{zg'(w)}{g(w)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ w \in \Delta), \right. \right)$$

where the function $g$ is given by (2).

**Definition 3.** Let $f(z)$ be given by (1), then $f(z)$ is said to be in the class $R^\beta_p(\beta, \gamma, \lambda, \nu)$ if the following conditions are satisfied:

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ z \in \Delta) \right. \right) \tag{24}$$

$$\left| \arg \left( \frac{zg'(w)}{g(w)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ w \in \Delta), \right. \right)$$

where the function $g$ is given by (2).

Putting $q \longrightarrow 1^-$, we obtain that

$$\lim_{q \longrightarrow 1^-} R^\beta_p(\beta, \gamma, \lambda, \nu) = R_p(\beta, \gamma, \lambda, \nu),$$

where

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ z \in \Delta) \right. \right) \tag{25}$$

$$\left| \arg \left( \frac{zg'(w)}{g(w)} \right) > \frac{\alpha\pi}{2} (0 < \alpha \leq 1, \ w \in \Delta), \right. \right)$$

where the function $g$ is given by (2).

To prove our results, we need the following lemma.

**Lemma 4** (see [19], Lemma 3). If $G \in P$ then $|c_k| \leq 2$ for each $k$, where $P$ is the family of all functions $G$ analytic in $\Delta$ for which

$$R(G) > 0, G(z) = 1 + c_1z + c_2z^2 + \cdots \quad \text{for } z \in \Delta.$$
Definition 5. Let \( f(z) \) be given by (1), \( h(z) \) and \( p(w) \) in \( P \) have the forms
\[
h(z) = 1 + h_1z + h_2z^2 + h_3z^3 + \cdots, \tag{27}
\]
\[
p(w) = 1 + u_1w + u_2w^2 + u_3w^3 + \cdots, \tag{28}
\]
then \( f(z) \) is said to be in the class \( Q_\Sigma \), unless otherwise mentioned, we assume throughout this paper that
\[
\frac{z \left( \mathcal{N}^\lambda_{\nu,q}f(z) \right)'}{(1 - \gamma) \left( \mathcal{N}^\lambda_{\nu,q}f(z) \right)'} + \frac{\gamma z^2 \left( \mathcal{N}^\lambda_{\nu,q}f(z) \right)''}{(1 - \gamma) \left( \mathcal{N}^\lambda_{\nu,q}f(z) \right)'} \in h(\Delta) (\nu > 0; \lambda > -1; 0 \leq q < 1; 0 \leq \alpha \leq 1; 0 \leq \gamma \leq 1, \ z \in \Delta), \tag{29}
\]
\[
\frac{w \left( \mathcal{N}^\lambda_{\nu,q}g(w) \right)'}{(1 - \gamma) \left( \mathcal{N}^\lambda_{\nu,q}g(w) \right)'} + \frac{\gamma w^2 \left( \mathcal{N}^\lambda_{\nu,q}g(w) \right)''}{(1 - \gamma) \left( \mathcal{N}^\lambda_{\nu,q}g(w) \right)'} \in p(\Delta) (\nu > 0; \lambda > -1; 0 \leq q < 1; 0 \leq \alpha \leq 1; 0 \leq \gamma \leq 1, \ w \in \Delta), \tag{30}
\]
where the function \( g \) is given by (2).

2. Coefficient Bounds for the Function Class \( Q_\Sigma^2(\alpha, \gamma, \lambda, \nu) \)

Unless otherwise mentioned, we assume throughout this paper that
\[
\begin{align*}
0 < & \alpha \leq 1, \\
0 & \leq \beta < 1, \\
0 & \leq \gamma \leq 1, \\
\nu & > 0, \\
\lambda & > -1, \\
0 & < q < 1, \\
z & \in \Delta.
\end{align*} \tag{31}
\]

Theorem 6. Let \( f(z) \) be given by (1) which belongs to the class \( Q_\Sigma^2(\alpha, \gamma, \lambda, \nu) \), then
\[
|a_2| \leq \frac{2\alpha}{\sqrt{\left| 4\alpha(1 + 2\gamma)\psi_3 + (1 - 3\alpha)(1 + \gamma)^2\psi_2 \right|}}, \tag{32}
\]
\[
|a_3| \leq \frac{\alpha}{(1 + 2\gamma)\psi_3} + \frac{4\alpha^2}{(1 + \gamma)^2\psi_2^2}, \tag{33}
\]
where \( \psi_k, \ k \in \{2, 3\}, \) are given by (15).
It follows from (42), (43), and (46) that
\[
a_3 = \frac{\alpha^2(p_3^2 + q_3^2)}{2(1 + y)^2\psi_2^2} + \frac{\alpha(p_2 - q_2)}{4(1 + 2y)\psi_3}.
\] (47)

Applying Lemma 4 once again for the coefficients \(p_1, p_2, q_1,\) and \(q_2,\) we immediately have
\[
|a_3| \leq \frac{4\alpha^2}{(1 + y)^2\psi_2^2} + \frac{\alpha}{(1 + 2y)\psi_3}.
\] (48)

This completes the proof of Theorem 6.

3. Coefficient Bounds for the Function Class \(\mathcal{P}_\Sigma^2(\beta, \gamma, \lambda, \nu)\)

**Theorem 7.** Let \(f(z)\) be given by (1) which belongs to the class \(\mathcal{P}_\Sigma^2(\beta, \gamma, \lambda, \nu),\) then
\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{2(1 + 2y)\psi_3 - (1 + y)^2\psi_2^2}},
\] (49)
\[
|a_3| \leq \frac{(1 - \beta)}{(1 + 2y)\psi_3} + \frac{4(1 - \beta)^2}{(1 + y)^2\psi_2^2},
\] (50)
where \(\psi_k, k \in \{2, 3\},\) are given by (15).

**Proof.** It follows from (24) and (25) that
\[
\frac{z\left(\mathcal{N}_{\lambda, \gamma}^\prime f(z)\right)'}{(1 - \gamma)\left(\mathcal{N}_{\lambda, \gamma} f(z)\right)'} + \frac{\gamma z^2\left(\mathcal{N}_{\lambda, \gamma}^\prime f(z)\right)''}{(1 - \gamma)\left(\mathcal{N}_{\lambda, \gamma} f(z)\right)''} = \beta + (1 - \beta)p(z)
\] (51)
\[
\frac{\omega\left(\mathcal{N}_{\lambda, \gamma} g(w)\right)'}{(1 - \gamma)\left(\mathcal{N}_{\lambda, \gamma} g(w)\right)'} + \frac{\gamma \omega^2\left(\mathcal{N}_{\lambda, \gamma}^\prime g(w)\right)''}{(1 - \gamma)\left(\mathcal{N}_{\lambda, \gamma}^\prime g(w)\right)''} = \beta + (1 - \beta)q(w),
\] (52)
where \(p(z)\) and \(q(w)\) have the forms (36) and (37), respectively. Equating the coefficients in (51) and (52), we get
\[
(1 + y)\psi_2 a_2 = (1 - \beta)p_1,
\] (53)
\[
2(1 + 2y)\psi_3 a_3 - (1 + y)^2\psi_2^2 a_2^2 = (1 - \beta)p_2,
\] (54)
\[
-2(1 + 2y)\psi_3 q_2 a_2 = (1 - \beta)q_1,
\] (55)
\[
2(1 + 2y)(2a_2^2 - a_1)\psi_3 - (1 + y)^2\psi_2^2 a_2^2 = (1 - \beta)q_2.
\] (56)

From (53) and (55), we get
\[
p_1 = -q_1,
\] (57)
\[
2(1 + y)^2\psi_2^2 a_2 = (1 - \beta)^2(p_1^2 + q_1^2).
\] (58)

From (54) and (56), we obtain
\[
a_2 = \frac{(1 - \beta)(p_2 - q_2)}{4(1 + 2y)\psi_3 - 2(1 + y)^2\psi_2^2}.
\] (59)

Applying Lemma 4 for the coefficients \(p_2\) and \(q_2,\) we immediately have
\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{2(1 + 2y)\psi_3 - (1 + y)^2\psi_2^2}}.
\] (60)

This gives the bound on \(|a_2|\) as asserted in (49).

Next, in order to find the bound on \(|a_3|\), by subtracting (56) from (54), we get
\[
4(1 + 2y)\psi_3 a_3 - 4(1 + 2y)\psi_2^2 a_2^2 = (1 - \beta)(p_2 - q_2).
\] (61)

It follows from (58) and (61) that
\[
a_3 = \frac{(1 - \beta)(p_2 - q_2)}{4(1 + 2y)\psi_3} + \frac{(1 - \beta)^2(p_1^2 + q_1^2)}{2(1 + y)^2\psi_2^2}.
\] (62)

Applying Lemma 4 once again for the coefficients \(p_1, p_2, q_1,\) and \(q_2,\) we immediately have
\[
|a_3| \leq \frac{(1 - \beta)}{(1 + 2y)\psi_3} + \frac{4(1 - \beta)^2}{(1 + y)^2\psi_2^2}.
\] (63)

This completes the proof of Theorem 6.

4. General Coefficient Bounds for the Function Class \(\mathcal{Q}_\Sigma^2(h, p, \gamma, \lambda, \nu)\)

This section begins by finding the estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in the class \(\mathcal{Q}_\Sigma^2(h, p, \gamma, \lambda, \nu),\) then

\[
|a_2| \leq \min \left\{ \frac{|h'(0)|^2 + |p'(0)|^2}{2(1 + y)^2\psi_2^2}, \frac{|h''(0)| + |u''(0)|}{4(1 + 2y)\psi_3 - 2(1 + y)^2\psi_2^2} \right\}
\] (64)
\[
|a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |u'(0)|^2}{2(1 + y)^2\psi_2^2}, \frac{|h''(0)| + |u''(0)|}{4(1 + 2y)\psi_3}, \frac{|4(1 + 2y)\psi_3 - (1 + y)^2\psi_2^2|}{|4(1 + 2y)\psi_3 - 2(1 + y)^2\psi_2^2|} \right\}
\] (65)

where \(\psi_k, k \in \{2, 3\},\) are given by (15).
Proof. It follows from (29) and (30) that

$$z\left(\frac{\mathcal{A}z_0 g(z)}{(1 - \gamma)\mathcal{A}z_0 g(z)}\right)' + \gamma z^2 \left(\frac{\mathcal{A}z_0 g(z)}{(1 - \gamma)\mathcal{A}z_0 g(z)}\right)'' = h(\Delta),$$

(66)

where $h(z)$ and $p(w)$ are given by (27) and (28). Now, equating the coefficients of $z^2$ and $z^3$ in (66) and (67), we get

$$2(1 + 2\gamma)\psi_3 a_2 - (1 + \gamma)^2 \psi_2^2 a_2^2 = h_2,$$

(69)

$$2(1 + 2\gamma)(2a_2^2 - a_3)\psi_3 - (1 + \gamma)^2 \psi_2 a_2^2 = u_2.$$

From (68) and (70), we get

$$h_1 = -u_1,$$

(72)

$$a_3^2 = \frac{h_1^2 + u_1^2}{2(1 + \gamma)^3 \psi_2^2}.$$  

(73)

From (69) and (71), we obtain

$$a_2^2 = \frac{h_2 + u_2}{4(1 + 2\gamma)\psi_3 - 2(1 + \gamma)^2 \psi_2^2}.$$  

(74)

Therefore, we obtain from the equations (73) and (74) that

$$|a_2| \leq \sqrt{\frac{\left|h'(0)\right|^2 + \left|u'(0)\right|^2}{2(1 + \gamma)^3 \psi_2^2}},$$

(75)

$$|a_3| \leq \sqrt{\frac{\left|h''(0)\right| + \left|u''(0)\right|}{4(1 + 2\gamma)\psi_3 - 2(1 + \gamma)^2 \psi_2^2}},$$

respectively, this gives the bound on $|a_3|$ as asserted in (64). Next, in order to find the bound on $|a_3|$, by subtracting (71) from (69), we get

$$4(1 + 2\gamma)a_3\psi_3 - 4(1 + 2\gamma)\psi_3 a_2^2 = h_2 - u_2.$$  

(76)

It follows from (73) into (76) that

$$a_3 = \frac{h_1^2 + u_1^2}{2(1 + \gamma)^3 \psi_2^2} + \frac{h_2 - u_2}{4(1 + 2\gamma)\psi_3}.$$  

(77)

We immediately have

$$|a_3| \leq \frac{|h'(0)|^2 + |u'(0)|^2}{2(1 + \gamma)^3 \psi_2^2} + \frac{|h''(0)| + |u''(0)|}{4(1 + 2\gamma)\psi_3^2}.  

(78)

On the other hand, upon substituting the value of $a_3^2$ from (74) into (76), we get

$$a_3 = \frac{4(1 + 2\gamma)\psi_3^2 - (1 + \gamma)^2 \psi_2^2}{4(1 + 2\gamma)\psi_3^2} \frac{h_2 + (1 + \gamma)^2 \psi_2^2 u_2}{2(1 + 2\gamma)\psi_3 - (1 + \gamma)^2 \psi_2^2}.$$  

(79)

Consequently, we get

$$|a_3| \leq \frac{|4(1 + 2\gamma)\psi_3^2 - (1 + \gamma)^2 \psi_2^2| \left|h''(0)\right| + (1 + \gamma)^2 \psi_2^2 |u''(0)|}{4(1 + 2\gamma)\psi_3^2 \left[2(1 + 2\gamma)\psi_3 - (1 + \gamma)^2 \psi_2^2\right]}.$$  

(80)

This completes the proof of Theorem 8.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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