Some basic results on fuzzy strong $\phi$-b-normed linear spaces

Abhishikta Das$^1$, T. Bag$^{2,*}$, and S. Chatterjee$^3$

$^1,2,3$Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan-731235, Birkharm, West-Bengal, India
E-mail$^1$: abhishikta.math@gmail.com
E-mail$^{2,*}$: tarapadavb@gmail.com
E-mail$^3$: shayani.mathvb10@gmail.com

Abstract
In this paper, definition of fuzzy strong $\phi$-b-normed linear space is given. Here the scalar function $|c|$ is replaced by a general function $\phi(c)$ where $\phi$ satisfies some properties. Some basic results on finite dimensional fuzzy strong $\phi$-b-normed linear space are studied.

Keywords: Fuzzy norm, t-norm, fuzzy normed linear space, fuzzy strong $\phi$-b-normed linear space.

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1 Introduction
The concept of a fuzzy set was introduced initially by Zadeh [1] in 1965. Since then, many authors have expansively developed the theory of fuzzy sets. Osmo Kaleva [2], Kramosil and Michalek [3], George and Veeramani [4] et al. introduced the concept of fuzzy metric spaces in different approaches. on the other hand, Katsaras [5], Felbin [6], Cheng and Mordeson [7], Bag and Samanta [8] gave the definition of fuzzy normed linear spaces in different way. Recently different types of generalized metric as well as norm (viz. 2-metric [11], b-metric [12], strong b-metric [13], G-metric [18], 2-norm [23], G-norm [24], etc.) and consequently generalized fuzzy metric and fuzzy norm (viz. fuzzy b-metric [21], strong fuzzy b-metric [22], fuzzy cone metric [19], fuzzy cone norm [20], G-fuzzy norm [25], etc.) have been introduced in different approaches.

In 2018, Oner [22] introduced the concept of fuzzy strong b-metric spaces and developed some topological results in such spaces. Following this definition of fuzzy strong b-metric spaces, in this paper we give a definition of fuzzy strong $\phi$-b-normed linear space whose induced fuzzy metric is Oner type. In fuzzy normed linear space, scalar multiplication is given by $N(cx, t) = N(x, \frac{1}{|c|})$. But in our definition of fuzzy strong $\phi$-b-norm, scalar multiplication is given by $N(cx, t) = N(x, \frac{1}{\phi(|c|)})$ where $\phi$ is a real valued function satisfying some properties.

We study some results on finite dimensional fuzzy strong $\phi$-b-normed linear spaces.

The organization of the paper is in the following.

Section 2 consists some preliminary results. In Section 3, we introduce a definition of fuzzy strong $\phi$-b-norm by using a special function $\phi$ in general t-norm settings and illustrate by examples. Some basic results of finite dimensional fuzzy strong $\phi$-b-normed linear spaces are established in Section 4.
2 Preliminaries

In this section some definitions and results are collected which are used in this paper.

**Definition 2.1.** [10] A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a \( t \)-norm if it satisfies the following conditions:

(i) \( * \) is associative and commutative;
(ii) \( \alpha * 1 = \alpha, \forall \alpha \in [0, 1] \);
(iii) \( \alpha \ast \gamma \leq \beta \ast \delta \) whenever \( \alpha \leq \beta \) and \( \gamma \leq \delta, \forall \alpha, \beta, \gamma, \delta \in [0, 1] \).

If \( * \) is continuous then it is called continuous \( t \)-norm.

The following are examples of some \( t \)-norms.

(i) Standard intersection: \( \alpha \ast \beta = \min\{\alpha, \beta\} \).
(ii) Algebraic product: \( \alpha \ast \beta = \alpha\beta \).
(iii) Bounded difference: \( \alpha \ast \beta = \max\{0, \alpha + \beta - 1\} \).

**Definition 2.2.** [9] Let \( X \) be a linear space over a field \( F \). A fuzzy subset \( N \) of \( X \times \mathbb{R} \) is called fuzzy norm on \( X \) if the following conditions hold:

(N1) \( \forall t \in \mathbb{R} \) with \( t \leq 0 \), \( N(x, t) = 0 \);
(N2) \( \forall t \in \mathbb{R}, t > 0, N(x, t) = 1 \) \iff \( x = \theta \);
(N3) \( \forall t \in \mathbb{R}, \text{ and } c \in \mathbb{R}, t > 0, N(cx, t) = N(x, \frac{t}{|c|}) \);
(N4) \( \forall s, t \in \mathbb{R}, N(x + y, s + t) \geq N(x, s) \ast N(y, t) \);
(N5) \( N(x, \cdot) \) is a non-decreasing function of \( t \) and \( \lim_{t \to \infty} N(x, t) = 1 \).

Then the pair \((X, N)\) is called fuzzy normed linear space.

**Definition 2.3.** [8] Let \((X, N)\) be a fuzzy normed linear space.

(i) A sequence \( \{x_n\} \) is said to be convergent if \( \exists x \in X \) such that \( \lim_{n \to \infty} N(x_n - x, t) = 1, \forall t > 0 \). Then \( x \) is called the limit of the sequence \( \{x_n\} \) and denoted by \( \lim x_n \).

(ii) A sequence \( \{x_n\} \) in a fuzzy normed linear space \((X, N)\) is said to be Cauchy if \( \lim_{n \to \infty} N(x_{n+p} - x_n, t) = 1, \forall t > 0 \) and \( p = 1, 2 \cdots \).

(iii) \( A \subseteq X \) is said to be closed if for any sequence \( \{x_n\} \) in \( A \) converges to \( x \in A \).

(iv) \( A \subseteq X \) is said to be the closure of \( A \), denoted by \( \bar{A} \) if for any \( x \in \bar{A} \), there is a sequence \( \{x_n\} \subseteq A \) such that \( \{x_n\} \) converges to \( x \).

(v) \( A \subseteq X \) is said to be compact if any sequence \( \{x_n\} \subseteq A \) has a subsequence converging to an element of \( A \).

**Definition 2.4.** [17] Let \((X, N)\) be a fuzzy normed linear space.

(i) A set \( B(x, \alpha, t), 0 < \alpha < 1 \) is defined as \( B(x, \alpha, t) = \{y : N(x - y, t) > 1 - \alpha\} \).
(ii) \( \tau = \{ G \subseteq X : x \in G, \exists t > 0 \text{ and } 0 < \alpha < 1 \text{ such that } B(x, \alpha, t) \subseteq G \} \) is a topology on \((X, N)\).

(iii) Members of \( \tau \) are called open sets in \((X, N)\).

**Definition 2.5.** A subset \( B \) of a fuzzy normed linear space \((X, N)\) is said to be fuzzy bounded if for each \( r, 0 < r < 1 \), \( \exists t > 0 \) such that \( N(x, t) > 1, \forall x \in B \).

**Lemma 2.6.** Let \((X, N)\) be a fuzzy normed linear space and \( N(x, \cdot)(x \neq 0) \). If the set \( M = \{ x : N(x, 1) > 0 \} \) is compact then \( X \) is finite dimensional.

### 3 Fuzzy strong \( \phi \)-b-normed linear space

In this section we give the definition of fuzzy normed linear space in a new approach.

**Definition 3.1.** Let \( \phi \) be a function defined on \( \mathbb{R} \) to \( \mathbb{R} \) with the following properties

(\( \phi_1 \)) \( \phi(-t) = \phi(t), \forall t \in \mathbb{R} \);

(\( \phi_2 \)) \( \phi(1) = 1 \);

(\( \phi_3 \)) \( \phi \) is strictly increasing and continuous on \((0, \infty)\);

(\( \phi_4 \)) \( \lim_{\alpha \to 0} \phi(\alpha) = 0 \) and \( \lim_{\alpha \to \infty} \phi(\alpha) = \infty \).

The followings are examples of such functions.

(i) \( \phi(\alpha) = |\alpha|, \forall \alpha \in \mathbb{R} \)

(ii) \( \phi(\alpha) = |\alpha|^p, \forall \alpha \in \mathbb{R}, p \in \mathbb{R}^+ \)

(iii) \( \phi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}, \forall \alpha \in \mathbb{R}, n \in \mathbb{N} \)

**Definition 3.2.** Let \( X \) be a linear space over a field \( F \) and \( K \geq 1 \) be a given real number. A fuzzy subset \( N \) of \( X \times \mathbb{R} \) is called fuzzy strong \( \phi \)-b-norm on \( X \) if \( \forall x, y \in X \) the following conditions hold:

(\( bN_1 \)) \( \forall t \in \mathbb{R} \text{ with } t \leq 0, \ N(x, t) = 0 \);

(\( bN_2 \)) \( \forall t \in \mathbb{R}, t > 0, \ N(x, t) = 1 \text{ iff } x = \theta \);

(\( bN_3 \)) \( \forall t \in \mathbb{R}, t > 0, \ N(cx, t) = N(x, \frac{t}{\phi(c)}) \text{ if } \phi(c) \neq 0 \);

(\( bN_4 \)) \( \forall s, t \in \mathbb{R}, N(x + y, s + Kt) \geq N(x, s) \ast N(y, t) \);

(\( bN_5 \)) \( N(x, \cdot) \) is a non-decreasing function of \( t \) and \( \lim_{t \to \infty} N(x, t) = 1 \).

Then \((X, N, \phi, K, \ast)\) is called fuzzy strong \( \phi \)-b-normed linear space.

**Remark 3.3.** If \( K = 1 \) and \( \phi(\alpha) = |\alpha| \) then \((X, N, \ast)\) is a B-S type fuzzy normed linear space.

**Example 3.4.** Consider the linear space \( \mathbb{R} \) and a fuzzy subset \( N \) of \( \mathbb{R} \times \mathbb{R} \) by

\[
N(x, t) = \begin{cases} 
\frac{t}{t + |x|^p} & t > 0 \\
0 & t \leq 0 
\end{cases}
\]
Consider the t-norm \( * \) by \( a * b = \min\{a, b\}, \forall a, b \in \mathbb{R} \).

We show that \( N \) is a fuzzy strong \( \phi \)-b-norm on \( \mathbb{R} \times \mathbb{R} \). For, 

(i) \( \forall t \in \mathbb{R} \) with \( t \leq 0 \), by definition we have, \( N(x, t) = 0 \). Thus, (bN1) holds.

(ii) \( \forall t \in \mathbb{R}, t > 0, \ N(x, t) = 1 \iff \frac{t}{|x| + |t|} = 1 \iff |x|^p = 0 \iff x = 0 \)

Therefore (bN2) holds.

(iii) \( \forall t > 0 \) and \( c \in \mathbb{R} \setminus \{0\} \), \( N(cx, t) = \frac{t}{|cx| + |t|} = \frac{t}{|c| + |t|} = N(x, \frac{t}{|c|}) \)

where \( \phi(c) = |c|^p, \ c \in \mathbb{R} \) and clearly \( \phi \) satisfies all the conditions of Definition 3.2.

Thus, (bN3) holds.

(iv) \( \forall s, t > 0 \) and \( x, y \in \mathbb{R} \), \( N(x + y, Ks + t) = \frac{Ks + t}{s + t + |x + y|^p} \) and

\( N(x, s) * N(y, t) = \min\{N(x, s), N(y, t)\} = \min\{\frac{s}{s + |x|^p}, \frac{t}{t + |y|^p}\} \).

We only prove the inequality for \( s, t > 0 \).

Let \( N(x, s) * N(y, t) = \min\{N(x, s), N(y, t)\} = N(x, s) \).

Then \( N(y, t) \geq N(x, s) \implies \frac{t}{|x|^p} \geq \frac{s}{s + |x|^p} \implies t|x|^p \geq s|y|^p. \)

Again, \( x, y \in \mathbb{R} \) and \( 0 < p \leq 1 \),

\[ |x + y|^p \leq 2^p|x|^p + |y|^p \]

If we take \( K = 2^p \) then

\[ N(x + y, 2^p s + t) - N(x, s) = \frac{2^p s + t}{2^p s + t + |x + y|^p} - \frac{s}{s + |x|^p} \]

\[ \geq \frac{2^p s + t}{2^p s + t + 2^p |x|^p + |y|^p} - \frac{s}{s + |x|^p} \]

\[ = \frac{t|x|^p - s|y|^p}{(2^p s + t + 2^p |x|^p + |y|^p)(s + |x|^p)} \geq 0 \]

Hence \( N(x + y, 2^p s + t) \geq N(x, s) = \min\{N(x, s), N(y, t)\} \).

Similarly, it can be shown that if \( \min\{N(x, s), N(y, t)\} = N(y, t) \) then \( N(x + y, 2^p s + t) \geq N(y, t) = \min\{N(x, s), N(y, t)\} \).

Therefore, (bN4) holds.

(v) From the definition of \( N(x, t) \) it is clear that \( N(x, .) \) is a non-decreasing function of \( t \) and \( \lim_{t \to \infty} N(x, t) = 1 \).

Hence \( (X, N, \phi, K, *) \) is a fuzzy strong \( \phi \)-b-normed linear space where \( K = 2^p(> 1) \) and \( \phi(\alpha) = |\alpha|^p, \forall \alpha \in \mathbb{R}, \ 0 < p \leq 1 \).

**Example 3.5.** Consider the linear space \( \mathbb{R} \) and a fuzzy subset \( N \) of \( \mathbb{R} \times \mathbb{R} \) by

\[ N(x, t) = \begin{cases} \exp\left(-\frac{|t|^p}{t}\right) & t > 0 \\ 0 & t \leq 0 \end{cases} \]

for all \( x \in \mathbb{R} \) and \( 0 < p \leq 1 \) and consider the t-norm \( * \) by \( a * b = ab, \forall a, b \in \mathbb{R} \). Now,

(i) Clearly (bN1) holds from the definition.

(ii) \( \forall t \in \mathbb{R}, t > 0, \ N(x, t) = 1 \iff \exp(-\frac{|t|^p}{t}) = 1 \iff |x|^p = 0 \iff x = 0 \)

Therefore (bN2) holds.
(iii) \( \forall t > 0 \) and \( c \in \mathbb{R} \setminus \{0\} \), \( N(cx, t) = \exp(-\frac{|cx|^p}{t}) = \exp(-\frac{|x|^p}{c^p}) = N(x, \phi(c)) \)

where \( \phi(c) = |c|^p, \; c \in \mathbb{R} \) and clearly \( \phi \) satisfies all the conditions of Definition 3.2.

Thus, (bN3) holds.

(iv) For \( s, t > 0 \) and \( x, y \in \mathbb{R} \), \( N(x + y, Ks + t) = \exp(-\frac{|x + y|^p}{Ks + t}) \) and

\( N(x, s) \ast N(y, t) = N(x, s) \cdot N(y, t) = \exp(-\frac{|x|^p}{s}) \cdot \exp(-\frac{|y|^p}{t}) \).

Using the inequality, \( |x + y|^p \leq 2^p|x|^p + |y|^p \), \( x, y \in \mathbb{R} \) and \( 0 < p \leq 1 \) and taking \( K = 2^p \), we obtain

\[
-\frac{|x + y|^p}{2^p s + t} \geq -\frac{2^p|x|^p + |y|^p}{2^p s + t} \geq -\frac{2^p|x|^p}{2^p s + t} - \frac{|y|^p}{2^p s + t} \geq -\frac{|x|^p}{s} - \frac{|y|^p}{t}
\]

which implies \( N(x + y, 2^p s + t) \geq N(x, s) \cdot N(y, t) \).

Thus (bN4): \( N(x + y, 2^p s + t) \geq N(x, s) * N(y, t) \) holds \( \forall s, t \in \mathbb{R} \) and \( \forall x, y \in \mathbb{R} \).

(v) Clearly \( N(x, \cdot) \) is a non-decreasing function of \( t \) and \( \lim_{t \to \infty} N(x, t) = 1 \).

Hence \( (X, N, \phi, K, \ast) \) is a fuzzy strong \( \phi \)-b-normed linear space where \( K = 2^p(> 1) \) and \( \phi(\alpha) = |\alpha|^p, \forall \alpha \in \mathbb{R}, \; 0 < p \leq 1 \).

**Remark 3.6.** The notions of converges, Cauchy sequences, boundedness, etc. are same as definitions in Bag and Samanta type fuzzy normed linear space [8].

## 4 Finite dimensional fuzzy strong \( \phi \)-b-normed linear spaces

In this section some basic results on finite dimensional fuzzy strong \( \phi \)-b-normed linear spaces are established.

**Lemma 4.1.** Let \( (X, N, \phi, K, \ast) \) be a fuzzy strong \( \phi \)-b-normed linear space with the underlying \( t \)-norm \( \ast \) continuous at \((1, 1)\) and \( \{x_1, x_2, \ldots, x_n\} \) be a linearly independent set of vectors in \( X \). Then \( \exists c > 0 \) and \( \delta \in (0, 1) \) such that for any set of scalars \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) with \( \sum_{i=1}^{n} |\alpha_i| \neq 0 \),

\[
N(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n, \frac{Kc}{\phi(\sum_{i=1}^{n} |\alpha_i|)}) < 1 - \delta
\]

(1)

**Proof.** The relation (1) is equivalent to the relation

\[
N(\beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n, Kc) < 1 - \delta
\]

(2)

for some \( c > 0 \) and \( \delta \in (0, 1) \) and for all set of scalars \( \{\beta_1, \beta_2, \ldots, \beta_n\} \) with \( \sum_{i=1}^{n} |\beta_i| = 1 \). If possible suppose that (2) does not hold. Thus for each \( c > 0 \) and \( \delta \in (0, 1) \), \( \exists \) a set of scalars \( \{\beta_1, \beta_2, \ldots, \beta_n\} \) with \( \sum_{i=1}^{n} |\beta_i| = 1 \) for which

\[
N(\beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n, Kc) \geq 1 - \delta
\]

Then for \( c = \delta = \frac{1}{m}, \; m = 1, 2, \ldots, \) \( \exists \) a set of scalars \( \{\beta_1^{(m)}, \beta_2^{(m)}, \ldots, \beta_n^{(m)}\} \) with \( \sum_{i=1}^{n} |\beta_i^{(m)}| = 1 \) such that \( N(y_m, \frac{Kc}{m}) \geq 1 - \frac{1}{m} \) where \( y_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + \ldots + \beta_n^{(m)} x_n \).

Since \( \sum_{i=1}^{n} |\beta_i^{(m)}| = 1 \), we have \( 0 \leq |\beta_i^{(m)}| \leq 1 \) for \( i = 1, 2, \ldots, n \). So for each fixed \( i \), the sequence \( \{\beta_i^{(m)}\} \) is bounded and hence \( \{\beta_i^{(m)}\} \) has a convergent subsequence. Let \( \beta_1 \) denotes the limit of that subsequence and let \( \{y_{1,m}\} \) denotes the corresponding subsequence of \( \{y_m\} \).
By the same argument \(\{y_{1,m}\}\) has a subsequence \(\{y_{2,m}\}\) for which the corresponding subsequence of scalars \(\{\beta_i^{(m)}\}\) converges to \(\beta_2\).

Continuing in this way, after \(n\) steps we obtain a subsequence \(\{y_{n,m}\}\) where \(y_{n,m} = \sum_{i=1}^{n} \gamma_i^{(m)} x_i\) with \(\sum_{i=1}^{n} |\gamma_i^{(m)}| = 1\) and \(\gamma_i^{(m)} \to \beta_i\) as \(m \to \infty\) for each \(i = 1, 2, \ldots, n\).

Let \(y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n\). Now,

\[
N(y_{n,m} - y, t) = N(\sum_{j=1}^{n} (\gamma_j^{(m)} - \beta_j) x_j, t)
\]

\[
= N(\sum_{j=2}^{n} (\gamma_j^{(m)} - \beta_j) x_j, \frac{t}{n} + K(n - 1) \frac{t}{n K})
\]

\[
\geq N(\sum_{j=2}^{n} (\gamma_j^{(m)} - \beta_j) x_j, (n - 1) \frac{t}{n K})
\]

\[
= N(\sum_{j=3}^{n} (\gamma_j^{(m)} - \beta_j) x_j, \frac{t}{n K} + K(1 - \frac{2}{n}) \frac{t}{K^2})
\]

\[
\geq N(\sum_{j=3}^{n} (\gamma_j^{(m)} - \beta_j) x_j, (1 - \frac{2}{n}) \frac{t}{K^2})
\]

\[
\geq \cdots
\]

\[
N(x_1, \frac{t}{n \phi(\gamma_1^{(m)} - \beta_1)}) \cdots N(x_n, \frac{t}{n K^{n-1} \phi(\gamma_n^{(m)} - \beta_n)})
\]

Taking limit as \(m \to \infty\) on both sides, we have \(\lim_{m \to \infty} N(y_{n,m} - y, t) \geq 1 \ast 1 \cdots 1, \forall t > 0\).

which implies \(\lim_{m \to \infty} N(y_{n,m} - y, t) = 1, \forall t > 0\).

Now for \(r > 0\), choose \(m\) such that \(\frac{1}{m} < \frac{r}{K^2}\). We have

\[
N(y_{n,m}, \frac{r}{K}) = N(y_{n,m} + \theta, \frac{K}{m} + K(\frac{r}{K^2} - \frac{1}{m})) \geq N(y_{n,m}, \frac{K}{m} \cdot \theta, (r - \frac{1}{m}) \geq 1 - \frac{K}{m} \ast 1
\]

\[
\Rightarrow \lim_{m \to \infty} N(y_{n,m}, \frac{r}{K}) \geq 1
\]

\[
\lim_{m \to \infty} N(y_{n,m}, \frac{r}{K}) = 1
\]

Again,

\[
N(y, 2r) = N(y - y_{n,m} + y_{n,m}, r + K \cdot \frac{r}{K}) \geq N(y - y_{n,m}, \frac{r}{K}) \ast N(y_{n,m}, \frac{r}{K})
\]

\[
\Rightarrow N(y, 2r) \geq \lim_{m \to \infty} N(y - y_{n,m}, \frac{r}{K}) \ast \lim_{m \to \infty} N(y_{n,m}, \frac{r}{K})
\]

\[
\Rightarrow N(y, 2r) \geq 1 \ast 1 = 1
\]

\[
\Rightarrow N(y, 2r) = 1
\]

Since \(r > 0\) is arbitrary, so \(y = \theta\).

Again since \(\sum_{i=1}^{n} |\beta_i^{(m)}| = 1\) and \(\{x_1, x_2, \cdots x_n\}\) is a linearly independent set of vectors so \(y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \neq \theta\).

Thus we arrive at a contradiction. Hence (2) holds and Lemma is proved.
Theorem 4.2. Every finite dimensional fuzzy strong $\phi$-$b$-normed linear space with the underlying $t$-norm $*$ continuous at $(1, 1)$ is complete.

Proof. Let $(X, N, \phi, K, *)$ be a fuzzy strong $\phi$-$b$-normed linear space where $K(> 1)$ is a real constant.

Let $d_{im}X = r$ and $\{e_1, e_2, \cdots, e_r\}$ be a basis for $X$.

Let $\{x_p\}$ be a Cauchy sequence in $X$. Then $x_n = \beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + \cdots + \beta_r^{(n)} e_r$ for suitable scalars $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_r^{(n)}$. So,

$$\lim_{m,n \to \infty} N(x_m - x_n, t) = 1, \forall t > 0 \quad (3)$$

Now from Lemma 4.1 it follows that $\exists > 0$ and $\delta \in (0, 1)$ such that

$$N\left(\sum_{i=1}^{r} (\beta_i^{(m)} - \beta_i^{(n)}) e_i, \frac{cK}{\phi(\sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}|)}\right) < 1 - \delta \quad (4)$$

If $\sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}| = 0$ then $\beta_i^{(m)} = \beta_i^{(n)}$, $\forall i$ implies $\{x_p\}$ is a constant sequence and hence follows the theorem. So we may assume $\sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}| \neq 0$.

Again for $0 < \delta < 1$, from (3), it follows that there exist a positive integer $n_0(\delta, t)$ such that

$$N\left(\sum_{i=1}^{r} (\beta_i^{(m)} - \beta_i^{(n)}) e_i, t \right) > 1 - \delta, \forall m, n \geq n_0(\delta, t) \quad (5)$$

Now from (4) and (5) we have,

$$N\left(\sum_{i=1}^{r} (\beta_i^{(m)} - \beta_i^{(n)}) e_i, \frac{cK}{\phi(\sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}|)}\right) < N\left(\sum_{i=1}^{r} (\beta_i^{(m)} - \beta_i^{(n)}) e_i, t \right), \forall m, n \geq n_0(\delta, t)$$

$$\Rightarrow \frac{cK}{\phi(\sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}|)} < t, \forall m, n \geq n_0(\delta, t) \quad (\text{Since } N(x, \cdot) \text{ is non-decreasing})$$

Since $t > 0$ is arbitrary, thus

$$\lim_{m,n \to \infty} \phi\left(\frac{cK}{\sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}|}\right) = 0$$

$$\Rightarrow \lim_{m,n \to \infty} \phi\left(\frac{1}{\sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}|}\right) = \infty$$

$$\Rightarrow \phi\left(\lim_{m,n \to \infty} \sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}|\right) = \infty \quad (\text{Since } \phi \text{ is continuous})$$

$$\Rightarrow \lim_{m,n \to \infty} \sum_{i=1}^{r} |\beta_i^{(m)} - \beta_i^{(n)}| = 0 \quad (\text{Since } \lim_{\alpha \to \infty} \phi(\alpha) = \infty)$$

Therefore, $\{\beta_i^{(m)}\}$ is a Cauchy sequence of scalars for each $i = 1, 2, \cdots, r$. So each sequence $\{\beta_1^{(m)}\}$ converges.

Let $\lim_{n \to \infty} \beta_i^{(n)} = \beta_i$ for $i = 1, 2, \cdots, r$. Define $x = \sum_{i=1}^{r} \beta_i e_i$. Clearly $x \in X$.

By similar calculation as in Lemma 4.1 it can be shown that $\lim_{n \to \infty} N(x_n - x, t) = 1, \forall t > 0$.

Hence $X$ is complete. $\square$
Theorem 4.3. Let \((X, N, \phi, K, \ast)\) be a finite dimensional fuzzy strong \(\phi\)-b-normed linear space in which the underlying \(t\)-norm \(\ast\) continuous at \((1, 1)\). Then a subset \(A\) of \(X\) is compact iff \(A\) is closed and bounded.

**Proof.** First we suppose that \(A\) is compact. We have to show that \(A\) is closed and bounded.

For, let \(x \in A\). Then there exist a sequence \(\{x_n\}\) in \(A\) such that \(\lim_{n \to \infty} x_n = x\).

Since \(A\) is compact, there exist a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) converges to a point in \(A\). Again \(x_n \to x\) so \(x_{n_k} \to x\) and hence \(x \in A\). So \(A\) is closed.

If possible suppose that \(A\) is not bounded.

Then \(\exists r_0, 0 < r_0 < 1, \) such that for each positive integer \(n, \exists x_n \in A\) for which \(N(x_n, n) \leq 1 - r_0\).

Since \(A\) is compact, there exist a subsequence \(\{x_{n_p}\}\) of \(\{x_n\}\) converging to some element \(x \in A\). Thus \(\lim_{p \to \infty} N(x_{n_p} - x, t) = 1, \forall t > 0\).

Again, \(N(x_{n_p}, n_p) \leq 1 - r_0\). Now,

\[
1 - r_0 \geq N(x_{n_p}, n_p) = N(x_{n_p} - x + x, t/K + K(n_p/t - t/K^2)) \geq N(x_{n_p} - x, t/K) * N(x, (n_p/t - t/K^2))
\]

On above inequality, taking limit as \(p \to \infty\), we obtain

\[
1 - r_0 \geq \lim_{p \to \infty} N(x_{n_p} - x, t/K) * \lim_{p \to \infty} N(x, (n_p/t - t/K^2))
\]

\[
\implies 1 - r_0 \geq 1 * 1 = 1
\]

\[
\implies r_0 \leq 0
\]

This is a contradiction. Hence \(A\) is bounded.

Conversely suppose that \(A\) is closed and bounded and we have to show that \(A\) is compact.

Let \(\text{dim}X = r\) and \(\{e_1, e_2, \ldots, e_r\}\) be a basis for \(X\).

Let us choose a sequence \(\{x_p\}\) in \(A\) and suppose \(x_p = \beta_1^{(p)} e_1 + \beta_2^{(p)} e_2 + \cdots + \beta_r^{(p)} e_r\) for suitable scalars \(\beta_1^{(p)}, \beta_2^{(p)}, \ldots, \beta_r^{(p)}\).

Now from Lemma [4.1] \(\exists c > 0\) and \(\delta \in (0, 1)\) such that

\[
N\left(\sum_{i=1}^{r} \beta_i^{(p)} e_i, \frac{Kc}{\phi(1)} \right) < 1 - \delta
\]

(6)

Again since \(A\) is bounded, for \(\delta \in (0, 1), \exists t > 0\) such that \(N(x, t) > 1 - \delta, \forall x \in A\). So

\[
N\left(\sum_{i=1}^{r} \beta_i^{(p)} e_i, t\right) > 1 - \delta
\]

(7)

From (6) and (7) we get,

\[
N\left(\sum_{i=1}^{r} \beta_i^{(p)} e_i, \frac{Kc}{\phi(1)} \right) < N\left(\sum_{i=1}^{r} \beta_i^{(p)} e_i, t\right)
\]

\[
\implies \frac{Kc}{\phi(1)} < t, \forall m, n \geq n_0(\delta, t) \quad (\text{Since } N(x, \cdot) \text{ is non-decreasing})
\]

8
Without loss of generality we may assume that $\sum_{i=1}^{n} |\beta_i^{(p)}| \neq 0$.

If $\sum_{i=1}^{n} |\beta_i^{(p)}| = 0$ then $\beta_i^{(p)} = 0$, for $i = 1, 2, \cdots$ and $\forall p$. Then $\{x_p\}$ is a constant sequence and the theorem follows.

Since $c, K, t$ are three fixed positive real numbers, it follows that $0 < \sum_{i=1}^{n} |\beta_i^{(p)}| < \infty$.

Therefore the sequence of scalars $\beta_i^{(p)}, p = 1, 2, \cdots$ and for $i = 1, 2, \cdots, n$ is bounded. So by Bolzano-Weierstrass theorem, there exist a convergent subsequence of $\{\beta_i^{(p)}\}$. Now, we follow the techniques of Lemma 4.1 to show that there exist a subsequence of $\{x_p\}$ that converges to some point in $A$.

Thus $A$ is compact and this proves the theorem.

**Conclusion:** Recently different types of generalized fuzzy metric spaces as well as generalized fuzzy normed linear spaces have been developed by several authors. Following the definition of fuzzy strong b-metric spaces, we introduce the idea of fuzzy strong $\phi$-b-normed linear spaces and study some results in finite finite dimensional fuzzy strong $\phi$-b-normed linear spaces. We think there is a huge scope of research to develop fuzzy strong $\phi$-b-normed linear spaces. Results on completeness and compactness, operator norms etc. are the open problems in such spaces.

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**References**

[1] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8, 1965, 338-353.

[2] O. Kaleva, S. Seikkala, On fuzzy metric spaces, *Fuzzy Sets and Systems*, 12, 1984, 215-229. DOI: https://doi.org/10.1016/0165-0114(84)90069-1.

[3] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica*, 11, 1975, 326-334.

[4] A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets Systems*, 64, 1994, 395-399. DOI: https://doi.org/10.1016/0165-0114(94)90162-7.

[5] A. K. Katsaras, Fuzzy topological vector spaces I, *Fuzzy Sets and Systems*, 12, 1984, 143-154. DOI: https://doi.org/10.1016/0165-0114(81)90082-8.

[6] C. Felbin, Finite dimensional fuzzy normed linear spaces, *Fuzzy Sets and Systems*, 48, 1992, 239-248. DOI: https://doi.org/10.1016/0165-0114(92)90338-5.

[7] S. C. Cheng, J. N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, *Bull. Cal. Math. Soc*, 86, 1994, 429-436.

[8] T. Bag, S. K. Samanta, Finite dimensional fuzzy normed linear spaces, *The Journal of Fuzzy Mathematics*, 11, 2003, 687-705.
[9] T. Bag, S. K. Samanta, Finite dimensional fuzzy normed linear spaces, *Annals of Fuzzy Mathematics and Informatics*, 6, 2013, 271-283.

[10] George. J. Klir, Bo Yuan, Fuzzy Sets and Fuzzy Logic, *Printice-Hall of India Private Limited, New Delhi-110001, 1997*

[11] S. Gahler, 2-metrische Raume und ihre topologische Struktur, Mathematische Nachrichten, 26, 1963, 115-118. DOI: https://doi.org/10.1002/mana.19630260109

[12] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math Inf Univ Ostraviensis*, 1, 1993, 5-11.

[13] W. Kirk, N. Shahzad, Fixed point theory in distance spaces, *Springer, Cham, 2014*.

[14] S. Nadaban, Fuzzy b-Metric Spaces, *International journal of computers communications and control*, 11, 2016, 273-281.

[15] F. Mehmood, R. Ali, C. Ionescu, T. Kamran, Extended fuzzy b-Metric Spaces, *Journal of Mathematical Analysis*, 8, 2017, 124-131.

[16] A. K. Katsaras, Fuzzy topological vector spaces, *Fuzzy Sets and Systems*, 12, 1984, 143-154.

[17] T. Bag, S. K. Samanta, Fuzzy bounded linear operators, *Fuzzy Sets and Systems*, 151, 2005, 513-547.

[18] Z. Mustafa, H. Obiedat, F. Awawdeh, Some Fixed Point Theorem for Mapping on Complete G-Metric Spaces, *Fixed Point Theory and Applications*, Volume 2008, Article ID 189870, 12 pages. DOI::10.1155/2008/189870.

[19] T. Oner, M. B. Kandemir, B. Tanay, Fuzzy cone metric spaces, *J. Nonlinear Sci. Appl*, 8, 2015, 610-616.

[20] T. Bag, Finite dimensional fuzzy cone normed linear spaces, *International Journal of Mathematics and Scientific Computing*, 3, 2013, 9-14.

[21] S. Nădăban, Fuzzy b-Metric Spaces, *International Journal of Computers Communications and Control*, 11, 2016, 273-281.

[22] T. Oner, On topology of fuzzy strong b-metric spaces, *J. New Theory*, 21, 2018, 59-67.

[23] S. Gahler, Lineare 2-normierte raume, *Mathematische Nachrichten*, 28, 1964, 1–43.

[24] K. A. Khan, Generalized normed spaces and fixed point theorems, *Journal of Mathematics and Computer Science*, 13, 2014, 157-167. DOI: https://doi.org/10.48550/arXiv.1809.09486.

[25] S. Chatterjee, T. Bag, S. K. Samanta, Some results on G-fuzzy normed linear space, *Int. J. Pure Appl. Math.*, 5, 2018, 1295-1320.