Algorithms that get old: the case of generative algorithms

Gabriel Turinici

CEREMADE, Université Paris Dauphine - PSL, Paris, France
gabriel.turinici@dauphine.fr
https://turinici.com

Abstract. Generative IA networks, like the Variational Auto-Encoders (VAE), and Generative Adversarial Networks (GANs) produce new objects each time when asked to do so. However, this behavior is unlike that of human artists that change their style as times go by and seldom return to the initial point.

We investigate a situation where VAEs are requested to sample from a probability measure described by some empirical set.

Based on recent works on Radon–Sobolev statistical distances, we propose a numerical paradigm, to be used in conjunction with a generative algorithm, that satisfies the two following requirements: the objects created do not repeat and evolve to fill the entire target probability measure.

Keywords: variational auto-encoder · generative adversarial network · statistical distance · vector quantization · neural network

1 Motivation

Consider a distribution \( \mu \) and an empirical sample of this distribution given as a collection of objects \( x_m, m = 1, \ldots, M \) where \( x_m \) belong to some space \( \mathbb{R}^N \) are independent and follow the law \( \mu_e = \sum \delta_{x_i} \); in some sense to be defined latter (cf. discussion on statistical metrics) \( \mu_e \) is close to \( \mu \); we focus on generative neural network architecture that are able to produce, given \( \mu_e \) a procedure to sample (as accurate as possible) from the distribution \( \mu \). One such neural network class are the Variational Auto-Encoders (cf. [7,12] for an introduction) that, after some training produce two functions: the encoder function \( E : x \in \mathbb{R}^N \mapsto z \in \mathbb{R}^L \) and the decoder function \( D : z \mapsto y \in \mathbb{R}^N \); the decoder function has the property that the image of a multi-dimensional Gaussian on the latent space \( \mathbb{R}^L \) through \( E \) is close to \( \mu_e \) thus to \( \mu \). Some recent proposals to construct such a VAE are presented in [13], which will also be our choice for statistical distance.

The quality of a VAE is given

1. by the proximity of the \( D \circ E \) to the identity operator on the support of the target measure \( \mu \)
2. and the small distance between target distribution \( \mu \) and \( D(\mathcal{N}(O, I_L)) \) (here \( \mathcal{N} \) is the \( L \)-dimensional standard Gaussian);
However, although in general the VAE (same thing applies to the Generative Adversarial Networks - GANs) obtains very good quality results by the previous criteria, the sampling performed at the exploitation phase is, because of the construction, done in an independent way: each time a new \( y \sim \mu \) is required, a \( z \sim \mathcal{N}_L \) is sampled and \( D(z) \) computed. But such a procedure is at odds with what we observe in real life: the painters do not paint the same landscape again, the musical composers’ productions vary in style over the years, etc, in general some evolution is witnessed with time. Such a phenomena is probably due to the **taking into account of previous works**. Our goal is to be able to mimic such an evolution and propose a generative algorithm that

1. is able to create new objects from some target distribution \( \mu \) (that may be also, for VAEs and GANs the latent distribution);
2. is able to ”recall” having created previous objects; **This second point will therefore synthetically induce an ”artificial age” for an AI because the process is non-reversible.**

A non-aging generative algorithm, when asked to produce, e.g. one new result will likely produce the same object (or similar) over and over again: think of the situation of a standard 1D-Gaussian: most likely the origin will be drawn over and over again, one has to wait a long time to obtain let’s say, a value at 3 standard deviations from the mean. The main goal of this paper is to speed up this waiting time. The advantages of such a process is to allow some ”maturation” for the results i.e. to be able to create new results, not the same ones again; this comes at the price of a non-reversibility and additional computation cost.

### 1.1 Relation to previous literature

Technically, our proposal has some similarities with different areas in computational statistics: first one can invoke the ”vector quantization” procedures (see [8,9,6] and references within) that, given a distribution, find a set of objects that represent it as a sum of Dirac masses. However, there the technical solution (Voronoi diagrams for instance) are naturally oriented to apply to probability measures (or more generally finite positive measures) which is not our situation (our effort involves signed measures); in the same vein see also [2] in the context of machine learning algorithms. On the other hand some efforts have been made to generalize the quantile idea to multi-dimensional distributions; in a one-dimensional situation our procedure and these techniques give similar results but they diverge as soon as the dimension is increased, see [5,III].

Continuing the works above, and given the discussion on the generative algorithms, we need a procedure that can incrementally find a good representation of a target measure \( \mu \) as a number of \( K \) Dirac masses (\( K \) is given and fixed) centered at some \( x_k, k = 1, ..., K \) while taking into account a set of points \( Y = (y_j)_{j=1}^{K_p} \) already available. To put it otherwise we want to find the point \( X = (x_k)_{k=1}^{K} \in \mathbb{R}^{N \times K} \) (\( k = 1, ..., K \)) that minimize the distance from the 'summed’ empirical distribution

\[
\sum_{k=1}^{K_p} \delta_{y_k} + \sum_{k=1}^{K} \delta_{x_k}
\]

to the target measure \( \mu \).
(here the points $y_k$ are not submitted to optimization); this can be written as minimizing the distance $d(\delta_X, \eta)^2$ from the distribution $\delta_X = \frac{1}{K} \sum_{k=1}^{K} \delta_{x_k}$ to the signed measure $\eta = \frac{(K_p + K)\mu - K_p \delta_Y}{K_p + K}$ where $\delta_Y = \frac{1}{K_p} \sum_{k}^{K_p} \delta_{y_k}$.

We present in section 2 our choice of distance $d$ and a theoretical result ensuring that, under appropriate hypotheses, the minimum with respect to $X$ exits. The algorithm to find such a minimum is presented in section 3 together with some numerical results. Final remarks are the object of section 4.

2 Theoretical results

In order to present the theoretical framework we need to make clear what we choose as distance $d$ between signed measures $\zeta$ (which is in fact a probability measure) and $\eta$.

We will take a kernelized measure given as follows: choose $h(\cdot)$ a negative definite kernel (see [10] for an introduction), which will here taken to be $\sqrt{\alpha^2 + |x|^2 - a}$ for some given constant $a \geq 0$ (see [3, 14] for some use cases in machine learning); then given $\eta_1, \eta_2$ two signed measures such that $\int (1 + |X|)\eta_i(dX) < \infty$ ($i = 1, 2$) then we define

$$d(\eta_1, \eta_2)^2 = \int \int h(|X - Y|)(\eta_1 - \eta_2)(dX)(\eta_1 - \eta_2)(dY) \quad (1)$$

Note that in particular, if both $\eta_1$ and $\eta_2$ are sums of (signed) Dirac masses $\eta_i = \sum_k^{K_i} p_i^k \delta_{z_k}$ (with $K_i < \infty$) then equation (1) can be written (see [14]) :

$$d(\eta_1, \eta_2)^2 = \sum_{k_1}^{K_1} \sum_{k_2}^{K_2} p_1^{k_1} p_2^{k_2} h(|z_1 - z_2|). \quad (2)$$

Once the distance is defined, a legitimate question is whether, given a target signed measure $\eta$ one can indeed find a uniform sum of Dirac masses $\zeta$ that minimizes $d(\zeta, \eta)^2$. This question is settled in the following

**Proposition 1.** Suppose $K$ is a fixed positive integer. Let $\eta$ be a signed measure such that $\int (1 + |X|)\eta(dX) < \infty$ ($i = 1, 2$); then the minimization problem :

$$X = (x_k)_{k=1}^{K} \in \mathbb{R}^{N \times K} \mapsto d\left(\frac{1}{K} \sum_{k=1}^{K} \delta_{x_k}, \eta\right)^2 \quad (3)$$

admits at least one solution.

**Remark 1.** The previous result only give the existence of a solution, the uniqueness is not necessarily true as one can observe by taking e.g. a rotation invariant measure: any solid rotation of a minimum will still be a minimum.
Proof. For any vector $Z = (z_j)_{j=1}^J \in \mathbb{R}^{N \times J}$ denote
\[
\delta_Z := \frac{1}{J} \sum_{j=1}^L \delta_{z_j}, \quad f(Z) := d(\delta_Z, \eta)^2.
\]

Let us further denote
\[
m_\eta := \inf_{X = (x_k)_{k=1}^K \in \mathbb{R}^{N \times K}} f(X).
\]

Take a point $X$ such that $f(X) \leq m_\eta + 1$. Then (denoting by 0 the null vector in $\mathbb{R}^{N \times K}$):
\[
m_\eta + 1 \geq f(X) = d(\delta_X, \eta)^2 \geq \frac{d(\delta_X, \delta_0)^2 - 2d(\delta_0, \eta)^2}{2},
\]
which implies
\[
d(\delta_X, \delta_0)^2 \leq 2(m_\eta + 1) + 2d(\delta_0, \eta)^2.
\]

But, using equation (2), we obtain:
\[
\begin{align*}
d(\delta_X, \delta_0)^2 &= \frac{1}{K} \sum_{k=1}^K h(|x_k|) - \frac{1}{K^2} \sum_{k,k'=1}^K h(|x_k - x_{k'}|) \\
&\geq \frac{1}{K} \sum_{k=1}^K h(|x_k|) - \frac{1}{K^2} \sum_{k,k'=1}^K h(|x_k - x_{k'}|) \\
&\geq \frac{1}{K} \sum_{k=1}^K h(|x_k|) - \frac{\sum_{k \neq k'} a + h(|x_k|) + h(|x_{k'}|)}{K^2} = \frac{1}{K^2} \sum_{k=1}^K h(|x_k|) - a,
\end{align*}
\]
where for the passage from (8) to (9) we used the inequality $h(|x - y|) \leq h(|x|) + h(|y|) + a$ true for any $x, y \in \mathbb{R}^N$. We obtain that, when $f(X) \leq m_\eta + 1$ there exists a constant $C_0 = K^2 \left(a + 2(m_\eta + 1) + 2d(\delta_0, \eta)^2\right)$ such that $\sum_{k=1}^K |x_k| \leq C_0$; therefore any minimizing sequence $(X_n)_{n \geq 1}$ (that is, any sequence such that $\lim_{n \to \infty} f(X_n) = m$) is bounded. This sequence will have a sub-sequence $(X_{n_k})_{k \geq 1}$ that is convergent to some $\bar{X}$; but since the distance is continuous, we obtain that $f(\bar{X}) = m$ which means that $\delta_{\bar{X}}$ is a solution of the minimization problem (3).

\[\square\]

3 Algorithm and numerical results

3.1 Algorithm formulation

Consider now a target distribution $\mu$ and a set of previous constructed points $Y$ (of cardinal $K_p$); we will construct an algorithm that, given a number $K$ of points to be constructed will find a proposal $X \in \mathbb{R}^{N \times K}$ such that the overall measure $\delta_{X \cup Y}$ minimizes the distance to the target measure $\mu$. 

Algorithm A1 History aware (signed measure) compression algorithm: HAC

1: procedure HAC
2: • set batch size \( B \), parameter \( a = 10^{-6} \),
3: • load previous generated points \( y_k, k = 1, ..., K \),
4: • initialize points \( x_k, k = 1, ..., K \) sampled at random from \( \mu \),
5: while (max iteration not reached) do
6: • sample \( z_1, ..., z_B \sim \mu \) (i.i.d).
7: • compute the global loss \( L(X) \) using formula (2):
\[
L(X) := d \left( \frac{1}{K} \sum_{k=1}^{K} \delta_{x_k}, \frac{K_{a+1}}{B} \sum_{b=1}^{B} \delta_{z_b} - \sum_{j=1}^{K_{a}} \delta_{y_j} \right)^2;
\]
8: • backpropagate the loss \( L(X) \) in order to minimize \( L(X) \) and update \( X \),
9: end while
10: end procedure

3.2 Numerical results

History unaware compression of a 2D Gaussian mix distribution

When the target measure is positive, the HAC algorithm A1 allows to compress any given (probability) measure, as illustrated in figure 1 for the situation of a uniform mixture of 16 lattice-centered 2D normal variables. Good results are obtained: without any previous knowledge, the algorithm can unveil the mixing structure and allow a coherent compression.

![Fig. 1. An example of compression for an uniform Gaussian mixture of 16 Gaussians centered on points of a **4 × 4** grid (red points are the centers of the Gaussians, blue points are the compressed points). We used \( K \) points to summarize the distribution: \( K = 48 \) (left image) or \( K = 3 \) (right image). Good quality results are obtained as the algorithm "understands" the mixing structure: for instance for \( K = 48 \) the algorithm allocates precisely 3 points per Gaussian mixture term.](image-url)
History aware multi-dimensional Gaussian compression and application to generative algorithms

We move now to a test where incremental compression is performed: we consider a 2D Gaussian centered at origin. First we compress it with a single point $u_1$; then we use $K_p = 1$ and $y_1 = u_1$ as history and compress the signed measure (initial Gaussian minus the first obtained point $u_1$) with another supplementary point $u_2$; then consider $K_p = 2$ and $y_i = u_i$ ($i = 1, 2$) and compress the Gaussian measure minus the sum of Dirac masses in $u_i$ with another point $u_3$; the procedure is then continued recursively, each step being an application of the algorithm A1. The results are presented in figure 2.

![Figure 2](image)

**Fig. 2.** An example of recursive compression of a 2D standard Gaussian (see text for details). Left image: the result of the compression after 10 iterations. Each point $u_i$ is labeled by its corresponding index $i$ when it was found.

In order to test these results on a practical case, we used the CVAE procedure from the Tensorflow documentation [1] with default parameters (except that we used 100 epochs instead of 10 because the results with 10 epochs are very fuzzy). The code was executed once in order to construct the encoder/decoder networks and then the sampling was done in the latent space using either a multi-dimensional sampling of 10 objects or an incremental sampling; once the sampling is done the data is propagated through the decoder network and the resulting images are presented in figure 3. We notice that the sampling retains a good diversity with respect to the uniform sampling and avoid some repetitions:

- the figures 1 and 2 that are repeated in the propagated random samples and only appear once in the incremental sampling;
- the figure 9 appears only very unclearly in the left sample and more clearly in the right sample;
- the incremental sampling avoids the symbol in row 2 column 1 (left image) which is not a figure.

Note that this out-of-the-box C-VAE is not good enough to make figures too precise which explains large numbers of fuzzy images - resembling to a 8, 6 or
9- present in both results). To improve the result, we re-run the CVAE for 20 epochs but increased, as recommenced in the documentation, all 'filters' numbers to 512 (instead of 32 or 64 in the initial setup). We obtain the results in figure 4: the quality of the VAE is indeed increased and the same conclusions hold for the comparison of the i.i.d. sampling with the incremental sampling.

4 Final remarks

We explored in this work the construction of objects from generative algorithms (like VAEs and GANs); more specifically the construction was incremental in the sense that each new sampling from the latent space considers the historical samples and tries to both respect the desired target distribution of the latent space but also stays away from the points already sampled. We described some theoretical properties of the procedure and tested it both on general data and on a C-VAE benchmark.

More experiments are required to characterize fully the applicability domain of the proposed procedure, but the present results provide encouraging arguments to do so.
References

1. https://www.tensorflow.org/tutorials/generative/cvae, cVAE, Tensorflow documentation, retrieved Jan 30, 2022
2. Chazal, F., Levrard, C., Royer, M.: Optimal quantization of the mean measure and applications to statistical learning (2021)
3. Deshpande, I., Hu, Y.T., Sun, R., Pyrros, A., Siddiqui, N., Koyejo, S., Zhao, Z., Forsyth, D., Schwing, A.G.: Max-Sliced Wasserstein Distance and Its Use for GANs. In: The IEEE Conference on Computer Vision and Pattern Recognition (CVPR) (June 2019)
4. Fraiman, R., Pateiro-Lopez, B.: Quantiles for finite and infinite dimensional data. Journal of Multivariate Analysis 108, 1–14 (2012), https://www.sciencedirect.com/science/article/pii/S0047259X12000176
5. Glazer, A., Lindenbaum, M., Markovitch, S.: q-OCSVM: A q-Quantile Estimator for High-Dimensional Distributions. In: Burges, C.J.C., Bottou, L., Welling, M., Ghahramani, Z., Weinberger, K.Q. (eds.) Advances in Neural Information Processing Systems. vol. 26. Curran Associates, Inc. (2013), https://proceedings.neurips.cc/paper/2013/file/81f46e52c25763a55cc64422644317-Paper.pdf
6. Graf, S., Luschgy, H.: Foundations of quantization for probability distributions. Springer (2007)
7. Kingma, D.P., Max, W.: An Introduction to Variational Autoencoders. Now Publishers Inc (Nov 2019)
8. Kohonen, T.: Learning Vector Quantization. In: Kohonen, T. (ed.) Self-Organizing Maps, pp. 175–189. Springer Berlin Heidelberg, Berlin, Heidelberg (1995)
9. R. Gray: Vector quantization. IEEE ASSP Magazine 1(2), 4–29 (Apr 1984). https://doi.org/10.1109/MASSP.1984.1162229
10. Sejdinovic, D., Sriperumbudur, B., Gretton, A., Fukumizu, K.: Equivalence of distance-based and RKHS-based statistics in hypothesis testing. The Annals of Statistics 41(5), 2263–2291 (2013), http://www.jstor.org/stable/23566550
11. Serfling, R.: Quantile functions for multivariate analysis: approaches and applications. Statistica Neerlandica 56(2), 214–232 (2002). https://doi.org/https://doi.org/10.1111/1467-9574.00195 http://onlinelibrary.wiley.com/doi/abs/10.1111/1467-9574.00195
12. Tabor, J., Knop, S., Spurek, P., Podolak, I.T., Mazur, M., Jastrzkebski, S.: Cramer-Wold AutoEncoder. CoRR abs/1805.09235 (2018), http://arxiv.org/abs/1805.09235
13. Turinici, G.: X-ray Sobolev variational auto-encoders (2020)
14. Turinici, G.: Radon–Sobolev Variational Auto-Encoders. Neural Networks 141, 294–305 (Sep 2021). https://doi.org/10.1016/j.neunet.2021.04.018 https://www.sciencedirect.com/science/article/pii/S0893608021001556