KARLSSON–MINTON TYPE HYPERGEOMETRIC FUNCTIONS ON THE ROOT SYSTEM $C_n$

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Abstract. We prove a reduction formula for Karlsson–Minton type hypergeometric series on the root system $C_n$ and derive some consequences of this identity. In particular, when combined with a similar reduction formula for $A_n$, it implies a $C_n$ Watson transformation due to Milne and Lilly.

1. Introduction

The Karlsson–Minton summation formula [Mi, Ka] is the hypergeometric identity

$$p+2F_{p+1} \left( \begin{array}{c} a, b, c_1 + m_1, \ldots, c_p + m_p \cr b+1, c_1, \ldots, c_p \end{array} ; 1 \right) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(1+b-a)} \prod_{i=1}^{p} \frac{(c_i - b)_{m_i}}{(c_i)_{m_i}},$$

which holds for $m_i$ non-negative integers and $\Re(a + |m|) < 1$. Accordingly, hypergeometric series with integral parameter differences have been called Karlsson–Minton type hypergeometric series; other results for such series may be found in [C, G1, G2, S1, S2].

In recent work [R], we have derived a very general reduction formula for series of Karlsson–Minton type. We recall it in its most general form as (4) below; here we only state a very degenerate case, namely,

$$p+2F_{p+1} \left( \begin{array}{c} a, b, c_1 + m_1, \ldots, c_p + m_p \cr d, c_1, \ldots, c_p \end{array} ; 1 \right) = \frac{\Gamma(d)\Gamma(d-a-b)}{\Gamma(d-a)\Gamma(d-b)} \prod_{i=1}^{p} \frac{(c_i + 1 - d)_{m_i}}{(c_i)_{m_i}} \times \prod_{x_1, \ldots, x_p = 0} \left( \prod_{1 \leq i < j \leq p} \frac{c_i + x_i - c_j - x_j}{c_i - c_j} \frac{(b+1-d)_{|x|}}{(1-|m| - a)_{|x|}} \right) \times \prod_{i=1}^{p} \frac{(c_i - a)_{x_i}}{(1+c_i - d)_{x_i}} \prod_{i,k=1}^{p} \frac{(c_i - c_k - m_k)_{x_i}}{(1+c_i - c_k)_{x_i}},$$

which holds for $m_i$ non-negative integers and $\Re(a + |m| + b - d) < 0$. Note that the case $d = b+1$ is (1); similarly, the more general identity (4) implies a large number of results for Karlsson–Minton type hypergeometric series from the papers mentioned above.

The right-hand side of (2) is a multivariable hypergeometric sum on the root system $A_n$, a type of series that were first introduced by Biedenharn, Holman and Louck [HBL], motivated by $6j$-symbols of the group SU($n$). During the last 25 years, hypergeometric series on root systems has been a very active area of research with many applications. In the more general identity (4), both the left- and the right-hand sides of (2) are

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side are hypergeometric series on $A_n$ (with different dimension $n$). The connection between Karlsson–Minton type series and hypergeometric series on $A_n$ encoded in (4) allows one to recover many identities also for the latter type of series; for instance, if we let $b = 0$ in (2), so that the left-hand side is 1, we obtain the case $q = 1$ of one of Milne’s multivariable $q$-Saalschütz summations [M, Theorem 4.1]. More generally, (4) implies multivariable $10W_9$ transformations due to Milne and Newcomb [MN] and Kajihara [K].

The purpose of the present paper is to find results analogous to those of [R] for the root system $C_n$. In [R], our starting point was an $A_n \psi_6$ summation of Gustafson [Gu1]; here we use instead Gustafson’s $C_n \psi_6$ sum from [Gu2]. Our main result is Theorem 3.1, which reduces a very general multilateral Karlsson–Minton type series on $C_n$ to a finite sum. In Section 4 we state some corollaries of Theorem 3.1. These include a transformation and a summation formula for Karlsson–Minton type series on $C_n$, Corollaries 4.1 and 4.2, respectively. One special case of Theorem 3.1 is a rather curious transformation formula between finite sums, Corollary 4.4. Another interesting case is when the Karlsson–Minton type series is one-variable, that is, connected to the root system $C_1$. In agreement with the coincidence of root systems $C_1 = A_1$, the same series may arise as a left-hand side of (4). This leads to a transformation formula relating finite hypergeometric sums on the root systems $A_n$ and $C_n$, which turns out to be a multivariable Watson transformation due to Milne and Lilly [ML], given here as Corollary 4.7.

2. Preliminaries

We will work with $q$-series rather than classical hypergeometric series, with $q$ a fixed complex number such that $0 < |q| < 1$. We will use the standard notation of [GR], but since $q$ is fixed we suppress it from the notation. Thus we write (this must not be confused with the standard notation for classical hypergeometric series used in the introduction)

$$ (a)_k = \prod_{j=0}^{k-1} \frac{1 - aq^j}{1 - aq^{j+k}} = \begin{cases} \frac{(1-a)(1-aq)\cdots(1-aq^{k-1})}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^k)}, & k \geq 0, \\ \frac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^k)}, & k < 0, \end{cases} $$

and analogously for infinite products $(a)_\infty = \prod_{j=0}^{\infty} (1 - aq^j)$.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we write $|z| = z_1 + \cdots + z_n$ and use the corresponding capital letter to denote the product of the coordinates: $Z = z_1 \cdots z_n$.

We let

$$ W(z) = \prod_{1 \leq j \leq k \leq n} (1 - z_j z_k) \prod_{1 \leq j < k \leq n} (z_j - z_k), $$

which may be viewed as the Weyl denominator for the root system $C_n$ or the Lie group $Sp(n)$. Basic hypergeometric series on $C_n$ are characterized by the factor

$$ \frac{W(zq^y)}{W(z)} = \prod_{1 \leq j \leq k \leq n} \frac{1 - z_j z_k q^{y_j + y_k}}{1 - z_j z_k} \prod_{1 \leq j < k \leq n} q^{y_j z_j - q^{y_k} z_k}, $$
where the $z_j$ are fixed parameters and the $y_j$ summation indices. It will be useful to note that

\[
W(z_1, \ldots, z_{n-1}, z_n) = \frac{1 - \lambda z_n}{1 - z_n^2} \prod_{k=1}^{n-1} \frac{(1 - \lambda z_n z_k)(1 - \lambda z_n/z_k)}{(1 - z_n z_k)(1 - z_n/z_k)}.
\]

We will also write

\[
\frac{\Delta(zq^y)}{\Delta(z)} = \prod_{1 \leq j < n} q^{y_j} z_j - q^{y_k} z_k;
\]

this factor characterizes hypergeometric functions on $A_n$.

For comparison we recall the main result of [R], namely, the identity

\[
\sum_{y_1, \ldots, y_n = -\infty}^{y_1 + \ldots + y_n = 0} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq i \leq p} \frac{(c_i z_k q^{m_i})_{y_k}}{(c_i z_k)_{y_k}} \prod_{i=1}^{n} \frac{(a_i z_k)_{y_k}}{(b_i z_k)_{y_k}}
\]

\[
= \frac{(q^{1-m}/AZ, q^{1-n}BZ)_{\infty}}{(q, q^{1-m-n}B/A)^{\infty}} \prod_{i=1}^{n} \frac{(b_i/a_k, qz_k/z_i)_{\infty}}{(q/a_k z_i, b_i z_k)_{\infty}} \prod_{1 \leq k \leq n} \frac{(q^{-m} b_k/c_i)_{m_i}}{(q^{1-m} c_i z_k)_{m_i}}
\]

\[
\times \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} \frac{\Delta(cq^x)}{\Delta(c)} q_{|x|} \frac{(BZ)^{|x|}}{(q^{1-m}/AZ)^{|x|}} \prod_{1 \leq k \leq p} \frac{(c_i/a_k)_{x_i}}{(q c_i/b_k)_{x_i}} \prod_{i=1}^{p} \frac{q^{-m} c_i / c_k)_{x_i}}{(q c_i / c_k)_{x_i}}.
\]

Here, $m_i$ are non-negative integers, $|q^{1-m}B/A| < 1$, and it is assumed that no denominators vanish. The case $m_i \equiv 0$ of (4) is Gustafson’s $A_n$ analogue of Bailey’s $6\psi_6$ summation [Gu3]. The proof of (4) is based on induction on the $m_i$, with Gustafson’s identity as the starting point.

In the present paper we will imitate the analysis of [R], starting from Gustafson’s $C_n$ Bailey summation [Gu2], which we write as

\[
\sum_{y_1, \ldots, y_n = -\infty}^{y_1 + \ldots + y_n = 0} \frac{W(zq^y)}{W(z)} \prod_{1 \leq j \leq n} \frac{(a_j z_k)_{y_k}}{(q z_k/a_j)_{y_k}} \left(\frac{q}{A}\right)^{|y|}
\]

\[
= \prod_{1 \leq j < k \leq 2n+2} \frac{(q z_j z_k, q/z_j z_k)_{\infty}}{(q z_k/a_j, q/a_j z_k)_{\infty}} \prod_{1 \leq j \leq n, 1 \leq j \leq 2n+2} \frac{(q/a_j a_k)_{\infty}}{(q/A)_{\infty}}
\]

This holds for $|q/A| < 1$, as long as no denominators vanish. The case $n = 1$ of (5) is Bailey’s $6\psi_6$ summation formula [GR, Equation (II.33)].

3. A Reduction Formula

Our main result is the following identity, which reduces a very general multilateral Karlsson–Minton type hypergeometric series on the root system $C_n$ to a finite sum.

**Theorem 3.1.** Let $m_i$ be non-negative integers and $a_j, c_j, z_j$ parameters such that $|q^{1-m}/A| < 1$ and none of the denominators in (5) vanishes. Then the following
identity holds:

\[
(6) \quad \sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{W(zq^y)}{W(z)} \prod_{1 \leq k \leq n} (q^{m_j} c_j z_k, q z_k / c_j)_{y_k} \prod_{1 \leq k \leq n} (a_j z_k)_{y_k} \left( \frac{1 - |m|}{A} \right)^{|y|} \\
= \prod_{1 \leq j \leq k \leq n} (q z_j z_k, q / z_j z_k)_{\infty} \prod_{j,k=1}^{n} (q z_k / z_j)_{\infty} \prod_{1 \leq j \leq k \leq 2n+2} (q / a_j a_k)_{\infty} \\
\times \prod_{1 \leq k \leq n, 1 \leq j \leq \infty} (c_j z_k, c_j / z_k)_{m_j} \prod_{j,k=1}^{p} (c_j c_k)_{m_j+m_k} \prod_{j,k=1}^{p} (q - 1 c_j c_k, q^{-m_k} c_j / c_k)_{x_j} \left( Aq^{|m|} \right)^{|x|}.
\]

The condition \(|q^{1-|m|}/A| < 1\) ensures that that the series on the left-hand side is absolutely convergent, so that the series manipulations occurring in the proof are justified.

**Proof.** To organize the computations, it will be convenient to write, for \(a \in \mathbb{C}^{2n+2}, z \in \mathbb{C}^n, y \in \mathbb{Z}^n, c \in \mathbb{C}^p\) and \(m, x \in \mathbb{N}^p\),

\[
P_y(a, z, c, m) = \frac{W(zq^y)}{W(z)} \prod_{1 \leq k \leq n} (q^{m_j} c_j z_k, q z_k / c_j)_{y_k} \prod_{1 \leq k \leq n} (a_j z_k)_{y_k} \left( \frac{1 - |m|}{A} \right)^{|y|},
\]

\[
U(a, z) = \prod_{1 \leq j \leq k \leq n} (q z_j z_k, q / z_j z_k)_{\infty} \prod_{j,k=1}^{n} (q z_k / z_j)_{\infty} \prod_{1 \leq j \leq k \leq 2n+2} (q / a_j a_k)_{\infty} \\
\times \prod_{1 \leq k \leq n, 1 \leq j \leq \infty} (c_j z_k, c_j / z_k)_{m_j} \prod_{j,k=1}^{p} (c_j c_k)_{m_j+m_k} \prod_{j,k=1}^{p} (q - 1 c_j c_k, q^{-m_k} c_j / c_k)_{x_j} \left( Aq^{|m|} \right)^{|x|},
\]

\[
V(a, z, c, m) = \prod_{1 \leq j \leq k \leq 2n+2, 1 \leq j \leq \infty} (c_j z_k, c_j / z_k)_{m_j} \prod_{j,k=1}^{p} (c_j c_k)_{m_j+m_k} \prod_{j,k=1}^{p} (q - 1 c_j c_k, q^{-m_k} c_j / c_k)_{x_j} \left( Aq^{|m|} \right)^{|x|},
\]

\[
Q_x(a, c, m) = \frac{W(q^{1-c}c q^{x})}{W(q^{1-c})} \prod_{1 \leq k \leq n} (c_j z_k)_{x_j} \prod_{j,k=1}^{p} (q^{-1} c_j c_k, q^{-m_k} c_j / c_k)_{x_j} \left( Aq^{|m|} \right)^{|x|},
\]

so that (6) may be written as

\[
(7) \quad \sum_{y_1, \ldots, y_n = -\infty}^{\infty} P_y(a, z, c, m) = U(a, z) V(a, z, c, m) \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} Q_x(a, c, m).
\]

We will prove the theorem by induction on \(|m|\), the case \(m_j \equiv 0\) being Gustafson’s identity (4). Thus we assume that (6) holds for fixed \(p\) and \(|m|\) and the other parameters free. Since both sides are invariant under simultaneous permutations of the \(m_j\) and \(c_j\), it is enough to prove that (7) also holds when \(m\) is replaced by

\[
m^+ = (m_1, \ldots, m_{p-1}, m_p + 1).
\]
It will be convenient to replace \( n \) by \( n + 1 \) in (6) and write

\[
a = (a_1, \ldots, a_{2n+2}), \quad z = (z_1, \ldots, z_n), \quad y = (y_1, \ldots, y_n),
\]

\[
a^+ = (a_1, \ldots, a_{2n+4}), \quad z^+ = (z_1, \ldots, z_{n+1}), \quad y^+ = (y_1, \ldots, y_{n+1}).
\]

We also specialize to the case \( a_{2n+4} = z_{n+1} \). Then the factor \( 1/(qz_{n+1}/a_{2n+4}) \) on the left-hand side vanishes unless \( y_{n+1} \geq 0 \), so that the series is supported on a half-space. Next we let \( a_{2n+3} = q/z_{n+1} \), which will cause most factors involving \( z_{n+1} \) to cancel. (In general it is not allowed to put \( a_{2n+3} = q/z_{n+1} \) in (6), but the choice of \( a_{2n+4} \) removes this singularity.) We denote the corresponding left-hand side of (7) by \( S \) and decompose it as

\[
S = \sum_{y_1, \ldots, y_{n+1} \in \mathbb{Z}, y_{n+1} \geq 0} P_{y^+}(a^+, z^+, c, m) = \sum_{y_{n+1} = 0} + \sum_{y_{n+1} \geq 1} = S_1 + S_2.
\]

Considering first \( S_1 \), we have

\[
\frac{W(z^+ q^{y_0})}{W(z^+)} = \frac{W(z q^y)}{W(z)} \prod_{k=1}^n \frac{(1 - q^{y_k} z_k z_{n+1})(1 - q^{y_k} z_k/z_{n+1})}{(1 - z_k z_{n+1})(1 - z_k/z_{n+1})}
\]

\[
= \frac{W(z q^y)}{W(z)} \prod_{k=1}^n (z_k z_{n+1}, z_k/z_{n+1})
\]

In particular, if we choose

\[
(8) \quad z_{n+1} = q^{-m_p}/c_p,
\]

the Weyl denominators combine with the factors involving \( m_p \) as

\[
\frac{W(z^+ q^{y_0})}{W(z^+)} \prod_{k=1}^n (q^{m_p c_p} z_k)_{y_k} = \frac{W(z q^y)}{W(z)} \prod_{k=1}^n (q^{m_p + 1} c_p z_k)_{y_k},
\]

which gives

\[
P_{y_0}(a^+, z^+, c, m) = P_y(a, z, c, m^+).
\]

Thus, \( S_1 \) is a sum as in the theorem, but with \( m \) replaced by \( m^+ \). To complete the induction, we must prove that

\[
S_1 = U(a, z) V(a, z, c, m^+) \sum_{x_1, \ldots, x_p = 0}^{m_{1, \ldots, m_{p-1}, m_p + 1}} Q_x(a, c, m^+).
\]

This will be achieved by verifying the two identities

\[
(9) \quad S = U(a, z) V(a, z, c, m^+) \sum_{x_1, \ldots, x_p = 0}^{m_{1, \ldots, m_p}} Q_x(a, c, m^+),
\]

\[
(10) \quad S_2 = -U(a, z) V(a, z, c, m^+) \sum_{x_1, \ldots, x_{p-1} = 0}^{m_{1, \ldots, m_{p-1}}} Q_{x, m_{p-1}}(a, c, m^+).
\]
Starting with (8), we know already that

\[ S = U(a^+, z^+) V(a^+, z^+, c, m) \sum_{x_1, \ldots, x_p=0} Q_x(a^+, c, m). \]

It is not hard to check that (as explained above, when substituting \( a^+ \) it is necessary to first let \( a_{2n+4} = z_{n+1} \) and afterwards \( a_{2n+3} = q/z_{n+1} \))

\[
\frac{U(a^+, z^+)}{U(a, z)} = \frac{1}{1 - A \prod_{k=1}^{n+1} (1 - z_{n+1}/a_k)} \prod_{k=1}^{2n+2} (1 - z_{n+1}/a_k) \\
= \frac{1}{1 - 1/A \prod_{k=1}^{n+1} (1 - z_{n+1}z_k)} \prod_{k=1}^{n} (1 - z_{n+1}/z_k) \\
\prod_{k=1}^{2n+2} (1 - a_k/z_{n+1}).
\]

(11)

\[
\frac{V(a^+, z^+, c, m)}{V(a, z, c, m)} = \frac{1 - A \prod_{j=1}^{p} (1 - q_j c_j/z_{n+1})}{1 - Aq^{m_1} \prod_{j=1}^{p} (1 - q_j c_j/z_{n+1})},
\]

(12)

\[
\frac{Q_x(a^+, c, m)}{Q_x(a, c, m)} = q^{c_1} \prod_{j=1}^{p} \frac{(1 - c_j/z_{n+1})(1 - q^{-c_j} z_{n+1})}{(1 - q^{c_j} c_j/z_{n+1})(1 - q^{c_j-1} c_j z_{n+1})},
\]

(13)

Combining this with the similarly derived identities

\[
\frac{V(a, z, c, m^+)}{V(a, z, c, m)} = \frac{1}{1 - Aq^{m_1} \prod_{j=1}^{p-1} (1 - q^{m_j+m_p} c_j c_p) \prod_{j=1}^{p} (1 - q^{m_j} c_j c_p) \prod_{j=1}^{p} (1 - q^{m_j+m_p} c_j c_p)} \\
\]

(14)

\[
\frac{Q_x(a, c, m^+)}{Q_x(a, c, m)} = q^{x_1} \prod_{j=1}^{p} \frac{(1 - q^{-m_j} c_j c_p)(1 - q^{m_j} c_j c_p)}{(1 - q^{x_j-m_j} c_j c_p)(1 - q^{x_j+m_j} c_j c_p)},
\]

and using \( z_{n+1} = q^{-m_p}/c_p \), we find that

\[ U(a^+, z^+) V(a^+, z^+, c, m) Q_x(a^+, c, m) = U(a, z) V(a, z, c, m^+) Q_x(a, c, m^+), \]

which proves (8).

Next we show that \( S_2 \) is a sum of the same type of \( S \). The choice (8) of \( z_{n+1} \) corresponds to a removable singularity of \( S_2 \). Namely, we must write

\[ \frac{(q^{m_p} c_p z_{n+1})_{y_{n+1}}}{(c_p z_{n+1})_{y_{n+1}}} = \frac{(q^{y_{n+1}} c_p z_{n+1})_{m_p}}{(c_p z_{n+1})_{m_p}} = \frac{(q^{y_{n+1}-m_p})_{m_p}}{(q^{-m_p})_{m_p}}, \]

(15)

which vanishes for \( 1 \leq y_{n+1} \leq m_p \). To obtain a sum with \( y_{n+1} \geq 0 \) we therefore replace \( y_{n+1} \) with \( y_{n+1} + m_p + 1 \) in the summation. This gives rise to factors of the form

\[ (\lambda z_{n+1})_{y_k+m_p+1} \prod_{k=1}^{n+1} (\lambda z_{k})_{y_k} = (\lambda z_{n+1})_{m_p+1} \prod_{k=1}^{n+1} (\lambda w_k)_{y_k}, \]

where \( w^+ = (w_1, \ldots, w_{n+1}) = (z_1, \ldots, z_n, q^{m_p+1} z_{n+1}) = (z_1, \ldots, z_n, q/c_p) \).
Thus, the change of summation variables gives

\[ S_2 = \sum_{y_1, \ldots, y_{n+1} \in \mathbb{Z}, y_{n+1} \geq 1} P_{y^+}(a^+, z^+, c, m) = M \sum_{y_1, \ldots, y_n \in \mathbb{Z}, y_{n+1} \geq 0} P_{y^+}(a^+, w^+, c, m), \]

where, using (15) with \( y_{n+1} \) replaced by \( 1 + m_p \),

\[
M = \frac{W(w^+)}{W(z^+)} \frac{(q)_{m_p}}{(q^{-m_p})_{m_p}} \prod_{j=1}^{p-1} \frac{(q^{m_j}c_j z_{n+1})_{m_p+1}}{(c_j z_{n+1})_{m_p+1}} \prod_{j=1}^{p} \frac{(q z_{n+1}/c_j)_{m_p+1}}{(q^{1-m_j} z_{n+1}/c_j)_{m_p+1}} \times \prod_{k=1}^{2n+2} \frac{(a_k z_{n+1})_{m_p+1}}{(q z_{n+1}/a_k)_{m_p+1}} \left( \frac{q^{-m_l}}{A} \right)^{m_p+1}.
\]

(16)

We first rewrite the multiplier \( M \). By (3),

\[
\frac{W(w^+)}{W(z^+)} = \frac{1 - q^2/c_p^2}{1 - q^{-2m_p}/c_p^2} \prod_{k=1}^{n} \frac{(1 - q z_k/c_p)(1 - q/z_k c_p)}{(1 - q^{-m_p}z_k/c_p)(1 - q^{-m_p}/z_k c_p)}.
\]

Plugging this into (16), and using (8) and the standard identities

\[
\frac{(a)_n}{(b)_n} = \left( \frac{a}{b} \right)^n \frac{(1-n)_n}{(1-n/b)_n}, \quad \frac{(q)_n}{(q^{-n})_n} = (-1)^n q^{n(n+1)/2},
\]

we obtain

\[
M = (-1)^{m_p} \left( Aq^{m_l-\frac{1}{2}m_p} \right)^{m_p+1} \frac{1 - q^{-2m_p}/c_p^2}{1 - q^{2m_p}/c_p^2} \prod_{k=1}^{n} \frac{(1 - q^{-1}c_p z_k)(1 - q^{-1}/z_k c_p)}{(1 - q^{m_p}c_p z_k)(1 - q^{-m_p}/z_k c_p)} \times \prod_{k=1}^{2n+2} \frac{(c_p/a_k)_{m_p+1}}{(q^{-1}c_p a_k)_{m_p+1}} \prod_{j=1}^{p-1} \frac{(q^{-m_j}c_p/c_j)_{m_p+1}}{(c_p/c_j)_{m_p+1}} \prod_{j=1}^{p} \frac{(q^{m_j}c_p/c_j)_{m_p+1}}{(q^{m_j+1}c_p/c_j)_{m_p+1}}.
\]

(17)

By our induction hypothesis, we have

\[
S_2 = M U(a^+, z^+) V(a^+, z^+, c, m) \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} Q_x(a^+, c, m),
\]

where \( a_{2n+3} \) and \( a_{2n+4} \) are related to \( z_{n+1} \) as above, but where instead of (8) we have that \( z_{n+1} = q/c_p \). Using (11), (12) and (14) with \( z_{n+1} = q/c_p \) gives

\[
\frac{U(a^+, z^+) V(a^+, z^+, c, m)}{U(a, z) V(a, z, c, m^+)} = \frac{1 - q^{2m_p}c_p^2}{1 - q^{-2m_p}/c_p^2} \prod_{k=1}^{2n+2} \frac{1 - q^{-1}a_k c_p}{1 - q^{m_p}a_k c_p} \times \prod_{k=1}^{n} \frac{(1 - q^{m_p}c_p z_k)(1 - q^{m_p}/z_k c_p)}{(1 - q^{-1}c_p z_k)(1 - q^{-1}/z_k c_p)} \prod_{j=1}^{p} \frac{(1 - q^{m_j}c_j c_p)(1 - q^{m_j+1}c_j c_p)}{(1 - q^{-1}c_j c_p)(1 - q^{m_j-1}c_j c_p)}.
\]

(19)

When \( z_{n+1} = q/c_p \), (13) vanishes unless \( x_p = 0 \), in which case

\[
\frac{Q_{(x,0)}(a^+, c, m)}{Q_{(x,0)}(a, c, m)} = q^x \prod_{j=1}^{p-1} \frac{(1 - c_j/c_p)(1 - q^{-1}c_j c_p)}{(1 - q^{x_j}c_j/c_p)(1 - q^{x_j-1}c_j c_p)}.
\]

(20)
Finally we want to compare \( Q_{(x,0)}(a,c,m) \) and \( Q_{(x,m_p+1)}(a,c,m^+) \). Again using (3), we have

\[
\frac{W(q^{-\frac{3}{2}}c q^{(x,m_p+1)})}{W(q^{-\frac{3}{2}}c q^{(x,0)})} = \frac{1 - q^{2m_p+1}c_p^2 p^{-1}}{1 - q^{-1}c_p^2} \prod_{j=1}^{p-1} \left( 1 - q^{x_j+c_p} (1 - q^{1-x_j+m_p c_p/c_j}) \right),
\]

which gives, after simplifications,

\[
\frac{Q_{(x,m_p+1)}(a,c,m^+)}{Q_{(x,0)}(a,c,m)} = (-1)^{m_p+1} q^{\left| y \right|} \left( Aq^{n-\frac{3}{2}m_p} \right)^{m_p+1} \frac{(c_p^2/m_p)}{(q^{m_p+1}c_p^2/m_p)}
\]

(21)

\[
\times \prod_{k=1}^{2n+2} (c_p/a_k)^{m_p+1} \prod_{j=1}^{p-1} \left( \frac{1 - q^{-1}c_p (1 - c_j/c_p)}{(1 - q^{-1}c_p) (1 - q^{1-c_j/c_p})} \right)
\]

Combining the equations (17), (18), (19), (20) and (21), we eventually obtain (10). This completes the proof. 

\[ \square \]

4. Corollaries

In this section we point out some interesting consequences and special cases of Theorem 3.1. Throughout, it is assumed that the \( n \) on the left. The case of the resulting identity is due to Schlosser \[S2, Corollary 4.2\].

One of the most conspicuous features of (3) is that the sum on the right is independent of the parameters \( z_j \). This implies a transformation formula for the series on the left. The case \( n = 1 \) of the resulting identity is due to Schlosser \[S2, Corollary 8.6\]. Schlosser also gave a generalization to the root system \( A_n \) \[S1, Theorem 4.2\]; cf. also \[R, Corollary 4.2\].

**Corollary 4.1.** For \( |q^{1-\left|m\right|}/A| < 1 \), the following identity holds:

\[
\sum_{y_1,\ldots,y_n=-\infty}^{\infty} \frac{W(z q^y)}{W(z)} \prod_{1 \leq k \leq n} \prod_{1 \leq j \leq p} \frac{(m_k c_j z_k, q z_k/c_j)_{y_k}}{(c_j z_k, q^{1-m_k z_k/c_j} y_k)} \prod_{1 \leq k \leq n} \prod_{1 \leq j \leq 2n+2} ^{\left| y \right|} \frac{(a_j z_k)_{y_k}}{(q z_k/a_j)_{y_k}} \left( q^{-\left|m\right|}/A \right)
\]

\[
\times \prod_{1 \leq k \leq n} \prod_{1 \leq j \leq 2n+2} \left( q z_k/a_j, q/a_j z_k \right) \prod_{1 \leq k \leq n} \prod_{1 \leq j \leq 2n+2} \left( c_j z_k, c_j z_k \right)_{m_j}
\]

\[
\times \sum_{y_1,\ldots,y_n=-\infty}^{\infty} \frac{W(w q^y)}{W(w)} \prod_{1 \leq k \leq n} \prod_{1 \leq j \leq p} \frac{(m_k c_j z_k, q z_k/c_j)_{y_k}}{(c_j z_k, q^{1-m_k z_k/c_j} y_k)} \prod_{1 \leq k \leq n} \prod_{1 \leq j \leq 2n+2} ^{\left| y \right|} \frac{(a_j w_k)_{y_k}}{(q w_k/a_j)_{y_k}} \left( q^{-\left|m\right|}/A \right)
\]

If we assume that \( a_{n+j} = a_j^{-1} \) for \( 1 \leq j \leq n \) and choose \( w_j = a_j \) in Corollary 4.1, the factor \( (w_{n+j}/a_j)_{y_j}/(q w_{j}/a_j)_{y_j} = (1)_{y_j}/(q)_{y_j} \) on the right vanishes for \( y_j \neq 0 \), so that the sum reduces to 1. Alternatively, we may in this situation use Corollary 4.3 below to compute the right-hand side of (6). Writing \( a_{2n+1} = b, a_{2n+2} = d \), either of these methods gives the following identity. When \( n = 1 \), it reduces to an identity of
For $A_n$ analogues of Chu’s identity, cf. [S1, Corollary 4.3], [R, Corollaries 4.3 and 4.4].

**Corollary 4.2.** For $|q^{1-m}|/bd < 1$, the following identity holds:

$$\sum_{y_1, y_2 = -\infty}^{\infty} \frac{W(zq^y)}{W(z)} \prod_{1 \leq k \leq n} \frac{(q^{m_j} c_j z_k, q z_k/c_j)_{y_k}}{(c_j z_k, q^{1-m_j} z_k/c_j)_{y_k}} \prod_{j,k=1}^{n} \frac{(a_j z_k, z_k/a_j)_{y_k}}{(q a_j z_k, q z_k/a_j)_{y_k}} \times \prod_{k=1}^{n} \frac{(b z_k, d z_k)_{y_k}}{(q z_k/b, q z_k/d)_{y_k}} \left(\frac{|y|}{bd}\right)$$

$$= \prod_{j,k=1}^{n} \frac{(q z_k/a_j, q/a_j z_k)_{\infty}}{(q z_k/b, q z_k/b, q z_k/d, q z_k/d)_{\infty}} \prod_{1 \leq j < k \leq n} \frac{(q a_j a_k, q/a_j a_k)_{\infty}}{(c_j a_k, c_j/a_k)_{m_j}}$$

To obtain an identity closer to the original Karlsson–Minton summation formula [P] one should specialize the parameters in Corollary 4.2 so that the summation indices are bounded from below. Essentially, this forces $n = 2$, when we may choose $b = z_1, d = z_2$. The resulting identity seems interesting enough to write out explicitly; it is a $C_2$ version of Gasper’s well-poised Karlsson–Minton type summation from [G2].

**Corollary 4.3.** For $|q^{1-m}|/z_1 z_2 < 1$, the following identity holds:

$$\sum_{y_1, y_2 = 0}^{\infty} \frac{W(zq^y)}{W(z)} \prod_{1 \leq k \leq 2} \frac{(q^{m_j} c_j z_k, q z_k/c_j)_{y_k}}{(c_j z_k, q^{1-m_j} z_k/c_j)_{y_k}} \prod_{j,k=1}^{2} \frac{(a_j z_k, z_k/a_j, z_j z_k)_{y_k}}{(q a_j z_k, q z_k/a_j, q z_k/z_j)_{y_k}} \left(\frac{|y|}{z_1 z_2}\right)$$

$$= \frac{(q z_1^2, q z_2 z_1, q z_2, q a_1 a_2, q/a_1 a_2)_{\infty}}{(q/z_1 z_2)_{\infty}} \prod_{j,k=1}^{2} \frac{(q a_k/a_j)_{\infty}}{(q z_k/a_j, q z_k/a_j)_{\infty}} \prod_{1 \leq k \leq 2} \frac{(c_j a_k, c_j/a_k)_{m_j}}{(c_j z_k, c_j/z_k)_{m_j}}.$$

More generally, one may choose the parameters so that the summation indices on the left-hand side of (3) are bounded from below or above. A particularly symmetric case arises when both these conditions hold, so that we have a finite sum. To this end we choose $a_j = z_j, a_{n+j} = q^{-l_j}/z_j, 1 \leq j \leq n$ in Theorem 3.1, and write $a_{2n+1} = b, a_{2n+2} = d$. Since we have a rational identity in $b, d$ the condition $|q^{1-\lambda}|/A = |q^{1-\lambda}|/bd < 1$ is then superfluous. The resulting identity is reminiscent of transformation formulas for $A_n$ hypergeometric series recently obtained by Kajihara [K].
Corollary 4.4. The following identity holds:

$$\sum_{y_1, \ldots, y_n=0}^{t_1, \ldots, t_n} \left( \frac{W(zq^y)}{W(z)} \prod_{1 \leq k \leq n}^{y_k} \frac{(q^{m_k}c_jz_k, qz_k/c_j)_{y_k}}{(c_jz_k, q^{1-m_k}z_k/c_j)_{y_k}} \prod_{j,k=1}^{n} \frac{(z_jz_k, q^{-k}z_k/z_j)_{y_k}}{(qz_k/z_j, q^{1+k}z_jz_k)_{y_k}} \right)$$

$$\times \prod_{k=1}^{n} \frac{(bz_k, dz_k)_{y_k}}{(qz_k/b, qz_k/d)_{y_k}} \left( \frac{q^{1-|m|+|l|}}{|y|} \right) \left( \frac{q}{bd} \right)^{|y|} = \prod_{j=1}^{p} \prod_{m=1}^{p} \frac{(q^{-k}c_jz_k, qz_k/c_j)_{x_j}}{(c_jz_k, q^{-k}c_jz_k)_{x_j}} \prod_{j,k=1}^{n} \frac{(q^{m_k}c_jz_k, qz_k/c_j)_{x_j}}{(q^{m_k}c_jz_k, qz_k/c_j)_{x_j}} \prod_{j,k=1}^{n} \frac{(c_jz_k, q^{-k}c_jz_k)_{x_j}}{(c_jz_k, q^{-k}c_jz_k)_{x_j}} \left( q^{m-|m|} |bd|^{x_j} \right).$$

Theorem 3.3 has some interesting consequences for low values of $n$. An inspection of the proof shows that it holds for $n = 0$, if the left-hand side of (3) is interpreted as 1. This leads to a $C_n$ analogue of the terminating $qW_5$ summation formula, which is in fact the special case $a_j = z_j$, $a_{n+j} = q^{-m_j}z_j$, $1 \leq j \leq n$ of Gustafson’s identity (5) (cf. also [LM]), or, equivalently, the case $m_j = 0$ of Corollary 4.4. We include it here as a first illustration of how Theorem 3.3 is related to known results for “classical” (i.e. not of Karlsson–Minton type) $C_n$ hypergeometric series. Compared to Theorem 3.3 we have replaced $p$ with $n$ and $c_j$, $a_1$, $a_2$ with $q^{1/2}z_j$, $q^{1/2}a$, $q^{1/2}b$, respectively.

Corollary 4.5 (Gustafson). The following identity holds:

$$\sum_{x_1, \ldots, x_n=0}^{m_1, \ldots, m_n} \frac{W(zq^x)}{W(z)} \prod_{j=1}^{n} \frac{(z_j/a, z_j/b)_{x_j}}{(qz_ja, qz_jb)_{x_j}} \prod_{j,k=1}^{n} \frac{(z_jz_k, q^{m_k}z_j/z_k)_{x_j}}{(qz_jz_k, q^{m_k+1}z_jz_k)_{x_j}} \left( abq^{m+1} \right)^{|x|}$$

$$= \frac{(qab)^{|m|}}{\prod_{j=1}^{n} (qaz_j, qbz_j)_{m_j}} \prod_{1 \leq j < k \leq n} (qz_jz_k)_{m_j+m_k}.$$
Corollary 4.6. For $|a^2q^{1-|m|}/bcde| < 1$, the following identity holds:

$$
\sum_{y=-\infty}^{\infty} \frac{1 - aq^{2y}}{1 - a} \frac{(b, c, d, e)_y}{(aq/b, aq/c, aq/d, aq/e)_y} \prod_{j=1}^{p} \frac{(f_j, aq^{1+m_j}/f_j)_y}{(q^{-m_j}f_j, aq/f_j)_y} \left( \frac{a^2q^{1-|m|}}{bcde} \right)^y
= \frac{(q, aq/q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de)_{\infty}}{(q/b, q/c, q/d, q/e, aq/b, aq/c, aq/d, aq/e, a^2q/bcde)_{\infty}} \frac{1}{(bcde/a^2)_{|m|}}
\prod_{j=1}^{p} \frac{(qb/f_j, qc/f_j, qd/f_j, qe/f_j)_{m_j}}{(aq/f_j, q/f_j)_{m_j}} \prod_{1 \leq j < k \leq p} (aq^2/f_j f_k)_{m_j+m_k}
\prod_{j<k} (aq/bf_j, aq/cf_j, aq/df_j, aq/ef_j)_{x_j}
\times \prod_{j,k=1}^{m_1,...,m_p} \frac{(aq/f_j f_k, q^{-m_k}f_k/f_j)_{x_j}}{(qf_k/f_j, aq^{2+m_k}/f_j f_k)_{x_j}} \left( \frac{bcde q^{m_k}}{a^2} \right)^{|x_j|}.
$$

(22)

We remark that the factor

$$
\frac{W(\sqrt{aq} q^x/f)}{W(\sqrt{aq}/f)} = \prod_{1 \leq j \leq k \leq p} \frac{1 - aq^{x_j+x_k+1}/f_j f_k}{1 - aq/f_j f_k} \prod_{1 \leq j < k \leq p} \frac{q^{x_j}/f_j - q^{x_k}/f_k}{1/f_j - 1/f_k}
$$

does not depend on the choice of square root.

Corollary 4.6 may be compared with Corollary 4.11 of [3], which is just the case $n = 2$ of [4]. It says that the left-hand side of (22) equals

$$
\frac{(q, aq/q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de)_{\infty}}{(q/b, q/c, q/d, q/e, aq/b, aq/c, aq/d, aq/e, a^2q/bcde)_{\infty}}
\times \prod_{j=1}^{p} \frac{(bq/f_j, cq/f_j)_{m_j}}{(aq/f_j, q/f_j)_{m_j}} \frac{(de/a)_{|m|}}{(bcde/a^2)_{|m|}}
\times \sum_{x_1,...,x_p=0}^{m_1,...,m_p} \frac{\Delta(q^{x_j}/f)}{\Delta(1/f)} \frac{(bc/a)_{|x|}}{(aq^{1-|m|}/de)_{|x|}} \prod_{j=1}^{p} \frac{(aq/df_j, aq/ef_j)_{x_j}}{(gb/f_j, qc/f_j)_{x_j}} \prod_{j,k=1}^{p} \frac{(q^{-m_k}f_k/f_j)_{x_j}}{(qf_k/f_j)_{x_j}} q^{x_j}.
$$

That this quantity equals the right-hand side of (22) is equivalent to a multivariable Watson transformation due to Milne and Lilly [ML, Theorem 6.6]. After replacing $p$ with $n$, $f_j$ with $\sqrt{aq}/z_j$ and $(b, c, d, e)$ with $\sqrt{aq}(b^{-1}, c^{-1}, d^{-1}, e^{-1})$, it takes the following form.
Corollary 4.7 (Milne and Lilly). One has the identity

\[
\sum_{x_1, \ldots, x_n=0}^{m_1, \ldots, m_n} \left( \frac{W(zq^x)}{W(z)} \prod_{j=1}^{n} \frac{(b z_j, c z_j, d z_j, e z_j)_{x_j}}{(q z_j/b, q z_j/c, q z_j/d, q z_j/e)_{x_j}} \right) \times \prod_{j,k=1}^{n} \frac{(z_j z_k, q^{-m_k} z_j / z_k)_{x_j}}{(q z_j/z_k, q^{1+m_k} z_j z_k)_{x_j}} \left( \frac{q^{m_j+2}}{bcde} \right)^{[x_j]} = \prod_{j=1}^{n} (q z_j z_k)_{m_j} \times \sum_{x_1, \ldots, x_n=0}^{m_1, \ldots, m_n} \frac{\Delta(zq^x)}{\Delta(z)} \Delta(z) \frac{(q/|de|)_{[x_j]}}{(q^{-m|de|})_{[x_j]}} \prod_{j=1}^{n} \frac{(d z_j, e z_j)_{x_j}}{(q z_j/b, q z_j/c)_{x_j}} \prod_{j,k=1}^{n} \frac{(q^{-m_k} z_j / z_k)_{x_j}}{(q z_j/z_k)_{x_j}} q^{[x_j]}.
\]

The proof of Corollary 4.7 obtained here gives a nice explanation of why such a transformation formula, relating a $C_n$ $8W_7$ series and an $A_n$ $4\phi_3$, exists: on the level of Karlsson–Minton type hypergeometric series it reflects the coincidence of root systems $A_1 = C_1$. It is appropriate to remark here that our proof of Theorem 3.1 depended on guessing the explicit expression for the right-hand side of \((\text{4})\). This task was much simplified by having access to the Milne–Lilly transformation, and thus (given also the results of \([\text{R}]\)) knowing the identity in advance for $n = 1$.

A generalization of Corollary 4.7 to the level of multivariable balanced $10W_9$ series has been obtained by Bhatnagar and Schlosser [BS, Theorem 2.1]. We have not been able to obtain this identity in our approach. Note that, when $n = 2$, the right-hand side of \((\text{4})\) is a $p$-variable $10W_9$, but to make it balanced one must let $A = q^{1-m}$, which corresponds to a pole of the left-hand side.

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