Emergent universe in theories with natural UV cutoffs

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Abstract
We investigate the realization of the emergent universe scenario in theories with natural UV cutoffs, namely a minimum length and a maximum momentum, quantified by a new deformation parameter in the generalized uncertainty principle. We extract the Einstein static universe solutions and we examine their stability through a phase-space analysis. As we show, the role of the new deformation parameter is crucial in a twofold way: firstly, it leads to the appearance of new Einstein static universe critical points, that are absent in standard cosmology. Secondly, it provides a way for a graceful exit from the Einstein static universe into the expanding thermal history, that is needed for a complete and successful realization of the emergent universe scenario.

Keywords: modified theories of gravity, cosmology, particle-theory and field-theory models of the early Universe

(Some figures may appear in colour only in the online journal)

1. Introduction

According to the concordance model of cosmology our universe has most probably begun from an initial singularity at a finite past. The introduction of the inflation paradigm as a
successful way to solve the horizon, flatness and magnetic monopole problems [1], did not affect the initial singularity issue, which is still considered as a potential, conceptual disadvantage [2]. Finally, in order to describe the observed late-time universe acceleration a cosmological constant was added, leading eventually to the $\Lambda$CDM cosmology, namely the standard model of the universe. Nevertheless, in spite of the remarkable successes of this paradigm, its physical content relating to the two accelerating phases at early and late times is still not satisfactory, and furthermore, the initial singularity problem remains open.

There are two ways one could follow in order to bypass the initial singularity problem. The first is to consider the scenario of bouncing cosmology, in which the current universe expansion followed a previous contracting phase, with the scale factor being always non-zero \cite{3, 4}. The second is to consider the scenario of ‘emergent universe’ \cite{5}, in which the universe originates from a static state, namely from the ‘Einstein static universe’, and then it enters the inflationary phase, without passing from any singularity. However, both these alternative cosmological scenarios cannot be obtained in the framework of general relativity. Concerning the Einstein static universe, which is a necessary ingredient of the emergent universe scenario, it can be shown that it is significantly affected by the initial conditions such as perturbations, which dominate at the Ultra-Violet (UV) limit, and hence it is indeed unstable against classical perturbations which eventually lead it to collapse to a singularity \cite{6}.

In order to alleviate the above problems one may follow the way to introduce new degree(s) of freedom, beyond the standard model of particle physics or/and general relativity. A first direction is to consider exotic forms of matter that could provide a successful description of the universe behavior in the framework of general relativity (see \cite{7, 8} and references therein). The second direction is to construct a gravitational modification whose extra degrees of freedom could describe the universe at large scales, while still possessing general relativity as a particular limit \cite{9, 10}. Concerning the initial singularity issue, modified gravity, amongst others, can trigger the cosmological bounce \cite{11}, or it can cure the emergent universe scenario by making Einstein static universe stable. In particular, the Einstein static universe and thus the emergent universe scenario, can be successfully realized in various gravitational modifications, such as in Einstein–Cartan theory \cite{12}, in $f(R)$ gravity \cite{13}, in $f(T)$ gravity \cite{14}, in loop quantum cosmology \cite{15}, in massive gravity \cite{16}, in Hořava–Lifshitz gravity \cite{17}, in braneworld models \cite{18, 19}, in null-energy-condition violated theories \cite{20} etc, although the successful exit from the Einstein static universe towards the subsequent expanding thermal history is not always achieved.

One interesting gravitational modification in the UV regime arises through the use of the ‘generalized uncertainty principle’ \cite{21}, which seen as a quantum gravity approach might be related with other quantum gravity models such as Double Special Relativity \cite{22} and string theory \cite{23}. Although one can have more than one generalizations of the uncertainty principle, the most interesting one is when one modifies the standard Heisenberg algebra by a linear and a quadratic term in Planck length and momentum respectively, which leads to the existence of two natural UV cutoffs, namely minimum length and maximum momentum \cite{24}. Hence, when applied to a cosmological framework, these natural cutoffs give rise to extra terms in the Friedmann equations, which can have interesting implications.

In the present work we are interested in investigating the Einstein static universe and the emergent universe scenario in the framework of theories with natural UV cutoffs. In particular, we show how the induced extra terms in the cosmological equations lead to the realization and stability of the Einstein static universe, as well as offering the mechanism to a phase transition to the inflationary era and the subsequent thermal history of the universe.

The plan of the manuscript is the following: in section 2, we briefly review generalized uncertainty principle with two UV cutoffs, and we apply it in a cosmological framework. In
section 3 we extract the Einstein static universe solutions. In section 4 we examine the stability of the Einstein static universe by performing a dynamical system analysis, studying its exit towards the inflationary era. Finally, in section 5 we provide the conclusions.

2. Theories with natural UV cutoffs and their cosmology

In this section we briefly present theories with natural UV cutoffs, and then we apply them in a cosmological framework. As we mentioned in the introduction, in general this kind of theories arise from the consideration of generalizations of the uncertainty principle [21]. Although one may have more than one such generalizations, in the present work we focus on the generalization with two natural UV cutoffs, namely a minimum length and a maximum momentum [24]. In order to achieve this, one starts by modifying the standard Heisenberg algebra at high energy scales, by a linear and a quadratic term in Planck length and momentum respectively, as

\[
[x_i, p_j] = i\hbar \left[ \delta_{ij} - \alpha \left( p^2 \delta_{ij} + 3 p_i p_j \right) \right],
\]

with \(i, j = 1, 2, 3\), where via the Jacobi identity [25] it is guaranteed that \([x_i, x_j] = 0 = [p_i, p_j]\). The parameter \(\alpha\) quantifies the quantum gravity deformation parameter, and can alternatively be written as \(\alpha = \tilde{\alpha} M_{pl} = \frac{\ell^2}{\hbar} \tilde{\alpha}\), with \(M_{pl}\) and \(\ell\) the Planck mass and Planck length respectively, \(c\) the speed of light, and \(\hbar\) the induced Planck constant. The dimensionless parameter \(\tilde{\alpha}\) according to experiments is bound to be smaller than \(10^{11}\), however theoretical arguments suggest that its value should be around 1, in order for minimal length effects to be important only around the Planck length and not introduce a new physical scale between the Planck and the electroweak scale [25–27].

Let us now apply the above generalized uncertainty principle in a cosmological framework. Since the quantum gravity deformation parameter \(\alpha\) is expected to have effects only at high energy scales, we will focus on the early-time phases of the cosmological evolution, which indeed will correspond to the realization of the emergent universe. We start by considering the homogeneous and isotropic Friedmann–Robertson–Walker (FRW) geometry, with metric

\[
ds^2 = -N^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),
\]

where \(a(t)\) is the scale factor and \(N\) the lapse function, and with \(k = 0, +1, -1\) corresponding to flat, close and open spatial geometry respectively.

One can extract the field equations in the above metric, i.e. the Friedmann equations, via the Hamiltonian constraint \(\mathcal{H}_E = 0\), namely [28]:

\[
\mathcal{H}_E = \frac{\kappa}{4} N p_a^2 + \frac{Na^3}{\kappa} \rho + \lambda \mathcal{P},
\]

with \(\kappa \equiv 1/3M^2_{pl} = 8\pi G/3\) the gravitational constant, and where \(\lambda\) and \(\mathcal{P}\) are the Lagrange multiplier and the momentum conjugate to the lapse function \(N\), respectively. In the above expression \(\rho\) is the energy density of the universe content, corresponding to a perfect fluid with equation-of-state parameter \(w\).

In general, for two typical variables \(A\) and \(B\), the Poisson brackets are defined as \(\{A, B\} = \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_j} \right) \{x_i, p_j\}\), where the canonical variables \(x_i\) and \(p_j\) in the cosmological context are replaced by \(a\) and \(p_a\), respectively. Although using the standard uncertainty principle they satisfy the usual relation \(\{a, p_a\} = 1\), considering the deformed Poisson algebra
that arises from the generalized uncertainty principle (1), up to first order in $\alpha$, the Poisson bracket between $a$ and $p_a$ becomes [24]

$$\{a, p_a\} = 1 - 2\alpha p_a.$$  \hfill (4)

Hence, using the Poisson algebra we obtain the following modified equations of motion

$$\dot{a} = \{a, \mathcal{H}_E\} = \frac{\partial \mathcal{H}_E}{\partial p_a} (1 - 2\alpha p_a),$$  \hfill (5)

$$\dot{p}_a = \{p_a, \mathcal{H}_E\} = -\frac{\partial \mathcal{H}_E}{\partial a} (1 - 2\alpha p_a).$$  \hfill (6)

Inserting $\mathcal{H}_E$ from (3) in the above equations, using its constraint value $\mathcal{H}_E = 0$, and combining them, we finally extract the first Friedmann equation, namely

$$\left(\frac{\dot{a}}{a}\right)^2 = \kappa \rho - \frac{k c^2}{a^2} - 2\sqrt{2\kappa \alpha c^2 a^2 \rho}^{3/2} \left(1 - \frac{k c^2}{\kappa a^2 \rho}\right)^{3/2}.$$  \hfill (7)

Additionally, taking the time-derivative of this equation, and using also the usual energy conservation relation

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \rho (1 + w) = 0,$$  \hfill (8)

we arrive at the second Friedmann equation, namely

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{2} (1 + 3w) \rho - 7\sqrt{\frac{\kappa}{2}} \alpha c^2 a^2 \rho^{3/2} \left(1 - \frac{k c^2}{\kappa a^2 \rho}\right)^{3/2}$$

$$+ 3\alpha c^2 a^2 \rho^{1/2} \left[\sqrt{\frac{\kappa}{2}} (1 + 3w) \rho - \sqrt{2} \frac{k c^2}{\kappa a^2}\right] (1 - \frac{k c^2}{\kappa a^2 \rho})^{1/2}.$$  \hfill (9)

As we observe, the two Friedmann equations (7) and (9) include terms with the quantum gravity deformation parameter $\alpha$, i.e. they have been modified by the generalized uncertainty principle. As expected, in the limit $\alpha \to 0$ they give rise to the standard Friedmann equations.

It is necessary to note that similar UV modified Friedman equations have been discussed earlier in the context of cosmology induced from other quantum gravity approaches such as loop quantum gravity (LQG)\(^7\) In [30, 31] it has been shown that the generalized uncertainty principle can be deduced in the context of LQG due to polymer quantization of the background spacetime geometry. Hence, one could consider that the corrections appearing in the dynamical equations, might arise from LQG with some higher spin representation. Snyder noncommutative geometry, see [32, 33] and [34].

Finally, as we have mentioned, we note that the above modified Friedmann equations have been extracted keeping terms up to first order in $\alpha$. If we additionally keep terms up to second

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\(^6\)In principle, the modified Friedmann equations should be derived from a full quantum gravitational action corresponding to the fundamental theory. However, since such a quantum-gravity modified action is still unknown, it is common in literature to consider quantum-gravitational effects phenomenologically, i.e. by deforming the standard commutation relations as has been mentioned above. Nevertheless, we mention that there is an inverse method to generate a canonical Hamiltonian structure, and subsequently an action, from arbitrary modifications of the dynamical equations, as it was formulated in detail in [29].

\(^7\)In [30, 31] it has been shown that the generalized uncertainty principle can be deduced in the context of LQG due to polymer quantization of the background spacetime geometry. Hence, one could consider that the corrections appearing in the dynamical equations, might arise from LQG with some higher spin representation.
order in $\alpha$ in the deformed Poisson bracket (4), the corresponding modified Friedmann equations will include terms such as $\alpha^2 a^4 \rho^2 \left(1 - \frac{k^2 c}{\alpha^2} \rho^2 \right)^2$. Although mathematically these extra terms will have an effect on the existence and stability of the critical points that will be analyzed in the following, in the energy scales that are of interest in the present work, namely those that correspond to pre-inflation epoch where $\rho \gg \frac{|k|}{\alpha^2}$, the effects of $\alpha^2$-dependent terms are very tiny and negligible in comparison with $\alpha$-dependent terms. Moreover, we notice that the commutator relation (1), even in the absence of $\alpha^2$ term matches with other models of generalized uncertainty principle and well-known approaches to quantum gravity, string theory and doubly spacial relativity, see for instance [24]. Hence, in summary, the first-order-in-$\alpha$ approximation is very efficient, and can capture the main effects of natural UV cutoffs.

3. Einstein static universe

In this section we show that in the cosmological application of the generalized uncertainty principle the Einstein static universe can be realized. Let us extract the Einstein static universe solutions. Inserting the conditions of the Einstein static universe, i.e. a constant scale factor $a = a_s$, with $\dot{a}|_{a=a_s} = \ddot{a}|_{a=a_s} = 0$, at an energy density $\rho = \rho_s$, in the two Friedmann equations (7) and (9), and focusing for mathematical convenience (although this is not necessary) on the regime $\rho \gg \frac{|k|}{\alpha^2}$ (which is a very robust approximation since in SI units it becomes $\rho \gg |k| \times 10^{27}$ kg m$^{-3}$, while we know that the energy density corresponding to (pre)-inflation scale is $\sim 10^{93}$ kg m$^{-3}$) we find

$$\kappa \rho_s - \frac{k^2}{a_s^2} + \frac{6\alpha k c^2}{\sqrt{2} \kappa} \rho_s^{3/2} - 2 \sqrt{2} \kappa \alpha c^2 a_s^2 \rho_s^{3/2} = 0,$$

(10)

$$\frac{3k^2 c^4}{\sqrt{2} \kappa^3 \rho_s} \frac{1}{a_s^2} + \left[ \frac{3k^2 c^4}{2 \sqrt{2} \kappa} (2 - w) \alpha - \frac{\kappa}{2} (1 + 3w) \rho_s \right] \frac{1}{a_s^2} - 2 \sqrt{2} \kappa \alpha \rho_s^{3/2} = 0.$$  

(11)

The solution of this system of algebraic equations will give the critical points of the cosmological scenario at hand, namely the pair of values for $\{a_s, \rho_s\}$ that correspond to Einstein static universe solutions. For convenience we study the flat and non-flat cases separately.

3.1. Flat universe ($k = 0$)

In the case of a flat geometry, and for a general $w \neq -1$, the system (10) and (11) accepts only the trivial solution $a_s \rightarrow \infty, \rho_s \rightarrow 0$, independently of the values of $\alpha$. However, in the special case where $w = -1$, i.e. where the universe is filled with a cosmological constant, we obtain an Einstein static universe solution for every $\rho_s$, with the corresponding scale factor being

$$a_s^2 = \frac{\sqrt{\rho_s}}{2 \alpha \sqrt{2 \kappa \rho_s}}.$$  

(12)

Note that the role of a non-zero quantum gravity deformation parameter $\alpha$ is crucial in making the above solution non-trivial, since in the limit $\alpha \rightarrow 0$ it becomes the aforementioned trivial solution.
Let us now investigate the non-flat universe. In this case, the system (10) and (11) accepts four solutions, i.e. four critical points (CP), namely

CP 1 : \( \left( \frac{1}{a_t^2} \right)_1 = 0, \)

CP 2 : \( \left( \frac{1}{a_t^2} \right)_2 = -\frac{3\alpha^2 c^6 k (2 - w)}{\kappa^2 (1 + 3w)} \times \left\{ -1 + \sqrt{1 + \frac{32}{3} \left[ \frac{1 + 3w}{(2 - w)^2} \right]} \right\}, \)

CP 3 : \( \left( \frac{1}{a_t^2} \right)_3 = -\frac{3\alpha^2 c^6 k (2 - w)}{\kappa^2 (1 + 3w)} \times \left\{ -1 - \sqrt{1 + \frac{32}{3} \left[ \frac{1 + 3w}{(2 - w)^2} \right]} \right\}, \)

CP 4 : \( \left( \frac{1}{a_t^2} \right)_4 = -2 \sqrt{\frac{q_1}{3}} \sinh \left( \frac{q_2}{3} \right), \)

with

\[ q_1 = -\frac{12k^2 c^{12} \alpha^4}{\kappa^4} \left[ w^2 + 20w + 12 \right] \left[ 1 + 3w \right]^2, \]
\[ q_2 = \sinh^{-1} \left( \frac{3q_1}{q_4 q_1} \right), \]
\[ q_3 = \frac{36k^4 c^6 \alpha^6}{\kappa^6} \left( 2 - w \right) \left[ 1 - \frac{4 (2 - w)^2}{9 [1 + 3w]} \right], \]
\[ q_4 = \frac{4k c^2 \alpha^2}{\kappa^2 (1 + 3w)} \sqrt{w^2 - 20w - 12}. \]

The critical point 1 is the trivial one with \( a_t \to \infty \). The critical point 2 is physical for \( k = +1 \), \( w > 2 \) or \( k = -1, -\frac{1}{3} < w < 2 \), with the second case being the realistic one. The critical point 3 is physical for \( k = +1, -\frac{1}{3} < w < 2 \) or \( k = -1, w > 2 \), with the first case being the realistic one. Finally, critical point 4 is physical for \( -10 - 2\sqrt{22} < w < -10 + 2\sqrt{22} \), which includes the cosmological constant value \( w = -1 \). We stress once again the crucial role of the deformation parameter \( \alpha \) that arises from the generalized uncertainty principle, in the existence of the three non-trivial critical points, since in the limit \( \alpha \to 0 \) all points coincide with the first, trivial one.

For each one of the above solutions for \( a_t \), the corresponding \( \rho_s \) is given by

\[ \rho_s^{1/2} = -\frac{1}{3A} \left( \kappa + \Delta_2 + \frac{\Delta_n}{\Delta_2} \right), \]

where
\[ \Delta_2 = \sqrt{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \]

\[ \Delta_0 \equiv \kappa^2 - 3AB, \]

\[ \Delta_1 \equiv 2\kappa^3 - 9\kappa AB + 27A^2C, \] \hspace{1cm} (19)

with

\[ A \equiv -2\sqrt{\frac{8\kappa}{c^4}\alpha^2}, \quad B \equiv \frac{6\alpha k e^4}{\sqrt{2\kappa}}, \quad C \equiv -\frac{ke^4}{\alpha^2}. \] \hspace{1cm} (20)

Relation (18) provides the energy density \( \rho_s \) corresponding to the derived critical points in the static configuration in terms of \( k, \kappa, w, c \) and \( \alpha \). In order to examine whether the resulting values are compatible with the energy of the inflationary epoch, in the following table 1 we present the approximate obtained values of \( \rho_s \). As we can see, we acquire a very good compatibility with the energy density at the inflationary phase.

### 4. Dynamical stability

In the previous section we showed that Einstein static universe can be a solution of the cosmological equations in the framework of the generalized uncertainty principle. In the present section we desire to analyze the dynamical stability of these solutions, i.e. to see whether the universe can remain in such a phase for very large time intervals.

In order to perform the dynamical analysis one usually expresses the cosmological equations as a dynamical system, and performing linear perturbations around the previously obtained solutions he proceeds to a detailed phase-space analysis, by examining the eigenvalues of the involved perturbation matrix, which reveals whether these solutions are stable or unstable [35]. In the following we will follow the alternative but equivalent (in cases of 2D equation systems) approach of [14, 19, 36, 37]. In particular, we perturb linearly the Friedmann equations (7) and (9) in the regime \( \rho \gg \frac{2\kappa e^4}{\alpha^2} \), around the obtained Einstein static universe solutions (12) and (13)–(16). The perturbations in the scale factor and matter density read as:

\[ a(t) \to a_s(1 + \delta a(t)), \]

\[ \rho(t) \to \rho_s(1 + \delta \rho(t)). \] \hspace{1cm} (21)

Inserting into the first Friedmann equation (7), using

\[ (1 + \delta a(t))^n \approx 1 + n\delta a(t), \]

\[ (1 + \delta \rho(t))^n \approx 1 + n\delta \rho(t), \] \hspace{1cm} (22)

and neglecting terms with two differentials, we obtain

\[ \textbf{Table 1.} \text{ The approximate values of } \rho_s \text{ corresponding to the non-trivial obtained critical points, calculated through (18).} \]

| Number of CP | \( \langle \rho_s \rangle_{k=+1} \text{ kg m}^{-3} \) | \( \langle \rho_s \rangle_{k=-1} \text{ kg m}^{-3} \) |
|-------------|---------------------------------|---------------------------------|
| CP 2        | \( \sim 10^{97} \)               | \( \sim 10^{97} \)               |
| CP 3        | \( \sim 10^{110} \)              | \( \sim 10^{97} \)               |
| CP 4        | \( \sim 10^{95}-10^{96} \)       | \( \sim 10^{95}-10^{96} \)       |
\[
\kappa \rho s(1 + \delta \rho) - k c^2 a_1^{-2} - 2\sqrt{2\kappa \alpha} \rho s^{3/2} + 2k c^2 a_1^{-2} \delta a \\
- \frac{3akc^2}{2\kappa} \left( \frac{1}{2} + \delta \rho \right) - \sqrt{2\kappa \alpha} \rho s^{3/2} (3\delta \rho + 4\delta a) = 0.
\]

Similarly, using (21) to perturb the second Friedmann equation (9), and neglecting terms with two differentials, we obtain

\[
\delta \ddot{a} = \left( 4\sqrt{2\kappa \alpha} \rho s^{3/2} + \frac{126\kappa c^4}{\sqrt{2\kappa}} a_1^{-2} \rho s^{1/2} \right) \delta a \\
- \left[ 3\sqrt{2\kappa \alpha} a_1^2 \rho s^{3/2} - \frac{3akc^2}{\sqrt{2\kappa}} (2 - w) \rho s^{1/2} \\
+ \frac{3akc^4}{\sqrt{2\kappa}} a_1^{-2} \rho s^{-1/2} + \frac{\kappa}{2} (3w + 1) \rho s \right] \delta \rho.
\]

In the following two subsections we examine the flat and non-flat cases separately.

### 4.1. Flat universe \((k = 0)\)

In the case of a flat universe, (23) leads to

\[
\left( \frac{\delta \rho}{\delta a} \right) = \left( \frac{4\sqrt{2\kappa \alpha} \rho s^{3/2}}{\kappa \rho s - 3\sqrt{2\kappa \alpha} \rho s^{3/2}} \right).
\]

Thus, inserting (25) into (24) and neglecting terms higher than \(O(\alpha^2)\), we acquire

\[
\delta \ddot{a} + \gamma \delta \dot{a} = 0,
\]

with

\[
\gamma = \frac{6\sqrt{2\alpha} \rho s^{3/2} a_1^{-2} (w + 1)}{\kappa - 3\sqrt{2\kappa \alpha} \rho s^{3/2}}.
\]

However, as we found in section 3.1, the non-trivial Einstein static solution (12) exists only for \(w = -1\), which leads to the limiting value \(\gamma = 0\). Hence, we deduce that the scenario at is stable.

In order to provide an additional verification of the above result, we apply the procedure of [38–40]. We introduce two variables, namely \(x_1 = a\) and \(x_2 = \dot{a}\), and hence the linear perturbations of the Friedmann equation (9), around the critical point (12) and with \(w = -1\), leads to

\[
\dot{x}_1 = x_2 \equiv O_1(x_1, x_2),
\]

\[
\dot{x}_2 = \kappa \rho s x_1 - 2\sqrt{2\kappa \rho s} x_1^{3/2} \equiv O_2(x_1, x_2).
\]

Hence, the eigenvalues square \(\lambda^2\) of the Jacobian matrix

\[
J\left( O_1(x_1, x_2), O_2(x_1, x_2) \right) = \begin{pmatrix}
\frac{\partial O_1}{\partial x_1} & \frac{\partial O_1}{\partial x_2} \\
\frac{\partial O_2}{\partial x_1} & \frac{\partial O_2}{\partial x_2}
\end{pmatrix},
\]

calculated at the critical point (12), is just

\[
\lambda^2 = -2\kappa \rho s.
\]

As we observe, the above eigenvalues square is negative for all physical cases \((\rho s > 0)\), independently of the value of \(\alpha\). Thus, the Einstein static universe in the flat geometry is always
stable, as we also found through (27). Nevertheless, as we mentioned above, we note that the presence of a non-zero $\alpha$ still has the crucial effect to make this critical point non-trivial.

In order to see the above effect more transparently, in the upper graph of figure 1 we present the phase-space behavior for the spatially flat cosmology with equation-of-state parameter $w = -1$, where the stable Einstein static universe critical point is clear (the physical critical point is the one with $x_1 > 0$). Moreover, in order to verify the stability of the Einstein static universe solution in an alternative way, in the lower graph of figure 1 we depict the evolution of the scale factor after a small perturbation around this solution. As can be seen, the universe exhibits small oscillations around the Einstein static universe, without deviating from it, as expected.

In summary, as we can see from the simple case of flat geometry, the effect of the quantum gravity deformation parameter that arise from the generalized uncertainty principle is twofold: Firstly, it leads to a non-trivial Einstein static universe solution that is absent in standard cosmological models, and secondly it leads to its stabilization. Note that although in some emergent universe scenarios quantum effects are responsible for destabilization [41], the present incorporation of quantum effects through natural UV cutoffs is the cause of stabilization. This is one of the main results of the present work, and will become more transparent in the more interesting solutions in the case of non-flat geometry.

4.2. Non-flat universe ($k \neq 0$)

In the case of a non-flat universe, (23) leads to

$$\left(\frac{\delta \rho}{\delta a}\right) = \left(\frac{4\sqrt{2}\kappa a_i^2 \rho_i^{3/2}}{\kappa \rho_s} - \frac{3\kappa a_i^2 \rho_i^{1/2}}{\sqrt{2\kappa}} - \frac{3\sqrt{2}\kappa a_i^2 \rho_i^{3/2}}{\kappa \rho_s}\right).$$

(32)

As expected, in the limit $\alpha \to 0$ we re-obtain the standard result, namely $\left(\frac{\delta \rho}{\delta a}\right) = -\frac{2\kappa c^2}{a^2 \kappa \rho_s}$.

Replacing (32) into (24) and neglecting terms higher than $O(\alpha^2)$, we acquire

$$\delta \ddot{a} + \gamma_{nf} \delta a = 0,$$

(33)

with

$$\gamma_{nf} = \left[\frac{\kappa \rho_s (f_1 + f_2) + f_3 + f_4 + f_5 + f_6 + f_7}{f_8}\right],$$

(34)

and

$$f_1 = 4\sqrt{2}\kappa a_i^2 \rho_i^{3/2},$$

$$f_2 = \frac{12\kappa a_i^2}{\sqrt{8\kappa}} a_i^{-1/2},$$

$$f_3 = 2\sqrt{2\kappa} \kappa a_i (3w + 1) \rho_i^{5/2},$$

$$f_4 = -6\sqrt{2\kappa} a_i \rho_i^{1/2},$$

$$f_5 = -6\kappa a_i \rho_s^{1/2},$$

$$f_6 = \frac{6\kappa a_i \rho_s^{1/2}(2 - w)}{\sqrt{2\kappa} a_i^2},$$

$$f_7 = -kc^2 \kappa (3w + 1) \rho_i a_i^{-2},$$

$$f_8 = -\frac{3}{4} f_1 + \frac{3\kappa c^2}{\sqrt{2\kappa}} \rho_i^{1/2} + \kappa \rho_s.$$
Figure 1. The phase diagram in \((a, \dot{a})\) or \((x_1, x_2)\) space (upper graph) and the evolution of the scale factor (lower graph), for the spatially flat cosmology, with equation-of-state parameter \(w = -1\), for the choice \(\alpha = 1\), in units where \(c = \ell_{\text{pl}} = 1\), \(\kappa = 1/3\). The value of \(\rho_s\) has been chosen as \(\rho_s = 10^2\), in order to be consistent with the condition \(\rho \gg \frac{|k|^2}{\kappa}\).
and where \(a_i\) and \(\rho_i\) have to be replaced from (13)–(16) for the four Einstein static universes respectively. Equation (33) is the perturbation equation of FRW cosmology in the case of generalized uncertainty principle. As expected, in the limit \(\alpha \to 0\) it reduces to the standard result, namely

\[
\delta \dot{a} - \frac{k c^2}{a^2} (3w + 1) \delta a = 0.
\]

Let us now examine whether we can obtain \(\gamma_{nf} > 0\). The form of \(\gamma_{nf}\) given in (34), calculated at the non-trivial critical points (14)–(16), is too complicated to accept any analytical treatment, and thus we will examine the value of \(\gamma_{nf}\) numerically. In figure 2 we present \(\gamma_{nf}\) versus \(\alpha\) and \(w\), for the case of critical point 2 of (14), in the case of open geometry (since for the open geometry this point exists for the more physically interesting \(\omega\)-interval, namely \(-\frac{1}{4} < w < 2\)). As we can see, for the regions of its existence, we obtain \(\gamma_{nf} > 0\) if \(\alpha\) acquires positive values. Hence, the scenario at hand can be stable. Similarly, in figure 3 we present \(\gamma_{nf}\) versus \(\alpha\) and \(w\), for the critical point 3 of (15), in the case of closed geometry (where \(-\frac{1}{4} < w < 2\)). As we observe, this point can be stable for suitable choices of \(\alpha\) and \(w\). Finally, in figure 4 we present \(\gamma_{nf}\) versus \(\alpha\) and \(w\), for the critical point 4 of (16), for a part of the range of its existence, namely for \(-10 - 2\sqrt{22} < w < -10 + 2\sqrt{22}\), in the case of closed and open geometry. Similarly to the previous critical points, we can see that for suitable values of \(\alpha\) and \(w\) this point, which is the most interesting one concerning the successful realization of the emergent universe scenario, is stable.

Let us verify the above results using the approach of [38–40], and express them in a more transparent way. We introduce the two variables \(x_1 = a\) and \(x_2 = \dot{a}\), and therefore the linear perturbations of the Friedmann equation (9) around the critical points (13)–(16) leads to

\[
\dot{x}_1 = x_2 \equiv O_1(x_1, x_2),
\]

\[
\dot{x}_2 = \left[ \frac{3k c^2}{2\sqrt{2\kappa}} (2 - w) \alpha - \frac{k}{2} (1 + 3w) \rho \right] x_1
- 2\sqrt{2\kappa} \rho_0 \sqrt{x_1}^3 + \frac{3k c^2 \alpha}{\sqrt{2\kappa^3} x_1} \equiv O_2(x_1, x_2).
\]

The eigenvalues square of the Jacobian matrix (30), for the above \(O_1(x_1, x_2), O_2(x_1, x_2)\), calculated at the non-trivial critical points 2, 3 and 4, read as follows.

For the critical points 2 and 3 we have

\[
\lambda^2 = -\frac{9\alpha k c^2}{16\kappa} \left[ \frac{(2 - w)^2}{1 + 3w} \right] \left[ 2f(w)^2 \pm f(w) \right] \\
+ \frac{2\kappa^4}{9\alpha^2 k c^2} \left[ \frac{1 + 3w}{2 - w} \right] f(w)^{-3},
\]

with \(f(w) \equiv -1 \pm \sqrt{1 + \frac{32}{3} \left[ \frac{1 + 3w}{2 - w} \right]}\), where the plus sign corresponds to critical point 2 and the minus sign to critical point 3. As we can see, for the range where they are physical, namely for \(k = +1, w > 2\) or \(k = -1, -\frac{1}{3} < w < 2\) for critical point 2, and for \(k = +1, -\frac{1}{3} < w < 2\) or \(k = -1, w > 2\) for critical point 3, we always get \(\lambda^2 < 0\), and thus both points are always stable.
Figure 2. The coefficient of the perturbation equation $\gamma_{nf}$ versus the quantum gravity deformation parameter $\alpha$ and the equation-of-state parameter $w$, for the case of critical point 2 of (14), in the case of open geometry, in units where $c = \ell_{pl} = 1$, $\kappa = 1/3$.

Figure 3. The coefficient of the perturbation equation $\gamma_{nf}$ versus the quantum gravity deformation parameter $\alpha$ and the equation-of-state parameter $w$, for the case of critical point 3 of (15), in the case of closed geometry, in units where $c = \ell_{pl} = 1$, $\kappa = 1/3$. 
Figure 4. The coefficient of the perturbation equation $\gamma_{nf}$ versus the quantum gravity deformation parameter $\alpha$ and the equation-of-state parameter $w$, for the case of critical point 4 of (16), in the case of closed (upper graph) and open (lower graph) geometry, in units where $c = \ell_{pl} = 1, \kappa = 1/3$. 
Let us now examine the stability of the critical point 4 given in (16). The corresponding eigenvalues square of the Jacobian matrix is found to be

$$\lambda^2 = -\frac{9\alpha^2}{2\sqrt{2}\kappa^2 q_1 \sinh^2\left(\frac{q_2}{3}\right)} + \frac{3kc^2(w - 2)}{16} \sqrt{\frac{q_1}{3}} \sinh\left(\frac{q_2}{3}\right)$$

$$-\frac{\kappa^2 q_1}{12\alpha^2} (1 + 3w) \sinh^2\left(\frac{q_2}{3}\right).$$

(40)

Hence, we deduce that the sign of $\lambda^2$ depends on both $\alpha$ and $w$ in a complicated way that does not allow for an analytical treatment, and thus in order to examine its behavior we will resort to numerical elaboration. In figure 5 we depict $\lambda^2$ versus $w$ for various values of $\alpha$, namely $\alpha = 0.88$ (solid curve) $\alpha = 0.92$ (dotted curve) $\alpha = 0.96$ (dashed curve) $\alpha = 1$ (dashed-dot curve), in units where $c = \ell_{\text{Pl}} = 1, \kappa = 1/3$.

Figure 5. The eigenvalues square $\lambda^2$ versus the equation-of-state parameter $w$ for the critical point 4 given in (16), for closed (upper graph) and open (lower graph) geometry, for various values of $\alpha$, namely $\alpha = 0.88$ (solid curve) $\alpha = 0.92$ (dotted curve) $\alpha = 0.96$ (dashed curve) $\alpha = 1$ (dashed-dot curve), in units where $c = \ell_{\text{Pl}} = 1, \kappa = 1/3$.  

Let us now examine the stability of the critical point 4 given in (16). The corresponding eigenvalues square of the Jacobian matrix is found to be

$$\lambda^2 = -\frac{9\alpha^2}{2\sqrt{2}\kappa^2 q_1 \sinh^2\left(\frac{q_2}{3}\right)} + \frac{3kc^2(w - 2)}{16} \sqrt{\frac{q_1}{3}} \sinh\left(\frac{q_2}{3}\right)$$

$$-\frac{\kappa^2 q_1}{12\alpha^2} (1 + 3w) \sinh^2\left(\frac{q_2}{3}\right).$$

(40)

Hence, we deduce that the sign of $\lambda^2$ depends on both $\alpha$ and $w$ in a complicated way that does not allow for an analytical treatment, and thus in order to examine its behavior we will resort to numerical elaboration. In figure 5 we depict $\lambda^2$ versus $w$ for various values of $\alpha$, for the case of closed and open geometry (as we mentioned earlier we consider the realistic values of $\tilde{\alpha}$, and thus of $\alpha$ in the units we use, to be around 1 in order for minimal length effects to be
important only around the Planck length and not introduce a new physical scale between the Planck and the electroweak scale [25–27]).

The critical point 4 exhibits a very interesting behavior concerning the realization of the exit from the static universe and of the emergent universe scenario. In particular, for both spatial geometries we can have a stable Einstein static universe (critical point 2 for \( k = -1 \) or critical point 3 for \( k = +1 \)) for very long time intervals (infinite in the past if \( w \) approaches 0 in the past). As time passes and the universe equation-of-state parameter decreases these critical points become unstable and are exchanged with their unstable counterpart critical point 4, offering a natural graceful exit from the Einstein static universe and an entering into the usual expanding thermal history (this procedure is more efficient for the closed geometry, since critical point 4 is always unstable for suitable values of \( \alpha \)). This behavior is also achieved in complicated models of the emergent universe in some various modified gravities [38], however in the present scenario it is obtained solely from the quantum gravity deformation parameter \( \alpha \).

Hence, from the above we can deduce the central role of the quantum gravity deformation parameter that arises from the generalized uncertainty principle: It leads to non-trivial Einstein static universe solutions that are absent in standard cosmological models, and it provides a mechanism for a successful exit from a stable Einstein static universe into the expanding thermal history, i.e. for a complete realization of the emergent universe scenario. This is one of the main results of the present work.

5. Discussions and conclusions

In the present work we investigated the realization of the emergent universe scenario in the framework of theories with natural UV cutoffs. In particular, we considered the generalized uncertainty principle, which includes a deformation parameter \( \alpha \) that arises from quantum gravity modifications corresponding to two natural cutoffs, namely a minimum length and a maximum momentum. Applying it in a cosmological framework we obtained the modified Friedmann equations and we studied them in detail to see whether we can acquire Einstein static universe solutions, which are the basic concept in the realization of the emergent universe scenario.

As a first step we extracted the Einstein static universe solutions, analyzing for convenience the flat and non-flat cases separately. Then, performing a dynamical analysis in the phase space we examined the dynamical stability of these solutions. As we showed, the role of the new deformation parameter \( \alpha \) is crucial in a twofold way. Firstly, it leads to the appearance of new Einstein static universe critical points, that are absent in standard cosmology, and this is true also in the flat case where standard cosmology does not accept an Einstein static universe.

Secondly, the deformation parameter \( \alpha \) plays a central role in providing a mechanism for a graceful exit from a stable Einstein static universe into the expanding thermal history, i.e. for a complete and successful realization of the emergent universe scenario. This double role of \( \alpha \), i.e. of the quantum gravity modifications arising from the natural UV cutoffs, is one of the main results of the present work.

In summary, we conclude that the emergent universe scenario can be successfully realized in the framework of cosmology with generalized uncertainty principle arising from the presence of natural UV cutoffs.
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