ASYMPTOTICS OF MAXWELL TIME IN THE PLATE-BALL PROBLEM

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Abstract. The problem on rolling of a sphere on a plane without slipping or twisting is considered. One should roll the sphere from one contact configuration to another so that the length of the curve traced by the contact point in the plane is the shortest possible. The asymptotics of Maxwell time for rolling of the sphere along small amplitude sinusoids is studied. A two-sided estimate for this asymptotics is obtained.

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1. Introduction

For the problem on rolling of a sphere on a plane without slipping or twisting, an optimal control problem is studied. The state of the system is described by the contact point of the sphere and the plane and orientation of the sphere in three-dimensional space. One should roll the sphere from one contact configuration to another so that the length of the curve traced by the contact point in the plane is the shortest possible. The problem has applications in robotics: rotation of a solid body in the robot’s hand. In this work we obtain a two-sided estimate of Maxwell time in the plate-ball problem in the asymptotic case.

The problem was stated in [6] by Hammersley. Then Arthur and Walsh [3] proved integrability of Hamiltonian system of PMP in elliptic functions. Jurdjevic in [7, 8] showed that projections of extremal curves (x(t), y(t)) are Euler elasticae (see [4, 9]). He gave a description of different qualitative types of extremal trajectories and obtained differential equations for evolution of Euler angles along extremal trajectories. Explicit formulas for the extremals were obtained in [11].

Optimality of extremals is still an open problem nowadays. Short arcs of extremal trajectories are optimal, but long arcs, in general, are not optimal. The point at which an extremal trajectory loses global optimality is called a cut point. A cut point is a conjugate point or a Maxwell point. A Maxwell point is a point in the state space, where an extremal trajectory crosses another one with the same value of the cost functional. Yu. Sachkov began to study cut points in the plate-ball problem (see [17]). He found continuous and discrete symmetries of the exponential mapping.

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Then he obtained equations that define Maxwell points as fixed points of the discrete symmetries, and formulated necessary optimality conditions in terms of Maxwell time (see Theorems 1 and 2 in Sec. 2). But the problem of optimality of extremal trajectories is still open because the equations that define the Maxwell points are not solved.

The papers [10, 11] present the asymptotics of extremal trajectories in a neighborhood of the stable equilibrium of a mathematical pendulum (see (2.6)), which appears in the adjoint subsystem of the Hamiltonian system of the maximum principle. In this case, the extremal curves on the plane are close to sinusoids of small amplitude.

This work continues to study the problem of optimality of extremal trajectories. It studies the problem in the asymptotic case, where the formulas defining the extremal trajectories and the Maxwell points are expressed via trigonometric functions. They are simpler than in the general case, where the formulas are expressed in elliptic functions. We study the behavior of Maxwell points MAX\(^1\) and MAX\(^2\) in this case and obtain two-sided estimates for the first Maxwell times \(t_1\) and \(t_2\), which correspond to the fixed points of the discrete symmetries \(\varepsilon^1\) and \(\varepsilon^2\) of the exponential mapping.

2. Statement of the Problem and Known Results

In this section, we formulate the plate-ball problem and recall some known results. Let \((x, y)\in\mathbb{R}^2\) be the contact point of the sphere and the plane. By \(q = (q_0, q_1, q_2, q_3)\in S^3\) denote the unit quaternion (see [14]) representing the rotation of three-dimensional space, which translates the current orientation of the sphere to the initial orientation. The problem on optimal rolling of a unit sphere on a plane is stated as follows:

\[
\dot{Q} = u_1 X_1(Q) + u_2 X_2(Q), \quad (2.1)
\]

\[
X_1(Q) = (1, 0, q_2, q_3, -q_0, -q_1)^T, \quad X_2(Q) = (0, 1, -q_1, q_0, q_3, -q_2)^T, \quad (2.2)
\]

\[
Q = (x, y, q_0, q_1, q_2, q_3) \in M = \mathbb{R}^2 \times S^3, \quad u = (u_1, u_2) \in \mathbb{R}^2, \quad (2.3)
\]

\[
Q(0) = Q_0 = (0, 0, 1, 0, 0, 0), \quad Q(t_1) = Q_1, \quad (2.4)
\]

\[
l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} \, dt \to \min. \quad (2.5)
\]

Admissible controls are measurable and essentially bounded. Admissible trajectories are Lipschitz. Problem (2.1)–(2.5) is a left-invariant sub-Riemannian problem on the Lie group \(M = \mathbb{R}^2 \times S^3\). The control system is completely controllable by the Rashevsky—Chow theorem (see the paper [2]). Existence of optimal controls follows from Filippov’s theorem (see [2]). The maximum principle is applied to study the optimal controls. In the abnormal case, the sphere rolls along a straight line in the plane \((x, y)\). In the normal case, the subsystem of the Hamiltonian system for the adjoint variables \((\theta, c, r, \alpha)\) satisfies the equations of a mathematical pendulum as follows:

\[
\dot{\theta} = c, \quad \dot{c} = -r \sin \theta, \quad \dot{\alpha} = \dot{r} = 0. \quad (2.6)
\]

Projections of extremal trajectories to the plane \((x, y)\) are Euler elasticae, i.e., stationary configurations of an elastic rod on a plane with fixed end points and fixed tangents at these points (see the paper [7]).

In the paper [17] the author describes continuous and discrete symmetries of the exponential mapping in the plate-ball problem

\[
\text{Exp}: (\lambda, t) \mapsto Q_t, \quad (\lambda, t) \in N = C \times \mathbb{R}_+, \quad Q_t \in M = \mathbb{R}^2 \times SO(3),
\]

\[
C = \{ \lambda \in T^*_0 M \mid H(\lambda) = 1/2 \} = \{ (\theta, c, \alpha, r) \mid \theta \in S^1, \quad c \in \mathbb{R}, \quad r \geq 0, \quad \alpha \in S^1 \}. \]