Quantum (Matrix) Geometry and Quasi-Coherent States

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Abstract

A general framework is described which associates geometrical structures to any set of $D$ finite-dimensional hermitian matrices $X^a$, $a = 1, \ldots, D$. This framework generalizes and systematizes the well-known examples of fuzzy spaces, and allows to extract the underlying classical space without requiring the limit of large matrices or representation theory. The approach is based on the previously introduced concept of quasi-coherent states. In particular, a concept of quantum Kähler geometry arises naturally, which includes the well-known quantized coadjoint orbits such as the fuzzy sphere $S_N^2$ and fuzzy $\mathbb{C}P_N^n$. A quantization map for quantum Kähler geometries is established. Some examples of quantum geometries which are not Kähler are identified, including the minimal fuzzy torus.

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1 Introduction

It is expected on general grounds that the classical description of space-time geometry is modified at very short length scales through quantum effects. An interesting approach towards quantum geometry is based on quantized symplectic spaces, whose structure is similar to quantum mechanical phase space. Many examples of this type have been studied, starting with the fuzzy sphere $S^2_N$ [1][2], the fuzzy torus $T^2_N$ and more elaborate 2-dimensional spaces [3], self-intersecting spaces such as squashed $\mathbb{CP}^2$ [4], and many more. A general class class is provided by quantized coadjoint orbits of compact semi-simple Lie groups. Many classical features of the underlying symplectic space are encoded in their quantized version, which is based on the algebra of matrices $\text{End}(\mathcal{H})$ acting on a finite-dimensional Hilbert space $\mathcal{H}$.

Of course, the notion of an algebra is not sufficient to define a geometry, which should also contain a metric structure. This extra structure arises in the context of Yang-Mills matrix models such as the IIB or IKKT model [5], which define a gauge theory on such fuzzy spaces. In this context, a fuzzy space is specified by a set of hermitian matrices $X^a$ for $a = 1, ..., D$. These matrices not only generate the algebra of “functions” $\text{End}(\mathcal{H})$, but also naturally define a matrix Laplacian $\Box = \delta^{ab}[X_a, [X_b, .]]$, and a Dirac-type operator $\slashed{D} = \Gamma_a [X^a, .]$ where $\Gamma_a$ are suitable
Clifford or Gamma matrices. However rather than focusing on the spectral geometry as in [7], we will emphasize a more direct approach based on (quasi-) coherent states defined through the matrices $X^a$, which provide a direct access to an underlying space $\mathcal{M}$.

The obvious question is how to recover or extract the classical geometry underlying these quantized or "fuzzy" spaces, defined by the matrices $X^a$. For special cases such as quantized coadjoint orbits, one can construct a sequence of similar matrices $X^a(N) \in \text{End}(\mathcal{H}_N)$, and show that the commutative description is recovered in the limit $N \to \infty$. This has led to the attitude that the geometrical content of fuzzy spaces can only be obtained in some semi-classical limit $N \to \infty$. However, such a limit is not satisfactory from a physics point of view, where one would like to attach geometrical meaning to a given set of matrices $X^a$. In particular, this is required to interpret numerical simulations of Yang-Mills matrix models [8, 9, 10, 11], which are viewed as candidates for a quantum theory of space-time and matter.

The purpose of the present paper is to establish a natural framework of "quantum geometry", which can be associated to any given set of $D$ hermitian matrices without requiring any limit, and which may admit a semi-classical or almost-local description in some regime. This is based on the previously introduced concept of quasi-coherent states [12, 13], which can be associated to any set of hermitian matrices. The concept is very much string-inspired [14], and the quantum geometries are naturally viewed as varieties or "branes" embedded in target space. Since the mathematical concepts are very close to those of quantum mechanics, the name "quantum geometry" seems justified, even if that name is perhaps already over-loaded with different meanings in the literature.

The framework nicely captures the standard examples of fuzzy spaces, but it is completely general. Moreover, it naturally leads to an intrinsic concept of quantum Kähler geometry, which is a special class of quantum geometries which satisfy certain conditions. There is no need to add any structure by hand. Of course for quantized coadjoint orbits, the coherent states are obtained easily from the representation theory. However, the present construction allows to reconstruct the full Kähler structure of the (quantum) space without resorting to representation theory, which is not known in general.

In the semi-classical limit, many of the structures and steps have been considered before, notably in work by Ishiki et al [12, 15] and in [13, 14, 16, 17]. However, the novelty is in introducing a more abstract point of view. We introduce the concept of an abstract quantum space $\mathcal{M}$, by considering the space of quasi-coherent states as a sub-variety of $CP^N$. This allows to make concise statements for finite $N$, and to give a clear conceptual correspondence between finite matrix configurations and geometry, based on a space $\text{Loc}(\mathcal{H}) \subset \text{End}(\mathcal{H})$ of almost-local operators. The semi-classical description applies in some infrared (IR) regime, while the UV regime of matrix geometry displays a very different and stringy nature, which is manifest in string states. This framework also allows to establish the existence of a surjective quantization map for quantum Kähler manifolds, and to make some non-trivial regularity statements about the abstract quantum space $\mathcal{M}$.

It is important to note that the proposed framework is more than just some ad-hoc procedure: by definition, the quasi-coherent states provide an optimal basis where all matrices have minimal joint uncertainty, i.e. they are simultaneously "almost-diagonal". Such almost-commuting configurations are expected to play a dominant role in Yang-Mills matrix models. The approach is well-suited to be implemented on a computer [18, 19], and should provide a powerful tool to understand and interpret the results of numerical simulations of Yang-Mills matrix models.

This paper comprises 3 main parts. In section 2 we define the quasi-coherent states $\ket{x}$ for $x \in \mathbb{R}^D$, and study their properties as functions of $x \in \mathbb{R}^D$. Much of this section is more-or-less

\footnote{see e.g. [6] for related work in that context.}

\footnote{This is in distinct from the approach in [15].}
known in some form, but at least the relation with solutions of the matrix-Yang-Mills equation is new. In section 3, we introduce the central concept of an abstract quantum space $M \subset \mathbb{CP}^N$. This offers a conceptually clear definition of almost-local operators and the semi-classical regime. It also leads to a natural concept of a real and complex quantum tangent space and quantum Kähler manifolds. Some consequences are developed in section 4, notably a quantization map for quantum Kähler manifolds. These concepts are illustrated in a number of examples in section 6, and the relation with physical Yang-Mills matrix models is briefly discussed in section 5.

2 Quasi-coherent states on $\mathbb{R}^D$

In this paper, a matrix configuration will be a collection of $D$ hermitian matrices $X^a \in \text{End}(H)$ acting on some (separable) Hilbert space $H$. To avoid technical complications, we will assume that $H \cong \mathbb{C}^N$ is finite-dimensional, apart from some illustrative infinite-dimensional examples. Such a matrix configuration will be called irreducible if the only matrix which commutes with all $X^a$ is the unit matrix. Equivalently, the algebra generated by the $X^a$ is the full matrix algebra $\text{End}(H)$. This will be assumed throughout.

By definition, such an irreducible matrix configuration does not admit any common eigenvectors $|\psi\rangle$, since otherwise $|\psi\rangle\langle\psi|$ would commute with all $X^a$. Nevertheless, we are mainly interested in matrix configurations which are "almost-commuting", in the sense that the commutators $[X^a, X^b]$ are "small"; these are expected to be the dominant configurations in Yang–Mills matrix models such as the IIB or IKKT model \cite{5}. We therefore wish to find a set of states which are optimally adapted to the matrix configuration, so that the $X^a$ are "as diagonal as possible". This may also be of interest in different contexts.

With this in mind, we associate to an irreducible matrix configuration $X^a$ and a point $x \in \mathbb{R}^D$ the following displacement Hamiltonian\footnote{As explained in section 5.2, $H_x$ can be interpreted in the IIB model as energy of a point–brane at $x$ on the background defined by the matrix configuration $X^a$.} (cf. \cite{12, 13})

$$H_x := \frac{1}{2} \sum_{a=1}^{D} (X^a - x^a \mathbb{1})^2.$$  \hspace{1cm} (1)

This is a positive definite\footnote{To see positive-definiteness, assume that $H_x|\psi\rangle = 0$; this implies $X^a|\psi\rangle = x^a|\psi\rangle$ for all $a$, but then $[H_x, |\psi\rangle\langle\psi|] = 0$ in contradiction with irreducibility.} hermitian operator on $H$, which can be thought of as an analog to the shifted harmonic oscillator. It allows to find optimally localized approximate eigenstates for the given matrix configuration as follows. Let $\lambda(x) > 0$ be the lowest eigenvalue of $H_x$. A quasi-coherent state $|x\rangle$ at $x$ is then defined following \cite{12, 13} as normalized vector $\langle x|x \rangle = 1$ in the eigenspace $E_x$ of $H_x$ with eigenvalue $\lambda(x)$,

$$H_x|x\rangle = \lambda(x)|x\rangle.$$  \hspace{1cm} (2)

We will assume for simplicity that $E_x$ is one-dimensional, except possibly on some singular set $\mathcal{K} \subset \mathbb{R}^D$. Clearly the quasi-coherent states $|x\rangle$ form a $U(1)$ bundle

$$\mathcal{B} \rightarrow \tilde{\mathbb{R}}^D \quad \text{over} \quad \tilde{\mathbb{R}}^D := \mathbb{R}^D \setminus \mathcal{K}.$$  \hspace{1cm} (3)

Standard theorems \cite{20, 21} ensure that $\lambda(x)$ and $E_x$ depend smoothly on $x \in \tilde{\mathbb{R}}^D$. We can then choose some local section of $\mathcal{B}$ near any given point $\xi \in \tilde{\mathbb{R}}^D$, denoted by $|\xi\rangle$. Thus $\mathcal{K}$ is the location where different eigenvalues of $H_x$ become degenerate. If $\lambda(x)$ can be extended smoothly...
at some point \( p \in \mathcal{K} \), different eigenvalues simply touch without crossing, and the sections \(|x\rangle\) and the bundle \( \mathcal{B} \) can be extended through \( p \); we can then basically remove \( p \) from \( \mathcal{K} \). Hence we can assume that \( \mathcal{K} \) contains only points where some eigenvalues cross, i.e. \( \lambda(x) \) cannot be continued. We denote this \( \mathcal{K} \) as \textbf{singular set}. The bundle is non-trivial if \( \mathcal{K} \neq 0 \).

For any operator in \( \Phi \in \text{End}(\mathcal{H}) \), we can define the \textbf{symbol} in \( \mathcal{C}(\tilde{\mathbb{R}}^D) \) through the map

\[
\text{End}(\mathcal{H}) \to \mathcal{C}(\tilde{\mathbb{R}}^D) \\
\Phi \mapsto \langle x|\Phi|x \rangle =: \phi(x).
\]

Elements of \( \text{End}(\mathcal{H}) \) will be indicated by upper-case letters, and functions by lower-case letters. The map \( (4) \) should be viewed as a de-quantization map, associating classical functions to non-commutative “functions” (or rather observables) in \( \text{End}(\mathcal{H}) \). In particular, the symbol of the matrices \( X^a \) provides a map

\[
x^a : \tilde{\mathbb{R}}^D \to \mathbb{R}^D \\
x \mapsto x^a(x) := \langle x | X^a | x \rangle.
\]

Genericaly \( x^a(x) \neq x^a \), and the deviation is measured by the \textbf{displacement}

\[
d^2(x) := \sum_a (x^a(x) - x^a)^2.
\]

The quality of the matrix configuration (or of the underlying quantum space) is measured by the \textbf{dispersion} or uncertainty

\[
\delta^2(x) := \sum_a (\Delta X^a)^2 \\
(\Delta X^a)^2 := \langle x | (X^a - x^a(x))^2 | x \rangle = \langle x | X^a X^a | x \rangle - x^a(x)x^a(x).
\]

If \( \delta^2(x) \) is small, then the \( X^a \) can be interpreted as operators or observables which approximate the functions \( x^a \) on \( \tilde{\mathbb{R}}^D \), and if \( d^2(x) \) is also small then \( X^a \approx x^a \). Note that \( (2) \) implies

\[
\lambda(x) = \delta^2(x) + d^2(x),
\]

hence a small \( \lambda(x) \) implies that both \( \delta^2(x) \) and \( d^2(x) \) are bounded by \( \lambda(x) > 0 \). \( d^2(x) \) will be understood in section \( 3 \) as displacement of \( x \) from the underlying quantum space or brane \( \mathcal{M} \). Hence quasi-coherent states should be viewed as the states with minimal dispersion and displacement for given \( x \in \tilde{\mathbb{R}}^D \), cf. \[13\] for a more detailed discussion.

\subsection{U(1) connection, would-be symplectic form and quantum metric}

Now we associate to any matrix configuration two unique tensors on \( \tilde{\mathbb{R}}^D \): the \textbf{would-be symplectic form} \( \omega_{ab} \) and \textbf{quantum metric} \( g_{ab} \). Since \( |x\rangle \in \mathcal{H} \), the bundle \( \mathcal{B} \) over \( \tilde{\mathbb{R}}^D \) naturally inherits a metric and a connection. We can define a connection 1-form \( A \) via

\[
P \circ d|x\rangle = |x\rangle iA, \quad iA := \langle x | d|x \rangle \in \Omega^1(\tilde{\mathbb{R}}^D)
\]

where \( P = |x\rangle\langle x| \) is the projector on \( E_x \). Here \( A \) is real because

\[
\langle (x | d|x) \rangle^* = d(\langle x| |x \rangle = -\langle x | d|x \rangle.
\]
and transforms like a $U(1)$ gauge field

$$|x⟩ → e^{iΛ(x)}|x⟩, \quad A_a → A_a + ∂_a Λ.$$  \hspace{1cm} (12)

In particular, we can parallel transport $|x⟩$ along a path $γ$ in $\mathbb{R}^D$. This connection is analogous to a Berry connection. It is encoded in the inner product

$$⟨x|y⟩ = e^{iϕ(x,y)−D(x,y)} ,$$  \hspace{1cm} (13)

which defines a distance function $D(x, y)$ and a phase function $ϕ(x, y)$ which satisfy

$$D(x, y) = D(y, x) ≥ 0, \quad D(x, y) = 0 ⇔ x = y,$$

$$ϕ(x, y) = −ϕ(y, x) .$$  \hspace{1cm} (14)

The phase clearly depends on the particular section $|x⟩$ of the bundle $\mathcal{B}$, while $D(x, y)$ does not. To understand these two functions, we differentiate \([13]\) w.r.t. $y$

$$⟨x|dy|y=x⟩ = id_yϕ(x, y)|y=x − d_yD(x, y)|y=x .$$  \hspace{1cm} (15)

Comparing with \([10]\) we conclude

$$id_yϕ(x, y)|y=x = iA = iA_a dx^a$$
$$d_yD(x, y)|y=x = 0 .$$  \hspace{1cm} (16)

Hence the phase $ϕ(x, y)$ encodes the connection $A$. For a contractible closed path $γ = ∂Ω$ in $\mathbb{R}^D$, the change of the phase of $|x⟩$ along $γ$ is hence given by the field strength via Stokes theorem

$$\oint_γ A = \int_Ω dA .$$  \hspace{1cm} (17)

If the connection is flat, the phase $ϕ(x, y)$ can be gauged away completely.

To proceed, consider the gauge-invariant hermitian $D × D$ matrix defined by

$$h_{ab} = ((∂_a + iA_a)|⟨x⟩(∂_b − iA_b)|x⟩|ξ = h_{ba}^*$$
$$= (∂_b + iA_b)(∂_a − iA_a)e^{iϕ(x,y)−D(x,y)}|ξ$$
$$= \frac{i}{2} (ω_{ab} + g_{ab})$$  \hspace{1cm} (18)

at some reference point $ξ ∈ \mathbb{R}^D$, which decomposes into the real symmetric and antisymmetric tensors $g_{ab}$ and $ω_{ab}$. The symmetric part

$$g_{ab} = ((∂_a + iA_a)|⟨x⟩(∂_b − iA_b)|x⟩ + (a ↔ b)$$
$$= (∂_a⟨x⟩|∂_b|x) − A_a A_b + (a ↔ b)$$  \hspace{1cm} (19)

(using\([10]\)) is the pull-back of the Riemannian metric\(^6\) on $\mathcal{H}$ (or equivalently of the Fubini–Study metric on $\mathbb{C}P^{N−1}$) through the section $|x⟩$. The antisymmetric part of $h_{ab}$ encodes a 2-form

$$iω_{ab} = i(∂_a A_b − ∂_b A_a) = (∂_a⟨x⟩|∂_b|x) − (∂_b⟨x⟩|∂_a|x)$$

$$iω = \frac{i}{2} ω_{ab} dx^a ∧ dx^b = d⟨x⟩ ∧ d|x⟩ = d(⟨x|d|x⟩) = idA$$  \hspace{1cm} (20)

\(^6\)Note that $g_{ab}$ is not related to the Euclidean metric $δ_{ab}$ on target space $\mathbb{R}^D$.  

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which is the \( U(1) \) field strength of the connection \( A \) and therefore closed,

\[
\omega = dA, \quad d\omega = 0. \tag{21}
\]

Assuming (local) translation invariance\(^7\), it follows that the expansion of \( \varphi(x,y) \) to quadratic order in \( x \) and \( y \) (setting \( \xi = 0 \)) is

\[
\varphi(x,y) = A_a(y^a - x^a) - \frac{1}{4} \omega_{ab}(x - y)^a(x - y)^b + \ldots. \tag{22}
\]

Similarly, the expansion of \( D(x,y) \) is given by

\[
D(x,y) = \frac{1}{4}(x - y)^a(x - y)^b g_{ab} + \ldots \tag{23}
\]

since \( D(x,y) \) is gauge invariant and satisfies \( D(x,x) = 0 \) and \( D(x,y) \geq 0 \). In fact viewing \( \mathcal{B}/U(1) \) as subset of \( \mathbb{C}P^{N-1} \), we can use the well-known formula

\[
\cos^2(\gamma(x,y)) = e^{-2D(x,y)} \tag{24}
\]

where \( \gamma(x,y) \) is the geodesic distance squared between \( |x\rangle \) and \( |y\rangle \) in the Fubini–Study metric on \( \mathbb{C}P^{N-1} \). Combining (13) and (23), we learn that the quasi-coherent states are localized within a region of size

\[
L^2_{\text{coh}} = \|g_{ab}\|^{-1} \tag{25}
\]

denoted as coherence scale. The \( |x\rangle \) are approximately constant below this scale due to (13). The relation with the uncertainty of \( X^a \) will be given in (68). Therefore \( g_{ab} \) will be denoted as quantum metric. We will see in section 5.2 that there is a different, effective metric which governs the low-energy physics on such quantum spaces in Yang-Mills matrix models. However, the intrinsic structure of the underlying quantum space is best understood using a more abstract point of view developed in section 3.

We will see that \( \omega \) typically arises from a symplectic form on an underlying space \( \mathcal{M} \). Therefore \( \omega \) will be denoted as would-be symplectic form. Since it is the curvature of a \( U(1) \) bundle, its flux is quantized for every 2-cycle \( S^2 \) in \( \mathbb{R}^D \) as

\[
\int_{S^2} \frac{1}{2\pi} \omega = n, \quad n \in \mathbb{Z}. \tag{26}
\]

This arises using (17) as consistency condition on the \( U(1) \) holonomy for the parallel transport along a closed path \( \gamma \) on \( S^2 \). In more abstract language, \( c_1 = -\frac{1}{2\pi} \omega \) is the first Chern class of \( \mathcal{B} \) viewed as line bundle, which is the pull-back of the first Chern class (or symplectic form) of \( \mathbb{C}P^{N-1} \) via the section \( |x\rangle \). The bundle \( \mathcal{B} \) is trivial if these numbers vanish for all cycles \( S^2 \), hence if \( H^2(\mathbb{R}^D) \) vanishes.

### 2.2 Differential structure of quasi-coherent states

Assume that \( |x\rangle \) is a local section of the quasi-coherent states, with

\[
H_x|x\rangle = \lambda(x)|x\rangle. \tag{27}
\]
Using Cartesian coordinates $x^a$ on $\mathbb{R}^D$, we observe that
\[ \partial_a H_x = -(X_a - x_a \mathbb{1}) . \] (28)

Thus differentiating (27) gives
\[ (H_x - \lambda(x))\partial_a |x\rangle = -\partial_a(H_x - \lambda)|x\rangle = (X_a - x_a + \partial_a \lambda)|x\rangle . \] (29)

Since lhs is orthogonal to $\langle x|$, it follows that
\[ 0 = \langle x|(X_a - x_a + \partial_a \lambda)|x\rangle \]
so that the expectation value or symbol of the basic matrices $X_a$ is given by
\[ X_a = \langle x|X_a|x\rangle = x_a - \partial_a \lambda . \] (31)

Furthermore, (29) gives (in the non-degenerate case under consideration)
\[ \partial_a |x\rangle = |x\rangle \langle x|\partial_a |x\rangle + (H_x - \lambda)^{-1}(X_a - x_a + \partial_a \lambda)|x\rangle \]
\[ (\partial_a - iA_a)|x\rangle = (H_x - \lambda)^{-1}(X_a - x_a + \partial_a \lambda)|x\rangle \] (32)

using (16). Even though the $(H_x - \lambda)^{-1}$ term is well-defined here, it is better to replace $(H_x - \lambda)^{-1}$ with an operator that is well-defined on $\mathcal{H}$. This is achieved using the “reduced resolvent”
\[ (H_x - \lambda(x))^{-1} := (\mathbb{1} - P_x)(H_x - \lambda(x))^{-1}(\mathbb{1} - P_x), \quad P_x := |x\rangle \langle x| \] (33)

which satisfies
\[ (H_x - \lambda)(H_x - \lambda)^{-1} = \mathbb{1} - P_x = (H_x - \lambda)^{-1}(H_x - \lambda), \]
\[ (H_x - \lambda)^{-1}|x\rangle = 0 . \] (34)

Observing $(H_x - \lambda)^{-1}(x_a - \partial_a \lambda)|x\rangle = 0$ due to (27), we can write (32) as
\[ (\partial_a - iA_a)|x\rangle = iX_a|x\rangle \] (35)

for $X_a = -i(H_x - \lambda)^{-1}X_a$. Since $(H_x - \lambda)^{-1}|x\rangle = 0$, this can be replaced by the hermitian generator
\[ X_a := -i(H_x - \lambda)^{-1}X_a = X_a^\dagger . \] (36)

Moreover, we note
\[ \langle x|X_a|x\rangle = 0 . \] (37)

Hence $X_a$ generates the gauge-invariant tangential variations of $|x\rangle$, which take value in the orthogonal complement of $|x\rangle$. This will be the basis for defining the quantum tangent space in section 3. The local section $|x\rangle$ over $\mathbb{R}^D$ can now be written as
\[ |x\rangle = P \exp \left( i \int_\xi^x (X_a + A_a)dx^a \right) |\xi\rangle \] (38)

near the reference point $\xi \in \mathbb{R}^D$. Here $P$ indicates path ordering, which is just a formal way of writing the solution of (35). In a small local neighborhood, the $X_a$ are approximately constant, and $A_a$ can be gauged away. Then (38) can be written as
\[ |x\rangle \approx e^{i(x - y)^aX_a}|y\rangle , \] (39)

which means that the $X_a$ generate the local translations on $\mathcal{M}$. 

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2.3 Relating the algebraic and geometric structures

Since the derivatives of $|x\rangle$ are spanned by the $X^a|x\rangle$, the $U(1)$ field strength $\omega_{ab}$ and the quantum metric $g_{ab}$ should be related to algebraic properties for the $X^a$. Indeed, starting from (18)

$$h_{ab} = \langle x|X_aX_b|x\rangle = \frac{i}{2} (\omega_{ab} + g_{ab}) ,$$

we obtain

$$i\omega_{ab} = h_{ab} - h_{ba} = \langle x|(X_aX_b - X_bX_a)|x\rangle,$$  \hspace{1cm} (41)

and

$$g_{ab} = h_{ab} + h_{ba} = \langle x|(X_aX_b + X_bX_a)|x\rangle .$$  \hspace{1cm} (42)

This provides a first link between the geometric and algebraic structures under consideration. Furthermore, it is useful to define the following hermitian tensor (similar as in [15])

$$P_{ab}(x) := \langle x|X_a(H_x - \lambda)\gamma X_b|x\rangle = P_{ba}(x)^* \quad \text{for } x \in \tilde{\mathbb{R}}^D ,$$

lowering indices with $\delta_{ab}$; for the antisymmetric part see (70). This will be recognized as projector on the embedded quantum space in (92), as obtained in the semi-classical limit in [15].

2.4 Almost-local operators

We would like to define a class $\text{Loc}(\mathcal{H}) \subset \text{End}(\mathcal{H})$ of almost-local operators which satisfy

$$\Phi|x\rangle \approx |x\rangle \langle x|\Phi|x\rangle = P_x \Phi|x\rangle = |x\rangle \phi(x) \quad \forall x \in \tilde{\mathbb{R}}^D$$

where $\phi(x) = \langle x|\Phi|x\rangle$ is the symbol of $\Phi$, and $P_x = |x\rangle\langle x|$ is the projector on the quasi-coherent state $|x\rangle$. The question is how to make the meaning of $\approx$ precise, without considering some limit as in [12]. We should certainly require that $\Phi|x\rangle \approx |x\rangle \phi(x)$ in $\mathcal{H}$ for every $x$, but it is not obvious yet how to handle the dependence on $x$, and how to specify bounds. The guiding idea is that it should make sense to identify $\Phi$ with its symbol

$$\Phi \sim \phi(x) = \langle x|\Phi|x\rangle ,$$

indicated by $\sim$ from now on. This will be made more precise in the section 3.1 by requiring that $\sim$ is an approximate isometry from $\text{Loc}(\mathcal{H})$ to $\mathcal{C}_{IR}(\mathcal{M})$, where $\mathcal{C}_{IR}(\mathcal{M})$ is a class of “infrared” functions on the abstract quantum space associated to the matrix configuration. The essence of almost-locality is then that the integrated deviations from classically are small compared with the classical values. With this in mind, we proceed to elaborate some consequences of (45) for fixed $x$ without specifying bounds.
Since \((1 - P_x)\) is a projector, we have the estimate
\[
\langle x | \Phi \dagger \Phi | x \rangle = \langle x | \Phi \dagger P_x \Phi | x \rangle + \langle x | \Phi \dagger (1 - P_x) \Phi | x \rangle \geq \langle x | \Phi \dagger P_x \Phi | x \rangle = |\phi(x)|^2.
\]
(47)
It follows that every hermitian almost-local operator \(\Phi = \Phi \dagger\) satisfies
\[
\langle x | \Phi \Phi | x \rangle \approx \langle x | \Phi | x \rangle^2 = |\phi(x)|^2 \quad \forall x \in \mathbb{R}^D,
\]
i.e. the uncertainty of \(\Phi\) is negligible,
\[
\langle x | (\Phi - \langle x | \Phi | x \rangle^2) | x \rangle \approx 0 \quad \forall x \in \mathbb{R}^D.
\]
(49)
This means that \((\Phi - \phi(x)) | x \rangle\) is approximately zero, which in turn implies (45). Therefore almost-locality is essentially equivalent to (49), up to global considerations and specific bounds. A more succinct global version of (49) is given in (105).

We also note that for two operators \(\Phi, \Psi \in \text{Loc}(\mathcal{H})\) the factorization properties
\[
\Phi \Psi | x \rangle \approx \Phi | x \rangle | x \rangle | x \rangle \approx | x \rangle \phi(x) \psi(x)
\]
\[
\langle x | \Phi \Psi | x \rangle \approx \phi(x) \langle x | \psi(x)
\]
(50)
follow formally. However this does not mean that \(\text{Loc}(\mathcal{H})\) is an algebra, since the specific bounds may be violated by the product. For some given matrix configuration, \(\text{Loc}(\mathcal{H})\) may be empty or very small. This happens e.g. for the minimal fuzzy spaces as discussed in section 6.3, and it is expected for random matrix configuration. But even in these cases, the associated geometrical structures still provide useful insights.

For interesting quantum geometries, we expect that all the \(X^a\) are almost-local, hence also polynomials \(P_n(X)\) up to some maximal degree \(n\) due to (50). \(\text{Loc}(\mathcal{H})\) can often be characterized by some bound on the eigenvalue of \(\Box\) (145), or the uncertainty scale \(L_{NC}\) (69). However, \(\text{Loc}(\mathcal{H})\) can never be more than a small subset of \(\text{End}(\mathcal{H})\).

### 2.5 Almost-local quantum spaces and Poisson tensor

To see how the Poisson structure arises, define the real anti-symmetric matrix-valued function
\[
\theta^{ab} := -i \langle x | [X^a, X^b] | x \rangle = -\theta^{ba}
\]
(51)
on \(\mathbb{R}^D\). To relate it to the previous structures, we shall loosely follow [15], starting from the identity
\[
[X^a, X^b](X_b - x_b) + (X_b - x_b)[X^a, X^b] = 2[X^a, H_x].
\]
(52)
Taking the expectation value, we obtain
\[
\langle x | [X^a, X^b](X_b - x_b) | x \rangle + \langle x | (X_b - x_b)[X^a, X^b] | x \rangle = 2 \langle x | [X^a, H_x] | x \rangle = 0.
\]
(53)
If \(X^a\) is almost-local\(^8\) then this implies
\[
0 \approx \langle x | [X^a, X^b] | x \rangle \langle x | (X_b - x_b) | x \rangle = -i \theta^{ab} \partial_b \lambda
\]
(54)
using (31). In section 3.1 we will see that this implies \(\lambda \sim \text{const}\) on the embedded quantum space \(\mathcal{M}\), and \(P_{ac} + P_{ca} \sim \partial_c x_a\) is its tangential projector.

\(^8\)This is expected from the definition of quasi-coherent states, as long as the uncertainty is sufficiently small.
We now define an almost-local quantum space to be a matrix configuration where all $X^a$ as well as all $[X^a, X^b]$ are almost-local operators. Then they approximately commute, and we can proceed following [15]

$$-2(H_x - \lambda)(X^a - x^a + \partial^a \lambda)|x\rangle = 2[X^a, H_x]|x\rangle \approx 2(X_b - x_b)[X^a, X^b]|x\rangle$$

$$\approx 2(X_b - x_b)|x\rangle[x][X^a, X^b]|x\rangle = 2i(X_b - x_b)|x\rangle \theta^{ab}$$

$$\approx 2i(X_b - x_b + \partial_b \lambda)|x\rangle \theta^{ab} .$$

(55)

using the factorization property, [54] and [27]. However the first approximation is subtle, since $< X_b - x_b | x > \approx 0$. This can be justified if $X^a$ is a solution of the Yang-Mills equations\(^9\)

$$[X_b, [X^b, X^a]] = 0$$

(56)

which are indeed the equations of motion for Yang-Mills matrix models [3]. Then (52) implies

$$[X^a, H_x] = (X_b - x_b)[X^a, X^b]$$

(57)

and the above steps become

$$-2(H_x - \lambda)X^a|x\rangle = 2[X^a, H_x]|x\rangle = 2(X_b - x_b)[X^a, X^b]|x\rangle$$

$$\approx 2(X_b - x_b)|x\rangle \theta^{ab}$$

$$\approx 2i(X_b - x_b + \partial_b \lambda)|x\rangle \theta^{ab} .$$

(58)

The rhs is indeed orthogonal to $|x\rangle$ due to (31), and we can conclude

$$- (X^a - x^a + \partial^a \lambda)|x\rangle \approx i(H_x - \lambda)^{-1}(X_b - x_b + \partial_b \lambda)|x\rangle \theta^{ab}$$

$$= -\theta^{ab}X_b|x\rangle = i\theta^{ab}(\partial_b - iA_b)|x\rangle$$

(59)

hence

$$(X^a - x^a + \partial^a \lambda)|x\rangle \approx -i\theta^{ab}(\partial_b - iA_b)|x\rangle$$

(60)

and by conjugating

$$|x\rangle(X^d - x^d + \partial^d \lambda) \approx i\theta^{dc}(\partial_c + iA_c)|x\rangle .$$

(61)

These relations are very useful. First, they imply the important relation

$$[X^a, |x\rangle\langle x|] \approx -i\theta^{ab}\partial_b(|x\rangle\langle x|) .$$

(62)

Furthermore, multiplying (60) with $(\partial_c + iA_c)|x\rangle$ gives

$$-i\theta^{ab}((\partial_c + iA_c)|x\rangle(\partial_b - iA_b)|x\rangle \approx -i|x\rangle[A_c, X^a - x^a + \partial^a \lambda]|x\rangle = -i|x\rangle[A_c, X^a]|x\rangle = P_c^a ,$$

(63)

and similarly from (61)

$$i\theta^{ac}((\partial_c + iA_c)|x\rangle(\partial_b - iA_b)|x\rangle \approx i|x\rangle[X^b, A_c]|x\rangle = P_b^a .$$

(64)

\(^9\)This argument also goes through for the generalized Yang-Mills equation $\Box X^a \equiv [X_b, [X^b, X^a]] = m X^a$ as long as $m$ is sufficiently small, where $\Box$ is defined in [143].
Adding these and using (14) and (18) gives

\[-\theta^{ab}\omega^{bc} \approx \partial_c x^a = \partial_c (x^a - \partial^a \lambda)\]

in the semi-classical regime, as in [15]. The rhs will be recognized as tangential projector on the embedded quantum space \(\tilde{M} \subset \mathbb{R}^D\). Therefore the above relation states that \(\theta^{ac}\) is tangential to \(\tilde{M}\), and the inverse of the would-be symplectic form \(\omega^{ab}\) on \(\tilde{M}\). This implies that \(\omega|_{\tilde{M}}\) is indeed non-degenerate i.e. symplectic, and \(\theta^{ac}\) is its associated Poisson structure. Together with (51) we obtain

\[[X^a, X^b]|_x \approx i\{x^a, x^b\}|_x = i\theta^{ab}|_x\] (65)

which can be written in the notation of section 3.1 as semi-classical relation

\[[X^a, X^b] \sim i\{x^a, x^b\} = i\theta^{ab}.\] (66)

Moreover, this means that almost-local quantum spaces \(\tilde{M}\) can be locally approximated by some Moyal-Weyl quantum plane \(\mathbb{R}^{2n}_\theta\). In particular, this implies that the almost-Kähler condition (71) holds at least approximately. Furthermore, taking the inner product of (60) and (61) we obtain

\[(\Delta X^a)^2 = \langle x|(X^a - x^a + \partial^a \lambda)(X^a - x^a + \partial^a \lambda)|x\rangle = \theta^{ab}\theta^{ac}g_{bc}\] (68)

(no sum over \(a\)), where \(g_{bc}\) is the quantum metric (19). Hence the uncertainty of \(X^a\) is characterized by the uncertainty length

\[L_{NC}^2 := \|\theta^{ab}\|^2 L_{\text{coh}}^{-2}.\] (69)

We also note the relation [15]

\[i\theta^{ac}g_{cb} = \delta^{ad}(P_{d'c} - P_{d'a}) = 2i\delta^{ad'}\text{Im}(P_{d'c})\] (70)

which is obtained by subtracting (63) and (64); in particular, \(\theta^{ac}g_{cb}\) is antisymmetric. Finally, by comparing (60) with (66) we obtain

\[\theta^{ab}(\partial_{b} - iA_b)|x\rangle \approx i\langle X^a - x^a + \partial^a \lambda| x\rangle = (H_x - \lambda)i(\partial^a - iA^a)|x\rangle ,\] (71)

which relates \(i(\partial^a - iA^a)|x\rangle\) and \(\theta^{ab}(\partial_{b} - iA_b)|x\rangle\), up to the action of \(H_x - \lambda\).

### 3 The abstract quantum space \(\mathcal{M}\)

In the previous section we considered the bundle \(\mathcal{B}\) of quasi-coherent states \(|x\rangle\) over \(\mathbb{R}^D\). However, these states often coincide for different \(x\). In this section we develop a general concept of quantum geometry which naturally captures such situations, and leads to a variety \(\mathcal{M} \subset \mathbb{C}^{D-1}\), which is naturally embedded in \(\mathbb{R}^D\).

Consider the union of the normalized quasi-coherent states for all \(x \in \mathbb{R}^D\)

\[\mathcal{B} := \bigcup_{x \in \mathbb{R}^D} U(1)|x\rangle \subset \mathcal{H} \cong \mathbb{C}^N\] (72)

10Recall that the Jacobi identity is a consequence of \(d\omega = 0\).

11On quantum Kähler manifolds, this reduces to the well-known form \(L_{NC}^2 = \|\theta^{ab}\|\).
as a subset of $H$; here the union need not be disjoint. $B$ can be viewed as a $U(1)$ bundle\textsuperscript{12} over $M$. We denote $M$ as abstract quantum space associated to $X^a$. Thus $M$ inherits the induced (subset) topology and metric from $\mathbb{C}P^{N-1}$. A matrix configuration will be denoted as quantum manifold if $M \subset \mathbb{C}P^{N-1}$ is a regular (real) submanifold. This is not far-fetched, since standard theorems \cite{20, 21} ensure the existence of (local) smooth maps

$$q: \ U \subset \hat{\mathbb{R}}^D \to M \subset \mathbb{C}P^{N-1}$$

$$x \mapsto |x|.$$  

Hence $M$ is “locally translation invariant”, with generators inherited from the $SU(N)$ symmetry of $\mathbb{C}P^{N-1}$. However, $q$ need not be injective. To understand this better, we note that

$$M \cong \hat{\mathbb{R}}^D/\sim$$

where the equivalence relation $\sim$ on $\hat{\mathbb{R}}^D$ is defined by identifying points $x \in \hat{\mathbb{R}}^D$ with identical eigenspace $E_x$. Denote the equivalence class through a point $x \in \hat{\mathbb{R}}^D$ with $\mathcal{N}_x$. Due to the identity

$$H_x = H_y + \frac{1}{2}(x^a x_a - y^a y_a) \mathbb{I} - (x^a - y^a) X_a,$$

$x \sim y$ implies that $|x|$ is an eigenvector of $(x^a - y^a) X_a$,

$$(x^a - y^a) X_a |x| \propto |x|.$$  

But this means that the equivalence classes $\mathcal{N}_x$ are always (segments of) straight lines or higher-dimensional planes\textsuperscript{13} and it follows using \cite{29} that

$$w_a X^a |x| = 0 = w_a (X^a - x^a + \partial^a \lambda) |x|, \quad w \in TN_x$$

along such directions. This implies via \cite{12} that $\mathcal{N}_x$ is a null space w.r.t. the quantum metric $g_{ab}$ induced from $\mathbb{C}P^{N-1}$. The quantum metric hence characterizes the dependence of the coherent states along the non-trivial directions of $M$. Moreover, kernel of $dq$ at $x$ is given by $TN_x$.

The above observations provide a remarkable link between local and global properties of $q$: whenever $q(x) = q(y)$ for $x \neq y$, a linear kernel $TN_x \ni (x - y)$ of $dq|_x$ arises. In particular if rank $dq = D$ i.e. $q$ is an immersion, $q$ must be injective globally, since otherwise $dq$ has some non-trivial kernel. This implies that $q$ can be extended to $\hat{\mathbb{R}}^D$, and

\textbf{Theorem 3.1.} If $q$ \textup{(}\textsuperscript{74}\textup{)} is an immersion, then $q: \hat{\mathbb{R}}^D \to M$ is bijective, and $M$ is a $D$-dimensional quantum manifold. Moreover, $x^a$ provide global coordinates.

An infinite-dimensional example is given by the Moyal-Weyl quantum plane, and the fuzzy disk \cite{22} is expected to provide a finite-dimensional example. However, there are many interesting examples (such as the fuzzy sphere, see section \textsuperscript{6.1} where the rank of $dq$ is reduced. We can still make non-trivial statements with some extra assumption:

A quantum space $M$ will be called \textbf{regular} if rank $dq = m$ is constant on $\hat{\mathbb{R}}^D$. Then the fibration $\hat{\mathbb{R}}^D/\sim$ is locally trivial, and according to the rank theorem \cite{23} we can choose functions $y^n, \mu = 1, \ldots, m$ on a neighborhood of $\xi \in U \subset \hat{\mathbb{R}}^D$ such that the image $q|_U \subset M \subset \mathbb{C}P^{N-1}$ is a submanifold of $\mathbb{C}P^{N-1}$. Since the only possible degeneracies of $q$ are the linear fibers $\mathcal{N}_x$, it follows that

\textsuperscript{12}in slight abuse of notation we use the same letter $B$ as in section \textsuperscript{2} hoping that no confusion arises.

\textsuperscript{13}The $\mathcal{N}_x$ either extend to infinity or end up at the singular set $K$, where the $|x|$ may turn into higher eigenstates.
Theorem 3.2. For regular quantum spaces i.e. for rank $dq = m$ constant, $\mathcal{M}$ is a $m$-dimensional quantum manifold.

In particular, there are no self-intersections of $\mathcal{M}$, and $\tilde{\mathbb{R}}^D$ has the structure of a bundle over $\mathcal{M}$. Clearly local versions of this statement can also be formulated; e.g. if the rank of $dq$ is reduced at some point, $\mathcal{M}$ may be “pinched”. Furthermore, it may seem natural to conjecture that $\mathcal{M}$ is compact, since $\mathcal{H}$ is finite-dimensional; however, the proper statement should be that $\mathcal{M}$ has a natural compactification: since $H_x \to -x_a X^a$ for $|x| \to \infty$, the state $|x\rangle$ approaches the lowest eigenspace of $e_a X^a$ for $e = \frac{x}{|x|} \in S^{D-1}$. Hence if $\mathcal{M}$ does not already contain these states, then $\mathcal{M}$ could be compactified by adding them (and possibly other states).

Now consider the following natural embedding map provided by the symbol of $X^a$:

$$x^a : \mathcal{M} \to \mathbb{R}^D$$
$$|x\rangle \mapsto x^a := \langle x|X^a|x\rangle = x^a - \partial^a \lambda$$

using (31). This is the quotient of the previously defined function $x^a$ on $\tilde{\mathbb{R}}^D$, which is constant on the fibers $N_x$. The image

$$\hat{\mathcal{M}} := x(\mathcal{M}) \subset \mathbb{R}^D$$

defines some variety in target space $\mathbb{R}^D$. In this way, we can associate to the abstract space $\mathcal{M}$ a subset $\hat{\mathcal{M}} \subset \mathbb{R}^D$, and $B$ can be considered as a $U(1)$ bundle over $\hat{\mathcal{M}}$. This structure defines the embedded quantum space or brane associated to the matrix configuration. The concept is very reminiscent of noncommutative branes in string theory, which is borne out in the context of Yang-Mills matrix models, cf. [24, 25, 26]. However the embedding might be degenerate, and the abstract quantum space is clearly a more fundamental concept.

If equivalence class $N_x$ of $x$ is non-trivial, further interesting statements can be made. Observe that $\lambda(x) = \delta^2(x) + d^2(x)$ reduces on $N_x$ to the displacement $d^2(x)$ plus a constant shift $c = \delta^2(x)$. Therefore there is a unique $x_0 \in N_x$ in each equivalence class where $\lambda$ assumes its minimum. This provides a natural representative of $\mathcal{M} \cong \tilde{\mathbb{R}}^D/\sim$, and another embedding function

$$x_0^a : \tilde{\mathbb{R}}^D \to \mathcal{M} \hookrightarrow \mathbb{R}^D$$

which is constant on the fibers $N$ and faithfully represents $\mathcal{M}$. It satisfies

$$w_a (x^a(x_0) - x_0^a) = w_a \partial^a \lambda|_{x_0} = 0 \quad \forall w \in T_N x_0$$

using (31), because $\lambda$ assumes its minimum on $N_{x_0}$ at $x_0$. Therefore $x^a(x) = x^a(x_0)$ provides the optimal estimator for $x_0$ in $N_x$, in the sense that

$$x_0^a = P_x^\perp x^a(x)$$

where $P_x^\perp$ is the orthogonal projector on $N_x$ w.r.t. the Euclidean metric on $\mathbb{R}^D$. This provides justification for the numerical “measuring algorithm” in [13, 18], and suggests further refinements.

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14This also provides the natural adapted coordinates implied by the constant rank theorem [23].
Quantum tangent space. From now on, we will assume that $\mathcal{M}$ is a quantum manifold. Since $\mathcal{M} \subset CP^{N-1}$ is a (sub)manifold, we can determine its tangent space. Choose some point $\xi \in \mathcal{M}$. The results of section 2.2 notably (35) imply that $T_\xi \mathcal{M}$ is spanned by the $D$ vectors

$$\langle \partial_a - i A_a | x \rangle = i \lambda_a | x \rangle \in T_\xi CP^{N-1};$$

(84)

note that $\langle x | (\partial_a - i A_a) | x \rangle = 0$, hence $i \lambda_a | x \rangle$ is indeed a tangent vector of $\mathcal{M} \subset CP^{N-1}$, and perpendicular to the “would-be vertical vector” $i | x \rangle$. According to (78), any $w \in T_\xi \mathcal{M}$ provides a non-trivial relation $w^a \lambda_a | x \rangle = 0$. Hence after a suitable $SO(D)$ rotation, we can choose among the Cartesian coordinates on $\mathbb{R}^D$ $m$ local coordinates $x^\mu$, which are perpendicular to $\lambda_a$, and can serve as local coordinates of $\mathcal{M}$ near $\xi$. We denote these as local “normal embedding” coordinates on $\mathcal{M}$. It follows that an explicit basis of the tangent vectors in $T_\xi \mathcal{M}$ is given by $(\partial_\mu - i A_\mu) | x \rangle = i \lambda_\mu | x \rangle$ for $\mu = 1, \ldots, m$. This provides a natural definition of the (real) quantum tangent space of $\mathcal{M}$:

$$T_\xi \mathcal{M} = \left\langle i \lambda_\mu | x \rangle \right\rangle \subset T_\xi CP^{N-1}$$

(85)

with basis $i \lambda_\mu | x \rangle$, $\mu = 1, \ldots, m$, so that $\dim T_\xi \mathcal{M} = m = \dim \mathcal{M}$.

One can now repeat the considerations in section 2.1 in terms of local coordinates $x^\mu$, $\mu = 1, \ldots, m$ on $\mathcal{M}$. Thus $\mathcal{M}$ is equipped with a $U(1)$ connection

$$i A = \langle x | d | x \rangle$$

(86)

and a closed 2-form (21)

$$i \omega_\mathcal{M} = d \langle x | d | x \rangle = \frac{i}{2} \omega_{\mu \nu} dx^\mu \wedge dx^\nu = i dA, \quad d \omega_\mathcal{M} = 0$$

(87)

as well as a quantum metric $g_{\mu \nu}$, which are simply the pull-back of the symplectic structure and the Fubini–Study metric on $CP^{N-1}$. These structures are intrinsic, and have nothing to do with target space $\mathbb{R}^D$. Given the basis $i \lambda_\mu | x \rangle$ of tangent vectors, we can evaluate the symplectic form and the quantum metric in local embedding coordinates as

$$i \omega_{\mu \nu} = \langle x | (\lambda_\mu \lambda_\nu - \lambda_\nu \lambda_\mu) | x \rangle$$

$$g_{\mu \nu} = \langle x | (\lambda_\mu \lambda_\nu + \lambda_\nu \lambda_\mu) | x \rangle .$$

(88)

It should be noted that the quantum tangent space $T_\xi \mathcal{M}$ of the abstract quantum space is a subspace of $CP^{N-1}$, and has a priori nothing to do with the embedding in target space $\mathbb{R}^D$. This is indicated by the attribute “quantum”. The embedding (79) in target space induces another metric on $\mathcal{M}$, which in turn is distinct from the effective metric discussed in section 5.3.

It is tempting to conjecture that for irreducible matrix configuration, $\omega_\mathcal{M}$ is always non-degenerate, and thus defines a symplectic form on $\mathcal{M}$. However this is not true, as demonstrated by the minimal fuzzy torus or minimal fuzzy $H^4$. But if there is a semi-classical regime, $\omega_\mathcal{M}$ is indeed non-degenerate and thereby a symplectic manifold.

Footnotes:

15 Since $A_a$ can be gauged away at any given point, these are derivatives of sections of the respective $U(1)$ bundles over $\mathcal{M}$ and $CP^{N-1}$, which can be taken as representatives of tangent vectors on $\mathcal{M}$ and $CP^{N-1}$, respectively. Although the $\lambda_a$ depend implicitly on $x$, the result is independent of the point $x \in N_\xi$ because $\mathcal{M}$ is a manifold.

16 Since $N_\xi$ is in one-to-one correspondence with $\xi \in CP^{N-1}$, we shall use this notation if appropriate.

17 For reducible matrix configuration $\omega_\mathcal{M}$ may be degenerate even in the semi-classical regime.
Embedded quantum space for almost-local quantum spaces. Now consider the tangent space $T\tilde{M}$ of the embedded brane $\tilde{M} \subset \mathbb{R}^D$ (80), which is spanned by $\partial_\mu x^a$ for any local coordinates on $\mathcal{M}$. This can be understood for almost-local quantum spaces, following the semi-classical analysis of [15]. Recall the relation (65)

$$\frac{\partial}{\partial x^c} x^a \approx -\theta^{ab}_{bc}$$

as tensors on $\tilde{R}^D$. It follows that $\theta^{ab}$ is non-degenerate on $\tilde{M}$. Then $0 \approx i\theta^{ab}\partial_b \lambda$ (54) implies that $\lambda$ is approximately constant on $\tilde{M}$, and the derivative of $\lambda$ along the transversal fiber $\mathcal{N}$ (approximately) vanish on $\tilde{M}$ due to (82). Then (31) implies

$$x^a(x) \approx x^a, \quad \partial_\mu x^a \approx \partial_\mu x^a$$

so that both tensors $\theta^{ab}$ and $\omega_{bc}$ are approximately tangential to $\tilde{M}$, and inverse of each other on $\tilde{M}$. This is particularly transparent in normal embedding coordinates. In particular, $\tilde{M}$ is the location where $\lambda$ assumes its “approximate” minimum, which was used in [13, 18] to numerically measure and picture such branes. Then the embedding map (79) is an immersion, but (the closure of) $\tilde{M} \subset \mathbb{R}^D$ may have self-intersections, as in the example of squashed $\mathbb{C}P^2$ [4]. Both $\omega_{ab}$ and $g_{ab}$ vanish along the directions $w^a$ along the fiber $\mathcal{N}$,

$$w^a \omega_{ab} = 0 = w^a g_{ab}, \quad w \in T\mathcal{N}.$$  \(91\)

Finally, we can recognize (44)

$$\partial^a x^a = P^{ab} + P^{ba}$$  \(92\)

as tangential projector on $\tilde{M} \subset \mathbb{R}^D$, since the rhs vanishes along the fibers $\mathcal{N}$. This was obtained in [15] in the semi-classical limit, but that relation holds in fact exactly.

3.1 Quantization map, symbol and semi-classical regime

Given the quasi-coherent states, we can define a quantization map

$$\mathcal{Q} : \mathcal{C}(\mathcal{M}) \rightarrow \text{End}(\mathcal{H})$$

$$\phi(x) \mapsto \int_{\mathcal{M}} \phi(x) \left| x \right\rangle \left\langle x \right|$$

which associates to every classical function on $\mathcal{M}$ an operator or observable in $\text{End}(\mathcal{H})$. The integral on the rhs is defined naturally via the symplectic volume form

$$\int_{\mathcal{M}} \phi(x) := \frac{1}{(2\pi \alpha)^n} \int_{\mathcal{M}} \Omega \phi(x), \quad \Omega := \frac{1}{n!} \omega^\wedge_n$$

(assuming $\dim \mathcal{M} = m = 2n$), where the normalization factor $\alpha$ is defined by

$$N = \text{Tr}(\mathbb{I}) = \int_{\mathcal{M}} 1.$$  \(95\)

\footnote{This is well-defined if (the closure of) $\mathcal{M}$ is a compact sub-manifold of $\mathbb{C}P^{N-1}$, which we shall assume. It is essential to use the abstract quantum space $\mathcal{M}$ here, otherwise the integral would typically not make sense.}
Semi-classical considerations suggest that $\alpha \approx 1$, however this cannot hold in full generality, since the symplectic form is degenerate for the minimal fuzzy torus and the integral vanishes. It would be desirable to find sufficient conditions for $\alpha \approx 1$, and a precise statement in particular for the quantum Kähler manifolds discussed below. In any case, the trace is related to the intragel via

$$\text{Tr} Q(\phi) = \int_M \phi(x) .$$

(96)

The map $Q$ cannot be injective, since $\text{End}(\mathcal{H})$ is finite-dimensional; the kernel is typically given by functions with high “energy”. It is not evident in general if this map is surjective, which will be established below for the case of quantum Kähler manifolds.

We can now re-define the symbol map (4) more succinctly as

$$\text{End}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{M})$$

$$\Phi \mapsto \langle x | \Phi | x \rangle =: \phi(x) .$$

(97)

Both sides have a natural norm and inner product, given by

$$\langle \Phi, \Psi \rangle = \text{Tr}(\Phi^\dagger \Psi) \quad \text{and} \quad \langle \phi, \psi \rangle = \int_M \phi(x)^* \psi(x)$$

(98)

leading to the Hilbert-Schmidt norm $\| \Phi \|_{HS}$ and the $L^2$ norm $\| \phi \|_2$, respectively. The symbol map can be viewed as de-quantization map, which makes sense for any quantum space in the present framework.

The concept of almost-local operators discussed in section 2.4 can now also be refined. We re-define $\text{Loc}(\mathcal{H}) \subset \text{End}(\mathcal{H})$ as a maximal (vector) space of operators such that the restricted symbol map

$$\text{Loc}(\mathcal{H}) \rightarrow \mathcal{C}_{\text{IR}}(\mathcal{M})$$

$$\Phi \mapsto \langle x | \Phi | x \rangle =: \phi(x)$$

(99)

is an “approximate isometry” with respect to the Hilbert-Schmidt norm on $\text{Loc}(\mathcal{H}) \subset \text{End}(\mathcal{H})$ and the $L^2$-norm on $\mathcal{C}_{\text{IR}}(\mathcal{M}) \subset L^2(\mathcal{M})$. We will then identify $\Phi \sim \phi$. Approximate isometry means that $\| \phi \|_2 - 1 < \varepsilon$ whenever $\| \Phi \|_{HS} = 1$ for some given $0 < \varepsilon < \frac{1}{2}$, depending on the context. Then the polarization identity implies

$$\langle \Phi, \Psi \rangle_{HS} \approx \langle \phi, \psi \rangle_2 ,$$

(100)

hence an ON basis of $\text{Loc}(\mathcal{H})$ is mapped to a basis of $\mathcal{C}_{\text{IR}}(\mathcal{M})$ which is almost ON. This defines the semi-classical regime, which can be made more precise in some given situation by specifying some $\varepsilon$. Accordingly, almost-local quantum spaces are (re)defined as matrix configurations where all $X^a$ and $[X^a, X^b]$ are in $\text{Loc}(\mathcal{H})$.

Of course some given matrix configuration may be far from any semi-classical space, in which case $\text{Loc}(\mathcal{H})$ is trivial. However we will see that for almost-local quantum space, $\text{Loc}(\mathcal{H})$ typically includes the almost-local operators in the sense of (15) up to some bound, and in particular polynomials in $X^a$ up to some order. Moreover, $Q$ is an approximate inverse of the symbol map (99) on $\text{Loc}(\mathcal{H})$. Then the semi-classical regime should contain a sufficiently large class of functions and operators to characterize the geometry to a satisfactory precision.

Let us try to justify these claims. The first observation is that $1 \in \text{Loc}(\mathcal{H})$, because its symbol is the constant function $1_M$, and the norm is preserved due to (95). Conversely, we should show the completeness relation

$$Q(1_M) = \int_M |x\rangle \langle x| \overset{!}{\approx} 1$$

(101)
which is equivalent\textsuperscript{19} to the trace identity
\[
Tr \Phi = \int_M \langle x|\Phi|x \rangle \quad \forall \Phi \in \text{End}(\mathcal{H}) .
\] (102)

This is not automatic, since the integral vanishes e.g. on minimal $T^2_2$. We can establish the completeness relation at least formally\textsuperscript{20} (or rather approximately) for almost-local quantum spaces. Indeed then (62) implies
\[
[X^a, \mathcal{Q}(\phi)] \approx -i \int_M \phi(x) \theta^{ab} \partial_b \langle x|\Phi|x \rangle = i \int_M \theta^{ab} \partial_b \phi \langle x|\Phi|x \rangle = \mathcal{Q}(i \theta^{ab} \partial_b \phi)
\] (103)
because the integration measure $\Omega$ (94) is invariant under Hamiltonian vector fields. In particular, $\mathcal{Q}(1_M)$ (approximately) commutes with all $X^a$, which by irreducibility implies $\mathcal{Q}(1_M) \propto 1$, and (101) follows using the trace (96).

Now assume that the completeness relation holds to a sufficient precision. Let $\Phi$ be an almost-local hermitian operator as defined in section 2.4, with symbol $\phi$. Then the trace relation gives
\[
\|\Phi\|_{\text{HS}}^2 \approx \int_M \langle x|\Phi\Phi|x \rangle \approx \int_M \phi(x)^2 = \|\phi\|_2^2
\] (104)
using (45). Therefore almost-local operators in the sense of (45) are indeed contained in $\text{Loc}(\mathcal{H})$, up to the specific bounds. Conversely, assume that $\|\Phi\|_{\text{HS}} \approx \|\phi\|_2$ for hermitian $\Phi$. Then the completeness relation implies
\[
\|\Phi\|_{\text{HS}}^2 \approx \int_M \langle x|\Phi\Phi|x \rangle \approx \int_M \phi(x)^2 = \|\phi\|_2^2
\]
\[
\int_M \langle x|\Phi - \phi(x)\rangle\Phi - \phi(x)\rangle|x \rangle \approx 0
\] (105)
which implies that $(\Phi - \phi(x))|x \rangle \approx 0 \ \forall x \in \mathcal{M}$. Hence they are approximately local in the sense of (45). In particular they approximately commute due to (50),
\[
\Phi \Psi \approx \Psi \Phi , \quad \Phi, \Psi \in \text{Loc}(\mathcal{H}) .
\] (106)

Hence the above definition of $\text{Loc}(\mathcal{H})$ is a refinement of the definitions in section 2.4 turning the local statements into global ones.

The image $\mathcal{C}_\text{IR}(\mathcal{M})$ is typically given by functions which are slowly varying on the length scale $L_{\text{coh}}$, corresponding to the semi-classical or infrared regime. To see that $\mathcal{Q}$ is approximately inverse to the symbol map, we note that the completeness relation implies
\[
|y \rangle \approx \int_M |x \rangle \langle x|y \rangle .
\] (107)

This means that
\[
\langle x|y \rangle \approx \delta_y(x)
\] (108)

\textsuperscript{19}The following considerations would also go through if these relations hold with some non-trivial density.

\textsuperscript{20}A more precise statement (140) will be shown for quantum Kähler manifold.
for any \( y \in \mathcal{M} \) w.r.t. the measure \( \mu \), consistent with \( |\langle x|y \rangle| \sim e^{-\frac{1}{2}(x-y)^2} \). Then
\[
\mathcal{Q}(\phi)|y\rangle \approx \int_{\mathcal{M}} \phi(x)|x\rangle\langle x|y \rangle \approx \phi(y)|y\rangle.
\] (109)

for functions \( \phi(x) \) which are slowly varying on \( L_{\text{coh}} \). Therefore \( \mathcal{Q}(\phi) \) is almost-local and hence \( \mathcal{Q}(\phi) \in \text{Loc}(\mathcal{H}) \) for slowly varying \( \phi \), and moreover \( \mathcal{Q} \) is approximately the inverse of the symbol map on \( \text{Loc}(\mathcal{H}) \), since (109) gives
\[
(y|\mathcal{Q}(\phi)|y \rangle \approx \phi(y) .
\] (110)

For almost-local quantum spaces, \( \text{Loc}(\mathcal{H}) \) contains in particular the basic matrices
\[
X^a \approx \int_{\mathcal{M}} x^a|x\rangle\langle x| .
\] (111)

The approximation is good as long as the classical function \( x^a \) is approximately constant on \( L_{\text{coh}} \). Moreover, (67) gives the approximate commutation relations on \( \mathcal{M} \)
\[
[X^a, X^b] \sim i\theta^{ab} = i\{x^a, x^b\} .
\] (112)

We have seen that \( \theta^{ab} \) is tangential to \( \mathcal{M} \) and the inverse of the symplectic form \( \omega \) on \( \mathcal{M} \), hence \( \{x^a, x^b\} \) are Poisson brackets on \( \mathcal{M} \). In this sense, the semi-classical geometry is encoded in the matrix configuration \( X^a \). These observations are summarized in table 1. This provides the

| \( \text{Loc}(\mathcal{H}) \subseteq \text{End}(\mathcal{H}) \) | \( \sim \) | \( \mathcal{C}_\text{IR}(\mathcal{M}) \subseteq L^2(\mathcal{M}) \) |
|-------------------|-----------------|-----------------|
| \( \Phi \) | \( \sim \) | \( \phi(x) = \langle x|\Phi|x \rangle \) |
| \( X^a \) | \( \sim \) | \( x^a(x) \) |
| \( [..,..] \) | \( \sim \) | \( i\{..,..\} \) |
| \( Tr \) | \( \sim \) | \( \int_{\mathcal{M}} \) |
| \( \Box \) | \( \sim \) | \( e^{-\sigma\Box} \) |

Table 1: Correspondence between almost-local operators and infrared functions on \( \mathcal{M} \) for almost-local quantum spaces. The metric structure is encoded in the Laplacian \( \Box \). (147)

starting point of the emergent geometry and gravity considerations in [27, 28], which will be briefly discussed in section 5.2.

The above Poisson structure extends trivially to \( \mathbf{R}^D \), which for \( D > \text{dim } \mathcal{M} \) decomposes into symplectic leaves of \( \omega_{ab} \) that are preserved by the Poisson structure. Functions which are constant on these leaves then have vanishing Poisson brackets, which leads to a degenerate effective metric as discussed in section 5.2.

In the UV or deep quantum regime, the above semi-classical picture is no longer justified, and in fact it is very misleading. In particular, consider string states which are defined as rank one operators built out of quasi-coherent states [29, 30]
\[
\psi_{x,y} := |x\rangle\langle y| \in \text{End}(\mathcal{H}) .
\] (113)

They are highly non-local for \( x \neq y \), and should not be interpreted as function but rather as open strings (or dipoles) linking \( |y\rangle \) to \( |x\rangle \) on the embedded brane \( \mathcal{M} \). These states provide a complete
and more adequate picture of $End(\mathcal{H})$, and exhibit the stringy nature of noncommutative field theory and Yang-Mills matrix models \cite{29}. This means that the physical content of Yang-Mills matrix models, and more generally of noncommutative field theory, is much richer than suggested by the semi-classical limit. In particular, string states arise as high-energy excitation modes, leading to UV/IR mixing in noncommutative field theory \cite{31}. This is a phenomenon which has no counterpart in conventional (quantum) field theory.

3.2 Complex tangent space and quantum Kähler manifolds

Now we return to the exact analysis. For any quantum manifold $\mathcal{M}$, the embedding $\mathcal{M} \rightarrow \mathbb{CP}^{N-1}$ induces the tangential map

$$T_{\xi} \mathcal{M} \rightarrow T_{\xi} \mathbb{CP}^{N-1}. \tag{114}$$

Now we take into account that $\mathbb{CP}^{N-1}$ carries an intrinsic complex structure $J$:

$$T_{\xi} \mathbb{CP}^{N-1} \rightarrow T_{\xi} \mathbb{CP}^{N-1}, \quad Jv = iv \tag{115}$$

for any $v \in T_{\xi} \mathbb{CP}^{N-1}$. Accordingly, $T\mathbb{CP}^{N-1} \cong T^{(1,0)}\mathbb{CP}^{N-1}$ can be viewed as holomorphic tangent bundle, thus bypassing an explicit complexification of its real tangent space. With this in mind, we define the complex quantum tangent space of $\mathcal{M}$ as

$$T_{\xi,\mathbb{C}} \mathcal{M} := \left\langle X_\mu |x\rangle \right\rangle_{\mathbb{C}} \subset T_{\xi} \mathbb{CP}^{N-1} \cong T_{\xi,\mathbb{C}}\mathbb{CP}^{N-1}, \tag{116}$$

which also carries the complex structure

$$JX_\mu |x\rangle := iX_\mu |x\rangle \in T_{\xi,\mathbb{C}} \mathcal{M}, \quad J^2 = -1. \tag{117}$$

Again, this complex tangent space is not necessarily the complexification of the real one. Using the basis $iX_\mu |x\rangle$, $\mu = 1, \ldots, m$ of $T_{\xi} \mathcal{M}$ which arises in normal embedding coordinates, there may be relations of the form

$$(iX_\mu - J_{\mu}^\nu X_\nu) |x\rangle = 0 \quad \text{for} \quad J_{\mu}^\nu \in \mathbb{R}, \tag{118}$$

so that $T_{\xi,\mathbb{C}} \mathcal{M}$ has reduced dimension over $\mathbb{C}$. We will see that for quantum Kähler manifolds as defined below, the complex dimension is half of the same as the real one.

Quantum Kähler manifolds. Consider the maximally degenerate case where the complex dimension of $T_{\xi,\mathbb{C}} \mathcal{M}$ is given by $n = \frac{m}{2} \in \mathbb{N}$ where $m = \dim_{\mathbb{R}} \mathcal{M}$. Then $T_{\xi} \mathcal{M}$ is stable under the complex structure operator $J$

$$T_{\xi,\mathbb{C}} \mathcal{M} = T_{\xi} \mathcal{M} \tag{119}$$

so that $T_{\xi} \mathcal{M}$ should be viewed as holomorphic tangent space of $\mathcal{M}$. But this implies that $\mathcal{M} \subset \mathbb{CP}^{N-1}$ is a complex sub-manifold (i.e. defined by holomorphic equations), cf. \cite{32} or Proposition 1.3.14 in \cite{33}. Such quantum manifolds $\mathcal{M}$ will be called quantum Kähler manifolds, for reasons explained below. Indeed, all complex sub-manifolds of $\mathbb{CP}^{N-1}$ are known to be Kähler. Note that this is an intrinsic property of a quantum space $\mathcal{M}$, and no extra structure is introduced here: $\mathcal{M}$ either is or is not of this type\cite{21}. We will see that this includes the well-known quantized or “fuzzy” spaces arising from quantized coadjoint orbits\cite{22}.

\cite{21}It is interesting to note that due to (71), $H_\xi$ preserves the complex tangent space $T_{\xi,\mathbb{C}} \mathcal{M}$, at least in the semi-classical regime. However, (71) is still weaker than the Kähler condition.

\cite{22}It is worth pointing out that that $\mathbb{CP}^{N-1}$ is itself a quantum Kähler manifold, as minimal fuzzy $\mathbb{CP}^{N-1}$.
Consider the quantum Kähler case in more detail. We can introduce a local holomorphic parametrization of \( \mathcal{M} \subset \mathbb{C}P^{N-1} \) near \( \xi \) in terms of \( z^k \in \mathbb{C}^n \). Then any local (!) holomorphic section of the tautological line bundle over \( \mathbb{C}P^{N-1} \) defines via pull-back a local holomorphic section of the line bundle
\[
\tilde{B} := \bigcup_{x \in \mathbb{R}^D} E_x \to \mathcal{M} \hookrightarrow \mathbb{C}P^{N-1}
\] (120)
over \( \mathcal{M} \), denoted by \( \|z\rangle \). This \( \|z\rangle \) can be viewed as holomorphic \( \mathbb{C}^N \)-valued function on \( \mathcal{M} \), which satisfies
\[
\frac{\partial}{\partial \bar{z}^k} \|z\rangle = 0, \quad \|z\rangle|_\xi = |\xi\rangle
\] (121)
where \( \bar{z}^k \) denotes the complex conjugate of \( z^k \). Hence \( \|z\rangle \) arises from \( |x\rangle \) through a re-parametrization and gauge transformation along with a non-trivial normalization factor; this is indicated by the double line in \( \|z\rangle \). In other words, the differential of the section
\[
d\|z\rangle = dz^k \frac{\partial}{\partial z^k} \|z\rangle \quad \in \Omega^{(1,0)}_z \mathcal{M}
\] (122)
is a \((1,0)\) one-form. Given this holomorphic one-form \( d\|z\rangle \) and the hermitian inner product on \( \mathcal{H} \), we naturally obtain a \((1,1)\) form
\[
\omega := (d\|z\rangle)^\dagger \wedge d\|z\rangle = \omega_{kl} d\bar{z}^k \wedge dz^l \quad \in \Omega^{(1,1)}_z \mathcal{M}
\] (123)
which is closed,
\[
d\omega = -(d\|z\rangle)^\dagger \wedge dd\|z\rangle + (dd\|z\rangle)^\dagger \wedge d\|z\rangle = 0
\] (124)
using holomorphicity of \( \|z\rangle \). This is the Kähler form, which encodes the \( \omega_{ab} \) in (20). As in (18), we can then define the hermitian metric
\[
h(X,Y) = \left( (d\|z\rangle)^\dagger \otimes d\|z\rangle \right)(X,Y) \quad \in T^{(1,1)}
\] (125)
whose imaginary and real part define the symplectic form and the quantum metric via
\[
\omega(X,Y) = -i(h(X,Y) - h(Y,X)^*) = -\omega(Y,X)
\]
\[
g(X,Y) = h(X,Y) + h(X,Y)^* = g(Y,X)
\] (126)
Since \( h \in T^{(1,1)} \), they satisfy the compatibility condition
\[
\omega(X,JY) = -i(h(X,JY) - h(JY,X)^*) = -i(ih(X,Y) + ih(Y,X)^*)
\]
\[
= g(X,Y)
\] (127)
(recall that \( J = -i \) on anti-holomorphic \((0,1)\) forms). This means that \( \mathcal{M} \) is a Kähler manifold, and the name “quantum Kähler manifold” indicates its origin from the matrices \( X^a \). In particular, the coherence length \( L_{\text{coh}} \) and the uncertainty scale \( L_{\text{NC}} \) coincide.

\[23\|z\rangle \text{ cannot be normalized, since e.g. } \langle y|\|z\rangle \text{ must be holomorphic in } z. \text{ Apart from that, } \tilde{B} \text{ is equivalent to } B.\]
Now we relate this to the local generators $X_\mu$ (35), (85). Introducing real coordinates $z^k = z^k(x^\mu)$ where $x^\mu$ are the local (Cartesian) embedding coordinates introduced above, the holomorphicity relation (121) can be expressed using (35) as

$$0 = \frac{\partial}{\partial z^k} \langle z \parallel z \rangle = \frac{\partial x^\mu}{\partial z^k} \frac{\partial}{\partial x^\mu} \langle z \parallel z \rangle = i \frac{\partial x^\mu}{\partial z^k} (X_\mu + A_\mu) \langle z \parallel z \rangle .$$

(128)

Similarly,

$$\frac{\partial}{\partial z^k} \langle z \parallel z \rangle = \frac{\partial x^\mu}{\partial z^k} \frac{\partial}{\partial x^\mu} \langle z \parallel z \rangle = i \frac{\partial x^\mu}{\partial z^k} (X_\mu + A_\mu) \langle z \parallel z \rangle .$$

(129)

We can now introduce new generators\footnote{The $A^k, \bar{A}_l$ are matrix-valued functions on $\mathcal{M}$ just like the $X_\mu$, while the $X_a$ are “constant” matrices.} $A^k, \bar{A}_l$ via

$$A^k = i \frac{\partial x^\mu}{\partial z^k} (X_\mu + A_\mu)$$

$$\bar{A}_k = i \frac{\partial x^\mu}{\partial \bar{z}^k} (X_\mu + A_\mu)$$

(130)

so that

$$A^k \parallel z \rangle = 0, \quad \bar{A}_k \parallel z \rangle = \frac{\partial}{\partial z^k} \langle z \parallel z \rangle .$$

(131)

These are clearly the analogs of the standard annihilation properties of coherent states. It is hence appropriate to denote the $\parallel z \rangle$ on quantum Kähler manifolds as coherent states. Then

$$T_{\xi, \mathcal{C} \mathcal{M}} = \langle \bar{A}_k \parallel z \rangle_\mathcal{C} \cong \mathbb{C}^n \quad k = 1, \ldots, n.$$ 

(132)

The metric tensor and the symplectic form are then determined as usual by the Kähler form

$$i \omega_{kl} = (d_k \parallel z \rangle)^\dagger d_l \parallel z \rangle = \langle z \parallel \bar{A}_l^\dagger \bar{A}_k \parallel z \rangle$$

(133)

which arises from a local Kähler potential,

$$\omega_{kl} = -\frac{1}{2} \hat{\partial}_k \hat{\partial}_l \rho$$

(134)

given by the restriction of the (Fubini–Study) Kähler potential on $\mathbb{C}P^N$.

This provides a rather satisfactory concept of quantum Kähler geometry, which arises in a natural way from the complex structure in the Hilbert space. There is no need to invoke any semi-classical or large $N$ limit. Not all quantum spaces are of this type, a counterexample being the minimal fuzzy torus $T^2$ as discussed in section 6.4. In [13], it is claimed that all quantum manifolds are Kähler in the semi-classical limit, based on (70). However this refers to a different almost-complex structure and metric which is not intrinsic. From the present analysis, there is no obvious reason why all quantum manifolds should be Kähler, even in the semi-classical limit.

Since for non-Kähler manifolds the complex tangent space $T_{\mathcal{C} \mathcal{M}}$ is higher-dimensional, quantum effects due to loops in Yang-Mills matrix models may be more significant, and the geometric trace formula (2.38) in [29] for string states would need to be replaced with some higher-dimensional analog. This suggests that quantum Kähler manifolds may be protected by some sort of non-renormalization theorems.
4 Coherent states and quantization map for quantum Kähler manifolds

We can establish the following lemma, which is well-known for standard coherent states:

**Lemma 4.1.** Let $|x\rangle$ be the coherent states of a quantum Kähler manifold $M$, and $H_0 \subset H$ their linear span. Assume $A \in \text{End}(H_0)$ satisfies $\langle x|A|x\rangle = 0$ for all $x \in M$. Then $A = 0$.

**Proof.** Consider the function

$$A(y, z) := \langle y\|A\|z \rangle$$

where $\|z\rangle, \|y\rangle$ are local holomorphic sections of the coherent states in a neighborhood of $\xi \in M$. Clearly this function is holomorphic in $z$ and in $\bar{y}$. By assumption, the restriction of $A(\bar{y}, z)$ to the diagonal $A(\bar{z}, z) = \langle z\|A\|z \rangle$ vanishes identically. But then the standard properties of holomorphic functions imply (cf. [34]) that $A(\bar{y}, z) \equiv 0$ identically. This argument applies near any given point $\xi \in M$, which implies that $A = 0$.

Using this lemma, we can establish the diagonal realization of operators via coherent states:

**Theorem 4.2.** Let $|x\rangle$ be the (normalized) coherent states of a quantum Kähler manifold $M$, and $H_0 \subset H$ their linear span. Then all operators $A \in \text{End}(H_0)$ can be written as

$$A = \int_M A(x)|x\rangle\langle x|$$

for some suitable complex-valued function $A(x)$ on $M$.

Note that if the holomorphic coherent states $\|x\rangle$ are used instead of the normalized $|x\rangle$, then $A(x)$ might have some singularities.

**Proof.** Assume that the subspace in $\text{End}(H_0)$ spanned by the rhs of (136) is smaller than $\text{End}(H_0)$. Let $B \in \text{End}(H_0)$ be in its orthogonal complement w.r.t. the Hilbert-Schmidt metric. Then

$$0 = \text{Tr}(AB) = \int_M A(x)\langle x|B|x \rangle \quad \forall A(x) \in C(M).$$

But this implies $\langle x|B|x \rangle = 0 \forall x \in M$, and then by Lemma 4.1 it follows that $B = 0$.

Consider again the span $H_0 \subset H$ of all quasi-coherent states $|x\rangle$. It is natural to conjecture

**Conjecture 1.** For every irreducible matrix configuration, $M$ is connected, and the quasi-coherent states are over-complete, i.e.

$$H_0 = \langle |x\rangle; x \in \tilde{R}^D \rangle_C = H.$$ (138)

In the semi-classical regime this follows from (101) and (103), which would give a central element for every connected component of $M$. A viable general strategy to show this more generally might be to show that the continuation of the $|x\rangle$ through the singular set $\mathcal{K}$ provides all eigenstates of $H_x$. However, this is left as a conjecture.
In any case, we can consider the following restricted form of the quantization map \( Q : \mathcal{C}(\mathcal{M}) \to \text{End}(\mathcal{H}_0) \)

\[
\phi(x) \mapsto \int_{\mathcal{M}} \phi(x) |x\rangle\langle x|
\]

associating to every classical function on \( \mathcal{M} \) an operator or observable in \( \text{End}(\mathcal{H}_0) \). The above theorem states that \( Q \) is surjective for quantum Kähler manifolds. This means that any given operator \( A \in \text{End}(\mathcal{H}_0) \) has a representation of that form, and in fact many. The kernel of \( Q \) is typically given by functions above some “energy cutoff”. Furthermore, it follows that the operators of the form \((136)\) form an algebra, and every operator can be viewed as quantized function on \( \mathcal{M} \).

Even though this is a very nice result, surjectivity of \( Q \) is rather surprising in light of the string states \((113)\), which are highly non-local. Nevertheless, even such string states can be represented in the above diagonal form \((136)\), but \( A(x) \) is then rapidly oscillating and in the UV or deep quantum regime. Therefore this diagonal representation should be used with caution, and a representation in terms of non-local string states is more appropriate in the UV regime. These can naturally be interpreted as open strings on the embedded quantum space or brane \( \tilde{\mathcal{M}} \).

**Completeness relation.** In particular, the theorem 4.2 implies that at least for quantum Kähler manifolds, the identity operator \( \mathds{1}_{\mathcal{H}_0} \) can be written in terms of coherent states:

\[
\mathds{1}_{\mathcal{H}_0} = \int_{\mathcal{M}} \mathds{1}(x)|x\rangle\langle x|,
\]

where the integral is defined as in \((94)\), and \( \mathds{1}(x) \) is some function on \( \mathcal{M} \). This gives

\[
\text{Tr} A = \int_{\mathcal{M}} \mathds{1}(x)\langle x|A|x\rangle,
\]

\[
\text{Tr}(Q(\phi(x))) = \int_{\mathcal{M}} \mathds{1}(x)\phi(x).\tag{141}
\]

The natural guess is

\[
\mathds{1}_{\mathcal{H}} = \int_{\mathcal{M}} |x\rangle\langle x|.
\]

This is well-known e.g. for the quantum spaces given by quantized coadjoint orbits of compact semi-simple Lie groups, where it follows immediately from Schur’s Lemma. It follows more generally from \((103)\) at least in the semi-classical regime, but is not evident if \( \mathds{1}(x) \propto 1_M \) for all quantum Kähler manifolds.

### 5 Remarks and discussion

The results and concepts discussed in this paper call for a number of remarks.

First, we only considered the case where the lowest eigenspace \( E_x \) of \( H_x \) is non-degenerate. This excludes many interesting examples such as fuzzy \( S^4_N \) and fuzzy \( H^4_n \) as discussed in section 6.3. If \( E_x \) is an \( k \)-dimensional (complex) vector space, then much of the above analysis would go through, replacing \( \mathcal{B} \) by an \( U(k) \) bundle and \( \omega \) by the field strength of its natural (Berry) connection. Sometimes the degeneracy may also be resolved by adding extra matrices \( X^i \). For
example, the abstract quantum space of $S^4_N$ is then recognized as $\mathbb{C}P^3$, and similarly in other examples, cf. section 6.3. In other words, such degenerate quantum spaces can be recognized as projections of non-degenerate ones, by dropping some $X^a$.

There are a number of issues which ask for a better understanding. One of them is the relation between the symplectic volume of $\mathcal{M}$ and the dimension of the Hilbert space (96). Even though equality holds in the standard examples, it is violated for the minimal fuzzy torus. Results from geometric quantization suggest a more complicated relation, and it would be desirable to have quantitative results for a large class of quantum spaces. Furthermore, it would be very important to have a more general derivation or qualification of the completeness relation (101).

Another open issue is the compactness of $\mathcal{M} \subset \mathbb{C}P^{N-1}$. It may be tempting to conjecture that all $\mathcal{M}$ are compact, but the fuzzy disk [22] is a candidate for a non-compact quantum space, which remains to be elaborated. However, the closure of $\mathcal{M}$ in $\mathbb{C}P^{N-1}$ is clearly compact, and it would be nice to understand this in more detail.

Small deformations of the basic quantum Kähler manifolds $\mathcal{M}_0$ of dimension $m < D$ typically lead to an “oxidation” $\mathcal{M}$ corresponding to some tubular neighborhood of $\mathcal{M}_0$. This leads to the idea of fuzzy extra dimensions [35, 36]. On the other hand, it is well-known that adding a small perturbation to some quantum manifold $\mathcal{M}$ can be viewed as a gauge field on $\mathcal{M}$, which becomes dynamic in Yang-Mills matrix models. Relating this field-theoretic point of view with the above geometric point of view provides useful insights, and one may hope to find further statements on stability and/or non-renormalization in this way. Similar considerations lead to the emergent gravity approach based on Yang-Mills matrix models [37, 27].

Finally, the present analysis is restricted to the case of irreducible matrix configurations. If the matrix configuration is reducible, $\mathcal{H} = \oplus \mathcal{H}_i$ decomposes into the orthogonal sum of irreducible subspaces, and the above considerations apply to all $\mathcal{H}_i$. This could be viewed as a stack of branes. In particular, commuting matrix configurations (cf. [28]) have a large stabilizer $U(1)^N$ under the adjoint action of $U(N)$, so that their $U(N)$ gauge orbit in Yang-Mills matrix models has smaller dimension than that of irreducible (noncommutative) matrix configurations. But then their contribution in the "path" integral over all matrices is negligible, which defines the quantum theory. Therefore irreducible matrix configurations as considered here are expected to play the central role in these models.

5.1 Dirac operator

The present framework has a natural extension to spinors and Dirac-type operators. Namely, for any matrix configuration $X^a$, $a = 1, \ldots, D$ we can consider [14] \[ \mathcal{D}_x = \Gamma_a(X^a - x^a), \quad x^a \in \mathbb{R}^D \] acting on $\mathcal{H} \otimes \mathbb{C}^s$. Here $\Gamma_a$ are the gamma matrices generating the Clifford algebra of $SO(D)$ on the irreducible representation $\mathbb{C}^s$. $\mathcal{D}_x$ arises as off-diagonal part of the matrix Dirac operator $\mathcal{D} = \Gamma_a [X^a, \cdot]$ in Yang-Mills matrix models such as the IIB or IKKT model, for the matrix configuration extended by a point brane $X^a \oplus x^a$. It describes a fermionic string stretched between the brane and the point $x^a$. Quite remarkably, numerical investigations [13] strongly suggest that the Dirac operator $\mathcal{D}_x$ always has exact zero modes \[ \mathcal{D}_x |x, s\rangle = 0 \] at $\mathcal{M}$, so that there is no need to introduce the lowest eigenvalue function $\lambda(x)$. This can be justified rigorously for 2-dimensional branes [14], and some heuristic reasons can be given also in...

\[24\]
more general cases; see [14, 16, 17] for further work. However, the presence of extra structure due to the spinors obscures the relation with the quasi-coherent states and $M$ as introduced here. This is certainly an interesting topic for further research.

5.2 Effective metric and relation with matrix models

The considerations in this paper are motivated by Yang-Mills matrix models, whose solutions are precisely matrix configurations as considered here. Fluctuations in these models are governed by the matrix Laplacian

$$\Box := \delta_{ab}[X^a, [X^b, .]] : \text{End}(\mathcal{H}) \rightarrow \text{End}(\mathcal{H}) .$$

(145)

The displacement Hamiltonian arises as off-diagonal part of the matrix Laplacian for a point or probe brane added to the matrix configuration [13], i.e. for $X^a \oplus x^a$ acting on $\mathcal{H} \oplus \mathbb{C}$. It describes a string stretched between the brane and the point $x^a$. This can also be viewed as a special case of intersecting branes [39], one brane being the point probe.

To understand the effective metric in matrix models, consider the inner derivations

$$[X_a, .] \sim i\theta^{a\mu}\partial_\mu$$

acting on $\text{End}(\mathcal{H})$ resp. $C_{\text{IR}}(M)$, which are (quantizations of) Hamiltonian vector fields on $M$ for almost-local quantum spaces. By considering the inner product $\langle \Phi, \Psi \rangle := Tr([X^a, \Phi^\dagger][X_a, \Psi])$ on $\text{Loc}(\mathcal{H})$, one can then show [27] that

$$\Box \sim e^\sigma \Box G$$

(147)

where $G$ is the effective metric on $M$ given by

$$G^{\mu\nu} = e^{-\sigma} \theta^{\mu\nu} \theta^{\mu\nu'} g_{\mu'\nu'} , \quad e^{-\sigma} = \left| G^{\mu\nu} \right|^{1/2} / \left| \theta^{\mu\nu} \right|^{1/2}$$

(148)

for $\dim M > 2$. This can be viewed as open-string metric, and it provides the starting point of the emergent geometry and gravity considerations in [27, 28]. In the two-dimensional case, the underlying Weyl invariance leads to a different interpretation of $\Box$, which is discussed in [40].

In the reducible case, $M$ decomposes into a foliation of symplectic leaves. Then the effective metric is non-vanishing only along this foliation, i.e. it vanishes along the transversal directions. In the context of Yang-Mills matrix models, this means that fluctuation modes on such backgrounds only propagate along the symplectic leaves, so that the resulting gauge theory is lower-dimensional. This happens on any superficially odd-dimensional quantum space, or e.g. on $\kappa$ Minkowski space [41] in dimensions larger than 2.

6 Examples

6.1 The fuzzy sphere

The fuzzy sphere $S^2_N$ [2, 1] is a quantum space defined in terms of three $N \times N$ hermitian matrices

$$X^a = \frac{1}{\sqrt{C_N}} J^a_{(N)}, \quad a = 1, 2, 3$$

(149)
where $J^r(N)$ are the generators of the $N$-dimensional irrep of $su(2)$ on $\mathcal{H} = \mathbb{C}^N$, and $C_N = \frac{1}{4}(N^2-1)$ is the value of the quadratic Casimir. They satisfy the relations

$$[X^a, X^b] = \frac{i}{\sqrt{C_N}}\epsilon^{abc} X^c , \quad \sum_{a=1}^{3} X^a X^a = \mathbb{1} \quad (150)$$

choosing the normalization [149] such that the radius is one. The displacement Hamiltonian is

$$H_x = \frac{1}{2} \sum_{a=1}^{3} (X^a - x^a)^2 = \frac{1}{2}(1 + |x|^2) - \sum_{a=1}^{3} x^a X^a \quad (151)$$

where $|x|^2 = \sum_a x_a^2$. Using $SO(3)$ invariance, it suffices to consider the north pole $x = (0, 0, x^3) =: n$ where

$$H_x = \frac{1}{2}(1 + |x|^2) - |x| X^3 \quad (152)$$

assuming $x^3 > 0$ to be specific. Hence the ground state of $H_x$ is given by the highest weight vector $|n⟩ := |\frac{N-1}{2}, \frac{N-1}{2}⟩$ of the $su(2)$ irrep $\mathcal{H}$, and the eigenvalue is easily found to be [13]

$$\lambda(x) = \frac{1}{2}(1 + |x|^2) - |x| \sqrt{\frac{N-1}{N+1}} \quad (153)$$

All other quasi-coherent states are obtained by $SO(3)$ acting on $|n⟩$, hence the abstract quantum space $\mathcal{M}$ is given by the group orbit

$$\mathcal{M} = SO(3) \cdot |n⟩ = SO(3)/U(1) \cong S^2 \subset \mathbb{C}P^{N-1}. \quad (154)$$

Note that the quasi-coherent states are constant along the radial lines in agreement with (78),

$$|x⟩ = |\alpha x⟩ \quad \text{for} \ \alpha > 0 \quad (155)$$

The equivalence classes $\mathcal{N}$ consist of the radial lines emanating from the origin, and the would-be symplectic form $\omega_{ab}$ and the quantum metric $g_{ab}$ vanish if any one component is radial. The minima of $\lambda(x)$ on $\mathcal{N}_x$ describe a sphere with radius $|x_0| = \sqrt{\frac{N-1}{N+1}} = 1 + O(\frac{1}{N})$. This coincides precisely with the embedded quantum space (80)

$$\tilde{\mathcal{M}} = \{⟨x|X^a|x⟩| \in \mathbb{R}^3 : |x| = \sqrt{\frac{N-1}{N+1}} \} \cong S^2 \quad (156)$$

defined by the expectation value $x^a$ (79), in accordance with (83). At the singular set $K = \{0\}$ the Hamiltonian is $H_0 = C^2 \mathbb{1}$, so that all energy levels become degenerate and cross. Following $|x⟩$ along the radial direction through the origin, it turns into the highest energy level. It is easy to see that the would-be symplectic form $\omega$ is the unique $SO(3)$-invariant 2-form on $\mathcal{M}$ which satisfies the quantization condition (26) with $n = N$. Moreover, the abstract quantum space $\mathcal{M} \cong S^2 \subset \mathbb{C}P^{N-1}$ is a quantum Kähler manifold, since the complex tangent space (116) is one-dimensional, spanned by

$$T_{n,c}\mathcal{M} = \langle J^-|n⟩⟩ \subset \mathcal{C} \quad (157)$$

(at $|n⟩ \in \mathcal{M}$). This holds because $|n⟩$ is the highest weight state, so that

$$J^+|n⟩ = 0 \quad (158)$$
therefore the two tangent vectors $X^1|n⟩, X^2|n⟩ ∈ T_nM$ (85) are related by $i$, while $X^3|n⟩$ vanishes at $n$. Indeed, it is well-known that the coherent states on $S^2_N$ form a Riemann sphere, and the (quasi-) coherent states coincide with the coherent states introduced in [34].

All this holds for any $N ≥ 2$. The coherence length is of order

$$ L_{coh} ≈ L_{NC} \sim \frac{1}{\sqrt{N}} $$

in the given normalization. Hence for sufficiently large $N$, the almost-local operators comprise all polynomials in $X^a$ up to order $O(\sqrt{N})$ (depending on some specific bound), so that $S^2_N$ is an almost-local quantum space. In contrast for the minimal fuzzy sphere $S^2_2$ with $N = 2$, the generators reduce to the Pauli matrices $X^a = σ^a$, and the (quasi)coherent states form the well-known Bloch sphere $M = S^2 \cong CP^1$. This is still a quantum Kähler manifold even though the semi-classical regime is trivial and contains only the constant functions $Loc(\mathcal{H}) = C\mathbb{1}$, since the coherence length is of the same order as the entire space $M$.

6.2 Quantized coadjoint orbits for compact semi-simple Lie groups

The above construction generalizes naturally to quantized coadjoint orbits for any compact semi-simple Lie group $G$ with Lie algebra $\mathfrak{g}$. For any irreducible representation $\mathcal{H}_\Lambda$ with highest weight $\Lambda = (n_1,...,n_k)$ labeled by Dynkin indices $n_j$, the matrix configuration

$$ X^a = c T^a, \quad a = 1,...,D $$

(160)

defines a quantum Kähler manifold $M \cong G/K$. Here $T^a$ are orthogonal generators of $\mathfrak{g} \cong \mathbb{R}^D$ acting on $\mathcal{H}_\Lambda$, $K$ is the stability group of the highest weight $\Lambda$, and $c$ is some normalization constant. Then the displacement Hamiltonian is

$$ H_x = C^2(\mathfrak{g}) + \frac{1}{2} x_a x^a - x_a T^a $$

(161)

where $C^2(\mathfrak{g}) \propto \mathbb{1}$ is the quadratic Casimir. Using $G$-invariance, we can assume that $x$ is in (the dual of) the Cartan subalgebra and has maximal weight. Then $|x⟩ = |\Lambda⟩$ is the highest weight state, so that the quasi-coherent states are the group orbit $M = G · |\Lambda⟩ \cong G/K$ of the highest weight state with stabilizer $K$. This is a quantum Kähler manifold due to the highest weight property, and the quantum metric $g_{ab}$ (19) and the symplectic form $ω$ (20) are the canonical group-invariant structures on the Kähler manifold $M$. For large Dynkin indices $n_j ≥ n ≫ 1$, the almost-local operators comprise all polynomials in $X^a$ up to some order $O(\sqrt{n})$, so that $M$ is an almost-local quantum space. This is essentially the well-known story of quantized coadjoint orbits, and the (quasi-) coherent states coincide with the coherent states introduced in [34], cf. [42]. Perhaps less known is the fact that if some of the $n_j$ are small, $M$ can be viewed as “oxidation” of some lower-dimensional brane, more precisely as a bundle over $M_0$ whose fiber is very “fuzzy”. For an application of such a structure see e.g. section 4.2 in [43].

This construction generalizes further to highest weight (discrete series) unitary irreducible representation of non-compact semi-simple Lie groups. A particularly interesting example is given by the “short” series of unitary irreps of $SO(4,2)$ known as singletons, which lead to the fuzzy 4-hyperboloids $H^4_\mathfrak{a}$ discussed below, and to quantum spaces which can be viewed as cosmological space-time [44].
(Minimal) fuzzy $CP^{N-1}_{N}$. As an example we consider minimal fuzzy $CP^{N-1}_{N}$, which is obtained using the above general construction for $G = SU(N)$ and its fundamental representation $\mathcal{H} = (1, 0, ..., 0)$, so that $G/K \cong CP^{N-1}$. This is the quantum Kähler manifold obtained from the matrix configuration

$$X^a = \lambda^a \in \text{End}(\mathcal{H}), \quad \mathcal{H} = \mathbb{C}^N$$

(162)

for $a = 1, ..., N^2 - 1$, where $\lambda^a$ are a (Gell-Mann) ON basis of $\mathfrak{su}(N)$ in the fundamental representation. Then $\text{End}(\mathcal{H}) \cong (0, ..., 0) \oplus (1, 0, ..., 0, 1)$ can be viewed as a minimal quantization of functions on $CP^{N-1}$. The quantization map

$$Q(\phi) = \int_{CP^{N-1}_{N}} |x\rangle \langle x| \phi(x)$$

(163)

is then the partial inverse of the symbol map, apart from the constant function:

$$Q(\langle x| \Phi |x \rangle) = c \Phi \quad \text{if} \quad Tr(\Phi) = 0$$

(164)

for some $c > 0$. Near $|\Lambda\rangle$, the quasi-coherent states $|x\rangle$ can be organized as holomorphic sections

$$\|z\rangle = \exp(z^k T^+_k) |\Lambda\rangle,$$

(165)

where the $T^+_k$, $k = 1, ..., N - 1$ are the rising operators of a Chevalley basis of $\mathfrak{su}(N)$. Hence fuzzy $CP^{N-1}_{N}$ is a quantum Kähler manifold which coincides with $CP^{N-1}$, with Kähler form

$$\omega_{kl} = \frac{\partial}{\partial z^k} \langle z| \frac{\partial}{\partial z^l} |z\rangle.$$

(166)

Squashed $CP^2_N$. Further quantum spaces can be obtained by projections of quantized coadjoint orbits. For example, starting from fuzzy $CP^2_N$ with $\mathcal{H} = (N, 0)$, consider the following matrix configuration

$$X^a = T^a, \quad a = 1, 2, 4, 5, 6, 7$$

(167)

dropping the Cartan generators $T_3$ and $T_8$ from the (Gell-Mann) basis of $\mathfrak{su}(3)$. Then the displacement Hamiltonian can be written as

$$H_x = \tilde{H}_x - \frac{1}{2}(X_3 - x_3)^2 - \frac{1}{2}(X_8 - x_8)^2$$

(168)

where $\tilde{H}_x$ is the displacement Hamiltonian for $CP^2_N$. Although the quasi-coherent states $|x\rangle$ are not known in this case, they are close to those of $CP^2_N$ in the large $N$ limit, cf. [13]. Indeed then the last two terms in (168) are small, and $0 < \lambda(x) \leq \bar{\lambda}(x)$ gives an upper bound for $\lambda$. This implies that the displacement is small, and

$$\mathcal{M} \cong CP^2 \subset CP^{N(N+3)/2}. $$

(169)

Again, the concept of the abstract quantum space is superior to the notion of an embedded brane, which is a complicated self-intersecting variety in $\mathbb{R}^3$ related to the Roman surface [4].
6.3 Degenerate cases

The fuzzy 4-sphere $S_N^4$. Now consider again the quantized coadjoint orbit of $SU(4) \cong SO(6)$ acting on the highest weight irrep $\mathcal{H}_\Lambda$ with $\Lambda = (N, 0, 0)$. We have seen just that the matrix configuration using all $\mathfrak{so}(6)$ generators $\mathcal{M}^{ab} = -\mathcal{M}^{ba}$ as in (160) would give fuzzy $\mathbb{C}P^3_N$, with coherent states acting on the highest weight state $|\Lambda\rangle$. Now instead of using all $\mathcal{M}^{ab}$, consider the matrix configuration defined by the following 5 hermitian matrices

$$X^a = \mathcal{M}^{a6} \in \text{End}(\mathcal{H}_\Lambda), \quad a = 1, \ldots, 5.$$  

Using $SO(5)$ invariance, it suffices to consider the displacement Hamiltonian at $x = (0, 0, 0, 0, x_5)$,

$$H_x = \frac{1}{2} \sum_{i=1}^{4} X_i^2 + \frac{1}{2} (X_5 - x_5)^2 = \frac{1}{2} (R^2 + x_5^2) \mathbb{1} - x_5 X_5$$

since $\sum_a X_a^2 = R^2 \mathbb{1}$ for $R^2 = \frac{1}{2} N(N + 4)$, cf. [45, 46]. Now $|\Lambda\rangle$ is by construction an eigenstate of $X^5$ which commutes with $SO(4)$, with maximal eigenvalue. Therefore the lowest eigenspace $E_x$ of $H_x$ is spanned by the orbit $SO(4) \cdot |\Lambda\rangle \cong S^2$, which spans a $N + 1$-dimensional complex vector space. This provides an example of a degenerate quantum space. The abstract quantum space $\mathcal{M}$ is obtained by acting with $SO(5)$ on this $S^2$, which is easily seen to recover

$$\mathcal{M} \cong \mathbb{C}P^3 \subset \mathbb{C}P^{\dim \mathcal{H} - 1}$$

which is an equivariant $S^2$ bundle over $S^4$. The $E_x$ naturally form a $SU(N + 1)$ bundle $\mathcal{B}$ over $S^4$, and $\omega$ is replaced by an $SU(N + 1)$ connection. Again the concept of an abstract quantum space greatly helps to understand the structure, as it resolves the degeneracy of the quasi-coherent states. Moreover $\mathcal{M}$ is clearly a Kähler manifold, and theorem 4.2 holds.

The fuzzy 4-hyperboloid $H^4_n$. Using an analogous construction for $SO(4, 2)$ and its singleton irreps $\mathcal{H}_n$, labeled by $n \in \mathbb{N}$, one obtains fuzzy $H^4_n$ [17, 43]. The corresponding matrix configuration is given by the following 5 hermitian operators

$$X^a = \mathcal{M}^{a5} \in \text{End}(\mathcal{H}_n), \quad a = 0, \ldots, 4.$$  

However, it is more appropriate here to define the displacement Hamiltonian using $\eta_{ab}$, so that $SO(4, 1)$ is preserved. Then we can assume that $x = (x_0, 0, 0, 0, 0)$, so that

$$H_x = \frac{1}{2} \sum_{i=1}^{4} X_i^2 - \frac{1}{2} (X_0 - x_0)^2 = \frac{1}{2} (R^2 - x_0^2) \mathbb{1} + x_0 X_0.$$  

Then the resulting quasi-coherent states form an abstract quantum space $\mathcal{M} \cong \mathbb{C}P^{1,2}$, which is an $S^2$ bundle over $H^4$. It is a Kähler manifold, and theorem 4.2 still holds in a weaker sense [37]. This in turn is the basis of the cosmological space-time solution $\mathcal{M}^{3,1}_n$ with an effective metric of FLRW type, as discussed in [44, 49].

Minimal fuzzy $H^4_0$. A particularly interesting example is obtained from $H^4_n$ for $n = 0$, which is not a quantized coadjoint orbit and not even symplectic. In that case $E_x$ is one-dimensional, and one can check that $\langle x | \partial_a | x \rangle = 0 = i A_a$ and $\langle x | [X^a, X^b] | x \rangle = 0$. Therefore the would-be symplectic form $\omega$ vanishes. The abstract quantum space is then

$$\mathcal{M} = H^4$$
but it carries a trivial line bundle $\mathcal{F}$. It still satisfies the quantum Kähler condition \cite{19} and theorem \ref{thm:4.2} should hold (using the $SO(4,1)$-invariant integral) in a weaker sense. However this is not an almost-local quantum space, and there is no semi-classical regime.

### 6.4 The minimal fuzzy torus

The minimal fuzzy torus $T^2_\theta$ turns out to be a quantum manifold which is not Kähler, and not even symplectic. It is defined in terms of

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X_1 + iX_2, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = X_3 + iX_4$$

(176)

which defines 4 hermitian matrices $X_i = X_i^\dagger \in \text{End}(\mathbb{C}^2)$. Noting that $[U, U^\dagger] = 0 = [V, V^\dagger]$ and

$$(U - z)(U - z)^\dagger = \begin{pmatrix} 1 + |z|^2 & -z - z^* \\ -z - z^* & 1 + |z|^2 \end{pmatrix}$$

$$(V - w)(V - w)^\dagger = \begin{pmatrix} |1 - w|^2 & 0 \\ 0 & |1 + w|^2 \end{pmatrix}$$

(177)

where $z = x_1 + ix_2$ and $w = x_3 + ix_4$, the displacement Hamiltonian is

$$H_y = \sum (X_i - x_i)^2 = \begin{pmatrix} 1 + |z|^2 + |1 - w|^2 & -z - z^* \\ -z - z^* & 1 + |z|^2 + |1 + w|^2 \end{pmatrix}.$$  

(178)

The lowest eigenvalue is

$$\lambda = 2 + |z|^2 + |w|^2 - \sqrt{|z + z^*|^2 + |w + w^*|^2}$$

(179)

and the corresponding quasi-coherent states are

$$|x\rangle \propto \left( \frac{\sqrt{|z + z^*|^2 + |w + w^*|^2} + w^* + w}{z^* + z} \right) \in \mathbb{R}_+ \times \mathbb{R} \subset \mathbb{C}^2.$$  

(180)

These clearly depend only on the real parts of $z, w$, and the normalized states describe a half circle in the upper half plane. However the two endpoints of this half-circle corresponding to $(z = 1, w = -\infty)$ and $(z = -1, w = -\infty)$ describe the same state $|x\rangle = \left( \begin{array}{c} 0 \\ \pm 1 \end{array} \right)$, and should hence be identified. Thus $\mathcal{M} = S^1$, which is clearly not a Kähler manifold any not even symplectic.

Now consider the equivalence classes $\sim (75)$ on $\mathbb{R}^4 \cong \mathbb{C}^2$. All points $(z, w) \sim (z', w') \in \mathbb{C}^2$ with the same real parts are identified, and also all $(z, w) \sim r(z, w) \in \mathbb{R}^2$ for $r > 0$. Among these, $\lambda$ assumes the minimum $\lambda = 1$ for $(z, w) = (x, y) \in S^1 \subset \mathbb{C}^2$, so that again $\mathcal{M} \cong \mathbb{C}^2/\sim \cong S^1$.

Therefore the minimal fuzzy torus $T^2_\theta$ should really be considered as a fuzzy circle. This shows the existence of "exotic" quantum spaces which are not quantized symplectic spaces, but do not have a semi-classical regime. There are also higher-dimensional such spaces as shown next, and the above example of minimal $H^4_\theta$.

\footnote{Note that $\dim \mathcal{M} = \infty$ here, so that we cannot conclude that $\mathcal{M}$ is Kähler in the usual sense.}

\footnote{It may seem that the state corresponding to the point $(z = 0, w = -1)$ vanishes, but this is just an artefact of the improper normalization. It is easy to see that in that case $H_y$ has indeed an eigenstate $(0, 1)$ for $\lambda = 1$.}

30
Non-Kähler quantum space from $T^2_2 \times T^2_2$. Now consider the Cartesian product of $T^2_2 \times T^2_2$, realized through 8 hermitian matrices $X^a_{(1)}, X^a_{(2)}$ acting on $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. All eigenstates of $H_x = H_x^{(1)} + H_x^{(2)}$ are given by the product states of the two eigenstates (180) of $T^2_2$, so that the ground states or quasi-coherent states are given by

$$|x_{(1)}, x_{(2)}\rangle \equiv |x_{(1)}\rangle \otimes |x_{(2)}\rangle \quad (181)$$

over $\mathbb{R}^8$. They are again degenerate, and inequivalent states are parametrized by $(x_{(1)}, x_{(2)}) \in S^1 \times S^1$. Hence the abstract quantum space is a torus $\mathcal{M} \cong S^1 \times S^1$. The quantum tangent space is spanned by two vectors

$$T^2_\mathcal{M} = \left\{ (\partial_1 |y_{(1)}\rangle) \otimes |y_{(2)}\rangle, |y_{(1)}\rangle \otimes (\partial_2 |y_{(2)}\rangle) \right\} \cong \mathbb{R}^2 \quad (182)$$

which are linearly independent from the two complexified vectors $i\partial_1 |y_{(1)}\rangle \otimes |y_{(2)}\rangle$ and $i|y_{(1)}\rangle \partial_2 \otimes |y_{(2)}\rangle$. Therefore $T^2_{\mathcal{M}} \cong \mathbb{C}^2 \cong \mathbb{R}^4$, and $\mathcal{M}$ is not a quantum Kähler manifold.

6.5 The Moyal-Weyl quantum plane

The Moyal-Weyl quantum plane is obtained for $X_1 = X$ and $X_2 = Y$ with $[X, Y] = i\mathbb{1}$. Then $\dim \mathcal{H} = \infty$, but all considerations can be carried over easily. The displacement Hamiltonian

$$2H_x = (X - x)^2 + (Y - y)^2 \quad (183)$$

is nothing but the shifted harmonic oscillator, with ground state

$$H_x|z\rangle = \frac{1}{2}|z\rangle \quad (184)$$

given by the standard coherent states

$$|z\rangle = U(z)|0\rangle, \quad z = \frac{1}{\sqrt{2}}(x + iy) \quad (185)$$

using the identification of $\mathbb{R}^2 \cong \mathbb{C}$. The translation operator is given as usual by

$$U(z) = \exp(i(yX - xY)) = \exp(za^\dagger - z^\ast a),$$

$$a = \frac{1}{\sqrt{2}}(X + iY), \quad a^\dagger = \frac{1}{\sqrt{2}}(X - iY). \quad (186)$$

$|0\rangle$ is the ground state of the harmonic oscillator $a|0\rangle = 0$, and more generally

$$(a - z)|z\rangle = 0 \quad (187)$$

implies

$$(z|X + iY)|z\rangle = x + iy. \quad (188)$$

The derivatives (32) are found to be

$$(\partial_x - iA_1)|z\rangle = -i(Y - y)|z\rangle = i\mathcal{X}_1|z\rangle$$

$$(\partial_y - iA_2)|z\rangle = i(X - x)|z\rangle = i\mathcal{X}_2|z\rangle \quad (189)$$
where the second expressions arise from \(35\), which are given explicitly by
\[
\begin{align*}
\mathcal{X}_1 &= -i((H_z - \frac{1}{2})' - 1) , X \\
\mathcal{X}_2 &= -i((H_z - \frac{1}{2})' - 1) , Y .
\end{align*}
\] (190)
The \(U(1)\) connection is found to be
\[
\begin{align*}
iA_1 &= \langle z | \partial_x | z \rangle = -i\langle z | (Y - \frac{1}{2}y) | z \rangle = -\frac{i}{2}y \\
iA_2 &= \langle z | \partial_y | z \rangle = i\langle z | (X - \frac{1}{2}x) | z \rangle = \frac{i}{2}x
\end{align*}
\] (191)
with field strength
\[
F_{12} = \partial_1 A_2 - \partial_2 A_1 = 1 .
\] (192)
Therefore \(38\) becomes
\[
|z\rangle = P \exp \left( i \int_0^z (\mathcal{X}_1 - y) dx + (\mathcal{X}_2 + x) dy \right) |0\rangle .
\] (193)
\(M \cong \mathbb{C}\) satisfies the quantum Kähler condition due to the constraint \((X + iY) |0\rangle = 0\), which states that \(iY |0\rangle = -X |0\rangle\), so that the complex tangent space \(T_{0,\mathbb{C}}M = T_0M\) coincides with the real one. The holomorphic coherent states are given by
\[
\|z\rangle = e^{za_\dag} |0\rangle = e^{za_\dag} e^{-\bar{z}a} |0\rangle = e^{\frac{1}{2}|z|^2} |z\rangle .
\] (194)
They cannot be normalized, since the map \(z \mapsto \langle w | z \rangle\) must be holomorphic and hence unbounded. Thus \(\|z\rangle\) should be viewed as holomorphic section of the line bundle \(\tilde{\mathcal{B}}\).

### 6.6 Commutative quantum spaces

In the infinite-dimensional case, one can also consider matrix configurations associated to commutative manifolds. The simplest example is the circle \(S^1\), which arises from the single operator
\[
X = -i\partial_\varphi
\] (195)
acting on \(C^\infty(S^1) \subset L^2(S^1) = \mathcal{H}\). The displacement Hamiltonian is
\[
H_x = \frac{1}{2}(-i\partial_\varphi - x)^2 , \quad x \in \mathbb{R} .
\] (196)
The quasi-coherent states for \(x = n \in \mathbb{Z}\) are clearly
\[
|n\rangle = e^{in\varphi} , \quad H_x |n\rangle = 0 , \quad n \in \mathbb{Z}
\] (197)
so that \(\lambda(\mathbb{Z}) = 0\). For any \(x \notin \mathbb{Z}\), all eigenstates of \(H_x |\psi\rangle = E |\psi\rangle\) are given by the above states \(|n\rangle\), with eigenvalue
\[
H_x |n\rangle = (-i\partial_\varphi - x)^2 e^{in\varphi} = (n - x)^2 e^{in\varphi} .
\] (198)
Therefore
\[ |x\rangle = |n\rangle, \quad |n - x| < \frac{1}{2}, \quad n \in \mathbb{Z} \] 
(199)
while for \( x \in \mathbb{Z} + \frac{1}{2} \) the space \( E_x \) is two-dimensional, containing both states \( |x \pm \frac{1}{2}\rangle \). Thus the abstract quantum space is the discrete lattice
\[ \mathcal{M} = \mathbb{Z} \subset \mathbb{R} \] 
(200)
and the quantum tangent space vanishes. This can be generalized to the higher-dimensional commutative torus \( T^n \) with commutative and reducible matrix configuration \( X_\mu = -i \partial_\mu \), which also leads to a discrete quantum space without further structure. Thus classical manifolds are not well captured in the present framework. This can of course be treated by adding extra structure as in [7], but such a description is not well suited for Yang-Mills matrix models.

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7 Conclusion

A general framework for quantum geometry was developed, based on general matrix configurations given in terms of \( D \) hermitian matrices \( X^a \). We have seen that a remarkably rich array of structures can be extracted from such a matrix configuration, which provide a semi-classical picture and geometric insights. Quasi-coherent states are an optimal set of states where the matrices are simultaneously "almost-diagonal". They form an abstract quantum space \( \mathcal{M} \subset \mathbb{C}P^N \), which allows to use geometric tools and even complex analysis. A class of almost-local operators \( \text{Loc}(\mathcal{H}) \) is characterized, which can be understood as quantized functions on \( \mathcal{M} \) in some IR regime. Moreover, a natural sub-class of matrix configurations is identified as quantum Kähler manifolds.

Although the present analysis is restricted to the case of finite-dimensional matrices, the concepts generalize to the case of selfadjoint operators on separable Hilbert spaces. This is illustrated for the Moyal-Weyl quantum plane and for the fuzzy hyperboloid. In these cases, the framework exhibits the finite number of degrees of freedom per unit volume, as well as the stringy nature in the deep quantum regime. It should also be useful to better understand other quantum spaces such as \( \kappa \) Minkowski space [11], and to resolve a hidden internal structure in other spaces such as [50] and in compact quantum spaces with infinite-dimensional \( \mathcal{H} \).

This framework for quantum geometry is particularly suited for Yang-Mills-type matrix models. Their description in terms of quantized symplectic spaces is now understood to be generic, rather than just an ad-hoc choice. This vindicates describing the low-energy regime of such matrix models via noncommutative field theory on the embedded quantum space or brane \( \mathcal{M} \), leading to dynamical emergent geometry and possibly gravity, cf. [27, 49]. However, it is important to keep in mind that semi-classical picture breaks down in the UV or deep quantum regime, where non-local string states become dominant. These are naturally interpreted as open strings on the brane \( \mathcal{M} \).

In particular, the new insights on the structure of \( \mathcal{M} \) should be very useful to interpret the results of numerical simulations of Yang-Mills matrix models [8, 9, 10, 11]. By definition,
the quasi-coherent states provide an optimal basis where the matrices are "almost-diagonal", which should improve upon simpler approaches based on block-matrices. They can be obtained numerically along the lines proposed in [13, 18], which can now be refined, notably using the abstract point of view as $M \subset \mathbb{C}P^{N-1}$. It should then be easier to disentangle the underlying geometry from the random noise.

The framework should also be useful for analytical computations in the context of noncommutative field theory. Given the natural role of quantum Kähler manifolds in this setting, one may hope that quantum Kähler manifolds play a special and preferred role not only from an analytical point of view, but also as preferred solutions or configurations in a matrix "path integral". For example, loop integrals analogous to (142) can be formulated in terms of the completeness relation for string states [29]. In particular, one may hope that some sort of non-renormalization statement can be made on such spaces.

Finally, it would be desirable to improve some of the technical results in this paper, notably related to the completeness relation and the regularity of $M$. In particular, one would like to know to which extent the results on quantum Kähler manifolds can be generalized to generic quantum manifolds with symplectic structure and a metric. It would also be interesting to develop an analogous approach based on the matrix Dirac operator as sketched in section [5] and to relate it to the present approach. All these are interesting directions for future work.

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