Hard Implicit Function Theorem via the DSM

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The aim of this paper is to demonstrate the power of the Dynamical Systems Method (DSM) as a tool for proving theoretical results. The DSM was systematically developed in [6] and applied to solving nonlinear operator equations in [6] (see also [7]), where the emphasis was on convergence and stability of the DSM-based algorithms for solving operator equations, especially nonlinear and ill-posed equations.
The DSM for solving an operator equation $F(u) = h$ consists of finding a nonlinear map $u \mapsto \Phi(t, u)$, depending on a parameter $t \in [0, \infty)$, that has the following three properties: (1) the Cauchy problem

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0 \quad (\dot{u} := \frac{du(t)}{dt})$$

has a unique global solution $u(t)$ for a given initial approximation $u_0$; (2) the limit $u(\infty) = \lim_{t \to \infty} u(t)$ exists; and (3) this limit solves the original equation $F(u) = h$, i.e., $F(u(\infty)) = h$. The operator $F : H \to H$ is a nonlinear map in a Hilbert space $H$. It is assumed that the equation $F(u) = h$ has a solution, possibly nonunique.
Introduction

The problem is to find a \( \Phi \) such that the properties (1), (2), and (3) hold. Various choices of \( \Phi \) for which these properties hold are proposed in [6], where the DSM is justified for wide classes of operator equations, in particular, for some classes of nonlinear ill-posed equations (i.e., equations \( F(u) = 0 \) for which the linear operator \( F'(u) \) is not boundedly invertible). By \( F'(u) \) we denote the Fréchet derivative of the nonlinear map \( F \) at the element \( u \). In this note the DSM is used as a tool for proving a new ”hard” implicit function theorem.
Let us first recall the usual implicit function theorem.

**Proposition:** If $F(U) = f$, $F$ is a $C^1$-map in a Hilbert space $H$, and $F'(U)$ is a boundedly invertible operator, i.e., $\| [F'(U)]^{-1} \| \leq m$, then the equation

$$F(u) = h$$

(1)

is uniquely solvable for every $h$ sufficiently close to $f$.

For convenience of the reader we include a proof of this known result.
First, one can reduce the problem to the case $u = 0$ and $h = 0$. This is done as follows. Let $u = U + z$, $h - f = p$, $F(U + z) - F(U) := \phi(z)$. Then $\phi(0) = 0$, $\phi'(0) = F'(U)$, and equation (1) is equivalent to the equation

$$\phi(z) = p,$$

with the assumptions

$$\phi(0) = 0, \lim_{z \to 0} \|\phi'(z) - \phi'(0)\| = 0, \|[\phi'(0)]^{-1}\| \leq m. \quad (3)$$
Proof of the preposition

We want to prove that equation (2) under the assumptions (3) has a unique solution $z = z(p)$, such that $z(0) = 0$, and $\lim_{p \to 0} z(p) = 0$. To prove this, consider the equation

$$z = z - \left[\phi'(0)\right]^{-1}(\phi(z) - p) := B(z),$$

and check that the operator $B$ is a contraction in a ball $B_\epsilon := \{z : \|z\| \leq \epsilon\}$ if $\epsilon > 0$ is sufficiently small, and $B$ maps $B_\epsilon$ into itself. If this is proved, then the desired result follows from the contraction mapping principle.
One has

\[ \| B(z) \| = \| z - [\phi'(0)]^{-1}(\phi'(0)z + \eta - p) \| \leq m\|\eta\| + m\|p\|, \tag{5} \]

where \( \|\eta\| = o(\|z\|) \). If \( \epsilon \) is so small that \( m\|\eta\| < \frac{\epsilon}{2} \) and \( p \) is so small that \( m\|p\| < \frac{\epsilon}{2} \), then \( \|B(z)\| < \epsilon \), so \( B : B_{\epsilon} \to B_{\epsilon} \).

Let us check that \( B \) is a contraction mapping in \( B_{\epsilon} \). One has:

\[ \| Bz - By \| = \| z - y - [\phi'(0)]^{-1}(\phi(z) - \phi(y)) \| \]

\[ = \| z - y - [\phi'(0)]^{-1} \int_{0}^{1} \phi'(y + t(z - y)) \, dt \, (z - y) \| \]

\[ \leq m \int_{0}^{1} \| \phi'(y + t(z - y)) - \phi'(0) \| \, dt \|z - y\|. \tag{6} \]
If $y, z \in B_\varepsilon$, then

$$\sup_{0 \leq t \leq 1} \|\phi'(y + t(z - y)) - \phi'(0)\| \leq \eta(\varepsilon) \to 0, \quad \varepsilon \to 0.$$  

Therefore, if $\varepsilon$ is so small that $m\eta(\varepsilon) < 1$, then $B$ is a contraction mapping in $B_\varepsilon$, and equation (2) has a unique solution $z = z(p)$ in $B_\varepsilon$, such that $z(0) = 0$. The proof is complete. □
The crucial assumptions, on which this proof is based, are assumptions (3).
Suppose now that $\phi'(0)$ is not boundedly invertible, so that the last assumption in (3) is not valid. Then a theorem which still guarantees the existence of a solution to equation (2) for some set of $p$ is called a ”hard” implicit function theorem. Examples of such theorems one may find, e.g., in [1], [2], [3], and [4].
Our goal is to establish a theorem of this type using a new method of proof, based on the Dynamical Systems Method (DSM). In [8] we have demonstrated a theoretical application of the DSM by establishing some surjectivity results for nonlinear operators. Our result is a new illustration of the applicability of the DSM as a tool for proving theoretical results.
To formulate the result, let us use the scale of Hilbert spaces $H_a$ (see [5]). Let $H_a \subset H_b$ and $\|u\|_b \leq \|u\|_a$ if $a \geq b$. Example of spaces $H_a$ is the scale of Sobolev spaces $H_a = W^{a,2}(D)$, where $D \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary. Consider equation (1). Assume that

$$F(U) = f, \quad F : H_a \to H_{a+\delta}, \quad u \in B(U, R), \quad (7)$$

where $B(U, R) := \{u : \|u - U\|_a \leq R\}$ and $\delta = const > 0$, and the operator $F : H_a \to H_{a+\delta}$ is continuous.
Furthermore, assume that $A := A(u) := F'(u)$ exists and is an isomorphism of $H_a$ onto $H_{a+\delta}$:

$$c_0 \|v\|_a \leq \|A(u)v\|_{a+\delta} \leq c'_0 \|v\|_a, \quad u, v \in B(U, R), \quad (8)$$

that

$$\|A^{-1}(v)A(w)\|_a \leq c, \quad v, w \in B(U, R), \quad (9)$$

and

$$\|A^{-1}(u)[A(u) - A(v)]\|_a \leq c\|u - v\|_a, \quad u, v \in B(U, R). \quad (10)$$

Here and below we denote by $c > 0$ various constants.
Assume that
\[ u_0 \in B_a(U, \rho), \quad h \in B_{a+\delta}(f, \rho), \]  
where
\[ B_a(U, \rho) := \{ u : \| u - U \|_a \leq \rho \}, \]
and \[ \rho \leq \frac{R}{1+c_0^{-1}(1+c'_0)}. \]
Consider the problem
\[ \dot{u} = -[F'(u)]^{-1}(F(u) - h), \quad u(0) = u_0. \]  

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Theorem

If the assumptions (7)-(11) hold, and \(0 < \rho \leq \rho_0 := \frac{R}{1+c_0^{-1}(1+c_0')},\)
where \(c_0, c_0'\) are the constants from (8), then problem (13) has a unique global solution \(u(t)\),
there exists \(V := u(\infty)\),

\[
\lim_{t \to \infty} \|u(t) - V\|_a = 0,
\]

and

\[
F(V) = h.
\]

Theorem 1 says that if \(F(U) = f\) and \(\rho \leq \rho_0\), then for any \(h \in B_{a+\delta}(f, \rho)\) equation (1) is solvable and the solution to (1) is \(u(\infty)\), where \(u(\infty)\) solves problem (13).
Proof of the theorem

Let us outline the ideas of the proof. The local existence and uniqueness of the solution to (13) will be established if one verifies that the operator $A^{-1}(u)[F(u) - h]$ is Lipschitz in $H_a$. The global existence of this solution $u(t)$ will be established if one proves the uniform boundedness of $u(t)$:

$$\sup_{t \geq 0} \|u(t)\|_a \leq c. \quad (16)$$

If $u(t)$ exists locally, then the function

$$g(t) := \|\phi\|_{a+\delta} := \|F(u(t)) - h\|_{a+\delta} \quad (17)$$

satisfies the relation

$$g\dot{g} = (F'(u(t))\dot{u}, \phi)_{a+\delta} = -g^2, \quad (18)$$

where equation (13) was used.
Since \( g \geq 0 \), it follows from (18) that

\[
g(t) \leq g(0)e^{-t}, \quad g(0) = \| F(u_0) - h \|_{a+\delta}. \tag{19}
\]

From (13), (18) and (8) one gets:

\[
\| \dot{u} \|_a \leq \frac{1}{c_0} \| \phi \|_{a+\delta} = \frac{g(0)}{c_0} e^{-t} := re^{-t}, \quad r := \frac{\| F(u_0) - h \|_{a+\delta}}{c_0}. \tag{20}
\]

Therefore,

\[
\lim_{t \to \infty} \| \dot{u}(t) \|_a = 0, \tag{21}
\]

and

\[
\int_0^\infty \| \dot{u}(t) \|_a dt < \infty. \tag{22}
\]

This inequality and the Cauchy criterion for the existence of the limit \( V := u(\infty) \) imply (14).
Assumptions (7) and (8) and relations (13), (14), and (21) imply (15). Integrating inequality (20) yields

$$\|u(t) - u_0\|_a \leq r,$$

(23)

and

$$\|u(t) - u(\infty)\|_a \leq re^{-t}.$$  

(24)

Inequality (23) implies (16).

Let us complete the proof of Theorem 1 by proving that the operator in (13) $A^{-1}(u)[F(u) - h]$ is Lipschitz in $H_a$. 
One has
\[ \|A^{-1}(u)(F(u) - h) - A^{-1}(v)(F(v) - h)\|_a \leq \|[A^{-1}(u) - A^{-1}(v)](F(u) + A^{-1}(v)(F(u) - F(v))\|_a := I_1 + I_2. \] (25)

Write
\[ F(u) - F(v) = \int_0^1 A(v + t(u - v))(u - v)dt, \] (26)
and use assumption (9) with \( w = v + t(u - v) \) to conclude that
\[ I_2 \leq c\|u - v\|_a. \] (27)

Write
\[ A^{-1}(u) - A^{-1}(v) = A^{-1}(u)[A(v) - A(u)]A^{-1}(v), \] (28)
and use the estimate
\[ \|A^{-1}(v)[F(u) - h]\|_a \leq c, \] (29)
which is a consequence of assumptions (7) and (8).
Then use assumption (10) to conclude that

\[ I_1 \leq c\|u - v\|_a. \]  

(30)

From (25), (27) and (30) it follows that the operator

\[ A^{-1}(u)[F(u) - h] \] is Lipschitz. Note that

\[ \|u(t) - U\|_a \leq \|u(t) - u_0\|_a + \|u_0 - U\|_a \leq r + \rho, \]  

(31)

\[ \|F(u(t)) - h\|_{a+\delta} \leq \|F(u_0) - h\|_{a+\delta} \]

\[ \leq \|F(u_0) - f\|_{a+\delta} + \|f - h\|_{a+\delta} \leq (1 + c'_0)\rho, \]

(32)

so, from (20) one gets

\[ r \leq \frac{(1 + c'_0)\rho}{c_0}. \]  

(33)
Choose

\[ R \geq r + \rho. \tag{34} \]

Then the trajectory \( u(t) \) stays in the ball \( B(U, R) \) for all \( t \geq 0 \), and, therefore, assumptions (7)-(10) hold in this ball for all \( t \geq 0 \). Condition (34) and inequality (33) imply

\[ \rho \leq \frac{R}{1 + c_0^{-1}(1 + c_0')} . \tag{35} \]

This is the "smallness" condition on \( \rho \).

Theorem 1 is proved. \( \square \)
Let

\[ F(u) = \int_0^x u^2(s) \, ds, \quad x \in [0, 1]. \]

Then

\[ A(u)q = 2 \int_0^x u(s)v(s) \, ds. \]

Let \( f = x \) and \( U = 1 \). Then \( F(U) = x \). Choose \( a = 1 \) and \( \delta = 1 \). Denote by \( H_a = H_a(0, 1) \) the usual Sobolev space. Assume that \( h \in B_2(x, \rho) := \{ h : \| h - x \|_2 \leq \rho \} \), and \( \rho > 0 \) is sufficiently small. One can verify that \( A^{-1}(u)\psi = \frac{\psi'(x)}{2u(x)} \) for any \( \psi \in H_1 \).
Let us check conditions (7)-(11) for this example. Condition (7) holds, because if $u_n \to u$ in $H_1$, then
\[ \int_0^x u_n^2(s) ds \to \int_0^x u^2(s) ds \text{ in } H_2. \] To verify this, it is sufficient to check that
\[ \frac{d^2}{dx^2} \int_0^x u_n^2(s) ds \to 2uu', \]
where $\to$ means the convergence in $L^2(0,1)$. In turn, this is verified if one checks that $u'_n(x)u_n \to u'u$ in $L^2(0,1)$, provided that $u'_n \to u'$ in $L^2(0,1)$.

One has
\[ I_n := \|u'_n u_n - u'u\|_{L^2(0,1)} \leq \|(u'_n - u')u_n\| + \|u'(u_n - u)\|. \]
Since \( \|u'_n\|_{L^2(0,1)} \leq c \), one concludes that \( \|u_n\|_{L^\infty(0,1)} \leq c_1 \) and \( \lim_{n \to \infty} \|u_n - u\|_{L^\infty} = 0 \). Thus, \( \lim_{n \to \infty} I_n = 0 \). Condition (8) holds because \( \|u\|_{L^\infty(0,1)} \leq c\|u\|_1 \), and

\[
\left\| \int_0^x u(s)q(s)ds \right\|_2 \leq c\|u'q + uq'\|_0 \\
\leq c(\|q\|_{L^\infty(0,1)}\|u\|_1 + \|u\|_{L^\infty(0,1)}\|q\|_1) \\
\leq c'_0\|u\|_1\|q\|_1,
\]

and

\[
\left\| \int_0^x uqds \right\|_2 \geq \|uq\|_1 \geq c_0\|q\|_1,
\]

provided that \( u \in B_1(1, \rho) \) and \( \rho > 0 \) is sufficiently small.
Condition (9) holds because

\[ \|A^{-1}(v)A(w)q\|_1 = \|\frac{1}{v(x)}w(x)q\|_1 \leq c\|q\|_1, \]

provided that \(u, w \in B_1(1, \rho)\) and \(\rho > 0\) is sufficiently small.

Condition (10) holds because

\[ \|A^{-1}(u)\int_0^x (u - v)qds\|_1 = \|\frac{u - v}{2u}q\|_1 \leq c\|u - v\|_1\|q\|_1, \]

provided that \(u, v \in B_1(1, \rho)\) and \(\rho > 0\) is sufficiently small.

By Theorem 1 the equation \(F(u) := \int_0^x u^2(s)ds = h\), where \(\|h - 1\|_2 \leq \rho\) and \(\rho > 0\) is sufficiently small has a solution \(V\), \(F(V) = h\). This solution can be obtained as \(u(\infty)\), where \(u(t)\) solves problem (13) and conditions (11) and (35) hold.
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