The Einstein-Podolsky-Rosen State Maximally Violates Bell’s Inequalities

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Abstract

In their well-known argument against the completeness of quantum theory, Einstein, Podolsky, and Rosen (EPR) made use of a state that strictly correlates the positions and momenta of two particles. We prove the existence and uniqueness of the EPR state as a normalized, positive linear functional of the Weyl algebra for two degrees of freedom. We then show that the EPR state maximally violates Bell’s inequalities.

1 Introduction

In their well-known argument against the completeness of quantum theory, Einstein, Podolsky, and Rosen (EPR) make use of a state that strictly correlates the positions and momenta of two particles [6]. However, since these strict correlations require the EPR state to be a common “eigenstate” of a pair of continuous spectrum observables, viz., $\hat{x}_1 - \hat{x}_2$ and $\hat{p}_1 + \hat{p}_2$, the problematic mathematical status of the EPR state has caused difficulties for two recent attempts to assess its foundational significance.

First, it has so far remained an open question whether the EPR state has nonlocal features similar to those of Bohm’s spin-1/2 singlet state. On the one hand, J. S. Bell [4] has shown that the correlations in the EPR state admit a local hidden variable model, at least for certain simple functions of the positions and momenta of the two particles. On the other hand, Cohen [5] has claimed to show that the EPR state does predict nonlocal correlations between more general observables associated with the respective particles.
Unfortunately, Cohen’s argument is unsatisfactory, since it attempts to use the fact that any entangled vector state violates Bell’s inequalities to draw a conclusion about the EPR state—which is not a vector state in the standard Schrödinger representation.

Second, Arens and Varadarajan [2] have recently given a characterization of states with EPR-type correlations. Ironically, however, their characterization does not extend to correlations between observables with continuous spectra; in particular, their characterization does not extend to the original EPR state.

In this letter, we resolve both difficulties noted above. First, we show that the EPR state has a mathematically unproblematic definition as a normalized, positive linear functional on the Weyl algebra for two degrees of freedom. Second, we show that the EPR state maximally violates Bell’s inequalities. Finally, we show that the original EPR state satisfies the characterization of EPR-type states recently given by Werner [15], which naturally extends the characterization given by Arens and Varadarajan.

2 Existence and uniqueness of the EPR state

Let $\mathfrak{A}[\mathbb{R}^2]$ denote the Weyl algebra over $\mathbb{R}^2$. That is, $\mathfrak{A}[\mathbb{R}^2]$ is the $C^*$-algebra generated by unitary operators $W(a)$, $a \in \mathbb{R}^2$, satisfying the Weyl relations

$$W(a)W(b) = \exp\{i\sigma(a, b)\}W(a + b),$$

(1)

with

$$\sigma((a, b), (a', b')) = \frac{1}{2}(ab' - ba'), \quad a, b \in \mathbb{R}. \quad (2)$$

(See [10], [11] for more details.) Let $\mathfrak{A}[\mathbb{R}^4]$ denote the Weyl algebra over $\mathbb{R}^4$. (In this case, the Weyl generators satisfy the analogue of Eq. 1 with $\sigma \oplus \sigma$ replacing $\sigma$.) Since $\mathfrak{A}[\mathbb{R}^{2n}]$ is simple $(n = 1, 2)$, all representations of $\mathfrak{A}[\mathbb{R}^{2n}]$ are faithful. We say that a linear functional $\omega$ of a $C^*$-algebra $\mathfrak{A}$ is a state just in case $\omega$ is positive and normalized.

The observables for a composite system consisting of a pair of one-dimensional particles are given by the self-adjoint elements of the spatial tensor product $\mathfrak{A}[\mathbb{R}^2] \otimes \mathfrak{A}[\mathbb{R}^2]$. If $\pi$ is a regular representation of $\mathfrak{A}[\mathbb{R}^2] \otimes \mathfrak{A}[\mathbb{R}^2]$, then Stone’s theorem entails the existence of self-adjoint operators $\hat{x}_i, \hat{p}_i (i = 1, 2)$ such
that
\[ \pi(W(a, 0) \otimes I) = e^{ia\hat{x}_1}, \quad \pi(W(0, b) \otimes I) = e^{ib\hat{p}_1}, \] (3)
\[ \pi(I \otimes W(c, 0)) = e^{ic\hat{x}_2}, \quad \pi(I \otimes W(0, d)) = e^{id\hat{p}_2}. \] (4)

Heuristically, the EPR state assigns a dispersion-free value \( \lambda \) to the relative position \( \hat{x}_1 - \hat{x}_2 \) of the two particles, and a dispersion-free value \( \mu \) to the total momentum \( \hat{p}_1 + \hat{p}_2 \) of the two particles. (EPR themselves chose the value \( \mu = 0 \).) Accordingly, the EPR state assigns the value \( e^{ia\lambda} \) to \( e^{i\alpha(\hat{x}_1 - \hat{x}_2)} = e^{ia\hat{x}_1}e^{-ia\hat{x}_2} \) and the value \( e^{ib\mu} \) to \( e^{ib(\hat{p}_1 + \hat{p}_2)} = e^{ib\hat{p}_1}e^{ib\hat{p}_2} \). We show now that these latter conditions suffice to define a unique EPR state of \( \mathfrak{A}[\mathbb{R}^2] \otimes \mathfrak{A}[\mathbb{R}^2] \).

**Theorem 1.** For each pair of real numbers \( \lambda, \mu \), there is a unique state \( \omega \) of \( \mathfrak{A}[\mathbb{R}^2] \otimes \mathfrak{A}[\mathbb{R}^2] \) such that
\[ \omega(W(a, 0) \otimes W(-a, 0)) = e^{ia\lambda}, \quad \omega(W(0, b) \otimes W(0, b)) = e^{ib\mu}, \] (5)
for all \( a, b \in \mathbb{R} \). Moreover, \( \omega \) is pure.

**Proof.** (Existence) We presuppose the canonical isomorphism between \( \mathfrak{A}[\mathbb{R}^2] \otimes \mathfrak{A}[\mathbb{R}^2] \) and \( \mathfrak{A}[\mathbb{R}^4] \) (see [11, 3.4.1]). Every state \( \omega \) on \( \mathfrak{A}[\mathbb{R}^4] \) gives rise to a function \( G : \mathbb{R}^4 \to \mathbb{C} \) via the equation \( G(x) = \omega(W(x)) \). Conversely, a function \( G : \mathbb{R}^4 \to \mathbb{C} \) gives rise to a state \( \omega \) on \( \mathfrak{A}[\mathbb{R}^4] \) just in case \( G(0) = 1 \) and the map \( x, y \mapsto G(x - y) \exp\{-i(\sigma \oplus \sigma)(x, y)\} \) is a positive definite kernel \([11, \text{p. 18}], \)

Define \( G \) by
\[ G(a, b, c, d) = \delta(a + c)\delta(b - d) e^{ia\lambda}e^{ib\mu}, \] (6)
where \( \delta \) is the characteristic function of \( \{0\} \). Obviously, \( G(0) = 1 \). Now define \( F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{C} \) by
\[ F(x, y) = G(x - y) \exp\{-i(\sigma \oplus \sigma)(x, y)\}. \] (7)

To see that \( F \) is positive-definite, let \( \{z_1, \ldots, z_n\} \subseteq \mathbb{C}\backslash\{0\} \) and let \( \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^4 \) with \( x_j = (a_j, b_j, c_j, d_j) \). Fix \( j, k \). It follows then that
\[ F(x_j, x_k) = \delta(a_j - a_k + c_j - c_k) \delta(b_j - b_k + d_k - d_j) \alpha_j\alpha_k, \] (8)
where
\[ \alpha_j = \exp\{i(a_j\lambda + b_j\mu)\} \exp\{i(b_j(a_j + c_j) - a_j(b_j - d_j))\}. \] (9)
Define a relation $R$ on $\{1, \ldots, n\}$ by

$$(j, k) \in R \iff F(x_j, x_k) \neq 0.$$  

An inspection of Eq. 8 shows that $R$ is an equivalence relation. Thus, there are disjoint subsets $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$ such that $R = \bigcup_{i=1}^{m} (S_i \times S_i)$, and

$$\sum_{j=1}^{n} \sum_{k=1}^{n} z_j \bar{z}_k F(x_j, x_k) = \sum_{(j, k) \in R} z_j \bar{z}_k \alpha_j \alpha_k \quad (10)$$

$$= \sum_{(j, k) \in S_1 \times S_1} z_j \alpha_j \bar{z}_k \alpha_k + \cdots + \sum_{(j, k) \in S_m \times S_m} z_j \alpha_j \bar{z}_k \alpha_k \quad (11)$$

$$= \sum_{j \in S_1} |z_j \alpha_j|^2 + \cdots + \sum_{j \in S_m} |z_j \alpha_j|^2 \geq 0. \quad (12)$$

Therefore, $F$ is positive-definite.

(Uniqueness) Let $\mathfrak{A} = \mathfrak{A}[\mathbb{R}^2] \otimes \mathfrak{A}[\mathbb{R}^2]$ and let $\rho$ be a state of $\mathfrak{A}$ such that

$$\rho(W(a, 0) \otimes W(-a, 0)) = e^{ia\lambda}, \quad \rho(W(0, b) \otimes W(0, b)) = e^{ib\mu}, \quad (13)$$

for all $a, b \in \mathbb{R}$. Fix $s, t \in \mathbb{R}$, let $A = W(s, 0) \otimes W(-s, 0)$, and let $B = W(0, t) \otimes W(0, t)$. Thus, $|\rho(A)| = 1 = ||A||$ and $|\rho(B)| = 1 = ||B||$. Since $A, B$ are unitary, it follows from [1, p. 305] that

$$\rho(AX) = \rho(XA) = \rho(X)\rho(A), \quad (14)$$

$$\rho(BX) = \rho(XB) = \rho(X)\rho(B), \quad (15)$$

for any $X \in \mathfrak{A}$. Let $W(a, b) \otimes W(c, d) \in \mathfrak{A}$. Then, using the Weyl relations, we have

$$[W(s, 0) \otimes W(-s, 0)][(W(a, b) \otimes W(c, d)] = e^{iasb}e^{-isd}[W(a, b) \otimes W(c, d)][W(s, 0) \otimes W(-s, 0)], \quad (16)$$

and

$$[W(0, t) \otimes W(0, t)][W(a, b) \otimes W(c, d)] = e^{ita}e^{itc}[W(a, b) \otimes W(c, d)][W(0, t) \otimes W(0, t)]. \quad (17)$$

Using Eqs. 13, 14, and 16 it follows that

$$\rho(W(a, b) \otimes W(c, d)) = e^{iasb}e^{-isd}\rho(W(a, b) \otimes W(c, d)). \quad (18)$$
Since this is true for all \( s \in \mathbb{R} \), it follows that \( \rho(W(a, b) \otimes W(c, d)) = 0 \) when \( d \neq b \). Similarly, Eqs. 13, 15, and 17 entail that
\[
\rho(W(a, b) \otimes W(c, d)) = e^{ita}e^{itc} \rho(W(a, b) \otimes W(c, d)).
\]
(19)

Since this is true for all \( t \in \mathbb{R} \), it follows that \( \rho(W(a, b) \otimes W(c, d)) = 0 \) when \( c \neq -a \). When \( d = b \) and \( c = -a \), we have
\[
\rho(W(a, b, c, d)) = \rho(W(a, b, -a, b))
\]
(20)
\[
= \rho(W(a, 0) \otimes W(-a, 0)[W(0, b) \otimes W(0, b)])
\]
(21)
\[
= \rho(W(a, 0) \otimes W(-a, 0)) \rho(W(0, b) \otimes W(0, b))
\]
(22)
\[
= e^{ia\lambda}e^{ib\mu}.
\]
(23)

Thus, \( \rho \) agrees with \( \omega \) on all Weyl operators. Since the values of a state on the Weyl operators fixes its values on \( \mathfrak{A} \), it follows that \( \rho = \omega \).

(Purity) Let \( \mathfrak{B} \) denote the abelian subalgebra of \( \mathfrak{A} \) generated by \( W(s, 0) \otimes W(-s, 0) \) and \( W(0, t) \otimes W(0, t) \), with \( s, t \in \mathbb{R} \). We have seen that \( \omega|_{\mathfrak{B}} \) is multiplicative, and hence is a pure state. Thus, \( \omega|_{\mathfrak{B}} \) has an extension to a pure state \( \rho \) on \( \mathfrak{A} \). On the other hand, we have shown that \( \omega|_{\mathfrak{B}} \) has a unique extension. Therefore, \( \omega = \rho \), and \( \omega \) is pure. \( \square \)

3 Maximal Bell correlation of the EPR state

We first recall some pertinent definitions concerning Bell correlation for an arbitrary pair of commuting \( C^* \)-algebras. Suppose then that \( \mathfrak{A}_1, \mathfrak{A}_2 \) are mutually commuting subalgebras of a \( C^* \)-algebra \( \mathfrak{A} \), and let \( \omega \) be a state of \( \mathfrak{A} \). We set
\[
\mathcal{T}(\mathfrak{A}_1, \mathfrak{A}_2) \equiv \{ \frac{1}{2}[A_1(B_1 + B_2) + A_2(B_1 - B_2)] : A_i = A_i^* \in \mathfrak{A}_1, B_i = B_i^* \in \mathfrak{A}_2, \|A_i\| \leq 1, \|B_i\| \leq 1 \}.
\]

Elements of \( \mathcal{T}(\mathfrak{A}_1, \mathfrak{A}_2) \) are called Bell operators for \( \mathfrak{A}_1, \mathfrak{A}_2 \). Following [13, p. 223], we say that the maximal Bell correlation of the pair \( \mathfrak{A}_1, \mathfrak{A}_2 \) in the state \( \omega \) is
\[
\beta(\omega, \mathfrak{A}_1, \mathfrak{A}_2) \equiv \sup\{\omega(R) : R \in \mathcal{T}(\mathfrak{A}_1, \mathfrak{A}_2)\}.
\]
(24)

It follows that \( \beta(\omega, \mathfrak{A}_1, \mathfrak{A}_2) \in [1, \sqrt{2}] \) [13, Prop. 5.2]. If \( \beta(\omega, \mathfrak{A}_1, \mathfrak{A}_2) > 1 \), we say that \( \omega \) violates a Bell inequality, or is Bell correlated across \( \mathfrak{A}_1, \mathfrak{A}_2 \). If
\( \beta(\omega, \mathbf{A}_1, \mathbf{A}_2) = \sqrt{2} \), we say that \( \omega \) is \textit{maximally Bell correlated} across \( \mathbf{A}_1, \mathbf{A}_2 \).

In this context, Bell’s theorem \(^3\) is the statement that a local hidden variable model of the correlations that \( \omega \) dictates between \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) is possible only if \( \beta(\omega, \mathbf{A}_1, \mathbf{A}_2) = 1 \).

In order to demonstrate that the state \( \omega \) is Bell correlated across \( \mathbf{A}_1, \mathbf{A}_2 \), we may wish to pass to the GNS representation \((\pi, \mathcal{H}, \Omega)\) induced by \( \omega \) (thereby giving us access to the tools of the theory of von Neumann algebras). If \( \pi \) is faithful, we may choose observables from the weak closures \( \pi(\mathbf{A}_1)'' \), \( \pi(\mathbf{A}_2)'' \) in computing the maximal Bell correlation.

**Lemma 1.** Let \((\pi, \mathcal{H}, \Omega)\) be the GNS representation of \( \mathbf{A} \) induced by \( \omega \), and let \( \mathcal{R}_i = \pi(\mathbf{A}_i)'' \). If \( \pi \) is faithful, then

\[
\beta(\omega, \mathbf{A}_1, \mathbf{A}_2) = \sup \{ \langle \Omega, R\Omega \rangle : R \in \mathcal{T}(\mathcal{R}_1, \mathcal{R}_2) \}. \tag{25}
\]

**Proof.** Since \( \pi \) is faithful, we have

\[
\beta(\omega, \mathbf{A}_1, \mathbf{A}_2) = \sup \{ \langle \Omega, R\Omega \rangle : R \in \mathcal{T}(\pi(\mathbf{A}_1), \pi(\mathbf{A}_2)) \}. \tag{26}
\]

Using the Kaplansky density theorem, and the fact that multiplication is jointly continuous (in the strong operator topology) on bounded sets, it follows that \( \mathcal{T}(\mathcal{R}_1, \mathcal{R}_2) \) is contained in the strong-operator closure of \( \mathcal{T}(\pi(\mathbf{A}_1), \pi(\mathbf{A}_2)) \). The conclusion then follows immediately. \( \square \)

In the case of present interest, we take \( \mathbf{A}_1 \equiv \mathbf{A}[\mathbb{R}^2] \otimes I \) and \( \mathbf{A}_2 \equiv I \otimes \mathbf{A}[\mathbb{R}^2] \), so that \( \mathbf{A}_1, \mathbf{A}_2 \) are mutually commuting subalgebras of \( \mathbf{A} \equiv \mathbf{A}[\mathbb{R}^2] \otimes \mathbf{A}[\mathbb{R}^2] \). We let \( \omega \) denote the EPR state of \( \mathbf{A} \). Let \((\pi, \mathcal{H}, \Omega)\) denote the GNS representation of \( \mathbf{A} \) induced by \( \omega \), and let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of bounded operators on \( \mathcal{H} \). Then, \( \mathcal{R}_1 \equiv \pi(\mathbf{A}_1)'' \) and \( \mathcal{R}_2 \equiv \pi(\mathbf{A}_2)'' \) are von Neumann subalgebras of \( \mathcal{B}(\mathcal{H}) \) such that \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \). We first note an important fact about the relationship between the vector \( \Omega \) and the algebras \( \mathcal{R}_1, \mathcal{R}_2 \).

**Lemma 2.** \( \Omega \) is a separating trace vector for \( \mathcal{R}_1 \) (respectively, for \( \mathcal{R}_2 \)).

**Proof.** We show first that \( \Omega \) is cyclic for \( \mathcal{R}_1 \). By the GNS construction, \( \Omega \) is cyclic for \( \pi(\mathbf{A}) \). Since \( \mathbf{A} \) is generated by products of Weyl operators, it follows that the set

\[
\mathcal{M} = \{ \pi(W(a) \otimes W(b))\Omega : a, b \in \mathbb{R}^2 \}. \tag{27}
\]
is a total set in \( \mathcal{H} \). Let
\[
N = \{ \pi(W(a) \otimes I)\Omega : a \in \mathbb{R}^2 \}.
\] (28)

Then it will suffice for our conclusion to show that \( N \) is a total set in \( \mathcal{H} \). Let \( \psi \in \mathcal{M} \). That is, \( \psi = \pi(W(a, b) \otimes W(c, d))\Omega \), for some quadruple \( a, b, c, d \) of real numbers. Let \( \phi = \pi(W(a + c, b - d) \otimes I)\Omega \in N \). Note that since Weyl operators are unitary, \( \| \psi \| = \| \phi \| = 1 \). Now,
\[
[W(a, b) \otimes W(c, d)]^*[W(a + c, b - d) \otimes I] = e^{it}W(c, -d) \otimes W(-c, -d),
\] (29)
where \( t = (ad + bc)/2 \). Thus,
\[
\langle \psi, \phi \rangle = \omega([W(a, b) \otimes W(c, d)]^*[W(a + c, b - d) \otimes I]) = e^{it}e^{ic\lambda}e^{-id\mu}.
\] (30)

Hence, \( |\langle \psi, \phi \rangle| = 1 \); that is, \( \phi \) is a scalar multiple of \( \psi \). Since \( \mathcal{M} \) is a total set in \( \mathcal{H} \) and since \( \psi \) was an arbitrary vector in \( \mathcal{M} \), it follows that \( N \) is a total set in \( \mathcal{H} \). Therefore, \( \Omega \) is cyclic for \( \mathcal{R}_1 \). By symmetry, \( \Omega \) is cyclic for \( \mathcal{R}_2 \). Since \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \), \( \Omega \) is separating for \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \).

In order to show that \( \Omega \) is a trace vector for \( \mathcal{R}_1 \), let \( a, b \in \mathbb{R}^2 \). If \( a = -b \), then \( W(a)W(b) \otimes I = W(b)W(a) \otimes I \). If \( a \neq -b \), then
\[
\omega(W(a)W(b) \otimes I) = \exp\{i\sigma(a, b)\} \omega(W(a + b) \otimes I) = 0.
\] (32)
Similarly, \( \omega(W(b)W(a) \otimes I) = 0 \). In either case,
\[
\omega(W(a)W(b) \otimes I) = \omega(W(b)W(a) \otimes I).
\] (33)

By taking linear combinations of Weyl operators and norm limits, it follows that \( \omega \) is a tracial state of \( A[\mathbb{R}^2] \otimes I \). By taking weak limits in the GNS representation, it follows that \( \Omega \) is a trace vector for \( \mathcal{R}_1 \). By symmetry, \( \Omega \) is a trace vector for \( \mathcal{R}_2 \).

The previous lemma shows that \( \mathcal{R}_i \) has a faithful numerical trace, and is therefore a finite von Neumann algebra \( \mathfrak{A} \) \( \text{p. 504} \). Thus, \( \pi|_{\mathfrak{A}} \) is the unique (up to quasi-equivalence) representation of \( \mathfrak{A}[\mathbb{R}^2] \) that generates a finite von Neumann algebra, and \( \mathcal{R}_i \) is a factor of type II \( \text{II}_1 \) \( \text{[12, Prop. 3.4] \text{]} \). It also follows that \( \mathcal{R}_2 = \mathcal{R}_1' \). Indeed, if the inclusion \( \mathcal{R}_2 \subseteq \mathcal{R}_1' \) were proper, then it would follow that \( \mathcal{R}_2 \) is of infinite type \( \mathfrak{A} \) \( \text{[Lemma 2] \text{]} \) (since \( \Omega \) is cyclic and separating for both \( \mathcal{R}_2 \) and \( \mathcal{R}_1' \)). Therefore, \( \mathcal{R}_2 = \mathcal{R}_1' \). 

7
Theorem 2. The EPR state is maximally Bell correlated across $\mathfrak{A}[\mathbb{R}^2] \otimes \mathfrak{A}[\mathbb{R}^2]$.

**Proof.** Let $(\pi, \mathcal{H}, \Omega)$ be the GNS representation of $\mathfrak{A}$ induced by $\omega$, and let $\mathfrak{R}_i = \pi(\mathfrak{A}_i)^\prime$. By Lemma 1, it will suffice to find a Bell operator $R$ for $\mathfrak{R}_1, \mathfrak{R}_2$ such that $\langle \Omega, R\Omega \rangle = \sqrt{2}$. Since $\mathfrak{R}_1$ is type II, there is a projection $P \in \mathfrak{R}_1$ such that $P$ is equivalent to $I - P$ [5, Lemma 6.6]. That is, there is a partial isometry $V \in \mathfrak{R}_1$ such that $VV^* = P$ and $V^*V = I - P$. [Note that $V^2 = (V^*)^2 = 0$.] For each $\theta \in \mathbb{R}$, the operator $A(\theta) \equiv \exp(i\theta)V + \exp(-i\theta)V^*$ is self-adjoint and unitary. Moreover, for $\theta_1, \theta_2 \in \mathbb{R}$, we have

$$A(\theta_1)A(\theta_2) = \exp\{i(\theta_1 - \theta_2)\}P + \exp\{-i(\theta_1 - \theta_2)\}(I - P).$$

(34)

Since $\Omega$ is a trace vector for $\mathfrak{R}_1$, it follows that $\langle \Omega, P\Omega \rangle = \langle \Omega, (I - P)\Omega \rangle$. Hence, $\langle \Omega, P\Omega \rangle = 1/2$, and

$$\langle \Omega, A(\theta_1)A(\theta_2)\Omega \rangle = \cos(\theta_1 - \theta_2).$$

(35)

Since $\Omega$ is also cyclic for $\mathfrak{R}_1$, there is a * anti-isomorphism $\gamma$ of $\mathfrak{R}_1$ onto $\mathfrak{R}_1'$ such that $\gamma(A)\Omega = A\Omega$ for all $A \in \mathfrak{R}_1$ [4, Theorem 7.2.15]. Defining self-adjoint unitaries $A_i \in \mathfrak{R}_1, B_i \in \mathfrak{R}_2' (i = 1, 2)$ by $A_1 = A(0), A_2 = A(\pi/2), B_1 = \gamma(A(\pi/4)), B_2 = \gamma(A(-\pi/4))$, one obtains (cf. [14, Theorem 2.1])

$$\frac{1}{2}\langle \Omega, (A_1(B_1 + B_2) + A_2(B_1 - B_2))\Omega \rangle = 2\cos(\pi/4) = \sqrt{2}.\quad (36)$$

Remark. Since representations of $\mathfrak{A} \equiv \mathfrak{A}[\mathbb{R}^2] \otimes \mathfrak{A}[\mathbb{R}^2]$ are faithful, there is in each representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ a state (perhaps non-normal) corresponding to the EPR state. That is, there is a state $\tau$ of $\pi(\mathfrak{A})$ such that

$$\tau(\pi(A)) = \omega(A), \quad A \in \mathfrak{A}.\quad (37)$$

In particular, this recipe may be applied to define an EPR state $\tau$ in the Schrödinger representation $\pi_s$ (cf. [4, Section 4.3]). However, since all states in the folium of $\pi_s$ are regular, and $\omega$ is non-regular, it follows that $\tau$ is non-normal. Nonetheless, since $\pi_s(\mathfrak{A}[\mathbb{R}^2] \otimes I) \subseteq \mathcal{B}(\mathcal{H}_1) \otimes I$ and $\pi_s(I \otimes \mathfrak{A}[\mathbb{R}^2]) \subseteq I \otimes \mathcal{B}(\mathcal{H}_2)$, the singular state $\tau$ is maximally Bell correlated across $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$. 

8
4 Characterizing EPR-type states

According to a recent scheme of Arens and Varadarajan [2], a state \( \rho \) of a joint quantum system \( S_1 \times S_2 \), with Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), is said to be an EPR-type state for an observable \( A \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{I} \) just in case there is some observable \( A' \in \mathcal{I} \otimes \mathcal{B}(\mathcal{H}_2) \) such that the joint distribution of \( A \) and \( A' \) with respect to \( \rho \) is concentrated on the diagonal; that is, \( \rho((A - A')^2) = 0 \). In such a case, we say that \( A' \) is a double of \( A \) relative to the state \( \rho \).

As they themselves acknowledge, Arens and Varadarajan’s characterization is not sufficiently general to cover the original EPR state. In particular, they assume that \( \rho \) is a normal state on a tensor product of type I factors—which the original EPR state is not—and it follows from this that any observable with a double relative to \( \rho \) must have a discrete spectrum [2, Theorem 4].

However, the characterization of EPR-type states by Arens and Varadarajan has been extended by Werner [15] to the case where the observables of the subsystems are given, respectively, by a von Neumann algebra \( \mathcal{R} \) and its commutant \( \mathcal{R}' \). Let \( D(\mathcal{R}, \mathcal{R}', \rho) \) denote the set of elements in \( \mathcal{R} \) for which a double exists in \( \mathcal{R}' \) relative to \( \rho \). Let \( C_\rho(\mathcal{R}) \) denote the centralizer of \( \rho \) in \( \mathcal{R} \). That is, \( C_\rho(\mathcal{R}) \) consists of those elements \( A \in \mathcal{R} \) such that \( \rho(AB) = \rho(BA) \) for all \( B \in \mathcal{R} \). We then have the following result:

**Theorem 3 ([15]).** Let \( \mathcal{R} \) be a von Neumann algebra with cyclic and separating vector \( \Omega \), and let \( \rho \) be the state induced by \( \Omega \). Then

\[
D(\mathcal{R}, \mathcal{R}', \rho) = C_\rho(\mathcal{R}).
\]

Moreover, the double \( A' \in \mathcal{R}' \) of any \( A \in C_\rho(\mathcal{R}) \) is unique.

It is possible, in general, for an observable \( A \in C_\rho(\mathcal{R}) \) to have a continuous spectrum (see, for example, the following paragraph). If, however, \( \mathcal{R} \) is type I, then \( C_\rho(\mathcal{R}) \) contains only discrete spectrum observables [14], re-establishing the conclusion of Arens and Varadarajan.

In the case of present interest, we let \( \mathcal{R} = \pi(\mathfrak{A}[[\mathbb{R}^2]] \otimes I)' \), where \( (\pi, \mathcal{H}, \Omega) \) is the GNS representation of \( \mathfrak{A}[[\mathbb{R}^2]] \otimes \mathfrak{A}[[\mathbb{R}^2]] \) induced by the EPR state. As we saw previously, it then follows that \( \mathcal{R}' = \pi(I \otimes \mathfrak{A}[[\mathbb{R}^2]])' \). Let \( \rho \) be the state of \( \mathcal{B}(\mathcal{H}) \) induced by \( \Omega \). Since \( \Omega \) is cyclic and separating for \( \mathcal{R} \), it follows from Theorem 3 that \( D(\mathcal{R}, \mathcal{R}', \rho) = C_\rho(\mathcal{R}) \). Moreover, since \( \Omega \) is a trace vector for \( \mathcal{R} \), it follows that \( C_\rho(\mathcal{R}) = \mathcal{R} \), and every element \( A \in \mathcal{R} \) has a unique double.
\( A' \in \mathcal{R}' \). [In fact, \( A' = \gamma(A^*), \) where \( \gamma \) is the \( * \) anti-isomorphism invoked in the proof of Theorem 4] Therefore, the original EPR state does qualify as an EPR-type state in the sense of Werner.

Finally, it is now clear that the heuristic analogy between the original EPR state and Bohm's singlet spin-1/2 state has more than a superficial basis. Indeed, in both cases the state \( \rho \) is induced by a vector that is cyclic and separating for the component observable algebras \( \mathcal{R}, \mathcal{R}' \). And in each case \( \rho \) restricts to the unique tracial state on the component algebras. These two facts, in turn, entail that \( \rho \) is maximally Bell correlated across \( \mathcal{R}, \mathcal{R}' \), and that for each observable \( A \in \mathcal{R} \) there is a unique observable in \( A' \in \mathcal{R}' \) such that \( A \) and \( A' \) are perfectly correlated in the state \( \rho \).

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