Weak stability of Lagrangian solutions to the semigeostrophic equations

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Received 25 January 2009, in final form 9 July 2009
Published 10 September 2009
Online at stacks.iop.org/Non/22/2521

Recommended by K Ohkitani

Abstract

In (Cullen and Feldman 2006 SIAM J. Math. Anal. 37 137–95), Cullen and Feldman proved the existence of Lagrangian solutions for the semigeostrophic system in physical variables with initial potential vorticity in $L^p$, $p > 1$. Here, we show that a subsequence of the Lagrangian solutions corresponding to a strongly convergent sequence of initial potential vorticities in $L^1$ converges strongly in $L^q$, $q < \infty$, to a Lagrangian solution, in particular extending the existence result of Cullen and Feldman to the case $p = 1$. We also present a counterexample for Lagrangian solutions corresponding to a sequence of initial potential vorticities converging in $BM$. The analytical tools used include techniques from optimal transportation, Ambrosio’s results on transport by $BV$ vector fields and Orlicz spaces.

Mathematics Subject Classification: 35Q35, 76U05

1. Semigeostrophic (SG) equations in physical and in dual variables

SG equations are simplified models for large-scale geophysical flows. These systems were introduced by Hoskins in [12], as part of a family of models for geophysical flows under approximate geostrophic balance, i.e. where Coriolis forces and horizontal gradients of pressure nearly balance. We refer the reader to [7] for a thorough account of semigeostrophy from the physical point of view.

In this work, we are concerned with two versions of the SG system—the incompressible 3D system and the shallow water system, both in a bounded domain in $\mathbb{R}^3$ with constant

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Coriolis force. We will first focus the discussion on the incompressible case, leaving the shallow water system to section 4. The incompressible SG equation, or SG equation, has a rich mathematical structure, closely related to optimal transport theory. Written in dual variables, it can be interpreted as a fully nonlinear active scalar equation, where the transported scalar, called potential vorticity, generates the transporting velocity by means of a Monge–Ampère equation. Several results are available for the SG system in dual form, beginning with the work of Benamou and Brenier, [3], on the existence of weak solutions, and including [9, 10, 14, 16]. In [13], Loeper proved the existence of weak solutions for potential vorticities in the space of Radon measures. Obtaining a solution in physical variables from weak solutions of the dual form is both delicate and physically relevant. For potential vorticities in $L^p$, $p > 1$, this problem was solved by Cullen and Feldman in [8], with the introduction of Lagrangian solutions to the system in physical variables.

This paper’s main concern is the weak stability of Lagrangian solutions with respect to perturbations in the initial potential vorticity, complementing the work of Cullen and Feldman. Our main result, see theorem 3.1, is the weak compactness in $L^p$, for any $p < \infty$, of sequences of Lagrangian solutions, obtained from sequences of initial potential vorticities converging strongly in $L^1$. In addition, we also include analysis of the compressible shallow water case, as in [8] (see theorem 4.2). Finally, we present a counterexample for the extension of our main result to initial potential vorticities in the space of measures.

Let $\Omega \subset \mathbb{R}^3$ be open and bounded. The 3D incompressible SG equations in physical variables are a system of four equations with three components of velocity $u = (u_1, u_2, u_3)$ and the pressure $p$ as unknown. Before we write down the system, we introduce the geostrophic velocity $v^g = (v^g_1, v^g_2, 0)$ as $v^g = (\partial_2 p, \partial_1 p, 0)$ and the density $\rho$ by assuming vertical hydrostatic balance $\rho = -\partial_3 p$. The SG equations have the form

\[
\begin{align*}
\partial_t (v^g_1, v^g_2) + (u_2, -u_1) &= (u_2, -u_1), \\
\partial_3 \rho &= 0, \\
\text{div } u &= 0,
\end{align*}
\]

where $\partial_t = \partial_t + u \cdot \nabla$ is the material derivative. The unknowns are functions of $(x, t) \in \Omega \times [0, T)$ with initial and boundary data given by

\[
\begin{align*}
u \cdot \nu &= 0 \quad \text{on } \partial\Omega \times [0, T), \\
p(x, 0) &= p_0(x) \quad \text{in } \Omega,
\end{align*}
\]

where $\nu$ is the unit outer normal to $\partial\Omega$. Taken together, these four equations make up a rather odd-looking system of PDEs. We have three transport equations for the components of $\nabla p$, in which $u$ enters only algebraically, together with the divergence-free condition. That this system turns out to be solvable only becomes apparent after a change in variables, which expresses its dual formulation.

In order to present the dual variable formulation of this system, we consider the modified pressure $P$, given by

\[
P(x, t) = p(x, t) + \frac{1}{2}(x^2_1 + x^2_2)
\]

and we rewrite the SG system as

\[
\begin{align*}
\partial_t X &= J(X - x), \\
\text{div } u &= 0, \\
X &= \nabla P, \\
u \cdot \nu &= 0 \quad \text{on } \partial\Omega \times [0, T), \\
P(x, 0) &= P_0(x) \quad \text{in } \Omega,
\end{align*}
\]
where
\[
J = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

The dual formulation is obtained by switching dependent and independent variables in the system above. More precisely, we assume that \( P \) and \( u \) are solutions of the system above, and, in addition, \( P(., t) \) is convex and smooth for all \( t \). A consequence of the convexity and smoothness of \( P \) is that \( \nabla P(., t) \) is a diffeomorphism between \( \Omega \) and some subset of \( \mathbb{R}^3 \). We introduce \( \alpha \equiv \nabla P^\sharp \chi_{/\Omega} \), where the sharp indicates measure pushforward and \( \chi_{/\Omega} \) denotes the Lebesgue measure in \( \Omega \). To be precise, if \( \Omega_1 \) and \( \Omega_2 \) are subsets of \( \mathbb{R}^n \), \( \mu \) is a measure on \( \Omega_1 \) and \( \nu \) is a measure on \( \Omega_2 \) and \( X \) maps \( \Omega_1 \) to a subset of \( \Omega_2 \), then the notation \( X^\sharp \mu = \nu \) means that
\[
\int_{\Omega_2} f(y) \, d\nu = \int_{\Omega_1} f(X(x)) \, d\mu,
\]
for any \( f \in C^0(\Omega_2) \).

The measure \( \alpha \) is called potential vorticity. We also introduce \( P^* \) the Legendre transform of \( P \), i.e.
\[
P^*(X, t) \equiv \sup_{x \in \Omega} \{ x \cdot X - P(x, t) \}.
\]
The potential vorticity \( \alpha \) satisfies the following system of equations:
\[
\begin{aligned}
\frac{\partial \alpha}{\partial t} + \nabla \cdot (U \alpha) &= 0, \quad \mathbb{R}^3 \times [0, T), \\
U(X, t) &= J[X - \nabla P^*(X, t)], \\
\nabla P(., t)^\sharp \chi_{/\Omega} &= \alpha(., t), \\
\alpha(X, 0) &= \alpha_0(X), \quad \text{a.e. } X \in \mathbb{R}^3.
\end{aligned}
\tag{4}
\]

From the definition of the pushforward measure, we can see that the statement \( \nabla P(., t)^\sharp \chi_{/\Omega} = \alpha(., t) \) amounts to a weak form of the equation \( \det(D^2 P^*) = \alpha \), with the condition that the image of \( \nabla P^* \) is \( \Omega \). This observation shows that (4) is an active scalar transport equation, where the transporting velocity is determined from the transported scalar by means of a Monge–Ampère equation. The derivation of the dual system from the physical system is a standard calculation, and it can be found, for example, in [3]. The key hypothesis for the validity of this derivation is the convexity of \( P \), something which is preserved under SG evolution, see [3].

Next we state a known result concerning existence of weak solutions for the SG system in dual variables. It collects together previous results by Benamou and Brenier, by two of the authors, and by Cullen and Feldman in [3, 8, 14] in a way that will be convenient for use in this work.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \), which we assume is contained in the ball of radius \( S > 0 \), centred at the origin and let \( T > 0 \). In fact, we consider \( \Omega \), the radius \( S > 0 \) and the time horizon \( T > 0 \) as fixed throughout the rest of this paper.

**Theorem 1.1.** Let \( P_0 = P_0(x) \) be a bounded, convex function in \( \Omega \) and let \( \alpha_0 := D P_0^\sharp \chi_{/\Omega} \). Suppose that \( \alpha_0 \in L^q(\mathbb{R}^3) \), for some \( q \geq 1 \), and assume that \( \alpha_0 \) is compactly supported. Let \( R_0 \) be such that the support of \( \alpha_0 \) is contained in the ball \( B(0, R_0) \) and set \( R(T) = R_0 e^T + (e^T - 1)S \).

Then, for any \( t > 0 \), there exist functions \( \alpha = \alpha(X, t) \in L^\infty([0, T), L^q(\mathbb{R}^3)) \), \( P = P(X, t) \in L^\infty([0, T), L^q(\mathbb{R}^3)) \), \( U = U(X, t) \in \mathbb{R}^3 \times [0, T) \), and \( \chi_{/\Omega} \) such that
\[
\begin{aligned}
\frac{\partial \alpha}{\partial t} + \nabla \cdot (U \alpha) &= 0, \quad \mathbb{R}^3 \times [0, T), \\
U(X, t) &= J[X - \nabla P^*(X, t)], \\
\nabla P(., t)^\sharp \chi_{/\Omega} &= \alpha(., t), \\
\alpha(X, 0) &= \alpha_0(X), \quad \text{a.e. } X \in \mathbb{R}^3.
\end{aligned}
\]
There exists a Borelian map $L^2(0, R(T))$, such that

(i) $\text{supp}(\alpha(\cdot, t)) \subset B(0, R(T)); \quad \forall t \in [0, T)$; \hspace{2cm} (5)

(ii) for each $0 \leq t < T$, $P(\cdot, t)$ is convex, and $\alpha, P$ satisfy

$$\alpha \in C([0, T), L^1(B(0, R(T)))) \quad \text{and} \quad P \in C([0, T), W^{1,r}(\Omega)),$$

for any real number $r \in [1, \infty)$;

(iii) $P^*(\cdot, t)$ is convex, pointwise in time, locally bounded in space time and

$$\nabla P^* \in L^\infty([0, T), L^\infty(B(0, R(T)))); \quad \forall t \in [0, T); \hspace{2cm} (6)$$

(iv) $(\alpha, P, P^*)$ satisfy (4), where the evolution equation and the initial data for $\alpha$ are understood in the weak sense, i.e. for each $\phi \in C^1_c([\mathbb{R}^3 \times [0, T))$

$$\int_{\mathbb{R}^3} [\partial_t \phi(X, t) + U(X, t) \cdot \nabla \phi(X, t)] \alpha(X, t) \, dX \, dt + \int_{\mathbb{R}^3} \alpha_0(X) \phi(X, 0) \, dX = 0. \hspace{2cm} (7)$$

One would like to find solutions for the SG system in physical space, which means solutions to (3). Our point of departure is the work of Cullen and Feldman in [8]. In that paper, Cullen and Feldman pointed out that

- concerning the Eulerian form of system (3)—a distributional formulation of this system requires making sense of products of components of $u$ with first derivatives of $P$;
- the formal expression for $u$ is given by

$$u(x, t) = \partial_t \nabla P^*(\nabla P(x, t), t) + D^2 P^*(\nabla P(x, t), t)[J(\nabla P(x, t) - x)];$$

- the known regularity for solutions of the dual problem has $P^*$ Lipschitz continuous.

Clearly, making sense of the physical velocity $u$ is complicated, given that $D^2 P^*$ is a measure while $\nabla P$ is only bounded, not to mention making sense of the product $u \cdot \nabla P$. As a consequence, seeking Eulerian solutions in physical variables is a difficult problem. This was the motivation for the introduction of the notion of Lagrangian solutions in definition 2.5 of [8].

**Definition 1.1 (Lagrangian solutions).** Let $P_0 \in W^{1,\infty}(\Omega)$ be a convex function and $r \in [1, \infty)$. Let $P : \Omega \times [0, T) \to \mathbb{R}$ satisfy

$$P \in L^\infty([0, T), W^{1,\infty}(\Omega)) \cap C([0, T), W^{1,r}(\Omega)), \hspace{2cm} (8)$$

$$P(\cdot, t) \text{ is convex in } \Omega, \forall t \in [0, T). \hspace{2cm} (9)$$

Let $F : \Omega \times [0, T) \to \Omega$ be a Borelian map such that

$$F \in C([0, T), L^r(\Omega)). \hspace{2cm} (10)$$

The pair $(P, F)$ is called a Lagrangian solution of (3) in $\Omega \times [0, T)$ if

(i) $F(x, 0) = x, P(x, 0) = P_0(x)$ for almost all $x \in \Omega$,

(ii) for all $0 \leq t < T$ the mapping $F_t = F(\cdot, t) : \Omega \to \Omega$ preserves Lebesgue measure, i.e.

$$F_t # \chi_\Omega = \chi_\Omega.$$

(iii) There exists a Borelian map $F^* : \Omega \times [0, T) \to \Omega$ such that, for all $t \in (0, T)$, the map

$$F_t^* = F^*(\cdot, t) : \Omega \to \Omega \text{ preserves Lebesgue measure, (i.e. } F_t^* # \chi_\Omega = \chi_\Omega \text{) and satisfies } F_t \circ F_t^*(x) = x \text{ and } F_t^* \circ F_t(x) = x \text{ for almost all } x \in \Omega,$$
(iv) the function

\[ Z = Z(x, t) = \nabla P(F_t(x), t) \]

(11)
is a weak solution of

\[
\begin{align*}
\partial_t Z(x, t) &= J[Z(x, t) - F(x, t)], \quad \text{in } \Omega \times [0, T) \\
Z(x, 0) &= \nabla P_0(x), \quad \text{in } \Omega,
\end{align*}
\]

(12)
that is, for any \( \varphi \in C^1_c(\Omega \times [0, T)) \), we have

\[
\int_{\Omega \times [0, T)} [Z(x, t) \cdot \partial_t \varphi(x, t) + J(Z(x, t) - F(x, t)) \cdot \varphi(x, t)] \, dx \, dt
\]
\[
+ \int_{\Omega} \nabla P_0(x) \cdot \varphi(x, 0) \, dx = 0.
\]

(13)

Given a Lagrangian solution \((P, F)\), the map \( F(\cdot, t) \) is called a Lagrangian flow in physical space, for each \( t \in [0, T) \).

Next we give the precise statement of existence of Lagrangian solutions. The case \( q > 1 \) of the theorem below is the main result in [8], and the case \( q = 1 \) is a consequence of the analysis in this paper together with the techniques developed in [14].

**Theorem 1.2.** Let \( P_0 \) be convex and bounded in \( B(0, S) \). Assume that \( P_0 \) satisfies

\[ D P_0^\#X_\Omega \in L^q(\mathbb{R}^3) \]

(14)
for some \( q \geq 1 \), and that \( D P_0^\#X_\Omega \) is compactly supported. Then there exists a Lagrangian solution \((P, F)\) of (3) in \( \Omega \times [0, T) \), for which (8)–(10) are satisfied for any \( r \in [1, \infty) \).

Moreover, the function \( Z = Z(x, t) \), defined by (11), satisfies \( Z(x, \cdot) \in W^{1,\infty}(\mathbb{R}^3) \) for almost all \( x \in \Omega \) and (12) is also satisfied in the following sense:

\[
\partial_t Z(x, t) = J[Z(x, t) - F(x, t)], \quad \text{for } (x, t) \in \Omega \times (0, T) \), \mathcal{L}^4 - \text{a.e.}
\]

\[ Z(x, 0) = \nabla P_0(x), \quad \text{for } x \in \Omega \), \mathcal{L}^3 - \text{a.e.} \]

Let us briefly examine the construction underlying the proof of theorem 1.2, as given in [8] for \( q > 1 \). Under the hypotheses of theorem 1.2, from the solution of the dual problem, and using the Polar factorization theorem (see [4]), one obtains \( P \). Given \( P \), it can be shown that the following expression gives rise to a Lagrangian flow in physical space:

\[ F = F(x, t) = \nabla P_0^* \circ \Phi_t \circ \nabla P_0(x), \]

(16)
where, for each \( t, \Phi_t \) is the Lagrangian flow in dual space, obtained using Ambrosio’s theorem as follows.

Consider the transport equation

\[ \partial_t \alpha + U \cdot \nabla \alpha = 0, \]

which is equivalent to the first equation in (4), since \( \text{div } U = 0 \). From the regularity of \( P^* \) we have

\[ U \in L^\infty_{\text{loc}}(\mathbb{R}^3 \times [0, T]), \quad \text{and } U \in L^\infty([0, T), BV_{\text{loc}}) \]

One uses Ambrosio’s theorem to obtain the Lagrangian flow associated with the transport equation above. To do so one must modify the velocity \( U \) near infinity without affecting the solution \( \alpha \); this can be achieved since \( \alpha \) has compact support in \( \mathbb{R}^3 \times [0, T) \). We have

\[ \text{supp } \alpha \subset \overline{B(0, R(T))} \times [0, T], \]

where \( R(T) \) was introduced in theorem 1.1(i).
Consider a modified velocity $\tilde{U}$ as follows: choose $\varrho \in C^\infty(R)$ such that
\begin{align}
\varrho &\equiv 1 \text{ in } |s| < R(T), \\
\varrho &\equiv 0 \text{ in } |s| > R(T) + 1, \\
0 &\leq \varrho \leq 1 \text{ in } R,
\end{align}
and define, for $X \in \mathbb{R}^3$,
\begin{equation}
H(X) = (\varrho(|X_1|)X_1, \varrho(|X_2|)X_2, \varrho(|X_3|)X_3).
\end{equation}

The modified velocity $\tilde{U}$ is then given by
\begin{equation}
\tilde{U}(X,t) = J[H(X) - \nabla P^*(X,t)],
\end{equation}
and, therefore, $\tilde{U}$ satisfies
\begin{align}
\tilde{U} &\in L^\infty(\mathbb{R}^3 \times [0, T]), \\
\tilde{U} &\in L^\infty([0, T), BV_{loc}), \\
\text{div } \tilde{U}(\cdot, t) &\equiv 0 \text{ in } \mathbb{R}^3, \quad \text{for all } t \in [0, T). 
\end{align}

Furthermore,
\begin{align}
U &\equiv \tilde{U} \quad \text{in } B(0, R(T)), \\
\|\tilde{U}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} &\leq S + R(T) + 1 \quad \text{for all } t \in [0, T). 
\end{align}

Therefore, using the results of Ambrosio in [2], the following proposition can be formulated.

**Proposition 1.1.** Let $\tilde{U}$ be given by (19). There exists a unique locally bounded and Borel measurable mapping $\Phi : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ satisfying
\begin{align}
(i) \quad &\Phi(X, \cdot) \in W^{1,\infty}([0, T)) \text{ for almost all } X \in \mathbb{R}^3; \\
(ii) \quad &\Phi(X, 0) = X, \quad X \in \mathbb{R}^3, \quad L^3 \text{ a.e.;} \\
(iii) \quad &\text{For almost all } (X, t) \in \mathbb{R}^3 \times (0, T) \quad \\
&\partial_t \Phi(X, t) = \tilde{U}(\Phi(X, t), t); \\
(iv) \quad &\Phi(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}^3 \text{ preserves Lebesgue measure in } \mathbb{R}^3 \text{ for all } t \in [0, T). \\
(v) \quad &\text{We have} \\
&\Phi(X, t) \subset B(0, R(T)) \text{ for almost all } (X, t) \in \nabla P_0(\Omega) \times [0, T). 
\end{align}

In particular,
\begin{equation}
\partial_t \Phi(X, t) = U(\Phi(X, t), t) \text{ for almost all } (X, t) \in \nabla P_0(\Omega) \times [0, T). 
\end{equation}

There exists a Borel mapping $\Phi^* : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$, such that for all $t \in (0, T)$ the map $\Phi_t^* : \mathbb{R}^3 \to \mathbb{R}^3$ preserves Lebesgue measure in $\mathbb{R}^3$, and satisfies $\Phi_t^* \circ \Phi_t(x) = x$ and $\Phi_t \circ \Phi_t^*(x) = x$, for almost all $x \in \mathbb{R}^3$.

Under the conditions of theorem 2.1, if $(\alpha, P)$ is a weak solution of (4), and if (i)–(vi) hold, then for any $t \in [0, T]$, 
\begin{equation}
\alpha_t = \Phi_t \# \alpha_0.
\end{equation}

Moreover, for any $t \in [0, T]$, 
\begin{equation}
\alpha_t(x) = \alpha_0(\Phi_t^*(x)) \text{ for almost all } x \in \mathbb{R}^3.
\end{equation}

**Remark 1.1.** This result is a summary of what was proved in lemmas 2.11, 2.12 and 2.13 in proposition 2.14 of [8], in the case $q > 1$. The adaptation of proposition 1.1 above to the case $q = 1$ can be done using the techniques developed in [14] in a straightforward manner.
2. Orlicz spaces

In the proof of our main result, weak stability of Lagrangian solutions in \( L^1 \), we will make use of several results concerning Orlicz spaces. For the convenience of the reader, in what follows we collect definitions together with the results we will use about Orlicz spaces. For details, see [1, 14].

Consider \( a : [0, \infty) \to [0, \infty) \) with the following properties:

(i) \( a(0) = 0, a(t) > 0 \) if \( t > 0 \) and \( \lim_{t \to \infty} a(t) = \infty \),

(ii) \( a \) is non-decreasing,

(iii) \( a \) is right-continuous.

The function \( A \), defined on \([0, \infty)\) by taking
\[
A(t) = \int_0^t a(\tau) \, d\tau,
\]
is called an \( N \)-function.

We note that \( N \)-functions are continuous on \([0, \infty)\), convex and strictly increasing.

An \( N \)-function is said to be \( \Delta_1 \)-regular if there exists a positive constant \( C \) and \( t_0 > 0 \) such that
\[
A(2t) \leq CA(t), \quad \forall t \geq t_0.
\]

Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and \( A \) an \( N \)-function. The Orlicz space \( L_A(\Omega) \) is the linear closure of the set of functions \( u \in L^1_{\text{loc}}(\Omega) \) such that \( A(|u|) \) is integrable. These are Banach spaces with norm given by
\[
\|u\|_A = \inf \left\{ k > 0, \int_{\Omega} A \left( \frac{|u(x)|}{k} \right) \, dx \leq 1 \right\}.
\]

They generalize the Lebesgue spaces \( L^p(\Omega) \), which are Orlicz spaces, with \( N \)-function \( A(t) = t^p \). We denote by \( E_A(\Omega) \) the closure, with respect to \( \| \cdot \|_A \), of the set of smooth, compactly supported functions in \( \Omega \). For every \( N \)-function \( A \), we have that \( E_A(\Omega) \) is separable. In general, \( L_A(\Omega) \) and \( E_A(\Omega) \) are distinct, and \( L_A(\Omega) \) is not separable. However, when \( A \) is \( \Delta \)-regular, \( L_A = E_A \).

Let \( A \) be an \( N \)-function. Its Legendre transform \( A^* \) is given by
\[
A^*(s) = \max_{t \geq 0} \{ st - A(t) \}.
\]

One can verify that \( A^* \) is also an \( N \)-function and that \( A^{**} = A \).

Finally, the following classical results will be relevant in our analysis.

**Theorem 2.1** ([1]). If \( A \) is an arbitrary \( N \)-function then the dual of \( E_A(\Omega) \) is \( L_A^*(\Omega) \).

**Lemma 2.1** ([6]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( f, f^k \in L^1(\Omega) \), for all \( k \). If \( f^k \to f \) strongly in \( L^1 \) then there exists a \( \Delta \)-regular \( N \)-function \( A \) such that \( \{ f^k \} \) and \( f \) are uniformly bounded in \( L_A(\Omega) \).

**Lemma 2.2** ([14]). Let \( \{ u_n \} \) be a sequence of functions uniformly bounded in \( L^\infty(\Omega) \). If \( u_n \to u \) in \( L^1(\Omega) \), then \( u_n \to u \) in \( L_A(\Omega) \), for any \( N \)-function \( A \).

3. Weak stability of the SG equations in physical space

We now turn to the main objective of this paper. We consider a sequence of initial potential vorticities, converging strongly in \( L^1 \) to a given limit vorticity. We would like to understand the convergence properties of the corresponding Lagrangian solutions in physical space. Our motivation for considering this problem was, originally, to try to extend Cullen and Feldman’s
construction to solutions of the SG equations with measures as potential vorticities. To do so we intended to approximate such solutions by smoother ones and, hence, we needed to understand how the corresponding Lagrangian solutions behaved. As it turns out, this approach to construct Lagrangian solutions for measure-valued potential vorticities does not work; this will be made clear by means of a counterexample, in section 5. Instead, we have established a weak stability, or continuity property, of Lagrangian solutions with respect to integrable perturbations of an $L^1$ initial potential vorticity. Weak stability of weak solutions of the SG equations, in dual formulation, has already been established by Loeper in [13].

Throughout the rest of this section we fix $\alpha_0 \in L^1(\mathbb{R}^3)$ together with a sequence $\{\alpha_n^0\} \subset L^1(\mathbb{R}^3)$ such that $\alpha_n^0 \rightharpoonup \alpha_0$ strongly in $L^1(\mathbb{R}^3)$. In addition, we assume that $\alpha_0$ and the sequence $\{\alpha_n^0\}$ are compactly supported, with supports contained in a ball $B(0, R_0)$.

Using lemma 2.1, as in [14], we have that there exists a $\Delta$-regular $N$-function $A$ such that $\alpha_n$, $\alpha_0$ are uniformly bounded in $L^A(\mathbb{R}^3)$. Let $\alpha_n(y, t) = \alpha_n(y, t)$ be a weak solution of the SG equations in dual formulation with initial potential vorticity $\alpha_n^0$. Consider the corresponding modified pressures $P_n$, $P_0$, defined in the physical space $\Omega$. Denote by $\Phi^\alpha$ the Lagrangian flow in dual space given in proposition 1.1. Finally, consider the corresponding Lagrangian flows in physical space, $F_t := \nabla (P^\alpha)_t \circ \Phi^\alpha \circ \nabla P^\alpha_0$, (28) as obtained in theorem 1.2, equation (16).

In the proof of theorem 1.1, case $q = 1$, (see [14]), it was shown that a subsequence of $\{\alpha^\alpha\}$, $\{P^n\}$, $\{(P^n)^*\}$ exists, which we do not re-label, together with a weak solution $\alpha = \alpha(x, t)$ of the SG equations in dual formulation (with initial potential vorticity $\alpha_0$), such that the following hold, for each $t \in [0, T)$:

$$\begin{align*}
\alpha^n(\cdot, t) &\rightharpoonup \alpha(\cdot, t) & w^* = L_A(B(0, R(T))) \\
\alpha^0(\cdot, t) &\rightharpoonup \alpha(\cdot, t) & \text{in } W^1,1(\Omega) \\
\alpha^0(\cdot, t) &\rightharpoonup \alpha(\cdot, t) & \text{in } L^1(\mathbb{R}^3) \\
\nabla (P^n)^*(\cdot, t) &\rightharpoonup \nabla P^*(\cdot, t) & \text{in } (E^\alpha)_\text{loc}(\mathbb{R}^3).
\end{align*}$$

(29)

In the proof of the convergence of $\nabla (P^n)^*(\cdot, t)$ to $\nabla P^*(\cdot, t)$ one uses theorem 2.1 and lemma 2.2.

We fix, throughout the rest of this section, such a subsequence. Above, $P(\cdot, t)$, $P_0$ are the modified pressures corresponding to $\alpha(\cdot, t)$ and $\alpha_0$ and $R(T)$ is given in theorem 1.1, item (i). Let $\Phi$ be the Lagrangian flow in dual space as in proposition 1.1 and let

$$F_t := \nabla P^*_t \circ \Phi \circ \nabla P_0,$$

(30)

be the Lagrangian flow in physical space obtained in theorem 1.2, see (16).

Our main result is the following.

**Theorem 3.1.** There exists a (further) subsequence $\{F_t^n\} \subset \{F_t\}$ such that for almost every $0 \leq t < T$ we have

$$F_t^n \rightharpoonup F_t, \quad \text{strongly in } L^r(\Omega), \text{ as } k \to \infty,$$

for all $r \in [1, \infty)$.

Before we present the proof of the theorem, we require the following auxiliary result.
Lemma 3.1. For each $R > 0$, we have
\[
\lim_{n \to \infty} \int_{B(0,R)} \sup_{t \in [0,T]} |\Phi^n(X,t) - \Phi(X,t)| \, dX = 0.
\]

Proof. For each $n$, recall that $\Phi^n$ is the Lagrangian flow in dual space associated with the vector field $\tilde{U}^n(X,t) = J[H(X) - \nabla (P^n_t)\ast \eta]$. Since we have that $(P^n_t)\ast \eta \to P\ast \eta$ in $W^{1,1}_\text{loc}(\mathbb{R}^3)$, it follows that
\[
\tilde{U}^n(X,t) \longrightarrow \tilde{U}(X,t) \text{ in } L^1_\text{loc}(\mathbb{R}^3).
\] (31)

However, such a condition is not enough to obtain the convergence of $\{\Phi^n\}$, due to the fact that we cannot control $\nabla \tilde{U}^m$. This is required in Ambrosio’s stability result, namely theorem 6.5 of [2].

We consider, instead, a family of approximations of $\tilde{U}^n$ given by
\[
\tilde{U}^{n,m}(X,t) = J[H(X) - (\nabla (P^n_t)\ast \eta^m)(X,t)],
\] (32)
where $\eta^m$ is a standard mollifier.

Now we have
\[
\tilde{U}^{n,m} \in C([0,T] \times \mathbb{R}^3, \mathbb{R}^3),
\]
\[
\sup_m \|\tilde{U}^{n,m}\|_{L^\infty([0,T],\mathbb{R}^3)} < \infty,
\]
\[
\text{div} \tilde{U}^{n,m} = 0,
\]
\[
\|\nabla \tilde{U}^{n,m}\|_{L^\infty([0,T] \times B(0,R),\mathbb{R}^3)} \leq C(n,m,R) < \infty,
\]
\[
\tilde{U}^{n,m} \longrightarrow \tilde{U}^n \text{ in } L^1_\text{loc}(\mathbb{R}^3 \times [0,T)).
\] (33)

Let $\Phi^{n,m}(X,t)$ be the Lagrangian flow associated with $\tilde{U}^{n,m}$. It follows from theorem 6.5 in [2] that
\[
\lim_{m \to \infty} \int_{B(0,R)} \sup_{t \in [0,T]} |\Phi^n(X,t) - \Phi^{n,m}(X,t)| \, dX = 0, \quad \forall R > 0.
\] (34)

Note that $\tilde{U}^{n,m}(X,t) \to \tilde{U}(X,t)$ in $L^1_\text{loc}(\mathbb{R}^3 \times [0,T))$ when $m, n \to \infty$. To see this it is enough to observe that, for any $R > 0$, we have
\[
\|\nabla (P^n_t)\ast \eta^m - \nabla P^n_t\|_{L^1(B(0,R))} \leq \|\eta^m\|_{L^1(B(0,R))} \|\nabla (P^n_t)\ast - \nabla P^n_t\|_{L^1(B(0,R))}
\]
\[+ \|\nabla P^n_t\ast \eta^m - \nabla P^n_t\|_{L^1(B(0,R))} \xrightarrow{n,m \to \infty} 0.
\]

Hence, we also have
\[
\lim_{m,n \to \infty} \int_{B(0,R)} \sup_{t \in [0,T]} |\Phi^n(X,t) - \Phi^{n,m}(X,t)| \, dX = 0, \quad \forall R > 0.
\] (35)

Given (34), it is possible to choose a subsequence $m = m(n) > n$ such that
\[
\lim_{n \to \infty} \int_{B(0,R)} \sup_{t \in [0,T]} |\Phi^n(X,t) - \Phi^{n,m(n)}(X,t)| \, dX = 0, \quad \forall R > 0.
\] (36)

We conclude, from (36) and (35), that
\[
\lim_{n \to \infty} \int_{B(0,R)} \sup_{t \in [0,T]} |\Phi^n(X,t) - \Phi(X,t)| \, dX = 0, \quad \forall R > 0,
\]
which concludes the proof. □
Remark 3.1. Once we take into account expression (16), proposition 1.1 (v) and (23), we see that we may assume in what follows that the flow $\Phi(X, t)$ is associated with the vector field $U(X, t) = J[X - NP^*(X, t)]$.

With this lemma we are now ready to give the proof of theorem 3.1.

Proof of theorem 3.1. Let us first prove our result for the case $r = 1$.

We note that, for each $0 \leq t < T$, we have

$$\int_{\Omega} |F_t^n(x) - F_t(x)| \, dx$$

$$= \int_{\Omega} \left| \nabla (P^n_t)^* \circ \Phi^n_t \circ \nabla P^n_0(x) - \nabla P^n_t \circ \Phi_t \circ \nabla P_0(x) \right| \, dx$$

$$\leq \int_{\Omega} \left| \nabla (P^n_t)^* \circ \Phi^n_t \circ \nabla P^n_0(x) - \nabla (P^n_t)^* \circ \Phi_t \circ \nabla P^n_0(x) \right| \, dx$$

$$+ \int_{\Omega} \left| \nabla P^n_t \circ \Phi_t \circ \nabla P^n_0(x) - \nabla P^n_t \circ \Phi_t \circ \nabla P_0(x) \right| \, dx$$

$$\equiv I_1 + I_2 + I_3. \quad (37)$$

We will show that each of these integrals vanish as $n$ tends to infinity, passing to subsequences as needed.

Let us begin by considering $I_1$. Using that $\nabla P^n_0 \chi_{\Omega} = a^n_0$ we have

$$\int_{\Omega} \left| \nabla (P^n_t)^* \circ \Phi^n_t \circ \nabla P^n_0(x) - \nabla (P^n_t)^* \circ \Phi_t \circ \nabla P^n_0(x) \right| \, dx$$

$$= \int_{\mathbb{R}^3} \left| \nabla (P^n_t)^* \circ \Phi^n_t(y) - \nabla (P^n_t)^* \circ \Phi_t(y) \right| \alpha^n_0(y) \, dy$$

$$\leq \int_{\mathbb{R}^3} \left| \nabla (P^n_t)^* \circ \Phi^n_t(y) - \nabla (P^n_t)^* \circ \Phi_t(y) \right| \alpha^n_0(y) \, dy$$

$$+ \int_{\mathbb{R}^3} \left| \nabla P^n_t \circ \Phi_t(y) - \nabla (P^n_t)^* \circ \Phi_t(y) \right| \alpha^n_0(y) \, dy$$

$$\equiv I_1^1 + I_1^2. \quad (38)$$

From proposition 1.1 (vii) we see that $\alpha^n_t = \Phi^n_t \# a^n_0$ and, since $L_A$ is a rearrangement invariant space, it follows that $\|a^n_0\|_{L_A} = \|\alpha^n_t\|_{L_1}$, for each $n$ and for each $t \in [0, T)$. Therefore, we have

$$\int_{\mathbb{R}^3} \left| \nabla (P^n_t)^* \circ \Phi^n_t(y) - \nabla P^n_t \circ \Phi^n_t(y) \right| \alpha^n_0(y) \, dy$$

$$\leq \|\nabla (P^n_t)^* - \nabla P^n_t\|_{E, A} \|\alpha^n_t\|_{L_1}$$

$$= \|\nabla (P^n_t)^* - \nabla P^n_t\|_{E, A} \|\alpha^n_0\|_{L_1} \overset{n \to \infty}{\longrightarrow} 0, \quad (39)$$

where we have used (29), theorem 2.1 and the boundedness of $\alpha^n_0$ in $L_A$. 
As for $I_2^1$, we have

$$I_2^1 = \int_{\mathbb{R}^3} |\nabla P_t^* \circ \Phi_t^*(y) - \nabla (P_t^*)^* \circ \Phi_t(y)| \alpha_0^e(y) \, dy$$

$$\leq \int_{\mathbb{R}^3} |\nabla P_t^* \circ \Phi_t^*(y) - \nabla P_t^* \circ \Phi_t(y)| \alpha_0^e(y) \, dy$$

$$+ \int_{\mathbb{R}^3} |\nabla P_t^* \circ \Phi_t(y) - \nabla (P_t^*)^* \circ \Phi_t(y)| \alpha_0^o(y) \, dy$$

$$= I_1^{2.1} + I_1^{2.2}. \tag{40}$$

Consider the integral $I_1^{2.1}$. Since $\alpha_0^o \to \alpha_0$ strongly in $L^1$, it is easy to see that the proof that $I_1^{2.1}$ tends to zero as $n \to \infty$ reduces, by Lebesgue’s dominated convergence theorem, to showing that, for all $t$,

$$\nabla P_t^* \circ \Phi_t(y) - \nabla P_t^* \circ \Phi_t(y) \to 0, \quad n \to \infty \text{ a.e. } y \in \mathbb{R}^3. \tag{41}$$

At this point we must pass to a further subsequence. We have, by lemma 3.1, that

$$g^n := g^e(y) = \sup_{0 < r < T} |\Phi^o(y, 0) - \Phi(y, t)| \to 0 \text{ strongly in } L^3.$$ 

From this it follows that there exists a subsequence, $\{g^{n_k}\}$, which converges, a.e. $y \in \mathbb{R}^3$, to 0 as $k \to \infty$. Hence, for every $0 \leq t < T$, we have

$$\Phi_t^{n_k}(y) - \Phi_t(y) \to 0 \text{ a.e. } y \in \mathbb{R}^3 \text{ as } k \to \infty. \tag{42}$$

Now, since $P_t^*$ is convex, it follows that $\nabla P_t^*$ is almost everywhere differentiable (see, for instance, [11]), and hence continuous except for a set of Lebesgue measure zero, say $N \subset \mathbb{R}^3$. Given (42) it is enough to show, therefore, that, for almost all $y \in \mathbb{R}^3$, $\nabla P_t^*$ is continuous at $\Phi_t(y)$. To see this we note that

$$|(y \in \mathbb{R}^3; \Phi_t(y) \in N)| = |(y \in \mathbb{R}^3; y \in \Phi_t(N))| = |N| = 0,$$

in view of the fact that $\Phi_t^*$ preserves Lebesgue measure. Therefore, $\Phi_t(y)$ is a continuity point for $\nabla P_t^*$, for almost all $y$, as desired.

Hence,

$$\int_{\mathbb{R}^3} |\nabla P_t^* \circ \Phi_t^{n_k}(y) - \nabla P_t^* \circ \Phi_t(y)| \alpha_0^e(y) \, dy \to 0. \tag{43}$$

The analysis of $I_1^{2.2}$ is similar. We have that $\nabla P_t^*(\mathbb{R}^3)$, $\nabla (P_t^*)^*(\mathbb{R}^3) \subset B(0, S)$, $\forall t, n$, $\nabla (P_t^*)^* \to \nabla P_t^*$ strongly in $L^1$, hence $\nabla (P_t^*)^* \to \nabla P_t^*$ strongly in $L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}_+)$. Thus, we may pass to a subsequence, chosen independently of $t$, and which we do not re-label, so that

$$\nabla (P_t^{n_k})^* \to \nabla P_t^* \text{ a.e. } y \in \mathbb{R}^3,$$

as $k \to \infty$, for almost every $0 \leq t < T$.

Using this, together with the fact that $\Phi_t$ is measure preserving, we may conclude as before that

$$\nabla (P_t^*)^{n_k} \circ \Phi_t(y) - \nabla P_t^* \circ \Phi_t(y) \to 0, \quad \text{a.e. } y \in \mathbb{R}^3,$$

a.e. $0 \leq t < T$. This, together with the strong convergence in $L^1$ of $\alpha_0^e \to \alpha_0$, and Lebesgue’s dominated convergence theorem, yield,

$$\int_{\mathbb{R}^3} |\nabla (P_t^*)^{n_k} \circ \Phi_t(y) - \nabla P_t^* \circ \Phi_t(y)| \alpha_0^e(y) \, dy \to 0. \tag{44}$$

From (43) and (44) we have that $I_2^1 \to 0$, which concludes the analysis of $I_1$. 

Next we consider $I_2$. Using the fact that $\nabla P^n_0 \# \chi_\Omega = \alpha^n_0$, we have that
\[
\int_\Omega |\nabla (P^n_0)^* \circ \Phi_r \circ \nabla P^n_0(x) - \nabla P^n_* \circ \Phi_r \circ \nabla P^n_0(x)| \, dx \\
= \int_{\mathbb{R}^d} |\nabla (P^n_0)^* \circ \Phi_r(y) - \nabla P^n_* \circ \Phi_r(y)| \alpha^n_0(y) \, dy,
\]
which is the same as $I_1^{2,2}$. Hence, from (44), it follows that, passing to the appropriate subsequence, $I_2 \to 0$.

Finally, we consider the last integral,
\[
I_3 = \int_\Omega |\nabla P^n_* \circ \Phi_r \circ \nabla P^n_0(x) - \nabla P^n_* \circ \Phi_r \circ \nabla P_0(x)| \, dx.
\]

Now, since $\nabla P^n_*$ is bounded, it is enough, by the Lebesgue dominated convergence theorem, to prove that
\[
\nabla P^n_* \circ \Phi_r \circ \nabla P^n_0(x) - \nabla P^n_* \circ \Phi_r \circ \nabla P_0(x) \to 0 \text{ a.e. } x \in \Omega.
\]

To this end we, once more, pass to a subsequence for which $\nabla P^n_0 \to \nabla P_0$ a.e. $x \in \Omega$, and we do not further re-label.

Recall that the support of $\alpha^n_0$ was assumed to be contained in the ball $B(0, R_0)$, for all $n$, and that $\nabla P^n_0(\Omega)$ is precisely the support of $\alpha^n_0$, hence contained in $B(0, R_0)$.

Next, we note that, passing to subsequences as needed, $\Phi_r \circ \nabla P^n_0 \to \Phi_r \circ \nabla P_0$ a.e. $x \in \Omega$. We will show this by proving the convergence of $\Phi_r \circ \nabla P^n_0 \to \Phi_r \circ \nabla P_0$ in $L^1$ and passing to a subsequence which converges a.e. $x \in \Omega$.

By Lusin’s theorem we have that $\Phi_r$ coincides with a continuous function up to a set of arbitrarily small Lebesgue measure. More precisely, let $\varepsilon > 0$ and consider $f^\varepsilon \in C^0(B(0, R_0))$ and $E^\varepsilon \subset B(0, R_0)$ such that $\Phi_r = f^\varepsilon$ outside of $E^\varepsilon$ and $\|E^\varepsilon\| < \varepsilon$. Since $\Phi_r$ is bounded, for each $r$, we may assume that $f^\varepsilon$ is also bounded, uniformly in $\varepsilon$. We use the fact that $\nabla P^n_0 \# \chi_\Omega = \alpha^n_0$, and the analogous fact for $P_0$, to estimate
\[
\limsup_{n_i \to \infty} \int_\Omega |\Phi_r \circ \nabla P^n_0 - \Phi_r \circ \nabla P_0| \, dx \leq \limsup_{n_i \to \infty} \int_\Omega |(\Phi_r - f^\varepsilon) \circ \nabla P^n_0| \, dx \\
+ \limsup_{n_i \to \infty} \int_\Omega |f^\varepsilon \circ \nabla P^n_0 - f^\varepsilon \circ \nabla P_0| \, dx + \int_\Omega |(f^\varepsilon - \Phi_r) \circ \nabla P_0| \, dx \\
= \limsup_{n_i \to \infty} \int_{E^\varepsilon} |\Phi_r - f^\varepsilon| \, d\alpha^n_0 + \limsup_{n_i \to \infty} \int_\Omega |f^\varepsilon \circ \nabla P^n_0 - f^\varepsilon \circ \nabla P_0| \, dx \\
+ \int_{E^\varepsilon} |f^\varepsilon - \Phi_r| \, d\alpha_0 \leq 2 \|\Phi_r - f^\varepsilon\|_{L^\infty(\alpha^n_0(E^\varepsilon))} + |\alpha_0(E^\varepsilon)| \\
+ \limsup_{n_i \to \infty} \int_{E^\varepsilon} |f^\varepsilon \circ \nabla P^n_0 - f^\varepsilon \circ \nabla P_0| \, dx.
\]

The first term can be made arbitrarily small since $\{\alpha^n_0\}$ is uniformly integrable, while the second term vanishes because $f^\varepsilon$ is continuous.

We have shown that $\Phi_r \circ \nabla P^n_0 \to \Phi_r \circ \nabla P_0$ a.e. $x \in \Omega$, passing to a further subsequence if needed. Next, recall that $\nabla P^n_*$ is continuous in $\mathbb{R}^3 \setminus N$, so that, passing to the subsequence above, to obtain (46) it is enough to show that
\[
|[x \in \Omega; \Phi_r \circ \nabla P_0(x) \in N]| = 0.
\]
Recall that $\Phi_t^\ast$ preserves Lebesgue measure, so that $|\Phi_t^\ast(N)| = |N| = 0$. With this we obtain, using again that $\nabla P_0 \chi_{\Omega^1} = \alpha_0$,

$$
|x \in \Omega; \Phi_t \circ \nabla P_0(x) \in N| = |x \in \Omega; \nabla P_0(x) \in \Phi_t^\ast(N)| = 0.
$$

With this we obtain, using again that $\nabla P_0 \chi_{\Omega^1} = \alpha_0$,

$$
\left\{ \begin{array}{l}
\nabla \alpha_0(y) dy = 0,
\end{array} \right.
$$

and therefore $I_3 \to 0$.

This establishes our result if $r = 1$.

Now, given that $\nabla (P_t^\ast)^\ast$ is uniformly bounded, we obtain the convergence in $L^r(\Omega)$, $1 < r < \infty$, by interpolation. This concludes the proof.

\[ \blacksquare \]

**Remark 3.2.** We commented above the statement of theorem 1.2 that the existence of Lagrangian solutions in the case $q = 1$ follows from the analysis in this work together with the techniques developed in [14]. We will now clarify this comment. Let $\alpha_0 = DP_0 \chi_{\Omega^1} \in L^1(\mathbb{R}^3)$ be compactly supported. We have, from theorem 1.1, the existence of a weak solution $\alpha \in C([0, T), L^1(\Omega))$ and a convex (modified pressure) $P \in C([0, T), W^{1, r}(\Omega))$, such that $\nabla P$ is uniformly bounded, and such that its Legendre transform $P^\ast$ is convex, pointwise in time, with uniformly bounded gradient. This is enough, as we have already observed (see Remark 1.1) to show that proposition 1.1 holds in the case $q = 1$. Following the construction in [8], one needs to show that the pair $(P, F)$, with $F = \nabla P_t^\ast \circ \Phi_t \circ \nabla P_0$, is a Lagrangian solution. This was established, in [8], by means of propositions 2.16, 2.17, 2.18, 2.20 and 2.21, for $\alpha_0 \in L^q, q > 1$. Of these propositions, the only one which is affected by the regularity of $\alpha$ is proposition 2.18, in which the continuity, in time, of $F$ is established. The proof that this result can be extended to allow for $\alpha_0 \in L^1$ consists of a simplified version of the estimates derived in the proof of our theorem 3.1.

\section*{4. Weak stability for the shallow water case}

The shallow water version of the SG equations can be written as an equation for $h = h(x, t)$, $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$, $t \in [0, T)$ and $v = (v_1, v_2)$. Here, $h$ is the height of fluid above $\Omega$ and $v$ is the velocity. We denote $D_t = \partial_t + v \cdot \nabla$ and we set

$$
P = P(x, t) = h(x, t) + \frac{1}{2} |x|^2 \quad \text{and} \quad X = \nabla P.
$$

With this notation, the shallow water SG equations take the form

$$
\left\{ \begin{array}{l}
D_t X = (X - x)^\perp, \quad \Omega \times (0, T), \\
\partial_t h + \text{div}(hv) = 0, \quad \Omega \times (0, T), \\
v \cdot v = 0, \quad \text{on } \partial \Omega \times (0, T), \\
h(x, 0) = h_0(x), \quad \text{in } \Omega.
\end{array} \right.
$$

Here, $(a, b)^\perp = (-b, a)$. 

In dual variables, this problem can be written as
\[
\begin{align*}
\partial_t \alpha + \nabla \cdot (U \alpha) &= 0, & \mathbb{R}^2 \times (0, T), \\
\nabla P \cdot h_t &= \alpha_t, & \text{a.e. in } t \in (0, T), \\
U(X, t) &= [X - \nabla P^*(X, t)]^\perp, & \mathbb{R}^2 \times [0, T), \\
P^*(X, t) &= \sup_{s \in \Omega} \{x \cdot X - P(x, t)\}, & \mathbb{R}^2 \times [0, T), \\
\alpha(\cdot, 0) &= \alpha_0 \equiv \nabla P_0 \cdot h_0.
\end{align*}
\]

(49)

For the modelling background concerning this system, see [8, 9]. A weak solution for this system was obtained by Cullen and Gangbo, see [9], in the case \( q > 1 \), and their existence result is similar to theorem 1.1. In [8], Cullen and Feldman also proved the existence of Lagrangian solutions in physical space for the system (48) for \( q > 1 \). The existence results, both for weak solutions in dual variables (from [9]) and for Lagrangian solutions in physical variables (from [8]), can be extended to \( q = 1 \). The proof for weak solutions is an easy adaptation of the work in [14], whereas the proof for Lagrangian solutions follows, as before, from the techniques of [14] together with the analysis developed in this work.

We are interested in Lagrangian solutions \((P, F)\), where \( F : \Omega \times [0, T) \to \Omega \) is a Lagrangian flow associated with \( v \), and \( P \) is obtained from a weak solution in dual variables. However, for the shallow water case, the vector field \( v \) is not divergence free. Nevertheless, the transport equation \( \partial_t h + \text{div}(h v) = 0 \) holds. Therefore, if \( F \) is a Lagrangian flow associated with \( v \), the solutions \( h \) of this equation satisfy \( F_t \cdot h_0 = h_t \), \( \forall t \in [0, T) \). This property replaces the fact that \( F \) preserves the Lebesgue measure in the incompressible case.

**Definition 4.1 (Lagrangian Solutions, shallow water case).** Let \( \Omega \subset \mathbb{R}^2 \) be open and bounded and let \( T > 0 \). Let \( P_0 = P_0(x) \) be a convex, bounded function in \( \Omega \) such that \( h_0(x) = P_0(x) - \frac{1}{2} |x|^2 \geq 0 \) in \( \Omega \). Let \( r \in [1, \infty) \) and \( P : \Omega \times [0, T) \to \mathbb{R} \) be such that
\[
P \in L^\infty([0, T), W^{1,\infty}(\Omega)) \cap C([0, T), W^{1,r}(\Omega))
\]
\[
P(\cdot, t) \text{ is convex in } \Omega \text{ for each } t \in [0, T).
\]

Let \( h(x, t) = P(x, t) - \frac{1}{4} |x|^2 \). Let \( F : \Omega \times [0, T) \to \Omega \) be a Borel map satisfying
\[
F \in C([0, T), L^r(\Omega, h_0 \text{d}x)).
\]

The pair \((P, F)\) is called a weak Lagrangian solution of (48) in \( \Omega \times [0, T) \) if

(i) \( F(x, 0) = x \), \( h_0 \)-a.e. in \( \Omega \), \( P(x, 0) = P_0(x) \) a.e. in \( \Omega \).

(ii) for every \( t > 0 \) the map \( F_t = F(\cdot, t) : \Omega \to \Omega \) is such that \( F_t \cdot h_0 = h_t \).

(iii) there exists a Borel map \( F^*_t : \Omega \times [0, T) \to \Omega \) such that, for each \( t \in (0, T) \) we have \( F^*_t = F^*_t \cdot h_0 = h_0 \) and \( F_t \circ F^*_t(x) = x \), \( h_t - a.e. \) in \( \Omega \) and \( F^*_t \circ F_t(x) = x \), \( h_0 - a.e. \) in \( \Omega \).

(iv) the function
\[
Z(x, t) = \nabla P(F_t(x), t)
\]
is a weak solution of
\[
\begin{align*}
\partial_t Z(x, t) &= [Z(x, t) - F(x, t)]^\perp, & \text{on supp } h_0 \text{ in } \Omega \times [0, T), \\
Z(x, 0) &= \nabla P_0(x), & \text{on supp } h_0 \text{ in } \Omega,
\end{align*}
\]
in the sense that, for every \( \varphi \in C^1_c(\Omega \times [0, T)) \), we have
\[
\int_{\Omega \times [0, T)} [Z(x, t) \cdot \partial_t \varphi(x, t) + (Z(x, t) - F(x, t))^\perp \cdot \varphi(x, t)] h_0(x) \text{d}x \text{d}t
\]
\[
+ \int_{\Omega} \nabla P_0(x) \cdot \varphi(x, 0) h_0(x) \text{d}x = 0.
\]

The map \( F_t \) is a Lagrangian flow in physical space.
With this definition in place we give the precise statement of the existence of Lagrangian solutions in the shallow water case. As noted, the case $q > 1$ was established in [8]; we will discuss the case $q = 1$ in a remark following the proof of theorem 4.2.

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^2$ be open and bounded, and assume that $\bar{\Omega} \subset B$, where $B$ is the open ball $B(0, S)$. Let $h_0(x) \geq 0$ be such that $P_0 = P_0(x) = h_0(x) + \frac{1}{2}|x|^2$ is a convex, bounded function in $B$, and assume that

$$D P_0 \# h_0 \in L^q(\nabla P_0(\Omega))$$

for some $q \geq 1$. Then, for each $T > 0$, there exists a Lagrangian solution $(P, F)$ of (48) in $\Omega \times [0, T)$, where (50)–(52) hold for all $r \in [1, \infty)$. Furthermore, the function $Z = Z(x, t)$ defined in (53) satisfies $Z(x, \cdot) \in W^{1,\infty}([0, T))$ $h_0$-almost everywhere in $\Omega$ and (54) is also satisfied in the following sense

$$\begin{align*}
\partial_t Z(x, t) &= (Z(x, t) - F(x, t))^{\perp}, \quad h_0 L^2 \times L^1 \text{ a.e. in } \Omega \times (0, T), \\
Z(x, 0) &= \nabla P_0(x), \quad h_0 L^2 \text{ a.e. in } \Omega.
\end{align*}$$

As before, one obtains $P$ from the dual problem; Cullen and Feldman showed that $F(\cdot, t)$, given by the expression below, is a Lagrangian flow in physical space:

$$F(x, t) = \nabla P^*_t \circ \Phi_t \circ \nabla P_0(x),$$

where, for each $t$, $\Phi_t(X)$ is the Lagrangian flow in dual space associated with the vector field $U(X, t) = [H(X) - \nabla P^*_t(X)]^{\perp}$, whose construction in $\mathbb{R}^2$ arises in the same manner as for the incompressible case.

Let us now address the stability of Lagrangian solutions. Consider $\alpha_0, \alpha_0^* \in L^1(\mathbb{R}^2)$ with $\alpha_0^* \to \alpha_0$ in $L^1$ and with supports contained in a single ball $B(0, R_0)$. Let $\alpha^n = \alpha^n(x, t)$ be weak solutions in dual variables with initial data $\alpha_0^n$. As before, there exists an Orlicz space $L_A$, with $A$-regular $N$-function $A$, such that $\{\alpha_0^n\}, \alpha_0$ is uniformly bounded in $L_A(\mathbb{R}^2)$.

Let $\alpha^n = \alpha^n(x, t)$ be a weak solution of (49) with initial data $\alpha_0^n$ and let $h^n$ be the corresponding height and $P^n$ be the corresponding modified pressure. It can be easily deduced, from the proofs of lemma 3.6, lemma 4.3 and theorem 4.4 of [9] that, since $\alpha_0^n \to \alpha_0$ strongly in $L^1$, there exists a subsequence such that

$$\begin{align*}
\alpha^n(\cdot, t) &\rightharpoonup \alpha(\cdot, t) \quad \text{weak in } \mathcal{B} \mathcal{M}, \\
h^n(\cdot, t) &\rightharpoonup h(\cdot, t) \quad \text{in } L^\infty(\Omega), \\
(P^n_t)^* &\rightharpoonup P^*_t \quad \text{strongly in } W^{1,1}, \\
\nabla (P^n_t)^* &\rightharpoonup \nabla P^*_t \quad \text{weak in } L^\infty,
\end{align*}$$

for each $0 \leq t < T$, with $\alpha, h, P^*$ a weak solution of the SG shallow water equations. Furthermore, we may assume that $\alpha^n(\cdot, t) \rightharpoonup \alpha(\cdot, t)$ weak in $L_A(\mathbb{R}^2)$, as we had in the incompressible case.

We use $\Phi^n$ to denote the Lagrangian flow in the dual space associated with $U^n$, and we denote by $F^n_t := \nabla (P^n_t)^* \circ \Phi^n_t \circ \nabla P_0^n$ the corresponding Lagrangian flow in physical space. Accordingly, let $\Phi$ denote the Lagrangian flow in dual variables, associated with the limit velocity $U$ and let $F_t := \nabla P^*_t \circ \Phi_t \circ \nabla P_0$ be the corresponding Lagrangian flow in physical space.

We note that the result in lemma 3.1 remains valid in this context.

**Theorem 4.2.** There exists a subsequence $\{F^n_t\} \subset \{F_t\}$ such that, for almost every $t \in [0, T)$, we have

$$\lim_{k \to \infty} \int_{\Omega} |F^n_k(x) - F_t(x)|^r \, h_0(x) \, dx = 0;$$

for any $r \in [1, \infty)$. 


Proof. Since $F^n$ is bounded uniformly in $L^\infty([0, T) \times \Omega)$, since $h^n_0 \to h_0$ uniformly and since $\Omega$ is bounded, it is clear enough to prove that
\[
\lim_{n \to \infty} \int_\Omega |F^n_t(x) - F_t(x)|^r h^n_0(x) \, dx = 0.
\] (60)

To show (60) we note that, as in the incompressible case, we need only analyse the case $r = 1$, as $r > 1$ follows by interpolation. We have
\[
\int_\Omega |F^n_t(x) - F_t(x)| h^n_0(x) \, dx
\]
\[
\leq \left\{ \int_\Omega |\nabla(P^n_{\alpha_0}) \ast \Phi^n_t \circ \nabla P^n_0(x) - \nabla(P^n_{\alpha_0}) \ast \Phi_t \circ \nabla P^n_0(x) h^n_0(x) \, dx
\right\}
\]
\[
+ \int_\Omega |\nabla P^n \ast \Phi_t \circ \nabla P^n_0(x) - \nabla P^n \ast \Phi_0 \circ \nabla P_0 h^n_0(x) \, dx
\]
\[
+ \int_\Omega |\nabla P \ast \Phi_t \circ \nabla P_0(x) - \nabla P \ast \Phi_0 \circ \nabla P_0(x) h^n_0(x) \, dx\}
\equiv \{\hat{I}_1 + \hat{I}_2 + \hat{I}_3\}. \tag{61}
\]

The analysis of each of these integrals closely follows the analysis performed on the analogous integrals in theorem 3.1, once we use the facts that $\nabla P^n \ast \alpha^n_0 = \alpha^n_0$, $\Phi^n_t \ast \alpha^n_0 = \alpha^n_t$.

\[\square\]

Remark 4.1. As it happened with the proof of existence of Lagrangian solutions in the incompressible case, for $\alpha^n_0 \in L^1$, the only nontrivial step in proving existence of Lagrangian solutions in the shallow water case, assuming only $DP_0 \ast h_0 \in L^1(\nabla P_0(\Omega))$, is the continuity in time of the Lagrangian flow in physical space. The proof is an easy adaptation of the proof of theorem 4.2, where one estimates similar, yet fewer, terms to those above.

5. An example in the space of measures

The purpose of this section is to describe a counterexample for theorem 3.1 for potential vorticities which are not absolutely continuous with respect to the Lebesgue measure.

For the discussion in this section we will ignore the vertical variable in the incompressible SG equations; the argument we will present can be easily adapted to accommodate the third direction.

We fix the physical space to be the planar disc $\Omega = B(0, 1)$. Let $z_0 = (1, 0)$ and set
\[
\alpha_0 = \pi \delta_{z_0},
\]
where $\delta_P$ denotes the Dirac measure at $P$. Let $\alpha(t) = \pi \delta_{z(t)}$, with $z(t) = (\cos t, \sin t)$. It can be checked that $\alpha$ is a weak solution of (4), in the sense of [13]. Next, observe that the unique (up to a constant) convex potential for the optimal transport map between $\chi_\Omega$ and $\alpha(t)$ is given by $P = P(x, t) = z(t) \cdot x$. Its Legendre transform is $P^*(y, t) = ||y - z(t)||$ and, consequently, $\nabla P^*(y, t) = y - z(t)/||y - z(t)||$. The Lagrangian flow in dual space, restricted to the support of $\alpha_0$, is precisely $z_0 \mapsto z(t)$. The Lagrangian flow in physical space cannot be computed by (16), since $\Phi_t \circ \nabla P_0(\cdot)$ is identically equal to $z(t)$, where $\nabla P^*$ is not defined.

One can use approximations as a strategy to circumvent the difficulty described above; as we will show, this does not work.
Proposition 5.1. Let \( z(t) = (\cos t, \sin t) \) and set

\[
\alpha^\varepsilon \equiv \frac{1}{\varepsilon^2} \chi_{B(z(t), \varepsilon)}.
\]

Then \( \alpha^\varepsilon \) is an exact weak solution of the SG equations in dual variables (4) with initial potential vorticity \( \alpha^\varepsilon(\cdot, 0) \).

**Proof.** Let us first establish the relation between a potential vorticity of the form \( \alpha^\varepsilon \) and the corresponding velocity \( U^\varepsilon \). To this end, we fix \( \bar{z} \in \mathbb{R}^2 \) and we consider

\[
\bar{\alpha}^\varepsilon \equiv \frac{1}{\varepsilon^2} \chi_{B(\bar{z}, \varepsilon)}.
\]

The optimal transport map between \( \chi/Omega_1 \) and \( \bar{\alpha}^\varepsilon \) is given by \( \nabla \bar{P}^\varepsilon \), where the convex potential \( \bar{P}^\varepsilon \) is, up to a constant,

\[
\bar{P}^\varepsilon(x) = \bar{z} \cdot x + \frac{\varepsilon |x|^2}{2},
\]

and hence \( \nabla \bar{P}^\varepsilon(x) = \bar{z} + \varepsilon x \). Indeed, it can be easily verified that \( \nabla \bar{P}^\varepsilon \# \chi/Omega_1 = \bar{\alpha}^\varepsilon \) and \( \bar{P}^\varepsilon \) is convex, so that the uniqueness part of Brenier’s polar factorization theorem, see [4], may be applied. The Legendre transform of \( \bar{P}^\varepsilon \) is

\[
(\bar{P}^\varepsilon)^*(y) = \begin{cases} \|y - \bar{z}\|^2 / 2\varepsilon, & \text{if } y \in B(\bar{z}, \varepsilon), \\ |y - \bar{z}| - \frac{\varepsilon}{2}, & \text{if } y \notin B(\bar{z}, \varepsilon). \end{cases}
\]

Therefore, we find that

\[
\nabla(\bar{P}^\varepsilon)^*(y) = \begin{cases} \frac{y - \bar{z}}{\varepsilon}, & \text{if } y \in B(\bar{z}, \varepsilon) \\ \frac{y - \bar{z}}{\|y - \bar{z}\|}, & \text{if } y \notin B(\bar{z}, \varepsilon). \end{cases}
\]

(62)

For each fixed \( t \), we have that \( \alpha^\varepsilon \) is of the form \( \bar{\alpha}^\varepsilon \) with \( \bar{z} = z(t) \). Therefore the corresponding SG velocity \( U^\varepsilon \), in dual variables, is given by

\[
U^\varepsilon = U^\varepsilon(y, t) = (y - \nabla P^\varepsilon)^*(y) \perp,
\]

where \( (P^\varepsilon)^* \) is given by the expression in (62) with \( \bar{z} = z(t) \).

Consider \( y_0 \in B(z_0, \varepsilon) \). Let \( y = y(t) \) be the solution of

\[
\begin{cases} y' = U^\varepsilon(y, t), \\ y(0) = y_0. \end{cases}
\]

As long as \( y(t) \in B(z(t), \varepsilon) \) we see that

\[
y' = \left( \frac{\varepsilon - 1}{\varepsilon} y + \frac{z(t)}{\varepsilon} \right) \perp.
\]

We also have

\[
z' = z \perp.
\]

Thus, subtracting these two equations, we deduce that

\[
\begin{cases} (y - z)' = \frac{\varepsilon - 1}{\varepsilon} (y - z) \perp, \\ (y - z)(0) = y_0 - z_0 \in B(0, \varepsilon). \end{cases}
\]
Therefore \( y - z \) rotates around the origin at the rate \( (\epsilon - 1)/\epsilon \); hence, in particular, \( y(t) \) rotates around \( z(t) \) and never leaves \( B(z(t), \epsilon) \). We have shown that the flow of \( U^\epsilon \) maps \( B(z_0, \epsilon) \) to \( B(z(t), \epsilon) \) through a rigid rotation.

This implies that \( \alpha^\epsilon \) is a weak solution of the transport equation \( \partial_t \alpha^\epsilon + U^\epsilon \cdot \nabla \alpha^\epsilon = 0 \), as desired.

\[ \Box \]

Remark 5.1. Note that \( \alpha^\epsilon(\cdot, t) \rightharpoonup \alpha(t) \) weak-* \( \mathcal{B} \mathcal{M} \), in accordance with [13].

Remark 5.2. From the proof above we obtain an explicit expression for the Lagrangian flow in dual variables for Lagrangian markers inside \( B(z_0, \epsilon) \), namely

\[
\Phi^\epsilon = \Phi^\epsilon_i(y_0) = z(t) + \begin{bmatrix}
\cos \left( \frac{\epsilon - 1}{\epsilon} t \right) - \sin \left( \frac{\epsilon - 1}{\epsilon} t \right) \\
\sin \left( \frac{\epsilon - 1}{\epsilon} t \right) \cos \left( \frac{\epsilon - 1}{\epsilon} t \right)
\end{bmatrix} (y_0 - z_0). \tag{63}
\]

Next we compute the Lagrangian flow in physical space associated with \( \alpha^\epsilon \) using expression (16). Let \( x \in \Omega \) and note that \( \nabla P_0^\epsilon(x) = z_0 + \epsilon x \in B(z_0, \epsilon) \). Hence we may use the Lagrangian map (63), together with the expression in (62) with \( \bar{z} = z(t) \), to obtain

\[
F^\epsilon_t = F^\epsilon_t(x) = \begin{bmatrix}
\cos \left( \frac{\epsilon - 1}{\epsilon} t \right) - \sin \left( \frac{\epsilon - 1}{\epsilon} t \right) \\
\sin \left( \frac{\epsilon - 1}{\epsilon} t \right) \cos \left( \frac{\epsilon - 1}{\epsilon} t \right)
\end{bmatrix} x. \tag{64}
\]

In other words, as \( \epsilon \to 0 \), \( F^\epsilon_t \) describes a rotation around the origin in physical space with arbitrarily large angular velocity. In short, a concentrated vortex in dual space corresponds to a Lagrangian fast eddy in physical space, but concentrating the dual space vortex into a point appears to produce an eddy which rotates at infinite speed. This shows that it is impossible to extend the weak stability theory we developed here in \( L^1 \) to the full space of measures, while keeping the strong convergence of sequences of Lagrangian flows as a conclusion. There are two possibilities for further work in this direction. One is to develop a theory of weak convergence of Lagrangian flows associated with converging sequences of potential vorticities in the space of measures. Another possibility is to try to extend the \( L^1 \) theory to spaces of continuous measures, considering that, even though it is not clear which physical flow is associated with Diracs in dual space, it is possible that other measures, such as potential vortex sheets, may be associated with physical space flows in the usual sense.

We conclude with the following observation. We established the convergence of Lagrangian flows a.e. in time, \( L' \) in space. However, this may not be optimal, and this leads to an interesting line of investigation. It was pointed out, by Brenier and Gangbo in [5], that the topology induced by \( L' \)-convergence in the space of diffeomorphisms is not very satisfactory. One may investigate, for instance, whether the convergence of Lagrangian flows can be improved for potential vorticities in Hölder spaces, using the regularity theory for optimal transport developed by Ma et al in [15].

Acknowledgments

The research presented here is part of the PhD thesis of J C O Faria, who was supported in part by CNPq grant #141.217/2004-9. The research of M C Lopes Filho is supported in part by CNPq grant #303.301/2007-4 and the research of H J Nussenzveig Lopes is supported in part by CNPq grant #302.214/2004-6. This work acknowledges the support of FAPESP grant #2007/51490-7.
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