Circular automata synchronize with high probability

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Abstract

In this paper we prove that a uniformly distributed random circular automaton \( A_n \) of order \( n \) synchronizes with high probability (whp). More precisely, we prove that

\[
P[A_n \text{ synchronizes}] = 1 - O \left( \frac{1}{n} \right).
\]

The main idea of the proof is to translate the synchronization problem into properties of a random matrix; these properties are then handled with tools of the probabilistic method. Additionally, we provide an upper bound for the probability of synchronization of circular automata in terms of chromatic polynomials of circulant graphs.

Keywords: Automata; Synchronization; Random Matrices; Circulant Graphs; Chromatic Polynomials.

1 Introduction

A complete deterministic finite automaton (DFA) is tuple \( A = (Q, L) \) where \( Q := \{q_1, q_2, \ldots, q_n\} \) is a finite set where \( q_i \) is a state and \( L := \{a_1, a_2, \ldots, a_k\} \) is a finite set of automorphisms of \( Q \) where \( a_i: Q \rightarrow Q \) is a letter; we say that \( A \) has order \( n \). We use the notation \( qa_1a_2 := a_2(a_1(q)) \) (similarly for a finite composition of elements); \( w = a_{i_1}a_{i_2} \ldots a_{i_h} \) is a word of \( L \) of length \( h \). We say that \( \{q_{i_1}, q_{i_2}, \ldots, q_{i_h}\} \subseteq Q \) collapses if there is a word \( w \) of \( L \) such that \( q_{i_1}w = q_{i_2}w = \ldots = q_{i_h}w \) (equivalently, we say that \( w \) collapses the states or that the states collapse). We say that \( A(Q, L) \) synchronizes if \( Q \) collapses; a word \( w \) in \( L \) that collapses \( Q \) is a synchronizing word of \( A \). The following criterion of synchronization is well known and very useful

Claim 1. \( A = (Q, L) \) synchronizes \( \iff \) every pair of states \( q, q' \in Q \) collapses.

Definition. \( \{q_{i_1}, q_{i_2}, \ldots, q_{i_h}\}w := \{q_{i_1}w, q_{i_2}w, \ldots, q_{i_h}w\} \)

Proof. \( \Rightarrow \) If \( w \) collapses \( Q \), then \( w \) collapses any pair of states in \( Q \).

\( \Leftarrow \) If \( Q \) does not collapse, then there is \( N \subset Q \) of minimal cardinality \( |N| \geq 3 \) such that \(|Nw| = |N|\) for any word \( w \) of \( L \). Let \( q, q' \in N \) such that \( q \neq q' \) then \( qw^* = q'w^* \) for some \( w^* \in L \); this implies that \( |Nw^*| < |N| \frac{1}{2} \). Therefore \( Q \) collapses.

The automaton \( A \) can also be seen as a directed graph \( D(A) \) with vertex set \( Q \) and arrows labeled by \( L \) where the arrow \( (q, q')a_i \) with label \( a_i \) belongs to \( D(A) \) if \( a_i(q) = q' \). Let \( q \in Q \) and let \( w = a_{i_1}a_{i_2} \ldots a_{i_h} \) be a word of \( L \), then \( w \) defines a directed path

\[
P(qw) := q, qa_{i_1}, qa_{i_1}a_{i_2}, \ldots, qa_{i_1}a_{i_2} \ldots a_{i_h}
\]

that begins in \( q \) and end in \( qw \). \( A \) synchronizes if there is a word \( w \) in \( L \) such that the paths \( \{P(qw) : q \in Q\} \) have a common endpoint.

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Synchronizing automata have been intensely studied in applications and pure mathematics since the 1960’s (see [Volkov, 2008] for a detailed introduction on synchronization of automata). A driving force in this research field is the Černý conjecture:

Conjecture 2. [Cerný et al., 1971] Let \( A \) be a synchronizing automaton of order \( n \), then a minimal synchronizing word has length at most \((n-1)^2\).

[Cerný, 1964] gives a series of cyclic automata \( C_2, C_3, \ldots \) such that \( C_n \) has order \( n \) and its minimal words have size \((n-1)^2\). This shows that the bound in Conjecture 2 is optimal.

Remark 3. The Černý series of cyclic automata \( C_2, C_3, \ldots \) is the only known infinite series of “extreme” automata [Ananichev et al., 2010].

The best known general upper bounds for the size of minimal words of an automaton of order \( n \) are \( O(n^3) \) [Pin, 1983][Szykuła, 2017]. Nevertheless, there are many classes of automata for which the Černý conjecture is valid (see [Volkov, 2008] for examples).

In the last decade, the probabilistic approach has gained the community’s attention due to remarkable results in the study of the Černý conjecture. Let \( n, k \in \mathbb{N} \) and let \( A(\{0, 1, \ldots, n-1\}, L) \) be a uniformly chosen DFA with \( k \) letters; here is a (non-comprehensive) list of recent achievements:

- [Skvortsov and Zaks, 2010] study random automata \( A \) where the number of letters \( k \) grow together with \( n \). In particular, they prove that \( A \) synchronizes whp when \( k(n) \) grows fast enough;
- [Berlinkov, 2013] proves that \( \mathbb{P}[A \text{ synchronizes}] = 1 - O(n^{-k/2}) \), for arbitrary but fixed \( k \geq 2 \). He also proves that \( \mathbb{P}[A \text{ synchronizes}] = 1 - \Theta(1/n) \) for \( k = 2 \);
- [Nicaud, 2014] proves that \( A \) admits whp a synchronizing word of length \( O(n \log^3 n) \) for arbitrary but fixed \( k \geq 2 \);
- [Berlinkov and Nicaud, 2018] prove that if \( A \) is uniformly chosen among the strongly-connected almost-group automata then \( A \) synchronizes with probability \( 1 - \Theta((2^{k-1} - 1) n^{-2(k-1)}) \) for arbitrary \( k \geq 2 \).

The rest of the article is organized as follows: in Section 2, we present the main result together with its proof and the enunciation of the two key lemmas for the proof. In Section 3, we study the dependence structure of the random matrix used in the proof of the main result; the result obtained in this section is crucial for the proof of the key lemmas. In Section 4 we prove the first lemma and in Section 5 we prove the second lemma. In Section 6, we present some interesting connections between synchronization of circular automata and chromatic polynomials of circulant graphs. In Section 7, we present some possible directions to generalize and improve the ideas presented in this article.

2 Main result

An automaton \( A(\{0, 1, \ldots, n-1\}, L) \) is a circular automaton if \( L \) contains a permutation that decomposes in exactly one cycle. In 1978, Pin proved with combinatorial methods that a circular automaton \( A(Q, L) \) of prime order synchronizes iff \( L \) contains a non-bijection letter. We restate Pin’s theorem in a probabilistic way:

Theorem 4 ([Pin, 1978]). Let \( A_p(b) := A(\mathbb{Z}_p, \{a, b\}) \) be a random circular automaton of prime order where \( \mathbb{Z}_p := \{0, 1, \ldots, p-1\} \) and \( a : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \) is the automorphism \( a(i) \equiv_p i + 1 \) and \( b : S \rightarrow S \) is a uniformly distributed random automorphism of \( \mathbb{Z}_p \). Then

\[
\mathbb{P}[A_p(b) \text{ synchronizes}] = 1 - \frac{p}{p^2} = 1 - \Theta\left(\frac{\sqrt{p}}{p^p}\right).
\]

Thus, a uniformly distributed random circular automaton of prime order \( p \) with \( k \geq 2 \) letters synchronizes with high probability (whp).

Remark 5. In the same paper, Pin proves that the Černý conjecture is true for the class circular automata.
A natural question arises: do random circular automata of order \( n \) (not necessarily prime) synchronize with high probability? We give a positive answer to this question in the following.

**Theorem 6 (Main result).** Let \( A_n(b) := (Z_n, \{a, b\}) \) be a random circular automaton of order \( n \in \mathbb{N} \) where \( Z_n := \{0, 1, \ldots, n - 1\} \) and \( a : Z_n \to Z_n \) is the circular permutation \( a(i) \equiv_n (i + 1) \) and \( b := (b_0, \ldots, b_{n-1}) \) is a random vector, where \( b_0, \ldots, b_{n-1} \) are independent and identically distributed (iid) random variables uniformly distributed over \( Z_n \). Then

\[
P[A_n(b) \text{ synchronizes}] = 1 - O \left( \frac{1}{n} \right) \xrightarrow{n \to \infty} 1
\]

i.e. \( A_n(b) \) synchronizes with high probability.

**Remark 7.** Theorem 6 does not follow from the results of Berlikov or Nicaud. In their models, they use a random \( A(Q, L) \) or order \( n \) where \( L \) is a collection of \( k \) automorphisms of \( Q \) i.i.d uniformly chosen. For fixed \( k \), the probability of hitting a permutation with exactly one cycle is \( \leq k \cdot \frac{\alpha}{n} \xrightarrow{n \to \infty} 0 \).

We use the following notation in the rest of the article:

**Definition.** Let \( r \) be an integer, we define

\[
(r)_n := r \pmod{n},
\]

where we always take \((r)_n \in \{0,\ldots, n - 1\}\).

**Definition.** Let \( r, s \) be integers and let \( n \) be a natural number. We say that

\[
|r - s|_n := \min\{(r - s)_n, (s - r)_n\}
\]

is the \( n \)-cyclic distance between \( r \) and \( s \).

**Remark 8.** Observe that \(|r - s|_n \in \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}\). We can interpret the numbers \( Z_n \) as the vertices of a cycle of length \( n \). The \( n \)-cyclic distance between two numbers is the length of the smallest path between them in the cycle.

The proof of Theorem 6 relies on the following technical lemmas:

**Lemma 9.** Let \( \epsilon > 0 \) be arbitrary but fixed and let \( b := (b_0, \ldots, b_{n-1}) \) be a random variable such that \( b_0, \ldots, b_{n-1} \) are iid uniformly distributed over \( Z_n \) and let

\[
R_i(b) := \# \left\{ |b_0 - b_{(i+1)}|_n, \ldots, |b_s - b_{(s+i)}|_n, \ldots, |b_{n-1} - b_0|_n \right\}
\]

for \( i \in 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \). Let \( \left( \frac{\epsilon}{\left\lfloor \frac{n}{2} \right\rfloor} \right)^c := \bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \{ R_i(b) < \left\lfloor \frac{n}{2} \right\rfloor (1 - \epsilon - \epsilon) \} \), then

\[
P\left[ \left( \frac{\epsilon}{\left\lfloor \frac{n}{2} \right\rfloor} \right)^c \right] = O \left( \frac{1}{n} \right).
\]

**Lemma 10.** Let \( \epsilon \in (0, 1) \) be arbitrary but fixed and let \( b := (b_0, \ldots, b_{n-1}) \) be a random variable such that \( b_0, \ldots, b_{n-1} \) are iid uniformly distributed over \( Z_n \) and let

\[
D_i(b) := \begin{cases} 
1, & \text{if the exist } u, v \in Z_n \text{ such that } |u - v|_n = i \text{ and } |b_u - b_v|_n = 0; \\
0, & \text{otherwise},
\end{cases}
\]

and let

\[
D(b) := \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} D_i(b)
\]

and let \( \left( \frac{\epsilon}{\left\lfloor \frac{n}{2} \right\rfloor} \right)^c := \{ D(b) < \left\lfloor \frac{n}{2} \right\rfloor (1 - \epsilon) \} \), then

\[
P\left[ \left( \frac{\epsilon}{\left\lfloor \frac{n}{2} \right\rfloor} \right)^c \right] = O \left( \frac{1}{n} \right).
\]
Proof of Theorem 6. Let $A_n(b) := (Z_n, \{a, b\})$ be a random automaton with $n$ states, where $a(i) = (i + 1)_n$ and let $b = (b_0, b_1, \ldots, b_{n-1})$ be a random vector, where $b_0, b_1, \ldots, b_{n-1}$ are i.i.d uniformly distributed random variables over $Z_n$. The main idea of the proof is to transform the question of synchronization of $A_n$ into properties of a random matrix $T_b$. The questions in the random-matrix problem are then solved with tools of the probabilistic method. Formally, we associate to $A_n(b)$ the following random matrix with $\lceil \frac{n}{2} \rceil$ rows and $n$ columns

$$T_b := \begin{bmatrix}
|b_0 - b_1|_n & |b_1 - b_2|_n & \ldots & |b_k - b_{k+1}|_n & \ldots & |b_{n-1} - b_0|_n \\
|b_0 - b_2|_n & |b_1 - b_3|_n & \ldots & |b_k - b_{k+2}|_n & \ldots & |b_{n-1} - b_1|_n \\
|b_0 - b_i|_n & |b_1 - b_{i+1}|_n & \ldots & |b_k - b_{k+i}|_n & \ldots & |b_{n-1} - b_{(n-1+i)}|_n \\
|b_0 - b_{1+|\frac{n}{2}|} & |b_1 - b_{1+|\frac{n}{2}|} & \ldots & |b_k - b_{k+1}|_n & \ldots & |b_{n-1} - b_{(n-1+|\frac{n}{2}|)}|_n
\end{bmatrix}$$

shortly written

$T_b(i, j) = |b_j - b_{(j+i)}|_n$ for $1 \leq i \leq \lfloor n/2 \rfloor$ and $0 \leq j \leq n - 1$.

where $b_i = b(i)$ i.e. $b_i$ is the image of the state $i$ under $b$.

Remark. The first row of $T_b$ is formed with the cyclic distances of the images of states $r, s$ such that $|r - s|_n = 1$; in general, the $i$'th row of $T_b$ is formed with the cyclic distances of the images of pairs of states $r, s$ at cyclic distance $i$. Notice that the columns are counted from zero to $n - 1$.

Let $b : Z_n \rightarrow Z_n$ be a realization of $b$ and $T_b$ be its corresponding associated matrix. We make two observations: a zero in row $i$, means that two states $r, s$ with cyclic distance $i$ collapse under $b$ (i.e. $b_i = b_a$), this implies that any other pair $r', s'$ with cyclic distance $i$ can be collapsed with a word of the type $a^i b'$ because $\{r', s'\}^{a^i} = \{r, s\}$ for some $l = l_{r,s,r',s'}$. Second observation: if the $i$'th row contains a number $|b_k - b_{(k+i)}|_n = j$ such that the $j$'th row of $T_b$ contains a zero, then every pair of states $r, s$ with cyclic distance $i$ can be collapsed with a word of the type $a^i b^2 b$, proceed as follows: $\{r, s\}^{a^i} \rightarrow \{k, (k + i)\}_n \rightarrow \{b_k, b_{(k+i)}\}$ where this last pair has n-cyclic distance $j$, then $b_k, b_{(k+i)}$ collapse with a word $a^2 b$ (for some $l_2$) because the $j$'th row has a zero, thus we can collapse $r, s$ with a word of the type $a^i b^2 b$. With these observations, we establish a sufficient conditions on $T_b$ for the synchronization of $A_n(b)$, namely

Claim 11. Let $b : Z_n^m \rightarrow Z_n$ be a realization of $b$ and let

$E_{\text{row}}^\alpha(b) := \text{Every row of } T_b \text{ has } \geq \alpha \cdot \lceil \frac{n}{2} \rceil$ different elements;

$E_{\text{zero}}^\beta(b) := \text{There are at least one zero row}$

If $E_{\text{row}}^\alpha(b), E_{\text{zero}}^\beta(b)$ both hold true for some $\alpha, \beta > 0$ such that $\alpha + \beta > 1$, then $A_n(b)$ synchronizes.

This claim follows from the two previous observations and the pigeonhole principle: let $r, s$ be a pair of different states. If row $|r - s|_n$ contains a zero, we can collapse $r, s$ with a word of the type $a^i b$; otherwise, row $i$ has a number $j \neq 0$ such that row $j$ contains a zero (because $\alpha + \beta > 1$), this implies that $r, s$ can be collapsed with a word of the type $a^i b^2 b$. Therefore every pair of different states collapses, thus $A_n(b)$ synchronizes by Claim 1.

The previous claim implies that for any fixed $\alpha, \beta > 0$ such that $\alpha + \beta > 1$, we have that

$$\mathbb{P}[A_n(b) \text{ synchronizes}] \geq \mathbb{P}[E_{\text{row}}^\alpha(b) \wedge E_{\text{zero}}^\beta(b)] = 1 - \mathbb{P}\left[\left(E_{\text{row}}^\alpha(b) \right)^c \vee \left(E_{\text{zero}}^\beta(b) \right)^c\right] \geq 1 - \mathbb{P}\left[\left(E_{\text{row}}^\alpha(b) \right)^c\right] - \mathbb{P}\left[\left(E_{\text{zero}}^\beta(b) \right)^c\right],$$

where we use the union bound for the last inequality. Using Lemmas 2,3 we get that

$$\mathbb{P}[A_n(b) \text{ synchronizes}] \geq 1 - \mathbb{P}\left[\left(E_{\text{row}}^\alpha(b) \right)^c\right] - \mathbb{P}\left[\left(E_{\text{zero}}^\beta(b) \right)^c\right] = 1 - O\left(\frac{1}{n}\right),$$

where $\alpha := 1 - e^{-1} - \epsilon' > 0$ and $\beta := 0.49(1 - \epsilon') > 0$ and $\alpha + \beta > 1$ for $\epsilon' = \frac{0.4}{1.09} < 1$. \qed
3 Dependence structure of $T_b$

In this section we get a simple-to-verify condition to know when a set of entries of $T_b$ is independent; this knowledge is fundamental to prove Lemmas 9,10. First, notice that not every subset of entries in $T_b$ is independent, e.g

$$T_b(1,0) = |b_0 - b_1|^n, \quad T_b(1,1) = |b_2 - b_2|^n, \quad T_b(2,0) = |b_0 - b_2|^n$$

are dependent: if the first two variables are zero then $b_0 = b_1 = b_2$, which implies that $|b_0 - b_2|^n = 0$. The problem is that these variables form a “cycle”; in fact, a set of entries of $T_b$ is independent iff the entries are “acyclic”. We formalize this in the following

**Definition.** We call $e(T_b(i,j)) := \{j, (j+i)\} \cup \{j, \ldots, s\}$ the associated edge of $T_b(i,j)$. Let

$$S = \{T_b(i_1,j_1), T_b(i_2,j_2), \ldots, T_b(i_k,j_k)\}$$

be a multi-set, the associated (multi-)graph $G(S)$ is the (multi-)graph with vertex set $\mathbb{Z}_n$ and edge (multi-)set

$$\{e(T_b(i_1,j_1)), e(T_b(i_2,j_2)), \ldots, e(T_b(i_k,j_k))\}.$$ 

We call $S$ acyclic if its associated multi-graph $G(S)$ is a forest.

The relation between acyclic variables and independent variables is stated in the following

**Proposition 12.** The variables $T_b(i_1,j_1), T_b(i_2,j_2), \ldots, T_b(i_k,j_k)$ are i.i.d $\iff$ they form an acyclic multi-set. Furthermore, if the variables are independent/acyclic then

$$\mathbb{P} \left[ \bigcap_{w=1}^k \{T_b(i_w,j_w) = s_w\} \right] = \frac{1}{n^k} \prod_{w=1}^k m_{s_w}, \quad \forall k \geq 1,$$

where $s_1, s_2, \ldots, s_w$ are arbitrary integer numbers and

$$m_s := \#\{d \in \mathbb{Z}_n : |1-d|^n = s\} = \left\{ \begin{array}{ll}
2, & \text{if } 0 < s < \frac{n}{2} \text{ and } s \in \mathbb{N}; \\
1, & \text{if } s = 0; \\
1, & \text{if } s = \frac{n}{2} \text{ and } \frac{n}{2} \in \mathbb{N}; \\
0, & \text{otherwise.}
\end{array} \right.$$

**Remark 13.**

- Note that different entries $T_b(i,j), T_b(i',j')$ may be associated with the same edge; this only happens when $n$ is even and $i = i' = n/2$ and $j \equiv j' \mod \frac{n}{2}$. Thus, for $n$ odd, a pair of different entries in $T_b$ are acyclic/independent;

- for the rest of the paper, we use the concepts “acyclic” and “independent” interchangeably, when we refer to a multi-set of acyclic/independent entries of $T_b$.

**Proof.** First, note that any two entries $T_b(i,j) = |b_j - b_{(j+i)}|^n, T_b(i',j') = |b_{j'} - b_{(j'+i')}|^n$ of $T_b$ are identically distributed because $b_0, b_1, \ldots, b_{n-1}$ are i.i.d. Therefore, we only need to prove that the independence is true iff the acyclic hypothesis is true:

$\Rightarrow$ (by contrapositive) if the multi-set $S = \{T_b(i_1,j_1), T_b(i_2,j_2), \ldots, T_b(i_k,j_k)\}$ is not acyclic, its associated multi-graph $G(S)$ has a cycle $C$ of length $l \geq 2$, let it be w.l.o.g

$$j_1, (j_1 + i_1)n = j_2, (j_2 + i_2)n = j_3, \ldots, (j_{l-2} + i_{l-2})n = j_{l-1}, (j_{l-1} + i_{l-1})n = j_1,$$

where \(\{j, (j+i)\}\) is the associated edge to $T_b(i,j)$. Note that $T_b(i,j) = 0$ iff $b_j = b_{(j+i)}$, therefore if

$$T_b(i_1,j_1) = T_b(i_2,j_2) = \ldots = T_b(i_{l-1},j_{l-1}) = 0$$

then $b_j = b_{j_2} = \ldots = b_{j_{l-1}}$, in particular $T_b(i_1,j_1) = |b_{j_1} - b_{(j_1+i_1)}|^n = |b_{j_1} - b_{j_1}|^n = 0$, then the variables in $S$ are not independent . We conclude that an independent multi-set must be acyclic.
\( \Leftrightarrow \) (by induction on \( k \)) Let \( S_k = \{ T_b(i_1, j_1), T_b(i_2, j_2), \ldots, T_b(i_k, j_k) \} \) be an acyclic multi-set. For \( k = 1 \), it easily follows that
\[
P \left[ \left\{ \left| b_p - b_{p+q} \right| = s \right\} \right] = \frac{n \cdot m_s}{n^2} = \frac{m_s}{n},
\]
for any entry of \( T_b \) because the variables \( b_0, b_1, \ldots, b_{n-1} \) are i.i.d uniformly distributed: choose \( b_p \) in \( n \) ways and then we can choose \( b_{p+q} \) in \( m_s \) ways. For \( k = 2 \), let \( T_b(i, j) = \left| b_j - b_{j+i} \right| \), \( T_b(i', j') = \left| b_{j'} - b_{j'+i'} \right| \) be acyclic variables; then, the cardinality of the set \( \{ b_i, b_{j+i}, b_{j'+i'}, b_{j'+i'} \} \) is three or four: if the cardinality is four, all variables are different and the independence of \( T_b(i, j), T_b(i', j') \) is clear since the \( b \)'s are i.i.d; furthermore
\[
P \left[ \left\{ \left| b_j - b_{(j+i)} \right| = s_1 \right\} \cap \left\{ \left| b_{j'} - b_{(j'+i')} \right| = s_2 \right\} \right] = \frac{n \cdot m_{s_1} \cdot n \cdot m_{s_2}}{n^2} = \frac{m_{s_1} \cdot m_{s_2}}{n^2},
\]
because the \( b \)'s are uniformly distributed: choose \( b_j \) in \( n \) possible ways, then choose \( b_{j+i} \) in \( m_{s_1} \) ways, then choose \( b_{j'} \) in \( n \) ways and finally choose \( b_{j'+i'} \) in \( m_{s_2} \) ways. For cardinality three, let us assume w.l.o.g. that \( b_{(j+i)} = b_{j'} \).
\[
\text{Then } P \left[ \left\{ \left| b_j - b_{(j+i)} \right| = s_1 \right\} \cap \left\{ \left| b_{j'} - b_{(j'+i')} \right| = s_2 \right\} \right] = \frac{n \cdot m_{s_1} \cdot m_{s_2}}{n^2} = \frac{m_{s_1} \cdot m_{s_2}}{n^2},
\]
this again follows because the \( b \)'s are i.i.d uniformly distributed and an easy counting argument. Using the case \( k = 1 \), the independence follows
\[
P \left[ \{ T_b(i, j) = s_1 \} \cap \{ T_b(i', j') = s_2 \} \right] = \frac{m_{s_1} \cdot m_{s_2}}{n^2} = P \left[ \{ T_b(i, j) = s_1 \} \right] \cdot P \left[ \{ T_b(i', j') = s_2 \} \right].
\]
This proves the base cases \( k = 1, 2 \). We now prove the induction step, let
\[
S = \{ T_b(i_1, j_1), T_b(i_2, j_2), \ldots, T_b(i_k, j_k), T_b(i_{k+1}, j_{k+1}) \}
\]
be an acyclic multi-set of cardinality \( k + 1 \geq 3 \), then its associated multi-graph \( G(S) \) is a forest with \( k + 1 \) edges, w.l.o.g let
\[
\epsilon_{k+1} := e(T_b(i_{k+1}, j_{k+1})) = \{ j_{k+1}, (j_{k+1} + i_{k+1}) \}
\]
be a leaf edge in \( G(S) \). We apply the induction hypothesis to the multi-graph \( G(S \setminus \{ \epsilon_{k+1} \}) \), which is a forest with \( k \) edges. We distinguish two cases: if the edge \( \epsilon_{k+1} \) is a connected component by itself in \( G(S) \), this means that any variable in \( \{ b_j - b_{(j+i)} \} \in S \} \) \( T_b(i_{k+1}, j_{k+1}) \) satisfies that \( b_j, b_{(j+i)}, b_{(j+i)} \) are i.i.d. Thus the variables in \( S \setminus \{ T_b(i_{k+1}, j_{k+1}) \} \) are independent of \( T_b(i_{k+1}, j_{k+1}) \), therefore
\[
P \left[ \bigcap_{w \in \mathcal{W} \cup \{k+1\}} \{ T_b(i_w, j_w) = s_w \} \right] = P \left[ \bigcap_{w \in \mathcal{W}} \{ T_b(i_w, j_w) = s_w \} \right] \cdot P \left[ \{ T_b(i_{k+1}, j_{k+1}) = s_{k+1} \} \right]
\]
\[= \prod_{w \in \mathcal{W} \cup \{k+1\}} m_w \frac{m_w}{n|\mathcal{W}|+1},
\]
where the last equality follows from the induction hypothesis and the case \( k = 1 \); this proves that the variables in \( S \) are i.i.d for the first case. For the second case, the edge \( \{ j_{k+1}, (j_{k+1} + i_{k+1}) \} \) is not a connected component in \( G(S) \). Let us assume w.l.o.g that \( j_{k+1} \) is a leaf vertex in \( G(S) \), meaning that any variable \( b_{j_{k+1}} \in S \} \) \( T_b(i_{k+1}, j_{k+1}) \) satisfies that \( b_{j_{k+1}}, b_{(j_{k+1}+i_{k+1})}, b_{(j_{k+1}+i_{k+1})} \) are i.i.d. We prove that Equation (2) is valid for \( S \) with an easy counting argument: let \( s_1, s_2, \ldots, s_{k+1} \) be arbitrary integers and let \( T = \{ b_1, b_{i_2}, \ldots, b_{i_k}, b_{j_{k+1}} \} \) be the set of variables on which the variables \( S \) depended on and \( T \setminus \{ b_{j_{k+1}} \} \) be the set of variables on which the variables \( S \setminus \{ T_b(i_{k+1}, j_{k+1}) \} \) depend on and let
\[
\text{Sol}(S) := \{ (b_1, b_{i_2}, \ldots, b_{i_k}, b_{j_{k+1}}) \in \mathbb{N}^{k+1} : T_b(i_w, j_w) = s_w, \forall 1 \leq w \leq k+1 \};
\]
\[
\text{Sol}(S \setminus \{ T_b(i_{k+1}, j_{k+1}) \}) := \{ (b_1, b_{i_2}, \ldots, b_{i_k}) \in \mathbb{N}^{k} : T_b(i_w, j_w) = s_w, \forall 1 \leq w \leq k \};
\]
then
\[
\Pr \left[ \bigcap_{w=1}^{k+1} \{ T_b(i_w, j_w) = s_w \} \right] = \frac{\#\text{Sol}(S)}{n^{k+1}} = \frac{\#\text{Sol}(S \setminus \{ T_b(i_{k+1}, j_{k+1}) \})}{n^k} \cdot \frac{m_{s_{k+1}}}{n} = \Pr \left[ \bigcap_{w=1}^{k} \{ T_b(i_w, j_w) = s_w \} \right] \cdot \frac{m_{s_{k+1}}}{n} = \prod_{w=1}^{k+1} m_w \frac{m_{s_{k+1}}}{n^{k+1}}.
\]

Using the case \( k = 1 \), we see that this last equality implies the independence of \( k + 1 \) variables in \( S \) because any other subset of \( S \) also satisfies Equation (2) by induction hypothesis since a subset of acyclic variables is also acyclic. We conclude that an acyclic multi-set of variables is an i.i.d multi-set of variables. This concludes the proof of Proposition 12. \( \square \)

4 Proof of Lemma 9

The overview of the proof is as follows: first, we prove that every row of \( T_b \) has a “large” number of i.i.d random variables, then we give a lower bound for the expected value of the number of elements for each row. Then we apply McDiarmid’s inequality to each row and finally we use the union bound together with the exponential decay delivered by McDiarmid’s inequality to guarantee that whp every row of \( T_b \) has at least \( \sim 0.63 \left\lfloor \frac{n}{2} \right\rfloor \) different elements.

Claim 14. Row \( i \) of \( T_b \) has at least \( n - \gcd(n, i) \) i.i.d random variables.

**Definition.** The circulant graph \( C_n(i) \) is a graph with vertex set \( Z_n \) where two vertices \( r, s \) are adjacent if \( |r - s|_n = i \).

**Proof.** The variables in row \( i \) are given by the multi-set
\[
E_i(b) := \{ |b_0 - b_{(0+i)}|_n, \ldots, |b_s - b_{(s+i)}|_n, \ldots, |b_{n-1} - b_0|_n \}.
\]

Let \( i \neq \frac{n}{2} \), by Remark 13 the multi-set \( E_i(b) \) does not have repeated elements and the associated multi-graph \( G(E_i(b)) \) is isomorphic to the circulant graph \( C_n(i) \) with \( n \) vertices and parameter \( i \), it is well know and easy to show that \( C_n(i) \) is a disjoint union of \( \gcd(n, i) \) cycles of length \( \frac{n}{\gcd(n,i)} \) [Boesch and Tindell, 1984]. We obtain an acyclic set of variables by removing variables that correspond to exactly one edge of each cycle in \( G(E_i(b)) \), the resulting set of variables is i.i.d by Proposition 12. For \( i = \frac{n}{2} \), the first \( \frac{n}{2} \) variables in row \( \frac{n}{2} \)
\[
E_{\frac{n}{2}}(b) = \{ |b_0 - b_{\frac{n}{2}}|_n, \ldots, |b_s - b_{(s+\frac{n}{2})}|_n, \ldots, |b_{\frac{n}{2}-1} - b_{n-1}|_n \}
\]
have an associated multi-graph that is isomorphic to the circulant graph \( C_n(\frac{n}{2}) \) which is a disjoint union of \( \frac{n}{2} = \gcd(n, \frac{n}{2}) \) edges. This last graph is acyclic, thus the variables are i.i.d by Proposition 12. \( \square \)

We prove the following lower bound

\[ \mathbb{E}[R_i(b)] \geq \left\lfloor \frac{n}{2} \right\rfloor (1 - e^{-1}) - 1, \] where
\[
R_i(b) = \#\{ |b_0 - b_{(0+i)}|_n, \ldots, |b_s - b_{(s+i)}|_n, \ldots, |b_{n-1} - b_0|_n \}
\]
is the cardinality of different elements in row \( i \) of \( T_b \).

**Proof.** First, we define the variables
\[
\delta_j^{(i)}(b, d) := 1 - \mathbb{1}\{ |b_j - b_{(j+i)}|_n = d \} = \begin{cases} 0, & \text{if } |b_j - b_{(j+i)}|_n = d; \\ 1, & \text{otherwise}. \end{cases}
\]
This proves Claim 15.

By Claim 14 there is a subset $I$ of $\mathbb{Z}_n$ of cardinality $n - \gcd(n, i)$ such that the variables $\{\delta_w : w \in I\}$ are i.i.d. Thus

$$
\mathbb{E} \left[ r_d^{(i)} \right] = \mathbb{E} \left[ \prod_{j \in \mathbb{Z}_n} \delta_j^{(i)}(b, d) \right] \leq \mathbb{E} \left[ \prod_{w \in I} \delta_w^{(i)}(b, d) \right] = \mathbb{E} \left[ \delta_0^{(i)}(b, d) \right]^{n-\gcd(n, i)};
$$

furthermore, by Proposition 12 we get that $\mathbb{E} \left[ \delta_0^{(i)}(b, d) \right] = 1 - \frac{m_d}{n}$, thus

$$
\mathbb{E} \left[ r_d^{(i)} \right] \leq \left( 1 - \frac{m_d}{n} \right)^{n-\gcd(n, i)} \leq \left( 1 - \frac{m_d}{n} \right)^{\frac{n}{2}} = \left\{ \begin{array}{ll}
(1 - \frac{2}{n})^{\frac{n}{2}}, & \text{if } d \neq 0, \frac{n}{2};
(1 - \frac{1}{n})^{\frac{n}{2}}, & \text{otherwise},
\end{array} \right.
$$

for $d \in \{0, 1, \ldots, \frac{n}{2}\}$. Using the inequality $1 - x \leq e^{-x}$ valid for all $x$ real, we get that

$$
\mathbb{E} \left[ \sum_{d=0}^{\frac{n}{2}} r_d^{(i)} \right] \leq \frac{n}{2} \left( 1 - \frac{2}{n} \right)^{\frac{n}{2}} + 2 \left( 1 - \frac{1}{n} \right)^{\frac{n}{2}} \leq \frac{n}{2} e^{-1} + 2;
$$

plugging this inequality in Equation (3) yields

$$
\mathbb{E} [R_i(b)] = \left( \frac{n}{2} + 1 \right) - \mathbb{E} \left[ \sum_{d=0}^{\frac{n}{2}} r_d^{(i)} \right] \geq \frac{n}{2} (1 - e^{-1}) - 1.
$$

This proves Claim 15. □

We introduce McDiarmid’s inequality to prove Claim 17.

**Definition.** Let $L : \mathcal{X}^n \to \mathbb{R}$ be a function. We say that $L$ has Lipschitz coefficient $r \in \mathbb{R}^+$ if

$$
|L(b_1) - L(b_2)| \leq r
$$

for every $b_1, b_2 \in \mathcal{X}^n$ such that $b_1(j), b_2(j)$ differ in at most one entry.

**Remark.** The definition of Lipschitz coefficient is more general than the one presented here.

**Proposition 16 (McDiarmid’s Inequality)** [McDiarmid, 1989]. Let $\mathcal{X} := (X_1, X_2, \ldots, X_n) \in (\mathbb{Z}_n)^n$ be a random vector where the variables $X_1, X_2, \ldots, X_n$ are independent and let $L : (\mathbb{Z}_n)^n \to \mathbb{R}$ be a function with bounded Lipschitz coefficient $r$. Then

$$
(\text{lower tail}) \quad \mathbb{P} \left[ L(\mathcal{X}) \leq \mathbb{E} [L(\mathcal{X})] - r\sqrt{n} \lambda \right] \leq e^{-2\lambda},
$$

for all $\lambda \geq 0$. 
Remark. Here we present a particular case of McDiarmid’s inequality. The general inequality also bounds the upper tail but we do not use this bound.

In the following claim, we use Proposition 16 to estimate the probability that row \( i \) of \( T_b \) has less than \( \sim 0.63 \left( \frac{n}{2} \right) \) pairwise different elements.

**Claim 17.** Let \( b \in (\mathbb{Z}_n)^n \) be as in Theorem 6 and let \( \epsilon > 0 \) be arbitrary but fixed, then

\[
P \left[ R_i(b) < \left( \frac{n}{2} \right)(1 - e^{-1} - \epsilon) \right] \leq e^{-\Theta(n)},
\]

for all \( i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) where \( \Theta(n) \) is independent from \( i \).

**Proof.** Let \( b = (b_0, b_1, \ldots, b_{n-1}) \) and \( b' = (b'_0, b'_1, \ldots, b'_{n-1}) \) be two arbitrary realizations of \( b \). The function \( R_i(b) = \#E_i \) has Lipschitz coefficient two: changing one \( b_j \) will only affect at most two variables of the set

\[
E_i = \left\{ \left| b_0 - b_{(0+i)n} \right|, \ldots, \left| b_n - b_{(s+i)n} \right|, \ldots, \left| b_{n-1} - b_{(n-1+i)n} \right| \right\},
\]

namely \( \left| b_j - b_{(j+i)n} \right| \) and \( \left| b_{j+i} - b_j \right| \). This means that if \( b \) and \( b' \) differ in at most one entry, then

\[
|R_i(b) - R_i(b')| = |\#E_i - \#E'_i| \leq 2
\]

i.e. \( R_i \) has Lipschitz coefficient two. Using McDiarmid’s inequality, we get that

\[
P \left[ R_i < \mathbb{E} [ R_i ] - 2\sqrt{\lambda n} \right] \leq e^{-2\lambda}, \quad \forall \lambda \geq 0.
\]

Using the lower bound \( \mathbb{E} [ R_i ] \geq \left( \frac{n}{2} \right)(1 - e^{-1}) \) of Claim 15 we deduce that

\[
P \left[ R_i < \left( \frac{n}{2} \right)(1 - e^{-1}) - 1 \right] - 2\sqrt{\lambda n} \leq P \left[ R_i < \mathbb{E} [ R_i ] - 2\sqrt{\lambda n} \right] \leq e^{-2\lambda}, \quad \forall \lambda \geq 0.
\]

Let \( \epsilon > 0 \) arbitrary but fixed and let

\[
\lambda_\epsilon(n) := \frac{1}{4n} \left( e \left( \frac{n}{2} \right) - 1 \right)^2 = \Theta(n);
\]

we observe that \( \lambda_\epsilon(n) \) is independent of \( i \). Let \( n > \frac{2}{\epsilon} \), then plugging \( \lambda = \lambda_\epsilon(n) \) in the previous inequality yields

\[
P \left[ R_i < \left( \frac{n}{2} \right)(1 - e^{-1} - \epsilon) \right] \leq e^{-2\lambda_\epsilon(n)} = e^{-\Theta(n)}.
\]

We recall that that \( E_{\text{row}}^\alpha(b) \) is the event that every row in \( T_b \) has at least \( \alpha \cdot \left\lfloor \frac{n}{2} \right\rfloor \) different elements, then

\[
\left( E_{\text{row}}^\alpha(b) \right)^c = \bigcup_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left\{ R_i(b) < \alpha \cdot \left\lfloor \frac{n}{2} \right\rfloor \right\}.
\]

Let \( \epsilon > 0 \) be arbitrary but fixed and let \( \alpha^* := 1 - e^{-1} - \epsilon \), then

\[
P \left[ \left( E_{\text{row}}^{\alpha^*}(b) \right)^c \right] = P \left[ \bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left\{ R_i(b) < \alpha^* \cdot \left\lfloor \frac{n}{2} \right\rfloor \right\} \right] \leq \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} P \left[ R_i(b) < \alpha^* \cdot \left\lfloor \frac{n}{2} \right\rfloor \right] \leq n \cdot e^{-\Theta(n)} = O \left( \frac{1}{n} \right),
\]

where we use the union bound for the first inequality and Claim 17 for the second inequality. This concludes the proof of Lemma 9.
5 Proof of Lemma 10

The overview of the proof is as follows: first, we find a new random variable \( Z_0(b) - Z_1(b) \) such that

\[
\begin{align*}
\text{• } & \mathbb{D}(b) \geq Z_0(b) - Z_1(b), \quad \forall b : \mathbb{Z}_n^m \to \mathbb{Z}_n^m; \\
\text{• } & \mathbb{E}[Z_0 - Z_1] \sim 0.5 \left[ \frac{n}{2} \right].
\end{align*}
\]

Then, \( \mathbb{D} \) concentrates if \( Z_0 \) and \( Z_1 \) concentrate, this follows from

Claim 18. Let \( \epsilon > 0 \) and let \( \delta \leq \mathbb{E}[Z_0 - Z_1] (1 - \epsilon) \). If \( Z_0 - Z_1 \leq \mathbb{D} \), then

\[
\mathbb{P}[\mathbb{D} < \delta] \leq \mathbb{P}[Z_0 - Z_1 < \mathbb{E}[Z_0 - Z_1] (1 - \epsilon)] + \mathbb{P}[Z_1 > \mathbb{E}[Z_1] (1 - \epsilon)],
\]

Proof. Let \( \delta_1 \leq \delta_2 \). Using the assumption \( Z_0 - Z_1 \leq \mathbb{D} \), we get that

\[
\{ \mathbb{D} < \delta_1 \} \subset \{ Z_0 - Z_1 < \delta_2 \}.
\]

In particular, we have that

\[
\mathbb{P}[\mathbb{D} < \delta] \leq \mathbb{P}[Z_0 - Z_1 < \mathbb{E}[Z_0 - Z_1] (1 - \epsilon)],
\]

for any \( \delta \leq \mathbb{E}[Z_0 - Z_1] (1 - \epsilon) \). Observing that

\[
\{ Z_0 - Z_1 < \mathbb{E}[Z_0 - Z_1] (1 - \epsilon) \} \subset \{ Z_0 < \mathbb{E}[Z_0] (1 - \epsilon) \} \cup \{ Z_1 > \mathbb{E}[Z_1] (1 - \epsilon) \},
\]

it follows that

\[
\begin{align*}
\mathbb{P}[\mathbb{D} < \delta] & \leq \mathbb{P}[Z_0 - Z_1 < \mathbb{E}[Z_0 - Z_1] (1 - \epsilon)] \\
& \leq \mathbb{P}\left[ \{ Z_0 < \mathbb{E}[Z_0] (1 - \epsilon) \} \cup \{ Z_1 > \mathbb{E}[Z_1] (1 - \epsilon) \} \right] \\
& \leq \mathbb{P}[Z_0 < \mathbb{E}[Z_0] (1 - \epsilon)] + \mathbb{P}[Z_1 > \mathbb{E}[Z_1] (1 - \epsilon)],
\end{align*}
\]

where the last inequality follows from the union bound. \( \Box \)

To prove concentration of \( Z_0 \) and \( Z_1 \) around their respective means, we use Chebyshev’s inequality.

Remark 19. \( \mathbb{D} : \mathbb{Z}_n^m \to \mathbb{Z}_n^m \) does not have a bounded Lipschitz coefficient. We can not use McDiarmid’s inequality to guarantee its concentration.

5.1 Lower bound for \( \mathbb{D}(b) \)

Let \( b : \mathbb{Z}_n \to \mathbb{Z}_n \) be an realization of \( b \). Let us recall that \( \mathbb{D}(b) \) counts the number of rows of \( T_b \) that contain at least one zero. Let

\[
z_i = z_i(b) := \#(\text{Zeros in row } i)
\]

and

\[
Z_0(b) := \#(\text{Zeros in } T_b) = \sum_{(i,j) \in [1, \lfloor \frac{n}{2} \rfloor] \times [0, n-1]} \mathbbm{1}\{ T_b(i,j) = 0 \},
\]

then

\[
\mathbb{D}(b) = Z_0 - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \max(z_i - 1, 0);
\]

it is easy to verify that

\[
\sum_{0 \leq j < j' \leq n-1} \mathbbm{1}\{ T_b(i,j) = 0 \} \mathbbm{1}\{ T_b(i,j') = 0 \} = \frac{z_i(z_i - 1)}{2} \geq \max(z_i - 1, 0), \quad \forall i.
\]
Therefore
\[ Z_1(b) := \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{0 \leq j < j' \leq n-1} 1 \{ T_b(i,j) = 0 \} 1 \{ T_b(i,j') = 0 \} \geq \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \max(z_i - 1, 0). \]

From this and Equation 6, we conclude that

Claim 20. \( \mathbb{D}(b) \geq Z_0(b) - Z_1(b), \forall b : Z_n \to Z_n. \)

5.2 Estimations of \( \mathbb{E}[Z_0], \mathbb{E}[Z_1], \mathbb{V}[Z_0], \mathbb{V}[Z_1] \)

In this subsection we prove that \( \mathbb{E}[Z_0 - Z_1] \sim 0.5 \left\lfloor \frac{n}{2} \right\rfloor \) and \( \mathbb{E}[Z_0] = O(n) \) and \( \mathbb{E}[Z_1] = O(n) \) and \( \mathbb{V}[Z_0] = O(n) \) and \( \mathbb{V}[Z_1] = O(n) \). For the rest of this subsection, we use the following notation

\[ y_{i,j} := 1 \{ T_b(i,j) = 0 \}, \]

for \( 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( 0 \leq j \leq n - 1 \).

Definition. The variables \( y_{i_1,j_1}, y_{i_2,j_2}, \ldots, y_{i_k,j_k} \) are called acyclic if the multi-set \( \bigcup_{w=1}^{k} \{ T_b(i_w,j_w) \} \) is acyclic. Let

\[ G(\{y_{i_1,j_1}, y_{i_2,j_2}, \ldots, y_{i_k,j_k}\}) = G(\bigcup_{w=1}^{k} \{ T_b(i_w,j_w) \}) \]

be the associated multi-graph of the multi-set \( \{y_{i_1,j_1}, y_{i_2,j_2}, \ldots, y_{i_k,j_k}\} \) and let \( e(y_{i,j}) := e(T_b(i,j)) = \{j, (j + i) \} \) be the associated edge to \( y_{i,j} \); the length of \( e(y_{i,j}) \) is \( |j - (j + i)||n| = i \).

Remark 21. If the variables \( y_{i_1,j_1}, y_{i_2,j_2}, \ldots, y_{i_k,j_k} \) are acyclic then they are i.i.d; this is an immediate consequence of Proposition 12.

We begin with the easy part: the estimation of the expected values.

Claim 22. Let \( n \in \mathbb{N} \), then \( \mathbb{E}[Z_0] = \Theta(n) \), \( \mathbb{E}[Z_1] = \Theta(n) \) and \( \mathbb{E}[Z_0 - Z_1] \geq 0.49 \left\lfloor \frac{n}{2} \right\rfloor. \)

Proof. Using the linearity of the expectation, we get that

\[ \mathbb{E}[Z_0] = \sum_{(i,j) \in [1, \left\lfloor \frac{n}{2} \right\rfloor] \times [0, n-1]} \mathbb{E}[y_{i,j}] = \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{n} = \left\lfloor \frac{n}{2} \right\rfloor = \Theta(n), \]  

where for the second equality, we use that

\[ \mathbb{E}[y_{i,j}] = \mathbb{P}[T_b(i,j) = 0] = \mathbb{P}[\{ b_j = b_{j+i} \}] = \frac{1}{n}. \]

We do an upper estimation of \( \mathbb{E}[Z_1] \) depending on the parity of \( n \).

Case 1: \( n \) odd. Every product \( y_{i,j}y_{i,j'} \) in the sum

\[ Z_1 = \left\lfloor \frac{n}{2} \right\rfloor \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{0 \leq j < j' \leq n-1} y_{i,j}y_{i,j'} \]

is formed with independent random variables \( y_{i,j}, y_{i,j'} \) by Remarks 13,21. Thus

\[ \mathbb{E}[Z_1] = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{0 \leq j < j' \leq n-1} \mathbb{E}[y_{i,j}y_{i,j'}] = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{0 \leq j < j' \leq n-1} \mathbb{E}[y_{i,j}] \mathbb{E}[y_{i,j'}] \]

\[ \leq \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n}\right) = \Theta(n). \]
We deduce from the previous cases that

\[ E = \sum_{0 \leq i < j \leq n-1} y_{i,j} y_{i,j'} + \sum_{0 \leq r < r' \leq n-1 \mod n/2} y_{n/2, r} y_{n/2, r'} + \sum_{s=0}^{n/2-1} y_{n/2, s}. \]

Every product \( y_{i,j} y_{i,j'} \) in the first sum is formed with independent variables \( y_{i,j}, y_{i,j'} \) by Remark 13 and the same is valid for the products \( y_{n/2, r} y_{n/2, r'} \) in the second sum, therefore

\[
E[Z_1] = \sum_{i=1}^{\frac{n}{2}-1} \sum_{0 \leq j' \leq n-1} E[y_{i,j}] E[y_{i,j'}] + \sum_{0 \leq r < r' \leq n-1 \mod n/2} E[y_{n/2, r}] E[y_{n/2, r'}] + \sum_{s=0}^{n/2-1} E[y_{n/2, s}]
\]

\[
= \sum_{i=1}^{\frac{n}{2}-1} \sum_{0 \leq j' \leq n-1} \frac{1}{n^2} + \sum_{0 \leq r < r' \leq n-1 \mod n/2} \frac{1}{n^2} + \sum_{s=0}^{n/2-1} \frac{1}{n}
\]

\[
= \left( \frac{n}{2} - 1 \right) \cdot \left( \frac{n}{2} \right) \cdot \frac{1}{n^2} + \left( \frac{n}{2} \right) \cdot \frac{1}{n^2} + n \cdot \frac{1}{n}
\]

\[
= \frac{1}{2} \cdot \frac{n}{2} \cdot \left( 1 - \frac{1}{n} + \frac{2}{n - 2/n^2} \right) = \Theta(n).
\]

We deduce from the previous cases that \( E[Z_1] = \Theta(n) \) and \( E[Z_1] \leq \frac{M}{2} + 1 \) for all \( n \). Using this last inequality and Equation 7, we conclude that

\[
E[Z_0] - E[Z_1] = \left( \frac{n}{2} \right) \cdot \frac{1}{n} + \left( \frac{n}{2} \right) \cdot \frac{1}{n^2} = \Theta(n).
\]

This concludes the proof of Claim 22.

Now we estimate the variance of \( Z_0 \) and \( Z_1 \).

**Claim 23.** Let \( n \in \mathbb{N} \), then \( \mathbb{V}[Z_0] = O(n) \) and \( \mathbb{V}[Z_1] = O(n) \).

**Proof.** Here we also divide the calculations according to the parity of \( n \).

**Case 1:** \( n \) odd. We expand the variance of \( Z_0 \) to get that

\[
\mathbb{V}[Z_0] = \sum_{1 \leq i \leq \frac{n}{2}} \mathbb{V}[y_{i,j}] + \sum_{1 \leq i, i' \leq \frac{n}{2}} \mathbb{Cov}[y_{i,j}, y_{i,j'}],
\]

where the covariances are calculated among pairs of independent variables \( y_{i,j}, y_{i,j'} \) due to the Remark 13. Thus

\[
\mathbb{V}[Z_0] = \sum_{1 \leq i \leq \frac{n}{2}} \mathbb{V}[y_{i,j}].
\]

We notice that \( y_{i,j}^2 = y_{i,j} \) because \( y_{i,j} \in \{0, 1\} \), therefore

\[
\mathbb{V}[y_{i,j}] = E[y_{i,j}^2] - E[y_{i,j}]^2 = \frac{1}{n} - \frac{1}{n^2}, \quad \forall n \in \mathbb{N}
\]

where we use we Equation 8 in the last equality. Then, for all \( n \) odd, we get that

\[
\mathbb{V}[Z_0] = \left( \frac{n}{2} \right) n \left( \frac{1}{n} - \frac{1}{n^2} \right) = \left( \frac{n}{2} \right) \left( 1 - \frac{1}{n} \right) = O(n).
\]
Now we calculate
\[ \mathbb{V}[Z_1] = \sum_{1 \leq i \leq \left[ \frac{4}{9} \right]} \mathbb{V}[y_{i,j}y_{i,j'}] + \sum_{1 \leq i,r \leq \left[ \frac{4}{9} \right], 0 \leq j < j' \leq n-1} \text{Cov} \left[ y_{i,j}y_{i,j'}, y_{r,s}y_{r,s'} \right]; \tag{11} \]

We first note that
\[ \mathbb{V}[y_{i,j}y_{i,j'}] = \mathbb{E}\left[ y_{i,j}^2 y_{i,j'}^2 \right] - \mathbb{E}[y_{i,j}y_{i,j'}]^2 = \frac{1}{n^2} - \frac{1}{n^4}, \quad \text{for } n \text{ odd and } \forall i, j \neq j'; \tag{12} \]
this follows since the variables \( y_{i,j} \) and \( y_{i,j'} \) are different and therefore independent (see Remark 13). Thus
\[ \sum_{1 \leq i \leq \left[ \frac{4}{9} \right]} \mathbb{V}[y_{i,j}y_{i,j'}] = \left( \frac{n}{2} \right) \left( \frac{n}{2} \right) \frac{1}{n^2} \left( 1 - \frac{1}{n^2} \right) = O(n). \tag{13} \]

For the sum of the covariances, we proceed as follows: if the variables \( y_{i,j}, y_{i,j'}, y_{r,s}, y_{r,s'} \) are acyclic then they are independent (see Proposition 12), therefore
\[ \text{Cov} \left[ y_{i,j}y_{i,j'}, y_{r,s}y_{r,s'} \right] = 0. \]

Let \( \mathcal{Y} := \{ y_{i,j}, y_{i,j'}, y_{r,s}, y_{r,s'} : (i, j, j') \neq (r, s, s'), j < j', s < s' \} \) and let
\[ Y = \{ y_{i,j}, y_{i,j'}, y_{r,s}, y_{r,s'} \} \in \mathcal{Y}, \]
then \( G(Y) \) is a multi-graph with four edges \( e(y_{i,j}), e(y_{i,j'}), e(y_{r,s}), e(y_{r,s'}) \) such that \( e(y_{i,j}) \neq e(y_{i,j'}) \) and \( e(y_{r,s}) \neq e(y_{r,s'}) \) (see remark 13). If \( G(Y) \) has at least one cycle, it is isomorphic to one of the multi-graphs in Figure 1. We make the following estimation.

![Figure 1: Possible non-acyclic multi-graphs for n odd.](image)

**Claim 24.** Let \( n \in \mathbb{N} \), then
\[ \# \{ Y \in \mathcal{Y} : G(Y) \cong G_c \} = \begin{cases} O(n^4), & \text{if } c = 1, 2, 3, 4, 6, 7; \\ O(n^3), & \text{if } c = 5. \end{cases} \]

**Proof.** The first four cases are easily bounded by \( O(n^4) \) and the fifth case by \( O(n^3) \); the last two cases require better estimations than \( O(n^3) \) respectively \( O(n^6) \). First, notice that for all cases, the four edges of the multi-graph \( G(\{ y_{i,j}, y_{i,j'}, y_{r,s}, y_{r,s'} \}) \) are divided into two pairs: \( e(y_{i,j}), e(y_{i,j'}) \) of length \( i \) and \( e(y_{r,s}), e(y_{r,s'}) \) of length \( r \). The case \( G_6 \) is bounded by \( \left( \frac{n}{3} \right) \ast 2n = O(n^4) \) because three vertices can be chosen freely to form a triangle whose edges have at most two different lengths \( i, r \), then we choose a vertex \( v \) for the free edge and finally we choose \( v \) such that \( |v - v'|_n = i \) or \( |v - v'|_n = r \) depending on the lengths of the edges in the triangle, therefore \( v \) has only two choices. The last case \( G_7 \) is bounded by \( O(n^4) \), we distinguish two subcases: if the multi-edge is formed by the associated edges of the same pair w.r.t. then the free edges are formed by the edges \( e(y_{r,s}), e(y_{r,s'}) \) which have length \( r \); we choose two vertices for the multi-edge and two more vertices \( v_1, v_2 \) (one for each of the free edges), then the two missing vertices \( v_1', v_2' \) have at most two options each because \( |v - v_1|_n = |v - v_2'|_n = r \), thus this subcase is bounded by \( O(n^3) \). The second subcase is when \( e(i,j) \neq e(y_{r,s}) \) and \( e(y_{r,s}) \neq e(y_{r,s'}) \), then w.r.t. the multi-edge if formed with \( e(y_{i,j}) = e(y_{r,s}) \) then
We have that
\[ \mathbb{E} \left[ y_{i,j} y_{i,j'} y_{r,s} y_{r,s'} \right] = \mathbb{P} \left[ y_{i,j} y_{i,j'} y_{r,s} y_{r,s'} = 1 \right] = \mathbb{P} \left[ b_{j,(j+i)_n} = b_{j',(j'+i)_n} = b_{s,(s+r)_n} = b_{s',(s'+r)_n} \right], \]
thus for \( Y = \{ y_{i,j}, y_{i,j'}, y_{r,s}, y_{r,s'} \} \in \mathcal{Y} \), we have that
\[ \mathbb{E} \left[ y_{i,j} y_{i,j'} y_{r,s} y_{r,s'} \right] = \begin{cases} \frac{1}{n^4}, & \text{if } G(Y) \cong G_{1,2,3,4,6,7}; \\ \\ \\ \frac{1}{n^3}, & \text{if } G(Y) \cong G_5. \end{cases} \tag{14} \]

This last equation combined with Claim 24, implies that
\[ \sum_{1 \leq i, r \leq \frac{n}{2}} Cov \left[ y_{i,j} y_{i,j'}, y_{r,s} y_{r,s'} \right] \leq \sum_{1 \leq i, r \leq \frac{n}{2}} \mathbb{E} \left[ y_{i,j} y_{i,j'} y_{r,s} y_{r,s'} \right] \]
\[ \leq 6 \cdot O(n^4) \frac{1}{n^3} + O(n^3) \frac{1}{n^2} = O(n). \]

Using the previous inequality and Equation 13 we get that
\[ \mathbb{V} \left[ Z_1 \right] = \sum_{1 \leq i \leq \frac{n}{2}} \mathbb{V} \left[ y_{i,j} y_{i,j'} \right] + \sum_{1 \leq i, r \leq \frac{n}{2}} Cov \left[ y_{i,j} y_{i,j'}, y_{r,s} y_{r,s'} \right] = O(n) + O(n) = O(n) \tag{15} \]
for all \( n \) odd.

Case 2: \( n \) even. We estimate the variances of \( Z_0 \) and \( Z_1 \). For \( n \) even, we can write \( Z_0 \) as
\[ Z_0 = \sum_{1 \leq i \leq \frac{n}{2}} y_{i,j} + 2 \sum_{j=0}^{\frac{n}{2}-1} y_{\frac{n}{2},j}, \]
where all variables involved in the sums are independent (see Remark 13). Thus
\[ \mathbb{V} \left[ Z_0 \right] = \sum_{1 \leq i \leq \frac{n}{2}} \mathbb{V} \left[ y_{i,j} \right] + 4 \sum_{j=0}^{\frac{n}{2}-1} \mathbb{V} \left[ y_{\frac{n}{2},j} \right]. \]
Using Equation 8, we deduce that
\[ \mathbb{V} \left[ Z_0 \right] = \left( \frac{n}{2} - 1 \right) n \left( \frac{1}{n} - \frac{1}{n^2} \right) + 4 \frac{n}{2} \left( \frac{1}{n} - \frac{1}{n^2} \right) = O(n), \tag{16} \]
for all \( n \) even. By Remark 13, we can write \( Z_1 \) as
\[ Z_1 = \sum_{1 \leq i \leq \frac{n}{2}} y_{i,j} y_{i,j'} + \sum_{s=0}^{\frac{n}{2}-1} y_{n/2,s}, \]
where
\[ i = r, \] thus all edges have the same length; we choose two vertices \( v, v' \) for the multi-edge and two more vertices \( v_1, v_2 \) (one for each of the free edges), the missing vertices \( v'_1, v'_2 \) have at most two choices each because \( |v_1 - v'_1|_n = |v_2 - v'_2|_n = |v - v'|_n \), this gives again a \( O(n^4) \) bound. Adding the bounds of the first and second subcases yield a \( O(n^4) \) bound for the last graph. \( \square \)
Therefore

\[
\mathbb{V}[Z_1] = \sum_{1 \leq i, j \leq \frac{n}{2}} \mathbb{V}[y_{i,j}y_{i,j'}] + \sum_{s=1}^{\frac{n}{2}-1} \mathbb{V}[y_{n/2,s}] + \sum_{1 \leq i, j \leq \frac{n}{2}} \sum_{0 \leq j', s, s' \leq n-1 \atop j \neq j', s \neq s', (i, j') \neq (r, s, s')} \text{Cov}[y_{i,j}y_{i,j'}, y_{r,s}y_{r,s'}]
\]

\[+ 2 \sum_{1 \leq u \leq \frac{n}{2}} \sum_{0 \leq v < v' < n-1 \atop u \neq \frac{n}{2}} \text{Cov}[y_{u,v}y_{u,v'}, y_{\frac{n}{2}, w}] + \sum_{0 \leq u, w' \leq \frac{n}{2} - 1 \atop w \neq w'} \text{Cov}[y_{\frac{n}{2}, w}, y_{\frac{n}{2}, w'}].
\]

We divide the work in three parts: the first two sums, the third sum and the fourth sum. Using Remark 13, we write the first two sums as

\[\sum_{1 \leq i, j \leq \frac{n}{2}} \mathbb{V}[y_{i,j}] \sum_{s=1}^{\frac{n}{2}-1} \mathbb{V}[y_{n/2,s}] \overset{\text{Eq. 8}}{=} \frac{n}{2} \left( \frac{1}{n} - \frac{1}{n^2} \right)^2 + n \left( \frac{1}{n} - \frac{1}{n^2} \right) = O(n). \tag{18}\]

The third sum can be upper bounded in the same way as in the odd case: the associated graphs of variables \(y_{i,j}, y_{i,j'}, y_{r,s}, y_{r,s'}\) with non-zero covariance in the third sum, are isomorphic to one of the graphs in Figure 1, thus we can use Claim 24 and Equation 14 to get that

\[\sum_{1 \leq i, j \leq \frac{n}{2}} \sum_{0 \leq j', s, s' \leq n-1 \atop j \neq j', s \neq s', (i, j') \neq (r, s, s')} \text{Cov}[y_{i,j}y_{i,j'}, y_{r,s}y_{r,s'}] = O(n). \tag{19}\]

In the fourth sum, the variables with covariance non-zero have an associated multi-graph isomorphic to one of the following multi-graphs

\[
\begin{align*}
G_8 & \quad G_9 & \quad G_{10}
\end{align*}
\]

Let \(\mathcal{X} := \{y_{u,v}y_{u,v'}, y_{\frac{n}{2}, w} : 1 \leq u \leq \frac{n}{2}; 0 \leq v < v' \leq n - 1; v \neq \frac{n}{2}; 0 \leq w \leq \frac{n}{2} - 1\}\). As in Claim 24, we can prove that

\[\# \{X \in \mathcal{X} : G(X) \cong G_c\} = O(n^3), \quad c = 8, 9, 10.\]

As in Equation 14, we can prove that \(E[y_{u,v}y_{u,v'}, y_{\frac{n}{2}, w}] = \frac{1}{n^2}\) for all \(X = \{y_{u,v}, y_{u,v'}, y_{\frac{n}{2}, w}\} \in \mathcal{X}\). Thus

\[\sum_{1 \leq u \leq \frac{n}{2}} \sum_{0 \leq v < v' < n-1} \sum_{0 \leq w \leq \frac{n}{2} - 1} \text{Cov}[y_{u,v}y_{u,v'}, y_{\frac{n}{2}, w}] \leq 3 \cdot O(n^3) \frac{1}{n^2} = O(n). \tag{20}\]

Plugging Equations 18, 19, 20 in Equation 17 yields

\[\mathbb{V}[Z_1] = O(n) + O(n) + 2 \cdot O(n) = O(n), \tag{21}\]

for all \(n\) even. Equations 10, 15, 16, 21 yield Claim 23.
5.3 $\mathcal{E}_{\text{zero}}^{0.49(1-\epsilon)}$ holds whp

Using Chebyshev’s inequality, we obtain that

$$
P[|Z_0 - E[Z_0]| \geq \lambda_0] \leq \frac{V[Z_0]}{\lambda_0^2}; \quad P[|Z_1 - E[Z_0]| \geq \lambda_1] \leq \frac{V[Z_1]}{\lambda_1^2},
$$

for every $\lambda_0, \lambda_1 > 0$. In particular, this implies that

$$
P[Z_0 < E[Z_0] - \lambda_0] \leq \frac{V[Z_0]}{\lambda_0^2}; \quad P[Z_1 > E[Z_1] - \lambda_1] \leq \frac{V[Z_1]}{\lambda_1^2}.
$$

Choosing $\lambda_0 = \epsilon \cdot E[Z_1]$ and $\lambda_2 = \epsilon \cdot E[Z_1]$ and using Claims 22,23 we get that

$$
P[Z_0 < E[Z_0] (1 - \epsilon)] \leq \frac{V[Z_0]}{\epsilon^2 \cdot E[Z_0]^2} = \frac{O(n)}{\epsilon^2 \cdot \Theta(n^2)} = O \left( \frac{1}{n} \right);
$$

$$
P[Z_1 > E[Z_1] (1 - \epsilon)] \leq \frac{V[Z_1]}{\epsilon^2 \cdot E[Z_1]^2} = \frac{O(n)}{\epsilon^2 \cdot \Theta(n^2)} = O \left( \frac{1}{n} \right),
$$

for any fixed $\epsilon > 0$. Let $\epsilon \in (0,1)$ be arbitrary but fixed, using Claim 18 with the relation

$$
\delta = 0.49 \left[ \frac{n}{2} \right] (1 - \epsilon) \leq E[Z_0 - Z_1] (1 - \epsilon),
$$

valid for $n$ large enough, we conclude that

$$
P \left( \left( \mathcal{E}_{\text{zero}}^{0.49(1-\epsilon)} \right)^C \right) = P \left( D < 0.49 \left[ \frac{n}{2} \right] (1 - \epsilon) \right) \leq P \left( Z_0 < E[Z_0] (1 - \epsilon) \right) + P \left( Z_1 > E[Z_1] (1 - \epsilon) \right) = O \left( \frac{1}{n} \right).
$$

This concludes the proof of Lemma 10.

6 Connections with chromatic polynomials of circulant graphs

As we already saw in the proof of Claim 14, the associated multi-graph to the variables in the row $i \neq \frac{n}{2}$ of $T_b$, is the circulant graph $C_n(i)$ and the same is valid for the variables in row $n/2$ if we consider the associated graph and not the associated multi-graph. Furthermore, we can express the probability of synchronization of circulant automata in terms of chromatic polynomials of circulant graphs: this is a consequence of the close connection of the moments of $D(b)$ with chromatic polynomials of circulant graphs. We formalize this in the following results.

**Definition.** The circulant graph $C_n(i_1, i_2, \ldots, i_k)$ is a graph with vertex set $\mathbb{Z}_n$ where two vertices $r, s$ are adjacent if $|r-s|_n = i_1, i_2, \ldots, i_k$.

**Definition.** Let $G$ be graph with vertex set $\{0, 1, \ldots, n-1\}$. The chromatic polynomial $P(G; x): \mathbb{N} \to \mathbb{N}$ of $G$ is defined by

$$
P(G; x) := \# \{b \in \{0, \ldots, x - 1\}^n : b \text{ is a proper coloring of } G\}.
$$

**Remark 25.** Let $G$ be of order $n$, then $P(G; x) = \sum_{j=1}^{\lambda_j} x^j$, where $\lambda_j \in \mathbb{Z}$ (e.g. [Fengming et al., 2005]).

**Claim 26.** Let $D$ and $b = (b_0, b_2, \ldots, b_{n-1})$ be as in Lemma 10, then

$$
\mathbb{E} \left[ D(b) \right] = \left[ \frac{n}{2} \right] - \sum_{i=1}^{\frac{n}{2}} \frac{P_i(n)}{n^n};
$$

$$
\mathbb{V} \left[ D(b) \right] = \sum_{i=1}^{\frac{n}{2}} \left( \frac{P_i(n)}{n^n} - \frac{P^2_i(n)}{n^{2n}} \right) + 2 \sum_{1 \leq i < j \leq \left[ \frac{n}{2} \right]} \left( \frac{P_{i,j}(n)}{n^n} - \frac{P_i(n)P_j(n)}{n^{2n}} \right),
$$

where $P_i$ is the chromatic polynomial of the circulant graph $C_n(i)$ and $P_{i,j}$ is the chromatic polynomial of the circulant graph $C_n(i, j)$.
Remark 27. • It is easy to derive that $P_i(x) = \left((x-1)^{l_i} + (-1)^{l_i}(x-1)^{\frac{n}{\gcd(n,i)}}\right)^{\frac{n}{\gcd(n,i)}}$, where $C_n(i)$ is a collection of $\gcd(n,i)$ disjoint cycles of length $\frac{n}{\gcd(n,i)}$ [Boesch and Tindell, 1984]. With this explicit expression, it is easy to estimate $\mathbb{E}(|\mathcal{D}|) \sim 0.63\left\lfloor \frac{3}{2} \right\rfloor$.

• We could not find an explicit expression for $P_{r,j}$. The calculation of the chromatic number of circulant graphs with an arbitrary number of parameters is an NP-Hard problem [Codenotti et al., 1998]. This implies that the calculation of chromatic polynomials of circulant graphs is also NP-Hard since $\chi(G) = \arg\min_{w \in \mathbb{N}} P(G; w) > 0$. We believe that our unfruitful attempts to estimate $\mathbb{V}(|\mathcal{D}|)$ are related to this. The variables $\mathcal{Z}_0$ and $\mathcal{Z}_1$ in Section 5 were defined to overcome these issues.

**Proof of Claim 26.** Let us recall that $\mathcal{D} = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \mathcal{D}_i$, where

$$\mathcal{D}_i(b) := \begin{cases} 1, & \text{if exist } u, v \in \mathbb{Z}_n \text{ such that } |u - v|_n = i \text{ and } |b_u - b_v|_n = 0; \\ 0, & \text{otherwise}, \end{cases}$$

then $\mathbb{D}_i = 1 - x_i$ where

$$x_i(b) := \prod_{j=0}^{n-1} 1\{|b_j - b_{(j+i)_n}|_n = 0\}.$$ We observe that $x_i = 1$ if every two numbers $r, s \in \mathbb{Z}_n$ at cyclic distance $i$ have different images under $b$ and $x_i = 0$ otherwise. Thus

$$\mathbb{E}[x_i(b)] = \mathbb{P}[x_i(b) = 1] = \frac{P_i(n)}{n^n},$$

where for the last equality we consider $b$ as a random coloration of $C_n(i)$, then $x_i(b) = 1$ iff $C_n(i)$ is properly colored by $b$. In a similar way

$$\mathbb{E}[x_i(b) \cdot x_j(b)] = \mathbb{P}[x_i(b) \cdot x_j(b) = 1] = \frac{P_{i,j}(n)}{n^n}.$$ Therefore

$$\mathbb{E}[\mathcal{D}] = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \mathbb{E}[\mathcal{D}_i] = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (1 - \mathbb{E}[x_i]) = \left\lfloor \frac{n}{2} \right\rfloor - \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{P_i(n)}{n^n};$$

and

$$\mathbb{V}[\mathcal{D}] = \mathbb{E}[\mathcal{D}^2] - \mathbb{E}[\mathcal{D}]^2 = \left(1 - \frac{P_i(n)}{n^n}\right) - \left(1 - \frac{P_i(n)}{n^n}\right)^2 = \frac{P_i(n)}{n^n} - \frac{P_i^2(n)}{n^{2n}};$$

$$\text{Cov}[\mathcal{D}_i, \mathcal{D}_j] = \mathbb{E}[\mathcal{D}_i \mathcal{D}_j] - \mathbb{E}[\mathcal{D}_i] \mathbb{E}[\mathcal{D}_j] = \mathbb{E}[(1 - x_i)(1 - x_j)] - \mathbb{E}[1 - x_i] \mathbb{E}[1 - x_j] = \mathbb{E}[x_i x_j] - \mathbb{E}[x_i] \mathbb{E}[x_j] = \frac{P_{i,j}(n)}{n^n} - \frac{P_i(n) P_j(n)}{n^{2n}}.$$ Plugging the two previous equations in

$$\mathbb{V}[\mathcal{D}] = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \mathbb{V}[\mathcal{D}_i] + 2 \sum_{1 \leq i < j \leq \left\lfloor \frac{n}{2} \right\rfloor} \text{Cov}[\mathcal{D}_i, \mathcal{D}_j]$$

yields the Claim 26. □

We get the following relation between chromatic polynomials of circulant graphs and synchronization of circular automata.
Theorem 28. Let \( A_n(b) := (\mathbb{Z}_n, \{a, b\}) \) be a random circular automaton with \( n \) states where \( \mathbb{Z}_n := \{0, 1, \ldots, n - 1\} \) and \( a \) is the circular permutation \( a(i) = (i + 1) \mod n \) and \( b := (b_0, \ldots, b_{n-1}) \) is a random vector, where \( b_0, \ldots, b_{n-1} \) are iid random variables uniformly distributed over \( \mathbb{Z}_n \). Let \( \epsilon \in (0, 0.13) \) and let \( n > \frac{2}{\epsilon} \), then

\[
\mathbb{P} [A_n(b) \text{ synchronizes}] \geq 1 - \left[ \frac{n}{2} \right] \exp \left\{ -\frac{1}{2n} \left( \epsilon \left( \frac{n}{2} \right) - 1 \right)^2 \right\} - \frac{\mathbb{V}[D(b)]}{(\frac{2}{\epsilon} - 1)^2},
\]

where \( \mathbb{V}[D(b)] \) is given by Claim 26.

Conjecture 29. \( \mathbb{V}[D(b)] = O(n) \).

Remark 30. The previous conjecture can be easily reduced to prove that \( P_{ij} = \frac{P_i P_j}{P_i P_j + \epsilon} \) for all \( i, j \) where the \( o(n^2) \) term should not depend on \( i, j \). In particular, a positive answer to this chromatic-polynomial question would give an alternative proof of Theorem 6.

Proof of Theorem 28. By Equations 4,5 we know that

\[
\mathbb{P} \left[ \left( \epsilon^{\alpha^*}(b) \right)^c \right] \leq \left[ \frac{n}{2} \right] \exp \left\{ -\frac{1}{2n} \left( \frac{3}{n} \right)^2 \right\},
\]

for all \( \epsilon > 0 \) and \( n > \frac{2}{\epsilon} \) where \( \alpha^* = 1 - e^{-1} - \epsilon \). Using the expression for \( P_i \) in Remark 27 , the well know inequality \( 1 - x \leq e^{-x} \) valid for \( x \in \mathbb{R} \) and elementary manipulations, it is easy to get that

\[
\mathbb{E}[D] \geq \eta_* := \left[ \frac{n}{2} \right] \left( 1 - \exp \left\{ \frac{n}{3(n-1)^2} \right\} - 1 \right).
\]

By Chebyshev’s inequality and elementary manipulations, we get that

\[
\mathbb{P} [D < \eta_* - \lambda] \leq \frac{\mathbb{V}[D(b)]}{\lambda^2},
\]

for all \( \lambda > 0 \). Replacing \( \lambda = \lambda'(n) := \left[ \frac{n}{2} \right] (1 - e^{-1} - \epsilon) > 0 \) for \( n > \frac{2}{\epsilon} \) with \( \epsilon > 0 \), we get that

\[
\mathbb{P} \left[ \left( \epsilon^{\tilde{\beta}}(b) \right)^c \right] = \mathbb{P} \left[ D < \left[ \frac{n}{2} \right] (1 - e^{-1} - \epsilon) \right] \leq \frac{\mathbb{V}[D(b)]}{(\lambda'(n))^2} \leq \frac{\mathbb{V}[D(b)]}{\left( \frac{n}{2} \right)^2 \epsilon - 1)^2},
\]

where \( \tilde{\beta} := \alpha^* = 1 - e^{-1} - \epsilon \). Using the previous inequalities, we conclude that

\[
\mathbb{P} [A_n(b) \text{ synchronizes}] \geq 1 - \mathbb{P} \left[ \left( \epsilon^{\alpha^*}(b) \right)^c \right] - \mathbb{P} \left[ \left( \epsilon^{\tilde{\beta}}(b) \right)^c \right] \geq 1 - \left[ \frac{n}{2} \right] \exp \left\{ -\frac{1}{2n} \left( \epsilon \left( \frac{n}{2} \right) - 1 \right)^2 \right\} - \frac{\mathbb{V}[D(b)]}{(\frac{2}{\epsilon} - 1)^2},
\]

for all \( n > \frac{2}{\epsilon} \) where the relations \( \alpha^*, \tilde{\beta} > 0 \) and \( \alpha^* + \tilde{\beta} > 1 \) are valid when \( \epsilon \in (0, 0.13) \). \( \square \)

7 Future work

Let \( A_n(a, b) \) be an automaton where \( a : \mathbb{Z}_n \to \mathbb{Z}_n \) is fixed and \( b : \mathbb{Z}_n \to \mathbb{Z}_n \) is a uniformly randomly chosen automorphism of \( \mathbb{Z}_n \). These are natural lines or work to extend and improve the results in this article:

- We want to explore in more detail the strengths and limitations in the ideas presented in this article. For example, we think that these ideas can extend Theorem 6 when \( a : \mathbb{Z}_n \to \mathbb{Z}_n \) is a finite number of pairwise disjoint cycles of almost-equal length.
- Theorem 4 has a decay rate in \( \Theta \left( \frac{\sqrt{n}}{n} \right) \). We believe that this can be extended:

Conjecture 31. Let \( A_n(b) \) be a uniformly distributed random circular automaton of order \( n \in \mathbb{N} \). Then

\[
\mathbb{P} [A_n(b) \text{ synchronizes}] = 1 - O(e^{-en}),
\]

for some \( e > 0 \).

The previous conjecture is true if the lower tail of \( D(b) \) decays exponentially: this reduction follows from Equation 22, the first part of Equation 23 and Equation 24.
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