Tubular topological bright-dark and dark laser solitons

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Abstract. We investigate, analytically and numerically, tubular topological laser solitons, which serve as intermediate link between 2D- and 3D- dissipative optical solitons in media with nonlinear amplification and absorption of light or in large-size laser with such a medium. Features of both 3D-dark solitons (with nonvanishing intensity at periphery for all spatial dimensions) and bright-dark solitons (vanishing in two of three dimensions) are studied, including their stability.

1. Introduction
Conservative dark optical solitons, the intensity of which does not vanish at the periphery, but tends to some nonzero value, have been considered in a large number of publications, see monograph [1]. Although such solitons’ power is infinitely high in an ideal model with an unlimited aperture, they are of interest for schemes with a large aperture and / or significant time duration of pulses due to a wider region of stability in the parameter space compared to more familiar bright solitons whose intensity tends to zero at the periphery. The purpose of this paper is to analyze such three-dimensional dissipative solitons in a laser medium of sufficiently large size, which can be placed inside a ring resonator.

In three-dimensional space, it makes sense to distinguish between properly dark solitons with non-vanishing intensity at the periphery for all three spatial dimensions, and solitons, in which the intensity at the periphery disappears only in some directions. We will call the latter light-dark solitons. The considered 3D-solitons are inevitably topological, that is, they contain one or several vortex lines, in the vicinity of which the radiation energy flux (Poynting vector) has a vortex character [2]. These solitons are generated by 2D-topological laser solitons, which were studied in sufficient detail earlier [3]. Various types of light topological 3D laser solitons are presented in [5-7] and in a number of other papers reviewed in [4].

2. The model
The radiation is assumed to be close to a plane monochromatic wave, which makes it possible to use the quasi-optical approximation, or the slowly varying envelope approximation, to describe its propagation. Accordingly, the electric field strength \( \mathbf{E}(\mathbf{r},t) \) is represented in the form
\[
\mathbf{E}(\mathbf{r},t) = \text{Re}[eE(\mathbf{r},t)\exp(i k_0 z - i \omega_0 t)],
\]
where \( \mathbf{r} \) is the radius vector, \( t \) is the time, \( e \) is the unit vector of polarization, \( E \) is the envelope, which changes slowly compared to the exponent appearing in this expression, in which \( \omega_0 \) is the carrier frequency, \( k_0 \) is the real part of the wave number at this frequency and \( z \) is the coordinate along the direction of the predominant propagation of radiation. Then, for radiation with a fixed (linear) polarization, Maxwell’s electrodynamic equations yield the following form
of the governing equation - the generalized complex Ginzburg-Landau equation in the dimensionless form [4]:

\[
\frac{\partial E}{\partial z} = \left[ (i + d_\perp) \nabla_\perp^2 + (i + d_\parallel) \frac{\partial^2}{\partial \tau^2} \right] E + f_m E. 
\]  

(1)

Here \( \nabla_\perp^2 = \delta^2 / \delta x^2 + \delta^2 / \delta y^2 \) is the transverse Laplacian, \( \mathbf{r}_\perp = (x, y) \) are the transverse coordinates, \( \tau = t - z / v_{gr} \) is the time in the accompanying coordinate system moving along the axis with the group velocity \( v_{gr} \), the "diffusion coefficients" \( d_\parallel \) and \( d_\perp \) are small, \( 0 < d_\perp / d_\parallel << 1 \). In this case, the linear operator \( d_\perp \nabla_\perp \) describes the angular selectivity of losses in the medium, and \( d_\parallel \delta^2 / \delta \tau^2 \) – the frequency dispersion of losses in the matrix. The last term on the right-hand side of (1) represents the centers’ response to radiation; this also includes the nonresonant absorption of the matrix. Here we restrict ourselves to a simpler case of the numerical equality of the "diffusion coefficients": \( d_\parallel = d_\perp = d \). Then the form of (1) is simplified:

\[
\frac{\partial E}{\partial z} = (i + d) \nabla_\perp^2 E + f_m E, \quad \nabla_\parallel^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial \tau^2}. 
\]  

(2)

The medium is considered to be a matrix with a non-resonant refractive index and absorption and with resonant amplification and absorption centers embedded in it. We suppose that the optical response of the medium is inertialess and depends only on the radiation intensity \( I = |E|^2 \), that is \( f_m = f_m(I) \) (class A lasers). For an effectively two-level scheme of centers with amplification and absorption in this approximation, one can obtain

\[
f_m(I) = -1 - \frac{a_0}{1 + I} + \frac{g_0}{1 + I / b}. 
\]  

(3)

Here, the term \(-1\) corresponds to the nonresonant absorption of the matrix (with the chosen normalization of the evolutionary variable \( z \)). When the frequencies of transitions at the centers and the radiation frequency are close, all quantities in (3) are real and have the meaning of the coefficients of unsaturated gain \( g_0 \) and absorption \( a_0 \), \( b \) is the ratio of the intensities of saturation of gain and absorption, and the intensity is normalized to the intensity of saturation of absorption. In basic calculations, we fix the values of the parameters \( a_0 = 2 \), \( g_0 = 2.114 \), \( b = 10 \) and \( d = 0.06 \).

The description of a ring cavity with a length \( L \) differs from the above in that the consideration is carried out in a laboratory coordinate system (without the transition to a moving coordinate system), time \( t \) serves as the evolutionary variable, and the periodicity condition is set

\[
\mathbf{E}(\mathbf{r}_\perp, z + L, t) = \mathbf{E}(\mathbf{r}_\perp, z, t). 
\]  

(4)

3. Bright-dark solitons

Let us first consider the symmetric distribution of the field with a rectilinear vortex line

\[
E(\mathbf{r}_\perp, \tau, z) = A_m(r) \exp(-iKz - i\Omega \tau + im\varphi). 
\]  

(5)

Here, a cylindrical coordinate system \((r, \varphi, z)\) with \( r = (x^2 + y^2)^{1/2} \) is adopted and \( m \) is the integer topological charge. Substitution of (5) to (2) leads to a nonlinear ordinary differential radial equation, and for bright in the transverse directions \( \mathbf{r}_\perp \) solitons we set \( A_m(r) \to 0 \) for \( r \to \infty \). For \( \Omega = 0 \) this equation coincides with the equation for 2D bright vortex solitons [4]. Thus, in this case, we know the eigenvalue \( K = K_0 \). Then, within the framework of perturbation theory, we obtain

\[
K(\Omega) = K_0 + \Omega^2. 
\]  

(6)
For an infinitely extended over $z$ medium, this implies a continuous spectrum of solutions of the form (5), that is, a family of 3D field distributions generated by a 2D single vortex soliton. For a resonator with condition (4), this spectrum becomes discrete. Numerical calculations show that solutions of the form (5) exist in a limited range of $\Omega$, $|\Omega|<1$. To carry out a linear stability analysis of solutions (5), we introduce a small perturbation

$$E(r_1, r, z) = [A_m(r) + \delta A(r, \tau, \varphi, z)] \exp(-iKz - i\Omega \tau + im\varphi).$$

We write the perturbation in the form

$$\delta A(r, \tau, \varphi, z) = a(r) \exp(i\omega \tau + i\delta m\varphi + \kappa z) + b^*(r) \exp(-i\omega \tau - i\delta m\varphi + \kappa^* z).$$

The sign of the real part of the eigenvalues $\kappa$ of the linearized problem indicates the stability ($\max \Re \kappa \leq 0$) or instability ($\max \Re \kappa > 0$) of solutions (5). An example of an unperturbed solution is shown in figure 1a, and in figure 1b,c examples of eigenvalue spectra are presented.

![Figure 1](image1.png)

**Figure 1.** (a): Isointensity surfaces at level $I = 0.6$ (blue) and 6 (yellow) for bright-dark vortex soliton with topological charge $m = 1$. The right plane shows lines $\Re E = 0$ in the section $(x, y)$; $L = 42$. (b), (c): maximum real part of eigenvalue $\kappa$ versus perturbation frequency $\omega$ for $\delta m = 0$ (b) and 1 (c). $\Omega = 0$.

The linear analysis shows that in a medium with an unlimited length, structures with form (5) are unstable with respect to perturbations modulated over $z$ with frequency $\Im \kappa$ and over $\tau$ with frequency $\omega$ (the values are taken for the maximum value of $\Re \kappa$). Such perturbations are not realized for a cavity with a shorter length, and then stable symmetric light-dark solitons form (Fig. 1a).

The direct solution of Eq. (2) indicates that at larger cavity lengths, the structure loses its symmetry and the vortex line of a soliton with a topological charge $m = 1$ bends (Fig. 2a). The structure generated by a 2D soliton with a larger charge $m$ behaves in approximately the same way, but for it, along with the curvature of the vortex line, it splits into $m$ ones with a unit charge.

![Figure 2](image2.png)

**Figure 2.** Surfaces of isointensity at level $I = 0.5$ for structures generated by 2D single vortex soliton with topological charge $m = 1$ (a) and strongly coupled pair of such solitons (b). $d = 0.047, L = 70$ (a), $d = 0.06, L = 100$ (b). $g_0 = 2.117, \Omega = 0$. 
If the generating 2D soliton is more complex, then the 3D structure of the field becomes more complicated. Figure 2b shows the structure of the field generated by a pair of strongly coupled vortex 2D solitons. One can see that in the second case, the vortex lines form a "double helix".

4. Dark solitons

The analysis of dark solitons is largely analogous to that carried out above for light-dark solitons. The difference lies in the conditions at the periphery. Now the intensity does not vanish there, and in numerical calculations it is necessary to set the condition of transparent boundaries.

According to simulations, field structures of the form (5) with a rectilinear vortex line turn out to be unstable. In fig. 3a, b we show the corresponding intensity distributions of a dark soliton with a unit topological charge, illustrating the bending of the vortex line.

![Intensity distribution for a dark soliton with topological charge m = 1 for z = L/2; in the lower plane, the colour or tone indicates the phase of the field. (b): single vortex line presented by the surface of isointensity at level I / I_{\text{max}} = 0.06, for z = L/2; L = 16.](image1)

![Intensity distribution for dark soliton with total topological charge m = 2 for z = L/2; the same as in figure 3a in the lower plane. (b): double vortex line (surface of isointensity at level I / I_{\text{max}} = 0.06); L = 25.6.](image2)

In an even wider range of cavity lengths, structures are stable in which the vortex lines are segments of a "double helix" with the same topological charges (Fig. 4). Depending on the length \( L \), these spirals make one (\( L \approx 12 \)) or two (\( L \approx 25 \)) turns. Structures with two spiral vortex lines are also rigid as the previous ones. Note that related structures were presented in [8] for a scheme of optical parametric oscillator.
5. Conclusion
The presented family of tubular topological laser solitons serves as an intermediate link between 2D and 3D solitons. Determination of their properties makes it possible to reveal the limitations of the popular mean-field approximation, and the significant width of the region of parameters in which they are stable indicates that the applications of such solitons are promising, including microparticles manipulating [9, 10].

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Disclosures
The authors declare no conflicts of interest.

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