Constructing Depth-Optimum Circuits for Adders and And-Or Paths

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Abstract

We examine the fundamental problem of constructing depth-optimum circuits for binary addition. More precisely, as in literature, we consider the following problem: Given auxiliary inputs \( t_0, \ldots, t_m \), the so-called generate and propagate signals, construct a depth-optimum circuit over the basis \{And2, Or2\} computing all \( n \) carry bits of an \( n \)-bit adder, where \( m \geq 2n-1 \). In fact, carry bits are And-Or paths, i.e., Boolean functions of the form \( t_0 \lor (t_1 \land (t_2 \lor \ldots \lor t_m) \ldots) \). Classical approaches construct so-called prefix circuits which do not achieve a competitive depth. For instance, the popular construction by Kogge and Stone [20] is only a 2-approximation. A lower bound on the depth of any prefix circuit is \( 1.44 \log_2 m + \text{const} \), while recent non-prefix circuits have a depth of \( \log_2 m + \log_2 \log_2 m + \text{const} \). However, it is unknown whether any of these polynomial-time approaches achieves the optimum depth for all \( m \in \mathbb{N} \).

We present a new exponential-time algorithm solving the problem optimally. The previously best exact algorithm by Hegerfeld [11] with a running time of \( O(2.45^m) \) is viable only for \( m \leq 29 \). Our algorithm is significantly faster: We achieve a theoretical running time of \( O(2.02^m) \) and apply sophisticated pruning strategies to improve practical running times dramatically. This allows us to compute optimum circuits for all \( m \leq 64 \). Combining these computational results with new theoretical insights, we derive the optimum depths of \( 2^k \)-bit adder circuits for all \( k \leq 13 \), previously known only for \( k \leq 4 \).

In fact, we solve a more general problem, namely delay optimization of generalized And-Or paths, which originates from late-stage logic optimization in VLSI design. Delay is a natural extension of circuit depth to prescribed input arrival times; and generalized And-Or paths are a generalization of And-Or paths where And and Or do not necessarily alternate. Our algorithm arises from our new structure theorem which characterizes delay-optimum generalized And-Or path circuits.

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1. Introduction

In this work, we construct fast circuits for binary addition and for related Boolean functions, so-called And-Or paths. An And-Or path is a function of the form \( t_0 \lor (t_1 \land (t_2 \lor \ldots \lor t_m) \ldots) \) for some \( m \in \mathbb{N} \); and a circuit for a Boolean function is a graph-based model for the computation of the function via elementary building blocks (called gates) on a computer chip. Here, we use And2 and Or2 as elementary building blocks, i.e., logical And and Or with two inputs each. Motivated from VLSI design, our objective function is circuit delay, a generalization of circuit depth to prescribed input arrival times \( a(t_i) \in \mathbb{N} \) for each input \( t_i \). The delay of a circuit is the maximum delay of any input \( t_i \), i.e., \( a(t_i) \) plus the maximum length of any directed path in the circuit starting in \( t_i \). In particular, when \( a \equiv 0 \), circuit delay is actually circuit depth, i.e., the maximum length of any directed path in the circuit. Given a specific And-Or path with input arrival times, we want to find a delay-optimum circuit for this Boolean function using only And2 and Or2 gates. Important secondary objective functions include circuit size (i.e., number of gates) and fanout (i.e., number of successors of a gate).

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And-Or paths occur as carry bits in the computation of adder circuits: Assume we compute the sum of two n-bit binary numbers \( \sum_{i=0}^{n-1} a_i 2^i \) and \( \sum_{i=0}^{n-1} b_i 2^i \). A circuit for this task can be constructed via *carry bits* which are defined recursively by \( c_0 = 0 \) and \( c_{i+1} = g_i \lor (p_i \land c_i) \) for \( 0 \leq i \leq n-1 \), where \( g_i = a_i \land b_i \) and \( p_i = a_i \oplus b_i \), see, e.g., Weinberger and Smith [32] or Knowles [19]. Using the carry bits, the sum \( \sum_{i=0}^{n} s_i 2^i \) can be computed via \( s_i = c_i \oplus p_i \) for \( i \in \{0, \ldots, n-1\} \) and \( s_n = c_n \).

The computation of all \( g_i \) and \( p_i \) as well as the computation of the sum from the carry bits only requires a constant depth and a linear number of gates. Therefore, we call a circuit computing all the And-Or paths

\[
c_{i+1} = g_i \lor (p_i \land c_i) = g_i \lor (p_i \land (g_{i-1} \lor (p_{i-1} \land c_{i-1})))
\]

\[
= g_i \lor (p_i \lor (g_{i-1} \lor (p_{i-1} \lor (g_{i-2} \lor \ldots (p_1 \lor g_0))))))
\]

for \( 0 \leq i \leq n-1 \) an *adder circuit*. A delay-optimum adder circuit may hence be obtained by separately computing all carry bits via an And-Or path. On the other hand, an \( n \)-bit adder circuit in particular computes the last carry bit \( c_n \), so it contains a circuit for the And-Or path with \( 2n - 1 \) inputs of the form \( g_i \) and \( p_i \). Therefore, one can show that, up to a small constant, the problem of constructing circuits for And-Or paths with minimum depth can be reduced to the problem of constructing circuits for addition with minimum depth. In Figure 1 we depict two logically equivalent adder circuits for \( n = 3 \) bits.

Hence, in the following, we will focus on the construction of fast circuits for And-Or paths. In fact, we consider generalized And-Or paths, i.e., a generalization of And-Or paths where And and Or do not necessarily alternate. We will see that this more general problem has a rich structure which we will exploit for our new results. To the best of our knowledge, we are the first to directly consider this generalized problem. Delay optimization of generalized And-Or paths can be applied to optimize the delay of critical combinatorial paths in VLSI design, but existing approaches (see, e.g., Werber et al. [33]) use a simple reduction to And-Or paths which leads to sub-optimal solutions.

### 1.1. Previous Work

We now review previous results on adder and And-Or path optimization. Recall from Equation (1) that an \( n \)-bit adder can be obtained from And-Or paths on \( 1, 3, \ldots, 2n - 1 \) inputs, so the optimum depth of an \( n \)-bit adder equals the optimum depth of an \( m \)-input And-Or path with \( m = 2n - 1 \), i.e., \( n = \frac{m+1}{2} \). Depth bounds for classical adder constructions are given in terms of \( n \) instead of \( m \).

Some of the following results only apply to depth optimization, some also to delay optimization for general arrival times. For general arrival times, a lower bound on the delay of a Boolean circuit on inputs \( t_0, \ldots, t_{m-1} \) is given by

\[
\sum_{i=0}^{m-1} \sum_{j=0}^{\lfloor \log_2 i \rfloor} \frac{t_i}{2^j}.
\]
circuit designs with And function is identified by its truth table, and circuits of larger depth are obtained by pairwise combinations of existing circuits. The contribution is the observation that a truth table of size $m^r$ can be optimized to depth $\log_2 m + \log_2 \log_2 m + 1.58$ up to an additive term of $5$. Using non-prefix circuits, Brent [3], Khrapchenko [17], Spirkl [30], and Held and Spirkl [12] achieve a guaranteed delay of the form $\log_2 W + O(r)$ for And-Or paths on $m$ inputs. For arbitrary arrival times, the best known delay guarantee of $\log_2 W + \log_2 \log_2 m + \log_2 \log_2 m + 1.58$ was first obtained by Brenner and Hermann [1].

In order to obtain good empirical results, the algorithm by Brenner and Hermann [2] combines the ideas from these theoretical constructions with practical improvements and generalizations. Although no better worst-case bound can be shown, the obtained circuits mostly have better – often optimal – delay. However, there are instances with only 6 inputs for which the algorithm does not find an optimum solution, see Section 6.3 in Hermann [14]. Still, regarding depth optimization, all instances with known optimum depth that can be solved by this algorithm are indeed solved optimally, and it is open whether this construction is always optimum in the depth case. There is also a depth-optimization heuristic by Grinchuk [8] which might actually be an exact algorithm (cf. the depths displayed in the table in Section 5 of [5]).

There are three previously known exact algorithms for depth optimization of And-Or paths. Apart from the aforementioned heuristic, Grinchuk [8] also provides an exact algorithm for depth optimization of And-Or paths with a running time of $\Omega(4^n)$. No explicit empirical results are given, but it is mentioned that the algorithm can only be used for up to 20 or 30 inputs. The idea of this algorithm is to compute the optimum achievable depth for all monotone Boolean functions on $m$ inputs in a bottom-up dynamic programming fashion. Each Boolean function is identified by its truth table, and circuits of larger depth are obtained by pairwise combinations of existing circuits with And or Or gates. Naively, the dynamic-programming table thus would have $2^m$ entries. Grinchuk’s main contribution is the observation that a truth table of size $m$ – called a passport in [8] – suffices to identify a monotone And-Or path circuit. This way, the size of the dynamic-programming table is reduced to $2^m$, which implies a running time of $\Omega(4^m)$ to compute all table entries and hence a depth-optimum And-Or path circuit.

Hegerfeld [11] proposes two enumeration algorithms constructing depth-optimum And-Or path circuits. In a first algorithm, Hegerfeld constructs all circuits that are size-optimum among all depth-optimum And-Or path circuits. This algorithm is based on tree enumeration and is viable for up to 19 inputs. The algorithm can also be used to enumerate And-Or path circuits with non-optimum depth with an increase in running time, which leads to optimum solutions with respect to delay for certain arrival time profiles.

Hegerfeld’s second algorithm is much faster, but is restricted to depth-optimum formula circuits (i.e., circuits where each gate has fanout 1) with a certain size guarantee (cf. Section 7). It has a provable running time of $O(2.45^m)$ and can be applied for up to 29 inputs. Hegerfeld does not enumerate formula circuits for And-Or paths directly, but so-called rectangle-good protocol trees for Karchmer-Wigderson games (see Karchmer and Wigderson [13]) for And-Or paths, which come from the area of communication complexity. From these, Hegerfeld derives the optimum formula circuits. This is the fastest previously known exact algorithm for depth optimization of And-Or paths.
1.2. Our Contributions

In this work, we present a new exact algorithm for constructing delay-optimum generalized And-Or path circuits. In the most prominent special case of depth optimization of And-Or path circuits (and thereby binary adders), our algorithm is significantly faster than previous approaches, both in theory and in practice. For the general problem, which occurs in late-stage timing optimization in VLSI design, we obtain the first known non-trivial exact algorithm.

We prove a new structure theorem which characterizes the structure of specific delay-optimum circuits for generalized And-Or paths. More precisely, we show how optimum solutions for generalized And-Or paths can be obtained by combining optimum circuits for certain smaller generalized And-Or paths in a recursive fashion, directly motivating a dynamic programming algorithm. We stress that an analogous statement does not hold for non-generalized And-Or paths, that is, generalized And-Or paths also occur as sub-functions of delay-optimum circuits for non-generalized And-Or paths.

In the general case, the running time of our new algorithm is at most $O(3^m)$. For And-Or paths, the bound is improved to $O(2.45^m)$. For the special case of delay optimization of And-Or paths, our algorithm has a running time of $O(2.02^m)$, significantly improving upon the previously best running time of $O(2.45^m)$ of the formula enumeration algorithm by Hegerfeld [11].

Our algorithm can either compute such a circuit or an arbitrary delay-optimum circuit, which can be done much faster. In contrast to Hegerfeld, in practice, we apply very efficient pruning techniques that drastically reduce empirical running times. The largest instance solved by Hegerfeld has 29 inputs, while our algorithm with size optimization can solve instances with up to 42 inputs; without size optimization even up to 64 inputs. Our running times on 26 inputs are 2.1 seconds with size optimization and 0.007 seconds without size optimization, while Hegerfeld’s running time is 17 hours. Our largest running time without size optimization on any of these instances is roughly 2.7 hours.

From our structure theorem, our computations and the results computed by the heuristic And-Or path optimization algorithm by Grinchuk [8], we deduce the optimum depths of adder circuits – i.e., circuits computing all the carry bits from the $g_i$- and $p_i$-signals as in Equation (1) – over the basis $\{\text{And2}, \text{Or2}\}$ on $2^k$ bits, where $k \leq 13$. To the best of our knowledge, we are the first to obtain such a result.

The rest of the paper is organized as follows: In Section 2, we formally introduce the problem and basic concepts. In Section 3, we present and prove our structure theorem. From this, in Section 4, we derive our exact algorithm, which is refined for the special case of depth optimization of And-Or paths in Section 5. Practical speedups are presented in Section 6. In Section 7, we show computational results, i.e., our practical running times and the computed optimum adder depths.

2. Preliminaries

2.1. Boolean Functions and Circuits

Our notation regarding Boolean functions and circuits is based on Savage [27]. We denote the set of natural numbers including 0 by $\mathbb{N}$. For an $n$-tuple $(x_0, \ldots, x_{n-1})$ and an index $i \in \{0, \ldots, n-1\}$, we use the standard notation $(x_0, \ldots, \hat{x}_i, \ldots, x_{n-1})$ to denote the $(n-1)$-tuple arising from $x$ by deleting the entry $x_i$.

For us, a Boolean function is a function $f : \{0, 1\}^m \rightarrow \{0, 1\}$ for some $m \in \mathbb{N}$. We often write $t = (t_0, \ldots, t_{m-1})$ for the input variables, short inputs, of $f$. For an input variable $t_i$ of $f$ and a value $\alpha \in \{0, 1\}$, the restriction of $f$ to $t_i = \alpha$ is the function $f|_{t_i=\alpha} : \{0, 1\}^{m-1} \rightarrow \{0, 1\}, (t_0, \ldots, \hat{t}_i, \ldots, t_{m-1}) \mapsto f((t_0, \ldots, t_{i-1}, \alpha, t_{i+1}, \ldots, t_{m-1}))$. An input $t_i$ is essential for $f$ if $f|_{t_i=0} \neq f|_{t_i=1}$.

In this work, a circuit $C$ is a connected acyclic digraph whose vertices can be partitioned into two sets: inputs with no incoming edges that each represent a Boolean variable or a constant 0 or 1, and gates with exactly 2 incoming edges that each represent an elementary Boolean function among logical And and Or. By $gt(g) \in \{\text{And}, \text{Or}\}$, we denote the gate type of a gate $g$, i.e., its associated Boolean function. There is a subset of the vertices called outputs. All vertices with no outgoing edges are outputs, but there might be others. If $C$ contains only a single output, we denote it by out($C$). The Boolean function computed at a vertex of a circuit $C$ can be read off recursively by combining the logical functions represented by the gates. If $C$ has a single output which computes a Boolean function $f$, we also say that $C$ computes or realizes $f$ and write $f = f(C)$ for the function computing $C$. 
Given a Boolean function \( f \), there are numerous circuits realizing \( f \). In order to evaluate the quality of different circuits, we introduce the following measures: By size(\( C \)), we denote the size of a circuit \( C \), i.e., its number of gates. The fanout of a vertex \( v \) of \( C \) is defined by fanout(\( v \)) := \( |\delta^+(v)| \). When each input \( t_i \) is associated with an arrival time \( a(t_i) \in \mathbb{R} \) at which the signal \( t_i \) is available, then the delay of \( C \) is

\[
\text{delay}(\mathcal{C}) := \max_{i \in \{0,\ldots,m-1\}} \left( a(t_i) + \max_{P} |P| \right),
\]

where the innermost maximum ranges over all directed paths \( P \) in \( C \) starting in \( t_i \). Note that when all arrival times are 0, the notion of delay coincides with the notion of circuit depth, i.e., the length of a longest directed path in \( C \), and we write depth(\( C \)) instead of delay(\( C \)). Figure 2 shows three different circuits computing the same function with their corresponding depths and delays. The circuits \( C_2 \) and \( C_3 \) are depth-optimum, while \( C_1 \) is delay-optimum for the indicated arrival times. The gates are shown in red (And gates) and in green (Or gates), and all edges are directed from top to bottom. The output gate is indicated by an arrow.

When analyzing the structure of delay-optimum circuits for a certain Boolean function, prime implicants play an important role. For a more elaborate introduction, see Section 1.7 of Crama and Hammer [3].

**Definition 2.1.** Let \( f : \{0,1\}^m \rightarrow \{0,1\} \) be a Boolean function with inputs \( t = (t_0, \ldots, t_{m-1}) \). A literal of \( f \) is a possibly negated input variable of \( f \), i.e., \( t_i \) or \( \overline{t_i} \) for some \( i \in \{0, \ldots, m-1\} \). Consider a conjunction \( t (t_0, \ldots, t_{m-1}) = l_1 \land \ldots \land l_k \), where \( l_1, \ldots, l_k \) are literals of \( f \). We write \( \text{lit}(t) = \{l_1, \ldots, l_k\} \) for the set of literals of \( t \). We call \( t \) an implicant of \( f \) if for any \( \alpha \in \{0,1\}^m \) with \( t(\alpha) = 1 \), we have \( f(\alpha) = 1 \). We call \( t \) a prime implicant of \( f \) if there is no implicant \( \pi \) of \( f \) with \( \text{lit}(\pi) \subseteq \text{lit}(t) \). The set of all prime implicants of \( f \) is denoted by \( \text{PI}(f) \).

We need several well-known statements about prime implicants for which a proof can be found, e.g., in Crama and Hammer [3].

**Proposition 2.2** (Theorem 1.13 of Crama and Hammer [3]). Every Boolean function can be represented by the disjunction of all its prime implicants. In particular, a Boolean function \( f \) is uniquely determined by \( \text{PI}(f) \). □

**Proposition 2.3** (Theorem 1.17 of Crama and Hammer [3]). Let \( f : \{0,1\}^m \rightarrow \{0,1\} \) be a Boolean function on inputs \( t_0, \ldots, t_{m-1} \). Then, \( f \) depends essentially on an input \( t_i \) if and only if there is a prime implicant \( \pi \) of \( f \) with \( t_i \in \text{lit}(\pi) \) or \( \overline{t_i} \in \text{lit}(\pi) \). □

Given \( \alpha, \beta \in \{0,1\}^m \), we write \( \alpha \leq \beta \) if \( \alpha_i \leq \beta_i \) for all \( i \in \{0, \ldots, m - 1\} \). A Boolean function \( f : \{0,1\}^m \rightarrow \{0,1\} \) is called monotone if for all \( \alpha, \beta \in \{0,1\}^m \) with \( \alpha \leq \beta \), we have \( f(\alpha) \leq f(\beta) \). In this work, we examine a sub-class of the class of monotone functions.

For monotone Boolean functions, function decomposition and (prime) implicants behave in a canonical way, see, e.g., Commentz-Walter [3], proof of Lemma 1.
Lemma 2.4. Consider Boolean functions \( h, f_1, f_2 : \{0, 1\}^m \rightarrow \{0, 1\} \) on input variables \( t_0, \ldots, t_{m-1} \) with \( h = f_1 \lor f_2 \).

(a) Any implicant of \( f_1 \) or \( f_2 \) is an implicant of \( h \).

(b) If \( h, f_1, f_2 \) are all monotone, then any (prime) implicant of \( h \) is a (prime) implicant of \( f_1 \) or \( f_2 \).

Proof. Using the definition of implicants, the first statement can be seen easily.

To show the second statement, assume that \( h, f_1, f_2 \) are monotone, and let \( \kappa \) be an implicant of \( h \). Assume that \( \kappa \) is not an implicant of \( f_1 \) or \( f_2 \). Then, there are \( a^{(1)}, a^{(2)} \in \{0, 1\}^m \) with

(i) \( \kappa(a^{(1)}) = \kappa(a^{(2)}) = 1 \) and

(ii) \( f_1(a^{(1)}) = f_2(a^{(2)}) = 0 \).

Define \( a \in \{0, 1\}^m \) by \( a_i = a^{(1)}_i \land a^{(2)}_i \) for \( i \in \{0, \ldots, m-1\} \). As \( \kappa \) is a conjunction of literals, (i) implies \( \kappa(a) = 1 \). Moreover, as \( f_1 \) and \( f_2 \) are monotone and \( \alpha \leq a^{(1)}, a^{(2)} \), (ii) implies \( f_1(\alpha) = f_2(\alpha) = 0 \) and thus \( h(\alpha) = f_1(\alpha) \lor f_2(\alpha) = 0 \), which contradicts \( \kappa \) being an implicant of \( h \). Hence, \( \kappa \) is an implicant of \( f_1 \) or \( f_2 \), w.l.o.g. of \( f_1 \).

Now assume additionally that \( \kappa \) is a prime implicant of \( h \) and consider an implicant \( \lambda \) of \( f_1 \) with \( \text{lit}(\lambda) \subseteq \text{lit}(\kappa) \). By the first statement of this lemma, \( \lambda \) is an implicant of \( h \). As \( \kappa \) is a prime implicant of \( h \), we have \( \lambda = \kappa \). Thus, \( \kappa \) is a prime implicant of \( f_1 \).

A formula circuit is a circuit where each gate has fanout at most 1. It is well known that for any Boolean function \( f \), there is a formula circuit which has optimum delay among all circuits for \( f \), see, e.g., Wegener [31]. Section 1.4. Hence, when computing a delay-optimum circuit for \( f \), we may restrict ourselves to formula circuits.

The following lower bound on the delay of general circuits has first been proven by Golumbic [7]. An alternative proof via Kraft’s inequality [21] is given by Rautenbach et al. [23].

Theorem 2.5 (Golumbic [7]). Consider any Boolean function \( f : \{0, 1\}^m \rightarrow \{0, 1\} \) on inputs \( t_0, \ldots, t_{m-1} \) with input arrival times \( a(t_0), \ldots, a(t_{m-1}) \in \mathbb{N} \) that depends essentially on all its inputs. Then, any circuit \( C \) over \( \{\text{And}, \text{Or}2\} \) computing \( f \) fulfills delay(\( C \)) \( \geq \lceil \log_2 W \rceil \), where \( W = \sum_{i=0}^{m-1} 2^{a(t_i)} \).

For the special case of depth optimization, Theorem 2.5 implies depth(\( C \)) \( \geq \lceil \log_2 m \rceil \) which was first proven by Winograd [34].

For our proofs, we need the concept of reduced circuits. Consider a circuit \( C \) over \( \{\text{And}, \text{Or}2\} \) with inputs \( t_0, \ldots, t_{m-1} \) realizing \( f \). The reduced circuit \( C \mid_{t=a} \) is a circuit on inputs \( t_0, \ldots, t_i, \ldots, t_{m-1} \) realizing the restricted function \( f \mid_{t=a} \) that arises from replacing \( t_i \) by \( a \) and then canonically reducing the circuit until it is trivial or until it does not contain any constants anymore. More formally, \( C \mid_{t=a} \) arises from \( C \) as follows: Replace \( t_i \) by \( a \) and apply the following to each gate \( g \) in topological order: Assume that there is a predecessor \( v \in \partial^-(g) \) which is a constant (otherwise, do nothing for \( g \)), and denote the other predecessor of \( g \) by \( w \).

Case 1: Assume that \( g(t) = \text{And} \) and \( v = 0 \), or \( g(t) = \text{Or} \) and \( v = 1 \). Replace each edge \( (g, y) \in \partial^+(g) \) by \( (v, y) \).

If \( g \) is an output, then let \( v \) be an output.

Case 2: Otherwise, replace each edge \( (g, y) \in \partial^+(g) \) by \( (w, y) \), and if \( g \) is an output, then let \( w \) be an output.

In both cases, remove \( g \) from \( C \). If the end, remove all gates from which no output is reachable.

Observation 2.6. Consider a Boolean function \( f : \{0, 1\}^m \rightarrow \{0, 1\} \) on input variables \( t_0, \ldots, t_{m-1} \) with arrival times \( a(t_0), \ldots, a(t_{m-1}) \in \mathbb{N} \), an index \( i \in \{0, \ldots, m-1\} \) and a value \( \alpha \in \{0, 1\} \). Consider a circuit \( C \) for \( f \) and the reduced circuit \( C \mid_{t=a} \). Then, \( C \mid_{t=a} \) is a circuit for the restricted function \( f \mid_{t=a} \). Moreover, we have delay(\( C \mid_{t=a} \)) \( \leq \) delay(\( C \)) and size(\( C \mid_{t=a} \)) \( \leq \) size(\( C \)). If fanout(\( t_i \)) \( > 0 \) in \( C \), then we have size(\( C \mid_{t=a} \)) \( < \) size(\( C \)).

2.2. Generalized And-Or Paths

We now introduce the special class of monotone Boolean functions considered in this work.

Definition 2.7. Let inputs \( t = (t_0, \ldots, t_{m-1}) \) and an \( (m - 1) \)-tuple \( \Gamma = (\gamma_0, \ldots, \gamma_{m-2}) \) of gate types \( \gamma_0, \ldots, \gamma_{m-2} \in \{\text{And}, \text{Or}2\} \) be given. We call a Boolean function of the form

\[
h(t; \Gamma) := t_0 \circ_0 (t_1 \circ_1 (t_2 \circ_2 \cdots (t_{m-2} \circ_{m-2} (t_{m-1}))))
\]

Equation (2) a generalized And-Or path. We call the circuit for \( h(t; \Gamma) \) arising from Equation (2) the standard circuit for \( h(t; \Gamma) \).
We can now formulate the main problem considered in this work.

**Delay Optimization Problem for Generalized AND-OR Paths**

*Instance:* $m \in \mathbb{N}$, inputs $t = (t_0, \ldots, t_{m-1})$, gate types $\Gamma = (\sigma_0, \ldots, \sigma_{m-2})$, arrival times $a(t_0), \ldots, a(t_{m-1}) \in \mathbb{N}$.  

*Task:* Compute a circuit over \{AND2, OR2\} realizing $h(t; \Gamma)$ with minimum possible delay.

In the special case where all arrival times $a(t_0), \ldots, a(t_{m-1})$ are identical, optimizing circuit delay means optimizing circuit depth and we call the problem **Depth Optimization Problem for Generalized AND-OR Paths**.

Due to the close connection to adder circuits (see Section 1), our main interest lies in the optimization of AND-OR paths, i.e., generalized AND-OR paths where the gate types alternate between AND and OR. Here, we call the respective optimization problems **Delay Optimization Problem for AND-OR Paths** and **Depth Optimization Problem for AND-OR Paths**. Figure 2 [Page 5] shows two logically equivalent circuits for the AND-OR path $t_0 \land (t_1 \lor (t_2 \land (t_3 \lor t_4)))$ on 5 inputs: the standard circuit and a circuit with a better depth. However, for the indicated blue input arrival times, the standard circuit has a better delay than the other circuit. Standard circuits for other generalized AND-OR paths are shown, e.g., in Figure 4(a) [Page 8] and Figure 6(a) [Page 10].

Figure 3: Two formula circuits for the AND-OR path $t_0 \land (t_1 \lor \ldots \lor t_{13})$ with optimum depth 5. They only differ in the left sub-circuit of the final output.

Given a Boolean function $f$ and a circuit $C$ for $f$ with prescribed input arrival times, we call $C$ **strongly delay-optimal** (or, in case of $a \equiv 0$, strongly depth-optimal) if for each vertex $v$, the Boolean function computed at $v$ is realized by a delay-optimal circuit in $C$. In Figure 3, we show two depth-optimum formula circuits for the AND-OR path $t_0 \land (t_1 \lor \ldots \lor t_{13})$. The circuit in Figure 3(a) is a circuit with optimum depth and, among all depth-optimum circuits, optimum size, while the circuit in Figure 3(b) is at least size-optimum among all strongly depth-optimum formula circuits. Note that in Figure 3(a) the left predecessor of the output gate computes an AND-OR path on 5 inputs with a non-optimum depth of 4 and a size of 5. In Figure 3(b) we instead use a depth-optimum circuit with depth 3 and size 5.

Note that we assume the input arrival times to be natural numbers. As shown in Hermann [14], this is not a restriction: When the arrival times are arbitrary fractional numbers, we can still solve the Delay Optimization Problem for Generalized AND-OR Paths optimally using a certain type of binary search on specific instances with all arrival times being natural numbers (see Theorem 5.1.5 of Hermann [14]). This increases the running time by at most a factor of $O(\log_2 m)$. Hence, during this work, we may assume all arrival times to be natural numbers.

In order to understand (generalized) AND-OR paths more thoroughly, it is helpful to divide the inputs into groups. Here, we use the notation $x \leftrightarrow y$ for the concatenation of two tuples $x$ and $y$.

**Definition 2.8.** Let $h(t; \Gamma)$ be a generalized AND-OR path with $m \geq 1$ inputs. For $i \in \{0, \ldots, m-2\}$, we call $o_i = gt(t_i)$ the gate type of $t_i$, and we call $t_i$ a $o_i$-signal. Given a gate type $o \in \{\text{AND}, \text{OR}\}$, we denote the set of all $o$-signals plus $t_{m-1}$ by $S^o$. We call $S^o$ the same-gate input set of the generalized AND-OR path $h(t; \Gamma)$ and the gate type $o$. By $D^o := \{t_0, \ldots, t_{m-1}\} \setminus S^o$, we denote the different-gate input set of $h(t; \Gamma)$ and $o$. The segment partition of $h(t; \Gamma)$ is the unique partition $(t_0, \ldots, t_{m-1}) = P_0 \oplus \ldots \oplus P_c$ of the inputs into maximal consecutive sub-tuples $P_0, \ldots, P_c$ called **input segments** such that for each $b \in \{0, \ldots, c\}$, we have $P_b \subseteq S^{AN} \lor P_b \subseteq S^{OR}$.
Figure 6(a) (Page 10) visualizes a generalized AND-OR path and its segment partition. Note that the last input $t_{m-1}$ does not have a gate type and, for $m \geq 2$, always is in the same input segment as $t_{m-2}$. In this example, we have $S^{\text{AND}} = \{t_0, t_5, t_6, t_8, t_9, t_{10}, t_{11}\}$, $D^{\text{AND}} = \{t_1, t_2, t_3, t_4, t_7\}$, $S^{\text{OR}} = \{t_1, t_2, t_3, t_4, t_7, t_{11}\}$, $D^{\text{OR}} = \{t_0, t_5, t_6, t_8, t_9, t_{10}\}$.

It is an easy exercise to characterize the prime implicants of generalized AND-OR paths as in the following proposition. Figure 4 shows all prime implicants for an AND-OR path on 6 inputs.

**Proposition 2.9.** The set of prime implicants of $h(t; \Gamma)$ is given by

$$
\left\{ t_1 \land \bigwedge_{j \in \text{gt}(t): t_j \in S^{\text{OR}}} t_j : t_j \in S^{\text{OR}} \right\}.
$$

![Prime Implicants Diagram](image)

**Figure 4:** All prime implicants of the AND-OR path $t_0 \lor \left( t_1 \land \left( t_2 \lor (t_3 \land t_5) \right) \right)$. Figures 4(a) to 4(d) illustrate one prime implicant each. The corresponding inputs are boxed.

Together with Proposition 2.3 this proposition implies the following important statement.

**Corollary 2.10.** Any generalized AND-OR path depends essentially on all of its inputs.

We can now give another basic lower bound on the delay of any circuit over $\{\text{AND}, \text{OR}\}$ realizing a given generalized AND-OR path.

**Proposition 2.11.** Let $m \in \mathbb{N}$ with $m \geq 2$. Let inputs $t = (t_0, \ldots, t_{m-1})$ with arrival times $a(t_0), \ldots, a(t_{m-1}) \in \mathbb{N}$ and gate types $\Gamma = (c_0, \ldots, c_{m-2})$ be given. Let $(t_0, \ldots, t_{m-1}) = P_{0}^+ \cdots + P_{r}$ be the segment partition of $h(t; \Gamma)$. Consider a circuit $C$ over $\{\text{AND}, \text{OR}\}$ realizing $h(t; \Gamma)$. Then, we have

$$
\text{delay}(C) \geq \max \left\{ \max_{t \in P_0} a(t) + 1, \max_{t \in P_r : b > 0} a(t) + 2 \right\}.
$$

**Proof.** Using Corollary 2.10 and $m \geq 2$, we immediately see that each input has depth at least 1 in $C$. Thus, it suffices to prove that each input $t_i$ contained in $P_b$ for some $b > 0$ has depth at least 2 in $C$. For such an input, one can show easily that for any $\alpha \in \{0, 1\}$, the function $f(C |_{t \leq \alpha})$ depends essentially on $t_0$ (note that $t \neq 0$). This is not the case if some directed path from $t_i$ to out$(C)$ contains only gates of the same type, so any path from $t_i$ to out$(C)$ contains at least one AND gate and one OR gate. Hence, $t_i$ has depth at least 2 in $C$.

Note that for fixed $m \in \mathbb{N}$, there are exactly 2 AND-OR paths on $m$ inputs. They can be turned into each other by exchanging all AND and OR operations. Even more, any circuit for the first AND-OR path can be turned into a circuit for the second AND-OR path with the same delay by exchanging all AND and OR gates. This process is called dualization and the statement is proven in a more general setting, for instance, in Crama and Hammer [6], Theorem 1.3.
3. Structure Theorem

Our structure theorem and our algorithm presented in Section 4 both reduce the problem of optimizing a given generalized AND-OR path to smaller instances of a specific form.

Definition 3.1. Consider a generalized AND-OR path \( h(t; \Gamma) \) with \( m \geq 1 \) inputs. Given indices \( 0 \leq i_0 < \ldots < i_{k-1} \leq m-1 \), the generalized AND-OR path \( t_{i_0} \circ \left( t_{i_1} \circ \ldots \circ t_{i_{k-1}} \right) \) is called a sub-path of \( h(t; \Gamma) \). Now, let a gate type \( \circ \in \{ \text{AND, OR} \} \) and a set \( S_0^+ \subseteq S^+ \) be given, and let \( t \) be maximum with \( t \in S_0^+ \). Then, the sub-path of \( h(t; \Gamma) \) containing all signals from \( S_0^+ \) and all signals \( t_j \in D^+ \) with \( j < i \) is denoted by \( h(t; \Gamma)_{S_0^+} \) and called a special sub-path of \( h(t; \Gamma) \).

Figure 6 shows a generalized AND-OR path and gives several examples for special sub-paths.

In Theorem 3.4, we will see that for any generalized AND-OR path \( h(t; \Gamma) \) with input arrival times, there is always a delay-optimum formula circuit \( C \) which arises from combining two circuits for special sub-paths \( h(t; \Gamma)_{S_0^+} \) and \( h(t; \Gamma)_{S_0^+} \) with a \( \circ \)-gate, where \( \circ \in \{ \text{AND2, OR2} \} \) and \( S^+ = S_0^+ \cup S_0^+ \). Our proof idea is based on Lemma 1 from Commentz-Walter [9]. However, only AND-OR paths are considered there, and not generalized AND-OR paths, and only a partial description of the structure of AND-OR path circuits is given, not a complete characterization. The main objects considered in both Commentz-Walter’s and our proof are prime implicants. An easy consequence of Proposition 2.9 is the following observation.

Observation 3.2. Let a generalized AND-OR path \( h(t; \Gamma) \) with \( t = (t_0, \ldots, t_{m-1}) \) be given. Consider any two prime implicants \( \pi \neq \rho \in \text{Pl}(h(t; \Gamma)) \). Choose \( i \in \{0, \ldots, m-1\} \) maximum with \( t_i \in \text{lit}(\pi) \). Then, we have \( t_i \notin \text{lit}(\rho) \).

The following lemma is the main ingredient of our structure theorem. Theorem 3.4. We consider a formula circuit \( C \) implementing a generalized AND-OR path \( h(t; \Gamma) \) with \( \text{out}(C) = \text{Or} \). If \( C \) is delay-optimum for the given arrival times and, among all delay-optimum circuits, size-optimum, then we will show that the two sub-circuits of \( \text{out}(C) \) compute special sub-paths of \( h(t; \Gamma) \), where each input of \( S_0^+ \) is contained in exactly one of the two sub-circuits. Figure 5 shows two examples for such circuits for the generalized AND-OR path from Figure 6(a). Figure 7 illustrates the proof of the lemma.
Figure 6: A generalized Asu-Or path $h(t; \Gamma)$ and three special sub-paths as in Definition 3.1. We also show the respective segment partitions, and the respective input set $S^\text{Sor}_1$ is marked blue.
Thus, assume that

Item (b) and the definition of prime implicants, we have

delay-optimum circuit for

Note that

We apply Lemma 2.4 to the monotone functions

If there are

We have

Case 2:

We have

By Item (b), we have

Example illustration for the proof of Lemma 3.3. Figure 7(b) depicts a circuit $C$ for the Ans-On path in Figure 7(a). The prime implicant $\rho = t_1 \land t_2 \land t_3$ of $f_1$ is highlighted. It contains the prime implicant $\pi = t_1 \land t_2$ of $h$, for which the highest index is $i = 2$. In Figure 7(c) we show an intermediate step to reduce the circuit w.r.t. $t_2 = 0$. The reduced circuit is again the circuit in Figure 7(a).

Lemma 3.3. Let $m \in \mathbb{N}_{\geq 2}$, inputs $t = (t_0, \ldots, t_{m-1})$ with arrival times $a(t_0), \ldots, a(t_{m-1}) \in \mathbb{N}$ and gate types $\Gamma = (\circ_0, \ldots, \circ_{m-2})$ be given. Consider a delay-optimum formula circuit $C$ for $h(t; \Gamma)$ with minimum number of gates. Assume that $\text{gt}(\text{out}(C)) = \mathbb{O}$. Denote the predecessors of $v := \text{out}(C)$ by $v_1$ and $v_2$. Write $h := h(t; \Gamma)$, and $f_1 := f(C(v_1))$, and $f_2 := f(C(v_2))$. Then, the following statements are fulfilled:

1. We have $\text{Pl}(h) = \text{Pl}(f_1) \cup \text{Pl}(f_2)$.

2. There exists a partition $S^\mathbb{O}_k = S^\mathbb{O}_1 \cup S^\mathbb{O}_2$ with $S^\mathbb{O}_1 \cap S^\mathbb{O}_2 \neq \emptyset$ such that for each $k \in \{1, 2\}$, the function $f_k$ depends essentially on all inputs of $S^\mathbb{O}_k$ and on no input of $S^\mathbb{O}_{-k}$.

3. Let $k \in \{1, 2\}$. Consider the special sub-path $h_k := h(t; \Gamma)|_{S^\mathbb{O}_k}$ of $h(t; \Gamma)$. Then, we have $f_k = h_k$.

Proof. We apply Lemma 2.4 to the monotone functions $h, f_1, f_2$. Item (b) implies $\text{Pl}(h) \subseteq \text{Pl}(f_1) \cup \text{Pl}(f_2)$. In order to show the first statement, it remains to prove that $\text{Pl}(f_1) \subseteq \text{Pl}(h)$ for each $k \in \{1, 2\}$ and that $\text{Pl}(f_1) \cap \text{Pl}(f_2) = \emptyset$.

By Item (a) any prime implicant $\rho$ of $f_1$ is an implicant of $h$ and must hence contain a prime implicant $\pi$ of $h$. By Item (b) and the definition of prime implicants, we have $\rho = \pi$ or $\pi$ is a prime implicant of $f_2$. Note that $\rho = \pi$ would imply $\rho \in \text{Pl}(h)$. Hence, to prove the first statement, it suffices to show the following claim.

Claim. If there are $\rho \in \text{Pl}(f_1)$ and $\pi \in \text{Pl}(h)$ with $\text{lit}(\pi) \subseteq \text{lit}(\rho)$ and $\pi \in \text{Pl}(f_2)$, then $C$ is not a size-minimum delay-optimum circuit for $h$.

Proof of claim: Choose $i \in \{0, \ldots, m - 1\}$ maximal such that $t_i$ is contained in $\pi$. As $C$ is a formula circuit, $C_1$ and $C_2$ do not share any gates. Consider the circuit $B$ arising from $C$ by replacing $C_1$ with the reduced circuit $C_1|_{\circ_0}$. Note that $B$ is again a formula circuit. Write $g := f(B)$, and for $k \in \{1, 2\}$, write $B_k := B_{t_k}$ and $g_k := f(B_k)$. As $\rho$ contains $t_i$, by Proposition 2.3, $f_1$ depends essentially on $t_i$. Hence, by Observation 2.6, we have delay($B$) $\leq$ delay($C$) and size($B$) $< size(C)$. It remains to show that $B$ and $C$ are logically equivalent.

Let $\alpha \in \{0, 1\}^m$. As $B$ is monotone and arises from $C$ by fixing an input to 0, we have $g(\alpha) = 0$ whenever $h(\alpha) = 0$. Thus, assume that $h(\alpha) = 1$. Then, there is $\psi \in \text{Pl}(h)$ with $\text{lit}(\psi) = 1$.

Case 1: We have $\psi \in \text{Pl}(f_2)$.

Here, as $B_2 = C_2$, we have $g_2(\alpha) = f_2(\alpha) = 1$ and thus $g(\alpha) = 1$.

Case 2: We have $\psi \notin \text{Pl}(f_2)$.

By Item (b) we have $\psi \notin \text{Pl}(f_1)$. As $\pi \in \text{Pl}(f_2)$, we must have $\psi \neq \pi$. By the choice of $t_i$, Observation 3.2 implies that $t_i \notin \text{lit}(\psi)$. As any implicant $\iota$ of $f_1$ with $\iota \notin \text{lit}()$ is an implicant of $g_1$, we have $\psi \in \text{Pl}(g_1)$. This implies $g(\alpha) = 1$.

Thus, $B$ is a delay-optimum formula circuit for $h$ with better size than $C$. □
Now, we show the second statement. For each \( k \in \{1, 2\} \), let \( S_k^{Ol} \) consist of the inputs among \( S^{Ol} \) that \( f_k \) depends on essentially. By Proposition 2.3, a Boolean function depends essentially on an input \( t \) if and only if \( t \) is contained in any of its prime implicants. By Observation 3.2, for each input \( t \in S_k^{Ol} \), there is exactly one prime implicant of \( h \) containing \( t \). Thus, the first statement implies \( S^{Ol} = S_1^{Ol} \cup S_2^{Ol} \).

Now, assume the conditions of the third statement. By Proposition 2.9 and the first two statements, the prime implicants of \( f_k \) are \( \{ t \land \bigwedge_{j \in \partial t} \alpha_j \} : t \in S_k^{Ol} \} \); and by Proposition 2.9 and the definition of \( h_k \), these are precisely the prime implicants of \( h_k \). By Proposition 2.2, we deduce \( h_k = f_k \), hence the third statement.

**Theorem 3.4** (Structure theorem). Let \( m \in \mathbb{N}_{\geq 2} \), inputs \( t = (t_0, \ldots, t_{m-1}) \) with arrival times \( a(t_0), \ldots, a(t_{m-1}) \in \mathbb{N} \) and gate types \( \Gamma = (\sigma_0, \ldots, \sigma_{m-2}) \) be given. Consider a delay-optimum formula circuit \( C \) for \( h(t; \Gamma) \) with minimum number of gates. Let \( \alpha := gt(out(C)) \). Denote the predecessors of \( v := out(C) \) by \( v_1 \) and \( v_2 \). Write \( f_1 := f(C_{v_1}) \), and \( f_2 := f(C_{v_2}) \). Then, there is a partition \( S^t = S_1^t \cup S_2^t \) into non-empty subsets such that for each \( k \in \{1, 2\} \), the function \( f_k \) depends essentially on the inputs of \( S_k^t \) but not on those of \( S_{3-k}^t \) and we have

\[ f_k = h(t; \Gamma)^{S_k^t}. \]

**Proof.** By duality, it suffices to consider the case \( gt(out(C)) = Ol \). In this case, the statements hold by Lemma 3.3. □

As a consequence of this theorem, we can derive an upper bound on the maximum number of inputs an And-Or path may have such that an And-Or path circuit with depth \( d \) exists. For this, we need the following notation.

**Notation 3.5.** Let \( h(t; \Gamma) \) with \( t = (t_0, \ldots, t_{m-1}) \) and \( \Gamma = (\sigma_0, \ldots, \sigma_{m-2}) \) be a generalized And-Or path. Given \( i \in \{0, \ldots, m-1\} \), we define \( h(t; \Gamma)^{t_i} \) as the generalized And-Or path arising from \( h(t; \Gamma) \) by removing \( t_i \), i.e.,

\[ h(t; \Gamma)^{t_i} := \begin{cases} h(t_0, \ldots, \widehat{t_i}, \ldots, t_{m-1}); (\sigma_0, \ldots, \sigma_{m-2}) & \text{if } i \leq m-2, \\
(h(t_0, \ldots, t_{m-2}); (\sigma_0, \ldots, \sigma_{m-3})) & \text{if } i = m-1. \end{cases} \]

We extend this notation to removal of a subset \( F \) as in Theorem 3.4.

The upper bound presented in the following corollary can be seen easily; we presume that much stronger bounds can be derived from Theorem 3.4.

**Corollary 3.6.** If an And-Or path \( h(t) \) on \( m \geq 3 \) inputs can be realized by a circuit \( C \) with depth \( d + 1 \), then an And-Or path on \( m + \left\lceil \frac{d}{d+1} \right\rceil \) inputs can be realized by a circuit with depth \( d \).

**Proof.** W.l.o.g., we may assume that \( C \) is a depth-optimum formula circuit with depth \( d + 1 \) with minimum number of gates for \( h(t) \). Dualization allows us to assume that \( out(C) = Ol \). By Theorem 3.4, there are circuits \( C_1 \) and \( C_2 \) with depth at most \( d \) each that realize generalized And-Or paths \( f_1 \) and \( f_2 \), respectively, such that \( C = C_1 \lor C_2 \). Consider the corresponding partition \( S^{Ol} = S_1^{Ol} \cup S_2^{Ol} \) of the same-gate signals of \( h(t) \) as in Theorem 3.4.

As \( h(t) \) is an And-Or path and \( m \geq 3 \), we have \( D^{Ol} \neq \emptyset \). As \( h(t) \) is an And-Or path and \( t_{m-1} \in S^{Ol} \), for every \( t_i \in D^{Ol} \), we have \( t_{i+1} \in S^{Ol} \). Hence, the function

\[ \theta : D^{Ol} \to S^{Ol}, \quad t_i \mapsto t_{i+1} \]

is well-defined. For \( k \in \{1, 2\} \), let \( D_k^{Ol} := \theta^{-1}(S_k^{Ol}) \). Note that \( D^{Ol} = D_1^{Ol} \cup D_2^{Ol} \).

Now, for each \( k \in \{1, 2\} \), let \( B_k \) denote the reduced circuit arising from \( C_k \) by fixing all inputs \( t_i \in D_{3-k} \) to \( \alpha := 1 \), and let \( g_k := f(B_k) \). Then, as all inputs in \( D_{3-k} \) are Ando-signals, by considering the standard circuit for \( h(t) \), we observe that \( g_k = (f_k)^{\partial(t)} \). By construction, the essential variables of \( g_k \) are the variables of \( S_k^{Ol} \) and \( D_k^{Ol} \). Let \( t_{j_k} \) be the essential variable of \( g_k \) with \( j_k \) maximum.

Consider \( k \in \{1, 2\} \). We show that \( g_k \) is an And-Or path. First note that by Definition 3.1 and the choice of \( \alpha \), every input of \( g_k \) except for \( t_j \) is an Ando-signal (Or-signal) of \( g_k \) if and only if it is an Ando-signal (Or-signal) of \( h(t) \). By definition of \( \theta \), for any two Or-signals \( t_i, t_j \) of \( g_k \) with \( i < j < j_k \), the Ando-signal \( t_{j+1} = \theta^{-1}(t_j) \) of \( h(t) \) is an input of \( g_k \). Furthermore, for any two Ando-signals \( t_i \neq t_j \) of \( g_k \) with \( i < j < j_k \), the Or-signal \( t_{i+1} = \theta(t_i) \) of \( h(t) \) is an input of \( g_k \). Hence, the inputs of \( g_k \) (except for \( t_j \)) are alternately Ando- and Or-signals and \( g_k \) is an And-Or path.
Let $m_1$ and $m_2$ be the numbers of inputs of $B_1$ and $B_2$, respectively. As

$$\{t_0, \ldots, t_{m-1}\} = S^0 \cup D^0 = S^0_1 \cup S^0_2 \cup D^0_1 \cup D^0_2,$$

we have $m_1 + m_2 = m$. Choose $i \in \{1, 2\}$ such that $m_i$ is maximum. Then, we have $m_i \geq \hat{m}$. As $B_i$ is an AND-Or path circuit on at least $\hat{m}$ inputs with depth at most $d$, the corollary is proven.

Note that this corollary is much weaker than our structure theorem (Theorem 3.4): We only use that the delay of the two sub-circuits of $C$ is at least the delay of two specific AND-Or paths, not the concrete structure of $C$. Hence, instead of deriving Corollary 3.6 from our structure theorem, we could also have generalized Lemma 1 from Commentz-Walter [4] – which is proven only for AND-Or paths with an even number of inputs – to AND-Or paths with an arbitrary number of inputs. From this generalized result, the corollary also follows.

For the special case when all input arrival times are equal, we conjecture that partitions of the same-gate inputs into two “non-overlapping” sets are always best for the delay.

**Conjecture 3.7.** Consider Theorem 3.4 for the case of depth optimization and let $S^0 = S^0_1 \cup S^0_2$ be a partition as in the theorem. Then, there is a $k \in \{1, 2\}$ such that for all inputs $t_i \in S^0_k$ and $t_j \in S^0_{3-k}$, we have $l < j$.

Note that in Figure 3(b) (Page 7), Conjecture 3.7 is fulfilled. For instance, for the outermost partition, we have $S^{\text{AND}} = \{t_0, t_2, t_4\} \cup \{t_6, t_8, t_{10}, t_{12}, t_{14}\}$. As mentioned in Section 1.1, we conjecture that the polynomial-time delay-optimization algorithm for AND-Or paths by Brenner and Hermann [2] is actually an exact algorithm for the special case of depth optimization, and in this algorithm, precisely those partitions described in Conjecture 3.7 are considered.

### 4. General Algorithm

The structure theorem from the previous section motivates an exact algorithm for the **Delay Optimization Problem for Generalized AND-Or Paths**: Consider a generalized AND-Or path $h(t; \Gamma)$ with prescribed input arrival times. Assume that we know a delay-optimum formula circuit for all strict sub-paths of $h(t; \Gamma)$. Then, by Theorem 3.4 there are $o \in \{\text{AND-Or}\}$ and a partition $S^0 = S^0_1 \cup S^0_2$ such that a delay-optimum circuit $C$ for $h(t; \Gamma)$ can be obtained from delay-optimum circuits $C_1$ for $h(t; \Gamma)S^0_1$ and $C_2$ for $h(t; \Gamma)S^0_2$ via $C = C_1 \circ C_2$. [Algorithm 4.1] describes the arising recursive algorithm for computing the optimum delay; by backtracking, an optimum formula circuit circuit can be computed. The sub-paths of $h(t; \Gamma)$ arising during the algorithm are identified with non-empty subsets $I$ of $\{t_0, \ldots, t_{m-1}\}$ via an injective map $\kappa$ which maps each sub-path to its essential inputs. It is not hard to see that $\kappa$ is actually a bijection. [Algorithm 4.1] recursively applies Theorem 3.4 and stores the computed delays $d(I)$ for non-empty subsets $I \subseteq \{t_0, \ldots, t_{m-1}\}$ in a dynamic-programming table of size at most $2^m - 1$.

In the following theorem, we estimate the running time of [Algorithm 4.1].

**Theorem 4.1.** Let inputs $t = (t_0, \ldots, t_{m-1})$ with arrival times $a(t_0), \ldots, a(t_{m-1}) \in \mathbb{N}$ and gate types $\Gamma = (\varphi_0, \ldots, \varphi_{m-2})$ be given. Then, [Algorithm 4.1] computes the optimum delay of any circuit realizing the generalized AND-Or path $h(t; \Gamma)$. The dynamic-programming table needed to store the delay of all sub-paths considered during the computation has exactly $2^m - 1$ entries. Denoting by $a$ and $o$ the number of AND-signals and OR-signals among $t_0, \ldots, t_{m-2}$, the algorithm can be implemented to run in time $O(3^a 2^o + 2^a 3^o)$. In particular, if $h(t; \Gamma)$ is an AND-Or path, then the running time is $O\left(\left(\sqrt{6}\right)^m\right)$. By backtracking, we can obtain a delay-optimum formula circuit for $h(t; \Gamma)$.

**Proof.** We have already argued that a sub-path of $h(t; \Gamma)$ arising from recursive application of Theorem 3.4 can be identified with the set of its essential inputs via a bijection $\kappa$. Hence, by induction on $m$ and Theorem 3.4 it is easy to see that [Algorithm 4.1] computes the optimum delay of any formula circuit – and thus of any circuit – for $h(t; \Gamma)$.

We assume a random-access machine model with unit costs that allows us to perform basic operations on integers of arbitrary size in constant time. This way, we can represent a set of inputs $I$ by an integer and access its table entry $d(I)$ in constant time.

First, we show that the algorithm can be implemented such that each execution of [Line 15] takes constant time: Before starting to enumerate partitions in [Line 13], for each input $t_i$, we precompute the set $D^i_0 \subseteq D^i$ of diff-gate
inputs of \( h((t_0, \ldots, t_{m-1}); \Gamma) \) with \( j < i \). Then, in Line 13 the set \( i_k \) can be computed as \( i_k = S_k^O \cup D_k^a \), where \( q := \max \{ i | t_i \in S_k^O \} \). Here, we assume that \( i_q \) is known by keeping track of the largest index contained in each occurring same-gate set while enumerating partitions. The precomputation only takes polynomial time and is dominated by the following exponential-time partition enumeration.

Now, let \( T := \{ t_0, \ldots, t_{m-1} \} \). By the above, the running time of Algorithm 4.1 is dominated by enumerating all partitions of the respective set \( S^O \) in Line 13 for each \( o \in \{ \text{And}, \text{Or} \} \) and for all subsets \( \emptyset \neq I \subseteq T \). A partition of \( S^O \) into \( 2 \) non-empty subsets corresponds to choosing a subset \( S^O_{1} \subseteq S^O \setminus \{ t_{i-1} \} \) and setting \( S^O_{2} := S^O \setminus S^O_{1} \). By Definition 3.1, the special sub-paths \( h(t; \Gamma)_{S^O_{1}} \) and \( h(t; \Gamma)_{S^O_{2}} \) are uniquely determined by \( I, S^O_{1} \) and \( S^O_{2} \).

Hence, it remains to bound the number of sets \( S^O_{1} \subseteq S^O \subseteq I \) considered during the algorithm. For fixed \( I, S^O_{1} \), and \( S^O_{2} \), the following holds: An And-signal of \( h(t; \Gamma) \) may be in \( I \) or in \( T \setminus I \). Each Or-signal of \( h(t; \Gamma) \) has three options: it is contained in \( S^O_{1} \), in \( S^O \setminus S^O_{1} \) or in \( \{ t_0, \ldots, t_{m-1} \} \) \( \setminus S^O \). By convention, \( t_{m-1} \) is either contained in \( S^O \setminus S^O_{1} \) or in \( \{ t_0, \ldots, t_{m-1} \} \) \( \setminus S^O \). Hence, there are at most \( 2 \cdot 3^2 2^e \) partitions for the case that the split gate is an Or.

Similarly, when \( o = \text{And} \), we have \( 2 \cdot 3^2 2^e \) partitions. Summing up yields the running time bound.

If \( h(t; \Gamma) \) is an And-Or path, we have \( a, o \in \left[ \frac{\log \min \{ m-1, \frac{\log m}{2} \}}{2} \right] \), so the running time follows directly. \( \square \)

Note that the formula circuit constructed by our algorithm is strongly delay-optimum. In our implementation of the algorithm, as an option, we can compute a size-optimum circuit among all strongly delay-optimum formula circuits by storing both delay and size for each sub-path in Line 17 and updating them accordingly. The algorithm could also be adapted to compute a size-optimum circuit among all delay-optimum formula circuits by storing a set of non-dominated candidate circuits regarding delay and size for each sub-path. As we see in Table 3 Page 21, already our restricted size optimization increases the running time a lot, so we did not implement this extension.
5. Improved Algorithm for Depth Optimization of And-Or Paths

In this section, we speed up Algorithm 4.1 for the special case of depth optimization of And-Or paths. For this, we partition all sub-paths considered during the algorithm into so-called sp-equivalence classes, where two sub-paths with segment partitions $P_0 + \ldots + P_c$ and $P'_0 + \ldots + P'_c$ are considered as sp-equivalent if and only if $c = c'$ and $|P_b| = |P'_b|$ for all $b \in \{0, \ldots, c\}$. Then, up to renaming of the input variables, any two sp-equivalent And-Or paths are either logically equivalent or dual to each other, i.e., the delays of optimum circuits for them coincide. Thus, for each sp-equivalence class, it suffices to compute the optimum delay only for one sub-path. Recall that we identify each sub-path with its essential inputs. Whenever we compute an optimum circuit for inputs $t'$ during the algorithm, we instead compute an optimum solution for its sp-representative $t$. We define $t$ by mapping each input of $t'$ to a certain input in $\{t_0, \ldots, t_{m-1}\}$.

Assume that $t' = (t_0, \ldots, t_k)$ with $0 \leq i_0 < \ldots < i_r \leq m - 1$. We always map $t_{i_r}$ to $t_0$. For $0 \leq j < r - 1$, assuming that input $t_{i_j}$ is mapped to $t_j$, the input $t_{i_{j+1}}$ is mapped to $t_{i_{j+1}}$ if the gate types of $t_j$ and $t_{i_{j+1}}$ are different, and to $t_{i_r}$ otherwise. As always, the last input $t_k$ has no specified gate type and is mapped to the next free input. For instance, given an And-Or path on inputs $t = (t_0, \ldots, t_{10})$ with gate types $\Gamma =\{\land, \lor, \ldots, \land\}$, the sub-path on inputs $t' = (t_2, t_5, t_6, t_9, t_{10})$ is mapped to its sp-representative $t := (t_0, t_1, t_2, t_3, t_6)$.

Furthermore, during partitioning, we avoid generating redundant partitions: For instance, consider again the sp-representative $t$ from above, and let $P_0 + \ldots + P_c$ be its segment partition. Note that $c = 3$ with $P_0 = (t_0), P_1 = (t_1), P_2 = (t_2, t_4), P_3 = (t_5, t_6)$. We have $S^{\text{And}} = \{4_0, 4_2, 4_3, 4_4\}$, $S^{\text{Or}} = \{4_0, 4_2\} \cup \{4_3, 4_4\}$, $S^{\text{And}} = \{4_0, 4_2\} \cup \{4_3, 4_4\}$ lead to the sub-paths $f_1$ on $(t_0, t_1, t_2)$ and $f_2$ on $(t_1, t_3, t_4, t_5)$, $g_1$ on $(t_0, t_1, t_2)$ and $g_2$ on $(t_1, t_2, t_3, t_4)$, respectively. For both $f_1$ and $g_1$, the sp-representative is the sub-path on $(t_0, t_1, t_2)$, and for both $f_2$ and $g_2$, it is the sub-path on $(t_0, t_1, t_2, t_4)$. More generally, two partitions $S^l = S^l_1 \cup S^l_2$ and $S^r = S^r_1 \cup S^r_2$ lead to the same sp-representatives if $(S^l_1 \cup P_0) = (S^r_1 \cup P_0)$ for every $b \in \{0, \ldots, c\}$. Hence, it suffices to consider those partitions $S^l = S^l_1 \cup S^l_2$ for which $S^l_0 \cup P_0$ is a prefix of $P_0$ for all $b \in \{0, \ldots, c\}$. (In the example, this is the partition $S^{\text{And}} = \{4_0, 4_2\} \cup \{4_3, 4_4\}$.) We call such partitions sp-conform.

The procedure to restrict all computations to sp-representatives and sp-conform partitions is called depth normalization. We will see that running Algorithm 4.1 with depth normalization leads to much better theoretical and practical running times. As a first step, we now estimate the number of sp-representatives.

Theorem 5.1. Let $m \geq 1$ and an And-Or path $h(t)$ on inputs $t = (t_0, \ldots, t_{m-1})$ with arrival times $a \equiv 0$ be given. Then, the number of sp-representatives is exactly $F_{m+1} \in O(\varphi^m)$, where $F_{m+1}$ is the $(m+1)$-th Fibonacci number and $\varphi := \frac{\sqrt{5} + 1}{2} \approx 1.618$ is the golden ratio.

Proof. We show that the indicator vectors $(x_0, \ldots, x_{m-1})$ of sp-representatives are exactly the 0-1 strings of length $m$ with the following properties:

1. We have $x_0 = 1$.
2. Whenever for some $i \in \{2, \ldots, m - 1\}$, we have $x_i = 1$, then $x_{i-1} = 0$ or $x_{i-2} = 0$.
3. Choose $i \in \{0, \ldots, m - 1\}$ maximum with $x_i = 1$. Then, we have $i = 0$ or $x_{i-1} = 1$.

For any sp-representative, all conditions are fulfilled: The first property is valid as all sp-representatives have at least 1 input and the first input is always mapped to $t_0$. The second property holds as otherwise, $t_{i-2}$ could have been chosen instead of $t_i$ during normalization. The last property holds as each sp-representative has at least one entry and as the last input is always mapped to the next free position, ignoring gate types.

For the other direction, it is easy to see that for any input vector $t'$ whose indicator vector satisfies the conditions above, the sp-representative is again $t'$.

Denote the set of 0-1 strings of length $m$ with the properties above by $G_m$. By induction, we will show that $|G_m| = F_{m+1}$. We have $|G_1| = |\{(1)\}| = 1 = F_2$ and $|G_2| = |\{(1,0),(1,1)\}| = 2 = F_3$.

Now, consider $m \geq 3$. We construct $G_m$ by adding prefixes to elements of $G_{m-1}$ and $G_{m-2}$. By prepending (1) to any element of $G_{m-1}$, we obtain all elements of $G_m$ that start with (1, 1). By prepending (1, 0) to elements of $G_{m-2}$, we obtain all elements of $G_m$ starting with (1, 0, 1) and, additionally, the invalid element (1, 0, 1, 0, 0, ..., 0) which violates...
the third rule. Moreover, so far, we miss all elements of $G_{m}$ starting with $(1, 0, 0)$. But by the second condition, the only such valid element is $(1, 0, 0, \ldots, 0)$. Together with the induction hypothesis, we obtain

$$|G_{m}| = |G_{m-1}| + (|G_{m-2}| - 1) + 1 = F_{m} + F_{m-1} = F_{m+1}.$$  

Hence, using depth normalization, we reduce the number of sub-paths for which an optimum circuit is computed during Algorithm 4.1 from $2^{m} - 1$ to $F_{m+1} \in O(1.619^{m})$. Using similar, but more involved techniques, we will estimate the running time of Algorithm 4.1 with depth normalization in Theorem 5.6. For this, we will prove that the $sp$-conform partitions considered in the algorithm essentially correspond to elements of the set $Q_{n}$ defined as follows (with $n \approx \frac{3}{2}$).

**Definition 5.2.** Let $Q_{1} := \{ (0, 1), (1, 0), (1, 1), (2, 0), (2, 1) \}$, and for $n \in \mathbb{N}_{\geq 2}$, let $Q_{n}$ be the set of 0-1-2 strings $(x_{0}, \ldots, x_{2n-1})$ of length $2n$ such that for all $i \in \{0, \ldots, n-1\}$, the following conditions are fulfilled:

1. We have $(x_{2i}, x_{2i+1}) \in Q_{1}.$
2. If $i > 0$, the entries $(x_{2i-2}, x_{2i-1}, x_{2i}, x_{2i+1})$ satisfy the extension rules from Table 1.

For instance, we have $(2, 1, 1, 0) \in Q_{2}$ and $(1, 0, 2, 0) \notin Q_{2}$. Note that the rules imply that there are no consecutive zeroes, and that for $x = (x_{0}, \ldots, x_{2n-1}) \in Q_{n-1}$ and $y = (y_{0}, y_{1}) \in Q_{1}$, we have $x \star y \in Q_{n}$ if and only if $(x_{2n-2}, x_{2n-1}, y_{0}, y_{1})$ fulfill the extension rules.

**Lemma 5.3.** Let $n \in \mathbb{N}_{\geq 1}$, Then, we have $|Q_{n}| = O(\beta_{1}^{n})$, where

$$\beta_{1} = \frac{1}{3} \left( 5 + \sqrt{\frac{1}{2} \left( 97 - 3 \sqrt{69} \right)} + \sqrt{\frac{1}{2} \left( 97 + 3 \sqrt{69} \right)} \right) < 4.08$$

is the unique real root of the polynomial $\pi(x) := x^{3} - 5x^{2} + 4x - 1$.

**Proof.** We will show that for $n \geq 4$, we have

$$|Q_{n}| = 5|Q_{n-1}| - 4|Q_{n-2}| + |Q_{n-3}|.$$  

(3)

From this, the statement can be deduced as follows: The characteristic polynomial $\pi(x)$ of $|Q_{n}|$ has three distinct roots $\beta_{1}, \beta_{2}, \beta_{3}$ with $\beta_{1} \approx 4.08, \beta_{2} \approx 0.46 - 0.18i, \beta_{3} \approx 0.46 + 0.18i$. Using a well-known statement about linear recurrence relations (see, e.g., Theorem 6.7.8 in Conradie and Goranko [3]), it follows that there are coefficients $\lambda_{1}, \lambda_{2}, \lambda_{3} \in \mathbb{C}$ such that for all $n \in \mathbb{N}_{\geq 1}$, we have $|Q_{n}| = \lambda_{1}\beta_{1}^{n} + \lambda_{2}\beta_{2}^{n} + \lambda_{3}\beta_{3}^{n}$. Since $|\beta_{2}| = |\beta_{3}| < \beta_{1}$, we obtain $|Q_{n}| \in O(\beta_{1}^{n})$.

Now, we show Equation (3). The idea is to construct $Q_{n}$ from $Q_{n-1}, Q_{n-2}$, and $Q_{n-3}$ by adding suffixes. Let $F := \{(1, 0, 0, 1), (2, 0, 0, 1), (1, 0, 2, 0), (1, 0, 2, 1)\}$. The extension rules imply $Q_{n} = (Q_{n-1} \times Q_{1}) \setminus (Q_{n-2} \times F)$. Apart from elements of $Q_{n-1} \times Q_{1}$, the set $Q_{n-2} \times F$ also contains the set $Q_{n-3} \times \{(1, 0, 2, 0, 0, 1)\}$, which is not contained in $Q_{n-1} \times Q_{1}$ as the extension rules for $Q_{n-1}$ do not allow $(1, 0, 2, 0)$. Since the extension rules allow any element before $(1, 0)$, we obtain

$$|Q_{n}| = |Q_{n-1} \times Q_{1}| - \left| Q_{n-2} \times F \right| - \left| Q_{n-3} \times \{(1, 0, 2, 0, 0, 1)\} \right|.$$  

This implies Equation (3) and thus the lemma.

In order to use the sets $Q_{n}$ for estimating our running time, we need the following intermediate construction.

**Definition 5.4.** For $n \in \mathbb{N}_{\geq 1}$, let $R_{n} := \bigcup_{i=0}^{n-1} (a \star \beta_{2}^{(2^{n-i})}) : a \in Q_{1}$ be the set of strings of length $2n$ arising from an element of any $Q_{i}$ with $i \in \{1, \ldots, n\}$ by appending $2(n-i)$ zeroes.

---

| $x_{2i}, x_{2i+1}$ | $(0, 1)$ | $(1, 0)$ | $(1, 1)$ | $(2, 0)$ | $(2, 1)$ |
|-------------------|---------|---------|---------|---------|---------|
| Allowed values for $(x_{2i-2}, x_{2i-1})$ | all but $(1, 0)$, $(2, 0)$ | all | all | all but $(1, 0)$ | all but $(1, 0)$ |

Table 1: Extension rules for $Q_{n}$. 

---
Observation 5.5. Note that by Definition 5.4 and Lemma 5.7, we have
\[ |R_n| = \sum_{i=1}^{n} |Q_i| \in O\left( \sum_{i=1}^{n} \beta_i \right) = O(\beta_1^5) \].

Theorem 5.6. Let an AND-OR path \( h(t) \) on \( m \geq 1 \) inputs \( t = (t_0, \ldots, t_{m-1}) \) with arrival times \( a \equiv 0 \) be given. Assume that Algorithm 4.1 with depth normalization is applied to compute a depth-optimum circuit for \( h(t) \). Then, the resulting running time is at most \( O(\alpha^m) \), where
\[ \alpha := \sqrt{\beta_1} = \sqrt{\frac{1}{3} + \sqrt{\frac{1}{2} \left( 97 - 3 \sqrt{69} \right) + \frac{1}{2} \left( 97 + 3 \sqrt{69} \right)}} \]
and \( \beta_1 \) is defined as in Lemma 5.3.

Proof. We again use a random-access machine model with unit costs.

We assume that for each sub-path, the \( sp \)-representative can be determined in constant time using a precomputed look-up table. The table can be computed in time \( O(m \cdot 2^m) \), which is dominated by the claimed running time. Thus, as in the proof of Theorem 4.1, it suffices to bound the number of \( sp \)-conform partitions of \( S^o \) considered by the algorithm for \( sp \)-representatives. As in Theorem 5.1, we encode this situation in strings with certain properties and then estimate the number of these strings.

Consider an \( sp \)-representative \( h'(t' ; \Gamma') \) with input set \( I = \{i_0, \ldots, i_{m-1}\} \) such that \( 0 \leq i_0 < \ldots < i_{m-1} \leq m-1 \). Let \( I = P_0 + \ldots + P_c \) be the segment partition of \( h'(t' ; \Gamma') \), and consider an \( sp \)-conform partition \( S^o = S^o_1 \cup S^o_2 \) of its same-gate inputs \( S^o \subseteq I \). By definition of \( sp \)-conformity, \( S^o_1 \cap P_b \) is a prefix of \( P_b \) for all \( b \in \{0, \ldots, c\} \). Let \( x_{t} \) denote the 0-1-2 string arising from \( x \) by mapping each input that is not contained in \( I \) to 0, each input \( i \in S^o_1 \) to 2 and each input \( i \) in \( S^o_2 \) + 1, and each other input of \( I \) to 1. Note that this is the same proof idea as in Theorem 5.1, where there were 3 possible states for the \( Ox \)-signals and 2 possible states for the \( Axn \)-signals. We define
\[ x' := \begin{cases} x\ast(0) & \text{if } t_0 \in S^o_1, \\ (0)\ast x & \text{otherwise}, \end{cases} \quad x'' := \begin{cases} x' & \text{if } m \text{ odd}, \\ x\ast(0) & \text{otherwise}. \end{cases} \]

Now, \( x'' \) has \( 2n \) entries, where \( n := \left\lceil \frac{m+1}{3} \right\rceil \). We will show that \( x'' \in R_n \). From this, the result follows: The mapping \( (I, S^o, S^o_1, S^o_2) \mapsto x'' \) is clearly injective. Hence, \( |R_n| \) is an upper bound on the number of partitions considered. We obtain a total running time of
\[ O(|R_n|) \leq O(\beta_1^5) = O(\beta_1^{m/2}) = O(\alpha^m) \]
Thus, it suffices to prove the following claim.

Claim. We have \( x'' \in R_n \).

Proof of claim: All elements of \( S^o \setminus \{i_0, \ldots, i_{m-1}\} \) correspond to even entries of \( x'' \). As \( x \) arises from an \( sp \)-representative, \( x \) does not contain two consecutive zeroes except for trailing zeroes. As \( t_0 \in I \) by normalization and hence \( x_0 \neq 0 \), the same holds for \( x' \) and \( x'' \).

Now, it remains to show that for any \( i \in \{1, \ldots, n-1\} \) with \( x_{2i} \neq 0 \) or \( x_{2i+1} \neq 0 \), the extension rules are fulfilled for \( (x_{2i-2}, x_{2i-1}, x_{2i}, x_{2i+1}) \). The case that \( (x_{2i-2}, x_{2i-1}, x_{2i}, x_{2i+1}) = (x_{2i-2}, 0, 0, 1) \) with \( x_{2i-2} \neq 0 \) is already excluded as there must not be consecutive zeroes before a 1. The case \( (x_{2i-2}, x_{2i-1}, x_{2i}, x_{2i+1}) = (1, 0, 2, x_{2i+1}) \) with \( x_{2i+1} \) arbitrary cannot occur as here, the entries in \( t' \) corresponding to \( x_{2i-2} \) and \( x_{2i} \) are in the same input segment \( P'_b \) for some \( b \in \{0, \ldots, c\} \), hence, by normalization, we have \( x_{2i-2} \geq x_{2i} \). All other configurations are permitted.

Hence, we have \( x'' \in R_n \).

This proves the theorem. \( \square \)

Apparently, the sequence \( \{Q_i\}_{i \in \mathbb{N}} \) is given by sequence A012814 in the OEIS [29], which consists of every 5th entry of the Padovan sequence, see sequence A000931 in the OEIS. The growth rate of the Padovan sequence is given by \( \rho := \sqrt[5]{3} \) which is also known as the plastic number. Hence, the running time of our algorithm with depth normalization can also be expressed as \( O\left( \rho^{5/2} \right)^m \).
6. Practical Implementation

We implemented Algorithm 4.1 in a C++ program, using 64-bit bit sets to encode the sub-paths via the bijection \( \kappa \) to subsets of \( \{i_0, \ldots, i_{m-1}\} \). In order to obtain good practical running times, we implemented several speedup techniques. On most instances, these in particular imply that we compute the delay for only a fraction of the sub-paths from our dynamic-programming table, see also Table 4. Hence, we store the table in a hash set, which violates the worst-case running time guarantee of Algorithm 4.1 but is much faster in practice and, more important, much less memory-consuming.

For describing our speedup techniques, assume that we apply Algorithm 4.1 to a generalized AND-Or path \( h(t; \Gamma) \) with \( m \) inputs and arrival times \( a(t_0), \ldots, a(t_{m-1}) \). Moreover, when the procedure compute_opt is applied to a subset \( I \subseteq \{i_0, \ldots, i_{m-1}\} \) with \( I = \{i_0, \ldots, i_{r-1}\} \) for \( 0 \leq i_0 < \ldots < i_{r-1} < m-1 \), we denote the corresponding sub-path by \( h(t' \Gamma') \) and its segment partition by \( P_0 + \ldots + P_r \).

When we apply our algorithm for depth optimization, we use the depth normalization as in Theorem 5.1. Most of the other speedups techniques are based on lower bounds and upper bounds on the delay. For a sub-path \( h(t'; \Gamma') \), we maintain not only the best delay of a circuit for \( h(t'; \Gamma') \) computed so far (and, in size-optimization mode, the best possible size for the best possible delay), but also a lower bound on its delay. Furthermore, when calling the procedure compute_opt, we assume that we are given an additional parameter \( D \) and are supposed to find a circuit with best delay among all solutions with delay at most \( D \). When applying compute_opt recursively to the sub-functions considered in partitioning, we hence may use an upper bound of \( D - 1 \).

Now, it is possible that we do not find a solution when applying the procedure compute_opt. In this case, we may update the lower bound to \( D + 1 \). On the other hand, if during partitioning, we find a solution with delay \( d \leq D \), in case of the non-size-optimization mode, we are only interested in another solution if it has delay strictly smaller than \( d \), and in case of the size-optimization mode, if it has delay at most \( d \). Hence, we may update the upper bound \( D \) to \( d - 1 \) or \( d \) in the respective mode for the remaining partitions to be considered. Note that if we allowed fractional arrival times, we would not be able to subtract 1 here.

We shall see later how we set \( D \) for the outermost call of the algorithm (cf. delay probing). We call the mechanism that handles upper bounds during the algorithm upper bound propagation. Having very good lower and upper bounds has a high impact on the running time, so we carefully use any information available to update our bounds.

Assume now that we apply compute_opt to compute a table entry, i.e., to find an optimum circuit for the sub-path \( h(t'; \Gamma') \) with input set \( I \) with delay at most \( D \). Before starting our partitioning process (see Section 6.2), we compute several lower bounds as in Section 6.1. If any of these is larger than \( D \), then we know that there is no circuit with delay at most \( D \) for \( h(t'; \Gamma') \) and need not start the partitioning process.

6.1. Lower Bounds

A basic lower bound that can be computed quickly for any generalized AND-Or path \( h(t'; \Gamma') \) arises from the lower bounds in Theorem 2.5 and Proposition 2.11, i.e.,

\[
\max \left\{ \log_2 \sum_{j=0}^{i_{r-1} - 1} 2^{a(t_j)}, \max_{i_j \in P_0} \left( \max_{i_j \in P_1} a(t_j) + 1, \max_{i_j \in P_2} a(t_j) + 2 \right) \right\}.
\]

Note that the first lower bound requires the arrival times to be natural numbers.

We use two other strong lower bounds that each consider a specific restricted sub-path \( h(t''; \Gamma'') \) of \( h(t'; \Gamma') \) with similar structural complexity. For \( h(t''; \Gamma'') \), we recursively apply the algorithm with depth bound \( D \). Either there is no solution, in which case \( D + 1 \) is a lower bound on the optimum delay for \( h(t''; \Gamma'') \), thus also for \( h(t'; \Gamma'); \) otherwise, we know the optimum delay for \( h(t''; \Gamma'') \), which is a lower bound for \( h(t'; \Gamma') \). This usually yields a strong lower bound, but is very time-consuming.

First, only in the special case of depth optimization, we consider the sub-path \( h(t''; \Gamma'') \) arising from \( h(t'; \Gamma') \) by keeping only the largest input segment in the segment partition completely and condensing each other input segment to a single input (except for the last segment, which keeps 2 inputs). In the case of depth optimization, only the input-segment sizes matter, so there are only \( O(m^3) \) of these sub-paths, and it is not harmful to solve them optimally.

Secondly, also in the case of delay optimization, we consider a restricted sub-path \( h(t''; \Gamma'') \) that arises from removing a single input of \( h(t'; \Gamma') \) in a way that hopefully the optimum delay of any circuit for \( h(t''; \Gamma'') \) is the
same as for \( h(t'; \Gamma') \). Hence, among all inputs with the minimum arrival time, we remove an input of the largest input segment. Empirically, we see that in the case of depth optimization, this lower bound is tight in 97% of its applications. This matches the observation that if we iteratively apply this lower bound \( m \) times, starting with a generalized And-Or path with optimum depth \( d \), the optimum depth changes only \( d \) times, where \( d \ll m \).

### 6.2. Partitioning the Same-Gate Inputs

For determining a solution with delay \( D \) for a sub-path \( h(t'; \Gamma') \) if it exists — we enumerate partitions \( S^\circ \) of its same-gate input set \( S \) for all \( \sigma \in \{ \text{And}, \text{Or} \} \) in Line 13 of Algorithm 4.1. In our implementation, we first choose \( \sigma := \sigma_0 \) because empirically, this more often yields a good circuit, and afterwards the other gate type. For both, we enumerate partitions of \( S^\circ \) and recursively try to find a solution with delay at most \( D \).

We avoid generating too many partitions of a set \( S^\circ \) by enumerating partitions in a specific order and skipping certain partitions that provably do not lead to a better solution. In a recursive approach, one by one, we assign the inputs to \( S^\circ_1 \) or to \( S^\circ_2 \). Here, just as in standard branch-and-bound algorithms, we follow the idea to make the most important decisions first. Recall from the proof of Theorem 4.1 that by convention, the last input \( t_{n-1} \) is always contained in \( S^\circ_2 \).

Now, we first enumerate the highest input index \( i_l \) for which input \( t_l \) is assigned to the other part, \( S^\circ_1 \). Once \( t_l \) is fixed, we have completely determined which of the diff-gate inputs are contained in both \( h(t'; \Gamma')_{S^\circ_1} \) and \( h(t'; \Gamma')_{S^\circ_2} \), or only in \( h(t'; \Gamma')_{S^\circ_2} \). Based on this, we compute another lower bound, the cross-partition lower bound, by applying Theorem 2.5 to all inputs of \( h(t'; \Gamma') \), where those inputs that are contained in both sub-functions are counted twice, and may stop when this lower bound exceeds \( D \).

As \( t_l \) is the input with the highest index in \( S^\circ_1 \), we already know that all inputs \( t_i \in S^\circ \) with \( i > i_l \) must be in \( S^\circ_2 \). It remains to enumerate those \( t_i \in S^\circ \) with \( i < i_l \). They are assigned to the sets \( S^\circ_1 \) and \( S^\circ_2 \) recursively, in the order of decreasing arrival time, and in case of ties, inputs with larger indices are considered first. For each input, we first assign it to \( S^\circ_2 \) and recursively continue with the other inputs; and then assign it to \( S^\circ_1 \) and go into recursion. This way, we in particular prioritize the construction of consecutive sets \( S^\circ_1 \) and \( S^\circ_2 \), which often allows finding an optimum solution quickly (cf. Table 1).

Now, assume that we try to compute a circuit for \( h(t'; \Gamma') \) with delay at most \( D \) via a fixed partition \( S^\circ = S^\circ_1 \cup S^\circ_2 \). Before computing a solution, we evaluate all lower bounds available for the two sub-instances, and stop if any of the lower bounds exceeds \( D \). Otherwise, we recursively compute the table entries of \( h(t'; \Gamma')_{S^\circ_1} \) and \( h(t'; \Gamma')_{S^\circ_2} \) with delay bound \( D - 1 \). As already mentioned, based on whether we did find a solution or not, we may update the lower bound for \( h(t'; \Gamma') \).

Note that the lower bound \( L \) on the best delay achievable for \( h(t'; \Gamma') \) is also a lower bound for all sub-paths on a superset of the inputs \( I \) of \( h(t'; \Gamma') \). Hence, if we have updated \( L \) for \( h(t'; \Gamma') \), in lower bound propagation, we also update the lower bound for certain sub-paths whose inputs are a superset of \( I \). Doing this for all supersets would be too costly; so we only update lower bounds of supersets which are already contained in our dynamic-programming table and arise from adding a single input. For those sets whose lower bounds are improved, we recursively repeat this procedure.

If we did not find a solution with delay at most \( D \) for the current partition, we might discard a part of our enumeration tree in subset enumeration pruning: Consider the inputs of \( S^\circ \) in the order \( t_{j_{k}}, \ldots, t_{j_{1}} \) in which we enumerate whether to assign them to \( S^\circ_1 \) or \( S^\circ_2 \); i.e., when considering input \( t_{j_{k}} \), we have already assigned the inputs \( t_{j_{k}}, \ldots, t_{j_{1}} \) to one of the two subsets. If we add \( t_{j_{k}} \) to \( S^\circ_2 \), the set \( S^\circ_1 \) is minimal among all sets that will arise from enumerating assignments for the elements \( t_{j_{k}}, \ldots, t_{j_{1}} \). The first assignment that will be tried for \( t_{j_{k}}, \ldots, t_{j_{1}} \) is to put them all into \( S^\circ_1 \). Hence, when the computation of a solution for this sub-path with delay at most \( D \) was not successful because the And-Or path \( h(t'; \Gamma')_{S^\circ_1} \) had too large delay, we already know that all other partitions with \( t_{j_{k}}, \ldots, t_{j_{1}} \) unchanged will also not lead to delay at most \( D \). Hence, we can skip this part of our enumeration tree. The same holds when adding \( t_{j_{k}} \) to \( S^\circ_1 \).

Finally, we note that the running time for the computation of a table entry highly depends on \( D \). Hence, when computing a table entry with a lower bound of \( L \), in delay probing, we in fact loop over all possible delays \( d \in \{ L, \ldots, D \} \) with increasing \( d \) and try to find a solution with delay \( d \). The first value \( d \) for which a solution is found is then the optimum delay of any circuit for \( h(t'; \Gamma') \).
7. Computational Results

In Section 7.1 we analyze results for delay optimization of And-Or paths and generalized And-Or paths. Then, in Section 7.2 we consider the Depth Optimization Problem for And-Or Paths. In particular, here we analyze all speedup techniques in detail, including their individual impact on the empirical running time. These speedups allow us to solve all instances of the Depth Optimization Problem for And-Or Paths with up to 64 inputs. For this problem, we also compare our running times with those of the previously best algorithm by Hegerfeld [11] which only allows to solve instances with up to 29 inputs. In Section 7.3 we derive the optimum depths of n-bit adder circuits for n that are a power of two for up to \( n = 8192 \) bits.

All our tests ran on a machine with two Intel(R) Xeon(R) CPU E5-2687W v3 processors, using a single thread.

### 7.1. Delay Optimization of And-Or Paths and Generalized And-Or Paths

In Table 2, we state the average running times of our algorithm on the following testbed: For each number \( m \in \{10, 20, 30, 40\} \) of inputs, we created 100 And-Or path instances with random integral arrival times uniformly distributed among \( \{0, \ldots, m-1\} \) and 100 generalized And-Or path instances where, additionally, the gate types are chosen uniformly among And2 and Or2. We show the respective running times of our algorithm both for the computation of the optimum delay ("No size opt.") and of the optimum delay and optimum size of a strongly delay-optimum formula circuit ("With size opt."). For lines where no running time is shown, the memory limit of 300 GB was attained on at least one instance.

We see that running times are much higher when size optimization is enabled, that is, our speedup techniques are particularly effective when size optimization is not required. For \( m = 20 \) inputs, average running times to solve And-Or path instances are 0.674 seconds with size optimization and only 0.002 seconds without size optimization. Without any speedups, our algorithm takes 13.255 seconds on such instances – cf. scenario 1 in Table 4 (Page 22) –, demonstrating the dramatic impact of our speedup techniques. Not surprisingly, the speedup is even more substantial for larger \( m \).

Finally, we note that the effectiveness of our pruning strategies varies drastically with the arrival time profile, thus also our running times. For instance, for the And-Or path runs with size optimization, the running times on instances with 30 inputs vary from 0.2 seconds up to 5.4 hours. By examining instances with high running times, we could most likely further improve our speedup techniques.

### 7.2. Depth Optimization of And-Or Paths

Now, we consider the Depth Optimization Problem for And-Or Paths. Note that up to duality, for this, there is exactly one instance for a fixed number of inputs. In Table 3, we give a comparison of our algorithm, i.e., Algorithm 4.1 with depth normalization and speedups from Section 6, with the formula enumeration algorithm by Hegerfeld [11].

Hegerfeld’s algorithm finds size-optimum formula circuits among all strongly depth-optimum formula circuits. Note that Hegerfeld erroneously states that the algorithm computes a size-optimum formula circuit among all depth-optimum formula circuits. For instance, for the And-Or path on 14 inputs, Hegerfeld reports a size of 18 (see Table 3 (Page 21)), but in Figure 3(a) (Page 7), we see a depth-optimum formula circuit with size 17. This circuit is not strongly delay-optimum.

| # inputs | And-Or paths [s] | Generalized And-Or paths [s] |
|----------|-----------------|-------------------------------|
|          | With size opt.  | No size opt.                  |
|          | With size opt.  | No size opt.                  |
| 10       | 0.001           | 0.000                         |
| 20       | 0.674           | 0.002                         |
| 30       | 1628.027        | 0.023                         |
| 40       | 12.944          | 12.944                        |

Table 2: Average running times for Algorithm 4.1 with speedups from Section 6. For each number of inputs, we tested 100 instances with randomly chosen integral arrival times and, in case of generalized And-Or paths, random gate types.
| m  | d  | s  | [11] [s] | Algorithm 4.1 [s] |
|----|----|----|---------|------------------|
|    |    |    | With size opt. | With size opt. | No size opt. |
| 5  | 3  | 5  | 0        | 0.000           | 0.000 |
| 6  | 3  | 6  | 0        | 0.000           | 0.000 |
| 7  | 4  | 7  | 0        | 0.000           | 0.000 |
| 8  | 4  | 9  | 0        | 0.000           | 0.000 |
| 9  | 4  | 10 | 0        | 0.000           | 0.000 |
| 10 | 4  | 13 | 0        | 0.000           | 0.000 |
| 11 | 5  | 13 | 0        | 0.001           | 0.000 |
| 12 | 5  | 14 | 0        | 0.002           | 0.000 |
| 13 | 5  | 16 | 0        | 0.004           | 0.000 |
| 14 | 5  | 18 | 0        | 0.005           | 0.000 |
| 15 | 5  | 20 | 1        | 0.007           | 0.000 |
| 16 | 5  | 21 | 2        | 0.008           | 0.000 |
| 17 | 5  | 24 | 4        | 0.008           | 0.000 |
| 18 | 5  | 25 | 11       | 0.009           | 0.000 |
| 19 | 5  | 29 | 27       | 0.015           | 0.002 |
| 20 | 6  | 27 | 71       | 0.234           | 0.005 |
| 21 | 6  | 28 | 180      | 0.358           | 0.007 |
| 22 | 6  | 31 | 463      | 0.588           | 0.008 |
| 23 | 6  | 32 | 1035     | 0.923           | 0.008 |
| 24 | 6  | 35 | 2893     | 1.259           | 0.007 |
| 25 | 6  | 36 | 7214     | 1.631           | 0.007 |
| 26 | 6  | 38 | 22661    | 2.097           | 0.007 |
| 27 | 6  | 40 | 60598    | 2.401           | 0.007 |
| 28 | 6  | 42 | 480960   | 2.680           | 0.007 |
| 29 | 6  | 44 | 2775000  | 2.763           | 0.007 |
| 30 | 6  | 47 | 2.927    | 0.008           |       |
| 31 | 6  | 49 | 2.991    | 0.008           |       |
| 32 | 6  | 53 | 3.068    | 0.009           |       |
| 33 | 6  | 57 | 3.159    | 0.010           |       |
| 34 | 7  | 51 | 1822     | 0.300           |       |
| 35 | 7  | 53 | 2921     | 0.861           |       |
| 36 | 7  | 55 | 5145     | 0.978           |       |
| 37 | 7  | 57 | 8064     | 0.958           |       |
| 38 | 7  | 59 | 13949    | 0.961           |       |
| 39 | 7  | 61 | 19539    | 0.957           |       |
| 40 | 7  | 63 | 33778    | 0.974           |       |
| 41 | 7  | 65 | 53287    | 0.954           |       |
| 42 | 7  | 67 | 87514    | 0.945           |       |
| 43 | 7  | 70 | 143409   | 0.939           |       |
| 44 | 7  | 73 | 1.285    | 0.941           |       |
| 45 | 7  | 76 | 1.285    | 0.958           |       |
| 46 | 7  | 77 | 1.285    | 0.941           |       |
| 47 | 7  | 83 | 1.285    | 0.941           |       |
| 48 | 7  | 84 | 1.399    | 1.406           |       |
| 49 | 7  | 84 | 1.399    | 1.406           |       |
| 50 | 7  | 85 | 1.404    | 1.406           |       |
| 51 | 7  | 89 | 1.404    | 1.406           |       |
| 52 | 7  | 90 | 1.405    | 1.406           |       |
| 53 | 7  | 93 | 1.405    | 1.406           |       |
| 54 | 7  | 94 | 1.409    | 1.406           |       |
| 55 | 7  | 98 | 1.415    | 1.410           |       |
| 56 | 7  | 99 | 1.415    | 1.410           |       |
| 57 | 7  | 104 | 1.410 |       |       |
| 58 | 7  | 105 | 1.395 |       |       |
| 59 | 7  | 109 | 1.413 |       |       |
| 60 | 7  | 110 | 1.425 |       |       |
| 61 | 8  | 111 | 4574 |       |       |
| 62 | 8  | 113 | 8468 |       |       |
| 63 | 8  | 114 | 9729 |       |       |
| 64 | 8  | 117 | 9037 |       |       |

Table 3: Single-threaded running times of our algorithm (with and without size optimization) and Hegerfeld’s formula enumeration algorithm [11]. Each line also shows the computed optimum depth $d$ of any Asp-Oz path circuit on $m$ inputs and the optimum size $s$ of any strongly-depth optimum formula circuit. Hegerfeld’s running times are taken from [11]; for 28 and 29 inputs, Hegerfeld ran his algorithm in parallel and displayed wall time multiplied by number of threads. Dashed lines separate instances with different $d$. 
| Scenario 1 | Scenario 2 | Scenario 3 | Scenario 4 | Scenario 5 |
|-----------|-----------|-----------|-----------|-----------|
| $m$       | $\log_2 E$ | $\log_2 P$ | $T$ [s]   | $\log_2 E$ | $\log_2 P$ | $T$ [s]   | $\log_2 E$ | $\log_2 P$ | $T$ [s]   |
| 20        | 20        | 27        | 13.4      | 13        | 22        | 0.8      | 10        | 18        | 0.1      | 10        | 15        | 0.0      | 8         | 13        | 0.0      |
| 21        | 21        | 29        | 35.4      | 14        | 23        | 1.6      | 10        | 19        | 0.1      | 10        | 15        | 0.0      | 9         | 13        | 0.0      |
| 22        | 22        | 30        | 94.6      | 15        | 24        | 3.5      | 11        | 19        | 0.1      | 10        | 16        | 0.0      | 9         | 14        | 0.0      |
| 23        | 23        | 31        | 237.9     | 16        | 25        | 6.9      | 11        | 20        | 0.2      | 11        | 16        | 0.0      | 9         | 14        | 0.0      |
| 24        | 24        | 32        | 630.6     | 16        | 26        | 15.4     | 11        | 20        | 0.2      | 11        | 16        | 0.0      | 9         | 14        | 0.0      |
| 25        | 25        | 34        | 1540.4    | 17        | 27        | 32.1     | 11        | 20        | 0.2      | 11        | 16        | 0.0      | 9         | 14        | 0.0      |
| 26        | 26        | 35        | 4055.7    | 18        | 28        | 69.7     | 11        | 20        | 0.2      | 11        | 16        | 0.0      | 9         | 14        | 0.0      |
| 27        | 27        | 36        | 10034.2   | 18        | 29        | 142.4    | 11        | 21        | 0.3      | 11        | 17        | 0.0      | 9         | 14        | 0.0      |
| 28        | 28        | 38        | 25055.1   | 19        | 30        | 315.9    | 11        | 21        | 0.4      | 11        | 17        | 0.0      | 9         | 14        | 0.0      |
| 29        | 20        | 31        | 642.0     | 12        | 22        | 0.8      | 11        | 17        | 0.0      | 9         | 14        | 0.0      |
| 30        | 20        | 32        | 1406.2    | 12        | 23        | 1.4      | 11        | 17        | 0.0      | 9         | 14        | 0.0      |
| 31        | 21        | 33        | 2939.7    | 12        | 24        | 2.6      | 11        | 17        | 0.0      | 9         | 14        | 0.0      |
| 32        | 22        | 34        | 6445.6    | 12        | 25        | 6.8      | 11        | 17        | 0.0      | 9         | 14        | 0.0      |
| 33        | 22        | 35        | 13062.4   | 12        | 26        | 14.3     | 11        | 17        | 0.0      | 9         | 14        | 0.0      |
| 34        | 17        | 31        | 623.0     | 16        | 26        | 14.1     | 11        | 19        | 0.3      |
| 35        | 18        | 33        | 1666.7    | 17        | 27        | 34.2     | 12        | 21        | 0.9      |
| 36        | 19        | 33        | 3066.7    | 18        | 27        | 45.4     | 12        | 21        | 1.0      |
| 37        | 19        | 34        | 6013.4    | 18        | 28        | 60.9     | 12        | 21        | 1.0      |
| 38        | 20        | 35        | 9211.6    | 19        | 28        | 75.8     | 12        | 21        | 1.0      |
| 39        | 20        | 36        | 15861.5   | 19        | 28        | 92.2     | 12        | 21        | 1.0      |
| 40        | 21        | 36        | 22140.3   | 19        | 29        | 109.8    | 12        | 21        | 1.0      |
| 41        | 20        | 36        | 22140.3   | 19        | 29        | 135.7    | 12        | 21        | 1.0      |
| 42        | 20        | 36        | 22140.3   | 19        | 29        | 145.1    | 12        | 21        | 0.9      |
| 59        | 21        | 29        | 224.7     | 12        | 22        | 1.4      |
| 60        | 21        | 29        | 226.6     | 12        | 22        | 1.4      |
| 61        | 18        | 32        | 4574.9    |
| 62        | 18        | 33        | 8468.1    |
| 63        | 18        | 33        | 9729.8    |
| 64        | 18        | 33        | 9037.3    |

Table 4: Comparison of speedup scenarios for computing depth-optimum Aso-On path circuits with $m = 20, \ldots, 64$ inputs. The number of table entries computed is denoted by $E$, the number of partitions computed by $P$, and the running time in seconds by $T$. Dashed lines separate ranges of instances with the same optimum depth.
For our algorithm, Table 3 again shows running times for the computation of the optimum depth and optimum size of a strongly depth-optimum formula circuit (“With size opt.”), and for the computation of the optimum depth only (“No size opt.”). Hegerfeld’s running times are taken from [11]. On any instance solved both by Hegerfeld’s algorithm and our algorithm, the computed optimum depths coincide; and using our size-optimization, we verified that Hegerfeld computes the optimum size of any strongly depth-optimum circuit on each instance.

For up to 14 inputs, Hegerfeld’s algorithm runs less than a second, and the largest solved instance has 29 inputs. Our algorithm with size optimization solves instances with up to 33 inputs within 3.2 seconds; the largest instance we can solve has 43 inputs. When we disregard circuit size and construct any depth-optimum circuit, we can solve any instance with up to 64 inputs within 3 hours; and any instance with up to 60 inputs even in up to 1.5 seconds. Note that the running time increases drastically with increasing depth. As our implementation uses 64-bit bit sets to encode the sub-paths, we currently cannot consider instances with more than 64 inputs. By adjusting the bit sets used, this technicality can be overcome. However, we do not expect to solve an instance with 110 inputs, where the next change in depth is likely, see Table 5.

In order to examine the impact of our speedup techniques, we define 5 scenarios: in scenario 1, we run the basic algorithm without any enhancements; in scenario 5, we enable all speedup techniques from Section 6. The intermediate scenarios all add a selection of speedups to the previous scenario:

- Scenario 1: No speedups.
- Scenario 2: Add depth normalization.
- Scenario 3: Add upper bound propagation, basic lower bound.
- Scenario 4: Add cross-partition lower bound, subset enumeration pruning.
- Scenario 5: Add strong lower bounds, lower bound propagation, delay probing.

For each scenario, we ran the algorithm on all depth optimization instances with at least 20 inputs – we only state results for an instance-scenario pair if the running time is at most 8 hours. For each run, we store the number $E$ of table entries for which the partitioning process has been started and the number $P$ of partitions considered. In Table 4, we show the logarithms of these numbers, rounded to the nearest integer, and the running times.

In general, for fixed $m$, the number of entries and partitions and the running time reduces significantly with increasing scenario number. From scenario 3 on, we can solve the instance with 34 inputs within the running time limit of 8 hours, which is the first instance with an optimum depth of 7. Only when using all pruning techniques in scenario 5, we can solve the instance with 61 inputs. In particular, note that in contrast to scenarios 1 - 4, in scenario 5, the running time does not necessarily increase with increasing $m$. In a range of inputs where the optimum depth does not increase (e.g., from 34 up to 60 inputs), our strong lower bounds have a high impact. Note that, as stated in Theorem 3.1 for each number $m$ of inputs, for scenario 1, we have $E \approx 2^m$, and that the running time increases by a factor of roughly $6^6$ when $m$ increases by 1. For scenario 2, we have checked that – as proven in Theorem 5.1 – the precise number of entries for $m$ inputs is exactly the Fibonacci number $F_{m+1}$. Note that from $m$ to $m+1$, the running time roughly doubles, matching the running time guarantee of $O(2.02^m)$ shown in Theorem 5.6.

### 7.3. Optimum Depths of Adder Circuits

In Table 3, we see the optimum depth of AND-OR path circuits for up to 64 inputs. From these, we will now deduce the optimum adder depths for all $2^k$-bit adders with $k \leq 13$.

As a first step, for fixed $d \in \mathbb{N}$, we want to determine ranges of inputs $m$ for which we can prove that the optimum depth of any AND-OR path circuit on $m$ inputs is $d$, see Table 5. For this, we use the empirically good results from the heuristic by Grinchuk [8]. There, a table in Section 5 displays the maximum $m$ such that the heuristic computes a circuit with depth $d$ for all $d \in \{0, \ldots, 32\}$. From this table, we obtain the upper bounds on $m$ in Table 5. Comparing with our results in Table 3, we directly see that Grinchuk’s circuits are depth-optimum for up to 109 inputs. But using Corollary 3.6 and our result that an AND-OR path on 61 inputs cannot be realized with depth 8, we deduce that an AND-OR path on 121 inputs has depth at least 9. Together with Corollary 3.6, this implies that an AND-OR path on 241 inputs has depth at least 10, and so on. This yields the lower bounds on $m$ in the right part of Table 3, while the lower bounds in the left part can be obtained by applying Corollary 3.6 to the results computed by Hegerfeld [11].
Using Hegerfeld [11] results
Using Algorithm 4.1 results

| d | \( \leq m \leq \) | \( \leq m \leq \) |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 3 | 3 |
| 3 | 4 | 6 |
| 4 | 7 | 10 |
| 5 | 11 | 19 |
| 6 | 20 | 33 |
| 7 | 39 | 60 |
| 8 | 77 | 109 |
| 9 | 153 | 202 |
| 10 | 305 | 241 |
| 11 | 609 | 481 |
| 12 | 1217 | 961 |
| 13 | 2433 | 1921 |
| 14 | 3841 | 2466 |
| 15 | 7681 | 4645 |
| 16 | 15361 | 8782 |
| 17 | 30721 | 16383 |

\(T_{able \ 5:} Numbers \ m \ of \ inputs \ for \ which \ we \ can \ show \ that \ the \ optimum \ depth \ of \ an \ A_{nd\text{-}O_{r}} \ circuit \ is \ d, \ for \ d \leq 17. \ In \ both \ columns, \ Grinchuk \ [8] \ yields \ the \ upper \ bounds \ on \ m; \ the \ lower \ bounds \ are \ derived \ from \ the \ results \ by \ Hegerfeld \ [11] \ and \ Algorithm \ 4.1, \ respectively, \ using \ our \ theoretical \ statement \ from \ Corollary \ 3.6.\)

| \(n\) | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 | 8192 |
| 2n−1 | 1 | 3 | 7 | 15 | 31 | 63 | 127 | 255 | 511 | 1023 | 2047 | 4095 | 8191 | 16383 |
| \(d\) | 0 | 2 | 4 | 5 | 6 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

\(T_{able \ 6:} Optimum \ depths \ d \ of \ n\)-bit adder circuits over the basis \(\{A_{nd2},O_{r2}\}\), where \(n\) is a power of 2. The middle row shows the number \(2n−1\) of inputs of the \(A_{nd\text{-}O_{r}}\) path computing the most significant carry bit.

Hegerfeld’s case, we apply Corollary 3.6 iteratively with the basic result that an \(A_{nd\text{-}O_{r}}\) path on 20 inputs cannot be realized with depth 5 and thus obtain significantly smaller ranges of \(m\) for which the optimum depth can be computed.

As a second step, recall from Equation (1) that the carry bits \(c_1,\ldots,c_n\) of an \(n\)-bit adder are \(A_{nd\text{-}O_{r}}\) paths on 1, 3, \ldots, \(2n−1\) inputs and that – when circuit size is not regarded – a depth-optimum adder circuit on \(n\) bits can be computed via depth-optimum \(A_{nd\text{-}O_{r}}\) path circuits computing each carry bit. Hence, in particular, Table 5 yields the optimum depths of all adder circuits with \(2^k\) inputs for \(k \leq 13\) over the basis \(\{A_{nd2},O_{r2}\}\). We show these in Table 6. Note that when the optimum depths computed by Hegerfeld were used instead of the results computed by our algorithm, the optimum adder depths could only be computed up to \(k = 4\).

Conclusions

We presented a new exact algorithm for constructing depth- and delay-optimum \(A_{nd\text{-}O_{r}}\) path and adder circuits over the basis \(\{A_{nd2},O_{r2}\}\). Our algorithm is much faster than previous approaches – both empirically and regarding provable worst-case running time – and hence can solve significantly larger instances. For all \(A_{nd\text{-}O_{r}}\) path instances with up to 64 inputs, the optimum depth was computed in reasonable time.

Using these empirical computations and new theoretical results, we derived the optimum depths for binary carry-propagate adders on \(2^k\) bits for all \(k \leq 13\), previously known only for \(k \leq 4\). Thus, for any practically relevant number of bits, the problem of constructing depth-optimum adder circuits over the basis \(\{A_{nd2},O_{r2}\}\), which for decades was a subject of research, is now settled. Further research may improve secondary objectives like fan-out and size or consider a trade-off between different objective functions.
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