ENTANGLEMENT BREAKING RANK

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Abstract. We introduce and study the entanglement breaking rank of an entanglement breaking channel. We show that the problem of computing the entanglement breaking rank of the channel

$$Z(X) = \frac{1}{d+1}(X + \text{Tr}(X)\mathbb{I}_d)$$

is equivalent to the existence problem of symmetric informationally-complete POVMs.

1. Introduction

The study of separable states is an important topic in quantum information theory and helps to shed light on the nature of entanglement. Recall that, a state $$\rho \in \mathbb{M}_m \otimes \mathbb{M}_n$$ is called separable if it can be written as a finite convex combination

$$\rho = \sum \lambda_i \sigma_i \otimes \delta_i,$$

where $$\sigma_i$$ and $$\delta_i$$ are pure states. In general, a separable state can have many such representations [16]. It is then natural to introduce the notion of optimal ensemble cardinality [9] or length [5] of a separable state by defining it to be the minimum number $$\ell(\rho)$$ of pure states $$\sigma_i \otimes \delta_i$$ required to write the separable state $$\rho$$ as their convex combination. In [9] it was shown that, in general, the inequality $$\text{rank}(\rho) \leq \ell(\rho)$$ can be strict for certain classes of separable states.

The notion of entanglement breaking maps was introduced and studied in [15,21]. We say that a linear map $$\Phi : \mathbb{M}_d \rightarrow \mathbb{M}_m$$ is entanglement breaking if the tensor product $$\Phi \otimes \mathbb{I}_n$$ maps states of $$\mathbb{M}_d \otimes \mathbb{M}_n$$ to separable states in $$\mathbb{M}_m \otimes \mathbb{M}_n$$, for all $$n \in \mathbb{N}$$. An equivalent criterion of entanglement breaking [15] is that $$\Phi$$ admits a Choi-Kraus representation of the form

$$\Phi(X) = \sum_{k=1}^{K} R_k X R_k^*,$$

where the $$R_k$$’s are rank one matrices in $$\mathbb{M}_{m,d}$$. Since Choi-Kraus representations are not unique, we define the entanglement breaking rank of $$\Phi$$ to be the minimum $$K$$ required in such an expression (1.1), and denote it by ebr($$\Phi$$). Clearly, the entanglement breaking rank of $$\Phi$$ is never less than its Choi-rank.

Another equivalent criterion [15] for a map to be entanglement breaking is that its Choi-matrix $$C_\Phi := [\Phi(E_{i,j})] \in \mathbb{M}_d \otimes \mathbb{M}_m$$ should be separable, where $$\{E_{i,j} : 1 \leq i, j \leq d\}$$ is the set of canonical matrix units of $$\mathbb{M}_d$$. If $$\Phi$$ is entanglement breaking, then it is easy to see that $$\ell(C_\Phi) = \text{ebr}(\Phi)$$ so that studying entanglement breaking
rank is the same as studying length. However, for our purposes it is more natural to study channels instead of states, so we will think in terms of entanglement breaking rank instead of length.

The existence problem of symmetric informationally-complete positive operator-valued measures (SIC POVM) in an arbitrary dimension \( d \) is an open problem and is an active area of research \[3,8,13,19,20\]. This problem has connections to frame theory and the theory of \( t \)-designs \[20\], and has also been verified numerically in finitely many dimensions \[1,12,23\]. This problem is equivalent to Zauner’s (weaker) conjecture \[26, 27\] about equiangular lines in \( \mathbb{C}^d \). Stated in terms of equiangular lines, the problem asks about the existence of \( d^2 \) unit vectors \( \{v_i : 1 \leq i \leq d^2\} \subseteq \mathbb{C}^d \) such that \(|\langle v_i, v_j \rangle|^2 = \frac{1}{d^2}\) for all \( i \neq j \). We refer the reader to \[11\] for a survey on this subject.

It can be shown \[4\] that if such a set of vectors \( \{v_i\}_{i=1}^{d^2} \) exists in a given dimension \( d \), then the rank one projections \( P_i \) onto the span of \( v_i \) satisfy:

\[
\frac{1}{d} \sum_{i=1}^{d^2} P_i XP_i = \frac{1}{d+1} (X + \text{Tr}(X)I_d), \quad X \in \mathbb{M}_d.
\]

Denoting this channel by \( \mathfrak{3} \), it is independently known to be entanglement breaking and its Choi-rank is \( d^2 \) \[14,18\]. Thus, if a SIC POVM exists in dimension \( d \), then the entanglement breaking rank of the channel \( \mathfrak{3} \) is \( d^2 \).

We prove the converse: if \( \text{ebr}(\mathfrak{3}) = d^2 \), then there exists a SIC POVM in dimension \( d \). This is a slight relaxation of the Zauner conditions, since the \( d^2 \) rank one matrices involved in the Choi-Kraus representation need not be positive. This leads to the conclusion (Corollary 3.5) that for \( d \geq 2 \), \( \text{ebr}(\mathfrak{3}) = d^2 \) if and only if a SIC POVM exists in dimension \( d \).

This result leads us to seek, and find, other channels with the property that a statement about their entanglement breaking rank is equivalent to Zauner’s conjecture. In particular, we show that determining if the Werner-Holevo channel \[24, \text{Example 3.36}\],

\[
\Phi_0(X) = \frac{1}{d+1} (X^T + \text{Tr}(X)I_d), \quad X \in \mathbb{M}_d,
\]

where \( X^T \) is the transpose of \( X \), has entanglement breaking rank equal to \( d^2 \) is also equivalent to Zauner’s conjecture. This Werner-Holevo channel also serves as an example of a channel for which the entanglement breaking rank is strictly bigger than its Choi rank.

This approach to Zauner’s conjecture also allows us to prove some perturbative results that would imply Zauner’s conjecture and leads naturally to a conjecture that appears to be stronger than Zauner’s conjecture. We prove that \( \text{ebr} \) is lower semicontinuous, so that if there are channels arbitrarily near to \( \mathfrak{3} \) with \( \text{ebr} \) bounded by \( d^2 \), then Zauner’s conjecture is true.

The channel \( \mathfrak{3} \) is a particular convex combination of the identity channel \( I_d \) and the completely depolarizing channel, \( \Psi_d(X) = \frac{\text{Tr}(X)}{d}I_d \). Namely,

\[
\mathfrak{3} = \frac{1}{d+1} I_d + \frac{d}{d+1} \Psi_d.
\]
The channel $tI_d + (1 - t)\Psi_d$ is entanglement breaking for all $0 \leq t \leq \frac{1}{7}$ [14, 18]. We conjecture that, for all $t$ with $0 \leq t \leq \frac{1}{d+1}$, the entanglement breaking rank of each of these channels is $d^2$.

We verify this stronger conjecture in dimensions $d = 2$ and $d = 3$. Moreover, we show, in these dimensions, that there is a continuous family of $d^2$ rank one matrices $R_i : [0, \frac{1}{d+1}] \rightarrow \mathcal{M}_d$ for $1 \leq i \leq d^2$ such that

$$(tI_d + (1 - t)\Psi_d)(X) = \sum_{i=1}^{d^2} R_i(t)X R_i(t)^*, \quad X \in \mathcal{M}_d.$$ 

In particular, when $t = \frac{1}{d+1}$, we get a Choi-Kraus representation of $\mathbb{3}$ consisting of $d^2$ rank one matrices, so that (by Theorem 3.3) we get the existence of a SIC POVM in that dimension. We conjecture that such continuous families of rank one matrices exist in all dimensions. Interestingly, we are able to prove that when such families do exist, then $R_i(t)$ cannot be positive semi-definite for $0 \leq t < \frac{1}{d+1}$.

Since determining if $\text{ebr}(\mathbb{3}) = d^2$ is equivalent to Zauner’s conjecture, it is interesting to know what sorts of bounds we can obtain on $\text{ebr}(\mathbb{3})$, that do not rely on Zauner’s conjecture. Similar to the SIC POVM existence problem, we have the existence problem of mutually unbiased bases (MUB) in dimension $d$. For a comprehensive survey and literature we refer the reader to [10]. The MUB existence problem asks whether for each $d \geq 2$, there exists a set of $d+1$ orthonormal bases $\{\mathcal{V}_i : 1 \leq i \leq d+1\}$ of $\mathbb{C}^d$, where $\mathcal{V}_i = \{v_{i,j} : 1 \leq j \leq d\}$, such that $|\langle v_{i,j}, v_{k,l} \rangle|^2 = \frac{1}{d}$ for all $i \neq k$. Unlike Zauner’s conjecture, there are infinitely many $d$ for which it is known that $d+1$ MUB exist [2, 17, 25]. We show that the existence of $d+1$ MUB in dimension $d$ implies $\text{ebr}(\mathbb{3}) \leq d(d+1)$.

2. Preliminaries

Let $d \in \mathbb{N}$. We shall let $\mathbb{C}^d$ denote the Hilbert space of $d$-tuples of complex numbers with the usual inner product. We shall denote the space of $d \times d$ complex matrices by $\mathcal{M}_d$. The trace of a matrix $A = [a_{i,j}] \in \mathcal{M}_d$ is defined by $\text{Tr}(A) = \sum_{i=1}^{d^2} a_{i,i}$. We let $I_d$ denote the identity matrix in $\mathcal{M}_d$. The space of all linear maps from $\mathcal{M}_d$ to $\mathcal{M}_m$ is denoted by $\mathcal{L}(\mathcal{M}_d, \mathcal{M}_m)$.

We review some of the definitions and known results that we shall be needing. We begin with stating the SIC POVM existence problem in terms of equiangular vectors in $\mathbb{C}^d$.

**Definition 2.1.** A set of $d^2$ matrices $\{R_i : 1 \leq i \leq d^2\} \subseteq \mathcal{M}_d$ is called a symmetric informationally-complete positive operator-valued measure (SIC POVM) if,

(a) it forms a positive operator-valued measure (POVM), that is, each $R_i$ is positive semi-definite (written $R_i \geq 0$) and $\sum_{i=1}^{d^2} R_i = I_d$,

(b) it is informationally-complete, that is, $\text{span}(\{R_i : 1 \leq i \leq d^2\}) = \mathcal{M}_d$,

(c) it is symmetric, that is, $\text{Tr}(R_i^2) = \lambda$ for all $1 \leq i \leq d^2$, and $\text{Tr}(R_i R_j) = \mu$ for all $i \neq j$, for some constants $\lambda$ and $\mu$, and,

(d) $\text{rank}(R_i) = 1$ for all $1 \leq i \leq d^2$.

The following proposition gives a characterization of a SIC POVM in terms of unit vectors.
Proposition 2.2. A set of matrices \( \{R_i\}_{i=1}^{d^2} \subseteq \mathcal{M}_d \) is a SIC POVM if and only if there exist \( d^2 \) unit vectors \( \{w_i\}_{i=1}^{d^2} \subseteq \mathbb{C}^d \) such that \( R_i = \frac{1}{d} w_i w_i^* \) for each \( 1 \leq i \leq d^2 \), and \( |\langle w_i, w_j \rangle|^2 = \frac{1}{d+1} \) for all \( i \neq j \).

Because of Proposition 2.2, we shall call a set \( \{w_i\}_{i=1}^{d^2} \subseteq \mathbb{C}^d \) of unit vectors a SIC POVM if \( |\langle w_i, w_j \rangle|^2 = \frac{1}{d+1} \) for all \( i \neq j \).

The constant \( \frac{1}{d+1} \) appearing in Proposition 2.2 can be derived independently.

Proposition 2.3. If \( \{w_i\}_{i=1}^{d^2} \subseteq \mathbb{C}^d \) is a set of unit vectors such that \( |\langle w_i, w_j \rangle|^2 = c \neq 1 \) for all \( i \neq j \), then \( c = \frac{1}{d+1} \).

We now state the SIC POVM existence conjecture in terms of equiangular vectors, which we shall call Zauner’s conjecture.

Conjecture 2.4 (Zauner’s conjecture). For each positive integer \( d \geq 2 \), there exist \( d^2 \) unit vectors \( \{w_i\}_{i=1}^{d^2} \subseteq \mathbb{C}^d \) such that \( |\langle w_i, w_j \rangle|^2 = \frac{1}{d+1} \) for all \( i \neq j \).

The existence of the SIC POVM for all dimension \( d \geq 2 \) is still an open question, however in some specific dimensions \( (d = 1, 21, 24, 28, 30, 31, 35 \text{ etc.}) \) the analytic solutions of SIC sets have been found. See [22] for the latest progress in finding these solutions. Many of the numerical solutions to Zauner’s conjecture have been found assuming a covariance property. We shall focus on a particular group covariance.

Definition 2.5. For a positive integer \( d \), let \( \mathbb{Z}_d = \{0, 1, 2, \ldots, d-1\} \) be the cyclic group of order \( d \). Let \( \omega = \exp \left( \frac{2\pi i}{d} \right) \) be a \( d \)-th root of unity. Define the following two \( d \times d \) matrices

\[
U = \begin{bmatrix}
0 & 0 & \cdots & 1 \\
1 & 0 & \cdots & 0 \\
0 & \ddots & 0 & 0 \\
0 & \cdots & 1 & 0 \\
\end{bmatrix}, \quad V = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \omega & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & \omega^{d-1} \\
\end{bmatrix}.
\]

Thus, \( U \) is the (forward) cyclic shift, and \( V \) is the diagonal matrix with all the \( d \)-th roots of unity on the diagonal. For any \( (i, j) \in \mathbb{Z}_d \times \mathbb{Z}_d \), define the discrete Weyl matrices as \( W_{i,j} = U^i V^j \). (For properties of the discrete Weyl matrices, we refer the reader to [24] Section 4.1.2.)

We now have the following stronger conjecture, also due to Zauner. It is not known whether Zauner’s conjecture implies this stronger conjecture.

Conjecture 2.6 (Strong Zauner conjecture). For each positive integer \( d \geq 2 \), there exists a unit vector \( w \in \mathbb{C}^d \) such that the \( d^2 \) vectors,

\[
\{W_{i,j}w : (i, j) \in \mathbb{Z}_d \times \mathbb{Z}_d \} \subseteq \mathbb{C}^d,
\]

where \( W_{i,j} \) are the discrete Weyl matrices, form a SIC POVM. In this case, \( w \) is called a fiducial vector.

We now review some of the standard results from the theory of quantum channels. The proofs may be found in [7,24].

Recall that a linear map \( \Omega : \mathcal{M}_d \to \mathcal{M}_m \) is completely positive and trace preserving (henceforth a quantum channel) if and only if \( \Omega \) can be expressed as

\[
(2.1) \quad \Omega(X) = \sum_{k=1}^{K} B_k X B_k^*, \quad X \in \mathcal{M}_d,
\]
for some $B_1, \ldots, B_K \in \mathbb{M}_{m,d}$ with $\sum_{k=1}^{K} B_k^* B_k = \mathbb{I}_d$. The expression (2.1) is called a Choi-Kraus representation of the quantum channel $\Omega$. Choi-Kraus representations of a quantum channel are not unique, and therefore the minimum $K$ possible in (2.1) is called the Choi-rank of $\Omega$, and is denoted by $\text{cr}(\Omega)$. The Choi-rank of $\Omega$ is equal to the rank of the $dm \times dm$ Choi-matrix $C_{\Omega} = [\Omega(E_{i,j})]_{i,j=1}^{d}$, where $\{E_{i,j} : 1 \leq i, j \leq d\}$ are the canonical matrix units of $\mathbb{M}_d$. If the value of $K$ in expression (2.1) equals the Choi-rank of $\Omega$, then $\{B_k\}_{k=1}^{K} \subseteq \mathbb{M}_{m,d}$ is a linearly independent set.

We identify two simple quantum channels and their convex combinations for our purpose.

**Definition 2.7.** The quantum channel $\mathcal{I}_d : \mathbb{M}_d \to \mathbb{M}_d$ defined by $\mathcal{I}_d(X) = X$, for all $X \in \mathbb{M}_d$, is called the identity channel. The quantum channel $\mathcal{I}_d : \mathbb{M}_d \to \mathbb{M}_d$ defined by $\mathcal{I}_d(X) = \frac{1}{d} \text{Tr}(X) \mathbb{I}_d$, for all $X \in \mathbb{M}_d$, is called the completely depolarizing channel. For $t \in [0, 1]$, we define $\Phi_t : \mathbb{M}_d \to \mathbb{M}_d$ to be $\Phi_t = t\mathcal{I}_d + (1 - t)\mathcal{I}_d$. When $t = \frac{1}{d+1}$, we set,

$$3 := \Phi = \frac{1}{d+1} \mathcal{I}_d + \frac{d}{d+1} \mathcal{I}_d.$$

A Choi-Kraus representation of the identity channel $\mathcal{I}_d$ is simply $\mathcal{I}_d(X) = \mathbb{I}_d X \mathbb{I}_d$, and hence its Choi-rank is 1. Since $C_{\mathcal{I}_d} = [\mathcal{I}_d(E_{i,j})] = \frac{1}{d^2} \mathbb{I}_d^2$, it follows that the Choi-rank of the completely depolarizing channel $\mathcal{I}_d$ is $d^2$.

Since the set of quantum channels $\Omega : \mathbb{M}_d \to \mathbb{M}_d$ forms a convex set in the space $\mathcal{L}(\mathbb{M}_d, \mathbb{M}_d)$, it follows that the map $\Phi_t = t\mathcal{I}_d + (1 - t)\mathcal{I}_d$ is also a quantum channel for every $t \in [0, 1]$. When $t = (0, 1)$, it is easy to see that the Choi-rank of $\Phi_t$ is $d^2$. Indeed, notice that the Choi-matrix of $\Phi_t$ is $C_{\Phi_t} = t[E_{i,j}] + \frac{1-t}{d} \mathbb{I}_d^2$. Let $\xi \in \mathbb{C}^{d^2}$. If $C_{\Phi_t}(\xi) = 0$, then $[E_{i,j}] \xi = \frac{1-t}{d} \xi$. But the only eigenvalues of $[E_{i,j}]$ are 0 and $d$, which implies that $\xi = 0$. Thus $\text{rank}(C_{\Phi_t}) = d^2$.

We now introduce the notion of entanglement breaking maps and then list some equivalent conditions on a map to be entanglement breaking suited for our purpose.

**Definition 2.8.** A linear map $\Phi : \mathbb{M}_d \to \mathbb{M}_m$ is called entanglement breaking if for all $n \in \mathbb{N}$, the $n$-th amplification map

$$\Phi \otimes \mathcal{I}_n : \mathbb{M}_d \otimes \mathbb{M}_n \to \mathbb{M}_m \otimes \mathbb{M}_n$$

maps all states (entangled or not) in $\mathbb{M}_d \otimes \mathbb{M}_n$ to separable states in $\mathbb{M}_m \otimes \mathbb{M}_n$.

**Theorem 2.9 (15).** Let $\Phi : \mathbb{M}_d \to \mathbb{M}_m$ be a linear map. Then the following are equivalent.

(a) The map $\Phi$ is entanglement breaking.

(b) The Choi-matrix $C_{\Phi} = [\Phi(E_{i,j})] \in \mathbb{M}_d \otimes \mathbb{M}_m$ is a separable state, where $\{E_{i,j} : 1 \leq i, j \leq d\}$ are the canonical matrix units of $\mathbb{M}_d$.

(c) The map $\Phi$ has a Choi-Kraus representation

$$\Phi(X) = \sum_{k=1}^{K} R_k X R_k^*, \quad X \in \mathbb{M}_d,$$

where each $R_k$ is a rank one matrix.

(d) The linear map $\Phi' \circ \Phi : \mathbb{M}_d \to \mathbb{M}_k$ is completely positive for every positive map $\Phi' : \mathbb{M}_m \to \mathbb{M}_k$. 
(e) The linear map $\Phi \circ \Phi' : M_k \to M_m$ is completely positive for every positive map $\Phi' : M_k \to M_d$.

Moreover, the set of all entanglement breaking maps $\Phi : M_d \to M_m$ forms a convex set in $\mathcal{L}(M_d, M_m)$.

Remark 2.10. If $\Phi : M_d \to M_m$ is an entanglement breaking map, then by Theorem 2.9(c), there exists a Choi-Kraus representation of $\Phi$,

$$\Phi(X) = \sum_{k=1}^{K} R_k X R_k^*, \quad X \in M_d,$$

where each $R_k$ is a rank one matrix. Then for each $1 \leq k \leq K$, there exist vectors $v_k \in \mathbb{C}^m$ and $w_k \in \mathbb{C}^d$ such that $R_k = v_k w_k^*$, which yields

$$\Phi(X) = \sum_{k=1}^{K} v_k w_k^* X w_k v_k^* = \sum_{k=1}^{K} \langle X w_k, w_k \rangle v_k v_k^*.$$

We shall use both these forms of $\Phi$ interchangeably.

We end this section with the following known result which characterizes when the maps $\Phi_t = t I_d + (1-t) \Psi_d$ as in Definition 2.7 are entanglement breaking.

**Proposition 2.11** ([14, 18]). Let $t \in \mathbb{R}$. The map $\Phi_t = t I_d + (1-t) \Psi_d$ is entanglement breaking if and only if $\frac{-1}{d} \leq t \leq \frac{1}{d+1}$.

### 3. Entanglement Breaking Rank

The main goal of this section is to establish the equivalence of computing entanglement breaking rank of the channel $\mathcal{Z}$ to the existence problem of SIC POVMs. Let us begin with the definition of the entanglement breaking rank of an entanglement breaking map.

**Definition 3.1.** Let $\Phi : M_d \to M_m$ be an entanglement breaking map. The **entanglement breaking rank** of $\Phi$, denoted by $\text{ebr}(\Phi)$, is the minimum number of rank one operators $R_k$ such that $\Phi$ can be written as $\Phi(X) = \sum_{k=1}^{K} R_k X R_k^*$.

Let $\text{cr}(\Phi)$ denote the Choi-rank of an entanglement breaking channel $\Phi : M_d \to M_m$. We have the following estimate:

$$d \leq \text{cr}(\Phi) \leq \text{ebr}(\Phi).$$

The first inequality follows from the fact that an entanglement breaking channel cannot have less than $d$ Choi-Kraus operators in its Choi-Kraus representation [15, Theorem 6].

By Proposition 2.11, it follows that the quantum channel $\mathcal{Z}$ (Definition 2.7) is entanglement breaking, and hence it has a Choi-Kraus representation consisting of rank one Choi-Kraus operators. Zauner’s conjecture can then be related to the problem of obtaining a *minimal* Choi-Kraus representation of the quantum channel $\mathcal{Z}$ consisting of rank one operators. First we establish a weaker proposition.

**Proposition 3.2.** Zauner’s conjecture is true if and only if for each positive integer $d \geq 2$, the quantum channel $\mathcal{Z}$ defined by $\mathcal{Z} = \frac{1}{d^2 + d} I_d + \frac{d}{d^2 + d} \Psi_d$, has a Choi-Kraus representation $\mathcal{Z}(X) = \sum_{i=1}^{d^2} B_i X B_i^*$, for all $X \in M_d$, where each $B_i \in M_d$ is a rank one positive operator.
Proof. The forward implication is a known result \[4]. However, we include a proof for the sake of completeness.

Suppose that Zauner’s conjecture is true. Then for each \(d \geq 2\), there exist \(d^2\) unit vectors \(\{w_i\}_{i=1}^{d^2} \subset \mathbb{C}^d\) such that \(\langle w_i, w_j \rangle = \frac{1}{d+1}\) for all \(i \neq j\). By Proposition 2.2, the set \(\{R_i\}_{i=1}^{d^2} \subset \mathbb{M}_d\), where \(R_i = \frac{1}{d}w_iw_i^*\), forms a SIC POVM. Note that \(R_i^2 = \frac{1}{d}R_i\) for every \(1 \leq i \leq d^2\). Define a map \(\Phi : \mathbb{M}_d \to \mathbb{M}_d\) by

\[
\Phi(X) = d \sum_{i=1}^{d^2} R_i X R_i^*, \quad X \in \mathbb{M}_d.
\]

Clearly, \(\Phi\) is completely positive, and the following shows that it is unital and trace preserving:

\[
d \sum_{i=1}^{d^2} R_i^* R_i = d \sum_{i=1}^{d^2} R_i^2 = d \sum_{i=1}^{d^2} \frac{1}{d} R_i = \sum_{i=1}^{d^2} R_i = \mathbb{I}_d.
\]

Thus, \(\Phi\) is a unital quantum channel. It remains to show that \(\Phi = 3\), from which the desired Choi-Kraus representation of \(3\) can be obtained by setting \(B_i = \sqrt{d}R_i\) for all \(1 \leq i \leq d^2\).

The set \(\{R_i\}_{i=1}^{d^2}\) being informationally complete spans \(\mathbb{M}_d\), and so each \(X \in \mathbb{M}_d\) may be written as \(X = \sum_{j=1}^{d^2} r_j R_j\) for some unique scalars \(r_j\). Taking trace on both sides of \(X = \sum_{j=1}^{d^2} r_j R_j\) yields

\[
\sum_{j=1}^{d^2} r_j = d \text{Tr}(X).
\]

Next observe that

\[
\Phi(R_j) = d \left( R_j^3 + \sum_{1 \leq i < j \leq d^2} R_i R_j R_i \right) = d \left( \frac{1}{d^2} R_j + \frac{1}{d^2(d+1)} \sum_{1 \leq i < j \neq k} R_i \right)
\]

\[
= d \left( \frac{1}{d^2} R_j + \frac{1}{d^2(d+1)} (\mathbb{I}_d - R_j) \right) = \frac{1}{d+1} \left( R_j + \frac{1}{d} \mathbb{I}_d \right),
\]

which implies

\[
\Phi(X) = \sum_{j=1}^{d^2} r_j \Phi(R_j) = \frac{1}{d+1} \sum_{j=1}^{d^2} r_j R_j + \frac{1}{d(d+1)} \sum_{j=1}^{d^2} r_j
\]

\[
= \frac{1}{d+1} X + \frac{1}{d(d+1)} \text{Tr}(X) = \frac{1}{d+1} \mathbb{I}_d(X) + \frac{d}{d+1} \Psi_d(X),
\]

where we used Equation (3.2) in the third equality. Hence \(\Phi = 3\).

Conversely, suppose that \(3\) has a Choi-Kraus representation given by \(3(X) = \sum_{i=1}^{d^2} B_i X B_i^*\), where each \(B_i \in \mathbb{M}_d\) is a rank one positive operator. Then for every \(1 \leq i \leq d^2\), \(B_i = v_i v_i^*\) for some \(v_i \in \mathbb{C}^d\). Since the channel \(3\) is unital, we have

\[
\mathbb{I}_d = \sum_{i=1}^{d^2} B_i^2 = \sum_{i=1}^{d^2} ||v_i||^2 B_i.
\]
Using this, and using \( 3 = \frac{1}{d+1} \mathbb{I}_d + \frac{d}{d+1} \Psi_d \), on one hand we get
\[
3(B_j) = \frac{1}{d+1} (B_j + \|v_j\|^2 \mathbb{I}_d) = \frac{1}{d+1} \left( B_j + \sum_{i=1}^{d^2} \|v_i\|^2 \|v_j\|^2 B_i \right)
\]
and on the other hand, using Choi-Kraus representation of \( 3 \), we get,
\[
3(B_j) = \sum_{i=1}^{d^2} B_i B_j B_i = B_j^2 + \sum_{1 \leq i \neq j \leq d^2} B_i B_j B_i = \|v_j\|^4 B_j + \sum_{1 \leq i \neq j \leq d^2} |\langle v_i, v_j \rangle|^2 B_i.
\]
Comparing \( 3(B_j) \) obtained in two ways, we get
\[
\left( \frac{1 + \|v_j\|^4}{d+1} - \|v_j\|^4 \right) B_j + \sum_{1 \leq i \neq j \leq d^2} \left( \frac{\|v_i\|^2 \|v_j\|^2}{d+1} - |\langle v_i, v_j \rangle|^2 \right) B_i = 0.
\]
Since \( \{B_i\}_{i=1}^{d^2} \) is a linearly independent set (because number of Choi-Kraus operators equals the Choi-rank; see the discussion after Definition 2.7), we must have
\[
\frac{1 + \|v_j\|^4}{d+1} - \|v_j\|^4 = 0, \quad \frac{\|v_i\|^2 \|v_j\|^2}{d+1} - |\langle v_i, v_j \rangle|^2 = 0, \forall i \neq j.
\]
The first one yields, \( \|v_j\|^4 = \frac{1}{d} \), which is constant for all \( 1 \leq j \leq d^2 \), and using this the second one yields \( |\langle v_i, v_j \rangle|^2 = \frac{1}{d(d+1)} \), for all \( i \neq j \). Then it is easy to see that the normalized vectors \( w_i := \frac{v_i}{\|v_i\|} \) satisfy \( |\langle w_i, w_j \rangle|^2 = \frac{1}{d(d+1)} \) for all \( i \neq j \), so that Zauner’s conjecture holds. \( \Box \)

The following result shows that the positivity condition on \( B_i \) in the Proposition 3.2 is redundant and can be dropped.

**Theorem 3.3.** Zauner’s conjecture is true if and only if for each positive integer \( d \geq 2 \), the quantum channel \( 3 \) defined by \( 3 = \frac{1}{d+1} \mathbb{I}_d + \frac{d}{d+1} \Psi_d \), has a Choi-Kraus representation \( 3(X) = \sum_{i=1}^{d^2} B_i X B_i^* \), for all \( X \in \mathbb{M}_d \), where each \( B_i \in \mathbb{M}_d \) is a rank one operator.

**Proof.** It suffices to prove the backward implication. Suppose that \( 3 \) has a Choi-Kraus representation given by \( 3(X) = \sum_{i=1}^{d^2} B_i X B_i^* \), where each \( B_i \in \mathbb{M}_d \) is a rank one operator. Then for every \( 1 \leq i \leq d^2 \), \( B_i = x_i y_i^* \) for some vectors \( x_i, y_i \in \mathbb{C}^d \). Without loss of generality, we may assume that each \( y_i \) is a unit vector. If we show that for every \( 1 \leq i \leq d^2 \), \( x_i = \lambda_i y_i \) for some \( \lambda_i \in \mathbb{C} \), then the Choi-Kraus representation of \( 3 \) reduces to a form which consists of rank one positive operators. Then the result follows from Proposition 3.2. We accomplish this in three steps.

We first prove that the set \( \{x_i y_i^*\}_{i=1}^{d^2} \subseteq \mathbb{M}_d \) is linearly independent. To do this we note that \( 3 \) is unital and hence we have
\[
(3.3) \quad \mathbb{I}_d = 3(\mathbb{I}_d) = \sum_{i=1}^{d^2} B_i B_i^* = \sum_{i=1}^{d^2} x_i^* y_i x_i = \sum_{i=1}^{d^2} x_i^* x_i.
\]
Using this equation, on one hand we get
\[
3(B_j) = \frac{1}{d+1} (B_j + \text{Tr}(B_j)I_d) = \frac{1}{d+1} \left( B_j + \langle x_j, y_j \rangle \left( \sum_{i=1}^{d^2} x_i x_i^* \right) \right),
\]
and on the other hand we get
\[
3(B_j) = \sum_{i=1}^{d^2} B_i B_i^* = \sum_{i=1}^{d^2} (x_i y_i^*) (x_j y_j^*)^* = \sum_{i=1}^{d^2} \langle x_j, y_i \rangle \langle y_i, y_j \rangle x_i x_i^*,
\]
for \(1 \leq j \leq d^2\). Comparing \(3(B_j)\) obtained in the above two ways, we get
\[
\frac{1}{d+1} \left( B_j + \langle x_j, y_j \rangle \left( \sum_{i=1}^{d^2} x_i x_i^* \right) \right) = \sum_{i=1}^{d^2} \langle x_j, y_i \rangle \langle y_i, y_j \rangle x_i x_i^*,
\]
which implies
\[
B_j = \sum_{i=1}^{d^2} ((d+1) \langle x_j, y_i \rangle \langle y_i, y_j \rangle - \langle x_j, y_j \rangle) x_i x_i^*.
\]

This shows that the set \(\{x_i x_i^*\}_{i=1}^{d^2}\) spans \(\{B_j\}_{j=1}^{d^2}\). But since the Choi-rank of \(3\) is \(d^2\), \(\{B_j\}_{j=1}^{d^2}\) is a basis for \(\mathbb{M}_d\). Consequently, the set \(\{x_i x_i^*\}_{i=1}^{d^2}\) is a minimal spanning set, and hence a basis for \(\mathbb{M}_d\).

We next prove that \(\|x_j\|^2 = \frac{1}{d}\) for every \(1 \leq j \leq d^2\). Using Equation (3.3), for each \(1 \leq j \leq d^2\), on one hand we get
\[
3(x_j x_j^*) = \frac{1}{d+1} \left( x_j x_j^* + \text{Tr}(x_j x_j^*)I_d \right) = \frac{1}{d+1} \left( x_j x_j^* + \|x_j\|^2 \left( \sum_{i=1}^{d^2} x_i x_i^* \right) \right),
\]
and on the other hand, we get
\[
3(x_j x_j^*) = \sum_{i=1}^{d^2} B_i (x_j x_j^*) B_i^* = \sum_{i=1}^{d^2} x_i y_i x_j^* y_i x_j^* = \sum_{i=1}^{d^2} \langle y_i, x_j \rangle \langle x_j, x_j \rangle |^2 x_i x_i^*.
\]
Comparing the two expressions of \(3(x_j x_j^*)\), we get
\[
\frac{1}{d+1} \left( x_j x_j^* + \|x_j\|^2 \left( \sum_{i=1}^{d^2} x_i x_i^* \right) \right) = \sum_{i=1}^{d^2} \langle y_i, x_j \rangle \langle x_j, x_j \rangle |^2 x_i x_i^*.
\]
Because of linear independence of the set \(\{x_i x_i^*\}_{i=1}^{d^2}\), comparing the coefficients of \(x_i x_i^*\), it follows that
\[
(d+1) \langle y_j, x_j \rangle |^2 = 1 + \|x_j\|^2, \quad \forall j,
\]
(3.4)
\[
(d+1) \langle y_i, x_j \rangle |^2 = \|x_j\|^2, \quad \forall i \neq j.
\]
Using the Cauchy-Schwarz inequality in Equation (3.4), we obtain
\[
1 + \|x_j\|^2 = (d+1) \langle y_j, x_j \rangle |^2 \leq (d+1) \|y_j\|^2 \|x_j\|^2 = (d+1) \|x_j\|^2,
\]
Then for $i$ the fact that $\text{cr}(Z)$ which implies $1 \\leq d$, we get $\sum_{i=1}^{d^2} \|x_i\|^2 = d$ which in turn yields

$$0 \leq \sum_{i=1}^{d^2} \left( \|x_i\|^2 - \frac{1}{d} \right) = \sum_{i=1}^{d^2} \|x_i\|^2 - \sum_{i=1}^{d^2} \frac{1}{d} = d - d^2 \frac{1}{d} = 0.$$ 

This shows that $\|x_j\|^2 = \frac{1}{d}$, for all $1 \leq j \leq d^2$.

We finish the proof by showing that each $x_i$ is some scalar multiple of $y_i$. Plugging $\|x_j\|^2 = \frac{1}{d}$ in Equations (3.4) and (3.5) readily shows that for $1 \leq i, j \leq d^2$,

$$\langle x_i, y_j \rangle = \begin{cases} \frac{1}{d}, & \text{if } i = j, \\ \frac{1}{d(d+1)}, & \text{if } i \neq j. \end{cases}$$

But

$$\frac{1}{d} = |\langle x_i, y_i \rangle|^2 \leq \|x_i\|^2 \|y_i\|^2 = \frac{1}{d},$$

so that equality holds everywhere. This implies (from the equality case in Cauchy-Schwarz inequality) that $x_i = \lambda_i y_i$, for some $\lambda_i \in \mathbb{C}$, and the assertion is proved. \(\square\)

**Remark 3.4.** The proof of Theorem 3.3 is constructive in the sense that the unit vectors \{\(y_i\)\}_{i=1}^{d^2} and \{\(\frac{x_i}{\|x_i\|}\)\}_{i=1}^{d^2} satisfy Zauner’s conjecture. First note that from Equation (3.6), it can be deduced that for all $i$,

$$\frac{1}{d} = |\langle x_i, y_i \rangle|^2 = |\langle \lambda_i y_i, y_i \rangle|^2 = |\lambda_i|^2.$$ 

Then for $i \neq j$, from Equation (3.6) we have

(a) $$\frac{1}{d(d+1)} = |\langle x_i, y_j \rangle|^2 = |\lambda_i|^2 |\langle y_i, y_j \rangle|^2 = \frac{1}{d} |\langle y_i, y_j \rangle|^2,$$

which implies that $|\langle y_i, y_j \rangle|^2 = \frac{1}{d+1}$, thereby showing that \(\{y_i\}_{i=1}^{d^2}\) satisfy Zauner’s conjecture; and,

(b) $$\left\langle \frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|} \right\rangle^2 = \frac{|\lambda_i|^2 |\lambda_j|^2}{\|x_j\|^2 \|x_j\|^2} |\langle y_i, y_j \rangle|^2 = |\langle y_i, y_j \rangle|^2 = \frac{1}{d+1},$$

which shows that \(\left\{\frac{x_i}{\|x_i\|}\right\}_{i=1}^{d^2}\) satisfy Zauner’s conjecture.

The following corollary is an immediate consequence of Theorem 3.3 along with the fact that $\text{cr}(Z) = d^2$.

**Corollary 3.5.** Zauner’s conjecture holds if and only if $\text{eb}(Z) = d^2$ for all $d \geq 2$.

Interestingly, for $t \in \left[\frac{1}{d(d+1)}, \frac{1}{d+1}\right)$, (see Proposition 2.11) the channels $\Phi_t : \mathbb{M}_d \to \mathbb{M}_d$ defined by $\Phi_t = t\mathbb{I}_d + (1-t)\Psi_d$ cannot have a Choi-Kraus representation with $d^2$ positive rank one Choi-Kraus operators, which we prove next.

**Proposition 3.6.** Let $t \in \left[\frac{1}{d(d+1)}, \frac{1}{d+1}\right)$. Let $\Phi_t : \mathbb{M}_d \to \mathbb{M}_d$ be the quantum channel given by $\Phi_t = t\mathbb{I}_d + (1-t)\Psi_d$. Then $\Phi_t$ cannot have a Choi-Kraus representation, $\Phi_t(X) = \sum_{i=1}^{d^2} R_i X R_i^*$, where each $R_i$ is a positive rank one operator.
Proof. We shall follow the arguments as in Proposition 3.2. Suppose \( \Phi_t \) has a Choi-Kraus representation, \( \Phi_t(X) = \sum_{i=1}^{d^2} R_i X R_i^* \), where \( R_i = v_i v_i^* \) for some vector \( v_i \in \mathbb{C}^d \). Since \( \Phi_t \) is unital, we have \( \mathbb{I}_d = \sum_{i=1}^{d^2} \| v_i \|^2 R_i \). Applying \( \Phi_t \) to \( R_j \),

\[
\Phi_t(R_j) = t R_j + \frac{1 - t}{d} \text{Tr}(R_j) \mathbb{I}_d
\]

\[
= \left( t + \frac{1 - t}{d} \| v_j \|^4 \right) R_j + \sum_{1 \leq i \leq d^2, i \neq j} \frac{1 - t}{d} \| v_j \|^2 \| v_i \|^2 R_i,
\]

and also

\[
\Phi_t(R_j) = \sum_{i=1}^{d^2} R_i R_j R_i^* = \| v_j \|^4 R_j + \sum_{1 \leq i \leq d^2, i \neq j} | \langle v_i, v_j \rangle |^2 R_i.
\]

Comparing both the expressions of \( \Phi_t(R_j) \) and using the linear independence of \( R_i \)'s, we get

\[
t + \frac{1 - t}{d} \| v_j \|^4 = \| v_j \|^4, \quad \frac{1 - t}{d} \| v_j \|^2 \| v_i \|^2 = | \langle v_i, v_j \rangle |^2, \quad \forall i \neq j.
\]

This implies

\[
\| v_j \|^4 = \frac{dt}{d + t - 1}, \quad | \langle v_i, v_j \rangle |^2 = \frac{t(1 - t)}{d + t - 1}, \quad \forall i \neq j.
\]

Let \( w_i = \frac{v_i}{\| v_i \|} \). Then for all \( i \neq j \), we have

\[
| \langle w_i, w_j \rangle |^2 = \frac{1}{\| v_i \|^2 \| v_j \|^2} | \langle v_i, v_j \rangle |^2 = \frac{d + t - 1}{dt} \frac{t(1 - t)}{d + t - 1} = \frac{1 - t}{d + t - 1}.
\]

But by Corollary 2.3, we must have \( \frac{1 - t}{d} = \frac{1}{t + d} \), which implies that \( t = \frac{1}{d+1} \), which is a contradiction. \( \Box \)

In what follows, we seek other entanglement breaking channels with the property that a statement about their entanglement breaking rank is equivalent to Zauner’s conjecture. If \( \Phi : \mathcal{M}_d \to \mathcal{M}_m \) is an entanglement breaking map, and if \( \Phi' : \mathcal{M}_m \to \mathcal{M}_k \) is a positive map, then \( \Phi' \circ \Phi \) is also entanglement breaking. This follows readily from Theorem 2.9(d). The following proposition states the relation between their entanglement breaking ranks.

**Proposition 3.7.** Let \( \Phi : \mathcal{M}_d \to \mathcal{M}_m \) be an entanglement breaking map and let \( \Phi' : \mathcal{M}_m \to \mathcal{M}_k \) be a positive map. Then \( \text{ebr}(\Phi' \circ \Phi) \leq \text{ebr}(\Phi) \text{ri}(\Phi') \), where \( \text{ri}(\Phi') = \max \{ \text{rank}(\Phi'(xx^*)) : \| x \| = 1 \} \).

**Proof.** Following Remark 2.10, the map \( \Phi \) may be expressed as

\[
\Phi(X) = \sum_{k=1}^{K} \langle X w_k, w_k \rangle v_k v_k^*, \quad X \in \mathcal{M}_d,
\]

for some set of vectors \( w_k \in \mathbb{C}^d \) and \( v_k \in \mathbb{C}^m \). Without loss of generality, we may assume that \( \| v_k \| = 1 \) for all \( 1 \leq k \leq K \). Then we have

\[
\Phi'(\Phi(X)) = \sum_{k=1}^{K} \langle X w_k, w_k \rangle \Phi'(v_k v_k^*), \quad X \in \mathcal{M}_d.
\]
Since $\Phi'$ is positive, it follows that $\Phi'(v_kv_k^*) \geq 0$ for all $1 \leq k \leq K$. The spectral decomposition then allows us to write each $\Phi'(v_kv_k^*)$ as a sum of (non-zero) rank one positive operators and each such sum will contain at most $\text{ri}(\Phi')$ terms. Counting the total number of terms in the resulting sum of $\Phi'(\Phi(X))$ yields the required inequality.

**Corollary 3.8.** Let $\Phi : \mathcal{M}_d \to \mathcal{M}_m$ be an entanglement breaking map. Set $K = \text{ebr}(\Phi)$ and suppose $\Phi(X) = \sum_{k=1}^{K} \langle Xw_k, w_k \rangle v_kv_k^*$, for all $X \in \mathcal{M}_d$, and for some $w_k \in \mathbb{C}^d$, $v_k \in \mathbb{C}^m$ with $\|v_k\| = 1$. If $\Phi' : \mathcal{M}_m \to \mathcal{M}_k$ is a unital quantum channel such that the projections $\{v_kv_k^*\}_{k=1}^{K}$ are in the multiplicative domain of $\Phi'$, then we have $\text{ebr}(\Phi' \circ \Phi) \leq \text{ebr}(\Phi)$.

**Proof.** If a projection $v_kv_k^*$ is in the multiplicative domain of $\Phi'$, then Choi’s theorem on multiplicative domain \[6\] asserts that $\Phi'(v_kv_k^*)$ is a rank one projection, for all $1 \leq k \leq K$. The result then follows from Proposition 3.7.

Let $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ be an entanglement breaking map, $T : \mathcal{M}_d \to \mathcal{M}_d$ be the transpose map, and for any given unitary $U \in \mathcal{M}_d$ let $Ad_U : \mathcal{M}_d \to \mathcal{M}_d$ be the map $Ad_U(X) = UXU^*$. Since $T$ and $Ad_U$ are both positive, it follows that $T \circ \Phi$ and $Ad_U \circ \Phi$ are entanglement breaking. The following result determines the entanglement breaking rank of these two channels.

**Corollary 3.9.** Let $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ be an entanglement breaking map. Let $T : \mathcal{M}_d \to \mathcal{M}_d$ be the transpose map. Given a unital $U \in \mathcal{M}_d$, let $Ad_U : \mathcal{M}_d \to \mathcal{M}_d$ be the map $Ad_U(X) = UXU^*$. Then $\text{ebr}(T \circ \Phi) = \text{ebr}(\Phi) = \text{ebr}(Ad_U \circ \Phi)$.

**Proof.** It is clear that $\text{ri}(T) = 1$ and $\text{ri}(Ad_U) = 1$. Using these and Proposition 3.7 we have

$$\text{ebr}(\Phi) = \text{ebr}(T \circ \Phi) \leq \text{ebr}(T \circ T \circ \Phi) \leq \text{ebr}(\Phi),$$

$$\text{ebr}(\Phi) = \text{ebr}(Ad_U \circ \Phi) \leq \text{ebr}(Ad_U \circ Ad_U \circ \Phi) \leq \text{ebr}(Ad_U \circ \Phi) \leq \text{ebr}(\Phi),$$

Thus, we have equalities everywhere and the corollary follows.

**Corollary 3.10.** Let $d \geq 2$ and $3 = \frac{1}{d+1}I_d + \frac{d}{d+1} \Psi_d$. Then the following are equivalent.

(a) Zauner’s conjecture is true.
(b) $\text{ebr}(3) = d^2$.
(c) $\text{ebr}(T \circ 3) = d^2$, where $T$ is the transpose map.
(d) $\text{ebr}(Ad_U \circ 3) = d^2$, for any unital $U \in \mathcal{M}_d$.

**Remark 3.11.** Note that the channel $T \circ 3$ is precisely the Werner-Holevo channel \[24\] Example 3.36]

$$\Phi_0(X) = \frac{1}{d+1} (X^T + \text{Tr}(X)I_d), \quad X \in \mathcal{M}_d,$$

where $X^T$ denotes the transpose of $X$. Thus the following statements are equivalent:

1. Zauner’s conjecture is true.
2. $\text{ebr}(\Phi_0) = d^2$.

The Werner-Holevo Channel $\Phi_0$ serves as an example of a quantum channel whose entanglement breaking rank is strictly greater than its Choi-rank. To see this first note that by Corollary 3.9 we have

$$\text{ebr}(\Phi_0) = \text{ebr}(3) \geq \text{rank}(C_3) = d^2.$$
Now it is not hard to see that the Choi-rank of $\Phi_0$ is $\frac{d(d+1)}{2}$. Indeed, the range of the Choi-matrix $C_{\Phi_0}$ is the subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$ spanned by the vectors

$$\frac{1}{\sqrt{2}} \{ e_i \otimes e_j + e_j \otimes e_i : i \neq j, 1 \leq i, j \leq d \} \cup \{ e_i \otimes e_i : 1 \leq i \leq d \},$$

where $\{ e_i : 1 \leq i \leq d \}$ is the canonical basis of $\mathbb{C}^d$. This is because $C_{\Phi_0}$ has same range as the projection $P = \frac{1}{2}(I_d \otimes I_d + W)$, where $W$ is the flip-operator defined on $\mathbb{C}^d \otimes \mathbb{C}^d$ by $W(\xi \otimes \eta) = \eta \otimes \xi$. The subspace spanned by the vectors mentioned in (3.7) is called the symmetric subspace and has dimension $\frac{d(d+1)}{2}$ [24, Example 6.10].

Since $d^2 > \frac{d(d+1)}{2}$ for all $d \geq 2$, we have

$$\text{ebr}(\Phi_0) \geq d^2 > \frac{d(d+1)}{2} = \text{cr}(\Phi_0),$$

and the assertion follows.

Note that since $\text{cr}(\Phi_0) = \frac{d(d+1)}{2}$ and $d^2 > \frac{d(d+1)}{2}$, its entanglement breaking rank “could” theoretically be strictly less than $d^2$. However, due to Corollary 3.9, we infer that $\text{ebr}(\Phi_0) = \text{ebr}(\Psi) \geq d^2$ and hence the entanglement breaking rank of $\Phi_0$ is at least $d^2$.

The Werner-Holevo channel is, of course, not the only channel with its entanglement breaking rank strictly greater than its Choi-rank. The following result generalizes [9, Theorem 1] and provides at least one way to find quantum channels with this property.

**Proposition 3.12.** Let $\Phi : M_d \rightarrow M_m$ be an entanglement breaking map and let $\Phi' : M_m \rightarrow M_k$ be a positive map. If $\text{ebr}(\Phi) = \text{ebr}(\Phi' \circ \Phi)$ and $\text{cr}(\Phi) \neq \text{cr}(\Phi' \circ \Phi)$, then either $\text{ebr}(\Phi) > \text{cr}(\Phi)$ or $\text{ebr}(\Phi' \circ \Phi) > \text{cr}(\Phi' \circ \Phi)$.

**Proof.** The proof is obvious. \qed

4. LOWER SEMICONtinuity OF ENTANGLEMENT BREAKING RANK

The following result shows that ebr is a lower semicontinuous function. Recall that if $X$ is a metric space, $x_0 \in X$, and $f : X \rightarrow \mathbb{R} \cup \{\pm \infty\}$, then we say that $f$ is lower semicontinuous at $x_0$ if and only if whenever $(x_n)_{n=1}^{\infty}$ is a sequence in $X$ which converges to $x_0$, we have $\liminf_{n \rightarrow \infty} f(x_n) \geq f(x_0)$.

**Proposition 4.1.** Let $(\Psi_n)_{n=1}^{\infty}$ be a sequence of entanglement breaking maps where $\Psi_n : M_d \rightarrow M_m$, and suppose that $\Psi_n \rightarrow \Psi$. Then

$$\text{ebr}(\Psi) \leq \lim \inf_n \text{ebr}(\Psi_n).$$

**Proof.** It is enough to show that if $\text{ebr}(\Psi_n) \leq k$ for all $n \in \mathbb{N}$, then $\text{ebr}(\Psi) \leq k$. With that assumption, for each $n \in \mathbb{N}$, there exist rank one operators $\{ R_{i,n} : 1 \leq i \leq k \}$ such that $\Psi_n(X) = \sum_{i=1}^{k} R_{i,n} X R_{i,n}^*$. Since $\Psi_n(I_d)$ converges to $\Psi(I_d)$ the sequence $(\Psi_n(I_d))$ is bounded and so there is $c > 0$ so that $\Psi_n(I_d) \leq c I_d$ for all $n \in \mathbb{N}$. Hence, $R_{j,n} R_{j,n}^* \leq \sum_{i=1}^{k} R_{i,n} R_{i,n}^* = \Psi_n(I_d) \leq c I_d$, and so these rank one operators are also bounded. By compactness, we have a subsequence $(n_m)_{m=1}^{\infty}$ so that, $\lim_m R_{i,n_m} = R_i$ which is also rank one. Thus,

$$\Psi(X) = \lim_m \Psi_{n_m}(X) = \lim_m \sum_{i=1}^{k} R_{i,n_m} X R_{i,n_m}^* = \sum_{i=1}^{k} R_i X R_i^*, $$
and the result follows. □

Proposition 4.1 yields the following corollary.

**Corollary 4.2.** Let \( 0 \leq t \leq \frac{1}{d+1} \), and let \( \Phi_t = t\mathcal{I}_d + (1-t)\Psi_d \). If

\[
\liminf_{t \to \frac{1}{d+1}} \text{ebr}(\Phi_t) \leq d^2,
\]

then Zauner’s conjecture is true.

This corollary motivates us to compute \( \text{ebr}(\Phi_t) \). We show that in dimensions \( d = 2 \) and \( d = 3 \), \( \text{ebr}(\Phi_t) = d^2 \) when \( 0 \leq t \leq \frac{1}{d+1} \). In fact, we have the following stronger conjecture than \( \text{ebr}(\Phi_t) = d^2 \) for all \( 0 \leq t \leq \frac{1}{d+1} \).

**Conjecture 4.3.** Let \( d \geq 2 \). For \( 0 \leq t \leq \frac{1}{d+1} \), consider the channel \( \Phi_t = t\mathcal{I}_d + (1-t)\Psi_d \). There exists a set of \( 2d^2 \) continuous functions \( x_i, y_i : \left[ 0, \frac{1}{d+1} \right] \to \mathbb{C}^d \), \( \|x_i(t)\| = \|y_i(t)\| = 1 \) for all \( 0 \leq t \leq \frac{1}{d+1}, 1 \leq i \leq d^2 \), such that

\[
\Phi_t(X) = \frac{1}{d} \sum_{i=1}^{d^2} (x_i(t)y_i(t)^*) X (x_i(t)y_i(t)^*)^*, \quad X \in \mathbb{M}_d,
\]

for all \( 0 \leq t \leq \frac{1}{d+1} \).

We show that for dimensions \( d = 2 \) and \( d = 3 \), we can indeed find such continuous families of vectors. To find them we assume a covariance property and some ad hoc assumptions. Indeed, instead of \( 2d^2 \) continuous functions, now we wish to find two continuous functions,

\[
x, y : \left[ 0, \frac{1}{d+1} \right] \to \mathbb{C}^d, \quad \|x(t)\| = \|y(t)\| = 1 \text{ for all } 0 \leq t \leq \frac{1}{d+1},
\]

such that the map defined by

\[
\tilde{\Phi}_t(X) = \frac{1}{d} \sum_{i,j=0}^{d-1} ((W_{i,j}x(t)) (W_{i,j}y(t)^*)^*) X ((W_{i,j}x(t)) (W_{i,j}y(t)^*)^*),
\]

satisfies \( \Phi_t = \tilde{\Phi}_t \), where of course \( \{W_{i,j} : 0 \leq i, j \leq d-1\} \) are the discrete Weyl matrices, and \( 0 \leq t \leq \frac{1}{d+1} \). To find the functions \( x \) and \( y \), we solve a set of simultaneous equations obtained by comparing their Choi-matrices:

\[
\Phi_t(E_{k,l}) = \tilde{\Phi}_t(E_{k,l}), \quad 0 \leq k, l \leq d-1,
\]

where \( \{E_{k,l} : 0 \leq k, l \leq d-1\} \) are the canonical matrix units of \( \mathbb{M}_d \).

We establish a couple of lemmas to help in the calculations for computing the functions \( x \) and \( y \).

**Lemma 4.4.** Let \( \{W_{i,j} : 0 \leq i, j \leq d-1\} \subseteq \mathbb{M}_d \) be the discrete Weyl matrices. If \( E_{p,q} \in \mathbb{M}_d \) is any canonical matrix unit with \( p \neq q \) and \( 0 \leq p, q \leq d-1 \), then

\[
\sum_{j=0}^{d-1} W_{i,j}^* E_{p,q} W_{i,j} = 0.
\]
Proof. Computation shows that the \((r,s)\)-entry of \(\sum_{j=0}^{d-1} W_{i,j}^* E_{p,q} W_{i,j}\) is
\[
\left( \sum_{j=0}^{d-1} \omega^j \right) \delta_{p,r} \delta_{q,s}.
\]
But, \(\sum_{j=0}^{d-1} \omega^j = 0\), when \(p \neq q\), and hence the result follows. \(\square\)

**Lemma 4.5.** Let \(x, y \in \mathbb{C}^d\) be unit vectors and let \(\{W_{i,j} : 0 \leq i, j \leq d - 1\} \subseteq \mathbb{M}_d\) be the discrete Weyl matrices. Let \(\Phi : \mathbb{M}_d \to \mathbb{M}_d\) be a linear map defined by
\[
\Phi(X) = \sum_{i,j=0}^{d-1} (W_{i,j} x)(W_{i,j} y)^* X (W_{i,j} y)(W_{i,j} x)^*
= \sum_{i,j=0}^{d-1} \langle W_{i,j}^* X W_{i,j} y, y \rangle W_{i,j} x x^* W_{i,j}^*.
\]
If \(E_{k,l} \in \mathbb{M}_d\) is a canonical matrix unit, then, \(\Phi(E_{k,k})\) is a diagonal matrix for all \(0 \leq k \leq d - 1\); and if \(k \neq l\), then the diagonal entries of \(\Phi(E_{k,l})\) are all zero.

Proof. Let \(x = (x_0, \ldots, x_{d-1})\) and \(y = (y_0, \ldots, y_{d-1})\). If \(\{e_j : 0 \leq j \leq d - 1\}\) is the canonical orthonormal basis of \(\mathbb{C}^d\), then a computation shows that the \((p,q)\)-entry of \(\Phi(E_{k,l})\) is given by
\[
\langle \Phi(E_{k,l}) e_q, e_p \rangle = \sum_{i,j=0}^{d-1} \langle W_{i,j} y, e_k \rangle \langle W_{i,j} y, e_l \rangle \text{Tr} (x x^* W_{i,j}^* E_{q,p} W_{i,j}).
\]
When \(0 \leq k \leq d - 1\), and \(p \neq q\), using Equation \((4.1)\) we have
\[
\langle \Phi(E_{k,l}) e_q, e_p \rangle = \sum_{i,j=0}^{d-1} |\langle W_{i,j} y, e_k \rangle|^2 \text{Tr} (x x^* W_{i,j}^* E_{q,p} W_{i,j})
= \sum_{i,j=0}^{d-1} |y_{k-i}|^2 \text{Tr} \left( x x^* \left( \sum_{j=0}^{d-1} W_{i,j}^* E_{q,p} W_{i,j} \right) \right) = 0,
\]
where the last equality follows from Lemma \(4.4\).

When \(k \neq l\), then using Equation \((4.1)\), the \((p,p)\)-entry of \(\Phi(E_{k,l})\) is,
\[
\langle \Phi(E_{k,l}) e_p, e_p \rangle = \sum_{i,j=0}^{d-1} \omega^{j-l} y_{k-i} y_{l-i} \text{Tr} (x x^* W_{i,j}^* E_{p,p} W_{i,j})
= \sum_{i,j=0}^{d-1} \omega^{j-l} y_{k-i} y_{l-i} |x_{p-i}|^2
= \sum_{i=0}^{d-1} y_{k-i} y_{l-i} |x_{p-i}|^2 \left( \sum_{j=0}^{d-1} \omega^{j-l} \right) = 0,
\]
where the last equality follows from \(\sum_{j=0}^{d-1} \omega^{j-l} = 0\), since \(k \neq l\). \(\square\)

We now prove Conjecture \(1.3\) for \(d = 2\) and \(d = 3\). We remark that there are many possible solutions for the functions \(x\) and \(y\), but we shall not be occupied with finding all of them.
Theorem 4.6. There exists continuous functions $x, y : [0, \frac{1}{3}] \rightarrow \mathbb{C}^2$ such that $\|x(t)\| = \|y(t)\| = 1$, and such that

$$
\Phi_t(X) = \sum_{i,j=0}^1 (W_{i,j}x(t))(W_{i,j}y(t))^*X(W_{i,j}y(t))(W_{i,j}x(t))^*, \quad X \in M_2
$$

for all $0 \leq t \leq \frac{1}{3}$, where $\Phi_t = tI_2 + (1-t)\Psi_2$ and $\{W_{i,j} : 0 \leq i, j \leq 1\}$ are the discrete Weyl matrices.

Proof. Without loss of generality we may assume that

$$
x = \left[ \begin{array}{c} r \\ \sqrt{1 - r^2} e^{i\theta} \end{array} \right], \quad y = \left[ \begin{array}{c} se^{i\theta} \\ \sqrt{1 - s^2} e^{i\theta} \end{array} \right],
$$

where $0 \leq r, s \leq 1$ and $\theta, \theta_1, \theta_2 \in [0, 2\pi)$, and where we have suppressed the dependence on $t$. Set

$$
(4.2) \quad a = rs, \quad b = \sqrt{(1 - r^2)(1 - s^2)}.
$$

With the help of Lemma 4.5, computation shows that

$$
\tilde{\Phi}_t(E_{1,1}) = \left[ \begin{array}{cc} a^2 + b^2 & 0 \\ 0 & 1 - (a^2 + b^2) \end{array} \right],
$$

$$
\tilde{\Phi}_t(E_{1,2}) = \left[ \begin{array}{cc} 0 & 2ab \cos(\theta + \theta_1 - \theta_2) \\ 2ab \cos(\theta - \theta_1 + \theta_2) & 0 \end{array} \right],
$$

$$
\tilde{\Phi}_t(E_{2,1}) = \left[ \begin{array}{cc} 0 & 2ab \cos(\theta - \theta_1 + \theta_2) \\ 2ab \cos(\theta + \theta_1 - \theta_2) & 0 \end{array} \right],
$$

$$
\tilde{\Phi}_t(E_{2,2}) = \left[ \begin{array}{cc} 1 - (a^2 + b^2) & 0 \\ 0 & a^2 + b^2 \end{array} \right].
$$

Comparing these with $\Phi_t(E_{k,l})$, we get the following simultaneous equations:

$$
(4.3) \quad \begin{cases}
\frac{1 + t}{2} = a^2 + b^2, \\
t = 2ab \cos(\theta + \theta_1 - \theta_2), \\
0 = 2ab \cos(\theta - \theta_1 + \theta_2).
\end{cases}
$$

If $t \neq 0$, then the last two equations in Equation (4.3) imply that $ab \neq 0$. Therefore the last equation in Equation (4.3) implies that $\cos(\theta - \theta_1 + \theta_2) = 0$, which happens only when $\theta - \theta_1 + \theta_2 = (2k + 1)\frac{\pi}{2}$ for some $k \in \mathbb{N}$. Thus we have $\theta_2 - \theta_1 = (2k + 1)\frac{\pi}{2} - \theta$, and hence

$$
\cos(\theta + \theta_1 - \theta_2) = \cos\left(2\theta - (2k + 1)\frac{\pi}{2}\right) = (-1)^k \sin(2\theta).
$$

Therefore the second equation of Equation (4.3) becomes

$$
(4.4) \quad t = (-1)^k 2ab \sin(2\theta).
$$

Now notice that by the AM-GM inequality, we have $\frac{a^2 + b^2}{2} \geq ab$. Plugging the values of $a^2 + b^2$ and $ab$ from Equations (4.3) and (4.4), respectively, and simplifying,

$$
(-1)^k \sin(2\theta) \geq \frac{2t}{1 + t}.
$$

While we can let $\theta$ depend on $t$, for keeping things simple, we choose a value of $\theta$ which satisfies the above inequality for all $0 \leq t \leq \frac{1}{3}$. Since $\frac{2t}{1 + t} \in [0, \frac{1}{2}]$ for
To get the other fiducial vector we choose
\[ \theta = \frac{\pi}{4}, \quad \theta_1 = 0, \quad \theta_2 = \frac{\pi}{4}. \]
Thus, the system of equations (4.3) reduces to
\[ a^2 + b^2 = \frac{1 + t}{2}, \]
\[ 2ab = t. \]
A bit of computation shows that one solution to Equations (4.6) is,
\[ a = \frac{1}{2} \left( \sqrt{\frac{1 + 3t}{2} + \sqrt{\frac{1 - t}{2}}} \right), \quad b = \frac{1}{2} \left( \sqrt{\frac{1 + 3t}{2} - \sqrt{\frac{1 - t}{2}}} \right). \]
Finally, we can invert equations (4.2) to express \( r \) and \( s \) in terms of \( a \) and \( b \). It is easy to see that \( r^2 + s^2 = 1 + a^2 - b^2 \) and \( 2rs = 2a \), so that one solution is,
\[ r = \frac{1}{2} \left( \sqrt{(1 + a)^2 - b^2} + \sqrt{(1 - a)^2 - b^2} \right), \]
\[ s = \frac{1}{2} \left( \sqrt{(1 + a)^2 - b^2} - \sqrt{(1 - a)^2 - b^2} \right). \]
With these values, we get an instance of the functions \( x(t) \) and \( y(t) \) which satisfy the system of equations (4.3). Clearly, \( x(t) \) and \( y(t) \) are continuous at \( t = \frac{1}{4} \).

Moreover, in the limit \( t \to 0 \), we get
\[ x(0) = \left[ \frac{1}{2}, \theta; \frac{1}{e^{\frac{i}{3}}} \right], \quad y(0) = \frac{1}{\sqrt{2}} \left[ \frac{1}{2}, \theta; \frac{1}{e^{\frac{i}{3}}} \right]. \]
It is easily checked that these vectors do satisfy the system of equations (4.3). Hence, \( x(t) \) and \( y(t) \) are continuous at \( t = 0 \).

While the proof is complete, we do few more calculations to show that when \( t = \frac{1}{4} \), we do recover a set of vectors which satisfy Zauner’s conjecture. We also show how the equiangular property depends on \( t \).

When \( t = \frac{1}{4} \), it is straightforward to compute from the above equations that
\[ x \left( \frac{1}{3} \right) = y \left( \frac{1}{3} \right) = \frac{1}{\sqrt{6}} \left[ \frac{\sqrt{3} + \sqrt{3}}{3 - \sqrt{3}} \right], \]
which is indeed one of the fiducial vectors in dimension two mentioned in [20].

To get the other fiducial vector we choose \( \theta = \frac{5\pi}{4} \) and replace \( r = \sqrt{\frac{1 + \sqrt{3}}{6}} \) by \( \sqrt{1 - r^2} = \sqrt{\frac{3 - \sqrt{3}}{6}} \).

Finally, we compute the angle \( |\langle W_{i,j}(t), x(t) \rangle| \) for all \((i, j) \neq (k, l)\). It suffices to compute \( |\langle W_{i,j}(x(t), x(t) \rangle|^2 \) for all \((i, j) \neq (0, 0)\). Computation shows that
\[ A_1(t) := |\langle W_{0,1}(t), x(t) \rangle|^2 = \frac{1 - 3t^2 + \sqrt{(1 - 9t^2)(1 - t^2)}}{2}, \]
\[ A_2(t) := |\langle W_{1,0}(t), x(t) \rangle|^2 = |\langle W_{1,1}(x(t), x(t) \rangle|^2 = \frac{1 + 3t^2 - \sqrt{(1 - 9t^2)(1 - t^2)}}{4}. \]
Thus we get two distinct angles as a function of \( t \). When \( t = \frac{1}{3} \), we see that both values are \( \frac{1}{3} \), which we would expect if the vectors are equiangular. We plot these two functions in Figure 1. We can do a similar analysis for \( y(t) \).

**Figure 1.** Plot of \(|\langle W_{i,j}x(t), x(t) \rangle|\)^2 for \((i,j) \neq (0,0)\) when \( d = 2 \).

For dimension 3, we simply state the vectors.

**Theorem 4.7.** There exists continuous functions \( x, y : [0, \frac{1}{4}] \to \mathbb{C}^3 \) such that \( \|x(t)\| = \|y(t)\| = 1 \), and such that

\[
\Phi_t(X) = \sum_{i,j=0}^{2} (W_{i,j}x(t))(W_{i,j}y(t))^*X(W_{i,j}y(t))(W_{i,j}x(t))^*, \quad X \in M_3,
\]

for all \( 0 \leq t \leq \frac{1}{4} \), where \( \Phi_t = tI_3 + (1-t)\Psi_3 \), and \( \{W_{i,j} : 0 \leq i,j \leq 2\} \) are the discrete Weyl matrices.

**Proof.** Consider the functions \( u, v : [0, \frac{1}{4}] \to \mathbb{C}^3 \) defined by

\[
u(t) = \begin{bmatrix} 1 \\ \lambda \\ \lambda \end{bmatrix}, \quad v(t) = \begin{bmatrix} \alpha \\ \beta \\ \beta \end{bmatrix},\]

where \( \lambda, \alpha, \beta \) are also functions of \( t \), but we have suppressed the dependence upon \( t \). The function \( \alpha \) is given by the positive square root of

\[
\alpha^2 = \frac{5 + 4t + 4\sqrt{1 + 7t - 8t^2}}{81}.
\]

The function \( \lambda \) is a solution of \( \lambda(\lambda + 1) = \rho \), say \( \lambda = \frac{-1 + \sqrt{1 + 4\rho}}{2} \), where \( \rho \) is given by

\[
\rho = \frac{1 + 2t - \sqrt{1 + 7t - 8t^2}}{1 - 4t}.
\]

Finally, \( \beta \) is given by \( \beta = -\alpha(\lambda + 1) \).

Now define \( x, y : [0, \frac{1}{4}] \to \mathbb{C}^3 \) as

\[
x(t) = \sqrt{3}\|v(t)\|u(t), \quad y(t) = \frac{v(t)}{\|v(t)\|},
\]
Clearly, $\|y(t)\| = 1$ and another string of calculations shows:

$$\|x(t)\|^2 = 3\alpha^2(1 + 2\lambda^2)(2\lambda^2 + 4\lambda + 3) = 3\alpha^2(4\rho^2 + 4\rho + 3) = 1.$$ 

Computations show that $x$ and $y$ are continuous on the domain. Moreover,

$$x\left(\frac{1}{4}\right) = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} = y\left(\frac{1}{4}\right).$$

This particular fiducial vector indeed matches with $[20]$ when $\rho_0 = \sqrt{\frac{2}{3}}$ and $\theta_1 = \theta_2 = \pi$.

There are again two values of $\|\langle W_{i,j}x(t), x(t)\rangle\|^2$ for all $0 \leq i,j \leq 2$ and $(i,j) \neq (0,0)$. Calling these values $A_1(t)$ and $A_2(t)$, we see that

$$A_1(t) := 9\|v(t)\|^2(1 - \lambda^2)^2 = 9\alpha^4(2\lambda^2 + 4\lambda + 3)^2(1 - \lambda^2)^2,$$

$$A_2(t) := 9\|v(t)\|^2(\lambda - \lambda^2)^2 = 9\alpha^4(2\lambda^2 + 4\lambda + 3)^2(\lambda - \lambda^2)^2,$$

which are graphed in Figure 2.

\[\Box\]

\textbf{5. Mutually unbiased bases}

The existence problem of $d + 1$ mutually unbiased bases in $\mathbb{C}^d$ ($d \geq 2$) is another major open problem in quantum information theory. In this final section we show that the vectors constituting the $d + 1$ mutually unbiased bases give rise to a Choi-Kraus representation of the channel 3.

\textbf{Definition 5.1.} Let $\mathcal{V} = \{v_1, \ldots, v_d\}$ and $\mathcal{W} = \{w_1, \ldots, w_d\}$ be two orthonormal bases of $\mathbb{C}^d$. We say that $\mathcal{V}$ and $\mathcal{W}$ are \textit{mutually unbiased} if $\|\langle v_i, w_j \rangle\|^2 = \frac{1}{d}$ for all $1 \leq i, j \leq d$. (It is easy to see that if $\|\langle v_i, w_j \rangle\|^2 = c$ is constant for all $1 \leq i, j \leq d$, then $c = \frac{1}{d}$.)
Let \( m(d) \) be the maximum number of possible mutually unbiased bases in \( \mathbb{C}^d \). It is an open question to decide how \( m(d) \) depends on \( d \). It is conjectured that \( m(d) = d + 1 \) for all \( d \geq 2 \). When \( d = p^n \) where \( p \) is prime and \( n \) is any natural number, then indeed \( m(p^n) = p^n + 1 \) \cite{[17]}.

However, even for \( d = 6 \) which is not of the form \( p^n \), it is not known whether \( m(6) = 7 \). In fact, only three mutually unbiased bases have been found so far for dimension six.

In what follows, our aim is to show that if there exist \( d + 1 \) mutually unbiased bases \( V_i = \{ v_{i,j} : 1 \leq j \leq d \} \) (\( 1 \leq i \leq d + 1 \)) for \( \mathbb{C}^d \), then the channel defined by

\[
\Phi(X) = \frac{1}{d+1} \sum_{i=1}^{d+1} \sum_{j=1}^{d} P_{i,j}XP_{i,j}, \quad X \in M_d
\]

where \( P_{i,j} \) is the projection onto \( \text{span}\{v_{i,j}\} \), is nothing but the channel \( \mathfrak{3} \).

**Lemma 5.2.** For \( 1 \leq i \leq d + 1 \), let \( V_i = \{ v_{i,j} : 1 \leq j \leq d \} \) be a collection of \( d + 1 \) mutually unbiased bases of \( \mathbb{C}^d \). If \( P_{i,j} = v_{i,j}v_{i,j}^* \) for all \( i, j \), then the set

\[
P = \{ P_{1,j} : 1 \leq j \leq d \} \cup \{ P_{i,j} : 2 \leq i \leq d + 1, 1 \leq j \leq d - 1 \} \subseteq M_d,
\]

is a basis for \( M_d \).

**Proof.** Since \( |P| = d^2 \), it is sufficient to show that it is linearly independent in \( M_d \). Using the fact that \( \{V_i\}_{i=1}^{d+1} \) are mutually unbiased bases, for \( 1 \leq i, k \leq d + 1 \) and \( 1 \leq j, l \leq d \), we have

\[
\text{Tr}(P_{i,j}P_{k,l}) = |\langle v_{i,j}, v_{k,l} \rangle|^2 = \begin{cases} 0 & \text{if } i = k \text{ and } j \neq l, \\ 1 & \text{if } i = k \text{ and } j = l, \\ \frac{1}{d} & \text{if } i \neq k. \end{cases}
\]

Suppose there exist scalars \( \{ \lambda_{i,j} : 1 \leq i \leq d + 1, 1 \leq j \leq d - 1 \} \cup \{ \lambda_{1,d} \} \) such that

\[
\lambda_{1,d}P_{1,d} + \sum_{i=1}^{d+1} \sum_{j=1}^{d-1} \lambda_{i,j}P_{i,j} = 0.
\]

Taking trace of Equation \( 5.2 \) yields

\[
\lambda_{1,d} + \sum_{i=1}^{d+1} \sum_{j=1}^{d-1} \lambda_{i,j} = 0.
\]

Multiplying Equation \( 5.2 \) by \( P_{1,l} \) for \( 1 \leq l \leq d \), taking trace and using Relations \( 5.1 \), we have

\[
0 = \lambda_{1,d}|\langle v_{1,d}, v_{1,l} \rangle|^2 + \sum_{i=1}^{d+1} \sum_{j=1}^{d-1} \lambda_{i,j}|\langle v_{i,j}, v_{1,l} \rangle|^2 = \lambda_{1,l} + \frac{1}{d} \sum_{i=2}^{d+1} \sum_{j=1}^{d-1} \lambda_{i,j},
\]

which implies

\[
\lambda_{1,l} = -\frac{1}{d} \sum_{i=2}^{d+1} \sum_{j=1}^{d-1} \lambda_{i,j}.
\]

Since the right hand side of Equation \( 5.4 \) is independent of \( l \), we conclude that

\[
\lambda_{1,1} = \lambda_{1,2} = \ldots = \lambda_{1,d} =: \lambda_1.
\]
Similarly multiplying Equation (5.2) by $P_{k,l}$, where $2 \leq k \leq d + 1$ and $1 \leq l \leq d - 1$, taking trace, and using Relations (5.1) we arrive at

\begin{equation}
\lambda_{k,l} = \frac{-1}{d} \left( \lambda_{1,d} + \sum_{i=1}^{d+1} \sum_{j=1}^{d+1-1} \lambda_{i,j} \right).
\end{equation}

Again since the right hand side of Equation (5.6) is independent of $l$, we conclude that

\begin{equation}
\lambda_{k,1} = \lambda_{k,2} = \ldots = \lambda_{k,d-1} =: \lambda_k.
\end{equation}

However, for any $2 \leq k \leq d + 1$, using Equation (5.3),

\begin{equation}
\lambda_k = \lambda_{k,l} = \frac{-1}{d} \left( \lambda_{1,d} + \sum_{i=1}^{d+1} \sum_{j=1}^{d+1-1} \lambda_{i,j} \right) = \frac{-1}{d} \left( \lambda_{1,d} + \sum_{i=1}^{d+1} \sum_{j=1}^{d+1-1} \lambda_{i,j} - \sum_{j=1}^{d-1} \lambda_{k,j} \right)
\end{equation}

which yields $\lambda_k = 0$. Then Equation (5.2) reduces to

\begin{equation}
0 = \lambda_{1,1} P_{1,1} + \ldots + \lambda_{1,d} P_{1,d} = \lambda_1 (P_{1,1} + \ldots + P_{1,d}) = \lambda_1 I_d,
\end{equation}

and thus $\lambda_1 = 0$. Hence we have shown that $\lambda_{i,j} = 0$ for all $1 \leq i \leq d + 1$ and $1 \leq j \leq d - 1$, and $\lambda_{1,d} = 0$.

**Theorem 5.3.** For $1 \leq i \leq d + 1$, let $V_i = \{v_{i,j} : 1 \leq j \leq d\}$ be a collection of $d + 1$ mutually unbiased bases of $\mathbb{C}^d$. If $P_{i,j} = v_{i,j} v_{i,j}^*$ for all $i, j$, then the linear map $\Phi : M_d \to M_d$ defined by

\begin{equation}
\Phi(X) = \frac{1}{d+1} \sum_{i=1}^{d+1} \sum_{j=1}^{d} P_{i,j} X P_{i,j}, \quad X \in M_d,
\end{equation}

is a unital quantum channel, and $\Phi = 3$.

**Proof.** Clearly, the map $\Phi$ is completely positive. It is trace preserving and unital since

\begin{equation}
\frac{1}{d+1} \sum_{i=1}^{d+1} \sum_{j=1}^{d} P_{i,j} = \frac{1}{d+1} \sum_{i=1}^{d+1} I_d = I_d.
\end{equation}

Using Relations (5.1), it follows that for $1 \leq k \leq d + 1$ and $1 \leq l \leq d$,

\begin{equation}
\Phi(P_{k,l}) = \frac{1}{d+1} \sum_{i=1}^{d+1} \sum_{j=1}^{d} |\langle v_{i,j}, v_{k,l} \rangle|^2 P_{i,j} = \frac{1}{d+1} (P_{k,l} + I_d).
\end{equation}

If $X \in M_d$, then by Lemma 5.2 we may write $X$ uniquely as

\begin{equation}
X = \sum_{j=1}^{d} \lambda_{1,j} P_{1,j} + \sum_{i=2}^{d+1} \sum_{j=1}^{d-1} \lambda_{i,j} P_{i,j},
\end{equation}

where $\lambda_{i,j}$ are the eigenvalues of $P_{i,j}$.
for some scalars $\lambda_{i,j}$. Taking trace of Equation (5.10),

\[
\text{Tr}(X) = \sum_{j=1}^{d} \lambda_{1,j} + \sum_{i=2}^{d-1} \sum_{j=1}^{d-1} \lambda_{i,j}.
\]

(5.11)

Using Equations (5.9) and (5.11) we have

\[
\Phi(X) = \sum_{j=1}^{d} \lambda_{1,j} \Phi(P_{1,j}) + \sum_{i=2}^{d-1} \sum_{j=1}^{d-1} \lambda_{i,j} \Phi(P_{i,j})
\]

\[
= \frac{1}{d+1} \left( \sum_{j=1}^{d} \lambda_{1,j} (P_{1,j} + \mathbb{I}_d) + \sum_{i=2}^{d-1} \sum_{j=1}^{d-1} \lambda_{i,j} (P_{i,j} + \mathbb{I}_d) \right)
\]

\[
= \frac{1}{d+1} \left( \left( \sum_{j=1}^{d} \lambda_{1,j} P_{1,j} + \sum_{i=2}^{d-1} \sum_{j=1}^{d-1} \lambda_{i,j} P_{i,j} \right) + \left( \sum_{j=1}^{d} \lambda_{1,j} + \sum_{i=2}^{d-1} \sum_{j=1}^{d-1} \lambda_{i,j} \right) \mathbb{I}_d \right)
\]

\[
= \frac{1}{d+1} (X + \text{Tr}(X) \mathbb{I}_d) = \mathcal{Z}(X),
\]

as required. □

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