The pressure of a weakly magnetized hot and dense deconfined QCD matter in one-loop
Hard-Thermal-Loop perturbation theory

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ABSTRACT: We consider our recently obtained general structure of two point (self-energy and propagator) functions of quarks and gluons in a nontrivial background like a heat bath and an external magnetic field. Based on this we have computed free energy and pressure of quarks and gluons for a magnetized hot and dense deconfined QCD matter in weak field approximation. For heat bath we have used hard thermal loop perturbation theory (HTLpt) in presence of finite chemical potential. For weak field approximations, the results are completely analytic and gauge independent but depends on the renormalization scale in addition to the temperature, chemical potential and the external magnetic field. We also discuss the modification of QCD Debye mass of such matter for an arbitrary magnetic field. An analytic expression for Debye mass is also obtained for both strong and weak field approximation. It is found to exhibit some interesting features depending upon the three different scales, i.e., the thermal quark mass, temperature and the strength of the magnetic field. The various divergences appearing in the quark and gluon free energies are regulated through appropriate counter terms. In weak field approximation, the low temperature behaviour of the pressure is found to strongly depend on the magnetic field than that at high temperature. We also discuss the specific problem with one-loop HTLpt associated with the over-counting of certain orders in coupling.
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1 Introduction

Quark Gluon Plasma (QGP) is a thermalized color deconfined state of nuclear matter in the regime of Quantum Chromo Dynamics (QCD) under extreme conditions such as very high temperature and/or density. For the past couple of decades, different high energy Heavy-Ion-Collisions (HIC) experiments are under way, e.g., RHIC @ BNL, LHC @ CERN and upcoming FAIR @ GSI, to study this novel state of QCD matter within the largely unknown QCD phase diagram. In recent years the focus has also shifted towards the noncentral HIC, where a very strong magnetic field is created in the direction perpendicular to the reaction plane due to the spectator particles that are not participating in the collisions [1–5]. Recent experimental evidences of photon anisotropy provided by the PHENIX Collaboration [6] has also challenged the present theoretical tools. By assuming a presence of large anisotropic magnetic field generated in HIC, eventually some explanations were made [7] in support of those experimental findings. In fact this has prompted that a theoretical study is much needed by considering the effects of intense background magnetic field on various aspects and observables of noncentral HIC. Also some of these studies have subsequently revealed that the strong magnetic field generated during the noncentral HIC is also time dependent. More specifically, it rapidly decreases with time [8, 9]. Nevertheless, the inclusion of an external magnetic field in QGP introduces also an extra energy scale in the system. At the time of the noncentral HIC, the value of the created magnetic field $B$ is very high compare to the temperature $T$ ($T^2 < q_f B$ where $q_f$ is the absolute charge of the quark with flavor $f$) associated with the system. It is estimated upto the order of $q_f B \sim 15m^2_{\pi}$ in the LHC at CERN [10]. On the other hand, neutron stars (NS), or more specifically magnetars are also known to possess strong enough magnetic field [11–13]. In this regime of study one usually works in the strong magnetic field approximation.

The presence of an external anisotropic field in the medium calls for the appropriate modification of the present theoretical tools to investigate various properties of QGP and a numerous activity is in progress. Over the last few years, several novel phenomena came into light, e.g, chiral magnetic effect [14–16], magnetic catalysis [17–19] and inverse magnetic catalysis [20–27] at finite temperature; chiral- and color-symmetry broken/restoration phase [28–32]; thermodynamic properties [31–35], refractive indices and decay constant of hadrons [36–40] and the equation of state (EoS) in holographic models [41, 42] in a hot magnetized medium; soft photon production from conformal anomaly [7, 43] in HIC; modification of dispersion properties in a magnetized hot QED [44] and QCD [45–48] medium; various transport coefficients [49–51], properties of quarkonia [52, 53], synchroton radiation [54], dilepton production from a hot magnetized QCD plasma [54–59] and in strongly coupled plasma in a strong magnetic field [60].

Thermodynamic properties of low lying hadrons in presence of magnetic field are studied in recent years within the various hadronic models [31–34]. Nevertheless, the EoS is a generic quantity and of phenomenological importance for studying the hot and dense QCD matter, QGP, created in HIC. At zero chemical potential and finite temperature Lattice QCD (LQCD) established itself as the most reliable method to calculate thermodynamic functions. Unfortunately at finite chemical potential LQCD faces the infamous sign prob-
lem. Information about the thermodynamic functions in LQCD can still be extracted by making a Taylor expansion of the partition function around zero baryonic chemical potential and extrapolating the result [61]. But due to the finite number of Taylor coefficients such extrapolation has its own limitations. On the other hand, naively, the asymptotic freedom of QCD leads us to expect that bare perturbation theory should be a reliable guide to calculate these properties of matter at high temperature and/or high density [62–68]. Although, it has been recognized early on that this is not so. Technically, infrared divergences plague the calculation of observables at finite temperature, preventing the determination of high order corrections. In order to cope with this difficulty, whose origin is the presence of massless particles, it has been suggested to reorganize perturbation theory, by performing the expansion around of a system of massive quasiparticles. The motivation for doing so is that thermal fluctuations can generate a mass. It amounts to a resummation of a class of loop diagrams, where the loop momenta are of the order of the temperature. Such diagrams are those which contribute to give the excitations a thermal mass. The Hard Thermal Loop perturbation theory (HTLpt) is one such state-of-the-art resummed perturbation theory [69]. In HTLpt the EOS of QCD in absence of magnetic field has systematically been computed within one-loop(Leading order (LO)) [69–77], two loop (next-LO (NLO)) [78–81] and three loop (next-to-NLO (NNLO)) [82–88] at finite temperature and chemical potential. Though the all loop order calculations are gauge invariant, the three loop results are complete in g^5, fully analytic that does not require any free fit parameter beside renormalization scale. The thermomagnetic correction to the quark-gluon vertex in the presence of a weak magnetic field within the HTL approximation has been computed recently [89, 90]. Recently the general structure of gluon [45, 46, 48] and quark [47] self-energy and propagator and their spectra have been obtained in a thermomagnetic medium within HTL approximation. Also the thermodynamic quantities in lowest Landau level (LLL) within the strong field approximation has been calculated in Ref. [35] using HTL approximation. In this calculation it has been assumed for the gluonic case without any justification that the only effect of the magnetic field is to shift the Debye mass, without any change in the general structure of two point functions at finite temperature. However, this is not the case as it has explicitly been shown in paper-I [45] that the presence of an external magnetic field breaks the rotational symmetry.

In view of this, presently, a systematic determination of EOS for magnetized hot QCD medium is of great importance. In this article (say paper-II) we make an effort to derive the pressure of a magnetized hot and dense deconfined QCD medium created in high energy HIC. Usually two kinds of approaches are taken in all the previous studies of EoS in presence of magnetic field. In first kind the pressure remains isotropic and the system can be easily described in terms of standard thermodynamic relations [12, 13]. In other one, the breaking of the spherical symmetry due to the anisotropic background magnetic field in a preferred direction [91–95] is taken into consideration. Subsequently, this results in an anisotropic pressure arising from the difference between pressure components that are transverse to and longitudinal to the background magnetic field direction. Eventually, the

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1Paper-I [45].
difference in stress causes the deformation of the fireball produced in heavy ion collisions or the NS. There are also some recent LQCD calculations, which incorporate both of these schemes [96]. However, it is also shown in [33], that pressure anisotropy decreases with the increase in temperature. Moreover, the magnetic field created in noncentral HIC is a fast decreasing function of time [8, 9], it is expected that by the time the quarks and gluons thermalize in a QGP medium, the magnetic field strength becomes sufficiently weak. By virtue of which temperature at that time becomes the largest energy scale of the system. In this regime, in principle, one can work within the weak magnetic field approximation which, in addition, leads to analytical simplicity. In this paper II we would work on weak \( (m_f^2 \sim m_{th}^2 \sim g^2 T^2 < q_f B < T^2) \) field limit where \( m_f \) is the current quark mass, \( m_{th} \) is thermal mass of a quark, \( T \) is the temperature, \( g \) is the strong coupling and \( B \) is the strength of an external magnetic field. For strong field case, the system is considered to be confined in the lowest landau level (LLL). On the other hand, a weak magnetic field case is different than that of a strong field one, as we shall see later. We here consider the first approach which states that the magnetic pressure adds isotropically\(^2\) to the total pressure by considering the general two point function of quark [47] and gluon [45] in a hot magnetized deconfined QCD medium. In this paper-II, for the first time, we compute the pressure of a hot and dense deconfined QCD medium in presence of weak magnetic field using one-loop HTLpt. As we would see that the calculation is very involve, nevertheless the expression free energy vis-a-vis pressure is completely analytic and gauge independent. We have used strong coupling that runs through renormalization scale and strength of the magnetic field in weak field domain. Sensitivity of the two scales on the pressure has also been discussed in details.

The paper-II has been organized as follows: in section 2 the basic computation of this manuscript has briefly been outlined. In section 3 we discuss the 1-loop quark free energy for a magnetized hot QCD medium within weak field\(^3\) and HTL approximation by considering a most general structure of the quark propagator [47]. Section 4 discusses gluon free energy for a magnetized hot and dense QCD medium in terms of the general structure of two point functions of gauge boson as obtained paper-I [45]. In subsection 4.1 we discuss the Debye screening mass both in strong and weak field approximation that clearly separates this two domains. In section 5 we obtain the finite contribution of weak field free energy in subsection 5.1 and pressure in subsection 5.2 of a magnetized hot medium. The magnetic field dependent strong coupling is discussed in subsection 6 and its validity in weak field domain is justified. Results are discussed in section 7 and finally, we conclude in section 8. The detailed calculations associated with various sections and subsections are given in appendices A, B and C and various subsections under them therein.

\(^2\)In Refs. [45, 47] it has been demonstrated that the various structure functions and correlators (propagator, self-energy etc) for both fermion and gauge boson become anisotropic due to presence of magnetic field which breaks the rotational invariance. The consequence of which makes various structure functions to depend on the angle between the gauge boson momentum and the magnetic field. However, at the end one integrates over the external momentum for obtaining free energy. This results in a isotropic free energy vis-a-vis pressure.

\(^3\)We note that the free energy in strong field approximation \( (m_f^2 \sim m_{th}^2 \sim g^2 T^2 < T^2 < q_f B) \) has already been computed in Ref. [35].
2 Setup

The total thermodynamic free energy up to one-loop order in HTLpt in presence of a background magnetic field, $B$, can be written as

$$F = F_q + F_g + F_0 + \Delta E_0,$$

(2.1)

where $F_q$ and $F_g$ are, respectively, the quark and gluon part of the free energy which will be computed in presence of magnetic filed with HTL approximation. $F_0 = \frac{1}{2} B^2$ is the tree level contribution due to the constant magnetic field and the $\Delta E_0$ is the HTL counter term given [80] as

$$\Delta E_0 = \frac{d_A}{128\pi^2 \epsilon} m_D^4,$$

(2.2)

with $d_A = N_c^2 - 1$, $N_c$ is the number of color in fundamental representation and $m_D$ is the Debye screening mass in HTL approximation.

The pressure of a system is defined as

$$P = -F.$$

(2.3)

3 Quark Free Energy in presence of magnetic field

In weak field one usually works in the domain ($m_f^2 \sim m_{th}^2 \sim g^2 T^2 < q_f B < T^2$), We will discuss about this mass scale later in details when one encounters magnetic mass.

3.1 General Structure of two-point fermionic function

![Figure 1. Diagramatic representation of the Dyson-Schwinger equation for one-loop effective fermion propagator.](image)

The inverse of the effective fermion propagator following the Dyson-Schwinger equation, as given in Fig. 1, can now be written as

$$S_{eff}^{-1}(P) = \slashed{P} - \Sigma(P).$$

(3.1)

The general structure of a fermionic two point function and its dispersion spectrum in a hot magnetized medium has recently been discussed in detail in Ref. [47] 4. The most general form of the fermion self-energy for CPT 5 and chirally invariant theory in a hot magnetized medium becomes

$$\Sigma(P) = - A \slashed{P} - B \gamma_5 \slashed{P} - B' \gamma_5 \gamma_5 \gamma_5 \slashed{P}. $$

(3.2)

4In which two of us, i.e., AB and MGM were involved
5Charge-conjugation, Parity and Time reversal.
where \( u_\mu \) is the four velocity of the heat bath and the direction of the magnetic field \([45, 47]\), \( n_\mu \) is given as
\[
n_\mu = \frac{1}{2B} \epsilon_{\mu\rho\lambda\gamma} u^\rho F^{\rho\lambda} = \frac{1}{B} u^\nu \tilde{F}_{\mu\nu}.
\] (3.3)

The background dual field tensor \( \tilde{F}_{\mu\nu} \) can be written in terms of field tensor \( F_{\mu\nu} \) as
\[
\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\rho\lambda\gamma} F^{\rho\lambda}.
\] (3.4)

Without any loss of generality we have considered the four velocity in the rest frame of the heat bath and the direction of the magnetic field \( B \) along \( z \)-direction, respectively, as
\[
u = (1, 0, 0, 0),
\] (3.5a)
\[
n_\mu = (0, 0, 1).
\] (3.5b)

The general form of the various structure functions can be obtained from Eq. (3.2) as
\[
A = \frac{1}{4} \frac{\text{Tr} \left[ \Sigma(P) \hat{P} \right] - (P \cdot u) \text{Tr} \left[ \Sigma(P) \hat{\gamma} \right]}{(P \cdot u)^2 - P^2},
\] (3.6a)
\[
B = \frac{1}{4} \frac{-(P \cdot u) \text{Tr} \left[ \Sigma(P) \hat{P} \right] + P^2 \text{Tr} \left[ \Sigma(P) \hat{\gamma} \right]}{(P \cdot u)^2 - P^2},
\] (3.6b)
\[
B' = -\frac{1}{4} \text{Tr} \left[ \hat{\gamma} \Sigma(P) \gamma_5 \right],
\] (3.6c)
\[
C' = \frac{1}{4} \text{Tr} \left[ \hat{\gamma} \Sigma(P) \gamma_5 \right],
\] (3.6d)

which are also Lorentz scalars. Beside \( T, \mu \) and \( B \), these structure functions would also depend on three Lorentz scalars due to the breaking of both Lorentz(boost) and rotational invariance defined by
\[
p_0 = \omega \equiv P^\mu u_\mu, \quad (3.7a)
\]
\[
p^3 = -P^\mu n_\mu = p_z, \quad (3.7b)
\]
\[
p_\perp \equiv \left[ (P^\mu u_\mu)^2 - (P^\mu n_\mu)^2 - (P^\mu P_\mu) \right]^{1/2}
\]
\[
= \left[ p_0^2 - p_3^2 - P^2 \right]^{1/2} = \left[ p_1^2 + p_2^2 \right]^{1/2}. \quad (3.7c)
\]

**Figure 2.** Self-energy diagram for a quark in weak magnetic field approximation. The double line indicates the modified quark propagator in presence of magnetic field.
All these structure functions in Eq. (3.6a) to (3.6d) in 1-loop order as shown in Fig. 2 within weak field and HTL approximations have been computed \(^6\) in Ref. [47] as

\[
\mathcal{A}(p_0, p_\perp, p_3) = -\frac{m_{th}^2}{p^2} \int \frac{d\Omega}{4\pi} \frac{p \cdot \hat{k}}{P \cdot \hat{K}}, \tag{3.8a}
\]

\[
\mathcal{B}(p_0, p_\perp, p_3) = \frac{m_{th}^2}{p^2} \int \frac{d\Omega}{4\pi} \frac{(P \cdot u)(P \cdot \hat{k}) - p^2}{P \cdot \hat{K}}, \tag{3.8b}
\]

\[
\mathcal{B}'(p_0, p_\perp, p_3) = -m_{\text{eff}}^2 \int \frac{d\Omega}{4\pi} \frac{\hat{K} \cdot u}{P \cdot \hat{K}}, \tag{3.8c}
\]

\[
\mathcal{C}'(p_0, p_\perp, p_3) = m_{\text{eff}}^2 \int \frac{d\Omega}{4\pi} \frac{\hat{K} \cdot u}{P \cdot \hat{K}}. \tag{3.8d}
\]

We emphasize that the structure functions in Eq. (3.8c) and (3.8d) have become anisotropic in nature due to the breaking of the rotational invariance in presence of magnetic field in a given direction. Also note that

\[
m_{\text{th}}^2 = \frac{g^2 C_F T^2}{8} (1 + 4\mu^2), \tag{3.9a}
\]

\[
m_{\text{eff}}^2 = 4g^2 C_F M_B^2(T, \mu, m_f, q_f B); \quad M_B^2 = \sum_f M_{B,f}^2(T, \mu, m_f, q_f B). \tag{3.9b}
\]

The magnetic mass\(^7\) for a given flavor \(f\) is given as

\[
M_{B,f}^2 = \frac{q_f B}{16\pi^2} \left[ -\frac{1}{4} \tilde{N}(z) - \frac{\pi T}{2m_f} - \frac{\gamma_E}{2} \right], \tag{3.11}
\]

where the function \(\tilde{N}(z)\) is defined in Eq. (A.13) in Appendix A. Now we note that in the limit of small current quark mass \((m_f \to 0)\), the magnetic mass in (3.11) diverges. One can regulate it by using the thermal mass \(m_{\text{th}}\) in (3.9a). In weak field approximation the domain of applicability now becomes \(m_{\text{th}}^2 < q_f B < T^2\) instead of \(m_{\text{th}}^2 < q_f B < T^2\).

Combining Eq. (3.2) with Eq. (3.8a)-(3.8d) the total quark self-energy contribution of \(O[q_f B]\) in presence of a weak magnetic field within HTL approximation can be written as

\[
\Sigma(P) = m_{\text{th}}^2 \int \frac{d\Omega}{4\pi} \frac{\hat{K}}{P \cdot \hat{K}} + m_{\text{eff}}^2 \int \frac{d\Omega}{4\pi} \frac{([\hat{K} \cdot n] \gamma_{\perp} - ([\hat{K} \cdot u] \gamma_{\perp})}{P \cdot \hat{K}}. \tag{3.12}
\]

Using the general structure of the quark self-energy and considering the external anisotropic weak magnetic field along the \(z\) or \(3\) direction [47] one can now write

\[
S_{\text{eff}}^{-1} = \mathcal{P} - \Sigma(P) = \left[ \mathcal{C}(p_0, p_\perp, p_3) p_0 \gamma_0 - \mathcal{D}(p_0, p_\perp, p_3) p_i \gamma_i + \mathcal{B}'(p_0, p_\perp, p_3) \gamma_5 \gamma_0 \right].
\]

\(^6\) In Ref. [47] those structure functions were computed for \(\mu = 0\) but we have modified it for \(\mu \neq 0\).

\(^7\) In case of finite chemical potential the expression for \(f_1\), the well known fermionic function also given in Eq.(34) of Ref [90], gets modified as

\[
f_1(y) = -\frac{1}{2} \ln \left( \frac{y}{4\pi} \right) + \frac{1}{4} \tilde{N}(z) + \cdots, \tag{3.10}
\]

which gets reflected in the expression of the magnetic mass in (3.11). For zero chemical potential the expression for magnetic mass becomes [90] \(M_{B,f}^2 = \frac{q_f B}{16\pi^2} \ln \left( 2 - \frac{\pi T}{2m_f} \right).\)
\[ C(p_0, p_\perp, p_3) = 1 + A(p_0, p_\perp, p_3) + \frac{B(p_0, p_\perp, p_3)}{p_0} = 1 - A'(p_0, p_\perp, p_3), \]
\[ D(p_0, p_\perp, p_3) = 1 - A(p_0, p_\perp, p_3), \]

with
\[ A(p_0, p_\perp, p_3) = -\frac{m_{th}^2}{p^2} \int \frac{d\Omega}{4\pi} \frac{p \cdot \hat{k}}{p_0 - p \cdot \hat{k}} = \frac{m_{th}^2}{p_0^2} [1 - T_P], \]
\[ A'(p_0, p_\perp, p_3) = \frac{m_{th}^2}{p_0} \int \frac{d\Omega}{4\pi} \frac{1}{p_0 - p \cdot \hat{k}} = \frac{m_{th}^2}{p_0^2} T_P, \]
\[ B'(p_0, p_\perp, p_3) = -\frac{m_{eff}^2}{p_0} \int \frac{d\Omega}{4\pi} \frac{\hat{k}_3}{p_0 - p \cdot \hat{k}} = \frac{m_{eff}^2}{p} [1 - T_P], \]
\[ C'(p_0, p_\perp, p_3) = m_{eff}^2 \int \frac{d\Omega}{4\pi} \frac{1}{p_0 - p \cdot \hat{k}} = \frac{m_{eff}^2}{p_0} T_P, \]

where we have written the coefficients in terms of
\[ T_P = \int \frac{d\Omega}{4\pi} \frac{p_0}{p_0 - p \cdot \hat{k}}, \]

which is an integral defined as the angular average over \( \angle p, \hat{k} \). For convenience, the arguments in all those structure functions will be omitted henceforth.

### 3.2 One-loop quark free energy

In statistical field theory the partition function \( Z \) can be represented as a functional determinant and by which the quark part of the free energy in one-loop order can be written as
\[ F_q = -N_cN_f \int \frac{d^4 P}{(2\pi)^4} \ln \left( \det \left[ S^{-1}_{\text{eff}}(P) \right] \right), \]

where \( P \equiv (p_0, p = |p|) \) is the four momentum of the external fermion with \( N_f \) flavor. For ideal gas of quarks the free energy reads as
\[ F_{q, \text{ideal}} = -2N_cN_f \int \frac{d^4 P}{(2\pi)^4} \ln (-P^2) \]
\[ = -\frac{7\pi^2 T^4}{180} N_cN_f \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right), \]

where \( \hat{\mu} = \mu/(2\pi T) \).

The quark free energy in terms of the inverse of general quark propagator is already defined in Eq. (3.21). In terms of those introduced notations in subsection 3.1, we evaluate the determinant of Eq. (3.13) as
\[ \det \left[ S^{-1}_{\text{eff}} \right] = (C^2 p_0^2 - D^2 p^2 + B'^2 - C'^2)^2 - 4(p_0 B' C + p_3 C'D) \]

\[ + C'(p_0, p_\perp, p_3) \gamma_5 \gamma_3 \]
with HTL approximation can be written as

\[ F_q = -N_c N_f \int \frac{d^4P}{(2\pi)^4} \ln \left( \frac{A_0^2 - A_s^2}{P^4} \right) \]

Combining Eqs. (3.21) and (3.23), the one-loop quark free energy in a weak magnetic field and HTL approximation can be written as

\[
\begin{align*}
F_q &= -N_c N_f \int \frac{d^4P}{(2\pi)^4} \ln \left( A_0^2 - A_s^2 \right) \\
&= -2N_c N_f \int \frac{d^4P}{(2\pi)^4} \ln (P^2) - N_c N_f \int \frac{d^4P}{(2\pi)^4} \ln \left( \frac{A_0^2 - A_s^2}{P^4} \right) \\
&= N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \mu^2 + \frac{240}{7} \mu^4 \right) \\
&\quad - \int \frac{d^4P}{(2\pi)^4} \ln \left( \frac{(A_0 + A_s)(A_0 - A_s)}{P^4} \right) \right].
\end{align*}
\]

Now the argument of the logarithm in Eq. (3.24) can be simplified using Eq. (3.23) as

\[
\frac{(A_0 + A_s)(A_0 - A_s)}{P^4} = 1 + 2 \left( \frac{A'(A' - 2)p_0^2 - A(A + 2)p^2 + B'^2 - C'^2}{P^2} \right) \\
+ \frac{(A'(A' - 2)p_0^2 - A(A + 2)p^2 + B'^2 - C'^2)^2 - 4(B'C' p_0 + C'D' p_3)^2}{P^4}.
\]

In high temperature limit, the logarithmic term in Eq. (3.24) can be expanded in series of coupling constant \( g \) keeping terms upto \( O(g^4) \) as

\[
\ln \left( \frac{(A_0 + A_s)(A_0 - A_s)}{P^4} \right) = 2 \left( \frac{A'^2 p_0^2 - A^2 p^2 + B'^2 - C'^2 - 2A' p_0^2 - 2A p^2}{P^2} \right) \\
- 4 \left( \frac{(A' p_0^2 + A p^2)^2 + (B' p_0 + C' p_3)^2}{P^4} \right) + O(g^6),
\]

with

\[
\begin{align*}
(A'^2 - C'^2) &= m_{\text{eff}}^4 \left[ \frac{1}{P^2} + \frac{T_P^2}{P^2} - \frac{T_P^2}{p_0^2} - \frac{2T_P}{p_0^2} \right], \\
(B' p_0 + C' p_3)^2 &= m_{\text{eff}}^4 \left[ \frac{p_0^2}{p_0^2} (1 + T_P^2 - 2T_P) + \frac{T_P^2}{p_0^2} p_3^2 \right], \\
(A' p_0^2 + A p^2) &= m_{\text{th}}^2, \\
(A'^2 p_0^2 - A^2 p^2) &= m_{\text{th}}^4 \left[ \frac{T_P^2}{p_0^2} - \frac{(1 - T_P)^2}{p^2} \right].
\end{align*}
\]

So, upto \( O(g^4) \) the one-loop free energy can be written as,

\[
\begin{align*}
F_q &= N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \mu^2 + \frac{240}{7} \mu^4 \right) + 4m_{\text{th}}^2 \int \frac{d^4P}{(2\pi)^4} \frac{1}{P^2} \right] \\
&\quad - m_{\text{th}}^4 \int \frac{d^4P}{(2\pi)^4} \left[ \frac{2T_P^2}{p_0^2 P^2} - \frac{4}{P^4} - \frac{2}{p^2 P^2} - \frac{2T_P^2}{p^2 P^2} + \frac{4T_P}{p^4 P^2} \right].
\end{align*}
\]
\[- \frac{m_\text{eff}^4}{N_f} \int \frac{d^4 P}{(2\pi)^4} \left[ \frac{2}{P^2} \left( \frac{1}{P^2} + \frac{T_P^2}{P^2} - \frac{2T_P}{P^2} \right) - \frac{4}{P^4} \left( \frac{p_0^2}{P^2} (1 + T_P^2 - 2T_P) + \frac{T_P^2}{P^2} p_3^2 \right) \right] \]

\[= N_c N_f \left[ - \frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \mu^2 + \frac{240}{7} \mu^4 \right) + 4m_\text{th}^4 \sum_{\{P\}} \frac{1}{P^2} \right] \]

\[- \frac{m_\text{eff}^4}{N_f} \left( \sum_{\{P\}} 2T_P^2 \frac{p_0}{P^2} - \sum_{\{P\}} \frac{4}{P^2} - 2 \sum_{\{P\}} 2T_P^2 \frac{p^2}{P^2} + \sum_{\{P\}} 4T_P \right) \]

\[+ \frac{2m_\text{eff}^4}{N_f} \left( \sum_{\{P\}} \frac{1}{p^2} - \sum_{\{P\}} \frac{T_P^2}{P^2} \right) \]

\[+ \frac{4m_\text{eff}^4}{N_f} \left( \sum_{\{P\}} \frac{1}{P^2} + \sum_{\{P\}} \frac{T_P^2}{P^2} + \sum_{\{P\}} \frac{T_P^2}{p_0^2 P^2} - \sum_{\{P\}} 2T_P \right) \]

\[= N_c N_f \left[ - \frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \mu^2 + \frac{240}{7} \mu^4 \right) + \frac{m_\text{th}^4 T^2}{6} (1 + 12\tilde{\mu}^2) \right. \]

\[+ 4m_\text{th}^4 \left[ 1 + \frac{1 - 2\Delta_3 + \Delta_4^\prime - \Delta_3^\prime}{2 - d} \right] \sum_{\{P\}} \frac{1}{P^4} \left. - \frac{4m_\text{eff}^4}{N_f} \sum_{\{P\}} \frac{1}{P^4} \right] \]

\[\times \left( 1 - \Delta_0'' - \frac{1 - 2\Delta_3 + \Delta_4^\prime - \Delta_3^\prime}{2 - d} + 2\Delta_{10} + \Delta_{11} \right) \frac{2}{2 - 5} - \frac{\Delta_0'' + \Delta_{11}}{d} \right) \], (3.31)

where all the sum-integrals are provided in the Appendix A. \(\Delta_i\)'s are the c-integrations arising due to angular integral \(T_P\) which are computed in the Appendix C. Using the expressions for various sum-integrals obtained in the Appendix A we can write

\[F_\text{q} = N_c N_f \left[ - \frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \mu^2 + \frac{240}{7} \mu^4 \right) \right. \]

\[+ \frac{m_\text{th}^4 T^2}{6} (1 + 12\tilde{\mu}^2) + 4m_\text{th}^4 \left[ \left( \frac{\pi^2}{3} - 2 \right \frac{2}{P^2} \right \frac{1}{P^4} \right] \]

\[+ \frac{4m_\text{eff}^4}{N_f} \left[ \frac{4}{9} (2\pi^2 - 15) - \frac{2}{27} (-72\zeta(3) + 7\pi^2 + 60) \frac{1}{P^4} \right] \]
\[ + \frac{g^4 C_F^2 T^4}{768 \pi^2} \left(1 + 4 \mu^2\right)^2 (\pi^2 - 6) + \frac{4g^4 C_F^2}{9N_f \pi^2} M_B^4 \]
\[ \times \left[ \left(2 \ln \frac{\Lambda}{4\pi T} - \Re(z)\right)(2\pi^2 - 15) + 12\zeta(3) - \frac{7\pi^2}{6} - 10 \right] \]
\[ + \frac{4N_c g^4 C_F^2}{9\epsilon} M_B^4 \left(2 - \frac{15}{\pi^2}\right). \]  

(3.32)

We note that the themomagnetic correction to quark part of the free energy in weak field has \(O(1/\epsilon)\) divergence, originating due to HTL approximation. To obtain finite contribution one needs an appropriate counter term which will be discussed later.

### 4 Gluon Free energy in presence of magnetic field

It is convenient to calculate the gluon partition function in Euclidean space. In general the QCD partition function for a gluon can be written in Euclidean space as

\[
Z_g = Z Z^\text{ghost}, \quad Z = N_\xi \prod_{n,p} \sqrt{\frac{(2\pi)^D}{\det D^{-1}_{\mu\nu,E}(P_E)}}, \quad Z^\text{ghost} = \prod_{n,p} P^2_E, \tag{4.1}
\]

where the product over \(n\) is for the discrete Bosonic Matsubara frequencies \((\omega_n = 2\pi n\beta; \ n = 0, 1, 2, \cdots)\) due to Euclidean time whereas \(p\) is for the spatial momentum, \(D\) is the space-time dimension of the theory, \(P^2_E = \omega_n^2 + p^2\) is the square of four momentum with \(D^{-1}_{\mu\nu,E}\) is the inverse gauge boson propagator in Euclidean space. \(N_\xi = 1/(2\pi \xi)^D/2\) is the normalization originates from the introduction of Gaussian integral at each location of position while averaging over the gauge condition function with a width \(\xi\), the gauge fixing parameter.

Gluon free energy can now be written as

\[
F_g = -(N_c^2 - 1) \frac{T}{V} \ln Z_g = (N_c^2 - 1) \left[ \frac{1}{2} \sum_{P_E} \ln \left[ \det \left(D^{-1}_{\mu\nu,E}(P_E)\right)\right] - \sum_{P} \ln P^2_E \right], \tag{4.2}
\]

where the gauge dependence explicitly cancels due to the presence of the normalization factor \(N_\xi\).

For ideal case \(\det \left(D^{-1}_{\mu\nu,E}(P)\right) = (P^2_E)^4/\xi\) and hence the free energy for \((N_c^2 - 1)\) massless spin one gluons yields as

\[
F_g^{\text{ideal}} = (N_c^2 - 1) \sum_{P_E} P^2_E = (N_c^2 - 1) \sum_{P} \ln (-P^2) = -(N_c^2 - 1) \frac{\pi^2 T^4}{45}, \tag{4.3}
\]

where \(P\) is four-momentum in Minkowski space and can be written as \(P^2 = p_0^2 - p^2\).

In presence of thermal background medium \([45, 100, 101]\) one can have

\[
\det \left(D^{-1}_{\mu\nu,E}(P_E)\right) = P^2_E \frac{\pi^2}{\xi} (P^2_E + P_T)^2 \left(P^2_E + \Pi_L\right), \tag{4.4}
\]
with four eigenvalues; respectively $P_E^2, (P_E^2 + \Pi_L)$ and two fold degenerate $(P_E^2 + \Pi_T)$. Here $\Pi_T$ and $\Pi_L$ are the transverse and longitudinal part of the gluon self-energy in thermal medium. Also we considered $D = 4$, and the spatial dimension, $d = 3$ throughout this manuscript\(^8\). From now on, we use Minkowski momentum $P$. Eventually the free energy becomes

$$F_{\text{th}}^g = \frac{1}{2} \left[ \sum_{\vec{p}} \ln (-P^2) + 2 \sum_{\vec{p}} \ln (-P^2 + \Pi_T) + \sum_{\vec{p}} \ln (-P^2 + \Pi_L) \right] - \sum_{\vec{p}} \ln (-P^2),$$

$$= \sum_{\vec{p}} \ln (-P^2 + \Pi_T) + \frac{1}{2} \sum_{\vec{p}} \ln \left( 1 - \frac{\Pi_T}{P^2} \right)$$

$$= (N_c^2 - 1) \left[ (d - 1) F_{\text{th}}^{T} + F_{\text{th}}^{L} \right],$$

(4.5)

Also, $F_{\text{th}}^{T}$ and $F_{\text{th}}^{L}$ are, respectively, the transverse and longitudinal part of the gluon free energy, both of which can be computed with the help of the general structure of gauge boson self-energy evaluated in Refs. [45, 100, 101]. Now, in presence of a hot magnetized medium the general structure of inverse propagator of a gauge boson is computed in Ref. [45] and reads as

$$\langle D_{\mu\nu} \rangle^{-1} = \frac{P_{\mu}^2}{\xi} \eta_{\mu\nu} + (P_m^2 - b) B_{\mu\nu} + (P_m^2 - c) R_{\mu\nu} + (P_m^2 - d) Q_{\mu\nu},$$

(4.7)

where

$$P_m^2 = \frac{P_{\mu}^2 \xi - 1}{\xi}. \tag{4.8}$$

The determinant of inverse of the gauge boson propagator can be evaluated from Eq. (4.7) as

$$\det \left( \langle D_{\mu\nu}^{-1} \rangle_{E}(P) \right) = -\frac{P_{\mu}^2}{\xi} ( -P^2 + b ) ( -P^2 + c ) ( -P^2 + d ),$$

(4.9)

with four eigenvalues: $-P^2/\xi$, $(-P^2 + b)$, $(-P^2 + c)$ and $(-P^2 + d)$. We note here that instead of a two fold degenerate transverse mode $(-P^2 + \Pi_T)$ in thermal medium in Eq. (4.4), now one has two distinct transverse modes, $(-P^2 + c)$ and $(-P^2 + d)$.

Using Eq. (4.9) in Eq. (4.2), the one-loop gluon free energy for hot magnetized medium is given by

$$F_g = (N_c^2 - 1) \left[ F_{\text{g}}^1 + F_{\text{g}}^2 + F_{\text{g}}^3 \right],$$

(4.10)

where

$$F_{\text{g}}^1 = \frac{1}{2} \sum_{\vec{p}} \ln \left( 1 - \frac{b}{P^2} \right), \tag{4.11a}$$

$$F_{\text{g}}^2 = \frac{1}{2} \sum_{\vec{p}} \ln (-P^2 + c), \tag{4.11b}$$

\(^8\) We will also use $d = 3 - 2\epsilon$ for dimensional regularization.
\( F_g^3 = \frac{1}{2} \sum_{P} \ln (-P^2 + d) \). \hfill (4.11c)

The various structure functions are obtained in Ref. [45] in both strong and weak field approximation. In the following sections we obtain the QCD Debye mass and the gluon free energy in weak magnetic field approximation.

In the following subsection we would discuss the Debye screening mass in a thermomagnetic QCD matter before moving into the gluon free energies in weakly magnetized medium.

4.1 QCD Debye mass in a magnetized hot and dense medium

The electromagnetic Debye mass in presence of magnetic field was computed in [45, 55, 102]. Generalizing this to QCD we obtain the expression for the modified QCD Debye mass at finite chemical potential and an arbitrary magnetic field as

\[
(m_B^D)^2 = \frac{g^2 N_c T^2}{3} + \sum_f \frac{g^2 q_f B}{2\pi^2} \int_0^\infty e^{-x} dx \times \sum_{l=1}^\infty (-1)^{l+1} \cosh (2l\pi \hat{\mu}) \coth \left( \frac{q_f B l^2}{4xT^2} \right) \exp \left( -\frac{m^2_f l^2}{4xT^2} \right). \hfill (4.12)
\]

Now, in the strong magnetic field limit \((m_{th}^2 \sim g^2 T^2 \leq T^2 \leq q_f B)\), i.e. in LLL, neglecting the current quark mass \(m_f\), from Eq. (4.12) we can straightway reach to a simplified expression\(^9\) given as

\[
(m_s^D)^2 = \frac{g^2 N_c T^2}{3} + \sum_f \frac{g^2 q_f B}{4\pi^2}. \hfill (4.13)
\]

We were able to get the same expression for Debye mass in the strong magnetic field limit as in Eq. (4.13) when we calculate gluon polarization tensor using quark propagator in strong field approximation and take the static limit of the zero-zero component of that tensor [45].

In the weak field approximation \((T^2 > q_f B > m_{th}^2)\), the square of Debye mass can be obtained from Eq. (4.12) by expanding \(\coth (q_f B l^2/4xT^2)\) as

\[
(m_w^D)^2 \simeq \frac{g^2 T^2}{3} \left[ (N_c + N_f \hat{\mu}^2) + 6N_f \hat{\mu}^2 \right] + \sum_f \frac{g^2 (q_f B)^2}{12\pi^2 T^2} \sum_{l=1}^\infty (-1)^{l+1} l^2 \cosh (2l\pi \hat{\mu}) K_0 \left( \frac{m_f l}{T} \right) + O[(q_f B)^4]
\]

\[
= m_w^2 + \delta m_w^2. \hfill (4.14)
\]

\(^9\)Our fermionic part of Debye mass is different from Ref. [52] by a factor of 2 which was somehow overlooked by the authors of the Ref. [52] in Matsurbara Sum. We also find the same mismatch with the Ref. [104].
where $m_D$ can be identified as the QCD Debye mass in a hot and dense medium in absence of any external magnetic field and $K_n(z)$ represents the modified Bessel function of the second kind. In Eq. (4.14) the first term is the Debye mass contribution in the absence of the external magnetic field whereas the second term is the thermomagnetic correction due to the presence of the weak external magnetic field.

In Fig. 3 the full expression in Eq. (4.12), the strong field expression in Eq. (4.13) and the weak field expression in Eq. (4.14) scaled with $m_D$ are displayed as magnetic field scaled with squared pion mass. In the strong field limit, (e.g., for $T = 200$ MeV, $|eB|/m_\pi^2 \geq 10$) the weak field result (red colored curve) starts to deviate from the full result (dashed line). However, there is no difference between the two in limit $|eB|/m_\pi^2 < 10$, so it defines the domain of weak field for $T = 200$ MeV and is indeed a good approximation to work with Eq. (4.14) in the weak field limit at that temperature. On the other hand, the LLL result (green colored line) agrees with that of full when, e.g., for $T = 200$ MeV, $|eB|/m_\pi^2 \geq 70$ MeV. And in between one should work with the full result for a given temperature. The right pane is for $T = 300$ MeV which shows same behaviour. However, these two plots indicate that the domain of applicability, for strong ($|eB| > T^2$) and weak ($|eB| < T^2$) field, changes quantitatively with the change in temperature. In between the weak and strong field domain in principle one should work with full expression but it is indeed a very involved and difficult task. However, we confine ourselves in weak magnetic field limit.

### 4.2 One-loop gluon free energy in weakly magnetized hot medium

The form factors for the gluonic self-energy in a weakly magnetized medium can be expressed as

$$b(T, \mu, B) = b_0(T, \mu) + b_2(T, \mu, B),$$
c(T, μ, B) = c_0(T, μ) + c_2(T, μ, B),

d(T, μ, B) = d_0(T, μ) + d_2(T, μ, B),

with

\[ b_0 = \frac{m_0^2}{u^2} (1 - \mathcal{T}_A), \]
\[ c_0 = d_0 = \frac{m_0^2}{2\mu^2} [p_0^2 - P^2\mathcal{T}_A], \]
\[ b_2 = \frac{\delta m_0^2}{u^2} + \sum_f \frac{g^2(q_f B)^2}{u^2\pi^2} \left[ \left( g_k + \frac{\pi m_f - 4T}{32 m_f^2 T} \right) (A_0 - A_2) + \left( f_k + \frac{8T - \pi m_f}{128 m_f^2 T} \right) \left( \frac{5}{3} A_0 - A_2 \right) \right], \]
\[ c_2 = - \sum_f \frac{4g^2(q_f B)^2}{3\pi^2} g_k + \sum_f \frac{g^2(q_f B)^2}{2\pi^2} \left( g_k + \frac{\pi m_f - 4T}{32 m_f^2 T} \right) \times \left[ -\frac{7}{3} p_0^2 + \left( 2 + \frac{3}{2} p_0^2 \right) A_0 \right] \]
\[ + \left( \frac{3}{2} + \frac{5p_0^2}{2\mu^2} + \frac{5p_0^3}{2\mu^2} \right) A_2 - \frac{3p_0 p_3}{\mu^2} A_1 - \frac{5}{2} \left( 1 - \frac{p_0^2}{\mu^2} \right) A_4 - \frac{5p_0 p_3}{\mu^2} A_3 \right], \]
\[ d_2 = - \sum_f \frac{g^2(q_f B)^2}{\pi^2} \frac{p_0^2 p_3}{\mu^2} \left[ g_k + \frac{\pi m_f - 4T}{32 m_f^2 T} \right] \left\{ \frac{A_0}{4} - \left( \frac{3}{2} + \frac{p_0^2}{\mu^2} \right) A_2 + \frac{5A_4}{4} \right\} \]
\[ - \frac{14}{3} \frac{f_k p_0^2 p_3}{\mu^2} + \left( f_k + \frac{8T - \pi m_f}{128 m_f^2 T} \right) \frac{p_0^2 p_3}{\mu^2} (5A_0 - A_2) \]
\[ + \sum_f \frac{g^2(q_f B)^2}{6\pi^2 m_f T} \frac{(3A_2 - A_3)}{1 + \cosh \frac{m_f}{T}} \frac{p_0 p_3}{\mu^2}, \]

where \( f_k, g_k \) and \( A_n \)'s are defined in paper-I [45] with \( A_0 = T_0 \). \(^{10}\)

Therefore, one can write the longitudinal and transverse parts of the gluonic free energy in a weakly magnetized medium respectively as

\[ F_g^1 = \frac{1}{2} \sum_p \ln \left( 1 - \frac{b}{P^2} \right) = - \frac{1}{2} \sum_p \left( \frac{b_0}{P^2} + \frac{b_0^3}{2P^4} \right) - \frac{1}{2} \sum_p \left( \frac{b_2}{P^2} + \frac{b_0 b_2}{P^4} \right) - \cdots \]

Thermal part: \( F_g^{1T} \)

Thermomagnetic part: \( F_g^{1B} \)

\[ = F_g^{1T} + F_g^{1B} \quad (4.16) \]

and

\[ F_g^2 + F_g^3 = \left[ \sum_p \ln \left( -P^2 \right) + \frac{1}{2} \sum_p \ln \left( 1 - \frac{c}{P^2} \right) + \frac{1}{2} \sum_p \ln \left( 1 - \frac{d}{P^2} \right) \right] \]
\[ = -\pi T^4 \frac{45}{45} - \sum_p \left( \frac{c_0}{P^2} + \frac{c_0^2}{2P^4} \right) - \frac{1}{2} \sum_p \left( c_2 + d_2 \right) \left( \frac{1}{P^2} + \frac{c_0}{P^4} \right) \]

Thermal part: \( F_g^{2T} + F_g^{3T} \)

Thermomagnetic part: \( F_g^{2B} + F_g^{3B} \)

\(^{10}\)We hereby note that though in paper-I [45] \( f_k \) and \( g_k \) are defined with zero chemical potential, but it has been checked that within the small \( m_f \) approximation (which we are going to use by the virtue of HTLpt.), the expressions for \( f_k \) and \( g_k \) do not change with the inclusion of finite chemical potential. (see Eq.(4.21))
where we have kept terms up to $O(q_f B)^2$.

The total thermal part (i.e., magnetic field independent terms) can straightforwardly be written from Eqs. (4.16) and (4.17) as

\[
\mathcal{F}^{1T}_g = \mathcal{F}^{2T}_g + \mathcal{F}^{3T}_g = -\frac{\pi T^4}{45} - \frac{1}{2} \sum_p \left( \frac{b_0}{p^2} + \frac{b_0^2}{2p^4} \right) - \sum_p \left( \frac{c_0}{p^2} + \frac{c_0^2}{2p^4} \right)
\]

Due to high temperature expansion within the HTL approximation, there also appear a soft contribution from the longitudinal part \cite{78} as

\[
\mathcal{F}^{1T}_g, \text{ soft} = -\frac{1}{12\pi} m_D^3 T.
\]

### 4.2.1 Longitudinal part - $\mathcal{F}^{1B}_g$

The thermomagnetic contribution of the longitudinal part can subsequently be expressed from Eqs. (4.16) and (4.17) as

\[
\mathcal{F}^{1B}_g = -\frac{1}{2} \sum_p \left( \frac{b_2}{p^2} + \frac{b_0 b_2}{p^4} \right) = \frac{m_D^3 \delta m_D^3}{2} \sum_p \frac{T_p}{p^2} + \sum_f g^2 (q_f B)^2 \frac{2}{2\pi^2} \times
\]

\[
\left[ \left( g_k + \frac{\pi m_f - 4T}{32m_f^2 T} \right) \left( \sum_p \frac{T_p}{p^2} - \sum_p \frac{A_2}{p^2} - m_D^2 \sum_p \frac{(1 - T_p)(T_p - A_2)}{p^4} \right) + \left( f_k + \frac{8T - \pi m_f}{128m_f^2 T} \right) \left( \frac{2}{3} \sum_p \frac{T_p}{p^2} - \frac{A_2}{p^2} - m_D^2 \sum_p \frac{(1 - T_p)(T_p - A_2)}{p^4} \right) \right]
\]

\[
= -\frac{m_D^3 \delta m_D^3}{4(4\pi)^2} \left( \frac{\Lambda\epsilon^\gamma T}{4\pi T} \right)^2 \left( \frac{2}{2\epsilon} + \ln 2 \right) - \sum_f g^2 (q_f B)^2 \frac{2}{2\pi^2} \left( \frac{\Lambda}{4\pi T} \right)^2 \left[ \left( g_k + \frac{\pi m_f - 4T}{32m_f^2 T} \right) \right.
\]

\[
\left. \left( \frac{T^2}{72} \left[ \frac{2}{\epsilon} + 11.046 \right] - \frac{2m_D^2 \epsilon^\gamma T}{9(4\pi)^2} \left[ \frac{1 - 4\ln 2}{\epsilon} - 5.326 \right] \right) + \left( f_k + \frac{8T - \pi m_f}{128m_f^2 T} \right) \right)
\]

\[
\left( \frac{T^2}{72} \left[ \frac{4}{\epsilon} + 21.759 \right] - \frac{2m_D^2 \epsilon^\gamma T}{9(4\pi)^2} \left[ \frac{2(1 - 4\ln 2)}{\epsilon} - 10.589 \right] \right). \quad (4.20)
\]

By the virtue of HTLpt, we now make a small $m_f$ approximation with

\[
g_k \approx -2f_k \approx \frac{1}{8m_f^2}. \quad (4.21)
\]
and subsequently \( \mathcal{F}_g^{1B} \) becomes
\[
\mathcal{F}_g^{1B} = - \frac{m_f^2 \delta m_f^2}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left( \frac{1}{2\epsilon} + \ln 2 + \gamma_E \right) - \sum_f g_2^2(q_f B)^2 \left( \frac{\pi T}{12\pi^2} \right)^{2\epsilon} \frac{3\bar{m}_D^2}{4} \left\{ \frac{1}{\epsilon} + 5.606 \right\} + \frac{3\bar{m}_D^2}{4} \left( \frac{8(4\ln 2 - 1)}{3\epsilon} + 19.7467 \right) \right),
\]
which has divergence of \( \mathcal{O}(1/\epsilon) \) in both HTL as well from thermomagnetic part.

### 4.2.2 Transverse part - \( \mathcal{F}_g^{2B} \) and \( \mathcal{F}_g^{3B} \)

To evaluate thermomagnetic contribution for transverse part of the gluonic free energy in a weakly magnetized medium one needs to compute the following sum integrals:
\[
\sum_p \frac{c_2 + d_2}{P^2} = \sum_f \frac{g_2^2(q_f B)^2}{\pi^2} \left( (14f_k - g_k) \sum_p \frac{p_0^2 p_3^2}{3p_1^4 p^2} - \frac{4}{3} g_k \sum_p \frac{1}{P^2} + \left( g_k + \frac{\pi m_f - 4T}{32m_f^2 T} \right) \sum_p \frac{p_0^2 - A_0 P^2}{P^2} \right)
\]
\[
\sum_p \frac{c_0(c_2 + d_2)}{P^4} = \sum_f \frac{g_2^2(q_f B)^2 m_f^2}{2\pi^2} \left( (14f_k - g_k) \sum_p \frac{p_0^2 p_3^2(p_0^2 - A_0 P^2)}{3p_1^4 p^4} - \frac{4}{3} g_k \sum_p \frac{p_0^2 - A_0 P^2}{p^2 P^4} \right)
\]
\[
+ \left( g_k + \frac{\pi m_f - 4T}{32m_f^2 T} \right) \sum_p \frac{p_0^2 - A_0 P^2}{P^2} \left\{ - \frac{7}{6} \frac{p_0^2}{p_1^2 P^2} + \frac{A_0}{P^2} \left( 1 + 3\frac{p_0^2 - p^2}{4p_1^2} \right) - \frac{5A_4}{2P^2} - \frac{p_0 p_3}{2p_1^2 P^2} (3A_1 + 5A_3) \right\}
\]
\[
+ \frac{A_2}{P^2} \left( \frac{5p_0^2 + 9p^2}{4p_1^2} + \frac{p_0 p_3^2}{p_1^2 p^2} \right) - \frac{5A_4}{2P^2} - \frac{p_0 p_3}{2p_1^2 P^2} (3A_1 + 5A_3) \right\} - \left( f_k + \frac{8T - \pi m_f}{128m_f^2 T} \right) \sum_p \frac{p_0^2 p_3^2(5A_0 - A_2)(p_0^2 - A_0 P^2)}{p_1^4 p^4} + \left( 1 + \cosh \frac{m_f}{T} \right)^{-1} \sum_p \frac{3A_1}{2} - A_3 \right) \frac{p_0 p_3 \left( p_0^2 - A_0 P^2 \right)}{p_1^2 p^2 P^4} \right). \tag{4.24}
\]

Finally, using various specific HTL sum-integrals listed in appendix B, the master sum-integrals listed in appendix B.2.2 along with angular intergrations listed in appendix C, one obtains
\[
\mathcal{F}_g^{2B} + \mathcal{F}_g^{3B} = - \frac{1}{2} \sum_p \left( c_2 + d_2 \right) \left( \frac{1}{P^2} + \frac{c_0}{P^4} \right)
\]
\[
- \sum_f \frac{g_2^2(q_f B)^2 T^2}{144\pi^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left( 14f_k - g_k \right) \left\{ \left( \frac{1}{\epsilon} + 24 \ln G \right) + 3\bar{m}_D^2 \left[ \frac{1 - \ln 2}{\epsilon^2} + \frac{1}{\epsilon} \left( 4 - \frac{\pi^2}{6} \right) \right] \left\{ - \ln^2(2) - 2\gamma_E (\ln 2 - 1) - \ln 2 \right\} + 4.38 \right\} + g_k \left\{ 8 + 3\bar{m}_D^2 \left[ \frac{1}{\epsilon} (4 - 4 \ln 4) - 2.79 \right] \right\}
\]
\[
- \sum_p \frac{p_0^2 p_3^2(5A_0 - A_2)(p_0^2 - A_0 P^2)}{p_1^4 p^4} + \left( 1 + \cosh \frac{m_f}{T} \right)^{-1} \sum_p \frac{3A_1}{2} - A_3 \right) \frac{p_0 p_3 \left( p_0^2 - A_0 P^2 \right)}{p_1^2 p^2 P^4} \right). \tag{4.24}
\]
\[ \begin{align*}
+ \left( g_k + \frac{\pi m_f - 4T}{32m_f^2T} \right) & \left\{ \frac{3}{8\epsilon^2} + \frac{1}{4\epsilon} \left( 36\ln G + 2 + 15\ln 2 \right) + 20.83 + 3\hat{m}_D^2 \left[ \frac{1}{\epsilon^2} \left( -\frac{319}{20} + \pi^2 \right) + \frac{89 \ln 2}{10} \right] + \frac{1}{600\epsilon} \left( 3600\zeta(3) - 37658 + 2900\pi^2 + 5340\ln^2(2) + 4736\ln 2 + 30\gamma_E\left( -638 \right) + 40\pi^2 + 356\ln 2 \right) + 7.18 \right\} - \left( f_k + \frac{8T - \pi m_f}{128m_f^2T} \right) \left\{ \frac{3}{2} \left( \frac{1 + 8\ln 2}{\epsilon} + 45.68 \right) + 3\hat{m}_D^2 \right. \\
& \left[ \frac{1}{10\epsilon^2} \left( 29 + 10\pi^2 - 128\ln 2 \right) + \frac{1}{25\epsilon} \left( 150\zeta(3) + 564 + \frac{125\pi^2}{6} + 5\gamma_E\left( 29 + 10\pi^2 - 128\ln 2 \right) - 4\ln 2 \left( 147 + 80\ln 2 \right) + 58.01 \right] \right\} + \left( 1 + \cosh \frac{m_f T}{2\epsilon} \right)^{-1} \left\{ \frac{3\ln 2}{2\epsilon} - 3.92 + 3\hat{m}_D^2 \right. \\
& \left[ \frac{1}{40\epsilon^2} \left( 11 + 5\pi^2 - 92\ln 2 \right) + \frac{1}{600\epsilon} \left( 450\zeta(3) + 4671 - 200\pi^2 - 1380\ln^2(2) - 4032\ln 2 \right) + 3\gamma_E\left( 11 + 5\pi^2 - 92\ln 2 \right) - 1.86 \right\} \right\} \\
& \left. \right). \quad (4.25) \\
\end{align*} \]

Similar to the longitudinal part, using small \( m_f \) approximations in Eq.(4.25), we get

\[ \mathcal{F}_B^{2B} + \mathcal{F}_B^{3B} = \sum_f \frac{g^2(q_f B)^2 T^2}{(12\pi)^2} \frac{m_f}{m^2} \left( \frac{1}{4\pi T} \right)^{2\epsilon} \]

\[ \times \left[ \frac{1}{\epsilon} + 4.97 + 3\hat{m}_D^2 \left\{ \frac{1 - \ln 2}{\epsilon^2} + \frac{1}{\epsilon} \left( \frac{7}{2} - \frac{\pi^2}{6} - \ln^2(2) - 2\gamma_E\left( \ln 2 - 1 \right) \right) + 4.73 \right\} \right] \]

\[ - \sum_f \frac{g^2(q_f B)^2}{(12\pi)^2} \frac{\pi T}{32m_f} \left( \frac{1}{4\pi T} \right)^{2\epsilon} \left\{ \frac{3}{8\epsilon^2} + \frac{1}{\epsilon} \left( \frac{13}{4} + \frac{3\zeta'(1)}{4\zeta(-1)} + \frac{27}{4} \ln 2 \right) + 37.96 \right. \\
+ \frac{3}{4} \hat{m}_D^2 \left[ \frac{1}{\epsilon^2} \left( 5\pi^2 - \frac{609}{10} + \frac{114\ln 2}{5} \right) + \frac{1}{\epsilon} \left( 30\zeta(3) - \frac{17137}{75} + \frac{121\pi^2}{6} + \frac{114}{5} \ln^2(2) \right) + \frac{604}{75} \ln 2 + \gamma_E\left( 10\pi^2 - \frac{609}{5} + \frac{228}{5} \ln 2 \right) + 86.73 \right] \right\} - \sum_f \frac{g^2(q_f B)^2}{(12\pi)^2} \frac{T}{12m_f} \left\{ \frac{3\ln 2}{2\epsilon} \right. \\
- 3.92 + 3\hat{m}_D^2 \left[ \frac{1}{40\epsilon^2} \left( 11 + 5\pi^2 - 92\ln 2 \right) + \frac{1}{600\epsilon} \left( 450\zeta(3) + 4671 - 200\pi^2 - 1380\ln^2(2) \right) - 4032\ln 2 + 3\gamma_E\left( 11 + 5\pi^2 - 92\ln 2 \right) - 1.86 \right\}. \quad (4.26) \]
5 Total free energy and pressure in weak Field approximation

5.1 Free Energy

Finally, the total one-loop free energy of a weakly magnetized hot medium as written in Eq. (2.1) reads as

\[ F = F_q + F_g + F_0 + \Delta \mathcal{E}_0, \]  

(5.1)

where the quark part of the free energy \( F_q \) has both HTL (viz., magnetic field independent) part as well as the thermomagnetic correction as obtained in Eq.(3.32). Similarly, the gluonic part has also HTL part \( F^\text{HTL}_g \) plus the thermomagnetic correction \( F^\text{B}_g \). The magnetic field independent renormalized gluonic part \( F^\text{HTL}_g \) can be written from Eq.(4.18) as

\[
(F^\text{HTL}_g)^r = d_A \left[ F^1_{\text{HTL}} + F^2_{\text{HTL}} + F^3_{\text{HTL}} + \Delta \mathcal{E}_0 \right],
\]

(5.2)

where we have also used the HTL counterterm [80] as given in Eq. (2.2) and \( d_A = N^2_c - 1 \) and \( \hat{m}_D = m_D/2\pi T \).

The magnetic field dependent gluonic part similarly can be extracted from Eq.(4.20) and Eq.(4.25) as

\[
F^\text{B}_g = d_A \left[ \mathcal{F}^1_g + \mathcal{F}^2_g + \mathcal{F}^3_g \right],
\]

(5.3)

where

\[
\mathcal{F}^1_g + \mathcal{F}^2_g + \mathcal{F}^3_g
\]

\[
= - \frac{m_D^2 \hat{m}_D^2}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left( \frac{3}{2\epsilon} + \ln 2 + \gamma_E \right) + \sum_f \frac{g^2(q_f B)^2 T^2}{(12\pi)^2 \hat{m}_f} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + 4.97 \right]
\]

\[
+ 3\hat{m}_D^2 \left( \frac{1 - \ln 2 - \frac{\pi^2}{6} - \ln^2(2) - 2\gamma_E(\ln 2 - 1)}{\epsilon^2} \right) + 4.73 \right] \right] \]

\[
+ \sum_f \frac{g^2(q_f B)^2}{(12\pi)^2 \hat{m}_f} \frac{\pi T}{32\hat{m}_f} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \left\{ \frac{3}{8\epsilon^2} + \frac{1}{\epsilon} \left( \frac{21}{8} + \frac{3\zeta(-1)}{4\zeta(-1)} + \frac{27}{4} \ln 2 \right) + 43.566 \right] \right.
\]

\[
+ \frac{3}{4\hat{m}_D^2} \left( \frac{1}{\epsilon^2} \left( \frac{5\pi^2}{10} - \frac{609}{10} + \frac{114\ln 2}{5} \right) + \frac{1}{\epsilon} \left( 30\zeta(3) - \frac{5779}{75} + \frac{121}{6} \pi^2 + \frac{114}{5} \ln^2(2) \right) \right)
\]

\[
+ \frac{468}{25} \ln 2 + \gamma_E \left( \frac{10\pi^2 - 609}{10} + \frac{228}{5} \ln 2 \right) \right) + 106.477 \right] \right) + \frac{8}{3\pi} \left( \frac{3\ln 2 - 4}{2\epsilon} - 3.92 \right)
\]

\[
+ 3\hat{m}_D^2 \left[ \frac{1}{40\epsilon^2} \left( 11 + 5\pi^2 - 92\ln 2 \right) + \frac{1}{\epsilon} \left( \frac{3}{4\zeta(3)} + \frac{1557}{200} - \frac{\pi^2}{3} - \frac{23}{10} \ln^2(2) \right)
\]

\[
- \frac{168}{25} \ln 2 + \gamma_E \left( \frac{11}{20} + \frac{\pi^2}{4} - \frac{23}{5} \ln 2 \right) \right) - 1.86 \right] \right] \right),
\]

(5.4)
which has both $O(1/\epsilon)$ (due to UV divergences) and $O(1/\epsilon^2)$ (due to both collinear and UV divergences) divergences. Now, the external magnetic field $B$ dependent divergence present in Eq. (3.32) and Eq. (5.4) can be removed [32] by redefining the magnetic field $B$ in the tree-level free energy as

$$
F_0 = \frac{B^2}{2} \rightarrow \frac{B^2}{2} \left[ 1 - \frac{8N_c g^4 C_F^2 M_B^4}{9\epsilon} \left( 2 - \frac{15}{\pi^2} \right) + \frac{m_H^2 \delta m_{H}^2}{\epsilon(4\pi)^2 B^2} \sum_f \frac{g^2 q_f^2}{(12\pi)^2} \frac{27\pi^2}{m_f^2} \frac{1}{\epsilon} + \frac{3\hat{m}_{D}^2}{\epsilon^2} \left\{ 1 - \frac{\ln 2}{\epsilon^2} + \frac{1}{\epsilon} \left( \frac{7}{2} - \frac{\pi^2}{6} - \ln^2(2) - 2 \left( \gamma_E + \ln \frac{\Lambda}{2} \right) (\ln 2 - 1) \right) \right\} \\
+ \frac{3}{4} \hat{m}_{D}^2 \left[ \frac{1}{\epsilon^2} \left( 5\pi^2 - \frac{609}{10} + \frac{114\ln 2}{5} \right) + \frac{1}{\epsilon} \left( 5779 - \frac{121}{6} \pi^2 + \frac{114}{5} \ln^2(2) \right) \\
+ \frac{468}{25} \ln 2 + \left( \gamma_E + \ln \frac{\Lambda}{2} \right) \left( 10\pi^2 - \frac{609}{5} + \frac{228}{5} \ln 2 \right) \right\} \right] \right) + \frac{8}{3\pi} \left( \frac{3\ln 2 - 4}{2\epsilon} \right) + \frac{3}{40\epsilon^2} \left( 11 + 5\pi^2 - 92 \ln 2 \right) + \frac{1}{\epsilon} \left( \frac{3}{4} \zeta(3) + \frac{17}{20} \ln 2 + \frac{\pi^2}{3} - \frac{23}{10} \ln^2(2) \right) \\
- \frac{168}{25} \ln 2 + \left( \gamma_E + \ln \frac{\Lambda}{2} \right) \left( \frac{11}{20} + \frac{\pi^2}{4} - \frac{23}{5} \ln 2 \right) \right\} \right].
$$

(5.5)

So, the renormalized total free energy becomes

$$
F = F_q^\tau + (F_g^{HTL})^\tau + F_g^\tau
$$

(5.6)

where,

$$
F_q^\tau = N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120\mu^2}{7} + \frac{240\mu^4}{7} \right) + \frac{g^4 C_F T^4}{48} (1 + 4\mu^2) (1 + 12\mu^2) + \frac{g^4 C_F T^4}{768\pi^2} (1 + 4\mu^2)^2 (\pi^2 - 6) + \frac{4g^4 C_F^2 M_B^4}{9N_f \pi^2} \right] \times \left[ \left( 2 \ln \frac{\Lambda}{4\pi T} - N(z) \right) (2\pi^2 - 15) + 12\zeta(3) - \frac{7\pi^2}{6} - 10 \right],
$$

(5.7)

and,

$$
\frac{F_g^\tau}{d_A} = -\pi^2 T^4 \hat{m}_{D}^2 \delta m_{D}^2 \left( \gamma_E + \ln \frac{\Lambda}{2} \right) + \sum_f \frac{g^2 (q_f B)^2 T^2}{(12\pi)^2 m_f} \left[ 4.97 + 2 \ln \frac{\Lambda}{2} \right] \\
+ \frac{3\hat{m}_{D}^2}{\epsilon^2} \left\{ 2 (1 - \ln 2) \ln \frac{\Lambda}{2} + \frac{7}{2} - \frac{\pi^2}{6} - \ln^2(2) - 2 \gamma_E (\ln 2 - 1) \right\} \ln \frac{\Lambda}{2} + 4.73 \right\} \\
- \sum_f \frac{g^2 (q_f B)^2 \pi T}{(12\pi)^2} \frac{T^2}{32 m_f} \left[ \frac{3}{4} \ln \frac{\Lambda}{2} + 2 \ln \frac{\Lambda}{2} \left( \frac{21}{8} + \frac{3}{4} \zeta(-1) + \frac{27}{4} \ln 2 \right) + 43.566 \right]
$$

(5.8)
\[
\begin{align*}
&+ \frac{3}{4} \tilde{m}_D^2 \left[ 2 \ln^2 \frac{\Lambda}{2} \left( 5\pi^2 - \frac{609}{10} + \frac{114}{5} \ln 2 \right) + 2 \ln \frac{\Lambda}{2} \left( 30 \zeta(3) - \frac{5779}{75} + \frac{121}{6} \pi^2 + \frac{114}{5} \ln^2(2) \right) \\
&+ \frac{468}{25} \ln 2 + \gamma_E \left( 10\pi^2 - \frac{609}{5} + \frac{228}{5} \ln 2 \right) \right] + 106.477 \right) \right] + \frac{8}{3\pi} \left( (3\ln 2 - 4) \ln \frac{\Lambda}{2} - 3.92 \right) \\
&+ 3\tilde{m}_D^2 \left[ \frac{1}{20} \ln^2 \frac{\Lambda}{2} \left( 11 + 5\pi^2 - 92 \ln 2 \right) + 2 \ln \frac{\Lambda}{2} \left( \frac{3}{4} \zeta(3) + \frac{1557}{200} - \frac{\pi^2}{3} - \frac{23}{10} \ln^2(2) \right) \\
&- \frac{168}{25} \ln 2 + \gamma_E \left( \frac{11}{20} + \frac{\pi^2}{4} - \frac{23}{5} \ln 2 \right) \right] \right] \right].
\end{align*}
\]

\(5.2\) Pressure

The expression for the pressure of hot and dense QCD matter in one-loop HTLpt in presence of a weak magnetic field can now be written directly from the one-loop free energy as

\[
P(T, \mu, B, \Lambda) = -F(T, \mu, B, \Lambda),
\]

whereas the ideal gas limit of the pressure reads as

\[
P_{\text{Ideal}}(T, \mu) = N_c N_f \frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \mu^2 + \frac{240}{7} \mu^4 \right) + (N_c^2 - 1) \frac{\pi^2 T^4}{45}.
\]

\(6\) Strong coupling and scales

The one-loop running coupling which evolves on both momentum transfer and the magnetic field is recently obtained in Ref. [105] as

\[
\alpha_s(\Lambda^2, |eB|) = \frac{\alpha_s(\Lambda^2)}{1 + b_1 \alpha_s(\Lambda^2) \ln \left( \frac{\Lambda^2}{\Lambda^2 + |eB|} \right)},
\]

in the domain \( |eB| < \Lambda^2 \) and the one-loop running coupling at renormalization scale reads as

\[
\alpha_s(\Lambda^2) = \frac{1}{b_1 \ln \left( \frac{\Lambda^2}{\Lambda^2_{\text{MS}}} \right)},
\]

where \( b_1 = \frac{11 N_c - 2 N_f}{12\pi} \), \( \Lambda_{\text{MS}} = 176 \) MeV [103] at \( \alpha_s(1.5 \text{GeV}) = 0.326 \) for \( N_f = 3 \). We note here that we choose separate renormalization scales for gluon \( \Lambda = \Lambda_g \), for quark \( \Lambda = \Lambda_q \), which are chosen at their central values, respectively as \( \Lambda_g = 2\pi T \) and \( \Lambda_q = 2\pi \sqrt{T^2 + \mu^2 / \pi^2} \). The renormalization scales can be varied by a factor of 2 with respect to its central value. On the other hand the magnetic field strength can also be varied as long as \( |eB| > \Lambda^2 \) for strong field and \( |eB| < \Lambda^2 \) for weak field approximation for a given temperature vis-a-vis the renormalization scale, as discussed in 3. The left panel of Fig. 4 displays running of \( \alpha_s \) with \( |eB| \) at the central value of the renormalization scale \( \Lambda_g = 2\pi T \) GeV for \( T = 0.4 \) GeV. This indicates a slow increase of \( \alpha_s \) with increase of \( |eB| \) in the domain \( |eB| < \Lambda^2 \). On the other hand the right panel of Fig. 4 exhibits running
of $\alpha_s$ with $T$ for $|eB| = m_\pi^2$. It also consolidates that $\alpha_s$ runs very slowly with $|eB|$. Now we are all set to discuss the thermodynamics of a hot magnetized deconfined QCD matter below.

*Figure 4.* Left panel: Variation of the one-loop QCD coupling with weak magnetic field, $|eB|$ for $T = 0.4\,\text{GeV}$. Right panel: Variation with temperature, $T$ for $|eB| = m_\pi^2$.

*Figure 5.* Variation of the scaled one-loop pressure with temperature for $N_f = 3$ with $\mu = 0$ (left panel) for $\mu = 300\,\text{MeV}$ (right panel) in presence of weak magnetic fields of various strengths, $|eB| = 0$, $m_\pi^2/2$, $m_\pi^2$ and $3m_\pi^2/2$. In the right panel for $\mu \neq 0$, the renormalization scales are defined in the text in subsection 6.
Figure 6. Variation of the scaled one-loop pressure with magnetic field for $N_f = 3$ with $\mu = 0$ (left panel) and $\mu = 300$ MeV (right panel) for $T = (0.3, 0.4, 0.6$ and $1)$ GeV. The renormalization scales are defined in the text in subsection 6.

7 Results

In Fig. 5 we displayed the temperature variation of scaled pressure with that of ideal gas value. The left panel is for chemical potential, $\mu = 0$ whereas the right panel is for $\mu = 0.3$ GeV for hot and dense magnetized QCD matter in one-loop HTLpt within weak field approximation for different values of field strengths, $|eB| = 0$, $m^2_{\pi}/2$, $m^2_{\pi}$ and $3m^2_{\pi}/2$. We note that for $|eB| = 0$ one gets back usual one-loop HTLpt pressure [69–77]. From both plots one observes that the low $T$ ($<0.8$ GeV) behaviour of the pressure is strongly affected by the presence of magnetic field whereas at high $T$ ($\geq 0.8$ GeV) it almost remains unaffected as the temperature becomes the dominant scale because of weak field approximation $m^2_{\text{th}} < |eB| < T^2$. Nevertheless, we also note a specific difficulty that one encounters with HTLpt ($|eB| = 0$). This has to do with the fact that the one-loop HTLpt introduces an over-counting of some contributions [69–76] in strong coupling. This is because the loop expansion and the coupling expansion are not symmetrical in HTLpt. So, at each loop order in HTLpt the result is an infinite series in $g$. At leading order in HTLpt one only gets the correct perturbative coefficients for $g^0$ and $g^3$ when one expands in power of $g$. Thus, for a given order in $g$ higher loop orders contribute and this is only corrected by making the calculation in higher loop-orders [78–88]. We also note that the pressure is slightly reduced in presence of $\mu$ (right panel) than that of $\mu = 0$, for a given $|eB|$.

It is seen from Fig. 6 that the slope of the curve decreases with the increase of $T$. So, Fig. 6 also consolidates the fact that in weak field approximation the effect of magnetic field diminishes with increase of $T$. This indicates that the magnetic field is the dominant scale at low $T$ and becomes negligible at high $T$. 
To check the sensitivity with the renormalization scale $\Lambda_{q,g}$ we have displayed in Fig. 7 the temperature variation of the scaled one-loop pressure in presence of a constant weak magnetic field by varying $\Lambda_{q,g}$ by a factor of two around its central value for both zero and finite chemical potential. It is found to depend moderately on the renormalization scale $\Lambda_{q,g}$. One may need higher loops calculation and log resummation to reduce further the renormalization scale dependent band.

8 Conclusion

In this paper we presented a systematic framework based on the general structure of two point functions of a fermion and a gauge boson to evaluate the QCD pressure in nontrivial backgrounds, viz., when both heat bath and magnetic field are considered together. This framework has been applied to the case when the heat bath is weakly magnetized. The total pressure of a magnetized hot and dense deconfined QCD matter is the sum of three contributions coming from (a) the quark part, (b) the gluonic part and (c) the tree level free energy due to the constant magnetic field. We note that the presence of an external magnetic field affects both the fermion and gluon effective two point (self-energy and propagator) functions. We have also used strong coupling that runs through both renormalization scale and magnetic field strength. Though gluons are electrically charge neutral, but they are mostly affected by the change in Debye mass through thermomagnetic correction and also through the quark loop. Based on the most general structure of the effective two point functions, the quark propagator in our earlier calculation [47] and gluon propagator in paper-I [45], we obtain QCD Debye screening mass, gluon and quark free energy in one-loop HTLpt in presence of weak field approximation. However
the divergences appeared therein are taken care of by redefining the magnetic field in the
tree level free energy term and through HTL counter term. We found finite results which
are also completely analytic and gauge independent but depends on renomalization scale
and magnetic field strength. We have also discussed in details the modification of QCD
Debye mass which depends on three scales, viz., the thermal quark mass, temperature and
the magnetic field. The weak field pressure is strongly affected at low $T(<0.8$ GeV) be-

Past the HTL result takes over. We have checked the sensitivity of the pressure on
the various scales, viz., the renormalization and magnetic field. The result is sensitive to
renomalization as it produces band while varying its value by a factor two. The sensitivity
of pressure on the magnetic field is strong at low $T$ and negligible at high $T$. We have
also outlined a general drawback with one-loop HTLpt that introduces an over-counting
of some contributions, as a remedy of which one needs to push the calculation to higher
loop-order.

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A Fermionic Sum-Integrals

The dimensionally regularized sum integrals are defined as,

$$
\sum_{\{P\}} \frac{1}{P^4} = \left( \frac{\epsilon \gamma_E \Lambda^2}{4\pi} \right) \frac{\epsilon}{T} \sum_{\omega_n = i\omega_n} \int \frac{d^{d-2}\epsilon}{(2\pi)^{d-2\epsilon}} \cdot 
$$

(A.1)

where $\Lambda$ can be identified as the $\overline{MS}$ renormalization scale which also introduces the factor
$(\frac{\epsilon \gamma_E}{4\pi})^\epsilon$ along with it, with $\gamma_E$ being the Euler-Mascheroni constant. Before listing all the
sum-integrals used in our purpose, we note that they are inter related among themselves via

$$
\sum_{\{P\}} \frac{1}{P^4} = -\frac{d-2}{2} \sum_{\{P\}} \frac{1}{p^2 P^2} = \frac{d-5}{d-4} \sum_{\{P\}} \frac{T_P}{P^4}. 
$$

(A.2)

A.1 Simple one-loop sum-integrals

The list of fermionic sum-integrals needed are

$$
\sum_{\{P\}} \frac{1}{P^2} = \frac{T^2}{24} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ 1 + 12\hat{\mu}^2 + 2\epsilon \left( 1 + 12\hat{\mu}^2 + 12\zeta(1, z) \right) \right], 
$$

(A.3)
\[
\sum_{\{P\}} \frac{1}{P^4} = \frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} - \Lambda(z) \right],
\]
(A.4)

\[
\sum_{\{P\}} \frac{p^2}{p^2_0} = -\frac{3}{4} \frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} - \frac{2}{3} - \Lambda(z) \right],
\]
(A.5)

\[
\sum_{\{P\}} \frac{1}{p^2 P^4} = -\frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + 2 - \Lambda(z) \right],
\]
(A.6)

\[
\sum_{\{P\}} \frac{p_3^2}{p^2 P^4} = \frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + \frac{8}{3} - \Lambda(z) \right],
\]
(A.7)

\[
\sum_{\{P\}} \frac{p_3^2}{p^2 P^2} = -\frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + \frac{8}{3} - \Lambda(z) \right].
\]
(A.8)

For some frequently occurring combinations of special functions we applied the following abbreviations

\[
\zeta'(x, y) \equiv \partial_x \zeta(x, y),
\]
(A.9)

\[
\Lambda(n, z) \equiv \zeta'(-n, z) + (-1)^{n+1} \zeta'(-n, z^*),
\]
(A.10)

\[
\Lambda(z) \equiv \Psi(z) + \Psi(z^*),
\]
(A.11)

where \( n \) is assumed to be an integer and \( z \) a general complex number, here \( z = 1/2 - i\hat{\mu} \).

Here \( \zeta \) denotes the Riemann zeta function, \( \Psi \) is the digamma function,

\[
\Psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)}.
\]
(A.12)

Below we enlist the values of the function \( \Lambda \) as required for our calculation. Though the following list are given at small values of \( \mu/T \), in the actual plot we calculate \( \Lambda \) for any value of \( \mu \) using Mathematica.

\[
\Lambda(z) = -2\gamma_E - 4\ln 2 + 14\zeta(3)\hat{\mu}^2 - 62\zeta(5)\hat{\mu}^4 + 127\zeta(7)\hat{\mu}^6 + O(\hat{\mu}^8),
\]
(A.13)

\[
\Lambda(1, z) = -\frac{1}{12} \left( \ln 2 - \frac{\zeta'(-1)}{\zeta(-1)} \right) - (1 - 2\ln 2 - \gamma_E)\hat{\mu}^2 - \frac{7}{6} \zeta(3)\hat{\mu}^4
\]
\[
+ \frac{31}{15} \zeta(5)\hat{\mu}^6 + O(\hat{\mu}^8)
\]
(A.14)

### A.2 HTL one-loop sum-integrals for weak field case

We also need some more difficult one-loop sum-integrals that involve the angular average defined earlier in Eq. (3.20). For brevity, henceforth we will use the notation \( c = \cos \theta \) for single angular average and \( c_i = \cos \theta_i \) for multiple angular averages. We list the sum-integrals below. The expressions for the respective angular averages appearing in the process and denoted by \( \Delta_i \)'s are given in Appendix C.

\[
\sum_{\{P\}} \frac{1}{P^4} = \frac{d - 4}{d - 5} \sum_{\{P\}} \frac{1}{P^4},
\]
(A.15)
\[ \sum_{\{P\}} \frac{1}{p^2 P^2} T_P = -\frac{2 \Delta^2}{d-2} \sum_{\{P\}} \frac{1}{P^4}, \quad (A.16) \]

\[ \sum_{\{P\}} \frac{1}{p^2 P^2} T_P^2 = -\frac{2 \Delta^2}{d-2} \sum_{\{P\}} \frac{1}{P^4}, \quad (A.17) \]

\[ \sum_{\{P\}} \frac{1}{p_0^2 P^2} T_P = -\frac{2 \Delta^2}{d-2} \sum_{\{P\}} \frac{1}{P^4}, \quad (A.18) \]

\[ \sum_{\{P\}} \frac{1}{p^4} T_P^2 = \left( \frac{d-4}{d-2} \Delta''_{0} - \frac{2}{d-2} \Delta_{10} \right) \sum_{\{P\}} \frac{1}{P^4}, \quad (A.19) \]

\[ \sum_{\{P\}} \frac{1}{p_0^4} T_P^2 = \left( \frac{\Delta''_{0}}{d} - \frac{2}{d(d-2)} \Delta_{11} \right) \sum_{\{P\}} \frac{1}{P^4}, \quad (A.20) \]

**B Bosonic Sum-Integrals**

**B.1 Simple one-loop sum-integrals**

To evaluate the sum-integrals over the external bosonic momenta, we use the following master formula

\[ \sum_{P} \frac{p_i p_j}{p^m P^m} = \left( c^2 (1 - c^2)^2 \right) \sum_{P} \frac{1}{p^{m_1} P^{m_1}}, \quad (B.1) \]

thus eventually requiring the following basis integrals

\[ \sum_{P} \frac{1}{P^2} = -\frac{T^2}{12} \left( \frac{\Lambda}{4 \pi T} \right)^{2\epsilon} \left[ 1 + 2\epsilon \left( 1 + \frac{\zeta'(-1)}{\zeta(-1)} \right) + \epsilon^2 \left( 4\gamma_E \left( 1 + \frac{\zeta'(-1)}{\zeta(-1)} \right) - \frac{\gamma_E}{2} - \ln 2\pi \right) \right] + O[\epsilon]^3, \quad (B.2) \]

\[ \sum_{P} \frac{1}{p^2 P^2} = -\frac{2}{(4\pi)^2} \left( \frac{\Lambda}{4 \pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + 2\gamma_E + 2 + \epsilon \left( 4 + 4\gamma_E + \frac{\pi^2}{4} - 4\gamma_1 \right) \right] + \epsilon^2 \left( 3 \left( 8 - 2\gamma_1 + \gamma_2 + 2 \right) + \pi^2 + \gamma_E \left( 16 + \pi^2 \right) - 14\zeta(3) \right) \right] + O[\epsilon]^3, \quad (B.3) \]

\[ \sum_{P} \frac{1}{p_0^2 P^2} = -\frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4 \pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + 2\gamma_E + \epsilon \left( \frac{\pi^2}{4} - 4\gamma_1 \right) + \epsilon^2 \left( 24\gamma_2 - 14\zeta(3) + 3\gamma_E \pi^2 \right) \right] + O[\epsilon]^3. \quad (B.4) \]

**B.2 HTL one-loop sum integrals**

Similarly as in the fermionic part, here also we list the one-loop bosonic HTL integrals required for our computation. We define the following master HTL integrals which is
largely required to compute the HTL sum integrals appearing in the expression for free energies.

\[
\sum_p \frac{p_i^j p_3^j p_0^k}{p^m} T_P = \langle c^{m-i-j-d} \rangle \sum_p \frac{p_i^j p_3^j p_0^k}{p^{m+2} P^2} = \langle c^{m-i-j-d} \rangle \sum_p \frac{p_i^j p_3^j}{p^{m-k-2} P^2} = \langle c_1^{m-i-j-d} \rangle c_1 (c_2^2 (1 - c_2^2)^2) c_2 \sum_p \frac{1}{p^{m-k-i-j-2} P^2} \tag{B.5}
\]

\[
\sum_p \frac{p_i^j p_3^j p_0^k}{p^{m+2} P^4} T_P = \left\langle \frac{1}{1 - c^2} \right\rangle \sum_p \frac{p_i^j p_3^j p_0^k}{p^{m+2} P^4} - \left\langle \frac{1 - c^{m+4-i-j-d}}{(1 - c^2)^2} \right\rangle \sum_p \frac{p_i^j p_3^j p_0^k}{p^{m+2} P^2} = \left\langle \frac{1}{1 - c^2} \right\rangle \sum_p \frac{p_i^j p_3^j}{p^{m-k} P^4} + \left\langle \frac{1}{1 - c^2} \right\rangle \sum_p \frac{k + 1}{p^{m+2-k} P^2} - \left\langle \frac{1 - c^{m+4-i-j-d}}{(1 - c^2)^2} \right\rangle \sum_p \frac{p_i^j p_3^j}{p^{m-k-2} P^2} = \left\langle \frac{1}{1 - c_1^2} \right\rangle c_1 (c_2^2 (1 - c_2^2)^2) c_2 \sum_p \frac{1}{p^{m-k-i-j-2} P^4} + \left\langle \frac{1}{1 - c_1^2} \right\rangle c_1 (c_2^2 (1 - c_2^2)^2) c_2 \sum_p \frac{k + 1}{p^{m+2-k-i-j-2} P^2} - \left\langle \frac{1 - c^{m+4-i-j-d}}{(1 - c_1^2)^2} \right\rangle \sum_p \frac{1}{p^{m-k-i-j-2} P^2} \tag{B.6}
\]

\[
\sum_p \frac{p_i^j p_3^j p_0^k}{p^{m} P^2} T_P^2 = \left\langle \frac{c_1^{m+2-i-j-d}}{c_1^2 - c_2^2} \right\rangle c_1 c_2 \sum_p \frac{1}{p^{m+2} P^2} \tag{B.7}
\]
B.2.1 HTL sum-integrals required for longitudinal part

1. \[
\sum_{p} \frac{A_2}{p^2} = \sum_{p} \frac{1}{2} \left[ T_P \left( \frac{1}{p^2} - \frac{p_3^2}{p^4} \right) + (1 - T_P) \left( \frac{p_0^2}{p^6} - \frac{3p_0^2p_3^2}{p^8} \right) \right] = \left[ \frac{1}{2} \langle c_1^2 - c_1^4 \rangle_{c_1} (1 - c_2^2)_{c_2} + \langle c_1^d \rangle_{c_1} \langle c_2^d \rangle_{c_2} \right] \sum_{p} \frac{1}{p^2} = -\left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \frac{T^2}{27} \left[ \frac{1}{\epsilon} + \frac{1}{3} + \frac{1}{2} \zeta'(1) + 2 \ln 2 \right]. \tag{B.10}
\]

2. \[
\sum_{p} \frac{A_2}{p^4} = \sum_{p} \frac{1}{2} \left[ T_P \left( \frac{1}{p^4} - \frac{p_3^2}{p^6} \right) + (1 - T_P) \left( \frac{p_0^2}{p^8} - \frac{3p_0^2p_3^2}{p^{10}} \right) \right] = \left[ \frac{1}{2} \langle c_1^{4-d} - c_1^{6-d} \rangle_{c_1} (1 - c_2^2)_{c_2} + \langle c_1^{d} \rangle_{c_1} \langle c_2^{d} \rangle_{c_2} \right] \sum_{p} \frac{1}{p^2 p^2} = -\left( \frac{\Delta c_{\gamma\nu}}{4\pi T} \right)^{2\epsilon} \frac{1}{3(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{1}{6} (1 + 12 \ln 2) \right]. \tag{B.11}
\]

3. \[
\sum_{p} \frac{T_P A_2}{p^4} = \sum_{p} \frac{1}{2} \left[ T_P^2 \left( \frac{1}{p^4} - \frac{p_3^2}{p^6} \right) + (T_P^2 - T_P^2) \left( \frac{p_0^2}{p^8} - \frac{3p_0^2p_3^2}{p^{10}} \right) \right] = \left[ \frac{1}{2} \langle c_1^{6-d} \rangle_{c_1} (1 - 3c_2^2)_{c_2} + \frac{1}{2} \left( \Delta_4' - \Delta_5'' \right) (1 - c_3^2)_{c_3} + \Delta_5'' (c_3^2)_{c_3} \right] \sum_{p} \frac{1}{p^2 p^2} = -\left( \frac{\Delta c_{\gamma\nu}}{4\pi T} \right)^{2\epsilon} \frac{2}{9(4\pi)^2} \left[ \frac{1 + 2 \ln 2}{\epsilon} + \frac{1}{60} ( -91 + 8 \ln 2 (59 + 15 \ln 2) ) \right]. \tag{B.12}
\]
Here different $\Delta$’s are the nontrivial angular averages given in Appendix C.

## B.2.2 HTL sum-integrals required for transverse part

1. \[
\sum_p \frac{A_1 p_0 p_3}{p^2_{\perp} P^2} = -\sum_p \frac{p^2_0 p^2_3}{p^2_{\perp} p^2} (1 - T_F) = [\Delta_3 \Delta_0 - \Delta_0] \sum_p \frac{1}{P^2} = \left( \frac{\Lambda}{4\pi T} \right)^2 \frac{T^2}{4} \left[ \frac{\ln 2 - \frac{1}{\epsilon}}{\epsilon} + \frac{\pi^2}{6} - 2 + (\ln 2)^2 + 2(\ln 2 - 1) \frac{\zeta'(-1)}{\zeta(-1)} \right]. \tag{B.13}
\]

2. \[
\sum_p \frac{A_3 p_0 p_3}{p^2_{\perp} P^2} = \sum_p \frac{p^2_0 p^2_3}{2p^2 p^2_{\perp} P^2} \left( 1 - \frac{5 p^2_3}{3 p^2} \right) - \sum_p \frac{3p^2_0 p^2_3 (1 - T_F)}{2p^2 p^2_{\perp} P^2} \left( \frac{p^2_0}{p^2} - \frac{p^2_0}{p^2} + \frac{5 p^2_0 p^2_3}{3 p^2} \right) = \left[ \frac{1}{2} \Delta_0 - \frac{11}{6} \Delta_1 - \frac{3}{2} \Delta_3 \Delta_0 - \frac{3}{2} \Delta_4 \Delta_0 - \frac{3}{2} \Delta_3 \Delta_1 + \frac{5}{2} \Delta_4 \Delta_1 \right] \sum_p \frac{1}{P^2} = \left( \frac{\Lambda}{4\pi T} \right)^2 \frac{T^2}{144} \left[ \frac{6 \ln 2 - 5}{\epsilon} + \pi^2 - \frac{55}{3} + 2 \ln 2(3 \ln 2 + 5) - 2(5 - 6 \ln 2) \frac{\zeta'(-1)}{\zeta(-1)} \right]. \tag{B.14}
\]

3. \[
\sum_p \frac{A_2 p^2_0 p^2_3}{p^2_{\perp} p^2 P^2} = \sum_p \frac{1}{2} \left[ T_F p^2_0 p^2_3 \left( 1 - \frac{p^2_3}{p^2} \right) + (1 - T_F) p^2_0 p^2_3 \left( \frac{p^2_0}{p^2} - \frac{3 p^2_0 p^2_3}{p^4} \right) \right] = \left( \Delta_0 - 3 \Delta_1 + \Delta_3 \Delta_0 - \Delta_4 \Delta_0 - \Delta_3 \Delta_1 + 3 \Delta_4 \Delta_1 \right] \sum_p \frac{1}{2P^2} = \left( \frac{\Lambda}{4\pi T} \right)^2 \frac{T^2}{48} \left[ \frac{2 \ln 2 - \frac{1}{\epsilon}}{\epsilon} + \frac{\pi^2}{3} - 5 + 2 \ln 2 \left( \ln 2 + \frac{5}{3} \right) + 2 \left( 2 \ln 2 - 1 \right) \frac{\zeta'(-1)}{\zeta(-1)} \right] + \mathcal{O}(\epsilon) \tag{B.15}
\]

4. \[
\sum_p \frac{A_2 (5p^2_0 + 9p^2)}{4p^2_{\perp} P^2} = \sum_p \frac{1}{2} \left[ T_F (5p^2_0 + 9p^2) \left( 1 - \frac{p^2_3}{p^2} \right) + (1 - T_F) (5p^2_0 + 9p^2) \left( \frac{p^2_0}{p^2} - \frac{3 p^2_0 p^2_3}{p^4} \right) \right]
\]

- 30 -
\[
\sum \int P - \sum = \left( \frac{\Lambda}{4\pi T} \right)^{2e} \frac{T^2}{96} \left[ \frac{2 \ln 2 - 1}{\epsilon} + \frac{\pi^2}{3} - \frac{79}{14} + 2 \ln 2 (\ln 2 + 1) \right] + \mathcal{O}(\epsilon)
\]

5.
\[
\sum \int \frac{A_1}{P^2} = \sum \int \frac{3}{8P^2} \left( 1 - \frac{p^2}{p^2} \right)^2 - \sum \int \frac{p_0^2}{8p^2P^2} \left( 1 - \frac{5p_0^2}{p^2} \right) + \sum \int \frac{5}{3} \frac{p_0^2}{p^2} \frac{p_1^4}{p^4}
\]
\[
= \sum \int \frac{3}{8} \left\{ \left( 1 - \frac{p_0^2}{p^2} \right)^2 - \frac{2p_0^2}{p^2} \left( 1 - \frac{3p_0^2}{p^2} \right)^2 + \frac{p_1^4}{p^4} \left( 1 - \frac{5p_0^2}{p^2} \right)^2 + \frac{8p_1^4}{p^4} \left( 1 - \frac{5p_0^2}{3p^2} \right) \right\} \left( \frac{1 - T_P}{P^2} \right)
\]
\[
= \frac{3}{8} \left[ \frac{2}{3} + \frac{4}{3} \Delta_0 - \frac{50}{9} \Delta_1 + \Delta_2 - 2 \Delta_3 + \Delta_4 - 2 \Delta_2 \Delta_0 - 10 \Delta_4 \Delta_0' + 12 \Delta_3 \Delta_0' \right]
\]
\[
+ \Delta_2 \Delta_1' + \frac{35}{3} \Delta_4 \Delta_1' - 10 \Delta_3 \Delta_1' \int \frac{1}{P^2}
\]
\[
= \left( \frac{\Lambda}{4\pi T} \right)^{2e} \frac{T^2}{120} \left[ \frac{1}{\epsilon} - \frac{13}{30} + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right]
\]

6.
\[
\sum \int \frac{A_1 p_0 p_3 (p_0^2 - T_P P^2)}{p_1^4 p^4 P^4} = \sum \int \frac{p_0^2 p_3^2 (p_0^2 - T_P P^2)}{p_1^4 p^4 P^4} (1 - T_P)
\]
\[
= - \sum \int \frac{p_0^2 p_3^2}{p_1^4 P^4} + \sum \int \frac{p_0 p_3}{p_1^4 P^4} T_P + \sum \int \frac{p_0^3 p_3}{p_1^4 P^4} T_P - \sum \int \frac{p_0^2 p_3^2}{p_1^4 P^4} T_P^2
\]
\[
= \left[ (1 + \Delta_0) \Delta_0 - \Delta_0 \right] \int \frac{1}{P^4} + \left[ -2 \Delta_0 + \Delta_4 \Delta_0 + 3(1 + \Delta_0) \Delta_0 - \Delta_5 \Delta_0 \right]
\]
\[
- \Delta_5' \Delta_0 \int \frac{1}{P^2}
\]
\[
= \left( \frac{\Lambda e^{\gamma_e}}{4\pi T} \right)^{2e} \frac{1}{12 (4\pi)^2} \left[ -13 + \frac{\pi^2 + 4 \ln 2}{e^2} + \frac{1}{\epsilon} \left\{ - \frac{137}{3} + 4\pi^2 + \frac{4}{3} \ln 2 (3 \ln 2 - 4) \right\} + 6 \zeta(3) \right] - 0.81947
\]
7. 

\[ \sum_p \frac{A_3 p_0 p_3 (p_0^2 - T_P P^2)}{p_1^2 p_1^4} = \sum_p \frac{p_0^2 p_3^2 (p_0^2 - T_P P^2)}{2 p_1^4 p_1^4} \left( 1 - \frac{5 p_0^2}{3 p_1^2} \right) \]

\[ - \sum_p \frac{3 p_0^2 p_3^2 (p_0^2 - T_P P^2) (1 - T_P)}{2 p_1^4 p_1^4} \left( 1 - \frac{p_0^2}{p_1^2} - \frac{p_0^2}{p_1^3} + \frac{5 p_0^2 p_3^2}{3 p_1^2 p_1^4} \right) \]

\[ = \left[ (1 + \Delta_0) \Delta_1 + \frac{1}{2} \Delta_0 - \frac{11}{6} \Delta_1 \right] \sum_p \frac{1}{p_1^2} + \left[ \frac{5}{2} \Delta_0 - \frac{37}{6} \Delta_1 - \frac{1}{2} \Delta_4 \left( \Delta_0 - \frac{5}{3} \Delta_1 \right) - \frac{3}{2} (\Delta_5' - \Delta_6') (\Delta_0 - \Delta_1) \right. \]

\[ + \Delta_5 \Delta_1 - \Delta_6' \Delta_1 \left] \sum_p \frac{1}{p_1^2 p_1^2} \right. \]

\[ = \left( \frac{\Lambda e^{7 \kappa \epsilon}}{4 \pi T} \right)^{2e} \frac{1}{60(4\pi)^2} \left[ -103 + 5 \pi^2 + 76 \ln 2 \right] + \frac{1}{15 \epsilon} \left\{ -7473 + 550 \pi^2 \right. \]

\[ + 12 \ln 2 (118 + 95 \ln 2) + 450 \zeta(3) \} + 21.3892 \right\} + \mathcal{O}[\epsilon]. \]  

(B.19)

8. 

\[ \sum_p \frac{A_2 p_0^2 p_3^2 (p_0^2 - T_P P^2)}{p_1^4 p_1^4} = \sum_p \frac{1}{2} \left[ T_P p_0^2 p_3^2 (p_0^2 - T_P P^2) \right. \left. \frac{1}{p_1^4 p_1^4} \left( 1 - \frac{p_0^2}{p_1^2} - \frac{p_0^2}{p_1^3} + \frac{3 p_0^2 p_3^2}{p_1^4} \right) \right] \]

\[ + \frac{p_0^2 p_3^2 (p_0^2 - T_P P^2)}{p_1^4 p_1^4} \left( \frac{p_0^2}{p_1^2} - \frac{3 p_0^2 p_3^2}{p_1^4} \right) \]

\[ = \frac{1}{2} \left[ (1 + \Delta_0) (3 \Delta_1 + \Delta_6' - \Delta_0) + \Delta_0 - 3 \Delta_1 \right] \sum_p \frac{1}{p_1^2} + \frac{1}{2} \left[ 3 \Delta_0 - 9 \Delta_1 - \Delta_5 (\Delta_0 - 3 \Delta_1) \right. \]

\[ + (1 + \Delta_0) (12 \Delta_1 + 3 \Delta_6' - 4 \Delta_0) - \Delta_5 \Delta_6' + \Delta_6 (\Delta_0 - 3 \Delta_1) - \Delta_5' \Delta_6' \]

\[ + \Delta_6' (\Delta_0 - 3 \Delta_1) \left] \sum_p \frac{1}{p_1^2 p_1^2} \right. \]

\[ = \left( \frac{\Lambda e^{7 \kappa \epsilon}}{4 \pi T} \right)^{2e} \frac{1}{60(4\pi)^2} \left[ -83 + 5 \pi^2 + 56 \ln 2 \right] + \frac{1}{15 \epsilon} \left\{ -5893 + 500 \pi^2 \right. \]

\[ + \ln 2 (556 + 840 \ln 2) + 450 \zeta(3) \} + 73.7496 \right\} + \mathcal{O}[\epsilon]. \]  

(B.20)

9. 

\[ \sum_p \frac{A_2 (5 p_0^2 + 9 p_2^2) (p_0^2 - T_P P^2)}{4 p_1^4 p_1^4} = \sum_p \frac{1}{2} \left[ T_P (p_0^2 - T_P P^2) (5 p_0^2 + 9 p_2^2) \right. \]

\[ \times \left. \frac{1}{4 p_1^4 p_1^4} \right] \]
\[
10.
\]
\[
\sum_{p} A_4(\frac{p^2_0 - T_P p^2}{p^2 p^4}) = \sum_{p} \frac{3(p^2_0 - T_P p^2)}{8p^2 p^4} (1 - \frac{p^2}{p^2})^2 - \sum_{p} \frac{p^2_0 (p^2_0 - T_P p^2)}{8p^4 p^4} (1 - \frac{5p^2}{p^2})^2 + \frac{5 p^2_0 (p^2_0 - T_P p^2)}{p^4 p^4} \frac{p^2}{p^4} \left(1 - \frac{5p^2}{p^2}\right)^2 \\
\times \left\{ \left(1 - \frac{p^2}{p^2}\right)^2 - \frac{2p^3}{p^2} \left(1 - \frac{3p^2_0}{p^2}\right)^2 + \frac{p^4_0}{p^3} \left(1 - \frac{5p^2}{p^2}\right)^2 + \frac{8p^4_0 p^2}{p^4 p^2} \left(1 - \frac{5p^2}{p^2}\right)^2 \right\}
\]
\[
= \left[ \frac{1}{4} + \frac{\Delta'_0}{2} - \frac{25\Delta'_0}{12} + (1 + \Delta_0)\Delta'_1 \right] \sum_{p} \frac{1}{p^4} + \left[ \frac{1}{8} + \frac{19\Delta'_0}{4} - \frac{205\Delta'_1}{24} + (1 + \Delta_0) \right]
\]
\[
(7\Delta'_1 - 3\Delta'_0) + \Delta_5 \left(\frac{3}{8} + \frac{15}{4} \Delta'_0 + \frac{35}{8} \Delta'_1 \right) - \Delta_4 \left(\frac{5}{8} - \frac{13}{4} \Delta'_0 + \frac{55}{24} \Delta'_1 \right)
\]
\[
-\frac{3}{8} \Delta'_4 \left(1 - 2\Delta'_0 + \Delta'_1 \right) + \frac{3}{8} \Delta'_5 \left(2 - 12\Delta'_0 + 10\Delta'_1 \right) - \frac{3}{8} \Delta'_6 \left(1 - 10\Delta'_0 + \frac{35}{3} \Delta'_1 \right)
\]
\[
-\frac{3}{8} \Delta''_4 \left(1 - 2\Delta'_0 + \Delta'_1 \right) + \frac{3}{8} \Delta''_5 \left(2 - 12\Delta'_0 + 10\Delta'_1 \right) - \frac{3}{8} \Delta''_6 \left(1 - 10\Delta'_0 + \frac{35}{3} \Delta'_1 \right)
\]
\[
= \left(\frac{\Delta e^{\gamma E}}{4\pi T}\right)^{2e} \frac{1}{30(4\pi)^2} \left[ 3 - \frac{\pi^2 + 12 \ln 2}{\epsilon} + \frac{16 \ln 2}{15} + \frac{12 \ln 2}{5} (5 \ln 2 - 7) - 6\zeta(3) \right] \quad (B.22)
\]
C c-integrations

In this section we note down the following angular averages, appeared throughout this manuscript due to the angular integrals $A_n$ ($A_0 \equiv T_P$). Symbol $\langle \rangle_c$ depicts the standard definition given in Ref [78].

\[
\Delta_0 = \left\langle \frac{c^2}{1-c^2} \right\rangle_c = -\frac{1}{2\epsilon} + O[\epsilon]^3
\]  
(C.1)

\[
\Delta'_0 = \langle c^2 \rangle_c = \frac{1}{3} + \frac{2\epsilon}{9} + \frac{4\epsilon^2}{27} + O[\epsilon]^3
\]  
(C.2)

\[
\Delta''_0 = \left\langle \frac{1}{(1-c^2)(1-c'^2)} \right\rangle_{c_1,c_2} = \frac{(d-2)^2}{(d-3)^2} = \frac{1}{4\epsilon^2} - \frac{1}{\epsilon} + 1
\]  
(C.3)

\[
\Delta_1 = \left\langle \frac{c^4}{1-c^2} \right\rangle_c = -\frac{1}{2\epsilon} - \frac{1}{3} - \frac{2\epsilon}{9} - \frac{4\epsilon^2}{27} + O[\epsilon]^3
\]  
(C.4)

\[
\Delta'_1 = \langle c^4 \rangle_c = \frac{1}{5} + \frac{16\epsilon}{75} + O[\epsilon]^2
\]  
(C.5)

\[
\Delta_2 = \left\langle \frac{1-c^2-d}{1-c^2} \right\rangle_c = 1 - \frac{1}{2\epsilon}
\]  
(C.6)

\[
\Delta_3 = \left\langle \frac{1-c^4-d}{1-c^2} \right\rangle_c = \ln 2 + \left( \frac{\pi^2}{6} - (2 - \ln 2) \ln 2 \right) \epsilon
\]  
\[+ \left\{ \frac{2}{3} (\ln 2)^2 (\ln 2 - 3) + \frac{\pi^2}{3} (\ln 2 - 1) + (\zeta(3)) \right\} \epsilon^2 + O[\epsilon]^3.
\]  
(C.7)

\[
\Delta'_3 = \left\langle \frac{1-c^4-d}{(1-c^2)^2} \right\rangle_c = -\frac{1}{4\epsilon} + \frac{3}{4} + \frac{3\epsilon^2}{4} + O[\epsilon]^3
\]  
(C.8)

\[
\Delta''_3 = \left\langle \frac{1-c^4-d}{(1-c^2)(1-c'^2)} \right\rangle_{c_1,c_2} = -\frac{\pi^2}{12} + \left( \frac{\pi^2}{3} - \zeta(3) \right) \epsilon + O(\epsilon^2).
\]  
(C.9)

\[
\Delta_4 = \left\langle \frac{1-c^6-d}{1-c^2} \right\rangle_c = \left( \frac{\pi^2}{6} - 1 - (1 - \ln 2) \ln 2 \right) \epsilon + \epsilon^2 \left( \zeta(3)
\right.
\]  
\[+ \frac{1}{3} \ln 2 \left\{ \ln 2 (2 \ln 2 - 3) - 6 \right\} + \frac{1}{6} \pi^2 (2 \ln 2 - 1) \right\} + O[\epsilon]^3
\]  
(C.10)

\[
\Delta'_4 = \left\langle \frac{1-c^6-d}{(1-c^2)^2} \right\rangle_c = -\frac{3}{4\epsilon} + \frac{5}{4} - \ln 2 + \epsilon \left( \frac{3}{4} - \frac{\pi^2}{6} - (\ln 2)^2 + \ln 4 \right) + \frac{1}{12} \epsilon^2 \left( -12 \zeta(3)
\right.
\]  
\[-9 - \ln^3(4) + 6 \ln^2(4) - 2 \pi^2 (\ln 4 - 2) \right\} + O[\epsilon]^3
\]  
(C.11)

\[
\Delta''_4 = \left\langle \frac{1-c^6-d}{(1-c^2)(1-c'^2)} \right\rangle_{c_1,c_2} = -\frac{\pi^2}{12} + \ln 4 + \left( \frac{\pi^2}{3} - \ln 4 (2 - \ln 2) - \frac{\zeta(3)}{2} \right) \epsilon + O(\epsilon^2)
\]  
(C.12)

\[
\Delta'''_4 = \left\langle \frac{c^6-d - c^6-d}{c^2 - c'^2} \right\rangle_{c_1,c_2}
\]  
\[= \frac{1}{3} (1 + 2 \ln 2) + \frac{2}{9} (-5 + \ln 2 (5 + 3 \ln 2)) \epsilon + O[\epsilon]^2
\]  
(C.13)

\[
\Delta_5 = \left\langle \frac{1-c^8-d}{1-c^2} \right\rangle_c = \frac{3}{4} + \ln 2 + \frac{\epsilon}{12} \left( 2 \pi^2 + 3 ((\ln(4) - 1) \ln(4) - 5) \right)
\]  
(C.14)
\[ \Delta_5' = \left\langle \frac{1 - c^{8-d}}{1 - c^2} \right\rangle_c = \frac{1}{12} \left( 4 - \pi^2 + 32 \ln 2 \right) + \frac{1}{18} \left( -20 + 6 \pi^2 - 52 \ln 2 + 48(\ln 2)^2 - 9\zeta(3) \right) \epsilon + 0.469927 \epsilon^2 + \mathcal{O}[\epsilon]^3 \] (C.15)

\[ \Delta_5'' = \left\langle \frac{1 - c^{8-d}}{1 - c^2} \right\rangle_{c_1,c_2} = \frac{1}{12} \left( 4 - \pi^2 + 32 \ln 2 \right) + \frac{1}{18} \left( -20 + 6 \pi^2 - 52 \ln 2 + 48(\ln 2)^2 - 9\zeta(3) \right) \epsilon + 0.469927 \epsilon^2 + \mathcal{O}[\epsilon]^3 \] (C.16)

\[ \Delta_5''' = \left\langle \frac{1 - c^{8-d}}{1 - c^2} \right\rangle_{c_1,c_2} = \frac{1}{12} \left( 4 - \pi^2 + 32 \ln 2 \right) + \frac{1}{18} \left( -20 + 6 \pi^2 - 52 \ln 2 + 48(\ln 2)^2 - 9\zeta(3) \right) \epsilon + 0.469927 \epsilon^2 + \mathcal{O}[\epsilon]^3 \] (C.17)

\[ \Delta_6' = \left\langle \frac{1 - c^{10-d}}{1 - c^2} \right\rangle_c = \frac{7}{4} \epsilon \left( 12 - 2 \pi^2 + (7 - 3 \ln 4) \ln 4 \right) - \ln 8 + \frac{1}{24} \epsilon^2 \left( -72\zeta(3) - 6 + 2 \pi^2 (7 - 6 \ln 4) + 3 \ln 4 (18 + (7 - 2 \ln 4) \ln 4) \right) + \mathcal{O}[\epsilon]^3 \] (C.18)

\[ \Delta_6'' = \left\langle \frac{1 - c^{10-d}}{1 - c^2} \right\rangle_{c_1,c_2} = \frac{7}{4} \epsilon \left( 12 - 2 \pi^2 + (7 - 3 \ln 4) \ln 4 \right) - \ln 8 + \frac{1}{24} \epsilon^2 \left( -72\zeta(3) - 6 + 2 \pi^2 (7 - 6 \ln 4) + 3 \ln 4 (18 + (7 - 2 \ln 4) \ln 4) \right) + \mathcal{O}[\epsilon]^3 \] (C.19)

\[ \Delta_7 = \left\langle c^{2+1/\epsilon} \right\rangle_c = \frac{1}{2} + \mathcal{O}[\epsilon] \] (C.20)

\[ \Delta_8 = \left\langle c^{2+3/\epsilon} \right\rangle_c = \frac{1}{4} + \mathcal{O}[\epsilon] \] (C.21)

\[ \Delta_9 = \left\langle c^{2-1/\epsilon} \right\rangle_c = \frac{1}{2} + 1 + \ln 2 + \mathcal{O}[\epsilon] \] (C.22)

\[ \Delta_{10} = \left\langle \frac{c_1^{3+2} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{3+2} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right\rangle_{c_1,c_2}. \] (C.23)

Unlike other \( \Delta \) functions, computation of \( \Delta_{10} \) is not straightforward and can not be done directly analytically in Mathematica in \( \epsilon \to 0 \). After calculating the angular average in Eq. (C.23), we end up with the following equation.

\[ \Delta_{10} = -\frac{1}{\pi \Gamma(1 - \epsilon)^2 \Gamma(3/2 - \epsilon)^2} \times \left[ \frac{1}{3} \left( 4 - 1 - e^{-i \pi \epsilon} (1 + 3 \epsilon + 2 e^2) \Gamma(1 + 2 \epsilon) \right) 3 e^{2 i \pi \epsilon} F \left( \frac{3}{2} + \epsilon, 2 + \epsilon \left| \frac{1}{2} \right. \right) + (2 + e^{2 i \pi \epsilon}) \left( 3 F \left( \frac{3}{2} + \epsilon, 2 + \epsilon \left| \frac{1}{2} \right. \right) - 6(3 + 2 \epsilon) F \left( 2 + \epsilon, \frac{5}{2} + \epsilon \left| \frac{3}{2} \right. \right) \right) \right] \]
\[ + (5 + 2\epsilon)(3 + 2\epsilon)F\left( \frac{2 + \epsilon}{2}, \frac{7}{2} + \epsilon \left| \frac{1}{2} \right| \right) - \frac{\pi}{\Gamma[-1/2 - \epsilon]} \left\{ - F\left( \frac{3}{2} + \epsilon \mid \frac{-1}{2} - \epsilon \mid 1 \right) \right\} \]

\[- \frac{3 + 2\epsilon}{1 + 2\epsilon} F\left( \frac{1}{2} + \epsilon \mid 1 \right) + \frac{1}{(1 + 2\epsilon)^2} F\left( \frac{3}{2} + \epsilon \mid -\frac{1}{2} - \epsilon \mid 1 \right) + \frac{3}{1 - 4\epsilon^2} F\left( \frac{1}{2}, \frac{3}{2} + \epsilon \mid 1 \right) \right\} \] \hspace{1cm} (C.24)

Eq. (C.24) can not be expanded directly at small \( \epsilon \) in Mathematica. So, we use the following technique to expand Eq. (C.24) at small \( \epsilon \). In Eq. (C.24), \( F \) represents the generalized hypergeometric function. The generalized hypergeometric function of type \( pF_q \) is an analytic function of one variable with \( p + q \) parameters. Here, the parameters are functions of \( \epsilon \), so the list of parameters sometimes gets so lengthy and the standard notation for these functions becomes cumbersome. We therefore introduce a more compact notation as

\[ F\left( \alpha_1, \alpha_2, \ldots, \alpha_n \mid \beta_1, \ldots, \beta_n \right) \]

It is not possible to directly expand \( pF_q \) at small \( \epsilon \). So, we will use the following procedure to expand \( pF_q \) in the series of \( \epsilon \).

In Eq. (C.24), there are two types of hypergeometric function viz.. \( 2F_1 \) and \( 3F_2 \). \( 2F_1 \) can be expanded in small \( \epsilon \) if one uses the following relation.

\[ F\left( \alpha_1, \alpha_2; \beta_1; 1 \right) = \frac{\Gamma(\beta_1) \Gamma(\beta_1 - \alpha_1 - \alpha_2)}{\Gamma(\beta_1 - \alpha_1) \Gamma(\beta_1 - \alpha_2)} \] \hspace{1cm} (C.26)

To expand \( 3F_2 \) type of hypergeometric function, we can try the following power series representation for the generalized hypergeometric function as

\[ F\left( \alpha_1, \alpha_2, \ldots, \alpha_p \mid \beta_1, \ldots, \beta_q \right) \left| z \right| = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n n!} z^n, \] \hspace{1cm} (C.27)

where \((a)_b\) is Pochhammer’s symbol:

\[ (a)_b = \frac{\Gamma(a + b)}{\Gamma(a)}. \] \hspace{1cm} (C.28)

The power series converges for \(|z| < 1\). For \( z = 1 \), it converges if \( \text{Re} \ s > 0 \), where

\[ s = \sum_{i=1}^{p-1} \beta_i - \sum_{i=1}^{p} \alpha_i. \] \hspace{1cm} (C.29)

In Eq. (C.24), both \( 3F_2 \) has negative \( s \) value for \( \epsilon \to 0 \). So, we will use the following relation to change the parameters to make \( s \) value positive.

\[ F\left( \alpha_1, \alpha_2, \alpha_3 \mid \beta_1, \beta_2 \right) = \frac{\Gamma(\beta_1) \Gamma(\beta_2) \Gamma(s)}{\Gamma(\alpha_1 + s) \Gamma(\alpha_2 + s) \Gamma(\alpha_3)} F\left( \beta_1 - \alpha_3, \beta_2 - \alpha_3, s \mid \alpha_1 + s, \alpha_2 + s \right) \] \hspace{1cm} (C.30)
• Expansion of \( F_3 \): \( \frac{3}{2}, \frac{1}{2} + \epsilon; -\frac{1}{2}, \frac{1}{2} - \epsilon; 1 \)

Here, \( s = \beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3 = -3 - 2\epsilon < 0 \) at \( \epsilon \to 0 \). So, we use Eq. (C.30) to change the parameter.

\[
F \left( \frac{1}{2}, \frac{3}{2} + \epsilon \left| 1 \right. \right) = \frac{\Gamma \left( -\frac{1}{2} \right) \Gamma \left( \frac{1}{2} - \epsilon \right) \Gamma \left( 3 - 2\epsilon \right)}{\Gamma \left( -2 - 2\epsilon \right) \Gamma \left( -\frac{3}{2} - 2\epsilon \right) \Gamma \left( \frac{1}{2} + \epsilon \right)} F \left( -1 - \epsilon, -2\epsilon, -3 - 2\epsilon \left| 1 \right. \right) \quad (C.31)
\]

For the hypergeometric function that appears in RHS of the above eqn, \( s = 1/2 + 9\epsilon > 0 \) at \( \epsilon \to 0 \) so the power series expansion will converge.

So,

\[
F \left( -1 - \epsilon, -2\epsilon, -3 - 2\epsilon \left| 1 \right. \right) = \sum_{n=0}^{\infty} \frac{\Gamma \left( -1 - \epsilon + n \right) \Gamma \left( -2\epsilon + n \right) \Gamma \left( -3 - 2\epsilon + n \right) \Gamma \left( -2 - 2\epsilon \right) \Gamma \left( -\frac{3}{2} - 2\epsilon \right) \Gamma \left( \frac{1}{2} + \epsilon \right) n!}{\Gamma \left( -2 - 2\epsilon + n \right) \Gamma \left( -\frac{3}{2} - 2\epsilon + n \right) \Gamma \left( -1 - \epsilon \right) \Gamma \left( -2\epsilon \right) \Gamma \left( -3 - 2\epsilon \right)} \quad (C.32)
\]

In the above sum, if one expand the term within summation at small \( \epsilon \), one ends up with an expression that diverges for \( n = 0 \) to \( n = 3 \). To avoid that we will do the summation before the expansion from \( n = 0 \) to \( n = 3 \). From \( n = 4 \), we will perform the summation after expansion in small \( \epsilon \).

So,

\[
\begin{align*}
3F_2 \left( -1 - \epsilon, -2\epsilon, -3 - 2\epsilon; -2 + 2\epsilon, \frac{3}{2} + 2\epsilon; 1 \right) &= \left( \sum_{n=0}^{3} + \sum_{n=4}^{\infty} \frac{\Gamma \left( -1 - \epsilon + n \right) \Gamma \left( -2\epsilon + n \right) \Gamma \left( -3 - 2\epsilon + n \right) \Gamma \left( -2 - 2\epsilon \right) \Gamma \left( -\frac{3}{2} - 2\epsilon \right) \Gamma \left( \frac{1}{2} + \epsilon \right) n!}{\Gamma \left( -2 - 2\epsilon + n \right) \Gamma \left( -\frac{3}{2} - 2\epsilon + n \right) \Gamma \left( -1 - \epsilon \right) \Gamma \left( -2\epsilon \right) \Gamma \left( -3 - 2\epsilon \right)} \right) \epsilon^2 \\
&= \left[ 1 - \frac{14}{3} \epsilon + \frac{64}{9} \epsilon^2 - \frac{652}{27} \epsilon^3 + O \left( \epsilon^4 \right) \right] + \sum_{n=4}^{\infty} \frac{8\sqrt{\pi} \Gamma \left( n - \frac{1}{2} \right) \Gamma \left( n - \frac{1}{2} \right)}{n(n-3)\Gamma \left( n - \frac{3}{2} \right)} \epsilon^2 \\
&= \left[ 1 - \frac{14}{3} \epsilon + \frac{64}{9} \epsilon^2 - \frac{652}{27} \epsilon^3 + 16 \sum_{n=1}^{\infty} 3F_3 \left( 1, 1, 3; 2, 5, \frac{5}{2}; 1 \right) \epsilon^2 \\
&+ \frac{1}{18} \left[ 32(3\gamma_E - 11 + 6\ln 2) 3F_2 \left( 1, 1, 4; \frac{5}{2}, 5; 1 \right) + 32(6\gamma_E - 11 + 12\ln 2) 3F_2 \left( 1, 1, 3; 3, 4; 1 \right) \\
+ 252\zeta(3) - 36\gamma_E \pi^2 - 51\pi^2 - 640\gamma_E + 824 - 72\pi^2 \ln 2 - 1280 \ln 2 \right] \epsilon^3 + O \left( \epsilon^4 \right) \right) \quad (C.33)
\end{align*}
\]

The hypergeometric function of type \( p+1F_{q+1} \) has an integral representation in terms of the hypergeometric function of type \( pF_q \):

\[
\int_0^1 dt \left( 1 - t \right)^{\mu-1} F_1 \left( \alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \ldots, \beta_q \left| tz \right. \right) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu + \nu)} F_1 \left( \alpha_1, \alpha_2, \ldots, \alpha_p, \nu; \beta_1, \ldots, \beta_q, \mu + \nu \left| z \right. \right) \quad (C.34)
\]
Using Eq. (C.34), Eq. (C.33) can be simplified as

\[
3F_2 \left( -1 - \epsilon, -2 \epsilon, -3 - 2 \epsilon; -2 + 2 \epsilon, -\frac{3}{2} + 2 \epsilon; 1 \right)
\]

\[
= 1 - \frac{14}{3} \epsilon + \frac{64}{9} \epsilon^2 - \frac{652}{27} \epsilon^3 + \frac{16}{3} \int_0^1 \frac{dt F \left( 1, 3, 4 \epsilon \right)}{t^{1/2}} \epsilon^2
\]

\[
+ \frac{1}{18} \left[ 32(3 \gamma_E - 11 + 6 \ln 2) \times 3 \int_0^1 \frac{dt t_2}{2} F_1 \left( 1, 1; \frac{3}{2} \epsilon; t \right)
\]

\[
+ 32(6 \gamma_E - 11 + 12 \ln 2) \times 4 \int_0^1 \frac{dt t_3}{2} F_1 \left( 1, 1; \frac{5}{2} \epsilon; t \right)
\]

\[
+ 252 \zeta(3) + 824 - 51 \pi^2 - 4(160 + 9 \pi^2)(\gamma_E + 2 \ln 2) \right] \epsilon^3 + O(\epsilon^4)
\]

\[
= 1 - \frac{14}{3} \epsilon + \frac{64}{9} \epsilon^2 - \frac{652}{27} \epsilon^3 + \left( \pi^2 + \frac{112}{9} \right) \epsilon^2 + \left( 14 \zeta(3) - \frac{2}{27}(746 + 63 \pi^2) \right) \epsilon^3 + O(\epsilon^4)
\]

So, Eq. (C.31) can be rewritten as

\[
F \left( 1, \frac{3}{2}, \frac{3}{2} + \epsilon; \frac{1}{2}, \frac{3}{2} - \epsilon; 1 \right) = \frac{\Gamma \left( -\frac{1}{2} \epsilon \right) \Gamma \left( \frac{1}{2} - \epsilon \right) \Gamma \left( 3 - 2 \epsilon \right)}{\Gamma (-2 - 2 \epsilon) \Gamma \left( \frac{3}{2} - 2 \epsilon \right) \Gamma \left( \frac{1}{2} + \epsilon \right)} \left[ 1 - \frac{14}{3} \epsilon + \left( \pi^2 + \frac{176}{9} \right) \epsilon^2 + \left( -\frac{2144}{27} - \frac{14 \pi^2}{3} + 14 \zeta(3) \right) \epsilon^3 + O(\epsilon^4) \right]
\]

**Expansion of** \( 3F_2 \left( 1, \frac{5}{2}, \frac{3}{2} + \epsilon; \frac{1}{2}, \frac{3}{2} - \epsilon; 1 \right) \)

Following the similar procedure, we can write

\[
3F_2 \left( 1, \frac{5}{2}, \frac{3}{2} + \epsilon; \frac{1}{2}, \frac{3}{2} - \epsilon; 1 \right) = \frac{\Gamma \left( \frac{5}{2} + \epsilon \right) \Gamma \left( \frac{3}{2} - \epsilon \right) \Gamma \left( 3 - 2 \epsilon \right)}{\Gamma (-2 - 2 \epsilon) \Gamma \left( \frac{1}{2} - 2 \epsilon \right) \Gamma \left( \frac{5}{2} + \epsilon \right)} \left[ 1 - \frac{10}{3} \epsilon + \left( \pi^2 + \frac{40}{3} \right) \epsilon^2 + \left( -\frac{160}{3} - \frac{10 \pi^2}{3} + 14 \zeta(3) \right) \epsilon^3 + O(\epsilon^4) \right]
\]

Adding all the contribution, we can write Eq. (C.24) as

\[
\Delta_{10} = \left( \frac{c_2^{3+2 \epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{3+2 \epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} \right)_{c_1, c_2}
\]

\[
= -\frac{1}{8 \epsilon^2} + \frac{1}{4 \epsilon} + \frac{1}{24} (18 - \pi^2) + \frac{1}{12} (2 \pi^2 - 3(9 + \zeta(3))) \epsilon + O(\epsilon^3)
\]

(C.38)

The remaining \( c \)-integration \( \Delta_{11} \) can be written as

\[
\Delta_{11} = \left( \frac{c_1^{1+2 \epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{1+2 \epsilon} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right)_{c_1, c_2}
\]

\[-38-\]
\[ \Delta_0'' + \Delta_3'' = \frac{1 - 3}{8e^2} - \frac{3}{4e} + \frac{1}{24} (\pi^2 + 42) + \frac{1}{12} (3(3/2 - 9) - 2\pi^2)e + O(e^2) \] (C.39)

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