ON GLOBAL SMOOTH SOLUTIONS OF 3-D COMPRESSIBLE EULER EQUATIONS WITH VANISHING DENSITY IN INFINITELY EXPANDING BALLS

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Abstract. In this paper, we are concerned with the global smooth solution problem for 3-D compressible isentropic Euler equations with vanishing density in an infinitely expanding ball. It is well-known that the classical solution of compressible Euler equations generally forms the shock as well as blows up in finite time due to the compression of gases. However, for the rarefactive gases, it is expected that the compressible Euler equations will admit global smooth solutions. We now focus on the movement of compressible gases in an infinitely expanding ball. Because of the conservation of mass, the fluid in the expanding ball becomes rarefied meanwhile there are no appearances of vacuum domains in any part of the expansive ball, which is easily observed in finite time. We will confirm this interesting phenomenon from the mathematical point of view. Through constructing some anisotropy weighted Sobolev spaces, and by carrying out the new observations and involved analysis on the radial speed and angular speeds together with the divergence and rotations of velocity, the uniform weighted estimates on sound speed and velocity are established. From this, the pointwise time-decay estimate of sound speed is obtained, and the smooth gas fluids without vacuum are shown to exist globally.

1. Introduction. In this paper, we are concerned with the 3-D compressible isentropic Euler equations:

\[ \begin{cases} 
  \partial_t \rho + \sum_{i=1}^{3} \partial_i (\rho v_i) = 0, \\
  \partial_t (\rho v_i) + \sum_{j=1}^{3} \partial_j (\rho v_i v_j) + \partial_i p = 0, \quad i = 1, 2, 3, 
\end{cases} \tag{1} \]

where \( x = (x_1, x_2, x_3) \), \( \rho, v = (v_1, v_2, v_3) \) and \( p \) represent the density, velocity and pressure of polytropic gases, respectively. In addition, the state equation \( p = \frac{\rho^\gamma}{\gamma-1} \) holds with \( \gamma > 1 \) being the adiabatic exponent (the appearance of the constant factor \( \frac{1}{\gamma-1} \) in the state equation is only for the notational convenience), and the sound speed is \( c(\rho) = \sqrt{\rho'(\rho)} = \sqrt{\frac{\rho^{\gamma-1}}{\gamma-1}} \).

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With respect to the initial data excluding vacuum
\[ \rho(0, x) = \rho_0(x), \quad v(0, x) = v_0(x) = (v^1_0(x), v^2_0(x), v^3_0(x)), \quad (2) \]
where \( \rho_0(x) > 0, \ (\rho_0(x), v_0(x)) \in C^\infty(\mathbb{R}^3), \) the author in [30] proves that the classical solution \((\rho, v)\) of (1) with (2) blows up in finite time when the initial velocity \(v_0\) is locally supersonic. The analogous result is also extended into the case of 2-D compressible Euler equation (see [27]). In addition, for the 2-D equations (1), when the rotationally invariant data are a perturbation of size \( \varepsilon > 0 \) of a rest state, S. Alinhac in [2] establishes that the smooth solution \((\rho, v)\) blows up and the lifespan \( T_\varepsilon \) satisfies \( \lim_{\varepsilon \to 0} \varepsilon^2 T_\varepsilon = v_0^2 > 0. \) Recently, under the suitable conditions of initial data, the authors in [6]-[7] and [23] prove that the multidimensional compressible Euler equations generally form the shocks as well as blow up in finite time due to the compressions of gases.

When the initial data (2) contain vacuum, the authors in [9]-[10], [18]-[19], [22] and [26] prove that (1) has a local solution \((\rho, v)\) under some distinct restrictions on the initial data. In the general case, the local solution blows up in finite time (see [32]). Recently, the author in [31] constructs a class of expanding global affine solutions \((\rho, v)\) with physical vacuum condition and finite degrees-of-freedom by solving the related nonlinear ODEs. For different adiabatic exponent \( \gamma, \) authors in [15] and [29] establish the global existence and stability of these solutions with small perturbation, where the physical vacuum boundary condition on the vacuum surface played the essential roles in deriving the weighted energy estimates and establishing the existence of the global solutions.

When the state equation of Chaplygin gases in (1) is given by \( p = A - B \varepsilon^r, \) where \( A > 0 \) and \( B > 0 \) are constants, so far there are many fundamental results on (1) with (2). For examples, if the Chaplygin gases are isentropic and irrotational, then the 2-D or 3-D compressible Euler equations can be changed into a second order quasilinear wave equation by introducing a velocity potential function \( \Phi \) with \( \nabla \Phi = v. \) In this case, the related null conditions hold, then it follows from the results in [3], [5] and [20] that the small perturbed smooth solution \((\rho, v)\) of (1) exists globally. In addition, for the full Euler equations (1), when the solution is symmetric and small perturbed, the authors in [11]-[13] and [16] have established the global existence of smooth solution.

When the initial states in (2) force particles to spread out, roughly speaking, \( v_0(x) \) is close to a linear field, which means \( \lim_{|x| \to \infty} |v_0(x)| = \infty, \) the authors in [14] and [28] have proved the global existence of smooth solution to (1).

In this paper, we are concerned with the global existence problem of the smooth polytropic gases in a 3-D infinitely expanding ball. The 3-D expanding ball is described by \( \Omega_0 = \{(t, x) : t \geq 0, |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq R_0(t)\}, \) where \( R_0(t) \in C^5[0, \infty) \) satisfies \( R_0(0) = 1, R'_0(0) = 0, R''_0(0) = 0, \) and \( R_0(t) = 1 + Lt \) for \( t \geq 1 \) with some positive constant \( L. \) From the expression of \( \Omega_0, \) we know that the ball \( B_0^t = \{x : |x| \leq R_0(t)\} \) at the time \( t \) is artificially formed by pulling out the initial unit ball \( B^0 = \{x : |x| \leq 1\} \) with a smooth speed and acceleration (See Fig.1 below). We impose the following initial-boundary conditions on (1)
\[ \rho(0, x) = \rho_0(x), \quad v(0, x) = v_0(x), \quad \text{for} \ x \in B^0, \quad (3) \]
\[ R'_0(t) = \sum_{i=1}^{3} \frac{x_i v_i}{|x|}, \quad \text{for} \ (t, x) \in \partial \Omega_0 = \{(t, x) : t \geq 0, |x| = R_0(t)\}, \quad (4) \]
where \( \rho_0(x) \in H^4(B^0) \), \( v_0(x) \in H^4(B^0) \), and \( \rho_0(x) > 0 \) for \( x \in B^0 \). Here, (4) represents the solid wall condition.

**Theorem 1.1.** Assume that \((\rho_0, v_0) \in H^4(B^0) \), and the compatibility conditions on \( \{(t, x) : t = 0, x \in \partial B^0 \} \) of \((\rho_0, v_0) \) hold. Then for \( 1 < \gamma < \frac{15}{14} \), there exist a constant \( h_0 > 0 \), and a small constant \( \varepsilon_0 > 0 \) depending only on \( h_0 \), such that when \( \sup_{0 \leq i \leq 1, 1 \leq k \leq 5} |R_0^{(k)}(t)| + \|\rho_0 - 1\|_{H^4(B^0)} + \|v_0\|_{H^4(B^0)} < \varepsilon_0 \), \( R_0(t) = 1 + Lt \) for \( t \geq 1 \), and \( 0 < L < h_0 \), problem (1) with (3)-(4) admits a global solution \((\rho, v) \) in \( \Omega_0 \) satisfying

\[
(\rho, v) \in C([0, \infty), H^4(B^1_0)) \cap C^1([0, \infty), H^3(B^1_0)),
\]

\[
\frac{1}{2R^3(t)} \leq \rho \leq \frac{3}{2R^3(t)} \quad \text{for} \quad t \geq 1
\]

and

\[
\sup_{x \in B^0_t} \left( |v - \frac{Lx}{R(t)}| + |R(t)\nabla(v - \frac{Lx}{R(t)})| \right) \to 0 \quad \text{as} \quad t \to +\infty.
\]

where \( B^0_t = \{x : |x| \leq R_0(t)\} \), and \( R(t) = 1 + Lt \) for \( t \geq 0 \).

**Remark 1.** Now we explicitly describe the the compatibility conditions of initial-boundary values for \((\rho, v) \) in Theorem 1.1. As in [25] and so on, the compatibility conditions mean that the tangent derivatives of boundary values are compatible with the initial data on the intersection sphere \( \{(t, x) : t = 0, x \in \partial B^0 \} \). Note that \( M = \{x_1 \partial_2 - x_2 \partial_1, x_1 \partial_3 - x_3 \partial_1, x_2 \partial_3 - x_3 \partial_2, x_1 \partial_3 + R(t)R'(t)\partial_1, x_2 \partial_3 + R(t)R'(t)\partial_2, x_3 \partial_3 + R(t)R'(t)\partial_3\} \) is the basis of smooth tangent fields to the surface \( \{t \geq 0, R^2(t) = |x|^2\} \), and for \( \Gamma \in M \) and \( |\alpha| \leq 4 \), denote by

\[
\Gamma^\alpha(t) = \sum_{i=1}^{3} \frac{x_i v_i}{|x|} = \sum_{k \leq |\alpha|} g_{\alpha k}(t, x) R^{(k+1)}(t) + \sum_{i=1}^{3} \sum_{\alpha_1 + \alpha_2 = |\alpha|} f_{\alpha_1 \alpha_2}^i(t, x) \partial_t^{\alpha_1} \partial_x^{\alpha_2} v_i.
\]

Since \( \partial_t^{\alpha_1} \partial_x^{\alpha_2} v_i \vert_{t=0} \) are all known from (1) with (3), the compatibility conditions of
initial-boundary values are
\[ \left( \sum_{k \leq |\alpha|} g_{\alpha k}(t,x) R^{(k+1)}(t) + \sum_{i=1}^{3} \sum_{\alpha_{t}+|\alpha_{x}| \leq |\alpha|} f_{\alpha_{t} \alpha_{x}}^{i}(t,x) \partial_{t}^{\alpha_{t}} \partial_{x}^{\alpha_{x}} v_{i} \right) \bigg|_{t=0, x \in \partial B^{0}} \equiv 0. \]

**Remark 2.** If the initial ball \( B^{0} \) is pulled outwards rapidly, namely, when the number \( L \) is large, then some parts of the region inside the ball may become vacuum in finite time (see [8] and so on).

**Remark 3.** For the compressible Navier-Stokes equations, when the initial density may vanish at infinity, and the smallness assumption on the total energy is posed, the authors in [17] establish the global existence and uniqueness of classical solutions. Our Theorem 1.1 is a somewhat analogous result to [17] for the compressible Euler equations.

**Remark 4.** Note that \( (\hat{\rho}(t,x), \hat{v}(t,x)) = (1 R^{3}(t), \frac{Lx}{R(t)}) \) with \( R(t) = 1 + Lt \) is a special solution to (1) with (4). In fact, Theorem 1.2 below will show the stability of this special solution. In addition, the smallness of \( L \) in Theorem 1.1 is only used to prove the local existence of the solution to (1) with (4) and to obtain the smallness of \( (\rho(t, x) - 1, v(t, x)) \).

**Remark 5.** For the n-dimensional \((n = 2, 3)\) steady supersonic Euler equations
\[
\begin{align*}
\sum_{i=1}^{n} \partial_{i} (\rho v_{i}) &= 0, \\
\sum_{j=1}^{n} \partial_{j} (\rho v_{i} v_{j}) + \partial_{i} p &= 0, \quad i = 1, ..., n,
\end{align*}
\]
where \( v_{2} > c(\rho) = \sqrt{p'(\rho)} \) for \( n = 2 \), or \( v_{3} > c(\rho) \) for \( n = 3 \) (in this case, (8) is hyperbolic with respect to \( x_{2} \)-direction for \( n = 2 \) or \( x_{3} \)-direction for \( n = 3 \)), motivated by the methods in the proof of Theorem 1.1, we can establish that the global smooth supersonic polytropic gases only with vacuum state at infinity exist in an n-D infinitely long divergent nozzle (see [34]). For the irrotational and isentropic polytropic gases in a 3-D infinitely long divergent nozzle, we have shown this phenomenon in [33]. For the clarity of readers, we give the following pictures on the supersonic flows in nozzles (one can also see more detailed physical backgrounds in [8]):

![Figure 2. Continuous transonic flow in an infinite long de Laval nozzle](image-url)
We now give some illustrations on Theorem 1.1 from the physical point of view. When the ball $B_0$ is pulled outwards slowly (i.e., the constant $L > 0$ is small in $R_0(t)$), due to the conservation of mass, the gases in the expanding ball will become rarefied and eventually tends to a vacuum state at infinite time, meanwhile there are no appearances of vacuum domains in any part of the expansive ball. This phenomenon can be strictly verified from the pointwise time-decay estimate $\rho \sim \frac{1}{t^{1/3}}$ in (6) of Theorem 1.1. For different models (including compressible Euler equations, Navier-Stokes equations and Boltzmann equation, respectively), such physical phenomenon has already been verified as follows:

(i) When viscosities of gases are neglected and the gases are isentropic and irrotational, the movement of the gases can be described by the potential flow equation, which is a second order quasilinear wave equation. In [35], we have proved the same results as in Theorem 1.1.

(ii) When viscosities of gases are considered, and the movement of the gases is described by the compressible Navier-Stokes equations, we have established the existence of global smooth gases and obtained the same result as in (6) of Theorem 1.1 (see [36]).

(iii) When microcosmic factors of fluid particles are considered, and the movement of the gases is described by the Boltzmann equation, in [37] we show that the Boltzmann equation has a global solution and the macroscopical density $\rho$ of gases globally satisfies (6).

In this paper, we will remove the restriction of irrotational assumption for (1), and show the global existence and stability in Theorem 1.1.
To prove Theorem 1.1, we first consider problem (1) together with (3)-(4) in domain $\Omega = \{(t, x) : t \geq 0, |x| \leq R(t)\}$, which means that $R_0(t)$ in Theorem 1.1 is replaced by $R(t)$. In this case, we introduce the following coordinate transformation

$$y = \frac{x}{R(t)}$$

and the unknown function transformation

$$\sigma = c^2(\rho) = \frac{\rho^{\gamma-1}}{\gamma-1}, \quad u(t, y) = v(t, x).$$

Then domain $\tilde{\Omega}$ is changed into $\Omega = \{(t, y) : t \geq 0, |y| \leq 1\}$, and (1) becomes

$$\begin{align*}
\partial_t \sigma + \sum_{j=1}^{3} \frac{u_j - Ly_j}{R(t)} \partial_j \sigma + \frac{(\gamma-1)\sigma}{R(t)} \partial_j u_j &= 0, \\
\partial_t u_i + \sum_{j=1}^{3} \frac{u_j - Ly_j}{R(t)} \partial_j u_i + \frac{\gamma}{R(t)} \partial_j \sigma &= 0, \quad i = 1, 2, 3,
\end{align*}$$

(9)

where $\partial_j$ stands for $\partial_{y_j}$ ($j = 1, 2, 3$). Meanwhile the boundary condition (4) becomes

$$R'(t) = \sum_{i=1}^{3} y_i u_i \quad \text{for} \quad (t, y) \in \partial\Omega = \{(t, y) : t \geq 0, |y| = 1\}. \quad (10)$$

In addition, we impose the following initial perturbed condition:

$$\sigma(0, y) = \frac{1}{\gamma-1} + \varepsilon \sigma_0(y), \quad u(0, y) = Ly + \varepsilon u_0(y), \quad (11)$$

where $y \in B^0$ and $(\sigma_0(y), u_0(y)) \in H^4(B^0)$.

**Theorem 1.2.** Under conditions (10)-(11), for $\gamma \in (1, \frac{17}{11})$, then there exists a constant $\varepsilon_0 > 0$ depending on $L$ and $\gamma$ such that problem (9) has a global solution $(\sigma, u) \in C([0, \infty), H^4(B^0)) \cap C^1([0, \infty), H^3(B^0))$ for $\varepsilon < \varepsilon_0$. Moreover, $\sigma > 0$ and $\lim_{t \to \infty} \sigma = 0$ hold.

Let us comment on the proof of Theorem 1.2. Since it follows from [24] that the local well-posedness of problem (9) with (10)-(11) is known as long as the vacuum does not appear, we will use the continuous induction method to prove Theorem 1.2. To achieve this objective, we need to establish the global energy estimates of $(\sigma, u)$ with suitable weights, where $\sigma$ is degenerate at infinity time. Inspired by our former paper [35], at first, we choose some suitable multipliers to derive the “weak” weighted energy estimates on $(\dot{\sigma}, \dot{u})$, where $(\dot{\sigma}, \dot{u})$ is the solution to the linearized equations of (9) (see (21)-(22) in Section 3), and the “weak” weighted energy estimate is referred to that the resulting weighted estimates on $(\dot{\sigma}, \dot{u})$ are weaker than the really required energies $E_T(\dot{\sigma}, \dot{u})$ and $S_T(\dot{\sigma}, \dot{u})$ (see their definitions in (27)-(28)) due to the weaker time-decay weights. Secondly, we decompose (21)-(22) into a system of rotations curl $u$ and a coupled degenerate hyperbolic system of $(\nabla \dot{\sigma}, \text{div} \dot{u})$. To estimate $(\text{curl} \dot{u}, \nabla \dot{\sigma}, \text{div} \dot{u})$, we will take the following measures:

- Since the main part of the system on curl $u$ is a linear ordinary differential equation (see (43) in Section 5), by choosing a suitable multiplier we can obtain the weighted energy estimates on curl $u$.

- For the coupled degenerate hyperbolic system of $(\nabla \dot{\sigma}, \text{div} \dot{u})$ (see (79)-(80) in Section 5), by choosing some multipliers we can establish the uniform weak weighted energy estimates of $(\nabla \dot{\sigma}, \text{div} \dot{u})$ and its derivatives. The key ingredients in the step
are to look for suitable anisotropic weights, and to find the available boundary
conditions of higher order derivatives of $(\nabla \sigma, \text{div} u)$ on the boundary $\partial \Omega$.

- Based on some key observations and delicate analysis on (21)-(22), we can
derive the required weighted energy estimates of $(\nabla \sigma, \text{div} u)$. Indeed, from
the linearized momentum equation (22), if the weighted energy estimates of $\sigma$ and its
derivatives are obtained, then we can take $\nabla \sigma$ as the known quantity. In this case,
the main linear part of (22) is regarded as an ODE of $\dot{u}$, and then we can find a
new multiplier to re-estimate $\dot{u}$. The advantage of this doing is that we can avoid
to utilize the integration by parts for the spatial derivatives of $\dot{u}$ and overcome the
difficulty arisen by the slip boundary condition (10) and by the lack of boundary
value of $\dot{u}$ on $\partial \Omega$.

- From the resulting estimates on third-order derivatives of $\sigma$, we can re-estimate
the derivatives of $\dot{u}$ up to second order. Based on this and Sobolev imbedding
theorem, we can derive the better decay rate of $\dot{u}$ than that in [35] and obtain the
estimates of $E_T(\dot{\sigma}, \dot{u})$ and $S_T(\dot{\sigma}, \dot{u})$.

In [35], we assume that the gases are isentropic and irrotational. For this situation,
the Euler equations can be simplified to a second order quasi-linear hyperbolic
equation of potential $\Phi$, whose linearized part admits the degenerate form
$\partial_t^2 \Phi - \frac{\gamma}{R^3(\gamma - 1)(t)} \Delta \Phi + \frac{3L(\gamma - 1)}{R(t)} \partial_3 \Phi$ with the development of time $t$. In addition, due to
the special form of such a second order quasi-linear equation on $\Phi$ and the vanishing
properties of rotations $\omega = (\omega_1, \omega_2, \omega_3) = (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)$ in (1),
we can conveniently derive some neat boundary conditions of $\partial^\alpha \Phi (|\alpha| \leq 4)$. From
this, the higher order weighted energy estimates of $\Phi$ can be successfully obtained
and further establish an analogous result to Theorem 1.1. However, in the present
paper, we have to treat the really Euler equations, which are more involved and
difficult for treating the rotations and deriving boundary conditions of higher order
derivatives of $(\sigma, u)$. The authors in [15] establish the global existence of classical
solutions to 3-D compressible Euler equations with the expanding physical vacuum
boundary, where it plays an crucial role in deriving the weighted energy estimates of
solutions by the vacuum boundary value ($\rho \equiv 0$ on the vacuum surface). Compared
with [15], in our paper, the complicated boundary conditions of $(\partial^\alpha \sigma, \partial^\alpha u)$ should
be concerned more carefully since only the fixed boundary condition (10) (a linear
combination of $u_1, u_2$ and $u_3$) is known. Moreover, more delicate observations on
the different time-decay properties of radial speed and angular speeds are required.

The paper is organized as follows. In Section 2, we derive some estimates on
the background solution, and list the Sobolev interpolation inequality and an el-
liptic estimate of $\nabla u$ by $\text{div} u$ and $\text{curl} u$. In Section 3, we reformulate problem
(9) with (10)-(11) by decomposing its solution as a sum of the background solution
and a small perturbation $(\dot{\sigma}, \dot{u})$. In addition, the required weighted Sobolev
norms are introduced in this section. In Section 4, we establish a uniform weak
weighted energy estimate for the resulting linear problem, where an appropriate
multiplier is constructed. In Section 5, at first, we give a main and basic weighted
energy result (Theorem 5.1), whose proof will be completed in the whole Section
5-7. To prove Theorem 5.1 partly, in Section 5, we will decompose the linearized
equations (21)-(22) into a system of $\text{curl} u$ and a coupled hyperbolic system of
$(\nabla \sigma, \text{div} u)$. Subsequently the higher-order weighted energy estimates of $\text{curl} u$ are established.
Meanwhile, the uniform zero-order and first-order weak weighted estimates
of $\nabla \sigma, \text{div} u$ are derived by delicate analysis on some radial derivatives and
tangent derivatives of \((\nabla \hat{s}, \text{div}\hat{u})\), where the domain composition techniques are applied near and away from \(|y| = 0\) respectively. Based on this, the weak weighted estimates on first-order and second-order derivatives of \((\hat{s}, \hat{u})\) can be obtained. In addition, due to the lengthy proofs on Lemma 5.3 and Lemma 5.5, we will put them in Section 6. In Section 7, at first we derive the uniform weak weighted energy estimates on the third-order derivatives of \((\hat{s}, \hat{u})\) in terms of the resulting higher-order boundary conditions of \((\hat{s}, \hat{u})\). In addition, by utilizing the obtained estimates on the higher-order derivatives, we can get the uniform weighted energy on the lower order derivatives of \(\hat{u}\). Synthesizing these conclusions, the weighted energy estimates of \(\hat{u}\) in \(\mathcal{E}_T(\hat{s}, \hat{u})\) and \(\mathcal{S}_T(\hat{s}, \hat{u})\) are derived. In last section, we complete the proof of Theorem 1.2 by applying the Sobolev embedding theorem and the continuation argument, and Theorem 1.1 follows from Theorem 1.2 immediately.

In the whole paper, as in [6] or [23], we shall use the following convention:

\[ |\nabla^\alpha f| \text{ represents } \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} |\partial_{\alpha_1}^1 \partial_{\alpha_2}^2 \partial_{\alpha_3}^3 f|, \text{ where } \alpha \in \mathbb{N}_0. \]

\[ |Z^k f| \text{ represents } \sum_{k_1+k_2+k_3=k} |Z_1^{k_1} Z_2^{k_2} Z_3^{k_3} f|, \text{ where } k \in \mathbb{N}_0, \text{ and } Z \in \{Z_1, Z_2, Z_3\} \]

with

\[ Z_1 = y_2 \partial_3 - y_3 \partial_2, \quad Z_2 = y_3 \partial_1 - y_1 \partial_3, \quad Z_3 = y_1 \partial_2 - y_2 \partial_1. \]

2. Background solution and preliminaries. In this section, we look for and analyze the background solution to (1) with (4) when the initial data (3) are replaced by

\[ \rho_0(x) = 1, \quad v_0(x) = Lx, \tag{12} \]

and \(R_0(t)\) is replaced by \(R(t)\). In this case, it is easy to know that the density \(\rho\) and velocity \(v\) take the form: \(\rho(t, x) = \hat{\rho}(t, r), \quad v(t, x) = \frac{x}{r} \hat{U}(t, r)\), here \(r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}\). Then problem (1) with (4) and (12) is equivalent to

\[
\begin{cases}
  r^2 \partial_r \hat{\rho} + \partial_r (r^2 \hat{r} \hat{U}) = 0, \\
  \partial_t (r^2 \hat{r} \hat{U}) + \partial_r (r^2 \hat{r} \hat{U}^2) + r^2 \partial_r \hat{\rho} = 0, \\
  \hat{r}(0, r) = 1, \quad \hat{U}(0, r) = Lr.
\end{cases}
\tag{13}
\]

One can directly check that (13) has a solution

\[ \hat{\rho}(t, r) = \frac{1}{R^3(t)}, \quad \hat{U}(t, r) = \frac{Lr}{R(t)}. \tag{14} \]

This yields \(\hat{s}(t, r) = \frac{\hat{\rho}^{\gamma-1}}{\gamma-1} = \frac{R(t)^{-3(\gamma-1)}}{\gamma-1} \) and \(\hat{v}(t, x) = \frac{Lx}{R(t)}\). Under the transformation \(y = \frac{x}{R(t)}\), we have \(\hat{u}(t, y) = \hat{v}(t, x) = Ly\). Introduce the basis of the smooth vector fields tangent to the sphere \(S^2\) as

\[ Z_1 = y_2 \partial_3 - y_3 \partial_2, \quad Z_2 = y_3 \partial_1 - y_1 \partial_3, \quad Z_3 = y_1 \partial_2 - y_2 \partial_1. \tag{15} \]

As in Lemma 4.4 of [21], it follows from direct computation that

Lemma 2.1.

(i) \([Z_1, Z_2] = -Z_3, [Z_2, Z_3] = -Z_1, [Z_3, Z_1] = -Z_2\).

(ii) \([Z_i, \partial_3] = 0, Z_i r = 0, [Z_i, \Delta] = 0\).

(iii) \(\nabla_y f \cdot \nabla_y g = \partial_y f \cdot \partial_y g + \frac{1}{r^2} \sum_{i=1}^{3} Z_i f \cdot Z_i g\) for any \(C^1\) smooth functions \(f\) and \(g\).
(iv) $|Zv| \leq r|\nabla_y v|$ for any $C^1$ smooth function $v$, here and below $Z \in \{Z_1, Z_2, Z_3\}$.

(v) \[ \partial_1 = \frac{y_1}{r} \partial_r + \frac{y_3}{r^2} Z_2 - \frac{y_2}{r^2} Z_3, \partial_2 = \frac{y_2}{r} \partial_r + \frac{y_1}{r^2} Z_3 - \frac{y_3}{r^2} Z_1, \]
\[ \partial_3 = \frac{y_3}{r} \partial_r + \frac{y_2}{r^2} Z_1 - \frac{y_1}{r^2} Z_2. \]

Next we cite the following two results.

**Lemma 2.2** (Gagliardo-Nirenberg Inequality, see [1]). Let $\Omega$ be any bounded domain in $\mathbb{R}^3$ with smooth boundary. Then
\[
\|D^\gamma f\|_p \leq C\|f\|_p^{1-\theta} \left( \sum_{|\alpha| = \ell} \|D^\alpha f\|_p^\theta \right)^\theta,
\]
where $1 \leq p_1, p_2 \leq \infty$, $0 \leq r \leq l$, \[ \frac{1}{p} - \frac{\xi}{n} = (1 - \theta)\frac{1}{p_1} + \theta\left(\frac{1}{p_2} - \frac{1}{n}\right), \]
and $\xi \leq \theta \leq 1$. Especially, choosing $r = 1$, $p = 4$, $p_1 = \infty$, $p_2 = 2$, $l = 2$, $\theta = \frac{1}{2}$, then
\[
\|Df\|_4 \leq C\|f\|_\infty \|\nabla^2 f\|_2.
\]

**Lemma 2.3** (Estimate of $\nabla u$ by $div u$ and $curl u$). Let $u \in H^s(\Omega)$ be a vector-valued function satisfying $u \cdot n|_{\partial \Omega} = 0$, where $n$ is the unit outer normal of the smooth boundary $\partial \Omega$, and $\Omega \subset \mathbb{R}^3$ is an bounded open set. Then for $s \geq 1$,
\[
\|u\|_{H^s} \leq C \left(\|\nabla u\|_{H^{s-1}} + \|div u\|_{H^{s-1}} + \|u\|_{H^{s-1}}\right),
\]
where the constant $C > 0$ depends only on $s$ and $|\Omega|$.

The proof of Lemma 2.3 can be found in [4], we omit it here.

3. Local existence and reformulation. Firstly, we state a local solvability result on problem (1) with (3)-(4).

**Lemma 3.1.** There exists a $T_0 > 0$ such that problem (1) with (3)-(4) has a local solution $(\rho, v) \in C([0, T_0]; H^4(B_0^0)) \cap C^1([0, T_0]; H^3(B_0^0))$ with $B_0^0 = \{x : |x| \leq R_0(t)\}$ under the compatibility conditions of initial boundary values in Theorem 1.1. Moreover,
\[
\|\rho - \tilde{\rho}\|_{C([0, T_0]; H^4(B_0^0))} + \|\rho - \tilde{\rho}\|_{C^1([0, T_0]; H^3(B_0^0))} + \|v - \tilde{v}\|_{C([0, T_0]; H^4(B_0^0))} + \|v - \tilde{v}\|_{C^1([0, T_0]; H^3(B_0^0))} \leq C\varepsilon,
\]
where $(\tilde{\rho}(t, r), \tilde{v}(t, x))$ is given in (13).

**Proof.** (1) is a symmetrizable hyperbolic system if the density $\rho$ is bounded below away from zero. Then by making use of the linearized iteration in Theorem 2.1 (a)-(b) of [24] together with the linear energy estimates in [12], Lemma 3.1 holds. \[ \square \]

Obviously, Lemma 3.1 means the local solvability of (9) with (10)-(11).

**Corollary 1.** There exists $T_0 > 0$, such that (9) with (10)-(11) has a local solution $(\sigma, u) \in C([0, T_0], H^4(B_0^0)) \cap C^1([0, T_0], H^3(B_0^0))$. Moreover, following estimates hold
\[
\|\sigma - \tilde{\sigma}\|_{C([0, T_0]; H^4(B_0^0))} + \|\sigma - \tilde{\sigma}\|_{C^1([0, T_0]; H^3(B_0^0))} + \|u - \tilde{u}\|_{C([0, T_0]; H^4(B_0^0))} + \|u - \tilde{u}\|_{C^1([0, T_0]; H^3(B_0^0))} \leq C\varepsilon.
\]
We now linearize (9) with (10)-(11). Let \( \hat{\sigma}(t, y) = \sigma(t, y) - \hat{\sigma}(t, y) \) and \( \hat{u}(t, y) = u(t, y) - \hat{u}(t, y) \). Then (9) can be reformulated as

\[
\mathcal{L}_0(\hat{\sigma}, \hat{u}) = \partial_t \hat{\sigma} + \frac{3(\gamma - 1)L}{R(t)} \hat{\sigma} + \frac{1}{R(t)^{3(\gamma - 1)}} \sum_{j=1}^{3} \partial_j \hat{u}_j = \hat{f}_0, \quad (21)
\]

\[
\mathcal{L}_i(\hat{\sigma}, \hat{u}) = \partial_t \hat{u}_i + \frac{L}{R(t)} \hat{u}_i + \frac{\gamma}{R(t)} \partial_t \hat{\sigma} = \hat{f}_i, \quad i = 1, 2, 3, \quad (22)
\]

where

\[
\begin{cases}
\hat{f}_0 = - \frac{1}{R(t)} \sum_{j=1}^{3} \hat{u}_j \partial_j \hat{\sigma} - \frac{\gamma - 1}{R(t)} \hat{\sigma} \sum_{j=1}^{3} \partial_j \hat{u}_j, \\
\hat{f}_i = - \frac{1}{R(t)} \sum_{j=1}^{3} \hat{u}_j \partial_j \hat{u}_i.
\end{cases}
\]

On the lateral boundary \( \partial \Omega \), \( \hat{u}(t, y) \) satisfies

\[
\sum_{i=1}^{3} y_i \hat{u}_i = 0. \quad (23)
\]

In addition, we have the following initial data of \( (\hat{\sigma}, \hat{u}) \) from (11)

\[
\hat{\sigma}(0, y) = \varepsilon \sigma_0(y), \quad \hat{u}(0, y) = \varepsilon u_0(y). \quad (24)
\]

For the later convenience, we introduce following notation of radial velocity \( \hat{u}_r \) and angular velocities \( \hat{u}_z = (\hat{u}_{z1}, \hat{u}_{z2}, \hat{u}_{z3}) \)

\[
\begin{cases}
\hat{u}_r = y_1 \hat{u}_1 + y_2 \hat{u}_2 + y_3 \hat{u}_3, \\
\hat{u}_{z1} = y_2 \hat{u}_3 - y_3 \hat{u}_2, \\
\hat{u}_{z2} = y_3 \hat{u}_1 - y_1 \hat{u}_3, \\
\hat{u}_{z3} = y_1 \hat{u}_2 - y_2 \hat{u}_1.
\end{cases} \quad (25)
\]

This yields

\[
\begin{cases}
\hat{u}_1 = \frac{y_2}{y_3} \hat{u}_r - \frac{y_2}{y_3} \hat{u}_{z2} - \frac{y_3}{y_2} \hat{u}_{z3}, \\
\hat{u}_2 = \frac{y_3}{y_1} \hat{u}_r + \frac{y_3}{y_1} \hat{u}_{z2} - \frac{y_1}{y_3} \hat{u}_{z3}, \\
\hat{u}_3 = \frac{y_1}{y_2} \hat{u}_r + \frac{y_1}{y_2} \hat{u}_{z1} - \frac{y_2}{y_1} \hat{u}_{z2}.
\end{cases} \quad (26)
\]

In order to state our results later on, we now introduce the following anisotropy weighted energy norms with the positive constant \( \mu = 3(\gamma - 1) \)

\[
\mathcal{E}_T(\hat{\sigma}, \hat{u}) = \sum_{\alpha + \beta \leq 2} \int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |\nabla^\alpha S_0^\beta \hat{\sigma}|^2 dt dy + \int_{\Omega_T} R(t)^{2\mu - 1 - 2\delta} |\nabla^3 \hat{\sigma}|^2 dt dy 
+ \sum_{\alpha + \beta + \nu = 3, \alpha < 3} \int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |\nabla^\alpha S_0^\beta Z^\nu \hat{u}|^2 dt dy 
+ \sum_{\alpha + \beta + \nu = 2, \alpha < 2} \int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |\nabla^\alpha S_0^\beta Z^\nu \hat{u}|^2 dt dy 
+ \sum_{\alpha + \beta + \nu = 3, \alpha < 3} \int_{\Omega_T} R(t)^{\mu - 1} |\nabla^\alpha S_0^\beta Z^\nu \hat{u}|^2 dt dy + \int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |\nabla^2 \hat{u}_z|^2 dt dy 
+ \int_{\Omega_T} R(t)^{2\mu - 1 - 2\delta} |\nabla^2 \hat{u}|^2 dt dy + \int_{\Omega_T} R(t)^{\mu - 1 - 2\delta} |\nabla^3 \hat{u}|^2 dt dy \quad (27)
\]
and

\[ S_T(\dot{\sigma}, \dot{u}) = \sum_{\alpha + \beta \leq 2} \int_{\Gamma_T} R(t)^{2\mu} |\nabla^\alpha S_0^\beta \dot{\sigma}|^2 \, dy + \int_{\Gamma_T} R(t)^{2\mu - 2\delta} |\nabla^3 \dot{\sigma}|^2 \, dy \]

\[ + \sum_{\alpha + \beta + \nu = 3, \alpha < 3} \int_{\Gamma_T} R(t)^{2\mu} |\nabla^\alpha S_0^\beta Z^\nu \dot{\sigma}|^2 \, dy \]

\[ + \sum_{\alpha + \beta + \nu \leq 2, \alpha < 2} \int_{\Gamma_T} R(t)^{2\mu - \delta} |\nabla^\alpha S_0^\beta Z^\nu \dot{u}|^2 \, dy \]

\[ + \int_{\Gamma_T} R(t)^{2\mu - 2\delta} |\nabla^2 \dot{u}|^2 \, dy + \int_{\Gamma_T} R(t)^{2\mu - 2\delta} |\nabla^3 \dot{u}|^2 \, dy \]

\[ + \sum_{\alpha + \beta + \nu = 3, \alpha < 3} \int_{\Omega_T} R(t)^{\mu} |\nabla^\alpha S_0^\beta Z^\nu \dot{u}|^2 \, dy + \int_{\Omega_T} R(t)^{2\mu - \delta} |\nabla^2 \dot{u}|^2 \, dy, \quad (28) \]

where \( S_0 = R(t) \partial_t \nabla = (\partial_{y_1}, \partial_{y_2}, \partial_{y_3}) \), \( \Omega_T = \Omega \cap \{0 \leq t \leq T\} \), \( \Gamma_T = \Omega \cap \{t = T\} \), and the constant \( \delta > 0 \) will be chosen later in a suitable range of values.

By the definitions of \( E_T(\dot{\sigma}, \dot{u}) \) and \( S_T(\dot{\sigma}, \dot{u}) \), \( \nabla^2 \dot{u}_z \) admits more rapid time-decay rate than \( \nabla^2 \dot{u} \). Indeed, if \( S_T(\dot{\sigma}, \dot{u}) \leq C \varepsilon^2 \) holds uniformly for \( T > 0 \), then by Sobolev imbedding theorem, \( |\dot{u}_z| \leq C \varepsilon R(t)^{-\mu + \frac{\delta}{2}} \) and \( |\dot{u}| \leq C \varepsilon R(t)^{-\mu + \delta} \) for any \( t > 0 \). Here we point out that the more rapid time-decay rate of \( \dot{u}_z \) than \( \dot{u} \) will play an important role in deriving the energy estimates of the third order radial derivatives of \( (\dot{\sigma}, \dot{u}) \) (one can see more details in Section 7).

4. Weak weighted energy estimate of \( (\dot{\sigma}, \dot{u}) \). In this section, we derive the weak weighted energy estimate of \( (\dot{\sigma}, \dot{u}) \) for the linear part (21)-(22) with (23)-(24). Set \( \mathcal{B}_T = \partial \Omega \cap \{0 \leq t \leq T\} = [0, T] \times S^2 \), \( \Omega_T \) and \( \Gamma_T \) are defined in (27) and (28) respectively. Then we have

**Theorem 4.1.** Let \( (\dot{\sigma}, \dot{u}) \in C^1(\Omega_T) \) satisfy (23) and (24). Then for \( 1 < \gamma < \frac{5}{2} \), there exist multipliers \( M_0 \dot{\sigma} = \gamma R(t)^{2\mu}(1 + R(t)^{-\delta}) \dot{\sigma} \) and \( M \dot{u}_i = R(t)^{\mu}(1 + R(t)^{-\delta}) \dot{u}_i \) \( (i = 1, 2, 3) \) with \( \mu = 3(\gamma - 1) \), such that

\[
\int_{\Omega_T} (R(t)^{2\mu} \dot{\sigma}^2 + R(t)^{\mu} |\dot{u}|^2) \, dt \, dy \leq \int_{\Omega_T} \left( \sum_{i=1}^{3} f_i \cdot M \dot{u}_i + \sum_{i=1}^{3} \dot{f}_i \cdot M \dot{u}_i \right) \, dt \, dy + C \varepsilon^2, \quad (29)
\]

where \( C > 0 \) is a generic positive constant, and \( 0 < \delta < 2 - \mu = 5 - 3\gamma \) is a small but fixed constant which will be determined later.

**Remark 6.** The choice of \( \mu = 3(\gamma - 1) \) in (29) is necessary due to the following two reasons: On the one hand, to guarantee the positivity of \( III \) in (30) below, \( \mu \leq 3(\gamma - 1) \) is required. On the other hand, only the estimate of \( |\dot{\sigma}| \leq M \varepsilon R(t)^{-\mu + 3(\gamma - 1)} \) can be obtained as shown in Section 8. Together with \( \dot{\sigma} = \frac{1}{\gamma - 1} R(t)^{-3(\gamma - 1)} \), this yields \( \frac{1}{\gamma - 1} \rho_1\gamma - 1 \geq \frac{1}{\gamma - 1} R(t)^{-3(\gamma - 1)} - M \varepsilon R(t)^{-\mu + 3(\gamma - 1)} \). In order to guarantee the absence of vacuum for any finite time \( t \), \( -\mu + 3(\gamma - 1) \leq -3(\gamma - 1) \) should be required, which yields \( \mu \geq 3(\gamma - 1) \). Thus, \( \mu = 3(\gamma - 1) \) has to be chosen.

**Proof.** Choosing multipliers \( M_0 \dot{\sigma} = \gamma R(t)^{\mu + 3(\gamma - 1)}(1 + R(t)^{-\delta}) \dot{\sigma} \) and \( M \dot{u}_i = R(t)^{\mu}(1 + R(t)^{-\delta}) \dot{u}_i \) \( (i = 1, 2, 3) \), where \( \mu, \delta > 0 \) will be determined later. Then it follows
It follows from (24) that

$$
I = \int_{\Omega_T} \left( L_0(\dot{\sigma}, \dot{u}) \cdot \mathcal{M}_0 + \sum_{i=1}^{3} L_i(\dot{\sigma}, \dot{u}) \cdot \mathcal{M}_i \right) dtdy = I + II + III,
$$

where

$$
I = \int_{\Gamma_T} \left( \gamma R(t)^{\mu-1} + R(t)^{-\delta} \right) dS,
$$

$$
II = II_1 - II_2,
$$

$$
III = \int_{\Omega_T} \left( \frac{L}{2} R(t)^{\mu-1} (2 - \mu + (2 - \mu - \delta) R(t)^{-\delta}) |\dot{\sigma}|^2 + \frac{\delta \gamma L}{2} - R(t)^{\mu-1+3(\gamma-1)-\delta} |\dot{\sigma}|^2 + \frac{\gamma L}{2} (3(\gamma - 1) - \mu) R(t)^{\mu-1+3(\gamma-1)} (1 + R(t)^{-\delta}) |\dot{\sigma}|^2 \right) dtdy,
$$

and

$$
II_1 = \int_{\Gamma_T} \left( \frac{\gamma R(t)^{\mu+3(\gamma-1)-1} (1 + R(t)^{-\delta}) |\dot{\sigma}|^2 + \frac{1}{2} R(t)^{\mu} (1 + R(t)^{-\delta}) |\dot{\sigma}|^2 \right) dy,
$$

$$
II_2 = \int_{\Gamma_T} \left( \frac{\gamma R(t)^{\mu+3(\gamma-1)-1} (1 + R(t)^{-\delta}) |\dot{\sigma}|^2 + \frac{1}{2} R(t)^{\mu} (1 + R(t)^{-\delta}) |\dot{\sigma}|^2 \right) dy.
$$

By the boundary condition (23), one has

$$
I = 0.
$$

In addition, direct computation yields

$$
II_1 \geq C \int_{\Gamma_T} \left( R(t)^{\mu+3(\gamma-1)-1} |\dot{\sigma}|^2 + R(t)^{\mu} |\dot{\sigma}|^2 \right) dy.
$$

(32)

It follows from (24) that

$$
II_2 \leq C \varepsilon^2.
$$

(33)

To guarantee III > 0, it requires

$$
\mu < 2 \quad \text{and} \quad \mu \leq 3(\gamma - 1)
$$

(34)

and

$$
\delta < 2 - \mu.
$$

As explained in Remark 4.1, we have to choose \( \mu = 3(\gamma - 1) \), which means \( 3(\gamma - 1) < 2 \). Thus \( 1 < \gamma < \frac{5}{3} \). Substituting (31)-(33) into (30) yields (29), and then Theorem 4.1 is proved.

**Lemma 4.2.** Let \( \dot{\sigma}, \dot{u} \in C^3(\Omega_T) \) be a solution of (21)-(22) with (23)-(24). Assume that for \( \mu = 3(\gamma - 1) \),

$$
\begin{cases}
|\dot{\sigma}| \leq M \varepsilon R(t)^{-\mu}, & |\dot{\sigma}| \leq M \varepsilon R(t)^{-\mu+\delta}, \\
|\nabla \dot{\sigma}| \leq M \varepsilon R(t)^{-\mu+\delta}, & |\nabla \dot{\sigma}| \leq M \varepsilon R(t)^{-\frac{3}{2}+\delta},
\end{cases}
$$

(35)

where \( M > 0 \) is a fixed constant, \( \varepsilon > 0 \) is sufficiently small, and \( 0 < \delta < \min\{5 - 3\gamma, \frac{3(\gamma-1)}{3} \} \). Then we have

$$
\int_{\Gamma_T} (R(t)^{2\mu} |\dot{\sigma}|^2 + R(t)^{\mu} |\dot{\sigma}|^2) dy + \int_{\Omega_T} (R(t)^{2\mu-1-\delta} |\dot{\sigma}|^2 + R(t)^{\mu-1} |\dot{\sigma}|^2) dtdy \leq C \varepsilon^2,
$$

(36)

where \( C > 0 \) is a generic positive constant independent of \( M \) and \( T \).
Proof. In terms of Theorem 4.1, it only suffices to treat the term \( \int_{\Omega_T} \left( \hat{f}_0 \cdot \mathcal{M}_0 \hat{\sigma} + \sum_{i=1}^{3} \hat{f}_i \cdot \hat{\mathcal{M}}_i \right) dtdy \) in (29). Direct computation yields

\[
\int_{\Omega_T} \hat{f}_0 \cdot \mathcal{M}_0 \hat{\sigma} dtdy = \int_{\Omega_T} \gamma R(t)^{2-\delta} \left( \sum_{j=1}^{3} \hat{u}_j \partial_j \hat{\sigma} - (\gamma - 1) \hat{\sigma} \text{div} \hat{u} \right) dtdy
\]

\[
= - \int_{\Omega_T} \frac{\gamma}{2} R(t)^{2-\delta} \hat{\sigma}^2 \sum_{j=1}^{3} u_j y_j dS + \int_{\Omega_T} \frac{\gamma}{2} R(t)^{2-\delta} \text{div} \hat{u} \hat{\sigma} dtdy.
\]

This follows from (23) and (35) that

\[
| \int_{\Omega_T} \hat{f}_0 \cdot \mathcal{M}_0 \hat{\sigma} dtdy | \leq C \epsilon \left( \int_{\Omega_T} R(t)^{2-\delta} \hat{\sigma}^2 dtdy, \right)
\]

here we have used \( \delta < \frac{3(\gamma - 1)}{4} \). Next, we estimate \( \int_{\Omega_T} \sum_{i=1}^{3} \hat{f}_i \cdot \hat{\mathcal{M}}_i dtdy \). Since

\[
\int_{\Omega_T} \sum_{i=1}^{3} \hat{f}_i \cdot \hat{\mathcal{M}}_i dtdy = - \int_{\Omega_T} \sum_{i,j=1}^{3} \hat{u}_i \hat{u}_j \partial_j \hat{u}_i dtdy
\]

\[
= - \int_{\Omega_T} \frac{1}{2} R(t)^{2-\delta} \text{div} \hat{u} \hat{u} dtdy
\]

by (23) and (35) we have

\[
| \int_{\Omega_T} \sum_{i=1}^{3} \hat{f}_i \cdot \hat{\mathcal{M}}_i dtdy | \leq C \epsilon \left( \int_{\Omega_T} R(t)^{2-\delta} \hat{u}^2 dtdy, \right)
\]

Substituting (38) and (40) into (29) yields (36).

From Lemma 4.2, we have got the weak weighted energy estimate of \( \dot{u} \) since we need to further improve the estimate of \( \dot{u} \) with the higher time-decay rate by comparing with the norms of \( \dot{u} \) in (27)-(28). This will be done in Section 7.

5. Main energy estimates and weak weighted energy estimates on some derivatives of \( (\dot{\sigma}, \dot{u}) \). In this section, at first, we give a main and basic weighted energy estimate \( (\dot{\sigma}, \dot{u}) \), which will be shown in the whole Section 5-Section 7. On the other hand, we will derive the weak weighted energy estimates on some derivatives of \( (\dot{\sigma}, \dot{u}) \).
where 0

\textbf{Theorem 5.1 (Main weighted energy estimate).} Let \( \hat{\sigma}, \hat{u} \in C^3(\bar{\Omega}_T) \) be a solution to (21)-(22) with (23)-(24). Assume that (35) holds for some \( \delta > 0 \). Moreover, the following a priori assumptions hold for \( \mu = 3(\gamma - 1) \) with \( 1 < \gamma < \frac{15}{11} \),

\[
\begin{align*}
\hat{u}_x &\leq M|z(t)|^{\mu+\frac{\beta}{2}}, \\
|S_0\hat{u}_x| &\leq M|z(t)|^{\mu+\frac{\beta}{2}}, \\
|S_0\hat{\sigma}| + |Z\hat{\sigma}| &\leq M|z(t)|^{-\mu}, \\
|S_0\hat{\sigma}| &\leq M|z(t)|^{-\mu+\delta}.
\end{align*}
\]  

(41)

Then for sufficiently small \( \varepsilon > 0 \) and \( 0 < \delta < \min\{5 - 3\gamma, \frac{\gamma - 1}{\gamma}\} \), we have

\[
E_T(\hat{\sigma}, \hat{u}) + S_T(\hat{\sigma}, \hat{u}) \leq Ce^2,
\]  

(42)

where \( C > 0 \) is independent of the constant \( M \) in (35) and \( T \).

\textbf{Remark 7.} Here we point out that the range of \( \gamma \) is reduced from \( (1, \frac{5}{3}) \) in Theorem 4.1 to \( (1, \frac{15}{11}) \), the reason comes from the requirement of better weighted energy estimates on \( curlu \) in Lemma 5.2) (see (47)).

Due to the slip boundary condition (23) and the lack of boundary values of the derivatives of \( (\hat{\sigma}, \hat{u}) \), the weighted energy estimates in Theorem 4.1 can not be applied directly to the higher order derivatives of \( (\hat{\sigma}, \hat{u}) \). To overcome this difficulty, we will decompose equations (21)-(22) into a system of ODEs, then we can derive the weighted energy estimates of \( curlu \). For the system of \( curlu \) (see (43) below), its main part can be regarded as the ODEs, then we can derive the weighted energy estimates of \( curlu \) directly. For the coupled hyperbolic system of \( (\nabla\sigma, div\hat{u}) \) (see (79)-(80)), the weak weighted energy estimates in Theorem 4.1 can be applied. Based on Lemma 2.3, we can establish all the weak weighted energy estimates of \( (\nabla\sigma, \nabla\hat{u}) \). Same idea can be used to treat the second and third order derivatives of \((\hat{\sigma}, \hat{u})\). To prove Theorem 5.1, we will take the following estimates:

(i) \( S_0^k
abla^{2-k}curlu \), where \( S_0 = R(t)\partial_t \), and \( 0 \leq k \leq 2 \). (see Lemma 5.2)

(ii) \( S_0^k\hat{u}, S_0^k\hat{\sigma} \), where \( 1 \leq k \leq 3 \). (see Lemma 5.3)

(iii) \( div\hat{u}, \nabla\hat{\sigma} \). (see Lemma 5.4)

(iv) Tangent derivatives of \( div\hat{u} \) and \( \nabla\hat{\sigma} \): \( S_0^kZ^{1-k}div\hat{u} \) and \( S_0^kZ^{1-k}\nabla\sigma \), where \( 0 \leq k \leq 1 \):

Radial derivatives of \( div\hat{u} \) and \( \nabla\hat{u} \): \( S_1div\hat{u} \) and \( S_1\nabla\hat{\sigma} \), where \( S_1 = \sum_{i=1}^{3} y_i\partial_i \). (in this case, together with (i)-(ii) and Lemma 4.2, all the weak weighted estimates on the second order derivatives of \( (\hat{\sigma}, \hat{u}) \) are obtained)

(v) Second order tangent derivatives of \( div\hat{u} \) and \( \nabla\hat{\sigma} \): \( S_0^kZ^{2-k}div\hat{u}, S_0^kZ^{2-k}\nabla\hat{\sigma} \), where \( 0 \leq k \leq 2 \);

First order tangent derivatives of \( S_1div\hat{u} (0 \leq l \leq 1) \) and \( S_1\nabla\hat{\sigma} \): \( S_1^lZ^{1-l}S_1div\hat{u}, S_1^lZ^{1-l}S_1\nabla\hat{\sigma} \), where \( 0 \leq l \leq 1 \);  

Second order radial derivatives of \( div\hat{u} \) and \( \nabla\hat{u} \): \( S_1^ldiv\hat{u}, S_1^l\nabla\hat{\sigma} \). (in this case, together with (i)-(iii) and Lemma 4.2, all the weak weighted estimates on the third order derivatives of \( (\hat{\sigma}, \hat{u}) \) are obtained)

Later on (iv) and (v) will be established in Lemmas 5.5-5.7 of Sections 5-6 and Lemmas 7.1-7.5 of Section 7 in turn. From (i)-(v), we can eventually complete the proof of Theorem 5.1 in Section 7.
In addition, for reader’s convenience, we list the following main boundary conditions on $B_T$ of some higher order derivatives of $(\dot{\sigma}, \dot{u})$, which will be derived in details and utilized later on.

(i) The boundary condition of $\nabla \dot{\sigma}$:

$$\gamma \sum_{i=1}^{3} y_i \partial_i \dot{\sigma} - |\dot{u}|^2 = 0 \quad \text{on } B_T.$$  
(see (82))

(ii) The boundary condition of $S_0^k \dot{u}$ ($k \in \mathbb{N}$):

$$\sum_{i=1}^{3} y_i S_0^k \dot{u}_i = 0 \quad \text{on } B_T.$$  
(see (119))

(iii) The boundary condition of $\text{div}\dot{u}$:

$$S_1(\text{div}\dot{u}) = \frac{R(t)^{\mu+1}}{1+(\gamma-1)\sigma R(t)} \left( -\frac{2}{7} \gamma R(t) \sum_{i=1}^{3} \dot{u}_i \partial_i \dot{R(t)} \dot{u}_i + \frac{3}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j \dot{\sigma} \right)$$
$$+ \frac{(5-3\gamma)L}{7 \gamma R(t)} |\dot{u}|^2 - \frac{1}{R(t)} \sum_{j=1}^{3} \sigma R_j \dot{u}_j \partial_j \dot{\sigma} - \frac{2}{7 R(t)} \sum_{i,j=1}^{3} \dot{u}_i \dot{u}_j \partial_i \partial_j \dot{u}_i - \frac{\gamma - 1}{7 R(t)} |\dot{u}|^2 \text{div}\dot{u}$$

$$\equiv \frac{R(t)^{\mu}}{1+(\gamma-1)\sigma R(t)} \cdot G \quad \text{on } B_T.$$  
(see (91))

(iv) The boundary condition of $S_1 \Delta \dot{\sigma}$:

$$S_1 \Delta \dot{\sigma} = -\frac{1}{7} \partial_t \left( \frac{R(t)^{\mu+1}}{1+(\gamma-1)\sigma R(t)} \cdot G \right) - \frac{1}{7} \left( \sum_{j=1}^{3} \frac{S_1(\partial_j \dot{u}_j \partial_j \dot{u}_i)}{} \right)$$
$$+ \frac{3}{R(t)} \sum_{j=1}^{3} S_1 \dot{u}_j \partial_j \text{div}\dot{u} + \frac{3}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j \text{div}\dot{u} \quad \text{on } B_T.$$  
(see (190))

We next derive the equation system of $\text{curl}\dot{u}$. Take the curl-operator to (22), and denote by $\omega = \text{curl}\dot{u} = (\omega_1, \omega_2, \omega_3) = (\partial_2 \dot{u}_3 - \partial_3 \dot{u}_2, \partial_3 \dot{u}_1 - \partial_1 \dot{u}_3, \partial_1 \dot{u}_2 - \partial_2 \dot{u}_1)$, then we have

$$\partial_t \omega_i + \frac{L}{R(t)} \omega_i = -\frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j \omega_i - \frac{1}{R(t)} \text{div}\omega_i + \frac{3}{R(t)} \sum_{j=1}^{3} \partial_i \omega_j, \quad i = 1, 2, 3.$$  
(43)

**Lemma 5.2** (Estimates on $S_0^k \nabla^{k_2} \omega$). Under the assumptions of Theorem 5.1, for $0 \leq k \leq 2$, then for $\mu = 3(\gamma - 1)$ with $1 < \gamma < \frac{15}{11}$,

$$\int_{\Gamma_T} \left( R(t)^{2\mu - \delta} |S_0^k \nabla^{k_2} \omega|^2 \, dy \right) \, dt + \int_{\Omega_T} \left( R(t)^{2\mu - 1 - \delta} |S_0^k \nabla^{k_2} \omega|^2 \, dt \, dy \right) \leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}).$$  
(44)

**Proof.** Acting $\frac{1}{R(t)} S_0^k \nabla^{k_2}$ on two sides of equation (43) yields

$$\partial_t (S_0^k \nabla^{k_2} \omega_i) + \frac{L}{R(t)} S_0^k \nabla^{k_2} \omega_i = -\frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j (S_0^k \nabla^{k_2} \omega_i) + R_{k_1, k_2},$$  
(45)
where
\[
R_{k_1,k_2} = \frac{1}{R(t)} \sum_{0 \leq l_1 + l_2 < k_1 + k_2} C_{l_1,l_2} S_0^{k_1-l_1} \nabla^{k_2-l_2} \tilde{u}_{k_1} \partial_j (S_0^{l_1} \nabla^{l_2} \omega_i) \\
+ \frac{1}{R(t)} \sum_{0 \leq l_1 + l_2 < k_1 + k_2} C_{l_1,l_2} S_0^{k_1-l_1} \nabla^{k_2-l_2} \left( \text{div} \tilde{u} \right) S_0^{l_1} \nabla^{l_2} \omega_i \\
+ \frac{1}{R(t)} \sum_{0 \leq l_1 + l_2 < k_1 + k_2} \sum_{j=1}^3 C_{l_1,l_2} S_0^{k_1-l_1} \nabla^{k_2-l_2} \left( \partial_j \hat{u}_j \right) S_0^{l_1} \nabla^{l_2} \omega_j.
\]

Multiplying \( R(t)^{2\mu-\delta} S_0^{k_1} \nabla^{k_2} \omega_i \) to (45) and integrating over \( \Omega_T \), we have that by the integration by parts
\[
\int_{\Omega_T} \left( \partial_t (S_0^{k_1} \nabla^{k_2} \omega_i) + \frac{L}{R(t)} S_0^{k_1} \nabla^{k_2} \omega_i \right) \cdot R(t)^{2\mu-\delta} S_0^{k_1} \nabla^{k_2} \omega_i dt dy \\
= \int_{\Omega_T} \partial_t \left( \frac{1}{2} R(t)^{2\mu-\delta} (S_0^{k_1} \nabla^{k_2} \omega_i)^2 \right) + \left( 1 - \frac{2\mu - \delta}{2} \right) LR(t)^{2\mu-\delta-1} (S_0^{k_1} \nabla^{k_2} \omega_i)^2 dt dy \\
= \int_{\Omega_T} \frac{1}{2} R(t)^{2\mu-\delta} (S_0^{k_1} \nabla^{k_2} \omega_i)^2 dy - \int_{\Omega_T} \frac{1}{2} R(t)^{2\mu-\delta} (S_0^{k_1} \nabla^{k_2} \omega_i)^2 dy \\
+ \int_{\Omega_T} \left( 1 - \frac{2\mu - \delta}{2} \right) LR(t)^{2\mu-\delta-1} (S_0^{k_1} \nabla^{k_2} \omega_i)^2 dt dy. \tag{46}
\]

In order to keep the positivity of last line in (46), one needs \( 1 - \frac{2\mu - \delta}{2} > 0 \). This yields \( \gamma < \frac{4}{3} + \frac{\delta}{6} \). By \( \delta < \min\{5 - 3\gamma, \frac{7-\gamma}{2} \} \), we have
\[
1 < \gamma < \frac{15}{11}. \tag{47}
\]

Thus, it follows from (46) and Lemma (3.1) that
\[
\int_{\Omega_T} R(t)^{2\mu-\delta} |S_0^{k_1} \nabla^{k_2} \omega|^2 dy + \int_{\Omega_T} R(t)^{2\mu-\delta-1} |S_0^{k_1} \nabla^{k_2} \omega|^2 dt dy \\
\leq C \varepsilon^2 - \int_{\Omega_T} R(t)^{2\mu-1-\delta} \sum_{j=1}^3 \partial_j (\hat{u}_j |S_0^{k_1} \nabla^{k_2} \omega|^2) dt dy \\
+ \int_{\Omega_T} R(t)^{2\mu-1-\delta} \text{div} \hat{u} |S_0^{k_1} \nabla^{k_2} \omega|^2 dt dy \\
+ \sum_{i=1}^3 \int_{\Omega_T} R_{k_1,k_2} R(t)^{2\mu-\delta} S_0^{k_1} \nabla^{k_2} \omega_i dt dy. \tag{48}
\]

Next, we treat the remainder terms in the right hand of (48). Direct computation yields
\[
\int_{\Omega_T} R(t)^{2\mu-1-\delta} \sum_{j=1}^3 \partial_j (\hat{u}_j |S_0^{k_1} \nabla^{k_2} \omega|^2) dt dy \\
= \int_{\partial_T} R(t)^{2\mu-1-\delta} \sum_{j=1}^3 y_j \hat{u}_j |S_0^{k_1} \nabla^{k_2} \omega|^2 dS = 0. \tag{49}
\]

In addition, it follows from (35) that
\[
| \int_{\Omega_T} R(t)^{2\mu-1-\delta} \text{div} \hat{u} |S_0^{k_1} \nabla^{k_2} \omega|^2 dt dy |
\]
In addition, by (35) and H"older inequality we arrive at
\[ \int_{\Omega_T} R(t)^{2\mu-1-\delta} |S_0^{k_2} \nabla^{k_2} \omega|^2 \, dtdy. \] (50)

In the end, we estimate the last term \( \sum_{i=1}^{3} \int_{\Omega_T} R^i_{k_1,k_2} \) \( R(t)^{2\mu-\delta} S_0^{k_1} \nabla^{k_2} \omega \, dtdy \) in (48). We divide this process into six cases for different \( k_1 \) and \( k_2 \).

Case (1) \( k_1 = k_2 = 0 \).

In this case,
\[ R_{0,0}^i = \frac{1}{R(t)} \sum_{j=1}^{3} \partial_t \hat{u}_j \omega_j. \]
Then it follows from (35) and H"older inequality that
\[ |\sum_{i=1}^{3} \int_{\Omega_T} R^i_{0,0} R(t)^{2\mu-\delta} \omega \, dtdy| = |\int_{\Omega_T} R(t)^{2\mu-1-\delta} \sum_{i,j=1}^{3} \partial_t \hat{u}_j \omega_i \omega_j \, dtdy| \]
\[ \leq C \varepsilon \int_{\Omega_T} R(t)^{2\mu-1-\delta} |\omega|^2 \, dtdy. \] (51)

This, together with (48)-(50), yields
\[ \int_{\Gamma_T} R(t)^{2\mu-\delta} |\omega|^2 \, dy + \int_{\Omega_T} R(t)^{2\mu-1-\delta} |\omega|^2 \, dtdy \leq C \varepsilon^2. \] (52)

Case (2) \( k_1 = 1, k_2 = 0 \).

Due to
\[ R_{1,0}^i = -\frac{1}{R(t)} \sum_{j=1}^{3} S_0 \hat{u}_j \partial_t \omega_i - \frac{1}{R(t)} S_0 \partial \omega \omega_i - \frac{1}{R(t)} \partial \omega S_0 \omega_i 
- \frac{1}{R(t)} \sum_{j=1}^{3} \partial_t S_0 \hat{u}_j \omega_j - \frac{1}{R(t)} \sum_{j=1}^{3} \partial_t \hat{u}_j S_0 \omega_j, \] (53)
one then has
\[ |R_{1,0}^i| \leq C \frac{1}{R(t)} \left( |S_0 \hat{u}| \| \nabla^2 \hat{u} \| + |S_0 \nabla \hat{u}| \| \nabla \hat{u} \| \right). \] (54)

In addition, by (35) and H"older inequality we arrive at
\[ \int_{\Omega_T} R(t)^{2\mu-1-\delta} |S_0 \hat{u}| \| \nabla^2 \hat{u} \| |S_0 \omega_1| \, dtdy \leq M \varepsilon \int_{\Omega_T} R(t)^{\mu-1} \| \nabla^2 \hat{u} \| |S_0 \omega_1| \, dtdy \]
\[ \leq C \varepsilon \left( \int_{\Omega_T} R(t)^{2\mu-1-2\delta} |\nabla^2 \hat{u}|^2 \, dtdy + \int_{\Omega_T} R(t)^{2\mu-1-2\delta} |S_0 \omega|^2 \, dtdy \right) \]
\[ \leq C \varepsilon \mathcal{E}_T(\sigma, \hat{u}). \] (55)

and
\[ \int_{\Omega_T} R(t)^{2\mu-1-\delta} |S_0 \nabla \hat{u}| \| \nabla \hat{u} \| |S_0 \omega| \, dtdy \leq M \varepsilon \int_{\Omega_T} R(t)^{2\mu-1} |S_0 \nabla \hat{u}| |S_0 \omega| \, dtdy \]
\[ \leq C \varepsilon \mathcal{E}_T(\sigma, \hat{u}). \] (56)

Then we have
\[ \sum_{i=1}^{3} \int_{\Omega_T} R_{1,0}^i R(t)^{2\mu-\delta} S_0 \omega_1 \, dtdy \leq C \varepsilon \mathcal{E}_T(\sigma, \hat{u}). \] (57)
It follows from (48)-(50) and (57) that
\[ \int_\Gamma R(t)^{2^{\mu_1^-\delta}} |S_0\omega|^2 dy + \int_{\Omega_T} R(t)^{2^{\mu_1^-\delta}} |S_0\omega|^2 dt dy \leq C\varepsilon^2 + C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (58)

Case (3) \( k_1 = 0, k_2 = 1 \).

In this case,
\[
R^i_{0,1} = -\frac{1}{R(t)} \sum_{j=1}^{3} \nabla u_j \partial_j \omega_i - \frac{1}{R(t)} \text{div} \nabla \dot{u} \omega_i - \frac{1}{R(t)} \text{div} \nabla \dot{\omega}.
\]
(59)

This yields
\[ |R^i_{0,1}| \leq \frac{C}{R(t)} |\nabla \dot{u}| |\nabla^2 \dot{u}|. \] (60)

In addition, it follows from (35) that
\[ \int_{\Omega_T} R(t)^{2^{\mu_1^-\delta}} |\nabla \dot{u}| |\nabla^2 \dot{u}| |\nabla \omega| dt dy \leq C\varepsilon \int_{\Omega_T} R(t)^{2^{\mu_1^-\delta}} |\nabla^2 \dot{u}|^2 dt dy \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) \] (61)
and
\[ \int_\Gamma R(t)^{2^{\mu_1^-\delta}} |\nabla \omega|^2 dy + \int_{\Omega_T} R(t)^{2^{\mu_1^-\delta}} |\nabla \omega|^2 dt dy \leq C\varepsilon^2 + C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (62)

Case (4) \( k_1 = 2, k_2 = 0 \).

In this case,
\[ |R^i_{2,0}| \leq \frac{C}{R(t)} \left( |S_0\dot{u}| |\nabla S_0\omega| + |S_0^2\dot{u}| |\nabla \omega| + |\nabla \dot{u}| |S_0^2\omega| + |\nabla \dot{u}| |\nabla S_0\dot{u}| \right). \]
(63)

We now treat \( \int_{\Omega_T} R(t)^{2^{\mu_1^-\delta}} |S_0^2\dot{u}|^2 |\nabla \omega|^2 dtdy \). Since we don’t obtain the weighted \( L_\infty^y \) estimates for \( S_0^2\dot{u} \) and \( \nabla^2 \dot{u} \), we will use the Gagliardo-Nirenberg interpolation inequality to overcome this difficulty. Before doing this, we give a new expression of \( S_0^2\dot{u} \). Taking \( S_0 \) to equation (22) yields
\[ S_0^2\dot{u}_i + L S_0\dot{u}_i + \gamma \partial_i S_0 \dot{\sigma} = -\sum_{j=1}^{3} \dot{u}_j \partial_j S_0 \dot{u}_i - \sum_{j=1}^{3} S_0 \dot{u}_j \partial_j \dot{u}_i, \quad i = 1, 2, 3. \] (64)

In addition, taking \( R(t) \partial_t \) to equation (21), we have
\[ \partial_t S_0 \dot{\sigma} = -3(\gamma - 1) L \partial_t \dot{\sigma} - R(t)^{-3(\gamma - 1)} \partial_t \text{div} \dot{u} - \sum_{j=1}^{3} \partial_t \dot{u}_j \partial_j \dot{\sigma} \]
(65)
Substituting (65) into (64) derives
\[
|S_0^2 \hat{u}| \leq C \left( |S_0 \hat{u}| + |\hat{u}| |\nabla S_0 \hat{u}| + |\nabla \hat{\sigma}| + R(t)^{-3(\gamma - 1)} |\nabla^2 \hat{u}| + |\hat{u}| |\nabla^2 \hat{\sigma}| \right). \tag{66}
\]
Then
\[
\int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |S_0^2 \hat{u}| |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy \leq C \int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |S_0 \hat{u}| |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy
\]
\[
+ C \int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |\hat{u}| |\nabla^2 \hat{\sigma}| |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy
\]
\[
+ C \int_{\Omega_T} R(t)^{\mu - 1 - \delta} |\nabla^2 \hat{u}|^2 |S_0 \omega| dt \, dy.
\]
\[
+ C \int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |\hat{u}| |\nabla S_0 \hat{u}| |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy. \tag{67}
\]
We next treat each term in the right side of (67). At first, it follows from (35), (41) and H"older inequality that
\[
\int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |S_0 \hat{u}| |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy \leq C \varepsilon \int_{\Omega_T} R^{\varepsilon - 1} |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy \leq C \varepsilon^2 \mathcal{E}_T(\hat{\sigma}, \hat{u}). \tag{68}
\]
In addition,
\[
\int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |\nabla S_0 \hat{u}| |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy \leq C \varepsilon \int_{\Omega_T} R^{\mu - 1} |\nabla S_0 \hat{u}| |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy
\]
\[
\leq C \varepsilon \int_0^T R(t)^{\mu - 1 - \frac{1}{2}} |\nabla S_0 \hat{u}|_{L^2(\Omega)} |\nabla^2 \hat{u}|_{L^2(\Omega)} |\nabla^2 \omega|_{L^2(\Omega)} dt. \tag{69}
\]
It follows from Gagliardo-Nirenberg inequality (17) that
\[
|\nabla S_0 \hat{u}|_{L^2(\Omega)} \leq C |S_0 \hat{u}|_{L^{\frac{2}{3}}(\Omega)} |\nabla^2 S_0 \hat{u}|_{L^{\frac{2}{3}}(\Omega)}, \tag{70}
\]
\[
|\nabla^2 \hat{u}|_{L^2(\Omega)} \leq C |\nabla \hat{u}|_{L^{\frac{2}{3}}(\Omega)} |\nabla^3 \hat{u}|_{L^{\frac{2}{3}}(\Omega)}. \tag{71}
\]
Substituting (70)-(71) into (69), and using (35), (41) and H"older inequality, we arrive at
\[
\int_{\Omega_T} R(t)^{2\mu - 1 - \delta} |\nabla S_0 \hat{u}| |\nabla^2 \hat{u}| |S_0^2 \omega| dt \, dy \leq C \varepsilon^2 \int_0^T R(t)^{\frac{7}{2} - 1 + \delta} |\nabla^2 S_0 \hat{u}|_{L^2(\Omega)} |\nabla^3 \hat{u}|_{L^2(\Omega)} |\nabla^2 \omega|_{L^2(\Omega)} dt
\]
\[
\leq C \varepsilon^2 \| R(t) \|_{L^\infty([0,T])} \left( \int_{\Omega_T} R(t)^{2\mu - 1 - 2\delta} |\nabla^2 S_0 \hat{u}|^2 dt \, dy \right)^{\frac{1}{2}}
\]
\[
\times \left( \int_{\Omega_T} R(t)^{2\mu - 1 - 2\delta} |\nabla^3 \hat{u}|^2 dt \, dy \right)^{\frac{1}{2}} \left( \int_{\Omega_T} R(t)^{2\mu - 1 - 2\delta} |S_0^2 \omega|^2 dt \, dy \right)^{\frac{1}{2}}.
\]
Similarly, combining (67)-(68) with (72)-(73) yields

\[
\int_{\Omega_T} R(t)^{2\mu-1-2\delta} \left| \nabla^2 S_0 \dot{u} \right|^2 + \left| \nabla^3 \dot{u} \right|^2 + \left| S_0^2 \omega \right|^2 dt dy \\
\leq C \varepsilon E_T (\dot{\sigma}, \dot{u}).
\]  

(72)

By the same methods for the third and fourth terms in (67), we can get

\[
\int_{\Omega_T} R(t)^{2\mu-1-\delta} |\dot{u}||\nabla S_0 \dot{u}| |\nabla^2 \dot{u}| |S_0^2 \omega| dt dy + \int_{\Omega_T} R(t)^{\mu-1-\delta} |\nabla^2 \dot{u}|^2 |S_0 \omega| dt dy \\
\leq C \varepsilon E_T (\dot{\sigma}, \dot{u}).
\]  

(73)

Combining (67)-(68) with (72)-(73) yields

\[
\int_{\Omega_T} R(t)^{2\mu-1-\delta} |S_0^2 \dot{u}| |\nabla^2 \dot{u}| |S_0^2 \omega| dt dy \leq C \varepsilon E_T (\dot{\sigma}, \dot{u}).
\]

Similarly,

\[
\sum_{i=1}^{3} \int_{\Omega_T} R(t)^{2\mu-\delta} |R_{2,i}^i| |S_0^2 \omega_i| dt dy \leq C \varepsilon E_T (\dot{\sigma}, \dot{u}).
\]

Thus, together with (48)-(50), we prove Lemma 5.2 for case (4).

Case (5) \( k_1 = 0, k_2 = 2 \).

In this case,

\[
|R_{0,2}^i| \leq C \frac{1}{R(t)} (|\nabla \dot{u}| |\nabla^3 \dot{u}| + |\nabla^2 \dot{u}|^2).
\]

(74)

In addition, it follows from (35) that

\[
\int_{\Omega_T} R(t)^{2\mu-1-\delta} |\nabla \dot{u}| |\nabla^3 \dot{u}| |\nabla^2 \omega| dt dy \leq C \varepsilon \int_{\Omega_T} R(t)^{\frac{\mu}{2}-1} |\nabla^3 \dot{u}| |\nabla^2 \omega| dt dy \\
\leq C \varepsilon \int_{\Omega_T} R(t)^{2\mu-1-2\delta} |\nabla^3 \dot{u}|^2 dt dy \leq C \varepsilon E_T (\dot{\sigma}, \dot{u}).
\]

(75)

Analogously to the treatment as in (72), we have

\[
\int_{\Omega_T} R(t)^{2\mu-1-\delta} |\nabla^2 \dot{u}|^2 |\nabla^2 \omega| dt dy \\
\leq C \int_0^T R(t)^{2\mu-1-\delta} \|\nabla^2 \dot{u}\|_{L^2_t(B^0)} \|\nabla^2 \omega\|_{L^2_t(B^0)} dt \\
\leq C \int_0^T R(t)^{2\mu-1-\delta} \|\nabla \dot{u}\|_{L^2_t(B^0)} \|\nabla^3 \dot{u}\|_{L^2_t(B^0)} \|\nabla^2 \omega\|_{L^2_t(B^0)} dt \\
\leq C \varepsilon \int_{\Omega_T} R(t)^{\frac{\mu}{2}-1} |\nabla^3 \dot{u}|^2 dt dy \leq C \varepsilon E_T (\dot{\sigma}, \dot{u}).
\]

(76)

Thus, together with (48)-(50), we get Lemma 5.2 for case (5).

Case (6) \( k_1 = k_2 = 1 \).

In this case, the treatments are very similar to those in cases (4) and (5), so we omit it here.

Consequently, from the above cases (1)-(6), (44) is proved. \( \square \)
Lemma 5.3 (Estimates on higher derivatives \((S_k^\ell \dot{u}, S_k^\ell \dot{\sigma})\) for \(1 \leq k \leq 3\)). Under the assumptions of Theorem 5.1, for \(1 \leq k \leq 3\), then for \(\mu = 3(\gamma - 1)\),
\[
\sum_{1 \leq k \leq 3} \left( \int_{\Gamma_T} R(t)^{\mu} |S_k^0 \dot{u}|^2 \, dy + \int_{\Omega_T} R(t)^{2\mu} |S_k^0 \dot{\sigma}|^2 \, dt \, dy \right) + \sum_{1 \leq k \leq 3} \left( \int_{\Gamma_T} R(t)^{\mu} |S_k^0 \dot{u}|^2 \, dy + \int_{\Omega_T} R(t)^{2\mu-1-\delta} |S_k^0 \dot{\sigma}|^2 \, dt \, dy \right) \leq C \varepsilon^2 + C \varepsilon E_T^2 (\dot{\sigma}, \dot{u}) + C \varepsilon E_T (\dot{\sigma}, \dot{u}).
\]
(77)

Proof. Its proof is lengthy and involved, we put it in Section 6.

Lemma 5.4 (Estimates on 1st order derivatives \((\nabla \dot{\sigma}, \text{div} \dot{u})\)). Under the assumptions of Theorem 5.1, then for \(\mu = 3(\gamma - 1)\),
\[
\int_{\Gamma_T} R(t)^{\mu} |\text{div} \dot{u}|^2 \, dy + \int_{\Omega_T} R(t)^{2\mu} |\nabla \dot{\sigma}|^2 \, dy + \int_{\Omega_T} R(t)^{\mu-1} |\text{div} \dot{u}|^2 \, dt \, dy + \int_{\Omega_T} R(t)^{2\mu-1-\delta} |\nabla \dot{\sigma}|^2 \, dt \, dy \leq C \varepsilon^2 + C \varepsilon E_T (\dot{\sigma}, \dot{u}).
\]
(78)

Proof. Acting \(\partial_t\) to (21) and computing \(\sum_{i=1}^3 \partial_i (22)^i\) yield
\[
\mathcal{L}_0 (\partial_t \dot{\sigma}, \partial_t (\text{div} \dot{u})) = \partial_t (\partial_t \dot{\sigma}) + \frac{\mu L}{R(t)} \partial_t (\text{div} \dot{u}) + R(t)^{-\mu-1} (1 + (\gamma - 1) \dot{\sigma} R(t)^{\mu}) \partial_t (\text{div} \dot{u}) = B_i, \quad (79)
\]
\[
\tilde{\mathcal{L}} (\dot{\sigma}, \text{div} \dot{u}) = \partial_t (\text{div} \dot{u}) + \frac{L}{R(t)} (\text{div} \dot{u}) + \frac{\gamma}{R(t)} \Delta \dot{\sigma} = B, \quad (80)
\]
where
\[
B_i = -\frac{1}{R(t)} \sum_{j=1}^3 \partial_j \partial_j \partial_t \dot{\sigma} - \frac{1}{R(t)} \sum_{j=1}^3 \partial_j \partial_j \partial_t \dot{\sigma} - \frac{\gamma - 1}{R(t)} \partial_t \dot{\sigma} \text{div} \dot{u}, \quad i = 1, 2, 3,
\]
\[
B = -\frac{1}{R(t)} \sum_{i,j=1}^3 \partial_i \partial_j \partial_i \dot{u} - \frac{1}{R(t)} \sum_{j=1}^3 \partial_j \partial_j \text{div} \dot{u}.
\]
At first, we derive the boundary conditions of \((\nabla \dot{\sigma}, \text{div} \dot{u})\) on \(\mathcal{B}_T\). Multiplying \(\partial_i (22)^i\) by \(y_i\) and summarizing them from \(i = 1\) to \(i = 3\), we then arrive at
\[
\partial_i \left( \sum_{i=1}^3 y_i \dot{u}_i \right) + \frac{L}{R(t)} \sum_{i=1}^3 y_i \dot{u}_i + \frac{\gamma}{R(t)} \sum_{i=1}^3 y_i \partial_t \dot{\sigma} = -\frac{1}{R(t)} \sum_{i,j=1}^3 y_i \partial_j \partial_j \dot{u}_i \quad \text{on } \mathcal{B}_T. \quad (81)
\]
Note that \(\partial_i \left( \sum_{i=1}^3 y_i \dot{u}_i \right) = 0\) and \(\sum_{i,j=1}^3 \partial_j \partial_j (y_i \dot{u}_i) = 0\) on \(\mathcal{B}_T\), then
\[
\gamma \sum_{i=1}^3 y_i \partial_t \dot{\sigma} - |\dot{u}|^2 = 0 \quad \text{on } \mathcal{B}_T. \quad (82)
\]
To apply the boundary condition (82), we change (80) into the following new form
\[
\tilde{\mathcal{L}} (\dot{\sigma}, \text{div} \dot{u}) = \partial_t (\text{div} \dot{u}) + \frac{L}{R(t)} (\text{div} \dot{u}) + \frac{1}{R(t)} \sum_{i=1}^3 \partial_i (\gamma \partial_t \dot{\sigma} - y_i |\dot{u}|^2) = \bar{B}, \quad (83)
\]
where 
\[ \bar{B} = B - \frac{1}{R(t)} \sum_{i=1}^{3} \partial_i(y_i|\dot{u}|^2). \]
As in Theorem 4.1, we choose a little different multipliers to get
\[
\int_{\Omega_T} \left( \sum_{i=1}^{3} \mathcal{L}_0(\partial_i\bar{\sigma}, \partial_i(\text{div}\dot{u})) \cdot \bar{\mathcal{M}}_0(\gamma\partial_i\bar{\sigma} - y_i|\dot{u}|^2) + \bar{\mathcal{L}}(\bar{\sigma}, \text{div}\dot{u}) \cdot \bar{\mathcal{M}}(\text{div}\dot{u}) \right) dt dy
= J_1 + J_2 + J_3,
\]
where \( \bar{\mathcal{M}}_0 = R(t)^2(1 + R(t)^{-\delta}), \bar{\mathcal{M}} = R(t)^{\mu}(1 + R(t)^{-\delta})(1 + (\gamma - 1)\sigma R(t)^{\mu}), \)
and
\[
J_1 = \int_{\Omega_T} \left( \sum_{i=1}^{3} y_i(\gamma\partial_i\bar{\sigma} + y_i|\dot{u}|^2) \text{div}\dot{u} \cdot R(t)^{\mu}(1 + R(t)^{-\delta})(1 + (\gamma - 1)\sigma R(t)^{\mu}) dt dy,
\]
\[
J_2 = J_{2,1} - J_{2,2},
\]
\[
J_3 = \int_{\Omega_T} \left( \frac{\delta \gamma L}{2} R(t)^{2\mu - \delta} |\nabla\bar{\sigma}|^2 + \frac{L}{2} (5 - 3\gamma + (5 - 3\gamma - \delta)R(t)^{-\delta})(1 + (\gamma - 1)\sigma R(t)^{\mu})R(t)^{\mu - 1} |\text{div}\dot{u}|^2 + (\partial_i(R(t)^{2\mu}(1 + R(t)^{-\delta})|\dot{u}|^2) + \mu LR(t)^{2\mu - 1} |\dot{u}|^2) \sum_{i=1}^{3} y_i \partial_i\bar{\sigma} \right) dt dy
\]
with
\[
J_{2,1} = \int_{\Gamma_T} \left( \frac{\gamma L}{2} R(t)^{2\mu}(1 + R(t)^{-\delta})|\nabla\bar{\sigma}|^2 + R(t)^{2\mu}(1 + R(t)^{-\delta})|\dot{u}|^2 \sum_{i=1}^{3} y_i \partial_i\bar{\sigma}
+ \frac{1}{2} R(t)^{\mu}(1 + R(t)^{-\delta})(1 + (\gamma - 1)\sigma R(t)^{\mu}) |\text{div}\dot{u}|^2 \right) dy,
\]
\[
J_{2,2} = \int_{B^0} \left( \frac{\gamma L}{2} R(t)^{2\mu}(1 + R(t)^{-\delta})|\nabla\bar{\sigma}|^2 + R(t)^{2\mu}(1 + R(t)^{-\delta})|\dot{u}|^2 \sum_{i=1}^{3} y_i \partial_i\bar{\sigma}
+ \frac{1}{2} R(t)^{\mu}(1 + R(t)^{-\delta})(1 + (\gamma - 1)\sigma R(t)^{\mu}) |\text{div}\dot{u}|^2 \right) dy.
\]
Thanks to boundary condition (82), it follows from (35), (41) and direct computation that
\[
\int_{\Gamma_T} \left( R(t)^{2\mu}|\nabla\bar{\sigma}|^2 + R(t)^{\mu}|\text{div}\dot{u}|^2 \right) dy
+ \int_{\Omega_T} \left( R(t)^{2\mu - 1 - \delta}|\nabla\bar{\sigma}|^2 + R(t)^{\mu - 1} |\text{div}\dot{u}|^2 \right) dt dy
\leq C\varepsilon^2 + C\varepsilon E_T(\bar{\sigma}, \dot{u})
+ \int_{\Omega_T} \left( \sum_{i=1}^{3} B_i \cdot \bar{\mathcal{M}}_0(\gamma\partial_i\bar{\sigma} - y_i|\dot{u}|^2) + \bar{B} \cdot \bar{\mathcal{M}}(\text{div}\dot{u}) \right) dt dy.
\]
In addition, by the similar computation as in Lemma 5.3, one has
\[
\left| \int_{\Omega_T} \left( B_i \cdot \bar{\mathcal{M}}_0(\gamma\partial_i\bar{\sigma} - y_i|\dot{u}|^2) + \bar{B} \cdot \bar{\mathcal{M}}(\text{div}\dot{u}) \right) dt dy \right| \leq C\varepsilon E_T(\bar{\sigma}, \dot{u}).
\]
Thus, combining (85) with (86) yields Lemma 5.4. \( \Box \)
Corollary 2. Based on Lemma 2.3 and Lemma 4.2, then combining Lemma 5.2 and Lemma 5.4 yields that for \( \mu = 3(\gamma - 1) \),
\[
\int_{\Gamma_T} \left( R(t)^{2\mu}(|\sigma|^2 + |\nabla \sigma|^2) + R(t)^{\mu}(|\dot{\sigma}|^2 + |\nabla \dot{\sigma}|^2) \right) dy \\
+ \int_{\Omega_T} \left( R(t)^{2\mu-1-\delta}(|\sigma|^2 + |\nabla \sigma|^2) + R(t)^{\mu-1-\delta}(|\dot{\sigma}|^2 + |\nabla \dot{\sigma}|^2) \right) dt dy \\
\leq C\varepsilon^2 + C\varepsilon\mathcal{E}_T(\sigma, \dot{u}).
\]  

Next we derive the estimates on the second order derivatives of \((\sigma, \dot{u})\). By Lemma 5.2, we have obtained the estimate of \((\nabla \omega, S_0\omega)\), and from Lemma 5.3, we get the estimate of \((S_0^2\dot{u}, S_0^2\dot{\sigma})\). In order to establish the energy estimate on the second order derivatives of \((\sigma, \dot{u})\), it is required to estimate \((S_0^{\text{div}}u, \nabla \text{div}u, \dot{\omega}, \dot{S}_0\dot{\sigma})\). Due to the restriction of boundary condition, we can’t estimate \(\nabla \text{div}u\) and \(\dot{\omega}\) directly. To overcome this difficulty, we will estimate the tangent derivatives \((\text{Zdiv}u, Z\dot{\sigma})\) and the radial derivatives \((S_1\dot{\text{div}}u, S_1\nabla \dot{\sigma})\) with \(S_1 = \frac{\sum_{i=1}^3 y_i\partial_i}{\sum_{i=1}^3 y_i^2} = r\partial_r\), respectively.

Lemma 5.5 (Estimates on 2nd tangent derivatives \((S_0\nabla \sigma, S_0\dot{\text{div}}u, \nabla Z\dot{\sigma}, Z\dot{\text{div}}u)\)). Under the assumptions of Theorem 5.1, then for \( \mu = 3(\gamma - 1) \),
\[
\int_{\Gamma_T} R(t)^{\mu}(|S_0\dot{\text{div}}u|^2 + |Z\dot{\text{div}}u|^2) dy + \int_{\Omega_T} R(t)^{2\mu}(|S_0\nabla \sigma|^2 + |Z\nabla \dot{\sigma}|^2) dy \\
+ \int_{\Omega_T} R(t)^{2\mu-1-\delta}(|S_0\dot{\text{div}}u|^2 + |Z\dot{\text{div}}u|^2) dt dy \\
\leq C\varepsilon^2 + C\varepsilon\mathcal{E}_T(\sigma, \dot{u}).
\]  

Proof. Its proof is lengthy and involved, we put it in Section 6. \( \square \)

Next we focus on the estimates on the radial derivatives \(\nabla S_1\dot{\sigma}\) and \(S_1\dot{\text{div}}u\).

Lemma 5.6 (Estimates on 2nd order radial derivatives \((\nabla S_1\dot{\sigma}, S_1\dot{\text{div}}u)\)). Under the assumption of Theorem 5.1, then for \( \mu = 3(\gamma - 1) \),
\[
\int_{\Gamma_T} R(t)^{\mu}(|S_1\dot{\text{div}}u|^2) dy + \int_{\Omega_T} R(t)^{2\mu}(|\nabla S_1\dot{\sigma}|^2) dy \\
+ \int_{\Omega_T} R(t)^{2\mu-1-\delta}(|S_1\dot{\text{div}}u|^2) dt dy + \int_{\Omega_T} R(t)^{2\mu-1-\delta}(|\nabla S_1\dot{\sigma}|^2) dt dy \\
\leq C\varepsilon^2 + C\varepsilon\mathcal{E}_T(\sigma, \dot{u}) + C\varepsilon\mathcal{E}_T(\dot{\sigma}, \dot{u}).
\]  

Proof. To derive the estimates on the radial derivatives of \(\nabla \dot{\sigma}\) and \(\dot{\text{div}}u\), we need to derive a boundary condition of \(\dot{\text{div}}u\) on \(B_T\). Multiplying \((79)\) by \(y_i\) and then summarizing them, we have
\[
S_1(\dot{\text{div}}u) = \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\sigma R(t)^{\mu}} \left( -\partial_i(S_1\dot{\sigma}) - \frac{\mu L}{R(t)} S_1\dot{\sigma} - \frac{1}{R(t)} \sum_{j=1}^3 \dot{y}_j \partial_j(S_1\dot{\sigma}) \\
+ \frac{1}{R(t)} \sum_{j=1}^3 \dot{y}_j \partial_j \dot{\sigma} - \frac{1}{R(t)} \sum_{j=1}^3 S_1\dot{y}_j \partial_j \dot{\sigma} - \frac{\gamma - 1}{R(t)} S_1\dot{\sigma}\dot{\text{div}}u \right).
\]
By the boundary condition (82), we know $S_1\dot{\sigma} = \frac{1}{\gamma}|\dot{u}|^2$ on $B_T$. Substituting this into (90) yields

$$S_1(div\dot{u}) = \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} \left( -\frac{2}{\gamma} \sum_{i=1}^{3} \dot{u}_i \partial_i \dot{u}_i - \frac{\mu L}{\gamma R(t)} |\dot{u}|^2 + \frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j \dot{\sigma} \right) + \frac{2}{\gamma R(t)} \sum_{i,j=1}^{3} \dot{u}_i \dot{u}_j \partial_j \dot{u}_i - \frac{1}{R(t)} \sum_{j=1}^{3} S_1 \dot{u}_j \dot{\sigma} - \frac{1}{\gamma R(t)} \frac{\gamma - 1}{\gamma |\dot{u}|^2 div\dot{u}} \right)$$

$$= \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} \left( -\frac{2}{\gamma} \sum_{i=1}^{3} \dot{u}_i \partial_i (R(t) \dot{u}_i) + \frac{(5 - 3\gamma)L}{\gamma R(t)} |\dot{u}|^2 
+ \frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j \dot{\sigma} - \frac{1}{R(t)} \sum_{j=1}^{3} S_1 \dot{u}_j \dot{\sigma} - \frac{2}{\gamma R(t)} \sum_{i,j=1}^{3} \dot{u}_i \dot{u}_j \partial_j \dot{u}_i \right)$$

$$= \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} \cdot G \quad \text{on} \quad B_T. \quad (91)$$

Acting $S_1$ to (79)-(80), we have

$$\mathcal{L}_0(\partial_i S_1 \dot{\sigma}, S_1 div\dot{u}) = \partial_i (\partial_i S_1 \dot{\sigma}) + \frac{\mu L}{R(t)} (\partial_i S_1 \dot{\sigma}) + \frac{R(t)^{\mu-1}(1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu})}{R(t)^{\mu+1}} \partial_i (S_1 div\dot{u}) = E, \quad (92)$$

$$\mathcal{L}(S_1 \dot{\sigma}, S_1 div\dot{u}) = \partial_i (S_1 div\dot{u}) + \frac{L}{R(t)} (S_1 div\dot{u}) + \frac{\gamma}{R(t)} \Delta S_1 \dot{\sigma} = E, \quad (93)$$

where

$$E_i = [\partial_i \dot{\sigma}, S_1] \dot{\sigma} + \frac{\mu L}{R(t)} [\partial_i, S_1] \dot{\sigma} + [\frac{1}{R(t)} + (\gamma - 1)\dot{\sigma} R(t)^{\mu}] \partial_i, S_1] \dot{u}_i$$

$$= \mathcal{L}_0(\partial_i \dot{\sigma}, div\dot{u}) - \frac{\gamma - 1}{R(t)} S_1 \dot{\sigma} \partial_i (div\dot{u}) + S_1 B_i$$

$$= \mathcal{L}_0(\partial_i \dot{\sigma}, div\dot{u}) - \frac{\gamma - 1}{R(t)} S_1 \dot{\sigma} \partial_i (div\dot{u}) + S_1 B_i$$

$$E = \frac{\gamma}{R(t)} [\Delta, S_1] \dot{\sigma} + S_1 B = \frac{2\gamma}{R(t)} \Delta \dot{\sigma} + S_1 B.$$

As in (84), one has that

$$\int_{\Omega_T} \left( \sum_{i=1}^{3} \mathcal{L}_0(\partial_i S_1 \dot{\sigma}, S_1 div\dot{u}) \cdot \mathcal{M}_0(\partial_i S_1 \dot{\sigma}) + \frac{\gamma}{R(t)} \sum_{i=1}^{3} y_i \partial_i S_1 \dot{\sigma} \mathcal{M}(S_1 div\dot{u} - \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} G) \right) dtdy$$

$$= K_1 + K_2 + K_3 + K_4, \quad (94)$$

where

$$K_1 = \int_{B_T} \left( \frac{\gamma}{R(t)} \sum_{i=1}^{3} y_i \partial_i S_1 \dot{\sigma} \mathcal{M}(S_1 div\dot{u} - \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} G) \right) dS = 0,$$

$$K_2 = K_{2,1} - K_{2,2},$$
Similar to the treatment in (163) in the proof of Lemma 5.5, we get
\[ G_{\mu} \] with
\[ K_3 = \int_{\Omega_T} \left( \frac{\delta_2 L}{2} R(t)^{2 \mu - 1 - \delta} |\nabla S_1 \dot{\sigma}|^2 + \frac{L}{2} \left( 5 - 3 \gamma + (5 - 3 \gamma - \delta) R(t)^{-\delta} \right) \cdot (1 + (\gamma - 1) \dot{\alpha} R(t)^{\mu}) R(t)^{\mu - 1} |S_1 \text{div}\dot{u}|^2 \right) dt dy. \]

Next, we deal with \( E \) of \( K_4 = \int_{\Omega_T} \mathcal{M}(S_1 \text{div}\dot{u})(\partial_t \left( \frac{R(t)^{\mu + 1}}{1 + (\gamma - 1) \dot{\alpha} R(t)^{\mu}} G \right)
\[ - \frac{L}{R(t)^{\mu}} \left( 1 + (\gamma - 1) \dot{\alpha} R(t)^{\mu} \right) \right) dt dy \]
\[ - \int_{\Omega_T} \sum_{i=1}^3 \gamma R(t)^{\mu} \partial_t \partial_i \left( \mathcal{M}(\frac{R(t)^{\mu + 1}}{1 + (\gamma - 1) \dot{\alpha} R(t)^{\mu}} G) \right) dt dy \]

with
\[ K_{2,1} = \int_{\Gamma_T} \left( \frac{1}{2} R(t)^{\mu} (1 + R(t)^{-\delta})(1 + (\gamma - 1) \dot{\alpha} R(t)^{\mu}) |S_1 \text{div}\dot{u}|^2 \right) dy, \]
\[ K_{2,2} = \int_{\Omega_T} \left( \frac{1}{2} R(0)^{\mu} (1 + R(0)^{-\delta})(1 + (\gamma - 1) \dot{\alpha} R(0)^{\mu}) |S_1 \text{div}\dot{u}|^2 \right) dy. \]

By the expression of \( G \) and (94), direct computation yields
\[ \int_{\Gamma_T} (R(t)^{2 \mu} |\nabla S_1 \dot{\sigma}|^2 + R(t)^{\mu} |S_1 \text{div}\dot{u}|^2) dy \]
\[ + \int_{\Omega_T} (R(t)^{2 \mu - 1 - \delta} |\nabla S_1 \dot{\sigma}|^2 + R(t)^{\mu - 1} |S_1 \text{div}\dot{u}|^2) dt dy \]
\[ \leq \int_{\Omega_T} \left( \sum_{i=1}^3 E_i \cdot \dot{M}_0(\partial_i S_1 \dot{\sigma}) + E \cdot \dot{M}(S_1 \text{div}\dot{u}) - \frac{R(t)^{\mu + 1}}{1 + (\gamma - 1) \dot{\alpha} R(t)^{\mu}} G \right) dt dy. \]
\[ + C\varepsilon^2 + C\varepsilon \dot{E}_T(\dot{\sigma}, \dot{u}) + |K_4| \]

(95)

As in (162) for the proof of Lemma 5.5, we also have
\[ |\int_{\Omega_T} \sum_{i=1}^3 E_i \cdot \mathcal{M}_0(\partial_i S_1 \dot{\sigma}) dt dy| \leq C\varepsilon \dot{E}_T(\dot{\sigma}, \dot{u}). \]

(96)

Next, we deal with \( \int_{\Omega_T} E \cdot \mathcal{M}(S_1 \text{div}\dot{u}) dt dy \) in (95). It follows from the expression of \( E \) that
\[ \int_{\Omega_T} E \cdot \mathcal{M}(S_1 \text{div}\dot{u}) dt dy \]
\[ = \int_{\Omega_T} S_1 B \cdot \mathcal{M}(S_1 \text{div}\dot{u}) dt dy + \int_{\Omega_T} \frac{2\gamma}{R(t)} \Delta \dot{\alpha} \cdot \dot{M}(S_1 \text{div}\dot{u}) dt dy. \]

(97)

Similar to the treatment in (163) in the proof of Lemma 5.5, we get
\[ |\int_{\Omega_T} S_1 B \cdot \mathcal{M}(S_1 \text{div}\dot{u}) dt dy| \leq C\varepsilon \dot{E}_T(\dot{\sigma}, \dot{u}). \]

(98)
We now handle the left term in (97). From (80), one has
\[ \gamma \Delta \dot{\sigma} = R(t)B - S_0(div \dot{u}) - Ldiv \dot{u}. \] (99)
Substituting (99) into the second term in (97) and using Lemma 5.5 derive
\[ \left| \int_{\Omega_T} \frac{2\gamma}{R(t)} \Delta \dot{\sigma} \cdot \mathcal{M}(S_1div \dot{u}) dtdy \right| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_T^2(\dot{\sigma}, \dot{u}). \] (100)
Combining (98) and (100) yields
\[ \left| \int_{\Omega_T} E \cdot \mathcal{M}(S_1div \dot{u}) dtdy \right| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_T^2(\dot{\sigma}, \dot{u}). \] (101)
In addition, it follows from the expression of \( G \) and the a priori bound (35) that
\[ \begin{cases} |G| & \leq C\varepsilon R(t)^{-\mu+\delta-1}(|S_0\dot{u}| + |\nabla \dot{u}| + |\dot{\sigma}|), \\ |\nabla G| & \leq C\varepsilon R(t)^{-\frac{\mu}{2}+\delta-1}(|S_0\dot{u}| + |\nabla \dot{u}| + |\nabla^2 \dot{u}|), \\ |S_0(R(t)G)| & \leq C\varepsilon R(t)^{-\mu+\delta}(|S_0\dot{u}| + |\nabla \dot{u}| + |S_0\nabla \dot{u}|), \\ +C\varepsilon R(t)^{-\frac{\mu}{2}+\delta}|\nabla S_0\dot{\sigma}|. \end{cases} \] (102)
This means
\[ \left| \int_{\Omega_T} \frac{2\gamma}{R(t)} \Delta \dot{\sigma} \cdot \mathcal{M}(\frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} G dtdy) \right| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (103)
On the other hand, from the expression of \( B, \) one has
\[ S_1B = -\frac{1}{R(t)} \sum_{j=1}^3 \dot{u}_j \partial_j (S_1div \dot{u}) \\
+ \frac{1}{R(t)} \left( \sum_{j=1}^3 \dot{u}_j \partial_j (S_1div \dot{u}) - \sum_{i,j=1}^3 S_1(\partial_i \dot{u}_j \partial_j \dot{u}_i) - \sum_{j=1}^3 S_1 \dot{u}_j \partial_j (div \dot{u}) \right) \\
eq B_I + B_{II}. \]
It follows from direct computation that
\[ \left| \int_{\Omega_T} B_I \cdot \mathcal{M}(\frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} G dtdy) \right| \]
\[ = - \int_{\Omega_T} \frac{1}{R(t)} \sum_{j=1}^3 \dot{u}_j \dot{u}_j (S_1div \dot{u}) \cdot \mathcal{M}(\frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} G) dS \quad (= 0) \\
+ \int_{\Omega_T} \frac{1}{R(t)} \sum_{j=1}^3 S_1div \dot{u}_j \partial_j (\dot{u}_j \mathcal{M}(\frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} G)) \\
\leq C\varepsilon \int_{\Omega_T} R(t)^{-\frac{\mu}{2}+1} |\nabla^2 \dot{u}| (|S_0\dot{u}| + |\nabla \dot{u}| + |\dot{\sigma}| + |\nabla^2 \dot{\sigma}|) dtdy \\
+ C\varepsilon \int_{\Omega_T} R(t)^{-\mu+\delta} |\nabla \dot{u}| (|\nabla S_0\dot{u}| + |\nabla \dot{u}| + |\nabla^2 \dot{u}|) dtdy \\
\leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) \] (104)
and
\[ \left| \int_{\Omega_T} B_{II} \cdot \mathcal{M}(\frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}} G dtdy) \right| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (105)
As in the treatment of (84), we have
\[ \int_{\Omega T} E \cdot \mathcal{M}(S_1 \text{div} \hat{u}) - \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)R(t)^{\mu}} G \, dt \, dy \leq C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) + C \varepsilon \mathcal{E}^{\frac{1}{2}}(\dot{\sigma}, \dot{u}). \] (106)

Finally, we deal with the term \( K_4 \). Since \( K_4 \) is expressed as
\[ K_4 = \int_{\Omega T} \mathcal{M}(S_1 \text{div} \hat{u}) \left( \partial_t \left( \frac{R(t)^{\mu}}{1 + (\gamma - 1)R(t)^{\mu}} \right) R(t) G - \frac{LR(t)^{\mu}}{1 + (\gamma - 1)R(t)^{\mu}} G \right) \, dt \, dy \]
\[ + \int_{\Omega T} R(t)\mu^{-1}(1 + R(t)^{-\delta}) \left( (S_1 \text{div} \hat{u}) S_\mu(R(t) G) - \sum_{i=1}^3 \gamma \partial_i S_1 \dot{\sigma} \partial_i \dot{R} \right) \, dt \, dy, \]
by (102), (35) and (41), one gets
\[ |K_4| \leq C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (107)
Thus, combining (96) and (106)-(107), we finish the proof of Lemma 5.6.

The vectors \( S_1 \) and \( Z \) have been used in our energy estimates. However, as shown in Lemma 2.1, \( \partial_i \) (1 ≤ i ≤ 3) are equivalent to \( S_1 \) and \( Z \) only for \(|y| \neq 0\). To get the energy estimates of \((\dot{\sigma}, \dot{u})\), we will take the domain decomposition techniques.

For this purpose, we choose a cut-off function \( \chi(s) \) as follows
\[ \chi(s) = \begin{cases} 0, & \text{for } \frac{2}{3} \leq s \leq 1, \\ 1, & \text{for } 0 \leq s \leq \frac{1}{3}, \\ \text{smooth connection}, & \text{for } \frac{1}{3} \leq s \leq \frac{2}{3}. \end{cases} \] (108)

**Lemma 5.7 (Estimates on \((\nabla^2 \dot{\sigma}, \nabla \text{div} \hat{u})\) near \(|y| = 0\)).** Under the assumptions of Theorem 5.1, then for \( \mu = 3(\gamma - 1) \),
\[ \int_{\Gamma T} \chi(|y|) \left( R(t)^{2\mu}|\nabla^2 \dot{\sigma}|^2 + R(t)^{\mu}|\nabla \text{div} \hat{u}|^2 \right) \, dy \]
\[ + \int_{\Omega T} \chi(|y|) \left( R(t)^{2\mu-1-\delta}|\nabla \dot{\sigma}|^2 + R(t)^{\mu-1}|\partial_k \text{div} \hat{u}|^2 \right) \, dt \, dy \]
\[ \leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) + C \varepsilon \mathcal{E}^{\frac{1}{2}}_T(\dot{\sigma}, \dot{u}). \] (109)

**Proof.** By direct computation, we have
\[ \mathcal{L}_0(\partial_k \partial_i \dot{\sigma}, \partial_k \dot{u}) = \partial_k B_i - \frac{\gamma - 1}{R(t)} \partial_k \dot{\sigma} \partial_i \dot{u} \],
\[ \mathcal{L}(\partial_k \dot{\sigma}, \partial_k \dot{u}) = \partial_k B. \] (111)

As in the treatment of (84), we have
\[ \int_{\Gamma T} \chi(|y|) R(t)^{2\mu}|\nabla \partial_k \dot{\sigma}|^2 + R(t)^{\mu}|\partial_k \text{div} \hat{u}|^2 \, dy \]
\[ + \int_{\Omega T} \chi(|y|) \left( R(t)^{2\mu-1-\delta}|\nabla \partial_k \dot{\sigma}|^2 + R(t)^{\mu-1}|\partial_k \text{div} \hat{u}|^2 \right) \, dt \, dy \]
\[ \leq \int_{\Omega T} \left( \sum_{i=1}^3 \mathcal{L}_0(\partial_k \partial_i \dot{\sigma}, \partial_k \dot{u}) \cdot \chi(|y|) \mathcal{M}(\partial_k \partial_i \dot{\sigma}) \right. \]
\[ + \mathcal{L}(\partial_k \dot{\sigma}, \partial_k \dot{u}) \cdot \chi(|y|) \mathcal{M}(\partial_k \dot{u}) \left. \right) \, dt \, dy \]
\[ + C \varepsilon^2 + C \int_{\Omega T} \chi'(|y|) \left( R(t)^{2\mu-1-\delta}|\nabla \partial_k \dot{\sigma}|^2 + R(t)^{\mu-1}|\partial_k \text{div} \hat{u}|^2 \right) \, dt \, dy. \] (112)
Note that the function $\chi(|y|)$ has a compact support away from $|y| = 0$, then by Lemma 2.1, the last term on the right hand side of (112) can be estimated as in Lemmas 5.4-5.6. On the other hand, by a similar argument for $\int_{\Omega_T} \left( \sum_{i=1}^{3} \tilde{B}_i \cdot \mathcal{M}_0(\gamma \partial_i Z_1 \tilde{\sigma} + y_i Z_1 |\tilde{u}|^2) + \tilde{B} \cdot \mathcal{M}(Z_1 \text{div} \tilde{u}) \right) dt dy$ in (158), we can obtain

$$
\int_{\Omega_T} \left( \sum_{i=1}^{3} \mathcal{L}_i(\partial_k \partial_i \tilde{\sigma}, \text{div} \tilde{u}) \cdot \chi(|y|) \mathcal{M}_0(\partial_k \partial_i \tilde{\sigma}) dt dy
+ \int_{\Omega_T} \tilde{E}(\partial_k \tilde{\sigma}, \partial_i \text{div} \tilde{u}) \cdot \chi(|y|) \mathcal{M}(\partial_k \text{div} \tilde{u}) dt dy \right)
\leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\tilde{\sigma}, \tilde{u}).
$$

(113)

Then (109) follows from (112)-(113) and Lemmas 2.1, 5.2, 5.5 and 5.6.

**Corollary 3.** By Lemma 2.3, Corollary 2, Lemma 5.2, 5.5 and 5.6, one can arrive at

$$
\int_{\Omega_T} \sum_{k=0}^{2} \left( R(t)^{2 \mu} |\nabla^k \tilde{\sigma}| + R(t)^{\mu} |\nabla^k \tilde{u}|^2 \right) dy
+ \int_{\Omega_T} \sum_{k=0}^{2} \left( R(t)^{2 \mu - 1 - \delta} |\nabla^k \tilde{\sigma}| + R(t)^{\mu - 1} |\nabla^k \tilde{u}|^2 \right) dt dy
\leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\tilde{\sigma}, \tilde{u}).
$$

(114)

**6. Proofs of Lemma 5.3 and Lemma 5.5.** In this section, we complete the proof of Lemma 5.3 and Lemma 5.5. First, we prove the Lemma 5.3.

**Proof.** Acting $\frac{1}{R(t)} S_k R(t)$ to equations (21) and (22) yields

$$
\mathcal{L}_i(S_k \tilde{\sigma}, S_k \tilde{u})
= \partial_i(S_k \tilde{\sigma}) + \frac{3(\gamma - 1)L}{R(t)}(S_k \tilde{\sigma}) + R(t)^{-\mu - 1}(1 + \gamma - 1)\tilde{\sigma} R(t)^{\mu} \text{div}(S_k \tilde{u})
= A_k^0,
$$

(115)

$$
\mathcal{L}_i(S_k \tilde{u})
= \partial_i(S_k \tilde{u}) + \frac{L}{R(t)}(S_k \tilde{u}) + \frac{\gamma}{R(t)} \partial_i(S_k \tilde{\sigma}) = A_k^i, \quad i = 1, 2, 3,
$$

(116)

where $A_k^0, A_k^i$ are

$$
A_k^0 = A_{k,1}^0 + A_{k,2}^0 + A_{k,3}^0,
$$

(117)

$$
A_{k,1}^0 = -\frac{1}{R(t)} \sum_{j=1}^{3} \tilde{u}_j \partial_j (S_k \tilde{\sigma}),
$$

$$
A_{k,2}^0 = \mu L R(t)^{-\mu - 1} \text{div}(S_k^{k-1} \tilde{\sigma}),
$$

$$
A_{k,3}^0 = \sum_{2 \leq l \leq k} C_l R(t)^{-\mu - 1} \text{div}(S_{k-l} \tilde{u}) + \frac{1}{R(t)} \sum_{1 \leq l \leq k} \sum_{j=1}^{3} C_j S_{k-l} \tilde{u}_j \partial_j S_{k-l} \tilde{\sigma}
+ \frac{1}{R(t)} \sum_{1 \leq l \leq k} C_j S_{k-l} \tilde{\sigma} \text{div}(S_{k-l} \tilde{u})
$$
and
\[ A_k^i = A_{k,1}^i + A_{k,2}^i, \quad i = 1, 2, 3, \quad (118) \]
\[ A_{k,1}^i = -\frac{1}{R(t)} \sum_{j=1}^{3} \hat{u}_j \partial_j (S_0^{k} \hat{u}_i), \]
\[ A_{k,2}^i = \frac{1}{R(t)} \sum_{1 \leq i \leq k} \sum_{j=1}^{3} C_i S_0^{j} \hat{u}_j \partial_j (S_0^{k-1} \hat{u}_i). \]

Motivated by the proof of Theorem 4.1, we will choose the modified multipliers \( M_0 \hat{\sigma} = \gamma R(t)^6(\gamma - 1)(1 + R(t)^{-\delta}) \hat{\sigma} \) and \( M_\hat{u} = R(t)^3(\gamma - 1)(1 + (\gamma - 1) \hat{\sigma} R(t)^3(\gamma - 1)) \) to derive Lemma 5.3. Note that \( S_0 \) is tangent to boundary \( B_T \), one then has
\[ \sum_{i=1}^{3} y_i S_0^{k} \hat{u}_i = S_0^{k} \left( \sum_{i=1}^{3} y_i \hat{u}_i \right) = 0 \quad \text{on} \quad B_T. \quad (119) \]

By
\[ \int_{\Omega_T} \left( L_0(S_0^{k} \hat{\sigma}, S_0^{k} \hat{u}) \cdot M_0 S_0^{k} \hat{\sigma} + \sum_{i=1}^{3} L_i(S_0^{k} \hat{\sigma}, S_0^{k} \hat{u}) \cdot M S_0^{k} \hat{u}_i \right) dtdy \]
\[ = \int_{\Omega_T} (A_k^0 \cdot M_0 S_0^{k} \hat{\sigma} + \sum_{i=1}^{3} A_k^i \cdot M S_0^{k} \hat{u}_i) dtdy, \quad (120) \]
direct computation yields
\[ \int_{\Omega_T} \left( R(t)^2 \mu |S_0^{k} \hat{\sigma}|^2 + R(t)^\mu |S_0^{k} \hat{u}|^2 \right) dtdy \]
\[ + \int_{\Omega_T} \left( R(t)^{2\mu-1-\delta} |S_0^{k} \hat{\sigma}|^2 + R(t)^{\mu-1} |S_0^{k} \hat{u}|^2 \right) dtdy \]
\[ \leq C \varepsilon^2 + \int_{\Omega_T} \left( A_k^0 \cdot M_0 S_0^{k} \hat{\sigma} + \sum_{i=1}^{3} A_k^i \cdot M S_0^{k} \hat{u}_i \right) dtdy. \quad (121) \]

Next, we treat \( \int_{\Omega_T} A_k^0 \cdot M_0 S_0^{k} \hat{\sigma} dtdy \) in (121). Due to \( A_k^0 = \sum_{i=1}^{3} A_{k,i}^0 \), we will divide this process into three parts.

**Part 1. Estimate of \( \int_{\Omega_T} A_{k,1}^0 \cdot M_0 S_0^{k} \hat{\sigma} dtdy. \)**

It follows from the integration by parts and direct computation that
\[ \int_{\Omega_T} A_{k,1}^0 \cdot M_0 S_0^{k} \hat{\sigma} dtdy \]
\[ = - \int_{\Omega_T} \gamma R(t)^{2\mu-1}(1 + R(t)^{-\delta}) \sum_{j=1}^{3} \hat{u}_j \partial_j S_0^{k} \hat{\sigma} \cdot S_0^{k} \hat{\sigma} dtdy \]
\[ = - \int_{B_T} \frac{\gamma}{2} R(t)^{2\mu-1}(1 + R(t)^{-\delta}) \sum_{j=1}^{3} y_j \hat{u}_j (S_0^{k} \hat{\sigma})^2 dS \quad (= 0) \]
\[ + \int_{\Omega_T} \frac{\gamma}{2} R(t)^{2\mu-1}(1 + R(t)^{-\delta}) \text{div}(S_0^{k} \hat{\sigma})^2 dtdy \]
= \int_{\Omega_T} \frac{\gamma}{2} R(t)^{2\mu-1}(1 + R(t)^{-\delta}) \text{div} u \cdot (S_0^k \dot{\sigma})^2 dt \, dy. \quad (122)

This, together with (35), yields

\[ | \int_{\Omega_T} A_{k,1}^0 \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dt \, dy | \leq C \varepsilon \int_{\Omega_T} R(t)^{2\mu-1-\delta} (S_0^k \dot{\sigma})^2 dt \, dy. \quad (123) \]

Part 2. Estimate of \( \int_{\Omega_T} A_{k,2}^0 \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dt \, dy \).

\[
\begin{align*}
\int_{\Omega_T} A_{k,2}^0 \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dt \, dy &= \int_{\Omega_T} \mu \gamma LR(t)^{-\mu-1}(1 + R(t)^{-\delta}) \text{div}(S_0^{k-1} \dot{u}) \cdot S_0^k \dot{\sigma} dt \, dy. \quad (124)
\end{align*}
\]

Note that

\[
S_0^k \dot{\sigma} = -\mu LS_0^{k-1} \dot{\sigma} - R(t)^{-\mu-1}(1 + (\gamma - 1) \dot{\sigma} R(t)^{\mu}) \text{div}(S_0^{k-1} \dot{u}) + R(t) A_{k-1}^0
\]

\[
\equiv -R(t)^{-\mu-1}(1 + (\gamma - 1) \dot{\sigma} R(t)^{\mu}) \text{div}(S_0^{k-1} \dot{u}) + A_{k,2}^0. \quad (125)
\]

Substituting (125) into (124) yields

\[
\begin{align*}
\int_{\Omega_T} A_{k,2}^0 \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dt \, dy &= -\int_{\Omega_T} \mu \gamma LR(t)^{-2}(1 + R(t)^{-\delta})(1 + (\gamma - 1) \dot{\sigma} R(t)^{\mu}) \text{div}(S_0^{k-1} \dot{u})^2 dt \, dy \\
&\quad + \int_{\Omega_T} \mu \gamma LR(t)^{-\mu-1}(1 + R(t)^{-\delta}) \text{div}(S_0^{k-1} \dot{u}) \cdot A_{k,2}^0 dt \, dy \\
&\leq \int_{\Omega_T} \mu \gamma LR(t)^{-\mu-1}(1 + R(t)^{-\delta}) \text{div}(S_0^{k-1} \dot{u}) \cdot A_{k,2}^0 dt \, dy. \quad (126)
\end{align*}
\]

Here we especially point out that due to the second line in (126) is negative, then we can remove it in the right hand of (126). In addition,

\[
\begin{align*}
| \int_{\Omega_T} \mu \gamma LR(t)^{-\mu-1}(1 + R(t)^{-\delta}) \text{div}(S_0^{k-1} \dot{u}) A_{k,2}^0 dt \, dy | \\
&\leq C \int_{\Omega_T} R(t)^{\mu-1} \left( | \text{div}(S_0^{k-1} \dot{u}) | | S_0^{k-1} \dot{\sigma} | + | \dot{u} \text{div} | S_0^{k-1} \dot{u} | | S_0^{k-1} \dot{\sigma} | \right) dt \, dy \\
&\quad + C \int_{\Omega_T} R(t)^{\mu-1} \sum_{1 \leq l \leq k-1} | S_0^{l-1} \dot{u} | | \nabla S_0^{k-1-l} \dot{\sigma} | | \text{div}(S_0^{k-1} \dot{u}) | dt \, dy \\
&\quad + C \int_{\Omega_T} R(t)^{\mu-1} \sum_{1 \leq l \leq k-1} | S_0^{l} \dot{\sigma} | \text{div} S_0^{k-1-l} \dot{u} | | \text{div} S_0^{k-1-1} \dot{u} | dt \, dy \\
&= II_1 + II_2 + II_3. \quad (127)
\end{align*}
\]

From now on and by the induction method, we assume that all the estimates of the derivatives up to \((k-1)\)-order are known before estimating the \(k\)-order derivatives. Then we have

\[
|II_1| \leq C \left( \int_{\Omega_T} R(t)^{2\mu-1-\delta} | S_0^{k-1} \dot{\sigma} |^2 dt \, dy \right)^{\frac{1}{2}} \left( \int_{\Omega_T} R(t)^{\mu-1-2\delta} | \text{div}(S_0^{k-1} \dot{u}) |^2 dt \, dy \right)^{\frac{1}{2}} \\
+ C \varepsilon \int_{\Omega_T} R(t)^{-1+\delta} | \nabla S_0^{k-1} \dot{u} |^2 dt \, dy.
\]
By the similar method for (128), we have
\[ |II_2| + |II_3| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (129)

Next, we estimate \(II_2\) and \(II_3\) in (127).

If \(k = 1\), then \(II_2 = II_3 = 0\);

If \(k = 2\), by (35), (41) and Hölder inequality, then
\[ |II_2| + |II_3| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (129)

If \(k = 3\), we also apply (35), (41) and Hölder inequality to get
\[ |II_2| + |II_3| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (130)

Thus, by (126)-(130), we arrive at
\[ \int_{\Omega_T} A_{k,2}^0 \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dtdy \leq C\varepsilon \mathcal{E}_T^\frac{3}{2}(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (131)

**Part 3. Estimate of \(\int_{\Omega_T} A_{k,3}^0 \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dtdy\).**

It follows from the definition of \(A_{k,3}^0\) and direct computation that
\[ |\int_{\Omega_T} A_{k,3}^0 \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dtdy| \leq C \sum_{2 \leq l \leq k} \int_{\Omega_T} R(t)^{\mu-1} |\text{div}(S_0^{k-l} \dot{u})||S_0^l \dot{\sigma}| dtdy + C \sum_{1 \leq l \leq k} \int_{\Omega_T} R(t)^{2\mu-1} |S_0^l \dot{u}||\nabla S_0^{k-l} \dot{\sigma}||S_0^l \dot{\sigma}| dtdy + C \sum_{1 \leq l \leq k} \int_{\Omega_T} R(t)^{2\mu-1} |S_0^l \dot{\sigma}||\nabla S_0^{k-l} \dot{u}||S_0^l \dot{\sigma}| dtdy = III_1 + III_2 + III_3. \] (132)

By the similar method for (128), we have
\[ |III_1| \leq C\varepsilon \mathcal{E}_T^\frac{3}{2}(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (133)

In addition, we obtain that

If \(k = 1\), then \(l = 1\) holds. Thus it follows from (41) and Hölder inequality that
\[ |III_2| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (134)

If \(k = 2\), then \(l = 1\) or \(l = 2\) holds. By (35), (41), Hölder inequality and (134), then
\[ |III_2| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (135)

If \(k = 3\), then \(l = 1, 2, 3\) hold, and we have
\[ III_2 = \int_{\Omega_T} R(t)^{2\mu-1} (|S_0^0 \dot{u}||\nabla S_0^0 \dot{\sigma}| + |S_0^0 \dot{u}||\nabla \dot{\sigma}|)|S_0^3 \dot{\sigma}| dtdy + \int_{\Omega_T} R(t)^{2\mu-1} |S_0^2 \dot{u}||\nabla S_0^3 \dot{\sigma}||S_0^3 \dot{\sigma}| dtdy = III_2^1 + III_2^2. \] (136)

Since there exists a first order derivative term \(S_0^0 \dot{u}\) in \(III_2^1\), by (35), (41) and Hölder inequality we have
\[ |III_2^1| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (137)
To treat $III_2\lambda$, we still use the similar method as in (72). For this purpose, it follows from (116) that
\[ S_0^2\ddot{u}_i = -LS_0\dot{u}_i - \gamma \partial_i (S_0\dot{\sigma}) + R(t)A^i, \quad (138) \]
which yields
\[ |S_0^2\dot{u}| \leq C(|S_0\dot{u}| + |\nabla (S_0\dot{\sigma})| + |\dot{u}|)|\nabla (S_0\dot{u})| + |S_0\dot{u}||\nabla \dot{u}|. \quad (139) \]
Thus
\[ |III_2\lambda| \leq C \int_{\Omega_T} R(t)^{2\mu-1}(1 + |\nabla \dot{u}|)|S_0\dot{u}||\nabla S_0\dot{\sigma}| |S_0^3\dot{\sigma}| dt dy \]
\[ + C \int_{\Omega_T} R(t)^{2\mu-1}(|\nabla S_0\dot{\sigma}| + |\dot{u}|)|\nabla (S_0\dot{u})|)|\nabla S_0\dot{\sigma}| |S_0^3\dot{\sigma}| dt dy. \quad (140) \]
In addition, by Hölder inequality and Gagliardo-Nirenberg inequality, as in (72), we know
\[ \int_{\Omega_T} R(t)^{2\mu-1} |S_0^2\dot{u}| |\nabla S_0\dot{\sigma}| |S_0^3\dot{\sigma}| dt dy \leq C \varepsilon \mathcal{E}_T (\dot{\sigma}, \dot{u}). \quad (141) \]
Thus, it follows from (134)-(141) that
\[ \sum_{1 \leq i \leq k} \int_{\Omega_T} R(t)^{2\mu-1} |S_0^2\dot{u}| |\nabla S_0\dot{\sigma}| |S_0^3\dot{\sigma}| dt dy \leq C \varepsilon \mathcal{E}_T (\dot{\sigma}, \dot{u}). \quad (142) \]
Analogously, by the same method as in (136)-(141), we can obtain
\[ |III_3| \leq C \varepsilon \mathcal{E}_T (\dot{\sigma}, \dot{u}). \quad (143) \]
Then combining (132)-(133) with (142)-(143) yields
\[ |\int_{\Omega_T} A^0_{k,3} \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dt dy| \leq C \varepsilon \mathcal{E}_T^2 (\dot{\sigma}, \dot{u}) + C \varepsilon \mathcal{E}_T (\dot{\sigma}, \dot{u}). \quad (144) \]
Together with (123), (131) and (144), we have
\[ \int_{\Omega_T} A^0_k \cdot \mathcal{M}_0 S_0^k \dot{\sigma} dt dy \leq C \varepsilon \mathcal{E}_T^2 (\dot{\sigma}, \dot{u}) + C \varepsilon \mathcal{E}_T (\dot{\sigma}, \dot{u}). \quad (145) \]
Next, we estimate $\sum_{i=1}^3 \int_{\Omega_T} A^i_k \cdot \mathcal{M} S_0^k \dot{u}_i dt dy$ in (121). Note that $A^i_k = A^i_{k,1} + A^i_{k,2}$.
At first, it follows from the integration by parts, (23) and (35) that
\[ |\sum_{i=1}^3 \int_{\Omega_T} A^i_{k,1} \cdot \mathcal{M} S_0^k \dot{u}_i dt dy| \]
\[ = \left| - \int_{\partial_T} \frac{1}{2} R(t)^{\mu-1}(1 + R(t)^{-\delta})(1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}) \sum_{j=1}^3 y_j \dot{u}_j |S_0^k \dot{u}|^2 dS \right| \]
\[ + \int_{\Omega_T} \frac{1}{2} R(t)^{\mu-1}(1 + R(t)^{-\delta}) \sum_{j=1}^3 \partial_j ((1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}) \dot{u}_j) |S_0^k \dot{u}|^2 dt dy | \]
\[ = \left| \int_{\Omega_T} \frac{1}{2} R(t)^{\mu-1}(1 + R(t)^{-\delta}) \sum_{j=1}^3 \partial_j ((1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}) \dot{u}_j) |S_0^k \dot{u}|^2 dt dy | \right| \]
\[ \leq C \varepsilon \mathcal{E}_T (\dot{\sigma}, \dot{u}). \quad (146) \]
In addition,

$$\left| \sum_{i=1}^{3} \int_{\Omega_T} A_{k,2}^i \cdot \mathcal{M} S_0^k \dot{u}_i dtdy \right| \leq C \sum_{i=1}^{k} \int_{\Omega_T} R(t)^{\mu-1} |S_0^k \dot{u}| \nabla S_0^k \dot{u} |dtdy. \quad (147)$$

For $k = 1, 2$, the integrand in the right hand of (147) includes the first order derivative term $|S_0 \dot{u}|$ or $|\nabla \dot{u}|$, then it follows from (35), (41) and Hölder inequality that

$$\sum_{1 \leq i \leq k} \int_{\Omega_T} R(t)^{\mu-1} |S_0^i \dot{u}| \nabla S_0^{k-i} \dot{u} |S_0^k \dot{u} |dtdy \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}), \quad k = 1, 2. \quad (148)$$

For $k = 3$, one has

$$\sum_{1 \leq i \leq 3} \int_{\Omega_T} R(t)^{\mu-1} |S_0^i \dot{u}| \nabla S_0^{k-i} \dot{u} |S_0^k \dot{u} |dtdy = C \int_{\Omega_T} R(t)^{\mu-1} \left( |S_0 \dot{u}| |\nabla S_0^2 \dot{u}| + |S_0^3 \dot{u}| |\nabla \dot{u}| |S_0^3 \dot{u}| \right) dtdy$$

$$+ C \int_{\Omega_T} R(t)^{\mu-1} |S_0^2 \dot{u}| \nabla S_0 \dot{u} |S_0^3 \dot{u} |dtdy$$

$$= A_1 + A_2. \quad (149)$$

Since the integrand in $A_1$ has the first order derivative term $|S_0 \dot{u}|$ or $|\nabla \dot{u}|$, as in (148), we have

$$\int_{\Omega_T} R(t)^{\mu-1} \left( |S_0 \dot{u}| |\nabla S_0^2 \dot{u}| + |S_0^3 \dot{u}| |\nabla \dot{u}| |S_0^3 \dot{u}| \right) dtdy \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}). \quad (150)$$

In addition, the integrand in $A_2$ includes the two second order derivative terms and one third order derivative term, then as in the treatments of (72) and (140)-(141), one can arrive at

$$\int_{\Omega_T} R(t)^{\mu-1} |S_0^2 \dot{u}| \nabla S_0 \dot{u} |S_0^3 \dot{u} |dtdy \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}). \quad (151)$$

Thus, collecting (147)-(151) yields

$$\left| \sum_{i=1}^{3} \int_{\Omega_T} A_{k,2}^i \cdot \mathcal{M} S_0^k \dot{u}_i dtdy \right| \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}). \quad (152)$$

By (146) and (151), one gets

$$\left| \sum_{i=1}^{3} \int_{\Omega_T} A_{k}^i \cdot \mathcal{M} S_0^k \dot{u}_i dtdy \right| \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}). \quad (153)$$

Combining (121), (145) with (153), we complete the proof of Lemma 5.3. \hfill \Box

Next we start to prove Lemma 5.5.

**Proof.** The proof will be divided into two parts.

**Part 1. Estimates on** $(\nabla Z \dot{\sigma}, Z \dot{div} \dot{u})$.

Acting $Z_1$ on (79) and (83), and noting $[Z_1, \Delta] = 0$, we then have

$$\mathcal{L}_0(\partial_t Z_1 \dot{\sigma}, \partial_t (Z_1 Z \dot{div} \dot{u})) = \partial_t (\partial_t (Z_1 \dot{\sigma})) + \frac{\mu L}{R(t)} (\partial_t (Z_1 \dot{\sigma}))$$

$$+ R(t)^{-\mu-1} (1 + (\gamma - 1) \dot{\sigma} R(t)^{\mu}) \partial_t (Z_1 Z \dot{div} \dot{u}) = \tilde{B}_i, \quad (154)$$
\[ \mathcal{L}(Z_1\dot{\sigma}, Z_1\text{div}\dot{u}) = \partial_t(Z_1\text{div}\dot{u}) + \frac{L}{R(t)}(Z_1\text{div}\dot{u}) + \frac{1}{R(t)} \sum_{i=1}^{3} \partial_i(\gamma \partial_i(Z_1\dot{\sigma})) \]

\[ + y_i(Z_1|\dot{u}|^2) = \dot{B}, \]  

(155)

where

\[ \dot{B}_i = \mathcal{L}_0(\partial_i Z_1\dot{\sigma}, \partial_i(Z_1\text{div}\dot{u})) - Z_1(\mathcal{L}_0(\partial_1\dot{\sigma}, \partial_i(\text{div}\dot{u}))) + Z_1B_i, \]

\[ \dot{B} = Z_1B + \frac{1}{R(t)} \sum_{i=1}^{3} \partial_i(y_i(Z_1|\dot{u}|^2)) \]  

(156)

with \( B \) and \( B_i \) being given in (79)-(80). We now derive the boundary condition of \( Z\dot{\sigma} \). Noticing that \( Z \) is tangent to the boundary \( \mathcal{B}_T \) and \( [S_1, Z] = 0 \) holds, then it follows from (82) that

\[ \gamma S_1 Z\dot{\sigma} - Z|\dot{u}|^2 = 0 \quad \text{on} \quad \mathcal{B}_T. \]  

(157)

Analogously to (85)-(86), one has

\[ \int_{\mathcal{B}_T} (R(t)^{2u}|\nabla Z_1\dot{\sigma}|^2 + R(t)^u|Z_1\text{div}\dot{u}|^2)dy \]

\[ + \int_{\Omega_T} \left( (R(t)^{2u-\delta}|\nabla Z_1\dot{\sigma}|^2 + R(t)^{u-1}|Z_1\text{div}\dot{u}|^2) dty \right) \]

\[ \leq C\varepsilon^2 + C\varepsilon E_T(\dot{\sigma}, \dot{u}) \]

\[ + \int_{\Omega_T} \left( \sum_{i=1}^{3} \dot{B}_i \cdot \mathcal{M}_0(\gamma \partial_i Z_1\dot{\sigma} + y_i Z_1|\dot{u}|^2) + \dot{B} \cdot \mathcal{M}(Z_1\text{div}\dot{u}) \right) dt dy, \]  

(158)

Next, we compute \( \int_{\Omega_T} \left( \sum_{i=1}^{3} \dot{B}_i \cdot \mathcal{M}_0(\gamma \partial_i Z_1\dot{\sigma} + y_i Z_1|\dot{u}|^2) + \dot{B} \cdot \mathcal{M}(Z_1\text{div}\dot{u}) \right) dt dy \) in (158). At first, direct computation yields

\[ \dot{B}_1 = Z_1B_1 - \frac{\gamma - 1}{R(t)} Z_1\dot{\sigma} \partial_1(\text{div}\dot{u}) \]

\[ = - \frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j(\partial_1 Z_1\dot{\sigma}) \]

\[ - \frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j Z_1 [\dot{\sigma} - \frac{\gamma - 1}{R(t)} Z_1(\partial_1\dot{\sigma} \text{div}\dot{u}) - \frac{\gamma - 1}{R(t)} Z_1 \dot{\sigma} \partial_1(\text{div}\dot{u})] \]

\[ = \dot{B}_{11} + \dot{B}_{12}, \]

\[ \dot{B}_2 = \partial_1\partial_3\dot{\sigma} + \frac{\mu L}{R(t)} \partial_3\dot{\sigma} + R(t)^{-u-1}(1 + (\gamma - 1) \dot{\sigma} R(t)^u) \partial_3(\text{div}\dot{u}) \]

\[ - \frac{\gamma - 1}{R(t)} Z_1\dot{\sigma} \partial_2(\text{div}\dot{u}) - \frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j(\partial_2 Z_1\dot{\sigma}) \]

\[ - \frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j Z_1 [\dot{\sigma} - \frac{\gamma - 1}{R(t)} Z_1(\partial_2\dot{\sigma} \text{div}\dot{u})] \]

\[ = - \frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j(\partial_2 Z_1\dot{\sigma}) \]
Thus, collecting (160) and (161) yields

\[ B_3 = 3 \sum_{j=1}^{3} \hat{B}_{21} + \hat{B}_{22}, \]

\[ \hat{B}_3 = -\partial_t \sigma \partial_\sigma - \frac{\mu L}{R(t)} \partial_\sigma \sigma - R(t)^{-\mu - 1} (1 + (\gamma - 1) \hat{\sigma} R(t)^{-\mu}) \partial_\sigma (\text{div} \hat{u}) \]

\[ \hat{B}_3 = \gamma - \frac{1}{R(t)} Z_1 \hat{\sigma} \partial_\sigma (\text{div} \hat{u}) - \frac{1}{R(t)} \sum_{j=1}^{3} \hat{u}_j \partial_j (\partial_\sigma Z_1 \hat{\sigma}) \]

\[ \hat{B}_3 = \gamma - \frac{1}{R(t)} Z_1 \hat{\sigma} \partial_\sigma (\text{div} \hat{u}) - \frac{1}{R(t)} \sum_{j=1}^{3} \hat{u}_j \partial_j (\partial_\sigma Z_1 \hat{\sigma}) \]

\[ -B_2 = \frac{1}{R(t)} Z_1 \hat{\sigma} \partial_\sigma (\text{div} \hat{u}) - \frac{1}{R(t)} \sum_{j=1}^{3} \hat{u}_j \partial_j (\partial_\sigma Z_1 \hat{\sigma}) \]

where \( \hat{B}_i (i = 1, 2, 3) \) only includes the third-order derivative terms. It follows from the divergence theorem that

\[ \int_{\Omega_T} \sum_{i=1}^{3} \hat{B}_{i1} \cdot \hat{M}_0 (\gamma \partial_t Z_1 \hat{\sigma} + y_i Z_1 |\hat{u}|^2) dt dy \]

\[ \int_{\Omega_T} \sum_{i=1}^{3} \hat{B}_{i1} \cdot \hat{M}_0 (\gamma \partial_t Z_1 \hat{\sigma} + y_i Z_1 |\hat{u}|^2) dt dy \]

\[ \int_{\Omega_T} \sum_{i=1}^{3} \hat{B}_{i1} \cdot \hat{M}_0 (\gamma \partial_t Z_1 \hat{\sigma} + y_i Z_1 |\hat{u}|^2) dt dy \]

By the boundary condition (23), the second line of (159) is 0. This, together with the a priori bound (35), yields

\[ | \int_{\Omega_T} \sum_{i=1}^{3} \hat{B}_{i1} \cdot \hat{M}_0 (\gamma \partial_t Z_1 \hat{\sigma} + y_i Z_1 |\hat{u}|^2) dt dy | \leq C \varepsilon E_T (\hat{\sigma}, \hat{u}). \]  

(160)

Since \( \hat{B}_{i2} \) only includes the second order error terms, it follows from (35) and direct computation that

\[ | \int_{\Omega_T} \sum_{i=1}^{3} \hat{B}_{i2} \cdot \hat{M}_0 (\gamma \partial_t Z_1 \hat{\sigma} + y_i Z_1 |\hat{u}|^2) dt dy | \leq C \varepsilon E_T (\hat{\sigma}, \hat{u}). \]  

(161)

Thus, collecting (160) and (161) yields

\[ | \int_{\Omega_T} \sum_{i=1}^{3} \hat{B}_i \cdot \hat{M}_0 (\gamma \partial_t Z_1 \hat{\sigma} + y_i Z_1 |\hat{u}|^2) dt dy | \leq C \varepsilon E_T (\hat{\sigma}, \hat{u}). \]  

(162)
Replacing (162) and (163) into (158) derives

\[ |\int_{\Omega_T} \sum_{i=1}^{3} \bar{B} : \mathcal{M}(Z_1 \text{div} \bar{u}) \, dtdy| \leq C \varepsilon \mathcal{E}_T(\sigma, \bar{u}). \quad (163) \]

Substituting (162) and (163) into (158) derives

\[ \int_{\Gamma_T} (R(t)^{2\mu}|\nabla Z_1 \dot{\sigma}|^2 + R(t)^{\mu}|Z_1 \text{div} \bar{u}|^2) \, dy \]
\[ + \int_{\Omega_T} (R(t)^{2\mu-1-\delta}|\nabla Z_1 \dot{\sigma}|^2 + R(t)^{\mu-1}|Z_1 \text{div} \bar{u}|^2) \, dtdy \]
\[ \leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\sigma, \bar{u}) \quad (164) \]

Replacing \( Z_1 \) by \( Z_2 \) or \( Z_3 \), we also have the similar estimate to (164), then

\[ \int_{\Gamma_T} (R(t)^{2\mu}|\nabla Z \dot{\sigma}|^2 + R(t)^{\mu}|Z \text{div} \bar{u}|^2) \, dy \]
\[ + \int_{\Omega_T} (R(t)^{2\mu-1-\delta}|\nabla Z \dot{\sigma}|^2 + R(t)^{\mu-1}|Z \text{div} \bar{u}|^2) \, dtdy \]
\[ \leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\sigma, \bar{u}) \quad (165) \]

**Part 2. Estimates on** \((S_0 \nabla \sigma, S_0 \text{div} \bar{u})\).

Multiplying \( R(t) \) on (5.76)-(5.77) and then taking \( \partial_t \) on the resulting equations, we arrive at

\[ \mathcal{L}_0(\partial_t S_0 \dot{\sigma}, \partial_t (S_0 \text{div} \bar{u})) \]
\[ = \partial_t (\partial_t (S_0 \dot{\sigma})) + \frac{\mu L}{R(t)} (\partial_t (S_0 \dot{\sigma})) + R(t)^{-\mu-1}(1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}) \partial_t (S_0 \text{div} \bar{u}) \]
\[ = D_t, \quad (166) \]
\[ \vec{L}(S_0 \dot{\sigma}, S_0 \text{div} \bar{u}) \]
\[ = \partial_t (S_0 \text{div} \bar{u}) + \frac{L}{R(t)} (S_0 \text{div} \bar{u}) + \frac{1}{R(t)} \sum_{i=1}^{3} \partial_i (\gamma \partial_i (S_0 \dot{\sigma}) + y_i (S_0 |\dot{u}|^2)) \]
\[ = D, \quad (167) \]

where

\[ D_t = \mu R(t)^{-\mu} \partial_t (\text{div} \bar{u}) - (\gamma - 1)\partial_t \dot{\sigma} \partial_t (\text{div} \bar{u}) + \partial_t (R(t) B_i), \]
\[ D = \partial_t (R(t) B) + \frac{1}{R(t)} \sum_{i=1}^{3} \partial_i (y_i S_0 |\dot{u}|^2). \]

Noting that \( S_0 \) is tangent to the boundary \( B_T \), then it follows from (82) that

\[ \gamma \sum_{i=1}^{3} y_i \partial_i (S_0 \dot{\sigma}) + S_0 |\dot{u}|^2 = 0 \quad \text{on} \quad B_T. \quad (168) \]

Similarly to (84)-(85), one has

\[ \int_{\Gamma_T} (R(t)^{2\mu}|\nabla S_0 \dot{\sigma}|^2 + R(t)^{\mu}|S_0 \text{div} \bar{u}|^2) \, dy \]
\[ + \int_{\Omega_T} (R(t)^{2\mu-1-\delta}|\nabla S_0 \dot{\sigma}|^2 + R(t)^{\mu-1}|S_0 \text{div} \bar{u}|^2) \, dtdy \]
Then
\[ \int_{\Omega_T} D_i \cdot \tilde{\mathcal{M}}_0(\gamma \partial_i S_0 \sigma + y_i S_0 |\dot{u}|^2) + D \cdot \tilde{\mathcal{M}}(S_0 \text{div}\dot{u}) \, dtdy. \] (169)

Here we point out that except the term \( \mu R(t)^{-\mu} \partial_i(\text{div}\dot{u}) \), all the other terms in \( D_i \) admit the analogous properties for \( \dot{B}_i \) in (158). We now firstly estimate \( \int_{\Omega_T} \mu R(t)^{-\mu} \partial_i(\text{div}\dot{u}) \cdot \tilde{\mathcal{M}}_0(\gamma \partial_i S_0 \sigma + y_i S_0 |\dot{u}|^2) \, dtdy \). Thanks to (79), we have
\[ \partial_i S_0 \sigma = -\mu L \partial_i \dot{\sigma} - R(t)^{-\mu}(1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}) \partial_i(\text{div}\dot{u}) + R(t)B_i. \] (170)

Then
\[
\int_{\Omega_T} \mu R(t)^{-\mu} \partial_i(\text{div}\dot{u}) \cdot \tilde{\mathcal{M}}_0(\gamma \partial_i S_0 \sigma) \, dtdy
\]
\[= - \int_{\Omega_T} \gamma \mu (1 + R(t)^{-\delta})(1 + (\gamma - 1)\dot{\sigma} R(t)^{\mu}) (\partial_i(\text{div}\dot{u})) \, dtdy \quad (\leq 0)
\]
\[+ \int_{\Omega_T} \gamma \mu R(t)^{\mu} (1 + R(t)^{-\delta}) \partial_i(\text{div}\dot{u}) (R(t)B_i - \mu L \partial_i \dot{\sigma}) \, dtdy
\]
\[\leq \int_{\Omega_T} \gamma \mu R(t)^{\mu} (1 + R(t)^{-\delta}) \partial_i(\text{div}\dot{u}) (R(t)B_i - \mu L \partial_i \dot{\sigma}) \, dtdy
\]
\[\leq C \varepsilon \mathcal{E}^\frac{1}{2}(\dot{\sigma}, \dot{u}) + C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (171)

In addition, \( S_0 |\dot{u}|^2 \) is a second order error term, then it is easy to get
\[ |\int_{\Omega_T} \mu R(t)^{-\mu} \partial_i(\text{div}\dot{u}) \cdot \tilde{\mathcal{M}}_0(y_i S_0 |\dot{u}|^2) \, dtdy| \leq C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (172)

Then by the similar method as in (161)-(165), and together with (171)-(172), we obtain
\[
\int_{\Omega_T} \left( \sum_{i=1}^{3} D_i \cdot \tilde{\mathcal{M}}_0(\gamma \partial_i S_0 \sigma + y_i S_0 |\dot{u}|^2) + D \cdot \tilde{\mathcal{M}}(S_0 \text{div}\dot{u}) \right) \, dtdy
\]
\[\leq C \varepsilon \mathcal{E}^\frac{1}{2}(\dot{\sigma}, \dot{u}) + C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \] (173)

Substituting this into (169) yields
\[
\int_{\Omega_T} (R(t)^{2\mu} |\nabla S_0 \dot{\sigma}|^2 + R(t)^{\mu} |S_0 \text{div}\dot{u}|^2) \, dy
\]
\[+ \int_{\Omega_T} (R(t)^{\mu-1-\delta} |\nabla S_0 \dot{\sigma}|^2 + R(t)^{\mu-1} |S_0 \text{div}\dot{u}|^2) \, dtdy
\]
\[\leq C \varepsilon^2 + C \varepsilon \mathcal{E}^\frac{1}{2}(\dot{\sigma}, \dot{u}) + C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) \] (174)

Collecting (165) and (174), then Lemma 5.5 is proved.

7. Weighted estimates on the third-order derivatives of \((\dot{\sigma}, \dot{u})\) and proof of Theorem 5.1. In this section, we will derive the estimates on all third-order derivatives of \((\dot{\sigma}, \dot{u})\). In fact, we have already got the estimates on some third-order derivatives of \((\dot{\sigma}, \dot{u})\) in Section 5, which include the derivatives \( S_k^0 \nabla S_0 \text{curl}\dot{u} \) (0 \( k \leq 2 \)), \( S_0^3 \dot{u} \) and \( S_0^3 \dot{\sigma} \). Next, we still need to estimate the following left terms:

- second order tangent derivatives of \( \nabla \dot{\sigma} \) and \( \text{div}\dot{u} \), i.e., \( \nabla (S_0^k Z^{2-k} \dot{\sigma}) \), \( S_0^k Z^{2-k} \text{div}\dot{u} \) (0 \( k \leq 2 \));
Lemma 7.1. Under the assumptions of Theorem 5.1, then for $\mu = 3(\gamma - 1)$,
\[
\int_{\Gamma_T} \left( R(t)^{2\mu} \left( \sum_{k=0}^{2} |\nabla(S_0^k Z^{2-k} \hat{\sigma})|^2 + \sum_{m=0}^{1} |\nabla(S_0^m Z^{1-m} \hat{\sigma})|^2 \right) + R(t)^{\mu} \left( \sum_{k=0}^{2} |S_0^k Z^{2-k} \hat{u}|^2 + \sum_{m=0}^{1} |S_0^m Z^{1-m} S_1 \hat{u}|^2 \right) \right) dy \\
+ \int_{\Omega_T} \left( R(t)^{2\mu - 1 - \delta} \left( \sum_{k=0}^{2} |\nabla(S_0^k Z^{2-k} \hat{\sigma})|^2 + \sum_{m=0}^{1} |\nabla(S_0^m Z^{1-m} \hat{\sigma})|^2 \right) + R(t)^{\mu - 1} \left( \sum_{k=0}^{2} |S_0^k Z^{2-k} \hat{u}|^2 + \sum_{m=0}^{1} |S_0^m Z^{1-m} S_1 \hat{u}|^2 \right) \right) dt dy \\
\leq C\varepsilon^2 + C\varepsilon E_T(\hat{\sigma}, \hat{u}) + C\varepsilon \mathcal{E}^{\frac{1}{2}}(\hat{\sigma}, \hat{u}). \tag{175}
\]

Proof. Since $S_0$ and $Z$ are tangent to the boundary $\mathcal{B}_T$, then from (82) and (91), we have the following boundary conditions on $\mathcal{B}_T$:
\[
S_0^k Z^{2-k}(S_1 \hat{\sigma} + \frac{1}{\gamma} \hat{u}) = S_1(S_0^k Z^{2-k} \hat{\sigma}) + \frac{1}{\gamma} S_0^k Z^{2-k} \hat{u} = 0, \quad 0 \leq m \leq 1, \quad 0 \leq k \leq 2 \quad \text{and} \quad 0 \leq l \leq 1. \tag{176}
\]

Based on this, completely analogously to the proof of Lemmas 5.5 and 5.6, we can complete the proof of Lemma 7.1. Here we omit the details. \hfill \Box

Similar to the proof of Lemma 5.7, we also have the following estimates on \(\nabla^\alpha S_0^\beta Z^{\nu} \nabla \hat{\sigma}\) and \(\nabla^\alpha S_0^\beta Z^{\nu} (\hat{u})\) (where $\alpha + \beta + \nu = 2$) near $|y| = 0$.

Lemma 7.2. Under the assumption of Theorem 5.1, then for $\mu = 3(\gamma - 1)$,
\[
\int_{\Gamma_T} \chi(|y|) \sum_{\alpha + \beta + \nu = 2, \alpha < 2} (R(t)^{2\mu} |\nabla^\alpha S_0^\beta Z^{\nu} (\nabla \hat{\sigma})|^2 + R(t)^{\mu} |\nabla^\alpha S_0^\beta Z^{\nu} (\hat{u})|^2) dy \\
+ \int_{\Omega_T} \chi(|y|) \sum_{\alpha + \beta + \nu = 2, \alpha < 2} (R(t)^{2\mu - 1 - \delta} |\nabla^\alpha S_0^\beta Z^{\nu} (\nabla \hat{\sigma})|^2 + R(t)^{\mu - 1} |\nabla^\alpha S_0^\beta Z^{\nu} (\hat{u})|^2) dt dy \\
\leq C\varepsilon^2 + C\varepsilon E_T(\hat{\sigma}, \hat{u}) + C\varepsilon \mathcal{E}^{\frac{1}{2}}(\hat{\sigma}, \hat{u}). \tag{178}
\]

Corollary 4. By Lemmas 2.1 and 2.3 together with Lemma 5.2-5.3, then it follows from Lemma 7.1-7.2 that
\[
\int_{\Gamma_T} \sum_{\alpha + \beta + \nu = 2, \alpha < 2} (R(t)^{2\mu} |\nabla^\alpha S_0^\beta Z^{\nu} (\nabla \hat{\sigma})|^2 + R(t)^{\mu} |\nabla^\alpha S_0^\beta Z^{\nu} (\hat{u})|^2) dy \\
+ \int_{\Omega_T} \sum_{\alpha + \beta + \nu = 2, \alpha < 2} (R(t)^{2\mu - 1 - \delta} |\nabla^\alpha S_0^\beta Z^{\nu} (\nabla \hat{\sigma})|^2 + R(t)^{\mu - 1} |\nabla^\alpha S_0^\beta Z^{\nu} (\hat{u})|^2) dt dy
\]
On the other hand, it follows from Lemma 2.1 and (184) that on $B_T$
In this case, the boundary conditions (82) can be rewrite as follows
We now have a crucial observation: $|\dot{u}|^2 = \sum_{i=1}^{3} |\dot{u}_i|^2$ holds on the boundary $B_T$, which will bring the better time-decay rates for $\dot{u}_z$. In fact, the boundary condition (23) means
\[
\dot{u}_r = \sum_{i=1}^{3} y_i \dot{u}_i = 0 \quad \text{on} \quad B_T.
\] (180)
In addition, it follows from (25)-(26) that
\[
\sum_{i=1}^{3} y_i \dot{u}_i = 0.
\] (181)
Thus, by (180)-(181) and direct computation, one has that
\[
|\dot{u}|^2 = \sum_{i=1}^{3} \dot{u}_i^2 \quad \text{on} \quad B_T.
\] (182)
For the later convenience, we denote
\[
|\dot{u}_z|^2 = \sum_{i=1}^{3} \dot{u}_i^2.
\] (183)
In this case, the boundary conditions (82) can be rewrite as follows
\[
S_1 \dot{\sigma} + \frac{1}{\gamma} |\dot{u}_z|^2 = 0 \quad \text{on} \quad B_T.
\] (184)
On the other hand, it follows from Lemma 2.1 and (184) that on $B_T$,
\[
\begin{aligned}
\{ & \partial_1 \dot{\sigma} = \frac{u}{\gamma} |\dot{u}_z|^2 + y_3 Z_2 \dot{\sigma} - y_2 Z_3 \dot{\sigma} \equiv h_1(|\dot{u}_z|^2, Z \dot{\sigma}), \\
& \partial_2 \dot{\sigma} = \frac{u}{\gamma} |\dot{u}_z|^2 + y_1 Z_3 \dot{\sigma} - y_3 Z_1 \dot{\sigma} \equiv h_2(|\dot{u}_z|^2, Z \dot{\sigma}), \\
& \partial_3 \dot{\sigma} = \frac{u}{\gamma} |\dot{u}_z|^2 + y_2 Z_1 \dot{\sigma} - y_1 Z_2 \dot{\sigma} \equiv h_3(|\dot{u}_z|^2, Z \dot{\sigma}).
\end{aligned}
\] (185)
Note that the boundary condition (91) on $B_T$ can be rewritten as
\[
S_1 (\text{div}\, \dot{u}) = \frac{R(t)^{\mu+1}}{1 + (\gamma - 1) \dot{\sigma} R(t)^{\mu}} \left(\frac{2}{\gamma} \sum_{i=1}^{3} \dot{u}_i \partial_i \dot{u}_i - \frac{\mu L}{\gamma R(t)} |\dot{u}|^2 + \frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j \dot{\sigma} \right)
\]
\[
- \frac{2}{\gamma R(t)} \sum_{i,j=1}^{3} \dot{u}_i \dot{u}_j \partial_i \dot{u}_i - \frac{1}{R(t)} \sum_{j=1}^{3} S_1 \dot{u}_j \partial_j \dot{\sigma} - \frac{\gamma - 1}{\gamma R(t)} |\dot{u}|^2 \text{div} \dot{u}
\]
\[
= \frac{R(t)^{\mu}}{1 + (\gamma - 1) \dot{\sigma} R(t)^{\mu}} \left(\frac{2}{\gamma} \sum_{i=1}^{3} \dot{u}_i S_0 \dot{u}_z - \frac{\mu L}{\gamma} |\dot{u}_z|^2 + \sum_{j=1}^{3} \dot{u}_j h_1(|\dot{u}_z|^2, Z \dot{\sigma}) \right)
\]
\[
- \frac{2}{\gamma} \sum_{i,j=1}^{3} \dot{u}_i \dot{u}_j \partial_i \dot{u}_z - \sum_{j=1}^{3} S_1 \dot{u}_j h_1(|\dot{u}_z|^2, Z \dot{\sigma}) - \frac{\gamma - 1}{\gamma} |\dot{u}_z|^2 \text{div} \dot{u}
\]
\[
= \frac{R(t)^{\mu}}{1 + (\gamma - 1) \dot{\sigma} R(t)^{\mu}} \cdot \mathcal{G}.
\] (186)
Due to (35) and (41), we have the estimates:

\[ |\tilde{G}| \leq C\left( |\dot{u}_z| |\dot{u}_z| + |S_0\dot{u}_z| + (|\dot{u}_z|^2 + |\nabla \sigma| (|\dot{u}| + |\nabla \dot{u}|) \right), \]

\[ |\nabla \tilde{G}| \leq C\varepsilon R(t)^{-\mu + \frac{1}{2}} \left( |\nabla \dot{u}_z| + |\nabla^2 \dot{u}| + |\nabla S_0 \dot{u}_z| \right) + C\varepsilon R(t)^{-\mu + \frac{1}{2}} |\nabla Z \dot{\sigma}|, \]

\[ |S_0 \tilde{G}| \leq C \left( |S_0 \dot{u}_z|^2 + |\dot{u}_z| \left( |S_0 \dot{u}_z| + |S_0^2 \dot{u}_z| \right) + |Z \dot{\sigma}| \left( |S_0 \dot{u}| + |\nabla S_0 \dot{u}| \right) + |\dot{u}|^2 \left( |S_0 \dot{u}| + |\nabla S_0 \dot{u}| \right) \right), \]

\[ |\nabla S_0 \tilde{G}| \leq C \left( |\dot{u}_z| + |S_0 \dot{u}_z| + |Z \dot{\sigma}| + |\nabla Z \dot{\sigma}| \right) \left( |\nabla S_0 \dot{u}| + C |\nabla \dot{u}_z| |S_0^2 \dot{u}_z| \right) + C\left( |S_0 \dot{u}| |\nabla Z \dot{\sigma}| + |\dot{u}| |\nabla S_0 \dot{u}| + |Z \dot{\sigma}| |\nabla^2 S_0 \dot{u}| \right), \]

\[ + C \left( |S_0 \dot{u}| |\nabla \dot{u}| + |S_0 \dot{u}| \right) \left( |\nabla S_0 \dot{u}| + |\nabla^2 S_0 \dot{u}| \right) + C |\dot{u}| \left( |\nabla \dot{u}| + |\nabla \dot{u}|^2 + |\nabla^2 \dot{u}| \right) \]

\[ + C |\nabla^2 \dot{u}| |S_0 \dot{u}|. \tag{187} \]

Based on the preparations above, we have

**Lemma 7.3 (Estimates on 3rd order radial derivatives $\nabla S^2_1 \dot{\sigma}$ and $S^2_1 \text{div} \dot{u}$).** Under the assumptions of Theorem 5.1, then for $\mu = 3(\gamma - 1)$,

\[ \int_{\Omega_T} \left( R(t)^{2\mu - 2\delta} |\nabla S^2_1 \dot{\sigma}|^2 + R(t)^{\mu - 2\delta} |S^2_1 \text{div} \dot{u}|^2 \right) dy \]

\[ + \int_{\Omega_T} \left( R(t)^{2\mu - 1 - 2\delta} |\nabla S^2_1 \dot{\sigma}|^2 + R(t)^{\mu - 1 - 2\delta} |S^2_1 \text{div} \dot{u}|^2 \right) dt dy \]

\[ \leq C\varepsilon^2 + C\varepsilon \mathcal{E}(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}^{\frac{1}{2}}(\dot{\sigma}, \dot{u}). \tag{188} \]

**Proof.** At first, we derive the boundary condition of the third derivatives of $\dot{\sigma}$ on $\mathcal{B}_T$. Acting $S_1$ to (80) yields

\[ S_1 \Delta \dot{\sigma} = -\frac{1}{\gamma} \left( R(t) \partial_t (S_1 \text{div} \dot{u}) + L S_1 \text{div} \dot{u} - R(t) S_1 B \right) \]
Direct estimate yields
\[ S_1 \Delta \hat{\sigma} = -\frac{1}{\gamma} \left( \partial_t (R(t)S_1 \text{div} \hat{u}) - R(t)S_1 B \right). \] (189)

Substituting (186) into (189), we have that on \( B_T \),
\[
S_1 \Delta \hat{\sigma} = -\frac{1}{\gamma} \partial_t \left( \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\sigma R(t)^{\mu}} \hat{G} \right)
+ \frac{1}{\gamma} \left( \sum_{i,j=1}^{3} S_1 (\partial_i \hat{u}_j \partial_j \hat{u}_i) + \sum_{j=1}^{3} \hat{S}_1 \hat{u}_j \partial_j \text{div} \hat{u} + \sum_{j=1}^{3} \hat{u}_j \partial_j \text{div} \hat{u} \right)
= N,
\]
which means
\[ S_1 \Delta \hat{\sigma} - N = 0 \quad \text{on} \quad B_T. \] (190)

Direct estimate yields
\[
|N| \leq \left| -\frac{1}{\gamma} \partial_t \left( \frac{R(t)^{\mu+1}}{1 + (\gamma - 1)\sigma R(t)^{\mu}} \hat{G} \right) \right|
+ \frac{1}{\gamma} \left| \sum_{i,j=1}^{3} S_1 (\partial_i \hat{u}_j \partial_j \hat{u}_i) + \sum_{j=1}^{3} \hat{S}_1 \hat{u}_j \partial_j \text{div} \hat{u} + \sum_{j=1}^{3} \hat{u}_j \partial_j \text{div} \hat{u} \right|
\leq CR(t)^{\mu} |\hat{G}| + CR(t)^{\mu} |S_0 \hat{G}| + C |\nabla \hat{u}| |\nabla^2 \hat{u}| \] (191)

and
\[
|\nabla N| \leq CR(t)^{2\mu} |\nabla \hat{\sigma}| |\hat{G}| + |S_0 \hat{G}| + CR(t)^{\mu} (|\nabla \hat{G}| + |\nabla S_0 \hat{G}|)
+CR(t)^{2\mu} |S_0 \hat{\sigma}| |\hat{G}| + C (|\nabla S_0 | |\nabla^2 \hat{u}| + |\nabla \hat{u}| |\nabla^3 \hat{u}|). \] (192)

In addition, taking \( \Delta \) to (79)-(80), one has
\[
\mathcal{L}_0(\partial_\sigma \Delta \hat{\sigma}, \partial_\sigma \Delta (\text{div} \hat{u}))
= \partial_t (\partial_\sigma \Delta \hat{\sigma}) + \frac{\mu L}{R(t)} \partial_\sigma (\partial_\sigma \Delta \hat{\sigma})
+ R(t)^{-\mu-1}(1 + (\gamma - 1)\sigma R(t)^{\mu}) \partial_\sigma (\Delta \text{div} \hat{u}) = \hat{B}_s, \] (194)
\[
\hat{\mathcal{L}}(\Delta \hat{\sigma}, \Delta \text{div} \hat{u}) = \partial_t (\Delta \text{div} \hat{u}) + \frac{L}{R(t)} (\Delta \text{div} \hat{u}) + \frac{\gamma}{R(t)} \Delta^2 \hat{\sigma} = \Delta B, \] (195)

here \( \hat{B}_s = \Delta B_s - \frac{\gamma-1}{R(t)} \) ( \( \sum_{j=1}^{3} \partial_\sigma \hat{u}_j (\text{div} \hat{u}) + \Delta \hat{\sigma} \partial_\sigma \text{div} \hat{u}) \).

As in (84)-(85), we have
\[
\int_{\Gamma_T} \left( R(t)^{2\mu-2\delta} |\nabla \Delta \hat{\sigma}|^2 + R(t)^{\mu-2\delta} |\Delta \text{div} \hat{u}|^2 \right) dy
+ \int_{\Omega_T} \left( R(t)^{2\mu-2\delta} |\nabla \Delta \hat{\sigma}|^2 + R(t)^{\mu-2\delta} |\Delta \text{div} \hat{u}|^2 \right) dt dy
\leq Ce^2 + C\varepsilon \mathcal{E}_T(\hat{\sigma}, \hat{u}) + \int_{\Omega_T} \left( \sum_{i=1}^{3} \mathcal{L}_0(\partial_\sigma \Delta \hat{\sigma}, \partial_\sigma \Delta (\text{div} \hat{u})) \cdot \hat{\mathcal{M}}_0(\partial_\sigma \Delta \hat{\sigma} - y_i N) \right).
where

\[ R(t)^{\mu - 1 - 2\delta} \sum_{i=1}^{3} y_i(\partial_i \Delta \sigma - y_i N) dS + \mathcal{R}, \] (196)

and here we use the multipliers \( \hat{\mathcal{M}}_0 = R(t)^{2\mu - 2\delta} \) and \( \hat{\mathcal{M}} = R(t)^{\mu - 2\delta} \) in (196) to replace \( \hat{\mathcal{M}}_0 \) and \( \hat{\mathcal{M}} \) in (84), which will play an important role in deriving the weighted energy estimate in Lemma 7.3 (see detailed explanations under (220)).

In addition, by the boundary condition (190), then the first term in the last line of (196) is

\[ \int_{\Omega_T} R(t)^{\mu - 1 - 2\delta} (1 + (\gamma - 1)^2 R(t)^{\mu}) \Delta \hat{\sigma} v \sum_{i=1}^{3} \partial_i(y_i N) dtdy \]

and \( \hat{\mathcal{M}}_0 = R(t)^{2\mu - 2\delta} \) and \( \hat{\mathcal{M}} = R(t)^{\mu - 2\delta} \) in (196) to replace \( \hat{\mathcal{M}}_0 \) and \( \hat{\mathcal{M}} \) in (84), which will play an important role in deriving the weighted energy estimate in Lemma 7.3 (see detailed explanations under (220)).

Thus it follows from (196) that

\[ \int_{\Omega_T} \left( R(t)^{2\mu - 2\delta} |\nabla \Delta \hat{\sigma}|^2 + R(t)^{\mu - 2\delta} |\Delta \hat{\sigma} v|^2 \right) dty \]

\[ + \int_{\Omega_T} \left( R(t)^{2\mu - 1 - 2\delta} |\nabla \Delta \hat{\sigma}|^2 + R(t)^{\mu - 1 - 2\delta} |\Delta \hat{\sigma} v|^2 \right) dtdy \]

\[ \leq \int_{\Omega_T} \left( \sum_{i=1}^{3} \hat{B}_i \cdot R(t)^{2\mu - 2\delta}(\partial_i \Delta \hat{\sigma} - y_i N) + \Delta B \cdot R(t)^{\mu - 2\delta}(\Delta \hat{\sigma} v) \right) dtdy \]

\[ + C\varepsilon^2 + C\varepsilon \mathcal{E}_T(\hat{\sigma}, \hat{u}) + |\mathcal{R}|. \] (199)

Next, we estimate each term in the right hand side of (199). Direct computation yields

\[ \hat{B}_i = -\frac{1}{R(t)} \sum_{j=1}^{3} \hat{u}_j \partial_j(\partial_i \Delta \hat{\sigma}) \]

\[ + \frac{1}{R(t)} \sum_{j,k=1}^{3} \left( \sum_{l=1}^{2} C_l \partial_k^l \hat{u}_j \partial_j(\partial_i \partial_k^{2-l} \hat{\sigma}) + \sum_{l=0}^{2} C_l \partial_k^l \partial_i \partial_k^{2-l}(\Delta \hat{\sigma} v) \right) \equiv F_i^1 + F_i^2 \] (200)
and
\[
\Delta B = -\frac{1}{R(t)} \sum_{j=1}^{3} \dot{u}_j \partial_j (\Delta \text{div} \dot{u}) + \frac{1}{R(t)} \left( \sum_{l=0}^{2} \sum_{i,j,k=1}^{3} C_l \partial_l \dot{u}_j \partial_j \partial^{2-l}_k (\partial_j \text{div} \dot{u}) + \sum_{l=1}^{2} \sum_{i,j,k=1}^{3} \partial_l \dot{u}_j \partial_j \partial^{2-l}_k (\partial_j \text{div} \dot{u}) \right)
\]
\[\equiv F_0^1 + F_0^2. \tag{201}\]

We start to treat the first term on the right hand side of (199). This procedure is divided into two parts.

**Part 1. The estimate on** \(\int_{\Omega_T} \left( \sum_{i=1}^{3} \Delta F_i^1 \cdot R(t)^{2\mu - 2\delta} (\partial_i \Delta \dot{\sigma} - y_i N) + F_0^1 \cdot R(t)^{\mu - 2\delta} (\Delta \text{div} \dot{u}) \right) dtdy\).

By the divergence theorem and the boundary condition (23), one has
\[
\int_{\Omega_T} \left( \sum_{i=1}^{3} \Delta F_i^1 \cdot R(t)^{2\mu - 2\delta} (\partial_i \Delta \dot{\sigma} - y_i N) + F_0^1 \cdot R(t)^{\mu - 2\delta} (\Delta \text{div} \dot{u}) \right) dtdy
\]
\[= \int_{\Omega_T} \frac{1}{R(t)} \sum_{j=1}^{3} y_j \dot{u}_j \left( R(t)^{2\mu - 2\delta} (S_1 \Delta \dot{\sigma} N - \frac{1}{2} |\nabla \Delta \dot{\sigma}|^2) - \frac{1}{2} R(t)^{\mu - 2\delta} (\Delta \text{div} \dot{u})^2 \right) dS \ (= 0)
\]
\[+ \int_{\Omega_T} \left( \frac{1}{2} R(t)^{2\mu - 1 - 2\delta} \text{div} \dot{u} |\nabla \Delta \dot{\sigma}|^2 - R(t)^{2\mu - 1 - 2\delta} \sum_{i,j=1}^{3} \partial_i \Delta \dot{\sigma} \partial_j (y_i \dot{u}_j N) \right) dtdy
\]
\[+ \frac{1}{2 R(t)} \sum_{j=1}^{3} \partial_j (R(t)^{\mu - 2\delta} \dot{u}_j) (\Delta \text{div} \dot{u})^2 dtdy
\]
\[= \int_{\Omega_T} \left( \frac{1}{2} R(t)^{2\mu - 1 - 2\delta} \text{div} \dot{u} |\nabla \Delta \dot{\sigma}|^2 + \frac{1}{2} R(t)^{\mu - 1 - 2\delta} \text{div} \dot{u} (\Delta \text{div} \dot{u})^2
\]
\[- R(t)^{2\mu - 1 - 2\delta} \sum_{i=1}^{3} \dot{u}_i \partial_i \Delta \dot{\sigma} N - R(t)^{2\mu - 1 - 2\delta} S_1 \Delta \dot{\sigma} \text{div} \dot{u} N \right) dtdy
\]
\[- \int_{\Omega_T} R(t)^{2\mu - 1 - 2\delta} \sum_{j=1}^{3} \dot{u}_j \partial_j NS_1 \Delta \dot{\sigma} dtdy
\]
\[\equiv H_1 + H_2. \tag{202}\]

It follows from (35), (41), (187) and (191) that
\[|H_1| \leq C \varepsilon E_T (\dot{\sigma}, \dot{u}). \tag{203}\]

In addition, by (35), (187) and (192), we arrive at
\[|H_2| = \left| \int_{\Omega_T} R(t)^{2\mu - 1 - 2\delta} \sum_{j=1}^{3} \dot{u}_j \partial_j NS_1 \Delta \dot{\sigma} dtdy \right|
\[\leq C \varepsilon \int_{\Omega_T} R(t)^{\mu - 1 - \delta} |\nabla N| |S_1 \Delta \dot{\sigma}| dtdy.\]
\[ \leq C \varepsilon \int_{\Omega_t} R(t)^{\mu - 1 - \delta} |S_1 \Delta \sigma| \left( (R(t)^{\mu + \delta} + R(t)^{2 \mu} |\nabla S_0 \tilde{\sigma}|) |\tilde{G}| + |\nabla \tilde{u}| |\nabla^3 \tilde{u}| + R(t)^{\mu} |\nabla \tilde{G}| \right) dt dy \\
\]
\[ + C \varepsilon \int_{\Omega_t} R(t)^{\mu - 1 - \delta} |S_1 \Delta \sigma| \left( |\nabla^2 \tilde{u}|^2 + R(t)^{\mu} |\nabla S_0 \tilde{G}| \right) dt dy \\
\equiv H_{21} + H_{22}. \quad (204) \]

By (35), (41) and (187), then
\[ |H_{21}| \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}). \quad (205) \]

We next deal with each term in \( H_{22} \). By Hölder inequality and Gagliardo-Nirenberg inequality (71), we have
\[ \int_{\Omega_t} R(t)^{\mu - 1 - \delta} |S_1 \Delta \sigma| |\nabla^2 \tilde{u}|^2 dt dy \]
\[ \leq C \int_0^T R(t)^{\mu - 1 - \delta} \| \nabla^2 \tilde{u} \|_{L^2(B^0)} \| S_1 \Delta \sigma \|_{L^2(B^0)} dt \]
\[ \leq C \int_0^T R(t)^{\mu - 1 - \delta} \| \nabla \tilde{u} \|_{L^2(B^0)} \| \nabla^3 \tilde{u} \|_{L^2(B^0)} \| S_1 \Delta \sigma \|_{L^2(B^0)} dt \]
\[ \leq C \varepsilon \int_0^T \left( R(t)^{\mu - 1 - 2\delta} |\nabla^3 \tilde{u}|^2 + R(t)^{2 \mu - 1 - 2 \delta} |S_1 \Delta \sigma| \right) dt dy \]
\[ \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}). \quad (206) \]

From the estimates on \( \nabla S_0 \tilde{G} \) in (187), one has
\[ \int_{\Omega_t} R(t)^{2 \mu - 1 - \delta} |S_1 \Delta \sigma| |\nabla S_0 \tilde{G}| dt dy \]
\[ \leq C \int_{\Omega_t} R(t)^{2 \mu - 1 - \delta} |S_1 \Delta \sigma| \left( |\dot{u}_z| + |S_0 \dot{u}_z| + |Z \sigma| \right) |\nabla S_0 \dot{u}| + |\nabla \dot{u}_z| |S_0^2 \dot{u}_z| \\
\]
\[ + |S_0 \dot{u}||\nabla Z \sigma| + |\nabla \dot{u}| |S_0 Z \sigma| + |\dot{u}| |\nabla S_0 Z \sigma| + |Z \sigma| |\nabla^2 S_0 \dot{u}| \\
\]
\[ + |\dot{u}| |\nabla \dot{u}| (|S_0 \dot{u}| + |\nabla S_0 \dot{u}|) \quad + |\dot{u}^2| (|\nabla S_0 \dot{u}| + |\nabla^2 S_0 \dot{u}|) \quad dtdy \\
\]
\[ + C \int_{\Omega_t} R(t)^{2 \mu - 1 - \delta} |S_1 \Delta \sigma| \left( |\nabla Z \sigma| |\nabla S_0 \dot{u}| + |\nabla \dot{u}^2| |S_0 Z \sigma| \right) dt dy \]
\[ \equiv H_{22}^1 + H_{22}^2. \quad (207) \]

For the term \( H_{22}^1 \), it follows from (35), (41) and Hölder inequality that
\[ |H_{22}^1| \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}). \quad (208) \]

For the term \( H_{22}^2 \), by the similar treatment as in (206), we have
\[ |H_{22}^2| \leq C \varepsilon \mathcal{E}_T(\sigma, \dot{u}). \quad (209) \]
Collecting (202)-(209) yields
\[
\left| \int_{\Omega_T} \left( \sum_{i=1}^{3} \Delta F_i^1 \cdot R(t)^{2\mu-2\delta} (\partial_i \Delta \sigma - y_i N) + F_0^1 \cdot R(t)^{\mu-2\delta} (\Delta \text{div} \bar{u}) \right) dtdy \right| 
\leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \tag{210}
\]

**Part 2. The estimate on** \( \int_{\Omega_T} \left( \sum_{i=1}^{3} \Delta F_i^2 \cdot R(t)^{2\mu-2\delta} (\partial_i \Delta \sigma - y_i N) + F_0^2 \cdot R(t)^{\mu-2\delta} (\Delta \text{div} \bar{u}) \right) dtdy \).

Since the estimation process is very similar to that for the term \( H_2 \) in Part 1, we will skip the details and only give the following estimate
\[
\left| \int_{\Omega_T} \left( \sum_{i=1}^{3} \Delta F_i^2 \cdot R(t)^{2\mu-2\delta} (\partial_i \Delta \sigma - y_i N) + F_0^2 \cdot R(t)^{\mu-2\delta} (\Delta \text{div} \bar{u}) \right) dtdy \right| 
\leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \tag{211}
\]

Combining (210) with (211) yields
\[
\left| \int_{\Omega_T} \left( \sum_{i=1}^{3} \Delta F_i^2 \cdot R(t)^{2\mu-2\delta} (\partial_i \Delta \sigma - y_i N) + F_0^2 \cdot R(t)^{\mu-2\delta} (\Delta \text{div} \bar{u}) \right) dtdy \right| 
\leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \tag{212}
\]

Finally, we estimate \( R \). At first, we estimate the first term in \( R \).
\[
\left| \int_{\Omega_T} R(t)^{\mu-1-2\delta} (1 + (\gamma - 1)\delta R(t)^{\mu}) \Delta \text{div} \bar{u} \sum_{i=1}^{3} \partial_i (y_i N) dtdy \right| 
\leq C \int_{\Omega_T} R(t)^{\mu-1-2\delta} |\Delta \text{div} \bar{u}|(|N| + |\nabla N|) dtdy
\leq C \int_{\Omega_T} R(t)^{\mu-1-2\delta} |\Delta \text{div} \bar{u}| \left( R(t)^{\mu+\delta}(|\bar{G}| + |S_0 \bar{G}|) + |\nabla \bar{u}|(|\nabla^2 \bar{u}| + |\nabla^3 \bar{u}|) 
\right.
\left. + R(t)^{\mu}(|\nabla G| + |\nabla S_0 G|) + R(t)^{2\mu} |\nabla S_0 \dot{\sigma}| |\bar{G}| \right) dtdy
\equiv I_1 + I_2. \tag{213}
\]

By the Hölder inequality, it follows from (35), (41) and (187) that
\[
|I_1| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \tag{214}
\]

Meanwhile, by the similar treatment as in (206), we have
\[
|I_2| \leq C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \tag{215}
\]

Since the estimates on the second term and forth term in \( R \) are similar to (214) and (215), one has
\[
\left| \int_{\Omega_T} (\gamma - 1)R(t)^{2\mu-1-2\delta} \sum_{i=1}^{3} y_i \partial_i \dot{\sigma} \Delta \text{div} \bar{u} N dtdy \right|
\]
In terms of (41) and Hölder inequality, we obtain in the first line of (220) will become

\[ \leq C \varepsilon E_T(\hat{\sigma}, \hat{u}). \]  

(216)

At last, we start to estimate the third term in $R$. Note that

\[ \left| \int_{\Omega_T} \left( \sum_{i=1}^{3} \partial_i \Delta \hat{\sigma} \partial_i \hat{R}(t) \Omega(\gamma-1)LR(t)^{6(\gamma-1)} S_1 \Delta \hat{\sigma} \right) dtdy \right| \]
\[ \leq C \int_{\Omega_T} R(t)^{2u-1-2\delta} |S_1 \Delta \hat{\sigma}||N| dtdy + C \int_{\Omega_T} R(t)^{2u-1-2\delta} |S_1 \Delta \hat{\sigma}| S_0 |N| dtdy. \]

(217)

In addition,

\[ \int_{\Omega_T} R(t)^{2u-1-2\delta} |S_1 \Delta \hat{\sigma}||N| dtdy \]
\[ \leq C \int_{\Omega_T} R(t)^{3u-1-2\delta} |S_1 \Delta \hat{\sigma}| |\hat{u}_z| \left( |\hat{u}_z| + |S_0 \hat{u}_z| + |S_0^2 \hat{u}_z| \right) dtdy \\
+ C \int_{\Omega_T} R(t)^{3u-1-2\delta} |S_1 \Delta \hat{\sigma}| S_0 Z |\hat{u}| \left( |\nabla \hat{u}| + |\nabla^2 \hat{u}| \right) dtdy + \tilde{R}, \]

(218)

where

\[ \tilde{R} = C \varepsilon \int_{\Omega_T} R(t)^{2u-1-2\delta} |S_1 \Delta \hat{\sigma}| \left( |\hat{u}| + |\nabla \hat{u}| + |\nabla^2 \hat{u}| \right) dtdy. \]

(219)

In terms of (41) and Hölder inequality, we obtain

\[ \int_{\Omega_T} R(t)^{3u-1-2\delta} |S_1 \Delta \hat{\sigma}| |\hat{u}_z| \left( |\hat{u}_z| + |S_0 \hat{u}_z| + |S_0^2 \hat{u}_z| \right) dtdy \]
\[ \leq C \varepsilon \int_{\Omega_T} R(t)^{2u-1-\frac{1}{2}\delta} |S_1 \Delta \hat{\sigma}| \left( |\hat{u}_z| + |S_0 \hat{u}_z| + |S_0^2 \hat{u}_z| \right) dtdy \]
\[ \leq C \varepsilon \int_{\Omega_T} \left( R(t)^{2u-1-2\delta} |S_1 \Delta \hat{\sigma}|^2 + R(t)^{2u-1-\delta} (|\hat{u}_z|^2 + |S_0 \hat{u}_z|^2 + |S_0^2 \hat{u}_z|^2) \right) dtdy \]
\[ \leq C \varepsilon E_T(\hat{\sigma}, \hat{u}). \]

(220)

Let’s give some explanations on the choice of the multipliers $M_0$ and $M$ in (196). If we still use the multipliers $M_0$ and $M$ in (84), then the first integration in the first line of (220) will become $\int_{\Omega_T} R(t)^{3u-1-\delta} |S_1 \Delta \hat{\sigma}| \hat{u}_z^2 dtdy$ other than $\int_{\Omega_T} R(t)^{3u-1-2\delta} |S_1 \Delta \hat{\sigma}| \hat{u}_z^2 dtdy$. Due to the time-decay rate of $|\hat{u}_z|$ is only of $R(t)^{-u+\frac{1}{2}\delta}$ (this has been better than the decay rate of $|\hat{u}|$ with $R(t)^{-\mu+\delta}$), then

\[ \int_{\Omega_T} R(t)^{3u-1-\delta} |S_1 \Delta \hat{\sigma}| \hat{u}_z^2 dtdy \]
\[ \leq C \varepsilon \int_{\Omega_T} R(t)^{2u-1-\frac{1}{2}\delta} |S_1 \Delta \hat{\sigma}| \hat{u}_z^2 dtdy \]
\[ \leq C \varepsilon \int_{\Omega_T} \left( R(t)^{2u-1-2\delta} |S_1 \Delta \hat{\sigma}|^2 + R(t)^{2u-1+\delta} |\hat{u}_z|^2 \right) dtdz \]

cannot be controlled by $E_T(\hat{\sigma}, \hat{u})$ since only the term $\int_{\Omega_T} R(t)^{2u-1-\delta} |\hat{u}_z|^2 dtdz$ other than $\int_{\Omega_T} R(t)^{2u-1+\delta} |\hat{u}_z|^2 dtdz$ appears in the energy $E_T(\hat{\sigma}, \hat{u})$. 

Meanwhile, it follows from (35), (41) and direct computation that
\[ |\tilde{R}| \leq C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}). \]  (221)

Next, we estimate the first term of the second line in the right hand of (218). Note that we cannot use Hölder inequality to treat this term directly since \( \dot{u} \) does not have enough time-decay rate. Otherwise, for example,
\[
\int_{\Omega_T} R(t)^{3\mu-1-2\delta} |\dot{u}| |S_1 \Delta \dot{\sigma}| |S_0 Z \dot{\sigma}| dt dy
\leq C \varepsilon \int_{\Omega_T} R(t)^{2\mu-1-\delta} |S_1 \Delta \dot{\sigma}| |S_0 Z \dot{\sigma}| dt dy
\leq C \varepsilon \int_{\Omega_T} R(t)^{2\mu-1-\delta} (|S_1 \Delta \dot{\sigma}|^2 + |S_0 Z \dot{\sigma}|^2) dt dy,
\]  (222)

which can not be controlled by \( \mathcal{E}_T(\dot{\sigma}, \dot{u}) \). To overcome this difficulty, we will use equation (20) to get the useful expression of \( S_0 Z \dot{\sigma} \). For this purpose, by acting \( R(t)Z \) to (20), we have
\[
S_0 Z \dot{\sigma} = -\mu L Z \dot{u} - R(t)^{-\mu} Z \text{div} \dot{u} + R(t) Z \hat{f}_t.
\]  (223)

In this case, then
\[
\int_{\Omega_T} R(t)^{3\mu-1-2\delta} |S_1 \Delta \dot{\sigma}| |S_0 Z \dot{\sigma}| (|\dot{u}| + |\nabla \dot{u}|) dt dy
\leq \int_{\Omega_T} R(t)^{3\mu-1-2\delta} |S_1 \Delta \dot{\sigma}| (|R(t)^{-\mu} Z \text{div} \dot{u}| + R(t)|Z \hat{f}_t|) (|\dot{u}| + |\nabla \dot{u}|) dt dy.
\]

On the other hand, it follows from (35) and (41) that
\[
\int_{\Omega_T} R(t)^{3\mu-1-2\delta} |S_1 \Delta \dot{\sigma}| |Z \dot{\sigma}| (|\dot{u}| + |\nabla \dot{u}|) dt dy
\leq C \varepsilon \int_{\Omega_T} R(t)^{2\mu-1-2\delta} |S_1 \Delta \dot{\sigma}| (|\dot{u}| + |\nabla \dot{u}|) dt dy
\leq C \varepsilon \int_{\Omega_T} R(t)^{2\mu-1-2\delta} \left(|S_1 \Delta \dot{\sigma}|^2 + |\dot{u}|^2 + |\nabla \dot{u}|^2 \right) dt dy
\leq C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}).
\]  (225)

And direct computation yields
\[
\int_{\Omega_T} R(t)^{3\mu-1-2\delta} |S_1 \Delta \dot{\sigma}| (|R(t)^{-\mu} Z \text{div} \dot{u}| + R(t)|Z \hat{f}_t|) (|\dot{u}| + |\nabla \dot{u}|) dt dy
\leq C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}).
\]  (226)

Collecting (218)-(221) and (224)-(226), we have
\[
\int_{\Omega_T} R(t)^{2\mu-1-2\delta} |S_1 \Delta \dot{\sigma}| N dt dy \leq C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}).
\]  (227)

By analogous analysis as in (218)-(227), one can get
\[
\int_{\Omega_T} R(t)^{2\mu-1-2\delta} |S_1 \Delta \dot{\sigma}| |S_0 N| dt dy \leq C \varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}).
\]  (228)
Combining (213)-(216), (221) with (227)-(228) yields
\[ |R| \leq C \varepsilon \mathcal{E}_T(\tilde{\sigma}, \tilde{u}). \]  
(229)

Thus, it follows from (199), (212) and (229) that
\[
\int_{\Gamma_T} \left( R(t)^{2\mu-2\delta} |\nabla \Delta \tilde{\sigma}|^2 + R(t)^{\mu-2\delta} |\Delta \nabla \tilde{u}|^2 \right) dy \\
+ \int_{\Omega_T} \left( R(t)^{2\mu-1-2\delta} |\nabla \tilde{\sigma}|^2 + R(t)^{\mu-1-2\delta} |\Delta \nabla \tilde{u}|^2 \right) dt dy \\
\leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\tilde{\sigma}, \tilde{u}). 
\]  
(230)

Note that
\[
S_i^2 = r^2 \Delta - S_1 - \sum_{i=1}^{3} Z_i^2. 
\]  
(231)

This means
\[
\begin{cases}
|\nabla S_i^2 \tilde{\sigma}| \leq C \left( |\nabla \Delta \tilde{\sigma}| + |\Delta \tilde{\sigma}| + |\nabla S_1 \tilde{\sigma}| + |\nabla Z^2 \tilde{\sigma}| \right), \\
|S_i^2 \nabla \nabla \tilde{u}| \leq C \left( |\Delta \nabla \tilde{u}| + |S_1 \nabla \nabla \tilde{u}| + |Z^2 \nabla \nabla \tilde{u}| \right).
\end{cases} 
\]  
(232)

Together with Corollary 3 and (230), this yields (188).

Similarly to Lemma 7.2, we also have the following estimate near \(|y| = 0\).

**Lemma 7.4** (Estimate of \((\nabla^3 \tilde{\sigma}, \nabla^2 \nabla \nabla \tilde{u})\) near \(|y| = 0\). Under the assumptions of Theorem 5.1, in terms of Lemma 2.1, then we have that for \(\mu = 3(\gamma - 1)\),
\[
\int_{\Gamma_T} \chi(|y|) \left( R(t)^{2\mu-2\delta} |\nabla^3 \tilde{\sigma}|^2 + R(t)^{\mu-2\delta} |\nabla^2 \nabla \nabla \tilde{u}|^2 \right) dy \\
+ \int_{\Omega_T} \chi(|y|) \left( R(t)^{2\mu-1-2\delta} |\nabla^3 \tilde{\sigma}|^2 + R(t)^{\mu-1-2\delta} |\nabla^2 \nabla \nabla \tilde{u}|^2 \right) dt dy \\
\leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\tilde{\sigma}, \tilde{u}) + C \varepsilon \mathcal{E}_T^{\frac{1}{2}}(\tilde{\sigma}, \tilde{u}), 
\]  
(233)

where the smooth function \(\chi\) is defined in (108).

**Corollary 5.** By Lemma 2.3 together with Lemmas 5.2-5.3, then Lemmas 7.1-7.4 yield the following estimate
\[
\int_{\Gamma_T} \left( R(t)^{2\mu-2\delta} |\nabla^3 \tilde{\sigma}|^2 + R(t)^{\mu-2\delta} |\nabla^3 \tilde{u}|^2 \right) dy \\
+ \int_{\Omega_T} \left( R(t)^{2\mu-1-2\delta} |\nabla^3 \tilde{\sigma}|^2 + R(t)^{\mu-1-2\delta} |\nabla^3 \tilde{u}|^2 \right) dt dy \\
\leq C \varepsilon^2 + C \varepsilon \mathcal{E}_T(\tilde{\sigma}, \tilde{u}) + C \varepsilon \mathcal{E}_T^{\frac{1}{2}}(\tilde{\sigma}, \tilde{u}). 
\]  
(234)

Next, we continue to improve the weighted estimates on the derivatives of \((\tilde{u}, \tilde{u}_z)\).

**Lemma 7.5.** Under the assumptions of Theorem 5.1, we have the following strong weighted estimate for \(\mu = 3(\gamma - 1)\),
\[
\int_{\Omega_T} R(t)^{2\mu-1-\delta} \left( \sum_{\alpha+\beta \leq 2} |\nabla^\alpha S_0^\beta \tilde{u}_z|^2 + |\nabla^2 \tilde{u}_z|^2 \right) dt dy \\
+ \sum_{\alpha+\beta+\gamma \leq 2, \alpha < 2} \int_{\Omega_T} R(t)^{2\mu-1-\delta} |\nabla^\alpha S_0^\beta Z^\gamma \tilde{u}_z|^2 dt dy 
\]
+ \int_{\Gamma_{T}} \left( \sum_{\alpha+\beta \leq 2} R(t)^{2\mu-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} + R(t)^{2\mu-2\delta} |\nabla^{2} \hat{u}|^{2} \right) dy \\
+ \sum_{\alpha+\beta+\gamma \leq 2, \alpha < 2} \int_{\Gamma_{T}} R(t)^{2\mu-\delta} |\nabla^{\alpha} S_{0}^{\beta} Z^{\gamma} \hat{u}|^{2} dy \\
\leq C \varepsilon^{2} + C \varepsilon \mathcal{E}_{T}(\hat{\sigma}, \hat{u}) + C \varepsilon \mathcal{E}^{\frac{3}{2}}(\hat{\sigma}, \hat{u}). \quad (235)

Proof. Acting \( R(t) \left( y_{2}(3.4^{3}) - y_{3}(3.4^{2}) \right) \) yields

\[ S_{0} \dot{u}_{z1} + L \dot{u}_{z1} + \gamma Z_{1} \dot{\sigma} = R(t) \left( y_{2} \dot{f}_{3} - y_{3} \dot{f}_{2} \right). \quad (236) \]

By taking \( \frac{1}{R(t)} \nabla^{\alpha} S_{0}^{\beta} \) to (236), we have

\[ \partial_{t} \left( \nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1} \right) + \frac{L}{R(t)} \nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1} + \frac{\gamma}{R(t)} \nabla^{\alpha} S_{0}^{\beta} Z_{1} \dot{\sigma} \]

\[ = \frac{1}{R(t)} \nabla^{\alpha} S_{0}^{\beta} \left( R(t)(y_{2} \dot{f}_{3} - y_{3} \dot{f}_{2}) \right), \quad (237) \]

where \( \alpha + \beta \leq 2 \). Multiplying \( R(t)^{2\mu-\delta} \nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1} \) on the two sides of (237), and then integrating over \( \Gamma_{T} \), one has that by the integration by parts,

\[ \int_{\Gamma_{T}} R(t)^{2\mu-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} dy + \int_{\Omega_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} dt dy \]

\[ \leq C \varepsilon^{2} + C \int_{\Omega_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} Z_{1} \dot{\sigma}| \nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1} |dt| dy \]

\[ + C \int_{\Omega_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} \left( R(t)(y_{2} \dot{f}_{3} - y_{3} \dot{f}_{2}) \right)| \nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1} |dt| dy. \quad (238) \]

Next we treat each term in the right side of (238). Direct computation yields

\[ \int_{\Omega_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} \left( R(t)(y_{2} \dot{f}_{3} - y_{3} \dot{f}_{2}) \right)||\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1} |dt| dy \leq C \varepsilon \mathcal{E}_{T}(\hat{\sigma}, \hat{u}). \quad (239) \]

In addition, by Corollary 4, it follows from Young inequality that

\[ C \int_{\Omega_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} Z_{1} \dot{\sigma}| \nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1} |dt| dy \]

\[ \leq C \int_{\Omega_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} Z_{1} \dot{\sigma}|^{2} dt dy + \frac{1}{2} \int_{\Gamma_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} dt dy \]

\[ \leq C \varepsilon^{2} + C \varepsilon \mathcal{E}_{T}(\hat{\sigma}, \hat{u}) + C \varepsilon \mathcal{E}^{\frac{1}{2}}(\hat{\sigma}, \hat{u}) + \frac{1}{2} \int_{\Gamma_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} dt dy, \quad (240) \]

Collecting (238)-(240) derives

\[ \int_{\Gamma_{T}} R(t)^{2\mu-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} dy + \int_{\Omega_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} dt dy \]

\[ \leq C \varepsilon^{2} + C \varepsilon \mathcal{E}_{T}(\hat{\sigma}, \hat{u}) + C \varepsilon \mathcal{E}^{\frac{1}{2}}(\hat{\sigma}, \hat{u}). \quad (241) \]

Analogously, we have

\[ \int_{\Gamma_{T}} R(t)^{2\mu-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} dy + \int_{\Omega_{T}} R(t)^{2\mu-1-\delta} |\nabla^{\alpha} S_{0}^{\beta} \hat{u}_{z1}|^{2} dt dy \]

\[ \leq C \varepsilon^{2} + C \varepsilon \mathcal{E}_{T}(\hat{\sigma}, \hat{u}) + C \varepsilon \mathcal{E}^{\frac{1}{2}}(\hat{\sigma}, \hat{u}). \quad (242) \]
Meanwhile, by the similar method, we also have
\[
\sum_{\alpha+\beta+\nu\leq 2, \alpha<2} \int_{\Gamma_T} R(t)^{2\mu-\delta} |\nabla^\alpha S_0^\beta Z^\nu \dot{u}|^2 dy + \sum_{\alpha+\beta+\nu\leq 2, \alpha<2} \int_{\Omega_T} R(t)^{2\mu-1-\delta} |\nabla^\alpha S_0^\beta Z^\nu \dot{u}|^2 dt dy \leq C\varepsilon^2 + C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_{1/2}^T(\dot{\sigma}, \dot{u}). \tag{243}
\]

Finally, we give the estimate on \(\nabla^2 \ddot{u}_i\). Acting \(\nabla^2\) to (22) yields
\[
\partial_t \nabla^2 \ddot{u}_i + \frac{L}{R(t)} \nabla^2 \ddot{u}_i + \frac{\gamma}{R(t)} \partial_i \nabla^2 \dot{\sigma} = \nabla^2 \dot{f}_i. \tag{244}
\]

Multiplying \(R(t)^{2\mu-2\delta} \nabla^2 \ddot{u}_i\) on two sides of (244), and integrating over \(\Omega_T\), then it follows from the integration by parts that
\[
\int_{\Gamma_T} R(t)^{2\mu-2\delta} |\nabla^2 \ddot{u}_i|^2 dy + \int_{\Omega_T} R(t)^{2\mu-1-2\delta} |\nabla^2 \ddot{u}_i|^2 dt dy \leq C\varepsilon^2 + C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_{1/2}^T(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_{1/2}^T(\dot{\sigma}, \dot{u}). \tag{245}
\]

Similarly to (239)-(240), but it is required to replace Corollary 4 by Corollary 5 in the proof of (240) since the weight in (245) is \(R(t)^{2\mu-2\delta}\) other than the weight \(R(t)^{2\mu-\delta}\) in (6.66), then we have
\[
\int_{\Gamma_T} R(t)^{2\mu-2\delta} |\nabla^2 \ddot{u}_i|^2 dy + \int_{\Omega_T} R(t)^{2\mu-1-2\delta} |\nabla^2 \ddot{u}_i|^2 dt dy \leq C\varepsilon^2 + C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_{1/2}^T(\dot{\sigma}, \dot{u}). \tag{246}
\]
Combining (243) with (246), we complete the proof of Lemma 7.5. \(\square\)

Based on Corollary 2- Corollary 3, Corollary 4-Corollary 5 and Lemma 7.5, we are ready to prove Theorem 5.1.

\textbf{Proof.} By the above Corollaries and Lemma 7.5, for sufficiently small \(\varepsilon > 0\), we have
\[
\mathcal{E}_T(\sigma, \dot{u}) + S(\sigma, \dot{u}) \leq C\varepsilon^2 + C\varepsilon \mathcal{E}_T(\dot{\sigma}, \dot{u}) + C\varepsilon \mathcal{E}_{1/2}^T(\dot{\sigma}, \dot{u}). \tag{247}
\]
This is easy to get
\[
\mathcal{E}_T(\sigma, \dot{u}) + S(\sigma, \dot{u}) \leq C\varepsilon^2. \tag{248}
\]
Thus, the proof of Theorem 5.1 is completed. \(\square\)

8. \textbf{Proof of Theorem 1.1 and Theorem 1.2.} At first, we prove Theorem 1.2.

\textbf{Proof.} By Sobolev imbedding theorem, one has
\[
|f(t,y)|^2 \leq C \int_{B_0} \sum_{0 \leq l \leq 2} |\nabla^l_y f(t,y)|^2 dy. \tag{249}
\]
This, together with Theorem 5.1, yields that for $\mu = 3(\gamma - 1)$,
\begin{align*}
|\dot{\sigma}| & \leq C\varepsilon R(t)^{-\mu}, \quad |\dot{u}| \leq C\varepsilon R(t)^{-\mu+\delta}, \quad |\nabla \dot{\sigma}| \leq C\varepsilon R(t)^{-\mu+\delta}, \\
|\nabla \dot{u}| & \leq C\varepsilon R(t)^{-\frac{\mu}{2}+\delta}, \quad |\dot{u}_z| \leq C\varepsilon R(t)^{-\mu+\frac{\mu}{2}}, \\
|S_0 \dot{\sigma}| + |Z \dot{\sigma}| & \leq C\varepsilon R(t)^{-\mu}, \quad |S_0 \dot{u}| \leq C\varepsilon R(t)^{-\frac{\mu}{2}+\delta}.
\end{align*}
(250)

But comparing with (41), we still need to improve the time-decay estimates on $|S_0 \dot{u}|$ and $|S_0 \dot{u}_z|$. In fact, from (236), we have
\begin{equation}
|S_0 \dot{u}_z| \leq C \left( |\dot{u}_z| + |Z \dot{\sigma}| + \sum_{i=1}^{3} |R(t) \dot{f}_i| \right).
\end{equation}
(251)

It follows from (251) and the expression of $\dot{f}_i$ that
\begin{equation}
|S_0 \dot{u}_z| \leq C\varepsilon R(t)^{-\mu+\frac{\mu}{2}}.
\end{equation}
(252)

Analogously, it follows from (21) that
\begin{equation}
|S_0 \dot{u}| \leq C \left( |\dot{u}| + |\nabla \dot{\sigma}| + \sum_{i=1}^{3} |R(t) \dot{f}_i| \right).
\end{equation}
(253)

This, together with the expression of $\dot{f}_i$, yields
\begin{equation}
|S_0 \dot{u}| \leq C\varepsilon R(t)^{-\mu+\delta}.
\end{equation}
(254)

Note that the generic positive constant $C$ appeared in (250), (252) and (254) depends only on the initial data. Then we can choose the constant $M = 2C$ in (35) and (41) for small $\varepsilon > 0$ so that (35) and (41) hold. In this case, as in Remark 4.1, we have $CR(t)^{3(1-\gamma)} - C\varepsilon R(t)^{3(1-\gamma)} < c^2(\rho) < CR(t)^{3(1-\gamma)} + C\varepsilon R(t)^{3(1-\gamma)}$. This derives $c^2(\rho) \sim R(t)^{3(1-\gamma)} > 0$ for any $t \geq 0$ and small $\varepsilon > 0$. Therefore, the proof of Theorem 1.2 is completed by the local existence result in Corollary 1 and continuation argument.

Finally, we start to prove Theorem 1.1.

**Proof.** Under the assumptions of Theorem 1.1, by going back to $(t, x)$-coordinates, it follows from Lemma 3.1 and the smallness of $L$ that (1) with (3)-(4) has a local solution $(\rho, u)$ which satisfies
\begin{equation}
||\rho_0(1, x) - 1||_{H^4(B^0_1)} + ||u(1, x)||_{H^4(B^0_1)} < \delta_0,
\end{equation}
(255)
where $B^0_1 = \{x : |x| \leq 1 + L\}$, $\delta_0 > 0$ is a small number depending only on $L$ and $\varepsilon_0$. This, together with Theorem 1.2 for the time $t \geq 1$, yields Theorem 1.1. \hfill $\square$

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