ABSTRACT. We consider second-order linear parabolic operators in non-divergence form that are intrinsically defined on Riemannian manifolds. In the elliptic case, Cabré proved a global Krylov-Safonov Harnack inequality under the assumption that the sectional curvature of the underlying manifold is nonnegative. Later, Kim improved Cabré’s result by replacing the curvature condition by a certain condition on the distance function. Assuming essentially the same condition introduced by Kim, we establish Krylov-Safonov Harnack inequality for nonnegative solutions of the non-divergent parabolic equation. This, in particular, gives a new proof for Li-Yau Harnack inequality for positive solutions to the heat equation in a manifold with nonnegative Ricci curvature.

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1. Introduction and Main Results

In this paper, we study Harnack inequalities for solutions of second-order parabolic equations of non-divergence type on Riemannian manifolds. Let \((M, g)\) be a smooth, complete Riemannian manifold of dimension \(n\). For \(x \in M\) and \(t \in \mathbb{R}\), let \(A_{x,t}\) be a positive definite symmetric endomorphism of \(T_x M\), where \(T_x M\) is the tangent space of \(M\) at \(x\). We denote \(\langle X, Y \rangle := g(X, Y)\) and \(|X|^2 := \langle X, X \rangle\) for some positive constants \(\lambda\) and \(\Lambda\). We consider a second-order, linear, uniformly parabolic operator \(\mathcal{L}\) defined by

\[
\mathcal{L} u = Lu - u_t := \text{trace}(A_{x,t} \circ D^2 u) - u_t \quad \text{in} \ M \times \mathbb{R},
\]

where \(\circ\) denotes composition of endomorphisms and \(D^2 u\) denotes the Hessian of the function \(u\) defined by

\[
D^2 u \cdot X = \nabla_X \nabla u,
\]

where \(\nabla u(x) \in T_x M\) is the gradient of \(u\) at \(x\). Notice that in the special case when \(A_{x,t} \equiv \text{Id}\), the equation \(\mathcal{L} u = 0\) simply becomes the usual heat equation \(u_t - \Delta u = 0\).

In the elliptic setting, Cabrè proved in a remarkable paper \([Ca]\) that if the underlying manifold \(M\) has nonnegative sectional curvature, then Krylov-Safonov type (elliptic) Harnack inequality holds for solutions of uniformly elliptic equations in non-divergence form. Later, Kim \([K]\) improved Cabrè’s result by removing the sectional curvature assumption and imposing a certain condition on distance function which, in the parabolic setting, should read as follows: For all \(p \in M\), we have

\[
\Delta d_p(x) \leq \frac{n - 1}{d_p(x)} \quad \text{for} \quad x \not\in \text{Cut}(p) \cup \{p\},
\]

and

\[
L d_p(x) \leq \frac{a_L}{d_p(x)} \quad \text{for} \quad x \not\in \text{Cut}(p) \cup \{p\}, \quad t \in \mathbb{R},
\]

where \(d_p(x) = d(p, x)\) is the geodesic distance between \(p\) and \(x\), \(\text{Cut}(p)\) denotes the cut locus of \(p\), and \(a_L\) is some positive constant that is fixed by the operator \(L\). We shall prove that if the above conditions \((3)\) and \((4)\) hold, then we have Krylov-Safonov Harnack inequality for the parabolic operator \(\mathcal{L}\); i.e., if \(u\) is a (smooth) nonnegative solution of \(\mathcal{L} u = f\) in a cylinder \(K_{2R} := B_{2R}(x_0) \times (t_0 - 4R^2, t_0)\), where \(x_0 \in M\) and \(t_0 \in \mathbb{R}\), then we have

\[
\sup_{K_{2R}} u \leq C \left\{ \inf_{K_R^+} u + R^2 \left( \frac{1}{\text{Vol}(K_{2R})} \int_{K_{2R}} |f|^{\frac{n+1}{n}} \right)^{\frac{n}{n+1}} \right\},
\]

where \(K_R^+ := B_R(x_0) \times (t_0 - 3R^2, t_0 - 2R^2)\), \(K_{2R}^+ := B_{2R}(x_0) \times (t_0 - R^2, t_0)\), \(\text{Vol}\) denotes the volume, and \(C\) is a uniform constant depending only on \(n, \lambda, \Lambda\) and \(a_L\). It is well known that the condition \((3)\) holds if the manifold \(M\) has nonnegative Ricci curvature. Also, as it is proved in \([K]\), the condition \((4)\) is satisfied, for example, if for all \(x \in M\) and any unit vector \(e \in T_x M\), we have \(M^\Gamma[R(e)] \geq 0\). Here, \(R(e)\) is the Ricci transformation of \(T_x M\) into itself given by \(R(e)X := R(X, e)e\), where \(R(X, Y)Z\) is the Riemannian curvature tensor, and

\[
M^\Gamma[R(e)] = M^\Gamma[R(e), \lambda, \Lambda] := \lambda \sum_{\kappa_i > 0} \kappa_i + \Lambda \sum_{\kappa_i < 0} \kappa_i,
\]

where \(\kappa_i\) are eigenvalues of the (symmetric) endomorphism \(R(e)\). In the case when \(\mathcal{L}\) is the heat operator and \(M\) has nonnegative Ricci curvature, then the condition \(M^\Gamma[R(e)] \geq 0\)
is satisfied and thus the Harnack inequality (5) holds; i.e., if $M$ has nonnegative Ricci curvature, then we have
\[
\sup_{K^*_n} u \leq C_n \left\{ \inf_{K^*_n} u + R^2 \left( \frac{1}{\operatorname{Vol}(K_{2R})} \int_{K_{2R}} |u_t - \Delta u| \right)^{\frac{n-1}{n}} \right\},
\]
where $C_n$ is a constant that depends only on the dimension $n$. This, in particular implies the Harnack inequality of Li and Yau [LY]. Also, in the case when $M$ has nonnegative sectional curvature, then the condition $M_i[R(e)] \geq 0$ is trivially satisfied and we have the inequality (5) with a constant $C$ depending only on $n, \lambda, \Lambda$, which especially reproduces the Harnack inequality by Krylov and Safonov [KS] in the Euclidean space.

One crucial ingredient in proving the Euclidean Krylov-Safonov Harnack inequality is the Krylov-Tso estimate, which is the parabolic counterpart of the Aleksandrov-Bakelman-Pucci (ABP) estimate. The Krylov-Tso estimate as well as the classical ABP estimate is proved using affine functions, which have no intrinsic interpretation in general Riemannian manifolds. In the elliptic case, Cabrè ingeniously overcame this difficulty by replacing the affine functions by quadratic functions; quadratic functions have geometric meaning as the square of distance functions. Following Cabrè’s approach, we introduce an intrinsically geometric version of Krylov-Tso normal map, namely,
\[
\Phi(x, t) := \left( \exp_t \nabla_x u(x, t), -\frac{1}{2} d \left( x, \exp_t \nabla u(x, t) \right)^2 - u(x, t) \right).
\]

The map $\Phi$ is called the parabolic normal map related to $u(x, t)$. A few remarks are in order regarding the normal map. In the classical ABP (and Krylov-Tso) estimate, an affine function concerning with the (elliptic) normal map $x \mapsto \nabla u(x)$ plays a role to bound the maximum of $u$ by estimating the measure of the image of the normal map. Since an affine function cannot be generalized naturally to an intrinsic object in Riemannian manifolds, Cabrè used paraboloids instead in [Ca]. The map
\[
p \mapsto \min_{\Omega} |u(x) - p \cdot x|
\]
is considered (up to a sign) as the Legendre transform of $u$. Krylov [Kr] discovered the parabolic version of the Aleksandrov-Bakelman maximum principle and Tso [T] later simplified his proof by using the map
\[
(x, t) \mapsto (\nabla_x u(x, t), \nabla_x u(x, t) \cdot x - u(x, t)).
\]

We end the introduction by stating our main theorems. The rest of the paper shall be devoted to their proof. Below and hereafter, we denote
\[
\int_Q f := \frac{1}{\operatorname{Vol}(Q)} \int_Q f
\]
and
\[
K_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0], \quad (x_0, t_0) \in M \times \mathbb{R}.
\]

**Theorem 1.1 (Harnack inequality).** Suppose conditions (3), (4) hold. Let $u$ be a nonnegative smooth function in $K_{2R}(x_0, 4R^2)$, where $x_0 \in M$ and $R > 0$. Then, we have
\[
\sup_{K_{4R}(x_0, 4R^2)} u \leq C \left\{ \inf_{K_{2R}(x_0, 4R^2)} u + R^2 \left( \int_{K_{2R}(x_0, 4R^2)} |u_t|^{\frac{n+1}{n-1}} \right) \right\},
\]
where $C$ is a uniform constant depending only on $n, \lambda, \Lambda$ and $\alpha_L$. 


Theorem 1.2 (Weak Harnack inequality). Suppose the conditions \([3], [4]\) hold. Let \(u\) be a nonnegative smooth function satisfying \(\mathcal{L}u \leq f\) in \(K_{2R}(x_0, 4R^2)\), where \(x_0 \in M\) and \(R > 0\). Then, we have

\[
\left( \int_{K_{2R}(x_0, 2R^2)} u^p \right)^{\frac{1}{p}} \leq C \left\{ \inf_{K_{2R}(x_0, 4R^2)} u + R^2 \left( \int_{K_{2R}(x_0, 4R^2)} [f^+]^{p+1} \right)^{\frac{1}{p+1}} \right\}; \quad f^+ := \max(f, 0),
\]

where \(p \in (0, 1)\) and \(C\) are uniform constants depending only on \(n, \lambda, \Lambda\) and \(a_L\).

2. Preliminaries

Let \((M, g)\) be a smooth, complete Riemannian manifold of dimension \(n\), where \(g\) is the Riemannian metric and \(\text{Vol} := \text{Vol}_g\) is the reference measure on \(M\). We denote \(\langle X, Y \rangle := g(X, Y)\) and \(|X|^2 := \langle X, X \rangle\) for \(X, Y \in T_xM\), where \(T_xM\) is the tangent space at \(x \in M\). Let \(d(\cdot, \cdot)\) be the distance function on \(M\). For a given point \(y \in M\), \(d_y(x)\) denotes the distance function from \(y\), i.e., \(d_x(y) := d(x, y)\).

We recall the exponential map \(\exp : TM \rightarrow M\). If \(\gamma_{x, X} : \mathbb{R} \rightarrow M\) is the geodesic starting from \(x \in M\) with velocity \(X \in T_xM\), then the exponential map is defined by

\[
\exp_x(X) := \gamma_{x, X}(1).
\]

We note that the geodesic \(\gamma_{x, X}\) is defined for all time since \(M\) is complete. Given two points \(x, y \in M\), there exists a unique minimizing geodesic \(\gamma_{x, y} : [0, 1] \rightarrow M\) joining \(x\) to \(y\) with \(y = \exp_x(x)\) and we will write \(X = \exp^{-1}_x(y)\).

For \(X \in T_xM\) with \(|X| = 1\), we define

\[
t_x(X) := \sup \{ t > 0 : \exp_x(tX) \text{ is minimizing between } x \text{ and } \exp_x(tX) \}.
\]

If \(t_x(X) < +\infty\), \(\exp_x(t_x(X))X\) is a cut point of \(x\). The cut locus of \(x\) is defined as the set of all cut points of \(x\), that is,

\[
\text{Cut}(x) := \{ \exp_x(t_x(X))X : X \in T_xM \text{ with } |X| = 1, \ t_x(X) < +\infty \}.
\]

Define

\[
E_x := \{ tX \in T_xM : 0 \leq t < t_x(X), \ X \in T_xM \text{ with } |X| = 1 \} \subset T_xM.
\]

One can show that \(\text{Cut}(x) = \exp_x(\partial E_x), M = \exp_x(E_x) \cup \text{Cut}(x)\) and \(\exp_x : E_x \rightarrow \text{exp}_x(E_x)\) is a diffeomorphism. We note that \(\text{Cut}(x)\) is closed and has measure zero. For any \(x \notin \text{Cut}(y)\) with \(x \neq y\), then \(d_t\) is smooth at \(x\) and the Gauss lemma implies that

\[
\nabla d_t(x) = -\frac{\exp^{-1}_x(y)}{|\exp^{-1}_x(y)|}
\]

and

\[
\nabla(\nabla^2 d_{t_x(X)}(X)) = -\exp^{-1}_x(y).
\]

Let the Riemannian curvature tensor be defined by

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]

where \(\nabla\) stands for the Levi-Civita connection. For a unit vector \(e \in T_xM\), \(R(e)\) will denote the Ricci transform of \(T_xM\) into itself given by \(R(e)X := R(X, e)e\).

For \(u \in C^\infty(M)\), the Hessian operator \(D^2u(x) : T_xM \rightarrow T_xM\) is defined by

\[
D^2u(x) \cdot X = \nabla_X \nabla u(x).
\]

Let \(M\) and \(N\) be Riemannian manifolds of dimension \(n\) and \(\phi : M \rightarrow N\) be smooth. The Jacobian of \(\phi\) is the absolute value of determinant of the differential \(d\phi\), i.e.,

\[
\text{Jac} \phi(x) := |\det d\phi(x)| \quad \text{for } x \in M.
\]
We quote the following lemma from Lemma 3.2 in [Ca], in which the Jacobian of the map \( x \mapsto \exp_x(\nabla v(x)) \) is computed explicitly.

**Lemma 2.1** (Cabré). Let \( v \) be a smooth function in an open set \( \Omega \) of \( M \). Define the map \( \phi : \Omega \to M \) by

\[
\phi(x) := \exp_x(\nabla v(x)).
\]

Let \( x \in \Omega \) and suppose that \( \nabla v(x) \in E_x \). Set \( y := \phi(x) \). Then we have

\[
\text{Jac} \phi(x) = \text{Jac} \exp_x(\nabla v(x)) \cdot \left| \det D^2 \left( v + d_s^2/2 \right)(x) \right|,
\]

where \( \text{Jac} \exp_x(\nabla v(x)) \) denotes the Jacobian of \( \exp_x \), a map from \( T_x M \) to \( M \), at the point \( \nabla v(x) \in T_x M \).

Under the condition (3), we have the estimate for Jacobian of the exponential map and Bishop’s volume comparison theorem as follows. We state the known results as a lemma. The proof can be found in [K, p. 286] (see also [L]).

**Lemma 2.2.** Suppose that \( M \) satisfies (3).

(i) For any \( x \in M \) and \( X \in E_x \),

\[
\text{Jac} \exp_x(X) = | \det d \exp_x(X) | \leq 1.
\]

(ii) (Bishop) For any \( x \in M \), \( \text{Vol}(B_R(x))/R^n \) is nonincreasing with respect to \( R \), where \( B_R(x) \) is a geodesic ball of radius \( R \) centered at \( x \). Namely,

\[
\frac{\text{Vol}(B_R(x))}{\text{Vol}(B_r(x))} \leq \frac{R^n}{r^n} \text{ if } 0 < r < R.
\]

In particular, \( M \) satisfies the volume doubling property; i.e., \( \text{Vol}(B_{2R}(x)) \leq 2^n \text{Vol}(B_R(x)) \).

The following is the area formula, which follows easily from the area formula in Euclidean space and a partition of unity.

**Lemma 2.3** (Area formula). For any smooth function \( \phi : M \times \mathbb{R} \to M \times \mathbb{R} \) and any measurable set \( E \subset M \times \mathbb{R} \), we have

\[
\int_E \text{Jac} \phi(x,t)dV(x,t) = \int_{M \times \mathbb{R}} H^0[E \cap \phi^{-1}(y,s)]dV(y,s),
\]

where \( H^0 \) is the counting measure.

**Notation.** Let us summarize the notations and definitions that will be used.

- Let \( r > 0, \rho > 0, z_o \in M \) and \( t_o \in \mathbb{R} \). We denote

\[
K_{r,\rho}(z_o,t_o) := B_r(z_o) \times (t_o - \rho, t_o],
\]

where \( B_r(z_o) \) is a geodesic ball of radius \( r \) centered at \( z_o \).

- We denote \( K_{r}(z_o,t_o) := K_{r,\rho}(z_o,t_o) \).

- We say that a constant \( C \) is uniform if \( C \) depends only on \( n, \Lambda, \alpha_\Lambda \) and \( a_L \).

- We denote \( \int_Q f := \frac{1}{\text{Vol}(Q)} \int_Q f \).

- We denote \( |Q| := \text{Vol}(Q) \).

- We denote the trace by \( \text{tr} \).
3. Key Lemma

In this section, we obtain Aleksandrov-Bakelman-Pucci-Krylov-Tso type estimate (Lemma 3.2) for parabolic Harnack inequalities. We begin with direct computation of the Jacobian of the parabolic normal map \( \Phi \) below, which is a parabolic analogue of Lemma 2.1.

**Lemma 3.1.** Let \( v \) be a smooth function in an open set \( K \) of \( M \times \mathbb{R} \). Define the map \( \phi : K \to M \) by

\[
\phi(x, t) := \exp_x \nabla_x v(x, t)
\]

and the map \( \Phi : K \to M \times \mathbb{R} \) by

\[
\Phi(x, t) := \left( \phi(x, t), -\frac{1}{2} d(x, \phi(x, t))^2 - v(x, t) \right).
\]

Let \( (x, t) \in K \) and assume that \( \nabla_x v(x, t) \in E_x \). Set \( y := \phi(x, t) \). Then

\[
\text{Jac } \Phi(x, t) = \text{Jac } \exp_x (\nabla_x v(x, t)) \cdot \left| \left( D_x^2 \left( v + \frac{d^2}{2} \right) \right) \right|.
\]

where \( \text{Jac } \exp_x (\nabla_x v(x, t)) \) denotes the Jacobian of \( \exp_x \) at the point \( \nabla_x v(x, t) \in T_x M \).

**Proof.** We may assume that \( \nabla_x v(x, t) \neq 0 \), which is equivalent to \( x \neq y \). Let \( (\xi, \sigma) \in T_x M \times \mathbb{R} \setminus \{(0,0)\} \) and let \( y = (\gamma_1, \gamma_2) \) be the geodesic with \( y(0) = (x, t) \) and \( y'(0) = (\xi, \sigma) \). We note that \( \gamma_1(\tau) = \exp_x \tau \xi \) and \( \gamma_2(\tau) = t + \sigma \tau \). Set

\[
Y(s, \tau) := \exp_{\gamma_1(\tau)} [s \nabla_x v(\gamma(\tau))].
\]

Consider the family of geodesics (in the parameter \( s \))

\[
\Pi(s, \tau) := \left( Y(s, \tau), \gamma_2(\tau) - s \left\{ \frac{1}{2} d(\gamma_1(\tau), \phi(\gamma(\tau)))^2 + v(\gamma(\tau)) + \gamma_2(\tau) \right\} \right)
\]

that joins \( \Pi(0, \tau) = \gamma(\tau) \) to \( \Pi(1, \tau) = \Phi(\gamma(\tau)) \). Then we define

\[
J(s) := \left. \frac{\partial}{\partial s} \right|_{s=0} \Pi(s, \tau),
\]

which is a Jacobi field along

\[
X(s) := \left( \exp_x (s \nabla_x v(x, t)), t - s \left\{ \frac{1}{2} d(x, \phi(x, t))^2 + v(x, t) + t \right\} \right).
\]

Simple computation says that

\[
J(0) = (\xi, \sigma) \quad \text{and} \quad J(1) = \left. \frac{\partial}{\partial t} \right|_{t=0} \Phi(\gamma(\tau)) = d \Phi(x, t) \cdot (\xi, \sigma).
\]

We also have

\[
D_x J(0) = \left( D_x^2 v(x, t) \xi + \sigma \nabla_x v_x(x, t), \sigma v_x(x, t) - \sigma \right) - \left( \nabla_x \left( d^2/2 \right)(\gamma), d \exp_x (\nabla_x v(x, t)) \cdot \left( D_x^2 \left( v + d^2/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_x(x, t) \right) \right).
\]
In fact, we have

\[
D_x J(0) = D_x \frac{\partial \Pi}{\partial t} \bigg|_{t=0, \tau=0} = D_x \frac{\partial \Pi}{\partial s} \bigg|_{s=0, \tau=0} \\
= D_x \left( \nabla_x v(\gamma(\tau)), -\frac{1}{2} d(\gamma_1(\tau), \phi(\gamma(\tau)))^2 - v(\gamma(\tau)) - \gamma_2(\tau) \right) \\
= \left( D_x^2 v(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t), \right.
\]

\[
- \left( \nabla_x (d^2_x/2)(x, \xi) - \nabla_x (d^2_x/2)(\phi(x, t)), \frac{\partial}{\partial \tau} \phi(\gamma(\tau)) \right)_{\tau=0} = \left( \nabla_x v(x, t), \xi - \sigma v_t - \sigma \right) \\
= \left( D_x^2 v \cdot \xi + \sigma \nabla_x v_t, \right.
\]

since \( \nabla_x (d^2_x/2)(x) = -\exp_x^{-1}(y) = -\nabla_x v(x, t) \). Then we use Lemma 2.1 to obtain

\[
\frac{\partial}{\partial \tau} \phi(\gamma(\tau)) \bigg|_{\tau=0} = d \exp_x(\nabla_x v(x, t)) \cdot \left( D_x^2 \left( v + d^2_x/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right) .
\]

On the other hand, consider the Jacobi field \( J_{\xi, \sigma} \) along \( X(s) \) satisfying

\[
J_{\xi, \sigma}(0) = (\xi, \sigma) \quad \text{and} \quad J_{\xi, \sigma}(1) = (0, 0).
\]

Then we can check that

\[
J_{\xi, \sigma}(s) = \frac{\partial}{\partial t} \psi \bigg|_{t=0} \quad \text{and} \quad D_x J_{\xi, \sigma}(0) = \left( -D_x^2 \left( d^2_x/2 \right)(x) \cdot \xi, -\sigma \right),
\]

where

\[
\Psi(s, \tau) := \left( \exp_{\gamma(t), \tau} s \exp_{\gamma(t), \tau} \phi(x, t), \gamma_2(\tau) - s \left( \frac{1}{2} d(x, \phi(x, t))^2 + v(x, t) + \gamma_2(\tau) \right) \right).
\]

(We refer [Ca, Lemma 3.2] for the proof.) Define \( J_{\xi, \sigma} := J - J_{\xi, \sigma} \). The Jacobi field \( J_{\xi, \sigma} \) along \( X(s) \) satisfying

\[
\tilde{J}_{\xi, \sigma}(0) = (0, 0) \quad \text{and} \quad D_x \tilde{J}_{\xi, \sigma}(0) = D_x J(0) - D_x J_{\xi, \sigma}(0)
\]
is written by

\[
d \exp_{(x,s)}(s X'(0)) \cdot \left( s D_x \tilde{J}_{\xi, \sigma}(0) \right).
\]

Therefore, we have

\[
J(1) = \tilde{J}_{\xi, \sigma}(1) = d \exp_{(x,t)}(\nabla_x v(x, t), -\frac{1}{2} d(x, y)^2 - v(x, t) - t) \cdot \left( D_x J(0) - D_x J_{\xi, \sigma}(0) \right),
\]

which means

\[
d \Phi(x, t) \cdot (\xi, \sigma) = d \exp_{(x,t)}(\nabla_x v(x, t), -\frac{1}{2} d(x, y)^2 - v(x, t) - t) \cdot \left( D_x^2 \left( v + d^2_x/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t), \right.
\]

\[
-\sigma v_t - \left( \nabla_x (d^2_x/2)(y), d \exp_{(x,v(x,t))} D_x^2 \left( v + d^2_x/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right) \\
= \left( d \exp_{(x,v(x,t))} D_x^2 \left( v + d^2_x/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right),
\]

\[
-\sigma v_t - \left( \nabla_x (d^2_x/2)(y), d \exp_{(x,v(x,t))} D_x^2 \left( v + d^2_x/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right).
\]
To calculate the Jacobian of $\Phi$, we introduce an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_xM$ and an orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ of $T_yM = T_{\exp_y \nu_t(x, t)}M$. By setting for $i, j = 1, \cdots, n$,

$$A_{ij} := \langle \tilde{e}_i, \exp_x (\nabla_x \nu_t(x, t) \cdot (D^2 \nu + d_x^2/2)(x) e_j) \rangle,$$

$$b_j := \langle \tilde{e}_i, \exp_x (\nabla_x \nu_t(x, t) \cdot \nabla v_t(x, t) e_j) \rangle,$$

$$c_j := \langle \tilde{e}_i, \nabla_x (d^2_x/2)(y) e_j \rangle,$$

the Jacobian matrix of $\Phi$ at $(x, t)$ is

$$\begin{pmatrix}
A_{ij} & b_i \\
-c_i A_{kj} & -v_i - b_k c_k
\end{pmatrix}.$$

Lastly, we use the row operations to deduce that

$$\text{Jac} \Phi(x, t) = \left| \det \begin{pmatrix} A_{ij} & b_i \\ 0 & -v_i \end{pmatrix} \right| = \left| (-v_i) \det(A_{ij}) \right|.$$ 

This completes the proof. \hfill \Box

The following lemma will play a key role to estimate sublevel sets of $u$ in Lemma 4.3 and then to prove a decay estimate of the distribution function of $u$ in Lemma 6.1. This ABP-type lemma corresponds to [Cal] Lemma 4.1.

**Lemma 3.2.** Suppose that $M$ satisfies the condition (4). Let $z_0 \in M$, $R > 0$, and $0 < \eta < 1$. Let $u$ be a smooth function in $K_{\alpha_1 R, \alpha_2 R}(z_0, 0) \subset M \times \mathbb{R}$ satisfying

$$u \geq 0 \text{ in } K_{\alpha_1 R, \alpha_2 R}(z_0, 0) \backslash K_{\beta_1 R, \beta_2 R}(z_0, 0) \text{ and } \inf_{K_{\alpha_1 R}(z_0, 0)} u \leq 1,$$

where $\alpha_1 := \frac{11}{q}$, $\alpha_2 := 4 + \eta^2 + \frac{\eta^4}{2}$, $\beta_1 := \frac{9}{q}$, and $\beta_2 := 4 + \eta^2$. Then we have

$$|B_R(z_0)| \cdot R^2 \leq C(\eta, n, \lambda) \int_{(x \in M) \cap K_{\alpha_1 R, \alpha_2 R}(z_0, 0)} \left( R^2 \mathcal{L} u + a_L + \Lambda + 1 \right)^{n+1}$$

where the constant $M_\eta > 0$ depends only on $\eta > 0$ and $C(\eta, n, \lambda) > 0$ depends only on $\eta, n$ and $\lambda$.

**Proof.** For any $\overline{y} \in B_R(z_0)$, we define

$$w_\overline{y}(x, t) := \frac{1}{2} R^2 u(x, t) + \frac{1}{2} d^2_x(x) - C_\overline{y} t, \quad C_\overline{y} := \frac{6}{\eta^2},$$

where $\overline{y}$ lies above $z_0$ and $n$ is the dimension of $M$.

**Figure 1.** $\alpha_1 := \frac{11}{q}, \alpha_2 := 4 + \eta^2 + \frac{\eta^4}{2}, \beta_1 := \frac{9}{q}, \beta_2 := 4 + \eta^2$. 

We recall that (10) Jac \( \Phi \) (and hence \( u \)) from below. Thus we have proved that for any \( (x, t) \in K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \backslash K_{\beta_1 R, \beta_2 R^2}(z_o, 0) \).

From the above observation, for any \( (\bar{y}, \bar{t}) \in B_R(z_o) \times \big( A_\eta R^2, (A_\eta + 1) R^2 \big) \), we can find a time \( \bar{t} \in (-\beta_2 R^2, 0) \) such that
\[
\bar{t} = \inf_{B_{\beta_2 R^2}(x) \times (-\beta_2 R^2, \bar{t})} w_T(y, t) = w_T(\bar{y}, \bar{t}).
\]

where the infimum is achieved at an interior point \( \bar{y} \) of \( B_{\beta_2 R^2}(z_o) \). By the same argument as in [Ca] pp. 637-638, we have the following relation:
\[
\bar{y} = \exp_{x} \nabla x \left( \frac{1}{2} R^2 u \right)(\bar{y}, \bar{t}).
\]

Now, we consider the map \( \Phi : K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \to M \times R \) (with \( v(x, t) = \frac{1}{2} R^2 u(x, t) - C_\eta t \) in Lemma 3.1) defined as
\[
\Phi(x, t) := \left\{ \exp_{x} \nabla x \left( \frac{1}{2} R^2 u \right)(x, t), -\frac{1}{2} d \left( x, \exp_{x} \nabla x \left( \frac{1}{2} R^2 u \right)(x, t) \right)^2 - \frac{1}{2} R^2 u(x, t) + C_\eta \right\}.
\]

Define a set
\[
E := \left\{ (x, t) \in K_{\beta_1 R, \beta_2 R^2}(z_o, 0) : \exists y \in B_R(z_o) \text{ s.t. } w_y(x, t) = \inf_{B_{\beta_2 R^2}(x) \times (-\beta_2 R^2, \bar{t})} w_T \leq (A_\eta + 1) R^2 \right\}.
\]

The set \( E \) is a subset of the contact set in \( K_{\beta_1 R, \beta_2 R^2}(z_o, 0) \) that contains a point \( (x, t) \), where a concave paraboloid \(-\frac{1}{2} d^2(x) + C_\eta t + C \) (for some \( C \)) touches \( \frac{1}{2} R^2 u \) from below. Thus we have proved that for any \((y, s) \in B_R(0) \times (-A_\eta - 1) R^2, -A_\eta R^2 \), there is at least one \((x, t) \in E \) such that \((y, s) \) satisfies \( \Phi(x, t) \), namely,
\[
B_R(z_o) \times \big( -A_\eta - 1 R^2, -A_\eta R^2 \big) \subset \Phi(E).
\]

So Area formula gives
\[
\int_{E} \text{Area} \big( E \cap \Phi^{-1}(y, s) \big) dV(y, s) = \int_{E} \text{Jac} \Phi(x, t) dV(x, t).
\]

We notice that for \((x, t) \in E \) and \( y \in B_R(z_o) \), \( w_y(x, t) = \frac{1}{2} R^2 u(x, t) + \frac{1}{2} d^2(x) - C_\eta t \leq (A_\eta + 1) R^2 \) and hence \( u(x, t) \leq 2(A_\eta + 1) =: M_\eta \) for \((x, t) \in E \).

Lastly, we claim that for \((x, t) \in E \),
\[
\text{Jac} \Phi(x, t) \leq \frac{1}{(n+1)^{\nu} \nu!} \left\{ \frac{1}{2} R^2 \mathcal{L} u(x, t) + a_L + \Lambda + C_\eta \right\}^{\nu+1}
\]

Fix \((x, t) \in E \) and \( y \in B_R(z_o) \) to satisfy
\[
w_y(x, t) = \inf_{B_{\beta_2 R^2}(x) \times (-\beta_2 R^2, \bar{t})} w_T.
\]

We recall that \( y = \exp_{x} \nabla x \left( \frac{1}{2} R^2 u \right)(x, t) \) (see [Ca] pp. 637-638).
If \( x \) is not a cut point of \( y \), then Lemma [3.1] (with \( v(x, t) = \frac{1}{2} R^2 u(x, t) - C \)) and Lemma 2.2(i) imply that

\[
\text{Jac } \Phi(x, t) \leq \left| \left( -\frac{1}{2} R^2 u + C \right) \det \left( D_x^2 \left( \frac{1}{2} R^2 u + \frac{1}{2} d^2 \right) \right) \right|.
\]

Since the minimum of \( w_y \) in \( B_{\beta, R}(z_0) \times (-\beta R^2, t) \) is achieved at \( (x, t) \), we have

\[
0 \leq D^2_x w(x, t) = D^2_x \left( \frac{1}{2} R^2 u + \frac{1}{2} d^2 \right) \quad \text{and} \quad 0 \geq \partial_i w(x, t) = \frac{1}{2} R^2 u_i - C_y,
\]

where \( D^2_x w(x, t) \geq 0 \) means that the Hessian of \( w_y \) at \( (x, t) \) is positive semidefinite. Therefore, by using the geometric and arithmetic means inequality, we get

\[
\text{Jac } \Phi(x, t) \leq \left( -\frac{1}{2} R^2 u + C \right) \det \left( D_x^2 \left( \frac{1}{2} R^2 u + \frac{1}{2} d^2 \right) \right) \leq \frac{1}{n+1} \left( \frac{1}{2} R^2 u + C \right) \det \left( D_x^2 \left( \frac{1}{2} R^2 u + \frac{1}{2} d^2 \right) \right) \leq \frac{1}{n+1} \left( \frac{1}{2} R^2 u + a_L + \Lambda + C \right)^{n+1} \leq \frac{1}{n+1} \left( \frac{1}{2} R^2 u + a_L + \Lambda + C \right)^{n+1},
\]

where we used

\[
L \left( d^2 / 2 \right) = d_L d_t + \langle A_{x, t} \nabla d_x, \nabla d_t \rangle \leq a_L + \Lambda | \nabla d_t |^2.
\]

When \( x \) is a cut point of \( y \), we make use of upper barrier technique due to Calabi [Cal]. Since \( y = \exp_x \nabla_x \left( \frac{1}{2} R^2 u \right)(x, t) \), \( x \) is not a cut point of \( y_\sigma := \phi_\sigma(x, t) := \exp_x \nabla_x \left( \frac{\sigma}{2} R^2 u \right)(x, t) \) for \( 0 \leq \sigma < 1 \). Now we consider

\[
\Phi_\sigma(z, \tau) := \left( \phi_\sigma(z, \tau), -\frac{\sigma^2}{2} R^2 u(z, \tau) - \frac{1}{2} d \langle z, \phi_\sigma(z, \tau) \rangle^2 + C \right)
\]

instead of \( \Phi \) since \( \text{Jac } \Phi(x, t) = \lim_{\sigma \rightarrow 1} \text{Jac } \Phi_\sigma(x, t) \). As before, we have

\[
\text{Jac } \Phi_\sigma(x, t) \leq \left| \left( -\frac{\sigma}{2} R^2 u + C \right) \det \left( D_x^2 \left( \frac{\sigma}{2} R^2 u + \frac{1}{2} d^2 \right) \right) \right|.
\]

We note that

\[
\lim_{\sigma \rightarrow 1} \left| \left( -\frac{\sigma}{2} R^2 u + C \right) \det \left( D_x^2 \left( \frac{\sigma}{2} R^2 u + \frac{1}{2} d^2 \right) \right) \right| = \lim_{\sigma \rightarrow 1} \left| \left( -\partial_i w_y \right) \det \left( D^2_x w_y \right) \right|
\]
for \( w_{y_\tau}(z, \tau) := \frac{1}{2} R^2 u(z, \tau) + \frac{1}{2} d_{y_\tau}(z) - C_\eta \tau \). According to the triangle inequality, we have
\[
w_{y_\tau}(z, \tau) \leq \frac{1}{2} R^2 u(z, \tau) + \frac{1}{2} \left( d_{y_\tau}(z) + d(y_{\tau, y}, y) \right)^2 - C_\eta \tau
\]
\[
= w_{y_\tau}(z, \tau) + d(y_{\tau, y}) d_{y_\tau}(z) + \frac{1}{2} d(y_{\tau, y})^2,
\]
where the equality holds at \((z, \tau) = (x, t)\). Since \( w_y \) has the minimum at \((x, t)\) in \( B_{\beta, R}(z_0) \times (-\beta_2 R^2, t] \), the minimum of \( w_{y_\tau}(z, \tau) + d(y_{\tau, y}) d_{y_\tau}(z) \) (in \( B_{\beta, R}(z_0) \times (-\beta_2 R^2, t] \)) is also achieved at \((x, t)\), that implies that
\[
D_y^2 \left( w_{y_\tau} + d(y_{\tau, y}) d_{y_\tau} \right)(x, t) \geq 0, \quad \partial_t w_{y_\tau}(x, t) \leq 0.
\]
To bound \( D^2 y_{\tau}(x) \) uniformly in \( \sigma \in [1/2, 1] \), we recall the Hessian comparison theorem (see [S], [SY]): Let \(-k^2 (k > 0)\) be a lower bound of sectional curvature along the minimal geodesic joining \( x \) and \( y \). Then for \( 0 < \sigma < 1 \),
\[
D^2 y_{\tau}(x) \leq k \cosh(kd_{y_\tau}(x))Id
\]
and hence we find a constant \( N > 0 \) independent of \( \sigma \) such that
\[
D^2 y_{\tau}(x) \leq NId \quad \text{for} \quad \frac{1}{2} \leq \sigma < 1.
\]
Following the above argument, for \( \frac{1}{2} \leq \sigma < 1 \), we obtain
\[
0 \leq \liminf_{\sigma \uparrow 1} \left( -\partial_t w_{y_\tau}(x, t) \right) \det \left( D^2 y_{\tau} + d(y_{\tau, y}) D^2 y_{\tau} \right)(x, t)
\]
\[
\leq \liminf_{\sigma \uparrow 1} \left( -\partial_t w_{y_\tau}(x, t) \right) \det \left( D^2 y_{\tau} + d(y_{\tau, y}) NId \right)(x, t)
\]
\[
\leq \liminf_{\sigma \uparrow 1} \frac{1}{(n + 1)^{n+1}} \left\{ \frac{1}{2} R^2 \mathcal{L} u + a_L + \Lambda + C_\eta + d(y_{\tau, y}) n \Lambda N \right\}^{n+1}
\]
\[
\leq \frac{1}{(n + 1)^{n+1}} \left\{ \left( \frac{1}{2} R^2 \mathcal{L} u + a_L + \Lambda + C_\eta \right) \right\}^{n+1}.
\]
Then we deduce that
\[
\text{Jac } \Phi(x, t) \leq \frac{1}{(n + 1)^{n+1}} \left\{ \left( \frac{1}{2} R^2 \mathcal{L} u(x, t) + a_L + \Lambda + C_\eta \right) \right\}^{n+1}
\]
since
\[
\liminf_{\sigma \uparrow 1} \left| \det \left( D^2 y_{\tau} \right)(x, t) \right| = \liminf_{\sigma \uparrow 1} \left| \det \left( D^2 y_{\tau} + d(y_{\tau, y}) NId \right)(x, t) \right|
\]
\[
= \liminf_{\sigma \uparrow 1} \det \left( D^2 y_{\tau} + d(y_{\tau, y}) NId \right)(x, t).
\]
We conclude that (10) is true for \((x, t) \in E\). Therefore the estimate (8) follows from (9) since \( E \subset \{ u \leq M_\sigma \} \cap K_{\beta_1 R, \beta_2 R}(z_0, 0) \).

4. Barrier functions

We modify the barrier function of [W] to construct a barrier function in the Riemannian case. First, we fix some constants that will be used frequently (see Figure 1): for a given \( 0 < \eta < 1 \),
\[
\alpha_1 := \frac{11}{\eta}, \quad \alpha_2 := 4 + \eta^2 + \frac{\eta^3}{4}, \quad \beta_1 := \frac{9}{\eta} \quad \text{and} \quad \beta_2 := 4 + \eta^2.
\]
Lemma 4.1. Suppose that $M$ satisfies the condition (4). Let $z_o \in M$, $R > 0$ and $0 < \eta < 1$. There exists a continuous function $v_o(x, t)$ in $K_{\alpha R^2}$, which is smooth in $(M \setminus \text{Cut}(z_o)) \cap K_{\alpha R^2}$ such that

(i) $v_o(x, t) \geq 0$ in $K_{\alpha R^2} \setminus K_{\alpha R^2} \setminus K_{\alpha R^2}$,

(ii) $v_o(x, t) \leq 0$ in $K_{\alpha R^2} \setminus K_{\alpha R^2}$,

(iii) $R^2 L v_o + a_L + \Lambda + 1 \leq 0$ a.e. in $K_{\alpha R^2} \setminus K_{\alpha R^2} \setminus K_{\alpha R^2}$,

(iv) $R^2 L v_o \leq C_o$ a.e. in $K_{\alpha R^2} \setminus K_{\alpha R^2}$,

(v) $v_o(x, t) \geq -C_o$ in $K_{\alpha R^2} \setminus K_{\alpha R^2}$.

Here, the constant $C_o > 0$ depends only on $\eta, n, \alpha, a_L$ (independent of $R$ and $z_o$).

Proof. Fix $0 < \eta < 1$. Consider

$$h(s, t) := -A e^{-mt} \left( 1 - \frac{s}{\beta^2} \right)^\frac{1}{4} \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{a}{t} \frac{s}{\beta^2} \right)$$

for $t > 0$, as in Lemma 3.22 of [W] and define

$$\psi(s, t) := h(s, t) + (a_L + \Lambda + 1)t \quad \text{in } [0, \beta^2] \times [0, \beta^2] \times [0, \frac{\eta}{R}],$$

where the positive constants $A, m, \alpha, \beta$ (depending only on $\eta, n, \alpha, a_L$) will be chosen later. In particular, $l$ will be an odd number in $\mathbb{N}$. We extend $\psi$ smoothly in $[0, \alpha^2] \times [-\frac{\eta}{R}, \beta^2]$ to satisfy

$$\psi \geq 0 \quad \text{on } [0, \alpha^2] \times [-\frac{\eta}{R}, \beta^2] \times [0, \beta^2],$$

$$\psi \geq -C_o \quad \text{on } [0, \alpha^2] \times [-\frac{\eta}{R}, \beta^2],$$

and

$$\sup_{[0, \alpha^2] \times [0, \beta^2]} \left[ 2a_L \| \partial_s \psi \| + \Lambda (2 \| \partial_t \psi \| + 4 s \| \partial_s \psi \|) + \| \partial_t \psi \| \right] < C_o$$

for some $C_o > 0$. We also assume that $\psi(s, t)$ is nondecreasing with respect to $s$ in $[0, \alpha^2] \times [-\frac{\eta}{R}, \beta^2]$. We define

$$v_o(x, t) = v(x, t) := \psi \left( \frac{d_{z_o}^2(x)}{R^2}, \frac{t}{R^2} \right) \quad \text{for } (x, t) \in K_{\alpha R^2 \setminus \alpha R^2 \setminus \alpha R^2},$$

where $d_{z_o}$ is the distance function to $z_o$. Properties (i) and (v) are trivial.

We denote $d_{z_o}(x)$ and $h\left( \frac{d_{z_o}^2(x)}{R^2}, \frac{t}{R^2} \right)$ by $d(x)$ and $\phi(x, t)$ for simplicity and we notice that

$$v(x, t) = h \left( \frac{d^2(x)}{R^2}, \frac{t}{R^2} \right) + (a_L + \Lambda + 1) \frac{t}{R^2} = \phi(x, t) + (a_L + \Lambda + 1) \frac{t}{R^2}$$

and $\phi(x, t)$ is negative in $K_{\alpha R^2 \setminus \alpha R^2 \setminus \alpha R^2}$.

Now, we claim that

$$L \phi \leq 0 \quad \text{a.e. in } K_{\alpha R^2 \setminus \alpha R^2 \setminus \alpha R^2 \setminus \alpha R^2}.$$

Once (11) is proved, then property (iii) follows from the simple calculation that $R^2 L \left[ (a_L + \Lambda + 1) \frac{t}{R^2} \right] = -(a_L + \Lambda + 1)$ in $K_{\alpha R^2 \setminus \alpha R^2 \setminus \alpha R^2 \setminus \alpha R^2}$. Now we use the identity

$$L \left[ \psi(u(x, t)) \right] = \partial_s \phi(u, t) L u + \partial_{uu} \phi(u, t) \partial_s \phi(u, t),$$

where $\partial_s \phi(u, t) = \partial_s \left( \psi \left( \frac{d^2(u)}{R^2}, \frac{t}{R^2} \right) \right).$
to obtain
\[ \mathcal{L} \phi = \frac{2d}{R^2} \frac{\partial}{\partial t} \left( \frac{d^2}{R^2}, \frac{t}{R^2} \right) Ld + \left( \frac{2}{R^2} \frac{\partial}{\partial t} + \frac{4d^2}{R^2} \frac{\partial}{\partial x_i} \right) \left( \frac{d^2}{R^2}, \frac{t}{R^2} \right) (A \phi \nabla d, \nabla d) - \frac{1}{R^2} \frac{\partial}{\partial t} \left( \frac{d^2}{R^2}, \frac{t}{R^2} \right). \]

Since \( d \cdot Ld \leq a_L \) and \( \lambda \leq (A \phi \nabla d, \nabla d) \leq \Lambda \) in \( M \setminus \text{Cut}(z_0) \), we have that
\[ \frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathcal{L} \phi \]
\[ = (\beta_1^2 R^2 - d^2) \left( 2l + (\beta_1^2 R^2 - d^2) \frac{2\alpha}{t} \right) d Ld \]
\[ - \left( (l - 1)4d^2 + 2(l \beta_1^2 R^2 - d^2) \frac{4\alpha d^2}{t} + (\beta_1^2 R^2 - d^2)^2 \frac{4\alpha d^2}{t^2} \right) (A \phi \nabla d, \nabla d) \]
\[ + (\beta_1^2 R^2 - d^2) \left( 2l + (\beta_1^2 R^2 - d^2) \frac{2\alpha}{t} \right) (A \phi \nabla d, \nabla d) \]
\[ + (\beta_1^2 R^2 - d^2) \frac{\alpha d^2}{t^2} - (\beta_1^2 R^2 - d^2)^2 \left( \frac{n}{2l} + \frac{m}{R^2} \right) \]
\[ \leq 2l(\beta_1^2 R^2 - d^2)(a_L + \Lambda) + (\beta_1^2 R^2 - d^2)^2 \left( \frac{2\alpha}{t}(a_L + \Lambda) + \frac{\alpha d^2}{t^2} \right) \]
\[ - l(l - 1)4d^2 \lambda - (\beta_1^2 R^2 - d^2)^2 \left( \frac{4\alpha d^2}{t^2} - \lambda + \frac{n}{2l} \right) \]
\[ - 2l(\beta_1^2 R^2 - d^2) \frac{4\alpha d^2}{t^2} \lambda - \frac{m}{R^2}(\beta_1^2 R^2 - d^2)^2 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R}(z_0, \beta_2 R^2). \]

By choosing
\[ \alpha := \frac{1}{4l}, \quad \frac{2\beta_1^2}{\eta^2} (a_L + \Lambda) + 1 \leq I := 2l' + 1 \quad \text{(for some } l' \in \mathbb{N}) \],
(12)
\[ m := 2 \cdot \max \left\{ \frac{8\alpha}{\eta^2} (a_L + \Lambda), \frac{2l(a_L + \Lambda)}{\beta_1^2 - \frac{\eta^2}{4}} \right\}, \]
we deduce
\[ \frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathcal{L} \phi \leq 0 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R}(z_0, \beta_2 R^2) \setminus K_{2R}(z_0, \frac{\eta^2}{4} R^2). \]

Indeed, we divide the domain \( K_{\beta_1 R, \beta_2 R}(z_0, \beta_2 R^2) \setminus K_{2R}(z_0, \frac{\eta^2}{4} R^2) \) into three regions such that
\[ K_{\beta_1 R, \beta_2 R}(z_0, \beta_2 R^2) \setminus K_{2R}(z_0, \frac{\eta^2}{4} R^2) =: A_1 \cup A_2 \cup A_3, \]
where \( A_1 := \{ 0 \leq \frac{l}{R^2} \leq \frac{\eta^2}{4} \leq \frac{d}{R} \leq \beta_1 \}, A_2 := \{ \frac{\eta^2}{4} \leq \frac{l}{R^2} \leq \beta_2, \frac{d}{R} \leq \beta_1 \} \) and \( A_3 := \{ \frac{\eta^2}{4} \leq \frac{l}{R^2} \leq \beta_2, 0 \leq \frac{d}{R} \leq \frac{\eta^2}{2} \} \). We can check that
\[ \frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathcal{L} \phi \leq 0 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R}(z_0, \beta_2 R^2) \setminus K_{2R}(z_0, \frac{\eta^2}{4} R^2) \]
by choosing \( \alpha \) and \( I \) large in \( A_1 \), \( m \) large in \( A_2 \) and \( A_3 \) as in (12). Therefore, we have proved (11).
From the assumption on \( \psi \), we have that for a.e. \((x, t) \in K_{\beta_{1}, R, R, 1}(z_{0}, \beta_{2} R^{2})\),
\[
R^{2} \langle \psi(x, t) \rangle = 2 \partial_{t} \psi \left( \frac{d^{2}}{R^{2}}, \frac{t}{R^{2}} \right) d \cdot L d + \left( 2 \partial_{t} \psi + 4 \frac{d^{2}}{R^{2}} \partial_{x} \psi \right) \left( \frac{d^{2}}{R^{2}}, \frac{t}{R^{2}} \right) \langle A_{\infty} \nabla d, \nabla d \rangle - \partial_{t} \psi \left( \frac{d^{2}}{R^{2}}, \frac{t}{R^{2}} \right) \leq \sup_{[0, \eta]} (2 \alpha_{1} \partial_{t} \psi + \Lambda (2 \partial_{t} \psi + 4 \partial_{x} \psi) + \partial_{t} \psi) (s, t) < C_{\eta}.
\]
This proves property (iv).

In order to show (ii), we take \( A > 0 \) large enough so that for \((x, t) \in K_{2R}(z_{0}, \beta_{2} R^{2})\),
\[
v(x, t) \leq -A e^{-\beta_{1}m} \left( 1 - \frac{4}{\beta_{1}^{2}} \right) \left( \frac{1}{(4 \pi \beta_{2})^{n/2}} e^{-4a_{L}/\eta^{2}} + (a_{L} + \Lambda + 1) \beta_{2} \right) \leq 0.
\]
This finishes the proof of the lemma. \( \square \)

Now we apply Lemma 3.2 to \( u + v_{\eta} \) with \( v_{\eta} \) constructed in Lemma 4.1 and translated in time. Since the barrier function \( v_{\eta}(x, t) = \psi_{\eta} \left( \frac{d_{\infty}(x)}{R}, \frac{t}{R} \right) \) is not smooth on \( \text{Cut}(z_{0}) \), we need to approximate \( v_{\eta} \) by a sequence of smooth functions as Cabré’s approach at \( \text{Ca} \). We recall that the cut locus of \( z_{0} \) is closed and has measure zero. It is not hard to verify the following lemma and we just refer to \( \text{Ca} \) Lemmas 5.3, 5.4.

**Lemma 4.2.** Let \( z_{0} \in M, R > 0 \) and let \( \psi : \mathbb{R}^{n} \times [0, T] \rightarrow \mathbb{R} \) be a smooth function such that \( \psi(s, t) \) is nondecreasing with respect to \( s \) for any \( t \in [0, T] \). Let \( v(x, t) := \psi \left( \frac{d_{\infty}(x)}{R}, t \right) \).

Then there exist a smooth function \( 0 \leq \zeta(x) \leq 1 \) on \( M \) satisfying
\[
\zeta \equiv 1 \quad \text{in} \quad B_{\beta_{1}}(z_{0}) \quad \text{and} \quad \text{supp} \ \zeta \subset B_{\frac{\beta_{1}}{2}}(z_{0})
\]
and a sequence \( \{w_{k}\}_{k=1}^{\infty} \) of smooth functions in \( M \times [0, T] \) such that
\[
\begin{align*}
w_{k} & \to \zeta v \quad \text{uniformly in} \ M \times [0, T], \\
\partial_{t} w_{k} & \to \zeta \partial_{t} v \quad \text{uniformly in} \ M \times [0, T], \\
D_{x}^{2} w_{k} & \leq C \text{Id} \quad \text{in} \ M \times [0, T], \\
D_{x}^{2} w_{k} & \to D_{x}^{2} v \quad \text{a.e. in} \ B_{\beta_{1}}(z_{0}) \times [0, T],
\end{align*}
\]
where the constant \( C > 0 \) is independent of \( k \).

**Lemma 4.3.** Suppose that \( M \) satisfies the conditions (3), (4). Let \( z_{0} \in M, R > 0 \), and \( 0 < \eta < 1 \). Let \( u \) be a smooth function such that \( \mathcal{L} u \leq f \) in \( K_{\alpha_{1}, R, \alpha_{2}, R}(z_{0}, 4 R^{2}) \) such that
\[
u \geq 0 \quad \text{in} \quad K_{\alpha_{1}, R, \alpha_{2}, R}(z_{0}, 4 R^{2}) \setminus K_{\beta_{1}, R, \beta_{2}, R}(z_{0}, 4 R^{2})
\]
and
\[
\inf_{K_{\alpha_{1}, R, \alpha_{2}, R}(z_{0}, 4 R^{2})} u \leq 1.
\]
Then, there exist uniform constants \( M_{\eta} > 1, 0 < \mu_{\eta} < 1 \), and \( 0 < \epsilon_{\eta} < 1 \) such that
\[
\left\| u - M_{\eta} \right\|_{K_{\alpha_{1}, R, \alpha_{2}, R}(z_{0}, 0)} \geq \mu_{\eta},
\]
provided
\[
R^{2} \left( \int_{K_{\alpha_{1}, R, \alpha_{2}, R}(z_{0}, 4 R^{2})} |f + \mu_{\eta}|^{p+1} \right)^{\frac{1}{p+1}} \leq \epsilon_{\eta},
\]
where \( M_{\eta} > 0, 0 < \epsilon_{\eta}, \mu_{\eta}, \eta \) depend only on \( \eta, n, \lambda, \Lambda \) and \( a_{L} \).
Proof. Let \( v_\eta \) be the barrier function in Lemma 4.1 after translation in time (by \(-\eta^2 R^2\)) and let \( \{w_k\}_{k=1}^{\infty} \) be a sequence of smooth functions approximating \( v_\eta \) as in Lemma 4.2. We notice that \( u + v_\eta \geq 0 \) in \( K_{z_0,R,R^2} (z_0, 4R^2) \setminus K_{\beta_1,R,R^2} (z_0, 4R^2) \) and \( \inf_{K_{z_0,R,R^2} (z_0, 4R^2)} (u + v_\eta) \leq 1 \).

Thanks to the uniform convergence of \( w_k \) to \( v_\eta \), we consider a sequence \( \{w_k\}_{k=1}^{\infty} \) converging to 0 such that \( \sup_{K_{z_0,R,R^2} (z_0, 4R^2)} w_k \leq \varepsilon_k \) and

\[
w_k \geq -\varepsilon_k \quad \text{in} \quad K_{z_0,R,R^2} (z_0, 4R^2) \setminus K_{\beta_1,R,R^2} (z_0, 4R^2),
\]

and define

\[
\overline{w}_k := \frac{u + w_k + \varepsilon_k}{1 + 2\varepsilon_k}.
\]

Then \( \overline{w}_k \) satisfies the hypotheses of Lemma [3.2] (after translation in time by \( 4R^2 \)). Now we replace \( u \) by \( \overline{w}_k \) in (8) and then the uniform convergence implies that for a given \( 0 < \delta < 1 \), we have

\[
|B_R(z_0)|R^2 \leq C(\eta, n, \lambda) \int_{|u + v_\eta| \leq M_\eta} \left( R^2 |\mathcal{L} \overline{w}_k + a_L + \Lambda + 1| \right)^{n+1}
\]

if \( k \) is sufficiently large. Since \( D_1^2 \overline{w}_k \leq C |\overline{w}_k| \) and \( |\overline{w}_k| \leq C \) uniformly in \( k \) on \( K_{\beta_1,R,R^2} (z_0, 4R^2) \), we use the dominated convergence theorem to let \( k \) go to \( +\infty \). Letting \( \delta \) go to 0, we obtain

\[
|B_R(z_0)|R^2 \leq C(\eta, n, \lambda) \int_{|u + v_\eta| \leq M_\eta} \left( R^2 |\mathcal{L} \overline{w}_k + a_L + \Lambda + 1| \right)^{n+1} = C(\eta, n, \lambda) \int_{E_1} \left( R^2 |\mathcal{L} \overline{w}_k + a_L + \Lambda + 1| \right)^{n+1},
\]

where \( E_1 := \{u + v_\eta \leq M_\eta\} \cap (K_{\beta_1,R,R^2} (z_0, 4R^2) \setminus K_{\eta R} (z_0, 0)) \) and \( E_2 := \{u + v_\eta \leq M_\eta\} \cap K_{\eta R} (z_0, 0) \). From properties (iii) and (iv) of \( \eta \) in Lemma 4.1 and Bishop's volume comparison theorem in Lemma 2.2, we deduce that

\[
|K_{1, R, R^2} (z_0, 4R^2)|^{\frac{1}{n+1}} \leq C_\eta \left( |R^2 \mathcal{L} u|^{\frac{1}{n+1}} \right)^{E_1} + C_\eta \left| \int_{E_2} x \mathcal{L} \overline{w}_k \right|_{n+1} (K_{\beta_1,R,R^2} (z_0, 4R^2)) + C_\eta \left| u + v_\eta \leq M_\eta \right| \cap K_{\eta R} (z_0, 0) \right|^{\frac{1}{n+1}},
\]

where \( C_\eta > 0 \) depends only on \( n, \lambda \) and \( \eta > 0 \). We note that \( \{u \leq M_\eta - v_\eta\} \subset \{u \leq M_\eta + C_\eta\} \) from (v) in Lemma 4.1. Therefore, by taking

\[
\varepsilon_\eta = \frac{1}{2C_\eta}, \quad M_\eta = M_\eta + C_\eta \quad \text{and} \quad \mu_\eta^{\frac{1}{n+1}} = \frac{1}{2C_\eta},
\]

we conclude that

\[
\frac{\{u \leq M_\eta\} \cap K_{\eta R} (z_0, 0)}{|K_{1, R, R^2} (z_0, 4R^2)|} \geq \mu > 0.
\]

Using iteration of Lemma 4.3, we have the following corollaries.

**Corollary 4.4.** Suppose that \( M \) satisfies the conditions [3], [4]. Let \( z_0 \in M \) and \( 0 < \eta < 1 \). For \( i \in \mathbb{N} \), let \( R_i := \left( \frac{\varepsilon_\eta}{\varepsilon_\eta} \right)^{-1} R \) and \( \tilde{t}_i := \sum_{j=1}^{k} 4R_j^2 \). Let \( u \) be a nonnegative smooth function such that \( \mathcal{L} u \leq f \) in \( \bigcup_{i=1}^{k} K_{1, R, R^2} (z_0, \tilde{t}_i) \) for some \( k \in \mathbb{N} \). We assume that for \( h > 0 \),
\[ \inf_{u \in K_{R^2}(z, \delta)} u \leq h \text{ and} \]
\[ R^2 \left( \int_{K_{R^2}(z, \delta)} |f^{+}|^{p+1} \right)^{\frac{1}{p+1}} \leq \varepsilon h M_{k+1}^{1}, \quad \forall 1 \leq i \leq k. \]

Then we have
\[ (15) \]
\[ \frac{\left| \{ u \leq h M_{k} \} \cap K_{R}(z_0, 0) \right|}{K_{R, R^2}(z_0, 4R^2)} \geq \mu_{\eta}, \]
where \( M_{k}, \varepsilon_{\eta}, \mu_{\eta} \) are the same uniform constants as in Lemma 4.3.

**Proof.** We may assume \( h = 1 \) since \( v := \frac{u}{h} \) satisfies \( \mathcal{L}v = \frac{1}{h} \mathcal{L}u \leq \frac{1}{h} \). We use the induction on \( k \) to show the lemma. When \( k = 1 \), it is immediate from Lemma 4.3.

Now suppose that (15) is true for \( k - 1 \). By assumption, we find a \( j_{0} \in \mathbb{N} \) such that \( 1 \leq j_{0} \leq k \) and
\[ \inf_{K_{R^2}(z, \delta)} u = \inf_{K_{R^2}(z, \delta)} u \leq 1. \] Define \( v := u/M_{k-j_{0}}^{1} \). Then \( v \) satisfies that
\[ \mathcal{L}v \leq f/M_{k-j_{0}}^{1}, \quad \inf_{K_{R^2}(z, \delta)} v \leq 1 \]

Applying Lemma 4.3 to \( v \) in \( K_{R, R^2}(z, \delta) \), we deduce
\[ \frac{\left| v \leq M_{k} \right| \cap K_{R^2}(z, \delta) - 4R^2}{K_{R^2}(z, \delta)} \geq \frac{\left| v \leq M_{k} \right| \cap K_{R^2}(z, \delta)}{K_{R, R^2}(z_0, 4R^2)} \geq \mu_{\eta} > 0 \]

which implies \( \inf_{K_{R^2}(z, \delta)} u \leq \inf_{K_{R, R^2}(z_0, 4R^2)} u \leq M_{k-J_{0}+1}^{1} \). Therefore, we use the induction hypothesis for \( j_{0} - 1 \) to conclude
\[ \frac{\left| u/M_{k-j_{0}+1}^{1} \leq M_{k-J_{0}+1}^{1} \right| \cap K_{R}(z_0, 0)}{K_{R, R^2}(z_0, 4R^2)} \geq \mu_{\eta} > 0, \]
which implies (15). \( \square \)

We remark that Lemma 4.3 and Corollary 4.4 hold for any \( M_{k} \geq M_{\eta} \). The following is a simple technical lemma that will be used in the proof of Proposition 4.6.

**Lemma 4.5.** Let \( A, D > 0 \) and \( \varepsilon > 0 \). Let \( u \) be a nonnegative smooth function such that \( \mathcal{L}u \leq f \) in \( B_{R}(z_0) \times (AR^2, 0] \) with
\[ R^2 \left( \int_{B_{R}(z_0) \times (-AR^2, 0]} |f^{+} |^{p+1} \right)^{\frac{1}{p+1}} \leq \varepsilon. \]

Then, there exists a sequence \( u_k \) of nonnegative smooth functions in \( B_{R}(z_0) \times (-AR^2, DR^2] \) such that \( u_k \) converges to \( u \) locally uniformly in \( B_{R}(z_0) \times (AR^2, 0] \) and \( \mathcal{L}u_k \leq g_k \) in \( B_{R}(z_0) \times (-AR^2, DR^2] \) with
\[ R^2 \left( \int_{B_{R}(z_0) \times (-AR^2, DR^2]} |g_k^{+} |^{p+1} \right)^{\frac{1}{p+1}} \leq \varepsilon. \]
Proof. First, we define for \((x, t) \in B_R(z_0) \times (-\infty, DR^2)\),
\[
\overline{u}(x, t) := \begin{cases} 
0 & \text{for } t \in (-\infty, -AR^2], \\
u(x, t) & \text{for } t \in (-AR^2, 0), \\
u(x, 0) + St & \text{for } t \in (0, DR^2], 
\end{cases}
\]
where \(S := \sup_{B_R(z_0)} \{(\mathcal{L}u)^+ + |u_t(x, 0)|\}\). Then \(\overline{u}\) is Lipschitz continuous with respect to time in \(B_R(z_0) \times (-AR^2, DR^2]\) and satisfies
\[
\mathcal{L}\overline{u}(x, t) \leq \overline{f}(x, t) := \begin{cases} 
0 & \text{for } t \in (-\infty, -AR^2), \\
f(x, t) & \text{for } t \in (-AR^2, 0), \\
f(x, 0) + u_t(x, 0) - S \leq 0 & \text{for } t \in (0, DR^2].
\end{cases}
\]
Let \(\varepsilon_k > 0\) converge to 0 as \(k \to +\infty\), and let \(\varphi\) be a nonnegative smooth function such that \(\varphi(t) = 0\) for \(t \notin (0, 1)\) and \(\int_{\mathbb{R}} \varphi(t) dt = 1\). We define \(\varphi_k(t) := \frac{1}{\varepsilon_k} \varphi \left( \frac{t}{\varepsilon_k} \right)\) and
\[
u_k(x, t) := \int_{\mathbb{R}} \overline{u}(x, s) \varphi_k(t - s) ds, \quad \forall (x, t) \in B_R(z_0) \times (-\infty, DR^2],
\]
where we notice that the above integral is calculated over \((t - \varepsilon_k, t) \subset \mathbb{R}\). Then, a smooth function \(\nu_k\) satisfies
\[
\mathcal{L}\nu_k(x, t) = \int_{\mathbb{R}} \mathcal{L}\overline{u}(x, s) \varphi_k(t - s) ds \leq g_k(x, t), \quad \forall (x, t) \in B_R(z_0) \times (-\infty, DR^2],
\]
where \(g_k(x, t) := \int_{\mathbb{R}} \overline{f}(x, s) \varphi_k(t - s) ds \geq 0\). We also have
\[
R^2 \left( \int_{B_R(z_0) \times (-AR^2, DR^2]} \left|g_k^+ + 1\right|^\frac{1}{\alpha} \right)^\frac{1}{\beta} \leq \frac{R^2}{[B_R(z_0)] \cdot (A + D)R^2} \left\|\overline{f}^+ \right\|_{L^{\alpha+1}(B_R(z_0) \times (-AR^2 - \varepsilon_k, DR^2 - \varepsilon_k))} \leq \frac{R^2}{[B_R(z_0)] \cdot (A + D)R^2} \left\|\overline{f}^+ \right\|_{L^{\alpha+1}(B_R(z_0) \times (-AR^2, 0))} \leq \left( \frac{A}{A + D} \right)^{\frac{1}{\alpha+1}} \varepsilon < \varepsilon,
\]
which finishes the proof. \(\square\)

**Proposition 4.6.** Suppose that \(M\) satisfies the conditions \([3], [4]\). Let \(z_0 \in M, R > 0, 0 < \eta < \frac{1}{2}\) and \(\tau \in [3, 16]\). Let \(u\) be a nonnegative smooth function such that \(\mathcal{L}u \leq f\) in \(B_{\frac{R^2}{\tau}}(z_0) \times \left(-3R^2, \frac{R^2}{\tau}\right]\). Assume that \(\inf_{B_R(z_0) \times \left(-3R^2, \frac{R^2}{\tau}\right]} u \leq 1\) and
\[
R^2 \left( \int_{B_{\frac{R^2}{\tau}}(z_0) \times \left(-3R^2, \frac{R^2}{\tau}\right]} \left|f^+ + 1\right|^\frac{1}{\alpha} \right)^\frac{1}{\beta} \leq \varepsilon' \eta
\]
for a uniform constant \(0 < \varepsilon' \eta < 1\). Let \(r > 0\) satisfy \(\left(\frac{r}{R}\right)^N R \leq r < \left(\frac{r}{R}\right)^{N-1} R\) for some \(N \in \mathbb{N}\) and let \((z_1, t_1)\) be a point such that \(d(z_0, z_1) < R\) and \(|t_1| < R^2\). Then there exists a uniform constant \(M' > 1\) (independent of \(r, N, z_1\) and \(t_1\)) such that
\[
\left[\left|u \leq M'^{N+2}\right] \cap K_{\alpha r, \alpha r^2}(z_1, t_1)\right] \geq \mu > 0,
\]
where \(0 < \mu_\eta < 1\) is the constant in Lemma 4.3.

**Proof.** (i) From Lemma 4.5, we approximate \(u\) by nonnegative smooth functions \(u_k\), which are defined on \(B_{\frac{\mu_\eta}{3}}(z_0) \times (-3R^2, \frac{64R^2}{(4-\eta^2)^2} + R^2)\). We find functions \(u_k\) and \(g_k\) such that \(u_k\) converges locally uniformly to \(u\) in \(B_{\frac{\mu_\eta}{3}}(z_0) \times (-3R^2, \frac{64R^2}{(4-\eta^2)^2} + R^2)\), and satisfies

\[
\mathcal{L} u_k \leq g_k \quad \text{in} \quad B_{\frac{\mu_\eta}{3}}(z_0) \times (-3R^2, \frac{64R^2}{(4-\eta^2)^2} + R^2),
\]

and

\[
R^2 \left( \int_{B_{\frac{\mu_\eta}{3}}(z_0) \times (-3R^2, \frac{64R^2}{(4-\eta^2)^2} + R^2)} |g_k^{\# +1}|^{\frac{1}{\alpha - 1}} \right)^{\frac{\alpha - 1}{\alpha}} \leq \left( \frac{49}{48} \right)^4 \eta^2 < 2 \epsilon_\eta'
\]

by using the volume comparison theorem and Lemma 4.5. For a small \(\delta > 0\), we consider \(w_k := \frac{u_k}{1 + \delta}\) and then for large \(k\), \(w_k\) satisfies

\[
\inf_{B_{\frac{\mu_\eta}{3}}(z_0) \times [-R, R]} w_k \leq 1, \quad \mathcal{L} w_k \leq g_k \quad \text{in} \quad B_{\frac{\mu_\eta}{3}}(z_0) \times (-3R^2, \frac{64R^2}{(4-\eta^2)^2} + R^2),
\]

and

\[
R^2 \left( \int_{B_{\frac{\mu_\eta}{3}}(z_0) \times (-3R^2, \frac{64R^2}{(4-\eta^2)^2} + R^2)} |g_k^{\# +1}|^{\frac{1}{\alpha - 1}} \right)^{\frac{\alpha - 1}{\alpha}} < 2 \epsilon_\eta',
\]

according to the local uniform convergence of \(u_k\) to \(u\) in Lemma 4.5. So if we show the proposition for \(w_k\), the local uniform convergence will imply that the result holds for \(u\) by letting \(k \to +\infty\) and \(\delta \to 0\). Now we assume that \(u\) is a nonnegative smooth function in \(B_{\frac{\mu_\eta}{3}}(z_0) \times (-3R^2, \frac{64R^2}{(4-\eta^2)^2} + R^2)\) satisfying the same hypotheses as \(w_k\).

(ii) We use Corollary 4.4 so we need to check the two hypotheses with \(k = N + 2\) and \(h = 1\). As in the corollary, we define for \(i \in \mathbb{N}\),

\[
\tilde{t}_i := \left( \frac{2^{i-1}}{\eta} \right) r \quad \text{and} \quad \tilde{t}_i := t_1 + \sum_{j=1}^i 4R^2.
\]

Using the conditions on \(r, z_1\), and \(t_1\), simple computation says that for \(0 < \eta < 1/2\),

\[
B_{2R}(z_1) \supset B_{2R}(z_0) \supset B_R(z_0),
\]

\[
\tilde{t}_N < R^2 + \frac{16R^2}{4 - \eta^2} < \frac{2R^2}{\eta^2} < \frac{16R^2}{\eta^2} < -R^2 + \frac{4(4 + \eta^2)R^2}{\eta^2} < \tilde{t}_{N+2}.
\]

Thus we have

\[
B_{2R}(z_1) \times (\tilde{t}_N, \tilde{t}_{N+2}) \supset B_R(z_0) \times \left[ \frac{2R^2}{\eta^2}, \frac{16R^2}{\eta^2} \right] \supset B_R(z_0) \times \left[ \frac{2R^2}{\eta^2}, \frac{R^2}{\eta^2} \right]
\]

for \(0 < \eta < 1/2\) and hence \(u \leq \inf_{\text{int} \tilde{t}_N} u \leq 1\). We remark that \(\tilde{t}_{N+2}\) is comparable to \(R\).

Now, it suffices to show for some large \(M'_\eta \geq M_\eta\), and small \(0 < \epsilon'_\eta < \epsilon_\eta\), we have

\[
\mathcal{L} \tilde{t}_i \left( \int_{K_{\frac{\mu_\eta}{3}}(z_{i+1}, \tilde{t}_i)} |f^{\# +1}|^{\frac{1}{\alpha - 1}} \right)^{\frac{\alpha - 1}{\alpha}} \leq \eta M'_\eta^{N+2-i}, \quad \forall 1 \leq i \leq N + 2,
\]

where \(M_\eta\) and \(\epsilon_\eta\) are the constants in Corollary 4.4. We notice that \(B_{\frac{\mu_\eta}{3}}(z_0) \subset B_{\tilde{t}_N}(z_0) \subset \bigcup_{i=1}^{N+2} K_{\frac{\mu_\eta}{3}}(z_{i+1}, \tilde{t}_i) \subset B_{\frac{\mu_\eta}{3}}(z_0) \times \left( -3R^2, \frac{64R^2}{(4-\eta^2)^2} + R^2 \right)\)
We select \(d(z_0, z_1) < R, |r| < R^2\) and \(\frac{2}{\eta} R \leq \tilde{r}_{N+2} < \frac{4}{\eta} R\). Then for \(i = 1, 2, \cdots, N + 2\), we have

\[
\tilde{r}_{i}^{2(n+1)} \int_{K_{n, \eta, r_{i}, r_{i}^{2}(z_1, t_1)}} |f^+|^{p+1} \leq \left(\frac{4}{\eta^2}\right)^{2(n+1)} \frac{R^{2(n+1)}}{|K_{n, \eta, r_{i}, r_{i}^{2}(z_1, t_1)}|} \left|f^+\right|^{p+1} \mathbb{L}^{n+1}\left(B_{\frac{2}{\eta} r_{i}^{2}(z_0)} \times \left(-3R^{2}, \frac{6\beta^2}{2 \eta^2} + R^2\right)\right)
\]

\[
\leq \left(\frac{4}{\eta^2}\right)^{2(n+1)} \left(2\varepsilon_{\eta}^{-p} \right)^{p+1} \frac{B_{\frac{2}{\eta} r_{i}^{2}(z_0)} \times \left(-3R^{2}, \frac{6\beta^2}{2 \eta^2} + R^2\right)}{|K_{n, \eta, r_{i}, r_{i}^{2}(z_1, t_1)}|} \leq C(n, \eta) \varepsilon_{\eta}^{p+1} \frac{|B_{\frac{2}{\eta} r_{i}^{2}(z_0)}| R^2}{|B_{r_{i}^{2}(z_1)}| R_i^2} \leq C(n, \eta) \varepsilon_{\eta}^{p+1} \frac{|B_{\frac{\tilde{r}_{N+2}^{2}(z_0)}{\tilde{t}_i}}| \tilde{r}_{N+2}^{2}}{|B_{r_{i}^{2}(z_1)}| R_i^2},
\]

where we use that \(\frac{2}{\eta} R \leq \tilde{r}_{N+2} < \frac{4}{\eta} R\) and the volume comparison theorem in the last inequality and the constant \(C(n, \eta) > 0\) depending only on \(n\) and \(\eta\), may change from line to line. Since \(d(z_0, z_1) < R\), we use the volume comparison theorem again to obtain

\[
\tilde{r}_{i}^{2(n+1)} \int_{K_{n, \eta, r_{i}, r_{i}^{2}(z_1, t_1)}} |f^+|^{p+1} \leq C(n, \eta) \varepsilon_{\eta}^{p+1} \frac{|B_{\frac{\tilde{r}_{N+2}^{2}(z_0)}{\tilde{t}_i}}| \tilde{r}_{N+2}^{2}}{|B_{r_{i}^{2}(z_1)}| R_i^2}
\]

\[
\leq C(n, \eta) \varepsilon_{\eta}^{p+1} \frac{|B_{\frac{\tilde{r}_{N+2}^{2}(z_0)}{\tilde{t}_i}}| \tilde{r}_{N+2}^{2}}{|B_{r_{i}^{2}(z_1)}| R_i^2} \leq C(n, \eta) \varepsilon_{\eta}^{p+1} \left(\frac{\tilde{r}_{N+2}}{R_i}\right)^{n+2} \leq C(n, \eta) \varepsilon_{\eta}^{p+1} \left(\frac{\tilde{r}_{N+2}}{R_i}\right)^{n+2}.
\]

We select \(M'_{\eta} > M_{\eta}\) large and \(0 < \varepsilon_{\eta} < \varepsilon_{\eta}\) small enough to satisfy

\[
C(n, \eta) \varepsilon_{\eta}^{p+1} \left(\frac{2}{\eta}\right)^{(n+2)/2N+2-i} \leq C(n, \eta) \varepsilon_{\eta}^{p+1} M_{\eta}^{(n+1)/2N+2-i}, \quad \forall 1 \leq i \leq N + 2,
\]

which proves (16). Therefore, Corollary 4.4 (after translation in time by \(t_1\)) gives

\[
\left|\left[u \leq M_{\eta}^{N+2}\left(K_{r_{i}^{2}}(z_1, t_1)\right)\right] \cap K_{r_{i}^{2}}(z_1, t_1 + 4R^2)\right| \geq \mu_{\eta} > 0.
\]

5. PARABOLIC VERSION OF THE CALDERÓN-ZYGUMUND DECOMPOSITION

Throughout this section, we assume that a complete Riemannian manifold \(M\) satisfies the condition (3). We introduce a parabolic version of the Calderón-Zygmund lemma (Lemma 5.7) to prove power decay of super-level sets in Lemma 6.1 (see [W, C1, C2]). Christ [C3] proved that the following theorem holds for so-called "spaces of homogeneous type”, which is a generalization of Euclidean dyadic decomposition. In harmonic analysis, a metric space \(X\) is called a space of homogeneous type when \(X\) equips a nonnegative Borel measure \(\nu\) satisfying the doubling property

\[
\nu(B_{2R}(x)) \leq A_1 \nu(B_R(x)) < +\infty, \quad \forall \, x \in X, \quad R > 0,
\]

for some constant \(A_1\) independent of \(x\) and \(R\). From Bishop’s volume comparison (Lemma 2.2), a complete Riemannian manifold \(M\) satisfying the condition (3) is a space of homogeneous type with \(A_1 = 2^n\).
**Theorem 5.1** (Christ). There exist a countable collection \( \{Q^{k,\alpha} \subset M : k \in \mathbb{Z}, \alpha \in I_k \} \) of open subsets of \( M \) and positive constants \( 0 < \delta_0 < 1, c_1 \) and \( c_2 \) (with \( 2c_1 \leq c_2 \)) that depend only on \( n \), such that

(i) \( |M \setminus \bigcup_{\alpha} Q^{k,\alpha}| = 0 \) for \( k \in \mathbb{Z} \),

(ii) if \( l \leq k, \alpha \in I_k \), and \( \beta \in I_l \), then either \( Q^{k,\alpha} \subset Q^{l,\beta} \) or \( Q^{k,\alpha} \cap Q^{l,\beta} = \emptyset \),

(iii) for any \( (k, \alpha) \) and any \( l < k \), there is a unique \( \beta \) such that \( Q^{k,\alpha} \subset Q^{l,\beta} \),

(iv) \( \text{diam}(Q^{k,\alpha}) \leq c_2 \delta_0^k \),

(v) any \( Q^{k,\alpha} \) contains some ball \( B_{c_1 \delta_0^k} (z^{k,\alpha}) \).

For convenience, we will use the following notation.

**Definition 5.2** (Dyadic cubes on \( M \)).

(i) The open set \( Q = Q^{k,\alpha} \) in Theorem 5.1 is called a dyadic cube of generation \( k \) on \( M \).

From the property (iii) in Theorem 5.1, for any \( (k, \alpha) \), there is a unique \( \beta \) such that \( Q^{k,\alpha} \subset Q^{k-1,\beta} \). We call \( Q^{k-1,\beta} \) the predecessor of \( Q^{k,\alpha} \). When \( Q := Q^{k,\alpha} \), we denote the predecessor \( Q^{k-1,\beta} \) by \( \tilde{Q} \) for simplicity.

(ii) For a given \( R > 0 \), we define \( k_R \in \mathbb{N} \) to satisfy

\[
2c_1 \delta_0^{k-R} < R \leq c_2 \delta_0^{k-2}.
\]

The number \( k_R \) means that a dyadic cube of generation \( k_R \) is comparable to a ball of radius \( R \).

For the rest of the paper, we fix some small numbers:

\[
\delta := \frac{2c_1 \delta_0}{c_2}, \quad \delta_1 := \frac{\delta_0(1-\delta_0)}{2} \in \left(0, \frac{\delta_0}{2}\right), \quad \eta := \min(\delta, \delta_1) \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad \kappa := \frac{\eta}{2} \sqrt{1-\delta_0^2}.
\]

By using the dyadic decomposition of a manifold \( M \), we have the following decomposition of \( M \times (T_1, T_2] \) in space and time. For time variable, we take the standard euclidean dyadic decomposition.

**Lemma 5.3.** There exists a countable collection \( \{K^{k,\alpha} \subset M \times (T_1, T_2] : k \in \mathbb{Z}, \alpha \in I_k \} \) of subsets of \( M \times (T_1, T_2] \subset M \times \mathbb{R} \) and positive constants \( 0 < \delta_0 < 1, c_1 \) and \( c_2 \) (with \( 2c_1 \leq c_2 \)) that depend only on \( n \), such that

(i) \( |M \times (T_1, T_2] \setminus \bigcup_{\alpha} K^{k,\alpha}| = 0 \) for \( k \in \mathbb{Z} \),

(ii) if \( l \leq k, \alpha \in I_k \), and \( \beta \in I_l \), then either \( K^{k,\alpha} \subset K^{l,\beta} \) or \( K^{k,\alpha} \cap K^{l,\beta} = \emptyset \),

(iii) for any \( (k, \alpha) \) and any \( l < k \), there is a unique \( \beta \) such that \( K^{k,\alpha} \subset K^{l,\beta} \),

(iv) \( \text{diam}(K^{k,\alpha}) \leq c_2 \delta_0^k \),

(v) any \( K^{k,\alpha} \) contains some cylinder \( B_{c_1 \delta_0^k} (z^{k,\alpha}) \times (t^{k,\alpha} - c_2 \delta_0^k, t^{k,\alpha}] \).

**Proof.** To decompose in time variable, for each \( k \in \mathbb{Z} \), we select the largest integer \( N_k \in \mathbb{Z} \) to satisfy

\[
\frac{1}{4} c_2^2 \delta_0^{2k} \leq \frac{T_2 - T_1}{2^{2N_k}} < c_2^2 \delta_0^{2k}.
\]

For \( k \)-th generation, we split the interval \( (T_1, T_2] \) into \( 2^{2N_k} \) disjoint subintervals which have the same length. Then we obtain \( |I_k| = |I_k| \cdot 2^{2N_k} \) disjoint subsets on \( M \times (T_1, T_2] \) satisfying properties (i)-(v). \( \square \)

For the rest of this section, let \( \{K^{k,\alpha} \subset M \times (T_1, T_2] : k \in \mathbb{Z}, \alpha \in I_k \} \) be the parabolic dyadic decomposition of \( M \times (T_1, T_2] \) as in Lemma 5.3.
Definition 5.4 (Parabolic dyadic cubes).
(i) $K = K^{k,a}$ is called a parabolic dyadic cube of generation $k$. If $K := K^{k,a} ⊂ K^{k-1,β} =: \overline{K}$, we say $\overline{K}$ is the predecessor of $K$.
(ii) For a parabolic dyadic cube $K$ of generation $k$, we define $l(k)$ to be the length of $K$ in time variable, namely, $l(k) = T_{k+1} - T_k$ for $M × (T_1, T_2]$ in Lemma 5.3.

We quote the following technical lemma proven by Cabré [Ca, Lemma 6.5].

Lemma 5.5 (Cabré). Let $z_o ∈ M$ and $R > 0$. Then we have the following.
(i) If $Q$ is a dyadic cube of generation $k$ such that

$$k ≥ k_R \quad \text{and} \quad Q ⊂ B_R(z_o),$$

then there exist $z_1 ∈ Q$ and $r_k ∈ (0, R/2)$ such that

$$(17) \quad B_{5r_k}(z_1) ⊂ Q ⊂ \widetilde{Q} ⊂ B_{2r_k}(z_1) ⊂ B_{\frac{5r_k}{2}}(z_1) ⊂ B_{\frac{3r_k}{2}}(z_1)$$

and

$$(18) \quad B_{\frac{3r_k}{2}}(z_o) ⊂ B_{\frac{3r_k}{2}}(z_1).$$

In fact, for $k ≥ k_R$, the above radius $r_k$ is defined by

$$r_k := \frac{1}{2} \cdot \frac{\sigma_0}{\sigma_0^k} = \frac{c_1}{\sigma_0^k},$$

(ii) If $Q$ is a dyadic cube of generation $k_R$ and $d(z_o, Q) ≤ \delta_1 R$, then $Q ⊂ B_R(z_o)$ and hence $(17)$ and $(18)$ hold for some $z_1 ∈ Q$ and $r_k ∈ \left[\frac{\delta_1 R}{2}, \frac{R}{2}\right]$. Moreover,

$$B_{5r_k}(z_o) ⊂ B_{2r_k}(z_1).$$

(iii) There exists at least one dyadic cube $Q$ of generation $k_R$ such that $d(z_o, Q) ≤ \delta_1 R$.

We remark that for $k ≥ k_R$,

$$η^2 r_k^2 ≤ 6 \sigma_0 r_k^2 = \frac{1}{4} \cdot \frac{\sigma_0^2}{\sigma_0^{2k}} ≤ l(k) < \frac{\sigma_0^2}{\sigma_0^{2k}} = 4r_{k+1}^2$$

and $(17)$ gives that for any $a ∈ \mathbb{R}$,

$$(19) \quad K_{5r_k}(z_1, a) ⊂ Q × (a - l(k), a] ⊂ K_{2r_k}(z_1, a)$$

Definition 5.6. Let $m ∈ \mathbb{N}$. For any parabolic dyadic cube $K := Q × (a - l(k), a]$ of generation $k$, the elongation of $K$ along time in $m$ steps (see [KL]), denoted by $\overline{K^m}$, is defined by

$$\overline{K^m} := \widetilde{Q} × (a, a + m · l(k - 1)],$$

where $l(k)$ is the length of a parabolic dyadic cube of generation $k$ in time and $\widetilde{Q}$ is the predecessor of $Q$ in space. The elongation $\overline{K^m}$ is the union of the stacks of parabolic dyadic cubes congruent to the predecessor of $K$.

Now we have a parabolic version of Calderón-Zygmund lemma. The proof of lemma is the same as Euclidean case so we refer to [W] for the proof.

Lemma 5.7 (Lemma 3.23, [W]). Let $K_1 = Q_1 × (a - l(k_0), a]$ be a parabolic dyadic cube of generation $k_0$ in $M × (T_1, T_2]$, and let $0 < η < 1$ and $m ∈ \mathbb{N}$. Let $A ⊂ K_1$ be a measurable set such that $|A ∩ K_1| ≤ α |K_1|$ and let

$$\mathcal{A}_m := \bigcup \overline{K^m} : |K ∩ A| > α |K|, K, a parabolic dyadic cube in $K_1$ ∩ (Q_1 × \mathbb{R})$$

Then, we have

$$|\mathcal{A}_m| ≥ \frac{m}{(m + 1)α} |A|.$$
6. Harnack inequality

In order to prove the parabolic Harnack inequality, we take the approach presented in [4] and iterate Lemma 4.3 with Christ decomposition (Theorem 5.1) and Calderón-Zygmund type lemma (Lemma 5.7). We begin this section with recalling that \( \eta \in \left( 0, \frac{1}{4} \right) \) is fixed as in the previous section. So the uniform constants \( \mu_\eta, \epsilon_\eta' \) and \( M_\eta' \) in Proposition 4.6 are also fixed and we denote them by \( \mu, \epsilon_0 \) and \( M_0 \) for simplicity.

We select an integer \( m > 1 \) large enough to satisfy

\[
\frac{m}{(m+1)(1-\mu)} > \frac{1}{1 - \frac{\mu}{2}},
\]

where \( 0 < \mu < 1 \) is the constant in Lemma 4.3. For \( T_1 := -3R^2 \) and \( T_2 := (\frac{16}{7} + 1 + m)R^2 \), we consider a parabolic dyadic decomposition of \( M \) that satisfies the conditions of Lemma 6.1.

6.1. Power decay estimate of super-level sets.

Lemma 6.1. Suppose that \( M \) satisfies the conditions (3), (4). Let \( z_o \in M, R > 0 \) and \( \tau \in [3, 16] \). Let \( u \) be a nonnegative smooth function such that \( \mathcal{L}u \leq f \) in \( B_{\frac{2R}{\tau^2}}(z_o) \times \left( -3R^2, \frac{\tau R^2}{\tau^2} \right) \). Assume that

\[
\inf_{B_{\tau R}(z_o) \times \left[ \frac{\delta R^2}{\tau^2}, \frac{2\delta R^2}{\tau^2} \right]} u \leq 1
\]

and

\[
R^2 \left( \int_{B_{\tau R}(z_o) \times \left( -3R^2, \frac{\tau R^2}{\tau^2} \right)} \left| f - \frac{\mu}{2} \right|^{\mu+1} \right)^{\frac{1}{\mu+1}} \leq \epsilon_1
\]

for a uniform constant \( 0 < \epsilon_1 < \epsilon_0 \). Let \( K_1 \) be a parabolic dyadic cube of generation \( k_R \) such that

\[
K_1 := Q_1 \times (t_1 - l(k_R), t_1] \subset Q_1 \times (-R^2, R^2),
\]

where \( Q_1 \) is a dyadic cube of generation \( k_0 \) such that \( d(z_o, Q_1) \leq \delta_1 R \). Then for \( i = 1, 2, \cdots \), we have

\[
\frac{\left| \{ u > M_i \} \cap K_1 \right|}{|K_1|} < \left( 1 - \frac{\mu}{2} \right)^i,
\]

where \( 0 < \epsilon_1 < \epsilon_0 \) and \( M_i > 0 \) depend only on \( n, \lambda, \Lambda, \) and \( a_L \).

Proof. (i) As Proposition 4.6, we use Lemma 4.5 to assume that a nonnegative smooth function \( u \) defined on \( B_{\frac{2R}{\tau^2}}(z_o) \times (T_1, T_2) \) satisfies that \( \inf_{B_{\tau R}(z_o) \times \left[ \frac{\delta R^2}{\tau^2}, \frac{2\delta R^2}{\tau^2} \right]} u \leq 1 \) and \( \mathcal{L}u \leq f \) in \( B_{\frac{2R}{\tau^2}}(z_o) \times (T_1, T_2) \) for some \( f \) with

\[
R^2 \left( \int_{B_{\tau R}(z_o) \times (T_1, T_2)} \left| f - \frac{\mu}{2} \right|^{\mu+1} \right)^{\frac{1}{\mu+1}} \leq \frac{50}{49} \epsilon_1 < 2 \epsilon_1.
\]

(ii) According to Lemma 5.5, there exists a dyadic cube \( Q_1 \subset B_R(z_o) \) of generation \( k_R \) such that \( d(z_o, Q_1) \leq \delta_1 R \). We find \( z_1 \in Q_1 \) and \( r_{k_0} \in \left[ \frac{\delta R^2}{\tau^2}, \frac{\delta R^2}{\tau} \right) \) satisfying (17), (18) and \( B_{\delta_1 R}(z_o) \subset B_{2r_{k_0}}(z_1) \). Since \( \eta^2 R^2 \leq l(k_R) < 4r_{k+1}^2 = 4\delta_0^2 r_{k+1}^2 < \delta_0^2 R^2 \), we find \( r_1 \in \)
\((-R^2+l(k_R), R^2)\) such that \(K_1 := Q_1 \times (t_1 - l(k_R), t_1)\) is a parabolic dyadic cube of generation \(k_R\) of \(M \times (T_1, T_2)\). From \([19]\), we also have that
\[K_{\eta k}(z_1, t_1) \subset K_1 \subset K_{2\eta k}(z_1, t_1).\]

We use the induction to prove \((20)\) so we first check the case \(i = 1\). We notice that
\[d(z_o, z_1) < R, r_{k_R} \in [\frac{\eta k}{2}, R, R) \subset (\frac{\eta k}{2}, R)\] and \(|t_1| < R^2\). We set \(\epsilon_1 := \left(\frac{3/\eta^2t_1}{16\eta^{-1} + t_1}\right)^{1/2} \epsilon_0\). Then, \(u\) satisfies the hypotheses of Proposition \([4.6]\) with \(r = r_{k_R}\) and \(N = 1\), so we deduce that
\[0 < \mu \leq \frac{|\{u > M_0^i\} \cap K_{\eta k}(z_1, t_1)|}{|K_{\eta k}(z_1, t_1)|} < 1 - \frac{\mu}{2}.\]
Thus, we have for \(M_1 \geq M_0^3\),
\[\frac{|\{u > M_1\} \cap K_1|}{|K_1|} \leq 1 - \mu < 1 - \frac{\mu}{2}.\]

(iii) Now, suppose that \((20)\) is true for \(i\), that is,
\[\frac{|\{u > M_i^j\} \cap K_1|}{|K_1|} < \left(1 - \frac{\mu}{2}\right)^{i}.\]
To show the \((i+1)\)-th step, define for \(h > 0\),
\[B_h := \{u > h\} \cap B_{\eta k}(z_o) \times (T_1, T_2).\]
We know
\[\frac{|B_{M_i^j} \cap K_1|}{|K_1|} \leq \left(1 - \frac{\mu}{2}\right)^i.\]
If \(h > 0\) is a constant such that
\[\frac{|\mathcal{A}|}{|K_1|} \geq \left(1 - \frac{\mu}{2}\right)^{i+1}\] for \(\mathcal{A} := B_{hM_i^j} \cap K_1\),
then we will show that \(h < M_1\) for a uniform constant \(M_1 > M_0 > 1\), that will be fixed later.

Suppose on the contrary that \(h \geq M_1\). From (ii), we have
\[\frac{|\mathcal{A}|}{|K_1|} \leq \frac{|B_{M_i^j} \cap K_1|}{|K_1|} \leq 1 - \mu\] for \(M_1 \geq M_0^3\) and \(h \geq 1\). Applying Lemma \([5.7]\) to \(\mathcal{A}\) with \(\alpha = 1 - \mu\), it follows that
\[|\mathcal{A}_{1-\mu}| \geq \frac{m}{(m+1)(1-\mu)}|\mathcal{A}| > \frac{1}{1-\frac{\mu}{2}}|\mathcal{A}|.
We claim that
\[(21)\]
\[\mathcal{A}_{1-\mu} \subset B_{\eta k}\]
for \(h > C_1M_0^3 > 1\), where a uniform constant \(C_1 > 0\) will be chosen. If not, there is a point \((x_1, s_1) \in \mathcal{A}_{1-\mu} \setminus B_{\eta k}\) and we find a parabolic dyadic cube \(K := Q \times (a - l(k), a) \subset K_1\) of generation \(k > k_R\) such that
\[|\mathcal{A} \cap K| > (1-\mu)|K|\] and \((x_1, s_1) \in \overline{K}_n\)
from the definition of \(\mathcal{A}_{1-\mu}\). According to Lemma \([5.5]\) there exist \(z_1 \in Q \subset Q_1 \subset B_R(z_o)\) and \(r_k \in (0, R/2)\) satisfying \((17), (18), K_{\eta k}(z_1, a) \subset K \subset K_{2\eta k}(z_1, a)\) and
\[(x_1, s_1) \in \overline{K}_n = \overline{Q} \times (a, a + m \cdot l(k-1)) \subset B_{44k}(z_1) \times (a, a + m \cdot 4r_k^2).\]
We note that
\[
\inf_{B_{a/r}(z_1) \cap (a, a + m \cdot 4r_k^2)} u \leq u(x_1, s_1) \leq \frac{hM_1^i}{M_0^m}
\]
and
\[
B_{a/r}(z_1) \times (a - (\eta^2 + \eta^4/4)r_k^2, a + m \cdot 4r_k^2) \subset B_{a,R}(z_0) \times (-3R^2, (1 + m)R^2),
\]
since \( r_k < R/2 \) and \( a \in (t_1 - l(k_R), t_1) \subset (-R^2, R^2) \). We also have that for \( j = 1, \ldots, m, \)
\[
(22) \quad r_k^2 \left( \int_{K_{a + m \cdot 4r_k^2}(z_1, a + m - j + 1) \cdot 4r_k^2} |f^{+p+1}| \right)^{\frac{1}{p+1}} \leq \varepsilon_0 \frac{hM_1^i}{M_0^{m-j+1}}.
\]
Indeed, the volume comparison theorem and the property (18) will give that
\[
r_k^2 \left( \int_{K_{a + m \cdot 4r_k^2}(z_1, a + m - j + 1) \cdot 4r_k^2} |f^{+p+1}| \right)^{\frac{1}{p+1}} = \frac{r_k^{\frac{2}{p+1}} \cdot r_k^{\frac{2}{p+1}}}{|B_{a/r}(z_1)|^{\frac{1}{p+1}} (\alpha_2 R_k^2)^{\frac{1}{p+1}}} \|f^{+p+1}\|_{L^{p+1}}
\leq \frac{R_k^{\frac{1}{p+1}} \cdot R_k^{\frac{1}{p+1}}}{|B_{a/r}(z_1)|^{\frac{1}{p+1}} (\alpha_2 R_k^2)^{\frac{1}{p+1}}} \|f^{+p+1}\|_{L^{p+1}} (B_{a/r}(z_1) \times (T_1, T_2))
\leq \frac{R_k^2}{C_1 R^2 / 2} \left( \frac{1}{2} \right) \cdot \frac{1}{|B_{2 \tilde{R}}(z_0) \times (T_1, T_2)|} \|f^{+p+1}\|_{L^{p+1}} (B_{2 \tilde{R}}(z_0) \times (T_1, T_2)) < C_1 \varepsilon_1,
\]
where a uniform constant \( C_1 > 1 \) depends only on \( \eta, n \) and \( m \). For \( h \geq C_1 M_0^m \) and \( M_1 > 1 \), we have that
\[
r_k^2 \left( \int_{K_{a + m \cdot 4r_k^2}(z_1, a + m - j + 1) \cdot 4r_k^2} |f^{+p+1}| \right)^{\frac{1}{p+1}} < C_1 \varepsilon_1 < \varepsilon_0 \frac{hM_1^i}{M_0^m} \leq \varepsilon_0 \frac{hM_1^i}{M_0^{m-j+1}},
\]
which proves (22). Thus, we can apply Lemma 4.3 iteratively to \( \bar{u}_j := \frac{M_0^{m-j+1}}{M_1^i} - u, \) for \( 1 \leq j \leq m, \) to deduce
\[
\mu \leq \frac{|u \leq hM_1^i| \cap K_{a/R}(z_1, a)}{|K_{a/r_k(\alpha_2 r_k^2)(z_1, a + 4R_k^2)}|} < \frac{|u \leq hM_1^i| \cap K}{|K|}.
\]
However, this contradicts to the fact that \( |\mathcal{A} \cap K| > (1 - \mu)|K| \). Therefore, we have proved that \( \mathcal{A}_{1-\mu}^m \subset B_{M_1^i}^{M_0^m} \) for \( h \geq C_1 M_0^m \).
(iv) Since \(|B_{M^i_1} \cap K_1| < \left(1 - \frac{\mu}{2}\right)^i |K_1|\), we have that \(|B_{M^i_1} \cap K_1| \leq |B_{M^i_1} \cap K_1| < \left(1 - \frac{\mu}{2}\right)^i |K_1| \leq \frac{1}{1 - \frac{\mu}{2}} |A|\) for \(h \geq C_1 M^m_1\). Then, by using (21), we obtain
\[
|\mathcal{A}_{1-\mu} \cap K_1| = |\mathcal{A}_{1-\mu} - |\mathcal{A}_{1-\mu} \cap K_1| \\
\geq \frac{m}{(m + 1)(1 - \mu)} |\mathcal{A}| - |B_{M^i_1} \cap K_1| \\
> \left(1 - \frac{\mu}{2}\right)^i |\mathcal{A}| = a |\mathcal{A}| \geq a \left(1 - \frac{\mu}{2}\right)^i |K_1|
\]
with \(a := \frac{m}{(m + 1)(1 - \mu)} - \frac{1}{1 - \frac{\mu}{2}} > 0\). We find a point \((x_1, s_1) \in \mathcal{A}_{1-\mu} \cap K_1\) and a parabolic dyadic cube \(K := Q \times (a - l(k), k) \subset K_1\) of generation \(k(> k_R)\) such that \((x_1, s_1) \in K\), and \(|\mathcal{A} \cap K| = (1 - \mu)|K|\). We may assume that
\[
s_1 > t_1 + \frac{\mu}{2} |l(k_R)|
\]
since \(\mathcal{A}_{1-\mu} \subset Q \times (t_1 - l(k_R), +\infty)\) and \(|\mathcal{A}_{1-\mu} \cap K_1| > a \left(1 - \frac{\mu}{2}\right)^i |l(k_R)|\). Using Lemma 5.5 again, there exist \(z_1 \in Q \subset Q_1 \subset B_{R}(z_o)\) and \(r_k \in (0, R/2)\) satisfying (17), (18), and \(K_{\eta_1}(z_1, a) \subset K \subset K_{2\eta_1}(z_1, a)\). Then we have
\[
s_1 \leq a + m \cdot l(k - 1) < t_1 + m \cdot 4r_k^2
\]
and hence
\[
r_k \geq \frac{\sqrt{a}}{\sqrt{8m}} \left(1 - \frac{\mu}{2}\right)^i |l(k_R)| \geq \frac{\sqrt{a} \alpha^2_0}{4 \sqrt{2m}} \left(1 - \frac{\mu}{2}\right)^i |l(k_R)| \geq \left(\frac{\eta}{2}\right)^{\eta_1} R
\]
for a uniform integer \(N > 0\) independent of \(i \in \mathbb{N}\). We apply Proposition 4.6 to \(u\) in order to get
\[
\mu \leq \frac{|\{u \leq M^{N+2}_0 \cap K_{\eta_1}(z_1, a)\}|}{K_{\eta_1, \eta_2^2}(z_1, a + 4r_k^2)} \leq \frac{|\{u \leq M^{N+2}_0 \cap K\}|}{|K|},
\]
since \(r_k \geq \left(\frac{\eta}{2}\right)^{\eta_1} R\), and \((z_1, a) \in K \subset K \subset B_{R}(z_o) \times (-R^2, R^2)\). If \(h \geq M_1 := \max\{C_1 M^m_1, M^{N+2}_0\}\), this implies
\[
1 - \mu > \frac{|\{u > M^{N+2}_0 \cap K\}|}{|K|} \geq \frac{|\{u > h M_1\} \cap K\}|}{|K|} = \frac{|\mathcal{A} \cap K|}{|K|},
\]
which is a contradiction to the fact that \(|\mathcal{A} \cap K| > (1 - \mu)|K|\). Thus, we have \(h < M_1\) for a uniform constant \(M_1 := \max\{C_1 M^m_1, M^{N+2}_0\}\). Therefore, we conclude that \(\frac{|\{u \geq h M_1\} |}{|K|} < \left(1 - \frac{\mu}{2}\right)^i\), completing the proof.

The following corollary is a direct consequence of Lemma 6.1 which estimates the distribution function of \(u\).

**Corollary 6.2.** Under the same assumption as Lemma 6.1, we have
\[
\frac{|\{u \geq h\} \cap K_1|}{|K_1|} \leq dh^{-\epsilon} \quad \forall h > 0,
\]
where \(d > 0\) and \(0 < \epsilon < 1\) depend only on \(n, \lambda, \Lambda,\) and \(a_L\).
Another consequence of Lemma 6.1 is a weak Harnack inequality for nonnegative supersolutions to $\mathcal{L}u = f$.

**Corollary 6.3.** Under the same assumption as Lemma 6.1 we have for $p_o := \frac{1}{2}$,

\begin{equation}
\left( \frac{1}{|K_{xR}(z_o,0)|} \int_{K_{xR}(z_o,0)} u^{p_o} \right)^{\frac{1}{p_o}} \leq C,
\end{equation}

where $C > 0$ depends only on $n, \lambda, \Lambda$, and $\alpha_L$.

**Proof.** Let $k = k_R$ and let $\{K^{k,\alpha}_\omega := Q^{k,\alpha} \times (S^{k,\alpha}, t^{k,\alpha})\}_{\alpha \in J'_k}$ be a family of parabolic dyadic cubes intersecting $K_{xR}(z_o,0)$. For $\alpha \in J'_k$, we have that $K^{k,\alpha}_\omega \subset B_{(k+\delta_0)^R}(z_o) \times (-R^2, R^2]$ since $d(z_o, Q^{k,\alpha}) \leq \kappa R < \delta_1 R$, diam$(Q^{k,\alpha}) \leq \epsilon \delta_0 \leq \delta_0 R$, and $-R^2 + l(k) < -\kappa^2 R^2 \leq t^{k,\alpha} \leq l(k) < \delta_0^2 R^2$. Since

\begin{align*}
|K_{xR}(z_o,0)| &\geq \left( \frac{k}{k+\delta_0} \right)^n |B_{(k+\delta_0)^R}(z_o)| \cdot \kappa^2 R^2 \geq \left( \frac{k}{k+\delta_0} \right)^n \sum_{\alpha \in J'_k} |Q^{k,\alpha}| \cdot \kappa^2 R^2 \\
&\geq \left( \frac{k}{k+\delta_0} \right)^n \sum_{\alpha \in J'_k} |B_{(k+\delta_0)^R}(z_o)| \cdot \kappa^2 R^2 \geq \left( \frac{k}{k+\delta_0} \right)^n \sum_{\alpha \in J'_k} |B_{(k+2\delta_0)^R}(z_o)| \cdot \kappa^2 R^2 \\
&\geq \left( \frac{k}{k+\delta_0} \right)^n \sum_{\alpha \in J'_k} |B_{xR}(z_o)| \cdot \kappa^2 R^2,
\end{align*}

the number $|J'_k|$ of parabolic dyadic cubes intersecting $K_{xR}(z_o,0)$ is uniformly bounded. Thus for some $K^{k,\alpha}$ with $\alpha \in J'_k$, we have

\begin{align*}
\int_{K_{xR}(z_o)} u^{p_o} \leq |J'_k| \cdot \int_{K^{k,\alpha}} u^{p_o} \\
&\leq |J'_k| \cdot \left( |K^{k,\alpha}| + p_o \int_0^h h^{p_o-1} |u \geq h| \cap K^{k,\alpha} \text{d}h \right) \\
&\leq |J'_k| \cdot \left( |K^{k,\alpha}| + p_o d |K^{k,\alpha}| \int_0^h h^{p_o-1-\epsilon} \text{d}h \right).
\end{align*}

from Corollary 6.2 where $d$ and $\epsilon$ are the constants in Corollary 6.2.

By using the volume comparison theorem, we conclude that

\begin{align*}
\frac{1}{|K_{xR}(z_o,0)|} \int_{K_{xR}(z_o,0)} u^{p_o} \leq C_0 \left( \frac{K^{k,\alpha}}{|K_{xR}(z_o,0)|} \right)^n \left( \frac{\kappa + \delta_0}{\kappa} \right)^n \cdot \frac{\delta_0^2}{k^2}
\end{align*}

for $C_0 := |J'_k| \cdot \left( 1 + p_o d \int_0^h h^{1-\epsilon/2} \text{d}h \right)$ since $K^{k,\alpha} \subset B_{(k+\delta_0)^R}(z_o) \times (t^{k,\alpha} - \delta_0 R^2, t^{k,\alpha}]$. $\square$

**6.2. Proof of Harnack Inequality.** So far, we have dealt with nonnegative supersolutions. Now, we consider a nonnegative solution $u$ of $\mathcal{L}u = f$. We apply Corollary 6.2 as in [Ca] (see also [Wi]) to solutions $C_1 - C_2 u$ for some constants $C_1$ and $C_2$.

**Lemma 6.4.** Suppose that $M$ satisfies the conditions (3), (4). Let $z_o \in M, R > 0$ and $\tau \in [3, 16]$. Let $u$ be a nonnegative smooth function such that $\mathcal{L}u = f$ in $B_{\frac{R}{\tau^2}}(z_o) \times (-3R^2, \frac{R^2}{\tau^2})$. Let $C_0 := |J'_k| \cdot \left( 1 + p_o d \int_0^h h^{1-\epsilon/2} \text{d}h \right)$ since $K^{k,\alpha} \subset B_{(k+\delta_0)^R}(z_o) \times (t^{k,\alpha} - \delta_0 R^2, t^{k,\alpha}]$. 

\begin{align*}
\int_{K_{xR}(z_o,0)} u^{p_o} \leq C_0 \left( \frac{K^{k,\alpha}}{|K_{xR}(z_o,0)|} \right)^n \left( \frac{\kappa + \delta_0}{\kappa} \right)^n \cdot \frac{\delta_0^2}{k^2}
\end{align*}
Assume that \( \inf_{B_r(x_0)} \frac{d(x,\partial D)}{|x|^{n-2}} u \leq 1 \) and
\[
R^2 \left( \int_{B_r(x_0)} \left| f \right|^{p+1} \right)^{\frac{1}{p+1}} \leq \frac{\epsilon_1}{4} =: \epsilon
\]
for a uniform constant \( 0 < \epsilon_1 < 1 \) as in Lemma 6.1.

Then there exist constants \( \sigma > 0 \) and \( \bar{M}_0 > 1 \) depending on \( n, \lambda, \Lambda \) and \( a_L \) such that for \( \nu := \frac{\bar{M}_0}{\bar{M}_0-1/2} > 1 \), the following holds:

If \( j \geq 1 \) is an integer and \( z_1 \in M \) and \( t_1 \in \mathbb{R} \) satisfy
\[
d(z_0, z_1) \leq \kappa R, \quad |t_1| \leq \kappa^2 R^2
\]
and
\[
u(1, t_1) \geq \nu^{j-1} \bar{M}_0,
\]
then

(i) \( K_{\sigma \bar{M}_0} L_j (z_1, t_1) \subset B \left( \frac{\sigma \bar{M}_0}{\nu^{j-1}}, \frac{\kappa R}{\nu^{j-1}} \right) \times \left( -3R^2, \frac{\sigma R}{\nu^{j-1}} \right) \),

(ii) \( \sup_{K_{\sigma \bar{M}_0} L_j (z_1, t_1)} u \geq \nu^{j-1} \bar{M}_0 \),

where \( L_j := \sigma \bar{M}_0 \nu^{-j} R \) and \( 0 < \epsilon < 1 \) as in Corollary 6.2.

**Proof.** We select \( \sigma > 0 \) and \( \bar{M}_0 > 1 \) large so that
\[
\sigma > \frac{c_2}{c_1 \delta_0} (2d^2) \nu^{\frac{1}{2}}
\]
and
\[
\sigma \nu^{\frac{1}{2}} \leq d \nu^{\frac{1}{2}} \leq \frac{k}{4},
\]
where \( d, \epsilon, c_1, c_2 \) and \( \delta_0 \) are the constants in Corollary 6.2 and Theorem 5.1. Since \( L_j \leq \frac{\sigma R}{\nu^{j-1}} < \frac{2R}{\nu^{j}}, d(z_0, z_1) \leq \kappa R < R \) and \( |t_1| \leq \kappa^2 R^2 < \frac{\nu^2 R^2}{\nu^{j-1}} \), we have
\[
B \left( \frac{\sigma \bar{M}_0}{\nu^{j-1}}, \frac{\kappa R}{\nu^{j-1}} \right) \times \left( t_1 - \frac{3}{\eta^2} \frac{\tau L_j^2}{\eta^2}, t_1 \right) \subset B \left( \frac{\sigma \bar{M}_0}{\nu^{j-1}}, \frac{\kappa R}{\nu^{j-1}} \right) \times \left( -3R^2, \frac{\sigma R}{\nu^{j-1}} \right)
\]
so (i) is true.

Now, suppose on the contrary that
\[
\sup_{K_{\sigma \bar{M}_0} L_j (z_1, t_1)} u < \nu^{j-1} \bar{M}_0.
\]
Let \( k_j := k_j \geq k_R \) with \( L_j \) in Definition 5.2. From Lemma 5.5 there exists a dyadic cube \( Q_{L_j} \) of generation \( k_j \) such that \( d(z_1, Q_{L_j}) \leq \sigma_1 L_j \). We also find a parabolic dyadic cube \( K_{L_j} \) of generation \( k_j \) such that
\[
K_{L_j} \subset Q_{L_j} \times \left( t_1 - \frac{\tau L_j^2}{\eta^2} - L_j, t_1 - \frac{\tau L_j^2}{\eta^2} + L_j \right)
\]
since \( l(k_j) < \delta_0^2 L_j^2 \). Let \( K_1 \) be the unique predecessor of \( K_{L_j} \) of generation \( k_R \), that is,
\[
K_{L_j} \subset K_1 := Q_j \times (a - l(k_R), a).
\]
Then we have
\[ d(z_0, Q_1) \leq d(z_0, Q_L) \leq d(z_0, z_1) + d(z_1, Q_L) \leq \kappa R + \delta L_j < \delta_1 R \]
and \((a - l(k_R), a) \subset (-R^2, R^2)\)

since
\[ l(k_R) + |t_1| + \frac{\tau L^2_j}{\eta^2} + L_j^2 \leq l(k_R) + |t_1| + \frac{16L^2_j}{\eta^2} + L_j^2 \]
\[ \leq \delta R^2 + \kappa^2 R^2 + \frac{16}{\eta^2} \frac{1}{16} \frac{k^2}{16} R^2 = \left\{ \delta^2 + \left( \frac{16}{\eta^2} + 17 \right) \frac{\eta^2(1 - \delta^2)}{64} \right\} R^2 < R^2. \]

Now, we apply Corollary [6.2] to \(u\) with \(K_1\) to obtain
\[
(25) \quad \left\{ \left| u \leq \nu \tilde{M}_0 \right\} \cap K_j \right\} \left| K_j \right| \leq \left| \left\{ u \geq \nu \tilde{M}_0 \right\} \cap K_1 \right| \leq d \left( \nu \tilde{M}_0, 2 \right)|K_1|.
\]

On the other hand, we consider the function
\[ w := \nu \tilde{M}_0 - u/\nu^{j-1}, \]
which is nonnegative and satisfies
\[ \mathcal{L} w = -\frac{f}{\nu^{j-1}(v - 1)M_0} \text{ in } K_{\frac{v}{v-1}} \left\{ \frac{3}{2} \right\} \left| \{ z_1, t_1 \} \right| \]
from the assumption. We also have \(w(z_1, t_1) \leq 1\) and
\[ \frac{|f|}{\nu^{j-1}(v - 1)M_0} \leq \frac{2(M_0 - 1/2)|f|}{M_0} \leq 2|f|. \]

By using the volume comparison theorem with \(L_j \leq \frac{\delta}{2} R < \frac{\eta R}{8}\) and \(B_{\frac{\eta}{8}} \left( z_0 \right) \subset B_{\frac{\eta}{8}} \left( z_1 \right)\), we get
\[
L_j \left( \int_{K_{\frac{v}{v-1}} \left\{ \frac{3}{2} \right\} \left| \{ z_1, t_1 \} \right|} \left| f \right| \right)^{\frac{1}{\nu^{j-1}}} = \frac{2L_j^2}{|B_{\frac{\eta}{8}} \left( \frac{3}{2} \right)(\eta R/8)|^{\frac{1}{\nu^{j-1}}}} \left( \left( \frac{3}{2} \right)(\eta R/8) \right)^{\frac{1}{\nu^{j-1}}} \left| f \right|_{L^{1+1}} \leq \frac{2(\eta R/8)^2}{|B_{\frac{\eta}{8}} \left( \frac{3}{2} \right)(\eta R/8)|^{\frac{1}{\nu^{j-1}}}} \left( \left( \frac{3}{2} \right)(\eta R/8) \right)^{\frac{1}{\nu^{j-1}}} \left| f \right|_{L^{1+1}} \leq \frac{2R^2}{|B_{\frac{\eta}{8}} \left( \frac{3}{2} \right)(\eta R/8)|^{\frac{1}{\nu^{j-1}}}} \left( \left( \frac{3}{2} \right)(\eta R/8) \right)^{\frac{1}{\nu^{j-1}}} \left| f \right|_{L^{1+1}} \left( B_{\frac{\eta}{8}} \left( z_0 \right) \times \left( -3R^2, \frac{\eta R}{8} \right) \right) \leq 2 \left( \frac{50}{44} \right)^{\frac{1}{\nu^{j-1}}} e_1 \leq e_1. \]

Applying Corollary [6.2] to \(w\) in \(K_{L_j}\), we deduce that \(|w \geq \tilde{M}_0| \cap K_{L_j} | \leq d \tilde{M}_0^{-1}|K_{L_j}|\), i.e.,
\[
\left\{ u \leq \nu \tilde{M}_0 \right\} \cap K_{L_j} \right\} \left| K_{L_j} \right| \leq d \tilde{M}_0^{-1}|K_{L_j}|. \]
Putting together with (25), we obtain

\[ |K_{L_j}| \leq 2d2^\gamma \bar{M}_0^{-\epsilon} |K_1| \]

since \( d\bar{M}_0^{-\epsilon} \leq \frac{\epsilon}{2} < 1/2 \). From Theorem 5.1, there is a point \( z_\ast \in Q_{L_j} \) such that \( B_{\frac{1}{c_1\delta_0}}(z_\ast) \subset Q_{L_j} \subset B_1 \). Then we have

\[
\left| B_{\frac{1}{c_1\delta_0}}(z_\ast) \right| \cdot 2^{2k_j} \leq \left| B_{\frac{1}{c_1\delta_0}}(z_\ast) \right| \cdot I(k_j) \leq |K_{L_j}|
\]

from the volume comparison theorem. This means

\[
2d2^\gamma \bar{M}_0^{-\epsilon} |K_1| = 2d2^\gamma \bar{M}_0^{-\epsilon} |Q_1| \cdot I(k_R)
\]

\[
< 2d2^\gamma \bar{M}_0^{-\epsilon} |B_{\frac{1}{c_1\delta_0}}(z_\ast)| \cdot c_2^2 \delta_{k_0}
\]

\[
\leq 2d2^\gamma \bar{M}_0^{-\epsilon} \bar{M}_0^{-\epsilon} \left( \frac{c_2 \delta_{k_0}}{c_1 \delta_0} \right)^{\frac{n}{\nu}} \left| B_{\frac{1}{c_1\delta_0}}(z_\ast) \right| c_2^2 \delta_{k_0}
\]

Thus we deduce the following lemma from Lemma 6.4.

\[ L_j \leq c_2 \delta_{k_0}^{-2} \leq \frac{c_2^2}{c_1 \delta_0^2} (2d2^\gamma \bar{M}_0^{-\epsilon} \bar{M}_0^{-\epsilon}) \nu^{-\frac{n}{\nu}} \delta_{k_0} \]

\[
< \frac{c_2}{c_1 \delta_0} (2d2^\gamma \bar{M}_0^{-\epsilon} \bar{M}_0^{-\epsilon}) \nu^{-\frac{n}{\nu}} R < \sigma \bar{M}_0^{-\epsilon} \bar{M}_0^{-\epsilon} R = L_j,
\]

in contradiction to the definition of \( L_j \). Therefore, (ii) is true. \( \square \)

Thus we deduce the following lemma from Lemma 6.4.

**Lemma 6.5.** Suppose that \( M \) satisfies the conditions (1)-(4). Let \( z_\ast \in M, R > 0 \) and \( \tau \in [3, 16] \). Let \( u \) be a nonnegative smooth function such that \( \mathcal{L}u = f \) in \( B_{\frac{R}{2}}(z_\ast) \times [0, \frac{R^\tau}{\nu}] \).

Assume that

\[
\inf_{B_{\frac{R}{2}}(z_\ast) \times \left[ \frac{R^\nu}{\nu+1}, \frac{R^\nu}{\nu} \right]} u \leq 1
\]

and

\[
R^2 \left( \int_{B_{\frac{R}{2}}(z_\ast) \times \left[ -R^\nu, \frac{R^\nu}{\nu} \right]} |f|^{\nu+1} \right)^{\frac{1}{\nu+1}} \leq \varepsilon
\]

for a uniform constant \( 0 < \varepsilon < 1 \) in Lemma 6.4. Then

\[
\sup_{B_{\frac{R}{2}}(z_\ast) \times \left[ -\frac{\nu}{\nu+1}, \frac{R^\nu}{\nu} \right]} u \leq C,
\]

where \( C > 0 \) depends only on \( n, \lambda, \Lambda \) and \( a_1 \).

**Proof.** We take \( j_0 \in \mathbb{N} \) such that

\[
\sum_{j=j_0}^{\infty} \frac{50}{\nu} L_j \leq \frac{kR}{2} \quad \text{and} \quad \sum_{j=j_0}^{\infty} \left( \frac{16}{\nu} \right) \nu L_j \leq \frac{k^2 R^2}{4}.
\]
We claim that \( \sup_{B_{\hat{g}}(z)} \mathcal{B}_u(z) \leq \frac{\kappa R}{4} \). If it does not hold, then there is a point \((z_{j,}, t_{j,}) \in B_{\hat{g}}(z) \times (-\frac{\kappa R^2}{4}, \frac{\kappa R^2}{4})\) such that \( u(z_{j,}, t_{j,}) > \mathcal{B}_u(z) \). Applying Lemma 6.5 with \((z_j, t_i) = (z_{j,}, t_{j,})\), we can find a point \((z_{j+1}, t_{j+1}) \in \mathcal{K} \) such that

\[ u(z_{j+1}, t_{j+1}) \geq \mathcal{B}_u(z). \]

According to the choice of \( j_0 \), we have

\[ d(z_0, z_{j+1}) \leq d(z_0, z_{j,}) + d(z_{j,}, z_{j+1}) < \frac{\kappa R}{2} < \kappa R \]

and

\[ |t_{j+1}| \leq |t_{j,}| + |t_{j,} - t_{j+1}| < \frac{\kappa R^2}{4} + \frac{\kappa R^2}{4} < \kappa R^2. \]

Thus we iterate this argument to obtain a sequence of points \((z_j, t_j)\) for \( j \geq j_0 \) satisfying

\[ d(z_0, z_j) \leq \kappa R, \quad |t_i| \leq \kappa R^2 \quad \text{and} \quad u(z_j, t_j) \geq \mathcal{B}_u(z) \]

since \( d(z_0, z_j) \leq d(z_0, z_{j,}) + \sum_{i=j}^{\infty} d(z_i, z_{i+1}) \leq \frac{\kappa R}{2} + \sum_{i=j}^{\infty} \frac{50}{\eta^2} L(t_i) < \kappa R \) and \( |t_i| \leq |t_{j,}| + \sum_{i=j}^{\infty} |t_i - t_{i+1}| \leq \frac{\kappa R^2}{4} + \sum_{i=j}^{\infty} \left(3 + \frac{\tau}{\eta^2}\right) L_i^2 < \kappa R^2 \) for \( j \geq j_0 \). This contradicts to the continuity of \( u \) and therefore we conclude that

\[ \sup_{B_{\hat{g}}(z)} \mathcal{B}_u(z) \leq \frac{\kappa R}{4}. \]

Now the Harnack inequality in Theorem 6.6 follows easily from Lemma 6.5 by using a standard covering argument and the volume comparison theorem.

**Theorem 6.6 (Harnack Inequality).** Suppose that \( M \) satisfies the conditions \([3,4] \). Let \( z_0 \in M \) and \( R > 0 \). Let \( u \) be a nonnegative smooth function in \( K_{2R}(0, 4R^2) \subset M \times \mathbb{R} \). Then

\[ \sup_{K_{2R}(z_0, 4R^2)} u \leq C \left\{ \inf_{K_{2R}(z_0, 4R^2)} u + R^2 \left( \int_{K_{2R}(z_0, 4R^2)} |\mathcal{L} u|^{\alpha+1} \right)^{\frac{1}{\alpha+1}} \right\}, \]

where \( C > 0 \) is a constant depending only on \( n, \lambda, \Lambda \) and \( a_L \).

**Proof.** According to Lemma 6.5 for \( \tau \in [3, 16] \), a nonnegative smooth function \( v \) in \( K_{2R}(\pi^2, \tau \pi^2, \pi^2) \) satisfies

\[ \sup_{K_{2R}(\pi^2, \tau \pi^2, \pi^2)} v \leq C \left\{ \inf_{K_{2R}(\pi^2, \tau \pi^2, \pi^2)} v + r^2 \left( \int_{K_{2R}(\pi^2, \tau \pi^2, \pi^2)} |\mathcal{L} v|^{\alpha+1} \right)^{\frac{1}{\alpha+1}} \right\}, \]

since \( \frac{\tau}{\eta^2} < 1 \).

Now, let \((x, t) \in K_R(z_0, 2R^2) = B_R(z_0) \times (R^2, 2R^2)\) and \((y, s) \in K_R(z_0, 4R^2) = B_R(z_0) \times (3R^2, 4R^2)\). We show that

\[ u(x, t) \leq C \left\{ u(y, s) + \frac{R^2}{\mathcal{L} u(K_{2R}(z_0, 4R^2))^{\alpha+1}} \left( \int_{K_{2R}(z_0, 4R^2)} |\mathcal{L} u|^{\alpha+1} \right)^{\frac{1}{\alpha+1}} \right\}. \]
for a uniform constant $C > 0$ depending only on $n, \lambda, \Lambda$ and $a_L$. We consider a piecewise $C^1$ path $\gamma : [0, l] \to M$, $\gamma(0) = x$, $\gamma(l) = y$, $l < 2R$, consisting of a minimal geodesic parametrized by arc length joining $x$ and $z_o$, followed by a minimal geodesic parametrized by arc length joining $z_o$ and $y$. We notice that $\gamma([0, l]) \subset B_R(z_o)$ and $d(\gamma(s_1), \gamma(s_2)) \leq |s_1 - s_2|$. 

We can select uniform constants $A > 0$ and $N \in \mathbb{N}$ such that 

$$A := \max \left\{ \frac{64}{3\eta^2}, \frac{50}{\eta^3} \right\} \quad \text{and} \quad \frac{3}{16} \eta^2 A^2 \leq N \leq \frac{1}{3} \eta^2 A^2$$

since $\frac{16 - 9}{16} \eta^2 A^2 \geq \frac{7}{3} \eta^2 A > 1$. For $i = 0, 1, \cdots, N$, we define 

$$(x_i, t) := \left( \gamma \left( \frac{i}{N} \frac{s - t}{N} + t \right), \frac{s - t}{N} + t \right) \in B_R(z_o) \times [R^2, 4R^2].$$

Then we have $(x_0, t_0) = (x, t), (x_N, t_N) = (y, s)$ and for $i = 0, \cdots, N - 1$, 

$$d(x_{i+1}, x_i) \leq \frac{2R}{N} \leq \frac{64}{3\eta^2 A^2} \leq \frac{\kappa R}{2A} \leq \frac{\kappa R}{2A},$$

$$\frac{3R^2}{\eta A^2} \leq \frac{R^2}{N} \leq t_{i+1} - t_i = \frac{s - t}{N} \leq \frac{3R^2}{N} \leq \frac{16R^2}{\eta^2 A^2}.$$ We also have that $K_{\frac{R}{A^2}} \leq \frac{3\frac{R}{A^2}}{\eta^3 A} \leq (x_i, t_i) \subset K_{2R}(z_o, 4R^2)$ for $i = 1, \cdots, N$ since $\frac{50}{\eta^2 A} \leq R$.

We apply the estimate (27) with $r = \frac{R}{A^2}, \tau = (t_{i+1} - t_i) \frac{2A^2}{R^2}$ and $(\bar{x}, \bar{t}) = (x_{i+1}, t_{i+1})$ for $i = 0, 1, \cdots, N - 1$ and use the volume comparison theorem to have 

$$u(x_i, t_i) \leq C \left\{ u(x_{i+1}, t_{i+1}) + \frac{(R/A)^2}{|K_{\frac{R}{A^2}} \leq \frac{3\frac{R}{A^2}}{\eta^3 A}(x_{i+1}, t_{i+1})|^{\frac{1}{\eta^3}}} \|\mathcal{L}u\|_{L^{\eta+1}(K_{2R}(z_o, 4R^2))} \right\}$$

$$\leq C \left\{ u(x_{i+1}, t_{i+1}) + \frac{(R/A)^2}{|B_{\frac{R}{A^2}}(x_{i+1})| \left( 3 + \frac{3}{\eta} \right) \frac{R^2}{A^2}} \|\mathcal{L}u\|_{L^{\eta+1}(K_{2R}(z_o, 4R^2))} \right\}$$

$$\leq C \left\{ u(x_{i+1}, t_{i+1}) + \frac{R^2}{|B_{2R}(x_{i+1}) \cdot 4R^2|^{\frac{1}{\eta^3}}} \|\mathcal{L}u\|_{L^{\eta+1}(K_{2R}(z_o, 4R^2))} \right\},$$

where a uniform constant $C > 0$ may change from line to line. Since $B_{3R}(x_{i+1}) \supset B_{2R}(z_o)$, we deduce that 

$$u(x_i, t_i) \leq C \left\{ u(x_{i+1}, t_{i+1}) + \frac{R^2}{|B_{2R}(z_o) \cdot 4R^2|^{\frac{1}{\eta^3}}} \|\mathcal{L}u\|_{L^{\eta+1}(K_{2R}(z_o, 4R^2))} \right\}.$$ Therefore, we conclude that 

$$u(x, t) \leq C \left\{ u(y, s) + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{\eta^3}}} \|\mathcal{L}u\|_{L^{\eta+1}(K_{2R}(z_o, 4R^2))} \right\}$$

for a uniform constant $C > 0$ since $N \in \mathbb{N}$ is uniform. 

Arguing in a similar way as Theorem 6.6, Corollary 6.3 gives the following weak Harnack inequality.

**Theorem 6.7** (Weak Harnack Inequality). Suppose that $M$ satisfies the conditions (3), (4). Let $z_o \in M, R > 0$. Let $u$ be a nonnegative smooth function such that $\mathcal{L}u \leq f$ in
Then we have 

\[
\left( \int_{K_2(z,2R^2)} u^{p_o} \right)^{\frac{1}{p_o}} \leq C \left\{ \inf_{K_2(z,4R^2)} u + R^2 \left( \int_{K_2(z,4R^2)} |f^{+ \rho+1}| \right)^{\frac{1}{\rho+1}} \right\},
\]

where \( 0 < p_o < 1 \) and \( C > 0 \) depend only on \( n, \lambda, \Lambda \) and \( a_L \).

**Proof.** Let \( \epsilon > 0 \) be the constant in Corollary 6.2 and let \( p_o := \frac{\epsilon}{2} \). We consider a parabolic decomposition of \( M \times (0,4R^2) \) according to Lemma 5.3. Let \( k := k_2(R) \) for the constant \( A > 0 \) in the proof of Theorem 6.6. Let \( \left\{ K_{k,i} := Q_{k,i} \times (L_{k,i} - l(k), L_{k,i}) \right\}_{i \in \mathbb{N}} \) be a family of parabolic dyadic cubes intersecting \( K_R(z,2R^2) \). We note that \( \text{diam}(Q_{k,i}) \leq c_0 \delta_0 \leq \delta_0 \cdot \frac{R}{A^2} \) and \( l(k) \leq \delta_0 \cdot \frac{2R^2}{A^2} \). Following the same argument as Corollary 6.3 we deduce that \( |J_i| \) is uniformly bounded and

\[
\int_{K_{k,i}} u^{p_o} \leq |J_i| \int_{K_{k,i}} u^{p_o}
\]

for some \( K_{k,i} \) with \( \alpha \in J_i \). Then we find \((x, t) \in K_{k,i} \cap B_R(z_0) \times [R^2, (2 + \delta_0)^2 R^2] \) such that \( K_{k,i} \subset K_{k,i}(x, t) \) since \( \text{diam}(Q_{k,i}) \leq \delta_0 \cdot \frac{R}{A^2} \) and \( l(k) \leq \delta_0 \cdot \frac{2R^2}{A^2} \). Since \( d(z, x) \leq R \) and \( B_{\frac{R}{A}}(x) \subset B_{1 + \frac{R}{A}}(z_0) \), we have

\[
\frac{1}{|K_{k,i}(z,2R^2)|} \int_{K_{k,i}(z,2R^2)} u^{p_o} \leq \frac{C_0}{|K_{k,i}(x, t)|} \int_{K_{k,i}(x, t)} u^{p_o}
\]

for \( C_0 := |J_i| \left( 1 + \frac{\epsilon}{2} \right)^n \cdot \frac{2R^2}{A^2} \) by using the volume comparison theorem.

We set

\[
\inf_{K_{k,i}(z,4R^2)} u =: u(y, s)
\]

for some \((y, s) \in K_{k,i}(z,4R^2) \). As in the proof of Theorem 6.6 we take a piecewise geodesic path \( y \) connecting \( x \) to \( y \). Let \( N \in \mathbb{N} \) be the constant in Theorem 6.6. For \( i = 0, 1, \cdots, N \), we define

\[
(x_i, t_i) := \left( \gamma \left( \frac{i}{N} \right), i \frac{s - t}{N} + t \right) \in B_R(z_0) \times [R^2, 4R^2].
\]

Then we have \((x_0, t_0) = (x, t), (x_N, t_N) = (y, s) \) and for \( i = 0, \cdots, N - 1 \),

\[
d(x_{i+1}, x_i) < \frac{\kappa}{2} \frac{R}{A} \quad \text{and} \quad \frac{3R^2}{\eta^2 A^2} \leq t_{i+1} - t_i \leq \frac{16R^2}{\eta^2 A^2}.
\]

It is easy to check that for any \( i = 0, 1, \cdots, N - 1 \), \( B_{\frac{R}{A}}(x_i) \cap B_{\frac{R}{A}}(x_{i+1}) \subset B_{\frac{R}{A}}(x_{i+1}) \) and hence

\[
\inf_{K_{k,i}(x_{i+1}, t_{i+1})} u \leq \inf_{K_{k,i}(x_i, t_i)} u \leq \left\{ \frac{1}{|K_{k,i}(x_{i+1}, t_{i+1})|} \int_{K_{k,i}(x_{i+1}, t_{i+1})} u^{p_o} \right\}^{\frac{1}{p_o}} \leq 2^{\frac{n+1}{p_o}} \left\{ \frac{1}{|K_{k,i}(x_i, t_i)|} \int_{K_{k,i}(x_i, t_i)} u^{p_o} \right\}^{\frac{1}{p_o}}.
\]

(29)
On the other hand, Corollary 6.3 says that for $i = 0, 1, \cdots, N - 1$
\[
\left\{ \frac{1}{|K_{\frac{R}{t}}(x, t)|} \int_{K_{\frac{R}{t}}(x, t)} u^p \right\}^{1/p_n} \leq C \left( \inf_{K_{\frac{R}{t}}(x, t)} u + \frac{(R/A)^2}{|K_{\frac{R}{t}}(x, t)|} \|f^+\|_{L^{1+1}(K_{\frac{R}{2}4R^2)}} \right)^p
\]
\[
\leq C \left( \inf_{K_{\frac{R}{t}}(x, t)} u + \frac{R^2}{|K_{2R}(z_0, 4R^2)|} \|f^+\|_{L^{1+1}(K_{2R}(z_0, 4R^2)}} \right)^p
\]
by using the same argument as Theorem 6.6 with the volume comparison theorem. Combining with (29), we deduce
\[
\left\{ \frac{1}{|K_{\frac{R}{t}}(x, t)|} \int_{K_{\frac{R}{t}}(x, t)} u^p \right\}^{1/p_n} \leq C \left( \inf_{K_{\frac{R}{t}}(x, t)} u + \frac{R^2}{|K_{2R}(z_0, 4R^2)|} \|f^+\|_{L^{1+1}(K_{2R}(z_0, 4R^2)}} \right)^p
\]
\[
\leq C \left( \inf_{K_{\frac{R}{t}}(x, t)} u + \frac{R^2}{|K_{2R}(z_0, 4R^2)|} \|f^+\|_{L^{1+1}(K_{2R}(z_0, 4R^2)}} \right)^p
\]
for a uniform constant $C > 0$ since $N \in \mathbb{N}$ is uniform. Therefore, the result follows from (28). \(\square\)

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