Second Virial Coefficient for Noncommutative Space

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Abstract

The second virial coefficient $B_{2c}^nc(T)$ for non-interacting particles moving in a two-dimensional noncommutative space and in the presence of a uniform magnetic field $\vec{B}$ is presented. The noncommutativity parameter $\theta$ can be chosen such that the $B_{2c}^nc(T)$ can be interpreted as the second virial coefficient for anyons of statistics $\alpha$ in the presence of $\vec{B}$ and living on the commuting plane. In particular, in the high temperature limit $\beta \to 0$, we establish a relation between the parameter $\theta$ and the statistics parameter $\alpha$. Moreover, $B_{2c}^nc(T)$ can also be interpreted in terms of composite fermions.
1 Introduction

One of the most interesting features of two-dimensional systems of charged particles is their exotic or fractional statistics \[1, 2\]. Particles with this property are known as anyons and represent an interpolation between bosons and fermions. In fact, quasiparticles of the fractional quantum Hall effect \[3\] and solitons of a nonlinear $\sigma$-model \[4\] are two important candidates for a realization of anyons, since they exhibit fractional statistics.

A thermodynamic way to capture and exhibit fractional statistics of two-dimensional systems of charged particles \[5\] is the so-called second virial coefficient $B_2(T)$ \[6\]. $B_2(T)$ represents a purely quantum mechanical effect and appears as the first correction to the ideal gas equation of state

$$\frac{P}{Nk_B T} = 1 + B_2(T) N + O(N^2), \quad (1)$$

where $N$ denotes the particle density. At the classical level this coefficient is missing for non-interacting particles. Moreover, $B_2(T)$ is negative for free bosons, which means that the pressure $P$ decreases from the classical value at fixed $N$ and temperature $T$. This effect is as a consequence of the tendency of bosons to overlap. However, $B_2(T)$ is positive for non-interacting fermions as a result of the Pauli exclusion principle.

We proceed to study the second virial coefficient and associated statistics for non-interacting particles in an external uniform magnetic field and moving in a two-dimensional noncommutative space. In fact, we will show that a description of the fractional statistics in terms of noncommutative geometry \[7\] is possible. Basically our aim is to clarify the role that noncommutativity can play in the statistics of particles. This work follows our previous investigations of the present system, where interesting phenomena like nonextensive statistics \[8\], orbital magnetism \[9\] and the Hall effect \[10\] are discussed.

In section 2, we give the energy spectrum and eigenfunctions of a particle moving on a two-dimensional noncommutative plane and exposed to a uniform external magnetic field. In section 3, after recalling the definition of the second virial coefficient, we compute its noncommutative expression. This can be interpreted as the usual second virial coefficient in terms of an effective magnetic field. In section 4, by making a specific choice of the parameter $\theta$, we offer an interpretation of $B_2^{nc}(\beta)$ as the second virial coefficient for anyons in the presence of
and living on the commuting plane. Moreover, in the high temperature limit, a relation between the parameter $\theta$ and the statistics $\alpha$ is obtained. In the final section we suggest another interpretation of $B_{2c}^{\alpha}(\beta)$ in terms of composite fermions.

2 Particle in a noncommutative space

Let us consider a non-interacting particle moving in a two-dimensional space $(x, y)$ under the influence of a perpendicular uniform magnetic field $\vec{B}$. In the symmetric gauge, this system is described by the Hamiltonian

$$H = \frac{1}{2m} \left[ \left( p_x - eB \frac{2c}{2c} y \right)^2 + \left( p_y + eB \frac{2c}{2c} x \right)^2 \right].$$

We would like to study (2) on a noncommutative plane. For that purpose we can accordingly assume that the coordinates of the plane are noncommuting,

$$[x, y] = i\theta,$$

where the parameter $\theta$ is a real constant. Noncommutativity can also be imposed by treating the coordinates as commuting, but requiring that the composition of their functions be given in terms of the star product

$$\ast \equiv \exp \frac{i\theta}{2} \left( \leftarrow \partial_x \rightarrow \partial_y - \leftarrow \partial_y \rightarrow \partial_x \right).$$

Following this route, we deal with commutative coordinates $x$ and $y$, but replace the ordinary products with the star product (4). For example, the commutator (3) is replaced by the expression

$$x \ast y - y \ast x = i\theta.$$

As usual, canonical quantization of this system is achieved by introducing the coordinate and momentum operators $x_i, p_i$ satisfying

$$[x_i, p_j] = i\hbar \delta_{ij}, \quad [p_i, p_j] = 0,$$

where $x_1 = x$ and $x_2 = y$. Thus, we treat our system in the framework of the algebra generated by the commutation relations (5)-(6), which implies that all subsequent products are replaced
by their star product counterparts as envisaged above. According to this prescription, the above Hamiltonian acts on an arbitrary function $\Psi(\vec{r}, t)$ as

$$H \star \Psi(\vec{r}, t) = \frac{1}{2m} \left[ (p_x - \frac{eB}{2c}y)^2 + (p_y + \frac{eB}{2c}x)^2 \right] \star \Psi(\vec{r}, t) \equiv H^{nc} \Psi(\vec{r}, t).$$

(7)

Therefore, the noncommutative version of (2) can be inferred to be

$$H^{nc} = \frac{1}{2m} \left[ (\hat{p}_x - \frac{eB}{2c}y)^2 + (\hat{p}_y + \frac{eB}{2c}x)^2 \right],$$

(8)

where the momentum operator $\hat{p}_i$ is a $\theta$-dependent function

$$\hat{p}_i = (1 - \theta l^{-2})p_i,$$

(9)

with $l = 2l_0$, $l_0 = \sqrt{\frac{\hbar c}{eB}}$ being the magnetic length. Notice that the standard Hamiltonian is recovered in the limit $\theta = 0$.

The eigenvalue problem

$$H^{nc} \Psi^{nc} = E^{nc} \Psi^{nc},$$

(10)

can be solved by introducing a set of creation and annihilation operators on the complex plane $(z, \bar{z})$ such that

$$\hat{a}^\dagger = -2i\hat{p}_z + \frac{m\omega_c}{2}z,$$

$$\hat{a} = 2i\hat{p}_\bar{z} + \frac{m\omega_c}{2}\bar{z},$$

(11)

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. These operators satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 2m\hbar\hat{\omega}_c,$$

(12)

with $\hat{\omega}_c = \omega_c(1 - \theta l^{-2})$. Now $H^{nc}$ can be expressed in terms of $\hat{a}$ and $\hat{a}^\dagger$

$$H^{nc} = \frac{1}{4m}(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger).$$

(13)

To distinguish between the degenerate eigenstates of (13), we define another set of creation and annihilation operators

$$\hat{b}^\dagger = 2i\hat{p}_z - \frac{m\omega_c}{2}\bar{z},$$

$$\hat{b} = -2i\hat{p}_\bar{z} - \frac{m\omega_c}{2}z.$$

(14)
They both commute with $\hat{a}$ and $\hat{a}^\dagger$ and their commutator is again
\[
[\hat{b}, \hat{b}^\dagger] = 2m\hbar\hat{\omega}_c. \tag{15}
\]

Therefore, we can show that the energy spectrum and the eigenstates are
\[
\psi_{nc}^{n(k,\theta)} = \sqrt{\frac{1}{(2m\hbar\hat{\omega}_c)^{n+k+n+k!}}} (\hat{a}^\dagger)^n (\hat{b}^\dagger)^k |0,0>, \tag{16}
\]
\[
E_n^{nc} = \hbar\hat{\omega}_c(n + \frac{1}{2}).
\]

Some remarks are in order at this stage. The eigenstates are labeled not only by the quantum numbers associated with $\hat{a}$ and $\hat{a}^\dagger$, but also depend on an additional degree of freedom. Moreover, from the expression of the eigenstates we distinguish three different cases: (i)- $\theta < l^2$, (ii)- $\theta > l^2$ and (iii)- $\theta = l^2$. The first two cases can be dealt with simultaneously, by simply considering the replacement $\theta \longleftrightarrow -\theta$. Furthermore, in the subsequent analysis we do not consider the case (iii) which would simply lead to a static critical point not relevant to our discussion.

## 3 Noncommutative second virial coefficient

The second virial coefficient for particles has been investigated on many occasions, in particular for anyons in the absence \[12,13\] as well as in the presence of a magnetic field \[5,14\]. Further study can be found in \[15\]. For instance in \[5\] it is found that $B_2(T)$ becomes a function of the fractional statistics $\alpha$. Now let us recall the basic definition of $B_2(T)$ in two dimensions \[5\]:
\[
B_2(T) = \lim_{S \to \infty} S \left[ \frac{1}{2} - \frac{Z_2}{Z_1^2} \right], \tag{17}
\]
where $S$ is the area of the system and $Z_1$ and $Z_2$ are the single-particle and two-particle partition functions, respectively
\[
Z_i = \text{Tr} \exp(-\beta H_i), \quad \beta = \frac{1}{k_B T}, \quad i = 1, 2. \tag{18}
\]

We now investigate the second virial coefficient for non-interacting particles moving on a noncommutative plane and in the presence of a magnetic field. This latter can be defined in the standard way as \[17\]
\[
B_2^{nc}(T) = \lim_{S \to \infty} S \left[ \frac{1}{2} - \frac{Z_2^{nc}}{(Z_1^{nc})^2} \right], \tag{19}
\]
where the partition function in the noncommutative coordinates is

\[ Z_{nc}^i = \text{Tr} \exp(-\beta H_{nc}^i). \]  

(20)

Thus, for two particles of energy \( E_{nc}^{(n_1, n_2)} = \hbar \omega_c (n_1 + n_2 + 1) \), we have

\[ Z_{nc}^2(\beta) = \sum_{n_1, n_2} \exp(-\beta E_{nc}^{(n_1, n_2)}). \]  

(21)

The calculation of (21) depends on what kind of particles we have. For non-interacting bosons in the presence of a magnetic field and living on a noncommutative plane, \( Z_{nc}^2(\beta) \) is

\[ Z_{nc}^2(\beta) = \frac{1}{2} \left( [Z_{nc}^1(\beta)]^2 + Z_{nc}^1(2\beta) \right). \]  

(22)

To obtain (22), we sum independently on \( n_1 \) and \( n_2 \), also we add to this sum the contribution of the diagonal states for which \( n_1 = n_2 \). Actually, the noncommutative second virial coefficient \( B_{nc}^2(T) \) becomes

\[ B_{nc}^2(T) = -\frac{1}{2} \lim_{S \to \infty} \frac{S}{Z_{nc}^1(2\beta)} Z_{nc}^1(\beta). \]  

(23)

To evaluate explicitly the single-particle partition function \( Z_{nc}^1(\beta) \), we introduce coherent states corresponding to the present system

\[ |\eta, \mu > = e^{-\frac{1}{2\hbar}(|\eta|^2 + |\mu|^2)} \exp\left( \frac{\eta \hat{a}^\dagger + \mu \hat{b}^\dagger}{2\hbar} \right) |0, 0 >. \]  

(24)

They satisfy the relations

\[ \hat{a}|\eta, \mu > = \frac{\hbar}{2\hbar} \eta|\eta, \mu >, \]
\[ \hat{b}|\eta, \mu > = \frac{\hbar}{2\hbar} \mu|\eta, \mu >. \]  

(25)

Now \( Z_{nc}^1(\beta) \) can be written as follows

\[ Z_{nc}^1(\beta) = \frac{4e^{-\beta \omega_c}}{\pi^2 \hbar^4} \int d^2 \eta d^2 \mu <\eta, \mu | e^{-\beta \omega_c} \hat{a}^\dagger \hat{a} | \eta, \mu >. \]  

(26)

Using the boson-operator identity

\[ e^{\xi a^\dagger a} = \sum_{n=0}^{\infty} \frac{(e^\xi - 1)^n}{n!} a^\dagger a^n, \]  

(27)

which holds for any operators \( a^\dagger \) and \( a \) satisfying the commutation relation \([a, a^\dagger] = 1\), we can show that (26) simplifies to

\[ Z_{nc}^1(\beta) = \frac{4e^{-\beta \omega_c}}{\pi^2 \hbar^4} \int d^2 \eta d^2 \mu e^{-2|\mu|^2/(1-e^{-\beta \hbar \omega_c})}. \]  

(28)
For a system of area $S$, the integral can be calculated to be

$$Z_{nc}^1(\beta) = \frac{m\hat{\omega}_c S}{4\pi \hbar} \frac{1}{\sinh \left( \frac{\beta\hat{\omega}_c}{2} \right)}.$$  \hfill (29)

Now, from (23) and (29) we obtain

$$B_{nc}^2(T) = -\frac{\pi \hbar}{m\hat{\omega}_c} \tanh \left( \frac{\beta\hat{\omega}_c}{2} \right).$$  \hfill (30)

This equation shows the dependence of $B_{nc}^2(T)$ on the noncommutativity parameter $\theta$. The free fermion case can be worked out in the same way as above by taking the Pauli exclusion principle into account,

$$B_{nc}^2(T)|_f = -B_{nc}^2(T)|_b = \frac{\pi \hbar}{m\omega_c} \tanh \left( \frac{\beta\hat{\omega}_c}{2} \right),$$  \hfill (31)

where $f$ and $b$ refer to fermions and bosons, respectively. Notice that the two expressions only differ by a minus sign. We would like to emphasize that the standard results are recovered if the noncommutativity parameter $\theta$ is switched off:

$$B_{nc}^2(T)|_f = -B_{nc}^2(T)|_b = \frac{\pi \hbar}{m\omega_c} \tanh \left( \frac{\beta\hat{\omega}_c}{2} \right).$$  \hfill (32)

Comparison of (30) with (32) suggests that one can interpret the noncommutative case as a theory of the second virial coefficient on a commuting plane with an effective magnetic field

$$B_{\text{eff}} = B - B\theta l^{-2}.$$  \hfill (33)

Some remarks are in order at this stage. One can measure the parameter $\theta$ in terms of the magnetic length $l^2$:

$$\theta_l = \frac{m}{n} l^2.$$  \hfill (34)

For $\theta > \theta_l^2$, $\frac{m}{n} > 1$ and we obtain a quantized effective magnetic field

$$B_{\text{eff}} = \left( \frac{n - m}{n} \right) B.$$  \hfill (35)

The quantization can be integer as well as fractional, depending on whether the ratio $\frac{m}{n}$ is integer or fractional. On the other hand, if we restrict ourselves to the high temperature limit ($\beta \to 0$), we find that $B_{nc}^2(T)$ can be approximated as

$$B_{nc}^2(T) \approx -\frac{\pi l^2}{8} \left[ x - \frac{x^3}{12} \right] + \frac{\pi l^2 x^3}{96} \left[ \theta^2 l^{-4} - 2\theta l^{-2} \right],$$  \hfill (36)
where $x = \beta \hbar \omega_c$. In the same limit, we have for $B_2(T)_b$

$$B_2(T)_b \approx -\frac{\pi l^2}{96} \left[ x - \frac{x^3}{12} \right].$$  \hfill (37)

Therefore, we obtain

$$B_2^{\text{nc}}(T) \approx B_2(T)_b + \frac{\pi l^2 x^3}{96} \left[ 2\theta l^{-4} - 2\theta l^{-2} \right].$$  \hfill (38)

From this relation it is clear that in the high temperature limit, a correction to the standard second virial coefficient is obtained in terms of the noncommutativity parameter $\theta$.

## 4 Fractional statistics and noncommutativity parameter

In this section we present an interpretation of the second virial coefficient for free bosons moving on a noncommutative plane. For that purpose, let us recall a result for the second virial coefficient obtained for anyons in the presence of a magnetic field [5, 14]. Based on bosons, Johnson and Canright [5] found that the coefficient is given by

$$B_\alpha^{(2,b)}(T) = -\frac{\lambda^2}{2x} \text{tanh} \left( \frac{x}{2} \right) + \frac{\lambda^2}{x} \left[ \frac{e^x}{\sinh x} (1 - e^{-\alpha x}) - \alpha \right],$$  \hfill (39)

where $0 \leq \alpha \leq 1$ and $\lambda = \left( \frac{2\pi \hbar^2}{mkT} \right)^{\frac{1}{2}}$. Moreover, they showed that the fermion-based result can be derived from the bosonic case, leading to

$$B_\alpha^{(2,f)}(T) = B_\alpha^{(2,b)}(T).$$  \hfill (40)

For this we can establish a relation between the noncommutativity parameter $\theta$ and the statistics parameter $\alpha$. Alternatively, we consider the possibility to describe exotic statistics in terms of noncommutative geometry. To do this, we can identify [30] with (39)

$$\frac{\lambda^2}{2x} \text{tanh} \left( \frac{\hat{x}_\alpha}{2} \right) = \frac{\lambda^2}{2x} \text{tanh} \left( \frac{x}{2} \right) - \frac{\lambda^2}{x} \left[ \frac{e^x}{\sinh x} (1 - e^{-\alpha x}) - \alpha \right],$$  \hfill (41)

where $\hat{x}_\alpha = x(1 - \theta_\alpha l^{-2})$. It is clear that this equation cannot be solved Explicitly. Some approximations are thus required to proceed analytically. Let us consider the high temperature limit, where $B_\alpha^{(2,b)}(T)$ can be expressed as

$$B_\alpha^{(2,b)}(T) \approx -\frac{\pi l^2}{8} \left[ x - \frac{x^3}{12} \right] + \frac{\pi l^2 \alpha}{2} \left[ x + \frac{x^2}{3} \right].$$  \hfill (42)
From equations (36) and (42), we find that $\theta_\alpha$ satisfies a second order equation

$$\theta_\alpha^2 - 2\theta_\alpha l^2 - 48 l^4 \alpha \left[ \frac{1}{x^2} + \frac{1}{3x} \right] = 0.$$  \hspace{1cm} (43)

According to the condition $\theta > l^2$, there is only one valid solution of this equation

$$\theta_\alpha = l^2 \left[ 1 + \sqrt{1 + 48 \alpha f(x)} \right],$$ \hspace{1cm} (44)

where we have set $f(x) = \frac{1}{x^2} + \frac{1}{3x}$ and it is always positive when $\beta \to 0$. Therefore, in the high temperature limit, and when $\theta$ is fixed to be $\theta_\alpha$, one can interpret $B^{nc}_2(T)$ as the second virial coefficient for anyons of statistics $\theta_\alpha$. We close this section by mentioning that for small $\theta$, the last equation becomes

$$\theta_\alpha = 24 l^2 \alpha \left[ \frac{1}{x^2} + \frac{1}{3x} \right].$$ \hspace{1cm} (45)

Clearly the condition $\theta > l^2$ implies the further constraint $24 \alpha \left[ \frac{1}{x^2} + \frac{1}{3x} \right] > 1$.

5 Concluding remarks

Complementary to the analysis above, we can also give an interpretation of $B^{nc}_2(T)$ in terms of composite fermions (CF) [18], which have been introduced as a new type of particle in condensed matter physics to provide an explanation of the fractional quantum Hall effect [19]. CF’s are particles carrying an even number $2p$ ($p = 1, 2, \cdots$) of flux quanta (vortices). They have the same charge, spin and statistics as particles, but they differ from them since they experience an effective magnetic field

$$B^* = B - 2pN\Phi_0,$$ \hspace{1cm} (46)

where $\Phi_0 = \frac{hc}{e}$ is the unit of flux. To interpret $B^{nc}_2(T)$ as the second virial coefficient for composite fermions we should choose $\theta$ such that

$$B_{\text{eff}}|_{\theta=\theta_c} = B^*.$$ \hspace{1cm} (47)

We solve this to obtain

$$\theta_c = \frac{4}{\pi \mu_B} \frac{pm}{n_B},$$ \hspace{1cm} (48)
where \( n_B = \frac{\Phi_B}{\Phi_0} \). Here \( \Phi_B \) is the flux due to \( \vec{B} \) and \( n = NS \) is the number of particles. Thus, this equation shows the possibility to express the noncommutativity parameter \( \theta \) in terms of the flux quanta.

In conclusion, non-interacting particles on a noncommutative plane and in the presence of a magnetic field have been considered. The second virial coefficient \( B_{nc}^2(T) \) corresponding to this system is obtained in terms of the noncommutativity parameter \( \theta \). By fixing the latter to be \( \theta_\alpha \), we interpreted \( B_{nc}^2(T) \) as the second virial coefficient for anyons of statistics \( \alpha \) moving on the commuting plane and experiencing a magnetic field. Moreover in the high temperature limit, a relation between the fractional statistics \( \alpha \) and the parameter \( \theta \) is obtained. These results illustrate the possibility to deal with the particle statistics in terms of noncommutative geometry.

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