EXISTENCE OF SELF-CHEEGER SETS ON RIEMANNIAN MANIFOLDS

IGNACE ARISTIDE MINLEND

Abstract. Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(N \geq 2\). We prove the existence of a family \((\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}\) of self-Cheeger sets in \((M, g)\). The domains \(\Omega_\varepsilon \subset M\) are perturbations of geodesic balls of radius \(\varepsilon\) centered at \(p \in M\), and in particular, if \(p_0\) is a non-degenerate critical point of the scalar curvature of \(g\), then the family \((\partial \Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}\) constitutes a smooth foliation of a neighborhood of \(p_0\).

1. Introduction and main results

Given a measurable set \(A\) and an open set \(\Omega\) in a compact Riemannian manifold \((M, g)\), the relative perimeter, \(P(A, \Omega)\), of \(A\) in \(\Omega\) is defined (see for instance [18]) by

\[
P(A, \Omega) := \sup \left\{ \int_A \text{div}_g(\xi) \, dv_g : \xi \in C^1_c(\Omega), |\xi| \leq 1 \right\},
\]

where \(dv_g\) is the Riemannian volume element and \(C^1_c(\Omega)\) is the set of \(C^1\) vectors fields with compact support in \(\Omega\). The perimeter, \(P(A)\), of a measurable set \(A \subset M\) is the relative perimeter \(P(A, M)\) of \(A\) in \(M\). If a set \(A\) is of finite perimeter, denoting by \(\partial^* A\) the reduced boundary of \(A\), \(P(A)\) and \(P(A, \Omega)\) are respectively the \((N-1)\)-dimensional Riemannian volume of \(\partial^* A\) and \(\partial^* A \cap \Omega\).

Let \(u \in L^1(\Omega)\), the total variation of \(u\) in \(\Omega\) is defined by

\[
|Du|(\Omega) = \sup \left\{ \int_\Omega u \text{div}_g(\xi) \, dv_g : \xi \in C^1_c(\Omega), |\xi| \leq 1 \right\}
\]

and we say that \(u\) is a function with bounded variation in \(\Omega\) if its total variation is finite. The space of functions with bounded variation in \(\Omega\) is denoted \(BV(\Omega)\). We notice that \(P(A, \Omega)\) is the total variation of the characteristic function \(1_A\) in \(\Omega\).

Let \(A\) be a set of finite perimeter in \(M\), then from De Giorgi’s structure Theorem [9, Theorem 2.2] (see also [11, Remark on p.161] or [1]), we have the Gauss-Green formula

\[
\int_A \text{div}_g(\xi) \, dv_g = \int_{\partial^* A} \langle \xi, \nu_A \rangle \, d\sigma_g \quad \text{for all } \xi \in C^1(M),
\]

(1.1)

where \(\nu_A\) is the unit outer normal to \(\partial^* A\) and \(d\sigma_g\) is the area element induced by the metric \(g\).

Key words and phrases. Over-determined problems, Cheeger sets, foliation.
Let $\Omega$ be an open set in $(\mathcal{M}, g)$ with smooth boundary. The Cheeger constant $h(\Omega)$ of $\Omega$ is defined by
\[
h(\Omega) := \inf_A \frac{P(A)}{|A|},
\]
where the domain $A$ varies over all measurable subsets of $\Omega$ with finite perimeter and positive $N$-dimensional Riemannian volume denoted by $|A|$. It was first introduced by J. Cheeger in [5] to obtain geometric lower bound of eigenvalues on $\Omega$. We refer the reader to [1] and [17] for an introductory survey on the Cheeger problem and to [2, 3] for further results on manifolds. Some isoperimetric estimates are also given in [13].

The Cheeger constant has physical applications in modeling of land slides [4], in image processing [12] and in fracture mechanics [14].

Any set $F$ in $\Omega$ which realizes the infimum (1.2) is called a Cheeger set in $\Omega$. If $\Omega$ is a minimizer for (1.2), we say that $\Omega$ is self-Cheeger. $\Omega$ will be called uniquely self-Cheeger, if any Cheeger set in $\Omega$ is equal to $\Omega$ up to a Riemann measure zero set.

The main result of this paper is the following.

**Theorem 1.1.** Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 2$. Then there exists a family of uniquely self-Cheeger sets $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ with
\[
h(\Omega_\varepsilon) = \frac{N}{\varepsilon} \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0).
\]
Moreover, if $p_0$ is a non-degenerate critical point of the scalar curvature $s$ of $g$, then the family $(\partial \Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ constitutes a smooth foliation of a neighborhood of $p_0$.

Theorem 1.1 relies on Theorem 1.3 due to M.M. Fall and the author in [8] together with Theorem 1.2 below. Indeed Theorem 1.2 gives a sufficient condition on the Ricci curvature for a domain $\Omega \subset \mathcal{M}$ to be a uniquely self-Cheeger while Theorem 1.3 provides a family of domains where this condition is always satisfied.

**Theorem 1.2.** Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 2$ and $\Omega$ a bounded domain in $\mathcal{M}$ with Lipschitz boundary. Assume that there exists a function $u \in C^2(\Omega)$ such that
\[
\begin{cases}
-\Delta_g u = 1 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega \\
g(\nabla u, \eta) = -\lambda & \text{on } \partial \Omega,
\end{cases}
\]
where $\lambda$ is a positive constant, $\Delta_g = \text{div}(\nabla_g)$ is the Laplace-Beltrami operator and $\eta$ is the unit outer normal of $\partial \Omega$. If
\[
\text{Ric}_g(\nabla u, \nabla u) > -\frac{1}{N} \quad \text{in } \Omega,
\]
then $\Omega$ is a uniquely self-Cheeger set with Cheeger constant equal to $\frac{1}{\lambda}$. 

Theorem 1.3. Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension $N \geq 2$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exist a smooth domain $\Omega_\varepsilon$ and a function $u_\varepsilon \in C^2(\overline{\Omega_\varepsilon})$ such that

$$
\begin{align*}
-\Delta_g u_\varepsilon &= 1 & \text{in } & \Omega_\varepsilon \\
u_\varepsilon &= 0 & \text{on } & \partial \Omega_\varepsilon \\
g(\nabla_g u_\varepsilon, \nu_\varepsilon) &= -\frac{\varepsilon}{N} & \text{on } & \partial \Omega_\varepsilon.
\end{align*}
$$

In particular if $p_0$ is a non-degenerate critical point of the scalar curvature $s$ of $g$, then the family $(\partial \Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ constitutes a smooth foliation of a neighborhood of $p_0$.

The domains $\Omega_\varepsilon$ in Theorem 1.3 are perturbations of geodesic balls $B^g_\varepsilon(p)$ centered at $p$ with radius $\varepsilon$. Indeed, we have that $\Omega_\varepsilon = (1 + v_{p, \varepsilon})B^g_\varepsilon(p)$, with $v_{p, \varepsilon} : \partial B^g_\varepsilon(p) \to \mathbb{R}$ satisfying

$$
||v_{p, \varepsilon}||_{C^{2, \alpha}(\partial B^g_\varepsilon(p))} \leq c\varepsilon^2,
$$

where the constant $c$ is independent on $\varepsilon$.

If $p_0$ is a non-degenerate critical point of the scalar curvature $s$ of $g$, we obtain a precise form of the boundary of the domains $\Omega_\varepsilon$: there exists a smooth function $\omega_\varepsilon : S^{N-1} \to \mathbb{R}_+$ such that

$$
\partial \Omega_\varepsilon = \left\{ \exp_{p_0} \left( \omega_\varepsilon(y) \sum_{i=1}^N y^i E_i \right) \right\}, \quad y \in S^{N-1}
$$

and moreover the map $\varepsilon \mapsto \omega_\varepsilon$ satisfies $\partial_\varepsilon \omega_\varepsilon|_{\varepsilon=0} = 1$.

From the result of Micheletti and Pistoia in [15], it is known that for a generic metric on a manifold, all critical points of the scalar curvature are non-degenerate. The result of Theorem 1.1 then implies that for a generic metric $g'$, a neighborhood of any critical point of the scalar curvature can be foliated by a family $(\partial \Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$, where $\Omega_\varepsilon$ is a uniquely self-Cheeger with Cheeger constant equal to $N/\varepsilon$. This can be seen as the analogue of Ye’s result, who proved in [20] that around a non-degenerate critical point of the scalar curvature function in a Riemannian manifold, there exists a unique regular foliation by constant mean curvature spheres. It is an open question to know whether the hypersurfaces constructed by Ye enclose self-Cheeger sets. Indeed, Nardulli [16] proved that the sets constructed by Ye are solutions to the isoperimetric problem in a small neighborhood of the non-degenerate maxima of the scalar curvature function. From the author’s knowledge, except in space of constant sectional curvature, a condition under which a solution to the isoperimetric problem is a self-Cheeger set is not yet known. The relation between solutions to the isoperimetric problem and self-Cheeger sets is discussed in Remark 3.1.
2. Proof of Theorem 1.2

Let \( u \) be the solution of (1.3) in \( \Omega \). We claim that

\[
|\nabla u|_g < \lambda \quad \text{in} \quad \Omega. \tag{2.1}
\]

Indeed, we consider the smooth function \( P : \mathcal{M} \to \mathbb{R} \) defined by

\[
P(q) := |\nabla u(q)|_g^2, \quad q \in \mathcal{M}.
\]

We recall Bochner’s formula (see [19]): for all \( u \in C^3(\mathcal{M}) \),

\[
\Delta_g |\nabla u|_g^2 = 2|\text{Hess}_g(u)|^2 + 2\langle \nabla u, \nabla (\Delta_g u) \rangle_g + 2\text{Ric}_g(\nabla u, \nabla u), \tag{2.2}
\]

where \( \text{Ric}_g \) is the Ricci curvature of \( g \), \( \text{Hess}_g(u) \) stands for the Hessian of \( u \) and we have denoted \( |A|^2 = \text{tr}(AA^t) \) for a bilinear form \( A \). By the Cauchy-Schwarz inequality,

\[
|\text{Hess}_g(u)|^2 \geq \frac{1}{N} (\Delta_g u)^2. \tag{2.3}
\]

Since \( -\Delta_g u = 1 \) in \( \Omega \), we get from (2.2) and (2.3) that

\[
\frac{1}{2} \Delta_g P(q) \geq \frac{1}{N} + \text{Ric}_g(\nabla u(q), \nabla u(q)), \quad q \in \Omega.
\]

Using the assumption \( \text{Ric}_g(\nabla u(q), \nabla u(q)) > -1/N \) in \( \Omega \), we get \( \Delta_g P > 0 \) in \( \Omega \). Thus by the strong maximum principle, \( P \) attains its maximum only on \( \partial \Omega \). Since \( P \equiv \lambda^2 \) on \( \partial \Omega \) by (1.3), the claim (2.1) follows.

Let \( A \subseteq \Omega \) be a set of finite perimeter. Then integrating (1.3) on \( A \) and applying the claim (2.1) with the Cauchy-Schwarz inequality, we get

\[
|A| = -\int_{\partial^* A} \langle \nabla u, \nu_A \rangle d\sigma_g \leq \int_{\partial^* A} |\nabla u| d\sigma_g \leq \lambda P(A). \tag{2.4}
\]

Recalling \( \Omega \) is Lipschitz, we integrate (1.3) on \( \Omega \) and use (2.4) to get the inequality

\[
\frac{P(\Omega)}{|\Omega|} = \frac{1}{\lambda} \leq \frac{P(A)}{|A|},
\]

which shows that \( \Omega \) is self-Cheeger with Cheeger constant equal to \( 1/\lambda \).

To complete the proof, it remains to show that \( \Omega \) is uniquely self-Cheeger. Let \( C \) be a Cheeger set inside \( \Omega \), we need to show that \( |\Omega \setminus C| = 0 \).

Since \( C \) is a Cheeger set we have \( |C| = \lambda P(C) \), and writing (2.4) with \( C \) in place of \( A \), we get

\[
\int_{\partial^* C} (\lambda - |\nabla u|) d\sigma_g = 0.
\]

From this and (2.1) we get

\[
|\nabla u| = \lambda \quad \text{on} \quad \partial^* C, \quad \sigma_g \quad \text{almost everywhere}
\]

and hence \( \partial^* C \subseteq \partial \Omega \).
We now prove that $|\Omega \setminus C| = 0$. This follows from Poincaré-Wirtinger’s inequality [10, Theorem 2.10]:

$$\forall w \in W^{1,1}(\Omega), \quad \|w - m(w)\|_{L^1(\Omega)} \leq k \int_{\Omega} |\nabla w| dv_g,$$

(2.5)

where $k > 0$ is a constant only depending on $\Omega$ and 

$$m(w) = \frac{1}{|\Omega|} \int_{\Omega} wdv_g.$$

Since $\Omega$ has Lipschitz boundary, for every $w \in BV(\Omega)$ there exists a sequence $(w_n) \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$ such that $w_n \rightarrow w$ in $L^1(\Omega)$ and $\int_{\Omega} |\nabla w_n| dv_g \rightarrow |Dw|(\Omega)$.

Inequality (2.5) then also holds in the space $BV(\Omega)$.

For $w = 1_C \in BV(\Omega)$, we have

$$|1_C - m(1_C)| = \frac{|\Omega \setminus C|}{|\Omega|} \quad \text{in} \quad C \quad \text{and} \quad |1_C - m(1_C)| = \frac{|C|}{|\Omega|} \quad \text{in} \quad \Omega \setminus C.$$

From this and (2.5), we get

$$\|1_C - m(1_C)\|_{L^1(\Omega)} = 2 \frac{|C|}{|\Omega|} |\Omega \setminus C| \leq kP(C, \Omega).$$

Since $\partial^* C \cap \Omega = \emptyset$, we have $P(C, \Omega) = |\partial^* C \cap \Omega| = 0$. Thanks to the above inequality, $|\Omega \setminus C| = 0$, and the proof of Theorem 1.2 is complete.

3. Proof of Theorem 1.1

In view of Theorem 1.2, the proof of Theorem 1.1 will be fulfilled once we show that the solution $u_\varepsilon$ to (1.5) satisfies

$$\text{Ric}_g(\nabla u_\varepsilon, \nabla u_\varepsilon) > -\frac{1}{N} \quad \text{in} \quad \Omega_\varepsilon.$$

Denote by $r_p(\cdot) := \text{dist}_g(p, \cdot)$ the Riemannian geodesic distance function to $p$. We recall that for all $\varepsilon \in (0, \varepsilon_0)$, the solution $u_\varepsilon$ to (1.5) is given (see [8, Section 3]) by

$$u_\varepsilon(q) = \frac{\varepsilon^2 - r_p^2(q)}{2N} + \varepsilon^4 w_\varepsilon(q) \quad \text{for all} \quad q \in \Omega_\varepsilon,$$

(3.1)

where

$$\|w_\varepsilon\|_{C^2(M)} \leq c$$

and $c$ is a positive constant independent on $\varepsilon$. Since $\mathcal{M}$ is compact, there exists a positive constant $C$ depending only on $\mathcal{M}$ such that

$$\text{Ric}_g(\nabla u_\varepsilon(q), \nabla u_\varepsilon(q)) \geq -C|\nabla u_\varepsilon(q)|^2_g,$$

(3.2)
Using (3.1) and the fact that $|\nabla r_p(\cdot)|_g = 1$, we get

$$|\nabla u_\varepsilon(q)|_g^2 \leq c\varepsilon^2,$$

where $c$ is a positive constant only depending on $\mathcal{M}$. This together with (3.2) implies that

$$\text{Ric}_g(\nabla u_\varepsilon(q), \nabla u_\varepsilon(q)) \geq -C\varepsilon^2,$$  (3.3)

where $C$ is a positive constant only depending on $\mathcal{M}$. Finally, it follows from (3.3) that for all $\varepsilon$ small enough

$$\text{Ric}_g(\nabla u_\varepsilon, \nabla u_\varepsilon) > -\frac{1}{N} \text{ in } \Omega_\varepsilon.$$

This ends the proof of Theorem 1.1.

**Remark 3.1.** It is an open problem to know if the sets enclosed by Ye’s hypersurfaces are self-Cheeger sets on $\mathcal{M}$. It is known (see [6, 16]) that solutions to the isoperimetric problem with small volume are near the maxima of the scalar curvature, and if this maximum is non-degenerate, then [16] implies that, solutions to the isoperimetric problem are among Ye’s sets.

As we shall see below, it might be possible to have some answers to the above open problem, provided higher order Taylor expansions of the perimeter of an isoperimetric set, as volume tends to zero, is available.

In what follows, $B$ denotes the unit Euclidean ball of $\mathbb{R}^N$ and we put $\omega_{N-1} := |\partial B|$.

The isoperimetric profile of a compact Riemannian manifold is given, see [6, 16] by

$$I_{\mathcal{M}}(v, g) = \inf_{E \subset \mathcal{M}, |E| = v} P(E)$$

while the Cheeger isoperimetric profile is defined [7, section 4] by

$$H_{\mathcal{M}}(v, g) = \inf_{\Omega \subset \mathcal{M}, |\Omega| = v} h(\Omega),$$  (3.4)

where $h(\Omega)$ is the Cheeger constant of the set $\Omega \subset \mathcal{M}$.

The profile $H_{\mathcal{M}}$ was derived in [7, Theorem 4.1] near $v = 0$, $(v > 0)$ and expands as

$$H_{\mathcal{M}}(v, g) = \left(1 + O(v^{\frac{4}{N}})\right) \frac{I_{\mathcal{M}}(v, g)}{v}$$  (3.5)

and the isoperimetric profile of the manifold $\mathcal{M}$ has the following expansion, [16, Theorem 8](assuming the absolute maxima of the scalar curvature $s$ are non-degenerate critical points),

$$I_{\mathcal{M}}(v, g) = c_N v^{\frac{N-1}{N}} \left(1 - \frac{S}{2N(N+2)} \left(\frac{v}{|B|}\right)^{\frac{N}{2N}} + O(v^{\frac{4}{N}})\right),$$

where

$$S = \max_{p \in \mathcal{M}} s(p) \quad \text{and} \quad c_N = \frac{\omega_{N-1}}{|B|^{\frac{N-1}{N}}}.$$
Let $E$ be a solution to the isoperimetric problem with volume $v$, then by (3.5) we get
\[ \mathcal{H}_M(v, g) = \left(1 + \mathcal{O}(v^{\frac{3}{N}})\right) \frac{P(E)}{|E|}. \tag{3.6} \]

If we further assume that $E$ is a self-Cheeger set so that
\[ h(E) = \frac{P(E)}{|E|}, \]
then by (3.6) and the definition of $\mathcal{H}_M$ in (3.4), we get
\[ \left(1 + \mathcal{O}(v^{\frac{3}{N}})\right) h(E) = \mathcal{H}_M(v, g) \leq h(E). \]

A higher order expansion of the perimeter of an isoperimetric set is then needed in order to derive a (necessary) condition under which a solution to the isoperimetric problem (or more generally the enclosure of a Ye’s hypersurface) is a self-Cheeger set.

**Acknowledgements:** The author thanks his supervisor Mouhamed Moustapha Fall for his helpful suggestions and insights throughout the writing of this paper. He is also grateful to the reviewer for his valuable comments. Part of this work was done when the author was visiting the Institute of Mathematics of the Goethe-University Frankfurt. This work is supported by the German Academic Exchange Service (DAAD).

**References**

[1] F. Alter, V. Caselles, and A. Chambolle. A characterization of convex calibrable sets in $\mathbb{R}^N$. Math. Ann., 332(2):329-366, 2005.
[2] B. Benson, The Cheeger constant, isoperimetric problems, and hyperbolic surfaces, preprint (2016), [http://arxiv.org/abs/1509.08993](http://arxiv.org/abs/1509.08993).
[3] L. Cadeddu, S. Gallot and A. Loi. Maximizing tortional rigidity on Riemannian manifolds, preprint (2013), [http://arxiv.org/abs/1309.7796](http://arxiv.org/abs/1309.7796).
[4] A. Chambolle, V. Caselles, D. Cremers, M. Novaga and T. Pock, An introduction to Total Variation for Image Analysis, Radon Series Comp. Appl. Math XX, 1-76, 2009.
[5] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian. In Problems in analysis (Papers dedicated to Salomon Bochner, 1969), pages 195-201. Princeton Univ. Press, Princeton, N. J., 1970.
[6] O. Druet, Asymptotic expansion of the Faber-Krahn profile of a compact Riemannian manifold. C. R. Math. Acad. Sci. Paris 346 (2008), no. 21-22, 1163-1167.
[7] M. M. Fall, Some local eigenvalue estimates involving curvatures. Calc. Var. Partial Differential Equations 36 (2009), no. 3, 437-451.
[8] M. M. Fall and I. A. Minlend, Serrin’s over-determined problem in Riemannian manifolds. Adv. Calc. Var. 8 (2015), 371-400.
[9] A. Figalli and Y. Ge, Isoperimetric-type inequalities on constant curvature manifolds, Adv. Calc. Var. 5 (2012), no. 3, 251-284.
[10] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Institute of Mathematical Sciences, Lecture Notes in Mathematics, 5, 1999. Second edition published by the American Mathematical Society, 2000.
[11] S. Hofmann, M. Mitrea, and M. Taylor, Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains, IMRN 2010 (2010), 2567-2865.
[12] I.R. Ionescu, Th. Lachand-Robert, Generalized Cheeger sets related to landslides, Calc. Var. 23 (2005) 227 7249.
[13] B. Kawohl and V.Fridman, Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant. Comment. Math. Univ. Carolin. 44 (2003) 659-667.
[14] J.B. Keller, Plate failure under pressure, SIAM Rev. 22 (1980) 2277228.
[15] A. M. Micheletti, A. Pistoia, Generic properties of critical points of the scalar curvature for a Riemannian manifold, Proc. Amer. Math. Soc. 138 (2010), no. 9, 3277-3284.
[16] S. Nardulli, Le profil isopérimétrique d’une variété Riemannienne compacte pour les petits volumes, Thèse de l’Université Paris 11 (2006), arXiv:0710.1849 and arXiv:0710.1396.
[17] E. Parini, An introduction to the Cheeger problem, Surveys Math. Appl. 6 (2011), 9722.
[18] M. Ritoré, Continuity of the isoperimetric profile of a complete Riemannian manifold under sectional curvature conditions, arXiv:1503.07014.
[19] G. Wei, UC Santa Barbara, Comparison Geometry for Ricci Curvature, August, 2008.
[20] R. Ye, Foliation by constant mean curvature spheres . Pacific J. Math. 147 (1991), no. 2, 381-396.