The quaternary Piatetski-Shapiro inequality
with one prime of the form \( p = x^2 + y^2 + 1 \)

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Abstract

In this paper we show that, for any fixed \( 1 < c < \frac{667}{805} \), every sufficiently large positive number \( N \) and a small constant \( \varepsilon > 0 \), the diophantine inequality

\[
|p_1^c + p_2^c + p_3^c + p_4^c - N| < \varepsilon
\]

has a solution in prime numbers \( p_1, p_2, p_3, p_4 \), such that \( p_1 = x^2 + y^2 + 1 \).

Keywords: Diophantine inequality · Exponential sum · Bombieri – Vinogradov type result · Primes.

2020 Math. Subject Classification: 11D75 · 11L07 · 11L20 · 11P32

1 Introduction and statement of the result

In 1960 Linnik [11] showed that there exist infinitely many prime numbers of the form \( p = x^2 + y^2 + 1 \), where \( x \) and \( y \) are integers. More precisely he proved the asymptotic formula

\[
\sum_{p \leq X} r(p - 1) = \pi \prod_{p > 2} \left( 1 + \frac{\chi_4(p)}{p(p - 1)} \right) \frac{X}{\log X} + O \left( \frac{X(\log \log X)^7}{(\log X)^{1+\theta_0}} \right),
\]

where \( r(k) \) is the number of solutions of the equation \( k = x^2 + y^2 \) in integers, \( \chi_4(k) \) is the non-principal character modulo 4 and

\[
\theta_0 = \frac{1}{2} - \frac{1}{4} \log 2 = 0.0289... .
\]

On the other hand in 1952 I. I. Piatetski-Shapiro [12] investigated the inequality

\[
|p_1^c + p_2^c + \cdots + p_4^c - N| < \varepsilon
\]
where $c > 1$ is not an integer, $\varepsilon$ is a fixed small positive number, and $p_1, \ldots, p_r$ are primes. He proved the existence of an $H(c)$, depending only on $c$, such that for all sufficiently large real $N$, the inequality (2) has a solution for $H(c) \leq r$. He established that

$$\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4$$

and also that $H(c) \leq 5$ if $1 < c < 3/2$.

In 2003 Zhai and Cao [16] solved (2) for $r = 4$. They proved that for any fixed $1 < c < \frac{81}{68}$, for every sufficiently large positive number $N$ and $\eta = \frac{1}{\log N}$, the diophantine inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c - N| < \eta$$

has a solution in prime numbers $p_1, p_2, p_3, p_4$.

Subsequently the result of Zhai and Cao was improved several times and the best result up to now belongs to Baker [1] with $1 < c < \frac{39}{29}$ and $\eta = N^{-\theta}$, where $\theta$ is a small positive number depending on $c$.

Let $P_l$ be a number with at most $l$ prime factors. In relation the solvability of inequality (3) with prime numbers of a special form, in 2017 the author [3] proved that (3) has a solution in primes $p_i$ such that $p_i + 2 = P_{32}, i = 1, 2, 3, 4$.

In this paper we continue to solve (3) with prime numbers of a special type. More precisely we shall prove the solvability of (3) with Linnik primes. We establish the following theorem.

**Theorem 1.** Let $1 < c < \frac{967}{805}$. For every sufficiently large positive number $N$, the diophantine inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c - N| < \frac{(\log \log N)^6}{(\log N)^{\theta_0}}$$

has a solution in prime numbers $p_1, p_2, p_3, p_4$, such that $p_1 = x^2 + y^2 + 1$. Here $\theta_0$ is defined by (1).

In addition we have the following task for the future.

**Conjecture 1.** Let $\varepsilon > 0$ be a small constant. There exists $c_0 > 1$ such that for any fixed $1 < c < c_0$, and every sufficiently large positive number $N$, the diophantine inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c - N| < \varepsilon$$

has a solution in prime numbers $p_1, p_2, p_3, p_4$, such that $p_1 = x_1^2 + y_1^2 + 1, p_2 = x_2^2 + y_2^2 + 1, p_3 = x_3^2 + y_3^2 + 1, p_4 = x_4^2 + y_4^2 + 1$. 


2 Notations

Assume that $N$ is a sufficiently large positive number. The letter $p$ with or without subscript denotes prime numbers. The notation $m \sim M$ means that $m$ runs through the interval $(M/2, M]$. Moreover $e(t) = \exp(2\pi it)$. We denote by $(m, n)$ the greatest common divisor of $m$ and $n$. The letter $\eta$ denotes an arbitrary small positive number, not the same in all appearances. As usual $\varphi(n)$ is Euler’s function and $\Lambda(n)$ is von Mangoldt’s function. We use the convention that a congruence, $m \equiv n \pmod{d}$ can be written as $m \equiv n (d)$. We denote by $r(k)$ the number of solutions of the equation $k = x^2 + y^2$ in integers. The symbol $\chi_4(k)$ means the non-principal character modulo 4. Throughout this paper unless something else is said, we suppose that $1 < c < \frac{967}{805}$.

Denote

$$X = \left( \frac{N}{c} \right)^{\frac{1}{2}};$$

$$D = \frac{X^{\frac{1}{2}}}{(\log N)^{\frac{6A + 34}{3}}}, \quad A > 3;$$

$$\Delta = X^{\frac{1}{2} - c};$$

$$\varepsilon = \frac{(\log \log X)^6}{(\log X)^{6\theta}};$$

$$H = \frac{\log^2 X}{\varepsilon};$$

$$S_{l,d,J}(t) = \sum_{p \in J, p \equiv l (d)} e(tp^c) \log p;$$

$$S(t) = S_{1,1,(X/2,X]}(t);$$

$$I_J(t) = \int_J e(ty^c) \, dy;$$

$$I(t) = I_{(X/2,X]}(t);$$

$$E(y, t, d, a) = \sum_{\mu y < n \leq y \atop n \equiv a (d)} \Lambda(n)e(tn^c) - \frac{1}{\varphi(d)} \int_{\mu y}^y e(tx^c) \, dx,$$

where $0 < \mu < 1$. 


3 Preliminary lemmas

Lemma 1. Let $a, \delta \in \mathbb{R}, 0 < \delta < a/4$ and $k \in \mathbb{N}$. There exists a function $\theta(y)$ which is $k$ times continuously differentiable and such that

$$
\theta(y) = 1 \quad \text{for} \quad |y| \leq a - \delta ;
$$

$$
0 < \theta(y) < 1 \quad \text{for} \quad a - \delta < |y| < a + \delta ;
$$

$$
\theta(y) = 0 \quad \text{for} \quad |y| \geq a + \delta .
$$

and its Fourier transform

$$
\Theta(x) = \int_{-\infty}^{\infty} \theta(y)e^{-xy}dy
$$

satisfies the inequality

$$
|\Theta(x)| \leq \min \left( 2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left( \frac{k}{2\pi|x|\delta} \right)^{k} \right).
$$

Proof. See ([12]).

Throughout this paper we denote by $\theta(y)$ the function from Lemma 1 with parameters $a = \frac{9\varepsilon}{10}, \delta = \frac{\varepsilon}{10}, k = \lceil \log X \rceil$ and by $\Theta(x)$ the Fourier transform of $\theta(y)$.

In this paper we need a Bombieri – Vinogradov type result for exponential sums over primes obtained very recently by the author.

Lemma 2. Let $1 < c < 3, c \neq 2, |t| \leq \Delta$ and $A > 0$ be fixed. Then the inequality

$$
\sum_{d \leq \sqrt{X/\log N}} \max_{y \leq X} \max_{(a, d) = 1} |E(y, t, d, a)| \ll \frac{X}{\log^{4} X}
$$

holds. Here $\Delta$ and $E(y, t, d, a)$ are denoted by (6) and (13).

Proof. See ([5], Lemma 18).

Lemma 3. Let $1 < c < 3, c \neq 2$ and $|t| \leq \Delta$. Then for the sum denoted by (10) and the integral denoted by (12) the asymptotic formula

$$
S(t) = I(t) + \mathcal{O} \left( \frac{X}{e(\log X)^{1/5}} \right)
$$

holds.
Proof. See ([15], Lemma 14).

Lemma 4. For the sum denoted by (10) and the integral denoted by (12) we have

(i) \[ \int_{-\Delta}^{\Delta} |S(t)|^2 dt \ll X^{2-c} \log^3 X, \]

(ii) \[ \int_{-\Delta}^{\Delta} |I(t)|^2 dt \ll X^{2-c} \log X, \]

(iii) \[ \int_{n}^{n+1} |S(t)|^2 dt \ll X \log^3 X. \]

Proof. It follows from the arguments used in ([15], Lemma 7).

Lemma 5. Assume that \( F(x), G(x) \) are real functions defined in \([a, b]\), \(|G(x)| \leq H\) for \( a \leq x \leq b \) and \( G(x)/F'(x) \) is a monotonous function. Set

\[ I = \int_{a}^{b} G(x)e(F(x))dx. \]

If \( F'(x) \geq h > 0 \) for all \( x \in [a, b] \) or if \( F'(x) \leq -h < 0 \) for all \( x \in [a, b] \) then

\[ |I| \ll H/h. \]

If \( F''(x) \geq h > 0 \) for all \( x \in [a, b] \) or if \( F''(x) \leq -h < 0 \) for all \( x \in [a, b] \) then

\[ |I| \ll H/\sqrt{h}. \]

Proof. See ([14], p. 71).

Lemma 6. For any complex numbers \( a(n) \) we have

\[ \left| \sum_{a<n \leq b} a(n) \right|^2 \leq (1 + \frac{b-a}{Q}) \sum_{|q|\leq Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{a<n, n+q \leq b} a(n+q)a(n), \]

where \( Q \geq 1 \).

Proof. See ([9], Lemma 8.17).
Lemma 7. Let $3 < U < V < Z < X$ and suppose that $Z - \frac{1}{2} \in \mathbb{N}$, $X \gg Z^2 U$, $Z \gg U^2$, $V^3 \gg X$. Assume further that $F(n)$ is a complex valued function such that $|F(n)| \leq 1$. Then the sum

$$\sum_{n \sim X} \Lambda(n) F(n)$$

can be decomposed into $O\left( \log^{10} X \right)$ sums, each of which is either of Type I

$$\sum_{m \sim M} a(m) \sum_{l \sim L} F(ml),$$

where

$$L \gg Z, \quad LM \asymp X, \quad |a(m)| \ll m^\eta,$$

or of Type II

$$\sum_{m \sim M} a(m) \sum_{l \sim L} b(l) F(ml),$$

where

$$U \ll L \ll V, \quad LM \asymp X, \quad |a(m)| \ll m^\eta, \quad |b(l)| \ll l^\eta.$$

Proof. See ([7], Lemma 3).

Lemma 8. Let $|f^{(m)}(u)| \asymp Y X^{1-m}$ for $1 \leq X < u < X_0 \leq 2X$ and $m \geq 1$. Then

$$\left| \sum_{X<n \leq X_0} e(f(n)) \right| \ll Y^\kappa X^\lambda + Y^{-1},$$

where $(\kappa, \lambda)$ is any exponent pair.

Proof. See ([6], Ch. 3).

Lemma 9. Let $\theta, \lambda$ be real numbers such that

$$\theta(\theta - 1)(\theta - 2)\lambda(\lambda - 1)(\theta + \lambda - 2)(\theta + \lambda - 3)(\theta + 2\lambda - 3)(2\theta + \lambda - 4) \neq 0.$$ 

Set

$$\Sigma_I = \sum_{m \sim M} a_m \sum_{l \in I_m} e(Bm^\lambda l^\theta),$$

where

$$B > 0, \quad M \geq 1, \quad L \geq 1, \quad |a_m| \leq 1, \quad I_m \subset (L/2, L].$$

Let

$$F = BM^\lambda L^\theta.$$
Then

\[ \Sigma_I \ll \left( F^2 M^2 L^2 + F^2 M^2 L^2 + F^2 M^2 L^2 + M^2 L + M M L + F M L \right) (M L) \eta . \]

Proof. See (\cite{2}, Theorem 2).

Lemma 10. Let \( \alpha, \beta \) be real numbers such that

\[ \alpha \beta (\alpha - 1)(\beta - 1)(\alpha - 2)(\beta - 2) \neq 0. \]

Set

\[ \Sigma_{II} = \sum_{m \sim M} a(m) \sum_{l \sim L} b(l) e \left( F \frac{m^\alpha l^\beta}{M L^\beta} \right), \]

where

\[ F > 0, \quad M \geq 1, \quad L \geq 1, \quad |a(m)| \leq 1, \quad |b(l)| \leq 1. \]

Then

\[ \Sigma_{II} (F M L)^{-\eta} \ll \left( F^4 M^3 L^4 \right)^{\frac{1}{31}} + \left( F^6 M^5 L^5 \right)^{\frac{1}{51}} + \left( F^6 M^6 L^6 \right)^{\frac{1}{41}} + \left( F^2 M^3 L^2 \right)^{\frac{1}{29}} + \left( F^3 M^3 L^3 \right)^{\frac{1}{43}} + \left( F M^2 L^2 \right)^{\frac{1}{11}} + \left( F M^6 L^6 \right)^{\frac{1}{4}} + M \frac{1}{L} + M L \frac{1}{8} + F M L. \]

Proof. See (\cite{13}, Theorem 9).

The next two lemmas are due to C. Hooley.

Lemma 11. For any constant \( \omega > 0 \) we have

\[ \sum_{p \leq X} \left| \sum_{d | p - 1} \chi_4(d) \right|^2 \ll \frac{X (\log \log X)^7}{\log X}, \]

where the constant in Vinogradov’s symbol depends on \( \omega > 0. \)

Lemma 12. Suppose that \( \omega > 0 \) is a constant and let \( F_\omega(X) \) be the number of primes \( p \leq X \) such that \( p - 1 \) has a divisor in the interval \( \left( \sqrt{X (\log X)^{-\omega}}, \sqrt{X (\log X)^{\omega}} \right) \). Then

\[ F_\omega(X) \ll \frac{X (\log \log X)^3}{(\log X)^{1 + 2\theta_0}}, \]

where \( \theta_0 \) is defined by (1) and the constant in Vinogradov’s symbol depends only on \( \omega > 0. \)
The proofs of very similar results are available in ([8], Ch.5).

**Lemma 13.** We have
\[ \int_{-\infty}^{\infty} I^4(t)\Theta(t)e(-Nt) \, dt \gg \varepsilon X^{4-c}. \]

**Proof.** See ([16], Lemma 8).

## 4 Outline of the proof

Consider the sum
\[ \Gamma(X) = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} r(p_1 - 1) \log p_1 \log p_2 \log p_3 \log p_4. \tag{14} \]

Obviously
\[ \Gamma(X) \geq \Gamma_0(X), \tag{15} \]
where
\[ \Gamma_0(X) = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} r(p_1 - 1)\theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \log p_1 \log p_2 \log p_3 \log p_4. \tag{16} \]

From ([16]) and well-known identity \( r(n) = 4 \sum_{\mathcal{d}|n} \chi_4(d) \) we obtain
\[ \Gamma_0(X) = 4(\Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X)), \tag{17} \]
where
\[ \Gamma_1(X) = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \left( \sum_{\substack{d|p_1-1 \\\ d \leq D}} \chi_4(d) \right) \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \prod_{k=1}^{4} \log p_k, \tag{18} \]
\[ \Gamma_2(X) = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \left( \sum_{\substack{d|p_1-1 \\\ D < d < X/D}} \chi_4(d) \right) \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \prod_{k=1}^{4} \log p_k, \tag{19} \]
\[ \Gamma_3(X) = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \left( \sum_{\substack{d|p_1-1 \\\ d \geq X/D}} \chi_4(d) \right) \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \prod_{k=1}^{4} \log p_k. \tag{20} \]
In order to estimate $\Gamma_1(X)$ and $\Gamma_3(X)$ we need to consider the sum
\[
I_{l,d,J}(X) = \sum_{X/2 < p_2, p_3, p_4 \leq X \atop p_1 \equiv l \pmod{d} \in J} \log p_1 \log p_2 \log p_3 \log p_4 \theta \left( p_1^c + p_2^c + p_3^c + p_4^c - N \right),
\]
where $l$ and $d$ are coprime natural numbers, and $J \subset (X/2, X]$-interval. If $J = (X/2, X]$ then we write for simplicity $I_{l,d}(X)$.

Using the inverse Fourier transform for the function $\theta(y)$ we deduce
\[
I_{l,d,J}(X) = \sum_{X/2 < p_2, p_3, p_4 \leq X \atop p_1 \equiv l \pmod{d} \in J} \log p_1 \log p_2 \log p_3 \log p_4 \int_{-\infty}^{\infty} \Theta(t) e \left( (p_1^c + p_2^c + p_3^c + p_4^c - N) t \right) dt
\]
\[
= \int_{-\infty}^{\infty} \Theta(t) S^3(t) S_{l,d,J}(t) e(-Nt) dt .
\]

We decompose $I_{l,d,J}(X)$ as follows
\[
I_{l,d,J}(X) = I_{l,d,J}(X)^{(1)} + I_{l,d,J}(X)^{(2)} + I_{l,d,J}(X)^{(3)} ,
\]
where
\[
I_{l,d,J}(X)^{(1)} = \int_{-\Delta}^{\Delta} \Theta(t) S^3(t) S_{l,d,J}(t) e(-Nt) dt ,
\]
\[
I_{l,d,J}(X)^{(2)} = \int_{\Delta \leq |t| \leq H} \Theta(t) S^3(t) S_{l,d,J}(t) e(-Nt) dt ,
\]
\[
I_{l,d,J}(X)^{(3)} = \int_{|t| > H} \Theta(t) S^3(t) S_{l,d,J}(t) e(-Nt) dt .
\]

We shall estimate $I_{l,d,J}(X)^{(1)}$, $I_{l,d,J}(X)^{(2)}$, $\Gamma_3(X)$, $\Gamma_2(X)$ and $\Gamma_1(X)$, respectively, in the sections 5, 6, 7, 8 and 9. In section 10 we shall finalize the proof of Theorem 1.
5 Asymptotic formula for $I_{l,d;J}^{(1)}(X)$

Denote

$$S_1 = S(t),$$
$$S_2 = S_{l,d;J}(t),$$
$$I_1 = I(t),$$
$$I_2 = \frac{I_{J}(t)}{\varphi(d)}.$$  
(26)

We use the identity

$$S_1^2 S_2 = I_1^3 I_2 + (S_2 - I_2)I_1^3 + S_2(S_1 - I_1)I_1^2 + S_1 S_2(S_1 - I_1)I_1 + S_1^2 S_2(S_1 - I_1).$$  
(30)

We also need the trivial estimations

$$S_1 \ll X, \quad S_2 \ll \frac{X \log X}{d}, \quad I_1 \ll X.$$  
(31)

Put

$$\Phi_{\Delta,J}(X, d) = \frac{1}{\varphi(d)} \int_{-\Delta}^{\Delta} \Theta(t) I_1^3 I_J(t) e(-Nt) \, dt,$$  
(32)

$$\Phi_J(X, d) = \frac{1}{\varphi(d)} \int_{-\infty}^{\infty} \Theta(t) I_1^3 I_J(t) e(-Nt) \, dt.$$  
(33)

Now \([11], [12], [23], [26] - [32],\) Lemma \([1],[3]\) and Lemma \([4]\) imply

$$I_{l,d;J}^{(1)}(X) - \Phi_{\Delta,J}(X, d) = \int_{-\Delta}^{\Delta} \Theta(t) \left( S_{l,d;J}(t) - \frac{I_J(t)}{\varphi(d)} \right) I_1^3(t) e(-Nt) \, dt$$
$$+ \int_{-\Delta}^{\Delta} \Theta(t) S_{l,d;J}(t) \left( S(t) - I(t) \right) I_1^2(t) e(-Nt) \, dt$$
$$+ \int_{-\Delta}^{\Delta} \Theta(t) S(t) S_{l,d;J}(t) \left( S(t) - I(t) \right) I(t) e(-Nt) \, dt$$
$$+ \int_{-\Delta}^{\Delta} \Theta(t) S^2(t) S_{l,d;J}(t) \left( S(t) - I(t) \right) e(-Nt) \, dt$$
$$+ \int_{-\Delta}^{\Delta} \Theta(t) S(t) S_{l,d;J}(t) \left( S(t) - I(t) \right) e(-Nt) \, dt.$$
\[ \ll \varepsilon \left( X \max_{|t| \leq \Delta} \left| S_{l,d,J}(t) - \frac{I_J(t)}{\varphi(d)} \right| \right)^\Delta \int_{-\Delta}^{\Delta} |I(t)|^2 \, dt \]
\[ + \frac{X^2 \log X}{d e(\log X)^{1/\gamma}} \int_{-\Delta}^{\Delta} |I(t)|^2 \, dt + \frac{X^2 \log X}{d e(\log X)^{1/\gamma}} \int_{-\Delta}^{\Delta} |S(t)||I(t)| \, dt \]
\[ + \frac{X^2 \log X}{d e(\log X)^{1/\gamma}} \int_{-\Delta}^{\Delta} |S(t)|^2 \, dt \]
\[ \ll \varepsilon \left( X^{3-c}(\log X) \max_{|t| \leq \Delta} \left| S_{l,d,J}(t) - \frac{I_J(t)}{\varphi(d)} \right| + \frac{X^4-c}{d e(\log X)^{1/\gamma}} \right). \] (34)

Using (11), (12) and Lemma 5 we deduce
\[ I_J(t) \ll \min \left( X, \frac{X^{1-c} \log X}{|t|} \right), \quad I(t) \ll \min \left( X, \frac{X^{1-c} \log X}{|t|} \right). \] (35)

From (11), (12), (32), (33), (35) and Lemma 1 it follows
\[ \Phi_{\Delta,J}(X,d) - \Phi_J(X,d) \ll \frac{1}{\varphi(d)} \int_{-\Delta}^{\Delta} |I(t)|^2 |\Theta(t)| \, dt \ll \frac{\varepsilon X^{4-4c}}{\varphi(d)} \int_{-\Delta}^{\Delta} \frac{dt}{t^2} \ll \frac{\varepsilon X^{4-4c}}{\varphi(d) \Delta^3} \]
and therefore
\[ \Phi_{\Delta,J}(X,d) = \Phi_J(X,d) + O \left( \frac{\varepsilon X^{4-4c}}{\varphi(d) \Delta^3} \right). \] (36)

Finally (6), (31), (36) and the identity
\[ I_{l,d;J}^{(1)}(X) = I_{l,d;J}^{(1)}(X) - \Phi_{\Delta,J}(X,d) + \Phi_{\Delta,J}(X,d) - \Phi_J(X,d) + \Phi_J(X,d) \]
yield
\[ I_{l,d;J}^{(1)}(X) = \Phi_J(X,d) + O \left( \varepsilon X^{3-c}(\log X) \max_{|t| \leq \Delta} \left| S_{l,d,J}(t) - \frac{I_J(t)}{\varphi(d)} \right| \right) + O \left( \frac{\varepsilon X^{4-c}}{d e(\log X)^{1/\gamma}} \right). \] (37)

6 Upper bound of $I_{l,d;J}^{(3)}(X)$

By (8), (9), (10), (25) and Lemma 1 we find
\[ I_{l,d;J}^{(3)}(X) \ll \frac{X^4 \log X}{d} \int_{-\delta}^{\delta} \frac{1}{t} \left( \frac{k}{2\pi \delta t} \right)^k dt = \frac{X^4 \log X}{d} \frac{k}{2\pi \delta H} \left( \frac{k}{2\pi \delta H} \right)^k \ll \frac{1}{d}. \] (38)
7 Upper bound of $\Gamma_3(X)$

Consider the sum $\Gamma_3(X)$.

Since

$$\sum_{\frac{d|p_1-1}{d \geq X/D}} \chi_4(d) = \sum_{\frac{m|p_1-1}{m \leq (p_1-1)D/X}} \chi_4\left(\frac{p_1-1}{m}\right) = \sum_{j=\pm 1} \chi_4(j) \sum_{\frac{m|p_1-1}{m \leq (p_1-1)D/X \mod 1 \equiv j}} 1,$$

then from (20) and (21) we obtain

$$\Gamma_3(X) = \sum_{m < D} \sum_{j=\pm 1} \chi_4(j) I_{1+jm,4m;J_m}(X),$$

where $J_m = \left( \max\{1 + mX/D, X/2\}, X \right]$. The last formula and (22) give us

$$\Gamma_3(X) = \Gamma^{(1)}_3(X) + \Gamma^{(2)}_3(X) + \Gamma^{(3)}_3(X),$$

where

$$\Gamma^{(i)}_3(X) = \sum_{m < D} \sum_{j=\pm 1} \chi_4(j) I_{1+jm,4m;J_m}^{(i)}(X), \quad i = 1, 2, 3.$$ (40)

7.1 Estimation of $\Gamma^{(1)}_3(X)$

From (37) and (40) we get

$$\Gamma^{(1)}_3(X) = \Gamma^* + \mathcal{O}\left(\varepsilon X^{3-c} (\log X) \Sigma_1 \right) + \mathcal{O}\left(\varepsilon X^{4-c} \frac{1}{e(\log X)^{1/6}} \Sigma_2 \right),$$ (41)

where

$$\Gamma^* = \sum_{m < D} \sum_{2|m} \phi_4(m, 4m) \sum_{j=\pm 1} \chi_4(j),$$ (42)

$$\Sigma_1 = \sum_{m < D} \max_{2|m} \left| S_{1+jm,4m;J_m}(t) - \frac{I_{j(t)}}{\phi(4m)} \right|,$$ (43)

$$\Sigma_2 = \sum_{m < D} \frac{1}{\phi(4m)}.$$ (44)

From the properties of $\chi(k)$ we have that

$$\Gamma^* = 0.$$ (45)
By (5), (9), (11), (43) and Lemma 2 we get
\[ \Sigma_1 \ll \frac{X}{\log^4 X} . \] (46)

It is well known that
\[ \Sigma_2 \ll \log X . \] (47)

Bearing in mind (41), (45), (46) and (47) we obtain
\[ \Gamma_3^{(1)}(X) \ll \varepsilon X^{4-c} \log X . \] (48)

### 7.2 Estimation of $\Gamma_3^{(2)}(X)$

Now we consider $\Gamma_3^{(2)}(X)$. From (24) and (40) we have
\[ \Gamma_3^{(2)}(X) = \int \Theta(t)S^3(t)K(t)e(-Nt) dt , \] (49)

where
\[ K(t) = \sum_{m<D} \sum_{j=\pm 1} \chi_4(j)S_{1+jm,4m,J_m}(t) . \] (50)

**Lemma 14.** For the sum denoted by (50) we have
\[ \int_{\Delta \leq |t| \leq H} |K(t)|^2|\Theta(t)| dt \ll X \log^7 X . \]

**Proof.** See ([5], Lemma 22).

**Lemma 15.** Assume that
\[ \Delta \leq |t| \leq H , \quad |a(m)| \ll m^n , \quad LM \asymp X , \quad L \gg X^{\frac{2}{5}} . \] (51)

Set
\[ S_I = \sum_{m=M} a(m) \sum_{l-L} e(tm^{\ell}l^{\ell'}) . \] (52)

Then
\[ S_I \ll \frac{X^{\frac{407}{2045} + \eta}}{\log \log X} . \]
Proof. We first consider the case when
\[ M \ll X^{\frac{6177}{12880}}. \]  
(53)

By (6), (8), (51), (52), (53) and Lemma 8 with the exponent pair \((\frac{1}{14}, \frac{11}{14})\) we obtain
\[ S_I \ll X^n \sum_{m \sim M} \left| \sum_{l \sim L} e(tm^c l^c) \right| \ll X^n \sum_{m \sim M} \left( (|t| X^c L^{-1})^{\frac{1}{14}} L^{\frac{11}{14}} + \frac{1}{|t| X^c L^{-1}} \right) \ll X^n \left( H^{\frac{11}{14}} X^{\frac{11}{14}} ML^{\frac{11}{14}} + \Delta^{-1} X^{1-c} \right) \ll X^n \left( X^{\frac{c+10}{14}} M^{\frac{11}{14}} + X^{\frac{3}{4}} \right) \ll X^{\frac{1207}{1288} + \eta}. \]  
(54)

Next we consider the case when
\[ X^{\frac{6177}{12880}} \ll M \ll X^{\frac{1}{5}}. \]  
(55)

Using (52), (56) and Lemma 9 we deduce
\[ S_I \ll X^{\frac{1207}{1288} + \eta}. \]  
(56)

Bearing in mind (54) and (56) we establish the statement in the lemma. \( \square \)

Lemma 16. Assume that
\[ \Delta \leq |t| \leq H, \ |a(m)| \ll m^n, \ |b(l)| \ll l^n, \ LM \ll X, \ X^{\frac{1}{5}} \ll L \ll X^{\frac{1}{5}}. \]  
(57)

Set
\[ S_{II} = \sum_{m \sim M} a(m) \sum_{l \sim L} b(l) e(tm^c l^c). \]  
(58)

Then
\[ S_{II} \ll X^{\frac{4207}{1288} + \eta}. \]

Proof. We first consider the case when
\[ X^{\frac{1}{5}} \ll L \ll X^{\frac{614}{1288}}. \]  
(59)

From (57), (58), Cauchy’s inequality and Lemma 6 with \( Q = X^{\frac{1}{5}} \) it follows
\[ |S_{II}|^2 \ll X^n \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q \leq Q} \sum_{l \sim L} \left| \sum_{m \sim M} e(f(l, m, q)) \right| \right). \]  
(60)
where $f(l, m, q) = tm^c((l + q)c - l^c)$. Now (6), (8), (57), (59), (60) and Lemma 8 with the exponent pair $\left(\frac{1}{6}, \frac{2}{3}\right)$ give us

$$S_{II} \ll X^\eta \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q \leq Q} \sum_{l \sim L} \left( \frac{|t|qX^{c-1}}{Q} \right)^{1/2} M^\frac{1}{2} + \frac{1}{|t|qX^{c-1}} \right)^{1/2}$$

$$\ll X^\eta \left( \frac{X^2}{Q} + \frac{X}{Q} \left( H^\frac{1}{2} X^\frac{c-1}{6} M^\frac{1}{2} Q^\frac{1}{2} L + \Delta^{-1} X^{1-c} X \log Q \right) \right)^{1/2}$$

$$\ll X^{\frac{1207}{1288} + \eta}. \quad (61)$$

Next we consider the case when

$$X^{\frac{261}{588}} \ll L \ll X^{\frac{1}{3}}. \quad (62)$$

Using (58), (62) and Lemma 10 we find

$$S_{II} \ll X^{\frac{1207}{1288} + \eta}. \quad (63)$$

Taking into account (61) and (63) we establish the statement in the lemma. \hfill \square

**Lemma 17.** Let $\Delta \leq |t| \leq H$. Then for the sum denoted by (10) we have

$$S(t) \ll X^{\frac{1207}{1288} + \eta}.$$ 

**Proof.** In order to prove the lemma we will use the formula

$$S(t) = S^*(t) + O \left( X^{\frac{1}{2} + \epsilon} \right), \quad (64)$$

where

$$S^*(t) = \sum_{X/2 < n \leq X} \Lambda(n)e(tn^c).$$

Let

$$U = X^{\frac{1}{4}}, \quad V = X^{\frac{1}{4}}, \quad Z = \left[ X^{\frac{3}{4}} \right] + \frac{1}{2}.$$ 

According to Lemma 7 the sum $S^*(t)$ can be decomposed into $O \left( \log^{10} X \right)$ sums, each of which is either of Type I

$$\sum_{m \sim M} a(m) \sum_{l \sim L} e(tm^c l^c),$$

where

$$L \gg Z, \quad LM \asymp X, \quad |a(m)| \ll m^\eta,$$
or of Type II
\[
\sum_{m \sim M} a(m) \sum_{l \sim L} b(l) e(t m \ell^c),
\]
where
\[
U \ll L \ll V, \quad LM \asymp X, \quad |a(m)| \ll m^\eta, \quad |b(l)| \ll l^\eta.
\]
Using Lemma 15 and Lemma 16 we deduce
\[
S^*(t) \ll X^{1207/1288 + \eta}.
\]
(65)

Bearing in mind (64) and (65) we establish the statement in the lemma.

**Lemma 18.** For the sum denoted by (10) we have
\[
\int_{\Delta} |S(t)|^2 |\Theta(t)| dt \ll X \log^4 X.
\]

**Proof.** According to Lemma 1 and Lemma 4 (iii) we find
\[
\int_{\Delta} |S(t)|^2 |\Theta(t)| dt \ll \varepsilon \int_{\Delta} |S(t)|^2 dt + \int_{1/\varepsilon}^H |S(t)|^2 dt
\]
\[
\ll \varepsilon \sum_{0 \leq n \leq 1/\varepsilon} \int |S(t)|^2 dt + \sum_{1/\varepsilon - 1 \leq n \leq H} \frac{1}{n} \int |S(t)|^2 dt
\]
\[
\ll X \log^4 X.
\]

**Lemma 19.** For the sum denoted by (10) we have
\[
\int_{\Delta} |S(t)|^4 |\Theta(t)| dt \ll X^{1167/322} \frac{H}{t} + \eta.
\]

**Proof.** Our argument is modification of Li’s and Cai’s [10] argument. Denote
\[
A(t) = \sum_{n \sim X} e(t n^c)
\]
(66)
For any continuous function $\Psi(x)$ defined in the interval $[-H, H]$ we have

$$\left| \int_{\Delta \leq |t| \leq H} S(t) \Psi(t) \, dt \right| = \left| \sum_{p \sim X} (\log p) \int_{\Delta \leq |t| \leq H} e(tp^c) \Psi(t) \, dt \right| \leq \sum_{p \sim X} (\log p) \left| \int_{\Delta \leq |t| \leq H} e(tp^c) \Psi(t) \, dt \right| \leq (\log X) \left| \sum_{n \sim X} \int_{\Delta \leq |t| \leq H} e(tn^c) \Psi(t) \, dt \right|. \quad (67)$$

By (67) and Cauchy’s inequality we obtain

$$\left| \int_{\Delta \leq |t| \leq H} S(t) \Psi(t) \, dt \right|^2 \leq X (\log X)^2 \left| \sum_{n \sim X} \int_{\Delta \leq |t| \leq H} e(tn^c) \Psi(t) \, dt \right|^2 = X (\log X)^2 \int_{\Delta \leq |y| \leq H} |\Psi(y)| \, dy \int_{\Delta \leq |t| \leq H} |\Psi(t)| A(t - y) \, dt \leq X (\log X)^2 \int_{\Delta \leq |y| \leq H} |\Psi(y)| \, dy \int_{\Delta \leq |t| \leq H} |\Psi(t)| |A(t - y)| \, dt. \quad (68)$$

From (66) and Lemma 8 with the exponent pair $\left( \frac{1}{2}, \frac{1}{2} \right)$ it follows

$$A(t) \ll \min \left( \left( |t|^c - 1 \right)^{\frac{1}{2}} X^{\frac{1}{2}} + \frac{1}{|t|^c - 1}, X \right). \quad (69)$$

Using (8) and (69) we write

$$\int_{\Delta \leq |t| \leq H} |\Psi(t)||A(t - y)| \, dt \ll \int_{\Delta \leq |t| \leq H} |\Psi(t)||A(t - y)| \, dt + \int_{\Delta \leq |t| \leq H} |\Psi(t)||A(t - y)| \, dt \ll X \int_{\Delta \leq |t| \leq H} |\Psi(t)| \, dt + \int_{\Delta \leq |t| \leq H} |\Psi(t)| \left( \left( |t - y|^c - 1 \right)^{\frac{1}{2}} X^{\frac{1}{2}} + \frac{1}{|t - y|^c - 1} \right) \, dt.$$
\[
\ll X \max_{\Delta \leq |t| \leq H} |\Psi(t)| \int_{|t-y| \leq X^{-c}} dt + X^{\frac{c}{2} + \eta} \max_{\Delta \leq |t| \leq H} |\Psi(t)| \int_{|t| \leq H} dt \\
+ X^{1-c} \max_{\Delta \leq |t| \leq H} |\Psi(t)| \int \frac{1}{|t-y|} dt \\
\ll X^{1-c} (\log X) \max_{\Delta \leq |t| \leq H} |\Psi(t)| + X^{\frac{c}{2} + \eta} \int |\Psi(t)| dt. 
\]

(70)

Now (68) and (70) imply

\[
\left| \int_{\Delta \leq |t| \leq H} S(t)|\Psi(t)| dt \right|^2 \leq X^{2-c+\eta} \max_{\Delta \leq |t| \leq H} |\Psi(t)| \int_{\Delta \leq |t| \leq H} |\Psi(t)| dt \\
+ X^{\frac{c+2}{2} + \eta} \left( \int_{\Delta \leq |t| \leq H} |\Psi(t)| dt \right)^2.
\]

(71)

Let’s put first

\[
\Psi_1(t) = \overline{S(t)}|S(t)||\Theta(t)|. 
\]

(72)

Bearing in mind (7), (71), (72), Lemma 1, Lemma 17 and Lemma 18 we get

\[
\int_{\Delta \leq |t| \leq H} |S(t)|^3|\Theta(t)| dt = \int_{\Delta \leq |t| \leq H} S(t)|\Psi_1(t)| dt \\
\ll \varepsilon \frac{1}{2} X^{\frac{1}{1288} - \frac{c}{2} + \eta} \left( \int_{\Delta \leq |t| \leq H} |S(t)|^2|\Theta(t)| dt \right)^{1/2} \\
+ X^{\frac{c+2}{4} + \eta} \int_{\Delta \leq |t| \leq H} |S(t)|^2|\Theta(t)| dt \\
\ll X^{\frac{4139}{1288} - \frac{c}{2} + \eta} + X^{\frac{24}{6} + \eta} \\
\ll X^{\frac{4139}{1288} - \frac{c}{2} + \eta}. 
\]

(73)

Next we put

\[
\Psi_2(t) = \overline{S(t)}|S(t)|^2|\Theta(t)|. 
\]

(74)
Taking into account (7), (71), (73), (74) and Lemma 17 we find
\[
\int_{\Delta \leq |t| \leq H} |S(t)|^4 |\Theta(t)| \, dt = \int_{\Delta \leq |t| \leq H} S(t) \Psi_2(t) \, dt
\]
\[
\ll \varepsilon^{\frac{1}{2}} X^{\frac{6167}{1288} - \frac{3}{4} + \eta} \left( \int_{\Delta \leq |t| \leq H} |S(t)|^3 |\Theta(t)| \, dt \right)^{\frac{1}{2}}
\]
\[
+ X^{\frac{3139}{1288} + \eta} \int_{\Delta \leq |t| \leq H} |S(t)|^3 |\Theta(t)| \, dt
\]
\[
\ll X^{\frac{4167}{1288} - \frac{3c}{4} + \eta} + X^{\frac{3139}{1288} + \frac{2-c}{4} + \eta}
\]
\[
\ll X^{\frac{4167}{1288} - \frac{3c}{4} + \eta}.
\]

The lemma is proved. \(\square\)

Finally (7), (49), Cauchy’s inequality, Lemma 14, Lemma 17 and Lemma 19 yield
\[
\Gamma_3^{(2)}(X) \ll \max_{\Delta \leq t \leq H} |S(t)| \left( \int_{\Delta} |S(t)|^4 |\Theta(t)| \, dt \right)^{1/2} \left( \int_{\Delta} |K(t)|^2 |\Theta(t)| \, dt \right)^{1/2}
\]
\[
\ll \frac{\varepsilon X^{4-c}}{\log X}.
\]  
(75)

### 7.3 Estimation of \(\Gamma_3^{(3)}(X)\)

From (38) and (40) we have
\[
\Gamma_3^{(3)}(X) \ll \sum_{m < D} \frac{1}{d} \ll \log X.
\]  
(76)

### 7.4 Estimation of \(\Gamma_3(X)\)

Summarizing (39), (48), (75) and (76) we get
\[
\Gamma_3(X) \ll \frac{\varepsilon X^{4-c}}{\log X}.
\]  
(77)
8 Upper bound of $\Gamma_2(X)$

In this section we need the following lemma.

**Lemma 20.** Let $1 < c < 3$, $c \neq 2$ and $N_0$ is a sufficiently large positive number. Then for the number of solutions $B_0(N_0)$ of the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N_0| < \varepsilon$$

in prime numbers $p_1, p_2, p_3 \in \left(\left(N_0/2\right)^{1/3}/2, (N_0/2)^{1/3}\right)$ we have that

$$B_0(N_0) \ll \frac{\varepsilon N_0^{\frac{c}{2}-1}}{\log^3 N_0}.$$  

**Proof.** Define

$$B(X_0) = \sum_{X_0/2 < p_1, p_2, p_3 \leq X_0 \atop |p_1^c + p_2^c + p_3^c - N_0| < \varepsilon} \log p_1 \log p_2 \log p_3,$$

where

$$X_0 = \left(\frac{N_0}{2}\right)^{\frac{1}{c}}.$$  

We take the parameters $a_0 = \frac{5\varepsilon}{4}, \delta_0 = \frac{\varepsilon}{4}$ and $k_0 = \lfloor \log X_0 \rfloor$. According to Lemma 1 there exists a function $\theta_0(y)$ which is $k_0$ times continuously differentiable and such that

$$0 < \theta_0(y) < 1$$

for $|y| < \frac{3\varepsilon}{2};$

$$\theta_0(y) = 0$$

for $|y| \geq \frac{3\varepsilon}{2},$

and its Fourier transform

$$\Theta_0(x) = \int_{-\infty}^{\infty} \theta_0(y)e(-xy)dy$$

satisfies the inequality

$$|\Theta_0(x)| \leq \min\left(\frac{5\varepsilon}{2}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left(\frac{2k_0}{|x|\varepsilon}\right)^{k_0}\right).$$
Using (79), the definition of $\theta_0(y)$ and the inverse Fourier transformation formula we get

$$B(X_0) \leq \sum_{X_0/2 < p_1, p_2, p_3 \leq X_0} \theta_0(p_1c_1 + p_2c_2 + p_3c_3 - N_0) \log p_1 \log p_2 \log p_3$$

$$= \int_{-\infty}^{\infty} \Theta_0(t)S_0^3(t)e(-N_0 t) \, dt = B_1(X_0) + B_2(X_0) + B_3(X_0),\quad (82)$$

where

$$S_0(t) = \sum_{X_0/2 < p \leq X_0} e(tpc) \log p, \quad (83)$$

$$\Delta_0 = \frac{\log X_0}{X_0^c}, \quad A_0 > 10, \quad (84)$$

$$B_1(X_0) = \int_{-\Delta_0}^{\Delta_0} \Theta_0(t)S_0^3(t)e(-N_0 t) \, dt, \quad (85)$$

$$B_2(X_0) = \int_{\Delta_0 \leq |t| \leq H} \Theta_0(t)S_0^3(t)e(-N_0 t) \, dt, \quad (86)$$

$$B_3(X_0) = \int_{|t| > H} \Theta_0(t)S_0^3(t)e(-N_0 t) \, dt. \quad (87)$$

We begin with the estimation of $B_1(X_0)$. Set

$$I_0(t) = \int_{X_0/2}^{X_0} e(tyc) \, dy, \quad (88)$$

$$\Psi_{\Delta_0}(X_0) = \int_{-\Delta_0}^{\Delta_0} \Theta_0(t)I_0^3(t)e(-N_0 t) \, dt, \quad (89)$$

$$\Psi(X_0) = \int_{-\infty}^{\infty} \Theta_0(t)I_0^3(t)e(-N_0 t) \, dt. \quad (90)$$
From (81), (83), (90) and Lemma 5 we obtain

\[ \Psi(X_0) = \int_{-X_0^{-c}}^{X_0^{-c}} \Theta_0(t) I_0^3(t) e(-N_0 t) \, dt + \int_{|t| > X_0^{-c}} \Theta_0(t) I_0^3(t) e(-N_0 t) \, dt, \]

\[ \ll \int_{-X_0^{-c}}^{X_0^{-c}} \varepsilon X_0^3 \, dt + \int_{X_0^{-c}}^{\infty} \varepsilon \left( \frac{X_0^{1-c}}{t} \right)^3 \, dt, \]

\[ \ll \varepsilon X_0^{3-c}. \quad (91) \]

By (81), (84), (85), (89), Lemma 3 and the trivial estimations

\[ S_0(t) \ll X_0, \quad I_0(t) \ll X_0 \quad (92) \]

it follows

\[ B_1(X_0) - \Psi_{\Delta_0}(X_0) \ll \int_{-\Delta_0}^{\Delta_0} |S_0^3(t) - I_0^3(t)||\Theta_0(t)| \, dt \]

\[ \ll \varepsilon \int_{-\Delta_0}^{\Delta_0} |S_0(t) - I_0(t)||S_0(t)|^2 + |I_0(t)|^2 \, dt \]

\[ \ll \varepsilon \frac{X_0}{e(\log X_0)^{1/5}} \left( \int_{-\Delta_0}^{\Delta_0} |S_0(t)|^2 \, dt + \int_{-\Delta_0}^{\Delta_0} |I_0(t)|^2 \, dt \right) \]

\[ \ll \varepsilon \frac{X_0^{3-c}}{e(\log X_0)^{1/6}}. \quad (93) \]

On the other hand (81), (84), (89), (90) and Lemma 5 give us

\[ |\Psi(X_0) - \Psi_{\Delta_0}(X_0)| \ll \int_{\Delta_0}^{\infty} |I_0(t)|^3|\Theta_0(t)| \, dt \ll \frac{\varepsilon}{X_0^{3(c-1)}} \int_{\Delta_0}^{\infty} \frac{dt}{t^3} \]

\[ \ll \frac{\varepsilon}{X_0^{3(c-1)} \Delta_0^2} \ll \varepsilon \frac{X_0^{3-c}}{\log X_0}. \quad (94) \]

From (91), (93) and (94) and the identity

\[ B_1(X_0) = B_1(X_0) - \Psi_{\Delta_0}(X_0) + \Psi_{\Delta_0}(X_0) - \Psi(X_0) + \Psi(X_0) \]

we deduce

\[ B_1(X_0) \ll \varepsilon X_0^{3-c}. \quad (95) \]
Further we estimate $B_2(X_0)$. Consider the integral

$$B_2^*(X_0) = \int_{\Delta_0}^H \Theta_0(t)S_0^3(t)e(-N_0t)\,dt.$$  (96)

Now (80), (81), (92), (96) and partial integration yield

$$B_2^*(X_0) = -\frac{1}{2\pi i} \int_{\Delta_0}^H \Theta_0(t)S_0^3(t)e(-N_0t)\,dN_0 + \frac{1}{2\pi i N_0} \int_{\Delta_0}^H e(-N_0t)\,d\left(\Theta_0(t)S_0^3(t)\right) \ll \varepsilon X_0^{3-c} + X_0^{-c}\Omega,$$  (97)

where

$$\Omega = \int_{\Delta_0}^H e(-N_0t)\,d\left(\Theta_0(t)S_0^3(t)\right).$$  (98)

Next we consider $\Omega$. Set

$$\Gamma : z = f(t) = \Theta_0(t)S_0^3(t), \quad \Delta_0 \leq t \leq H.$$  (99)

Now (98) and (99) imply

$$\Omega = \int_{\Gamma} e\left(-N_0f^{-1}(z)\right)\,dz.$$  (100)

Using (81), (92), (99) and that the integral (100) is independent of path we get

$$\Omega = \int_{\Gamma} e\left(-N_0f^{-1}(z)\right)\,dz \ll \int_{\Gamma} |dz| \ll |f(\Delta_0)| + |f(H)| \ll \varepsilon X_0^3,$$  (101)

where $\Gamma$ is the line segment connecting the points $f(\Delta_0)$ and $f(H)$. Bearing in mind (80), (96), (97) and (101) we obtain

$$B_2(X_0) \ll \varepsilon X_0^{3-c}.$$  (102)

Finally we estimate $B_3(X_0)$. Using (8), (81), (83), (87), (92) we deduce

$$B_3(X_0) \ll X_0^3 \int_{\Pi} \frac{\alpha_0}{t} \left(\frac{2k_0}{\pi \varepsilon}\right)^{k_0} dt \ll X_0^3 \left(\frac{k_0}{\pi \varepsilon H}\right)^{k_0} \ll 1.$$  (103)
Summarizing (82), (93), (102) and (113) we find

\[ B(X_0) \ll \varepsilon X_0^{3-c}. \]  

(104)

Taking into account (79), (80) and (104), for the number of solutions \( B_0(N_0) \) of the diophantine inequality (78) we establish

\[ B_0(N_0) \ll \frac{\varepsilon N_0^{3-1}}{\log^3 N_0}. \]

The lemma is proved.

Consider the sum \( \Gamma_2(X) \). We denote by \( \mathcal{F}(X) \) the set of all primes \( X/2 < p \leq X \) such that \( p - 1 \) has a divisor belongs to the interval \((D, X/D)\). The inequality \( xy \leq x^2 + y^2 \) and (19) give us

\[
\Gamma_2(X)^2 \ll (\log X)^8 \sum_{\substack{X/2 < p_1, \ldots, p_8 \leq X \\ \left| p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2 + p_7^2 + p_8^2 - N \right| < \varepsilon}} \chi_4(d) \quad \sum_{\substack{d | p_1 - 1 \\ D < d < X/D}} \sum_{t | p_2 - 1} \chi_4(t) | t | \leq X/D \]

\[
\ll (\log X)^8 \sum_{\substack{X/2 < p_1, \ldots, p_8 \leq X \\ \left| p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2 + p_7^2 + p_8^2 - N \right| < \varepsilon}} \chi_4(d) \quad \sum_{\substack{d | p_1 - 1 \\ D < d < X/D}} \sum_{t | p_2 - 1} \chi_4(t) | t | \leq X/D \]

The summands in the last sum for which \( p_1 = p_5 \) can be estimated with \( O(X^{5+\varepsilon}) \).

Therefore

\[
\Gamma_2(X)^2 \ll (\log X)^8 \Sigma_0 + X^{5+\varepsilon}, \tag{105}
\]

where

\[
\Sigma_0 = \sum_{X/2 < p \leq X} \left| \sum_{d | p - 1} \chi_4(d) \quad \sum_{X/2 < p_2, \ldots, p_8 \leq X \atop p_5 \notin \mathcal{F}(X)} \sum_{p_5 \notin \mathcal{F}(X)} 1. \right| \tag{106}
\]

Now (106) and Lemma 20 imply

\[
\Sigma_0 \ll \frac{X^{6-2c}}{\log^6 X} \Sigma_0' \Sigma_0' \tag{107}
\]

where

\[
\Sigma_0' = \sum_{X/2 < p \leq X} \left| \sum_{d | p - 1} \chi_4(d) \quad \sum_{X/2 < p_2, \ldots, p_8 \leq X \atop p_5 \notin \mathcal{F}(X)} \right|^2, \quad \Sigma_0'' = \sum_{X/2 < p \leq X} 1.
\]
Applying Lemma 11 we get
\[ \Sigma_0' \ll \frac{X (\log \log X)^7}{\log X}. \]  (108)

Using Lemma 12 we obtain
\[ \Sigma_0'' \ll \frac{X (\log \log X)^3}{(\log X)^{1+2\theta_0}}, \]  (109)

where \( \theta_0 \) is denoted by (11).

Finally (105), (107), (108) and (109) yield
\[ \Gamma_2(X) \ll \frac{X^{4-c}(\log \log X)^5}{(\log X)^{\theta_0}} = \varepsilon X^{4-c}. \]  (110)

9 Lower bound for \( \Gamma_1(X) \)

Consider the sum \( \Gamma_1(X) \). From (18), (21) and (22) it follows
\[ \Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X), \]  (111)

where
\[ \Gamma_1^{(i)}(X) = \sum_{d \leq D} \chi_4(d) I_{1,d}^{(i)}(X), \quad i = 1, 2, 3. \]  (112)

9.1 Estimation of \( \Gamma_1^{(1)}(X) \)

First we consider \( \Gamma_1^{(1)}(X) \). Using formula (37) for \( J = (X/2, X] \), (112) and treating the reminder term by the same way as for \( \Gamma_3^{(1)}(X) \) we find
\[ \Gamma_1^{(1)}(X) = \Phi(X) \sum_{d \leq D} \frac{\chi_4(d)}{\varphi(d)} + O\left( \frac{\varepsilon X^{4-c}}{\log X} \right), \]  (113)

where
\[ \Phi(X) = \int_{-\infty}^{\infty} \Theta(t) I(t) e(-Nt) dt. \]

By Lemma 13 we get
\[ \Phi(X) \gg \varepsilon X^{4-c}. \]  (114)

According to [4] we have
\[ \sum_{d \leq D} \frac{\chi_4(d)}{\varphi(d)} = \frac{\pi}{4} \prod_p \left( 1 + \frac{\chi_4(p)}{p(p-1)} \right) + O\left( X^{-1/20} \right). \]  (115)
From (113) and (115) we obtain
\[
\begin{align*}
\Gamma^{(1)}_1(X) &= \frac{\pi}{4} \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)}\right) \Phi(X) + O\left(\frac{\varepsilon X^{4-c}}{\log X}\right) + O\left(\Phi(X) X^{-1/20}\right). 
\end{align*}
\]
(116)

Now (114) and (116) imply
\[
\Gamma^{(1)}_1(X) \gg \varepsilon X^{4-c}.
\]
(117)

### 9.2 Estimation of $\Gamma^{(2)}_1(X)$

Arguing as in the estimation of $\Gamma^{(2)}_3(X)$ we find
\[
\Gamma^{(2)}_1(X) \ll \frac{\varepsilon X^{4-c}}{\log X}.
\]
(118)

### 9.3 Estimation of $\Gamma^{(3)}_1(X)$

By (38) and (112) it follows
\[
\Gamma^{(3)}_1(X) \ll \sum_{m < D} \frac{1}{d} \ll \log X.
\]
(119)

### 9.4 Estimation of $\Gamma_1(X)$

Summarizing (111), (117), (118) and (119) we obtain
\[
\Gamma_1(X) \gg \varepsilon X^{4-c}.
\]
(120)

### 10 Proof of the Theorem

Taking into account (7), (15), (17), (77), (110) and (120) we deduce
\[
\Gamma(X) \gg \varepsilon X^{4-c} = \frac{X^{4-c}(\log \log X)^6}{(\log X)^{\theta_0}}.
\]

The last lower bound yields
\[
\Gamma(X) \to \infty \quad \text{as} \quad X \to \infty.
\]
(121)

Bearing in mind (14) and (121) we establish Theorem 1.
References

[1] R. Baker, Some diophantine equations and inequalities with primes, Funct. Approx. Comment. Math., 64 (2), (2021), 203 – 250.

[2] R. Baker, A. Weingartner, A ternary diophantine inequality over primes, Acta Arith., 162, (2014), 159 – 196.

[3] S. I. Dimitrov, A quaternary diophantine inequality by prime numbers of a special type, Proc. Techn. Univ.-Sofia, 67, 2, (2017), 317 – 326.

[4] S. I. Dimitrov, Diophantine approximation with one prime of the form $p = x^2 + y^2 + 1$, Lith. Math. J., 61, 4, (2021), 445 – 459.

[5] S. I. Dimitrov, A ternary diophantine inequality by primes with one of the form $p = x^2 + y^2 + 1$, Ramanujan J., 59, 2, (2022), 571 – 607.

[6] S. W. Graham, G. Kolesnik, Van der Corput’s Method of Exponential Sums, Cambridge University Press, New York, (1991).

[7] D. R. Heath-Brown, The Piatetski-Shapiro prime number theorem, J. Number Theory, 16, (1983), 242 – 266.

[8] C. Hooley, Applications of sieve methods to the theory of numbers, Cambridge Univ. Press, (1976).

[9] H. Iwaniec, E. Kowalski, Analytic number theory, Colloquium Publications, 53, Amer. Math. Soc., (2004).

[10] S. Li, Y. Cai, On a binary Diophantine inequality involving prime numbers, Ramanujan J., 54, (2021), 571 – 589.

[11] Ju. Linnik, An asymptotic formula in an additive problem of Hardy and Littlewood, Izv. Akad. Nauk SSSR, Ser. Mat., 24, (1960), 629 – 706 (in Russian).

[12] I. Piatetski-Shapiro, On a variant of the Waring-Goldbach problem, Mat. Sb., 30, (1952), 105 – 120, (in Russian).

[13] P. Sargos, J. Wu, Multiple exponential sums with monomials and their applications in number theory, Acta Math. Hungar., 87, (2000), 333 – 354.
[14] E. Titchmarsh, *The Theory of the Riemann Zeta-function* (revised by D. R. Heath-Brown), Clarendon Press, Oxford (1986).

[15] D. Tolev, *On a diophantine inequality involving prime numbers*, Acta Arith., 61, (1992), 289 – 306.

[16] W. Zhai, X. Cao, *On a diophantine inequality over primes*, Adv. Math. (China), 32(1), (2003), 63 – 73.

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