MIXED WEAK ESTIMATES OF SAWYER TYPE FOR COMMUTATORS OF SINGULAR INTEGRALS AND RELATED OPERATORS

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Abstract. We study mixed weak type inequalities for the commutator \([b,T]\), where \(b\) is a \(BMO\) function and \(T\) is a Calderón-Zygmund operator. More precisely, we prove that for every \(t > 0\)

\[
 uv\left\{ x \in \mathbb{R}^n : \left| \frac{[b,T](fv)(x)}{v(x)} \right| > t \right\} \leq C \int_{\mathbb{R}^n} \phi\left( \frac{|f(x)|}{t} \right) u(x)v(x) \, dx,
\]

where \(\phi(t) = t(1 + \log^+ t)\), \(u \in A_1\) and \(v \in A_\infty(u)\). Our technique involves the classical Calderón-Zygmund decomposition, which allow us to give a direct proof. We use this result to prove an analogous inequality for higher order commutators. We also obtain a mixed estimation for a wide class of maximal operators associated to certain Young functions of \(L \log L\) type which are in intimate relation with the commutators. This last estimate involves an arbitrary weight \(u\) and a radial function \(v\) which is not even locally integrable.

Introduction

In [6] the authors considered weighted weak type norm inequalities given by

\[
(0.1) \quad uv\left\{ x \in \mathbb{R}^n : \frac{T(fv)(x)}{v(x)} > t \right\} \leq C \int_{\mathbb{R}^n} |f(x)|u(x)v(x) \, dx, \quad t > 0
\]

for some positive constant \(C\), where \(T\) is either the Hardy-Littlewood maximal operator or any Calderón-Zygmund operator. The authors proved that (0.1) holds if \(u, v\) are weights such that \(u, v \in A_1\), or \(u \in A_1\) and \(v \in A_\infty(u)\). This result proves the conjecture given by Sawyer in [19], where (0.1) is proved in \(\mathbb{R}\) for the Hardy-Littlewood maximal operator \(M\) and \(u, v \in A_1\). The author also conjectured that the inequality holds if \(T\) is the Hilbert transform. The motivation of Sawyer for consider (0.1) yields a new proof of the classical Muckenhoupt’s Theorem concerning to the boundedness of \(M\) in \(L^p(w)\), for \(1 < p < \infty\) and \(w \in A_p\). Indeed, given \(w \in A_p\), from the P. Jones factorization Theorem we have that \(w = uv^{1-p}\), with \(u, v \in A_1\), so that the operator \(S(f) = M(fv)/v\) is bounded on \(L^\infty(uv)\). Hence, the Muckenhoupt’s Theorem is obtained from the usual Marcinkiewicz interpolation Theorem provided that \(S\) is of weak type \((1,1)\) with respect to the measure \(uvdx\), which is precisely (0.1) with \(T = M\) (see [20]).

In this paper we study inequalities of the type described in (0.1) for higher order commutators of Calderón-Zygmund operators with \(BMO\) symbols, generalizing the results obtained in [6]. However, our techniques are quite different of those given in this article. As far as we know, this type of estimates are new even for the case of the first order commutator.

We also obtain an analogous mixed estimation for generalized maximal operators associated to higher order commutators which are defined by means of a Young function. This estimate extends the results given in [13] to a wide class of maximal operators involving Luxemburg averages.

There is a close relationship between the boundedness properties of commutators acting on different functional spaces and partial differential equations, and it is well known that the continuity properties of such operators provides us with regular solutions of certain PDE’s.

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Several authors were working in this direction, (see, for example \[2, 3, 4, 5, 8\] and \[18\] between a vast amount of articles). Therefore, it seems appropriate to explore the weighted inequalities for that operators and, particularly, we shall be concerned with the mixed estimates mentioned above.

Recall that a linear operator $T$ is a Calderón-Zygmund operator if $T$ is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel $K$ such that for $f \in L^2$ with compact support,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy, \quad x \notin \text{supp} f.$$ 

We say that $K : \mathbb{R}^n \times \mathbb{R}^n \Delta \rightarrow \mathbb{C}$ is a standard kernel if it satisfies a size condition given by

$$|K(x,y)| \leq \frac{C}{|x - y|^n},$$

and the smoothness conditions

$$|K(x,y) - K(x,z)| \leq C \frac{|x - z|}{|x - y|^{n+1}}, \quad \text{if } |x - y| > 2|y - z|,$$

$$|K(x,y) - K(w,z)| \leq C \frac{|x - w|}{|x - y|^{n+1}}, \quad \text{if } |x - y| > 2|x - w|.$$ 

Recall, as well, that the commutator operator $[b,T]$ is formally defined, for adequate functions $f$, by

$$[b,T]f = bT(f) - T(bf).$$

We are now in position to state our main result.

**Theorem 1.** Let $u, v$ be weights such that $u \in A_1$ and $v \in A_{\infty}(u)$. Let $T$ be any Calderón-Zygmund operator and let $b \in \text{BMO}$. Then, for every $t > 0$ we have that

$$uw\left(\{x \in \mathbb{R}^n : \left|\frac{[b,T]fv(x)}{v(x)}\right| > t\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{\|b\|_{\text{BMO}} \|f(x)\|}{t}\right) u(x)v(x) dx,$$

where $\Phi(t) = t(1 + \log^+ t)$.

The theorem above is a starting point to prove by induction an analogous inequality for higher order commutators, denoted by $T^m_b$, for a non negative integer $m$ and defined by induction as follows: $T^0_b = T$ and $T^m_b = [b,T^{m-1}_b]$ for $m \geq 1$. If $x$ is not in the support of $f$ then it is clear that

$$T^m_b f(x) = \int (b(x) - b(y))^m K(x,y) f(y) dy.$$

Thus we obtain the following result.

**Theorem 2.** Let $u, v$ be weights such that $u \in A_1$ and $v \in A_{\infty}(u)$. Let $T$ be any Calderón-Zygmund operator and let $b \in \text{BMO}$. Then, for every $t > 0$ and every positive integer $m$ we have that

$$uw\left(\{x \in \mathbb{R}^n : \left|\frac{T^m_bfv(x)}{v(x)}\right| > t\}\right) \leq C \int_{\mathbb{R}^n} \Phi_m\left(\frac{\|b\|_{\text{BMO}} \|f(x)\|}{t}\right) u(x)v(x) dx,$$

where $\Phi_m(t) = t(1 + \log^+ t)^m$. 

Observe that since $\Phi_m$ is submultiplicative, that is, $\Phi_m(ab) \leq C\Phi_m(a)\Phi_m(b)$ for $ab \geq 0$, we have that (0.3) implies that

$$uw\left(\left\{ x \in \mathbb{R}^n : \frac{T_b^m(fv)(x)}{v(x)} > t \right\}\right) \leq C\Phi_m(\|b\|_{\text{BMO}}^m)\int_{\mathbb{R}^n} \Phi_m\left(\frac{|f(x)|}{t}\right) u(x)v(x)\,dx.$$

When $m = 0$, that is $T_b^0 = T$, the same estimate holds and our proof also works for this case. This estimation was first proved in [6]. However, our proof is quite different from that one since we shall not use the control of $T$ by the Hardy-Littlewood maximal operator, but the proof is straightforwardly related with the classical Calderón-Zygmund decomposition.

We can relax the hypotheses on the weights in both theorems above in order to obtain mixed inequalities for other operators. For example, in [13] the authors give a mixed estimation for the Hardy-Littlewood maximal operator on $\mathbb{R}^n$ for the case in which $u$ is a weight and $v$ is a power that is not even locally integrable. We wonder if an analogous estimation holds for $M^2$. Indeed, we have proved a more general result, involving the operator $M_\Phi$ for the case $\Phi(t) = t^r(1 + \log^+ t)^\delta$, with $r \geq 1$ and $\delta \geq 0$. Observe that, when $r = 1$ and $\delta = 1$ this is the desired result, since it is well known that $M^2$ is equivalent to $M_{L^{\log L}}$ (see next section). The result that we obtain is the following.

**Theorem 3.** Let $u$ be a weight and $v(x) = |x|^\beta$, where $\beta < -n$. Define $w(x) = 1/\Phi(v^{-1}(x))$, where $\Phi(t) = t^r(1 + \log^+ t)^\delta$ with $r \geq 1$ and $\delta \geq 0$. Then, for every $t > 0$,

$$(0.4) \quad uw\left(\left\{ x \in \mathbb{R}^n : \frac{M_{\Phi}(fv)}{v} > t \right\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{fv}{t}\right) Mu(x)\,dx.$$

By using the submultiplicativity of $\Phi$ the formula given in (0.4) can be rewritten as follows

$$\left\| \frac{M_{\Phi}(fv)}{v} \right\|_{\mathcal{L}^\Psi(uw)} := \sup_{t > 0} \Psi(t) \quad uw\left(\left\{ x : \frac{M_{\Phi}(fv)}{v} > t \right\}\right) \leq C \int_{\mathbb{R}^n} \Phi(fv) Mu(x)\,dx,$$

where $\Psi(t) = 1/\Phi(1/t)$ and $\|f\|_{\mathcal{L}^\Psi}$ denotes the weak Orlicz norm associated to $\Psi$.

Let us observe that if $u \in A_1$ then we have that

$$uw\left(\left\{ x \in \mathbb{R}^n : \frac{M_{\Phi}(fv)}{v} > t \right\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{fv}{t}\right) u(x)\,dx.$$

On the other hand, if $\Phi(t) = t$, we get $M_{\Phi} = M$ and $w = v$, and thus we obtain the same estimation given in [13].
1. Preliminaries and definitions

Let us recall that a weight \( w \) is a locally integrable function defined on \( \mathbb{R}^n \), such that \( 0 < w(x) < \infty \) a.e. \( x \in \mathbb{R}^n \). For \( 1 < p < \infty \) the Muckenhoupt \( A_p \) class is defined as the set of all weights \( w \) for which there exists a positive constant \( C \) such that the inequality
\[
\left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} \leq C
\]
holds for every cube \( Q \subset \mathbb{R}^n \), with sides parallel to the coordinate axes. For \( p = 1 \), we say that \( w \in A_1 \) if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q w \leq C \inf_Q w(x),
\]
for every cube \( Q \subset \mathbb{R}^n \). The smallest constant \( C \) for which the Muckenhoupt condition holds is called the \( A_p \)-constant of \( w \), and denoted by \([w]_{A_p}\). The \( A_\infty \) class is defined by the collection of all the \( A_p \) classes. It is easy to see that if \( p < q \) then \( A_p \subseteq A_q \). Given \( 1 < p < \infty \), we use \( p' \) to denote the conjugate exponent \( p/(p-1) \). For \( p = 1 \) we take \( p' = \infty \). Some classical references for the basic theory of Muckenhoupt weights are for example [9] and [10].

An important property of Muckenhoupt weights is the reverse Hölder’s condition. This means that given \( w \in A_p \), for some \( 1 \leq p < \infty \), there exists a positive constant \( C \) and \( s > 1 \) that depends only on the dimension \( n \), \( p \) and \([w]_{A_p}\), such that for every cube \( Q \)
\[
\left( \frac{1}{|Q|} \int_Q w^s(x) \, dx \right)^{1/s} \leq C \frac{1}{|Q|} \int_Q w(x) \, dx.
\]
We write \( w \in RH_s \) to point out that the inequality above holds, and we denote by \([w]_{RH_s}\) the smallest constant \( C \) for which this condition holds. A weight \( w \) belongs to \( RH_\infty \) if there exists a positive constant \( C \) such that
\[
\sup_Q w \leq C \frac{1}{|Q|} \int_Q w,
\]
for every \( Q \subset \mathbb{R}^n \). Let us observe that \( RH_\infty \subseteq RH_s \subseteq RH_q \), for every \( 1 < q < s \).

We shall use the next result.

**Lemma 4.** [6] Lemma 2.4] The following statements hold.

1. \( w \in A_\infty \) if and only if \( w = w_0 w_1 \), with \( w_0 \in A_1 \) and \( w_1 \in RH_\infty \).
2. If \( w \in A_1 \) then \( w^{-1} \in RH_\infty \).
3. If \( u, v \in RH_\infty \) then \( uv \in RH_\infty \).

A locally integrable function \( f \) is of bounded mean oscillation if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq C
\]
for every cube \( Q \subset \mathbb{R}^n \), where \( f_Q \) denotes the average \( |Q|^{-1} \int_Q f(y) \, dy \). In this case we write \( f \in BMO \), and we consider the norm
\[
\|f\|_{BMO} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |f - f_Q| \, dx.
\]
In fact, the function \( \| \cdot \|_{BMO} \) is not properly a norm since constant functions have BMO norm equal to zero, but it is a norm on quotient space of BMO functions modulo the space of constant functions. It is well known that every function \( f \in BMO \) satisfies the John-Nirenberg inequality.
More precisely, there exist two positive constants $C_1$ and $C_2$, depending only on the dimension, such that for any cube $Q$ in $\mathbb{R}^n$ and any $\lambda > 0$ we have

\begin{equation}
|x \in Q : |f(x) - f_Q| > \lambda| \leq C_1 |Q| e^{-c_2 \lambda^{-1} \|f\|_{\text{BMO}}}.
\end{equation}

As a consequence of (1.1) we obtain that for every $1 < p < \infty$, the quantity

$$\|f\|_{\text{BMO}, p} := \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{1/p}$$

is a norm on BMO equivalent to $\| \cdot \|_{\text{BMO}}$ (see for example [9]).

We shall also consider the following version of weighted BMO space. Let $w$ be a weight. We say that a locally integrable function $f$ belongs to $\text{BMO}_w^*$, if

$$\|f\|_{\text{BMO}_w^*} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{w(Q)} \int_Q |f(x) - f_Q| w(x) \, dx < \infty.$$  

Note that $f_Q$ is defined as above, that is, $f_Q = |Q|^{-1} \int_Q f(y) \, dy$, and $w(Q) = \int_Q w(x) \, dx$. We shall prove a relationship between BMO and $\text{BMO}_w^*$ for $w \in A_1$ in Lemma 8.

We say that $\Phi : [0, \infty) \to [0, \infty]$ is a *Young function* if it is strictly increasing, convex, $\Phi(0) = 0$ and $\Phi(t) \to \infty$ when $t \to \infty$. Given a Young function $\Phi$ and a Muckenhoupt weight $w$, the *generalized maximal operator* $M_{\Phi,w} = M_{\Phi(L),w}$ is defined by

$$M_{\Phi,w} f(x) := \sup_{Q} \|f\|_{\Phi,Q,w},$$

where $\|f\|_{\Phi,Q,w}$ denotes the weighted $\Phi$-average over $Q$ defined by means of the Luxemburg norm

\begin{equation}
\|f\|_{\Phi,Q,w} := \inf \left\{ \lambda > 0 : \frac{1}{w(Q)} \int_Q \Phi \left( \frac{|f|}{\lambda} \right) w \, dx \leq 1 \right\}.
\end{equation}

It can be proved that

\begin{equation}
\frac{1}{w(Q)} \int_Q \Phi \left( \frac{|f|}{\|f\|_{\Phi,Q,w}} \right) w \, dx \leq 1.
\end{equation}

By following the same arguments as in the result of Krasnosel’skiǐ and Rutickiǐ ([12], see also [17]), since $d\mu(x) = w(x) \, dx$ is a doubling measure for $w \in A_\infty$, we can get that $\|f\|_{\Phi,Q,w}$ is equivalent to the following quantity

$$\inf_{\tau > 0} \left\{ \tau + \frac{\tau}{w(Q)} \int_Q \Phi \left( \frac{|f|}{\tau} \right) w \, dx \right\}.$$  

If $w = 1$ we simply write $M_\Phi$ and $\|f\|_{\Phi,L}$. For example, when $\Phi(t) = t$, $M_\Phi$ is the Hardy-Littlewood maximal operator $M$. The function $\Phi(t) = t(1 + \log^+ t)^m$, $m \in \mathbb{N}$, plays an important role in the estimations for commutators of singular integrals. In this case, the corresponding maximal function is denoted by $M_{L(\log L)^m}$, which satisfies

\begin{equation}
M_{L(\log L)^m} f(x) \approx M^{m+1} f(x),
\end{equation}

where $M^{m+1}$ denotes the composition of the maximal operator $m + 1$ times with itself (see [15] and [1]).

The next result is a well known fact about a relation between the $\|b - bQ\|_{\exp L,Q}$ and $\|b\|_{\text{BMO}}$ when $b$ is a BMO function and its proof can be found in [14], where $\| \cdot \|_{\exp L,Q}$ denotes the $\Phi$-average over $Q$ when $\Phi(t) = e^t - 1$.
Lemma 5. Given \( f \in \text{BMO} \), there exists a positive constant \( C \) such that
\[
\| f - f_Q \|_{\exp L, Q} \leq C \| f \|_{\text{BMO}}.
\]

Given a Young function \( \Phi \), we use \( \bar{\Phi} \) to denote the \emph{complementary Young function} associated to \( \Phi \), defined for \( t \geq 0 \) by
\[
\bar{\Phi}(t) = \sup\{ ts - \Phi(s) : s \geq 0 \}.
\]
It is well known that \( \bar{\Phi} \) satisfies
\[
t \leq \Phi^{-1}(t)\bar{\Phi}^{-1}(t) \leq 2t, \quad \forall t > 0.
\]

When \( \Phi(t) = t(1 + \log^+ t)^\alpha \), \( \alpha > 0 \) we have that \( \bar{\Phi}(t) \approx \exp(t^{1/\alpha}) - 1 \), with the corresponding maximal function denoted by \( M_{\exp L, 1/\alpha} \).

The following \emph{generalized Hölder inequality}
\[
\frac{1}{w(Q)} \int_Q |fg| w \, dx \leq 2 \| f \|_{\Phi, Q, w} \| g \|_{\Phi, Q, w}
\]
holds.

Given weights \( u \) and \( v \), by \( v \in A_p(u) \) we mean that \( v \) satisfies the \( A_p \) condition with respect to the measure \( \mu \) defined as \( d\mu = u \, dx \). More precisely, for \( 1 < p < \infty \), we say that \( v \in A_p(u) \) if there exists a positive constant \( C \) such that
\[
\left( \frac{1}{u(Q)} \int_Q v(x) u(x) \, dx \right) \left( \frac{1}{u(Q)} \int_Q v(x)^{-\frac{1}{p-1}} u(x) \, dx \right)^{p-1} \leq C,
\]
for every cube \( Q \subset \mathbb{R}^n \). A weight \( v \) belongs to \( A_1(u) \) if
\[
\frac{1}{u(Q)} \int_Q v(x) u(x) \, dx \leq C \inf_Q v(x).
\]
We denote the union of all the \( A_p(u) \) classes by \( \text{A}_\infty(u) \). We shall use the following result.

Lemma 6. \([6] \text{Lemma 2.1}\) If \( u \in A_1 \) and \( v \in \text{A}_\infty(u) \), then \( uv \in \text{A}_\infty \). Particularly, if \( v \in A_p(u) \) with \( 1 \leq p < \infty \), then \( uv \in A_p \).

Finally, we shall state a result concerning to a Coifman type inequality for commutators of Calderón-Zygmund operators, which is proved in \([10]\).

Lemma 7. Let \( 0 < p < \infty \), \( w \in \text{A}_\infty \) and \( b \in \text{BMO} \). Then there exists a positive constant \( C \) such that
\[
\int_{\mathbb{R}^n} |T^k_b f(x)|^p w(x) \, dx \leq C \| b \|_{\text{BMO}}^kp \| w \|_{\text{BMO}}^{(k+1)p} \int_{\mathbb{R}^n} M_{\text{BMO}}^{k+1} f(x)^p w(x) \, dx.
\]

Note that when \( w \in A_p \), by applying \( k + 1 \) times Muckenhoupt’s Theorem we obtain the well known fact that the higher order commutators are bounded on \( L^p(w) \).

2. Auxiliary Lemmas

In this section we prove four lemmas that we shall use in the proof of our results. The first one states that the spaces \( \text{BMO} \) and \( \text{BMO}^*_w \) coincide when \( w \in A_1 \).

Lemma 8. Let \( w \in A_1 \). Then \( \| f \|_{\text{BMO}} \) and \( \| f \|_{\text{BMO}_w} \) are equivalent.

Proof. Since \( w \) belongs to \( A_1 \), we have that
\[
\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx = \frac{1}{w(Q)} \frac{w(Q)}{|Q|} \int_Q |f(x) - f_Q| \, dx 
\leq \frac{1}{w(Q)} \frac{[w]_{A_1}}{w(Q)} \int_Q |f(x) - f_Q| w(x) \, dx
\]
\[ \frac{1}{w(Q)} \int_Q |f(x) - f_Q| w(x) \, dx \leq \frac{|Q|}{w(Q)} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^s \right)^{1/s'} \left( \frac{1}{|Q|} \int_Q w^s \right)^{1/s} \]

\[ \leq C[w]_{\text{RH}_s} \|f\|_{\text{BMO}} \frac{|Q|}{w(Q)} \frac{1}{|Q|} \int_Q w(x) \, dx \]

\[ = C[w]_{\text{RH}_s} \|f\|_{\text{BMO}}. \]

□

The next lemma gives us a way to deal with the weighted Orlicz norms, controlling them by the same non-weighted norms.

**Lemma 9.** Let \( w \) be a weight such that \( w \in \text{RH}_s \) for some \( s > 1 \). Then

\[ \|f\|_{\text{expL}, w} \leq 2^{1/s'} [w]_{\text{RH}_s} s' \|f\|_{\text{expL}, Q}. \]

**Proof.** Fix \( \lambda = s' \|f\|_{\text{expL}, Q} \). In order to show that \( \|f\|_{\text{expL}, w} \leq 2^{1/s'} [w]_{\text{RH}_s} \lambda \), it is enough to prove that

\[ \frac{1}{w(Q)} \int_Q \left( e^{\frac{|f(x)|}{\lambda}} - 1 \right) w(x) \, dx \leq 2^{1/s'} [w]_{\text{RH}_s}, \]

for every cube \( Q \). Indeed, since \( C = 2^{1/s'} [w]_{\text{RH}_s} > 1 \), from (2.1) we obtain that

\[ \frac{1}{w(Q)} \int_Q \left( e^{\frac{|f(x)|}{\lambda}} - 1 \right) w(x) \, dx \leq \frac{1}{Cw(Q)} \int_Q \left( e^{\frac{|f(x)|}{\lambda}} - 1 \right) w(x) \, dx \leq 1, \]

where we have used that \( \psi(\alpha t) \leq \alpha \psi(t) \), for every convex function \( \psi \) with \( \psi(0) = 0 \) and every \( \alpha \in [0, 1] \). Then we conclude that \( \|f\|_{\text{expL}, w} \leq C \lambda. \)

Then, let us prove that (2.1) holds. From Hölder’s inequality and the reverse Hölder condition \( \text{RH}_s \), we get that

\[ \frac{1}{w(Q)} \int_Q \left( e^{\frac{|f(x)|}{\lambda}} - 1 \right) w(x) \, dx \leq \frac{1}{w(Q)} \int_Q e^{\frac{|f(x)|}{\lambda}} w(x) \, dx \]

\[ \leq \frac{|Q|}{w(Q)} \left( \frac{1}{|Q|} \int_Q e^{\lfloor f \rfloor_{\text{expL}, Q}} \, dx \right)^{1/s'} \left( \frac{1}{|Q|} \int_Q w^s \, dx \right)^{1/s} \]

\[ \leq [w]_{\text{RH}_s} \left( \frac{1}{|Q|} \int_Q e^{\lfloor f \rfloor_{\text{expL}, Q}} \, dx \right)^{1/s'} \]

\[ = [w]_{\text{RH}_s} \left( \frac{1}{|Q|} \int_Q \left( e^{\lfloor f \rfloor_{\text{expL}, Q}} - 1 \right) \, dx + 1 \right)^{1/s'} \]

\[ \leq 2^{1/s'} [w]_{\text{RH}_s}, \]

where in the last inequality we have used (1.3). We are done. □

The following result is useful in order to prove our main result.

**Lemma 10.** Given \( f \in \text{BMO} \), there exists a positive constant \( C \) such that

\[ |f_Q - f_{2^k Q}| \leq C k \|f\|_{\text{BMO}}, \]

for every \( k \in \mathbb{N} \) and every cube \( Q \).
Proof. Fix a cube \( Q \) and a positive integer \( k \). Then

\[
|f_Q - f_{2^kQ}| \leq \sum_{j=0}^{k-1} |f_{2^{j+1}Q} - f_{2^jQ}|
\]

\[
\leq \sum_{j=0}^{k-1} \frac{1}{|2^jQ|} \int_{2^jQ} |f(x) - f_{2^j+1Q}| \, dx
\]

\[
\leq C \sum_{j=0}^{k-1} \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |f(x) - f_{2^j+1Q}| \, dx
\]

\[
\leq Ck \|f\|_{\text{BMO}}.
\]

For the proof of Theorem \( \square \) we shall use the following lemma which will be a fundamental tool. It states that if \( u \in A_1 \) and \( uv \in A_\infty \), then \( u \in A_1(v) \).

Lemma 11. Let \( u \) and \( v \) be weights such that \( u \in A_1 \) y \( uv \in A_\infty \). Then there exists a positive constant \( C \) such that

\[
(2.2) \quad \frac{(uv)(Q)}{v(Q)} \leq C \inf_Q u,
\]

for every cube \( Q \).

Proof. Fix a cube \( Q \). From (1) in Lemma \( \square \) there exist two weights \( w_0, w_1 \) such that \( uv = w_0w_1 \), with \( w_0 \in A_1 \) and \( w_1 \in \text{RH}_\infty \). From the hypothesis on \( u \) and (2), \( u^{-1} \in \text{RH}_\infty \), so that from (3) we can conclude that \( w_1u^{-1} \in \text{RH}_\infty \). Let \( s > 1 \) such that \( w_0 \in \text{RH}_s \). Then, we have that

\[
\frac{(uv)(Q)}{v(Q)} = \frac{(w_0w_1)(Q)}{(w_0w_1u^{-1})(Q)} \int_Q w_0w_1 \, dx
\]

\[
\leq \left[ w_1u^{-1}\right]_{\text{RH}_\infty} \frac{1}{\inf_Q w_0} \sup_Q (w_1u^{-1}) \frac{1}{|Q|} \int_Q w_0w_1 \, dx
\]

\[
\leq \left[ w_1u^{-1}\right]_{\text{RH}_\infty} \frac{1}{\inf_Q w_0} \sup_Q (w_1u^{-1}) \left( \frac{1}{|Q|} \int_Q w_0' \right)^{1/s} \left( \frac{1}{|Q|} \int_Q w_1' \right)^{1/s'}
\]

\[
\leq \left[ w_1u^{-1}\right]_{\text{RH}_\infty} \frac{1}{\inf_Q w_0} \sup_Q (w_1u^{-1}) \frac{[w_0]\text{RH}_s w_0(\cdot)[w_1]\text{RH}_s w_1(\cdot)}{|Q|}
\]

\[
\leq \left[ w_0\right]_{A_1} \frac{[w_0]\text{RH}_s w_0(\cdot)[w_1]\text{RH}_s w_1(\cdot)}{|Q|} \int_Q \frac{w_1u^{-1}u \, dx}{\sup_Q (w_1u^{-1})}
\]

\[
\leq \left[ w_0\right]_{A_1} \frac{[w_0]\text{RH}_s w_0(\cdot)[w_1]\text{RH}_s w_1(\cdot)}{|Q|} \frac{u(Q)}{|Q|} \inf_Q u.
\]

\( \square \)

3. Proof of the main results

We shall use the following result about Calderón-Zygmund operators.
Theorem 12. [6] Thm. 1.3] If \( u, v \) are weights such that \( u, v \in A_1 \), or \( u \in A_1 \) and \( v \in A_\infty(u) \), then there exists a positive constant \( C \) such that for every \( t > 0 \),

\[
uv \left( \left\{ x \in \mathbb{R}^n : \frac{T(fv)(x)}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} |f(x)|u(x)v(x) \, dx,
\]

where \( T \) is any Calderón-Zygmund operator.

Proof of Theorem 7. Note that since \([b, T](f/\|b\|_{BMO}) = [b/\|b\|_{BMO}, T]f\), we can assume that \( \|b\|_{BMO} = 1 \). Without loss of generality, we can assume that \( f \) is a bounded, non-negative function with compact support. Fix \( t > 0 \), and form the Calderón-Zygmund decomposition of \( f \) at height \( t > 0 \) with respect to the doubling measure \( \mu \) given by \( d\mu(x) = v(x) \, dx \) (\( \mu \) is doubling since \( v \in A_\infty(u) \) implies \( v \in A_\infty \) (see [6] Lemma 2.1)). This yields a collection of disjoint dyadic cubes \( \{Q_j\}_{j=1}^\infty \), such that \( t < f_{Q_j}^v \leq Ct \) for some \( C > 1 \), where \( f_{Q_j}^v \) is defined by

\[
f_{Q_j}^v = \frac{1}{v(Q_j)} \int_{Q_j} f(y)v(y) \, dy.
\]

From this decomposition we have that if \( \Omega = \bigcup_{j=1}^\infty Q_j \) then \( f(x) \leq t \) in almost every \( x \in \mathbb{R}^n \setminus \Omega \).

We decompose \( f \) as \( f = g + h \), where

\[
g(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{R}^n \setminus \Omega; \\ f_{Q_j}^v, & \text{if } x \in Q_j, \end{cases}
\]

and \( h(x) = \sum_{j=0}^\infty h_j(x) \), with

\[
h_j(x) = \left(f(x) - f_{Q_j}^v\right) \chi_{Q_j}(x),
\]

where \( \chi_E \) denotes the indicator function in the set \( E \). It follows that \( g(x) \leq Ct \) almost everywhere, each \( h_j \) is supported on \( Q_j \) and

\[
\int_{Q_j} h_j(y)v(y) \, dy = 0.
\]

With \( cQ, c > 0, \) we will denote the cube concentric with \( Q \) whose side length is \( c \) times the side length of \( Q \). So let \( Q_j^* = 3Q_j \) and \( \Omega^* = \bigcup_j Q_j^* \). Then

\[
uv \left( \left\{ x \in \mathbb{R}^n : \frac{[b, T](fv)(x)}{v(x)} > t \right\} \right) \leq uv \left( \left\{ x \in \mathbb{R}^n : \left| \frac{[b, T](gv)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right)
\]

\[
+ uv \left( \left\{ x \in \mathbb{R}^n : \left| \frac{[b, T](hv)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right)
\]

\[
\leq uv \left( \left\{ x \in \mathbb{R}^n : \left| \frac{[b, T](gv)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right)
\]

\[
+ uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| \frac{[b, T](hv)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right)
\]

\[
+ uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| \frac{[b, T](hv)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right)
\]

\[
= I + II + III.
\]

We shall estimate each term separately. Since \( v \in A_\infty(u) \), there exists \( q' > 1 \) such that \( v \in A_{q'}(u) \), so that \( v^{1-q} \in A_q(u) \) and by Lemma [6] we have that \( uv^{1-q} \in A_q \). By applying Tchebychev’s inequality with \( q > 1 \) we obtain

\[
I = uv \left( \left\{ x \in \mathbb{R}^n : \left| \frac{[b, T](gv)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right)
\]

\[
\leq \frac{C}{t^q} \int_{\mathbb{R}^n} \left| [b, T](gv)(x) \right|^q u(x)v^{1-q}(x) \, dx
\]
since $uv^{1-q} \in A_q$ implies that the commutator $[b,T]$ is bounded on $L^q(uv^{1-q})$ (see remark after Lemma 7) and $g(x) \leq Ct$. From the definition of $g$ and Lemma 11 we get that

$$I \leq \frac{C}{t} \int_{\mathbb{R}^n} f(x)u(x)v(x) \, dx + \frac{C}{t} \sum_{j=1}^{\infty} \frac{(uv)(Q_j)}{v(Q_j)} \int_{Q_j} f(y)v(y) \, dy$$

$$\leq \frac{C}{t} \int_{\mathbb{R}^n} f(x)u(x)v(x) \, dx + \frac{C}{t} \sum_{j=1}^{\infty} \int_{Q_j} f(x)u(x)v(x) \, dx$$

$$\leq \frac{C}{t} \int_{\mathbb{R}^n} f(x)u(x)v(x) \, dx.$$

In order to estimate $II$, since $uv$ is doubling and by applying Lemma 11 we get

$$II = (uv)(\Omega^*) = \sum_j (uv)(Q_j^*)$$

$$\leq C \sum_j v(Q_j) \frac{(uv)(Q_j)}{v(Q_j)}$$

$$\leq C \sum_j (\inf u) \frac{1}{t} \int_{Q_j} f(x)v(x) \, dx$$

$$\leq \frac{C}{t} \int_{\mathbb{R}^n} f(x)u(x)v(x) \, dx.$$

By observing that

$$\frac{[b,T](hv)}{v} = \sum_j \frac{[b,T](h_jv)}{v} = \sum_j \frac{(b-bQ_j)T(h_jv)}{v} = \sum_j \frac{T((b-bQ_j)h_jv)}{v},$$

the estimate of $III$ can be made as follows

$$III \leq uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| \sum_j \frac{(b-bQ_j)T(h_jv)}{v} \right| > \frac{t}{4} \right\} \right)$$

$$+ uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| \sum_j \frac{T((b-bQ_j)h_jv)}{v} \right| > \frac{t}{4} \right\} \right)$$

$$= A + B.$$

Let us first estimate $A$. From the Tchebychev’s inequality, Tonelli’s theorem and (3.1) we have that

$$A \leq \frac{C}{t} \int_{\mathbb{R}^n} \sum_j |b(x) - bQ_j| |T(h_jv)(x)|u(x) \, dx$$

$$\leq \frac{C}{t} \sum_j \int_{\mathbb{R}^n} |b(x) - bQ_j| \left| \int_{Q_j} h_j(y)v(y)K(x-y) \, dy \right| u(x) \, dx$$

$$= \frac{C}{t} \sum_j \int_{\mathbb{R}^n} |b(x) - bQ_j| \left| \int_{Q_j} h_j(y)v(y) \left[ K(x-y) - K(x-xQ_j) \right] \, dy \right| u(x) \, dx$$

$$\leq \frac{C}{t} \sum_j \int_{Q_j} |h_j(y)|v(y) \int_{\mathbb{R}^n} |b(x) - bQ_j| \left| K(x-y) - K(x-xQ_j) \right| u(x) \, dx \, dy.$$
Given a cube $Q_j$, denote $x_{Q_j}$ its center, $\ell(Q_j)$ the length of its side, $r_j = 2^{-1} \ell(Q_j)$ and $A_{j,k} = \{x : 2^k r_j \leq |x - x_{Q_j}| < 2^{k+1} r_j \}$. Then, for each $y \in Q_j$, from (1.2) we have
\[
\int_{\mathbb{R}^n \setminus Q_j} |b(x) - b_{Q_j}| |K(x - y) - K(x - x_{Q_j})| u(x) \, dx
\leq \sum_{k=1}^{\infty} \int_{A_{j,k}} |b(x) - b_{Q_j}| \frac{|y - x_{Q_j}|}{|x - x_{Q_j}|^{n+1}} u(x) \, dx
\leq \sum_{k=1}^{\infty} \frac{r_j}{(2^k r_j)^{n+1}} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}| u(x) \, dx.
\]

Then, from Lemmas 8 and 10 and the fact that $u \in A_1,
\[
\sum_{k=1}^{\infty} \frac{r_j}{(2^k r_j)^{n+1}} \int_{2^{k+1} Q_j} |b - b_{Q_j}| u \, dx \leq C \sum_{k=1}^{\infty} \frac{2^{-k}}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b - b_{Q_j}| u \, dx
\leq C \sum_{k=1}^{\infty} \frac{2^{-k}}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b - b_{2^{k+1} Q_j}| u \, dx
+ C \sum_{k=1}^{\infty} \frac{2^{-k}}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b_{2^{k+1} Q_j} - b_{Q_j}| u \, dx
\leq C \sum_{k=1}^{\infty} 2^{-k} \frac{u(2^{k+1} Q_j)}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b - b_{2^{k+1} Q_j}| u \, dx
+ C \sum_{k=1}^{\infty} 2^{-k} C (k + 1) u(y)
\leq C u(y),
\]

Hence, from Lemma 11 we obtain that
\[
A \leq \frac{C}{t} \sum_{j} \int_{Q_j} |h_j(y)| u(y) v(y) \, dy
\leq \frac{C}{t} \sum_{j} \left( \int_{Q_j} f(y) u(y) v(y) \, dy + \int_{Q_j} f_{Q_j}^v u(y) v(y) \, dy \right)
\leq \frac{C}{t} \sum_{j} \left( \int_{Q_j} f(y) u(y) v(y) \, dy + \frac{(uv)(Q_j)}{v(Q_j)} \int_{Q_j} f(y) v(y) \, dy \right)
\leq \frac{C}{t} \int_{\mathbb{R}^n} f(y) u(y) v(y) \, dy,
\]

We shall finally estimate $B$. By applying Theorem 12 we get
\[
B \leq \frac{C}{t} \int_{\mathbb{R}^n} \left| \sum_{j} (b(x) - b_{Q_j}) h_j(x) \right| u(x) v(x) \, dx
\leq \frac{C}{t} \sum_{j} \int_{Q_j} |b(x) - b_{Q_j}| f(x) u(x) v(x) \, dx + \frac{C}{t} \sum_{j} \int_{Q_j} |b(x) - b_{Q_j}| f_{Q_j}^v u(x) v(x) \, dx
= B_1 + B_2.
\]

From the generalized Hölder’s inequality with respect to the doubling measure $w = uv$, Lemma 9 and Lemmas 5 and 11 we obtain
\[
B_1 \leq \frac{C}{t} \sum_{j} (uv)(Q_j) \|b - b_{Q_j}\|_{\exp L, Q_j, uv} \|f\|_{L^{\log L, Q_j}, uv}
\]
and now we proceed as in the proof of Theorem 1 to obtain the desired estimate. The estimation applying Hölder’s inequality we get

\[ \leq \frac{C}{t} \sum_{j} (uv)(Q_j) \| b - b_{Q_j} \|_{\exp L, Q_j} \inf_{\tau > 0} \left\{ \tau + \frac{\tau}{(uv)(Q_j)} \int_{Q_j} \Phi \left( \frac{f}{\tau} \right) uv \, dx \right\} \]

\[ \leq C \sum_{j} \left( (uv)(Q_j) + \int_{Q_j} \Phi \left( \frac{f}{t} \right) uv \, dx \right) \]

\[ \leq C \sum_{j} \left( v(Q_j) \inf_{Q_j} u + \int_{Q_j} \Phi \left( \frac{f}{t} \right) uv \, dx \right) \]

\[ \leq C \sum_{j} \left( \inf_{Q_j} u \frac{1}{t} \int_{Q_j} f v \, dx + \int_{Q_j} \Phi \left( \frac{f}{t} \right) uv \, dx \right) \]

\[ \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{f}{t} \right) uv \, dx. \]

In order to estimate \( B_2 \), let \( s \) be the reverse Hölder exponent of the weight \( uv \). Then, by applying Hölder’s inequality we get

\[ B_2 \leq \frac{C}{t} \sum_{j} \frac{|Q_j|}{v(Q_j)} \left( \int_{Q_j} f(y)v(y) \, dy \right) \left( \frac{1}{|Q_j|} \int_{Q_j} |b - b_{Q_j}|^s \right)^{1/s'} \left( \frac{1}{|Q_j|} \int_{Q_j} (uv)^s \right)^{1/s} \]

\[ \leq C \frac{1}{t} \sum_{j} \frac{(uv)(Q_j)}{v(Q_j)} \int_{Q_j} f(y)v(y) \, dy \]

\[ \leq C \frac{1}{t} \sum_{j} \inf_{Q_j} u \int_{Q_j} f(y)v(y) \, dy \]

\[ \leq C \frac{1}{t} \int_{\mathbb{R}^n} f(y)u(y)v(y) \, dy, \]

and the result is proved.

**Proof of Theorem**

Again, since \( T^m_b(f/\|b\|_{\text{BMO}}) = T^m_b(f) \), we assume that \( \|b\|_{\text{BMO}} = 1 \). Without loss of generality, we can also assume that \( f \) is a bounded, non-negative function with compact support. Fix a positive integer \( m \). We will use an induction argument. The case \( m = 1 \) is proved in Theorem. Assume that the result holds for every \( 1 \leq k \leq m - 1 \). For a fixed \( t > 0 \), we consider again the Calderón-Zygmund decomposition of \( f \) at height \( t \) with respect to the doubling measure \( \mu \) given by \( d\mu(x) = v(x) \, dx \). Then, with the same notation as in the proof of Theorem, we have that

\[ uv \left( \left\{ x \in \mathbb{R}^n : \frac{T^m_b(fv)(x)}{v(x)} \right\} > \frac{t}{2} \right) \]

\[ \leq uv \left( \left\{ x \in \mathbb{R}^n : \frac{T^m_b(gv)(x)}{v(x)} \right\} > \frac{t}{2} \right) \]

\[ \quad + uv(\Omega^*) + uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \frac{T^m_b(hv)(x)}{v(x)} \right\} > \frac{t}{2} \right) \]

\[ = I + II + III. \]

We shall first estimate \( I \). Under the hypothesis on \( u \) and \( v \), there exists \( q > 1 \) such that \( uv^{1-q} \in A_q \). From Tchebychev’s inequality and the remark after Lemma we obtain

\[ I \leq \frac{C}{t^q} \int_{\mathbb{R}^n} |T^m_b(gv)(x)|^q u(x)v(x) \, dx \]

\[ \leq \frac{C}{t^q} \int_{\mathbb{R}^n} |g(x)|^q u(x)v(x) \, dx \]

\[ \leq \frac{C}{t} \int_{\mathbb{R}^n} |g(x)| u(x)v(x) \, dx, \]

and now we proceed as in the proof of Theorem to obtain the desired estimate. The estimation of \( II \) is obtained exactly as in that theorem.
Then, we shall focus on $III$. Observe that $h_j v$ is supported on $Q_j$, so that if $x \notin Q_j$

$$T^m_b(h_j v)(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x - y) h_j(y) v(y) \, dy$$

$$= \sum_{\ell = 0}^{m} C_{\ell,m}(b(x) - b_{Q_j})^{m-\ell} \int_{\mathbb{R}^n} (b(y) - b_{Q_j})^{\ell} K(x - y) h_j(y) v(y) \, dy$$

$$= C_{0,m}(b(x) - b_{Q_j})^m T(h_j v)(x) + C_{m,m} T((b - b_{Q_j})^m h_j v)(x)$$

$$+ \sum_{\ell = 1}^{m-1} C_{\ell,m}(b(x) - b_{Q_j})^{m-\ell} \int_{\mathbb{R}^n} (b(y) - b_{Q_j})^{\ell} K(x - y) h_j(y) v(y) \, dy.$$

Note that, expanding as before the binomial expression $(b(x) - b_{Q_j})^{m-\ell}$, we obtain

$$\sum_{\ell = 1}^{m-1} C_{\ell,m}(b(x) - b_{Q_j})^{m-\ell} \int_{\mathbb{R}^n} (b(y) - b_{Q_j})^{\ell} K(x - y) h_j(y) v(y) \, dy$$

$$= \sum_{\ell = 1}^{m-1} C_{\ell,m} \sum_{i=1}^{m-\ell} C_{i,\ell,m} \int_{\mathbb{R}^n} (b(x) - b(y))^i (b(y) - b_{Q_j})^{m-i} K(x - y) h_j(y) v(y) \, dy$$

$$= \sum_{i=1}^{m-1} \sum_{\ell = 1}^{m-i} C_{\ell,m} C_{i,m,\ell} \int_{\mathbb{R}^n} (b(x) - b(y))^i (b(y) - b_{Q_j})^{m-i} K(x - y) h_j(y) v(y) \, dy$$

$$+ \sum_{\ell = 1}^{m-1} C_{\ell,m} C_{0,m,\ell} \int_{\mathbb{R}^n} (b(y) - b_{Q_j})^m K(x - y) h_j(y) v(y) \, dy.$$

Thus we can write

$$\sum_{i=0}^{m-1} C_{i,m} \int_{\mathbb{R}^n} (b(x) - b(y))^i (b(y) - b_{Q_j})^{m-i} K(x - y) h_j(y) v(y) \, dy$$

$$= C_{0,m} T((b - b_{Q_j})^m h_j v)(x) + \sum_{i=1}^{m-1} C_{i,m} T_b^i ((b - b_{Q_j})^{m-i} h_j v)(x),$$

so that

$$\left| \sum_j T^m_b(h_j v)(x) \right| = \left| C_{0,m} \sum_j (b(x) - b_{Q_j})^m T(h_j v)(x) \right|$$

$$+ \left| C_{m,m} \sum_j T((b - b_{Q_j})^m h_j v)(x) \right|$$

$$+ \left| \sum_{i=1}^{m-1} C_{i,m} T_b^i ((b - b_{Q_j})^{m-i} h_j v)(x) \right|.$$ 

Then we can estimate $III$ as follows:

$$III \leq uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| C_{0,m} \sum_j (b(x) - b_{Q_j})^m T(h_j v)(x) \right| > \frac{t}{6} \right\} \right)$$

$$+ uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| C_{m,m} \sum_j T((b - b_{Q_j})^m h_j v)(x) \right| > \frac{t}{6} \right\} \right)$$

$$+ uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| \sum_{i=1}^{m-1} C_{i,m} T_b^i ((b - b_{Q_j})^{m-i} h_j v)(x) \right| > \frac{t}{6} \right\} \right)$$

$$= I_1 + I_2 + I_3.$$
In order to estimate $I_1$, we use Tchebychev's inequality to obtain

$$I_1 \leq \frac{C}{t} \int_{\mathbb{R}^n \setminus Q^*} \left| \sum_j (b(x) - b_{Q_j})^m T(h_j v)(x) \right| u(x) \, dx$$

$$\leq \frac{C}{t} \sum_j \int_{\mathbb{R}^n \setminus Q_j^*} |b(x) - b_{Q_j}|^m \left| \int_{Q_j} (K(x - y) - K(x - x_{Q_j})) h_j(y) v(y) \, dy \right| u(x) \, dx$$

$$\leq \frac{C}{t} \sum_j \int_{Q_j} |h_j(y)| v(y) \int_{\mathbb{R}^n \setminus Q_j^*} |b(x) - b_{Q_j}|^m |K(x - y) - K(x - x_{Q_j})| u(x) \, dx \, dy$$

$$\leq \frac{C}{t} \sum_j \int_{Q_j} |h_j(y)| v(y) \sum_{k=1}^{\infty} \int_{A_{j,k}} |b(x) - b_{Q_j}|^m \frac{|y - x_{Q_j}|}{|x - x_{Q_j}|^{n+1}} u(x) \, dx \, dy$$

$$\leq \frac{C}{t} \sum_j \int_{Q_j} |h_j(y)| v(y) \sum_{k=1}^{\infty} \frac{2^{-k}}{2^{k+1} Q_j} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}|^m u(x) \, dx \, dy,$$

where $A_{j,k}$ is the set defined in the proof of Theorem 1. For $y \in Q_j$ we can bound the sum over $k$ by

$$\sum_{k=1}^{\infty} \frac{2^{-k}}{2^{k+1} Q_j} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}|^m u(x) \, dx$$

$$\leq 2^m \sum_{k=1}^{\infty} \frac{2^{-k}}{2^{k+1} Q_j} \int_{2^{k+1} Q_j} |b(x) - b_{2^{k+1} Q_j}|^m u(x) \, dx$$

$$+ 2^m \sum_{k=1}^{\infty} \frac{2^{-k}}{2^{k+1} Q_j} \int_{2^{k+1} Q_j} |b_{Q_j} - b_{2^{k+1} Q_j}|^m u(x) \, dx.$$

With a change of variables we easily get that

$$\|g^m\|_{\exp L^{1/m}, Q} = \|g\|_{\exp L^{1/m}, Q},$$

for every cube $Q$. From this fact and the generalized Hölder’s inequality we obtain

$$2^m \sum_{k=1}^{\infty} \frac{2^{-k}}{2^{k+1} Q_j} \int_{2^{k+1} Q_j} |b(x) - b_{2^{k+1} Q_j}|^m u(x) \, dx$$

$$\leq C \|b - b_{2^{k+1} Q_j}\|_{\exp L^{1/m}, 2^{k+1} Q_j}^m \|u\|_{L(\log L)^m, 2^{k+1} Q_j}$$

$$\leq C \|b - b_{2^{k+1} Q_j}\|_{\exp L^{1/m}, 2^{k+1} Q_j}^m M_{L(\log L)^m} u(y)$$

$$\leq CM^{m+1} u(y)$$

$$\leq C u(y),$$

where we have used (1.4) and that $u \in A_1$.

On the other hand, from Lemma 10 we have that

$$2^m \sum_{k=1}^{\infty} \frac{2^{-k}}{2^{k+1} Q_j} \int_{2^{k+1} Q_j} |b_{Q_j} - b_{2^{k+1} Q_j}|^m u(x) \, dx$$

$$\leq C \sum_{k=1}^{\infty} \frac{2^{-k}}{2^{k+1} Q_j} \int_{2^{k+1} Q_j} (k + 1)^m u(x) \, dx$$

$$\leq C u(y) \sum_{k=1}^{\infty} 2^{-k} (k + 1)^m$$

$$= C u(y).$$

Hence

$$I_1 \leq \frac{C}{t} \sum_j \int_{Q_j} |h_j(y)| u(y) v(y) \, dy,$$
and then we proceed as in the proof of Theorem 11. In order to estimate \( I_2 \), we use Theorem 12 to get that

\[
I_2 \leq \frac{C}{t} \int_{\mathbb{R}^n} \left| \sum_j (b(x) - b_{Q_j})^{m} h_j(x) \right| u(x)v(x) \, dx
\]

\[
\leq \frac{C}{t} \sum_j \int_{Q_j} |b(x) - b_{Q_j}|^{m} |h_j(x)| u(x)v(x) \, dx
\]

\[
\leq \frac{C}{t} \sum_j \int_{Q_j} |b(x) - b_{Q_j}|^{m} f(x) u(x)v(x) \, dx
\]

\[
+ \frac{C}{t} \sum_j \int_{Q_j} |b(x) - b_{Q_j}|^{m} f^{v}_{Q_j} u(x)v(x) \, dx
\]

\[
= I_{2,1} + I_{2,2}.
\]

From the generalized Hölder’s inequality with respect to the measure \( \mu \) defined by \( d\mu = uv \, dx \), and by applying (3.2) we have that

\[
I_{2,1} \leq \frac{C}{t} \sum_j (uv)(Q_j) \| b - b_{Q_j} \|_{\exp L^{1/m}, Q_j, uv} f \|_{L(\log L)^m, Q_j, uv}
\]

\[
\leq \frac{C}{t} \sum_j (uv)(Q_j) \| b - b_{Q_j} \|_{\exp L^{1/m}, Q_j, uv} \left\{ t + \frac{t}{uv(Q_j)} \int_{Q_j} \Phi_m \left( \frac{f}{t} \right) uv(x) \, dx \right\}
\]

\[
\leq C \sum_j \left( \frac{(uv)(Q_j)}{v(Q_j)} \right)^{\frac{1}{s'}} v(Q_j) + \int_{Q_j} \Phi_m \left( \frac{f}{t} \right) u(x)v(x) \, dx
\]

\[
\leq C \sum_j \frac{1}{t} \int_{Q_j} f(x)u(x)v(x) \, dx + \int_{Q_j} \Phi_m \left( \frac{f}{t} \right) u(x)v(x) \, dx
\]

\[
\leq C \int_{\mathbb{R}^n} \Phi_m \left( \frac{f}{t} \right) u(x)v(x) \, dx,
\]

where we have used Lemma 11.

For \( I_{2,2} \), we apply Hölder’s inequality with exponent \( s \) and \( s' \), where \( s > 1 \) is such that \( uv \in \text{RH}_s \). Then,

\[
I_{2,2} \leq \sum_j \frac{|Q_j|}{v(Q_j)} \left\{ \frac{1}{|Q_j|} \int_{Q_j} |b(x) - b_{Q_j}|^{ms'} \right\}^{1/s'} \left( \frac{1}{|Q_j|} \int_{Q_j} (u(x)v(x))^s \right)^{1/s} \int_{Q_j} f(x)v(x) \, dx
\]

\[
\leq \frac{C}{t} \sum_j (uv)(Q_j) \int_{Q_j} f(x)v(x) \, dx
\]

\[
\leq \frac{C}{t} \int_{\mathbb{R}^n} f(x)u(x)v(x) \, dx,
\]

by applying again Lemma 11. It only remains the estimation of \( I_3 \).

\[
I_3 \leq \sum_{i=1}^{m-1} uv \left\{ \frac{T^i_b \left( \sum_j (b - b_{Q_j})^{m-i} h_j v \right)(x)}{v(x)} > \frac{t}{C} \right\}
\]

\[
\leq \sum_{i=1}^{m-1} uv \left\{ \frac{T^i_b \left( \sum_j (b - b_{Q_j})^{m-i} f_{Q_j} v \right)(x)}{v(x)} > \frac{t}{C} \right\}
\]

\[
+ \sum_{i=1}^{m-1} uv \left\{ \frac{T^i_b \left( \sum_j (b - b_{Q_j})^{m-i} f_{Q_j} v \right)(x)}{v(x)} > \frac{t}{C} \right\}
\]

\[
= I_{3,1} + I_{3,2}.
\]
In order to estimate $I_{3,1}$ and $I_{3,2}$ we will use the inductive hypothesis and the following fact: if $A$, $B$ and $C$ are Young functions satisfying $A^{-1}(t)B^{-1}(t) \leq C^{-1}(t)$ for every $t > 0$, it is easy to see that

$$C(st) \leq A(s) + B(t),$$

holds for every $0 \leq s, t < \infty$.

Let us take $\alpha$ such that $\alpha s' < C_2$, where $C_2$ is the constant that appears in (1.1) for $b$ and $s'$ is the conjugated exponent of $s$, that verifies $uv \in \text{RH}_s$. Thus, if $\Psi_k(t) = e^{c\lambda t/k} - 1$ then $\Psi_k^{-1}(t) \approx (\log(e + t))^k$. Since $\Phi_m^{-1}(t) \approx t/(\log(e + t))^m$, then $\Phi_m^{-1}(t)\Psi_m^{-1}(t) \lesssim \Phi_1^{-1}(t)$, so that by the inductive hypothesis we obtain

$$I_{3,1} \leq C \sum_{i=1}^{m-1} \int_{\mathbb{R}^n} \Phi_i \left( \frac{\sum_j (b(x) - b_{Q_j})^{m-i} f(x)\mathcal{X}_{Q_j}(x)}{t} \right) u(x)v(x) \, dx$$

$$\leq C \sum_{i=1}^{m-1} \sum_j \int_{Q_j} \Phi_i \left( \frac{(b(x) - b_{Q_j})^{m-i} f(x)}{t} \right) u(x)v(x) \, dx$$

$$\leq C \sum_{h=1}^{m} \sum_{j=1}^{m-1} \int_{Q_j} \left( \Phi_m \left( \frac{f(x)}{t} \right) + \Psi_{m-i}((b(x) - b_{Q_j})^{m-i}) \right) u(x)v(x) \, dx.$$ 

Now applying Hölder’s inequality, (1.1) and Lemma 11 we have that

$$\int_{Q_j} \Psi_{m-i}((b(x) - b_{Q_j})^{m-i}) u(x)v(x) \, dx$$

$$\leq \int_{Q_j} e^{c\lambda b(x) - b_{Q_j}} u(x)v(x) \, dx$$

$$\leq \left( \frac{1}{|Q_j|} \int_{Q_j} e^{b(x) - b_{Q_j}} |x|^{\alpha s'} \, dx \right)^{1/(\alpha s')} \left( \frac{1}{|Q_j|} \int_{Q_j} (u(x)v(x))^s \, dx \right)^{1/s} |Q_j|$$

$$\leq \left( \int_0^{\infty} \alpha s' e^{\alpha s' \lambda} C_1 \lambda e^{-\lambda C_2} d\lambda \right)^{1/(\alpha s')} (uv)(Q_j) = C(uv)(Q_j)$$

$$\leq C \frac{(uv)(Q_j)}{v(Q_j)} v(Q_j)$$

$$\leq C \int_{Q_j} f(x)u(x)v(x) \, dx.$$ 

Then,

$$I_{3,1} \leq C \int_{\mathbb{R}^n} \Phi_m \left( \frac{f(x)}{t} \right) u(x)v(x) \, dx,$$

as desired. In the case of $I_{3,2}$, let us first note that

$$\frac{f_{Q_j}^u}{t} = \frac{1}{t} \frac{1}{v(Q_j)} \int_{Q_j} f(y)v(y) \, dy \leq \frac{1}{t} C t = C,$$

so that $\Phi_m \left( \frac{f_{Q_j}^u}{t} \right) \leq \Phi_m(C)$. Hence

$$I_{3,2} \leq C \sum_{i=1}^{m-1} \sum_j \int_{Q_j} \Phi_i \left( \frac{(b(x) - b_{Q_j})^{m-i} f_{Q_j}^u}{t} \right) u(x)v(x) \, dx$$

$$\leq C \sum_{i=1}^{m-1} \sum_j \left( \int_{Q_j} \Phi_m \left( \frac{f_{Q_j}^u}{t} \right) u(x)v(x) \, dx + \int_{Q_j} \Psi_{m-i}((b(x) - b_{Q_j})^{m-i}) u(x)v(x) \, dx \right)$$
\[ \leq C \sum_{i=1}^{m-1} \sum_{j} \frac{(uv)(Q_j)}{v(Q_j)}v(Q_j) \]

\[ \leq \frac{C}{t} \int_{\mathbb{R}^n} f(x)uv \, dx, \]

which is the desired estimation for \( k = m \), and the result is proved. \( \square \)

For the proof of Theorem 3 concerning to a mixed inequality for \( M \Phi \) we shall use the following technical lemma whose proof is given in [13].

Lemma 13. Let \( f \) be a positive and locally integrable function. Then for each \( \gamma, \lambda > 0 \) there exists a number \( a \in \mathbb{R}^+ \) which depends on \( f \) and \( \lambda \) that satisfies

\[ \left( \int_{|y| \leq a \gamma} f(y) \, dy \right) a^n = \lambda. \]

We are now in position to prove the mentioned theorem.

Proof of Theorem 3. As in the proof given in [13], we define the sets \( G_k = \{ x : 2^k < |x| \leq 2^{k+1} \} \), \( I_k = \{ x : 2^{k-1} < |x| \leq 2^{k+2} \} \), \( L_k = \{ x : 2^{k+2} < |x| \} \) and \( C_k = \{ x : |x| \leq 2^{k-1} \} \). Without loss of generality we can write \( g = f \) and we will assume \( t = 1 \) by homogeneity. Then

\[ uw \left( \{ x \in \mathbb{R}^n : M \Phi(g)(x) > v(x) \} \right) = \sum_{k \in \mathbb{Z}} uw \left( \{ x \in G_k : M \Phi(g \chi_k)(x) > v(x) \} \right) \]

\[ + \sum_{k \in \mathbb{Z}} uw \left( \{ x \in G_k : M \Phi(g \chi_L_k)(x) > v(x) \} \right) \]

\[ + uw \left( \{ x \in \mathbb{R}^n : M \Phi(g \chi_C_k)(x) > v(x) \} \right) \]

\[ = I + II + III. \]

We shall begin by estimating \( I \). Recalling that \( w = 1/\Phi(1/v) \) and \( v = |x|^\beta \), if \( x \in G_k \) we have

\[ \frac{1}{\Phi \left( \frac{1}{2^{(k+1)\beta}} \right)} \leq w(x) < \frac{1}{\Phi \left( \frac{1}{2^{k\beta}} \right)}, \]

and also

\[ 2^{(k+1)\beta} \leq v(x) < 2^{k\beta}. \]

Using these estimates and the weak modular type of \( M \Phi \) with weight \( u \) (see [11]) we get

\[ I \leq \sum_{k \in \mathbb{Z}} \frac{1}{\Phi \left( \frac{1}{2^{k\beta}} \right)} u \left( \{ x \in G_k : M \Phi(g \chi_k)(x) > 2^{(k+1)\beta} \} \right) \]

\[ \leq \sum_{k \in \mathbb{Z}} \frac{1}{\Phi \left( \frac{1}{2^{k\beta}} \right)} \int_{\mathbb{R}^n} \Phi \left( \frac{g \chi_k(x)}{2^{(k+1)\beta}} \right) Mu(x) \, dx \]

\[ \leq C \sum_{k \in \mathbb{Z}} \frac{1}{\Phi \left( \frac{1}{2^{k\beta}} \right)} \Phi \left( \frac{1}{2^{k\beta}} \right) \int_{I_k} \Phi(g(x)) Mu(x) \, dx \]

\[ \leq C \sum_{k \in \mathbb{Z}} \int_{I_k} \Phi(g(x)) Mu(x) \, dx \]

\[ = C \int_{\mathbb{R}^n} \Phi(g(x)) Mu(x) \, dx, \]

where we have used the submultiplicativity of \( \Phi \). This gives the desired estimation for \( I \).
In order to estimate $II$, we define, for $x \in G_k$

$$F(x) = C_n \int_{|y|>|x|} \frac{\Phi(g(y))}{|y|^n} dy$$

where $C_n = c_n 4^n$ and $c_n$ is the measure of the surface area of the unit sphere $S^{n-1}$. Fix $x \in G_k$ and let $B = B(x_0, r)$ be a ball containing $x$. We want to obtain an upper bound for $\|g\mathcal{X}_{L_k}\|_{\Phi, B}$. Note that if $y \in L_k \cap B$, since $x \in G_k$ we have that $\frac{|y|}{2} > |x|$, and then

$$2r \geq |y - x| > |y| - |x| > \frac{|y|}{2}.$$  

Since $\Phi$ is submultiplicative, this leads to

$$\frac{1}{|B|} \int_B \Phi \left( \frac{g\mathcal{X}_{L_k}(y)}{(1/\Phi^{-1}(1/F(x)))} \right) dy \leq \frac{1}{|B|} \int_B \Phi(\Phi^{-1}(1/F(x))) \Phi(g\mathcal{X}_{L_k}(y)) dy$$

$$\leq \frac{1}{F(x)} \frac{1}{|B|} \int_{B \cap L_k} \Phi(g(y)) dy$$

$$\leq \frac{c_n 4^n}{F(x)} \int_{|y|>|x|} \frac{\Phi(g(y))}{|y|^n} dy$$

$$= \frac{1}{F(x)} F(x) = 1.$$  

Thus, we get that

$$\|g\mathcal{X}_{L_k}\|_{\Phi, B} \leq \frac{1}{\Phi^{-1}(1/F(x))}$$

and we can proceed as follows

$$II \leq \sum_{k \in \mathbb{Z}} uw \left( \left\{ x \in G_k : \frac{1}{\Phi^{-1}(1/F(x))} > v(x) \right\} \right)$$

$$= \sum_{k \in \mathbb{Z}} uw \left( \left\{ x \in G_k : v(x) \Phi^{-1}(1/F(x)) < \frac{1}{v(x)} \right\} \right)$$

$$= \sum_{k \in \mathbb{Z}} uw \left( \left\{ x \in G_k : F(x) > w(x) \right\} \right)$$

$$\leq \sum_{k \in \mathbb{Z}} \Phi \left( \frac{1}{2^k n} \right) \left( \left\{ x \in G_k : F(x) > \frac{1}{\Phi \left( \frac{1}{2^k (k+1)n} \right)} \right\} \right)$$

$$\leq C \int_0^\infty u (\{ x \in \mathbb{R}^n : F(x) > t \}) dt$$

$$= C \int_{\mathbb{R}^n} F(x) u(x) dx$$

$$= C \int_{\mathbb{R}^n} \Phi(g(y)) \frac{1}{|y|^n} \int_{|y|>|x|} u(x) dx dy$$

$$\leq C \int_{\mathbb{R}^n} \Phi(g(y)) M u(y) dy,$$

giving the estimation for the second term.

In order to estimate $III$, we define, for $x \in G_k$

$$G(x) = \frac{C_n}{|x|^n} \int_{|y|<\frac{|x|}{2}} \Phi(g(y)) dy.$$  

For a fixed $x \in G_k$, let us take $B = B(x_0, r)$ a ball containing $x$. If $y \in C_k$, we get $|y| \leq \frac{|x|}{2}$. By following the same arguments as in the estimation of $II$ we obtain that $\|g\mathcal{X}_{C_k}\|_{\Phi, B} \leq$
III \leq uw \left( \{ x \in \mathbb{R}^n : G(x) > w(x) \} \right).

Let $\gamma = n/(-n - r\beta)$. Note that $\gamma > 0$ since, by hypothesis, $\beta < -n$. Now applying Lemma 13 with $\gamma$ and $\lambda = 1$, there exists $a > 0$ which verifies

\begin{equation}
(3.3) \quad \left( \int_{|y| \leq a^\gamma} \Phi(g(y)) \, dy \right) a^n = 1.
\end{equation}

Then,

$$uw \left( \{ x \in \mathbb{R}^n : G(x) > w(x) \} \right) = uw \left( \left\{ x : |x| \leq a^\gamma, \frac{C_n}{|x|^n} \int_{|y| \leq a^{\frac{\gamma}{n}}} \Phi(g(y)) \, dy > \frac{1}{\Phi \left( \frac{1}{|x|^n} \right)} \right\} \right)$$

$$+ \sum_{k=0}^{\infty} uw \left( \left\{ x : 2^k a^\gamma < |x| \leq 2^{k+1} a^\gamma, \frac{C_n}{|x|^n} \int_{|y| \leq a^{\frac{\gamma}{n}}} \Phi(g(y)) \, dy > \frac{1}{\Phi \left( \frac{1}{|x|^n} \right)} \right\} \right)$$

$$= A + B.$$

Note that

\begin{equation}
(3.4) \quad \left\{ x : \frac{C_n}{|x|^n} \int_{|y| \leq a^\gamma} \Phi(g(y)) \, dy > \frac{1}{\Phi \left( \frac{1}{|x|^n} \right)} \right\} = \left\{ x : \frac{\Phi \left( \frac{1}{|x|^n} \right)}{|x|^n} > C^{-1}_n a^n \right\}.
\end{equation}

If we set $z = |x|^{-\beta}$ then

$$|x|^{-n} \Phi \left( \frac{1}{|x|^\beta} \right) = z^{r+n/\beta} (1 + \log^+ z) = z^\alpha (1 + \log^+ z)^\delta = : \varphi(z),$$

where $\alpha = r + n/\beta$, which is positive because $\beta < -n$. It can be proved (see for example [7, Lemma 1.1.27]) that there exists a constant $D \geq 1$ such that

$$\frac{1}{D} z^{1/\alpha} (1 + \log^+ z)^{-\delta/\alpha} \leq \varphi^{-1}(z) \leq D z^{1/\alpha} (1 + \log^+ z)^{-\delta/\alpha}.$$

With this in mind, we can write (3.4) as follows

$$\{ x : \varphi(|x|^{-\beta}) > C^{-1}_n a^n \} = \{ x : |x|^{-\beta} > \varphi^{-1}(C^{-1}_n a^n) \}$$

$$\subseteq \{ x : |x|^{-\beta} > \frac{\left( C^{-1}_n a^n \right)^{1/\alpha}}{D (1 + \log^+ (C^{-1}_n a^n))^{\delta/\alpha}} \}$$

$$= \left\{ x : D \left( \frac{1 + \log^+ (C^{-1}_n a^n)\delta}{C^{-1}_n a^n \delta} \right)^{1/\alpha} > |x|^\beta \right\}$$

$$= \left\{ x : D^{1/\beta} \left( \frac{1 + \log^+ (C^{-1}_n a^n)\delta}{C^{-1}_n a^n \delta} \right)^{1/(\alpha\beta)} < |x| \right\}$$

$$= \left\{ x : D^{1/\beta} \left( \frac{C^{-1}_n}{1 + \log^+ (C^{-1}_n a^n)\delta} \right)^{-1/(\alpha\beta)} a^\gamma < |x| \right\}.$$

Since $D \geq 1$, we have that $D^{1/\beta} \left( \frac{C^{-1}_n}{(1 + \log^+ (C^{-1}_n a^n))\gamma} \right)^{-1/(\alpha\beta)} = : C_0 < 1$. Thus, we get

$$A \leq uw \left( \{ x : C_0 a^\gamma < |x| \leq a^\beta \} \right)$$

$$\leq \int_{|x| > C_0 a^\gamma} u(x) \nu'(x) \, dx$$

$$= \sum_{k=1}^{\infty} \int_{C_0 2^{k-1} a^\gamma \leq |x| < C_0 2^k a^\gamma} u(x) \nu'(x) \, dx$$
\[
\begin{align*}
&\leq \sum_{k=1}^{\infty} \frac{1}{(C_02^{k-1}a^\gamma)^{-r\beta}} \int_{|x|<C_02^{k}a^\gamma} u(x) \, dx \\
&= \sum_{k=1}^{\infty} 2^n c^{r\beta} 2^{r(1)(n+r\beta)} \int_{|y|\leq a^\gamma} \Phi(g(y)) \left( \frac{1}{(2^{k}a^\gamma)^n} \int_{|x|<2^{k}a^\gamma} u(x) \, dx \right) \, dy \\
&\leq C \int_{\mathbb{R}^n} \Phi(g(y)) M_u(y) \, dy.
\end{align*}
\]

To finish the proof, it only remains to estimate part \( B \).
\[
\begin{align*}
B &\leq \sum_{k=0}^{\infty} uv^r \left\{ \{ x : 2^k a^\gamma < |x| \leq 2^{k+1} a^\gamma \} \right\} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{(2^k a^\gamma)^{-r\beta}} \int_{|x|\leq 2^{k+1}a^\gamma} u(x) \, dx \\
&\leq \sum_{k=0}^{\infty} \frac{(2^{k+1}a^\gamma)^n}{(2^k a^\gamma)^{n-r\beta}} \frac{1}{(2^{k+1}a^\gamma)^n} \int_{|x|\leq 2^{k+1}a^\gamma} u(x) \, dx \\
&= C \sum_{k=0}^{\infty} 2^n 2^{n+1} \Phi(g(y)) \left( \frac{1}{(2^{k+1}a^\gamma)^n} \int_{|x|\leq 2^{k+1}a^\gamma} u(x) \, dx \right) \, dy \\
&\leq C \int_{\mathbb{R}^n} \Phi(g(y)) M_u(y) \, dy \\
&\leq C \int_{\mathbb{R}^n} \Phi(g(y)) M_u(y) \, dy,
\end{align*}
\]

which completes the proof. \( \square \)

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