Finitely presented groups and homotopy of presentations of triangular algebras.

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Abstract: Given any finitely presented group $G$ we find a triangular algebra such that has two presentations, one with fundamental group $G$ and another with trivial group. Thus proving that given a collection $G_1, \ldots, G_n$ of finitely presented groups there exist a triangular algebra $A$ such that all $G_i$ appear as fundamental group of some presentation of $A$, extending one of the result of [6].

1 Introduction

Let $A$ be a basic, connected, finite dimensional algebra over a closed field $k$.

There exists a unique quiver $Q$ and a two-side ideal $I$ of the path algebra $kQ$, such that $A = kQ/I$. Since, in general, $I$ is not unique, the morphism $v : kQ \to A = kQ/I$, as well as $(Q, I)$ is called a presentation of $A$. Each presentation has associated a fundamental group $\pi_1(Q, I)$. In [7] it is proved that given a finitely presented group $G$, there exists a bounded quiver $(Q, I)$ such that $\pi_1(Q, I) = G$. In [6] this results it is generalized, namely: for any collection of finitely presented groups $\{G_i\}_{i=1}^n$ there exists an algebra $A$ and a collections of presentations $\{(Q, I_i)\}_{i=1}^n$ of this algebra such that $\pi_1(Q, I_i) = G_i$ for $i = 1, \ldots, n$. Also it is proved a similar result by triangular algebras, but, in this case, the involved groups are obtained from cyclic groups by performing finite free and direct products. In the present work we leave this restriction over the groups and, thus, we proved the Theorem:
Theorem 1 Let $G_1, \ldots, G_n$ be finitely presented groups. Then there exists a triangular algebra $A$ having presentations $(\hat{Q}, I_i)$ for $i = 1, \ldots, n$, such that $\pi_1(\hat{Q}, I_i) = G_i$.

To prove this we build a special quiver $\hat{Q}$ and show the appropriated ideals $I_i$. In the first place, for any finitely presented group $G_i$ we built a quiver $Q_{G_i}$ and two presentations $(Q_{G_i}, J_i)$ and $(Q_{G_i}, \bar{J}_i)$ such that $\pi_1(Q_{G_i}, J_i) = G_i$ and $\pi_1(Q_{G_i}, \bar{J}_i) = \{1\}$. In the second place, we make a co-product, (topologically, it is a type of amalgamate of bounded quivers)

$$(\hat{Q}, I_i) := \left( \bigsqcup_{j \neq i}(Q_{G_j}, \bar{J}_j) \right) \sqcup (Q_{G_i}, J_i). \quad (1)$$

Is not difficult prove that the fundamental group of a co-product is the free product of the fundamental groups of each bonded quivers (see [?]). Thus we have $\pi_1(\hat{Q}, I_i) = (\bigsqcup_{j \neq i}\{1\}) \sqcup G_i = G_i$. This is basically the same way made in [?]. Our principal achievement is obtain each $Q_{G_i}$ be triangular and find the appropriated $J_i$ and $\bar{J}_i$. The constructions of $Q_{G_i}$ and $J_i$ are based over the presentations of the groups by generators and relations.

In section 3 given a group $G$, we write $G = \mathbb{Z}^\text{free} \sqcup H$ where $H$ has not free generator. After that find a convenient presentation $\langle S'' | R'' \rangle$ of $H$. The relations of $R''$ are the kind of $w_i = w_j$ or $w_k \ldots w_l = 1$. In section 4 we used this presentation to built a quiver $Q_H$ and ideals $I$ and $\bar{I}$ such that $\pi_1(Q_H, I) = H$ and $\pi_1(Q_H, \bar{I}) = \{1\}$. The quiver $Q_H$ is essentially a chain of triangles $T_i$ where each triangle represents an element of $S''$ (Sec. 4.1). Each relation in $R''$ is in correspondence with a minimal relation in $I$ in this approximated way: if $w_i$ is involved in the relation of $R''$, then $T_i$ is involved in the minimal relation (4.2.2). To get $\bar{I}$, the idea is kill all triangles. For this we built $\bar{I}$ as the image of $I$ by a automorphism (Sec. 4.2.3). In the Sec. 4.3 to recuperated $G$, we make appropriates co-product with the bounder quivers of the Lemma 10 getting the mentioned $(Q_{G_i}, J_i)$ and $(Q_{G_i}, \bar{J}_i)$.

2 Preliminaries

2.1 Quivers and algebras

A quiver $Q$ is a quadruple $(Q_0, Q_1, s, t)$ where $Q_0$ and $Q_1$ are sets and them elements are called vertices and arrows respectively; $s, t : Q_1 \to Q_0$ are
functions which indicate the source and the target of each arrow respectively. A path $w$ is a sequence of arrows $w = \alpha_1\alpha_2\cdots\alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n$. When $s(\alpha_1) = x$ and $t(\alpha_n) = y$, the fact is symbolized $w : x \rightarrow y$, and two paths with this proprieties are called parallel. A quiver $Q$ is said finite if $Q_0$ and $Q_1$ are finite, and $Q$ is said connected when the underlying graph is connected. We will only consider finite and connected quivers.

Given a commutative field $k$ and a quiver $Q$, the path algebra $kQ$ is the $k$-vector space whose base is the set of paths of $Q$, including one stationary path $e_x$ for each vertex $x$ of $Q$. The multiplication of two basis elements of $kQ$ is their composition whenever it is possible, and 0 in otherwaise. Let $F$ be the two-sided ideal of $kQ$ generated by the arrows of $Q$. A two-sided ideal $I$ of $kQ$ is called admissible if there exists an integer $m \geq 2$ such that $F^m \subset I \subset F^2$. The pair $(Q, I)$ is called a bound quiver. When a distinguished vertex $x \in Q_0$ is consider, $(Q, I, x)$ is called pointed bound quiver.

Conversely, if $A$ is a basic, connected and finite dimensional algebra over $k$, then there exists a unique finite connected quiver $Q$ and a surjective morphism of $k$-algebras $\nu : kQ \rightarrow A$, which is not unique, with $I = \ker \nu$ an admissible ideal (see [8] and [4]). The morphism $\nu$ and the pair $(Q, I)$ are called presentations of the algebra $A$. Remark that a morphism $\nu : kQ \rightarrow A$ is a presentation of $A$ whenever $\{\nu(e_x) | x \in Q_0\}$ is a complete set of primitive orthogonal idempotents and, for any fixed $x, y \in Q_0$, we have that $\{\nu(\alpha) + \operatorname{rad}^2A\alpha : x \rightarrow y \in Q_1\}$ is basis of $\nu(e_x)(\operatorname{rad}A/\operatorname{rad}^2A)\nu(e_y)$. An algebra $A$ is triangular whenever $Q$ has not oriented cycles. In this work we will say that the quiver is triangular as well. For further reference of bound quivers in the representation theory of algebras can see [2] and [3].

**Remark 2** Let $\nu$ be a morphisms $\nu : kQ \rightarrow kQ/I \simeq A$ defined by $\nu(e_x) = e_x + I$ for $x \in Q_0$, and, $\nu(\alpha) = \alpha + \rho_\alpha + I$ for $\alpha \in Q_1$ where $\rho_\alpha$ is a linear combination of paths parallel to $\alpha$ and the paths have length at least 2, then $(Q, \ker \nu)$ is a presentation of $A$. In particular if one only arrow $\beta$ is transformed and $\rho_\beta$ has one term, then $\nu$ is a transvection.

**2.2 Fundamental group of a bounder quiver**

Given a bound quiver $(Q, I)$, its fundamental group is defined as follows (see [10]). For $x, y \in Q_0$, is defined the set $I(x, y) = e_x(kQ)e_y \cap I$. A relation
\[ \rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y) \] (where \( \lambda_i \in k^* \), and \( w_i \) are different paths from \( x \) to \( y \)) is said to be minimal if \( m \geq 2 \), and, for every proper subset \( J \subset \{1, \ldots, m\} \), we have \( \sum_{i \in J} \lambda_i w_i \notin I(x, y) \). For a given arrow \( \alpha : x \to y \), let \( \alpha^{-1} : y \to x \) be its formal inverse. A walk \( w \) in \( Q \) from \( x \) to \( y \) is a composition \( w = \alpha \varepsilon_1 \alpha \varepsilon_2 \cdots \alpha \varepsilon_n \) such that the \( s(\alpha \varepsilon_i) = x \), \( t(\alpha \varepsilon_n) = y \), and, \( s(\alpha_i \varepsilon_i) = t(\alpha_{i-1} \varepsilon) \) for \( i = 1, \ldots, n \). Define the homotopic relation \( \sim \) on the set of walks on \((Q, I)\), as the smallest equivalence relation satisfying the following conditions:

1. For each arrow \( \alpha : x \to y \), we have \( \alpha \alpha^{-1} \sim e_x \) and \( \alpha^{-1} \alpha \sim e_y \).

2. For each minimal relation \( \sum_{i=1}^{m} \lambda_i w_i \), we have \( w_i \sim w_j \) for all \( i, j \) in \( \{1, \ldots, m\} \).

3. If \( u, v, w \) and \( w' \) are walks, and \( u \sim v \) then \( wuw' \sim wvw' \), whenever these compositions are defined.

We denote by \( \tilde{w} \) the homotopic class of a walk \( w \). Let \( v_0 \) be a fixed point in \( Q_0 \), and consider the set \( W(Q, v_0) \) of walks of source and target \( v_0 \). On this set, the product of walks is everywhere defined. Because of the first and the third conditions in the definition of the relation \( \sim \), one can form the quotient group \( W(Q, v_0)/\sim \). This group is called the fundamental group of the bound quiver \((Q, I)\) with base point \( v_0 \), denoted by \( \pi_1(Q, I, v_0) \). It follows from the connectedness of \( Q \) that this group does not depend on the base point \( v_0 \), and we denote it simply by \( \pi_1(Q, I) \). This group has a clear geometrical interpretation as the first homotopic group of a C.W. complex \( B(Q, I) \) associated to \((Q, I)\), see [5].

**Remark 3** If \( I_0 \) is null or monomial It is know that \( \pi_1(Q, I_0) \) is the free product of \( \chi(Q) \) copies of \( \mathbb{Z} \), where \( \chi(Q) = |Q_1| - |Q_0| + 1 \) is the Euler characteristic of the underlying graph. Thus, \( \pi_1(Q, I_0) = \mathbb{Z}^{\chi(Q)} \).

**Example 4** Consider the quiver

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\text{b} \\
\uparrow \\
\bullet \\
\text{v}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{z}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\text{a}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\text{d}
\end{array}
\end{array}
\]

and the clearly admissible ideal \( I_1 = \langle ad \rangle \). We defined the morphism \( \nu : kO \to kO/I_1 \simeq A \) such that \( \nu(a) = a - bc + I_1, \) and \( \nu(\gamma) = \gamma + I_1 \) for any other
path $\gamma$. We defined $I_2 = \ker(\nu)$. By the Remark 2, $\nu$ is a presentation of $A$ and therefore $kO/I_1 \simeq kO/I_2 \simeq A$. On the other hand, since $I_1$ is generated by monomials relations and $\chi(Q) = 4 - 4 + 1$, we have $\pi_1(O, I_1) = \mathbb{Z}^{\leq 1} = \mathbb{Z}$. Since $I_2 = \langle ad + bcd \rangle$, we have $ad \sim bcd$ and $a \sim bc$. Therefore, it is clear that $\pi_1(O, I_2) = \{1\}$.

### 2.3 Co-products

Given two pointed bounder quivers $(Q', I', v')$ and $(Q'', I'', v'')$ without any vertex or arrow in common, we defined the bounded quiver $(Q, I) := (Q', I', v') \sqcup (Q'', I'', v'')$ in the following way: $Q_0 := Q'_0 \cup Q''_0$ but we identify $v'$ and $v''$ to single vertex $v$, and $Q_1 := Q'_1 \cup Q''_1$. Thus $Q'$ and $Q''$ are identify with a full convex sub-quivers of $Q$, and therefore $I'$ and $I''$ are ideals of $Q$. Finally we defined $I := I' + I''$. Note that if $I'$ and $I''$ are admissible then $I$ as well. The vertex $v'$ and $v''$ are not relevant in general, therefore we will omit him and the word pointed.

We will use the symbol $\sqcup$ as much to mention the co-product between bounder quiver as mention the free product between groups. Note that this operations are associative and commutative.

In [?] is proved two important result for us. The first say that the co-product behaves well under changes of presentations:

**Proposition 5** Let $A$ be an algebra with two presentations $(Q_A, I_A)$ and $(Q_A, I'_A)$ and let $B$ be an algebra with two presentations $(Q_B, I_B)$ and $(Q_B, I'_B)$. If

\[
(Q, I) = (Q_A, I_A) \sqcup (Q_B, I_B) \quad \text{and} \quad (Q, I') = (Q_A, I'_A) \sqcup (Q_B, I'_B)
\]

then $(Q, I)$ and $(Q, I')$ are presentation of the same algebra.

The second is analogous to the result to Van Kampen’s theorem for topological spaces:

**Proposition 6** If $(Q, I) = (Q', I') \sqcup (Q'', I'')$ then

\[
\pi_1(Q, I) = \pi_1(Q', I') \sqcup \pi_1(Q'', I'')
\]

The two previous propositions and the next clear proposition will be very used in our work.
Proposition 7 If \((Q, I) = (Q', I') \sqcup (Q'', I'')\) and the algebras \(kQ'/I'\) and \(kQ''/I''\) are triangular, then \(kQ/I\) is triangular.

Remark 8 It is important to note that the propositions 6 and 7 are easily generalized for a finite amount of bounder quivers. In proposition 5 we have two pair of presentation of the same algebra respectively. This result can be generalizable for any finite amount of pairs in this sense: If we have collections of pairs \(\{(Q_{A_i}, I_{A_i}), (Q_{A_i}', I_{A_i}')\}\) such that the elements of each pair is a presentation of the same algebra \(A_i\), then any co-product of \(n\) presentations taking one, and only one, presentation of each pair, is a presentation of the same algebra.

Example 9 Consider the bounded quivers \((O', I'_1)\) and \((O'', I''_1)\) isomorphous to the first bounded quiver in the example 4. The co-product \((O, I)\) between \((O', I'_1, z')\) and \((O'', I''_1, v'')\) is

\[
\begin{array}{c}
b' & \rightarrow & u' & \rightarrow & y' & \rightarrow & c' & \rightarrow & b'' & \rightarrow & u'' & \rightarrow & y'' & \rightarrow & z'' \\
\end{array}
\]

Note that by the Proposition 6, \(\pi_1(O, L) = \mathbb{Z} \sqcup \mathbb{Z} = \mathbb{Z} \sqcup 2\). On the other hand we make the co-product \((O, \bar{L})\) between \((O', I'_2, z')\) and \((O'', I''_2, v'')\) isomorphous to the second bounded quiver in example 4. Again, \(\pi(O, \bar{L}) = \{1\} \sqcup \{1\} = \{1\}\). Like in example \(\mathbb{A}\) \((O', I'_1)\) and \((O', I'_2)\) are presentations of the same algebra and \((O''', I''_1)\) and \((O''', I''_2)\) as well; hence we have the hypothesis of proposition 5. Thus \((O, L)\) and \((O, \bar{L})\) are presentations of the same algebra. More, \(O'\) and \(O''\) are triangular, therefore, by proposition 7 \(O\) is triangular.

Summarizing: For the group \(\mathbb{Z} \sqcup 2\) we got a triangular algebra \(A := kO/L\) with two presentation \((O, L)\) and \((O, \bar{L})\), such that \(\pi_1(O, L) = \mathbb{Z} \sqcup 2\) and \(\pi_1(O, \bar{L}) = \{1\}\).

Is clearly that for get a bounded quiver with group \(\mathbb{Z}^{\text{lim}}\) and triangular algebra we can built by induction an analogous construction of example 9. We can use more triangles and the remark 8. Therefore we are in conditions to statement the following lemma:

Lemma 10 For each \(m\) natural, there exits a triangular algebra \(A\) with two presentation \((O, L)\) and \((O, \bar{L})\) such that \(\pi_1(O, L) = \mathbb{Z}^{\text{lim}}\) and \(\pi_1(O, \bar{L}) = \{1\}\).

In the next section we will put our effort to prove the analogous for any finitely presented group instead of \(\mathbb{Z}^{\text{lim}}\).
3 Fundamental group without free generator.

If \( G \) is a finitely presented group, a \textbf{presentation} of \( G \) consists of a finite set of generators \( S \), and a finite set of relations among these generators \( R \) such that \( G = \langle S \mid R \rangle \) (see [9]). If \( x \in S \) but neither \( x \) nor a power of \( x \) appear in any relation of \( R \), we say that \( x \) is a \textbf{free generator}. Let \( \mathbb{H} \) be the set of all finitely presented group without free generators. It is possible to write \( G \) as free product between a group generated by its free generator and a group in \( \mathbb{H} \). In fact, we consider \( S_F = \{ x \in S : x \) is a free generator \} \), \( m \) the amount of element of \( S_F \) and \( H = \langle S - S_F \mid R \rangle \). We have \( G = \mathbb{Z}^\mathbb{H} \square H \). In this way, we divided the problem to find a algebra with presentation whose group is \( G \) in two problem: to find a algebra with a presentation whose group is \( \mathbb{Z}^\mathbb{H} \) (solved by lemma [10]) and to find a algebra with a presentation whose group is \( H \). Hereafter we can handle to prove the next key proposition:

\textbf{Proposition 11} Let \( H \) be a finitely presented without free generators group, there exits a triangular algebra \( A \) with two presentation \( (Q_H, I) \) and \( (Q_H, \bar{I}) \) such that \( \pi_1(Q_H, I) = H \) and \( \pi_1(Q_H, \bar{I}) = \{1\} \).

The idea for the proof of Proposition 11 is the following. First, we obtain \( H \) generated by \( w_1, w_2, \ldots, w_n \) with two type of relations: \( w_x \cdots w_{x+t} = 1 \) (lasso) and \( w_a = w_b \) (cross-lasso). The quiver will be a chain of consecutive triangles, one for each generator \( w_i \). The relations in the groups will be associated to the minimal relations in \( kQ_H \), that will generate \( I \). Namely, a lasso relation \( w_x \cdots w_{x+t} = 1 \) can be thought as a lasso which catches the triangles \( x, x+1, \ldots, x+t \). The cross-lasso relations \( w_a = w_b \) can be thought as a twist-lasso which catches the triangles \( a \) and \( b \). The idea is more flexible than the one showed throughout this work, in fact, with this idea it is possible to write a presentation with smaller (less vertices and arrows) quivers that we present here.

3.1 A particular presentation of a group

Let \( H \) be group in \( \mathbb{H} \) with \( H = \langle S \mid R \rangle \). For \( H \) we will to give two special presentations \( \langle S' \mid R' \rangle \) and \( \langle S'' \mid R'' \rangle \) with special proprieties. The first will be only auxiliary to get the second.

The general procedure to obtain the first alternative presentations is the next. We consider an original presentation \( \langle S \mid R \rangle \). Without lost of generality
we can suppose that each relations of $R$ is a products of generators in the left side without powers and the neutral element in the right side:

$$y_1^1 \cdots y_{m_1}^1 = 1 \quad (2)$$

$$\vdots$$

$$y_s^1 \cdots y_{m_s}^s = 1$$

Note that two or more variables can represent the same generator. For example, the group $\mathbb{Z}_2 \oplus \mathbb{Z}$ has the presentations $\langle \{a, b\} \mid \{aba^{-1}b^{-1} = 1, \ a^2 = 1\} \rangle$.

We can choice $\langle \{a, b, c, d\} \mid \{abcd = 1, ac = 1, \ bd = 1, aa = 1\} \rangle$.

Where $y_3^1 = y_2^2 = c$. We also impose that there are not relation with only one variable, e.i., $y_a^1 = 1$.

Note that since $H \in \mathbb{H}$, every generator appear at least in one relation of $R$.

The next step is created $S'$ and $R'$. The generators in $S'$ will be $g_1^1, \ldots, g_{m_s}^s$, all different. To construct the new relations in $R'$ we proceed in this way. In (2) we replace $y_{k_i}^i$ by $g_{k_i}^i$ and, thus, we have another $s$ relations.

$$g_1^1 \cdots g_{m_1}^1 = 1$$

$$\vdots$$

$$g_1^s \cdots g_{m_s}^s = 1$$

If some $y_a^b$ is the same element that $y_c^d$, then we add the relation $g_a^b = g_c^d$. Some of these relations can be redundant and we can suppress these. Is evident that this presentation is equivalent to the forward.

**Example 12** For $H = \mathbb{Z} \oplus \mathbb{Z}_2$ we choice the presentation

$$\langle \{a, b, c, d\} \mid \{abcd = 1, \ ac = 1, \ bd = 1, aa = 1\} \rangle.$$  

The four relations produce the four new relations

$$g_1^1g_2^1g_3^1g_4^1 = 1, \ g_1^2g_2^2 = 1, \ g_1^3g_2^3 = 1, \ g_1^4g_2^4 = 1.$$  

The aggregate relations are $g_1^1 = g_2^1, g_1^2 = g_2^2 = g_1^4, g_1^3 = g_2^3 \ (for \ a), g_1^4 = g_2^3 \ (for \ b), g_3^1 = g_2^2 \ (for \ c), g_4^1 = g_2^3 \ (for \ d)$. Note that the equality $g_1^1 = g_2^4$ involved two elements of the same relation. This fact is not convenient for the construction which we want to do. Hence, we need a one more type of presentation.
For built $\langle S'' | R'' \rangle$ we procedure in this way: Let $n$ be the amount os variables in $S'$. We simply rename all variables $g_1, \ldots, g_{m_1}, \ldots, g_{m_s}$ by $w_1, \ldots, w_n$ respecting the order. Note that $n = m_1 + \cdots + m_s$. This presentation is equivalent to the second (and, therefore, equivalent to the original) because we only made a rename of variables.

The special proprieties which $R''$ has are:

(a) Note that all the relations in $R''$ has the form $w_i \cdots w_{i_k} = 1$ (we will call lasso) or $w_i = w_j$ (cross-lasso).

(b) Each $w_i$ appear at most ones in each lasso and cross-lasso relations.

(c) In the relations $w_i \cdots w_{i_k} = 1$ the index are consecutive, e.i., $i_{k+1} = i_k + 1$, therefore this relations has the form $w_i, w_{i+1}, \ldots, w_{i+k} = 1$

4 The quivers and the ideals

Let $H \in \mathbb{H}$. We consider the presentation $\langle S'' | R'' \rangle$ made in the section ??.

Based on this presentation we will build a quiver $Q$ and a ideal $I$ such that the homotopic group of $(Q_H, I)$ will be $H$. The quiver is based in $S''$ and the ideal on $R''$. After we will build another ideal $\bar{I}$ such that $\pi_1(Q_H, \bar{I}) = \{1\}$.

4.1 The quiver

We defined the quiver $Q_H := (Q_0, Q_1)$ where

$$Q_0 = \{ x_1, \ldots, x_{n+1}, y_1, \ldots, y_n \}$$

and $Q_1$ is formed by following collections of arrows

- $r_i : x_i \to y_i$ for $i = 1, \ldots, n$
- $l_i : y_i \to x_{i+1}$ for $i = 1, \ldots, n$
- $a_i : x \to x_{i+1}$ for $i = 1, \ldots, n$.

Remark: Let $n$ be the amount of elements of $S''$. The quiver $Q_H$ is

$$\begin{align*}
r_1 \quad y_1 & \quad \downarrow l_1 \quad \cdots \quad r_i \quad y_i & \quad \downarrow l_1 \quad \cdots \quad r_n \quad y_n & \quad \downarrow l_n \\
x_1 \quad a_1 & \quad x_2 & \quad \quad & \quad \quad & \quad \quad & \quad \quad x_{i+1} \quad a_i & \quad x_{i+1} & \quad \quad & \quad \quad & \quad \quad x_{n+1} \quad a_n & \quad x_{n+1}
\end{align*}$$

We can described this as a chain of $n$ triangles.
Notation 13 We take the following notations

\[ m_{ij} = a_{i+1} \cdots a_{j-1} \]

\[ A_i = r_i l_i \]

\[ T_i = A_i a_i^{-1} \]

\[ \alpha_i = m_0 T_i m_0^{-1} \]

4.2 The ideals and the homotopy

4.2.1 The homotopy of the bounder quivers

We will study the homotopy of the quiver \( Q_H \) defined in ???. Consider the null ideal \( I_0 \). Note that \( \chi(Q_H) = 3n - (2n + 1) - 1 = n \). We choose the base point \( x_1 \). Thus \( \pi_1(Q_H, I_0, x_1) = Z \sqcup n \). We choice the base point \( x_1 \). Thus \( \pi_1(Q_H, I_0, x_1) = \langle \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n | \emptyset \rangle \). More, if we now consider a new ideal \( I \) in \( kQ_H \), then \( \pi_1(Q_H, I_0, x_1) = \langle \{ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n | R \rangle \) where the relations in \( R \) must be generated by homotopic relations generates by the minimal relations in \( I \). We will rely on this fact to make the computation of the fundamental groups of the bounder quivers of the subsection 4.2.2.

4.2.2 The ideal for \( H \).

We will express the ideal \( I \) generated by relations in the algebra \( kQ_H \). For each relations in \( R'' \) we consider a relations in \( I \) in the next way. If the relations in \( R'' \) has the form \( w_i w_{i+1} \cdots w_j = 1 \), then we consider the relation \( a_i a_{i+1} \cdots a_j A_i A_{i+1} \cdots A_j \) (we call lasso because the induced homotopy is \( A_i A_{i+1} \cdots A_j a_j^{-1} \cdots a_{i+1} a_i \sim 1 \), which can be thought as a lasso that envelops the triangles \( T_i, \ldots, T_j \)). Note that this it is possible because the index of \( w_i w_{i+1} \cdots w_j \) are consecutive (property (c)). On a other hand, if the relations has the form \( w_i = w_j \) with \( i < j \) then we put \( a_i m_{ij} A_j + A_i m_{ij} a_j \) (we call cross-lasso because the induced homotopy is \( a_i m_{ij} A_j \sim A_i m_{ij} a_j \), which can be thought as a lasso that envelops the triangles \( T_i \) and \( T_j \)).

Note that we never has a monomial relation as \( a_i + A_i \), therefore we have \( I \subset F^2 \). And, since \( Q_H \) has not oriented cycled for some \( m \), one has \( F^m = 0 \subset I \). Therefore \( I \) is admissible.
**Example 14** Continuing with $\mathbb{Z} \oplus \mathbb{Z}_2$. The elements of $S''$ are $w_1, \ldots, w_{10}$. Hence the quiver has ten triangles. For example, the relations $w_1w_2w_3w_4 = 1$ in $R''$ is in correspondence with the relation $a_1a_2a_3a_4 + A_1A_2A_3A_4$ in $kQ_H$, and the relation $w_1 = w_5$ corresponding to $a_1m_{15}A_5 + A_1m_{15}a_5$. The ideal $I$ is generated by this relations and the other six of example [12].

Now we will to prove that all this relations together are minimal. The unique sub-algebras which has some relations, are $I(x_ix_j)$. And the unique possible relation in this sub-algebra are $a_im_{ij}A_j + A_im_{ij}a_j$ or $a_i \cdots a_j + A_i \cdots A_j$. Note that the four terms involved are different, therefore, the two relations are minimal in the ideals generated by themselves.

It remains to prove that the homotopic group of $(Q_H, I)$ is isomorphous to $H$. For this we will to prove that the classes of the loops $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ check exactly the same relations that the elements $w_1, \ldots, w_n$ check in $H$.

**Lemma 15** The relation $a_im_{ij}A_j + A_im_{ij}a_j$ generates in $\pi_1(Q_H, I, x_1)$ the homotopic relation $\tilde{\alpha}_i \sim \tilde{\alpha}_j$.

**Proof.** Since $a_im_{ij}A_j + A_im_{ij}a_j$ is minimal in $I$ then $a_im_{ij}A_j \sim A_im_{ij}a_j$. For simplicity we denote $m_{ij}$ by $m$. Therefore

\[
\begin{align*}
A_im_{ij} & \sim a_imA_j \\
A_ia_i^{-1}a_im_{ij} & \sim a_imA_j \\
A_ia_i^{-1} & \sim a_imA_ja_i^{-1}m^{-1}a_i^{-1} \\
T_i & \sim a_imT_j(ma_i)^{-1} \\
m_{0i}T_im_{0i}^{-1} & \sim m_{0i}a_imT_j(ma_i)^{-1}m_{0i}^{-1} \\
\alpha_i & \sim \alpha_j
\end{align*}
\]

\[\square\]

**Lemma 16** The relation $a_i a_{i+1} \cdots a_{i+k} + A_i A_{i+1} \cdots A_{i+k}$ generates in $\pi_1(Q_H, I, x_1)$ the homotopic relation $\tilde{\alpha}_i \tilde{\alpha}_{i+1} \cdots \tilde{\alpha}_{i+k} \sim 1$.

**Proof.** For simplicity, we denote $a_{i+j}$, $A_{i+j}$, $m_{0i}$ by $b_j$, $B_j$ and $m$, respectively. Since $b_0 \cdots b_k + B_0 \cdots B_k$ is minimal in $I$ then

\[
\begin{align*}
B_0 \cdots B_k & \sim b_0 \cdots b_k \\
B_0 \cdots B_k(b_0 \cdots b_k)^{-1} & \sim e_{x_i} \\
mB_0 \cdots B_k(b_0 \cdots b_k)^{-1}m^{-1} & \sim e_{x_1} \\
mB_0(m_{0})^{-1}(m_{0})B_1 \cdots B_k(mb_1 \cdots b_k)^{-1} & \sim e_{x_1}
\end{align*}
\]
Note that \( mB_0(mb_0)^{-1} \sim \alpha_i \). Thus,
\[
\alpha_i(mb_0)B_1 \cdots B_k(mb_0 \cdots b_k)^{-1} \sim e_{x_1}
\]
\[
\alpha_i(mb_0)B_1(mb_0b_1)^{-1}(mb_0b_1) \cdots B_k(mb_0 \cdots b_k)^{-1} \sim e_{x_1}
\]
Note that \( (mb_0)B_1(mb_0b_1)^{-1} \sim \alpha_{i+1} \). Hence we repeat the procedure until the last step
\[
\alpha_i \alpha_{i+1} \cdots \alpha_{i+k-1}(mb_0b_1 \cdots b_{k-1})B_k(mb_0 \cdots b_k)^{-1} \sim e_{x_1}
\]
\[
\alpha_i \alpha_{i+1} \cdots \alpha_{i+k-1} \alpha_{i+1} \sim e_{x_1}
\]

Finally, note that for each relation in \( R'' \) of the kind \( w_i \cdots w_j = 1 \), we have \( \bar{\alpha}_i \cdots \bar{\alpha}_j \sim 1 \) in \( \pi_1(Q_H,I,x_1) \). And for each relation in \( R'' \) of the kind \( w_i = w_j \), we have \( \bar{\alpha}_i = \bar{\alpha}_j \) in \( \pi_1(Q_H,I,x_1) \). Therefore \( H = \pi_1(Q_H,I,x_1) = \pi_1(Q_H,I) \). Thus we proofed a first part of the Proposition.

4.2.3 The ideal for the trivial group.

The new ideal is the image of the previous ideal \( I \) by composition of trasversions. Actually, each transvection \( \varphi_j \) kill the element \( \bar{\alpha}_i \) in the fundamental group \( \pi_1(Q_H, \varphi_j(I)) \). We not will to prove exactly that, however for alternative understanding and realization of this, it can be interested see [11].

Remark 17 If \( \bar{\eta} : kQ_H \to kQ_H \) is an automorphism of algebra such that \( \bar{\eta}^{-1} \) is a trasversion and \( J \) is an admissible ideal, then \( (Q_H, \eta(J)) \) and \( (Q_H, J) \) are presentations of the same algebra, because, when consider the projection \( \eta : kQ_H \to kQ_H/J \), we have that \( \bar{\eta}(J) = \ker \eta^{-1} \).

Notation 18 Let \( s \) be the amount of lassos. If the lasso \( a_i \cdots a_{i+h} + A_i \cdots A_{i+h} \) is the \( k \)-th lasso, we rename the walks such that this relation result \( a_1^k \cdots a_{nk}^k + A_1^k \cdots A_{nk}^k \), we abbreviate this by \( W^k \). Let \( \varphi_i^k : kQ_H \to kQ_H \) be automorphism of algebras defined by \( \varphi_i^k(a_i^k) = a_i^k + A_i^k \) and the identity at other vertex and arrow. We defined \( \gamma^k := \varphi_{nk}^k \circ \cdots \circ \varphi_1^k \) and \( \gamma = \gamma^k \circ \cdots \circ \gamma^1 \).

Lemma 19 With the above notation
\[
\bar{\gamma}(W^k) = a_1^k \cdots a_{nk}^k + \sum_{i=1}^{nk} a_i^k \cdots a_{i-1}^k A_i^k a_{i+i}^k \cdots a_{nk}^k + \Gamma_k
\]
where \( \Gamma_k \) is a sum such that each term has two or more capital letters.
Proof. Note $\bar{\gamma}(W^k) = \bar{\gamma}^* \circ \cdots \circ \bar{\gamma}^1(W^k) = \bar{\gamma}^k(W^k)$, because $\bar{\gamma}_k$ the identity over $I(x^k_1,x^k_{n_k-1})$ when $l \neq k$. We fix a $k$ and for to make more clear the notations we replace: $n_k$ for $n$, $a$ by $b$ and $A$ by $B$, and we omit the upper-index $k$ in the letters $b$, $B$ and $\varphi$. Thus the inductive hypothesis over $i$ is

$$\varphi_i \circ \cdots \circ \varphi_1(W^k) = b_1 \cdots b_n + \sum_{j=1}^i b_1 \cdots B_j \cdots b_n + \Gamma^i_k$$ (4)

where $\Gamma^i_k$ is a sum such that each term has two or more capital letters. For $i = 1$ is clearly, in fact $\varphi_1(W^k) = b_1 \cdots b_n + B_1 b_2 \cdots b_n + B_1 \cdots B_n$, and $n \geq 2$. Suppose that the hypothesis is true for $i$. Applying $\varphi_{i+1}$ in (4) and reorganizing we have:

$$\begin{align*}
= & \quad b_1 \cdots b_n + b_1 \cdots B_{i+1} \cdots b_n + \sum_{j=1}^i (b_1 \cdots B_j \cdots b_n + b_1 \cdots B_j \cdots B_{i+1} \cdots b_n) \\
+ & \quad \varphi_{i+1}(\Gamma^i_k) \\
= & \quad b_1 \cdots b_n + \sum_{j=1}^i (b_1 \cdots B_j \cdots b_n) + b_1 \cdots B_{i+1} \cdots b_n + \\
+ & \quad \sum_{j=1}^i (b_1 \cdots B_j \cdots B_{i+1} \cdots b_n) + \varphi_{i+1}(\Gamma^i_k) \\
= & \quad b_1 \cdots b_n + \sum_{j=1}^{i+1} (b_1 \cdots B_j \cdots b_n) + \Gamma^{i+1}_k
\end{align*}$$

where it is clear that $\Gamma^{i+1}_k$ is a sum such that each term has two or more capital letters because: $\Gamma^i_k$ has the same propriety and when $\varphi_{i+1}$ is applied over it, the amount of capital letter can never decrease, and, on another hand, it is clearly that $\sum_{i=1}^j (b_1 \cdots B_j \cdots B_{i+1} \cdots b_n)$ has two capital letter. So, applying induction until $n$, we have that $\bar{\gamma}^k(W^k) = \varphi_n \circ \cdots \circ \varphi_1(W^k)$ has the required form. $\blacksquare$

Now we can finish the proof of the proposition $\blacksquare$

Proof. Defined $\bar{I} := \bar{\gamma}(I)$. It is easy check that $\bar{I}$ is admissible as well. Since $\bar{\gamma}$ is automorphism, the minimal relation in $I$ are minimal relations in $\bar{I}$, in particular the relation (3). By the lemma we can ensure that the homotopy

$$a^k_1 \cdots a^k_i \cdots a^k_{n_k} \sim a^k_1 \cdots A^k_i \cdots a^k_{n_k}$$
is verified. And from this we have $a_k^i \sim A_k^i$ for all $k = 1, \ldots, s$ and $i = 1, \ldots, n_k$. Since all $w_j$ is in some lasso, we have $a_j \sim A_j$ for all $j = 1, \ldots n$. Manipulating this, we get $e_{x_1} \sim m_0 T_j m_{0j}^{-1} \sim 2$. Thus, the fundamental group of $\pi_1(Q, \bar{I}, x_1)$ verify $\bar{\alpha}_j \sim 1$ for all $j$. But $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ is a set of generators, therefore $\pi_1(Q, \bar{I}) \approx \{1\}$. ■

4.3 The presentation for $G$

Let $G$ a finitely presented group such that $G = \mathbb{Z} \sqcup H$ with $H \in \mathbb{H}$. We defined the bounder quivers

\[
(Q_G, J_G) : = (O, L) \sqcup (Q, I)
\]

\[
(Q_G, \bar{J}) : = (O, \bar{L}) \sqcup (Q, \bar{I})
\]

where $O$, $L$ and $\bar{L}$ are the same of Lemma 11 and $Q$, $I$ and $\bar{I}$ are the same that in the Proposition 11. By Proposition 5 these are presentation of the same algebra. By Proposition 6

$$
\pi_1(Q_G, J_G) = \pi_1(O, L) \sqcup \pi_1(Q, I) = H \sqcup \mathbb{Z} \sqcup m = G
$$

and $\pi_1(Q_G, \bar{J}_G) = \{1\} \sqcup \{1\} = \{1\}$. By Proposition 7 the quiver $Q_G$ is triangular. Thus we proved the next proposition.

**Proposition 20** Let $G$ be a finitely presented group, there exits a triangular algebra $A$ with two presentation $(Q_G, J_G)$ and $(Q_G, \bar{J}_G)$ such that $\pi_1(Q_G, J_G) = G$ and $\pi_1(Q_G, \bar{J}_G) = \{1\}$.

Now we prove the Theorem 1.

**Proof.** Let $G_1, \ldots, G_n$ be a finitely presented groups. By the formula (11) we defined $(\hat{Q}, I_1)$. Similarly to do with $(Q_G, J_G)$ and $(Q_G, \bar{J}_G)$, by the remark 8 we have, first:

$$
\pi_1(\hat{Q}, I_i) = (\sqcup_{j \neq i} \pi_1(Q_{G_j}, \bar{J}_j)) \sqcup \pi_1(Q_{G_i}, J_i)
$$

$$
= (\sqcup_{j \neq i} \{1\}) \sqcup G_i = G_i
$$

Second: we have pairs of presentation $((Q_{G_i}, J_{G_i}), (Q_{G_i}, \bar{J}_{G_i}))^{n}_{i=1}$ of the same algebras. Each $(\hat{Q}, I_i)$ is made taking one and only one, presentation of each par, hence these are presentations of the same algebra $A$. Third, all quivers involved in the co-product are triangular, therefore the quivers $\hat{Q}$ is triangular, and thus $A$ is triangular. ■
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