On improving Popov’s criterion for nonlinear feedback system stability

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For the $L_2$-stability of a nonlinear single-input–single-output (SISO) feedback system, described by an integral equation and with the forward block transfer function $G(j\omega)$ and a first- and third-quadrant non-monotone nonlinearity $\phi(\cdot) \in \mathcal{N}$ in the feedback path, we derive an interesting generalization of the celebrated criterion of Popov [(1962). Absolute stability of nonlinear systems of automatic control. Automation and Remote Control, 22(8), 857–875]: $\Re(1 + j \omega)G(j\omega) > 0, 0 \leq \omega < \infty$, where $\omega > 0$ is a constant. The generalization entails the addition of a general causal + anticausal O’Shea–Zames–Falb multiplier function whose time-domain $L_1$-norm is constrained by certain characteristic parameters (CPs) of the nonlinearity obtained from certain novel algebraic inequalities. If the nonlinearity is monotone or belongs to any prescribed subclass of $\mathcal{N}$, its CPs are reduced, thereby relaxing the time-domain constraint on the multiplier. An important special feature of the new stability results is a partial bridging of the significant gap between the Popov criterion and the stability results that appeared post-Popov in the form of considering monotone and other subclasses of nonlinearities in exchange for weakening the restrictions on the phase angle behaviour of $G(j\omega)$. Extensions to time-varying nonlinearities more general than those in the literature are also presented. Numerical examples are given to illustrate the theorems and to demonstrate their superiority over the existing literature.

Keywords: K–Y–P lemma; $L_2$-stability; Lur’e problem; Popov criterion; time-varying feedback systems

1. Introduction

In the analysis of problems arising in diverse areas of dynamical system design (such as satellite control, communication networks, and chemical plants), stability theory of nonlinear time-varying systems is an invaluable tool. Nonlinear differential and integral equations, which are typically used as mathematical models to describe such systems, are linearized (or perturbed) around, for instance, a periodic solution. The perturbed behaviour of a dynamical system is found to be more accurately modelled by nonlinear time-varying differential and integral equations, of which the latter are more general than the former, since they can describe infinite-dimensional systems and include the former as a special case. In this context, the feedback system of Figure 1(a) plays an important role in the analysis and synthesis of dynamical systems which are in practice an interconnection of subsystems subject to switching operations. The system of Figure 1(a) consists of a linear time-invariant part in the forward path and a nonlinear time-varying gain in the feedback path. A special case of Figure 1(a) is Figure 1(b) in which the feedback block is a time-invariant nonlinear gain. Primarily inspired by the classic papers of Nyquist and Bode on the frequency-domain analysis (and synthesis) of systems modelled by a special case of Figure 1(b), namely, a system with a constant linear gain in the feedback path, research workers have considered extension of similar ideas to the analysis of stability (and instability) of the system of Figure 1(a), the main subject of this paper, described by

$$\dot{v}(t) = f(t) - k(t)\phi(\sigma(t));$$

$$\sigma(t) = \sum_{m=0}^{\infty} g_m v(t - \tau_m) + \int_0^\infty g(\tau) v(t - \tau) \, d\tau, \quad (1)$$

where $\delta(t - \tau_m)$ is the Dirac delta function at instant $t = \tau_m$; $\sum_{m=0}^{\infty} g_m \delta(t - \tau_m) + g(t)$ is the impulse response of the time-invariant forward block with constant real sequences $\{g_m\}, \{\tau_m\}$, in which $\tau_m \geq 0, \forall m$; $g(\cdot)$ is a real-valued function in $[0, \infty)$; and $\sum_{m=0}^{\infty} |g_m| + \int_0^\infty |g(t)| \, dt < \infty$, i.e. $\{g_m\} \in \ell_1$, and $g(\cdot) \in L_1$; real-valued gain $k(\cdot)$ assumes values in $[0, \infty)$; $f(\cdot), v(\cdot), \sigma(\cdot)$ are, respectively, the input, ‘error’ signal and output of the system; and $\phi(\cdot), a$ real-valued function on $(-\infty, \infty)$, is a memoryless, first- and third-quadrant nonlinearity having the following basic properties: $\varphi(0) = 0$; there exist positive constants $q_1$ and $q_2$ such that $q_1 \sigma^2 \leq \varphi(\sigma) \sigma \leq q_2 \sigma^2, \sigma \neq 0$. We call the class of such nonlinearities $\mathcal{N}$, with its subclass of monotone nonlinearities denoted by $\mathcal{M}$. If the monotone nonlinearity also has the
The class of (closed-loop) control systems initiated by Lur’e and Postnikov (1944) in their pioneering Lyapunov method-based work on a special case of Equation (1), namely, a differential equation with \( k(t) \) replaced by a constant gain \( K \in [0, \infty) \). In this case, the system is governed by the following time-invariant nonlinear integral equation:

\[
v(t) = f(t) - K \phi(\sigma(t)); \\
\sigma(t) = \sum_{m=0}^{\infty} g_m v(t - \tau_m) + \int_{0}^{t} g(\tau)v(t - \tau) \, d\tau. \tag{2}
\]

We consider the problems of \( L_2 \)-stability of the systems governed by Equation (1) and by Equation (2). To this end, let \( L_2[0, \infty) \) be the linear space of real-valued functions \( x(\cdot) \) on \([0, \infty)\) with the property that \( \int_{0}^{\infty} |x(t)|^2 \, dt < \infty \), and equipped with the norm, \( \| x(\cdot) \| = \left( \int_{0}^{\infty} |x(t)|^2 \, dt \right)^{1/2} \). The nonlinear system described by Equation (1) is \( L_2 \)-stable if \( v \in L_2[0, \infty) \) for \( f \in L_2[0, \infty) \), and an inequality of the type \( \| v \| \leq C \| f \| \) holds where \( C \) is a constant.

### 1.1. Brief survey of literature

The class of (closed-loop) control systems initiated by Lur’e (1951) is governed, in the notation used by Kalman (1963), by the following equations:

\[
\frac{dx}{dt} = Fx - g \phi(\sigma); \quad \frac{d\xi}{dt} = -g \phi(\sigma); \quad \sigma = h'x + \rho \xi, \tag{3}
\]

where the prime denotes transpose; \( \sigma, \xi, \rho \) are real scalars; \( x, g, h \) are real \( n \)-vectors; and \( F \) is real \( n \times n \) matrix. The nonlinearity \( \phi(\sigma) \) is a real-valued continuous function which belongs to the class \( A_k: \phi(0) = 0, 0 < \phi(\sigma) \sigma < \kappa \sigma^2 \). The problem then is to establish conditions for the global asymptotic stability — also called ‘absolute stability’ — of Equation (3) for any \( \phi \in A_k \). It is well known that V. M. Popov established the following frequency-domain inequality for the absolute stability of Equation (3) in his celebrated paper (Popov, 1962):

\[
\Re(2\alpha \rho + j \omega \beta) \left[ k'(j \omega I - F)^{-1} g + \frac{\rho}{j \omega} \right] \geq 0
\]

for all real \( \omega \) \tag{4}

holds for \( 2\alpha \rho = 1 \) and some \( \beta \geq 0 \).

Earlier Lur’e (1951) had formulated the algebraic problem of finding necessary and sufficient conditions on \( \rho, g, h \) and \( F \) for the existence of a Lyapunov function comprising a quadratic form in \( (\chi, \sigma) \) and an integral of \( \phi(\sigma) \) to guarantee absolute stability of Equation (3) for class \( A_k \), where \( \kappa = \infty \). It was left to Kalman (1963) to provide the complete answer in the form of a lemma, now known as the Kalman–Yakubovich–Popov (K–M–Y) lemma, containing certain matrix equations (and inequalities), to the question of the existence of a Lyapunov function which guarantees absolute stability whenever Popov’s criterion (4) is satisfied. See, in this context, Aizerman and Gantmacher (1964), LaSalle (1962), Lefschetz (1965) and Yakubovich (1962). For a generalization of this result (and of the circle criterion (Narendra and Goldwyn, 1964; Sandberg, 1964)) to input–output stability in an abstract setting, see the pioneering work of Zames (1966).

In the first 10 years or so after the publication of Popov’s paper, a very large number of papers concentrated on the ramifications of Popov’s criterion, as applied mainly to systems described by differential equations. In the course of analysing the challenge posed by the conjecture of Aizerman (1948) in light of Popov’s criterion, Brockett and Willems (1965) relaxed the assumptions made on the phase angle behaviour of \( G(j\omega) \) in exchange for (i) the choice of nonlinearity classes which are subsets of \( \mathcal{N} \) and (ii) the
use of multiplier functions more general than \((1 + j\omega)\) so that the ‘real-part’ conditions (in the frequency domain) are satisfied.

Brockett and Willems (1965) presented a list of third- and fourth-order \(G(s)\) for which the null solutions of the linearized versions of the (time-invariant) system are asymptotically stable in the large for all positive gains \(K\), but the Popov criterion cannot predict stability for \(\phi(\cdot) \in \mathcal{N}\). ‘regardless of any additional assumptions placed on the coefficients of the system.’ To deal with such problems, Brockett and Willems (1965) assumed monotone and odd monotone nonlinearities, and used RC-RL impedance functions. On the other hand, Lakshmi Thathachar, Srinath, and Ramapriyan (1966) explored the use of more general impedance (and hence positive real) functions with complex poles and zeros, and invoked the K–Y–P lemma. In this context, see also Narendra and Neuman (1966), Thathachar (1970) and Dewey and Jury (1966) for related results. However, the most significant result is due to O’Shea (1967) who proposed causal + anticausal functions with a time-domain \(L_1\)-norm constraint on the multiplier. See Zames and Falb (1968) for the infinite dimensional system counterpart based on the theory of positive operators. The multiplier used by O’Shea and Zames and Falb (hereafter called the OZF multiplier) is the most general multiplier function known in stability literature.

\textbf{Remark 1} The system we consider is governed by the (infinite-dimensional) integral equation (1) which is more general than the (finite-dimensional) differential equation (3). When the following replacements are made, Equation (1) \textit{effectively} reduces to the special case of Equation (3): (a) \(k(t)\) is replaced by a constant gain \(K \in [0, \bar{K}]\), if there exists a frequency function, \(Z(j\omega)\) such that \(-\pi/2 \leq \arg[Z(j\omega)] \leq \pi/2\) and \(-\pi/2 < \arg[Z(j\omega)(G(j\omega) + 1/\bar{K})] < \pi/2\), where \(\arg\) denotes ‘the phase angle of.’ Alternatively, \(\Re Z(j\omega) > 0\), and \(\Re Z(j\omega)(G(j\omega) + 1/\bar{K}) > 0\), \(\omega \in (-\infty, \infty)\), where \(\Re\) denotes ‘the real part of.’

\textbf{1.2. Motivation}\n
A motivation to generalize the Popov criterion is the multiplier form of the Nyquist criterion which in the most general form reads: the system (2) with \(\psi(\sigma) \equiv \sigma \) is asymptotically stable for all constant gains \(K \in [0, \bar{K}]\), if there exists a frequency function, \(Z(j\omega)\) such that \(-\pi/2 \leq \arg[Z(j\omega)] \leq \pi/2\) and \(-\pi/2 < \arg[Z(j\omega)(G(j\omega) + 1/\bar{K})] < \pi/2\), where \(\arg\) denotes ‘the phase angle of.’ Alternatively, \(\Re Z(j\omega) > 0\), and \(\Re Z(j\omega)(G(j\omega) + 1/\bar{K}) > 0\), \(\omega \in (-\infty, \infty)\), where \(\Re\) denotes ‘the real part of.’ The last two frequency-domain inequalities are called below (for convenience) as merely \textit{real-part conditions}. See Brockett and Willems (1965) and Thathachar (1970). Note that neither a restriction of causality nor a time-domain \(L_1\)-norm (or otherwise) constraint has been imposed on \(Z(j\omega)\). In particular, the chasm that exists between what we recognize as a very general frequency function \(Z(j\omega)\) of the Nyquist criterion for linear time-invariant system (obtained from Equation (2) by setting \(\psi(\sigma) \equiv \sigma\) and the Popov multiplier function \((1 + j\omega)\) for the (absolute stability of the) nonlinear time-invariant system (2) is too big. In fact, a possible way of resolving Aizerman’s conjecture Aizerman (1948) is to isolate classes of \(G(j\omega)\) functions which obey Nyquist’s criterion and for which \((1 + j\omega)\) for some \(\alpha > 0\) is also a multiplier function. In this context, it is to be noted that the \textit{transition} from the class \(\mathcal{N}\) of nonlinearities to the class of linear gains is too abrupt.

In the present paper, we adopt a converse approach to deal with Aizerman’s conjecture. More, specifically, in order to derive \(L_2\)-stability criteria for the system (2) with \(\psi(\cdot) \in \mathcal{N}\) we explore the use of the OZF multiplier itself. The results can be strengthened to be applicable to systems governed by differential equations of the type considered by Popov and others. It is known that under certain general conditions, \(L_2\)-stability of systems governed by differential equations can be shown to be equivalent to their absolute stability.

Another motivation (to generalize the Popov criterion) arose from some recent stability results for multi-input–multi-output (MIMO) time-varying continuous- and discrete-time nonlinear systems, which constitute a generalizations of the system (1). See Huang, Venkatesh, Xiang, and Lee (2014b) and Venkatesh (2014). It was found that for continuous-time MIMO \(L_2\)-stability as et al. (1966). For others, see the book Boyd, El Ghauui, Feron, and Balakrishnan (1994) (pp. 123–128, 131–135). However, the K–Y–P lemma as such is not used in this paper because (i) we consider infinite-dimensional systems for which no counterparts of the K–Y–P lemma seem to exist; and, more importantly, (ii) for time-varying systems and for exploring anti-causal multiplier functions, it is not clear how to generate candidate Lyapunov functions, using the K–Y–P lemma or otherwise.
also for discrete-time MIMO $\ell_2$-stability, certain algebraic inequalities involving multi-variable nonlinearities in generalized two-variable polynomial forms (extending quadratic and bi-quadratic forms) can be exploited to modify (and partially dispense with) the existing constraints on multiplier functions commonly used in the literature. Characteristic parameters (CPs) of the nonlinearity obtained from the extremal values of the ratios of polynomial forms in two variables govern the time-domain constraints on the multiplier.

1.3. Main contributions

We establish two stability theorems: Theorem 1 for the time-invariant system (2) with $\varphi(\cdot) \in \mathcal{N}$ and Theorem 2 for the time-varying system (1) for the same class $\mathcal{N}$ of nonlinearities. In Theorem 1, the stability conditions are expressed in terms of a causal + anticausal multiplier function $Z(j\omega)$ subject to the following requirements: [C1-1] - $\Re [Z(j\omega)G(j\omega)] > 0$, $\omega \in (-\infty, \infty)$; and [C1-2] - a time-domain constraint on the inverse Fourier transform of $Z(j\omega)$, depending on certain CPs of the nonlinearity to be defined later below. On the other hand, in Theorem 2, the stability requirements are [C2-1] - same as [C1-1] of Theorem 1; [C2-2] - a global bound on positive and negative lobes of the normalized rate of variation $\theta(t) \equiv \frac{dx(t)}{dt}/k(t)$; and [C2-3] - this is a modified version of [C1-2] of Theorem 1 that depends on the choice of the global bounds in [C2-2].

1.4. Organization of the paper

The next section (Section 2) is concerned with the main assumptions, problem formulation, and mathematical preliminaries, including the definitions of the CPs of two classes $\mathcal{N}$ and $\mathcal{M}$ of nonlinearities needed in the proof of the stability theorems. Section 3 presents the first main preliminaries, including the definitions of the CPs of the nonlinearity obtained in two variables govern the time-domain constraints on the multiplier. Examples are given in this section with a view to illustrate the application of the theorems and to exhibit the superiority of the stability conditions over those in the literature. In Section 6, the new stability results are critically examined for a theoretical comparison with the literature, and some open problems are presented. Section 7 concludes the paper, followed by appendices which contain proofs of some of the lemmas used in the proofs of the theorems.

2. Assumptions, problem formulation, and preliminaries

For the system (1), the impulse response of the linear block is assumed to be in $L_1 \cap L_2$, and when the nonlinearity $\varphi(\sigma)$ is replaced by $q^2\sigma$ (with constant $q_2 > 0$) and time-varying gain $k(t)$ by the constant gain $K \in [0, \infty)$, its solutions are in $L_1 \cap L_2$, which implies that the zeros of $(1 + KG(s))$ for $K \in [0, \infty)$ lie strictly in the left-half ($\Re s < -\delta \leq 0$) of the complex plane.

Problem formulation: Given $G(s)$ satisfying the above assumptions, find conditions (i) on $\varphi(\cdot) \in \mathcal{N}$ for the $\ell_2$-stability of the system (2), and (ii) on $\varphi(\cdot) \in \mathcal{N}$ and $k(t)$ for the $\ell_2$-stability of the system (1).

Preliminaries: For any real-valued function $x(\cdot)$ on $[0, \infty)$ and any $T \geq 0$, we define the truncated function $x_T(\cdot)$ by: $x_T(\cdot) = x(\cdot)$ for $0 \leq t \leq T$; and $x_T(\cdot) = 0$ for $t < 0$ and $t > T$. Let $L_{2c}$ be the space of those real-valued functions $x(\cdot)$ on $[0, \infty)$ whose truncations $x_T(\cdot)$ belong to $L_2(0, \infty)$ for all $T \geq 0$. In order to establish stability of the system under consideration, we first assume infinite escape time for the solution of the system with $f \in L_2$ and the solution belongs to $L_{2c}$. We then show that, under certain conditions on $\varphi(\cdot), k(t)$ and $G(j\omega)$, the solution actually belongs to $L_2(0, \infty)$.

Consider the class of operators $\mathcal{Z} \in L_{2c} \rightarrow L_{2c}$, satisfying an equation of the type

$$
\mathcal{Z}[\sigma(t)] = \sigma(t) + \frac{d\sigma}{dt} + \sum_{m=1}^{\infty} [z_m \sigma(t - \tau_m) + z'_m \sigma(t + \tau'_m)] + \int_{-\infty}^{\infty} z(\tau) \sigma(t - \tau) d\tau,
$$

(5)

where (real) constant $\alpha > 0$; the real sequences $\{z_m\}$ and $\{z'_m\}$ are in $\ell_1$, i.e. $\sum_{m=1}^{\infty} |z_m| + |z'_m| < \infty$; sequences $\{\tau_m\}$ and $\{\tau'_m\}$ are in $[0, \infty)$; $z(\cdot)$ is a real-valued function on $(-\infty, \infty)$, and is in $L_1(-\infty, \infty)$, i.e. $\int_{-\infty}^{\infty} |z(t)| dt < \infty$.

Its Fourier transform is given by

$$
Z(j\omega) = 1 + j\alpha \omega + \sum_{m=1}^{\infty} [z_m (e^{-j\tau_m}) + z'_m (e^{j\tau'_m})] + \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt.
$$

(6)

The multiplier function used by O’Shea (1967) and Zames and Falb (1968) for dealing with the class $\mathcal{M}$ of monotone nonlinearities is Equation (6), and called earlier (above) as the OZF multiplier. For convenience in later manipulations, let

(1) $z^+_m$ and $-z^-_m$ denote, respectively, the positive and negative coefficients from the set $\{z_m\}$, i.e. $z_m = z^+_m - z^-_m$, $m = 1, 2, \ldots$; $z^+_m$ and $-z^-_m$ denote, respectively, the positive and negative coefficients from the set $\{z'_m\}$. i.e. $z'_m = z^+_m - z^-_m$, $m = 1, 2, \ldots$;

(2) $z_+(t) = z(t)$ for $t \geq 0$; and $z_-(t) = z(t)$ for $t < 0$, so that $z(t) = z_+(t) + z_-(t)$ for $t \in (-\infty, \infty)$;

(3) $z^+_m(t) = z_+(t)$ if $z_+(t) > 0, t \in [0, \infty)$; else $z^+_m(t) = 0, t \in [0, \infty)$; $-z^-_m(t) = z_-(t)$ if $z_-(t) < 0, t \in [0, \infty)$;
else $z_i^-(t) = 0, t \in [0, \infty)$, i.e. $z_i(t) = z_i^+(t) - z_i^-(t)$, $t \in [0, \infty)$; and

$$z_i^+(t) = z_i(t) \text{ if } z_i(t) > 0, t \in (-\infty, 0]; \text{ else } z_i^+(t) = 0, t \in (-\infty, 0].$$

Similarly, $z_i^-(t) = z_i(t) \text{ if } z_i(t) < 0, t \in (-\infty, 0]; \text{ else } z_i^-(t) = 0, t \in (-\infty, 0], \text{ i.e. } z_i(t) = z_i^+(t) - z_i^-(t), t \in (-\infty, 0].$

Theorems 1 and 2 (and their proofs) are based on new algebraic inequalities, involving the nonlinearity $\phi(\cdot)$, which are motivated by the well-known algebraic inequality $-\frac{1}{2}(a^2 + b^2) \leq ab \leq \frac{1}{2}(a^2 + b^2)$ for arbitrary, real scalars $a$ and $b$. Based on these inequalities, the characteristic parameters (CPs) of $\phi(\cdot) \in \mathcal{N}, \mathcal{M}$ are defined as follows. Let $\Phi(\sigma) = \int_0^\infty \phi(\xi) d\xi > 0 \text{ for } \sigma \neq 0,$ and for (real) $x, y \in (-\infty, \infty),$ let $\Psi(x, y) = \{\phi(x) + \phi(y)\}.$ Then (i) for $\phi(\cdot) \in \mathcal{N}, \psi_1 \phi(x) \sigma \leq \Phi(\sigma) \leq \psi_1 \phi(x) \phi(\sigma) \sigma$ and $-\mu_i \Psi(x, y) \leq \phi(x) \leq \mu_i \Psi(x, y),$ where the CPs $\psi_1, \psi_2 > 0, \mu_i > 0, \text{ and } \mu_2 > 0$ are defined by

$$\psi_1 \equiv \inf_{x \neq 0} \frac{\Phi(x)}{\phi(x)x}; \quad \psi_2 \equiv \sup_{x \neq 0} \frac{\Phi(x)}{\phi(x)x};$$

$$-\mu_i \equiv \inf_{x \neq 0} \frac{\phi(x)y}{\Psi(x, y)}; \quad \mu_i \equiv \sup_{x \neq 0} \frac{\phi(x)y}{\Psi(x, y)} \quad \text{and (7)}$$

(ii) for $\phi(\cdot) \in \mathcal{M}, \eta_i \phi(\sigma) \sigma \leq \Phi(\sigma) \leq \eta_i \phi(\sigma) \phi(\sigma) \sigma$ and $-\gamma_i \Psi(x, y) \leq \phi(x) \leq \gamma_i \Psi(x, y),$ where the CPs $\eta_0, \eta_1 > 0, \gamma_0, \gamma_1 > 0,$ and $\gamma_0 > 0$ are defined by

$$\eta_0 \equiv \inf_{x \neq 0} \frac{\Phi(x)}{\phi(x)x}; \quad \eta_1 \equiv \sup_{x \neq 0} \frac{\Phi(x)}{\phi(x)x};$$

$$-\gamma_0 \equiv \inf_{x \neq 0} \frac{\phi(x)y}{\Psi(x, y)}; \quad \gamma_0 \equiv \sup_{x \neq 0} \frac{\phi(x)y}{\Psi(x, y)}. \quad \text{(8)}$$

Note that $\eta_0 \leq 1$ for $\phi(\cdot) \in \mathcal{M}.$ See Table 1 for typical values of $\mu_i$ and $\mu_2$ for $\phi(\cdot) \in \mathcal{N}$ and $\mathcal{M}.$ In the last item of Table 1, the slope of $\phi(\sigma)$ is negative for $x \in [1.214, 2.1275].$ It is conjectured that, for an arbitrary $\phi(\cdot) \in \mathcal{N},$ the upper limit of $\mu_i = \mu_2 = 1.$

3. Main results-1

With the preliminaries settled, we now present the first main result of the paper.

Table 1. Typical values of $\mu_i$ and $\mu_2$ for $\phi(\cdot) \in \mathcal{N}$ and $\mathcal{M}.$

| No. | Nonlinearity $\phi(\cdot)$ | Class of $\phi(\cdot)$ | $\mu_i$ | $\mu_2$ |
|-----|-----------------|-----------------|-------|-------|
| 1   | $(x - 0.3x^3 + 0.03x^5)$ | $\mathcal{N}$    | 0.7892| 0.7892|
| 2   | $(x + 0.3x^3 + 0.03x^5)$ | $\mathcal{M}$    | 0.636 | 0.636 |
| 3   | $(x - 0.3x^3 + 0.1x^5)$ for $x < 0$ | $\mathcal{N}$    | 0.9389| 0.7892|
|     | $(x - 0.3x^3 + 0.03x^5)$ for $x \geq 0$ | $\mathcal{M}$    | 0.636 | 0.636 |
Invoking the condition [H-1] of Theorem 1, \(\forall Z(j\omega)\) \(G(j\omega) \geq \delta > 0\) for some \(\delta > 0\), we obtain the following inequality:

\[
\int_0^T v_T(t) Z[G(v_T(t))] \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(j\omega) G(j\omega) |V_T(j\omega)|^2 \, d\omega \geq \frac{\delta}{2\pi} ||v_T||^2.
\]

(13)

By virtue of Lemma 1, the second integral right-hand side of Equation (11),

\[
\int_0^T K \varphi(\sigma_T(t)) Z[\sigma_T(t)] \, dt \geq \alpha(\Phi(\sigma_T(T)) - \Phi(\sigma_T(0))),
\]

(14)

where, to recall, \(\Phi(\sigma) = \int_0^\sigma \varphi(\xi) \, d\xi\). By applying the Parseval theorem to Equation (10), combining the result with Equations (13) and (14), and noting that \(\Phi(\sigma) \geq 0, \forall \sigma\), we obtain

\[
\delta ||v_T||^2 \leq \alpha(\Phi(\sigma_T(0)) + 2\pi \int_0^T f_T(t) Z[G(v_T(t))] \, dt = \int_{-\infty}^{\infty} F_T(-j\omega) Z(j\omega) G(j\omega) V_T(j\omega) \, d\omega.
\]

(15)

Now using the Cauchy–Schwarz inequality in Equation (15) leads to

\[
\int_{-\infty}^{\infty} F_T(-j\omega) Z(j\omega) G(j\omega) V_T(j\omega) \, d\omega \leq \sup_{-\infty < \omega < \infty} |Z(j\omega)G(j\omega)||\|f_T||||v_T||.
\]

(16)

Note that \(\sup_{-\infty < \omega < \infty} |Z(j\omega)G(j\omega)|\) is finite by virtue of the assumptions on \(Z(\cdot)\) and \(G(\cdot)\). Let \(C = \sup_{-\infty < \omega < \infty} |Z(j\omega)G(j\omega)| \) and \(a_0 = \alpha(\Phi(\sigma_T(0)))\), which is independent of \(T\) and can be assumed bounded. Then, from Equations (15) and (16), we obtain the inequality \(\delta ||v_T||^2 \leq a_0 + C ||f_T|| ||v_T||\) which is valid for all \(T > 0\). Since \(\delta, C\) and \(a_0\) are independent of \(T\), we conclude that \(||v|| \leq C ||f|| + \sqrt{a_0^2 + C^2/4 ||f||^2}\), where \(a_0 = a_0/\delta\) and \(C = C/\delta\). The theorem is proved.

4. Main results-2

We now derive \(L_2\)-stability conditions for the system (1). To this end, let \(h(t)\) be a nonnegative, integrable and bounded function on \(t \in [0, \infty)\), and let \(\sigma(t) = e^{-\int_0^t h(r) \, dr}\). Assume that the integral \(\int_0^t h(r) \, dr \leq M < \infty, t \in [0, \infty)\) and \(0 < \epsilon \leq \lim_{t \to \infty} \int_0^t h(r) \, dr \leq M < \infty\). Then, \(\sigma(t)\) is a bounded positive function. Note that \((d\sigma/dt)/\sigma(t) = -h(t)\), which is non-positive. We assume that (i) \(k(t) \in [\epsilon, \infty)\) for \(t \geq 0\), where (constant) \(\epsilon > 0\), is a piecewise-continuous function of bounded variation with first-order (i.e. jump-) discontinuities in \(\ell_1\); and (ii) \(k(t)\) is made up of the continuous part, \(k_c(t)\), and the discontinuous part \(k_d(t)\): discontinuities at instants \(t_m^+\) correspond to positive jumps, \(\alpha_{m^+}\); and instants \(t_m^-\) corresponding to negative jumps, \(\alpha_{m^-}\).

The derivative of \(\theta(t)\) is then given by

\[
\frac{dk}{dt} = \frac{dk_c}{dt} + \sum_m \{\alpha_{m^+} \delta(t-t_{m^+}) + \alpha_{m^-} \delta(t-t_{m^-})\}.
\]

(17)

Furthermore, let \(\theta(t) = dk/dt/k(t)\). At the positive discontinuities, \(t_{m^+}\), of \(k(t)\), the value of \(k(t)\) is, by convention, taken as \(k(t_{m^+})\) where \(t_{m^+}\) is the instant just to the left of \(t_{m^+}\). Similarly, at the negative discontinuities, \(t_{m^-}\), of \(k(t)\), the value of \(k(t)\) is taken as \(k(t_{m^-})\) where \(t_{m^-}\) is the instant just to the left of \(t_{m^-}\). Note that, based on the assumptions on \(k(t), k(t_{m^+}) \neq 0, k(t_{m^-}) \neq 0, t \geq 0\). Furthermore, let \(\theta^+(t) = \theta(t), \forall t > 0; \theta^+(t) = 0, \forall t \leq 0; \theta^-(t) = \theta(t), \forall t < 0; \theta^-(t) = 0, \forall t \geq 0\). Also, let \(\theta(t) = dk_c/dt/k_c(t)\). Note that \(\theta(t) = \theta^+(t) + \theta^-(t)\), where

\[
\theta^+(t) = \theta^+_c(t) + \sum_m \psi_{m^+} \delta(t-t_{m^+}); \quad \text{and}
\]

\[
\theta^-(t) = \theta^-_c(t) + \sum_m \psi_{m^-} \delta(t-t_{m^-}).
\]

(18)

where \(\psi_{m^+} = \alpha_{m^+}^2/k(t_{m^+})\) and \(\psi_{m^-} = \alpha_{m^-}^2/k(t_{m^-})\). We need some additional preliminaries. By the statement that \(\sigma(t)e^{-\int_0^t}k(t)\) for \(t \in [0, \infty)\) is nonincreasing, we mean that

\[
\frac{d(\sigma(t)k(t)e^{-\int_0^t})}{dt} = e^{-\int_0^t} \left\{ \frac{d\sigma}{dt} - \xi(\sigma(t)) \right\} k(t) + e^{-\int_0^t} \sigma(t)
\]

And, similarly, by the statement that \(\sigma(t)e^{\int_0^t}k(t)\) for \(t \in [0, \infty)\) is nondecreasing, we mean that

\[
\frac{d(\sigma(t)k(t)e^{\int_0^t})}{dt} = e^{\int_0^t} \left\{ \frac{d\sigma}{dt} + \xi(\sigma(t)) \right\} k(t) + e^{\int_0^t} \sigma(t)
\]

(19)

In view of the assumptions on \(\sigma(t)\) and \(k(t)\), it can be shown that (19) and (20) together reduce to the following inequality:

\[
-\xi \leq \left( \frac{d\sigma}{dt}/\sigma(t) \right) + \theta(t) \leq \xi.
\]

(21)

THEOREM 2. The system (1) with \(\varphi(\cdot) \in N; k(t) \in [\epsilon, \infty)\) for \(t \geq 0\), where (constant) \(\epsilon > 0\); and \(||v||^2 \leq \int_0^\infty e^{2\epsilon t}(v(t))^2 \, dt\), where \(\epsilon > 0\) is an arbitrarily small number, is exponentially \(L_2\)-stable in the sense that \(||v|| \leq C_1 ||f|| + C_2 ||\xi(\sigma(t))||\), where \(C_1, C_2\) are constants, if there exist a multiplier function \(Z(j\omega)\) defined
by Equation (6), a bounded positive function $\sigma(\cdot)$ as defined above; and nonnegative constants, $\xi, \zeta$, such that

$$H^{-1}$$

for the $\hat{H}$ defined above sup, and nonnegative $\omega, \epsilon$.)

$\|G(j\omega - \epsilon)\| < \infty$; and

$H^{3}$: $\sigma(t)e^{-\epsilon t}k(t)$ is nonincreasing and $\sigma(t)e^\epsilon t k(t)$ is nondecreasing for all $t \in [0, \infty)$; $H^{2}$: $\alpha \epsilon t + \sum_{m=1}^\infty ((1 + e^{i\epsilon t})(\mu \xi + \mu \zeta m) + (1 - e^{-i\epsilon t})(\mu \xi + \mu \zeta m)) \frac{1}{\epsilon t} d\epsilon < 1$, where $\nu, \mu, \xi, \zeta$ are defined in Equation (7).

The proof of Theorem 2 depends on Lemma 2 given below. The proofs of both are given in Appendix 3. (Also see in this context Venkatesh, 1978, pp. 572–573.)

**Lemma 2** With the operator $Z$ defined by Equation (5), the integral

$$\lambda(T) = \int_0^T \sigma t (k(t)\Phi(\sigma_T(t))) Z \sigma_T(t) dt$$

for all $\sigma_T$ in the domain of $Z$ and for all $T \geq 0$, if conditions $[H\neg 2]$ and $[H\neg 3]$ of Theorem 4 are satisfied.

**Corollary L2-1** With $\theta^+(t)$ and $\theta^-(t)$ as defined above, if $\sigma(t)e^{-\epsilon t}k(t)$ is nonincreasing and $\sigma(t)e^\epsilon t k(t)$ is nondecreasing for $t \in [0, \infty)$, then, for some positive constants $N_1$ and $N_2$, for all finite $T > 0$ and for all $t_0 \geq 0$,

$$\frac{1}{T} \int_{t_0}^{T+t_0} \theta^+(t) dt \leq N_1, \quad -N_2 \leq \frac{1}{T} \int_{t_0}^{T+t_0} \theta^-(t) dt; \quad (23)$$

for

$$\lim_{t \to \infty} \frac{1}{T} \int_{t_0}^{T+t_0} \theta^-(t) dt \leq \xi, \quad \xi \neq \zeta,$$

but for $\xi = \zeta, \theta^+(t)$ and $\theta^-(t)$ are unrestricted.

If $k(t)$ is piecewise constant, then $k_c(t)$ is identically zero. Let $M$ denote the number of discontinuities in a finite interval $T > 0$. Then, by invoking Equations (18), (24), becomes

$$\frac{1}{T} \int_{m_0}^{m_0+M} \psi_{m+} \leq N_1, -N_2 \leq \frac{1}{T} \int_{m_0}^{m_0+M} \psi_{m-}$$

and

$$\lim_{T \to \infty} \frac{1}{T} \int_{m}^{m+M} \psi_{m+} \leq \xi, -\xi \leq \lim_{T \to \infty} \frac{1}{T} \int_{m}^{m+M} \psi_{m-}; \quad (25)$$

but for $\xi = \zeta, \sum \psi_{m+} \text{ and } \sum \psi_{m-}$ are unrestricted. The proof of Lemma 2 is similar to that of Lemma 2 of Venkatesh (1978) (pp. 572, 577–579), and is hence omitted.

When $k(t)$ is periodic with period $p$, then Equations (23)–(25) reduce respectively to

$$\frac{1}{p} \int_0^p \theta^-(t) dt \geq -\xi, \quad \frac{1}{p} \int_0^p \theta^+(t) dt \leq -\xi; \quad (26)$$

and

$$\left( \frac{1}{p} \right) \sum_{m_{j_k}} \psi_m(t_{m_k}^+) \geq -\xi, \quad \left( \frac{1}{p} \right) \sum_{m_{j_k}} \psi_m(t_{m_k}^-) \leq \xi, \quad (27)$$

where $D$ denotes the semi-closed interval $(0, p]$.

### 4.1. Switching nonlinearities

We now consider the case of switching between nonlinearities in Equation (1). To this end, let the nonlinear gain consists of two nonlinearities $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$, both belonging to class $N$, which switch from one to the other periodically, with the apparent dwell times of $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ given, respectively, by $d_1$ and $d_2$. The fundamental period of switching is $p = d_1 + d_2$. Let $k_1(t), k_2(t) \in \{0, \infty\}$ for $t \geq 0$ and for some $\epsilon > 0$, be periodic with the same period $p$ and active, respectively, during the regimes of $\varphi_1(\cdot)$, and $\varphi_2(\cdot)$. In Equation (1), with $u(t)$ denoting the step function, let

$$k(t)\sigma(\cdot) = k_1(t)\varphi_1(\cdot) \sum_{m=0}^{\infty} (u(t - mp) - u(t - mp - d_1)) + k_2(t)\varphi_2(\cdot) \sum_{m=0}^{\infty} (u(t - mp - d_1) - u(t - (m + 1)p)). \quad (28)$$

For simplicity in later manipulations, let $U_1(t) = \sum_{m=0}^{\infty} (u(t - mp) - u(t - mp - d_1))$ and $U_2(t) = \sum_{m=0}^{\infty} (u(t - mp - d_1) - u(t - (m + 1)p))$, so that Equation (28) becomes $k(t)\psi(\cdot) = k_1(t)U_1(t)\varphi_1(\cdot) + k_2(t)U_2(t)\varphi_2(\cdot)$. Note that $U_1(t)$ and $U_2(t)$ are periodic with period $p$. We now establish a corollary to Theorem 2 using a special case of the multiplier operator $Z$ defined by Equation (5). Let

$$Z_p \{ \sigma(t) \} = \sigma(t) + \sum_{m=1}^{\infty} (z_m \sigma(t - mp) + z_m' \sigma(t + mp)), \quad (29)$$

where, as before, the real sequences $\{z_m\}$ and $\{z_m'\}$ are in $\xi_1$, i.e. $\sum_{m=1}^{\infty} |z_m| + |z_m'| < \infty$; sequences $\{r_m\}$ and $\{r_m'\}$ are in $[0, \infty)$. Its Fourier transform is given by

$$Z_p (j\omega) = 1 + \sum_{m=1}^{\infty} (z_m e^{-jmp\omega} + z_m' e^{jmp\omega}). \quad (30)$$

For (real) $x, y \in (-\infty, \infty)$ and with $r = 1, 2$, let $\Phi_r(x, y) \doteq (\varphi_r(x) + \varphi_r(y))y$. Then (i) for $\varphi_r(\cdot) \in N$, $-\mu_r \Phi_r(x, y) \leq$
\[ \phi_r(x)(y) \leq \mu_{s,r}, \Psi_r(x,y), \text{ where the CPs } \mu_{s,r} > 0 \text{ and } \mu_{s,r} > 0 \text{ are defined by} \]
\[ \mu_{s,r} \neq \inf_{\phi_r(x) \neq 0} \frac{\phi_r(x)(y)}{\Psi_r(x,y)}, \mu_{s,r} \neq \sup_{\phi_r(x) \neq 0} \frac{\phi_r(x)(y)}{\Psi_r(x,y)}, \]
\[ r = 1, 2, \quad \mu_{s,r} \neq \max(\mu_{s,1}, \mu_{s,2}); \]
\[ \mu_{s,r} \neq \max(\mu_{s,1}, \mu_{s,2}); \]
\[ (31) \]

(ii) for \( \phi_r(.) \in \mathcal{M}, -\gamma_{s,r} \Psi_r(x,y) \leq \phi_r(x)(y) \leq \gamma_{s,r} \Psi_r(x,y), \)
where the CPs \( \gamma_{s,r} > 0 \) and \( \gamma_{s,r} > 0 \) are defined by
\[ \gamma_{s,r} \neq \inf_{\phi_r(.) \in \mathcal{M}} \frac{\phi_r(x)(y)}{\Psi_r(x,y)}, \gamma_{s,r} \neq \sup_{\phi_r(.) \in \mathcal{M}} \frac{\phi_r(x)(y)}{\Psi_r(x,y)}, \]
\[ r = 1, 2, \quad \gamma_{s,r} \neq \max(\gamma_{s,1}, \gamma_{s,2}), \]
\[ \gamma_{s,r} \neq \max(\gamma_{s,1}, \gamma_{s,2}); \]
\[ (32) \]

**Corollary T2-1** The system (1) with \( \phi_1(.) \neq \phi_2(.) \in \mathcal{N}, \text{ and } k_1(t), k_2(t) \text{ as defined above, is } L_2-\text{stable, if there exist a multiplier operator } Z_p \text{ defined by Equation (29) such that} \]
\[ [H-1]: \inf_{\omega \in \mathbb{R}, \varphi \neq 0} \| Z_p(\omega) G(\omega) \| < \infty; \text{ and } \| Z_p(\omega) G(\omega) \| > 0, \omega \in (\mathbb{R}, \mathbb{R}); \text{ and } [H-2]: \]
\[ \sum_{m=1}^{\infty} \mu_{s,1}(\mu_{s,1}^+ + \mu_{s,2}^+) \mu_{s,2}(\mu_{s,1}^+ + \mu_{s,2}^+) < \frac{1}{2}, \text{ where } \mu_{s,1} \text{ and } \mu_{s,2} \text{ are as defined in Equation (31)}. \]

Its proof is similar to the proof of Theorem 1 and depends on the following corollary which is proved in Appendix 4.

**Corollary L2-2** With the operator multiplier \( Z_p \) defined by (29), the integral
\[ \lambda_{Z_p}(T) \leq \int_0^T k(t) \psi(\sigma(t)) Z_p \psi(t) \ dt \]
\[ (33) \]
satisfies the inequality \( \lambda_{Z_p}(T) \geq 0, \text{ for all } \psi(t) \text{ in the domain of } Z_p \text{ and for all } T \geq 0 \text{ if condition } [H-2] \) of Corollary T2-1 is satisfied.

### 5. Examples

The difference (in form) between the Popov criterion and the circle criterion is that the former uses the (frequency domain) multiplier function \((1 + j \omega \phi_0)\), where the real constant \( q > 0 \), but the latter (i.e. the circle criterion) uses none. The starting point, then, for the illustrations to follow is that the phase angle behaviour of \( G(\omega) \) is such that \( L_2-\text{stability cannot be established by either the circle criterion as applied to the nonlinear time-varying systems with} \)
\( \psi(.) \notin \mathcal{N}, \text{ i.e. } \| G(\omega) \phi \| < 0, \omega \in (\mathbb{R}, \mathbb{R}), \text{ or the Popov criterion as applied to the nonlinear time-invariant system, i.e. there does not exist a (real)} \]
\( \text{constant } q > 0 \text{ such that } \| G(\omega) \phi \| < 0, \omega \in (\mathbb{R}, \mathbb{R}). \text{ For an application of the new theorems to the examples, we recall that the CPs} \)
\( \psi, \mu_i, \mu_s \text{ of the nonlinearity } \phi(.) \in \mathcal{N} \text{ are defined by} \)
\[ \text{Equation (7)}; \text{ the CPs } \gamma_1 \text{ and } \gamma_s \text{ for } \psi(.) \in \mathcal{M} \text{ are defined by}\]
\[ \text{Equation (8)} \text{; and } K \text{ denotes the upper limit of the time-varying gain } k(t) \text{ in the case of Equation (1), and of the constant gain } K \text{ in Equation (2). Here, we implicitly assume, without loss of generality, that } 0 < \psi(\sigma) \sigma \leq \sigma^2 \text{ for all } \sigma \neq 0. \]

**Example 1** In the sixth-order system of O’Shea (1967) (pp. 726–727), the \( k_2 \) (in O’Shea’s notation) corresponds to the sector limit of the nonlinearity. Since we assume gain-transformed system in our theorems, the function
\[ G_1(s) = \frac{1}{K_2} = \frac{(s + 0.05)(s + 0.1)(s + 1000)^2}{(s + 0.0001)(s + 2)(s + 50)} \]
\[ (34) \]
(where we have used subscript 1 for the transfer function of the forward block to distinguish it from our use of the notation) corresponds to our \( G(s) \). We choose a multiplier function of the form
\[ Z(j \omega) = 1 + j \alpha \phi_0 \omega + \frac{\alpha_1}{(j \omega + \beta_1)} + \frac{\alpha_2}{(j \omega + \beta_2)}, \]
\[ (35) \]
where the parameters \( \alpha_0, \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are to be computed subject to the constraint that the real part condition \([H-1]\) of Theorems 1 and 2 is to be satisfied. O’Shea’s multiplier function corresponds to \( \alpha_0 = 0.01927; \alpha_1 = 1.8992; \beta_1 = -2.0; \alpha_2 = -0.000247; \beta_2 = 0.05. \text{ This set can be slightly improved – by way of (subsequently) relaxing the constraints on the CPs of } \psi(.) \text{ or/and on the normalized rate of variation of } k(t) \text{ when applied to system (1), but without attempting an explicit cancellation of poles and zeros – to } \alpha_0 = 0.01927; \alpha_1 = 1.684; \beta_1 = -2.0; \alpha_2 = -0.00025; \beta_2 = 0.05. \text{ It is found that an entirely different and improved set of parameters is } \alpha_0 = 0.0049; \alpha_1 = 1.576; \beta_1 = 1.9; \alpha_2 = 1.0; \beta_2 = 0.0. \]

**Application of Theorem 1:** If we use O’Shea’s multiplier function, we can conclude that the system (2) with \( \psi(.) \notin \mathcal{N} \text{ is } L_2-\text{stable if, from condition } [H-2], \text{ the inequality} \]
\( \mu_s < 0.5005 \text{ is satisfied. The conclusions of using the slightly improved O’Shea’s multiplier are almost the same. In contrast, if we use the new parameters, condition } [H-2] \text{ leads to the inequality } \mu_s < 0.6028 \text{ which is an improvement over the earlier inequality. A contribution of Theorem 1 is that the nonlinear time-invariant system (2), while being absolutely stable, according to OShea, for} \]
\( \psi(.) \in \mathcal{M}, \text{ is, in fact, stable for } \psi(.) \notin \mathcal{N}, \text{ if the CP } \mu_s \text{ of the nonlinearity obeys the inequality given above.} \)

**Application of Theorem 2:** From the same O’Shea’s multiplier function, we can conclude that the system (1) with \( \psi(.) \in \mathcal{N} \) is (exponentially) \( L_2-\text{stable if there exist nonnegative constants } \xi \text{ and } \xi \text{ such that} \]
\( \text{condition } [H-2] \text{ is satisfied; and with CPs } \psi \text{ and } \mu_s \text{ defined by Equation (7) and } 0 \leq \xi < 0.005 \text{ and } 0 \leq \xi < 2, \text{ the inequality } (0.01927 \nu_\xi + 0.999 \mu_s + \mu_s (0.0002477 \int_0^\infty e^{(\xi - 0.005)t} \ dt + 1.899 \int_0^\infty e^{(\xi - 0.005)t} \ dt) < 1, \)
obtained from condition [H-3], is satisfied. Evidently, we can strike a trade-off among $\xi, \zeta, \nu$, and $\mu_s$ to satisfy the last inequality. For instance, suppose we set $\xi = 0.003$ and $\zeta = 1.5$, then the inequality becomes $(0.00006v_1 + 4.87943L_s) < 1$. In other words, the system (1) is (exponentially) $L_2$-stable if (i) the nonlinearity with $\phi(\cdot) \in \mathcal{N}$ has CPs $v_1$ and $\mu_s$, which together satisfy the last inequality; and, with $\xi = 0.003$ and $\zeta = 1.5$, (ii) the positive lobes $\theta^+(t)$ and negative lobes $\theta^-(t)$ of the normalized rate of variation of the time-varying gain $k(t)$ satisfy the inequalities (23), (24), and/or (25), depending on the nature of $k(t)$. The optimized parameters of O’Shea’s multiplier function improve the above conclusions marginally.

In contrast, the consequences of using the new set of parameters are as follows. With $\xi \geq 0$ and $0 \leq \zeta < 1.9$, the inequality obtained from condition [H-3] is $(0.0049v_1\xi + \mu_s(0.8295 + 1.576) + \mu_s(0.00015v_1 + 4.7695\mu_s) < 1)$. Suppose we set $\xi = 0.003$ and $\zeta = 1.5$ as before, then the last inequality becomes $(0.000015v_1 + 4.7695\mu_s) < 1$, which allows for a larger sub-class of nonlinearities in $\mathcal{N}$ than the inequality obtained above for O’Shea’s multiplier function. Note that the constraints on $\theta^+(t)$ and $\theta^-(t)$ of the normalized rate of variation of $k(t)$ are the same as with the use of the O’Shea multiplier function (because the values of $\xi$ and $\zeta$ have been retained).

Note that Corollary T2-1 (meant for the case of a periodic switching of nonlinearities) of Theorem 2 cannot be applied to the present problem because the multiplier function does not have the periodic structure of Equation (30).

**Example 2** O’Shea (1967) also considers the example of Dewey and Jury (1966) with the transfer function $G_1(s) = 40/(s(s + 1)(s^2 + 0.8s + 16))$. The time-invariant system (2) with $\phi(\sigma) \equiv \sigma$ with the above $G_1(s)$ is asymptotically stable for $\epsilon \geq 6.7$ where constant $\epsilon > 0$. For an application of our theorems, we deal with the gain-transformed

$$G(s) \doteq \left( G_1(s) + \left( \frac{1}{1.76} \right) \right). \quad (36)$$

According to O’Shea (1967), the multiplier function

$$Z(j\omega) = 1 + j10^{-14}\omega + j0.999 \sin (1.1118\omega) \quad (37)$$

satisfies the real-part condition as well as the time-domain constraint on it as required by his Theorem 2 (on p. 725 of the quoted reference) for the absolute stability of the system (2) with odd-monotone (i.e. class $\mathcal{M}_o$) nonlinearities having slopes in the sector $(\epsilon, 1.76)$, where $\epsilon$ is any positive real number.

In contrast, from condition [H-2] of our Theorem 1, the system is $L_2$-stable, in fact, for $\phi(\cdot) \in \mathcal{N}$, if its CPs $\mu_i$ and $\mu_s$ satisfy the inequality $(\mu_i + \mu_s) < 1.001$. When we use this multiplier in our Theorem 2, meant for the time-varying system (1), we need to satisfy the following inequality: $10^{-14}v_1\xi + 0.4995(\mu_i\varepsilon^1 + \mu_s\varepsilon^2) < 0.001,$

on the basis of which any desired trade-off can be struck among (i) the CPs $\nu_1, \mu_i, \mu_s$ of $\phi(\cdot) \in \mathcal{N}$; and (ii) the (global) upper bound $\xi$ on the positive lobes $\theta^+(t)$ and (global) lower bound $\zeta$ on the negative lobes $\theta^-(t)$ of the normalized rate of variation of $k(t)$. In the application of our Theorems 1 and 2, note that there is no explicit bound on the slope of the nonlinearity. Note further that Corollary T2-1 of Theorem 2 cannot be used because the term $j10^{-14}\omega$ in the multiplier function affects the periodic structure of the multiplier function as required by Equation (30). Hence, nonlinear switching cannot be handled by this multiplier.

Interestingly, the following multiplier functions without the $(j\omega)$ term also satisfy not only the real-part condition [H-1] of Theorem 2, but also the periodicity requirement of its Corollary T2-1:

$$Z_1(j\omega) = 1 + j1.109 \sin (1.1\omega); \quad (38)$$

$$Z_2(j\omega) = 1 + j1.62 \sin (1.082\omega); \quad \text{and} \quad (39)$$

$$Z_3(j\omega) = 1 + j2.6 \sin (1.067\omega). \quad (40)$$

We can use these three multipliers separately in applying Theorem 2 to the system (1) with $\phi(\cdot) \in \mathcal{N}$ and $k(t)$ periodic with the fundamental period $p$. Note that such an application corresponds to a special case of Corollary T2-1 (meant for the case of a periodic switching of nonlinearities), since we consider only one of each $k(t)$ and $\phi(\cdot)$. As a consequence, we find that the system (1) is $L_2$-stable if (i) $p = 1.1$, and $(\mu_i + \mu_s) < 0.9091$; or (ii) $p = 1.082$, and $(\mu_i + \mu_s) < 0.6713$; or (iii) $p = 1.067$, and $(\mu_i + \mu_s) < 0.3846$. A by-product of the results is that the smaller the period of $k(t)$, the more severe is the constraint on the CPs of the nonlinearity. There are no restrictions on the rate of variation of $k(t)$. To apply Corollary T2-1 in full, let us now assume that, in the system (1), two nonlinearities $\phi_1(\cdot)$ and $\phi_2(\cdot)$, both belonging to class $\mathcal{N}$, switch from one to the other periodically with period $p$, and the associated time-varying gains $k_1(t)$ and $k_2(t)$ also have the same period, then the system (1) is $L_2$-stable, if, in the last three inequalities (above in this paragraph), $\mu_i$ and $\mu_s$ are replaced, respectively, by $\mu_{i,s}$ and $\mu_{s,s}$, defined by Equation (31). It may be further noted that, in the applications of Theorem 2 and Corollary T2-1, no constraints are imposed on the dwell-time characteristics of $k(t)$ or of switching nonlinearities.

**Example 3** In the course of illustrating integral quadratic constraint-based system analysis, which includes absolute stability, Megretski and Rantzer (1997) (pp. 822–824) consider a third-order system with

$$G(s) = \frac{s^2}{(s^3 + 2s^2 + 2s + 1)}, \quad (41)$$

a saturation nonlinearity and a unit-gain element characterized by delay parameter $\theta_0$, as a result of which the
transfer function of the forward block becomes $G(s)e^{-\theta s}$. The authors analyse the absolute stability of the system without and with delay.

The linear system obtained from Equation (2) by setting $\phi(\sigma) \equiv \sigma$ is asymptotically stable for $k \in [0, \infty)$. For nonlinear time-varying systems, the circle criterion gives the upper bound ($\bar{K}$) on the gain as approximately 8.13; and for a successful application of the Popov criterion to the system (2), this gain is approximately 8.9.

In this example of Megretski and Rantzer (1997), the aspect relevant to the application of our theorems is the reference to an odd monotone (i.e. class $\mathcal{M}_o$ of) nonlinearities (of which saturation nonlinearity is an example). In this case, $\bar{K}$ is allowed to be arbitrarily large. To establish the absolute stability of the system without delay, they suggest the use of an OZF multiplier function of the form

$$Z(j\omega) = 1 + \frac{\alpha}{(\beta - j\omega)}.$$  \hspace{1cm} (42)

It is found (from our experimental work) that $\alpha = -1.3704$ and $\beta = 1.4$. The assumption of an odd monotone nonlinearity is crucial to an application of either Theorem 2 of O’Shea (1967) (p. 725) (or, equivalently, the corresponding theorem in Zames and Falb (1968)). Furthermore, when there is non-zero delay, there is a need to arrive at the Routh–Hurwitz limit for the feedback gain of the system (2) after setting $\phi(\sigma) \equiv \sigma$.

Our stability Theorems 1 and 2 do not restrict the sign of the impulse response function of the multiplier; and the same multiplier is applicable to the nonlinear class $\mathcal{N}$ for both the systems (2) and (1). In this particular case, the consequences of applying our theorems are as follows. From Theorem 1, the system (2) with $\phi(\cdot) \in \mathcal{N}$ is $L_{2\text{-}}$stable, if $\mu_s < 0.5108$. From Theorem 2, the system (1) with $\phi(\cdot) \in \mathcal{N}$ is $L_{2\text{-}}$stable, if, for $0 \leq \xi < 1.4$, the inequality $(0.7143 + 1/(1.4 - \xi))\mu_s < 1$ is satisfied. In the latter case, we can trade-off between $\mu_s$ and $\xi$. Suppose we set $\xi = 1$, then the last inequality reduces to $\mu_s < 0.3111$. Since $\xi$ can be allowed to be arbitrarily large, there is no upper bound on the positive lobes $\theta^+(t)$ of the normalized rate of variation of the time-varying gain $k(t)$. On the other hand, since $\xi = 1$, the (global) lower bound on the negative lobes $\theta^-(t)$ of the normalized rate of variation of the time-varying gain $k(t)$ must satisfy the inequalities (23), (24), and/or (25), depending on the nature of $k(t)$.

For the same problem, typically and without any attempt to optimize the parameters of the multiplier function, we find the following: (i) by setting $\bar{K} = 11.4$, the multiplier function

$$Z_1(j\omega) = (1 - j0.199 \sin 3.57\omega);$$ \hspace{1cm} (43)

and (ii) by setting $\bar{K} = 16.0$, the multiplier function

$$Z_2(j\omega) = (1 - j0.46 \sin 2.7\omega),$$ \hspace{1cm} (44)

satisfy separately the condition [H-1] of our Theorem 2 (as also of Theorem 1) with $G(j\omega)$ replaced by $(G(j\omega) + 1/\bar{K})$. The corresponding stability results are as follows. With $\phi(\cdot) \in \mathcal{N}$ and $k(t)$ periodic with the fundamental period $p$, the system (1) is $L_{2\text{-}}$stable if (i) for $\bar{K} = 11.4$, $p = 3.57$, and $(\mu_s + \mu_\theta) < 0.0251$; and (ii) for $\bar{K} = 16.0$, $p = 2.7$, and $(\mu_s + \mu_\theta) < 2.1739$. From these results, it is evident that, for the $L_{2\text{-}}$stability of Equation (1), the larger the upper bound $\bar{K}$ is, the smaller will be the period of $k(t)$ and the more severe the constraint on the CPs of the nonlinearity.

It is found that, when there is delay in the system, $\bar{K}$ is to be restricted. Even with such a restriction, designing a suitable multiplier is quite complicated. A typical result is as follows. With $\theta_0 = 0.2$ and $\bar{K} = 8.8$, the Popov-multiplier function $Z(j\omega) = 1 + j0.0375\omega$ satisfies condition [H-1] of Theorem 2, in which we replace $G(j\omega)$ by $e^{-\theta_0 \omega}G(j\omega)$. Therefore, the system (1) with $G(s)$ replaced by $e^{-\theta_0 s}G(s)$ and with $\phi(\cdot) \in \mathcal{N}$ is (exponentially) $L_{2\text{-}}$stable if (from condition H-3) the inequality $0.0375(\nu_\xi^2) < 1$, or $\nu_\xi^2 < 26.6667$, is satisfied, in which we can strike a trade-off between $\xi$ and $\nu_\xi$. (Note that $\xi$ can assume arbitrarily large values.) Suppose we set $\xi = 50$, then the CP $\nu_\xi < 0.5333$. The (global) upper bound on the positive lobes $\theta^+(t)$ of the normalized rate of variation of the time-varying gain $k(t)$ must satisfy the inequalities (23), (24), and/or (25), depending on the nature of $k(t)$, with $\xi = 50$. Note that Corollary T2-1 of Theorem 2 cannot be considered for the system (1) with switching nonlinearities, because the multiplier does not have the structure of Equation (29).

Example 4 The fourth-order system considered by Brockett and Willems (1965) (Part 2, p. 410) has the transfer function

$$G(s) = \frac{(10s + 1)(2s + 1)}{(s^2 + 20s + 400)(s^2 + 5s + 4)},$$ \hspace{1cm} (45)

(which is Equation (22) in the same reference). Brockett and Willems (1965) conclude that the time-invariant nonlinear feedback system (2) with $\phi(\cdot) \in \mathcal{M}_o$ is absolutely stable. However, it is found that there does exist the Popov-multiplier function $(1 + 0.0401s)$ using which the Popov criterion $\Re((1 + j0.0401\omega)G(j\omega)) > 0$, $\omega < \infty$ is verified. Therefore, the time-invariant nonlinear feedback system is absolutely stable for $\phi(\cdot) \in \mathcal{N}$. In other words, there is no need to use a multiplier function meant for monotone nonlinearities as found in Brockett and Willems (1965) (Part 2).

With the above choice of the Popov-multiplier function, we apply our Theorem 2 to the time-varying nonlinear feedback system with $\phi(\cdot) \in \mathcal{N}$. Note that (i) the condition [H-1] of Theorem 2 is satisfied for an arbitrarily small $\varepsilon > 0$; (ii) and the multiplier function $Z(j\omega)$ defined by Equation (6) now has only one term, namely $j\xi\omega$. As a
Table 2. Application of Theorem 1. See text for details.

| Example No. | G(s) | Multiplier function | Multiplier parameters | CPs of \( \varphi(\cdot) \) | Constraints on CPs |
|-------------|------|---------------------|-----------------------|------------------|------------------|
| 1           | (34) | (35)                | \((\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2)\) | \(\mu_s\) | – |
|             |      |                     | (0.01927, 1.8992, -2.0, 0.000247, 0.005) (O'Shea) | \(\mu_s\) | \(\mu_s < 0.5005\) |
|             |      |                     | (0.01927, 1.684, -2.0, 0.00025, 0.005) (Improved) | \(\mu_s\) | \(\mu_s < 0.5006\) |
|             |      |                     | (0.0049, 1.576, -1.9, -0.0, 0.0) (New) | \(\mu_s\) | \(\mu_s < 0.6028\) |
| 2           | (36) | (37)                | \((\alpha, \beta)\) | \(\mu_s\) | \(\mu_s < 1\) |
| 3           | (41) | (42)                | \((-1.3704, 1.4)\) | \(\mu_s\) | \(\mu_s < 0.5108\) |

Table 3. Application of Theorem 2. See Table 2 for the \(G(s)\) and corresponding multiplier functions, and the text for details.

| Example no. | A typical choice of \((\xi, \zeta)\) | Constraint on CPs of \(\varphi(\cdot)\) | Constraints\(^a\) on \((\theta^+(t), \theta^-(t))\) |
|-------------|--------------------------------------|------------------------------------------|------------------------------------------|
| 1           | \(\xi = 0.003, \zeta = 1.5\) | \(0.00006v_\xi + 4.87943\mu_s < 1\) | (23) and (24), or (25) and (26) and (27) Same as above |
|             | Same as above | \(0.000052v_\xi + 4.69\mu_s < 1\) | Same as above Same as above |
| 2           | \(\xi, \zeta\) | \(10^{-14}v_\xi + 0.4995(\mu_s e^\xi + \mu_s e^\xi) < 1\) | (23) and (24), or (25) \(\theta^+(t) < \infty; \theta^-(t)\) obeys |
| 3           | \(\zeta = 1; 0 \leq \xi < \infty\) | \(\mu_s < 0.3111\) | (23) and (24), or (25) with \(\zeta = 1\). |

\(^a\) See text for details.

Consequence, in the condition [H-3] of Theorem 2, the summation involving \(z_+(\tau)\) and \(z_-(\tau)\) vanishes, and \(\xi\) can be allowed to be arbitrarily large. Therefore, the system (1) is (exponentially) \(L_2\)-stable if (from condition H-3) the inequality \(0.0401(v_\xi e^\xi) < 0.03, v_\xi < 24.9376\), is satisfied. Now invoking Corollary A17 of Theorem 2, we can strike a trade-off between the (global) bound \(\xi\) on the positive lobes \(\theta^+(t)\) of the normalized rate of variation of \(k(t)\) and the CP \(v_\xi\) of the nonlinearity \(\varphi(\cdot) \in N\) for the (exponential) \(L_2\)-stability of the system (1). Suppose we set \(\xi = 25\), then the CP \(v_\xi < 0.9975\). Note that Corollary T2-1 of Theorem 2 cannot be considered for the system (1) with switching nonlinearities, because the multiplier does not have the structure of Equation (29).

Tables 2-4 contain a summary of the computational results obtained from the above Examples 5.1–5.3 for the \(L_2\)-stability of:

(i) system (2), using Theorem 1 and with \(\varphi(\cdot) \in N\) whose CPs are defined by Equation (7); and

(ii) system (1), using Theorem 2 with the same \(\varphi(\cdot)\) as for system (2), and with \(k(t)\) aperiodic if the multiplier function is aperiodic (in the frequency domain), and \(k(t)\) periodic if the multiplier function is periodic (in the frequency domain).

For Example 5.4 and also for Example 5.3 with delay, the Popov multiplier can be used to satisfy the real-part condition of Theorems 1 and 2, i.e., there is no need for extra terms in the multiplier function. Therefore, these two cases are not included in Tables 2 and 3. Table 4 is meant for an application of Theorem 2 to the special case of a periodic \(k(t)\) with period \(\pi\). See text for details.

Table 4. Application of Theorem 2 to Examples 5.2 and 5.3 for periodic \(k(t)\) with period \(\pi\). See text for details.

| Example no. | Multiplier function | \(p\) | Constraint on CPs of \(\varphi(\cdot)\) |
|-------------|---------------------|------|----------------------------------|
| 2           | (38)                | 1.1  | \((\mu_1 + \mu_3) < 0.9091\) |
|             | (39)                | 1.082| \((\mu_1 + \mu_3) < 0.6713\) |
|             | (40)                | 1.067| \((\mu_1 + \mu_3) < 0.3846\) |
| 3, \(\bar{K} = 11.4\) | (43) | 3.57 | \((\mu_1 + \mu_3) < 5.0251\) |
| 3, \(\bar{K} = 16.0\) | (44) | 2.7  | \((\mu_1 + \mu_3) < 2.1739\) |

6. Comparisons and critique

(1) There seem to be no results in the literature on the stability of nonlinear time-invariant/time-varying feedback systems with \(\varphi(\cdot) \in N\) in which a multiplier distinct from the Popov multiplier \((1 + j\omega)\) has been used.

(2) The \(L_2\)-stability conditions of Venkatesh (1978) (Theorem 1, p. 571) are a special case of Theorem 2. In fact, as applied to the class \(\mathcal{M}\) of monotone nonlinearities, Theorem 2 of the present paper can be considered as an alternative form of Theorem 1 of Venkatesh (1978), in the
sense that the CPs $\gamma_1$ and $\gamma_2$ obtained from Equation (8) are distinct from the CPs $\delta_1$ and $\delta_2$ that appear in Theorem 1 of Venkatesh (1978). The problem of establishing which CPs of the class $\mathcal{M}$ of nonlinearities lead to more relaxed constraints on the (normalized) rate of variation of $k(t)$ seems to be open.

(3) In Huang et al. (2014b), the only theorem that has some tangible relationship with Theorem 2 of the present paper is Theorem 4B on page 25. However, the multiplier function in Theorem 4B of that reference is the counterpart of the Popov multiplier, i.e. it is a special case of Equation (6). A generalization of that theorem, namely, Theorem 4B, to correspond to Theorem 2 of the present paper is an open problem. The other theorems of Huang et al. (2014b) meant for nonlinear MIMO systems apply to nonlinearities which are monotone (and its variations). Even in these cases, the multiplier function is a special case of (the matrix counterpart of) Equation (6). More specifically, the former does not contain the matrix counterparts of the causal and anti-causal functions $z_c(t)$ and $z_a(t)$ of Equation (6).

(4) As far as Example 5.2 above is concerned, if we were to use the matrix counterpart of the O’Shea multiplier in Theorem 4B of Huang et al. (2014b), then the stability theorems are applicable only to those systems in which the vector nonlinearities have the property of path independence for line integrals involving them, and also possess the property of monotonicity (or its variations). However, if we were to use the (matrix) counterparts of the new multipliers, Equations (38)–(40), of Example 5.2 in the theorems of Huang et al. (2014a), then the vector nonlinearities in Huang et al. (2014b) need to belong to the monotone class (or its variations). In other words, the stability conditions of Huang et al. (2014a) cannot be applied to the case when the vector nonlinearities belong to (the vector counterpart of) class $\mathcal{N}$. A generalization of the theorems of Huang et al. (2014a) to hold for the case of vector nonlinearities belonging to class $\mathcal{N}$ is an open problem.

(5) While we can include any type of restriction on nonlinearities in the definition of their CPs, it would be highly desirable to improve Theorems 1 and 2 by invoking more effective CPs of the class $\mathcal{N}$ of nonlinearities such that, when we specialize the theorems to apply to a linear time-invariant system, the bound on the time-domain $L_1$-norm of the multiplier tends to an arbitrarily large value. In effect, the goal is to arrive at the Nyquist criterion as a limiting case of a stability result for nonlinear systems having a nonlinearity belonging to class $\mathcal{N}$. In this context, it is interesting to compare this with the introduction of the class of power-law (monotone) nonlinearities in Brockett and Willems (1965) and its exploitation in Thathachar (1970). Note that the problem posed above is of a different nature.

(6) In the framework used in the paper, we cannot estimate the domain of attraction of the origin. In fact, the relationship between $L_2$-stability and the domain of attraction as studied in the literature (using Lyapunov functions) does not seem to be known. See Hu, Huang, and Lin (2004) for an application of linear matrix inequalities (in a Lyapunov framework) to compute domains of attraction. From another point of view, the implications of Theorem 2 for robust absolute stability need further study. In this context, see, for instance, Liu and Molchanov (2002) in which a Lyapunov framework has been used to derive some robust absolute stability criteria for certain types of time-varying uncertainties and multiple time-varying nonlinearities. Similarly, it would be interesting to study the relationship between Theorems 1 and 2 and the results in, for instance, Wada, Ikeda, Ohta, and Siljak (1998) on parametric absolute stability to deal with parametric uncertainties and input reference values.

(7) To recall, Section 2 and its subsection on switching nonlinearities deal with the $L_2$-stability of the system (1) under the general assumption that the system is asymptotically stable when $\phi(\sigma) \equiv \sigma$ and $k(t) \equiv K \in [0, \infty)$. It is not known how to modify the framework adopted here to deal with the possibility of stabilizing an unstable time-invariant system by switching operations on either the time-varying gain or the nonlinearity (or both).

(8) As mentioned in the introduction, the KYP-lemma proves that Popov’s stability criterion for the system (2) with $\phi(\cdot) \in \mathcal{N}$ is equivalent to the existence of a Lyapunov function comprising a quadratic form and an integral of the nonlinearity. There exist various generalization of this lemma. See, for instance, Iwasaki and Hara (2000). An interesting open problem is to find possible Lyapunov function candidates for Theorem 1, as also their generalization for Theorem 2. Such candidates, if they exist, facilitate computation of finite domains of attraction on the basis of Theorems 1 and 2.

(9) One of the goals of researchers is to find a graphical interpretation (in the frequency domain) of the real-part condition of Theorems 1 and 2. In contrast with the simple, elegant and completely graphical version of Popov’s theorem for time-invariant nonlinear systems, the variations on that theorem (using more general multiplier functions) not only lack simplicity in the frequency domain, but also require satisfaction of time-domain integral inequalities involving the inverse Fourier transforms of the multiplier functions. See Venkatesh (1978) (pp. 573–575) for one of the few attempts in this genre for time-varying nonlinear feedback systems with $\phi(\cdot) \in \mathcal{M}$. A similar procedure can be adopted for $\phi(\cdot) \in \mathcal{N}$, but is more complex. It is possible, however, to convert the real-part condition and the time-domain constraints on the multiplier function of Theorems 1 and 2 to a non-convex optimization problem. Details are omitted due to lack of space.
(10) Since we deal with periodically switching nonlinearities (with periodic gains having the same period) as one of the applications of the main results, the very interesting survey paper of Shorten, Wirth, Mason, Wulff, and King (2007) on the stability of switching and hybrid systems is relevant here. Note, however, that Shorten et al. (2007) do not consider generalization of the Popov theorem to systems with time-varying nonlinearities with \( \varphi(\cdot) \in \mathcal{N} \), and described by integral equations. Zevin and Pinsky (2005) present frequency-domain absolute (asymptotic) stability (and instability) conditions for a system, which is described by a Volterra equation and controlled by a nonlinear sector-restricted feedback having a time-varying delay, to be absolutely (asymptotically) stable (and unstable). The stability conditions are independent of the delay. Interestingly, these authors provide examples of systems satisfying the Aizerman conjecture. However, they do not consider time-varying feedback systems of the type (1). On the other hand, Dehghan and Ong (2012) introduce the concepts of dwell-time invariance and maximal constraint admissible dwell-time-invariant set for discrete-time switching systems under dwell-time switching, and derive a necessary and sufficient condition for asymptotic stability of the origin of the switching systems under dwell-time switching. In contrast, we consider continuous-time systems and dispense with dwell-time considerations for the \( L_2 \)-stability of time-varying nonlinear systems described by Equation (1). See Venkatesh (2014) for more general results (dispensing with dwell-time considerations) for the \( L_2 \)-stability of discrete-time MIMO systems.

(11) Based on the computational experiments that gave typical values of \( \mu_i \) and \( \mu_j \) for specific nonlinearities, as listed in Table 1, it is conjectured that, when nothing is known about the precise structure of \( \varphi(\cdot) \in \mathcal{N} \), the upper limit of both \( \mu_i \) and \( \mu_j \) is 1. A consequence of this is the following conjecture:

**Generalized Popov Theorem: The nonlinear system** (1) **with** \( \varphi(\cdot) \in \mathcal{N} \) **and** \( k(t) \) **replaced by a constant gain** \( K \in [0, \infty) \) **is** \( L_2 \)-stable, if there exists a multiplier function \( Z(j\omega) \) of the form Equation (6) such that [H-1] for some positive constant \( \delta \), \( \Re Z(j\omega)G(j\omega) \geq \delta > 0, \omega \in (-\infty, \infty) \); and [H-2] \( \sum_{m=1}^{\infty} (|z_m| + |z_m'|) + \int_{0}^{\infty} |z_m(\tau)|d\tau + \int_{-\infty}^{0} |z_m(\tau)|d\tau \leq \frac{1}{\delta} \).

Conjectured **Generalization of Theorem 1 of Venkatesh (1978)** (p. 571): The system (1) with \( \varphi(\cdot) \in \mathcal{N} \); \( k(t) \in [\epsilon, \infty) \) for \( t \geq 0 \), where (constant) \( \epsilon > 0 \); and \( \|v_0\|^2 \leq \int_{0}^{\infty} e^{i\omega t}(v(t))^2 d\tau \), where \( \omega > 0 \) is an arbitrarily small number, is exponentially \( L_2 \)-stable in the sense that \( \|v_0\| \leq C_1 \|f\| + \sqrt{C_0 + (C_1^2/4)\|f\|^2} \), where \( C_1, C_2 \) are constants, if there exist a multiplier function \( Z(j\omega) \) defined by Equation (6); a bounded positive function \( \sigma(\cdot) \) as defined above; and nonnegative constants \( \xi, \zeta \), such that [H-1]: (for the \( \bar{f} \) defined above) \( \sup_{-\infty < \omega < \infty} \|Z(j\omega - \epsilon)G(j\omega - \epsilon)\| < \infty \); and \( \Re Z((j\omega - \epsilon)G(j\omega - \epsilon)) > 0, \omega \in (-\infty, \infty) \); [H-2]: \( \sigma(t)e^{-\xi t}k(t) \) is nonincreasing and \( \sigma(t)e^{-\xi t}k(t) \) is nondecreasing for all \( t \in [0, \infty) \); and [H-3]: \( \alpha v_0 + \sum_{m=1}^{\infty} [(1 + e^{i\tau})|z_m| + (1 + e^{-i\tau})|z_m'|] + \int_{0}^{\infty} (1 + e^{-i\tau})|z_m(\tau)|d\tau + \int_{-\infty}^{0} (1 + e^{-i\tau})|z'_m(\tau)|d\tau \leq 1 \), where \( v_0 \) is defined by (7).

### 7. Conclusions

For the \( L_2 \)-stability of time-invariant and time-varying single-input–single-output feedback systems with nonmonotone nonlinearities, we have derived new frequency-domain criteria in terms of the transfer function of the linear time-invariant part and a general multiplier function originally employed for monotone nonlinearities. The results provide a preliminary bridge between the Popov criterion (for first- and third-quadrant nonlinearities) and the results of the literature on monotone and other nonlinearities for time-invariant nonlinear systems. This bridge is established via certain CP of the nonlinearities obtained from new algebraic inequalities. In some sense, the results can be treated as a quantified (and somewhat baroque) improvement of Popov’s criterion whose necessity or otherwise cannot be established from the presented results. Without doubt, Popov’s criterion has an everlasting and apparently impregnable beauty. Examples are given not only to illustrate the theorems, but also to demonstrate their superiority over the existing stability conditions of the literature.

In common with many of the results in the literature, a limitation of the framework used in the paper is that it appears to be impossible to find \( L_2 \)-stabilization conditions in the frequency domain for an unstable transfer function of the forward block or for a feedback system in which the gain is in the unstable Routh–Hurwitz sector. On the other hand, the same framework in an extended form has been used to derive new frequency-domain \( L_2 \)-stability conditions for continuous-time MIMO systems in Huang et al. (2014b), and \( L_2 \)-stability conditions for discrete-time MIMO systems in Venkatesh (2014) for (in both the cases) aperiodic/periodic time-varying and nonlinear feedback gains.

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### Appendix 1. Parseval’s theorem

Suppose $f_1(t)$ and $f_2(t)$ are real-valued functions defined on $[0, \infty)$, and belong to the class of $L_1 \cap L_2$ functions. Then

$$\int_0^\infty f_1(t)f_2(t) \, dt = \frac{1}{2\pi} \int_0^{\infty} F_1(j\omega)F_2(-j\omega) \, d\omega,$$

where $F_1, F_2$ are Fourier transforms of $f_1(t)$ and $f_2(t)$.

### Appendix 2. Proof of Lemma 1

The integral of Equation (9) can be rewritten as

$$\lambda_1(T) = \int_0^T \sigma_\tau(t) \left( \sigma_\tau(t) + a \frac{d\sigma_\tau}{dt} + \sum_{m=1}^{\infty} z_m \sigma_\tau(t - \tau_m) + z_m^2 \sigma_\tau(t + \tau_m') + \int_{-\infty}^{\infty} \tau(t) \sigma_\tau(t - \tau) \, dt \right) \, dt \quad \text{(A1)}$$

We split Equation (A1) into its components and simplify wherever required as follows. Let

$$\lambda_{1-1}(T) = a \int_0^T \left( \frac{d\sigma}{dt} \right) \varphi(\sigma(t)) \, dt = \alpha \int_{\sigma(T_0)}^{\sigma(T)} \varphi(\sigma) \, d\sigma \quad \text{(A2)}$$

where $\varphi(\sigma) = \int_0^\sigma \varphi(\xi) \, d\xi > 0$ for $\sigma \neq 0$.

We assume that interchanges of the operations of summation and integration and of two integrals (one with respect to $\tau$ and the
other with respect to \( t \) in Equation (A1) are valid. Let

\[
\lambda_{1-2}(T) = \int_0^T \sum_{m=1}^\infty \int_0^\infty (z_m^+ - z_m^-) \sigma(t) \psi(\sigma(t)) \, dt
\]

\[
= \sum_{m=1}^\infty \int_0^\infty (z_m^+ - z_m^-) \sigma(t) \psi(\sigma(t)) \, dt,
\]

(A3)

\[
\lambda_{1-3}(T) = \int_0^T \sum_{m=1}^\infty \int_0^\infty (z_m^+ + z_m^-) \psi(\sigma(t)) \, dt
\]

\[
= \sum_{m=1}^\infty \int_0^\infty (z_m^+ + z_m^-) \psi(\sigma(t)) \, dt,
\]

(A4)

\[
\lambda_{1-4}(T) = \int_0^T \sigma(t) \left\{ \int_0^\infty z_c(\tau) \sigma(t - \tau) \, d\tau \right\} \, dt
\]

\[
= \int_0^\infty z_c(\tau) \left\{ \int_0^\infty \psi(\sigma(t)) \sigma(t - \tau) \, d\tau \right\} \, d\tau,
\]

(A5)

\[
\lambda_{1-5}(T) = \int_0^T \sigma(t) \left\{ \int_{-\infty}^0 z_a(\tau) \sigma(t - \tau) \, d\tau \right\} \, dt
\]

\[
= \int_{-\infty}^0 z_a(\tau) \left\{ \int_0^\infty \psi(\sigma(t)) \sigma(t - \tau) \, d\tau \right\} \, d\tau.
\]

(A6)

We invoke Equation (7) defining the CPs of \( \psi(\cdot) \) to reduce Equation (A3) to the following inequality:

\[
\lambda_{1-2}(T) \geq - \sum_{m=1}^\infty (\mu z_m^+ + \mu z_m^-)
\]

\[
\times \int_0^T [\psi(\sigma(t)) \sigma(t) + \psi(\sigma(t) - \tau)] \sigma(t - \tau) \, dt,
\]

(A7)

in which the integral involving the integrand \( \psi(\sigma(t) - \tau) \) can be simplified as follows by changing the variable of integration to \( \eta = (t - \tau) \):

\[
\int_0^T \psi(\sigma(t) - \tau) \sigma(t - \tau) \, d\tau = \int_{-\tau}^{T-\tau} \psi(\sigma(\eta)) \sigma(\eta) \, d\eta,
\]

(A8)

which, on noting that \( \tau = 0 \) for \( \eta < 0 \) and changing the (dummy) variable of integration back to \( t \), becomes

\[
\int_0^T \psi(\sigma(t) - \tau) \sigma(t - \tau) \, d\tau = \int_{-\tau}^{T-\tau} \psi(\sigma(\eta)) \sigma(\eta) \, d\eta.
\]

(A9)

We now recall the property of the nonlinearity that \( \psi(\sigma) \sigma > 0 \), \( \forall \sigma \neq 0 \) to reduce Equation (A9) to the inequality

\[
\int_0^T \psi(\sigma(t) - \tau) \sigma(t - \tau) \, d\tau \leq \int_0^T \psi(\sigma(t)) \sigma(t) \, d\tau.
\]

(A10)

which in combination with Equation (A7) gives

\[
\lambda_{1-2}(T) \geq - \sum_{m=1}^\infty (\mu z_m^+ + \mu z_m^-)
\]

\[
\times \int_0^T \psi(\sigma(t)) \sigma(t) \, d\tau.
\]

(A11)

We now consider \( \lambda_{1-4}(T) \) as defined in Equation (A5), and adopt (with minor obvious modifications) the procedure employed above for \( \lambda_{1-2}(T) \). With respect to the integral with the integrand \( \psi(\sigma(t) - \tau) \sigma(t - \tau) \, d\tau \), we change the variable of integration to \( \eta = (t - \tau) \), and simplify as before (by invoking the property of truncated functions) to arrive at the following inequality:

\[
\lambda_{1-4}(T) \geq - \sum_{m=1}^\infty (\mu z_m^+ + \mu z_m^-)
\]

\[
\times \int_0^T \psi(\sigma(t)) \sigma(t) \, d\tau.
\]

(A13)

Similarly, we can show that

\[
\lambda_{1-5}(T) \geq -2 \int_0^\infty \left( \mu z_a^+(\tau) + \mu z_a^-(\tau) \right) \, d\tau
\]

\[
\times \int_0^T \psi(\sigma(t)) \sigma(t) \, d\tau.
\]

(A14)

Using the inequalities (A11)–(A14) along with Equation (A2) in Equation (A1), we obtain

\[
\lambda_{1}(T) \geq \int_0^T \left\{ 1 - 2 \sum_{m=1}^\infty (\mu z_m^+ + \mu z_m^-) + (\mu z_m^+ + \mu z_m^-) \right\}
\]

\[
\times \int_0^\infty \left( \mu z_a^+(\tau) + \mu z_a^-(\tau) \right) \, d\tau
\]

\[
\times \int_0^T \psi(\sigma(t)) \sigma(t) \, d\tau,
\]

(A15)

from which we conclude that \( \lambda_{1}(T) \geq \alpha (\Phi(\sigma(T)) - \Phi(\sigma(0))) \)

if condition [H-2] of Theorem 1 is satisfied. The lemma is proved.

**Appendix 3. Proof of Lemma 2**

The integral of Equation (22) can be rewritten as

\[
\lambda_2(T) = \int_0^T \psi(t) k(t) \phi(t) \, dt
\]

\[
\times \left\{ \sigma(t) + \alpha \frac{d\sigma(t)}{dt} + \sum_{m=1}^\infty \phi_m \sigma(t - \tau_m) \right\}
\]

\[
+ z_m^+ \sigma(t + \tau_m^+) + \int_{-\infty}^\infty z(\tau) \sigma(t - \tau) \, d\tau \right\} \, dt.
\]

(A16)

Let \( \lambda_2(T) \equiv \lambda_{2-1}(T) + \lambda_{2-2}(T) \), where

\[
\lambda_{2-1}(T) = \int_0^T \sigma(t) k(t) \phi(t) \, dt
\]

\[
\times \left\{ \beta \sigma(t) + \alpha \frac{d\sigma(t)}{dt} + \sum_{m=1}^\infty \phi_m \sigma(t - \tau_m) \right\}
\]

\[
+ \int_0^\infty z_c(\tau) \sigma(t - \tau) \, d\tau \right\} \, dt ; \quad \text{and} \quad (A17)
\]
where positive constants $\beta_1$ and $\beta_2$ are chosen such that $(\beta_1 + \beta_2) = 1$ a $\lambda$ As in the proof of Lemma 1, we assume that interchanges of summation and integration operations; and of two integrals (one with respect to $\tau$ and the other with respect to $t$) are valid. We split Equation (A17) into its components and simplify wherever required as follows:

$$
\lambda_{2-1a}(T) = \beta_1 \int_0^T \sigma(t)k(t)\varphi(\sigma_T(t)) \, dt, \quad \lambda_{2-2a}(T)
$$

$$
\lambda_{2-2a}(T) = \alpha \int_0^T \sigma(t)k(t)\varphi(\sigma_T(t)) e^{-\xi t} \left\{ e^{\xi t} \varphi(\sigma_T(t)) \right\} \, dt.
$$

$$
\lambda_{2-3a}(T) = \sum_{m=1}^{\infty} \int_0^T \sigma(t)k(t)\varphi(\sigma_T(t)) e^{-\xi t} \left\{ e^{\xi t} \varphi(\sigma_T(t)) \right\} \, dt.
$$

$$
\lambda_{2-4a}(T) = \sigma(0)k(0) \int_0^\infty \xi \, \varphi(\sigma_T(t)) \sigma_T(t) \, dt.
$$

The second integral in Equation (A25) can be simplified by integrating by parts and by invoking the definition of the CP $\nu_\xi$ in Equation (7) to obtain

$$
\int_0^T e^{\xi t} \varphi(\sigma_T(t)) \, dt.
$$

As far as integrals (A26) and (A27) are concerned, by changing the variables of integration in them respectively to $u = (t - \tau_m)$ and $u = (\tau - t)$, exploiting the property of (time-)truncated functions, invoking the definition of CPS $\mu_1$ and $\mu_2$ in Equation (7), and recalling the nonnegativity of $\varphi(\sigma_T)$ as was done in the proof of Lemma 1, we get the following inequalities:

$$
\lambda_{2-3a}(T) \geq \sigma(0)k(0) \int_0^\infty \left\{ (1 + e^{\xi t})(\mu_1 \varphi + \mu_2 \varphi) \right\} \, dt.
$$

$$
\lambda_{2-4a}(T) \geq \sigma(0)k(0) \int_0^\infty \left\{ (1 + e^{\xi t})(\mu_1 \varphi + \mu_2 \varphi) \right\} \, dt.
$$

We use Equations (A25) and (A28)–(A31) in Equation (A17) to obtain the inequality

$$
\lambda_{2-1}(T) \geq \alpha \sigma(0)k(0) e^{\xi T}(\Phi(\sigma_T(T)) - \Phi(\sigma_T(0))) + \sigma(0)k(0)
$$

$$
\times \left\{ \beta_1 - \alpha \nu_\xi \right\} \int_0^T e^{\xi t} \varphi(\sigma_T(t)) \sigma_T(t) \, dt.
$$
from which we conclude that \( \lambda_2 - 1(T) > -\alpha \pi(0)k(0)\Phi(\sigma) \) if
\[
\begin{align*}
\alpha v_k + \sum_{m=1}^{\infty} \left( (1 + e^{i\tau}) (\mu M^+ + \mu m^+) \right) + \int_0^\infty (1 + e^{-i\tau}) (\mu L_1^+ (\tau) + \mu L_2^+ (\tau)) \, d\tau < \beta_1. \tag{A33}
\end{align*}
\]

We now consider \( \lambda_2 - 2(T) \) defined by Equation (A18). Since \( \pi(t)k(t)e^{it} \) is nondecreasing, by the (extended) second mean value theorem, there is a point \( T' \) in \([0, T]\) for which the integrals of Equations (A1)–(A3) respectively become
\[
\begin{align*}
\lambda_2 - 1_b(T) &= \beta_2 \pi(T)k(T)e^{iT} \int_{T'}^T e^{-i\tau} \phi(\sigma(T)) \sigma(T) \, d\tau; \tag{A34}
\end{align*}
\]
\[
\begin{align*}
\lambda_2 - 3_b(T) &= \pi(T)k(T)e^{iT} \sum_{m=1}^\infty \left( (1 + e^{i\tau}) (\mu M^m + \mu m^m) \right) \\
&\times \left\{ \int_{T'}^T e^{-i\tau} \phi(\sigma(T)) \sigma(T + t) \, d\tau \right\}; \quad \text{and} \ (A35)
\end{align*}
\]
\[
\begin{align*}
\lambda_2 - 4_b(T) &= \pi(T)k(T)e^{iT} \int_{T'}^T \left( v_m(t) + e^{iT} \phi(\sigma(T)) \sigma(T - t) \, d\tau \right). \tag{A36}
\end{align*}
\]

In the integral of Equation (A35), let \( u = (T + t_m) \), invoke the property of (time-truncated) functions, recall the definitions of the CPs \( \mu_1 \) and \( \mu_2 \) in Equation (7), and follow the line of simplication adopted above for Equation (A26) to get
\[
\begin{align*}
\lambda_2 - 3_b(T) &\geq -\pi(T)k(T)e^{iT} \sum_{m=1}^\infty \left( (1 + e^{i\tau}) (\mu M^m + \mu m^m) \right) \\
&\times \int_{T'}^T e^{-i\tau} \phi(\sigma(T)) \sigma(T) \, d\tau. \tag{A37}
\end{align*}
\]

Similarly, from Equation (A36), by changing the variable of integration in its integral to \( v = (T - t) \), and following a procedure similar to the above, we obtain
\[
\begin{align*}
\lambda_2 - 4_b(T) &\geq -\pi(T)k(T)e^{iT} \left\{ \int_{T'}^T \left( v_m(t) + e^{iT} \phi(\sigma(T)) \sigma(T - t) \, d\tau \right) \right\} \\
&\times \int_{T'}^T \left( v_m(t) + e^{iT} \phi(\sigma(T)) \sigma(T - t) \, d\tau \right). \tag{A38}
\end{align*}
\]

We combine Equations (A34), (A37) and (A38) with Equation (A18) to get the inequality
\[
\begin{align*}
\lambda_2 - 2(T) &\geq \pi(T)k(T)e^{iT} \left\{ \beta_2 - \sum_{m=1}^\infty \left( (1 + e^{i\tau}) (\mu M^m + \mu m^m) \right) \\
- \int_{T'}^T \left( v_m(t) + e^{iT} \phi(\sigma(T)) \sigma(T - t) \, d\tau \right) \right\} \\
&\times \int_{T'}^T \left( v_m(t) + e^{iT} \phi(\sigma(T)) \sigma(T - t) \, d\tau \right). \tag{A39}
\end{align*}
\]

from which we conclude that \( \lambda_2 - 2(T) > 0 \) if
\[
\begin{align*}
\sum_{m=1}^\infty \left( (1 + e^{i\tau}) (\mu M^m + \mu m^m) \right) \\
+ \int_0^\infty (1 + e^{-i\tau}) (\mu L_1^m (\tau) + \mu L_2^m (\tau)) \, d\tau < \beta_1. \tag{A33}
\end{align*}
\]

Since \( \lambda_2(T) = \lambda_2 - 1(T) + \lambda_2 - 2(T) \), from Equations (A33) and (A40) we conclude that
\[
\lambda_2(T) > -\alpha \pi(0)k(0)\Phi(\sigma), \tag{A41}
\]
if conditions [H-2] and [H-3] of Theorem 2 are satisfied. Lemma 2 is proved.

**Proof** Consider the integral, for any \( T > 0 \),
\[
\begin{align*}
\rho_2(T) &\equiv \int_0^T \pi(T)G[v(t)] \, dt, \tag{A42}
\end{align*}
\]
where \( G[v(t)] = \int_0^\infty G(\tau) v(t - \tau) \, d\tau \). It follows from \( f_T(t) = v_T(t) + k(t)\phi(\sigma(t)) \) in Equation (1) that
\[
\begin{align*}
\rho_2(T) &= \int_0^T \pi(t)G[v(t)] \, dt \\
&\quad + \int_0^T \pi(t)k(t)\phi(\sigma(t))G[v(t)] \, dt. \tag{A43}
\end{align*}
\]

From condition [H-1], there exists an \( \varepsilon > 0 \), however small, such that \( \Re [Z(\omega - \varepsilon)] > \delta > 0 \) for \( \omega \in (-\infty, \infty) \). There exists a \( \varepsilon > 0 \) from condition [H-1] of the theorem on the basis of which the first integral on the right-hand side of Equation (A43) is rewritten as follows:
\[
\begin{align*}
\int_0^T \pi(t)G[v(t)] \, dt \\
= \int_0^T \pi(t)e^{-2it}G[v(t)](e^{it}G[v(t)]) \, dt. \tag{A44}
\end{align*}
\]

Now, from the assumptions made on \( \pi(t), e^{-2it}\pi(t) \) is non-increasing. From the second mean value theorem as applied to Equation (A44), there exists a \( T' \in [0, T] \) such that
\[
\begin{align*}
\int_0^T \pi(t)G[v(t)] \, dt \\
= \pi(0) \int_0^T (v_T(t)e^{it})(e^{-it}ZG[v(t)]) \, dt. \tag{A45}
\end{align*}
\]

We let \( V_T(j\omega) \) denote, as before, the Fourier transform of the integral of the time-truncated function \( v_T(t) \), and apply the Parseval theorem to the integral on the right-hand side of Equation (A45). For this process, we note that there is no loss of generality in assuming that the upper limit \( T' \) of the integral can be replaced by \( T \) itself, and \( v_T(t) \) can be set to zero in the interval \( (T', T) \). We get
\[
\begin{align*}
\int_0^T \pi(t)G[v(t)] \, dt \\
= \frac{\pi(0)}{2\pi} \int_{-\infty}^\infty V_T(-j\omega - \varepsilon)Z(j\omega - \varepsilon) \, d\omega. \tag{A46}
\end{align*}
\]

Invoking the condition [H-1] of the theorem, namely, for some \( \delta > 0, \Re [Z(\omega - \varepsilon)] > \delta > 0, \omega \in (-\infty, \infty), \) the
following inequality holds:
\[
\int_0^T \sigma(t) v_T(t) Z[G(v_T(t))] \, dt
\]
\[
= \frac{\sigma(0)}{2\pi} \int_0^\infty Z(jw - \varepsilon)G(jw - \varepsilon) V_T(jw - \varepsilon) \, dw \leq \frac{\sigma(0) \delta}{2\pi} \int_0^T e^{2\pi \varepsilon} \, dt.
\]  
(A47)

By virtue of Lemma 2, the second integral right-hand side of Equation (A43),
\[
\int_0^T \sigma(t) k(t) \psi(\sigma(t)) Z[\sigma(t)] \, dt \geq -\alpha \sigma(0) k(0) \Phi(\sigma(0))
\]
where, to recall, \(\Phi(\sigma) = \int_0^\pi \psi(\xi) \, d\xi\). With (an arbitrarily small) \(\varepsilon > 0\), we rewrite Equation (A42) as follows:
\[
\rho_2(T) = \int_0^T \sigma(t)e^{-\varepsilon t} f_T(t)(e^{\varepsilon t} ZG[v_T(t)]) \, dt.
\]  
(A49)

Based on the assumed property of \(\sigma(t)\), we infer that \(e^{-\varepsilon t} \sigma(t)\) is nonincreasing. Invoking the second mean value theorem in Equation (A48), there exists a \(T' \in [0, T]\) such that
\[
\rho_2(T) = \sigma(0) \int_0^T f_T(t)(e^{\varepsilon t} ZG[v_T(t)]) \, dt.
\]  
(A50)

Noting that there is no loss of generality, as before, in assuming that \(f_T(t) = 0\) and \(v_T(t) = 0\) for \(t \in [T', T]\), we apply the Parseval theorem to Equation (A49) to obtain
\[
\rho_2(T) = \sigma(0) \frac{\int_{-\infty}^\infty F_T(-jw) Z(jw - \varepsilon)G(jw - \varepsilon) V_T(jw - \varepsilon) \, dw}{2\pi}
\]
which can be reduced to an inequality (in steps) as follows:
\[
\rho_2(T) \leq \sigma(0) \frac{1}{2\pi} \sup_{-\infty < \omega < \infty} Z(jw - \varepsilon)G(jw - \varepsilon)
\]
\[
\times \int_{-\infty}^\infty f_T(t)(e^{\varepsilon t} V_T(t)) \, dt
\]
\[
\leq \sigma(0) \frac{1}{2\pi} \sup_{-\infty < \omega < \infty} Z(jw - \varepsilon)G(jw - \varepsilon) \|f_T\|
\]
\[
\times \int_{-\infty}^{\infty} e^{2\pi \varepsilon} \, dt
\]
(A51)

the last step being based on an application of the Cauchy–Schwartz inequality. With the relationship (A43) in mind, we now combine the inequalities (A50), (A47), and (A46) to get
\[
\frac{\sigma(0) \delta}{2\pi} \int_0^T e^{2\pi \varepsilon} \, dt
\]
\[
\leq \sigma(0) k(0) \Phi(\sigma(0)) + \frac{\sigma(0)}{2\pi} \sup_{-\infty < \omega < \infty} Z(jw - \varepsilon)G(jw - \varepsilon) \|f_T\| \int_{-\infty}^{\infty} e^{2\pi \varepsilon} \, dt
\]  
(A53)

Noting that \(\sup_{-\infty < \omega < \infty} |Z(jw - \varepsilon)G(jw - \varepsilon)| \leq C\) is finite by virtue of the assumptions on \(Z(\cdot)\) and \(G(\cdot)\), Equation (A51) can be simplified to give
\[
\int_0^T e^{2\pi \varepsilon} \, dt
\]
\[
\leq \frac{2\pi \alpha}{\delta} k(0) \Phi(\sigma(0)) + \frac{C}{\delta} \|f_T\| \left( \int_0^T e^{2\pi \varepsilon} \, dt \right)^{1/2}
\]
\[
\leq C_0 + C_1 \|f_T\| \left( \int_0^T e^{2\pi \varepsilon} \, dt \right)^{1/2},
\]
(A54)

where \(C_0 \equiv (2\pi \alpha/\delta) k(0) \Phi(\sigma(0))\) and \(C_1 \equiv (C/\delta)\) are finite. From Equation (A52), we get the following inequality:
\[
\int_0^T e^{2\pi \varepsilon} \, dt \leq C_0 + C_1 \|f_T\| \left( \int_0^T e^{2\pi \varepsilon} \, dt \right)^{1/2},
\]
(A55)

which is valid for all \(T > 0\). Since \(C_0\) and \(C_1\) are independent of \(T\), we conclude that \(\|v_T\| \leq C \|f\| + \sqrt{C_0 + C_1^2/\delta} \|f\|^2\), where \(\|v_T\|^2 \equiv \int_0^\infty e^{2\pi \varepsilon} \, dt\).

The theorem is proved.

**Appendix 4. Proof of Corollary L2-1**

The integral of Equation (33) can be rewritten as
\[
\lambda_{2p}(T) = \int_0^T k(t) \psi(\sigma(t)) \psi(\sigma(t)) \right) dt.
\]  
(A56)

As before, we assume that an interchange of the operations of summation and integration in Equation (A55) is valid. In Equation (101), we replace \(k(t) \psi(\sigma(t))\) by the right-hand side of Equation (28), and split \(\lambda_{2p}(T)\) in terms of its components and simplify as follows.

Let
\[
\lambda_{2p-1}(T) \equiv \int_0^T \left( \sum_{m=1}^\infty z_m \sigma(t - mp) k(t) \psi(\sigma(t)) \right) dt
\]
\[
= \sum_{m=1}^\infty (z_m - \overline{z}_m) \int_0^T \sigma(t - mp) k(t) \psi(\sigma(t)) dt
\]
\[
= \sum_{m=1}^\infty (z_m^* - \overline{z}_m^*) \int_0^T \sigma(t - mp) k_1(t) U_1(t) \psi_1(\sigma(t)) dt + k_2(t) U_2(t) \psi_2(\sigma(t)) dt.
\]  
(A57)

In the last integral of the right-hand side of Equation (A56), we change the variable of integration from \(t\) to \(\xi = t - mp\), invoke Equation (31) defining the CPs of \(\psi_1(t)\) and \(\psi_2(t)\), use the periodicity property of \(U_1(t)\) and \(U_2(t)\) along with the properties of the time-truncated function—the last step being similar to what was done in the proof of Lemma 1—and recall that \(k(t) \psi(\sigma(t)) = [k_1(t) \psi_1(\sigma(t)) + k_2(t) \psi_2(\sigma(t))]\) to reduce Equation (A56) to the
following inequality:

$$\lambda_{2p-1}(T) \geq -\sum_{m=1}^{\infty} \left( \mu_{i,m} z_{m}^{-} + \mu_{s,m} z_{m}^{+} \right)$$

$$\times \int_{0}^{T} \left[ k_{1}(t) U_{1}(t) \left( \psi_{1}(\sigma_{T}(t)) \sigma_{T}(t) + \psi_{1}(\sigma_{T}(t-m)) \right) \right] dt + \left( \mu_{i,2} z_{m}^{-} + \mu_{s,2} z_{m}^{+} \right)$$

$$\times \int_{0}^{T} \left[ k_{2}(t) U_{2}(t) \left( \psi_{2}(\sigma_{T}(t)) \sigma_{T}(t) + \psi_{2}(\sigma_{T}(t-m)) \right) \right] dt$$

$$\geq -\sum_{m=1}^{\infty} \left( \mu_{i,s} z_{m}^{+} + \mu_{s,s} z_{m}^{-} \right) \int_{0}^{T} k(t) \psi(\sigma_{T}(t)) \sigma_{T}(t) dt.$$  \tag{A58}

Similarly, let

$$\lambda_{2p-2}(T) = \sum_{m=1}^{\infty} \left( z_{m}^{+} - z_{m}^{-} \right) \int_{0}^{T} \sigma_{T}(t) + \tau_{m} k(t) \psi(\sigma_{T}(t)) dt$$

$$= \sum_{m=1}^{\infty} \left( \sigma_{m}^{+} - \sigma_{m}^{-} \right) \int_{0}^{T} k(t) \psi(\sigma_{T}(t)) \sigma_{T}(t) dt.$$ \tag{A59}

Following the line of simplification and reduction of $\lambda_{2p-1}(T)$ above, we can show that $\lambda_{2p-2}(T)$ obeys the following inequality:

$$\lambda_{2p-2}(T) \geq -2 \sum_{m=1}^{\infty} \left( \mu_{i,s} z_{m}^{+} + \mu_{s,s} z_{m}^{-} \right) \int_{0}^{T} k(t) \psi(\sigma_{T}(t)) \sigma_{T}(t) dt.$$ \tag{A60}

Using the inequalities (A58) and (A59) in (A55), we obtain

$$\lambda_{2p}(T) \geq \left\{ 1 - 2 \sum_{m=1}^{\infty} \left( \mu_{i,s} (z_{m}^{+} + z_{m}^{-}) + \mu_{s,s} (z_{m}^{+} + z_{m}^{-}) \right) \right\} \int_{0}^{T} k(t) \psi(\sigma_{T}(t)) \sigma_{T}(t) dt.$$ \tag{A61}

from which we conclude that $\lambda_{2p}(T) \geq 0$, if condition [H-2] of Corollary T2-1 is satisfied. The lemma is proved.