A NOVEL NONCOMMUTATIVE KDV-TYPE EQUATION, ITS RECURSION OPERATOR, AND SOLITONS.

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Abstract. A noncommutative KdV-type equation is introduced extending the Bäcklund chart in \cite{4}. This equation, called meta-mKdV here, is linked by Cole-Hopf transformations to the two noncommutative versions of the mKdV equations listed in \cite[Theorem 3.6]{22}. For this meta-mKdV, and its mirror counterpart, recursion operators, hierarchies and an explicit solution class are derived.

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1. Introduction

The study of integrable systems in \cite{14} was based on a Bäcklund chart comprising the KdV and modified KdV equations, the KdV singularity manifold equation (also known as UrKdV or Schwarz-KdV \cite{25,26}), the KdV interacting soliton equation \cite{12} and the Harry-Dym equation. In the recent article \cite{4} a noncommutative interpretation of this chart is suggested.

We mention two major features contrasting with the commutative model:

1.) At the place of the usual mKdV, we proceed via two different noncommutative interpretations\footnote{Throughout the text, $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$ denote commutator and anti-commutator of $A$ and $B$, respectively.}. The first is the mKdV

\begin{equation}
V_t = V_{xxx} - 3\{V^2, V_x\},
\end{equation}

see \cite{1} for an early appearance. The second noncommutative version of the mKdV equation, which is abbreviated by alternative mKdV in the sequel (amKdV in \cite{4}),

\begin{equation}
\dot{V}_t = \dot{V}_{xxx} + 3[\dot{V}, \dot{V}_{xx}] - 6\dot{V}\ddot{V}_x\dot{V},
\end{equation}

was first described by Khalilov and Khruslov \cite{17}.

2.) The noncommutative Bäcklund chain in \cite{4} does not include the Harry-Dym equation. The difficulty is that the interpretation of the reciprocal transformation between the KdV interacting soliton and the Harry-Dym equation\footnote{This transformation is an extended hodograph transformation, which intertwines dependent and independent variables.} requires substantially new ideas.
The relation between (1) and (2), as presented in [4], builds on work by Liu and Athorne [19], who showed that the associated scattering problems are related by a gauge transformation. The fundamental observation of the present work is that, on the level of evolution equations, this correspondence can be made more explicit by introducing an intermediate equation

\[ Q_t = Q_{xxx} - 3Q_{xx}Q^{-1}Q_x, \]

which we call meta-mKdV equation, to emphasize that it is somewhat hidden behind (1) and (2). Analogously we obtain the mirror version \(^3\) of (3) by reversing the order of multiplication in the nonlinear term. Those are linked to (1), (2) by various Cole-Hopf transformations. Including also the mirror meta-mKdV, we obtain the symmetric picture in Figure 1.

The extended Bäcklund chart is then used in the further study of the novel equations. More precisely, recursion operators are derived, and it is explained why these operators are hereditary. Note that hereditariness is much harder to verify directly in the noncommutative setting [24]. This leads to hierarchies of commuting symmetries with the Cole-Hopf links extending to each level of the respective hierarchies.

Moreover, we construct an explicit solution class of the meta-mKdV. Applying the Cole-Hopf link towards the mKdV, one recovers solutions which are already obtained in [6] and can be interpreted as noncommutative analogs of 1-soliton solutions. In contrast, the solutions derived by applying the Cole-Hopf link towards the alternative mKdV are new to the best of the authors’ knowledge. The fact that the gauge transformation in [19] becomes completely explicit for these solution classes is used to extend the construction to the entire hierarchies.

The article is organised as follows. In Section 2 it is shown that two different noncommutative interpretations of the Cole-Hopf transformation map solutions of the meta-mKdV to solutions of the noncommutative mKdV and alternative mKdV, respectively. In Section 3 we make the comparison with the commutative case, where the picture is slightly different since the two noncommutative mKdV’s specialise to one equation, the standard mKdV. Building on the present work, the commutative case was further elaborated in [3]. It is interesting that the commutative meta-mKdV was also derived in the recent [2], from a completely different approach.

In Section 4 we give an explicit solution of the nc meta-mKdV, considered as an equation with values in some Banach algebra. As a corollary we obtain solutions of the nc mKdV’s \(^1\) and \(^2\), which can be understood as algebra-valued analogues of the familiar 1-soliton solutions. In Section 5 we use the Cole-Hopf link between \(^1\) and \(^2\) together with a structure theorem of Fokas and Fuchssteiner to derive a recursion operator for the nc meta-mKdV. In a way comparable to earlier work in [6], we then treat the induced hierarchy in Section 6 and extend the solutions from Section 4 to all members of the hierarchy.

In Appendix A we give the computationally involved proof of Theorem 1. Appendix B is a concise introduction to general methods for recursion operators and Bäcklund transformations. In Appendix C the recursion operator of the nc mirror meta-mKdV is derived. Moreover, an alternative derivation of the recursion operators in Theorem 9 is given.

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\(^3\) The term mirror has been used in a comparable way for the noncommutative Burgers equations in [15], see also [6, 8].
2. A novel equation “behind” the two noncommutative mKdV equations

In this section we are concerned with the mathematical justification of Figure 1. Throughout the article, we understand evolution equations as (3) as equations for an unknown function $Q(x, t)$ taking values in a (possibly noncommutative) Banach algebra $A$. If we consider Bäcklund transformations, like the two noncommutative interpretations of the Cole-Hopf transformation

\[
C(Q) = Q_x Q^{-1},
\]

\[
\tilde{C}(Q) = Q^{-1} Q_x,
\]

we will tacitly assume that the Banach algebra is the same for the involved equations.

The following is fundamental for the sequel.

**Theorem 1.** Let $Q$ be a solution of the meta-mKdV (3). Then

a) $V = C(Q)$ is a solution of the mKdV (1),
b) $\tilde{V} = \tilde{C}(Q)$ is a solution of the alternative mKdV (2).

The proof requires longer computations and is postponed to Appendix A.

One gets a more complete picture by including the mirror meta-mKdV equation,

\[
\tilde{Q}_t = \tilde{Q}_{xxx} - 3\tilde{Q}_x \tilde{Q}^{-1} \tilde{Q}_{xx},
\]

in which the factors of the nonlinear term in (3) appear in reversed order.

The next proposition establishes a direct link between the meta-mKdV (3) and the mirror meta-mKdV (4).

**Proposition 2.** An invertible algebra-valued function $Q$ is a solution of (3) if and only if $\tilde{Q} = Q^{-1}$ is a solution of (4).

**Proof.** Let $Q$ be a solution of the meta-mKdV (3), and $\tilde{Q} = Q^{-1}$. Then

\[
\begin{align*}
\tilde{Q}_x &= -Q^{-1} Q_x Q^{-1}, \\
\tilde{Q}_{xx} &= -Q^{-1} Q_{xx} Q^{-1} + 2Q^{-1} Q_x Q^{-1} Q_x Q^{-1} \\
\tilde{Q}_{xxx} &= -Q^{-1} Q_{xxx} Q^{-1} + 3Q^{-1} Q_x Q^{-1} Q_{xx} Q^{-1} - 3Q^{-1} Q_{xx} Q^{-1} Q_x Q^{-1} \\
&\quad - 6Q^{-1} Q_x Q^{-1} Q_x Q^{-1} Q_x Q^{-1}
\end{align*}
\]
This shows $Q(\tilde{Q}_{xxx} - 3\tilde{Q}_{xx}\tilde{Q}^{-1} \tilde{Q}_x)Q = -(Q_{xxx} - 3Q_xQ^{-1}Q_{xx})$, and similarly one verifies $Q\tilde{Q}_t Q = -Q_t$. As a result

$$\tilde{Q}_{xxx} - 3\tilde{Q}_{xx}\tilde{Q}^{-1} \tilde{Q}_x = -Q^{-1}(Q_{xxx} - 3Q_xQ^{-1}Q_{xx})Q^{-1} = -Q^{-1}Q_tQ^{-1} = \tilde{Q}_t,$$

showing that $\tilde{Q}$ solves (8). The proof of the converse is symmetrical. \hfill \square

Now we can prove the counterpart to Theorem 1 for the mirror meta-mKdV (13).

**Proposition 3.** Let $\tilde{Q}$ be a solution of the mirror meta-mKdV (13). Then

a) $V = -\tilde{C}(\tilde{Q})$ is a solution of the mKdV (11),

b) $\tilde{V} = -C(Q)$ is a solution of the alternative mKdV (2).

**Proof.** Let $\tilde{Q}$ be a solution of (13), and set $Q := \tilde{Q}^{-1}$. Then $\tilde{Q}_x = -Q^{-1}Q_xQ^{-1}$, and therefore the relations

$$-C(\tilde{Q}) = -\tilde{Q}_x\tilde{Q}^{-1} = -(Q^{-1}Q_xQ^{-1})Q = Q^{-1}Q_x = C(Q),$$

$$-\tilde{C}(\tilde{Q}) = -\tilde{Q}^{-1}\tilde{Q}_x = -\tilde{Q}^{-1}(-Q^{-1}Q_xQ^{-1}) = Q_xQ^{-1} = C(Q)$$

hold. Furthermore, $Q$ is a solution of (13) by Proposition 2. Now Proposition 3 follows from Theorem 1. \hfill \square

3. The commutative versus the noncommutative Bäcklund chart

Including the results from Section 2 into the noncommutative Bäcklund chart from [13], we arrive at the chart depicted in Figure 2. This chart starts with the KdV equation

$$U_t = U_{xxx} + 3\{U, U_x\},$$

and proceeds via (1), (3) and (2) – or (1), (6) and (2) – to the noncommutative counterparts of the KdV interacting soliton equation

$$S_t = S_{xxx} - 3S_S^{-1}S_x,$$

and of the KdV singularity manifold equation

$$\phi_t = \phi_x\{\phi; x\},$$

where $\{\phi; x\}$ denotes the noncommutative Schwarzian derivative

$$\{\phi; x\} = (\phi^{-1}\phi_{xx})_x - \frac{1}{2}(\phi^{-1}\phi_{xx})^2.$$

The connecting Bäcklund links are given by

a) $U = -(V^2 + V_x)$,
b) $\tilde{V} = \frac{1}{2}S^{-1}S_x$, c) $S = \phi_x$.

Note that (a) is the noncommutative Miura transformation; the links (d), (\tilde{d}) stand for the Cole-Hopf transformation (11) and its mirror [5]. Compare also Theorem 1 and Proposition 3.

Since in the commutative case mKdV (11) and alternative mKdV (2) both reduce to the usual mKdV equation $v_t = v_{xxx} - 6v^2v_x$, it is instructive to spell out how the meta-mKdV

$$q_t = q_{xxx} - 3q_xq_{xx}/q$$

fits into the Bäcklund chart in [14].

By Theorem 1 the Cole-Hopf transformation $v = q_x/q$ maps solutions of (10) to solutions of the mKdV. Moreover one sees that $s = q^2$ transfers solutions of (10) to solutions of the KdV.
interacting soliton equation (and vice versa as long as $\sqrt{s}$ is defined). This means that we can split the Bäcklund link (b) into two parts, namely

\[(b1) \quad v = \frac{q_x}{q}, \quad (b2) \quad s = q^2.\]

Observe that

\[\frac{1}{2} s_x = \frac{1}{2} (\frac{q_x}{q})_x = \frac{q_x}{q} = v.\]

Hence we get the extension of the Bäcklund chart in [14] as depicted in Figure 3.

**Figure 2.** KdV-type equations and their Bäcklund links: the non-commutative case.

Building on the present work, the commutative case is further elaborated in [3]. For an occurrence of (10) in a completely different context, we refer to [2].

### 4. Noncommutative analogs of solitons

In the present section we obtain an explicit solution of the meta-mKdV (3) depending on two parameters in $\mathcal{A}$, the Banach algebra where solutions take values. Using Theorem 1 this solution is first transferred to the mKdV (1), giving new access to the soliton found in [9], see also [6] for the mKdV hierarchy. Secondly, we obtain the soliton for the alternative mKdV (2).

**Proposition 4.** Let $A, B \in \mathcal{A}$ with $A$ invertible, and let $L(x,t) = \exp(Ax + A^3t)B$. Then a solution of the meta-mKdV equation (3) is given by

\[(11) \quad Q = (I - L)^{-1}A^{-1}(I + L)\]

on $\Omega = \{(x, t) \in \mathbb{R}^2 \mid I \pm L(x, t) \text{ are invertible}\}$. 

**Figure 3.** KdV-type equations and their Bäcklund links: the commutative case.
We will first show that
\begin{equation}
\tilde{Q} = (I + L)^{-1}A(I - L),
\end{equation}
which is the inverse of (11), solves the mirror meta-mKdV (6), and then use the link between meta-mKdV and its mirror in Proposition 2. To this end, we need two ingredients. First we observe

**Lemma 5.** The function (12) satisfies the equation \( \tilde{Q}_{xx} = \tilde{Q}\tilde{Q}_x \).

**Proof.** Straightforward computations using the noncommutative product and derivation rules for inverses give
\begin{equation}
\tilde{Q}_x = ((I + L)^{-1})_x A(I - L) + (I + L)^{-1}A(I - L)_x
= -(I + L)^{-1}L_x(I + L)^{-1} A(I - L) + (I + L)^{-1}A(-L_x)
= -(I + L)^{-1}AL(I + L)^{-1}A(I - L) + (I + L)^{-1}A(-AL)
= -(I + L)^{-1}A(I + L)^{-1}(LA(I - L) + (I + L)AL)
\end{equation}
and then
\begin{equation}
\tilde{Q}_{xx} = (I + L)^{-1}AL(I + L)^{-1}A(I + L)^{-1}(AL + LA)
\end{equation}
\begin{equation}
+ (I + L)^{-1}A(I + L)^{-1}AL(I + L)^{-1}(AL + LA)
- (I + L)^{-1}A(I + L)^{-1}A(AL + LA)
= (I + L)^{-1}A(I + L)^{-1}(LA + AL - A(I + L))(I + L)^{-1}(AL + LA)
= -(I + L)^{-1}A(I + L)^{-1} \cdot (I - L)A \cdot (I + L)^{-1}(AL + LA)
= -(I + L)^{-1}A(I - L) \cdot (I + L)^{-1}A(I + L)^{-1}(AL + LA)
= \tilde{Q}\tilde{Q}_x,
\end{equation}
which is the claim.

The second ingredient is the close relation between (12) and the soliton
\begin{equation}
W = (I + L)^{-1}(AL + LA)
\end{equation}
of the potential KdV equation
\begin{equation}
W_t = W_{xxx} + 3W_x^2
\end{equation}
derived in [6, Corollary 9]. More precisely, the following lemma holds.

**Lemma 6.** For \( \tilde{Q} , W \) as in (12), (14), we have \( \tilde{Q} = A - W \).

We are now in the position to prove Proposition 4.

**Proof of Proposition 4.** Note that Lemma 5 implies that (12) satisfies the potential KdV (15) with a minus sign in front of the nonlinear term, \( \tilde{Q}_t = \tilde{Q}_{xxx} - 3\tilde{Q}_x^2 \). To prove that \( \tilde{Q} \) solves the mirror meta-mKdV (6), it hence remains to rewrite the nonlinear term \( \tilde{Q}_x^2 = \tilde{Q}_x\tilde{Q}^{-1}\tilde{Q}_{xx} \) using Lemma 5. Finally, since \( Q = \tilde{Q}^{-1} \), Proposition 4 follows from Proposition 2.

Using the Cole-Hopf transformations (4), (5) and Theorem 10 it is straightforward to derive solutions of the mKdV (11) and the alternative mKdV (2) from the solution in Proposition 4.

**Corollary 7.** Let \( A, B \in A \) with \( A \) invertible, and let \( L(x, t) = \exp(Ax + A^4t)B \). Then
\begin{enumerate}[(a)]
\item \( V = (I - L^2)^{-1}(AL + LA) \) is a solution of the mKdV (11).
\end{enumerate}
Proof. Let \( \tilde{Q} \) be given as in (12). Starting with the expression for \( \tilde{Q}_x \) derived in (13), cf. the proof of Lemma 5, we have

\[
\tilde{Q}_x = -(I + L)^{-1}A(I + L)^{-1}(AL + LA)
\]

On the other hand,

\[
\tilde{Q}_x = -(I + L)^{-1}A(I + L)^{-1}(AL + LA)Q^{-1} \cdot Q
\]

The proof is complete upon use of Proposition 3. \qed

5. Recursion Operators

The aim of this section is to derive strong symmetries [11] (or recursion operators in the sense of [21]) of the meta-mKdV [3] and its mirror [4]. A short summary of the related terminology is given in Appendix B. For more background information we refer to [10, 11], see also [24].

Recall that the recursion operator

\[
\Psi(V) = (D - C_V D^{-1} C_V)(D - A_V D^{-1} A_V)
\]

of the mKdV (1) and the recursion operator

\[
\tilde{\Psi}(\tilde{V}) = (D + 2C_{\tilde{V}})(D - 2R_{\tilde{V}})(D + C_{\tilde{V}})^{-1}(D + 2L_{\tilde{V}})D(D + C_{\tilde{V}})^{-1}
\]

of the alternative mKdV (2) were derived in (16), see also (10). In (17) occurs the derivation \( \tilde{D} := D + C_{\tilde{V}} \).

As usual, \( D \) stands for derivation with respect to \( x \), \( C_U \) and \( A_U \) denote the commutator and anti-commutator with respect to \( U \), and we will use \( L_U \), \( R_U \) for left- and right multiplication with \( U \), respectively.

Using a structure theorem by Fokas and Fuchssteiner [10] about transferring strong symmetries along Bäcklund transformations, we deduce the meta-mKdV recursion operator [3] from the Cole Hopf link in Theorem 1. Consider

\[
B(Q, V) = VQ - Q_x,
\]

which links the mKdV (1) and the meta-mKdV (3) according to Theorem 1. Computing the Fréchet derivatives

\[
B_Q[\tilde{Q}] = \frac{\partial}{\partial \epsilon}|_{\epsilon=0}(V(Q + \epsilon\tilde{Q}) - (Q + \epsilon\tilde{Q})_x) = V\tilde{Q} - \tilde{Q}_x = -(D - L_V)\tilde{Q},
\]

\[
B_V[\tilde{V}] = \frac{\partial}{\partial \epsilon}|_{\epsilon=0}((V + \epsilon\tilde{V})Q - Q_x) = \tilde{V}Q = R_Q \tilde{V},
\]

we obtain the transformation operator

\[
\Pi = B_Q^{-1}B_V = -(D - L_V)^{-1}R_Q.
\]

We will need another representation of \( \Pi \).

Lemma 8. If \( B(Q, V) = 0 \), then the identity \( (D - L_V)^{-1}R_Q = R_Q(D - C_V)^{-1} \) holds.

\( \text{That} \ D \ (\text{or} \ \tilde{D}) \ (\text{is} \ \text{a} \ \text{derivation} \ \text{means} \ \text{that} \ \tilde{D}(UV) = \tilde{D}(U)V + U\tilde{D}(V)). \)
Proof. Using the product rule $DR_Q = R_Q D + R_Q$, and the fact that left- and right multiplication commute, we get

$$(D - L_V)R_Q = DR_Q - LV R_Q = R_Q D + R_Q L_V = R_Q(D - L_V) + R_Q.$$  

Since $B(Q, V) = 0$, we have $Q_x = V Q$ and therefore $R_Q = R_V Q R_V$. This shows

$$(D - L_V)R_Q = R_Q(D - L_V) + R_Q R_V = R_Q(D - C_V),$$

which implies the lemma.

By the structure theorem,

$$\Phi(Q) = \Pi \Psi(V)\Pi^{-1}$$

$$= (D - L_V)^{-1} R_Q (D - C_V D^{-1} C_V)(D - A_V D^{-1} A_V) R_Q^{-1} (D - L_V)$$

$$= R_Q(D - C_V)^{-1} (D - C_V D^{-1} C_V)(D - A_V D^{-1} A_V) (D - C_V) R_Q^{-1}$$

$$= R_Q D^{-1}(D + C_V)(D - A_V D^{-1} A_V)(D - C_V) R_Q^{-1}$$

$$= R_Q D^{-1}(D + C_{Q, Q^{-1}})(D - A_{Q, Q^{-1}} D^{-1} A_{Q, Q^{-1}})(D - C_{Q, Q^{-1}} R_Q^{-1}).$$

This is a recursion operator of the meta-mKdV \(^3\). In the calculation above we have used Lemma \(12\) and the factorisation $(D - C_V D^{-1} C_V) = (D - C_V) D^{-1}(D + C_V)$.

This implies part a) of the following theorem. For a verification of part b) see Appendix \(12\).

**Theorem 9.** For the operators

$$\Phi(Q) = R_Q D^{-1}(D + C_{Q, Q^{-1}})(D - A_{Q, Q^{-1}} D^{-1} A_{Q, Q^{-1}})(D - C_{Q, Q^{-1}}) R_Q^{-1},$$

$$\hat{\Phi}(\hat{Q}) = L_{\hat{Q}} D^{-1}(D - C_{\hat{Q}, \hat{Q}^{-1}})(D - A_{\hat{Q}, \hat{Q}^{-1}} \hat{Q}^{-1} D^{-1} A_{\hat{Q}, \hat{Q}^{-1}} \hat{Q}) (D - C_{\hat{Q}, \hat{Q}^{-1}}) L_{\hat{Q}^{-1}},$$

it holds that

a) $\Phi(Q)$ is a strong symmetry for the meta-mKdV \(3\),

b) $\hat{\Phi}(\hat{Q})$ is a strong symmetry for the mirror meta-mKdV \(3\).

The next proposition explains the connection between $D$ and the derivation $\hat{D}$ in \(18\).

Denote by $K_Q$ conjugation with $Q$, i.e.

$$K_Q = L_{Q^{-1}} R_Q.$$  

**Proposition 10** (\[4\] Proposition 3.3). For $V = Q_x Q^{-1}$ and $\hat{V} = Q^{-1} Q_x$, we have

a) $K_Q D K_{\hat{Q}}^{-1} = \hat{D}$,

b) $K_Q C_{\hat{V}} K_{Q^{-1}} = C_V$ and $K_Q A_V K_{Q^{-1}} = A_{\hat{V}}$.

In particular, $K_Q D^{-1} K_{\hat{Q}}^{-1} = \hat{D}^{-1}$ and $K_Q(D + C_V) K_{\hat{Q}}^{-1} = \hat{D} + C_V$, implying

$$K_Q D^{-1}(D + C_V) K_{\hat{Q}}^{-1} = K_Q D^{-1} K_{Q^{-1}} K_Q D + C_V K_{\hat{Q}}^{-1} = \hat{D}(\hat{D} + C_V).$$

As a consequence,

$$\Phi(Q) = R_Q D^{-1}(D + C_V)(D - A_V D^{-1} A_V)(D - C_V) R_Q^{-1}$$

$$= L_Q \left( K_Q D^{-1}(D + C_V)(D - A_V D^{-1} A_V) D^{-1} (D + A_V)(D - C_V) K_Q^{-1} \right) L_Q^{-1}$$

$$= L_Q \left( \hat{D}^{-1}(\hat{D} + C_{\hat{V}})(\hat{D} - A_{\hat{V}}) \hat{D}^{-1} (\hat{D} + A_{\hat{V}})(\hat{D} - C_{\hat{V}}) \right) L_Q^{-1}$$

$$= L_Q \hat{D}^{-1}(\hat{D} + C_{\hat{V}})(\hat{D} - A_{\hat{V}} \hat{D}^{-1} A_{\hat{V}})(\hat{D} - C_{\hat{V}}) L_Q^{-1}.$$
Proposition 11. Define the derivations $\mathcal{D} = D + C_{Q^{-1}Q_x}$ and $\hat{\mathcal{D}} = D - C_{Q^{-1}Q_x}$. Then the operators in Theorem 13 can be rewritten as

$$\Phi(Q) = L_Q^{-1}(\mathcal{D} + C_{Q^{-1}Q_x})(\mathcal{D} - A_{Q^{-1}Q_x}\mathcal{D}^{-1}A_{Q^{-1}Q_x})(\mathcal{D} - C_{Q^{-1}Q_x})L_Q^{-1},$$

$$\hat{\Phi}(\hat{Q}) = R_Q^{-1}(\hat{\mathcal{D}} - C_{Q^{-1}Q_x})(\hat{\mathcal{D}} - A_{Q^{-1}Q_x}\hat{\mathcal{D}}^{-1}A_{Q^{-1}Q_x})(\hat{\mathcal{D}} + C_{Q^{-1}Q_x})R_Q^{-1}.$$  

Remark. It is worth mentioning that the strong symmetries one obtains transferring the recursion operator (17) to the meta-mKdV (3) and the mirror meta-mKdV (6) are precisely those given in the above proposition, see Appendix C.

Finally, we turn to hereditariness, an important concept introduced in [11]. Hereditariness of the noncommutative KdV recursion operator

$$\Phi_{KdV}(U) = D^2 + 2AU + AD^{-1} + CuD^{-1}C_uD^{-1}$$

is the main result of [24]. Another result from [10] ensures that hereditariness is preserved by Bäcklund transformations. Applying first the Miura transformation and then the respective Cole-Hopf link yields

Theorem 12. The operators in Theorem 9 are hereditary symmetries.

6. Solving the meta-mKdV hierarchy

Consider the noncommutative meta mKdV hierarchy,

$$(E_{2n-1}) \quad Q_{t_{2n-1}} = \Phi(Q)^{n-1}Q_x,$$

for $n \geq 1$, where $\Phi(Q)$ is the recursion operator given in Theorem 9. The base member ($n = 1$) is $Q_{t_1} = Q_x$, and the following argument shows that for $n = 2$ the meta-mKdV (3) is obtained:

Abbreviate $V = Q_xQ^{-1}$. As a first step, $(D - CV)R_{Q^{-1}}Q_x = (D - CV)V = V_x$. Second, since $D^{-1}AVx = D^{-1}\{V, V_x\} = V^2$, we find $(D - AVD^{-1}AV)(D - CV)R_{Q^{-1}}Q_x = V_{xx} - 2V^3$. This shows

$$\Phi(Q)Q_x = R_QD^{-1}(D + CV)(D - AVD^{-1}AV)(D - CV)R_{Q^{-1}}Q_x$$

$$= R_QD^{-1}(D + CV)(V_{xx} - 2V^3)$$

$$= R_QD^{-1}(V_{xx} - 2V^3) - [V, V_{xx}]Q.$$

Reinserting $V = Q_xQ^{-1}$ and computing the term in the large brackets explicitly, one obtains $Q_{xxx} - 3Q_{xx}Q^{-1}Q_x$, i.e. the right-hand side of (3).

Remark. Observe that this gives a heuristic access to the meta-mKdV (3).

The equations $(E_{2n-1})$, $1 \leq n \leq N$, are regarded as a system of partial differential equations in the variables $t_1, t_3, \ldots, t_{2N-1}$ (note that we as usual identify $t_1 = x, t_3 = t$). In the literature, it is customary to consider $\{E_{2n-1}\}_{n \geq 1}$ as an infinite system for formal functions in infinitely many variables $t_1, t_3, \ldots$. In our context we prefer to work with truncated expressions in finitely many variables.

The following theorem generalises Proposition 4.

Theorem 13. Let $A, B \in A$ with $A$ invertible. Consider

$$L_N(t_1, \ldots, t_{2N-1}) = \exp\left(\sum_{k=1}^{N} A^{2k-1}t_{2k-1}\right)B.$$
Then, for all \( N \in \mathbb{N} \), a solution of the system of the first \( N \) equations of the meta-mKdV hierarchy is given by

\[
Q_N = (I - L_N)^{-1} A^{-1}(I + L_N)
\]
on \( \Omega = \{(x,t) \mid I \pm L_N \text{ are invertible}\} \).

The proof heavily relies on [5, Theorem 12], where it is shown that

\[
V_N = (I - L_N^2)^{-1}(AL_N + L_N A)
\]
solves the system of the first \( N \) equations of the noncommutative mKdV hierarchy \( V_{t_{2n-1}} = \Psi(V)^{n-1}V_x, 1 \leq n \leq N \), where \( \Psi(V) \) is the recursion operator in [10]. Then the Cole-Hopf transformation (4) is used to transfer this solution to the meta-mKdV hierarchy.

**Proof of Theorem 13.** To simplify notation, we suppress the index \( V \) solves the system of the first \( N \) of the system of the first \( N \) on \( \Omega = \{(x,t) \mid I \pm L_N \text{ are invertible}\} \).

By [5, Theorem 12], \( V \) is a solution of the mKdV hierarchy. Furthermore,

\[
V_{t_{2n-1}} = (Q_xQ^{-1})_{t_{2n-1}} = Q_xt_{2n-1}Q^{-1} - Q_xQ^{-1}Q_{t_{2n-1}}Q^{-1} = R_{Q^{-1}}(D - L_V)Q_{t_{2n-1}}
\]

where \( \Pi = (D - L_V)^{-1}R_Q \). Hence,

\[
\Pi^{-1} Q_{t_{2n-1}} = V_{t_{2n-1}} = \Psi(V)^{n-1}V_x = \Psi(V)^{n-1} \Pi^{-1} Q_x,
\]

Recall from the construction of the meta-mKdV recursion operator preceding Theorem 9 that \( \Phi(Q) = \Pi \Psi(V) \Pi^{-1} \). As a result,

\[
Q_{t_{2n-1}} = \Pi \left( \Pi^{-1} \Phi(Q) \Pi \right)^{n-1} \Pi^{-1} Q_x = \Phi(Q)^{n-1}Q_x,
\]

which completes the proof. \( \square \)

Finally, the solution given in Theorem 13 is mapped to a solution of the alternative mKdV hierarchy,

\[
\tilde{V}_{t_{2n-1}} = \Psi(\tilde{V})^{n-1}\tilde{V}_x,
\]

\( 1 \leq n \leq N \), where \( \tilde{\Psi}(\tilde{V}) \) is the recursion operator in [10]. This leads to

**Corollary 14.** Let the assumptions of Theorem 13 be satisfied. Then, for all \( N \in \mathbb{N} \), a solution of the system of the first \( N \) equations of the alternative mKdV hierarchy is given by

\[
\tilde{V}_N = (I + L_N)^{-1}A(I + L_N)^{-1}(AL_N + L_N A)(I - L_N)^{-1}A^{-1}(I + L_N)
\]
on \( \Omega = \{(x,t) \mid I \pm L_N \text{ are invertible}\} \).

---

5 Observe first that the Cole-Hopf transformations (4), (5) also map solutions of the base member of the meta-mKdV hierarchy, \( Q_t = Q_x \), to solutions of the base members \( V_t = V_x \) and \( \tilde{V}_t = \tilde{V}_x \) of the mKdV and the alternative mKdV hierarchies. Since the recursion operators are hereditary, a result in [10] then ensures that the Cole-Hopf links extend to all levels of the hierarchies.

Note that this also yields an alternative (less direct) proof of Theorem 1.
Appendix A. Proof of Theorem 1

In this appendix we give a direct verification that the Cole-Hopf transformations (1) and (5) map solutions of the meta-mKdV (3) to solutions of the mKdV (1) and the alternative mKdV (2), respectively.

Proof of Theorem 1. Let $V$ be a solution of the meta-mKdV (3). As for (a), we need to verify that $V = C(Q) = Q_x Q^{-1}$ solves the mKdV (1). We compute:

\[
\begin{align*}
V_x &= (Q_x Q^{-1})_x = Q_{xx} Q^{-1} - (Q_x Q^{-1})^2, \\
V_{xx} &= (Q_{xx} Q^{-1} - (Q_x Q^{-1})^2)_x \\
&= Q_{xxx} Q^{-1} - 2Q_{xx} Q^{-1} Q_x Q^{-1} - Q_x Q^{-1} Q_{xxx} Q^{-1} + 2(Q_x Q^{-1})^3, \\
V_{xxx} &= Q_{xxxx} Q^{-1} \\
&= -3Q_{xxx} Q^{-1} Q_x Q^{-1} - 3Q_{xx} Q^{-1} Q_{xx} Q^{-1} - Q_x Q^{-1} Q_{xxx} Q^{-1} \\
&\quad + 6Q_{xx} Q^{-1} Q_x Q^{-1} Q_{xx} Q^{-1} + 3Q_x Q^{-1} Q_{xx} Q^{-1} Q_{xx} Q^{-1} + 3Q_x Q^{-1} Q_x Q^{-1} Q_{xxx} Q^{-1} \\
&\quad - 6(Q_x Q^{-1})^4.
\end{align*}
\]

For the nonlinear term in (1) we get

\[
\{V^2, V_x\} = Q_x Q^{-1} Q_x Q^{-1} Q_{xx} Q^{-1} + Q_{xx} Q^{-1} Q_x Q^{-1} Q_{xx} Q^{-1} - 2(Q_x Q^{-1})^4,
\]

and hence

\[
V_{xxx} - 3\{V^2, V_x\} = Q_{xxxx} Q^{-1} \\
&\quad - 3Q_{xxx} Q^{-1} Q_x Q^{-1} - 3Q_{xx} Q^{-1} Q_{xx} Q^{-1} - Q_x Q^{-1} Q_{xxx} Q^{-1} \\
&\quad + 3Q_{xx} Q^{-1} Q_x Q^{-1} Q_{xx} Q^{-1} + 3Q_x Q^{-1} Q_{xx} Q^{-1} Q_{xx} Q^{-1}.
\]

On the other hand, we find

\[
\begin{align*}
V_t &= Q_x Q^{-1} - Q_x Q^{-1} Q_t Q^{-1} \\
&= (Q_{xxx} - 3Q_{xx} Q^{-1} Q_x) Q^{-1} - Q_x Q^{-1} (Q_{xxx} - 3Q_{xx} Q^{-1} Q_x) Q^{-1},
\end{align*}
\]

since $Q$ solves (3), and, consequently,

\[
\begin{align*}
V_t &= (Q_{xxxx} - 3Q_{xxx} Q^{-1} Q_x - 3Q_{xx} Q^{-1} Q_{xx} + 3Q_{xx} Q^{-1} Q_x Q^{-1} Q_x) Q^{-1} \\
&\quad - Q_x Q^{-1} (Q_{xxx} - 3Q_{xx} Q^{-1} Q_x) Q^{-1} \\
&= Q_{xxxx} Q^{-1} - 3Q_{xxx} Q^{-1} Q_x Q^{-1} - 3Q_{xx} Q^{-1} Q_{xx} Q^{-1} - Q_x Q^{-1} Q_{xxx} Q^{-1} \\
&\quad + 3Q_{xx} Q^{-1} Q_x Q^{-1} Q_{xx} Q^{-1} + 3Q_x Q^{-1} Q_{xx} Q^{-1} Q_{xx} Q^{-1} \\
&\quad = V_{xxx} - 3\{V^2, V_x\}.
\end{align*}
\]

As for (b), one has to check that $\dot{V} = \dot{C}(Q) = Q^{-1} Q_x$ solves the alternative mKdV (2). Here we compute

\[
\begin{align*}
\dot{V}_x &= (Q^{-1} Q_x)_x = -(Q^{-1} Q_x)^2 + Q^{-1} Q_{xx}, \\
\dot{V}_{xx} &= 2(Q^{-1} Q_x)^3 - Q^{-1} Q_{xx} Q^{-1} Q_x - 2Q^{-1} Q_x Q^{-1} Q_{xx} + Q^{-1} Q_{xxx}, \\
\dot{V}_{xxx} &= -6(Q^{-1} Q_x)^4 \\
&\quad + 3Q^{-1} Q_{xx} Q^{-1} Q_x Q^{-1} Q_x + 3Q^{-1} Q_x Q^{-1} Q_{xx} Q^{-1} Q_x + 6Q^{-1} Q_x Q^{-1} Q_x Q^{-1} Q_{xx} \\
&\quad - Q^{-1} Q_{xxx} Q^{-1} Q_x - 3Q^{-1} Q_{xx} Q^{-1} Q_{xx} - 3Q^{-1} Q_x Q^{-1} Q_{xxx} + Q^{-1} Q_{xxxx}.
\end{align*}
\]

\[\text{In the subsequent computations only the noncommutative product rule and derivation rule for inverses,} \\
\text{(Q^{-1})_x = Q^{-1} Q_x Q^{-1}, \text{ are used.}\]
For the nonlinear terms in (2) we therefore get
\[
[\tilde{V}, \tilde{V}_{xx}] = Q^{-1}Q_xQ^{-1}Q_{xxx}Q^{-1}Q_x + Q^{-1}Q_{xx}Q^{-1}Q_xQ^{-1}Q_{xx}Q^{-1}Q_x - 2Q^{-1}Q_xQ^{-1}Q_xQ^{-1}Q_{xx},
\]
\[-2\tilde{V} \tilde{V}_x \tilde{V} = 2(Q^{-1}Q_x)^4 - 2Q^{-1}Q_xQ^{-1}Q_{xx}Q^{-1}Q_x.
\]
This shows
\[
\tilde{V}_{xxx} + 3[\tilde{V}, \tilde{V}_{xx}] - 6\tilde{V}_x \tilde{V} = 6Q^{-1}Q_{xx}Q^{-1}Q_xQ^{-1}Q_x - 4Q^{-1}Q_{xx}Q^{-1}Q_xQ^{-1}Q_{xx} + Q^{-1}Q_{xxxx}.
\]
On the other hand,
\[
\tilde{V}_t = (Q^{-1}Q_x)_t = -Q^{-1}Q_xQ^{-1}Q_x + Q^{-1}Q_{xt}
\]
\[= -Q^{-1}(Q_{xx} - 3Q_{xx}Q^{-1}Q_x)Q^{-1}Q_x + Q^{-1}(Q_{xxx} - 3Q_{xx}Q^{-1}Q_x)_x.
\]
since \(Q\) solves (19). Hence
\[
\tilde{V}_t = Q^{-1}Q_{xxxx} - 4Q^{-1}Q_{xxx}Q^{-1}Q_x - 3Q^{-1}Q_{xx}Q^{-1}Q_{xx}Q^{-1}Q_x + 6Q^{-1}Q_{xx}Q^{-1}Q_xQ^{-1}Q_x
\]
\[= \tilde{V}_{xxx} + 3[\tilde{V}, \tilde{V}_{xx}] - 6\tilde{V}_x \tilde{V}.
\]
This completes the proof.

\[\square\]

**Appendix B. Background on symmetries and Bäcklund transformations**

In this appendix, we will concisely review some general concepts for evolution equations, which are used throughout the article. The essence is to view an evolution equation as an ordinary differential equation with values in a suitable function space and to apply methods from differential topology and dynamical systems. We refer to [20] for an introduction to the general theory, and to [24] and the references therein for more details on applications relevant for the present article.

Our phase space will be a topological algebra \(F\) of functions depending on \(x \in \mathbb{R}\) and with values in some Banach algebra \(A\). An evolution equation can be viewed as an ordinary differential equation

(19) \[U_t = K(U),\]

where the unknown function \(U = U(t)\) takes values in \(F\) and \(K\) is a vector field on \(F\), i.e., a mapping that associates to \(U \in F\) a vector \(K(U)\) in the tangent space \(T_U F\) of \(F\) at \(U\). In the sequel, all vector fields are assumed to be smooth. The Lie bracket \([K, L]_{\text{Lie}}\) of two vector fields should not be confused with their commutator \([K, L]\), defined pointwise by \([K, L](U) = [K(U), L(U)],\) where we identify \(T_U F\) with \(F\) and use the commutator in \(A\).

A vector field \(L\) is a *symmetry* of (19) if \([K, L]_{\text{Lie}} = 0\). By an operator we mean a mapping \(\Phi\) that maps \(U \in F\) to a bounded linear operator \(\Phi(U) : T_U F \to T_U F\) on the tangent space at \(U\). Such a \(\Phi\) is a *strong symmetry* of (19) (or recursion operator) if \(\Phi\) maps symmetries to symmetries. As proved in [14], \(\Phi\) is a strong symmetry of (19) if

\[\Phi'[K]V = K'[\Phi V] - \Phi'K'[V]\]

is valid for every vector field \(V\). An operator \(\Phi\) is *hereditary* if

\[\Phi \Phi'[V]W - \Phi'[\Phi V]W\]

is symmetric in the vector fields \(V\) and \(W\). Note that hereditariness does not depend on the underlying evolution equation.

If \(\Phi\) is a strong symmetry, so also the powers \(\Phi^n, n \in \mathbb{N}\), and we get a hierarchy of vector fields \(K_n = \Phi^n K\) which all commute with \(K\). For \(\Phi\) hereditary, it was proved in [13] that these
vector fields all commute pairwise. In all cases considered in the present article, the hierarchy can be extended taking as a base member $K_0(U) = U_x$, the symmetry expressing translation invariance.

Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be function spaces as above. In addition to (19), consider a second evolution equation

$$V_t = L(V), \tag{20}$$

for $V = V(t)$ with values in $\mathcal{G}$. Following [10], a smooth mapping

$$B : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$$

defines a Bäcklund transformation if

1) for any solutions $U(t)$, $V(t)$ of (19), (20), respectively, which are defined on an open interval $I$ and satisfy $B(U(t_0), V(t_0)) = 0$ for some $t_0 \in I$, we have $B(U(t), V(t)) = 0$ for all $t \in I$.

Such a Bäcklund transformation is called locally invertible, if

2) at every point $(U, V) \in \mathcal{F} \times \mathcal{G}$, the partial derivatives $B_U$ and $B_V$ restrict to invertible linear mappings $T_U \mathcal{F} \times \{0\} \to T_{B(U,V)} \mathcal{H}$ and $\{0\} \times T_V \mathcal{G} \to T_{B(U,V)} \mathcal{H}$, respectively. Here we use the obvious identification $T_{B(U,V)} \mathcal{H} \simeq T_U \mathcal{F} \times T_V \mathcal{G} \simeq \mathcal{F} \times \mathcal{G}$.

Let $\Phi$ be a strong symmetry of (19) and let $B$ define a locally invertible Bäcklund transformation. Near $(U_0, V_0)$ with $B(U_0, V_0) = 0$, we can solve $B(U, V) = 0$ for $V$ by $V = V(U)$ with differential $dV = \Pi(U, V(U))$, $\Pi = -B_V^{-1}B_U$. Transforming $\Phi$ in the natural way gives an operator $\Psi$ defined near $V_0$ by

$$\Psi(V) = \Pi(U(V), V) \Phi(U(V)) \left( \Pi(U(V), V) \right)^{-1}.$$

Fokas and Fuchssteiner proved in [10] that $\Psi$ is a strong symmetry of (20). Furthermore, if $\Phi$ in addition is hereditary, $\Psi$ is hereditary, too. Finally, the correspondence generated by a Bäcklund transformation extends to hierarchies\(^7\).

**Remark.** a) In Section 5 and 6, we make formal use of these results in a mild sense. Instead of requiring local invertibility, we compute under the assumption that inverses of operators like $D$ and $D + CV$ exist.

b) Our short introduction to Bäcklund transformations is adapted to our needs. For a broader view on the topic we refer to the monographs [15, 23].

**Appendix C. Derivation of the mirror meta-mKdV recursion operator**

In this appendix the proof of Theorem 2) b) is provided, and an alternative derivation of the meta-mKdV recursion operator is given. We start with the following lemma.

**Lemma 15.**

a) For $T = \mp S_x S^{-1}$, the identity $(D \pm L_T)^{-1} R_S = R_S (D \pm C_T)^{-1}$ holds.

b) For $T = \mp S^{-1} S_x$, the identity $(D \pm R_T)^{-1} L_S = L_S (D \mp C_T)^{-1}$ holds.

**Proof.** a) is Lemma 8)

b) Since $DL_S = L_S D + L_{S_x}$, and since left- and right multiplication commute, $(D \pm R_T)L_S = L_S (D \pm R_T) + L_{S_x}$. Using $T = \mp S^{-1} S_x$, we can replace $L_{S_x}$ by $L_{\mp ST} = \mp L_S L_T$, and the lemma follows. \(\square\)

**Proof of Theorem 2 b).** Consider the Cole-Hopf link

$$B(\hat{Q}, V) = \hat{Q}V + \hat{Q}_x$$

\(^7\) Actually, the arguments in [10] are given in the classical setting ($A = C$). However, it is easily seen that everything remains valid for evolution equations with noncommuting dependent variables.
between the mKdV \cite{1} and the mirror meta-mKdV \cite{2}, see Proposition \cite{3}. Computing the Fréchet derivatives
\[ B_{\tilde{Q}}[\tilde{Q}] = \tilde{Q}V + \tilde{Q}_x, \quad B_{\tilde{V}}[\tilde{V}] = \tilde{Q}V, \]
we get \( \tilde{\Phi}(\tilde{Q}) = \Pi \Psi(V)\Pi^{-1} \) with \( \Pi = B_{\tilde{Q}}^{-1}B_{\tilde{V}} = (D + R_{\tilde{V}})^{-1}L_{\tilde{Q}} = L_{\tilde{Q}}(D - C_{\tilde{V}})^{-1} \), according to Lemma \cite{5}. Hence,
\[ \tilde{\Phi}(\tilde{Q}) = L_{\tilde{Q}}(D - C_{\tilde{V}})^{-1}((D - C_{\tilde{V}}D^{-1}C_{\tilde{V}})(D - A_{\tilde{V}}D^{-1}A_{\tilde{V}})(D - C_{\tilde{V}})L_{\tilde{Q}}^{-1} = L_{\tilde{Q}}D^{-1}(D + C_{\tilde{V}})(D - A_{\tilde{V}}D^{-1}A_{\tilde{V}})(D - C_{\tilde{V}})L_{\tilde{Q}}^{-1} \]
with \( V = \tilde{Q}^{-1}\tilde{Q}_x \).

As a supplement we give an alternative derivation of the meta-mKdV recursion operator starting with the Bäcklund transformation
\[ B(Q, V) = Q\tilde{V} - Q_x, \]
which links the meta-mKdV \cite{3} to the alternative mKdV \cite{2}, see Theorem \cite{1}. In this case the Fréchet derivatives are
\[ B_{Q}[\tilde{Q}] = \tilde{Q}V - \tilde{Q}_x, \quad B_{\tilde{V}}[\tilde{V}] = Q\tilde{V}, \]
yielding the transformation operator \( \Pi = B_{\tilde{Q}}^{-1}B_{\tilde{V}} = -(D - R_{\tilde{V}})^{-1}L_{\tilde{Q}} = -L_{\tilde{Q}}(D + C_{\tilde{V}})^{-1} \), by Lemma \cite{5}. Hence a recursion operator for the meta-mKdV follows from
\[ \Phi(Q) = \Pi \tilde{\Psi}(\tilde{V})\Pi^{-1} = L_{\tilde{Q}}D^{-1}((\tilde{D} + C_{\tilde{V}})(\tilde{D} - A_{\tilde{V}})\tilde{D}^{-1}(\tilde{D} + A_{\tilde{V}})(\tilde{D} - C_{\tilde{V}})\tilde{D}^{-1} \cdot \tilde{D}L_{Q}^{-1} = L_{\tilde{Q}}D^{-1}(\tilde{D} + C_{\tilde{V}})(\tilde{D} - A_{\tilde{V}})\tilde{D}^{-1}A_{\tilde{V}})(\tilde{D} - C_{\tilde{V}})L_{Q}^{-1}, \]
where \( \tilde{D} = D + C_{\tilde{V}} \). Note that this is the representation of the meta-mKdV recursion operator given in Proposition \cite{1}.

An analogous derivation of the mirror meta-mKdV recursion operator from the alternative mKdV \cite{2} using Proposition \cite{3} gives
\[ \Phi(\tilde{Q}) = R_{\tilde{Q}}\tilde{D}^{-1}(\tilde{D} + C_{\tilde{V}})(\tilde{D} - A_{\tilde{V}})\tilde{D}^{-1}A_{\tilde{V}})(\tilde{D} - C_{\tilde{V}})R_{\tilde{Q}}^{-1}, \]
compare again Proposition \cite{1}.

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