Topological phase transitions in four dimensions

Nicolò Deleu,1 Andrea Trombettoni,2,3 and Dario Zappalà4

1Institute for Theoretical Physics, ETH Zurich, Wolfgang-Pauli-Str. 27, 8093 Zurich, Switzerland
2Department of Physics, University of Trieste, Strada Costiera 11, I-34151 Trieste, Italy
3CNR-ION DEMOCRITOS Simulation Center and SISSA, Via Bonomea 265, I-34136 Trieste, Italy
4INFN, Sezione di Catania, Via Santa Sofia 64, 95123 Catania, Italy

We show that four-dimensional systems may exhibit a topological phase transition analogous to the well-known Berezinskii-Kosterlitz-Thouless vortex unbinding transition in two-dimensional systems. The realisation of an engineered quantum system, where the predicted phase transition shall occur, is also presented. We study a suitable generalization of the sine-Gordon model in four dimensions and the renormalization group flow equation of its couplings, showing that the critical value of the frequency is the square of the corresponding value in 2D. The value of the anomalous dimension at the critical point is determined ($\eta = 1/32$) and a conjecture for the universal jump of the superfluid stiffness ($4/\pi^2$) is presented.

Introduction: The Berezinskii-Kosterlitz-Thouless (BKT) transition is a paradigmatic example of a topological phase transition occurring in absence of spontaneous symmetry breaking and therefore not characterized by a local order parameter. Its observation is tied to the behaviour of the correlation functions, which displays the typical power law behaviour at all temperatures in the superfluid, low-temperature, phase and exponential in the disordered, high-temperature, phase. A related property is the presence of a line of fixed points, associated to the the power-law behaviour in absence of spontaneous symmetry breaking – namely the presence of a line of fixed points, the hallmark of phase transitions in 2D systems. A complete understanding of this critical behaviour can be obtained by mapping in 2D the XY model – or more precisely, its low temperature limit, the Villain model, which is in the same universality class – into the 2D Coulomb gas [17], which in turn can exactly be mapped onto the 2D sine-Gordon model [15]. The latter is a field theory with an interaction term proportional to $\cos(\beta \phi)$ where $\beta$ is the frequency. The 1+1 sine-Gordon model has been thoroughly investigated by several techniques, including bosonization [19] [20], functional renormalization group [21] [22] and integrable approaches [23] [24]. The main result is that there is a phase transition occurring at a critical value of $\beta$, given by $\beta_c^2 = 8\pi$ [25], which corresponds to the BKT superfluid transition.

These mappings are specific of two dimensions ($d = 2$), and despite the sine-Gordon model and the Coulomb gas can be mapped between them in any dimension [29], it is their mapping to the XY model or to interacting bosons that is no longer valid in $d > 2$. So, the properties of the BKT transition – namely the presence of a line of fixed points, the absence of magnetization, the presence of superfluidity in absence of condensation, and the universal jump of the superfluid fraction are considered the hallmarks of phase transitions in 2D systems.

In this paper we want to investigate how to obtain a BKT phase transition in $d > 2$. For the reasons above illustrated, and despite BKT-like deconfinement properties in $d = 3$ [30] and isotropic Lifshitz
points in $d = 4$ have been considered and discussed, to the best of our knowledge the peculiar features of the BKT universality class, such as the jump of the superfluid fraction and the universality of the critical exponent $\eta$ at the end-point of the fixed points line have not been discussed in $d > 2$ or related to any feasible microscopic model. Here, we focus on $d = 4$ and show that a sine-Gordon model in $3 + 1$ dimensions does exhibit a BKT topological phase transition in $4D$.

The microscopic model: One of the most celebrated realization of BKT critical behaviour is the XY model on a square lattice. Here we will focus on its second neighbours generalization

$$H = -K \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) - \tilde{K} \sum_{\langle\langle i,j \rangle\rangle} \cos(\theta_i - \theta_j),$$

where $i,j$ denote the sites of a 4D lattice and $K = J/k_BT$, $\tilde{K} = J'/k_BT$ with $J,J'$ respectively the nearest-neighbour (n.n.) and next-nearest-neighbour (n.n.n.) couplings. The partition function is $Z = \int \prod_i d\theta_i e^{-H}$. In the continuum limit, the action will contain both quadratic and quartic momentum contributions, due to the presence of n.n.n. couplings. However, with the choice $\tilde{K} = -K/6, (K > 0)$, at mean field level one cancels in [1] the quadratic momentum contributions, so that the interacting $3 + 1$ field theory near to the critical point can be described by [33]:

$$S[\varphi] = \int \left\{ \frac{(\Delta \varphi)^2}{2} + g_0(1 - \cos(\beta \varphi)) \right\} d^4x,$$

where $\Delta$ indicates the four-dimensional laplacian and $\varphi(x)$ is a real scalar field. The action [2] will be the one studied in the following.

We pause here to comment on an obvious objection: 4D models do not exist in a laboratory. One may think to circumvent this issue by observing that, at variance, 3D quantum models at $T = 0$ do exist. So one at first sight could take a 3D network of quantum Josephson junctions and add to them n.n.n. interactions to emulate the model [1] and therefore [2]. A very clear discussion of this for $1 + 1$ quantum chains is done in [35], and reviewed in [36]. The result of this analysis is that one has fourth derivatives in the three spatial directions, but usual second derivative in the imaginary time direction. If from one side this is a case interesting in itself, possibly in connection with tuning mechanisms of couplings along the imaginary time axis, from the other side it clarifies that using quantum Josephson junctions with n.n.n. interactions appears not the best way to realize [1]. One may anyway resort to the proposal of implementing lattices in synthetic dimensions [37], experimentally realized with cold $^{174}$Yb atoms [38]. In these schemes, the fourth direction is realized by a large number of internal degrees of freedom, such as the $^{174}$Yb levels. To implement [1], one needs to have interacting bosons on a lattice, with n.n.n. hoppings. Remind that the Bose-Hubbard model can be mapped in the quantum phase model, and that in a suitable range of parameters (in which interactions are not vanishing, but negligible with respect to Josephson energy), one gets the XY model [39, 40]. Therefore, in order to have [1], one needs a term of the form $b^\dagger b$ acting on pairs of n.n. sites, and this as well in the extra, synthetic dimension. Despite being certainly challenging to be realized in current-day experiments, this provides a platform to study topological transitions in $4D$.

Field theory study: The action in Eq. (2) contains only a periodic local potential term in analogy with the usual sine-Gordon theory used to describe BKT physics in low dimensions [20, 41]. Within this framework, the parameter $\beta$ is related to the phase stiffness of the model, while the parameter $g_0$ describes the fugacity of the topological excitations. It is worth noting that in $d = 2$ a formal mapping is possible only at low temperatures between the traditional $O(2)$ model and the quadratic 2D sine-Gordon model [13]. In the next section we are going to show how the theory in Eq. (2) can be connected with the 4D quartic $O(2)$ via the introduction of certain singular phase configurations.

In order to construct the RG study of the action in Eq. (2) we will employ the functional RG approach. This modern RG technique derives from the possibility to write an exact RG equation for the effective action [12, 14], which may then be solved by projecting it on a restricted theory space parametrised by a proper ansatz [15, 19]. In the present case we will rely on an ansatz analogous to the bare action

$$\Gamma_k[\varphi] = \int \left\{ \frac{\hbar k}{2}(\Delta \varphi)^2 + g_k(1 - \cos \varphi) \right\} d^4x,$$

but with the bare coefficients substituted by scale dependent ones. An ansatz analogous to the one in Eq. (2) has been proven to reproduce all the qualitative features of the BKT transition, including the universal jump of the superfluid stiffness [22, 47] and to yield consistent results for the computation of the $c$-function [38]. More complicated ansatz were also shown to yield quantitative insight into the spectrum of the model [24].

By projecting the functional RG equation for the effective action on the restricted theory space
parametrised by the ansatz in Eq. (3) one obtains

$$\partial_t V_k(\phi) = \int \frac{d^d q}{(2\pi)^d} G(q) \partial_t R_k(q),$$

$$\partial_t w_k = \lim_{p \to 0} \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} \int \frac{d^d q}{(2\pi)^d} \partial_t R_k(q) G(q)^2 \times

V''_k(\phi)^2 \frac{d^d G(p + q)}{dp^d} (5)$$

where \( t = -\log(k/\Lambda) \) is the RG logarithmic scale, \( V_k(\phi) = g_k(1 - \cos \phi) \) the local potential and \( G(q) \) the propagator in momentum space

$$G(q) = \frac{1}{w_k q^4 + V'_k(\phi) + R_k(q)}. (6)$$

The function \( R_k(q) \) is a regulator function which introduces a finite mass for long wave-length fluctuations \( R_k(q \approx 0) \approx k^4 \). The computation can be carried in \( d \) dimensions, leading to the introduction of the generalized flow equations

$$\partial_t w_k = \beta w(w, g, d), \quad (7)$$

$$\partial_t g_k = \beta g(w, g, d). (8)$$

In order to obtain an explicit form for the functions \( \beta w \) and \( \beta g \), it is convenient to introduce the regulator function \( R_k(q) = k^4 \), which allows to calculate the integrals in Eqs. (4) and (5) analytically. However, this choice for the regulator generates ultraviolet divergencies of the momentum integrals in \( d = 4 \). These divergencies are regularised by pursuing the computation for \( d > 4 \) and, then taking the \( d \to 4^+ \) limit. The explicit calculation is shown in the Appendix.

After introducing the rescaled variable \( \tilde{g}_k = g_k/k^4 \) the flow equations can be written similarly to the well-known \( d = 2 \) case \( \ref{11} \) \( \ref{12} \) \( \ref{50} \).

$$\partial_t w_k = -\frac{9}{160\pi^2} \frac{\tilde{g}_k^2}{(1 - \tilde{g}_k^2)^2}, \quad (9)$$

$$4 + \partial_t \tilde{g}_k = \frac{1}{8\pi^2 w_k \tilde{g}_k} \left( 1 - \sqrt{1 - \tilde{g}_k^2} \right). (10)$$

The phase diagram in Fig. 1, obtained by the set of Eqs. (9) and (10) displays a line of attractive Gaussian fixed points for \( w_k > \beta_c^2 \) with \( g_k = 0 \), while for \( w_k < \beta_c^2 \) the \( \cos(\phi) \) perturbation becomes relevant and the flow is driven at an infrared point with exponential correlations. The critical value of the frequency in \( d = 4 \), obtained from Eq. (10), is

$$\beta_c^2 = 64\pi^2$$(11)

in agreement with the heuristic arguments given in the next section. This value is universal and independent from the choice of the regulator, as it can be proven by expanding Eqs. (4) and (5) around \( g_k = 0 \). Remarkably, the result \( \ref{11} \) is found to be the square of the corresponding standard result for the \( 2D \) sine-Gordon model, reading \( \beta^2 = 8\pi \) \( \ref{28} \).

The action in Eq. (2) does not contain any quadratic momentum terms, as they vanish in the Hamiltonian in Eq. \( \ref{1} \) for \( K = -6\beta^2 \). Indeed, in order for the system to attain BKT behaviour, one has to tune two parameters: the temperature, which controls the \( \beta \) parameter, and the nearest neighbour coupling \( K \). Then, the BKT line of fixed points described by Eqs. (9) and (10) is actually a line of third order critical points, in analogy with the case of an isolated Lifshitz point \( \ref{51} \). Yet, the actual critical value for the coupling \( K > 0 \), may differ from the mean field value \( K_c = -6\beta^2 \) and, possibly, become temperature dependent. This specific critical value \( K_c \) in the microscopic model described by Eq. (1) is not a universal quantity and cannot be estimated by the continuum theory. Its determination by numerical simulations of the lattice Hamiltonian is left for future investigations. In the following, we show how the sine-Gordon theory described here can be connected with the \( 4D \) quartic \( U(1) \) model by a suitable identification of the topological excitations.

**FIG. 1:** The phase diagram of the model obtained by the evolution Eqs. (9) and (10) in the space of the dimensionless running parameters \( w_k \) and \( \tilde{g}_k \). The similarities with the traditional BKT picture are evident: for \( w_k > 1/\beta_c^2 \) one has a line of fixed points with \( g_k = 1 \), where the system is massless. Conversely, if \( w_k < 1/\beta_c^2 \) the effective theory becomes massive and the flow is attracted to a spinodal point at \( \tilde{g}_k = 1 \) and \( w_k = 0 \).

**Topological configurations:** Now we illustrate the
example of a specific field configuration of a 4D low energy effective hamiltonian for a \( U(1) \) symmetric model with four derivatives of the field, that realizes the above picture. The effective hamiltonian is

\[
H[\vartheta(r)] = \frac{K}{2} \int d^4r \left[ \Delta \vartheta(r) \Delta \vartheta(r) \right], \quad (12)
\]

where the field \( \vartheta \) is the phase of a complex scalar field \( \Phi \), represented in polar components by \( \vartheta \) and its radial component (\( \rho = \sqrt{\Phi^* \Phi} \)). Fluctuations of \( \rho \) are absent in Eq. (12) because they are suppressed in the infrared region by the presence of a radial mass. We notice that this suppression is warranted by the presence of a \( \partial \Phi \partial \Phi^* \) term which, in turn, yields a square momentum contribution in the propagator \([52] \); however, in analogy with the criterion adopted for Eq. (2), we discarded in Eq. (12) the quadratic contribution \( \partial \Phi \partial \Phi^* \), as this operator, if suitably taken on the critical manifold, is expected to be driven to zero by the RG flow in the low energy regime \([31] \). In addition, we did not include the term \( (\partial \Phi \partial \Phi^*)^2 \), as it is possible to arrange the complex field four-derivative sector in such a way that only quadratic terms in \( \vartheta \) are left.

We expect that the desired configuration \( \vartheta(r) = G(r-r') \), associated to a particular point \( r' \) in the 4D space, is such that \( \Delta_{\vartheta} G(r-r') = -(r-r')^{-2} \), as it produces a logarithmic scaling of the energy. Then, from the solution of the Laplace equation \(-\Delta_{\vartheta} (r-r')^{-2} = (2\pi)^2 \delta^4(r-r') \), \([32] \), we find

\[
G(r-r') = \int \frac{d^4r''}{(2\pi)^2 (r-r'')^2 (r''-r')^2} = \frac{1}{\ln |r-r'|^2}, \quad (13)
\]

where \( R \) is a large distance cutoff.

Such scaling is also realized by the field configuration \( \vartheta(r) = A_{\vartheta}(r) \) which has the following expression in terms of spatial coordinates, \( A_{\vartheta}(r) = (1/2)(\alpha_4 - \pi/2) \cot(\alpha_4) \), where \( \alpha_4 \) is the angle between \( r-r' \) and one of the the coordinate axes, e.g. \( x_4 \). We find that \( A_{\vartheta}(r) \) is a solution of the equation \( \Delta_{\vartheta} A_{\vartheta}(r) = -(r-r')^{-2} \) and therefore, when inserted in Eq. (12), it produces equivalent effects to those of \( G(r-r') \) (see Appendix).

Consequently, we get \( \Delta_{\vartheta}^2 A_{\vartheta}(r) = (2\pi)^2 \delta^4(r-r') \), i.e. \( A_{\vartheta}(r) \), which is singular at the point \( r' \), provides an extremum of the Hamiltonian \([12] \). The corresponding energy is \( H[A_{\vartheta}] = K \pi^2 \ln (R/r_0) \), \( r_0 \) being a short distance cutoff. Then, similarly to the 2-dimensional BKT transition, by estimating the entropy as the logarithm of the number of ways to place \( A_{\vartheta} \) (i.e. the point \( r' \)) in the \( d = 4 \) space with cut-offs \( R \) and \( r_0 \): \( S[A_{\vartheta}] = \ln (R^4/r_0^4) \), the free energy of the system exhibits a change of sign at

\[
K_c = \frac{4}{\pi^2}, \quad (14)
\]

which is to be associated with a measurable discontinuous jump of \( K \) from \( 4/\pi^2 \) to 0.

Finally, by following a heuristic procedure already developed in the 2D case, we can map the sine-Gordon model in Eq. (3) onto Eq. (12), computed for \( \vartheta(r) = G(r-r') \), and derive the relation \( (2\pi)^4 K = w^{-1} \) between the respective couplings. Details are displayed in the Appendix. Then, \( K_c \) in \([14] \) corresponds to \( \beta_c^2 = 64\pi^2 \), in agreement with Eq. (11).

We are now able to determine the universal exponent \( \eta \), associated to the critical value \( K_c \). In fact, the fixed point action in the low temperature phase \( (K < K_c) \) is simply Gaussian (see Fig. 4), and one can explicitly obtain the correlation functions of the vertex operator \( V(r) = \exp(i\vartheta(r)) \)

\[
(V(r)V(0)) = \exp \left( -\frac{\eta^2}{2} ((\vartheta(r) - \vartheta(0))^2) \right). \quad (15)
\]

From the correlation functions above one obtains the scaling of the vertex operator \( \Delta_{\vartheta}= (8\pi^2 K_c)^{-1} \), which can be compared with the conventional \( d = 2 \) result \( \Delta_{\vartheta}^{2D} = (2\pi K_{2D})^{-1} \). \([54] \). Then, as for the 2D case, the scaling of the vertex operator can be connected with the power law decay of the correlation functions of the model, and this leads to the anomalous dimension exponent associated to \( K_c \)

\[
\eta = \frac{1}{32}, \quad (16)
\]

which has to be compared with the traditional BKT result \( \eta_{BD} = 1/4 \).

Conclusions: We showed that four-dimensional systems may exhibit a topological phase transition which extends to higher dimensions the celebrated Berezinskii-Kosterlitz-Thouless (BKT) transition. A discussion of an experimental setup which may realise the effective action in Eq. (2) is presented. We introduced a suitable generalization of the sine-Gordon model in four dimensions and we performed a renormalization group flow equation of its couplings. The critical value of the sine-Gordon frequency \( (\beta_c^2 = 64\pi^2) \) and the value of the anomalous dimension at the critical point \( (\eta = 1/32) \) are determined. A delicate point is to put in relation the 4D sine-Gordon model and a suitable \( O(2) \) model. In two dimensions this duality \([13] \) is at the heart of the whole BKT theory, based on the identification of the vertex degrees of freedom with Coulomb charges and on the exact mapping between the sine-Gordon model and the Coulomb gas. In the considered 4D case we presented a discussion of the topological configurations and, relying on this analysis, we presented a conjecture for the universal jump of the superfluid stiffness. The investigation of 4 dimensional models is crucial to the understanding of
cosmological problems in space-time, where the introduction of higher derivatives terms in the action has already been suggested as a solution to gravity quantization [55]. As future work, we mention that it would be interesting to consider the anisotropic limit, where no fourth derivative terms in the action in the fourth direction are present, which will, possibly, allow an extension of the present theory to zero temperature quantum systems in $d = 3$.

Acknowledgements: We thank T. Enss and I. Nandori for useful discussions. This work is supported by the CNR / HAS (Italy-Hungary) project "Strongly interacting systems in confined geometries" and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC-2181/1-390900948 (the Heidelberg STRUCTURES Excellence Cluster).

Derivation of the FRG flow equations

Let us rewrite our effective action ansatz, see Eq. (3) in the main text, 

$$\Gamma[\varphi] = \int \left\{ \frac{w_k}{2} \Delta\varphi(x)\Delta\varphi(x) + g_k(1 - \cos(\varphi)) \right\} d^d x. \quad (17)$$

In the following we will derive the RG flow equations for the two couplings $g_k$ and $w_k$ within the FRG formalism. The flow equations of the potential and the two point function for a single field action, obtained by the flow of the effective action, [44–46], read

$$\partial_t V_k(\varphi) = \int \partial_t R_k(q) G(q) \left( \frac{2}{\pi} \right)^{d/2} \int d^d q \left\{ \partial_t R_k(q) G(q)^2 V''_k(\varphi)^2 \right\} d^d p G(p + q). \quad (19)$$

where $P_0 = \frac{1}{2} \int_\pi^{2\pi} \cdots d\varphi$ is a projector on the field independent space and the regularized single field propagator reads

$$G(q) = \frac{1}{w_k q^4 + V''_k(\varphi) + k^4} \quad (20)$$

which has been obtained by the introduction of the purely massive regulator

$$R_k(q) = k^4. \quad (21)$$

It is convenient to introduce the variable $y = |q + p|^2$ and rewrite the momentum derivative in Eq. (19) according to the transformations

$$\frac{1}{2} \frac{d^2}{dp^2} = \frac{1}{2} \left( \frac{d^2 y}{dp^2} \frac{d}{dy} + \left( \frac{dy}{dp} \right)^2 \frac{d^2}{dy^2} \right), \quad (22)$$

$$\frac{1}{24} \frac{d^4}{dp^4} = \frac{1}{24} \left( 3 \left( \frac{d^2 y}{dp^2} \right)^2 \frac{d^2}{dy^2} + 6 \left( \frac{dy}{dp} \right)^2 \frac{d^2 y}{dp^2} \frac{d^3}{dy^3} + \left( \frac{dy}{dp} \right)^4 \frac{d^4}{dy^4} \right), \quad (23)$$

where the derivatives of the $y$ variable at $p = 0$ read

$$\left. \frac{dy}{dp} \right|_0 = 2q \cos \theta, \quad (24)$$

$$\left. \frac{d^2 y}{dp^2} \right|_0 = 2. \quad (25)$$

Inserting the expression in Eq. (23) into Eq. (19), the $\beta$-function for the $w_k$ splits into three contributions

$$\partial_t w_k = P_0 \left[ T_1 + T_2 + T_3 \right] \quad (26)$$
with

\[ T_1 = \frac{d}{2} c_d \int q^{d-1} dq \partial_t R_t(q) G(q) G(\varphi)^2 V_k''(\varphi)^2 G(\varphi)^2(q), \]  
\[ T_2 = 2c_d \int q^{d+1} dq \partial_t R_t(q) G(q) G(\varphi)^2 V_k''(\varphi)^2 G(\varphi)^3(q), \]  
\[ T_3 = 2 \frac{c_d}{d+2} \int q^{d+3} dq \partial_t R_t(q) G(q) G(\varphi)^2 V_k''(\varphi)^2 G(\varphi)^4(q). \] 

After obtaining the derivatives of the regularised propagator in Eq. (20) with respect to the \( y \) variable and inserting them into Eq. (27) one obtains

\[ T_1 = \frac{d}{2} c_d \int q^{d-1} dq \partial_t R_t(q) G(q) G(\varphi)^2 (8w_k^2 q^4 G^3 - 2w_k G^2) = \]
\[ = -2d c_d k^4 \int q^{d-1} dq V_k''(\varphi)^2 (8w_k^2 q^4 G^3 - 2w_k G^4) = \]
\[ = -2d c_d k^4 V_k''(\varphi)^2 (8w_k^2 k^{d+3} - 2w_k k^{d-1}) \]  

where

\[ \hat{I}_m^n = \int \frac{q^n dq}{(w_k g^4 + V_k''(\varphi) + k^4)^m} = \frac{\Gamma \left( \frac{n+1}{4} \right) \Gamma \left( m - \frac{n}{4} - \frac{1}{4} \right)}{4 \Gamma(m)w_k^{\frac{n+1}{4}}} (V_k''(\varphi) + k^4)^{\frac{n+1}{4} - \frac{m}{4}} \] 

The same procedure can be followed for the second term

\[ T_2 = 2c_d \int q^{d+1} dq \partial_t R_t(q) G(q) G(\varphi)^2 V_k''(\varphi)^2 G(\varphi)^3(q) = \]
\[ = 2c_d \int q^{d+1} dq \partial_t R_t(q) G(q) G(\varphi)^2 V_k''(\varphi)^2 (24w_k^2 q^2 G^3 - 48w_k^3 q^5 G^4) = \]
\[ = -8c_d k^4 V_k''(\varphi)^2 (24w_k^2 k^{d+3} - 48w_k^3 k^{d+7}) \]  

and the third term

\[ T_3 = 2 \frac{c_d}{d+2} \int q^{d+3} dq \partial_t R_t(q) G(q) G(\varphi)^2 V_k''(\varphi)^2 G(\varphi)^4(q) = \]
\[ = -8 \frac{c_d}{d+2} k^4 \int q^{d+3} dq \partial_t R_t(q) G(q) G(\varphi)^2 V_k''(\varphi)^2 (384w_k^4 q^8 G^5 + 24w_k^5 G^3) = \]
\[ = -8 \frac{c_d}{d+2} k^4 V_k''(\varphi)^2 (384w_k^4 k^{d+11} + 24w_k^5 k^{d+3}) \]  

Let us define the two flow equations as follows

\[ \partial_t w_k = \beta w(w, g, d), \]  
\[ \partial_t g_k = \beta g(w, g, d). \]  

In order to pursue the computation of \( \beta w \) we have to insert the parametrisation \( V_k(\varphi) = g_k(1 - \cos(\varphi)) \) and take the integral in \( \varphi \) from \(-\pi\) to \(\pi\). All terms have the same form and can be easily computed by defining the new quantities

\[ I_m^n = \int_{-\pi}^{\pi} \frac{V_k''(\varphi)}{2\pi} I_m^n = \frac{g_k^2}{2\pi} \frac{\Gamma \left( \frac{n+1}{4} \right) \Gamma \left( m - \frac{n}{4} - \frac{1}{4} \right)}{4 \Gamma(m)w_k^{\frac{n+1}{4}}} \int_{-\pi}^{\pi} \frac{\sin^2 \theta}{(k^4 + g_k \cos \theta)^{\frac{n+1}{4} - \frac{m}{4}}} = \frac{\Gamma \left( \frac{n+1}{4} \right) \Gamma \left( m - \frac{n}{4} - \frac{1}{4} \right)}{4 \Gamma(m)w_k^{\frac{n+1}{4}}} \frac{2 \left( k^8 - k^4 g_k \right) {}_2F_1 \left( -\frac{1}{2}, m - \frac{n}{4} - \frac{1}{4}; \frac{1}{2}; -\frac{2g_k}{k^4 - g_k} \right) - \left( k^4 + g_k \right) \left( 2k^4 + g_k(4m - n - 5) \right) {}_2F_1 \left( \frac{1}{2}, m - \frac{n}{4} - \frac{1}{4}; \frac{1}{2}; -\frac{2g_k}{k^4 - g_k} \right)}{(4m - n - 9)(4m - n - 5) (k^4 - g_k)^{\frac{n+1}{4}}} \]  

\[ (36) \]
Irrespectively of the choice of \( m \) and \( n \) we can define the rescaled parameter \( \tilde{g}_k = g_k/k^4 \) leading to

\[
\Gamma^n_m = 4 \frac{\Gamma \left( \frac{n+1}{4} \right) \Gamma \left( m - \frac{n}{4} - \frac{1}{4} \right)}{4\Gamma(m) \Gamma \left( \frac{n+1}{4} \right)} k^{5+n-4m} \\
\frac{2(1 - \tilde{g}_k)}{2F\left( -\frac{1}{2}, m - \frac{n}{4} - \frac{1}{4}; 1; -\frac{2g_k}{1 - g_k} \right) \left( 1 + \tilde{g}_k \right) \left( 2 + \tilde{g}_k (4m - n - 5) \right) 2F\left( \frac{1}{2}, m - \frac{n}{4} - \frac{1}{4}; 1; -\frac{2g_k}{1 - g_k} \right)}{(4m - n - 9)(4m - n - 5) (1 - \tilde{g}_k)^{4m-n-1}} \tag{37}
\]

The \( \beta \)-function for the \( w_k \) parameter in general dimension \( d \) reads,

\[
\beta w(w, g, d) = 4\nu c(d) \left( \frac{d^4}{d^4} - \frac{96w^2(8w^2g + 16w(d + 14) + 36)I_d^4}{d + 2} \right) \tag{38}
\]

The flow of the coupling \( g_k \) can be derived by noticing that, if the potential is parametrized as \( V_k(\varphi) = g_k(1 - \cos(\varphi)) \), then \( g_k \) is obtained with the help of the following projector \( P_1 \), as \( g_k = P_1[V_k(\varphi)] = \frac{1}{\pi} \int_\varphi [V_k(\varphi)] \cos(\varphi) d\varphi \), and therefore \( \beta g \) is derived by applying \( P_1 \) to both sides of Eq. (18):

\[
\beta g = -\frac{1}{\pi} \int_\varphi \partial_\varphi V_k(\varphi) \cos(\varphi) = \frac{4k^4 s_d}{\pi} \int_\varphi \cos(\varphi) d\varphi \int_0^{\infty} \frac{q^d dq}{w_k q^4 + g_k \cos(\varphi) + k^4} = \frac{4k^4 s_d}{\pi} \int_\varphi^{d-1} \cos(\varphi) d\varphi
\]

\[
= - \frac{k^4}{w_k \sin \left( \frac{\pi d}{4} \right)} \int_\varphi \left( \frac{w_k}{k^4 + g_k \cos(\varphi)} \right)^{1-\frac{d}{4}} \cos(\varphi) d\varphi = \frac{k^4}{(g_k - k^4) \sin \left( \frac{\pi d}{4} \right)} \left( k^4 - g_k \right)^{d/4} \tag{39}
\]

Our focus is the description of the BKT scaling, which appears in case of marginal scaling of the couplings, then we take the \( d \to 4^+ \) limit of the general \( \beta \)-functions, yielding

\[
\partial_d w_k = \lim_{d \to 4^+} \beta w(w, g, d) = \frac{9k^4}{16\pi^2} \frac{g_k^2}{(k^8 - g_k^2)^2}, \tag{40}
\]

\[
\partial_d g_k = \lim_{d \to 4^+} \beta g(w, g, d) = \frac{k^4}{8\pi^2 w_k g_k} \left( k^4 - \sqrt{k^8 - g_k^2} \right). \tag{41}
\]

According to the transformation \( g_k = k^4 \tilde{g}_k \), the dimensionless flow equations read

\[
\partial_t w_k = - \frac{9}{16\pi^2} \frac{\tilde{g}_k^2}{(1 - \tilde{g}_k^2)^2}, \tag{42}
\]

\[
(4 + \partial_t) \tilde{g}_k = \frac{1}{8\pi^2 w_k \tilde{g}_k} \left( 1 - \sqrt{1 - \tilde{g}_k} \right). \tag{43}
\]

in perfect correspondence with Eqs. (9) and (10) in the main text.

**Topological configurations in d=4**

In \( d = 4 \), the coordinates in spherical representation are given by

\[
\begin{align*}
&x_1 = r \sin(\phi_1) \sin(\phi_3) \sin(\phi_2) \\
&x_2 = r \sin(\phi_1) \sin(\phi_3) \cos(\phi_2) \\
&x_3 = r \sin(\phi_3) \cos(\phi_2) \\
&x_4 = r \cos(\phi_4)
\end{align*}
\]

where \( r = \sqrt{x_1^2 + x_2^2} \) and the angles \( \phi_1, \phi_3 \) are defined in the range \([0, \pi]\) while \( \phi_2 \) in the range \([0, 2\pi]\) and, by inverting the last line in Eq. (44), \( \phi_4 \) is

\[
\phi_4 = \text{ArcCos} \left( \frac{x_4}{r} \right). \tag{45}
\]
Within this representation, the laplacian $\Delta = \partial_i \partial_i$ has the following expression

$$\Delta = \frac{\partial^2}{r^4} + \frac{\partial^2_{\theta_2}}{r^2 \sin^2(\phi_4) \sin^2(\phi_3)} + \frac{\partial_{\phi_3}[\sin(\phi_4) \partial_{\phi_3}]}{r^2 \sin^2(\phi_4) \sin(\phi_3)} + \frac{\partial_{\phi_4}[\sin^2(\phi_4) \partial_{\phi_4}]}{r^2 \sin^2(\phi_4)}. \tag{46}$$

Therefore, it is easy to verify that the scalar configuration $A(r)$,

$$A(r) = \frac{1}{2} \left( \phi_4 - \frac{\pi}{2} \right) \frac{\cos(\phi_4)}{\sin(\phi_4)}, \tag{47}$$

which is essentially related to the angle $\phi_4$ between $r$ and the coordinate axis $\hat{x}_4$, yields

$$\Delta A(r) = -\frac{1}{r^2}. \tag{48}$$

The configuration $A$ depends on $\phi_4$ only, it is defined for $0 < \phi_4 < \pi$, and the shift by $-\pi/2$ in the definition makes it symmetric in the interval $[0, \pi]$ with respect to the point $\pi/2$. The concavity of $A$ turns downward and $A < 0$ everywhere, except at its maximum in $\phi_4 = \pi/2$, where $A = 0$. The factor $\cos(\phi_4)/(2\sin(\phi_4))$ makes $A$ divergent to $-\infty$, both in $\phi_4 = 0$ and $\phi_4 = \pi$, but it is essential to recover the spherical symmetry of $\Delta A(r)$ shown in Eq. (48). In fact, when dealing with an hamiltonian that contains the laplacian of the (real) field $\phi$

$$H[\phi(r)] = \frac{K}{2} \int d^4r \left[ \Delta \phi(r) \Delta \phi(r) \right], \tag{49}$$

the configuration $\phi(r) = A(r)$ does not induce any singularity along the axis $\hat{x}_4$ in the integrand in Eq. (49), with the exception of the point $r = 0$. In addition, $A(r)$ corresponds to an extremal field configuration, as it is verified with the help of Eq. (48) and by recalling the solution of the Laplace equation in $d = 4$, [53]:

$$\Delta \frac{-1}{(r - r')^2} = (2\pi)^2 \delta^4(r - r'), \tag{50}$$

which imply

$$\Delta^2 A(r) = (2\pi)^2 \delta^4(r) \tag{51}$$

i.e. $\Delta^2 A$ vanishes everywhere, with the exception of the point $r = 0$.

Therefore, due to Eq. (51), the solution in Eq. (47) can be regarded as the potential generated by a charge located at the origin, but with the standard laplacian replaced by the square laplacian. The integration of the left hand side of Eq. (51), extended to any volume containing the origin $r = 0$, gives $(2\pi)^2$, while it vanishes if $r = 0$ is external. Obviously, one can introduce a general configuration without modifying the results of the above analysis, with the singularity of Eq. (47) in $r = 0$, shifted to the generic point $r'$,

$$A_{\alpha}(r) = \frac{1}{2} \left( \alpha_4 - \frac{\pi}{2} \right) \frac{\cos(\alpha_4)}{\sin(\alpha_4)} \tag{52}$$

and now $\alpha_4$ indicates the angle between $(r - r')$ and $\hat{x}_4$.

The scaling displayed in Eq. (48) by the configuration $A(r)$, is also observed for

$$G(r - r') = \int d^4r'' \frac{1}{(2\pi)^2} \frac{1}{(r - r'')^2} \frac{1}{(r'' - r')^2} \tag{53}$$

where $r'$ indicates the location of the singularity. In fact, from Eq. (50) one finds

$$\Delta_r G(r - r') = \frac{-1}{(r - r')^2} \tag{54}$$

and

$$\Delta^2_r G(r - r') = (2\pi)^2 \delta^4(r - r'), \tag{55}$$
as observed for $A(r)$ in Eqs. (18) and (51). This indicates that $G(r - r')$ and $A_r(r)$ can be interchanged in the hamiltonian in Eq. (49) with no consequence.

In addition, by introducing a large distance spatial cut-off $R$, the integral in Eq. (54) can be solved:

$$G(r - r') = \frac{1}{4} \ln \left( \frac{R^2}{(r - r')^2} \right),$$

indicating that $G(r - r')$ decreases from large positive values to zero, when the distance $|r - r'|$ grows up to the cut-off $R$. Clearly when the limit $r \to r'$ is taken, the logarithm diverges and one must require the validity of the expression in (56) only up to a minimum distance $r_0$ from the singularity in $r'$. Then, from Eqs. (55) and (56), it is easy to calculate the energy of a single configuration (where, again, we make use of the ultraviolet cutoff $r_0$)

$$H[G] = \frac{K}{2} \pi^2 \ln \frac{R^2}{r_0^2}.$$

The same result is obtained by directly computing $H$ with the help of Eq. (54).

After computing the energy associated to a single charge, we consider the configuration associated to a distribution of charges located at different points

$$G^C(r) = \sum_i n_i G(r - r_i),$$

where $n_i \in \mathbb{Z}$ indicates the number of (positive or negative) charges at the point $r_i$ and the hamiltonian is

$$H[G^C] = \frac{K}{2} \int d^4 r \sum_{i,j} n_i n_j \left[ \Delta G(r - r_i) \right] \left[ \Delta G(r - r_j) \right] = \sum_i n_i^2 \epsilon_s + \frac{K}{2} \int d^4 r \sum_{i \neq j} n_i n_j \left[ \Delta G(r - r_i) \right] \left[ \Delta G(r - r_j) \right]$$

where we isolated the contribution due to the self-energy, indicated with $\epsilon_s$, related to the cases in which $i = j$ in the sum. Then, it is straightforward to compute the second term in the right hand side of Eq. (59), for the elementary case of two distinct charges, one located in $r_a$ and the other in $r_b$ ($n_{a,b} = \pm 1$), with the help of Eqs. (55) and (56).

$$H_{a,b} = \frac{K}{2} \int d^4 r 2n_a n_b \left[ \Delta G(r - r_a) \right] \left[ \Delta G(r - r_b) \right] =$$

$$\frac{K}{2} \int d^4 r 2n_a n_b \left[ 2\pi \right] \delta^4(r - r_a) \frac{1}{4} \ln \frac{R^2}{(r - r_b)^2} =$$

$$\frac{K}{2} \ln \frac{R^2}{|r_a - r_b|}$$

Apart from the self-energy contribution the energy coming from the interaction of two distinct charges is positive (negative) if the product $n_a n_b$ is positive (negative), according to the expectations, because the distance between the two charges $|r_a - r_b|$ is smaller than the large distance cut-off $R$.

It must be noticed that the presence of the four derivatives in Eq. (49), implies that the logarithmic scaling is peculiar of a $d = 4$ space. In fact, if we repeat the above considerations in $d = 3$, by replacing the Green function of the laplacian in Eq. (50) with the three-dimensional $-1/|r - r'|$, we find that the corresponding function $G$ grows linearly with $R$ (instead of the logarithmic growth of Eq. (56)), as it can be checked by simple dimensional analysis. Moreover, in the $d = 3$ case, the linear dependence on $R$ is also found in the computation of the energy of the configuration $G$, instead of the logarithmic dependence found in Eq. (57).

Relation with the sine-Gordon model

We now establish a relation between the hamiltonian in Eq. (59) and the sine-Gordon model in $d = 4$.

For the moment we neglect the self-energy part, $\sum_i n_i^2 \epsilon_s$, and focus on the remaining part that, according to the result in Eq. (60), can be written as

$$H_i = -\frac{K}{2} \sum_{i \neq j} n_i n_j \left( 2\pi \right)^2 \frac{1}{4} \ln \frac{|r_i - r_j|^2}{R^2} = \frac{K}{2} \left( 2\pi \right)^2 \int d^4 r_1 \int d^4 r_2 \ n(r_1) \ G(r_1 - r_2) \ n(r_2),$$

(61)
where we introduced the charge density real field

\[ n(r) = \sum_i n_i \delta^4(r - r_i). \tag{62} \]

By introducing the Fourier Transform (FT) of \( \tilde{G}(r_1 - r_2) \) according to

\[ \tilde{G}(p) = \int \frac{d^4r}{(2\pi)^2} e^{ir \cdot p} G(r) = \left( \frac{1}{p^2} \right)^2, \tag{63} \]

and also the FT \( \tilde{n}(p) \) of the density field, the Hamiltonian in Eq. (61) becomes

\[ H_s = \frac{K}{2} (2\pi)^4 \int d^4p \tilde{n}(p) \tilde{G}(p) \tilde{n}(-p) = \frac{1}{2} \int d^4p \tilde{n}(p) \left( \frac{(2\pi)^4K}{p^4} \right) \tilde{n}(-p). \tag{64} \]

The partition function associated to \( H_s \) can be written by introducing a Gaussian functional integration over an auxiliary field \( \tilde{\phi}(p) \), suitably inserted in the exponent \( (N \) is the normalization factor) :

\[ \exp[-H_s] = N \int D\tilde{\phi} \exp \left[-\frac{1}{2} \int d^4p \frac{p^4(\tilde{\phi}(p))^2}{(2\pi)^4K} + \frac{i}{2} \int d^4p \left( \tilde{\phi}(p) \tilde{n}(-p) - \tilde{\phi}(-p) \tilde{n}(p) \right) \right] \tag{65} \]

where it is understood that the inverse temperature factor \( T^{-1} \) in the partition function is absorbed here by the redefinition \( T^{-1}K \to K \). Then, by taking the inverse FT, one finds

\[ \exp[-H_s] = N \int D\phi \exp \left[-\frac{1}{2} \int d^4\phi(\Delta \phi(\Delta \phi + i \int d^4r \phi(n(r)) \right] \tag{66} \]

The full partition function includes the additional term of the Hamiltonian neglected so far

\[ Z = \exp \left[-H_s - \sum_i n_i^2 \epsilon_s \right] \tag{67} \]

and we signal the dependence of the latter and of the second integral in the exponent in Eq. (66) on \( n_i \), while the first integral does not depend on the number of charges.

Then, one can sum in the partition function over all possible configurations with either zero charge or one positive or one negative charge, located in a generic point \( r_s \), and discard the other configurations with two or more charges, that are negligible because exponentially suppressed :

\[
Z = N \int D\phi \exp \left[-\frac{1}{2} \int d^4r \Delta \phi(\Delta \phi + i \int d^4r_s e^{-\epsilon_s - i\phi(r_s)} + \int d^4r_s e^{-\epsilon_s - i\phi(r_s)} \right] \tag{68}
\]

where the two terms in the the curly bracket are summed to an exponential, and we defined

\[ w = \frac{1}{(2\pi)^4K} \tag{69} \]

\[ y = e^{-\epsilon_s} \tag{70} \]

The last line of Eq. (68) can be regarded as the partition function of the sine-Gordon model in \( d = 4 \), with four derivatives and with the parameters \( w \) and \( y \) related to \( K \) and \( \epsilon_s \) of the original model by Eqs. (69) and (70).

[1] D. J. Bishop and J. D. Reppy, Phys. Rev. Lett. 40,
[2] K. Epstein, A. M. Goldman, and A. M. Kadin, 1727 (1978).
