Quasi-Carousel Tournaments

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Abstract

A tournament is called locally transitive if the outneighbourhood and the inneighbourhood of every vertex are transitive. Equivalently, a tournament is locally transitive if it avoids the tournaments $W_4$ and $L_4$, which are the only tournaments up to isomorphism on four vertices containing a unique 3-cycle. On the other hand, a sequence of tournaments $(T_n)_{n \in \mathbb{N}}$ with $|V(T_n)| = n$ is called almost balanced if all but $o(n)$ vertices of $T_n$ have outdegree $(1/2 + o(1))n$. In the same spirit of quasi-random properties, we present several characterizations of tournament sequences that are both almost balanced and asymptotically locally transitive in the sense that the density of $W_4$ and $L_4$ in $T_n$ goes to zero as $n$ goes to infinity.

A balanced tournament $T$ is a tournament with an odd number of vertices $2n + 1$ such that every vertex of $T$ has outdegree $n$. On the other hand, a locally transitive tournament $T$ is a tournament such that the outneighbourhood $N^+(v) = \{w \in V(T) : vw \in A(T)\}$ and the inneighbourhood $N^-(v) = \{w \in V(T) : wv \in A(T)\}$ of every vertex $v$ are both transitive. With these definitions, there is only one up to isomorphism balanced locally transitive tournament $R_{2n+1}$ (see Figure 1) of order $2n + 1$ for each $n \in \mathbb{N}$.

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1This is a direct consequence of a result of Brouwer [Bro80], which is on Section 1 of this paper.
which we call the *carousel tournament*\(^2\) of order \(2n + 1\). This tournament is given by

\[
V(R_{2n+1}) = \{0, 1, \ldots, 2n\}; \quad A(R_{2n+1}) = \{(x, (x + i) \mod (2n + 1)) : i \in [n]\};
\]

where \([n] = \{1, 2, \ldots, n\} \text{ (and } [0] = \emptyset\).

\[\text{Figure 1: The tournament } R_{2n+1} \text{ for } n = 2, 3, 4.\]

Given the well-organized structure of the carousel tournaments, it is natural to expect nice asymptotic properties to hold for the sequence \((R_{2n+1})_{n \in \mathbb{N}}\) and in this note we begin studying this sequence asymptotically in two directions. In the first direction, we are simply interested in what are the asymptotic properties of \((R_{2n+1})_{n \in \mathbb{N}}\). But, in the more stimulating second direction, we are interested in the question: when does a sequence of tournaments \((T_n)_{n \in \mathbb{N}}\) “look like” the sequence \((R_{2n+1})_{n \in \mathbb{N}}\)?

Although this seems a rather vague question, it turns out that there is a notion of similarity of sequences of combinatorial objects that yields a very rich field of study. Namely, we say that two sequences of tournaments \((T_n)_{n \in \mathbb{N}}\) and \((T'_n)_{n \in \mathbb{N}}\) are equivalent if for every fixed tournament \(T\) the density of \(T\) in \(T_n\) is asymptotically equal to the density of \(T\) in \(T'_n\), that is, we have

\[
\lim_{n \to \infty} p(T, T_n) - p(T, T'_n) = 0,
\]

where \(p(T, U)\) denotes the unlabelled density of \(T\) as a subtournament of \(U\).

\(^2\)This is because in a carousel, each horse is beating half of the other horses in a circular structure.
This notion of similarity can be traced back to the theory of quasi-randomness, originated with the study of graphs sequences (by comparing with the sequence of Erdős–Rényi graphs \( G_{n,1/2} \)) in the seminal papers by Thomason [Tho87] and Chung, Graham, Wilson [CGW89] (see [KS06] for a survey) and now a field with branches in several combinatorial objects such as uniform hypergraphs [CG90, Chu12, BR13], graph orientations [Gri13], permutations [Coo04, KP13] and tournaments [CG91, KS13, CR15].

Such notion of similarity also yields a very useful notion of convergence, namely, we say that \( (T_n)_{n \in \mathbb{N}} \) is convergent if \( (|V(T_n)|)_{n \in \mathbb{N}} \) is increasing and \( (p(T, T_n))_{n \in \mathbb{N}} \) is convergent for every tournament \( T \). With this notion of convergence, one can define limit objects that codify these densities. One approach is to define the limit object to be semantically close to the underlying combinatorial objects, that is, to find a limit object that resembles the definition of the combinatorial objects (such approach was originated with the definition of graphons [LS06] and has also been taken in the definition of hypergraphons [ES12], permutons [HKM+13] and digraphons [DJ08, Section 9]). Another approach is to study the limit object syntactically, that is, to see what kind of properties the sequence \( (\phi(T))_T \) must satisfy if we have \( \phi(T) = \lim_{n \to \infty} p(T, T_n) \). This latter approach is precisely the thrust of the theory of flag algebras [Raz07] and in what follows, we will mostly use this language.

In the particular case of quasi-random tournaments, we are interested in comparing with the sequence \( (T_{1/2}(n))_{n \in \mathbb{N}} \), where \( T_{1/2}(n) \) is the random tournament of order \( n \) where each arc orientation is picked independently at random with probability \( 1/2 \). It is a straightforward exercise on distribution concentration to prove that \( (T_{1/2}(n))_{n \in \mathbb{N}} \) is a convergent sequence with probability 1 and we call its limit \( \phi_{qr} \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R}) \) in the flag algebra language the quasi-random homomorphism. It is also straightforward to prove that the sequence of carousel tournaments \( (R_{2n+1})_{n \in \mathbb{N}} \) is convergent\(^3\) and we call its limit \( \phi_R \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R}) \) the carousel homomorphism.

The theory of quasi-random tournaments was inaugurated by Chung and Graham in [CG91], where they presented not only some quasi-random tournament properties (their \( P \) properties), but also showed another class of properties (their \( Q \) properties) that were equivalent to each other but were strictly weaker than the quasi-random properties.

\(^3\)In Section 4, we also offer an alternative proof of this convergence that does not involve computing the limit of the densities \( p(T, R_{2n+1}) \).
For every $k \in \mathbb{N}$, let $\text{Tr}_k$ denote the transitive tournament of order $k$ and $\vec{C}_3$ denote the directed 3-cycle. We are particularly interested in the following $Q$ properties of a sequence of tournaments $(T_n)_{n \in \mathbb{N}}$ with $|V(T_n)| = n$.

- $Q_1$: $\lim_{n \to \infty} p(\text{Tr}_3, T_n) = 3/4$ and $\lim_{n \to \infty} p(\vec{C}_3, T_n) = 1/4$;
- $Q_2$: $p(\vec{C}_3, T_n)$ is asymptotically maximized by the sequence $(T_n)_{n \in \mathbb{N}}$;
- $Q_3$: The sequence of tournaments $(T_n)_{n \in \mathbb{N}}$ of increasing orders is \textit{almost balanced}, that is, all but $o(n)$ vertices of $T_n$ have outdegree $(1/2+o(1))n$.

Now, consider the extremal problem of minimizing the density of a fixed tournament $T$ asymptotically in a sequence of tournaments $(T_n)_{n \in \mathbb{N}}$ of increasing orders. In the language of flag algebras, this can be cleanly stated as minimizing $\phi(T)$ for $\phi \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R})$.

If $T$ is non-transitive, this problem is trivial because we can take $T_n$ to be the transitive tournament $\text{Tr}_n$ of size $n$ and we will have $p(T, T_n) = 0$ for every $n \in \mathbb{N}$.

For the transitive case, Chung and Graham’s [CG91] property $Q_2$ implies that $\phi(\text{Tr}_3)$ is minimized if and only if $\phi$ is the limit of an almost balanced sequence. Later, Griffiths [Gri13] proved that $\phi(\text{Tr}_4)$ is minimized if and only if $\phi$ is the quasi-random homomorphism $\phi_{qr}$. Finally, the minimization problem for a single tournament was closed when Griffith’s result was extended in [CR15]: for $k \geq 4$, the density $\phi(\text{Tr}_k)$ is minimized if and only if $\phi$ is the quasi-random homomorphism $\phi_{qr}$.

Now, if we consider the analogous maximization problem, the gears completely reverse: the transitive case becomes the trivial case (since $p(\text{Tr}_k, \text{Tr}_n) = 1$ for every $k \leq n$) and property $Q_2$ of Chung and Graham says that $\phi(\vec{C}_3)$ is maximized if and only if $\phi$ is the limit of an almost balanced sequence. However, this leaves the maximization problem open for every non-transitive tournament of order at least 4, thus making the maximization problem much more meaningful.

In this note, we begin studying this maximization problem by proving that for the unique tournament $R_4$ with outdegree sequence $(1, 1, 2, 2)$, the density $\phi(R_4)$ is maximized if and only if $\phi$ is the carousel homomorphism $\phi_R$. Furthermore, in the same spirit of the quasi-randomness theory, we present several properties that a sequence of tournaments has if and only if it is
equivalent to \((R_{2n+1})_{n \in \mathbb{N}}\) (and we call a sequence having these properties a \textit{quasi-carousel sequence}).

In the same flavour of the carousel tournaments, one of these properties implies that \(\phi_R\) is the only balanced locally transitive homomorphism after we extend the notions of balancedness and local transitivity to homomorphisms.

Let us also highlight another set of properties of the carousel homomorphism \(\phi_R\) that have nice analogues for the quasi-random homomorphism \(\phi_{qr}\). If \(\langle u, v \rangle \) is an arc of a tournament \(T\), all other vertices \(w \in V(T) \setminus \{u, v\}\) can be classified into four classes ("flags"):

1. \(\langle u, w \rangle, \langle v, w \rangle \in E(T)\),
2. \(\langle w, u \rangle, \langle w, v \rangle \in E(T)\),
3. \(\langle u, w \rangle, \langle w, v \rangle \in E(T)\),
4. \(\langle v, w \rangle, \langle w, u \rangle \in E(T)\).

Following and expanding a bit the notation in [Raz13], we let \(O^A(u, v), I^A(u, v), \Tr^A_3(u, v)\) and \(\vec{C}^A_3(u, v)\) denote the numbers of vertices in the four classes (taken in this order, see also Figure 2) divided by \(|V(T)| - 2\). A set of interesting characterizations of quasi-randomness says that if \(F\) is any of \(O^A, I^A, \Tr^A_3\) or \(\vec{C}^A_3\), then a sequence of tournaments \((T_n)_{n \in \mathbb{N}}\) is quasi-random if and only if \(F(u, v)\) is "nearly" 1/4 for "almost all" arcs \(\langle u, v \rangle\) (the theorems for \(O^A\) and \(I^A\) are from [CG91] and the theorems for the other two classes are from [CR15]). This can be stated formally and cleanly\(^4\) by saying that if \(\langle u_n, v_n \rangle\) is a random arc of \(T_n\) picked uniformly at random, then the sequence of random variables \((F(u_n, v_n))_{n \in \mathbb{N}}\) converges almost surely to 1/4.

In the case of the carousel homomorphism, we prove an interesting analogous characterization: a sequence \((T_n)_{n \in \mathbb{N}}\) converges to \(\phi_R\) if and only if the sequence of random variables \(F(u_n, v_n)\) converges in distribution to the uniform random variable on \([0, 1/2]\).

The note is organized as follows. In Section 1, we review some basic properties of locally transitive tournaments. In Section 2, we remind some concepts of the theory of flag algebras and of the tournament quasi-randomness theory. We also establish some basic lemmas on the flag algebra of tournaments in the same section. In Section 3, we present the main theorem.

\(^4\)But can be stated even more cleanly in the language of flag algebras using extensions of homomorphisms [Raz07, §3.2].
that characterizes the carousel homomorphism $\phi_R$, but we defer the proof of convergence of the sequence $(R_{2n+1})_{n \in \mathbb{N}}$ to Section 4. Finally, in Section 5, we present some related open problems.

1  Locally Transitive Tournaments

In this section, we remind some basic properties of locally transitive tournaments.

A tournament $T$ is called locally transitive if for every vertex $v \in V(T)$, the outneighbourhood $N^+(v) = \{w \in V(T) : vw \in A(T)\}$ and the inneghbourhood $N^-(v) = \{w \in V(T) : wv \in A(T)\}$ of $v$ are both transitive.

Let $W_4$ and $L_4$ denote the (unique) tournaments with outdegree sequences $(1, 1, 1, 3)$ and $(0, 2, 2, 2)$ respectively (i.e., these are precisely the tournaments of order 4 that have a unique copy of a directed 3-cycle $\vec{C}_3$). The following characterization follows immediately from the definition of local transitivity.

**Proposition 1.1.** A tournament $T$ is locally transitive if and only if $T$ has no copy of $W_4$ nor of $L_4$.

Note that if $v$ is a vertex of a locally transitive tournament $T$, then the arcs of $T$ induce linear orders on $N^+(v)$ and $N^-(v)$ (that is, defining $w <_T z \iff wz \in A(T)$, the restriction of the relation $<_T$ to either of these sets is a linear order). With this observation, Brouwer obtained the following properties.

**Proposition 1.2** (Brouwer [Bro80]). If $v$ is a vertex of a locally transitive tournament $T$ and $a \in N^+(v)$, then $N^+(a)$ is the union of a terminal interval of $N^+(v)$ and an initial interval of $N^-(v)$ (in the order induced by the arcs of $T$).

**Proof.** From the order induced on $N^+(v)$, it follows that $N^+(a) \cap N^+(v)$ is a terminal interval of $N^+(v)$. This means that if the proposition is false, there must exist $b, c \in N^-(v)$ such that $bc \in A(T)$, $c \in N^+(a)$ and $b \notin N^+(a)$. This implies that $a, c, v \in N^+(b)$ and $ac, cv, va \in A(T)$, hence $N^+(b)$ is not transitive, a contradiction. 

**Proposition 1.3** (Brouwer [Bro80]). A tournament $T$ is locally transitive if and only if it can be cyclically ordered in a way such that

(i) For every vertex $v \in V(T)$, the sets $N^+(v) \cup \{v\}$ and $N^-(v) \cup \{v\}$ are intervals of the cyclic order (with one endpoint being $v$);
For every vertices $v, a \in V(T)$ with $a \in N^+(v)$, the set $N^+(a)$ is the union of a terminal interval of $N^+(v)$ and an initial interval of $N^-(v)$ (in the cyclic order).

**Proof.** Suppose $T$ is a locally transitive tournament of order $n$ and let $w_0$ be one of its vertices. Let $w_1, w_2, \ldots, w_k$ be the vertices in $N^+(w_0)$ in the order induced by the arcs of $T$ and let $w_{k+1}, w_{k+2}, \ldots, w_{n-1}$ be the vertices in $N^-(w_0)$ in the order induced by the arcs of $T$.

Consider the cyclic order induced by the mapping $\mathbb{Z}_n \ni i \mapsto w_i \in V(T)$, where $\mathbb{Z}_n = \mathbb{Z}/(n\mathbb{Z})$ denotes the cyclic group of order $n$.

Trivially item (i) holds for $v = w_0$. Note also that to prove item (i) for a vertex $v$, it is enough to prove just the assertion regarding the set $N^+(v) \cup \{v\}$.

Now, since the orders on $N^+(w_0)$ and $N^-(w_0)$ induced by the arcs of $T$ coincide with the orders induced by the cyclic order defined, if $v \in N^+(w_0)$, then Proposition 1.2 implies that $N^+(v)$ is of the form

$$\{w_i, w_{i+1}, \ldots, w_k\} \cup \{w_{k+1}, w_{k+2}, \ldots, w_j\},$$

for some $i \leq j$, hence an interval of the cyclic order. Furthermore, the definition of the cyclic order implies that $w_{i-1} = v$, hence is $N^+(v) \cup \{v\}$ an interval of cyclic order with one endpoint being $v$.

Finally, suppose that $v \in N^-(w_0)$. From the definition of the cyclic order, we know that $(N^+(v) \cup \{v\}) \cap N^-(w_0)$ is an interval with endpoints $v$ and $v_{n-1}$. On the other hand, Proposition 1.2 implies that $N^+(v) \cap N^-(w_0)$ must be a terminal interval of $N^+(v)$ in the order induced by the arcs of $T$, but since this order coincides with the one induced by the cyclic order in $N^-(w_0)$, we have that $N^+(v) \cup \{v\}$ is an interval with an endpoint being $v$.

Now that item (i) is proved, we know that for every vertex $v \in V(T)$ the order induced by the arcs of $T$ in the sets $N^+(v)$ and $N^-(v)$ coincide with the ones induced by the cyclic order. With this observation, item (ii) follows directly from Proposition 1.2.

Suppose now that $T$ is not locally transitive. By Proposition 1.1, there must be a set $X$ of four vertices of $T$ that induces an occurrence of either $W_4$ or $L_4$ in $T$.

Note that any cyclic order satisfying items (i) and (ii) in $T$ must induce a cyclic order on $X$ that satisfies these items in the tournament induced by this set.

Since neither $W_4$ nor $L_4$ have a cyclic ordering satisfying both items (i) and (ii), the proof is complete. ■
Recalling that a balanced tournament is a tournament of odd order $2n + 1$ such that every vertex has outdegree $n$, we get the following corollary.

**Corollary 1.4.** For every $n \in \mathbb{N}$, there is exactly one up to isomorphism balanced locally transitive tournament $R_{2n+1}$ (see Figure 1) of order $2n + 1$ and it is given by

$$V(R_{2n+1}) = \{0, 1, \ldots, 2n\}; \quad A(R_{2n+1}) = \{(x, (x + i) \mod (2n + 1)) : i \in [n]\};$$

where $[n] = \{1, 2, \ldots, n\}$ (and $[0] = \emptyset$).

**Proof.** Trivially $R_{2n+1}$ is a balanced locally transitive tournament.

On the other hand, if $T$ is a balanced locally transitive tournament of order $2n + 1$, Proposition 1.3 gives us a cyclic ordering $f : \mathbb{Z}_{2n+1} \to V(T)$, where $\mathbb{Z}_{2n+1} = \mathbb{Z}/((2n + 1)\mathbb{Z})$ denotes the cyclic group of order $2n + 1$. It is easy to see that $f$ is an isomorphism between $R_{2n+1}$ and $T$. ■

We call $R_{2n+1}$ the **carousel tournament** of order $2n + 1$.

**Remark 1.5.** Although we define the carousel tournament $R_n$ only for odd values of $n$, our choice of notation $R$ comes from analogy with the structure of $R_4$, which is the locally transitive tournament of order 4 closest to being balanced.

## 2 Almost Balanced Tournament Sequences in Flag Algebras

In this section, we translate the results of the theory of quasi-random tournaments regarding almost balanced tournament sequences to the language of flag algebras. We also add another characterization that will be useful later on. We assume the reader has some familiarity with the basic setting of flag algebras and with the notion of extensions of homomorphisms [Raz07, §3.2].

Following the notation of [Raz07, Raz13], we consider the theory of tournaments $T_{\text{Tournaments}}$ (and we will drop this from notation when it is clear from the context). We let $0$ denote the trivial type of order 0 and 1 denote the (unique) type of order 1 as usual. We also define $A$ to be the type of order 2 such that the vertex labelled with 1 beats the other (labelled) vertex.
(see Figure 2). For a type $\sigma$, we denote the unity of the algebra $A^\sigma$ by $1_\sigma$, and, as always, the element $1_0$ is abbreviated to 1.

We have already introduced the notation $\text{Tr}_k$ to denote the transitive tournament of order $k$ and the notation for all the other tournaments of orders 3 and 4, but we repeat them below for the readers convenience.

- The tournament $\vec{C}_3$ is the 3-directed cycle;
- The tournament $R_4$ is the (unique) tournament of order 4 that has outdegree sequence $(1, 1, 2, 2)$;
- The tournament $W_4$ is the (unique) non-transitive tournament of order 4 that has a vertex with outdegree 3 (that is, there is a “winner” in $W_4$);
- The tournament $L_4$ is the (unique) non-transitive tournament of order 4 that has a vertex with indegree 3 (that is, there is a “loser” in $L_4$).
We define the 1-flag $\alpha$ as the (unique) 1-flag of order 2 in which the labelled vertex beats the unlabelled vertex and $\beta$ as the other 1-flag of order 2. We also define the following $A$-flags of order 3.

- The flag $O^A$, in which the only unlabelled vertex is beaten by both labelled vertices;
- The flag $I^A$, in which the only unlabelled vertex beats both labelled vertices;
- The flag $\text{Tr}_3^A$, which is the only remaining $A$-flag whose underlying model is $\text{Tr}_3$;
- The flag $\vec{C}_3^A$, which is the only $A$-flag whose underlying model is $\vec{C}_3$.

This is the complete list of $A$-flags of order 3.

We also follow the original notation of flag algebras when using the downward operator $\lfloor \cdot \rfloor_\sigma$ to the 0-algebra or when using $\sigma$-extensions of homomorphisms $\phi \in \text{Hom}^+(A^0, \mathbb{R})$ (which are denoted by $\phi^\sigma$). We remind that $\phi^\sigma$ can be conveniently viewed [Raz07, Definition 10] as the unique $\text{Hom}^+(A^\sigma, \mathbb{R})$-valued random variables satisfying the identities

\[
E[\phi^\sigma(F)] = \frac{\phi([F]_\sigma)}{\phi([1]_\sigma)} \tag{1}
\]

for every $F \in F^\sigma$.

Finally, we recall a very useful way to obtain the probability measure of $\phi^\sigma$.

If $F$ is a 0-flag and $\sigma$ is a type such that $p(\sigma, F) > 0$ (when regarding $\sigma$ as a 0-flag), then we consider the following random experiment. Choose uniformly at random an embedding $\theta$ of $\sigma$ in $F$ and for every Borel subset $A$ of $[0, 1]^{F^\sigma}$, define (see [Raz07, Definition 9])

\[
\mathbb{P}^\sigma_F(A) = \mathbb{P}(p^{(F, \theta)} \in A),
\]

where $p^F$ denotes the linear functional $p(\cdot, F)$, which can be regarded as a point of $[0, 1]^{F^\sigma}$.

Recall [Raz07, Theorem 3.12] that if $(F_n)_{n \in \mathbb{N}}$ is a convergent sequence converging to $\phi$, then the sequence of probability measures $(\mathbb{P}^\sigma_{F_n})_{n \in \mathbb{N}}$ on Borel subsets of $[0, 1]^{F^\sigma}$ weakly converges to the probability measure $\mathbb{P}^\sigma$ of $\phi^\sigma$. 

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We will not need these concepts in the more complicated scenario when
the smaller type is also non-trivial.

In this note, the most useful property of weak convergence of probability
measures is the following.

**Proposition 2.1.** If $X$ is a metrizable space, $\mathbb{P}$ is a Borel probability measure
on $X$ and $(\mathbb{P}_n)_{n \in \mathbb{N}}$ is a sequence of Borel probability measures on $X$, then
the following are equivalent.

- The sequence $(\mathbb{P}_n)_{n \in \mathbb{N}}$ weakly converges to $\mathbb{P}$;
- For every $A \subset X$ with $\mathbb{P}(\delta A) = 0$ (where $\delta A$ is the boundary of $A$), we have
  $$\lim_{n \to \infty} \mathbb{P}_n(A) = \mathbb{P}(A);$$
- For every $A \subset X$ open, we have
  $$\liminf_{n \to \infty} \mathbb{P}_n(A) \geq \mathbb{P}(A).$$

We have already introduced the notation $\phi_{qr}$ to denote the homomor-
phism of $\text{Hom}^+(A^0, \mathbb{R})$ corresponding to the random tournament, that is, it
is the almost sure limit of the sequence of random tournaments $(T_{1/2}(n))_{n \in \mathbb{N}}$
(where each arc orientation is picked independently at random with proba-
bility $1/2$) when the number of vertices goes to infinity.

As we said in the introduction, the $Q$ properties of Chung–Graham [CG91]
of a sequence of tournaments $(T_n)_{n \in \mathbb{N}}$ with $|V(T_n)| = n$ that we are interested
in are the following.

- $Q_1$: $\lim_{n \to \infty} p(\text{Tr}_3, T_n) = 3/4$ and $\lim_{n \to \infty} p(\vec{C}_3, T_n) = 1/4$;
- $Q_2$: $p(\vec{C}_3, T_n)$ is asymptotically maximized by the sequence $(T_n)_{n \in \mathbb{N}}$;
- $Q_3$: The sequence of tournaments $(T_n)_{n \in \mathbb{N}}$ of increasing orders is almost
  balanced, that is, all but $o(n)$ vertices of $T_n$ have outdegree $(1/2+o(1))n$.

If we assume that this sequence converges to a homomorphism $\phi \in \text{Hom}^+(A^0, \mathbb{R})$, then these properties are translated to the following properties
of $\phi$.

- $Q_1$: $\phi(\text{Tr}_3) = 3/4$ and $\phi(\vec{C}_3) = 1/4$;
• $Q_2$: $\phi(\vec{C}_3)$ is maximum, i.e., we have $\phi(\vec{C}_3) = \max\{\psi(\vec{C}_3) : \psi \in \text{Hom}^+(A^0, \mathbb{R})\}$;

• $Q_3$: $\phi^1(\alpha) = 1/2$ a.s.

Note that since $\vec{C}_3 + \text{Tr}_3 = 1_0$, it is enough to check only one of the values in $Q_1$. Furthermore, since $\alpha + \beta = 1_1$, we immediately get that $Q_3$ is equivalent to $\phi^1(\beta) = 1/2$ a.s. and equivalent to $\phi^1(\alpha) = \phi^1(\beta)$ a.s.

We call a homomorphism $\phi \in \text{Hom}^+(A^0, \mathbb{R})$ balanced if it satisfies any (and therefore all) of these properties.

We now prove a small lemma that adds one other item to this list of properties.

**Lemma 2.2.** In the theory of tournaments, if $\phi \in \text{Hom}^+(A^0, \mathbb{R})$, then $\phi(\text{Tr}_4) \geq \phi(R_4)$ with equality if and only if $\phi$ is balanced.

**Proof.** It is easy to check the following flag algebra identity.

$$\vec{C}_3 = \frac{1}{4} + \frac{1}{4}R_4 - \frac{1}{4}\text{Tr}_4.$$

From $Q_1$ and $Q_2$, we know that $\phi(\vec{C}_3) \leq 1/4$, with equality if and only if $\phi$ is balanced; this directly implies that $\phi(\text{Tr}_4) \geq \phi(R_4)$, with equality if and only if $\phi$ is balanced. $\blacksquare$

### 3 The Carousel Homomorphism

Stemming from Proposition 1.1, let us call a homomorphism $\phi \in \text{Hom}^+(A^0, \mathbb{R})$ locally transitive if we have $\phi(W_4 + L_4) = 0$.

Note that the fact that a sequence of tournaments $(T_n)_{n \in \mathbb{N}}$ converges to a locally transitive homomorphism does not imply that the tournaments are locally transitive. Rather, it only implies that the density of $W_4$ and $L_4$ go to zero as $n$ goes to infinity, that is, the sequence is only asymptotically locally transitive.

However, every locally transitive homomorphism $\phi$ is also an algebra homomorphism in the theory of locally transitive tournaments (i.e., the theory of tournaments that avoid both $W_4$ and $L_4$), hence there exists a sequence of locally transitive tournaments converging to $\phi$.

Now we claim that the sequence of carousel tournaments $(R_{2n+1})_{n \in \mathbb{N}}$ is convergent, but we defer the proof of this claim to Section 4. We will call
the limit of this sequence the *carousel homomorphism* and we will denote it by \( \phi_R \).

We now list a series of properties of a homomorphism \( \phi \in \text{Hom}^+ (\mathcal{A}^0, \mathbb{R}) \) that we will prove to hold if and only if \( \phi = \phi_R \). Property \( S_1 \) is stated just for practical reasons and the equivalence of properties \( S_1 \) and \( S_2 \) implies that \( \phi_R \) is the only homomorphism that is both balanced and locally transitive.

- \( S_1 \): \( \phi = \phi_R \);
- \( S_2 \): \( \phi \) is balanced and locally transitive;
- \( S_3 \): \( \phi \) maximizes the density of \( R_4 \), i.e., we have
  \[
  \phi(R_4) = \max\{\psi(R_4) : \psi \in \text{Hom}^+ (\mathcal{A}^0, \mathbb{R})\};
  \]
- \( S_4 \): \( \phi \) maximizes the second moment of \( \phi^A (\overline{C}_3^A) \).

For the next properties, it will be more practical to state them with free parameters \( F \) and \( q \), which will be respectively an \( A \)-algebra element and a real number (not any element and real number!).

- \( S_5(F, q) \): \( \phi^A (F) \sim U(0, q) \) (that is, the random variable \( \phi^A (F) \) is uniformly distributed in \([0, q] \));
- \( S_6(F, q) \): \( \phi \) maximizes the second moment of \( \phi^A (F) \) restricted to \( \mathbb{E} [\phi^A (F)] = q \), i.e., we have \( \mathbb{E} [\phi^A (F)] = q \) and
  \[
  \mathbb{E} [\phi^A (F)^2] = \max\{\mathbb{E} [\psi^A (F)^2] : \psi \in \text{Hom}^+ (\mathcal{A}^0, \mathbb{R}) \text{ with } \mathbb{E} [\psi^A (F)] = q \}.
  \]

We can now state the theorem.

**Theorem 3.1.** If \( F \) is an \( A \)-flag of order 3 and \( G \) is either \( O^A + I^A \) or \( \overline{C}_3^A + \text{Tr}_3^A \), then

\[
S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4 \Rightarrow S_5(F, 1/2) \Rightarrow S_6(F, 1/4) \Rightarrow S_5(G, 1) \Rightarrow S_6(G, 1/2) \Rightarrow S_1.
\]

We will establish Theorem 3.1 through a series of lemmas, enlarging the family of properties known to be equivalent after each lemma.

**Lemma 3.2.** We have \( S_1 \Leftrightarrow S_2 \).
Proof. Since $R_{2n+1}$ is both balanced and locally transitive for every $n \in \mathbb{N}$, it follows that $\phi_R$ is balanced and locally transitive.

Suppose that $\phi \in \text{Hom}^+(\mathcal{A}_0^0[T_{\text{Tournaments}}], \mathbb{R})$ satisfies $S_2$ and let $T_{-(W_4, L_4)}$ be the theory of tournaments without any occurrence of $W_4$ or $L_4$ (i.e., the theory of locally transitive tournaments). Note that $\phi$ is also an element of $\text{Hom}^+(\mathcal{A}_0^0[T_{-(W_4, L_4)}], \mathbb{R})$, hence there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of tournaments in $T_{-(W_4, L_4)}$ converging to $\phi$ and we can take this sequence to be such that $|V(T_n)|$ is odd for every $n \in \mathbb{N}$.

Since $\phi$ is balanced, we know that all but $o(|V(T_n)|)$ vertices of $T_n$ have outdegree $(1/2 + o(1))|V(T_n)|$ hence, considering the cyclic ordering of $T_n$ given by Proposition 1.3, we see that we can obtain $R_{|V(T_n)|}$ from $T_n$ by flipping $o(|V(T_n)|^2)$ arcs of $T_n$. Since this flipping operation does not change the limit homomorphism, we have that $(T_n)_{n \in \mathbb{N}}$ converges to the same limit as a subsequence of $(R_{2n+1})_{n \in \mathbb{N}}$. Therefore, we have $\phi = \phi_R$. ■

**Lemma 3.3.** We have $S_1 \iff S_3$.

Proof. Let us prove first that $\phi_R$ satisfies $S_3$.

Note that Lemma 2.2 immediately gives that $\phi(R_4) \leq 1/2$ for every $\phi \in \text{Hom}^+(\mathcal{A}_0^0, \mathbb{R})$.

Since $S_1 \iff S_2$ by Lemma 3.2, we have that $\phi_R$ is balanced, hence Lemma 2.2 gives $\phi_R(\text{Tr}_4) = \phi_R(R_4)$. But also, we have $\phi_R(W_4 + L_4) = 0$ by $S_2$, hence $\phi(\text{Tr}_4 + R_4) = 1$, which implies $\phi_R(R_4) = 1/2$.

Therefore $S_1 \implies S_3$.

Suppose now that $\phi \in \text{Hom}^+(\mathcal{A}_0^0, \mathbb{R})$ maximizes $\phi(R_4)$. Then we must have $\phi(R_4) = 1/2$. On the other hand, since $\phi(\text{Tr}_4 + R_4) \leq 1$, a double application of Lemma 2.2 implies that $\phi(\text{Tr}_4) = 1/2$ and that $\phi$ is balanced, hence $\phi$ satisfies $S_2$ (since $\phi(W_4 + L_4) = 1 - \phi(\text{Tr}_4 + R_4)$).

Therefore $S_3 \implies S_1$ (by Lemma 3.2). ■

Note that the proof of Lemma 3.3 also established the following corollary.

**Corollary 3.4.** In the theory of tournaments, if $\phi \in \text{Hom}^+(\mathcal{A}_0^0, \mathbb{R})$, then $\phi(R_4) \leq 1/2$, with equality if and only if $\phi = \phi_R$.

Let us continue with the proof of Theorem 3.1.

**Lemma 3.5.** We have $S_1 \iff S_4$. 14
Proof. Note that

\[ E\left[ \phi^A(C^A_3)^2 \right] = \frac{\phi(R_4)}{6}, \]

hence \( \phi \) maximizes the second moment of \( \phi^A(C^A_3) \) if and only if \( \phi \) maximizes the density of \( R_4 \), so the result follows from Corollary 3.4.

\[ \Box \]

Lemma 3.6. If \( F \) is an \( A \)-flag of order 3, then \( S_1 \Leftrightarrow S_5(F,1/2) \Leftrightarrow S_6(F,1/4) \).

Proof. Let us first prove that \( S_1 \) implies \( S_5(F,1/2) \).

Let \( P^A \) be the Borel probability measure of \( \phi^A \) and for every \( a \leq b \), let

\[ B_{a,b}(F) = \{ x \in [0,1]^F : a < x_F < b \}. \]

Note that \( B_{a,b} \) is an open subset of \([0,1]^F\). Since \((P^A_{R_{2n+1}})_{n \in \mathbb{N}}\) weakly converges to \( P^A \), by Proposition 2.1, it is enough to prove that

\[ \liminf_{n \to \infty} P^A_{R_{2n+1}}(B_{a,b}(F)) = 2(b-a), \]

for every \( 0 \leq a \leq b \leq 1/2 \); and

\[ \liminf_{n \to \infty} P^A_{R_{2n+1}}(B_{a,b}(F)) = 1 - 2a, \]

for every \( 0 \leq a \leq 1/2 \leq b \leq 1 \).

Recall the definition of \( P^A_{R_{2n+1}} \): consider the random experiment where we pick at random an embedding \( \theta \) of \( A \) in \( R_{2n+1} \), then we have

\[ P^A_{R_{2n+1}}(B_{a,b}(F)) = P(a < p(F,L_{2n+1}) < b), \]

where \( L_{2n+1} \) is the random \( A \)-flag \((R_{2n+1}, \theta)\).

Note that since \( \theta \) is an embedding of \( A \) in \( R_{2n+1} \), we must have

\[ \theta(2) = (\theta(1) + i) \mod (2n + 1), \]

for some (random) \( i \in [n] \). Note also that from the symmetry of \( R_{2n+1} \), the variable \( i \) has uniform distribution in \([n]\).

Let \( j \in [2n] \) and \( J = \{ \theta(1), \theta(2), (\theta(1) + j) \mod (2n + 1) \} \). Note that we have the following (see Figure 3).
If $j < i$, then $J$ induces an occurrence of $\text{Tr}_3^A$;

- If $i < j \leq n$, then $J$ induces an occurrence of $O^A$;

- If $n < j \leq i + n$, then $J$ induces an occurrence of $\vec{C}_3^A$;

- If $i + n < j$, then $J$ induces an occurrence of $I^A$.

This implies that

\[
\begin{align*}
    p(\text{Tr}_3^A, L_{2n+1}) &= \frac{i - 1}{2n - 1}; \\
    p(O^A, L_{2n+1}) &= \frac{n - i}{2n - 1}; \\
    p(\vec{C}_3^A, L_{2n+1}) &= \frac{i}{2n - 1}; \\
    p(I^A, L_{2n+1}) &= \frac{n - i}{2n - 1}.
\end{align*}
\]

Hence, since $i$ has uniform distribution over $[n]$, we get that $p(\text{Tr}_3^A, L_{2n+1})$, $p(O^A, L_{2n+1})$ and $p(I^A, L_{2n+1})$ have uniform distribution over $\{t/(2n - 1) : t \in \{0, 1, \ldots, n - 1\}\}$. Moreover $p(\vec{C}_3^A, L_{2n+1})$ has uniform distribution over $\{t/(2n - 1) : t \in [n]\}$.

Letting $n \to \infty$, it follows that

\[
\liminf_{n \to \infty} \mathbb{P}_{R_{2n+1}}^A (B_{a,b}(F)) = 2(b - a),
\]
for every $0 \leq a \leq b \leq 1/2$; and
\[
\liminf_{n \to \infty} \mathbb{P}^{A}_{R_{2n+1}}(B_{a,b}(F)) = 1 - 2a,
\]
for every $0 \leq a \leq 1/2 \leq b \leq 1$ as desired.

Therefore $S_1 \implies S_5(F, 1/2)$.

Now let us prove that $S_5(F, 1/2) \implies S_6(F, 1/4)$.
Suppose $\psi$ is such that $E[\psi^A(F)] = 1/4$.
If $F = \vec{C}_3^A$, then we have
\[
\frac{1}{4} = E[\psi^A(F)] = \psi(\vec{C}_3),
\]
hence $\psi$ is balanced.

If $F$ is one of $O^A$, $I^A$ or $\text{Tr}_3^A$, then we have
\[
\frac{1}{4} = E[\psi^A(F)] = \frac{\psi(\text{Tr}_3)}{3},
\]
which yields $\psi(\text{Tr}_3) = 3/4$, hence $\psi$ is balanced.

Therefore every $\psi$ with $E[\psi^A(F)] = 1/4$ must be balanced.
Since the second moment of a $U(0, 1/2)$-random variable is $1/12$, it is
enough to prove that if $\psi$ is balanced, then $\psi^A(F) \leq 1/12$.
If $F = \vec{C}_3^A$, then we have
\[
E[\psi^A(F)^2] = \frac{\psi(R_4)}{6} \leq \frac{1}{12},
\]
since the maximum value of $\psi(R_4)$ is $1/2$ (Corollary 3.4).

On the other hand, if $F$ is one of $O^A$, $I^A$ or $\text{Tr}_3^A$, then we have
\[
E[\psi^A(F)^2] = \frac{\psi(\text{Tr}_3)}{6} = \frac{\psi(R_4)}{6} \leq \frac{1}{12},
\]
by Lemma 2.2 and Corollary 3.4.

Therefore $S_5(F, 1/2) \implies S_6(F, 1/4)$.

Finally, let us prove that $S_6(F, 1/4)$ implies $S_1$.
If $\phi$ satisfies $S_6(F, 1/4)$, we have already proved that it must be balanced
(since $E[\phi^A(F)] = 1/4$) and from the equation part of (2) and (3) and the
fact that the second moment of a $U(0, 1/2)$-random variable is $1/12$, we have
that $\phi(R_4) \geq 1/2$, hence $\phi = \phi_R$ by Corollary 3.4.

\[\blacksquare\]
Lemma 3.7. If \( G \) is either \( O^A + I^A \) or \( \vec{C}_3^A + \text{Tr}_3^A \), then \( S_1 \iff S_5(G, 1) \iff S_6(G, 1/2) \).

Proof. (The proof is somewhat analogous to the proof of Lemma 3.6.)
To prove that \( S_1 \implies S_5(G, 1) \), repeat the part \( S_1 \implies S_5(F, 1/2) \) of the proof of Lemma 3.6 and note that since
\[
\begin{align*}
p(\text{Tr}_3^A, L_{2n+1}) &= \frac{i-1}{2n-1}; \\
p(O^A, L_{2n+1}) &= \frac{n-i}{2n-1}; \\
p(\vec{C}_3^A, L_{2n+1}) &= \frac{i}{2n-1}; \\
p(I^A, L_{2n+1}) &= \frac{n-i}{2n-1};
\end{align*}
\]
we have that \( p(O^A + I^A, L_{2n+1}) \) has uniform distribution on \( \{2t/(2n-1) : t \in \{0, 1, \ldots, n-1\}\} \) and that \( p(\vec{C}_3^A + \text{Tr}_3^A, L_{2n+1}) \) has uniform distribution on \( \{(2t-1)/(2n-1) : t \in [n]\} \).

Letting \( n \to \infty \), it follows that
\[
\lim \inf_{n \to \infty} \mathbb{P}_n^A( a < p(F, L_{2n+1}) < b ) = b - a,
\]
for every \( 0 \leq a \leq b \leq 1 \), which implies \( S_5(G, 1) \).

Now let us prove that \( S_5(G, 1) \implies S_6(G, 1/2) \).
Suppose \( \psi \) is such that \( \mathbb{E}[\psi^A(G)] = 1/2 \).
If \( G = O^A + I^A \), then we have
\[
\frac{1}{2} = \mathbb{E}[\psi^A(G)] = \frac{2\psi(\text{Tr}_3)}{3},
\]
which yields \( \psi(\text{Tr}_3) = 3/4 \), hence \( \psi \) is balanced.
If \( G = \vec{C}_3^A + \text{Tr}_3^A \), then we have
\[
\frac{1}{2} = \mathbb{E}[\psi^A(G)] = \psi(\vec{C}_3) + \frac{\psi(\text{Tr}_3)}{3} = \frac{1}{3} + \frac{2\psi(\vec{C}_3)}{3},
\]
which yields \( \psi(\vec{C}_3) = 1/4 \), hence \( \psi \) is balanced.

Therefore every \( \psi \) with \( \mathbb{E}[\psi^A(G)] = 1/2 \) must be balanced.
Since the second moment of a \( U(0, 1) \)-random variable is \( 1/3 \), it is enough to prove that if \( \psi \) is balanced, then \( \psi^A(G) \leq 1/3 \).
But note that, if \( G = O^A + I^A \), then we have
\[
\mathbb{E}[\psi^A(G)^2] = \frac{\text{Tr}_4}{2} + \frac{R_4}{6} = \frac{2R_4}{3} \leq \frac{1}{3},
\]
(4)
by Lemma 2.2 and Corollary 3.4.

Furthermore, if $G = \bar{C}_3^A + \text{Tr}_3^A$, then we have

$$
\mathbb{E}[\phi^A(G)^2] = \frac{\text{Tr}_4}{6} + \frac{R_4}{3} = \frac{2R_4}{3} \leq \frac{1}{3},
$$

also by Lemma 2.2 and Corollary 3.4.

Therefore $S_6(G, 1) \implies S_6(G, 1/2)$.

Finally, let us prove that $S_6(G, 1/2)$ implies $S_1$.

If $\phi$ satisfies $S_6(G, 1/2)$, we have already proved that it must be balanced (since $\mathbb{E}[\phi^A(G)] = 1/2$) and from equation part of (4) and (5) and the fact that the second moment of a $U(0, 1)$-random variable is 1/3, we have that $\phi(R_4) \geq 1/2$, hence $\phi = \phi_R$ by Corollary 3.4.

This finishes the proof of Theorem 3.1.

4 Convergence of the Sequence $(R_{2n+1})_{n \in \mathbb{N}}$

We present now the proof that the sequence of carousel tournaments $(R_{2n+1})_{n \in \mathbb{N}}$ is convergent. The proof can be obtained by reinterpreting the proof of Lemma 3.2.

Proposition 4.1. The sequence $(R_{2n+1})_{n \in \mathbb{N}}$ is convergent.

Proof. From compactness of $[0, 1]^{\mathcal{F}_0}$, we know that $(R_{2n+1})_{n \in \mathbb{N}}$ must have a convergent subsequence, so for every infinite set $I \subset \mathbb{N}$ of indexes such that the subsequence $(R_{2i+1})_{i \in I}$ converges, let $\phi_I \in \text{Hom}(\mathcal{A}_0, \mathbb{R})$ be its limit. For convenience, let $\mathcal{C}$ be the set of all $I \subset \mathbb{N}$ such that $(R_{2i+1})_{i \in I}$ converges.

Now we repeat the proof of Lemma 3.2 using an arbitrary $I \in \mathcal{C}$.

For the forward implication $S_1 \implies S_2$, since $R_{2n+1}$ is both balanced and locally transitive, we have that $\phi_I$ is balanced and locally transitive for every $I \in \mathcal{C}$.

The proof of implication $S_2 \implies S_1$ proceeds a little bit differently: we pick the sequence $(T_n)_{n \in \mathbb{N}}$ of locally transitive tournaments converging to $\phi$ to be such that

$$
\{|V(T_n)| : n \in \mathbb{N}\} \subset \{2i + 1 : i \in I\}.
$$

To see that this can be done, recall [Raz07, Theorem 3.3b] that if we define the probability measure $\mathbb{P}_n$ over $\mathcal{F}_n$ as $\mathbb{P}_n(F) = \phi(F)$ and we pick
independently at random for every \( n \in \mathbb{N} \) the 0-flag \( F_n \) according to the measure \( \mathbb{P}_{f(n)} \), where \( f(n) = \Omega(n^2) \), then the sequence \( (F_n)_{n \in \mathbb{N}} \) converges almost surely to \( \phi \). Since \( I \) infinite, we can certainly pick \( f \) such that both \( f(n) = \Omega(n^2) \) and \( f(\mathbb{N}) \subset \{2i + 1 : i \in I\} \) hold. Thus almost every sample of \( (F_n)_{n \in \mathbb{N}} \) is a desired sequence \( (T_n)_{n \in \mathbb{N}} \).

Again, since \( \phi \) is balanced, we know that we can obtain \( R_{|V(T_n)|} \) from \( T_n \) by flipping \( o(|V(T_n)|^2) \) arcs of \( T_n \) and since this flipping operation does not change the limit homomorphism, we have that the sequence \( (T_n)_{n \in \mathbb{N}} \) converges to the same limit as a subsequence of \( (R_{2i+1})_{i \in I} \), hence \( \phi = \phi_I \).

But this means that, if \( I, J \in \mathcal{C} \), then, we have

\[
S_2(\phi_J) \implies \phi_J = \phi_I,
\]

hence every convergent subsequence of \( (R_{2n+1})_{n \in \mathbb{N}} \) converges to the same homomorphism, therefore it must be a convergent sequence from compactness of \([0, 1]^{\mathbb{F}_0}\). ■

We remark that the convergence of \( (R_{2n+1})_{n \in \mathbb{N}} \) can also be proved directly and that a limit of this sequence in the theory of digraphons (see [DJ08, Section 9]) can be constructed as follows.

**Proposition 4.2.** Using the quintuple definition of digraphons, let \( W_{00}, W_{11} : [0, 1]^2 \to [0, 1] \) be the identically zero functions on \([0, 1]^2\) and \( w : [0, 1] \to [0, 1] \) be the identically zero function on \([0, 1] \). Furthermore, define the functions \( W_{01}, W_{10} : [0, 1]^2 \to [0, 1] \) as follows (see Figure 4).

\[
W_{01}(y, x) = W_{10}(x, y) = \begin{cases} 
1, & \text{if } (x - y) \mod 1 < 1/2; \\
0, & \text{if } (x - y) \mod 1 \geq 1/2.
\end{cases}
\]

Under these definitions, the sequence \( (R_{2n+1})_{n \in \mathbb{N}} \) converges to the digraphon \((W_{00}, W_{01}, W_{10}, W_{11}, w) \in \mathcal{W}_5\), that is, for every tournament \( T \) with \( V(T) = [k] \), we have

\[
\lim_{n \to \infty} p(T, T_n) = \frac{k!}{|\text{Aut}(T)|} \int_{[0, 1]^k} \prod_{i,j \in A(T)} W_{10}(x_i, x_j) dx_1 dx_2 \cdots dx_k,
\]

where \( \text{Aut}(T) \) denotes the group of automorphisms of \( T \).

**Remark.** The factor \( k!/|\text{Aut}(T)| \) comes from the fact that \( p \) measures unlabelled subtournament density and the integral on the right-hand side measures labelled subtournament density.
Figure 4: The function $W_{10}$ of Proposition 4.2. The gray area represents where the function has value 1, the white area represents where the function has value 0.

5 Concluding Remarks and Open Problems

As we mentioned in the introduction, the problem of minimizing $\phi(T)$ for a fixed tournament $T$ is completely closed but the analogous maximization problem is still open for very small tournaments. Corollary 3.4 completely solves the maximization of $\phi(R_4)$, this leaves only one case of order 4 still open since maximizing $\phi(W_4)$ is analogous to maximizing $\phi(L_4)$ by flipping all arcs.

For the particular problem of maximizing $\phi(W_4)$, consider the following construction (see Figure 5). Let $N$ be an arbitrarily large integer and $t \in (0,1)$. Define recursively the sequence $A_0, A_1, \ldots$ by taking $A_0 = [N]$ and by letting $A_i$ be a subset of $A_{i-1}$ with size $t |A_i|$ (rounded to the nearest integer) for every $i > 0$. Define the random tournament $S_{N,t}$ through the following procedure: let $V(S_{N,t}) = [N] = A_0$, for every $i > 0$, every $v \in A_i$ and every $w \in A_{i-1} \setminus A_i$, let $(v, w) \in A(S_{N,t})$ and pick all the remaining arc orientations independently at random with probability $1/2$. That is, for every $i > 0$, if $k = |A_{i-1} \setminus A_i|$, then the set $A_{i-1} \setminus A_i$ spans $T_{1/2}(k)$.

It is (somewhat) easy to see that $(S_{N,t})_{N \in \mathbb{N}}$ converges almost surely to a
Figure 5: Typical structure of the random tournament $S_{N,t}$. The arcs in the picture represent arcs between vertices in distinct parts $A_{i-1} \setminus A_i$. The arcs completely contained any part $A_{i-1} \setminus A_i$ have their orientation picked independently at random with probability $1/2$ for each orientation. This figure uses $t = 0.65$, which makes it easier to see the structure of the construction but is far from the value of $t$ that maximizes $\phi_t(W_4)$.

The limit homomorphism $\phi_t$ such that

$$\phi_t(W_4) = (1 - t)^3 \left( t + \frac{1 - t}{8} \right) / (1 - t^4).$$

Certainly, every value of $\phi_t(W_4)$ for $t \in (0, 1)$ is a lower bound for the maximization problem for $W_4$. The maximum of $\phi_t(W_4)$ (which can be computed with standard calculus arguments) is

$$\max\{\phi_t(W_4) : t \in (0, 1)\} = 1 + \frac{3^{5/3} - 3^{7/3}}{8} \approx 0.157501,$$

attained when $t$ is equal to

$$\frac{2 \cdot 3^{2/3} - 3^{1/3} - 2}{5} \approx 0.143584.$$

We conjecture that this is actually the maximum value of $\phi(W_4)$ for $\phi \in \text{Hom}^+(A^0, \mathbb{R})$. 

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Conjecture 5.1. In the theory of tournaments, we have

\[
\max \{ \phi(W_4) : \phi \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R}) \} = 1 + \frac{3^{5/3} - 3^{7/3}}{8}.
\]

Using the flag algebra semidefinite method, we were able to obtain the bound

\[
\forall \phi \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R}), \phi(W_4) \leq 0.157516,
\]

subject to floating point rounding errors. This is suggests that the conjecture is true and that there may be a straightforward (but numerically intensive) proof using the semidefinite method and rounding techniques (see [BHL+13, CKP+13, DHM+13, FRV13, PV13] for some examples).

The intuition of the recursive construction of \( S_{N,t} \) is that at every step we have one part \( A_{i-1} \setminus A_i \) that maximizes the density of \( \vec{C}_3 \) (hence is almost balanced) and another part \( A_i \) whose vertices all beat the first part. This maximizes the occurrences of \( W_4 \) with exactly one vertex in the latter part, and since only one vertex is being selected in it, we might as well repeat this structure recursively in \( A_i \).

In this particular construction, we chose the almost balanced part to be quasi-random. However, one might wonder if this is the best we can do in the class of almost balanced tournaments to maximize the density of \( W_4 \), but the following couple of lemmas show that this is indeed the case.

Lemma 5.2. In the theory of tournaments, if \( \phi \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R}) \) is balanced, then \( \phi(W_4) = \phi(L_4) \).

Proof. Since \( \phi \) is balanced, we have \( \phi^1(\alpha) = \phi^1(\beta) \) a.s. In particular, this means that

\[
\frac{\phi(\text{Tr}_4 + W_4)}{4} = \mathbb{E}[\phi^1(\alpha)^3] = \mathbb{E}[\phi^1(\beta)^3] = \frac{\phi(\text{Tr}_4 + L_4)}{4},
\]

hence \( \phi(W_4) = \phi(L_4) \). \( \blacksquare \)

Lemma 5.3. In the theory of tournaments, if \( \phi \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R}) \) is balanced, then \( \phi(W_4) \leq 1/8 \) with equality if and only if \( \phi \) is the quasi-random tournament \( \phi_{qr} \).
Proof. Property $P_2$ of Chung–Graham [CG91] says\footnote{In their paper, Chung and Graham work with labelled non-induced densities instead of unlabelled induced densities, so a straightforward translation is necessary to get this value.} that if $\psi \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R})$, then $\psi(\text{Tr}_4 + R_4) \geq 3/4$ with equality if and only if $\psi = \phi_{qr}$, hence $\psi(W_4 + L_4) \leq 1/4$ with equality if and only if $\psi = \phi_{qr}$.

On the other hand, since $\phi$ is balanced, Lemma 5.2 implies that $\phi(W_4) = (\phi(W_4 + L_4))/2 \leq 1/8$.

Since $\phi_{qr}$ is also balanced, the result follows.  \hfill \blacksquare

Focusing back on the carousel homomorphism, as we mentioned on Remark 1.5, the choice of the notation $R_{2n+1}$ comes from the similarity of the structure of these tournaments with the structure of $R_4$. Given this structural similarity, the following conjecture is natural.

**Conjecture 5.4.** For every $n \in \mathbb{N}$, the carousel homomorphism $\phi_R$ maximizes the density of $R_{2n+1}$, that is, we have

$$\max\{\phi(R_{2n+1}) : \phi \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R})\} = \phi_R(R_{2n+1}).$$

And if the above conjecture is true, then naturally the following conjecture arises.

**Conjecture 5.5.** For every $n \geq 2$, a homomorphism $\phi \in \text{Hom}^+(\mathcal{A}^0, \mathbb{R})$ maximizes the density of $R_{2n+1}$ if and only if $\phi = \phi_R$.

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**References**

[BHL+13] József Balogh, Ping Hu, Bernard Lidický, Oleg Pikhurko, Balázs Udvari, and Jan Volec. Minimum number of monotone subsequences of length 4 in permutations. 2013. Pre-print available at http://homepages.warwick.ac.uk/ maskat/Papers/monoSeq.pdf.
[BR13] Vindya Bhat and Vojtěch Rödl. Note on upper density of quasi-random hypergraphs. *Electron. J. Combin.*, 20(2):Paper 59, 8, 2013.

[Bro80] A. E. Brouwer. *The enumeration of locally transitive tournaments*, volume 138 of *Afdeling Zuivere Wiskunde [Department of Pure Mathematics]*. Mathematisch Centrum, Amsterdam, 1980.

[CG90] F. R. K. Chung and R. L. Graham. Quasi-random hypergraphs. *Random Structures Algorithms*, 1(1):105–124, 1990.

[CG91] F. Chung and R. Graham. Quasi-random tournaments. *J. Graph Theory*, 15(2):173–198, 1991.

[CGW89] F. Chung, R. Graham, and R. Wilson. Quasi-random graphs. *Combinatorica*, 9:345–362, 1989.

[Chu12] Fan Chung. Quasi-random hypergraphs revisited. *Random Structures Algorithms*, 40(1):39–48, 2012.

[CKP+13] James Cummings, Daniel Král’, Florian Pfender, Konrad Sperfeld, Andrew Treglown, and Michael Young. Monochromatic triangles in three-coloured graphs. *J. Combin. Theory Ser. B*, 103(4):489–503, 2013.

[Coo04] Joshua N. Cooper. Quasirandom permutations. *J. Combin. Theory Ser. A*, 106(1):123–143, 2004.

[CR15] Leonardo Nagami Coregliano and Alexander Razborov. On the density of transitive tournaments. 2015. Submitted. Pre-print available at [http://arxiv.org/abs/1501.04074](http://arxiv.org/abs/1501.04074).

[DHM+13] Shagnik Das, Hao Huang, Jie Ma, Humberto Naves, and Benny Sudakov. A problem of Erdős on the minimum number of k-cliques. *J. Combin. Theory Ser. B*, 103(3):344–373, 2013.

[DJ08] Persi Diaconis and Svante Janson. Graph limits and exchangeable random graphs. *Rend. Mat. Appl. (7)*, 28(1):33–61, 2008.

[ES12] Gábor Elek and Balázs Szegedy. A measure-theoretic approach to the theory of dense hypergraphs. *Adv. Math.*, 231(3-4):1731–1772, 2012.
[FRV13] Victor Falgas-Ravry and Emil R. Vaughan. Applications of the semi-definite method to the Turán density problem for 3-graphs. *Combin. Probab. Comput.*, 22(1):21–54, 2013.

[Gri13] Simon Griffiths. Quasi-random oriented graphs. *J. Graph Theory*, 74(2):198–209, 2013.

[HKM+13] Carlos Hoppen, Yoshiharu Kohayakawa, Carlos Gustavo Moreira, Balázs Ráth, and Rudini Menezes Sampaio. Limits of permutation sequences. *J. Combin. Theory Ser. B*, 103(1):93–113, 2013.

[KP13] Daniel Král’ and Oleg Pikhurko. Quasirandom permutations are characterized by 4-point densities. *Geom. Funct. Anal.*, 23(2):570–579, 2013.

[KS06] M. Krivelevich and B. Sudakov. Pseudo-random graphs. In *More Sets, Graphs and Numbers*, Bolyai Society Mathematical Studies 15, pages 199–262. Springer-Verlag, 2006.

[KS13] S. Kalyanasundaram and A. Shapira. A note on even cycles and quasirandom tournaments. *J. Graph Theory*, 73(3):260–266, 2013.

[LS06] László Lovász and Balázs Szegedy. Limits of dense graph sequences. *J. Combin. Theory Ser. B*, 96(6):933–957, 2006.

[PV13] Oleg Pikhurko and Emil R. Vaughan. Minimum number of $k$-cliques in graphs with bounded independence number. *Combin. Probab. Comput.*, 22(6):910–934, 2013.

[Raz07] A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.

[Raz13] A. Razborov. On the Caccetta-Haggkvist conjecture with forbidden subgraphs. *J. Graph Theory*, 74(2):236–248, 2013.

[Tho87] A. Thomason. Pseudo-random graphs. *Ann. of Discrete Math.*, 33:307–331, 1987.