Quantum limited sensitivity of SET-based displacement detectors

D. Mozyrsky, I. Martin, and M. B. Hastings

Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
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We consider a model of a quantum-mechanical resonator capacitively coupled to a single electron transistor (SET). The tunnel current in the SET is modulated by the vibrations of the resonator, and thus the system operates as a displacement detector. We analyze the effect of the back-action noise of charge fluctuations in the SET onto the dynamics of the resonator and evaluate the displacement sensitivity of the system. The relation between the “classical” and “quantum” parts of the SET charge noise and their effect on the measured system are also discussed.

Micro-mechanical resonators have been used as ultra-sensitive force detectors in a number of experimental applications, ranging from Atomic Force Microscopy to Magnetic Resonance Force Microscopy [1] to experiments on Casimir force detection [2]. Recently, mechanical resonators with vibrational eigenfrequencies (ν) of the order of 1 GHz have been fabricated [3]. At low temperatures (hν > k_BT ∼ 50 mK), these resonators provide an example of a man-made system that can be used to test the basic principles of quantum mechanics at the macroscopic level.

The standard cantilever displacement measurement schemes are based on laser interferometry, and can reach the levels of sensitivity of the order 10^-4 Å/√Hz. This level of sensitivity requires, however, high laser power that may not be compatible with the ultra-low temperature operation. This limitation provided the motivation to explore alternative electrical measurement schemes [4–6]. In particular, Blencowe and Wyborne [4] have suggested based on a semiclassical analysis that by capacitively coupling the cantilever to a Single Electron Transistor (SET), it is possible to achieve the sensitivity better than the zero-point-motion uncertainty. More recently, two of us [6] have found based on a fully quantum mechanical description of the quantum measurement of a cantilever using a quantum point contact (QPC), that the apparatus back action (current shot noise that induces force noise on the cantilever) fundamentally limits the displacement sensitivity and leads to a quantum-to-classical transition in the oscillator dynamics. Due to the resonant nature of transport through SET, it is expected that the device has a single resonant level. The mechanical system, in the following referred to as an oscillator, which can be either a micro-mechanical resonator or a localized phonon mode, is capacitively coupled to the resonant level. Therefore, the displacement of the oscillator can alter the position of the resonant level with respect to the chemical potentials in the leads and thus can affect the tunnel current through the device.

The model is schematically presented in Fig. 1. The measuring apparatus is a single electron transistor: a quantum dot coupled to the leads via tunnel junctions. For simplicity we assume that the dot contains a single resonant level. The mechanical system, in the following referred to as an oscillator, which can be either a micro-mechanical resonator or a localized phonon mode, is capacitively coupled to the resonant level. Therefore, the displacement of the oscillator can alter the position of the resonant level with respect to the chemical potentials in the leads and thus can affect the tunnel current through the device.

The Hamiltonian of the model can be written as

\[ H = H_{\text{leads}} + H_{\text{osc}} + H', \]

where the first two terms are the Hamiltonians of the electrons in the leads and the oscillator respectively, \( H_{\text{leads}} = \sum_{q,n=L,R} e_{qn} c_{qn}^\dagger c_{qn} \) and \( H_{\text{osc}} = (m/2) \partial_x^2 + V(x) \). Here \( c_{q,n=L(R)}^\dagger \) (\( c_{q,n=L(R)} \)) creates (annihilates) an electron with a quantum number \( q \) in the left (right) lead. And the electron charge \( e \) and the oscillator coordinate \( x \) is equal to unity, unless stated otherwise. The single par-
particle states $E_{qn}$ are filled up to chemical potentials in the leads, $\mu_L$ and $\mu_R$, which are biased by external voltage, $\mu_L - \mu_R = V$, see Fig. 1. For simplicity we assume zero external temperature. The Hamiltonian $H'$ includes both electron tunneling and modulation of the position of the resonant level by the oscillator:

$$H' = \sum_{q,n=L,R} T_n (d^\dagger c_{qn} + c^\dagger_{qn} d) + E_0(x)\hat{n}_0.$$  \hspace{1cm} (2)

In the first, tunneling, term of the Hamiltonian (2), the operator $d^\dagger (d)$ creates (annihilates) an electron in the resonant level, $\hat{n}_0 = d^\dagger d$, and the tunneling amplitude $T_n$ is assumed to be independent of the single particle states in the leads. We assume that the energy of the resonant level $E_0(x)$ depends linearly on the oscillator’s coordinate $x$, i.e. $E_0(x) = e_0 + \lambda x$. The unperturbed position of the resonant level $e_0$ can be varied by appropriately adjusting the gate voltage $V_0$, see Fig. 1. The parameter $\lambda$ physically represents an effective electric field in the capacitor formed by the oscillator and the quantum dot.

We use the Keldysh-Feynman-Vernon formalism \[8,9\] to determine the evolution of the oscillator under the influence of the tunneling electrons. We define a scattering operator for the oscillator alone, i.e. with electronic degrees of freedom traced out:

$$S_{osc} = \text{Tr}_{el} [\rho_{el} T_c S(-\infty, \infty)S(\infty, -\infty)] / \text{Tr} [\rho_{el}].$$  \hspace{1cm} (3)

In Eq. (3) $S(-\infty, -\infty)$ and $S(\infty, -\infty)$ are scattering operators for the full system, $S(\infty, -\infty) = \exp[-i \int_{-\infty}^{\infty} H dt]$, where $H$ is defined in Eqs. (1,2), and the operator $T_c$ denotes time ordering along the Keldysh contour. The density matrix of the unperturbed electrons is the direct product of the uncoupled density matrices of electron reservoirs in the leads ($\rho_D = d^\dagger d$) with an empty electron state in the resonant level ($\rho_{el} = \rho_L \otimes \rho_R \otimes \rho_D$). Eq. (3) implies that at $t = -\infty$ the leads, the resonant level, and the oscillator are uncoupled and that the interaction, $H'$, is switched on adiabatically at $t > -\infty$.

In what follows we assume that the coupling constant $\lambda$ is small, while the tunneling amplitudes $T_L$ and $T_R$ need not be small. Then, $S_{osc}$ can be written explicitly as a functional integral over the oscillator coordinate as

$$S_{osc} = \int D x e^{\int_c i dt' \mathcal{L}_{osc}'} \times \exp \left[ -\frac{\lambda^2}{2} \int_c dt_1 dt_2 x(t_1)x(t_2)K(t_1 - t_2) + \ldots \right].$$  \hspace{1cm} (4)

In this work we limit ourselves to $O(\lambda^2)$ contribution to the effective action of the oscillator. The higher orders in the expansion (denoted by $\ldots$ in Eq. (4)) are unimportant in the limit of strong tunneling as will be seen below.

The first order contribution in $\lambda$, i.e. interaction of the oscillator with average charge $\langle \hat{n}_0 \rangle$ in the dot, is included in the Lagrangian of the oscillator in Eq. (4), $\mathcal{L}_{osc} = \mathcal{L}_{osc}^{(bare)} - \lambda \langle \hat{n}_0 \rangle_{el} x$. The charge $\langle \hat{n}_0 \rangle$ is related to the Fourier transform of the renormalized single particle Green’s function $G_D(t' - t) = -i \langle \mathcal{T} d(t) d^\dagger(t') \rangle_{el}$ as $\langle \hat{n}_0 \rangle_{el} = (1/2\pi i) \int d\omega G_D^{-\dagger}(\omega)$. The averaging denoted by $\langle \cdot \rangle_{el}$ is taken with respect to the exact stationary state of the electronic subsystem alone, i.e. decoupled from the oscillator. The renormalization of $G_D(t)$ by the tunneling transitions can be obtained by a standard calculation \[11\]. Following the notation of Ref. \[8,10\] we define a matrix Green’s function

$$G^{ij}_{D}(t_2 - t_1) = -i \langle d^\dagger (t'_1) d^\dagger (t'_2) \rangle_{el},$$

where $t'_1$ and $t'_2$ can either be on the same or different Keldysh contours, i.e. $i,j = \pm$. We also introduce the unperturbed Green’s functions of the electrons in the left and the right leads

$$G^{ij}_{D,0}(t_1 - t_2) = -i \langle d^\dagger (t'_1) d^\dagger (t'_2) \rangle_{el,0},$$

where $n = L, R$, and the unperturbed Green’s function of the dot electron

$$G^{ij}_{D,0}(t_1 - t_2) = -i \langle d^\dagger (t'_1) d^\dagger (t'_2) \rangle_{el,0}.$$  \hspace{1cm} (5a)

The time ordered and anti-time ordered Green’s functions, i.e. with time arguments on forward and return branches respectively can be expressed in terms of the Green’s functions with time arguments on different branches as $G^{++}(t) = \Theta(t) G^{++}(t) + \Theta(-t) G^{--}(t)$ and $G^{--}(t) = \Theta(t) G^{++}(t) + \Theta(-t) G^{--}(t)$, where $\Theta(t)$ is a unit step function \[8,10\]. Then, by solving the Dyson equation

$$G^{ij}_{D}(\omega) = G^{ij}_{D,0}(\omega) + \int_{-\infty}^{\infty} dt' G^{ij}_{D,0}(\omega - i\epsilon_d) G^{kj}_{D}(\omega - i\epsilon_d) G^{ij}_{D,0}(\omega - i\epsilon_d),$$

where $G^{ij}_{D}(\omega) = \int G^{ij}_{D}(t) dt (\omega + i\epsilon_d) d\omega$. In Eq. (5) we introduced tunnelling rates $\Gamma_{L(R)} = \pi T^2_{L(R)} \rho_{L(R)}$, where the densities of states in the leads $\rho_{L(R)}$ are assumed constant for simplicity.

The $O(\lambda^2)$ contribution to the effective action in Eq. (4) is generated by the integral kernel

$$K(t_1 - t_2) = 2i \int_{-\infty}^{\infty} dt_1 dG^{ij}_{D}(\omega) \left[ \frac{1}{(\omega - e_0)^2 + (\Gamma_L + \Gamma_R)^2} \right],$$

where we have introduced the “rotated” Keldysh variables $x(t) = x(t^+) - x(t^-)$, $x(t) = x(t^+) + x(t^-)$). The kernels

$$A(t_1 - t_2) = \text{Im} \left[ G^{ii}_{D,0}(t_2 - t_1) G^{ii}_{D,0}(t_1 - t_2) \right]$$

and

$$S(t_1 - t_2) = \text{Re} \left[ G^{ii}_{D,0}(t_2 - t_1) G^{ii}_{D,0}(t_1 - t_2) \right],$$

with $G^{ii}_{D}$ and $G^{ii}_{D}$ given by Eqs. (4), are related to anti-symmetric (quantum) and symmetric (classical) parts the charge correlation function $K(t_2 - t_1)$.  \hspace{1cm} (5b)
If we assume that the tunneling through the resonant level is fast compared to the motion of the oscillator, i.e., the kernel $K(t_2 - t_1)$ is nonzero on a time scale which is much smaller than the time scale of the oscillator, then the oscillator trajectory between $t_1$ and $t_2$ is close to a straight line, i.e., $x(t_2) = x(t_1) + x(t_1)(t_1 - t_2)$. Using this approximation in Eq. (6) gives an action which is local in time:

$$\mathcal{F} = -\frac{\lambda^2}{2} \int dt \{ iR \dot{x}^0 \dot{x}^0 + iA \dot{x}^0 \dot{x} + S \dot{x} \dot{x}^0 \} + ..., \quad (7)$$

where the coefficients $R$, $A$ and $S$ can be expressed in terms of single particle Green’s functions as follows:

$$S = \frac{1}{2\pi} \int d\omega G_D^{+}(\omega)G_D^{-}(\omega),$$

$$A = \frac{1}{2\pi} \int d\omega G_D^{+}(\omega) \frac{\partial}{\partial \omega} G_D^{+}(\omega),$$

$$R = \frac{1}{\pi} \mathcal{P} \int d\omega_1 d\omega_2 \frac{G_D^{+}(\omega_1)G_D^{-}(\omega_2)}{\omega_1 - \omega_2}. \quad (8c)$$

The effective action in Eq. (7) is exactly of the form of the Caldeira and Leggett action [12], derived for a bosonic heat bath at high temperature. The first term in the effective action $\mathcal{F}$ is a renormalization of the oscillator potential. In contrast to the infinite renormalization in the Caldeira-Leggett model, here $R$ is finite. Evaluating the integral in Eq. (8c) yields $R = 2\delta \langle n_0 \rangle_{el}/\partial \epsilon_0$, where $\langle n_0 \rangle_{el}$ is the average occupation of the electron in the dot given by Eq. (6). Physically this corresponds to the back action of tunnel current, which is perturbed by the displacement of the oscillator. Combining this second order renormalization with the first order renormalization in Eq. (4) the effective potential of the oscillator can be written as $V(x) \sim V^{bare}(x) + \lambda x \langle n_0(x) \rangle_{el}$, where $\langle n_0(x) \rangle_{el}$ is the occupation number of the resonant level for a fixed position of the oscillator. The last term in the exponent of Eq. (7) provides dephasing of the system. Its effect at the classical level corresponds to a white noise force $f(t)$ exerted by the tunnel current on the oscillator. The second term causes energy damping. The classical equation of motion for the oscillator [12] can be written as $m \ddot{x} + m\gamma \dot{x} + \partial_x V = f(t)$, where, in our case, $m\gamma = \lambda^2 A$ and $(f(t))(t') = \lambda^2 S \delta(t - t')$. Thus the classical and quantum parts of the resonant level charge correlation function determine fluctuations and dissipation for the oscillator respectively. One can therefore define an effective temperature, $T_{eff}$, using a fluctuation-dissipation relation, giving $\mathcal{S}/A = 2T_{eff}$. The effective temperature, $T_{eff}$, is not determined by the reservoir’s actual temperature, as in the Caldeira-Leggett model [12], but rather by the coupling to the tunnel current.

$T_{eff}$ determines the fluctuations of the oscillator coordinate due to the tunnel current induced noise. In the case of a linear oscillator, $V(x) = m\omega^2_{osc} x^2/2$, the dispersion of the oscillator coordinate is $\langle x^2 \rangle = T_{eff}/(m\omega_{osc}^2)$. We can now check the validity of our expansion in $\lambda$. From the structure of Eqs. (7,8), we see that higher order terms in Eq. (4) will be smaller by powers of the dimensionless parameter $\lambda \sqrt{\langle x^2 \rangle}/(\Gamma_L + \Gamma_R)$. Physically, if the oscillator induced shift of the resonant level is small compared to the width of the level, the back action on the oscillator is weakly dependent on the position of the oscillator and the higher order nonlinearities are unimportant. Thus, for sufficiently large $\Gamma$, we only need to consider the leading, quadratic, terms in the effective action (7).

It is instructive to evaluate $T_{eff}$ explicitly, using Eqs. (5,8) in two limiting cases—the threshold and co-tunneling regimes. Suppose first that the resonant level is in the vicinity of one of the chemical potentials in the leads, say $\epsilon_0 = \mu_R = 0$ and the bias between the chemical potentials is large, $\mu_L = \infty$. In this regime, the current through the device is very sensitive to the energy of the resonant level, and hence to the displacement of the cantilever. In this case we obtain: $\gamma_{thr} = \hbar \lambda^2 \Gamma_R/(\pi m \Gamma^3)$, $T_{thr} = \pi \Gamma_L/4$. Therefore, the effective temperature of the oscillator in the threshold regime is essentially defined by the tunneling-induced width $\Gamma$ of the resonant level. For a practical Si nano-mechanical resonator with dimensions $3 \mu m \times 0.1 \mu m \times 0.1 \mu m$, coupled to SET with $\Gamma \sim 10^5$ s$^{-1}$ by a coupling strength $\lambda \sim 10^{-14}$ V/Å, this corresponds to an effective temperature $T_{thr} \sim 0.1$ K, and the damping coefficient $\gamma \sim 10^7$ s$^{-1}$. The effect of back action in this case will limit the lowest achievable oscillator temperature to 0.1 K, and the maximum quality factor to about 100.

Another important limiting regime corresponds to the situation when the resonant level is far above or below the chemical potentials in the leads. This is the so-called co-tunneling regime. The tunneling electrons can now occupy the level only virtually and the effective coupling constant between the leads is small as it is suppressed by the large energy separation between the chemical potentials in the leads and the resonant level. Assuming $\mu_L - \mu_R \approx V \ll (\mu_L + \mu_R)/2 \equiv \mu_F$, and $\mu_L(R) \equiv \Gamma, \epsilon_0$, we obtain $\gamma_{cot} = \hbar \lambda^2 \Gamma^2/(\pi \mu_F^2)$, $T_{cot} = \Gamma_L \Gamma_R V/\Gamma^2$. Compared to the threshold regime, in the co-tunneling both the oscillator damping and the back-action noise are significantly reduced, with the bias voltage across the leads determining the effective temperature of the oscillator. This result is consistent with Ref. [6,13].

We are now in a position to analyze the sensitivity of the system. Suppose that the oscillator (which will be assumed linear from now on) is perturbed by an external force $F(t)$, say, a short kick of duration $\tau_F \ll \omega_0^{-1}$, so that $F(t) \approx F \tau_F \delta(t)$. This kick results in the variation of the oscillator’s amplitude by the amount $\delta \tilde{x} = F \tau_F/(m\omega_0)$. What minimum $\delta \tilde{x}$ can be detected by observing the tunnel current, given the noise $\langle \langle I_x \rangle^2 \rangle$ in the current?
The ability to measure a signal can be represented by the integrated signal-to-noise ratio [14]

$$s/n = (1/2\pi) \int d\omega \langle |S_F(\omega)|^2 / |I_0|^2 \rangle,$$  \hspace{1cm} (9)

where $S_F(\omega)$ is the Fourier transform of the detector’s response to an external perturbation, i.e., the force $F$. The variation of the current through the structure due to the variation of the oscillator coordinate is $\delta I = (\partial I/\partial x) \delta x$, and therefore the response can be written as $S_F(\omega) = (\partial I/\partial x) F(\omega)/[m(\omega^2 + i\gamma(\omega - \omega_0^2))]$.

The current and the noise can be easily evaluated if we recall that the dynamics of the oscillator is slow compared to the dynamics of the tunneling electrons. The current in the adiabatic approximation, i.e., for a fixed position of the oscillator is given by $I(x) = (1/2\pi) \int_{\mu_R}^{\mu_L} d\omega T(\omega, x)$, where the transmission coefficient is $T(\omega, x) = 4\Gamma L/[\omega - E_0(x)]^2 + \Gamma^2$ [11].

The noise at low frequencies (of order $\omega_0$) is given by $\langle |I_0|^2 \rangle \simeq \langle |I_0|^2 \rangle + (\partial I/\partial x)^2 \langle |\Delta x_\omega|^2 \rangle$, where the shot noise is related to the transmission coefficient as $\langle |I_0|^2 \rangle = (1/2\pi) \int_{\mu_R}^{\mu_L} d\omega T(\omega, 0)[1 - T(\omega, 0)] [15]$, and $\langle |\Delta x_\omega|^2 \rangle = 2\gamma T_{th}/m[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega_0^2]$ is the fluctuation spectrum for the oscillator. Substituting these expressions into Eq. (9) and setting the $s/n = 1$ as a criterion for a successful measurement, one obtains a criterion for a detection of a minimum force by our apparatus. By expressing this minimum force in terms of the minimum displacement $\delta x_{\text{min}}$ that it causes, this criterion reduces to

$$x_0^2/\delta x_{\text{min}}^2 \simeq |\partial I/\partial \xi_0|/[4\sqrt{S(|I_0|^2)}],$$  \hspace{1cm} (10)

where $x_0^2 = h/(2m\omega_0)$ is zero point displacement for the oscillator, and $S$ is given by Eq. (8b).

The sensitivity defined by Eq. (10) can be evaluated and is presented in Fig. 2 as a function of chemical potentials for a symmetric structure $(\Gamma_L = \Gamma_R = \Gamma/2)$. The axes are $\mu_L/\Gamma$ and $\mu_R/\Gamma$.

FIG. 2. Sensitivity in dimensionless units $(x_0^2/\delta x_{\text{min}}^2)$ as a function of chemical potentials for a symmetric structure $(\Gamma_L = \Gamma_R = \Gamma/2)$. The axes are $\mu_L/\Gamma$ and $\mu_R/\Gamma$.

In summary, we have analyzed the quantum measurement of a mechanical oscillator coupled to an electronic resonant level that models a single electron transistor. We determined the back action effects of the detector on the quantum system, which lead to a measurement-induced effective temperature and damping coefficient. We also determined the fundamental sensitivity limits of the scheme in all operation regimes.

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