Inflationary parameters and primordial perturbation spectra

David Wands

Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth, PO1 2EG, United Kingdom

Abstract

I discuss how parameters describing inflation in the very early universe may be related to primordial perturbation spectra. Precision observations of anisotropies in the cosmic microwave background (CMB) such as those provided by the WMAP satellite offer an unprecedented window onto the physics of the very early universe. To theorists exploring speculative models of physics at high energies, the CMB offers us the chance to put our ideas to the test.

Key words: cosmology, early universe, origin of structure

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1 Introduction

Inflation was originally proposed to solve classical problems of the hot big bang “standard model”. Inflation sought to explain the remarkable homogeneity on large scales, negligible spatial flatness, the large entropy of the observable universe and the absence of monopoles and other dangerous relics from phase transitions in the young universe. But it was the realisation that it also provides a quantum origin for the large scale structure of the universe that has proved to be its greatest success. This was an unexpected bonus that puts models of inflation in the very early universe into the realm of testable science and not simply a matter of philosophical prejudice.

The series of acoustic peaks in the CMB angular power spectrum is strong evidence for the existence of primordial density perturbations on super-horizon...
scales before the last-scattering of the CMB photons. The WMAP satellite has given further support by detecting an anticorrelation on degree scales in the temperature-polarisation (TE) cross-correlation, as expected for a spectrum of density perturbations on super-horizon scales at last-scattering [Peiris et al., 2003]. The COBE satellite gave us the first direct measurement of the amplitude of density fluctuations at last-scattering, and WMAP has now taken another leap forward in determining not only the overall amplitude but also the scale-dependence of the spectrum.

In this talk I will review how observations of primordial perturbation spectra can constrain the parameters of different theoretical models of the early universe.

2 Origin of perturbations

In the standard (radiation-dominated) hot big bang there is no way to explain the origin of matter perturbations on scales greater than the causal particle horizon, which is simply related to the Hubble length at that time, \( \ell_{\text{hor}} = cH^{-1} \). A period of inflation changes the causal structure of the very early universe. During inflation the causally connected region can grow arbitrarily large.

Zero-point vacuum fluctuations of a free field \( \phi \), with comoving wavenumber \( k \gg aH \), behave like an under-damped harmonic oscillator:

\[
\delta \phi_k = \frac{1}{a\sqrt{2k}} e^{-ikt/a},
\]

where the overall normalisation is set by the canonical commutation relations. Linear evolution for a massless field during de Sitter inflation stretches these small-scale fluctuations up to large scales, \( k \ll aH \), leading to a scale-invariant spectrum of perturbations on large scales

\[
\langle \delta \phi^2 \rangle = \left[ \frac{4\pi k^3}{(2\pi)^3} |\delta \phi_k|^2 \right]_{k=aH} = \left( \frac{H}{2\pi} \right)^2.
\]

More generally, a quasi-de Sitter expansion produces an almost scale-invariant spectrum of fluctuations in any light, minimally coupled scalar field. Light fields, with an effective mass less than the Hubble scale, become over-damped in the long-wavelength limit and, as the decaying mode rapidly decays, the perturbations are well described by a classical Gaussian random field. Equation (2) then gives an estimate of the perturbation in the amplitude of the
asymptotic solution (as $k/aH \to 0$) in terms of quantities at Hubble-crossing ($k = aH$). Heavy fields, with mass greater than the Hubble scale, remain under-damped with a steep blue ($k^3$) spectrum, producing no classical perturbations on large scales.

Zero-point quantum fluctuations in the free gravitational field, stretched by inflation to super-Hubble scales, yield a nearly scale-invariant spectrum of gravitational waves. These tensor modes remain decoupled from matter perturbations (to first-order) and thus provide a direct probe of the inflationary dynamics. On large scales (super-Hubble during the radiation era) we have

$$\langle T^2 \rangle = \frac{16}{\pi} \left( \frac{H}{M_{\text{Pl}}} \right)^2. \quad (3)$$

If we can detect tensor perturbations in the CMB anisotropies then this would be a direct probe of the expansion rate during inflation, but as yet there is no evidence that gravitational waves contribute to the observed anisotropies of the CMB.

A detection of a spectrum of gravitational waves would be a great triumph for inflation - a genuine prediction before the fact - but there is no guarantee that gravitational waves will be produced at a detectable level. The amplitude of gravitational waves is a direct measure of how close inflation occurs to the Planck scale, $M_{\text{Pl}}$. In models driven by a slowly rolling scalar field the expansion rate is principally determined by the potential energy of the field

$$\langle T^2 \rangle \approx \frac{128}{3} \left( \frac{V}{M_{\text{Pl}}^4} \right). \quad (4)$$

Such a spectrum of primordial gravitational waves could be distinguished from scalar-type perturbations via B-mode polarisation of the CMB. For example, Knox and Song (2002) have argued that intrinsic B-mode polarisation from gravitational waves at last-scattering would be distinguishable from that generated by secondary effects only $V^{1/4} > 3.2 \times 10^{15}\text{GeV}$. This is close to GUT scales and certainly possible in some models of inflation, but inflation could take place at much lower energy scales.

3 Primordial density perturbation from inflation

Around the time of last-scattering the cosmic fluid is composed of (at least) photons, baryons, neutrinos and cold dark matter. The primordial density per-
perturbation can be characterised by an overall density/curvature perturbation

\[ R = H \frac{\delta \rho}{\dot{\rho}}, \]  

(5)

and a relative density/isocurvature perturbation in the different matter components

\[ S_m = 3H \left( \frac{\delta \rho_\gamma}{\dot{\rho}} - \frac{\delta \rho_m}{\dot{\rho}_m} \right), \]  

(6)

which describes the perturbed matter-to-photon number ratio, \( n_m/n_\gamma \). Thus \( S_m \) vanishes for the commonly considered case of a purely adiabatic primordial density perturbation. But in general it is possible to have isocurvature perturbations and these may be correlated with the curvature perturbations. We define the correlation angle

\[ \cos \Theta \equiv \frac{\langle RS \rangle}{\langle R^2 \rangle^{1/2} \langle S^2 \rangle^{1/2}}. \]  

(7)

If there is only one light, slowly-rolling scalar field during inflation (the inflation) then Eq. (2) describes an overall density/curvature perturbation

\[ \langle R^2 \rangle = \langle \left( \frac{H \delta \phi}{\dot{\phi}} \right)^2 \rangle_{k=\alpha H}, \]  

(8)

which remains constant for adiabatic perturbations on large scales. The existence of an almost scale-invariant spectrum of adiabatic density perturbations is a generic prediction of single-field inflation models. The amplitude of the perturbations is not in general predicted, but rather matching the observed amplitude of density perturbations on large scales is imposed as a constraint on the model parameters.

In general, multiple scalar fields, \( \varphi_i \), can produce both curvature and isocurvature perturbations. During inflation, analogously to the primordial era, we define instantaneous adiabatic and entropy field perturbations ([Gordon et al., 2001])

\[ R_\ast = H \frac{\delta \phi}{\dot{\phi}}, \quad S_\ast = H \frac{\delta \chi}{\dot{\phi}}, \]  

(9)

where \( \phi \) describes the evolution along the background (homogeneous) trajectory in field space, and \( \chi \) is orthogonal to it. Of course one can have as many
isocurvature perturbation modes during inflation as one has additional light fields, but for simplicity I will consider only two fields. I have picked the (arbitrary) normalisation of the isocurvature perturbation during inflation so that $R_s$ and $S_s$ have the same power at Hubble-crossing, $\langle R_s^2 \rangle = \langle S_s^2 \rangle$, although they are independent random fields $\langle R_s S_s \rangle = 0$.

We can describe the evolution of curvature and isocurvature perturbations from Hubble-crossing during inflation to primordial density perturbation via a transfer matrix (Amendola et al., 2002; Wands et al., 2002)

$$\begin{pmatrix} R \\ S \end{pmatrix} = \begin{pmatrix} 1 & T_{RS} \\ 0 & T_{SS} \end{pmatrix} \begin{pmatrix} R_s \\ S_s \end{pmatrix}$$

where $T_{RS}$ and $T_{SS}$ are functions of $k$ to be determined from the (observable) primordial perturbations:

$$T_{SS} = \frac{\langle S_s^2 \rangle}{\langle R_s^2 \rangle},$$

$$T_{RS} = \cot \Theta,$$

where $\Theta$ is defined in Eq.(7). $T_{SS}$ determines the amplitude of the surviving isocurvature perturbation in the primordial era, and $T_{RS}$ quantifies how much of the primordial curvature perturbation is due to non-adiabatic field perturbations during inflation.

### 3.1 Scale-dependence

The weak time-dependence (relative to the Hubble rate) of quantities calculated at Hubble-crossing, $k = aH$, during slow-roll inflation yields a weak scale-dependence in the spectra after inflation:

$$\Delta n_x \equiv \frac{d \ln \langle x^2 \rangle}{d \ln k} \sim \left( H^{-1} \frac{d \ln \langle x^2 \rangle}{dt} \right)_{k=aH},$$

where the right-hand-side is evaluated at Hubble crossing.

Taking the logarithmic derivative of tensor power spectrum (3) we obtain

$$\Delta n_T \sim -2\epsilon.$$
This is always negative and determined by the slow-roll parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{M_{Pl}^2}{16\pi} \left(\frac{V'}{V}\right)^2.$$  \hspace{1cm} (15)

This same slow-roll parameter appears as the ratio between the amplitude
of tensor perturbations (3) to scalar curvature perturbations (5) at Hubble-crossing:

$$\frac{\langle T^2 \rangle}{\langle R^2 \rangle} = 16\epsilon.$$ \hspace{1cm} (16)

This holds regardless of the number of light fields present during inflation. In
terms of the observable (primordial) tensor and scalar perturbations and the
tensor tilt we have a two-field consistency relation \cite{Wands et al. 2002}

$$\frac{\langle T^2 \rangle}{\langle R^2 \rangle} = -8\Delta n_T \sin^2 \Theta.$$ \hspace{1cm} (17)

Taking the derivative of the three scalar perturbation spectra (curvature, isocurvature and their cross-correlation) we obtain \cite{Wands et al. 2002}\footnote{Note that the scale-dependence of the primordial scalar curvature perturbation is more usually written as $n = 1 + \Delta n_R$.}

$$\Delta n_R \simeq -(6 - 4\cos^2 \Theta)\epsilon$$

$$+ 2(\eta_{\phi\phi} \sin^2 \Theta + 2\eta_{\phi\chi} \sin \Theta \cos \Theta + \eta_{\chi\chi} \cos^2 \Theta),$$ \hspace{1cm} (18)

$$\Delta n_S \simeq -2\epsilon + 2\eta_{\chi\chi},$$ \hspace{1cm} (19)

$$\Delta n_{RS} \simeq -2\epsilon + 2\eta_{\chi\chi} + 2\eta_{\phi\chi} \tan \Theta,$$ \hspace{1cm} (20)

where the dimensionless mass matrix for the fields is given by three more slow-roll parameters

$$\eta_{ij} = \frac{M_{Pl}^2}{8\pi V} \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}.$$ \hspace{1cm} (21)

Although the overall amplitude of $T_{RS}$ and $T_{SS}$ depends upon the details
of reheating and other physical processes from the end of inflation up until
the primordial (radiation-dominated) era, their the scale dependence is deter-
mined by the evolution around Hubble-crossing during inflation \cite{Wands et al. 2002}.\footnote{Note that the scale-dependence of the primordial scalar curvature perturbation is more usually written as $n = 1 + \Delta n_R$.}
3.2 Inflaton scenario

If the primordial curvature perturbations results solely from the overall density/curvature perturbation at Hubble-crossing during inflation (i.e., $T_{RS} \ll 1$) then we have

$$\langle R^2 \rangle = \langle R^2_\star \rangle,$$  \hspace{1cm} (22)

and hence we have the usual single-field tensor-scalar consistency relation

$$\frac{\langle T^2 \rangle}{\langle R^2 \rangle} = 16\epsilon = -8\Delta n_T.$$  \hspace{1cm} (23)

The tilt of the primordial curvature perturbation is given by

$$\Delta n_R \simeq -6\epsilon + 2\eta_{\phi\phi},$$  \hspace{1cm} (24)

Any isocurvature perturbation must arise from a second light field during inflation (e.g., the axion) and is uncorrelated with the curvature perturbation ($\cos \Theta = 0$). The spectral tilt of such an isocurvature spectrum is given by

$$\Delta n_S \simeq -2\epsilon + 2\eta_{\chi\chi}.$$  \hspace{1cm} (25)

Because the curvature and isocurvature perturbations originate from completely unrelated fields there is, a priori, no reason to expect them to have comparable amplitude, i.e., $T_{SS} \sim 1$, which is required for primordial isocurvature modes to be detectable.

3.3 Curvaton scenario

In the curvaton scenario [Enqvist and Sloth, 2002; Lyth and Wands, 2002; Moroi and Takahashi, 2001] the primordial density perturbation is supposed to arise from quantum fluctuations during inflation in a light scalar field, other than the inflaton, which then decays some time after inflation. It is one example of a model where the primordial perturbation comes entirely from an isocurvature perturbation at Hubble-crossing (i.e, $T_{RS} \gg 1$):

$$\langle R^2 \rangle = T^2_{RS} \langle S^2_\star \rangle.$$  \hspace{1cm} (26)
The curvature perturbation at Hubble-exit during inflation is then negligible, and the tensor-scalar consistency relation becomes

\[
\frac{\langle T^2 \rangle}{\langle R^2 \rangle} \ll -8\Delta n_T = 16\epsilon < 16.
\] (27)

The curvaton scenario provides a class of inflationary models where gravitational waves must have a negligible effect on the CMB anisotropies.

The scalar tilt in the curvaton scenario is given by

\[
\Delta n_R \simeq -2\epsilon + 2\eta_{\chi\chi},
\] (28)

Because the original perturbations that seed the primordial curvature perturbation are non-adiabatic at Hubble-exit during inflation it is possible for the curvaton to leave residual isocurvature perturbations after the curvaton decays, depending upon the relative timescales of the matter-decoup ling and curvaton decay \cite{Lyth, Ungarelli and Wands, 2003}. In any case the curvature and residual isocurvature perturbations must be completely correlated (\(\cos \Theta = 1\)) and share the same spectral tilt \(\Delta n_R = \Delta n_S\). There is no lower limit on the amplitude of the primordial isocurvature perturbation (it could be negligible) but the curvaton does provide a simple physical model where primordial curvature and isocurvature perturbations have similar amplitude. In the simplest such case, where the matter decouples while the curvaton density is negligible, before the curvaton decays, then we find \(T_{SS} = 3T_{RS}\), yielding \(S = 3R\) \cite{Lyth, Ungarelli and Wands, 2003}. Unfortunately such a large isocurvature perturbation is no longer compatible with current CMB data when considering cold dark matter or baryon isocurvature modes \cite{Gordon and Lewis, 2002}.

### 3.4 Scale-dependent tilt?

If the WMAP data as yet gives no hint of gravitational waves or non-adiabatic modes, it offers the tantalising prospect of detecting some scale-dependence of the scalar tilt, \(d\Delta n_R/d\ln k\) \cite{Peiris et al, 2003}. Combining CMB data on large scales with survey data on smaller scales may favour a spectrum with less power on very large scales (suggested by the “low quadrupole” of the CMB) and less power on smaller scales than expected for a scale-invariant spectrum. This may not fit that easily with our models of slowly evolving, slow-roll inflation.

In the single-field scenario the running of the tilt is given in the slow-roll...
approximation by

\[
\frac{d\Delta n_R}{d \ln k} \simeq -2\xi_{\phi\phi} - 8\epsilon(3\epsilon - 2\eta_{\phi\phi}),
\]

(29)

while in the curvaton scenario one obtains

\[
\frac{d\Delta n_R}{d \ln k} \simeq -2\xi_{\chi\phi} - 4\epsilon(2\epsilon - \eta_{\phi\phi} - \eta_{\chi\chi}),
\]

(30)

The parameters

\[
\xi_{\phi\phi} = \frac{M^4_{Pl} \partial V \partial^3 V}{64\pi^2 V^2 \partial \phi \partial^3 \phi},
\]

(31)

\[
\xi_{\chi\phi} = \frac{M^4_{Pl} \partial V \partial^2 V}{64\pi^2 V^2 \partial \phi \partial \phi \partial \chi^2},
\]

(32)

are a dimensionless measure of how fast the scalar field’s effective mass is changing on a Hubble time. In particular for the inflaton we have

\[
\frac{d\eta_{\phi\phi}}{dt} \simeq 2H(\epsilon\eta_{\phi\phi} - \xi_{\phi\phi}).
\]

(33)

In either inflaton or curvaton scenarios the \( \xi \)-parameters involve four derivatives of the potential and are expected to be of the same order as \( \epsilon^2 \). Hence the small tilt observed suggests that the running should also be small. If not there is a new coincidence problem as to why the primordial power spectrum should reach a maximum on scales around 100Mpc.

In the inflaton case a value for \( \xi_{\phi\phi} \sim \eta_{\phi\phi} \) suggests that slow-roll inflation may break down in a few Hubble times. We may have to move away from the notion that inflation is a slow-rolling, weakly time-dependent process, and consider more transient bursts of inflation and/or generation of perturbations. In the curvaton case \( \xi_{\chi\phi} \sim \eta_{\chi\chi} \) would just signal that the curvaton field becomes heavy in a few Hubble times, which is not inconsistent with slow-roll inflation continuing.

4 Discussion

Increasingly precise constraints upon the spectral tilt and possible running of the spectral tilt will place increasingly tight constraints upon models of inflation. These are usually characterised by the slow-roll parameters \( \epsilon, \eta \) and \( \xi \)
which give a useful qualitative understanding of inflationary dynamics. But an unexpectedly large running of the tilt might be an indication of the limitations of the slow-roll description. Observations may require more sophisticated models of inflation if their parameters can be related to observables. For instance non-adiabatic perturbations produced when there is more than one light field during inflation, provide an additional source of primordial density perturbations. These might provide features in the primordial power spectrum or non-Gaussianity without violating slow-roll.

Precise observations offer the hope of seeing more than just the primordial curvature perturbation. Evidence of primordial gravitational waves would be an important test of inflation. Even in multi-field models of inflation there are consistency conditions constraining the tensor-scalar ratio. Indeed a large amplitude of gravitational waves could falsify any slow-roll model of inflation. Primordial isocurvature modes and/or non-Gaussianity would also provide valuable additional information about the physics of inflation and the very early universe.

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