\( \mathcal{N} = 8 \) Gauged Supergravity Theory and \( \mathcal{N} = 6 \) Superconformal Chern-Simons Matter Theory

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Abstract

By studying the previously known holographic \( \mathcal{N} = 4 \) supersymmetric renormalization group flow(Gowdigere-Warner) in four dimensions, we find the mass deformed Chern-Simons matter theory which has \( \mathcal{N} = 4 \) supersymmetry by adding the four mass terms among eight adjoint fields. The geometric superpotential from the eleven dimensions is found and provides the M2-brane probe analysis. As second example, we consider known holographic \( \mathcal{N} = 8 \) supersymmetric renormalization group flow(Pope-Warner) in four dimensions. The eight mass terms are added and similar geometric superpotential is obtained.
1 Introduction

The three dimensional $\mathcal{N} = 6$ Chern-Simons matter theories with gauge group $U(N) \times U(N)$ and level $k$ have been studied in [1]. This theory is described as the low energy limit of $N$ M2-branes at $\mathbb{C}^4/\mathbb{Z}_k$ singularity. Since the coupling of this theory may be thought of as $1/k$ and so this is weakly coupled for large $k$. In particular, for $k = 1, 2$, the full $\mathcal{N} = 8$ supersymmetry ($SO(8)$ R-symmetry) is preserved and the theory is strongly coupled.

The renormalization group (RG) flow between the ultraviolet (UV) fixed point and the infrared (IR) fixed point of the three dimensional field theory has a close connection with a gauged supergravity solution in four dimensions. There exist holographic RG flow equations connecting $\mathcal{N} = 8$ $SO(8)$ fixed point to $\mathcal{N} = 2$ $SU(3) \times U(1)$ fixed point [2, 3] (See also [4]). Moreover, the other holographic RG flow equations from $\mathcal{N} = 8$ $SO(8)$ fixed point to $\mathcal{N} = 1$ $G_2$ fixed point also exist [3, 5, 6] (See also [7, 8, 9]). The exact solutions to the $M$-theory lift of these RG flows are known in [10, 5].

The mass deformed $U(2)^2$ Chern-Simons matter theory with level $k = 1$ or $k = 2$ preserving $\mathcal{N} = 2$ $SU(3) \times U(1)$ symmetry was studied in [11, 12, 13]. For $\mathcal{N} = 1$ $G_2$ symmetric case, the corresponding mass deformation was described in [14]. Recently [15] the nontrivial nonsupersymmetric flow equations preserving $SO(7)\pm$ have been described in the context of $\mathcal{N} = 6$ superconformal Chern-Simons matter theory. In [15], the possibility having the flow equations from $\mathcal{N} = 1$ $G_2$ fixed point to $\mathcal{N} = 2$ $SU(3) \times U(1)$ fixed point, by realizing 1) the negativity of mass term around $G_2$ fixed point and 2) the natural symmetry breaking $G_2 \to SU(3)$, was proposed. Very recently, this possibility was shown in [16] by introducing a new superpotential.

As mentioned and suggested in [15], it is known that there exist $\mathcal{N} = 4$ supersymmetric flows with equal mass terms for four real scalars in gauged supergravity theory discovered by Gowdigere and Warner [17]. All the previous flow equations were obtained from $SU(3)$-invariant sectors of gauged $\mathcal{N} = 8$ supergravity in four dimensions. Due to the $SU(2) \times SU(2)$ R-symmetry of $\mathcal{N} = 4$ supersymmetry, this $\mathcal{N} = 4$ supersymmetric flows are not contained in $SU(3)$-invariant sectors. We would like to see what is the gauge dual to $\mathcal{N} = 4$ supersymmetric flows in four dimensions [17]. Moreover, there exist $\mathcal{N} = 8$ supersymmetric flows with equal mass terms for eight real scalars discovered by Pope and Warner [18]. In this case, the R-symmetry is given by $SO(4) \times SO(4)$. We also find out the gauge dual to $\mathcal{N} = 8$ supersymmetric flows in four dimensions [18].

In this paper, starting from the first order differential equations, that are the $\mathcal{N} = 4$ supersymmetric flow solutions in four dimensional $\mathcal{N} = 8$ gauged supergravity interpolating
between an exterior $AdS_4$ region with maximal $\mathcal{N} = 8$ supersymmetry and an interior region with $\mathcal{N} = 4$ supersymmetry, we would like to interpret this as the RG flow in Chern-Simons matter theory broken to the mass-deformed Chern-Simons matter theory by the addition of mass terms for the adjoint superfields. An exact correspondence may be obtained between fields of bulk supergravity in the $AdS_4$ region in four-dimensions and composite operators of the IR field theory in three-dimensions. The three dimensional analog of Leigh-Strassler \cite{19} RG flow in the context of mass-deformed Chern-Simons matter theory is expected but the present RG flows do not reveal the marginality of Leigh-Strassler because there are no fixed critical points. As analyzed in \cite{20} where they observed the work of \cite{19}, the beta function description in the context of 3-dimensional Chern-Simons matter theory can be done. We present the results of probing the eleven-dimensional supergravity solution corresponding to RG flows.

In section 2, we review the supergravity solution in four dimensions in the context of RG flow and describe the scalar potential and the superpotential. The analysis for the supersymmetry variations on the spin $\frac{1}{2}$ and $\frac{3}{2}$ fields is new. In section 3, we deform BL theory \cite{21} \cite{22} \cite{23} by adding four mass terms and write down the $[SU(2) \times U(1)]^2$-invariant superpotential in $\mathcal{N} = 2$ superfields. We also describe the corresponding mass-deformed Chern-Simons matter theory. In section 4, the 11-dimensional geometric superpotential which reduces to the usual $AdS_4$ superpotential for the particular internal coordinate is computed and this is needed to analyze the M2-brane probe analysis.

In section 5, we review the supergravity solution in four dimensions in the context of RG flow. The analysis for the supersymmetry variations on the spin $\frac{1}{2}$ and $\frac{3}{2}$ fields is made. In section 6, we deform BL theory and Chern-Simons matter theory by adding eight mass terms and write down the $SO(4) \times SO(4)$-invariant superpotential in $\mathcal{N} = 2$ superfields. In section 7, the 11-dimensional geometric superpotential is described for the M2-brane probe analysis.

In section 8, we present the future directions. In the appendices, we present the details which are needed for the previous sections.

2 The holographic $\mathcal{N} = 4$ RG flow in four dimensions

The invariant scalar manifold by Gowdigere-Warner \cite{17} consisting of eight scalars has a factor $SO(6,1) \times SL(2,\mathbb{R})$ in $E_{7(7)}$ which has a maximal non-compact subgroup $SL(8,\mathbb{R})$. The special gauge choice reduces this scalar manifold to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and eight scalars

\footnote{There are other related works \cite{24} \cite{25} \cite{26} \cite{27} \cite{28} \cite{29} which discuss about the different supersymmetric theories in the context of Chern-Simons matter theory.}
go to four. Each $SL(2, \mathbb{R})$ has two supergravity scalars. The noncompact generators of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ can be written as \[ \phi_{ijkl} = \frac{1}{2}(\alpha \cos \phi + i \alpha \sin \phi)\delta^{1234}_{ijkl} + \frac{1}{2}(\alpha \cos \phi - i \alpha \sin \phi)\delta^{5678}_{ijkl} + \frac{1}{2}(\chi \cos \varphi + i \chi \sin \varphi)(\delta^{1256}_{ijkl} + \delta^{3456}_{ijkl}) + \frac{1}{2}(\chi \cos \varphi - i \chi \sin \varphi)(\delta^{1278}_{ijkl} + \delta^{3478}_{ijkl}), \quad (2.1) \] where $\alpha e^{i\phi}$ parametrizes the one $SL(2, \mathbb{R})$ and $\chi e^{i\varphi}$ parametrizes the other $SL(2, \mathbb{R})$. The real parts of these correspond to the 35 of $SO(8)$ and the imaginary parts of these correspond to the 35c of $SO(8)$. Then 56-beins $\mathcal{V}(x)$ can be written as $56 \times 56$ matrix of $\mathcal{N} = 8$ supergravity in four-dimensions whose elements are some functions of scalar, pseudo-scalars, $\phi$ and $\varphi$ out of seventy fields \[ \mathcal{V}(x) = \exp \left( \begin{array}{cc} 0 & 2\phi_{ijkl}(x) \\ 2\phi^{ijkl}(x) & 0 \end{array} \right). \quad (2.2) \] On the other hand, 28-beins $u$ and $v$ of $\mathcal{N} = 8$ supergravity are elements of this $\mathcal{V}(x)$ according to \[ \mathcal{V}(x) = \left( \begin{array}{cc} u_{ij}^{IJ}(x) & v_{ij}^{KL}(x) \\ v_{kl}^{IJ}(x) & u_{kl}^{KL}(x) \end{array} \right). \quad (2.3) \] One can construct these 28-beins $u$ and $v$ in terms of $\alpha, \chi, \phi$ and $\varphi$ using (2.1), (2.2) and (2.3) explicitly and they are given in (A.1) of the appendix A. Now it is ready to get the complete expression for $A_1$ and $A_2$ tensors in terms of $\alpha, \chi, \phi$ and $\varphi$ from T-tensor \[ \mathcal{A}_1 = \text{diag} \left( z_1, z_1, z_1, z_1, z_2, z_2, z_3, z_3 \right), \] where the eigenvalues are functions of $\alpha, \chi, \phi$ and $\varphi$:

\[
\begin{align*}
    z_1 &= \cosh \alpha \cosh^2 \chi + e^{-i\phi} \sinh \alpha \sinh^2 \chi, \\
    z_2 &= \cosh \alpha \cosh^2 \chi + e^{-i(2\varphi - \phi)} \sinh \alpha \sinh^2 \chi, \\
    z_3 &= \cosh \alpha \cosh^2 \chi + e^{i(2\varphi + \phi)} \sinh \alpha \sinh^2 \chi. 
\end{align*}
\] (2.4)
Similarly, $A_2$ tensor can be obtained from the T-tensor (triple product of $u$ or $v$ fields: the explicit form is given by (A.3)). It turns out that they are written as seven-kinds of fields $y_i$ where $i = 1, \cdots, 7$ and are given in (A.4) and (A.5) of the appendix A where some of these are related to the derivatives of eigenvalues of $A_1$ tensor $z_1, z_2$ or $z_3$ with respect to $\alpha$ and $\chi$.

Finally, the scalar potential \cite{17} from the general expression of \cite{31} can be written, by adding all the components of $A_1, A_2$ tensors, as

\begin{equation}
V(\alpha, \chi) = -g^2 \left( \frac{3}{4} |A_1^ij|^2 - \frac{1}{24} |A_2^i jkl|^2 \right) = -2g^2 [\cosh(2\alpha) + 2 \cosh(2\chi)]
= g^2 \left( 2 \left| \frac{\partial W}{\partial \alpha} \right|^2 + \left| \frac{\partial W}{\partial \chi} \right|^2 - 6 |W|^2 \right),
\end{equation}

where we consider the particular case

\begin{equation}
\phi = 0, \quad \varphi = \frac{\pi}{2}, \quad \rho \equiv e^\alpha.
\end{equation}

The other four components for $A_1$ are given by $\frac{1}{2} (\rho - \rho^{-1} \cosh 2\chi) = z_2 = z_3$. This implies half-maximal supersymmetry (that is, $N = 4$ supersymmetry: this will be clear when we understand the variations of spin $\frac{1}{2}$ and spin $\frac{3}{2}$ fields later). The scalar potential at the $SO(8)$ UV critical point where $\alpha = 0$ and $\chi = 0$ becomes $V = -6g^2$ where $g = \frac{1}{\sqrt{2L}}$ and the superpotential $W$ becomes 1. Note that the supergravity scalar potential (2.5) is independent of the angles $\phi$ and $\varphi$ even without the conditions (2.6). This critical point is common to both a scalar potential which has a maximum value and a superpotential (2.7) which has a minimum value and is depicted in Figure 1. The choice $\phi = 0 = \varphi$ where all the $z_i$'s in (2.4) are equal preserves $N = 8$ supersymmetry.

The resulting Lagrangian of the scalar-gravity sector can be obtained by finding out the scalar kinetic terms appearing in the action in terms of $\alpha, \chi, \phi$ and $\varphi$. By taking the product of $A_\mu$ given in (A.6) of appendix A and its complex conjugation and taking into account the multiplicity four, we arrive at the following scalar kinetic term \cite{17} with (2.6)

\begin{equation}
(\partial_\mu \alpha)^2 + 2(\partial_\mu \chi)^2.
\end{equation}

By substituting the domain-wall ansatz

\begin{equation}
\begin{aligned}
\Delta s^2 &= e^{2A} \eta_{\mu \nu} dx^\mu dx^\nu + dr^2
\end{aligned}
\end{equation}
with $\eta_{\mu\nu} = (-, +, +)$ into the Lagrangian, the Euler-Lagrange equations are given in terms of the functional $E[A, \alpha, \chi]$ \cite{33}. Then the energy-density per unit area transverse to $r$-direction is given by

$$E[A, \alpha, \chi] = \int_{-\infty}^{\infty} d\alpha d\chi dA \left[ -3 \left( 2(\partial_r A)^2 + \partial_r^2 A \right) - (\partial_r \alpha)^2 - 2 (\partial_r \chi)^2 - V(\alpha, \chi) \right].$$

Then $E[\alpha, \chi]$ is extremized by the BPS domain-wall solutions. The first order differential equations are the gradient flow equations of a superpotential defined on a restricted two-dimensional slice of the scalar manifold and simply related to the potential of gauged supergravity on this slice via (2.5) and (2.7). The half-maximal ($\mathcal{N} = 4$) supersymmetric flow satisfying the first order differential equations for given superpotential $W$ in (2.7) leads to the solutions \cite{17} for $\alpha(r), \chi(r), A(r)$:

$$\frac{d\alpha}{dr} = -\sqrt{2} g \partial_\alpha W, \quad \frac{d\chi}{dr} = -\sqrt{2} g \partial_\chi W, \quad \frac{dA}{dr} = \sqrt{2} g W. \quad (2.9) \{?\}$$

The flow equations imply that

$$\alpha = \frac{1}{2} \log \left[ e^{-2\chi} + \gamma \sinh(2\chi) \right] \quad \text{and} \quad e^A = \frac{k\rho}{\sinh(2\chi)}, \quad (2.10) \{?\}$$

where $\gamma$ and $k$ are constants of the integration \cite{17}. In particular, for $\gamma = 0$, $\alpha = -\chi$ from the first equation of (2.10) all along the flow and $e^A$ from the second equation of (2.10) goes to
zero as $\chi \to \infty$. Note that the superpotential (2.7) is invariant under $\chi \to -\chi$. There exists similar behavior in [34] from ten-dimensional type IIB string theory.

One can understand the BPS bound, so-called inequality of the energy-density as a consequence of supersymmetry preserving bosonic background. For the supersymmetric bosonic backgrounds, the variations of spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fields should vanish. The gravitational and scalar parts of these variations are [31]:

$$\delta \psi^i = 2D_\mu \psi^i - i\sqrt{2}gA_1^{ij}\gamma^i \epsilon^j, \quad \delta \chi^{ijk} = -\gamma^\mu A_\mu^{ijk} \epsilon_1 - 2gA_2^{ijk} \epsilon^1,$$

(2.11)

where the covariant derivative on supersymmetry parameter is given by

$$D_\mu \epsilon^i = \partial_\mu \epsilon^i - \frac{1}{2} \omega_{\alpha}^{ab} \sigma^{ab} \epsilon^i + \frac{1}{2} B_{\mu}^i \epsilon^j, \quad B_{\mu}^i \epsilon^j \equiv \frac{2}{3} (u^{ik}_{IJ} \partial_{\mu} u^{lj}_{IJ} - u^{ikIJ} \partial_{\mu} v_{ijkl}) .$$

(2.12)

Here $\epsilon_i$ and $\epsilon^j$ are complex conjugates each other under the chiral basis. In this basis, the $\gamma$ matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $g_{\mu\nu} = \text{diag}(-e^{2A}, e^{2A}, e^{2A}, 1)$ and $\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}$ where $\sigma^i$ are Pauli matrices and $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. Also there is $\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The field $B_{\mu}^i \epsilon^j$ is a $SU(8)$ gauge field for a local $SU(8)$ invariance, $\omega$ a spin connection, $\sigma$ a commutator of two $\gamma$ matrices $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$. Under the projection operators $(1 \pm \gamma_5)/2$ the supersymmetry parameter $\epsilon^i$ has the four column components as $(\eta_1, \eta_2, 0, 0)$ where $\eta_1, \eta_2$ are complex spinor fields. Moreover, complex conjugate $\epsilon_i$ is the charge conjugate spinor of $\epsilon^i$ and satisfies $\epsilon_i = C(\gamma^0)^T \epsilon_i$ and has the 4 column components as $(0, 0, \eta_3 = i\eta_2, \eta_4 = -i\eta_1)$ with $C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}$. The explicit form for $B_{\mu}^i \epsilon^j$ is presented in appendix A (A.8).

The variation of 56 Majorana spinors $\chi^{ijk}$ gives rise to the first order differential equation of $\alpha$ and $\chi$ by exploiting the explicit forms of $A_{\mu}^{ijkl}$ (A.6) and $A_{2l}^{ijk}$ (A.4) and (A.5) in the appendix A. Although there is a summation over the last index $l$ appearing in $A_{\mu}^{ijkl}$ and $A_{2l}^{ijk}$ in the right hand side of (2.11), this structure implies that summation runs over only one index. The vanishing of variation of $\chi^{ijk}$ for supersymmetry parameter $\epsilon^i$ where $i = 1, 2, 3, 4$ leads to

$$\sqrt{2}e^{-i\varphi} \left[ \partial_\mu \alpha - \frac{i}{2} \sinh (2\alpha) \partial_\mu \phi \right] \begin{pmatrix} \eta_2^* \\ \eta_1^* \end{pmatrix} = -2ge^{i\varphi} \frac{\partial z_{1l}^*}{\partial \alpha} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

$$\sqrt{2}e^{-i\varphi} \left[ \partial_\mu \chi - \frac{i}{2} \sinh (2\chi) \partial_\mu \varphi \right] \begin{pmatrix} \eta_2^* \\ \eta_1^* \end{pmatrix} = -ge^{-i\varphi} \frac{\partial z_{1l}^*}{\partial \chi} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

$$\sqrt{2}e^{i\varphi} \left[ \partial_\mu \chi + \frac{i}{2} \sinh (2\chi) \partial_\mu \varphi \right] \begin{pmatrix} \eta_2^* \\ \eta_1^* \end{pmatrix} = -ge^{i\varphi} \frac{\partial z_{1l}^*}{\partial \chi} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$
Moreover, the variation of gravitinos $\psi^i_{\mu=0,1,2}$ will lead to
\[
\partial_r A \left( \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right) = \sqrt{2} g z_1 \left( \begin{array}{c} \eta_1^* \\ \eta_2^* \end{array} \right).
\] (2.14) 

Finally, one of the variation of gravitinos $\psi^i_{\mu=3}$ gives rise to
\[
2 \partial_r \left( \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right) = i \partial_{\mu} \phi \sinh^2 \alpha \left( \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right) - \sqrt{2} g z_1 \left( \begin{array}{c} \eta_1^* \\ \eta_2^* \end{array} \right). \] (2.15) 

Let us denote the complex spinor fields in terms of their magnitudes and phases as follows:
\[\eta_1 = |\eta_1(r)| e^{i\beta(r)} \text{ and } \eta_2 = |\eta_2(r)| e^{i\delta(r)}.\] By multiplying the $\eta_2$ into the first row and $\eta_1$ into the second row in the first equation (2.13) and combining these two relations, one obtains $|\eta_1(r)| = |\eta_2(r)| \equiv |\eta(r)|$. Let us consider the equation (2.15) and substitute the $\eta_1$ and $\eta_2$. By multiplying $e^{-i\beta}$ into the first row and $e^{-i\delta}$ into the second row and subtracting these two, then one obtains $(\partial_r \beta - \partial_r \delta) \eta$ which implies that $\beta = \delta + \text{const}$. Now let us analyze the equation (2.14). From the first row, one gets $\partial_r A = \sqrt{2} g z_1 e^{-2i\beta}$. This implies that $z_1 e^{-2i\beta}$ should be real because $\partial_r A$ is real.

Let us compare the second and third equations of (2.13). One gets
\[
\sqrt{2} e^{-i\phi} \left[ \partial_{\mu} \chi - \frac{i}{2} \sinh (2\chi) \partial_{\mu} \varphi \right] = -g e^{-i\phi + 2i\beta} \frac{\partial z_1^*}{\partial \chi},
\]
\[
\sqrt{2} e^{i\phi} \left[ \partial_{\mu} \chi + \frac{i}{2} \sinh (2\chi) \partial_{\mu} \varphi \right] = -g e^{i\phi + 2i\beta} \frac{\partial z_1^*}{\partial \chi}. \] (2.16) 

Then we obtain $e^{2i\beta} \frac{\partial z_1^*}{\partial \chi} = e^{-2i\beta} \frac{\partial z_1}{\partial \chi}$ because the left hand side of second equation of (2.16) is a complex conjugation of the left hand side of first equation. This leads to the fact that $e^{2i\beta} \frac{\partial z_1^*}{\partial \chi}$ is real. Therefore the expression of the bracket in (2.16) should be real. So one has $\partial_r \varphi = 0$. Moreover by plugging the expression of $z_1$, one has $\cosh \alpha \sin(2\beta) + \sinh \alpha \sin(2\beta + \phi) = 0$. One simple solution for this gives rise to $\beta = 0$ and $\phi = 0$.

By collecting all the informations so far, one obtains $\partial_r A = \sqrt{2} g W$ because $z_1$ becomes $W$. From (2.13), one also obtains $\partial_r \alpha = -\sqrt{2} g \partial_\alpha W$ and $\partial_r \chi = -\frac{1}{\sqrt{2}} g \partial_\chi W$. These are exactly the flow equations (2.9) we explained before. From (2.15), there exists $2 \partial_r |\eta| = -\sqrt{2} g |\eta| W$. It is straightforward to check that all other supersymmetry parameters $\epsilon^i$ where $i = 5, 6, 7, 8$ vanish. So we have checked these for $\varphi = \frac{\pi}{2}$ explicitly.

According to the branching rule of 6 representation corresponding to spin $\frac{3}{2}$ field of $SU(4)$ under the $SU(2) \times SU(2)$, $6 \rightarrow [(1, 1) \oplus (1, 1)] \oplus (2, 2)$, the two singlets in square bracket correspond to the component of massless graviton of the $\mathcal{N} = 4$ theory. From the branching rule of 15 representation corresponding to spin 1 field of $SU(4)$ under the $SU(2) \times SU(2)$,
Let us contract the $\phi J$ gamma matrices $\Gamma^N$ The corresponds to the remaining component of massless vector multiplet of the $\mathcal{N} = 4$ theory. Finally, spin 2 field with the breaking $1 \to (1, 1)$ is located at the remaining component of $\mathcal{N} = 4$ massless graviton multiplet.

From the decomposition of spin 1 field above, the representation $(1, 3) \oplus (3, 1)$ corresponds to the massless vector multiplet of $\mathcal{N} = 4$ theory. According to the branching rule of 10 representation corresponding to spin $\frac{1}{2}$ field of $SU(4)$ under the $SU(2) \times SU(2)$, $10, \overline{10} \to [(1, 3) \oplus (3, 1)] \oplus (2, 2)$, the representations in square bracket correspond to the component of massless vector multiplet of the $\mathcal{N} = 4$ theory. Moreover, the branching rule for spin zero field provides that each representation $(1, 3) \oplus (3, 1)$ from the two 15’s of $SU(4)$, as above, corresponds to the remaining component of massless vector multiplet of the $\mathcal{N} = 4$ theory.

3 The $\mathcal{N} = 4$ supersymmetric membrane flows in three dimensions

Let us contract the $\phi_{IJKL}$ (2.1) with the gamma matrices $\Gamma^I$. We use $\Gamma^1 = 1_{8 \times 8}$ and $SO(7)$ gamma matrices $\Gamma^J$ where $J = 2, 3, \cdots, 8$ [35]. We multiply $\bigg( \begin{array}{cc} 0 & 1_{4 \times 4} \\ 1_{4 \times 4} & 0 \end{array} \bigg)$ with $\Gamma^I$ and construct new gamma matrices $\bigg( \begin{array}{cc} 0 & 1_{4 \times 4} \\ 1_{4 \times 4} & 0 \end{array} \bigg) \Gamma^I \bigg( \begin{array}{cc} 0 & 1_{4 \times 4} \\ 1_{4 \times 4} & 0 \end{array} \bigg)$. Let us define the following quantity which was introduced in [17] with new gamma matrices

$$S_{AB} \equiv \phi_{IJKL} (\Gamma^I_{JKL})_{AB},$$

and one obtains$^2$

$$S = \text{diag}(\alpha, \alpha, \alpha, \alpha, -\alpha + i2\chi, -\alpha + i2\chi, -\alpha - i2\chi, -\alpha - i2\chi). \quad (3.1) \quad (?$$

We want to see how the supergravity scalar fields $(\alpha, \chi)$ map onto the corresponding boundary field theory object.

Let us consider the BL theory [21] with $SO(4)$ gauge group and matter fields. Although this is not directly connected to AdS/CFT correspondence in the large $N$ limit due to the

$^2$For the $\mathcal{N} = 2$ $SU(3) \times U(1)_R$ invariant supersymmetric flow, one can also obtain $S = \text{diag}(\lambda - i\frac{\lambda^\prime}{3}, \lambda + i\frac{\lambda^\prime}{3}, \lambda, \lambda, \lambda, -3\lambda, -3\lambda)$ where $\lambda$ is a scalar field and $\lambda^\prime$ is a pseudo scalar field. The supergravity scalar field corresponds to the mass terms of (77) plus (88) components in the boundary theory. The supergravity pseudo-scalar field corresponds to the mass terms of (22) minus (11) components in the boundary theory. For $\mathcal{N} = 1$ $G_2$ invariant supersymmetric flow one has $S = \text{diag}(\lambda \cos \alpha - i\frac{\lambda^\prime}{3} \sin \alpha, \lambda \cos \alpha + i\frac{\lambda^\prime}{3} \sin \alpha, \lambda \cos \alpha + i\frac{\lambda^\prime}{3} \sin \alpha, \lambda \cos \alpha + i\frac{\lambda^\prime}{3} \sin \alpha, -7\lambda \cos \alpha + i\frac{\lambda^\prime}{3} \sin \alpha)$ where $\lambda$ and $\alpha$ are two supergravity fields. In this case, the scalar field $\lambda \cos \alpha$ corresponds to the mass term of (88) component in the boundary theory and pseudo-scalar field $\lambda \sin \alpha$ does the mass term of (11) component.
fact that there exist only two M2-branes, the approach to BL theory will give us some hints for the gravity duals. We impose the constraint on the \( \epsilon \) parameter that satisfies the \((1/2)\) BPS condition (the number of supersymmetries is eight) \([36]\): 
\[
\Gamma_{78910} \epsilon = -\epsilon
\]
with eleven-dimensional gamma matrices. The variation for the bosonic mass term
\[
\mathcal{L}_{b.m.} = -\frac{1}{2} h_{ab} X^a_I (m^2)_{IJ} X^b_J,
\]
plus the fermionic mass term with equal masses
\[
\mathcal{L}_{f.m.} = -i h_{ab} \bar{\Psi}^a \left( m \Gamma^{4589} - m \Gamma^{45710} \right) \Psi^b,
\]
in the BL theory leads to \([15]\)
\[
\delta \mathcal{L} = i h_{ab} X^a_I (m^2)_{IJ} \bar{\Psi}^b \Gamma_J \epsilon - i h_{ab} \bar{\Psi}^a \left( m \Gamma^{4589} - m \Gamma^{45710} \right)^2 X^b_I \Gamma_I \epsilon.
\]
Then the bosonic mass term \((m^2)_{IJ} \Gamma_J\) should take the form \(4m^2(\Gamma_7 + \Gamma_8 + \Gamma_9 + \Gamma_{10})\). Then the diagonal bosonic mass term has nonzero component only for \((7,7), (8,8), (9,9),\) and \((10,10)\) and other components \((3,3), (4,4), (5,5), (6,6)\) are vanishing. The degeneracy 4 is related to the \(\mathcal{N} = 4\) supersymmetry. Then one obtains the bosonic mass term which appears in \((3.2)\)
\[
(m^2)_{IJ} = \text{diag}(0, 0, 0, 0, 4m^2, 4m^2, 4m^2, 4m^2).
\]
Of course, there exist the quartic terms for bosonic field \(X^a_I\) in the deformed full Lagrangian. This can be seen from the component expansion in \(\mathcal{N} = 2\) superspace later.

Then how do we understand this analysis in \(\mathcal{N} = 2\) superspace approach? From the superpotential \([12]\) of the BL theory in \(\mathcal{N} = 2\) superspace with \(SU(2) \times SU(2) = SO(4)\) gauge group, by adding the quadratic mass deformation \((3.2)\), we expect to have the full superpotential \([3]\)
\[
- \frac{1}{8 \cdot 4!} \epsilon_{ABCD} \epsilon^{abcd} Z^A_a Z^B_b Z^C_c Z^D_d - 2m^2 (Z^3_a)^2 - 2m^2 (Z^4_b)^2,
\]
where \(Z^A_a\) \([12]\) is an \(\mathcal{N} = 2\) chiral superfield with \(SU(4)\) index \(A = 1, 2, \cdots, 4\) and with \(SO(4)\) gauge group index \(a = 1, 2, \cdots, 4\). The global symmetry \(SO(8)_R\) of the theory is broken to \([SU(2) \times U(1)]^2\) symmetry. The mass terms, the last two terms in \((3.3)\), break \(\mathcal{N} = 8\) down to \(\mathcal{N} = 4\). The mass-deformed theory has matter multiplet in two flavors \(Z^1\) and \(Z^2\) transforming in the adjoint of the gauge group. We’ll describe the moduli space characterized
\[\text{One cannot conclude that the dimension of monomials in the superpotential is the sum of the dimensions of each component. There is no conformal fixed point in the IR (for the bulk theory, look at the Figure 1) and the RG flow does not terminate at a conformal field theory fixed point, contrary to the case of [11, 12, 13].} \]
by these massless $Z^1$ and $Z^2$ in next section. The $SO(8)_R$ symmetry of the $\mathcal{N} = 8$ gauge theory is broken to $[SU(2) \times SU(2)]_R \times U(1)^2$ where the $[SU(2) \times SU(2)]_R$ is the R-symmetry of the $\mathcal{N} = 4$ theory and one of them $SU(2)$ is a flavor symmetry. Therefore, we turn on the mass perturbation in the UV and flow to the IR. This maps to turning on the scalar fields $\alpha$ and $\chi$ in the $AdS_4$ supergravity where the scalars approach to zero in the UV ($r \to \infty$) and develop a nontrivial profile as a function of $r$ becoming more significantly different from zero as one goes to the IR ($r \to -\infty$). Similar flow but nonsupersymmetric $SU(2) \times SU(2) \times U(1)$ critical point was studied in [37].

Motivated by the fact that the two M2-branes theory of BL theory is equivalent to $U(2) \times U(2)$ Chern-Simons matter theory with level $k = 1$ or $k = 2$ (there is further enhancement from $\mathcal{N} = 6$ to $\mathcal{N} = 8$ supersymmetry and two extra supersymmetries transform as $SO(6) = SU(4)$ singlets), it is natural to ask what happens for Chern-Simons matter theory when we turn on mass perturbation in the gauged supergravity? Let us consider the $U(2) \times U(2)$ Chern-Simons matter theory. From the superpotential [13] of $U(2) \times U(2)$ Chern-Simons matter theory, the quadratic mass deformations are added as follows:

\[ T^{-4} \frac{1}{4!} \epsilon_{ABCD} \text{Tr} Z^A Z^l B Z^C Z^l D - 2m^2 T^{-2} \text{Tr} Z^l Z^l 3 - 2m^2 T^{-2} \text{Tr} Z^l Z^l 4, \]  

(3.4)

where $Z^A$ is also an $\mathcal{N} = 2$ chiral superfield with $SU(4)$ index $A = 1, 2, \ldots, 4$ and an operation $\dagger$ is defined by $Z^l A = -i \sigma_2 (Z^A)^T i \sigma_2$. The relation of $Z^A$ to $SO(4)$ notation $Z^A_a$ is described in [12]. Note that there exist relations $Z^3 = W^l_1$ and $Z^4 = W^l_2$. The $T^2$ in (3.4) is a monopole operator [38] which creates two units of magnetic flux when the level $k = 1$ for the $U(1)$ field in the $U(1) \times U(1)$ Chern-Simons matter theory. Note that the $SU(4)$ subgroup of $SO(8)$ consists of complex rotation on the four component complex $SO(8)$ vector $\vec{v}$. Also there exists $U(1)$ subgroup which transforms this complex vector as itself with an extra overall phase. Then the independent two $SU(2)$ transformations, acting on the first two components of a complex vector and on the last two components of a complex vector respectively and the overall phase rotation gives a manifest $[SU(2) \times SU(2)]_R \times U(1)$ symmetry. When the superpotential

\[ \frac{1}{4!} \epsilon_{ABCD} \text{Tr} Z^A Z^l B Z^C Z^l D \]  

which is proportional to $\epsilon_{ABCD} \epsilon^{abcd} Z^a Z^b Z^c Z^d$ in (3.4) is written as $\frac{1}{4} \epsilon_{AC} \epsilon^{BD} \text{Tr} Z^A W_B Z^C W_D$, the $[SU(2) \times SU(2)]_R \times U(1)$ symmetry is manifest where the baryonic $U(1)$ acts as $Z^A \to e^{\frac{2\pi i}{k}} Z^A$ and $W_B \to e^{-\frac{2\pi i}{k}} W_B$ with level $k$ [12, 13].

In order to see the relation between the deformed Lagrangian from BL theory and $\mathcal{N} = 2$ superspace description, we need to integrate the superpotential over the fermionic coordinates. After the integration over the superspace explicitly, the quadratic deformation, the last two
terms in \((3.4)\), leads to

\[
 m^2T^{-2} \int d^2\theta \text{Tr}(Z^3Z^\dagger + Z^4Z^\dagger)
 = -m^2T^{-2} \text{Tr}(\zeta^3\zeta^\dagger + \zeta^4\zeta^\dagger) + 2m^2T^{-2} \text{Tr}(F^3Z^3 + F^4Z^4)
 = -m^2T^{-2} \text{Tr}(\zeta^3\zeta^\dagger + \zeta^4\zeta^\dagger) + \frac{m^2L}{3}(\epsilon^{ABC} \text{Tr} \tilde{Z}_A \tilde{Z}_B^\dagger \tilde{Z}_C Z^3 + \epsilon^{\hat{A}\hat{B}\hat{C}} \text{Tr} \tilde{Z}_A \tilde{Z}_B^\dagger \tilde{Z}_C Z^4),
\]

where the auxiliary field \(F\) is replaced using the equation of motion in the last line: \(F^A = -\frac{L}{6}\epsilon^{ABCD}Z_B Z_C^\dagger Z_D\) and the component expansions for \(\mathcal{N} = 2\) superfields are used. Here the \(L\) is a coefficient of the superpotential term in \(\mathcal{N} = 2\) superspace. The mass terms for the fermions correspond to the action of \(S\) introduced in \((3.1)\) where \(\zeta^3\) is related to a complex combination of 5-th component \(2\chi\) and 7-th component \(-2\chi\) of \((3.1)\) and \(\zeta^4\) is related to a complex combination of 6-th component \(2\chi\) and 8-th component \(-2\chi\). Note that in component expansion, there exist quartic terms as well as the mass terms for the fermions.

There are also similar constructions \([39, 40, 41]\) which have \(\mathcal{N} = 4\) supersymmetry. It would be interesting to see how they are related to the present mass-deformed BL or Chern-Simons matter theories.

\section{The M2-brane probe}

Now we want to understand the eleven-dimensional solution from the scalar field configuration. By applying the formula of \([12]\) to the \(\mathcal{N} = 4\) gauged supergravity, the 11-dimensional metric is given by \([17]\)

\[
ds_{11}^2 = \Omega^2 \left( e^{2A(r)} dx^\mu_2 + dr^2 \right) + \frac{4L^2\Omega^2}{c} d\theta^2 + L^2\Omega^2 \rho^2 \cos^2 \theta \left( \frac{1}{X_1^1} \sigma^2_1 + \frac{1}{X_1^2} \sigma^2_2 + \frac{1}{cX_1^3} \sigma^2_3 \right)
 + L^2\Omega^2 \sin^2 \theta \left( \frac{1}{X_2^1} \tau^2_1 + \frac{1}{X_2^2} \tau^2_2 + \frac{1}{cX_2^3} \tau^2_3 \right),
\]

(4.1) \hspace{1cm} \text{(4.1)}

where we introduce the following quantities

\[
c \equiv \cosh(2\chi), \quad \rho \equiv e^\alpha, \quad X_1 \equiv \cos^2 \theta + \rho^2 \sin^2 \theta \cos(2\chi),
 X_2 \equiv \cos^2 \theta \cosh(2\chi) + \rho^2 \sin^2 \theta, \quad \Omega \equiv \left[ \frac{X_1 X_2 \cosh(2\chi)}{\rho^2} \right]^\frac{1}{6}.
\]

(4.2) \hspace{1cm} \text{(4.2)}

The \(\sigma_i\) and \(\tau_i\) are independent sets of left-invariant one-forms on \(SU(2)\). This metric \([4.1]\) has a manifest symmetry of \(SU(2)_\sigma \times U(1)_\sigma\) and \(SU(2)_\tau \times U(1)_\tau\). The \(U(1)_\sigma\) rotates \(\sigma_1\) into \(\sigma_2\) while the \(U(1)_\tau\) rotates \(\tau_1\) into \(\tau_2\). The Killing spinors are singlets under \(SU(2)_\tau \times U(1)_\sigma\) and transform as \(2_{\pm 1}\) under \(SU(2)_\sigma \times U(1)_\tau\). We’ll see that \(SU(2)_\tau\) acts on the complex
structure in the hyper-Kahler moduli space. Note that the additional isometries of the metric transverse to the M2-brane amount to additional global symmetries of boundary field theory on the M2-brane.

The 3-form potential \[17\] takes the form of

\[
A^{(3)} = \tilde{W} e^{3A} \, dt \wedge dx_1 \wedge dx_2 + L^3 \tanh(2 \chi) \sin \theta \, d\theta \wedge \sigma_3 \wedge \tau_3 \\
+ \frac{L^3 \rho^2}{2X_1} \sinh(2\chi) \, \sin^2 \theta \, \cos^2 \theta \wedge \sigma_1 \wedge \sigma_2 \wedge \tau_3 \\
+ \frac{L^3}{2X_2} \sinh(2\chi) \, \sin^2 \theta \, \cos^2 \theta \wedge \sigma_3 \wedge \tau_1 \wedge \tau_2. \tag{4.3}
\]

Here \(\tilde{W}\) is called “geometric” superpotential \[43\] and it is given by

\[
\tilde{W} = \frac{X_1}{2\rho} = \frac{1}{2\rho} \left[ \cos^2 \theta + \rho^2 \sin^2 \theta \cosh(2\chi) \right], \tag{4.4}
\]

which is exactly the same as a half of the superpotential \[27\]

\[
\tilde{W} = \frac{W}{2},
\]

at \(\theta = \frac{\pi}{4}\).

Let us describe the \(\mathcal{N} = 8\) four-dimensional gauged supergravity. Now we go to the \(\text{SL}(8, \mathbb{R})\) basis \[43\] and introduce the rotated vielbeins

\[
U_{ijIJ} = u_{ijab}(\Gamma_{IJ})_{ab}, \quad V_{ijIJ} = v_{ijab}(\Gamma_{IJ})_{ab},
\]

where all indices \(i, j\) and \(a, b\) run from 1 to 8 and correspond to the realization of \(E_{7(7)}\) in the \(\text{SU}(8)\) basis and \(\Gamma_{IJ}\) are the \(\text{SO}(8)\) generators in \[5\]. We also define the following quantities

\[
A_{ijIJ} = \frac{1}{\sqrt{2}} (U_{ijIJ} + V_{ijIJ}), \quad B_{ijIJ} = \frac{1}{\sqrt{2}} (U_{ijIJ} - V_{ijIJ}),
\]

\[
C_{ijIJ} = \frac{1}{\sqrt{2}} (U_{ijIJ} + V_{ijIJ}), \quad D_{ijIJ} = \frac{1}{\sqrt{2}} (-U_{ijIJ} + V_{ijIJ}). \tag{4.5}
\]

The full expressions for these are in \[B.2, B.3, B.4, B.5\] and \[B.6\] from the appendix B.

Then the “geometric” T tensor \[43\] can be written as

\[
\tilde{T}^{kij}_l = \frac{1}{21\sqrt{2}} C^{kij}_{LM} \left( A_{lmJK} D^{kmKI} \delta^L_M x_M + B_{lmJK} C^{km}_K \delta^M_J x_L \right), \tag{4.6}
\]
where we have a relation between $x_I$ and $Y_I$ that is a coordinate for $\mathbb{R}^8$ where $\sum_{I=1}^{8}(Y_I)^2 = 1$ and that is introduced in order to absorb the cross terms between $x_I$'s:

\[
\begin{align*}
Y_1 &\equiv \frac{1}{\sqrt{2}}(x_2 - x_6), & Y_2 &\equiv -\frac{1}{\sqrt{2}}(x_3 - x_7), \\
Y_3 &\equiv \frac{1}{\sqrt{2}}(x_4 - x_8), & Y_4 &\equiv -\frac{1}{\sqrt{2}}(x_1 - x_5), \\
Y_5 &\equiv \frac{1}{\sqrt{2}}(x_2 + x_6), & Y_6 &\equiv \frac{1}{\sqrt{2}}(x_3 + x_7), \\
Y_7 &\equiv \frac{1}{\sqrt{2}}(x_4 + x_8), & Y_8 &\equiv \frac{1}{\sqrt{2}}(x_1 + x_5).
\end{align*}
\]

From this, the corresponding “geometric” $\tilde{A}_1$ tensor is given by $\tilde{A}_{ij} = \tilde{T}_{imj}$ and we present them in $\text{[B.7], [B.8], [B.9], [B.10], [B.11]}$ from appendix B. The idea of $\text{[43]}$ is to introduce geometric analogue of the T-tensor by replacing $\delta^{IJ}$ by $x^I x^J$.

By computing the real part of 11-component of the $\tilde{A}_1$ tensor $\text{[B.7]}$ with $\phi = 0$ and $\varphi = \frac{\pi}{2}$, one obtains the geometrical superpotential $W_{gs}$ as follows:

\[
W_{gs} = e^{-\alpha} \left( Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 \right) + e^\alpha \cosh(2\chi) \left( Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2 \right). \tag{4.7}\]

Note that half of the transverse coordinates has common factor and the remaining half of them has other common factor. We want to see how this geometric superpotential is related to the superpotential $\text{(2.7)}$ or $\text{(4.4)}$. There is a chance to compare this $\text{(4.7)}$ with $\text{(4.4)}$ by looking at the $\alpha$ and $\chi$ dependence. Through the relations

\[
\begin{align*}
Y_1 &= \cos \theta \cos\left(\frac{\alpha_1}{2}\right) \cos\left(\frac{\alpha_2 + \alpha_3}{2}\right), & Y_2 &= \cos \theta \cos\left(\frac{\alpha_1}{2}\right) \sin\left(\frac{\alpha_2 + \alpha_3}{2}\right), \\
Y_3 &= \cos \theta \sin\left(\frac{\alpha_1}{2}\right) \cos\left(\frac{\alpha_2 - \alpha_3}{2}\right), & Y_4 &= \cos \theta \sin\left(\frac{\alpha_1}{2}\right) \sin\left(\frac{\alpha_2 - \alpha_3}{2}\right), \\
Y_5 &= \sin \theta \cos\left(\frac{\beta_1}{2}\right) \cos\left(\frac{\beta_2 + \beta_3}{2}\right), & Y_6 &= \sin \theta \cos\left(\frac{\beta_1}{2}\right) \sin\left(\frac{\beta_2 + \beta_3}{2}\right), \\
Y_7 &= \sin \theta \sin\left(\frac{\beta_1}{2}\right) \cos\left(\frac{\beta_2 - \beta_3}{2}\right), & Y_8 &= \sin \theta \sin\left(\frac{\beta_1}{2}\right) \sin\left(\frac{\beta_2 - \beta_3}{2}\right),
\end{align*}
\]

this geometric superpotential $\text{(4.7)}$ reduces to

\[
W_{gs} = 2\tilde{W},
\]

with $\text{(4.4)}$. Here $\alpha_i$ and $\beta_i$ are Euler angles that parametrize two independent sets of $SU(2)$ left-invariant one-forms respectively. Also the usual spherical coordinates for three-sphere can be used to write any element of $SU(2)$. This is consistent with the definition of geometric superpotential through the superpotential $8\tilde{W} = \frac{1}{2} \sum_{I=1}^{8} \frac{\partial^2}{\partial Y_I \partial Y_I} W_{gs} \text{[43]}$. 

13
Furthermore, by calculating the $\tilde{A}_2$ tensor (we do not present in this paper) obtained from the geometric T tensor, one arrives at the geometric scalar potential eventually

$$V_{gp}(\alpha, \chi, \theta) = -g^2 \left( \frac{3}{4} |\tilde{A}^{ij}_{1}|^2 - \frac{1}{24} |\tilde{A}^{i}_{jkl}|^2 \right)$$

$$= \frac{g^2}{4} \cosh(2\chi) [(3 + \cos[4\theta]) \cosh[2(\alpha - \chi)] + (3 + \cos[4\theta]) \cosh[2(\alpha + \chi)]$$

$$+ (3 + \cos[4\chi]) \sin^2(2\theta) - 8 \cos(2\theta) \cosh(2\chi) \sinh(2\alpha)] .$$

This has a critical point at $\alpha = 0 = \chi$ and $\theta = \frac{\pi}{4}$.

The potential seen by the M2-brane probe \[44, 43, 45\] has a factor $e^{3A}(\Omega^3 - 2\tilde{W})$. \(\text{(4.8)}\) \{1\}

The moduli spaces of the brane probe are given by the loci where the potential vanishes. One sees that this potential \(\text{(4.8)}\) vanishes at

$$\cos \theta = 0 \rightarrow \theta = \frac{\pi}{2},$$

where we use \(\text{(4.2)}\) and \(\text{(4.4)}\) or \(\text{(4.7)}\). On this subspace, the four-dimensional moduli space from \(\text{(4.1)}\), by multiplying the factor $e^{A}\Omega^{-\frac{1}{2}}$ into the seven-dimensional internal metric, is given by

$$ds^2|_{\text{moduli}} = \rho \cosh(2\chi) e^{A} L^2 \left[ \frac{1}{L^2} dr^2 + \frac{1}{\rho^2} (\tau_1^2 + \tau_2^2) + \frac{1}{\rho^2 \cosh^2(2\chi)} \tau_3^2 \right]. \quad (4.9) \{2\}$$

It is evident that the explicit one $SU(2)$ from $[SU(2) \times SU(2)]_R$ invariance is that of the flavor symmetry. Note that the other $SU(2)$ acts on the geometry transverse to both the M2-branes and moduli space. The vacuum expectation values of the massless scalars are denoted by $Z^1$ and $Z^2$(that are $\theta$-independent components of $Z^1$ and $Z^2$ respectively) and $Z^i$'s transform in the fundamental representation of $SU(2)$ and their complex conjugates transform in the anti-fundamental representation of $SU(2)$. That is, $Z^1$ and $Z^2$ parametrize the Cartesian coordinates $x^3 \equiv r, x^8, x^9, x^{10}$ while $Z^3$ and $Z^4$ parametrize the remaining Cartesian coordinates $x^4, x^5, x^6, x^7$. The $SU(2)$ flavor symmetry implies that the Kahler potential is a function of $u^2$ where let us define $u^2 \equiv Z^1 \bar{Z}_1 + Z^2 \bar{Z}_2$. Specifying the coordinates and indices into holomorphic and anti-holomorphic and if the Kahler structure exists, then the metric is given by $ds^2 = \partial_\mu \partial_\nu K(u^2) dz^\mu dz^\nu$. Then the metric turns out to be \[45\]

$$ds^2 = (K' + u^2 K'') du^2 + u^2 \left[ K' (\tau_1^2 + \tau_2^2) + (K' + u^2 K'') \tau_3^2 \right]. \quad (4.10) \{3\}$$
Is there any connection between (4.9) and (4.10)? Let us compare (4.9) with (4.10). Then we get

\[(K' + u^2 K'') du^2 = \rho \cosh(2\chi)e^A dr^2,\]

\[u^2(K' + u^2 K'') = \frac{L^2 e^A}{\rho \cosh(2\chi)},\]

\[u^2 K' = \frac{L^2 \cosh(2\chi)e^A}{\rho} .\]  (4.11)  (?)

From the last equation of (4.11),

\[u^2 K' = u^2 \frac{dK}{d(u^2)} = \frac{L^2 \cosh(2\chi)e^A}{\rho} .\]  (4.12)  (?)

We need to check the second equation of (4.11). Since

\[u^2(K' + u^2 K'') = u^2 \frac{d}{d(u^2)}(u^2 K'),\]  (4.13)  (?)

by substituting (4.12) into (4.13), one has

\[u^2 \frac{d}{d(u^2)}(u^2 K') = \frac{L}{2\rho \cosh(2\chi)} \frac{d}{dr} \left( \frac{L^2 \cosh(2\chi)e^A}{\rho} \right).\]

Here we use the following change of variable which can be obtained by using the first and second equations of (4.11) \[u^2 \frac{d}{d(u^2)} = \frac{L}{2\rho \cosh(2\chi)} \frac{d}{dr} .\] Now after using the flow equations (2.9) explicitly we obtain

\[\frac{L}{2\rho \cosh(2\chi)} \frac{d}{dr} \left( \frac{L^2 \cosh(2\chi)e^A}{\rho} \right) = \frac{2L^2 e^A}{\rho \cosh(2\chi)},\]

which is not the right hand side of second equation (4.11). Therefore, the metric (4.9) cannot be written as (4.10) due to the coefficient one of \(\tau_3^2\) in (4.9): shortened by a factor \(\frac{1}{2}\). If the coefficient of \(\tau_3^2\) in (4.9) is two, then one can express the metric in terms of Kahler potential through (4.10).

On the other hand, one can write the metric (4.9) in terms of \(\chi\) variable instead of \(r\) using the flow equation (2.9) as follows [17]:

\[ds^2_{\text{moduli}} = \frac{\cosh(2\chi)}{\sinh^3(2\chi)} d\chi^2 + \frac{\cosh(2\chi)}{4 \sinh(2\chi)} (\tau_1^2 + \tau_2^2) + \frac{1}{4 \cosh(2\chi) \sinh(2\chi)} \tau_3^2.\]  (4.14)  (?)

There is no \(\rho\) or \(\alpha\) dependence on this moduli space metric. As \(\chi \to \infty\), the 2-sphere by \(\tau_1\) and \(\tau_2\) limits a sphere of radius \(\frac{1}{2}\). Moreover, by introducing a new radial variable \(\mu\), the metric (4.14) can be written as [17]

\[ds^2_{\text{moduli}} = \frac{d\mu^2}{(1 - \frac{1}{\mu^2})} + \frac{\mu^2}{4} (\tau_1^2 + \tau_2^2) + \frac{\mu^2}{4} \left(1 - \frac{1}{\mu^4}\right) \tau_3^2, \quad \mu \equiv \sqrt{\frac{\cosh(2\chi)}{\sinh(2\chi)}},\]
As \( \chi \to \infty \) or \( \mu \to 1 \), the branes are spread out over finite two sphere. In this parametrization one can do similar analysis of (4.10)-(4.13) and obtains the same result: the difference in the coefficient of the metric. For \( \theta = \frac{\pi}{2} \), the three-form potential (4.3) becomes \([17]\)

\[
A^{(3)} \sim H^{-1} dt \wedge dx_1 \wedge dx_2, \quad H \equiv e^{-3A\Omega^{-3}} \big|_{\theta = \frac{\pi}{2}} = \frac{\sinh^2(2\chi)}{\rho^4 \cosh(2\chi)}.
\]

One can easily check that the behavior of the function \( H \) gives the right asymptotics for the uniform distribution of branes spread over the two sphere at \( e^{-2\chi} = 0 \) which is another good radial coordinate.

5 The holographic \( N = 8 \) flow in four dimensions

The invariant scalar manifold by Pope-Warner \([18]\) consisting of two scalars has a factor \( SL(2, \mathbb{R}) \) in \( E_{7(7)} \) which has a maximal non-compact subgroup \( SL(8, \mathbb{R}) \). Then the noncompact generators of \( SL(2, \mathbb{R}) \) can be written as \([18]\)

\[
\phi_{ijkl} = (\alpha \cos \zeta + i\alpha \sin \zeta)\delta_{ijkl}^{1234} + (\alpha \cos \zeta - i\alpha \sin \zeta)\delta_{ijkl}^{5678}.
\]

This can be obtained from (2.1) by taking \( \phi \to \zeta \) and \( \chi \to 0 \). See also \([35, 46]\) for the \( SO(p) \times SO(8-p) \) invariant generator of \( SL(8, \mathbb{R}) \). As we did before, the 56-beins \( \mathcal{V}(x) \) can be written as (2.2) or (2.3) and 28-beins \( u \) and \( v \) are summarized in (C.1) in the appendix C. It turns out that \( A_1 \) tensor has one real eigenvalue, \( z_1 \) with degeneracies 8 and has the following form

\[
A_1 = \text{diag} (z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_1),
\]

with \( z_1 = \cosh \alpha \).

The scalar potential can be written, by combining all the components of \( A_1, A_2 \) tensors where (C.2) gives the data of \( A_2 \) tensor, as

\[
V(\alpha) = -2g^2 [2 + \cosh(2\alpha)] = 4g^2 \left[ \left( \frac{\partial W}{\partial \alpha} \right)^2 - 3W^2 \right]. \tag{5.1}
\]

The superpotential is given by

\[
W(\alpha) = \cosh \alpha. \tag{5.2}
\]

It turns out that the \( A_1 \) tensor that appears in the gravitino transformation rule has \( \cosh \alpha \) with degeneracy 8. This implies maximal supersymmetry \( (N = 8) \). We’ll check this explicitly
The scalar potential at the $SO(8)$ UV critical point where $\alpha = 0$ (or $r \to \infty$) becomes $V = -6g^2$ where $g = \frac{1}{\sqrt{2}L}$ and the superpotential $W$ becomes 1. Note that the supergravity scalar potential (5.1) is independent of the angle $\zeta$. This critical point is common to both a scalar potential (5.1) and a superpotential (5.2) and is depicted in Figure 2.

The resulting Lagrangian of the scalar-gravity sector can be obtained by finding out the scalar kinetic term. We arrive at the following scalar kinetic term $(\partial_{\mu}\alpha)^2$ from (C.3). By substituting the domain-wall ansatz into the Lagrangian, the energy-density per unit area transverse to $r$-direction is obtained. Then $E[\alpha, \chi]$ is extremized by the BPS domain-wall solutions. The maximally supersymmetric flow satisfying the first order differential equations for given superpotential leads to the solutions for $\alpha(r), A(r)$ and $\zeta(r) = \text{const}$ explicitly. The dilaton and the axion are function of $\alpha(r)$ and $\zeta$ which remains fixed along the flow:

$$
\frac{d\alpha}{dr} = -\sqrt{2} g \partial_{\alpha} W, \quad \frac{d\zeta}{dr} = 0, \quad \frac{dA}{dr} = \sqrt{2} g W.
$$

The solutions for flow equations are given by

$$
e^\alpha = \coth \left( \frac{r}{2L} \right), \quad e^A = \sinh \left( \frac{r}{L} \right) = \frac{1}{\sinh \alpha}, \quad \zeta = \text{const}.
$$

Note that the flow is maximally supersymmetric for all choices of $\zeta$ and the potential (5.1), superpotential (5.2) and the flow (5.3) do not depend on $\zeta$.

The vanishing of variation of $\chi^{ijk}$ for supersymmetry parameter $\epsilon^i$ where $i = 1, \cdots, 8$ leads
to
\[
\sqrt{2}e^{i\zeta} \left[ \partial_\mu \alpha + \frac{i}{2} \sinh(2\alpha) \partial_\mu \zeta \right] \begin{pmatrix} \eta_2^* \\ \eta_1^* \end{pmatrix} = -2g e^{i\zeta} \frac{\partial W}{\partial \alpha} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},
\]
\[
\sqrt{2}e^{-i\zeta} \left[ \partial_\mu \alpha - \frac{i}{2} \sinh(2\alpha) \partial_\mu \zeta \right] \begin{pmatrix} \eta_2^* \\ \eta_1^* \end{pmatrix} = -2g e^{-i\zeta} \frac{\partial W}{\partial \alpha} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.
\]
(5.4) \?

By multiplying the \( \eta_2 \) into the first row and \( \eta_1 \) into the second row in the first equation (5.4) and combining these two relations, one obtains \(|\eta_1(r)| = |\eta_2(r)| \equiv |\eta(r)|\). Let us consider the equation
\[
2\partial_r \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \pm i\partial_\mu \zeta \sinh^2 \alpha \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \sqrt{2}g W \begin{pmatrix} \eta_2^* \\ \eta_1^* \end{pmatrix},
\]
where the upper sign is on the supersymmetry parameter \( \epsilon^{i=1,2,3,4} \) and the lower sign is on the supersymmetry parameter \( \epsilon^{i=5,6,7,8} \) and substitute the \( \eta_1 \) and \( \eta_2 \). We used the result (C.4) in the appendix C. By multiplying \( e^{-i\beta} \) into the first row and \( e^{-i\delta} \) into the second row and subtracting these two, then one obtains \((\partial_r \beta - \partial_r \delta)\eta \) which implies that \( \beta = \delta + \text{const.} \). Now let us analyze the equation
\[
\partial_r A \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \sqrt{2}g W \begin{pmatrix} \eta_2^* \\ \eta_1^* \end{pmatrix}.
\]
From the first row, one gets \( \partial_r A = \sqrt{2}g W e^{-2i\beta} \). This implies that \( W e^{-2i\beta} \) should be real because \( \partial_r A \) is real.

Let us compare the first and second equations of (5.3). Then we obtain \( e^{2i\beta} \frac{\partial W}{\partial \alpha} = e^{-2i\beta} \frac{\partial W}{\partial \alpha} \) because the left hand side of second equation of (5.4) is a complex conjugation of the left hand side of first equation. This leads to the fact that \( \beta = 0 \). Therefore the expression of the bracket in (5.4) should be real. So one has \( \partial_r \zeta = 0 \). By collecting all the informations so far, one obtains the flow equations (5.3) we explained before. We have checked that all the supersymmetry parameters \( \epsilon^i \) where \( i = 1, \cdots, 8 \) do not vanish explicitly.

According to the branching rule [17, 18, 19] of \( 8_s \) representation corresponding to spin \( \frac{3}{2} \) field of \( SO(8) \) under the \([SO(4)]^2 = [SU(2)]^4\), \( 8_s \rightarrow (1, 2, 1, 2) \oplus (2, 1, 2, 1) \), the two singlets correspond to the component of massless graviton of the \( N = 8 \) theory. From the branching rule of \( 28 \) representation corresponding to spin 1 field of \( SO(8) \) under the \([SU(2)]^4\), \( 28 \rightarrow (2, 2, 2, 2) \oplus (1, 3, 1, 1) \oplus [(1, 1, 1, 3)] \oplus (1, 1, 3, 1) \oplus (3, 1, 1, 1) \), the singlet in square bracket corresponds to the component of the massless graviton of the \( N = 8 \) theory. Finally, spin 2 field with the breaking \( 1 \rightarrow (1, 1, 1, 1) \) is located at the remaining component of \( N = 8 \) massless graviton multiplet.

From the decomposition of spin 1 field above, the representation \( (1, 1, 3, 1) \oplus (1, 3, 1, 1) \oplus (3, 1, 1, 1) \) correspond to the massless vector multiplet of \( N = 8 \) theory. According to the
branching rule of $56_s$ representation corresponding to spin $\frac{1}{2}$ field of $SO(8)$ under the $[SU(2)]^4$, $56_s \rightarrow [(2,3,2,1) \oplus (1,2,3,2) \oplus (3,2,1,2)] \oplus (2,1,2,3) \oplus (1,2,1,2) \oplus (2,1,2,1)$, the subset of the representations in square bracket corresponds to the component of massless vector multiplet of the $\mathcal{N} = 8$ theory. Moreover, the branching rules for spin zero field $35_v \rightarrow [(3,3,1,1) \oplus (2,2,2,2) \oplus (1,1,3,3) \oplus (1,1,1,1)]$ and $35_c \rightarrow [(1,3,3,1) \oplus (2,2,2,2) \oplus (3,1,1,3) \oplus (1,1,1,1)]$ provide the remaining component of massless vector multiplet of the $\mathcal{N} = 8$ theory.

6 The $\mathcal{N} = 8$ supersymmetric membrane flows in three dimensions

Let us consider the BL theory with $SO(4)$ gauge group and matter fields. The variation for the bosonic mass term (3.2) plus the fermionic mass term

$$\mathcal{L}_{f.m.} = - \frac{i}{2} h_{ab} \bar{\Psi}^a m^{3456} \Psi^b,$$

leads to the following variation

$$\delta \mathcal{L} = i h_{ab} X^a_I (m^2)_{IJ} \bar{\Psi}^b \Gamma_J \epsilon - i h_{ab} \bar{\Psi}^a (m^{3456})^2 X^b_I \Gamma_J \epsilon.$$

Then the bosonic mass term $(m^2)_{IJ} \Gamma_J$ should take the form $m^2 \sum_{I=3}^{10} \Gamma_I$. Then the diagonal bosonic mass term has nonzero components for all eight elements. The degeneracy 8 is related to the $\mathcal{N} = 8$ supersymmetry [50, 36, 51]. Then one obtains the bosonic mass term which appears in (3.2)

$$(m^2)_{IJ} = \text{diag}(m^2, m^2, m^2, m^2, m^2, m^2, m^2, m^2).$$

(6.1) {?}

From the superpotential of the BL theory in $\mathcal{N} = 2$ superspace with $SU(2) \times SU(2) = SO(4)$ gauge group, by adding the quadratic mass deformation (6.1), we have the full superpotential

$$- \frac{1}{8 \cdot 4!} \epsilon_{ABCD} \epsilon^{abcd} \mathcal{Z}_a^A \mathcal{Z}_b^B \mathcal{Z}_c^C \mathcal{Z}_d^D - \frac{m^2}{2} \sum_{A=1}^{4} (\mathcal{Z}_a^A)^2,$$

(6.2) {?}

where $\mathcal{Z}_a^A$ is an $\mathcal{N} = 2$ chiral superfield with $SU(4)$ index $A = 1, 2, \cdots, 4$ and with $SO(4)$ gauge group index $a = 1, 2, \cdots, 4$ as in previous section. The global symmetry $SO(8)_R$ of the theory is broken to $[SO(4) \times SO(4) \times \mathbb{Z}_2]$ symmetry. The mass terms, the last terms in (6.2),

\footnote{As in previous case, one cannot conclude that the dimension of monomials in the superpotential is the sum of the dimensions of each component. There is no conformal fixed point in the IR and the RG flow does not terminate at a conformal field theory fixed point.}

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do not break $\mathcal{N} = 8$. The theory does not have matter multiplets transforming in the adjoint of the gauge group because all of them are massive. The $SO(8)_{R}$ symmetry of the $\mathcal{N} = 8$ gauge theory is broken to $[SO(4) \times SO(4)]_{R}$ which is nothing but the R-symmetry. We turn on the mass perturbation in the UV and flow to the IR. This maps to turning on the scalar field $\alpha$ in the $AdS_4$ supergravity where the scalar approaches to zero in the UV ($r \to \infty$) and develop a nontrivial profile as a function of $r$ as one goes to the IR ($r \to -\infty$) through (5.3).

From the superpotential of $U(2) \times U(2)$ Chern-Simons matter theory, the quadratic mass deformations are added as follows:

$$
\frac{T^{-4}}{4!} \epsilon_{ABCD} \text{Tr} \; Z^A \bar{Z}^B \bar{Z}^C \bar{Z}^D - \frac{m^2}{2} T^{-2} \sum_{A=1}^{4} \text{Tr} \; Z^A \bar{Z}^A.
$$

Let us consider a vector $\vec{v}$ on which $SO(8)$ acts. Then the first $SO(4)$ acts on the first four components of the vector while the second $SO(4)$ acts on the last four components of the vector. Here $Z_2$ is the transformation which replaces the first four components of the vector with the last four components of the vector. Then the $SU(4)$ subgroup above consists of complex rotation on the four component “complex” vector. Also there exists $U(1)$ subgroup which transform this complex vector as itself with an extra overall phase as in previous section. The independent two $SU(2)$ transformations, acting on the first two components of a complex vector and on the last two components of a complex vector respectively and the overall phase rotation give a manifest $SU(2) \times SU(2) \times U(1) \times Z_2$ symmetry. Note that the subgroup of $SO(8)$ common to both $SU(4) \times U(1)$ for undeformed $\mathcal{N} = 6$ Chern-Simons matter theory and $SO(4) \times SO(4)$ for the R-symmetry of the theory is nothing but $SU(2) \times SU(2) \times U(1)$.

After the integration over the superspace, the quadratic deformation leads to

$$
m^2 T^{-2} \int d^2 \theta \text{Tr} \sum_{A=1}^{4} Z^A \bar{Z}^A
$$

$$
= -m^2 T^{-2} \text{Tr} \sum_{A=1}^{4} \zeta^A \zeta^A + 2m^2 T^{-2} \text{Tr} \sum_{A=1}^{4} F^A Z^A
$$

$$
= -m^2 T^{-2} \text{Tr} \sum_{A=1}^{4} \zeta^A \zeta^A + T^{-4} m^2 L \sum_{D=1}^{4} \epsilon_{ABCD} \text{Tr} \; \bar{Z}^A \bar{Z}^B \bar{Z}^C \bar{Z}^D,
$$

where the auxiliary field $F$ is replaced using the equation of motion in the last line: $F^A = -\frac{L}{6} \epsilon^{ABCD} Z_B \bar{Z}_C \bar{Z}_D$ as in [12]. One sees that there are also quartic terms above.

It would be interesting to study whether this mass-deformed theory can be realized in terms of $\mathcal{N} = 4$ superspace formalism explicitly.
7 The M2-brane probe

It is known in [52] that the four-dimensional $\mathcal{N} = 4$ $SO(4)$ gauged supergravity can be obtained from the eleven-dimensions. The 11-dimensional metric is given by [18]

$$ds_{11}^2 = \Omega^2 \left( e^{2A(r)} dx_\mu^2 + dr^2 \right) + 4L^2 \Omega^2 d\theta^2 + \frac{L^2 \Omega^2}{Y} \cos^2 \theta \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) + \frac{L^2 \Omega^2}{Y} \sin^2 \theta \left( \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \tilde{\sigma}_3^2 \right) .$$

(7.1)

Here the left-invariant one-forms parametrize the three sphere $S^3$ and similarly other left-invariant one-forms parametrize the other three sphere $S^3$. These are invariant under the action of the $SO(4) \times SO(4)$ R-symmetry. An interchange symmetry $\theta \rightarrow \tilde{\theta} - \theta$ and $\alpha \rightarrow - \alpha$ appears also. The various functions appearing in this metric (7.1) are defined by

$$\Omega = \left( Y \tilde{Y} \right)^{1/8} ,$$

$$Y = \cos^2 \theta \left[ \cosh(2\alpha) + \cos \zeta \sinh(2\alpha) \right] + \sin^2 \theta ,$$

$$\tilde{Y} = \sin^2 \theta \left[ \cosh(2\alpha) - \cos \zeta \sinh(2\alpha) \right] + \cos^2 \theta .$$

(7.2)

The three-form potential is given by

$$A^{(3)} = \frac{k^3Z}{\sinh^3 \alpha} dt \wedge dx_1 \wedge dx_2 + L^3 \sin \zeta \sinh(2\alpha) \left( \frac{\cos^4 \theta}{Y} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 - \frac{\sin^4 \theta}{Y} \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 \right) ,$$

where the function is defined as

$$Z = \frac{1}{2 \cosh \alpha} \left( Y + \tilde{Y} \right) = \cosh \alpha + \cos \zeta \cos(2\theta) \sinh \alpha .$$

(7.3)

When $\theta = \frac{\pi}{4}$, this $Z$ is identical to $W$ in (5.2). As we did before, using the definitions (4.5), the corresponding 28-beins are obtained in (D.1), (D.2), (D.3) and (D.4) in appendix D.

By computing the real part of 11 component of the $\tilde{A}_1$ tensor (D.5), one obtains the geometrical superpotential $W_{gs}$ as follows:

$$W_{gs} = \cosh \alpha \left( Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2 \right)$$

$$+ \cos \zeta \sinh \alpha \left( -Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 + Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2 \right) .$$

(7.4)

We want to see how this geometric superpotential is related to the superpotential (5.2). One compares this (7.4) with (7.3) by looking at the $\alpha$ and $\zeta$ dependence. Through the relations

$$Y_1 = \sin \theta \cos \left( \frac{\alpha_1}{2} \right) \cos \left( \frac{\alpha_2 + \alpha_3}{2} \right) , \quad Y_2 = \sin \theta \cos \left( \frac{\alpha_1}{2} \right) \sin \left( \frac{\alpha_2 + \alpha_3}{2} \right) ,$$

$$Y_3 = \sin \theta \sin \left( \frac{\alpha_1}{2} \right) \cos \left( \frac{\alpha_2 - \alpha_3}{2} \right) , \quad Y_4 = \sin \theta \sin \left( \frac{\alpha_1}{2} \right) \sin \left( \frac{\alpha_2 - \alpha_3}{2} \right) ,$$

$$Y_5 = \cos \theta \cos \left( \frac{\beta_1}{2} \right) \cos \left( \frac{\beta_2 + \beta_3}{2} \right) , \quad Y_6 = \cos \theta \cos \left( \frac{\beta_1}{2} \right) \sin \left( \frac{\beta_2 + \beta_3}{2} \right) ,$$

$$Y_7 = \cos \theta \sin \left( \frac{\beta_1}{2} \right) \cos \left( \frac{\beta_2 - \beta_3}{2} \right) , \quad Y_8 = \cos \theta \sin \left( \frac{\beta_1}{2} \right) \sin \left( \frac{\beta_2 - \beta_3}{2} \right) ,$$

we obtain

$$W_{gs} = \cos \alpha \left( Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2 \right)$$

$$+ \cosh \alpha \cos \zeta \sinh \alpha \left( -Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 + Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2 \right) .$$

(7.5)

We can now turn to the superpotential (5.2). The comparison between (7.4) and (7.5) gives

$$W_{gs} = \cosh \alpha \left( \frac{1}{2} \cos \theta \cos \left( \frac{\beta_1}{2} \right) \cos \left( \frac{\beta_2 + \beta_3}{2} \right) \right)$$

$$+ \cosh \alpha \cos \zeta \sinh \alpha \left( -\frac{1}{2} \sin \theta \cos \left( \frac{\alpha_1}{2} \right) \cos \left( \frac{\alpha_2 + \alpha_3}{2} \right) \right) .$$

(7.6)
this geometric superpotential (7.4) reduces to
\[ W_{gs} = Z, \]
with (7.3) and \( \alpha_i \) and \( \beta_i \) are Euler angles that parametrize two independent sets of \( SU(2) \) left-invariant one-forms respectively.

The potential seen by the M2-brane probe has a factor \( e^{3A}(\Omega^3 - 2Z) \) with (7.2) and (7.3). The moduli spaces of the brane probe are given by the loci where the potential vanishes. There is no solution for internal coordinates satisfying the vanishing of potential. This is due to the fact that there are no massless scalars in this maximal case. All of them are massive.

Furthermore, by calculating the \( \tilde{A}_2 \) tensor obtained from the geometric \( \tilde{T} \) tensor, one arrives at the geometric scalar potential
\[
V_{gp}(\alpha, \zeta, \theta) = \frac{g^2}{4} \left[ 7 + \cos(2\zeta) + 2 \cosh(4\alpha) \sin^2 \zeta - 4 \cosh(2\alpha)(-6 + \cos[4\theta] \sinh^2 \alpha) \right. \\
+ \left. 32 \cos \zeta \cos(2\theta) \sinh(2\alpha) + 2 \cos(4\theta)(6 \sinh^2 \alpha + \cos[2\zeta] \sinh^2[2\alpha]) \right].
\]
This has a critical point at \( \alpha = \zeta = 0 = \theta = \frac{\pi}{4} \). When \( \zeta = 0 \), the transverse components of three-form potential vanish and \( A^{(3)} \) due to the factor \( \sin \zeta \) reduces to
\[ A^{(3)} = k^3 H^{-1} dt \wedge dx \wedge dy, \quad H \equiv \frac{\sinh^3 \alpha}{e^{-\alpha} \sin^2 \theta + e^\alpha \cos^2 \theta}. \]
Moreover, by the change of variables appropriately the metric can be simplified as the standard harmonic form where the M2-branes are spread out into a solid four-ball. This is consistent with the case where \( \chi = 0 \) for \( \mathcal{N} = 4 \) supersymmetric case before.

8 Conclusions and outlook

We have found the gauge duals in the context of BL theory (3.3) and (6.2) and Chern-Simons matter theory (3.4) and (6.3) to the holographic \( \mathcal{N} = 4 \) supersymmetric RG flow and the holographic \( \mathcal{N} = 8 \) supersymmetric RG flow.

As pointed out in [17], when the parameters satisfy \( \phi = \pm 2\varphi \), then there exist four supersymmetries. In other words, \( \mathcal{N} = 2 \) supersymmetry. In this case, the corresponding superpotential will be either \( z_2 \) or \( z_3 \) which has multiplicities 2, 2, respectively. One can construct 11-dimensional solution by using the formula given by [42]. According to [17], there exists a family of solutions that interpolates between the pure metric flow with \( \phi = \varphi = 0 \) and the flow with \( \phi = 0, \varphi = \frac{\pi}{2} \). How one can realize this explicitly?

There are further developments [53] which generalizes the work of [17], using the Killing spinors. So it would be interesting to study [53] in the present context. Moreover, further
generalization of [18] arises in the work of [54]. It would be interesting to find out how the gauge dual appears. The regularity of the solutions was proved in [55] and it would be interesting to find out the implication between the flux configuration and gauge theory configuration.

Besides the two supersymmetric critical points $\mathcal{N} = 2 \ SU(3) \times U(1)$ invariant point, $\mathcal{N} = 1 \ G_2$ invariant point of four-dimensional $\mathcal{N} = 8$ gauged supergravity, there exist also three nontrivial nonsupersymmetric critical points as well as the trivial $\mathcal{N} = 8 \ SO(8)$ critical point for the scalar potential: $SO(7)^+, SO(7)^-$ and $SU(4)^-$. It would be interesting to discover any flow equations connecting any two (non)supersymmetric critical points.

Recall that the set of seventy scalars in $\mathcal{N} = 8$ gauged supergravity has six singlets of $SU(3)$. Three singlets from $35_c$ and three singlets from $35_c$. Under the $SO(3) = SU(2)$ subgroup of $SU(3)$, the irreducible representation $6$ of $SU(3)$ breaks into $5$ plus $1$. Totally the $SO(3)$-singlet space has four more fields and may be parametrized by ten fields. It would be interesting to find out any new $AdS_4$ critical points, if any, in the $SO(3)$-invariant sector of $\mathcal{N} = 8$ gauged supergravity in four-dimensions and see how the gauge duals appear.

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Appendix A \ The $28 \times 28$ matrices $u$ and $v$, $A_2$ tensor, kinetic terms and $SU(8)$ connection with $[SU(2) \times U(1)]^2$ symmetry in $SU(8)$ basis

The 28-beins $u^{IJ}_{KL}$ and $v^{IJKL}$ fields, which are elements of $56 \times 56 \ V(x)$ of the fundamental 56-dimensional representation of $E_7(7)$ through (2.3), can be obtained by exponentiating the vacuum expectation values $\phi_{ijkl}$ (2.1) via (2.2). These 28-beins have the following seven $4 \times 4$ block diagonal matrices $u_i$ and $v_i$ where $i = 1, 2, \cdots, 7$ respectively:

$$u^{IJ}_{KL} = \text{diag}(u_1, u_2, u_3, u_4, u_5, u_6, u_7),$$
$$v^{IJKL} = \text{diag}(v_1, v_2, v_3, v_4, v_5, v_6, v_7).$$

Each hermitian (for example, $(u_1)^{78}_{12} = ((u_1)^{12}_{78})^* = \frac{1}{2} e^{i(\phi + \phi)} E$) submatrix is $4 \times 4$ matrix and we denote antisymmetric index pairs $[IJ]$ and $[KL]$ explicitly for convenience. For simplicity,
we make an empty space corresponding to lower triangle elements that can be read off from
the corresponding upper triangle elements by hermiticity. Then the $4 \times 4$ submatrix leads to

\[
\begin{pmatrix}
12 & 34 & 56 & 78 \\
12 & A & B & \frac{1}{2} e^{-i(\phi - \varphi)} E \\
34 & A & \frac{1}{2} e^{-i(\phi - \varphi)} E & \frac{1}{2} e^{-i(\phi + \varphi)} E \\
56 & \frac{1}{2} e^{-i(\phi - \varphi)} E & A & e^{-2i\varphi} B \\
78 & \frac{1}{2} e^{-i(\phi + \varphi)} E & e^{-2i\varphi} B & A
\end{pmatrix},
\]

\[
u_1 = \begin{pmatrix}
12 & 34 & 56 & 78 \\
12 & e^{-i\phi} D & e^{-i\phi} C & \frac{1}{2} e^{-i\phi} F \\
34 & e^{-i\phi} D & \frac{1}{2} e^{-i\phi} F & \frac{1}{2} e^{i\phi} F \\
56 & \frac{1}{2} e^{-i\phi} F & e^{i(\phi - 2\varphi)} D & e^{i\phi} C \\
78 & \frac{1}{2} e^{i\phi} F & e^{i(\phi - 2\varphi)} D & e^{i\phi} C
\end{pmatrix},
\]

\[
u_2 = \begin{pmatrix}
13 & 24 & 57 & 68 \\
13 & -e^{-i\phi} H & 0 & 0 \\
24 & -e^{-i\phi} H & 0 & 0 \\
57 & -e^{i\phi} H & 0 & 0 \\
68 & -e^{i\phi} H & 0 & 0
\end{pmatrix} = -\nu_3,
\]

\[
u_4 = \begin{pmatrix}
15 & 26 & 37 & 48 \\
15 & -e^{-i\varphi} J & 0 & 0 \\
26 & -e^{-i\varphi} J & 0 & 0 \\
37 & -e^{i\varphi} J & 0 & 0 \\
48 & -e^{i\varphi} J & 0 & 0
\end{pmatrix} = -\nu_5,
\]

\[
u_6 = \begin{pmatrix}
17 & 28 & 35 & 46 \\
17 & -e^{i\varphi} J & 0 & 0 \\
28 & -e^{i\varphi} J & 0 & 0 \\
35 & -e^{-i\varphi} J & 0 & 0 \\
46 & -e^{-i\varphi} J & 0 & 0
\end{pmatrix} = -\nu_7,
\]

where we introduce some quantities that are functions of $\alpha$ and $\chi$ as follows:

\[
A \equiv \cosh \alpha \cosh^2 \chi, \quad B \equiv \cosh \alpha \sinh^2 \chi, \quad C \equiv \sinh \alpha \cosh^2 \chi,
\]

\[
D \equiv \sinh \alpha \sinh^2 \chi, \quad E \equiv \sinh \alpha \sinh(2\chi), \quad F \equiv \cosh \alpha \sinh(2\chi),
\]

\[
G \equiv \cosh \alpha, \quad H \equiv \sinh \alpha, \quad I \equiv \cosh \chi, \quad J \equiv \sinh \chi.
\]

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The lower triangle part can be read off from the upper triangle part by hermitian property. Also, the other kinds of 28-beins \( u_{IJ}^{KL} \) and \( v_{IJKL} \) are obtained by taking a complex conjugation of (A.1). The complex conjugation operation can be done by raising or lowering the indices: \((u_{IJ}^{KL})^* = u_{IJ}^{KL}^*\) and so on.

The components of \( A_2 \) tensor, \( A_{2L}^{IK} \), are obtained from the defining equation

\[
T_l^{kij} = \left(u^{ij}_{lJ} + v^{ij}_{IJK}\right) \left(u_{lm}^{JK} u^{km}_{KL} - v_{lmJK} v^{kmKL}\right),
\]

and the well-known tensors of the \( \mathcal{N} = 8 \) gauged supergravity are

\[
A_1^{ij} = -\frac{4}{21} T_{m}^{ijm}, \quad A_2^{ijk} = -\frac{4}{3} T_{l}^{[ijk]},
\]

by substituting (A.1) into (A.2) and (A.3) and they are classified by seven different fields with degeneracies 4, 4, 2, 2, 4, 4, 4 respectively and given by:

\[
\begin{align*}
A_{2,1}^{278} &= A_{2,2}^{187} = A_{2,3}^{478} = A_{2,4}^{387} \equiv y_1 = -\frac{1}{2} e^{i\varphi} \frac{\partial z_1^*}{\partial \chi}, \\
A_{2,2}^{234} &= A_{2,3}^{143} = A_{2,4}^{124} \equiv y_2 = -e^{-i\phi} \frac{\partial z_2^*}{\partial \alpha}, \\
A_{2,6}^{568} &= A_{2,7}^{576} \equiv y_3 = -e^{i\phi} \frac{\partial z_3^*}{\partial \alpha}, \\
A_{2,5}^{678} &= A_{2,6}^{587} \equiv y_4 = -e^{i\phi} \frac{\partial z_2^*}{\partial \alpha}, \\
A_{2,7}^{128} &= A_{2,8}^{348} = A_{2,172} = A_{2,8}^{374} \equiv y_5 = -\frac{1}{2} e^{i\varphi} \frac{\partial z_3^*}{\partial \chi}, \\
A_{2,1}^{256} &= A_{2,2}^{165} = A_{2,3}^{456} = A_{2,4}^{365} \equiv y_6 = -\frac{1}{2} e^{-i\varphi} \frac{\partial z_2^*}{\partial \chi}, \\
A_{2,5}^{126} &= A_{2,6}^{346} = A_{2,2}^{152} = A_{2,6}^{354} \equiv y_7 = -\frac{1}{2} e^{-i\varphi} \frac{\partial z_2^*}{\partial \chi},
\end{align*}
\]

where the redefined functions are given by

\[
\begin{align*}
y_1 &= -e^{i\varphi} \cosh \chi \left( \cosh \alpha + e^{i\phi} \sinh \alpha \right) \sinh \chi, \\
y_2 &= -e^{-i\phi} \cosh^2 \chi \sinh \alpha - \cosh \alpha \sinh^2 \chi, \\
y_3 &= -e^{i\phi} \cosh^2 \chi \sinh \alpha - e^{-2i\varphi} \cosh \alpha \sinh^2 \chi, \\
y_4 &= -e^{i\phi} \cosh^2 \chi \sinh \alpha - e^{2i\varphi} \cosh \alpha \sinh^2 \chi, \\
y_5 &= -\frac{1}{2} e^{-i(\phi+\varphi)} \left( e^{i(\phi+2\varphi)} \cosh \alpha + \sinh \alpha \right) \sinh (2\chi), \\
y_6 &= -e^{-i\varphi} \cosh \chi \left( \cosh \alpha + e^{i\phi} \sinh \alpha \right) \sinh \chi, \\
y_7 &= -\frac{1}{2} e^{-i(\phi+\varphi)} \left( e^{i\phi} \cosh \alpha + e^{2i\varphi} \sinh \alpha \right) \sinh (2\chi).
\end{align*}
\]
We write these in terms of the derivatives of the eigenvalues of $A_1$ tensor (2.4) with respect to the $\alpha$ and $\chi$. It is manifest that $A_{2,IJK} = -A_{2,IKJ}$, by definition. Moreover there exists a symmetry between the upper indices: $A_{2,IJK} = A_{2,JKI} = A_{2,KIJ}$. Recall that in the supersymmetric transformation rules, the $A_{2,IJK}$ appears in the equation satisfied by spin-$\frac{1}{2}$ field (2.11). This is the reason why we rewrite it in terms of superpotential.

The kinetic terms coming from the definition

$$A_{ikl} = 2\sqrt{2} \left( u_{ij}^{il} \partial_{\mu} v_{kl}^{ij} - v_{ij}^{il} \partial_{\mu} u_{kl}^{ij} \right),$$

can be summarized as following seven $4 \times 4$ block diagonal hermitian matrices like as 28-beins $u^{ij}_{KL}$ and $v^{ij}_{KIJ}$ as above:

$$A_{iJKL}^{ijkl} = \text{diag} \left( A_{\mu,1}, A_{\mu,2}, A_{\mu,3}, A_{\mu,4}, A_{\mu,5}, A_{\mu,6}, A_{\mu,7} \right),$$

where each hermitian submatrix can be written as follows:

$$A_{\mu,1} = \begin{pmatrix}
[12] & 0 & -a^* & -b^* & -b
[34] & 0 & -b^* & -b
[56] & 0 & -a
[78] & 0
\end{pmatrix},$$

$$A_{\mu,2} = \begin{pmatrix}
[13] & 0 & a^* & 0 & 0
[24] & 0 & 0 & 0
[57] & 0 & a
[68] & 0
\end{pmatrix} = -A_{\mu,3},$$

$$A_{\mu,4} = \begin{pmatrix}
[15] & 0 & b^* & 0 & 0
[26] & 0 & 0 & 0
[37] & 0 & b
[48] & 0
\end{pmatrix} = -A_{\mu,5} = (A_{\mu,6})^* = (-A_{\mu,7})^*, \quad (A.6) \{?\}$$

where we introduce

$$a \equiv \sqrt{2} e^{i\phi} \left( \partial_{\mu} \alpha + \frac{i}{2} \sinh (2\alpha) \partial_{\mu} \phi \right), \quad b \equiv \sqrt{2} e^{i\varphi} \left( \partial_{\mu} \chi + \frac{i}{2} \sinh (2\chi) \partial_{\mu} \varphi \right).$$

The lower triangle part of (A.6) can be read off from the upper triangle part by hermitian property. Then the final expression of kinetic terms by counting the correct multiplicities is given by

$$\frac{1}{96} |A_{ijkl}^{ijkl}|^2 = (\partial_{\mu} \alpha)^2 + \frac{1}{4} \sinh^2 (2\alpha) (\partial_{\mu} \phi)^2 + 2 \left[ (\partial_{\mu} \chi)^2 + \frac{1}{4} \sinh^2 (2\chi) (\partial_{\mu} \varphi)^2 \right].$$

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In particular, when \( \phi = 0 \) and \( \varphi = \frac{x}{2} \), this becomes \( (\partial_\mu \alpha)^2 + 2 (\partial_\mu \chi)^2 \) as in \([2,8]\).

The quantity in the covariant derivative \([2,12]\) was defined as

\[
B_\mu^{i\ j} \equiv \frac{2}{3} (u^i_{\ kJ} \partial_\mu u^J_{\ jk} - v^i_{\ kJ} \partial_\mu v^J_{\ jk}) ,
\]

and it turns out, by substituting the 28-beins \([A.1]\) into \([A.7]\), that it is diagonal and is given by

\[
B_\mu^{i\ j} = \begin{cases} 
\text{diag}(-i\partial_\mu \phi \sinh^2 \alpha, -i\partial_\mu \phi \sinh^2 \alpha, -i\partial_\mu \phi \sinh^2 \alpha, -i\partial_\mu \phi \sinh^2 \alpha, \\
i (\partial_\mu \phi \sinh^2 \alpha - 2\partial_\mu \varphi \sinh^2 \chi), i (\partial_\mu \phi \sinh^2 \alpha - 2\partial_\mu \varphi \sinh^2 \chi), \\
i (\partial_\mu \phi \sinh^2 \alpha + 2\partial_\mu \varphi \sinh^2 \chi), i (\partial_\mu \phi \sinh^2 \alpha + 2\partial_\mu \varphi \sinh^2 \chi)) .
\end{cases}
\]

Appendix B  The 28 \times 28 matrices \( U \) and \( V \) and \( A_1 \) tensor with \([SU(2) \times U(1)]^2\) symmetry in \( SL(8,\mathbb{R})\) basis

The 28-beins \( A_{IJKL}, B_{IJ}^{KL}, C_{IJ}^{KL}, D_{IJ}^{KL}\) fields \([4,5]\), that can be obtained from the \( SU(8)\) basis to the \( SL(8,\mathbb{R})\) basis using gamma matrices, have the following four kinds of seven \( 4 \times 4 \) block diagonal matrices \( a_i, b_i, c_i \) and \( d_i \) where \( i = 1, 2, \ldots, 7 \) respectively:

\[
A_{IJKL} = \text{diag}(a_1, a_2, a_3, a_4, a_5, a_6, a_7), \quad B_{IJ}^{KL} = \text{diag}(b_1, b_2, b_3, b_4, b_5, b_6, b_7),
\]
\[
C_{IJ}^{KL} = \text{diag}(c_1, c_2, c_3, c_4, c_5, c_6, c_7), \quad D_{IJ}^{KL} = \text{diag}(d_1, d_2, d_3, d_4, d_5, d_6, d_7) .
\]

It turns out that the 28-bein \( A_{IJKL} \) with \([B.1]\) are given by

\[
a_2 = \begin{pmatrix} 
13 & 24 & 57 & 68 \\
13 & -a & a & -a \\
24 & b & -b & b \\
57 & -a & -a & -a \\
68 & b^* & b^* & b^* 
\end{pmatrix} , \quad a_4 = \begin{pmatrix} 
15 & 26 & 37 & 48 \\
15 & -c & -d & d & c \\
26 & c & -d & d & -c \\
37 & -c^* & d^* & d^* & -c^* \\
48 & c^* & d^* & d^* & c^* 
\end{pmatrix} ,
\]

\[
a_6 = \begin{pmatrix} 
17 & 28 & 35 & 46 \\
17 & -f & -f & -f \\
28 & g & g & g \\
35 & -f & f & -f \\
46 & -g & g & -g 
\end{pmatrix} , \quad a_1 = 0 = a_3 = a_5 = a_7 .
\]

where we introduce

\[
a \equiv \sqrt{2} \cosh \alpha, \quad b \equiv \sqrt{2} e^{i\phi} \sinh \alpha, \quad c \equiv \sqrt{2} \left(\cosh \chi + e^{i\varphi} \sinh \chi\right),
\]
\[
d \equiv \sqrt{2} \left(\cosh \chi - e^{i\varphi} \sinh \chi\right), \quad f \equiv \sqrt{2} \cosh \chi, \quad g \equiv \sqrt{2} e^{i\varphi} \sinh \chi .
\]
Compared with the previous 28-beins, there is no hermiticity property in $SL(8, \mathbb{R})$ basis. Similarly, one gets the 28-beins $B_{IJK}^{KL}$ with \((\text{B.1})\) and \((\text{B.3})\)

$$b_2 = \begin{pmatrix} [13] & [24] & [57] & [68] \\ [13] & a & -a & a \\ [24] & b & -b & a \\ [57] & -a & -a & -a \\ [68] & -b^* & -b^* & -b^* \end{pmatrix}, \quad b_4 = \begin{pmatrix} [15] & [26] & [37] & [48] \\ [15] & -d & -c & c \\ [26] & d & -c & c \\ [37] & -d^* & c^* & c^* \\ [48] & d^* & c^* & c^* \end{pmatrix},$$

$$b_6 = \begin{pmatrix} [17] & [28] & [35] & [46] \\ [17] & -f & -f & -f \\ [28] & -g^* & -g^* & -g^* \\ [35] & f & -f & f \\ [46] & g & -g & g \end{pmatrix}, \quad b_1 = 0 = b_3 = b_5 = b_7. \quad (\text{B.4})$$

Moreover, one has the 28-beins $C_{IJK}^{KL}$ with \((\text{B.1})\) and \((\text{B.3})\)

$$c_2 = \begin{pmatrix} [13] & [24] & [57] & [68] \\ [13] & a & -a & a \\ [24] & -b^* & b^* & -b^* \\ [57] & -a & -a & -a \\ [68] & b & b & b \end{pmatrix}, \quad c_4 = \begin{pmatrix} [15] & [26] & [37] & [48] \\ [15] & -c^* & -d^* & d^* \\ [26] & c^* & -d^* & d^* \\ [37] & -c & d & d \\ [48] & c & d & d \end{pmatrix},$$

$$c_6 = \begin{pmatrix} [17] & [28] & [35] & [46] \\ [17] & -f & -f & -f \\ [28] & g & g & g \\ [35] & f & -f & f \\ [46] & -g^* & g^* & g^* \end{pmatrix}, \quad c_1 = 0 = c_3 = c_5 = c_7. \quad (\text{B.5})$$

Finally, we have the 28-beins $D_{IJK}^{KL}$ with \((\text{B.1})\) and \((\text{B.3})\)

$$d_2 = \begin{pmatrix} [13] & [24] & [57] & [68] \\ [13] & -a & a & a \\ [24] & -b^* & b^* & -b^* \\ [57] & a & a & a \\ [68] & b & b & b \end{pmatrix}, \quad d_4 = \begin{pmatrix} [15] & [26] & [37] & [48] \\ [15] & d^* & c^* & -c^* \\ [26] & -d^* & c^* & -c^* \\ [37] & d & -c & -c \\ [48] & -d & -c & -c \end{pmatrix},$$

$$d_6 = \begin{pmatrix} [17] & [28] & [35] & [46] \\ [17] & f & f & f \\ [28] & g & g & g \\ [35] & f & -f & f \\ [46] & -g^* & g^* & g^* \end{pmatrix}, \quad d_1 = 0 = d_3 = d_5 = d_7. \quad (\text{B.6})$$

From the definition of \((\ref{4.6})\), one obtains the components of $\tilde{A}_1$ tensor by substituting...
\( \tilde{A}_{11} \) provides the geometric superpotential (B.7) when we restrict to the case \( \phi = 0 \) and \( \varphi = \frac{\pi}{2} \) and moreover there are

\[
\tilde{A}_{155} = \tilde{A}_{166} = 2e^{-i\varphi}(\sinh(2\chi)(Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2) + \sinh(2\chi)(-Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2) + \cosh(2\chi)(Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2))\]

\[
\tilde{A}_{177} = -2e^{-i\varphi}(\cosh(2\chi)Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2) + 2\sinh(2\chi)(Y_1^2 + Y_2^2 - Y_3^2 - Y_4^2) + \cosh(2\chi)(Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2))\]

\[
\tilde{A}_{112} = \tilde{A}_{134} = \tilde{A}_{156} = \tilde{A}_{178} = 0,
\]

\[
\tilde{A}_{113} = \tilde{A}_{124} = 2e^{-i(\phi+\varphi)}(-1 + e^{2i\varphi})(\sinh(2\chi)(Y_1Y_3 + Y_2Y_4),
\]

\[
\tilde{A}_{114} = \tilde{A}_{133} = 2e^{-i(\phi+\varphi)}(-1 + e^{2i\varphi})(\cosh(2\chi)(Y_5Y_7 + Y_6Y_8),
\]

\[
\tilde{A}_{158} = \tilde{A}_{167} = 2e^{-i\varphi}(-1 + e^{2i\varphi})(\cosh(2\chi)(Y_5Y_8 + Y_6Y_7),
\]

\[
\tilde{A}_{157} = -\tilde{A}_{168} = 2e^{-i\varphi}(-1 + e^{2i\varphi})(\cosh(2\chi)(Y_5Y_7 - Y_6Y_8),
\]

The off-diagonal components are given by

\[
\tilde{A}_{111}^{(B.2), (B.3), (B.4), (B.5) and (B.6)} and the diagonal components are
\]

\[
\tilde{A}_{11}^{(B.2)} = 2e^{-i\varphi}(\sinh(2\chi)(Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2) + \sinh(2\chi)(-i\sin(2\chi)(Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2)) + \cosh(2\chi)(Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2))
\]

\[
\tilde{A}_{11}^{(B.3)} = -2e^{-i\varphi}(\cosh(2\chi)Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2) + 2\sinh(2\chi)(Y_1^2 + Y_2^2 - Y_3^2 - Y_4^2) + \cosh(2\chi)(Y_5^2 + Y_6^2 - Y_7^2 - Y_8^2))
\]

\[
\tilde{A}_{11}^{(B.4)} = 2e^{-i\varphi}(-1 + e^{2i\varphi})(\sinh(2\chi)(Y_1Y_3 + Y_2Y_4),
\]

\[
\tilde{A}_{11}^{(B.5)} = 2e^{-i\varphi}(-1 + e^{2i\varphi})(\cosh(2\chi)(Y_5Y_7 + Y_6Y_8),
\]

\[
\tilde{A}_{11}^{(B.6)} = 2e^{-i\varphi}(-1 + e^{2i\varphi})(\cosh(2\chi)(Y_5Y_8 + Y_6Y_7),
\]

\[
\tilde{A}_{11}^{(B.7)} = 2e^{-i\varphi}(-1 + e^{2i\varphi})(\cosh(2\chi)(Y_5Y_7 - Y_6Y_8),
\]
and moreover we have

\[ \tilde{A}_1^{15} = 2i(cosh \chi - e^{-i\phi} sinh \chi)(\sin \phi sinh(2\alpha) + \sin \varphi sinh(2\chi))(Y_1Y_5 + Y_2Y_6) \]
\[ -2i(cosh \chi + e^{-i\phi} sinh \chi)(\sin \phi sinh(2\alpha) - \sin \varphi sinh(2\chi))(Y_3Y_7 + Y_4Y_8), \]

\[ \tilde{A}_1^{26} = 2i(cosh \chi - e^{-i\phi} sinh \chi)(\sin \phi sinh(2\alpha) + \sin \varphi sinh(2\chi))(Y_1Y_5 + Y_2Y_6) \]
\[ +2i(cosh \chi + e^{-i\phi} sinh \chi)(\sin \phi sinh(2\alpha) - \sin \varphi sinh(2\chi))(Y_3Y_7 + Y_4Y_8), \]

\[ \tilde{A}_1^{37} = -2i(cosh \chi - e^{i\phi} sinh \chi)(\sin \phi sinh(2\alpha) - \sin \varphi sinh(2\chi))(Y_1Y_5 - Y_2Y_6) \]
\[ +2i(cosh \chi + e^{i\phi} sinh \chi)(\sin \phi sinh(2\alpha) + \sin \varphi sinh(2\chi))(Y_3Y_7 - Y_4Y_8), \]

\[ \tilde{A}_1^{48} = -2i(cosh \chi - e^{i\phi} sinh \chi)(\sin \phi sinh(2\alpha) - \sin \varphi sinh(2\chi))(Y_1Y_5 - Y_2Y_6) \]
\[ -2i(cosh \chi + e^{i\phi} sinh \chi)(\sin \phi sinh(2\alpha) + \sin \varphi sinh(2\chi))(Y_3Y_7 - Y_4Y_8), \]  (B.10)
\[
\tilde{A}_1^{16} = 2i(\cosh \chi - e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_1Y_6 - Y_2Y_5)
\]
\[
-2i(\cosh \chi + e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_3Y_8 - Y_4Y_7),
\]
\[
\tilde{A}_1^{25} = -2i(\cosh \chi - e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_1Y_6 - Y_2Y_5)
\]
\[
-2i(\cosh \chi + e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_3Y_8 - Y_4Y_7),
\]
\[
\tilde{A}_1^{38} = -2i(\cosh \chi - e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_1Y_6 + Y_2Y_5)
\]
\[
+2i(\cosh \chi + e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_3Y_8 + Y_4Y_7),
\]
\[
\tilde{A}_1^{47} = 2i(\cosh \chi - e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_1Y_6 + Y_2Y_5)
\]
\[
+2i(\cosh \chi + e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_3Y_8 + Y_4Y_7),
\]
\[
\tilde{A}_1^{17} = 2i(\cosh \chi + e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_1Y_7 - Y_2Y_8)
\]
\[
+2i(\cosh \chi - e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_3Y_5 - Y_4Y_6),
\]
\[
\tilde{A}_1^{28} = -2i(\cosh \chi + e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_1Y_7 - Y_2Y_8)
\]
\[
+2i(\cosh \chi - e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_3Y_5 - Y_4Y_6),
\]
\[
\tilde{A}_1^{35} = 2i(\cosh \chi + e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_1Y_7 + Y_2Y_8)
\]
\[
+2i(\cosh \chi - e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_3Y_5 + Y_4Y_6),
\]
\[
\tilde{A}_1^{46} = -2i(\cosh \chi + e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_1Y_7 + Y_2Y_8)
\]
\[
+2i(\cosh \chi - e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_3Y_5 + Y_4Y_6),
\]
\[
\tilde{A}_1^{18} = 2i(\cosh \chi + e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_1Y_8 + Y_2Y_7)
\]
\[
+2i(\cosh \chi - e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_3Y_6 + Y_4Y_5),
\]
\[
\tilde{A}_1^{27} = 2i(\cosh \chi + e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_1Y_8 + Y_2Y_7)
\]
\[
-2i(\cosh \chi - e^{i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_3Y_6 + Y_4Y_5),
\]
\[
\tilde{A}_1^{36} = 2i(\cosh \chi + e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_1Y_8 - Y_2Y_7)
\]
\[
+2i(\cosh \chi - e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_3Y_6 - Y_4Y_5),
\]
\[
\tilde{A}_1^{45} = 2i(\cosh \chi + e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) - \sin \varphi \sinh(2\chi))(Y_1Y_8 - Y_2Y_7)
\]
\[
-2i(\cosh \chi - e^{-i\varphi} \sinh \chi)(\sin \phi \sinh(2\alpha) + \sin \varphi \sinh(2\chi))(Y_3Y_6 - Y_4Y_5). (B.11) \{?\}
\]

Note that using the symmetric property \(\tilde{A}_1^{ij} = \tilde{A}_1^{ji}\), one can obtain the lower triangular parts of \(\tilde{A}_1\) tensor.
Appendix C  The 28 × 28 matrices $u$ and $v$, $A_2$ tensor, kinetic terms, and $SU(8)$ connection with $SO(4) \times SO(4)$ symmetry in $SU(8)$ basis

The 28-beins $u^{IJ}_{KL}$ and $v^{IJKL}$ fields can be obtained by exponentiating the vacuum expectation values $\phi_{ijkl}$ (2.1) via (2.2). The 28-beins have the following seven $4 \times 4$ block diagonal matrices $u_i$ and $v_i$ where $i = 1, 2, \cdots, 7$ respectively as before. Each hermitian submatrix is 4 × 4 matrix and we denote antisymmetric index pairs $[IJ]$ and $[KL]$ explicitly:

$$
\begin{align*}
  u_1 &= \cosh \alpha \mathbf{1}_{4 \times 4} = u_2 = u_3, \\
  u_4 &= u_5 = u_6 = u_7 = \mathbf{1}_{4 \times 4}, \\
  v_1 &= \begin{pmatrix}
    [12] & [34] & [56] & [78] \\
    0 & e^{-i \zeta} \sinh \alpha & 0 & 0 \\
    [34] & 0 & 0 & 0 \\
    [56] & 0 & e^{i \zeta} \sinh \alpha & 0 \\
    [78] 
  \end{pmatrix} = -v_2 = v_3, \\
  v_4 &= v_5 = v_6 = v_7 = 0.
\end{align*}
$$

(C.1)

See also [52] for previous construction of these 28-beins from the $\mathcal{N} = 4$ gauged $SO(4) = SU(2) \times SU(2)$ truncation of the full $\mathcal{N} = 8$ gauged $SO(8)$ theory. The components of $A_2$ tensor, $A_{2,L}^{IJK}$, are obtained similarly and they are classified by two different fields with degeneracies 4, 4 respectively and given by:

$$
A_{2,1}^{34} = A_{2,2}^{143} = A_{2,3}^{214} = A_{2,4}^{32} = -e^{-i \zeta} \sinh \alpha = -e^{-i \zeta} \frac{\partial W}{\partial \alpha}, \\
A_{2,5}^{678} = A_{2,6}^{587} = A_{2,7}^{568} = A_{2,8}^{76} = -e^{i \zeta} \sinh \alpha = -e^{i \zeta} \frac{\partial W}{\partial \alpha}.
$$

(C.2)

We also rewrite them in terms of the derivative of the superpotential with respect to $\alpha$. The kinetic terms can be summarized as following seven $4 \times 4$ block diagonal hermitian matrices where each hermitian submatrix can be written as follows:

$$
A_{\mu,1} = \begin{pmatrix}
  [12] & [34] & [56] & [78] \\
  [12] & 0 & -a^* & 0 \\
  [34] & 0 & 0 & 0 \\
  [56] & 0 & -a & 0 \\
  [78] 
\end{pmatrix} = -A_{\mu,2} = A_{\mu,3}, \\
A_{\mu,4} = 0 = A_{\mu,5} = A_{\mu,6} = A_{\mu,7}, \\
a \equiv \sqrt{2} e^{i \zeta} \left[ \partial_\mu \alpha + \frac{i}{2} \sinh(2 \alpha) \partial_\mu \zeta \right].
$$

(C.3)

The quantity in the covariant derivative in (2.12) is given by

$$
B_{\mu}^i = \text{diag}(-i \partial_\mu \zeta \sinh^2 \alpha, -i \partial_\mu \zeta \sinh^2 \alpha, -i \partial_\mu \zeta \sinh^2 \alpha, -i \partial_\mu \zeta \sinh^2 \alpha, \\
i \partial_\mu \zeta \sinh^2 \alpha, i \partial_\mu \zeta \sinh^2 \alpha, i \partial_\mu \zeta \sinh^2 \alpha, i \partial_\mu \zeta \sinh^2 \alpha).
$$

(C.4)
Note that the above results (C.1), (C.2), (C.3), and (C.4) can be read off from the previous (A.1), (A.4), (A.6) and (A.8) in appendix A respectively by taking $\phi \rightarrow \zeta$ and $\chi \rightarrow 0$.

**Appendix D**  
*The $28 \times 28$ matrices $U$ and $V$ and $A_1$ tensor with $SO(4) \times SO(4)$ symmetry in $SL(8, \mathbb{R})$ basis*

Following the notation given in (B.1), one obtains the 28-beins $A_{ijkl}$

$$a_2 = \begin{pmatrix} 13 & 24 & 57 & 68 \\ 13 & -a & a & -a \\ 24 & -b & b & -b \\ 57 & -a & -a & -a \\ 68 & b^* & b^* & b^* \end{pmatrix}, \quad a_4 = \begin{pmatrix} 15 & 26 & 37 & 48 \\ 15 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 26 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 37 & -\sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 48 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix},$$

$$a_6 = \begin{pmatrix} 17 & 28 & 35 & 46 \\ 17 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 28 & 0 & 0 & 0 & 0 \\ 35 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 46 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad a_1 = 0 = a_3 = a_5 = a_7. \quad (D.1) \{?\}$$

where we introduce

$$a \equiv \sqrt{2} \cosh \alpha, \quad b \equiv \sqrt{2} e^{i\zeta} \sinh \alpha.$$

Also one has the 28-beins $B_{IJKL}^{KL}$

$$b_2 = \begin{pmatrix} 13 & 24 & 57 & 68 \\ 13 & -a & a & -a \\ 24 & -b & b & -b \\ 57 & -a & -a & -a \\ 68 & -b^* & -b^* & -b^* \end{pmatrix}, \quad b_4 = \begin{pmatrix} 15 & 26 & 37 & 48 \\ 15 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 26 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 37 & -\sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 48 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix},$$

$$b_6 = \begin{pmatrix} 17 & 28 & 35 & 46 \\ 17 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 28 & 0 & 0 & 0 & 0 \\ 35 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 46 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad b_1 = 0 = b_3 = b_5 = b_7. \quad (D.2) \{?\}$$

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Moreover one has the following results $C_{KL}^{IJ}$

$$c_2 = \begin{pmatrix} [13] & [24] & [57] & [68] \\ 13 & a & -a & a & -a \\ 24 & -b^* & b^* & -b^* & b^* \\ 57 & -a & b & -a & -a \\ 68 & b & b & -a & -a \end{pmatrix}, \quad \quad c_4 = \begin{pmatrix} [15] & [26] & [37] & [48] \\ 15 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 26 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 37 & -\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 48 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix},$$

$$c_6 = \begin{pmatrix} [17] & [28] & [35] & [46] \\ 17 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 28 & -\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 35 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 46 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad c_1 = 0 = c_3 = c_5 = c_7. \quad (D.3)$$

Finally, one gets the 28-beins $D_{KL}^{IJ}$

$$d_2 = \begin{pmatrix} [13] & [24] & [57] & [68] \\ 13 & -a & a & a & a \\ 24 & b^* & b^* & -b^* & b^* \\ 57 & a & b & b & b \\ 68 & a & a & a & a \end{pmatrix}, \quad d_4 = \begin{pmatrix} [15] & [26] & [37] & [48] \\ 15 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 26 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 37 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 48 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix},$$

$$d_6 = \begin{pmatrix} [17] & [28] & [35] & [46] \\ 17 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 28 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 35 & 0 & 0 & 0 & 0 \\ 46 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_1 = 0 = d_3 = d_5 = d_7. \quad (D.4)$$

The $\tilde{A}_1$ tensor has the following form:

$$\tilde{A}_1^{11} = \tilde{A}_1^{22} = \tilde{A}_1^{33} = \tilde{A}_1^{44} = (\tilde{A}_1^{55})^* = (\tilde{A}_1^{66})^* = (\tilde{A}_1^{77})^* = (\tilde{A}_1^{88})^* = \cosh \alpha + e^{-i\zeta} \sinh \alpha (-Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 + Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2),$$

$$\tilde{A}_1^{15} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_5 + Y_2 Y_6 - Y_3 Y_7 - Y_4 Y_8),$$

$$\tilde{A}_1^{26} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_5 + Y_2 Y_6 + Y_3 Y_7 + Y_4 Y_8),$$

$$\tilde{A}_1^{37} = i \sin \zeta \sinh (2\alpha) (-Y_1 Y_5 + Y_2 Y_6 + Y_3 Y_7 - Y_4 Y_8),$$

$$\tilde{A}_1^{48} = i \sin \zeta \sinh (2\alpha) (-Y_1 Y_5 + Y_2 Y_6 - Y_3 Y_7 - Y_4 Y_8),$$

$$\tilde{A}_1^{16} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_6 - Y_2 Y_5 - Y_3 Y_8 + Y_4 Y_7),$$

$$\tilde{A}_1^{25} = i \sin \zeta \sinh (2\alpha) (-Y_1 Y_6 + Y_2 Y_5 - Y_3 Y_8 + Y_4 Y_7),$$

$$\tilde{A}_1^{38} = i \sin \zeta \sinh (2\alpha) (-Y_1 Y_6 - Y_2 Y_5 + Y_3 Y_8 + Y_4 Y_7),$$
\[ \tilde{A}_{1}^{47} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_6 + Y_2 Y_5 + Y_3 Y_8 + Y_4 Y_7), \]
\[ \tilde{A}_{1}^{17} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_7 - Y_2 Y_8 + Y_3 Y_5 - Y_4 Y_6), \]
\[ \tilde{A}_{1}^{28} = i \sin \zeta \sinh (2\alpha) (-Y_1 Y_7 + Y_2 Y_8 + Y_3 Y_5 - Y_4 Y_6), \]
\[ \tilde{A}_{1}^{35} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_7 + Y_2 Y_8 + Y_3 Y_5 + Y_4 Y_6), \]
\[ \tilde{A}_{1}^{46} = i \sin \zeta \sinh (2\alpha) (-Y_1 Y_7 - Y_2 Y_8 + Y_3 Y_5 + Y_4 Y_6), \]
\[ \tilde{A}_{1}^{18} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_8 + Y_2 Y_7 + Y_3 Y_6 + Y_4 Y_5), \]
\[ \tilde{A}_{1}^{27} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_8 + Y_2 Y_7 - Y_3 Y_6 - Y_4 Y_5), \]
\[ \tilde{A}_{1}^{36} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_8 - Y_2 Y_7 + Y_3 Y_6 - Y_4 Y_5), \]
\[ \tilde{A}_{1}^{45} = i \sin \zeta \sinh (2\alpha) (Y_1 Y_8 - Y_2 Y_7 - Y_3 Y_6 + Y_4 Y_5), \]
\[ \tilde{A}_{1}^{12} = \tilde{A}_{1}^{13} = \tilde{A}_{1}^{14} = \tilde{A}_{1}^{23} = \tilde{A}_{1}^{24} = \tilde{A}_{1}^{34} = 0. \] (D.5)

There is the symmetric property \( \tilde{A}_{1}^{ij} = \tilde{A}_{1}^{ji} \). Note that the above results (D.1), (D.2), (D.3), (D.4) and (D.5) can be read off from the previous (B.2), (B.4), (B.5), (B.6), (B.7), (B.8), (B.9), (B.10) and (B.11) in appendix B by taking \( \phi \to \zeta \) and \( \chi \to 0 \).

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