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ABSTRACT. There is an error in the computation of the truncated point schemes $V_d$ of the degenerate Sklyanin algebra $S(1,1,1)$. We are grateful to S. Paul Smith for pointing out that $V_d$ is larger than was claimed in Proposition 3.13. All 2 or 3 digit references are to the above paper, while 1 digit references are to the results in this corrigendum. We provide a description of the correct $V_d$ in Proposition 4 below. Results about the corresponding point parameter ring $B$ associated to the schemes $\{V_d\}_{d \geq 1}$ are given afterward.

1. CORRECTIONS

The main error in the above paper is to the statement of Lemma 3.10. Before stating the correct version, we need some notation.

Notation. Given $\zeta = e^{2\pi i/3}$, let $p_a := [1 : 1 : 1]$, $p_b := [1 : \zeta : \zeta^2]$, and $p_c := [1 : \zeta^2 : \zeta]$. Also, let $\tilde{\mathbb{P}}_A^1 := \mathbb{P}_A^1 \setminus \{p_b, p_c\}$, $\tilde{\mathbb{P}}_B^1 := \mathbb{P}_B^1 \setminus \{p_a, p_c\}$, and $\tilde{\mathbb{P}}_C^1 := \mathbb{P}_C^1 \setminus \{p_a, p_b\}$.

We also require the following more precise version of Lemma 3.9; the original result is correct though there is a slight change in the proof as given below.

Lemma 1. (Correction of Lemma 3.9) Let $p = (p_0, \ldots, p_{d-2}) \in V_{d-1}$ with $p_{d-2} \in \tilde{\mathbb{P}}_A^1, \tilde{\mathbb{P}}_B^1, \text{ or } \tilde{\mathbb{P}}_C^1$. If $p' = (p, p_{d-1}) \in V_d$, then $p_{d-1} = p_a, p_b, \text{ or } p_c$ respectively.

Proof. The proof follows from that of Lemma 3.9, except that there is a typographical error in the case when $p_{d-2} = [0 : y_{d-2} : z_{d-2}]$. Here, we require that $(p_{d-2}, p_{d-1})$ satisfies the system of equations:

\[
\begin{align*}
    f_{d-2} &= g_{d-2} = h_{d-2} = 0, \\
    u_{d-2}^3 + z_{d-2}^3 &= 0, \\
    x_{d-1}^3 + y_{d-1}^3 + z_{d-1}^3 - 3x_{d-1}y_{d-1}z_{d-1} &= 0.
\end{align*}
\]

This implies that either $y_{d-2} = z_{d-2} = 0$ or $x_{d-1} = y_{d-1} = z_{d-1} = 0$, which produces a contradiction. \hfill $\square$

Now the correct version of Lemma 3.10 is provided below. The present version is slightly weaker than the original result, where it was claimed that $p_{d-1} \in \mathbb{P}_s^1$ instead of $p_{d-1} \in \mathbb{P}_s^1$. Here, $\mathbb{P}_s^1$ denotes either $\mathbb{P}_A^1, \mathbb{P}_B^1, \text{ or } \mathbb{P}_C^1$.

Lemma 2. (Correction of Lemma 3.10) Let $p = (p_0, \ldots, p_{d-2}) \in V_{d-1}$ with $p_{d-2} = p_a, p_b, \text{ or } p_c$. If $p' = (p, p_{d-1}) \in V_d$, then $p_{d-1} \in \mathbb{P}_A^1, \mathbb{P}_B^1, \text{ or } \mathbb{P}_C^1$ respectively.
\textbf{Proof.} The proof follows from that of Lemma 3.10 with the exception that there is a typographical error in the definition of the function $\theta$; it should be defined as:

$$\theta(y_{d-1}, z_{d-1}) = \begin{cases} -(y_{d-1} + z_{d-1}) & \text{if } p_{d-2} = p_a, \\ -(\zeta^2 y_{d-1} + \zeta z_{d-1}) & \text{if } p_{d-2} = p_b, \\ -(\zeta y_{d-1} + \zeta^2 z_{d-1}) & \text{if } p_{d-2} = p_c. \end{cases}$$

\hfill \Box

\textbf{Remark 3.} There are two further minor typographical corrections to the paper.

(1) (Correction of Figure 3.1) The definition of the projective lines $\mathbb{P}_B^1$ and $\mathbb{P}_C^1$ should be interchanged. More precisely, the curve $E_{111}$ is the union of three projective lines:

- $\mathbb{P}_A^1 : x + y + z = 0,$
- $\mathbb{P}_B^1 : x + \zeta^2 y + \zeta z = 0,$
- $\mathbb{P}_C^1 : x + \zeta y + \zeta^2 z = 0.$

(2) (Correction to Corollary 4.10) The numbers 57 and 63 should be replaced by 24 and 18 respectively.

2. Consequences

The main consequence of weakening Lemma 3.10 to Lemma 3 is that the truncated point schemes $\{V_d\}_{d \geq 1}$ of $S = S(1,1,1)$ are strictly larger than the truncated point schemes computed in Proposition 3.13 for $d \geq 4$. We discuss such results in §2.1 below. Furthermore, the corresponding point parameter ring associated to the correct point scheme data of $S$ is studied in §2.2.

\textbf{Notation.} (i) Let $W_d := \bigcup_{i=1}^6 W_{d,i}$ with $W_{d,i}$ defined in Proposition 3.13.
(ii) Let $B := \bigoplus_{d \geq 0} H^0(V_d, \mathcal{O}_V(1))$ be the point parameter ring of $S(1,1,1)$ as in Definition 1.8.
(iii) Likewise let $P := \bigoplus_{d \geq 0} H^0(W_d, \mathcal{O}_{W_d}(1))$ be the point parameter ring associated to the schemes $\{W_d\}_{d \geq 1}$.

The results of §4 of the paper are still correct; we describe the ring $P$, and we show that it is a factor of $S(1,1,1)$. Unfortunately, the ring $P$ is not equal to the point parameter ring $B$ of $S(1,1,1)$. More precisely, the following corrections should be made.

\textbf{Remark 4.} (1) The scheme $V_d$ should be replaced by $W_d$ in Theorem 1.7, in Proposition 3.13, in Remark 3.14, and in all §4 after Definition 4.1.
(2) The ring $B$ should be replaced by $P$ in §1 after Definition 1.8, and in all §4 with the exception of the second paragraph.

2.1. On the truncated point schemes $\{V_d\}_{d \geq 1}$. We provide a description of the truncated point schemes $\{V_d\}_{d \geq 1}$ as follows.

\textbf{Notation.} Let $\{V_{d,i}\}_{i \in I_d}$ denote the $|I_d|$ irreducible components of the $d^{\text{th}}$ truncated point scheme $V_d$.

\textbf{Proposition 5.} (Description of $V_d$) For $d \geq 2$, the length $d$ truncated point scheme $V_d$ is realized as the union of length $d$ paths of the quiver $Q$ below. With $d = 2$, for example, the path $\mathbb{P}_A^1 \rightarrow p_a$ corresponds to the component $\mathbb{P}_A^1 \times p_a$ of $V_2$. 

Proof. We proceed by induction. Considering the $d = 2$ case, Lemma 3.12 still holds so $V_2 = W_2$, the union of the irreducible components:

$$\mathbb{P}_A^1 \times p_a, \quad \mathbb{P}_B^1 \times p_b, \quad \mathbb{P}_C^1 \times p_c.$$ 

One can see these components correspond to length 2 paths of the quiver $Q$. Conversely, any length 2 path of $Q$ corresponds to a component that lies in $V_2$.

We assume the proposition holds for $V_{d-1}$, and recall that Lemmas 2 and 3 provide the recipe to build $V_d$ from $V_{d-1}$. Take a point $(p_0, \ldots, p_{d-2}) \in V_{d-1,i}$, where the irreducible component $V_{d-1,i}$ of $V_{d-1}$ corresponds to a length $d-1$ path of $Q$. Let $\{V_{d,ij}\}_{j \in J}$ be the set of $|J|$ irreducible components of $V_d$ with

$$(p_0, \ldots, p_{d-2}, p_{d-1}) \in V_{d,ij} \subseteq V_d$$

for some $p_{d-1} \in \mathbb{P}^2$. There are two cases to consider.

Case 1: We have that $(p_{d-3}, p_{d-2})$ lies in one of the following products:

$$\mathbb{P}_A^1 \times p_a, \quad \mathbb{P}_B^1 \times p_b, \quad \mathbb{P}_C^1 \times p_c,$$

For the first three choices, Lemma 2 implies that $pr_d(V_{d,ij}) = \mathbb{P}_A^1, \mathbb{P}_B^1, \text{ or } \mathbb{P}_C^1$, respectively. For the second three choices, $p_{d-2}$ belongs to $\mathbb{P}_A^1, \mathbb{P}_B^1, \text{ or } \mathbb{P}_C^1$, and Lemma 1 implies that $pr_d(V_{d,ij}) = p_a, p_b, \text{ or } p_c$, respectively. We conclude by induction that the component $V_{d,ij}$ yields a length $d$ path of $Q$.

Case 2: We have that $(p_{d-3}, p_{d-2})$ is equal to one of the following points:

$$p_a \times p_b, \quad p_a \times p_c, \quad p_b \times p_a, \quad p_b \times p_c, \quad p_c \times p_a, \quad p_c \times p_b.$$ 

Now Lemma 2 implies that:

$$pr_d(V_{d,ij}) = \begin{cases} 
\mathbb{P}_A^1 & \text{if } p_{d-2} = p_a, \\
\mathbb{P}_B^1 & \text{if } p_{d-2} = p_b, \\
\mathbb{P}_C^1 & \text{if } p_{d-2} = p_c.
\end{cases}$$

Again we have that in this case, the component $V_{d,ij}$ yields a length $d$ path of $Q$. 
Conversely (in either case), let $P$ be a length $d$ path of $Q$. Then, by induction, the embedded length $d - 1$ path $P'$ ending at the $d - 1$st vertex $v'$ of $P$ yields a component $X'$ of $V_{d-1}$. Say $v$ is the $d$th vertex of $P$. If $v'$ is equal to $\mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$, then $v$ must be $p_a$, $p_b$, or $p_c$ by the definition of $Q$, respectively. Lemma 2 then ensures that $P$ yields a component $X$ of $V_d$ so that $pr_{1\ldots d-1}(X) = X'$. On the other hand, if $v'$ is equal to $p_a$, $p_b$, or $p_c$, then $v$ lies in $\mathbb{P}_{1\ldots d}^1$, $\mathbb{P}^1_A$, or $\mathbb{P}^1_C$, respectively. Likewise, Lemma 3 implies that $P$ yields a component $X$ of $V_d$ so that $pr_{1\ldots d-1}(X) = X'$. □

**Corollary 6.** We have that $V_d = W_d$ for $d = 1, 2, 3$, and that $V_d \supseteq W_d$ for $d \geq 4$.

**Proof.** First, $V_1 = \mathbb{P}^2 = W_1$. Next, as mentioned in the proof of Proposition 5, $V_2 = W_2$ is the union of the irreducible components:

$$\mathbb{P}^1_A \times p_a, \quad \mathbb{P}^1_B \times p_b, \quad \mathbb{P}^1_C \times p_c$$

$$p_a \times \mathbb{P}^1_A, \quad p_b \times \mathbb{P}^1_B, \quad p_c \times \mathbb{P}^1_C.$$

By Proposition 5, we have that $V_3 = X_{3,1} \cup X_{3,2}$ where $X_{3,1}$ consists of the irreducible components:

$$\mathbb{P}^1_A \times p_a \times \mathbb{P}^1_A, \quad \mathbb{P}^1_B \times p_b \times \mathbb{P}^1_B, \quad \mathbb{P}^1_C \times p_c \times \mathbb{P}^1_C,$$

$$p_a \times \mathbb{P}^1_A \times p_a, \quad p_b \times \mathbb{P}^1_B \times p_b, \quad p_c \times \mathbb{P}^1_C \times p_c,$$

and $X_{3,2}$ is the union of:

$$\mathbb{P}^1_A \times p_a \times p_b, \quad \mathbb{P}^1_A \times p_a \times p_c, \quad p_a \times p_b \times \mathbb{P}^1_B, \quad p_a \times p_c \times \mathbb{P}^1_C,$$

$$p_a \times p_b \times p_a, \quad p_a \times p_b \times p_c, \quad p_a \times p_c \times p_a, \quad p_a \times p_c \times p_b,$$

$$\mathbb{P}^1_B \times p_b \times p_c, \quad \mathbb{P}^1_B \times p_b \times p_a, \quad p_b \times p_c \times \mathbb{P}^1_B, \quad p_b \times p_a \times \mathbb{P}^1_A,$$

$$p_b \times p_c \times p_b, \quad p_b \times p_c \times p_a, \quad p_b \times p_a \times p_b, \quad p_b \times p_a \times p_c,$$

$$\mathbb{P}^1_C \times p_c \times p_a, \quad \mathbb{P}^1_C \times p_c \times p_b, \quad p_c \times p_a \times \mathbb{P}^1_A, \quad p_c \times p_b \times \mathbb{P}^1_B,$$

$$p_c \times p_a \times p_c, \quad p_c \times p_a \times p_b, \quad p_c \times p_b \times p_c, \quad p_c \times p_b \times p_a.$$

Note that $X_{3,2}$ is contained in $X_{3,1}$; hence $V_3 = X_{3,1} = W_3$. Furthermore, one sees that $W_d \subseteq V_d$ for $d \geq 4$ as follows. The components of $W_d$ are read off the subquiver $Q'$ of $Q$ below.

$$\begin{array}{c}
\mathbb{P}^1_A \\
p_a
\end{array}$$

$$\begin{array}{c}
\mathbb{P}^1_B \\
p_b
\end{array}
\begin{array}{c}
\mathbb{P}^1_C \\
p_c
\end{array}$$

The quiver $Q'$

On the other hand, for $d \geq 4$, the length $d$ path containing

$$\mathbb{P}^1_A \rightarrow p_a \rightarrow p_b \rightarrow \mathbb{P}^1_B$$

corresponds to a component of $V_d$ not contained in $W_d$. □
2.2. On the point parameter ring $B(\{V_d\})$. The result that there exists a ring surjection from $S = S(1,1,1)$ onto the ring $P(\{W_d\})$ remains true. However, by Lemma 7 below, $B$ is a larger ring than $P$, and whether there is a ring surjection from $S$ onto $B$ is unknown. We know that there is a ring homomorphism from $S$ to $B$ with $S_1 \cong B_1$ by [H Proposition 3.20], and computational evidence suggests that $S \cong B$. The details are given as follows.

**Lemma 7.** The $k$-vector space dimension of $B_d$ is equal to dim$_k S(1,1,1)_d$ for $d = 0,1,\ldots,4$. In particular, dim$_k B_4 \neq$ dim$_k P_4$.

It is believed that analogous computations will show that dim$_k B_d =$ dim$_k S(1,1,1)_d = 3 \cdot 2^{d-1}$ for $d = 5,6$.

**Proof of Lemma 7.** By Corollary 6 we know that $V_d = W_d$ for $d = 1,2,3$; hence

$$\dim_k B_d = 3 \cdot 2^{d-1} = \dim_k S(1,1,1)_d$$

for $d = 0,1,2,3$.

To compute dim$_k B_4$, note that by Proposition 5 $V_4$ equals the union $X_{4,1} \cup X_{4,2} \subseteq (\mathbb{P}^2)^4$ as follows. Here, $X_{4,1}$ consists of the following irreducible components

$$\mathbb{P}_A^1 \times p_a \times \mathbb{P}_A^1 \times p_a, \quad p_a \times \mathbb{P}_A^1 \times p_a \times \mathbb{P}_A^1,$$

$$\mathbb{P}_B^1 \times p_b \times \mathbb{P}_B^1 \times p_b, \quad p_b \times \mathbb{P}_B^1 \times p_b \times \mathbb{P}_B^1,$$

$$\mathbb{P}_C^1 \times p_c \times \mathbb{P}_C^1 \times p_c, \quad p_c \times \mathbb{P}_C^1 \times p_c \times \mathbb{P}_C^1,$$

and $X_{4,2}$ is the union of

$$\mathbb{P}_A^1 \times p_a \times p_b \times \mathbb{P}_B^1, \quad \mathbb{P}_A^1 \times p_a \times p_c \times \mathbb{P}_C^1,$$

$$\mathbb{P}_B^1 \times p_b \times p_a \times \mathbb{P}_A^1, \quad \mathbb{P}_B^1 \times p_b \times p_c \times \mathbb{P}_C^1,$$

$$\mathbb{P}_C^1 \times p_c \times p_a \times \mathbb{P}_A^1, \quad \mathbb{P}_C^1 \times p_c \times p_b \times \mathbb{P}_B^1.$$

We consider a component such as $\mathbb{P}_A^1 \times p_a \times p_b \times p_a$ contained in $\mathbb{P}_A^1 \times p_a \times p_b \times \mathbb{P}_B^1$ to be included as part of $X_{4,2}$.

Since $X_{4,1} = W_4$ we get that $h^0(\mathcal{O}_{X_{4,1}}(1,1,1,1)) = 6 \cdot 4 - 6 = 18$ by Proposition 4.3. Moreover, $h^0(\mathcal{O}_{X_{4,2}}(1,1,1,1)) = 6 \cdot 4 = 24$ as $X_{4,2}$ is a disjoint union of its irreducible components.

Consider the finite morphism

$$\pi_1 : X_{4,1} \cup X_{4,2} \longrightarrow V_4 = X_{4,1} \cup X_{4,2},$$

which by twisting by $\mathcal{O}_{(\mathbb{P}^2)^4}(1,1,1,1)$, we get the exact sequence:

$$0 \longrightarrow \mathcal{O}_{V_4}(1,1,1,1) \longrightarrow [(\pi_1)_* \mathcal{O}_{X_{4,1} \cup X_{4,2}}](1,1,1,1) \longrightarrow \mathcal{O}_{X_{4,1} \cap X_{4,2}}(1,1,1,1) \longrightarrow 0.$$  \(\dag\)

Here, $X_{4,1} \cap X_{4,2}$ is the union of the following irreducible components:

$$\mathbb{P}_A^1 \times p_a \times p_b \times p_a, \quad p_b \times p_a \times p_b \times \mathbb{P}_B^1,$$

$$\mathbb{P}_A^1 \times p_a \times p_c \times p_a, \quad p_c \times p_a \times p_c \times \mathbb{P}_C^1,$$

$$\mathbb{P}_B^1 \times p_b \times p_a \times p_a, \quad p_a \times p_b \times p_a \times \mathbb{P}_A^1,$$

$$\mathbb{P}_B^1 \times p_b \times p_c \times p_b, \quad p_b \times p_c \times p_b \times \mathbb{P}_B^1,$$

$$\mathbb{P}_C^1 \times p_c \times p_a \times p_c, \quad p_a \times p_c \times p_a \times \mathbb{P}_A^1,$$

$$\mathbb{P}_C^1 \times p_c \times p_b \times p_c, \quad p_b \times p_c \times p_b \times \mathbb{P}_B^1.$$
a union that is not disjoint. Let \((X_{4,1} \cap X_{4,2})'\) be the disjoint union of these twelve components and consider the finite morphism

\[ \pi_2 : (X_{4,1} \cap X_{4,2})' \to X_{4,1} \cap X_{4,2}. \]

Again by twisting by \(\mathcal{O}_{P^2}(1, 1, 1, 1)\), we get the exact sequence:

\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_{X_{4,1} \cap X_{4,2}}(1, 1, 1, 1) \\
& \longrightarrow [(\pi_2)_* \mathcal{O}_{(X_{4,1} \cap X_{4,2})'}](1, 1, 1, 1) \\
& \longrightarrow \mathcal{O}_S(1, 1, 1, 1) \\
& \longrightarrow 0,
\end{align*}
\]

where \(S\) is the union of the following six points:

\[
p_a \times p_b \times p_a \times p_b, \quad p_b \times p_a \times p_b \times p_a, \quad p_a \times p_c \times p_a \times p_c, \quad p_b \times p_c \times p_b \times p_c, \quad p_c \times p_b \times p_c \times p_b.
\]

**Claim 1.** \(H^1(\mathcal{O}_{X_{4,1} \cap X_{4,2}}(1, 1, 1, 1)) = 0.\)

Note that \(H^0([\pi_2)_* \mathcal{O}_{(X_{4,1} \cap X_{4,2})'}](1, 1, 1, 1) \cong H^0(\mathcal{O}_{(X_{4,1} \cap X_{4,2})'}(1, 1, 1, 1))\) as \(k\)-vector spaces since \(\pi_2\) is an affine map [2 Exercise III 4.1]. Hence, if Claim 1 holds, then by (†):

\[
h^0(\mathcal{O}_{X_{4,1} \cap X_{4,2}}(1, 1, 1, 1)) = h^0(\mathcal{O}_{(X_{4,1} \cap X_{4,2})'}(1, 1, 1, 1)) = 12 \cdot 2 - 6 = 18.
\]

**Claim 2.** \(H^1(\mathcal{O}_{V_4}(1, 1, 1, 1)) = 0.\)

Note that \(H^0([\pi_1)_* \mathcal{O}_{(X_{4,1} \cap X_{4,2})'}](1, 1, 1, 1) \cong H^0(\mathcal{O}_{X_{4,1} \cap X_{4,2}}(1, 1, 1, 1))\) as \(k\)-vector spaces since \(\pi_1\) is an affine map [2 Exercise III 4.1]. Hence, if Claim 2 is also true, then by (†) and the computation above, we note that:

\[
\dim_k B_4 = h^0(\mathcal{O}_{V_4}(1, 1, 1, 1)) \\
= h^0(\mathcal{O}_{X_{4,1} \cap X_{4,2}}(1, 1, 1, 1)) - h^0(\mathcal{O}_{(X_{4,1} \cap X_{4,2})'}(1, 1, 1, 1)) \\
= 18 + 24 - 18 = 24.
\]

Therefore,

\[
\dim_k B_4 = \dim_k S(1, 1, 1, 4) = 24 \neq 18 = \dim_k P_4.
\]

Now we prove Claims 1 and 2 above. Here, we refer to the linear components of \((\mathbb{P}^2)^4\) of dimensions 1 or 2 by “lines” or “planes”, respectively.

**Proof of Claim 1.** It suffices to show that

\[ \theta : H^0(\mathcal{O}_{(X_{4,1} \cap X_{4,2})'}(1, 1, 1, 1)) \to H^0(\mathcal{O}_S(1, 1, 1, 1)) \]

is surjective. Say \(S = \{v_1\}_{i=1}^6\), the union of points \(v_i\). Each point \(v_i\) is contained in two lines of \((X_1 \cap X_2)'\), and each of the twelve lines of \((X_1 \cap X_2)'\) contains a unique point of \(S\).

Choose a basis \(\{t_i\}_{i=1}^6\) for \(H^0(S(1, 1, 1, 1))\), where \(t_i(v_j) = \delta_{ij}\). For each \(i\), there exists a unique line \(L_i\) of \((X_{4,1} \cap X_{4,2})'\) containing \(v_i\) so that \(pr_{234}(L_i) = pr_{234}(v_i)\). Now we define a preimage of \(t_i\) by first extending \(t_i\) to a global section \(s_i\) of \(\mathcal{O}_{L_i}(1, 1, 1, 1)\). Moreover, extend \(s_i\) to a global section
\(\tilde{s}_i\) on \(O_{(X_{4,1} \cap X_{4,2})/(1,1,1,1)}\) by declaring that \(\tilde{s}_i = s_i\) on \(L_i\) and zero elsewhere. Now \(\theta(\tilde{s}_i) = t_i\) for all \(i\), and \(\theta\) is surjective. \(\square\)

**Proof of Claim 2.** It suffices to show that

\[
\tau : H^0(O_{X_{4,1}\cup X_{4,2}}(1,1,1,1)) \rightarrow H^0(O_{X_{4,1}\cap X_{4,2}}(1,1,1,1))
\]

is surjective.

Recall that \(X_{4,1} \cap X_{4,2}\) is the union of twelve lines \(\{L_i\}\), and \(X_{4,1} \cup X_{4,2}\) is the union of twelve planes \(\{P_i\}\). Here, each line \(L_i\) of \(X_{4,1} \cap X_{4,2}\) is contained in precisely two planes of \(X_{4,1} \cup X_{4,2}\), and each plane \(P_i\) of \(X_{4,1} \cup X_{4,2}\) contains precisely two lines of \(X_{4,1} \cap X_{4,2}\).

Choose a basis \(\{t_i\}_{i=1}^{12}\) of \(H^0(O_{X_{4,1}\cap X_{4,2}}(1,1,1,1))\) so that \(t_i(L_j) = \delta_{ij}\). For each \(i\), we want a preimage of \(t_i\) in \(H^0(O_{X_{4,1}\cup X_{4,2}}(1,1,1,1))\).

Say \(P_i\) is a plane of \(X_{4,1} \cup X_{4,2}\) that contains \(L_i\), and \(L_j\) is the other line that is contained in \(P_i\). Since \(O_{P_i}(1,1,1,1)\) is very ample, its global sections separate the lines \(L_i\) and \(L_j\). In other words, there exists \(s_i \in H^0(O_{P_i}(1,1,1,1))\) so that \(s_i(L_k) = \delta_{ik}\). Extend \(s_i\) to \(\tilde{s}_i \in H^0(O_{X_{4,1}\cup X_{4,2}}(1,1,1,1))\) by declaring that \(\tilde{s}_i = s_i\) on \(L_i\), and zero elsewhere. Now \(\tau(\tilde{s}_i) = t_i\) for all \(i\), and \(\tau\) is surjective. \(\square\)

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