RIBBON-MOVE-UNKNOTTING-NUMBER-TWO 2-KNOTS, PASS-MOVE-UNKNOTTING-NUMBER-TWO 1-KNOTS, AND HIGH DIMENSIONAL ANALOGUE

EIJI OGASA

ABSTRACT. The (ordinary) unknotting-number of 1-dimensional knots, which is defined by using the crossing-change, is a very basic and important invariant. It is very natural to consider the ‘unknotting-number’ associated with other local-moves on \( n \)-dimensional knots \((n \in \mathbb{N})\). In this paper we prove the following facts. For the ribbon-move on 2-knots, which is a kind of local-move on knots, we have the following: There is a ribbon-move-unknotting-number-two 2-knot. The ribbon-move-unknotting-number of 2-knots is unbounded. For the pass-move on 1-knots, which is a kind of local-move on knots, we have the following: There is a pass-move-unknotting-number-two 1-knot whose (ordinary) unknotting-number is 4. For any natural number \( n \), there is a 1-knot whose pass-move-unknotting-number is \( n \) and whose (ordinary) unknotting-number is \( 4n \). For the high-dimensional-pass-move on high-dimensional knots, which is a kind of local-move on knots, we have the following: There is a (\( 2k+1, 2k+2 \))-pass-move-unknotting-number-two (4\( k+2 \))-knot. The (\( 2k+1, 2k+2 \))-pass-move-unknotting-number of (4\( k+2 \))-knot is unbounded. There is a (\( 2k+1, 2k+1 \))-pass-move-unknotting-number-two (4\( k+1 \))-knot. The (\( 2k+1, 2k+1 \))-pass-move-unknotting-number of (4\( k+1 \))-knot is unbounded. There is a (4\( k+1 \))-knot whose twist-move-unknotting-number is \( n \) for any natural number \( n \).

1. Introduction

The (ordinary) unknotting-number of 1-dimensional knots is a very basic and important invariant of 1-dimensional knots, has been studied for a long time, and has still many topics to investigate. It is well-known that there is a 1-knot whose (ordinary) unknotting-number is \( n \) for any natural number \( n \). The (ordinary) unknotting-number is defined by using the crossing change, which is a local move on knots. A local move means as follows: When we change a 1-knot \( K \) into a 1-knot \( J \) by a crossing-change in a 3-ball \( B \), we make a change only in \( B \) and that we do not impose any requirement on diffeomorphism type or homeomorphism type of \( J \) other than the change only in \( B \). See also Note after Definition 2.2.

Keywords: the ribbon-move on 2-knots, the pass-moves on 1-knots, the \((p, q)\)-pass-move on \((p + q - 1)\)-knots, the twist-move on \((2p + 1)\)-knots.
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By the way we know other local moves on knots. In this paper we discuss the following local-moves:

the ribbon-move on 2-dimensional knots, which is defined in [21],
the pass-move on 1-knots, which is defined in [6],
high-dimensional pass-moves on high-dimensional knots, which is defined in [19,23], and
the twist-move on high-dimensional knots, which is defined in [23].
(We review their definitions in this paper.)

It is very natural to ask whether there is a knot whose ‘unknotting-number’ associated with each of the local moves is two and whether the ‘unknotting-number’ is unbounded. (Problems 1.4, 2.6, 6.4, 6.6, and 9.4.) In this paper we give answers to these questions. The each answer is our main theorem. Our main results are the following:
Theorem 1.5 about the pass-move on 1-knots,
Theorem 2.7 about the ribbon-move on 2-knots,
Theorems 6.5 and Theorem 6.7 about high-dimensional pass-moves on high-dimensional knots, and
Theorem 9.5 about the twist-move on high-dimensional knots.
The statements and the proofs of the first four theorems are different-dimensional analogues of each other.

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We review the definitions of the local-moves and we state our main theorems. We begin by explaining the pass-move on 1-knots and the pass-move-unknotting-number of 1-knots.

We work in the smooth category unless we indicate otherwise. Let $n \in \mathbb{N}$. If an $n$-(dimensional) oriented submanifold $K \subset S^{n+2}$ is orientation-preserving PL-homeomorphic to the standard sphere $S^n$, $K$ is called an $n$-(dimensional) (spherical)-knot.

Note the following: We usually define $n$-knots as above (see e.g. [2]). Not all $n$-knots are diffeomorphic to the standard $n$-sphere although all $n$-knots are PL homeomorphic to
the standard $n$-sphere. The reason for this is the fact that many exotic $n$-spheres, which are not diffeomorphic to the standard $n$-sphere, can be embedded smoothly in $S^{n+2}$ (see [14, 15, 17] for the proof of this fact.)

Let $id : S^{n+2} \to S^{n+2}$ be the identity map. We say that $n$-knots $K$ and $K'$ are identical if $id(K) = K'$ and $id|_K : K \to K'$ is an orientation-preserving diffeomorphism map. We say that $n$-knots $K$ and $K'$ are equivalent if there exists an orientation-preserving diffeomorphism $f : S^{n+2} \to S^{n+2}$ such that $f(K) = K'$ and $f|_K : K \to K'$ is an orientation-preserving diffeomorphism. An $n$-knot $K$ is called a trivial $n$-knot if $K$ is equivalent to the boundary of an $(n + 1)$-ball embedded in $S^{n+2}$.

**Definition 1.1.** ([6].) Two 1-knots are pass-move-equivalent if one is obtained from the other by a sequence of pass-moves. See Figure 1.1 for an illustration of the pass-move.

If $K$ and $J$ are pass-move-equivalent and if $K$ and $K'$ are equivalent, then we also say that $K'$ and $J$ are pass-move-equivalent.

**Note.** [6] proved the following: Let $K$ be a 1-knot. $K$ is pass-move equivalent to the trivial knot if and only if $Arf_K = 0$.

**Definition 1.2.** Let $K$ be a 1-knot which is pass-move-equivalent to the trivial 1-knot. The pass-move-unknotting-number of $K$ is the minimal number of pass-moves which we change $K$ to the trivial 1-knot by.

We call the (ordinary) unknotting-number of 1-knots the crossing-change-unknotting-number in order to avoid the confusion of notations from now on.

**Proposition 1.3.** There is a 1-knot whose pass-move-unknotting-number is one.

**Proof of Proposition 1.3.** Let $R$ be the trefoil knot (We do not suppose that $R$ is the right-hand trefoil knot or the left-hand one). Then $R\sharp(-R^*)$ is obtained form the trivial knot by a single pass-move. **Reason:** See Figure 1.2. The uppermost knot in Figure 1.2 is $R\sharp(-R^*)$. See the middle figure in Figure 1.2. We move the part $*$ by isotopy into the
Figure 1.2. $R\#(-R^*)$ is obtained from the trivial knot by a single pass-move.
direction ⇒ to the part #. We carry out one pass-move on the resultant 1-knot and we obtain the lowermost knot in Figure 1.2. Note that the pass-move is done near the part ♮. The readers can check easily by using isotopy that the bottom knot is the trivial knot. Hence the pass-move-unknotting-number of \( R \) is no more than one.

\( R \) is a nontrivial knot because its Alexander polynomial is nontrivial. (See Definition 4.4 for the Alexander polynomial.) Hence its pass-move-unknotting-number is nonzero. Therefore the pass-move-unknotting-number of \( R \) is one.

This completes the proof of Proposition 1.3.

It is very natural to submit the following problem as we state in the first part before ‘Table of contents’ of this section.

**Problem 1.4.** (1) Is there a pass-move-unknotting-number-two 1-knot whose crossing-change-unknotting-number is \( \leq 4 \)?

(2) For any natural number \( n \), is there a 1-knot \( K \) whose pass-move-unknotting-number is \( > n \) and whose crossing-change-unknotting-number is \( \leq 4n \)?

**Note.** It is easy to prove that if the crossing-change-unknotting-number of \( K \) is \( > 4n \) and the Arf invariant is zero, the pass-move-unknotting-number is \( > n \). Hence we impose the condition on the crossing-change-unknotting-number in Problem 1.4.

We give a positive answer to Problem 1.4(1) (resp. 1.4(2)). The answers make one of our main theorems.

**Theorem 1.5.** (1) There is a pass-move-unknotting-number-two 1-knot whose crossing-change-unknotting-number is 4.

(2) For any natural number \( n \), there is a 1-knot whose pass-move-unknotting-number is \( > n \) and whose crossing-change-unknotting-number is \( 4n \).

2. The ribbon-move-unknotting-number of 2-knots

We use the terms ‘handle’ and ‘surgeries’ in this paper. See [1, 12, 16, 27, 28, 29] for the definition of handles (resp. surgeries, the attaching parts of handles, the attached part, other related terms to handles). Note that an \( a \)-dimensional \( q \)-handle \( h^q \) is diffeomorphic to \( B^q \times B^{a-q} \) (resp. \( B^a \)), where \( B^r \) denotes the \( r \)-ball, and that the attaching part of \( h^q \) is diffeomorphic \( S^{q-1} \times B^{a-q} \).

**Definition 2.1.** Let \( x, m \in \mathbb{N} \) and \( x < m \). Let \( X \) be an \( x \)-dimensional submanifold of an \( m \)-dimensional manifold \( M \). Suppose that we can embed \( X \times [0, 1] \) in \( M \) so that \( X \times \{0\} = X \). Suppose that an \( (x+1) \)-dimensional handle \( h^p \) is embedded in \( M \) and is attached to \( X \times [0, 1] \) (\( p \in \mathbb{N} \cup \{0\}, 0 \leq p \leq x \)). Suppose that the attaching part of \( h^p \) is embedded in \( X \times \{1\} \). See Figure 2.1. Suppose that \( h^p \cap (X \times [0, 1]) \) is only the
attaching part of $h^p$. Let $X' = \partial(h^p \cup (X \times [0,1])) - (X \times \{0\})$. Note that there are two cases, $\partial X = \phi$ and $\partial X \neq \phi$. Then we say that $X'$ is obtained from $X$ by the surgery by using the embedded handle $h^p$. We do not say that we use $X \times [0,1]$ if there is no danger of confusion.

**Note.** Of course we can define ‘embedded surgery’ even if we cannot embed $X \times [0,1]$ in $M$. However we do not need the case in this paper.

We review the definition of the ribbon-move on 2-knots. We begin by showing an example. Embed the disjoint union of two copies of $S^2$ in $\mathbb{R}^3 \times \{t = 0\} \subset \mathbb{R}^4 \subset S^4$, where we regard $\mathbb{R}^4 = \mathbb{R}^3 \times \{t \in \mathbb{R}\}$, as drawn in Figure 2.2(i). Attach an embedded 3-dimensional 1-handle $h^1$ to $S^2 \sqcup S^2$ so that the result of the surgery by using this 1-handle is one $S^2$. The 1-handle ‘rotates’ the two $S^2$ as drawn in Figure 2.2(ii). If a part of the 1-handle is drawn over (resp. under) a part of the new $S^2$, then it means the part of the handle exists in $\mathbb{R}^3 \times \{t > 0\}$ (resp. $\mathbb{R}^3 \times \{t < 0\}$) as usual. The new embedded $S^2$ is embed nontrivially in $\mathbb{R}^4$ because the Alexander polynomial is not trivial. (See Definition 4.4 for the Alexander polynomial.) Note a dotted circle in Figure 2.2(iii), which represents the boundary of a 3-ball $B^3$ embedded there. We can suppose that $(\text{the new } S^2) \cap B^3 = (\text{an annulus}) \sqcup (\text{a disc})$. If we change the over-under of the annulus and the disc in $B^3$, then the new $S^2$ becomes a trivial 2-knot as drawn in Figure 2.2(iv).

**Definition 2.2.** Let $K_+$ and $K_-$ be (not necessarily connected or spherical) smooth closed oriented 2-dimensional submanifolds $\subset S^4$. We say that $K_-$ is obtained from $K_+$ by one ribbon-move if there is a 4-ball $B$ embedded in $S^4$ with the following properties.

$(1)$ $K_+$ and $K_-$ differ only in $B$. 

\[ h^p \phantom{0} [0,1] \phantom{0} \]

**Figure 2.1.** A handle $h^p$ is attached to $X \times [0,1]$. 

\[ X \phantom{0} \]
Figure 2.2. An example of ribbon-moves on 2-knots
(2) $B \cap K_+$ (resp. $B \cap K_-$) is diffeomorphic to $D^2 \Pi (S^1 \times [0, 1])$, where $\Pi$ denotes the disjoint union. $B \cap K_+$ (resp. $B \cap K_-$) satisfies the following conditions.

We regard $B$ as (a closed 2-disc) $\times [0, 1] \times \{t \mid -1 \leq t \leq 1\}$. Let $B_t = (a \text{ closed 2-disc}) \times [0, 1] \times \{t\}$. Note that $B = \bigcup B_t$. In Figure 2.3. (1) (resp. 2.3. (2)), we draw $B_{-0.5}$ with $B_{-0.5} \cap K_+$, $B_0$ with $B_0 \cap K_+$, and $B_{0.5}$ with $B_{0.5} \cap K_+$ (resp. $B_{-0.5}$ with $B_{-0.5} \cap K_-$, $B_0$ with $B_0 \cap K_-$, and $B_{0.5}$ with $B_{0.5} \cap K_-$).

We draw $B_* \cap K_+$ and $B_\# \cap K_-$ by the bold line, where $*, \# \in \{0.5, 0, -0.5\}$. We draw $\partial B_t$ by the fine line.

$B \cap K_+$ has the following properties: $B_t \cap K_+$ is empty for $-1 \leq t < 0$ and $0.5 < t \leq 1$. $B_0 \cap K_+$ is diffeomorphic to $D^2 \Pi (S^1 \times [0, 0.3]) \Pi (S^1 \times [0.7, 1])$. $B_{0.5} \cap K_+$ is diffeomorphic to $(S^1 \times [0.3, 0.7])$. $B_t \cap K_+$ is diffeomorphic to $S^1 \Pi S^1$ for $0 < t < 0.5$. (Here we draw $S^1 \times [0, 1]$ to have the corner in $B_0$ and in $B_{0.5}$. However we can let $B \cap K_+$ in $B$ be a smooth submanifold by making the corner smooth naturally.)

$B \cap K_-$ has the following properties: $B_t \cap K_-$ is empty for $-1 \leq t < -0.5$ and $0 < t \leq 1$. $B_0 \cap K_-$ is diffeomorphic to $D^2 \Pi (S^1 \times [0, 0.3]) \Pi (S^1 \times [0.7, 1])$. $B_{-0.5} \cap K_-$ is diffeomorphic to $(S^1 \times [0.3, 0.7])$. $B_t \cap K_-$ is diffeomorphic to $S^1 \Pi S^1$ for $-0.5 < t < 0$.

In Figure 2.3. (1) (resp. 2.3. (2)) there are an oriented cylinder $S^1 \times [0, 1]$ and an oriented disc $D^2$ as we stated above. We do not make any assumption about the orientation of the cylinder and the disc. (Of course it holds that this orientation of (the cylinder)$\Pi$(the disc) coincides with the orientation of $B \cap K_+$ (resp. $B \cap K_-$).)

Suppose that $K_-$ is obtained from $K_+$ by one ribbon-move and that $K'_-$ is equivalent to $K_-$. Then we also say that $K'_-$ is obtained from $K_+$ by one ribbon-move. If $K_+$ is obtained from $K_-$ by one ribbon-move, then we also say that $K_-$ is obtained from $K_+$ by one ribbon-move. $K_+$ and $K_-$ are said to be ribbon-move equivalent if there are 2-knots $K_+ = \bar{K}_1, \bar{K}_2, ..., \bar{K}_r, \bar{K}_r = K_-$, where $r$ is a natural number, such that $\bar{K}_i$ is obtained from $\bar{K}_{i-1}$ ($1 < i \leq r$) by one ribbon-move.

Note. When we change a spherical 2-knot $K$ into a closed oriented submanifold $J$ of $S^4$ by a ribbon-move in a 4-ball $B$, we make a change only in $B$ and that we do not impose any requirement on diffeomorphism type or homeomorphism type of $J$ other than the change only in $B$. Note that there are two cases: $J$ is diffeomorphic to $S^2$ (resp. $S^2 \Pi T^2$). This is a reason why we use a term ‘local’ in the term ‘local-moves’ as we state in the first part of §1.

We explain a derivation of the ribbon-move of 2-knots after we review the definition of ribbon 2-knots. A 2-knot $K$ is called a ribbon 2-knot if $L$ satisfies the following properties.

(1) There is a self-transverse immersion $f : D^3 \rightarrow S^4$ such that $f(\partial D^3) = K$.
(2) The singular point set $C$ (⊂ $S^4$) of $f$ consists of double points. $C$ is a disjoint union of 2-discs $D^2_i(i = 1, ..., k)$.
Figure 2.3 (1). Ribbon-move

Figure 2.3 (2). Ribbon-move
Figure 2.4. An example of spun-knots of 1-knots

(3) Let \( j \in \{1, ..., k\} \). Let \( f^{-1}(D_j^2) = D_{jB}^2 \amalg D_{jS}^2 \). The 2-disc \( D_{jS}^2 \) is embedded in the interior of the 3-disc \( D^3 \). The circle \( \partial D_{jB}^2 \) is embedded in the boundary of \( D^3 \). The 2-disc \( D_{jB}^2 \) is embedded in \( D^3 \).

It is well-known that it is trivial that ribbon 2-knots are changed into the trivial 2-knot by a sequence of ribbon-moves. Thus we call the operation defined in Definition 2.2 the ribbon-move.

The author proved the following.

**Theorem 2.3.** ([21].) (1) Not all spherical 2-knots are ribbon-move-equivalent to the trivial 2-knot.

(2) There is a nonribbon spherical 2-knot which is ribbon-move-equivalent to the trivial 2-knot.

**Definition 2.4.** Let \( K \) be a 2-knot which is ribbon-move-equivalent to the trivial 2-knot. The **ribbon-move-unknotting-number** of \( K \) is the minimal number of ribbon-moves which we change \( K \) to the trivial 2-knot by.

**Proposition 2.5.** There is a 2-knot whose pass-move-unknotting-number is one.

**Proof of Proposition 2.5.** Figure 2.2(ii) is an example. We give another example. It is the spun-knot \( S \) of the trefoil knot. See Figure 2.4. See [32] for spun-knots. This 2-knot \( S \) is a nontrivial knot because the Alexander polynomial is nontrivial. (See Definition
for the Alexander polynomial.) See the curve $C$ in $R^2 \times \{z \geq 0\}$. Rotate $C$ around $R^2 \times \{z = 0\}$ as the axis. The result is $S$. Note that $C \cap (R^2 \times \{z = 0\})$ consists of two points and is the boundary of $C$. Note $B^3$ in $R^2 \times \{z \geq 0\}$ which is represented by a dotted curves. Note that $B^3 \cap C$ is a disjoint union of two curved segments and that (two curved segments)$\cap (R^2 \times \{z = 0\})$ is one point. Rotate $B^3$ around $R^2 \times \{z = 0\}$ as the axis. The result is a 4-ball $B^4$. Note that $C \cap B^3$ becomes $S \cap B^4$ when we rotate $C$ (resp. $B^3$) around $R^2 \times \{z = 0\}$. Note that $S \cap B^4$ is (a 2-disc)$\Pi$(an annulus). We can carry out the ribbon-move in $B^4$. This operation changes $S$ to the trivial 2-knot. \hfill $\Box$

It is very natural to submit the following problem as we state in the first part of §1

**Problem 2.6.** (1) Is there a ribbon-move-unknotting-number-two 2-knot?
(2) For any natural number $n$, is there a 2-knot whose ribbon-unknotting-number is $> n$?

We give a positive answer to Problem 2.6(1) (resp. 2.6(2)). The answers make one of our main theorems.

**Theorem 2.7.** (1) There is a ribbon-move-unknotting-number-two 2-knot.
(2) For any natural number $n$, there is a 2-knot whose ribbon-move-unknotting-number is $> n$.

3. The (1,2)-pass-move on 2-knots

In order to prove Theorem 2.7 we review the definition of another local move on 2-knots, which is the (1,2)-pass-move on 2-knots defined by the author in [21]. Why we need the (1,2)-pass-move on 2-knots is because we have Proposition 3.2 which the author proved.

**Definition 3.1.** Let $K_+$ and $K_-$ be 2-links in $S^4$. We say that $K_+$ (resp. $K_-$) is obtained from $K_-$ (resp. $K_+$) by one (1,2)-pass-move if $K_+$ and $K_-$ differ only in a 4-ball $B$ embedded in $S^4$ with the following properties: $B \cap K_+$ is drawn as in Figure 3.1(1). $B \cap K_-$ is drawn as in Figure 3.1(2). If $K$ is equivalent to $K'$ and if $K'$ is obtained from $K''$ by a sequence of (1,2)-pass-moves, we say that $K$ is (1,2)-pass-move-equivalent to $K''$.

We draw $B$ as in Definition 3.2.

$B \cap L_+$ (resp. $B \cap L_-$) is diffeomorphic to $D^2 \amalg D^2 \amalg (S^1 \times [0,1])$, where $\amalg$ denotes the disjoint union. $B \cap L_+$ has the following properties: $B_i \cap L_+$ is empty for $-1 \leq t < 0$ and $0.5 < t \leq 1$. $B_0 \cap L_+$ is $(D^2 \times \{0.4\}) \amalg (D^2 \times \{0.6\}) \amalg (S^2 \times [0.3,0.7]) \amalg (S^1 \times [0.7,1])$. $B_{0.5} \cap L_+$ is $S^1 \times [0.3,0.7]$. $B_t \cap L_+$ is diffeomorphic to $S^1 \amalg S^1$ for $0 < t < 0.5$.

$B \cap L_-$ has the following properties: $B_t \cap L_-$ is empty for $-1 \leq t < -0.5$ and $0 < t \leq 1$. $B_0 \cap L_-$ is $(D^2 \times \{0.4\}) \amalg (D^2 \times \{0.6\}) \amalg (S^1 \times [0.3,0.7]) \amalg (S^1 \times [0.7,1])$. $B_{-0.5} \cap L_-$ is $S^1 \times [0.3,0.7]$. $B_t \cap L_-$ is diffeomorphic to $S^1 \amalg S^1$ for $-0.5 < t < 0$. 

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Figure 3.1.(1). The $(1,2)$-pass-move on 2-knots

Figure 3.1.(2). The $(1,2)$-pass-move on 2-knots
In Figure 3.1.(1) (resp. 3.1.(2)) there are an oriented cylinder $S^1 \times [0,1]$ and two oriented discs $D^2$. We do not make any assumption about the orientation of the cylinder. We suppose that each arrow $\vec{x}$, $\vec{y}$ in Figure 3.1.(1) (resp. 3.1.(2)) is a tangent vector of each disc at a point. (Note we use the same notations $\vec{x}$ (resp. $\vec{y}$) for different arrows.) The orientation of each disc in Figure 3.1.(1) (resp. 3.1.(2)) is determined by the each ordered set $(\vec{x}, \vec{y})$. The orientation of $B \cap L_+$ (resp. $B \cap L_-$) coincides with that of the cylinder and that of the disc.

**Proposition 3.2.** ([21 Proposition 4.3.(1)].) If a 1-knot $K$ is obtained from $J$ by one ribbon-move, then $K$ is obtained from $J$ by one $(1,2)$-pass-move.

4. PROOF OF THEOREM 2.7

**Proposition 4.1.** Let $K$ be a 2-knot $\subset S^4$ whose ribbon-move-unnknotting-number is one. Let $M_3(K)$ be the 3-fold branched covering space of $S^4$ along $K$. Then there are three elements $\in H_1(M_3(K); \mathbb{Z})$ which generate $H_1(M_3(K); \mathbb{Z})$.

**Proof of Proposition 4.1.** By Proposition 3.2 $K$ is obtained from the trivial 2-knot $T$ by one $(1,2)$-pass-move in a 4-ball $B^4 \subset S^4$. See Figure 4.1. Note that $K \cap B^4 = (S^0 \times B^2) \amalg (S^1 \times B^1)$, where $\amalg$ denotes the disjoint union. Take a 2-ball $B^2$ in the 4-ball $B^4$ such that $B^2 \cap (S^0 \times B^2)$ is two points and such that $B^2 \cap (S^1 \times B^1) = \phi$. Call $\partial B^2$, $Y$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4.1.png}
\caption{The $(1,2)$-pass-move carried out by surgeries}
\end{figure}
Take a 3-ball $B^3$ in the 4-ball $B^4$ such that $B^3 \cap (S^1 \times B^1)$ is a circle trivially embedded in $B^3$ and that $B^3 \cap (S^0 \times B^2) = \emptyset$. Call $\partial B^3$, $Z$. Suppose that the linking number of $Y$ and $Z$ is one. Attach a 5-dimensional $2$-handle to $B^4$ along $Y$ (resp. $Z$) with the trivial framing. Note that these two handles are attached to $S^4$ on time. Carry out surgeries by using these two handles on $S^4$. Then the new manifold which we obtain is the 4-sphere again, and call it $S^4$. Furthermore the new submanifold $\subset S^4$ which is made from $K$ is the trivial 2-knot $T$.

Note that now we have a compact oriented 5-dimensional manifold $W$ with a handle decomposition

$$W = (S^4 \times [0, 1]) \cup (a 5\text{-dimensional 2-handle})$$

$$\cup (a 5\text{-dimensional 3-handle}) \cup (S^4 \times [0, 1]).$$

Note that $\partial W = (S^4 \times \{0\}) \cup (S^4 \times \{1\})$. Note that there is an embedding map $f : S^2 \times [0, 1] \hookrightarrow W$ with the following properties:

1. $f(S^2 \times [0, 1] \cap (S^4 \times \{0\}))$ is $f(S^2 \times \{0\})$. $f(S^2 \times [0, 1] \cap (S^4 \times \{1\}))$ is $f(S^2 \times \{1\})$.
2. $f(S^2 \times \{0\})$ in $(S^4 \times \{0\})$ is $K$.
   $f(S^2 \times \{1\})$ in $(S^4 \times \{1\})$ is $T$.

Take a 3-fold branched covering space $\tilde{W}$ of $W$ along $f(S^2 \times [0, 1])$. Note the circle which is the core of the attaching part of the 2-handle in the above handle decomposition of $W$. The circle is null-homologous in $S^4 - N(K)$, where $N(K)$ is the tubular neighborhood of $K$ in $S^4$. Therefore we obtain a compact oriented 5-dimensional manifold $\tilde{W}$ with a handle decomposition

$$\tilde{W} = (M_3(K) \times [0, 1]) \cup (\text{three 5-dimensional 2-handles}, h^2_1, h^2_2, \text{and } h^2_3)$$

$$\cup (\text{three 5-dimensional 3-handles}, h^3_1, h^3_2, \text{and } h^3_3) \cup (S^4 \times [0, 1]).$$

Here, note that the 3-fold branched covering space of $S^4$ along $T$ is the standard 4-sphere, and call it $S^4$ again.

We prove that $\tilde{W}$ is simply connected. *Reason.* Take the dual handle decomposition

$$\tilde{W} = (S^4 \times [0, 1]) \cup (\text{three 5-dimensional 2-handles}, \overline{h}^2_1, \overline{h}^2_2, \overline{h}^2_3)$$

$$\cup (\text{three 5-dimensional 3-handles}, \overline{h}^3_1, \overline{h}^3_2, \overline{h}^3_3) \cup (M_3(K) \times [0, 1]),$$

of the above handle decomposition, where $\overline{h}^n_#$ is the dual handle of $h^{n-\#}_*$. Take a manifold which is represented by the sub-handle-decomposition

$$(S^4 \times [0, 1]) \cup (\text{the three 5-dimensional 2-handles}, h^2_1, h^2_2, \text{and } h^2_3)$$

of the dual decomposition of $\tilde{W}$. Since $S^4 \times [0, 1]$ is simply-connected, this manifold is simply-connected. Recall that if we attach 3-handles to a manifold $E$ and we obtain a new manifold $E'$, then $\pi_1 E \cong \pi_1 E'$. Therefore $\pi_1(\tilde{W}) = 1$.
Therefore the manifold which is represented by the sub-handle-decomposition \((M_3(K) \times [0,1]) \cup (\text{the three 5-dimensional 2-handles, } h_1^3, h_2^3, \text{ and } h_3^3)\)
of the above handle decomposition is simply-connected.

Therefore the cores of the attaching parts of \(h_1^3, h_2^3\), and \(h_3^3\) generate \(H_1(M_3(K); \mathbb{Z})\).
This completes the proof of Proposition \(\blacksquare\)

In a similar fashion we can prove the following.

**Proposition 4.2.** Let \(n \in \mathbb{N}\). Let \(K \subset S^4\) be a 2-knot whose ribbon-move-unknotting-number is \(\leq n\). Let \(M_3(K)\) be the 3-fold branched covering space of \(S^4\) along \(K\). Then there are \(3n\) elements \(\in H_1(M_3(K); \mathbb{Z})\) which generate \(H_1(M_3(K); \mathbb{Z})\).

**Definition 4.3.** Let \(n \in \mathbb{N}\). Let \(K\) be an \(n\)-knot \(\subset S^{n+2}\). If \(V\) is a connected, compact, oriented, \((n+1)\)-dimensional submanifold \(\subset S^{n+2}\) whose boundary is \(K\), we call \(V\) a **Seifert hypersurface** for \(K\). Let \(p, n + 1 - p \in \mathbb{N}\). Let \(x_1, \ldots, x_\mu\) be \(p\)-cycles in \(V\) which compose a basis of \(H_p(V; \mathbb{Z})/\text{Tor}\), where \(\mu \in \mathbb{N} \cup \{0\}\). Let \(y_1, \ldots, y_\nu\) be \((n + 1 - p)\)-cycles in \(V\) which compose a basis of \(H_{n+1-p}(V; \mathbb{Z})/\text{Tor}\), where \(\nu \in \mathbb{N} \cup \{0\}\). By Poincaré duality, we have \(\nu = \mu\). Push \(y_i\) into the positive (resp. negative) direction of the normal bundle of \(V\). Call it \(y_i^+\) (resp. \(y_i^-\)). A \((p, n + 1 - p)\)-**positive Seifert matrix** for the above submanifold \(K\) associated with \(V\) represented by an ordered basis, \((x_1, \ldots, x_\mu)\), and an ordered basis, \((y_1, \ldots, y_\mu)\), is a \((\mu \times \mu)\)-matrix

\[
S = (s_{ij}) = (\text{lk}(x_i, y_j^+)).
\]

A \((p, n + 1 - p)\)-**negative Seifert matrix** for the above submanifold \(K\) associated with \(V\) represented by an ordered basis, \((x_1, \ldots, x_\mu)\), and an ordered basis, \((y_1, \ldots, y_\mu)\), is a matrix

\[
N = (n_{ij}) = (\text{lk}(x_i, y_j^-)).
\]

We have the following: Let \(S\) and \(N\) be as above. Then \(S - N\) represents the map \(\{H_p(V; \mathbb{Z})/\text{Tor}\} \times \{H_{n+1-p}(V; \mathbb{Z})/\text{Tor}\} \to \mathbb{Z}\) which is defined by the intersection product. We call \(t \cdot S - N\) the \((p, n + 1 - p)\)-**Alexander matrix** for \(K\) associated with \(V\) represented by an ordered basis, \((x_1, \ldots, x_\mu)\), and an ordered basis, \((y_1, \ldots, y_\mu)\). \(S\) and \(N\) (resp. \(S\) and \(t \cdot S - N\)) are said to be related if \(S\) and \(N\) (resp. \(S\) and \(t \cdot S - N\)) are defined by using the same \(V\), the same ordered basis \((x_1, \ldots, x_\mu)\), and the same ordered basis \((y_1, \ldots, y_\mu)\). We sometimes abbreviate \((p, n + 1 - p)\)-positive Seifert matrix (resp. \((p, n + 1 - p)\)-negative Seifert matrix, \((p, n + 1 - p)\)-Alexander matrix) to \(p\)-Seifert matrix (resp. \(p\)-negative Seifert matrix, \(p\)-Alexander matrix) when it is clear from the context.

**Definition 4.4.** Let \(n, p \in \mathbb{N}\). Let \(K\) be an \(n\)-knot \(\subset S^{n+2}\). Let \(S_p\) (resp. \(N_p\)) be a \(p\)-positive (resp. negative) Seifert matrix for \(K\) associated with \(V\) represented by an ordered basis, \((x_1, \ldots, x_\mu)\), and an ordered basis, \((y_1, \ldots, y_\mu)\), where \(\mu \in \mathbb{N} \cup \{0\}\). Thus \(S_p\) and \(N_p\) are related.
Two polynomials, \( f(t) \) and \( g(t), \in \mathbb{Q}[t, t^{-1}] \) are said to be \( \mathbb{Q}[t, t^{-1}] \)-balanced if there is an integer \( \xi \) and a nonzero rational number \( r \) such that \( f(t) = r \cdot t^{\xi} \cdot g(t) \).

We define the \( p \)-\( \mathbb{Q}[t, t^{-1}] \)-Alexander polynomial to be the \( \mathbb{Q}[t, t^{-1}] \)-balanced class of ‘the determinant of \( p \)-Alexander matrix’

\[
\det(t \cdot S_p - N_p).
\]

**Note.** This definition is equivalent to the spherical-knot-case of Definition 3.1 of [26] because of Proposition 3.2 of [26].

**Proposition 4.5.** Let \( n, p, n+1-p \in \mathbb{N} \). Let \( N_p \) be a \((p, n+1-p)\)-negative Seifert matrix for \( K \) associated with \( V \) represented by an ordered basis, \((x_1, \ldots, x_\mu)\), and an ordered basis, \((y_1, \ldots, y_\mu)\), where \( \mu \in \mathbb{N} \cup \{0\} \). Let \( S_{n+1-p} \) be a \((n+1-p, p)\)-positive Seifert matrix for \( K \) associated with \( V \) represented by an ordered basis, \((y_1, \ldots, y_\mu)\), and an ordered basis, \((x_1, \ldots, x_\mu)\). Then we have

\[
N_p = (-1)^{p(n+1)} \cdot S_{n+1-p}.
\]

**Proof of Proposition 4.5** By the definition of \( x_i^+ \) and \( y_i^- \), \( \text{lk}(y_i, x_i^+) = \text{lk}(y_i^-, x_i) \). By [13], page 541, \( \text{lk}(y_i^-, x_j) = (-1)^{p(n+1-p)+1} \text{lk}(x_j, y_i^-) \). Note that \( p(1-p) \) is an even number.

Proposition 4.5 implies Proposition 4.6.

**Proposition 4.6.** Let \( m \in \mathbb{N} \cup \{0\} \). Let \( K \) be a \((2m+1)\)-dimensional closed oriented submanifold \( \subset S^{2m+3} \). Let \( S \) be an \((m+1,m+1)\)-Seifert matrix. Then we have

\[
S = (-1)^m \cdot !S.
\]

We use the following proposition, which the Mayer-Vietoris sequence implies.

**Proposition 4.7.** (Folklore.) Let \( n \in \mathbb{N} \). Let \( A \) be an \( n \)-knot. Let an \((l \times l)\)-matrix \( Z \) be a positive-\( p \)-Seifert matrix for \( A \), where \( l \in \mathbb{N} \cup \{0\} \) and \( p \in \mathbb{N} \). Suppose that \( Z \) is invertible as \( \mathbb{Z} \)-valued matrix. Let \( k \in \mathbb{N} \). Let \( X_k(A) \) be the \( k \)-fold branched covering space of \( S^{n+2} \) along \( A \). Then \( H_p(N_k(A); \mathbb{Z}) \) is generated by \((\mathbb{Z} \mathbb{Z}^{-1})^k - I\), where \( I \) is the \((l \times l)\)-identity matrix.

Let \( R_i \) be the trefoil knot for \( i = 1, 2 \) (We do not suppose whether \( R_i \) is the right-hand trefoil knot or the left-hand one for each \( i \), nor whether \( R_1 \) is equivalent to \( R_2 \)). Let \( P \) be a spun-knot of \( R_1 \oplus R_2 \). Note that \( P \) is a 2-knot \( \subset S^4 \). It is well-known that spun-knots are ribbon-knots. Hence \( P \) is ribbon-move equivalent to the trivial 2-knot.

There is a Seifert surface \( V \) for \( R_i \) \((i = 1, 2)\) with the following properties:

1. \( H_1(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \).
(2) There is an ordered set \((x_1, x_2)\) of basis of \(H_1(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\). The intersection matrix \((x_k \cdot x_l)\) \((k, l \in \{1, 2\})\) on \(H_1(V; \mathbb{Z}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\). Note that Poincaré dual of \(x_1\) is \(x_2\).

(3) The Seifert matrix \((lk(x_k, x_i^+))\) for \(R_i\) \((i = 1, 2)\) is \(X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\).

Therefore we have the following:

\[
X^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad XX^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad (XX^{-1})^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad (XX^{-1})^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

By the definition of the spun-knot, \(P\) has a Seifert hypersurface \(V\) as follows:

1. \(H_1(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\). \(H_2(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\).
2. There is an ordered set \((x_1, x_2)\) of basis of \(H_1(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\). There is an ordered set \((y_1, y_2)\) of basis of \(H_2(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\). The intersection matrix \((x_k \cdot y_l)\) \((k, l \in \{1, 2\})\) on \(H_1(V; \mathbb{Z})\) (resp. \(H_2(V; \mathbb{Z})\)) is \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\). Note that Poincaré dual of \(x_1\) (resp. \(x_2\)) is \(y_2\) (resp. \(-y_1\)).

(3) The Seifert matrix \((lk(x_k, y_i^+))\) for \(R_i\) \((i = 1, 2)\) is \(\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\).

Let \(M_3(P)\) be the 3-fold branched covering space of \(S^4\) along \(P\).

By Proposition \(2.7\) we have \(H_1(M_3(P); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\). Hence we need no less than 4 generators in order to generate \(H_1(M_3(P); \mathbb{Z})\).

Suppose that the ribbon-move-unlinking-number of \(P\) is \(\leq 1\). By Proposition \(4.1\), we can take three generators in order to generate \(H_1(M_3(P); \mathbb{Z})\). We arrived at a contradiction. Hence the ribbon-move-unlinking-number of \(P\) is \(\geq 2\).

The ribbon-move-unlinking-number of \(P\) is \(\leq 2\). \textit{Reason:} See Proof of Proposition \(2.6\).

Hence the ribbon-move-unlinking-number of \(P\) is two.

This completes the proof of Theorem \(2.7\)(1).

Let \(n \in \mathbb{N}\). Let \(m \in \mathbb{N}\) and \(\frac{2m}{3} > n\). Let \(#^m P\) be the connected-sum of \(m\)-copies of \(P\). Since \(P\) is ribbon-move equivalent to the trivial 2-knot, \(#^m P\) is ribbon-move equivalent to the trivial 2-knot.

Let \(N_3(#^m P)\) be the 3-fold branched covering space of \(S^4\) along \(#^m P\). By Proposition \(4.7\) we have \(H_1(N_3(#^m P); \mathbb{Z}) \cong \mathbb{Z} \oplus 2m\mathbb{Z} \oplus \mathbb{Z}\). Hence we need no less than 2\(m\) generators in order to generate \(H_1(N_3(#^m P); \mathbb{Z})\).

Suppose that the ribbon-move-unlinking-number of \(#^m P\) is \(\leq n\). By Proposition \(4.2\) \(H_1(N_3(#^m P); \mathbb{Z})\) can take 3\(n\) generators. Since \(2m > 3n\), we arrived at a contradiction. Therefore the ribbon-move-unlinking-number of \(#^m P\) is \(> n\).
Figure 5.1. The pass-move carried out by surgeries

This completes the proof of Theorem 2.7.(2).

This completes the proof of Theorem 2.7. □

5. Proof of Theorem 1.5

Proposition 5.1. Let $J$ be a 1-knot $\subset S^3$ whose pass-move-unknotting-number is one. Let $N_3(J)$ be a 3-fold branched covering space of $S^3$ along $J$. Then there are six elements $\in H_1(N_3(J); \mathbb{Z})$ which generate $H_1(N_3(J); \mathbb{Z})$.

Proof of Proposition 5.1. Take a 3-ball $B^3 \subset S^3$ where we carry out the pass-move which changes $J$ into $T$. See Figure 5.1. Note that $J \cap B^3$ is regarded as $(S^0 \times B^1) \cup (S^0 \times B^1)$. Call one of the two $S^0 \times B^1$, $A_1$, and the other $A_2$. Take two 2-balls, $B^2_1$ and $B^2_2$, in the 3-ball $B^3$ such that $B^2_i \cap A_i$ is two points and such that $B^2_i \cap A_j = \emptyset$ ($i = 1, 2$, and $i \neq j$). Call $\partial B^2_i$, $Y_i$ ($i = 1, 2$). Suppose that the linking number of $Y_1$ and $Y_2$ is one. Attach a 4-dimensional 2-handle to $B^3$ along $Y_i$ with the trivial framing ($i = 1, 2$). Note that these two handles are attached to $S^3$ on time. Carry out surgeries by using these two handles on $S^3$. Then the new manifold which we obtain is the 3-sphere again, and call it $S^3$. Furthermore the new submanifold $\subset S^3$ which is made from $J$ is the trivial 1-knot $T$.

Note that we now have a compact oriented 4-dimensional manifold $U$ with a handle decomposition

$$U = (S^3 \times [0, 1]) \cup (\text{two 3-dimensional 2-handles}) \cup (S^2 \times [0, 1]).$$

Note that $\partial U = (S^3 \times \{0\}) \cup (S^2 \times \{1\})$. There is an embedding map $f : S^1 \times [0, 1] \hookrightarrow U$ with the following properties:
Take a 3-fold branched covering space $\tilde{U}$ of $U$ along $f(S^1 \times [0, 1])$. Note the circle which is the core of the attaching part of each of the two 2-handles in the above handle decomposition of $U$. Each of the two circles is null-homologous in $S^3 - N(J)$, where $N(J)$ is the tubular neighborhood of $J$ in $S^3$. Therefore we obtain a compact oriented 4-dimensional manifold $\tilde{U}$ with a handle decomposition

$$\tilde{U} = (N_3(J) \times [0, 1]) \cup (\text{six 4-dimensional 2-handles}, h_1^2, \ldots, h_6^2) \cup (S^3 \times [0, 1]).$$

Here, note that a 3-fold branched covering space of $S^3$ along $T$ is the standard 3-sphere, and call it $S^3$ again.

We prove that $\tilde{U}$ is simply connected. 
\textit{Reason.} Take the dual handle decomposition

$$\tilde{U} = (S^3 \times [0, 1]) \cup (\text{six 4-dimensional 2-handles}, \overline{h}_1^2, \ldots, \overline{h}_6^2) \cup (N_3(J) \times [0, 1]),$$

of the above handle decomposition, where $\overline{h}_\#^2$ is the dual handle of $h_\#^2$. Since $S^3 \times [0, 1]$ is simply-connected, $\tilde{U}$ is simply-connected.

Therefore the cores of the attaching parts of $h_1^2, \ldots, h_6^2$ generate $H_1(N_3(J); \mathbb{Z})$.

This completes the proof of Proposition \ref{prop:pass_move_unknotting_number}.

\begin{proof}
In a similar way, we can prove the following.

\textbf{Proposition 5.2.} Let $n \in \mathbb{N}$. Let $J \subset S^3$ be a 1-knot whose pass-move-unknotting-number is $\leq n$. Let $N_3(J)$ be a 3-fold branched covering space of $S^3$ along $J$. Then there are $6n$ elements $\in H_1(N_3(J); \mathbb{Z})$ which generate $H_1(N_3(J); \mathbb{Z})$.

Let $R$ be the trefoil knot (We do not suppose that $R$ is the right-hand trefoil knot or the left-hand one). Let $C = (R\#(-R^*))\#(R\#(-R^*))$. Note that $\arf C = 0$. By Note to Definition \ref{def:pass_move_unknotting_number}, $C$ is pass-move equivalent to the trivial 1-knot.

By Proposition \ref{prop:pass_move_unknotting_number} and the calculations right after Proposition \ref{prop:pass_move_unknotting_number}, $H_1(N_3(C); \mathbb{Z}) \cong \mathbb{Z} \oplus 8\mathbb{Z}_2$. Hence we need no less than eight generators to generate $H_1(N_3(C); \mathbb{Z})$.

Suppose that the pass-move-unknotting-number of $C$ is $\leq 1$. By Proposition \ref{prop:pass_move_unknotting_number} $H_1(N_3(C); \mathbb{Z})$ can take six generators. We arrived at a contradiction.

Therefore the pass-move-unknotting-number of $C$ is $\geq 2$.

The pass-move-unknotting-number of $C$ is $\leq 2$. 
\textit{Reason:} See Proof of Proposition \ref{prop:pass_move_unknotting_number}.

Therefore the pass-move-unknotting-number of $C$ is two.

It is trivial to prove that the crossing-change-unknotting-number of $R$ is one.

The crossing-change-unknotting-number of $C$ is 4 because of \cite[Proof of Theorem 10.1 in page 420 and (2.4) in page 389]{paper}.

This completes the proof of Theorem \ref{thm:pass_move_unknotting_number} (1).
Let \( n \in \mathbb{N} \). Let \( \#^n C \) be the connected-sum of \( n \) copies of \( C \). Since \( C \) is pass-move equivalent to the trivial 1-knot, \( \#^n C \) is pass-move equivalent to the trivial 1-knot.

Let \( N_3(\#^n C) \) be the 3-fold branched cyclic covering space of \( S^3 \) along \( \#^n C \). By Proposition 4.7, \( H_1(N_3(\#^n C); \mathbb{Z}) \cong \oplus^{8n} \mathbb{Z} \). Hence we need no less than \( 8n \) generators which generate \( H_{2k+1}(N_3(\#^n C); \mathbb{Z}) \).

Suppose that the pass-move-unknotting-number of \( \#^n C \) is \( \leq n \). By Proposition 5.2 we can prove that \( H_1(N_3(\#^n C); \mathbb{Z}) \) can take \( 6n \) generators. We arrived at a contradiction.

Therefore the pass-move-unknotting-number of \( \#^n C \) is \( > n \).

The crossing-change-unknotting-number of \( \#^n C \) is \( 4n \) because of [18, Proof of Theorem 10.1 in page 420 and (2.4) in page 389].

This completes the proof of Theorem 1.5. (2).

This completes the proof of Theorem 1.5. □

6. High-dimensional-pass-moves on high-dimensional knots and their associated ‘unknotting-number’

Local moves on high dimensional knots were defined in [19, 21, 23]. They have been researched in [8, 9, 10, 19, 20, 21, 22, 23, 24, 25]. We show an example of them before we review the definition of high-dimensional pass-moves on high dimensional knots.

**Lemma.** Let \( p \in \mathbb{N} \). Letting \( B^p \) denote a \( p \)-dimensional ball, we can write

\[
S^p = B^p_u \cup B^p_d
\]

\[
S^p \times S^q = (B^p_u \cup B^p_d) \times (B^q_u \cup B^q_d).
\]

Thus

\[
S^p \times S^q = (B^p_u \times B^q_u) \cup (B^p_u \times B^q_d) \cup (B^p_d \times B^q_u) \cup (B^p_d \times B^q_d).
\]

**Proof.** Use the fact

\[
(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z).
\]

Let \( p, q \in \mathbb{N} \). Let

\[
F = (S^p \times S^q) - \text{Int}(B^p_u \times B^q_u).
\]

We indicate \( F \) in the figure below and abbreviate \( B^*_s \) to \( B_s \).
$F$ is drawn in another way as below. Note that we can bend the corner of $B_p \times B_q$ and change it into the $(p + q)$-dimensional ball. Let $p + q = n + 1$. Hence the boundary of $F$ is $S^n$.

We can regard $B_d \times B_d$ as a $(p + q)$-dimensional 0-handle, $B_u \times B_u$ as a $(p + q)$-dimensional $p$-handle, and $B_d \times B_u$ as a $(p + q)$-dimensional $q$-handle.

Embed $F \subset S^{n+2}$ as follows: Embed $B^{p+1} \subset S^{n+2}$. Take the tubular neighborhood $N(\partial B^{p+1})$ of $\partial B^{p+1}$ in $S^{n+2}$. Take $\partial(N(\partial B^{p+1}))$. Note that $\partial(N(\partial B^{p+1}))$ is diffeomorphic to $S^p \times S^q$. Embed $F \subset S^p \times S^q$ as above.
The boundary of $F$ in $S^{n+2}$ is an $n$-knot. Furthermore it is the trivial $n$-knot. Carry out a ‘local move’ on this $n$-knot in an $(n + 2)$-ball, which is denoted by a dotted circle in the following figure.

![Figure 6.2: A trivial $n$-knot](image1)

![Figure 6.3: A local move will be carried out in the dotted $(n + 2)$-ball. The resulting $n$-knot $K$ is a nontrivial $n$-knot.](image2)
We can prove that \( K \) is nontrivial by using Seifert matrices and the Alexander polynomial. (See Definition 4.3 for Seifert matrices, and Definition 4.4 for the Alexander polynomial.) We use the fact that \( S^p \) and \( S^q \) can be ‘linked’ in \( S^{p+q+1} \). Recall that \( p + q + 1 = n + 2 \). Note that \( S^q \) and \( S^p \) are included in \( F \) as shown below.

Note that the above operation is done only in an \((n + 2)\)-ball. This operation is an example of the \((p, q)\)-pass-moves, whose definition we review in Definition 6.1.

Local moves on high dimensional submanifolds are exciting ways of explicit construction of high dimensional figures. They are also generalization of local moves on 1-links.
They are useful to research link cobordism, knot cobordism, and the intersection of sub-manifolds (see [19]) etc. There remain many exciting problems. Some of them are proper in high dimensional case and others are analogous to 1-dimensional one. For example, we do not know a local move on high dimensional knots which is an unknotting operation.

Let $p, q \in \mathbb{N}$ and $p + q = n + 1$. We review the definition of the $(p, q)$-pass-move on $n$-knots, which was defined in [19] and which has been studied in [8, 19, 20, 21, 22, 23, 24, 25]. If $p = 1$ and $q = 1$, the $(p, q)$-pass-move on $(p + q - 1)$-knots is the pass-move on 1-knots, whose definition we reviewed in [11]. If $p = 1$ and $q = 2$, the $(p, q)$-pass-move on $(p + q - 1)$-knots is the (1,2)-pass-move on 2-knots, whose definition we reviewed in [12].

**Definition 6.1.** Let $n, p \in \mathbb{N}$. Let $n + 1 - p > 0$. Regard an $(n + 2)$-ball $B = D^{n+2}$ as $D^1 \times D^p \times D^{n+1-p}$. (See Figure 6.7.) Let $D^1 = [-1, 1] = \{t \mid -1 \leq t \leq 1\}$. Take a p-ball $D^p_s$ (resp. an $(n + 1 - p)$-ball $D^{n+1-p}_s$) embedded in $\text{Int}D^p$ (resp. $\text{Int}D^{n+1-p}$.) Let $S^{p-1}$ (resp. $S^{n-p}$) denote the $(p - 1)$-sphere $\partial D^p$ (resp. the $(n - p)$-sphere $\partial D^{n+1-p}$). Let a submanifold $\{0\} \times D^p \times D^{n+1-p}_s$ (resp. $\{0\} \times D^{n+1-p}_s \times D^p$) ⊂ $B$ be called $h^p$ (resp. $h^{n+1-p}$). We give an orientation to $h^p$ (resp. $h^{n+1-p}$). Note that $h^p \cap h^{n+1-p} \neq \phi$. Move $h^p$ in $B$ by using an isotopy with keeping $h^p \cap \partial B$, let ‘the resultant submanifold—$B$’ be put in $\{t > 0\} \times D^p \times D^{n+1-p}$ (resp. $\{t < 0\} \times D^p \times D^{n+1-p}$), and call the submanifold $h^p_\ast$ (resp. $h^{n+1-p}_\ast$). (See Figure 6.6.) Note that $h^p_\ast \cap h^{n+1-p}_\ast = \phi$, that $h^p_\ast \cap h^{n+1-p}_\ast = \phi$, and that $h^p_\ast \cap h^p_\ast = h^p_\ast \cap \partial B = h^p_\ast \cap \partial B$. (Each of Figure 6.6 and Figure 6.7, which consists of the two figures (1) and (2), is a diagram of the $(p, q)$-pass-move, where $q = n + 1 - p$.)

Let $K_+$ and $K_-$ be $n$-dimensional closed oriented submanifolds ⊂ $S^{n+2}$. Embed the $(n + 2)$-ball $B$ in $S^{n+2}$. Let $K_+$ and $K_-$ differ only in $B$. Let $K_+$ (resp. $K_-$) satisfy the condition

\[ K_+ \cap \text{Int}B = (\partial h^p_\ast - \partial B) \cup (\partial h^{n+1-p}_\ast - \partial B) \]
\[ (\text{resp. } K_- \cap \text{Int}B = (\partial h^p_\ast - \partial B) \cup (\partial h^{n+1-p}_\ast - \partial B)), \]

where we suppose that there is not $h^p_\ast$ (resp. $h^{n+1-p}_\ast$) in $B$. Then we say that $K_+$ (resp. $K_-$) is obtained from $K_-$ (resp. $K_+$) by one $(p, n + 1 - p)$-pass-move in $B$.

In Definition 6.1 we have the following: Let $z \in \{+, -\}$, there is a Seifert hypersurface $V_z \subset S^{n+2}$ for $K_z$ such that $V_z \cap B = h^p_\# \cup h^{n+1-p}_\#$. (The idea of the proof is Thom-Pontrjagin construction.) We say that $V_+$ (resp. $V_-$) is obtained from $V_+$ (resp. $V_-$) by one $(p, n + 1 - p)$-pass-move in $B$.

In Definition 6.1 note the following: Let $V_0 = V_z - \text{Int}B$

= ‘the closure of $(V_z - (h^p_\# \cup h^{n+1-p}_\#))$ in $S^{n+2}$’. We can say that we attach an embedded $(n+1)$-dimensional $p$-handle $h^p_\#$ and an embedded $(n+1)$-dimensional $(n+1-p)$-handle $h^{n+1-p}_\#$ to the submanifold $V_0 \subset S^{n+2}$, and obtain the submanifold $V_\# \subset S^{n+2}$.
$$s^p - D^n + 1 - p = \partial h^n + 1 - p - \partial B$$

Figure 6.2: The $(p, n + 1 - p)$-pass-move on an $n$-dimensional closed submanifold $\subset S^{n+2}$. Note $B = B^{n+2} = D^{n+2} \subset S^{n+2}$.

**Definition 6.2.** Let $p, q, p + q - 1 \in \mathbb{N}$. Let $K$ be a $(p + q - 1)$-knot $\subset S^{p+q+1}$ which is $(p, q)$-pass-move-equivalent to the trivial $(p + q - 1)$-knot. The $(p, q)$-pass-move-unknotting-number of $K$ is the minimal number of $(p, q)$-pass-moves which we change $K$ to the trivial $(p + q - 1)$-knot by.

**Proposition 6.3.** Let $p, q, p + q - 1 \in \mathbb{N}$. There is a $(p + q - 1)$-knot whose $(p, q)$-pass-move-unknotting-number is one. 

**Proof of Proposition 6.3.** See the nontrivial $(p + q - 1)$-knot $K$ in Figures 6.4 of this section: A Seifert hypersurface $V$ for $K$ is diffeomorphic to $S^p \times S^q - \text{Int}B^{p+q}$. We supposed the following: $x$ (resp. $y$) is a generator of $H_p(V; \mathbb{Z})$ (resp. $H_p(V; \mathbb{Z})$). The intersection matrix associated with the base $\{x\}$ and $\{y\}$ is a $1 \times 1$-matrix (1). The Seifert matrix associated with the base $\{x\}$ and $\{y\}$ is a $1 \times 1$-matrix (2).

Hence the $(p, q)$-pass-move-unknotting-number of $K$ is $\geq 1$.

$K$ is obtained from the trivial $(p + q - 1)$-knot by one $(p, q)$-pass-move as drawn in Figures 6.2-4.

Therefore the $(p, q)$-pass-move-unknotting-number of $K$ is one. $\square$

We consider the following problem.

**Problem 6.4.** Let $k \in \mathbb{N} \cup \{0\}$.

(1) Is there a $(2k + 1, 2k + 2)$-pass-move-unknotting-number-two $(4k + 2)$-knot?
$D = [-1, 1]$  

This cube is $B = D^{n+2} = D^1 \times D^p \times D^{n+1-p}$ 

$B \cap K_+$  

**Figure 6.7.(1):** The $(p, n + 1 - p)$-pass-move  

$B \cap K_-$  

**Figure 6.7.(2):** The $(p, n + 1 - p)$-pass-move
(2) For any natural number $n$, there is a $(4k + 2)$-knot whose $(2k + 1, 2k + 2)$-pass-move-unknotting-number is $\succ n$.

We give a positive answer to Problem 6.4 (1) (resp. 6.4 (2)). The answers make one of our main theorems.

**Theorem 6.5.** Let $k \in \mathbb{N} \cup \{0\}$.

(1) There is a $(2k + 1, 2k + 2)$-pass-move-unknotting-number-two $(4k + 2)$-knot.

(2) For any natural number $n$, there is a $(4k + 2)$-knot whose $(2k + 1, 2k + 2)$-pass-move-unknotting-number is $\succ n$.

We consider the following problem.

**Problem 6.6.** Let $k \in \mathbb{N} \cup \{0\}$.

(1) Is there a $(2k + 1, 2k + 1)$-pass-move-unknotting-number-two $(4k + 1)$-knot?

(2) For any natural number $n$, is there a $(4k + 1)$-knot whose $(2k + 1, 2k + 1)$-pass-move-unknotting-number is $\succ n$?

We give a positive answer to Problem 6.6 (1) (resp. 6.6 (2)). The answers make one of our main theorems.

**Theorem 6.7.** Let $k \in \mathbb{N} \cup \{0\}$.

(1) There is a $(2k + 1, 2k + 1)$-pass-move-unknotting-number-two $(4k + 1)$-knot.

(2) For any natural number $n$, there is a $(4k + 1)$-knot whose $(2k + 1, 2k + 1)$-pass-move-unknotting-number is $\succ n$.

7. Proof of Theorem 6.5

**Proposition 7.1.** Let $k \in \mathbb{N} \cup \{0\}$. Let $K$ be a $(4k + 2)$-knot $\subset S^{4k+4}$ whose $(2k + 1, 2k + 2)$-pass-move-unknotting-number is one. Let $M_3(K)$ be the 3-fold branched covering space of $S^{4k+4}$ along $K$. Then there are three elements $\in H_{2k+1}(M_3(K); \mathbb{Z})$ which generate $H_{2k+1}(M_3(K); \mathbb{Z})$.

**Proof of Proposition 7.1.** Take a $(4k + 4)$-ball $B^{4k+4} \subset S^{4k+4}$ where we carry out the $(2k + 1, 2k + 2)$-pass-move which changes $K$ into $T$. See Figure 7.1. Note that $K \cap B^{4k+4} = (S^{2k} \times B^{2k+2}) \cup (S^{2k+1} \times B^{2k+1})$. Take a $(2k + 2)$-ball $B^{2k+2}$ in the $(4k + 4)$-ball $B^{4k+4}$ such that $B^{2k+2} \cap (S^{2k} \times B^{2k+2})$ is the $2k$-sphere trivially embedded in $B^{2k+2}$. and such that $B^{2k+2} \cap (S^{2k+1} \times B^{2k+1}) = \emptyset$. Call $\partial B^{2k+2}$, $Y$. Take a $(2k + 3)$-ball $B^{2k+3}$ in the $(4k + 4)$-ball $B^{4k+4}$ such that $B^{2k+3} \cap (S^{2k+1} \times B^{2k+1})$ is the $(2k + 1)$-sphere trivially embedded in $B^{2k+3}$ and such that $B^{2k+3} \cap (S^{2k} \times B^{2k+2}) = \emptyset$. Call $\partial B^{2k+3}$, $Z$. Suppose that the linking number of $Y$ and $Z$ is one. Attach a $(4k + 5)$-dimensional $(2k + 2)$-(resp.
(2k+1)-handle to $B^{4k+4}$ along $Y$ (resp. $Z$) with the trivial framing. Note that these two handles are attached to $S^{4k+4}$ on time. Carry out surgeries by using these two handles on $S^{4k+4}$. Then the new manifold which we obtain is the $(4k+4)$-sphere again, and call it $S^{4k+4}$. Furthermore the new submanifold $\subset S^{4k+4}$ which is made from $K$ is the trivial $(4k+2)$-knot $T$.

Note that now we have a compact oriented $(4k+5)$-dimensional manifold $W$ with a handle decomposition

$W = (S^{4k+4} \times [0,1]) \cup (a (4k + 5)-dimensional (2k + 2)-handle)$

$\cup (a (4k + 5)-dimensional (2k + 3)-handle) \cup (S^{4k+4} \times [0,1]).$

Note that $\partial W = (S^{4k+4} \times \{0\}) \cap (S^{4k+4} \times \{1\}).$ Note that there is an embedding map $f : S^{4k+2} \times [0,1] \hookrightarrow W$ with the following properties:

1. $f(S^{4k+2} \times [0,1]) \cap (S^{4k+4} \times \{0\})$ is $f(S^{4k+2} \times \{0\}).$ $f(S^{4k+2} \times [0,1]) \cap (S^{4k+4} \times \{1\})$ is $f(S^{4k+2} \times \{1\}).$

2. $f$ is transverse to $\partial W.$

Take a 3-fold branched covering space $\tilde{W}$ of $W$ along $f(S^{4k+2} \times [0,1]).$ Note the $(2k+1)$-sphere which is the core of the attaching part of the $(2k+2)$-handle in the above
handle decomposition of $W$. The $(2k + 1)$-sphere is null-homologous in $S^{4k+4} - N(K)$, where $N(K)$ is the tubular neighborhood of $K$ in $S^{4k+4}$. Therefore we obtain a compact oriented $(4k + 5)$-dimensional manifold $\widehat{W}$ with a handle decomposition

$$\widehat{W} = (M_3(K) \times [0, 1])$$

$$\cup (\text{three } (4k + 5)\text{-dimensional } (2k + 2)\text{-handles}, h_1^{2k+2}, h_2^{2k+2}, h_3^{2k+2})$$

$$\cup (\text{three } (4k + 5)\text{-dimensional } (2k + 3)\text{-handle}, h_1^{2k+3}, h_2^{2k+3}, h_3^{2k+3})$$

$$\cup (S^{4k+4} \times [0, 1]).$$

Here, note that the 3-fold branched covering space of $S^{4k+4}$ along $T$ is the standard $(4k + 4)$-sphere, and call it $S^{4k+4}$ again.

We prove that $H_{2k+1}(\widehat{W}; \mathbb{Z}) \cong 0$. 

**Reason.** Take the dual handle decomposition

$$\widehat{W} = (S^{4k+4} \times [0, 1])$$

$$\cup (\text{the three } (4k + 5)\text{-dimensional } (2k + 2)\text{-handles}, \overline{h}_1^{2k+2}, \overline{h}_2^{2k+2}, \overline{h}_3^{2k+2})$$

$$\cup (\text{the three } (4k + 5)\text{-dimensional } (2k + 3)\text{-handle}, \overline{h}_1^{2k+3}, \overline{h}_2^{2k+3}, \overline{h}_3^{2k+3})$$

$$\cup (M_3(K) \times [0, 1]),$$

of the above handle decomposition, where $\overline{h}_\#$ is the dual handle of $h_\#^{4k+5-\ast}$. Take a manifold $Q_S$ which is represented by the sub-handle-decomposition $Q_S$

$$= (S^{4k+4} \times [0, 1]) \cup (\text{the three } (4k + 5)\text{-dimensional } (2k + 2)\text{-handles}, \overline{h}_1^{2k+2}, \overline{h}_2^{2k+2}, \overline{h}_3^{2k+2})$$

of the dual handle decomposition of $\widehat{W}$. Since $H_{2k+1}(S^{4} \times [0, 1]; \mathbb{Z}) \cong 0$, we have $H_{2k+1}(Q_S; \mathbb{Z}) \cong 0$. Recall that if we attach $(2k + 3)$-handles to a manifold $E$ and we obtain a new manifold $E'$, then $H_{2k+1}(E; \mathbb{Z}) \cong H_{2k+1}(E'; \mathbb{Z})$.

Therefore the manifold $R_S$ which is represented by the sub-handle-decomposition $R_S$

$$(M_3(K) \times [0, 1]) \cup (\text{the three } (4k + 5)\text{-dimensional } (2k + 2)\text{-handles}, h_1^{2k+2}, h_2^{2k+2}, h_3^{2k+2})$$

of the above handle decomposition satisfies the condition $H_{2k+1}(R_S; \mathbb{Z}) \cong 0$.

Therefore the cores of the attaching parts of $h_1^{2k+2}, h_2^{2k+2}$ and $h_3^{2k+2}$ generate $H_{2k+1}(M_3(K); \mathbb{Z})$.

This completes the proof of Proposition 7.1.

In a similar fashion we can prove the following.

**Proposition 7.2.** Let $k \in \mathbb{N} \cup \{0\}$. Let $n \in \mathbb{N}$. Let $K$ be a $(4k + 2)$-knot $\subset S^{4k+4}$ whose $(2k+1, 2k+2)$-pass-move-unknotting-number of $K$ is $\leq n$. Let $M_3(K)$ be the 3-fold branched covering space of $S^{4k+4}$ along $K$. Then there are $3n$ elements $\in H_{2k+1}(M_3(K); \mathbb{Z})$ which generate $H_{2k+1}(M_3(K); \mathbb{Z})$.

**Claim 7.3.** Let $k \in \mathbb{N} \cup \{0\}$. There is a $(4k + 2)$-knot $P \subset S^{4k+4}$ as follows.

1. A Seifert hypersurface $V$ for $P$ is diffeomorphic to

$$((S^{2k+1} \times S^{k+2})_\#(S^{2k+1} \times S^{2k+2})) - \text{open } B^{4k+3}.$$
\( h^0 \) denotes a \((4k + 2)\)-dimensional 0-handle. 
\( h^*_i (i = 1, 2 \text{ and } * = 2k + 1, 2k + 2) \) denotes a \((4k + 2)\)-dimensional \(*\)-handle. 
\( h^{2k+1}_i (i = 1, 2) \) corresponds \( x_i \).
\( h^{2k+2}_i (i = 1, 2) \) corresponds \( y_i \).

**Figure 7.2. A Seifert hypersurface \( V_T \) for a \((4k + 2)\)-knot \( T \)**

For an ordered set \((x_1, x_2)\) of basis of \( H_{2k+1}(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \) and an ordered set \((y_1, y_2)\) of basis of \( H_{2k+2}(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \), the intersection matrix \((x_k \cdot y_l) (k, l \in \{1, 2\})\) on \( H_{2k+1}(V; \mathbb{Z}) \) (resp. \( H_{2k+2}(V; \mathbb{Z}) \)) is \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]
We can suppose that Poincaré dual of \( x_1 \) (resp. \( x_2 \)) is \( y_2 \) (resp. \(-y_1\)).

(2) The Seifert matrix \((\text{lk}(x_k, y_l^+)\)) for \( P \) is \( X = \begin{pmatrix}
-1 & 1 \\
0 & -1
\end{pmatrix}.\)

**Note.** The negative Seifert matrix related to \( X \) is the transposed matrix of \( X \). Recall that a \((2k + 1)\)-positive Seifert matrix of \((4k + 2)\)-knot is not the transposed matrix of its related negative Seifert matrix in general (see Definitions 4.3 and 4.4, and Propositions 4.5 and 4.6). Note that we have the following:
\[
X^{-1} = \begin{pmatrix}
-1 & -1 \\
0 & -1
\end{pmatrix}, \quad XX^{-1} = \begin{pmatrix}
1 & 1 \\
-1 & 0
\end{pmatrix}, \quad (XX^{-1})^2 = \begin{pmatrix}
0 & 1 \\
-1 & -1
\end{pmatrix}, \quad (XX^{-1})^3 = \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}.
\]

**Proof of Claim 7.3** See Figure 7.2. Embed \(((B^{2k+2} \times S^{2k+2}) \# (B^{2k+2} \times S^{2k+2}))\) in \( S^{4k+4} \). Note that its boundary is diffeomorphic to \(((S^{2k+1} \times S^{2k+2}) \# (S^{2k+1} \times S^{2k+2}))\). Remove
an open $B^{4k+3}$ from it. We can suppose that this
\((S^{2k+1} \times S^{2k+1}) \# (S^{2k+1} \times S^{2k+1})\) is a Seifert hypersurface $V_T$ for the trivial
\((4k+2)\)-knot. We can take an ordered set of basis $\{x_1, x_2\}$ (resp. $\{y_1, y_2\}$) of $H_{2k+1}(V_T; \mathbb{Z})$
(resp. $H_{2k+2}(V_T; \mathbb{Z})$) which satisfies (1) of Claim 7.3. Furthermore we can suppose
that the Seifert matrix $(\text{lk}(x_k, y_i^\pm))$ associated with $V_T$ and this ordered set of basis is

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. $$

Note that $V_T$ has a handle decomposition

\((0\text{-handle}) \cup (\text{two} \ (2k+3)\text{-dimensional} \ (2k+1)\text{-handles, } h_1^{2k+1} \text{ and } h_2^{2k+1}) \\cup (\text{two} \ (2k+3)\text{-dimensional} \ (2k+2)\text{-handles, } h_1^{2k+2} \text{ and } h_2^{2k+2}).$$

By two times of $(2k+1, 2k+2)$-pass-move we change the submanifold $V_T$ into the
submanifold $V$ so that $V$ satisfies (1) and (2) of Claim 7.3.

This completes the proof of Claim 7.3 \[\Box\]

**Note.** We can say that the $(4k+2)$-knot $P$ in Claim 7.3 is the knot product of the
$2$-knot $P$ in [4] and $k$ copies of the Hopf link. See [8] for the knot product.

In Proof of Claim 7.3 we also prove that the $(2k+1, 2k+2)$-pass-move-unknotting-
number of $P$ in Proof of Claim 7.3 is $\leq 2$. (Another proof is given by using Main Theorem
2.6 of [9].)

Let $M_3(P)$ be the $3$-fold branched covering space of $S^{4k+4}$ along $P$.

By Proposition 4.7 we have $H_{2k+1}(M_3(P); \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence we need no
less than four generators in order to generate $H_{2k+1}(M_3(P); \mathbb{Z})$.

Suppose that the $(2k+1, 2k+2)$-pass-move-unknotting-number of $P$ is $\leq 1$. By Proposition 7.1 we can take three generators in order to generate $H_{2k+1}(M_3(P); \mathbb{Z})$. We arrived at a contradiction.

Therefore the $(2k+1, 2k+2)$-pass-move-unknotting-number of $P$ is $\geq 2$.

Therefore the $(2k+1, 2k+2)$-pass-move-unknotting-number of $P$ is two.

This completes the proof of Theorem 6.5(1).

Let $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $\frac{2m}{3} > n$. Let $\#^m P$ be the connected-sum of $m$-copies of $P$.

Since $P$ is $(2k+1, 2k+2)$-pass-move equivalent to the trivial $(4k+2)$-knot, $\#^m P$ is
$(2k+1, 2k+2)$-pass-move equivalent to the trivial $(4k+2)$-knot.

Let $N_3(\#^m P)$ be the $3$-fold branched covering space of $S^4$ along $\#^m P$. By Proposition 4.7 we have $H_1(M_3(\#^m P); \mathbb{Z}) \cong 3^m \mathbb{Z}_2$. Hence we need no less than $2m$ generators in order to generate $H_1(M_3(P); \mathbb{Z})$.

Suppose that the $(2k+1, 2k+2)$-pass-move-unknotting-number of $\#^m P$ is $\leq n$. By Proposition 7.2 $H_1(M_3(\#^m P); \mathbb{Z})$ can take $3n$ generators. Since $2m > 3n$, we arrived at a contradiction.

Therefore the $(2k+1, 2k+2)$-pass-move-unknotting-number of $\#^m P$ is $> n$. 31
This completes the proof of Theorem 6.5. (2).
This completes the proof of Theorem 6.5. □

8. Proof of Theorem 6.7

Proposition 8.1. Let $k \in \mathbb{N} \cup \{0\}$. Let $J$ be a $(4k+1)$-knot $\subset S^{4k+3}$ whose $(2k + 1, 2k + 1)$-pass-move-unknotting-number is one. Let $N_3(J)$ be a 3-fold branched covering space of $S^{4k+3}$ along $J$. Then there are six elements $\in H_{2k+1}(N_3(J); \mathbb{Z})$ which generate $H_{2k+1}(N_3(J); \mathbb{Z})$.

Proof of Proposition 8.1. Take a $(4k + 3)$-ball $B^{4k+3} \subset S^{4k+3}$ where we carry out the pass-move which changes $J$ into $T$. See Figure 8.1. Note that $J \cap B^{4k+3}$ is regarded as $(S^{2k} \times B^{2k+1}) \amalg (S^{2k} \times B^{2k+1})$. Call one of the two $S^{2k} \times B^{2k+1}, A_1$, and the other $A_2$. Take two $(2k + 2)$-balls, $B_1^{2k+2}$ and $B_2^{2k+2}$ in the $(4k + 3)$-ball $B^{4k+3}$ such that $B_i^{2k+2} \cap A_i$ is a 2$k$-sphere trivially embedded in $B_i^{2k+2}$ and such that $B_i^{2k+2} \cap A_j = \phi$ ($i = 1, 2$, and $i \neq j$). Call $\partial B_i^{2k+2}, Y_i$ $(i = 1, 2)$. Suppose that the linking number of $Y_1$ and $Y_2$ is one. Attach a $(4k + 4)$-dimensional $(2k + 2)$-handle to $B^3$ along $Y_i$ with the trivial framing $(i = 1, 2)$. Note that these two handles are attached to $S^{4k+3}$ on time. Carry out surgeries by using these two handles on $S^{4k+3}$. Then the new manifold which we obtain is the $(4k + 3)$-sphere again, and call it $S^{4k+3}$. Furthermore the new submanifold $\subset S^{4k+3}$ which is made from $J$ is the trivial $(4k + 1)$-knot $T$.

Note that we now have a compact oriented $(4k + 4)$-dimensional manifold $U$ with a handle decomposition.
\[ U = (S^{4k+3} \times [0, 1]) \cup \text{(two (4k + 3)-dimensional (2k + 2)-handles)} \cup (S^{4k+3} \times [0, 1]). \]

Note that \( \partial U = (S^{4k+3} \times \{0\}) \cup (S^{4k+3} \times \{1\}) \). There is an embedding map \( f: S^{4k+1} \times [0, 1] \hookrightarrow U \) with the following properties:

1. \( f(S^{4k+1} \times [0, 1]) \cap (S^{4k+3} \times \{0\}) = f(S^{4k+1} \times [0, 1]) \cap (S^{4k+3} \times \{1\}) = f(S^{4k+1} \times \{1\}) \).
2. \( f \) is transverse to \( \partial U \).

Therefore we obtain a compact oriented \((4k + 4)\)-dimensional manifold \( \tilde{U} \) with a handle decomposition

\[ \tilde{U} = (N_3(J) \times [0, 1]) \cup (\text{(six (4k + 4)-dimensional (2k + 2)-handles,} h_1^{2k+2}, ..., h_6^{2k+2}) \cup (S^{4k+3} \times [0, 1]). \]

Here, note that a 3-fold branched covering space of \( S^{4k+3} \) along \( T \) is the standard \((4k + 3)\)-sphere, and call it \( S^{4k+3} \) again.

We prove that \( H_{2k+1}(\tilde{U}; \mathbb{Z}) \cong 0 \). \textit{Reason.} Take the dual handle decomposition

\[ \tilde{U} = (S^{4k+3} \times [0, 1]) \cup (\text{(six (4k + 4)-dimensional (2k + 2)-handles,} \overline{h_1^{2k+2}}, ..., \overline{h_6^{2k+2}}) \cup (N_3(J) \times [0, 1]), \]

of the above handle decomposition, where \( \overline{h_1^{2k+2}} \) is the dual handle of \( h_1^{2k+2} \). Since \( H_{2k+1}(S^{3} \times [0, 1]; \mathbb{Z}) \cong 0 \), we have \( H_{2k+1}(\tilde{U}; \mathbb{Z}) \cong 0 \).

Therefore the cores of the attaching parts of \( h_1^{2k+2}, ..., h_6^{2k+2} \) generate \( H_{2k+1}(N_3(J); \mathbb{Z}) \).

This completes the proof of Proposition \[\] 8.1.

In a similar way, we can prove the following.

\textbf{Proposition 8.2.} Let \( k \in \mathbb{N} \cup \{0\} \). Let \( n \in \mathbb{N} \). Let \( J \) be a \((4k + 1)\)-knot \( \subset S^{4k+3} \) whose \((2k + 1, 2k + 1)\)-pass-move-unknotting-number is \( \leq n \). Let \( N_3(J) \) be a 3-fold branched covering space of \( S^{4k+3} \) along \( J \). Then there are \( 6n \) elements \( \in H_{2k+1}(N_3(J); \mathbb{Z}) \) which generate \( H_{2k+1}(N_3(J); \mathbb{Z}) \).

Let \( R \) be a \((4k + 1)\)-knot \( \subset S^{4k+3} \) whose Seifert hypersurface \( V \) is diffeomorphic to \( (S^{2k+1} \times S^{2k+1}) - B^{4k+2} \). See Figure 8.2. Take an ordered set \( (x_1, x_2) \) of basis of \( H_{2k+1}(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \) with the following properties:
Figure 8.2. A $(4k+1)$-knot $R$

(1) The intersection matrix $(x_k \cdot x_l)$ $(k, l \in \{1, 2\})$ on $H_{2k+1}(V; \mathbb{Z})$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that Poincaré dual of $x_1$ is $y_2$.

(2) The Seifert matrix $(\text{lk}(x_k, y_l^+))$ for $R$ is $X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

Therefore we have the following:

\[
X^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad (X^{-1})^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (X^{-1})^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

It is true that this knot exists. See e.g. [14]. Furthermore we can say that this $(4k+1)$-knot $R$ is the knot product of the trefoil knot and $k$ copies of the Hopf link. See [8] for the knot product.

Let $C = (R\#(-R^*))\#(R\#(-R^*))$. Note that Arf $C = 0$. By [19], $C$ is $(2k+1, 2k+1)$-pass-move-equivalent to the trivial $(4k+1)$-knot.

By Proposition 4.7 $H_{2k+1}(N_3(C); \mathbb{Z}) \cong \mathbb{Z}^8$. Hence we need no less than eight generators to generate $H_{2k+1}(N_3(C); \mathbb{Z})$.

Suppose that the $(2k+1, 2k+1)$-pass-move-unknotting-number of $C$ is $\leq 1$. By Proposition 8.1 $H_{2k+1}(N_3(C); \mathbb{Z})$ can take six generators. We arrived at a contradiction.

Therefore the $(2k+1, 2k+1)$-pass-move-unknotting-number of $C$ is $\geq 2$.

We prove that the $(2k+1, 2k+1)$-pass-move-unknotting-number of $R\#(-R^*)$ is one. **Reason:** A $(2k+1)$-Seifert matrix of the $(4k+1)$-knot $R\#(-R^*)$ is the same as a 1-Seifert matrix of the uppermost 1-knot in Figure 1.2. (We have $R\#(-R^*)$)
\((\text{the uppermost } 1\text{-knot in Figure } 1.2)^{\otimes k}(\text{the Hopf link}), \text{ where } \otimes \text{ denotes the knot product which is defined in } [5, 7].\) One \((2k + 1, 2k + 1)\)-pass-move can change \(R\#(-R^*)\) into a \((4k + 1)\)-knot \(J_T\) whose Seifert matrix is the same as that of the lower most 1-knot in Figure 1.2. Since the lower most 1-knot in Figure 1.2 is the trivial 1-knot, the Seifert matrix of \(J_T\) is \(S\)-equivalent to that of the trivial \((4k + 1)\)-knot (See [15] for \(S\)-equivalence.) By [15], \(J_T\) is the trivial \((4k + 1)\)-knot. Hence the \((2k + 1, 2k + 1)\)-pass-move-unknotting-number of \(R\#(-R^*)\) is one. (Another proof is given by using Main Theorem 2.4 of [9].)

Therefore the \((2k + 1, 2k + 1)\)-pass-move-unknotting-number of \(C\) is \(\leq 2\).

Therefore the \((2k + 1, 2k + 1)\)-pass-move-unknotting-number of \(C\) is two.

This completes the proof of Theorem 6.7.(1).

Let \(n \in \mathbb{N}\). Let \(\#^n C\) be the connected-sum of \(n\) copies of \(C\). Since \(C\) is \((2k + 1, 2k + 1)\)-pass-move-equivalent to the trivial \((4k + 1)\)-knot, \(\#^n C\) is \((2k + 1, 2k + 1)\)-pass-move-equivalent to the trivial \((4k + 1)\)-knot.

Let \(N_3(\#^n C)\) be the 3-fold branched cyclic covering space of \(S^{4k+3}\) along \(\#^n C\). By Proposition 4.7, \(H_{2k+1}(N_3(\#^n C); \mathbb{Z}) \cong \oplus^{8n} \mathbb{Z}_2\). We need no less than \(8n\) generators which generate \(H_{2k+1}(N_3(\#^n C); \mathbb{Z})\).

Suppose that the \((2k + 1, 2k + 1)\)-pass-move-unknotting-number of \(\#^n C\) is \(\leq n\). By Proposition 8.2 we can prove that \(H_{2k+1}(N_3(\#^n C); \mathbb{Z})\) can take \(6n\) generators. We arrived at a contradiction.

Therefore the \((2k + 1, 2k + 1)\)-pass-move-unknotting-number of \(\#^n C\) is \(> n\).

This completes the proof of Theorem 6.7.(2).

This completes the proof of Theorem 6.7. \(\square\)

**Note 8.3.** Theorem 6.7 is a high-dimensional analogue of Theorem 1.5. Since Theorem 1.5 has a condition on the crossing-change-number, we naturally hope to impose a condition of a local-move which is a generalization of the crossing-change, on Theorem 6.7. This is discussed in Note 9.6.

### 9. The twist-move on high-dimensional knots

Let \(p \in \mathbb{N} \cup \{0\}\). We review the definition of the twist-move on \((2p + 1)\)-dimensional closed oriented submanifold \(\subset S^{2p+3}\), which is defined in [23] and which is researched in [8, 10, 23]. ([23] calls the twist-move the \(XXII\)-move.) If \(p = 0\), the twist-move on \((4p + 1)\)-dimensional closed oriented submanifold \(\subset S^{2p+3}\) is the crossing-change on 1-links.

**Definition 9.1.** Let \(p \in \mathbb{N} \cup \{0\}\). Regard a \((2p + 3)\)-ball \(B = D^{2p+3}\) as \(D^1 \times D^{p+1} \times D^{p+1}\). Let \(D^1 = [-1, 1] = \{t | -1 \leq t \leq 1\}\). Take a \(p\)-ball \(D^{p+1}_S\) embedded in \(\text{Int}D^{p+1}\). Let a submanifold \(\{0\} \times D^{p+1} \times D^{p+1}_S \subset B\) be called \(h_+\). We give an orientation to \(h_+\). Take
a submanifold \( h_- \subset B \) which is diffeomorphic to \( h_+ \). Let \( h_+ \cap h_- = h_+ \cap (\partial B) \). Let \( h_- - (\partial B) \subset \{ t < 0 \} \times D^{p+1} \times D^{p+1} \). (See Note (1) below.) We give an orientation to \( h_- \) so that \( h_+ \cup h_- \) is an oriented submanifold \( \subset B \) if we give the opposite orientation to \( h_- \). We can regard \( h_+ \cup h_- \) as a Seifert hypersurface for \( \partial (h_+ \cup h_-) \). We can suppose that a \((p + 1)\)-Seifert matrix for a \((2p + 1)\)-dimensional closed oriented submanifold \( \partial (h_+ \cup h_-) \subset B \) associated with a Seifert hypersurface \( h_+ \cup h_- \) is (1). (We can define Seifert hypersurfaces in \( B \) and their Seifert matrices in the same fashion as ones in the \( S^n \) case. Each of Figure 9.1 and Figure 9.2 draws a diagram of the twist-move. See Note (2) below.)

Let \( p \in \mathbb{N} \cup \{0\} \). Let \( K_+ \) and \( K_- \) be \((2p + 1)\)-dimensional closed oriented submanifold \( \subset S^{2p+3} \). Take \( B \) in \( S^{2p+3} \). Let \( K_+ \) and \( K_- \) differ only in \( B \). Let \( K_+ \) (resp. \( K_- \)) satisfy the condition \( K_+ \cap \text{Int} B = (\partial h_+ - \partial B) \) (resp. \( K_- \cap \text{Int} B = (\partial h_- - \partial B) \)), where we suppose that there is not \( h_- \) (resp. \( h_+ \)) in \( B \). Then we say that \( K_+ \) (resp. \( K_- \)) is obtained from \( K_- \) (resp. \( K_+ \)) by one \((\text{positive-})\text{twist-move} \) (resp. \((\text{negative-})\text{twist-move} \) in \( B \).

In Definition 9.1 we have the following: Let \( \sharp \in \{+, -\} \). there is a Seifert hypersurface \( V_\sharp \subset S^{2p+3} \) for \( K_\sharp \) such that \( V_\sharp \cap B = h_\sharp \). (The idea of the proof is Thom-Pontrjagin construction.) We say that \( V_- \) (resp. \( V_+ \)) is obtained from \( V_+ \) (resp. \( V_- \)) by one \((\text{positive-})\text{twist-move} \) (resp. \((\text{negative-})\text{twist-move} \) in \( B \).

**Note.** (1) \([3, 4, 30, 31]\) etc. imply that we can move the core of \( h_- \) to the core of \( h_+ \) in \( B \) by an isotopy keeping \( \partial (\text{the core of } h_-) \).

(2) Figure 9.1 which consists of the two figures (1) and (2), is a diagram of the twist-move. In Figure 9.1(2), we move \( \partial h_- - \partial B \) by isotopy and draw \( \partial h_- - \partial B \). The upper half of Figure 9.2 is another diagram of the twist-move. Compare the upper half of Figure 9.2 and the lower half. If \( p = 0 \) (hence \( n = 2p + 1 = 1 \)), the left figure in the upper half and that in the lower half are the same. That is, if \( p = 0 \), the twist-move on \((2p + 1)\)-closed oriented submanifold \( \subset S^{2p+3} \) is the crossing-change on 1-links. Note that \( 'B \cap K_0 \in the left B in the upper half of Figure 9.2 in the p = 0 case' and 'B \cap K_0 in the left B in the lower half of Figure 9.2 are the same (\text{Reason. Use an isotopy.}) See also Figure 9.3."

In Definition 9.1 note the following: Let \( \sharp \in \{+, -\} \). Let \( V_0 = V_\sharp - \text{Int} B \) = ‘the closure of \((V_\sharp - (h_\sharp)) \) in \( S^{2p+3} \)’. We can say that we attach an embedded \((2p + 2)\)-dimensional \((p+1)\)-handle \( h_\# \) to the submanifold \( V_0 \subset S^{2p+3} \), and obtain the submanifold \( V_\# \subset S^{2p+3} \).
This cube is $D^{2p+3} = B$. 

$B \cap K_+ 

\textbf{Figure 9.1.}(1) \textbf{The twist-move-triple}

\textbf{Figure 9.1.}(2) \textbf{The twist-move-triple}
the pair of two \( \bigcirc \) makes a crossing-change on a 1-dimensional link.

**Figure 9.2.** The twist-move on 1-knots is the crossing-change-triple on 1-knots.

**Definition 9.2.** Let \( p \in \mathbb{N} \cup \{0\} \). Let \( K \) be a \((2p+1)\)-knot \( \subset S^{2p+3} \) which is twist-move-equivalent to the trivial \((2p+1)\)-knot. The *twist-move-unknotting-number* of \( K \) is the minimal number of twist-moves which we change \( K \) to the trivial \((2p+1)\)-knot by.

**Proposition 9.3.** There is a \((2p+1)\)-knot whose twist-move-unknotting-number is one for a natural number \( p \).

**Proof of Proposition 9.3.** Let \( k \in \mathbb{N} \cup \{0\} \). Take a \((4k+1)\)-knot \( K \) with a \((2k+1)\)-Seifert matrix \[ \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \]. See Figure 9.3.
Figure 9.3. **One twist-move changes a nontrivial \((4k+1)\)-knot into the trivial \((4k+1)\)-knot**

Use the van Kampen theorem, the Mayor-Vietoris exact sequence, and \([28]\). \(K\) is PL homeomorphic to the standard sphere. (Note: Let \(J\) be a \((2p+1)\)-knot with a \((p+1)\)-Seifert matrix \(\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\). Then \(J\) is not a homology sphere if \(p\) is an odd natural number.

**Reason.** Use the Mayor-Vietoris exact sequence.)

By \([15]\), \(K\) is a nontrivial spherical knot.

We carry out one twist-move and obtain a \((4k+1)\)-knot \(T\) with a \((2k+1)\)-Seifert matrix \(\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\). See Figure 9.3. By \([15]\), \(T\) is the trivial knot.

This completes the proof of Proposition 9.3.

**Note.** By \([11]\), we have the following: \(K\) in Figure 9.3 is diffeomorphic to the standard sphere if \(bP_{4k+2}\) is the trivial group. \(K\) in Figure 9.3 is diffeomorphic to an exotic sphere if \(bP_{4k+2}\) is nontrivial. See \([11]\) for the bP-subgroup.

We consider the following problem.

**Problem 9.4.** Let \(k \in \mathbb{N} \cup \{0\}\).

1. Is there a twist-move-unknotting-number-two \((4k+1)\)-knot?
2. For any natural number \(n\), is there a \((4k+1)\)-knot whose twist-move-unknotting-number is \(> n\)?

We give a positive answer to Problem 9.4(1) (resp. 9.4(2)). The answers make one of our main theorems.
Theorem 9.5. Let \( k \in \mathbb{N} \cup \{0\} \). Let \( n \in \mathbb{N} \). There is a \((4k + 1)\)-knot whose twist-move-unknotting-number is \( n \).

Proof of Theorem 9.5. The \( 4k + 1 = 1 \) case holds because of \cite{15} Theorem 10.1 in page 420. Let \( n \in \mathbb{N} \). The ordinary-unknotting-number, which is the twist-move-number, of the connected-sum of \( n \)-copies of the trefoil knot is \( n \).

The \( 4k + 1 \geq 5 \) case is proved in the same fashion as one in \cite{15} Proof of Theorem 10.1 in page 420 and (2.4) in page 389. Let \( k \in \mathbb{N} \cup \{0\} \). Take the \((4k + 1)\)-knot \( K \) in Figure 9.3 of Proof of Proposition 9.3. The twist-move-number of the connected-sum of \( n \)-copies \( \sharp nK \) of \( K \) is \( n \). \( \square \)

Note. By \cite{11}, we have the following: Let \( n \in \mathbb{N} \). Let \( k \in \mathbb{N} \cup \{0\} \). In the case where \( bP_{4k+2} \) is nontrivial and \( n \) is odd, \( \sharp nK \) is diffeomorphic to an exotic sphere. In the other case, \( \sharp nK \) is diffeomorphic to the standard sphere.

Note 9.6. We continue Note 8.3. By the discussion in this section, we can prove that the twist-move-unknotting-number of \( C \) in the previous section is 4 and that of \( \sharp nC \) is \( 4n \). It is natural to ask whether the twist-move-unknotting-number of any \((4k + 1)\)-knot \( K \) is \( \leq 4n \) if the \((2k + 1, 2k + 1)\)-pass-move-unknotting-number of \( K \) is \( n \).

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Eiji Ogasa: Computer Science, Meijigakuin University, Yokohama, Kanagawa, 244-8539, Japan pqr100pqr100@yahoo.co.jp ogasa@mail1.meijigakuin.ac.jp