Matrix Theory for the DLCQ of Type IIB String Theory on the AdS/Plane-wave

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Abstract

We propose a recipe to construct the DLCQ Hamiltonian of type IIB string theory on the AdS (and/or plane-wave) background. We consider a system of $J$ number of coincident unstable non-BPS D0-branes of IIB theory in the light-cone gauge and on the plane-wave background with a compact null direction, the dynamics of which is described by the world-line $U(J)$ gauge theory. This configuration suffers from tachyonic instabilities. Having instabilities been cured through the process of open string tachyon condensation, by expanding the theory about true minima of the effective potential and furthermore taking low energy limit to decouple the heavy modes, we end up with a 0+1-dimensional supersymmetric $U(J)$ gauge theory, a Matrix Theory. We conjecture that the Hamiltonian of this Matrix Theory is just the DLCQ Hamiltonian of type IIB string theory on the AdS or equivalently plane-wave background in a sector with $J$ units of light-cone momentum. We present some pieces of evidence in support of the proposal.

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1 Introduction and Motivation

Over the course of time and through revolutionary progress, the question of what is string theory, has been changed to what is M-theory. M-theory has not been fully formulated. The only proposal yet for its quantum definition, \textit{i.e.} Matrix theory, is based on describing the theory in terms of its Hamiltonian in the discrete light-cone quantization (DLCQ), which is a simple $J$-body quantum mechanical Hamiltonian $H_{J}^{\text{DLCQ}}$.

There exists a 1-parameter family of 0+1-dimensional $U(J)$ gauge theories in the form of supersymmetric Matrix quantum mechanics which is conjectured to give the DLCQ description of M-theory in the sector with $J$ units of light-cone momentum on the plane-wave background, called the BMN Matrix theory \cite{1}. The action is basically describing or described by the dynamics of $J$ BPS D0-branes of type IIA, which are gravitons from the eleven dimensional viewpoint \cite{2}. The parameter $\mu$, coming from the background, characterizes a homotopy where at its starting point $\mu = 0$, sits the seminal BFSS Matrix theory as the DLCQ of M-theory on the flat background which is the 0+1-dimensional SYM theory with 16 supercharges \cite{3}. The extra adjustable parameter makes the theory fascinatingly tractable which is useful in taking limits or performing perturbative expansion. It removes all the flat directions in the effective potential and puts barriers which makes it possible to distinguish single and multi-particle states. It also implies discrete spectrum and isolated set of normalizable vacua in the form of fuzzy spheres which in the M-theory (continuum) limit $J \to \infty$, have interpretation of spherical M2-brane giant gravitons \cite{4}.

Albeit the DLCQ procedure was used to give a non-perturbative and second-quantized definition of M-theory, in principle it can be employed to define any string theory by giving a recipe how to construct the DLCQ Hamiltonian, $H_{J}^{\text{DLCQ}}$ \cite{5}. Furthermore, it is believe that the idea of DLCQ is most natural in the context of Matrix theory \cite{6}.

The DLCQ is in fact the light-cone quantization of a theory while the null direction is compactified. In the light-cone frame the basic coordinates are $X^+, X^-, X^I$ with conjugate momenta $P^-, p^+, P^I$. The effect of compactification $X^+ \sim X^+ + 2\pi R_-$ is to discretize the spectrum of its conjugate momentum $p^+ = J/R_-$. It provides a convenient IR regulator for the theory. Moreover, if $p^+$ is positive and conserved, the Fock space of the system splits into an infinite number of superselection sectors characterized by $J$. In each sector states with $p^+ = J/R_-$ can have at most $J$ particles in them each carrying at least one unit of momentum. Thus, the DLCQ of a theory in
a given sector reduces to a quantum mechanics with fixed number of particles \[5, 7\]. Hence if we have a theory governing the dynamics of these \(J\) partons, by definition, it gives the dynamics of the original theory in the sector \(J\) of its DLCQ. It has been argued that the dynamics of these \(J\) partons is mapped into the dynamics of \(J\) KK modes of the same theory compactified on a space-like circle of radius \(R\) once the limit \(R \to 0\) is taken \[5\]. What would be needed, inspired by some theory of partons, is to give a recipe for constructing this \(J\)-body quantum mechanical Hamiltonian.

The aim of this note is to propose a recipe to construct the DLCQ Hamiltonian \(H_{\text{DLCQ}}^J\) of type IIB string theory on the AdS/plane-wave backgrounds. In the light of the above facts and inspired by the web of dualities, we consider a system of \(J\) number of non-BPS D0-Branes of type IIB string theory in the light-cone gauge on the plane-wave background with a compact null direction, the dynamics of which is described by the world-line \(U(J)\) gauge theory.

Besides usual stable BPS \(Dp\)-branes for odd \(p\), spectrum of type IIB theory contains unstable non-BPS \(Dp\)-branes with even \(p\) \[8\]. They do not carry any net RR charge, so because of their tension they are unstable to decay to lighter neutral closed string states. Fluctuations of the brane are captured by open strings ending on it, so it is a natural expectation that instability of the brane can also be encoded in open strings. In fact, unstable branes contain tachyon in the open string spectrum on their world-volume that is not removed by the usual GSO projection. The tachyonic mode is a Higgs type excitation which spontaneously acquires a vacuum expectation value and develops a stable state.

Although our system of interest is unstable, we show it could be tractably stabilized. Indeed, the collection of non-BPS D0-branes in the plane-wave background develops a non-trivial effective potential for transverse coordinates and tachyon field which demonstrates tachyonic instabilities. In fact this setup suffers from two sorts of tachyonic instabilities which originate from each non-BPS D0-brane separately and collectively a bunch of them in the presence of the RR flux. They are respectively represented by tachyon field in the spectrum of an open string connecting a brane to itself and tachyonic modes of off-diagonal elements of the transverse scalars which represent open strings stretching between the branes. These instabilities are cured through the process of open string tachyon condensation and the system dynamically stabilizes and falls into true minima of the effective potential. By expanding the theory about the true minima and taking a low energy limit to decouple the heavy fluctuations, we end up with a 1-parameter family of 0+1-dimensional supersymmetric \(U(J)\) gauge theories, a Matrix theory.
We propose that the Hamiltonian of this Matrix Theory is just the DLCQ Hamiltonian of type IIB string theory on the AdS or equivalently plane-wave backgrounds in a sector with $J$ units of light-cone momentum. This Matrix theory provides a non-perturbative and second-quantized formulation of type IIB string theory and would give an alternative definition and a better understanding of this theory.

There is yet another proposal to construct the DLCQ Hamiltonian of type IIB string theory on the AdS/plane-wave backgrounds. It has been noted that the right probe in the presence of RR flux is spherical D3-brane giant gravitons. Thus it is a natural expectation that quantum completion of type IIB on this background comes from quantum D3-brane theory. In [9] a prescription is given to regularize light-cone D3-brane theory via discretizing it world-volume. It has been conjectured that time-independent volume-preserving diffeomorphism is isomorphic to $U(\infty)$ and can be truncated to $U(J)$. Thus DLCQ of a D3-brane on the plane-wave background leads to a 0+1-dimensional supersymmetric $U(J)$ gauge theory, a Matrix theory, which is proposed to be the DLCQ of type IIB string theory in the sector with $J$ units of light-cone momentum on the AdS/plane-wave backgrounds. It is named tiny graviton Matrix theory (TGMT). we will see that these two proposal exactly coincide.

This paper is organized as follows; in section 2 we study the system of interest, a system of $J$ number of coincident non-BPS D0-branes of type IIB string theory on plane-wave background with a null compact direction. Having fixed gauge redundancies, we derive conjugate momenta of the system. In section 3 we analyze stabilization processes via open string tachyon condensation. Then by expanding the theory about true minima accompanied with taking low energy limit we are led to a physically well-defined Matrix theory. Having this theory at hand, in section 4 we solve its equations of motions and derive various solutions. Taking all into account, we write the light-cone Hamiltonian of this Matrix Theory and proposed to be the DLCQ Hamiltonian of type IIB string theory on the AdS/plane-wave backgrounds. In section 5 we elaborate on our proposal. We introduce yet another proposal of Matrix theory for DLCQ of type IIB string theory on AdS/plane-wave. It will be shown that these two proposals are exactly the same and thus it is a natural expectation that they share in evidence. We present some pieces of evidence in support of the proposal. They are basically based on symmetry structure of the theory, its vacuum structure, spectrum of fluctuations about the vacua and studying the BPS states. For physical applications, we will show how this Matrix theory behaves under the string theory or decompactification limit. Finally, in the last section we conclude and based on present ideas give an outlook for future works.
2 The Setup

In this section we consider a system of $J$ number of coincident non-BPS D0-brane of type IIB string theory on the maximally supersymmetric ten dimensional plane-wave background\footnote{In \cite{10}, system of non-BPS D0-branes on the non-supersymmetric flat background with 4-form RR potential has been studied.} which has a null circle.

2.1 The action

Dynamics of massless bosonic and fermionic as well as tachyonic modes of the non-BPS $Dp$-branes of type II is suitably described by generalized DBI and CS world-volume actions \cite{11, 12, 13, 14, 15}. Supersymmetry is of course not manifest due to presence of the tachyon and is regarded as spontaneously broken. Furthermore the relationship between BPS and non-BPS branes implies that T-duality must also hold for non-BPS branes, so we require the action to be T-dual covariant. The low energy effective world-volume action has been obtained as a function of the tachyonic and massless mode where all the infinite massive mode is integrated out.

The non-Abelian extension of this action for a system of non-BPS $Dp$-branes has also been worked out in the literature. Demanding the $U(J)$ invariance and T-duality covariance\footnote{It is also interesting to require its consistency under S-duality for IIB case.} it takes a unique form. Indeed the guiding principle in constructing such an action is its consistency with the rules of T-duality á la Myers \cite{16} and the starting point is the action of space filling non-BPS D9-brane of type IIA. Dynamics of bosonic sector is given by \cite{13, 18}

\begin{equation}
S = -\beta T_9 \int d^{10} \sigma \text{Tr} \left[ e^{-\Phi} V(T) \right] - \text{det} \left( G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} + 2\pi\alpha' D_\mu T D_\nu T \right)^{1/2} \\
+ \beta Q_9 \int \sum C_n \wedge D \text{Tr} \left[ T \wedge e^{2\pi\alpha' F + B} \right],
\end{equation}

where $\beta$ is a constant, $G$ and $B$ are the metric and the Kalb-Ramond field of the bulk respectively and $F$ is the field strength of the gauge field on the brane. $T$ is the tachyon field with the tachyon mass $m^2 = -1/2\alpha'$ and its potential is $V(T) = e^{2\pi\alpha' m^2 T^2}$ which is zero at the minimum $V(T_0) = 0$. Tension and charge of the brane is $T_9 = Q_9 = l_s^{-9}$. Covariant derivative is defined as $D_\mu = \partial_\mu + i [A_\mu, \cdot \cdot \cdot]$, $\mu = 0, 1, \ldots 9$. The gauge field $A_\mu$ and $T$ sit in the adjoint representation of the gauge group. Trace is taken completely
symmetrized between all non-Abelian expressions in $U(J)$ representation. Determinant is taken over $SO(10)$ representations.

In order to construct the non-Abelian low energy effective action of $J$ non-BPS D0-branes, we apply T-duality transformation rules to D9-brane in 9 longitudinal directions. The final form of the action is

$$S = -\beta T_0 \int d\tau \text{Tr} [e^{\phi} V(T) \det^{1/2} Q^{J'} - \left( P [G_{00} + G_{0I} (Q^{-1} - \delta)^{IJ} K G^{KJ} G_{0J}] + T_{00} \right)^{1/2}]$$

$$+ \beta Q_0 \int d\tau \text{Tr} \left[ P \left[ e^{i/(2\pi \alpha')} \chi_{X^I} \sum_n C_n \wedge e^B \cdot \left( -i/(2\pi \alpha')^{1/2} \epsilon^{[X,T]} + \wedge \partial T \right) \right] \right].$$

(2.2)

Fluctuations of the transverse collective coordinates is manifested as another potential and is encoded in the matrix $Q$

$$Q^{I,J} = \delta^{I,J} - \frac{i}{2\pi \alpha'} [X^I, X^K] E_{KJ} - \frac{1}{2\pi \alpha'} [X^I, T] [X^K, T] E_{KJ},$$

(2.3)

and contribution of tachyonic modes to the dynamics is entering through matrix $T_{00}$

$$T_{00} = (2\pi \alpha' - [X^I, T] (Q^{-1})_{IJ} [X^J, T]) \mathcal{D}_0 T \mathcal{D}_0 T$$

$$- \frac{i}{2\pi \alpha'} [X^I, T] \mathcal{D}_0 T - i \mathcal{D}_0 T [X^I, T] (Q^{-1})_{IJ} E_{J0}$$

$$- i \mathcal{D}_0 X^I (Q^{-1})_{IJ} [X^J, T] \mathcal{D}_0 T - i \mathcal{D}_0 T [X^I, T] (Q^{-1})_{IJ} \mathcal{D}_0 X^J,$$

(2.4)

where we have defined $E_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu}$ with which we raise or lower indices. $E^{IJ}$ denotes the inverse of $E_{IJ}$. $I$'s are transverse indices to the D0-branes. Note that there is no contribution from field strength, and the gauge field contributes only through covariant derivative. Also, we use covariant derivatives in pulling back the bulk fields. The potential $\det^{1/2} Q$ is equal to one for the Abelian case.

**The plane-wave background**

We are interested in the ten dimensional plane-wave background which is maximally supersymmetric and $\alpha'$-exact solution of the IIb supergravity, specified by

$$ds^2 = -2dX^+ dX^- - \mu^2 X^I X^J (dX^+)^2 + dX^J dX^I$$

(2.5a)

$$F_{+ijkl} = 4 \frac{\mu}{g_s} \epsilon_{ijkl}, \quad F_{+abcd} = 4 \frac{\mu}{g_s} \epsilon_{abcd}$$

(2.5b)

$$e^\phi = g_s = \text{constant.}$$

(2.5c)

The RR 4-form potential in a gauge which maintains translational symmetry along $X^+$ reads as

$$C_{+ijk} = -\frac{\mu}{g_s} \epsilon_{ijkl} X^l, \quad C_{+abc} = -\frac{\mu}{g_s} \epsilon_{abcd} X^a.$$
This background has a globally defined light-like Killing vector $\partial/\partial X$ and one dimensional light-like causal boundary. The parameter $\mu$ whose value is arbitrary has dimension of energy. For a detailed discussion regarding this background see \cite{19}.

**The gauge fixing**

Next we fix gauge redundancies. There are two of them; The first one is 1-dimensional diffeomorphism along the world-line. Due to symmetries of the plane-wave background and in particular translational symmetries along the light-like directions, fixing the light-cone gauge will considerably simplify the action. We do this by identifying world-line time with one of the light-cone coordinates, using and fixing worldline reparametrization

$$\tau \sim X^+. \quad (2.7)$$

$X^+$ and $X^-$ is no longer dynamical variables. $X^-$ is a cyclic coordinate, so its conjugate momentum $p^+$ is a constant of motion.

The second gauge redundancy is the internal gauge symmetry. We use temporal gauge

$$\mathcal{A}_0 = 0 \quad (2.8)$$

to fix it. Then, we impose its equation of motion as constraint on the system or physical condition for the states.

**2.2 The conjugate momenta of the system**

Putting back the background fields in the action and fixing the light-cone gauge we uncover the effective action of this configuration as

$$S[X^+, X^-, X^I, \mathcal{A}_0, T] = \int dX^+ \left( L_{DBI} + L_{CS} \right), \quad (2.9)$$

$$L_{DBI} = \text{Tr} \left[ -\frac{1}{l_s g_s} V(T) \det^{1/2} Q \times ight.$$}

$$\times \left[ \mu^2 X^I (\partial_0 X^I)^2 + 2 \partial_0 X^+ \partial_0 X^- - \mathcal{D}_0 X_I (Q^{-1})^{IJ} \mathcal{D}_0 X_J \right.$$

$$\left. - \mathcal{D}_0 T (2\pi \alpha' - [X_I, T] (Q^{-1})^{IJ} [X_J, T]) \mathcal{D}_0 T ight. + 

$$

$$\left. 2i (\mathcal{D}_0 X_I (Q^{-1})^{IJ} [X_J, T] \mathcal{D}_0 T + \mathcal{D}_0 T [X_I, T] (Q^{-1})^{IJ} \mathcal{D}_0 X_J) \right]^{1/2}] \right] ,$$

$$L_{CS} = \frac{\mu}{(2\pi)^3 g_s l_s^4} \text{Tr} \left[ [X^I, X^I] \epsilon^{ijkl} X^k [X^i, T] + [X^b, X^a] \epsilon^{abcd} X^c [X^d, T] \right] \partial_0 X^+ ,$$

where the matrix $Q$ is of the form

$$Q_{IJ} = \delta_{IJ} - \frac{i}{2\pi \alpha'} [X_I, X_J] - \frac{1}{2\pi \alpha'} [X_I, T] [X_J, T], \quad (2.10)$$
and \( Q^{-1} \) is its inverse. \( Q \) is an \( 8 \times 8 \), as well as a \( J \times J \) matrix. The \( \text{det} \, Q \) is however, only on \( 8 \times 8 \) indices and can be expanded as
\[
\text{det} \, Q = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \text{Tr} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{k=0}^{m} C_k^m (-1)^m \chi \left[ [X, X]^k \left( [X, T] [X, T]^m \right)^{m-k} \right] \right) n,
\]
(2.11)
where we have to consider all possible permutations \( \chi \) of commutators, because they are matrices rather than \( c \)-numbers. Now having the action at hand, we derive conjugate momenta associated with non-dynamical \( X^+, X^- \) and dynamical variables \( X^I, T, A_0 \).

The Light-cone momentum, momentum conjugate to \( X^- \) is defined as \( p^+ = -P_- = -\partial L/\partial \dot{X}^- \), and can explicitly written as
\[
p^+ = \frac{1}{J l_s g_s} \text{Tr} \left[ V(T) \det^{1/2} Q \times \left( \mu^2 X^2 + 2 \partial_0 X^- - D_0 X_I (Q^{-1})^{IJ} D_0 X^J \\
- D_0 T (2 \pi \alpha' - [X_I, T] (Q^{-1})^{IJ} [X^J, T]) D_0 T \\
+ 2i (D_0 X_I (Q^{-1})^{IJ} [X_J, T] D_0 T + D_0 T [X_I, T] (Q^{-1})^{IJ} D_0 X_J) \right) \right]^{-1/2},
\]
(2.12)
As proposed in the introduction, the light-cone momentum should be distributed among \( J \) D0-branes, strictly speaking each D0-brane carries one unit of light-cone momentum. Hence, the eigenvalue of this operator should be somehow discretized. It can directly be obtained by compactifying the longitudinal light-cone coordinate \( X^- \sim X^- + 2 \pi R_- \) which results in discretized eigenvalues for corresponding conjugate momenta as
\[
p^+ = J/R_-. \tag{2.13}
\]
We have chosen the radius of compactification \( R_- \) in such a way that \( J \) appears in the numerator.

Transverse momenta, momenta conjugate to \( X^I \) are \( P_I = \partial L/\partial \dot{X}^I \)
\[
P^I = \frac{1}{2} p^+ \left[ D_0 X_J (Q^{-1})^{IJ} + (Q^{-1})^{IJ} D_0 X_J - 2i (Q^{-1})^{IJ} [X_J, T] D_0 T - 2i D_0 T [X_J, T] (Q^{-1})^{IJ} D_0 X_J \right].
\]
(2.14)
Momentum conjugate to the tachyon field is \( P_T = \partial L/\partial \dot{T} \),
\[
P_T = 2 \pi \alpha' p^+ \left[ D_0 T + 2 \pi \alpha' D_0 T [X_I, T] (Q^{-1})^{IJ} [X_J, T] \\
- 2 \pi \alpha' ([X_I, T] (Q^{-1})^{IJ} D_0 X_J - D_0 X_I (Q^{-1})^{IJ} [X_J, T]) \right].
\]
(2.15)
The light-cone Hamiltonian, that is the momentum conjugate to $X^+$, is $H = P^- = -p_+ = -\partial L/\partial \dot{X}^+$. Explicitly,

$$JH = p^+ \operatorname{Tr} \left[ \frac{1}{2(\mu^+ l_s g_s)^2} V^2(T) \det Q + \frac{1}{2} \mu^2 X_I^2 + \frac{1}{2} \partial_0 X_I (Q^{-1})^{IJ} \partial_0 X_J 
+ \frac{1}{2} \partial_0 T \left( 2\pi \alpha' - [X_I, T] (Q^{-1})^{IJ} [X_J, T] \right) \partial_0 T 
- i \left( \partial_0 X_I (Q^{-1})^{IJ} [X_J, T] \partial_0 T + \partial_0 T [X_I, T] (Q^{-1})^{IJ} \partial_0 X_J \right) \right]$$

(2.16)

Finally, momentum conjugate to the gauge field is zero $P_{A_0} = 0$. We fix this gauge redundancy in temporal gauge $A_0 = 0$ and impose its equation of motion

$$\Phi = \frac{\partial L}{\partial A_0} = \left[ X_I, \frac{\partial L}{\partial \partial_0 X_I} \right] + \left[ T, \frac{\partial L}{\partial \partial_0 T} \right] = [X^I, P_I] + [T, P_T],$$

(2.17)

as the Gauss law constraint on the system.

The light-cone Lagrangian is defined through Legendre transformation

$$L = p_- \dot{X}^- + P_+ \dot{X}^I + P_T \dot{T} - H - A_0 \Phi,$$

(2.18)

and explicitly is

$$JL = p^+ \operatorname{Tr} \left[ \frac{1}{2} \partial_0 X_I (Q^{-1})^{IJ} \partial_0 X_J - \frac{1}{2(\mu^+ l_s g_s)^2} V^2(T) \det Q - \frac{1}{2} \mu^2 X_I^2 
+ \frac{1}{2} \partial_0 T \left( 2\pi \alpha' + [X_I, T] (Q^{-1})^{IJ} [X_J, T] \right) \partial_0 T 
+ i \left( \partial_0 X_I (Q^{-1})^{IJ} [X_J, T] \partial_0 T + \partial_0 T [X_I, T] (Q^{-1})^{IJ} \partial_0 X_J \right) \right]$$

(2.19)

Having introduced our system of interest, in the following section we continue to study its dynamics.

### 3 Stabilization of the System

We now analyze stabilization of the theory through the processes of open string tachyon condensation. In order to analyze the dynamics, we rewrite the action (2.9) as $S = S_{\text{kin.}} + S_{\text{pot.}}$, where

$$S_{\text{pot.}} = - \int dX^+ V(X, T, A_0).$$

(3.1)
The collection of $J$ non-BPS D0-branes in the plane-wave background develops a non-trivial effective potential $V$. It is a functional of $U(J)$ matrix valued fields $X^I, T, A_0$ and possible matrix commutators of them $[X, X], [X, T], [A_0, X], [A_0, T]$ as well as a set of parameters $\alpha', g_s, \mu, R_-, g_s$. It reads

$$V(X, T, A_0) = R_- \text{Tr} \left[ \frac{1}{2} \left( \frac{\mu}{R_-} \right)^2 X^2_I + \frac{1}{2(Jl_s g_s)^2} V^2(T) \det Q ight]$$

$$- \frac{1}{2R_-^2} D_0 T [X_I, T] (Q^{-1})^{IJ} [X_J, T] D_0 T$$

$$- \frac{i}{R_-^2} (D_0 X_I (Q^{-1})^{IJ} [X_J, T] D_0 T + D_0 T [X_I, T] (Q^{-1})^{IJ} D_0 X_J)$$

$$- \frac{\mu}{(2\pi)^3 R_- g_s l_s^4 R_-} \left( [X^j, X^i] \epsilon^{ijkl} X^k [X^l, T] + [X^b, X^a] \epsilon^{abcd} X^c [X^d, T] \right) \right].$$

(3.2)

The first term, the mass term of transverse coordinates, comes from the metric through DBI part of the action and is due to the back-reaction of the 5-form RR flux in the background. The rest (except for trivial part of the $\det Q$ which gives the mass term of the tachyon) are due to collective behavior of D0-branes as is manifested in the commutators of the fields. The next three terms are the potential for fluctuations of the transverse coordinates and the tachyon coming also via DBI action. The last term originates from the coupling of the transverse/tachyon fluctuations to the 4-form gauge potential through CS part of the action.

This system suffers from two sorts of tachyonic instabilities. One is due to tachyon field in the spectrum of open string connecting a non-BPS D0-brane to itself. It is encoded in (diagonal part of) the $J \times J$ matrix $T$. It gives further contribution to the effective potential through $[X, T]$ terms besides the $V(T)$ term. The other instability stems from tachyonic modes of the off-diagonal modes of the scalars $X$’s. They are agents of coupling of transverse fluctuations (represented by open strings stretching between D0-branes) to the bulk RR flux. It gives contribution via $[X, X]$ and $[X, T]$ commutators.

Footnote:

4Throughout this note we are using temporal gauge for the gauge field, as $A_0 = 0$

5See similar analysis for the BPS D0-branes of type IIA theory in the presence of 4-form flux has been elaborated in detail in [20].
3.1 Open string tachyon condensation

Eventually, upon the process of open string tachyon condensation, this system non-trivially stabilizes and falls into true extrema of the effective potential. The tachyon field as well as matrix commutators as agents of instabilities spontaneously acquire non-zero expectation values

\[ \langle T \rangle \neq 0, \quad \langle [X, X] \rangle \neq 0, \quad \langle [X, T] \rangle \neq 0 \]  

and all the modes condensate at the minima of the potential. In the next section we will see that in fact there is a set of minima and study the theory at these points. It is considerable noting that commutators are not quantum commutation relations but just matrix commutators and so is the expectation values, they are statistical (average values) not quantum mechanical.

In the absence of RR flux the preferred configuration is the one with \([X, X] = 0\) and \([X, T] = 0\). It defines a moduli space on which \(X\) and \(T\) matrices are simultaneously diagonalizable. Before the process of open string tachyon condensation, it has interpretation of positions for the non-BPS D0-branes. However, in the presence of the flux, they no longer commute and the classical interpretation of D0-brane positions breaks down. There is a fuzziness in description of their positions and we define the mean-square value of the \(i\)th coordinate to be

\[ \langle X_i^2 \rangle \sim 1/J \]  

when averaged over \(J\) D0-branes. Upon tachyon condensation, the values of commutators become specific in such a way that extremize the effective potential.

3.2 Expansion about true minima

At the minima of the potential there is a stable theory, the potential of which can by obtained be expanding the above effective potential about each of the minima in the form of a harmonic oscillator potential. As we discussed there are two different physical processes which are governed by \([X, X]\), \([X, T]\) as well as \(T\) in the interaction part of the above action. They can be seen as orthogonal directions in the graph of effective potential (there is yet another orthogonal direction related to mass term of \(X\), although disjoined from the others). We extremize the potential with respect to all of them and expand appropriately about the minima of them. To have a potential of the form of harmonic oscillator, we keep terms second-order in derivatives, as

\[ V = V_0 + \left. \frac{\delta^2 V}{\delta \Phi \delta \Phi} \right|_{\Phi = \Phi_0} (\Phi - \Phi_0)^2 + \cdots , \]  

where \(\Phi\) could be either of \(X^I\), \([X^I, X^J]\)[\(X^K, T\)] or \(T\). As we will see in the next section, they are really true minima and not saddle points.
3.3 The decoupling limit

Furthermore, we decouple the heavy modes and fluctuations about these minima, by taking low energy limit $\alpha' \sim \epsilon \rightarrow 0$ while keeping the physical light-cone momentum $\mu p^+ \alpha'$, string coupling $g_s$, $J$ and $R_-$ fixed. To be consistent, we send dimensionful parameter of the background $\mu \sim \epsilon^{-1} \rightarrow \infty$. Prior to taking the limit, we rescale parameters properly to bring them in energy dimension as

$$X \rightarrow \alpha' X, \quad \alpha'^{-1}[X, X] \rightarrow \alpha'[X, X], \quad \alpha'^{-1/2}[X, T] \rightarrow \alpha'^{1/2}[X, T],$$

(3.5)

and leave $T$ unchanged. In the low energy regime, we just keep terms in the determinant of $Q$ to order $O(\alpha'^2)$ and higher order terms decouple from the theory.

It is a non-trivial possibility to take a limit where $\langle T \rangle$ remains finite while the fluctuations of $T$ about it become very massive and decouple. From now on we refer to matrix of vacuum expectation values of tachyon as $\langle T \rangle = \mathcal{T}$ which can take various values, and of course matrix form. Furthermore, we set $P_T = 0$ and $V(T)$ becomes a constant matrix.

3.4 The action for the stabilized phase of the system

Finally having done all this, the harmonic oscillator potential about the true minima takes this form

$$V = R_- \text{Tr} \left[ \frac{1}{2} \left( \frac{\mu}{R_-} \right)^2 X_I^2 \right. + \left. \frac{1}{2(J_8 g_s)^2} V^2(\mathcal{T}) \chi \left[ [X^I, X^J] [X^K, X^L] [X^M, \mathcal{T}] [X^N, \mathcal{T}] \right] - \frac{\mu}{(2\pi)^3 R_- g_s (\alpha'^c)} \left( [X^j, X^i] \epsilon^{ijkl} X^k [X^l, \mathcal{T}] + [X^b, X^a] \epsilon^{abcd} X^c [X^d, \mathcal{T}] \right) \right].$$

(3.6)

For later use, we also derived the light-cone Lagrangian of this Matrix theory. It reads

$$L = R_- \text{Tr} \left[ \frac{1}{2 R_-^2} (\mathcal{D}_0 X_I)^2 - \frac{1}{2} \left( \frac{\mu}{R_-} \right)^2 X_I^2 \right. \right. - \frac{1}{2.3! (\alpha'^c)} V^2(\mathcal{T}) [X^I, X^J, X^L, \mathcal{T}] [X^I, X^J, X^L, \mathcal{T}] \left] \right. \right. + \left. \frac{\mu}{31! R_- g_s^{14}} \left( \epsilon^{ijkl} X^i [X^j, X^k, X^l, \mathcal{T}] + \epsilon^{abcd} X^a [X^b, X^c, X^d, \mathcal{T}] \right) \right],$$

(3.7)

where the 4-commutator is defined as

$$[A, B, C, \mathcal{T}] = \frac{1}{4!} \left( [A, B][C, \mathcal{T}] - [A, C][B, \mathcal{T}] + [A, \mathcal{T}][B, C] \right. \right. + \left. \left. [C, \mathcal{T}][A, B] - [B, \mathcal{T}][A, C] + [B, C][A, \mathcal{T}] \right) \right) \right),$$

(3.8)
and we have used the identity
\[ \varepsilon^{ijkl} X^i X^j X^k X^l = \frac{1}{4!} \varepsilon^{ijkl} [X^i, X^j, X^k, X^l] = \frac{1}{2.4!} \varepsilon^{ijkl} [X^i, X^j][X^k, X^l]. \] (3.9)

The action
\[ S = \int dX^+ \mathbf{L} \] (3.10)
gives the dynamics of a physically well-defined and stable 0+1-dimensional $U(J)$ gauge theory, a Matrix theory. Furthermore, as is noted before, the supersymmetry in the unstable theory with tachyon field is regarded as spontaneously broken. Upon tachyon condensation the supersymmetry is restored and we end up with a supersymmetric gauge theory. If, besides tachyonic and massless bosonic sector, we had also added massless fermionic sector to the action we started with (2.1), the above action for the stable theory would be manifestly supersymmetric.

4 The Proposal

4.1 Equations of motion

We now study this Matrix theory by solving its equation of motion for dynamical fields. We will see that the solutions are in the form of concentric fuzzy 3-spheres, which form various vacuum configurations of the theory.

The equation of motion of transverse adjoint scalars $X^i$ (and similarly for $X^a$) is
\[ D_0 P_i - i \frac{\partial \mathbf{L}}{\partial D_0 X^j} \left( \frac{\partial [A_0, X^j]}{\partial X^i} - \frac{\partial \mathbf{L}}{\partial [X^j, X^k, X^l, T]} \frac{\partial [X^j, X^k, X^l, T]}{\partial X^i} - \frac{\partial \mathbf{L}}{\partial X^i} \right) = 0, \] (4.1)
which can be written as
\[ \partial_0 P^i + \frac{1}{2.3! \, g_s^2 l_s^2 J^2} [X^j, X^k, [X^i, X^j, X^k, T], T] + \left( \frac{\mu}{R_-} \right)^2 X^i - \frac{\mu}{3! l_s^4 R_- g_s J} \varepsilon^{ijkl} [X^j, X^k, X^l, T] = 0. \] (4.2)

Although tachyon is condensed at the minima of the tachyon potential, we can treat it as a variable and compute its equation of motion, it reads as
\[ - \frac{\partial \mathbf{L}}{\partial [X^I, X^J, X^K, T]} \frac{\partial [X^I, X^J, X^K, T]}{\partial T} - \frac{\partial \mathbf{L}}{\partial T} = 0, \] (4.3)
which can explicitly be written as
\[ \frac{1}{2.3! g_s^2 l_s^2 J^2} \left( [X^I, X^J, X^K, [X^I, X^J, X^K, T]] + 4 T V^2 (T) [X^I, X^J, X^K, T]^2 \right) - \frac{\mu}{2.3! l_s^4 R_- g_s J} (\varepsilon^{ijkl} [X^i, X^j, X^k, X^l] + \varepsilon^{abcd} [X^a, X^b, X^c, X^d]) = 0. \] (4.4)
Equation of motion of the tachyon can be interpreted as a constraint on the system. Namely, having solved $X$ equations of motion, the configurations should also satisfy $T$ constraint equation.

Equation of motion for $A_0$ reads

$$i[X^I, P_I] = 0, \quad (4.5)$$

which is the Gauss’ law and should be satisfied by all the physical configurations.

We are looking for static solutions $P^I = 0$ of the equations of motion for which the potential is extremum. Furthermore, we consider a class of solutions for which

$$[X^i, X^j, X^a, T] = [X^a, X^b, X^i, T] = 0. \quad (4.6)$$

With this, equations mixed between $i$ and $a$ directions decouple. Hence, the equations of motion reduce to (the same set of equation holds for $X^a$)

$$X^i :$$

$$\frac{1}{2.3!g_s^2J^2} [X^j, X^k, [X^i, X^j, X^k, T], T] + \left( \frac{\mu}{R_-} \right)^2 X^i$$

$$- \frac{\mu}{2.3!l_s^4R_-g_sJ} \epsilon^{ijkl} [X^j, X^k, X^l, T] = 0 \quad (4.7)$$

$$T :$$

$$\frac{1}{2.3!g_s^2J^2} \left( [X^i, X^j, X^k, [X^i, X^j, X^k, T]] + 8T[X^i, X^j, X^k, T]^2 \right)$$

$$- \frac{\mu}{2.3!l_s^4R_-g_sJ} (\epsilon^{ijkl} [X^i, X^j, X^k, X^l]) = 0. \quad (4.8)$$

### 4.2 The anzats

The solution to the $X$ equation of motion (4.7), is

$$[X^i, X^j, X^k, T] = -\frac{\mu g_s}{R_-} \alpha^2 J \epsilon^{ijkl} X^l. \quad (4.9)$$

put it back in $T$ equation of motion (constraint equation) (4.8), consistency requires

$$[X^i, X^j, X^k, T] = \left( \frac{\mu g_s}{R_-} \alpha^2 J \right)^2 \epsilon^{ijkl} T, \quad (4.10)$$

together with

$$\sum_{i=1}^{4} \delta^{ij} X^i X^j = \frac{\mu g_s}{R_-} \alpha^2 J. \quad (4.11)$$

Consistency among equations (4.9)-(4.11) requires $T$ anticommutes with $X$ and furthermore, squares to 1. Hermiticity and tracelessness together with the fact that $T$
could become diagonalized, fix it to be a $J \times J$ matrix with equal number of +1 and -1 eigenvalues. More details about realization of the solutions to matrix equations (4.9)-(4.11) is given in appendix A.

Solution to the equations of motions, equations (4.9) and (4.11), define fuzzy 3-sphere (see appendix A), the radius $R$ and fuzziness $l$ of which is defined as

$$R_{\text{fuzzy}} = \left( \frac{\mu g_s}{R_-} J \right)^{1/2} \alpha' ; \quad l = \left( \frac{\mu g_s}{R_-} \right)^{1/2} \alpha'. \quad (4.12)$$

Equations (4.7) and (4.8) are matrix equations and we have to take into account matrix form of $X$’s and $T$. Hence, we must consider all the possible forms of the matrices as well as coefficients which solve the above equations. If one considers block-diagonal set of matrices $X$’s and $T$ then the classical equations of motion for the blocks are separable. One can think of these blocks as describing different matrix theory objects which obeying classically independent equations of motions. This gives an implication how Matrix theory can encode a configuration of multiple objects [21].

As we argued in previous section, there is a set of minima for the effective potential at which the tachyons can be condensed and spontaneously take nonzero vacuum expectation values and Matrix form. These extrema corresponds to different configurations of concentric fuzzy 3-spheres. Generically, a solution could be of the following form when $J \times J$ matrices $X$ and $T$ is partitioned into $k$ blocks of size $J_i$ in such a way that $\sum_{i=1}^k J_i = J$, as

$$X^i = \begin{pmatrix} \frac{1}{J_1} & & & \frac{1}{J_2} \\ & \ddots & & \\ & & \ddots & \\ \frac{1}{J_{i-1}} & & & \frac{1}{J_i} \end{pmatrix}, \quad T = J$$

Each block in the matrix $T$ has $J_i/2$ number +1 and $J_i/2$ number -1 eigenvalues. $X$’s have the same pattern of block-diagonalization as $T$. They obey independent equation of motion and consistency demands

$$\sum_{i=1}^4 \delta^{ij} X^i X^j = \frac{\mu g_s}{R_-} J_i. \quad (4.14)$$
for each block inside $X$’s. These solutions are in the form of $k$ concentric fuzzy 3-spheres. Actually, solutions can be classified and labeled in terms of irreducible or reducible $J \times J$ representation of $\text{spin}(4)$ which respectively correspond to single and multi concentric fuzzy 3-spheres of various radii extended in $X^i$ and/or $X^a$ directions. Multi solutions are related to all the possible partitions of $J$ and the radius of each sphere is given by the fraction of the total light-cone momentum $J_i$ it carries

$$R_{fuzzy} = \left(\frac{\mu g_s}{R_-} J_i\right)^{1/2} \alpha'.$$ \hspace{1cm} (4.15)

The fuzziness, which is characterized by the parameters of the background and string coupling constant, is a unique parameter for all the solutions and given by

### 4.3 Statement of the proposal: Matrix theory Hamiltonian as the DLCQ Hamiltonian

Taking all these into account, now we rewrite bosonic part of the light-cone Hamiltonian of this Matrix theory (once we rescale $\mathcal{T} \rightarrow \mathcal{T}/J$) as

$$H = R_- \text{Tr} \left[ \frac{1}{2} P_i^2 + \frac{1}{2} \left(\frac{\mu}{R_-}\right)^2 X_i^2 + \frac{1}{2} \cdot \frac{3! l^4 g_s^2}{4} [X^i, X^j, X^L, \mathcal{T}]^2 - \frac{\mu}{3! l^4 R_- g_s} \left( \epsilon^{ijkl} X^i [X^j, X^k, X^l, \mathcal{T}] + \epsilon^{abcd} X^a [X^b, X^c, X^d, \mathcal{T}] \right) \right],$$ \hspace{1cm} (4.16)

where now $\mathcal{T}$ is fixed to take only its vacuum values. This Hamiltonian well defines a 0+1-dimensional $U(J)$ gauge theory in the form of a Matrix theory. We propose that it is just the DLCQ Hamiltonian, $H_{\text{DLCQ}}^J$ of type IIB string theory with $J$ unit of light-cone momentum on the AdS/plane-wave backgrounds.

As is evident, it is just the bosonic sector of the DLCQ Hamiltonian. The fermionic sector could similarly be derived. We insist that upon tachyon condensation and decoupling of the extra tachyonic degree of freedom, the supersymmetry is restored and the stable theory is fully supersymmetric.

Another crucial point is that we have really expanded the theory about the true minima of the effective potential (3.2), not its saddle points. It can be justified by noting that all the terms in the potential of our Matrix theory is positive definite

$$V = R_- \text{Tr} \left[ \frac{1}{2} \left(\frac{\mu}{R_-} X^i + \frac{1}{3! l^4 g_s} \epsilon^{ijkl} [X^i, X^j, X^k, \mathcal{T}] \right)^2 \right] + \frac{1}{4.3! l^4 g_s^2} [X^i, X^j, X^a, \mathcal{T}]^2 \right] + \frac{1}{4.3! l^4 g_s^2} [X^i, X^a, X^b, \mathcal{T}]^2 \right].$$ \hspace{1cm} (4.17)
5  Considerations on the Proposal

This section is devoted to tighten our conjecture by giving some pieces of evidence in support of.

5.1 Another proposal: Tiny Graviton Matrix Theory

For completeness we review another proposal to construct the DCLQ Hamiltonian of type IIB string theory on the AdS/planewave backgrounds.

It has been noted that the right probe in the presence of RR flux is spherical D3-brane giant gravitons [4]. It is a natural expectation that quantum completion of type IIB on this background comes from quantum D3-brane theory. Due to large amount of unphysical degrees of freedom and gauge redundancies, it is difficult to study its quantum mechanical properties. However, there is an alternative formulation of the D3-brane as a gauge theory of the volume-preserving transformations of the brane hypersurface [22]. It is inspired by the fact that these transformations are residual invariance of a D3-brane theory when formulated in the light-cone gauge. Fixing the light-cone gauge fixes a part of the diffeomorphism invariance of the action which mixes the world-volume time with the world-volume spatial directions. The part of diffeomorphisms which only act on the spatial directions are still present and not fixed. This supersymmetric gauge theory provides a convenient framework for study the quantum mechanical properties of the D3-brane. It is possible to consider truncations of this gauge theory by truncating the infinite harmonic expansion of the D3-brane coordinates. At least for the D3-brane with the topology of a sphere this can be done in such a way that the supersymmetry remains preserved. These truncations lead to a class of matrix models in supersymmetric quantum mechanics and D3-brane can be viewed as a limiting case of them.

Consider the low energy effective action of the D3-brane

\[
S_{D3} = -T \int d^4 e^{-\phi} \left( 1 - \det \left( P[G + B] + F \right) \right)^{1/2} + Q \int C_{(n)}, \tag{5.1}
\]

where \( T = Q = (2\pi)^3 l_s^{-1} \). The world-volume theory of D3-brane consists of two gauge symmetries; 4-dimensional diffeomorphisms and internal gauge symmetry. We use light-cone gauge to fix a part of redundancies. For that end, we use reparametrization of the world-volume to identify the temporal direction of the brane with the light-cone time \( \tau \sim X^+ \). Furthermore, in the low energy limit the gauge field decouples but to make sure supersymmetry at the quantum level in the spectrum of fluctuations, we
need to give an appropriate prescription for quantization. In the light-cone frame the action reads as
\[ S_{D3} = \int dX^+ L. \]

The light-cone Hamiltonian is defined as
\[ H = P^- - \partial L / \partial \partial_r X^+ \] which can be written explicitly
\[ H = \int d^3 \sigma \left[ \frac{1}{2p^+} (P_i^2 + P_a^2) + \frac{1}{2} \mu^2 p^+ (X_i^2 + X_a^2) + \frac{1}{2p^+((2\pi)^3 l_s g_s)^2} \det g_{rs} \right. \\
- \frac{\mu}{3!(2\pi)^3 l_s^4 g_s} (\epsilon^{ijkl} X^i X^j X^k X^l) + \epsilon^{abcd} X^a X^b X^c X^d) \\
+ \frac{1}{2g_s} (\theta^{\alpha \beta} (\sigma^{ij})^{\delta} \{ X^i, X^j, \theta_{\delta \beta} \} + \theta^{\dagger \alpha \beta} (\sigma^{ab})^{\delta} \{ X^a, X^b, \theta_{\delta \beta} \}) \\
+ \frac{1}{2g_s} \left( \theta^{\dagger \alpha \beta} (\sigma^{ij})^{\delta} \{ X^i, X^j, \theta_{\delta \beta} \} + \theta^{\dagger \alpha \beta} (\sigma^{ab})^{\delta} \{ X^a, X^b, \theta_{\delta \beta} \} \right), \]

where \( p^+ \) is the light-cone momentum which is defined as \( p^+ = -\partial L / \partial \partial_r X^- \), and \( P_I \)'s are transverse momenta \( P_I = \partial L / \partial \partial_r X^I \). We have collected everything in the form of Nambu bracket
\[ \epsilon^{rst} \partial_r X^l \partial_s X^j \partial_t X^K \equiv \{ X^I, X^J, X^K \}, \]
and the determinant can be expanded as
\[ \det g_{rs} = \frac{1}{3!} (\{ X^i, X^j, X^k \} \{ X^i, X^j, X^k \} + \{ X^a, X^b, X^c \} \{ X^a, X^b, X^c \}) \\
+ \frac{1}{2} (\{ X^i, X^j, X^a \} \{ X^i, X^j, X^a \} + \{ X^a, X^b, X^i \} \{ X^a, X^b, X^i \}). \]

Another comment is in order. As can be seen from the light-cone Hamiltonian, there is a unphysical pole for \( p^+ = 0 \) and so have to be excluded. It can simply be done just by quantizing this continuous variable, via \( X^- \) compactification as
\[ X^- \sim X^- + 2\pi R_+ \Rightarrow p^+ = \frac{J}{R_+}. \]

In [9] it has been proposed that time-independent volume-preserving diffeomorphism on connected, orientable and compact manifolds is isomorphic to unitary group \( U(\infty) \). In other words, Nambu bracket on Reimannian 3 dimensional surfaces forms the algebra \( su(\infty) \). The whole idea is that this gauge group can be truncated and approximated by \( SU(J) \). It is performed by truncating infinite-dimensional harmonic expansion of all the dynamical fields (3-brane coordinates) and setting them in the adjoint

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6 Explicit derivation is given in [23, 9].
7 with \( U(1) \) factor as a trivial diffeomorphism.
matrix representation. The precise discretization (regularization) prescription made in [9] is as follows; replace functions with matrices or operators, integration over spatial coordinates with trace of matrices and Nambu brackets with four-commutators, i.e.

\[
X^I \leftrightarrow X^I_{J\times J}, \theta \leftrightarrow \theta_{J\times J}, \bar{P}^I \leftrightarrow P^I_{J\times J}
\]

\[
p^+ \int d^3\sigma \leftrightarrow \frac{1}{R_-} \text{Tr}
\]

\[
\{\mathcal{F}, \mathcal{F}, \mathcal{F}\} \leftrightarrow J[\mathcal{O}, \mathcal{O}, \mathcal{O}, \mathcal{L}_5].
\]

Then the regularized Hamiltonian for discretized D3-brane reads

\[
H = R_- \text{Tr} \left[ \frac{1}{2} (P_1^2 + P_3^2) + \frac{1}{2} \left( \frac{\mu}{R_-} \right)^2 (X_1^2 + X_3^2) \right]
\]

\[
+ \frac{1}{2} \cdot 3! g_s \left( [X^i, X^j, X^k, \mathcal{L}_5][X^i, X^j, X^k, \mathcal{L}_5] + [X^a, X^b, X^c, \mathcal{L}_5][X^a, X^b, X^c, \mathcal{L}_5] \right)
\]

\[
+ \frac{1}{2} \cdot 2 g_s^2 \left( [X^i, X^j, X^a, \mathcal{L}_5][X^i, X^j, X^a, \mathcal{L}_5] + [X^a, X^b, X^i, \mathcal{L}_5][X^a, X^b, X^i, \mathcal{L}_5] \right)
\]

\[
- \frac{\mu}{3! R_- g_s} (\epsilon^{ijkl} X^i X^j X^k \mathcal{L}_5 + \epsilon_{abcd} X^a X^b X^d \mathcal{L}_5)
\]

\[
+ \frac{1}{2 g_s} \left( \theta^\alpha_{\beta\gamma} (\sigma_5^i)_\alpha^\delta [X^i, X^j, \theta_{\delta\beta}, \mathcal{L}_5] + \theta^\alpha_{\beta\gamma} (\sigma_5^a)_\alpha^\delta [X^a, X^b, \theta_{\delta\beta}, \mathcal{L}_5] \right)
\]

\[
+ \frac{1}{2 g_s} \left( \theta_{\alpha\beta\gamma} (\sigma_5^i)_\alpha^\delta [X^i, X^j, \theta_{\delta\beta}, \mathcal{L}_5] + \theta_{\alpha\beta\gamma} (\sigma_5^a)_\alpha^\delta [X^a, X^b, \theta_{\delta\beta}, \mathcal{L}_5] \right).
\]

(5.7)

In the prescription (5.6), it is proposed that in order to quantize Nambu 3-bracket one should introduce by hand a fix (non-dynamical) matrix \( \mathcal{L}_5 \) and convert it to a well-defined 4-commutator. \( \mathcal{L}_5 \) is closely related to chirality operator of \( SO(4) \) and make it possible to survives trace and by-part integration properties [9]. Starting from a Nambu 4-bracket we can non-trivially fix one of the functionals in such a way that it would not contribute and Nambu 4-bracket effectively reduces to 3-bracket but still with four functionals. In looking for a unique and basis independent definition of \( \mathcal{L}_5 \), we can always diagonalize \( \mathcal{L}_5 \) using \( U(J) \) rotation. Then Hermiticity and tracelessness together with the fact that it squares to identity matrix, fix the form of \( \mathcal{L}_5 \) to be a matrix with +1 and -1 eigenvalues up to permutation \( S_J \). It is worth noting that although \( \mathcal{L}_5 \) is a non-dynamical matrix, it can take various form or values [9, 25].

The Hamiltonian (5.6) of the 0+1-dimensional gauge theory is written in the temporal gauge \( A_0 = 0 \). To ensure \( SU(J) \) invariance, all the physical configurations must satisfy \( A_0 \) equation of motion, which are \( J^2 - 1 \) independent Gauss’ law conditions.

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8If one fixes it to identity matrix \( \mathbf{1} \), 4-commutator vanishes as opposed the case of 3-commutator. This is one of the reasons why we refer to the former well- and latter ill-defined.
Thus, the DLCQ of a D3-brane on the plane-wave background with a compact null direction leads to a 0+1-dimensional supersymmetric $U(J)$ gauge theory, a Matrix theory. It is conjectured to be the DLCQ of type IIB string theory in the sector with $J$ units of light-cone momentum on the AdS/plane-wave backgrounds [9]. This theory, called tiny graviton Matrix theory (TGMT), is proposed to be theory of $J$ tiny gravitons, the giant gravitons carrying minimum light-cone momentum, which show brane structure and gauge enhancement when some number of them sit on top of each other.

5.2 Comparing the two proposals

Thus far, we have presented two recipe to construct the DLCQ Hamiltonian $H_{J}^{DLCQ}$ of type IIB string theory in the sector $J$ on the AdS/plane-wave backgrounds. Both recipes result in an unique Hamiltonian; the Hamiltonian coming from non-BPS D0-branes (4.16) and the one coming from BPS D3-brane (5.7) have exactly the same form. In particular, the matrix $L_5$ in the TGMT plays the same role as $T$ in our construction.

The similarity between these two different approaches for the construction of the $H$, shed light on the quantization (regularization) of volume-preserving diffeomorphism on 3-dimensional Reimannian surfaces. Specifically, it gives an strong evidence in favor of the discretization prescription introduced in [9] and reviewed above, as the way the gauge group of diffeomorphism is approximated by the unitary group $U(J)$.

5.2.1 The symmetry structure

The Matrix theory has a large number of local and global symmetries. Although working in the temporal gauge, the Hamiltonian still enjoys the time-independent part of the $U(J)$ gauge symmetry, which appears as a global symmetry. Moreover, it has $PSU(2|2) \times PSU(2|2) \times U(1)_{H} \times U(1)_{p^+}$ supersymmetry group, which is the supergroup of the plane-wave background, with the minimal generators $Q_{\alpha \dot{\beta}}, Q_{\dot{\alpha} \beta}, J^{i j}, J^{a b}, H, p^+$. The Hamiltonian is invariant under expected dynamical supersymmetries. The superalgebra is a big one with 16 supercharges and 30 isometries. It puts severe restriction on the form of Hamiltonian. A supersymmetric quantum mechanics with 16 supercharges is uniquely determined once we specify superalgebra and the gauge group. Furthermore, its superalgebra naturally contains some of the extensions. In particular it has a 4-form corresponding to the dipole moment of the self dual RR 5-form, needed for stabilization of the vacuum. For the supersymmetry algebra and its matrix realized generators see [24].
The bosonic part of which is \( SO(4)_i \times SO(4)_a \times U(1)_H \times U(1)_p^+ \times \mathbb{Z}_2 \times \mathbb{Z}_2 \). The two \( SO(4)_i \) and \( SO(4)_a \) rotations act on \( i \) and \( a \) vector indices of the bosonic \( X^i \) and \( X^a \) fields and on the spinor (Weyl) indices of fermionic \( \theta_{\alpha\beta} \) as

\[
X^i_{rs} \rightarrow \tilde{X}^i_{rs} = R^i_j X^j_{rs} \\
(\theta_{\alpha\beta})_{rs} \rightarrow (\tilde{\theta}_{\alpha\beta})_{rs} = R_{\alpha\gamma}(\theta_{\gamma\beta})_{rs}
\]

\( \mathcal{T} \rightarrow \mathcal{T} \), (5.8)

where \( R_{ij} = e^{i\omega_{ij}^\gamma\gamma} \), \( R_{\alpha\gamma} = e^{i\omega_{ij}\sigma^{ij}} \) are respectively \( 4 \times 4 \) and \( 2 \times 2 \) \( SO(4) \) rotation matrices and \( r, s \) are \( J \times J \) indices. \( i \) and \( a \) transverse directions are exchanged by a \( \mathbb{Z}_2 \) symmetry. There is another \( \mathbb{Z}_2 \) symmetry which changes the orientation of the \( X^i \) and \( X^a \) simultaneously (i.e. \( \epsilon_{ijkl}, \epsilon_{abcd} \rightarrow -\epsilon_{ijkl}, -\epsilon_{abcd} \) ) together with sending \( \mathcal{T} \rightarrow -\mathcal{T} \).

Under the \( U(J) \) rotations all the dynamical fields as well as the \( \mathcal{T} \) are in the adjoint representation:

\[
X_I \rightarrow UX^I U^{-1} ; \quad \mathcal{T} \rightarrow U\mathcal{T}U^{-1},
\]

where \( U \in U(J) \). There is a \( U(1) \) subgroup of \( U(J) \), \( U(1)_a \) which is generated by \( \mathcal{T} \):

\[
U_\alpha = e^{i\alpha\mathcal{T}}.
\]

(5.10)

This subgroup has the interesting property that keeps the \( \mathcal{T} \) invariant.

### 5.3 Evidence for the proposal

Here we give some pieces of evidence in support of our Matrix theory proposal. Having shown similarities with TGMT, it is a natural expectation that they share in evidence.

#### 5.3.1 The vacuum structure

The Hamiltonian relevant for the zero energy solutions, which are static and bosonic, takes the form (4.17). Each term in that expression is positive-definite, hence the zero energy solutions are obtained when each of the four terms are vanishing, i.e.

\[
[X^i, X^j, X^k, \mathcal{T}] = -\frac{\mu g_s}{R_-} \epsilon^{ijkl} X^l \\
[X^a, X^b, X^c, \mathcal{T}] = -\frac{\mu g_s}{R_-} \epsilon^{abcd} X^d \\
[X^a, X^b, X^i, \mathcal{T}] = [X^a, X^i, X^j, \mathcal{T}] = 0.
\]

(5.11)

The first class of solutions to the above equations is the trivial \( X = 0 \) solution:

\[
X^i = 0 ; \quad X^a = 0
\]

(5.12)
Although mathematically trivial, as we will see this vacuum is physically quite non-trivial.

The next class of solutions is obtained when either \( X^i = 0 \) or \( X^a = 0 \). In this case equations (5.11c) and either of (5.11a) or (5.11b) are trivially satisfied. Since there is a \( Z_2 \) symmetry in the exchange of \( X^i \) and \( X^a \), here we only focus on the \( X^a = 0 \) case and the \( X^i = 0 \) solutions have essentially the same structure. Therefore, this class of vacua are solutions to

\[
X^a = 0, \quad [X^i, X^j, X^k, \mathcal{T}] = -\frac{\mu g_s}{R_-} \epsilon^{ijkl} X^l.
\]  

(5.13)

In [25] we gave the most general solutions to (5.13). One should, however, note that if we choose to expand the theory around either of these vacua the \( Z_2 \) symmetry is spontaneously broken. As we will see these solutions are generically of the form of concentric fuzzy three spheres in either of the \( SO(4) \)’s.

There is yet another class of solutions where both \( X^i \) and \( X^a \) are non-zero. These are non-trivial solutions which in the string theory limit correspond to giant gravitons grown in both \( X^a \) and \( X^i \) directions.

Noting the supersymmetry algebra of this Matrix theory [24], it can be shown that these zero energy solutions, either trivial or non-trivial, are half-BPS, i.e. they preserve all of the dynamical supercharges (half out of whole kinematical and dynamical ones), as they have \( H = 0 \) and \( J_{ij} = J_{ab} = 0 \) [25].

### 5.3.2 Spectrum of fluctuations about the vacua

The next evidence for our proposal comes from study of fluctuations about above vacua. Fluctuations about the simplest vacuum, the single fuzzy 3-sphere solution, has been worked out [9]. It has been shown that it exactly matches that of a spherical D3-brane giant graviton in the plane-wave background [23]. The effective coupling of these fluctuation modes has also been worked. It is

\[
geff = \frac{R_-}{J \mu \sqrt{g_s}} = \frac{1}{p^+ \mu \sqrt{g_s}}.
\]  

(5.14)

It is again what we obtain for spherical D3-brane giant graviton.

The fluctuations about the trivial vacuum has also been worked out [9]. It has been shown that the spectrum of small BPS fluctuations exactly matches with the spectrum of IIB supergravity modes, BPS states of strings, on the plane-wave background [26]. The coupling of these fluctuation is

\[
geff = \frac{f^3}{(\mu p^+)^2 g_s}.
\]  

(5.15)
Based on this evidence, it has been conjectured that fundamental type IIB strings are just non-perturbative objects about this vacuum [9].

5.3.3 Analysis of BPS solutions

All the BPS states of this Matrix theory have been studied and classified. Half-BPS states are just vacua of the theory and are in the form of various configuration of fuzzy 3-spheres or trivial ones [25]. Less-BPS stated have also been analyzed in details [27]. It has been shown that there is a one-to-one correspondence between these BPS states and those of $D = 4, N = 4, U(N)$ SYM theory [28] and bubbling AdS geometries of IIB supergravity [33].

5.4 String theory limit

Physical applications requires that the limit $R_-, J \to \infty$ be taken while keeping fixed physical momentum $p^+$. It is called string theory or decompactification limit. In this limit fuzzy 3-sphere generically goes over to round 3-sphere or spherical 3-brane giant graviton of radius

$$R_{\text{giant}}^2/\alpha' = \left(\frac{g_s}{N}\right)^{1/2} J,$$

This relation between radius and angular momentum comes from requirement of stability of topologically spherical BPS 3-brane. The radius of fuzzy sphere comes from stable state of non-BPS D0-branes. Together with

$$R_-/\mu = (g_s N)^{1/2} \alpha',$$

these two relations coincide. In this picture a BPS spherical D3-brane of radius $R$ is the string theory limit of fuzzy 3-sphere, a state into which $J$ non-BPS D0-branes are blown up.

Based on the study of fluctuation modes about the trivial vacua, it has been conjectured that in the string theory limit this vacuum quantum mechanically becomes the vacuum for strings on the plane-wave background [9]. In this limit the coupling of fluctuations become large and the fundamental type IIB closed strings appear as non-perturbative objects in this vacuum.

It is interesting noting that the plane-wave background has two $SO(4)$ isometries. D0-branes are singlets of $SO(4)$ which eventually decay to other representations of this group which could be either spherical or trivial(singlet) configurations. In fact, this system stabilizes to either the configurations of fuzzy 3-spheres or trivial ones which in the string theory limit, go over to giant gravitons or fundamental strings respectively.
Conclusion and Outlook

Motivated by the tempting idea of discrete light-cone quantization (DLCQ), in this note we proposed how to construct the DLCQ Hamiltonian of type IIB string theory on the AdS/planewave backgrounds in the sector with \(J\) units of light-cone momentum. We conjectured it is just the Hamiltonian of a 0+1-dimensional supersymmetric \(U(J)\) gauge theory, a Matrix theory. It is itself the light-cone Hamiltonian for the stabilized phase of a system of \(J\) coincident non-BPS D0-branes of type IIB string theory on the plane-wave background with a null circle. We presented some evidence in support of this proposal.

Furthermore, on the one hand through the strong form of the Maldacena’s conjecture, quantum type IIB string theory on the AdS background is equivalently described by a superconformal gauge theory [29, 30]. From practical point of view, our computational ability does not go beyond classical supergravity on the AdS background. On the other hand, in this note we propose a Matrix theory which governs the quantum dynamics of the very string theory by giving its DLCQ Hamiltonian. It is now interesting to investigate the correspondence between quantum string theory (quantum gravity) and gauge theory. Parts of this analysis has been done [25, 27] and lots remains to be done to complete this dictionary. It would be called under the rubric MT-AdS/CFT correspondence.

The collection of unstable non-BPS D0-branes in the presence of RR flux, stabilize via blowing up to a fuzzy 3-sphere, going over to spherical D3-branes or giant gravitons in the string theory (continuum) limit. The net RR charge of a spherical D3-brane is zero but there is a non-zero electric or magnetic dipole moment of the 4-form field developed by the vacuum expectation value of the tachyon, which stabilizes it against its tension. In fact, the vacuum structures of this Matrix theory are configurations of fuzzy 3-spheres.

Recall that type IIB is related to M-theory through \(T^2\) compactification. From eleven dimensional viewpoint, D3-branes are just M5-branes wrapped on \(T^2\). Following the same logic, we propose that D0-branes are just M2-branes compactified on \(T^2\), in such a way that the resulting state is a unstable brane [31]. On the other hand, a class of half-BPS solutions to eleven dimensional supergravity has been constructed in [32] which correspond to M2-branes dielectrically polarized into M5-branes and asymptotes

\[\text{For quantum string theory on the plane-wave background we can go beyond this limitation [1]. However the question of quantum string theory on AdS background is still there, which was the subject of this note.}\]
to eleven dimensional AdS space. We propose that this system is related to ours via $T^2$ compactification when one of the circles of the torus is interpreted as the tachyon field. In this sense tachyon can be thought of as one extra spatial direction. It has also been argued that fuzzy 3-sphere topologically is $S^3$ times two points. The extra factor, controlled by $T$, is reminiscent of the 11th circle. The eleventh dimension of M-theory which is hidden in perturbative string theory, manifests itself in non-perturbative string theory through tachyon. This issue is still under further study.

Furthermore, M2-branes can also be compactified in a proper way to lead to stable states, BPS KK gravitons. Hence, it is a natural expectation that fuzzy 3-sphere, as regularized spherical D3-brane giant graviton, could also be seen as bound state of point-like gravitons of type IIB theory. Furthermore, as a theory of quantum gravity, we suppose that the same Matrix theory could be derived from a system of $J$ number of BPS gravitons or gravitational waves of type IIB theory on the plane-wave background. In a paper in preparation, we show that this configuration stabilizes again to a fuzzy 3-sphere upon graviton condensation. In fact a microscopic description of giant gravitons has been given in [34] but it does not have the complete symmetry structure $SO(4)$.

In the sense of brane construction, one of the manifestation of the brane democracy is the ascending and descending relation between D-branes. By justifying the existence of non-BPS branes besides BPS ones and treating them on equal footing, we may extend this democracy to encompass non-BPS branes. Consider general form of the WZ part of the low energy effective action

$$S_{WZ} \sim \int_{M_{p+1}} \text{Tr} \mathcal{P} e^{i \sum X_i} C_n \left( \tau_{[X,T]} + \wedge DT \right) \wedge e^{DA+B}.$$ 

It manifests coupling of gauge field $A$ and tachyon field $T$ living on and scalars $X$ transverse to the brane, to RR fields of the bulk. Lower dimensional branes can be described in terms of degrees of freedom of higher dimensional ones through $DA$ and/or $DT$ and, conversely higher dimensional branes can be described by a system of lower dimensional one in terms of $[X,T]$ and/or $[X,X]$. Hence, D-branes are dynamically related by the process of open string tachyon condensation and/or transverse fluctuations. In the sense of ascending/descending relations among the branes, $A$ and $X$ fluctuations relate branes with even steps and $T$ fluctuations does with odd step. One would say that these processes lead to BPS branes, but on the other hand non-BPS ones can be realized as decay products in the process of annihilation of BPS branes and anti branes [8]. In this note starting from the lowest dimensional branes we have

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$^{10}$See also [33], where solutions with regular boundary conditions were found.

$^{11}$We would like to avoid Wick rotation to D-instantons, we prefer Minkowskian space rather than
constructed spherical D3-brane\textsuperscript{12}.

It is conceivable that both the BPS and non-BPS D\textit{p}-branes of type II theories have the same eleven dimensional origin and can be treated on equal footing, \textit{i.e.} they are BPS graviton, M2 or M5 branes which do or do not wrap around the extra spatial dimension seen as tachyon. Hence, the existence of non-BPS D\textit{p}-branes suggests the presence of at least one more dimension in critical string theory which is invisible in perturbation theory. In non-BPS D\textit{p}-branes tachyon looks as an extra transverse dimension to the brane [38].

Hilbert space of Matrix quantum mechanics naturally contains multi particle states. We may consider block-diagonal set of matrices which obey classically independent equations of motion, as describing two matrix theory objects [21]. This implies how matrix theory encodes a configuration of multiple objects. In this sense we can think of any matrix theory as a second-quantized theory from the point of view of the target space. Furthermore, since this Matrix theory is describing D3-branes, it is describing part of the moduli which is non-perturbative with respect to fundamental string theory [9]. Hence, this Matrix theory provides us with a second-quantized non-perturbative formulation of type IIB string theory.

As an equivalent description of a theory of quantum gravity, this Matrix theory should also provide a realization of the holographic principle. The plane-wave background has a 1-dimensional light-like boundary. This 0+1-dimensional gauge theory can be a reliable candidate for the holographic theory living on the boundary. It has been proposed that Bekenstein’s bound in terms of the DLCQ of M-theory on backgrounds with a null Killing direction with light-cone momentum \( p^+ \) reads [39]

\[ S_{DLCQ} \leq 2\pi^2 p^+. \]

It is interesting to test this proposal for the DLCQ of type IIB theory on the ten-dimensional plane-wave background. Closely related, it is also fascinating to study this Matrix theory at finite temperature due to its relation to Hagedorn behavior and black hole studies.

In the BFSS Matrix model for M-theory on the flat background, space arises as moduli space of vacua of scalar field in the 0+1-dimensional D0-brane gauge theory [40]. Vacuum expectation values of which correspond to transverse coordinates of the D0-brane. In this sense space is an emergent concept from more fundamental

\textsuperscript{12}Construction of D\textit{p}-branes from non-BPS D0-branes in flat background has been studied in the earlier works [36, 37].
degrees of freedom. We would like to extend this idea to the case of Matrix model for type IIB theory on the AdS background, where space arises from a 0+1-dimensional massive gauge theory. Transverse coordinates arise properly, longitudinal light-cone coordinate arises as the size of the matrices and as already proposed, tachyon arises as eleventh dimension. Furthermore, it also provides us with a suitable arena to study time-dependent phenomena and may shed light on the concept and role of time.

In the context of AdS/CFT correspondence it has been shown that non-BPS D0-branes are mapped to sphaleron solutions in the dual gauge theory [41]. They are unstable solutions located at the saddle point of the potential in the configuration space, at the top of a non-contractible loop. Here we showed that they stabilize to spherical D3-brane giant gravitons. It would be interesting to look for parallel process of tachyon condensation in the gauge theory side.

For the type IIB string theory on the flat space, there is a proposal for the Matrix formulation called IKKT Matrix model [42]. It is based on a system of D-instantons and derived as discretized Green-Schwarz action for fundamental string in the Schild gauge. In order to relate our Matrix theory to IKKT model we propose that in the absence of RR flux, non-BPS D0-branes decay and leave out D-instantons as decay products. In the presence of the flux, instead, they stabilized to another vacua which are fuzzy 3-spheres or discretized spherical D3-branes.

Finally, the most immediate and indispensable step to be taken in favor of our Matrix theory conjecture, is to show that non-BPS D0-brane is the only remaining and dominating degree of freedom for the DLCQ of type IIB theory on the AdS space. This parallels Seiberg-Sen argument for the DLCQ of M-theory on the flat background [2]. We would like to generalize it to the non-flat (highly curved) eleven and ten dimensional AdS backgrounds. It may also shed light on the relation between corresponding plane-wave limit and the DLCQ description. Parts of this argument is given in [43, 24] and appendix B, where it is argued that how the DLCQ of a theory reduces its underlying geometric and algebraic structure to corresponding Penrose limit and Inonü-Wigner contraction, respectively. Further evidence in support of this conjecture will be given in a future work [31].

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A  Fuzzy 3-Sphere

In this appendix we review fuzzy 3-sphere and the way of construction. More details can be found in [25].

Precise definition of round d-sphere is given in terms of constraints on embedding coordinates containing two $SO(d+1)$ invariant tensors $\delta_{mn}$, $\epsilon^{m_1m_2...m_{d+1}}$, as

$$\sum_{m=1}^{d+1} \delta_{mn} X^m X^n = R^2_d$$

(A.1a)

$$\{X^{m_1}, X^{m_2}, \ldots, X^{m_d}\}_{N.B.} = R^{-1}_{d-1} \epsilon^{m_1m_2...m_{d+1}} X^{m_{d+1}}.$$  

(A.1b)

Then fuzzy d-sphere, for even $d$, is directly defined by substituting continuous embedding functions with operator or matrices and Nambu bracket with commutator, as

$$\sum_{m=1}^{d+1} \delta_{mn} X^m X^n = R^2_d$$

(A.2a)

$$[X^{m_1}, X^{m_2}, \ldots, X^{m_d}]_{D.B.} = i^{[d+1/2]} J^{d-1} \epsilon^{m_1m_2...m_{d+1}} X^{m_{d+1}}.$$  

(A.2b)

For the case of odd-spheres, to overcome the difficulty of dealing with odd-brackets, we prescribed to add one fixed matrix to convert it into an even-bracket. For instance for fuzzy 3-sphere we have [9, 25]

$$\sum_{i=1}^{4} \delta_{ij} X^i X^j = R^2_3 1$$

(A.3a)

$$[X^i, X^j, X^k, \mathcal{L}_5] = -i^2 J^{ijkl} X^l.$$  

(A.3b)

The matrix equations (A.3) may be solved by the harmonic oscillator approach, according to which the embedding coordinates of the fuzzy spheres are related to the coordinates of a Moyal plane, $z_\alpha$ with harmonic non-commutative algebra through Hopf map.

$$V_{\infty \times \infty}^m = \bar{z}_\alpha (\gamma^m)_{\alpha \beta} z_\beta$$

(A.4)

$$N_{\infty \times \infty} = \bar{z}_\alpha \delta_{\alpha \beta} z_\beta .$$

(A.5)

where $\gamma$‘s and $\delta$ are matrix Glebsch-Gordon coefficients. First we would like to construct more handleable fuzzy 4-sphere by restricting to irreducible representation of $SO(5)$ which singles out $N \times N$ matrices inside infinite dimensional matrices. It is done by the projection matrix $\mathcal{P}_N$ as

$$W_{N \times N}^m = \mathcal{P}_N \bar{z}_\alpha (\gamma^m)_{\alpha \beta} z_\beta \mathcal{P}_N .$$

(A.6)
One can easily check that the above coordinates truly satisfy constraints of fuzzy 4-sphere \([A.2]\). Before moving to the construction of fuzzy 3-sphere, we would like to comment on the above construction of the 4-sphere. By definition an \(S^4\) is a four dimensional manifold with \(so(5)\) isometries. In the above we have given a specific embedding of a four sphere in an eight dimensional (noncommutative) space. More specifically, noting that \(N = \text{const.}\) defines an \(S^7\) in the eight dimensional space, we have an embedding of \(S^4\) into \(S^7\). This embedding is a (noncommutative) realization of the Hopf fibration with \(S^4\) as the base. Out of the \(so(8)\) isometries of the \(S^7\) there is a \(u(4)\) subgroup which is compatible with the holomorphic structure on \(\mathbb{C}^4 \simeq \mathbb{R}^8\). Note also that in the noncommutative Moyal case of \(\mathbb{C}_\theta^4\), that is this \(u(4) \subset so(8)\) which does not change the noncommutative structure. The \(X^\mu\) behaves as a vector under \(so(5) \subset su(4)\) and the generators of the full \(su(4)\) are \(X^\mu\) and \([X^\mu, X^\nu]\). (The generator of the \(u(1) \subset u(4)\) is \(N\)).

Finally, we construct fuzzy 3-sphere via fixing one direction and reducing one dimension. One should construct irreducible representation of \(SO(4)\) out of highly reducible one, by projecting out to \(J \times J\) matrices using the projector \(P_J\) as

\[
X^i_{J \times J} = P_J P_N \bar{z}_\alpha (\gamma^i)_{\alpha\beta} z_\beta \quad P_N P_J
\]

\[
L_5 = P_J P_N \bar{z}_\alpha (\gamma^5)_{\alpha\beta} z_\beta \quad P_N P_J.
\]

One can easily check that above definition of coordinates satisfies \([A.3]\). These equations fully contain \(SO(4)\) invariant matrices namely \(\delta_{ij}, \epsilon^{ijkl}, L_5\).

Note that there is yet another equivalent solution to the matrix equation. Suppose we take coordinates

\[
Y^i_{J \times J} = P_J P_N \bar{z}_\alpha (i \gamma^5 \gamma^i)_{\alpha\beta} z_\beta \quad P_N P_J \equiv i L_5 X^i_{J \times J}
\]

we can easily show that above definition of coordinates also satisfies \([A.3]\).

**B DLCQ on the AdS Space**

The DLCQ of a physical theory acts as Penrose limit and Inonu-Wigner contraction on its underlying geometrical and algebraic structures respectively \([43, 9, 24]\). Indeed the parameter \(\mu\) characterizes a homotopy indicating a plane-wave setup, at starting point of which \((\mu = 0)\) we have flat space.

One of the ingredients of the DLCQ is going to the frame of light-cone (freely-falling) observer who gives the simplest possible description. The other ingredient is the presence of a light-like circle. The whole idea is that the compact null direction
defined as a limit of a space-like circle. In this appendix we see that how these ideas apply on the AdS space and are encompassed in the Penrose limiting process on the geometry. Parallel process at the level of the algebra has been done in [24].

For that end consider $Z_M$ orbifolded $AdS_5 \times S^5$ space in the global coordinates with metric

$$ds^2 = R^2 \left( - \cosh^2 \rho \tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_5^2 + \cos^2 \theta d\phi^2 + d\theta^2 + \sin^2 \theta d\tilde{\Omega}_3^2 \right)$$

$$d\Omega_5^2 = d\alpha_1^2 + \sin^2 \alpha_1 (d\alpha_2^2 + \sin^2 \alpha_2 d\alpha_3^2) \quad , \quad d\tilde{\Omega}_3^2 = \cos^2 \psi d\chi^2 + d\psi^2 + \sin^2 \psi d\omega^2$$

$$\chi \rightarrow \chi + \frac{2\pi}{M} \quad , \quad \omega \rightarrow \omega - \frac{2\pi}{M}. \quad \text{(B.1)}$$

We focus on a null geodesic parameterized and defined by $\rho = 0$, $\psi = 0$, $\alpha = \frac{\pi}{2}$, $\chi = \tau$, and define the light-cone coordinates $x^\pm = 1/2(\tau \mp \chi)$. Orbifolding with $Z_M$ on and boosting with rapidity $\beta$ along the two isometric direction $\tau$ and $\chi$ changes light-cone coordinates as

$$x^\pm \rightarrow e^{\mp \beta} (x^\pm + \pi/M), \quad \text{(B.2)}$$

then we rescale coordinate and take the limit $R \rightarrow \infty$, $\beta \rightarrow \infty$, $M \rightarrow \infty$ while keeping following combinations fixed

$$X^\pm \equiv Rx^\pm, \quad x \equiv R\rho, \quad y \equiv R\theta, \quad z \equiv R\psi, \quad \frac{e^\beta}{R} \equiv \mu, \quad \frac{e^\beta R}{M} \equiv 2R_-. \quad \text{(B.3)}$$

With this limiting process, Penrose limit, neighborhood of the geodesic is stretched to become the whole space and the spacetime is then reduced to its corresponding plane-wave background

$$ds^2 = -2dX^+ dX^- - \mu^2 (x^2 + y^2 + z^2) (dX^+)^2 + dx^2 + x^2 d\Omega_5^2 + dy^2 + d\phi^2 + dz^2 + z^2 d\omega^2, \quad \text{(B.4)}$$

it can be rewritten in a more convenient form as

$$ds^2 = -2dX^+ dX^- - \mu^2 (X_1^2 + X_2^2) (dX^+)^2 + dX_1^2 + dX_2^2, \quad \text{(B.5)}$$

at the same time $X^- \rightarrow X^- + 2\pi R_-$ compactifies and we get a null circle as a limit of space-like circle. Similar limiting process is applied to the form field. In this frame, spacetime is parameterized by $X^+$, $X^-$ and $X^I$, the light-cone time, light-cone longitudinal and transverse coordinates respectively. In this basis momenta takes the form $p^+, P^-, P^I$ which are light-cone momentum, light-cone Hamiltonian and transverse momenta.
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