Counting small subgraphs in multi-layer networks

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Abstract

Motivated by the prevalence of multi-layer network structures in biological and social systems, we investigate the problem of counting the number of occurrences of (small) subgraphs or motifs in multi-layer graphs in which each layer of the graph has useful structural properties. Making use of existing meta-theorems, we focus on the parameterised complexity of motif-counting problems, giving conditions on the layers of a graph that yield fixed-parameter tractable algorithms for motif-counting in the overall graph. We give a dichotomy showing that, under some restricting assumptions, either the problem of counting the number of motifs is fixed-parameter tractable, or the corresponding decision problem is already \text{W}[1]-hard.

1 Introduction

A multilayer network includes edges that may be qualitatively different, and describe different types of interaction: for example, different varieties of social interaction, or physical as compared to electronic contact [23]. The capacity of multi-layer networks to richly represent physical and social systems has made their study one of the leading areas of research in network science. A desire to contribute algorithms to a multilayer setting and therefore build further links between algorithmic graph theory and application-focussed network science partially motivates our interest in motif embedding in layered graphs.

Some algorithmic problems related to multilayer networks could be addressed using traditional techniques by “flattening” all the layers to a single layer and solving the problem on this standard graph. However, in doing so we lose structural information about the original network that may be of use in solving the problem: it may be much easier to understand or predict the structure of individual layers than that of the flattened graph. In Section 2 we discuss a specific motivating example for this approach, based on the study of the spread of disease through a livestock contact network. We are interested in determining the circumstances under which we can exploit algorithmically useful (but different) structure in each of the layers to solve problems in the combined multilayer network.

For our first exploration of this question, we have chosen to investigate the problem of counting small subgraphs. The problem of counting the number of small substructures or motifs with certain properties in a large graph is of very general interest, having been linked to applications ranging from network security tools [15, 29, 30] to the analysis of biological networks [23]. Typically, the goal is to compare two networks or to monitor the evolution of a network over time by considering the number of specific motifs in the network.

The complexity of counting small subgraphs within a large host graph has also received a lot of attention from a theoretical perspective. Several of the problems introduced in the seminal paper by Flum and Grohe on parameterised counting complexity [12] are of this form, and very recently Curticapean, Dell and Marx [5] gave a dichotomy for the parameterised complexity of counting so-called network motif parameters, based on the structure of the motifs under consideration.
In this paper we focus instead on structural properties of the large host graph, while allowing arbitrary (small) motifs; the idea is to exploit the structure that is often present in real-world networks or the layers thereof. A number of celebrated meta-theorems (discussed in more detail in Section 1.3) give tractability results for the parameterised complexity of counting small subgraphs in graphs where certain width parameters are bounded, and our goal is to establish the conditions under which this tractability is inherited by a graph formed of two or more layers in each of which the problem is tractable. Under certain assumptions on the graph classes to which the layers of the network belong, we are able to prove a dichotomy result about the tractability of counting small subgraphs; in fact, we prove that in every case either the counting problem belongs to FPT or the decision version of the problem is already \( W[1] \)-hard.

1.1 Problem definitions

In this section we give formal problems of the subgraph counting problems we consider. An embedding of a graph \( H \) into a graph \( G \) is a mapping \( \theta \) from \( V(H) \) to \( V(G) \) such that, whenever \( uv \) is an edge in \( H \), we have that \( \theta(u)\theta(v) \) is an edge in \( G \). We consider the following problem.

\[ \text{p-\#Emb} \]

\textit{Input:} Two graphs \( G \) and \( H \).

\textit{Parameter:} \( k = |H| \).

\textit{Question:} How many embeddings are there of \( H \) into \( G \)?

We refer to \( G \) as the \textit{host graph} and \( H \) as the \textit{pattern graph}. We are also interested in particular in a generalisation of \( p\text{-Emb} \), in which each vertex of \( H \) must map to a specific subset of \( V(G) \) (but these subsets are not necessarily disjoint for distinct vertices of \( H \)):

\[ \text{p-\#Restricted-Emb} \]

\textit{Input:} Two graphs \( G = (V_G, E_G) \) and \( H = (\{v_1, \ldots, v_k\}, E_H) \), and subsets \( V_1, \ldots, V_k \subseteq V_G \).

\textit{Parameter:} \( k = |H| \).

\textit{Question:} How many embeddings \( \theta \) of \( H \) into \( G \) have the property that \( \theta(v_i) \in V_i \) for each \( i \)?

Note that \( \text{Emb} \) can be regarded as a special case of \( \text{Restricted-Emb} \) when \( V_1 = \cdots = V_k = V_G \). We will also consider the corresponding decision problems, which are defined as follows.

\[ \text{p-Emb} \]

\textit{Input:} Two graphs \( G \) and \( H \).

\textit{Parameter:} \( k = |H| \).

\textit{Question:} Does there exist an embedding of \( H \) into \( G \)?

\[ \text{p-Restricted-Emb} \]

\textit{Input:} Two graphs \( G = (V_G, E_G) \) and \( H = (\{v_1, \ldots, v_k\}, E_H) \), and subsets \( V_1, \ldots, V_k \subseteq V_G \).

\textit{Parameter:} \( k = |H| \).

\textit{Question:} Does there exist an embedding \( \theta \) of \( H \) into \( G \) such that \( \theta(v_i) \in V_i \) for each \( i \)?

We are interested in determining whether these problems admit fpt-algorithms, that is whether there is an algorithm running in time \( f(k) \cdot n^c \) where \( n \) is the total input size, \( k \) is the parameter, \( f \) is any (computable) function, and \( c \) is a fixed constant that does not depend on \( n \). In order to demonstrate that a problem is unlikely to admit an fpt-algorithm, it suffices...
to demonstrate that it is complete for the complexity class \( W[1] \). For further background on the theory of parameterised complexity we refer the reader to \[9, 13\].

Note that the “hardest” of these problems is \( p\text{-}\#\text{RESTRICTED-Emb} \), and the “easiest” is \( p\text{-Emb} \): the existence of an fpt-algorithm for \( p\text{-}\#\text{RESTRICTED-Emb} \) when restricted to host graphs from the class \( C \), implies the existence of such an algorithm for the other three problems in the same setting, whereas if any of the four problems defined above admits an fpt-algorithm when restricted to host graphs from \( C \) then there must be an fpt-algorithm for \( p\text{-Emb} \) under the same restriction. Thus, when proving tractability, the strongest result is to demonstrate the existence of an fpt-algorithm for \( p\text{-}\#\text{RESTRICTED-Emb} \), whereas the strongest hardness result is one for \( p\text{-Emb} \).

1.2 Notation

In this section we introduce the main notation which will be used throughout the paper.

The vertex cover number of a graph is the smallest number of vertices that must be deleted from \( G \) to obtain an independent set; a set of vertices whose deletion leaves an independent set is called a vertex cover. We recall that the problem of finding a vertex cover of size \( k \) is well known to be in FPT when parameterised by \( k \). We say that a class \( C \) of graphs has bounded vertex cover number if there exists a constant \( \ell \) such that every graph in \( C \) has vertex cover at most \( \ell \).

We say that a class \( C \) of graphs has almost bounded degree if there exists some constant \( \ell \) such that every element of \( C \) is has at most \( \ell \) vertices whose degree is strictly greater than \( \ell \).

Given two graph classes \( C_1 \) and \( C_2 \), we write \( \text{layer}(C_1, C_2) \) to denote the class of graphs of the form \( G = (V, E_1 \cup E_2) \) where \((V, E_1) \in C_1 \) and \((V, E_2) \in C_2 \); we will assume that an explicit partition of the edges is given for graphs belonging to \( \text{layer}(C_1, C_2) \). For \( s > 2 \), we write \( \text{layer}_s(C_1, \ldots, C_s) \) for the class of graphs of the form \( G = (V, E_1 \cup \cdots \cup E_s) \), where \((V, E_i) \in C_i \) for each \( i \).

Given any graph \( G = (V, E) \), and a vertex \( v \in V \), we write \( d_G(v) \) for the degree of \( v \) in \( G \). A graph \( G \) is \( \ell \)-regular if every vertex in \( G \) has degree exactly \( \ell \). Given a subset \( U \subseteq V \), we write \( G[U] \) for the subgraph of \( G \) induced by \( U \). If \( u, v \in V \), the distance between \( u \) and \( v \) in \( G \) is the number of edges on a shortest path between \( u \) and \( v \) in \( G \). A star is a graph isomorphic to the complete bipartite graph \( K_{1,p} \) for some \( p \in \mathbb{N} \). A star forest is an acyclic graph in which every connected component is a star.

1.3 Existing results on subgraph counting in restricted graph classes

As discussed above, there is a rich literature concerning the (parameterised) complexity of finding and counting specific small pattern graphs in a large host graph. For the purposes of our work, the most important results are those which determine classes of graphs on which the counting problems belong to FPT. The most general results of this kind are corollaries to two celebrated meta-theorems on the complexity of counting problems in restricted classes of graphs. Note that \( p\text{-}\#\text{RESTRICTED-Emb} \) can easily be expressed in first-order logic (and hence also in monadic second-order logic). We can therefore deduce the following results.

\textbf{Theorem 1.1} (Follows from \[13\]). \( p\text{-}\#\text{RESTRICTED-Emb} \) is in FPT when restricted to any class of graphs of bounded local treewidth.

\textbf{Theorem 1.2} (Follows from \[3\]). \( p\text{-}\#\text{RESTRICTED-Emb} \) is in FPT when restricted to any class of graphs of bounded cliquewidth.

Graphs of bounded degree have bounded local treewidth, so we have already established that \( p\text{-}\#\text{RESTRICTED-Emb} \) is in FPT on such graphs; however, a superior bound on the time needed to solve \( p\text{-}\text{Emb} \) can be obtained by considering the problem directly.
Lemma 1.3 ([27]). Given a connected graph $H$ on $k$ vertices and a graph $G$ on $n$ vertices with maximum degree $\Delta$, it is possible to determine exactly the number of induced copies of $H$ in $G$ in time

$$O(k^2n^2\Delta^{2(k-1)}).$$

This algorithm can easily be adapted to the case of $p$-#RESTRICTED-Emb.

1.4 Some simple observations on multi-layer graphs

As a prelude to our results, we make some initial observations about the relationship between structural properties of layers $(C_1,\ldots,C_s)$ and those of $C_1,\ldots,C_s$. First of all, we consider conditions on the maximum degree.

**Observation 1.** Suppose that, for each $i$, all graphs in $C_i$ have maximum degree at most $\Delta_i$. Then the maximum degree of any graph in layer $(C_1,\ldots,C_s)$ is at most $\sum_{i=1}^{s} \Delta_i$. In particular, if each of $C_i$ has bounded degree and $s$ is a fixed constant, then layer $(C_1,\ldots,C_s)$ has bounded degree.

We can make a similar observation regarding the vertex cover number; this relies on the fact that if, for each $i$, $W_i$ is a vertex cover for $G_i = (V_i,E_i)$, then $\bigcup_{1 \leq i \leq s} W_i$ is a vertex cover for $G = (V,E_1 \cup \cdots \cup E_s)$.

**Observation 2.** Suppose that, for each $i$, all graphs in $C_i$ have vertex cover number at most $x_i$. Then the vertex cover number of any graph in layer $(C_1,\ldots,C_s)$ is at most $\sum_{i=1}^{s} x_i$. In particular, if each of $C_i$ has bounded vertex cover number and $s$ is a fixed constant, then layer $(C_1,\ldots,C_s)$ has bounded vertex cover number.

However, for some other parameters we cannot draw a similar conclusion: treewidth is one such example, as two layers which both have treewidth one can give rise to a graph with arbitrarily high treewidth.

**Observation 3.** Let $\mathcal{P}$ be the set of all paths. Then layer($\mathcal{P},\mathcal{P}$) contains the $n \times n$ grid for each for each $n \in \mathbb{N}$, and hence has unbounded treewidth.

1.5 Our contribution

We consider the complexity of $p$-#RESTRICTED-Emb when restricted to classes of layered graphs, where each layer is drawn from a class on which $p$-#RESTRICTED-Emb belongs to FPT. On the positive side, we show in Section 3 that if either

1. all but one of the layers are drawn from classes of bounded vertex cover number, or
2. all of the layers have almost bounded degree,

then $p$-#RESTRICTED-Emb belongs to FPT. We show in Section 4 that in fact these conditions are not only sufficient but also necessary to ensure tractability if the classes under consideration are closed under deletion of both edges and vertices; moreover, even the decision problem $p$-Emb becomes hard in this case if the conditions are not met.

2 A motivating application

Many real-world systems that can be modelled by graphs consist of multiple types of connections that are qualitatively different, and produce graphs with different properties. Recent work in multi-layer network science[23,28,8] notes the importance and prevalence of this type of network
across the natural and social sciences, and the potential advantages of fully embracing the multi-layer approach when modelling real-life systems. This potential is a primary motivation for our work on motif-counting in multi-layer networks. As a motivating example of a multi-layer graph, we consider contact networks capable of spreading livestock disease, give evidence that the multi-layer nature of the contact graph is important to understanding disease spread, and outline how a multi-layer approach to motif-counting might be used.

In Great Britain (as elsewhere in the EU), cattle and sheep trading between farms and markets is recorded and reported to a central repository, as are geographic locations and adjacencies of farms [22]. These two types of contact, the long-range hub-and-spoke trading network, and the planar local contact network, can be considered as two different layers in the overall contact graph of British livestock farming.

There is significant evidence that both the long-distance animal trades and local geographic spread contribute to the spread of livestock disease in Britain, including the serious and economically-damaging 2001 outbreak of foot-and-mouth disease [19, 21]. Modelling these two types of contacts separately is a key feature of many successful models of livestock disease, including models used to understand and control foot-and-mouth disease [19, 16, 20, 21], bovine tuberculosis [1], blue-tongue virus [32], and the emerging Schmallenberg virus [31].

When considering these two main layers of the livestock contact system in Britain, it becomes immediately clear that they are very different, but that both have potentially useful characteristics. The geographically-local contact graph will necessarily be planar, and will have limited degree due to physical constraints: realistically-shaped pastures and farms can only neighbour a limited number of other farms, and cannot physically neighbour farms that are geographically far away. The long-range trading network depends on a relatively small number of markets that intermediate almost all trades [22]. If we consider both agricultural holdings and markets as vertices in the trading graph with animal movements as edges, then that trading graph has a small number of high-degree vertices and small vertex cover number - this is common for trading or contact networks, which often have power-law degree distributions [26]. Work on this 2-layer network, and similar networks describing human animal contacts, is a major focus of many epidemiological research groups.

With this 2-layer graph structure in mind, we describe an algorithm to solve \( p\text{-}\text{#RESTRICTED-EMB} \) on geometrically-embedded graphs with limited local vertex density and limited edge length. Note that such graphs will have maximum degree bounded by a function of the local vertex density and edge length, so inclusion in \( \text{FPT} \) follows from Lemma 1.3, but we are able to improve somewhat on the running time by exploiting the richer structure in this setting. In Section 3 we describe a general technique which allows us to exploit this method to count motifs in the overall 2-layer graph, using the fact that the second layer has limited vertex cover number.

First, we define more precisely our notion of density. For technical reasons, we will be interested in the density of vertices in semi-circular areas in which the straight-line segment of the semi-circle is vertical: we call such a semi-circle a vertical semi-circle. Given an arrangement of a graph in the plane, the density of a vertical semi-circle is the number of vertices contained within it divided by the area of the vertical semi-circle. The maximum vertical semi-circular density of a graph arranged in the plane is the maximum such density over all vertical semi-circles.

**Lemma 2.1.** If \( G \) is a graph on \( n \) vertices with an arrangement in the plane such that the longest edge is of geometric length \( \ell \), and maximum vertical semi-circular density \( \rho \), then there is an algorithm in time \( O(n|H|^2(\rho(|H|\ell)^2)\text{O}(|H|)) \) to count the embeddings of \( H \) in \( G \).

**Proof.** Let \( G = (V, E) \) be a graph with an arrangement in the plane such that the longest edge is of length \( \ell \), and with maximum vertical semi-circular density \( \rho \).

We will use an approach of scanning vertical semi-circular “windows” of the arrangement of \( G \), looking for embeddings of \( H \). We first define a polynomial number of vertical semi-circles, each with its vertical line segment centered on a vertex of the graph, and then argue that we need only search within each semi-circle for embeddings of \( H \).
If $\delta$ is the diameter of $H$, then for each vertex $v$ in $G$, consider the vertical semicircle of radius $\delta \ell$ centered at $v$. Let $W_v$ be the set of vertices that are contained in that vertical semicircle, including $v$ and those on the perimeter, except for those exactly vertically above $v$. We denote the set of all $n$ such vertex sets as $W$.

We say that an embedding occurs in a vertex set $W_v$ if it includes a mapping to $v$, and all mappings are to vertices in $W_v$. We now argue that each embedding of $H$ occurs in exactly one vertex set in $W$: the key idea is that every embedding of the $H$ in the arranged graph must have an uppermost leftmost vertex, and the embedding will occur in exactly the window anchored at that vertex.

Firstly, because motif $H$ has diameter $\delta$, and each edge in the arrangement is of length at most $\ell$, then certainly any embedding of $H$ with vertex $v$ as its uppermost leftmost vertex will fall entirely within the described semi-circular window used to produces $W_v$, as the geometrically farthest vertex will be at most $\delta \ell$ away from $v$, and must be non-left of $v$, nor vertically above it. Because we produced a vertex set $W_v$ for every vertex $v$, and every copy of the motif must have exactly one uppermost leftmost vertex, each copy will fall in at least one of our vertex sets.

On the other hand, because every embedding of a motif $H$ in the arranged graph has a unique uppermost leftmost vertex $v_a$, it occurs in at most one of the anchored windows in $W$, specifically in $W_{v_a}$.

Given a vertex set $W_v$ and a motif $H = (V_H, E_H)$, we can exhaustively check for each vertex $w \in V_H$ for all copies of $H$ in which $w$ is mapped to $v$ in time $O(|H|^2 |W_v|^{O(|H|)})$.

We can bound $|W_v|$ by the product of the maximum vertical semi-circular density of the graph arrangement $\rho$ and the size of the vertical semi-circles used to search for embeddings of $H$. We know the area of the vertical semicircle producing $W_v$ is $\frac{\pi(\delta \ell)^2}{2}$, therefore $|W_v| \leq \rho \frac{\pi(\delta \ell)^2}{2}$, and the running time is $O(|H|^2 (\rho \frac{\pi(\delta \ell)^2}{2})^{O(|H|)}) = O(|H|^2 (\rho(\delta \ell)^2)^{O(|H|)})$, or, as $\delta \leq |H|$, we can express this as $O(|H|^2 (\rho(|H|\ell)^2)^{O(|H|)})$. Because we must perform this search for each of $n$ anchored vertical semi-circular windows, this approach gives an overall running time of $O(n|H|^2 (\rho(|H|\ell)^2)^{O(|H|)})$. In some applications, a better bound might be obtained by considering densities appropriate for the size of semicircles in use: these densities will be upper bounded by $\rho$, but might sometimes be significantly smaller.

3 Tractable cases for counting

In this section we identify a number of situations in which $p$-\#\textsc{Restricted-Emb} belongs to \textsc{FPT}. We begin by showing that, if $p$-\#\textsc{Restricted-Emb} is in \textsc{FPT} when restricted to graphs from some class $C$, we can still solve the problem efficiently on any graph obtained from an element of $C$ by adding a constant number of layers each of which has bounded vertex cover number.

**Theorem 3.1.** Suppose that, when the host graph belongs to the class $C_1$, $p$-\#\textsc{Restricted-Emb} can be solved in time $f(k) \cdot n^c$ for some fixed constant $c$ and a computable function $f$, where $k$ and $n$ are the number of vertices in the pattern and host graphs respectively. For some fixed constant $s$, let $C_2, \ldots, C_s$ be classes of graphs of bounded vertex cover number. Then, when restricted to host graphs from $C = \text{layer}_s(C_1, \ldots, C_s)$, $p$-\#\textsc{Restricted-Emb} can be solved in time $g(k) \cdot n^c$ for some computable function $g$.

**Proof.** Note first that if $\ell$ is the maximum vertex cover number of any graph in $C_2 \cup \cdots \cup C_s$, then the vertex cover number of any element of $\text{layer}_{s-1}(C_2, \ldots, C_s)$ is at most $s\ell$, and hence is bounded by a constant. Thus it suffices to prove the result in the case that $s = 2$.

Suppose that the input to $p$-\#\textsc{Restricted-Emb} is $(G, H, V_1, \ldots, V_k)$, where $G = (V, E_1 \cup E_2) \in C$ with $G_1 = (V, E_1) \in C_1$ and $G_2 = (V, E_2) \in C_2$. We will assume that $G_2$ has a vertex cover $U$, where $|U| \leq \ell$. Then there are at most $(\ell + 1)^k$ possibilities for which vertices of $H$ map to elements of $U$ and the mapping restricted to this subset; we will consider each such possibility in turn. Note that for each of these possible partial mappings we can determine in
time $O(k^2)$ whether it does indeed define a partial embedding of a subgraph of $H$ into $G[U]$ such that each vertex $v_i$ in the domain maps to an element of $V_i$.

Suppose we have fixed a set $W \subseteq V(H)$ and an embedding $\theta_U$ of $H[W]$ into $G[U]$. Assume without loss of generality that $v_1, \ldots, v_r$ are the elements of $V(H) \setminus W$. For each $1 \leq i \leq r$, define $X_i$ to be the set of vertices in $V_i \setminus U$ whose neighbourhood contains the set $\{\theta_U(w) : v_iw \in E(H)\}$; we can compute each $X_i$ in time $O(kn)$. It is then clear that the number of ways to extend $\theta_U$ to an embedding $\theta$ of $H$ into $G$ such that $\theta(v_i) \in V_i$ for all $i$ is precisely equal to the number of embeddings $\pi$ of $H \setminus W$ into $G \setminus U$ such that $\pi(v_i) \in X_i$ for each $1 \leq i \leq r$. Note that such embeddings, as they do not use vertices of $U$, cannot use any edges of $G_2$, so we can equivalently consider the number of embeddings into $G_1 \setminus U$; moreover, as none of the sets $V_i$ intersects $U$, this quantity is the same whether we consider embeddings into $G_1 \setminus U$ or into $G_1$.

Thus it suffices to solve at most $(\ell + 1)^k$ instances of $p$-$\text{Restricted-Emb}$ in which the host graph belongs to $C_1$; as we are assuming that we can solve instances of $p$-$\text{Restricted-Emb}$ where the host graph comes from $C_1$ and the pattern graph has order $k$, it follows that we can solve $p$-$\text{Restricted-Emb}$ on host graphs from $C$ in time $O((\ell + 1)^k (k^2 + k^2n + f(k)n^c))$, as required.

It now follows easily that $p$-$\text{Restricted-Emb}$ belongs to FPT when restricted to the class of graphs of almost bounded degree.

**Corollary 3.2.** Let $C$ be a class of graphs of almost bounded degree. Then, when restricted to host graphs from $C$, $p$-$\text{Restricted-Emb}$ is in FPT.

**Proof.** Let $G = (V, E)$ be a graph in $C$, and suppose that $G$ has at most $\ell$ vertices of degree greater than $\ell$; let $U$ be the set of vertices of degree greater than $\ell$. We now set $E_2$ to be the set of edges in $E$ with at least one endpoint in $U$, and set $E_1 = E \setminus E_2$. Then we have an explicit decomposition of $G$ as $(V, E_1 \cup E_2)$ such that $G_1 = (V, E_1)$ has maximum degree at most $\ell$ and $G_2 = (V, E_2)$ has vertex cover number at most $\ell$. Since $p$-$\text{Restricted-Emb}$ is in FPT when the host graph has bounded degree by Lemma 1.3, the result now follows immediately from Theorem 3.1.

Finally, we observe that if each layer has almost bounded degree then the resulting graph also has almost bounded degree, and hence $p$-$\text{Restricted-Emb}$ is in FPT whenever each layer has almost bounded degree.

**Corollary 3.3.** Let $s$ be a fixed constant, and suppose that $C_1, \ldots, C_s$ are classes of almost bounded degree. Then, when restricted to host graphs from $C = \text{layer}_s(C_1, \ldots, C_s)$, $p$-$\text{Restricted-Emb}$ is in FPT.

**Proof.** We will argue that $C$ is in fact a class of graphs of almost bounded degree, and so the result follows immediately from Corollary 3.2.

By definition, there exist constants $\ell_1, \ldots, \ell_s$ such that every element of $C_i$ has at most $\ell_i$ vertices of degree greater than $\ell_i$. We now set $\ell = \sum_{i=1}^s \ell_i$, and claim that every element of $C$ has at most $\ell$ vertices of degree greater than $\ell$. Let $G$ be an element of $C$, where $G = (V, E_1 \cup \cdots \cup E_s)$ and each $G_i = (V, E_i)$ belongs to $C_i$. For each $i$, we set

$$U_i = \{u \in V : d_{G_i}(u) > \ell_i\},$$

and we set $U = \bigcup_{1 \leq i \leq s} U_i$. Note that

$$|U| \leq \sum_{i=1}^s |U_i| \leq \sum_{i=1}^s \ell_i = \ell.$$

Now suppose that $v \in V \setminus U$. Since $v \notin U_i$ for any $i$, we know that $d_{G_i}(v) \leq \ell_i$ for each $i$. Thus

$$d_G(v) \leq \sum_{i=1}^s d_{G_i}(v) \leq \sum_{i=1}^s \ell_i = \ell.$$

Therefore we have that at most $\ell$ vertices in $G$ have degree greater than $\ell$, as required. \qed
4 Hard cases for decision

In contrast with the tractability results above we prove that, for many graph classes, if the conditions given above for the existence of an fpt-algorithm for \( p \)\#RESTRICTED-EMB are not met, then in fact the corresponding decision problem is hard. Specifically, in this section we prove the following result.

**Theorem 4.1.** Let \( C_1 \) and \( C_2 \) be recursively enumerable classes of graphs of unbounded vertex cover number which are closed under the deletion of vertices and edges, and suppose further that \( C_1 \) does not have almost bounded degree. Then \( p\text{-EMB} \) (and hence \( p\text{-RESTRICTED-EMB} \)) is \( W[1] \)-complete when restricted to host graphs from \( C = \text{layer}(C_1, C_2) \).

The proof of Theorem 4.1 relies heavily on the following result.

**Theorem 4.2.** Let \( C_1 \) be the class of star forests and \( C_2 \) the class of 1-regular graphs. Then \( p\text{-EMB} \) is \( W[1] \)-hard even if the host graph is restricted to \( C = \text{layer}(C_1, C_2) \).

We give a reduction from the following problem, which was shown to be \( W[1] \)-complete in [11].

\[ p\text{-MULTICOLOUR CLIQUE} \]

**Input:** A graph \( G = (V, E) \) and a partition of \( V \) into \( k \) sets \( V_1, \ldots, V_k \)

**Parameter:** \( k \)

**Question:** Does \( G \) contain a clique with exactly one vertex in each set \( V_1, \ldots, V_k \)?

Let \( (G', \{V_1, \ldots, V_k\}) \) be the input to an instance of \( p\text{-MULTICOLOUR CLIQUE} \). We adapt a strategy which has previously been used in several contexts [15, 17, 2, 3, 10, 24, 6] to encode a \( k \)-clique with a \( k \times \binom{k}{2} \) grid. We begin by constructing two graphs \( \tilde{G} \) and \( \tilde{H} \) so that there is a restricted embedding of \( \tilde{H} \) into \( \tilde{G} \) if and only if \( G' \) contains a multicolour clique; we then show how to decorate \( \tilde{G} \) and \( \tilde{H} \) to obtain graphs \( G \) and \( H \) respectively so that there is an embedding of \( H \) into \( G \) if and only if there is an embedding of \( \tilde{H} \) into \( \tilde{G} \) which satisfies the restrictions. Finally, we will demonstrate that the edges of \( G = (V, E) \) can be partitioned into two sets \( E_1 \) and \( E_2 \) so that \( G_1 = (V, E_1) \) is a star forest and \( G_2 = (V, E_2) \) has maximum degree one.

We begin by defining \( \tilde{G} \) and \( \tilde{H} \). To help do so, we fix an ordered list of all unordered pairs of distinct elements of the set \( \{1, \ldots, k\} \), and write \( (i, j) \) for the \( i^{th} \) element in this list.

\( \tilde{H} \) is now defined as follows. \( \tilde{H} \) consists of \( k \) paths, each on \( 6 \binom{k}{2} \) vertices, with some additional edges: for each \( 1 \leq \ell \leq \binom{k}{2} \), if \( (i, j) \) is an edge between the \( (6(\ell - 1) + 3)^{th} \) vertices on the \( i^{th} \) and \( j^{th} \) path. Notice that this means that, between any two of the \( k \) paths, there is precisely one edge. Note that \( |V(H)| = 4k\binom{k}{2} = O(k^3) \). The structure of \( \tilde{H} \) is illustrated in Figure 1. It will be useful in the arguments that follow to refer to certain distinguished vertices of \( \tilde{H} \): we will refer to the \( (6j)^{th} \) vertex of the \( i^{th} \) path as \( u_{i,j} \).

We now define \( \tilde{G} \). The vertices of \( \tilde{G} \) can be partitioned into \( k \binom{k}{2} \) sets, which we denote \( W_{i,j} \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq \binom{k}{2} \). Each set \( W_{i,j} \) contains two kinds of vertices. For each vertex \( v \in V_i \) we call these anchor vertices. Additionally, for each pair \( (v, e) \) such that \( v \in V_i \) and \( e \in E(G') \) is incident with \( v \), we call the vertices of these paths path vertices.

The construction of \( \tilde{G} \) is illustrated in Figure 2. Note that \( |V(G)| \leq \binom{k}{2} 

\[ |V(G)| = \binom{k}{2} \cdot |V(G)|(|5|E(G)| + 1) = O(k^2|V(G)|^3). \]

We now argue that \( \tilde{H} \) and \( \tilde{G} \) have the desired properties; note that Lemma [4,3] alone demonstrates that \( p\text{-RESTRICTED-EMB} \) is \( W[1] \)-hard when the host graph is restricted to \( C \).
Let\( G \) be a multicolour graph with \( |V(G)| = n \) and \( |E(G)| = m \). Suppose \( G \) contains a multicolour clique. It is straightforward to see that there is an embedding \( \theta \) from \( V(\tilde{H}) \) to the vertex set
\[
\{w_i^j : 1 \leq j \leq \binom{k}{2}\} \cup \{P_{w_i,w_j}^k : \binom{k}{2}[\ell] = \{i,j\}\} \cup \{P_{w_i,e_i}^k : i \notin \binom{k}{2}[\ell]\}
\]
in which \( \theta(u_{i,j}) \) is an anchor vertex in \( W_{i,j} \) for each \( i \) and \( j \).

Conversely, suppose there is an embedding \( \theta \) of \( \tilde{H} \) into \( \tilde{G} \) such that \( \theta(u_{i,j}) \) is an anchor vertex in \( W_{i,j} \) for each \( i \) and \( j \). We define a mapping \( \phi : \{u_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \binom{k}{2}\} \rightarrow V(G') \) by setting \( \phi(u_{i,j}) \) to be the unique vertex \( w \in V_i \) such that \( \phi(u_{i,j}) = w_j \). We now set
\[
X = \{\phi(u_{i,j}) : 1 \leq i \leq k, 1 \leq j \leq \binom{k}{2}\}.
\]

We begin by arguing that \( X \) contains precisely one vertex from each colour class \( V_i \). It is clear that \( X \) contains at least one vertex from each colour class, as \( \phi(u_{i,1}) \in V_i \) for each \( i \). We now claim that, for each \( i \), \( X \cap V_i = \{\phi(u_{i,1})\} \). To see that this is true, suppose for a contradiction that this is not true for some \( i \). Then there is some \( j \in \{1,\ldots,\binom{k}{2}\} \) such that \( \phi(u_{i,j}) \neq \phi(u_{i,1}) \); fix the smallest \( j \) for which this is true. Note that the distance in \( \tilde{H} \) from \( u_{i,j-1} \) to \( u_{i,j} \) is six; however, the only element of \( W_{i,j-1} \) at distance six from \( \phi(u_{i,j-1}) = \phi(u_{i,j-1})^j \), so if \( \phi(u_{i,j}) \neq \phi(u_{i,j-1}) = \phi(u_{i,1}) \) then this contradicts the fact that \( \theta \) is an embedding of \( \tilde{H} \) into \( \tilde{G} \).

We now argue that \( X \) induces a clique in \( G' \). To do this, it suffices to show that every pair of vertices in \( X \) is adjacent. Fix \( r, s \in [k] \), and suppose that \( \binom{k}{2}[\ell] = \{r,s\} \). Set \( w_r = \phi(u_{r,1}) = \phi(u_{r,\ell}) \) and \( w_s = \phi(u_{s,1}) = \phi(u_{r,\ell}) \). Let \( x \) be the unique vertex of \( \tilde{H} \) at distance three from both \( u_{r,\ell} \) and \( u_{r,\ell-1} \), and \( y \) the unique vertex of \( \tilde{H} \) at distance three from both \( w_{s,\ell} \) and \( w_{s,\ell-1} \). Note that \( \theta(x) \) must be of the form \( P_{w_{r,e},[3]} \) for some edge \( e_r \), and \( \theta(y) \) of the form \( P_{w_{s,e},[3]} \) for some edge \( e_s \). Since \( r, s \in \binom{k}{2}[\ell] \), it follows from the definition of \( \tilde{G} \) that \( w_r \) is incident with \( e_r \) and \( w_s \) is incident with \( e_s \).

Notice that \( x \) and \( y \) are adjacent in \( \tilde{H} \), so \( \theta(x) \) and \( \theta(y) \) must be adjacent in \( \tilde{G} \). By definition, this edge is only present if in fact \( e_r = e_s \). Hence this edge is incident with both \( w_r \) and \( w_s \), as required.

We now show how to decorate \( \tilde{G} \) and \( \tilde{H} \) so that we can omit the restrictions on where certain vertices are mapped in the embedding.
Figure 2: An example of the construction of $\tilde{G}$ (bottom) from $G'$ (top). A subgraph of $\tilde{G}$ corresponding to the clique induced by $x_2, y_2$ and $z_1$ is highlighted.
Observe that $\tilde{G}$ is bipartite, and hence does not contain any cycles of odd length. The idea is to attach odd-length cycles of suitably chosen lengths to certain vertices of both $\tilde{G}$ and $\tilde{H}$ so that certain vertices of the new motif graph can only map to specific subsets of the new host graph.

Specifically, we define $H$ and $G$ as follows. We obtain $H$ from $\tilde{H}$ by, for each $1 \leq i \leq k$ and $1 \leq j \leq \binom{k}{2}$, adding a cycle of length $2\left((i-1)\binom{k}{2} + j\right) + 1$ which contains $u_{i,j}$ and $2\left((i-1)\binom{k}{2} + j\right)$ new vertices. Similarly, we obtain $G$ from $\tilde{G}$ by, for each $1 \leq j \leq \binom{k}{2}$ and $v \in V_i$, adding a cycle of length $2\left((i-1)\binom{k}{2} + j\right) + 1$ which contains $v'$ and $2\left((i-1)\binom{k}{2} + j\right)$ new vertices. The construction of $H$ and $G$ is illustrated in Figure 3.

**Lemma 4.4.** There is an embedding of $H$ into $G$ if and only if $G'$ contains a multicolour clique.

**Proof.** By Lemma 4.3, it suffices to show that there is an embedding of $H$ into $G$ if and only if there is an embedding $\theta$ of $\tilde{H}$ into $\tilde{G}$ such that $\theta(u_{i,j})$ is an anchor vertex in $W_{i,j}$ for all $1 \leq i \leq k$ and $1 \leq j \leq \binom{k}{2}$.

Suppose first that there is an embedding $\theta$ of $\tilde{H}$ into $\tilde{G}$ such that $\theta(u_{i,j})$ is an anchor vertex in $W_{i,j}$ for all $1 \leq i \leq k$ and $1 \leq j \leq \binom{k}{2}$. We define a embedding $\theta'$ of $H$ into $G$ by extending $\theta$ as follows: for any vertex $w \in V(H) \setminus V(\tilde{H})$ which belongs to a cycle containing $u_{i,j}$, we define $\theta'(w)$ to be the corresponding vertex on the cycle in $G$ that includes $\theta(u_{i,j})$. It is immediate from the construction of $G$ that such a cycle, of the correct length, exists.

Conversely, suppose that there is an embedding $\theta'$ of $H$ into $G$. Observe that $u_{i,j}$ belongs to a cycle of length $2\left((i-1)\binom{k}{2} + j\right) + 1$, and has degree at least four. Since $\tilde{G}$ is bipartite, the only odd length cycles in $G$ are those added in the construction of $\tilde{G}$ and in particular the only cycles in $G$ of length $2\left((i-1)\binom{k}{2} + j\right) + 1$ are those that contain an anchor vertex in $W_{i,j}$. Moreover, the only vertices belonging to such cycles that have degree greater than two are precisely the anchor vertices in $W + i, j$. Thus it must be that $\theta'(u_{i,j})$ is an anchor vertex of $W_{i,j}$ for each $i, j$.

Observe also that, for each $i, j$, the image $\theta'(V(H))$ cannot contain any anchor vertex of $W_{i,j}$ other than $\theta'(u_{i,j})$: the distance in $G$ between any two anchor vertices in $G$ is at least $W_{i,j}$ or between an anchor vertex of $W_{i,j}$ and one of $W_{i',j'}$ is at least six, but no vertex in $H \setminus \{u_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \binom{k}{2}\}$ is at distance more than three from some vertex $u_{i,j}$. Thus, as $H$ is connected, we can also rule out the possibility that $\theta'(V(\tilde{H}))$ includes any vertex of $G \setminus \tilde{G}$.

We therefore see that restriction $\theta$ of $\theta'$ to $V(H)$ is an embedding of $\tilde{H}$ into $\tilde{G}$ such that $\theta(u_{i,j})$ is an anchor vertex of $W_{i,j}$ for each $i$ and $j$, completing the proof.

Finally, it remains to show that we can decompose the edge-set of $G$ into two sets with the required properties.

**Lemma 4.5.** There exist two sets of edges $E_1$ and $E_2$ such that $G = (V_G, E_1 \cup E_2)$, $G_1 = (V_G, E_1)$ is a forest and $G_2 = (V_G, E_2)$ has maximum degree one.

**Proof.** We begin by defining our edge partition. The set $E_2$ contains the following edges:

- all edges with one endpoint in $W_{i,j}$ and the other in $W_{i',j}$ where $i \neq i'$;
- the edges $P^i_{v,e}[1]P^i_{v,e}[2]$ and $P^i_{v,e}[4]P^i_{v,e}[5]$ for each $v, e, i$;
- for each cycle in $E(\tilde{G}) \setminus E(G)$, every edge with an even index when the edges of the cycle are numbered consecutively and an edge incident with the vertex belonging to $\tilde{G}$ is numbered one.

All remaining edges are assigned to $E_1$. This partition of the edges is illustrated in Figure 4. It is straightforward to verify that $E_1$ and $E_2$ have the desired properties.
Figure 3: The construction of $H$ (top) and $G$ (bottom).
Figure 4: The partition of the edge-set of $G$ into $E_1$ and $E_2$: edges from $E_2$ (highlighted in the diagram) are disjoint, while the remaining edges induce a star forest.
Together, Lemmas 4.3, 4.4 and 4.5 complete the proof of Theorem 4.2. We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We will argue that $C_1$ contains all finite star forests and that $C_2$ contains all finite 1-regular graphs; the result will then follow immediately from Theorem 4.2.

First, let $G_1$ be an arbitrary star forest; we will argue that $G_1 \in C_1$. Let $\Delta$ be the maximum degree of $G_1$, and suppose that $G_1$ has exactly $c$ connected components. We will show that $cK_1, \Delta$, the star forest consisting of $c$ identical connected components, each isomorphic to $K_{1, \delta}$, belongs to $C_1$; the fact that $G_1 \in C_1$ will then follow from the fact that $C_1$ is closed under the deletion of vertices and edges. Since $C_1$ does not have almost bounded degree, there must be some graph $G_1 \in C_1$ which has at least $c(\Delta + 1)$ vertices of degree at least $c(\Delta + 1)$. In $G_1$ we can therefore find a collection of $c$ vertex-disjoint copies of $K_{1, \Delta}$ greedily. To do so we pick any vertex of degree at least $\Delta$ and delete it together with $\Delta$ of its neighbours; the deleted vertex set induces a graph which contains $K_{1, \Delta}$ as a subgraph, while the degree of any vertex in the rest of $G_1$ decreases by at most $\Delta + 1$. Thus, we will be able to repeat this process $c$ times to obtain our disjoint copies of $K_{1, \Delta}$; the fact that $cK_1, \Delta$ belongs to $C_1$ follows from the closure of $C_1$ under deletion of vertices and edges.

To see that $C_2$ contains all finite graphs of maximum degree one, fix some 1-regular graph $G_2$; suppose that $G_2$ has exactly $m$ edges and $2m$ vertices. It is well known that the size of the smallest vertex cover in a graph is equal to the size of the largest matching so, as $C_2$ does not have bounded vertex cover number it must contain graphs with arbitrarily large matchings and in particular there exists $G'_2 \in C_2$ which contains $m$ disjoint edges. Since $C_2$ is closed under the deletion of vertices and edges, it follows that the graph consisting of precisely $m$ disjoint edges belongs to $C_2$; but this graph is precisely $G_2$.

5 Conclusions and future work

We have determined some sufficient conditions for $p\text{-}\#\text{Restricted-Emb}$ to be in FPT when restricted to graphs from a layered class of graphs; moreover, we have demonstrated that, if we restrict our attention to layers drawn from graph classes which are closed under the deletion of both vertices and edges, then these same conditions are necessary for the existence of an FPT algorithm for even the decision problem $p\text{-Emb}$. These results are summarised in Table 1.

| $C_1$ | $C_2$ |
|-------|-------|
| Bounded vertex cover | FPT | FPT | FPT |
| Almost bounded degree | FPT | FPT | p-Emb is W[1]-hard |
| Not almost bounded degree | FPT | p-Emb is W[1]-hard | p-Emb is W[1]-hard |

Table 1: The complexity of $p\text{-}\#\text{Restricted-Emb}$ when restricted to host graphs from $C = \text{layer}(C_1, C_2)$, where $C_1$ and $C_2$ are classes of graphs, closed under deletion of edges and vertices, on which $p\text{-}\#\text{Restricted-Emb}$ is in FPT.

We note that our positive results easily extend to the setting where we wish to ensure that specific subsets of the edges of the pattern occur in distinguished layers of the host graph, or
indeed the case of directed or mixed pattern graphs when one or more layers include directed edges.

However, this work forms only the first step in understanding how the structural properties of different layers in a multilayer network contribute to the complexity of computational problems on the network. A natural next step would be to apply this approach to other computational problems that are well-understood in the single-layer setting.

Another potentially fruitful direction for future research would be to consider the effect of placing restrictions on how the layers can interact. In our work, we only considered structural restrictions on each layer individually, but we could equally enforce local restrictions that involve more than one layer (for example “every vertex must have the same degree in Layers 1 and 2”). It seems likely that in many applications there will additionally be constraints of this form, where the way in which we can learn something about how a vertex must interact with one layer by observing its interaction with another, so it would be interesting to explore what restrictions of this form give rise to new tractable cases.

Finally, in order to apply these methods as widely as possible, it would be useful to develop methods for decomposing a graph into two or more layers which have useful structural properties. This would be essential in the situation where we can only observe the existence or otherwise of some connection between two vertices, in spite of the fact that our understanding of the underlying system that creates the graph tells us that edges arise from several distinct processes, each with their own structural properties.

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