THE HORIZONFUNCTION COMPACTIFICATION OF THE ARC METRIC ON TEICHMÜLLER SPACE

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Abstract. The arc metric is an asymmetric metric on the Teichmüller space $\mathcal{T}(S)$ of a surface $S$ with nonempty boundary. In this paper we study the relation between Thurston’s compactification and the horofunction compactification of $\mathcal{T}(S)$ endowed with the arc metric. We prove that there is a natural homeomorphism between the two compactifications.

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Contents

1. Introduction 2
2. Preliminaries 5
3. Measured laminations and Thurston’s compactification 7
  3.1. Measured laminations 7
  3.2. Rational measured laminations are dense in $\mathcal{ML}(S)$. 9
  3.3. Thurston’s compactification 11
  3.4. Topology of the boundary 13
4. Thurston’s asymmetric metric 14
5. Geometry of the arc metric 18
  5.1. The arc metric 18
  5.2. The horofunction compactification 18
  5.3. Convergence in Thurston’s compactification 20
6. An inequality for length functions 21
  6.1. Estimation of arc length in a pair of pants 22
  6.2. Key inequality 29
7. Proof of Theorem 1 30
References 37

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1. Introduction

The horofunction compactification of a metric space \((X, d)\) was introduced by Gromov in his paper \([10]\). The idea is to consider the space \(C(X)\) of all real-valued continuous functions on \(X\), equipped with the topology of uniform convergence on compact sets, up to the equivalence relation which identifies two functions when they differ by an additive constant. The space \(X\) is then embedded into \(C(X)\) by the map that sends each point to (the equivalence class of) the map

\[ x \mapsto (d(\cdot, x) - d(x_0, x)). \]

The horofunction boundary is obtained by taking the closure of this embedding. A horofunction is an element of the horofunction boundary. Early work of Busemann contains the idea of a horofunction \([3]\). After the appearance of Gromov’s paper, the question of identifying the horofunctions and the horofunction boundaries of various spaces became a central question in metric and Riemannian geometry and it was solved for various spaces, including non-compact symmetric spaces, Riemannian spaces of negative curvature, the Heisenberg group, Hilbert geometry, finite-dimensional normed spaces, and others. See e.g. \([1], [11], [23], [24]\); see also the recent results in \([15]\) and the references there.

The Teichmüller space of a surfaces is equipped with various metrics, Riemannian and non-Riemannian, and many important results and problems were formulated with the aim of finding properties of that space which are analogous to properties of non-compact symmetric spaces or spaces of negative (or non-positive) curvature. The question of understanding horofunctions and the horofunction boundary of that space can be considered as part of this project. The horofunction boundary of the Thurston metric has been identified recently by Walsh \([25]\), and that of the Teichmüller metric by Liu and Su \([9]\). In this paper, we identify the horofunction boundary of the arc metric. This metric is defined on the Teichmüller space of a surface with nonempty boundary.

We now state our result more precisely and we provide some background.

Let \(R\) be an oriented surface of finite type genus \(g\) with \(n\) punctures and let \(T_{g,n}\) be the Teichmüller space of hyperbolic metrics on \(R\). Thurston introduced a compactification of \(T_{g,n}\) which he used in his classification of diffeomorphisms of surfaces \([22]\). The boundary of this compactification is the space of projective classes of measured foliations on \(R\). The action of the mapping class group of \(R\) on \(T_{g,n}\) extends continuously to Thurston’s boundary.

Thurston defined in \([21]\) an asymmetric Finsler metric on \(T_{g,n}\). The geodesics of this metric are families of extremal Lipschitz maps between hyperbolic surfaces. The space \(T_{g,n}\) endowed with Thurston’s metric is a complete (asymmetric) geodesic metric space. Unlike the classical Teichmüller metric, Thurston’s metric is not uniquely geodesic. A special class
of geodesics for this metric, called stretch lines, are constructed by “stretching” along complete geodesic laminations of hyperbolic surfaces, that is, geodesic laminations whose complementary regions are all ideal triangles. The introduction of this metric paved the way to a whole set of new interesting questions on the geometry of Teichmüller space [14, 16].

Thurston’s compactification and Thurston’s metric are closely related to each other. A connection between Thurston’s compactification and the geodesic rays of Thurston’s metric was pointed out by Papadopoulos [12]. To state things more precisely, let \( \mu \) be a complete geodesic lamination. (Note that we do not assume that \( \mu \) carries a transverse invariant measure of full support.) Associated to \( \mu \) is a global parametrization of \( \mathcal{T}_{g,n} \), called the cataclysm coordinates, sending \( \mathcal{T}_{g,n} \) to the set of measured foliations transverse to \( \mu \). The cataclysm coordinates extend continuously to Thurston’s boundary (see [12, Theorem 4.1] for a more precise statement). In particular, a stretch line is determined by a measured foliation \( F \) that is transverse to \( \mu \), called the horocyclic foliation associated to the stretch line. This stretch line converges to the projective class of \( F \) in Thurston’s boundary [12].

Walsh showed in [25] that Thurston’s compactification of \( \mathcal{T}_{g,n} \) can be naturally identified with the horofunction compactification with respect to Thurston’s metric. Horofunction boundaries have the property that each geodesic ray converges to a point on the boundary. As a corollary, every geodesic ray for Thurston’s metric converges to a point in Thurston’s boundary. Another corollary of Walsh’s result is that any isometry of \( \mathcal{T}_{g,n} \) equipped with Thurston’s metric induces a self-homeomorphism of Thurston’s boundary.

On the other hand, there is a “detour cost” distance defined on Thurston’s boundary, which is also asymmetric, which may take the value infinity and which is preserved by the isometries of \( \mathcal{T}_{g,n} \) equipped with Thurston’s metric. By calculating the detour cost between any two projective measured foliations, Walsh proved in [25] that, with some exceptional cases, the isometry group of \( \mathcal{T}_{g,n} \) equipped with Thurston’s metric is the extended mapping class group. These results are only valid in the case of surfaces without boundary. The reason is that several of Thurston’s fundamental results in [21] either fail or are unknown in the case of surfaces with boundary. In the case with boundary, the theory needs a serious modification and the definition of Thurston’s metric needs to be replaced by another metric.

In this paper, we compare Thurston’s compactification for Teichmüller spaces of surfaces with boundary with the horofunction boundary with respect to the arc metric introduced in [8].

Thurston’s asymmetric metric can be defined by a formula which compares lengths of simple closed curves computed with the metrics representing the two elements in Teichmüller space (see the definition in [4]). The arc metric for surfaces with boundary uses proper arcs instead of closed curves. This passage from curves to arcs is very natural but it addresses geometric questions which are far from obvious. For instance, it is unknown
whether the arc metric is Finsler, or whether it realizes the extremal Lipschitz constant of homeomorphisms between hyperbolic surfaces, as in the case of Thurston’s metric on Teichmüller spaces of surfaces without boundary. We also do not know whether two points in Teichmüller space are joined by a concatenation of stretch lines. Working with arcs, on a surface with boundary, instead of simple closed curves, involves several complications and requires new topological and geometrical tools, and this is what makes this subject interesting.

We now present our results in more detail. Let $S$ be a hyperbolic surface of finite area with totally geodesic boundary and let $\mathcal{T}(S)$ be the Teichmüller space of $S$. We describe (§3) an analogue of Thurston’s compactification of $\mathcal{T}(S)$ defined using hyperbolic length and intersection number with simple closed curves and simple arcs on $S$. The boundary of such a compactification is identified with the space of projective measured laminations on $S$, which is homeomorphic to a sphere (see Theorem 3.8 and Proposition 3.9).

We recall the definition of the arc metric in §5 and prove the following:

**Theorem 1.** Thurston’s compactification of $\mathcal{T}(S)$ is identified with the horofunction compactification of the arc metric on $\mathcal{T}(S)$ by a natural homeomorphism.

The proof of Theorem 1 depends on the study of the asymptotic behaviour of the geodesic lengths of simple closed curves and arcs along certain paths on $\mathcal{T}(S)$. In particular, we will show (Lemma 6.8) that there exists a path $X_t, t \in [0, +\infty)$ in $\mathcal{T}(S)$ such that each simple closed curve or simple arc $\alpha$ on $S$ satisfies

$$e^{t}i(\mu, \alpha) - C \leq \ell_{\alpha}(X_t) \leq e^{t}i(\mu, \alpha) + C_{\alpha},$$

where $C > 0$ is a uniform constant and $C_{\alpha} > 0$ is a constant depending on $\alpha$.

**Remark 1.1.** It is reasonable to conjecture that, in the case where $S$ has boundary, the isometry group of $\mathcal{T}(S)$ endowed with the arc metric is the (extended) mapping class group $\text{Mod}(S)$. In fact, let $S^d = S \cup \bar{S}$ be the double of $S$, obtained by taking the mirror image $\bar{S}$ of $S$ and by identifying the corresponding boundary components by an orientation-reversing homeomorphism. Then $S^d$ is a surface without boundary. We know that such a doubling induces an isometric embedding from $\mathcal{T}(S)$ to $\mathcal{T}(S^d)$ (see [2]). As a result, one may hope that Walsh’s argument can be applied. However, the proof of Walsh depends on Thurston’s construction of stretch maps, which does not apply to $\mathcal{T}(S)$ when the surface $S$ has boundary components. A further understanding of Thurston’s compactification of $\mathcal{T}(S)$ and the action of isometry group may require some generalized notion of (appropriately defined) “stretch maps” for surfaces with boundary.

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Throughout this paper, we denote by $S = S_{g,n,p}$ a connected orientable surface of finite type, of genus $g$ with $n$ punctures and $p$ boundary components. We always assume that the Euler characteristic $\chi(S) = 2 - 2g - n - p < 0$ and that the boundary of $S$, denoted by $\partial S$, is nonempty.

A hyperbolic structure on $S$ is a complete metric of constant curvature $-1$ such that

(i) each puncture has a neighborhood which is isometric to a cusp, i.e.,

\[ \text{to the quotient of } \{ z = x + iy \in \mathbb{H}^2 \mid y > a \}, \text{ for some } a > 0, \text{ by the group generated by the translation } z \mapsto z + 1; \]

(ii) each boundary component is a closed geodesic.

A marked hyperbolic surface is a pair $(X, f)$, where $X$ is a hyperbolic structure on $S$ and $f : S \to X$ an orientation-preserving homeomorphism. The map $f$ (or any homeomorphism homotopic to it) is called a marking. Two marked hyperbolic surfaces $(X_1, f_1)$ and $(X_2, f_2)$ are said to be equivalent if there exists an isometry $h : X_1 \to X_2$ which is homotopic to $f_2 \circ f_1^{-1}$ (note that in our setting, homotopies fix each boundary component setwise but they do not need to fix it pointwise). The reduced Teichmüller space $T(S)$ is the set of equivalence classes of marked hyperbolic structures on $S$.

**Remark 2.1.** Since all Teichmüller spaces that we consider are reduced, we shall omit the word “reduced” in our exposition. Furthermore, we shall sometimes denote an equivalence class of $(X, f)$ in $T(S)$ by $X$, without explicit reference to the marking or to the equivalence relation.

Let $S^d$ be the double of $S$ and $T(S^d)$ the Teichmüller space of $S^d$. Note that $S^d$ is a surface of genus $2g + p - 1$ with $2n$ punctures, without boundary. We construct a natural embedding of $T(S)$ into $T(S^d)$.

For any equivalence class of marked hyperbolic structures $[(X, f)] \in T(S)$, we let $\overline{X}$ be the isometric mirror image of $X$. Denote the mirror map by $J : X \to \overline{X}$. This map is an orientation-reversing isometry and satisfies $J^2 = \text{id}$. The surface $X^d$ is obtained in the following way: we take the disjoint union of $X$ and $\overline{X}$, and glue $\partial X$ with $\partial \overline{X}$ by the restriction of $J$ to the boundary. The map $J$ extends to an involution of $X^d$ and we still denote this extension by $J$. Taking the double of a Riemann surface is a well-known operation, and was already considered in Teichmüller’s paper [17].

To determine a point in $T(S^d)$, we have to choose a marking for $X^d$. Note that we can modify the marking $f : S \to X$ in its homotopy class in such a way that $f = \text{id}$ in a small collar neighborhood of each boundary component. We extend $f$ to a marking $\tilde{f} : S^d \to X^d$ by setting
\[ \tilde{f}(x) = J \circ f \circ J(x) \]
when \( x \in \overline{X} \). It is easy to check that the equivalence class \( [(X^d, \tilde{f})] \) is independent of the choice of \( (X, f) \in [(X, f)] \).

We set \( \Psi([(X, f)]) = [(X^d, \tilde{f})] \) and we use the notation \( \Psi(X) = X^d \) for simplicity. Then we have

**Proposition 2.2.** The map

\[
\Psi : \mathcal{T}(S) \rightarrow \mathcal{T}(S^d),
\]

\[
X \mapsto \Psi(X) = X^d.
\]

is an embedding.

**Proof.** An efficient way to see that \( \Psi \) is continuous and injective is to present \( \Psi \) in terms of Fenchel-Nielsen coordinates. We choose a maximal set \( \{\alpha_i\}_{i=1}^{3g-3+n+p} \) of mutually disjoint and non homotopic simple closed curves in the interior of \( S \), all of them non-trivial and not homotopic to boundary components. Denote the boundary components of \( S \) by \( \{\beta_j\}_{j=1}^p \). The map

\[
\mathcal{T}(S) \rightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n+p} \times (\mathbb{R}_+)^p,
\]

\[
X \mapsto (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)) \times \ell_{\beta_j}(X),
\]

where \( \ell_{\alpha_i}, \ell_{\beta_j} \) are the length coordinates and \( \tau_{\alpha_i} \) are the twist coordinates, defines the Fenchel-Nielsen coordinates of \( \mathcal{T}(S^d) \) by

\[
(\ell_{\alpha_i}, \tau_{\alpha_i}) \times (\ell_{\beta_j}, \tau_{\beta_j}) \times (\ell_{\overline{\alpha}_i}, \tau_{\overline{\alpha}_i}).
\]

Then the map \( \Psi \) can be written in the Fenchel-Nielsen coordinates as

\[
(\ell_{\alpha_i}, \tau_{\alpha_i}) \times \ell_{\beta_j} \mapsto (\ell_{\alpha_i}, \tau_{\alpha_i}) \times (\ell_{\beta_j}, 0) \times (\ell_{\overline{\alpha}_i}, -\tau_{\overline{\alpha}_i}).
\]

Note that \( \tau_{\overline{\alpha}_i} = -\tau_{\alpha_i} \) since the mirror image of a right twist deformation on \( X \) becomes a left twist deformation on \( \overline{X} \). \( \Box \)

The map \( \Psi \) will be an isometric embedding if we equip \( \mathcal{T}(S) \) with the arc metric and \( \mathcal{T}(S^d) \) with Thurston’s metric \( [8] \). We shall recall the definition of Thurston’s metric in \( \S 4 \) and the arc metric in \( \S 5 \).

We consider the involution \( J : S^d \rightarrow S^d \) as an element of the extended mapping class group of \( S^d \) (that is, we identity when needed a map with its homotopy class). We set

\[
\mathcal{T}^{sym}(S^d) := \{ R \in \mathcal{T}(S^d) \mid J(R) = R \}.
\]

It not hard to show that \( \Psi(\mathcal{T}(S)) = \mathcal{T}^{sym}(S^d) \).
3. Measured laminations and Thurston’s compactification

In this section, we shall recall the notion of measured lamination space and the Thurston compactification of Teichmüller space, and their extensions to hyperbolic surfaces with geodesic boundaries. Part of our results here is a continuation of work done in [8].

3.1. Measured laminations. In the setting of surfaces with boundary, we need to be precise on the definition of measured geodesic laminations that we deal with.

We endow $S$ with a fixed hyperbolic structure. A geodesic lamination $\lambda$ on $S$ is a closed subset of $S$ which is the union of disjoint simple geodesics called the leaves of $\lambda$. With such a definition, a leaf $L$ of $\mu$ may be a boundary component of $S$. It may also be a geodesic ending at a cusp or a boundary component of $S$ ($L$ may meet a boundary component of $S$ or spiral along it). If $L$ is a geodesic with some end at a point $p \in \partial S$, we require that $L$ is perpendicular to $\partial S$ at $p$.

Let $\lambda$ be a geodesic lamination on $S$ with compact support. A transverse measure for $\lambda$ is an assignment, for each embedded arc $k$ on $S$ transverse to $\lambda$ and with endpoints contained in the complement of $\lambda$, of a finite Borel measure $\mu$ on $k$ with the following properties:

1. The support of $\mu$ is $\lambda \cap k$.
2. For any two transverse arcs $k$ and $k'$ that are homotopic through embedded arcs which move their endpoints within fixed complementary components of $\mu$, the assigned measures satisfy $\mu(k) = \mu(k')$.

A measured geodesic lamination is a geodesic lamination $\lambda$ together with a transverse measure. To lighten notation, we shall sometimes talk about a “measured lamination” instead of a “measured geodesic lamination”. We shall denote such a measured lamination by $(\lambda, \mu)$ or, sometimes, $\mu$ for simplicity.

All the measured lamination are assumed to have compact support. An example of a measured lamination is a weighted simple closed geodesic, that is, a simple closed geodesic $\alpha$ equipped with a positive weight $a > 0$. The measure deposited on a transverse arc $k$ is then the sum of the Dirac masses at the intersection points between $k$ and $\alpha$ multiplied by the weight $a$. In the general case, a lamination is a finite union of uniquely defined minimal sub-laminations, called its components. They are of the following three sorts:

(i) a simple closed geodesic in $S$ (such a simple closed geodesic can be a boundary component);
(ii) a geodesic arc meeting $\partial S$ at right angles;
(iii) a measured geodesic lamination in the interior of $S$, in which every leaf is dense.
This follows from our definition and from the corresponding result for surfaces without boundary.

Let \( M_L(S) \) be the space of measured geodesic laminations on \( S \). We shall equip \( M_L(S) \) with the weak\(^*\)-topology, following Thurston [20] in the case of surfaces without boundary. We can choose a finite collection of generic geodesic arcs \( k_1, \ldots, k_m \) on \( S \) such that \( \mu_n \in M_L(S) \) converges to \( \mu \) if and only if

\[
\max_{i=1,\ldots,m} \left| \int_{k_i} d\mu_n - \int_{k_i} d\mu \right| \to 0.
\]

Here a geodesic arc is called generic if it is transverse to any simple geodesic on \( S \). Note that almost every geodesic arc on \( S \) is generic [2].

We also recall that there are natural homeomorphisms between the various measured lamination spaces when the hyperbolic structure on the surface varies. Using this fact, it is possible to talk about a measured geodesic lamination on the surface without referring to a specific hyperbolic structure on it.

Let \( S^d \) be the double of \( S \) and \( M_L(S^d) \) the space of measured geodesic laminations on \( S^d \). As before, denote the natural involution of \( S^d \) by \( J \). For any subset \( A \subset S \) or \( A \subset \overline{S} \), we denote by \( \overline{A} = J(A) \). Moreover, if \( \mu \) is a measure on an arc \( I \) on \( S \) or \( \overline{S} \), then we set \( \overline{\mu}(I) = \mu(J(I)) \). From the above definition of measured geodesic lamination on \( S \), there is a natural inclusion \( \psi : M_L(S) \to M_L(S^d) \) defined by

\[
\psi : M_L(S) \to M_L(S^d)
\]

\[ (\lambda, \mu) \mapsto (\lambda \cup \overline{\lambda}, \mu + \overline{\mu}) \]

We will use the notation \( \mu^d = (\lambda \cup \overline{\lambda}, \mu + \overline{\mu}) \) for simplicity. Note that if \( \mu \) is a weighted simple closed geodesic \( (\alpha, a) \) where \( \alpha \) is a boundary component of \( S \) and \( a \) the weight it carries, then \( \mu^d = (\alpha, 2a) \).

A measured lamination (respectively, hyperbolic structure, simple closed curve, etc.) on \( S^d \) is said to be symmetric if it is invariant by the canonical involution \( J \). Denote the subset of all symmetric measured laminations in \( M_L(S^d) \) by \( M_L^{sym}(S^d) \).

**Lemma 3.1.** The map \( \psi : M_L(S) \to M_L(S^d) \) is continuous and

\[ M_L^{sym}(S^d) = \psi(M_L(S)). \]

**Proof.** It is obvious that all elements in \( \psi(M_L(S)) \) are symmetric.

Conversely, let \( \overline{\mu} \) be a symmetric measured lamination in \( M_L(S^d) \). Every component of \( \mu \) which meets the fixed point locus of the involution \( J \) is, if it exists, a simple closed geodesic. Indeed, such a component must intersect the fixed point locus perpendicularly, and no component which is not a simple closed geodesic can intersect the fixed point locus in this way, because of the recurrence of leaves in the closed surface \( S^d \). It follows that any sublamination of \( \overline{\mu} \) which is connected (that is, which has only one component) that
intersects $\partial S$ is either a boundary component of $S$ or a symmetric closed geodesic meeting $\partial S$ at right angles. As a result, the restriction of $\tilde{\mu}$ to $S$ defines a unique measured lamination $\mu \in \mathcal{ML}(S)$ such that $\tilde{\mu} = \psi(\mu) = \mu^d$.

The continuity of $\psi$ follows directly from the definition of the weak$^*$-topology on measured lamination spaces. □

3.2. **Rational measured laminations are dense in $\mathcal{ML}(S)$**. We say that a simple closed curve on a surface is *essential* if it is neither homotopic to a puncture nor homotopic to a point (but it can be homotopic to a boundary component). We let $\mathcal{C}(S)$ be the set of homotopy classes of essential simple closed curves on $S$.

In the case where $\partial S$ is nonempty, an *arc* in $S$ is the homeomorphic image of a closed interval which is properly embedded in $S$ (the interior of the arc is in the interior of $S$ and the endpoints of the arc are on the boundary of $S$). All homotopies of arcs that we consider are relative to $\partial S$, that is, they keep the endpoints of arcs on the set $\partial S$ (but they do not necessarily fix pointwise the points on $\partial S$). An arc is said to be *essential* if it is not homotopic to a subset of $\partial S$. We let $\mathcal{A}(S)$ be the set of homotopy classes of essential arcs on $S$.

Endowing $S$ with a hyperbolic structure $X$, for any $\gamma \in \mathcal{A}(S) \cup \mathcal{C}(S)$, there is a unique geodesic $\gamma^X$ in its homotopy class. It is orthogonal to $\partial X$ at each intersection point, in the case where $\gamma$ is an equivalence class of arc. We denote by $\ell_\gamma(X)$ the length of $\gamma^X$, and we call it the *geodesic length* of $\gamma$ on $X$. This length only depends on the equivalence class of $X$ in Teichmüller space.

The geodesic representation $\gamma \mapsto \gamma^X$ defines a correspondence between $\mathbb{R}_+ \times (\mathcal{A}(S) \cup \mathcal{C}(S))$ and the set of weighted simple closed geodesics union weighted simple geodesic arcs on $S$.

A measured lamination $\mu$ is *rational* if the support of $\mu$ consists of simple closed geodesics or simple geodesic arcs. Let us denote a rational measured lamination by

$$\sum_{i \in I} a_i \gamma_i,$$

where $I$ is some finite set, $a_i > 0$ and the $\gamma_i \in \mathcal{A}(S) \cup \mathcal{C}(S)$ are pairwise disjoint.

The set of weighted simple closed curves on $S^d$ is dense in the space $\mathcal{ML}(S^d)$, and the geodesic length function, defined on weighted simple closed geodesics, extends to a continuous function on the space $\mathcal{ML}(S^d)$ [20]. The situation is different for surfaces with boundary.

In general, the set $\mathbb{R}_+ \times \mathcal{A}(S) \cup \mathbb{R}_+ \times \mathcal{C}(S)$ is not dense in $\mathcal{ML}(S)$. For example, if $\mu = \alpha + \beta$ where $\alpha$ is a simple closed curve in the interior of $S$ and $\beta$ is a boundary component of $S$, then $\mu$ cannot be approximated by any sequence in $\mathbb{R}_+ \times \mathcal{A}(S) \cup \mathbb{R}_+ \times \mathcal{C}(S)$. However, using multiple curves and arcs instead of curves and arcs suffices, and we have the following:
Lemma 3.2. The set of rational measured laminations on $S$ is dense in $\mathcal{ML}(S)$.

Proof. Let $\mu \in \mathcal{ML}(S)$. Each component of $\mu$ is either a simple closed geodesic, a geodesic arc or a minimal measured lamination in the interior of $S$. Since any minimal component of $\mu$ can be approximated by a sequence in $\mathbb{R}_+ \times \mathcal{C}(S)$, we are done. □

Proposition 3.3. For every $X$ and $Y$ in $\mathcal{T}(S)$, we have

\[ \sup_{\gamma \in \mathcal{C}(S) \cup \mathcal{A}(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \sup_{\mu \in \mathcal{ML}(S)} \frac{\ell_\mu(Y)}{\ell_\mu(X)}. \]

Proof. It is obvious that

\[ \sup_{\gamma \in \mathcal{C}(S) \cup \mathcal{A}(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} \leq \sup_{\mu \in \mathcal{ML}(S)} \frac{\ell_\mu(Y)}{\ell_\mu(X)}. \]

Let us set $\mathcal{ML}_1(S) = \{ \mu \in \mathcal{ML}(S) \mid \ell_\mu(X) = 1 \}$ and $\mathcal{ML}_2(S^d) = \{ \bar{\mu} \in \mathcal{ML}(S^d) \mid \ell_{\bar{\mu}}(X) = 2 \}$. The map $\psi$ sends $\mathcal{ML}_1(S)$ into $\mathcal{ML}_2(S^d)$.

Since $\mathcal{ML}_2(S^d)$ is compact and $\psi(\mathcal{ML}_1(S))$ is a closed subset of $\mathcal{ML}_2(S^d)$, $\mathcal{ML}_1(S)$ is a compact subset of $\mathcal{ML}(S)$. Therefore, there is a measured lamination $\mu_0 \in \mathcal{ML}_1(S)$ that realizes the maximum:

\[ \sup_{\mu \in \mathcal{ML}(S)} \frac{\ell_\mu(Y)}{\ell_\mu(X)} = \frac{\ell_{\mu_0}(Y)}{\ell_{\mu_0}(X)}. \]

Consider the decomposition of $\mu_0$ into minimal components,

\[ \mu_0 = \sum_i a_i \nu_i. \]

Let $K$ be the value of the supremum in (2). We have $\ell_{\mu_0}(Y) = K \ell_{\mu_0}(X)$, that is, since the length function is positively homogeneous,

\[ \sum_i a_i \ell_{\nu_i}(Y) = K \sum_i a_i \ell_{\nu_i}(X). \]

Since $\ell_{\nu_i}(Y) \leq K \ell_{\nu_i}(X)$ (from the definition), it follows that $\ell_{\nu_i}(Y) = K \ell_{\nu_i}(X)$ for each $\nu_i$. As a result, any component of $\mu_0$ also realizes the supremum $L$.

As before, since each component of $\mu_0$ is either a simple closed geodesic, a geodesic arc or a minimal measured lamination in the interior of $S$, each of which can be approximated by a sequence in $\mathbb{R}_+ \times (\mathcal{A}(S) \cup \mathcal{C}(S))$, we conclude that

\[ \sup_{\gamma \in \mathcal{C}(S) \cup \mathcal{A}(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \sup_{\mu \in \mathcal{ML}(S)} \frac{\ell_\mu(Y)}{\ell_\mu(X)}. \]

□
Denote by $B$ the set of all boundary components of $S$. In the paper [8], the following was shown:

**Proposition 3.4.**

$$\sup_{\gamma \in C(S) \cup A(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \sup_{\gamma \in B(S) \cup A(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} \geq 1,$$

and the inequality becomes an equality if and only if $X = Y$.

**3.3. Thurston’s compactification.** We need to recall some fundamental results of Thurston from [6].

Let $R$ be a surface of genus $g$ with $n$ punctures and without boundary components. Let $\mathbb{R}_+^C(R)$ be the set of all nonnegative functions on $C(R)$ and $P\mathbb{R}_+^C(R)$ the projective space of $\mathbb{R}_+^C(R)$ (that is, its quotient by the action of positive reals), and denote by $\pi : \mathbb{R}_+^C(R) \to P\mathbb{R}_+^C(R)$ the natural projection. We endow $\mathbb{R}_+^C(R)$ with the product topology and $P\mathbb{R}_+^C(R)$ with the quotient topology. There is a mapping $L$ defined by

$$L : \mathcal{T}(R) \to \mathbb{R}_+^C(R),$$

and

$$X \mapsto (\ell_\alpha(X))_{\alpha \in C(R)}.$$  

The map $\pi \circ L : \mathcal{T}(R) \to P\mathbb{R}_+^C(R)$ is an embedding.

There is also a mapping $I$, defined by

$$I : \mathcal{ML}(R) \to \mathbb{R}_+^C(R),$$

and

$$\mu \mapsto (i(\mu, \alpha))_{\alpha \in C(R)},$$

where

$$i(\mu, \alpha) = \inf_{\alpha' \in [\alpha]} \int_{\alpha'} d\mu$$

is the intersection number. Then $I$ is also an embedding.

Thurston showed that the closure of $\pi \circ L(\mathcal{T}(R))$ is compact and coincides with

$$\pi \circ L(\mathcal{T}(R)) \cup \pi \circ I(\mathcal{ML}(R)).$$

We denote this closure by $\overline{\mathcal{T}}(R)$. This is **Thurston’s compactification** of $\mathcal{T}(R)$. In the following, we shall identify $\mathcal{T}(R)$ with its image and the boundary of $\overline{\mathcal{T}}(R)$ with $\mathcal{PM}\mathcal{L}(R)$, the space of projective classes of measured laminations on $R$.

Now we introduce an analogue of Thurston’s compactification for the Teichmüller space $\mathcal{T}(S)$, where $S$ is a surface with boundary. For simplicity, let $C = C(S)$ and $A = A(S)$.

Consider the map

$$\mathcal{T}(S) \xrightarrow{L} \mathbb{R}_+^{C \cup A} \xrightarrow{\pi} P\mathbb{R}_+^{C \cup A}.$$  

**Lemma 3.5.** The map defined in (3) is injective.
Proof. Suppose that $X, Y \in \mathcal{T}(S)$ are mapped to the same point in $P_{\mathbb{R}_+^{C \cup A}}$. Then there exists a constant $K > 0$ such that

$$\ell_{\gamma}(X) = K \ell_{\gamma}(Y)$$

for all $\gamma \in C \cup A$. Without loss of generality, we may assume that $K \geq 1$. This implies that

$$\sup_{\gamma \in C \cup A} \left\{ \ell_{\gamma}(Y) / \ell_{\gamma}(X) \right\} \leq 1.$$ 

It follows from Proposition 3.4 that $X = Y$. □

Similarly, we consider $\mathcal{ML}(S) \xrightarrow{I} \mathbb{R}_+^{C \cup A}$.

Lemma 3.6. The map defined in (4) is injective.

Proof. Suppose that $\mu, \nu \in \mathcal{ML}(S)$ are mapped to the same point in $\mathbb{R}_+^{C \cup A}$. Let

$$\mu = \mu_0 + \mu_1, \nu = \nu_0 + \nu_1,$$

where $\mu_0, \nu_0$ are the union of components contained in the interior of $S$ and $\mu_1, \nu_1$ are the union of components that belong to $\partial S$ or intersect $\partial S$ (some of these components might be empty). Since $\mu_1$ and $\nu_1$ (if they exist) are simple geodesic arcs or boundary components of $S$, it is easy to see that $\mu_1 = \nu_1$. For otherwise, if $\mu_1$ exists and $\nu_1 \neq \mu_1$, there exists some element $\gamma$ in $C \cup A$ such that $i(\mu_1, \gamma) \neq 0$, while $i(\nu_1, \gamma) = 0$.

On the other hand, since $\mu_0$ and $\nu_0$ are contained in the interior of the surface, by the same argument as for a surface without boundary (may be with punctures), we have $\mu_0 = \nu_0$. It follows that $\mu = \nu$. □

Remark 3.7. Both Lemmas 3.5 and 3.6 can be proved directly by the same arguments as [6, Exposés 6 and 7]. Note that the images of $\mathcal{T}(S)$ and $\mathcal{ML}(S)$ in $\mathbb{R}_+^{C \cup A}$ are disjoint. This follows from the fact that for each $X \in \mathcal{T}(S)$, the set of lengths $\ell_{\gamma}(X), \gamma \in C \cup A$ is bounded below by a strictly positive constant (only depending on $X$), while for each $\mu \in \mathcal{ML}(S)$ and for any $\epsilon > 0$, there is some $\gamma \in C \cup A$ such that

$$i(\mu, \gamma) < \epsilon.$$ 

Here $\gamma$ can be taken to be a simple closed curve, a simple arc belonging to a component of $\mu$ (if it exists) or a simple closed curve quasi-transverse to $\mu$ (see [6, Proposition 8.1] for details).

By Lemma 3.5, Lemma 3.6 and Remark 3.7, we have an embedding

$$\mathcal{T}(S) \cup \mathcal{PM}(S) \rightarrow P_{\mathbb{R}_+^{C \cup A}}.$$ 

We have already identified $\mathcal{T}(S)$ with the subset $\mathcal{T}^{sym}(S^d)$ of $\mathcal{T}(S^d)$ by the map $\Psi$ and $\mathcal{PM}(S)$ with the subset $\mathcal{PM}^{sym}(S^d)$ of $\mathcal{PM}(S^d)$ by the map $\psi$. To give an idea of the image of $\mathcal{T}(S) \cup \mathcal{PM}(S)$ in $P_{\mathbb{R}_+^{C \cup A}}$,
we shall show that the convergence of sequences in $\mathcal{T}(S)$ in the topology of $P^{\mathcal{C}(S)}_+$ is equivalent to the convergence in the topology of $P^{\mathcal{C}(S)}_{+\cup A}$.

Let $\{X_n\}$ be a sequence in $\mathcal{T}(S)$ and let $\{X^d_n\}$ the corresponding sequence in $\mathcal{T}^{sym}(S^d)$.

Assume that $X^d_n$ converges to a point $\bar{\mu} \in \mathcal{PML}(S^d)$ in the topology of $P^{\mathcal{C}(S^d)}_+$. An element in $P^{\mathcal{C}(S^d)}_+$ is in the image of $\mathcal{T}^{sym}(S^d)$ or of $\mathcal{ML}^{sym}(S^d)$ if and only if as a function on the set of homotopy classes of curves $\mathcal{C}(S^d)$ it has the same values on pairs of curves that are images of each other by the involution $J$ of $S^d$. Thus, since $X^d_n$ is symmetric, $\bar{\mu}$ is also symmetric. It follows that $X_n$ converges to $\mu$ (which satisfies $\bar{\mu} = \mu^d$) in the topology of $P^{\mathcal{C}(S^d)}_+$.

Conversely, assume that $X_n$ converges to a point $P$ in $P^{\mathcal{C}(S^d)}_+\cup A(S)$. Let $\bar{\mu}$ be any accumulation point of $X^d_n$ in $\mathcal{PML}^{sym}(S^d)$. By definition, there exists a sequence $c_n > 0$ such that (up to a subsequence)

$$c_n \ell_\gamma(X^d_n) \to i(\bar{\mu}, \gamma)$$

for any $\gamma \in \mathcal{C}(S^d)$. Setting $\bar{\gamma} = J(\gamma)$, we have

$$c_n \ell_\gamma(X^d_n) = c_n \ell_{\bar{\gamma}}(X^d_n) \to i(\mu, \gamma) = i(\mu, \bar{\gamma}).$$

In particular, we have

$$i(\bar{\mu}, \gamma) = i(\bar{\mu}, \bar{\gamma})$$

for any $\gamma \in \mathcal{C}(S^d)$. Such a $\bar{\mu}$ must be symmetric and unique (the restriction of $\bar{\mu}$ on $S$ is identified with $P$).

In conclusion, we have

**Theorem 3.8.** $\mathcal{PML}(S)$ is identified with the boundary of $\mathcal{T}(S)$ in $P^{\mathcal{C}(S)}_+\cup A$. The embedding $\Psi : \mathcal{T}(S) \to \mathcal{T}(S^d)$ extends to $\mathcal{T}(S) \cup \mathcal{PML}(S)$ such that $\Psi|_{\mathcal{PML}(S)} = \psi$.

### 3.4. Topology of the boundary

Let $S$ be a surface of genus $g$, with $p$ punctures and with $b$ boundary components, denoted by $\{B_1, \ldots, B_b\}$. A pants decomposition of $S$ contains $3g - 3 + b + p$ pairwise disjoint interior curves, denoted by $\{C_1, \ldots, C_{3g-3+b+p}\}$, which decompose the surface into $2g - 2 + b + p$ pairs of pants. Such a pants decomposition induces a symmetric pants decomposition of the double $S^d$, with $6g - 6 + 3b + 2p$ curves denoted by

$$\{C_1, \ldots, C_{3g-3+b+p}, B_1, \ldots, B_b, \bar{C}_1, \ldots, \bar{C}_{3g-3+b+p}\},$$

dividing $S^d$ into $4g - 4 + 2b + 2p$ pairs of pants.

The space of measured laminations $\mathcal{ML}(S^d)$ can be parametrized by the Dehn-Thurston coordinates associated with a pants decomposition.

Given a measured lamination $\mu$, for every curve $C$ in the symmetric pant decomposition of $S^d$, there are two associated coordinates, the length coordinate $i(\mu, C) \in \mathbb{R}_{\geq 0}$ and the twist coordinate $\theta(\mu, C) \in \mathbb{R}$ (see Dylan Thurston [19] for details). This gives an element $(i(\mu, C), \theta(\mu, C)) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$.
Consider the quotient $\mathbb{R}^2 = \mathbb{R}_{\geq 0} \times \mathbb{R} / \sim$, where $(0, t) \sim (0, -t)$, and denote by $DT(\mu, C)$ the equivalent class of $(i(\mu, C), \theta(\mu, C))$ in $\mathbb{R}^2$. Notice that $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$. The Dehn-Thurston coordinates give a homeomorphism

$$ML(S^d) \rightarrow ([\mathbb{R}^2])^{6g-6+3b+2p}$$

$$\mu \rightarrow (DT(\mu, C_1), \ldots, DT(\mu, C_{3g-3+b+p}), DT(\mu, B_1),$$

$$\ldots, DT(\mu, B_b), DT(\mu, \tilde{C}_1), \ldots, DT(\mu, \tilde{C}_{3g-3+b+p}))$$

The subspace $ML(S) \subset ML(S^d)$ is characterized by equations imposing symmetry of the coordinates:

$$\forall j : i(\mu, C_j) = i(\mu, \tilde{C}_j),$$

$$\forall j : \theta(\mu, C_j) = -\theta(\mu, \tilde{C}_j),$$

$$\forall j : \theta(\mu, B_j) = 0 \text{ if } i(\mu, B_j) \neq 0.$$ 

The minus sign in the equation for the twist comes from the fact that the sign of the twist parameter depends on the orientation of the surface, and mirror symmetry changes the orientation.

The first two equations mean that, for symmetric laminations, the coordinates associated with the curves $\tilde{C}_i$ can be recovered from the coordinates associated to $C_i$, so we can neglect the curves $\tilde{C}_i$ in the coordinates.

The third equation shrinks every factor $\mathbb{R}^2$ corresponding to a boundary curve $B_j$ into a line. So we define the coordinate $\breve{\theta}(\mu, B_j)$ as $i(\mu, B_j)$ if $i(\mu, B_j) \neq 0$, and as $-|\theta(\mu, B_j)|$ if $i(\mu, B_j) = 0$.

This proves the following:

**Proposition 3.9.** The following map is a homeomorphism

$$ML(S) \ni \mu \mapsto (DT(\mu, C_1), \ldots, DT(\mu, C_{3g-3+b+p}), \breve{\theta}(\mu, B_1), \ldots, \breve{\theta}(\mu, B_b)) \in ([\mathbb{R}^2])^{3g-3+b+p} \times \mathbb{R}^b$$

In particular, $ML(S)$ is homeomorphic to $\mathbb{R}^{6g-6+3b+2p}$, and $PML(S)$ is homeomorphic to $S^{6g-7+3b+2p}$.

### 4. Thurston’s asymmetric metric

Given a set $M$, a nonnegative function $d$ defined on $M \times M$ is said to be a weak metric if it satisfies all the axioms of a distance function except the symmetry axiom, saying that $d(x, y) = d(y, x)$. A weak metric $d$ is said to be asymmetric if it is strictly weak, that is, if there exist two points $x$ and $y$ in $M$ such that $d(x, y) \neq d(y, x)$.

In this section, we first review Thurston’s metric and stretch maps on Teichmüller spaces of surfaces without boundary (with or without punctures).

We consider a surface $R$ of genus $g$ with $n$ punctures and without boundary. We may consider the Teichmüller $T_{g,n} = \mathcal{R}$ as a space of marked
hyperbolic structures on $R$, or, in an equivalent matter, as a space of homotopy classes of hyperbolic metrics on the surface $S$. Thurston [21] defined an asymmetric metric $d_{Th}$ on $T_{g,n}$ by setting

$$d_{Th}(X,Y) = \inf_f \log L_f(X,Y),$$

where the infimum is taken over all homeomorphisms $f : X \to Y$ homotopic to the identity map of $R$ and where $L_f(X,Y)$ is the Lipschitz constant of $f$, that is,

$$L_f(X,Y) = \sup_{x \neq y \in S} \frac{d_Y(f(x), f(y))}{d_X(x,y)}.$$

An important result of Thurston in [21] is that

$$d_{Th}(X,Y) = \log \sup_{\gamma \in \mathcal{C}(S)} \frac{\ell_{\gamma}(Y)}{\ell_{\gamma}(X)}.$$

The asymmetric metric defined in (5) is Finsler, that is, it is a length metric which is defined by integrating an asymmetric norm on the tangent bundle of $T_{g,n}$ along paths in $T_{g,n}$, and taking the infimum of lengths over all piecewise $C^1$-paths. Thurston [21] gave an explicit formula for the weak norm of a tangent vector $V$ at a point $X$ in $T_{g,n}$, namely,

$$\|V\|_{Th} = \sup_{\lambda \in \mathcal{ML}} \frac{d\ell_{\lambda}(V)}{\ell_{\lambda}(X)}.$$

Here, $\mathcal{ML}$ is the space of measured laminations on the surface, $\ell_{\lambda} : T_{g,n} \to \mathbb{R}$ is the geodesic length function on Teichmüller space associated to the measured lamination $\lambda$ and $d\ell_{\lambda}$ is the differential of $\ell_{\lambda}$ at the point $X \in T_{g,n}$.

There is a (non-necessary unique) extremal Lipschitz homeomorphism that realizes the infimum in (5). Related to the extremal Lipschitz homeomorphism, there is a class of geodesics for Thurston’s metric called stretch lines, which we will describe below.

Let $X$ be again a hyperbolic surface on $R$. A geodesic lamination $\lambda$ on $X$ is said to be complete if its complementary regions are all isometric to ideal triangles. Associated with $(X, \lambda)$ is a measured foliation $F_\lambda(X)$, called the horocyclic foliation, whose equivalence class is characterized by the following three properties:

(i) $F_\lambda(X)$ intersects $\lambda$ transversely, and in each cusp of an ideal triangle in the complement of $\lambda$, the leaves of the foliation are pieces of horocycles that make right angles with the boundary of the triangle;
(ii) on the leaves of $\lambda$, the transverse measure for $F_\lambda(X)$ agrees with hyperbolic arc length;
(iii) there is a non-foliated region at the center of each ideal triangle of $X \setminus \lambda$ whose boundary consists of three pieces of horocycles that are pairwise tangent (see Figure 1).
2. Lipschitz Norm

This section contains results of Thurston from his paper [32] that we will use later in this paper. We have provided proofs because at times Thurston’s proofs in [32] are considered as sketchy.

Thurston [32] proved the following fundamental result.

We denote by $\mathcal{MF}(\lambda)$ the space of measured foliations that are transverse to $\lambda$. Note that by the definition of a horocyclic foliation, we require the measured foliation in $\mathcal{MF}(\lambda)$ to be standard in a neighborhood of any cusp of the surface. This means that its leaves are circles, and the measure of any arc converging to the cusp is infinite. Thurston [21] proved the following fundamental result.

**Theorem 4.1.** The map $\phi_{\lambda}: T_{g,n} \to \mathcal{MF}(\lambda)$ defined by $X \mapsto F_{\lambda}(X)$ is a homeomorphism.

The stretch line $g_{\lambda}^t(X)$ directed by $\lambda$ and passing through $X \in T(R)$ is the curve in Teichmüller space parameterized by

$$\mathbb{R} \ni t \mapsto g_{\lambda}^t(X) = \phi_{\lambda}^{-1}(e^t F_{\lambda}(X)).$$

We call a segment of a stretch line a stretch path. We also have a natural notion of stretch ray.

Stretch rays are geodesics for the Thurston metric: Suppose that $\lambda_0$ is a measured lamination whose support is contained in a complete geodesic lamination $\lambda$. Let $\Gamma(t) = g_{\lambda}^{t}(X)$. Then, for any two points $\Gamma_s, \Gamma_t, s \leq t$ on the stretch line, the distance $d_{\text{Th}}(\Gamma_s, \Gamma_t)$ is equal to $t - s$, and this distance is realized by

$$\log \frac{\ell_{\lambda_0}(\Gamma_t)}{\ell_{\lambda_0}(\Gamma_s)}.$$

It was observed by Thurston [21] that any measured lamination that realizes the maximum of

$$\sup_{\mu \in \mathcal{ML}} \frac{\ell_{\mu}(X_t)}{\ell_{\mu}(X_s)}$$

is supported by $\lambda$. The union of all the measured geodesic laminations that realize this maximum is also a measured geodesic lamination, called the stump of $\lambda$.  

Thurston proved that any two points in Teichmüller space can be joined by a geodesic which is a finite concatenation of stretch paths, but in general such a geodesic is not unique. There also exist geodesics for Thurston’s metric that are not concatenations of stretch paths. Some of them are made explicit in [14]. This contrasts with Teichmüller’s theorem establishing the existence and uniqueness of Teichmüller geodesics joining any two distinct points.

Given \( X \in \mathcal{T}_{g,n} \) and a complete geodesic lamination \( \lambda \) on \( X \), we consider the map

\[
\Gamma(t) : \mathbb{R}_{\geq 0} \to \mathcal{T}_{g,n} \quad t \mapsto \Gamma^t_\lambda(X).
\]

\( \Gamma(t) \) is the stretch ray directed by \( \lambda \) starting from \( X \). Note that \( \Gamma(0) = X \). There is a unique measured lamination \( \mu \) which is equivalent to the horocylic foliation \( F_\lambda(X) \). In fact, there is a one-to-one correspondence between measured laminations and (equivalence class of) measured foliations on \( X \). The measured lamination \( \mu \) equivalent to \( F_\lambda(X) \) is totally transverse to \( \lambda \) (see Thurston [21, Proposition 9.4]).

In the following, we assume that \( \lambda \) has no closed leaves. Then \( \lambda \) is obtained from its stump by adding finitely many infinite geodesics. It follows that any simple closed geodesic or any geodesic arc (connecting two simple closed geodesics \( \beta_1, \beta_2 \) with \( i(\beta_1, \beta_2) = 0 \) perpendicularly) is transverse to \( \lambda \). We shall use this fact. Papadopoulos [12] proved the following:

**Lemma 4.2.** For any simple closed curve \( \gamma \) on \( R \), there is a constant \( C_\gamma \) that depends only on \( \gamma \) such that

\[
e^t i(\mu, \gamma) \leq \ell_\gamma(\Gamma(t)) \leq e^t i(\mu, \gamma) + C_\gamma.
\]

This implies that, as \( t \to +\infty \), \( \Gamma(t) \) converges to \([\mu]\), the projective class of \( \mu \) on the boundary of Thurston’s compactification.

When \( i(\mu, \gamma) = 0 \), Lemma 4.2 says that \( \ell_\gamma(\Gamma(t)) \) is bounded above by a constant \( C_\gamma \) (depending on \( \gamma \)). The following result of Théret [18] gives a further estimate for \( \ell_\gamma(\Gamma(t)) \).

**Lemma 4.3.** Let \( \gamma \) be a simple closed curve on \( R \) with \( i(\mu, \gamma) = 0 \). If \( \gamma \) is a leaf of \( \mu \) with weight equal to \( \omega_\gamma \), then

\[
\ell_\gamma(\Gamma(t)) \leq \frac{3|\chi(R)|}{\sinh(e^t \omega_\gamma/2)}.
\]

If \( \gamma \) is not a leaf of \( \mu \) (in this case we set \( \omega_\gamma = 0 \)), then

\[
B_\gamma \leq \ell_\gamma(\Gamma(t)) \leq C_\gamma,
\]

where \( B_\gamma \) and \( C_\gamma \) are constants that depend only on \( \gamma \).
5. Geometry of the arc metric

In this section, we prove our main theorem. We first recall the definition of the arc metric. Then we introduce the horofunction compactification of the arc metric. Finally, we show that Thurston’s compactification \( \mathcal{T}(S) \) is naturally homeomorphic to the horofunction compactification of the arc metric.

5.1. The arc metric. For any \( \gamma \in \mathcal{A}(S) \cup \mathcal{C}(S) \) and for any hyperbolic structure \( X \) on \( S \), we let \( \gamma^X \) be the geodesic representative of \( \gamma \) (that is, the curve of shortest length in the homotopy class relative to \( \partial S \)). In the case where \( \gamma \) is an equivalence class of arcs, the geodesic \( \gamma^X \) is unique, and it is orthogonal to \( \partial X \) at each intersection point. We denote by \( \ell_\gamma(X) \) the length of \( \gamma^X \) with respect to the hyperbolic metric considered. This length only depends on the equivalence class of \( X \) in Teichmüller space.

Let \( S \) be a hyperbolic surface with geodesic boundary. Let \( \mathcal{C} = \mathcal{C}(S) \) and \( \mathcal{A} = \mathcal{A}(S) \). In the paper [8], the authors defined an asymmetric metric, the arc metric, on \( \mathcal{T}(S) \) by

\[
d(X, Y) = \log \sup_{\gamma \in \mathcal{C} \cup \mathcal{A}} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.
\]

Relations between the arc metric and the Teichmüller metric are studied in the same paper.

Remark 5.1. Note that the arcs are necessary in order to have a metric because if we use only the closed curves, then on any surface \( S \) there exist \( X, Y \) such that (see [13])

\[
\log \sup_{\gamma \in \mathcal{C}} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} < 0.
\]

The definition of the arc metric is a natural generalization of Thurston’s formula [4].

Proposition 5.2 ([8]). The map \( \Psi \) (defined in Section 2) gives an isometric embedding

\[
(\mathcal{T}(S), d) \hookrightarrow \left( \mathcal{T}(S^d), d_{\text{Th}} \right),
\]

that is,

\[
d(X, Y) = d_{\text{Th}}(X^d, Y^d).
\]

5.2. The horofunction compactification. Let \( \mathcal{T}(S) \) be the Teichmüller space of \( S \) endowed with the arc metric \( d \). We set \( \bar{d}(X, Y) = d(Y, X) \). Then, \( \bar{d} \) is also an asymmetric metric on \( \mathcal{T}(S) \). The topology of \( \mathcal{T}(S) \) induced by the arc metric \( d \) is the same as the one induced by \( d \); it is defined as the topology induced by the genuine metric \( d + \bar{d} \) or \( \delta = \max\{d, \bar{d}\} \), and it is the usual topology on Teichmüller space (see [8, Theorem 4.4]).
Fix a base point $X_0 \in \mathcal{T}(S)$. To each $X \in \mathcal{T}(S)$ we assign a function $\Phi_X : \mathcal{T}(S) \to \mathbb{R}$, defined by

$$\Phi_X(Y) = d(Y, X) - d(X_0, X).$$

Let $C(\mathcal{T}(S))$ be the space of continuous functions on $\mathcal{T}(S)$ endowed with the topology of locally uniform convergence. Then the map

$$\Phi : \mathcal{T}(S) \to C(\mathcal{T}(S)), \quad X \mapsto \Phi_X$$

is an embedding. The closure $\overline{\Phi(\mathcal{T}(S))}$ is compact (this follows from the fact that $\mathcal{T}(S)$ is locally compact and the Arzelá-Ascoli theorem) and it is called the horofunction compactification of $\mathcal{T}(S)$. The horofunction boundary is defined to be

$$\overline{\Phi(\mathcal{T}(S))} - \Phi(\mathcal{T}(S)),$$

and its elements are called horofunctions.

**Remark 5.3.** For a general locally compact metric space $(M, d)$, the horofunction compactification was defined by Gromov in [10]. A good property of the horofunction compactification is that the action of the isometry group $\text{Isom}(M, d)$ of $M$ extends continuously to a homeomorphism on the horofunction boundary.

Note that our definition depends on the choice of a base point $X_0$. However, if we let

$$\Phi_X(Y) = d(\cdot, X) - d(Y_0, X)$$

for another base point $Y_0$, then the relation between $\Phi_X$ and $\Phi_X$ is described by

$$\Phi_X(Y) = \Phi_X(Y) - \Phi_X(Y_0). \tag{8}$$

Equation (8) induces a natural homeomorphism between $\Psi(\mathcal{T}(S))$ and $\tilde{\Psi}(\mathcal{T}(S))$ and it induces a homeomorphism between the corresponding horofunction boundaries. As a result, we can embed the Teichmüller space $\mathcal{T}(S)$ into the quotient of $C(\mathcal{T}(S))$ obtained by identifying two functions whenever they differ by an additive constant. In the following, we shall fix a base point.

In the remaining part of this paper, we shall make the identification

$$\mathcal{PML} \cong \{ \eta \in \mathcal{ML}(S) \mid \ell_\eta(X_0) = 1 \}.$$ 

Suppose that $X \in \mathcal{T}(S)$. From the definition,

$$\Phi_X(Y) = \log \sup_{\eta \in \mathcal{PML}} \frac{\ell_\eta(X)}{\ell_\eta(\cdot)} - \log \sup_{\eta \in \mathcal{PML}} \frac{\ell_\eta(X)}{\ell_\eta(X_0)}.$$ 

For any $\gamma \in \mathcal{ML}$, we set

$$\mathcal{L}_\gamma(X) = \ell_\gamma(X) / \sup_{\eta \in \mathcal{PML}} \frac{\ell_\eta(X)}{\ell_\eta(X_0)}.$$
Then
\[ \Phi_X(\cdot) = \log \sup_{\gamma \in \mathcal{PM}\mathcal{L}} \frac{L_\gamma(X)}{\ell_\gamma(\cdot)}. \]

5.3. Convergence in Thurston’s compactification. Let \((X_n)\) be a sequence in \(\mathcal{T}(S)\) that converges to \(\mu \in \mathcal{PM}\mathcal{L}\). From the definition, there exists a sequence of numbers \((c_n)\), \(c_n > 0\), such that for any \(\eta \in \mathcal{ML}\),
\[ c_n \ell_\eta(X_n) \to i(\mu, \eta) \quad \text{as } n \to \infty. \]

Lemma 5.4. With the above notation, we have:
\[ L_\gamma(X_n) \to i(\mu, \gamma)/\sup_{\nu \in \mathcal{PM}\mathcal{L}} i(\mu, \nu) \quad \text{as } n \to \infty. \]

Proof. We have
\[ L_\gamma(X_n) = \ell_\gamma(X_n)/\sup_{\eta \in \mathcal{PM}\mathcal{L}} \frac{\ell_\eta(X_n)}{\ell_\eta(X_0)} = c_n \ell_\gamma(X_n)/\sup_{\eta \in \mathcal{PM}\mathcal{L}} \frac{c_n \ell_\eta(X_n)}{\ell_\eta(X_0)}. \]

By assumption, \(c_n \ell_\eta(X_n) \to i(\mu, \eta) \quad \text{(as } n \to \infty)\) for all \(\eta \in \mathcal{PM}\mathcal{L}\). Moreover, since \(\mathcal{PM}\mathcal{L}\) is compact and \(\ell_\eta(X) : \mathcal{ML} \times \mathcal{T}(S) \to \mathbb{R}\) is continuous (with respect to \(\eta\) and \(X\)), we have \(c_n \ell_\eta(X_n) \to i(\mu, \eta)\) uniformly on \(\mathcal{PM}\mathcal{L}\) (ref. [25 Lemma 3.1]). This implies that
\[ \lim_{n \to \infty} \sup_{\eta \in \mathcal{PM}\mathcal{L}} \frac{c_n \ell_\eta(X_n)}{\ell_\eta(X_0)} = \sup_{\eta \in \mathcal{PM}\mathcal{L}} i(\mu, \eta) \frac{i(\mu, \eta)}{\ell_\eta(X_0)}. \]

Since \(c_n \ell_\gamma(X_n) \to i(\mu, \gamma) \quad \text{as } n \to \infty\), we are done. \(\square\)

For \(\gamma\) and \(\mu\) in \(\mathcal{ML}\), we set
\[ L_\gamma(\mu) = i(\mu, \gamma)/\sup_{\nu \in \mathcal{PM}\mathcal{L}} i(\mu, \nu). \]

Note that the value \(L_\gamma(\mu)\) is invariant by multiplication of \(\mu\) by a positive constant, therefore we can also define \(L_\gamma(\mu)\) by the same formula for \(\mu\) in \(\mathcal{PM}\mathcal{L}\).

Proposition 5.5. A sequence \((X_n)\) in \(\mathcal{T}(S)\) converges to \(\mu \in \mathcal{PM}\mathcal{L}\) if and only if \(L_\gamma(X_n)\) converges to \(L_\gamma(\mu)\) for all \(\gamma \in \mathcal{ML}\).

Proof. We already showed that if \((X_n)\) converges to \(\mu\), then \(L_\gamma(X_n)\) converges to \(L_\gamma(\mu)\) for all \(\gamma \in \mathcal{ML}\).

Conversely, assume that \(L_\gamma(X_n)\) converges to \(L_\gamma(\mu)\) for all \(\gamma \in \mathcal{ML}\). Then \((X_n)\) is unbounded in \(\mathcal{T}(S)\). Let \((Y_n)\) be any subsequence of \(X_n\) that
converges to \( \mu' \in \mathcal{PML} \). Then \( \mathcal{L}_\gamma(Y_n) \) converges to \( \mathcal{L}_\gamma(\mu') \) for all \( \gamma \in \mathcal{ML} \).

By assumption, \( \mathcal{L}_\gamma(\mu') = \mathcal{L}_\gamma(\mu) \), therefore

\[
i(\mu', \gamma)/\sup_{\nu \in \mathcal{PML}} i(\mu', \nu)/\ell_\nu(X_0) = i(\mu', \gamma)/\sup_{\nu \in \mathcal{PML}} i(\mu', \nu)/\ell_\nu(X_0).
\]

Therefore, if we set

\[
C = \sup_{\nu \in \mathcal{PML}} i(\mu, \nu)/\ell_\nu(X_0)/\sup_{\nu \in \mathcal{PML}} i(\mu', \nu)/\ell_\nu(X_0),
\]

then \( i(\mu, \gamma) = Ci(\mu', \gamma) \) for all \( \gamma \in \mathcal{ML} \). This implies that \( \mu = \mu' \) in \( \mathcal{PML} \).

Since \( (Y_n) \) is arbitrary, \( (X_n) \) converges to \( \mu \).

\[\Box\]

**Corollary 5.6.** A sequence \( (Z_n) \) in \( \overline{T(S)} \) converges to \( Z \in \overline{T(S)} \) if and only if \( \mathcal{L}_\gamma(Z_n) \) converges to \( \mathcal{L}_\gamma(Z) \) for all \( \gamma \in \mathcal{ML} \).

**Proof.** This follows from Proposition 5.5 and a usual continuity argument. \[\Box\]

For \( \mu \in \mathcal{PML}(S) \), let \( \Phi : \mathcal{PML} \to \mathcal{C}(\mathcal{T}(S)) \) be the function defined by

\[
\Phi(\mu)(\cdot) = \log \sup_{\gamma \in \mathcal{PML}} \mathcal{L}_\gamma(\mu).
\]

The maps on \( \mathcal{PML} \) defined by Equations (9) and (10) combine together and define a map

\[
\Phi : \overline{T(S)} \to \mathcal{C}(\mathcal{T}(S)),
\]

\[
Z \mapsto \Phi_Z.
\]

By Corollary 5.6 and the compactness of \( \mathcal{PML} \), this map is continuous.

In the next sections, we prove that \( \Phi \) is injective on \( \overline{T(S)} \). The same result for surface without boundary was proved by Walsh [25] by a direct method which does not apply here. Our proof is based on the inequality (Lemma 6.8) in next section.

---

6. **An Inequality for Length Functions**

For any \( \mu \in \mathcal{PML} \), let \( \mu^d \) be the double of \( \mu \) on \( S^d \). We endow \( S^d \) with the hyperbolic structure \( X_0^d \) and we choose a complete geodesic lamination \( \lambda \) which contains no closed leaves and which is totally transverse to \( \mu^d \). (Recall that this is equivalent to saying that \( \mu^d \) can be represented by a measured foliation transverse to \( \lambda \) and trivial around each puncture.)

Denote by \( \Gamma(t) \) the stretch line in \( \mathcal{T}(S^d) \) directed by \( \lambda \) and converging to \( \mu^d \) in the positive direction, that is,

\[
\Gamma(t) = \phi^{-1}_{\lambda}(e^t \mu^d)
\]
where $\phi_\lambda$ is the map in Theorem 4.1. For $t \geq 0$, the hyperbolic structure $\Gamma(t)$ might not be symmetric, and this is the reason for the technical work that follows.

Consider any $\alpha \in A$. We realize $\alpha$ as a geodesic arc $\alpha_t$ on $\Gamma(t)$ whose endpoints are on two simple closed geodesics $\beta_1, \beta_2$ and which meets them perpendicularly. These closed geodesics are homotopic to the images in the hyperbolic surface $\Gamma(t)$ of the boundary curves of $S$ which contain the endpoints of $\alpha$. They can either coincide in $\Gamma(t)$ or be distinct, depending on whether they come from curves that coincide or are distinct in $S$.

Similarly, we can realize $\mu$ as a measured geodesic lamination $\mu_t$ on $\Gamma(t)$. The support of $\mu_t$ lies on a totally geodesic subsurface of $\Gamma(t)$ which is homeomorphic to $S$. The intersection number $i(\mu, \alpha)$ is realized by the total mass of the intersection of $\alpha_t$ with $\mu_t$. Thus, we have:

$$i(\mu, \alpha) = I(\mu_t, \alpha_t)$$

where

$$I(\mu_t, \alpha_t) = \int_{\alpha_t} d\mu_t.$$

We wish to prove an inequality similar to [12, Lemma 4.9]. The first step is to show that there is a constant $C > 0$ (depending only on the stretch line) such that for all $\alpha \in A$,

$$e^t i(\mu, \alpha) - C \leq \ell(\Gamma(t)).$$

This is obtained in Corollary 6.4 below.

We fix $\alpha$ in $A$ and the hyperbolic structure $\Gamma(t)$. We will use the same notation $\alpha$ to denote the geodesic representation of $\alpha$ on $\Gamma(t)$. We suppose that $\alpha$ joins two simple closed geodesics $\beta_1, \beta_2$ perpendicularly. We set $\ell(\alpha) = \ell(\Gamma(t))$ and so on.

**Remark 6.1.** It seems that the constant $C > 0$ is necessary when $\alpha \in A$. This is due to the fact that the horocyclic foliation $F_t$ equivalent to $e^t \mu^d$ is not symmetric. A similar argument as in [12, Lemma 4.9] shows that for any $\alpha \in C$, $e^t i(\mu^d, \alpha) \leq \ell(\Gamma(t))$. This can be done by showing that $\ell(\alpha) \geq I(F_t, \alpha_t)$, where $\alpha_t$ is the geodesic representation of $\alpha$ on $\Gamma(t)$.

### 6.1. Estimation of arc length in a pair of pants.

The three geodesics $\beta_1, \beta_2, \alpha$ determine a geodesic pair of pants, denoted by $\mathcal{P}$, which is isotopic to a tubular neighborhood of $\alpha \cup \beta_1 \cup \beta_2$.

When $\beta_1 = \beta_2$ (and in this case we denote both curves by $\beta$), the boundary of $\mathcal{P}$ has three connected components: one is $\beta$ and the other two will be denoted by $\gamma_1, \gamma_2$. It may happen that $\gamma_1$ and $\gamma_2$ coincide on the surface $S^d$.

If $\beta_1 \neq \beta_2$, the boundary of $\mathcal{P}$ has three connected components, two of them are $\beta_1$ and $\beta_2$. We denote by $\gamma$ the third one, so that $\partial \mathcal{P} = \beta_1 \cup \beta_2 \cup \gamma$. 
**Remark 6.2.** In both cases, some boundary component of \(\mathcal{P}\) (such as \(\gamma, \gamma_1\) or \(\gamma_2\)) may be degenerate to a puncture. We always consider a puncture to be a boundary component with length equal to zero.

The intersection numbers of the three boundary components of \(\mathcal{P}\) with \(\mu\) are three positive numbers satisfying some equation. To simplify notation, we will always assume that indices are chosen so that \(i(\mu, \gamma_1) \geq i(\mu, \gamma_2)\) in the first case, and \(i(\mu, \beta_1) \geq i(\mu, \beta_2)\) in the second case.

![Figure 2](image)

**Figure 2.** The pair of pants containing the arc \(\alpha\) falls into two types. For each type, there are three cases illustrated in (A)-(C).

As indicated to the left of Figure 2, the case where \(\beta_1 = \beta_2\) is separated into three different subcases.

(A) The intersection number of \(\mu\) with one boundary component of \(\mathcal{P}\) is less than the sum of the intersection number of \(\mu\) with the two others. (That is, the triangle inequality for the triple of intersection numbers does not hold.)

(B) \(i(\mu, \gamma_1) > i(\mu, \beta) + i(\mu, \gamma_2)\).

(C) \(i(\mu, \beta) > i(\mu, \gamma_1) + i(\mu, \gamma_2)\).

In each subcase, we have the following corresponding equation:

(A) \(i(\mu, \alpha) = \frac{1}{2} (i(\mu, \gamma_1) + i(\mu, \gamma_2) - i(\mu, \beta)) + \omega_\beta\).

(B) \(i(\mu, \alpha) = i(\mu, \gamma_1) - i(\mu, \beta) + \omega_\beta\).

(C) \(i(\mu, \alpha) = 0\).
Here \( \omega_\beta \) is the weight of \( \beta \) in \( \mu^d \). We clearly have:

\[
i(\mu^d, \gamma_1) = i(\mu, \gamma_1), \quad i(\mu^d, \gamma_2) = i(\mu, \gamma_2), \quad i(\mu^d, \beta) = 2i(\mu, \beta).
\]

Now we give a lower bound of \( \ell(\alpha) \) in terms of \( \ell(\beta), \ell(\gamma_1), \ell(\gamma_2) \) for all cases (A)-(C). We need the following formula, which can be proved by combining the hyperbolic pentagon and hexagon formulae.

\[
\cosh^2 \left( \frac{1}{2} \ell(\alpha) \right) = \frac{-1 + \cosh^2 \left( \frac{1}{2} \ell(\beta) \right) + 2 \cosh \left( \frac{1}{2} \ell(\beta) \right) \cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right)}{\sinh^2 \left( \frac{1}{2} \ell(\beta) \right)}
\]

We also need some elementary estimates:

(i) For \( x \geq 0, \frac{1}{2} e^x \leq \cosh(x) \leq e^x; \frac{1}{4} e^{2x} \leq \cosh^2(x) \leq e^{2x}. \)

(ii) If \( x > A > 0, \) then

\[
\frac{1}{2} (1 - e^{2A}) e^x \leq \sinh(x) \leq \frac{1}{2} e^x;
\]

if \( 0 < x < 1, \) then

\[
x < \sinh(x) < 2x.
\]

(iii) For each \( \gamma \in C(S^d), \) we have (recalling that we denote by \( \ell(\gamma) \) the geodesic length of \( \gamma \) on \( \Gamma(t) \))

\[
\frac{1}{2} \exp \left( \frac{1}{2} e^{i(\mu^d, \gamma)} \right) \leq \cosh \left( \frac{1}{2} \ell(\gamma) \right) \leq \exp \left( \frac{1}{2} \ell(\gamma) \right) \leq \exp \left( \frac{C_\gamma}{2} \right) \exp \left( \frac{1}{2} e^{i(\mu^d, \gamma)} \right).
\]

(iv) If \( \gamma \in C(S^d) \) is a leaf of \( \mu^d \) (that is, \( \omega_\gamma > 0 \)), then

\[
\ell(\gamma) \leq 12 |c(S^d)| \exp(-\frac{1}{2} \omega_\gamma e^t).
\]

The inequality in (iii) follows from Lemma 4.2. The inequality in (iv) follows from Lemma 4.3 and the fact that \( 1/\sinh(x) \leq 4/e^x \) for \( x > 0. \)

**Case (A).** We rewrite Formula (11) in the following way:

\[
\cosh^2 \left( \frac{1}{2} \ell(\alpha) \right) = 2 \coth \left( \frac{1}{2} \ell(\beta) \right) \frac{\cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right)}{\sinh \left( \frac{1}{2} \ell(\beta) \right)} (1 + R_A)
\]

where

\[
R_A = \frac{-1 + \cosh^2 \left( \frac{1}{2} \ell(\beta) \right) + 2 \cosh \left( \frac{1}{2} \ell(\beta) \right) \cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right)}{2 \cosh \left( \frac{1}{2} \ell(\beta) \right) \cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right)}
\]

It is easy to see that that \( R_A > 0. \)

If \( \omega_\beta = 0, \) then \( \ell(\beta) \geq B_\beta \) (see lemma 4.3), and we have

\[
1 < \coth \left( \frac{1}{2} \ell(\beta) \right) \leq \coth \left( \frac{1}{2} B_\beta \right).
\]

To give a lower bound for \( \ell(\alpha), \) note that
\[
\exp (\ell(\alpha)) \geq \cosh^2 \left( \frac{1}{2} \ell(\alpha) \right) \\
\geq 2 \coth \left( \frac{1}{2} \ell(\beta) \right) \frac{\cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right)}{\sinh \left( \frac{1}{2} \ell(\beta) \right)} \\
\geq \frac{\cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right)}{\sinh \left( \frac{1}{2} \ell(\beta) \right)} \\
\geq \frac{e^{\frac{1}{2}\ell(\gamma_1)} e^{\frac{1}{2}\ell(\gamma_2)}}{e^{\frac{1}{2}\ell(\beta)}} = \exp \left( \frac{\ell(\gamma_1)}{2} + \frac{\ell(\gamma_2)}{2} - \frac{\ell(\beta)}{2} \right).
\]

By taking the logarithm of each side and applying Lemma 4.2, we have

\[
\ell(\alpha) \geq \frac{\ell(\gamma_1)}{2} + \frac{\ell(\gamma_2)}{2} - \frac{\ell(\beta)}{2} \\
\geq e^{\frac{1}{2}} \left( i(\mu, \gamma_1) + i(\mu, \gamma_2) - i(\mu, \beta) \right) - C_\beta \\
= e^{\frac{1}{2}} i(\mu, \alpha) - C_\beta.
\]

If \( \omega_\beta > 0 \), we have \( i(\mu, \beta) = 0 \). Moreover, the length \( \ell(\beta) \) is less than \( C_\beta \) and it is less than 1 when \( t \) is sufficiently large. As a result, we may assume (using the second inequality in (ii)) that

\[
\sinh \left( \frac{1}{2} \ell(\beta) \right) \leq \ell(\beta).
\]

Then we have

\[
\exp (\ell(\alpha)) \geq \cosh^2 \left( \frac{1}{2} \ell(\alpha) \right) \\
\geq 2 \coth \left( \frac{1}{2} \ell(\beta) \right) \frac{\cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right)}{\sinh \left( \frac{1}{2} \ell(\beta) \right)} \\
\geq \frac{\cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right)}{\sinh^2 \left( \frac{1}{2} \ell(\beta) \right)} \\
\geq \frac{1}{2} \exp \left( \frac{1}{2} \ell(\gamma_1) + \frac{1}{2} \ell(\gamma_2) \right) \frac{e^{\frac{1}{2}\ell(\beta)}}{\ell(\beta)^2}.
\]

Applying (iv), we get

\[
\exp (\ell(\alpha)) \geq \frac{1}{2} \exp \left( \frac{1}{2} \ell(\gamma_1) + \frac{1}{2} \ell(\gamma_2) \right) \frac{e^{\frac{1}{2}\ell(\beta)}}{(12 \chi(S^d))^2} \exp (-\omega_\gamma e^t).
\]
By taking the logarithm of each side, we have
\[
\ell(\alpha) \geq \frac{1}{2} \ell(\gamma_1) + \frac{1}{2} \ell(\gamma_2) + \omega_\gamma e^t - \log(288|\chi(S^d)|^2)
\]
\[
\geq \frac{1}{2} \left( i(\mu, \gamma_1) + i(\mu, \gamma_2) \right) e^t + \omega_\gamma e^t - \log(288|\chi(S^d)|^2)
\]
\[
= e^t i(\mu, \alpha) - \log(288|\chi(S^d)|^2).
\]

**Case (B).** We can rewrite formula [11] in the following way:
\[
\cosh^2 \left( \frac{1}{2} \ell(\alpha) \right) = \frac{\cosh^2 \left( \frac{1}{2} \ell(\gamma_1) \right)}{\sinh^2 \left( \frac{1}{2} \ell(\beta) \right)} (1 + R_B)
\]
where the term \( R_B > 0 \) is given by
\[
-1 + \cosh^2 \left( \frac{1}{2} \ell(\beta) \right) + \cosh^2 \left( \frac{1}{2} \ell(\gamma_2) \right) + 2 \cosh \left( \frac{1}{2} \ell(\beta) \right) \cosh \left( \frac{1}{2} \ell(\gamma_1) \right) \cosh \left( \frac{1}{2} \ell(\gamma_2) \right) \cosh^2 \left( \frac{1}{2} \ell(\gamma_1) \right)
\]
\[
\cosh^2 \left( \frac{1}{2} \ell(\gamma_1) \right)
\]

Now, if \( w_\beta = 0 \), we have
\[
\exp(\ell(\alpha)) \geq \cosh^2 \left( \frac{1}{2} \ell(\alpha) \right)
\]
\[
\geq \cosh^2 \left( \frac{1}{2} \ell(\gamma_1) \right)
\]
\[
\geq \sinh^2 \left( \frac{1}{2} \ell(\beta) \right)
\]
\[
\geq \exp \left( \ell(\gamma_1) - \ell(\beta) \right).
\]

It follows that
\[
\ell(\alpha) \geq \ell(\gamma_1) - \ell(\beta)
\]
\[
\geq e^t (i(\mu, \gamma_1) - i(\mu, \beta)) - C_\beta
\]
\[
= e^t i(\mu, \alpha) - C_\beta.
\]

If, instead, \( w_\beta > 0 \), we have \( i(\mu, \beta) = 0 \) and \( \ell(\beta) \) is going to zero as \( t \) tends to infinity. Applying (iv), we have (for \( t \) sufficiently large)
\[
\exp(\ell(\alpha)) \geq \frac{\cosh^2 \left( \frac{1}{2} \ell(\gamma_1) \right)}{\sinh^2 \left( \frac{1}{2} \ell(\beta) \right)}
\]
\[
\geq \frac{1}{2} \exp \left( \ell(\gamma_1) \right)
\]
\[
\geq \frac{1}{(\ell(\beta))^2}
\]
\[
\geq \frac{1}{576|\chi(S^d)|^2} \exp \left( \ell(\gamma_1) + \omega_\beta e^t \right).
\]
Thus

\[
\ell(\alpha) \geq \ell(\gamma_1) + \omega \gamma e^t - \log \left(576|\chi(S^d)|^2\right)
\]

\[
\geq e^t (i(\mu, \gamma_1) + \omega \beta) - \log \left(576|\chi(S^d)|^2\right)
\]

\[
= e^t i(\mu, \alpha) - \log \left(576|\chi(S^d)|^2\right).
\]

**Case (C).** Since \(i(\mu, \alpha) = 0\), the inequality \(\ell(\alpha) \geq i(\mu, \alpha)\) is trivial.

Now we consider the case where \(\beta_1 \neq \beta_2\). As we did before, we separate the intersection pattern into three different cases:

(A') the intersection number of \(\mu\) with one boundary component of \(P\) is less than the sum of the intersection number of \(\mu\) with the two others.

(B') \(i(\mu, \beta_1) > i(\mu, \beta_2) + i(\mu, \gamma)\).

(C') \(i(\mu, \gamma) > i(\mu, \beta_1) + i(\mu, \beta_2)\).

Each of the above cases corresponds respectively to

(A') \(i(\mu, \alpha) = \frac{1}{2} (\omega \beta_1 + \omega \beta_2)\).

(B') \(i(\mu, \alpha) = \frac{1}{2} (\omega \beta_1 + \omega \beta_2)\).

(C') \(i(\mu, \alpha) = \frac{1}{2} (i(\mu, \gamma) - i(\mu, \beta_1) - i(\mu, \beta_2)) + \omega \beta_1 + \omega \beta_2\).

Recall the following formula:

\[
\cosh \left(\frac{1}{2} \ell(\beta_1)\right) = \frac{\cosh \left(\frac{1}{2} \ell(\beta_1)\right) + \cosh \left(\frac{1}{2} \ell(\beta_2)\right) \cosh \left(\frac{1}{2} \ell(\beta_2)\right)}{\sinh \left(\frac{1}{2} \ell(\beta_1)\right) \sinh \left(\frac{1}{2} \ell(\beta_2)\right)}.
\]

**Case (A') or (B').** We can rewrite formula (12) in the following way:

\[
\cosh \left(\ell(\alpha)\right) = \frac{\cosh \left(\frac{1}{2} \ell(\gamma)\right) - \cosh \left(\frac{1}{2} \ell(\beta_1)\right) \cosh \left(\frac{1}{2} \ell(\beta_2)\right)}{\sinh \left(\frac{1}{2} \ell(\beta_1)\right) \sinh \left(\frac{1}{2} \ell(\beta_2)\right)}.
\]

where the term \(S_{A,B} > 0\) is given by

\[
\frac{\cosh \left(\frac{1}{2} \ell(\gamma)\right)}{\cosh \left(\frac{1}{2} \ell(\beta_1)\right) \cosh \left(\frac{1}{2} \ell(\beta_2)\right)}.
\]

Now, if \(w_{\beta_1} = w_{\beta_2} = 0\), we have \(i(\mu, \alpha) = 0\). Then it is obvious that \(\ell(\alpha) \geq i(\mu, \alpha)\).

If \(w_{\beta_1} = 0\) and \(w_{\beta_2} > 0\), we have \(1 < \coth(\frac{1}{2} \ell(\beta_1)) \leq \coth(\frac{1}{2} B_{\beta_1})\); while \(\ell(\beta_2)\) goes to zero as \(t\) tends to infinity. We may assume (by considering \(t\) sufficiently large) that

\[
\coth(\frac{1}{2} \ell(\beta_2)) \geq \frac{1}{\sinh(\frac{1}{2} \ell(\beta_2))} \geq \frac{1}{\ell(\beta_2)}.
\]

Applying (iv), we have
\[ \exp (\ell(\alpha)) \geq \cosh (\ell(\alpha)) \]
\[ \geq \frac{1}{\ell(\beta_2)} \]
\[ \geq \frac{1}{12|\chi(S^d)|} \exp \left( \frac{1}{2} \omega_{\beta_2} e^t \right). \]

Thus
\[ \ell(\alpha) \geq \frac{1}{2} \omega_{\beta_2} e^t - \log \left( 12|\chi(S^d)| \right) \]
\[ = e^{t}{i(\mu, \alpha)} - \log \left( 12|\chi(S^d)| \right). \]

The above argument applies also to the case where \( w_{\beta_1} > 0 \) and \( w_{\beta_2} = 0 \).

Now if \( w_{\beta_1} > 0 \) and \( w_{\beta_2} > 0 \), we have (for \( t \) sufficiently large)
\[ \coth\left( \frac{1}{2} \ell(\beta_i) \right) \geq \frac{1}{\sinh\left( \frac{1}{2} \ell(\beta_i) \right)} \geq \frac{1}{\ell(\beta_i)}, \quad i = 1, 2. \]

Applying (iv) again, we have
\[ \exp (\ell(\alpha)) \geq \cosh (\ell(\alpha)) \]
\[ \geq \frac{1}{\ell(\beta_1)\ell(\beta_2)} \]
\[ \geq \frac{1}{144|\chi(S^d)|^2} \exp \left( \frac{1}{2} \omega_{\beta_1} e^t + \frac{1}{2} \omega_{\beta_2} e^t \right). \]

Thus
\[ \ell(\alpha) \geq \frac{1}{2} \left( \omega_{\beta_1} + \omega_{\beta_2} \right) e^t - \log \left( 144|\chi(S^d)|^2 \right) \]
\[ = e^{t}{i(\mu, \alpha)} - \log \left( 144|\chi(S^d)|^2 \right). \]

**Case (C').** we can rewrite formula 12 in the following way:
\[ \cosh \left( \ell(\alpha) \right) = \frac{\cosh \left( \frac{1}{2} \ell(\gamma) \right)}{\sinh \left( \frac{1}{2} \ell(\beta_1) \right) \sinh \left( \frac{1}{2} \ell(\beta_2) \right)} \left( 1 + S_C \right) \]
where the term \( S_C > 0 \) is given by
\[ \frac{\cosh \left( \frac{1}{2} \ell(\beta_1) \right) \cosh \left( \frac{1}{2} \ell(\beta_2) \right)}{\cosh \left( \frac{1}{2} \ell(\gamma) \right)}. \]

In this case, we have a lower bound for \( \ell(\alpha) \):
\[ \exp (\ell(\alpha)) \geq \cosh (\ell(\alpha)) \]
\[ \geq \frac{\cosh \left( \frac{1}{2} \ell(\gamma) \right)}{\sinh \left( \frac{1}{2} \ell(\beta_1) \right) \sinh \left( \frac{1}{2} \ell(\beta_2) \right)} \].
By comparing the above inequality with the estimates in Case (A)-(C) and using a similar argument, one can show that $\ell(\alpha)$ is larger than
\[ e^t \left( \frac{1}{2} (i(\mu, \gamma) - i(\mu, \beta_1) - i(\mu, \beta_2)) + \omega_{\beta_1} + \omega_{\beta_2} \right) \]
up to some constant (only depending on $\beta_1, \beta_2$). We omit the details.

We arrive to the following conclusion:

**Proposition 6.3.** For any $\alpha \in A$, considered as a geodesic arc in $\Gamma(t)$ connecting two simple closed geodesics $\beta_1, \beta_2$, there are constants $C, T > 0$ (depending on $\beta_1$ and $\beta_2$) such that when $t > T$, $\ell_\alpha(\Gamma(t)) \geq e^t i(\mu, \alpha) - C$.

**Corollary 6.4.** There is a constant $C > 0$ depending only on the stretch line such that for all $\alpha \in A$,
\[ e^t i(\mu, \alpha) - C \leq \ell_\alpha(\Gamma(t)). \]

*Proof.* Since there are finitely many choices of the pair $\beta_1, \beta_2$ (note that here $\beta_1, \beta_2$ are boundary components of $S$), we can choose a uniform constant $C$ such that the above conclusion holds for all $\ell_\alpha(\Gamma(t))$.

**Remark 6.5.** One can apply the above argument to give a upper bound for $\ell_\alpha(\Gamma(t))$. That is, one can show that for each $\alpha \in A$, there is a constant $C_\alpha > 0$ (depending on $\alpha$) such that
\[ \ell_\alpha(\Gamma(t)) \leq e^t i(\mu, \alpha) + C_\alpha. \]

To avoid long calculations, we will adopt an indirect method to certify (14) in the next section.

**Remark 6.6.** Our method is close in spirit to [6, Exposé 6, Appendix D]. It can be adapted to the case of a general measured lamination and general stretch line, by specifying an appropriate definition for the intersection number between a measured lamination and an arc.

### 6.2. Key Inequality

Let $\Gamma(t)$ be a stretch line in $\mathcal{T}(S^d)$ as we have constructed above. If we restrict each hyperbolic structure $\Gamma(t)$ to the subsurfaces $S$ and $\overline{S}$, then we have two families of hyperbolic structures on $\mathcal{T}(S)$ and $\mathcal{T}(\overline{S})$, respectively. We call them $\Gamma_U(t)$ and $\Gamma_L(t)$.

It follows directly from Corollary 6.4 that there is a constant $C > 0$ such that for any $\alpha \in A$,
\[ e^t i(\mu, \alpha) - C \leq \ell_\alpha(\Gamma_U(t)), \]
\[ e^t i(\bar{\mu}, \bar{\alpha}) - C \leq \ell_\alpha(\Gamma_L(t)). \]

Note that the above inequalities also hold for any simple closed curve $\alpha \in C$. In this case, we can take $C = 0$ (this is a consequence of Lemma 4.2).

Denote by $\Gamma_U(t)$ and $\Gamma_L(t)$ the mirror image of $\Gamma_U(t)$ and $\Gamma_L(t)$ respectively. Note that $\Gamma_U(t) \subset \mathcal{T}(S)$ and $\Gamma_L(t) \subset \mathcal{T}(S)$. 


Lemma 6.7. With the above notation, for any $\alpha \in \mathcal{C} \cup \mathcal{A}$, the following inequalities hold:

\begin{align}
\begin{cases}
    e^t i(\mu, \alpha) - C & \leq \ell_{\alpha}(\Gamma_U(t)) \\
    e^t i(\mu, \alpha) - C & \leq \ell_{\bar{\alpha}}(\Gamma_U(t)) \\
    e^t i(\mu, \alpha) - C & \leq \ell_{\alpha}(\Gamma_L(t)) \\
    e^t i(\mu, \alpha) - C & \leq \ell_{\bar{\alpha}}(\Gamma_L(t))
\end{cases}
\end{align}

Lemma 6.7 provides a lower bound of the geodesic length of a simple closed curve or simple arc on $S$ along the path $\Gamma_U(t)$. In the following, we will give an upper bound.

Consider $\alpha \in \mathcal{C} \cup \mathcal{A}$. Denote by $\alpha^d$ the double of $\alpha$. Then $\alpha^d$ is either a simple closed curve or the union of two symmetric simple closed curves on $S^d$. Using Lemma 4.2, we have a constant $C_{\alpha}$ such that

\[\ell_{\alpha^d}(\Gamma(t)) \leq e^t i(\mu^d, \alpha^d) + C_{\alpha}.\]

Note that the sum of the lengths of the two arcs $\ell_{\alpha}(\Gamma(t))$ and $\ell_{\bar{\alpha}}(\Gamma(t))$ is less than $\ell_{\alpha^d}(\Gamma(t))$. It follows that

\[\ell_{\alpha}(\Gamma(t)) + \ell_{\bar{\alpha}}(\Gamma(t)) \leq e^t i(\mu^d, \alpha^d) + C_{\alpha} = 2e^t i(\mu, \alpha) + C_{\alpha}.\]

Combining the above inequalities with Lemma 6.7, we have

\begin{align}
\begin{cases}
    \ell_{\alpha}(\Gamma_U(t)) = \ell_{\alpha}(\Gamma(t)) \leq i(\mu, \alpha) + C_{\alpha} + C \\
    \ell_{\bar{\alpha}}(\Gamma_L(t)) = \ell_{\bar{\alpha}}(\Gamma(t)) \leq i(\mu, \alpha) + C_{\alpha} + C
\end{cases}
\end{align}

We summarize the above results in the following key lemma, which is a generalization of [12, Lemma 4.9].

Lemma 6.8. There exists a path $X_t, t \in [0, +\infty) \text{ in } \mathcal{T}(S)$ such that each $\alpha \in \mathcal{C} \cup \mathcal{A}$ satisfy

\[e^t i(\mu, \alpha) - C \leq \ell_{\alpha}(X_t) \leq e^t i(\mu, \alpha) + C_{\alpha},\]

where $C > 0$ is a uniform constant and each $C_{\alpha} > 0$ is a constant depending only on $\alpha$.

Note that the path $X_t$ converges to the point $\mu$ in Thurston’s compactification.

7. Proof of Theorem 1

Before we prove the main theorem, we need a generalization of [25, Lemma 6.4].

Recall that each measured lamination $\mu$ on $S$ can be decomposed into a finite union of components, each of which is either a simple closed geodesic, a simple geodesic arc or a minimal component where each half-leaf is dense.
A measured lamination $\mu$ is said to be *uniquely ergodic* if the transverse measure of $\mu$ is the unique measure on the same support up to a scalar multiple.

More generally, let $\mu$ be an arbitrary minimal measured lamination on $S$. There exist invariant transverse measures $\mu_1, \ldots, \mu_p$ on $\lambda$ such that

- $\mu_i$ is uniquely ergodic for each $i$.
- Any transverse invariant measure $\nu$ on $\lambda$ can be written as $\nu = \sum_i a_i \nu_i$ for $a_i \geq 0$.

It follows that any measured lamination $\mu$ has a unique decomposition as

$$
\mu = \sum a_j \mu_j, a_j \geq 0
$$

where each $\mu_j$ is either a simple closed curve, a simple geodesic arc or a uniquely ergodic measured lamination. Such a decomposition is called the *ergodic decomposition* of $\mu$.

The following lemma is proved by Walsh [25, Lemma 6.4] for surfaces without boundary. His proof works as well for surfaces with boundary.

**Lemma 7.1.** Let $\mu = \sum_j \mu_j$ be the ergodic decomposition of $\mu \in \mathcal{PML}$. Then

$$
\sup_{\gamma \in \mathcal{C} \cup \mathcal{A}} \frac{i(\nu, \gamma)}{i(\mu, \gamma)} = \max \{ f_j \}
$$

if $\nu = \sum_j f_j \mu_j$. If $\nu$ cannot expressed as $\sum_j f_j \mu_j, f_j \geq 0$, then the supremum is $+\infty$.

**Proposition 7.2.** The map

$$
\Phi : \overline{T(S)} \mapsto C(T(S)),
$$

$$
Z \mapsto \Phi_Z
$$

is injective.

**Proof.** We separate the proof into three steps.

**I.** Note that $\Phi|_{T(S)}$ is injective, since for any $X \in T(S)$, we have

$$
\inf_{Y \in T(S)} \Phi_X(Y) = -d(X_0, X)
$$

and the infimum is exactly obtained at $X$.

**II.** It is easy to see that for any $Y \in T(S)$ and $\mu \in \mathcal{PML}$, we have $\Phi_Y \neq \Phi_\mu$. In fact, by definition,

$$
\Phi_\mu(X) = \log \sup_{\gamma \in \mathcal{PML}} \frac{i(\mu, \gamma)}{\ell_{\gamma}(X)} - \log \sup_{\gamma \in \mathcal{PML}} \frac{i(\mu, \gamma)}{\ell_{\gamma}(X_0)}.
$$

Let $(X_n)$ be a sequence in $T(S)$ converging to $\mu$. There is a sequence of numbers $a_n > 0$ such that $\lim_{n \to \infty} a_n = 0$ and $a_n \ell_{\gamma}(X_n) \to i(\mu, \gamma)$ for all $\gamma \in \mathcal{PML}$. It follows that
\[ \Phi_\mu(X_n) = \log a_n + \log \sup_{\gamma \in \mathcal{PML}} \frac{i(\mu, \gamma)}{a_n \ell_\gamma(X_n)} - \log \sup_{\gamma \in \mathcal{PML}} \frac{i(\mu, \gamma)}{\ell_\gamma(X_0)} \]
tends to \(-\infty\) as \(n \to \infty\). In particular, we have\[ \inf_{X \in T(S)} \Phi_\mu(X) = -\infty. \]

On the other hand, for any \(Y \in T(S)\),\[ \inf_{X \in T(S)} \Phi_Y(X) = -d(X_0, Y) > -\infty. \]
It follows that \(\Phi_Y \neq \Phi_\mu\).

**III.** It remains to show that for any \(\mu \neq \nu\), \(\Phi_\mu \neq \Phi_\nu\). Recall that we made the identification

\[ \mathcal{PML} \equiv \{ \eta \in \mathcal{ML} \mid \ell_\eta(X_0) = 1 \}. \]

Without loss of generality, we assume that\[ \log \sup_{\eta \in \mathcal{PML}} \frac{i(\nu, \eta)}{\ell_\eta(Y)} > \log \sup_{\eta \in \mathcal{PML}} \frac{i(\mu, \eta)}{\ell_\eta(X)}. \]

Then for any \(X \in T(S)\), we have

(17) \[ \Phi_\nu(X) - \Phi_\mu(X) \geq \log \sup_{\eta \in \mathcal{PML}} \frac{i(\nu, \eta)}{\ell_\eta(X)} - \log \sup_{\eta \in \mathcal{PML}} \frac{i(\mu, \eta)}{\ell_\eta(X)}. \]

We conclude the proof by showing the following lemma, which implies that \(\Phi_\mu \neq \Phi_\nu\).

**Lemma 7.3.** There exists a point \(Y\) in \(T(S)\) such that

\[ \log \sup_{\eta \in \mathcal{PML}} \frac{i(\nu, \eta)}{\ell_\eta(Y)} > \log \sup_{\eta \in \mathcal{PML}} \frac{i(\mu, \eta)}{\ell_\eta(Y)}. \]

To prove Lemma 7.3 we will use our construction in Section 6 and some observations on the fine structure of the measured lamination \(\mu\).

Let \(\mu = \sum \mu_i\) be the ergodic decomposition of \(\mu\). We choose a measured lamination \(\hat{\mu}\) which contains \(\mu\) as a sublamination, by using the following steps:

(I) If \(\beta\) is a boundary component of \(S\) disjoint from \(\mu\), then we add \(\beta\) to \(\mu\). We get a measured lamination \(\mu_0\) such that

\[ \mu_0 = \mu + \sum_{j=1}^{m} \beta_j \]

where \(\beta_j, j = 1, \cdots, m\) are boundary components of \(S\) which are disjoint from \(\mu\).
(II) By definition, $S$ has $p \geq m$ boundary components. The numbering is such that for each $m + 1 \leq j \leq p$, there is at least an arc contained in $\mu$ that intersects $\beta_j$. We construct a new measured lamination $\mu_1$ by adding to $\mu_0$ an arc $\alpha_1$ disjoint from $\mu_0$ (if such an arc exists). Inductively, we construct a new measured lamination $\mu_j$ by adding to $\mu_{j-1}$ an arc $\alpha_j$ disjoint from $\mu_{j-1}$. After a finite number of steps, we get a measured lamination $\mu_k$ with the following property:

*any arc $\alpha \in \mathcal{A}$ not contained in $\mu_k$ either intersects a simple leaf (an arc or boundary component) of $\mu_k$ or intersect $\mu$.*

(III) By cutting the surface $S$ along all the arcs contained in $\mu_k$, we get a finite union of connected components, each of which is either a polygon (may be a punctured polygon) or a surface with piecewise geodesic boundary components. Let $G$ be a such a component with piecewise geodesic boundaries. Let $C$ be a boundary component of $G$. Then $C$ is either a simple closed geodesic contained in $\mu_k$ or a finite concatenations of geodesic arcs.

In the latter case, each geodesic segment of $C$ either comes from an arc in $\mu_k$ (contained as a leaf) or a boundary component of $S$. Note that $C$ is homotopic to a simple closed curve $\gamma$ on $S$, and the geodesic representation of $\gamma$ is contained in or disjoint from $\mu_k$. We will add such a $\gamma$ to $\mu_k$. The resulting measured lamination, denote by $\hat{\mu}$, is the one we want. Note that $\hat{\mu}$ is not necessarily unique.

---

**Figure 3.** The measured lamination $\hat{\mu}$ is an extension of $\mu$ such that any geodesic arc on $S$ not contained in $\hat{\mu}$ is transverse to some simple leaf of $\hat{\mu}$.

It follows from the above construction that for any $\alpha \in \mathcal{A}$, either $\alpha$ is a leaf of $\hat{\mu}$ or $\alpha$ intersects a simple leaf of $\hat{\mu}$.

We write $\hat{\mu}$ as
\[ \hat{\mu} = \mu + \zeta. \]

Suppose that \( \ell_{\zeta}(X_0) = L. \) Then we set \[ \hat{\mu}_\epsilon = (1 - \epsilon)\mu + \frac{\epsilon}{L} \zeta. \]

It follows that \( \hat{\mu}_\epsilon \in \mathcal{PML} \) for each \( 0 \leq \epsilon \leq 1. \)

**Proof of Lemma 7.3.** We first claim that we can find \( 0 < \epsilon < 1 \) and some \( \gamma_0 \in C \cup A \) such that

\[
\frac{i(\nu, \gamma_0)}{i(\hat{\mu}_\epsilon, \gamma_0)} > \frac{1}{1 - \epsilon}.
\]

We now prove this claim. As above, we assume that \[ \hat{\mu}_\epsilon = (1 - \epsilon) \sum_j \mu_j + \epsilon \sum_k \zeta_k. \]

There are two cases. If \( \nu \) cannot be expressed as \( \sum_j f_j \mu_j + \sum_k g_k \zeta_k, f_j \) with \( g_k \geq 0, \) then (by Lemma 7.1)

\[
\sup_{\gamma \in C \cup A} \frac{i(\nu, \gamma)}{i(\hat{\mu}_\epsilon, \gamma)} = \infty.
\]

In this case, for any given \( 0 < \epsilon < 1, \) there is some \( \gamma_0 \in C \cup A \) such that

\[
\frac{i(\nu, \gamma_0)}{i(\hat{\mu}_\epsilon, \gamma_0)} > \frac{1}{1 - \epsilon}.
\]

Otherwise,

\[
\nu = \sum_j f_j \mu_j + \sum_k g_k \zeta_k, f_j \] for some \( g_k \geq 0.
\]

If there is some \( g_k > 0, \) then we choose \( 0 < \epsilon < 1 \) sufficiently small such that \( \frac{g_k}{\epsilon} > \frac{1}{1 - \epsilon}. \) It follows from Lemma 7.1 that there is some \( \gamma_0 \in C \cup A \) such that

\[
\frac{i(\nu, \gamma_0)}{i(\hat{\mu}_\epsilon, \gamma_0)} > \frac{1}{1 - \epsilon}.
\]

In the case where \( \nu = \sum_j f_j \mu_j, \) since we assumed that \( \ell_{\mu}(X_0) = \ell_{\nu}(X_0) = 1, \) we have

\[
\sum_j f_j \ell_{\mu_j}(X_0) = \sum_j \ell_{\mu_j}(X_0) = 1.
\]

There exists some \( f_j > 1. \) It follows again from Lemma 7.1 that there exists some \( \gamma_0 \in C \cup A \) such that

\[
\frac{i(\nu, \gamma_0)}{i(\hat{\mu}_\epsilon, \gamma_0)} > \frac{f_j}{1 - \epsilon} > \frac{1}{1 - \epsilon}.
\]

Fix \( 0 < \epsilon < 1 \) as above. We assume that
for some sufficiently constant $\delta > 0$.

As we did in Section 6, we define by

$\Gamma_{\epsilon} : t \in \mathbb{R}_{+} \rightarrow T(S^d)$

the stretch line converging to the double of $\hat{\mu}_{\epsilon}$. We set $X_t = \Gamma_{\epsilon}^U(t)$.

By Lemma 6.8, there exists a constant $C > 0$ such that for each $\alpha \in \mathcal{C} \cup \mathcal{A}$,

$e^t i(\hat{\mu}_{\epsilon}, \alpha) \leq \ell_\alpha(X_t) + C.

When $\alpha \in \mathcal{C}$, we can take $C$ to be zero.

As a result, we have

$\sup_{\eta \in \mathcal{PML}} i(\hat{\mu}_{\epsilon}, \eta) e^{-t\ell_\eta(X_t)} = \sup_{\alpha \in \mathcal{C} \cup \mathcal{A}} i(\hat{\mu}_{\epsilon}, \eta) e^{-t\ell_\alpha(X_t)}$

$\leq \max\{1, \sup_{\alpha \in \mathcal{A}, i(\hat{\mu}_{\epsilon}, \alpha) > 0} \frac{i(\hat{\mu}_{\epsilon}, \eta)}{e^{-t\ell_\alpha(X_t)}}\}.$

Let $N > 0$ be a sufficiently large constant such that $C/N < \frac{\delta}{3}$. Since $\ell_{\hat{\mu}_{\epsilon}}(X_t) \rightarrow 0$ as $t \rightarrow \infty$, by the Collar Lemma, the length of any geodesic arc intersecting some simple leaf of $\hat{\mu}_{\epsilon}$ is uniformly large as soon as $t$ is sufficiently large. By the construction of $\hat{\mu}_{\epsilon}$, we have

$\ell_\alpha(X_t) \geq N, \ \forall \ \alpha \in \mathcal{A}, i(\hat{\mu}_{\epsilon}, \alpha) > 0, t \geq T(N).$

It follows that

$\sup_{\eta \in \mathcal{PML}} \frac{i(\mu, \eta)}{e^{-t\ell_\eta(X_t)}} \leq \frac{1}{1 - \epsilon} \max\{1, \sup_{\alpha \in \mathcal{A}, i(\hat{\mu}_{\epsilon}, \alpha) > 0} \frac{i(\hat{\mu}_{\epsilon}, \eta)}{e^{-t\ell_\alpha(X_t)}}\}$

$\leq \frac{1}{1 - \epsilon} \max\{1, \sup_{\alpha \in \mathcal{A}, i(\hat{\mu}_{\epsilon}, \alpha) > 0} \frac{e^{-t\ell_\alpha(X_t)} + e^{-tC}}{e^{-t\ell_\alpha(X_t)}}\}$

$\leq \frac{1 + \frac{C}{N}}{1 - \epsilon}$

As a result, for $t \geq T(N)$ we have
\[ \log \sup_{\eta \in PML} \ell_{\eta}(X_t) - \log \sup_{\eta \in PML} \ell_{\eta}(X_t) = \log \sup_{\eta \in PML} e^{-t\ell_{\eta}(X_t)} - \log \sup_{\eta \in PML} e^{-t\ell_{\eta}(X_t)} \geq \log \frac{i(\nu, \gamma_0)}{i(\hat{\mu}, \gamma_0) + e^{-tC\gamma_0}} - \log \frac{1 + \frac{\delta}{3}}{1 - \epsilon}. \]

By (18), when \( t \) is sufficiently large, we have
\[ \log \frac{i(\nu, \gamma_0)}{i(\hat{\mu}, \gamma_0) + e^{-tC\gamma_0}} > \log \frac{1 + \frac{2\delta}{3}}{1 - \epsilon}. \]

This implies Lemma 7.3.

**Remark 7.4.** The proof of Proposition 7.2 applies to the Teichmüller space of surfaces without boundary. This is based again on Lemma 4.2. Thus we get a new proof for [25, Theorem 3.6]. However, the argument in [25] does not work for surfaces with boundary. Note that in contrast with surfaces without boundary, the set of uniquely ergodic measured laminations on a surface \( S \) with boundary is not dense in \( ML(S) \).

**Theorem 7.5.** The map \( \Phi \) establishes a homeomorphism between Thurston’s compactification \( \overline{T}(S) \) and the horofunction compactification.

**Proof.** We showed that \( \Phi : \overline{T}(S) \to C(\overline{T}(S)) \) is injective and continuous. Any embedding from a compact space to a Hausdorff space is a homeomorphism onto its image. As a result, \( \Phi(\overline{T}(S)) \) is a compact subset of \( C(\overline{T}(S)) \). Since the horofunction compactification is the closure of \( \Phi(\overline{T}(S)) \), it must be equal to \( \Phi(\overline{T}(S)) \). □

**Remark 7.6.** As we mentioned in the introduction, one of the remaining questions is to understand the isometry group of the arc metric. One step to handle this question is to calculate the “detour cost” distance between any two measured laminations on Thurston’s boundary of a surface with boundary. We will go into details of this calculation in the future work.

In conclusion, we collect some open questions:

**Questions 7.7.** (a) Is the arc metric Finsler? If yes, what is the Finsler norm?

(b) Construct families of geodesic between any two points on Teichmüller space, analogous to concatenations of stretch lines in the case without boundary.

(c) What is the relation between the arc metric and the extremal Lipschitz maps between hyperbolic structures?

Finally, we note that by recent works of Danciger, Guérin, and Kassel, the deformation theory of surfaces with boundary is related to Margulis...
spacetimes in Lorentz geometry [7]. Extremal Lipschitz maps are generalized to geometrically finite hyperbolic manifolds of dimension $n \geq 2$, see [7].

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