We relate the derived category of a relative Ginzburg algebra of an $n$-angulated surface to the geometry of the underlying surface. Results include the description of a subset of the objects in the derived category in terms of curves in the surface and their Hom’s in terms of intersection. By using the description of such a derived category as the global sections of perverse schober, we arrive at the geometric model through gluing local data. Nearly all results also hold for the perverse schober defined over any commutative ring spectrum.

As a direct application of the geometric model, we categorify the extended mutation matrices of a class of cluster algebras with coefficients, associated to multi-laminated marked surfaces by Fomin-Thurston [FT18].

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1 Introduction

This paper continues the study of Ginzburg algebras of triangulated surfaces, or more generally \( n \)-angulated surfaces, using their description in terms of perverse schobers started in \([\text{Chr21}]\).

In his study of Calabi-Yau algebras \([\text{Gin06}]\), Ginzburg introduced a class of dg-algebras constructed from a quiver with potential, called Ginzburg algebras. They have been used for, among other things, the categorification of cluster algebras, see \([\text{Kel12}]\) for a survey, and the algebraic description of Fukaya categories \([\text{Smi15, Smi20, IS20}]\). Particularly relevant for this work are a class of Ginzburg algebras obtained from a quiver with potential constructed from an oriented marked surface with an ideal triangulation. We are further concerned with a generalization of this class of Ginzburg algebras which are called relative Ginzburg algebras, c.f. \([\text{Chr21}]\). The difference to the non-relative versions is that these dg-algebras incorporate additional information about the boundary of the surface, and thus exhibit a gluing behavior. These gluing properties lead to a description of the derived \( \infty \)-category of such a relative Ginzburg algebra in loc. cit. as the \( \infty \)-category of global sections of a perverse schober, that is a constructible sheaf or cosheaf of \( \infty \)-categories with certain properties defined on a ribbon graph embedded in the surface. The first result of this paper is a generalization of the perverse schober description to include the relative (higher) Ginzburg algebra \( \mathcal{G}_T \) associated to ideal \( n \)-angulation \( T \) with \( n \geq 3 \), see Theorem 4.23. The number \( n \) is the Calabi-Yau dimension of the non-relative Ginzburg algebra. The corresponding perverse schober is denoted by \( \mathcal{F}_T(R) \), where \( R \) is the commutative ground ring \( (R = k) \) or the ground \( \mathbb{E}_{\infty} \)-ring spectrum.

Geometric models

By describing the derived category of a dg-algebra as a Fukaya category, one can obtain much information about the objects and morphisms in the category: they are formed by suitable Lagrangian subspaces and their intersections. A much studied class of examples are formed by the topological Fukaya categories of surfaces, which encompass the class of derived categories of gentle algebras, see \([\text{HKK17, LP20}]\). In subsequent work using purely algebraic methods, all indecomposable modules over a gentle algebra and their Hom’s were realized in terms of curves in the surface and their intersections, see \([\text{BS19, OPS18}]\). This description forms the basis of a geometric model for the derived category, which also allows the description of other features of these derived categories in terms of the surface, e.g. certain autoequivalences.

In this paper, we develop a geometric model for a full subcategory of the \( \infty \)-category of global sections of the perverse schober \( \mathcal{F}_T(R) \). The case \( R = k \) then gives the geometric model for the derived category of the relative Ginzburg algebra \( \mathcal{G}_T \). The main novelty in our approach is the application of the sheaf properties of the \( \infty \)-category of global sections of \( \mathcal{F}_T(R) \), which allow us to glue the objects and morphisms from local data.

We associate a global section \( M^L_\gamma \) to each suitable curve \( \gamma \) in the surface, called a matching curve, and 'local value' \( L \), which in the case \( R = k \) corresponds to a module over the polynomial algebra \( k[t_{n-2}] \) with generator in degree \( |t_{n-2}| = n - 2 \). By choosing different values of \( L \), we can realize in the geometric model classes of finite \( \mathcal{G}_T \)-modules, such as the simple \( n \)-spherical modules, as well as non-finite \( \mathcal{G}_T \)-modules, such as the perfect modules, associated to the vertices of the underlying quiver of \( \mathcal{G}_T \). We further describe the \( R \)-linear morphism objects between the \( M^L_\gamma \)'s in terms of intersections of the curves, see Theorems 6.2 and 6.3. Finally, we also express the derived equivalences associated to the flips of the \( n \)-angulations in the geometric model via rotations of parts of the surface, see Theorem 8.1.
The Jacobian algebra $\mathcal{J}$ is defined as the 0-th homology algebra of the relative Ginzburg algebra $\mathcal{G}_\mathcal{T}$. We will see that $\mathcal{J}$ is always a gentle algebra, generalizing an observation from [ABCJP10]. However, if $\mathcal{S}$ contains internal marked points, the Jacobian algebra may be infinite dimensional, i.e. a non-proper gentle algebra. In the cases of surfaces with no interior marked points, we apply the geometric model for $\mathcal{D}(\mathcal{G}_\mathcal{T})$ to show that the homology algebra $H_* (\mathcal{G}_\mathcal{T})$ is equivalent to the tensor algebra $\mathcal{J} \otimes k[t_{n-2}^{-1}]$, see Proposition 7.4.

The derived $\infty$-category of the non-relative Ginzburg algebra of the $n$-angulation can be realized as a full subcategory of $\mathcal{D}(\mathcal{G}_\mathcal{T})$ consisting of global sections of $\mathcal{F}_\mathcal{T}(k)$ with certain support constraints. Our results for $\mathcal{D}(\mathcal{G}_\mathcal{T})$ thus restrict to a geometric model for (a full subcategory of) the derived category of the non-relative Ginzburg algebra which overlaps considerably with results from the series of papers [Qiu16, Qiu18, QZ19, IQZ20]. The setup of this paper, using perverse schobers and $\infty$-categories, allows us to extend these previous results in generality. For example, the previous geometric models were restricted to marked surfaces whose marked points lie on the boundary and, for $n \geq 4$, to finite modules.

**Interpretation in terms of Fukaya categories**

We discuss how we expect the results of this paper to relate to Fukaya categories.

Perverse schobers are expected to provide a framework for the description of Fukaya-categories ‘with coefficients’ in terms of sheaves or cosheaves of higher categories, see [KS14]. The main motivation for the perverse schober description of the relative Ginzburg algebras of triangulated surfaces thus arises from Ivan Smith’s embedding of the finite derived category of the (non-relative) Ginzburg algebra in the Fukaya category of a Calabi-Yau 3-fold $Y$ with a Lefschetz fibration to the surface with typical fiber $T^* S^2$, see [Smi15]. The derived categories of Ginzburg algebras of $n$-angulated surfaces have for $n > 3$ so far not been related to Fukaya categories. We expect them to describe some versions of partially wrapped Fukaya categories of Calabi-Yau $n$-folds equipped with Lefschetz fibrations to the surfaces with typical fiber $T^* S^{n-1}$.

Contrary to the situation for topological Fukaya categories mentioned before, our geometric model does not describe the objects in terms of some half-dimensional subspaces of the conjectural Calabi-Yau $n$-fold $Y$. Instead, we expect that these curves are the images under the Lefschetz fibration of Lagrangian subspaces of $Y$. The global section $M^L_\gamma$ associated to a matching curve $\gamma$ takes as input a local system $L \in \text{Fun}(S^{n-1}, R\text{Mod}_R)$. In the case $R = k$, the choice of $L$ is by [Abo11] equivalent to a choice of element of the Ind-completion of the wrapped Fukaya category of $T^* S^{n-1}$. If $L$ corresponds to a Lagrangian submanifold of $T^* S^{n-1}$, we expect $L$ to describe the typical fiber of the Lagrangian subspace of $Y$ being mapped to the curve. Indeed, particularly interesting choices of $L$ are the constant local system with value $k$, corresponding to the Lagrangian zero section of $T^* S^{n-1}$, and $L$ a certain non-constant local system corresponding to the Lagrangian fiber of the projection $T^* S^{n-1} \to S^{n-1}$. The former and latter choices of $L$ give rise to the simple $n$-spherical and projective $\mathcal{G}_\mathcal{T}$-modules, respectively.

In this picture, the non-singular matching curves, see Definition 5.4, would correspond to non-compact Lagrangians which do not intersect the singular fibers, whereas the matching curves which begin and end in singularities would correspond to compact Lagrangians in $Y$.

**Categorification of the extended mutation matrix**

Ginzburg algebras can be used for the categorification of cluster algebras. Central in this regard is Amoit’s quotient construction [Ami09], which exhibits a 2-CY cluster category in terms of the Ginzburg algebra. However, there are also direct links between Ginzburg algebras and the combinatorics of cluster algebras: the mutation matrix of a cluster algebra can be recovered via the Euler-characteristics of the Ext-complexes of the simple 3-spherical modules over the Ginzburg algebra associated to the vertices of the underlying quiver. This observation is made, formulated in the more general setting of cluster collections, in [KS08, Section 8.1]. As an application of the geometric model for relative Ginzburg algebras, we extend the relation
between the mutation matrices and Ginzburg algebras to extended mutation matrices and relative Ginzburg algebras of triangulated surfaces. The extended mutation matrix consists of the mutation matrix and the $c$-matrix, the latter encodes the coefficients of the cluster algebra.

We consider the class of cluster algebras with coefficients introduced in [FT18]. The input is an oriented marked surface with an ideal triangulation $\mathcal{T}$ and a multi-lamination, that is a collection of laminations which are collections of certain disjoint curves in the surface which we call lamination curves. The cluster variables of the cluster algebra are the lambda lengths, i.e. certain coordinates on the decorated Teichmüller space of the surface. Fomin and Thurston defined the $c$-matrix in terms of the shear coordinates of the surface, which describe a signed count of intersections of the laminations and the edges of $\mathcal{T}$. The construction in terms of lamination allows great flexibility in the choice of coefficients.

Each lamination curve can be considered as a curve in our geometric model and thus defines an object in $D(\mathcal{H}_T)$ (with the local system $L$ chosen to be constant with value $k$). We can thus associate a $\mathcal{H}_T$-module $M_\lambda$ to a lamination $\lambda$ as the direct sum of the modules associated to the lamination curves. The categorical model for the $c$-matrix consists of the Euler characteristics of the Ext-groups between the 3-spherical modules associated to vertices of the quiver and the $M_\lambda$'s. Using our description of the morphism objects in terms of intersections, we can match the shear coordinates and the Euler characteristic of the Ext-groups, which leads us to the following.

**Theorem 1** (Theorem 2.12). Consider an oriented marked surface $S$ which is not the three punctured sphere or once-punctured monogon, equipped with an ideal triangulation $\mathcal{T}$ and multi-lamination $\Lambda$. The extended mutation matrix of [FT18] agrees with the categorical extended mutation matrix.

In the acyclic case, the Ext-groups between the simples of higher Ginzburg algebras have also been shown in [KQ15] to categorify the combinatorics of colored quivers of [BT09].

**Structure of the paper**

We begin in Section 2 by surveying part of the geometric model for relative Ginzburg algebras of triangulated surfaces, followed by the application to the categorification of the mutation matrices of cluster algebras. The main technical constructions and proofs appear in the later sections. In Section 3, we introduce some preliminaries from higher category theory, most importantly the Grothendieck construction. In Section 4, we recall the notion of a parametrized perverse schober from [Chr21] and discuss the perverse schobers describing relative Ginzburg algebras. In Sections 5 and 6, we construct the global sections associated to curves and describe the morphisms objects in terms of intersections. In Section 7, we discuss the relation with the Jacobian gentle algebra. In the final Section 8, we give the geometric description of the derived equivalences associated to flips of the $n$-angulations.

**Notation and conventions**

This paper is formulated in the language of stable $\infty$-categories, as developed in the extensive works [Lur09, Lur17]. We generally follow the terminology and notation used in [Lur09, Lur17]. In particular, we use the homological grading convention.

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2 Categorification of the extended mutation matrix

In this section we always assume that the oriented marked surface $S$ (with or without boundary) is not the sphere with three marked points nor the monogon with one marked point. By an ideal 3-angulation $T$ we mean a trivalent spanning ribbon graph, whose dual thus describes a decomposition of the surfaces into ideal triangles (also known as an ideal triangulation), see Definition 4.6.

2.1 The geometric definition of the extended mutation matrix

In the following we recall, in the conventions of this paper, the definition of the extended mutation matrix of the cluster algebra with coefficients of a surface equipped with an ideal 3-angulation $T$ and a multi-lamination $\Lambda$, due to Fomin-Thurston [FT18].

Definition 2.1 ([FST08, Definition 4.1]). Let $T$ be an ideal 3-angulation of an oriented marked surface. We denote the internal edges of $T$ by $e_1, \ldots, e_m$. We define the quiver $Q_T$ as follows.

- The vertices are the internal edges of $T$.
- Let $e_i \neq e_j$ be two not self-folded edges. We add an arrow $a : e_i \to e_j$ for each vertex $v$ of $T$ incident to halfedges of $e_i, e_j$ at which the halfedge of $e_j$ precedes the halfedge of $e_i$ in the cyclic (counterclockwise) order. The arrows $Q_T$ thus go in the clockwise direction.
- For each self-folded edge $e_i$, we add further arrows obtained as follows. Consider the unique edge $e_j$ that such $e_i$ and $e_j$ are incident to the same vertex of $T$, i.e. $e_j$ is the outer edge of the self-folded ideal triangle containing $e_i$. For $k \neq i, j$, we add an arrow $e_k \to e_i$ for each arrow $e_k \to e_j$ and an arrow $e_i \to e_k$ for each arrow $e_j \to e_k$, respectively.

The signed adjacency matrix of $T$ is the skew-symmetric $m \times m$-matrix $B(T) = (b_{i,j})$ where $b_{i,j}$ is the number of arrows from $e_i$ to $e_j$ minus the number of arrows from $e_j$ to $e_i$ in $Q_T$.

Definition 2.2 ([FT18, Definition 12.1]). Let $S$ be an oriented surface with marking $M$. A lamination curve is a non-selfintersecting curve $\gamma : U \to S \setminus M$ with $U = S^1, [0, 1], [0, \infty), (-\infty, \infty)$ such that

- all endpoints of $\gamma$ lie on $\partial S \setminus M$.
- the curve does not bound any unpunctured disc or once-punctured disc (possibly with the puncture on the boundary) in $S$.
- if $U$ is not compact, then at the infinite ends the curve spirals around an interior marked point.
- if $U = (-\infty, \infty)$ then both ends do not spiral around the same marked point in the same direction without the curve enclosing a further marked point.

Laminations curves are considered as equivalence classes under homotopies fixing endpoints. A lamination $\lambda$ on $S$ is a collection of pairwise non-intersecting lamination curves in $S$. A multi-lamination $\Lambda = (\lambda_1, \ldots, \lambda_k)$ on $S$ is a collection of $k \geq 1$ laminations on $S$.

Example 2.3. A lamination (in blue) with one spiraling curve in a surface with boundary (in green) with 7 marked points (also in green).
Definition 2.4. Let $S$ be an oriented marked surface with an ideal 3-angulation $\mathcal{T}$. Denote the internal edges of $\mathcal{T}$ by $e_1, \ldots, e_m$.

- Let $\gamma_i$ be a laminating curve and $e_j$ a not self-folded edge. We call a crossing of $\gamma_i$ with $e_j$ positive and negative, if their local arrangement is as depicted on the left, respectively, right in Figure 1. We denote the signed count of such crossings of $\gamma_i$ and $e_j$ by $(e_j, \gamma_i)$.

- Let $\gamma_i$ be a laminating curve and $e_j$ a self-folded edge. Let $e'_j$ be the not self-folded edge incident to the same vertex of $\mathcal{T}$ as $e_j$. We denote $(e_j, \gamma_i) = (e'_j, \tilde{\gamma}_i)$, where $\tilde{\gamma}_i$ is the lamination curve obtained by replacing each infinite end of $\gamma_i$ spiraling around an interior marked point $p$ by the infinite end spiraling around $p$ in the opposite direction.

- The shear coordinates of a lamination $\lambda = (\gamma_1, \ldots, \gamma_l)$ with respect to $\mathcal{T}$ are given by the $m$-tuple $v_{\lambda, \mathcal{T}} \in \mathbb{Z}^n$ whose $j$-th entry is given by

$$(v_{\lambda, \mathcal{T}})_j = \sum_{\gamma_i \in L} (e_j, \gamma_i).$$

![Figure 1: A crossing of a lamination curve $\gamma_i$ (in blue) with an edge $e_j$ of the 3-angulation contributing +1 on the left and −1 on the right to the shear coordinates.](image)

Definition 2.5. Let $S$ be an oriented marked surface with an ideal 3-angulation $\mathcal{T}$ with edges $e_1, \ldots, e_m$. Let $\Lambda = (\lambda_1, \ldots, \lambda_k)$ be a multi-lamination on $S$. The extended mutation matrix $B(\mathcal{T}, \Lambda)$ is given by the $m \times (m+k)$-matrix with

- the upper $m \times m$-submatrix is given by the signed adjacency matrix $B(\mathcal{T})$,
- the $(m+i)$-th row of $B(\mathcal{T}, \Lambda)$ is given by the shear coordinates $v_{\lambda_i, \mathcal{T}}$ of $\lambda_i$ with respect to $\mathcal{T}$, for $1 \leq i \leq k$.

The $c$-matrix $C(\mathcal{T}, \Lambda)$ is the $m \times k$ submatrix of $B(\mathcal{T}, \Lambda)$ consisting of the rows $m+1, \ldots, m+k$.

Theorem 2.6 ([FT18, Theorem 13.5]). If two ideal 3-angulations $\mathcal{T}, \mathcal{T}'$ of $S$ are related by the flip of an edge, then $B(\mathcal{T}, \Lambda)$ and $B(\mathcal{T}', \Lambda)$ are related by matrix mutation.
2.2 The geometric model

We fix an oriented marked surface $S$ with an ideal $3$-angulation $T$ and a base commutative ring $k$. Let $\mathcal{G}_T$ be the relative Ginzburg algebra of Definition 4.17. We describe the part of the geometric model for $\mathcal{D}(\mathcal{G}_T)$ obtained from pure matching curves with local value $f^*(k) \in \text{Fun}(S^2, \text{RMod}_k)$. Such a curve $\gamma$ in $S\setminus M$ is composed of (possibly infinitely many) segments of the following two types such that $\gamma$ intersects $\partial S\setminus M$ and the vertices of $T$ only at the endpoints (possibly none).

![Diagram]

The segments are depicted in blue, they are each contained in exactly one ideal triangle which is depicted in green. The dual ribbon graph is depicted in black. Two edges of the ideal triangle may coincide in the case of a self-folded triangle. The segments of the first type start at a vertex of the ribbon graph and end on an edge of the $3$-angulation. The segments of the second type start and end on edges of the $3$-angulation and only wrap around the vertex of the ribbon graph by one step in the clockwise or counterclockwise direction.

Homotopy classes of pure matching curves relative $(\partial S\setminus M) \cup T_0$ are in bijection with homotopy classes of curves in $S$ relative $\partial S\setminus M$ which do not cut out any discs in $S\setminus M$, see Lemma 5.7.

Example 2.7. Let $e$ be an internal edge of the ribbon graph $T$. If $e$ is not self-folded, we denote by $\gamma_e$ the pure matching curve which traces along $e$. If $e$ is self-folded, we denote by $\gamma_e$ the pure matching curve which traces along $e$ (in any direction, e.g. clockwise) and then traces along the other edge of the self-folded ideal triangle. We depict $\gamma_e$ in the these two cases as follows.

![Diagram]

Proposition 2.8. There exists a full subcategory of $\mathcal{D}(\mathcal{G}_T)$ whose objects are labeled $M_\gamma$, where $\gamma$ is a pure matching curve considered modulo orientation. Given two such curves $\gamma_1 \neq \gamma_2$, the $k$-module $\text{Ext}^*_{\mathcal{D}(\mathcal{G}_T)}(M_{\gamma_1}, M_{\gamma_2})$ is given by the direct sum of the free $k$-modules obtained by counting intersections as below.

Proof. Using the equivalence of $\infty$-categories $\mathcal{D}(\mathcal{G}_T) \simeq \mathcal{H}(T, \mathcal{F}_T(k))$ of Theorem 4.23, up to the degrees of the morphisms, the proposition is shown in more generality in Sections 5 and 6. If $\gamma$ is closed, we also need to specify a $k[x]$-module $h$ to define $M_\gamma$. For concreteness, we set $h = k$ with multiplication matrix the identity matrix, but any other choice of multiplication matrix would also work. The degrees of the morphisms are determined by inspecting their construction in Section 6. \qed
Endpoint intersections
For each endpoint intersection between $\gamma_1$ and $\gamma_2$ as below, where the orange arrow goes in the counterclockwise direction, we have $k \subset \text{Ext}^1(\gamma_1, \gamma_2)$ and $k \subset \text{Ext}^2(\gamma_2, \gamma_1)$.

![Endpoint Intersection Diagram]

If $\gamma_1$ and $\gamma_2$ have an endpoint intersection such that both $\gamma_1$ and $\gamma_2$ exit the ideal triangle at the same boundary edge $f$ the above description of the Ext-groups is adapted as follows. Choosing $\gamma_1$ and $\gamma_2$ with the minimal number of intersections, we instead have $k \subset \text{Ext}^0(\gamma_1, \gamma_2)$ and $k \subset \text{Ext}^3(\gamma_2, \gamma_1)$, if the intersection of $\gamma_2$ with $f$ follows the intersection of $\gamma_1$ with $f$ in the counterclockwise order.

Crossings
For each crossing of $\gamma_1$ and $\gamma_2$ as below, where the orange arrows go in the clockwise direction, we have $k \subset \text{Ext}^0(\gamma_1, \gamma_2)$, $\text{Ext}^2(\gamma_1, \gamma_2)$ and $k \subset \text{Ext}^1(\gamma_2, \gamma_1)$, $\text{Ext}^3(\gamma_2, \gamma_1)$.

![Crossing Diagram]

Directed boundary intersections
For each intersection of $\gamma_1, \gamma_2$ with the same boundary component of $S \setminus M$ as below, where the orange arrow goes in the clockwise direction, we have $k \subset \text{Ext}^0(\gamma_1, \gamma_2)$, $\text{Ext}^2(\gamma_1, \gamma_2)$.

![Directed Boundary Intersection Diagram]

2.3 The categorical description
We fix a commutative ring $k$ as the base ring.

Notation 2.9. Let $S$ be an oriented marked surface with an ideal 3-angulation $\mathcal{T}$ and a lamination $\lambda$ consisting of lamination curves $\gamma_1, \ldots, \gamma_l$. Using Lemma 5.7, we can consider each lamination curve as a pure matching curve. We denote by $M_\lambda = \bigoplus_{i=1}^l M_{\gamma_i} \in \mathcal{D}(\mathcal{G}_T)$ the direct sum of the objects associated to the laminations curves, see Proposition 2.8. For $e$ an edge of $\mathcal{T}$, we denote by $M_{\gamma_e} \in \mathcal{D}(\mathcal{G}_T)$ the object associated to the pure matching curve $\gamma_e$, see Example 2.7.
**Definition 2.10.** Let $S$ be an oriented marked surface with an ideal 3-angulation $T$ with edges $e_1, \ldots, e_m$. Let $\Lambda = (\lambda_1, \ldots, \lambda_k)$ be a multi-lamination on $S$. The categorical extended mutation matrix $\hat{B}(T, \Lambda) = (\hat{b}_{i,j})$ is given by the $m \times (m + k)$-matrix with

- $\hat{b}_{i,j} = \chi \Ext^s(M_{\gamma_e}, M_{\gamma_f})$ for $1 \leq i, j \leq m$ and
- $\hat{b}_{i,m+j} = \frac{1}{2} \chi \Ext^s(M_{\gamma_e}, M_{\lambda_k})$ for $1 \leq i \leq m$ and $1 \leq j \leq k$,

where $\chi$ denotes the Euler-characteristic, see Section 3.2. The categorical $c$-matrix $\hat{C}(T, \Lambda)$ is the $m \times k$ submatrix of $\hat{B}(T, \Lambda)$ consisting of the rows $m + 1, \ldots, m + k$.

**Remark 2.11.** We will see below that in the setting of Definition 2.10

$$\chi \Ext^s(M_{\gamma_e}, M_{\gamma_f}) = \dim_k \Ext^2(M_{\gamma_e}, M_{\gamma_f}) - \dim_k \Ext^1(M_{\gamma_e}, M_{\gamma_f})$$

and

$$\frac{1}{2} \chi \Ext^s(M_{\gamma_e}, M_{\lambda_k}) = \dim_k \Ext^2(M_{\gamma_e}, M_{\lambda_k}) - \dim_k \Ext^1(M_{\gamma_e}, M_{\lambda_k}).$$

**Theorem 2.12.** Let $S$ be an oriented marked surface with a multi-lamination $\Lambda$ and let $T$ be an ideal 3-angulation of $S$. The extended mutation matrices $\hat{B}(T, \Lambda)$ and $B(T, \Lambda)$ are identical.

**Corollary 2.13.** Let $S$ be an oriented marked surface with a multi-lamination $\Lambda$. Suppose that two ideal 3-angulations $T, T'$ of $S$ are related by the flip of an edge. The categorical extended mutation matrices $\hat{B}(T, \Lambda)$ and $\hat{B}(T', \Lambda)$ are related by matrix mutation.

**Proof.** Combine Theorem 2.6 and Theorem 2.12. ■

**Remark 2.14.** If the surface $S$ has internal marked points, the relative Ginzburg algebra $\mathcal{H}_T$ is the relative version of a Ginzburg algebra with a potential which is in most cases degenerate. It is thus generally expected that the relation between such a Ginzburg algebra and the cluster combinatorics breaks down when flipping to an ideal 3-angulation $T'$ such that $Q_T$ has loops or 2-cycles. It is thus surprising that Theorem 2.12 holds in full generality for any marked surface and ideal 3-angulation.

**Proof of Theorem 2.12.** We begin by showing that the upper $m \times m$-submatrix of $\hat{B}(T, \Lambda)$ agrees with the signed adjacency matrix $B(T)$. Let $e_i, e_j$ be two edges of $T$. If $e_i = e_j$, it is obvious that $\hat{b}_{i,i} = 0 = b_{i,i}$. We can thus assume that $e_i \neq e_j$. Assume that $e_i, e_j$ are not the two edges of a self-folded ideal triangle. By Proposition 2.8, we know that $\dim_k \Ext^2(M_{\gamma_e}, M_{\gamma_f})$ counts the number of endpoint intersections where $\gamma_e$ follows $\gamma_f$ in the clockwise order. This number is equal to the number of arrows from $e_i$ to $e_j$ in $Q_T$. Similarly, $\dim_k \Ext^1(M_{\gamma_e}, M_{\gamma_f})$ is equal to the number of arrows from $e_j$ to $e_i$ in $Q_T$. All other $\Ext$-groups vanish and it follows that $\hat{b}_{i,j} = \chi \Ext^s(M_{\gamma_e}, M_{\gamma_f}) = b_{i,j}$. In the case that $e_i, e_j$ are two edges of a self-folded ideal triangle with $e_i$ self-folded, we have $\hat{b}_{i,j} = b_{j,i} = 0$ and $\Ext^s(\gamma_{e_i}, \gamma_{e_j}) = k \oplus k[-1]$ and $\Ext^s(\gamma_{e_j}, \gamma_{e_i}) = k[-2] \oplus k[-3]$ so that also $\hat{b}_{i,j} = b_{i,j} = 0$.

We continue by showing that the $c$-matrices are identical. Using the additivity of $\Ext$, it suffices to verify that for each lamination curve $\gamma$ and each edge $e_i$ there exists an equality

$$\frac{1}{2} \chi \Ext^s(M_{\gamma_e}, M_{\gamma_f}) = (e_i, \gamma).$$  

(1)

We begin with the case that $e_i$ is not self-folded. Proposition 2.8 shows that $\Ext^s(M_{\gamma_e}, M_{\gamma_f})$ is the direct sum of contributions arising from crossings of $e_i$ and $\gamma$. If a crossing of $\gamma$ and $e_i$ is as on the left in Figure 1, then $\Ext^s(M_{\gamma_e}, M_{\gamma_f}) \simeq k \oplus k[-2]$ and $\frac{1}{2} \chi \Ext^s(M_{\gamma_e}, M_{\gamma_f}) = 1$ and the intersection thus contributes the same amount to both sides of (1). Similarly, if the crossing of $\gamma$ and $\gamma_{e_i}$ is as on the right in Figure 1, then $\Ext^s(M_{\gamma_e}, M_{\gamma_f}) = k[-1] \oplus k[-3]$ and the intersection also contributes with $-1$ to both sides of Figure 1.
Consider now the case that $e_i$ is self-folded and let $e'_i$ be the unique other edge of $\mathcal{T}$ incident to $e_i$. If $\gamma$ does not have an infinite end spiraling around the marked point at which $e_i$ lies, then both sides of (1) vanish. We thus assume that such a spiraling infinite end exists. Consider the vertex $v$ incident to $e'_i$ at which $e_i$ does not lie and consider the two edges $e_1 \neq e_2$ incident to $v$ such their cyclic order is given by $e'_i, e_1, e_2, e'_i$. There are four possible arrangements: the end of $\gamma_i$ either arrives at $e'_i$ first passing along $e_1$ or $e_2$ and the infinite end either spirals clockwise or counterclockwise. In the clockwise case, one finds $\Ext^*(M_{\gamma_i}, M_{\gamma}) \simeq k \oplus k[-2]$ if $\gamma$ passes along $e_1$ and $\Ext^*(M_{\gamma_i}, M_{\gamma}) \simeq 0$ if $\gamma$ passes along $e_2$. In the counterclockwise case, one finds $\Ext^*(M_{\gamma_i}, M_{\gamma}) \simeq 0$ if $\gamma$ passes along $e_1$ and $\Ext^*(M_{\gamma_i}, M_{\gamma}) \simeq k[-1] \oplus k[-3]$ if $\gamma$ passes along $e_2$. In each case, we thus find as desired
\[
\frac{1}{2} \chi \Ext^*(M_{\gamma_i}, M_{\gamma}) = \frac{1}{2} \chi \Ext^*(M_{\gamma_i}, M_{\gamma}) = (e'_i, \gamma) = (e_i, \gamma),
\]
where $\gamma$ is as in Definition 2.4. We have thus shown that the $c$-matrices are also identical, concluding the proof. \hfill \blacksquare

3 Background from higher category theory

In Section 3.1, we recall the description of limits in the $\infty$-category of $\infty$-categories in terms of coCartesian sections of the Grothendieck construction, which will be used heavily throughout the paper. We also discuss the computation of limits in such a limit $\infty$-category. In Section 3.2, we discuss morphisms objects in linear $\infty$-categories.

3.1 Limits of $\infty$-categories

An $\infty$-category if called presentable if it is the Ind-completion of a small $\infty$-category and admits all colimits. Examples include (unbounded) derived $\infty$-categories of dg-algebras and $\infty$-categories of modules over $E_\infty$-ring spectra. A remarkable property of presentable $\infty$-categories is the adjoint functor theorem and the duality for limits and colimits it entails. Namely, a functor $F : \mathcal{C} \to \mathcal{D}$ between stable $\infty$-categories admits

- a right adjoint if and only if $F$ preserves colimits.
- a left adjoint if and only if $F$ preserves limits and filtered colimits.

We denote by $\mathcal{P}r_L$ and $\mathcal{P}r_R$ the $\infty$-categories with objects the presentable $\infty$-categories and morphisms the functors which preserve colimits, respectively, perverse limits and filtered colimits.

**Theorem 3.1** ([Lur09, 5.5.3.4]). There exists a canonical equivalence of $\infty$-categories $\mathcal{P}r_L \simeq (\mathcal{P}r_R)^{\text{op}}$, restricting to the identity on objects and mapping each functor in $\mathcal{P}r_L$ to its right adjoint.

Theorem 3.1 implies that the colimit of a diagram in $\mathcal{P}r_L$ is equivalent to the limit of the right adjoint diagram in $\mathcal{P}r_R$.

The inclusions $\mathcal{P}r_L, \mathcal{P}r_R \subset \text{Cat}_\infty$ into the $\infty$-category of $\infty$-categories preserve all limits. Limits in $\text{Cat}_\infty$ can be computed as follows. Consider a 1-category $C$ and a diagram $D : C \to \text{Set}_\Delta$ of simplicial sets taking values in $\infty$-categories. Let $p : \Gamma(D) \to N(C)$ be the relative nerve construction of [Lur09, 3.2.5.2], where $N(C)$ denotes the nerve of $C$. We call $p$ or $\Gamma(D)$ the (covariant) Grothendieck construction of $D$. The objects and morphisms in the Grothendieck construction $\Gamma(D)$ can be described as follows.

The fiber of $p$ over $x \in C$, i.e. the pullback $\infty$-category $\Gamma(D) \times_C \{x\}$, is given by $D(x)$. The set of objects of $\Gamma(D)$ is thus the disjoint union of the sets of objects of the $\infty$-categories $D(x)$ with $x \in C$. Given $x, y \in C$ and two objects $X \in D(x)$ and $Y \in D(y)$, a morphism $\alpha : X \to Y$ in $\Gamma(D)$ consists of
• a morphism \( f : x \to y \) in \( C \) and
• a morphism \( D(f)(X) \to Y \) in \( D(y) \).

If \( D(f)(X) \to Y \) is an equivalence, we call the morphism \( \alpha \) a \textit{coCartesian} morphism and write \( \alpha : X \xrightarrow{\sim} Y \). If \( D(f) \) admits a right adjoint, we call the morphism \( \alpha \) a \textit{Cartesian} morphism if \( D(f)(X) \to Y \) is a counit morphism of the adjunction and write \( \alpha : X \xrightarrow{\sim} Y \).

A choice of limit of the functor of \( \infty \)-categories \( \mathcal{D} : N(C) \to \text{Cat}_\infty \) (given by considering \( D \) as a diagram of \( \infty \)-categories) is given by the \( \infty \)-category of coCartesian sections of \( p : \Gamma(D) \to N(C) \), that is the full subcategory of the \( \infty \)-category

\[
\text{Fun}_{N(C)}(N(C), \Gamma(D)) := \text{Fun}(N(C), \Gamma(D)) \times_{\text{Fun}(N(C), N(C))} \{ \text{id}_{N(C)} \},
\]

spanned by coCartesian sections, i.e. those sections \( s : N(C) \to \Gamma(D) \) satisfying that \( s(e) \) is a coCartesian morphism for each morphism \( e \) in \( N(C) \). We can thus describe objects in the limit of \( D \) as coCartesian sections and morphisms as natural transformations between coCartesian sections.

In the cases of interest for us, the diagram \( D' \) takes values in stable and presentable \( \infty \)-categories, which we thus assume in the following. Limits in the limit of \( D' \) can be computed in the respective fibers of \( p \). This can be seen as follows: the computation of limits in the \( \infty \)-category of sections of \( p : \Gamma(D') \to N(C) \) is equivalent to the computation of \( p \)-relative limits in the \( \infty \)-category \( \text{Fun}(N(C), \Gamma(D')) \). By [Lur09, 4.3.1.10] and [Lur09, 5.1.2.3], these limits are computed pointwise in the respective fibers of \( p \). It thus suffices to note that the property of a section of \( p \) being coCartesian is preserved under limits in the \( \infty \)-category of sections of \( p \), which follows from \( D'(\alpha) \) being a limit preserving functor for each \( 1 \)-simplex \( \alpha \) of \( N(C) \).

Finally, we also introduce the following terminology used later on.

**Definition 3.2.** Let \( p : \Gamma(D') \to N(C) \) be as above. We define the support of a morphism \( \beta : s \to s' \) between (not necessarily coCartesian) sections of \( p \) as the subset of vertices \( x \) of \( N(C) \) such that \( \beta(x) : s(x) \to s'(x) \) is not zero.

### 3.2 Linear \( \infty \)-categories and morphism objects

Given an \( \mathbb{E}_\infty \)-ring spectrum \( R \), the \( \infty \)-category \( \text{RMod}_R \) of right \( R \)-module spectra admits the structure of a symmetric monoidal \( \infty \)-category such that the monoidal product preserves colimits separately in both variables. The \( \infty \)-category thus defines a commutative algebra object in the symmetric monoidal \( \infty \)-category \( \mathcal{P} \mathcal{T}^L \), see [Lur17, Section 4.8], and we can consider its left modules. Note that if \( R = k \) is a commutative ring, then \( \text{RMod}_k \) is equivalent to a symmetric monoidal \( \infty \)-category to the (unbounded) derived \( \infty \)-category \( \mathcal{D}(k) \).

**Definition 3.3.** Let \( R \) be an \( \mathbb{E}_\infty \)-ring spectrum. The \( \infty \)-category \( \text{LinCat}_R = \text{LMod}_{\text{RMod}_R}(\mathcal{P} \mathcal{T}^L) \) of \( R \)-linear \( \infty \)-categories is defined as the \( \infty \)-category of left modules over \( \text{RMod}_R \) in \( \mathcal{P} \mathcal{T}^L \).

**Definition 3.4** ([Lur17, 4.2.1.28]). Let \( R \) be an \( \mathbb{E}_\infty \)-ring spectrum. Let \( \mathcal{C} \) be an \( R \)-linear \( \infty \)-category and let \( X, Y \in \mathcal{C} \). A morphism object is an \( R \)-module \( \text{Mor}_\mathcal{C}(X, Y) \in \text{RMod}_R \) equipped with a map \( \alpha : \text{Mor}_\mathcal{C}(X, Y) \otimes_R X \to Y \) in \( \mathcal{C} \) such that for every object \( C \in \text{RMod}_R \) composition with \( \alpha \) induces an equivalence of spaces

\[
\text{Map}_{\text{RMod}_R}(C, \text{Mor}_\mathcal{C}(X, Y)) \to \text{Map}_\mathcal{C}(C \otimes X, Y).
\]

We will also denote \( \text{Mor}_\mathcal{C}(X, X) \) by \( \text{End}(X) \).

**Remark 3.5.** The formation of morphism objects forms a functor

\[
\text{Mor}_\mathcal{C}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \text{RMod}_R
\]

which preserves limits in both entries.
Given a stable ∞-category \( \mathcal{C} \) and two objects \( A, B \in \mathcal{C} \), the \( n \)-th Ext-group is defined as

\[
\text{Ext}_n^\mathcal{C}(A, B) := \pi_n \text{Map}_\mathcal{C}(A, B[n]) \cong \pi_{-n} \text{Mor}_\mathcal{C}(A, B).
\]

If \( \mathcal{C} \) is \( k \)-linear, \( \text{Mor}_\mathcal{C}(A, B) \in \text{RMod}_k \cong \mathcal{D}(k) \) describes a chain complex and \( \text{Ext}_n^\mathcal{C}(A, B) \) is its \( -n \)-th homology group. The abelian group \( \text{Ext}_n^\mathcal{C}(A, B) \) thus inherits the structure of a (discrete) \( k \)-module. If all involved Ext-groups are free \( k \)-modules, we denote by

\[
\chi \text{ Ext}_c^*(A, B) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}_k \text{Ext}_c^i(A, B)
\]

the Euler characteristic.

For later, we record the following observations:

The \( \infty \)-category \( \text{Fun}(S^{n-1}, \text{RMod}_R) \) with values in \( \text{RMod}_R \) carries canonically the structure of an \( R \)-linear \( \infty \)-category. The pullback functor \( f^* \) along \( S^{n-1} \to * \) gives the following spherical adjunction of \( R \)-linear functors, see [Chr20, Cor. 3.9].

\[
f^* : \text{RMod}_R \leftrightarrow \text{Fun}(S^{n-1}, \text{RMod}_R) : f_*
\]

Lemma 3.6 describes the \( R \)-linear endomorphism object \( \text{End}(f^*(R)) \).

**Lemma 3.6.** There exists an equivalence of \( R \)-modules

\[
\text{End}(f^*(R)) = \text{Mor}_{\text{Fun}(S^{n-1}, \text{RMod}_R)}(f^*(R), f^*(R)) \cong R \oplus R[1 - n].
\]

**Proof.** We begin by recalling notation from proof of the sphericalness of the functor \( f^* \) in [Chr20]. For \( n \geq 0 \), we define the simplicial set \( P_n \) by \( P_0 = \Delta^0 \sqcup \Delta^0 \) and \( P_{i+1} = P_i \sqcup_i P_i \), where \( P_i = P_i \sqcup \Delta^0 \) denotes the join of simplicial sets. The geometric realization of \( \tilde{P}_n \) is homeomorphic to the \( n \)-sphere, so that we can find a morphism of simplicial sets \( \iota : P_n \to S^n \). The pullback functor \( \iota^* \) along \( \iota \) induces an equivalence between \( \text{Fun}(S^{n-1}, \text{RMod}_R) \) and the full subcategory \( \text{Fun}^\text{lc}(P_{n-1}, \text{RMod}_R) \) of \( \text{Fun}(P_{n-1}, \text{RMod}_R) \) of locally constant functors, i.e. spanned by functors which map each \( 1 \)-simplex in \( P_{n-1} \) to an equivalence in \( \text{RMod}_R \). This follows inductively by using the following equivalences between pullback squares in \( \text{LinCat}_R \).

\[
\begin{array}{ccc}
\text{Fun}(S^{i+1}, \text{RMod}_R) & \xrightarrow{\iota^*} & \text{Fun}^\text{lc}(P_{i+1}, \text{RMod}_R) \\
\downarrow_{\text{ev}_x} & & \downarrow_{\text{ev}_x} \\
\text{RMod}_R & \cong & \text{Fun}^\text{lc}(P_i, \text{RMod}_R) \\
\downarrow_{f^*} & & \downarrow_{f^*} \\
\text{Fun}(S^i, \text{RMod}_R) & \xrightarrow{\iota^*} & \text{Fun}^\text{lc}(P_i, \text{RMod}_R)
\end{array}
\]

We thus find an equivalence of \( R \)-modules

\[
\text{Mor}_{\text{Fun}(S^{n-1}, \text{RMod}_R)}(f^*(R), f^*(R)) \cong \text{Mor}_{\text{Fun}(P_{n-1}, \text{RMod}_R)}(\iota^* f^*(R), \iota^* f^*(R)).
\]

We denote \( X_{n-1} = \iota^* f^*(R) \). For \( 1 \leq i \leq n - 2 \), we denote by \( X_i \) the right Kan extension along \( P_i \to P_{n-1} \) of the restriction of \( X_{n-1} \) to \( P_i \). \( X_i \) is thus identical to the functor \( X_{n-1} \) on \( P_i \) and vanishes everywhere else. Similarly, we denote by \( X_i^{\dagger} \) and \( X_i^\circ \) the right Kan extensions along the two inclusions \( P_i \to P_{n-1} \) of the restrictions of \( X_{n-1} \). The \( X_i \)'s are related to each other via biCartesian squares in \( \text{Fun}(P_{n-1}, \text{RMod}_R) \) as follows.

\[
\begin{array}{ccc}
X_i & \xrightarrow{\square} & X_i^{\dagger} \\
\downarrow & & \downarrow \\
X_{i-1}^{\circ} & \xrightarrow{X_{i-1}} & X_{i-1}
\end{array}
\]
Using the universal property of the Kan extension, one can show that for $i > j$ and $l = 1, 2$

\[ \text{Mor}_{\text{Fun}}(P_{n-1}, \text{RMod}_R)(X_i, X_j) \simeq \text{Mor}_{\text{Fun}}(P_{n-1}, \text{RMod}_R)(X_j, X_j) \]

\[ \text{Mor}_{\text{Fun}}(P_{n-1}, \text{RMod}_R)(X_i, X_j^l) \simeq R. \]

Inductively, one shows that these equivalences assemble into equivalences of biCartesian squares in $\text{RMod}_R$,

\[ \text{Mor}_{\text{Fun}}(P_{n-1}, \text{RMod}_R)(X_{n-1}, X_i) \simeq R \oplus R[-i] \]

\[ \text{Mor}_{\text{Fun}}(P_{n-1}, \text{RMod}_R)(X_{n-1}, X_{1-i}) \simeq R \]

\[ \text{Mor}_{\text{Fun}}(P_{n-1}, \text{RMod}_R)(X_{n-1}, X_{i-1}) \simeq R \oplus R[1-i] \]

yielding the desired equivalence $\text{Mor}_{\text{Fun}}(P_{n-1}, \text{RMod}_R)(X_{n-1}, X_{n-1}) \simeq R \oplus R[1-n]$. \[\square\]

### 4 Parametrized perverse schobers

After discussion some background material on ribbon graphs and surfaces in Section 4.1, we give in Section 4.2 an overview over the notion of a parametrized perverse schober from [Chr21]. We then proceed in Section 4.3 by describing how the relation between perverse schobers and relative Ginzburg algebras of 3-angulated surfaces extends to $n$-angulated surfaces.

#### 4.1 Ribbon graphs and $n$-angulated surfaces

**Definition 4.1.**

- A graph $\mathcal{T}$ consists of two finite sets $\mathcal{T}_0$ of vertices and $H$ of halfedges together with an involution $\tau : H \to H$ and a map $\sigma : H \to \mathcal{T}_0$.
- Let $\mathcal{T}$ be a graph. We call the orbits of $\tau$ the edges of $\mathcal{T}$. An edge is called internal if the orbit contains two elements and called external if the orbit contains a single element.
- A ribbon graph consists of a graph $\mathcal{T}$ together with a cyclic order on the set $H(v)$ of halfedges incident to $v$ for all $v \in \mathcal{T}_0$.
- A graph (or ribbon graph) with singularities $(\mathcal{T}, V)$ consists of a graph (or ribbon graph) $\mathcal{T}$ together with a subset $V \subset \mathcal{T}_0$ of the vertices of $\mathcal{T}$ called singularities.

**Definition 4.2.** Let $\mathcal{T}$ be a graph. We define the poset $\text{Exit}(\mathcal{T})$ with

- elements the vertices and edges of $\mathcal{T}$ and
- non-identity morphisms from vertices to their incident edges.

The geometric realization $|\mathcal{T}|$ of $\mathcal{T}$ is defined as the geometric realization of $\text{Exit}(\mathcal{T})$ (as a simplicial set).

**Definition 4.3.**

- By an oriented marked surface $S$, we refer to a compact connected oriented 2-dimensional smooth manifold with boundary $\partial S$ together with a finite subset of marked points $M$.
- Let $S$ be an oriented marked surface and let $\mathcal{T}$ be a graph with an embedding $f : |\mathcal{T}| \to S \setminus M$. We call $f$, or by abuse of notation $\mathcal{T}$, a spanning graph for $S$ if
the embedding $f$ is a homotopy equivalence,
(2) $f(\partial|\mathcal{T}|) \subset \partial S \setminus M$ and $f$ restricts to a homotopy equivalence $\partial|\mathcal{T}| \to \partial S \setminus M$.

**Remark 4.4.** A spanning graph inherits a canonical structure of a ribbon graph, where the cyclic order at each vertex is the counterclockwise order induced by the orientation of the surface. We call also ribbon graphs of this form spanning.

**Definition 4.5.** A graph $\mathcal{T}$ is called $n$-valent if each vertex of $\mathcal{T}$ has valency $n$, i.e. $n$ incident halfedges.

**Definition 4.6.** An ideal $n$-angulation $\mathcal{T}$ of an oriented surface $S$ with marking $M$ consists of an $n$-valent spanning ribbon graph $\mathcal{T}$ for $S$. In this case, we also consider $\mathcal{T}$ as a ribbon graph with singularities $V_T = T_0$ the set of all vertices of $\mathcal{T}$.

**Definition 4.7.**
- We call an internal edge of an ideal $n$-angulation $\mathcal{T}$ self-folded if it is incident to only a single vertex.
- We call an ideal $n$-angulation $\mathcal{T}$ regular if $\mathcal{T}$ contains no self-folded edges.
- We call an ideal $n$-angulation $\mathcal{T}$ semi-regular if there exist a regular ideal $n$-angulation $\mathcal{T}'$ such that $\mathcal{T}'$ can be obtained from $\mathcal{T}$ by repeated flips of not self-folded internal edges of $\mathcal{T}$. For the definition of flip, see Section 8.

**Remark 4.8.** Suppose that $S$ is not sphere with three marked points or the monogon with one marked point in the interior. Any ideal 3-angulation $\mathcal{T}$ of $S$ is semi-regular. This follows from the following picture

and the observation that there is no ideal 3-angulation of $S$ for which every vertex is incident to a self-folded edge.

**Remark 4.9.** In the case $n = 3$, the datum of an ideal 3-angulation is equivalent to the datum of an ideal triangulation, as defined in [FST08]. As for example observed in [DK18, Prop. 3.3.7], the equivalence is given by passing to the dual triangulation of the ribbon graph. The datum of a regular ideal $n$-angulation $\mathcal{T}$ is equivalent to the datum of an $n$-angulation in terms of a decomposition of $S$ into $n$-gons whose vertices lie at the marked points.

**Remark 4.10.** Let $\mathcal{T}$ be a spanning ribbon graph of an oriented marked surface $S$. To each vertex $v$ of $\mathcal{T}$ of valency $n$ we associate an oriented (non-compact) surface $\Sigma_v$ with an embedding of $v$ and its $n$ incident edges. We depict $\Sigma_v$ as follows (in green). The dotted lines correspond to open ends, whereas the solid lines indicate the boundary.
Formally, \( \Sigma_v \) can be defined as the real blow-up of an \( n \)-gon. We define the thickening of \( \mathcal{T} \) to be the oriented surface \( \Sigma_\mathcal{T} \), obtained from gluing the surfaces \( \Sigma_v \) at their boundaries whenever two vertices are incident to the same edge. Note that there are homotopy equivalences \( \Sigma_\tau \simeq |\mathcal{T}| \simeq S\backslash M \). Further, we can choose an embedding of \( \Sigma_\tau \) into \( S\backslash M \) and a retraction \( S\backslash M \to \Sigma_\tau \).

### 4.2 Definition and properties

Let \( F : A \leftrightarrow B : G \) be an adjunction between stable \( \infty \)-categories. We define

- the twist functor \( T_A : A \to A \) as the cofiber \( T_A = \text{cof}(\text{id}_A \to GF) \) of the unit of \( F \dashv G \) in the stable \( \infty \)-category \( \text{Fun}(A,A) \) and

- the cotwist functor as the fiber \( T_B = \text{fib}(FG \to \text{id}_B) \) of the counit of \( F \dashv G \) in the stable \( \infty \)-category \( \text{Fun}(B,B) \).

The adjunction \( F \dashv G \) is called spherical if \( T_A \) and \( T_B \) are equivalences. In this case, we also call \( F \) a spherical functor. The notion of a spherical functor in the context of dg-categories is due to [AL17] and generalizes the notion of a spherical object. For treatments of spherical adjunctions in the setting of stable \( \infty \)-categories, we refer to [DKSS21, Chr20]. We proceed by introducing the local model for a parametrized perverse schober, based on the datum of a spherical functor.

Let \( F : A \to B \) be a spherical functor and consider the diagram \( D(F) : \Delta^{n-1} \to \text{Set}_\Delta \) obtained from the \( n-1 \) composable functors

\[
A \xrightarrow{F} B \xrightarrow{id} \ldots \xrightarrow{id} B.
\]

We define the stable \( \infty \)-category \( \mathcal{V}_F^n \) as the \( \infty \)-category of sections of the Grothendieck construction of the diagram \( D(F) \). The objects of \( \mathcal{V}_F^n \) are therefore diagrams of the form

\[
a \xrightarrow{\alpha_1} b_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{n-1}} b_{n-1}
\]

with \( a \in A, b_i \in B \) and the morphism \( \alpha_1 \) encoding the datum of a morphism \( F(a) \to b \) in \( B \).

The \( \infty \)-category \( \mathcal{V}_F^n \) comes with a semiorthogonal decomposition (SOD) \( (A, B, \ldots, B) \) of length \( n \), see [Chr21, Section 2.6] for background. The \( i \)-th component of the SOD is spanned by objects of the form (3) satisfying that \( b_l = 0 \) for all \( l \) if \( i = 1 \) and \( a = 0 = b_l \) for all \( l \neq i-1 \) for \( 2 \leq i \leq n \).

We consider an \( n \)-valent vertex \( v \) of a ribbon graph \( \mathcal{T} \) with incident edges \( e_1, \ldots, e_n \), ordered compatibly with their cyclic order. We define a poset \( C_v \) with objects \( v, e_1, \ldots, e_n \) and morphisms \( v \to e_i \), which is a full subcategory of \( \text{Exit}(\mathcal{T}) \). The local model at \( v \) of a parametrized perverse schober consists of a functor

\[
\mathcal{F}_v(F) : C_v \to \text{St}
\]

defined as follows. It is defined on objects via \( \mathcal{F}_v(F)(v) = \mathcal{V}_F^n \) and \( \mathcal{F}_v(F)(e_i) = N_F := B \). It is defined on morphisms via

\[
\mathcal{F}_v(F)(v \to e_i) = g_i := \begin{cases} 
\pi_n & i = 1 \\
\text{cof}_{n-i,n-i+1}[i-2] & 2 \leq i \leq n-1 \\
\text{rcof}_{1,2}[n-2] & i = n
\end{cases}
\]

where

- \( \pi_n \) denotes the projection functor to the last component of the SOD (i.e. maps (3) to \( b_{n-1} \),
• cof_{i,i+1} denotes the composite of the projection to the i-th and (i + 1)-th components of the SOD with the cofiber functor (i.e. maps (3) to the cofiber of α_i)
• and rcof_1,2 denotes the composite of the projection to the first two components of the SOD with the relative cofiber functor (i.e. maps (3) to the cofiber of F(a) → b_1).

Remark 4.11. The functor ρ_1, ..., ρ_n in (5) admit right adjoints ζ_i which from a sequence of adjunctions
ζ_i T_B^{-1}[1 - n] \dashv g_1 \dashv g_n \dashv ... \dashv g_{n-1} \dashv ζ_{n-1} \dashv ... \dashv ζ_1,
(6)
see [Chr21, Section 3].

We are now ready to give the full definition of a parametrized perverse schober.

Definition 4.12. Let (T, V) be a ribbon graph with singularities. A (T, V)-parametrized perverse schober is a functor
\[ F : \text{Exit}(T) \longrightarrow \text{St} \]
such that
• for each vertex v of T there exists a spherical functor F : A → B such that the restriction of F to C_v is equivalent to F(v)(F) and
• if v /∈ V, then A = 0.
We often leave the set V implicit, referring to F as a T-parametrized perverse schober.

Definition 4.13. Let F be a T-parametrized perverse schober.
• The stable ∞-category of global sections \( H(T, F) \) is defined as the limit of F in Cat_{∞}.
• The stable ∞-category \( L \) of sections of F is the ∞-category of (all) sections of the Grothendieck construction \( p : \Gamma(F) \longrightarrow \text{Exit}(T) \).

The definition of a T-parametrized perverse schober involves for each vertex of T a choice of a total order on the a priori cyclically ordered set of incident edges. Different choices lead to different local models \( F_v(F) \), which are related by natural equivalences obtained from powers of the equivalence \( T_V F \) described in Proposition 4.14.

Proposition 4.14 ([Chr21]). Let F : A ↔ B : G be a spherical adjunction. The functor
(\varrho_1, ..., \varrho_n) : V_F^p \longrightarrow N_F^x
is spherical. The twist functor \( T_V^p \) satisfies
\[ \varrho_i \circ T_V^p = \begin{cases} \varrho_{i+1} & 1 \leq i \leq n - 1 \\ T_B[n - 1] \circ \varrho_1 & i = n \end{cases} \]
where \( T_B \) is the cotwist functor of F \dashv G.

Remark 4.15. We call \( T_V^p \) the paracyclic twist functor as it realizes the paracyclic symmetry of perverse schobers at their vertices.

A further way to produce, morally equivalent, parametrized perverse schobers from given ones is via contractions of ribbon graphs. A contraction c : (T, V) \longrightarrow (T', V') of ribbon graphs with singularities collapses finitely many edges of T such that no loops nor edges incident to two singularities are collapsed. The set of singularities V' consists of all vertices in V (not counting any of the collapsed vertices) and the vertices arising from the collapse of edges with one incident singular vertex. A more formal definition of a contraction of ribbon graphs can be found in [Chr21, Def. 4.21].

Assume that c collapses a single edge e of T, connecting two vertices v, v'. Given a T-parametrized perverse schober F, one can show that there exists a T'-parametrized perverse schober c_*F satisfying the following.
• \( c_\ast \mathcal{F}(\tilde{v}) = \mathcal{F}(\tilde{v}) \) for each vertex \( \tilde{v} \neq v, v' \) of \( \mathcal{T}' \).
• \( c_\ast \mathcal{F}(\tilde{e}) = \mathcal{F}(\tilde{e}) \) for each edge \( \tilde{e} \neq e \) of \( \mathcal{T}' \).
• \( c_\ast \mathcal{F}(\tilde{v} \to \tilde{e}) = \mathcal{F}(\tilde{v} \to \tilde{e}) \) for each edge \( \tilde{e} \) incident to a vertex \( \tilde{v} \) such that \( \tilde{v} \neq v, v' \) and \( \tilde{e} \neq e \).
• For the vertex \( v = v' \) of \( \mathcal{T}' \), there exists an equivalence of \( c_\ast \mathcal{F}(v) \) and the pullback in \( \text{Cat}_\infty \) of the following diagram.

\[
\begin{array}{c}
\mathcal{F}(v) \\
\downarrow \mathcal{F}(v \to e) \\
\mathcal{F}(v') \\
\mathcal{F}(v') \mathcal{F}(v \to e) \mathcal{F}(e)
\end{array}
\]

• For each morphism \( v' = v \to \tilde{e} \) in \( \text{Exit}(\mathcal{T}') \) arising from a morphism \( v \to \tilde{e} \) in \( \text{Exit}(\mathcal{T}) \), there exists an equivalence of functors between \( c_\ast \mathcal{F}(v \to \tilde{e}) \) and the composite

\[
\mathcal{F}(v) \xrightarrow{\mathcal{F}(v \to \tilde{e})} \mathcal{F}(\tilde{e}) \simeq c_\ast \mathcal{F}(\tilde{e}).
\]

An analogous statement holds with \( v \) replaced by \( v' \).

Since each contraction can be written as the composite of contractions which only collapse a single edge, we can associate to each contraction \( c \) a functor \( c_\ast \) which takes \( (\mathcal{T}, V) \)-parametrized perverse schobers to \( (\mathcal{T}', V') \)-parametrized perverse schobers. As shown in [Chr21, Prop. 4.25], the functor \( c_\ast \) commutes with taking global sections.

**Proposition 4.16.** Let \( c : (\mathcal{T}, V) \to (\mathcal{T}', V') \) be a contractions of ribbon graphs with singularities and let \( \mathcal{F} \) be a \( (\mathcal{T}, V) \)-parametrized perverse schober. Then there exists an equivalence of \( \infty \)-categories

\[
\mathcal{H}(\mathcal{T}, \mathcal{F}) \simeq \mathcal{H}(\mathcal{T}', c_\ast \mathcal{F}).
\]

### 4.3 Relative Ginzburg algebras and perverse schobers

We begin with the definition of the relative Ginzburg algebra of an ideal \( n \)-angulation, generalizing the definition in the case \( n = 3 \) from [Chr21].

**Definition 4.17.** Let \( \mathcal{T} \) be an ideal \( n \)-angulation of an oriented marked surface. We define a graded quiver \( \bar{Q}_\mathcal{T} \) with objects the edges of \( \mathcal{T} \) and the following graded arrows.

- An arrow \( a_{v,i,j} : i \to j \) of degree \( l - 1 \) for each singularity \( v \in \mathcal{T}_0 \) at which a halfedge of \( i \) follows a halfedge of \( j \) in the cyclic order after \( 1 \leq l \leq n - 1 \) steps. The arrows thus go in the clockwise direction and the self-folded edges of \( \mathcal{T} \) give rise to loops.
- A loop \( l_i : i \to i \) of degree \( n - 2 \) for each internal edge \( i \).

Given two edges \( i, j \in \mathcal{T}_1 \) incident to \( v \in \mathcal{T}_0 \), we denote by \( j - i \in \{0, \ldots, n - 1\} \) the number of steps after which \( j \) follows \( i \) in the cyclic order at \( v \).

The relative Ginzburg algebra \( \mathcal{G}_\mathcal{T} \) is defined as the dg-algebra with underlying graded algebra \( k\bar{Q}_\mathcal{T} \) and with differential \( d \) determined on the generators as follows.

For the generators \( a_{v,i,j} \), we set

\[
d(a_{v,i,j}) = \sum_{i<k<j} (-1)^{j-k} a_{v,k,j} a_{v,i,k},
\]

where the sum runs over all edges \( k \) appearing between \( i \) and \( j \) in the cyclic order.

If \( i \) is an external edge incident to the singularity \( v_1 \in \mathcal{T}_0 \), we set

\[
d(l_i) = \sum_{j \neq i} (-1)^{j-i} a_{v_1,j,i} a_{v_1,i,j}.
\]
If \( i \) is an internal edge incident to two (possibly identical) singularities \( v_1, v_2 \in \mathcal{T}_0 \) (with arbitrarily chosen labeling), we set

\[
d(l_i) = \sum_{j \neq i} (-1)^{j-i} a_{v_1,j,i} a_{v_1,i,j} + (-1)^{n-1} \sum_{j \neq i} (-1)^{j-i} a_{v_2,j,i} a_{v_2,i,j}.
\]

(7)

Note that a different choice of labeling of \( v_1, v_2 \) in (7) changes \( \mathcal{G}_\mathcal{T} \) only up to isomorphism of dg-algebras.

**Remark 4.18.** The formula for the differential of \( a_{v,i,j} \) in Definition 4.17 can be considered as the graded cyclic derivative of the potential

\[
W'_\mathcal{T} = \sum_{p \in \mathcal{T}_0} \sum_{i < j < k} a_{v,k,i} a_{v,j,k} a_{v,i,j}.
\]

We note further that if an edge \( i \) is self-folded, then its incident singularities \( v_1, v_2 \) are identical but \( a_{v_1,j,i} a_{v_1,i,j} \neq a_{v_2,j',i} a_{v_2,i,j'} \) for all \( j, j' \), as the arrows arise from the two different halfedges of \( i \). The differential (7) is thus always nonzero.

**Example 4.19.** For \( n = 3, 4 \), let \( S \) be the \( n \)-gon and \( \mathcal{T} \) the unique ideal \( n \)-angulation. The graded algebras underlying the relative Ginzburg algebras \( \mathcal{G}_\mathcal{T} \) are given by the path algebras of the following graded quivers.

![Diagram of graded quivers](image_url)

The differentials of the arrows in degree 0 vanish and the differential of an arrow \( a : x \to y \) of higher degree consists modulo signs of a sum of all paths composed of two arrows of lower degrees which compose to a path \( x \to y \).

**Construction 4.20.** We fix an oriented marked surface \( S \) with an ideal \( n \)-angulation \( \mathcal{T} \) and a choice of \( E_\infty \)-ring spectrum \( R \). In this construction, we describe the \( (\mathcal{T}, V_\mathcal{T}) \)-parametrized perverse schober \( \mathcal{F}_\mathcal{T}(R) \), with the property that its \( \infty \)-category of global sections \( \mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R)) \) is equivalent to \( \mathcal{D}(\mathcal{G}_\mathcal{T}) \) if \( R = k \) is a commutative ring, see Theorem 4.23.

Locally at each singularity of \( \mathcal{T} \), the perverse schober \( \mathcal{F}_\mathcal{T}(R) \) is described by the spherical adjunction

\[
f^*: \text{RMod}_R \leftrightarrow \text{Fun}(S^{n-1}, \text{RMod}_R) : f_*
\]

between the pullback functor \( f^* \) along \( S^{n-1} \to * \) and its right adjoint \( f_* \). The sphericalness of \( f^* \dashv f_* \), was shown in [Chr20, Prop. 3.9]. Note that our notation for \( f^* \dashv f_* \) suppresses the choices of \( n \) and \( R \). We thus want to define \( \mathcal{F}_\mathcal{T}(R) \) as the gluing of the perverse schobers \( \mathcal{F}_v(f^*) : C_v \to \text{St} \), see (4), i.e. as the diagram \( \text{Exit}() \) \( \to \text{St} \) which restricts at \( C_v \) to \( \mathcal{F}_v(f^*) \).

Note that this definition involves making choices, see Remark 4.22. In the case that \( n \) is odd, we must choose the gluing diagram more carefully by slightly modifying the perverse schobers \( \mathcal{F}_v(f^*) \), to match the signs in the differentials of the Ginzburg algebras.

For each edge \( e \) of \( \mathcal{T} \) we consider its two incident (possibly identical) vertices \( v_1, v_2 \). We denote by \( i_1 \in \{1, \ldots, n\} \) the position of the halfedge of \( e \) lying at \( v_1 \) in the chosen total order of the \( n \) halfedges incident to \( v_1 \). We similarly denote by \( i_2 \in \{1, \ldots, n\} \) the position of the halfedge
of $e$ at $v_2$ in the chosen total order of halfedges at $v_2$. If $i_1 - i_2$ is even, we change $\mathcal{F}_{v_1}(f^*)$ by composing $\mathcal{F}_{v_1}(f^*)(v_1 \to e)$ with the autoequivalence $T$ making the following diagram commute.

$$\begin{array}{ccc}
N_{f^*} & \xrightarrow{T} & N_{f^*} \\
\downarrow^{(9) \simeq} & & \downarrow^{(9) \simeq} \\
R\text{Mod}_{R[t_{n-2}]} & \xrightarrow{\phi^*} & R\text{Mod}_{R[t_{n-2}]}
\end{array}$$

Above $\phi^*$ is the pullback functor along the map of ring spectra $R[t_{n-2}] \xrightarrow{t_{n-2} \mapsto (-1)^n t_{n-2}} R[t_{n-2}]$. Note that for $R = k$, the functor $T$ is equivalent to the cotwist functor of $f^* \circ f_*$, see [Chr21, Prop. 5.7] and further $T \simeq \text{id}_{N_f}$ if $n$ is even. If $i_1 - i_2$ is odd, we do nothing. The perverse schober $\mathcal{F}_\mathcal{T}(R)$ is now defined as the gluing of the above modifications of the $\mathcal{F}_v(f^*)$.

**Remark 4.21.** Structurally, the $\infty$-category of global sections of $\mathcal{F}_\mathcal{T}(R)$ does not change very much by the modifications of the pieces $\mathcal{F}_v(f^*)$ used in the gluing in Construction 4.20. In fact, the only result of this paper which is affected by the modifications is the description of the endomorphisms objects of the global sections associated to closed matching curves in Theorem 6.3.

**Remark 4.22.** The definition of $\mathcal{F}_\mathcal{T}(R)$ in Construction 4.20 involves a choice of total order of the halfedges incident to each singularity $v$, which are a priori only carry a cyclic order. Different choices of total orders lead to a different definition of $\mathcal{F}_\mathcal{T}(R)$ in Construction 4.20. It is shown in [Chr20, Section 7.1], that in the case $R = k$ and $n = 3$ the equivalence class of the perverse schober $\mathcal{F}_\mathcal{T}(k)$ does not depend on these choices. The proof given there generalizes to any $n \geq 3$. In the case that $n$ is even, instead of a choice of spin structure on $S \setminus M$, only a choice of orientation is required as input for the construction of $\mathcal{F}_\mathcal{T}(k)$.

**Theorem 4.23.** Let $S$ be an oriented marked surface with an ideal $n$-angulation $\mathcal{T}$. There exists an equivalence of $\infty$-categories

$$\mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(k)) \simeq D(\mathcal{G}_\mathcal{T})$$

between the $\infty$-category of global sections of $\mathcal{F}_\mathcal{T}(k)$ and the derived $\infty$-category of the relative Ginzburg algebra $\mathcal{G}_\mathcal{T}$.

Section 4.4 is dedicated to the proof of Theorem 4.23.

**Remark 4.24.** The perverse schober $\mathcal{F}_\mathcal{T}(R)$ canonically factors through LinCat$_R \to \text{St}$, so that its limit $\mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R))$ inherits a canonical $R$-linear structure. Theorem 4.23 can be strengthened to the statement that (8) is an equivalence of $k$-linear $\infty$-categories.

**Proposition 4.25.** Let $\mathcal{T}$ be an ideal $n$-angulation of an oriented marked surface. Given an edge $e$ of $\mathcal{T}$, we denote by $ev_e : \mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R)) \to \mathcal{F}_\mathcal{T}(R)(e) = N_f$, $\simeq \text{RMod}_{R[t_{n-2}]}$ the evaluation functor and by $ev^*_e$ its left adjoint.

1. The direct sum $\bigoplus_e ev^*_e(N_{f_{t_{n-2}}})$ over all edges of $\mathcal{T}$ is a compact generator of $\mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R))$.

2. If $R = k$, the projective $\mathcal{G}_\mathcal{T}$-module $pe_{\mathcal{G}_\mathcal{T}}$, were $p_e \in k\mathcal{Q}_\mathcal{T}$ is the lazy path at $e$, is identified under the equivalence (8) with $ev^*_e[k[t_{n-2}]]$.

**Proof.** For part (1), it suffices to observe that the functor $(q_1, \ldots, q_n) : \mathcal{V}_e^{t_{n-2}} \to N_{f_e}^{t_{n-2}}$ is conservative, so that a global section $X \in \mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R))$ is zero if and only if $ev_e(X) = 0$ for all edges $e$ of $\mathcal{T}$.

For part (2), the corresponding proof from [Chr21, Prop. 6.7] for the case $n = 3$ directly generalizes to any $n \geq 3$. 

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4.4 The proof of Theorem 4.23

For the entirety of this section, we fix an integer $n \geq 3$ and a commutative ring $k$. Let $\mathcal{J}$ be an ideal $n$-angulation of an oriented marked surface. The goal of this section is to show that the ∞-category of global sections $\mathcal{H}(\mathcal{J}, \mathcal{F}_\mathcal{J})$ is equivalent to the derived ∞-category of the relative Ginzburg algebra of $\mathcal{J}$. This approach very much follows the approach of Sections 5 and 6 of [Chr21]. We will refer at many points to results from loc. cit..

In the following, we very briefly recall the relation between dg-categories and stable, presentable ∞-categories and some sign conventions. More details can be found in [Chr21, Section 2]. We then give an algebraic description of the perverse schober on the $n$-gon obtained from the spherical adjunction $f^* \dashv f_*$, which we apply in finding the equivalence $\mathcal{H}(\Gamma, \mathcal{F}_\mathcal{J}) \simeq \mathcal{D}(\mathcal{G}_\mathcal{J})$.

We denote by $\text{dgCat}_k$ the category of $k$-linear dg-categories. A dg-functor $F : A \to B$ is called a quasi-equivalence if for all $a, a' \in A$, the map between morphism complexes $F(a, a') : \text{Hom}(A(a, a')) \to \text{Hom}(B(F(a), F(a')))$. A quasi-equivalence of categories exists if the induced functor $F_{\text{perf}} : A_{\text{perf}} \to B_{\text{perf}}$ on dg-categories of perfect modules is a quasi-equivalence. As shown by Tabuada, $\text{dgCat}_k$ admits the quasi-equivalence model structure whose weak equivalences are the quasi-equivalences. There is a functor $\mathcal{D}(-) : N(\text{dgCat}_k) \to \mathcal{P}_{\text{r}L}$ from the nerve of $\text{dgCat}_k$, which assigns to a dg-algebra (dg-category with a single object) its unbounded derived ∞-category, c.f. [Chr21, Section 2.5]. The functor $\mathcal{D}(-)$ maps homotopy colimits with respect to the quasi-equivalence structure to colimits in $\mathcal{P}_{\text{r}L}$.

Proposition 4.26 ([Chr21, Proposition 5.5]). Let $n \geq 3$. There exists an equivalence of ∞-categories

$$N_{f^*} = \text{Fun}(S^{n-1}, \mathcal{D}(k)) \simeq \mathcal{D}(k[t_{n-2}]),$$

where $k[t_{n-2}]$ denotes the polynomial algebra with generator in degree $|t_{n-2}| = n - 2$.

Lemma 4.27. Let

$$A_m = \begin{pmatrix}
    k & k[t_{n-2}] & 0 & \ldots & 0 & 0 \\
    0 & k[t_{n-2}] & k[t_{n-2}] & 0 & \ldots & 0 \\
    0 & 0 & k[t_{n-2}] & k[t_{n-2}] & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & k[t_{n-2}] & k[t_{n-2}] \\
    0 & 0 & 0 & \ldots & 0 & k[t_{n-2}]
\end{pmatrix}$$

be the upper triangular dg-algebra. There exists an equivalence of ∞-categories

$$\mathcal{V}_{f^*} \simeq \mathcal{D}(A_m)$$

Proof. The proof of [Chr21, 5.13] directly generalizes. □

Remark 4.28. Consider the graded quiver $Q_m$

$$x_0 \xrightarrow{a_{0,1}} x_1 \xrightarrow{a_{1,2}} \ldots \xrightarrow{a_{m-2,m-1}} x_{m-1}$$

(11)
with \( |a_{i,i+1}| = 0 \) and \( |l_i| = n - 2 \). The dg-algebra \( A_m \) is Morita-equivalent to the dg-category \( B_m \) with objects the vertices of \( Q_m \) and morphisms freely generated by the arrows of \( Q_m \) subject to the relations \( l_1 \circ a_{0,1} = 0 \), \( a_{i+1,i+2} \circ a_{i,i+1} = 0 \) for \( i \geq 0 \) and \( a_{i,i+1}l_i = l_{i+1}a_{i,i+1} \) for \( i \geq 1 \).

For \( m \geq 3 \), we define by \( D_m \) the dg-category with objects \( z_1, \ldots, z_m \) and morphisms freely generated by \( b_{i,j} : z_i \to z_j \) for all \( i \neq j \) in degree \( j - i - 1 \) if \( j > i \) and \( n + j - i - 1 \) if \( j < i \) and with differentials determined by

\[
d(b_{i,j}) = \begin{cases} 
\sum_{i<k<j}(-1)^{j-k+1}b_{k,j}b_{i,k} & \text{if } j > i, \\
\sum_{i<k\leq m}(-1)^{j-k+n+1}b_{k,j}b_{i,k} + \sum_{1 \leq k < j}(-1)^{j-k+1}b_{k,j} \circ b_{i,k} & \text{if } j < i.
\end{cases}
\]

Note that if \( m = n \), the dg-category \( D_m \) is Morita equivalent to the relative Ginzburg algebra of the \( n \)-gon and depicted in the cases \( m = n = 3 \) and \( m = n = 4 \) in Example 4.19.

**Lemma 4.29.** The homology of the mapping complexes in \( D_m \) is given by

\[
H_* \text{Hom}_{D_m}(z_i, z_j) \simeq \begin{cases} 
0 & j \neq i, i + 1 \\
k[t_{n-2}] & j = i \\
k[t_{n-2}] & j = i + 1 \\
k[t_{n-2}]n & j = 1, i = m
\end{cases}
\]

**Proof.** Let \( c \) be a cycle in a morphism complex in \( D_m \). We can decompose \( c \) into the \( k \)-linear sum of morphisms composed of the generating morphisms of \( D_m \). We call the length of \( c \) the maximal number of generating morphisms appearing in a summand of \( c \). We show the following statements via an induction over the length of \( c \).

1) If \( c : z_i \to z_i \), then \( c \) is homologous to

\[
\lambda \left( \sum_{j<i}(-1)^{n+j-i}b_{j,i}b_{i,j} + \sum_{i<j}(-1)^{j-i}b_{j,i}b_{i,j} \right)^l
\]

with \( \lambda \in k \) and \( l \in \mathbb{Z} \).

2) If \( c : z_i \to z_{i+1} \), with \( i + 1 \) considered modulo \( m \), then \( c \) is homologous to

\[
\lambda b_{i,i+1} \left( \sum_{j<i}(-1)^{n+j-i}b_{j,i}b_{i,j} + \sum_{i<j}(-1)^{j-i}b_{j,i}b_{i,j} \right)^l
\]

with \( \lambda \in k \) and \( l \in \mathbb{Z} \).

3) Otherwise, \( c \) is nullhomologous.

Observing that (12) and (13) define non-zero homology classes, the assertion follows.

We continue with showing 1), 2) and 3). We denote the cycles of the form (12) by \( l_i \) and the cycles of the form (13) by \( b_{i,i+1}l_i \). We consider all indices \( i, j \) of the \( b_{i,j} \) modulo \( m \).

We begin with 2), as it is the easiest case. We consider a cycle \( c : z_i \to z_{i+1} \). Since the morphisms \( b_{j,l} \) freely generate \( D_m \), we can write \( c = \sum_{k \neq i+1} b_{k,i+1}u_{i,k} \) for some chains \( u_{i,k} \). The condition \( d(c) = 0 \) implies \( d(u_{i,i+2}) = 0 \). By the induction assumption there exists a chain \( v_{i,i+2} \) with \( d(v_{i,i+2}) = u_{i,i+2} \). We thus find

\[
c + (-1)^{n-1}d(b_{i+2,i+1}u_{i,i+2}) = \sum_{k \neq i+1,i+2} b_{k,i+1} \left( u_{i,k} + (-1)^{s_{k,i+1}+i-k}b_{i+2,k}v_{i,i+2} \right)
\]

with \( s_{k,i} = 0 \) if \( i + 2 < k \leq m \) and \( s_{k,i} = n \) if \( 1 \leq k \leq i \). This shows that \( c \) is homologous to a cycle \( c_2 = \sum_{k \neq i+1,i+2} b_{k,i+1}u_{i,k} \) for some other chains also denoted \( u_{i,k} \). Repeating this
argument $m - 2$ times, we see that $c$ is homologous to a cycle of the form $b_{i,i+1}u_{i,i}$ and by the induction hypothesis we find $u_{i,i} = l + d(v_{i,i})$ for some chain $v_{i,i}$. It follows that $c$ is homologous to $b_{i,i+1}l_i$, showing 2).

For 3), we consider a cycle $c : z_i \to z_j$ with $j \neq i, i + 1$ and assume without loss of generality that $i < j$. We write $c = \sum_{k \neq j} b_{k,j}u_{i,k}$ for some chains $u_{i,k}$ with $d(u_{i,j}+1) = 0$. If $j + 1 \neq i$, then by the induction assumption $u_{i,j+1} = d(v_{i,j+1})$. As in the case ii), we thus find that $c$ is homologous a cycle of the form $\sum_{k \neq j} b_{k,j}u_{i,k}$ for some chains also labeled $u_{i,k}$. Repeating this process a few times, we find that $c$ is homologous to a cycle of the form $\sum_{k \neq j, \ldots, i-1} b_{k,j}u_{i,k}$ with $d(u_{i,i}) = 0$. Applying the induction assumption we find that $u_{i,i} = l_i + d(v_{i,i})$ for some chain $v_{i,i}$. The condition $d(c) = 0$ then implies that $(-1)^{j-i}b_{i+1,j}b_{i,i+1}l_i = b_{i+1,j}d(u_{i,i+1})$. Since $b_{i,i+1}l_i$ is not a boundary unless $l_i = 0$, it follows that $l_i = 0$. We obtain that $c$ is homologous to $\sum_{k \neq j, \ldots, i-1} b_{k,j}u_{i,k}$, for some chains also labeled $u_{i,k}$ with $d(u_{i,i+1}) = 0$. If $j = i + 2$, the assertion now follows and otherwise we argue as before to obtain that $c$ is homologous to $\sum_{k \neq j, \ldots, i+2} b_{k,j}u_{i,k}$ for some chains also labeled $u_{i,k}$. Repeating this argument a few times, we can finally conclude that $c$ is a boundary.

For 1), we consider a cycle $c : z_i \to z_i$, which we can write as $c = \sum_{k \neq i} b_{k,i}u_{i,k}$ for some chains $u_{i,k}$ with $d(u_{i,i+1}) = 0$. Using the induction assumption, we find a chain $v_{i,i+1}$ with $u_{i,i+1} = d(v_{i,i+1}) - b_{i,i+1}l_i$. It follows that $c$ is homologous to a cycle of the form $c_1 = \sum_{k \neq i, i+1} b_{k,i}u_{i,k} - b_{i,i+1}b_{i,i+1}l_i$ for some other chains also labeled $u_{i,k}$. This constitutes the base case for an induction over $j$ of the following assertion.

For all $1 \leq j \leq m - 1$, the cycle $c$ is homologous to

$$c_j = \sum_{k \neq i} b_{k,i}u_{i,k} + \left( \sum_{1 \leq k \leq i+j-m<i} (-1)^{n+k-i}b_{k,i}b_{i,k} + \sum_{i<k<i+j-m} (-1)^{k-i}b_{k,i}b_{i,k} \right) l_i,$$

where $I$ is the set of $1 \leq k \leq m$ such that $k > i+j$ or $k < i+j-m$ and the $u_{i,k}$ are some chains.

For the induction step, we consider the case $i+j < m$. The case $i+j > m$ is dealt with analogously. Suppose that $c$ is homologous to $c_j$. Evaluating the condition $d(c_j) = 0$ at the summands beginning with $b_{i+j+1,i}$ yields

$$0 = (-1)^{n-j}b_{i+j+1,i}d(u_{i,i+j+1}) + \left( \sum_{i<k<i+j} (-1)^{n-j+k-i}b_{i+j+1,i}b_{i+j+1}b_{i,k} \right) l_i,$$

so that $u_{i,i+j+1} = d(v_{i,i+j+1}) + (-1)^{j+1}b_{i,i+j+1}l_i$ for some chain $v_{i,i+j+1}$. It follows that $c$ is homologous to $c_{j+1}$. This completes the induction step. Setting $j = m - 1$, we obtain that $c$ is homologous to

$$\left( \sum_{k<i} (-1)^{n+k-i}b_{k,i}b_{i,k} + \sum_{i<k} (-1)^{k-i}b_{k,i}b_{i,k} \right) l_i,$$

and thus of the form (12). This concludes the proof. □

**Proposition 4.30.**

1. There exists an equivalence of ∞-categories

$$\mathcal{V}_j^m \simeq \mathcal{D}(D_m).$$

(14)
The \( \text{dg-functor} \) \( \mu: k[t_{n-2}] \to D_m \) is determined by

\[
\mu_i(t_{n-2}) = (-1)^m i + m \left( \sum_{j < i} (-1)^{n+j-i} b_{i,j} + \sum_{i < j} (-1)^{j-i} b_{i,j} \right).
\]

**Proof.** We recursively define objects \( y_{i+1} = \text{cone}(y_i \xrightarrow{\alpha_i} x_i) \) for \( i \geq 0 \) in \( \text{dgMod}(B_m) \), where \( y_1 = x_0 \) and for \( i \geq 1 \)

\[
\alpha_i = (0, \ldots, 0, a_{i-1,i}) \in \bigoplus_{j=0}^{i-1} \text{Hom} \( B_m, \langle x_j[i-j-1], x_i \rangle \simeq \text{Hom}_{\text{dgMod}(B_m)}(y_i, x_i) \)
\]

where the splitting only holds on the level of graded \( k \)-modules. We denote by \( \langle x_1, \ldots, x_{m-1}, y_m \rangle \subset \text{dgMod}(B_m) \) the full \( \text{dg} \)-subcategory spanned by \( x_1, \ldots, x_{m-1}, y_m \). Note that \( x_1, \ldots, x_{m-1}, y_m \)

compactify \( \text{dgMod}(B_m) \) so that there exists an equivalence of \( \infty \)-categories

\[
\text{D}(\langle x_1, \ldots, x_{m-1}, y_m \rangle) \simeq \text{D}(B_m).
\]

A direct computation shows that \( \langle x_1, \ldots, x_{m-1}, y_m \rangle \) is quasi-equivalent to the \( \text{dg} \)-category \( C_m \) with objects \( x_1, \ldots, x_{m-1}, y_m \), generated by the morphisms

- \( a_{i-1,i} : x_i \to x_{i+1} \) in degree 0,
- \( a_{i,m} : x_i \to y_m \) in degree \( m - i - 1 \) and
- \( a_{m,i} : y_m \to x_i \) in degree \( n - m + i - 1 \)

subject to the relations \( a_{i-1,i} a_{i-1,i} = 0 \) for \( 2 \leq i \leq m - 2 \) and \( a_{m,i} a_{j,m} = 0 \) for \( i \neq j \) and with differentials determined on generators by

- \( d(a_{i, i+1}) = 0 \) for \( 1 \leq i \leq m - 1 \) and \( d(a_{m, 1}) = 0 \),
- \( d(a_{i, m}) = (-1)^{m-i} a_{i+1, m} a_{i, i+1} \) for \( i \neq m - 1 \),
- \( d(a_{m, i}) = a_{i-1, i} a_{m, i-1} \) for \( i \neq 1 \).

Under the quasi-equivalence \( \langle x_1, \ldots, x_{m-1}, y_m \rangle \to C_m \), the morphisms \( a_{i,m} \) and \( a_{m,i} \) correspond to

\[
(0, \ldots, \text{id}_{x_1}, \ldots, 0) \in \bigoplus_{j=0}^{m-1} \text{Hom} \( B_m, (x_i, x_j[m-j-1]) \simeq \text{Hom}_{\text{dgMod}(B_m)}(x_i, y_m) \)
\]

and

\[
(0, \ldots, (-1)^{j(n-1)} l_i, \ldots, 0) \in \bigoplus_{j=0}^{m-1} \text{Hom} \( B_m, (x_j[m-j-1], x_i) \simeq \text{Hom}_{\text{dgMod}(B_m)}(y_m, x_i) \),
\]

respectively. For example, for \( m = 4 \), we can depict the generating morphisms of \( C_m \) as follows.

![Diagram](attachment:image.png)

The \( \text{dg-functor} \) \( \mu_m : D_m \to C_m \) determined by
• \( \mu_m(\zeta_i) = x_i \) for \( i \neq m \) and \( \mu_m(\zeta_m) = y_m \),

• \( \mu_m(b_{i,j}) = \begin{cases} a_{i,j} & \text{if } j = i + 1 \text{ or } i = m \text{ or } j = m \\ 0 & \text{else} \end{cases} \)

is using Lemma 4.29 easily seen to be a quasi-equivalence. We thus find equivalences of \( \infty \)-categories

\[
\mathcal{D}(D_m) \xrightarrow{\mathcal{D}(\mu_m)} \mathcal{D}(C_m) \simeq \mathcal{D}(B_m) \simeq V^m_H,
\]

showing part (1).

For part (2), we observe, that for \( 1 \leq i \leq m - 1 \), the functor \( \varsigma_i \) is modeled by the dg-functor \( k[t_{n-2}] \to B_m \), determined by mapping \( t_{n-2} \) to \( l_i \). The commutative diagram of dg-categories

\[
\begin{array}{ccc}
D_m & \xrightarrow{\iota_i} & k[t_{n-2}] \\
& \downarrow^{t_{n-2}} & \downarrow^{l_i} \\
C_m & \to & \text{dgMod}(B_m)
\end{array}
\]

whose horizontal morphisms are Morita equivalences hence shows that \( \varsigma_i \) is modeled by \( \iota_i \); note that the sign in \( \iota_i \) follows from the sign \((-1)^{m-1}\) of the summand \( b_{m,i}b_{i,m} \) in (12) and the sign \((-1)^{(n-1)}\) in (14). In the case \( n = m \), the remaining assertion that \( \varsigma_m \) is modeled by \( \iota_m \) follows from the cyclic symmetry of \( D_m \) and the sequence of adjunction (6).

\[ \square \]

Proof of Theorem 4.23. The proof of [Chr21, Theorem 6.1] directly translates to relative Ginzburg algebras of ideal \( n \)-angulations by using Proposition 4.30. \[ \square \]

5 Objects from curves

In Section 5.1 we introduce a class of curves used for the geometric model called matching curves. We proceed in Section 5.2 with the construction of the global sections associated to matching curves. In Section 5.3, we show that the projective \( \mathcal{G}_\mathcal{T} \)-modules associated to the vertices of the underlying quiver can be realized in terms of global sections associated to pure matching curves.

5.1 Matching curves

We fix an oriented marked surface \( S \) with an ideal \( n \)-angulation \( \mathcal{T} \). For each vertex \( v \) of \( \mathcal{T} \), we have an immersion \( \Sigma_v \subset \Sigma_\mathcal{T} \), see Remark 4.10, which is an embedding if no edge of \( \mathcal{T} \) incident to \( v \) is self-folded.

Definition 5.1. A segment is an embedded curve \( \delta : [0, 1] \to \Sigma_v \subset \Sigma_\mathcal{T} \) with \( v \in V_\mathcal{T} \) a singularity, which does not intersect \( V_\mathcal{T} \) away from the endpoints and which is of one of the following two types.

(1) One end lies at \( v \), the other on the boundary of \( \Sigma_v \).

(2) Both ends lie on the boundary of \( \Sigma_v \). In this case, we require that the segment wraps \( 1 \leq a \leq n \) steps around the vertex \( v \).

We consider segments as equivalence classes under homotopy relative \( \partial \Sigma_v \cup \{v\} \).

The two types of segments are depicted in Figure 2.

We can always assume that a given end of a segment which does not lie at a singularity ends on an edge \( e \) of \( \mathcal{T} \). This will be useful to specify at which boundary component of \( \Sigma_v \) the segments begins or ends. If the segment is of the first type and ends at \( e \), we also say that the segment exits the singularity through \( e \).
Definition 5.2. Let $\delta$ be a segment in $\Sigma_T$. If $\delta$ is of the first type, we define its degree as $d(\delta) = 0$ and if $\delta$ is of the second type, we define its degree as $d(\delta) = a - 1$ if $\delta$ goes in the counterclockwise direction and as $d(\delta) = 1 - a$ if $\delta$ goes in the clockwise direction. We call a segment pure if $d(\delta) = 0$.

Definition 5.3. We say that an immersed curve $\gamma : U \to \Sigma_T$ with

$$U = S^1, [0, 1], [0, \infty), (-\infty, \infty)$$

is composed of segments if it is the reparametrization of a composite of segments in $\Sigma_T$ such that

- the segments are not composed at endpoints which lie in $V_T$.
- $\gamma$ does not cut out a once-punctured disc.
- $\gamma$ is not composed of only two identical segments of the first type lying at the same singularity.
- $\gamma$ consists of the minimal number of segments, i.e. is not homotopic to a different curve relative $(\partial \Sigma_T \setminus M) \cup V_T$ with a smaller number of segments. This excludes certain composites of segments lying at the same singularity.

Definition 5.4. Let $\gamma$ be a curve in $\Sigma_T$ composed of segments.

1. We say that $\gamma$ encircles no singularities if no segment of the second type appearing in $\gamma$ warps around a singularity by $n$ steps.
2. We call $\gamma$ non-singular if it is composed only of segments of the second type and encircles no singularities. If $\gamma$ is not non-singular, we call $\gamma$ singular.
3. Suppose that $\gamma$ is composed of finitely many segments $\delta^1, \ldots, \delta^m$. We define the degree of $\gamma$ as

$$d(\gamma) = \sum_{i=1}^{m} d(\delta^i).$$

Definition 5.5. Consider an immersed curve $\gamma : U \to \Sigma_T$ composed of segments, see Definition 5.3. The curve $\gamma$ is called a matching curve in $\Sigma_T$ if

- for all $x \in \partial U$, $\gamma(x)$ lies in $V_T$ or in $\partial \Sigma_T$.
- $d(\gamma) = 0$, if $U = S^1$. 

Figure 2: A segment of the first type (in blue, on the left) with unspecified direction and two segments of the second type (in blue, on the right), going around $v$ in the counterclockwise and clockwise direction by $a = 1$ and $a = n - 2$ steps, respectively.
We consider matching curves as equivalence classes under homotopies relative $\partial \Sigma_T \cup V_T$.

If $\gamma$ is closed, we additionally include the datum of an indecomposable $\pi_0(R)[x]$-module whose underlying $\pi_0(R)$-module is free of finite rank, where $R$ is the base $E_\infty$-ring spectrum.

A matching curve in the marked surface $S$ equipped with the ideal $n$-angulation $\mathcal{T}$ is defined to be a homotopy class relative $(\partial S \setminus M) \cup V_T$ of curves $U \to S \setminus M$ which contains a representative given by the composite of a matching curve in $\Sigma_T$ with the embedding $\Sigma_T \to S \setminus M$ of Remark 4.10.

A matching curve is called finite if it is composed of finitely many segments.

A matching curve is called pure if it is composed of pure segments.

Note that matching curves do not intersect $V_T$ nor the boundary of the surface except at the endpoints. We further introduce the following notation.

**Notation 5.6.** Let $\gamma$ be a curve in $S$ composed of segments and let $\delta$ be a segment of $\gamma$. We denote by $\gamma < \delta$ the curve obtained by the composite of the segments of $\gamma$ appearing before $\delta$ and by $\gamma \leq \delta$ the curve obtained by the composite of $\gamma < \delta$ and $\delta$. We similarly denote $\delta < \gamma$ and $\delta \leq \gamma$, and given two segments $\delta, \delta'$, the curves $\delta < \gamma < \delta', \delta < \gamma < \delta', \delta < \gamma < \delta'$ and $\delta < \gamma < \delta'$.

**Lemma 5.7.** Let $S$ be an oriented surface with marking $M$ and an ideal $3$-angulation $\mathcal{T}$. There exists a bijection between

1. pure matching curves in $S \setminus M$
2. curves $U \to S \setminus M$ which do not cut out any discs in $S \setminus M$ and whose endpoints lie in $V_T$ or $\partial S \setminus M$, considered modulo homotopies relative $\partial S \setminus M$ which fix the endpoints in $V_T$.

**Proof.** Each curve which cuts out no discs in $S \setminus M$ is homotopy equivalent, relative $\partial S \setminus M$ and fixing endpoints in $V_T$, to a unique pure matching curve (this homotopy is allowed to cross singularities). We can thus produce from each curve as in (2) a pure matching curve. Conversely, any pure matching curve clearly defines a curve as in (2). These assignments are inverse bijections. 

### 5.2 Objects from matching curves

We fix an ideal $n$-angulation $\mathcal{T}$ of an oriented marked surface $S$ and an $E_\infty$-ring spectrum $R$. In this section, we associate to each segment $\delta$ and $L \in \text{Fun}(S^{n-1}, \text{RMod}_R)$ as in Remark 5.8 a section $M^L_\delta$ of the $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_\mathcal{T}(R)$ of Construction 4.20, which we then combine via gluing to produce for each matching curve $\gamma$ a global section $M^L_\gamma$, see Proposition 5.14.

**Remark 5.8.** We always assume that $L \in \text{Fun}(S^{n-1}, \text{RMod}_R)$ satisfies $T_{\text{Fun}(S^{n-1}, \text{RMod}_R)}(L) \simeq L[1-n]$, where $T_{\text{Fun}(S^{n-1}, \text{RMod}_R)}$ is the cotwist functor of the spherical adjunction $f^* \dashv f_*$. If $R = k$ is a commutative ring, then it follows from [Chr20, Remark 3.3], that this is satisfied for any $L$. If $R$ is not a commutative ring, it is easy to see that for example $f^*(R)$, $R[[t_{n-2}]]$ and even $R[[t_{n-2}]]$ (the module of Laurent polynomials) all satisfy the requirement. Conjecture 7.16 from [Chr21] would also directly imply that any $L$ satisfies the requirement.

**Remark 5.9.** We note that we can extend the paracyclic twist functor at $v$, see Remark 4.15, to an autoequivalence $T_v$ of the parametrized perverse schober $\mathcal{F}_\mathcal{T}(R)$. The autoequivalence $T_v$ induces an autoequivalence of the $\infty$-category $\mathcal{L}$ of (all) sections of $\mathcal{F}_\mathcal{T}(R)$, see Definition 4.13, which we also denote by $T_v$.

**Construction 5.10.** Let $p : \Gamma(\mathcal{F}_\mathcal{T}(R)) \to \text{Exit}(\Gamma)$ be the Grothendieck construction of $\mathcal{F}_\mathcal{T}(R)$ and $\mathcal{L}$ the $\infty$-category of sections of $\mathcal{F}_\mathcal{T}(R)$.

We consider a singularity $v \in V_T$ and label the incident edges by $e_1, \ldots, e_n$, compatibly with their cyclic order.
1) For $1 \leq i \leq n$, we denote by $\delta_i$ the segment of the first type at $v$ ending at $e_i \cap \partial \Sigma_v$. We define the section $M_{\delta_i} = M^{f^*(R)}_{\delta_i}$ of $\mathcal{F}_T(R)$ as the $p$-relative left Kan extension along the inclusion $\Delta^0 \hookrightarrow \text{Exit}(\mathcal{T})$ of the functor $\Delta^0 \to \mathcal{F}_T(R)(v) \subset \mathcal{L}$ with value

$$M_{\delta_i}(v) = (R \to R \to \cdots) [1 - n] \in \mathcal{V}_f^p = \mathcal{F}_T(R)(v).$$

We define $M_{\delta_i} = M^{f^*(R)}_{\delta_i}$ via

$$M_{\delta_i} = T_v^{i-n} M_{\delta_i}$$

using inverses of the autoequivalence $T_v$ of $\mathcal{L}$ of Remark 5.9. The sections $M_{\delta_i}$ thus satisfy $T_v M_{\delta_i} = M_{\delta_{i+1}}$, with $i + 1$ considered modulo $n$.

Spelling out the definition, one sees that the section $M_{\delta_i}$ is concentrated at the elements $v, e_i \in \text{Exit}(\mathcal{T})$ and takes up to equivalence the values

$$M_{\delta_i}(v) = \left(R \xrightarrow{f^*(R)} \cdots \xrightarrow{id} \cdots \xrightarrow{id} \xrightarrow{0 \to \cdots \to 0} [n-1+1] \in \mathcal{V}_f^p = \mathcal{F}_T(R)(v) \right) (16)$$

and assigns to the edge $v \to e_i$ a coCartesian morphism, describing an apparent equivalence $\varrho_i(M_{\delta_i}(v)) \simeq f^*(R) \simeq M_{\delta_i}(e_i)$. The notation $\xrightarrow{\varphi}$ and $\xrightarrow{\varrho}$ in (16) and below refer to Cartesian and coCartesian morphisms, respectively, see also Section 3.1.

2) For $1 \leq i, j \leq n$, we denote by $\delta_{i,j}$ the segment at $v$ which starts at $e_i \cap \partial \Sigma_v$ and ends at $e_j \cap \partial \Sigma_v$ going in the counterclockwise direction. For $1 < i \leq n$, and $L \in \text{Fun}(S^{n-1}, \text{RMod}_R)$, we define $M^L_{\delta_{i,j}}$ as the $p$-relative left Kan extension along $\Delta^0 \hookrightarrow \text{Exit}(\mathcal{T})$ of the functor $\Delta^0 \to \mathcal{F}_T(R)(v) \subset \mathcal{L}$ with value

$$M^L_{\delta_{i,j}}(v) = \left(0 \to \cdots \to 0 \to \xrightarrow{L} \cdots \xrightarrow{id} \cdots \xrightarrow{id} \xrightarrow{L} \right) \in \mathcal{V}_f^p = \mathcal{F}_T(R)(v).$$

If $L = f^*(R)$, we further allow the case $i = 1$, for which we define $M^L_{\delta_{1,j}}$ as the $p$-relative left Kan extension as above of

$$M^L_{\delta_{1,j}}(v) = \left(f_* f^*(R) \xrightarrow{f_* f^*(R)} \cdots \xrightarrow{id} \cdots \xrightarrow{id} \xrightarrow{f_* f^*(R)} \right) \in \mathcal{V}_f^p = \mathcal{F}_T(R)(v).$$

We further agree to make a specific choice of left Kan extension to define $M^L_{\delta_{i,j}}$, see Remark 5.11.

For arbitrary $1 \leq i, j \leq n$, we define

$$M^L_{\delta_{i,j}} = T_v^{i-1} M^L_{\delta_{i,j-i+1}}.$$  

The sections thus satisfy $T_v M^L_{\delta_{i,j}} = M^L_{\delta_{i+1,j+1}}$. We further denote $M_{\delta_{i,j}} = M^{f^*(R)}_{\delta_{i,j}}$.

Spelling out the definition, one sees that the section $M^L_{\delta_{i,j}}$ is concentrated at $e_i, e_j$ and $v$. To the edges it assigns

$$M^L_{\delta_{i,j}}(e_k) \simeq \begin{cases} 
L & k = i \neq j, \\
L \oplus L[n-1] & k = i = j, \\
L[j-i-1] & k = j > i, \\
L[n+j-i-1] & k = j < i, \\
0 & \text{else}.
\end{cases}$$

This uses that $T_{\text{Fun}(S^{n-1}, \text{RMod}_R)}(L) \simeq L[1-n]$, see Remark 5.8.
The value of $M_{\delta_{i,j}}$ at $v$ is given as follows. If $i < j$, we have
\[
M_{\delta_{i,j}}(v) \simeq \left( 0 \to \cdots \to 0 \to L_{(n-j+1)} \xrightarrow{\sim} \cdots \xrightarrow{\sim} L_{(n-i+1)} \to 0 \to \cdots \to 0 \right) \{ \text{w} \},
\]
if $j < i < n$
\[
M_{\delta_{i,j}}(v) \simeq \left( f_*L \xrightarrow{1} f^*f_*L \xrightarrow{\sim} \cdots \xrightarrow{\sim} f^*f_*L \xrightarrow{\text{cu}_L} L \xrightarrow{\sim} \cdots \xrightarrow{\sim} L \to 0 \to \cdots \to 0 \right) \{ \text{w} \},
\]
where $\text{cu}_L$ denotes the counit map of the adjunction $f^* \dashv f_*$ at $L$, and in the case $j < i = n$ we have
\[
M_{\delta_{i,n}}(v) \simeq \left( f_*L \xrightarrow{1} f^*f_*L \xrightarrow{\sim} \cdots \xrightarrow{\sim} L \to 0 \to \cdots \to 0 \right).
\]
In the final cases $i = j < n$ and $i = j = n$, we have assumed $L = f^*(R)$ and get
\[
M_{\delta_{i,i}}(v) \simeq \left( f_*f^*(R) \xrightarrow{1} f^*f_*f^*(R) \xrightarrow{\sim} \cdots \xrightarrow{\sim} f^*f_*f^*(R) \to 0 \to \cdots \to 0 \right) \{ \text{w} \}
\]
and
\[
M_{\delta_{n,n}}(v) \simeq \left( f_*f^*(R) \to 0 \to \cdots \to 0 \right).
\]

**Remark 5.11.** In the definition of the section $M_{\delta_{i,i}}$, we need to take additional care, to later ensure that the global sections associated to any matching curve containing a segment $\delta_{i,i}$ is not automatically decomposable. We agree to choose $M_{\delta_{i,i}}(f^*(R))$ so that $M_{\delta_{i,i}}(e) = f^*(R) \oplus f^*(R)[1-n]$ and such that the morphism $M_{\delta_{i,i}}(v \to e_1)$ is given by
\[
\varrho_1(M_{\delta_{i,i}})(v) = f^*(R) \oplus f^*(R)[1-n] \xrightarrow{\begin{pmatrix} \text{id} & 0 \\ \alpha & \text{id} \end{pmatrix}} f^*(R) \oplus f^*(R)[1-n] \simeq M_{\delta_{i,i}}(e_1),
\]
where $\alpha : f^*(R)[1-n] \to f^*(R)$ is the adjoint of the morphism $R[1-n] \xrightarrow{(0, \text{id})} R \oplus R[1-n] \simeq f_*f^*(R)$.

**Remark 5.12.** Consider a segment $\delta_{i,j}$ of the second type. The suspensions arising in the definition of the $M_{\delta_{i,j}}$ in (18) correspond to the degree of $\delta_{i,j}$. Namely, one finds $M_{\delta_{i,j}}(e_j) = f^*(R)[d(\delta_{i,j})]$ if $i \neq j$ and $M_{\delta_{i,j}}(e_j) = f^*(R) \oplus f^*(R)[d(\delta_{i,j})]$ if $i = j$.

**Construction 5.13.** Let $\gamma : U \to \Sigma \Sigma$ be a matching curve or a non-closed curve composed of segments. If $U = S^1$, then we include a choice of an indecomposable $\pi_0(R)[x]$ module of dimension $a$ over $\pi_0(R)$, whose multiplication matrix $h$ (obtained from the action of $x$) is invertible. Here $\pi_0(R)$ is the 0-th homotopy group of $R$, thus in the case $R = k$, we have $\pi_0(k) = k$. We further include a choice of local value $L \in \text{Fun}(S^{n-1}, \text{RMod}_R)$ such that $L = f^*(R)$ if $\gamma$ is singular, see Definition 5.4. In the following we give the construction of a section $M_\gamma \in \mathcal{L}$ of $\Sigma \Sigma(R)$. For $L = f^*(R)$, we also use the notation $M_\gamma = M_\gamma^{f^*(R)}$.

**The case $\gamma$ is not closed.** Suppose $\gamma$ is not closed. If $\gamma$ consists of finitely many segments $N$, we choose $I = \{1, \ldots, N\}$, if $\gamma$ has one infinite end (i.e. $U = [0, \infty)$), we choose $I = \mathbb{N} = \{1, 2, \ldots\}$ and if $\gamma$ has two infinite ends (i.e. $U = (-\infty, \infty)$), we choose $I = \mathbb{Z}$. In the first case, we denote $I' = I \setminus \{N\}$ and $I'' = I \setminus \{1\}$, in the second case we denote $I' = I$ and $I'' = \mathbb{N} \setminus \{1\}$ and in the third case we denote $I'' = I' = I$. We denote the segments of $\gamma$ by $\delta^i$ with $i \in I$ (ordered compatibly with
their appearance in $\gamma$) and the edges where $\delta^i$ begins and ends by $e^i$, respectively, $e^{i+1}$. For later use, we also denote by $\psi$ the singularity at which $\delta^i$ lies. We define $M^L_\delta$ as the colimit of a diagram $D_\gamma$ in the $\infty$-category $\mathcal{L}$ of sections of $\mathcal{F}_\gamma(R)$, see Definition 4.13, which is given as follows.

The domain of $D_\gamma$ is the coequalizer $E_\gamma$ in the 1-category of simplicial sets of the diagram

$$\Pi_{i \in I'} \Delta^0 \xrightarrow{\Pi_{i \in I'} \Delta^{(i) \times \{i\}}} \Pi_{i \in I'} \Lambda^2_0 \times \{i\}$$

where the horn $\Lambda^2_0 \times \{i\} \simeq \Lambda^2_0$ is the poset with objects $(0, i), (1, i), (2, i)$ and morphisms $(0, i) \rightarrow (1, i), (2, i)$, i.e., a span. For $l = 1, 2$ and $j \in I'$, the morphism $\Delta^{(l)} \times \{j\}$ is the inclusion $\Delta^0 \rightarrow \Pi_{i \in I'} \Lambda^2_0 \times \{i\}$ determined by mapping $0 \in (\Delta^0)_0$ to $(l, j) \in \Lambda^2_0 \times \{j\} \subset \Pi_{i \in I'} \Lambda^2_0 \times \{i\}$.

Recall that to each section $\delta$, we have associated a section $M^L_\delta$ in Construction 5.10. We further denote by $Z^L_\delta \in \mathcal{L}$ the section concentrated at $e^1$ with value $L$. The diagram $D_\gamma$ is determined via its restrictions to $\Lambda^2_0 \times \{i\}$ with $i \in I'$, which is for $i \geq 1$ given by

$$\begin{align*}
M^L_\delta[d(\delta^i \leq \gamma < \delta^{i+1})] & \xrightarrow{\alpha} Z^L_e[d(\delta^i \leq \gamma < \delta^1)] & \xrightarrow{\beta} M^L_\delta[d(\delta^i \leq \gamma < \delta^{i+1})]
\end{align*}$$

and for $i \leq 0$ given by

$$\begin{align*}
M^L_\delta[-d(\delta^i \leq \gamma < \delta^1)] & \xrightarrow{\alpha} Z^L_e[-d(\delta^{i+1} \leq \gamma < \delta^1)] & \xrightarrow{\beta} M^L_\delta[-d(\delta^{i+1} \leq \gamma < \delta^1)]
\end{align*}$$

where $\alpha$ and $\beta$ are the apparent (pointwise in $\text{Exit}(\mathcal{T})$) inclusions.

**The case $\gamma$ is closed.**

Suppose that $\gamma$ is a closed matching curve, i.e., $U = S^1$. We chose a homeomorphism between $[0, 1]/(0 \sim 1)$ and $U$ mapping 0 to $x$ such that the composite

$$\eta: [0, 1] \rightarrow [0, 1]/(0 \sim 1) \simeq S^1 \rightarrow \Sigma$$

defines a curve composed of segments. Let $e$ be the edge of $\mathcal{T}$ where the curve $\eta$ starts and ends. We construct a section $M^L_\eta$ as above and define $M^L_e$ as the coequalizer in $\mathcal{L}$ of

$$\begin{align*}
(Z^L_e)^{\oplus a} & \xrightarrow{\iota \circ h} (M^L_\eta)^{\oplus a}
\end{align*}$$

where $\iota$ and $\iota'$ are the morphisms with support at $e$ determined by the inclusions of $L^{\oplus a}$ into $M^L_\eta(e)$ arising from the two ends of $\eta$ at $e$ and $h : L^{\oplus a} \rightarrow L^{\oplus a}$ is the equivalence given by the multiplication matrix of $h$. We note that the section $M^L_e$ is independent of the choice of homeomorphism between $[0, 1]/(0 \sim 1)$ and $U$.

**Proposition 5.14.** Let $\gamma$ be matching curve and $L \in \text{Fun}(S^{n-1}, R\text{Mod}_R)$ as in Remark 5.8, such that $L = f^*(R)$ if $\gamma$ is singular. The section $M^L_\gamma$ of $\mathcal{F}_\gamma(R)$ associated to $\gamma$ with local value $L$ defined in Construction 5.13 is a global section.

**Proof.** We use the notation from Construction 5.13 in the following. We begin with the case that $\gamma$ is not closed. Showing that $M^L_\gamma$ is a global section means by definition that $M^L_\gamma$ is a
coCartesian section. If $I = [N]$, for $1 < i < N$, the segment $\delta^i$ is of the second type and does not begin or end at the boundary of $S$. The section $M^I_{\delta^i}$ thus assigns a coCartesian morphism to each morphism $v \to e$ in $\text{Exit}(\mathcal{J})$, except for the morphisms $v^{i-1} \to e^i$ and $v^{i+1} \to e^{i+1}$. Similarly, the sections $M^I_{\delta^i}$ and $M^{I_{\delta^i}}$ assign coCartesian morphisms to all morphisms in $\text{Exit}(\mathcal{J})$ except $v^2 \to e^1$ and $v^{m-1} \to e^m$, respectively. Unraveling the definition of $M^L_{\gamma}$, one obtains an equivalence $M^L_{\gamma}(v \to e) \simeq \bigoplus M^L_{\delta_i}(v \to e)$, where the sum runs over all segments $\delta^i$ of $\gamma$ which lie at $v$. It follows that $M^L_{\gamma}(v \to e)$ is the direct sum of coCartesian morphisms and thus a coCartesian morphism. This shows that $M^L_{\gamma}$ is a global section if $I$ is finite.

We proceed with the case that $I$ is infinite. We denote by $\gamma_m$ the curve composed of the segments $\delta^1, \ldots, \delta^m$ if $I = \mathbb{N}\setminus 1$ and composed of the segments $\delta^{-m}, \ldots, \delta^0, \ldots, \delta^m$ if $I = \mathbb{Z}$. We observe that $M^L_{\gamma}$ is equivalent to the direct limit $\lim \gamma_m$ of the polynomial algebra with $\mathbb{C}[\gamma_m]$. Suppose that $e$ is not a self-folded edge, enclosing a marked point into which an end of $\gamma$ spirals. We find for every morphism $\alpha : v \to e$ in $\text{Exit}(\mathcal{J})$ a final functor $q : Z \to \mathbb{N}^{op}$, by choosing $Z$ to consist of those $i \in I$ such that $\gamma_{\delta^i}$ and $\gamma_{\delta^{-i}}$ are segments which do not lie at $v$. The restriction of the diagram $(M^L_{\gamma_m})_{m \in \mathbb{N}^{op}}$ along $q$ consists of sections which all assign a coCartesian morphism to $\alpha$. As colimits are computed pointwise in $\mathcal{L}$, it follows that $M^L_{\gamma}(\alpha)$ is a coCartesian morphism. In the case that $e$ is a self-folded edge along which $\gamma$ spirals, the above argument does not apply, but it can be easily be verified directly that $M^L_{\gamma}(v \to e)$ is a coCartesian morphism. We conclude that $M^L_{\gamma}$ is a coCartesian section.

Suppose now that $\gamma$ is closed. The same argument as in the previous case shows that for each singularity $v$ and morphism $v \to e'$ with $e' \neq e$, the morphism $M^L_{\gamma}(v \to e') = M^L_{\gamma}(v \to e')$ is coCartesian. The assertion that $M^L_{\gamma}(v \to e)$ is also coCartesian for each morphism $v \to e$ can be easily checked. It follows that $M^L_{\gamma}$ is also a global section in this case, concluding the proof.

**Example 5.15.** We illustrate Construction 5.13 in an example with $R = k$ and $n = 3$. Consider the once-punctured 3-gon (in green), with the interior marked point called $m$ and the ideal 3-angulation depicted in black.

![Diagram](Diagram.png)

The perverse schober $\mathcal{F}_T(k)$ is up to natural equivalence given by the following diagram,

$$
\begin{array}{cccccc}
\mathcal{N}_{f^*} & \xrightarrow{\alpha_1} & \mathcal{N}_{f^*} & \xleftarrow{\alpha_1} & \mathcal{V}^3_{f^*} & \xrightarrow{\alpha_2} & \mathcal{N}_{f^*} \\
\mathcal{V}^3_{f^*} & \xleftarrow{\alpha_1} & \mathcal{N}_{f^*} & \xrightarrow{\alpha_1} & \mathcal{V}^3_{f^*} & \xrightarrow{\alpha_2} & \mathcal{N}_{f^*} \\
\end{array}
$$

where $\mathcal{V}^3_{f^*}$ denotes the value of $\mathcal{F}_T(k)$ at the singularities and $\mathcal{N}_{f^*} = \text{Fun}(S^2, \text{RMod}_k)$ denotes the value of $\mathcal{F}_T(k)$ at the edges of $\mathcal{T}$. Algebraically, one can describe $\mathcal{N}_{f^*}$ as the derived $\infty$-category $\mathcal{D}(k[1_1])$ of the polynomial algebra with $|1_1| = 1$ and $\mathcal{V}^3_{f^*}$ as the derived $\infty$-category.
of the relative Ginzburg algebra of the 3-gon, which we denote in the following by $G_\Delta$. For a depiction of $G_\Delta$ see Example 4.19.

The singular matching curve $\gamma_c$ given by the edge $e$ connecting $v_1$ and $v_2$ gives rise to the following global section $M_{\gamma_c}$, which describes a 3-spherical object.

$$
\begin{array}{c}
0 \\
\uparrow g_3 \quad \uparrow g_2 \\
\swarrow g_1 \\
\downarrow g_3 \quad \downarrow g_2 \\
0 \\
\end{array}
$$

0 \xleftarrow{g_3} s_1 \xrightarrow{g_2} 0 \\
\uparrow g_1 \\
0 \xleftarrow{g_3} s_2 \xrightarrow{g_1} 0 \xleftarrow{g_2} 0 \xrightarrow{g_3} 0

Above, we denote $s_1 = (k \overset{\sim}{\to} f^*(k) \overset{\text{id}}{\to} f^*(k))$, $s_2 = (k \overset{\sim}{\to} f^*(k) \to 0) \in V^3_{3}$.

Algebraically, $s_1$ and $s_2$ describe simple $G_\Delta$-modules associated to a vertex each and $f^*(k)$ is the $k[t_1]$-module with value $k$.

The matching curve $\gamma_1$ which begins and starts at $v_1$ and encircles $v_2$ gives rise to a global section $M_{\gamma_1}$ of the following form.

$$
\begin{array}{c}
0 \\
\uparrow g_3 \\
\swarrow g_1 \\
\downarrow g_3 \\
0 \\
\end{array}
$$

0 \xleftarrow{g_3} s_1 \oplus s_1[2] \xrightarrow{g_2} 0 \\
\uparrow g_1 \\
0 \xleftarrow{g_3} s_2 \oplus s_2[2] \xrightarrow{g_1} 0 \xleftarrow{g_2} 0 \xrightarrow{g_3} 0

Note that $M_{\gamma_1}$ is not isomorphic to the direct sum $M_{\gamma_c} \oplus M_{\gamma_c}[2]$, even though this holds for the evaluation at each element of $\text{Exit}(T)$. This is because of the non-trivial equivalence $\varphi_1(s_1 \oplus s_1[2]) \simeq f^*(k) \oplus f^*(k)[2]$ used in the construction of $M_{\gamma_1}$, see also Remark 5.11. We further note that $M_{\gamma_1}$ is isomorphic to the section $M_{\gamma_2}$ associated to the matching curve $\gamma_2$ which starts and ends at $v_2$ and encircles $v_1$.

Consider any local system $L \in \text{Fun}(S^2, \text{RMod}_k)$, i.e. $k[t_1]$-module, and the closed pure matching curve $\gamma_3$ wrapping around $m$ with any irreducible $k[x]$-module $h$ of rank 1. The associated global section $M_{\gamma_3}$ is of the form

$$
\begin{array}{c}
0 \\
\uparrow g_3 \\
\swarrow g_1 \\
\downarrow g_3 \\
0 \\
\end{array}
$$

0 \xleftarrow{g_3} t_3(L) \xrightarrow{g_2} L \\
\uparrow g_1 \\
0 \xleftarrow{g_3} t_3(L) \xrightarrow{g_1} L \xleftarrow{g_2} t_3(L) \xrightarrow{g_3} 0

where $t_3(L) = (0 \to 0 \to L) \in V^3_{3}$ is the object concentrated in the third component of the semiorthogonal decomposition of $V^3_{3}$, with value $L$. Algebraically, one can describe $t_3(L)$ as the derived tensor product $L \otimes_{k[t_1]} p_*G_\Delta$ with the projective $G_\Delta$-module associated with some vertex of $G_\Delta$ or equivalently if $L = f^*(k)$ as an extension of two simple $G_\Delta$-modules associated to two of its vertices.

**Remark 5.16.** An alternative perspective on the global sections associated to non-closed matching curves with local value $f^*(R)$ from Proposition 5.14 is given in [IQZ20, Section
This rests on the following observation: if two matching curves $\gamma_1$ and $\gamma_2$ end, respectively, begin at a singularity $v$, then there are nonzero morphisms $M_{\gamma_1} \to M_{\gamma_2}[i + d(\gamma_1)]$ and $M_{\gamma_2} \to M_{\gamma_1}[j - d(\gamma_1)]$ with $i, j \geq 0$ and $i + j = n$, generating the free rank 1 summands of $\text{Mor}(M_{\gamma_1}, M_{\gamma_2}), \text{Mor}(M_{\gamma_2}, M_{\gamma_1}) \in R\text{Mod}_R$ associated to the endpoint intersection at $v$ described in Theorem 6.2. The cofibers of these two morphisms describe the global sections associated to two matching curves each obtained as a smoothing of curve composed of $\gamma_1$ and $\gamma_2$. The two different smoothings correspond to the two different sides at which one can warp around $v$. It follows that each section $M_{\gamma}$ with $\gamma$ a non-closed matching curve with domain $U = [0, 1]$ can be obtained via repeated cofibers of morphisms between suspensions of objects of the form $M_{\gamma_e}$ for $e$ an edge of $\mathcal{T}$.

A similar observation can be made for the global sections associated to non-singular matching curves and arbitrary coefficient system $L$. Namely, if there is a directed boundary intersection from $\gamma$ to $\gamma'$, the cofiber of the morphisms from $M_{\gamma}^L$ to (a suspension or delooping of) $M_{\gamma'}^L$ corresponding to $\text{id}_L \in \text{End}(L)$ is equivalent to the global section $M_{\tilde{\gamma}}^L$, where $\tilde{\gamma}$ is the matching curve obtained from tracing along $\gamma$, part of the intersected boundary component and $\gamma'$ (and choosing a homotopic curve which does not intersect $\partial S \setminus M$), under the additional assumption that $\tilde{\gamma}$ defines a non-singular matching curve.

### 5.3 Projective modules via pure matching curves

We fix an oriented marked surface $S$ with an ideal $n$-angulation $\mathcal{T}$ and let $R$ be an $E_\infty$-ring spectrum.

Let $e$ be an edge of $\mathcal{T}$ and $v_1, v_2$ the vertices incident to $e$. Consider the curve composed of segments $c_e^i$ with $i = 1, 2$ whose first segment lies at $v_i$, which begins at $e$ and whose segments are all pure of the second type, wrapping exactly one step in the counterclockwise direction around a singularity. We define the curve $c_e$ as the composite of $c_e^1$ with the curve obtained by reversing the orientation of $c_e^2$. Note that $c_e$ is a pure non-singular matching curve.

**Example 5.17.** Below, we depict the 4-gon with an ideal 3-angulation $\mathcal{T}$ with edges $e_1, \ldots, e_5$ and associated matching curves $c_{e_1}, \ldots, c_{e_5}$.

![Diagram of a 4-gon with matching curves](image)

**Proposition 5.18.** For each $L \in \text{Fun}(S^{n-1}, R\text{Mod}_R)$, there exists an equivalence in $\mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R))$

$$M_{e_5}^L \simeq \text{ev}_{e_5}^*(L)$$

with $\text{ev}_{e_5}^*$ the functor from Proposition 4.25.

In particular, for $L$ the image of $R[t_{n-2}]$ under the equivalence $\text{Fun}(S^{n-1}, R\text{Mod}_R) \simeq R\text{Mod}_R[t_{n-2}]$, the section $M_{e_5}^L$ is the direct summand of the compact generator of $\mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R))$ associated to the edge $e$. 

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Proof. Consider the curve $c_x$ as above and label its segments by the set $I \subset \mathbb{Z}$. The two first segments of the curves $c_1^x$ and $c_2^x$, lying at $v_1$, respectively, $v_2$, yield segments $\delta^x_1$ and $\delta^x_2$ of $c_x$ with $x \in I$.

Consider the $R$-linear $\infty$-categories $\mathcal{C} = \mathcal{H}(\mathcal{U}, \mathcal{F}_\mathcal{U}(R))$ and $\mathcal{L}$ of global sections, respectively, all sections of $\mathcal{F}_\mathcal{U}$. Let $v \in \mathcal{U}$ be a singularity with incident edges labeled $e_1, \ldots, e_n$ and let $\delta_{i+1}$ be the pure segment of the second type lying at $v$ passing from $e_i$ to $e_{i+1}$. The functor $\text{Mor}_\mathcal{C}(M_{\delta_i+1}^L) : \mathcal{L} \to \text{RMod}_R$ is equivalent to the functor

$$\tilde{e}_{V, i : \mathcal{L}} \to \mathcal{N}_{f \ast} \simeq \text{RMod}_{R[t_{i-1}]}(\text{Mor}(L, -)) \to \text{RMod}_R,$$

as is easily seen using that $M^L_{\delta_i+1}$ is a $p$-relative left Kan extension of its restriction to $v$. Similarly, for each edge $e$ of $\mathcal{T}$, we find the functor $\text{Mor}_\mathcal{C}(Z^L_e, -) : \mathcal{L} \to \text{RMod}_R$ to be equivalent to

$$\tilde{e}_{V, e} : \mathcal{L} \to \mathcal{N}_{f \ast} \simeq \text{RMod}_{R[t_{i-1}]}(\text{Mor}(L, -)) \to \text{RMod}_R.$$

Note that the composites of $\tilde{e}_{V, i}$ and $\tilde{e}_{e_i}$ with the inclusion $\mathcal{C} \to \mathcal{L}$ are equivalent, we denote this functor by $\tilde{e}_{V, i}^L$.

Using the definition of $M^L_{\delta_i+1}$ and that $\text{Mor}_\mathcal{C}(-, -)$ preserves limits in the first entry, we obtain that $\text{Mor}_\mathcal{C}(M^L_{\delta_i+1}) : \mathcal{L} \to \text{RMod}_R$ is given by the limit of the diagram $\text{Mor}_\mathcal{C}(D_{e_i}, -) : E_{op} \to \text{Fun}(\mathcal{L}, \text{RMod}_R)$, Composing with the limit preserving pullback functor $\text{Fun}(\mathcal{L}, \text{RMod}_R) \to \text{Fun}(\mathcal{C}, \text{RMod}_R)$ along the inclusion $\mathcal{C} \to \mathcal{L}$, one obtains the diagram in $\text{Fun}(\mathcal{C}, \text{RMod}_R)$ which assigns to $\Lambda_0^2 \times \{x\}$ the constant diagram with value $\tilde{e}_{V, i}^L$, to $(0, j) \to (2, j)$ for $j > 0$ and to $(0, j) \to (1, j)$ for $j < 0$ the constant diagram with value $\tilde{e}_{V, i}^L$, where $e^{j-1}$ is the edge where $\delta^{j-1}$ ends and $\delta^j$ begins. The limit $\text{Mor}_\mathcal{C}(M^L_{\delta_i+1}, -)$ is thus equivalent to the functor $\tilde{e}_{V, i}^L$. Evaluating the left adjoints at $R$ shows the desired equivalence (20).

\section{Morphisms from intersections}

For the entirety of Section 6, we fix an $\mathbb{E}_{\infty}$-ring spectrum $R$ and an oriented marked surface $\mathbf{S}$ with a semi-regular $n$-angulation $\mathcal{T}$, see Definition 4.7.

**Definition 6.1.** Let $\gamma : U \to \mathbf{S}$, $\gamma' : U' \to \mathbf{S}$ be two matching curves. We choose representatives of $\gamma$ and $\gamma'$ with the minimal number of intersections. We define

- the number of crossings $i^{ct}(\gamma, \gamma')$ as the number of intersections of $\gamma|_{U'}$ and $\gamma'|_{U'}$. If $\gamma = \gamma'$, then $i^{ct}(\gamma, \gamma)$ counts each self-crossing twice.

- the endpoints intersection number $i^{end}(\gamma, \gamma')$ as the number of intersections of $\gamma|_{\partial U}$ and $\gamma'|_{\partial U'}$ in $V_T$. If $\gamma = \gamma'$ with distinct endpoints, we define $i^{end}(\gamma, \gamma)$ as the number of endpoints of $\gamma$. If $\gamma = \gamma'$ with two identical endpoints, we set $i^{end}(\gamma, \gamma) = 4$.

- the directed boundary intersection number $i^{bdry}(\gamma, \gamma')$ as the number of intersections of $\gamma$ and $\gamma'$ with the same boundary component of $\partial \mathbf{S}\setminus M$ such that the intersection of $\gamma$ and $\partial \mathbf{S}\setminus M$ precedes the intersection of $\gamma'$ and $\partial \mathbf{S}\setminus M$ in the orientation of $\partial \mathbf{S}\setminus M$ induced by the clockwise orientation of $\mathbf{S}$.

Given two local systems $L, L' \in \text{Fun}(S^{n-1}, \text{RMod}_R)$, we denote the morphism object by $\text{Mor}(L, L') = \text{Mor}_{\text{Fun}(S^{n-1}, \text{RMod}_R)}(L, L')$.

**Theorem 6.2.** Let $\gamma \neq \gamma'$ matching curves in $\mathbf{S}$ with no common infinite ends, see Remark 6.7. If $\gamma$ and $\gamma'$ are closed, they include choices of irreducible $\pi_0(R)[x]$-modules $h_1$ and $h_2$ of rank $a_1$, respectively, $a_2$ over $\pi_0(R)$. Otherwise, we set $a_1 = 1$ or $a_2 = 1$. Consider the associated global sections $M^L_\gamma, M^L_{\gamma'} \in \mathcal{C} = \mathcal{H}(\mathcal{U}, \mathcal{F}_\mathcal{U}(R))$ with local values $L, L' \in \text{Fun}(S^{n-1}, \text{RMod}_R)$ of Proposition 5.14.
i) Suppose that $\gamma$ or $\gamma'$ is non-singular, see Definition 5.4. The morphism object $\text{Mor}_C(M^L_\gamma, M^L_{\gamma'})$ consists of the direct sum of 

$$a_1a_2(i^{\text{cr}}(\gamma, \gamma') + i^{\text{bdry}}(\gamma, \gamma'))$$

copies of suspensions or deloopings of $\text{Mor}(L, L') \in \text{RMod}_R$.

ii) Suppose that $L = L' = f^*(R)$. The morphism object $\text{Mor}_C(M_\gamma, M_{\gamma'})$ is a free $R$-module of rank

$$a_1a_2 \left(2i^{\text{cr}}(\gamma, \gamma') + i^{\text{end}}(\gamma, \gamma') + 2i^{\text{bdry}}(\gamma, \gamma') \right).$$

**Theorem 6.3.**

i) For each finite non-closed non-singular matching curve $\gamma$ in $S$ and $L, L' \in \text{Fun}(S^{n-1}, \text{RMod}_R)$, the morphism objects $\text{Mor}_C(M^L_\gamma, M^L_{\gamma'})$ is the direct sum of $1 + i^{\text{cr}}(\gamma, \gamma')$-many copies of suspensions (or deloopings) of $\text{Mor}(L, L')$.

ii) For each finite non-closed matching curve $\gamma$ in $S$ the $R$-module $\text{Mor}_C(M_\gamma, M_{\gamma'})$ is free of rank

$$\begin{cases} 
1 + 2i^{\text{cr}}(\gamma, \gamma) & \text{if } i^{\text{end}}(\gamma, \gamma) = 1 \\
2 + 2i^{\text{cr}}(\gamma, \gamma) & \text{if } i^{\text{end}}(\gamma, \gamma) = 0, 2 \\
4 + 2i^{\text{cr}}(\gamma, \gamma) & \text{if } i^{\text{end}}(\gamma, \gamma) = 4.
\end{cases}$$

iii) Let $R = k$ be a commutative ring, $\gamma$ be a pure closed matching curve (thus automatically non-singular) and $L, L' \in \text{Fun}(S^{n-1}, \text{RMod}_k)$. Assume that the indecomposable $k[x]$-module $h$ is of rank 1 over $k$. The morphism object $\text{Mor}_C(M^L_\gamma, M^L_{\gamma'})$ is the direct sum of $2 + i^{\text{cr}}(\gamma, \gamma')$-many copies of suspensions (or deloopings) of $\text{Mor}(L, L')$.

**Example 6.4.** Let $\gamma$ be a finite matching curve in $S$ without self-intersections.

1. If both ends of $\gamma$ lie at singularities, then

$$\text{Mor}_C(M_\gamma, M_{\gamma'}) \simeq R \oplus R[-n],$$

meaning that $M_\gamma$ is an $n$-spherical object.

2. If $\gamma$ begins at a singularity and ends on the boundary of $S$, then $M_\gamma$ is an exceptional object.

3. If $\gamma$ begins and ends on the boundary of $S$, then

$$\text{Mor}_C(M_\gamma, M_{\gamma'}) \simeq R \oplus R[1 - n],$$

meaning that $M_\gamma$ is an $(n-1)$-spherical object.

4. If $\gamma$ is pure and closed, $R = k$ a commutative ring and the $k[x]$-module $h$ has rank 1 over $k$, then

$$\text{Mor}_C(M_\gamma, M_{\gamma'}) \simeq k \oplus k[-1] \oplus k[1 - n] \oplus k[-n].$$

**Remark 6.5.** Except for pure matching curves in 3-angulated surfaces with $L = f^*(k)$, see Section 2.2, we do not systematically compile the degrees of the morphisms described in Theorems 6.2 and 6.3. In the case of pure matching curves, it is not difficult to give a simple description. For arbitrary matching curves, the degrees would most naturally be described in terms of gradings of the surface and the matching curves, see [IQZ20].

**Remark 6.6.** The proof of Theorem 6.3.iii) directly generalizes from the case $R = k$ to $R$ arbitrary if one assumes Conjecture 7.16 from [Chr21] on the cotwist functor of the adjunction $f^* \dashv f_*$. Theorem 6.2 and Theorem 6.3 do not cover the case that $\gamma = \gamma'$ is closed and equipped with two different irreducible $\pi_0(R)[x]$-modules $h_1$ and $h_2$ of degrees $a_1$, respectively, $a_2$, nor the case
that \( h_1 = h_2 \) is of degree \( a \geq 2 \). In these cases, every self-crossing of \( \gamma \) still produces a summand given by suspensions (or deloopings) of \( \text{Mor}(L, L') \) of \( \text{Mor}_\mathcal{C}(M^L_k, M^L_{k'}) \). Describing the entirety of \( \text{Mor}_\mathcal{C}(M^L_k, M^L_{k'}) \) leads to questions about the representation theory of \( \pi_0(R)[x] \).

**Remark 6.7.** Consider two different matching curves \( \gamma : U \to \Sigma \) and \( \gamma' : U' \to \Sigma \). We say that \( \gamma \) and \( \gamma' \) have a common infinite end if there exist immersions \( I_1 : \mathbb{R}_{>0} \to U \), \( I_2 : \mathbb{R}_{>0} \to U' \) such that the curves \( \gamma|_{I_1} \) and \( \gamma|_{I_2} \) are composed of infinitely many identical segments. Note that this definition allows that one of the two curves \( \gamma \) and \( \gamma' \) is closed. If both \( \gamma \) and \( \gamma' \) are infinite with a common infinite end then the \( R \)-module \( \text{Mor}_\mathcal{C}(M^L_k, M^L_{k'}) \) consists of infinitely many copies of suspensions (or deloopings) of \( \text{Mor}(L, L') \).

**Remark 6.8.** For the proof of Theorems 6.2 and 6.3 it suffices to consider the case that \( \mathcal{T} \) is regular. This follows from Theorem 8.1 and the observation that the intersection number is invariant under composition of matching curves with the automorphisms \( D_c \left( \frac{1}{n-1} \pi \right) \) of \( \mathcal{S} \setminus M \).

The proofs of Theorems 6.2 and 6.3 consist of gluing arguments. We decompose \( \gamma \) and \( \gamma' \) into segments and begin in Section 6.1 by describing all morphisms between the associated sections. In Section 6.2, we then describe \( \text{Mor}_\mathcal{C}(M^L_k, M^L_{k'}) \) via the colimit of a diagram of morphisms objects between the sections associated to the segments. In Section 6.3 we combine the findings of Section 6.1 and Section 6.2 to prove Theorems 6.2 and 6.3.

### 6.1 Intersections locally

Let \( \mathcal{L} \) denote the \( R \)-linear \( \infty \)-category of (all) sections of \( \mathcal{F}_\mathcal{T}(R) \), see Definition 4.13. In this section, we describe the morphism objects \( \text{Mor}_\mathcal{L}(M^L_k, M^L_{k'}) \) and \( \text{Mor}_\mathcal{L}(Z^L_\mathcal{V}, M^L_{k'}) \) where \( L, L' \in \text{Fun}(S^{n-1}, \text{RMod}_R) \), \( \delta \) is any segment \( S \), \( e \) is an edge of \( \mathcal{T} \) and \( \gamma' \) is any matching curve in \( S \) or a non-closed curve composed of segments, see also Construction 5.13 for the notation. If \( L \neq f^*(R) \) and \( L' \neq f^*(R) \), we require as always that \( \delta \), respectively, \( \gamma' \) is non-singular.

We begin by determining the morphism objects between sections \( M^L_k \) and \( M^L_{k'} \) associated to segments \( \delta, \delta' \).

We first consider the case that \( \delta \) and \( \delta' \) are located at the same singularity \( v \). We choose representatives of \( \delta \) and \( \delta' \) with the minimal number of intersections. We show that \( \text{Mor}_\mathcal{L}(M^L_k, M^L_{k'}) \) is the direct sum of \( R \)-modules given as follows.

- Each crossing of \( \delta \) and \( \delta' \) contributes a suspensions (or delooping) of the \( R \)-linear morphism object \( \text{Mor}_{\mathcal{N}_v}(L, L') \) to \( \text{Mor}_\mathcal{L}(M^L_k, M^L_{k'}) \), corresponding to morphisms with support at \( v \).
  
  In the following, we denote \( \text{Mor}(L, L') = \text{Mor}_{\mathcal{N}_v}(L, L') \).

- Each directed boundary intersection of \( \delta \) and \( \delta' \) in \( \mathcal{S}_v \) contributes a copy of a suspensions of \( \text{Mor}(L, L') \) to \( \text{Mor}_\mathcal{L}(M^L_k, M^L_{k'}) \). The corresponding morphisms have support at \( v, e_i \) if \( \delta \neq \delta' \), where \( e_i \) is the edge of \( \mathcal{T} \) intersecting same the component of \( \partial \mathcal{S}_v \) as \( \delta, \delta' \), and support at \( v, e_i, e_j \) if \( \delta = \delta' = \delta_{i,j} \). The case \( \delta = \delta' = \delta_{i,j} \) a segment of the first type is excluded.

- If \( L = f^*(R) \), then each endpoint intersection of \( \delta \) and \( \delta' \) contributes a single (suspension of a) copy of \( R \) to \( \text{Mor}_\mathcal{L}(M^L_k, M^L_{k'}) \). The support of the corresponding morphisms is at \( v \) if \( \delta \neq \delta' \) and at \( v, e_i, e_j \) if \( \delta = \delta' = \delta_{i,j} \).

The above assertions follow from the universal properties of the involved Kan extensions and some basic computations, as we now explain. We denote the Grothendieck construction of \( \mathcal{F}_\mathcal{T}(R) \) by \( p : \Gamma(\mathcal{F}_\mathcal{T}(R)) \to \text{Exit}(\mathcal{T}) \) and its pullback to \( C_0 \subset \text{Exit}(\mathcal{T}) \) by \( p' \). The sections \( M^L_k \) and \( M^L_{k'} \) were defined as the \( p \)-relative left Kan extensions of their restrictions to \( \{v\} \subset C_0 \subset \text{Exit}(\mathcal{T}) \), Applying [Lur09, 4.3.2.15], we obtain a trivial fibration (in particular equivalence) from a full subcategory of \( \mathcal{L} \), containing both \( M^L_k \) and \( M^L_{k'} \), to \( \mathcal{V}_{\mathcal{T}}^j \). We thus find an equivalence of \( R \)-modules

\[
\text{Mor}_\mathcal{L}(M^L_k, M^L_{k'}) \simeq \text{Mor}_{\mathcal{V}_{\mathcal{T}}^j}(M^L_k(v), M^L_{k'}(v)).
\]
Mor_{\mathcal{L}}(M^{L}_{\delta},M^{L'}_{\eta}) \simeq \bigoplus_{\delta'>\eta} \text{Mor}_{\mathcal{V}^v}(\text{Mor}_{\mathcal{L}}(M^{L}_{\delta},M^{L'}_{\eta}[d(\delta' \leq \eta < \delta')]) + \bigoplus_{\delta'<\eta} \text{Mor}_{\mathcal{L}}(M^{L}_{\delta},M^{L'}_{\eta}[d(\delta' \leq \eta < \delta')]).

The resulting morphisms objects in \mathcal{V}^v can be very directly determined using again universal properties of Kan extensions. We collect the resulting morphisms objects for all possible pairs \delta, \delta' in Tables 1 to 3. For the description of \delta, \delta', we use the notation of Construction 5.10. In Table 1, we consider the cases \delta = \delta_i and \delta' = \delta_j with 1 \leq i \leq n and \text{L} = f^*(R). In Table 2, we consider the cases \delta = \delta_i and \delta' = \delta_{i,n} with 1 \leq i \leq n and \text{L} = f^*(R). In Table 3, we consider (a subset of) the cases \delta = \delta_{i,j} and \delta' = \delta_{i',j'} with 1 \leq i', j' \leq n. Any unordered pair \delta, \delta' is up to the cyclic symmetry of \Sigma_v given by rotation by \frac{2\pi}{|\Sigma_v|}, on the categorical level realized by the paracyclic twist functor, see Remark 5.9, described by one of the pairs considered below.

If the segments \delta and \delta' are not located at the same singularity, one finds Mor_{\mathcal{L}}(M^{L}_{\delta},M^{L'}_{\eta}) \simeq 0, see also Lemma 6.9.

Lemma 6.9. Let \eta be a curve in S composed of segments. As in Construction 5.13, we denote by I \subset \mathbb{Z} the set which parametrizes the segments \delta_i, i \in I, of \eta.

1. Let \delta be a segment lying at a singularity v and let K \subset I be the set of segments of \eta lying at v. Then there exists an equivalence of R-modules

Mor_{\mathcal{L}}(Z^{L}_{\eta},M^{L'}_{\eta}) \simeq \text{Mor}_{\mathcal{L}}(Z^{L}_{\eta},M^{L'}_{\eta}(\eta)).

2. Let e be an edge of T. Then there exists an equivalence of R-modules

Mor_{\mathcal{L}}(Z^{L}_{\eta},M^{L'}_{\eta}) \simeq \text{Mor}_{\mathcal{L}}(L,M^{L'}_{\eta}(e)).

Denote

N = \bigoplus_{\delta \in K' \cap I \geq 1} \text{Mor}(L,L'[d(\delta \leq \gamma' < \delta')]) + \bigoplus_{\delta \in K' \cap I \leq \eta} \text{Mor}(L,L'[d(\delta' \leq \gamma' < \delta')]),

where K' \subset I is the set of segments of \eta which begin at e.

If \eta does not end at e, there exists an equivalence of R-modules

Mor_{\mathcal{L}}(Z^{L}_{\eta},M^{L'}_{\eta}) \simeq N

(21)

If \eta ends at e, then there exists an equivalence of R-modules

Mor_{\mathcal{L}}(Z^{L}_{\eta},M^{L'}_{\eta}) \simeq N \oplus \text{Mor}(L,L'[d(\delta \leq \eta \leq \hat{\delta}])]

(22)

where \hat{\delta} is the last segment of \eta.

Proof. Note that M^{L}_{\delta} and Z^{L}_{\eta} are left Kan extensions relative the Grothendieck construction of \mathcal{F}_\mathcal{T}(R) of their restrictions to v, respectively, e. Using the universal property of Kan extensions, see [Lur09, 4.3.2.17], it follows that for any section X \in \mathcal{L} the restriction morphisms of R-modules

Mor_{\mathcal{L}}(M^{L}_{\delta},X) \rightarrow Mor_{\mathcal{V}^v}(M^{L}_{\delta}(v),X(v))

Mor_{\mathcal{L}}(Z^{L}_{\eta},X) \rightarrow Mor_{\mathcal{V}^v}(L,M^{L'}_{\eta}(e))

restrict to equivalences on all homotopy groups so that they are equivalences of R-modules.
Table 1: All possible pairs of two segments \( \delta_1, \delta_i \) of the first type up to cyclic symmetry.

| Intersections | \( \delta_i, \delta_{j,n} \) | \( \text{Mor}(\mathcal{M}_{\delta_1}, \mathcal{M}_{\delta_{j,n}}) \) | \( \text{Mor}(\mathcal{M}_{\delta_{j,n}}, \mathcal{M}_{\delta_i}) \) | Support |
|---------------|-----------------|-----------------|-----------------|--------|
| none          | \( i < j < n \) | 0               | 0               | /      |
| 1x crossing   | \( j < i < n \) | Mor(\( f^*(R), L \)[i - j - 1]) Mor(\( L, f^*(R) \)[j - i]) | at \( v \) |
| 1x boundary   | \( j < i = n \) | Mor(\( f^*(R), L \)[i - 1]) Mor(\( L, f^*(R) \)[-i]) | at \( v \) |
| 1x boundary   | \( j = i < n \) | 0               | Mor(\( L, f^*(R) \)[j - i]) | at \( v, e_i \) |
| 1x boundary   | \( j < i = n \) | Mor(\( f^*(R), L \)[n - j - 1]) | 0 | at \( v, e_n \) |
| 2x boundary   | \( i = j = n \) | Mor(\( f^*(R), L \)[n - 1]) Mor(\( L, f^*(R) \)) | at \( v, e_n \) |

Table 2: All possible pairs of one segment \( \delta_i \) of the first type and one segment \( \delta_{j,n} \) of the second type up to cyclic symmetry.

| Intersections | \( \delta_{1,j}, \delta_{i',j'} \) | \( \text{Mor}(\mathcal{M}_{\delta_{1,j}}, \mathcal{M}_{\delta_{i',j'}}) \) | \( \text{Mor}(\mathcal{M}_{\delta_{i',j'}}, \mathcal{M}_{\delta_{1,j}}) \) | Support |
|---------------|-----------------|-----------------|-----------------|--------|
| none          | \( 1 < j < i' \leq j' \) | 0               | 0               | /      |
| none          | \( 1 < i' < j' < j \) | 0               | 0               | /      |
| 1x crossing   | \( 1 < i' < j < j' \) | Mor(\( L, L' \)[1 - i']) Mor(\( L', L \)[i' - 2]) | at \( v \) |
| 2x crossing   | \( 1 < j' \leq i' < j \) | Mor(\( L, L' \)[1 - i'] \oplus Mor(\( L, L' \)[n - i']) | at \( v \) |
| 2x crossing   | \( 1 = j < i' = j' \) | Mor(\( L, L' \)[1 - i'] \oplus Mor(\( L', L \)[i' - 2]) | at \( v \) |
| 1x boundary   | \( 1 < i' < j = j' \) | Mor(\( L, L' \)[1 - i']) | 0 | at \( v, e_j \) |
| 1x boundary   | \( 1 < i' = j < j' \) | 0               | Mor(\( L', L \)[i - 2]) | at \( v, e_j \) |
| 1x boundary   | \( 1 = i' < j < j' \) | Mor(\( L, L' \)) | 0 | at \( v, e_j \) |
| 2x boundary   | \( 1 = j < j' = j' \) | Mor(\( L, L' \)[-i' + 1]) Mor(\( L', L \)[i' - 2]) | at \( v, e_j \) |
| 2x boundary   | \( 1 = i' \leq j = j' \) | Mor(\( L, L' \)) Mor(\( L', L \)) | at \( v, e_j \) |
| 2x boundary   | \( 1 = i' = j' \leq j \) | Mor(\( L, L' \) \oplus Mor(\( L, L' \)[n - 1]) | at \( v, e_j \) |
| 2x boundary   | \( 1 < j = i' = j' \) | Mor(\( L, L' \)[n - i']) | Mor(\( L', L \)[i' - 2]) | at \( v, e_j \) |
| 2x boundary   | \( 1 < j < i' = j' \) | Mor(\( L, L' \)[n - i']) | Mor(\( L', L \)[i' - 2]) | at \( v, e_j \) |
| 1x crossing   | \( 1 < j' < j = i' \) | Mor(\( L, L' \)[n - i']) | Mor(\( L', L \)[i' - 2]) | at \( v, e_j \) |
| 1x boundary   | \( 1 < j = i' = j' \) | Mor(\( L, L' \)[n - i']) | Mor(\( L', L \)[i' - 2]) | at \( v, e_j \) |

Table 3: All possible pairs of two segments \( \delta_{i,j} \) and \( \delta_{i',j'} \) of the second type up to a swap and cyclic symmetry. Note that if \( j = 1 \) or \( i' = j' \), we require \( L = f^*(R) \).
By construction of $M^L_\eta$, we find an equivalence in $\mathcal{V}_f^i$.

$$M^L_\eta(v) \simeq \bigoplus_{\delta' \in K \cap I_{\geq 1}} M^L_{\delta'}(v)[d(\delta^1 \leq \eta < \delta')] \oplus \bigoplus_{\delta' \in K \cap I_{\leq 0}} M^L_{\delta'}(v)[d(\delta' \leq \eta < \delta^1)]$$

showing statement (1). Similarly, there exists an equivalence in $\mathcal{N}_f^i$.

$$M^L_\eta(e) \simeq \bigoplus_{\delta' \in K \cap I_{\geq 1}} L'[d(\delta^1 \leq \gamma' < \delta')] \oplus \bigoplus_{\delta' \in K \cap I_{\leq 0}} L'[d(\delta' \leq \gamma' < \delta^1)]$$

if $\eta$ does not end at $e$ and

$$M^L_\eta(e) \simeq \bigoplus_{\delta' \in K \cap I_{\geq 1}} L'[d(\delta^1 \leq \gamma' < \delta')] \oplus \bigoplus_{\delta' \in K \cap I_{\leq 0}} L'[d(\delta' \leq \gamma' < \delta^1)] \oplus L'[d(\delta^1 \leq \eta \leq \delta')]$$

if $\eta$ ends at $e$, see Remark 5.12. This shows statement (2).

\[\square\]

**Remark 6.10.** The support of the morphisms objects in Tables 1 to 3 refers to the support of the corresponding morphisms between sections in the sense of Section 3.1. It has the following further interpretation: let $\delta, \delta'$ be two segment in $S$ both lying at a singularity $v$. Suppose that $\delta$ starts at an edge $e^1$ and ends at an edge $e^2$. Let $\text{Mor}(L, L')[m]$, with $m \in \mathbb{Z}$, be a summand of $\text{Mor}_*(M^L_1, M^L_2)$ identified in Tables 1 to 3, corresponding to morphisms with support at a singularity $v$ and some edges $e^i$ with $i \in I \subset \{1, 2\}$ ($I = \emptyset$ is possible). The composite $c$ of the inclusion morphisms of $R$-modules

$$\text{Mor}(L, L')[m] \rightarrow \text{Mor}_*(M^L_1, M^L_2)$$

with the morphism

$$\text{Mor}_*(M^L_1, M^L_2) \rightarrow \text{Mor}_*(Z^L_{e^i}[d_1], M^L_{\delta'})$$

obtained from precomposing with the pointwise inclusion of sections $Z^L_{e^i}[d_1] \rightarrow M^L_\delta$ for $i = 1, 2$ with $d_1 = 0, d_2 = d(\delta)$ can be described as follows.

- If $i \notin I$, then $c$ is zero.
- If $i \in I$, then $c$ is the inclusion of a direct summand $\text{Mor}(L, L')[m] \rightarrow \text{Mor}_*(Z^L_{e^i}[d_1], M^L_{\gamma'})$ under the equivalence (21) or (22).

### 6.2 Intersections globally

We fix two curves $\gamma \neq \gamma'$ in $S$ composed of segments and choose representatives with the minimal number of intersections. We also assume in this section that $\gamma$ is not closed. As in Construction 5.13, we denote the segments of $\gamma$ and $\gamma'$ by $\delta$, respectively, $\delta'$ with $i \in I \subset \mathbb{Z}$ and $j \in J \subset \mathbb{Z}$. $M^L_\gamma$ was defined as the colimit of the diagram $D_\gamma : E_\gamma \rightarrow \mathcal{L}$. Using that the functor

$$\text{Mor}_*(\cdot, M^L_\gamma) : \mathcal{L}^{\text{op}} \rightarrow R\text{Mod}_R$$

preserves limits, it follows that $\text{Mor}_*(M^L_\gamma, M^L_{\gamma'})$ is the limit of the diagram

$$\text{Mor}_*(\cdot, M^L_{\gamma'}) \circ D_\gamma^{\text{op}} : E_\gamma^{\text{op}} \rightarrow R\text{Mod}_R.$$  \hfill (23)

In the following we fully describe the diagram (23). We will see that the diagram (23) is equivalent to the direct sum in the stable $\infty$-category $\text{Fun}(E_\gamma^{\text{op}}, R\text{Mod}_R)$ of a collection of very manageable diagrams. A subset of these diagrams correspond to the intersections of $\gamma$ and $\gamma'$ which we will show in the case that $\gamma, \gamma'$ are matching curves to be the only summands with nonzero limits in $R\text{Mod}_R$.  

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We proceed with the constructions of the summands of (23) associated to the different types of intersections.

**Endpoint intersections**

Assume that the endpoints of $\gamma \neq \gamma'$ intersect in a singularity $v \in \mathcal{T}_0$. Note that in this case $\gamma, \gamma'$ are necessarily singular and thus $L = L' = f^*(R)$. Reversing the orientation of a matching curve $\gamma$ leads to a $-2d(\gamma)$-fold suspension of $M_{\gamma}$. Up to suspensions of $M_\gamma$ and $M_{\gamma'}$, we can thus assume that $\gamma$ and $\gamma'$ both start at $v$.

We distinguish two cases. Either the first segments $\delta^1 = \delta_1$ and $\tilde{\delta}^1 = \delta_i$ of $\gamma$ and $\gamma'$, respectively, are identical, i.e. $i = 1$, or they are not. We begin with the case $i \neq 1$. By Table 1, we have $\text{Mor}_\mathcal{L}(M_{\delta_i}, M_{\delta_i}) \simeq R[1-i]$ and the corresponding morphisms have support at $v$. Using Remark 6.10, it follows that there is a direct summand of the diagram (23) which restricts at $(\Lambda_0^2 \times \{1\})^{\mathcal{O}_p}$ to the diagram

$$
\begin{array}{c}
R[1-i] \\
\Downarrow \simeq \\
R \oplus R[1-n]
\end{array}
$$

and vanishes on $(\Lambda_0^2 \times \{i\})^{\mathcal{O}_p}$ for $1 < i \in I'$. Passing to limits, we get a direct summand $R[1-i]$ of $\text{Mor}_\mathcal{L}(M_\gamma, M_{\gamma'})$.

We now consider the case $i = 1$. In that case, the matching curves $\gamma$ and $\gamma'$ are composed of $m$ identical segments $\delta^1 = \delta^2 = \ldots = \delta^m = \delta^1$, starting at $v$. The $(m+1)$-th segments of $\gamma$ and $\gamma'$ are denoted $\delta^{m+1}_i$ and $\tilde{\delta}^{m+1}_i$ respectively, and lie at a singularity $v'$. We choose $\delta^{m+1}_i, \tilde{\delta}^{m+1}_i$ such that they have a minimal number of intersections.

We distinguish the following two cases.

1) $\delta^{m+1}_i(0) \in \partial \Sigma_{v'}$ precedes $\tilde{\delta}^{m+1}_i(0) \in \partial \Sigma_{v'}$ in the clockwise orientation of $\partial \Sigma_{v'}$.
2) $\delta^{m+1}_i(0) \in \partial \Sigma_{v'}$ follows $\tilde{\delta}^{m+1}_i(0) \in \partial \Sigma_{v'}$ in the clockwise orientation of $\partial \Sigma_{v'}$

We find in either case for $2 \leq i \leq m$

$$\text{Mor}_\mathcal{L}(M_{\delta_i}, M_{\delta_i}) \simeq R$$

$$\text{Mor}_\mathcal{L}(M_{\delta_i}, [d(\gamma < \delta^i)], M_{\delta_i}, [d(\gamma' < \tilde{\delta}^i)]) \simeq R \oplus R[1-n]$$

In case 1) we find

$$\text{Mor}_\mathcal{L}(M_{\delta^{m+1}_i}[d(\gamma < \delta^{m+1})], M_{\delta^{m+1}_i}[d(\gamma' < \tilde{\delta}^{m+1})]) \simeq R \oplus R[1-n] ,$$

whereas in case 2) we find

$$\text{Mor}_\mathcal{L}(M_{\delta^{m+1}_i}[d(\gamma < \delta^{m+1})], M_{\delta^{m+1}_i}[d(\gamma' < \tilde{\delta}^{m+1})]) \simeq 0 ,$$

see Section 6.1. By part (1) of Lemma 6.9, each of the above $R$-modules also gives for each segment $\delta$ of $\gamma$ a direct summand of the morphism object $\text{Mor}_\mathcal{L}(M_\delta, [d(\gamma < \delta)], M_{\gamma'})$. In the case 1), using again Remark 6.10, we thus find a summand of (23) which restricts on $(\Lambda_0^2 \times \{1\})^{\mathcal{O}_p}$ to the diagram

$$
\begin{array}{c}
R \xleftarrow{(id,0)} \simeq R \oplus R[1-n] \\
\Downarrow \simeq \\
R \oplus R[1-n]
\end{array}
$$

on $(\Lambda_0^2 \times \{i\})^{\mathcal{O}_p}$ for $2 \leq i \leq m$ to the diagram

$$
\begin{array}{c}
R \oplus R[1-n] \\
\Downarrow \simeq \\
R \oplus R[1-n]
\end{array}
$$

(24)

(25)
and vanishes on the remaining parts of $E_{\gamma}^{op}$. The limit of this summand gives us a rank 1 free direct summand $R \subset \text{Mor}_\mathcal{L}(M_{\gamma}, M_{\gamma'})$, as desired. In the case 2), we analogously find a summand of (23) which restricts on $(\Lambda_0^3 \times \{1\})^{op}$ to the diagram (24), on $(\Lambda_0^3 \times \{i\})^{op}$ for $2 \leq i < m$ to the diagram (25), on $(\Lambda_0^3 \times \{m\})^{op}$ to the diagram

$$R \oplus R[1-n] \xrightarrow{\sim} R \oplus R[1-n] \xrightarrow{\sim} 0$$

and vanishes on the remaining parts of $E_{\gamma}^{op}$. Passing to limits, we obtain a direct summand $R[-n] \subset \text{Mor}_\mathcal{L}(M_{\gamma'}, M_{\gamma})$ of rank 1.

**Crossings**

Assume that $\gamma$ and $\gamma'$ have a crossing and consider the segments $\delta$ and $\tilde{\delta}$ of $\gamma$ and $\gamma'$, respectively, describing the curves at the crossing. The segments $\delta$ and $\delta'$ are located at a singularity $v$. We distinguish two cases.

**Case 1:** There are no representatives of the equivalence classes of the segments $\delta$ and $\tilde{\delta}$ as matching curves in $n$-gon $\Sigma_v$ which do not share the crossing in question. This means that $\delta$ and $\tilde{\delta}$ already have a crossing. We call such a crossing **localized** at $v$.

We denote $\delta = \delta^i$ and $\tilde{\delta} = \tilde{\delta}^j$ with $i \in I$ and $j \in J$. We assume without loss of generality that $i, j \geq 1$. As shown in Section 6.1, for each crossing of $\delta^i$ and $\tilde{\delta}^j$, there exist a summand

$$N \subset \text{Mor}_\mathcal{L}(M_{\delta}^L | \delta^i \leq \gamma < \delta^j], M_{\tilde{\delta}^j}^{L'} | \tilde{\delta}^j \leq \gamma < \tilde{\delta}^j])$$

with $N$ a suspension of $\text{Mor}(L, L')$ corresponding to morphisms located at $v$. Using Lemma 6.9, we thus find a summand of the diagram (23) which vanishes everywhere except at

$$\text{Mor}_\mathcal{L}(-, M_{\gamma'}) \circ D_{\gamma}^{op}(\cdot, j) = \text{Mor}_\mathcal{L}(M_{\delta}^L | \delta^i \leq \gamma < \delta^j], M_{\gamma'}^{L'})$$

with

$$(x, j) = \begin{cases} (2, i-1) \in E_{\gamma}^{op} & \text{if } I = [N] \text{ and } i = N, \\ (1, i) \in E_{\gamma}^{op} & \text{else}, \end{cases}$$

where it takes, up to equivalence, the value $N$. Passing to limits, we obtain the summand

$$N \subset \text{Mor}_\mathcal{L}(M_{\delta}^L, M_{\gamma'}^{L'})$$

**Case 2:** There are representatives of the equivalence classes of the segments $\delta$ and $\delta'$ as matching curves in $\Sigma_v$ which do not share the crossing in question. This means that the crossing between $\gamma$ and $\gamma'$ does not arise as a crossing between segments in $\Sigma_v$. We call such a crossing **not localized**.

Before and after the crossing, the two curves are composed of $m \geq 0$ identical segments. If $m$ were infinite, we could find different representatives for $\gamma, \gamma'$ with one intersection less which would contradict our assumptions. We can thus assume $m$ to be finite.

We can choose representatives of $\gamma$ and $\gamma'$ such that the crossing lies on an edge connecting two singularities $v, v'$. We assume that $\gamma$ and $\gamma'$ are oriented such that locally around the crossing, they both first pass through $\Sigma_v$ and then $\Sigma_{v'}$. We consider all segments $\delta^{x+i}$ and $\tilde{\delta}^{y+i}$ with $x \in I, y \in J$ and $0 \leq i \leq m+1$ for which there exist representatives of $\gamma$ and $\gamma'$, such that the induced representatives of the segments $\delta^{x+i}$ and $\tilde{\delta}^{y+i}$ share the crossing in question. Up to suspensions of the sections, we can assume that $x, y \geq 1$. Note also that by assumption $\delta^{x+i} = \tilde{\delta}^{y+i}$ for $1 \leq i \leq m$.

The segments $\delta^{x}$ and $\delta^{y}$ both lie in the neighborhood of a singularity $v_1$ and the segments $\delta^{x+m+1}$ and $\tilde{\delta}^{y+m+1}$ also both lie in the neighborhood of a singularity $v_2$. We distinguish the following two cases.
1) The point $\tilde{\delta}^y(1) \in \partial\Sigma_{v_1}$ follows the point $\delta^y(1) \in \partial\Sigma_{v_1}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_1}$ and the point $\tilde{\delta}^{y+m+1}(1) \in \partial\Sigma_{v_2}$ follows the point $\delta^{x+m+1}(1) \in \partial\Sigma_{v_2}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_2}$.

2) The point $\tilde{\delta}^y(1) \in \partial\Sigma_{v_1}$ precedes the point $\delta^y(1) \in \partial\Sigma_{v_1}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_1}$ and the point $\tilde{\delta}^{y+m+1}(1) \in \partial\Sigma_{v_2}$ precedes the point $\delta^{x+m+1}(1) \in \partial\Sigma_{v_2}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_2}$.

It follows from Section 6.1 that there exist direct summands

$$N \subset \text{Mor}_\mathcal{L}(M_{\delta^y}^L, [d(delta \leq \gamma < \delta^{x+y}], M_{\tilde{\delta}^y}^L, [d(delta \leq \gamma' < \tilde{\delta}^{y+i}])$$

with $N$ a suspension of $\text{Mor}(L, L')$, $0 \leq i \leq m + 1$ in the case 1) and $1 \leq i \leq m$ in the case 2).

In the case 1), we thus find a direct summand of the diagram (23) which restricts on $(\Lambda^2_{\delta} \times \{x + i\})^{op}$ for $0 \leq i \leq m$ up to equivalence to the constant diagram with value $N$ and vanishes on the remaining parts of $E_{\gamma}^{op}$. Passing to limits, we obtain the desired summand $N \subset \text{Mor}_\mathcal{L}(M_{\gamma}^L, M_{\gamma'}^L)$. In the case 2), we similarly find a direct summand of the diagram (23) which restricts on $(\Lambda^2_{\delta} \times \{x + i\})^{op}$ to the diagram

$$0 \xrightarrow{\sim} N \xrightarrow{\sim} N$$

on $(\Lambda^2_{\delta} \times \{x + i\})^{op}$ for $1 \leq m - 1$ to the constant diagram with value $N$, on $(\Lambda^2_{\delta} \times \{x + m\})^{op}$ to the diagram

$$N \xrightarrow{\sim} 0$$

and vanishes on the remaining parts of $E_{\gamma}^{op}$. Passing to limits, we find the direct summand $N[-1] \subset \text{Mor}_\mathcal{L}(M_{\gamma}^L, M_{\gamma'}^L)$.

**Boundary intersections**

We assume that $\gamma$ and $\gamma'$ both intersect a boundary component $B$ of $S \setminus M$ and distinguish two cases.

1) The intersection of $\gamma'$ and $B$ follows the intersection of $\gamma$ and $B$ in the orientation of $B$ induced by the clockwise orientation of $S$.

2) The intersection of $\gamma'$ and $B$ precedes the intersection of $\gamma$ and $B$ in the orientation of $B$ induced by the clockwise orientation of $S$.

We can assume that both $\gamma$ and $\gamma'$ start at $B$ and denote their shared segments by $\delta^1 = \tilde{\delta}^1, \ldots, \delta^m = \tilde{\delta}^m$, starting at $B$. We denote the $(m + 1)$-th segments of $\gamma$ and $\gamma'$ by $\delta^{m+1}$ and $\tilde{\delta}^{m+1}$, respectively. The segments $\delta^{m+1} \neq \tilde{\delta}^{m+1}$ both lie at a singularity $v$. In the case 1), we find that $\delta^{m+1}(0) \in \partial\Sigma_v$ follows $\tilde{\delta}^{m+1}(0) \in \partial\Sigma_v$ on a boundary component of $\partial\Sigma_v$ in the clockwise direction. In the case 2), we find that $\delta^{m+1}(0) \in \partial\Sigma_v$ precedes $\tilde{\delta}^{m+1}(0) \in \partial\Sigma_v$ in the clockwise orientation of $\partial\Sigma_v$.

We thus find direct summands

$$\text{Mor}(L, L') \subset \text{Mor}_\mathcal{L}(M_{\delta^y}^L[d(\gamma < \delta^y)], M_{\tilde{\delta}^y}^L[d(\gamma' < \tilde{\delta}^y)])$$

with $1 \leq i \leq m + 1$ in the case 1) and $1 \leq i \leq m$ in the case 2).
In the case 1), the summands (26) assemble using Lemma 6.9 and Remark 6.10 into a direct summand of (23) which has constant value \( \text{Mor}(L, L') \) on \( (A_0^2 \times \{i\})_{\text{op}} \) for \( 1 \leq i \leq m \) and vanishes on the remainder of \( E_{\gamma}^{\text{op}} \). Passing to limits, we obtain the desired summand \( \text{Mor}(L, L') \subset \text{Mor}_\text{loc} (M^L_\gamma, M'^L_\gamma) \). In the case 2), the summands (26) assemble using Lemma 6.9 into a direct summand of (23) which has constant value \( \text{Mor}(L, L') \) on \( (A_0^2 \times \{i\})_{\text{op}} \) for \( 1 \leq i \leq m - 1 \), takes the value

\[
\text{Mor}(L, L') \xrightarrow{\approx} \text{Mor}(L, L') \xrightarrow{\approx} 0
\]

on \( (A_0^2 \times \{m\})_{\text{op}} \) and vanishes on the remainder of \( E_{\gamma}^{\text{op}} \). The limit of this summand vanishes.

**Remark 6.11.** The above discussion of boundary intersections can be generalized as follows.

We consider two curves \( \eta \) and \( \gamma' \) in \( S \) which are composed of segments and such that \( \eta \) begins or ends at an edge \( e \) at which a segment \( \delta \) of \( \gamma' \) also lies. Reorienting \( \eta \) if necessary, we assume that \( \eta \) starts at \( e \). We denote by \( \delta \) the first segment of \( \eta \) and by \( \nu \) the singularity at which \( \delta \) is located. We assume that \( \delta \) also lies at \( \nu \) and, reorienting \( \gamma' \) if necessary, that \( \delta \) also begins at \( e \). We choose \( \delta \) in such a way that it has the minimal number of intersections with \( \delta \). This arrangement roughly looks as follows (for \( n = 3 \)).

If \( \tilde{\delta}(0) \in \partial \Sigma_v \) follows \( \delta(0) \in \partial \Sigma_v \) in the clockwise direction in the boundary component of \( \partial \Sigma_v \), we find \( \text{Mor}_\text{loc}(M^L_\delta, M'^L_\delta) \simeq \text{Mor}(L, L') \), if not then \( \text{Mor}_\text{loc}(M^L_\delta, M'^L_\delta) \simeq 0 \). Assuming that we are in the former case, the above construction translates to this situation and provides us with a direct summand of \( \text{Mor}_\text{loc}(M^L_\eta, M'^L_\gamma) \) given by a suspension of \( \text{Mor}(L, L') \).

**Non-intersections**

A relevant non-intersection appears every time both curves \( \gamma, \gamma' \) pass through the neighborhood of a singularity \( \nu \) so that the corresponding sections \( \delta, \tilde{\delta} \) at \( \nu \) satisfy \( \text{Mor}_\text{loc}(M^L_\delta, M'^L_\delta) \neq 0 \) or \( \text{Mor}_\text{loc}(M^L_{\tilde{\delta}}, M'^L_{\tilde{\delta}}) \neq 0 \), even though \( \delta \) and \( \tilde{\delta} \) do not have an endpoint or boundary intersection or crossing and are neither part of a not localized crossing.

Since \( \delta \) and \( \tilde{\delta} \) do not have a crossing, we find by Section 6.1 that they must exist a boundary component \( B \) of \( \Sigma_v \) which intersects both \( \gamma \) and \( \gamma' \). We choose to orient \( \gamma \) and \( \gamma' \) so that both \( \delta \) and \( \tilde{\delta} \) start at \( B \). Before and after \( \delta \) and \( \tilde{\delta} \), the curves \( \gamma \) and \( \gamma' \) are composed of \( m \) identical segments. If \( m \) is infinite, \( \gamma \) and \( \gamma' \) have a common infinite end. We thus assume that \( m \) is finite. We denote the \( m \) common segments of \( \gamma \) and \( \gamma' \) by \( \delta^{x+y} = \delta^{y+j} \) with \( x \in I, y \in J \) and \( 1 \leq i \leq m \). We assume that \( x, y \geq 1 \). The segments \( \delta^x \) and \( \delta^y \) both lie at a singularity \( v_1 \) and the segments \( \delta^{x+m+1} \) and \( \delta^{y+m+1} \) both lie at a singularity \( v_2 \). The discussion now resembles the discussion in the case of a not localized crossing above. However in contrast to the situation there, we find the following two possibilities.

1) The point \( \delta^y(1) \in \partial \Sigma_{v_1} \) follows the point \( \delta^x(1) \in \partial \Sigma_{v_1} \) in the clockwise direction on the intersected boundary component of \( \partial \Sigma_{v_1} \) and the point \( \delta^{y+m+1}(0) \in \partial \Sigma_{v_2} \) precedes the point \( \delta^{x+m+1}(0) \in \partial \Sigma_{v_2} \) in the clockwise direction on the intersected boundary component of \( \partial \Sigma_{v_2} \).

2) The point \( \delta^x(1) \in \partial \Sigma_{v_1} \) precedes the point \( \delta^y(1) \in \partial \Sigma_{v_1} \) in the clockwise direction on the intersected boundary component of \( \partial \Sigma_{v_1} \) and the point \( \delta^{x+m+1}(0) \in \partial \Sigma_{v_2} \) follows the
point \( \tilde{\delta}^{y+m+1}(0) \in \partial \Sigma v_2 \) in the clockwise direction on the intersected boundary component of \( \partial \Sigma v_2 \).

We continue with the case 1), the case (2) is analogous. The direct summand of (23) corresponding to the non-intersection is given by the diagram which vanishes on each \( \Lambda^2_0 \times \{j\} \) for \( j \in I \) with \( j \neq x, \ldots, x + m + 2 \), restricts on each \( \Lambda^2_0 \times \{x + i\} \) for \( 0 \leq i \leq m \) to the diagram

\[
\begin{array}{ccc}
\text{Mor}(L, L')[d] & \cong & \text{Mor}(L, L')[d] \\
\cong & & \\
\text{Mor}(L, L')[d] & \cong & \text{Mor}(L, L')[d]
\end{array}
\]

where \( d = d(\tilde{\delta}^1 \leq \gamma' < \tilde{\delta}^y) - d(\tilde{\delta}^1 \leq \gamma < \delta^x) \) and restricts on \( \Lambda^2_0 \times \{x + m + 1\} \) to the following diagram.

\[
\begin{array}{ccc}
\text{Mor}(L, L')[d] & \cong & 0 \\
\cong & & \\
\text{Mor}(L, L')[d] & \cong & \text{Mor}(L, L')[d]
\end{array}
\]

The limit of this summand thus vanishes, as desired.

### 6.3 The proofs of Theorems 6.2 and 6.3

**Proof of Theorem 6.2.** In the following, \( \gamma \neq \gamma' \) are any two matching curves and \( L, L' \) arbitrary (so long \( \gamma \) and \( \gamma' \) are non-singular, see Proposition 5.14). Statement ii) can be obtained from the following arguments by applying Lemma 3.6.

We distinguish the following two cases.

**Case 1: \( \gamma \) is not closed.**

In Section 6.2 we have associated to each intersection (or relevant non-intersection) of \( \gamma, \gamma' \) a direct summand of the diagram (23), which passing to limits gave direct summands of \( \text{Mor}_\mathcal{C}(M^L_\gamma, M^L_\gamma') \) given by suspensions of \( \text{Mor}(L, L') \). The number of these summands matches the number described Theorem 6.2. Note that if \( \gamma' \) is closed, then it is locally equivalent to the \( a_2 \)-fold direct sum of the section associated to the underlying curve. In this case, each direct summand of \( \text{Mor}_\mathcal{C}(M^L_\gamma, M^L_\gamma') \) thus appears with multiplicity \( a_2 \).

By Lemma 6.9, each value of the diagram (23) corresponds to morphisms between segments of \( \gamma \) and \( \gamma' \) or instances where \( \gamma \) and \( \gamma' \) pass along an identical edge \( e \) of \( \mathcal{T} \). These morphisms and instances are each accounted for in exactly one of the described direct summands of the diagram (23). These direct summands thus describe the entirety of the diagram (23) and we conclude that Theorem 6.2 holds for \( \gamma \) not closed.

**Case 2: \( \gamma \) is closed.**

The global section \( M^L_{\gamma} \) is given by the coequalizer of the diagram (19), so that the \( R \)-module \( \text{Mor}_\mathcal{C}(M^L_{\gamma}, M^L_{\gamma'}) \) is equivalent to the equalizer of the following diagram in \( \text{RMod}_R \).

\[
\begin{array}{ccc}
\text{Mor}_\mathcal{C}((M^L_{\gamma})^{\oplus a_1}, M^L_{\gamma'}) & \cong & \text{Mor}_\mathcal{C}((Z^L_e)^{\oplus a_1}, M^L_{\gamma'}) \\
\cong & & \\
\text{Mor}_\mathcal{C}((M^L_{\gamma}, M^L_{\gamma'})^{\oplus a_1}) & \cong & 0
\end{array}
\]

(27)

The \( R \)-module \( \text{Mor}_\mathcal{C}((M^L_{\gamma})^{\oplus a_1}, M^L_{\gamma'}) \simeq \text{Mor}_\mathcal{C}(M^L_{\gamma}, M^L_{\gamma'})^{\oplus a_1} \) can be determined using its description as the limit of the \( a_1 \)-fold direct sum of the diagram (23). All direct summands of (23) associated to intersections between \( \eta \) and \( \gamma' \) yield direct summands of the diagram (27) of the form

\[
N \longrightarrow 0
\]
so that passing to limits yields the direct summands \( N \subset \text{Mor}_C(M^L_\gamma, M^{L'}_\gamma) \). However, not all summands of \( \text{Mor}_C(M^L_\gamma, M^{L'}_\gamma) \) have to be of this form, because there can be crossings between \( \gamma \) and \( \gamma' \) which do not restrict to a crossing between \( \eta \) and \( \gamma' \). These remainder of this proof consists of an account of these summands.

Since \( \eta \) is not a matching curve, the direct summands associated to the intersections of \( \eta \) and \( \gamma' \) in general do not describe the entirety of \( \text{Mor}_C((M^L_\eta)^{\oplus a}, M^{L'}_\gamma) \). We additionally have to include the direct summands described in Remark 6.11 to obtain the entire morphism object \( \text{Mor}_C((M^L_\eta)^{\oplus a}, M^{L'}_\gamma) \). The closed curve \( \gamma \) was opened at \( e \) to the curve \( \eta \). We denote the first and last segment of \( \eta \) by \( \delta^2 \), respectively, \( \delta^1 \). We denote the compose of \( \delta^1 \) and \( \delta^2 \) at \( e \) by \( \mu \).

We now show the following.

a) Any not localized crossing of \( \gamma \) and \( \gamma' \), where there exist representatives of \( \gamma \) and \( \gamma' \) such that the crossing is between \( \mu \) and \( \gamma' \), leads to \( a_1 \) many direct summands of suspensions of \( \text{Mor}(L, L') \) in the equalizer of \( (27) \).

b) Direct summands of \( \text{Mor}_C((M^L_\eta)^{\oplus a}, M^{L'}_\gamma) \) as described in Remark 6.11 and direct summands of \( \text{Mor}_C((Z^L_\eta)^{\oplus a_1}, M^{L'}_\gamma) \) do not persist in the equalizer of \( (27) \) if they cannot be accounted for by a crossing as above.

If \( \gamma' \) is closed, statement a) needs to be modified, as in the previous case where \( \gamma \) is not closed, to include the \( a_2 \)-fold multiplicity. Together with the previous discussion, the statements a) and b) then imply that \( \text{Mor}_C(M^L_\gamma, M^{L'}_\gamma) \) is the direct sum of the desired number of suspensions of \( \text{Mor}(L, L') \), concluding this proof.

We begin by showing part a). The curves \( \gamma \) and \( \gamma' \) can be chosen so that their crossing restricts to an intersection between \( \mu \) and the composite of two segments \( \delta^y \) and \( \delta^{y+1} \) of \( \gamma' \) which end, respectively, begin at \( e \). Here \( y \in J \) lies in the set indexing the segments of \( \gamma' \) and we can assume that \( y \geq 1 \). We denote the two vertices incident to \( e \) by \( v_1 \) and \( v_2 \) and, reorienting \( \eta \) is necessary, can assume that both \( \delta^y \) and \( \delta^1 \) lie at \( v_1 \) and both \( \delta^{y+1} \) and \( \delta^2 \) lie at \( v_2 \). We distinguish the following two cases.

1) \( \delta^1(1) \in \partial \Sigma_{v_1} \) precedes \( \delta^y(1) \in \partial \Sigma_{v_3} \) in the clockwise orientation on the intersected boundary component of \( \partial \Sigma_{v_1} \) and \( \delta^2(0) \in \partial \Sigma_{v_2} \) precedes \( \delta^{y+1}(0) \in \partial \Sigma_{v_2} \) in the clockwise orientation on the intersected boundary component of \( \partial \Sigma_{v_2} \).

2) \( \delta^1(1) \in \partial \Sigma_{v_3} \) follows \( \delta^y(1) \in \partial \Sigma_{v_3} \) in the clockwise orientation on the intersected boundary component of \( \partial \Sigma_{v_1} \) and \( \delta^2(0) \in \partial \Sigma_{v_2} \) follows \( \delta^{y+1}(0) \in \partial \Sigma_{v_2} \) in the clockwise orientation on the intersected boundary component of \( \partial \Sigma_{v_2} \).

In the case 1), we find by Remark 6.11 a suspension \( N \) of \( \text{Mor}(L, L') \) and a direct summand \( N^{\oplus a_1} \oplus N^{\oplus a_1} \subset \text{Mor}_C((M^L_\eta)^{\oplus a_1}, M^{L'}_\gamma) \), where the first copy of \( N^{\oplus a_1} \) arises from the boundary intersection of \( \delta^2 \) and \( \delta^{y+1} \) and the second copy of \( N^{\oplus a_1} \) arises from the boundary intersection of \( \delta^1 \) and \( \delta^y \). In terms of the diagram \( (27) \), the crossing corresponds to a direct summand of \( (27) \) of the form

\[
N^{\oplus a_1} \oplus N^{\oplus a_1} \xrightarrow{(id, 0)} \xrightarrow{(0, id)} N^{\oplus a_1}
\]

whose equalizer gives a direct summand \( N^{\oplus a_1} \subset \text{Mor}_C(M^L_\gamma, M^{L'}_\gamma) \).

In the case 2), there are no morphisms in \( \text{Mor}_C((M^L_\eta)^{\oplus a_1}, M^{L'}_\gamma) \) associated to the crossing. Using part (2) of Lemma 6.9, we thus find a direct summand of \( (27) \) corresponding to the crossing of the following form.

\[
0 \xrightarrow{} (\text{Mor}(L, L'))^{\oplus a_1} \mid [d(\delta^1 \leq \gamma' \leq \delta^y)]
\]
Passing to equalizers, we thus obtain the direct summand \((\text{Mor}(L, L'))^{\oplus a_1}[d(\tilde{\delta}^1 \leq \gamma' \leq \tilde{\delta}^y) - 1]\) of \(\text{Mor}_\mathcal{L}(M^L_N, M^L_N')\). This concludes the proof of a).

For part b), we assume that the part of \(\gamma'\) passes along \(e\) but does not have a crossing with \(\gamma\) restricting to an intersection between \(\gamma\) and \(\mu\). It follows that either \(\tilde{\delta}^1(1) \in \partial \Sigma_{v_1}\) and \(\tilde{\delta}^{y+1}(0) \in \partial \Sigma_{v_2}\) both follow or both precede \(\tilde{\delta}^y(1) \in \partial \Sigma_{v_1}\) and \(\tilde{\delta}^y(0) \in \partial \Sigma_{v_2}\). The corresponding direct summand of (27) is thus of the form

\[
(\text{Mor}(L, L'))^{\oplus a_1}[d] \xrightarrow{\sim} (\text{Mor}(L, L'))^{\oplus a_1}[d]
\]

with \(d = d(\tilde{\delta}^1 \leq \gamma' \leq \tilde{\delta}^y)\). The equalizer of (28) vanishes, showing b). \(\square\)

Proof of Theorem 6.3. The proof of Theorem 6.3 goes along the same lines as the proof of Theorem 6.2.

Case 1: \(\gamma\) is not closed.
As shown in Section 6.2, each self-crossing gives rise to a direct summand of \(\text{Mor}_\mathcal{L}(M^L_N, M^L_N')\) given by a suspension of \(\text{Mor}(L, L')\).

Assume that all segments of \(\gamma\) are of the second type. The morphism object in \(\mathcal{L}\) of the sections associated to a segment of the second type with local values \(L, L'\) is \(\text{Mor}(L, L')\), see Table 3. Similarly, the morphism object of the sections \(Z^L_{\gamma}, Z^L_{\gamma}\) is equivalent to \(\text{Mor}(L, L')\).

The constant diagram with value \(\text{Mor}(L, L')\) thus defines a direct summand of (23). Passing to limits, we obtain the direct summand \(\text{Mor}(L, L') \subset \text{Mor}_\mathcal{L}(M^L_N, M^L_N')\).

Assume that exactly one segment of \(\gamma\) is of the first type. The curve \(\gamma\) is thus singular and exactly one end lies at a singularity. Reorienting \(\gamma\) if necessary, we can assume that \(\gamma\) begins at the singularity. The endomorphisms of the segments of \(\gamma\) thus yield a direct summand of (23) which assigns to \((\Lambda^2)^{\text{op}} \times \{1\}\) the diagram (24) and is constant on the remainder of \(E^{\text{op}}\) with value \(R \oplus R[1-n]\). The limit of this direct summand is given by \(R \subset \text{Mor}_\mathcal{L}(M_\gamma, M_\gamma)\).

If exactly two segments of \(\gamma\) are of the first type, then \(\gamma\) is singular, and begins and ends at singularities. An analogous argument as above shows that the endomorphisms of the segments of \(\gamma\) assemble into a direct summand \(R \oplus R[1-n] \subset \text{Mor}_\mathcal{L}(M_\gamma, M_\gamma)\). If the endpoints of \(\gamma\) furthermore coincide, then the singular intersections produce two further free rank 1 direct summands of \(\text{Mor}_\mathcal{L}(M_\gamma, M_\gamma)\).

The above identified direct summand of \(\text{Mor}_\mathcal{L}(M^L_N, M^L_N')\) account for the entire morphism object and match the count given in Theorem 6.3, see also Lemma 3.6. This thus concludes the proof in the case that \(\gamma\) is not closed.

Case 2: \(\gamma\) is closed.
We need to compute the equalizer of (27) with \(a_1 = 1\) and \(R = k\). Showing that each self-crossing of \(\gamma\) contributes a direct summand given by suspensions of \(\text{Mor}(L, L')\) to the equalizer of (27) is analogous to the discussion in the proof of Theorem 6.2 in the case that \(\gamma\) is closed. A novel argument is required to determine the endomorphisms not corresponding to self-crossings.

The morphism from \(M^L_N\) to \(M^L_N\) arising from the morphisms between the sections associated to the common segments of \(\eta, \gamma\) (all of the second type) contribute a direct summand \(\text{Mor}(L, L') \subset \text{Mor}(M^L_N, M^L_N')\). Each of the two composites with the pointwise inclusion \(Z^L_{\gamma} \to M^L_N\) in \(\mathcal{L}\) arising from an end of \(\eta\) at \(e\) yields an equivalence between the direct summand \(\text{Mor}(L, L')\) of both \(\text{Mor}_\mathcal{L}(M^L_N, M^L_N')\) and \(\text{Mor}_\mathcal{L}(Z^L_{\gamma}, M^L_N')\). We claim that these equivalences are equivalent, so that we obtain the following direct summand of (27).

\[
\text{Mor}(L, L') \xrightarrow{(\text{id}, 0)} \text{Mor}(L, L') \xrightarrow{(0, \text{id})} \text{Mor}(L, L')
\]
The equalizer of (29) then gives a direct summand \( \text{Mor}(L, L') \oplus \text{Mor}(L, L')[-1] \subset \text{Mor}_\zeta(M^L_\eta, M^L_\gamma) \). We have again determined the entire morphism object \( \text{Mor}_\zeta(M^L_\eta, M^L_\gamma) \), showing Theorem 6.3.

To show the above claim, we need to describe the equivalence of the direct summand \( \text{Mor}(L, L') \) obtained from composing the two equivalences contained in the following diagram of \( R \)-modules.

\[
\begin{array}{ccc}
\text{Mor}(L, L') & \simeq & \text{Mor}(L, L') \\
\downarrow & & \downarrow \\
\text{Mor}(L, L') & \simeq & \text{Mor}(L, L') \\
\text{Mor}_\zeta(M^L_\eta, M^L_\gamma) & & \text{Mor}_\zeta(M^L_\eta, M^L_\gamma) \\
\downarrow & & \downarrow \\
\text{Mor}_\zeta(Z^L_\nu, M^L_\gamma) & & \text{Mor}_\zeta(Z^L_\nu, M^L_\gamma)
\end{array}
\] (30)

As we explain below, this equivalence is obtained by applying to \( \text{Mor}(L, L') \) the monodromy of \( \mathcal{F}_\tau(k) \) along \( \gamma \) defined as follows.

Given a pure segment \( \delta \) of the second type in \( S \) lying at a singularity \( v' \) going from the edge \( e' \) to the edge \( e'' \), we define the monodromy of \( \mathcal{F}_\tau(k) \) along \( \delta \) as the following autoequivalence of \( \mathcal{N}_\tau \).

- If \( \delta \) wraps around \( v' \) in the counterclockwise direction, we define the monodromy as the composite of the left adjoint of \( \mathcal{F}_\tau(k)(v \to e') \) with the functor \( \mathcal{F}_\tau(k)(v \to e'') \).
- If \( \delta \) wraps around \( v' \) in the clockwise direction, we define the monodromy as the composite of the right adjoint of \( \mathcal{F}_\tau(k)(v \to e') \) with the functor \( \mathcal{F}_\tau(k)(v \to e'') \).

The monodromy of \( \mathcal{F}_\tau(k) \) along \( \gamma \) is defined as the autoequivalence of \( \mathcal{N}_\tau \) obtained from composing the monodromy of all segments of \( \gamma \).

The morphism object \( \text{Mor}(M^L_\eta, M^L_\gamma) \) is given by the limit of (23). Instead of directly computing the equivalence in (30), we can thus equivalently compute the equivalences obtained by tracing along the segments of \( \eta \), i.e. compose the autoequivalences of \( \text{Mor}(L, L') \) contained in the commutative diagrams

\[
\begin{array}{ccc}
\text{Mor}(L, L') & \simeq & \text{Mor}(L, L') \\
\downarrow & & \downarrow \\
\text{Mor}_\zeta(M^L_{e^i}(e^i), M^L_{e^{i+1}}(e^i)) & \simeq & \text{Mor}_\zeta(M^L_{e^i}(e^i), M^L_{e^{i+1}}(e^i)) \\
\downarrow & & \downarrow \\
\text{Mor}_\zeta(Z^L_{e^i+1}, M^L_\gamma) & & \text{Mor}_\zeta(Z^L_{e^i+1}, M^L_\gamma)
\end{array}
\]

with \( 1 \leq i \leq m \), where \( \gamma \) has \( m \) segments \( \delta^i \) lying at \( e^i \) and beginning and ending at the edge \( e^{i+1} \), respectively. The functors \( \mathcal{F}_\tau(v^i \to e^i), \mathcal{F}_\tau(v^i \to e^{i+1}) : \mathcal{N}_\tau \to \mathcal{N}_\tau \) give rise to the morphisms in the middle part of the diagram and are up to composition with the autoequivalence \( T \) of Construction 4.20 given by \( g_j \) and \( g_k \) for some \( 1 \leq k \leq n \). If \( \delta^i \) wraps counterclockwise, then \( k = j + 1 \), otherwise \( k = j - 1 \), modulo \( n \). The description of the autoequivalence of \( \text{Mor}(L, L') \) in terms of the monodromy along \( \delta^i \) follows now from the observations \( M^L_{e^i}(v^i) \simeq \zeta_j(L) \simeq \zeta_j(M^L_{e^i}(e^i)) \) and \( M^L_{e^{i+1}}(v^i) \simeq \zeta_j(L) \simeq \zeta_j(M^L_{e^i}(e^{i+1})) \), Remark 4.11 and the observation that \( T \simeq \text{id} \) on objects.

What is left to determine is the monodromy of \( \mathcal{F}_\tau(k) \). Consider a segment \( \delta^i \) of \( \gamma \) lying at \( v^i \) and connecting \( e^i \) and \( e^{i+1} \). We assume that \( \delta^i \) turns counterclockwise, the clockwise case is analogous. Up to the action of the paracyclic twist functor \( T_{v^i} \), see Remark 4.15 and Proposition 4.14, we can assume that \( \mathcal{F}_\tau(k)(v^i \to e^i) \) and \( \mathcal{F}_\tau(k)(v^i \to e^{i+1}) \) are given by \( T_1 \circ g_1 \),

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respectively, $T_2 \circ \varrho_2$, with $T_1, T_2$ each given by one of the two autoequivalences id, $T$, using the notation from Construction 4.20. The left adjoint $\varrho_2$ of $\varrho_1$ is right adjoint to $\varrho_2$, see Remark 4.11. Since $\varrho_2$ is a fully faithful functor, it thus follows that $\varrho_2 \circ \varrho_2 \simeq \text{id}_{\mathcal{N}^\ast}$. The above shows that the total monodromy of $\gamma$ is given by $i$-th power of the involution $T$ for some $i \in \mathbb{Z}$. The integer $i$ is even, as follows form the construction of $\mathcal{F}_\tau(k)$ and from the observation that there are an equal number of halfedges being transversed by $\gamma$ which carry an even or odd labeling in the chosen total orders.

\section{The Jacobian gentle algebra}

We fix an oriented marked surface $S$ with an ideal $n$-angulation $\mathcal{T}$.

\textbf{Definition 7.1.} Let $k$ be the commutative ground ring. The Jacobian algebra $\mathcal{F}_\tau$ is the 0-th homology $k$-algebra $\mathcal{F}_\tau = H_0(\mathcal{G}_\tau)$ of the relative Ginzburg algebra.

Consider the sub-quiver $P_\tau$ of the quiver $\tilde{Q}_\tau$ of Definition 4.17 consisting of all vertices and the arrows $a_{v,i,i+1}$, i.e. all arrows in degree 0. It is immediate from the definition of $\mathcal{G}_\tau$ that $\mathcal{F}_\tau \simeq kP_\tau/I$, where the ideal $I = \{a_{v,i,i+1}, a_{v,i-1,i}\}$ consists of paths of length two where $v \in \mathbb{T}_0$ and $i$ is considered modulo $n$. The Jacobian algebra $\mathcal{F}_\tau$ is therefore an (in general non-proper) gentle algebra, as defined in [LP20, Def. 3.1]. If $S$ has interior marked points, the Jacobian algebra $\mathcal{F}_\tau$ is in most, if not all, cases infinite dimensional, because there are ‘permitted cycles’, see Lemma 3.3 of [LP20].

The goal of this section is to explore the relation between the global sections associated to pure matching curves with local value $f^*(R)$ and discrete modules over the Jacobian algebra, i.e. modules concentrated in degree 0. The main result of this section is the description of $H_n(\mathcal{G}_\tau)$ in Proposition 7.4.

If $R$ is connective, meaning that its homotopy groups are concentrated in positive degrees, the compact generator identified in Proposition 4.25 is also connective and we obtain a $t$-structure on the $\infty$-category $\mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))$ by using [Lur17, 7.1.2.1, 7.1.1.13]. In the case $R = k$, this is the standard $t$-structure on $\mathcal{D}(\mathcal{G}_\tau)$ in Proposition 7.4.

\textbf{Lemma 7.2.} Let $R$ be a connective $\mathbb{E}_\infty$-ring spectrum.

(1) The heart $\mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))^\vee$ is equivalent to the abelian category $\mathcal{D}(\mathcal{F}_\tau)^\vee$ of discrete modules over the Jacobian algebras $\mathcal{F}_\tau$ with base ring $\pi_0(\mathcal{R})$.

(2) The heart $\mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))^\vee$ consists of global sections whose image under the evaluation functor

$$\hat{\ev}_e : \mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R)) \xrightarrow{\hat{\ev}} N_{\mathcal{F}_\tau} \simeq \text{RMod}_R[t_{n-2}] \xrightarrow{\text{Mor}(\mathcal{R}[t_{n-2}], \to)} \text{RMod}_R$$

lies in $\text{RMod}_R^\vee$ for each edge $e$ of $\mathcal{T}$.

\textbf{Proof.} Statement (1) directly follows from part (3) of Prop. 7.1.1.13 in [Lur17]. For statement (2), we note that a global sections $X$ lies in $\mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))^\vee$ if and only if $\text{Mor}_{\mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))}(\hat{\ev}_e^*(R[t_{n-2}]), X) \in \text{RMod}_R^\vee$ for all edges $e$ of $\mathcal{T}$. Using that $\text{Mor}_{\mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))}(\hat{\ev}_e^*(R[t_{n-2}]), X) \simeq \hat{\ev}_e(X)$, the statement follows.

\textbf{Proposition 7.3.}

(1) Let $R$ be a connective $\mathbb{E}_\infty$-ring spectrum and $\gamma$ a matching curve. Consider a nonvanishing $L \in \text{Fun}(S^{n-1}, \text{RMod}_R^\vee) \simeq \text{RMod}_R^\vee[t_{n-2}]$, with $L = f^*(R)$ if $\gamma$ is singular. The matching curve $\gamma$ is pure if and only if $M^\gamma_L \in \mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))^\vee$.

(2) The direct sum $\bigoplus_e M^\gamma_e \in \mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))^\vee$ indexed by the edges of $\mathcal{T}$ forms a generator of the heart, i.e. $\text{Map}_{\mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))^\vee}(\bigoplus_e M^\gamma_e, X)$ is zero if and only if $X \in \mathcal{H}(\mathcal{T}, \mathcal{F}_\tau(R))^\vee$ is zero.
Proof. A curve $\gamma$ is by definition pure if and only if it is composed of pure segments. An inspection of (17) and (18) shows that exactly the pure matching curves lead to global sections whose evaluations at the edges of $\mathcal{I}$ lie in $\text{RMod}_R^\ominus$. Part (1) of the proposition thus follows from part (2) of Lemma 7.2.

Part (1) directly follows from Proposition 5.18 in the case $L = f^*(R)$ using that $\text{Mor}_{\mathcal{N}(f^*)}(f^*(R), X) \simeq 0$ for $X \in \text{Fun}(S^{n-1}, \text{RMod}_R^\ominus)$ if and only if $X \simeq 0$. \hfill $\square$

**Proposition 7.4.** Let $R = k$ be a commutative ring and suppose that $S$ has no interior marked points.

(1) There exists an isomorphism of dg-algebras with vanishing differentials between $H_* (\mathcal{D}_\mathcal{T})$ and the tensor algebra $\mathcal{F}_\ell \otimes_k k[t_{n-2}]$.

(2) The discrete endomorphism algebra $H_0 \text{End}(\bigoplus_e M_{c_e})$ is isomorphic to $\mathcal{F}_\ell$.

**Remark 7.5.** Part (1) of Proposition 7.4 shows a striking similarity between $\mathcal{D}(\mathcal{D}_\mathcal{T})$ and the 3-CY categories constructed in [Hai21], as anticipated in Section 1.4 in loc. cit..

**Remark 7.6.** If $S$ has internal marked points, the curves $c_e$ associated to the edges of $\mathcal{I}$ have common infinite ends. We expect, that a generalization of Theorem 6.2 to include common infinite ends would allow to extend Proposition 7.4 to arbitrary surfaces.

**Proof of Proposition 7.4.** We denote by $L$ be the image of $k[t_{n-2}]$ under the equivalence of $k$-linear $\infty$-categories $\text{Fun}(S^{n-1}, \text{RMod}_k) \simeq \text{RMod}_{k[t_{n-2}]}$. By Proposition 5.18, there exists an isomorphism of dg-algebras

$$\mathcal{D}_\mathcal{T} \simeq \text{End} \left( \bigoplus_e M_{c_e}^L \right),$$

so that it suffices for part (1) to construct an isomorphism between $H_* \text{End} \left( \bigoplus_e M_{c_e}^L \right)$ and $\mathcal{F}_\ell \otimes_k k[t_{n-2}]$. Given two edges $e_1, e_2$ of $\mathcal{I}$, the associated pure matching curves $c_{e_1}, c_{e_2}$ do not intersect, except for directed boundary intersections. Applying Theorem 6.2, we obtain for each directed boundary intersection a direct summand

$$k[t_{n-2}] \simeq \text{Mor}(k[t_{n-2}, k[t_{n-2}]] \subset H_* \text{End} \left( \bigoplus_e M_{c_e}^L \right) \simeq \text{End} \left( \bigoplus_e M_{c_e}^L \right).$$

Note that the direct summands all lie in the same degree because $c_{e_1}, c_{e_2}$ are pure. Since $k[t_{n-2}] \simeq \bigoplus_{i \geq 0} k[i(n-2)]$ as $k$-modules, this shows that

$$H_* \text{End} \left( \bigoplus_e M_{c_e}^L \right) \simeq \bigoplus_{i \geq 0} H_0 \text{End} \left( \bigoplus_e M_{c_e}^L \right)[i(n-2)].$$

(32)

Applying $H_0$ to the equivalence (31), we obtain in combination with (32) an isomorphism of $k$-modules

$$\alpha : H_* \text{End}(\bigoplus_e M_{c_e}) \simeq \mathcal{F}_\ell \otimes_k k[t_{n-2}].$$

(33)

To see that $\alpha$ is also an isomorphism of dg-algebras (with vanishing differentials), we need to compare the composition in the two dg-algebras.

We call two directed boundary intersections from $c_{e_1}$ to $c_{e_2}$ and $c_{e_2}$ to $c_{e_3}$ composable if they lie at the same boundary component $B$ of $S \setminus M$. In this case, starting at $B$, the curves are composed of identical segments such that $c_{e_1}$ shares the same segments with both $c_{e_2}$ and $c_{e_3}$ and the two curves $c_{e_2}$ and $c_{e_3}$ share at least as many segments with each other as with $c_{e_1}$. Let $a, b \in \{1, 2, 3\}$ with $a \leq b$. Each generating morphisms given by $t_{n-2} \in k[t_{n-2}] \subset \text{End}(\bigoplus_e M_{c_e}^L)$ with $i \geq 0$ associated to the boundary intersections of $c_{e_a}, c_{e_b}$ at $B$, or the endomorphisms

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of \( M_{c_{ea}}^L \) if \( a = b \), corresponds to a morphism between the sections \( M_{c_{ea}}^L, M_{c_{eb}}^L \) which restricts for each shared segment \( \delta \) of \( c_{ea}, c_{eb} \) to the endomorphism \( t_{n-2}^a \in k[t_{n-2}] \approx \text{End}(M_{\delta}^L) \). We thus see, that the composite of \( t_{n-2}^a : M_{c_{ea}}^L \rightarrow M_{c_{eb}}^L \) with \( t_{n-2}^b : M_{c_{eb}}^L \rightarrow M_{c_{ec}}^L \) is given by \( t_{n-2}^{a+b} : M_{c_{ea}}^L \rightarrow M_{c_{ec}}^L \) for all \( a \leq b \leq c \in \{1, 2, 3\} \).

We also note that if two boundary intersections are not composable, then the corresponding endomorphisms of \( \bigoplus_e M_{c_{ea}}^L \) compose to zero.

Comparing the two sides of (33) in degree 0, one obtains that there is a directed boundary intersection from \( c_{e_1} \) to \( c_{e_2} \) if and only if there is a nonzero path from \( e_1 \) to \( e_2 \) in \( J_T \). Using that by construction \( H_0(\alpha) \) is an isomorphism of \( k \)-algebras, we further obtain that two boundary intersections are composable if and only if the corresponding paths in \( J_T \) are composable with nonzero composite. The description of the product of the generating morphisms of \( \text{End}(M_{c_{ea}}^L) \) given above implies that \( \alpha \) commutes with the multiplications and is thus an isomorphism of dg-algebras.

Part (2) follows from the above identified relation between composable paths in \( J_T \) and composable directed boundary intersections between the different \( c_{c_e} \)'s and the observation that each directed boundary intersection gives rise to a direct summand \( k = H_0(k \oplus k[1-n]) \subset H_0(\text{End}(\bigoplus_e M_{c_e})) \).

If \( R = k \) is a field, we conjecture that the pure matching curves with local value \( f^*(k) \) exactly describe the indecomposable objects in the heart.

**Conjecture 7.7.** Assume that the ground ring \( k \) is a field and let

\[
\nu : \mathcal{D}(J_T)^\vee \simeq \mathcal{D}(G_T)^\vee \rightarrow \mathcal{D}(G_T)
\]

be the inclusion of the heart. The functor \( \nu \) restricts to a bijection between the sets of

- isomorphisms classes of \( G_T \)-modules of the form \( M_\gamma \) with \( \gamma \) a finite pure matching curve.
- isomorphism classes of finite discrete indecomposable \( J_T \)-modules.

### 8 Derived equivalences associated to flips of the \( n \)-angulation

Given a decomposition of a surface into \( n \)-gons and an edge \( e \) of one of the \( n \)-gons which is not self-folded, there are \( n-2 \) possible flips of the decomposition, which are obtained by replacing \( e \) by a different diagonal contained in the \( (2n-2) \)-gon formed by the two adjacent \( n \)-gons of \( e \). For example, the local pictures of the two possible flips of a decomposition into 4-gons are depicted in Figure 3.

![Fig 3](image)

**Figure 3:** The two possible flips of a decomposition into 4-gons (in green) at an edge and the corresponding change in the dual ribbon graphs with singularities (in black, in the graphical notation defined below).

Starting with a flip of a decomposition into \( n \)-gons of the \( 2n-2 \)-gon and passing to the dual ribbon graphs, we obtain the local description of a flip of a ribbon graph at an edge \( e \). The
flip of an $n$-valent ribbon graph $T$ at a not-self folded internal edge $e$ is defined by locally at $e$ changing $T$ as above and away from $e$ not changing $T$. The operation of flipping extends to ribbon graphs with singularities, such as ideal $n$-angulations as in Definition 4.6, by keeping the set of singularities constant.

We proceed with describing flips of $n$-valent ribbon graphs with singularities in terms of contractions of ribbon graphs with singularities. It suffices to restrict to a flip at an edge by one step in the counterclockwise direction. We use the following graphical notation for ribbon graphs with singularities. The edges (interior and exterior) of a ribbon graph are denoted by straight lines. The singular vertices are denoted by $\times$ and the non-singular vertices are denoted by $\cdot$. If an external edge is labelled by an integer, that means that this edge represents that number of edges.

The flip by one step is realized by the contraction which is everywhere trivial, except near the edge which is being flipped, where it can be depicted as follows.

Applying Proposition 4.16, we can use the above contractions of ribbon graphs with singularities to produce an equivalence between $\infty$-categories of global sections of perverse schobers parametrized by the involved ribbon graphs. To this end, we describe below a collection of parametrized perverse schobers, also based on a graphical notation. Each vertex of the underlying ribbon graph is decorated with a spherical functor, i.e. in either case $f^*:\text{RMod}_R \to N_{f^*} = \text{Fun}(S^{n-1}, \text{RMod}_R)$ or the zero functor $0_{N_{f^*}}: 0 \to N_{f^*}$. To a vertex labeled $F$, the perverse schober assigns the $\infty$-category $\mathcal{V}_n^F$. Each incidence of an edge with a vertex in the underlying ribbon graph is decorated with a functor from (5), possibly composed with an autoequivalence of $N_{f^*}$, which describes the functor $\mathcal{V}_n^F \to N_F$ assigned by the perverse schober to the given incidence. Below, $T$ denotes the autoequivalence of $N_{f^*}$ from Construction 4.20.

From now on we fix an oriented marked surface $S$ and an $E_\infty$-ring spectrum $R$. Let $\mathcal{T}, \mathcal{T}'$ be two ideal $n$-angulations of $S$ which differ by a flip at any not self-folded edge $e$ of $\mathcal{T}$ by one step in the counterclockwise direction. We find a collection of parametrized perverse schobers, related by equivalences and contractions, which are everywhere identical except at $e$ and its two incident vertices $v, v'$, where they are given as follows, starting with $\mathcal{F}_{\mathcal{T}}(R)$ and ending with $\mathcal{F}_{\mathcal{T}'}(R)$. For better readability, we do not depict all edges below.
The above equivalences of parametrized perverse schober are each nontrivial only at one or two vertices with label $0_{N_f}$, where they are each given by a power of the paracyclic twist functor $T_{V^*_{0_{N_f}}}$, with $i = 3, 4$, see Section 4.2, except for the equivalence between the parametrized perverse schober in (36) and the left parametrized perverse schober in (37) and the equivalence between the right parametrized perverse schober of (39) and the parametrized perverse schober of (41). The former is nontrivial only at the lower vertex labeled $f^*$, where it is given by the autoequivalence $e$ of $V^*_{0_{N_f}}$ which restricts on each of the $n - 1$ components $N_f$ of the semiorthogonal decomposition to $T$ and on the component $\text{RMod}_R$ of the semiorthogonal decomposition to the identity functor. The latter equivalence of parametrized perverse schobers is nontrivial at the three objects of $\text{Exit}(T')$ corresponding to $e$ and the two incident singularities. At the left singularity, the equivalence is given by [3], at the right singularity by $T^{-1} \circ [2]$ and at $e$ by $[-2]$.

We thus obtain an equivalence

$$\mu_e^{1} : \mathcal{H}(\mathcal{F}^1(\mathcal{F}_T(R))) \rightarrow \mathcal{H}(\mathcal{F}^{1'}(\mathcal{F}_{T'}(R))),$$

which we call the mutation equivalence. We denote the repeated mutation by $\mu_e^{i} := (\mu_e^{1})^i$ for $i \in \mathbb{Z}$. 

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In the remainder of this section, we give a geometric description of $\mu_1^e$ in terms of a homeomorphism

$$D_e\left(\frac{1}{n-1}\pi\right) : S \setminus M \simeq \Sigma_{\mathcal{T}} \xrightarrow{D} \Sigma_{\mathcal{T}'} \simeq S \setminus M$$

which we now describe.

Let $v, v'$ be the two vertices of $\mathcal{T}$ and $\mathcal{T}'$ incident to $e$. Note that $\Sigma_{\mathcal{T}}$ and $\Sigma_{\mathcal{T}'}$ are by construction identical away from $\Sigma_v \cup \Sigma_{\mathcal{T}} v$ and $\Sigma_{\mathcal{T}'} v$. The latter two subspaces are both clearly homeomorphic to the unit disc in $\mathbb{R}^2$ with $2n-2$-marked points on the boundary and two singularities at $v = \left(-\frac{1}{4}, 0\right)$, $v' = \left(\frac{1}{4}, 0\right)$.

We define $D$ to be any homeomorphism that

- restricts to a homeomorphisms between $\Sigma_v \cup \Sigma_{\mathcal{T}} v$ and $\Sigma_v' \cup \Sigma_{\mathcal{T}'} v$ which under the above homeomorphisms with the unit disc is an automorphism of the disc which keeps the boundary fixed and rotates the convex hull of $v$ and $v'$ by $\frac{1}{n-1}\pi$.
- is constant on the remainder of $\Sigma_{\mathcal{T}}$ and $\Sigma_{\mathcal{T}'}$.

Repeated rotation gives a homeomorphism $D_e\left(\frac{1}{n-1}\pi\right) : \Sigma_{\mathcal{T}} \rightarrow \Sigma_{\mathcal{T}'}$, where $\mathcal{T}^j$ is obtained from flipping the edge $e$ by $i$ steps in the counterclockwise direction. Note that $\mathcal{T}^{(n-1)} = \mathcal{T}$ for all $j$.

![Figure 4: The 4-gon with two singularities $v, v'$, the edge $e$ and some matching curves (in blue) on the left and their image under $D_e\left(\frac{1}{n-1}\pi\right)$ on the right.](image)

**Theorem 8.1.** Let $\mathcal{T}$ be an ideal $n$-angulation of $S$ and let $\mathcal{T}'$ be the ideal $n$-angulation of $S$ obtained by a flip of a not self folded edge $e$ of $\mathcal{T}$ by one step in the counterclockwise direction. Let $\gamma$ be a matching curve and $L \in \text{Fun}(S^{n-1}, \text{RMod}_R)$. There exists an equivalence in $\mathcal{H}(\mathcal{T}', \mathcal{F}_\mathcal{T}(R))$

$$\mu_1^e(M_L^L) \simeq M_L^{L, D_{\left(\frac{1}{n-1}\pi\right)}}[m],$$

where

- $m = 1$ if $\gamma$ begins or ends at a singularity $v$ incident to $e$ and the first or last segment of $\gamma$ is a segment of the first type exiting $v$ through the edge $e$.
- $m = 0$ if $\gamma$ is not as above.

**Remark 8.2.** Repeated application of Theorem 8.1 shows that for all $i \in \mathbb{Z}$

$$\mu_i^e(M_L^L) \simeq M_L^{L, D_{\left(\frac{i}{n-1}\pi\right)}}[m]$$

for an integer $m$. 52
Proof of Theorem 8.1. We first note that it suffices to show that
\[
\mu^1_e(M^L_{\gamma}) \simeq M^L_{D^2((n-\pi)\circ \gamma)[m]} \tag{43}
\]
for each matching curve \(\gamma\) in the \((2n - 2)\)-gon. The theorem then follows, using that \(\mu^1_e\) and the object \(M^L_{\gamma}\) associated to the matching curve \(\gamma\) in \(S\) are defined via gluing.

Denote the vertices incident to \(e\) by \(v, v'\). We denote the edge \(e\) by \(e_1 = e'_1\) and label the other edges of \(v\) and \(v'\) by \(e_2, \ldots, e_n\) and \(e'_2, \ldots, e'_n\), respectively, compatible with the given cyclic orders. The segments of the first type in the \((2n - 2)\)-gon which are matching are indexed by the edges \(\{e_2, e'_2, \ldots, e_n, e'_n\}\) along which they exit \(v\) or \(v'\). We denote the corresponding segments by \(\delta_i\) and \(\delta'_i\), respectively. Note that \(\delta_i\) is obtained from \(\delta_i\) by a rotation of of the \((2n - 2)\)-gon by \(\pi\).

Using Remark 5.16 we can further reduce the number of matching curves which we need to consider. In the case \(L = f^*(R)\), it suffices to consider the matching curves given by the segments \(\delta_2, \ldots, \delta_n, \delta'_2, \ldots, \delta'_n\) and \(\gamma_e\). In the case \(L\) arbitrary and \(\gamma\) non-singular, it suffices to consider the matching curves in the \((2n - 2)\)-gon \(c_e = c_{e_1}, c_{e_2}, \ldots, c_{e_n}, c'_{e'_2}, \ldots, c'_{e'_n}\) using the notation introduced in the beginning of Section 5.3. Using the involved symmetry under rotation by \(\pi\), it furthermore suffices to verify (43) for \(\delta_2, \ldots, \delta_n, \gamma_e\) for \(L = f^*(R)\) and \(c_{e_1}, \ldots, c_{e_n}\) for \(\gamma\) non-singular. To do so, we simply trace through the equivalences between the global sections of the parametrized perverse schobers defining \(\mu^1_e\). A direct computation, case by case, shows that

\[
\mu^1_e(M_{\gamma_e}) \simeq M_{\gamma_e}[1], \tag{44}
\]
\[
\mu^1_e(M_{\delta_i}) \simeq M_{D^2((n-\pi)\circ \delta_i)[m]}, \tag{45}
\]
\[
\mu^1_e(M^L_{c_{e_i}}) \simeq M^L_{D^2((n-\pi)\circ c_{e_i})}. \tag{46}
\]

To help the reader appreciate the appearance and absence of suspensions in the above equations without having to trace through the definition of \(\mu^1_e\), we offer the following hints. To verify the appearance of the suspension in (44), one computes

\[
\mu^1_e(M_{\gamma_e})(v) \simeq \mu^1_e(M_{\gamma_e})(v') \simeq \left( R \xrightarrow{1} f^*(R) \xrightarrow{id} \ldots \xrightarrow{id} f^*(R) \xrightarrow{0} 0 \right) \in V^p_f, 
\]

which is the suspension of (16) (for \(i = 2\)). For (45),(46), we first observe that the values of the corresponding sections at the boundary edges of the \((2n - 2)\)-gon remain unchanged under \(\mu^1_e\) and in particular do not acquire any suspensions. In combination with the fact that the \(\delta_i\) and \(c_{e_i}\) are pure matching curves, so that we are free to choose their orientations so that they start at the boundary of the \((2n - 2)\)-gon, this shows that no suspensions can appear in (45),(46). \(\square\)

Remark 8.3. Descriptions of derived equivalences in terms of rotations of a disc by fractions of \(\pi\) also appear in [DJL21, Prop. 3.5.1] and for gentle algebras associated to unpunctured \(n\)-gons in [OPS18, Thm. 5.1]. A similar geometric description for \(\mu^1_e\) in the case \(n = 3\) also appears in [Qiu16]. The automorphism \(\mu^{n-1}_e\) acts objectwise as the cotwist functor of the \(n\)-spherical object \(M_{\gamma_e}\).

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