Astrophysically relevant bound trajectories around a Kerr black hole

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Abstract

We derive alternate and new closed-form analytic solutions for the non-equatorial eccentric bound trajectories, \( \{ \phi (r, \theta), t (r, \theta), r (\theta) \} \), around a Kerr black hole by using the transformation \( 1/r = \mu (1 + e \cos \chi) \). The application of the solutions is straightforward and numerically fast. We obtain and implement translation relations between the energy and angular momentum of the particle, \((E, L)\), and eccentricity and inverse-latus rectum, \((e, \mu)\), for a given spin, \(a\), and Carter’s constant, \(Q\), to write the trajectory completely in the \((e, \mu, a, Q)\) parameter space. The bound orbit conditions are obtained and implemented to select the allowed combination of parameters \((e, \mu, a, Q)\). We also derive specialized formulae for equatorial, spherical and separatrix orbits. A study of the non-equatorial analog of the previously studied equatorial separatrix orbits is carried out where a homoclinic orbit asymptotes to an energetically bound spherical orbit. Such orbits simultaneously represent an eccentric orbit and an unstable spherical orbit, both of which share the same \(E\) and \(L\) values. We present exact expressions for \(e\) and \(\mu\) as functions of the radius of the corresponding unstable spherical orbit, \(r_s\), \(a\), and \(Q\), and their trajectories, for \((Q \neq 0)\) separatrix orbits; they are shown to reduce to the equatorial case. These formulae have applications to study the gravitational waveforms from extreme-mass ratio inspirals (EMRIs) using adiabatic progression of a sequence of Kerr geodesics, besides relativistic precession and phase space explorations. We obtain closed-form expressions of the fundamental frequencies of non-equatorial eccentric trajectories that are equivalent to the previously obtained quadrature forms and also numerically match with the equivalent formulae previously derived. We sketch non-equatorial eccentric, separatrix, zoom-whirl, and spherical orbits, and discuss their astrophysical applications.
Keywords: exact solutions, physics of black holes, classical black holes, Hamiltonian mechanics, numerical relativity

(Some figures may appear in colour only in the online journal)

1. Introduction

It has now been established with observational evidence that black holes with masses ranging from $4-20M_\odot$ in x-ray binaries, to $10^6-10^9M_\odot$ in galactic nuclei, are ubiquitous. One among such important evidences is the recent detection of gravitational waves from the black hole binary merger [1] and more such events are awaited to be detected by the planned LISA mission [2]. One of the major objectives of the LISA mission is the detection of gravitational waves from EMRIs, most probably to be sourced from the compact objects spiralling and finally plunging onto the super-massive black hole (SMBH) in galactic nuclei. The dynamics of EMRIs is widely accepted as representative of test-particle motion, evolving adiabatically, in the spacetime of a rotating black hole. Understanding of such strong gravity regimes involves using the Kerr metric [3], which is a vacuum solution of Einstein’s equation for a rotating black hole. The study of trajectories around the black holes is critical to our understanding of the physical processes and their observational consequences [4].

The trajectories in Kerr and Schwarzschild [5] geometries have been studied extensively. Some of these results are covered in a pioneering work by [6] in an elegant manner. The key idea that the general trajectory in Kerr background can be expressed in terms of quadratures, was first given in [7]. In [8], the energy, $E$, and angular momentum, $L$, were expressed in terms of the circular orbit radius and the spin parameter $a$; the specific solution for the radius of the innermost stable circular orbit (ISCO) was also derived. The necessary conditions for bound geodesics for spherical orbits and the dragging of nodes along the direction of spin of a black hole was discussed [9]. The formulae have proved to be extremely useful in predicting observables in astrophysical applications like accretion around the black holes. For example, a general solution for a star in orbit around a rotating black hole was expressed in terms of quadratures [10] using the formulation given by [7]; the resulting integrals have been calculated numerically. The general expression in terms of quadratures for fundamental orbital frequencies $\nu_\theta$, $\nu_\varphi$ and $\nu_r$, for a general eccentric orbit, were first derived by [11], where different cases for circular and equatorial orbits are also discussed but complete analytic trajectories were not calculated. An exact solution for non-spherical polar trajectories in Kerr geometry and an exact analytic expression for $t(r)$ for eccentric orbits in the equatorial plane were derived [12]. These were used to obtain the expressions for the periapsis advance and Lense–Thirring frequencies. The time-like geodesics were expressed in terms of quadratures involving hyper-elliptic, elliptic and Abelian integrals for Kerr and Kerr–(anti) de Sitter spacetimes including cosmological constant [13] and applied in a semi-analytic treatment of the Lense–Thirring effect.

A time-like orbital parameter $\lambda$ called Mino time [14] was introduced to decouple the $r$ and $\theta$ equations, which was then used to express a wider class of trajectory functions in terms of the orbital frequencies $\nu_\theta$, $\nu_\varphi$ and $\nu_r$ [15]. These methods are applied to calculate closed-form solutions of the trajectories and their orbital frequencies [16], using the roots of the effective potential. However, the solutions are expressed in terms of Mino time. The commensurability of radial and longitudinal frequencies, their resonance conditions for orbits in Kerr geometry, and their location in terms of spin and orbital parameter values were studied using numerical implementations of Carlson’s elliptic integrals [17]. Considering the problem
of the precession of a test gyroscope in the equatorial plane of Kerr geometry, the analytic expressions to transform energy, angular momentum of the orbiting test particle, and spin of the black hole \((E, L, a)\) to eccentricity, inverse-latus rectum of the bound orbit \((e, \mu, a)\) parameters were obtained \([18]\). The expressions for radial and orbital frequencies are obtained to the order \(e^2\) for bound orbits and analytically for the marginally bound homoclinic orbits \([19]\). The dynamical studies of an important family of Kerr orbits called separatrix or homoclinic orbits are important for computing the transition of spiralling to plunge in EMRIs emitting gravitational waves \([20, 21]\). The test particles (compact objects in this case) transit through an eccentric separatrix orbit in EMRIs, while progressing adiabatically, before they merge with the massive black hole.

In this paper, we study the generic bound trajectories, which are eccentric and inclined, around a Kerr black hole, and then we investigate the non-equatorial separatrix orbits as a special case. We have solved the equations of motion and produce alternate and new closed-form solutions for the trajectories in terms of elliptic integrals without using Mino time, \(\{\phi(r, \theta), t(r, \theta), r(\theta)\ \text{or} \ \theta(r)\}\), which makes them numerically faster. We also implement the essential bound orbit conditions to choose the parameters \((e, \mu, a, Q)\) of an allowed bound orbit, derived from the essential conditions on the parameters for the elliptic integrals involved in the trajectory solutions. We find that the \(e^{-\mu}\) space is more convenient for integrating the equations of motion as the turning points of the integrands are naturally specified in terms of the bound orbit conditions. The exact solutions for the trajectories are found in terms of not overly long expressions involving elliptic functions. We implement the translation formulae between \((E, L)\) and \((e, \mu)\) parameters that help us to express the trajectory solutions completely in the \((e, \mu, a, Q)\) parameter space which we call the conic parameter space. We then study the case of non-equatorial separatrix trajectories in the conic parameter space. First, we write the essential equations for the important radii like innermost stable spherical orbit (ISSO), marginally bound spherical orbit (MBSO), and spherical light radius. Similar to the equatorial separatrix orbits, the non-equatorial separatrix or homoclinic trajectories asymptote to an energetically bound unstable spherical orbit, where the spherical orbit radius can vary between MBSO and ISSO. We show that the formulae for \((e, \mu)\) for the non-equatorial separatrix orbits can be expressed as functions of the radius of the corresponding spherical orbit, \(r_s, a,\) and \(Q\), which also reduce to their equatorial counterpart \([22]\) by implementing the limit \(Q \rightarrow 0\). These formulae are obtained by using the expressions of \(E\) and \(L\) for the spherical orbits. Next, we derive the exact solutions for the non-equatorial separatrix trajectories by reducing our general eccentric trajectory solutions to this case. These solutions are important for investigating the behaviour of gravitational waveforms emitted by inspiralling and inclined test objects near non-equatorial separatrix trajectories in the case of EMRIs.

The \textit{ab initio} specification of the allowed geometry of bound orbits in the parameter space is crucial for the calculation of the orbital trajectories and its frequencies. These criteria are used in building, studying and sketching different types of trajectories around a Kerr black hole: for instance, spherical, non-equatorial eccentric, non-equatorial separatrix and zoom-whirl orbits, using our closed-form expressions for trajectories are constructed. We also derive closed-form analytic expressions for the fundamental frequencies of the general non-equatorial trajectories as functions of elliptic integrals around the Kerr black holes. We use a time-averaging method on the first-order equations of motion to derive these frequencies and show that our closed-form analytic expressions of frequencies match with the formulae given by \([11]\) which were left in quadrature forms. We also reduce the general forms to the equatorial case, which is also a new form that is easier to implement and faster by a factor of \(\sim 20\).

This paper is organized as follows (see figure 1): in section 2, we review the basic equations describing \(\{r, \theta, \phi, t\}\) motion around the Kerr black hole using Hamiltonian
2. Integrals of motion and bound orbits around Kerr black hole

In this section, we first set up the basic equations defining the integrals of motion of the general eccentric orbit with $Q \neq 0$ around a Kerr black hole. We then write the formulae defining the transformation from conic parameters $(e, \mu)$ to dynamical parameters $(E, L)$ for bound orbits and also provide the conditions for the selection of the parameters $(e, \mu, a, Q)$ for the bound orbits. These results are essential for expressing the integrals of motion in $(e, \mu, a, Q)$ space for bound orbits. Finally, we derive and present an alternate and simple form of the analytic solutions for the integrals of motion in terms of standard elliptic integrals using the transformation $1/r = \mu (1 + e \cos \chi)$. Such transformations lead to a compact and useful dynamics. In section 2.1, we write the translation formulae from $(e, \mu, a, Q)$ to $(E, L, a, Q)$ parameter space. In section 2.2, we derive the exact closed-form solutions for the trajectories by solving all involving integrals and writing them in terms of elliptic integrals, $\{\phi (r, \theta), t (r, \theta), r (\theta) \text{ or } \theta (r)\}$. In section 2.3, we give the essential bound orbit conditions on $(e, \mu, a, Q)$ parameters applicable to the astrophysical situations. In section 2.4, we reduce the analytic solutions to the case of equatorial plane. In section 3.1, we derive the formulae for $E$ and $L$ for spherical orbits as a function of radius $r_s$, $a$, and $Q$. In section 3.2, we write the equations for the radii ISSO, MBSO, and spherical light radius. We then derive the exact expressions for $e$ and $\mu$ for the non-equatorial separatrix trajectories. In section 3.3, we derive the trajectory solutions for the non-equatorial separatrix orbits. In section 4, we sketch and discuss various bound trajectories around the Kerr black hole. In section 5, we derive the closed-form expressions of the fundamental frequencies in terms of elliptic integrals by the long time averaging method without using Mino time. In section 6, we conduct consistency checks by reducing the separatrix trajectories to the equatorial case, and also match the azimuthal to polar frequency ratio, $\nu_\phi/\nu_\theta$, with the spherical orbits case derived by [9]. We discuss possible applications of our trajectory solutions and frequency formulae in section 7. We summarize and conclude our results in sections 8 and 9 respectively. The mathematical details of the derivations are given in the appendices of the arxiv version of this paper [23]. In table 1 a glossary of symbols is provided.
trajectory solution for the non-equatorial and eccentric orbits around a rotating black hole. Later, we reduce these results to a simpler form for equatorial eccentric trajectories.

Considering the Kerr metric [3] for a rotating black hole of mass M in the Boyer–Lindquist coordinates, \( x^a = (t, r, \theta, \phi) \), one can write the relativistic and conservative Hamiltonian for the geodesic motion of a test particle [7, 24, 25]. A canonical transformation can be implemented such that the Hamiltonian is cyclic and the new set of momenta are conserved along the world-line of the particle. A characteristic function is obtained to generate such a transformation and Hamilton’s equations are used to obtain the first-order equations of motion [7, 24, 25], given by

\[
\begin{align*}
    m_0 \frac{dr}{d\tau} &= \frac{r^2 + a^2}{(r^2 + a^2 - 2r)} [E (r^2 + a^2) - aE] - a (aE \sin^2 \theta - L), \\
    m_0 \frac{dr}{d\tau} &= \pm \sqrt{\rho}, \\
    m_0 \frac{d\theta}{d\tau} &= \pm \sqrt{\Theta}, \\
    m_0 \frac{d\phi}{d\tau} &= \frac{a}{(r^2 + a^2 - 2r)} [E (r^2 + a^2) - aE + \frac{L}{\sin^2 \theta}], \\
    \Theta &= Q - \left[ (m_0^2 - E^2) a^2 + \frac{L^2}{\sin^2 \theta} \right] \cos^2 \theta.
\end{align*}
\]

where geometrical units \( G = c = 1 \) are implemented and \( m_0 \) is the particle’s rest mass, \( a \) is in units of \( J/M^2 \), is the specific angular momentum of the black hole, \( \rho^2 = r^2 + a^2 \cos^2 \theta \), and

\[
R = [(r^2 + a^2) E - aL] - (r^2 + a^2 - 2r) \left[ m_0^2 r^2 + (L - aE)^2 + Q \right],
\]

\[
\Theta = Q - \left[ (m_0^2 - E^2) a^2 + \frac{L^2}{\sin^2 \theta} \right] \cos^2 \theta.
\]

We have written the variables \( \rho, r, \) and \( t \) in the units of \( M \). The integrals of motion have also been derived to be [7, 11]

\[
\begin{align*}
    \tau - \tau_0 &= \int_{r_0}^r \frac{r^2 dr'}{\sqrt{R}} + \int_{\theta_0}^0 a^2 \cos^2 \theta' d\theta', \\
    \phi - \phi_0 &= -\frac{1}{2} \int_{r_0}^r \frac{1}{\Delta \sqrt{R}} \frac{\partial R}{\partial L} dr' - \frac{1}{2} \int_{\theta_0}^0 \frac{1}{\sqrt{\Theta}} \frac{\partial \Theta}{\partial L} d\theta' = -\frac{1}{2} I_1 - \frac{1}{2} H_1, \\
    t - t_0 &= \frac{1}{2} \int_{r_0}^r \frac{1}{\Delta \sqrt{R}} \frac{\partial R}{\partial E} dr' + \frac{1}{2} \int_{\theta_0}^0 \frac{1}{\sqrt{\Theta}} \frac{\partial \Theta}{\partial E} d\theta' = \frac{1}{2} I_2 + \frac{1}{2} H_2, \\
    \int_{r_0}^r \frac{dr'}{\sqrt{R}} &= \int_{\theta_0}^0 \frac{d\theta'}{\sqrt{\Theta}} \Rightarrow I_8 = H_3,
\end{align*}
\]

where \( \Delta = r^2 - 2r' + a^2 \) and \( I_1, I_2, H_1, H_2 \) are integrals defined above and solved in section 2.2.

The equation for the radial motion around the Kerr black hole, equation (1b), can be expressed in the form
\[
\frac{(E^2 - m_0^2)}{2} = \frac{m_0^2 \mu^4}{2r^4} \left( \frac{dr}{d\tau} \right)^2 + V_{\text{eff}}(r, a, E, L, Q),
\]

where the term on the lhs represents the total energy, the first term on the rhs represents the radial kinetic energy and the second term on the rhs represents the radial effective potential given by

\[
V_{\text{eff}}(r, a, E, L, Q) = -\frac{m_0^2}{r} + \frac{L^2 - a^2 (E^2 - m_0^2) + Q}{2r^2} - \frac{(L - aE)^2 + Q}{r^3} + \frac{a^2 Q}{2r^2}.
\]

2.1. Translation relations between \((E, L)\) and \((e, \mu)\)

We present the transformation of energy, angular momentum, and Carter’s constant \((E, L, Q)\) space of the test particle to the eccentricity, inverse-latus rectum \((e, \mu, Q)\) space of its corresponding bound orbit. These relations can be derived if \(R(r)\) is factorized and the periastron \(r_p\) and apastron \(r_a\) of the orbit are substituted as \(1/\mu (1 + e)\) and \(1/\mu (1 - e)\) respectively. Hence, the formulae connecting \((E, L)\) and \((e, \mu)\) parameters for bound orbits (a derivation of these formulae is given in appendix A of [23]) are

\[
E(e, \mu, a, Q) = \left[ 1 - \mu^3 (1 - e^2)^2 (\mu a^2 Q - Q - x^2) - \mu (1 - e^2) \right]^{1/2},
\]

where \(x = L - aE\) and it can be written in terms of conic parameters as

\[
x^2(e, \mu, a, Q) = \frac{-S - \sqrt{S^2 - 4PR}}{2P},
\]

where

\[
P(e, \mu, a, Q) = \frac{1}{4a^2} \left[ (3 + \mu^2) \mu - 1 \right]^2 - \mu^3 (1 - e^2)^2,
\]

\[
S(e, \mu, a, Q) = \mu (1 - e^2) + \mu^3 (1 - e^2)^2 (\mu a^2 Q - Q) - 1 + \frac{1}{2a^2} \left[ (3 + \mu^2) \mu - 1 \right] \cdot \left[ \frac{1}{\mu} - a^2 - Q + a^2 Q \mu^2 (1 - e^2) - \mu (3 + e^2) (\mu a^2 Q - Q) \right],
\]

\[
R(e, \mu, a, Q) = \frac{1}{4a^2} \left[ \frac{1}{\mu} - a^2 - Q + a^2 Q \mu^2 (1 - e^2) - \mu (3 + e^2) (\mu a^2 Q - Q) \right]^2.
\]

These expressions are used to derive analytic results for the integrals of motion, given in section 2.2, completely in the \((e, \mu, a, Q)\) parameter space.

2.2. Analytic solutions of integrals of motion

Next, we solve for the integrals of motion, i.e. equations (2c)–(2d) and reduce them to a simple form involving elliptic integrals. We first derive the expressions for the radial integrals \(I_1\) and \(I_2\). We assume the starting point of the radial motion to be apastron point of the bound orbit, \(r_0 = r_a\). The steps taken to obtain the reduced form of the radial integrals are as follows:

(i) We make the substitution \(1/r' = \mu (1 + e \cos \chi)\) and implement the method of partial fractions.
(ii) Then make the substitutions, $\cos \chi = 2 \cos^2 \frac{\chi}{2} - 1$ and $\psi = \frac{\chi}{2} - \frac{\pi}{2}$.

(iii) Implement the variable transformation given by $\sin \alpha = \frac{\sqrt{1-m^2 \sin \psi}}{\sqrt{1-m^2 \sin^3 \psi}}$ where $m^2$ is defined by equation (7h).

As a result the integrals of motion are expressed as functions of standard elliptic integrals, given by

$$ I_1 (\alpha, e, \mu, a, Q) = - [C_1 I_3 (\alpha, e, \mu, a, Q) + C_4 I_4 (\alpha, e, \mu, a, Q)] , \quad (6a) $$

$$ I_2 (\alpha, e, \mu, a, Q) = [C_5 I_3 (\alpha, e, \mu, a, Q) + C_6 I_6 (\alpha, e, \mu, a, Q) + C_7 I_5 (\alpha, e, \mu, a, Q) \] , \quad (6b) $$

$$ I_3 (\alpha, e, \mu, a, Q) = \frac{1}{\sqrt{1-m^2 (m^2 + p_1^2)}} \left[ m^2 F (\alpha, k^2) + p_1^2 \Pi \left( \frac{- (p_1^2 + m^2)}{1-m^2}, \alpha, k^2 \right) \right] , \quad (6c) $$

$$ I_4 (\alpha, e, \mu, a, Q) = \frac{1}{\sqrt{1-m^2 (m^2 + p_1^2)^2}} \left[ m^2 F (\alpha, k^2) + p_1^2 \Pi \left( \frac{- (p_1^2 + m^2)}{1-m^2}, \alpha, k^2 \right) \right] , \quad (6d) $$

$$ I_5 (\alpha, e, \mu, a, Q) = \frac{1}{\sqrt{1-m^2}} \left[ m^2 F (\alpha, k^2) + p_1^2 \Pi (s^2, \alpha, k^2) \right] + p_1^2 I_5 (\alpha, e, \mu, a, Q) , \quad (6e) $$

$$ I_6 (\alpha, e, \mu, a, Q) = \frac{1}{\sqrt{1-m^2 (m^2 + p_1^2)^2}} \left[ m^2 F (\alpha, k^2) + p_1^2 \Pi (s^2, \alpha, k^2) \right] , \quad (6f) $$

$$ I_7 (\alpha, e, \mu, a, Q) = \frac{s^4 \sin \alpha \cos \alpha \sqrt{1-k^2 \sin^2 \alpha}}{2(1-s^2)(k^2-s^2)/(1-s^2 \sin^2 \alpha)} - \frac{s^2}{2(1-s^2)(k^2-s^2)} K (\alpha, k^2) $$

$$ \quad - \frac{1}{2(1-s^2)} F (\alpha, k^2) + \frac{[s^4-2s^2(1+k^2)+3k^2]}{2(1-s^2)(k^2-s^2)} \Pi (s^2, \alpha, k^2) , \quad (6g) $$

$$ I_8 (\alpha, e, \mu, a, Q) = \frac{2\mu (1-e^2)}{\sqrt{C - A + \sqrt{B^2 - 4AC}}} F (\alpha, k^2) , \quad (6h) $$

where

$$ C_3 = \frac{2 (1-e^2) \mu [La^2 - 2x_+]}{\sqrt{(A - B + C)(1 - a^2)(a^2 \mu - a^2 \mu e - r_+)}}, \quad (7a) $$

$$ C_4 = \frac{2 (1-e^2) \mu [-La^2 + 2x_+]}{\sqrt{(A - B + C)(1 - a^2)(a^2 \mu - a^2 \mu e - r_-)}}, \quad (7b) $$

$$ C_5 = \frac{4E (1+e)}{\mu \sqrt{(A - B + C)(1 - e)}}, \quad C_6 = \frac{8E (1+e)}{\sqrt{(A - B + C)}}, \quad (7c) $$
\[ C_7 = \frac{4a^2 \mu (1 - e^2) (-La + 2Er_\cdot)}{r_- \sqrt{(A - B + C) (1 - a^2)}} (a^2 \mu - a^2 \mu e - r_+), \quad (7d) \]
\[ C_8 = \frac{4a \mu (1 - e^2) \left(-2Lr_- \sqrt{1 - a^2} - 2Ear_\cdot + La^2\right)}{r_- \sqrt{(A - B + C) (1 - a^2)}} (a^2 \mu - a^2 \mu e - r_+), \quad (7e) \]
\[ A = Qa^2 e^2 \mu^4 (1 - e^2)^2, \quad (7f) \]
\[ B = 2e^2 (1 - e^2)^2 [2Qa^2 \mu - x^2 - Q], \quad (7g) \]
\[ C = \mu^3 (1 - e^2)^2 [3\mu Qa^2 - 2x^2 - 2Q] + (1 - E^2) (1 - e^2), \quad (7h) \]
\[ n^2 = \frac{4A}{2A - B - \sqrt{B^2 - 4AC}}, \quad m^2 = \frac{4A}{2A - B + \sqrt{B^2 - 4AC}}, \quad (7i) \]
\[ k^2 = \frac{n^2 - m^2}{1 - m^2}, \quad s^2 = \frac{-p_1^2 - m^2}{1 - m^2}, \quad (7j) \]
\[ p_1^2 = \frac{2e}{1 - e}, \quad p_2^2 = \frac{2ea^2 \mu}{a^2 \mu - a^2 \mu e - r_+}, \quad p_3^2 = \frac{2ea^2 \mu}{a^2 \mu - a^2 \mu e - r_-}, \quad (7k) \]
\[ x_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \quad (7l) \]

and where the variables \( E, L \) and \( x \) can be written as functions of \( (e, \mu, a, Q) \) using equations (5a)–(5e), which makes all the integrals to be only functions of \( (e, \mu, a, Q) \). The definition of the elliptical integrals involved, is given below [26]:

\[ F (\alpha, k^2) = \int_0^\alpha \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \quad (8a) \]
\[ K (\alpha, k^2) = \int_0^\alpha \sqrt{1 - k^2 \sin^2 \alpha} \cdot d\alpha, \quad (8b) \]
\[ \Pi (s^2, \alpha, k^2) = \int_0^\alpha \frac{d\alpha}{(1 - s^2 \sin^2 \alpha) \sqrt{1 - k^2 \sin^2 \alpha}}, \quad (8c) \]

A complete derivation of these integrals is given in appendix B of [23]. Next, to solve the integrals \( H_1, H_2, \) and \( H_3 \) of equations (2b)–(2d), we make the substitutions \( z = \cos \theta' \) and \( z = z_- \sin \beta \) [16] which reduces these integrals to

\[ H_1 (\theta, \theta_0, e, \mu, a, Q) = \frac{2L}{z_+ a \sqrt{1 - E}} \left\{ F \left( \arcsin \left( \frac{\cos \theta_0}{z_-} \right), \frac{z_-^2}{z_+^2} \right) - F \left( \arcsin \left( \frac{\cos \theta}{z_-} \right), \frac{z_-^2}{z_+^2} \right) \right\} 
+ \Pi \left( z_-^2, \arcsin \left( \frac{\cos \theta}{z_-} \right), \frac{z_-^2}{z_+^2} \right) - \Pi \left( z_-^2, \arcsin \left( \frac{\cos \theta_0}{z_-} \right), \frac{z_-^2}{z_+^2} \right), \quad (9a) \]
\[ H_2 (\theta, \theta_0, e, \mu, a, Q) = \frac{2 E a z_+}{\sqrt{1 - E^2}} \left\{ K \left( \arcsin \left( \frac{\cos \theta}{z_-} \right), \frac{z_+^2}{z_-^2} \right) - F \left( \arcsin \left( \frac{\cos \theta}{z_-} \right), \frac{z_+^2}{z_-^2} \right) \right. \\

\left. - K \left( \arcsin \left( \frac{\cos \theta_0}{z_-} \right), \frac{z_+^2}{z_-^2} \right) + F \left( \arcsin \left( \frac{\cos \theta_0}{z_-} \right), \frac{z_+^2}{z_-^2} \right) \right\}, \quad (9b) \]

\[ H_3 (\theta, \theta_0, e, \mu, a, Q) = \frac{1}{a \sqrt{1 - E^2} z_+} \left\{ F \left( \arcsin \left( \frac{\cos \theta_0}{z_-} \right), \frac{z_+^2}{z_-^2} \right) - F \left( \arcsin \left( \frac{\cos \theta}{z_-} \right), \frac{z_+^2}{z_-^2} \right) \right\}. \quad (9c) \]

where

\[ z_\pm^2 = \frac{-P' \pm \sqrt{P'^2 - 4Q'}}{2}, \quad P' = \frac{-L^2 - Q - a^2 (1 - E^2)}{a^2 (1 - E^2)}, \quad Q' = \frac{Q}{a^2 (1 - E^2)} \quad (9d) \]

Hence, the equations of motion can now be written in short as

\[ \phi - \phi_0 = \frac{1}{2} [C_3 I_3 (\alpha, e, \mu, a, Q) + C_4 I_4 (\alpha, e, \mu, a, Q)] - H_1 (\theta, \theta_0, e, \mu, a, Q) \]

\[ t - t_0 = \frac{1}{2} [C_5 I_5 (\alpha, e, \mu, a, Q) + C_6 I_6 (\alpha, e, \mu, a, Q) + C_7 I_7 (\alpha, e, \mu, a, Q) + C_8 I_8 (\alpha, e, \mu, a, Q) + H_2 (\theta, \theta_0, e, \mu, a, Q)], \quad (10b) \]

\[ I_k (\alpha, e, \mu, a, Q) = H_3 (\theta, \theta_0, e, \mu, a, Q), \quad (10c) \]

where \( I_3, I_4, I_5, I_6, I_7, I_8, H_1, H_2 \) and \( H_3 \) are given by equations (6c)-(6h) and (9) respectively. Hence, all the integrals are written explicitly as functions of parameters \((e, \mu, a, Q)\) through variables \(\alpha (e, \mu, a, Q; \chi)\) and \(\beta (e, \mu, a, Q; \theta)\) which are directly used to calculate \((r, \theta, t)\) through equations (10a)-(10c). The radial motion, which varies with the \(\alpha\), is assumed to have the starting point at the apastron distance, \(r_\alpha\) or \(\alpha = 0\), of the orbit and the starting point of the polar motion, \(\beta_0\) or \(\theta_0\), is an extra variable which can be chosen in the range \(\{\theta_-, \pi - \theta_-\}\). This is tantamount to shifting the starting point of the motion in time or adjusting the initial value of the observed time, \(t_0\).

Once the initial points are fixed \((\alpha = 0, \theta = \theta_0)\), equations (10a) and (10b) are used to calculate \(\phi (r, \theta)\) and \(t (r, \theta)\) respectively, whereas equation (10c) gives \(r (\theta)\) or \(\phi (r)\), which can be used to obtain \(t (r)\) or \(t (\theta)\) and \(\phi (r)\) or \(\phi (\theta)\).

The elegant alternate forms presented here help us to write useful and simpler expressions of \((\phi, t)\) for the equatorial eccentric trajectories, as shown later in section 2.4. Also, these results can be used to reduce the radial integrals for non-equatorial separatrix trajectories in the form of logarithmic and trigonometric functions, presented in section 3.3, which are useful in the study of gravitational waves from EMRIs.

### 2.3. Bound orbit conditions in conic parameter space

The bound orbit regions have been studied and divided in the \((E, L)\) space according to the different types of possible \(r\) motion [27]. The most relevant astrophysical bound orbit region corresponds to the case where \(E < 1\) and there are four real roots of \(R (r)\), \(r_1 > r_2 > r_3 > r_4 > 0\), such that the bound orbit either exists between \(r_1\) and \(r_2\) or \(r_3\) and \(r_4\), this has been defined as region III in the \((E, L)\) plane by [27]. Since \(r_1\) and \(r_2\) are the outer most turning points of the
effective potential, the bound orbit should exist between these two in the astrophysical situations. We can implement this constraint in the \((e, \mu, a, Q)\) space by imposing the condition \(k^2 < 1\) on the parameter \(k\) used in the radial integrals in section 2.2, where we have assumed that a bound orbit exists between \(r_1\) and \(r_2\), which requires \(k^2 < 1\) as an essential condition for the elliptic integrals to have real values, equations (8a)–(8c). This further implies
\[
n^2 < 1; \tag{11a}
\]
where the substitution of equation (7i) in the above expression yields
\[
(A + B + C) > 0, \tag{11b}
\]
and by using equations (7f)–(7h) and (5a)–(5e) this implies
\[
\left[\mu^2 a^2 Q (1 + e)^2 + \mu^2 (\mu a^2 Q - x^2 - Q) (3 - e) (1 + e) + 1\right] > 0. \tag{11c}
\]
Another necessary condition is that the periastron of the orbit \(r_2 = 1/|\mu (1 + e)|\) is outside the horizon, which gives
\[
\left[\mu (1 + e) \left(1 + \sqrt{1 - a^2}\right)\right] < 1. \tag{11d}
\]
Hence, the necessary and independent conditions for this region can be collectively given as
\[
\mu^3 a^2 Q (1 + e)^2 + \mu^2 (\mu a^2 Q - x^2 - Q) (3 - e) (1 + e) + 1 > 0, \tag{12a}
\]
\[
\mu (1 + e) \left(1 + \sqrt{1 - a^2}\right) < 1, \tag{12b}
\]
\[
E(e, \mu, a, Q) < 1. \tag{12c}
\]

There exists unstable bound orbits for \(E > 1\) specified as region IV in the \((E, L)\) plane by [27], where the bound orbit exists between \(r_2\) and \(r_3\). Such a situation is not important from the astrophysical point of view, because the particle will follow a bound trajectory between the outermost turning points, i.e. \(r_1\) and \(r_2\), and hence the above conditions, equation (12), together represent a necessary and sufficient condition for the existence of bound orbits.

2.4. Equatorial bound orbits

In this section, we apply the integrals of motion, equations (10a) and (10b), to the eccentric equatorial trajectories, where \(Q = 0 (\theta = \pi/2)\). We show that the forms derived in section 2.2 reduce to very compact expressions of \((\phi, \tau)\) involving trigonometric functions and elliptic integrals for the equatorial eccentric orbits. We implement the limit, \(Q \to 0\) which leads to \(A \to 0\) and reduces the factors \((1 + x_1), A (1 + x_2)\) to
\[
(1 + x_1) \to \left(1 - \frac{C}{B}\right), \text{ and } A (1 + x_2) \to -B, \tag{13a}
\]
which gives
\[
A (1 + x_1) (1 + x_2) = C - B = \mu \left(1 - e^2\right)^2 \left[1 - \mu^2 x^2 (3 - e^2 - 2e)\right], \tag{13b}
\]
where the translation equation given by equation (5a) for \(Q = 0\) is used to substitute for \(E^2\). Also, \(m^2\) and \(n^2\) reduce to
\[ m^2 = \frac{2B}{B - C} = \frac{4 \mu^2 e^2}{1 - \mu^2 x^2 (3 - e^2 - 2e)}, \quad n^2 = \frac{4AB^2}{2B^2 (A - B) + 2AC} = 0. \]

(13c)

The substitution of these reduced expressions of \( m^2 \) and \( n^2 \) further simplifies the integrals \( I_3 \), \( I_5 \), and \( I_6 \), as shown in appendix C of [23], which finally yields the expressions for azimuthal angle and time coordinate for equatorial trajectories to be given by

\[ \phi - \phi_0 = \frac{1}{2} I_1 = a_1 \Pi (-p_3^2, \psi, m^2) + b_1 \Pi (-p_3^2, \psi, m^2), \]

(14a)

\[ t - t_0 = \frac{1}{2} I_2 = a_2 I_5 + b_2 I_6 + c_2 I_3 + d_2 I_4, \]

(14b)

where

\[ a_1 = \frac{C_3}{2} = \frac{\mu^{1/2} [La^2 - 2x_r_+]}{\sqrt{1 - a^2 (a^2 - a^2 \mu e - r_+)}} \sqrt{1 - \mu^2 x^2 (3 - e^2 - 2e)}, \]

(14c)

\[ b_1 = \frac{C_4}{2} = \frac{\mu^{1/2} [-La^2 + 2x_r_-]}{\sqrt{1 - a^2 (a^2 - a^2 \mu e - r_-)}} \sqrt{1 - \mu^2 x^2 (3 - e^2 - 2e)}, \]

(14d)

\[ a_2 = \frac{C_5}{2} = \frac{\mu^{1/2}}{2} \frac{1}{1 - e^2} \sqrt{1 - \mu^2 x^2 (3 - e^2 - 2e)}, \]

(14e)

\[ b_2 = \frac{C_6}{2} = \frac{4E}{\mu^{1/2} (1 - e^2)} \sqrt{1 - \mu^2 x^2 (3 - e^2 - 2e)}, \]

(14f)

\[ c_2 = \frac{C_7}{2} = \frac{2a \mu^{1/2} (-La + 2Er_-)}{r_- \sqrt{1 - \mu^2 x^2 (3 - e^2 - 2e) (1 - a^2)}} \frac{(a^2 \mu - a^2 \mu e - r_+)}{(a^2 - a^2 \mu e - r_-)}, \]

(14g)

\[ d_2 = \frac{C_8}{2} = \frac{2a \mu^{1/2} \left(-2Lr_- \sqrt{1 - a^2 - 2Er_- a + La^2}\right)}{r_- \sqrt{1 - \mu^2 x^2 (3 - e^2 - 2e) (1 - a^2)}} \frac{(a^2 \mu - a^2 \mu e - r_-)}{(a^2 - a^2 \mu e - r_-)}, \]

(14h)

The corresponding fundamental frequency formulae for the equatorial trajectories are

\[ \nu_{\phi} = \frac{c^3}{2 \pi GM \cdot [r (\psi = \pi/2) - t_0]}; \quad \nu_r = \frac{c^3}{GM \cdot r}; \quad \nu_\psi = \frac{c^3}{2GM \cdot [r (\psi = \pi/2) - t_0]}. \]

(15)

These compact expressions, equations (14) and (15), for the equatorial eccentric trajectories have their importance in various astrophysical studies, in addition to, gyroscope precession and phase space studies.
3. Non-equatorial separatrix trajectories

The separatrix orbits have been studied for the equatorial plane around a Kerr black hole [20, 28]. They have been shown as homoclinic orbits which asymptote to an energetically bound and unstable circular orbit. Here, we discuss the non-equatorial counterpart of these separatrix trajectories where these orbits asymptote to an energetically bound, unstable spherical orbit. These non-equatorial homoclinic trajectories are critical in calculating the evolution of test objects transiting from inspiral to plunge, which is not always confined to the equatorial plane, as in EMRIs emitting gravitational radiation.

In this section, we first deduce the expressions of $E$ and $L$ for the spherical orbits as functions of the radius $r_s$, and $(\alpha, Q)$. We then derive the exact expressions for the conic parameters $(e, \mu)$ for non-equatorial separatrix orbits as a function of the radius of the corresponding spherical orbit, $r_s$, and $(\alpha, Q)$. We also show that these formulae reduce to the equatorial case, previously derived in [20], when $Q \to 0$ is applied. Next, we derive the exact analytic expressions for the non-equatorial separatrix trajectories by reducing it from the general trajectory formulae, equations (10a)–(10c). We find that in this case, the radial part of the solutions can be reduced to a form that involves only logarithmic and trigonometric functions.

3.1. Energy and angular momentum of spherical orbits

Spherical orbits are the non-equatorial counterparts of circular orbits and set a crucial signpost in the dynamical study of non-equatorial and separatrix trajectories. The exact expressions for energy and angular momentum for the spherical orbits can be derived by substituting $e = 0$ and $\mu = 1/r_s$, where $r_s$ is the radius of the orbit, in the expressions for $E$, $L$, and $x$ given by equations (5a)–(5e), which yields

$$E = \frac{\left\{ 2a^4 Q + (r_s - 3) (r_s - 2)^2 r_s^4 - a^2 r_s \left[ r_s^2 (3r_s - 5) + Q (r_s^2 (r_s - 4) + 5) \right] \right\}^{1/2}}{r_s^2 \left[ r_s (r_s - 3)^2 - 4a^2 \right]^{1/2}} \cdot (16a)$$

$$x = \frac{\left\{ -2a^4 Q + r_s^2 (r_s - 3) \left[ r_s^2 - (r_s - 3) Q \right] + a^2 r_s \left( r_s^2 + r_s^2 - 2Qr_s + 8Q \right) \right\}^{1/2}}{r_s^{1/2} \left[ r_s (r_s - 3)^2 - 4a^2 \right]^{1/2}} \cdot (16b)$$

and

$$L = x + aE. \quad (16c)$$

Similar formulae were derived in terms of inclination angle using an approximation in [29], whereas we have written the exact form in terms of the fundamental parameters and constant of motion $Q$. These formulae reduce to the energy and angular momentum formulae for circular orbits when $Q = 0$ is substituted [8]:

$$E = \frac{r_e^2 - 2r_e + a \sqrt{r_e}}{r_e \left( r_e^2 - 3r_e + 2a \sqrt{r_e} \right)^{1/2}}, \quad L = \frac{\sqrt{r_e} \left( r_e^2 - 2a \sqrt{r_e} + a^2 \right)}{r_e \left( r_e^2 - 3r_e + 2a \sqrt{r_e} \right)^{1/2}}. \quad (17)$$
3.2. Exact expressions of conic variables for non-equatorial separatrix orbits

Similar to the case of equatorial plane, the non-equatorial separatrix trajectories can be parametrized by the radius of unstable spherical orbits, \( r_s \), for a given combination of \( a \) and \( Q \), where \( r_s \) varies from the MBSO to ISSO. The energy and angular momentum of the separatrix orbits can be determined by equations (16a)–(16c) by varying \( r_s \) between the extrema MBSO and ISSO radii. In the \((e, \mu)\) plane, these homoclinic orbits form the boundary (other than \( e = 0 \) and \( e = 1 \) curves) of the allowed bound orbit region defined by equation (12) for a fixed \( a \) and \( Q \); see red curve in figure 2(a). The locus of this boundary in the \((e, \mu)\) plane is obtained when equality is applied to the inequality equation (12a), which results in

\[
\left[ \mu^3 a^2 Q (1 + e)^2 + \mu^2 (\mu a^2 Q - x^2 - Q) (3 - e) (1 + e) + 1 \right] = 0. \tag{18}
\]

ISSO is a homoclinic orbit with \( e = 0 \) and MBSO is a homoclinic orbit with \( e = 1 \); hence the endpoints of the separatrix curve (red curve in figure 2(a)) represents the ISSO and MBSO radii. At these endpoints, the parameter \( \mu \) takes values as described below:

For ISSO, \( e = 0 \) for the homoclinic orbit gives \( \mu = \frac{2r_a}{2r_p} = \frac{1}{r_p} = \frac{1}{r_s} \).

For MBSO, \( e = 1 \) for the homoclinic orbit gives \( \mu = \frac{1 + r_p/r_a}{2r_p} = \frac{1}{2r_p} = \frac{1}{2r_s} \).

The equations for ISSO and MBSO radii can be obtained using the equation of separatrix curve, equation (18), by plugging in \((e = 0, \mu = 1/r_s)\) and \((e = 1, \mu = 1/2r_s)\) to derive ISSO and MBSO respectively (as shown in appendix D of [23]). Hence, the equations for these radii are given by

\[
\begin{align*}
& r_s^3 - 12r_s^8 - 6a^2 r_s^7 + 36a^2 r_s^6 + 8a^2 Q r_s^5 - 28a^2 r_s^4 + 9a^4 r_s^3 - 4a^4 Q r_s^2 \\
& + 48a^2 Q r_s^4 + 16a^4 Q r_s^3 - 8a^4 Q r_s^2 - 48a^4 Q^2 r_s - 16a^6 Q^2 = 0, \tag{19}
\end{align*}
\]

for ISSO and

\[
\begin{align*}
& r_s^8 - 8r_s^7 - 2a^2 r_s^6 + 16a^2 r_s^5 + 2a^2 Q r_s^4 - 8a^2 Q r_s^3 - 6a^3 Q r_s^2 + a^4 r_s^3 \\
& + 8a^2 Q r_s^4 + a^4 Q r_s^3 + 2a^2 Q^2 r_s - 2a^4 Q^2 r_s + a^4 Q^2 = 0 \tag{20}
\end{align*}
\]

for MBSO. The light radius for the spherical orbits can be obtained by equating the denominator of equation (16a) to zero, so that \( E \to \infty \), which has the well known form for the equatorial light radius [8] given by

\[
X = 2 \left\{ 1 + \cos \left[ \frac{2}{3} \arccos (-a) \right] \right\}. \tag{21}
\]

Figure 3 shows the contours of these radii in the \((r_s, a)\) plane for various \( Q \) values.

The effective potential diagram for the non-equatorial separatrix orbits shows double roots \((r_2 = r_3)\) of \( R(r) \) at the periastron of the eccentric orbit and it also represents the spherical orbit radius, \( r_s \) (see figure 2(b)). One of the remaining two roots of \( R(r) \) represents the apastron \((= r_1 > r_s)\) of the eccentric orbit and the other inner root \((= r_4 < r_s)\) is not the part of bound trajectory.

Now, following a similar method used in [20], we derive the expressions for \( e \) and \( \mu \) for separatrix orbits with \( Q \neq 0 \). We write \( R(r) = 0 \) in the form
\[ u^4 + a' u^3 + b' u^2 + c' u + d' = 0, \]  

(22a)

where \( u = 1/r \) and

\[
\begin{align*}
& a' = -\frac{2}{a'Q} \left[ x^2 + Q \right], \\
& b' = \left( \frac{x^2 + 2aEx + a^2 + Q}{a'Q} \right), \\
& c' = -\frac{2}{a'Q}, \\
& d' = \frac{1 - E^2}{a'^2Q}. 
\end{align*}
\]

(22b)

For the separatrix orbits, equation (22a) can be written as

\[
(u - u_s)^2 \cdot [u^2 - (u_1 + u_4) u + u_1 u_4] = 0,
\]

(23)

where \( u_s = 1/r_s, u_1 = 1/r_1 \) are the apsides of the orbit, and \( u_4 = 1/r_4 \) corresponds to the innermost root of \( R(r) \). The comparison of \( u^3 \) and constant term of the above equation with those of equation (22a) further gives the expression of \( u_1 \). The conic parameters for such an orbit are given by

\[
\begin{align*}
& e_s = \frac{u_s - u_1}{u_s + u_1}, \\
& \mu_s = \frac{u_s + u_1}{2},
\end{align*}
\]

(24)

where the substitution of \( u_1 \) and \( u_s = 1/r_s \) yields

\[
\begin{align*}
& e_s = \frac{4 + a'r_s + \sqrt{(r_s a' + 2)^2 - 4d'r_s^4}}{-a'r_s - \sqrt{(r_s a' + 2)^2 - 4d'r_s^4}}, \\
& \mu_s = \frac{1}{4r_s} \left[ -a'r_s - \sqrt{(r_s a' + 2)^2 - 4d'r_s^4} \right].
\end{align*}
\]

(25a)
since a homoclinic orbit has same energy and angular momentum of the unstable spherical orbit, as shown in figure 2(b); hence \( a' \) and \( d' \) can be rewritten using the formulae of \( E \) and \( L \) for the spherical orbits, equations (16a)–(16c), to be

\[
a' = \frac{2 \left\{ 2a^4 Q - r_s^2 (r_s - 3) \right\} - a^2 r_s \left( r_s^2 - 2Qr_s + 8Q \right)}{a^2 Qr_s \left( r_s (r_s - 3)^2 - 4a^2 \right)},
\]

\[
d' = \frac{\left\{ -2a^4 Q - (r_s - 3) (r_s - 2)^2 r_s^2 + a^2 r_s \left[ r_s^2 (3r_s - 5) + Q (r_s - 4) + 5 \right] \right\}}{a^2 Qr_s^4 \left( r_s (r_s - 3)^2 - 4a^2 \right)}.\tag{25c}
\]

These expressions reduce to the \((e, \mu)\) formulae for the equatorial separatrix orbits (see appendix E of [23] for the details) when the limit \( Q \to 0 \) is implemented, to the forms previously derived by [20]:

\[
e_s = -\frac{r_c^2 - 6r_c - 3a^2 + 8a \sqrt{r_c}}{r_c^2 + a^2 - 2r_c}, \quad \mu_s = \frac{r_c^2 + a^2 - 2r_c}{4r_c \left( \sqrt{r_c} - a \right)^2}. \tag{26}
\]
3.3. Exact forms for the non-equatorial separatrix trajectories

In this section, we show the reduction of our general trajectory solutions, equation (10), for the case of separatrix orbits with \( Q \neq 0 \) to simple expressions. The separatrix or homoclinic orbits represent a curve in the \((e, \mu)\) plane for a fixed \(a\) and \(Q\) combination, figure 2, which is also the boundary of the bound orbit region defined by equation (12). This separatrix curve is defined by equation (18), which gives us the relation

\[
x^2 + Q = \frac{1 + 4\mu^3 a^2 Q (1 + e)}{\mu^2 (3 - e) (1 + e)}; \quad (27)
\]

this further reduces the expressions of \(A, B, C\) (equations (7f)–(7h)) and correspondingly the expressions of \(n^2\) and \(m^2\) to

\[
n^2 = 1 \quad \text{or} \quad k^2 = 1, \quad (28a)
\]

\[
m^2 = \frac{a^2 Q\mu^3 e (1 + e) (3 - e)}{[1 + 2a^2 (-1 + e^2) Q\mu^2]}, \quad (28b)
\]

The integrals governing the vertical motion (\(\theta\) integrals) given by equations (9a)–(9c) retain their same form as they do not involve \(k^2 = 1\), whereas, the radial integrals given by equations (6a)–(6h) reduce further, when \(k^2 = 1\) is substituted. The elliptic integrals reduce to forms involving trigonometric and logarithmic functions using the following identities given here

\[
\Pi (q^2, \alpha, 1) = \frac{1}{1 - q^2} \left[ \ln (\tan \alpha + \sec \alpha) - q \ln \sqrt{\frac{1 + q \sin \alpha}{1 - q \sin \alpha}} \right], \quad \text{where} \quad q^2 > 0, \quad q^2 \neq 1,
\]

\[
= \frac{1}{1 - q^2} \left[ \ln (\tan \alpha + \sec \alpha) + |q| \tan^{-1} (|q| \sin \alpha) \right], \quad \text{where} \quad q^2 < 0, \quad (29a)
\]

\[
F (\alpha, 1) = \ln (\tan \alpha + \sec \alpha), \quad (29b)
\]

\[
K (\alpha, 1) = \sin \alpha. \quad (29c)
\]

The final and simple expressions for the azimuthal angle, \((\phi - \phi_0)\), \((t - t_0)\), and the equation relating \(r - \theta\) motion for the non-equatorial separatrix trajectories (see appendix F of [23] for the derivation) are given by

\[
\phi - \phi_0 = \frac{1}{2} \left\{ \frac{\sqrt{\mu (1 + e) (3 - e)}}{\sqrt{e[1 + 2ar (-1 + e^2) Q\mu^2](1 - e)}} \right\} \left[ \frac{[La^2 - 2sr_{+}]}{(a^2 \mu - a^2 \mu e - r_{+}) S_{5} + \frac{-La^2 + 2sr_{+}}{(a^2 \mu - a^2 \mu e - r_{+}) S_{5}}} - H_{1} \right], \quad (30a)
\]

\[
t - t_0 = \frac{\sqrt{[1 + e] (3 - e)}}{\sqrt{e[1 + 2ar (-1 + e^2) Q\mu^2]}} \left\{ \frac{E \mu (1 - e)^2 S_{5} + \frac{a^2 \mu (2E \mu r_{+})}{r_{+} \sqrt{(1 - a^2) (a^2 \mu - a^2 \mu e - r_{+}) S_{5} + \frac{2E}{(1 - e) S_{5}}}}}{r_{+} \sqrt{(1 - a^2) (a^2 \mu - a^2 \mu e - r_{+}) S_{5}}} \right\} + \frac{1}{2} H_{2}, \quad (30b)
\]

\[
\frac{2 \mu (1 - e^2) a_{+} \sqrt{1 - E^2}}{\sqrt{C - A + \sqrt{B^2 - 4AC}}} \ln (\tan \alpha + \sec \alpha) = \left\{ F \left( \arcsin \left( \frac{\cos \theta_{0}}{t_{-}} \right) \frac{z_{+}}{z_{-}} \right) - F \left( \arcsin \left( \frac{\cos \theta_{0}}{t_{-}} \right) \frac{z_{+}}{z_{-}} \right) \right\}, \quad (30c)
\]

\[
\]
where integrals $S_3 - S_7$ are summarized in table 2, and $H_1 \left( \theta, \theta_0, e, \mu, a, Q \right)$, $H_2 \left( \theta, \theta_0, e, \mu, a, Q \right)$ are given by equations (9a) and (9b) respectively.

These expressions have their utility in evaluating the trajectory evolution of inspiralling objects near the separatrix, and just before plunging, for extreme mass ratio inspirals (EMRIs) in gravitational wave astronomy [30–32].

4. Trajectories

The analytic solution of the integrals of motion presented in this paper in section 2.2 provides a direct and exact recipe to study bound trajectories without involving numerical integrations. We now discuss various kinds of bound geodesics around Kerr black hole using our analytic solution for the integrals of motion. We use the translation formulae, equations (5a)–(5e), to obtain the integrals of motion only in terms of $(e, \mu, a, Q)$ parameters. To sketch the trajectories, we have chosen the starting point for the trajectories to be $(\beta_0 = \pi/2, \alpha = 0)$ as it follows from equation (2d). We use equation (10c) to calculate corresponding small change in $\theta$ or $\beta$ with the small change in $r$ or $\alpha$ and substitute corresponding $(r, \theta)$ or $(\alpha, \beta)$ values in equations (10a) and (10b) to calculate $(\phi, t)$.

There are various possible kinds of bound orbits. Here, we take up each case and sketch these trajectories for different combinations of $(a, Q)$, where the parameters values are tabulated in table 3. We take up slow rotating $(a = 0.2)$ and fast rotating $(a = 0.8)$ black hole situations for various $Q$ values. The various features of these orbits are enumerated below:

(i) Eccentric orbits: figure 4 represents eccentric bound prograde trajectories, where the parameter values are depicted in the table 3. The particle periodically oscillates between the periastron and the apastron, and is also bound between $\theta = \arccos \left( \frac{z_0}{r} \right)$ and $\theta = \arccos \left( \frac{-z_0}{r} \right)$ as shown in $(t - r)$ and $(t - \theta)$ plots in figure 4, whereas $(t - \phi)$ plots depict that $\phi$ varies between $0$ to $2\pi$.

(ii) Homoclinic/Separatrix orbits: homoclinic orbits are the separatrices between eccentric bound and plunge orbits, where the particle asymptotically approaches the unstable spherical/circular orbit in both the distant past and the distant future. The energy and angular momentum of the orbiting particle simultaneously correspond to a stable eccentric bound orbit and an unstable spherical/circular orbit. Separatrix orbits in the equatorial plane of a Kerr black hole are well studied, [20, 28, 33]. The homoclinic orbits form an important group in Kerr dynamics as they represent the transition between inspiral and plunge orbits and hence, have their significance in the study of gravitational wave spectrum under the adiabatic approximation. The homoclinic or separatrix orbits correspond to the boundary of the region in $(e, \mu, a, Q)$ space, defined by equation (18). Figure 5 show prograde non-equatorial homoclinic/separatrix orbits (see table 3 for parameter values). The orbit initially follows an eccentric path and asymptotically approaches the periastron radius which also corresponds to the unstable spherical orbit radius as shown in $(t - r)$ plots of figure 5.

(iii) Spherical orbits: figure 6 shows a prograde innermost stable spherical orbit (ISSO), which is also the homoclinic orbit with $e = 0$. All the spherical stable orbits exist outside ISSO, whereas unstable spherical orbits are found between ISSO and MBSO.

(iv) Zoom-whirl orbits: zoom whirl orbits are orbits where the particle takes a finite number of revolutions at the periastron before going back to the apastron, which is an extreme form of the periastron precession. Their significance in gravitational astronomy has been studied for the case of equatorial Kerr orbits [21]. Here, we discuss zoom-whirl orbits with $Q \neq 0$ as shown in figure 7, where the particle takes finite revolutions with...
varying $\theta$ at the periastron before turning back to the apastron. We have chosen the value of $\mu$ very near to the separatrix, where usually the zoom whirl behavior is seen, for different values of $(e, a, Q)$ combinations. As expected, the particle spends more time at the periastron, compared to the time taken at apastron, to take a finite number of revolutions which is making the $t-r$ plots appear flatter near the periastron (see figure 7). Homoclinic or Separatrix orbit family is the limiting case of the zoom-whirl orbit family where the particle takes infinite revolutions as it asymptotes to the unstable spherical orbit.

| Table 1. Glossary of symbols used. |
|------------------------------------|
| **Boyer Lindquist coordinates**    |
| $t$ | Time coordinate | $r$ | Radial distance from the black hole |
| $\theta$ | Polar angle | $\phi$ | Azimuthal angle |
| $\rho^2$ | $r^2 + a^2 \cos^2 \theta$ | $a$ | Spin of the black hole |
| **Common physical parameters**    |
| $u$ | $\frac{1}{2}$ | $\tau$ | Proper time |
| $r_+$ | Horizon radius | $Q$ | Carter’s constant |
| $E$ | Energy per unit rest mass of the test particle | $L_z$ | $z$ component of Angular momentum per unit rest mass of the test particle |
| $p_t$ | Generalized momentum for $t$ coordinate | $p_\phi$ | Generalized momentum for $\phi$ coordinate |
| $p_r$ | Generalized momentum for $r$ coordinate | $m_0$ | $= 0$ for photon orbits and $= 1$ for particle orbits |
| $V_{eff}$ | Radial effective potential for an eccentric test particle trajectory | $\mathcal{H}$ | Relativistic Hamiltonian for the geodesic motion |
| $r_1$ | Apastron distance ($=r_a$) | $r_2$ | Periastron distance ($=r_p$) |
| $r_3$ | Third turning point of the test particle | $r_4$ | Innermost turning point of the test particle |
| $e$ | Eccentricity parameter | $\mu$ | Inverse latus-rectum parameter |
| **Integrals of motion**            |
| $\chi$ | Defined by $u = \mu (1 + e \cos \chi)$ | $\psi$ | $\frac{1}{2} - \frac{\mu}{r}$ |
| $y$ | $1 + e \cos \chi$ | $I$ | Terminology used for radial integrals |
| $H$ | Terminology used for $\theta$ integrals | |
| **Spherical and separatrix orbits**|
| $r_s$ | Radius of spherical orbit | $r_c$ | Radius of circular orbit |
| $e_s$ | Eccentricity of the separatrix orbits | $\mu_s$ | Inverse latus-rectum of the separatrix orbits |
| $Z$ | ISCO radius | $X$ | Light radius |
| **Fundamental frequencies**        |
| $\nu_\phi$ | Azimuthal frequency | $\nu_r$ | Radial frequency |
| $\nu_\theta$ | Vertical oscillation frequency | |

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Table 2. This table summarizes the trajectory solution derived in section 3.3 for the non-equatorial separatrix orbits.

| Analytic solutions | |
|---------------------|------------------------|
| $S_3 = \frac{1}{(1+p^2)} \left[ -p^2 \ln \sqrt{1-a^2} + \frac{(1-p^2)}{(1+p^2)} \ln \left( \sqrt{1-a^2} + \sqrt{1-a^2 \sin^2 \alpha} \right) + \frac{\ln(a+\sec \alpha)}{\sqrt{1-a^2}} \right]$ | |
| $S_4 = \frac{1}{(1+p^2)} \left[ -p^2 \ln \sqrt{1-a^2} + \frac{(1-p^2)}{(1+p^2)} \ln \left( \sqrt{1-a^2} + \sqrt{1-a^2 \sin^2 \alpha} \right) + \frac{\ln(a+\sec \alpha)}{\sqrt{1-a^2}} \right]$ | |
| $S_5 = \frac{1}{(1+p^2)} \left[ a^2 \ln(\tan \alpha + \sec \alpha) + p^2 S_5 + \frac{2a^2(a^2-1)}{(1+p^2)} | s | \ln | s | \sin \alpha \right]$ | |
| $S_6 = \frac{\ln(\tan \alpha + \sec \alpha)}{\sqrt{1-a^2}(1+p^2)} + \frac{p^2 \sqrt{1-a^2}}{(1+p^2)^2} | s | \tan^{-1} | s | \sin \alpha \right]$ | |
| $S_7 = \frac{1}{(1+p^2)} \left[ s \ln(\tan \alpha + \sec \alpha) - s^2 \sin \alpha + (3-s^2) | s | \ln | s | \sin \alpha \right]$ | |

Now, we discuss how different kinds of orbits are distributed in the bound orbit region in the $(e, \mu)$ plane defined by the equation (12) for a fixed combination of $(a, Q)$. We fix $a = 0.5$ and $Q = 5$ and show the shaded bound orbit region in figure 8, that represents the eccentric orbits allowed. The black curve which is the boundary of the shaded region represents homoclinic or separatrix orbits. The curve defined by $e = 0$ represents all the spherical orbits with its end point at ISSO, which intersects with the separatrix line. We fix $e = 0.5$ depicted by the red curve in figure 8 and take different values of $\mu$, as depicted by the black dots on the red curve, and plot the corresponding trajectories and study their corresponding behavior.

We see from figures 9 and 10, that for a fixed $e = 0.5$, as $\mu$ is increased, the trajectory shows zoom-whirl behavior as it gets closer to the separatrix or homoclinic orbit for the corresponding $e$ value. It can be seen in the $r$ vs $t$ plot of figure 10(b) that the particle spends some time at the periastron which clearly depicts the zoom-whirl behavior. Hence, it is seen that zoom-whirl behavior is a near separatrix phenomenon and can occur at any eccentricity.

5. Fundamental frequencies

In this section, we derive the expressions for fundamental frequencies ($\nu_s$, $\nu_r$, $\nu_\theta$) in terms of the integrals derived analytically in section 2.2. We take a long time average of equation (2d) on both the sides so that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{dr}{\sqrt{R}} = \lim_{T \to \infty} \frac{1}{T} \int_0^\theta \frac{d\theta}{\sqrt{G}}.$$  \hfill (31a)

As $T \to \infty$, there exists a large integer solutions, which can be found with arbitrary precision, so that $N_r t_r = N_\theta t_\theta = T$, where $N_r$ and $N_\theta$ are the number of radial and vertical oscillations; hence equation (31a) reduces to

$$\lim_{N_r \to \infty} \frac{2N_r}{N_r \cdot t_r} \int_0^r \frac{dr}{\sqrt{R}} = \lim_{N_\theta \to \infty} \frac{2N_\theta}{N_\theta \cdot t_\theta} \int_\theta^\pi \frac{d\theta}{\sqrt{G}}.$$ \hfill (31b)

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Table 3. This following table summarizes the values of conic parameters \((e, \mu)\) chosen in the listed orbit simulations to study eccentric, homoclinic and spherical orbits for different \((a, Q)\) combinations for the prograde cases constructed using equation (10).

| Type of orbit   | Orbit # | Inverse latus-rectum of the orbit | Eccentricity of the orbit \(e\) | Spin of the black hole \(a\) | Carter’s constant \(Q\) |
|-----------------|---------|-----------------------------------|---------------------------------|----------------------------|--------------------------|
| Eccentric orbits| E1      | 0.1                               | 0.6                             | 0.2                        | 8                        |
|                 | E2      | 0.1                               | 0.6                             | 0.8                        | 8                        |
| Homoclinic orbits| H1    | 0.153                             | 0.6                             | 0.2                        | 8                        |
|                 | H2      | 0.153                             | 0.6                             | 0.2                        | 3                        |
| Spherical orbits| S1     | 0.222                             | 0                               | 0.5                        | 3                        |
| Zoom-whirl      | Z1      | 0.226                             | 0.5                             | 0.8                        | 5                        |
|                 | Z2      | 0.142                             | 0.8                             | 0.2                        | 5                        |

Figure 4. The figure shows prograde eccentric bound orbits (a) E1, and (b) E2 in the table 3, for various combinations of \((e, \mu, a, Q)\) satisfying equation (11c) and also presents the evolution of corresponding \(\theta, \phi\) and \(r\) with coordinate time, \(t\).

Figure 5. The figure shows the prograde homoclinic orbits (a) H1, and (b) H2 in the table 3, for various combinations of \((e, \mu, a, Q)\) and also presents the evolution of corresponding \(\theta, \phi\) and \(r\) with coordinate time, \(t\).
where \( r_p \) and \( r_a \) are the periastron and apastron of the orbit and \( \theta_- \) corresponds to the starting point of the vertical oscillation, and where

\[
\theta_- = \arccos(z_-) \quad \text{and} \quad \pi - \theta_- = -\arccos(z_-),
\]

which results in \( \beta \) varying from \( -\pi/2 \) to \( \pi/2 \). Hence, using equations (6h) and (9c) we find

\[
\nu = \frac{\int_{r_p}^{r_a} \frac{dE}{\sqrt{R}}}{\int_{\theta_-}^{\pi - \theta_-} \frac{dE}{\sqrt{\Theta}}} = \frac{a \sqrt{1 - \dot{E}^2 z_+ I_8 \left( \frac{\pi}{2}, e, \mu, a, Q \right)}}{2 \cdot F \left( \frac{\pi}{2}, \frac{z}{2} \right)}.
\]

The similar expression can also be derived using formulae given in [16]. Again, we take a long time average of equation (2e), so that

\[
\lim_{T \to \infty} \frac{T - t_0}{T} = \lim_{T \to \infty} \frac{1}{T} \left[ \frac{1}{2} \int_{t_0}^{t} \frac{1}{\Delta \sqrt{R}} \frac{\partial R}{\partial E} \, dt' + \frac{1}{2} \int_{\theta_0}^{\theta} \frac{1}{\Delta \sqrt{\Theta}} \frac{\partial \Theta}{\partial E} \, d\theta' \right].
\]

where using the same argument, again, of large possible integer solutions, so that \( N \dot{t}_r = N_\theta t_\theta = T \) to find

\[
1 = \frac{2N_r \int_{r_p}^{r_a} \frac{1}{\Delta \sqrt{R}} \frac{\partial R}{\partial E} \, dr'}{2N \dot{t}_r} + \frac{2N_\theta \int_{\theta_-}^{\pi - \theta_-} \frac{1}{\Delta \sqrt{\Theta}} \frac{\partial \Theta}{\partial E} \, d\theta'}{2N_\theta t_\theta} = \nu_r I_2 + \nu_\theta H_2,
\]

which gives

\[
\nu_r (e, \mu, a, Q) = \frac{1}{I_2 \left( \frac{\pi}{2}, e, \mu, a, Q \right) + \frac{\nu_\theta}{\nu_r} H_2 \left( -\frac{\pi}{2}, \frac{\pi}{2}, e, \mu, a, Q \right)}.
\]
\[ \nu \left( e, \mu, a, Q \right) = \frac{1}{\nu} I_2 \left( \frac{\pi}{2}, e, \mu, a, Q \right) + H_2 \left( -\frac{\pi}{2}, \frac{\pi}{2}, e, \mu, a, Q \right). \] (32d)

The limits of integral \( I_2 \) are \( \alpha = \{0, \pi/2\} \), and that of \( H_2 \) are \( \beta_0 = \{\pi/2, -\pi/2\} \). The substitution of \( H_2 \left( -\frac{\pi}{2}, \frac{\pi}{2}, e, \mu, a, Q \right) \) and \( \frac{\nu}{\nu} \), from equations (9b) and (31c) in the above equations give

\[ \nu_0 \left( e, \mu, a, Q \right) = \frac{1}{\nu} I_2 \left( \frac{\pi}{2}, e, \mu, a, Q \right) + H_2 \left( -\frac{\pi}{2}, \frac{\pi}{2}, e, \mu, a, Q \right). \] (33a)

Figure 8. The shaded region depicts the bound orbit region defined by equation (11c) in the \((e, \mu)\) plane for \( a = 0.5 \) and \( Q = 5 \). The black curve represents the homoclinic orbits where the end points depict \( e = 0 \) and \( e = 1 \) homoclinic orbits corresponding to the ISSO and MBSO respectively. The red curve represents \( e = 0.5 \) and we study orbits with different \( \mu \) values as depicted by the dots on this curve.

Figure 9. The figure shows the eccentric trajectories on the red curve of figure 8 \((e = 0.5, a = 0.5, Q = 5)\) for (a) \( \mu = 0.05 \), and (b) \( \mu = 0.1 \).
Similarly, taking the long time average of equation (2b) and the substitution of $H_1$ and $H_2$ from equations (9a) and (9b) yields

$$\nu_\theta (e, \mu, a, Q) = \frac{a \sqrt{1 - E^2}}{\pi} \left\{ \frac{|I_2 \left( \frac{\pi}{2}, e, \mu, a, Q \right) + 2a^2z^2^1 E\xi \left( \frac{\pi}{2}, e, \mu, a, Q \right)| F \left( \frac{\pi}{2}, \frac{z^2_1}{z^2} \right)}{2} - 2a^2z^2_1 E\xi \left( \frac{\pi}{2}, e, \mu, a, Q \right) K \left( \frac{\pi}{2}, \frac{z^2_1}{z^2} \right)} \right\}, \quad (33b)$$

Similarly, taking the long time average of equation (2b) and the substitution of $H_1$ and $H_2$ from equations (9a) and (9b) yields

$$\nu_\phi (e, \mu, a, Q) = \frac{a \sqrt{1 - E^2}}{\pi} \left\{ \frac{|-I_1 \left( \frac{\pi}{2}, e, \mu, a, Q \right) - 2LI_8 \left( \frac{\pi}{2}, e, \mu, a, Q \right)| F \left( \frac{\pi}{2}, \frac{z^2_1}{z^2} \right)}{2} - 2a^2z^2_1 E\xi \left( \frac{\pi}{2}, e, \mu, a, Q \right) K \left( \frac{\pi}{2}, \frac{z^2_1}{z^2} \right)} \right\}, \quad (33c)$$

where $I_1$, $I_2$, and $I_8$ are given by equations (6a)–(6g) and (6h). Hence, the fundamental frequencies are explicit functions of input parameters $(e, \mu, a, Q)$, which can be chosen using the bound orbit conditions presented in section 2.3. These frequency formulae also match with the quadrature formulae derived in [11]; but here we have explicitly solved the integrals $I_1$, $I_2$, and $I_8$ in section 2.2.

6. Consistency check with previous results

In order to verify our results, we have reduced our formulae for the non-equatorial separatrix trajectories, equation (30), to the case of equatorial separatrix orbits and found that they are consistent with earlier results derived in [22]. We also found that the frequency ratio, $\nu_\phi/\nu_\theta$, from equations (33b) and (33c), reduce to the case of maximally rotating black hole, $a = 1$, for spherical orbits previously derived in [9]. See appendix G of the arxiv version of this paper for these derivations [23].
7. Applications

There are various important applications of our analytic solutions of the general non-equatorial trajectories and the fundamental frequencies for astrophysical studies as discussed below:

(i) **Gravitational waves**: one of the crucial applications of our trajectory solution is the case of gravitational waves from the extreme-mass ratio inspirals (EMRIs). Our analytic formulae are directly applicable for the frequency domain calculation of the gravitational waves using the Teukolsky formalism, [34] or Kludge scheme [35], and the orbits can be computed more accurately than the numerical calculations [36]. Also, the homoclinic orbits, which are the separatrix between plunge and bound geodesics [22, 28], have their importance to study the zoom-whirl behavior of inspirals near separatrix [21, 37]. In this paper, we provide the analytic formulae for eccentricity and inverse-latus rectum, \((e, \mu)\), for non-equatorial separatrix orbits which are crucial for the selection of these orbits for the study of gravitational waveforms in the Kerr geometry.

(ii) **Relativistic precession**: the exact analytic formula for azimuthal angle, \(\phi - \phi_0\), is useful to find the precession of the orbits in the astrophysical systems like planets, black hole, and double pulsar systems. PSR J0737-3039 is one example of a double pulsar system having two pulsars, PSR J0737-3039A and PSR J0737-3039B having 23 ms [38] and 2.8 s [39] period respectively, which is useful to study the relativistic precession phenomenon valid in a strong gravitational field. The periastron advance was estimated in this source using the first PK parameter, \(\dot{\omega}\) [40]. Our exact analytic results can be used to make a more accurate estimation of the relativistic advance of the periastron in pulsar systems where one component is having a major spin contribution.

(iii) **Quasi-periodic oscillations (QPOs)**: QPOs are broad peaks seen in the Fourier power spectrum of the Neutron star x-ray binaries (NSXRB) and black hole x-ray binaries (BHXRB). The relativistic precession (RP) model was introduced [41] to explain the kHz QPOs in NSXRB and later applied to BHXRB [42]. The RP model can be used to calculate the black hole parameters assuming a circular or eccentric orbit is giving rise to a pair of observed high-frequency QPOs and a singular and nearly simultaneous corresponding Type-C QPO, [43, 44], where our exact formulae for the fundamental frequencies are applicable.

(iv) **Gyroscope precession**: the calculation of precession of spin of a test gyroscope is another application for the test of general relativity. In previous studies, approximate expressions were used for the fundamental frequencies as a series expansion in terms of eccentricity up to order \(e^2\) around a Kerr black hole for the stable bound orbits in the equatorial plane [18]. Our exact analytic results are useful to estimate more accurate results which are useful to explain the reported results of geodetic drift rate and frame-dragging drift rate by the gravity probe B (GP-B) [45].

(v) **Phase space study**: study of dynamics of Kerr orbits by Poincaré maps is also well discussed [22]. Our closed-form solutions are directly applicable to the study of the extreme chaotic behavior of orbits like Zoom-whirl orbits, which are extreme forms of perihelion precession [21], and their phase space structures.

8. Summary

The summary of this paper is given below:

(i) We first translate the parameters \((E, L, a, Q)\) to \((e, \mu, a, Q)\) using the translation formulae, equation (5) to completely describe the trajectory solution in the \((e, \mu, a, Q)\) space.
We then select the allowed bound orbit by choosing the parameters $(e, \mu, a, Q)$ using the bound orbit conditions, equation (12).

(ii) We have derived the closed-form analytic solutions of the general eccentric trajectory in the Kerr geometry as function of elliptic integrals, $\{\phi (r, \theta), t (r, \theta), r (\theta)\}$, equations (10a)–(10c). These trajectories around a Kerr black hole were previously derived in terms of Mino time [16], $\lambda$, subject to the initial conditions on $dr(0)/d\lambda$ and $d\theta(0)/d\lambda$. The application of our trajectory solution to the various possible studies is numerically faster and does not require any selection of initial conditions. We choose the starting point of the trajectory as the apastron of the orbit, $r_a$ or $\alpha = 0$, and the initial polar angle, $\theta_0$ or $\beta_0$, is an extra parameter which can be arbitrarily chosen between maximum and minimum allowed $\theta$ range for a given $Q$. The input variables for plotting the trajectories are $\alpha$ and $\beta$ which define the range of $r$ and $\theta$ for a fixed combination of $(e, \mu, a, Q)$. These results are summarized in table 2 of the arxiv version of this paper [23].

(iii) We have derived the formulae for $E$ and $L$ for the spherical orbits as functions of radius $r_s$, $a$, and $Q$, given by equation (16).

(iv) We have derived the equations for ISSO, MBSO, and spherical light radius, equations (19)–(21). The light radius derived is the same as that for the equatorial case.

(v) We discussed the non-equatorial separatrix orbits, which asymptote to the unstable spherical radius sharing the same $E$ and $L$ values with the eccentric bound orbit. The radius of this unstable spherical radius for the separatrix orbit exists between MBSO and ISSO. We write the exact forms for the eccentricity and inverse-latus rectum $(e_s, \mu_s)$ for the non-equatorial separatrix orbits as functions of $r_s$, $a$, and $Q$, given by equation (25).

(vi) We use our general trajectory solutions to derive the equations of motions for non-equatorial separatrix orbits, given by equation (30), and find that the radial part of the solutions can be completely reduced to the form containing only trigonometric and logarithmic functions. We also show the reduction of these trajectories to the equatorial case which is also a new and useful form and match the solutions with the previously known result derived in [20]. Separatrix trajectories are essential in the study of gravitational waves from EMRIs, where our analytic solutions are directly applicable. These results are summarized in table 2.

(vii) We discuss families of allowed bound orbit trajectories like non-equatorial eccentric, non-equatorial separatrix, zoom whirl, and spherical orbits around a rotating black hole using our analytic solution for the trajectories. Homoclinic trajectories have their applications in the gravitational wave astronomy as these trajectories are the boundaries between bound eccentric and plunge orbits. Separatrix/homoclinic orbits were studied for the equatorial case in [20, 28]. In this paper, we describe non-equatorial homoclinic and zoom-whirl trajectories, which is the more generalized case for the application to gravitational astronomy.

(viii) We derived the closed-form expressions for the fundamental frequencies in terms of elliptic integrals, equations (33a)–(33c), using the long time average method and without using Mino time, $\lambda$. We show that these expressions match with those derived in [11] using Hamilton–Jacobi formulation, which were left in the quadrature form, and we have obtained a closed form using elliptic integrals. These expressions are summarized in table 5 of the arxiv version of this paper [23]. We present the consistency of our trajectory solution by reducing it to the equatorial separatrix case and also show that the frequency ratio, $\nu_\phi/\nu_\theta$, matches with the standard expression derived [9] for the spherical orbits.

The results include novel aspects given in (i) and (iii)–(vii), listed above, and alternate new forms of the known formulae, given in the points (ii) and (viii) above. The equations and tables providing
these results are indicated in the points above. The appendices for the derivations of the results presented in sections 2, 3 and 6 are given in the arxiv version of this paper [23].

9. Discussion and conclusions

There are several notable results in the vast literature discussing various aspects of dynamics in Kerr geometry such as the quadrature formulae for the trajectories [6, 7], circular orbit formulae [8], conditions for spherical orbits [9], expressions in terms of quadratures for the oscillation frequencies [11], formulae for trajectories in terms of quadratures for spherical polar motion [13], trajectories for non-spherical polar motion [12], and expressions for the trajectories and oscillation frequencies [16] in terms of Mino time [14]. Besides these key results there are other useful expressions reported for example on separatrix orbits [20], and on eccentric equatorial bound orbits [6, 18].

We discuss below the utility of the results in our paper:

The recipe for calculating frequencies and trajectories by [16] is as follows: the operative equations are \{φ(λ), t(λ), r(λ)\}, equations (6), (23)–(33), and (35)–(45) which require linear combinations of many other equations. The analogy to \(r(\chi)\) or \(\chi(r)\) is \(r(\lambda)\) or \(\lambda(r)\) (equations (26) and (27)); the latter is non-trivial, whereas the former is simple. Given \(\lambda, (\phi, t)\) are calculated subsequently inverting linear combinations of many other elliptic integrals. We have numerically matched our frequency formulae with that given in [16] and we find that there is a minor typo in their expression of \(\Gamma\) below equation (19) in section 3.3, where there is a factor of \(E/2\) missing in the term \((r_1 − r_3)(r_2 − r_4)E(k_e)\). However, the correct factor has been applied to calculate the numbers in their tables 1–3 given in [16]. By using the set of equations in [16] and comparing with our expressions, it is found that our calculation is easier to implement and numerically faster by \(\sim 20\), in the equatorial case, for example.

The novel results listed in (i) and (iii)–(vii) of the summary: translation conditions of \(\{E, L\} \to \{e, \mu\}\), bound orbit conditions, \(\{E(r_s), L(r_s)\}\), ISSO, MBSO, and light radius formulae, besides new form for equatorial trajectories, are useful for various applications and simulations related to astrophysical scenarios involving relativistic precession like QPOs and accretion disks. We have also derived the locus of the \(Q \neq 0\) separatrix curve in the \(e − \mu\) plane besides providing the form of the trajectories. Using this, further studies can be carried for chaotic motion and study of gravitational waves from zoom-whirl orbits which can be set-up by locating them near the separatrix locus, in the same spirit, as was done for the equatorial case [20, 21].

The analytic results presented in this paper have direct applications in astrophysics for example, the study of non-equatorial separatrix orbits which has not been discussed before. They also help in understanding the highly eccentric behaviour of trajectories seen in numerical simulations [30] just before plunging onto the massive black hole in the case of EMRIs which is possibly related to the eccentric and inclined homoclinic orbits, besides relativistic precession in other astrophysical systems like binary pulsars and black holes, spin precession of gyroscopes around rotating black holes for the test of general relativity, and the study of chaotic orbits in the phase space.

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