Recent developments of analysis for hydrodynamic flow of nematic liquid crystals

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The study of hydrodynamics of liquid crystals leads to many fascinating mathematical problems, which has prompted various interesting works recently. This article reviews the static Oseen–Frank theory and surveys some recent progress on the existence, regularity, uniqueness and large time asymptotic of the hydrodynamic flow of nematic liquid crystals. We will also propose a few interesting questions for future investigations.

1. Static theory

Liquid crystal is an intermediate phase between the crystalline solid state and the isotropic fluid state. It possesses no or partial positional order but displays an orientational order. There are roughly three types of liquid crystals commonly referred to in the literature: nematic, cholesteric and smectic. The nematic liquid crystals are composed of rod-like molecules with the long axes of neighbouring molecules approximately aligned to one another. The simplest continuum model to study the equilibrium phenomena for nematic liquid crystals is the Oseen–Frank theory, proposed by Oseen [1] in 1933 and Frank [2] in 1958. A unifying programme describing general liquid crystal materials is the Landau–de Gennes theory [3,4], which involves the orientational order parameter tensors.

In the absence of surface energy terms and applied fields, the Oseen–Frank theory seeks unit-vector fields $d : \Omega \subset \mathbb{R}^3 \to S^2 = \{ y \in \mathbb{R}^3 : |y| = 1 \}$, representing mean...
orientations of a molecule’s optical axis, that minimize the Oseen–Frank bulk energy functional $\mathcal{E}_{\text{OF}}(d) = \int_{\Omega} \mathcal{W}(d, \nabla d) \, dx$, where

$$2\mathcal{W}(d, \nabla d) = k_1(\text{div} \, d)^2 + k_2(\nabla d \cdot \text{curl} \, d)^2 + k_3|d \times \text{curl} \, d|^2$$

$$+ (k_2 + k_4)[\text{tr}(\nabla d)^2 - (\text{div} \, d)^2],$$

(1.1)

where $k_1, k_2, k_3 > 0$ are splaying, twisting and bending constants, $k_2 \geq |k_4|$, and $2k_1 \geq k_2 + k_4$. Using the null-Lagrangian property of the last term in $\mathcal{W}(d, \nabla d)$, it can be shown that, under strong anchoring condition (or Dirichlet boundary condition) $d|_{\partial \Omega} = d_0 \in H^1(\Omega, S^2)$, there always exists a minimizer $d \in H^1(\Omega, S^2)$ of the Oseen–Frank energy functional $\mathcal{E}_{\text{OF}}$. Furthermore, such a $d$ solves the Euler–Lagrange equation

$$\frac{\delta \mathcal{W}}{\delta d} \times d := -\text{div} \left( \frac{\partial \mathcal{W}}{\partial \nabla d}(d, \nabla d) \right) + \frac{\partial \mathcal{W}}{\partial d}(d, \nabla d) \times d = 0.$$  

(1.2)

The basic question for such a minimizer $d$ concerns both the regularity property and the defect structure. A fundamental result by Hardt et al. [5] (see also [6]) asserts the following.

**Theorem 1.1.** If $d \in H^1(\Omega, S^2)$ is a minimizer of the Oseen–Frank energy $\mathcal{E}_{\text{OF}}$, then $d$ is analytic on $\Omega \setminus \text{sing}(d)$ for some closed subset $\text{sing}(d) \subseteq \Omega$, whose Hausdorff dimension is smaller than one. If, in addition, $d_0 \in C^{k,\alpha}(\Omega, S^2)$ for some $k \in \mathbb{N}_+$, then there exists a closed subset $\Sigma_1 \subseteq \partial \Omega$, with $\mathcal{H}^1(\Sigma_1) = 0$, such that $d \in C^{k,\alpha}(\Omega \setminus (\text{sing}(d) \cup \Sigma_1), S^2)$.

Concerning the interior singular set $\text{sing}(d)$ in theorem 1.1, an outstanding open question is the following.

**Question 1.2.** How large is the singular set $\text{sing}(d)$ for minimizers of Oseen–Frank energy $\mathcal{E}_{\text{OF}}$? Or equivalently, what is the optimal estimate of size of $\text{sing}(d)$?

Note that when $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$, the Oseen–Frank energy density function $\mathcal{W}(d, \nabla d) = \frac{1}{2}|\nabla d|^2$ is the Dirichlet energy density. Hence, the problem reduces to minimizing harmonic maps into $S^2$ and (1.2) becomes the equation of harmonic maps

$$\frac{\delta \mathcal{W}}{\delta d} := \Delta d + |\nabla d|^2 d = 0.$$  

(1.3)

Harmonic maps have been extensively studied in the past several decades, which are much better understood (see [7–11]). In particular, it is well known that in dimension $n = 3$, the singular set of minimizing harmonic maps is at most a non-empty set of finitely many points (see [7]). A typical example is the following hedgehog singularity.

**Example 1.3 ([10,12]).** $x/|x| : \mathbb{R}^3 \to S^2$ is a minimizing harmonic map.

Motivated by the isolated point singularity for minimizing harmonic maps, it is natural to ask the following.

**Question 1.4.** Is the set $\text{sing}(d)$ for a minimizer of Oseen–Frank energy $\mathcal{E}_{\text{OF}}$ in theorem 1.1 a set of finite points?

The main difficulty in investigating the size of singular set of minimizers $d$ to the Oseen–Frank energy functional $\mathcal{E}_{\text{OF}}$ is that it is an open question whether an energy monotonicity inequality, similar to that of harmonic maps [7], holds for $d$. In fact, the following seemingly simple question remains open.

**Question 1.5.** Assume $k_4 = -k_2$ and $\max_{1 \leq i \leq 3} |k_i - 1| \ll 1$. Suppose $d \in H^1(\Omega, S^2)$ is a minimizer of the Oseen–Frank energy $\mathcal{E}_{\text{OF}}$. Is the following monotonicity inequality true?

$$r^{-1} \int_{B_r} |\nabla d|^2 + \int_{\mathbb{R}^3 \setminus B_r} |x|^{-1} \left| \frac{\partial d}{\partial |x|} \right|^2 \leq (1 + \omega(R - r))r^{-1} \int_{B_r} |\nabla d|^2$$

(1.4)

for $0 < r \leq R$, with $\lim_{r \to 0} \omega(r) = 0$. 

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Note that when \(k_4 = -k_2\) and \(\max_{1 \leq i \leq 3} |k_i - 1|\) is sufficiently close to 0, it has been shown by Hardt et al. [6] that \(\text{sing}(d)\) for any minimizer \(d\) of \(\mathcal{E}_{OF}\) has locally finite points. Readers can consult the survey article by Hardt & Kinderlehrer [13] for some earlier developments of Oseen–Frank theory. The book by Virga [14] and the lecture note by Brezis [15] provide detailed exposition of Oseen–Frank theory. Partly motivated by these mathematical studies and physical experiments where line and surface defects of nematic liquid crystals have been observed (see [16]), Ericksen [17] proposed the so-called Ericksen model of liquid crystals with variable degree of orientation. It assumes that the bulk energy density depends on both the orientation vector \(d\), with \(|d| = 1\), and orientational order \(s \in \{-\frac{1}{2}, 1\}\), which is given by

\[
w(s, d, \nabla s, \nabla d) = w_0(s) + w_2(s, d, \nabla s, \nabla d),
\]

where \(w_0(s)\) is roughly a W-shape potential function with \(w_0(-\frac{1}{2}) = w_0(1) = +\infty\). The term \(w_2\) is given by

\[
2w_2(s, \nabla s, d, \nabla d) = 2\mathcal{W}(d, \nabla d) + K_5|\nabla s - (\nabla s \cdot d) d - \alpha(\nabla d)d|^2
+ K_6(\nabla s \cdot d - \beta \text{ div } d)^2,
\]

where \(\mathcal{W}(d, \nabla d)\) is given by (1.1), and \(K_5 > 0, K_6 > 0, \alpha, \beta\) are coefficients that may depend on \(s\). We can view the Oseen–Frank model as a special case of Ericksen’s model by imposing the constraint on the orientational order \(s = s^*\), with \(w_0(s^*) = \min\{w_0(s) : s \in [-\frac{1}{2}, 1]\}\). The simplest form of Ericksen’s bulk energy density is

\[
w(s, d) = k|\nabla s|^2 + s^2|\nabla d|^2 + w_0(s).
\]

One can translate the problem of minimizers \((s, d)\) of Ericksen’s energy functional (1.6) into the problem of minimizing harmonic maps \((s, u)\), with potentials \(w_0\),

\[
\min \int_{\Omega} (|k| - 1)|\nabla s|^2 + |\nabla u|^2 + w_0(s)) \, dx
\]

subject to the constraint \(|s| = |u|\) and \((s, u) = (s_0, u_0)\) on \(\partial \Omega\). Then the problem (1.7) is essentially a minimizing harmonic map into a circular cone \(C_k = \{(s, u) \in \mathbb{R} \times \mathbb{R}^3 : |s| = \sqrt{k - 1}|u|\} \subset \mathbb{R}^4\) for \(k > 1\) or \(C_k = \{(s, u) \in \mathbb{R} \times \mathbb{R}^3 : |s| = \sqrt{1 - k}|u|\} \subset \mathbb{R}^{3,1}\)–the Minkowski space for \(0 < k < 1\), Lin has proved in [18] (see also [19–23] for further related works).

**Theorem 1.6.** For any \((s_0, u_0) \in H^{1/2}(\partial \Omega, C_k)\), there exists a minimizer \((s, u)\) of (1.6) such that the following properties hold:

1. If \(k > 1\), then \((s, u) \in C^\alpha(\Omega)\) for some \(0 < \alpha < 1\), and \((s, u)\) is analytic away from \(s^{-1}[0]\). If \(s \not\equiv 0\), then \(s^{-1}[0]\) has Hausdorff dimension at most 1.
2. If \(0 < k < 1\), then \((s, u) \in C^{0,1}(\Omega)\), and \((s, u)\) is analytic away from \(s^{-1}[0]\). If \(s \not\equiv 0\), then \(s^{-1}[0]\) has Hausdorff dimension at most 2.

As a liquid crystal molecule is indistinguishable between its head and tail, i.e. \(d \approx -d\), its odd order moments must vanish and the second-order moment is believed to be most important to the energy. In general, one can use a class of \(3 \times 3\) symmetric, traceless matrices \(Q\) to describe the second-order moments

\[
\mathcal{M} = \left\{ Q \in \mathbb{R}^{3 \times 3} : Q = \int_{S^2} \left( p \otimes p - \frac{1}{3}I_3 \right) \, d\mu(p), \, \mu \in \mathcal{P}(S^2) \right\},
\]

where \(\mathcal{P}(S^2)\) denotes the space of probability measures on \(S^2\). It is clear that for any order parameter tensor \(Q \in \mathcal{M}\), its three eigenvalues \(\lambda_i(Q) \in [-\frac{1}{2}, \frac{2}{3}]\) \((1 \leq i \leq 3)\) and \(\sum_{i=1}^{3} \lambda_i(Q) = 0\). If \(Q\) has two equal eigenvalues, it becomes a uniaxial nematic or cholesteric phase so that it
can be expressed as $Q = s(d \otimes d - \frac{1}{3} I_3)$, with $|d| = 1$ and $s \in [-\frac{1}{2}, 1]$. If $Q \in \mathcal{M}$ has three distinct eigenvalues, then the liquid material is biaxial. The Landau–de Gennes bulk energy for nematic liquid crystal material was derived by de Gennes [3,4]. A simplified form is given by

$$E_{DG}(Q) = \frac{1}{2} \left( L_1 |\nabla Q|^2 + L_2 (\text{div} Q)^2 + L_3 \text{tr}(\nabla Q)^2 + L_4 \nabla Q \otimes \nabla Q : Q \right)$$

$$+ \left[ \frac{1}{2} A \text{tr}(Q^2) - \frac{1}{3} B \text{tr}(Q^3) + \frac{1}{4} C \text{tr}(Q^2)^2 \right].$$

(1.8)

It is well known that if $Q$ is uniaxial, then the Landau–de Gennes energy can be essentially reduced to the Ossen–Frank energy or the Ericksen energy. We mention that it is not hard to show that there exists at least one minimizer $Q \in \mathcal{M}$ of the Landau–de Gennes energy functional $E_{DG}$, provided we remove the constraint on the eigenvalues $-\frac{1}{3} \leq \lambda_i(Q) \leq \frac{2}{3}, 1 \leq i \leq 3$. Moreover, the defect set of biaxial minimizers $Q \in \mathcal{M}$ of the Landau–de Gennes energy $E_{DG}$ is contained in the set on which at least two of its eigenvalues are equal. The study of such defect sets is extremely challenging and remains largely open. Here, we mention that there have been several recent interesting works on the static case of the Landau–de Gennes model by Ball and collaborators, and others (see [24–32]).

2. Hydrodynamic theory

(a) Ericksen–Leslie system modelling nematic liquid crystal flows

The Ericksen–Leslie system modelling the hydrodynamics of nematic liquid crystals, reducing to the Oseen–Frank theory in the static case, was proposed by Ericksen and Leslie during the period between 1958 and 1968 [33,34]. It is a macroscopic continuum description of the time evolution of materials under the influence of both the flow velocity field $u(x,t)$ and the macroscopic description of the microscopic orientation configuration $d(x,t)$ of rod-like liquid crystals, i.e. $d(x,t)$ is a unit vector in $\mathbb{R}^3$. The full Ericksen–Leslie system can be described as follows. The hydrodynamic equation takes the form

$$\partial_t u + u \cdot \nabla u + \nabla P = \nabla \cdot \sigma,$$

(2.1)

where the stress $\sigma$ is modelled by the phenomenological constitutive relation

$$\sigma = \sigma^L(u,d) + \sigma^E(d).$$

Here, $\sigma^L(u,d)$ is the viscous (Leslie) stress given by

$$\sigma^L(u,d) = \mu_1 (d \otimes d : A) d \otimes d + \mu_2 d \otimes N + \mu_3 N \otimes d + \mu_4 A$$

$$+ \mu_5 d \otimes (A \cdot d) + \mu_6 (A \cdot d) \otimes d,$$

(2.2)

with six Leslie coefficients $\mu_1, \ldots, \mu_6$, and

$$A = \frac{1}{2} (\nabla u + (\nabla u)^T), \quad \omega = \frac{1}{2} (\nabla u - (\nabla u)^T) \quad \text{and} \quad N = \partial_t d + u \cdot \nabla d - \omega \cdot d$$

representing the velocity gradient tensor, the vorticity field and rigid rotation part of the changing rate of the director by fluid, respectively. Assuming the fluid is incompressible, we have

$$\nabla \cdot u = 0.$$  

(2.3)

While $\sigma^E(d)$ is the elastic (Ericksen) stress

$$\sigma^E(d) = -\frac{\partial W}{\partial (\nabla d)} \cdot (\nabla d)^T,$$
where \( \mathcal{W} = \mathcal{W}(d, \nabla d) \) is the Oseen–Frank energy density given by (1.1). The dynamic equation for the director field takes the form
\[
d \times \left( -\frac{\delta \mathcal{W}}{\delta d} - \lambda_1 N - \lambda_2 A \cdot d \right) = 0,
\]
(2.4)
where \( \delta \mathcal{W} / \delta d \) is the Euler–Lagrangian operator given by (1.2). We also have the compatibility condition
\[
\lambda_1 = \mu_2 - \mu_3 \quad \text{and} \quad \lambda_2 = \mu_5 - \mu_6,
\]
(2.5)
and Parodi’s condition (see [35])
\[
\mu_2 + \mu_3 = \mu_6 - \mu_5.
\]
(2.6)

(b) Energetic variational approach

The derivation of the Ericksen–Leslie system in [33,34] is based on the conservation of mass, the incompressibility of fluid and conservation of both linear and angular momentums. Here, we will sketch a modern approach called the energy variational approach, prompted by Liu and collaborators (see [36,37]), which is based on both the least action principle and the maximal dissipation principle. In the context of hydrodynamics of nematic liquid crystals, the basic variable is the flow map \( x = x(X, t): \Omega_0^X \rightarrow \Omega_t^X \), with \( X \) the Lagrangian (or material) coordinate and \( x \) the Eulerian (or reference) coordinate. For a given velocity field \( u(x, t) \), the flow map \( x(X, t) \) solves
\[
x_t(x(X, t), t) = u(x(X, t), t) \quad \text{and} \quad x(X, 0) = X \in \Omega_0^X.
\]
The deformation tensor \( F \) of the flow map \( x(X, t) \) is \( F = (\partial x_i / \partial X_j)_{1 \leq i, j \leq 3} \). By the chain rule, \( F \) satisfies the following transport equation:
\[
\partial_t F + u \cdot \nabla F = \nabla u F.
\]
The kinematic transport of the director field \( d \) represents the molecules moving in the flow [37,38]. It can be expressed by
\[
d(x(X, t), t) = Ed_0(X),
\]
(2.7)
where the deformation tensor \( E \) carries information of microstructures and configurations. It satisfies the following transport equation:
\[
\partial_t E + u \cdot \nabla E = \omega E + (2\alpha - 1)AE,
\]
(2.8)
where \( 2\alpha - 1 = (r^2 - 1)/(r^2 + 1) \in [-1, 1] \) \((r \in \mathbb{R})\) is related to the aspect ratio of the ellipsoid-shaped liquid crystal molecules (see [39]). Note that if there is no internal damping, then the transport equation of \( d \) from (2.7) and (2.8) is given by
\[
\partial_t d + v \cdot \nabla d - \omega \cdot d - (2\alpha - 1)Ad = 0.
\]
(2.9)

The total energy of the liquid crystal system is the sum of kinetic energy and internal elastic energy and is given by
\[
\mathcal{E} = \mathcal{E}^\text{kinetic} + \mathcal{E}^\text{int}, \quad \mathcal{E}^\text{kinetic} = \frac{1}{2} \int |u|^2 \quad \text{and} \quad \mathcal{E}^\text{int} = \mathcal{E}_{OF}(d) = \int \mathcal{W}(d, \nabla d).
\]

The action functional \( \mathcal{A} \) of particle trajectories in terms of \( x(X, t) \) is given by
\[
\mathcal{A}(x) = \int_0^T (\mathcal{E}^\text{kinetic} - \mathcal{E}^\text{int}) dt.
\]
As the fluid is assumed to be incompressible, the flow map \( x(X, t) \) is volume preserving, which is equivalent to \( \nabla \cdot u = 0 \). The least action principle asserts that the action functional \( \mathcal{A} \) minimizes among all volume preserving flow map \( x(X, t) \), i.e. \( \delta_x \mathcal{A}(x) = 0 \) subject to the constraint \( \nabla \cdot u = 0 \). To simplify the derivation, we consider one constant approximation of Oseen–Frank energy density
function and relax the condition \(|d| = 1\) by setting \(\mathcal{W}(d, \nabla d) = \frac{1}{2}|\nabla d|^2 + F(d)\) for \(d : \Omega \rightarrow \mathbb{R}^3\). Then direct calculations (similar to \([37, 7.1]\)) yield that

\[
0 = \delta_x \mathcal{A} = \frac{d}{de} \mathcal{A}(x^e) = \int_0^T \int_{\Omega} \left( \partial_t u + u \cdot \nabla u + \nabla \cdot (\nabla d \otimes \nabla d) - \nabla \cdot \tilde{\sigma} \right) y \, dx \, dt, \tag{2.10}
\]

where \(y = d/de |_{e=0} x^e\) satisfies \(\nabla \cdot y = 0\), and

\[
\tilde{\sigma} = -\frac{1}{2} \left(1 - \frac{\lambda_2}{\lambda_1}\right) (\Delta d - f(d)) \otimes d + \frac{1}{2} \left(1 + \frac{\lambda_2}{\lambda_1}\right) d \otimes (\Delta d - f(d)), \quad f(d) = \nabla F(d).
\]

Hence, it follows from (2.10) that

\[
\partial_t u + u \cdot \nabla u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d) + \nabla \cdot \tilde{\sigma}, \tag{2.11}
\]

where the pressure \(P\) serves as a Lagrangian multiplier for the incompressibility of fluid.

It follows from (2.8) that the total transport equation of \(d\), without internal microscopic damping, is

\[
\partial_t d + u \cdot \nabla d - \omega d + \frac{\lambda_2}{\lambda_1} Ad = 0. \tag{2.12}
\]

If we take the internal microscopic damping into account, then we have

\[
\partial_t d + u \cdot \nabla d - \omega d + \frac{\lambda_2}{\lambda_1} Ad = \frac{1}{\lambda_1} \delta \phi^{\text{int}} \frac{\delta d}{\delta d} = -\frac{1}{\lambda_1} (\Delta d - f(d)). \tag{2.13}
\]

It is known that the dissipation functional \(\mathcal{D}\) to the system (2.21) is

\[
\mathcal{D} = \int_{\Omega} \left[ \mu_1 |d| Ad|^2 + \frac{1}{2} \mu_4 |\nabla u|^2 + (\mu_5 + \mu_6) |Ad|^2 + \lambda_1 |N|^2 \right] + (\lambda_2 - \mu_2 - \mu_3) (N, Ad). \tag{2.14}
\]

By (2.13) and Parodi’s condition (2.6), \(\mathcal{D}\) can be rewritten as

\[
\mathcal{D} = \int_{\Omega} \left[ \mu_1 |d| Ad|^2 + \frac{1}{2} \mu_4 |\nabla u|^2 + \left(\mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1}\right) |Ad|^2 - \lambda_1 \left|N + \frac{\lambda_2}{\lambda_1} Ad\right|^2 \right]. \tag{2.15}
\]

According to the maximal dissipation principle, one has that first-order variation of \(\mathcal{D}\) with respect to the rate function \(u\) must vanish, i.e. \(\delta \mathcal{D}/\delta u = 0\) subject to the constraint \(\nabla \cdot u = 0\). Direct calculations (similar to \([37, 7.2]\)) yield

\[
0 = \frac{\delta \mathcal{D}}{\delta u} = 2 \int_{\Omega} \langle u, \nabla \cdot (\nabla d \otimes \nabla d) - \nabla \cdot \sigma^L(u, d) \rangle,
\]

where \(\sigma^L(u, d)\) is the Leslie stress tensor given by (2.2). Hence, we arrive at

\[
0 = -\nabla P - \nabla \cdot (\nabla d \otimes \nabla d) + \nabla \cdot (\sigma^L(u, d)). \tag{2.16}
\]

Combining the dissipative part derived from the maximal dissipation principle with the conservative part derived from the least action principle, we arrive at equation (2.1). This, together with (2.13) and (2.3), yields the Ginzburg–Landau approximated version of the Ericksen–Leslie system (2.1), (2.3) and (2.4).

(c) Brief descriptions of relationship between various models

There are basically three different kinds of theories to model nematic liquid crystals: Doi–Onsager theory, Landau–de Gennes theory and Ericksen–Leslie theory. The first is the molecular kinetic theory, and the latter two are the continuum theory. Kuzuu & Doi [40] and Weinan & Zhang [41] have formally derived the Ericksen–Leslie equation from the Doi–Onsager equation by taking small Deborah number limit. Wang et al. [42] have rigorously justified this formal derivation before the first singular time of the Ericksen–Leslie equation. Wang et al. [43] have rigorously derived the Ericksen–Leslie equation from the Beris–Edwards model in the Landau–de Gennes
theory. There are several dynamic Q-tensor models describing hydrodynamic flows of nematic liquid crystals, which are derived either by closure approximations (see [44–46]) or by variational methods such as the Beris–Edwards model [47] and the Qian–Sheng model [48]. See [49,50] for the well-posedness of some dynamic Q-tensor models. Furthermore, a systematical approach to derive the continuum theory for nematic liquid crystals from the molecular kinetic theory in both static and dynamic cases was proposed in [46,51]. We also mention the derivation of liquid crystal theory from the statistical point of view by Seguin & Fried [52,53].

(d) Analytic issues of simplified Ericksen–Leslie system

When we consider one constant approximation for the Oseen–Frank energy, i.e. \( \mathcal{W}(d, \nabla d) = \frac{1}{2} |\nabla d|^2 \), the general Ericksen–Leslie system reduces to

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla P &= -\nabla \cdot (\nabla d \otimes \nabla d) + \nabla \cdot (\sigma^+(u, d)), \\
\nabla \cdot u &= 0 \\
\end{aligned}
\]

and

\[
\begin{aligned}
\partial_t d + u \cdot \nabla d - \omega d + \frac{\lambda_2}{\lambda_1} A d &= \frac{1}{|\lambda_1|} (\Delta d + |\nabla d|^2 d) + \frac{\lambda_2}{\lambda_1} (d^T A d) d.
\end{aligned}
\]

Direct calculations, using both (2.5) and (2.6), show that smooth solutions of (2.17), under suitable boundary conditions, satisfy the following energy law (see [54]):

\[
\frac{d}{dt} \int (|u|^2 + |\nabla d|^2) + \left[ \frac{\mu_4}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \right] 
= -2 \left[ \left( \mu_1 - \frac{\lambda_2^2}{\lambda_1} \right) |A : d \otimes d|^2 + \left( \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \right) |A : d|^2 \right].
\]

In particular, we have the following energy dissipation property.

Lemma 2.1. Assume both (2.5) and (2.6). If Leslie’s coefficients satisfy the algebraic condition

\[
\lambda_1 < 0, \quad \mu_1 - \frac{\lambda_2^2}{\lambda_1} \geq 0, \quad \mu_4 > 0 \quad \text{and} \quad \mu_5 + \mu_6 \geq -\frac{\lambda_2^2}{\lambda_1}, \tag{2.19}
\]

then any smooth solution \((u, d)\) of (2.17), under suitable boundary conditions, satisfies the energy dissipation inequality

\[
\frac{d}{dt} \int (|u|^2 + |\nabla d|^2) + \left[ \frac{\mu_4}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \right] \leq 0. \tag{2.20}
\]

A fundamental question related to the general Ericksen–Leslie system (2.17) is as follows.

Question 2.2. For dimensions \( n = 2 \) or \( 3 \), smooth domains \( \Omega \subset \mathbb{R}^n \) (or \( \Omega = \mathbb{R}^n \)), establish

(i) the existence of global Leray–Hopf type weak solutions of (2.17) under generic initial and boundary values \((u_0, d_0) \in H \times H^1(\Omega, \mathbb{S}^2)^2\)

(ii) partial regularity and uniqueness properties for certain restricted classes of weak solutions of (2.17), and

(iii) an optimal global (or local) well-posedness of (2.17) for rough initial data \((u_0, d_0)\), and the long-time behaviour of global solutions of (2.17).

While both parts (i) and (ii) remain open for dimensions \( n = 3 \) in general, the problem has been essentially solved in dimensions \( n = 2 \) recently. Furthermore, there has been some interesting progress towards the problem in dimensions \( n = 3 \).

\[ \text{Throughout this paper, } H = \text{Closure of } \{ v \in C^0_0(\Omega, \mathbb{R}^n) \mid \text{div } v = 0 \} \text{ in } L^2(\Omega, \mathbb{R}^n) \text{ and } J = \text{Closure of } \{ v \in C^0_0(\Omega, \mathbb{R}^n) \mid \text{div } v = 0 \} \text{ in } H_0^1(\Omega, \mathbb{R}^n). \]
In order to rigorously analyse (2.17), Lin [55] first proposed a simplified version of (2.17) that preserves both the nonlinearity and the energy dissipation mechanism, i.e. \((u, d) : \Omega \times (0, +\infty) \rightarrow \mathbb{R}^n \times S^2\) solves
\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P = \mu \Delta u - \nabla \cdot (\nabla d \otimes \nabla d), \\
\nabla \cdot u = 0
\end{cases}
\] (2.21)
and
\[
\begin{cases}
\partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d.
\end{cases}
\]

It is readily seen that under suitable boundary conditions, (2.21) enjoys the following energy dissipation property:
\[
\frac{d}{dt} \int (|u|^2 + |\nabla d|^2) = -2 \int [\mu |\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2] \leq 0. 
\] (2.22)

Because of the supercritical nonlinearity \(\nabla \cdot (\nabla d \otimes \nabla d)\) in (2.21), Lin & Liu [56–58] have studied the Ginzburg–Landau approximation (or orientation of variable degrees in Ericksen’s terminology [59]) of (2.21): for \(\epsilon > 0\), \((u, d) : \Omega \times (0, +\infty) \rightarrow \mathbb{R}^n \times \mathbb{R}^3\) solves
\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P = \mu \Delta u - \nabla \cdot (\nabla d \otimes \nabla d), \\
\nabla \cdot u = 0
\end{cases}
\] (2.23)
and
\[
\begin{cases}
\partial_t d + u \cdot \nabla d = \Delta d + \frac{1}{\epsilon^2} (1 - |d|^2) d.
\end{cases}
\]

Under the initial and boundary conditions
\[
(u, d)|_{t=0} = (u_0, d_0), \quad x \in \Omega \quad \text{and} \quad (u, d)|_{\partial \Omega} = (0, d_0), \quad t > 0,
\] (2.24)
we have the following basic energy law:
\[
\int_{\Omega} \left( |u|^2 + |\nabla d|^2 + \frac{1}{2\epsilon^2} (1 - |d|^2)^2 \right) (t) + 2 \int_0^t \int_{\Omega} (\mu |\nabla u|^2 + |\partial_t d + u \cdot \nabla d|^2) \leq \int_{\Omega} \left( |u_0|^2 + |\nabla d_0|^2 + \frac{1}{2\epsilon^2} (1 - |d_0|^2)^2 \right). 
\] (2.25)

Besides stability and long-time asymptotic, the following existence and partial regularity were proven in [56,57].

**Theorem 2.3.** For any \(\epsilon > 0\) fixed, the following holds:

(a) For \(u_0 \in H^1(\Omega) \cap H^{3/2}(\partial \Omega)\) and \(d_0 \in H^1(\Omega) \cap H^{3/2}(\partial \Omega)\), there exists a global weak solution \((u, d)\) of (2.23) and (2.24) satisfying (2.25) and
\[
\begin{align*}
&u \in L^2(0, T; J) \cap L^\infty(0, T; H), \\
d &\in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), \quad \forall T \in (0, +\infty).
\end{align*}
\]

(b) There exists a unique global classical solution \((u, d)\) to (2.23) and (2.24) provided \((v_0, d_0) \in J \times H^2(\Omega)\) and either \(n = 2\) or \(n = 3\) and \(\mu \geq \mu(u_0, d_0)\).

(c) If \((u, d)\) is a global suitable weak solution of (2.23), then \((u, d) \in C^\infty(\Omega \times (0, +\infty) \setminus \Sigma)\), with \(\mathcal{H}^1(\Sigma) = 0\).

A long outstanding open problem pertaining to the Ginzburg–Landau approximation equation (2.23) and the original nematic liquid crystal flow equation (2.21) is the following.

**Question 2.4.** Whether weak limits \((u, d)\) of weak solutions \((u_\epsilon, d_\epsilon)\) of (2.23) and (2.24) are weak solutions of (2.21) and (2.24), as \(\epsilon \rightarrow 0\).
It follows from (2.25) that there exists \((u, d) \in (L^2(0, T; J) \cap L^\infty(0, T; H)) \times L^\infty(0, T; H^1(\Omega, S^2))\) such that, up to a subsequence,

\[
(u_\epsilon, d_\epsilon) \to (u, d) \quad \text{in} \quad L^2(0, T; L^1(\partial \Omega)) \times L^2(0, T; H^1(\Omega)),
\]

\[
e_\epsilon(d_\epsilon) \, dx \, dt := \left( |\nabla d_\epsilon|^2 + \frac{1}{2\epsilon^2} (1 - |d_\epsilon|^2)^2 \right) \, dx \, dt \to \mu := |\nabla d|^2 \, dx \, dt + \eta,
\]

\[
(\nabla d_\epsilon \circ \nabla d_\epsilon) \, dx \, dt \to (\nabla d \circ \nabla d) \, dx \, dt + \mathcal{M},
\]

for some non-negative Radon measure \(\eta\) and a positive semi-definite \(n \times n\) symmetric matrix-valued function \(\mathcal{M}\) with each entry being a Radon measure, as convergence of Radon measures in \(\Omega \times [0, T]\). Both \(\nu\) and \(\mathcal{M}\) are called defect measures, and \(\mathcal{M} \ll \nu\). It is readily seen that

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla P &= \mu \Delta u - \nabla \cdot (\nabla d \circ \nabla d + \mathcal{M}), \\
\nabla \cdot u &= 0
\end{aligned}
\]

and

\[
\begin{aligned}
\partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 \, d.
\end{aligned}
\]

Thus, we need to study the following difficult question.

**Question 2.5.** Let \(\eta\) and \(\mathcal{M}\) be the defect measures.

1. How large are the supports of \(\nu\) and \(\mathcal{M}\)?
2. Under what conditions does \(\nu\) or \(\mathcal{M}\) vanish?

Before describing a very recent progress towards questions 2.4 and 2.5, we present slightly earlier works on the simplified Ericksen–Leslie system (2.21) in dimension \(n = 2\) by Lin et al. [60], through a different approach that is based on priori estimates under small energy condition. More precisely, we have established in [60] the following.

**Lemma 2.6.** For \(n = 2\) and \(0 < r \leq 1\), let \(P_r = B_r \times [-r^2, 0]\) be the parabolic ball of radius \(r\). There exists \(\epsilon_0 > 0\) such that if \(u \in L_1^\infty L_2^2 \cap L_1^2 H_\mathcal{M}^1(P_r, \mathbb{R}^2), d \in L_1^2 H_\mathcal{N}^2(P_r, S^2)\) and \(P \in L^2(P_r)\) solves (2.21), satisfying

\[
\Phi(u, d, P, r) := \|u\|_{L^2(P_r)} + \|\nabla u\|_{L^2(P_r)} + \|\nabla d\|_{H^1(P_r)} + \|\nabla^2 d\|_{L^2(P_r)} + \|P\|_{L^2(P_r)} \leq \epsilon_0,
\]

then \((u, d) \in C(H_\mathcal{M}^1(P_r, \mathbb{R}^2 \times S^2)), \) and

\[
\|(u, \nabla d)\|_{C(P_r^l)} \leq C(l)\epsilon_0 r^{-l}, \quad l \geq 0.
\]

Approximating initial and boundary data \((u_0^\epsilon, d_0^\epsilon)\) by smooth \((u_0^0, d_0^0),\) and employing lemma 2.6 to the short-time smooth solutions \((u^\epsilon, d^\epsilon)\) of (2.21) under the initial and boundary value \((u_0^0, d_0^0),\) we have established the following result on existence and uniqueness of (2.21) in [60] and [61], respectively.

**Theorem 2.7.** For any \(u_0 \in H\) and \(d_0 \in H^1(\Omega, S^2),\) with \(d_0 \in C^2(\partial \Omega, S^2)\) for some \(\alpha \in (0, 1),\)

1. There exists a global weak solution \(u \in L^\infty(0, +\infty; H) \cap L^2(0, +\infty; J)\) and \(d \in L^\infty(0, +\infty; H^1(\Omega, S^2))\) of (2.21) and (2.24) such that the following properties hold:

   1a. There exists a non-negative integer \(L\) depending only on \(u_0, d_0\) and \(0 < T_1 < \cdots < T_L < +\infty\) such that

   \[
   (u, d) \in C^\infty(\Omega \times [0, +\infty) \setminus \{T_i\}_{i=1}^L) \cap C^2(\Omega \times [(0, +\infty) \setminus \{T_i\}_{i=1}^L])
   \]
(1b) At each time singularity $T_j$, $j = 1, \ldots, L$, it holds

$$\liminf_{t \uparrow T_j} \frac{1}{\varepsilon^2 \ln \left( \frac{1}{\varepsilon} \right)} \int_{\Omega} \left( |u|^2 + |\nabla d|^2 \right)(y, t) \, dy \geq 8\pi, \quad \forall \varepsilon > 0.$$ 

There exist $x^j_m \to x_0^j \in \Omega$ and $t^j_m \uparrow T_j$, $t^j_m \downarrow 0$, and a non-trivial harmonic map $\omega_j : \mathbb{R}^2 \to \mathbb{S}^2$ with finite energy such that

$$(u^j_m, d^j_m) := (u^j_m(x^j_m + t^j_m x^j_m, t^j_m + (t^j_m)^2), \nabla (x^j_m + t^j_m x^j_m + (t^j_m)^2, d(x^j_m + t^j_m x^j_m + (t^j_m)^2)))$$

$$\to (0, \omega_j) \text{ in } C^1_{\text{loc}}(\mathbb{R}^2 \times [-\infty, 0]).$$

(1c) There exist $t_k \uparrow +\infty$ and a harmonic map $d_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^2_{d_0}(\Omega, \mathbb{S}^2)$ such that $u(t_k) \to 0$ in $H^1(\Omega)$, $d(t_k) \to d_\infty$ in $H^1(\Omega)$, and there exist $\{x_i^j\}_{i=1}^l \subset \Omega$ and $\{m_i^j\}_{i=1}^l \subset \mathbb{N}$ such that

$$|\nabla d(t_k)|^2 \, dx \to |\nabla d_\infty|^2 \, dx + \sum_{i=1}^l 8\pi m_i \delta_{x_i}.$$ 

(1d) If either $d_0^2 \geq 0$ or $\int_{\Omega} (|u_0|^2 + |\nabla d_0|^2) \leq 8\pi$, then $(u, d) \in C^\infty(\Omega \times (0, +\infty)) \cap C^2_{d_0}(\Omega \times (0, +\infty))$. Furthermore, there exists $t_k \uparrow +\infty$ and a harmonic map $d_\infty \in C^\infty(\Omega) \cap C^2_{d_0}(\Omega, \mathbb{S}^2)$ such that $(u(t_k), d(t_k)) \to (0, d_\infty)$ in $C^2(\Omega)$.

(2) The global weak solution $(u, d)$ obtained in (1) is unique in the same class of weak solutions, i.e. if $(u, d) \in L^\infty((0, T_1); H) \cap L^2((0, T_1); \mathbb{S}^2)$, then $(u, d) \equiv (0, d)$ in $\Omega \times [0, T_1)$.

Some related works on the existence of global weak solutions of (2.21) in $\mathbb{R}^2$ have also been considered by Hong [62], Hong & Xin [63], Xu & Zhang [64] and Lei et al. [65]. There are a few interesting questions related to theorem 2.7, namely, the following.

**Question 2.8.**

(1) Does the global weak solution $(u, d)$ obtained in theorem 2.7 have at most finitely many singularities?

(2) Does there exist a smooth initial value $(u_0, d_0)$ such that the short-time smooth solution $(u, d)$ to (2.21) and (2.24) develops finite time singularity?

(3) Does the solution $(u, d)$ obtained in theorem 2.7 have a unique limit $(0, d_\infty)$ as $t \uparrow +\infty$?

It is not hard to check that the example of finite time singularity of harmonic map heat flow in dimension two by Chang et al. [66] satisfies (2.21) with $u \equiv 0$. It is desirable to construct an example with a non-trivial velocity field. There have been some partial results on the uniform limit at $t = \infty$ for (2.21) on $\mathbb{S}^2$ (see [67]).

**Remark 2.9.** For the simplified Ericksen–Leslie system (2.21) in $\mathbb{R}^n$ for $n \geq 3$, the uniqueness was proven for

(i) the class of weak solutions $(u, d) \in C([0, T], L^2(\mathbb{R}^n, \mathbb{S}^2)) \times W^{1, n}_{0, \text{loc}}(\mathbb{R}^n, \mathbb{S}^2)$, $\varepsilon_0 \in \mathbb{S}^2$, by Lin & Wang [61] and

(ii) the class of weak solutions $(u, d) \in L^p(0, T; L^q(\mathbb{R}^n, \mathbb{S}^2)) \times L^p(0, T; W^{1, q}_{0, \text{loc}}(\mathbb{R}^n, \mathbb{S}^2))$, with $p > 2$ and $q > n$ satisfying Serrin’s condition $2/p + n/q = 1$, by Huang [68] (it is interesting to ask whether the uniqueness holds for the endpoint case $(p, q) = (\infty, n)$).

It turns out that the $\varepsilon_0$-regularity lemma 2.6 for the simplified Ericksen–Leslie system (2.21) can also be proved by a blow-up type argument. Furthermore, with some delicate analysis to establish smoothness of the limiting linear coupling system resulting from the blow-up argument, such a
blow-up argument also works for the general Ericksen–Leslie system (2.17). Indeed, it was shown by Huang et al. [54] that lemma 2.6 also holds for (2.17). As a consequence, we have extended in [54] theorem 2.7 to the general Ericksen–Leslie system (2.17) in $\mathbb{R}^2$.

**Theorem 2.10.** For $u_0 \in H$ and $d_0 \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{S}^2)$, $e_0 \in \mathbb{S}^2$, assume the conditions (2.5), (2.6) and (2.19) hold. Then there exists a global weak solution $(u, d) : \mathbb{R}^2 \times [0, +\infty) \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ to (2.17) in $\mathbb{R}^2$, with $u \in L^\infty(0, +\infty; H) \cap L^2(0, +\infty; J)$, $d \in L^\infty(0, +\infty; H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{S}^2))$, under the initial condition $(u, d)|_{t=0} = (u_0, d_0)$ in $\mathbb{R}^2$. Furthermore, $(u, d)$ enjoys the properties (1a), (1b), (1c), (1d) in theorem 2.7 (with $\Omega$ replaced by $\mathbb{R}^2$).

See also Wang & Wang [69] for some related work. It is an interesting question to ask whether the uniqueness theorem for (2.21) by Lin & Wang [61] also holds for (2.21) in $\mathbb{R}^2$.

Now we outline a programme to use the $\epsilon$-version of Ericksen–Leslie system (2.23) in the study of defect motion of nematic liquid crystals in dimension 2 (see [70,71]). For a bounded smooth domain $\Omega \subset \mathbb{R}^2$, let $(u_\epsilon, d_\epsilon) : \Omega \times [0, +\infty) \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ be solutions of (2.23), and $d_\epsilon(x, t) = g(x) \cdot \partial \Omega \rightarrow \mathbb{S}^1$ is smooth with $\deg(g) = m \in \mathbb{N}$. Assume that the initial data $(u^{\epsilon}_0, d^{\epsilon}_0)$ satisfy
\[
\int_{\Omega} |d^{\epsilon}_0|^2 \leq A, \quad E_\epsilon(d^{\epsilon}_0) = \int_{\Omega} \left( |\nabla d^{\epsilon}_0|^2 + \frac{1}{2\epsilon^2} |1 - |d^{\epsilon}_0|^2|^2 \right) \leq 2\pi m \log \frac{1}{\epsilon} + B.
\] (2.29)

Then the following facts hold:

(a) $E_\epsilon(d_\epsilon(t)) + \int_{\Omega} |u_\epsilon(t)|^2 \leq 2\pi m \log(1/\epsilon) + A + B$ for $t \geq 0$.
(b) $E_\epsilon(d_\epsilon(t)) \geq 2\pi m \log(1/\epsilon) - C_\epsilon \int_{\Omega} |u_\epsilon(t)|^2$ for $t \geq 0$ and
\[
\int_{\Omega} \left( \mu |\nabla u_\epsilon|^2 + |\Delta d_\epsilon + \frac{1}{\epsilon^2} (1 - |d_\epsilon|^2) d_\epsilon \right)^2 \leq C(A, B, C).
\]
(c) $\|\nabla d_\epsilon(t)\|_{L^p(\Omega)} \leq C(p, \Omega, g, A, B, C)$ for $1 \leq p < 2$.
(d) $u_\epsilon(t) \rightarrow u(t)$ in $L^2(\Omega)$ and weakly in $H^1(\Omega)$, and $\exists h(t) \in H^1(\Omega)$, with $\|h(t)\|_{H^1(\Omega)} \leq C$, and $\{a_j(t)\}_{j=1}^m \subset C^1([0, T), \Omega)$ such that, up to a subsequence,
\[
d_\epsilon(t) \rightarrow \Pi_{j=1}^m \frac{x - a_j(t)}{|x - a_j(t)|} e^{ih(x, t)} = e^{i\Theta_d + h}
\]
in $L^2(\Omega) \cap H^1_{\text{loc}}(\Omega \setminus \{a_j(t)\}_{j=1}^m)$.
(e) $\nabla d_\epsilon \circ \nabla d_\epsilon \rightarrow (\nabla \Theta_d + \nabla h) \circ (\nabla \Theta_d + \nabla h) + \eta$ in $D'(\Omega)$ for some positive semi-definite matrix valued function $\eta$ with each entry being Radon measures in $\Omega$. Moreover, $\supp(\eta(t)) \subset \{a_j(t)\}_{j=1}^m$.

Then we have
\[
\frac{d a_j}{d t} = u(a_j(t), t); \quad a_j(0) = a_{0j}, \quad j = 1, \ldots, m,
\] (2.30)
and $(u, h, \Theta_d)$ satisfies
\[
\begin{aligned}
ht + u \cdot \nabla h &= \Delta h + (u(a(t), t) - u) \cdot \nabla \Theta_d, \\
u_t + u \cdot \nabla u + \nabla P &= \mu \Delta u - \nabla h \cdot \Delta h - \nabla \Theta_d(\Delta h - \Delta h(a(t), t)),
\end{aligned}
\] (2.31)
and
\[
\nabla \cdot u = 0.
\]

An important question is to show that the system (2.30) and (2.31) admits a smooth solution $(a(t), u, h, P)$. Once this has been established, the consistency of energy laws will yield the defect measure $\eta \equiv 0$.

Next, we turn to discuss some recent progress towards the simplified Ericksen–Leslie system (2.21) in dimensions $n \geq 3$. For (2.21), it is standard (see [72,73]) that for any initial data $(u_0, d_0) \in H^2(\mathbb{R}^n, \mathbb{R}^n) \times H^1_{\text{loc}}(\mathbb{R}^n, \mathbb{S}^2)$, $e_0 \in \mathbb{S}^2$, there exist $T_\ast > 0$ and a unique strong solution $(u, d) : \mathbb{R}^n \times [0, T_\ast) \rightarrow \mathbb{R}^n \times \mathbb{S}^2$ of (2.21) and (2.24); see [74–76] for related works on the existence of global strong solutions to (2.21) for small initial data in suitable function spaces. Furthermore, a criterion
on possible breakdown for local strong solutions of (2.21), analogous to the BKM criterion for the Navier–Stokes equation [77], was obtained by Huang & Wang [78]: if $0 < T_0 < +\infty$ is the maximum time, then
\[
\int_0^{T_0} (\|\nabla \times u\|_{L^\infty(\mathbb{R}^n)} + \|\nabla d\|_{L^\infty(\mathbb{R}^n)}) \, dt = +\infty.
\] (2.32)

See Wang et al. [79] and Hong et al. [80] for related works on the general Ericksen–Leslie system (2.17).

For (2.21), inspired by Koch & Tataru [81] the following well-posedness result has been obtained by Wang [82].

**Theorem 2.11.** There exists $\epsilon_0 > 0$ such that if $(u_0, d_0) : \mathbb{R}^n \to \mathbb{R}^n \times S^2$, with $\nabla \cdot u_0 = 0$, satisfies
\[
[u_0]_{\text{BMO}^{-1}(\mathbb{R}^n)} + |d_0|_{\text{BMO}(\mathbb{R}^n)} \leq \epsilon_0,
\]
then there exists a unique global smooth solution $(u, d) \in C^\infty(\mathbb{R}^n \times (0, +\infty), \mathbb{R}^n \times S^2)$ of (2.21), under the initial condition $(u_0, d_0)$, which enjoys the decay estimate
\[
t^{1/2} (\|u(t)\|_{L^\infty(\mathbb{R}^n)} + \|\nabla d(t)\|_{L^\infty(\mathbb{R}^n)}) \lesssim \epsilon_0, \quad t > 0.
\] (2.33)

Higher order decay estimates for the solution given by theorem 2.11 have been established by Lin [83] and Du & Wang [84]. Local well-posedness of (2.21) in $\mathbb{R}^3$ for initial data $(u_0, d_0)$, with $(u_0, \nabla d_0) \in L^3_{\text{loc}}(\mathbb{R}^3)$ (the uniformly locally $L^3$-space in $\mathbb{R}^3$ having small norm $\|(u_0, \nabla d_0)\|^3_{L^1_{\text{loc}}(\mathbb{R}^3)}$) has been established by Hineman & Wang [85]. We mention some interesting works on both new modelling and analysis of the hydrodynamics of non-isothermal nematic liquid crystals by Feireisl et al. [86,87] and Li & Xin [88].

Now we describe a very recent work by Lin & Wang [89] on the existence of global weak solutions of (2.21) in dimensions three by performing blow-up analysis of the Ginzburg–Landau approximation system (2.23). More precisely, we have proved the following.

**Theorem 2.12.** For $\Omega \subset \mathbb{R}^3$ either a bounded domain or the entire $\mathbb{R}^3$, assume $u_0 \in H$ and $d_0 \in H^1(\Omega, S^2)$ satisfy $d_0(\Omega) \subset S^2_+$, the upper half sphere. Then there exists a global weak solution $(u, d) : [0, +\infty) \to \mathbb{R}^3 \times S^2$ to the initial and boundary value problem of (2.21) and (2.24) such that

(i) $u \in L^\infty(0, +\infty; H) \cap L^2(0, +\infty; L^\infty(\mathbb{R}^3))$.
(ii) $d \in L^\infty(0, +\infty; H^1(\Omega, S^2))$ and $d(\Omega) \subset S^2_+$ for $L^1$ a.e. $t \in [0, +\infty)$.
(iii) $(u, d)$ satisfies the global energy inequality: for $L^1$ a.e. $t \in [0, +\infty)$,
\[
\int_\Omega (|u|^2 + |\nabla d|^2)(t) + 2 \int_0^t \int_\Omega (\mu |\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) \leq \int_0^t \int_\Omega (|u_0|^2 + |\nabla d_0|^2).
\] (2.34)

The proof of theorem 2.12 in [89] is very delicate, which is based on suitable extensions of the blow-up analysis scheme that has been developed in the context of harmonic maps by Lin [90] and harmonic map heat flows by Lin & Wang [91–93]. A crucial ingredient is the following compactness result.

**Theorem 2.13.** For any $0 < a \leq 2$, $L_1 > 0$ and $L_2 > 0$, set $X(L_1, L_2, a; \Omega)$ consisting of maps $d_\epsilon \in H^1(\Omega, \mathbb{R}^3)$, $0 < \epsilon \leq 1$, that are solutions of
\[
\Delta d_\epsilon + \frac{1}{\epsilon^2} (1 - |d_\epsilon|^2) d_\epsilon = \tau_\epsilon \quad \text{in } \Omega
\] (2.35)
such that the following properties hold:

(i) $|d_\epsilon| \leq 1$ and $d_\epsilon^2 \geq -1 + a$ for a.e. $x \in \Omega$.
(ii) $E_\epsilon(d_\epsilon) = \int_\Omega e_\epsilon(d_\epsilon) \, dx \leq L_1$.
(iii) $\|\tau_\epsilon\|_{L^2(\Omega)} \leq L_2$.

Then $X(L_1, L_2, a; \Omega)$ is precompact in $H^1_{\text{loc}}(\Omega, \mathbb{R}^3)$, i.e. if $\{d_\epsilon\} \subset X(L_1, L_2, a; \Omega)$, then there exists $d_0 \in H^1(\Omega, S^2)$ such that, up to a subsequence, $d_\epsilon \to d_0$ in $H^1_{\text{loc}}(\Omega, \mathbb{R}^3)$ as $\epsilon \to 0$. 

\[\]
as convergence of Radon measures, for some non-negative Radon measure \( \nu \), called defect measure. We claim that \( \nu \equiv 0 \). This will be done in several steps.

**Step 1** (almost monotonicity). \( d_\varepsilon \in X(L_1, L_2; \Omega) \) satisfies

\[
\Phi_\varepsilon(R) \geq \Phi_\varepsilon(r) + \int_{B_r \setminus B_{r/2}} \left| x^{-1} \left| \frac{\partial d_\varepsilon}{\partial |x|} \right| \right|^2, \quad \forall 0 < r \leq R,
\]

where

\[
\Phi_\varepsilon(r) := \frac{1}{r} \int_{B_r} \left( e_\varepsilon(d_\varepsilon) - (x \cdot \nabla d_\varepsilon, f_\varepsilon) \right) + \frac{1}{2} \int_{B_r} |x| |f_\varepsilon|^2.
\]

**Step 2** (\( \delta_0 \)-strong convergence). \( \exists \delta_0 > 0, \alpha_0 \in (0, 1), \) and \( C_0 > 0 \) such that if \( d_\varepsilon \in X(L_1, L_2; \Omega) \) satisfies

\[
\Phi_\varepsilon(r_0) \leq \delta_0,
\]

then \( |d_\varepsilon| \geq \frac{1}{2} \) in \( B_{r_0/2} \), and

\[
\left[ \frac{d_\varepsilon}{|d_\varepsilon|} \right]_{C^0(B_{r_0/2})} \leq C_0.
\]

In particular, \( d_\varepsilon \to d_0 \) in \( H^1(B_{r_0/2}) \) and \( \nu = 0 \) in \( B_{r_0/2} \).

**Step 3** (almost monotonicity for \( \mu \)).

\[
\Theta^1(\mu, r) := \frac{1}{r} \mu(B_r) \leq \Theta^1(\mu, R) + C_0(R - r), \quad \forall 0 < r \leq R.
\]

Thus, \( \Theta^1(\mu) = \lim_{r \to 0} \Theta^1(\mu, r) \) exists and is upper semi-continuous.

**Step 4** (concentration set). \( d_\varepsilon \to d_0 \) in \( H^1_{\text{loc}}(\Omega \setminus \Sigma) \), where

\[
\Sigma := \{ x \in \Omega : \Theta^1(\mu) \geq \delta_0 \}
\]

is a 1-rectifiable, closed subset, with \( H^1(\Sigma) < +\infty \). Moreover,

\[
\text{supp}(\nu) \subseteq \Sigma \quad \text{and} \quad \delta_0 \leq \Theta^1(\nu, x) = \Theta^1(\mu, x) \leq C_0 h^1 \text{a.e. } x \in \Sigma.
\]

**Step 5** (stratification and blow-up). If \( \nu \neq 0 \), then \( H^1(\Sigma) > 0 \). We can pick up a generic point \( x_0 \in \Sigma \) such that

\[
\lim_{r_i \to 0} \int_{B_{r_i}(x_0)} |\nabla d_0|^2 = 0, \quad \Theta^1(\nu, \cdot) \text{ is } H^1 \text{ approximately continuous at } x_0.
\]

Define blow-up sequences \( d_i(x) = d_\varepsilon(x_0 + r_i x) \) and \( v_i(A) = r_i^{-1} v(x_0 + r_i A) \) for \( A \subset \mathbb{R}^3 \). Then

\[
v_i \to v_0 := \theta_0 h^1 Y \quad \text{and} \quad e_\varepsilon(d_i) \to v_0(= \theta_0 h^1 Y)
\]

as convergence of Radon measures, for some \( \theta_0 > 0 \) and \( Y = \{ (0, 0, x_3) : x_3 \in \mathbb{R} \} \). Applying (2.36), one has

\[
\int_{B_1} \left| \frac{\partial d_i}{\partial x_3} \right|^2 \to 0.
\]

Now we can perform another round of blow-up process to extract a non-trivial smooth harmonic map \( \omega : \mathbb{R}^2 \to S^2_{-1+\delta} \) that has finite energy (see [89] for the detail). This is impossible, since any such \( \omega \) has non-zero topological degree.

**e) Compressible flow of nematic liquid crystals**

When the fluid is assumed to be compressible, the Ericksen–Leslie system becomes more complicated and there seem very few analytic works available yet. We mention that there have been both modelling study (see [94]) and numerical study (see [95]) on the hydrodynamics...
of compressible nematic liquid crystals under the influence of temperature gradient or electromagnetic forces. Here, we briefly describe some recent studies on a simplified version of compressible Ericksen–Leslie system in $\Omega \subseteq \mathbb{R}^n$, which is given by

$$
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla (P(\rho)) &= \mathcal{L} u - \nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_n), \\
\partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d \\
(\rho, u, d)|_{t=0} &= (\rho_0, u_0, d_0) \quad \text{and} \quad (u, d)|_{\partial \Omega} = (u_0, d_0),
\end{align*}
$$

(2.38)

where $\rho : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_+$ is the fluid density and $P(\rho) : \Omega \rightarrow \mathbb{R}_+$ denotes the pressure of the fluid, and $\mathcal{L}$ denotes the Lamé operator

$$
\mathcal{L} u = \mu \Delta u + (\mu + \lambda) \nabla (\nabla \cdot u),
$$

where $\mu$ and $\lambda$ are the shear viscosity and the bulk viscosity coefficients of the fluid, satisfying the following physical condition:

$$
\mu > 0, \quad 3\lambda + 2\mu \geq 0.
$$

The system (2.38) is a strong coupling between the compressible Navier–Stokes equation and the transported harmonic map heat flow to $S^2$. As it is a challenging question to establish the existence of global weak solutions to the compressible Navier–Stokes equation itself (see [96]), it turns out to be more difficult to show the existence of global weak solutions to (2.38) than the incompressible nematic liquid crystal flow (2.21). See [97] for a rigorous proof of convergence of (2.38) to (2.21) under suitable conditions. The local existence of strong solutions to (2.38) can be established under suitable regularity and compatibility conditions on initial data. For example, we have proven the following in [98, 99].

**Theorem 2.14.** For $n = 3$, assume $P \in C^{0,1}(\mathbb{R}_+), \ 0 \leq \rho_0 \in W^{1,q}_0 \cap H^1 \cap L^1$ for some $3 < q < 6$, $u_0 \in H^1 \cap H^2$ and $d_0 \in H^2(\Omega, S^2)$ satisfies

$$
\mathcal{L} u_0 - \nabla (P(\rho_0)) - \Delta d_0 \cdot \nabla d_0 = \rho_0^{1/2} g, \quad g \in L^2.
$$

Then there exist $T_0 > 0$ and a unique strong solution $(\rho, u, d)$ to (2.38) in $\Omega \times [0, T_0)$. Furthermore, if $T_0 < +\infty$ is the maximal time interval, then

$$
\| \nabla u + (\nabla u)^2 \|_{L^1(0, T_0; L^\infty)} + \| \nabla d \|_{L^2(0, T_0; L^\infty)} = +\infty
$$

or

$$
\| \rho \|_{L^\infty(0, T_0; L^\infty)} + \| \nabla d \|_{L^1(0, T_0; L^\infty)} = +\infty \quad (\text{if } 7\mu > 9\lambda).
$$

(2.39)

The local strong solution established in theorem 2.14 has been shown to be global for small initial-boundary data $(\rho_0, u_0, d_0)$ by Jiang et al. [100] and Li et al. [101].

The global existence of weak solutions to (2.38) is more challenging than (2.21) in dimensions $n \geq 2$; see [102, 103] for $n = 1$. When relaxing the condition $|d| = 1$ in (2.38) to $d \in \mathbb{R}^3$ by the Ginzburg–Landau approximation, similar to (2.23), the global existence of weak solutions has been established by Wang & Yu [104] and Liu & Qing [105] in dimensions $n = 3$. Finally, we mention that by employing the precompactness theorem 2.13, the following global existence of weak solutions to (2.38) has been established by Lin et al. [106] in dimensions $n = 2, 3$ (see [100] for a different proof for $n = 2$).

**Theorem 2.15.** Assume $P(\rho) = \rho^\gamma$ for $\gamma > 1, 0 \leq \rho_0 \in L^\gamma, m_0 \in L^{2\gamma/(\gamma+1)}$ and $(|m_0|^2/\rho_0)\chi_{(\rho_0>0)} \in L^1$ and $d_0 \in H^1(\Omega, S^2)$ with $d_0^3 \geq 0$. Then there exists a global renormalized weak solution $(\rho, u, d)$ to (2.38).

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