EXTENSION COMPLEXITY AND REALIZATION SPACES OF HYPERSIMPLEXES

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Abstract. The \((n,k)\)-hypersimplex is the convex hull of all 0/1-vectors of length \(n\) with coordinate sum \(k\). We explicitly determine the extension complexity of all hypersimplices as well as of certain classes of combinatorial hypersimplices. To that end, we investigate the projective realization spaces of hypersimplices and their (refined) rectangle covering numbers. Our proofs combine ideas from geometry and combinatorics and are partly computer assisted.

1. Introduction

1.1. The extension complexity of hypersimplices. The extension complexity or nonnegative rank \(\text{rk}_+(P)\) of a convex polytope \(P\) is the minimal number of facets (i.e., describing linear inequalities) of an extension, a polytope \(\tilde{P}\) that linearly projects onto \(P\). The motivation for this definition comes from linear optimization: The computational complexity of the simplex algorithm is intimately tied to the number of linear inequalities and hence it can be advantageous to optimize over \(\tilde{P}\). As a complexity measure, the nonnegative rank is an object of active research in combinatorial optimization; see [KLTT15]. There are very few families of polytopes for which the exact nonnegative rank is known. Besides simplices, examples are cubes, crosspolytopes, Birkhoff polytopes and bipartite matching polytopes [FKPT13] as well as all \(d\)-dimensional polytopes with at most \(d+4\) vertices [Pad16]. Determining the nonnegative rank is non-trivial even for polygons [FRT12, PP15, Shi, Shi14]. For important classes of polytopes exponential lower bounds obtained in [FMP+15, Rot13, Rot14] are celebrated results.

In the first part of the paper we explicitly determine the nonnegative rank of the family of hypersimplices. For \(0 < k < n\), the \((n,k)\)-hypersimplex is the convex polytope

\[
\Delta_{n,k} = \text{conv}\{x \in \{0,1\}^n : x_1 + \cdots + x_n = k\}.
\]

Hypersimplices were first described (and named) in connection with moment polytopes of orbit closures in Grassmannians (see [GGMS87]) but, of course, they are prominent objects in combinatorial optimization, appearing in connection with packing problems and matroid theory; see also below. This marks hypersimplices as polytopes of considerable interest and naturally prompts the question as to their extension complexity.

Note that \(\Delta_{n,k}\) is affinely isomorphic to \(\Delta_{n,n-k}\). The hypersimplex \(\Delta_{n,1} = \Delta_{n-1}\) is the standard simplex of dimension \(n-1\) and \(\text{rk}_+(\Delta_{n-1}) = n\). Our first result concerns the extension complexity of the proper hypersimplices, that is, the hypersimplices \(\Delta_{n,k}\) with \(2 \leq k \leq n-2\).

Theorem 1.1. The hypersimplex \(\Delta_{4,2}\) has extension complexity 6, the hypersimplices \(\Delta_{5,2} \cong \Delta_{5,3}\) have extension complexity 9. For any \(n \geq 6\) and \(2 \leq k \leq n-2\), we have \(\text{rk}_+(\Delta_{n,k}) = 2n\).

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It is straightforward to check that
\[ (2) \quad \Delta_{n,k} = [0,1]^n \cap \{ x \in \mathbb{R}^n : x_1 + \cdots + x_n = k \} \]
and that for \( 1 < k < n-1 \), all \( 2n \) inequalities of the \( n \)-dimensional cube are necessary. The nonnegative rank of a polytope is trivially upper bounded by the minimum of the number of vertices and the number of facets. We call a polytope \( P \) **extension maximal** if it attains this upper bound. Cubes as well as their duals, the crosspolytopes, are known to be extension maximal; see also Corollary 2.5. Theorem 1.1 states that in addition to simplices, cubes, and crosspolytopes, all proper hypersimplices except for \( \Delta_{5,2} \) are extension maximal.

1.2. **Psd rank and 2-level matroids.** Our original motivation for studying the nonnegative rank of hypersimplices comes from matroid theory [Oxl11]. For a matroid \( M \) on the ground set \( [n] := \{1, \ldots, n\} \) and bases \( B \subseteq 2^{[n]} \), the associated **matroid base polytope** is the polytope
\[
P_M := \text{conv}\{1_B : B \in B\},
\]
where \( 1_B \in \{0,1\}^n \) is the characteristic vector of \( B \subseteq [n] \). Hence, the \((n,k)\)-hypersimplex is the matroid base polytope of the uniform matroid \( U_{n,k} \). In [GS], the first and third author studied **2-level matroids**, which exhibit extremal behavior with respect to various geometric and algebraic measures of complexity. In particular, it is shown that \( M \) is 2-level if and only if \( P_U \) is **psd minimal**. The **psd rank** \( \text{rk}_{psd}(P) \) of a polytope \( P \) is the smallest size of a spectrahedron (an affine section of the positive definite cone) that projects onto \( P \). In [GRT13] it is shown that \( \text{rk}_{psd}(P) \geq \dim P + 1 \) and polytopes attaining this bound are called psd minimal. Our starting point was the natural question whether the class of 2-level matroids also exhibits an extremal behavior with respect to the nonnegative rank. We recall from [GS, Theorem 1.2] the following synthetic description of 2-level matroids: A matroid \( M \) is 2-level if and only if it can be constructed from uniform matroids by taking direct sums or 2-sums. So, the right starting point are the hypersimplices.

To extend Theorem 1.1 to all 2-level matroids, it would be necessary to understand the effect of taking direct and 2-sums on the nonnegative rank. The direct sum of matroids translates into the Cartesian product of matroid polytopes. Two out of three authors of this paper believe in the following conjecture, first asked during a Dagstuhl seminar in 2013 [BKLTL13].

**Conjecture 1.** The nonnegative rank is additive with respect to Cartesian products, that is,
\[
\text{rk}_+(P_1 \times P_2) = \text{rk}_+(P_1) + \text{rk}_+(P_2),
\]
for polytopes \( P_1 \) and \( P_2 \).

We provide evidence in favor of Conjecture 1 by showing it to hold whenever one of the factors is a simplex (cf. Corollary 2.4). By taking products of extensions it trivially follows that the nonnegative rank is subadditive with respect to Cartesian products. As for the 2-sum \( M_1 \oplus_2 M_2 \) of two matroids \( M_1 \) and \( M_2 \), it follows from [GS, Lemma 3.4] that \( P_{M_1 \oplus_2 M_2} \) is a codimension-1 section of \( P_{M_1 \times P_{M_2}} \) and the extension complexity is therefore dominated by that of the direct sum. Combined with Theorem 1.1 and [GS, Theorem 1.2] we obtain the following simple estimate.

**Corollary 1.2.** If \( M \) is a 2-level matroid on \( n \) elements, then \( \text{rk}(P_M) \leq 2n \).

1.3. **Extension complexity of combinatorial hypersimplices.** The extension complexity is not an invariant of the combinatorial type. That is, two combinatorially isomorphic polytopes do not necessarily have the same extension complexity. For example, the extension complexity of a hexagon is either 5 or 6 depending on the incidences of the facet-defining lines [PP15, Prop. 4]. On the other hand, the extension complexity of any polytope combinatorially isomorphic to the \( n \)-dimensional cube is always \( 2n \); cf. Corollary 2.5. The close connection to simplices and cubes and Theorem 1.1 raises the following question for **combinatorial** \((n,k)\)-hypersimplices.
Question 1. Is \( \text{rk}_+(P) = 2n \) for any combinatorial \((n,k)\)-hypersimplex \( P \) with \( n \geq 6 \) and \( 2 \leq k \leq n-2 \)?

For \( n = 6 \) and \( k \in \{2,3\} \) this is true due to Proposition 3.3 but we suspect that the answer is no for some \( n > 6 \) and \( k = 2, n-2 \). The rectangle covering number \( \text{rc}(P) \) of a polytope \( P \) is a combinatorial invariant that gives a lower bound on \( \text{rk}_+(P) \); see Section 3. While the rectangle covering number of the small hypersimplices \( \Delta_{6,2} \) and \( \Delta_{6,3} \) is key to our proof of Theorem 1.1, it is not strong enough to resolve Question 1 (see Proposition 3.4).

We introduce the notions of \( F^- \), \( G^- \), and \( FG^- \)-genericity of combinatorial hypersimplices, that are defined in terms of the relative position of certain facets and that play a crucial role. We show that all \( FG^- \)-generic hypersimplices are extension maximal (Theorem 4.1). Unfortunately, \( FG^- \)-genericity is not a property met by all hypersimplices, which is confirmed by the existence of a non-\( FG^- \)-generic realization of \( \Delta_{6,2} \); see Proposition 5.2. On the other hand, we show that hypersimplices with \( n \geq 6 \) and \( \left\{ \frac{n}{2} \right\} \leq k \leq \left\lceil \frac{n}{2} \right\rceil \) are \( FG^- \)-generic, which ensues the following.

Corollary 1.3. If \( P \) is a combinatorial \((n,k)\)-hypersimplex with \( n \geq 6 \) and \( 2 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \), then

\[
\text{rk}_+(P) \geq \begin{cases} 
    n + 2k + 1 & \text{if } k < \left\lceil \frac{n}{2} \right\rceil, \\
    2n & \text{otherwise}.
\end{cases}
\]

We do not know of any realization of a \((n,k)\)-hypersimplex with \( n \geq 6 \) of extension complexity less than \( 2n \), but we do not dare to conjecture that every combinatorial \((n,k)\)-hypersimplex with \( n \geq 6 \) and \( 2 \leq k \leq n \) is extension maximal.

1.4. Realization spaces of hypersimplices. The projective realization space \( R_{n,k} \) of combinatorial \((n,k)\)-hypersimplices parametrizes the polytopes combinatorially isomorphic to \( \Delta_{n,k} \) up to projective transformation. (Projective) realization spaces of polytopes are provably complicated objects. The universality theorems of Mnév [Mnë88] and Richter-Gebert [RG96] assert that realization spaces of polytopes of dimension \( \geq 4 \) are as complicated as basic open semialgebraic sets defined over the integers. In contrast, for a 3-dimensional polytope \( P \) with \( e \geq 9 \) edges, it follows from Steinitz’ theorem that the projective realization space is homeomorphic to an open ball of dimension \( e - 9 \); see also [RG96, Thm. 13.3.3].

For our investigation of the extension complexity of combinatorial hypersimplices, we study their realization spaces. The observation that every hypersimplex is either \( F^- \) or \( G^- \)-generic (Lemma 4.2) turns out to be instrumental in our study. For \( k = 2 \), we are able to give a full description.

Theorem 1.4. For \( n \geq 4 \), \( R_{n,2} \) is rationally equivalent to the interior of a \( \left( \frac{n-1}{2} \right) \)-dimensional cube. In particular, \( R_{n,2} \) is homeomorphic to an open ball and hence contractible.

Rationally equivalent means that the homeomorphism as well as its inverse are given by rational functions (c.f. [RG96, Sect. 2.5]).

A key tool in the context of the Universality Theorem is that the projective realization of a facet of a high-dimensional polytope can not be prescribed in general; see, for example, [Zie95, Sect. 6.5]. In contrast, the shape of any single facet of a 3-polytope can be prescribed [BG70]. This description of \( R_{n,2} \) allows us to show that facets of \((n,2)\)-hypersimplices can be prescribed (Corollary 5.4), but also allows us to construct hypersimplices that are not \( FG^- \)-generic, which implies that facets of hypersimplices cannot be prescribed in general (Corollary 5.3).

For \( 2 < k < n-2 \), the realization spaces are more involved and, in particular, related to the algebraic variety of \( n \times n \) matrices with vanishing principal \( k \)-minors that was studied by Wheeler [Whe]. In Theorem 4.4, we show that certain facets of \( \Delta_{n,k} \) completely determine the realization, which then gives an upper bound on the dimension of the realization space. However, we can currently not exclude that \( R_{n,k} \) is disconnected and has components of different dimensions.
The extension complexity is invariant under (admissible) projective transformations and hence $\text{rk}_+$ is well-defined on $\mathcal{R}_{n,k}$. The locus $E_{n,k} \subseteq \mathcal{R}_{n,k}$ of extension maximal $(n,k)$-hypersimplices is open and Theorem 1.1 implies that $E_{n,k}$ is non-empty for $n \geq 6$ and $2 \leq k \leq n - 2$. For $k = 2$, we can say considerably more.

**Corollary 1.5.** For $n \geq 5$, the combinatorial $(n,2)$-hypersimplices with extension complexity $2n$ are dense in $\mathcal{R}_{n,2}$.

Our results on $FG$-generic hypersimplices, which are characterized by the non-vanishing of a determinantal condition on $\mathcal{R}_{n,k}$, strongly suggest that Corollary 1.5 extends to all the cases.

**Conjecture 2.** For $n \geq 5$ and $2 \leq k \leq n - 2$, the combinatorial hypersimplices of nonnegative rank $2n$ form a dense open subset of $\mathcal{R}_{n,k}$.

1.5. **Structure of the paper.** Theorem 1.1 is proved in Sections 2 and 3. In Section 2 we investigate the discrete geometry of extensions and we set up an induction that deals with the large hypersimplices $\Delta_{n,k}$ with $n > 6$. In particular, we devise general tools for upper bounding the extension complexity. For the small hypersimplices $\Delta_{6,2}$ and $\Delta_{6,3}$, we make use of rectangle covering numbers in Section 3. We show that most of the geometric tools of Section 2 have combinatorial counterparts for rectangle covering numbers. Section 4 is devoted to the study of combinatorial hypersimplices and the associated realization spaces. In Section 5 we focus on the combinatorial $(n,2)$-hypersimplices.

## 2. The Geometry of Extensions and Large Hypersimplices

In this section we develop some useful tools pertaining to the geometry of extensions. These will be used to give an inductive argument for the large hypersimplices $\Delta_{n,k}$ with $n > 6$ and $1 < k < n - 1$. The small hypersimplices are treated in the next section.

For a polytope $P$, we write $v(P)$ for the number of vertices of $P$ and $f(P)$ for the number of facets. Moreover, $\hat{P}$ will typically denote an extension of $P$, and the linear projection that takes $\hat{P}$ to $P$ is denoted by $\pi$. We start with the simple observation that the nonnegative rank is strictly monotone with respect to taking faces.

**Lemma 2.1.** Let $P$ be a polytope and $F \subset P$ a facet. Then

$$\text{rk}_+(P) \geq \text{rk}_+(F) + 1.$$  

**Proof.** Let $\hat{P}$ be a minimal extension of $P$. The preimage $\hat{F} = \pi^{-1}(F) \cap \hat{P}$ is an extension of $F$. Every facet of $\hat{F}$ is the intersection of a facet of $\hat{P}$ with $\hat{F}$. Moreover, since $\hat{F}$ is a proper face of $\hat{P}$, there are at least $c \geq 1$ facets of $\hat{P}$ that contain $\hat{F}$ and hence do not contribute facets to $\hat{F}$. It follows that

$$\text{rk}_+(P) = f(\hat{P}) \geq f(\hat{F}) + c \geq \text{rk}_+(F) + 1,$$

which proves the claim. 

By induction, this extends to lower dimensional faces.

**Corollary 2.2.** Let $P$ be a polytope and $F \subset P$ a face. Then

$$\text{rk}_+(P) \geq \text{rk}_+(F) + \dim(P) - \dim(F).$$

We can strengthen this observation if we take into consideration more than one facet.

**Lemma 2.3.** Let $P$ be a polytope and let $F_1$ and $F_2$ be two disjoint facets of $P$. Then

$$\text{rk}_+(P) \geq \min\{\text{rk}_+(F_1),\text{rk}_+(F_2)\} + 2.$$
Proof. If \( \text{rk}_+(F_1) > \text{rk}_+(F_2) \), the claim follows from Lemma 2.1. Hence, we can assume that \( \text{rk}_+(F_1) = \text{rk}_+(F_2) = k \). Extending the argument of Lemma 2.1, let \( \hat{P} \) be a minimal extension of \( P \) and \( \hat{F}_i \) the preimage of \( F_i \) for \( i = 1, 2 \). Let \( c_i \) be the number of facets of \( \hat{P} \) containing \( \hat{F}_i \). Since \( f(\hat{P}) \geq k + c_i \), the relevant case is \( c_1 = c_2 = 1 \). Now, \( \pi(\hat{F}_1 \cap \hat{F}_2) \subseteq F_1 \cap F_2 = \emptyset \) implies that \( \hat{F}_1 \) and \( \hat{F}_2 \) are disjoint facets of \( \hat{P} \). Hence, \( \text{rk}_+(P) = f(\hat{P}) \geq k + 2 \). \( \square \)

We cannot replace \( \min \) with \( \max \) in Lemma 2.3: The convex hull of the 12 columns of the matrix

\[
\begin{pmatrix}
1 & -1 & -1 & 1 & 2 & 2 & -2 & -2 & 1 & -1 & 1 & -1 \\
2 & 2 & -2 & -2 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}
\]

gives rise to a 4-dimensional polytope \( Q \) combinatorially isomorphic to product of a triangle and a quadrilateral, and consequently has 7 facets. If we project onto the first three coordinates we obtain a 3-dimensional polytope \( P \) with two parallel facets, \( F_1 \) and \( F_2 \), that are an octagon and a square, with nonnegative ranks 6 and 4, respectively. Thus, \( \text{rk}_+(P) \leq 7 < \max\{\text{rk}_+(F_1), \text{rk}_+(F_2)\} + 2 = 8 \). Figure 1 gives an idea of the geometry underlying \( Q \).

![Figure 1](image)

Figure 1. The left figures gives a sketch (not a Schlegel diagram) of the geometric idea underlying the construction of \( Q \). It is a union of three facets that yield the projection on the right. We highlighted the structure as a product of polygons, that makes it more visible how the two square faces of \( Q \) yield the octagonal face of \( P \).

Combining Lemma 2.1 and Lemma 2.3 yields the following result pertaining to Conjecture 1.

**Corollary 2.4.** Let \( P \) be a non-empty convex polytope and \( k \geq 1 \). Then

\[
\text{rk}_+(P \times \Delta_k) = \text{rk}_+(P) + k + 1.
\]

**Proof.** Let \( \hat{P} \) be a minimal extension of \( P \) with \( \text{rk}_+(P) \) facets. Since the number of facets of a product add up, \( \hat{P} \times \Delta_k \) is an extension of \( P \times \Delta_k \) with \( \text{rk}_+(P) + k + 1 \) facets. Thus, we need to show that \( \text{rk}_+(P) + k + 1 \) is also a lower bound.

For \( k = 1 \), the polytope \( P \times \Delta_1 \) is a prism over \( P \) with two distinct facets isomorphic to \( P \) and the claim follows from Lemma 2.3. If \( k > 1 \), note that \( P \times \Delta_{k-1} \) is a facet of \( P \times \Delta_k \), and an application of Lemma 2.1 yields the claim by induction on \( k \). \( \square \)

Another byproduct is a simple proof that every combinatorial cube is extension maximal (see [FKPT13, Proposition 5.9]).

**Corollary 2.5.** If \( P \) is combinatorially equivalent to the \( n \)-dimensional cube \( C_n = [0,1]^n \), then \( \text{rk}_+(P) = 2n \).

**Proof.** Since \( f(P) = f(C_n) = 2n \), we only need to prove \( \text{rk}_+(P) \geq 2n \). For \( n = 1 \), \( P \) is a 1-dimensional simplex for which the claim is true. For \( n \geq 2 \) observe that \( P \) has two
disjoint facets $F_1, F_2$ that are combinatorially equivalent to $(n-1)$-cubes. By induction and Lemma 2.3 we compute $\text{rk}_+(P) \geq \text{rk}_+(C_{n-1}) + 2 = 2n$. □

With these tools, we are ready to prove Theorem 1.1 for the cases with $n > 6$. The case $n = 6$ and $1 < k < n - 1$ will be treated in Proposition 3.3 in the next section. A key property, inherited from cubes, that allows for an inductive treatment of hypersimplices is that for $1 < k < n - 1$, the presentation (2) purports that

\begin{align}
F_i &:= \Delta_{n,k} \cap \{x_i = 0\} \cong \Delta_{n-1,k}, \\
G_i &:= \Delta_{n,k} \cap \{x_i = 1\} \cong \Delta_{n-1,k-1},
\end{align}

are disjoint facets for any $1 \leq i \leq n$. We call these the $F$-facets and $G$-facets, respectively.

**Proposition 2.6.** Assume that $\text{rk}_+(\Delta_{6,2}) = \text{rk}_+(\Delta_{6,3}) = 12$. Then $\text{rk}_+(\Delta_{n,k}) = 2n$ for all $n > 6$ and $1 < k < n - 1$.

**Proof.** Let $n \geq 7$. For $2 < k < n - 2$, the pairs of disjoint facets (3) allow us to use Lemma 2.3 together with induction on $n$ and $k$ to establish the result. Hence, the relevant cases are $n \geq 7$ and $k = 2$ (which is equivalent to $k = n - 2$).

For $k = 2$, let

$$\hat{P} = \{y \in \mathbb{R}^m : \ell_i(y) \geq 0 \text{ for } i = 1, \ldots, M\}$$

be an extension of $\Delta_{n,k}$ given by affine linear forms $\ell_1, \ldots, \ell_M$ and $M = \text{rk}_+(\Delta_{n,k})$. For convenience, we can regard $\Delta_{n,k}$ as a full-dimensional polytope in the affine hyperplane $\{x \in \mathbb{R}^n : x_1 + \cdots + x_n = k\} \cong \mathbb{R}^{n-1}$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^{n-1}$ the linear projection that takes $\hat{P}$ to $\Delta_{n,k}$. If for some $1 \leq i \leq n$, the preimage $\hat{F}_i = \pi^{-1}(F_i) \cap \hat{P}$ is not a facet then $f(\hat{P}) \geq \text{rk}_+(\hat{F}_i) + 2 = 2n$ by induction and we are done. So, we have to assume that $\hat{F}_i = \{y \in \hat{P} : \ell_i(y) = 0\}$ is a facet of $\hat{P}$ for all $i = 1, \ldots, n$.

It is sufficient to show that the polyhedron $\hat{Q} := \{y \in \mathbb{R}^m : \ell_i(y) \geq 0 \text{ for } i = n + 1, \ldots, M\}$ is bounded and hence has $f(\hat{Q}) \geq m + 1 \geq n$ facets. Since $f(\hat{P}) = n + f(\hat{Q})$ this implies the result. The key observation is that the polyhedron $\hat{Q} \subset \mathbb{R}^{n-1}$ bounded by the hyperplanes defining the facets $G_i$ of $\Delta_{n,k}$ is a full-dimensional simplex and hence bounded. We claim that $\pi(\hat{Q}) \subset \hat{Q}$. For this it is sufficient to show that if $H_i$ is the unique hyperplane containing $G_i$, then $\pi^{-1}(H_i)$ supports a face of $\hat{Q}$. By construction, $\pi^{-1}(H_i)$ supports the face $\hat{G}_i := \pi^{-1}(G_i) \cap \hat{P}$ of $\hat{P}$. Now, if $\hat{G}_i \not\subset \hat{F}_j$ for some $1 \leq j \leq n$, this would imply $G_i \not\subset F_j$.

This, however, cannot happen as $G_i = \pi(\hat{G}_i)$ and $F_j$ are distinct facets of $\Delta_{n,k}$. Thus, $\hat{G}_i = \{y \in \hat{P} : \ell_j(y) = 0 \text{ for } j \in J\}$ for some $J \subseteq \{n + 1, \ldots, M\}$ and consequently $H_i$ supports a face of $\hat{Q}$. Moreover, $\hat{Q} \subseteq \pi^{-1}(Q)$ and hence, the lineality space of $\hat{Q}$ is contained in $\ker \pi$. However the hyperplanes $\{y : \ell_i(y) = 0\}$ with $1 \leq i \leq n$ are parallel to $\ker \pi$, because they are preimages of the hyperplanes supporting the facets $F_i$. Therefore, $\hat{Q}$ is bounded since we assumed that $\hat{P} = \hat{Q} \cap \{y : \ell_i(y) \geq 0 \text{ for } i = 1, \ldots, n\}$ is bounded. □

### 3. Rectangle covering numbers and small hypersimplices

In this section we treat the small hypersimplices $4 \leq n \leq 6$ and $1 < k < n - 1$. We will do this by way of rectangle covering numbers. The rectangle covering number, introduced in [FKPT13], is a very elegant, combinatorial approach to lower bounds on the nonnegative rank of a polytope. For a polytope $P = \{x \in \mathbb{R}^d : \ell_1(x) \geq 0, \ldots, \ell_M(x) \geq 0\} = \text{conv}(v_1, \ldots, v_N)$, where $\ell_1(x), \ldots, \ell_M(x)$ are affine linear forms, the slack matrix is the nonnegative matrix $S_P \in \mathbb{R}^{M \times N}_{\geq 0}$ with $(S_P)_{ij} = \ell_i(v_j)$. A rectangle of $S_P$ is an index set $R = I \times J$ with $I \subseteq [M]$, $J \subseteq [N]$ such that $(S_P)_{ij} > 0$ for all $(i, j) \in R$. The rectangle covering number $\text{rc}(S_P)$ is
the smallest number of rectangles $R_1, \ldots, R_s$ such that $(S_P)_{ij} > 0$ if and only if $(i, j) \in \bigcup_t R_t$. As explained in [FKPT13, Section 2.4]

$$\text{rc}(S_P) \leq \text{rk}_+(P).$$

There are strong ties between the geometry of extensions and rectangle covering numbers. In particular our geometric tools from Section 2 have independent counterparts for rectangle covering numbers. Note that although the results are structurally similar they do not imply each other and even the proofs are distinct.

**Lemma 3.1.** Let $P$ be a polytope and $F \subset P$ a facet. Then

$$\text{rc}(S_P) \geq \text{rc}(S_F) + 1.$$ 

Moreover, if there is a facet $G \subset P$ disjoint from $F$, then

$$\text{rc}(S_P) \geq \min\{\text{rc}(S_F), \text{rc}(S_G)\} + 2.$$ 

**Proof.** In the first case, part of the slack matrix of $S_P$ is of the form

$$\begin{bmatrix}
0 & \cdots & 0 & a \\
S_F & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix},$$

and since $F$ is a facet, $a > 0$. There are at least $\text{rc}(S_F)$ rectangles necessary to cover $S_F$. None of these rectangles can cover $a$ as this is obstructed by the zero row above $S_F$.

For the second case, we may assume that $r = \text{rc}(S_F) = \text{rc}(S_G)$. Similarly, we can assume that parts of $S_P$ look like

$$\begin{bmatrix}
0 & \cdots & 0 & a_1 \cdots a_l \\
b_1 \cdots b_k & 0 & \cdots & 0 \\
S_F & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
S_G & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix},$$

with $a_1, \ldots, a_l, b_1, \ldots, b_k > 0$. There are $r$ rectangles necessary to cover $S_F$. None of these rectangles can cover the first row. If the first row is covered with $\geq 2$ rectangles, we are done. If, however, a single rectangle covers the first row, then it cannot cover any row of $S_G$. Indeed, every row of $S_G$ corresponds to a facet of $G$ and contains at least one vertex of $G$. Hence, every row of $S_G$ has one zero entry. Since also $S_G$ needs at least $r$ rectangles to be covered, by the same token we obtain that the second row must be covered by a unique rectangle which does not extend to $S_F$ or $S_G$. Consequently, at least $r + 2$ rectangles are necessary. □

The example from Section 2 shows that similar to Lemma 2.3, we cannot replace min with max. It can be checked that the rectangle covering number of an octagon is 6.

As direct consequence we obtain a lower bound on rectangle covering numbers (cf. [FKPT13, Prop. 5.2]).

**Corollary 3.2.** Let $P$ be a $d$-polytope, then $\text{rc}(S_P) \geq d + 1$.

It was amply demonstrated in [KW15, FMP+15] that the rectangle covering number is a very powerful tool. We use it to compute the nonnegative rank of small hypersimplices. For a given polytope $P$ with slack matrix $S = S_P$ the decision problem of whether there is a rectangle covering with $r$ rectangles can be phrased as a satisfiability problem: For every rectangle $R_l$ and every $(i, j)$ with $S_{ij} > 0$ we designate a Boolean variable $X_{ij}$. If $X_{ij}$ is true, this signifies that $(i, j) \in R_l$. Every $(i, j)$ has to occur in at least one rectangle. Moreover, for $(i, j)$ and $(i', j')$ if $S_{ij} \cdot S_{i'j'} > 0$ and $S_{ij} \cdot S_{i'j'} = 0$, then $(i, j)$ and $(i', j')$ cannot be in the same rectangle.

The validity of the resulting Boolean formula can then be verified using a SAT solver. For
the hypersimplices $\Delta_{n,k}$ with $1 < k < n - 1$, the sizes of the slack matrix is $2n \times \binom{n}{k}$. For $n \leq 6$ these sizes are manageable and the satisfiability problem outlined above can be decided by a computer. For example, for $(n, k) = (6, 3)$ this yields 1320 Boolean variables and 55566 clauses in a conjunctive normal form presentation. The attached python script produces a SAT instance for all $(n, k, r)$ and we used lingeling [BHJ14] for the verification. This gives a computer-aided proof for the small cases which also completes our proof for Theorem 1.1.

**Proposition 3.3.** For $n \leq 6$, $\text{rc}(\Delta_{n,k}) = \text{rk}_+ (\Delta_{n,k})$ for all $1 \leq k \leq n$. In particular, $\text{rk}_+ (\Delta_{4,2}) = 6$, $\text{rk}_+ (\Delta_{5,2}) = \text{rk}_+ (\Delta_{5,3}) = 9$, and $\text{rk}_+ (\Delta_{6,2}) = \text{rk}_+ (\Delta_{6,3}) = 12$.

**Proof.** The hypersimplex $\Delta_{4,2}$ is a 3-dimensional polytope with 6 vertices and, more precisely, affinely isomorphic to the octahedron. Since the nonnegative rank is invariant under taking polars, Corollary 2.5 asserts that the nonnegative rank is indeed 6. The polytope $\Delta_{5,2}$ is a 4-dimensional polytope with 10 vertices and facets. Its nonnegative rank is 9. It was computed in [OVW, Table 3] under the ID 6014. Alternatively it can be computed with the python script in the appendix. Using, for example, polymake [GJ00], removing two non-adjacent vertices of $\Delta_{5,2}$ yields a 4-dimensional polytope $Q$ with 8 vertices and 7 facets. Taking a 2-fold pyramid over $Q$ gives an extension of $\Delta_{5,2}$ with 9 facets. Finally, $\Delta_{6,2}$ and $\Delta_{6,3}$ are 5-polytopes with 12 facets and the SAT approach using the attached python script yields the matching lower bound on the rectangle covering number. □

The hypersimplex $\Delta_{5,2}$ is special. We will examine it more closely in Section 5.1 and we will, in particular, show that up to a set of measure zero all realizations have the expected nonnegative rank 10.

It is tempting to think that Proposition 2.6 might hold on the level of rectangle covering numbers. Indeed, such a result would imply that all combinatorial hypersimplices are extension maximal. As can be checked with the python script in the appendix, Proposition 3.3 extends at least to $n = 8$. In fact, the results above imply that $\text{rc}(\Delta_{n,k}) = 2n$ when $\max \{2, n - 6\} \leq k \leq \min \{n - 2, 6\}$. However, the same script also shows that $\text{rc}(\Delta_{10,2}) \leq 19$ and the following result (a corollary of [FKPT13, Lemma 3.3]) shows just how deceiving the situation is in small dimensions.

**Proposition 3.4.** The rectangle covering number of the $(n, k)$-hypersimplex satisfies

$$n \leq \text{rc}(\Delta_{n,k}) \leq n + \lceil e(k + 1)^2 \log(n) \rceil.$$

**Proof.** The lower bound follows from Corollary 3.2. For the upper bound, consider the matrix $\mathcal{G}(n, k)$ whose columns are the 0/1 vectors with $k$ zeros. If $2 \leq k \leq n - 2$, the rows of the slack matrix of $\Delta_{n,k}$ induced by the $G_i$ facets provide a copy of $\mathcal{G}(n, k)$, and the rows induced by $F_i$ facets a copy of $\mathcal{G}(n, n - k)$. Thus,

$$\text{rc}(\Delta_{n,k}) \leq \text{rc}(\mathcal{G}(n, n - k)) + \text{rc}(\mathcal{G}(n, k)).$$

Observing that $\text{rc}(\mathcal{G}(n, n - k))$ is trivially bounded from above by $n$ (take a rectangle for each row), it suffices to see that $\text{rc}(\mathcal{G}(n, k)) \leq \lceil e(k + 1)^2 \log(n) \rceil$, which is shown in [FKPT13, Lemma 3.3].

We reproduce their nice argument for completeness. The rows and columns of $\mathcal{G}(n, k)$ are indexed by the sets $[n]$ and $\binom{[n]}{n - k}$, respectively. The non-zero elements are the pairs $(x, S) \in [n] \times \binom{[n]}{n - k}$ with $x \in S$. The inclusion-maximal rectangles are of the form

$$R_I := \left\{ (x, S) \in [n] \times \binom{[n]}{n - k} : x \in I \text{ and } I \subseteq S \right\},$$

for $I \subseteq [n]$. We can pick an $I$ at random by selecting every element in $I$ independently with probability $p = \frac{1}{1 + k}$. The probability then that an entry $(x, S)$ with $x \in S$ is covered by $R_I$ is $p(1 - p)^k$. Hence, if we choose $r = \lceil e(k + 1)^2 \log(n) \rceil$ such rectangles $R_I$ independently, then probability that an entry is not covered by any of the rectangles is $(1 - p(1 - p)^k)^r$. 

The total number of non-zero entries of \( G(n, k) \) is \( (n-k) \binom{n}{k} < n^{k+1} \). Therefore, the logarithm of the expected number of non-zero entries of \( G(n, k) \) that are not covered by any rectangle is at most

\[
\log \left( (1 - p(1 - p))^r n^{k+1} \right) = r \log (1 - p(1 - p)) + (k + 1) \log(n)
\]

\[
\leq -rp(1 - p) + (k + 1) \log(n) = -r \frac{k}{(k + 1)^{k+1}} + (k + 1) \log(n).
\]

If this is negative, then there is at least one covering with \( r \) rectangles. That is, whenever

\[
r > \frac{(k + 1)^{k+2}}{k^k} \log(n).
\]

Observing that

\[
\frac{(k + 1)^{k+2}}{k^k} \log(n) = (k + 1)^2 \left( \frac{k + 1}{k} \right)^k \log(n) < e (k + 1)^2 \log(n)
\]

concludes the proof. \( \square \)

4. COMBINATORIAL HYPERSIMPLECTES AND REALIZATION SPACES

Typically, the extension complexity is not an invariant of the combinatorial isomorphism class of a polytope (see, for example, the situation with hexagons [PP15]). However, Corollary 2.5 states that every combinatorial cube, independent of its realization, has the same extension complexity. The proximity to cubes and the results in Sections 2 and 3 raised the hope that this extends to all hypersimplices. A **combinatorial** \((n, k)\)-**hypersimplex** is any polytope whose face lattice is isomorphic to that of \( \Delta_{n, k} \). One approach would have been through rectangle covering numbers but Proposition 3.4 refutes this approach in the strongest possible sense.

We extend the notions of \( F \)- and \( G \)-facets from (3) to combinatorial hypersimplices. The crucial property that we used in the proof of Proposition 2.6 was that in the standard realization of \( \Delta_{n, k} \), the polyhedron bounded by hyperplanes supporting the \( G \)-facets is a full-dimensional simplex. We call a combinatorial hypersimplex **\( G \)-generic** if the hyperplanes supporting the \( G \)-facets are not projectively concurrent, that is, if the hyperplanes supporting combinatorial \((n-1, k-1)\)-hypersimplices do not meet in a point and are not parallel to a common line. We define the notion of **\( F \)-generic** hypersimplices likewise and we simply write \( \text{FG-generic} \) if a hypersimplex is \( F \)- and \( G \)-generic.

Now, if a combinatorial hypersimplex \( P \) is \( G \)-generic, then there is an admissible projective transformation that makes the polyhedron induced by the \( G \)-facets bounded. To find such a transformation, one can proceed as follows: translate \( P \) so that it contains 0 in the interior, then take the polar \( P^o \) and translate it so that the origin belongs to the interior of the convex hull of the \( G \)-vertices. This is possible because \( G \)-genericity implies that these vertices span a full-dimensional simplex. Taking the polar again yields a polytope \( P' \) that is projectively equivalent to \( P \). Since projective transformations leave the extension complexity invariant, the proof of Proposition 2.6, almost verbatim, carries over to \( \text{FG-generic} \) hypersimplices.

Indeed, with the upcoming Lemma 4.2, it is straightforward to verify that \( F \)-facets of an \( \text{FG-generic} \) \((n, k)\)-hypersimplex with \( 2k \geq n \) are again \( \text{FG-generic} \); and the same works with \( G \)-facets when \( 2k \leq n \). Hence, one can apply the inductive reasoning of the \( k = 2 \) case of Proposition 2.6 and together with Proposition 3.3, this proves the following theorem.

**Theorem 4.1.** If \( P \) is an \( \text{FG-generic} \) combinatorial \((n, k)\)-hypersimplex with \( n \geq 6 \) and \( 2 \leq k \leq n - 2 \), then \( \text{rk}_+(P) = 2n \).

The following lemma states that every combinatorial hypersimplex is either \( F \)-generic or \( G \)-generic.
Lemma 4.2. Every combinatorial \((n,k)\)-hypersimplex is

\begin{enumerate}[(i)]
\item \(F\)-generic if \(2k < n + 2\), and
\item \(G\)-generic if \(2k > n - 2\).
\end{enumerate}

In particular, every combinatorial \((n,k)\)-hypersimplex is \(FG\)-generic for \(n - 2 < 2k < n + 2\).

Proof. The two statements are dual under the affine equivalence \(\Delta_{n,k} \cong \Delta_{n,n-k}\). Hence, we only prove the second statement. For this, let \(P^0\) be polar to a combinatorial \((n,k)\)-hypersimplex with \(2k > n - 2\). Thus, \(P^0\) is a polytope of dimension \(n - 1\) with \(2n\) vertices \(f_1, \ldots, f_n\) and \(g_1, \ldots, g_n\) corresponding to the \(F\)- and \(G\)-facets. In this setting, \(G\)-genericity means that the polytope \(Q = \text{conv}(g_1, \ldots, g_n)\) is of full dimension \(n - 1\). From the combinatorics of \((n,k)\)-hypersimplices, we infer that for every \(I \subseteq [n]\) with \(|I| = k\), the set

\[
\text{conv}(\{g_i : i \in I\} \cup \{f_i : i \notin I\})
\]

is a face of \(P^0\) and hence \(\text{conv}(g_i : i \in I)\) is a face of \(Q\). Thus \(Q\) is a \(k\)-neighborly polytope with \(2k \geq n - 1 \geq \dim Q\). It follows from [Grü03, Thm. 7.1.4] that \(Q\) is a simplex and thus of dimension \(n - 1\). \qed

Although we will later see that not every combinatorial hypersimplex is \(FG\)-generic (cf. Proposition 5.2), this has some immediate consequences for the extension complexity of combinatorial hypersimplices. The following corollary can be deduced from Figure 2, using Corollary 2.2 to navigate along the arrows to the (thick) diagonal.

Corollary 1.3. If \(P\) is a combinatorial \((n,k)\)-hypersimplex with \(n \geq 6\), then

\[
\text{rk}_+(P) \geq \begin{cases} 
  n + 2k + 1, & \text{if } 2 \leq k < \left\lfloor \frac{n}{2} \right\rfloor, \\
  2n, & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \leq k \leq \left\lceil \frac{n}{2} \right\rceil, \\
  n + 2(n-k) + 1, & \text{if } \left\lceil \frac{n}{2} \right\rceil < k \leq n - 2.
\end{cases}
\]

Proof. For \(\left\lfloor \frac{n}{2} \right\rfloor \leq k \leq \left\lceil \frac{n}{2} \right\rceil\), we get the result as a combination of Theorem 4.1 with Lemma 4.2. If \(P\) is a combinatorial \((n,k)\)-hypersimplex with \(k < \left\lfloor \frac{n}{2} \right\rfloor\), then \(P\) has a \(2k\)-dimensional face \(Q\) isomorphic to \(\Delta_{2k+1,k}\) (by successively taking \(F\)-facets). By the previous case, \(\text{rk}_+(Q) = 2(2k+1)\). By Corollary 2.2, \(\text{rk}_+(P) \geq n + 2k + 1\). The case \(k > \left\lceil \frac{n}{2} \right\rceil\) follows symmetrically. \qed

4.1. Realization spaces of hypersimplices. A combinatorial \((n,k)\)-hypersimplex is a polytope \(P \subseteq \mathbb{R}^{n-1}\) given by \(2n\) linear inequalities \(f_i(x) = f_{i0} + \sum_j f_{ij}x_j \geq 0\) and \(g_i(x) = g_{i0} + \sum_j g_{ij}x_j \geq 0\) for \(i = 1, \ldots, n\) such that \(P\) is combinatorially isomorphic to \(\Delta_{n,k}\) under the correspondence

\[
F_i = \{x \in \Delta_{n,k} : x_i = 0\} \quad \text{and} \quad \{x \in P : f_i(x) = 0\},
\]

\[
G_i = \{x \in \Delta_{n,k} : x_i = 1\} \quad \text{and} \quad \{x \in P : g_i(x) = 0\}.
\]

Of course, the inequalities are unique only up to a positive scalar and hence the group \((\mathbb{R}_{\geq 0}, \cdot)\) acts on ordered collections of linear inequalities furnished by all combinatorial \((n,k)\)-hypersimplices in \(\mathbb{R}^{n-1}\). We only want to consider realizations that are genuinely distinct and it is customary to identify two realizations of \(\Delta_{n,k}\) that differ by an affine transformation or an (admissible) projective transformation; see, for example, [RG96, Sect. 2.1] or [BLS+99, Sect. 8.1]. We do the latter. However, care has to be taken as the projective linear group does not act on the realization space. To that end, we identify \(P\) with its homogenization

\[
\text{hom}(P) := \text{cone}(\{1\} \times P) = \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^n : g_0x_0 + \cdots + g_nx_n \geq 0 \quad \text{for } i = 1, \ldots, n \right\}.
\]

Under this identification, one verifies that two \((n,k)\)-hypersimplices \(P\) and \(P'\) are projectively equivalent if and only if \(\text{hom}(P)\) and \(\text{hom}(P')\) are linearly isomorphic. The projective realization space \(\mathcal{R}_{n,k}\) of combinatorial \((n,k)\)-hypersimplices is the set of matrices \((g_1, \ldots, g_n; f_1, \ldots, f_n) \in \mathbb{R}^{n \times 2n}\) that yield cones isomorphic to the homogenization of the
Figure 2. The smaller hypersimplices. Arrows represent the structure of the $F$- and $G$-facets. Those in the upper half are $F$-generic, those in the lower half are $G$-generic and those in the middle are $FG$-generic.

Let us fix $1 < k \leq n^2$ and let $P$ be a combinatorial $(n,k)$-hypersimplex. By Lemma 4.2, the $F$-facets are generic and hence bound a simplex $Q$ up to projective transformation. That is, $\text{hom}(Q) = \{ x \in \mathbb{R}^n : f_0x_0 + \cdots + f_nx_n \geq 0 \text{ for } i = 1, \ldots, n \} \cong \mathbb{R}^n_{\geq 0}$ by a linear transformation. Hence, we can choose a matrix representing $\text{hom}(P)$ of the form

$$
\begin{pmatrix}
| & | & | \\
g_1 & g_2 & \cdots & g_n \\
| & | & | \\
1 & 1 & \cdots & 1
\end{pmatrix}.
$$

Modulo positive column and row scaling, the matrix $(g_1, \ldots, g_n)$ uniquely determines $P$ up to projective transformations. Indeed, by using suitable positive column scaling on the $f_i$’s, the effect of positively scaling rows of (4) leaves the identity matrix of (4) invariant.

For example, a representative of the standard realization of $\Delta_{n,k}$ is given by the $G$-matrix:

$$
\begin{pmatrix}
-k + 1 & 1 & \cdots & 1 \\
1 & -k + 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & -k + 1
\end{pmatrix}.
$$

The $G$-matrix also gives us a condition for $G$-genericity. The hyperplanes to $G$-facets are projectively concurrent if there is a nonzero element in the kernel of the $G$-matrix. This proves the following useful criterion.

**Lemma 4.3.** Let $P$ be a combinatorial $(n,k)$-hypersimplex with $k \leq \frac{n^2}{2}$. Then $P$ is $G$-generic if and only if $\det(g_1, \ldots, g_n) \neq 0$.

Notice that the principal $k$-minors of (5) vanish. This is common to all combinatorial hypersimplices. Indeed, the combinatorics of hypersimplices dictates that for any $I \subseteq [n]$ with
The complex strict inequalities coming vertex-facet-nonincidences. Notice that these are the only equality constraints for $R_{n,k}$. The remaining conditions are only strict inequalities coming vertex-facet-nonincidences.

The complex variety of $n$-by-$n$ matrices with vanishing principal $k$-minors was studied by Wheeler in [Whe], which turns out to be a rather complicated object. For example, it is not known if the variety is irreducible and even the dimension is only known in certain cases. The following result yields an upper bound on the dimension of $R_{n,k}$.

**Theorem 4.4.** For $2 < k < n - 2$, every $P \in R_{n,k}$ is completely identified by a realization of $\Delta_{n-1,k-1}$ unique up to affine transformation. In particular, for $2 \leq k \leq n - 2$, the dimension of $R_{n,k}$ is at most $\binom{n-1}{2}$.

**Proof.** Let $P$ be a combinatorial $(n,k)$-hypersimplex. By a suitable projective transformation, we can assume that the facet-defining hyperplanes of $F_1$ and $G_1$ are parallel. This assumption fixes the intersection of $\text{aff}(G_1)$ with the hyperplane at infinity and hence fixes $G_1$ up to an affine transformation.

Since $F_1$ and $G_1$ lie in parallel hyperplanes, the corresponding facets of $F_1$ and $G_1$ are parallel (because they are induced by the intersection of the same supporting hyperplanes of $\Delta_{n,k}$ with these two parallel hyperplanes).

A result of Shephard (see [Grü03, Thm. 15.1.3, p.321]) states that if all the 2-dimensional faces of a polytope $R \subset \mathbb{R}^d$ are triangles, then for any representation $R = R_1 + R_2$ of $R$ as Minkowski sum, there are $t_i \in \mathbb{R}^d$ and $\lambda_i \geq 0$ such that $R_i = t_i + \lambda_i R$ for $i = 1, 2$. Now, if $Q$ and $Q'$ are normally equivalent polytopes, i.e. combinatorially equivalent and corresponding facets are parallel, and $Q$ has only triangular 2-faces, then, by [Zie95, Prop. 7.12], all 2-faces of $Q + Q'$ are triangles as well. It follows that $Q$ and $Q'$ are positively homothetic.

Since every face of a hypersimplex is a hypersimplex and 2-dimensional hypersimplices are triangles, this shows that realizations of hypersimplices are determined up to positive homothety once their facet directions are determined. In particular, this shows that given $G_1$, $F_1$ is determined up to a positive homothety. Hence, given $G_1$, $P$ is unique up to projective transformations.

The bound on the dimension follows by induction on $k$. We will see in Theorem 1.4 that $\dim R_{n,2} = \binom{n-1}{2}$, settling the base case. The affine group of $\mathbb{R}^d$ is a codimension $d$ subgroup of the projective group. Hence, by induction,

$$\dim R_{n,k} \leq \dim R_{n-1,k-1} + (n - 2) \leq \binom{n-2}{2} + (n - 2) = \frac{(n-1)!}{2!}.$$ $lacksquare$

### 5. The $(n,2)$-hypersimplices

Although realization spaces are notoriously difficult objects and it is generally difficult to access different realizations of a given polytope, in the case of $(n,2)$-hypersimplices we have a simple construction and a nice geometrical interpretation. Let us denote by $\Delta_{n-1} = \text{conv}(e_1, \ldots, e_n) \subset \mathbb{R}^n$ the standard simplex of dimension $n - 1$.

**Theorem 5.1.** For $n \geq 4$, let $p_{ij}$ be a point in the relative interior of the edge $[e_i, e_j] \subset \Delta_{n-1}$ for $1 \leq i < j \leq n$. Then

$$P := \text{conv}\{p_{ij} : 1 \leq i < j \leq n\}$$

is a combinatorial $(n,2)$-hypersimplex. Up to projective transformation, every combinatorial $(n,2)$-hypersimplex arises this way.
Proof. Since $\Delta_{n-1}$ is a simple polytope, the polytope $P$ is the result of truncating every vertex $e_i$ of $\Delta_{n-1}$ by the unique hyperplane spanned by $\{p_{ij} : j \neq i\}$. Hence, $P$ has $\binom{n}{2}$ vertices and every $p_{ij}$ is incident to exactly two facets isomorphic to $\Delta_{n-1,1} \cong \Delta_{n-2}$. If $n = 4$, then it is easily seen that $P$ is an octahedron. For $n > 4$, we get by induction on $n$ that the remaining facets $P \cap \{x : x_i = 0\}$ are isomorphic to $\Delta_{n-1,2}$, which implies the first claim.

For the second statement, let $P$ be a combinatorial $(n,2)$-hypersimplex. We know from Lemma 4.2 that the $F$-facets bound a projective simplex. By a suitable projective transformation, we may assume that this is exactly $\Delta_{n-1}$. Each vertex of $P$ lies in all the $F$-facets except for two. So every vertex of $P$ lies in the relative interior of a unique edge of $\Delta_{n-1}$. □

Note that the representation given in Theorem 5.1 is not unique up to projective transformation. The simplest way to factor out the projective transformations is to perform a first truncation at the vertex $e_1$ of $\Delta_{n-1}$. Then $\text{conv}\{p_{12}, \ldots, p_{1n}, e_2, \ldots, e_n\}$ is a prism over a simplex, which is projectively unique (cf. [Gri03, Ex. 4.8.30]). It then only remains to choose $(\binom{n}{2}-1)$ points in the interior of every edge of the base of the prism. Each choice produces a projectively distinct $(n,2)$-hypersimplex, and every $(n,2)$-hypersimplex arises this way, up to projective transformation. This completes the proof of Theorem 1.4.

**Theorem 1.4.** For $n \geq 4$, $\mathcal{R}_{n,2}$ is rationally equivalent to the interior of a $(\binom{n-1}{2})$-dimensional cube. In particular, $\mathcal{R}_{n,2}$ is homeomorphic to an open ball and hence contractible.

To recover the description as an $n \times n$ matrix, we can proceed as follows. Set the (projective) simplex $\Delta_F$ bounded by the $F$-facets to be the standard simplex. Now, for every (oriented) edge $[e_i, e_j]$ of $\Delta_F$, consider the ratio $\rho_{ij} = \frac{|e_i - p_{ij}|}{|e_j - p_{ij}|}$ for $i \neq j$. It is not hard to see that the diagonal entries of the $G$-matrix are negative and that, if we scale its columns so that they are $-1$, then we are left with the matrix

$$
\begin{pmatrix}
-1 & \rho_{12} & \cdots & \rho_{1n} \\
\rho_{21} & -1 & \cdots & \rho_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{n1} & \rho_{n2} & \cdots & -1
\end{pmatrix}
$$

containing $-1$ in the diagonal and the ratios $\rho_{ij}$ in the remaining entries. Notice how the condition on the vanishing $2 \times 2$ principal minors coincides with the relation $\rho_{ij} = \rho_{ji}^{-1}$. By Theorem 5.1, any choice of positive ratios fulfilling this relation gives rise to a realization of an $(n,2)$-hypersimplex.

All non-diagonal entries are positive by construction. Multiplying the $i$th column by $\rho_{ii}$ and the $i$th row by $\rho_{1i}$ for $2 \leq i \leq n$, which corresponds to a projective transformation, we are left with an equivalent realization. Relabelling the ratios, it is of the form:

$$
\begin{pmatrix}
-1 & 1 & \cdots & 1 \\
1 & -1 & \cdots & \rho_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \rho_{n2} & \cdots & -1
\end{pmatrix}
$$

Choosing a realization of $\Delta_{n,2}$ up to projective transformation amounts to choosing $(\binom{n-1}{2})$ positive ratios and we recover the description of Theorem 1.4. Actually, this is the transformation that we also used before, which transforms the truncated simplex into a prism over a simplex. The $(\binom{n-1}{2})$ remaining ratios correspond to the edges of the basis of the prism.

Notice that, although a similar argumentation provides the realization space up to affine transformation, that setup is slightly more delicate. While a choice of a point on each of the $(\binom{n}{2})$ edges of a standard simplex gives a unique affine realization of $\Delta_{n,2}$, not every affine realization can be obtained this way. The $F$-simplex might be unbounded for some realizations of $\Delta_{n,2}$. Hence, there are several “patches” in the affine realization space, each one corresponding to
a different relative position of the $F$-simplex with respect to the hyperplane at infinity. For
every patch, realizations are parametrized by the position of the points in the edges of $\Delta_F$.

**Example 1.** To show an example, we will work with $\Delta_{3,2}$, which we look at as two nested
projective simplices, the second being the convex hull of a point in each edge of the first. Even
though this is not strictly a hypersimplex, it provides simpler figures than $\Delta_{4,2}$. Figure 3
depicts four such realizations.

![Figure 3](image)

**Figure 3.** Four (projectively equivalent) realizations of $\Delta_{3,2}$ as two nested
(projective) simplices.

In the first three, the outer simplex is a standard simplex. Computing the ratios of (7) we
recover the matrices

\[
\begin{pmatrix}
-1 & 3 & 1 \\
\frac{1}{3} & -1 & 3 \\
1 & \frac{1}{3} & -1 \\
\end{pmatrix}
, \quad
\begin{pmatrix}
-1 & 3 & \frac{1}{3} \\
\frac{1}{3} & -1 & 1 \\
3 & 1 & -1 \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-1 & 3 & 3 \\
\frac{1}{3} & -1 & 9 \\
\frac{1}{3} & \frac{1}{9} & -1 \\
\end{pmatrix}
.\]

We can transform the first into the second (resp. third) by multiplying the third row by $3$
(resp. $\frac{1}{3}$) and the third column by $\frac{1}{3}$ (resp. 3), which means that they represent projectively
equivalent realizations. To fix a unique projective representative, we send $e_1$ to infinity, and
impose that $p_{12}, p_{13}, e_2, e_3$ form a prism over a standard simplex (in this case a square), as
in the fourth figure. The ratios that we get on the base of the prism give the entries of the
matrix corresponding to (8):

\[
\begin{pmatrix}
-1 & 1 & 1 \\
1 & -1 & 9 \\
1 & \frac{1}{3} & -1 \\
\end{pmatrix}
.\]

In this example, this projective transformation corresponds to multiplying the second row and
the third column of the first matrix by 3, and its third row and second column by $\frac{1}{3}$.

With the aid of this description, we can produce our first example of a non $FG$-generic
hypersimplex. The following is due to Francisco Santos (personal communication), who found
nicer coordinates than our original example.

**Proposition 5.2.** There are $(6,2)$-hypersimplices that are not $G$-generic.

**Proof.** The following matrix corresponds to a non-$G$-generic realization of $\Delta_{6,2}$

\[
\begin{pmatrix}
-1 & 2\sqrt{6} + 5 & -2\sqrt{6} + 5 & 1 & 1 & 1 \\
-2\sqrt{6} + 5 & -1 & 2\sqrt{6} + 5 & 1 & 1 & 1 \\
2\sqrt{6} + 5 & -2\sqrt{6} + 5 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 \\
\end{pmatrix}
.\]

It can be checked that the determinant is zero and Lemma 4.3 finishes the proof. \qed
Corollary 5.3. Not every combinatorial \((n,k)\)-hypersimplex is a facet of a combinatorial \((n + 1, k + 1)\)-hypersimplex.

Proof. Every combinatorial \((7,3)\)-hypersimplex is \(FG\)-generic by Lemma 4.2, a property that is inherited by its \(G\)-facets. Hence, the combinatorial \((6,2)\)-hypersimplex of Proposition 5.2 is not a facet of any combinatorial \((7,3)\)-hypersimplex. \(\square\)

In contrast, the shape of facets of \((n,2)\)-hypersimplices can be prescribed. This is a direct corollary of the construction from Theorem 5.1.

Corollary 5.4. Every combinatorial \((n,2)\)-hypersimplex is a facet of a combinatorial \((n + 1,2)\)-hypersimplex.

5.1. The \((5,2)\)-hypersimplex. The \((5,2)\)-hypersimplex is special. As we argued in the proof of Proposition 3.3, the nonnegative rank of the hypersimplex \(\Delta_{5,2}\) in its defining realization \((1)\) is equal to 9. In light of Theorem 1.1, this deviates from the ‘expected’ nonnegative rank. The goal of this section is to show that generic realizations of \(\Delta_{5,2}\) have nonnegative rank 10.

We just described the (projective) realization space of \(\Delta_{5,2}\) obtained by choosing an interior point in every edge of the base of \(\Delta_3 \times \Delta_1\). That means that \(\dim R_{5,2} = 6\).

We now claim that the realization \(P\) of \(\Delta_{5,2}\) given as the convex hull of the columns

\[
\begin{pmatrix}
35 & 35 & 35 & 35 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 0 & 0 & 0 & 50 & 42 & 20 & 0 & 0 & 0 \\
0 & 35 & 0 & 0 & 20 & 0 & 0 & 56 & 60 & 0 \\
0 & 0 & 35 & 0 & 0 & 28 & 0 & 14 & 0 & 42 \\
0 & 0 & 0 & 35 & 0 & 0 & 50 & 0 & 10 & 28
\end{pmatrix}
\]

has nonnegative rank 10. To prove this, we again use a computer to compute the refined rectangle covering number from [OVW]. Like the ordinary rectangle covering number, the refined rectangle covering number yields lower bounds on the nonnegative rank of a polytope \(P\) but instead of only the support pattern of the slack matrix \(S_P\), it takes into account simple relations between the actual values. A covering of \(S = S_P\) by rectangles \(R_1, \ldots, R_s\) is refined if for any pair of indices \((i,k), (j,l)\) such that

\[
S_{ik}, S_{jl}, S_{il}, S_{jk} > 0 \text{ and } S_{ik}S_{jl} \neq S_{il}S_{jk},
\]

then the number of rectangles containing \(S_{ik}\) or \(S_{jl}\) is at least two. (necessarily positive) entries \(S_{ik}, S_{jl}\) are contained in at least two rectangles. Of course, if \(S_{il}\) or \(S_{jk}\) are zero, then this reduces to the condition for ordinary coverings by rectangles. The refined rectangle covering number \(rcc(S_P)\) is the least size of a refined covering. It is shown in [OVW, Theorem 3.4] that \(rcc(S_P)\) lies between \(rc(S_P)\) and \(rk_+(P)\) and thus yields a possibly better lower bound on the nonnegative rank. We will work with the following relaxation that we call the generic refined rectangle covering number. We consider coverings by ordinary rectangles with the additional condition that for any pair of indices \((i,k), (j,l)\) such that

\[
S_{ik}, S_{jl}, S_{il}, S_{jk} > 0 \text{ and } S_{ik}S_{jl} \neq S_{il}S_{jk},
\]

the four entries \(S_{ik}, S_{jl}, S_{il}, S_{jk}\) are covered by at least two rectangles. The denomination ‘generic’ is explained in the proof of Theorem 5.5 below.

As for the rectangle covering number, to determine if there is a generic refined covering of a given size can be phrased as a Boolean formula. For the example given above and the number of rectangles set to 9, a python script in the appendix produces such a formula with 450 Boolean variables and 16796 clauses. Any suitable SAT solver verifies that this formula is unsatisfiable which proves that the \((5,2)\)-hypersimplex given in \((9)\) has nonnegative rank 10.

Theorem 5.5. The combinatorial \((5,2)\)-hypersimplices with nonnegative rank 10 form a dense open subset of \(R_{5,2}\).
Proof. Let $I$ be the collection of all pairs of indices $(i, k; j, l)$ satisfying (11) in the realization of $\Delta_{5,2}$ given in (9). The set of realizations $X \subseteq \mathcal{R}_{5,2}$ for which (11) is not satisfied for some $(i, k; j, l) \in I$ form an algebraic subset of $\mathcal{R}_{5,2}$. Since the realization (9) is not in $X$, this shows that $X$ is a proper subset and hence of measure zero in the open set $\mathcal{R}_{5,2}$. □

The same result extends easily to the remaining $(n, 2)$-hypersimplices.

**Corollary 1.5.** For $n \geq 5$, the combinatorial $(n, 2)$-hypersimplices with extension complexity $2n$ are dense in $\mathcal{R}_{n,2}$.

**Proof.** For $n = 5$, this is the previous result. For $n \geq 6$, this follows from Theorem 4.1 with the fact that $FG$-generic hypersimplices form a dense open subset of $\mathcal{R}_{n,2}$, because $G$-genericity is equivalent to the determinant of (8) being non-zero (Lemma 4.3). □

As of now, we are unable to extend this result to higher values of $k$. Nevertheless, we conjecture that except for a subset of zero measure, all $(n, k)$-hypersimplices have nonnegative rank $2n$ (for $n \geq 5$). (Notice that the existence of particular instances already ensures the existence of open neighborhoods of hypersimplices of nonnegative rank $2n$.)

**Conjecture 2.** For $n \geq 5$ and $2 \leq k \leq n − 2$, the combinatorial hypersimplices of nonnegative rank $2n$ form a dense open subset of $\mathcal{R}_{n,k}$.

Theorem 4.1 implies that combinatorial hypersimplices of nonnegative rank $2n$ are an open subset of $\mathcal{R}_{n,k}$. However, notice that $\mathcal{R}_{n,k}$ is a quotient of the variety of vanishing principal $k$-minors, which has irreducible components of different dimensions [Whe]. We cannot certify that the nonnegative rank $2n$ subset is dense because it could skip a whole component.

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