The weakly perturbed Schwarzschild lens in the strong deflection limit

V. Bozza\textsuperscript{a,b}, M. Sereno\textsuperscript{c}
\begin{itemize}
  \item \textsuperscript{a} Dipartimento di Fisica ‘E.R. Caianiello’, Università di Salerno, via Allende, I-84081 Baronissi (SA), Italy.
  \item \textsuperscript{b} Istituto Nazionale di Fisica Nucleare, Sezione di Napoli.
  \item \textsuperscript{c} Institut für Theoretische Physik der Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
\end{itemize}

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We investigate the strong deflection limit of gravitational lensing by a Schwarzschild black hole embedded in an external gravitational field. The study of this model, analogous to the Chang &Refsdal lens in the weak deflection limit, is important to evaluate the gravitational perturbations on the relativistic images that appear in proximity of supermassive black holes hosted in galactic centers. By a simple dimensional argument, we prove that the tidal effect on the light ray propagation mainly occurs in the weak field region far away from the black hole and that the external perturbation can be treated as a weak field quadrupole term. We provide a description of relativistic critical curves and caustics and discuss the inversion of the lens mapping. Relativistic caustics are shifted and acquire a finite diamond shape. Sources inside the caustics produce four sequences of relativistic images. On the other hand, retro-lensing caustics are only shifted while remaining point-like to the lowest order.

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I. INTRODUCTION

Gravitational lensing in the Strong Deflection Limit (SDL) is emerging as a promising tool for black hole investigation. It is well known that photons making one or more complete loops around the black hole before approaching the observer produce two infinite series of images very close to the shadow of the black hole [1, 2, 3]. However, this fact became an astrophysical breakthrough only when the super-massive black hole supposed to be hosted in the radio-source Sgr A* in the Galactic center was proposed as an ideal candidate for detecting relativistic images of sources passing behind it [4]. SDL investigations strongly benefitted from expansion techniques of the deflection angle [5], which were then generalized to arbitrary spherically symmetric metrics [6]. Theoretical properties of SDL lensing are now well assessed for a large class of black holes, since effects of charge [7, 8], rotation [9, 10] and several extended theories of gravitation [6, 11, 12, 13, 14, 15, 16, 17, 18] have been considered.

On the observational side, lensing effects are expected to play a significant role in all high-resolution observations. However, a clean detection of a deeply relativistic image of some source is by no way an easy task, since Sgr A* must be resolved at least at microarcsecond level, and a source is needed with a surface brightness much higher than Sgr A* itself in the spectral band in which observations are lead. The development of Very Long Baseline Interferometry (VLBI) techniques in different bands of the spectrum is fast approaching the microsecond resolution required to distinguish such relativistic images (for an updated discussion on observational perspectives of relativistic images, see e.g. Ref. [19]). As discussed in Ref. [10], the best prospect comes from relativistic images of low mass X-ray binaries around Sgr A* [20], which should be detected by future X-ray interferometry missions such as MAXIM (http://maxim.gsfc.nasa.gov).

Supermassive black holes should be hosted in galactic bulges and the surrounding environment could perturb the space-time of the main lens. As the black hole is part of a larger system, the gravitational field around it can not be exactly spherically symmetric. In some cases, large clusters of stars or secondary black holes are known to lie close to supermassive black holes. In this paper, we want to discuss the effect of external perturbations on the trajectories of light rays that are highly deflected. In the weak deflection limit, lenses with perturbed symmetry are usually modelled as quadrupole lenses [21]. If the gravitational field of the perturber changes very little over the relevant length scale of the black hole, it is enough to expand the potential of the external macro-lens up to the first non trivial term, i.e. the quadratic term. The perturbed Schwarzschild lens in the weak deflection limit is known as Chang-Refsdal lens after its first investigators [22, 23]. It was also recently revisited in Ref. [24]. Here, we offer a complementary treatment in the SDL, which may be relevant to clarify the role of external fields in the formation of relativistic images. The paper is structured as follows. In Section \[II\] we introduce the general lens equation for an unperturbed Schwarzschild black hole and specify it for the two cases of standard lensing and retro-lensing. In Section \[III\] we analyze the perturbation on the deflection angle exerted by an external weak field, showing that it mainly occurs in the weak field regime. In Section \[IV\] we analyze the perturbed lens equation in the standard lensing geometry and in the retro-lensing configuration, finding critical curves, caustics and images. Section \[V\] is devoted to some final considerations.
II. THE SCHWARZSCHILD LENS IN THE STRONG DEFLECTION LIMIT

If the lens is spherically symmetric, as in the case of a Schwarzschild black hole, the motion of the photon takes place on a single plane. Then the lens equation relating the angular position of the source $\beta$ and the angular position of a lensed image $\theta$, relative to the position of the lens in the sky, can be found by simple trigonometry, looking at Fig. 1.

We denote by $D_L$, $D_S$ and $D_{LS}$ the distance between observer and lens, observer and source, and lens and source respectively. They are related by Carnot theorem $D_{LS}^2 = D_L^2 + D_S^2 - 2D_L D_S \cos \beta$. We also introduce the angle $\gamma$ as the angular position of the source as seen from the lens. The angle $\theta$ is the emission angle of the light ray from the source w.r.t the lens. The deflection angle $\alpha_{BH}$ is the angle between the asymptotic incident and outgoing directions of the photon.

The impact parameter $u$ can be taken as the distance between the lens and the incident direction or the outgoing direction. The two distances are indeed the same by time-reversal symmetry. This implies the equality

$$D_L \sin \theta = D_{LS} \sin \gamma.$$  \hspace{1cm} (1)

Let us consider the triangle $OLS$. Indicating the angle $\angle OSL$ by $\zeta$, the sum of the internal angles gives

$$\zeta + \beta + (\pi - \gamma) = \pi.$$  \hspace{1cm} (2)

Considering the triangle $OCS$, the sum of the internal angles gives

$$(\theta - \beta) + (\gamma - \zeta) + (\pi - \alpha_{BH}) = \pi.$$  \hspace{1cm} (3)

Summing the two equations, we get the relation

$$\gamma = \theta + \gamma - \alpha_{BH},$$  \hspace{1cm} (4)

which has already been introduced as lens equation in Refs. [9, 10, 25]. Substituting this expression for $\gamma$ in the equality $D_S \sin \beta = D_{LS} \sin \gamma$, we get

$$D_S \sin \beta = D_{LS} \sin \gamma \cos(\alpha_{BH} - \theta) - D_{LS} \cos \gamma \sin(\alpha_{BH} - \theta).$$  \hspace{1cm} (5)

Using Eq. (1) to eliminate $\gamma$, we finally get the general form of the lens equation

$$D_S \sin \beta = D_L \sin \theta \cos(\alpha_{BH} - \theta) - \sqrt{D_{LS}^2 - D_L^2 \sin^2 \theta} \sin(\alpha_{BH} - \theta).$$  \hspace{1cm} (6)

This equation is more general than the lens equation derived in Ref. [4], which was derived assuming that $C$ belongs to the lens plane. That assumption is only valid for small $\alpha_{BH}$, making that equation inadequate for retro-lensing or other cases when the incident direction differs very much from the outgoing direction. Our lens equation is completely general and can be used to describe any configuration.
If the lens is an unperturbed Schwarzschild black hole of mass $m$, the deflection angle takes the classical weak field form $4m/u$ for large impact parameters $u$, where we are using geometrized units (gravitational constant $G = 1$, speed of light in vacuum $c = 1$). For small impact parameters, the deflection angle grows more and more, diverging at $u = u_m = 3\sqrt{3}m$. The calculation of the exact deflection angle as a function of $u$ has been first carried out by Darwin [1] analytically in terms of elliptic integrals. A good approximation, valid for small impact parameters (large deflection angles), is the so-called Strong Deflection Limit (SDL), firstly introduced by Darwin himself and lately generalized by Bozza [6] to any spherically symmetric black hole. It reads

$$\hat{\alpha}_{BH} \simeq -\hat{a} \ln \left( \frac{u}{u_m} - 1 \right) + \hat{b},$$

(7)

For a Schwarzschild black hole, $\hat{a} = 1$, $\hat{b} = -0.40002$. Note that $\theta_m \simeq u_m/D_L$ corresponds to the angular radius of the apparent shadow of the black hole, which is typically very small in all cases of interest. For example, it is expected to be $\sim 23 \mu$as for the supermassive black hole at the center of our Galaxy. This formula is strictly valid for $D_{LS}, D_L \gg 2m$. For sources or observers very close to the black hole it needs to be corrected.

As shown in several papers [1, 3, 4, 5, 6, 26, 27], relativistic images formed by light rays experiencing large deflections are prominent only when the source is in proximity of a caustic point. The caustic points are behind the lens and in front of the lens. The first class of caustic points is relevant for standard gravitational lensing configurations, in which the source is on the same side of the observer and the light rays must turn around the black hole in order to hit the observer. In the following, we shall specify the general lens equation for sources close to these two classes of caustic points.

### A. Lens equation for standard lensing geometry

Considering sources very close to the optical axis behind the black hole, we have $\beta, \gamma \ll 1$, while the relation $D_L \gg 2m$ implies $\theta, \vartheta \ll 1$. Then, Eq. (4), interpreted as an equation valid modulo $2\pi$, implies $\hat{\alpha}_{BH} = 2n\pi + \Delta \hat{\alpha}_{BH}$, with $\Delta \hat{\alpha}_{BH} \ll 1$. In consequence of this, all trigonometric functions in Eq. (6) can be expanded for small angles. Taking the first order terms of these expansions, we get

$$\beta \simeq \theta - \frac{D_{LS}}{D_S} \Delta \hat{\alpha}_{BH},$$

(8)

where we have also used that $D_S \simeq D_L + D_{LS}$. This is the lens equation for standard lensing configurations. The weak field lens equation is included for $n = 0$, i.e. when the deflection angle is very close to zero. Then $\Delta \hat{\alpha}_{BH} \equiv 0$ coincides with $\hat{\alpha}_{BH}$. On the other hand, for small impact parameters, the deflection angle grows very large, making the photon complete one or more loops around the black hole. In such cases, in the lens equation one has to consider the offset $\Delta \hat{\alpha}_{BH} = \hat{\alpha}_{BH} - 2n\pi$.

If the source is behind the lens, we have images for each integer $n$, corresponding to a deflection angle that is close to a multiple of $2\pi$. In consequence of this, we are mainly interested in those values of $\theta$ such that $\hat{\alpha}_{BH}$ is a multiple of $2\pi$. Indicating these values by $\theta_n^0$, we have $\hat{\alpha}_{BH}(D_L\theta_n^0) = 2n\pi$. From Eq. (7), we get a simple expression for $\theta_n^0$

$$\theta_n^0 \simeq \theta_m(1 + e_n), \quad e_n \equiv e^{(b-2n\pi)/\hat{a}},$$

(9)

Then, letting $\epsilon \equiv (\theta - \theta_n^0)/\theta_n^0$ and expanding the off-set around $\theta_n^0$, we get

$$\Delta \hat{\alpha}_{BH}^n = -\frac{\hat{a}}{e_n \theta_m^0} \epsilon.$$  

(10)

The lens equation reduces to

$$\beta \simeq \theta_n^0 \left( 1 - \frac{\epsilon}{\epsilon_n^E} \right),$$

(11)

where

$$\epsilon_n^E \equiv \frac{D_S}{D_{LS}} \frac{e_n}{\hat{a} \theta_m}.$$  

(12)

and we have already taken into account that $\epsilon, \epsilon_n^E \ll 1$ to get rid of terms of order $\epsilon$. 


Images on the same side of the source solve Eq. [12] for a positive $\beta$. The solution of the same equation with the source placed in $-\beta$ gives the position of the images on the opposite side. If $\beta = 0$, the source is lensed in an infinite sequence of relativistic Einstein rings at $\theta = \theta_n^0(1 + \epsilon_n^E)$. For $\beta \neq 0$, two infinite sequences of relativistic images form on opposite sides of the black hole at

$$
\epsilon = \left(1 + \frac{\beta}{\theta_n^0}\right) \epsilon_n^E.
$$

(13)

Note that for any relativistic image $\epsilon \approx \epsilon_n^E = \frac{D_B}{D_L} \theta_n^0$. Thus, unless $D_{LS} \approx 2m$, we have $\epsilon \ll 1$, so that $\theta_n^0$ already represents an excellent approximation to the position of the $n$-th relativistic image.

### B. Lens equation for retro-lensing geometry

The treatment presented up to now only covers standard lensing geometry, with a source almost aligned behind the black hole. The modifications needed to account for the case in which the source is in front of the lens are very simple. In fact, redefining $\theta_n^0$ by the equation $\tilde{\alpha}_{BH}(D_L \theta_n^0) = (2n - 1)\pi$, we get

$$
\theta_n^0 = \theta_m(1 + \epsilon_n), \quad \epsilon_n = e[(2n-1)\pi/\tilde{a}].
$$

(14)

The expansion of Eq. (6) for small angles and $\tilde{\alpha}_{BH} = (2n - 1)\pi + \Delta \tilde{\alpha}_{BH}^n$ leads to

$$
\beta \simeq -\left(1 + 2 \frac{D_{LS}}{D_S} \right) \theta + \frac{D_{LS}}{D_S} \Delta \tilde{\alpha}_{BH}^n,
$$

(15)

which differs from Eq. (8) for two signs coming from the expansions of the trigonometric functions and from the fact that now $D_S = D_L - D_{LS}$. Since the magnification privileges sources close to the black hole, here we only consider $D_{LS} < D_L$, while Ref. [27] considers the other case $D_{LS} > D_L$, with a slightly different expression for the lens equation.

The expression for $\Delta \tilde{\alpha}_{BH}^n$ remains the same given in Eq. (10), so that the lens equation can be written as

$$
\beta \simeq -\left(1 + 2 \frac{D_{LS}}{D_S} \right) \theta_n^0 \left(1 - \frac{\epsilon}{\epsilon_n^E}\right),
$$

(16)

where now

$$
\epsilon_n^E = -\frac{D_S}{D_{LS}} \frac{e_n \theta_m}{\tilde{a}} \left(1 + 2 \frac{D_{LS}}{D_S}\right)^{-1}.
$$

(17)

The relativistic Einstein rings form at $\epsilon = \epsilon_n^E$. For $\beta \neq 0$, the relativistic images form at

$$
\epsilon = \left[1 \pm \frac{\beta}{\theta_n^0} \left(1 + 2 \frac{D_{LS}}{D_S}\right)^{-1}\right] \epsilon_n^E.
$$

(18)

### III. DEFLECTION ANGLE IN A PERTURBED SCHWARZSCHILD GEOMETRY

We assume that the perturbation is weak, so that the gravitational potential $\varphi_Q$ induced by the tidal field is small throughout the photon trajectory. The order of the curvature radius $R$ of the external field at the black hole position is then determined by the scale of the variations in the gravitational potential induced by the perturber. Indicating this scale by $b$, we can thus establish the relation

$$
R^{-2} \approx \partial_i \partial_j \varphi_Q \approx b^{-2}|\varphi_Q|.
$$

(19)

If the scale of the variations $b$ is much larger than the Schwarzschild radius of the black hole $2m$, this relation automatically implies $R \gg 2m$. For example, if the external field is due to a second point-mass, we have $\varphi_Q = -2m_2/r_2$, where $r_2$ is the distance of the generic point from the second mass. In this case, the scale of the variation of such a potential around the first black hole is dictated by the distance $b$ between the two black holes and $R^{-2} \approx m_2/b^3$, coherently with what we know from the curvature invariants of the Schwarzschild metric.

Coming back to the general situation, the metric of a spherically symmetric black hole distorted by an external gravitational field can be calculated with an expansion in inverse powers of the local radius of curvature $R$ of the
external spacetime. The first corrections, proportional to $R^{-2}$, have been calculated by Alvi [28], who found an approximate form of the metric of a binary system of black holes. In particular, he divided the spacetime in four regions: two of them surround the two black holes, an intermediate region is then encircled by the asymptotic region far from the two black holes. For each region, he found an approximated expression of the metric, nicely matching the metrics of the neighbor regions in the overlap zones. These approximate solutions were proposed as initial conditions for numerical programs designed to find the best approximation to the whole spacetime. Recently, Poisson [29] gave the general expression for the tidal distortion of a Schwarzschild black hole embedded in an arbitrary spacetime up to terms proportional to $R^{-3}$. In his work, the external spacetime determines the tidal fields through the electric and magnetic components of the Weyl tensor. These fields are then multiplied by radial functions to be determined solving the Einstein equations. The metric found by Poisson reproduces Alvi’s metric when the tidal fields are generated by a second Schwarzschild black hole. What matters for us, as we shall see in a while, is that the first corrections close to the black hole are always of order of $R^{-2}$.

In an asymptotically flat spacetime the deflection vector is simply given by the difference between the incoming and outgoing tangent vectors to the photon path. Integrating the geodesics equation, we get

$$\hat{\alpha} = \hat{x}_i^i - \hat{x}_\text{out} = \int \Gamma^i_{\mu\nu} \dot{x}^\mu \dot{x}_\nu \, d\tau,$$

(20)

where $\Gamma^i_{\mu\nu}$ is the affine connection and $\tau$ is an affine parameter. The integral can be split into three pieces: one covering the approach to the black hole, where the effects due to the black hole and the perturber are both weak; a second piece where the photon winds around the tidally distorted black hole under its strong gravitational field; a third piece where the photon goes away from the black hole and the deflections due to black hole and perturber are again weak. Denoting by $-\tau_{\text{Strong}}$ and $\tau_{\text{Strong}}$ the values of the affine parameter where the transitions between the different regimes take place, we have

$$\hat{\alpha} = \hat{\alpha}_\text{in} + \hat{\alpha}_{\text{Strong}} + \hat{\alpha}_\text{out},$$

(21)

$$\hat{\alpha}_\text{in} = \int_{-\infty}^{-\tau_{\text{Strong}}} \Gamma^i_{\mu\nu} \dot{x}^\mu \dot{x}_\nu \, d\tau,$$

(22)

$$\hat{\alpha}_{\text{Strong}} = \int_{-\tau_{\text{Strong}}}^{\tau_{\text{Strong}}} \Gamma^i_{\mu\nu} \dot{x}^\mu \dot{x}_\nu \, d\tau,$$

(23)

$$\hat{\alpha}_\text{out} = \int_{\tau_{\text{Strong}}}^{\infty} \Gamma^i_{\mu\nu} \dot{x}^\mu \dot{x}_\nu \, d\tau.$$

(24)

The first and the third piece can be calculated in the weak field regime, where $\Gamma^i_{\mu\nu} \dot{x}^\mu \dot{x}_\nu \simeq 2(\nabla_\perp \varphi)^i$, with $\nabla_\perp \varphi$ being the projection of the gradient onto the plane orthogonal to the direction of the light ray and $\varphi$ the Newtonian potential, given by the sum of the black hole and perturber contributions $\varphi = \varphi_{\text{BH}} + \varphi_Q$. The central piece, $\hat{\alpha}_{\text{Strong}}$, can be expanded in inverse powers of the tidal radius $R$. Following Poisson [29], the first order correction $\hat{\alpha}_{\text{Strong}}^{(1)}$ is of order of $R^{-2}$. We can thus write

$$\hat{\alpha} \simeq \hat{\alpha}_{\text{in}} - \hat{\alpha}_{\text{out}} + \int_{-\infty}^{-\tau_{\text{Strong}}} \nabla_\perp \varphi_{\text{BH}} \, d\tau + \int_{\tau_{\text{Strong}}}^{\infty} \nabla_\perp \varphi_{\text{BH}} \, d\tau$$

$$+ 2 \int_{-\infty}^{-\tau_{\text{Strong}}} \nabla_\perp \varphi_{Q} \, d\tau + \int_{\tau_{\text{Strong}}}^{\infty} \nabla_\perp \varphi_{Q} \, d\tau.$$

(25)

The first three pieces reconstruct the unperturbed Schwarzschild deflection vector $\hat{\alpha}_{\text{BH}}$, related to the scalar deflection angle introduced in the previous section by

$$|\hat{\alpha}_{\text{BH}}| = 2 \left| \sin \frac{\hat{\alpha}_{\text{BH}} - 2n\pi}{2} \right|.$$

(26)

Note that the deflection vector, as defined by Eq. (20), is independent of the number of loops performed by the photon around the black hole, since it only refers to the asymptotic directions.
Now let us consider the terms introduced by the perturber. The perturbation in the metric of a tidally distorted black hole is of the order of $R^{-2}$, i.e. $\delta g_{\mu\nu} \approx (r/R)^2$. Then, $\delta \Gamma^i_{\mu\nu} \approx (r/R^2)$. The contribution to the deflection angle due to the external perturbation is then given by an integral over the strong field region, whose size is governed by the Schwarzschild radius of the black hole $2m$, of the perturbation to the affine connection in the tidally distorted Schwarzschild metric,

$$\hat{\alpha}^{(1)}_{\text{Strong}} \approx \int_{-2m}^{2m} \frac{\tau}{R^2} d\tau \approx \left(\frac{2m}{R}\right)^2$$

(27)

This is what expected on a dimensional analysis since $\alpha^{(1)}_{\text{Strong}} \propto R^{-2}$ and, apart from the local curvature radius of the external field providing the scale relevant for tidal perturbations, the only other length is $2m$ itself.

On the other hand, the two weak field contributions contain integrals of $\nabla_{\perp} \varphi_Q \approx \frac{1}{b} \varphi_Q$ and cover the whole photon trajectory, safe for a small piece whose length is of order $2m$. Since the scale of variations $b$ determines the size of the region where $\varphi_Q$ is relevant, the two weak field contributions are of order

$$|\hat{\alpha}_Q| = \left| 2 \int_{-\infty}^{-\tau_{\text{Strong}}} \nabla_{\perp} \varphi_Q d\tau + 2 \int_{\tau_{\text{Strong}}}^{\infty} \nabla_{\perp} \varphi_Q d\tau \right| \approx |\varphi_Q|.$$  

(28)

Recalling Eq. (13), we see that the contribution to the deflection due to the perturber from the weak field region is much larger than the strong field contribution.

We can thus conclude that the perturbation of the trajectory of a photon passing close to a black hole due to an external field is dominated by the weak field contribution and that the deviations arising in the strong regime can be safely neglected. If we replace the two integrals in the range $[-\infty, -\tau_{\text{Strong}}] \cup [\tau_{\text{Strong}}, \infty]$ by the integral covering the whole dominion $[-\infty, \infty]$, we commit a relative error of order $2m/b \ll 1$. We can finally express the whole deflection vector as

$$\hat{\alpha} = \hat{\alpha}_{\text{BH}} + \hat{\alpha}_Q$$

(29)

$$\hat{\alpha}_Q \approx 2 \int_{-\infty}^{\infty} \nabla_{\perp} \varphi_Q d\tau.$$  

(30)

So we have recovered a superposition principle between the strong deflection due to the black hole and the weak deflection due to the external field. This has been possible since the perturbation of the photon trajectory arising in the strong field regime is very small compared to that arising in the weak field regime. As we shall see in the following sections, the consequence of this crucial fact is that the lens equation preserves a very simple structure, allowing a straightforward analysis of the effects of the external field on the relativistic images.

IV. PERTURBED SCHWARZSCHILD LENS

A. Standard configuration

Since the strong deflection angle due to the black hole, $\hat{\alpha}_{\text{BH}}$, and the weak deflection angle due to the external field, $\hat{\alpha}_Q$, can be added, the lens equation will take a very simple form. However, the lens equation (9) and its specifications in the standard [6] and retro-lensing [13] geometries can not be directly generalized to the presence of an external field. The superposition principle was derived assuming that total deflection occurs by steps. After the photons wind around the black hole, they quickly reach the region where the only weak gravitational field is effective and eventually approach the asymptotic trajectory to the observer with impact parameter $u$. In other words, the impact parameter $u_{\text{BH}}$ which enters in the expression for the deflection angle due to the black hole in Eq. (11) differs from the impact parameter $u$ as seen from the observer after the action of the perturber. Equation (11) was explicitly derived assuming $u_{\text{BH}} = D_L \theta$, which is no longer true when an external deflection is present, and can not be directly applied.

Let us define a coordinate system centered on the black hole and with the $z$-axis oriented towards the observer. Then let us consider a photon coming out from the black hole with an impact parameter $u_{\text{BH}}$. We shall indicate its projection on the lens plane by the vector $u_{\text{BH}}$. As the photon travels towards the observer, it will still be deflected...
by the external field. Its trajectory can be thus parameterized as
\[ x(\tau) = x_0 + \dot{x}_O(\tau - \tau_O) - 2 \int_{\tau}^{\tau_0} d\tau' \int_{\tau'}^{\tau_0} \nabla_\perp \varphi_Q d\tau'', \tag{31} \]

where \( x_0 = (0, 0, D_L) \) is the observer position; \( \dot{x}_0 = (-\theta_1, -\theta_2, 1) \) is the tangent vector when the photon reaches the observer (to first order in \( \theta \)) and \( \tau_O \approx D_L \) is the value of the affine parameter in which the photon reaches the observer. Note that here we are only including the weak field shift due to the external field, as the black hole deflection is nearly ineffective in the weak field region. We will come back later to this point for its full clarification. Furthermore, we are only interested in how the perturber affects the relation between the asymptotic trajectory and the impact parameter at the black hole position.

The integrals in Eq. (31) can be calculated along the unperturbed trajectory \( x^{(0)}(\tau) = (u_1^{BH}, u_2^{BH}, \tau). \) Moreover, the integrand is significant only for \( \tau \lesssim b \) so the result does not change if we push the upper integration limit \( \tau_O \) to infinity. So, evaluating \( x(\tau) \) at \( \tau = 0 \) and identifying \( u_{BH} \) with \( x(0) \), we finally have
\[ u_{BH} = x(0) = D_L \theta - \Delta u_O, \tag{32} \]
\[ \Delta u_O = 2 \frac{1}{b^2} \int_0^{\infty} dz' \int_{z'}^{\infty} \nabla_\perp \varphi_Q(u_1^{BH}, u_2^{BH}, z', z'') dz''. \tag{33} \]

Since \( u = D_L \theta \), we see how the presence of an external perturbation modifies the relation between angles and impact parameters. The lens and the relativistic images it produces are shifted by \( \Delta u_O / D_L \) in the observer sky w.r.t. their true position relative to the black hole.

Let us estimate the order of magnitude of the shift term in Eq. (33). By definition, \( \Delta u_O \) is the spatial shift due to the deflection action of the perturber between the black hole and the observer. As before, the transverse gradient introduces a factor \( b^{-1} \), while the two integrals multiply the result by \( b^2 \). We thus have \( |\Delta u_O| \approx \varphi_Q b \). This quantity might be comparable with \( u_{BH} \) itself. To understand this, as perturbing body let us consider a second black hole with mass \( m_2 \) at distance \( b \). Then \( |\Delta u_O| \approx 2 m_2 \) and if the second black hole has a mass similar to the first one, \( u_{BH} \approx 2 m_1 \approx 2 m_2 \approx |\Delta u_O| \).

The reader may wonder why in the standard weak field limit we do not need to evaluate this kind of shift of the lens plane. Indeed, there is a big difference. In fact, in the weak field, the scale of variations in the deflection angle is what we have called \( b \) and is very large if the photons pass very far from the black hole. Then, a shift of order \( \varphi_Q b \) in the lens plane is absolutely negligible on the scale of the deflection \( b \). It is practically a higher order term in a weak deflection expansion. For relativistic images, where we need a resolution of the order of the Schwarzschild radius \( 2 m_1 \), this shift of the apparent lens position is of the same order of the relative positions of the images.

Another point to clarify is the fact that we are assuming that the whole deflection by the black hole takes place in a region of order \( 2 m_1 \), neglecting any weak field deflection by the black hole. Indeed, the weak field deflection induced by the black hole is governed by the transverse gradient of its potential. Yet the photon is coming out with a very small impact parameter, so the transverse gradient is of order \( 2 m_1 u_{BH} / b^3 \approx (2 m_1)^2 / b^3 \) while the transverse gradient of the potential of the perturber is \( \varphi_Q / b \), so that the first is of higher order in the gravitational potential w.r.t. the second. So, we can safely assume that the photon coming out from the black hole reaches the asymptotic trajectory in a small impact parameter, so the transverse gradient is of order \( (2 m_1)^2 / b^3 \) while the transverse gradient of the potential of the perturber is \( \varphi_Q / b \), so that the first is of higher order in the gravitational potential w.r.t. the second. So, we can safely assume that the photon coming out from the black hole reaches the asymptotic trajectory in few Schwarzschild radii and suffers no more bending by the black hole. This can be explicitly verified plotting the trajectory of a photon experiencing a very large deflection.

To get the lens equation, we must relate \( \theta \) to \( \beta \). This can be done adding the information on the photon path between the source and the black hole in analogy to Eq. (32). In fact, the photon trajectory from the source to the black hole can be parameterized by
\[ x(\tau) = x_S + \dot{x}_S(\tau - \tau_S) - 2 \int_{\tau}^{\tau_0} d\tau' \int_{\tau_S}^{\tau_0} \nabla_\perp \varphi_Q d\tau'', \tag{34} \]

where \( x_S = (D_S \beta_1, D_S \beta_2, -D_{LS}) \) is the source position; \( \dot{x}_S = (\dot{\alpha}_1 - \dot{\theta}_1, \dot{\alpha}_2 - \dot{\theta}_2, 1) \) is the tangent vector when the photon leaves the source (to first order in \( \theta \) and \( \dot{\alpha} \)) and \( \tau_S \approx -D_{LS} \) is the value of the affine parameter in which the photon leaves the source. Note that the deflection vector entering here is the total deflection given by Eq. (20).

Let us determine the point where the photon hits the lens plane. Since we are neglecting the effects of the external field when the photon winds around the black hole, the motion around the black hole is still planar. Then, the
incidence point coincides with the exit point \( u_{BH} \). We thus set \( x(0) = u_{BH} \) in Eq. (34), so that

\[
u_{BH} = x(0) = D_S \beta + D_{LS}(\alpha - \theta) - \Delta u_S \tag{35}\]

\[
\Delta u_S \equiv 2 \int_0^\infty dz' \int_{-\infty}^{\infty} \nabla_\perp \varphi_Q(u_1^{BH}, u_2^{BH}, z') dz''. \tag{36}\]

Now, eliminating \( u_{BH} \) between Eqs. (32) and (35), we get the perturbed Schwarzschild lens equation in the standard geometry

\[
\beta = \theta - \frac{D_{LS}}{D_S} (\alpha_{BH} + \alpha_Q) + \frac{\Delta u_S - \Delta u_O}{D_S}. \tag{37}\]

Apart from the presence of the perturber deflection angle \( \alpha_Q \), we have the two shifts \( \Delta u_S \) and \( \Delta u_O \). By their definition, it is easy to verify that

\[
\Delta u_S - \Delta u_O = D_{LS} \hat{\alpha}_Q - 2 \int_{-D_{LS}}^{D_L} dr' \int_{-D_{LS}}^{D_L} \tau' \nabla_\perp \varphi_Q d\tau'' \tag{38}\]

The total shift is related to the discrepancy between the position of the center of the perturber and the plane of the black hole. It can be effective if the external field mostly intervenes before or after the black hole and it is of higher order and can be neglected in the lens equation only if the distance between black hole and perturber along the line of sight is of the same order of the Schwarzschild radius of the black hole. If the deflector potential has the symmetry \( \varphi_Q(u_1^{BH}, u_2^{BH}, z) = \varphi_Q(u_1^{BH}, u_2^{BH}, -z) \), then the shift is null.

The perturbed lens equation (37) reproduces the vector generalization of Eq. (8) when the external field vanishes.

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At this point it is worth to eliminate \( \beta \) throughout the lens equation and use \( u_{BH} \) instead. However, we must take care of the spatial dependence of \( \Delta u_O \) and \( \Delta u_S \). We can expand them around \( u_{BH} = 0 \) and write

\[
\Delta u_O = \Delta u_O(0) + \hat{J}_O u_{BH} \tag{39}\]

\[
\Delta u_S = \Delta u_S(0) + \hat{J}_S u_{BH} \tag{40}\]

where \( \hat{J}_O \) and \( \hat{J}_S \) are the Jacobian matrices of \( \Delta u_O \) and \( \Delta u_S \) evaluated at \( u_{BH} = 0 \).

In the same way, we can express the perturber deflection angle as in the classical Chang & Refsdal lens, expanding to first order in \( u \)

\[
\alpha_Q = \frac{D_{LS}}{D_S} \hat{\alpha}_Q = \alpha_Q(0) + \hat{J}_Q u_{BH} \tag{41}\]

\[
\hat{J}_Q = \frac{1}{D_L} \begin{pmatrix} \kappa_Q + \gamma_Q & 0 \\ 0 & \kappa_Q - \gamma_Q \end{pmatrix}, \tag{42}\]

where \( \kappa_Q \) and \( \gamma_Q \) are the convergence and the shear of the perturber at the location of the black hole, respectively, and the orientation is chosen to diagonalize the tidal matrix \( \hat{J}_Q \).

After all these considerations on the terms of Eq. (37), we can write the lens equation as

\[
\beta = D_L^{-1}u_{BH} + \Delta u_O(0) + \hat{J}_O u_{BH} - \frac{D_{LS}}{D_S} \Delta \alpha_n^{BH}(u_{BH}) - \alpha_Q(0) - \hat{J}_Q u_{BH} + D_S^{-1} \Delta u_S(0) + \hat{J}_S u - \Delta u_O(0) - \hat{J}_O u \tag{43}\]

We can englobe all constant terms in \( \beta \), redefining the origin of the source plane. Rather than introducing a new notation, for simplicity we will continue to use \( \beta \) and drop the constant quantity \( \Delta \beta \) from the right hand side of Eq. (33).

\[
\Delta \beta = \frac{D_{LS}}{D_L D_S} \Delta u_S(0) - \alpha_Q(0) + \frac{1}{D_S} \Delta u_S(0) \simeq -\alpha_Q(0). \tag{44}\]

Note that the shift in the source plane is largely dominated by \( \alpha_Q(0) \), which is of order \( \varphi_Q \), while all other shift terms are at most of order \( \varphi_Q b/D_S \) and we remind that we are assuming the scale of variation of the external field \( b \) to be much smaller than all distances among source, lens and observer.
Similarly, all spatial variations of the shift $\Delta u_0$ are negligible w.r.t. the deflection angle spatial variation, so that they too can be neglected from the lens equation, which reduces to

$$\beta = D_L^{-1} u_{BH} - \frac{D_L}{D_S} \Delta \alpha_{BH}^{\text{BH}}(u_{BH}) - \hat{J}_Q u_{BH}. \quad (45)$$

As anticipated before, the shifts play no role in the lens equation. However, $\Delta u_0$ intervenes in the relation between $\theta$ and $u_{BH}$ and must be taken into account to describe the correct position of the images in the sky.

$\Delta \alpha_{BH}^{\text{BH}}$ can be expressed by Eq. (10) where now $\epsilon = (u_{BH} - D_L\theta_n^0)/(D_L\theta_n^0)$ and $\theta_n^0$ is still given by Eq. (2). Writing the components of $u_{BH}$ as $D_L\theta_n^0(1 + \epsilon)(\cos \phi, \sin \phi)$ and those of $\beta$ as $(\beta_1, \beta_2)$, the lens equation can be written to zero order in $\epsilon$ as

$$\beta_1 = \theta_n^0 \left[ 1 - \kappa_Q - \frac{\epsilon}{\epsilon_n} \right] \cos \phi, \quad \beta_2 = \theta_n^0 \left[ 1 - \kappa_Q + \gamma_Q - \frac{\epsilon}{\epsilon_n} \right] \sin \phi, \quad (46, 47)$$

where $\epsilon_n^E$ is given by Eq. (12).

The perturbed lens equation has an amazingly simple form, which encourages its complete investigation. Let us study critical curves and caustics. The determinant of the Jacobian matrix is

$$J = \frac{1}{(\epsilon_n^E)^2} \left[ \epsilon - (1 - \kappa_Q + \gamma_Q \cos 2\phi)\epsilon_n^E \right]. \quad (48)$$

Setting $J = 0$, the critical curves are found as

$$\epsilon = (1 - \kappa_Q + \gamma_Q \cos 2\phi)\epsilon_n^E, \quad (49)$$

which, in terms of $\theta$, correspond to

$$\theta = D_L^{-1} \Delta u_0^{(0)} + \theta_n^0 |1 + (1 - \kappa_Q + \gamma_Q \cos 2\phi)\epsilon_n^E(\hat{I} + \hat{J}_O) \left( \frac{\cos \phi}{\sin \phi} \right), \quad (50)$$

where we have denoted the two-dimensional identity matrix by $\hat{I}$. The circular symmetry of the Einstein rings is broken both by $\gamma_Q$ and by $\hat{J}_O$, which give contributions of similar relevance here. The convergence $\kappa_Q$ only acts as an isotropic focusing factor. Note that the perturbations to the critical curves stay small even if $\kappa_Q$ and $\gamma_Q$ are larger than 1. This is quite different from the weak field Chang &Refsdal lens, where the critical curves and the caustics change their topology when the convergence and the shear become larger than 1.

The perturbed critical curves are mapped in symmetric diamond-shaped caustics centered at $\beta = 0$, with parametric equations

$$\left\{ \begin{array}{l}
\beta_1 = -2\gamma_Q \theta_n^0 \cos 3\phi \\
\beta_2 = 2\gamma_Q \theta_n^0 \sin 3\phi,
\end{array} \right. \quad (51)$$

The half-width of the caustics $\gamma_Q \theta_n^0$ decreases with the number of loops, tending to the asymptotic value $\gamma_Q \theta_m$. In principle, a point-like source might be inside the first caustics but outside the caustics corresponding to large values of $n$. The shift of the caustics from the optical axis is given by Eq. (44) and is the same for all of them (it even coincides at the first order with the shift of the weak field caustic) and amounts to the zero-order term in the expansion of the deflection angle due to the external field $\alpha_Q(0)$. This is very different from the effect of an intrinsic angular momentum of the black hole $\mathbb{R}^4$. So, the degeneracy between external shear and intrinsic spin of the lens can be broken, in principle, observing relativistic images of different order in $n$.

The solutions of the general lens equation can be found by reducing it to a one-dimensional form. Adding the squares of $\cos \phi$ and $\sin \phi$ yields a fourth-order equation for $\epsilon$ that can be solved by standard methods. For each number of loops $n$ around the black hole, the maximum number of images is four. The expressions of the solutions in the general case are lengthy and we do not report them here. Basic properties of image multiplicity can be obtained by considering images for a source on one of the axes, for example $\beta_2 = 0$. We have the following possibilities. If $\sin \phi = 0$, then $\beta_1 = \pm \theta_n^0 (1 - k_Q - \gamma_Q - \epsilon/\epsilon_n^E)$. Two images form at $\phi = 0$ and $\phi = \pi$, with

$$\epsilon_{0,\pi} = \left( 1 - k_Q - \gamma_Q \pm \frac{\beta_1}{\theta_n^0} \right) \epsilon_n^E. \quad (52)$$
This couple of series of images always exists and matches the unperturbed solutions when the external perturbation is switched off. If \( \sin \phi \neq 0 \), then it must be \( (1 - k_Q + \gamma_Q - \epsilon_\phi^E) = 0 \). For a source inside a caustic, when the condition \( \beta_1 < 2\gamma_Q \theta_n^E \) is fulfilled, an additional couple of images is produced, whereas sources outside it have only two. The new sequences of images form at

\[
\epsilon = (1 - k_Q + \gamma_Q)\epsilon_n^E, \\
\phi = \pm \arccos \left( -\frac{\beta_1}{2\gamma_Q \theta_n^E} \right).
\]

Note that the caustic is fully inverted w.r.t. the critical curve. This means that if the source enters from the top-right side, the new images form on the left-bottom side. This is different from what happens in Kerr lensing \([10]\), since the transformation from \( \phi \) quasi-tangential direction. We can thus distinguish a radial and a tangential magnification, which respectively read

\[
\mu_r = -\epsilon_n^E, \\
\mu_t = -\frac{\epsilon_n^E}{1 - \kappa_Q + \gamma_Q \cos 2\phi} \epsilon_n^E.
\]

These expressions nicely reproduce those given in Refs. \([3, 5, 6]\), in the limit \( \kappa_Q, \gamma_Q \to 0 \), once we substitute the expression of \( \epsilon_n^E \) \([12]\) and the definition of \( \epsilon \). Note that the radial magnification is always positive (\( \epsilon_n^E < 0 \) by its definition), while the tangential magnification is positive if the image is outside the critical curve and negative otherwise. It diverges for degenerate images on the critical curve. Thus, as in all other cases in which relativistic images are highly magnified, they appear as elongated and very thin tangential arcs \([10]\).

### B. Retro-lensing configuration

The generalization of the retro-lensing equation \([15]\) to the presence of an external field proceeds in the same way. Equation \([22]\) is still valid, while Eq. \([33]\) must be revised because the source is on the same side of the lens. Starting from Eq. \([41]\), we now have \( x_S = (D_S \beta_1, D_S \beta_2, D_L S) \), \( \dot{x}_S = (\dot{\alpha}_1 - \theta_1, \dot{\alpha}_2 - \theta_2, -1) \) and \( \tau_S = -D_L S \). Note that the \( z \)-component of the deflection vector \( \dot{\alpha}_3 \approx -2 \) is responsible of the sign flip in the \( z \)-component of the tangent vector.

Moreover, the photon must hit the lens plane at the point \( -u_{BH} \) in order to come back from the point \( u_{BH} \). Then we need to integrate along the unperturbed trajectory \( x(0)(\tau) = (-u_{1BH}^B, -u_{2BH}^B, -\tau) \). Evaluating Eq. \([34]\) at \( \tau = 0 \), we have

\[
- u_{BH} = x(0) = D_S \beta + D_L S (\dot{\alpha}_\perp - \theta) - \Delta u_S \\
\Delta u_S \equiv 2 \int_0^\infty d\zeta' \int_{\zeta'}^\infty d\zeta'' \nabla_\perp \varphi_Q (-u_{1BH}^B, -u_{2BH}^B, z') dz' dz''
\]

where we have indicated by \( \dot{\alpha}_\perp \) the projection of the deflection vector on a plane perpendicular to the \( z \)-axis. Compare these relations with Eqs. \([35]\) and \([36]\).

Combining Eq. \([16]\) with Eq. \([33]\), we get the perturbed Schwarzschild retro-lens equation

\[
\beta = -\left(1 + 2 \frac{D_L S}{D_S}\right) \theta - \frac{D_L S}{D_S} \left[ \tilde{\alpha}_\perp^{BH} + \dot{\alpha}_Q \right] + \frac{\Delta u_S + \Delta u_O}{D_S},
\]

where now \( \tilde{\alpha}_\perp^{BH} = -\Delta \alpha_n^{BH} \). The minus sign is easily understood in the following way: the projected deflection vector increases its modulus as \( 0 < \dot{\alpha}_{BH} < \pi/2 \). Then, it has modulus equal to unity at \( \dot{\alpha}_{BH} = \pi/2 \) and then starts to decrease as \( \pi/2 < \dot{\alpha}_{BH} < \pi \). At \( \dot{\alpha}_{BH} = \pi \), \( \alpha_\perp^{BH} = 0 \); with a positive \( \Delta \alpha_n^{BH} \) it starts to increase its modulus again but
its new direction is opposite to the original one. After this consideration, we can appreciate how the perturbed lens equation (60) reduces to the unperturbed retro-lens equation (15) when the perturbation vanishes.

Now let us expand the shifts \( \Delta u_S \) and \( \Delta u_O \) for small values of \( u \). Comparing Eq. (59) with Eq. (63), we get

\[
\Delta u_O = \Delta u_O^{(0)} + \tilde{J}_O u_{BH}
\]

\[
\Delta u_S = \Delta u_S^{(0)} - \tilde{J}_O u_{BH}.
\]

In fact, the two shifts are calculated along very similar paths. Thus, their first order expansion in \( u_{BH} \) has the same zero order term and opposite first order terms.

Something similar happens when we calculate the deflection angle of the perturber \( \alpha_Q \), since the incident and exit paths are very close each other

\[
\alpha_Q = 4 \int_0^\infty \nabla_{\perp} \varphi_Q(0,0,z) dz + 2 \int_0^\infty \frac{\partial \nabla_{\perp} \varphi_Q}{\partial u_{BH}} u_{BH} dz - 2 \int_0^\infty \frac{\partial \nabla_{\perp} \varphi_Q}{\partial u_{BH}} u_{BH} dz,
\]

where the symbol \( \partial \nabla_{\perp} \varphi_Q/\partial u_{BH} \) indicates the Jacobian of the transverse gradient of the potential w.r.t. to \( u_{BH} \).

While the zero-order contributions in the approach to the black hole and the way back sum up, the first order contributions in \( u_{BH} \) cancel out. We thus conclude that, up to octopole terms, the perturbed Schwarzschild lens in the retro-lensing geometry is equivalent to a shifted Schwarzschild lens. Alas, the octopole contributions are of the same order of the contribution coming from the strong field region. Their complete investigation requires a careful evaluation of the perturbation of the photon trajectory in the strong field regime, unnecessary in standard lensing. This higher order investigation is beyond the scope of this work. So, we can conclude this section saying that, up to perturbations of the order \( \varphi_Q(2m)/b \), retro-lensing caustics are shifted but remain point-like. The shift is generally different from that suffered by standard lensing caustics and it is approximately given by the first term on the right hand side of Eq. (63). As regards the images and the critical curves, everything proceeds as in the unperturbed case. The only difference is that \( \theta \) is given as a function of \( \epsilon \) and \( \phi \) by Eq. (65).

V. CONCLUSIONS

Super-massive black holes are supposed to be hosted by galactic bulges, where the gravitational field of the dense stellar environment may perturb gravitational lensing phenomenology. While the perturbation of external fields on the Schwarzschild lens in the weak deflection limit is well-known, our paper has clarified the effects of a tidal perturbation on relativistic images generated by photons winding one or more times around a Schwarzschild black hole.

We have firstly given a form of the unperturbed lens equation more general than that given in Ref. 11.

By a dimensional argument, we have then demonstrated that the perturbation to the photon trajectory mainly arises in a weak field regime, since the perturbation in the strong regime is of higher order in the local curvature radius produced by the perturbing body. On the other hand, the black hole deflection in the weak field regime can be safely neglected, since the photon trajectory is almost face-on and the transverse gradient of the black hole potential is very small.

In this way, we have practically decoupled the two deflections. This justifies the construction of the lens equation as a three-steps process. First we have the weak deflection by the perturber before the photon reaches the black hole, second we have the strong deflection by the black hole and last we have another weak deflection by the perturber. We have explicitly built the lens equation for the two most interesting cases: standard and retro-lensing.

For the standard case (source behind the black hole) the perturbed lens equation reveals that the infinite sequence of relativistic Einstein rings near the photon sphere is slightly distorted and the point-like caustics get a finite size, being shifted by a constant amount. The size of the caustics decreases with the number of loops, being of order of the size of the shadow of the black hole multiplied by the external shear. A source inside a diamond-shaped caustic generates two additional relativistic images, as usual.

In the retro-lensing geometry, the perturbation of the external field cancels out at the lowest order. In order to evaluate the effect, it is necessary to go to the next order in \( 1/b \), which also involves the perturbation of the photon trajectory in the strong field region around the central black hole.

The present work certainly does not exhaust all possible cases of tidal distortion, since we have restricted to weak perturbers in a static situation. However, the most physically relevant cases are clearly included in this class. In fact, the gravitational field at the Galactic center generated by the environment surrounding the supermassive black hole should be reasonably described in the weak field approximation. Gravitational lensing by Sgr A* could be a very accurate tool to track the existence of a non-vanishing shear in the field generated by the environment. In other
galaxies, where sometimes the bulge hosts more than one supermassive black hole, gravitational lensing may help to detect the mass ratios of different black holes. In any case, given the ubiquity of shear fields, the study presented here fills an important gap in the literature about gravitational lensing in strong gravitational fields.

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