A NOTE ON MEASURES VANISHING AT INFINITY

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Abstract. In this paper, we review the basic properties of measures vanishing at infinity and prove a version of the Riemann–Lebesgue lemma for Fourier transformable measures.

1. Introduction

Physical diffraction plays a central role in the study of the atomic structure of solids. The discovery of quasicrystals by Shechtman [43] in 1984 led to an increased interest into structures which fail to be periodic, but still exhibit long range order, typically shown by a large Bragg spectrum.

Mathematically, the diffraction is described as follows: given a point set $\Lambda$, representing the positions of atoms in an ideal crystal, or more generally a measure $\mu$, we construct a new measure $\gamma$ called the autocorrelation of $\Lambda$ or $\mu$ respectively. This measure is positive definite and hence Fourier transformable [1, 16, 24]. Its Fourier transform $\hat{\gamma}$ is a positive measure which models the diffraction of our structure [21, 4].

As any measure $\hat{\gamma}$ has a Lebesgue decomposition

$$\hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{ac} + \hat{\gamma}_{sc},$$

with respect to the Haar measure $\theta_G$ into the pure point, absolutely continuous and singular continuous components.

In many particular cases, we know quite a bit about the pure point and continuous spectrum. One way to get information about these components is via the Eberlein decomposition: the measure $\gamma$ can be decomposed [21, 34, 47] into two positive definite measures $\gamma_s$ and $\gamma_0$, called the strong and null weakly almost periodic parts of $\gamma$, such that

$$\hat{\gamma}_s = (\hat{\gamma})_{pp} \quad \text{and} \quad \hat{\gamma}_0 = (\hat{\gamma})_c.$$

By using this approach, progress has been made towards understanding the pure point and continuous spectra of measures with lattice support [3, 4], and with Meyer set support [30, 43, 44, 48, 49] and few examples of measures with FLC support [30, 47]. More recently, this decomposition has been studied via spectral decomposition of dynamical systems [2].

These methods also led to a good understanding of models with pure point diffraction. Pure point diffraction was characterised in terms of almost periodicity of the autocorrelation measure [15, 19, 21, 34, 46], in terms of almost periodicity of the underlying structure [33] or in terms of the pure point dynamical spectrum [12, 25]. Various systems with pure point diffraction, such as regular model sets and weighted Dirac combs with Meyer set support...
weak model sets of maximal density \[11, 23\], stationary processes \[17, 27\] and various deformations \[13, 29\], or almost periodic measures \[30\] have been studied. For more general overviews of these results we recommend \[5, 4, 26, 14, 7\].

In contrast, little is known in general about the continuous spectra and systems with purely continuous spectrum. The decomposition
\[
\hat{\gamma}_c = \hat{\gamma}_{ac} + \hat{\gamma}_{sc},
\]
is only understood in few particular examples, and almost nothing is known in general about systems which don’t have some underlying lattice structure. For example, the pinwheel tiling is known to have only one trivial Bragg peak \[32\], therefore the diffraction is essentially continuous, but nothing else is known about the structure of this diffraction, except its circular symmetry.

One interesting example for which the continuous spectrum is well understood is the Thue–Morse model, see \[7, 36\] for details.

The study of the Thue–Morse example is simplified by the fact that this model leads to a class of weighted combs with lattice support. Moreover, by choosing the right weights, we can get a measure \(\omega\) whose diffraction is exactly the continuous diffraction spectra of the Thue–Morse. This also implies that the autocorrelation \(\gamma\) of \(\omega\) is a null-weakly almost periodic measure with lattice support. A relatively simple computation shows that the autocorrelation coefficients of \(\gamma\) are not vanishing at infinity \[5, 7\]. Since \(\gamma\) is supported on a lattice, the Riemann–Lebesgue lemma can be applied to conclude that the diffraction cannot be absolutely continuous, and a relatively simple argument can be then used to conclude that \(\hat{\gamma}_{ac} = 0\) \[5, 7\].

The Thue–Morse example can be generalised to a large class of similar models \[22\]. By a similar argument, all these models have been shown to have purely singularly continuous diffraction for a suitable choice of the weights \[8, 9, 4\]. Other similar models with mixed singular spectrum have been studied \[6, 10\].

In all these examples, the key is the fact that, for a measure supported on a lattice with coefficients not vanishing at infinity, the Riemann–Lebesgue lemma can be applied to conclude that the Fourier transform cannot be absolutely continuous.

It is worth emphasising that this approach does not work for Fourier transformable measures which are not supported on lattices. It is our goal to give a version of the Riemann–Lebesgue lemma for arbitrary Fourier transformable measures, and to show that for measures with lattice support this version is equivalent to the standard one.

The paper is organised as it follows: In Section 3 we recall the notion of measures vanishing at infinity and study the basic properties of this class. Next, in Section 4, we provide the main theorem of the paper, the Riemann–Lebesgue lemma for measures together with few consequences. We also recall a classical result about absolutely continuous positive definite measures, which complements our main result. Further examples are provided in Section 5. We complete the paper by taking a close look, in Section 6, at the Fourier duality between Fourier transformable measures vanishing at infinity and Rajchman measures.
2. Preliminaries

For the entire paper, $G$ denotes a $\sigma$-compact LCA group. We will denote by $C_U(G)$ the space of uniformly continuous and bounded functions on $G$. $C_0(G)$ will represent the subspace of $C_U(G)$ consisting of functions vanishing at infinity. Let us recall that $f$ is said to be vanishing at infinity if, for each $\varepsilon > 0$, there exists a compact set $K_\varepsilon$ such that

$$|f(x)| < \varepsilon \quad \text{for all } x \notin K_\varepsilon.$$  

$C_U(G)$ is a Banach space with respect to the $\| \cdot \|_\infty$-norm, and $C_0(G)$ is a closed subspace of $C_U(G)$.

As usual, $C_c(G)$ denotes the subspace of $C_U(G)$ of functions with compact support. This is not closed with respect to $\| \cdot \|_\infty$, but it is closed with respect to the inductive topology, which is defined as follows: For each compact set $K \subseteq G$, the space

$$C(G : K) := \{ f \in C_c(G) \mid \text{supp}(f) \subseteq K \}$$

is a Banach subspace of $C_U(G)$. As we have

$$C_c(G) = \bigcup_{K \text{ compact}} C(G : K),$$

$C_c(G)$ can be seen as the inductive limit of the spaces $C(G : K)$. The topology defined by this inductive limit is called the inductive topology.

By the Riesz–Markov theorem (see [39, Appendix] for details), the space of regular Radon measures on $G$ can be identified with the dual space of $C_c(G)$, when this is equipped with the inductive limit topology. Because of this, we will think of measures as continuous linear functionals on $C_c(G)$, and denote the duality between a measure $\mu$ and a function $f \in C_c(G)$ by

$$\langle f, \mu \rangle := \int_G f d\mu.$$  

The convolution between two functions $f, g \in C_c(G)$ is defined via

$$(f * g)(x) = \int_G f(x - t) g(t) dt = \int_G f(t) g(x - t) dt.$$  

Similarly, for a measure $\mu$ and a function $f \in C_c(G)$, we define the convolution $\mu * f$ by

$$(\mu * f)(x) = \int_G f(x - t) d\mu(t).$$  

Let us next recall the definition of a translation bounded measure.

**Definition 1.** A measure $\mu$ is called translation bounded if for all $f \in C_c(G)$ we have $\mu * f \in C_U(G)$.

We denote the space of translation bounded measures by $\mathcal{M}^\infty(G)$.
Remark 2. By \cite{1}, a measure $\mu$ is translation bounded if and only if, for all compact sets $K \subseteq G$, we have

$$\|\mu\|_K := \sup_{x \in G} |\mu|(x + K) < \infty,$$

where $|\mu|$ denotes the variation measure for $\mu$ (see \cite{35} Sec. 6.5 for details).

In \cite{15}, the authors showed that it suffices to check that $|\mu|_K \leq 8$ for a single compact set with non-empty interior.

Next, we give a brief review of Fourier transformability for measures. For a more detailed overview, we refer the reader to \cite{34}.

As usual, $K_2(G) = \text{Span}\{f \ast \tilde{f} | f \in C_c(G)\}$.

Definition 3. A measure $\mu$ is called Fourier transformable if there exists a measure $\tilde{\mu}$ on $\hat{G}$ such that, for all $f \in C_c(G)$ we have $\tilde{f} \in L^2(\tilde{\mu})$ and

$$\langle f \ast \tilde{f}, \mu \rangle = \langle |\tilde{f}|^2, \tilde{\mu} \rangle .$$

Note that the definition is equivalent to $\langle g, \mu \rangle = \langle \tilde{g}, \tilde{\mu} \rangle$.

We complete this section by briefly reviewing the notions of strongly, weakly and null weakly-almost periodic measures and their relation to the Fourier transform of measures. For a comprehensive review of this, we refer the reader to \cite{34}.

As the definitions for measures are obtained by convolutions from the equivalent notions for functions, we start by introducing these concepts for functions.

Definition 4. A function $f \in C_U(G)$ is called strongly almost periodic or Bohr almost periodic if the hull $\{T_t f | t \in G\}$ is pre-compact in $(C_U(G), \| \cdot \|_\infty)$, where the translation $T_t f$ is defined via

$$T_t f(x) := f(x - t) .$$

Further, $f \in C_U(G)$ is called weakly almost periodic if the hull $\{T_t f | t \in G\}$ is pre-compact in the weak topology of $(C_U(G), \| \cdot \|_\infty)$.

We denote by $SAP(G)$ and $WAP(G)$ the spaces of strongly respectively weakly almost periodic functions.

It is obvious that $SAP(G) \subseteq WAP(G)$.

To define null weakly almost periodicity, we first need to define the concept of a mean.

Proposition 5. \cite{18, 34} Let $f \in WAP(G)$ and $(A_n)_{n \in \mathbb{N}}$ be a van Hove sequence. Then, the following limit exists:

$$\lim_{n \to \infty} \frac{1}{|A_n|} \int_{A_n} f(x) \, dx =: M(f) .$$
Definition 6. For $f \in WAP(G)$, the number $M(f)$ is called the mean of $f$.

A function $f$ is called null weakly almost periodic if $f \in WAP(G)$ and $M(|f|) = 0$. We denote the space of null weakly almost periodic functions by $WAP_0(G)$.

Now, we can carry these definitions to measures.

Definition 7. A measure $\mu \in M^\infty(G)$ is called strongly, weakly respectively null weakly almost periodic if, for all $f \in C_c(G)$, the function $\mu \ast f$ is strongly, weakly respectively null weakly almost periodic.

We denote by $SAP(G), WAP(G)$ and $WAP_0(G)$ the spaces of strongly, weakly respectively null weakly almost periodic measures.

The importance of the spaces of almost periodic measures to diffraction theory is given by the next two theorems.

Theorem 8. [24, 34] 

$$WAP(G) = SAP(G) \oplus WAP_0(G).$$

In particular, any measure $\mu \in WAP(G)$ can be written uniquely as

$$\mu = \mu_s + \mu_0 \quad \text{with } \mu_s \in SAP(G) \text{ and } \mu_0 \in WAP_0(G).$$

We will refer to the decomposition $\mu = \mu_s + \mu_0$ as the Eberlein decomposition for weakly almost periodic measures.

Theorem 9. [34] Let $\mu \in M^\infty(G)$ be Fourier transformable. Then $\mu \in WAP(G)$, $\mu_s, \mu_0$ are Fourier transformable, and

$$\widehat{\mu_s} = (\hat{\mu})_p; \quad \widehat{\mu_0} = (\hat{\mu})_c.$$

3. Measures vanishing at infinity

In this section, we define and study the basic properties of measures vanishing at infinity. Later, we will need these measures for the formulation of the Riemann–Lebesgue lemma for measures. Some of the results of this section are stated (without proof) in [16, pp. 5-6].

Definition 10. [16, Def. 1.14] A measure $\mu \in M(G)$ is vanishing at infinity if

$$\mu \ast f \in C_0(G) \quad \text{for all } f \in C_c(G).$$

In this case, we write $\mu \in M_0^\infty(G)$.

The following observation is an exercise in [16] and is trivial to prove.

Remark 11. [16, Exercise 1.17] Let $\mu$ be a positive measure. Then $\mu \in M_0^\infty(G)$ if and only if for all compact sets $K \subset G$ the function $x \mapsto \mu(x + K)$ is vanishing at infinity.

Exactly as for translation boundedness, it is immediate to see that it suffices to check the condition from Remark 11 for one compact sets $K$ with non-empty interior.

We next look at a basic example of a measure vanishing at infinity, which can be found in the literature in various places [3, 34, 45, 48]:
Example 12. Let
\[ \mu = \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \delta_{\frac{1}{n}} - \delta_n \right). \]
Then \( \mu \in \mathcal{M}_0^\infty(G) \). For a proof we refer to \([14, \text{Ex. 5.8}]\).

When we deal with the Fourier transform of measures, we need to deal with test functions in \( K_2(G) \) instead of \( C_c(G) \). We start by proving that the definition is the same if one uses \( K_2(G) \) as the space of test functions.

Lemma 13. Let \( \mu \in \mathcal{M}^\infty(G) \). Then, \( \mu \in \mathcal{M}_0^\infty(G) \) if and only if
\[ (2) \quad \mu * f \in C_0(G) \quad \text{for all } f \in K_2(G). \]

Proof. It is trivial that (2) is necessary because \( K_2(G) \subseteq C_0(G) \). To prove that it is also sufficient, let \( f \in C_c(G) \). By assumption, for all \( g \in C_c(G) \) we have
\[ \mu * (f * g) \in C_0(G). \]
Let \( (h_\alpha)_\alpha \) be an approximate identity for \( (C_U(G), *) \), consisting of functions from \( C_c(G) \). Then,
\[ (3) \quad (\mu * f) * h_\alpha = \mu * (f * h_\alpha) \in C_0(G). \]
The translation boundedness of \( \mu \) implies \( \mu * f \in C_U(G) \), therefore
\[ (4) \quad \lim_{\alpha} (\mu * f) * h_\alpha = \mu * f \quad \text{uniformly}. \]
Now, (3) and (4) together with the completeness of \( C_0(G) \) imply \( \mu * f \in C_0(G) \). \( \square \)

The rest of this section is devoted to some basic properties of measures vanishing at infinity. As to that, the next two lemmas state useful examples of sets that contain \( \mathcal{M}_0^\infty(G) \). First, we state a simple, but important consequence of vanishing at infinity (compare \([16, \text{Remark 1.15}]\)).

Lemma 14. Every measure \( \mu \) that is vanishing at infinity is translation bounded.

Proof. The claim follows immediately from \( C_0(G) \subseteq C_U(G) \). \( \square \)

Next, we show that for non-compact groups the measures vanishing at infinity are automatically null weakly almost periodic.

Lemma 15. Let \( G \) be non-compact. Then, \( \mathcal{M}_0^\infty(G) \subseteq WAP_0(G) \).

Proof. Due to \([18, \text{Thm 11.1}]\) we know that
\[ C_0(G) \subseteq WAP(G). \]
Hence, we are left to show that
\[ M(|f|) = 0 \quad \text{for all } f \in C_0(G). \]
Let \((A_n)_{n \in \mathbb{N}}\) be a van Hove sequence in \(G\), see [41]. As \(f \in C_0(G)\), for all \(\varepsilon > 0\) there is a compact set \(K_\varepsilon\) such that
\[
|f(x)| < \varepsilon \quad \text{for all } x \notin K_\varepsilon.
\]
Thus,
\[
M(|f|) = \lim_{n \to \infty} |A_n|^{-1} \int_{A_n} |f| \, d\theta
\leq \lim_{n \to \infty} |A_n|^{-1} \int_{K_\varepsilon} |f| \, d\theta + \lim_{n \to \infty} |A_n|^{-1} \int_{A_n \setminus K_\varepsilon} |f| \, d\theta
\leq 0 + \varepsilon \cdot \lim_{n \to \infty} \frac{|A_n \setminus K_\varepsilon|}{|A_n|} \leq \varepsilon,
\]
since \(\lim_{n \to \infty} |A_n|^{-1} = 0\) follows from \(G\) non-compact. \(\square\)

On the other hand, we can find sufficient conditions for a measure to be vanishing at infinity, which are often easy to check. For instance, every finite measure is vanishing at infinity. This produces a large class of examples, which includes \(L^p(G)\) for \(p \geq 1\).

**Proposition 16.** Every finite measure \(\mu \in \mathcal{M}(G)\) is vanishing at infinity.

**Proof.** Let \(f \in C_c(G)\) and \(\varepsilon > 0\). Since \(\mu\) is regular, there is a compact set \(\widetilde{K}_\varepsilon \subseteq G\) such that
\[
\mu(G \setminus \widetilde{K}_\varepsilon) < \frac{\varepsilon}{1 + \|f\|_{\infty}}.
\]
Set \(K_f := \text{supp}(f)\). Then,
\[
|\mu \ast f(t)| = \left| \int_{G} f(t-s) \, d\mu(s) \right|
= \left| \int_{\widetilde{K}_\varepsilon} f(t-s) \, d\mu(s) + \int_{G \setminus \widetilde{K}_\varepsilon} f(t-s) \, d\mu(s) \right|
< \left| \int_{\widetilde{K}_\varepsilon} f(t-s) \, d\mu(s) \right| + \|f\|_{\infty} \cdot \frac{\varepsilon}{1 + \|f\|_{\infty}}
= 0 + \varepsilon = \varepsilon
\]
for all \(t \notin K_\varepsilon := \widetilde{K}_\varepsilon + K_f\). \(\square\)

An immediate consequence of this is the following result.

**Corollary 17.**

(i) Every compactly supported measure is vanishing at infinity.

(ii) Let \(G\) be compact. Then, \(\mathcal{M}(G) = \mathcal{M}_0^G(G)\).

The second part of Corollary [17] gives rise to the following result.

**Proposition 18.** Let \(G\) be compact. Then, the space \((\mathcal{M}_0^G(G), \|\cdot\|_G)\) is a Banach space, where \(\|\mu\|_G := |\mu|(G)\).

**Proof.** It is a known fact that \((\mathcal{M}(G), \|\cdot\|_G)\) is a Banach space if \(G\) is compact. Now, the claim follows from Corollary [17](ii). \(\square\)
The next question is whether we can obtain similar results for arbitrary locally compact Abelian groups $G$. Let us start by reviewing the product topology for measures [24, Sec. 2].

Given a translation bounded measure $\mu$ and a function $f \in C_c(G)$, we have $f \ast \mu \in C_U(G)$. This allows us to embed $\mathcal{M}^\infty(G) \hookrightarrow [C_U(G)]^{C_c(G)}$ via

$$\mu \mapsto (\mu \ast g)_{g \in C_c(G)}.$$ 

Since $(C_U(G), \| \cdot \|_\infty)$ is a Banach space, we can equip $[C_U(G)]^{C_c(G)}$ with the product topology. The topology induced on $\mathcal{M}^\infty(G)$ via the embedding $\mathcal{M}^\infty(G) \hookrightarrow [C_U(G)]^{C_c(G)}$ is called the product topology for measures.

As observed in [24], the product topology is a locally convex topology given by the family of semi-norms $\{p_g\}_{g \in C_c(G)}$, where

$$p_g(\mu) := \| \mu \ast g \|_\infty.$$ 

Let us emphasise here that while $[C_U(G)]^{C_c(G)}$ is a complete locally convex topological vector space, the authors of [24] do not know whether $\mathcal{M}^\infty(G)$ is complete in $[C_U(G)]^{C_c(G)}$. They could only prove that $\mathcal{M}^\infty(G)$ is bounded closed [24, Thm. 2.4] and quasi-complete [24, Cor. 2.1].

We show that $\mathcal{M}_0^\infty(G)$ is complete in $\mathcal{M}^\infty(G)$ and therefore, also bounded closed in $[C_U(G)]^{C_c(G)}$ and quasi-complete.

**Theorem 19.** $\mathcal{M}_0^\infty(G)$ is complete in $\mathcal{M}^\infty(G)$ with respect to the product topology.

**Proof.** Let $(\mu_\alpha)_\alpha$ be a Cauchy net in $\mathcal{M}_0^\infty(G)$, which converges to some $\mu \in \mathcal{M}^\infty(G)$. Then, for each $g \in C_c(G)$ we have

$$\mu_\alpha \ast g \to \mu \ast g \quad \text{in} \quad (C_U(G), \| \cdot \|_\infty).$$

Since $\mu_\alpha \ast g \in C_0(G)$ and $(C_0(G), \| \cdot \|_\infty)$ is complete in $(C_U(G), \| \cdot \|_\infty)$, it follows that $\mu \ast g \in C_0(G)$. Since $g \in C_c(G)$ is arbitrary, this proves the claim. \hfill \Box

**Remark 20.** If $(\mu_\alpha)_\alpha$ is an arbitrary Cauchy net in $\mathcal{M}_0^\infty(G)$, then it is straightforward to prove that there exists a linear function $L : C_c(G) \to \mathbb{C}$ such that, for all $g \in C_c(G)$, we have

$$\mu_\alpha \ast g \to L \ast g \quad \text{in} \quad (C_U(G), \| \cdot \|_\infty).$$

Moreover, as $C_0(G)$ is closed in $C_U(G)$, it follows that $L \ast g \in C_0(G)$ for all $g \in C_c(G)$.

The boundedness of the net $(\mu_\alpha)_\alpha$ is necessary in order to prove that $L$ is continuous and hence a measure. Indeed, for this we need to show that, for each compact set $K \subseteq G$, there exists some constant $C_K$ such that, for all $f \in C_c(G)$ with $\text{supp}(f) \subseteq K$, we have

$$|L(f)| \leq C_K \| f \|_\infty.$$ 

Note that, since $\mu_\alpha$ is a measure, there exists a constant $C_K(\alpha)$ such that

$$|\mu_\alpha(f)| \leq C_K(\alpha) \| f \|_\infty.$$ 

The boundedness of the net $(\mu_\alpha)_\alpha$ is important for this proof as it means that, for each $K$, the set $\{C_K(\alpha)\}$ is bounded and hence we can pick the supremum of this set as $C_K$, which yields the continuity of $L$. \hfill \Diamond
Let us note now that, for a function \( f \in C_U(G) \), vanishing at infinity could mean vanishing at infinity as a measure or as a function. We will see that the two concepts are equivalent.

**Proposition 21.**

\[
\mathcal{M}_0^\infty(G) \cap C_U(G) = C_0(G).
\]

**Proof.** (i) Obviously, \( C_0(G) \subseteq C_U(G) \). Furthermore, \( C_0(G) \subseteq \mathcal{M}_0^\infty(G) \) because \( f \in C_0(G) \) and \( g \in C_c(G) \) imply \( f \ast g \in C_0(G) \).

(ii) Let \( f \in \mathcal{M}_0^\infty(G) \cap C_U(G) \). As before, let \( (g_\alpha)_\alpha \) be an approximate identity consisting of functions from \( C_c(G) \). Since \( f \in \mathcal{M}_0^\infty(G) \), we have

\[
(6) \quad f \ast g_\alpha \in C_0(G).
\]

Now, as \( f \in C_U(G) \) we have

\[
(7) \quad \lim_{\alpha} f \ast g_\alpha = f \quad \text{uniformly.}
\]

Hence, by (6) and (7), we obtain \( f \in C_0(G) \).

\[ \square \]

Let us emphasise that the uniform continuity of \( f \) is crucial in the proof of Proposition 21. For this purpose, we provide an example of a bounded and continuous function which is not uniformly continuous, which is vanishing at infinity as a measure but not as a function.

**Example 22.** Consider the sequence of functions \( (f_n)_{n \in \mathbb{N}} \) defined by

\[
f_n : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1 - 2^n \cdot |x - n|, & |x - n| < 2^{-n}, \\ 0, & \text{otherwise}, \end{cases}
\]

and the function \( f(x) := \sum_{n=1}^\infty f_n(x) \). A short calculation shows \( \int_\mathbb{R} f_n(x) \, dx = (\frac{1}{2})^n \), hence

\[
\int_\mathbb{R} f(x) \, dx = \sum_{n=1}^\infty \int_\mathbb{R} f_n(x) \, dx = 1.
\]

Since \( f \in L^1(\mathbb{R}) \), the measure \( \mu := f \lambda \) is finite and, by Proposition 16, \( \mu \in \mathcal{M}_0^\infty(\mathbb{R}) \). Due to Proposition 21 \( f \) cannot be uniformly continuous because \( f \notin C_0(\mathbb{R}) \).

We complete this section by giving a simple criterion for measures with uniformly discrete support to be vanishing at infinity.

**Proposition 23.** Let \( \mu \in \mathcal{M}^\infty(G) \) and \( \Lambda := \text{supp}(\mu) \) be uniformly discrete. Then, \( \mu \in \mathcal{M}_0^\infty(G) \) if and only if, for all \( \varepsilon > 0 \), there is a compact set \( K_\varepsilon \subseteq G \) such that

\[
|\mu(\{x\})| < \varepsilon \quad \text{for all } x \in \Lambda \setminus K_\varepsilon.
\]

**Proof.** (i) necessary part. Due to the uniform discreteness, there is an open set \( U \) such that

\[
(U + x) \cap (U + y) = \emptyset \quad \text{for all distinct } x, y \in \Lambda.
\]

Let \( \varepsilon > 0 \). \( \mu \in \mathcal{M}_0^\infty(G) \) implies that there is a compact set \( K_\varepsilon \subseteq G \) such that

\[
|\mu \ast f(t)| < \varepsilon \quad \text{for all } t \notin K_\varepsilon.
\]
This is true for all \( f \in C_c(G) \). Now, let \( f \in C_c(G) \) with \( f(0) = 1 \) and \( \text{supp}(f) \subseteq U \). We obtain

\[
|\mu(s)| = \left| \sum_{x \in \Lambda \cap (-U + s)} \mu(\{x\}) \right| = \left| \sum_{x \in \Lambda \cap (-U + s)} \mu(\{x\}) f(s - x) \right| = \left| \sum_{x \in \Lambda} \mu(\{x\}) f(s - x) \right| = |\mu * f(s)| < \varepsilon
\]

for all \( s \in \Lambda \setminus K_\varepsilon \).

(ii) sufficient part. Let \( f \in C_c(G) \) with \( K_f := \text{supp}(f) \). Let \( \varepsilon > 0 \). By assumption, there is a compact set \( K_\varepsilon \) such that

\[
|\mu(\{x\})| < \varepsilon \cdot \left( \|f\|_\infty \cdot \sup_{t \in G} |\Lambda \cap (-K_f + t)| \right)^{-1}.
\]

For \( t \notin \tilde{K}_\varepsilon := K_f + K_\varepsilon \), we obtain

\[
|\mu * f(t)| = \left| \sum_{s \in \Lambda} \mu(\{s\}) f(t - s) \right| \leq \sum_{s \in \Lambda \cap (-K_f + t) \setminus K_\varepsilon} |\mu(\{s\})| |f(t - s)| + \sum_{s \in \Lambda \cap (-K_f + t) \setminus (\Lambda \setminus K_\varepsilon)} |\mu(\{s\})| |f(t - s)| \leq 0 + \varepsilon \cdot \left( \|f\|_\infty \cdot \sup_{t \in G} |\Lambda \cap (-K_f + t)| \right)^{-1} \sum_{s \in (-K_f + t) \cap \Lambda} |f(t - s)| < \varepsilon.
\]

\( \Box \)

Proposition 23 suggests the following definition:

**Definition 24.** A pure point measure \( \mu = \sum_{x \in \Lambda} \mu(\{x\}) \delta_x \) is said to have *coefficients vanishing at infinity* if, for all \( \varepsilon > 0 \), there is a compact set \( K_\varepsilon \) such that

\[
|\mu(\{x\})| < \varepsilon \quad \text{for all } x \in \Lambda \setminus K_\varepsilon.
\]

**Remark 25.** By Proposition 23, for measures with uniformly discrete support, vanishing at infinity is equivalent to coefficients vanishing at infinity.

For pure point measures without uniformly discrete support, there is no relation in general between these two concepts. Indeed, Example 12 gives a measure \( \mu \) which is vanishing at infinity, but for which the coefficients are not vanishing at infinity. On the other hand, we will provide a measure for which the coefficients are vanishing at infinity, but the measure is not, in the next example.

**Example 26.** Let

\[
\nu := \sum_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \frac{1}{\eta} \delta_{\frac{n+k}{n}}.
\]

It is obvious that this measure has coefficients vanishing at infinity.
However, if \( f \in C_c(\mathbb{R}) \) is supported inside \((0, 1)\), it is easy to see that, by the convergence of Riemann sums, we have
\[
\lim_{x \to \infty} (\nu * f)(x) = \int_0^1 f(t)dt.
\]
Therefore, if we pick an \( f \) with \( \int_0^1 f(t)dt \neq 0 \), we get that \( \nu * f \notin C_0(\mathbb{R}) \). This shows that \( \nu \) is not vanishing at infinity.

4. The Riemann–Lebesgue Lemma for Measures

In this section, we prove a version of the Riemann–Lebesgue lemma for a Fourier transformable measure \( \mu \). We then review a similar result of [16] which complements our result.

The version we prove is important for diffraction theory. While this result does not seem to have been proved explicitly before, the idea of the proof combined with Proposition 23 has been exploited in some places to prove the existence of singular continuous spectrum [4, 6, 10, 20].

**Theorem 27.** [Riemann–Lebesgue lemma for measures] Let \( \mu \in \mathcal{M}(G) \) be a Fourier transformable measure. If \( \hat{\mu} \) is absolutely continuous, then \( \mu \in \mathcal{M}_0^c(G) \).

**Proof.** Let \( g \in K_2(G) \). By [1, Thm. 3.1], \( \mu * g \) is Fourier transformable and
\[
\widehat{\mu * g} = \hat{\mu} \hat{g}.
\]
The measure \( \hat{\mu} \) is absolutely continuous, i.e.
\[
\hat{\mu} = f \, d\theta \quad \text{with } f \in L^1_{\text{loc}}(G)
\]
due to the Radon–Nikodym theorem. Furthermore, by the definition of the transformability of \( \mu \), we have
\[
\int_G \hat{g} f \, d\theta = \int_G \hat{g} \, d\hat{\mu} < \infty.
\]
Hence, \( \hat{\mu} \hat{g} \) is a finite measure and therefore \( f \hat{g} \in L^1(G) \). In particular, \( f \hat{g} \) is Fourier transformable as \( L^1 \)-function. This implies [11, 34]
\[
(\mu * g)\uparrow = \hat{f} \hat{g},
\]
where for a function \( h \) we denote by \( h\uparrow \) the reflection
\[
h\uparrow(x) := h(-x).
\]

Now, by the Riemann–Lebesgue lemma for functions, we have \( (\mu * g)\uparrow \in C_0(G) \). The claim follows by an application of Lemma [13]. \( \square \)

**Remark 28.** If \( \mu \in \mathcal{M}_0^c(G) \), its Fourier transform is not necessarily absolutely continuous. Consider the measure
\[
\mu := 2\pi J_0(2\pi \|x\|) \lambda^2,
\]
where \( J_0 \) is the Bessel function of the first kind of order zero. By [32, p. 154], we obtain \( \lambda^2 |_{S^1} = \mu \) and, consequently,
\[
\hat{\mu} = \hat{\lambda^2 |_{S^1}}.
\]
As $2\pi J_0(2\pi \|\cdot\|) \in C_0(\mathbb{R}^2)$, Proposition 21 implies $2\pi J_0(2\pi \|\cdot\|) \lambda^2 \in \mathcal{M}_0^\infty(\mathbb{R}^2)$. But $\hat{\mu}$ is singular continuous. \hfill \Box

Let us start by looking at some immediate consequences of Theorem 27.

First, by combining Theorem 27 with Proposition 23 we get

**Corollary 29.** Let
\[
\mu := \sum_{x \in \Lambda} \mu(\{x\}) \delta_x,
\]
be a Fourier transformable measure. If $\Lambda$ is uniformly discrete and $\hat{\mu}$ is absolutely continuous, then the coefficients of $\mu$ are vanishing at infinity. \hfill \Box

**Corollary 30.** Let
\[
\mu := \sum_{x \in \Lambda} \mu(\{x\}) \delta_x,
\]
be a Fourier transformable measure with lattice support. If $\hat{\mu}$ is absolutely continuous, then the coefficients of $\mu$ are vanishing at infinity. \hfill \Box

**Corollary 31.** Let
\[
\mu := \sum_{x \in \Lambda} \mu(\{x\}) \delta_x,
\]
be a Fourier transformable measure with Meyer set support. If $\hat{\mu}_{sc} = 0$, then the coefficients of $\mu_0$ are vanishing at infinity.

*Proof.* Since $\mu$ has Meyer set support, so has $\mu_0$. Therefore, $\mu_0$ is a measure vanishing at infinity with uniformly discrete support. The claim now follows from Corollary 30. \hfill \Box

We look next at a Fourier dual version of Theorem 27.

**Corollary 32.** Let $\mu$ be a measure which is twice Fourier transformable. If $\mu$ is absolutely continuous, then $\hat{\mu} \in \mathcal{M}_0^\infty(\hat{G})$. \hfill \Box

We complete the section by recalling a result of [16]. This result shows that in Corollary 32 twice Fourier transformability can be replaced by positive definiteness, and can also be seen as the Fourier dual of a particular case in Theorem 27.

**Theorem 33.** [16, Prop. 4.9] Let $\mu$ be a positive definite measure. If $\mu$ is absolutely continuous, then $\hat{\mu} \in \mathcal{M}_0^\infty(\hat{G})$. \hfill \Box
5. Few more examples

In this section, we will look at few more examples and provide a simple method of creating measures vanishing at infinity.

We start by reviewing an example from [16].

Example 34. [16, Remark 1.15] For each positive integer \( n \) and each integer \( 0 \leq k \leq 2^n - 1 \), define \( f(x) = (-1)^k \) for all \( x \in \left[ n + \frac{k}{2^n}, n + \frac{k+1}{2^n} \right) \). Next, define \( f(x) = 0 \) at all other \( x \in \mathbb{R} \).

Then \( \mu = f \lambda \in \mathcal{M}_0^c (\mathbb{R}) \).

Example 35. Let \( \mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{n + \frac{k}{n}} \). Define

\[
\mu = \lambda - \sum_{n=1}^{\infty} (\mu_n + \tilde{\mu}_n)
\]

Then \( \mu \in \mathcal{M}_0^c (\mathbb{R}) \). This is true for the following reason.

By the convergence of the Riemann Sums for the integrals of continuous function, in the vague topology we have

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} \delta_{\frac{k}{n}} \right) = \lambda_{ [0,1] }.
\]

This shows that \( \lim_{x \to \infty} \mu * f(x) = 0 \), which proves the claim.

Example 36. Consider the sequence of measures \((\mu_n)_{n \in \mathbb{N}}\) defined by

\[
\mu_n := \delta_0 + \lambda_{[n,n+1]}.
\]

Applying [10] Thm. 1.3.3(b)], we obtain

\[
(\mu_n * \tilde{\mu}_n) = \tilde{\mu}_n * \mu_n = |\mu_n|^2 = |\delta_0 + \lambda_{[n,n+1]}|^2
\]

\[
= |1 + e^{-\pi i x(2n+1)} \text{sinc}(\pi x)|^2 \lambda
\]

\[
= (1 + 2 \cos \left( \pi x(2n + 1) \right) \text{sinc}(\pi x) + \text{sinc}^2(\pi x)) \lambda.
\]

Now, if we define

\[
\mu := \sum_{n=0}^{\infty} \frac{1}{2^n} \mu_n * \tilde{\mu}_n,
\]

we get

\[
\hat{\mu} = \sum_{n=0}^{\infty} \frac{1}{2^n} (1 + 2 \cos \left( \pi x(2n + 1) \right) \text{sinc}(\pi x) + \text{sinc}^2(\pi x)) \lambda
\]

Thus, \( \hat{\mu} \) is absolutely continuous and, by Theorem [27] \( \mu \in \mathcal{M}_0^c (\mathbb{R}) \).

We complete the section by introducing a large class of measures vanishing at infinity, which contains the measures from Example [12], Example 34 and Example 35.

Proposition 37. Let \((\mu_n)_{n \in \mathbb{N}}\) a sequence of measures, \( K \subseteq G \) be a compact set, and \((t_n)_{n \in \mathbb{N}}\) a sequence in \( G \) with the following properties:

(i) \( \text{supp}(\mu_n) \subseteq K \) for all \( n \).

(ii) \( |\mu_n|(K) \) is bounded.
(iii) In the vague topology, we have
\[ \lim_{n \to \infty} \mu_n = 0. \]

(iv) The \( t_n \) are distinct and the set \( \{t_n | n \in \mathbb{N}\} \) is uniformly discrete.

Then, the measure
\[ \mu := \sum_{n=1}^{\infty} T_{t_n} \mu_n, \]
is vanishing at infinity, where, for a measure \( \mu \), the translation \( T_t \mu \) is defined via
\[ [T_t(\mu)](f) := \mu(T_{-t} f). \]

Proof. First, it follows immediately from the assumptions that
\[ \|\mu\|_K < \infty \]
and hence \( \mu \) is a translation bounded measure.

Fix some arbitrary \( f \in C_c(G) \) and let \( \varepsilon > 0 \). Let \( K_0 \) be a compact set such that \( \text{supp}(f) \subseteq K_0 = -K_0 \). By (iv), there exists some \( M \geq 0 \) such that for all \( x \in G \) there are at most \( M \) elements in
\[ \{t_n | n \in \mathbb{N}\} \cap (x + K_0 - K). \]

Since \( \mu \) is translation bounded, \( \mu * f \) is uniformly continuous. Therefore, there exists some open neighbourhood \( U \) of 0 such that, for all \( t, s \in G \) with \( t - s \in U \), we have
\[ |\mu * f(t) - \mu * f(s)| < \frac{\varepsilon}{2M}. \]

Next, by the compactness of \( K + K_0 \), we can find a finite set \( x_1, ..., x_m \) such that
\[ K + K_0 \subseteq \bigcup_{j=1}^{m} x_j + U. \]

Finally, by the vague convergence of \( \mu_n \), there exists some \( N_\varepsilon \) such that, for all \( n > N_\varepsilon \) and all \( 1 \leq j \leq m \), we have
\[ |\mu_n(T_{x_j} f^t)| < \frac{\varepsilon}{2M}. \]

Let \( K_\varepsilon := \bigcup_{n=1}^{N_\varepsilon} t_n + (K + K_0) \). Then, \( K_\varepsilon \) is a compact set. We show that \( |\mu * f| < \varepsilon \) outside \( K_\varepsilon \). Let \( x \notin K_\varepsilon \). Then
\[ (8) \quad |\mu * f(x)| \leq \sum_n |T_{t_n} \mu_n * f(x)| = \sum_{n \in F} |T_{t_n} \mu_n * f(x)| \]
where \( F := \{n | T_{t_n} \mu_n * f(x) \neq 0\} \). Since \( \{t_n | n \in F\} \subseteq \{t_n\} \cap (K_0 - K) \), we know that \( |F| \leq M \).

Moreover, for each \( n \in F \), we have \( n > N_\varepsilon \), therefore
\[ |\mu_n * f(x_j)| = |\mu_n(T_{x_j} f^t)| < \frac{\varepsilon}{2M}. \]

Finally, if \( n \in F \), we have \( x - t_n \in K + K_0 = \bigcup_{j=1}^{m} x_j + U \). Therefore, there exists some \( j \) and \( u \in U \) such that
\[ x - t_n = x_j + u. \]
Hence,
\[
|T_n,\mu_n * f(x)| = |\mu * f(x_j + u)| \leq |\mu * f(x_j)| + \frac{\varepsilon}{2M} < \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} = \frac{\varepsilon}{M}.
\]

Therefore, by (8), we get
\[
|\mu * f(x)| < \frac{\varepsilon}{M} |F| \leq \varepsilon.
\]
This completes the proof. \(\square\)

Remark 38. (i) Setting \(t_n = n, K = [0,1]\) and
\[
\mu_n = \sum_{k=0}^{2^n-1} (-1)^k \lambda_{\left\{ n + \frac{k}{2^n}, n + \frac{k+1}{2^n} \right\}},
\]
in Proposition 37 we get exactly the Example 34.

(ii) Setting \(t_{2n} = n, t_{2n+1} = -n, K = [-1,1]\) and
\[
\mu_{2n} = \delta_{\frac{1}{\pi}} - \delta_0; \quad \mu_{2n+1} = \delta_{\frac{1}{\pi}} - \delta_0,
\]
in Proposition 37 we get exactly the Example 12.

(iii) Let us denote by \(\lambda_{[0,1]}\) the restriction of the Lebesgue measure to \([0,1]\). Setting exactly as in (i) \(t_{2n} = n, t_{2n+1} = -n, K = [-1,1]\) and
\[
\mu_{2n} = \lambda_{[0,1]} - \left( \frac{1}{n} \sum_{k=1}^{n} \delta_{\frac{k}{n}} \right); \quad \mu_{2n+1} = \mu_{2n},
\]
for \(n \geq 0\) in Proposition 37 we get exactly the Example 35. \(\diamond\)

6. CONNECTION TO RAJCHMAN MEASURES

In this section, we will see that there is a strong connection between measures vanishing at infinity and Rajchman measures. Therefore, let us recall the definition of a Rajchman measure [31].

Definition 39. A finite measure \(\mu\) is called a Rajchman measure if \(\hat{\mu} \in C_0(\hat{G})\).

We denote the class of Rajchman measures by \(\mathcal{R}(G)\).

Remark 40. It is well-known that every finite measure which is absolutely continuous is a Rajchman measure. The converse is not true. There are Rajchman measures that are not absolutely continuous, see [31] and references therein. \(\diamond\)

A stronger version of Theorem 27 is given by the following theorem, which shows the connection between measures vanishing at infinity and Rajchman measures.

Theorem 41. Let \(\mu\) be a Fourier transformable measure. Then, \(\mu \in \mathcal{M}_0^\infty(G)\) if and only if for all \(f \in C_c(G)\) we have
\[
|\hat{f}|^2 \hat{\mu} \in \mathcal{R}(\hat{G}).
\]

Proof. Let \(f \in C_c(G)\). Then, since \(\mu\) is Fourier transformable, \(\mu * f \ast \hat{f}\) is the inverse Fourier transform of the finite measure \(|\hat{f}|^2 \hat{\mu}|\). Therefore, by the definition of \(\mathcal{R}(\hat{G})\) we get that \(\mu * f \ast \hat{f} \in C_0(G)\) if and only if \(|\hat{f}|^2 \hat{\mu} \in \mathcal{R}(\hat{G})\). The claim follows now from Lemma 13. \(\square\)
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