FREE BOOLEAN TOPOLOGICAL GROUPS

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Abstract. Known and new results on free Boolean topological groups are collected. An account of properties which these groups share with free or free Abelian topological groups and properties specific of free Boolean groups is given. Special emphasis is placed on the application of set-theoretic methods to the study of Boolean topological groups.

This is a revised and expanded version of [64].

1. Introduction

In the very early 1940s, A. A. Markov [42, 43] introduced the free topological group \( F(X) \) and the free Abelian topological group \( A(X) \) on an arbitrary completely regular Hausdorff topological space \( X \) as a topological-algebraic counterpart of the abstract free and the Abelian groups on a set; he also proved the existence and uniqueness of these groups. During the next decade, Graev [22, 23], Nakayama [49], and Kakutani [34] simplified the proofs of the main statements of Markov’s theory of free topological groups, generalized Markov’s construction, and proved a number of important theorems on free topological groups. In particular, Graev generalized the notion of the free and the free Abelian topological group on a space \( X \) by identifying the identity element of the free group with an (arbitrary) point of \( X \) (the free topological group on \( X \) in the sense of Markov coincides with Graev’s group on \( X \) plus an isolated point), described the topology of free topological groups on compact spaces, and extended any continuous pseudometric on \( X \) to a continuous invariant pseudometric on \( F(X) \) (and on \( A(X) \)) which is maximal among all such extensions [22].

This study stimulated Mal’tsev, who believed that the most appropriate place of the theory of abstract free groups was in the framework of the general theory of algebraic systems, to introduce general free topological algebraic systems. In 1957, he published the large paper [39], where the basics of the theory of free topological universal algebras were presented.

Yet another decade later, Morris initiated the study of free topological groups in the most general aspect. Namely, he introduced the notion of a variety of topological group\(^1\) and a full variety of topological groups and studied free objects of these

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\(^1\)A definition of a variety of topological groups (determined by a so-called varietal free topological group) was also proposed in 1951 by Higman [28]; however, it is Morris’ definition which has proved viable and developed into a rich theory.
varieties [45][37] (see also [38]). Varieties of topological groups and their free objects were also considered by Porst [34], Comfort and van Mill [12], Kopperman, Mislove, Morris, Nicholas, Pesty, and Svetlichny [35], and other authors. Special mention should be made of Dikranjan and Tkachenko’s detailed study of varieties of Abelian topological groups with properties related to compactness [14].

The varieties of topological groups in which free objects have been studied best are, naturally, the varieties of general and Abelian topological groups; free and free Abelian precompact groups have also been considered (see, e.g., [8]). However, there is yet another natural variety—Boolean topological groups. Free objects in this variety and its subvarieties have been investigated much less extensively, although they arise fairly often in various studies (especially in the set-theoretic context). The author is aware of only two published papers considering free Boolean topological groups from a general point of view: [18], where the topology of the free Boolean topological group on a compact metric space was explicitly described, and [19], where the free Boolean topological groups on compact initial segments of ordinals were classified (see also [20]). The purpose of this paper is to draw attention to these very interesting groups and give a general impression of them. We collect some (known and new) results on free Boolean topological groups, which describe both these very interesting groups and give a general impression of the m. We collect some

2. Preliminaries

All topological spaces and groups considered in this paper are assumed to be completely regular and Hausdorff.

The notation \( \omega \) is used for the set of all nonnegative integers and \( \mathbb{N} \), for the set of all positive integers. By \( \mathbb{Z}_2 \) we denote the group of order 2. The cardinality of a set \( A \) is denoted by \( |A| \), and the closure of a set \( A \) in an ambient topological space is denoted by \( \overline{A} \) and its interior, by \( \text{Int} A \). For any set \( A \), we put

\[
[A]^k = \{ S \subset A : |S| = k \} \quad \text{for} \quad k \in \mathbb{N},
\]

\[
[A]^{<\omega} = \bigcup_{k \in \mathbb{N}} [A]^k = \{ S \subset A : |S| < \omega_0 \}, \quad \text{and} \quad [A]^\omega = \{ S \subset A : |S| = \omega_0 \}.
\]

Given \( s, t \in [\omega]^{<\omega} \), \( s \sqsubset t \) means that \( s \) is an initial segment of \( t \) with respect to the order induced by \( \omega \). For \( g \in [\omega]^{<\omega} \setminus \{ \emptyset \} \) by \( \max g \) we mean the greatest element of the finite set \( g \) in the ordering of \( \omega \). We also set \( \max \emptyset = -1 \).

We denote the disjoint union of spaces \( X \) and \( Y \) by \( X \oplus Y \). The same symbol \( \oplus \) is used for direct sums of groups (hopefully, this will cause no confusion).

A seminorm \( \| \cdot \| \) on a group (or \( \mathbb{Z}_2 \)-vector space) \( G \) with identity element \( e \) is a function \( G \to \mathbb{R} \) such that \( \| e \| = 0 \), \( \| g \| \geq 0 \) and \( \| g^{-1} \| = \| g \| \) for any \( g \in G \), and \( \| gh \| \leq \| g \| + \| h \| \) for any \( g, h \in G \). A seminorm satisfying the condition \( \| g \| = 0 \iff g = e \) is called a norm.

The main object of study in this paper is Boolean topological groups. A Boolean group is a group in which all elements are of order 2. Any Boolean group is Abelian: \( xy = (yx)^2xy = yxxy = yxy^2 = yx \). Algebraically, all Boolean groups are free, because any Boolean group is a linear space over the field \( \mathbb{F}_2 = \{ 0, 1 \} \) and must have a basis (a maximal linearly independent set) by Zorn’s lemma. This basis freely generates the given Boolean group. Moreover, any Boolean group (linear space) with basis \( X \) is isomorphic to the direct sum \( \bigoplus_{X} \mathbb{Z}_2 \) of \( |X| \) copies of \( \mathbb{Z}_2 \),
i.e., the set of finitely supported maps \( g : X \to \mathbb{Z}_2 \) with pointwise addition (in the field \( \mathbb{F}_2 \)). Of course, such an isomorphic representation depends on the choice of the basis.

Given a set or space \( X \), by \( F(X) \), \( A(X) \), and \( B(X) \) we denote, respectively, the free, free Abelian, and free Boolean group on \( X \) with or without a topology (depending on the context).

Topological spaces \( X \) and \( Y \) are said to be \( M \)-equivalent (\( A \)-equivalent) if their free (free Abelian) topological groups are topologically isomorphic. We shall say that \( X \) and \( Y \) are \( B \)-equivalent if \( B(X) \) and \( B(Y) \) are topologically isomorphic.

Given \( X \supset Y \), we use \( B(Y|X) \) to denote the topological subgroup of \( B(X) \) generated by \( Y \).

Whenever \( X \) algebraically generates a group \( G \), we set the length of the identity element to 0, define the length of any nonidentity \( g \in G \) with respect to \( X \) as the least (positive) integer \( n \) such that \( g = x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n} \) for some \( x_i \in X \) and \( e_i = \pm 1 \), \( i = 1, 2, \ldots, n \), and denote the set of elements of length at most \( k \) by \( G_k \) for \( k \in \omega \); then \( G = \bigcup G_k \). Thus, we use \( F_k(X) \) \((A_k(X), B_k(X))\) to denote the sets of words of length at most \( k \) in \( F(X) \) (respectively, in \( A(X) \) and \( B(X) \)).

Let \( X \) be a space, and let \( X_n, n \in \omega \), be its subspaces such that \( X = \bigcup X_n \). Suppose that any \( Y \subset X \) is open in \( X \) if and only if each \( Y \cup X_n \) is open in \( X_n \) (replacing “open” by “closed,” we obtain an equivalent condition). Then \( X \) is said to have the \textit{inductive limit topology} (with respect to the decomposition \( X = \bigcup X_n \)). When talking about inductive limit topologies on \( F(X) \), \( A(X) \), and \( B(X) \), we always mean the decompositions \( F(X) = \bigcup F_k(X) \), \( A(X) = \bigcup A_k(X) \), and \( B(X) = \bigcup B_k(X) \) and always assume the sets \( F_k(X) \), \( A_k(X) \), and \( B_k(X) \) to be endowed with the topology induced by the respective free topological groups.

By a zero-dimensional space we mean a space \( X \) with \( \text{ind} \, X = 0 \) and by a strongly zero-dimensional space, a space \( X \) with \( \text{dim} \, X = 0 \).

**Filters and ultrafilters.** A special place in the theory of Boolean topological groups is occupied by free Boolean groups on almost discrete spaces, which are closely related to filters. Recall that a filter on a set \( X \) is a nonempty family of subsets of \( X \) closed under taking finite intersections and supersets. A maximal (by inclusion) filter is called an \textit{ultrafilter}. A filter on \( X \) is an ultrafilter if, given any \( A \subset X \), it contains either \( A \) or \( X \setminus A \). We largely deal with filters on \( \omega \). We assume all filters \( \mathcal{F} \) on \( \omega \) to be free, i.e., to contain the \textit{Fréchet} filter of all cofinite sets.

An important role in our study is played by Ramsey, or selective, ultrafilters.

The notion of a Ramsey ultrafilter is closely related to Ramsey’s theorem, which says that if \( n \in \mathbb{N} \) and the set \([\omega]^n\) of \( n \)-element subsets of \( \omega \) is partitioned into finitely many pieces, then there is an infinite set \( H \subset \omega \) homogeneous with respect to this partition, i.e., such that \([H]^n\) is contained in one of the pieces \([50] \). An ultrafilter \( \mathcal{U} \) on \( \omega \) is called a \textit{Ramsey ultrafilter} if, given any positive integers \( n \) and \( k \), every partition \( F : [\omega]^n \to \{1, \ldots, k\} \) has a homogeneous set \( H \in \mathcal{U} \). In what follows, we use the following well-known characterizations of Ramsey ultrafilters.

**Theorem 2.1** (see [9]). The following conditions on a free ultrafilter \( \mathcal{U} \) on \( \omega \) are equivalent:

(i) \( \mathcal{U} \) is Ramsey;
(ii) for any partition \( \{C_n : n \in \omega\} \) of \( \omega \) such that \( C_n \notin \mathcal{U} \) for \( n \in \omega \), there exists a selector in \( \mathcal{U} \), that is, a set \( A \in \mathcal{U} \) such that \( |A \cap C_n| = 1 \) for all \( n \);

(iii) for any sequence \( \{A_n : n \in \omega\} \), where \( A_n \in \mathcal{U} \), there exists an \( A \in \mathcal{U} \) such that \( A = \{a_n : n \in \omega\} \) and \( a_n \in A_n \) for all \( n \);

(iv) for any family \( \{A_n : n \in \omega\} \), where \( A_n \in \mathcal{U} \), there exists a diagonal intersection in \( \mathcal{U} \), that is, a set \( D \in \mathcal{U} \) such that \( j \in A_i \) whenever \( i, j \in D \) and \( i < j \);

(v) for any \( A_n \in \mathcal{U} \), \( n \in \omega \), there exists a strictly increasing function \( f : \omega \rightarrow \omega \) such that \( f(n+1) \in A_{f(n)} \) for each \( n \in \omega \) and the range of \( f \) belongs to \( \mathcal{U} \).

Ultrafilters with property (ii) are said to be selective; thus, the Ramsey ultrafilters on \( \omega \) are precisely the selective ultrafilters, and the terms “Ramsey” and “selective” are often used interchangeably in the literature. Any Ramsey ultrafilter is P-point, but not vice versa.

We also mention P-point ultrafilters and P-filters.

An ultrafilter \( \mathcal{U} \) on \( \omega \) is a P-point ultrafilter, or simply a P-ultrafilter, if, for any family of \( A_i \in \mathcal{U} \), \( i \in \omega \), the ultrafilter \( \mathcal{U} \) contains a pseudointersection of this family, i.e., there exists an \( A \in \mathcal{U} \) such that \( |A \setminus A_i| < \omega \) for all \( i \in \omega \). The P-point ultrafilters are precisely those which are P-points in the remainder \( \beta\omega \setminus \omega \) of the Stone–Čech compactification of the discrete space \( \omega \).

By analogy, a filter \( \mathcal{F} \) on \( \omega \) is called a P-filter if any family of \( A_i \in \mathcal{F} \), \( i \in \omega \), has a pseudointersection in \( \mathcal{F} \). There are models of ZFC with no P-point ultrafilters (see [58]; Shelah’s original proof is presented in [74]), while P-filters always exist: the simplest example is the Fréchet filter.

By analogy with P-filters, we might define Ramsey filters as filters satisfying condition (i) in Theorem 2.1 but this would not yield new objects: it is easy to see that any such filter is an ultrafilter. The situation with selective filters is not so obvious. First, conditions (ii) and (iii), which are trivially equivalent for ultrafilters, become potentially different. Secondly, although the proof of the implication (iii) \( \Rightarrow \) (v) given in [9] (as well as the trivial equivalence (iv) \( \iff \) (v) and the obvious implication (iv) \( \iff \) (iii)) remains valid for filters, the standard proof of (v) \( \Rightarrow \) (i) (see, e.g., [32] Lemma 9.2) uses \( \mathcal{U} \) being an ultrafilter. The author found several mentions (without proof) in the literature that any selective filter is an ultrafilter, but it was never clear from the context what exactly was meant by “selective.” Anyway, results of Section 8 on free Boolean topological groups imply that any filter satisfying any of the equivalent conditions (iii)–(v) is a Ramsey ultrafilter.

Finally, a filter \( \mathcal{F} \) on \( \omega \) is said to be rapid if every function \( \omega \rightarrow \omega \) is majorized by the increasing enumeration of some element of \( \mathcal{F} \). Clearly, any filter containing a rapid filter is rapid as well; thus, the existence of rapid filters is equivalent to that of rapid ultrafilters. Rapid ultrafilters are also known as semi-Q-point, or weak Q-point, ultrafilters. In [14] Miller proved that the nonexistence of rapid ultrafilters is consistent with ZFC (as well that of P-point ultrafilters, as mentioned above). However, it is still unknown whether the nonexistence of both rapid and P-point ultrafilters is consistent with ZFC.

Moreover, interpreting \( A \in \mathcal{U} \) as \( \omega \setminus A \notin \mathcal{U} \) (this is the same thing for ultrafilters) in condition (ii), we obtain the definition of \( \rightarrow \)-selective filters [11], which are not necessarily ultrafilters.
3. Varieties of Topological Groups and Free Topological Groups

A variety of topological groups is a class of topological groups closed with respect to taking topological subgroups, topological quotient groups, and Cartesian products of groups with the Tychonoff product topology. Thus, the abstract groups \( \tilde{G} \) underlying the topological groups \( G \) in a variety \( V \) of topological groups (that is, all groups \( G \in V \) without topology) form a usual variety \( \tilde{V} \) of groups. A variety \( V \) of topological groups is full if any topological group \( G \) for which \( \tilde{G} \in \tilde{V} \) belongs to \( V \). The notions of a variety and a full variety of topological groups were introduced by Morris in [45, 46], who also proved the existence of the free group of any full variety on any completely regular Hausdorff space \( X \).

Free objects of varieties of topological groups are characterized by the corresponding universality properties (we give a somewhat specific meaning to the word “universality,” but we use this word only in this meaning here). Thus, the free topological group \( F(X) \) on a space \( X \) admits the following description: \( X \) is topologically embedded in \( F(X) \) and, for any continuous map \( f \) of \( X \) to a topological group \( G \), there exists a unique continuous homomorphism \( \tilde{f} : F(X) \to G \) for which \( f = \tilde{f} \restriction X \). As an abstract group, \( F(X) \) is the free group on the set \( X \). The topology of \( F(X) \) can be defined as the strongest group topology inducing the initial topology on \( X \). On the other hand, the free topological group \( F(X) \) is the abstract free group generated by the set \( X \) (which means that any map of the set \( X \) to any abstract group can be extended to a homomorphism of \( F(X) \)) endowed with the weakest topology with respect to which all homomorphic extensions of continuous maps from \( X \) to topological groups are continuous. The free Abelian topological group \( A(X) \) on \( X \), the free Boolean topological group \( B(X) \) on \( X \), and free (free Abelian, free Boolean) precompact groups are defined similarly; instead of continuous maps to any topological groups, continuous maps to topological Abelian groups, topological Boolean groups, and precompact (Abelian precompact, Boolean precompact) groups should be considered.

For any space \( X \), the free Abelian topological group \( A(X) \) is the quotient topological group of \( F(X) \) by the commutator subgroup, and the free Boolean topological group \( B(X) \) is the quotient of \( A(X) \) by the subgroup of squares \( A(2X) \) (which is generated by all words of the form \( 2x, x \in X \)). (The universality of free objects in varieties of topological groups implies that the corresponding homomorphisms are continuous and open.) Thus, \( B(X) \) is the image of \( A(X) \) (and of \( F(X) \)) under a continuous open homomorphism.

Linear topological groups. There is yet another family of varieties of topological groups, which are not full but still interesting and useful. Following Malykhin (see also [3]), we say that a topological group is linear if it has a base of neighborhoods of the identity element which consists of open subgroups. The classes of all linear groups, all Abelian linear groups, and all Boolean linear groups are varieties of topological groups. As mentioned, these varieties are not full, but for any zero-dimensional space \( X \), there exist free groups of all of these three varieties on \( X \). Indeed, a free group of a variety of topological groups on a given space exists if this space can be embedded as a subspace in a group from this variety [45, Theorem 2.6]. The following lemma ensures the existence of the required embeddings for the three varieties under consideration (although it would suffice to embed any
Lemma 3.1.  
(i) For any space $X$ with $\text{ind} X = 0$, there exists a Hausdorff linear topological group $F'(X)$ such that $F'(X)$ is the algebraically free group on $X$, $X$ is a closed subspace of $F'(X)$, and all sets $F_n(X)$ of words of length at most $n$ are closed in $F'(X)$.

(ii) For any space $X$ with $\text{ind} X = 0$, there exists a Hausdorff Abelian linear topological group $A'(X)$ such that $A'(X)$ is the algebraically free Abelian group on $X$, $X$ is a closed subspace of $A'(X)$, and all sets $A_n(X)$ of words of length at most $n$ are closed in $A'(X)$.

(iii) For any space $X$ with $\text{ind} X = 0$, there exists a Hausdorff Boolean linear topological group $B'(X)$ such that $B'(X)$ is the algebraically free Boolean group on $X$, $X$ is a closed subspace of $B'(X)$, and all sets $B_n(X)$ of words of length at most $n$ are closed in $B'(X)$.

Proof. Assertion (i) was proved in [61, Theorem 10.5]. Let us prove (ii). Given a disjoint open cover $\gamma$ of $X$, consider the subgroup

$$H(\gamma) = \{ \sum_{i=1}^{n} (x_i - y_i) : n \in \mathbb{N} \text{ and for each } i \leq n, \text{there exists an } U_i \in \gamma \text{ for which } x_i, y_i \in U_i \}.$$ 

clearly, we can assume that all words in $H(\gamma)$ are reduced (if $x_i$ is canceled with $y_j$, then $U_i = U_j$, because $U_i \cap U_j \ni x_i = y_j$ and $\gamma$ is disjoint, and we can replace $x_i - y_i + x_j - y_j$ by $x_j - y_i$). All such subgroups generate a group topology on the free Abelian group on $X$; we denote the free Abelian group with this topology by $A'(X)$. (We might as well take only finite covers.)

The space $X$ is indeed embedded in $A'(X)$: given any clopen neighborhood $U$ of any point $x \in X$, we have $x + H ((U, X \setminus U)) \cap X = U$.

Let us show that $A_n(X)$ is closed in $A'(X)$ for any $n$. Take any reduced word $g = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_k x_k$ with $k > n$, where $\varepsilon_i = \pm 1$ and $x_i \in X$ for $i \leq k$. Let $U_i$ be clopen neighborhoods of $x_i$ such that $U_i$ and $U_j$ are disjoint if $x_j \neq x_i$ and coincide if $x_j = x_i$. We set

$$\gamma = \{ U_1, \ldots, U_k, X \setminus \bigcup_{i \leq k} U_i \}.$$ 

Take any reduced word $h = \sum_{i=1}^{m} (y_i - z_i)$ in $H(\gamma)$ and consider $g + h$. If, for some $i \leq m$, both $y_i$ and $-z_i$ are canceled in $g + h$ with some $x_j$ and $x_l$, then, first, $x_j = x_l$ (because any different letters in $g$ are separated by the cover $\gamma$, while $y_i$ and $z_i$ must belong to the same element of this cover), and secondly, $\varepsilon_j = -\varepsilon_l$ (because $y_i$ and $z_i$ occur in $h$ with opposite signs). Hence $\varepsilon_j x_j = -\varepsilon_l x_l$, which contradicts $g$ being reduced. Thus, among any two letters $y_i$ and $-z_i$ in $h$ only one can be canceled in $g + h$, so that $g + h$ cannot be shorter than $g$. In other words, $g + H(\gamma) \cap A'_n(X) = \emptyset$.

The proof that $X$ is closed in $A'(X)$ is similar: given any $g \notin X$, we construct precisely the same $\gamma$ as above (if $g \notin -X$) or set $\gamma = \{ X \}$ (if $g \in -X$) and show that $g + H(\gamma)$ must contain at least one negative letter.

The Hausdorffness of $A'(X)$ is equivalent to the closedness of $A_0(X)$.

The proof of assertion (iii) is similar. \qed
Lemma 3.1 immediately implies the following theorem.

**Theorem 3.1.** For any space $X$ with $\text{ind } X = 0$, the free, free Abelian, and free linear topological groups $F_{\text{lin}}(X)$, $A_{\text{lin}}(X)$, and $B_{\text{lin}}(X)$ are defined. They are Hausdorff and contain $X$ as a closed subspace, and all sets $F_n(X)$, $A_n(X)$, and $B_n(X)$ are closed in the respective groups.

By definition, the free linear groups of a zero-dimensional space $X$ have the strongest linear group topologies inducing the topology of $X$, that is, any continuous map from $X$ to a linear topological group (Abelian linear topological group, Boolean linear topological group) extends to a continuous homomorphism from $F_{\text{lin}}(X)$ ($A_{\text{lin}}(X)$, $B_{\text{lin}}(X)$) to this group.

4. **Descriptions of the Free Boolean Group Topology**

The topology of free groups can be described explicitly; all descriptions of the topology of free and free Abelian topological groups of which the author is aware are given in [61]. The descriptions of the free topological group topology are very cumbersome (except in a few special cases); the topology of free Abelian and Boolean topological groups looks much simpler. Thanks to the fact that $B(X) = A(X)/A(2X)$, the descriptions of the free Abelian topological group topology given in [61] immediately imply the following descriptions of the free topology of $B(X)$.

**I** For each $n \in \mathbb{N}$, we fix an arbitrary entourage $W_n \in \mathcal{U}$ of the diagonal of $X \times X$ in the universal uniformity of $X$ and set

$$
\tilde{W} = \{W_n\}_{n \in \mathbb{N}},
$$

$$
U(W_n) = \{x + y: (x, y) \in W_n\},
$$

and

$$
U(\tilde{W}) = \bigcup_{n \in \mathbb{N}} (U(W_1) + U(W_2) + \cdots + U(W_n)).
$$

The sets $U(\tilde{W})$, where $\tilde{W}$ ranges over all sequences of uniform entourages of the diagonal, form a neighborhood base at zero for the topology of the free Boolean topological group $B(X)$.

**II** For each $n \in \mathbb{N}$, we fix an arbitrary normal (or merely open) cover $\gamma_n$ of the space $X$ and set

$$
\Gamma = \{\gamma_n\}_{n \in \mathbb{N}},
$$

$$
U(\gamma_n) = \{x + y: (x, y) \in U \in \gamma_n\},
$$

and

$$
U(\Gamma) = \bigcup_{n \in \mathbb{N}} (U(\gamma_1) + U(\gamma_2) + \cdots + U(\gamma_n)).
$$

The sets $U(\Gamma)$, where $\Gamma$ ranges over all sequences of normal (or arbitrary open) covers, form a neighborhood base at zero for the topology of $B(X)$.

**III** For an arbitrary continuous pseudometric $d$ on $X$, we set

$$
U(d) = \left\{x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n: n \in \mathbb{N}, x_i, y_i \in X, \sum_{i=1}^{n} d(x_i, y_i) < 1\right\}.
$$

The sets $U(d)$, where $d$ ranges over all continuous pseudometrics on $X$, form a neighborhood base at zero for the topology of $B(X)$. 
Topology of free linear groups. It follows directly from the second description that the base of neighborhoods of zero in $\mathcal{B}\mathcal{L}(X)$ (for zero-dimensional $X$) is formed by the subgroups

$$\langle U(\gamma) \rangle = \left\{ \sum_{i=1}^{n} (x_i + y_i) : n \in \mathbb{N}, (x_i, y_i) \in U_i \in \gamma \text{ for } i \leq n \right\}$$

generated by the sets $U(\gamma)$ with $\gamma$ ranging over all normal covers of $X$. By definition, any normal cover of a strongly dimensional space has a disjoint open refinement. Therefore, for $X$ with $\dim X = 0$, the covers $\gamma$ can be assumed to be disjoint, and for disjoint $\gamma$, we have

$$\langle U(\gamma) \rangle = \left\{ \sum_{i=1}^{n} (x_i + y_i) : n \in \mathbb{N}, (x_i, y_i) \in U_i \in \gamma \text{ for } i \leq n, \text{ the word } \sum_{i=1}^{n} (x_i + y_i) \text{ is reduced} \right\}$$

(see the proof of Lemma 3.1). A similar description is valid for the Abelian groups $A\mathcal{L}(X)$ (the pluses must be replaced by minuses). This leads to the following statement.

**Theorem 4.1.** For any strongly zero-dimensional space $X$ and any $n \in \omega$, the topology induced on $A_n(X)$ (on $B_n(X)$) by $A\mathcal{L}(X)$ (by $B\mathcal{L}(X)$) coincides with that induced by $A(X)$ (by $B(X)$).

**Proof.** We can assume without loss of generality that $n$ is even. Given any neighborhood $U$ of zero in $A(X)$ (in $B(X)$), it suffices to take a sequence $\Gamma = \{ \gamma_k \}_{k \in \mathbb{N}}$ of disjoint covers such that $\frac{n}{2} \cdot U(\Gamma) \subset U$ and note that $\langle U(\gamma_1) \rangle \cap A_n(X) \subset \frac{n}{2} \cdot U(\gamma_1) \subset U$. □

Free topological groups in the sense of Graev and Graev’s extension of pseudometrics. In [22] Graev proposed a procedure for extending any continuous pseudometric $d$ on $X$ to a maximal invariant pseudometric $\hat{d}$ on $F(X)$, which is easy to adapt to the Boolean case. Following Graev, we first consider free topological groups in the sense of Graev, in which the identity element is identified with a point of the generating space and the universality property is slightly different: only continuous maps of the generating space to a topological group $G$ that take the distinguished point to the identity element of $G$ must extend to continuous homomorphisms [22]. Graev showed that the free topological and Abelian topological groups $F_G(X)$ and $A_G(X)$ in the sense of Graev are unique (up to topological isomorphism) and do not depend on the choice of the distinguished point; moreover, the free topological group in the sense of Markov is nothing but the Graev free topological group on the same space to which an isolated point is added (and identified with the identity element). By analogy with $F_G(X)$ and $A_G(X)$ the *Graev free Boolean topological group* $B_G(X)$ can be defined: we fix a point $x \in X$, identify it with the zero element of $B(X)$, and endow the resulting group with the strongest group topology inducing the initially existing topology on $X$, or, equivalently, with the coarsest group topology such that any continuous map of $X$ to a Boolean topological group $G$ that take the distinguished point $x$ to the zero element of $G$ extends to a continuous homomorphism.
The subgroup $B_G(X)$ thus obtained is unique (up to a topological isomorphism) and does not depend on the choice of the distinguished point. Indeed, let $B_{G'}(X)$ and $B_{G''}(X)$ be the Graev free Boolean topological groups on $X$ in which the zero elements are identified with $x' \in X$ and $x'' \in X$, respectively. The map $\varphi: X \to B_{G''}(X)$ taking each point of $X \subset B_{G'}(X)$ to the point $x + x' \in B_{G''}(X)$ is continuous, and the image of $x'$ is the zero of $B_{G''}(X)$. Therefore, $\varphi$ can be extended to a continuous homomorphism $\tilde{\varphi}: B_{G'}(X) \to B_{G''}(X)$. Similarly, the map $\psi: X \to B_{G'}(X)$ taking each point of $X \subset B_{G''}(X)$ to the point $x + x'' \in B_{G'}(X)$ is continuous, and the image of $x''$ is the zero of $B_{G'}(X)$. Therefore, $\psi$ can be extended to a continuous homomorphism $\tilde{\psi}: B_{G''}(X) \to B_{G'}(X)$. For each point $x \in X \subset B_{G'}(X)$, we have

$$\tilde{\psi}\tilde{\varphi}(x) = \tilde{\psi}(x + x') = \tilde{\psi}(x) + \tilde{\psi}(x') = x + x'' + x' + x'' = x + x' = x,$$

because $x'$ is the zero element of $B_{G'}(X)$. Thus, the continuous self-homomorphism $\tilde{\psi}\tilde{\varphi}$ of $B_{G'}(X)$ is the identity map on $X$. Since $X$ generates the group $B_{G'}(X)$, it follows that $\tilde{\psi}\tilde{\varphi}$ is the identity automorphism of $B_{G'}(X)$, and hence $\tilde{\varphi}$ is an isomorphism of $B_{G'}(X)$ onto $B_{G''}(X)$.

The extension of a continuous pseudometric $d$ on $X$ to a maximal invariant continuous pseudometric $\tilde{d}$ on the Graev free Boolean topological group $B_G(X)$ is defined by setting

$$\tilde{d}(g, h) = \inf \left\{ \sum_{i=1}^{n} d(x_i, y_i) : n \in \mathbb{N}, x_i, y_i \in X, g = \sum_{i=1}^{n} x_i, h = \sum_{i=1}^{n} y_i \right\}$$

for any $g, h \in B_G(X)$. The infimum is taken over all representations of $g$ and $h$ as (reducible) words of equal lengths. The corresponding Graev seminorm $\| \cdot \|_d$ (defined by $\|g\|_d = \tilde{d}(g, 0)$ for $g \in B_G(X)$, where 0 is the zero element of $B_G(X)$) is given by

$$\|g\|_d = \inf \left\{ \sum_{i=1}^{n} d(x_i, y_i) : g = \sum_{i=1}^{n} (x_i + y_i), x_i, y_i \in X \right\}.$$

The infimum is attained at a word representing $g$ which may contain one 0 (if the length of $g$ is odd) and is otherwise reduced. Indeed, if the sum representing $g$ contains terms of the form $x + z$ and $z + y$, then these terms can be replaced by one term $x + y$; the sum $\sum_{i=1}^{n} d(x_i, y_i)$ does not increase under such a change thanks to the triangle inequality.

For the usual (Markov’s) free Boolean topological group $B(X)$, which is the same as $B_G(X \oplus \{0\})$ (where 0 is an isolated point identified with zero), the Graev metric depends on the distances from the points of $X$ to the isolated point (they are usually set to 1 for all $x \in X$). The corresponding seminorm $\| \cdot \|_d$ on the subgroup $B_{\text{even}}(X)$ of $B(X)$ consisting of words of even length does not change. The subgroup $B_{\text{even}}(X)$ is open and closed in $B(X)$, because this is the kernel of the continuous homomorphism $\tilde{f}: B(X) \to \{0, 1\}$ extending the constant continuous map $f: X \to \{0, 1\}$ taking all $x \in X$ to 1. Thus, in fact, it does not matter how to
extend \( \| \cdot \|_d \) to \( B(X) \setminus B_{\text{even}}(X) \); for convenience, we set

\[
\| g \|_d = \begin{cases} 
\min \left\{ \sum_{i=1}^n d(x_i, y_i) : g = \sum_{i=1}^n (x_i + y_i), \ x_i, y_i \in X, \
\text{the word } \sum_{i=1}^n (x_i + y_i) \text{ is reduced} \right\} & \text{if } g \in B_{\text{even}}(X), \\
1 & \text{if } g \in B(X) \setminus B_{\text{even}}(X).
\end{cases}
\]

All open balls of radius 1 (or of any radius smaller than 1) in all seminorms \( \| \cdot \|_d \) for \( d \) ranging over all continuous pseudometrics on \( X \) form a base of open neighborhoods of zero in \( B(X) \).

**Boolean groups generated by almost discrete spaces.** A special role in the theory of topological groups and in set-theoretic topology is played by Boolean topological groups generated by almost discrete spaces, that is, spaces having only one nonisolated point. Clearly, for any almost discrete space \( X \cup \{ * \} \), where * is the only nonisolated point, the punctured neighborhoods of * form a filter on \( X \). Conversely, each free filter \( \mathcal{F} \) on any set \( X \) is naturally associated with the almost discrete space \( X_{\mathcal{F}} = X \cup \{ * \} \) (\( * \) is a point not belonging to \( X \)); all points of \( X \) are isolated and the neighborhoods of * are \( \{ * \} \cup A, A \in \mathcal{F} \). For a space \( X \) with infinitely many isolated points, there is no difference between the canonical definition of the groups \( F(X), A(X) \), and \( B(X) \) and Graev's generalizations \( F_G(X), A_G(X), \) and \( B_G(X) \); as mentioned above, Markov’s group \( B(X) \) is topologically isomorphic to \( B_G(Y) \), where \( Y \) is \( X \) plus an extra isolated point. Thus, when dealing with spaces \( X_{\mathcal{F}} \) associated with filters, we can identify \( B(X_{\mathcal{F}}) \) with \( B_G(X_{\mathcal{F}}) \) and assume that the only nonisolated point is the zero of \( B(X_{\mathcal{F}}) \); the descriptions of the neighborhoods of zero and the Graev seminorm are altered accordingly. To understand how they change, take the new (but in fact the same) space \( \tilde{X}_{\mathcal{F}} = X_{\mathcal{F}} \cup \{ 0 \} \), where 0 is one more isolated point, represent \( B(X_{\mathcal{F}}) \) as the Graev free Boolean topological group \( B_G(\tilde{X}_{\mathcal{F}}) \) with distinguished point (zero of \( B_G(\tilde{X}_{\mathcal{F}}) \)) 0, and consider the topological isomorphism \( g \mapsto g + 0 \) between this group and the similar group with distinguished point (zero) *.

For example, since any open cover of \( X_{\mathcal{F}} \) can be assumed to consist of a neighborhood of * and singletons, the description II reads as follows in this case: For each \( n \in \mathbb{N} \), we fix an arbitrary neighborhood \( V_n \) of *, that is, \( A_n \cup \{ * \} \), where \( A_n \in \mathcal{F} \), and set \( W = \{ V_n \}_{n \in \mathbb{N}}, \ U(V_n) = \{ x : x \in V_n \} = \{ x : x \in A_n \} \) (\( * \) is zero in \( B_G(X) \)), and

\[
U(W) = \bigcup_{n \in \mathbb{N}} (U(V_1) + U(V_2) + \cdots + U(V_n)) = \bigcup_{n \in \mathbb{N}} \{ x_1 + \cdots + x_n : x_i \in A_i \text{ for } i \leq n \}.
\]

The sets \( U(W) \), where the \( W \) range over all sequences of neighborhoods of *, form a neighborhood base at zero for the topology of \( B(X_{\mathcal{F}}) \). Strictly speaking, to obtain a full analogy with the description II of the Markov free group topology, we should set

\[
U(W) = \bigcup_{n \in \mathbb{N}} (2U(V_1) + 2U(V_2) + \cdots + 2U(V_n))
\]

\[
= \bigcup_{n \in \mathbb{N}} \{ x_1 + y_i \cdots + x_n + y_n : x_i, y_i \in V_i \text{ for } i \leq n \},
\]
but this would not affect the topology: the former $U(W)$ equals the latter for a sequence of smaller neighborhoods, say $V^i_n = \bigcap_{i < 2^n} V_i$ (remember that some of the $x_i$ and $y_i$ in the expression for $U(W)$ may equal $*$, that is, vanish).

Similarly, the base neighborhoods of zero in description III take the form

$$U(d) = \left\{ x_1 + x_2 + \cdots + x_n : n \in \mathbb{N}, x_1, \ldots, x_n \in X, \sum_{i=1}^{n} d(x_i, *) < 1 \right\},$$

where $d$ ranges over continuous pseudometrics on $X$. (Again, we should set $U(d) = \left\{ x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n : n \in \mathbb{N}, x_i, y_i \in X \cup \{ * \}, \sum_{i=1}^{n} d(x_i, y_i) < 1 \right\}$, but this would not make any difference.)

It is also easy to see that the isomorphism between $B_G(\bar{X}_\mathcal{F})$ (with distinguished point $*$) and $B(X)$ does not essentially affect the sets of words of length at most $n$; in particular, they remain closed, and $B_G(\bar{X}_\mathcal{F})$ is the inductive limit of these sets with the induced topology if and only if $B(X)$ has the inductive limit topology. In what follows, we identify $B_G(\bar{X}_\mathcal{F})$ with $B(X)$ and use the notation $B(X)$ for the Graev free Boolean topological group on $X$ with zero $*$.

Thus, $B(X)$ is naturally identified with the set $[X]^{<\omega}$ of all finite subsets of $X$ with the operation $\triangle$ of symmetric difference $(A \triangle B = (A \setminus B) \cup (B \setminus A))$. The point $*$, which is the zero element of $B(X)$, is identified with the empty set $\emptyset$, which belongs to $[X]^{<\omega}$ as the zero element. Sets of the form $[X]^{<\omega}$ often arise in set-theoretic topology and in forcing. The role of $X$ is usually played by $\omega$, and the filter $\mathcal{F}$ is often an ultrafilter with certain properties.

In the context of free Boolean groups on almost discrete spaces we identify each $n \in \omega$ with the one-point set $\{ n \} \in [\omega]^{<\omega}$.

5. **A Comparison of Free, Free Abelian, and Free Boolean Topological Groups**

**Similarity.** There are a number of known properties of free and free Abelian topological groups which automatically carry over to free Boolean topological groups simply because they are preserved by taking topological quotient groups or, more generally, by continuous maps. Thus, if $F(X)$ (and $A(X)$) is separable, Lindelöf, ccc, and so on, then so is $B(X)$. It is also quite obvious that $X$ is discrete if and only if so are $F(X)$, $A(X)$, and $B(X)$.

Let $X$ be a space, and let $Y$ be its subspace. The topological subgroup $B(Y|X)$ of $B(X)$ generated by $Y$ is not always the free Boolean topological group on $Y$ (the induced topology may be coarser). Looking at the description I of the free group topology on $B(X)$, we see that $X$ and $Y$ equipped with universal uniformities $\mathcal{U}_X$ and $\mathcal{U}_Y$ are uniform subspaces of $B(X)$ and $B(Y)$ with their group uniformities $\mathcal{U}_{B(X)}$ and $\mathcal{U}_{B(Y)}$ (generated by entourages of the form $W(U) = \{(g, h) : h \in g + U\}$, where $U$ ranges over all neighborhoods of zero in the corresponding group), which completely determine the topologies of $B(X)$ and $B(Y)$. Thus, if the topology of $B(Y|X)$ coincides with that of $B(Y)$, then, like in the case of free and free Abelian topological groups, $(Y, \mathcal{U}_Y)$ must be a uniform subspace of $(X, \mathcal{U}_X)$,
which means that any bounded continuous pseudometric on \( Y \) can be extended to a continuous pseudometric on \( X \) (in this case, \( Y \) is said to be \( P \)-embedded in \( X \) \([55]\)). The converse has been proved to be true for free Abelian (announced in \([67]\), proved in \([73]\)) and even free \([60]\) topological groups. This immediately implies the following theorem.

**Theorem 5.1.** Let \( X \) be a space, and let \( Y \) be its subspace. The topological subgroup of the free Boolean groups \( B(X) \) generated by \( Y \) is the free topological group \( B(Y) \) if and only if each closed continuous pseudometric on \( Y \) can be extended to a continuous pseudometric on \( X \).

Any space \( X \) is closed in its free Boolean topological group \( B(X) \), as well as in \( F(X) \) and \( A(X) \) (see, e.g., \([40]\) Theorems 2.1 and 2.2). Moreover, all \( F_n(X) \), \( A_n(X) \), and \( B_n(X) \) (the sets of words of length at most \( n \)) are closed in their respective groups as well. The most elegant proof of this fact was first proposed by Arkhangel’skii in the unavailable book \([3]\) (for respective groups as well. The most elegant proof of this fact was first proposed by Arkhangel’skii in the unavailable book \([3]\) (for \( F(X) \), but the argument works for \( A(X) \) and \( B(X) \) without any changes): note that all \( F_n(\beta X) \subset F(\beta X) \) are compact, since these are the continuous images of \( (X + \{e\} + X^{-1})^n \) under the natural multiplication maps \( t_n: (x_1, \ldots, x_n) \mapsto x_1 \cdot \ldots \cdot x_n \). (Here \( e \) denotes the identity element of \( F(X) \), \( e_i = \pm 1 \), and the word \( x_1 \cdot \ldots \cdot x_n \) may be reducible, i.e., have length shorter than \( n \).) Therefore, the \( n_n(\beta X) \) are closed in \( F(\beta X) \), and hence the sets \( F_n(X) = F_n(\beta X) \cap F(X|\beta X) \) are closed in \( F(X|\beta X) \). It follows that these sets are also closed in \( F(X) \), which is the same group as \( F(X|\beta X) \) but has stronger topology.

The topological structure of a free group becomes much clearer when this group has the inductive limit topology (or, equivalently, when the inductive limit topology is a group topology). The problem of describing all spaces for which \( F(X) \) (or \( A(X) \)) possesses this property has proved extremely difficult (and is still unsolved). Apparently, the problem was first stated explicitly by Pestov and Tkachenko in 1985 \([72]\), but it was tackled as early as in 1948 by Graev \([22]\), who proved that the free topological group of a compact space has the inductive limit topology. Then Mack, Morris, and Ordman \([35]\) proved the same for \( k_\omega \)-spaces. Apparently, the strongest result in this direction was obtained by Tkachenko \([68]\), who proved that if \( X \) is a \( P \)-space or a \( C_\omega \)-space (the latter means \( X \) is the inductive limit of an increasing sequence \( \{X_n\} \) of its closed subsets such that all finite powers of each \( X_n \) are countably compact and strictly collectionwise normal), then \( F(X) \) has the inductive limit topology. All these sufficient conditions are also valid for \( A(X) \) and \( B(X) \) by virtue of the following simple observation.

**Proposition 5.1.** Suppose that \( X = \bigcup_{n \in \omega} X_n \), \( Y = \bigcup_{n \in \omega} Y_n \), \( X \) is the inductive limit of its subspaces \( X_n \), \( n \in \omega \), and \( f: X \to Y \) is an open continuous map such that \( f(X_n) = Y_n \) for each \( n \in \omega \). Then \( Y \) is the inductive limit of its subspaces \( Y_n \).

**Proof.** Let \( U \subset Y \) be such that all \( U_n = U \cap Y_n \) are open in \( Y_n \). Consider \( V = f^{-1}(U) \) and \( V_n = f^{-1}(U_n) \cap X_n \) for \( n \in \omega \). For each \( n \), fix open \( W_n \subset Y_n \) for which \( W_n \cap Y_n = U_n \). We have

\[ V_n = f^{-1}(W_n \cap Y_n) \cap X_n = (f^{-1}(W_n) \cap f^{-1}(Y_n)) \cap X_n = f^{-1}(W_n) \cap X_n; \]

hence each \( V_n \) is open in \( X_n \). On the other hand,

\[ V_n = f^{-1}(U \cap Y_n) \cap X_n = (f^{-1}(U) \cap f^{-1}(Y_n)) \cap X_n = V \cap X_n; \]

---

3See also \([61]\), where a minor misprint in the condition 3° on p. 186 of \([60]\) is corrected.
therefore, $V$ is open in $X$. Since the map $f$ is open, it follows that $U = f(V)$ is an open set.

For $X$ of the form $\omega_\mathcal{F}$ (where $\mathcal{F}$ is a filter on $\omega$), not only the sufficient conditions mentioned above but also a necessary and sufficient condition for $F(X)$ and $A(X)$ to have the inductive limit topology is known. This condition is also valid for $B(X)$.

**Theorem 5.2.** Given a filter $\mathcal{F}$ on $\omega$, $B(\omega_\mathcal{F})$ has the inductive limit topology if and only if $\mathcal{F}$ is a $P$-filter.

**Proof.** This theorem is true for free and free Abelian topological groups \[59\]. Therefore, by Proposition 5.1, $B(\omega_\mathcal{F})$ has the inductive limit topology for any $P$-filter. It remains to prove that if $\mathcal{F}$ is not a $P$-filter, then $B(\omega_\mathcal{F})$ is not the inductive limit of the $B_n(\omega_\mathcal{F})$.

Thus, suppose that $\mathcal{F}$ is not a $P$-filter (or, equivalently, there exist a decreasing sequence of $A_n \in \mathcal{F}$, $n \in \omega$, such that, for any $A \in \mathcal{F}$, there is an $i$ for which the intersection $A \cap A_i$ is infinite) but $B(\omega_\mathcal{F})$ is the inductive limit of the $B_n(\omega_\mathcal{F})$. As usual, we assume that the zero element of $B(\omega_\mathcal{F})$ is the nonisolated point $* \in \omega_\mathcal{F}$.

Without loss of generality, we can assume that $A_0 = \omega$ and all sets $A_n \setminus A_{n+1}$ are infinite. We enumerate these sets as

$$A_n \setminus A_{n+1} = \{x_{ni} : i \in \omega\}$$

and put

$$D_n = \{x_{nm} + x_{i_1j_1} + x_{i_2j_2} + \cdots + x_{i_nj_n} : n < i_1 < i_2 < \cdots < i_n < j_1 < j_2 < \cdots < j_n < m\}$$

for all $n \in \omega$. Let us show that each $D_n$ is a closed discrete subset of $B(\omega_\mathcal{F})$. Fix $n$ and consider $X = \{\ast\} \cup \{x_{nm} : m \in \omega\}$ and the retraction $r: \omega_\mathcal{F} \to X$ that maps $\omega_\mathcal{F} \setminus X$ to $\{\ast\}$. Clearly, $X$ is discrete and the map $r$ is continuous. Let $\hat{r}: B(\omega_\mathcal{F}) \to B(X)$ be the homomorphic extension of $r$; then $\hat{r}$ continuously maps $B(\omega_\mathcal{F})$ onto the discrete group $B(X)$. For any $g \in B(\omega_\mathcal{F})$, the set $\hat{r}^{-1}(g) \cap D_n$ is finite: if $\hat{r}^{-1}(g) \cap D_n$ is nonempty, then we have $g = \hat{r}(x_{nm_0} + x_{i_0j_0} + x_{i_2j_2} + \cdots + x_{i_nj_n})$ for some $m_0, i_0, j_0 \in \omega$ such that $n < i_0 < i_2 < \cdots < i_n < j_0 < j_2 < \cdots < j_n < m$, whence $g = x_{nm_0}$ and

$$\hat{r}^{-1}(g) \cap D_n = \{x_{nm_0} + x_{i_1j_1} + x_{i_2j_2} + \cdots + x_{i_{n-1}j_{n-1}} : n < i_1 < i_2 < \cdots < i_{n-1} < j_1 < j_2 < \cdots < j_{n-1} < m_0\}. $$

Since the sets $\hat{r}^{-1}(g), g \in B(X)$, form an open cover of $B(\omega_\mathcal{F})$, it follows that $D_n$ is a closed discrete subspace of $B(\omega_\mathcal{F})$.

The length of each word in $D_n$ equals $n + 1$. Therefore, $D = \bigcup_n D_n$ is closed in the inductive limit topology. It remains to show that $*$ (the zero of $B(\omega_\mathcal{F})$) belongs to the closure of $D$ in the free group topology, i.e., that $U(d) \cap D \neq \emptyset$ for any continuous pseudometric $d$ on $\omega_\mathcal{F}$ (see the description III of the topology of $B(\omega_\mathcal{F})$).

Take an arbitrary (continuous) pseudometric $d$ on $\omega_\mathcal{F}$. In $\omega_\mathcal{F}$ the ball $B_\mathcal{F}(*, \frac{1}{n})$ of radius $\frac{1}{n}$ centered at $*$ with respect to $d$ is a neighborhood of $*$; that is, the punctured ball (with $*$ removed) belongs to $\mathcal{F}$. By assumption, the set $M = \{m \in \omega : d(*, x_{nm}) < \frac{1}{n}\}$ is infinite for some $n \in \omega$. Since $B_\mathcal{F}(*, \frac{1}{2n}) \cap A_{n+1}$ is a punctured neighborhood of $*$, it follows that the sets $J_i = \{j \in \omega : d(*, x_{ij}) < \frac{1}{2n}\}$
are infinite for infinitely many \(i > n\). Choose \(i_1 < i_2 < \cdots < i_n\) greater than \(n\) so that all \(J_{i_k}\) are infinite, in each \(J_{i_k}\) choose \(j_k\) so that \(i_n < j_1 < \cdots < j_n\), and take \(m \in M\) such that \(m > j_n\). We have \(g = x_{nm} + x_{i_1j_1} + x_{i_2j_2} + \cdots + x_{i_nj_n} \in D_n\). We also have \(g \in U(d)\), because

\[
d(\ast, x_{nm}) + \sum_{k=1}^{n} d(\ast, x_{i_kj_k}) < \frac{1}{2} + n \frac{1}{2n} = 1.
\]

Therefore, \(g \in D_n \cap U(d)\). \(\square\)

In [11] Tkachuk proved that the free Abelian topological group of a disjoint union of two spaces \(X\) and \(Y\) is topologically isomorphic to the direct sum \(A(X) \bigoplus A(Y) = A(X) \times A(Y)\). His argument carries over to varieties of Abelian topological groups closed under direct sums (or, in topological terminology, \(\sigma\)-products with respect to the zero elements of factors) with the box topology. We denote such sums by \(\sigma\).

**Theorem 5.3.** For any family \(\{X_\alpha : \alpha \in A\}\) of spaces,

\[
A\left(\bigoplus_{\alpha \in A} X_\alpha\right) \cong \sigma \bigoplus_{\alpha \in A} A(X_\alpha) \quad \text{and} \quad B\left(\bigoplus_{\alpha \in A} X_\alpha\right) \cong \sigma \bigoplus_{\alpha \in A} B(X_\alpha).
\]

If all \(X_\alpha\) are zero-dimensional, then

\[
A^{\text{lin}}\left(\bigoplus_{\alpha \in A} X_\alpha\right) \cong \sigma \bigoplus_{\alpha \in A} A^{\text{lin}}(X_\alpha) \quad \text{and} \quad B^{\text{lin}}\left(\bigoplus_{\alpha \in A} X_\alpha\right) \cong \sigma \bigoplus_{\alpha \in A} B^{\text{lin}}(X_\alpha).
\]

**Proof.** Let \(T\) stand for \(A\), \(B\), \(A^{\text{lin}}\), or \(B^{\text{lin}}\), and let \(0_\alpha\) denote the zero element of \(T(X_\alpha)\). For each \(\alpha \in A\), we set \(X'_\alpha = \sigma \bigoplus_{\beta \in A} Y_\beta\), where \(Y_\alpha = X_\alpha\) and \(Y_\beta = \{0_\beta\}\) for \(\beta \neq \alpha\). Every \(X'_\alpha\) is embedded in the group \(T'_n(X_\alpha)\) defined accordingly as a product of \(T(X_\alpha)\) and zeros. Clearly, the union \(\bigcup_{\alpha \in A} X'_\alpha\) algebraically generates \(\sigma \bigoplus_{\alpha \in A} T(X_\alpha)\) and is homeomorphic to \(\bigoplus_{\alpha \in A} X_\alpha\). It remains to show that the homomorphic extension of any continuous map of this union to any topological group from the corresponding variety is continuous. Let \(f : \bigcup_{\alpha \in A} X'_\alpha \to G\) be such a map. For each \(\alpha \in A\), the homomorphic extension \(\hat{f}_\alpha : T'_n(X_\alpha) \to G\) of the restriction of \(f\) to \(X'_\alpha\) is continuous. We define \(\hat{f} : \sigma \bigoplus_{\alpha \in A} T(X_\alpha) \to G\) by setting \(\hat{f}(\langle y_\alpha \rangle_{\alpha \in A}) = \sum_{\alpha \in A} \hat{f}_\alpha(y_\alpha)\) for each \(\langle y_\alpha \rangle_{\alpha \in A} \in \sigma \bigoplus_{\alpha \in A} T(X_\alpha)\); the sum is defined, because any element of \(\sigma \bigoplus_{\alpha \in A} T(X_\alpha)\) has only finitely many nonzero components. Let us show that \(\hat{f}\) is continuous. It suffices to check continuity at the zero element of \(\sigma \bigoplus_{\alpha \in A} T(X_\alpha)\). Take any neighborhood \(U\) of zero in \(G\). Its preimages \(V_\alpha\) under the component maps \(\hat{f}_\alpha\) are open neighborhoods of zero in \(T'_n(X_\alpha)\). The product \(\sigma \bigoplus_{\alpha \in A} V_\alpha\) is the preimage of \(U\) under \(\hat{f}\), and it is open in the box topology. \(\square\)

The free Boolean topological group of a nondiscrete space is never metrizable (as well as the free and free Abelian topological groups). Indeed, if \(B(X)\) is metrizable and \(X\) is nondiscrete, then \(X\) contains a convergent sequence \(S\) with limit point \(*\), and \(B(S) = B(S|X)\) (see Theorem 5.1); thus, it suffices to show that \(B(S)\) is nonmetrizable. Suppose that it is metrizable. Then the topology of \(B(S)\) is generated by a continuous norm \(\|\cdot\|\). For all pairs of positive integers \(n\) and \(m \leq n\) choose different \(s_{nm} \in S\) so that \(\|s_{nm} + *\| < \frac{1}{n^2}\). Clearly, the set

\[
D = \{(s_{n_1} + *) + (s_{n_2} + *) + \cdots + (s_{n_n} + *) : n \geq 0\}
\]
The list of properties shared by free, free Abelian, and free Boolean topological groups that can be proved without much effort is very long. Many of these properties are proved for Boolean groups by analogy, but sometimes their proofs are drastically simplified. We conclude our brief excursion by one of such examples. The proof of the following theorem for free topological groups given in \[60\] is extremely complicated (it is based on a more general construction). The proof given in \[61\] is much shorter but still very cumbersome. In the Boolean case, the proof becomes almost trivial.

**Theorem 5.4.** If \( \dim X = 0 \), then \( \ind B(X) = 0 \).

**Proof.** Any continuous pseudometric \( d \) on \( X \) is majorized by a non-Archimedean pseudometric\(^4\) \( \rho \) taking only values of the form \( \frac{1}{n} \). To see this, it suffices to consider the elements \( V_0, V_1, \ldots \) of the universal uniformity on \( X \) which are determined by decreasing disjoint open refinements \( \gamma_0, \gamma_1, \ldots \) of the covers of \( X \) by balls of radii \( \frac{1}{n}, \frac{1}{2}, \ldots \) with respect to \( d \) and apply the construction in the proof of Theorem 8.1.10 of \[15\] (see also \[24\]). Since the covers \( \gamma_n \) determining the entourages \( V_n \) are disjoint and each \( \gamma_{i+1} \) is a refinement of \( \gamma_i \), it follows that the function \( f \) in this construction has the property \( f(x, z) \leq \max\{f(x, y), f(y, z)\} \) and, therefore, the pseudometric \( \rho \) constructed there from \( f \) is non-Archimedean and takes the values \( \frac{1}{n} \). Clearly, it majorizes \( d \).

Each value \( \|g\|_\rho \) for the Graev extension \( \| \cdot \|_\rho \) of \( \rho \) is either 1 or a finite sum of values of \( d \) (recall that the minimum in the expression for \( \|g\|_\rho \) is attained at the irreducible representation of \( g \)). Hence \( \| \cdot \|_\rho \) takes only rational values, and the balls with irrational radii centered at zero in this norm are open and closed. They form a base of neighborhoods of zero, and their translates, a base of the entire topology on \( B(X) \).

**Difference.** Pestov gave an example of a space \( X \) for which \( F(X) \) is not homeomorphic to \( A(X) \) \[52\]. Spaces for which \( A(X) \) is not homeomorphic to \( B(X) \) exist, too.

**Proposition 5.2.** The free Abelian topological group of any connected space has infinitely many connected components. The free Boolean topological group of any connected space has two connected components.

**Proof.** Consider a connected space \( X \). The connected component of zero in \( A(X) \) is the subgroup \( A^c(X) \) consisting of all words \( \sum_{i=1}^{n} x_\varepsilon_i \) with \( \sum_{i=1}^{n} \varepsilon_i = 1 \) (see \[8\] Lemma 7.10.2). Clearly, all words in this subgroup are of even length, and the canonical homomorphism \( A(X) \to B(X) \) takes \( A^c(X) \) to the subgroup \( B^c(X) \) of \( B(X) \) consisting of all words of even length. Since the canonical homomorphism is continuous, the subgroup \( B^c(X) \) is connected, and it has index 2 in \( B(X) \). Thus, \( B(X) \) has at most two (in fact, precisely two) connected components, while

\(^4\)A pseudometric \( \rho \) is said to be non-Archimedean if \( \rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\} \) for any \( x, y, z \in X \).
$A(X)$ has infinitely many connected components, because $A(X)/A^c(X) \cong \mathbb{Z}$ (see [8, Lemma 7.10.2]).

There is a fundamental difference in the very topological-algebraic nature of free, free Abelian, and free Boolean groups. Thus, nontrivial free and free Abelian groups admit no compact group topologies (see [69]); this follows from the well-known algebraic description of infinite compact Abelian groups [26, Theorem 25.25]. On the other hand, for any infinite cardinal $\kappa$, the direct sum $\bigoplus_{\kappa} \mathbb{Z}_2$ of $2^\kappa$ copies of $\mathbb{Z}_2$ (that is, the free Boolean group of rank $2^\kappa$) is algebraically isomorphic to the Cartesian product $(\mathbb{Z}_2)^\kappa$ [27, Lemma 4.5] and, therefore, admits compact group topologies (e.g., the product topology).

The free and free Abelian groups are never finite, while the free Boolean group of any finite set is finite.

The free and the free Abelian topological group of any completely regular Hausdorff topological space $X$ contain all finite powers $X^n$ of $X$ as closed subspaces. Thus, each $X^n$ is homeomorphic to the closed subset $\{x_1 \ldots x_n : x_i \in X \text{ for } i = 1 \leq n\}$ of $F(X)$ (see [24]) and to the closed subset $\{x_1 + 2x_2 + \cdots + nx_n : x_i \in X \text{ for } i = 1 \leq n\}$ of $A(X)$ [66]. However, the situation with free Boolean topological groups is much more complicated. For example, consider extremally disconnected free topological groups.

Extremally disconnected groups are discussed in the next section. Here we only mention that nondiscrete $F(X)$ and $A(X)$ are never extremally disconnected, while $B(X)$ may be nondiscrete and extremally disconnected under certain set-theoretical assumptions (e.g., under CH) even for countable $X$ of the form $\omega_\omega$, and that any hereditarily normal, in particular, countable, extremally disconnected space is hereditarily extremally disconnected (this is shown in the next section). It follows that if $X$ is a nondiscrete countable space for which $B(X)$ is extremally disconnected, then $B(X)$ does not contain $X^2$ as a subspace. Indeed, otherwise, $X^2$ is extremally disconnected (and nondiscrete), and the existence of such spaces is prohibited by the following simple observation; it must be known, although the author failed to find a reference.

**Proposition 5.3.** If $X \times X$ is extremally disconnected, then $X$ is discrete.

This immediately follows from Frolik’s general theorem that the fixed point set of any surjective self-homeomorphism of an extremally disconnected space is clopen [3, 17]; it suffices to consider the self-homeomorphism of $X \times X$ defined by $(x, y) \mapsto (y, x)$.

5Arkhangelskii announced this result in [2] and proved it in [3] by considering the Stone–Čech compactification $\beta X$ of $X$ and its free topological group; details can be found in [8, Theorem 7.1.13]. Unfortunately, the book [3], which is a rotaprint edition of a lecture course, is (and always was) virtually unavailable, even in Russia. Thus, the result was rediscovered by Joiner [33], and the idea of proof, by Morris [46] (see also [25]). In fact, both Arkhangelskii and Joiner proved a stronger statement; namely, they gave the same complete description of the topological structure of all $F_n(X)$, although obtained by different methods (Arkhangelskii proof is much shorter).

6Frolik proved this theorem for compact extremally disconnected spaces and not necessarily surjective self-homeomorphisms; in the surjective case, the theorem is extended to noncompact spaces by considering their Stone–Čech compactifications, which are always extremally disconnected for extremally disconnected spaces (this and other fundamental properties of extremally disconnected spaces can be found in the book [21]).
Thus, there exist (under CH) filters \( F \) on \( \omega \) for which \((\omega F)^2\) is not contained in \(B(\omega F)\) as a subspace. However, in the simplest case where \( F \) is the Fréchet filter (i.e., \( \omega F \) is a convergent sequence), not merely does \( B(\omega F) \) contain \((\omega F)^n\) but it is topologically isomorphic to \(B(\omega F)^n\) for all \( n \) by virtue of Theorem 5.3 and the fact that a convergent sequence is \( B \)-equivalent to the disjoint union of two convergent sequences, which can be demonstrated as follows.

Any \( M \)-equivalent spaces are \( A \)-equivalent, and any \( A \)-equivalent spaces are \( B \)-equivalent, because \( A(X) \) (\( B(X) \)) is the quotient of \( F(X) \) (\( A(X) \)) by an algebraically determined subgroup not depending on \( X \). Therefore, all known sufficient conditions for \( M \)- and \( A \)-equivalence (see, e.g., [5, 22, 23, 51, 70]) remain valid for \( B \)-equivalence. In particular, if \( X_0 \) is a space, \( K \) is a retract of \( X_0 \), \( X \) is the space obtained by adding an isolated point to \( X_0 \), and \( Y = X_0/K \oplus K \), then \( X \) and \( Y \) are \( M \)-equivalent [51, Theorem 2.4]. This immediately implies the required \( B \)-equivalence of a convergent sequence \( S \) and the disjoint union \( S \oplus S \) of two convergent sequences: it suffices to take \( S \oplus S \) for \( X_0 \) and \( X \) and the two-point set of the two limit points in \( S \oplus S \) for \( K \).

However, there exist \( B \)-equivalent spaces which are neither \( F \)- nor \( A \)-equivalent. Genze, Gul’ko, and Khmyleva obtained necessary and sufficient conditions for infinite initial segments of ordinals to be \( F \)-, \( A \)-, and \( B \)-equivalent [19] (see also [20]). It turned out that the criteria for \( F \)- and \( A \)-equivalence are the same, and the criterion for \( B \)-equivalence differs from them; see [19] for details.

Finally, the following theorem shows that there is also a fundamental difference between free groups of the varieties of Abelian and Boolean linear topological groups.

**Theorem 5.5.** The free Boolean linear topological group of any strongly zero-dimensional pseudocompact space is precompact.

**Proof.** Let \( X \) be a strongly zero-dimensional pseudocompact space. As mentioned in the preceding section, a base of neighborhoods of zero in \( B\lim(X) \) is formed by subgroups of the form

\[
\langle U(\gamma) \rangle = \left\{ \sum_{i=1}^{n} (x_i + y_i) : n \in \omega, \ (x_i, y_i) \in U_i \in \gamma \text{ for } i \leq n \right\},
\]

where \( \gamma \) in a (finite) disjoint open cover of \( X \). Clearly,

\[
\langle U(\gamma) \rangle = \left\{ \sum_{i=1}^{2n} x_i : n \in \omega, \ |\{i \leq 2n : x_i \in U\}| \text{ is even for each } U \in \gamma \right\}.
\]

Every such subgroup has finite index. Therefore, \( B(X) \) is precompact. \( \square \)

This theorem is not true for Abelian groups; moreover, free Abelian linear groups are never precompact. Indeed, the group

\[
A^e(X) = \left\{ \sum_{i=1}^{n} x_i^{\varepsilon_i} : n \in \mathbb{N}, \sum_{i=1}^{n} \varepsilon_i = 1 \right\}
\]

considered above is always open, being the preimage of the isolated point 0 under the homomorphism \( A(X) \to \mathbb{Z}_2 = \{0, 1\} \) which extends the constant map \( X \to \{0, 1\} \) taking everything to 1. As already mentioned, \( A^e(X) \) has infinite index in \( A(X) \).
6. Extremally Disconnected Groups

There is an old problem of Arkhangel’skii on the existence in ZFC of a nondiscrete Hausdorff extremally disconnected topological group; it was posed in 1967 [11] and has been extensively studied since then. Various authors also posed the countable version of Arkhangel’skii’s problem (see, e.g., [53] Problem 6 and [13] Question 6.1): Does there exist a ZFC example of a countable nondiscrete extremally disconnected topological group? The countable version of the problem has been solved in the negative in the quite recent joint paper [57] by Reznichenko and the author; the uncountable case still persists.

The first consistent example of a nondiscrete extremally disconnected group was constructed as early as in 1969 by Sirota [65]; more examples were constructed in [37, 40, 41, 75, 76]. We refer the reader interested in extremally disconnected groups and, in general, topological groups with extremal properties (maximal, nodec, and so on) to Zelenyuk’s book [78].

A space $X$ is said to be extremally disconnected if the closure of each open set in this space is open, or, equivalently, if any two disjoint open sets have disjoint closures. In particular, the space $X_{\mathcal{F}}$ associated with a filter $\mathcal{F}$ is extremally disconnected if and only if $\mathcal{F}$ is an ultrafilter. Clearly, all extremally disconnected spaces are zero-dimensional. The most fundamental properties of extremally disconnected spaces can be found in the book [21]. Much useful information (especially in the topological-algebraic context) is contained in [6]. The central place in the theory of extremally disconnected topological groups is occupied by Boolean topological groups because of the following beautiful theorem of Malykhin.

**Theorem 6.1** (Malykhin [40]). Any extremally disconnected group contains an open (and therefore closed) Boolean subgroup.

This theorem follows from Frolík’s fixed-point theorem mentioned at the end of the preceding section. In [40] Malykhin reproved Frolík’s theorem for the particular self-homeomorphism $g \mapsto g^{-1}$; its fixed point set $U$ is an open neighborhood of the identity element, and the subgroup $H$ generated by an open neighborhood $V$ of the identity for which $V^2 \subset U$ is as required. Indeed, any $u, v \in V$ commute: $uv \in U$ and hence $vu = vu(uvu) = (v(uu)v)uv = uv$. Therefore, each $h \in H$ belongs to $U$, because $h = v_1v_2 \ldots v_n$ for some $v_1, v_2, \ldots, v_n \in V$, and the $v_i$ commute (and belong to $U$), so that $h^2 = v_1^2v_2^2 \ldots v_n^2$ equals the identity element.

Thus, in the theory of extremally disconnected groups only Boolean groups matter. However, as we shall see later on, a nondiscrete free Boolean topological group cannot be extremally disconnected in ZFC [62]; moreover, in the case of countable groups, this it true for any (not necessarily free) group topologies [57].

Extremal disconnectedness in groups is closely related to properties of countable discrete subsets of these groups. Thus, the presence of a countable nonclosed discrete set in an extremally disconnected group implies the existence of a $P$-ultrafilter [77], and the proof of the nonexistence in ZFC of countable nondiscrete extremally disconnected groups is based on the construction of two two disjoint discrete sequences with the same unique limit point under the assumption that there are no rapid ultrafilters [57]. Below we present yet another observation concerning discrete subsets, more relevant to the context of free topological groups.
Theorem 6.2.  
(i) If $G$ is a hereditarily extremally disconnected Boolean group, then any closed linearly independent subset of $G$ contains at most one nonisolated point.

(ii) If $G$ is an extremally disconnected Boolean group, then any countable closed linearly independent subset of $G$ contains at most one nonisolated point.

A source of examples of hereditarily extremally disconnected spaces is provided by the following simple observation.

Remark 6.1 (see [15] Exercise 6.2.G(c)). Any hereditarily normal extremally disconnected space is hereditarily extremely disconnected. Indeed, suppose that $X$ is a hereditarily normal extremally disconnected space. Let us show that any $Y \subset X$ is extremally disconnected. We must show that the closures (in $Y$ or in $X$, there is no difference) of any disjoint sets $U$ and $V$ which are open in $Y$ are disjoint. Note that the closures $U'$ and $V'$ of $U$ and $V$ in the open subspace $Z = X \setminus (\overline{U} \cap \overline{V})$ of $X$ are disjoint.

Since $X$ is hereditarily normal, there exist disjoint open (in $Z$ and, therefore, in $X$) sets $U'' \supset U'$ and $V'' \supset V'$. Their closures in $X$ cannot intersect, because $X$ is extremally disconnected, and hence the closures in $Y$ of the smaller sets $U$ and $V$ do not intersect either.

In the proof of Theorem 6.2 and later on we use the following obvious fact.

Remark 6.2. If countable sets $A$ and $B$ in a regular space $X$ are separated (i.e., each of them is disjoint from the other’s closure), then they have disjoint open neighborhoods. Indeed, numbering the elements of $A$ and $B$ as $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, respectively, we can construct neighborhoods $U_i$ of $a_i$ and $V_i$ of $b_i$ so that each $U_i$ is disjoint from $B$ and from all $V_j$ with $j \leq i$ and each $V_i$ is disjoint from $A$ and from all $U_j$ with $j \leq i$. Clearly, $\bigcup U_i$ and $\bigcup V_i$ are disjoint open neighborhoods of $A$ and $B$.

Proof of Theorem 6.2 (i) Let $A \subset G$ be a linearly independent subset of $G$. Suppose that $a \in A$ and $b \in A$ are distinct limit points of $A$. Take their disjoint closed neighborhoods $U \ni a$ and $V \ni b$. Since $A$ is linearly independent and closed, it follows that $a + (V \cap A) \cap b + (V \cap A) = \{a + b\}$ and the sets $a + (V \cap A)$ and $b + (V \cap A)$ are closed in $G$. Therefore, each of the disjoint sets $A' = a + ((V \setminus \{b\}) \cap A)$ and $B' = b + ((U \setminus \{a\}) \cap A)$ is open in the subspace $a + (V \cap A) \cup b + (U \cap A)$ of $G$; obviously, $a + b$ belongs to the closure of each of them, which contradicts the hereditary extremal disconnectedness of $G$.

(ii) Arguing as in (i), we obtain countable sets $A' = a + ((V \setminus \{b\}) \cap A)$ and $B' = b + ((U \setminus \{a\}) \cap A)$, which are separated in $G$ and, therefore, have disjoint open neighborhoods $U$ and $V$ (see Remark 6.2). We have $U \cap V \supset A' \cap B' \ni a + b$, which contradicts the extremal disconnectedness of $G$. □

Corollary 6.1. If $X$ is a nondiscrete countable space for which the free Boolean topological group $B(X)$ is extremally disconnected, then $X$ is the space $\omega_\mathcal{U}$ associated with an ultrafilter $\mathcal{U}$ on $\omega$.

Indeed, $X$ must be a filter space by Theorem 6.2 and $X$ must be extremally disconnected, because $B(X)$ is countable and, hence, hereditarily extremally disconnected. Therefore, the filter associated with $X$ must be an ultrafilter.
In fact, $\mathcal{F}$ must be a Ramsey ultrafilter (see the next theorem and Theorem 8.2 in the last section).

A detailed insight into extremally disconnected free Boolean topological groups on filter spaces can be gained from Section 8; here we present more general considerations.

The following theorem is a direct consequence of Theorem 1 in [62].

**Theorem 6.3.** Let $X$ be a topological space satisfying any of the following two conditions:

(i) the free Boolean topological group $B(X)$ is extremally disconnected;
(ii) $\text{ind} X = 0$ and the free Boolean linear topological group $B^{\text{lin}}(X)$ is extremally disconnected.

Then either $X$ is a $P$-space or there exists a Ramsey ultrafilter on $\omega$.

**Proof.** Let $\tau$ be the free group topology on $B(X)$ in case (i) and the free linear group topology on $B(X)$ in case (ii). Suppose that $X$ is not a $P$-space, that is, some point $x_0 \in X$ has a countable family of open neighborhoods $U_i$ ($i \in \mathbb{N}$) such that the interior of their intersection does not contain $x_0$. We can assume that $U_1 = X$, $U_{i+1} \subset U_i$ for $i \in \mathbb{N}$, and, moreover, all neighborhoods $U_i$ are clopen, because $X$ is zero-dimensional (as a subspace of the extremally disconnected space $(B(X), \tau)$).

Consider the countable space $Y$ obtained as the quotient image of $X$ under the map contracting the closed set $C_0 = \bigcap_{i \in \mathbb{N}} U_i$ and the clopen sets $C_i = U_i \setminus U_{i+1}$, $i \in \mathbb{N}$, to points; namely, $Y$ is the countable set $Y = \{y_i : i \in \omega\}$ with the quotient topology induced by the map $\varphi: X \to Y$ defined as

$$\varphi(x) = \begin{cases} y_0 & \text{if } x \in \bigcap_{i \in \mathbb{N}} U_i = C_0, \\ y_i & \text{if } x \in U_i \setminus U_{i+1} = C_i, i \in \mathbb{N}. \end{cases}$$

Clearly, the point $y_0$ is not isolated in the space $Y$ (while all of the other points $y_i$ are isolated). Thus, $Y$ is a space of the form $\omega_{\mathcal{F}}$ for some filter $\mathcal{F}$ on $\omega$.

Let $\tau'$ be the free group topology on $B(Y)$ in case (i) and the free linear group topology on $B(Y)$ in case (ii). The quotient map $\varphi: X \to Y$ extends to a continuous homomorphism $\tilde{\varphi}: (B(X), \tau) \to (B(Y), \tau')$. Since $\tau'$ is the strongest group (or linear group) topology inducing the topology of $Y$ on $Y$, it follows that this homomorphism is quotient and, hence, open (as any quotient homomorphism). Open maps preserve extremal disconnectedness (see [7, Problem 177]); therefore, the group $(B(Y), \tau')$ is extremally disconnected. It remains to apply Theorem 8.2 of the last section.

This theorem was proved in [62] in a more general situation (for $B(X)$ with any topology such that all continuous maps $X \to \mathbb{Z}_2$ extend to continuous homomorphisms), and the proof given there is much more complicated, because the continuity of $\tilde{\varphi}$ is not automatic and Theorem 8.2 does not apply in this general situation.

Theorem 6.3 has the following immediate consequence.

**Corollary 6.2.** The nonexistence of nondiscrete extremally disconnected free Boolean topological and linear topological groups is consistent with ZFC.
Indeed, as is known, any extremally disconnected $P$-space of nonmeasurable cardinality is discrete (see [31]). On the other hand, the nonexistence of measurable cardinals and Ramsey ultrafilters is consistent with ZFC (see [16]).

7. Free Boolean Groups on Filters on $\omega$

We have already seen in the preceding sections that free Boolean groups on almost discrete countable spaces (associated with filters on $\omega$) exhibit quite interesting behavior. Moreover, they are encountered more often than it may seem at first glance.

Consider any Boolean group $B(X)$ with countable basis $X$. As mentioned in Section 2 this group is (algebraically) isomorphic to the direct sum (or, in topological terminology, $\sigma$-product) $\bigoplus^{\aleph_0} \mathbb{Z}_2$ of countably many copies of $\mathbb{Z}_2$. There is a familiar natural topology on this $\sigma$-product, namely, the usual product topology; let us denote it by $\tau_{\text{prod}}$. This topology induces the topology of a convergent sequence on $X \cup \{0\}$ (where 0 denotes the zero element of $B(X)$) and is metrizable; therefore, it never coincides with the topology $\tau_{\text{free}}$ of the free Boolean topological group on $X$. Moreover, $\tau_{\text{prod}}$ is contained in $\tau_{\text{free}}$ only when $X = \omega_\beta$ for some filter (recall that we assume all filters to be free, i.e., contain the filter of cofinite sets, and identify the nonisolated points of the associated spaces with the zeros of their free Boolean groups). On the other hand, any countable space is zero-dimensional; therefore, any countable free Boolean topological group contains a sequence of subgroups with trivial intersection (see Theorem 3.1). The following statement shows that the topology of any group with this property contains the product topology $\tau_{\text{prod}}$ associated with some basis.

**Theorem 7.1** ([63] Lemma 2]). Let $G$ be a countable nondiscrete Boolean topological group which contains a family of open subgroups $G_i$ with trivial intersection. Then there exists a basis of $G$ such that the isomorphism $G \to \bigoplus^{\aleph_0} \mathbb{Z}_2$ taking this basis to the canonical basis of $\bigoplus^{\aleph_0} \mathbb{Z}_2$ is continuous with respect to the product topology on $\bigoplus^{\aleph_0} \mathbb{Z}_2 = \sigma(\mathbb{Z}_2)^{\aleph_0}$.

**Proof.** We treat $G$ as a vector space over the field $\mathbb{Z}_2$ and the $G_i$ as its subspaces.

To prove the lemma, it suffices to construct a basis $E = \{e_n : n \in \omega\}$ such that, for every $i \in \omega$, there exists a $J_i \subset \omega$ for which $G_i = \langle e_n : n \in J_i \rangle$. Indeed, if $E$ is such a basis, then the assumption $\bigcap G_i = \{0\}$ implies $\bigcap J_i = \emptyset$, and all linear spans $\langle e_k : k \geq n \rangle$ (which form a base of neighborhoods of zero in the product topology associated with the basis $E$) are open as subgroups with nonempty interior.

In each (nontrivial) quotient space $G_i/G_{i+1}$, we take a basis $\{e_\gamma : \gamma \in I_i\}$, where $|I_i| = \dim G_i/G_{i+1}$, and let $e_\alpha$ be representatives of $e_\gamma$ in $G_i$. We assume the (at most countable) index sets $I_i$ to be well ordered and disjoint, let $I = \bigcup \gamma \in I_i I_i$, and endow $I$ with the lexicographic order (for $\alpha, \beta \in I$, we say that $\alpha < \beta$ if $\alpha \in I_i, \beta \in I_j$, and either $i < j$ or $i = j$ and $\alpha < \beta$ in $I_i$). For any $i \in \omega$ and $\alpha \in I_i$, we define $H_\alpha$ to be the subspace of $G$ spanned by $\{e_\beta : \alpha \in I_i, \beta \geq \alpha\}$ and $G_{i+1}$. Thus, $H_\beta$ is defined for each $\beta \in I$; moreover, if $\beta, \gamma \in I$ and $\beta < \gamma$ in $I$, then $H_\beta \supset H_\gamma$, and if $\alpha$ is the least element of $I_i$, then $H_\alpha = G_i$. This means that the subspaces $H_\alpha$ form a decreasing (with respect to the order induced by $I$) chain of

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7Recall that a cardinal $\kappa$ is measurable if there exists an ultrafilter with the countable intersection property on a set of cardinality $\kappa$. Measurable cardinals do not exist in, e.g., ZFC + ($V = L$), while the consistency of their existence with ZFC has not been proved.
subspaces refining the chain $G_0 \supset G_1 \supset \ldots$. Note that the lexicographic order on $I$ is a well-order, so for each $\alpha \in I$, its immediate successor $\alpha + 1$ is defined; by construction, we have $\dim H_\alpha / H_{\alpha + 1} = 1$ for every $\alpha \in I$.

Clearly, it suffices to construct a basis $E' = \{ e'_\alpha : \alpha \in I \}$ with the property

$$e'_\alpha \in H_\alpha \setminus H_{\alpha + 1} \quad \text{for every} \quad \alpha \in I;$$

the required basis $E$ is then obtained by reordering $E'$. Moreover, property (\star) ensures the linear independence of $E'$, so it is sufficient to construct a set of vectors spanning $G$ with this property.

Take any basis $E'' = \{ e''_n : n \in \omega \}$ in $G$. We construct $E'$ by induction on $n$.

Let $\alpha_0$ be the (unique) element of $I$ for which $e''_0 \in H_{\alpha_0} \setminus H_{\alpha_0 + 1}$. We set $e'_0 = e''_0$.

Suppose that $k$ is a positive integer and we have already defined elements $\alpha_i \in I$ and vectors $e'_\alpha$, for $i < k$ so that $e'_\alpha \in H_\alpha \setminus H_{\alpha + 1}$ and $\{ e'_0, \ldots, e'_{\alpha_i-1} \} = \langle e'_0, \ldots, e'_{\alpha_i-1} \rangle$. Take the (unique) $\alpha \in I$ for which $e'_k \in H_\alpha \setminus H_{\alpha + 1}$. If $\alpha$ is unoccupied, that is, $\alpha \neq \alpha_i$ for any $i < k$, then we set $\alpha_k = \alpha$ and $e'_{\alpha_k} = e'_\alpha$. If $\alpha$ is already occupied, i.e., $\alpha = \alpha_m$ for some $m < k$, then we take the (unique) $\beta < k$ for which $e'_k + e'_m \in H_\beta \setminus H_{\beta + 1}$. (Clearly, $\beta > \alpha$, because $\dim H_\alpha / H_{\alpha + 1} = 1$ and $e'_k + e'_m \in H_\alpha \setminus H_{\alpha + 1}$). If $\beta$ is unoccupied, then we set $\alpha_k = \beta$ and $e'_{\alpha_k} = e'_k + e'_m$; if $\beta = \alpha_i$ for some $i < k$, then we take the (unique) $\gamma < I$ for which $e'_k + e'_m + e'_\gamma \in H_\gamma \setminus H_{\gamma + 1}$ (clearly, $\gamma > \beta > \alpha$), and so on. Only finitely many $(k - 1)$ indices from $I$ are occupied; therefore, after finitely many steps, we obtain $e'_k + e'_m + e'_n + \cdots + e'_{\alpha_i} \in H_\delta \setminus H_{\delta + 1}$ for an unoccupied index $\delta$. We set $\alpha_k = \delta$ and $e'_{\alpha_k} = e'_k + e'_m + e'_n + \cdots + e'_{\alpha_i}$.

As a result, we obtain a set of vectors $E' = \{ e'_n : n \in \omega \}$ such that $e'_n \in H_{\alpha_0} \setminus H_{\alpha_0 + 1}$ and $\langle e'_0, \ldots, e'_n \rangle = \langle e''_0, \ldots, e''_n \rangle$ for every $n \in \omega$. The latter means that $E'$ spans $G$, because so does the basis $E''$.

Formally, it may happen that not all of the indices $\alpha \in I$ are occupied, that is, $\{ \alpha_n : n \in \omega \} = J \subseteq I$. In this case, we take arbitrary vectors $e'_\alpha \in H_\alpha \setminus H_{\alpha + 1}$ for $\alpha \in I \setminus J$ and put them to $E'$. The set $E'$ thus enlarged satisfies condition (\star) and is therefore linearly independent; thus, it cannot differ from the initial $E'$, because the latter spans $G$, and $J$ in fact coincides with $I$, i.e., $E' = \{ e'_n : n \in \omega \} = \{ e'_\alpha : \alpha \in I \}$. This completes the proof of the lemma.

Theorem 7.1 implies that any countable Boolean topological group containing a family of open subgroups with trivial intersection (in particular, any free Boolean topological or linear topological group on a countable space) has a discrete or almost discrete closed basis. It turns out that this assertion holds for all countable Boolean topological groups. Namely, the following theorem is valid.

**Theorem 7.2.** Any countable Boolean topological group $G$ has either a discrete closed basis or a discrete basis for which 0 is a unique limit point. In the latter case, $G$ is a continuous isomorphic image of the free Boolean topological group $B(\omega, \mathcal{F})$ on the space $\omega, \mathcal{F}$ associated with a filter $\mathcal{F}$ on $\omega$.

**Proof.** On any countable topological group a continuous norm can be defined (see, e.g., [23] or [1]). Take any basis $\{ e_1, e_2, \ldots \}$ in $G$ and let $\| \cdot \|$ be a continuous norm on $G$. Consider a new basis $\{ e'_1, e'_2, \ldots \}$ defined by induction as follows:

- $e'_1 = e_1$;
- if $n \in \mathbb{N}$ and $e'_1, e'_2, \ldots, e'_n$ are already defined, then $e'_{n+1}$ is a word in the alphabet $\{ e'_1, e'_2, \ldots, e'_n, e_{n+1} \}$ with minimum norm (if there are several such words, then we take any of them).
For $k \in \omega$, we denote the set of all words in $\{e'_1, e'_2, \ldots \}$ of reduced length precisely $k$ by $G_{=k}$ and use the old notation $G_k$ for the set of all words of reduced length at most $k$; thus, $G_k = \bigcup_{m \leq k} G_m$.

**Lemma 7.1.** Let $w = e'_{i_1} + e'_{i_2} + \cdots + e'_{i_n}$, where $n, i_1, \ldots, i_n \in \mathbb{N}$ and $i_1 < i_2 < \cdots < i_n$. Then
\[
\|e'_{i_{n-k}}\| \leq 2^k \|w\| \quad \text{for each} \quad k = 0, \ldots, n-1.
\]

**Proof.** We argue by induction on $k$. The required inequality for $k = 0$ follows from the definition of $e'_{i_n}$. Suppose that $l > 0$ and the inequality holds for all $k < l$. We have $\|e'_{i_n}\| \leq \|w\|$, $\ldots$, $\|e'_{i_{n-l+1}}\| \leq 2^{l-1} \|w\|$; by the triangle inequality for norms, we have
\[
\|e'_{i_{n-l+1}} + e'_{i_{n-l+2}} + \cdots + e'_{i_n}\| \leq (2^{l-1} + 2^{l-2} + \cdots + 1) \|w\|
\]
and again applying the triangle inequality, we obtain
\[
\|e'_{i_1} + e'_{i_2} + \cdots + e'_{i_n}\| \leq \|w\| + \|e'_{i_{n-l+1}} + e'_{i_{n-l+2}} + \cdots + e'_{i_n}\| \leq 2^k \|w\|.
\]

**Lemma 7.2.** Each set $G_n$ is discrete in $G$ with respect to the norm $\| \cdot \|$.

**Proof.** For $n = 0$, the assertion is trivial. Suppose that $n > 0$ and take any $w \in G_n$; we have $w = e'_{i_1} + \cdots + e'_{i_n}$, where $i_1 < \cdots < i_n$. Let $d = \min_{k < n} \left\{ \frac{\|e'_{i_k}\|}{2^k} \right\}$. Suppose that $w' \in G_n$, i.e., $w' = e'_{j_1} + \cdots + e'_{j_n}$ for some $j_1 < \cdots < j_n$, and $w'$ belongs to the $d$-neighborhood of $w$ with respect to the norm $d$, i.e., $\|w + w'\| < d$. If the word $w'' = w + w'$ contains a letter $x$, then $x$ equals $e'_{i_r}$ or $e'_{j_s}$ for some $r \leq n$, and the length of $w''$ does not exceed $2n$; thus, by Lemma 7.1, we have $\|x\| \leq 2^{2n} \|w''\|$, whence $\|w''\| \geq \frac{\|e'_{i_r}\|}{2^{2n}}$. By the definition of $d$ $x$ cannot equal $e'_{i_r}$, i.e., all letters $e'_{i_r}$ must be cancelled in the word $w'' = w + w'$, which means that $w = w'$. Since the norm $\| \cdot \|$ on $G$ is continuous, it follows that the $d$-neighborhood of $w$ contains no elements of $G_n$ except $w$.

**Lemma 7.3.** Each set $G_n$ is closed in $G$ with respect to the norm $\| \cdot \|$.

**Proof.** As in the proof of Lemma 7.2, suppose that $n > 0$, take any $w = e'_{i_1} + \cdots + e'_{i_n} \in G_n$, where $i_1 < \cdots < i_n$, and let $d = \min_{k < n} \left\{ \frac{\|e'_{i_k}\|}{2^k} \right\}$. We show that, for $k < n$, the $d$-neighborhood of $w$ with respect to $d$ does not intersect $G_k$. Take any $w' = e'_{j_1} + \cdots + e'_{j_n}$, where $j_1 < \cdots < j_k$. The word $w'' = w + w'$ contains at least one letter $e'_{i_r}$ with $r \leq n$, because $k < n$, and the length of $w''$ does not exceed $2n$; by Lemma 7.1 we have $\|e'_{i_r}\| \leq 2^{2n} \|w''\|$, whence $\|w''\| \geq \frac{\|e'_{i_r}\|}{2^{2n}} \geq d$. This means that the $d$-neighborhood of $w$ contains no elements of $G_k$.

We proceed to prove the theorem. Note that the words of length 1 with respect to any given basis are precisely the basis elements. Therefore, by Lemma 7.2, the set $G_{=1} = \{e'_1, e'_2, \ldots \}$ is discrete in $G$, and by Lemma 7.3 the set $G_1 = \{e'_1, e'_2, \ldots \} \cup \{0\}$ is closed. This means that either $\{e'_1, e'_2, \ldots \}$ is closed (and discrete) in $G$ or the subspace $\{e'_1, e'_2, \ldots \} \cup \{0\}$ of $G$ is homeomorphic to a space of the form $\omega_{\mathcal{F}}$, where $\mathcal{F}$ is a filter on $\omega$, and the homeomorphism $\{e'_1, e'_2, \ldots \} \cup \{0\} \cong \omega_{\mathcal{F}}$ induces an algebraic isomorphism between $G$ and $B(\omega_{\mathcal{F}})$. Since the free group topology
of \( B(\omega^\mathcal{F}) \) is the strongest group topology inducing the given topology on \( \omega^\mathcal{F} \), it follows that the topology of \( G \) (which induces the same topology on the topological copy \( \{e'_1, e'_2, \ldots \} \cup \{0\} \) of \( \omega^\mathcal{F} \) in \( G \)) is coarser than that of \( B(\omega^\mathcal{F}) \).

\[ \square \]

**Remark 7.1.** Lemma 7.1 (with obvious modifications) and Lemmas 7.2 and 7.2 are valid for any normed countable-dimensional vector space over a finite field.

Spaces of the form \( \omega^\mathcal{F} \) are one of the rare examples where the free Boolean topological group is naturally embedded in the free and free Abelian topological groups as a closed subspace. The embedding of \( B(\omega^\mathcal{F}) \) into \( A(\omega^\mathcal{F}) \) is defined simply by \( x_1 + x_2 + \cdots + x_n \mapsto x_1 + x_2 + \cdots + x_n \) (for the Graev free groups, which coincide with Markov ones for such spaces), and the embedding into \( F(\omega^\mathcal{F}) \) is \( x_1 + x_2 + \cdots + x_n \mapsto x_1x_2\ldots x_n \), provided that \( x_1 < x_2 < \cdots < x_n \) (in \( \omega \)). These embeddings take \( B(\omega^\mathcal{F}) \) to

\[
A = \{ x_1 + x_2 + \cdots + x_n = (x_1 - *) + (x_2 - *) + \cdots + (x_n - *) \colon n \in \mathbb{N}, \ x_i \in \omega \} \subset A(\omega^\mathcal{F})
\]

and

\[
F = \{ x_1x_2\ldots x_n = x_1^{-1} x_2^{-1} \cdots x_n^{-1} \colon n \in \mathbb{N}, \ x_i \in \omega, \ x_1 < x_2 < \cdots < x_n \} \subset F(\omega^\mathcal{F}).
\]

The topologies induced on \( A \) and \( F \) by \( A(\omega^\mathcal{F}) \) and \( F(\omega^\mathcal{F}) \) are easy to describe; the restrictions of base neighborhoods of the zero (identity) element to these sets are determined by sequences of open covers of \( \omega^\mathcal{F} \) (i.e., of neighborhoods of the nonisolated point \(*\)) in the same manner as in description II (see [61]). The rigorous proof of the homeomorphism of \( A, F, \) and \( B(\omega^\mathcal{F}) \) is obvious but tedious, and we omit it.

8. **Free Boolean Topological Groups and Forcing**

As mentioned at the end of Section 4, for any filter \( \mathcal{F} \), the free Boolean group on \( \omega^\mathcal{F} \) is simply \( \omega^\mathcal{F} \hat{\omega} \) with zero \( \emptyset \). Generally, a topology on any set is a partially ordered (by inclusion) family of subsets. Partial orderings of subsets of \( \omega^\mathcal{F} \) have been extensively studied in forcing, and countable Boolean topological groups turn out to be closely related to them. In this section we shall try to give an intuitive explanation of this relationship. The basic definitions and facts related to forcing can be found in Kunen’s book [36] and Jech’s book [32].

Forcing is a method for extending models of set theory so as to include an object with desired properties. This is done by means of a partially ordered set \( (\mathcal{P}, \leq) \), referred to as a notion of forcing, and a generic set \( G \) consisting of compatible elements of \( \mathcal{P} \) and meeting all dense subsets of \( \mathcal{P} \) (such sets never belong to the ground model, except in trivial cases of no interest). The method of forcing yields the minimal extension of \( M \) containing \( G \), called the generic extension. The object with desired properties, which is the goal of the construction, is often simply \( \bigcup G \) or \( \bigcap G \). Figuratively, the desired properties are, so to speak, a frame for an infinite picture, the elements of \( \mathcal{P} \) are finite jigsaw pieces (which can never be fit together to make a picture large enough in the space where \( \mathcal{P} \) lives), and \( G \) is a set of compatible pieces that form a picture filling the frame but in a higher-dimensional space. Note that the design of \( \mathcal{P} \) is as important as that of \( G \), because \( \mathcal{P} \) is responsible for preventing undesirable destructive effects of forcing, such as collapse of cardinals.
Given two conditions \( p, q \in \mathbb{P} \), \( p \) is said to be stronger than \( q \) if \( p \leq q \). A partially ordered set \((\mathbb{P}, \leq)\) is separative if, whenever \( p \not\leq q \), there exists an \( r \leq p \) which is incompatible with \( q \). Thus, any topology is a generally nonseparable notion of forcing, and the family of all regular open sets in a topology is a separative notion of forcing. Any separative forcing notion \((\mathbb{P}, \leq)\) is isomorphic to a dense subset of a complete Boolean algebra. Indeed, consider the set \( \mathbb{P} \downarrow p = \{ q : q \leq p \} \) for each \( p \in \mathbb{P} \). The family \( \{ X \subset \mathbb{P} : (\mathbb{P} \downarrow p) \subset X \text{ for every } p \in X \} \) generates a topology on \( \mathbb{P} \). The complete Boolean algebra mentioned above is the algebra \( \text{RO}(\mathbb{P}) \) of regular open sets in this topology.

Two notions of forcing \( \mathbb{P} \) and \( \mathbb{Q} \) are said to be forcing equivalent if the algebras \( \text{RO}(\mathbb{P}) \) and \( \text{RO}(\mathbb{Q}) \) are isomorphic, or, equivalently, if \( \mathbb{P} \) can be densely embedded in \( \mathbb{Q} \) and vice versa (which means that \( \mathbb{P} \) and \( \mathbb{Q} \) produce the same generic extensions).

In the context of free Boolean groups on filters most interesting are two well-known notions of forcing, Mathias forcing and Laver forcing relativized to filters on \( \omega \).

In Mathias forcing relative to a filter \( \mathcal{F} \) the forcing poset, denoted \( \mathcal{M}(\mathcal{F}) \), is a pair \( (s, A) \) consisting of a finite set \( s \subset \omega \) and an (infinite) set \( A \in \mathcal{F} \) such that every element of \( s \) is less than every element of \( A \) in the ordering of \( \omega \) (i.e., \( \max s < \min A \)). A condition \( (t, B) \) is stronger than \( (s, A) \) \(( (t, B) \leq (s, A) ) \) if \( s \subset t \), \( B \subset A \), and \( t \setminus s \subset A \).

The poset in Laver forcing consists of subsets of the set \( \omega^{<\omega} \) of ordered finite sequences in \( \omega \). However, it is more convenient for our purposes to consider its modification consisting of subsets of \( [\omega]^{<\omega} \). Thus, we restrict the Laver forcing poset to the set \( \omega^{<\omega} \) of strictly increasing finite sequences (this restricted poset is forcing equivalent to the original one) and note that the latter is naturally identified with \( [\omega]^{<\omega} \). Below we give the definition of the corresponding modification of Laver forcing.

The definition of Laver forcing uses the notion of a Laver tree. A Laver tree is a set \( p \) of finite subsets of \( \omega \) such that

(i) \( p \) is a tree (i.e., if \( t \in p \), then \( p \) contains any initial segment of \( t \)),

(ii) \( p \) has a stem, i.e., a maximal node \( s(p) \in p \) such that \( s(p) \subset t \) or \( t \subset s(p) \) for all \( t \in p \), and

(iii) if \( t \in p \) and \( s(p) \subset t \), then the set \( \text{succ}(t) = \{ n \in \omega : n > \max t, t \cup \{ n \} \in p \} \) is infinite.

In Laver forcing relative to \( \mathcal{F} \) the poset, denoted \( \mathcal{L}(\mathcal{F}) \), is the set of Laver trees \( p \) such that \( \text{succ}(t) \in \mathcal{F} \) for any \( t \in p \) with \( s(p) \subset t \), ordered by inclusion.

The Mathias and Laver forcings \( \mathcal{M}(\mathcal{F}) \) and \( \mathcal{L}(\mathcal{F}) \) determine two natural topologies on \( [\omega]^{<\omega} \): the Mathias topology \( \tau_M \) generated by the base

\[ \{ [s, A] : s \in [\omega]^{<\omega}, A \in \mathcal{F}, \max s < \min A \}, \]

where \( [s, A] = \{ t \in [\omega]^{<\omega} : s \subset t, t \setminus s \subset A \} \),

and the Laver topology \( \tau_L \) generated by all sets \( U \subset [\omega]^{<\omega} \) such that

\[ t \in U \implies \{ n > \max t : t \cup \{ n \} \in U \} \in \mathcal{F}. \]

It is easy to see that the Mathias topology is nothing but the topology of the free Boolean linear topological group on \( \omega_{\mathcal{F}} \) (recall that linear groups are those with topology generated by subgroups): a base of neighborhoods of zero \( \emptyset \) is formed by the sets \( \{ \emptyset, A \} \) with \( A \in \mathcal{F} \), that is, by all subgroups generated by elements of \( \mathcal{F} \).
The neighborhoods of zero in the Laver topology are not so easy to describe explicitly; their recursive definition immediately follows from that given above for general open sets (the only additional condition $\emptyset \in U$ must be added). Thus, $U$ is an open neighborhood of zero if, first, $\emptyset \in U$; by definition, $U$ must also contain all $n \in A(\emptyset)$ for some $A(\emptyset) \in \mathcal{F}$ (moreover, $U$ may contain no other elements of size 1); for each of these $n$, there must exist an $A(n) \in \mathcal{F}$ such that $\min A(n) > n$ and $U$ contains all $\{n, m\} \in \mathcal{F}$ with $m \in A(n)$ (moreover, $U$ may contain no other elements of size 2); for any such $\{n, m\}$ (note that $m > n$) there must exist an $A(\{n, m\}) \in \mathcal{F}$ such that $\min A(\{n, m\}) > m$ and $U$ contains all $\{n, m, l\} \in \mathcal{F}$ with $l \in A(\{n, m\})$, and so on. Thus, each neighborhood of zero is determined by a family $\{A(s) : s \in [\omega]^\omega\}$ of elements of $\mathcal{F}$. Clearly, the topology $\tau_L$ is invariant with respect to translation by elements of $[\omega]^\omega$; upon a little reflection it becomes clear that $\tau_L$ is the maximal invariant topology on $[\omega]^\omega$ in which the filter $\mathcal{F}$ converges to zero. Since the free group topology is invariant as well, it is coarser than $\tau_L$.

The Mathias topology is, so to speak, the uniform version of the Laver topology: a neighborhood of zero in the Laver topology determined by a family $\{A(s) \in \mathcal{F} : s \in [\omega]^{<\omega}\}$ is open in the Mathias topology if and only if there exists a single $A \in \mathcal{F}$ such that $A(s) = A \setminus \{0, 1, \ldots, \max s\}$ for each $s$. (In [13] the corresponding relationship between Mathias and Laver forcings was discussed from a purely set-theoretic point of view.) Hence $\tau_M \subset \tau_L$.

The topology of the free Boolean topological group on $\omega$ occupies an intermediate position between the Mathias and the Laver topology: it is not so uniform as the former but more uniform than the latter. Neighborhoods of zero are determined not by a single element of the filter (like in the Mathias topology) but by a family of elements of $\mathcal{F}$ assigned to $s \in [\omega]^{<\omega}$ (like in the Laver topology), but these elements depend only on the lengths of $s$.

The following theorem shows that the Laver topology is a group topology only for special filters. This theorem was proved in 2007 by Egbert Thümmel, who kindly communicated it, together with a complete proof, to the author. The symbols $\tau_{\text{free}}$ and $\tau_{\text{indlim}}$ in its statement denote the topology of the free topological group $B(\omega, \mathcal{F})$ and the inductive limit topology of $B(\omega, \mathcal{F})$, respectively. We conventionally use the term selective filter for a filter satisfying any of the equivalent conditions (iii)–(v) in Theorem 2.4. Recall that, according to condition (iv), if $\mathcal{F}$ is a selective filter, then any family $\{A_i : i \in \omega\}$, where $A_i \in \mathcal{F}$, has a diagonal intersection in $\mathcal{F}$, that is, there exists a set $D \in \mathcal{F}$ such that $j \in A_i$ whenever $i, j \in D$ and $i < j$.

**Theorem 8.1** (Thümmel, 2007). For any filter on $\omega$, the following conditions are equivalent:

(i) $\mathcal{F}$ is selective;
(ii) $\tau_M = \tau_{\text{free}} = \tau_{\text{indlim}} = \tau_L$;
(iii) $\tau_L$ is a group topology;
(iv) for any sequence of $A_i \in \mathcal{F}$ with $\min A_i > i$, $i \in \omega$, the set $U = \{\emptyset\} \cup \bigcup_{i \in \omega} [i, A_i]$ is open in $\tau_{\text{free}}$.

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8An invariant topology is a topology with respect to which the group operation is separately continuous; Boolean groups with an invariant topology are precisely quasi-topological Boolean groups. The convergence of $\mathcal{F}$ to zero means that $\mathcal{F}$ contains all neighborhoods of zero in $\tau_L$ restricted to $\omega$, i.e., that $\tau_L$ induces the initially given topology on $\omega$. 

Thm"ommel mentioned that, in his proof of the implication (i) $\implies$ (ii) in this theorem, he used an argument known from forcing theory. The proof given below only slightly differs from Thm"ommel’s.

**Proof.** First, note that $\tau_M \subset \tau_{\text{tree}} \subset \tau_{\text{indim}} \subset \tau_L$. Indeed, the first two inclusions are obvious (recall that $\tau_M$ coincides with the free linear group topology), and the third one follows from Theorem 4.1 (or from the inclusion $\tau_{\text{tree}} \subset \tau_L$ noted above) and the observation that $\tau_L$ is the inductive limit of its restrictions to $B_n(\omega,\mathcal{F})$.

Thus, to prove the implication (i) $\implies$ (ii), it suffices to show that $\tau_M = \tau_L$ for any selective filter. Let $U$ be a neighborhood of $\emptyset$ in $\tau_L$. For each $i \in \omega$, we set

$$A_i = \bigcap \left\{ \{ n > \max s : s \cup \{ n \} \in U \} : s \in U, \max s \leq i \right\}.$$

Since the number of $s \in [\omega]^{<\omega}$ with $\max s \leq i$ is finite, it follows that $A_i \in \mathcal{F}$. Take a diagonal intersection $D \in \mathcal{F}$ of the family $\{ A_i : i \in \omega \}$. We can assume that $D \subset A_0$. Clearly, $[\emptyset, D] \subset U$, whence $U \in \tau_M$.

The implication (ii) $\implies$ (iii) is trivial.

Let us prove (iii) $\implies$ (iv). Note that it follows from (iii) that $\tau_{\text{tree}} = \tau_L$, because $\tau_{\text{tree}} \subset \tau_L$ and $\tau_{\text{tree}}$ is the strongest group topology inducing the initially given topology on $\omega,\mathcal{F}$. It remains to note that any set of the form $\{ \emptyset \} \cup \bigcup_{i \in \omega} [i, A_i]$, where $A_i \in \mathcal{F}$ and $\min A_i > i$, is open in $\tau_L$.

We proceed to the last implication (iv) $\implies$ (i). Take any sequence of $A_i \in \mathcal{F}$ with $\min A_i > i$, $i \in \omega$, and consider the set $U$ defined as in (iv). Since this is an open neighborhood of zero in the group topology $\tau_{\text{tree}}$, there exists an open neighborhood $V$ of zero (in $\tau_{\text{tree}}$) such that $V + V \subset U$. The set $D = \{ i \in \omega : i \in V \}$ belongs to $\mathcal{F}$ (because $\tau_{\text{tree}}$ induces the initially given topology on $\omega,\mathcal{F}$) and is a diagonal intersection of $\{ A_i : i \in \omega \}$. Indeed, if $i < j$ and $i,j \in D$, then $i + j = \{ i,j \} \in U$; thus, there exists a $k$ for which $\{ i,j \} \subset [k, A_k]$. The conditions $\min A_k > k$ and $i < j$ imply $k = i$. Therefore, $j \in A_i$. \(\square\)

Theorem 5.1 is worth comparing with Ihoda and Shelah’s theorem that if $\mathcal{F}$ is a Ramsey ultrafilter, then $\mathcal{M}(\mathcal{F})$ is forcing equivalent to $L(\mathcal{F})$ [30 Theorem 1.20 (i)] (Mathias forcing is referred to as Silver forcing in [30]).

Thommel also noticed that Theorem 5.1 combined with Sirota’s construction of a CH example of an extremely disconnected group, implies that, given a filter $\mathcal{F}$ on $\omega$, the free Boolean topological group $B(\omega,\mathcal{F})$ is extremally disconnected if and only if $\mathcal{F}$ is a Ramsey ultrafilter. Below we prove this statement in a formally stronger form: we do not assume $\mathcal{F}$ to be an ultrafilter in the if part. (Amazingly, the most immediate consequence of this stronger statement is that is not actually stronger.)

Thommel has never published these results, and the statement that, for an ultrafilter $\mathcal{U}$ on $\omega$, $B(\omega,\mathcal{U})$ is extremally disconnected if and only if $\mathcal{U}$ is Ramsey was rediscovered by Zelenyuk, who included it, among other impressive results, in his book [78] (see Theorem 5.1 in [78]).

**Theorem 8.2.**

(i) For any selective filter $\mathcal{F}$ on $\omega$, the free Boolean linear topological group $B^{\text{lin}}(\omega,\mathcal{F})$ (and hence the free Boolean topological group $B(\omega,\mathcal{F})$) is extremally disconnected.

(ii) If $\mathcal{F}$ is a filter on $\omega$ for which $B^{\text{lin}}(\omega,\mathcal{F})$ or $B(\omega,\mathcal{F})$ is extremally disconnected, then $\mathcal{F}$ is a Ramsey ultrafilter.
Proof. The proof of (i) is essentially contained in Sirota’s construction of a (consistent) example of an extremally disconnected group [63]. In [63] Sirota introduced the notion of a k-ultrafilter on $\omega$ and proved that $B^{lin}(\omega_\mathcal{F})$ is extremally disconnected for any k-ultrafilter $\mathcal{F}$. A k-ultrafilter is defined as an ultrafilter satisfying two conditions, one of which is precisely the selectivity condition (v) in Theorem [27]. We use only selectivity and do not assume our filter to be an ultrafilter; at that, thanks to Theorem [83] the proof presented below is far simpler than Sirota’s original proof.

**Lemma 8.1.** If $U$ is an open set in $B^{lin}(\omega_\mathcal{F})$ and $\emptyset \in \overline{\omega \cap U}$, then $\{\emptyset\} \cup U$ is open, i.e., $\emptyset \in \text{Int } U$.

*Proof.* Note that $\emptyset \in \overline{\omega \cap U}$ if and only if $U \supset A$ for some $A \in \mathcal{F}$. Since $U$ is open in $\tau_M$, it must contain each $i \in A \cap \omega$ together with its open neighborhood. Thus, $U$ contains a set of the the form $[i, A]$, $A_i \in \mathcal{F}$, for each $i \in A$, and any set of the form $\{\emptyset\} \cup \bigcup_{i \in A} \{i, A_i\}$, where $A_i \in \mathcal{F}$, is open in $\tau_M = \tau_\mathcal{L}$.

**Lemma 8.2.** If $X \subset B^{lin}(\omega_\mathcal{F})$ and $\emptyset \in \overline{X \setminus X}$, then $\emptyset \in \overline{\omega \cap X}$.

*Proof.* Suppose that, on the contrary, $\emptyset \notin \overline{\omega \cap X}$. Then $\emptyset \notin \overline{\omega \cap X}$ (because $\emptyset \in \mathcal{F}$). Let $U = B^{lin}(\omega_\mathcal{F}) \setminus X$. We have $\omega \cap U = \omega \setminus X$, so that by Lemma [8.1] $W = U \cup \{\emptyset\}$ is an open neighborhood of $\emptyset$. We also have $\emptyset \notin X$, $U \cap X = \emptyset$, and $W \cap X = \emptyset$, which contradicts the assumption $\emptyset \in \overline{X}$. □

**Lemma 8.3.** If $U$ is an open subset of $B^{lin}(\omega_\mathcal{F})$, $k \in \mathbb{N}$, and $\emptyset \in B_k(\omega_\mathcal{F}) \cap U$, then $\emptyset \in \text{Int } U$.

*Proof.* We prove the lemma by induction on $k$. For $k = 0$, $B_k(\omega_\mathcal{F}) = \{\emptyset\}$, and we have $\emptyset \in U \subset \text{Int } U$. Suppose that $n > 0$ and the required assertion holds for all $k < n$. Let us prove that if $\emptyset \in B_n(\omega_\mathcal{F}) \cap U$, then $\emptyset \in \text{Int } U$.

First, we show that $\omega \cap B_n(\omega_\mathcal{F}) \cap U \subset \text{Int } U$. Take any $i \in \omega$ such that $i \in \overline{B_n(\omega_\mathcal{F}) \cap U}$. An arbitrary open neighborhood of $i$ contains a neighborhood of the form $[i, A]$, where $A \in \mathcal{F}$, $\min A > i$, and we have $i \in [i, A] \cap B_n(\omega_\mathcal{F}) \cap U$. Note that $(i + [i, A] \cap B_n(\omega_\mathcal{F})) \subset B_{n-1}(\omega_\mathcal{F})$, because $\min A > i$ and $i + i = \emptyset$. On the other hand, $\emptyset = i + i \in (i + [i, A] \cap B_n(\omega_\mathcal{F})) \cap (i + U)$. By the induction hypothesis, we have $\emptyset \in \text{Int } i + U$, whence $i \in \text{Int } U$.

Thus, $\omega \cap B_n(\omega_\mathcal{F}) \cap U \subset \text{Int } U$. If $\emptyset \notin U$, there is nothing to prove. Suppose that $\emptyset \notin U \cap B_n(\omega_\mathcal{F}) \cap U$. Then, by Lemma [8.2] we have $\emptyset \in \omega \cap B_n(\omega_\mathcal{F}) \cap U \subset \omega \cap \text{Int } U$, and Lemma [8.1] implies $\emptyset \notin \text{Int } U = \text{Int } U$. □

To complete the proof of (i), it remains to recall that $\tau_M = \tau_{\text{indlim}}$ for selective filters and, therefore, $\overline{X} = \bigcup_{k \in \omega} B_k(\omega_\mathcal{F}) \cap X$ for any $X \subset B^{lin}(\omega_\mathcal{F})$. Thus, by Lemma [8.3] we have $\emptyset \in \text{Int } U$ whenever $U$ is an open set and $\emptyset \in \overline{U}$. Since $B^{lin}(\omega_\mathcal{F})$ is homogeneous, it follows that, for any open $U \subset B^{lin}(\omega_\mathcal{F})$ and any $x \in U$, we have $x \in \text{Int } U$, i.e., $\overline{U}$ is open. Thus, the free Boolean linear topological group $B^{lin}(\omega_\mathcal{F})$ is extremally disconnected, and hence so is the free Boolean topological group $B(\omega_\mathcal{F})$, because these groups coincide for selective filters by Theorem [83](ii).

The proof of (ii) is based on the implication (iv) $\iff$ (i) of Theorem [8.1] for any sequence of $A_i \in \mathcal{F}$ with $\min A_i > i$, $i \in \omega$, the set $U = \bigcup_{i \in \omega} [i, A_i]$ is by definition open in both $\tau_M$ and $\tau_{\text{free}}$, and $U' = \{\emptyset\} \cup \bigcup_{i \in \omega} [i, A_i]$ is closed in each of these.
topologies. Indeed, suppose that \( s = \{i_1, i_2, \ldots, i_n\} \notin U' \) and \( i_1 < i_2 < \cdots < i_n \). Then \( i_k \notin A_{i_1} \) for some \( k \in \{2, \ldots, n\} \). Let \( s' = s \setminus A_{i_1} \). Then \( [s, A_{i_1}] \) is an open (in both topologies) neighborhood of \( s \), and it does not intersect \( U' \), because the least letter of each word in \([s, A_{i_1}]\) is \( i_1 \), and any such word contains \( i_k \notin A_{i_1} \).

Thus, \( U' \) is the closure of \( U \) in \( \tau_M \) and \( \tau_{\text{free}} \). Therefore, \( U' \) must be open in \( \tau_{\text{free}} \) (if \( B(\omega,\mathcal{F}) \) is extremally disconnected) or even in \( \tau_M \supset \tau_{\text{free}} \) (if \( B_{\text{lin}}(\omega,\mathcal{F}) \) is extremally disconnected). In any case, the assertion (iv) \( \Leftrightarrow \) (i) of Theorem 8.1 implies that \( \mathcal{F} \) is a selective filter. It remains to apply Corollary 6.1. \( \square \)

Theorem 8.2 immediately implies the following (most likely, known) statement concerning filters.

**Corollary 8.1.** Each filter satisfying any of the equivalent conditions (iii)–(v) in Theorem 2.1 is a Ramsey ultrafilter.

Free Boolean topological and free Boolean linear (that is, Mathias) topological groups on spaces associated with filters, as well as Boolean groups with other topologies determined by filters, are the main tool in the study of topological groups with extreme topological properties (see [78] and the references therein). However, free Boolean (linear) topological groups on filters arise also in more “conservative” domains. We conclude with mentioning an instance of this kind.

The most elegant (in the author’s opinion) example of a countable nonmetrizable Fréchet–Urysohn group was constructed by Nyikos in [50] under the relatively mild assumption \( p = b \) (Hrušák and Ramos-García have recently proved that such an example cannot be constructed in ZFC [29]).

It is clear from general considerations that test spaces most convenient for studying convergence properties which can be defined pointwise (such as the Fréchet–Urysohn property and the related \( \alpha \)-properties) are countable almost discrete spaces (that is, spaces of the form \( \omega,\mathcal{F} \)), and the most convenient test groups for studying such properties in topological groups are those generated by such spaces, simplest among which are free Boolean linear topological groups. Thus, it is quite natural that Nyikos’ example is \( B_{\text{lin}}(\omega,\mathcal{F}) \) for a very cleverly constructed filter \( \mathcal{F} \).

In fact, he constructed it on \( \omega \times \omega \) (which does not make any difference, of course) as the set of neighborhoods of the only nonisolated point in a \( \Psi \)-like space defined by using graphs of functions \( \omega \to \omega \) from a special family. In the same paper Nyikos proved many interesting convergence properties of groups \( B_{\text{lin}}(\omega,\mathcal{F}) \) for arbitrary filters \( \mathcal{F} \) on \( \omega \). We do not give any more details here: the interested reader will gain much more benefit and pleasure from reading Nyikos’ original paper.

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