MOD 2 HOMOLOGY OF THE STABLE SPIN MAPPING CLASS GROUP

SØREN GALATIUS

Abstract. We compute the mod 2 homology of spin mapping class groups in the stable range. In earlier work [G] we computed the stable mod $p$ homology of the oriented mapping class group, and the methods and results here are very similar. The forgetful map from the spin mapping class group to the oriented mapping class groups induces a homology isomorphism for odd $p$ but for $p = 2$ it is far from being an isomorphism. We include a general discussion of tangential structures on 2-manifolds and their mapping class groups and then specialise to spin structures. As in [MW], on which [G] is based, the stable homology is the homology of the zero space of a certain Thom spectrum.

1. Introduction and statement of results

The main result of this paper is a calculation of the mod 2 homology of the “spin mapping class groups” in a stable range, in the spirit of [G], which rests heavily on [MW]. The paper consists of two parts. In the second part we adapt the proof in [MW] to the case of surfaces with a spin structure\(^1\). The result is that these groups have the same homology, in a stable range, as the infinite loop space $\Omega^\infty \text{Th}(\mathcal{U}_{\text{Spin}(2)})$ where $\mathcal{U}_{\text{Spin}(2)} = E\text{Spin}(2) \times_{\text{Spin}(2)} \mathbb{R}^2$ is the canonical Spin(2)-vectorbundle over $B\text{Spin}(2)$ and $-\mathcal{U}_{\text{Spin}(2)}$ is the $-2$-dimensional virtual inverse. $\text{Th}(\mathcal{U}_{\text{Spin}(2)})$ is the Thom spectrum with Thom class in dimension $-2$ (see section 2 for a more precise definition). In the first part of the paper we calculate the mod 2 homology of the infinite loop space $\Omega^\infty \text{Th}(\mathcal{U}_{\text{Spin}(2)})$. Let us introduce some notation before giving a more precise description of our results.

Let $\theta: U_3 \to B_3$ be a three-dimensional real vector bundle, and let $P_3 \to B_3$ be the underlying principal $\text{Gl}_3(\mathbb{R})$-bundle. For the moment it can be arbitrary, but we shall later specialise to the case $\theta = \theta_{\text{Spin}}: E\text{Spin}(3) \times_{\text{Spin}(3)} \mathbb{R}^3 \to B\text{Spin}(3)$. Let $U_2 = P_3 \times_{\text{Gl}_2(\mathbb{R})} \mathbb{R}^2$. This is a 2-dimensional real vectorbundle over the space $B_2 = P_3/\text{Gl}_2(\mathbb{R})$. $B_2$ is a fibre bundle over $B_3$ with fibre $\text{Gl}_3(\mathbb{R})/\text{Gl}_2(\mathbb{R}) \simeq S^2$, so $B_2$ is fibre homotopy equivalent to the sphere bundle of $U_3$. We have a canonical homeomorphism

$$\text{Bun}(V, U_2) = \text{Bun}(V \times \mathbb{R}, U_3)$$

where $\text{Bun}$ denotes the space of bundle maps.

\(^1\)The version of [MW] on which this manuscript is based, only treats oriented surfaces. A newer version of [MW] that treats surfaces with a more general tangential structure, similar to the “$\theta$-structures” considered in the second part of this paper, has since become available.
Definition 1.1. Let $F$ be a surface, possibly with boundary, and let $\theta : U_3 \to B_3$ be as above. Then the space of $\theta$-structures on $F$ is the space $\text{Bun}(TF, U_2)$ of bundle maps (these are supposed to be standard near the boundary if $F$ has boundary). This has a left action of $\text{Diff}(F)$. The space of $(F, \theta)$-surfaces is the space $\mathcal{M}_\theta(F, \gamma) := \text{E} \text{Diff}(F, \gamma) \times \text{Diff}(F, \gamma) \text{Bun}_\gamma(TF, U_2)$.

This is a connected space, and in general we have

$$\mathcal{M}_\theta(F) \simeq \bigsqcup_{\gamma} \mathcal{M}_\theta(F, \gamma),$$

where the disjoint union is over one $\gamma \in \pi_0 \text{Bun}(TF, U_2)$ in each $\text{Diff}(F)$-orbit.

There is a fibration sequence

$$\text{Bun}_\gamma(TF, U_2) \to \mathcal{M}_\theta(F, \gamma) \to \text{BDiff}(F, \gamma).$$

(1.1)

In particular (for genus $\geq 2$), $\mathcal{M}_\theta(F, \gamma)$ is a $K(\pi, 1)$ if and only if $\text{Bun}_\gamma(TF, U_2)$ is. In this case we have $\mathcal{M}_\theta(F, \gamma) \simeq \text{B} \Gamma_\theta(F, \gamma)$ where $\Gamma_\theta(F, \gamma) = \pi_1 \mathcal{M}_\theta(F, \gamma)$ is what we could call the mapping class group of $(F, \gamma)$.

The parametrised Pontryagin-Thom construction defines a map

$$\alpha : \mathcal{M}_\theta(F, \gamma) \to \Omega^\infty \text{Th}(-U_2)$$

and in favorable cases this will be an isomorphism in $H_n(-; \mathbb{Z})$ when $n$ is small compared to the genus of $F$.

The case $\theta = \theta_{SO} : E\text{SO}(3) \times \text{SO}(3) \mathbb{R}^3 \to B\text{SO}(3)$ is equivalent to the case considered in [MW]: An element $\gamma \in \pi_0 \text{Bun}(TF, U_2)$ is an orientation of $F$, and $\mathcal{M}_\theta(F, \gamma) \simeq \text{B} \text{Diff}(F, \gamma)$ is the classifying space of the group of orientation preserving diffeomorphisms. Furthermore $\Omega^\infty \text{Th}(-U_2) = \Omega^\infty \mathbb{C}P^\infty_1$. The homology of this space is calculated in [G].

Now specialise to the case $\theta = \theta_{\text{Spin}} : E\text{Spin}(3) \times \text{Spin}(3) \mathbb{R}^3 \to B\text{Spin}(3)$. Then an element $\gamma \in \pi_0 \text{Bun}(TF, U_2)$ is a “spin structure” on $F$, given equivalently by a “quadratic refinement of the intersection form” on $H_1(F, \mathbb{F}_2)$ cf [J]. Any two spin structures on $F$ differ by an element in $H^1(F, \mathbb{F}_2)$ so there are $4^g$ spin structures.
There are only two \(\text{Diff}(F)\)-orbits, however. They are distinguished by the Arf invariant of the quadratic form. Therefore
\[
\mathcal{M}^\theta(F) = \mathcal{M}^\theta(F; \gamma_0) \amalg \mathcal{M}^\theta(F; \gamma_1)
\]
where \(\gamma_0\) is an Arf invariant 0 spin structure and \(\gamma_1\) is an Arf invariant 1 spin structure.

The fibration sequence (1.1) specialises to
\[
\mathbb{R}P^\infty \to \mathcal{M}^\theta(F, \gamma) \to \text{BDiff}(F, \gamma)
\]
and in case \(F\) has genus \(\geq 2\) these are all \(K(\pi, 1)\)-spaces. The fundamental groups of \(\mathcal{M}^\theta(F, \gamma)\) and \(\text{BDiff}(F, \gamma)\) could both be called “spin mapping class groups”. Both are studied in [B] who uses the notation \(G_\gamma(F) = \pi_1\text{BDiff}(F, \gamma) = \pi_0\text{Diff}(F, \gamma)\) and \(\tilde{G}_\gamma(F) = \pi_1\mathcal{M}^\theta(F, \gamma)\) and attributes the latter to Gregor Masbaum. [H] and [B] proves homological stability of these groups: If \(F'\) is obtained from \(F\) by glueing along boundaries of \(F\), then the natural maps \(G_\gamma(F) \to G_\gamma(F')\) and \(\tilde{G}_\gamma(F) \to \tilde{G}_\gamma(F')\) are both isomorphisms in \(H_k(\; ; \mathbb{Z})\) when \(g \geq 2k^2 + 6k - 2\), where \(g\) is the genus of \(F\). [H] proves this in the case where \(F'\) has boundary, and [B] extends Harer’s proof to the case where \(\partial F' = \emptyset\), and proves homological stability for the groups \(\tilde{G}_\gamma(F)\).

The parametrised Pontryagin-Thom construction defines a map
\[
\alpha: \mathcal{M}^\theta(F, \gamma) \to \Omega^\infty\text{Th}(-U_{\text{Spin}(2)})
\]
and we prove that (on components) it is an isomorphism in \(H_k(\; ; \mathbb{Z})\) whenever \(g \geq 2k^2 + 6k - 2\). The number \(2k^2 + 6k - 2\) is the stability range of \(\tilde{G}_\gamma(F)\) in [B], and an improvement of the stability range would give an improvement of the isomorphism range of the map (1.2).

It is easily seen (using e.g. the fibration sequence (1.3) below) that \(\pi_0\Omega^\infty\text{Th}(-U_{\text{Spin}(2)}) \cong \mathbb{Z} \times \mathbb{Z}/2\). One may verify that the image of the map (1.2) is in the component given by the genus of \(F\) and the Arf invariant of \(\gamma\). We conclude this introduction by stating the theorems about the homology of \(\Omega^\infty\text{Th}(-U_{\text{Spin}(2)})\), and thus the homology in a stable range, of \(\mathcal{M}^\theta(F, \gamma)\).

The starting point of the calculation is a fibration sequence of infinite loop spaces
\[
\Omega^\infty\text{Th}(-U_{\text{Spin}(2)}) \xrightarrow{\Omega\omega} Q(B\text{Spin}(2)_+) \xrightarrow{\Omega\partial} \Omega Q(S(U_{\text{Spin}(2)})_+).
\]
Here \(S(U_{\text{Spin}(2)})\) is the sphere bundle of \(U_{\text{Spin}(2)}\) and \(Q\) denotes the functor \(\Omega^\infty\Sigma^\infty\). If we identify \(B\text{Spin}(2)\) with \(\mathbb{C}P^\infty\) then \(U_{\text{Spin}(2)} = L \otimes_{\mathbb{C}} L\), where \(L\) is the canonical complex line bundle, and \(S(U_{\text{Spin}(2)}) = \mathbb{R}P^\infty\). We give a concrete description of (1.3) in Section 2.

For brevity we shall write \(U\) for \(U_{\text{Spin}(2)}\). In the following, all Hopf algebras are commutative and cocommutative. Recall that any map \(f: A \to B\) of such Hopf algebras have a kernel denoted \(A \parallel f\) and a cokernel \(B \parallel f\) in the category of Hopf algebras. Homology and cohomology is always with coefficients in \(\mathbb{F}_2\).
Theorem 1.2. The fibration sequence (1.3) induces a short exact sequence of Hopf algebras

\[ H_*(\Omega^\infty \Sigma^1(S^2)) \langle \Omega \partial_* = H_*(\Omega^\infty \Sigma^2(S^2)) \langle H_*(Q(BSpin(2))) \langle \Omega \partial_* \]

and dually

\[ H^*(Q_0(BSpin(2))) \langle \Omega \partial_* \langle H^*(\Omega^\infty \Sigma^2(S^2)) \langle \Omega \partial_* \]

It remains to determine the Hopf algebras \( H_*(Q(BSpin(2))) \langle \Omega \partial_* \) and \( H_*(\Omega^\infty \Sigma^2(S^2)) \langle \Omega \partial_* \). The next theorem determines the Hopf algebra \( H_*(Q(BSpin(2))) \langle \Omega \partial_* \). We also produce an explicit splitting of the sequence (1.4), although the splitting is only as algebras, not as Hopf algebras.

The action of \( Spin(3) = SU(2) \) on \( S^2 \) gives an \( S^2 \)-bundle \( ESpin(3) \times Spin(3) S^2 \rightarrow BSpin(3) \). The vertical tangent bundle \( ESpin(3) \times Spin(3) T S^2 \) has a canonical spin-structure, and the classifying map \( ESpin(3) \times Spin(3) S^2 \rightarrow BSpin(2) \) is a homotopy equivalence. Consequently we get a map \( BSpin(3) \rightarrow \mathcal{M}^0(S^2) \). The composition

\[ BSpin(3) \rightarrow \mathcal{M}^0(S^2) \rightarrow \Omega^\infty \Sigma^2(S^2) \rightarrow Q(BSpin(2)) \]

is the Becker-Gottlieb transfer for the fibration sequence \( S^2 \rightarrow BSpin(2) \rightarrow BSpin(3) \).

Before stating the next theorem, let us recall that for a Hopf algebra \( A \) over \( \mathbb{F}_2 \) there is a Frobenius map \( \xi: A \rightarrow A \) given by \( \xi x = x^2 \) which is a morphism of Hopf algebras. Write \( a_i \in H_2i(BSpin(2)) \) and \( b_i \in H_4i(BSpin(3)) \) for the generators, \( i \geq 0 \). Recall that \( H_*(Q(BSpin(2))) \) is the free commutative algebra on the set \( T_2 \) of generators given by

\[
T_2 = \{ Q^I a_i \mid i \geq 0, \text{ \( I \) admissible, } e(I) > 2i \},
\]

where \( Q^I \) are the iterated Dyer-Lashof operations (see [CLM] for definitions and proofs). Similarly \( H_*(Q(BSpin(3))) \) is the free commutative algebra on the set of generators given by

\[
T_3 = \{ Q^I b_i \mid i \geq 0, \text{ \( I \) admissible, } e(I) > 4i \}.
\]

Theorem 1.3.

(i) We have \( H_*(Q(BSpin(2))) \langle \Omega \partial_* = \xi H_*(Q(BSpin(2))) \). Both the algebra \( H_*(Q(BSpin(2))) \langle \Omega \partial_* \) and the dual algebra \( H^*(Q_0(BSpin(2))) \langle \Omega \partial_* \) are free commutative.

(ii) The composition

\[
H_*(Q(BSpin(3))) \rightarrow H_*(\Omega^\infty \Sigma^2(S^2)) \rightarrow H_*(Q(BSpin(2))) \langle \Omega \partial_*
\]

is surjective. It maps \( b_i \) to \( a^2_i \) and more generally it maps \( Q^I b_i \) to \( (Q^I a_i)^2 \).
It remains to describe the Hopf algebra $H_*(\Omega^\infty \text{Th}(-U)) \| \Omega \omega_*$ in Theorem 1.2. This is done by first describing the (co-)homology of $\Omega Q(\mathbb{R}P^\infty_+)$ and $\Omega^2 Q(\mathbb{R}P^\infty_+)$. To state the results about $\Omega Q(\mathbb{R}P^\infty_+)$ and $\Omega^2 Q(\mathbb{R}P^\infty_+)$, let us recall a certain functor from [MM]. It is called $V$ in [MM, definition 6.2], but we shall call it $A$.

**Definition 1.4 ([MM]).** Let $V$ be a graded vectorspace and $\xi: V \to V$ a linear map such that $\xi V_n \subseteq V_{2n}$. Let $SV$ denote the free commutative (i.e. polynomial) algebra generated by $V$, and let $I \subseteq SV$ be the ideal generated by the elements $x^2 - \xi x, x \in V$. Let $AV = A(V, \xi) = SV/I$.

The functor $A$ satisfies $A(V \oplus V') = AV \otimes AV'$ and therefore the diagonal $V \to V \oplus V$ induces a comultiplication on $AV$ making it a Hopf algebra. The vectorspace of primitive elements is $V$ itself, $PAV = V$.

**Theorem 1.5.**

(i) The suspension

$$\sigma^*: QH^*(Q_0 \mathbb{R}P^\infty_+) \to PH^*(\Omega Q\mathbb{R}P^\infty_+)$$

is an isomorphism (of degree $-1$).

(ii) The suspension $\sigma^*$ above induces an isomorphism

$$A(s^{-1}QH^*(Q_0 \mathbb{R}P^\infty_+), s^{-1}\text{Sq}_1) \cong H^*(\Omega Q\mathbb{R}P^\infty_+).$$

Here $s^{-1}$ denotes desuspension of graded vector spaces and $\text{Sq}_1$ is the Steenrod operation given by $\text{Sq}_1(x) = \text{Sq}^{k-1}(x)$ if $\deg(x) = k$.

(iii) The Hopf algebra $H^*(\Omega Q(\mathbb{R}P^\infty_+))$ is primitively generated and polynomial.

(iv) The suspension induces an isomorphism

$$\sigma^*: \text{Coker}(\text{Sq}_1) \to QH^*(\Omega Q\mathbb{R}P^\infty_+).$$

**Theorem 1.6.**

(i) The suspension

$$\sigma^*: QH^*(\Omega_0 Q \mathbb{R}P^\infty_+) \to PH^*(\Omega^2 Q \mathbb{R}P^\infty_+)$$

is an isomorphism (of degree $-1$).

(ii) The suspension $\sigma^*$ above induces an isomorphism

$$A(s^{-2}\text{Coker}(\text{Sq}_1), s^{-2}\text{Sq}_2) \cong H^*(\Omega^2 Q\mathbb{R}P^\infty_+).$$

Here $\text{Sq}_2: \text{Coker}(\text{Sq}_1) \to \text{Coker}(\text{Sq}_1)$ is the Steenrod operation given by $\text{Sq}_2(x) = \text{Sq}^{k-2}(x)$ if $\deg(x) = k$.

(iii) The Hopf algebra $H^*(\Omega_0 Q(\mathbb{R}P^\infty_+))$ is primitively generated but not polynomial.

(iv) The suspension induces an isomorphism

$$\sigma^* \circ \sigma^*: \text{Coker}(\text{Sq}_2) \to QH^*(\Omega^2 Q\mathbb{R}P^\infty_+)$$

of degree $-2$. 
Using this description of $H^*(\Omega^2 Q \mathbb{R} P^\infty_+)\cap \omega^*$ we describe the Hopf algebra $H^*_s(\Omega^\infty \text{Th}(-U))\cap \omega^*_s$ and its dual $H^*(\Omega^\infty \text{Th}(-U))/\omega^*$.

**Theorem 1.7.**
(i) The Hopf algebra $H^*_s(\Omega^\infty \text{Th}(-U))\cap \omega^*_s$ is precisely the image of $H^*_s(\Omega^2 Q \mathbb{R} P^\infty_+) \to H^*_s(\Omega^\infty \text{Th}(-U))$.
(ii) $H^*(\Omega^\infty \text{Th}(-U))/\omega^*$ injects into $H^*(\Omega^2 Q(\mathbb{R} P^\infty_+))$ and is primitively generated.
(iii) Under the isomorphism in Theorem 1.6.(ii), $H^*(\Omega^\infty \text{Th}(-U))/\omega^*$ injects into $H^*_s(\Omega^2 Q \mathbb{R} P^\infty_+)\cap \omega^*_s$ and is primitively generated.

Finally we can combine the above to conclude the following corollary.

**Corollary 1.8.** The infinite loop map

$$\Omega^2 Q(\mathbb{R} P^\infty_+) \times Q(B\text{Spin}(3)_+) \to \Omega^\infty \text{Th}(-U),$$

which on the first factor is the map $\Omega^2 Q(\mathbb{R} P^\infty_+) \to \Omega^\infty \text{Th}(-U)$ induced by (1.3) and which on the second factor is the map $Q(B\text{Spin}(3)_+) \to \Omega^\infty \text{Th}(-U)$ from Theorem 1.3, induces an injection

$$H^*(\Omega^\infty \text{Th}(-U)) \to H^*(\Omega^2 Q(\mathbb{R} P^\infty_+)) \otimes H^*(Q(B\text{Spin}(3)_+)).$$

### 1.1. Acknowledgements

This paper is part of my thesis at the University of Aarhus. It is a great pleasure to thank my thesis advisor Ib Madsen for his help and encouragement during my years as a graduate student. I also thank M. Bökstedt, J. Tornehave and N. Wahl for many useful conversations and Lars Madsen for excellent technical assistance.

## 2. A cofibration sequence

Let us first describe a concrete model for the maps of spectra underlying the fibration sequence (1.3).

Let $q: \mathbb{C}P^n \to \mathbb{C}P^n$ denote the map $q([z_0 : \cdots : z_n]) = [z_0^2 : \cdots : z_n^2]$. Let $L_n$ denote the canonical complex line bundle over $\mathbb{C}P^n$ and $L_n^\perp$ its orthogonal complement. There is a bundle map

$$L_n \otimes L_n \xrightarrow{\hat{q}} L_n$$

$$\mathbb{C}P^n \xrightarrow{q} \mathbb{C}P^n$$

where $\hat{q}: (z_0, \ldots, z_n) \otimes (w_0, \ldots, w_n) \mapsto (z_0w_0, \ldots, z_nw_n)$. Thus $\hat{q}$ identifies $L_n \otimes L_n$ with $q^* L_n$. We shall write $L_n^2 = q^* L_n$ and $L_n^\perp = q^* L_n^\perp$. 
There is an obvious bundle map

\[ L_{n-1}^\perp \times \mathbb{C} \longrightarrow L_n^\perp \]

\[ \mathbb{C} \mathbb{P}^{n-1} \longrightarrow \mathbb{C} \mathbb{P}^n \]

and an induced bundle map

\[ L_{n-1}^\perp \times \mathbb{C} \longrightarrow L_n^\perp \]

\[ \mathbb{C} \mathbb{P}^{n-1} \longrightarrow \mathbb{C} \mathbb{P}^n \]

These gives maps of Thom spaces \( \text{Th}(L_{n-1}^\perp \wedge S^2) \rightarrow \text{Th}(L_n^\perp) \) and \( \text{Th}(L_{n-1}^2 \perp \wedge S^2) \rightarrow \text{Th}(L_n^2 \perp) \). Therefore we get spectra \( \text{Th}(-L) \) and \( \text{Th}(-L^2) \) with \((2n+2)\)-nd space \( \text{Th}(L_n^\perp) \) and \( \text{Th}(L_n^2 \perp) \), respectively. The associated infinite loop spaces are

\[ \Omega^\infty \text{Th}(-L) = \text{colim} \Omega^{2n+2} \text{Th}(L_n^\perp) \]

and

\[ \Omega^\infty \text{Th}(-L^2) = \text{colim} \Omega^{2n+2} \text{Th}(L_n^2 \perp) \] (2.1)

The bundle \( L \rightarrow \mathbb{C} \mathbb{P}^\infty \) above is isomorphic to \( U_{SO(2)} = \text{ESO}(2) \times_{\text{SO}(2)} \mathbb{R}^2 \rightarrow \text{BSO}(2) \) and \( L^2 \rightarrow \mathbb{C} \mathbb{P}^\infty \) is isomorphic to \( U_{\text{Spin}(2)} = \text{ESpin}(2) \times_{\text{Spin}(2)} \mathbb{R}^2 \rightarrow \text{BSpin}(2) \). The map \( q \) above is induced from the double cover \( \text{Spin}(2) \rightarrow \text{SO}(2) \). Therefore we shall write \( \Omega^\infty \text{Th}(-U_{SO(2)}) \) and \( \Omega^\infty \text{Th}(-U_{\text{Spin}(2)}) \) for the spaces of (2.1).

For a vector bundle \( \xi \rightarrow X \), let \( \text{Th}(\xi) = \xi \cup \{\infty\} \) be the one-point compactification of the total space.

**Lemma 2.1.** Let \( \xi \) and \( \eta \) be vector bundles over \( X \). Then there is a cofibration sequence

\[ \text{Th}(\xi) \xrightarrow{z} \text{Th}(\xi \oplus \eta) \xrightarrow{\partial} \text{Th}(\mathbb{R} \oplus \xi|_{S(\eta)}) \]

where \( z \) is induced from the zero section of \( \eta \) and \( \xi|_{S(\eta)} \) denotes pullback of \( \xi \) to the sphere bundle of \( \eta \). If \( \xi \oplus \eta = \mathbb{R}^n \times X \), then \( \partial \) is the parametrised Pontryagin-Thom construction of the sphere bundle \( S(\eta) \rightarrow X \).

**Proof.** The normal bundle of the embedding \( S(\eta) \rightarrow \eta \) is \( \mathbb{R} \times S(\eta) \). This embeds via “polar coordinates” onto \( \eta - X \). Therefore the normal bundle of the composition \( S(\eta) \rightarrow \eta \rightarrow \eta \oplus \xi \) is \( \mathbb{R} \oplus \xi|_{S(\eta)} \) and this embeds onto \( \xi \oplus \eta - \xi \subset \xi \oplus \eta \). This defines a homeomorphism

\[ \text{Th}(\xi \oplus \eta)/\text{Th}(\eta) = (\xi \oplus \eta - \eta) \cup \{\infty\} \cong \text{Th}(\mathbb{R} \oplus \xi|_{S(\eta)}) \]

If \( \xi \oplus \eta = X \times \mathbb{R}^n \), then \( \partial \) is exactly the Thom-Pontryagin construction applied to the embedding \( S(\eta) \subset X \times \mathbb{R}^n \) over \( X \). \( \square \)
Lemma 2.2. The map $\mathbb{R}P^{2n+1} \to \mathbb{C}P^n \times \mathbb{C}^{n+1}$ given by 
\[ [x_0 : y_0 : \cdots : x_n : y_n] \mapsto ([z_0 : \cdots : z_n], (z_0^2, \ldots, z_n^2)), \]
where $z_j = x_j + iy_j$, is a homeomorphism onto $S(L_n^2) \subseteq \mathbb{C}P^n \times \mathbb{C}^{n+1}$. Thus $S(L_n^2) \to \mathbb{C}P^n$ is identified with the quotient map $\mathbb{R}P^{2n+1} = S^{2n+1}/\{\pm 1\} \to S^{2n+1}/S^1 = \mathbb{C}P^n$.

Corollary 2.3. There is a cofibration sequence
\[ \text{Th}(\mathbb{R} \oplus L_n^2) \to \Sigma^{2n+3}\mathbb{C}P_+ \to \Sigma^{2n+2}\mathbb{R}P^{2n+1} \] (2.2)

Proof. Let $\xi = \mathbb{R} \oplus L_n^2$ and $\eta = L_n^2$ in Lemma 2.1. Then $\xi \oplus \eta = \mathbb{C}P^n \times \mathbb{C}^{n+1} \times \mathbb{R}$ and $\mathbb{R} \oplus \xi|_{S(0)} = \mathbb{C} \oplus L_n^2|_{S(L_n^2)} = L_n^2 \oplus L_n^2 = S(L_n^2) \times \mathbb{C}^{n+1}$, using the canonical trivialisation of $L_n^2|_{S(L_n^2)}$. Now lemmas 2.1 and 2.2 gives the desired result.

Corollary 2.4. There is a cofibration sequence of spectra
\[ \Sigma \text{Th}(-L^2) \to \Sigma^{\infty+1}(\mathbb{C}P_+) \to \Sigma^{\infty}\mathbb{R}P^\infty \]
and associated fibration sequences
\[ \Omega^{\infty}\Sigma \text{Th}(-L^2) \xrightarrow{\omega} Q\Sigma(\mathbb{C}P^\infty_+) \xrightarrow{\partial} Q\mathbb{R}P^\infty \] (2.3)
and
\[ \Omega^{\infty}\text{Th}(-L^2) \xrightarrow{\Omega\omega} Q(\mathbb{C}P^\infty_+) \xrightarrow{\Omega\partial} \Omega Q\mathbb{R}P^\infty \] (2.4)

Proposition 2.5. The map
\[ \partial : Q\Sigma\mathbb{C}P^\infty_+ \to Q\mathbb{R}P^\infty_+ \]
is the “$S^1$-transfer” denoted $t_1$ in [MMM]

Proof. The map $t_1$ in [MMM] is exactly the pretransfer of the $S^1$-bundle $ES^1 \times_{S^1} (S^1/\{\pm 1\}) \to BS^1$, and this is $\partial$.

Theorem 2.6 ([MMM]). Let $\bar{a}_r \in H_{2r+1}(\Sigma\mathbb{C}P^\infty_+)$ and $e_r \in H_r(\mathbb{R}P^\infty)$ be the generator. Then
\[ \partial_*(\bar{a}_r) \equiv e_{2r+1} + Q^{r+1}e_r \]
modulo decomposable elements.

Proof. This follows from [MMM, Theorem 4.4] by ignoring the decomposable terms.

Corollary 2.7. The map
\[ \partial_* : H_* Q\Sigma\mathbb{C}P^\infty_+ \to H_* Q\mathbb{R}P^\infty_+ \]
is injective.
Proof. This follows from Theorem 2.6 and the known structure of $H_*(Q\Sigma\mathbb{C}P^\infty_+)$ and $H_*(Q\mathbb{R}P^\infty_+)$, cf [CLM]. \hfill \boxrule 3. Cohomology of $\Omega Q\mathbb{R}P^\infty_+$ and $\Omega^2Q\mathbb{R}P^\infty_+$

The goal of this section is to prove Theorems 1.5 and 1.6. This is done via the following proposition.

Proposition 3.1. Let $X$ be a simply connected, homotopy commutative, homotopy associative $H$-space. Assume that $H_*(X)$ and $H_*(\Omega X)$ are of finite type. Then $H^*(X)$ is a polynomial algebra is and only if $\xi: PH^*(X) \to PH^*(X)$ is injective. In this case we have

(i) The suspension

$$\sigma^*: QH^*(X) \to PH^*(\Omega X)$$

is an isomorphism (of degree $-1$).

(ii) The suspension $\sigma^*$ above induces an isomorphism

$$A[s^{-1}QH^*(X), s^{-1}Sq_1] \cong H^*(\Omega X).$$

Here $s^{-1}$ denotes desuspension of graded vector spaces and $Sq_1: QH^*(X) \to QH^*(X)$ is the Steenrod operation given by $Sq_1(x) = Sq^{k-1}(x)$ if $\deg(x) = k$.

(iii) The Hopf algebra $H^*(\Omega X)$ is primitively generated. It is polynomial if and only if $Sq_1: QH^*(X) \to QH^*(X)$ is injective.

Proof. It follows from Borel’s structure theorem that $H^*(X)$ is polynomial if and only if $\xi: H^*(X) \to X^*(X)$ is injective. And this happens if and only if $\xi: PH^*(X) \to PH^*(X)$ is injective. The proof of the proposition is based on the Eilenberg-Moore spectral sequence, see [EM] or the review in [G]. The $E_2$-term is $\text{Tor}_{H^*(X)}(\mathbb{F}_2, \mathbb{F}_2)$ and it converges to $H^*(\Omega X)$.

When $H^*(X)$ is a polynomial algebra, the $E_2$-term of the spectral sequence is

$$\text{Tor}_{H^*(X)}(\mathbb{F}_2, \mathbb{F}_2) = E[s^{-1}QH^*(X)]$$

which has generators and primitives concentrated on the line $E_2^{-1,*}$. Therefore it must collapse, because it is a spectral sequence of Hopf algebras. The suspension can be identified with the map

$$QH^*(X) \cong \text{Tor}_{H^*(X)}^{-1,*}(\mathbb{F}_2, \mathbb{F}_2) = E_2^{-1,*} \to E_2^{-1,*} \subseteq \widetilde{H}^*(\Omega X)$$

and the image is within the vector space of primitive elements. Therefore $\sigma^*$ is injective because $E^2 = E^\infty$.

The image of $\sigma^*$ generates the algebra $H^*(\Omega X)$ because it generates the $E^\infty$-term. In particular we have proved that $H^*(\Omega X)$ is primitively generated.

The suspension $\sigma^*$ commutes with Steenrod operations. In particular we have

$$(\sigma^*(x))^2 = \sigma^*(Sq_1 x)$$
so the image of $\sigma^*$ is closed under the Frobenius map $\xi : x \mapsto x^2$. That $\sigma^*$ is surjective now follows from the Milnor-Moore exact sequence,

$$0 \to P\xi H^*(\Omega X) \to PH^*(\Omega X) \to QH^*(\Omega X) \to 0.$$ 

Namely, if $\sigma^*$ were not surjective, there would be an element of minimal degree in $PH^*(\Omega X)$ not in the image of $\sigma^*$. This element would have to map to zero in $QH^*(\Omega X)$ because the image of $\sigma^*$ generates. Hence, by the exact sequence, it would have to be a square of some other element. But this contradicts minimality because the image of $\sigma^*$ is closed under $\xi$.

We have proved (i) and the first part of (iii). Now (ii) follows from the fact that $H^*(\Omega X)$ is primitively generated and that $\xi : PH^*(\Omega X) \to PH^*(\Omega X)$ corresponds under $\sigma^*$ to $S_{Q_1}$. Finally, by (ii) we have that $\xi : H^*(\Omega X) \to H^*(\Omega X)$ is injective if and only if $S_{Q_1} : QH^*(X) \to QH^*(X)$ is injective. □

Remark 3.2. Without the assumption on simple connectivity the above proposition is generally false. It does hold in the following very special case, however. Namely, if $\pi_1 X$ is an $\mathbb{F}_2$-vectorspace and $X$ splits as $X \simeq \tilde{X} \times B\pi_1 X$. In this case we have $PH_1(X) = H_1(X) = \pi_1(X)$ and $QH_0(\Omega X) = \pi_0(\Omega X) = \pi_1(X)$, and for $k \geq 2$ we have $PH_k(X) = PH_k(\tilde{X})$ and $QH_{k-1}(\Omega X) = QH_{k-1}(\Omega \tilde{X})$.

Let $e_r \in H_r(\mathbb{R}P^\infty)$ be the generator. Recall from [CLM] that $H_*(\mathbb{Q}\mathbb{R}P^\infty)$ is the free commutative algebra on the set

$$T = \{Q^i e_r \mid r \geq 0, I \text{ admissible}, e(I) > r\}$$

We shall also need a basis for $PH_*(\mathbb{Q}\mathbb{R}P^\infty)$

Definition 3.3. Let $p_{2r+1} \in PH_*(\mathbb{Q}\mathbb{R}P^\infty)$ be the unique primitive class with $p_{2r+1} - e_{2r+1}$ decomposable. For an admissible sequence of the form $I = (2s + 1, 2I')$ with $e(I) \geq 2i$, let $p_{(I,2i)}$ be the unique primitive class with $p_{(I,2i)} - Q^i e_{2i}$ decomposable. For an admissible sequence $I = (I', 2s + 1, 2I'')$ with $e(I) \geq 2i$, let $p_{(I,2i)} = Q^i p_{(2s+1,2I'',2i)}$.

Thus $p_{(I,i)} \in PH_*(\mathbb{Q}\mathbb{R}P^\infty)$ is defined for all $(I,i)$ with $2 \nmid (I,i)$.

Lemma 3.4. The set

$$\{p_{(I,i)} \mid i \geq 0, I \text{ admissible}, e(I) \geq i, 2 \nmid (I,i)\}$$

is a basis of $PH_*(\mathbb{Q}\mathbb{R}P^\infty) = PH_*(\mathbb{Q}_0\mathbb{R}P^\infty)$.

Proof. This is well known. That $p_{(I,i)}$ spans all of $PH_*(\mathbb{Q}\mathbb{R}P^\infty)$ follows from the Milnor-Moore exact sequence. See [G] for more details □
Definition 3.5. Define operations $H_*(\mathbb{RP}^\infty) \to H_*(\mathbb{RP}^\infty)$ by
\[
\lambda x = Sq^k x, \quad \deg(x) = 2k, \\
\lambda' x = Sq^k x, \quad \deg(x) = 2k + 1 \\
\lambda'' x = Sq^k x, \quad \deg(x) = 2k + 2
\]

We write $\lambda, \lambda'$ and $\lambda''$ for the induced operations on $PH_*(\mathbb{RP}^\infty)$ and $QH_*(\mathbb{RP}^\infty)$ also. These are dual to $\xi = Sq_0, Sq_1,$ and $Sq_2$ on cohomology, respectively.

Lemma 3.6. In $H_*(\mathbb{RP}^\infty)$ we have
\[
\lambda e_{2r} = e_r \\
\lambda' e_{2r-1} = re_r \\
\lambda'' e_{2r-2} = \left(\frac{r}{2}\right)e_r
\]

Proof. This is dual to the formula $Sq^k w^n_s = \binom{n}{k} w^{n+k}_s \in H^*(\mathbb{RP}^\infty)$. □

Lemma 3.7. The operations $\lambda, \lambda'$ and $\lambda''$ satisfy the relations
\[
\lambda Q^{2s} x = Q^s \lambda x \tag{3.1} \\
\lambda' Q^{2s} x = Q^s \lambda' x \tag{3.2} \\
\lambda' Q^{2s-1} x = (\deg Q^s \lambda x) Q^s \lambda x \tag{3.3} \\
\lambda'' Q^{2s} x = Q^s \lambda'' x, \quad \text{if } \lambda x = 0 \tag{3.4} \\
\lambda'' Q^{2s-1} x = (1 + \deg Q^s \lambda' x) Q^s \lambda' x \tag{3.5}
\]

Proof. This follows from the Nishida relations (cf [CLM]). □

Proposition 3.8. $\lambda: QH_*(\mathbb{RP}^\infty) \to QH_*(\mathbb{RP}^\infty)$ is surjective.

Proof. This is because $\lambda: H_*(\mathbb{RP}^\infty) \to H_*(\mathbb{RP}^\infty)$ is surjective. Explicitly, (3.1) and Lemma 3.6 implies that
\[
\lambda Q^{2l} e_{2r} = Q^l e_r
\]
so the basis $T$ of $QH_*(\mathbb{RP}^\infty)$ is hit. □

Proposition 3.9. $\lambda': PH_*(\mathbb{RP}^\infty) \to PH_*(\mathbb{RP}^\infty)$ is surjective.

Proof. Lemma 3.6 and Lemma 3.7 imply that
\[
\lambda' e_{4r+1} = e_{2r+1}
\]
and that
\[
\lambda'(Q^{4s+1}Q^{4l'} e_{2l}) = Q^{2s+1}Q^{2l'} e_{2l}
\]
Hence, since \( \lambda' \) preserves decomposables
\[
\lambda' p_{4r+1} = p_{2r+1}
\]
and
\[
\lambda' p_{(4s+1,4i,4i)} = p_{(2s+1,2i,2i)}
\]
and hence
\[
\lambda' p_{(2i,4s+1,4i,4i)} = p_{(i,2s+1,2i,2i)}
\]
Therefore, by Lemma 3.4, \( \lambda' \) is surjective. \( \square \)

**Proposition 3.10.** \( \lambda'' : PH_*(QRP_+^\infty) \to \ker(\lambda') \) is not surjective.

**Proof.** The element \( p_3 = e_3 + e_1e_2 + e_1^3 \) satisfies \( \lambda'(p_3) = Q^2 e_1 = p_{(1,1)} \) and the element \( p_{(2,1)} \) satisfies \( \lambda'(p_{(2,1)}) = p_{(1,1)} \). So \( p_{(2,1)} + p_3 \in \ker(\lambda') \). But \( PH_4(QRP_+^\infty) \) has basis \( \{Q^3 e_1, Q^2 Q e_1\} \) and \( \lambda''(Q^3 e_1) = \lambda''(Q^2 Q e_1) = 0 \), so \( p_{(2,1)} + p_3 \) is not hit by \( \lambda'' \). \( \square \)

**Proof of Theorem 1.5.** This follows from Proposition 3.1, using Propositions 3.8 and 3.9. \( \square \)

**Proof of Theorem 1.6.** This follows from Proposition 3.1, using Theorem 1.5. That \( H^*(\Omega_0^\infty Q(RP_+^\infty)) \) is not polynomial follows from proposition 3.10. Indeed there must be an \( a \) generator of degree two with square zero. \( \square \)

### 4. The spectral sequence

The aim of this section is to prove theorems 1.2 and 1.7. The starting point is the fibration (1.3). None of the spaces in the fibration are connected. In fact we have
\[
\pi_0(\Omega QRP_+^\infty) = \mathbb{Z}/2 \times \mathbb{Z}/2, \quad \pi_0 Q(BSpin(2)_+) = \mathbb{Z}, \quad \pi_0 \Omega^\infty Th(-U) = \mathbb{Z} \times \mathbb{Z}/2
\]
and
\[
\pi_1 Q(BSpin(2)_+) = \mathbb{Z}/2, \quad \pi_1(\Omega Q(RP_+^\infty)) = \mathbb{Z}/2 \times \mathbb{Z}/2
\]
The claim in theorem (1.3) is clearly equivalent to the claim that the sequence
\[
H_*(\Omega^\infty Th(-U)) \xrightarrow{\Omega}\omega_* H_*(Q(BSpin(2)_+)) \xrightarrow{\Omega\partial_*} H_*(\Omega Q(RP_+^\infty)) \tag{4.1}
\]
is short exact (both means that \( \Omega\omega_* \) maps onto the kernel of \( \Omega\partial_* \)). This is equivalent to proving that the sequence
\[
H_*(\Omega_0^\infty Th(-U)) \xrightarrow{\Omega}\omega_* H_*(Q_0(BSpin(2)_+)) \xrightarrow{\Omega\partial_*} H_*(\Omega_0 Q(RP_+^\infty)) \tag{4.2}
\]
is short exact. Here \( \Omega_0 Q(RP_+^\infty) \) is the double cover of \( \Omega_0 Q(RP_+^\infty) \) corresponding to the image of \( \Omega\partial \) in \( \pi_1 \). This is equivalent because there is a map from (4.2) to (4.1), the kernel of which is the sequence
\[
H_0(\Omega^\infty Th(-U)) \xrightarrow{\Omega}\omega_* H_0(Q(BSpin(2)_+)) \xrightarrow{\Omega\partial_*} H_0(\Omega Q(RP_+^\infty)) \otimes H_*(RP_+^\infty)
\]
which is exact.

Now (4.2) corresponds to the following modified version of (1.3)

$$\Omega_0^{\infty} \mathrm{Th}(-U_{\text{Spin}(2)}) \xrightarrow{\Omega_0} Q_0(B\text{Spin}(2)_+) \xrightarrow{\hat{\Omega}_0} \hat{\Omega} Q(\mathbb{R}P^\infty_+).$$

To this fibration there is an associated Eilenberg-Moore spectral sequence

$$E^2 = \text{Cotor}^{H_*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+)}(H_*(Q_0 \mathbb{C}P^\infty_+), \mathbb{F}_2)$$

$$\cong \text{Cotor}^{H_*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+)/\Omega_0}(\mathbb{F}_2, \mathbb{F}_2) \otimes H_*(Q(B\text{Spin}(2)_+)) \\hat{\Omega}_0$$

$$\Rightarrow H_*(\Omega_0^{\infty} \mathrm{Th}(-U))$$

(4.3)

**Lemma 4.1.** The dual algebra $H^*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+)/\Omega_0^{\infty} \hat{\Omega}$ is polynomial.

**Proof.** It is a subalgebra of $H^*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+)$ which again is a subalgebra of $H^*(\Omega_0 Q \mathbb{R}P^\infty_+) \simeq \mathbb{R}P^\infty_+ \times \hat{\Omega}_0 Q(\mathbb{R}P^\infty_+)$. Therefore the lemma follows from Theorem 1.5. □

**Proof of Theorem 1.2.** From Lemma 4.1 we get that

$$\text{Cotor}^{H_*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+)/\Omega_0}(\mathbb{F}_2, \mathbb{F}_2) = E^2_{s-1} P(H_*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+)/\Omega_0)$$

Therefore the spectral sequence (4.3) has primitives and generators concentrated in $E^2_{0, *}$ and $E^2_{-1, *}$. Since it is a spectral sequence of Hopf algebras, it must collapse. Therefore the map

$$\Omega \omega_* : H_*(\Omega_0 \mathrm{Th}(-U)) \rightarrow H_*(Q_0(B\text{Spin}(2)_+)) \\hat{\Omega}_0$$

is surjective. □

We next prove Theorem 1.7. We need a lemma.

**Lemma 4.2.** The map

$$PH_*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+) \rightarrow P(H_*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+)/\Omega_0)$$

is surjective.

**Proof.** First note that $\hat{\Omega}_0 Q \mathbb{R}P^\infty_+ \simeq \hat{\Omega}_0 Q \mathbb{R}P^\infty_+ \times \mathbb{R}P^\infty_+$. The one-dimensional class in $PH_*(\mathbb{R}P^\infty_+)$ is in the image of $P(\Omega_0)$, by definition of the double cover $\hat{\Omega}_0 Q \mathbb{R}P^\infty_+$, so we may substitute $\hat{\Omega}_0 Q \mathbb{R}P^\infty_+$ for $\hat{\Omega}_0 Q \mathbb{R}P^\infty_+$ in the statement. The functor $P$ is left exact and has a right derived functor $\hat{P}$. See [G] for a survey and references. The important property is that it vanishes when the dual algebra is polynomial. There is an exact sequence of Hopf algebras

$$\mathbb{F}_2 \rightarrow \text{Im}(\Omega\partial_*) \rightarrow H_*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+) \rightarrow H_*(\hat{\Omega}_0 Q \mathbb{R}P^\infty_+)/\Omega_0 \rightarrow \mathbb{F}_2$$

Now $(\text{Im}(\Omega\partial_*))^* = \text{Im}(\Omega\partial^*)$ is a subalgebra of $H^*(Q_0(B\text{Spin}(2)_+))$ and hence is polynomial. Therefore $\hat{P}(\text{Im}(\Omega\partial_*)) = 0$ and the lemma follows. □
Corollary 4.3. The map
\[ \text{Cotor}^H_*(\hat{\Omega}_0 Q R P_+^\infty)(F_2, F_2) \to \text{Cotor}^H_*(\hat{\Omega}_0 Q R P_+^\infty)/\partial_*(F_2, F_2) \]
is surjective.

Proof. This is because \( \text{Cotor}^A(F_2, F_2) = E[s^{-1} PA] \) when \( A^* \) is polynomial. □

Proof of Theorem 1.7. The spectral sequence gives a filtration \( F_0 \supseteq F_{-1} \supseteq \ldots \) of \( H_*(\Omega_0^\infty \text{Th}(-U)) \) which restricts to a filtration of \( H_*(\Omega_0^\infty \text{Th}(-U))/\Omega \omega_* \). With respect to this filtration we have
\[ E^0(H_*(\Omega_0^\infty \text{Th}(-U))/\Omega \omega_*) \cong \text{Cotor}^H_*(\hat{\Omega}_0 Q R P_+^\infty)/\Omega \partial_*(F_2, F_2) \]

There is a map of fibrations
\[
\begin{array}{ccc}
\Omega_0^2 Q R P_+^\infty & \to * & \hat{\Omega}_0 Q R P_+^\infty \\
\downarrow & & \downarrow \\
\Omega_0^\infty \text{Th}(-U) & \xrightarrow{\text{Th}} & Q_0(B\text{Spin}(2)_+) & \xrightarrow{\Omega \partial} & \hat{\Omega}_0 Q R P_+^\infty
\end{array}
\]

and an associated map of spectral sequences which on the \( E^2 \)-term is
\[
\text{Cotor}^H_*(\hat{\Omega}_0 Q R P_+^\infty)(F_2, F_2) \to \text{Cotor}^H_*(\hat{\Omega}_0 Q R P_+^\infty)/\Omega \partial_*(F_2, F_2) \otimes H_*(Q(B\text{Spin}(2)_+))/\partial_*
\]
Since both spectral sequences collapse, we get that the map
\[ H_*(\Omega_0^2 Q R P_+^\infty) \to H_*(\Omega_0^\infty \text{Th}(-U))/\Omega \omega_* \quad (4.4) \]
is filtered and on filtration quotients the map is identified with
\[
\text{Cotor}^H_*(\hat{\Omega}_0 Q R P_+^\infty)(F_2, F_2) \to \text{Cotor}^H_*(\hat{\Omega}_0 Q R P_+^\infty)/\Omega \partial_*(F_2, F_2)
\]
Since this is surjective by Lemma 4.2, then also the map (4.4) is surjective.

This proves (i). (ii) is just the dual statement of (i). To prove (iii) we see that the quotient
\[ QH_*(\Omega^2 Q(R P_+^\infty)) \to Q(H_*(\Omega^\infty \text{Th}(-U))/\Omega \omega_*) \]
is identified under suspension with
\[ PH_*(\Omega Q(R P_+^\infty)) \to P(H_*(\Omega Q(R P_+^\infty))/\Omega \partial_*) \]
which again by suspension is mapped to
\[ PH_*(Q(R P_+^\infty)) \to P(H_*(Q(R P_+^\infty))/\partial_*) = \text{Coker}(P \partial_*). \]
By dualising we get \( \text{Ker}(Q \partial_*) \) as claimed. □
5. Proof of Theorem 1.3

We know from theorem 1.5 that $H_*(\Omega Q(\mathcal{P}_{\infty}^\perp))$ is an exterior algebra. We also know that $H_*(Q\text{BSpin}(2)_+)$ is a polynomial algebra. We have the following commutative diagram

$$
\begin{array}{ccc}
QH_*(Q(B\text{Spin}(2)_+)) & \xrightarrow{Q(\Omega \partial_\ast)} & QH_*(\Omega Q(\mathcal{P}_{\infty}^\perp)) \\
\cong & & \cong \\
PH_*(Q\Sigma(B\text{Spin}(2)_+)) & \xrightarrow{P(\partial_\ast)} & PH_*(Q(\mathcal{P}_{\infty}^\perp))
\end{array}
$$

It follows that $Q(\Omega \partial_\ast)$ is injective. These three facts prove that $H_*(Q\mathcal{P}_{\infty}^\perp)\wr \Omega \partial_\ast = \xi H_*(Q\mathcal{P}_{\infty}^\perp)$. We calculate the Becker-Gottlieb transfer of the bundle

$$E\text{Spin}(2) \times_{\text{Spin}(2)} S^2 \to B\text{Spin}(2).$$

**Lemma 5.1.** Let $N, S: B\text{Spin}(2) \to E\text{Spin}(2) \times_{\text{Spin}(2)} S^2$ denote the sections at the north and south pole, respectively. Then the Becker-Gottlieb transfer is

$$\tau = N + S \in [B\text{Spin}(2), Q(E\text{Spin}(2) \times_{\text{Spin}(2)} S^2_{\ast})].$$

**Proof.** This is similar to the Becker-Gottlieb calculations in [GMT]: $S^2$ is the Spin(2)-equivariant pushout of $D^2 \leftarrow S^1 \to D^2$ and therefore the bundle $E\text{Spin}(2) \times_{\text{Spin}(2)} S^2$ is the fibrewise pushout of $E\text{Spin}(2) \times_{\text{Spin}(2)} D^2 \leftarrow E\text{Spin}(2) \times_{\text{Spin}(2)} S^1 \to E\text{Spin}(2) \times_{\text{Spin}(2)} D^2$. Then properties (A1)-(A3) in [GMT, p. 15] proves the proposition. Indeed the transfer of $E\text{Spin}(2) \times_{\text{Spin}(2)} S^1$ vanishes by (A3) and the transfer of $E\text{Spin}(2) \times_{\text{Spin}(2)} D^2$ is the section at the center of $D^2$ by (A1). Then the additivity (A2) proves that the transfer of the whole bundle is $N + S$. \qed

**Corollary 5.2.** Let $E\text{Spin}(2) \times_{\text{Spin}(2)} S^2 \to B\text{Spin}(2)$ classify the vertical tangent bundle. Then

$$\alpha = \iota + c \in [B\text{Spin}(2), Q(B\text{Spin}(2)_+)]$$

where $\iota$ is the usual inclusion of $B\text{Spin}(2)$ and $c$ is the orientation reversal map.

**Proof of theorem 1.3.** We have $\iota_\ast a_i = a_i$ and $c_\ast a_i = (-1)^i a_i$. Therefore

$$(\iota + c)_\ast a_i = \sum_{r+s=i} (-1)^s a_r a_s$$

Reducing mod 2 we get

$$(\iota + c)_\ast (a_{2i}) = a_i^2 \quad \text{and} \quad (\iota + c)_\ast (a_{2i+1}) = 0$$

Since $B\text{Spin}(2) \to B\text{Spin}(3)$ maps $a_{2i} \mapsto b_i$, we have proved that the composition in theorem 1.3 maps $b_i$ to $a_i^2$ as claimed.

Then it will also map $Q^{2l} b_i$ to $(Q^l a_i)^2$ and hence the composition is surjective.
Both $H_\ast(Q(B\text{Spin}(2)_+))$ and $H^\ast(Q_0(B\text{Spin}(2)_+))$ are free commutative. This follows from the fact that $\lambda: H_\ast(B\text{Spin}(2)) \to H_\ast(B\text{Spin}(2))$ is surjective, similarly to the case of $Q(\mathbb{R}P^\infty_+)$. But then
\[
\xi: H_\ast(Q(B\text{Spin}(2)_+)) \to \xi H_\ast(Q(B\text{Spin}(2)_+))
\]
is an isomorphism so the same holds for $\xi H_\ast(Q(B\text{Spin}(2)_+))$. 

Proof of Corollary 1.8. By the exact sequence in Theorem 1.2 and by Theorem 1.7, the kernel of
\[
H^\ast(\Omega^\infty_0 \text{Th}(-U)) \to H^\ast(\Omega^2_0 Q(\mathbb{R}P^\infty_+))
\]
is exactly $H^\ast(Q_0(B\text{Spin}(2)_+))///\Omega^\ast$. By theorem 1.3 (ii), this injects into $H^\ast(Q(B\text{Spin}(3)_+))$. 

□
6. Adapting [MW]

This is the second part of the paper, and the aim is to adapt the proof in [MW]. As explained in the introduction we can let $\theta = \theta_{SO}$ and then for genus $\geq 2$ we have $\mathcal{M}^\theta(F, \gamma) = B\Gamma(F, \gamma)$, where $\gamma$ is an orientation of $F$ and $\Gamma(F, \gamma) = \pi_0\text{Diff}(F, \gamma)$ is the oriented mapping class group of $F$. Then we can let $F = F_{g,2}$ and let $\Gamma_{\infty,2} = \text{colim} \Gamma_{g,2}$ where the colimit is over gluing an oriented torus. Then [MW] proves that there is a homology equivalence

$$\mathbb{Z} \times B\Gamma_{\infty,2} \to \Omega^\infty \text{Th}(-U_{SO})$$

For $\theta = \theta_{\text{Spin}}$ we can again let $F = F_{g,2}$ and let

$$\mathcal{M}^\theta(F_{\infty,2}) := \text{hocolim} \mathcal{M}^\theta(F_{g,2})$$

where the hocolim is over gluing a torus. There are two essentially different ways of doing this because we can choose either an Arf invariant 0 torus or an Arf invariant 1 torus. Which one we use is not important however, because the composition of two tori will be a surface of genus 2 and with an Arf invariant 0 spin structure anyhow.

Then we adapt the proof to showing that there is a homology equivalence

$$\mathbb{Z} \times \mathcal{M}^\theta(F_{\infty,2}) \to \Omega^\infty \text{Th}(-U_{\text{Spin}(2)})$$

Since the $\mathcal{M}^\theta(-)$ satisfies Harer stability, we also get that $\mathcal{M}^\theta(F, \gamma)$ has the same homology as $\Omega^\infty \text{Th}(-U_{\text{Spin}(2)})$ in a stable range.

Most of the modifications are straightforward and the proofs are valid for any vectorbundle $\theta: U_3 \to B_3$. Only at the very end shall we specialise to the case $\theta = \theta_{\text{Spin}}$. The idea is roughly as follows. All the sheaves in [MW] are made out of either submersions $\pi: E \to X$ with oriented three-dimensional fibres, or surface bundles $q: M \to X$ with oriented two-dimensional fibres, with some extra structure. Then we can modify the definition by removing the word “oriented” and instead include a bundle map $T\pi E \to U_3$ or $Tq M \to U_2$. The original case in [MW] can the be recovered by setting $\theta = \theta_{SO}$ (the sheaves will be slightly fattened versions of those in [MW]).

This procedure works very well, and for most of the chapters we shall just give the modified definitions and claim that the proofs work in our more general situation as well. There is one point that needs attention, however. Namely the definition of $E^{\pi\xi}$ and the sheaf map $\mathcal{L}_T \to \mathcal{W}_T$ in [MW, Chapter 5]. To do this properly in the added generality we shall need to give a new definition of fibrewise surgery. Also [MW, Chapter 6] about the “connectivity problem” need some attention.

7. The sheaves

This section defines the appropriate generalisations of the sheaves on $\mathcal{X}$ defined in [MW, Section 2]. Let $\theta: U_3 \to B_3$ be a 3-dimensional real vectorbundle. Let
$q_0: T(S^1 \times [0, 1] \times \mathbb{R}) \to U_3$ be a fixed bundle map, constant in the $[0, 1] \times \mathbb{R}$-directions.

**Definition 7.1.** Let $\mathcal{V}^\theta$ be the sheaf on $\mathcal{X}$ defined such that $\mathcal{V}(X)$ is the set of $(\pi, f, q)$ such that $(\pi, f): E^{k+3} \to X^k \times \mathbb{R}$ is a proper smooth map, $\pi: E \to X$ is a graphic submersion, $f$ is fibrewise regular, and $q: T_\pi E \to U_3$ is a bundle map. We assume that near the boundary of $E$, $(\pi, f)$ agrees over $X \times \mathbb{R}$ with $S^1 \times [0, 1] \times \mathbb{R}$ and $q$ agrees with $q_0$.

Define $h\mathcal{V}^\theta$, $\mathcal{W}^\theta$, $h\mathcal{W}^\theta$, $\mathcal{W}_{\text{loc}}^\theta$ and $h\mathcal{W}_{\text{loc}}^\theta$ similarly.

**Remark 7.2.** For $\theta = \text{ESO}(3) \times \text{SO}(3)/\mathbb{R}^3 \to B\text{SO}(3)$, the map $q: T_\pi E \to U$ induces an orientation on the fibres of $\pi: E \to X$. Thus for this $\theta$ there is a sheaf map $\mathcal{V}^\theta \to \mathcal{V}$ and this is a weak equivalence. Thus $\mathcal{V}^\theta$ is a fat version of the sheaf $\mathcal{V}$ in [MW].

Following [MW] we get a diagram of classifying spaces

$$
\begin{array}{ccc}
|\mathcal{V}_c^\theta| & \longrightarrow & |\mathcal{W}^\theta| \\
\downarrow & & \downarrow \\
|h\mathcal{V}^\theta| & \longrightarrow & |h\mathcal{W}^\theta|
\end{array}
$$

where the vertical maps are induced by taking the 2-jet prolongation of $f$.

We aim at generalising [MW] to the statement $\Omega B|\mathcal{V}_c^\theta| \simeq |h\mathcal{V}^\theta|$.

**Lemma 7.3.** Let $\mathbb{T}h(-U_2)$ denote the Thom spectrum of the virtual bundle $-U_2$ over $B(2)$. Then

$$|h\mathcal{V}^\theta| \simeq \Omega^\infty \mathbb{T}h(-U_2)$$

**Lemma 7.4.** We have

$$|\mathcal{V}_c^\theta| \simeq \coprod_F \mathcal{M}^\theta(F)$$

where the disjoint union is taken over surfaces with two boundary components, one in each diffeomorphism class.

If $U \to B$ is orientable, this means that the disjoint union is over the surfaces $F_{g,2}$, $g \geq 0$.

**Proof.** This is proved similarly to the case considered in [MW]. \qed

8. Adjusting the proof

Most of the proof given in [MW] goes through with little or no change also in this more general situation. We describe the necessary changes chapter for chapter.
8.1. **Chapter 3.** [MW] determines the homotopy types of \( |h \mathcal{V}|, |h \mathcal{W}| \) and \( |h \mathcal{W}_{\text{loc}}| \) and proves that
\[
|h \mathcal{V}| \to |h \mathcal{W}| \to |h \mathcal{W}_{\text{loc}}|
\]
is a homotopy fibre sequence.

Let \( \text{Bun}(\mathbb{R}^3, U_3) \) denote the space of bundle maps from \( \mathbb{R}^3 \), considered as a bundle over a point, to the bundle \( U_3 \). As in [MW] we let \( S(\mathbb{R}^3) \) be the vector space of quadratic forms on \( \mathbb{R}^3 \) and \( \Delta \subseteq S(\mathbb{R}^3) \) be the subset of degenerate quadratic forms.

Define an \( O(3) \)-space \( A^\theta(\mathbb{R}^3) \) by
\[
A^\theta = ((\mathbb{R}^3)^* \times S(\mathbb{R}^3) - \{0\} \times \Delta) \times \text{Bun}(\mathbb{R}^3, U_3)
\]
and define
\[
GW^\theta(3, n) = O(n + 3) \times_{O(n) \times O(3)} A^\theta(\mathbb{R}^3).
\]
Thus a point in \( GW(3, n) \) is a quadruple \( (V, l, q, \xi) \) where \( V \subseteq \mathbb{R}^{3+n} \) is a three-dimensional subspace, \( l : V \to \mathbb{R} \) is a linear map, \( q : V \to \mathbb{R} \) is a quadratic map, and \( \xi : V \to U_3 \) is a bundle map, subject to the condition that \( q \) is non-degenerate if \( l = 0 \).

**Example 8.1.** For \( U_3 = EO(3) \times_{O(3)} \mathbb{R}^3 \), the space \( A^\theta(\mathbb{R}^3) \) has the same (equivariant) homotopy type as the \( A(\mathbb{R}^3) \) of [MW]. For \( U_3 = ESO(3) \times_{SO(3)} \mathbb{R}^3 \), the space \( GW^\theta(3, n) \) has the same homotopy type as \( GW(3, n) \) of [MW].

Let \( \Sigma^\theta(3, n) \subseteq GW^\theta(3, n) \) be the subspace corresponding to \( \{0\} \times (S(\mathbb{R}^3) - \Delta) \times \text{Bun}(\mathbb{R}^3, U_3) \subseteq A^\theta(\mathbb{R}^3) \), and let
\[
GV^\theta(3, n) = GW^\theta(3, n) - \Sigma^\theta(3, n)
\]

Let \( U^\theta_n \to GW^\theta(3, n) \) be the universal bundle. We get a cofibration sequence
\[
\text{Th}(U^\theta_n^{-1}|GV^\theta(3, n)) \to \text{Th}(U^\theta_n^{-1}) \to \text{Th}(U^\theta_n^{-1} \oplus U^\theta_n^{-1} | \Sigma^\theta(3, n))
\]
and an associated fibration sequence of infinite loop spaces
\[
\Omega^\infty h\mathcal{V}^\theta \to \Omega^\infty h\mathcal{W}^\theta \to \Omega^\infty h\mathcal{W}_{\text{loc}}^\theta
\]
as in [MW, Paragraph 3.1].

We have the following generalisations of [MW]:

**Theorem 8.2.**

(i) \( |h\mathcal{W}| \simeq \Omega^\infty h\mathcal{W}^\theta \)

(ii) \( |h\mathcal{V}| \simeq \Omega^\infty h\mathcal{V}^\theta \)

(iii) \( |h\mathcal{W}_{\text{loc}}| \simeq \Omega^\infty h\mathcal{W}_{\text{loc}}^\theta \)

(iv) \( |W^\theta_{\text{loc}}| \simeq \Omega^\infty hW^\theta_{\text{loc}} \)

**Proof.** Similar to [MW]. \( \square \)
8.2. Chapter 4. In 4.2, we define $\mathcal{W}_θ^{\sigma}$ and $h\mathcal{W}_θ^{\sigma}$ in the obvious way. These are sheaves of posets.

In 4.3, we define a sheaf $\mathcal{T}_θ^{\sigma}$ as in [MW, Definition 4.3.1], but with the added data of a bundle map $q: T^\sigma E \to U_3$. Notice that this is a small errata to [MW]: Their $\mathcal{T}_θ$ should consist of $(\pi, \psi): E \to X \times \mathbb{R}$ such that $\pi: E \to X$ is a submersion with oriented fibres.

With these modifications, the proof in [MW, Section 4.3] goes through without further difficulties. Thus we get

**Theorem 8.3.** $|\mathcal{W}_θ| \simeq |h\mathcal{W}_θ|$

8.3. Chapter 5: Surgery. [MW, Chapter 5.2] is about fibrewise surgery. The idea is roughly as follows. Given a bundle $q: M \to X$ of manifolds, a finite set $T$, a Riemannian vectorbundle $\omega: V \to X \times \mathbb{R}$ with isometric involution $\rho: V \to V$, and an embedding $e: D(V^\rho) \times_{T \times X} S(V^\rho) \to M - \partial M$, then one performs surgery by removing the interior of the embedded $D(V^\rho) \times_{T \times X} S(V^\rho)$ and replacing it with $S(V^\rho) \times_{T \times X} D(V^\rho)$.

In our generalised setting, $M$ will be equipped with a bundle map $\xi: T^\theta M \to U_2$. We would like to perform surgery in a way that we end up with a bundle $\bar{q}: \bar{M} \to X$, equipped with a bundle map $\bar{\xi}: T^\theta \bar{M} \to U_2$. We describe how to do this.

8.3.1. Saddles. Choose once and for all a smooth function $\tau: [0, 1] \to [0, 1]$ which is 0 near 0 and 1 near 1. Let $Y$ be a manifold and $\omega: V \to Y$ a Riemannian vectorbundle with isometric involution $\rho: V \to V$. Let $g: Y \to \mathbb{R}$ be smooth. As in [MW] we define the saddle of $V$ to be the subset

$\text{Sad}(V) = \{v \in V \mid |v_+||v_-| \leq 1\}$

Define three smooth functions by

$$f_0(v) = g\omega(v) + |v_+|^2 - |v_-|^2$$
$$f_+(v) = g\omega(v) + \frac{1}{|v_-|^2} \left(|v_+|^2|v_-|^2\tau(|v_+||v_-|) + (1 - \tau(|v_+||v_-|))\right) - |v_-|^2$$
$$f_-(v) = g\omega(v) + |v_+|^2 - \frac{1}{|v_+|^2} \left(|v_+|^2|v_-|^2\tau(|v_+||v_-|) + (1 - \tau(|v_+||v_-|))\right)$$

The map $f_0$ is defined on all of $\text{Sad}(V)$ and is fibrewise regular except at the zero section of $V$, where it has a Morse singularity with critical value given by $g\omega$. The maps $f_\pm$ is defined on $\text{Sad}(V) - V^\pm \rho$, is fibrewise regular and proper, and agrees with $f_0$ near $\partial \text{Sad}(V)$. The following picture shows the level curves of $f_0$ in $V$. $\text{Sad}(V) \subseteq V$ is the shaded area.
This should be compared with the level curves of $f_+$ and $f_-$, shown in the following pictures.

Moreover, $f_+$ defines a diffeomorphism

\[ \text{Sad}(V) - V^\rho \to D(V^\rho) \times_Y S(V^{-\rho}) \times \mathbb{R} \]
\[ v \mapsto (|v_-|v_+, |v_-|^{-1}v_-, f_+(v)) \] \hfill (8.1)

Similarly, $f_-$ defines a diffeomorphism

\[ \text{Sad}(V) - V^{-\rho} \to S(V^\rho) \times_Y D(V^{-\rho}) \times \mathbb{R} \]
\[ v \mapsto (|v_+|v_+, |v_+|^{-1}v_-, f_- (v)) \] \hfill (8.2)

Remark 8.4. Comparing (8.1) to equation [MW, equation (5.3)] we see that, up to diffeomorphism, the process of removing $V^\rho$ and replacing $f$ with $f_+$ is equivalent to gluing $D(V^\rho) \times_Y S(V^{-\rho}) \times \mathbb{R}$ to $\text{Sad}(V) - V^\rho$ along [MW, equation (5.3)]. Similarly for (8.2) and [MW, equation (5.4)].
Definition 8.5. Given a vectorbundle $\omega: V \to Y$ and a smooth $g: Y \to \mathbb{R}$ as above, we let

$$M_+(V, g) = f_+^{-1}(0) \subseteq \text{Sad}(V)$$
$$M_-(V, g) = f_-^{-1}(0) \subseteq \text{Sad}(V)$$

By our earlier remarks we see that both $M_+(V, g)$ and $M_-(V, g)$ agrees near $\partial \text{Sad}(V)$ with $f_0^{-1}(0)$. By restriction of (8.1) we get a diffeomorphism over $Y$

$$M_+(V, g) \to D(V^\rho) \times_Y S(V^{-\rho})$$

and the fibrewise differential induces an isomorphism

$$T^\omega M_+(V, g) \times \mathbb{R} \to T^\omega V_{M_+(V, g)}$$

Similarly for $M_-(V, g)$.

This gives an alternative description of surgery. Namely, given a surface bundle $q: M \to X$, a finite set $T$, a Riemannian vectorbundle $V \to T \times X$ with isometric involution $\rho: V \to V$, a smooth function $g: T \times X \to \mathbb{R}$, and an embedding over $X \lambda: M_+(V, g) \to M - \partial M$, then one performs surgery by replacing the embedded $M_+(V, g)$ by $M_-(V, g)$. Since $M_+(V, g)$ and $M_-(V, g)$ agree near their boundary, this gives a welldefined smooth bundle $\tilde{q}: \tilde{M} \to X$. Moreover the following is true. If $M$ is equipped with a bundle map $\xi: T^q M \to U_2$ and $V$ is equipped with $\xi: T^\omega V_{\text{Sad}(V)} \to U_3$, and the fibrewise differential of $\lambda$ is over $U_2$, then $\tilde{M}$ gets a canonical map $T^q \tilde{M} \to U_2$.

8.3.2. The sheaves. Keeping these remarks in mind, we make the following definitions. $\mathcal{W}_{\text{loc}, T}(X)$ is the set of

(i) $\omega: V \to T \times X$ a Riemannian vector bundle with isometric involution $\rho$, as in [MW].
(ii) $g: T \times X \to \mathbb{R}$ a smooth function.
(iii) $\xi: T^\omega V_{\text{Sad}(V)} \to U_3$ a vectorbundle map

and $\mathcal{W}_{T}(X)$ is the set of

(1) $(V, g, \xi) \in \mathcal{W}_{\text{loc}, T}(X)$
(2) $q: M \to X$ a bundle of surfaces
(3) $\xi: T^q M \to U_2$ a bundle map
(4) $e: M_+(V, g) \to M - \partial M$ an embedding over $X$ such that the fibrewise differential $De$ is over $U_2$.

8.3.3. The proofs. We go through the definitions and proofs in [MW, Chapter 5] and describe what modifications are needed in this more general situation. Again
this is summarised in the diagram

\[
\begin{array}{ccc}
\mathcal{W}^\theta & \longrightarrow & \mathcal{W}^\theta_{\text{loc}} \\
\uparrow & & \uparrow \\
\mathcal{L}^\theta & \longrightarrow & \mathcal{L}^\theta_{\text{loc}} \\
\uparrow & & \uparrow \\
hocolim \mathcal{L}^\theta_T & \longrightarrow & hocolim \mathcal{L}^\theta_{\text{loc},T} \\
\downarrow & & \downarrow \\
hocolim \mathcal{W}^\theta_T & \longrightarrow & hocolim \mathcal{W}^\theta_{\text{loc},T}
\end{array}
\]

(8.3)

8.3.4. Second row. Define \( \mathcal{L}^\theta_{\text{loc}}(X) \) to be the set of

(i) \((p, g): Y \rightarrow X \times \mathbb{R} \) proper smooth maps such that \( p \) is etale and graphic
and such that \( g \) is smooth.

(ii) \( \omega: V \rightarrow Y \) is a Riemannian vectorbundle with isometric involution \( \rho: V \rightarrow V \).

(iii) \( \xi: T^\omega V_{\text{Sad}(V)} \rightarrow U_3 \) a bundle map

and let \( \mathcal{L}^\theta(X) \) be the set of

(i) \((\pi, f, \xi) \in \mathcal{W}^\theta(X) \) with \((\pi, f): E \rightarrow X \times \mathbb{R}, \xi: T^\pi E \rightarrow U_3 \).

(ii) \((p, g, V, \xi) \in \mathcal{L}^\theta_{\text{loc}}(X) \)

(iii) \( \lambda: \text{Sad}(V) \rightarrow E - \partial E \) an embedding over \( X \times \mathbb{R} \) such that the fibrewise differential \( D\lambda \) is over \( U_3 \).

The proofs given in [MW] of [MW, Proposition 5.3.3] and [MW, Proposition 5.3.7] goes through with the obvious changes and proves that the sheaf maps \( \mathcal{L}^\theta_{\text{loc}} \rightarrow \mathcal{W}^\theta_{\text{loc}} \) and \( \mathcal{L}^\theta \rightarrow \mathcal{W}^\theta \) are weak equivalences.

8.3.5. Third row. Let \( \mathcal{L}^\theta_{\text{loc},T}(X) \) be the set of

(i) \((p, g, V, \xi) \in \mathcal{L}^\theta_{\text{loc}}(X) \)

(ii) \( h: S \times X \rightarrow Y \) an embedding over \( 3 \times X \)

(iii) \( \delta: Y - \text{Im}(h) \rightarrow \{\pm 1\} \) continuous

and let \( \mathcal{L}^\theta_T(X) \) be the set of

(i) \((p, g, V, \xi, h, \delta) \in \mathcal{L}^\theta_{\text{loc},T}(X) \)

(ii) \((\pi, f, \xi) \in \mathcal{W}^\theta(X) \)

(iii) \( \lambda: \text{Sad}(V) \rightarrow E - \partial E \) embedding over \( X \times \mathbb{R} \) such that the fibrewise differential \( D\lambda \) is over \( U_3 \).

The proofs given in [MW] of [MW, Proposition 5.4.2] and [MW, Proposition 5.4.4] goes through with the obvious changes and proves that the sheaf maps \( \hocolim \mathcal{L}^\theta_{\text{loc},T} \rightarrow \mathcal{L}^\theta_{\text{loc}} \) and \( \hocolim \mathcal{L}^\theta_T \rightarrow \mathcal{L}^\theta \) are weak equivalences.
8.3.6. Fourth row, right hand column. [MW, Lemma 5.5.2] and [MW, Corollary 5.5.3] goes through as in [MW].

8.3.7. Fourth row left hand column. This is more technical, and more changes are needed to adapt the proof in [MW]. The modified definitions of $\mathcal{W}_T^\theta$ and $\mathcal{W}_{\text{loc},T}^\theta$ were made with this in mind. The problem is to give a definition of $E^{\text{rg}}$ and to define a sheaf map $\mathcal{L}_T^\theta \to \mathcal{W}_T^\theta$, natural in $T \in \mathcal{K}$. 

Take an element of $\mathcal{L}_T^\theta(X)$. This consists of $(p,g,V,ξ,h,δ) \in \mathcal{L}_\text{loc},T(X)$, $(π,f,ξ) \in \mathcal{W}_T^\theta(X)$, and $λ: \text{Sad}(V) → E − \partial E$. Define $Y_0,Y_+,Y_- \subseteq Y$ and $V_+,V_-,V_0 \subseteq V$ as in [MW]. Define $E^{\text{rg}}, f^{\text{rg}}$ in the following way

- On the embedded $\text{Sad}(V_+)$, remove $V_+^ρ$ and replace $f$ by $f_+$.
- On the embedded $\text{Sad}(V_-)$, remove $V_-^ρ$ and replace $f$ by $f_-$.
- On the embedded $\text{Sad}(V_0)$, remove $V_0^ρ$ and replace $f$ by $f_+$.

This defines a bundle $(π^{\text{rg}}, f^{\text{rg}}): E^{\text{rg}} → X × R$ of smooth compact surfaces. Now let $M = (f^{\text{rg}})^{-1}(0)$. This is a bundle of smooth compact surfaces over $X$, and is equipped with the following extra structure

(i) A bundle map $ξ: T^½ M → U_2$
(ii) A Riemannian vectorbundle $ω: h^* V_0 → T × X$ with isometric involution $ρ$.
(iii) A bundle map $ξ: T^½ (h^* V_0)|_{\text{Sad}(V)} → U_3$.
(iv) A smooth function $g: T × X → Y → \mathbb{R}$.
(v) An embedding (over $X$) $e: M_+(V,g) → M$ such that the fibrewise differential is over $U_2$.

That is, we have an element of $\mathcal{W}_T^\theta(X)$. This defines a sheaf map $\mathcal{L}_T^\theta → \mathcal{W}_T^\theta$ which is natural in $T \in \mathcal{K}$. Just as in [MW] one proves that $\mathcal{L}_T^\theta → \mathcal{W}_T^\theta$ is an equivalence.

8.3.8. Using the concordance lifting property. To prove that the sheaf maps $\mathcal{W}_T^\theta → \mathcal{W}_{\text{loc},T}^\theta$ has the concordance lifting property we need the following lemma

**Lemma 8.6.** Let $A \subseteq X$ be a cofibration and let $V → [0,1] × X$ be a vectorbundle. Let $U → B$ be another vectorbundle. Then any bundle map $ξ: V_{|[0],[0,1]∩X} → U$ extends to a bundle map $V → U$.

**Proof.** Choose a retraction $r: [0,1] × X → \{0\} × X ∪ [0,1] × A$. Now the fibre bundle $\text{Iso}(V,r^* V) → [0,1] × A$ has a canonical section over $\{0\} × X ∪ [0,1] × A$. This section extends over all of $[0,1] × X$ because $\{0\} × X ∪ [0,1] × A → [0,1] × X$ is a trivial cofibration. This section defines a bundle map

\[
\begin{array}{ccc}
V & \xrightarrow{r} & V_{|[0],[0,1]∩X} \\
\downarrow & & \downarrow \\
[0,1] × X & \xrightarrow{r} & \{0\} × X ∪ [0,1] × A
\end{array}
\]

and we can compose $ξ$ with $r$. □
Proposition 8.7. The map $\mathcal{W}_T^\theta \to \mathcal{W}_{\text{loc},T}^\theta$ has the concordance lifting property.

Proof. Let $\chi \in \mathcal{W}_T^\theta$ be an element given by

- $(V, \xi) \in \mathcal{W}_{\text{loc},T}^\theta(X)$ with $\omega: V \to T \times X$ and $\xi: T^\omega V|_{\text{Sad}(V)} \to U_3$
- $q: M \to X$ a surface bundle (with certain boundary conditions).
- $\tilde{q}: T^q M \to U_2$ a bundle map.
- $e: M_+(V, g) \to M$ an embedding over $X$ such that the fibrewise differential $De$ is over $U_2$.

Suppose given a concordance of $(V, \xi)$. This will be given by a vectorbundle $\tilde{\omega}: \tilde{V} \to (0, 1) \times T \times X$ and $\tilde{\xi}: T^\omega \tilde{V}|_{\text{Sad}(\tilde{V})} \to U$. We can choose an isomorphism $\tilde{V} \cong (0, 1) \times V$ over $(0, 1) \times T \times X$. Put $\tilde{M} = (0, 1) \times M$ and $\tilde{g} = (0, 1) \times g$. Let $\tilde{\tilde{g}} = g \circ \text{pr}_{T \times X}: (0, 1) \times T \times X \to \mathbb{R}$. Then we have the isomorphism $M_+(\tilde{V}, \tilde{g}) \cong (0, 1) \times M_+(V, g)$ and we can set $\tilde{e} = (0, 1) \times e: M_+(\tilde{V}, \tilde{g}) \cong (0, 1) \times M_+(V, g) \to (0, 1) \times M$.

It remains to define a bundle map $T^q \tilde{\tilde{M}} \to U_2$ which is specified on $T^q \tilde{\tilde{M}}|_{\{0\} \times M \cup \{0, 1\} \times M_+(V, g)}$. This can be done by the previous lemma, using that $M_+(V, g) \to M$ is a cofibration. $\square$

8.4. Chapter 6: The connectivity problem. We describe how to adapt the definition of the sheaf $\mathcal{C}_M$ and prove that it is contractible. Let $\mathbb{R}^2 \times \mathbb{R}$ have the standard euclidean metric and involution $\rho = \text{diag}(1, 1, -1)$. For any finite set $T$ and a manifold $X$ we have the trivial vectorbundle $V = \mathbb{R}^2 \times \mathbb{R} \times T \times X$ over $T \times X$ and we have canonical identifications

- $\text{Sad}(V) = \text{Sad}(\mathbb{R} \times \mathbb{R}) \times T \times X$.
- $T^\omega V|_{\text{Sad}(V)} = \mathbb{R} \times T \times \text{Sad}(V)$
- $D^2 \times S^0 \times T \times X \cong M_+(0) \subseteq \text{Sad}(V)$

Thus to promote $V$ to an element of $\mathcal{W}_{\text{loc},T}^\theta$ with $T \to \{1\}$ we must specify a bundle map $T^\omega V|_{\text{Sad}(V)} \to U_3$, or equivalently a map $\text{Sad}(V) \to \text{Bun}(\mathbb{R}^2 \times \mathbb{R}, U_3)$.

Definition 8.8. Let $M$ be a surface and $TM \to U_2$ a bundle map. Let $\mathcal{C}_{\theta,M}$ be the sheaf whose value at a connected manifold $X$ is the set of

- A finite set $T$
- A map $\text{Sad}(V) \to \text{Bun}(\mathbb{R}^2 \times \mathbb{R}, U_3)$, where $V = \mathbb{R}^2 \times T \times X$ as above
- An embedding $e_T: M_+(0) \to (M - \partial M) \times X$ over $X$ such that the fibrewise differential $De_T$ is over $U_2$ and such that surgery along $e_T$ results in a connected surface bundle over $X$.

We want to prove that $B|\mathcal{C}_{\theta,M}| \simeq |\beta \mathcal{C}_{\theta,M}^{\text{op}}|$ is contractible. We proceed as in [MW]: Given a closed set $A \subseteq X$ and a germ $s_0 \in \text{colim}_U \beta \mathcal{C}_{\theta,M}^{\text{op}}(U)$ we extend this germ to an element of $\beta \mathcal{C}_{\theta,M}^{\text{op}}(X)$. The germ $s_0$ consists of a locally finite open cover $(U_j)_{j \in J}$ of $U$ and objects $\varphi_{RR} \in \mathcal{C}_{\theta,M}(U_R)$ for each finite non-empty $R \subseteq J$, and for each $R \subseteq S$ a morphism $\varphi_{RS}: \varphi_{SS} \to \varphi_{RR}|_{U_S}$ satisfying the
cocycle condition. Each of the \( \varphi_{RR} \) defines an embedding
\[
D^2 \times S^0 \times T_R \times U_R \cong M_+(0) \to (M - \partial M) \times U_R
\]
(really there should be one finite set \( T_R \) for each component of \( U_R \), but we will suppress this from the notation).

[MW] shows how to extend this to an element of their \( \beta C_M^{\text{op}}(X) \) by choosing contractible open sets \( V_j \subseteq X \) and embeddings
\[
D^2 \times S^0 \times Q_j \times V_j \to (M - \partial M) \times V_j
\]
and by taking coproducts they get an element of their \( \beta C_M^{\text{op}}(X) \) which restricts to the given germ. To finish the proof that our \( \beta C_{\theta,M}^{\text{op}}(X) \) is contractible we have to promote (*) to an object of our \( \beta C_{\theta,M}^{\text{op}}(V_j) \). This can be done by the next lemma.

**Lemma 8.9.** Let \( M \) be a surface and \( TM \to U_2 \) a bundle map. Let \( X \) be contractible and let \( V = \mathbb{R}^2 \times \mathbb{R} \times T \times X \) be the trivial vector bundle over \( T \times X \). Then for any embedding
\[
e: D^2 \times S^0 \times T \times X \to (M - \partial M) \times X
\]
over \( X \) there exists a bundle map \( T^a V_{\text{Sad}(V)} \to U_3 \) and a diffeomorphism \( h: M_+(0) \to D^2 \times S^0 \times T \times X \) such that the fibrewise differential of \( e \circ h \) is over \( U_2 \).

**Proof.** First let \( h \) be the inverse of the standard diffeomorphism given by (8.1). The requirement that \( D(e \circ h) \) is over \( U_2 \) defines a unique bundle map \( T^a V_{M_+(0)} \to U_3 \), or equivalently a map \( M_+(0) \to \text{Bun}(\mathbb{R}^2 \times \mathbb{R}, U_3) \). After possibly composing \( h \) with an orientation preserving diffeomorphism of \( D^2 \) we can extend this to \( M_+(0) \cup (\{0\} \times D^1 \times T \times X) \). Now the inclusion
\[
M_+(0) \cup (\{0\} \times D^1 \times T \times X) \to \text{Sad}(V)
\]
is a trivial cofibration so we can extend to all of \( \text{Sad}(V) \). \( \square \)

8.5. **Chapter 7: Stabilisation.** This is almost as in [MW]. Start by choosing an element \( z \in W_0(*) \) of genus 2. This is a torus with two boundary components and with a spin structure. As already explained in paragraph 6, there are two essentially different choices of such tori, but which one we pick is not important for stabilisation.

As in [MW] we get a fibration sequence
\[
|z^{-1}h\mathcal{V}| \to \text{hocolim} \ z^{-1}|\mathcal{W}_T| \to \text{hocolim} \ z^{-1}|\mathcal{W}_{\text{loc},T}|
\]
where \( |z^{-1}h\mathcal{V}| \cong \Omega^\infty \text{Th}(-U) \) and where
\[
\text{hofib}(z^{-1}|\mathcal{W}_T| \to z^{-1}|\mathcal{W}_{\text{loc},T}|) \cong \mathbb{Z} \times \mathcal{M}^\theta(F_{\infty,2+2|T|}).
\]
When the spaces \( \mathcal{M}^\theta(F_{\infty,2+2|T|}) \) satisfies Harer stability, i.e. if any morphism \( S \to T \) in \( \mathcal{K} \) induces homology equivalences \( \mathcal{M}^\theta(F_{\infty,2+2|T|}) \to \mathcal{M}^\theta(F_{\infty,2+2|S|}) \), then the proof in [MW] goes through and proves that
\[
\mathbb{Z} \times \mathcal{M}^\infty(F_{\infty,2}) \to \Omega^\infty \text{Th}(-U)
\]
is a homology equivalence.

And we know from [H] and [B] that for $\theta = \theta_{\text{Spin}}$ this Harer stability indeed does hold.

References

[B] T. Bauer: An infinite loop space structure on the nerve of spin bordism categories, Quart. J. Math. 55 (2004), 117–133.

[CLM] F. R. Cohen, T. J. Lada, J. P. May: The Homology of Iterated Loop Spaces, Lecture Notes in Mathematics 533, Springer-Verlag, 1976.

[EM] S. Eilenberg, J. C. Moore: Homology and Fibrations I Coalgebras, cotensor products and its derived functors, Comm. Math. Helv. 40 (1965), 199–236.

[G] S. Galatius: Mod $p$ homology of the stable mapping class group, to appear in Topology.

[GMT] S. Galatius, I. Madsen, U. Tillmann: Divisibility of the stable Miller-Morita-Mumford classes, preprint.

[H] J. Harer: Stability of the homology of moduli spaces of Riemann surfaces with spin structure, Math. Ann. 287 (1990), 323–334.

[J] Johnson: Spin structures and quadratic forms on surfaces, J. London Math. Soc. (2) 22 (1980), 365–373.

[MM] J. Milnor, J. C. Moore: On the Structure of Hopf algebras, Ann. Math. 81 (1965), 211–264.

[MMM] B. M. Mann, E. Y. Miller, H. R. Miller: $S^1$-equivariant function spaces, Trans. Amer. Math. Soc. 295 (1989), 233–256.

[MW] I. Madsen, M. Weiss: The stable moduli space of Riemann surfaces: Mumford’s conjecture, arXiv:math.AT/0212321, Version 2, October 2003.

Stanford University, Stanford, USA
E-mail address: galatius@imf.au.dk