On relations for the $q$-multiple zeta values

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Abstract

We prove some relations for the $q$-multiple zeta values ($q$MZV). They are $q$-analogues of the cyclic sum formula, the Ohno relation and the Ohno-Zagier relation for the multiple zeta values (MZV). We discuss the problem to determine the dimension of the space spanned by $q$MZV’s over $\mathbb{Q}$, and present an application to MZV.

1 Introduction

In this paper we prove some relations for a certain class of $q$-series called $q$-multiple zeta values ($q$MZV, for short).

Let us recall the definition of $q$MZV [10]. We call a sequence of ordered positive integers $k = (k_1, \ldots, k_r)$ an index. The weight, depth and height of the index are defined by $|k| := k_1 + \cdots + k_r$, $\text{dep } k := r$ and $\text{ht } k := \# \{j | k_j \geq 2\}$ respectively. The index is said to be admissible if and only if $k_1 \geq 2$.

**Definition 1.** For $0 < q < 1$ and an admissible index $k$, the $q$-multiple zeta value ($q$MZV) is defined by

$$\zeta_q(k) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{q^{n_1(k_1-1)+n_2(k_2-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1}[n_2]^{k_2}\cdots[n_r]^{k_r}},$$

where $[n]$ is the $q$-integer

$$[n] := \frac{1-q^n}{1-q}.$$

Note that $0 < 1/[n] \leq 1$ for any positive integer $n$. Hence the right hand side of (1) is absolutely convergent if $k_1 > 1$. In particular it is well defined as a $q$-series if $k_1 > 1$.

By taking the limit $q \to 1$ of $q$MZV, we obtain the multiple zeta value (MZV, for short):

$$\lim_{q\uparrow 1} \zeta_q(k) = \zeta(k) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{1}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}}.$$

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For MZV’s there are many linear relations and algebraic ones over \( \mathbb{Q} \). The examples are the cyclic sum formula [3], the Ohno relation [4] and the Ohno-Zagier relation [5]. These relations do not suffice to give all relations of MZV’s, and as a time the proof is much technical. However these relations have quite beautiful structures and it is valuable to construct explicit relations.

In the present paper, we give a \( q \)-anologue of the relations for MZV’s and prove it.

First we show the \( \mathbb{Q} \)-linear relations for \( q \)MZV’s which are \( q \)-analogues of the cyclic sum formula and the Ohno relation:

**Theorem 1 (The cyclic sum formula).** For any index \( k \) with some \( k_i \geq 2 \),

\[
\sum_{i=1}^{r} \zeta_q(k_i + 1, k_i+1, \ldots, k_r, k_1, \ldots, k_i-1) = \sum_{i=1}^{r} \sum_{j=0}^{k_i-2} \zeta_q(k_i - j, k_i+1, \ldots, k_r, k_1, \ldots, k_i-1, j + 1).
\]

**Theorem 2 (The Ohno relation).** For any admissible index \( k = (k_1, \ldots, k_r) \), there exist positive integers \( a_1, b_1, a_2, b_2, \ldots, a_s, b_s \) such that

\[
k = (a_1 + 1, 1, \ldots, 1, a_2 + 1, 1, \ldots, 1, \ldots, a_s + 1, 1, \ldots, 1).
\]

Then the dual index \( k' = (k'_1, \ldots, k'_r) \) for \( k \) is defined by

\[
k' := (b_s + 1, 1, \ldots, 1, b_2 + 1, 1, \ldots, 1, b_1 + 1, 1, \ldots, 1).
\]

For any admissible index \( k \) and non-negative integer \( l \)

\[
\sum_{c_1 + \cdots + c_r = l} \zeta_q(k_1 + c_1, \ldots, k_r + c_r) = \sum_{c'_1 + \cdots + c'_r = l} \zeta_q(k'_1 + c'_1, \ldots, k'_r + c'_r).
\]

These relations have the same form as the corresponding ones for MZV’s.

Secondly we show a \( q \)-anologue of the Ohno-Zagier relation:

**Theorem 3 (The Ohno-Zagier relation).** We define a generating function of \( q \)MZV’s as follows:

\[
\Phi_0(x, y, z) := \sum_{k, r, s=0}^{\infty} \left\{ \sum_{|k|=s, \text{dep } k=r \text{, let } k=s} \zeta_q(k) \right\} x^{k-r-s} y^{r-s} z^{s-1}.
\]

Then

\[
1 + (z - xy)\Phi_0 = \exp \left( \sum_{n=2}^{\infty} \zeta_q(n) \sum_{m=0}^{\infty} \frac{(q - 1)^m}{m+n} (x^{m+n} + y^{m+n} - (\alpha^{m+n} + \beta^{m+n})) \right).
\]
Here $\alpha^{m+n} + \beta^{m+n}$ is a polynomial in $x, y$ and $z$ determined by

$$\alpha + \beta = x + y + (q - 1)(z - xy) \quad \text{and} \quad \alpha \beta = z.$$  

By taking the limit $q \to 1$ we obtain the Ohno-Zagier relation for MZV’s. The Ohno-Zagier relation for MZV’s explains when the combinations of MZV’s are in the algebra generated by Riemann’s zeta values $\zeta(k)$ over $\mathbb{Q}$. Unfortunately the product of qMZV’s is not closed in $\mathbb{Q}$-vector space spanned by qMZV’s, however it is closed in $\mathbb{Q}[1 - q]$-module and preserves the weights by counting the weight of $(1 - q)$ by 1 [10]. From the consideration above we define the modified qMZV $\zeta_q(k)$ by

$$\zeta_q(k) := (1 - q)^{-|k|} \zeta_q(k).$$ (5)

Then the product of modified qMZV’s is closed in the $\mathbb{Q}$-vector space spanned by them. Now the relation (4) is rewritten as follows:

$$1 + (z - xy) \Phi_0 = \exp \left( \sum_{n=2}^{\infty} \zeta_q(n) \sum_{m=0}^{\infty} (-1)^m \frac{1}{m+n} (x^{m+n} + y^{m+n} - (\alpha^{m+n} + \beta^{m+n})) \right).$$

Here $\Phi_0$ is defined by (3) with $\zeta_q(k)$ replaced by $\zeta_q(k)$, and

$$\bar{\alpha} + \bar{\beta} = x + y - (z - xy) \quad \text{and} \quad \bar{\alpha} \bar{\beta} = z.$$ 

Thus the relation (4) explains when the linear combinations of the modified qMZV’s are in the algebra generated by $\zeta_q(n)$ ($n \ge 2$) over $\mathbb{Q}$. 

In preparation of this paper we found [1]. In [1], Theorem [1], Theorem [2] and Theorem [3] of $z = 0$ case are proved independently.

The rest of the paper is organized as follows. In section 2 we prove the theorems above. Since the proofs are similar to the case of MZV, we omit some details presenting new features in the case of qMZV. In section 3 we discuss the problem to determine dimensions of certain spaces spanned by modified qMZV’s over $\mathbb{Q}$. In the study of MZV it is an important problem to determine the dimension of the $\mathbb{Q}$-vector space spanned by MZV’s of a fixed weight (see, e.g. [9]). We consider a similar problem in the case of qMZV and present some observations.

2 Proofs

2.1 The Cyclic Sum Formula

Set

$$T(k_1, \ldots, k_r) := \sum_{n_1 > \cdots > n_r > n_{r+1} \ge 0} \frac{q^{n_1 - n_{r+1}} \cdot q^{n_1(k_1 - 1) + \cdots + n_r(k_r - 1)}}{[n_1 - n_{r+1}] [n_1]^{k_1} \cdots [n_r]^{k_r}}$$

$$S(k_1, \ldots, k_r, k_{r+1}) := \sum_{n_1 > \cdots > n_r > n_{r+1} > 0} \frac{q^{n_1} \cdot q^{n_1(k_1 - 1) + \cdots + n_r(k_r - 1) + n_{r+1}(k_{r+1} - 1)}}{[n_1 - n_{r+1}] [n_1]^{k_1} \cdots [n_r]^{k_r} [n_{r+1}]^{k_{r+1}}}$$
It is easy to see that $T(k_1, \ldots, k_r)$ converges absolutely if all $k_i$’s are positive integers, and $S(k_1, \ldots, k_{r+1})$ converges absolutely if $\forall k_i \geq 1$, or $k_{r+1} = 0$ and some of $k_i \geq 2$ ($i = 1, \ldots, r$).

**Lemma 1.** Let $(k_1, \ldots, k_r)$ be an index with some $k_i \geq 2$. Then we have

$$T(k_1, k_2, \ldots, k_r) - \zeta_q(k_1 + 1, k_2, \ldots, k_r) = T(k_2, \ldots, k_r, k_1) - \sum_{j=0}^{k_i-2} \zeta_q(k_1 - j, k_2, \ldots, k_r, j + 1).$$  \hfill (6)

We get Theorem 1 by summing Lemma 1 over all cyclic permutations of the sequence $(k_1, \ldots, k_r)$.

**Proof.** The left hand side of (6) is

$$T(k_1, \ldots, k_r) - \zeta_q(k_1 + 1, k_2, \ldots, k_r) = S(k_1, \ldots, k_r, 0).$$

Let us prove that $S(k_1, \ldots, k_r, 0)$ is equal to the right hand side of (6).

We use the following partial fractional expansions:

$$\frac{q^{n_1-n_{r+1}}}{[n_1 - n_{r+1}] [n_1]} = \begin{cases} \frac{1}{[n_1 - n_{r+1}]} - \frac{1}{[n_1]} & (7a) \\ \frac{q^{n_1}}{[n_1 - n_{r+1}]} - \frac{q^n}{[n_1]} & (7b) \end{cases}$$

If $k_1 \geq 2$, by using (6a) we have

$$S(k_1, k_2, \ldots, k_r, k_{r+1}) = \sum_{n_1 > \ldots > n_r > n_{r+1} > 0} q^{n_1} \frac{q^{n_1(k_1-1)+\ldots+n_r(k_r-1)+n_{r+1}(k_{r+1}-1)}}{[n_1 - n_{r+1}] [n_1]^{k_1-1} [n_2]^{k_2} \ldots [n_r]^{k_r} [n_{r+1}]^{k_{r+1}}} \frac{1}{[n_1]}$$

$$= \sum_{n_1 > \ldots > n_r > n_{r+1} > 0} \left\{ \frac{1}{[n_1 - n_{r+1}]} - \frac{1}{[n_1]} \right\} \frac{q^{n_1(k_1-1)+\ldots+n_r(k_r-1)+n_{r+1}(k_{r+1}+1)}}{[n_1]^{k_1-1} [n_2]^{k_2} \ldots [n_r]^{k_r} [n_{r+1}]^{k_{r+1}+1}}$$

$$= S(k_1 - 1, k_2, \ldots, k_r, k_{r+1} + 1) - \zeta_q(k_1, k_2, \ldots, k_r, k_{r+1} + 1).$$

By using this equality repeatedly we find

$$S(k_1, k_2, \ldots, k_r, 0) = S(1, k_2, \ldots, k_r, 1 - 1) - \sum_{j=0}^{k_1-2} \zeta_q(k_1 - j, k_2, \ldots, k_r, j + 1).$$  \hfill (8)

The equality above holds also in the case of $k_1 = 1$. 

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Now we consider the first term in the right hand side of (8):

\[
S(1, k_2, \ldots, k_r, k_{r+1})
= \sum_{n_1 > \cdots > n_r > n_{r+1} > 0} \frac{q^{n_1}}{n_1} \frac{q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+1}(k_{r+1}-1)}}{n_2[k_2] \cdots [n_r][k_{r+1}][k_{r+1}+1]}
\]

\[
= \sum_{n_2 > \cdots > n_r > n_{r+1} > 0} \frac{q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+1}k_{r+1}}}{n_2[k_2] \cdots [n_r][k_{r+1}]^{k_{r+1}+1}} \sum_{n_1 = n_2+1}^{\infty} \left\{ \frac{q^{n_1-n_{r+1}}}{n_1-n_{r+1}} - \frac{q^{n_1}}{n_1} \right\}.
\]

Here we used (7b). Note that the sum \( \sum_{n=1}^{\infty} q^n/[n] \) is convergent. Hence we have

\[
\sum_{n_1 = n_2+1}^{\infty} \left\{ \frac{q^{n_1-n_{r+1}}}{n_1-n_{r+1}} - \frac{q^{n_1}}{n_1} \right\} = \sum_{n_{r+1} > j > 0} \frac{q^{n_2-j}}{n_2-j}.
\]

Therefore we find

\[
S(1, k_2, \ldots, k_{r+1}) = T(k_2, \ldots, k_r, k_{r+1}+1).
\]

(9)

From (8) and (9) we obtain (6). \( \square \)

### 2.2 The Ohno Relation

The proof progresses as same as [7].

For an admissible index \( \mathbf{k} = (k_1, \ldots, k_r) \), a sequence \((a_1, b_1, \ldots, a_s, b_s)\) of positive integers is determined by the rule (2). We call the sequence a code of \( \mathbf{k} \).

Let \((a_1, b_1, \ldots, a_s, b_s)\) be the code of an admissible index \( \mathbf{k} \). We define generating functions of the left hand side and the right hand side of the Ohno relation as follows:

\[
f(a_1, b_1, \ldots, a_s, b_s; \lambda) := \sum_{l=0}^{\infty} \sum_{c_1+\cdots+c_r = l} \zeta_q(k_1 + c_1, \ldots, k_r + c_r) \lambda^l,
\]

\[
g(a_1, b_1, \ldots, a_s, b_s; \lambda) := \sum_{l=0}^{\infty} \sum_{c_1'+\cdots+c_r' = l} \zeta_q(k_1' + c_1', \ldots, k_r' + c_r') \lambda^l
\]

\[
= f(b_s, a_s, \ldots, b_1, a_1; \lambda).
\]

Here \( \mathbf{k}' = (k_1', \ldots, k_r') \) is the dual index of \( \mathbf{k} \). These functions converge at \(|\lambda| < 1\), and can be analytically continued to \( \mathbb{C} \setminus \Omega \), where

\[
\Omega := \{ q^{-n}[n] | n \in \mathbb{Z}_{\geq 1} \},
\]
by

\[ f(a_1, b_1, \ldots, a_s, b_s; \lambda) = \sum_{c_1, \ldots, c_r=0}^{\infty} \sum_{n_1 > \cdots > n_r > 0} q^{n_1(k_1-1)+\cdots+n_r(k_r-1)} [n_1]^{k_1+c_1} \cdots [n_r]^{k_r+c_r} \lambda^{c_1+\cdots+c_r} \]

where \( c_1 = 1 \) and \( c_i = b_1 + \cdots + b_{i-1} + 1 \) (2 \( \leq \) i \( \leq \) s).

Using these generating functions, Theorem 2 is stated as

\[ f(a_1, b_1, \ldots, a_s, b_s; \lambda) = g(a_1, b_1, \ldots, a_s, b_s; \lambda) \]

(11)

for any code \( (a_1, b_1, \ldots, a_s, b_s) \). To prove this equality, we prepare two propositions.

**Proposition 1.**

\[ f(a_1, b_1, \ldots, a_s, b_s; \lambda) = \sum_{p=1}^{\infty} C_p \left| \prod_{i=1}^{s} q^{n_{c_i}a_i} \prod_{j=1}^{r} \left[ n_j \right] - q^{n_{j}-p} \right| \]

where

\[ C_p = \sum_{d=1}^{r} \sum_{n_1 > \cdots > n_d > 0, n_{d+1} > \cdots > n_r} q^{n_{c_i}a_i} \prod_{j=1}^{r} \left[ n_j \right] - q^{n_{j}-p} \]

**Proof.** From (10) we have

\[ f(a_1, b_1, \ldots, a_s, b_s; \lambda) = \sum_{n_1 > \cdots > n_r > 0} \prod_{i=1}^{s} q^{n_{c_i}a_i} \prod_{j=1}^{r} \left[ n_j \right] - q^{n_{j}-n_i} \left[ n_i \right] \]

If \( \lambda \) is in a compact set in \( \mathbb{C} \setminus \Omega \), we have

\[ \sum_{n_1 > \cdots > n_r > 0} \prod_{i=1}^{s} q^{n_{c_i}a_i} \prod_{j=1}^{r} \left[ n_j \right] - q^{n_{j}-n_i} \left[ n_i \right] \left| n_d - q^{n_{d}} \lambda \right| \]

\[ \leq C \sum_{n_1 > \cdots > n_r > 0} \prod_{i=1}^{s} q^{n_{c_i}a_i} \prod_{j=1}^{r} \left[ n_j \right] - q^{n_{j}-n_i} \left[ n_i \right] \left| n_d - q^{n_{d}} \right| \]

for a positive constant \( C \). Hence we can change the order of the summation and obtain the lemma. \( \square \)
Before we state the second proposition (Proposition 2 below) we introduce some conventions. We allow 0’s appear in the code \((c_1, \ldots, c_{2s})\) with the identification
\[
(\ldots, c_{i-1}, 0, c_{i+1}, \ldots) = (\ldots, c_{i-1} + c_{i+1}, \ldots)
\]
for \(2 \leq i \leq 2s - 1\). It is consistent with the rule (2). By definition we set
\[
f(a_1, b_1, \ldots, a_s, b_s; \lambda) = 0 \quad \text{if} \quad a_1 = 0 \text{ or } b_s = 0.
\]

**Proposition 2.** Set \(I := \{(0,0), (0,1), (1,0), (1,1)\}\). For \(\eta = \{(\varepsilon_i, \delta_i)\}_{i=1}^s \in I^s\) we set
\[
|\eta| = \sum_{i=1}^r (\varepsilon_i + \delta_i), \quad h(\eta) = \# \{i \mid (\varepsilon_i, \delta_i) = (1,1)\}.
\]
Then we have
\[
\sum_{\eta \in I^s} (1 - q)^{h(\eta)}(-\lambda)^{|\eta|-h(\eta)} f(a_1 - \varepsilon_1, b_1 - \delta_1, \ldots, a_s - \varepsilon_s, b_s - \delta_s; \lambda)
\]
\[
= \sum_{\eta' \in I^{s-1}} \sum_{\varepsilon'_1, \delta'_1 = 0}^1 \sum_{1} (1 - q)^{h(\eta')}(-\lambda')^{s-|\eta'|-\varepsilon'_1-\delta'_1} f(a_1 - \varepsilon'_1, b_1 - \delta'_1, a_2 - \varepsilon'_2, \ldots, b_{s-1} - \delta'_s, a_s - \varepsilon'_s, b_s - \delta'_s+1; \lambda'),
\]
where \(\lambda' := q\lambda - 1\). In the left hand side \(\eta = \{(\varepsilon_i, \delta_i)\}_{i=1}^s\), and in the right hand side \(\eta' = \{\varepsilon'_i, \delta'_i\}_{i=1}^{s-1}\).

To prove this proposition, we use the following function:
\[
\rho((a_1, d_1), b_1, \ldots, (a_s, d_s), b_s; \lambda)
:= \sum_{n_1 > \cdots > n_s > 0} \prod_{a_j} q^{(n_i-a_j)a_j} \prod_{b_j} r \frac{1}{n_j} q^{n_j} \lambda^r,
\]
where \(r = \sum_{i=1}^s (a_i + b_i), c_1 = 1 \text{ and } c_i = b_1 + \cdots + b_{i-1} + 1 (2 \leq i \leq s)\). Then the generating function \(f\) is given by
\[
f(a_1, b_1, \ldots, a_s, b_s; \lambda) = \rho((a_1, 0), b_1, \ldots, (a_s, 0), b_s; \lambda).
\]

In the following we use the identification
\[
(\ldots, b_{i-1}, (0, d_i), b_i, \ldots) = (\ldots, b_{i-1} + b_i, \ldots),
(\ldots, (a_i-1, d), 0, (a_i, d), \ldots) = (\ldots, (a_i-1 + a_i, d), \ldots).
\]
It is consistent with the definition of \(\rho\) (18).

**Lemma 2.** The function \(\rho\) satisfies following relations:
\[
1. (a) \text{ If } a_1 \geq 2,
\]
\[
\lambda \rho((a_1, 0), b_1, (a_2, 0), \ldots; \lambda)
= \rho((a_1 - 1, 0), b_1, (a_2, 0), \ldots; \lambda) - \rho((a_1, 0), b_1 - 1, (a_2, 0), \ldots; \lambda)
= (1 - q) \rho((a_1 - 1, 0), b_1 - 1, (a_2, 0), \ldots; \lambda)
\]
\[
= \lambda' \rho((a_1, 1), b_1, (a_2, 0), \ldots; \lambda) - \rho((a_1 - 1, 1), b_1, (a_2, 0), \ldots; \lambda).
\]
(b) If $a_1 = 1$,
\[ \lambda \rho((1, 0), b_1, \ldots ; \lambda) - \rho((1, 0), b_1 - 1, \ldots ; \lambda) = \lambda' \rho((1, 1), b_1, \ldots ; \lambda). \]

2. For $2 \geq i \geq s - 1$ or $i = s$ with $b_s \geq 2$,
\[ \lambda \rho(\ldots, (a_{i-1}, 1), b_{i-1}, (a_i, 0), b_i, \ldots; \lambda) - \rho(\ldots, (a_{i-1}, 1), b_{i-1}, (a_i - 1, 0), b_i, \ldots; \lambda) - (1 - q) \rho(\ldots, (a_{i-1}, 1), b_{i-1}, (a_i, 0), b_i, \ldots; \lambda) = \lambda' \rho(\ldots, (a_{i-1}, 1), b_{i-1}, (a_i - 1, 0), b_i, \ldots; \lambda) - (1 - q) \rho(\ldots, (a_{i-1}, 1), b_{i-1} - 1, (a_i, 1), b_i, \ldots; \lambda). \]

3. (a) If $b_s \geq 2$,
\[ \rho((a_1, 1), b_1, \ldots, (a_s, 1), b_s; \lambda) = \rho((a_1, 0), b_1, \ldots, (a_s, 0), b_s; \lambda') - \frac{1}{\lambda} \rho((a_1, 0), b_1, \ldots, (a_s, 0), b_s - 1; \lambda'). \]

(b) If $b_s = 1$,
\[ \lambda \rho((a_1, 1), b_1, \ldots, (a_{s-1}, 1), b_{s-1}, (a_s, 0), 1; \lambda) - \rho((a_1, 1), b_1, \ldots, (a_{s-1}, 1), b_{s-1}, (a_s - 1, 0), 1; \lambda) = \lambda' \rho((a_1, 0), b_1, \ldots, (a_{s-1}, 0), b_{s-1} - 1, (a_s, 0), 1; \lambda') - \rho((a_1, 0), b_1, \ldots, (a_{s-1}, 0), b_{s-1} - 1, (a_s - 1, 0), 1; \lambda') - (1 - q) \rho((a_1, 0), b_1, \ldots, b_{s-1} - 1, (a_s - 1, 0), 1; \lambda'). \]

**Proof.** Here we prove (14). The proofs of the other relations are similar.

We have
\[ \lambda \rho((a_1, 0), b_1, \ldots ; \lambda) - \rho((a_1 - 1, 0), b_1, \ldots ; \lambda) = \sum_{n_1 > \cdots > n_r > 0} \left\{ \frac{q^{n_1} \lambda}{[n_1]^{a_1}} - \frac{q^{n_1(a_1 - 1)}}{[n_1]^{a_1 - 1}} \right\} \frac{1}{[n_1] - q^{n_1} \lambda} \prod_{i=2}^{s} \frac{q^{(n_i - d_i) a_i}}{[n_i - 1] - q^{(n_i - d_i) a_i}} \prod_{j=2}^{r} \frac{1}{[n_j] - q^{n_j} \lambda}. \]

Now we use the formula
\[ \left\{ \frac{q^n \lambda}{[n]^a} - \frac{q^{n(a-1)}}{[n]^{a-1}} \right\} \frac{1}{[n] - q^n \lambda} = \left\{ \frac{q^{(n-1)a} \lambda'}{[n-1]^a} - \frac{q^{(n-1)(a-1)}}{[n-1]^{a-1}} \right\} \frac{1}{[n-1] - q^n \lambda} + \frac{q^{(n-1)(a-1)}}{[n-1]^a} - \frac{q^{n(a-1)}}{[n]^a}. \]
Then we have

\[ (14) = \lambda' \rho((a_1, 0), b_1, \ldots; \lambda) - \rho((a_1 - 1, 0), b_1, \ldots; \lambda) \]

\[ + \sum_{n_2 > \cdots > n_r > 0} \prod_{i=2}^r \frac{1}{[n_i]_a} \prod_{j=2}^r \frac{g^{(n_i - d_j)a_i}}{n_j} - q^{a_j} \lambda \]

\[ \times \sum_{n_1 = n_2 + 1}^{\infty} \left\{ \frac{g^{(n_1 - 1)(a_1 - 1)}}{[n_1 - 1]_a} - \frac{q^{a_1}}{[n_1]_a} \right\}. \]

From the equality

\[ \sum_{n_1 = n_2 + 1}^{\infty} \left\{ \frac{g^{(n_1 - 1)(a_1 - 1)}}{[n_1 - 1]_a} - \frac{q^{a_1}}{[n_1]_a} \right\} = \frac{q^{a_1}}{[n_2]_a} = \frac{q^{a_2}}{[n_2]_a} + (1 - q) \frac{q^{a_2}}{[n_2]_a}, \]

we obtain

\[ (15) = \lambda' \rho((a_1, 0), b_1, \ldots; \lambda) - \rho((a_1 - 1, 0), b_1, \ldots; \lambda) \]

\[ + \rho((a_1, 0), b_1 - 1, (a_2, 0), \ldots; \lambda) \]

\[ + (1 - q) \rho((a_1 - 1, 0), b_1 - 1, (a_2, 0), \ldots; \lambda). \]

This completes the proof.

**Proof of Proposition**

The left hand side of (12) is equal to

\[ \sum_{\eta \in I^s} (1 - q)^{h(\eta)}(-\lambda)^{s-|\eta|+h(\eta)} \rho((a_1 - \epsilon_1, 0), b_1 - \delta_1, \ldots, (a_s - \epsilon_s, 0), b_s - \delta_s; \lambda). \]

Take the sum over \((\epsilon_1, \delta_1) \in I\) by using Lemma 2 (3a) and (3b). Then we have

\[ \sum_{\eta \in I^s} \sum_{\epsilon'_1, \delta'_1=0,1} (1 - q)^{h(\eta)}(-\lambda)^{s-|\eta'|+h(\eta)} \]

\[ \times \rho((a_1 - \epsilon'_1, 0), b_1, (a_2 - \epsilon'_2, 0), b_2 - \delta_2, \ldots, (a_s - \epsilon_s, 0), b_s - \delta_s; \lambda), \]

where \(\eta = \{(\epsilon_i, \delta_i)\}_{i=1}^s\). Next use (2) for \(i = 2, \ldots, s\) repeatedly, and apply (3a). In the case of \(b_s = 1\), use (3b) at the last step. As a result we obtain

\[ \sum_{\eta' \in I^s} \sum_{\epsilon'_1, \delta'_1=0,1} (1 - q)^{h(\eta')}(\lambda)^{s-|\eta'|+h(\eta') - \epsilon'_1 - \delta'_{s+1} + h(\eta')} \]

\[ \times f(a_1 - \epsilon'_1, b_1 - \delta'_2, a_2 - \epsilon'_2, \ldots, b_{s-1} - \delta'_s, a_s - \epsilon'_s, b_s - \delta'_s + \lambda'), \]

where \(\eta' = \{(\delta'_i, \epsilon'_i)\}_{i=1}^s\). This is equal to the right hand side of (12).

**Proof of Theorem**

We prove (11) by induction on the value \(\sum_i (a_i + b_i)\). If \((a_1, b_1) = (1, 1), \)

\[ f(1, 1; \lambda) = g(1, 1; \lambda) \]

is obvious from the definition.
Recall that
\[ g(a_1, b_1, \ldots, a_s, b_s) = f(b_s, a_s, \ldots, b_1, a_1). \]

Hence the function \( g \) also satisfies the equation \((12)\). Assume that \((11)\) holds for the codes less than \((a_1, b_1, \ldots, a_s, b_s)\). From the induction hypothesis, the difference of \((12)\) for \( f \) and \( g \) gives
\[
\lambda^s \left( f(a_1, b_1, \ldots, a_s, b_s; \lambda) - g(a_1, b_1, \ldots, a_s, b_s; \lambda) \right) = \lambda^s \left( f(a_1, b_1, \ldots, a_s, b_s; \lambda') - g(a_1, b_1, \ldots, a_s, b_s; \lambda') \right),
\]
i.e. the left hand side is invariant under the variable transform \( \lambda \mapsto q\lambda - 1 \). On the other hand, because of Proposition 1, we have
\[
\lambda^s \left( f(a_1, b_1, \ldots, a_s, b_s; \lambda) - g(a_1, b_1, \ldots, a_s, b_s; \lambda) \right) = \lambda^s \sum_{p=1}^{\infty} \frac{\tilde{C}_p}{|p| - q^p\lambda}
\]
for certain constants \( \tilde{C}_p \). Using the invariance under \( \lambda \mapsto q\lambda - 1 \), we find \( \tilde{C}_p = 0 \) for all \( p \).

### 2.3 The Ohno-Zagier Formula

First we introduce the \( q \)-multiple polylogarithms:

**Definition 2.** For an index \( k = (k_1, \ldots, k_r) \) the \( q \)-multiple polylogarithm (of one variable) \( \text{Li}_k(t) \) is defined by
\[
\text{Li}_k(t) := \sum_{n_1 > \cdots > n_r > 0} \frac{t^{n_1}}{[n_1]^{k_1} \cdots [n_r]^{k_r}}. \tag{16}
\]

The right hand side of \((16)\) is absolutely convergent if \(|t| < 1\).

The \( q \)-multiple polylogarithms are related to \( q \text{MZV} \) as follows:
\[
\text{Li}_{k_1, \ldots, k_r}(q) = \sum_{a_1=2}^{k_1} \sum_{a_2=1}^{k_2} \cdots \sum_{a_r=1}^{k_r} \frac{(k_1 - 2)}{a_1 - 2} \left( \prod_{j=2}^{r} \frac{(k_j - 1)}{a_j - 1} \right) \times (1 - q)^{\sum_{j=1}^{r}(k_j - a_j)} \zeta_q(a_1, \ldots, a_r). \tag{17}
\]

Here \((k_1, \ldots, k_r)\) is an admissible index.

Now let us prove Theorem 3. Denote by \( I(k, r, s) \) the set of indices of weight \( k \), depth \( r \) and height \( s \), and by \( I_0(k, r, s) \) the subset consisting of admissible ones. Set
\[
G(k, r, s; t) := \sum_{k \in I(k, r, s)} \text{Li}_k(t), \quad G_0(k, r, s; t) := \sum_{k \in I_0(k, r, s)} \text{Li}_k(t).
\]
By definition we set $G(0,0,0; t) = 1$, and $G(k, r, s; t) = 0$ unless $k \geq r + s$ and $r \geq s \geq 0$. Consider the following generating functions

$$\Phi := \sum_{k, r, s \geq 0} G(k, r, s; t) u^{k-r-s} v^{r-s} w^s,$$

$$\Phi_0 := \sum_{k, r, s \geq 0} G_0(k, r, s; t) u^{k-r-s} v^{r-s} w^s-1.$$

The function $\Phi_0$ is related to $qMZV$ as follows:

**Lemma 3.**

$$\Phi_0|_{t=q} = \frac{1}{1-(1-q)u} \sum_{k, r, s \geq 0} \left( \sum_{k \in I_{0(k,r,s)}} \zeta_q(k) \right) x^{k-r-s} y^{r-s} z^{s-1},$$

where $x$, $y$ and $z$ are given by

$$x = \frac{u}{1-(1-q)u}, \quad y = \frac{v + (1-q)(w - uv)}{1-(1-q)u}, \quad z = \frac{w}{(1-(1-q)u)^2}.$$  \hfill (18)

It is easy to prove this lemma from (17).

To prove Theorem 3 we derive a $q$-difference equation satisfied by $\Phi_0 = \Phi_0(t)$. Denote by $D_q$ the $q$-difference operator

$$(D_qf)(t) := \frac{f(t) - f(qt)}{(1-q)t}.$$  \hfill (19)

Then the $q$-multiple polylogarithms satisfy

$$D_q \text{Li}_{k_1, \ldots, k_r}(t) = \begin{cases} \frac{1}{t} \text{Li}_{k_1-1,k_2,\ldots,k_r}(t), & k_1 \geq 2, \\ \frac{1}{1-t} \text{Li}_{k_2,\ldots,k_r}(t), & k_1 = 1. \end{cases}$$

These relations can be rewritten in terms of $G(k, r, s; t)$ and $G_0(k, r, s; t)$ as follows:

$$D_q G_0(k, r, s; t) = \frac{1}{t} (G(k-1, r, s-1; t) - G_0(k-1, r, s-1; t) + G_0(k-1, r, s; t)),$$

$$D_q (G(k, r, s; t) - G_0(k, r, s; t)) = \frac{1}{1-t} G(k-1, r-1, s; t),$$

or, in terms of generating functions,

$$D_q \Phi = \frac{1}{vt} (\Phi - 1 - w\Phi_0) + \frac{u}{t} \Phi_0, \quad D_q (\Phi - w\Phi_0) = \frac{v}{1-t} \Phi.$$

Eliminate $\Phi$ using the formula

$$D_q(tf(t)) = qt \cdot D_q f(t) + f(t).$$

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Then we obtain
\[ q(t)(1-t)D_q^2\Phi_0 + ((1-u)(1-t)-vt) D_q\Phi_0 + (uv-w)\Phi_0 = 1. \] (20)

This is an equation for the power series \( \Phi_0 = \Phi_0(t) \). Note that \( \Phi_0(0) = 0 \). There is a unique solution to (20) vanishing at \( t = 0 \). To write down the solution we introduce the \( q \)-hypergeometric function
\[ \phi(a, b, c; t) := \sum_{n=0}^{\infty} t^n \prod_{j=1}^{n} \frac{(1-aq^{j-1})(1-bq^{j-1})}{(1-q^j)(1-cq^{j-1})}. \]

Then the solution is given by
\[ \Phi_0(u, v, w; t) = \frac{1}{uv-w} \left( 1 - \phi(a, b, c; ct/ab) \right). \] (21)

Here \( a, b \) and \( c \) are defined in terms of \( u, v \) and \( w \) as follows:
\[ a = \frac{1}{1 - (1-q)(u-\alpha_0)}, \quad b = \frac{1}{1 - (1-q)(u-\beta_0)}, \quad c = \frac{q}{1 - (1-q)u}, \] (22)
where \( \alpha_0 \) and \( \beta_0 \) are determined by
\[ \alpha_0 + \beta_0 = u + v, \quad \alpha_0\beta_0 = w. \]

Now we use Heine’s \( q \)-analogue of Gauss’ summation formula (see, e.g. [2]):
\[ \phi(a, b, c; \frac{c}{ab}) = \prod_{n=0}^{\infty} \frac{(1-a^{-1}cq^n)(1-b^{-1}cq^n)}{(1-cq^n)(1-a^{-1}b^{-1}cq^n)}. \]

In our case, substituting (22), we have
\[ \phi(a, b, c; \frac{c}{ab}) = \prod_{n=1}^{\infty} \frac{1-q^n}{(1-q^n)(1-a^{-1}b^{-1}cq^n)}. \] (23)

Here \( x \) and \( y \) are given by (18), and \( \alpha \) and \( \beta \) are defined by
\[ \alpha = \frac{\alpha_0}{1 - (1-q)u}, \quad \beta = \frac{\beta_0}{1 - (1-q)u}, \]
or equivalently determined by
\[ \alpha + \beta = x + y + (q-1)(z-xy), \quad \alpha\beta = z. \]

From Lemma 3. (21), (23) and the formula
\[ \log \prod_{n=1}^{\infty} \left( 1 - \frac{q^n}{[n]} s \right) = \frac{1}{q-1} \log(1-s(q-1)) \sum_{n=1}^{\infty} \frac{q^n}{[n]} \]
\[ - \sum_{n=2}^{\infty} \zeta_q(n) \sum_{m=0}^{\infty} \frac{(q-1)^m}{m+n} s^{m+n}, \]
we obtain Theorem 3.
3 Discussion

Now we consider the space spanned by the modified qMZV’s (5). We regard the modified qMZV as a formal power series of $q$, and define subspaces of $Q[[q]]$ by

$$Z_k := \sum_{|k|=k} Q\zeta_q(k), \quad Z_{\leq k} := \sum_{2 \leq |k| \leq k} Q\zeta_q(k).$$

Let us consider the problem to determine the dimension of the space $Z_k$ over $Q$. In principle, a lower bound for the dimension can be obtained as follows. Expand $qMZV \zeta_q(k)$ as a power series of $q$:

$$\zeta_q(k) = \sum_{n=0}^{\infty} a_n(k)q^n, \quad a_n(k) \in \mathbb{Z}_{\geq 0}.$$

Note that $a_n(k) = 0$ if $n < |k| - 1$. Recall that the number of admissible indices of weight $k$ is equal to $2^{k-2}$. Now consider the following square matrices

$$A_k := (a_n(k))_{k-1 \leq n \leq k+2^{k-2}-2}, \quad A_{\leq k} := (a_n(k))_{1 \leq n \leq k+2^{k-2}-2}.$$

Then we have

$$\text{rank } A_k \leq \dim Z_k, \quad \text{rank } A_{\leq k} \leq \dim Z_{\leq k}.$$

Let $d_k$ be the conjectured dimension of the space of MZV’s of weight $k$ over $Q$ in [9]. In [8] it is shown that $d_k$ gives the upper bound of the dimension of MZV’s.

The values of rank $A_k$ and rank $A_{\leq k}$ for $2 \leq k \leq 10$ are given as follows:

| weight | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------|---|---|---|---|---|---|---|---|----|
| $d_k$  | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 |    |
| rank $A_k$ | 1 | 1 | 2 | 3 | 6 | 9 | 18 | 29 | 54 |
| By cyclic and Ohno | 1 | 1 | 2 | 3 | 6 | 9 | 18 | 30 | 56 |
| $\sum_{j<k} d_j$ | 1 | 2 | 3 | 5 | 7 | 10 | 14 | 19 | 26 |
| rank $A_{\leq k}$ | 1 | 2 | 4 | 7 | 11 | 18 | 27 | 42 | 63 |
| $\sum_{j<k} \text{rank } A_j$ | 1 | 2 | 4 | 7 | 13 | 22 | 40 | 69 | 123 |

Here the third row shows the upper bound for the dimension of $Z_k$ which follows from the cyclic sum formula and the Ohno relation [4]. We show some data of
is correct by substituting $q$ with $\zeta$. It is linearly independent of the cyclic sum formula and the Ohno relation, and following relations seem to hold:

$$\zeta \q q_0 0 0 0 0 0 0 1 1 4 9 14 23$$

Any qMZV of weight less than 7 is given by a linear combination of ones in the list above from the cyclic sum formula and the Ohno relation.

Now we discuss two points. First, at weight 9 there is a gap between rank $A_9$ and the upper bound. This shows a possibility that our linear relations are not sufficient to determine the dimension of $Z_k$. In fact, the following equality holds up to $q^{269}$:

$$4\zeta_q(7, 2) + 6\zeta_q(6, 3) - \zeta_q(5, 4) - \zeta_q(4, 5) - 6\zeta_q(6, 2, 1) - 6\zeta_q(6, 1, 2)$$

$$-2\zeta_q(5, 3, 1) - 7\zeta_q(5, 2, 2) - 3\zeta_q(5, 1, 3) + 2\zeta_q(4, 4, 1) - \zeta_q(4, 3, 2)$$

$$+\zeta_q(3, 5, 1) + \zeta_q(3, 2, 4) - 3\zeta_q(2, 5, 2) + 2\zeta_q(5, 2, 1, 1)$$

$$+2\zeta_q(5, 1, 2, 1) + 2\zeta_q(5, 1, 1, 2) + \zeta_q(3, 3, 1, 2) - \zeta_q(3, 2, 3, 1)$$

$$-4\zeta_q(3, 2, 2, 2) - \zeta_q(3, 2, 1, 3) - 2\zeta_q(2, 2, 3, 2) + \zeta_q(2, 1, 3, 3) \equiv 0$$

It is linearly independent of the cyclic sum formula and the Ohno relation, and is correct by substituting $\zeta_q$ with $\zeta$.

Next, if the weight is greater than 5, the equality rank $A_{\leq k} = \sum_{j=2}^{k} \text{rank } A_j$ breaks. This shows that there may exist linear relations among the modified qMZV’s of different weights. For example, as observed in the data above, the following relations seem to hold:

$$-\zeta_q(3, 1) + 3\zeta_q(4, 1) - 3\zeta_q(3, 2)$$

$$-\zeta_q(6) - 3\zeta_q(4, 2) + 6\zeta_q(3, 3) \equiv 0,$$

$$-2\zeta_q(3, 1) + 2\zeta_q(5) - 6\zeta_q(4, 1) - 9\zeta_q(3, 2)$$

$$+\zeta_q(6) - 12\zeta_q(4, 2) - 3\zeta_q(4, 1, 1) + 3\zeta_q(3, 2, 1) \equiv 0.$$
above. Multiply \((1 - q)^6\) and take the limit \(q \to 1\) in (24). Then the terms of weight less than 6 vanish and as a result we find

\[
-\zeta(6) - 3\zeta(4, 2) + 6\zeta(3, 3) = 0,
\]

\[
\zeta(6) - 12\zeta(4, 2) - 3\zeta(4, 1, 1) + 3\zeta(3, 2, 1) = 0.
\]

These relations for MZV’s are correct [4].

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