BEYOND LOCAL MAXIMAL OPERATORS

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ABSTRACT. We obtain (essentially sharp) boundedness results for certain generalized local maximal operators between fractional weighted Sobolev spaces and their modifications. Concrete boundedness results between well known fractional Sobolev spaces are derived as consequences of our main result. We also apply our boundedness results by studying both generalized neighbourhood capacities and the Lebesgue differentiation of fractional weighted Sobolev functions.

1. Introduction

In this paper we study the boundedness of a centered maximal-type operator $M_R$ between fractional $A_p$ weighted Sobolev spaces and their $R$-modifications; the well known fractional Sobolev spaces $W^{s,p}(G)$ for open sets $G \subset \mathbb{R}^n$ are special cases of the aforementioned spaces, see [3]. The operator $M_R$ depends on a given measurable function $R : G \to \mathbb{R}$ which satisfies the condition $0 \leq R(x) \leq \text{dist}(x, \partial G)$ whenever $x \in G$. Here, and throughout the paper, we agree that $\text{dist}(x, \partial G) = \infty$ if $x \in G = \mathbb{R}^n$. For any $f \in L^1_{\text{loc}}(G)$ and every $x \in G$ we define

$$M_R(f)(x) = \sup_r \int_{B(x,r)} |f(y)| \, dy,$$

where the supremum is taken over all radii $0 \leq r \leq R(x)$ and we have used the notational convention

$$\int_{B(x,0)} |f(y)| \, dy = |f(x)|.$$

Even though special cases of this maximal-type operator have been studied earlier, cf. below, we are not aware of previous studies in this generality and in connection with Sobolev spaces. There is a parallel problem of fixing the appropriate Sobolev spaces where the boundedness is to be studied; to illustrate, let us remark that $M_R$ need not preserve the smoothness of order $0 < s \leq 1$, unless $R$ is (say) a Lipschitz function.

Our main result shows that fractional $A_p$ weighted Sobolev spaces and their $R$-modified counterparts, [4], are well-suited for studying the boundedness properties of $M_R$; this result can be found in [1]. The main result will be applied to the study of certain neighbourhood capacities (see [7]) and the Lebesgue differentiation of fractional weighted Sobolev functions (see [8]). We expect that there are other applications in fractional weighted potential theory; indeed, an operator $M_R$ that is given by an application specific $R$-function provides a flexible tool that can be used to estimate ‘size’ in terms of ‘smoothness’. This is especially true when combined with fractional Sobolev or Hardy inequalities [6, 18, 19].

More specifically, our main result is Theorem 1.1. This theorem is a ‘fractional Sobolev analogue’ of the celebrated Muckenhoupt’s theorem which, in turn, is a boundedness result for the Hardy–Littlewood maximal operator on the $A_p$ weighted $L^p$-spaces (for a detailed formulation, we refer to Proposition 2.1). In order to avoid technicalities at this stage, let
us formulate Theorem 1.1 that is a consequence of our main result (when applied with a Muckenhoupt $A_p$ weight that is defined by $\omega(x) = |x|^{\varepsilon-n}$ for the given $0 < \varepsilon < np$).

**Theorem 1.1.** Let $\emptyset \neq G \subset \mathbb{R}^n$ be an open set, $0 < \varepsilon, s < 1$ and $1 < p < \infty$. Fix a measurable function $R : G \to \mathbb{R}$ satisfying inequality $0 \leq R(x) \leq \text{dist}(x, \partial G)$ for every $x \in G$. Then there exists a constant $C = C(n, p, \varepsilon) > 0$ such that inequality

$$
\int_G \int_G \frac{|M_R(f)(x) - M_R(f)(y)|^p}{|x-y|^{n+s\varepsilon}} \, dy \, dx \leq C \int_G \int_G \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dy \, dx \tag{1.3}
$$

holds for every $f \in L^p(G)$.

We remark that if $R$ is a Lipschitz function, e.g., if $R = \text{dist}(-, \partial G)$ in case of a proper open subset $G$ of $\mathbb{R}^n$, then the left-hand side of inequality (1.3) is comparable to

$$
\int_G \int_G \frac{|M_R(f)(x) - M_R(f)(y)|^p}{|x-y|^{n+sp}} \, dy \, dx.
$$

In particular, Theorem 1.1 generalizes a recently obtained boundedness result for the local Hardy–Littlewood maximal operator $M_{\text{dist}(\cdot, \partial G)}$ on fractional Sobolev spaces $W^{s,p}(G)$, see [37, Theorem 1.1]. Another interesting case is when $R$ is an $\alpha$-Hölder function ($0 < \alpha < 1$) on a bounded open set $G$ such that $0 \leq R(x) \leq \text{dist}(x, \partial G)$ for each $x \in G$. Corollary 5.6 then implies that

$$
M_R : W^{s,p}(G) \to W^{\sigma,p}(G), \quad 0 < \sigma < \alpha s,
$$

is a bounded operator whenever $0 < s < 1$ and $1 < p < \infty$; with the aid of a fractional Hardy inequality we show in Lemma 5.7 that this result is essentially sharp, in that we cannot allow $\sigma > \alpha s$ in general (however, we do not know if $\sigma = \alpha s$ is allowed). In particular, our main result (Theorem 1.1) is also essentially sharp in its generality.

We close this introduction with a brief overview on related results for the maximal and local maximal operators. The maximal operators $M_B$ that are defined by (differentiation) bases $B$ have been extensively studied, e.g., in connection with differentiability properties of functions, we refer to [8, 11, 21, 23, 33].

Concerning the boundedness of maximal operators on the Sobolev-type spaces, previous research has mainly focused on the Hardy–Littlewood maximal operator $M$ and the local maximal operator $M_{\text{dist}(\cdot, \partial G)}$ for a given open set $G \subset \mathbb{R}^n$; see [12, 23, 27, 35]. In particular, the boundedness of the local maximal operator on the first order Sobolev spaces $W^{1,p}(G)$ is proved by Kinnunen and Lindqvist [23]. Their main result states that if $1 < p \leq \infty$ and $f \in W^{1,p}(G)$, then $M_{\text{dist}(\cdot, \partial G)}(f) \in W^{1,p}(G)$ and

$$
|\nabla (M_{\text{dist}(\cdot, \partial G)}(f))(x)| \leq 2M_{\text{dist}(\cdot, \partial G)}(|\nabla f|)(x) \tag{1.4}
$$

for almost every $x \in G$; observe that inequality (1.4) and boundedness of the local maximal operator on $L^p(G)$ yields boundedness of the local maximal operator on $W^{1,p}(G)$. We will prove a fractional weighted counterpart of inequality (1.4) in Proposition 1.14. Korry [28] studied boundedness of the Hardy–Littlewood maximal operator on the Triebel–Lizorkin spaces of the (fractional) order smoothness $0 < s < 1$. The first author established in [36] the boundedness and continuity properties of $M_{\text{dist}(\cdot, \partial G)}$ on the (non-intrinsically defined) Triebel–Lizorkin spaces $F^s_{pq}(G)$ for $0 < s < 1$ and $1 < p, q < \infty$. Boundary results for the discrete analogues of maximal operators in metric spaces can be found, e.g., in [14, 15, 24].

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2. Notation and preliminaries

The open ball centered at \( x \in \mathbb{R}^n \) and with radius \( r > 0 \) is \( B(x, r) \). The Euclidean distance from \( x \in \mathbb{R}^n \) to a set \( E \) in \( \mathbb{R}^n \) is denoted by \( \text{dist}(x, E) \). Here we agree that \( \text{dist}(x, \emptyset) = \infty \).

The Euclidean diameter of \( E \) is \( \text{diam}(E) \). The characteristic function of a set \( E \) is written as \( \chi_E \). The Lebesgue \( n \)-measure of a measurable set \( E \) is denoted by \( |E| \). If \( 0 < |E| < \infty \), the integral average of a function \( f \in L^1(E) \) is \( f_E = \int_E f \, dx = |E|^{-1} \int_E f \, dx \). If \( G \) is an open set in \( \mathbb{R}^n \), then \( C_0(G) \) denotes the space of continuous functions \( f \) in \( G \) whose support

\[
\text{supp}(f) = \{ x \in G : f(x) \neq 0 \}
\]

is a compact set contained in \( G \); the closure above is taken in \( \mathbb{R}^n \). If there exists a constant \( C > 0 \) such that \( a \leq C b \), we write \( a \preceq b \), and if \( a \preceq b \preceq a \) we write \( a \simeq b \) and say that \( a \) and \( b \) are comparable. We let \( C(\ast, \cdots, \ast) \) denote a positive constant which depends on the quantities appearing in the parentheses only.

Function \( \omega \in L^1_{\text{loc}}(\mathbb{R}^n) \) is a weight if \( \omega(x) > 0 \) for almost every \( x \in \mathbb{R}^n \). Let \( 1 < p < \infty \). A weight \( \omega \) is an \( A_p \) weight if there exists \( A > 0 \) such that, for every cube \( Q \subset \mathbb{R}^n \),

\[
\left( \int_Q \omega \, dx \right) \left( \int_Q \omega^{-1/(p-1)} \, dx \right)^{p-1} \leq A.
\]

The infimum over all such constants \( A \) is called the \( A_p \) constant of \( \omega \), written as \( [\omega]_{A_p} \). The Hardy–Littlewood maximal function \( Mf \) for a function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) is defined by

\[
Mf(x) = \sup_{r>0} \int_{B(x,r)} |f(y)| \, dy, \quad x \in \mathbb{R}^n.
\]

Muckenhoupt’s theorem is the following well known result, see [3] §IV.2 Theorem 2.8 for a proof and further details.

**Proposition 2.1.** Let \( 1 < p < \infty \) and let \( \omega \) be an \( A_p \) weight. Then there exists a constant \( C > 0 \) such that

\[
\int_{\mathbb{R}^n} (Mf(x))^p \omega(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx
\]

whenever \( f \) is a measurable function for which the integral on the right-hand side is finite. Moreover, the constant \( C \) depends only on \( n \), \( p \) and the \( A_p \) constant of \( \omega \).

When \( A \subset \mathbb{R}^n \) is bounded and \( r > 0 \), we let \( N(A, r) \) denote the minimal number of (open) balls of radius \( r \) and centered at \( A \) that are needed to cover the set \( A \). For any set \( E \subset \mathbb{R}^n \), the (upper) Assouad dimension of \( E \) is defined by setting

\[
\overline{\dim}_A(E) = \inf \left\{ \lambda \geq 0 : N(E \cap B(x,R), r) \leq C_\lambda \left( \frac{r}{R} \right)^{-\lambda} \text{ for all } x \in E, \ 0 < r < R < \text{diam}(E) \right\}.
\]

This is the ‘usual’ Assouad dimension found in the literature, e.g. in [38], often denoted \( \dim_A(E) \). If \( E \subset \mathbb{R}^n \) is a (sufficiently) regular set, for instance, Ahlfors \( d \)-regular, then the upper Assouad dimension of \( E \) coincides with its Hausdorff dimension; we refer to [29].

A set \( E \subset \mathbb{R}^n \) is \( \kappa \)-porous \((0 < \kappa < 1)\) if for each \( x \in E \) and every \( 0 < r < \text{diam}(E) \) there exists a point \( y \in \mathbb{R}^n \) such that \( B(y, \kappa r) \subset B(x, r) \setminus E \). We remark that a set \( E \subset \mathbb{R}^n \) is \( \kappa \)-porous for some \( 0 < \kappa < 1 \) if and only if \( \overline{\dim}_A(E) < n \), see [38, Theorem 5.2].
3. Fractional weighted Sobolev spaces

We present the fractional weighted Sobolev seminorms and the associated function spaces that are used throughout this paper. Moreover, we consider the density of smooth functions in these spaces by adapting the argument given in [17]. Incidentally, density properties for other fractional weighted Sobolev spaces have recently been studied in [4, 9]. Since our weights are always translation invariant, the density arguments are quite straightforward and (eventually) based upon the continuity of translations in the classical Lebesgue spaces. Whereas a similar approach is used in the work [9], a more refined approximation scheme is developed in [4] to handle weights that are not translation invariant.

The fractional weighted Sobolev seminorm $|f|_{W^{s,p,\omega}(G)}$ given in Definition 3.1 has been previously studied, e.g., in connection with fractional weighted Hardy-type inequalities, extension problems and variational problems, we refer to [3, 5, 9].

**Definition 3.1.** Let $s > 0$ and $1 \leq p < \infty$, and let $\omega$ be a weight in $\mathbb{R}^n$ (see [2]). Fix an open set $G \subset \mathbb{R}^n$. Then $W^{s,p,\omega}(G)$ is the fractional weighted Sobolev space of functions $f \in L^p(G)$ satisfying

$$|f|_{W^{s,p,\omega}(G)} = \left( \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx \right)^{1/p}$$

is the fractional weighted Sobolev seminorm.

We remark that the global norm is translation invariant, i.e., for each $f \in W^{s,p,\omega}(\mathbb{R}^n)$ and every $h \in \mathbb{R}^n$ we have

$$\|f(\cdot + h)\|_{W^{s,p,\omega}(\mathbb{R}^n)} = \|f\|_{W^{s,p,\omega}(\mathbb{R}^n)}.$$  

Hence, our framework is most likely not the nearest fractional analogue of the first order $A_p$ weighted Sobolev space that is not generally translation invariant, see [22, 15].

There is an $R$-modification of the seminorm (3.5) that will also be relevant to us. Namely, given a measurable function $R : G \to \mathbb{R}$, we will often encounter the following (often translation invariantless) seminorm

$$\left( \int_G \int_G \frac{|f(x) - f(y)|^p}{(|x - y| + |R(x) - R(y)|)^{sp}} \omega(x - y) \, dy \, dx \right)^{1/p}.$$  

Theorem [41] and the supporting counterexample given in [5] indicate that if $\varphi \in W^{s,p,\omega}(G)$, then the right way to measure the smoothness of $f = M_R(\varphi)$ is to use (3.6). This quantity can be viewed as a weighted seminorm that measures ‘variable fractional smoothness’ of $f \in L^p(G)$. Indeed, assuming that $R$ is a Lipschitz function on $G$, the last seminorm (3.6) is comparable to $|f|_{W^{s,p,\omega}(G)}$. On the other hand, if $R$ oscillates more significantly then

$$|x - y| + |R(x) - R(y)|$$

(3.7)

can be much larger than $|x - y|$. We remark that (3.7) is comparable to Euclidean distance between $(x, R(x))$ and $(y, R(y))$ that belong to the graph $\{(w, R(w)) : w \in G\} \subset \mathbb{R}^{n+1}$.

We will apply the fractional $A_p$ weighted Sobolev spaces and their $R$-modifications. Both of these spaces arise naturally in the proof of our main result and, moreover, the well known fractional Sobolev spaces are their special cases:

**Example 3.2.** Consider the well known and widely used fractional Sobolev space $W^{s,p}(G)$, whose survey can be found in [41]. For any given $\varepsilon > 0$ this space can be represented as $W^{s+\varepsilon/p,p,\omega}(G)$ when the weight is given by $w = |\cdot|^{\varepsilon-n}$. In particular, the fractional Sobolev
We remark that $|\varepsilon^{-n}$ is an $A_p$ weight if, and only if, inequality $0 < \varepsilon < np$ holds; we refer to [33] p. 229, p. 236.

We turn to density of continuous functions in $W^{s,p,\omega}(\mathbb{R}^n)$; this will be needed in [6] when studying Lebesgue differentiation and quasicontinuous representatives of fractional weighted Sobolev functions. Our density argument seems to require that $\omega$ has a sufficient decay at infinity that is quantified by inequality (3.8) below. This decay inequality turns out to be quite natural: it is equivalent to the requirement that $C_0^\infty(\mathbb{R}^n)$ is a subset of $W^{s,p,\omega}(\mathbb{R}^n)$. We also remark that when $sp < n$ inequality (3.8) for a given $\rho > 0$ fails even for the $A_p$ weight that is defined by $\omega(x) = 1$ for every $x \in \mathbb{R}^n$.

**Lemma 3.3.** Let $0 < s < 1$ and $1 \leq p < \infty$, and let $\omega$ be a weight in $\mathbb{R}^n$. Then $C_0^\infty(\mathbb{R}^n)$ is a subset $W^{s,p,\omega}(\mathbb{R}^n)$ if, and only if,

$$\int_{\mathbb{R}^n \setminus B(0,\rho)} \frac{\omega(x)}{|x|^{sp}} \, dx < \infty \quad (3.8)$$

for every $\rho > 0$ (or, equivalently, for some $\rho > 0$).

**Proof.** Let us first assume that $C_0^\infty(\mathbb{R}^n)$ is a subset of $W^{s,p,\omega}(\mathbb{R}^n)$. Fix $\rho > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$ such that supp$(f) \subset B(0, \rho/2)$ and $f(0) = 2$. Fix $0 < \delta < \rho/2$ such that $f(x) \geq 1$ if $|x| < \delta$. Then

$$\int_{\mathbb{R}^n \setminus B(0,\rho)} \frac{\omega(x)}{|x|^{sp}} \, dx = \int_{B(0,\delta)} \int_{\mathbb{R}^n \setminus B(0,\rho)} \frac{\omega(x)}{|x|^{sp}} \, dx \, dy$$

$$\leq \int_{B(0,\delta)} \int_{\mathbb{R}^n \setminus B(0,\rho/2)} \frac{1}{|x-y|^{sp}} \omega(x-y) \, dx \, dy$$

$$\leq \frac{1}{|B(0,\delta)|} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{sp}} \omega(x-y) \, dx \, dy = \frac{|f|_{W^{s,p,\omega}(\mathbb{R}^n)}^p}{|B(0,\delta)|} < \infty.$$

Hence, inequality (3.8) holds.

Conversely, let us assume that inequality (3.8) holds for some $\rho > 0$. Fix $f \in C_0^\infty(\mathbb{R}^n)$ and choose $R > \rho$ such that supp$(f) \subset B(0, R)$. Suppose that $x$ and $y$ are in the ball $B(0, 2R)$, $x \neq y$. Then, by the mean-value theorem,

$$\frac{|f(x) - f(y)|^p}{|x-y|^{sp}} \leq \|\nabla f\|_{L^\infty(\mathbb{R}^n)} |x-y|^{p(1-s)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} (4R)^{p(1-s)} = M.$$

By assumption $\omega$ is a weight. In particular, it is locally integrable, see [32]. Hence,

$$\int_{B(0,2R)} \int_{B(0,2R)} \frac{|f(x) - f(y)|^p}{|x-y|^{sp}} \omega(x-y) \, dy \, dx$$

$$\leq M \int_{B(0,2R)} \int_{B(0,2R)} \omega(x-y) \, dy \, dx \leq M |B(0, 2R)| \int_{B(0, 4R)} \omega(z) \, dz < \infty. \quad (3.9)$$

Furthermore, by the fact that $R > \rho$ and inequality (3.8),

$$\int_{\mathbb{R}^n \setminus B(0,2R)} \int_{B(0,2R)} \frac{|f(x) - f(y)|^p}{|x-y|^{sp}} \omega(x-y) \, dy \, dx$$

$$\leq \int_{\mathbb{R}^n \setminus B(0,2R)} \int_{B(0,2R)} \frac{|f(y)|^p}{|x-y|^{sp}} \omega(x-y) \, dy \, dx \leq \|f\|_{L^p(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(0,\rho)} \frac{\omega(z)}{|z|^{sp}} \, dz < \infty. \quad (3.10)$$
A similar computation shows that
\[
\int_{B(0,2R)} \int_{\mathbb{R}^n \setminus B(0,2R)} \frac{|f(x) - f(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx < \infty.
\] (3.11)

Inequalities (3.9)–(3.11) and the fact \( \text{supp}(f) \subset B(0, R) \) yield that \( f \in W^{s,p,\omega}(\mathbb{R}^n) \). □

Next we focus on weights \( \omega \) satisfying \( C_0^\infty(\mathbb{R}^n) \subset W^{s,p,\omega}(\mathbb{R}^n) \). Under this restriction it is now straightforward to show that continuous functions are dense in \( W^{s,p,\omega}(\mathbb{R}^n) \). For this purpose, we let \( \varphi \in C_0^\infty(B(0, 1)) \) be a non-negative bump function such that \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 1 \). For \( j \in \mathbb{N} \) and \( x \in \mathbb{R}^n \), we write \( \varphi_j(x) = 2^j \varphi(2^j x) \). Recall that
\[
f * \varphi_j(x) = \int_{\mathbb{R}^n} f(x - z) \varphi_j(z) \, dz
\]
defines a smooth function in \( \mathbb{R}^n \) and \( \lim_{j \to \infty} \| f - f * \varphi_j \|_{L^p(\mathbb{R}^n)} = 0 \) whenever \( f \in L^p(\mathbb{R}^n) \) and \( 1 \leq p < \infty \), see e.g. [12].

**Lemma 3.4.** Let \( 0 < s < 1 \) and \( 1 \leq p < \infty \), and let \( \omega \) be a weight in \( \mathbb{R}^n \) such that \( C_0^\infty(\mathbb{R}^n) \) is a subset \( W^{s,p,\omega}(\mathbb{R}^n) \). Then for every \( f \in W^{s,p,\omega}(\mathbb{R}^n) \) we have
\[
\| f - f * \varphi_j \|_{W^{s,p,\omega}(\mathbb{R}^n)} \xrightarrow{j \to \infty} 0.
\] (3.12)

In particular, the set \( C_0^\infty(\mathbb{R}^n) \cap W^{s,p,\omega}(\mathbb{R}^n) \) is dense in \( W^{s,p,\omega}(\mathbb{R}^n) \).

**Proof.** The basic ideas for the proof are from [17]. Since the convolutions \( f * \varphi_j \) converge to \( f \) in \( L^p(\mathbb{R}^n) \) when \( j \to \infty \), it suffices to show that \( \| f - f * \varphi_j \|_{W^{s,p,\omega}(\mathbb{R}^n)} \to 0 \) when \( j \to \infty \). Fix \( \varepsilon > 0 \). We write
\[
|f - f * \varphi_j|^p_{W^{s,p,\omega}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y) + f * \varphi_j(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx.
\]
Since \( |f|_{W^{s,p,\omega}(\mathbb{R}^n)} < \infty \), we may apply the monotone convergence theorem in \( \mathbb{R}^n \times \mathbb{R}^n \) in order to obtain a number \( \rho = \rho(\varepsilon, f, \omega) > 0 \) such that
\[
\int_{\mathbb{R}^n} \int_{B(x, \rho)} \frac{|f(x) - f(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx < \varepsilon. 
\] (3.13)

Now, for any \( j \in \mathbb{N} \)
\[
\int_{\mathbb{R}^n} \int_{B(x, \rho)} \frac{|f * \varphi_j(x) - f * \varphi_j(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx \\
\lesssim 2^{-jn(p-1)} \int_{\mathbb{R}^n} \varphi_j(z)^p \int_{\mathbb{R}^n} \int_{B(z, \rho)} \frac{|f(x - z) - f(y - z)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx \, dz \\
= 2^{-jn(p-1)} \int_{\mathbb{R}^n} \varphi_j(z)^p \int_{\mathbb{R}^n} \int_{B(z, \rho)} \frac{|f(x) - f(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx \, dz.
\]

Hence, we obtain that
\[
\int_{\mathbb{R}^n} \int_{B(x, \rho)} \frac{|f * \varphi_j(x) - f * \varphi_j(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx \\
\lesssim \int_{\mathbb{R}^n} \int_{B(x, \rho)} \frac{|f(x) - f(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx < \varepsilon. 
\] (3.14)

From (3.13) and (3.14) it follows that
\[
\sup_{j \in \mathbb{N}} \int_{\mathbb{R}^n} \int_{B(x, \rho)} \frac{|f(x) - f * \varphi_j(x) - f(y) + f * \varphi_j(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx \lesssim \varepsilon. 
\] (3.15)
On the other hand, since \( C_0^\infty(\mathbb{R}^n) \subset W^{s,p,\omega}(\mathbb{R}^n) \) by assumptions, Lemma 3.3 yields
\[
\int_{\mathbb{R}^n \setminus B(0, \rho)} \frac{\omega(x)}{|x|^s} dx < \infty.
\]
Moreover, by assumptions, we have \( f \in L^p(\mathbb{R}^n) \) and therefore
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(x, \rho)} \frac{|f(x) - f \ast \varphi_j(x) - f(y) + f \ast \varphi_j(y)|^p}{|x-y|^s \omega(x-y)} dy dx \lesssim \left( \int_{\mathbb{R}^n \setminus B(0, \rho)} \frac{\omega(x)}{|x|^s} dx \right) \int_{\mathbb{R}^n} |f(x) - f \ast \varphi_j(x)|^p dx \xrightarrow{j \to \infty} 0; \tag{3.16}
\]
here we again used the fact that \( f \ast \varphi_j \) converges to \( f \) in \( L^p(\mathbb{R}^n) \) when \( j \to \infty \). By combining the estimates (3.15) and (3.16), we find that \( |f - f \ast \varphi_j|_{W^{s,p,\omega}(\mathbb{R}^n)} \to 0 \) when \( j \to \infty \). \( \square \)

4. A BOUNDEDNESS RESULT FOR \( M_R \)

We formulate and prove our main result, i.e., Theorem 4.1 that provides a boundedness result for the maximal operator \( M_R \) (see (1.1)) from a fractional \( A_p \) weighted Sobolev space to its \( R \)-modification, both of which are defined in \( \S 3 \).

The main result is akin to the Muckenhoupt’s theorem, i.e., Proposition 2.1, in that both sides of inequality (1.17) incorporate an \( A_p \) weight. Another interesting aspect is how the left-hand side of inequality (1.17) depends on the given \( R \)-function; from the viewpoint of applications, such a dependence is both flexible and straightforward to work with. Moreover, as we will see in \( \S 3 \) the \( R \)-dependence is essentially the best possible in this generality.

Recall our notational convention \( \text{dist}(x, \partial G) = \infty \) if \( x \in G = \mathbb{R}^n \).

**Theorem 4.1.** Assume that \( \emptyset \neq G \subset \mathbb{R}^n \) is an open set, \( 0 \leq s \leq 2 \), and \( 1 < p < \infty \). Fix a measurable function \( R : G \to \mathbb{R} \) satisfying inequality \( 0 \leq R(x) \leq \text{dist}(x, \partial G) \) for every \( x \in G \). Then, if \( \omega \) is an \( A_p \) weight in \( \mathbb{R}^n \), there exists a constant \( C > 0 \) such that inequality
\[
\int_G \int_G \frac{|M_R(f)(x) - M_R(f)(y)|^p}{(|x-y| + |R(x) - R(y)|)^s} \omega(x-y) dy dx \leq C \int_G \int_G \frac{|f(x) - f(y)|^p}{|x-y|^s} \omega(x-y) dy dx \tag{4.17}
\]
holds for every \( f \in L^p(G) \). The constant \( C \) depends only on \( n, p \) and the \( A_p \) constant of \( \omega \).

This result is a far-reaching extension of \( \S 7 \) Theorem 1.1) whose proof, in turn, applies ideas from \( \S 6 \) Theorem 3.2). Here delicate modifications are required in the proofs due to the \( A_p \) weight and the \( R \)-function. In the sequel, we follow outline of the proof in \( \S 7 \); in particular, we repeat many details therein without further notice.

The proof of Theorem 4.1 will be completed at the end of this section. The main technical tool is a pointwise inequality that is given in Proposition 4.4. Moreover, some implications of the Muckenhoupt’s theorem are also needed, see Proposition 4.2. In order to state the latter proposition, we first need some preparations.

Let us fix \( i, j \in \{0, 1\} \). For a measurable function \( F \) on \( \mathbb{R}^{2n} \) we write
\[
M_{ij}(F)(x, y) = \sup_{r > 0} \int_{B(0, r)} |F(x + iz, y + jz)| dz \tag{4.18}
\]
whenever the right-hand side is well-defined, i.e., for almost every \( (x, y) \in \mathbb{R}^{2n} \) by Fubini’s theorem. Observe that \( M_{00}(F) = |F| \). The measurability of \( M_{ij}(F) \) can be checked by first noting that the supremum in (4.18) can be restricted to the rational numbers \( r > 0 \) and then adapting the proof of [12] Theorem 8.14] with each \( r \) separately.
By applying Fubini’s theorem in appropriate coordinates and \( L^p(\mathbb{R}^n) \)-boundedness of the Hardy–Littlewood maximal operator \( f \mapsto Mf \) we find that \( M_{ij} = (F \mapsto M_{ij}(F)) \) is a bounded operator on \( L^p(\mathbb{R}^{2n}) \) whenever \( 1 < p < \infty \). Furthermore, we need the following \( A_p \) weighted norm inequalities that eventually rely on Muckenhoupt’s theorem.

**Proposition 4.2.** Let \( 1 < p < \infty \). Then, if \( \omega \) is an \( A_p \) weight in \( \mathbb{R}^n \), there exists a constant \( C = C(n, p, [\omega]_{A_p}) > 0 \) such that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (M_{kl}(F)(x,y))^p \omega(x-y) \, dy \, dx \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x,y)|^p \, dx \, dy \tag{4.19}
\]

whenever \( F \) is a measurable function in \( \mathbb{R}^{2n} \) and \( k,l \in \{0,1\} \) are such that \( kl = 0 \).

**Proof.** We focus on the case \((k,l) = (0,1)\); the case \((k,l) = (1,0)\) is analogous, and the claim is trivial when \( k = 0 = l \). Let us consider a measurable function \( F \) on \( \mathbb{R}^{2n} \) for which the double integral on the right-hand side of \( (4.19) \) is finite. By dilation and translation invariance of the \( A_p \)-condition, we find that the function \( y \mapsto \omega_x(y) := \omega(x-y) \), \( x \in \mathbb{R}^n \), belongs to \( A_p \), and its \( A_p \) constant coincides with \([\omega]_{A_p}\). Hence, by Proposition 2.1,

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (M_{01}(F)(x,y))^p \omega(x-y) \, dy \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (M(F(x,\cdot))(y))^p \omega_y(y) \, dy \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x,y)|^p \omega_x(y) \, dy \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x,y)|^p \omega(x-y) \, dx \, dy,
\]

and the proof is complete. \( \square \)

**Remark 4.3.** The directional maximal operators \( M_{ij} \) are dominated by the so-called strong maximal operator, whose certain weighted norm inequalities can be found in \[IV.6\].

The following proposition gives a certain extension of inequality \((4.1)\). But first let us introduce further convenient notation that is used in the remaining part of this section. We write \( \omega_0(x,y) = \omega(x-y)^{1/p} \) and \( \omega_1(x,y) = \omega(y-x)^{1/p} \) if \( x,y \in \mathbb{R}^n \) and \( \omega \) is an \( A_p \) weight in \( \mathbb{R}^n \). For \( f \in L^p(G) \) we denote

\[
S_R(f)(x,y) = S_{R,G,s}(f)(x,y) = \frac{\chi_G(x)\chi_G(y)|f(x) - f(y)|}{(|x-y| + |R(x) - R(y)|)^s}
\]

for almost every \((x,y) \in \mathbb{R}^{2n}\); we also abbreviate \( S(f) = S_0(f) \).

**Proposition 4.4.** Assume that \( \emptyset \neq G \subset \mathbb{R}^n \) is an open set, \( 0 \leq s \leq 2 \), and \( 1 < p < \infty \). Let \( \omega \) be an \( A_p \) weight in \( \mathbb{R}^n \) and let \( R : G \to \mathbb{R} \) be a measurable function such that

\[
0 \leq R(x) \leq \text{dist}(x, \partial G)
\]

for every \( x \in G \). Then there exists a constant \( C = C(n) > 0 \) such that, for almost every \((x,y) \in \mathbb{R}^{2n}\), inequality

\[
\omega(x-y)^{1/p}S_R(M_R(f))(x,y)
\]

\[
\leq C \sum_{i,j,k,l,m \in \{0,1\}} \sum_{kl=0} (M_{ij}(\omega_m M_{kl}(Sf))(x,y) + M_{ij}(\omega_m M_{kl}(Sf))(y,x)) \tag{4.20}
\]

holds whenever \( f \in L^p(G) \) satisfies the condition \( \{\omega_0 Sf, \omega_1 Sf\} \subset L^p(\mathbb{R}^{2n}) \).

**Proof.** By replacing the function \( f \) with \(|f|\) we may assume that \( f \geq 0 \). Since \( f \in L^p(G) \) and, hence, \( M_R(f) \in L^p(G) \) we may restrict ourselves to \((x,y) \in G \times G\) for which both \( x \) and \( y \) are Lebesgue points of \( f \) and both \( M_R(f)(x) \) and \( M_R(f)(y) \) are finite. By symmetry, and
changing the weight to $\tilde{\omega}$ if necessary, we may further assume that $M_R(f)(x) > M_R(f)(y)$.
These reductions allow us to find

$$0 \leq r(x) \leq R(x) \quad \text{and} \quad 0 \leq r(y) \leq R(y)$$

such that

$$S_r(M_R(f))(x,y) = \frac{|M_R(f)(x) - M_R(f)(y)|}{(|x-y| + |R(x) - R(y)|)^s} = \frac{|f_B(x,r(x)) - f_B(y,r(y))|}{(|x-y| + |R(x) - R(y)|)^s}.$$  

Moreover, since $M_R(f)(x) > M_R(f)(y)$, we find that inequality

$$S_r(M_R(f))(x,y) \leq \frac{|f_B(x,r(x)) - f_B(y,r(y))|}{(|x-y| + |R(x) - R(y)|)^s}$$  

(4.21)

is valid for any number $0 \leq r_2 \leq R(y)$; this number will be chosen in a convenient manner in the two case studies below.

**Case** $r(x) \leq |x-y| + |R(x) - R(y)|$. Let us denote $r_1 = r(x)$ and choose

$$r_2 = 0.$$  

(4.22)

If $r_1 = 0$, then we get from (4.21) and (4.22)—and our notational convention (1.2)—that

$$\omega_0(x,y)S_r(M_R(f))(x,y) \leq \omega_0(x,y)S(f)(x,y).$$

Suppose then that $r_1 > 0$. Now, by (4.21),

$$\omega_0(x,y)S_r(M_R(f))(x,y) \leq \frac{\omega_0(x,y)}{(|x-y| + |R(x) - R(y)|)^s} \left| \int_{B(x,r_1)} f(z) dz - \int_{B(y,r_2)} f(z) dz \right|$$

$$= \frac{\omega_0(x,y)}{(|x-y| + |R(x) - R(y)|)^s} \left| \int_{B(x,r_1)} (f(z) - f(y)) dz \right|$$

$$\leq \omega_0(x,y) \int_{B(0,r_1)} \frac{\chi_G(x+z)\chi_G(y) |f(x+z) - f(y)|}{|x + z - y|^s} dz \leq \omega_0(x,y) M_{10}(Sf)(x,y).$$

We have shown that, in the case under consideration,

$$\omega_0(x,y)S_r(M_R(f))(x,y) \leq \omega_0(x,y)S(f)(x,y) + \omega_0(x,y) M_{10}(Sf)(x,y).$$

It is clear that inequality (4.20) follows; recall that $M_{00}$ is the identity operator when restricted to non-negative functions.

**Case** $r(x) > |x-y| + |R(x) - R(y)|$. Let us denote $r_1 = r(x) > 0$ and choose

$$0 < r_2 = r(x) - |x-y| - |R(x) - R(y)| \leq R(y).$$

We then have

$$\left| \int_{B(x,r_1)} f(z) dz - \int_{B(y,r_2)} f(z) dz \right| = \left| \int_{B(0,r_1)} \left( f(x+z) - f(y + \frac{r_2}{r_1} z) \right) dz \right|$$

$$= \left| \int_{B(0,r_1)} \left( f(x+z) - \int_{B(y+\frac{r_2}{r_1} z,|x-y|+|R(x) - R(y)|)} f(a) da \right) dz \right|$$

$$+ \left| \int_{B(y+\frac{r_2}{r_1} z,|x-y|+|R(x) - R(y)|)} f(a) da - f(y + \frac{r_2}{r_1} z) \right| dz \right|$$

$$\leq E_1 + E_2,$$
where we have written
\[
E_1 = \int_{B(0,r_1)} \left( \int_{B(y+\frac{r_2}{r_1}z,[x-y]+|R(x)-R(y)|)^G} |f(x+z) - f(a)| \, da \right) \, dz,
\]
\[
E_2 = \int_{B(0,r_1)} \left( \int_{B(y+\frac{r_2}{r_1}z,[x-y]+|R(x)-R(y)|)^G} |f(y+\frac{r_2}{r_1}z) - f(a)| \, da \right) \, dz.
\]

We estimate both of these terms separately, but first we need certain auxiliary estimates.

Recall that \( r_2 = r_1 - |x-y| - |R(x)-R(y)| \). Hence, for every \( z \in B(0,r_1) \),
\[
|y+\frac{r_2}{r_1}z - (x+z)| = |y-x + \frac{(r_2-r_1)}{r_1}z| \leq 2|x-y| + |R(x)-R(y)|.
\]
This, in turn, implies that
\[
B(y+\frac{r_2}{r_1}z,[x-y]+|R(x)-R(y)|) \subset B(x+z,3|x-y|+2|R(x)-R(y)|) \tag{4.23}
\]
if \( z \in B(0,r_1) \). Since \( r_1 > |x-y|+|R(x)-R(y)| \) and \( \{y+\frac{r_2}{r_1}z,x+z\} \subset B(x,r_1) \subset G \) if \( |z| < r_1 \), we obtain the two equivalences
\[
|B(y+\frac{r_2}{r_1}z,[x-y]+|R(x)-R(y)|) \cap G| \simeq (|x-y|+|R(x)-R(y)|)\varepsilon
\]
\[
\simeq |B(x+z,3|x-y|+2|R(x)-R(y)|) \cap G| \tag{4.24}
\]
for every \( z \in B(0,r_1) \). Here the implied constants depend only on \( n \).

**An estimate for** \( E_1 \). The inclusion \( 4.23 \) and equivalences \( 4.24 \) show that, in the definition of \( E_1 \), we can replace the set over which the inner integral its taken by the set
\[
B(x+z,3|x-y|+2|R(x)-R(y)|) \cap G
\]
and, at the same time, control the error term while integrating on average. That is,
\[
E_1 \lesssim \int_{B(0,r_1)} \left( \int_{B(x+z,3|x-y|+2|R(x)-R(y)|)^G} |f(x+z) - f(a)| \, da \right) \, dz.
\]
By observing that \( x+z \) and \( a \) in the last double integral belong to \( G \), and using \( 4.24 \) again, we can continue as follows:
\[
\int_{B(0,r_1)} \left( \omega_0(x,y) \int_{B(x+z,3|x-y|+2|R(x)-R(y)|)^G} \frac{\chi_G(x+z)\chi_G(a)|f(x+z) - f(a)|}{|x+z-a|^s} \, da \right) \, dz
\]
\[
\lesssim \int_{B(0,r_1)} \left( \omega_0(x,y) \int_{B(y+z,4|x-y|+2|R(x)-R(y)|)^G} S(f)(x+z,a) \, da \right) \, dz.
\]
Since \( \omega_0(x,y) = \omega_0(x+z,y+z) \), we may apply the maximal operators defined in \( 12 \) in order to find that
\[
\int_{B(0,r_1)} \frac{\omega_0(x+z,y+z)M_{01}(Sf)(x+z,y+z)}{|x-y|+|R(x)-R(y)|} \, dz \leq M_{11}(\omega_0M_{01}(Sf))(x,y). \tag{4.25}
\]
An estimate for $E_2$. We use the inclusion $y + \frac{r_2}{r_1} z \in G$ for all $z \in B(0, r_1)$ and then apply the first equivalence in (4.21) to obtain

$$E_2 = \int_{B(0, r_1)} \left( \int_{B(y + \frac{r_2}{r_1} z, |x-y| + |R(x)-R(y)| \cap G} \chi_G(y + \frac{r_2}{r_1} z) \chi_G(a) |f(y + \frac{r_2}{r_1} z) - f(a)| \, da \right) \, dz \leq \int_{B(0, r_1)} \left( \int_{B(y + \frac{r_2}{r_1} z, |x-y| + |R(x)-R(y)|)} \chi_G(y + \frac{r_2}{r_1} z) \chi_G(a) |f(y + \frac{r_2}{r_1} z) - f(a)| \, da \right) \, dz.$$  

Hence, a change of variables yields

$$E_2 \omega_0(x, y) \left( \frac{1}{|x-y| + |R(x)-R(y)|} \right)^s \leq \int_{B(0, r_1)} \omega_0(x, y) \left( \int_{B(y + \frac{r_2}{r_1} z, |x-y| + |R(x)-R(y)|)} \chi_G(y + \frac{r_2}{r_1} z) \chi_G(a) |f(y + \frac{r_2}{r_1} z) - f(a)| \, da \right) \, dz \leq \int_{B(0, r_1)} \omega_0(x, y) \left( \int_{B(x + z, 2|x-y| + |R(x)-R(y)|)} S(f)(y + z, a) \, da \right) \, dz.$$  

Let us observe that $\omega_0(x, y) = \omega_1(y + z, x + z)$. Hence, by applying operators $M_{01}$ and $M_{11}$ from (2) we can proceed as follows

$$E_2 \omega_0(x, y) \left( \frac{1}{|x-y| + |R(x)-R(y)|} \right)^s \leq \int_{B(0, r_1)} \omega_1(y + z, x + z) M_{01}(Sf)(y + z, x + z) \, dz \leq M_{11}(\omega_1 M_{01}(Sf))(y, x).$$  

By combining the estimates (4.25) and (4.26), we obtain that

$$\omega_0(x, y) S_R(M_R(f))(x, y) \leq \frac{(E_1 + E_2) \omega_0(x, y)}{\left( \frac{1}{|x-y| + |R(x)-R(y)|} \right)^s} \leq M_{11}(\omega_0 M_{01}(Sf))(x, y) + M_{11}(\omega_1 M_{01}(Sf))(y, x),$$

where the implied constant depends only on $n$. As a consequence, inequality (4.20) follows also in the second case $r(x) > |x-y| + |R(x)-R(y)|$ that is now under our consideration.  

We are finally ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $f \in L^p(G)$. We may assume that the double integral on the right hand side of (4.17) is finite, and therefore $\omega_m S f \in L^p(\mathbb{R}^n)$ if $m \in \{0, 1\}$. Observe that $\omega_1(x, y)^p = \tilde{\omega}(x, y)$, where $\tilde{\omega}(z) = \omega(-z)$ is also an $A_p$ weight such that $[\tilde{\omega}]_{A_p} = [\omega]_{A_p}$. Hence, inequality (4.17) is a consequence of Proposition 4.2, the boundedness of operators $M_{ij}$ on $L^p(\mathbb{R}^n)$, and Proposition 4.2 applied with the two $A_p$ weights $\omega$ and $\tilde{\omega}$.  

5. Powers of distance as weights

In this section we apply Theorem 4.1 with $\omega = \text{dist}(-, E)^{\varepsilon-n}$, where the set $E \subset \mathbb{R}^n$ and $\varepsilon > 0$ are chosen such that $\omega$ is an $A_p$ weight in $\mathbb{R}^n$; we refer to Theorem 5.2. The important special case $E = \{0\}$ and $\omega = |\cdot|^{\varepsilon-n}$ yields boundedness results for the operators $f \mapsto M_R(f)$ between fractional Sobolev spaces. These results with Lipschitz and Hölder functions $R$ are formulated in Corollaries 5.5 and 5.6 respectively. The sharpness of Corollary 5.5 in terms of the Hölder exponent is considered in Lemma 5.7. Furthermore, this lemma shows that Theorem 4.1, i.e., our main result, is essentially sharp in its generality.

The following proposition can be found in [20] (see also [32] and [16, Lemma 2.2]). The straightforward proof relies on a characterization of the Assouad dimension in terms of the so-called Aikawa dimension, we refer to [31].
Proposition 5.1. Let $E \subset \mathbb{R}^n$ be a (non-empty) closed set and let $\omega = \text{dist}(\cdot, E)^{\epsilon-n}$ for a fixed $\epsilon > 0$. Then the following statements are true.

(A) If $\overline{\text{dim}}(E) < \epsilon \leq n$, then $\omega$ is an $A_p$ weight in $\mathbb{R}^n$ for all $1 < p < \infty$.

(B) If $\epsilon > n$ and $1 < p < \infty$ are such that

$$\overline{\text{dim}}(E) < n - \frac{\epsilon - n}{p - 1},$$


then $\omega$ is an $A_p$ weight in $\mathbb{R}^n$.

The following result illustrates the flexibility of our main result, and it is an immediate consequence of Theorem 4.1 and Proposition 5.1.

Theorem 5.2. Assume that $\emptyset \neq G \subset \mathbb{R}^n$ is an open set, $0 \leq s \leq 2$, and $1 < p < \infty$. Let $\epsilon > 0$ and $E \neq \emptyset$ be a closed set in $\mathbb{R}^n$ such that

$$\overline{\text{dim}}(E) < \epsilon < n + (n - \overline{\text{dim}}(E))(p - 1).$$

Fix a measurable function $R : G \to \mathbb{R}$ satisfying inequality $0 \leq R(x) \leq \text{dist}(x, \partial G)$ for every $x \in G$. Then there exists a constant $C = C(n, p, \epsilon, E) > 0$ such that inequality

$$\int_G \int_G \frac{|M_R(f)(x) - M_R(f)(y)|^p}{(x - y) + |R(x) - R(y)|)^{n-p}} \frac{dy \, dx}{\text{dist}(x, \partial G)^{n-\epsilon}} \leq C \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{p}} \frac{dy \, dx}{\text{dist}(x, \partial G)^{n-\epsilon}}$$

holds for every $f \in L^p(G)$.

Next we turn to an important special case, where $E = \{0\}$ and $\omega = \text{dist}(\cdot, E)^{\epsilon-n} = |\cdot|^{\epsilon-n}$. The following convenient result is a reformulation of Theorem 1.1 for the definition of the seminorm appearing in the right-hand side of (5.27), we refer to Example 3.2.

Proposition 5.3. Let $\emptyset \neq G \subset \mathbb{R}^n$ be an open set, $0 < \epsilon, s < 1$ and $1 < p < \infty$. Fix a measurable function $R : G \to \mathbb{R}$ satisfying inequality $0 \leq R(x) \leq \text{dist}(x, \partial G)$ for every $x \in G$. Then there exists a constant $C = C(n, p, \epsilon) > 0$ such that inequality

$$\int_G \int_G \frac{|M_R(f)(x) - M_R(f)(y)|^p}{(x - y) + |R(x) - R(y)|)^{s+\epsilon}} \frac{dy \, dx}{|x - y|^{s+\epsilon}} \leq C \|f\|_{W^{s,p}(G)}$$

(5.27)

holds for every $f \in L^p(G)$.

Proof. Since $0 < \epsilon < 1$, we find that the function $|\cdot|^{\epsilon-n}$ is an $A_p$ weight; see [13, p. 236] or Proposition 5.1(A). Moreover, the $A_p$ constant of this weight depends only on $n$, $p$ and $\epsilon$. Observe also that $\epsilon/p + s < 2$. Hence, inequality (5.27) follows from Theorem 4.1.

Remark 5.4. Observe that Proposition 5.3 is related to the case $E = \{0\}$ of Theorem 5.2. Indeed, we have that $\overline{\text{dim}}(\{0\}) = 0$.

The following boundedness result, which applies for Lipschitz $R$-functions, is a corollary of Proposition 5.3. Let us fix $L \geq 0$ and recall that $R$ is an $L$-Lipschitz function on $G$ if $|R(x) - R(y)| \leq L|x - y|$ whenever $x, y \in G$.

Corollary 5.5. Let $\emptyset \neq G \subset \mathbb{R}^n$ be an open set, $0 < \epsilon, s < 1$ and $1 < p < \infty$. Fix $L \geq 0$ and an $L$-Lipschitz function $R : G \to \mathbb{R}$ satisfying inequality $0 \leq R(x) \leq \text{dist}(x, \partial G)$ for every $x \in G$. Then there exists a constant $C = C(n, p, \epsilon) > 0$ such that inequality

$$|M_R(f)|_{W^{s,p}(G)} \leq C(1 + L)^{s/p + s} \|f\|_{W^{s,p}(G)}$$

holds for every function $f \in L^p(G)$. 

The case of Hölder functions $R$ is addressed in Corollary 5.6 below. Let us recall that a function $R$ is $\alpha$-Hölder on $G$ (for a given $0 < \alpha < 1$) if there exists $L \geq 0$ such that inequality $|R(x) - R(y)| \leq L|x - y|^\alpha$ holds whenever $x, y \in G$.

**Corollary 5.6.** Let $\emptyset \neq G \subset \mathbb{R}^n$ be a bounded open set, $0 < s, \alpha < 1$ and $1 < p < \infty$. Fix an $\alpha$-Hölder function $R$ on $G$ such that $0 \leq R(x) \leq \text{dist}(x, \partial G)$ for every $x \in G$. Then, if $0 < \sigma < \alpha s$, there exists a constant $C = C(\sigma, s, \alpha, n, p, L, \text{diam}(G)) > 0$ such that inequality

$$|M_R(f)|_{W^{s,p}(G)} \leq C|f|_{W^{s,p}(G)} \quad (5.28)$$

holds for every $f \in L^p(G)$.

We omit the proof of Corollary 5.6 that is quite a straightforward but tedious reduction to Proposition 5.3; it is worthwhile to emphasize that the open set $G$ is assumed to be bounded. Hence, the case when $\sigma$ is close to $\alpha s$ is a difficult one to establish.

It is unknown to the authors, whether inequality (5.28) holds also when $\sigma = \alpha s$ is the endpoint. However, the following Lemma 5.7 shows that Corollary 5.6 is essentially sharp, in that we cannot allow $\sigma > \alpha s$ in general.

**Lemma 5.7.** Fix $\alpha = 1/M$ for any number $M \in \{2, 3, \ldots\}$. Let $0 < s < 1$ and $1 < p < \infty$ be such that $\alpha s p \geq 1$. Then there exists a bounded open set $G$ in $\mathbb{R}$ and an $\alpha$-Hölder function $R : G \to \mathbb{R}$ which satisfies the inequality $0 \leq R(x) \leq \text{dist}(x, \partial G)$ whenever $x \in G$ and which has the following property: for any given $\sigma \in (\alpha s, 1)$ there does not exist a constant $C > 0$ such that

$$|M_R(f)|_{W^{s,p}(G)} \leq C|f|_{W^{s,p}(G)}$$

for all functions $f \in L^p(G)$.

**Proof.** Let us fix $\alpha s < \sigma < 1$ and first sketch the proof that relies on a fractional $(\sigma, p)$-Hardy inequality: there exists a constant $C(p, \sigma) > 0$ such that

$$\int_I \int_I \frac{|f(x) - f(y)|^p}{|x - y|^{1 + \sigma p}} \, dy \, dx \geq C(p, \sigma) \int_I \frac{|f(x)|^p}{\text{dist}(x, \partial I)^{\sigma p}} \, dx \quad (5.29)$$

whenever $f \in L^p(I)$ is compactly supported in an open interval $I \subset \mathbb{R}$, [34, Corollary 2.7]. Actually, this corollary is formulated only for $C^\infty_0(I)$ functions, and therefore approximation by such functions is required. This can easily be done by a straightforward combination of Lemma 3.4 and [17, Lemma 4.4]; we omit the details.

We take $G = (-8, 9)$ and construct $R$ and test functions $\psi_N$ ($N \geq 1$) that are supported in an interval $I_N \subset G$ such that $M_R(\psi_N)$ has a compactly supported bump in many dyadic subintervals $I_{N,j} \subset I_N$. Hence, the fractional $(\sigma, p)$-Hardy inequality applies to the restriction of $M_R(\psi_N)$ in each of the subintervals. The resulting estimates, when combined with an upper bound for $|\psi_N|_{W^{s,p}(G)}$, will yield that $|M_R(\psi_N)|_{W^{s,p}(G)}/|\psi_N|_{W^{s,p}(G)} \to \infty$ as $N \to \infty$.

Let us now turn to the details. We set

$$E = G \setminus \left( \bigcup_{N \in \mathbb{N}} \bigcup_{j=1}^{2^{(M-1)N}} I_{N,j} \right),$$

where

$$I_{N,j} = (2^{-N} + (j - 1)2^{-MN}, 2^{-N} + j2^{-MN}), \quad j = 1, \ldots, 2^{(M-1)N}.$$ 

Define an $\alpha$-Hölder function $R = 2^{2^{M+1}} \text{dist}(\cdot, E)^\alpha$. It is now straightforward to check that inequality $0 \leq R(x) \leq \text{dist}(x, \partial G)$ holds for every $x \in G$.

Let $\psi \in C^\infty_0((0, 1))$ be such that $\int_{\mathbb{R}} |\psi| \, dx = 4$. Fix $N \in \mathbb{N}$ and write $\psi_N(x) = \psi(2^N x)$ if $x \in \mathbb{R}$. Now $\psi_N$ is supported in $(0, 2^{-N})$ and, by a change of variables, we find that

$$|\psi_N|_{W^{s,p}(G)}^p \leq |\psi_N|_{W^{s,p}(\mathbb{R})}^p = 2^{N(s^{p-1})}|\psi|_{W^{s,p}(\mathbb{R})}^p. \quad (5.30)$$
Next we turn to establishing a lower bound for $|M_R(\psi_N)|_{W^{s,p}(G)}$. Let us fix $j = 1, \ldots, 2^{(M-1)N}$ and $x \in \tilde{I}_{N,j} = 2^{-1}I_{N,j}$. Since $(0,2^{-N}) \subset B(x,2^{-N+1})$ and $2^{-N+1} \leq R(x)$, we obtain that

$$M_R(\psi_N)(x) \geq \int_{B(x,2^{-N+1})} |\psi_N(y)| \, dy \geq 2^{-N+2}/2^{N+2} = 1, \quad x \in \tilde{I}_{N,j}. \quad (5.31)$$

Moreover, the restriction $M_R(\psi_N)|_{I_{N,j}} \in L^p(I_{N,j})$ has a compact support in $I_{N,j}$. Hence, by the fractional $(\sigma,p)$-Hardy inequality \cite{[5,29]} followed by inequality \eqref{5.31},

$$|M_R(\psi_N)|_{W^{s,p}(G)}^p \geq \sum_{j=1}^{2^{(M-1)N}} \int_{I_{N,j}} \int_{I_{N,j}} \frac{|M_R(\psi_N)(x) - M_R(\psi_N)(y)|^p}{|x-y|^{1+\sigma p}} \, dy \, dx$$

$$\geq C(p,\sigma) \sum_{j=1}^{2^{(M-1)N}} \int_{I_{N,j}} \frac{|M_R(\psi_N)(x)|^p}{\text{dist}(x, \partial I_{N,j})^{\sigma p}} \, dx$$

$$\geq C(p,\sigma) \sum_{j=1}^{2^{(M-1)N}} |\tilde{I}_{N,j}|^{-\sigma p} \geq C(p,\sigma) \sum_{j=1}^{2^{(M-1)N}} (2^{-MN})^{-\sigma p} = C(p,\sigma)2^{N(\sigma pM-1)}.$$

By combining the estimate above with \eqref{5.30}, we obtain that

$$\frac{|M_R(\psi_N)|_{W^{s,p}(G)}^p}{|\psi_N|_{W^{s,p}(G)}^p} \geq C(p,\sigma)||\psi||_{W^{s,p,0}(\mathbb{R}^n)}^{2pN(\sigma M-1)} = C(p,\sigma)||\psi||_{W^{s,p,0}(\mathbb{R}^n)}^{2pN(\sigma/\alpha - s)}.$$

Since $\sigma > \alpha s$, the lower bound above tends to infinity as $N \to \infty$. \hfill \Box

### 6. Sobolev capacity and Lebesgue differentiation

We apply our main result by studying the Lebesgue differentiation of a Sobolev function $f \in W^{s,p,\omega}(\mathbb{R}^n)$ outside a set of zero Sobolev capacity, see Definition \ref{def-6.1}. The outline of our treatment is based on the work \cite{[24]} of Kinnunen–Latvala, who obtain Lebesgue point results for (first order) Sobolev functions on metric spaces. We adapt their treatment to the present setting when $\omega$ is an $A_p$ weight that is subject to the condition $C_0^\infty(\mathbb{R}^n) \subset W^{s,p,\omega}(\mathbb{R}^n)$. Hence, the key ingredients for the proof of our Theorem \ref{thm-6.2} are: the density property of continuous functions (Lemma \ref{lem-5.3}) and the boundedness of (an appropriate) maximal operator, both in $W^{s,p,\omega}(\mathbb{R}^n)$; the boundedness property follows from our main result, i.e., Theorem \ref{thm-4.1}.

**Definition 6.1.** Suppose that $0 < s < 1$ and $1 \leq p < \infty$. Let $\omega$ be a weight in $\mathbb{R}^n$. For a set $E \subset \mathbb{R}^n$ we define its Sobolev capacity

$$C_{s,p,\omega}(E) = \inf_{\varphi \in \mathcal{A}(E)} ||\varphi||_{W^{s,p,\omega}(\mathbb{R}^n)}^p,$$

where the infimum is taken over all admissible functions

$$\mathcal{A}(E) = \{ \varphi \in W^{s,p,\omega}(\mathbb{R}^n) : \varphi \geq 1 \text{ in an open set containing } E \}.$$

If $\mathcal{A}(E) = \emptyset$, we set $C_{s,p,\omega}(E) = \infty$.

The unweighted fractional Sobolev capacity (corresponding to $W^{s,p}(\mathbb{R}^n) = W^{s+p/\omega}(\mathbb{R}^n)$ with $\omega = |\cdot|^{-n}$) is well known and extensively studied, see, e.g., \cite{[11,10,46]}. Let us remark that $W^{s,p}(\mathbb{R}^n)$ coincides with the Besov space $B_{p,0}^s(\mathbb{R}^n)$ and their norms are comparable, if $1 < p < \infty$ and $0 < s < 1$; we refer to \cite{[14]} pp. 6–7.

We prove the following result that is concerned with the Lebesgue differentiation and a quasicontinuous representative $f^*$ of a function $f$ in $W^{s,p,\omega}(\mathbb{R}^n)$.
Theorem 6.2. Suppose that $0 < s < 1$ and $1 < p < \infty$. Let $\omega$ be an $A_p$ weight in $\mathbb{R}^n$ such that $C_0^\infty(\mathbb{R}^n)$ is a subset of $W^{s,p,\omega}(\mathbb{R}^n)$. Then, for every $f \in W^{s,p,\omega}(\mathbb{R}^n)$, there is a $G_\delta$-set $E \subset \mathbb{R}^n$ such that $C_{s,p,\omega}(E) = 0$ and the limit
\[
\lim_{r \to 0^+} \int_{B(x,r)} f(y) \, dy = f^*(x)
\]
exists for every $x \in \mathbb{R}^n \setminus E$. Moreover, for every $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ such that $C_{s,p,\omega}(U) < \varepsilon$ and $f^*|_{\mathbb{R}^n \setminus U}$ is well-defined and continuous on $\mathbb{R}^n \setminus U$.

An analogue of Theorem 6.2 for the first order $A_p$ weighted Sobolev spaces is known, see \[22\] \[45\]. The unweighted case $W^{s,p}(\mathbb{R}^n) = W^{s+\varepsilon/p,p,\omega}(\mathbb{R}^n)$ with $\omega = |\cdot|^{-n}$ of Theorem 6.2 is also known, we refer to \[10\] or \[2, \S 6\] when $p = 2$. The local aspects of quasicontinuity (in the unweighted case) have been studied in \[16\] Theorem 3.7; however, the Lebesgue differentiation is not explicitly considered therein.

If all singletons have a positive Sobolev capacity, then $f^* : \mathbb{R}^n \to \mathbb{R}$ is continuous. This is a corollary of Theorem 6.2 and the translation invariance of $\|\cdot\|_{W^{s,p,\omega}(\mathbb{R}^n)}$.

Corollary 6.3. Let $0 < s < 1$ and $1 < p < \infty$. Let $\omega$ be an $A_p$ weight in $\mathbb{R}^n$ such that $C_0^\infty(\mathbb{R}^n)$ is a subset of $W^{s,p,\omega}(\mathbb{R}^n)$ and
\[
C_{s,p,\omega}(\{x\}) > 0
\]
for every $x \in \mathbb{R}^n$ (or, equivalently, for some $x \in \mathbb{R}^n$). Then every function $f \in W^{s,p,\omega}(\mathbb{R}^n)$ has a continuous representative. That is, the function $f^* : \mathbb{R}^n \to \mathbb{R}$ defined by (6.32) is continuous and satisfies $f = f^*$ pointwise almost everywhere in $\mathbb{R}^n$.

The proof of Theorem 6.2 is given in the end of this section; first we state and prove several auxiliary results. A key result among these is the following capacitary weak type estimate, which is a counterpart of \[24\] Lemma 4.4. For every $f \in L^p(\mathbb{R}^n)$, we define
\[
\widehat{M} f(x) = \sup_{0 < r \leq 1} \frac{1}{B(x,r)} \int_{B(x,r)} |f(y)| \, dy, \quad x \in \mathbb{R}^n.
\]
Write $R(x) = 1$ whenever $x \in \mathbb{R}^n$. Then $\widehat{M} f(x) = M_R(f)(x)$ in the Lebesgue points $x \in \mathbb{R}^n$ of $|f|$, that is, almost everywhere. Moreover, we clearly have that $\widehat{M} f \leq M f$, where $M$ is the Hardy–Littlewood maximal operator.

Lemma 6.4. Let $0 < s < 1$ and $1 < p < \infty$, and let $\omega$ be an $A_p$ weight in $\mathbb{R}^n$. Suppose that $f \in W^{s,p,\omega}(\mathbb{R}^n)$. Then, for every $\lambda > 0$, we have
\[
C_{s,p,\omega}(\{x \in \mathbb{R}^n : \widehat{M} f(x) > \lambda\}) \leq C \lambda^{-p} \|f\|_{W^{s,p,\omega}(\mathbb{R}^n)}^p,
\]
where $C = C(n, p, [\omega]_{A_p})$.

Proof. Fix $f \in W^{s,p,\omega}(\mathbb{R}^n)$ and $\lambda > 0$. If $0 < r \leq 1$, then the function
\[
x \mapsto \int_{B(x,r)} |f(y)| \, dy
\]
is continuous in $\mathbb{R}^n$ by the dominated convergence theorem. As a consequence, the function $\widehat{M} f(x)$ is lower semicontinuous in $\mathbb{R}^n$. Hence, $E_\lambda = \{x \in \mathbb{R}^n : \widehat{M} f(x) > \lambda\}$ is an open set in $\mathbb{R}^n$ and $\lambda^{-1} \widehat{M} f(x) \geq 1$ holds if $x \in E_\lambda$. Theorem 4.1 and the boundedness of the Hardy–Littlewood maximal operator in $L^p(\mathbb{R}^n)$ imply that
\[
C_{s,p,\omega}(E_\lambda) \leq \|\lambda^{-1} \widehat{M} f\|_{W^{s,p,\omega}(\mathbb{R}^n)}^p \leq C(n, p, [\omega]_{A_p}) \lambda^{-p} \|f\|_{W^{s,p,\omega}(\mathbb{R}^n)}^p.
\]
This concludes the proof. \[\square\]
The following lemma is an adaptation of [26, Theorem 3.2].

**Lemma 6.5.** Let $0 < s < 1$ and $1 < p < \infty$, and let $\omega$ be a weight in $\mathbb{R}^n$. Then $C_{s,p,\omega}$ is an outer measure on $\mathbb{R}^n$.

**Proof.** By definition, $C_{s,p,\omega}(\emptyset) = 0$ and $C_{s,p,\omega}$ is monotone, that is, $C_{s,p,\omega}(E) \leq C_{s,p,\omega}(F)$ whenever $E \subset F$. To prove subadditivity, we suppose that $E_i, 1 = 1, 2, \ldots$, are subsets of $\mathbb{R}^n$. We need to establish the inequality

$$C_{s,p,\omega} \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} C_{s,p,\omega}(E_i). \quad (6.33)$$

We may clearly assume that $\sum_{i=1}^{\infty} C_{s,p,\omega}(E_i) < \infty$. Let us fix $\varepsilon > 0$. For every $i = 1, 2, \ldots$ it holds that $A(E_i) \neq \emptyset$ and, therefore, we can choose $\varphi_i \in A(E_i)$ such that

$$\|\varphi_i\|_{W^{s,p,\omega}(\mathbb{R}^n)} \leq C_{s,p,\omega}(E_i) + \varepsilon 2^{-i}. \quad (6.34)$$

By replacing each function $\varphi_i$ with $\min\{1, |\varphi_i|\}$ we may assume that $0 \leq \varphi_i \leq 1$ everywhere and $\varphi_i = 1$ in an open set containing $E_i$.

Define $\varphi = \sup_i \varphi_i$. Then $\varphi = 1$ in an open set containing $\bigcup_{i=1}^{\infty} E_i$. Let us fix $x, y \in \mathbb{R}^n$.

If $\varphi(x) \geq \varphi(y)$ then, for every $\delta > 0$, there is $j = j(\delta, x) \in \mathbb{N}$ such that

$$|\varphi(x) - \varphi(y)| = \varphi(x) - \varphi(y) \leq \varphi_j(x) + \delta - \varphi(y) \leq \varphi_j(x) + \delta - \varphi_j(y) \leq \delta + |\varphi_j(x) - \varphi_j(y)| \leq \delta + \left( \sum_{i=1}^{\infty} |\varphi_i(x) - \varphi_i(y)|^p \right)^{1/p}.$$ 

Taking $\delta \to 0$ we obtain that

$$|\varphi(x) - \varphi(y)| \leq \left( \sum_{i=1}^{\infty} |\varphi_i(x) - \varphi_i(y)|^p \right)^{1/p}. \quad (6.35)$$

By repeating the previous argument if $\varphi(x) < \varphi(y)$, with the obvious changes, we find that inequality (6.35) holds for all $x, y \in \mathbb{R}^n$. Therefore,

$$|\varphi|_{W^{s,p,\omega}(\mathbb{R}^n)}^p \leq \sum_{i=1}^{\infty} |\varphi_i|_{W^{s,p,\omega}(\mathbb{R}^n)}^p. \quad (6.36)$$

Clearly, we also have

$$\|\varphi\|_{L^p(\mathbb{R}^n)}^p \leq \sum_{i=1}^{\infty} \|\varphi_i\|_{L^p(\mathbb{R}^n)}^p. \quad (6.37)$$

By first combining inequalities (6.36) and (6.37), and then using (6.34), we obtain that

$$\|\varphi\|_{W^{s,p,\omega}(\mathbb{R}^n)}^p \leq \sum_{i=1}^{\infty} \|\varphi_i\|_{W^{s,p,\omega}(\mathbb{R}^n)}^p \leq \sum_{i=1}^{\infty} \left( C_{s,p,\omega}(E_i) + \varepsilon 2^{-i} \right) = \varepsilon + \sum_{i=1}^{\infty} C_{s,p,\omega}(E_i).$$

Hence, $C_{s,p,\omega}(\bigcup_{i=1}^{\infty} E_i) \leq \varepsilon + \sum_{i=1}^{\infty} C_{s,p,\omega}(E_i)$. Inequality (6.33) follows by taking $\varepsilon \to 0$. \[\square\]

**Proof of Theorem 6.2.** We follow the proof of [24, Theorem 4.5] very closely; the details are provided for completeness and convenience of the reader.

Fix a function $f \in W^{s,p,\omega}(\mathbb{R}^n)$ and $i \in \mathbb{N}$. By the assumptions and Lemma 3.4 we may choose $f_i \in C(\mathbb{R}^n) \cap W^{s,p,\omega}(\mathbb{R}^n)$ such that

$$\|f - f_i\|_{W^{s,p,\omega}(\mathbb{R}^n)}^p \leq 2^{-i(p+1)}. \quad (6.38)$$

Denote $A_i = \{x \in \mathbb{R}^n : \hat{M}(f - f_i)(x) > 2^{-i}\}$. Lemma 3.4 implies that

$$C_{s,p,\omega}(A_i) \leq C2^{ip}\|f - f_i\|_{W^{s,p,\omega}(\mathbb{R}^n)}^p \leq C2^{-i},$$

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where \( C = C(n, p, [\omega]_{A^p}) \). Now (say) for every \( x \in \mathbb{R}^n \) and \( 0 < r \leq 1 \),
\[
|f_i(x) - f_{B(x,r)}| \leq \int_{B(x,r)} |f_i(x) - f_i(y)| \, dy + \int_{B(x,r)} |f(y) - f_i(y)| \, dy
\]
which (by the continuity of \( f_i \)) implies that
\[
\limsup_{r \to 0} |f_i(x) - f_{B(x,r)}| \leq \tilde{M}(f - f_i)(x) \leq 2^{-i}, \quad x \in \mathbb{R}^n \setminus A_i.
\]

Let us fix \( k \in \mathbb{N} \) and write \( B_k = \bigcup_{i=k} \mathbb{N} A_i \). An application of both subadditivity of the Sobolev capacity, given by Lemma 6.5 and inequality (6.38) yields
\[
C_{s,p,\omega}(B_k) \leq \sum_{i=k}^{\infty} C_{s,p,\omega}(A_i) \leq C \sum_{i=k}^{\infty} 2^{-i}.
\]
(6.39)

If \( x \in \mathbb{R}^n \setminus B_k \) and \( i, j \geq k \), then
\[
|f_i(x) - f_j(x)| \leq \limsup_{r \to 0} |f_i(x) - f_{B(x,r)}| + \limsup_{r \to 0} |f_j(x) - f_{B(x,r)}| \leq 2^{-i} + 2^{-j}.
\]
It follows that \( \{ f_i \}_{i \in \mathbb{N}} \) converges uniformly in \( \mathbb{R}^n \setminus B_k \) to a continuous function \( g_k \) on \( \mathbb{R}^n \setminus B_k \). Moreover, if \( x \in \mathbb{R}^n \setminus B_k \) and \( i \geq k \), we have
\[
\limsup_{r \to 0} |g_k(x) - f_{B(x,r)}| \leq |g_k(x) - f_i(x)| + \limsup_{r \to 0} |f_i(x) - f_{B(x,r)}| \leq |g_k(x) - f_i(x)| + 2^{-i}.
\]
Hence, by taking \( i \to \infty \), we obtain that
\[
g_k(x) = \lim_{r \to 0} \int_{B(x,r)} f(y) \, dy = f^*(x)
\]
for every \( x \in \mathbb{R}^n \setminus B_k \).

Let us define \( E = \bigcap_{k=1}^{\infty} B_k \). Then, by monotonicity of the Sobolev capacity and (6.39),
\[
C_{s,p,\omega}(E) \leq \lim_{k \to \infty} C_{s,p,\omega}(B_k) = 0
\]
and the limit
\[
\lim_{r \to 0} \int_{B(x,r)} f(y) \, dy = f^*(x)
\]
does exist for every \( x \in \mathbb{R}^n \setminus E \). Finally, we fix \( \varepsilon > 0 \) and choose \( k \) large enough so that \( C_{s,p,\omega}(B_k) \leq \varepsilon \). By arguing as in the proof of Lemma 6.4, we find that \( B_k = \bigcup_{i=k}^{\infty} A_i \) is a union of open sets in \( \mathbb{R}^n \), hence \( B_k \) is open. Since \( f^* = g_k \) in \( \mathbb{R}^n \setminus B_k \), we find that \( f^* \) is continuous on \( \mathbb{R}^n \setminus B_k \). Accordingly, we can choose \( U = B_k \). \( \square \)

7. Comparison of neighbourhood capacities

As another application of our main result, Theorem 4.1, we prove a capacitary comparison inequality that is formulated as Theorem 7.3 below; this inequality extends the work [30] of Lehrbäck. To briefly explain our inequality, let us fix a compact \( \kappa \)-porous set \( E \) (see [2]) that is contained in a bounded open set \( G \subset \mathbb{R}^n \). We write
\[
E_{t,R} = \{ x \in G : R(x) < t \}, \quad t > 0,
\]
where \( R : G \to \mathbb{R} \) is a continuous function such that \( R = 0 \) on \( E \). Hence, the set \( E_{t,R} \) is an open neighborhood of \( E \) in \( G \). We focus on small values of \( t > 0 \) and the underlying open set \( G \) serves for the purpose of an ‘ambient space’. In particular, the structure of \( E_{t,R} \) near
to the boundary \( \partial G \) will be irrelevant to us. Our ‘frame of reference’ in comparison is the \( t \)-neighbourhood

\[
E_t = E_t,\text{dist}(.,E) = \{x \in G : \text{dist}(x, E) < t\}, \quad t > 0.
\]

Namely, our capacitary comparison inequality is that

\[
cap_{s,p,\omega,R}(E, E_t \cap E_{4t/s, R}) \leq C \cap_{s,p,\omega}(E, E_t, G)
\]

for all small \( t > 0 \) with a constant \( C = \kappa^{-np} C(n, p, [\omega]_{A_p}) > 0 \); this is inequality (7.40). Observe that an \( R \)-modified relative capacity is used in the left-hand side above, whereas an relative \( (s, p, \omega) \)-capacity is used in the right-hand side; these are defined as follows.

**Definition 7.1.** Let \( 0 < s < 1 \) and \( 1 < p < \infty \), and let \( \omega \) be a weight in \( \mathbb{R}^n \). Suppose that \( G \subset \mathbb{R}^n \) is an open set and \( R : G \to \mathbb{R} \) is a measurable function. Let \( E \subset \mathbb{R}^n \) be a compact set that is contained in an open set \( H \subset G \). Then we write

\[
\cap_{s,p,\omega,R}(E, H, G) = \inf_{\varphi} \int_G \int_G \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{n+sp}} \omega(x-y) \, dy \, dx,
\]

where the infimum is taken over all real-valued functions \( \varphi \in C_0(G) \) such that \( \varphi(x) \geq 1 \) for every \( x \in E \) and \( \varphi(x) = 0 \) for every \( x \in G \setminus H \). If \( R(x) = 0 \) for every \( x \in G \), then we abbreviate \( \cap_{s,p,\omega,R}(E, H, G) = \cap_{s,p,\omega}(E, H, G) \).

Let us still clarify the previous definition;

**Remark 7.2.** Observe that \( \cap_{s,p,\omega,R}(E, H, G) \) need not coincide with \( \cap_{s,p,\omega,R}(E, H, H) \); cf. [32] p. 598 and [9]. This non-locality contributes to our heavy notation, involving several parameters. However, the number of parameters reduces when we look at special cases: In the light of Example 7.2, the ‘relative \( (s, p) \)-capacity’

\[
\cap_{s,p}(E, H, G) = \cap_{s+\varepsilon/p,p,|\cdot|^{-n}}(E, H, G) = \inf_{\varphi} \int_G \int_G \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{n+sp}} \, dy \, dx
\]

is obtained as a special case of our general framework. This relative \( (s, p) \)-capacity, in turn, generalizes the following ‘fractional \( (s, p) \)-capacity’

\[
\cap_{s,p}(E, G) = \cap_{s,p}(E, G, G), \quad E \subset G \text{ compact}.
\]

These fractional \( (s, p) \)-capacities have recently found applications, e.g., in connection with the fractional Hardy inequalities, we refer to [7] [37].

The following is our capacity comparison result.

**Theorem 7.3.** Fix \( 0 < s < 1 \), \( 1 < p < \infty \), and an \( A_p \) weight \( \omega \) in \( \mathbb{R}^n \). Suppose that \( \bar{E} \neq \emptyset \) is a compact \( \kappa \)-porous set, contained in a bounded open set \( G \subset \mathbb{R}^n \), and that \( R : G \to \mathbb{R} \) is a continuous function satisfying both \( R(x) = 0 \) for every \( x \in E \) and \( 0 \leq R(x) \leq \text{dist}(x, \partial G) \) for every \( x \in G \). Then, if \( 0 < t < \kappa \text{diam}(E)/4 \) is such that \( E_t \subset G \), we have

\[
\cap_{s,p,\omega,R}(E_t, E_t \cap E_{4t/\kappa, R}) \leq C \cap_{s,p,\omega}(E_t, G),
\]

where \( C = \kappa^{-np} C(n, p, [\omega]_{A_p}) \).

Before proving this result, let us illustrate the special case of relative \( (s, p) \)-capacity while working in the setting of Theorem [7.3] together with a fixed \( 0 < \alpha < 1 \). Since \( E_{t^{1/\alpha}} \subset E_t \) for all small \( t > 0 \), we have

\[
\cap_{s,p}(E, E_t, G) \leq \cap_{s,p}(E, E_{t^{1/\alpha}}, G).
\]

Because the set \( E_{t^{1/\alpha}} \) can be much smaller than \( E_t \), it is reasonable to expect—unless both of the above capacities vanish—that the converse of inequality [7.41] with a \( t \)-independent
constant cannot hold for all small $t > 0$. For a more precise statement, we need the following non-trivial example.

**Example 7.4.** Let $E \subset G$ be a compact Ahlfors $\lambda$-regular set \cite{30} with $0 < \lambda < n$; then $E$ is $\kappa$-porous for some $0 < \kappa < 1$. Assume that $0 < s < 1$ and $1 < p < \infty$ satisfy $n - sp < \lambda < n$. Then there exists $t_0 > 0$ such that, whenever $0 < t < t_0$, we have

$$\text{cap}_{s,p}(E, E_t, G) \simeq t^{n-\lambda-sp}$$

(7.42)

and the constants of comparison are independent of $t$. Indeed, this comparison estimate can be obtained by adapting the arguments that are given in \cite{30}; we omit the details here.

Let us continue our discussion (before the example) and suppose that $\alpha s < \sigma < 1$. If we take $n - \alpha sp < \lambda < n$ to be sufficiently large, then Example 7.4 shows that inequality

$$\text{cap}_{\sigma,p}(E, E_{t^{1/\alpha}}, G) \leq C \text{cap}_{s,p}(E, E_t, G), \quad t > 0 \text{ small},$$

(7.43)

fails for some compact $\kappa$-porous set $E \subset G$ if $c$ and $C$ are not allowed to depend on the parameter $t > 0$ (but are allowed to depend on $E$ and the other parameters).

On the other hand, if we assume that $\sigma \leq \alpha s$, then inequality (7.43) holds for any fixed compact Ahlfors $\lambda$-regular set $E \subset G$ given that $n - \alpha sp < \lambda < n$; see Example 7.4. If $\sigma < \alpha s$ (we now exclude the ‘critical’ case $\sigma = \alpha s$), then the last conclusion can be independently obtained with our results: by straightforward estimates and Theorem 7.3 we find that, for small $t > 0$,

$$\text{cap}_{\sigma,p}(E, E_{t^{1/\alpha}}, G) \lesssim \text{cap}_{s+\varepsilon/p,\omega,R}(E, E_t \cap E_{t^{1/\alpha}}, G)$$

\begin{equation}
\leq \text{cap}_{s+\varepsilon/p,\omega}(E, E_t, G) = \text{cap}_{s,p}(E, E_t, G).
\end{equation}

Here $c = (4/\kappa)^{1/\alpha}$, $\omega = |x|^{-n}$ (for a sufficiently small $\varepsilon > 0$) is an $A_p$ weight, and

$$R(x) = \min \{\text{dist}(x, E)^\alpha, \text{dist}(x, \partial G)\}, \quad x \in G,$

defines an $\alpha$-Hölder function on the bounded open set $G$. We also have $R(x) = \text{dist}(x, E)^\alpha$ if $x$ is sufficiently close to a fixed $\kappa$-porous compact set $E \subset G$. In particular, this set is allowed to be a compact Ahlfors $\lambda$-regular set with $0 < \lambda < n$.

We turn our focus to the proof of Theorem 7.3. To this end, we first consider the following modification of \cite{7} Lemma 2.3.

**Lemma 7.5.** Suppose that $R : G \to \mathbb{R}$ is a continuous function on an open set $\emptyset \neq G \subseteq \mathbb{R}^n$ such that $0 \leq R(x) \leq \text{dist}(x, \partial G)$ for every $x \in G$. Assume that $f : G \to \mathbb{R}$ has a continuous extension to $\mathbb{R}^n$. Then $M_R(f) = M_R(f \chi_G)$ is a continuous function on $G$.

**Proof.** We first observe that the function defined by $F(x, 0) = |f(x)|$ and

$$F(x, r) = \int_{B(x,r)} |f(y)| \, dy$$

for $r > 0$ is continuous on $\mathbb{R}^n \times [0, \infty)$ (in this definition the function $|f|$ is continuously extended to the whole $\mathbb{R}^n$, which is possible due to assumptions).

Let us fix $x \in G$ and $\varepsilon > 0$. By the uniform continuity of $F$ on $B(x, 1) \times [0, R(x) + 1]$, there exists $0 < \eta < 1$ such that $|F(y, s) - F(x, t)| < \varepsilon$ whenever $|y - x| + |s - t| < \eta$ and $0 \leq s, t \leq R(x) + 1$. Moreover, by continuity of $R$ at $x$, there exists $0 < \delta < \eta/2 \wedge \text{dist}(x, \partial G)$ such that

$$|R(x) - R(y)| < \frac{\eta}{2}$$

whenever $|x - y| < \delta$. To prove the continuity of $M_R(f)$ at the point $x$, let us consider a point $y \in G$ such that $|x - y| < \delta$. Now, for some $0 \leq r(y) \leq R(y)$, we have

$$M_R(f)(y) = F(y, r(y)) \leq F(x, r(y) \wedge R(x)) + \varepsilon \leq M_R(f)(x) + \varepsilon,$$
Hence, we obtain that
\[ M_R(f)(x) = F(x, r(x)) \leq F(y, r(x) \wedge R(y)) + \varepsilon \leq M_R(f)(y) + \varepsilon . \]
This proves continuity of \( M_R(f) \).

**Proof of Theorem 7.3** Fix \( t > 0 \) as in the statement of the theorem. Let \( \varphi \in C_0(G) \) be such that \( \varphi \geq 1 \) on \( E \) and \( \varphi = 0 \) on \( G \setminus E_t \). We define
\[ f = 1 - \min\{1, \varphi\} \in C(G) . \]
Then \( f(x) = 0 \) for every \( x \in E \) and \( f(x) = 1 \) if \( x \in G \setminus E_t = \{x \in G : \text{dist}(x, E) \geq t\} \). Let us consider a point \( x \in G \) which satisfies \( R(x) \geq 4t/\kappa \). By using the \( \kappa \)-porosity of \( E \), it is rather straightforward to find \( y \in B(x, 4t/\kappa) \) such that \( B(y, t) \subset B(x, 4t/\kappa) \subset G \) and \( \text{dist}(B(y, t), E) \geq t \). Hence,
\[ M_R(f)(x) \geq \int_{B(x, 4t/\kappa)} |f(z)| \, dz \geq \frac{|B(y, t)|}{|B(x, 4t/\kappa)|} \int_{B(y, t)} |f(z)| \, dz \geq \frac{|B(y, t)|}{|B(x, 4t/\kappa)|} = 4^{-n}\kappa^n . \]
It follows that \( 4^n\kappa^{-n}M_R(f) \geq 1 \) on the set \( G \setminus E_{4t, \kappa,R} \). We also have \( M_R(f) \geq |f| = 1 \) on the set \( G \setminus E_t \). Moreover, if \( x \in E \), then \( R(x) = 0 \) and hence
\[ M_R(f)(x) = |f(x)| = 0 . \]
Since \( f : G \to \mathbb{R} \) can be continuously extended to the whole \( \mathbb{R}^n \) by setting \( f(x) = 1 \) for every \( x \in \mathbb{R}^n \setminus G \), by Lemma 7.3 we find that \( M_R(f) \) is continuous on \( G \).

By using the previous facts, we find that the function \( g = 1 - \min\{1, 4^{n+1}\kappa^{-n}M_R(f)\} \) is an admissible test function for the capacity in the left-hand side of inequality (7.40); in particular, the support condition follows from the chain of inclusions
\[ \text{supp}(g) \subset \{x \in G : 4^{n+1}\kappa^{-n}M_R(f)(x) \leq 1\} \subset E_t \cap E_{4t, \kappa,R} \subset \overline{E_t} \subset G . \]
Hence, we obtain that
\[ \text{cap}_s,p,\omega,R(E, E_t \cap E_{4t, \kappa,R}, G) \leq \int_G \int_G \frac{|g(x) - g(y)|^p}{(|x - y| + |R(x) - R(y)|)^{sp}} \omega(x - y) \, dy \, dx \]
\[ \leq (4^{n+1}\kappa^{-n})^p \int_G \int_G \frac{|M_R(f)(x) - M_R(f)(y)|^p}{(|x - y| + |R(x) - R(y)|)^{sp}} \omega(x - y) \, dy \, dx . \]
Observe that \( f \in L^\infty(G) \subset L^p(G) \) since \( G \) is assumed to be bounded. By Theorem 4.1, we then obtain that
\[ \text{cap}_s,p,\omega,R(E, E_t \cap E_{4t, \kappa,R}, G) \leq \kappa^{-np}C(n, p, [\omega]_{A_p}) \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx \]
\[ \leq \kappa^{-np}C(n, p, [\omega]_{A_p}) \int_G \int_G \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{sp}} \omega(x - y) \, dy \, dx . \]
The required inequality (7.40) follows by taking infimum over all of the functions \( \varphi \) that are considered above.

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