From Super Poincaré to Weighted Log-Sobolev and Entropy-Cost Inequalities

Feng-Yu Wang

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
Present: Department of Mathematics, University of Swansea, Singleton Park, SA2 8PP, UK

February 2, 2008

Abstract

We derive weighted log-Sobolev inequalities from a class of super Poincaré inequalities. As an application, the Talagrand inequality with larger distances are obtained. In particular, on a complete connected Riemannian manifold, we prove that the log$^δ$-Sobolev inequality with $δ ∈ (1, 2)$ implies the $L^{2/(2−δ)}$-transportation cost inequality

$$W_{2/(2−δ)}^ρ(μ_1, μ_2)^{2/(2−δ)} ≤ Cμ(f \log f), \quad μ(f) = 1, f ≥ 0$$

for some constant $C > 0$, and they are equivalent if the curvature of the corresponding generator is bounded below. Weighted log-Sobolev and entropy-cost inequalities are also derived for a large class of probability measures on $\mathbb{R}^d$.

AMS subject Classification: 60J60, 58G32.
Keywords: Entropy-cost inequality, super Poincaré inequality, weighted log-Sobolev inequality.

1 Introduction

Let $(E, ρ)$ be a Polish space and $μ$ a probability measure on $E$. For $p ≥ 1$ we define the $L^p$-Wasserstein distance (or the $L^p$-transportation cost) by

$$W_p^ρ(μ_1, μ_2) := \left\{ \inf_{π \in P(μ_1, μ_2)} \int_{E \times E} ρ(x, y)^p π(dx, dy) \right\}^{1/p}$$

*Supported in part by NNSFC(10121101) and the 973-Project in China.
for probability measures $\mu_1, \mu_2$ on $E$, where $C(\mu_1, \mu_2)$ is the class of probability measures on $E \times E$ with marginal distributions $\mu_1$ and $\mu_2$.

According to [4, Corollary 4],

$$W^p_{\rho}(f\mu, \mu)^{2p} \leq C \mu(f \log f), \quad f \geq 0, \mu(f) = 1$$

holds for some $C > 0$ provided $\mu(e^{\lambda \rho(o, \cdot)^{2p}}) < \infty$ for some $\lambda > 0$, where $o \in E$ is a fixed point. See also [8] for $p = 1$. Furthermore, it is easy to derive from [14, Theorem 1.15] that for any $q \in [1, 2p)$, there exists $C > 0$ such that

$$(1.1) \quad W^p_q(f\mu, \mu)^{2p} \leq C \mu(f \log f), \quad f \geq 0, \mu(f) = 1$$

if and only if $\mu(e^{\lambda \rho(o, \cdot)^2}) < \infty$ for some $\lambda > 0$. In general, however, this concentration of $\mu$ does not imply (1.1) for $q = 2p$. Indeed, there exist a plentiful examples where $\mu(e^{\lambda \rho(o, \cdot)^2}) < \infty$ for some $\lambda > 0$ but there is no any constant $C > 0$ such that the Talagrand inequality

$$(1.2) \quad W^p_2(f\mu, \mu)^2 \leq C \mu(f \log f), \quad f \geq 0, \mu(f) = 1$$

holds, see e.g. [1] for examples with $\mu(e^{\lambda \rho(o, \cdot)^2}) < \infty$ for some $\lambda > 0$ but the Poincaré inequality does not hold, which is weaker than (1.2) (see [17, Section 7] or [2, Section 4.1]).

Therefore, to derive (1.1) with $q = 2p$, one needs something stronger than the corresponding concentration of $\mu$. In fact, it is now well known in the literature that, the Talagrand inequality follows from the log-Sobolev inequality for a class of local Dirichlet forms, see [21, 17, 2, 25, 20] and references within.

In this paper, we aim to derive (1.1) with $q = 2p$, i.e.

$$W^p_2(f\mu, \mu)^{2p} \leq C \mu(f \log f), \quad f \geq 0, \mu(f) = 1 \tag{1.3}$$

by using functional inequalities stronger than the log-Sobolev one.

To this end, in Section 2 we study the weighted log-Sobolev inequality

$$\mu(f^2 \log f^2) \leq C \mu(\alpha \circ \rho(o, \cdot) \Gamma(f, f)), \quad \mu(f^2) = 1$$

for a positive function $\alpha(r) \to 0$ as $r \to \infty$ and a nice square field $\Gamma$. Combining this with known results on log-Sobolev and the Talagrand inequality, we derive (1.2) with the original distance $\rho$ replaced by a larger one, which is induced by the weighted square field $\alpha \circ \rho(o, \cdot) \Gamma$. In particular, we have the following result on a Riemannian manifold.

Let $M$ be a connected complete Riemannian manifold, and $\mu(dx) = e^{V(x)}dx$ a probability measure on $M$ for some $V \in C(M)$. We shall use the following super Poincaré inequality (see [23])
(1.4) \[ \mu(f^2) \leq r\mu(|\nabla f|^2) + \beta(r)\mu(|f|^2), \quad r > 0 \]
to establish the corresponding weighted log-Sobolev inequality

(1.5) \[ \mu(f^2 \log f^2) \leq C\mu(\alpha \circ \rho(o, \cdot)|\nabla f|^2), \quad \mu(f^2) = 1. \]
By [25, Theorem 1.1], (1.5) implies

(1.6) \[ W_{2}^{\rho_{\alpha}}(f\mu, \mu)^{2} \leq C\mu(f \log f), \quad f \geq 0, \mu(f^2) = 1, \]
where \( \rho_{\alpha} \) is the Riemannian distance induced by the metric

(1.7) \[ \langle X, Y \rangle' := \frac{1}{\alpha \circ \rho(o, x)} \langle X, Y \rangle, \quad X, Y \in T_xM, \ x \in M. \]
The main result of the paper is the following.

**Theorem 1.1.** Assume that (1.4) holds for some positive decreasing \( \beta \in C((0, \infty)) \) such that

\[ \eta(s) := (\log(2s))(1 \wedge \beta^{-1}(s/2)), \quad s \geq 1 \]
is bounded, where \( \beta^{-1}(s) := \inf\{t \geq 0 : \beta(t) \leq s\} \). Then (1.5) holds for some \( C > 0 \) and

\[ \alpha(s) := \sup_{t \geq \mu(\rho(o, \cdot) \geq s^{-2})^{-1}} \eta(t), \quad s \geq 0. \]
Consequently, (1.6) holds.

The following consequences show that the above result is sharp in specific situations.

**Corollary 1.2.** Let \( \delta \in (1, 2) \).

(a) (1.4) with \( \beta(r) = \exp[c(1 + r^{-1/\delta})] \) implies (1.5) with

\[ \alpha(s) := (1 + \rho(o, \cdot))^{-2(\delta-1)/(2-\delta)} \]
and (1.6) with \( \rho_{\alpha}(x, y) \) replaced by

\[ \rho(x, y)(1 + \rho(o, x) \vee \rho(o, y))^{(\delta-1)/(2-\delta)}. \]
Consequently, it implies

(1.8) \[ W_{2}^{\rho}((2-\delta)/(2-\delta)}(f\mu, \mu)^{2/(2-\delta)} \leq C\mu(f \log f), \quad \mu(f) = 1, \ f \geq 0 \]
for some constant \( C > 0 \).

(b) If \( V \in C^{2}(M) \) with \( \text{Ric} - \text{Hess}_{V} \) bounded below, then the following are equivalent to each other:
(1) \((1.4)\) with \(\beta(r) = \exp[c(1 + r^{1/\delta})]\) for some constant \(c > 0\);

(2) \((1.5)\) with \(\alpha(s) := (1 + \rho(o, \cdot))^{-2(\delta-1)/(2-\delta)}\) for some \(C > 0\);

(3) \((1.6)\) for some \(C > 0\) and \(\rho_\alpha(x, y)\) replaced by \(\rho(x, y)(1 + \rho(o, x) \vee \rho(o, y))^{(\delta-1)/(2-\delta)}\);

(4) \((1.8)\) for some \(C > 0\);

(5) \(\mu(\exp[\lambda \rho(o, \cdot)^{2/(2-\delta)}]) < \infty\) for some \(\lambda > 0\).

We remark that \((1.4)\) with \(\beta(r) = \exp[c(1 + r^{1/\delta})]\) for some \(c > 0\) is equivalent to the following log\(\delta\)-Sobolev inequality mentioned in the abstract (see \[23, 24, 13, 26\] for more general results on \((1.4)\) and the \(F\)-Sobolev inequality)

\[\mu(f^2 \log f) \leq C_1 \mu(|\nabla f|^2) + C_2, \quad \mu(f^2) = 1.\]

Since due to \[24\] Corollary 5.3 if \((1.4)\) holds with \(\beta(r) = \exp[c(1 + r^{1/\delta})]\) for some \(\delta > 2\) then \(M\) has to be compact, as a complement to Corollary \(1.2\) we consider the critical case \(\delta = 2\) in the next Corollary.

**Corollary 1.3.** \((1.4)\) with \(\beta(r) = \exp[c(1 + r^{1/\delta})]\) for some \(c > 0\) implies \((1.5)\) with \(\alpha(s) := e^{-c_1 s}\) for some \(c_1 > 0\) and \((1.6)\) with \(\rho_\alpha(x, y)\) replaced by

\[\rho(x, y)e^{c_2 |\rho(o, x) \vee \rho(o, y)|} \geq e^{c_3 \rho(x, y)} - 1\]

for some \(c_2, c_3 > 0\). If \(\text{Ric} - \text{Hess}_V\) is bounded below, they are all equivalent to the concentration \(\mu(\exp[e^{\lambda \rho(o, \cdot)}]) < \infty\) for some \(\lambda > 0\).

**Example 1.1.** Let \(\text{Ric}\) be bounded below. Let \(V \in C(M)\) be such that \(V + a \rho(o, \cdot)\theta\) is bounded for some \(a > 0\) and \(\theta \geq 2\). By \[23\] Corollaries 2.5 and 3.3, \((1.4)\) holds for \(\delta = 2(\theta - 1)/\theta\). Then Corollary \(1.2\) implies

\[W_\theta^\rho(f, \mu, \mu)^\theta \leq C \mu(f \log f), \quad f \geq 0, \mu(f) = 1\]

for some constant \(C > 0\).

In this inequality \(\theta\) could not be replaced by any larger number, since \(W_\theta^\rho \geq W_1^\rho\) and by Proposition \(3.1\) below for any \(p \geq 1\) the inequality

\[W_1^\rho(f, \mu, \mu)^p \leq C \mu(f \log f), \quad f \geq 0, \mu(f) = 1\]

implies \(\mu(e^{\lambda \rho(o, \cdot)^p}) < \infty\) for some \(\lambda > 0\), which fails when \(p > \theta\) for \(\mu\) specified above.
Example 1.2. In the situation of Example 1.1 but let $V + \exp[\sigma \rho(o, \cdot)]$ be bounded for some $\sigma > 0$. Then by [23] Corollaries 2.5 and 3.3, (1.4) holds with $\beta(r) = \exp[c(1+r^{-1/2})]$ for some $c > 0$. Hence, by Corollary 1.3,

$$\inf_{\pi \in \mathcal{P}(\mu, \nu)} \int_{M \times M} \rho(x, y)^2 e^{c_1 \rho(x, y)} \pi(dx, dy) \leq C \mu(f \log f), \quad f \geq 0, \mu(f) = 1$$
holds for some $c_1, C > 0$.

On the other hand, it is easy to see from Jensen’s inequality that the left hand side is larger than

$$(\exp[c_2 W_0^p(\mu, f\mu)] - 1)^2$$
for some $c_2 > 0$. So, by Proposition 3.1 below (1.9) implies $\mu(\exp[\exp(\lambda \rho(o, \cdot))]) < \infty$ holds for any $\lambda > 0$, which is the exact concentration property of the given measure $\mu$.

In the next section we study the super Poincaré and the weighted log-Sobolev inequality in an abstract framework, and complete proofs of the above results are presented in Section 3. Finally, weighted log-Sobolev and transportation cost inequalities are also studied for probability measures on $\mathbb{R}^d$ by using concentrations.

2 From super Poincaré to weighted log-Sobolev inequalities

We shall work with a diffusion framework as in [1]. Let $(E, \mathcal{F}, \mu)$ be a separable complete probability space, and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a conservative symmetric local Dirichlet form on $L^2(\mu)$ with domain $\mathcal{D}(\mathcal{E})$ in the following sense. Let $\mathcal{A}$ be a dense subspace of $\mathcal{D}(\mathcal{E})$ under the $E_1/2$-norm ($\mathcal{E}_1(f, f) = \| f \|^2_2 + \mathcal{E}(f, f)$) which is composed of bounded functions, stable under products and composition with Lipschitz functions on $\mathbb{R}$. Let $\Gamma : \mathcal{A} \times \mathcal{A} \to \mathcal{M}_b$ be a bilinear mapping, where $\mathcal{M}_b$ is the set of all bounded measurable functions on $E$, such that

1. $\Gamma(f, f) \geq 0$ and $\mathcal{E}(f, g) = \mu(\Gamma(f, g))$ for $f, g \in \mathcal{A}$;
2. $\Gamma(\phi \circ f, g) = \phi'(f) \Gamma(f, g)$ for $f, g \in \mathcal{A}$ and $\phi \in C_0^\infty(\mathbb{R})$;
3. $\Gamma(fg, h) = g \Gamma(f, h) + f \Gamma(g, h)$ for $f, g, h \in \mathcal{A}$ with $fg \in \mathcal{A}$.

It is easy to see that the positivity and the bilinear property imply $\Gamma(f, g)^2 \leq \Gamma(f, f) \Gamma(g, g)$ for all $f, g \in \mathcal{A}$. For simplicity we set below $\Gamma(f, f) = \Gamma(f)$ and $\mathcal{E}(f, f) = \mathcal{E}(f)$.

We shall denote by $\mathcal{A}_{\text{loc}}$ the set of functions $f$ such that for any integer $n$, the truncated function $f_n = \min(n, \max(f, -n))$ is in $\mathcal{A}$. For such functions, the bilinear map $\Gamma$ automatically extends and shares the same properties than for functions in $\mathcal{A}$.

Next, let $\rho \in \mathcal{A}_{\text{loc}}$ be positive such that $\Gamma(\rho, \rho) \leq 1$. We shall start from the super Poincaré inequality
To derive the desired weighted log-Sobolev inequality

$$
\mu(f^2) \leq r\mathcal{E}(f, f) + \beta(r)\mu(|f|^2), \quad r > 0.
$$

we shall also need the following Poincaré inequality

$$
\mu(f^2) \leq C_0\mathcal{E}(f, f) + \mu(f)^2
$$

for some $C_0 > 0$. Here and in what follows, the reference function $f$ is taken from $\mathcal{A}$.

**Theorem 2.1.** Assume (2.3) holds for some $C_0 > 0$. Then (2.2) implies (2.2) for some constant $C > 0$ and $\alpha$ given in Theorem 1.1.

**Proof.** (a) Let $\Phi(s) = \mu(\rho \geq s)$ which decreases to zero as $s \to \infty$. We may take $r_0 > 0$ such that

$$
r_0(1 + \sup_{s \geq 1} \eta(s)) \leq \frac{1}{32}
$$

and

$$
\beta^{-1}(e^{\rho^{-1}}/4) \leq 1.
$$

For a fixed number $r \in (0, r_0]$ we define

$$
\begin{align*}
    h_n &= (\rho - \Phi^{-1}(2e^{-r-1}) - n)_+ \land 1) \left( (n + 2 + \Phi^{-1}(2e^{-r-1}) - \rho)_+ \land 1 \right), \\
    \delta_n &= \left( \log \frac{2}{\Phi(n + \Phi^{-1}(2e^{-r-1}))} \right) \beta^{-1} \left( \frac{1}{2\Phi(n + \Phi^{-1}(2e^{-r-1}))} \right), \\
    B_n &= \{ n \leq \rho - \Phi^{-1}(2e^{-r-1}) \leq n + 2 \}, \quad n \geq 0.
\end{align*}
$$

Then

$$
\sum_{n=0}^{\infty} h_n^2 \geq \frac{1}{2} 1_{\{ \rho \geq \phi^{-1}(2e^{-r-1}) \}}.
$$

By (2.1) and noting that
\[ \mu(|f|h_n)^2 \leq \mu(f^2h_n^2)(\varrho > n + \Phi^{-1}(2e^{-r^{-1}})) \leq \mu(f^2h_n^2)\Phi(n + \Phi^{-1}(2e^{-r^{-1}})), \]

we have

\[
\sum_{n=0}^{\infty} \mu(f^2h_n^2) \leq \sum_{n=0}^{\infty} \{r_n\mu(\Gamma(fh_n, fh_n)) + \beta(r_n)\mu(|f|h_n)^2\} \\
\leq \sum_{n=0}^{\infty} \left\{ \frac{2r_n}{\delta_n} \mu(\Gamma(f, f)\delta_n1_{B_n}) + 2r_n\mu(f^21_{B_n}) + \beta(r_n)\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))\mu(f^2h_n^2) \right\}
\]

for \( r_n > 0 \). Since by (2.5) and the definition of \( \alpha \)

\[ \alpha(s) \geq \delta_n \quad \text{for} \quad s \geq n + 2 + \Phi^{-1}(2e^{-r^{-1}}), \]

letting \( r_n = \delta_n r \) we obtain

\[
\sum_{n=0}^{\infty} \mu(f^2h_n^2) \leq \sum_{n=0}^{\infty} \left\{ 2r\mu(\Gamma(f, f)\alpha \circ \varrho 1_{B_n}) + 2r\delta_n\mu(f^21_{B_n}) \\
+ \beta(r\delta_n)\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))\mu(f^2h_n^2) \right\}.
\]  

(2.7)

Noting that

\[ A := r \log \frac{2}{\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))} \geq r \log \frac{2}{\Phi(\Phi^{-1}(2e^{-r^{-1}}))} = 1, \]

we have

\[ \beta(\delta_n r) = \beta \left( A^{-1} \left( \frac{1}{2\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))} \right) \right) \leq \frac{1}{2\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))}. \]

Thus, by (2.7) and (2.4) and the fact that \( \delta_n \leq \sup \eta \), we arrive at

\[ \sum_{n=0}^{\infty} \mu(f^2h_n^2) \leq \sum_{n=0}^{\infty} \left\{ 2r\mu(\Gamma(f, f)\alpha \circ \varrho 1_{B_n}) + \frac{1}{8}\mu(f^2) + \frac{1}{2} \sum_{n=0}^{\infty} \mu(f^2h_n^2) \right\}. \]

It follows from this and (2.6) that

\[
\mu(f^21_{\{\varrho \geq n + \Phi^{-1}(2e^{-r^{-1}})}}) \leq 8r\mu(\Gamma(f, f)\alpha \circ \varrho) + \frac{1}{2}\mu(f^2).
\]  

(2.8)

(b) On the other hand, since \( \alpha \) is decreasing
\[
\mu(f^2 \mathbb{1}_{\{\varrho \leq 1 + \Phi^{-1}(2e^{-r-1})\}}) \leq \mu(f^2 \{(2 + \Phi^{-1}(2e^{-r-1}) - \varrho)^2 \wedge 1\})
\]
\[
\leq 2s\mu(\Gamma(f, f) \mathbb{1}_{\{\varrho \leq 2 + \Phi^{-1}(2e^{-r-1})\}}) + 2s\mu(f^2) + \beta(s)\mu(|f|^2)
\]
\[
\leq \frac{2s}{\alpha(2 + \Phi^{-1}(2e^{-r-1}))} \mu(\Gamma(f, f) \alpha \circ \varrho) + 2s\mu(f^2) + \beta(s)\mu(|f|^2), \quad s > 0.
\]
Taking
\[
s = r\alpha(2 + \Phi^{-1}(2e^{-r-1})) \leq \frac{1}{32}
\]
due to (2.4), we obtain
\[
\mu(f^2 \mathbb{1}_{\{\varrho \leq 1 + \Phi^{-1}(2e^{-r-1})\}}) \leq 2r\mu(\Gamma(f, f) \alpha \circ \varrho) + \frac{1}{16}\mu(f^2) + \beta(r\alpha(2 + \Phi^{-1}(2e^{-r-1})))\mu(|f|^2).
\]
Since by (2.5) and the definition of \(\alpha\)
\[
r\alpha(2 + \Phi^{-1}(2e^{-r-1})) \geq \left(r \log \frac{2}{\Phi(\Phi^{-1}(2e^{-r-1}))}\right)\beta^{-1} \left(\frac{1}{2\Phi(\Phi^{-1}(2e^{-r-1}))}\right)
\]
\[
= \beta^{-1}\left(\frac{e^{r-1}}{4}\right),
\]
we obtain
\[
\mu(f^2 \mathbb{1}_{\{\varrho \leq 1 + \Phi^{-1}(2e^{-r-1})\}}) \leq 2r\mu(\Gamma(f, f) \alpha \circ \varrho) + \frac{1}{16}\mu(f^2) + \frac{e^{r-1}}{4}\mu(|f|^2).
\]
Combining this with (2.3) we conclude that
\[
\mu(f^2) \leq 40r\mu(\Gamma(f, f) \alpha \circ \varrho) + e^{r-1}\mu(|f|^2), \quad r \in (0, r_0].
\]
Therefore, there exists a constant \(c > 0\) such that
\[
(2.9) \quad \mu(f^2) \leq r\mu(\Gamma(f, f) \alpha \circ \varrho) + e^{c(1-r^{-1})}\mu(|f|^2), \quad r > 0.
\]
According to e.g. [24, Corollary 1.3], this is equivalent to the defective weighted log-Sobolev inequality
\[
(2.10) \quad \mu(f^2 \log f^2) \leq C_1\mu(\Gamma(f, f) \alpha \circ \varrho) + C_2, \quad \mu(f^2) = 1.
\]
(c) Finally, for any \(f\) with \(\mu(f) = 0\), it follows from (2.3) that
\[
\mu(f^2) \leq \mu(f^2\{(1 + R - \rho)^2 \wedge 1\}) + \|f\|_\infty^2 \mu(\rho \geq R) \\
\leq 2C_0 \mu(\Gamma(f, f)1_{(\rho \leq 1+R)}) + (2C_0 + 1)\|f\|_\infty^2 \mu(\rho \geq R) + \mu(f\{(\rho - R)^+ \wedge 1\})^2 \\
\leq \frac{2C_0}{\alpha(1 + R)} \mu(\Gamma(f, f)\alpha \circ \rho) + 2(C_0 + 1)\|f\|_\infty^2 \mu(\rho \geq R), \quad R > 0.
\]

Since \( \mu(\rho \geq R) \to 0 \) as \( R \to \infty \), the weighted weak Poincaré inequality

\[
\mu(f^2) \leq \tilde{\beta}(r)\mu(\Gamma(f, f)\alpha \circ \rho) + r\|f\|_\infty^2, \quad r > 0, \mu(f) = 0
\]

holds for some positive function \( \tilde{\beta} \) on \((0, \infty)\). By [19, Proposition 1.3], this and (2.9) implies the weighted Poincaré inequality

\[
\mu(f^2) \leq C'\mu(\Gamma(f, f)\alpha \circ \rho) + \mu(f)^2
\]

for some constant \( C' > 0 \). Combining this with (2.10) we obtain the desired weighted log-Sobolev inequality (2.2). \( \square \)

3 Proofs of Theorem 1.1 and Corollaries

**Proof of Theorem 1.1.** Since \( \alpha \) is bounded, the completeness of the original metric implies that of the weighted one given by (1.7). So, (1.6) follows from (1.5) due to [23, Theorem 1.1] with \( p \to 2 \). Thus, by Theorem 2.1 with \( E = M \) and \( \Gamma(f, f) = |\nabla f|^2 \), it suffices to prove that (1.4) implies the Poincaré inequality (2.3) for some \( C_0 > 0 \). Due to [23] the super Poincaré inequality (1.4) implies that the spectrum of \( L \) is discrete. Moreover, since \( M \) is connected, the corresponding Dirichlet form is irreducible so that 0 is a simple eigenvalue. Therefore, \( L \) possesses a spectral gap, which is equivalent to the desired Poincaré inequality. \( \square \)

To complete the proof of Corollary 1.2, in the spirit of [16, 3] we introduce below a deviation inequality induced by the \( L^1 \)-transportation cost inequality.

**Proposition 3.1.** Let \( \tilde{\rho} : M \times M \to [0, \infty) \) be measurable. For any \( r > 0 \) and measurable set \( A \subset M \) with \( \mu(A) > 0 \), let

\[
A_r = \{x \in M : \tilde{\rho}(x, y) \geq r \text{ for some } y \in A\}, \quad r > 0.
\]

If

\[
W_1^\rho(f, \mu) \leq \Phi \circ \mu(f \log f), \quad f \geq 0, \mu(f) = 1
\]

(3.1)

holds for some positive increasing \( \Phi \in C([0, \infty)) \), then
(3.2) \[ \mu(A_r) \leq \exp\left[ -\Phi^{-1}(r - \Phi \circ \log \mu(A)^{-1}) \right], \quad r > \Phi \circ \log \mu(A)^{-1}, \]

where \( \Phi^{-1}(r) := \inf\{s \geq 0 : \Phi(s) \geq r\}, \quad r \geq 0. \)

Proof. It suffices to prove for \( \mu(A_r) > 0. \) In this case, letting \( \mu_A = \mu(\cdot \cap A)/\mu(A) \) and \( \mu_{A_r} = \mu(\cdot \cap A_r)/\mu(A_r), \) we obtain from (3.1) that

\[ r \leq W_1^\beta(\mu_A; \mu_{A_r}) \leq W_1^\beta(\mu_A; \mu) + W_1^\beta(\mu_{A_r}; \mu) \leq \Phi \circ \log \mu(A)^{-1} + \Phi \circ \log \mu(A_r)^{-1}. \]

This completes the proof. \( \square \)

**Proof of Corollary 1.2.** (a) Let \( \beta(r) = e^{c(1+r^{-1/\delta})} \) for some \( c > 0 \) and \( \delta > 1. \) It is easy to see that

\[ 1 \wedge \beta^{-1}(s/2) \leq c_1 \log^{-\delta}(2s), \quad s \geq 1 \]

holds for some constant \( c_1 > 0. \) Next, by [24 Corollary 5.3], (1.4) with this specific function \( \beta \) implies

\[ \mu(\rho(o, \cdot) \geq s - 2) \leq c_2 \exp[-c_3 s^{2/(2-\delta)}], \quad s \geq 0 \]

for some constants \( c_2, c_3 > 0. \) Therefore,

\[ (3.3) \quad \alpha(s) \leq c_4 (1 + s)^{-2(\delta-1)/(2-\delta)}, \quad s \geq 0 \]

holds for some constant \( c_4 > 0. \)

On the other hand, for any \( x_1, x_2 \in M \) let \( i \in \{1, 2\} \) such that \( \rho(o, x_i) = \rho(o, x_1) \vee \rho(o, x_2). \) Define

\[ f(x) = (\rho(x, x_i) \wedge \frac{\rho(o, x_i)}{2})(1 + \rho(o, x_i))^{(\delta-1)/(2-\delta)}, \quad x \in \mathbb{R}^d. \]

Then

\[ \alpha \circ \rho(o, \cdot) |\nabla f|^2 \leq c_4 (1 + \rho(o, \cdot))^{-2(\delta-1)/(2-\delta)} |\nabla f|^2 \]

\[ \leq c_4 (\rho(o, x_i)/2 \leq \rho(o, \cdot) \leq 3\rho(o, x_i)/2) (1 + \rho(o, \cdot))^{-2(\delta-1)/(2-\delta)} (1 + \rho(o, x_i))^{2(\delta-1)/(2-\delta)} \leq c_5 \]

for some constant \( c_5 > 0. \) Since by the triangle inequality \( \rho(o, x_i) \geq \frac{1}{2} \rho(x_1, x_2), \) this implies that the intrinsic distance \( \rho_\alpha \) satisfies

\[ \rho_\alpha(x_1, x_2)^2 \geq \frac{|f(x_1) - f(x_2)|^2}{c_5} \]

\[ \geq c_6 \rho(x_1, x_2)^2 (1 + \rho(o, x_1) \vee \rho(o, x_2))^{2(\delta-1)/(2-\delta)} \geq c_7 \rho(x_1, x_2)^{2/(2-\delta)} \]

\[ \geq c_7 \rho(x_1, x_2)^{2/(2-\delta)} \]
for some constant $c_0, c_7 > 0$. Hence the proof of (a) is completed by Theorem 1.1.

(b) Now, assume that

$$\text{Ric} - \text{Hess}_V \geq -K$$

for some $K \geq 0$. By (a) and Proposition 3.1, which ensures the implication from (4) to (5), it suffices to deduce (1) from (5). Let

$$h(r) = \mu(e^{r\rho(o, \cdot)^2}), \quad r > 0.$$ 

By [24, Theorem 5.7], the super Poincaré inequality (1.4) holds with

$$\beta(r) := c_0 \inf_{0 < r_1 < r} \inf_{s > 0} \frac{1}{s} h(2K + 12s^{-1})e^{s/r_1 - 1}, \quad r > 0$$

for some constant $c_0 > 0$. Since for any $\lambda > 0$ there exists $c(\lambda) > 0$ such that

$$rt^2 \leq \lambda t^{2/(2-\delta)} + c(\lambda)r^{1/(\delta-1)}, \quad r > 0,$$

it follows from (5) that

$$h(r) \leq c_1 \exp[c_1 r^{1/(\delta-1)}], \quad r > 0$$

for some constants $c_1 > 0$. Therefore,

$$\beta(r) \leq c_2 \inf_{0 < r_1 < r} \inf_{s > 0} \frac{1}{s} \exp[c_2 s^{-1/(\delta-1)}] + s/r_1], \quad r > 0$$

for some $c_2 > 0$. Taking $s = r^{(\delta-1)/2}$ and $r_1 = r$, we conclude that

$$\beta(r) \leq c^{(1+r^{-1/\delta})}, \quad r > 0$$

for some $c > 0$. Thus, (1) holds.

Proof of Corollary 1.3. The proof is similar to that of Corollary 1.2 by noting that (1.4) with $\beta(r) = \exp[c(1+r^{-1/2})]$ implies $\mu(\rho(o, \cdot) \geq s) \leq \exp[-c e^{c_1 s}]$ for some $c_1 > 0$, see [24, Corollary 5.3].

4 Weighted log-Sobolev and transportation cost inequalities on $\mathbb{R}^d$

Our main purpose of this section is to establish the weighted log-Sobolev inequality for an arbitrary probability measure using the concentration of this measure. We shall also prove the HWI inequality introduced in [2] for the corresponding weighted Dirichlet form. The main point is to find square fields (resp. cost functions) for a given probability measure to satisfy the log-Sobolev inequality (resp. the Talagrand transportation cost inequality).
So, the line of our study is exactly opposed to existed references in the literature, see e.g. \cite{9, 10, 11} and references within, which provided conditions on the reference measure such that the log-Sobolev (resp. transportation cost) inequality holds for a given square field (resp. the corresponding cost function).

The basic idea of the study comes from Caffarelli \cite{5} which says that for any probability measure $\mu(dx) := e^{V(x)}dx$ on $\mathbb{R}^d$, there exists a convex function $\psi$ on $\mathbb{R}^d$ such that $\nabla \psi$ pushes $\mu$ forward to the standard Gaussian measure $\gamma$; that is, letting $y(x) := \nabla \psi(x)$, $x \in \mathbb{R}^d$, which is one-to-one, one has $\gamma = \mu \circ y^{-1}$. Furthermore, $\nabla \psi$ is uniquely determined and $\text{Hess}_\psi$ is non-degenerate with

$$\text{det}(\text{Hess}_\psi) = (2\pi)^{d/2}e^{V + |\nabla \psi|^2/2}.$$  

Let

$$\rho(x_1, x_2) := |y(x_1) - y(x_2)|, \quad x_1, x_2 \in \mathbb{R}^d.$$  

Let $W_2$ be the $L^2$-Wasserstein distance induced by the usual Euclidean metric. Due to Talagrand \cite{21}

$$(4.1) \quad W_2(\gamma, f^2 \gamma)^2 \leq 2\gamma(f^2 \log f^2), \quad \gamma(f^2) = 1.$$  

Since $\pi \in C(\mu \circ y^{-1}, (f^2 \circ y^{-1}) \mu \circ y^{-1})$ if and only if $\pi \circ (y \otimes y) \in C(\mu, f^2 \mu)$, we obtain from (4.1) and the change of variables theorem that

$$W_2^p(\mu, f^2 \mu)^2 = W_2(\gamma, (f^2 \circ y^{-1}) \gamma)^2 \leq 2\gamma(f^2 \circ y^{-1} \log f^2 \circ y^{-1}) = 2\mu(f^2 \log f^2), \quad \mu(f^2) = 1.$$  

Similarly, since

$$\nabla(f \circ y^{-1}) = (Dy^{-1})(\nabla f) \circ y^{-1} = [(Dy) \circ y^{-1}]^{-1}(\nabla f) \circ y^{-1} = [(\text{Hess}_\psi)^{-1}\nabla f] \circ y^{-1},$$  

where $Dy := (\partial_i y_j)_{d \times d}$, by Gross’ log-Sobolev inequality for $\gamma$ (see \cite{12}) we obtain

$$\mu(f^2 \log f^2) \leq 2\mu((\text{Hess}_\psi)^{-1}\nabla f)^2), \quad f \in C^\infty_0(\mathbb{R}^d), \mu(f^2) = 1.$$  

On the other hand, however, since the transportation $\nabla \psi$ is normally inexplicit, it is hard to estimate the distance $\rho$ and the matrix $\text{Hess}_\psi$. So, to derive transportation and log-Sobolev inequalities with explicit distances and Dirichlet forms, we shall construct, instead of $\nabla \psi$, an explicit map using the concentration of $\mu$, which transports the measure into the standard Gaussian measure with a perturbation. In many cases this perturbation is bounded and hence, does not make much trouble to derive the desired inequalities.
4.1 Main results

In this subsection we provide an explicit positive function $\alpha$ and an explicit distance $\rho$ on $\mathbb{R}^d$ such that the log-Sobolev inequality

$$(4.2) \quad \mu(f^2 \log f^2) \leq 2\mu(\alpha |\nabla f|^2), \quad f \in C_0^\infty(\mathbb{R}^d), \ \mu(f^2) = 1$$

and the transportation-cost inequality

$$(4.3) \quad W_2^p(\mu, f^2 \mu)^2 \leq 2\mu(f^2 \log f^2), \quad \mu(f^2) = 1$$

hold. In a special case, we are also able to present the HWI inequality stronger than (4.2).

Let us first consider a probability measure $\mu(dx) := e^{V(x)} dx$ on $[\delta, \infty)$ for some $\delta \in [-\infty, \infty)$, where $[-\infty, \infty)$ is regarded as $\mathbb{R}$. Let $\Phi_\delta(r) := \frac{1}{c_\delta} \int_\delta^r e^{-s^2/2} ds, \quad \varphi(r) := \mu([\delta, r)) = \int_\delta^r e^{V(x)} dx, \quad r \geq \delta,$

where $c_\delta := \int_\delta^\infty e^{-x^2/2} dx$ is the normalization.

Theorem 4.1. Let $\mu(dx) := 1_{[\delta, \infty)}(x)e^{V(x)} dx$ be a probability measure on $[\delta, \infty)$. For the above defined $\Phi_\delta$ and $\varphi$, (4.2) and (4.3) hold with $\mathbb{R}^d$ replaced by $[\delta, \infty)$ for

$$\alpha := \left( \frac{\Phi_\delta' \circ \Phi_\delta^{-1} \circ \varphi'}{\varphi'} \right)^2,$$

$$\rho(x, y) := |\Phi_\delta^{-1} \circ \varphi(x) - \Phi_\delta^{-1} \circ \varphi(y)|, \quad x, y \geq \delta.$$

Furthermore,

$$(4.4) \quad \mu(f^2 \log f^2) + W_2^p(\mu, f^2 \mu)^2 \leq 2\sqrt{2\mu(\alpha f'^2) W_2^p(\mu, f^2 \mu)}, \quad f \in C_0^\infty([\delta, \infty)), \mu(f^2) = 1.$$

The inequality (4.4), linking the Wasserstein distance, the relative entropy and the energy, is called the HWI inequality in [2] and [18].

To extend this result to $\mathbb{R}^d$ for $d \geq 2$, we consider the polar coordinate $(r, \theta) \in [0, \infty) \times S^{d-1}$, where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ with the induced metric. Then $\mu$ can be represented as

$$d\mu = c(d)r^{d-1}e^{V(r\theta)}drd\theta =: G(r, \theta)drd\theta,$$

where $d\theta$ is the normalized volume measure on $S^{d-1}$, and $c(d)/d$ equals to the volume of the unit ball in $\mathbb{R}^d$. Let $B(0, r) := \{x \in \mathbb{R}^d : |x| < r\}$ and
\[
\Phi_0(r) := \int_{B(0,r)} \frac{e^{-|x|^2/2} \mathrm{d}x}{(2\pi)^{d/2}}, \quad r \geq 0,
\]
\[
h(\theta) := \int_{0}^{\infty} s^{d-1} e^{V(s\theta)} \mathrm{d}s, \quad \theta \in \mathbb{S}^{d-1},
\]
\[
\varphi_\theta(r) := \frac{1}{h(\theta)} \int_{0}^{r} s^{d-1} e^{V(s\theta)} \mathrm{d}s, \quad \theta \in \mathbb{S}^{d-1}, r \geq 0.
\]

Since \(\mu(\mathbb{R}^d) = 1\), we have \(h(\theta) \in (0, \infty)\) for a.e. \(\theta \in \mathbb{S}^{d-1}\).

We shall prove that the map
\[
x \mapsto \Phi_0^{-1} \circ \varphi_\theta(|x|) \frac{x}{|x|}
\]
transforms \(\mu\) into a Gaussian measure with density \(h \circ \theta\). Thus, to derive the desired inequalities for \(\mu\), we need a regularity property of this transportation specified in the following result.

**Theorem 4.2.** Let \(r(x) := |x|, \theta(x) := \frac{x}{|x|}, x \in \mathbb{R}^d\). If \(C(h) := \sup_{\theta_1, \theta_2 \in \mathbb{S}^{d-1}} h(\theta_1 \theta_2) < \infty\), then (4.5) holds for
\[
\rho(x_1, x_2) := C(h)^{-1/2}|(\Phi_0^{-1} \circ \varphi_\theta(r)(\theta))(x_1) - (\Phi_0^{-1} \circ \varphi_\theta(r)(\theta))(x_2)|, \quad x_1, x_2 \in \mathbb{R}^d.
\]
If moreover \(\varphi_\theta(r)\) is differentiable in \(\theta\) then (4.2) holds for
\[
\alpha := C(h) \inf_{\varepsilon > 0} \max \left\{ \frac{(1 + \varepsilon)^2}{(\Phi_0^{-1} \circ \varphi_\theta(r))^2}, \frac{\Phi_0' \circ \Phi_0^{-1} \circ \varphi_\theta(r)^2}{(\varphi_\theta'(r))^2} + \frac{(1 + \varepsilon^{-1}) |\nabla \varphi_\theta(r)|^2}{(\varphi_\theta'(r))^2} \right\}.
\]
If, in particular, \(h\) is constant (it is the case if \(V(x)\) depends only on \(|x|\)), then the following HWI inequality holds:
\[
(4.5) \mu(f^2 \log f^2) + W_2^p(\mu, f^2 \mu)^2 \leq 2 \sqrt{2} \mu(\alpha |\nabla f|^2) W_2^p(\mu, f^2 \mu), \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1,
\]
for
\[
\alpha := \max \left\{ \frac{r^2}{(\Phi_0^{-1} \circ \varphi(r))^2}, \frac{\Phi_0' \circ \Phi_0^{-1} \circ \varphi(r)^2}{(\varphi'(r))^2} \right\}
\]
and \(\varphi = \varphi_\theta\) is independent of \(\theta\).

Note that if \(V\) is locally bounded and \(\zeta(r) := \sup_{|x| = r} V(x)\) satisfies \(\int_{0}^{\infty} r^{d-1} e^{\zeta(r)} \mathrm{d}r < \infty\), then \(C(h) < \infty\). Thus, Theorem 4.2 applies to a large number of probability measures. In particular, we have the following concrete result.
Corollary 4.3. Let $V$ be differentiable such that $\mu(dx) := e^{V(x)} dx$ is a probability measure and

$$-c_1|x|^{\delta-1} \leq \langle \nabla V(x), \nabla |x| \rangle \leq -c_2|x|^{\delta-1}$$

holds for some constants $\delta, c_1, c_2 > 0$ and large $|x|$. If there exists a constant $c_3 > 0$ such that

$$|\nabla \theta V| \leq c_3,$$

where $\nabla \theta$ is the gradient on $S^{d-1}$ at point $\theta$, then there exists a constant $c > 0$ such that

$$\mu(f^2 \log f^2) \leq c\mu((1 + |x|^{\delta-1} \nabla f)^2), \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1.$$

Consequently,

$$W_2^\rho(\mu, f^2 \mu)^2 \leq c'\mu(f^2 \log f^2), \quad \mu(f^2) = 1$$

holds for some constant $c' > 0$ and

$$\tilde{\rho}(x, y) := \frac{|x - y|}{(1 + |x| \vee |y|)^{1-\delta/2}}, \quad x, y \in \mathbb{R}^d.$$

Remark. (a) The inequalities presented in Corollary 4.3 are sharp in the sense that (4.9) (and hence also (4.8)) implies $\mu(e^{\lambda r}) < \infty$ for some $\lambda > 0$, which is the exact concentration of $\mu$. This follows from [3, Corollary 3.2] and the fact that $\tilde{\rho}(0, x) \approx |x|^{\delta/2}$ for large $|x|$.

(b) When $V$ is strictly concave, the matrix

$$\Lambda[v_1, v_2] := \int_0^1 s(-\text{Hess}_V)((1 - s)v_1 + sv_2)ds$$

is strictly positive definite for any $v_1, v_2 \in \mathbb{R}^d$. It is proved by Kolesnikov (see [15, Corollary 3.1]) that

$$\mu(f^2 \log f^2) \leq \int_{\mathbb{R}^d} \langle \Lambda[T_f, -1]^{-1} \nabla f, \nabla f \rangle d\mu, \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1,$$

where $x \mapsto T_f(x)$ is the optimal transport of $f^2 \mu$ to $\mu$. In particular, for $V(x) := -|x|^\delta + c$ with $\delta > 2$ and a constant $c$, [15, Example 3.2] implies (4.8) for even smooth function $f^2$. But Corollary 4.3 works for more general $V$ and all smooth function $f$. 

15
Recently, Gentil, Guillin and Miclo \cite{9} (see \cite{10, 11} for further study) established a Talagrand type inequality for $V(x) = -|x|^{\delta} + c$ with $\delta \in [1, 2]$ and a constant $c$. Precisely, there exist constants $a, D > 0$ such that

$$
(4.11) \inf_{\pi \in C(\mu, f^2 \mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} L_{a,D}(x - y) \pi(dx, dy) \leq D \mu(f^2 \log f^2), \quad \mu(f^2) = 1,
$$

where

$$
L_{a,D}(x) := \begin{cases} 
\frac{|x|^2}{a^2 - \delta}, & \text{if } |x| \leq a, \\
\frac{a^2 - \delta}{\delta} |x|^\delta + \frac{a^2(\delta - 2)}{2a}, & \text{otherwise.}
\end{cases}
$$

Since $L_{a,D}(x - y) \geq \epsilon \tilde{\rho}(x, y)^2$ for some constant $\epsilon > 0$, this inequality implies (4.9) for $\delta \in [1, 2]$. But (4.11) is yet unavailable for $\delta / \in [1, 2]$ while (4.9) holds for more general $V$. In particular, if $\delta > 2$ then (4.9) with $\tilde{\rho}(x, y) \geq c(|x - y| \vee |x - y|^{\delta/2})$ for some $c > 0$, which is new in the literature.

### 4.2 Proofs

We first briefly prove for the one-dimensional case (i.e. Theorem 4.1), then extend the argument to high dimensions. It turns out, comparing with the one-dimensional case, that the difficulty point of the proof for high dimensions comes from the angle part. So, a restriction concerning the angle part was made in Theorem 4.2.

**Proof of Theorem 4.1.** Let $y(x) := \Phi^{-1}_{\delta} \circ \varphi(x), \ x \geq \delta$. We have

$$
\frac{d\mu}{dy} = \frac{d\mu}{dx} \cdot \frac{dx}{dy} = e^{V(x)} \frac{d\varphi^{-1} \circ \Phi_{\delta}(y)}{dy} = \frac{e^{V(x)}\Phi'_{\delta}(y)}{\varphi' \circ \varphi^{-1} \circ \Phi_{\delta}(y)} = \frac{e^{V(x)}\Phi'_{\delta}(y)}{\varphi'(x)} = \Phi'_{\delta}(y).
$$

Therefore, $\mu$ is the standard Gaussian measure under the new coordinate $y \in [\delta, \infty)$. In other words, one has

$$
\gamma(dx) := (\mu \circ y^{-1})(dx) = Z_{1|\delta,\infty}(x)e^{-x^2/2}dx,
$$

where $Z$ is the normalization constant. By the HWI inequality proved in \cite{2, 17, 18} and the Gross log-Sobolev inequality which implies the Talagrand inequality, we have

$$
(4.12) \gamma(g^2 \log g^2) + W_2(\gamma, g^2 \gamma)^2 \leq 2 \sqrt{2 \gamma((g')^2)} W_2(\gamma, g^2 \gamma),
$$

$$
W_2(\gamma, g^2 \gamma)^2 \leq 2 \gamma(g^2 \log g^2), \quad \gamma(g^2) = 1.
$$
We remark that although the HWI and Gross’s log-Sobolev inequalities are stated in the above references for the global Gaussian measure, they are also true on a regular convex domain \( \Omega \), since the stronger gradient estimate

\[
|\nabla P_t f| \leq e^{-t} P_t |\nabla f|, \quad f \in C^1_b(\Omega)
\]

holds for the Neumann heat semigroup on \( \Omega \) (cf. [22] and references within).

For any \( f \in C^1_0([\delta, \infty)) \) with \( \mu(f^2) = 1 \), let \( g := f \circ y^{-1} \). We have

\[
\frac{dg}{dx} = (f' \circ y^{-1}) \frac{dy^{-1}}{dx} = (f' \circ y^{-1}) \left( \frac{\Phi_\delta' \circ \Phi_{\delta}^{-1} \circ \varphi'}{y'} \right) \circ y^{-1}.
\]

Since \( \gamma = \mu \circ y^{-1} \), this and (4.12) imply (4.3) and (4.4). Finally, (4.2) is implied by (4.4).

**Proof of Theorem 4.2.** Let \((r, \theta)\) be the polar coordinate introduced in Section 2, and let \( \nabla_\theta \) denote the gradient operator on \( \mathbb{S}^{d-1} \) for the standard metric induced by the Euclidean metric on \( \mathbb{R}^d \). By the orthogonal decomposition of the gradient, we have

\[
(4.13) \quad \nabla f = (\partial_r f) \frac{\partial}{\partial r} + r^{-1} \nabla_\theta f, \quad |\nabla f|^2 = (\partial_r f)^2 + r^{-2} |\nabla_\theta f|^2.
\]

Let us introduce a new polar coordinate \((\bar{r}, \theta)\), where

\[
\bar{r}(r, \theta) := \Phi_{\delta}^{-1} \circ \varphi_{\theta}(r), \quad r \geq 0, \theta \in \mathbb{S}^{d-1}.
\]

We have

\[
d\mu := G(r, \theta) dr d\theta = G(r, \theta) \frac{\partial}{\partial \bar{r}} \bar{r} d\theta = c(d) h(\theta) \Phi_{\delta}(\bar{r}) d\bar{r} d\theta = c(d) h(\theta) d\mu_0,
\]

where \( d\mu_0 := \Phi_{\delta}^{-1} d\bar{r} d\theta \) is the standard Gaussian measure under the new polar coordinate \((\bar{r}, \theta)\). Thus, letting

\[
y(x) := \bar{r}(x) \theta(x) = \Phi_{\delta}^{-1} \circ \varphi_{\frac{|x|}{|x|}}(\theta(x), \quad x \in \mathbb{R}^d,
\]

we have

\[
(\mu \circ y^{-1})(dx) = c(d) h(x/|x|)(\mu_0 \circ y^{-1})(dx) = c(d) h(x/|x|) \gamma(dx),
\]

where \( \gamma \) is the standard Gaussian measure on \( \mathbb{R}^d \). By Gross’ log-Sobolev inequality one has

\[
\gamma(g^2 \log g^2) \leq 2 \gamma(|\nabla g|^2), \quad g \in C^\infty_0(\mathbb{R}^d), \mu_0(g^2) = 1.
\]

Thus, by the perturbation of the log-Sobolev inequality (cf. [7]), we have
(4.14) \((\mu \circ y^{-1})(g^2 \log g^2) \leq 2C(h)(\mu \circ y^{-1})(|\nabla g|^2), \quad g \in W^{2,1}(\gamma), (\mu \circ y^{-1})(g^2) = 1.\)

Moreover, by [2, Corollary 3.1], (4.14) implies

(4.15) \(W_2(\mu \circ y^{-1}, g^2 \mu \circ y^{-1}) \leq 2C(h)(\mu \circ y^{-1})(g^2 \log g^2), \quad (\mu \circ y^{-1})(g^2) = 1.\)

This implies (4.3) for the desired distance \(\rho\) by using the change of variables theorem as explained above.

Similarly, to prove (4.2) we intend apply (4.14) for \(g := f \circ y^{-1}\), where \(f \in C^\infty_0(\mathbb{R}^d)\) with \(\mu(f^2) = 1\).

Since \(y^{-1} = (\varphi^{-1}_\theta \circ \Phi_0(r), \theta)\) under the polar coordinate, by the chain rule we have

\[\nabla_\theta(f \circ y^{-1}) = \nabla_\theta f(\varphi^{-1}_\theta \circ \Phi_0(r), \theta) = ((\nabla_\theta f) \circ y^{-1} + (\partial_r f) \circ y^{-1}) \nabla_\theta \varphi^{-1}_\theta \circ \Phi_0(r).\]

But \(\varphi_\theta \circ \varphi^{-1}_\theta \circ \Phi_0 = \Phi_0\) implies

\[(\nabla_\theta \varphi_\theta)(\varphi^{-1}_\theta \circ \Phi_0(r)) + \varphi'_\theta \circ \varphi^{-1}_\theta \circ \Phi_0(r) \cdot \nabla_\theta(\varphi^{-1}_\theta \circ \Phi_0(r)) = 0,
\]

where \((\nabla_\theta \varphi_\theta)(\varphi^{-1}_\theta \circ \Phi_0(r)) := \nabla_\theta \varphi_\theta(s)|_{s = \varphi^{-1}_\theta \circ \Phi_0(r)}\), we arrive at

\[
|\nabla_\theta(f \circ y^{-1})|^2 \\
\leq (1 + \varepsilon)(\partial_r f)^2 \circ y^{-1} \left(\frac{|\nabla_\theta \varphi_\theta(r)|(|\varphi^{-1}_\theta \circ \Phi_0(r)|)}{\varphi'_\theta \circ \varphi^{-1}_\theta \circ \Phi_0(r)}\right)^2 + (1 + \varepsilon^{-1})|\nabla_\theta f|^2 \circ y^{-1} \\
= (1 + \varepsilon)(\partial_r f)^2 \circ y^{-1} \left(\frac{|\nabla_\theta \varphi_\theta(r)|}{\varphi'_\theta(r)}\right)^2 \circ y^{-1} + (1 + \varepsilon^{-1})|\nabla_\theta f|^2 \circ y^{-1}
\]

for any \(\varepsilon > 0\).

On the other hand,

\[\partial_r(f \circ y^{-1}) = (\partial_r f) \circ y^{-1} \frac{\Phi'_0(r)}{\varphi'_\theta \circ \varphi^{-1}_\theta \circ \Phi_0(r)}.\]

Since

(4.17) \(r = \Phi_0^{-1} \circ \varphi_\theta(r(y^{-1})) = \Phi_0^{-1} \circ \varphi_\theta(r) \circ y^{-1},\)

we have

\[\Phi'_0(r) = (\Phi'_0 \circ \Phi_0^{-1} \circ \varphi_\theta(r)) \circ y^{-1}, \quad \varphi'_\theta \circ \varphi^{-1}_\theta \circ \Phi_0(r) = \varphi'_\theta(r) \circ y^{-1}.
\]

Thus,
\[ |\partial_r(f \circ y^{-1})|^2 = \left( (\partial_r f) \frac{\Phi'_0 \circ \Phi^{-1}_0 \circ \varphi_\theta(r)}{\varphi'_\theta(r)} \right)^2 \circ y^{-1}. \]

Combining this with (4.13), (4.16) and (4.17), we obtain

\[ |\nabla(f \circ y^{-1})|^2 = (\partial_r(f \circ y^{-1}))^2 + r^{-2} |\nabla_\theta(f \circ y^{-1})|^2 \]

\[ \leq \left( (\partial_r f) \frac{\Phi'_0 \circ \Phi^{-1}_0 \circ \varphi_\theta(r)}{\varphi'_\theta(r)} \right)^2 \circ y^{-1} \]

\[ + (\Phi^{-1}_0 \circ \varphi_\theta(r))^{-2} \circ y^{-1} \left\{ (1 + \varepsilon) (\partial_r f)^2 \left( \frac{|\nabla_\theta \varphi_\theta(r)|}{\varphi'_\theta(r)} \right)^2 + (1 + \varepsilon^{-1}) |\nabla_\theta f|^2 \right\} \circ y^{-1} \]

\[ = (\partial_r f)^2 \circ y^{-1} \left\{ \frac{(\Phi'_0 \circ \Phi^{-1}_0 \circ \varphi_\theta(r))^2}{(\varphi'_\theta(r))^2} + \frac{(1 + \varepsilon)|\nabla_\theta \varphi_\theta(r)|^2}{(\varphi'_\theta(r))^2(\Phi^{-1}_0 \circ \varphi_\theta(r))^2} \right\} \circ y^{-1} \]

\[ + (r \circ y^{-1})^{-2} |\nabla_\theta f|^2 \circ y^{-1} \left( \frac{(1 + \varepsilon^{-1}) r^2}{(\Phi^{-1}_0 \circ \varphi_\theta(r))^2} \right) \circ y^{-1} \]

\[ \leq |\nabla f|^2 \circ y^{-1} \max \left\{ \frac{(1 + \varepsilon^{-1}) r^2}{(\Phi^{-1}_0 \circ \varphi_\theta(r))^2}, \frac{(\Phi'_0 \circ \Phi^{-1}_0 \circ \varphi_\theta(r))^2}{(\varphi'_\theta(r))^2} + \frac{(1 + \varepsilon)|\nabla_\theta \varphi_\theta(r)|^2}{(\varphi'_\theta(r))^2(\Phi^{-1}_0 \circ \varphi_\theta(r))^2} \right\} \circ y^{-1} \]

for any \( \varepsilon > 0 \). Therefore,

\[ (4.18) \quad |\nabla(f \circ y^{-1})|^2 \leq (\alpha |\nabla f|^2) \circ y^{-1} \]

and hence (4.2) follows from (4.14) by letting \( g = f \circ y^{-1} \).

Finally, if \( h \) is constant then \( \mu \circ y^{-1} \) is the standard Gaussian measure. Hence, by [2, Theorem 4.3] one has

\[ W_2(\mu \circ y^{-1}, (f^2 \circ y^{-1})\mu \circ y^{-1})^2 + (\mu \circ y^{-1})(f^2 \circ y^{-1} \log f^2 \circ y^{-1}) \]

\[ \leq 2 \sqrt{2} (\mu \circ y^{-1}) |\nabla(f \circ y^{-1})|^2 W_2(\mu \circ y^{-1}, (f^2 \circ y^{-1})\mu \circ y^{-1}). \]

By combining this with (4.18) we prove (4.5). \( \square \)

**Proof of Corollary 4.3.** Since there exists a constant \( c_0 > 0 \) such that

\[ \Phi'_0(r) = c_0 r^{d-1} e^{-r^2/2} = \begin{cases} \Theta(r^{d-1}) & \text{as } r \to 0, \\ \Theta(r(1 - \Phi_0(r))) & \text{as } r \to \infty, \end{cases} \]

where \( f = \Theta(g) \) means that the two positive functions \( f \) and \( g \) are asymptotically bounded by each other up to constants, there exists a constant \( c \geq 1 \) such that

\[ \frac{1}{c} \Phi'_0(r) \leq \min \{ r, r^{d-1} \} (1 - \Phi_0(r)) \leq c \Phi'_0(r), \quad r \geq 0. \]

Equivalently,
Next, it is easy to see from (4.6) that

\[ \Phi^{-1}_0 \circ \varphi_\theta(r) = \begin{cases} \Theta(r^{\delta/2}) & \text{as } r \to \infty, \\ \Theta(r) & \text{as } r \to 0, \end{cases} \]

and

\[ \frac{1 - \varphi_\theta(r)}{\varphi'_\theta(r)} = \int_r^\infty s^{d-1} e^{V(s\theta)} ds \leq c r^{1-\delta} \]

for some constant \(c > 0\) and all \(r \geq 1\). Combining (4.19), (4.20) and (4.21) we obtain

\[ \max \left\{ \frac{r^2}{(\Phi^{-1}_0 \circ \varphi_\theta(r))^2}, \frac{(\Phi'_0 \circ \Phi^{-1}_0 \circ \varphi_\theta(r))^2}{(\varphi'_\theta(r))^2} \right\} \leq c (1 + r)^{2-\delta} \]

for some constant \(c > 0\).

If (4.7) holds then

\[ |\nabla_\theta \varphi_\theta(r)| = |\nabla_\theta (1 - \varphi_\theta(r))| \leq c_4 \min \left\{ r^d, \int_r^\infty s^{d-1} e^{V(s\theta)} ds \right\}, \]

so that due to (4.20) and (4.21)

\[ \frac{|\nabla_\theta \varphi_\theta(r)|^2}{(\varphi'_\theta(r))^2(\Phi^{-1}_0 \circ \varphi_\theta(r))^2} \leq c_5 \left( \frac{\min \left\{ r^d, \int_r^\infty s^{d-1} e^{V(s\theta)} ds \right\}}{(r^1_{r<1} + r^{\delta/2} 1_{r \geq 1}) r^{d-1} e^{V(r\theta)}} \right)^2 \leq c_6 (1 + r)^{2-3\delta} \]

for some constants \(c_5, c_6 > 0\). Combining this with (4.22) and Theorem 4.2 we prove (4.8).

Finally, for any \(x_1, x_2 \in \mathbb{R}^d\) let \(i \in \{1, 2\}\) such that \(|x_i| = |x_1| \vee |x_2|\). Similarly to the proof of Corollary 1.2 define

\[ f(x) = \frac{|x - x_i| \wedge |x_i|}{(1 + |x_i|)^{1-\delta/2}}, \quad x \in \mathbb{R}^d. \]

Then

\[ \Gamma(f, f) := (1 + |\cdot|)^{2-\delta} |\nabla f|^2 \leq \frac{1}{(1 + |x_i|)^{2-\delta}} \frac{1 + |x_i|^2}{(1 + |x_i|)^{1-\delta/2}} \leq C(\delta) \]
for some constant $C(\delta) > 0$. Since $|x_i| \geq \frac{1}{2}|x_1 - x_2|$, this implies that the intrinsic distance $\rho$ induced by $\Gamma$ satisfies

$$\rho(x_1, x_2)^2 \geq \frac{|f(x_1) - f(x_2)|^2}{C(\delta)} \geq C_1(\delta)\bar{\rho}(x_1, x_2)^2$$

for some constant $C_1(\delta) > 0$, and hence is complete. Thus, by [25, Theorem 1.1] or [26, Theorem 6.3.3], (4.9) follows from (4.8).

Acknowledgement. The author would like to thank the referees for useful comments.

References

[1] D. Bakry, M. Ledoux and F.-Y. Wang, Perturbations of functional inequalities using growth conditions, J. Math. Pures Appl. 87(2007), 394-407.

[2] S. G. Bobkov, I. Gentil and M. Ledoux, Hypercontractivity of Hamilton-Jacobi equations, J. Math. Pures Appl. 80(2001), 669–696.

[3] S. G. Bobkov, and F. Götze, Exponential integrability and transportation cost related to logarithmic Sobolev inequalities, J. Funct. Anal. 163(1999), 1–28.

[4] F. Bolley and C. Villani, Weighted csiszár-Kullback-Pinsker inequalities and applications to transportation inequalities, Ann. Fac. Sci. Toulouse Math. (6), 14(2005), 331–352.

[5] L. A. Caffarelli, The regularity of mappings with a convex potential, J. Amer. Math. Soc. 5(1992), 99–104.

[6] J. D. Deuschel and D. W. Stroock, Large Deviations, Pure and Appl. Math. Ser. 137, Academic Press, San Diego, 1989.

[7] J. D. Deuschel and D. W. Stroock, Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising models, J. Funct. Anal. 92(1990), 30–48.

[8] H. Djellout, A. Guillin and L.-M. Wu, Transportation cost-information inequalities for random dynamical systems and diffusions, Ann. Probab. 32(2004), 2702–2732.

[9] I. Gentil, A. Guillin and L. Miclo, Modified logarithmic Sobolev inequalities and transportation inequalities, Probab. Theory Relat. Fields 133(2005), 409–436.

[10] I. Gentil, A. Guillin and L. Miclo, Modified logarithmic Sobolev inequalities in null curvature, Revista Mat. Ibero. 23(2007).

[11] N. Gozlan, Characterization of Talagrand’s like transportation-cost inequalities on the real line, J. Funct. Anal. 250(2007), 400–425.
[12] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97(1976), 1061–1083.

[13] F.-Z. Gong and F.-Y. Wang, *Functional inequalities for uniformly integrable semigroups and application to essential spectrums*, Forum Math. 14(2002), 293–313.

[14] N. Gozlan, *Integral criteria for transportation cost inequalities*, Electron. Comm. Probab. 11(2006), 64–77.

[15] A. V. Kolesnikov, *Convexity inequalities and optimal transport of infinite dimensional measures*, J. Math. Pures Appl. 83(2004), 1373–1404.

[16] K. Marton, *A simple proof of the blowing-up lemma*, IEEE Trans. Inform. Theory, 32(1986), 445–446.

[17] F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. 173(2000), 361–400.

[18] F. Otto and C. Villani, *Comment on: “Hypercontractivity of Hamilton-Jacobi equations”, by S. Bobkov, I. Gentil and M. Ledoux*, J. Math. Pures Appl. 80(2001), 697–700.

[19] M. Röckner and F.-Y. Wang, *Weak Poincaré inequalities and L²-convergence rates of Markov semigroups*, J. Funct. Anal. 185(2001), 564–603.

[20] J. Shao, *Hamilton-Jacobi semigroup in infinite dimensional spaces*, Bull. Sci. Math. 130(2006), 720–738.

[21] M. Talagrand, *Transportation cost for Gaussian and other product measures*, Geom. Funct. Anal. 6(1996), 587–600.

[22] F.-Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields 109 (1997), 417–424.

[23] F.-Y. Wang, *Functional inequalities for empty essential spectrum*, J. Funct. Anal. 170(2000), 219–245.

[24] F.-Y. Wang, *Functional inequalities, semigroup properties and spectrum estimates*, Infin. Dimens. Anal. Quant. Probab. Relat. Topics 3(2000), 263–295.

[25] F.-Y. Wang, *Probability distance inequalities on Riemannian manifolds and path spaces*, J. Funct. Anal. 206(2004), 167–190.

[26] F.-Y. Wang, *Functional Inequalities, Markov Processes, and Spectral Theory*, Science Press, Beijing 2005.