Asymptotics for minimizers of a Donaldson functional and mean curvature 1-immersions of surfaces into hyperbolic 3-manifolds

Gabriella Tarantello

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Abstract

It has been shown in [17] that, for given $|c| < 1$, the moduli space of constant mean curvature (CMC) $c$-immersions of a closed orientable surface of genus $g \geq 2$ into a hyperbolic 3-manifold can be parametrized by elements of the tangent bundle of the corresponding Teichmüller space. This is attained by showing the unique solvability of the Gauss-Codazzi equations governing (CMC) $c$-immersions. The corresponding unique solution is identified as the global minimum (and only critical point) of the Donaldson functional $D_t$ (introduced in [11]) given in (1.3) with $t = 1 - c^2$.

When $|c| \geq 1$ (i.e. $t \leq 0$), so far nothing is known about the existence of analogous (CMC) $c$-immersions. Indeed, for $t \leq 0$ the functional $D_t$ may no longer be bounded from below and evident non-existence situations do occur.

Already the case $|c| = 1$ (i.e. $t = 0$) appears rather involved and actually (CMC) 1-immersions can be attained only as "limits" of (CMC) $c$-immersions for $|c| \to 1^-$ (see Theorem 2). To handle this situation, here we analyse the asymptotic behaviour of minimizers of $D_t$ as $t \to 0^+$.

We use an accurate asymptotic analysis (see [34]) to describe possible blow-up phenomena. In this way, we can relate the existence of (CMC) 1-immersions to the Kodaira map defined in (1.1). As a consequence, when the genus $g = 2$, we obtain the first existence and uniqueness result about (CMC) 1-immersions of surfaces into hyperbolic 3-manifolds.

1 Introduction

Let $S$ be an oriented closed surface with genus $g \geq 2$ and denote by $\mathcal{T}_g(S)$ the Teichmüller space of conformal structures on $S$, modulo biholomorphisms in the homotopy class of the identity.

The Teichmüller space has proved to be particularly useful in the description of various moduli spaces, and here we explore such possibility for constant mean curvature (CMC) $c$-immersions of $S$ into hyperbolic 3-manifolds. We recall (from [36,11]) that, for a given conformal class $X \in \mathcal{T}_g(S)$, such immersions
are governed by solutions of the Gauss-Codazzi equations relating the pull-back metric on $X$ with the second fundamental form $II$ of the immersion. Actually, the Codazzi equation just states that the $(2,0)$-part of $II$ is a holomorphic quadratic differential $\alpha$ (i.e. the Hopf differential) which completely identifies $II$. In particular for minimal (i.e. $c = 0$) immersions we have $II = \text{Re}(\alpha)$. Thus, Uhlenbeck in [36] proposed a parametrization of such minimal immersions in terms of elements of the cotangent bundle of $Tg(S)$, as described by the pairs: $(X, \alpha) \in Tg(S) \times C_2(X)$, where $C_2(X)$ is the space of holomorphic quadratic differentials on $X$. The role of holomorphic quadratic differentials in Teichmüller theory has emerged naturally also in relation with harmonic maps.

Thus, to pursue Uhlenbeck’s parametrization, it appears reasonable to seek minimal immersions with assigned second fundamental form, that is to set: $II = \text{Re}(\alpha)$, for a given $\alpha \in C_2(X)$. In this way, one is reduced to solve the Gauss equation (of Liouville type) for the pull-back metric in terms of the (given) second fundamental form.

However, as discussed in [15], [16], a minimal immersion with assigned second fundamental form may not exist, or when it exists, it may not be unique (see also [14]). So, although suggestive, this approach does not yield a one-to-one correspondence between a minimal immersion and a solution of the Gauss-Codazzi equations, with respect to a given pair $(X, \alpha)$.

On the contrary, it has proved more successful the approach adopted by Goncalves and Uhlenbeck in [11] where, more generally, the authors propose to parametrize constant mean curvature (CMC) immersions of $S$ into 3-manifolds of constant sectional curvature $-1$, in terms of elements of the tangent bundle of $Tg(S)$. By recalling the isomorphism: $C_2(X) \simeq (H^{0,1}(X, E))^*$, where $E = T^1,0_X$ is the holomorphic tangent bundle of $X$ and $H^{0,1}(X, E)$ is the Dolbeault cohomology group of $(0,1)$-forms valued in $E$ (see (2.2)), we find a parametrization of the tangent bundle of $Tg(S)$ by the pairs: $(X, [\beta]) \in Tg(S) \times H^{0,1}(X, E)$.

Accordingly, it has been proved in [17] that (as anticipated by [11]) the following holds:

**Theorem A ([11],[17]).** For given $c \in (-1,1)$ there is a one-to-one correspondence between the space of constant mean curvature $c$ immersions of $S$ into a 3-manifold of constant sectional curvature $-1$ and the tangent bundle of $Tg(S)$, parametrized by the pairs: $(X, [\beta]) \in Tg(S) \times H^{0,1}(X, E)$, $E = T^1,0_X$.

As we shall discuss in Section 1.1 (see Remark 1.1), the datum $(X, [\beta])$ characterizes the Donaldson functional $D_t$ given in (1.3)-(1.4), whose global minimum (and only critical point) uniquely identifies the corresponding (CMC) $c$-immersion. We recall that, the functional $D_t$ was introduced in [11] as the appropriate "lagrangean" of the Gauss-Codazzi equations governing the immersion.

At this point it is natural to ask whether, for a given pair $(X, [\beta])$, such (CMC)-immersions do exist also for $|c| \geq 1$.

We face an evident non-existence situation when $[\beta] = 0$ (see Section 2 for details), while for $[\beta] \neq 0$ such question is completely open to investigation. In
this note, we start to tackle the case $|c| = 1$ and refer to such immersions as (CMC) 1-immersions. Starting with the work of [8], (CMC) 1-immersions of surfaces into the hyperbolic space $\mathbb{H}^3$ have played a relevant role in hyperbolic geometry, in view of their striking analogies with minimal immersions into the euclidean space $\mathbb{E}^3$, see also [31], [37].

Since here we deal with compact surfaces, in view of those analogies, we expect that for $|c| = 1$ such immersions should "favor" the presence of "punctures" at a finite number of points. Indeed, in our analysis those points occur as the "blow-up" points, and form the so called "blow-up set" appearing in Theorem 3 and Theorem 10 below. But surprisingly, when the genus $g = 2$, we are able to find "regular" (CMC) 1-immersions. Actually, their existence will be formulated in terms of the Kodaira map discussed in section 12.1.3 of [10], and specified as follows:

$$\tau : X \to \mathbb{P}(V^*), \quad V = C_2(X);$$

which defines a holomorphic map of $X$ into the projective space of $V^* = \mathcal{H}^{0,1}(X, E)$ with $E = T_X^{1,0}$, see [10] and Section 2 for details.

The role of the projective space $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$, $E = T_X^{1,0}$, is readily explained. Indeed, if to a pair $(X, [\beta])$ there corresponds a (CMC) 1-immersion, then $[\beta] \neq 0$ and it exists a (CMC) 1-immersion also corresponding to the data $(X, \lambda[\beta])$, for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Recall that, for $E = T_X^{1,0}$ we have: $\mathbb{P}(\mathcal{H}^{0,1}(X, E)) \simeq \mathbb{P}^{3g-4}$ and $3g - 4 \geq 2$ for $g \geq 2$, therefore $\dim_{\mathbb{C}} \mathbb{P}(\mathcal{H}^{0,1}(X, E)) \geq 2$. On the other hand, we shall see in Lemma 2.3 that the image $\tau(X)$ defines a complex curve (hence of complex dimension one) into $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$, whence $\tau(X) \subseteq \mathbb{P}(\mathcal{H}^{0,1}(X, E))$, and actually $\tau(X)$ is a "tiny" subset of $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$. If for $[\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}$ we let $[\beta]^p$ denote the element of the complex line $\lambda[\beta], \lambda \in \mathbb{C} \setminus \{0\}$, then we prove:

**Theorem 1.** If $g = 2$, then to every $(X, [\beta]) \in T_g(X) \times (\mathcal{H}^{0,1}(X, E) \setminus \{0\})$ with projective representative $[\beta]^p \notin \tau(X)$, there corresponds a unique (CMC) 1-immersion of $X$ into a 3-manifold $M(\simeq S \times \mathbb{R})$ with sectional curvature $-1$.

Let us emphasize once more that, since $\tau(X)$ is a complex curve into a complex space of dimension 2 (for $g = 2$) then Theorem 1 applies for a "generic" class $[\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}$, and it furnishes the first existence result about (CMC) 1-immersions.

We obtain such an existence result "variationally", from a strict global minimum (and unique critical point) of the Donaldson functional $D_0$ in [16]. Based on "stability" arguments, we expect existence also of (CMC) $c$-immersions with $|c| > 1$ but close to 1.

It would be interesting to inquire if (CMC) 1-immersions as described in Theorem 1 could exist also when $[\beta]^p \in \tau(X)$. 

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To establish Theorem 1, firstly for any genus \( g \geq 2 \) we provide a detailed asymptotic analysis about the (CMC)-immersions given in Theorem A (with \(|c| < 1\)) as \(|c| \to 1^-\). Indeed, by Theorem 2, (CMC) 1-immersions can be attained only as "limits" in this way.

For this purpose, we need to handle possible "blow-up" situations for the pull-back metrics. This analysis becomes particularly delicate when "blow-up" occurs around points of "collapsing" zeroes of the holomorphic quadratic differentials governing the second fundamental form. Recall that indeed, any holomorphic quadratic differential admits \( 4(g-1) \) zeroes in \( X \), counted with multiplicity.

By the recent estimates obtained in [34] for solutions of Liouville equations with "collapsing" Dirac sources, we are able to overcome such difficulties when "blow-up" occurs with the least possible "blow-up mass" (see Section 3.1 for details). This property is always guaranteed for genus \( g = 2 \). But we expect it to hold also for any genus \( g \geq 2 \), by virtue of the "minimising" character of the (CMC) c-immersions of Theorem A, from the point of view of the Donaldson functional (1.3).

We refer to the following sections for details and more specifically to our main Theorem 3, Theorem 5 and Theorem 10 (and their corollaries) for precise statements of our results. They contain useful information in case of "blow-up", including a crucial "orthogonality" condition imposed on the class \([\beta]\), as stated in (1.16), (1.19) and (3.70).

This enables us to relate the "blow-up" phenomenon to the image of the Kodaira map and to establish Theorem 1. Equivalently, we refer to Theorem 6 concerning the existence for extremals of the associated (non-coercive) Donaldson function \( D_0 \) (in (1.6)).

1.1 Statement of the Main Results.

We take a slightly more general point of view.

For a given conformal class \( X \in \mathcal{T}_g(S) \) (identified with the corresponding Riemann surface), we recall that, \( T^{1,0}_X \) denotes the holomorphic tangent bundle of \( X \), whose dual \((T(X)^{1,0})^* = K_X\) coincides with the canonical bundle \( K_X \) of \( X \). Then, for \( \kappa \geq 2 \), we define the holomorphic line bundle: \( E = \otimes^{\kappa-1} T^{1,0}_X \), equipped with the corresponding holomorphic structure \( \bar{\partial} = \bar{\partial}_E \).

We let \( A^0(E) \) be the space of smooth sections of \( E \) and \( A^{0,1}(X, E) = A^{0,1}(X, \mathbb{C}) \otimes E \) be the space of \((0,1)\)-forms of \( X \) valued on \( E \). By considering the \( d\)-bar operator \( \bar{\partial} : A^0(E) \to A^{0,1}(X, E) \), we may define the \((0,1)\)-Dolbeault cohomology group: \( \mathcal{H}^{0,1}(X, E) = A^{0,1}(X, E)/\bar{\partial}(A^0(E)) \) so that, every cohomology class \([\beta]\) in \( \mathcal{H}^{0,1}(X, E) \) is formed by \((0,1)\)-forms valued in \( E \) of the type: \( \beta + \bar{\partial} \eta, \forall \eta \in A^0(E) \).

Furthermore, we can use the induced complex structure \( \bar{\partial} \) over the holomorphic line bundle \( \otimes^k(T^{1,0}_X)^* \), to define the space \( C_\kappa(X) \) of holomorphic \( \kappa \)-differentials on \( X \) as given by the holomorphic sections of \( E^* \otimes K_X = \otimes^\kappa (T^{1,0}_X)^* \),
that is:

\[ C_\kappa(X) = H^0(X, E^* \otimes K_X) := \{ \alpha \in A^0(X, \otimes^k(T^{1,0}_X)^*) : \bar{\partial}\alpha = 0 \}. \]

So \( C_\kappa(X) \subset A^0(E^* \otimes K_X) = A^{1,0}(X, E^*) \), and by the Riemann Roch theorem, we know that:

\[ \dim_C C_\kappa(X) = (2\kappa - 1)(g - 1), \quad \text{for } \kappa \geq 1 \]

see [12].

In particular, if \( \kappa = 2 \) then \( \dim_C C_2(X) = 3(g - 1) \), and since \( T_\kappa(S) \) has the structure of a differential cell of real dimension \( 6(g - 1) \) (see [13]), as already mentioned, we can parametrize the cotangent bundle of \( T_\kappa(S) \) by the pairs \((X, \alpha) \in T_\kappa(X) \times C_2(X)\). Similarly, in view of the isomorphism: \( C_\kappa(X) \cong (H^{0,1}(X,E))^* \) (see Section 2 for details), then for \( \kappa = 2 \), we see also that the tangent bundle of \( T_\kappa(S) \) can be parametrized by the pairs \((X, [\beta]) \in T_\kappa(S) \times H^{0,1}(X,E)\).

Next, we consider on \( X \) the unique hyperbolic metric \( g_X \) (from the "uniformization" of \( X \)) with induced scalar product \( \langle \cdot, \cdot \rangle \), norm \( \| \cdot \| \) and volume form \( dA \). Unless specified otherwise, we shall always consider \( g_X \) as the background metric on \( X \).

By using the Hermitian metric induced by \( g_X \) on \( X \), we can define an Hermitian structure on \( E \), with induced fiberwise Hermitian product \( \langle \cdot, \cdot \rangle_E \) and corresponding (fiberwise) norm \( \| \cdot \|_E \) of sections and forms valued in \( E \).

Also, as detailed in Section 2, we can introduce the wedge product: \( \beta \wedge \alpha \), for \( \beta \in A^{0,1}(X,E) \) and \( \alpha \in A^{1,0}(X,E^*) \), and then define the Hodge star operator \( *_E : A^{0,1}(X,E) \to A^{1,0}(X,E^*) \) in terms of the Hermitian product \( \langle \cdot, \cdot \rangle \). We know that, \( *_E \) is an isomorphism with inverse \( *^{-1}_E \), and actually it defines an isometry with respect to the "dual" (fiberwise) Hermitian product on \( E^* \):

\[ \langle \cdot, \cdot \rangle_{E^*} = \langle *^{-1}_E \cdot, *^{-1}_E \cdot \rangle_E \quad \text{with norm } \| \cdot \|_{E^*} = \| *^{-1}_E \cdot \|_E. \]

In the sequel we shall drop the lower script \( E^* \) and \( E \) for the Hermitian product and norm, unless there is some ambiguity.

By Dolbeault decomposition, any \( \beta \in A^{0,1}(X,E) \) can be expressed uniquely as follows:

\[ \beta = \beta_0 + \bar{\partial}\eta \quad \text{with harmonic } \beta_0 \in A^{0,1}(X,E) \quad \text{(with respect to } g_X), \quad \eta \in A^0(E). \]

So \( \beta_0 \in [\beta] \), and every cohomology class in \( H^{0,1}(X,E) \) is uniquely identified by its harmonic representative. More importantly, we have:

\[ \beta_0 \in A^{0,1}(X,E) \quad \text{is harmonic } \iff *_E \beta_0 \in C_\kappa(X) \]

see Section 2 for details.

In particular, if for a smooth function \( u \) in \( X \) we have: \( \bar{\partial} *_E e^{2u} \beta = 0 \) then \( \beta \) is harmonic with respect to the metric \( h = e^u g_X \).
For any pair \((X, [\beta])\), with \(\beta_0 \in [\beta] \) the harmonic representative of the class \([\beta]\), we define (in the terminology of [11]) the Donaldson functional:

\[
D_t (u, \eta) = \int_X \left( \frac{\vert \nabla u \vert^2}{4} - u + te^u + 4e^{(\kappa - 1)u} \| \beta_0 + \bar{\partial} \eta \|_2^2 \right) dA,
\]

\(t \in \mathbb{R}\), with “natural” (convex) domain:

\[
\Lambda = \{ (u, \eta) \in H^1 (X) \times W^{1,2} (X, E) : \int_X e^{(\kappa - 1)u} \| \beta_0 + \bar{\partial} \eta \|^2 dA < \infty \},
\]

with the usual Sobolev spaces \(H^1 (X)\) of \(X\), and \(W^{1,2} (X, E)\) of sections of \(E\) (see (2.3)). Clearly, for \(t > 0\), the functional \(D_t\) is bounded from below in \(\Lambda\).

As observed in [11], at least locally, it is possible to construct a (CMC) immersion of \(X\) with constant \(c\) into a 3-manifold having sectional curvature \(-1\), directly from a critical point of the Donaldson functional \(D_t\), when we take:

\[
t = (1 - c^2) \quad \text{and} \quad \kappa = 2.
\]

Indeed, if \((u, \eta)\) is a (weak) critical point of \(D_t\) (in the sense of (3.1) below), then it is smooth and satisfies:

\[
\begin{align*}
\Delta u + 2 - 2te^u - 8(\kappa - 1)e^{(\kappa - 1)u} \| \beta_0 + \bar{\partial} \eta \|_2^2 &= 0 \quad \text{in} \ X, \\
\bar{\partial} (e^{(\kappa - 1)u} \star_E (\beta_0 + \bar{\partial} \eta)) &= 0.
\end{align*}
\]

From the second equation in (1.5) we see that, \(\beta_0 + \bar{\partial} \eta \in [\beta]\) is harmonic with respect to the metric \(h = e^{\kappa u - 1} g_X\).

More importantly, from [36] we have:

**Remark 1.1.** If \((u, \eta)\) satisfies (1.5) with \(\kappa = 2\) and \(t = (1 - c^2)\), then \((X, e^u g_X)\) can be immersed as a (CMC) surface with constant \(c\) into a suitable 3-manifold \(M^3 \cong S \times \mathbb{R}\) of sectional curvature \(-1\). Furthermore, the \((2, 0)\)-part of the second fundamental form \(\mathcal{H}\) of the immersion is given by: \(\alpha = e^u (\beta_0 + \bar{\partial} \eta) \in C^{2} (X)\).

We refer to [11], [10], and [17] for details.

Actually, the system (1.5) can be formulated as the Hitchin selfduality equations [13] for a suitable nilpotent \(SL(2, \mathbb{C})\) Higgs bundle, we refer to [1], [17] for details, see also [22] for a similar formulation concerning Harmonic maps in relation to minimal immersions. Under this point of view, it becomes clear why we refer to \(D_t\) as a Donaldson functional.

Interestingly, (as also anticipated in [11]) for \(t > 0\), it is possible to show the unique solvability of (1.5).

**Theorem B** ([17]). For given \((X, [\beta]) \in T_0 (S) \times \mathcal{H}^{0,1} (X, E)\) and \(t > 0\), the functional \(D_t\) admits a unique critical point \((u_t, \eta_t)\) which corresponds to the global minimum of \(D_t\) in \(\Lambda\). In particular, \((u_t, \eta_t)\) is smooth and it is the only solution of (1.5).
Such a uniqueness result yields also to several interesting algebraic consequences. For example (as already mentioned) for $\kappa = 2$ and $c = 0$, we derive a parametrization for the moduli space of minimal immersions of $S$ into a "germ" of a hyperbolic 3-manifold (cf. [35]) by elements of $T_\theta(S) \times H^{0,1}(X, E), E = T^1_{X,0}$.

We can lift such information to minimal immersions of the Poincaré disk $\mathbb{D}$ into the hyperbolic space $\mathbb{H}^3$ which are equivariant with respect to an irreducible representation: $\rho : \pi_1(S) \rightarrow PSL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$ is the (orientation preserving) isometry group of $\mathbb{H}^3$ (see [36]). As a consequence, we obtain an analogous parametrization for such irreducible representations as well, see [17] and [26], [27].

By taking $\kappa = 3$, a similar conclusion may be attained for equivariant minimal Lagrangian immersions of $\mathbb{D}$ into $CH^2$. Again we refer to [17], [26], [27] and the references therein for details.

At this point, to find (CMC) immersions of $X$ with constant $c : |c| \geq 1$, we need to see whether $D_t$ admits a critical point when we take $t \leq 0$. As we shall see, this is not an easy problem to tackle, even for $t = 0$. Indeed, the functional:

$$D_0(u, \eta) = \int X \left( \frac{1}{4} |\nabla u|^2 - u + 4e^{(\kappa-1)u} \|\beta_0 + \partial \eta\|^2 \right) dA, \quad (1.6)$$

may no longer be bounded from below or coercive in $\Lambda$, and actually the system [1.5] may not admit a solution for $t = 0$. This occurs for example when we take $[\beta] = 0$ (i.e. $\beta_0 = 0$), where we find: $u_t = \ln \frac{1}{t} \rightarrow +\infty$, $\eta_t = 0$, $D_t(u_t, \eta_t) \rightarrow -\infty$, as $t \rightarrow 0^+$, so $D_0$ is unbounded from below in $\Lambda$, and in fact (1.5) cannot be solved for $t = 0$ and $\beta_0 = 0$.

On the other hand, it is clear that, if for $[\beta] \in H^{0,1}(X, E) \setminus \{0\}$, the Donaldson functional $D_0$ attains its global minimum in $\Lambda$ (i.e. [1.5] at $t = 0$ is solvable), then, by a simple scaling argument, also the functional corresponding to $\lambda[\beta] = [\lambda\beta], \lambda \in \mathbb{C} \setminus \{0\}$, will have the same property.

Furthermore, if we assume the existence of a solution for the system [1.5] at $t = 0$, then we can prove that the corresponding functional $D_0$ preserves the same properties of $D_t$, for $t > 0$, in the sense that the following holds:

**Theorem 2.** If $(u_0, \eta_0)$ is a (smooth) solution for the system [1.5] with $t = 0$ then,

(i) $(u_t, \eta_t) \rightarrow (u_0, \eta_0)$, as $t \rightarrow 0^+$, in $V_p := H^1(X) \times W^{1,p}(X, E), \forall p > 2$.

(ii) $D_0$ is bounded from below in $\Lambda$ and attains its global minimum at $(u_0, \eta_0)$.

Furthermore $(u_0, \eta_0)$ is the only critical point of $D_0$ and hence the only solution of [1.5] for $t = 0$.

In view of Theorem 2 to identify possible critical points for $D_0$, we must investigate whether $(u_t, \eta_t)$ survives the passage to the limit, as $t \rightarrow 0^+$. 


Besides the existence of (CMC)-immersions with $c = \pm 1$, when $\kappa \geq 2$, such an asymptotic analysis permits to follow the behaviour of the global minimizer $(u_\lambda, \eta_\lambda)$ of the Donaldson functional:

$$D(u, \eta) = \int_X \left( \frac{\|\nabla u\|^2}{4} - u + e^{u} + 4e^{(\kappa - 1)u}\|\lambda \beta_0 + \nabla \eta\|^2 \right) dA$$

(1.7)

along the ray of cohomology classes: $[\lambda \beta]$, with $\lambda \in \mathbb{C} \setminus \{0\}$ and $[\beta] \neq 0$ fixed in $H^{0,1}(X, E)$. Indeed, via the transformations:

$$t = |\lambda|^{-\frac{2}{k-1}}, \ u_t = u_\lambda + \frac{2}{k-1} \log |\lambda|, \ \eta_t = \frac{1}{\lambda} \eta_\lambda,$$

$$D_t(u_t, \eta_t) = D(u_\lambda, \eta_\lambda) - 4\pi(g - 1) \log |\lambda|^{\frac{2}{k-1}}$$

we can recast the analysis of $(u_\lambda, \eta_\lambda)$ (the global minimum of $D$ in (1.7)), as $|\lambda| \to +\infty$, to the analysis of $(u_t, \eta_t)$ (the global minimum of $D_t$ in (1.3)), as $t \to 0^+$. We begin our asymptotic analysis by using the strict positivity of the Hessian $D''_t$ at $(u_t, \eta_t)$ (see [11], [17]) and the Implicit Function Theorem to show the $C^2$-dependence of $(u_t, \eta_t)$ with respect to $t \in (0, +\infty)$. More interestingly, we show that the expression:

$$t \int_X e^{u_t} dA$$

is increasing as a function of $t \in (0, +\infty)$, see Lemma 3.6. Since (after integration over $X$ of the first equation in (1.5)) we have:

$$t \int_X e^{u_t} dA + 4(\kappa - 1) \int_X e^{(k-1)u_t}\|\beta_0 + \nabla \eta_t\|^2 dA = 4\pi(g - 1)$$

we may conclude that

$$\rho_t([\beta]) := 4(\kappa - 1) \int_X e^{(k-1)u_t}\|\beta_0 + \nabla \eta_t\|^2 dA \in (0, 4\pi(g - 1))$$

is decreasing in $t \in (0, +\infty)$. Thus, it is well defined the value:

$$\rho([\beta]) := \lim_{t \to 0^+} \rho_t([\beta]) = \sup_{t>0} \rho_t([\beta]) \leq 4\pi(g - 1), \quad (1.8)$$

which we wish to identify in terms of the fixed cohomology class $[\beta] \in H^{0,1}(X, E)$, or equivalently in terms of its harmonic representative $\beta_0 \in [\beta]$. To this purpose we start by showing the following:

**Proposition 1.1.**

(i) $\rho([\beta]) = 0$ if and only if $[\beta] = 0$; and if $[\beta] \neq 0$ then $\rho([\beta]) \geq \frac{4\pi}{\kappa - 1}$.

(ii) If $D_0$ is bounded from below on $\Lambda$ then $\rho([\beta]) = 4\pi(g - 1)$.

From (i) it follows in particular that, for $[\beta] \neq 0$ the interval $(0, \rho([\beta]))$ gives the range of $\rho_t([\beta])$, for $t \in (0, +\infty)$. Furthermore, from (1.8) and Proposition 1.1 we derive:

$$\rho([\beta]) = 4\pi, \ \forall [\beta] \neq 0. \quad (1.9)$$

Also Proposition 1.1 prompts us to ask the following questions:
(1) if \( \rho([\beta]) = 4\pi(g - 1) \) then is it true that \( D_0 \) is bounded from below in \( \Lambda \)?

(2) if \( D_0 \) is bounded from below in \( \Lambda \), then for which class \([\beta] \neq 0\) is the infimum attained?

In view of (1.3), in the sequel we shall provide an affirmative answer to question (1) and (2) when the genus \( g = 2 \) (see Theorem 5 and Theorem 9). In addition, when \( g \geq 3 \), we shall give strong indications towards an affirmative answer to question (1).

On the ground of Theorem 2 to explore question (2) for \( g \geq 3 \), we shall introduce appropriate tools to investigate the asymptotic behavior of the minimizers of \( D_t \), as \( t \rightarrow 0^+ \).

To be more precise, for fixed \([\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}\) with harmonic representative \( \beta_0 \in [\beta] \neq 0 \) we set:

\[
\beta_t = \beta_0 + \overline{\partial} \eta_t \in A^{0,1}(X, E) \quad \text{and} \quad \alpha_t = e^{(\kappa-1)u_t} \ast_E \beta_t \in C_\kappa(X) \setminus \{0\}.
\]

We recall in particular that, \( \beta_t \) is harmonic with respect to the metric \( h = e^{\frac{\kappa-1}{2}u_t} g_X \) on \( X \).

It is a consequence of the Riemann-Roch theorem [18], that any holomorphic \( \kappa \)-differential in \( X \) admits \( 2\kappa(g - 1) \) zeroes counted with multiplicity. Thus, we let \( Z_t \) be the finite set of distinct zeroes of \( \alpha_t \) whose multiplicities add up to \( 2\kappa(g - 1) \). Hence, in terms of the fiberwise norm of \( \alpha_t \) we have:

\[
\|\alpha_t\|(q) = \|\alpha_t\|_{E^*(q)} > 0, \forall \, q \in X \setminus Z_t.
\]

Since \( C_\kappa(X) \) is finite dimensional (see (1.2)), all norms of \( \alpha_t \) are equivalent, and it is usual (recall the Weil-Petterson form [18]) to consider the (well defined) \( L^2 \)-norm:

\[
\|\alpha\|_{L^2} = \left( \int_X \|\alpha\|^2 \, dA \right)^{\frac{1}{2}}, \quad \text{for} \quad \alpha \in C_\kappa(X).
\]

We let,

\[
s_t \in \mathbb{R} : e^{(\kappa-1)s_t} = \|\alpha_t\|_{L^2}^2 \quad \text{and} \quad \hat{\alpha}_t = \frac{\alpha_t}{\|\alpha_t\|_{L^2}} = e^{-(\kappa-1)s_t} \alpha_t.
\]

In order to attain an accurate asymptotic description about the behavior of \((u_t, \eta_t)\), as \( t \rightarrow 0^+ \), we need to account for possible blow-up phenomena of

\[
\xi_t := -u_t + s_t,
\]

which occur when there holds: \( \max_X \xi_t \rightarrow \infty \), as \( t \rightarrow 0^+ \).

In this respect, it will not suffice to use the well known blow-up analysis developed in [7], [21], [5] for solutions of Liouville equations. Indeed, we face a particularly delicate situation in what we call the "collapsing" case, namely when, along a sequence \( t_k \rightarrow 0^+ \), we have that \( \xi_k = \xi_{t_k} \) "blows-up" around a point where different zeroes of \( \hat{\alpha}_k = \hat{\alpha}_{t_k} \) collapse together. To be more precise, by (1.10) we may suppose that:

\[
\hat{\alpha}_k = \hat{\alpha}_{t_k} \rightarrow \hat{\alpha}_0, \quad \text{as} \quad k \rightarrow \infty, \quad \text{with} \quad \hat{\alpha}_0 \in C_\kappa(X) \quad \text{and} \quad \|\hat{\alpha}_0\|_{L^2} = 1,
\]

and the \( 2\kappa(g - 1) \) zeroes (counted with multiplicity) of \( \hat{\alpha}_0 \) correspond to the limit points of the zeroes of \( \hat{\alpha}_k \) in \( Z_k = Z_{t_k} \), as \( k \rightarrow \infty \).
In particular, we find a suitable integer $1 \leq N \leq 2\kappa(\mathfrak{g} - 1)$ such that (for $k$ large) the set $Z_k$ of distinct zeroes of $\hat{\alpha}_k$ is given by:

$$Z_k = \{z_{1,k}, \ldots, z_{N,k}\} \quad \text{and} \quad z_{j,k} \neq z_{l,k}, \ l \neq j \in \{1, \ldots, N\},$$

and every $z_{j,k}$ admits multiplicity $n_j \in \mathbb{N}$ with $\sum_{j=1}^{N} n_j = 2\kappa(\mathfrak{g} - 1)$. Furthermore, $z_{j,k} \to z_j$, as $k \to \infty$, and the set

$$Z = \{z_1, \ldots, z_N\}$$

is the zero set of $\hat{\alpha}_0$. However, now we cannot guarantee that the points in $Z$ are distinct. So we denote by $Z_0$ the subset (possibly empty) of $Z$ where different zeroes of $\hat{\alpha}_k$ collapse together. Namely,

$$Z_0 = \{z \in Z : \exists \ s \geq 2, 1 \leq j_1 < \ldots < j_s \leq N \text{ such that } z = z_{j_1} = \ldots = z_{j_s} \text{ and } z \notin Z \setminus \{z_{j_1}, \ldots, z_{j_s}\}\} \quad (1.12)$$

The "blow-up" analysis of $\xi_k$ needs a particular attention when blow-up occurs around a point in $Z_0$.

To reduce technicalities, we handle such a delicate "collapsing" situation only in case: $\kappa = 2$.

After [33], a scenario of blow-up in the "collapsing" situation was first illustrated in [19] and [25], where the new phenomenon of "blow-up without concentration" was recorded. See [20], [21] for a description of similar phenomena in the context of systems.

A more detailed blow-up analysis was recently presented in [34]. A first fact, explicitly stated in Theorem D below, allows us to ensure that (even in the "collapsing" case) blow-up can occur around at most a finite number of "blow-up points" with quantized "blow-up mass" of at least $8\pi$, see [19], [20] and [34].

Interestingly, in case blow-up occurs with the least possible blow-up mass $8\pi$, the pointwise estimates obtained in the "collapsing" case in [34] (stated explicitly in Theorem E below), are in striking analogy with the sharp "single bubble" estimates obtained in [9] and [23] for the non-vanishing (hence non-collapsing) case. Observe that no "bubble" profile is available in the "collapsing" situation. We refer the reader to [34] for details.

For the sequence $\xi_k = \xi_{t_k}$, we can use Theorem D to conclude that,

(i) either (compactness) : $\xi_k \to \xi_0$ in $C^2(X)$, as $k \to +\infty$, and the functional $D_0$ is bounded from below and attains its infimum in $\Lambda$;

(ii) or (blow-up) : $\xi_k$ admits a finite blow-up set

$$S = \{q_1, \ldots, q_m\} \quad \text{with} \quad m \in \{1, \ldots, \mathfrak{g} - 1\},$$

(i.e. $\xi_k^+$ is bounded uniformly on compact sets of $X \setminus S$) with blow-up mass:

$$\sigma(q) := \lim_{r \to 0^+} \left( \lim_{k \to +\infty} 8 \int_{B_r(q)} e^{u_{t_k}} \|\beta_0 + \overline{\partial} \eta_{t_k}\|^2 dA \right) \in 8\pi \mathbb{N}, \ \forall \ q \in S; \quad (1.13)$$
(recall that \( \kappa = 2 \)). Furthermore, we may have "blow-up with concentration", or "blow-up without concentration", as described respectively in part (ii)-(a) or (ii)-(b) of Theorem \( \text{[1]} \) below.

Thus, we focus on the "blow-up" case. Due to the minimizing property of \((u_t, \eta_t)\), we expect that the corresponding "blow-up mass" should be the least possible (namely \(8\pi\)). In fact, it is likely that \(\xi_k\) admits only one blow-up point (see Remark \( \text{[3.3]} \)) and it cannot occur in \(Z \setminus Z_0\) since, by \([5]\), it would have a "blow-up mass" larger than \(8\pi\). This latter possibility could also be excluded by the "vanishing condition", recently identified by Wei and Zhang in \([39], [40]\).

Therefore, we expect that \(\xi_k\) blows up either at a point in \(X \setminus Z\) (not a zero of \(\hat{\alpha}_0\)) or it must be a point of "collapsing" zeroes in \(Z_0\). So, next we shall analyze the blow-up behavior of \(\xi_k\) in those situations.

To this purpose, for given distinct points \(\{x_1, \ldots, x_\nu\} \subset X\) we introduce the following subspace of \(C_2(X)\):

\[
Q_2[x_1, \ldots, x_\nu] = \{\alpha \in C_2(X) : \alpha \text{ vanishes at } x_1, \ldots, x_\nu\},
\]

and, by the Riemann-Roch theorem, we know that,

\[
\dim_C Q_2[x_1, \ldots, x_\nu] = 3(g - 1) - \nu, \text{ for } 1 \leq \nu < 2(g - 1),
\]

(1.14)

see Section 2 for details.

We start by considering the case \(S \cap Z = \emptyset\), where we know that only "blow-up with concentration" occurs \([2], [23]\) and so, for \(\alpha_k = \alpha_{t_k}\), we have:

\[
Se^{u_k}||\alpha_k||^2 = 8||\hat{\alpha}_k||^2 e^{\xi_k} \rightharpoonup 8\pi \sum_{l=1}^{m} \delta_{q_l} \text{ weakly in the sense of measures},
\]

and \(\rho([\beta]) = 4\pi m\). We have:

**Theorem 3.** Let \(\kappa = 2\), \([\beta] \neq 0\) and \(S = \{q_1, \ldots, q_m\}\) with \(m \in \{1, \ldots, g - 1\}\) be the (non empty) blow-up set of \(\xi_k = \xi_{t_k}\). If \(S \cap Z = \emptyset\) then (along a subsequence), as \(k \to +\infty\), we have:

\[
\alpha_k \to \alpha_0 \in C_2(X) \text{ (in any norm) with } \alpha_0 \neq 0 \text{ vanishing exactly at } Z
\]

\[
e^{-u_{t_k}} \to \pi \sum_{q \in S} \frac{1}{||\alpha_0||^2(q)} \delta_q \text{ weakly in the sense of measures}
\]

\[
c_k = D_{t_k}(u_{t_k}, \eta_{t_k}) = -4\pi(g - 1 - m)d_k + O(1), \text{ with } d_k = \int_X u_{t_k} dA \to +\infty.
\]

(1.15)

\[
\int_X \beta_0 \wedge \alpha = 0, \forall \alpha \in Q_2[q_1, \ldots, q_m].
\]

(1.16)

Furthermore, \(\rho([\beta]) = 4\int_X \beta_0 \wedge \alpha_0 = 4\pi m\).
Since $\dim \mathbb{C} Q_2[q_1, \ldots, q_m] = 3(g - 1) - m$, then the orthogonality condition \[ (1.16) \] together with the estimate \[ (1.15) \] for the global minimum value of $D_{t_k}$ support the fact that $\xi_k$ should admit only one blow-up point ($m = 1$), where the holomorphic quadratic differential $\ast E\beta_0$ does not vanish. This would also match with the asymptotic analysis in [16] concerning minimal immersions with fixed second fundamental form.

Furthermore, the estimate \[ (1.15) \] allows us to answer question (1), posed above, in case $\mathcal{S} \cap \mathcal{Z} = \emptyset$. Indeed, if $\rho(\beta) = 4\pi(g - 1)$ then $m = g - 1$ and therefore, by using \[ (1.16) \], we find that $D_0$ is bounded from below in $\Lambda$. However, it remains as a challenging open question to see whether $D_0$ attains its infimum in this case. The orthogonality condition \[ (1.16) \], seems to suggest that, for "almost all" classes $[\beta]$ the infimum must be attained.

This is confirmed here for the case $m = 1$ (i.e. $g = 2$) in Theorem 6, where the orthogonality condition \[ (1.16) \] is nicely interpreted in terms of the Kodaira map of $X$ into the projective space $\mathbb{P}(V^*)$, with $V = C_2(X)$. Extension of such a fact for $m \geq 2$ will be discussed in future work.

Next, we wish to acquire some useful information about the blow-up behavior of $(u_{t_k}, \eta_{t_k})$ in the "collapsing" case and when blow-up occurs with "least" blow-up mass $8\pi$. Thus, we assume that, in \[ (1.13) \], there holds:

$$\sigma(q) = 8\pi, \quad \forall \ q \in \mathcal{S}. \quad (1.17)$$

When we assume \[ (1.17) \] then every blow-up point $q \in \mathcal{S} \cap \mathcal{Z}$ must correspond to a "collapsing" of zeroes, i.e. $q \in \mathcal{Z}_0$ and so $\mathcal{S} \cap \mathcal{Z} = \mathcal{S} \cap \mathcal{Z}_0$. As a consequence, under \[ (1.17) \] the conclusion of Theorem 3 holds under the (weaker) assumption that, $\mathcal{S} \cap \mathcal{Z}_0 = \emptyset$, see Corollary 3.1.

To avoid technicalities, but still give a flavour of the blow-up behaviour in the "collapsing" case, we shall state here our result in the case of a single blow-up point (i.e. $\mathcal{S} = \{q\}$). Thus we find:

$$x_k \in X : x_k \to q \quad \text{and} \quad \xi_k(x_k) = \max_X \xi_k \to \infty, \quad \text{as} \quad k \to \infty.$$ 

In particular, notice that if $q \in \mathcal{Z}$, then $\|\hat{\alpha}_{t_k}\|(x_k) \to 0$, as $k \to \infty$, and we prove:

**Theorem 4.** Assume \[ (1.17) \] and let $\mathcal{S} = \{q\}$. If $q \in \mathcal{Z}$ (i.e. $\hat{\alpha}_0(q) = 0$) then $q \in \mathcal{Z}_0$ and (along a subsequence), as $k \to +\infty$, we have:

\[(i) \quad s_k \to +\infty, \quad \|\alpha_{t_k}\|^2(x_k) = e^{s_k}\|\hat{\alpha}_{t_k}\|^2(x_k) \to \mu > 0 \]
\[(ii) \quad e^{-u_{t_k}} \to \frac{1}{\mu} \delta_q, \text{ weakly in the sense of measures;} \]
\[(iii) \quad c_k = -4\pi(g - 2)d_k + O(1) \text{ with } d_k = \int_X u_{t_k} dA \to +\infty, \]
\[(iv) \quad \int_X \beta_0 \wedge \alpha = 0, \quad \text{if and only if} \quad \alpha \in Q_2[q], \quad (1.18) \]

In particular, $\int_X \beta_0 \wedge \hat{\alpha}_0 = 0.$
We refer to Theorem 10 and Corollary 3.2 for complete statements of our results in case $S$ contains more than one blow-up point.

Theorem 3 and Theorem 4 apply particularly well in case $κ = 2$ and $g = 2$, where by (1.9) we know that, $ρ(β) = 4π$, for every $[β] ∈ H^{0,1}(X,E) \setminus \{0\}$. This implies that the blow-up set $S$ (if not empty) must contain one single blow-up point ($m = 1$). Consequently, we can summarize the above results into the following:

**Theorem 5.** Let $κ = 2$ and genus $g = 2$. Then for any $[β] ≠ 0$ we have $ρ([β]) = 4π$ and the Donaldson functional $D_0$ is bounded from below in $Λ$.

Furthermore, if $D_0$ does not attain its infimum in $Λ$ then there exists $q ∈ X$ such that, $*_{E}β_0(q) ≠ 0$ and

$$\int_X β_0 ∧ α = 0 \iff α ∈ Q_2[q]. \tag{1.19}$$

Note that, Theorem 5 contains a nontrivial information, since for $[β] = 0$ the functional $D_0$ is always unbounded below.

Furthermore, from (1.14) we know: $\dim C_{Q_2[q]} = \dim C_{C_2(X)} - 1$ so that the complex vector space $V = C_2(X) = H^0(X, ⊗_{k=1}^2(T^1,0))^*$ is base-point free. Hence, by following section 12.1.3 of [10], for any $q ∈ X$ we can identify a one dimensional subspace of $V^* = H^{0,1}(X,E)$, $E = T^1,0_X$: namely an element of $P(H^{0,1}(X,E))$ given by the ray of functionals in $V^*$ which admit $Q_2[q]$ as their kernel. Thus, it is well defined the Kodaira map:

$$τ : X → P(H^{0,1}(X,E)), \quad E = T^1,0_X \tag{1.20}$$

and it is easy to check that $τ$ is holomorphic, see [10],[12]. More importantly, $\tag{1.19}$ holds $\iff [β]_p ∈ τ(X)$ with $[β]_p$ the representative in $P(H^{0,1}(X,E))$ of the class $[β] ∈ H^{0,1}(X,E) \setminus \{0\}$ identified by the harmonic $β_0 ∈ [β]$.

We shall show in Lemma 2.3 that $τ(X)$ defines a complex curve (i.e. of complex dimension 1) into $P(H^{0,1}(X,E))$ $≃ P^{3g-4}$ (recall $\tag{1.21}$ with $κ = 2$) and we have: $3g - 4 ≥ 2$ for $g ≥ 2$. Therefore, $τ(X)$ is a ”tiny” subset of $P(H^{0,1}(X,E))$, and $[β]_p ∉ τ(X)$ for ”generic” $[β] ∈ H^{0,1}(X,E) \setminus \{0\}$.

**Remark 1.2.** In case $[β]_p ∉ τ(X)$, it follows from Theorem 4 that, if $S$ is not empty and $\tag{1.17}$ holds, then $S$ must contain at least two points.

More interestingly, from Theorem 5 we conclude:

**Theorem 6.** Let $κ = 2$ and $g = 2$. Then for any $[β] ∈ H^{0,1}(X,E) \setminus \{0\}$, $E = T^1,0_X$ such that $[β]_p ∉ τ(X)$, the functional $D_0$ attains its infimum in $Λ$. 

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Clearly, Theorem 1 of the introduction is a direct consequence of Theorem 2 and Theorem 6.

Finally, we wish to comment about the case left out by our analysis. Namely when blow-up occurs at a zero of $\hat{\alpha}$ corresponding to the limit of “non-collapsing” zeroes of $\hat{\alpha}_k$, and hence preserving their same multiplicity, say $n \in \{1, \ldots, 4(g-1)\}$. More specifically, in this case we need to deal (after scaling) with possible “multiple” bubble profiles “symmetrically” placed at the $(n+1)$-roots of the identity, as described in [6] and [39], [40].

Such symmetry is at the origin of crucial cancellations in the "blow-up" estimates, which prevent to obtain useful information, for example about the sequence $s_k$. The nontrivial new estimates established recently by Wei and Zhang in [39] and [40] for such a non-simple "blow-up" situation, may help to overcome those difficulties and to accomplish analogous conclusions as in Corollary 3.2 below.

2 Preliminaries

In this section we introduce the basic notations and recall some facts useful in the sequel.

Let $X \in \mathcal{T}_g(S)$ be a given Riemann surface with hyperbolic metric $g_X$ and induced scalar product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$ and volume element $dA$. We let

$$T_{X,0}^{1,0} = \text{holomorphic tangent bundle of } X$$

equipped with the corresponding complex, holomorphic and Hermitian structure. For given $\kappa \geq 2$, we consider the holomorphic line bundle:

$$E = \otimes^{\kappa-1} T_{X,0}^{1,0}$$

with holomorphic structure $\bar{\partial} = \bar{\partial}_E$ and the corresponding spaces:

$$A^0(E) = \{ \text{smooth sections of } E \},$$
$$A^{0,1}(X, E) = \{ (0,1)\text{-forms valued on } E \},$$
$$A^{1,0}(X, E^*) = \{ (1,0)\text{-forms valued on } E^* \}.$$ We can use the d-bar operator $\bar{\partial}_E : A^0(X) \rightarrow A^{0,1}(X, E)$ to define the $(0,1)$-Dolbeault cohomology group:

$$\mathcal{H}^{0,1}(X, E) = A^{0,1}(X, E)/\bar{\partial}(A^0(E))$$

with cohomology class $[\beta] \in H^{0,1}(X, E)$ given as follows:

$$[\beta] = \{ \beta + \bar{\partial}\eta \in A^{0,1}(X, E), \forall \eta \in A^0(E) \}.$$ Furthermore we use the induced Hermitian structure on $E$, to define the fiberwise Hermitian product $\langle \cdot, \cdot \rangle_E$ and norm $\| \cdot \|_E$ for sections of $A^0(E)$ and
(0, 1)-forms in $A^{0,1}(X, E)$. So, for $p > 1$, we obtain the $L^p$-spaces of sections and forms valued on $E$, respectively as follows:

$$L^p(X, E) = \{ \eta : X \to E : \| \eta \|_{L^p} := \left( \int_X \| \eta \|^p_E dA \right)^{\frac{1}{p}} < +\infty \},$$

$$L^p(A^{0,1}(X, E)) = \{ \beta \in A^{0,1}(X, E) : \| \beta \|_{L^p} := \left( \int_X \| \beta \|^p_E dA \right)^{\frac{1}{p}} < +\infty \},$$

which define a Banach space equipped with the given norm: $\| \cdot \|_{L^p}$. Thus, for $p \geq 1$, we obtain also the Sobolev space:

$$W^{1,p}(X, E) = \{ \eta \in L^p(X, E) : \bar{\partial} \eta \in L^p(A^{0,1}(X, E)) \},$$

again a Banach space equipped with the norm:

$$\| \eta \|_{W^{1,p}} = \| \eta \|_{L^p} + \| \bar{\partial} \eta \|_{L^p}, \forall \eta \in W^{1,p}(X, E).$$

Actually, for the holomorphic line bundle $E$ in (2.1), the following Poincaré inequality holds:

**Lemma 2.1** ([17]). Let $E = \otimes^{k-1} T^{1,0}_X$ and $p > 1$. Then there exists a suitable constant $C_p > 0$ such that,

$$\| \eta \|_{L^p} \leq C_p \| \bar{\partial} \eta \|_{L^p}, \forall \eta \in W^{1,p}(X, E).$$

**Proof.** See proposition 2.2 in [17].

In view of (2.4), we obtain the equivalence of the following norms:

$$\| \eta \|_{W^{1,p}} \simeq \| \eta \|_{L^p} + \| \bar{\partial} \eta \|_{L^p}, \eta \in W^{1,p}(X, E), p > 1.$$

In general, the sharp value of the constant $C_p$ in (2.4) is not known, unless we take $p = 2$, where the following holds:

**Lemma 2.2** ([17]). Let $E = \otimes^{k-1} T^{1,0}_X$ and $h$ be a metric on $X$ with Gaussian curvature $K_h$. Then

$$\int_X \langle \bar{\partial} \eta, \bar{\partial} \eta \rangle_h dA_h \geq -(\kappa - 1) \int_X K_h \langle \eta, \eta \rangle_h dA_h, \forall \eta \in W^{1,2}(X, E)$$

with $\langle \cdot, \cdot \rangle_h = \langle \cdot, \cdot \rangle_{E,h}$ the fiberwise Hermitian product on $E$ and $dA_h$ the volume form induced by the metric $h$.

**Proof.** See proposition 2.1 in [17].

In particular, if $h = e^{2\phi} g_X$ with $\phi$ a smooth function in $X$, then by the following transformation rules:

$$\langle \eta, \eta \rangle_h = e^{4(\kappa-1)\phi} \langle \eta, \eta \rangle_h, \quad \langle \bar{\partial} \eta, \bar{\partial} \eta \rangle_h = e^{4(\kappa-1)\phi} \langle \bar{\partial} \eta, \bar{\partial} \eta \rangle_h e^{-2\phi},$$

$$dA_h = e^{2\phi} dA, \quad K_h = e^{-2\phi} (-\Delta \phi - 1),$$

$$\int_X \langle \bar{\partial} \eta, \bar{\partial} \eta \rangle_h dA_h \geq -(\kappa - 1) \int_X K_h \langle \eta, \eta \rangle_h dA_h, \forall \eta \in W^{1,2}(X, E)$$

with $\langle \cdot, \cdot \rangle_h = \langle \cdot, \cdot \rangle_{E,h}$ the fiberwise Hermitian product on $E$ and $dA_h$ the volume form induced by the metric $h$.
we deduce the following:

\[
\int_X e^{4(\kappa-1)\phi} \langle \bar{\partial} \eta, \bar{\partial} \eta \rangle dA \geq (\kappa - 1) \int_X (\Delta \phi + 1) e^{4(\kappa-1)\phi} \langle \eta, \eta \rangle dA \quad (2.5)
\]

In particular, for \( \phi = 0 \), that is \( h = g_X \), then (2.5) yields to the following sharp Poincaré inequality, valid for the hyperbolic metric \( g_X \) on \( X \):

\[
\int_X \langle \bar{\partial} \eta, \bar{\partial} \eta \rangle dA \geq (\kappa - 1) \int_X \langle \eta, \eta \rangle dA, \quad \eta \in W^{1,2}(X, E).
\]

Interestingly, from (2.5), we obtain the following estimate for solutions of (1.5).

**Corollary 2.1.** If \((u, \eta)\) is a solution of (1.5) with \( t \geq 0 \), then

\[
\int_X \langle \bar{\partial} \eta, \bar{\partial} \eta \rangle e^{2(\kappa-1)u} dA \geq 2(\kappa - 1) \int_X \| \beta_0 + \bar{\partial} \eta \|_E^2 \| \eta \|_E^2 e^{2(\kappa-1)u} dA + \frac{\kappa - 1}{2} \int_X \| \eta \|_E^2 e^{(\kappa-1)u} dA, \quad \forall \ \eta \in A^0(E).
\] (2.6)

**Proof.** It suffices to take \( \phi = \frac{u}{4} \) in (2.5). \( \square \)

Next, we recall the (natural) wedge product defined on forms valued on \( E \) and \( E^* \) respectively. For \( \alpha \in A^{1,0}(X, E^*) = A^{1,0}(X, \mathbb{C}) \otimes E^* \), let \( \alpha = \alpha_0 \otimes e^* \) with \( \alpha_0 \in A^{1,0}(X, \mathbb{C}) \) and \( e^* \in A^0(E^*) \), and for \( \beta \in A^{0,1}(X, E) = A^{0,1}(X, \mathbb{C}) \otimes E \), let \( \beta = \beta_0 \otimes e \) with \( \beta_0 \in A^{0,1}(X, \mathbb{C}) \) and \( e \in A^0(E) \), then

\[
\beta \wedge \alpha = e^*(e) \beta_0 \wedge \alpha_0 \in A^{1,1}(X, \mathbb{C}),
\]

where \( \beta_0 \wedge \alpha_0 \) is the usual wedge product of complex valued forms.

By the well known properties of the wedge product we can define the bilinear form:

\[
A^{1,0}(X, E^*) \times A^{0,1}(X, E) \rightarrow \mathbb{C} : (\alpha, \beta) \rightarrow \int_X \beta \wedge \alpha, \quad (2.7)
\]

which, by Serre duality (see [38]), is non-degenerate and induces the isomorphism:

\[
A^{1,0}(X, E^*) \simeq (A^{0,1}(X, E))^*. \quad (2.8)
\]

Furthermore, by the given (fiberwise) Hermitian product on \( E \), we have the duality map:

\[
A^{0,1}(X, E) \rightarrow (A^{0,1}(X, E))^* : \beta \rightarrow \beta^* : \beta^*(\xi) = \int_X \langle \xi, \beta \rangle dA,
\]

expressing the isomorphism between \( A^{0,1}(X, E) \) and its dual. In turn, by (2.8), we obtain the isomorphism \( A^{1,0}(X, E^*) \simeq A^{0,1}(X, E) \), explicitly expressed by the Hodge star operator:

\[
^*_E : A^{0,1}(X, E) \rightarrow A^{1,0}(X, E^*),
\]
where for $\beta \in A^{0,1}(X, E)$ we obtain $*_E \beta$ according to the following relation:

$$\xi \wedge *_E \beta = \langle \xi, \beta \rangle_E dA, \ \forall \xi \in A^{0,1}(X, E).$$

We denote by $*_E^{-1}$ the inverse operator, and observe that $*_E$ defines an isometry with respect to the (fiberwise) Hermitian product on $E^*$:

$$\langle \cdot, \cdot \rangle_{E^*} = \langle *_E^{-1} \cdot, *_E^{-1} \cdot \rangle_E \text{ and corresponding norm } \| \cdot \|_{E^*} = \| *_E^{-1} \cdot \|_E.$$  

As a relevant (finite dimensional) subspace of $A^{1,0}(X, E^*)$ we consider the space of holomorphic $\kappa$-differentials, namely the holomorphic sections of $E^* \otimes K_X = \otimes^\kappa K_X = \otimes^\kappa (T^1,0_X)^*$.

$$C_\kappa(X) = H^0(X, E^* \otimes K_X) := \{ \alpha \in A^0(X, \otimes^\kappa (T^1,0_X)^*) : \bar{\partial} \alpha = 0 \} \subset A^{1,0}(X, E^*)$$

with $\bar{\partial}$ the holomorphic structure of the line bundle $\otimes^\kappa (T^1,0_X)^*$. We recall that, $\dim_C C_\kappa(X) = (2\kappa - 1)(g - 1)$.

Since, by Stokes theorem, for $\alpha \in C_\kappa(X)$ we have: $\int_X \bar{\partial} \eta \wedge \alpha = 0, \ \forall \eta \in A^0(E)$, we see that the bilinear form (2.7) is well defined and non degenerate when considered on the space: $C_\kappa(X) \times H^{0,1}(X, E)$, and so it induces the isomorphism:

$$C_\kappa(X) \simeq (H^{0,1}(X, E))^*. \quad (2.9)$$

Since $C_\kappa(X)$ is finite dimensional, then we also get the "dual" isomorphism: $(C_\kappa(X))^* \simeq H^{0,1}(X, E)$, and to identify the linear (complex) functional on $C_\kappa(X)$ associated (by (2.7)) to a class $[\beta] \in H^{0,1}(X, E)$, we recall that, by Dolbeault decomposition, every $\beta \in A^{0,1}(X, E)$ can be uniquely written as follows:

$$\beta = \beta_0 + \bar{\partial} \eta \text{ with } \beta_0 \text{ harmonic (with respect to } g_X) \text{ and } \eta \in A^0(E).$$

Hence, every class $[\beta] \in H^{0,1}(X, E)$ is uniquely identified by its harmonic representative $\beta_0 \in [\beta]$. Furthermore, for $[\beta] \in H^{0,1}(X, E)$ with harmonic $\beta_0 \in [\beta]$, we obtain an element of $(C_\kappa(X))^*$ as follows:

$$C_\kappa(X) \twoheadrightarrow \mathbb{C} : \alpha \mapsto \int_X \beta_0 \wedge \alpha, \quad (2.10)$$

which is well defined independently from the chosen element in $[\beta]$.

In addition, harmonic $(0,1)$-forms valued in $E$ are characterized by the property: $\int_X \langle \bar{\partial} \eta, \beta_0 \rangle_E dA = 0, \ \forall \eta \in A^0(E)$ and so,

$$\beta_0 \in A^{0,1}(X, E) \text{ harmonic } \iff \ast_E \beta_0 \in C_\kappa(X).$$

So, every harmonic $\beta_0 \in A^{0,1}(X, E)$ also identifies an element of $(H^{0,1}(X, E))^*$ via the linear map:

$$H^{0,1}(X, E) \twoheadrightarrow \mathbb{C} : [\xi] \mapsto \int_X \xi \wedge \ast_E \beta_0 = \int_X \langle \xi, \beta_0 \rangle_E dA,$$
and (by (2.9)) we identify the isomorphism:
\[ H^{0,1}(X, E) \rightarrow C_\kappa(X) : [\beta] \rightarrow \ast_E \beta_0. \]

In other words, for \( \alpha \in C_\kappa(X) \subset A^{1,0}(X, E^*) \), \( \ast_E^{-1} \alpha \) is given by the unique harmonic \((0,1)\)-form \( \beta_0 \) valued in \( E : \ast_E \beta_0 = \alpha \).

Since all norms on \( C_\kappa(X) \) are equivalent, it is usual (recall the Weil-Patterson form [18]) to consider the following \( L^2 \)-norm
\[ \|\alpha\|_{L^2} := \left( \int_X (\ast_E^{-1} \alpha, \ast_E^{-1} \alpha) E dA \right)^{1/2} \text{ for } \alpha \in C_\kappa(X). \]

Indeed, it is conveniently computed with respect to a fixed basis in \( C_\kappa(X) \) given as follows:
\[ \{s_1, \ldots, s_\nu\} \subset C_\kappa(X) \text{ with } \nu = (2\kappa - 1)(g - 1), \int_X (\ast_E^{-1} s_j, \ast_E^{-1} s_k) E dA = \delta_{j,k} \]
and \( \delta_{j,k} \) are the Kronecker symbols. So, for \( \alpha \in C_\kappa(X) \), we may write:
\[ \alpha = \sum_{j=1}^{\nu} a_j s_j, \quad a_j \in \mathbb{C}, \quad \text{and } \beta_0 = \ast_E^{-1} \alpha = \sum_{j=1}^{\nu} a_j \ast_E^{-1} s_j \]
(\( \beta_0 \) the associated harmonic \((0,1)\)-form valued on \( E \)) and compute:
\[ \|\alpha\|^2_{L^2} = \|\beta_0\|^2_{L^2} = \sum_{j=1}^{\nu} |a_j|^2. \]

Conveniently, we have ”compactness” for any sequence \( \alpha_n \in C_\kappa(X) \) with uniformly bounded \( L^2 \)-norm. Thus, for example, if \( \alpha_n \) satisfies \( \|\alpha_n\|_{L^2} = 1 \) then it admits a convergent subsequence \( \alpha_n \rightarrow \alpha_0 \in C_\kappa(X) \) with \( \|\alpha_0\|_{L^2} = 1 \).

Next, we recall some well known consequences of the Riemann Roch theorem [18], [32] that will be useful in the sequel. To this purpose, given a holomorphic line bundle \( L \), we denote by \( H^0(X, L) \) the space of holomorphic sections of \( L \) and by \( \operatorname{deg} L \) the degree of \( L \). Then the Riemann Roch theorem states that
\[ \dim \mathbb{C} H^0(X, L) - \dim \mathbb{C} H^0(X, K_X - L) = \operatorname{deg} L + 1 - g, \]  
(2.12)
with \( K_X \) the canonical bundle of \( X \).

It is well known that \( \deg K_X = 2(g - 1) \) (see [28]), and therefore by (2.12) we deduce: \( \dim \mathbb{C} H^0(X, K_X) = g \). As a consequence, using again (2.12) we find in particular that,
\[ \text{if } g \geq 2 \text{ then } \dim \mathbb{C} H^0(X, K_X - q) = \dim \mathbb{C} H^0(X, K_X) - 1, \]  
(2.13)
for any \( q \in X \). Hence, \( H^0(X, K_X) \) is base-point free.

On the other hand, if we let \( L = E^* \otimes K_X = \otimes^c K_X \), then \( \operatorname{deg} L = 2\kappa(g - 1) \), and we may conclude:
Remark 2.1. Every non-trivial holomorphic \( \kappa \)-differential

\[ \alpha \in C_\kappa(X) = H^0(X, L), \quad \alpha \neq 0 \]

vanishes exactly at \( 2\kappa(g - 1) \) points in \( X \), counted with multiplicity.

For given distinct points \( \{x_1, \ldots, x_\nu\} \subset X \) we consider the subspace:

\[ Q_\kappa[x_1, \ldots, x_\nu] = \{ \alpha \in C_\kappa(X) : \alpha(x_j) = 0, \ j = 1, \ldots, \nu \} \subset C_\kappa(X). \quad (2.14) \]

We are going to show that,

\[ \dim \mathbb{C} Q_\kappa[x_1, \ldots, x_\nu] = (2\kappa - 1)(g - 1) - \nu, \quad \text{for } 1 \leq \nu < 2(\kappa - 1)(g - 1). \quad (2.15) \]

Indeed, \( Q_\kappa[x_1, \ldots, x_\nu] \) is given by the space of holomorphic sections of the line bundle \( L = \otimes^\kappa K_X - \{x_1, \ldots, x_\nu\} \), and we know that \( \deg(L) = 2\kappa(g - 1) - \nu \).

While, if \( 1 \leq \nu < 2(\kappa - 1)(g - 1) \), then for the line bundle \( K_X - L = L^{-1} \otimes K_X \) (with \( L^{-1} = L^* \)) we obtain:

\[ \deg(K_X - L) = -2(\kappa - 1)(g - 1) + \nu < 0, \]

and so \( \dim \mathbb{C} H^0(X, K_X - L) = 0 \). Therefore, by the Riemann-Roch theorem, we obtain

\[ \dim \mathbb{C} H^0(X, L) - \dim \mathbb{C} H^0(X, K_X - L) = \deg(L) + 1 - g = (2\kappa - 1)(g - 1) - \nu, \]

and (2.15) is established, since \( \dim \mathbb{C} Q_\kappa[x_1, \ldots, x_\nu] = \dim \mathbb{C} H^0(X, L) \).

Remark 2.2. From (2.15) follows in particular that we can always find \( \alpha \in C_\kappa(X) \) that vanishes at all but one point of \( \{x_1, \ldots, x_\nu\} \). Furthermore

\[ \dim \mathbb{C} Q_\kappa[q] = \dim \mathbb{C} C_\kappa(X) - 1, \quad (2.16) \]

for every \( q \in X \), namely also \( H^0(X, \otimes^\kappa K_X) \) is base-point free.

By using (2.15), we can define the Kodaira map:

\[ \tau : X \longrightarrow \mathbb{P}(\mathcal{H}^{0,1}(X, E)) \simeq \mathbb{P}^{(2\kappa - 1)(g - 1) - 1} \quad (2.17) \]

simply by associating to any \( q \in X \) the element of \( \mathbb{P}(\mathcal{H}^{0,1}(X, E)) \) identified by the ray of classes generated by \( [\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\} \) with harmonic representative \( \beta_0 \in [\beta] \) and satisfying:

\[ \int_X \beta_0 \wedge \alpha = 0, \ \forall \ \alpha \in Q_2[q]. \quad (2.18) \]

Indeed, by recalling (2.10), in this way we can identify the ray of functionals in \( (C_\kappa(X))^* \) which admit \( Q_2[q] \) as their kernel. Such a map is holomorphic [10], [12], and we have:
Lemma 2.3. The image $\tau(X)$ defines a complex curve into the projective space: $\mathbb{P}(H^{0,1}(X,E))$ of complex dimension $(2\kappa - 1)(g - 1) - 1 \geq 2$.

Proof. In case $(\kappa, g) \neq (2, 2)$ then $2(\kappa - 1)(g - 1) > 2$ and so we can apply (2.15) with $\nu = 2$ to show that $\tau$ is injective and defines an embedding in this case (see proposition 4.20 in [28]). Therefore, $\tau(X)$ is a (regular) complex curve with the same complex dimension of $X$, namely one.

Next, let us consider the case: $\kappa = 2$ and $g = 2$ where (2.15) is valid only with $\nu = 1$, and so the argument above does not apply. Indeed, for $g = 2$ the map $\tau$ is no longer injective, and instead we are going to show that it is "generically" a map two-to-one.

To this purpose, let $L = \bigotimes_{k=1}^{2}K_{X}$, and for given $q \in X$ let $q^{*} \in X$ be such that: $\tau(q) = \tau(q^{*})$. That is $Q_{2}[q] = Q_{2}[q^{*}] = Q_{2}(q, q^{*})$, and consequently (by (2.16)) we have:

$$\dim_{C} H^{0}(X, L - q - q^{*}) = \dim_{C}(X, L - q) = \dim_{C}(X, L - q^{*}) = 3(g - 1) - 1 = 2 \quad \text{(for } g = 2).$$

Hence, by using (2.19) together with the Riemann Roch Theorem (2.12), we find: $\dim_{C} H^{0}(X, -K_{X} + q + q^{*}) = 1$. In other words, the holomorphic line bundle: $-K_{X} + q + q^{*}$ (of degree zero) must be trivial. So there must exist a holomorphic section of $K_{X}$ which vanishes exactly at $q$ and $q^{*}$. We shall use this information for the (well-defined) holomorphic Kodaira map $\tau_{1} : X \rightarrow \mathbb{P}(V^{*}_{1})$ relative to the space $V_{1} = H^{0}(X, K_{X})$, see (2.13). Indeed, $\deg(K_{X}) = 2$ (for $g = 2$), and so $\tau_{1}$ is "generically" a two-to-one map and (by the information above) necessarily: $\tau_{1}(q) = \tau_{1}(q^{*})$. Furthermore, for $g = 2$, the Riemann-Hurwitz formula implies that, $q = q^{*}$ only for six points which must coincide exactly with the Weierstrass points of $X$. Knowing that $\tau_{1}(X)$ defines a compact, smooth complex submanifold of $\mathbb{P}(V^{*}_{1}) \simeq \mathbb{P}^{1}$ and it cannot reduce to a point, then necessarily: $\tau_{1}(X) = \mathbb{P}(V^{*}_{1}) \simeq \mathbb{P}^{1}$. Namely $\tau_{1}(X)$ defines a smooth complex curve (a conic) in $\mathbb{P}^{1}$.

As a consequence of the above discussion, also $\tau$ must be "generically" a two-to-one map, with the same preimages as $\tau_{1}$, in the sense that:

$$\forall q \in X : \{ p \in X : \tau(p) = \tau(q) \} = \{ p \in X : \tau_{1}(p) = \tau_{1}(q) \}.$$ 

Therefore, it is well defined the embedding: $\mathbb{P}^{1} \simeq \mathbb{P}(V^{*}_{1}) \xrightarrow{\phi} \mathbb{P}(V^{*}) \simeq \mathbb{P}^{2}$ such that $\tau(q) = \phi(\tau_{1}(q))$, for all $q \in X$. And so $\tau(X)$ defines a complex curve into $\mathbb{P}^{2}$ as claimed. 

Finally, we recall that around any given $q \in X$, we can introduce holomorphic coordinates $\{ z \}$ centered at the origin, so that, for $z = x + iy \in B_{r}$ with $r > 0$
small, we have:
\[
\begin{align*}
\partial &= \frac{\partial}{\partial z} = \frac{1}{2} (\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}), \\
\bar{\partial} &= \frac{\partial}{\partial \bar{z}} = \frac{1}{2} (\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}),
\end{align*}
\]
d\bar{z} = dx + idy, \ d\bar{z} = dx - idy,
\[
g_X = e^{2u_0}|dz|^2, \ u_0 \text{ smooth and } u_0(0) = 0,
\]
d\bar{z} = ie^{2u_0}d\bar{u}, \ *d\bar{z} = -ie^{2u_0}dz.
\tag{2.20}
\]

So we can express the Laplace-Beltrami operator: \(\Delta_g = \Delta = 4e^{-2u_0}\partial\bar{\partial}\) and obtain \(4\partial\bar{\partial}u_0 = e^{2u_0}\).

Actually (with abuse of notation) in the sequel we also denote the flat Laplacian by \(\Delta = 4\partial\bar{\partial}\), unless confusion arises.

Moreover, in such local coordinates, the (fiberwise) Hermitian product operates essentially as the usual Hermitian product on \(\mathbb{C}\), and for the local expression of the (fiberwise) norm of sections and forms, we record that in \(B_r\) there holds:
\[
\|\eta\|_E \simeq |\eta(\bar{z})|e^{2(\kappa-1)u_0} \eta \in A^0(E), \quad \|\bar{\beta}\|_E \simeq |\bar{\beta}(\bar{z})|e^{2(\kappa-2)u_0} \beta \in A^{0,1}(X, E).
\]

In local holomorphic coordinates, any holomorphic \(\kappa\)-differential \(\alpha \in C_\kappa(X)\), takes the expression: \(\alpha = h(\bar{z})^\kappa\) with \(h\) holomorphic in \(B_r\).

In this way, it is clear what we mean by a zero of \(\alpha\) and corresponding multiplicity, since such notions are independent of the chosen holomorphic coordinates. In particular, if \(g\) is a zero of \(\alpha\) with multiplicity \(n\), then in local coordinates we have: \(\|\alpha\|_E^r \simeq |\alpha|^n |h(\bar{z})|e^{2(\kappa-1)u_0}\) with the function \(h\) holomorphic and never vanishing in \(B_r\). In particular, \(\partial\bar{\partial}\ln|h(\bar{z})|^2 = 0\) in \(B_r\), a property we shall use in the sequel.

3 Asymptotics

From now on we shall use the (fiberwise) Hermitian product \(\langle \cdot, \cdot \rangle\) and norm \(\|\cdot\|\) of sections and forms valued on \(E\) and \(E^*\) without the subscripts \(E\) and \(E^*\) respectively, unless some confusion arises.

For given \((X, [\beta]) \in \mathcal{T}_0(S) \times H^{0,1}(X, E)\) and \(t \geq 0\), we consider the Donaldson functional:
\[
D_t(u, \eta) = \int_X \left(\frac{\nabla u}{4} - u + te^u + 4\|\bar{\beta}\| + \bar{\partial}\eta\right)^2e^{(\kappa-1)u}dA
\]
with domain
\[
\Lambda = \{(u, \eta) \in H^1(X) \times W^{1,2}(X, E) : \int_X e^{(\kappa-1)u}\|\bar{\beta}\| + \bar{\partial}\eta\|dA < +\infty\},
\]
where \(H^1(X) = \{u \in L^2(X) : |\nabla u| \in L^2(X)\}\) is the Sobolev space with usual scalar product and norm: \(\|u\|_{H^1} = (\int_X (u^2 + |\nabla u|^2)dA)^{1/2}\) and \(W^{1,2}(X, E)\) is the Sobolev space defined in \([21]\).
It is readily verified that, for $t > 0$, the functional $D_t$ is bounded from below in $\Lambda$, while this is not always the case for $t = 0$, as we shall discuss below.

We notice also that the functional $D_t$ admits Gateaux derivatives at a point $(u, \eta) \in \Lambda$ only along smooth directions $(v, l) \in C^1(X) \times A^0(E)$. In fact, the troublesome term with respect to differentiability is given by:

$$T(u, \eta) = \int_X e^{(\kappa - 1)u} \| \beta_0 + \bar{\partial} \eta \| dA, \; (u, \eta) \in \Lambda.$$ 

On the other hand, if $(u, \eta) \in \Lambda$ is a weak critical point of $D_t$ in the sense that:

$$D_t'(u, \eta)[v, l] = 0, \; \forall (v, l) \in C^1(X) \times A^0(E) \quad (3.1)$$

then,

$$\int_X \nabla u \nabla v + 2v(-1 + te^u + 4(\kappa - 1)e^{(\kappa - 1)u}\| \beta_0 + \bar{\partial} \eta \|^2) dA = 0, \; \forall v \in C^1(X) \quad (3.2)$$

$$\int_X e^{(\kappa - 1)u} \langle \beta_0 + \bar{\partial} \eta, \bar{\partial} l \rangle dA = 0, \; \forall l \in A^0(E). \quad (3.3)$$

The elliptic operators involved (in a weak form) in (3.2) and (3.3) allow us to gain all the regularity we need about $(u, \eta) \in \Lambda$, starting with the following result established in [17].

**Lemma 3.1** ([17]). Let $(u, \eta) \in \Lambda$.

(i) If (3.3) holds then $\eta \in W^{1,p}(X, E)$ for all $p \geq 2$, and we can allow $l \in W^{1,2}(X, E)$ in (3.3).

(ii) If (3.2) and (3.3) hold (i.e. $(u, \eta)$ is a "weak" critical point of $D_0$) then $(u, \eta)$ is smooth and satisfies, in the classical sense, the following system of equations:

$$(P)_t \begin{cases} \Delta u + 2 - 2te^u - 8(\kappa - 1)e^{(\kappa - 1)u}\| \beta_0 + \bar{\partial} \eta \|^2 = 0 & \text{in } X \\ \bar{\partial}(e^{(\kappa - 1)u} \ast_E (\beta_0 + \bar{\partial} \eta)) = 0. \end{cases} \quad (3.4)$$

**Proof.** By using Weil’s theorem, from (3.3) we get that $e^{(\kappa - 1)u} \ast_E (\beta_0 + \bar{\partial} \eta) \in C^\infty(X)$. So we can use the basis in (2.11) to write

$$e^{(\kappa - 1)u}(\beta_0 + \bar{\partial} \eta) = \sum_{j=1}^\nu a_j s_j \quad \text{for suitable } a_j \in \mathbb{C}, \; \text{and } \nu = (2\kappa - 1)(g - 1).$$

As a consequence, since $e^{-(\kappa - 1)u} \in L^p(X), \; \forall p > 1$, we find

$$\bar{\partial} \eta = e^{-(\kappa - 1)u}(\sum_{j=1}^\nu a_j \ast_E s_j) - \beta_0 \in L^q(X), \; \forall q > 1.$$
Therefore, we can use elliptic regularity and the Poincaré inequality for the elliptic operator $\bar{\partial}$, to conclude that $\eta \in W^{1,p}(X,E)$ for all $p > 1$, as claimed in (i). At this point (ii) follows easily, since by using part (i) and elliptic regularity theory together with well known boot-strap arguments, we obtain that $(u, \eta)$ is smooth and satisfies (3.4).

Therefore, to find solutions to (3.4), we may more conveniently consider $D_t$ in the stronger space:

$$V_p = H^1(X) \times W^{1,p}(X,E), \quad p > 2,$$

as we check easily that, $T \in C^1(V_p)$ (see [17]) and so $D_t \in C^1(V_p)$. Summarizing:

$(u, \eta)$ is a (classical) solution of problem $(\mathcal{P})_t$ if and only if it is a weak critical point of $D_t$ in $\Lambda$ (in the sense of (3.1)) or equivalently, it is a (usual) critical point of $D_t$ in $V_p$.

As already mentioned in the introduction, in [17] it has been proved the following uniqueness result which we recall for the convenience:

**Theorem 7 ([17]).** For any $t > 0$, the functional $D_t$ admits a unique critical point $(u_t, \eta_t) \in V_p$ for every $p > 2$, and it corresponds to the global minimum of $D_t$ in $\Lambda$. In particular, problem $(\mathcal{P})_t$ admits the unique solution $(u_t, \eta_t)$.

As already mentioned, for $t = 0$, the functional:

$$D_0(u, \eta) = D_{t=0}(u, \eta) = \int_X \frac{|\nabla u|^2}{4} - u + 4e^{(\kappa-1)u} \|\beta_0 + \bar{\partial}\eta\|_2^2 dA =: c_0(u).$$

may no longer be bounded from below in $\Lambda$ and actually problem $(\mathcal{P})_{t=0}$ in (3.4) may not admit a solution. For example, if $[\beta] = 0$ (i.e. $\beta_0 = 0$), then we easily check that $(\mathcal{P})_{t=0}$ admits no solution and actually:

$$u_t = \ln \frac{1}{t}, \eta_t = 0 \quad \text{and} \quad D_0(u_t, \eta_t) \leq D_t(u_t, \eta_t) = \ln t \to -\infty,$$

as $t \to 0^+$, and so $D_0$ is unbounded from below in $\Lambda$. Such an example illustrates the only possible obstruction to the solvability of $(\mathcal{P})_{t=0}$, in the following sense.

**Theorem 8.** If $(u_0, \eta_0)$ is a solution for $(\mathcal{P})_{t=0}$ in (3.4), then

(i) $(u_t, \eta_t) \to (u_0, \eta_0)$ in $V_p$, $p > 2$, as $t \to 0^+$;

(ii) $(u_0, \eta_0)$ is the only solution of $(\mathcal{P})_{t=0}$ and so the only critical point of $D_0$. Furthermore, $D_0$ is bounded from below in $\Lambda$ and attains its global minimum at $(u_0, \eta_0)$.

To establish Theorem 8 we need few preliminaries. To start, for a fixed $u \in H^1(X)$, we are going to account for (3.4) by considering the minimization problem:

$$\inf_{\eta \in W^{1,2}(X,E)} \int e^{(\kappa-1)u} \|\beta_0 + \bar{\partial}\eta\|^2 dA := c_0(u).$$

We have:
Lemma 3.2. For every $u \in H^1(X)$ there exists a unique global minimum $\eta(u)$ for (3.5) with $\eta(u) \in W^{1,p}(X, E)$, $\forall \ p > 2$. Furthermore,

$$\eta \in W^{1,2}(X, E)$$ satisfies (3.5) if and only if $\eta = \eta(u)$.

Proof. We will be sketchy, as this result was essentially pointed out in [17]. Let $\eta_n \in W^{1,2}(X, E)$ be a minimizing sequence for (3.5), that is

$$T(u, \eta_n) \rightarrow c_0(u), \ \text{as} \ n \rightarrow +\infty.$$  

Since $e^{-u} \in L^q(X), \ \forall \ q > 1$, we can use Hölder inequality to check that $\eta_n$ is uniformly bounded in $W^{1,a}(X,E)$ for $a \in (0, 1)$. Therefore, along a subsequence, we find:

$$\eta_n \rightharpoonup \eta \ \text{weakly in} \ W^{1,a}(X,E) \ \text{and therefore,}$$

$$\hat{X} e^{(\kappa - 1)u} \langle \dot{\eta}_n - \eta \rangle dA \rightarrow 0, \ \text{as} \ n \rightarrow +\infty.$$ 

In particular, by taking $\xi = \eta e^{(\kappa - 1)u}$ we can derive:

$$c_0(u) = \lim_{n \rightarrow +\infty} \int_X e^{(\kappa - 1)u} \langle \dot{\eta}_n \rangle dA \geq 0, \ \forall \ \xi \in L^b(A^{0,1}(X,E))$$

so $\eta$ is a minimum for (3.5) and $\eta \in W^{1,p}(X, E), \ \forall \ p > 2$. Since for fixed $u \in H^1(X)$ the operator $T(u, \cdot)$ is strictly convex, then $\eta = \eta(u)$ is the only minimum point for (3.5). In addition, for any $\eta$ satisfying (3.5), we have: $\eta \in W^{1,p}(X, E), \ \forall \ p > 2$, and

$$\int_X e^{(\kappa - 1)u} \langle \hat{\eta} \rangle dA \geq \int_X e^{(\kappa - 1)u} \langle \dot{\eta}_n \rangle dA \geq \int_X e^{(\kappa - 1)u} \langle \hat{\eta} \rangle dA + \int_X e^{(\kappa - 1)u} \langle \dot{\eta}_n \rangle dA,$$

and necessarily $\eta = \eta(u)$, as claimed. \ \square

We wish to point out some useful properties about the map:

$$H^1(X) \rightarrow W^{1,p}(X, E) : u \rightarrow \eta(u).$$

To this purpose, for $u \in H^1(X)$, we let

$$\beta(u) = e^{(\kappa - 1)u} (\hat{\beta}_0 + \dot{\eta}(u)) \in W^{1,p}(X, E), \ p > 2.$$
Since \( *_E \beta(u) \in C_c(X) \), using the frame \( \{ s_1, \ldots, s_r \} \) of \( C_c(X) \) with \( \nu = (2\kappa - 1)(g - 1) \), as given in (2.11), we may write: 

\[
*_{\mathcal{E}} \beta(u) = \sum_{j=1}^{\nu} a_j(u) s_j \quad \text{with suitable } a_j(u) \in \mathbb{C}. 
\]

Consequently,

\[
\beta(u) = \sum_{j=1}^{\nu} a_j(u) *_{\mathcal{E}}^{-1} s_j, \quad a_j(u) = \int_{X} \langle \beta(u), *_{\mathcal{E}}^{-1} s_j \rangle\, dA, \quad j = 1, \ldots, \nu. \tag{3.7}
\]

For \( u \) and \( u_0 \in H^1(X) \) we point the following simple (but useful) identities:

\[
\tilde{\partial} \eta(u) - \tilde{\partial} \eta(u_0) = e^{(\kappa-1)u}(\beta(u) - \beta(u_0)) + (e^{(\kappa-1)u} - u_0)(\beta(u_0)). \tag{3.8}
\]

or equivalently:

\[
\beta(u) - \beta(u_0) = e^{(\kappa-1)u}(\tilde{\partial} \eta(u) - \tilde{\partial} \eta(u_0)) + (e^{(\kappa-1)u} - e^{(\kappa-1)u_0})\beta(u_0). \tag{3.9}
\]

Lemma 3.3. (i) If \( u_n \rightharpoonup u \) weakly in \( H^1(X) \) then \( \eta(u_n) \longrightarrow \eta(u) \) strongly in \( W^{1,p}(X, E) \), \( p > 2 \). In particular, the map in (3.6) takes bounded sets of \( H^1(X) \) into bounded sets of \( W^{1,p}(X, E) \), \( p > 2 \).

(ii) For given \( u_0 \in H^1(X) \) and \( p > 2 \) there exists a positive constant \( \sigma_p = \sigma_p(u_0) \) (depending only on \( p \) and \( u_0 \)) such that

\[
\| \tilde{\partial} \eta(u) - \tilde{\partial} \eta(u_0) \|_{L^p}^2 \leq \sigma_p(\| \tilde{\partial} \eta(u) - \tilde{\partial} \eta(u_0) \|_{L^2}^2 + \| u - u_0 \|_{H^1}^2). \tag{3.10}
\]

Proof. To establish (i), we observe that \( u_n \) is uniformly bounded in \( H^1(X) \), and in particular, \( e^{\pm u_n} \rightharpoonup e^{\pm u} \) in \( L^q(X) \), \( \forall \, q > 1 \). As a consequence, by setting \( \eta_n = \eta(u_n) \), from (3.5) we have:

\[
\int_{X} e^{(\kappa-1)u_n} \| \tilde{\partial} \eta_0 + \tilde{\partial} \eta_n \|^2\, dA \leq \int_{X} e^{(\kappa-1)u_n} \| \tilde{\partial} \eta_0 \|^2 \leq C.
\]

So, for \( 1 < a < 2 \), we can use Hölder inequality to see that \( \eta_n \) is uniformly bounded in \( W^{1,a}(X, E) \). Thus, along a subsequence, we have: \( \eta_n \rightharpoonup \eta \) weakly in \( W^{1,a}(X, E) \), as \( n \rightarrow +\infty \). Consequently, for any \( \xi \in L^b(A^{0,1}(X, E)) \), \( b = \frac{a}{q-1} \), we have:

\[
\int_{X} \langle \tilde{\partial} \eta_n - \tilde{\partial} \eta, \xi \rangle_E\, dA \rightarrow 0, \quad \text{as } n \rightarrow +\infty.
\]

Furthermore, if we take \( \xi \in L^q(A^{0,1}(X, E)) \) with \( q > b \) and \( p = \frac{bq}{q-b} \), we can estimate

\[
\int_{X} | e^{(\kappa-1)u_n} - e^{(\kappa-1)u} | \langle \tilde{\partial} \eta_n, \xi \rangle\, dA | \leq C \| e^{(\kappa-1)u_n} - e^{(\kappa-1)u} \|_{L^p} \rightarrow 0,
\]

as \( n \rightarrow +\infty \). Hence, as \( n \rightarrow +\infty \),

\[
\int_{X} e^{(\kappa-1)u_n} \langle \tilde{\partial} \eta_0 + \tilde{\partial} \eta_n, \xi \rangle\, dA \rightarrow \int_{X} e^{(\kappa-1)u} \langle \tilde{\partial} \eta_0 + \tilde{\partial} \eta, \xi \rangle\, dA,
\]

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for any $\xi \in L^q(A^{0,1}(X, E))$ with $q > b = \frac{n}{n-1}$. Consequently, the property
\[
\int_X e^{(k-1)u_{n}} \langle \beta_0 + \bar{\bar{\eta}}_n, \bar{\bar{\partial}}l \rangle \, dA = 0, \quad \forall \, l \in A^0(E)
\]
passes to the limit, as $n \to +\infty$, and we conclude that $\eta$ satisfies (3.3).

Hence, by Lemma 3.2 we conclude that, $\eta = \eta(u) \in W^{1,p}(X, E), \quad \forall \, p > 2$. Moreover, as $n \to +\infty$,
\[
a_{j,n} := a_j(u_n) = \int_X e^{(k-1)u_{n}} \langle \beta_0 + \bar{\bar{\eta}}_n, \bar{\bar{\partial}}s_j \rangle \, dA
\]
\[
\rightarrow \int_X e^{(k-1)u_{n}} \langle \beta_0 + \bar{\bar{\eta}}, \bar{\bar{\partial}}s_j \rangle \, dA = a_j(u), \quad \forall \, j = 1, \ldots, \nu
\]
and therefore,
\[
\beta(u_n) = \sum_{j=1}^{\nu} a_{j,n} s_j \longrightarrow \beta(u) = \sum_{j=1}^{\nu} a_j(u) s_j \quad \text{in} \quad L^q(A^{0,1}(X, E)), \quad \forall \, q > 1.
\]

At this point we can use (3.8) to conclude that,
\[
\|\bar{\bar{\eta}}(u_n) - \bar{\bar{\eta}}(u)\|_{L^p} \leq \|e^{(k-1)u_{n}}\|_{L^{2p}} \|\beta(u_n) - \beta(u)\|_{L^{2p}}
\]
\[
+ \|e^{(k-1)(u_n - u)} - 1\|_{L^{2p}} \|\beta(u)\|_{L^{2p}}
\]
\[
\leq C(\|\beta(u_n) - \beta(u)\|_{L^{2p}} + \|e^{(k-1)(u_n - u)} - 1\|_{L^{2p}}) \to 0,
\]
as $n \to +\infty$. Since any other convergent subsequence of $\eta(u_n)$ admits the same limit $\eta(u)$, we conclude that the full sequence $\eta(u_n) \to \eta(u)$ in $W^{1,p}(X, E)$, $p > 2$, as claimed.

To establish (3.10), we can assume without loss of generality that
\[
\|\bar{\bar{\eta}}(u) - \bar{\bar{\eta}}(u_0)\|_{L^2}^2 + \|u - u_0\|_{H^1}^2 = 1. \quad (3.11)
\]
In particular $\|u\|_{H^1} \leq 1 + \|u_0\|_{H^1}$, and so $\|e^{(k-1)u} - e^{(k-1)u_0}\|_{L^\infty} \leq C_q \|u - u_0\|_{H^1}$ and $\|e^{\pm u}\|_{L^q} \leq C_q, \; \forall \; q > 1$ and with suitable $C_q > 0$ depending only on $q$ and $u_0$.

Firstly, by (3.7), we see that $|a_j(u) - a_j(u_0)| \leq C \|\beta(u) - \beta(u_0)\|_{L^1}$, for every $j = 1, \ldots, \nu$. Thus,
\[
\|\beta(u) - \beta(u_0)\|_{L^p} \leq C \left( \sum_{j=1}^{\nu} |a_j(u) - a_j(u_0)|^p \right)^{\frac{1}{p}}
\]
\[
\leq C_p \sum_{j=1}^{\nu} |a_j(u) - a_j(u_0)| \leq C_p \|\beta(u) - \beta(u_0)\|_{L^1}.
\]
By combining (3.8) and (3.9), for $p > 2$ we derive:

$$
\|\tilde{\eta}(u) - \tilde{\eta}(u_0)\| \leq \|e^{(\kappa - 1)u}\|_{L^p} \|\beta(u) - \beta(u_0)\|_{L^p} \\
+ \|e^{(\kappa - 1)u - u_0} - 1\|_{L^p} \|\beta(u_0)\|_{L^p} \\
\leq C_p(\|\beta(u) - \beta(u_0)\|_{L^1} + \|u - u_0\|_{H^1}) \\
\leq C_p(\|e^{(\kappa - 1)u}\|_{L^2} \|\tilde{\eta}(u) - \tilde{\eta}(u_0)\|_{L^2} \\
+ \|e^{(\kappa - 1)u - e^{(\kappa - 1)u_0}}\|_{L^2} \|\beta(u_0)\|_{L^2} + \|u - u_0\|_{H^1}) \\
\leq C_p(\|\tilde{\eta}(u) - \tilde{\eta}(u_0)\|_{L^2} + \|u - u_0\|_{H^1}) \leq \sigma_p
$$

with a suitable constant $\sigma_p > 0$ (depending only on $p$ and $u_0$) obtained in view of (3.11), and the proof is completed.

By Lemma 3.3 for $t > 0$, we readily get a minimum of $D_t$ in $\Lambda$ simply by taking (without loss of generality) a minimizing sequence of the form: $(u_n, \eta_n(u_n)) \in H^1(X) \times W^{1,p}(X,E)$, $p > 2$. Indeed, for $t > 0$, we can take advantage of the estimate:

$$
D_t(u, \eta) \geq \int_X \left( \frac{\|\nabla u\|^2}{4} + 4e^{(\kappa - 1)u} \|\beta_0 + \tilde{\eta}\|^2 \right) dA + 4\pi(g - 1)(\ln t + 1),
$$

(3.12)

which holds for every $(u, \eta) \in \Lambda$, to show that $(u_n)$ is uniformly bounded in $H^1(X)$. Then we obtain convergence (along a subsequence) to the desired minimum from part (i) of Lemma 3.3 see [17].

More in general, we can use analogous arguments to extend the convergence property in (i) of Lemma 3.3 and obtain the following weaker form of the Palais-Smale (PS)-condition, valid for $D_t$, when $t > 0$.

**Lemma 3.4 ([17])**. Let $t > 0$ and assume that $(u_n, \eta_n) \in \mathcal{V}_p$, $p > 2$ satisfies:

$$
\|\eta_n\|_{W^{1,p}} \leq C, \quad D_t(u_n, \eta_n) \to c \quad \text{and} \quad D'_t(u_n, \eta_n) \|\mathcal{V}_p^* \to 0,
$$

(3.13)

as $n \to +\infty$. Then there exist $(u, \eta) \in \mathcal{V}_p$ such that (along a subsequence):

(i) $u_n \to u$ in $H^1(X)$, $\eta_n \to \eta$ in $W^{1,2}(X, E)$, as $n \to \infty$,

(ii) $D_t(u, \eta) = c$ and $D_t'(u, \eta) = 0$.

Namely $(u, \eta)$ is a critical point for $D_t$ with corresponding critical value $c$.

**Proof.** See lemma 5.1 of [17].

Such a "compactness" property cannot be extended for $t = 0$. Indeed, $D_0$ no longer enjoys any sort of "coercivity" property with respect to the variable $u$, as instead ensured by (3.12) for $t > 0$. This is also the reason for the possible unboundedness of $D_0$ in $\Lambda$.

So, while we cannot guarantee that $D_0$ admits a (weak) critical point (namely that $(P)_{t=0}$ in (3.10) admits a solution), we see that, when a critical point of $D_0$
does exist, then it shares the exact same properties of \((u_t, \eta_t)\), the (only) critical point of \(D_t\) for \(t > 0\).

For example, the smoothness of \((u_t, \eta_t)\) allows us to compute the Hessian \(D''_t\) of \(D_t\) at \((u_t, \eta_t)\) as follows (see [17]):

\[
D''_t[v, l] = t \int_X e^{u''} v^2 dA + A[v, l] + B_t[v, l], \quad \text{for every } (v, l) \in \mathcal{V}_p \tag{3.14}
\]

with

\[
A[v, l] = 4 \int_X \| (\kappa - 1) v \beta_t + \bar{\partial} l \|^2 e^{(\kappa-1)u} dA \geq 0, \tag{3.15}
\]

\[
B_t[v, l] = 2 \int_X (|\bar{\partial} v|^2 - 4(\kappa - 1) Re(\beta_t, \bar{\partial} v \otimes l)e^{(\kappa-1)u}) dA + 4 \int_X \| \bar{\partial} l \|^2 e^{(\kappa-1)u} dA \leq \frac{\kappa - 1}{2} \int_X e^{(\kappa-1)u} \| l \|^2 dA, \tag{3.16}
\]

where the last estimate in (3.16) follows by completing the square and applying Corollary 2.1 to \((u_t, \eta_t)\), see [17]. Also we shall give the details about the estimate (3.16) for the case \(t = 0\) in (3.17).

Clearly, if we assume that \(D_0\) admits a critical point \((u_0, \eta_0)\), then we can take \(t = 0\) in the expressions (3.14), (3.15), (3.16), and for \(\beta_t = \beta_0 + \bar{\partial} \eta_0\) and \(u_t = u_0\), we obtain the Hessian \(D''_0\) at \((u_0, \eta_0)\).

Also the last inequality in (3.16) carries over to the case \(t = 0\), since again we can use, for the solution \((u_0, \eta_0)\) the estimate (2.8) of Corollary 2.1 and obtain that,

\[
B_{t=0}[v, l] \geq 4 \int_X (\| \bar{\partial} l \|^2 e^{(\kappa-1)u_0} - 2(\kappa - 1)^2 \| \beta_{t=0} \|^2 e^{2(\kappa-1)u_0} \| l \|^2) dA \geq \frac{\kappa - 1}{2} \int_X \| l \|^2 e^{(\kappa-1)u_0} dA. \tag{3.17}
\]

By virtue of (3.17), we will deduce that any critical point of \(D_0\) is a strict local minimum in \(\mathcal{V}_p\) for \(p > 2\), a property already established in [17] for \(D_t, t > 0\).

**Lemma 3.5.** Assume that the functional \(D_0\) admits a critical point \((u_0, \eta_0)\). Then \(\exists \gamma_0 > 0, \delta_0 > 0\) such that

\[
D_0(u, \eta) \geq D_0(u_0, \eta_0) + \int_X e^{(\kappa-1)u} \| \bar{\partial} \eta - \bar{\partial} \eta(u) \|^2 dA + \gamma_0(\| u - u_0 \|^2_{H^1} + \| \eta(u) - \eta_0 \|^2_{W^{1,p}}) \tag{3.18}
\]

for all \((u, \eta) \in \mathcal{V}_p : \| u - u_0 \|_{H^1} < \delta_0\). In particular, \((u_0, \eta_0)\) is a strict local minimum for \(D_0\) in \(\mathcal{V}_p, p > 2\).

**Proof.** Firstly, we observe that necessarily: \(\beta_0 \neq 0\), since for \([\beta] = [\beta_0] = 0\), problem \((\mathcal{P})_{t=0}\) admits no solution, and hence \(D_0\) cannot admit a critical point.
Claim: There exists \( \tau_0 > 0 \) such that
\[
\min_{v \in H^1 + \|v\|_{W^{1,2}}} D'_0[v, l] \geq \tau_0.
\] (3.19)

To establish (3.19), we argue by contradiction and suppose that there exists \((v_n, l_n) \in H^1(X) \times W^{1,2}(X, E)\) such that:
\[
\|v_n\|_{H^1} + \|l_n\|_{W^{1,2}} = 1 \quad \text{and} \quad D'_0[v_n, l_n] \to 0, \quad \text{as} \quad n \to +\infty.
\] (3.20)

As a consequence
\[
A_n = A_{t=0}(v_n, l_n) \to 0, \quad B_n = B_{t=0}(v_n, l_n) \to 0, \quad \text{as} \quad n \to +\infty.
\]

By using (3.17), we derive: \( \int_X \epsilon^{(k-1)\mu_0} \|l_n\|^2 \to 0 \) and thus (as \( u_0 \) is smooth in \( X \)) \( \|l_n\|_{L^2} \to 0 \), as \( n \to +\infty \).

As a consequence we have: \( \int_X \|\beta_0\|^2 \epsilon^{2(k-1)\mu_0} \|l_n\|^2 dA \to 0 \), and we can use such information in (3.17) to deduce that, \( \int_X \|\partial l_n\|^2 dA \to 0 \), as \( n \to +\infty \).

In conclusion we have shown that, \( \|l_n\|_{W^{1,2}} \to 0 \) as \( n \to +\infty \). So, from the explicit expression of \( B_n \), we find also that, \( \int_X \|\nabla v_n\|^2 dA = 4 \int_X |\partial v_n|^2 dA \to 0 \), as \( n \to +\infty \). Finally, we decompose: \( v_n = w_n + c_n \) with \( \int_X w_n dA = 0 \) and \( c_n = f_X v_n \). We know that: \( \|w_n\|_{L^2} \to 0 \), and by means of (3.15) (with \( t = 0 \)) we deduce that, \( \int_X \|\beta_{t=0}\|^2 \epsilon^{2} v_n^2 \to 0 \), and so \( \epsilon^2 \int_X \|\beta_{t=0}\|^2 dA \to 0 \), as \( n \to +\infty \). But \( \int_X |\beta_{t=0}|^2 \leq \int_X \|\beta_0\|^2 + \int_X \|\partial \eta_0\|^2 \geq \int_X \|\beta_0\|^2 > 0 \) and therefore, also \( c_n \to 0 \), as \( n \to +\infty \).

In conclusion we have obtained:
\[
\|v_n\|_{H^1}^2 + \|l_n\|_{W^{1,2}} \to 0, \quad \text{as} \quad n \to +\infty,
\]
and this is in contradiction with (3.20). Thus (3.19) is established.

At this point, we can use Taylor expansion for \( D_0 \) around \((u_0, \eta_0)\) in \( V_p \) and the continuity of the map \( \eta(u) \) in (3.6) to find a suitable \( \delta_0 > 0 \) sufficiently small, such that, for every \( u \in H^1(X) : \|u - u_0\|_{H^1} < \delta_0 \), we have:
\[
D_0(u, \eta(u)) = D_0(u_0, \eta_0) + \frac{1}{2} D''_0(u - u_0, \eta(u) - \eta_0)
+ o(\|u - u_0\|_{H^1}^2 + \|\partial \eta(u) - \eta_0\|_{L^2}^2)
\geq D_0(u_0, \eta_0) + \frac{\gamma_0}{2} (\|u - u_0\|_{H^1}^2 + \|\partial \eta(u) - \eta_0\|_{L^2}^2)
+ o(\|u - u_0\|_{H^1}^2 + \|\partial \eta(u) - \eta_0\|_{L^2}^2).
\]

Since \( \eta_0 = \eta(u_0) \), we can use the estimate (3.10) to conclude that, for every \( u \in H^1(X) : \|u - u_0\|_{H^1} < \delta_0 \), there hold:
\[
D_0(u, \eta(u)) \geq D_0(u_0, \eta_0) + \gamma_0 (\|u - u_0\|_{H^1}^2 + \|\eta(u) - \eta_0\|_{W^{1,2}}^2)
\]
with suitable \( \gamma_0 > 0 \). Consequently, if \((u, \eta) \in V_p \) and \( \|u - u_0\|_{H^1} < \delta_0 \) then (by
we find:

\[
D_0(u, \eta) = D_0(u, \eta(u)) + \int_X e^{(\kappa-1)u} \|\tilde{\eta} - \tilde{\eta}(u)\|^2 \, dA
\geq D_0(u_0, \eta_0) + \gamma_0 \left( \|u-u_0\|_{H^1}^2 + \|\eta(u) - \eta_0\|_{W^{1,p}}^2 \right)
+ \int_X e^{(\kappa-1)u} \|\tilde{\eta} - \tilde{\eta}(u)\|^2 \, dA,
\]

and (3.18) is established. In particular, if \( \|u-u_0\|_{H^1} < \delta \) and \((u, \eta) \neq (u_0, \eta_0)\), then \( D_0(u, \eta) > D_0(u_0, \eta_0) \) and the proof is completed.

However, to know that any critical point of \( D_0 \) is a strict local minimum in \( V_p \) is not enough to ensure that \( D_0 \) admits only one critical point. In fact we could be facing a situation similar to the function \( f(z) = |e^z - 1|^2 \) which admits infinitely many strict local minima at \( z = 2\pi in, n \in \mathbb{Z} \) and no other critical point.

Nonetheless, the presence of a strict local minimum for \( D_0 \) away from \((u_t, \eta_t)\) (for \( t > 0 \) small) allows us to exhibit a "mountain pass" structure (see [2]) for the functional \( D_t \), when \( t > 0 \) is sufficiently small. As shown in [17], this fact will contradict the uniqueness of \((u_t, \eta_t)\), as claimed in Theorem 7. In this way we can finally obtain,

**The Proof of Theorem 8** By using Lemma 3.5 for the critical point \((u_0, \eta_0)\) of \( D_0 \) we have:

**Claim 1:** \( \forall \delta \in (0, \delta_0) \ \exists d_\delta > 0 \) and \( t_\delta > 0 \), such that for \( t \in (0, t_\delta) \)
we have:
\[
D_t(u, \eta) \geq D_t(u_0, \eta_0) + d_\delta
\]
\[
\forall (u, \eta) \in V_p : \|u-u_0\|_{H^1}^2 + \|\eta(u) - \eta_0\|_{W^{1,p}}^2 = \delta.
\]

To establish (3.21), we simply apply (3.18) as follows:

\[
D_t(u, \eta) = t \int_X e^u \, dA + D_0(u, \eta)
\geq t \int_X e^u \, dA + D_0(u_0, \eta_0) + \int_X e^{(\kappa-1)u} \|\tilde{\eta} - \tilde{\eta}(u)\|^2 \, dA
+ \gamma_0 \left( \|u-u_0\|_{H^1}^2 + \|\eta(u) - \eta_0\|_{W^{1,p}}^2 \right)
\geq D_t(u_0, \eta_0) + \gamma_0 \delta + t \int_X (e^u - e^{(\kappa-1)u}) \, dA
\geq D_t(u_0, \eta_0) + \gamma_0 \delta - C_\delta t
\]
with a suitable constant \( C_\delta > 0 \). Clearly, the estimate above readily implies (3.21).

Next, we argue by contradiction and assume:

\( \exists \varepsilon_0 > 0 \) and \( t_n \to 0^+ : \|u-u_{t_n}\|_{H^1}^2 + \|\eta_0 - \eta_{t_n}\|_{W^{1,p}}^2 \geq \varepsilon_0 \),
for all $n \in \mathbb{N}$.

So, we fix $0 < \delta < \min\{\frac{\epsilon}{2}, \delta_0\}$ and take $n_0 = n_0(\delta) \in \mathbb{N}$ sufficiently large, so that $t_0 := t_{n_0} \in (0, t_d)$. Consequently,

$$\|u_0 - u_{t_0}\|_{H^1}^2 + \|\eta_0 - \eta_{t_0}\|_{W^{1,p}}^2 \geq \epsilon_0$$  \hspace{1cm} (3.22)

and

$$D_{t_0}(u, \eta) \geq D_{t_0}(u_0, \eta_0) + d_\delta \text{ for } \|u - u_{t_0}\|_{H^1}^2 + \|\eta(u) - \eta_{t_0}\|_{W^{1,p}}^2 = \delta. \hspace{1cm} (3.23)$$

Also recall that $D_{t_0}(u_0, \eta_0) \geq D_{t_0}(u_{t_0}, \eta_{t_0})$. So, from (3.22) and (3.23), we see that $D_{t_0}$ admits a mountain pass structure in the sense of [2]. But, as in [17], we show that this is impossible, since we can deduce the existence of another critical point for $D_{t_0}$ different from $(u_{t_0}, \eta_{t_0})$, in contradiction to Theorem 7. To be more precise, let $P_0 = (u_0, \eta_0)$ and $P_1 = (u_{t_0}, \eta_{t_0})$. We know that $P_0 \neq P_1$ (see (3.22)), and that $\eta_0 = \eta(u_0)$ and $\eta_{t_0} = \eta(u_{t_0})$. We define the family of paths

$$\mathcal{P} = \{\gamma \in C^0([0, 1], \mathcal{V}_p) : \gamma(0) = P_0, \gamma(1) = P_1\}.$$  

Clearly, $\mathcal{P}$ is not empty, as $\gamma(s) = (1 - s)P_0 + sP_1 \in \mathcal{P}$. Moreover, by setting: $d(\gamma_1, \gamma_2) = \max_{s \in [0, 1]} ||\gamma_1(s) - \gamma_2(s)||_{\mathcal{V}_p}$, we see that $(\mathcal{P}, d)$ defines a complete metric space.

**Claim 2:**

$$c = \inf_{\gamma \in \mathcal{P}} \max_{s \in [0, 1]} D_{t_0}(\gamma(s)) \geq D_{t_0}(u_0, \eta_0) + d_\delta \hspace{1cm} (3.24)$$

Indeed, if we take $\gamma \in \mathcal{P}$ with $\gamma(s) = (u(s), \eta(s))$, $s \in [0, 1]$ and we define

$$f(s) = ||u(s) - u_{t_0}||_{H^1}^2 + ||\eta(u(s)) - \eta_{t_0}||_{W^{1,p}}^2 \in C^0([0, 1]),$$

we see that: $f(0) = 0$ while $f(1) = ||u_{t_0} - u_0||_{H^1}^2 + ||\eta_{t_0} - \eta_0||_{W^{1,p}}^2 \geq \epsilon_0 > \delta$. So, by continuity, there exists $s_0 \in [0, 1]$ such that $f(s_0) = \delta$. Therefore, by (3.23), we find: $\max_{s \in [0, 1]} D_{t_0}(\gamma(s)) \geq D_{t_0}(\gamma(s_0)) \geq D_{t_0}(u_0, \eta_0) + d_\delta$, and (3.24) follows.

We are going to show that $c$ in (3.24) defines a critical value for $D_{t_0}$, and since $c > D_{t_0}(u_0, \eta_0) \geq D_{t_0}(u_{t_0}, \eta_{t_0})$, the corresponding critical point must be different from $(u_{t_0}, \eta_{t_0})$. In this way we reach a contradiction to Theorem 7.

To this purpose we note first that, if $\gamma \in \mathcal{P}$ with $\gamma(s) = (u(s), \eta(s))$, $s \in [0, 1]$, then setting $\tilde{\gamma}(s) = (u(s), \eta(u(s)))$, $s \in [0, 1]$, we easily check that also $\tilde{\gamma} \in \mathcal{P}$ and $D_{t_0}(\tilde{\gamma}(s)) \geq D_{t_0}(\gamma(s)) \geq D_{t_0}(u_0, \eta_0) + d_\delta$, and (3.24) follows.

Theorem 32. Let $(Y, d)$ be a complete metric space and $F : Y \rightarrow \mathbb{R}$ a non-negative and lower semi-continuous functional. For every $\epsilon > 0$ let $\gamma^0_\epsilon \in Y$ be such that: $F(\gamma^0_\epsilon) \leq \epsilon + \inf F$. Then there exists $\gamma_\epsilon \in Y$ such that

$$F(\gamma_\epsilon) \leq F(\gamma^0_\epsilon), \quad d(\gamma_\epsilon, \gamma^0_\epsilon) \leq \sqrt{\epsilon} \quad \text{and} \quad F(\gamma) \geq F(\gamma_\epsilon) - \sqrt{\epsilon}d(\gamma, \gamma_\epsilon), \quad \forall \gamma \in Y.$$
We are going to apply Theorem \[\text{[X]}\] with \((Y, d) = (P, d)\) and

\[
F(\gamma) = \max_{s \in [0,1]} D_{t_0}(\gamma(s)).
\] (3.25)

Therefore, for given \(\epsilon > 0\), without loss of generality we can take a path \(\gamma_0^\epsilon \in P\) of the form \(\gamma_0^\epsilon(s) = (u_0^\epsilon(s), \eta(u_0^\epsilon(s)))\) and satisfying: \(F(\gamma_0^\epsilon) < \epsilon + \inf_{\gamma \in P} F(\gamma)\) with \(F \) in \(\text{(3.25)}\). As a consequence, we find a path \(\gamma_0 \in P\) such that,

\[
c \leq \max_{s \in [0,1]} D_{t_0}(\gamma_0(s)) < c + \epsilon, \quad \max_{s \in [0,1]} \|\gamma_0 - \gamma_0^\epsilon\|_{V_p} \leq \sqrt{\epsilon},
\]

and,

\[
\max_{s \in [0,1]} D_{t_0}(\gamma(s)) \geq \max_{s \in [0,1]} D_{t_0}(\gamma_0(s)) - \sqrt{\epsilon} \max_{s \in [0,1]} \|\gamma(s) - \gamma_0(s)\|_{V_p},
\]

for every \(\gamma \in P\). Furthermore, in view of \(\text{(3.24)}\), the set

\[
T_\epsilon := \{ \hat{s} \in [0,1] : D_{t_0}(\gamma_\epsilon(\hat{s})) = \max_{s \in [0,1]} D_{t_0}(\gamma_\epsilon(s)) \}
\]

is relatively compact in the open interval \((0, 1)\), that is \(T_\epsilon \subset \subset (0, 1)\). As a consequence, by Lemma 5.4 of \([17]\):

\[
\exists s_\epsilon \in T_\epsilon : \|D'_{t_0}(\gamma_\epsilon(s_\epsilon))\|_{V_p} \leq \sqrt{\epsilon}.
\]

So, along a sequence \(\epsilon_n \rightarrow 0\), we find \((u_n, \eta_n) = \gamma_\epsilon_n(s_\epsilon_n)\) and \((u_0^\epsilon, \eta(u_0^\epsilon)) = \gamma_0^\epsilon(s_\epsilon_n) \in V_p\) such that, as \(n \rightarrow \infty\),

\[
D_{t_0}(u_n, \eta_n) \rightarrow c, \quad \|D'_{t_0}(u_n, \eta_n)\|_{V_p^*} \rightarrow 0 \quad (3.26)
\]

\[
\|u_n - u_0^\epsilon\|_{H^1} + \|\eta_n - \eta(u_0^\epsilon)\|_{W^{1,p}} \rightarrow 0. \quad (3.27)
\]

As before, from the first limit in \(\text{(3.26)}\), we deduce that \(u_n\) is uniformly bounded in \(H^1(X)\), and so, by \(\text{(3.27)}\), also \(u_0^\epsilon\) is uniformly bounded in \(H^1(X)\).

As a consequence of (i) in Lemma \([3.3]\) we deduce that necessarily \(\eta(u_0^\epsilon)\) is uniformly bounded in \(W^{1,p}(X, E)\), and (by \(\text{(3.27)}\)) we find \(\|\eta_n\|_{W^{1,p}} \leq C\) for suitable \(C > 0\).

Therefore the (PS)-sequence \((u_n, \eta_n)\) satisfies \(\text{(3.14)}\) and so we can apply Lemma \([3.4]\) to conclude that \(c\) is a critical value for \(D_{t_0}\). Hence we reach the desired contradiction and conclude that:

\[
(u_t, \eta_t) \rightarrow (u_0, \eta_0), \quad \text{as} \quad t \rightarrow 0^+, \quad \text{in} \quad V_p. \quad (3.28)
\]

Since any other critical point of \(D_0\) must satisfy \(\text{(3.25)}\), by the uniqueness of \((u_t, \eta_t)\), we deduce that \((u_0, \eta_0)\) must be the only critical point of \(D_0\). Finally, for every \((u, \eta) \in \Lambda\), we have:

\[
D_0(u, \eta) = \lim_{t \rightarrow 0^+} D_t(u, \eta) \geq \lim_{t \rightarrow 0^+} D_t(u_t, \eta_t) = D_0(u_0, \eta_0).
\]

Consequently, \(D_0\) is bounded from below in \(\Lambda\) and \((u_0, \eta_0)\) is its global minimum point. This concludes the proof of Theorem \([8]\) \(\square\)
Next, we notice that, by the strict positivity of the Hessian $D''_t$ at $(u_t, \eta_t)$, as pointed out by the estimates in 3.14-3.16, we can use the Implicit Function Theorem (cf. 30) for the map:

$$ F : \mathbb{R}^+ \times H^1(X) \times W^{1,p}(X,E) \longrightarrow (H^1(X) \times W^{1,p}(X,E))^*, \ p > 2, $$
given by:

$$ F(t,u,\eta) = (\frac{\partial}{\partial u} D_t(u,\eta), \frac{\partial}{\partial \eta} D_t(u,\eta)) $$
in order to show the $C^2$-dependence of $(u_t, \eta_t)$ with respect to the parameter $t \in (0, +\infty)$. Furthermore, by setting:

$$ c_t := D_t(u_t, \eta_t) \leq D_t(u, \eta), \ \forall (u, \eta) \in \Lambda, $$
we have, $c_t \in C^2([0, +\infty[)$ and we may compute:

$$ \dot{c}_t = \frac{d}{dt} c_t = \frac{\partial}{\partial u} D_t(u, \eta_t) \dot{u}_t + \frac{\partial}{\partial \eta} D_t(u, \eta_t) \dot{\eta}_t + \int_X u_t = \int_X u_t, \quad (3.29) $$
and 3.29 confirms the fact that $c_t$ is increasing for $t \in (0, +\infty)$. Furthermore,

**Lemma 3.6.** (i) $c_t = D_t(u_t, \eta_t)$ is concave in $(0, +\infty)$.
(ii) The function: $t \longrightarrow t \int_X e^{u_t} dt$ is increasing in $(0, +\infty)$.

**Proof.** By straightforward calculations we find:

$$ \ddot{c}_t = \int_X e^{u_t} \ddot{u}_t dA = \frac{d^2}{dt^2} D_t(u_t, \eta_t) = 2 \int_X e^{u_t} \dot{u}_t dA + D''_t[\dot{u}_t, \dot{\eta}_t], $$
and so, \( \int_X e^{u_t} \dot{u}_t dA = -D''_t[\dot{u}_t, \dot{\eta}_t]. \) By 3.14-3.16, also we know that:

$$ D''_t[\dot{u}_t, \dot{\eta}_t] > t \int_X e^{u_t} \dot{u}_t^2 dA $$
and we obtain:

$$ \ddot{c}_t = \int_X e^{u_t} \ddot{u}_t dA = -D''_t[\dot{u}_t, \dot{\eta}_t] \leq -t \int_X e^{u_t} \dot{u}_t^2 dA \leq 0. \quad (3.30) $$
Hence, $c_t$ is concave, and we have:

$$ \int_X e^{u_t} \dot{u}_t dA + t \int_X e^{u_t} \dot{u}_t^2 dA \leq 0. \quad (3.31) $$
Therefore, by 3.30, 3.31 and Jensen’s inequality, we deduce:

$$ t(\int_X e^{u_t} \dot{u}_t dA)^2 \leq t \int_X e^{u_t} (\dot{u}_t)^2 dA \leq \int_X e^{u_t} \dot{u}_t dA, $$
namely: \( t | \int_X e^{u_t} \dot{u}_t dA | \leq 1 \) or equivalently: \( \int_X e^{u_t} dA + t \int_X e^{u_t} \dot{u}_t dA \geq 0. \)
Thus, we have proved that, $\frac{d}{dt}(t \int_X e^{u_t} dA) > 0$ and (ii) follows. \( \square \)
On the other hand (by integration of the first equation in (3.4)) we have:

\[ t \int_X e^{u_t} dA + 4(\kappa - 1) \int_X e^{(\kappa - 1)u_t} \| \beta_0 + \bar{\partial} \eta_t \|^2 dA = 4\pi(g - 1) \]  
(3.32)

and so, the following integral term:

\[ 4(\kappa - 1) \int_X e^{(\kappa - 1)u_t} \| \beta_0 + \bar{\partial} \eta_t \|^2 dA = 4\pi(g - 1) - t \int_X e^{u_t} dA \]

is decreasing as a function of \( t \in (0, +\infty) \). Therefore, it is well defined the value:

\[ \rho([\beta]) = \rho([\beta_0]) = 4(\kappa - 1) \lim_{t \to 0^+} \int_X e^{(\kappa - 1)u_t} \| \beta_0 + \bar{\partial} \eta_t \|^2 dA \]

(3.33)

and naturally we wish to identify the value of \( \rho([\beta]) \) in terms of the given cohomology class \([\beta] \in H^{0,1}(X, E)\).

So far, concerning the value \( \rho([\beta]) \) in (3.33), we know the following:

**Proposition 3.1.** For given \([\beta] \in H^{0,1}(X, E)\) with harmonic representative \( \beta_0 \in A^{0,1}(X, E)\) we have:

(i) \( \rho([\beta]) \in [0, 4\pi(g - 1)] \) and \( \rho([\beta]) = 0 \iff [\beta] = 0 \)

(ii) If \([\beta] \neq 0\) then for every \( \rho \in (0, \rho([\beta])) \) there exits a unique \( t \in (0, +\infty) \) such that,

\[ \rho = 4(\kappa - 1) \int_X e^{(\kappa - 1)u_t} \| \beta_0 + \bar{\partial} \eta_t \|^2 dA, \]

where \((u_t, \eta_t)\) is the global minimum (and unique critical point) of the functional \( D_t \).

It is also clear that, \( D_0 \) is bounded from below on \( \Lambda \), if and only if

\[ \inf_{t > 0} c_t = \lim_{t \to 0^+} c_t := c_0 > -\infty \quad \text{and} \quad c_0 = \inf_{\Lambda} D_0. \]

More importantly the following holds:

**Proposition 3.2.** If \( D_0 \) is bounded from below on \( \Lambda \) then \( \rho([\beta]) = 4\pi(g - 1) \).

*Proof.* Argue by contradiction and assume that \( 0 \leq \rho([\beta]) < 4\pi(g - 1) \). Hence

\[ \lim_{t \to 0^+} t \int_X e^{u_t} dA = 4\pi(g - 1) - \rho([\beta]) := \mu > 0. \]

On the other hand (by de L'Hopital rule)

\[ \lim_{t \to 0^+} \frac{c_t}{\ln t} = \lim_{t \to 0^+} \frac{c_t}{t} = \lim_{t \to 0^+} t \int_X e^{u_t} dA = \mu > 0 \]

and this implies that, \( c_t \to -\infty \) as \( t \to 0^+ \), a contradiction. \( \square \)
Now the delicate questions to ask are the following:

if \( \rho([\beta]) = 4\pi(\mathfrak{g} - 1) \) is it true that \( D_0 \) is bounded from below? \hfill (3.34) 
if \( D_0 \) is bounded from below then is the infimum attained? \hfill (3.35)

In order to investigate the questions raised above, for \((u_t, \eta_t)\), the global minimum of \( D_t \) in \( \Lambda \) (given in Theorem B) we let:

\[
    u_t = w_t + d_t, \quad \text{with} \quad \int_X w_t dA = 0 \quad \text{and} \quad d_t = \int_X u_t dA
\]

\[
    \beta_t = \beta_0 + \overline{\partial} \eta_t \in \mathcal{A}^{0,1}(X, E) \quad \text{and} \quad \alpha_t = e^{u_t} \star E \beta_t \in C_*(X)
\]

and set

\[
    s_t \in \mathbb{R} : e^{(\kappa - 1)s_t} = \|\alpha_t\|^2_{L^2}.
\]

With the notation above, we check the following easy properties:

**Lemma 3.7.** For given \([\beta] \in \mathcal{H}^{0,1}(X, E)\) with harmonic representative \(\beta_0 \in [\beta]\) and \(t > 0\) there holds

\[
    \begin{align*}
        (i) & \quad \forall \, q \in [1, 2) \quad \exists \, C_q > 0 : \|w_t\|_{W^{1,q}(X)} \leq C_q \quad (3.36) \\
        (ii) & \quad w_t \leq C \quad \text{in} \quad X \\
        (iii) & \quad t e^{u_t} \leq 1 \quad \text{in} \quad X \\
        (iv) & \quad s_t \leq d_t + C \quad \text{for a suitable constant} \quad C > 0 \\
        (v) & \quad \text{if} \ [\beta] \neq 0, \ \text{there exists a constant} \quad \gamma = \gamma(\kappa) > 0 \\
        & \quad \text{(depending on} \ \kappa \ \text{only) such that,} \\
        & \quad \int_X e^{-u_t} dA \geq \gamma \int_X \|\beta_0\|^2 dA \\
    \end{align*}
\]

*Proof.* The inequality \((3.36)\) is a direct consequence of the fact that the right hand side of the first equation in \((3.4)\) is uniformly bounded in \(L^1(X)\) (see \((3.32)\)), while \((3.38)\) follows easily by the maximum principle.

Next, to get \((3.37)\) we use the Green's representation formula for \(w_t\), with \(G(p, q)\) the Green function of the Laplace-Beltrami operator on \(X\) satisfying:

\[
    \int_X G(p, q) dA(q) = 0 \quad \text{and} \quad G(p, q) \leq a \quad \text{in} \quad (X \times X) \quad \text{(see \[3\])}.
\]

We have:

\[
    \begin{align*}
        w_t(p) &= 2 \int_X G(p, q)(te^{u_t(q)} + 4(\kappa - 1)e^{(\kappa - 1)u_t(q)}\|\beta_t\|^2(q) - 1)dA(q) \\
        &= 2 \int_X G(p, q)(te^{u_t(q)} + 4(\kappa - 1)e^{(\kappa - 1)u_t(q)}\|\beta_t\|^2(q))dA(q) \\
        &\leq 2a \int_X te^{u_t} + 4(\kappa - 1)e^{(\kappa - 1)u_t}\|\beta_t\|^2 dA \leq 8\pi(\mathfrak{g} - 1)a.
    \end{align*}
\]

To establish \((iv)\) we estimate:

\[
    \begin{align*}
        e^{(\kappa - 1)s_t} = \|\alpha_t\|^2_{L^2} &= \int_X ||\alpha_t||^2 dA = \int_X e^{2(\kappa - 1)u_t}\|\beta_t\|^2 dA \\
        &\leq C e^{(\kappa - 1)d_t} \int_X e^{(\kappa - 1)u_t}\|\beta_t\|^2 dA \leq C e^{(\kappa - 1)d_t} \frac{\rho([\beta])}{4(\kappa - 1)} \leq C e^{(\kappa - 1)d_t}
    \end{align*}
\]
with a suitable constant $C > 0$, and (3.39) follows.

Finally, to get (3.40), we recall first that, if $\beta_0 \in A^{0,1}(X, E)$ is harmonic then we have the estimate: $\|\beta_0\|_{L^\infty} \leq C\|\beta_0\|_{L^2}$ for suitable $C > 0$. Consequently,

$$\int_X \|\beta_0\|^2 dA = \int_X \langle \beta_0, \partial \eta \rangle dA \leq \|\beta_0\|_{L^\infty} \int_X \|\beta_0 + \partial \eta\| dA$$

$$\leq C\|\beta_0\|_{L^2} \left( \int_X e^{-\mu t} dA \right)^{\frac{1}{2}} \left( \int_X e^{u_t} \|\beta_0 + \partial \eta\|^2 dA \right)^{\frac{1}{2}}$$

$$\leq C\|\beta_0\|_{L^2} \left( \frac{\rho(\|\beta_0\|)}{4(\kappa - 1)} \right)^{\frac{1}{2}} \left( \int_X e^{-\mu t} \right)^{\frac{1}{2}} dA.$$

Hence, if $[\beta] \neq 0$, then $0 < \rho([\beta]) \leq 4\pi(g - 1)$ and we easily deduce (3.40) from the above estimate. \[ \square \]

**Remark 3.1.** By (3.32) and Jensen’s inequality, we have:

$$te^{d_t} \leq 1.$$

According to the fixed basis for $C_\kappa(X)$ in (2.11), we can decompose:

$$\alpha_t = \sum_{j=1}^\nu a_{j,t} s_j \quad \text{with} \quad \|\alpha_t\|_{L^2} = \sum_{j=1}^\nu |a_{j,t}|^2 = e^{(\kappa - 1)s_t}.$$

Furthermore, along a sequence $t_k \to 0^+$, we may assume that, for $d_k := d_{t_k}$ and $u_k := u_{t_k}$, there holds, as $k \to +\infty$:

$$w_k := w_{t_k} \to w_0 \quad \text{and} \quad e^{w_k} \to e^{w_0} \quad \text{pointwise and in} \quad L^p(X), \quad p > 1;$$

$$t_k e^{d_k} \to \mu \geq 0 \quad \text{and} \quad t_k e^{u_k} \to \mu e^{w_0} \quad \text{pointwise and in} \quad L^p(X). \quad (3.41)$$

In addition, in view of Remark 2.1 also we may assume that, for suitable $1 \leq N \leq 2\kappa(g - 1)$ and $k$ large:

$$\alpha_{t_k} \in C_\kappa(X) \setminus \{0\} \quad \text{admits} \quad N \quad \text{distinct zeroes}: \quad \{z_{1,k}, \ldots, z_{N,k}\} \quad \text{with}$$

$$\text{corresponding multiplicity} \quad \{n_1, \ldots, n_N\} \subset \mathbb{N} \quad \text{and} \quad \sum_{j=1}^N n_j = 2\kappa(g - 1).$$

Moreover,

$$z_{j,k} \to z_j, \quad \text{as} \quad k \to +\infty, \quad j \in \{1, \ldots, N\}. \quad (3.42)$$

Next, we set:

$$\hat{\alpha}_t = \frac{\alpha_t}{\|\alpha_t\|_{L^2}} = e^{-\frac{\kappa - 1}{\kappa} s_t} \alpha_t = \sum_{j=1}^\nu \hat{\alpha}_{j,t} s_j \quad \text{with} \quad \hat{\alpha}_{j,t} = e^{-\frac{\kappa - 1}{\kappa} s_t} a_{j,t}. \quad (3.43)$$

Since $|\hat{\alpha}_{j,t}| \leq 1, \quad \forall \ j = 1, \ldots, N$, also we may suppose that, as $k \to +\infty$,

$$\hat{\alpha}_{j,t_k} \to \hat{\alpha}_j \quad \text{and for} \quad \hat{\alpha}_0 := \sum_{j=1}^\nu \hat{\alpha}_j s_j \in C_\kappa(X) : \quad \hat{\alpha}_{t_k} \to \hat{\alpha}_0, \quad (3.44)$$
with \( \|\hat{\alpha}_0\|_{L^2} = 1 \), and so \( \hat{\alpha}_0 \neq 0 \). Therefore, \( \hat{\alpha}_0 \) vanishes exactly at the set:

\[
Z := \{z_1, \ldots, z_N\}
\]

with \( z_j \) given in (3.42). Since the total multiplicity of each \( z_j \) in \( Z \) adds up to the value: \( 2\kappa(g - 1) \), by Remark 2.1 we know that, \( \hat{\alpha}_0 \) cannot vanish anywhere else. Observe that the points in \( Z \) may not be distinct.

We define:

\[
\xi_k = -(\kappa - 1)(u_{tk} - s_{tk})
\]

and

\[
R_k = 8(\kappa - 1)^2\|\hat{\alpha}_t\|^2
\]

such that

\[
- \Delta \xi_k = R_k e^{\xi_k} - f_k \text{ in } X
\]

with \( f_k := 2(\kappa - 1)(1 - t_k e^{u_{tk}}) \) satisfying: \( \|f_k\|_{L^\infty(X)} \leq C \) and

\[
f_k \rightharpoonup f_0 =: 2(\kappa - 1)(1 - \mu e^{u_0}) \text{ in } L^p(X), \quad p > 1,
\]

\[
\int_X f_0 = 2(\kappa - 1)\rho([\beta]) > 0, \quad \text{for } [\beta] \neq 0
\]

see (3.38)-(3.41) and (3.32)-(3.33). Notice that,

\[
G_k(z) = 8(\kappa - 1)^2\prod_{j=1}^N d_{g_k}(z, z_{j,k})^{2n_j} G_k(z), \quad z \in X,
\]

where

\[
z_{j,k} \neq z_{l,k} \quad \text{for } j \neq l \in \{1, \ldots, N\}, \quad n_j \in \mathbb{N} : \sum_{j=1}^N n_j = 2\kappa(g - 1)
\]

\[
G_k \in C^1(X) : 0 < a \leq G_k \leq b \quad \text{and} \quad |\nabla G_k| \leq A \text{ in } X
\]

with suitable positive constants \( a, b \) and \( A \). Hence (by taking a subsequence if necessary) we may assume that,

\[
G_k \rightharpoonup G_0 \text{ in } C^0(X) \quad \text{and so } R_k \rightharpoonup R_0 \text{ in } C^0(X), \quad k \to +\infty
\]

with

\[
R_0(z) = 8(\kappa - 1)^2\prod_{j=1}^N (d_{g_k}(z, z_j))^{n_j} G_0(z) = 8(\kappa - 1)^2\|\hat{\alpha}_0\|^2.
\]

Next, in (3.42) we have identified the subset \( Z_0 \subseteq Z \) (possibly empty) given by "collapsing" zeroes, namely by those zeroes of \( R_0 \) (or equivalently \( \hat{\alpha}_0 \)) corresponding to the limit points of different zeroes of \( R_k \) (or equivalently \( \hat{\alpha}_k \)), namely:

\[
Z_0 = \{z \in Z : \exists s \geq 2, \ 1 \leq j_1 < \ldots < j_s \leq N \text{ such that} \quad z = z_{j_1} = \ldots = z_{j_s} \quad \text{and} \quad z \notin Z \setminus \{z_{j_1}, \ldots, z_{j_s}\}\}
\]

With the information above, we see that Theorem 3 in [34] applies and yields to the following alternatives about the asymptotic behavior of \( \xi_k \):
**Theorem D** (34). Let \( \xi_k \) satisfy (3.46) and assume (3.47)-(3.51). Then with the above notation one of the following alternatives holds (along a subsequence):

(i) (compactness) : \( \xi_k \to \xi_0 \) in \( C^2(\mathcal{X}) \) with 
\[
- \Delta \xi_0 = R_0 e^{\xi_0} - f_0, \quad \text{in } \mathcal{X} \quad (3.52)
\]

(ii) (blow-up) : There exists a finite blow-up set 
\( \mathcal{S} = \{ q \in \mathcal{X} : \exists q_k \to q \text{ and } \xi_k(q_k) \to +\infty, \text{ as } k \to +\infty \} \)

such that, \( \xi_k \) is uniformly bounded from above on compact sets of \( \mathcal{X} \setminus \mathcal{S} \) and, as \( k \to +\infty \),

(a) either (blow-up with concentration) :
\[
\xi_k \to -\infty \text{ uniformly on compact sets of } \mathcal{X} \setminus \mathcal{S},
\]
\[
R_k e^{\xi_k} \rightharpoonup \sum_{q \in \mathcal{S}} \sigma(q)\delta_q \text{ weakly in the sense of measures, with } \sigma(q) \in 8\pi\mathbb{N}.
\]

In particular, \( \int_X f_0 \, dA \in 8\pi\mathbb{N} \), and
\[
\sigma(q) = 8\pi \text{ if } q \notin \mathbb{Z} \text{ and } \sigma(q) = 8\pi(1 + n_i) \text{ if } q = z_i \in \mathbb{Z} \setminus \mathbb{Z}_0.
\]

Moreover, such an alternative always holds when \( \mathcal{S} \setminus \mathbb{Z}_0 \neq \emptyset \).

(b) or (blow-up without concentration) :
\[
\xi_k \to \xi_0 \text{ in } C_{loc}^2(\mathcal{X} \setminus \mathcal{S}),
\]
\[
R_k e^{\xi_k} \rightharpoonup R_0 e^{\xi_0} + \sum_{q \in \mathcal{S}} \sigma(q)\delta_q \text{ weakly in the sense of measures,}
\]
\[
\sigma(q) \in 8\pi\mathbb{N};
\]

and 
\[
- \Delta \xi_0 = R_0 e^{\xi_0} + \sum_{q \in \mathcal{S}} \sigma(q)\delta_q - f_0 \text{ in } \mathcal{X}.
\]

Furthermore, if alternative (b) of (ii) holds, then \( \mathcal{S} \subset \mathbb{Z}_0 \) and so, in this case, blow-up occurs at points of "collapsing" zeroes.

By virtue of Theorem [19] we start to derive the following consequences:

**Lemma 3.8.** If \( \xi_k \) in (3.45) satisfies alternative (i), then \( D_0 \) is bounded from below in \( \Lambda \) and \( (u_t, \eta_t) \to (u_0, \eta_0) \), as \( t \to 0^+ \), in \( V_p \), \( p > 2 \) (and in any other relevant norm) with \( (u_0, \eta_0) \) the only critical point of \( D_0 \) corresponding to its global minimum. In particular, \( \rho([\beta]) = 4\pi(g - 1) \) in this case.
Proof. Recall that we have set, \( u_t = u_k = d_k + w_k \). By hypothesis, \( \xi_k \leq C \) in \( X \) and therefore, by elliptic estimates we derive that \( w_k \) is uniformly bounded and actually (along a subsequence) converges strongly in \( C^{2,\alpha}(X) \). Thus we can estimate:

\[
e^d_k \int_X |\beta_0 + \bar{\partial} \eta_k|^2 dA \leq C \int_X e^{u_k} |\beta_0 + \bar{\partial} \eta_k|^2 dA \leq C
\]

and \( \int_X |\beta_0 + \bar{\partial} \eta_k|^2 dA \geq \int_X |\beta_0|^2 dA > 0 \). As a consequence, \( d_k \) is uniformly bounded from above. On the other hand, since \( c_t = D_t(u_t, \eta_t) \leq c_1 \), for all \( t \in (0, 1) \), we obtain:

\[
\frac{1}{4} \int_X |\nabla w_k|^2 - 4\pi (\kappa - 1) d_k = c_t + O(1) \leq C,
\]

showing that \( d_k \) is also uniformly bounded from below. Hence, along a subsequence, we find that, \( u_k \to u_0 \) strongly in \( C^{2\alpha}(X) \), as \( k \to +\infty \), and, by virtue of Proposition 3.2, we have:

\[
\eta_k = \eta(u_k) \to \eta(u_0) = \eta_0 \quad \text{in} \quad W^{1,p}(X, E), \quad \text{as} \quad k \to +\infty, \quad p > 1.
\]

In conclusion, \( D_0(u_0, \eta_0) = \lim_{k \to +\infty} D_t(u_t, \eta_t) = \min \Lambda D_0 \).

As an immediate consequence of Lemma 3.8 we have:

**Proposition 3.3.** For every \( [\beta] \in H^{0,1}(X) \setminus \{0\} \) we have: \( \rho([\beta]) \geq \frac{4\pi}{(\kappa - 1)} \).

**Proof.** If by contradiction we assume that: \( \rho([\beta]) < \frac{4\pi}{(\kappa - 1)} \), then:

\[
\lim_{k \to +\infty} \int_X R_k e^{\xi_k} = 8(\kappa - 1)^2 \lim_{k \to +\infty} \int_X e^{(\kappa - 1)u_k} |\beta_0 + \bar{\partial} \eta_k|^2 dA = 2(\kappa - 1) \rho([\beta]) < 8\pi.
\]

Therefore, we see that necessarily \( \xi_k \) must satisfy the "compactness" alternative (i) in Theorem D. Thus, by Lemma 3.8 we find \( 4\pi (\kappa - 1) = \rho([\beta]) = \frac{4\pi}{(\kappa - 1)} \), a contradiction.

**Remark 3.2.** By combining part (i) of Proposition 3.1 and Proposition 3.3 we conclude that, if \( \kappa = 2 \) and \( \kappa = 2 \) then \( \rho([\beta]) = 4\pi \), \( \forall [\beta] \in H^{0,1}(X, E) \setminus \{0\} \).

So far we have established Theorem 2 and Proposition 1.1.

To proceed further, in the following section we are going to analyze what happens when \( \xi_k \) blows-up, in the sense of alternative (ii) of Theorem D.

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3.1 Blow-up Analysis for Minimizers

In order to simplify technicalities, from now on we shall focus to the case:

\[ \kappa = 2. \tag{3.54} \]

As above, we write

\[ u_t = u = d_k + w_k, \quad s = s_t, \quad d_k = d_t, \quad \text{and we consider the} \]

sequence

\[ \xi_k = -(u_k - s_k), \tag{3.55} \]

satisfying:

\[ -\Delta \xi_k = 8\|\hat{\alpha}_k\|^2 e^{\xi_k} - f_k \quad \text{in} \quad X \tag{3.56} \]

(see (3.43) and (3.44)) with \( \hat{\alpha}_k = e^{-\frac{s_k}{2}}\alpha_k \) and, as \( k \to +\infty \),

\[ f_k = 2(1 - t_k e^{u_k}) \to f_0 = 2(1 - \mu e^{w_0}) \quad \text{in} \quad L^p(X), \quad p > 1, \tag{3.57} \]

We suppose that \( \xi_k \) blows up, in the sense of (ii) in Theorem D, and that

\[ S = \{ q_1, \ldots, q_m \}, \quad 1 \leq m \leq g - 1 \tag{3.58} \]

is the corresponding (non empty) blow-up set. By recalling that, \( \|w_k\|_{L^2(X)} \leq C \) (see (3.36)) and by using elliptic estimates, we easily derive that the sequence \( w_k \) is uniformly bounded away from the blow-up set \( S \), and therefore,

\[ \xi_k = -(d_k - s_k) + O(1) \quad \text{on compact sets of} \quad X \setminus S. \tag{3.59} \]

Remark 3.3. Since

\[ c_k = D_{t_k}(v_k, \eta_k) = \frac{1}{4} \int_X |
abla w_k|^2 dA - 4\pi(g-1)d_k + O(1), \]

when blow-up occurs then, \( d_k \to +\infty \), as \( k \to +\infty \).

From (3.59) we see that, "blow-up with concentration" in Theorem D occurs if and only if \( d_k - s_k \to +\infty \).

We start our investigation with the case where \( \hat{\alpha}_0 \) in (3.44) does not vanish on \( S \), namely: \( \hat{\alpha}_0(q_l) \neq 0, \ \forall \ l \in \{1, \ldots, m\} \). In this case we can use the blow-up analysis available in [7],[24],[9] to show Theorem 3 which for convenience we restate as follows:

**Theorem 9.** Assume (3.54)-(3.57) and suppose that the blow-up set \( S \neq \emptyset \) of \( \xi_k \) in (3.58) satisfies:

\[ S \cap Z = \emptyset. \tag{3.60} \]

Then (along a subsequence), as \( k \to +\infty \):

\[ \alpha_k \to \alpha_0 \in C_2(X) \quad \text{with} \quad \alpha_0 \neq 0 \quad \text{vanishing exactly at} \quad Z, \tag{3.61} \]

\[ e^{-w_k} \to \pi \sum_{q \in S} \frac{1}{\|\alpha_0\|^2(q)} \delta_q, \quad \text{weakly in the sense of measures}, \tag{3.62} \]

\[ c_k = D_{t_k}(u_k, \eta_k) = -4\pi(g-1-m)d_k + O(1), \quad \text{with} \quad d_k = \int_X u_k dA \to +\infty, \tag{3.63} \]

\[ \int_X \beta_0 \wedge \alpha = 0 \quad \text{for} \quad \alpha \in Q_2[q_1, \ldots, q_m] \tag{3.64} \]

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(recall (2.14)). Furthermore, \( \rho([\beta]) = 4 \int_X \beta_0 \wedge \alpha_0 = 4\pi m \).

**Remark 3.4.** Since \( \dim \mathbb{Q} Q_2[q_1, \ldots, q_m] = 3(\vartheta - 1) - m \) (see (2.15)), then the orthogonality condition (3.64) together with the estimate (3.63) for the global minimum of \( D_{t_k} \) seem to indicate that \( \xi_k \) should admit only one blow-up point \( (m = 1) \), where the holomorphic quadratic differential \( \ast E \beta_0 \) does not vanish.

**Proof.** From (3.60) we see that \( \xi_k \) satisfies alternative (a) of (ii) in Theorem D, i.e. blow-up occurs with the “concentration” property, and furthermore \( \| \alpha_0 \| (q) > 0, \forall q \in \mathcal{S} \). More precisely, (along a subsequence) as \( k \rightarrow +\infty \), we have:

\[
8e^{\xi_k} \| \hat{\alpha}_k \|^2 \rightharpoonup 8\pi \sum_{l=1}^m \delta_{q_l}, \text{ weakly in the sense of measures,}
\]

\[
w_k \rightharpoonup w_0 \text{ in } C^2(\alpha(X \setminus \mathcal{S}), -\Delta w_0 = 8\pi \sum_{l=1}^m \delta_{q_l} + 2\mu e^{w_0} - 2 \text{ in } X.}
\]

Furthermore (by (3.59)) we know that, \( d_k - s_k \rightarrow +\infty \), and from (3.60), we derive also that,

\[
\int_X e^{\xi_k} \leq C. \tag{3.65}
\]

Hence

\[
e^{\xi_k} \rightarrow \pi \sum_{l=1}^m \frac{1}{\| \alpha_0 \|^2(q_l)} \delta_{q_l}, \text{ as } k \rightarrow +\infty, \tag{3.66}
\]

weakly in the sense of measures.

**Claim:** \( s_k = O(1) \). \tag{3.67}

To establish (3.67), we observe that,

\[
\int_X \| \beta_0 \|^2 dA = \int_X (\beta_0 + \tilde{\eta}_k) dA = \int_X (e^{-u_k} \beta_0) \wedge \alpha_k \\
\leq \| \beta_0 \|_{L^\infty} \| \alpha_k \|_{L^\infty} \int_X e^{-u_k} dA \\
\leq C e^{-\frac{\vartheta}{2}} \int_X e^{-u_k} dA \leq C e^{-\frac{\vartheta}{2}} \int_X e^{\xi_k} dA \leq C e^{-\frac{\vartheta}{2}}, \tag{3.68}
\]

where in the last inequality we have used (3.65). Since \( \beta_0 \neq 0 \), then (3.68) implies that, \( s_k \leq C \) for suitable \( C > 0 \). In order to obtain a lower bound for \( s_k \), we take \( \alpha \in C_2(X) \) and compute:

\[
\int_X \beta_0 \wedge \alpha = \int_X (\beta_0 + \tilde{\eta}_k) \wedge \alpha = \int_X (\beta_0 + \tilde{\eta}_k, \ast E^{-1} \alpha) dA \\
= e^{-\frac{\vartheta}{2}} \int_X e^{\xi_k} (\ast E^{-1} \hat{\alpha}_k, \ast E^{-1} \alpha) dA. \tag{3.69}
\]
For $r > 0$ sufficiently small, clearly we have
\[ |\int_{X \setminus \bigcup_{l=1}^{m} B_r(q_l)} e^{\xi_k} (\ast_{E}^{-1} \hat{\alpha}_k, \ast_{E}^{-1} \alpha) dA| \leq C \int_{X \setminus \bigcup_{l=1}^{m} B_r(q_l)} e^{\xi_k} dA \rightarrow 0, \]
as $k \rightarrow +\infty$. Next, for $l \in \{1, \ldots, m\}$, around $q_l$ we introduce holomorphic coordinates $\{z\}$, centered at the origin, so that:
\[ \hat{\alpha}_k = h_k^{(l)}(dz)^2 \quad \text{and} \quad \alpha = \varphi^{(l)}(dz)^2 \quad \text{in} \quad B_r, \]
where $h_k^{(l)}$ and $\varphi^{(l)}$ are holomorphic in $B_r$. Furthermore, $h_k^{(l)} \rightarrow h^{(l)}$ uniformly in $B_r(0)$ and $\hat{\alpha}_0 = h^{(l)}(dz)^2$. Since $\|\hat{\alpha}_0\|(q_l) > 0$, then (for $r > 0$ sufficiently small) we have that $\hat{\alpha}_0$ never vanishes in a neighbourhood of $q_l$, that is the holomorphic function: $h^{(l)}(z) \neq 0$ for all $z \in B_r$. Therefore, in view of (3.66), for $r > 0$ sufficiently small, we find, as $k \rightarrow +\infty$,
\[ \int_{B_r(q_l)} e^{\xi_k} (\ast_{E}^{-1} \hat{\alpha}_k, \ast_{E}^{-1} \alpha) dA \rightarrow \pi \frac{h^{(l)}(0)}{|h^{(l)}(0)|^2} \varphi^{(l)}(0) := H_l \varphi_l \in \mathbb{C}, \quad (3.70) \]
with $H_l \neq 0$ and $|\varphi_l| = \|\alpha\|(q_l)$, $l = 1, \ldots, m$.

By Remark 2.2, we know that there exists a quadratic holomorphic differential $\alpha \in C_{\kappa}(X)$, which vanishes at all but one point in $S$. For example, we may choose
\[ \alpha \in C_2(X) : \|\alpha\|(q_1) = 1 \quad \text{while} \quad \|\alpha\|(q) = 0, \quad \forall \ q \in S \setminus \{q_1\}, \]
and by this choice, according to the estimates above, we find:
\[ \int_X \beta_0 \wedge \alpha = e^{-u_k} (H_1 + o(1)), \quad \text{as} \quad k \rightarrow +\infty, \quad (3.71) \]
with $H_1 \neq 0$. Since we have already shown that $s_k$ is uniformly bounded from above, then (3.71) implies that,
\[ |\int_X \beta_0 \wedge \alpha| \geq L > 0, \]
and in turn we deduce that, $s_k \geq -c$ and (3.67) is established.

As a consequence of Claim (3.67), we have that, $C^{-1} \leq \|\alpha_k\|_{L^2} \leq C$ for suitable $C > 0$, and then (along a subsequence): $\alpha_k \rightarrow \alpha_0 \in C_2(X)$, as $k \rightarrow +\infty$, (in any relevant norm), and $\alpha_0 \neq 0$. So, $\alpha_0$ vanishes exactly at $Z$ and moreover, by (3.60), $\|\alpha_0\|(q_l) \neq 0$, $\forall \ l = 1, \ldots, m$. So we can reformulate (3.60) as follows:
\[ e^{-u_k} \rightarrow \pi \sum_{l=1}^{m} \left( \frac{1}{\|\alpha_0\|^2(q_l)} \right) \delta_{q_l}, \quad \text{as} \quad k \rightarrow +\infty, \ \text{weakly in the sense of measures}, \]
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and (3.61) and (3.62) are established. Moreover, for \( r > 0 \) sufficiently small, we can use the first identity in (3.69) and argue as above to obtain

\[
\lim_{k \to \infty} \left( \sum_{l=1}^{m} \hat{B}_{r} \left( q_{l} \right) e^{-u_{k}} \right) = \lim_{k \to \infty} \hat{X}_{\beta_{0} \wedge \alpha_{0}} = \lim_{k \to \infty} e^{-u_{k}} \| \alpha_{k} \|^{2} = \pi m.
\]

Since \( \int_{X} e^{-u_{k}} \| \alpha_{k} \|^{2} dA = \int_{X} e^{u_{k}} \| \beta_{0} + \tilde{\partial} \eta_{0} \|^{2} dA \), we obtain in particular, that \( \rho([\beta]) = 4\pi m \).

Similarly if \( \alpha \in C_{2}(X) \) vanishes at each point of \( S \), then from (3.69) and (3.70) we have:

\[
\hat{X}_{\beta_{0} \wedge \alpha_{0}} = 0 \quad \text{as claimed in (3.64)}.
\]

Finally, under the assumption (3.60), we can use well known \( \text{sup} + \text{inf} \) estimates and gradient estimates for \( \xi_{k} \) around \( q_{l} \in S \) (see [9]) and obtain (for \( r > 0 \) sufficiently small)

\[
\lambda_{k} = \max_{B_{r}(q_{l})} \xi_{k} = - \min_{\partial B_{r}(q_{l})} \xi_{k} + O(1), \quad \forall \ l = 1, \ldots, m
\]

\[
\int_{B_{r}(q_{l})} |\nabla w_{k}|^{2} = \int_{B_{r}(q_{l})} |\nabla \xi_{k}|^{2} = 16\pi \lambda_{k} + O(1).
\]

But, from (3.59) and (3.67), we see that \( \lambda_{k} = d_{k} + O(1) \), and we deduce that

\[
\int_{X} |\nabla w_{k}|^{2} dA = (16\pi m)d_{k} + O(1).
\]

Consequently,

\[
c_{k} = D_{t_{k}}(u_{k}, \eta_{k}) = \int_{X} \left( \frac{|\nabla w_{k}|^{2}}{4} - u_{k} + t_{k} e^{u_{k}} + 4e^{u_{k}} \| \beta_{0} + \tilde{\partial} \eta_{0} \|^{2} \right) dA
\]

\[
= \int_{X} \left( \frac{|\nabla w_{k}|^{2}}{4} - \pi(g - 1)d_{k} + O(1) = -4\pi(g - 1 - m)d_{k} + O(1)
\]

and also (3.63) is established.

**Remark 3.5.** Under the assumption (3.60), the estimate (3.63) allows us to give a positive answer to (3.34) posed above. Indeed if \( \rho([\beta]) = 4\pi(g - 1) \), then necessarily \( m = g - 1 \) and according to (3.63) we deduce that \( D_{0} \) is bounded from below in \( \Lambda \).

When (3.60) holds then Theorem [9] gives a reasonable description about the blow-up behaviour of minimizers for the Donaldson functional, as \( t \to 0^{+} \).
Although we cannot yet guarantee that (3.60) always holds, by the minimizing property of the sequence $(u_k, \eta_k)$ for $D_i$ we expect, that blow-up for $(u_k, \eta_k)$ should occur with the least possible blow-up mass $8\pi$. So, next, we shall focus to this situation and assume that:

$$\lim_{r \to 0^+} \lim_{k \to +\infty} 8 \int_{B_r(q)} e^{u_k} \| \beta_0 + \partial \eta_k \|^2 dA = 8\pi, \forall q \in S. \quad (3.72)$$

Indeed, (3.72) is always ensured under the assumption (3.60) or when $g = 2$, see (3.53). On the other hand, when $q \in S \cap Z$ and we assume (3.72), then (by Theorem D), necessarily $q \in Z_0$, that is blow-up must occurs at a point of "collapsing" zeroes of $\hat{\alpha}_k$. In other words, if (3.72) holds then $S \cap Z = S \cap Z_0$. Therefore,

**Corollary 3.1.** Assume (3.72) and that $S \cap Z_0 = \emptyset$. Then the conclusion of Theorem 9 holds.

To deal with the case where $S \cap Z_0 \neq \emptyset$, we introduce the following notation. For $q_l \in S$ and $r > 0$ sufficiently small, we let

$$x_{k,l} \in B_r(q_l) : \xi_k(x_{k,l}) = \max_{B_r(q_l)} \xi_k \longrightarrow +\infty \text{ and } x_{k,l} \longrightarrow q_l, \quad (3.73)$$

as $k \longrightarrow +\infty$, and set

$$\mu_{k,l} = \| \alpha_k \|^2(x_{k,l}), \quad (3.74)$$

for $l = 1, \ldots, m$. Since "locally" in holomorphic $z$-coordinates $\| \alpha_k \|^2(z)$ coincides (essentially) with the norm of a holomorphic function, we see that

$$\text{if } \| \hat{\alpha}_0 \|(q_l) \neq 0 \text{ then } \| \alpha_k \|^2(x_{k,l}) = O(e^{u_k}). \quad (3.75)$$

We have:

**Theorem 10.** Assume (3.72) and suppose that the blow-up set $S$ of $\xi_k$ in (3.58) satisfies: $S \cap Z_0 \neq \emptyset$. Then (along a subsequence):

$$s_k \longrightarrow +\infty, \text{ as } k \longrightarrow +\infty. \quad (3.76)$$

Moreover, there exists a set of indices $J \subseteq \{1, \ldots, m\}$ such that, as $k \longrightarrow +\infty$,

(i) $\forall l \in J : q_l \in S \cap Z_0$ and $\mu_{k,l} \longrightarrow \mu_l > 0$:

$$e^{-u_k} \rightharpoonup \pi \sum_{l \in J} \frac{1}{\mu_l} \delta_{q_l} \text{ weakly in the sense of measures}, \quad (3.76)$$

(ii) $\int X \beta_0 \wedge \alpha = 0, \forall \alpha \in C_2(X)$ vanishing at any point of $S_0 = \{q_l \in S : l \in J \} \subset Z_0$. In particular, $\int X \beta_0 \wedge \hat{\alpha}_0 = 0.$
\((iii) \; \mu_{k,l} \rightarrow +\infty, \; \text{as} \; k \rightarrow +\infty, \; \forall \; l \in \{1, \ldots, m\} \setminus J \) (if not empty)

\[
c_k = D_{t_k}(u_k, \eta_k) = -4\pi \left((g - 1 - m)d_k + \sum_{l \in \{1, \ldots, m\} \setminus J} \log(\mu_{k,l})\right) + O(1). \tag{3.77}
\]

with \(d_k = \int_X u_k \rightarrow +\infty\).

Proof. It is understood that the summation in (3.77) is dropped when \(J = \{1, \ldots, m\}\). As above, for given \(\alpha \in C_2(X)\) and \(r > 0\) small, from (3.79) we have:

\[
\int_X \beta_0 \wedge \alpha = \int_X e^{-u_t} \langle *_{E} \alpha_t, *_{E} \alpha \rangle dA = e^{-\frac{2}{s}} \left( \int_X e^{\xi_k} \langle *_{E} \alpha_t, *_{E} \alpha \rangle dA \right)
\]

\[
e^{-\frac{2}{s}} \left( \sum_{l=1}^{m} \int_{B_{r}(x_{k,l})} e^{\xi_k} \langle *_{E} \alpha_t, *_{E} \alpha \rangle dA + O(e^{-d_k + x}) \right)
\]

with \(x_{k,l}\) as given in (3.73). Since \(s_k \leq d_k + O(1)\), see (3.39), then \(-d_k + \frac{n}{2} \rightarrow -\infty\), and we may still conclude (as before) that,

\[
\int_X \beta_0 \wedge \alpha = e^{-\frac{2}{s}} \left( \sum_{l=1}^{m} \int_{B_{r}(x_{k,l})} e^{\xi_k} \langle *_{E} \alpha_t, *_{E} \alpha \rangle dA \right) + o(1) \tag{3.78}
\]

In case \(x_{k,l} \rightarrow q_l \in S \setminus Z_0\), as \(k \rightarrow +\infty\), then \(q_l \notin Z\) (as \(S \cap Z_0 = S \cap Z\)) and so,

\[
0 < a < \|\alpha_k\|(x_{k,l}) \leq b \; \text{in} \; B_{r}(x_{k,l})
\]

So we can argue exactly as in (3.70) to obtain:

\[
e^{-\frac{2}{s}} \int_{B_{r}(x_{k,l})} e^{\xi_k} \langle *_{E} \alpha_t, *_{E} \alpha \rangle dA = \frac{1}{\|\alpha_k\|(x_{k,l})} (H_{1} \bar{\varphi}_{l} + o(1)), \tag{3.79}
\]

as \(k \rightarrow +\infty\), for suitable \(H_l, \varphi_l \in \mathbb{C}\), with \(H_l \neq 0\) and \(|\varphi_l| = \|\alpha\|(q_l)\).

Hence suppose that, \(x_{k,l} \rightarrow q_l \in S \cap Z_0\), as \(k \rightarrow +\infty\). In this situation we are going to apply the blow-up analysis developed in [34]. For this purpose it is convenient to introduce local holomorphic \(z\)-coordinates around \(x_{k,l}\) centered at the origin. So, for \(r > 0\) sufficiently small, we can always assume that,

\[
\hat{\alpha}_k = (\prod_{j=1}^{s} (z - p_{j,k})^{n_j}) \psi_k(z)(dz)^2 \; \text{in} \; B_r, \; s \geq 2, \; n_j \in \mathbb{N};
\]

\[
0 \leq |p_{1,k}| \leq |p_{2,k}| \leq \ldots \leq |p_{s,k}| \rightarrow 0, \; \text{as} \; k \rightarrow \infty, \; p_{i,k} \neq p_{j,k}, \; i \neq j; \tag{3.80}
\]

\[
\tilde{\alpha}_0(z) = z^n \psi(z)(dz)^2 \; \text{in} \; B_r, \; n = \sum_{j=1}^{s} n_j;
\]

\(\psi_k\) and \(\psi\) holomorphic and never vanishing in \(B_r\),

\[
\psi_k \rightarrow \psi \; \text{uniformly in} \; B_r, \; \text{as} \; k \rightarrow \infty; \tag{3.81}
\]

\(\alpha = \varphi(dz)^2\) in \(B_r\) and \(\varphi\) is holomorphic in \(B_r\).
Thus, if (by abusing notation) we identify a function with its local expression in local $z$-coordinates, we see that $\xi_k = \xi_k(z)$ satisfies:

\[
\begin{cases}
-\Delta \xi_k(z) = (\Pi_{j=1}^s |z - p_{j,k}|^{2n_j})h_k(z)e^{\xi_k(z)} + g_k(z) \quad \text{in } B_r \\
\max_{\partial B_r(0)} \xi_k - \min_{\partial B_r} \xi_k \leq C \\
\xi_k(0) = \max_{B_r(0)} \xi_k \rightarrow \infty, \quad \text{as} \quad k \rightarrow \infty,
\end{cases}
\]

where

\[
h_k(z) = 8|\psi_k|^2(z)\lambda^{-1}(z) \rightarrow h(z) := 8|\psi|^2(z)\lambda^{-1}(z) \quad \text{uniformly in } B_r;
\]

\[
g_k(z) = -f_k(z)\lambda(z) \quad \text{is convergent in } L^p(B_r), \quad p > 1,
\]

with $\lambda(z) = e^{2u_0(z)} > 0$ the conformal factor of the metric $g_X$ in the $z$-coordinates, taken with the normalization $\lambda(0) = 1$.

We recall that (3.83) is by now a well known consequence of the Green representation formula for the function $\xi_k$ on $X$ (see [4]). Also it follows from (3.32) that,

\[
\int_{B_r} W_ke^{\xi_k} \leq C, \quad W_k(z) := (\Pi_{j=1}^s |z - p_{j,k}|^{2n_j})h_k(z).
\]

By taking $r > 0$ smaller if necessary, we can assume in addition that the origin it the only blow-up point for $\xi_k$ in $B_r$, namely

\[
\forall \, \delta \in (0, r) \, \exists \, C_\delta > 0 : \max_{B_{\delta}} \xi_k \leq C_\delta.
\]

Finally, around a point $q \in S \cap Z_0$, the assumption (3.72) can be stated as follows:

\[
m := \lim_{\delta \searrow 0} \lim_{k \rightarrow +\infty} \frac{1}{2\pi} \int_{B_\delta} W_ke^{\xi_k} = 4.
\]

At this point the blow-up analysis developed in [34] can be applied and we deduce the following:

**Theorem E** ([34]). Let $\xi_k$ be a solution of (3.82) and assume (3.80) - (3.81) and (3.83) - (3.84). If (3.83) holds then $p_{j,k} \neq 0$, $j = 1, \ldots, s$ and there exists $s_1 \in \{2, \ldots, s\}$ such that (along a subsequence)

\[
\begin{align*}
\lim_{k \rightarrow +\infty} \frac{p_{j,k}}{|p_{s_1,k}|} &= z_j \neq 0, \quad \forall \, j = 1, \ldots, s_1, \\
\lim_{k \rightarrow +\infty} \frac{p_{j,k}}{|p_{s_1,k}|} &= q_j \neq 0 \quad \text{and} \quad \frac{|p_{j,k}|}{|p_{s_1,k}|} \rightarrow +\infty, \quad \forall \, j = s_1 + 1, \ldots, s.
\end{align*}
\]
Moreover,

\[
\xi_k(0) + 2 \ln |p_{s_1,k}| + \ln(W_k(0)) \longrightarrow +\infty, \quad \text{as} \quad k \longrightarrow +\infty,
\]

\[
\xi_k(0) = -\left( \min_{\partial B} \xi_k + 2 \ln(W_k(0)) \right) + O(1) \quad (3.86)
\]

\[
\xi_k(x) = \ln \left( \frac{e^{\xi_k(0)}}{1 + \frac{W_k(0)}{s} e^{\xi_k(0)} |x|^2} \right) + O(1) \quad (3.87)
\]

\[
\int_{B_r} |\nabla \xi_k|^2 dx = 16\pi \left( \xi_k(0) + \ln(W_k(0)) \right) + O(1). \quad (3.88)
\]

**Proof.** See Theorem 5 in [34]. \qed

**Remark 3.6.** Incidentally, for the original sequence \( u_k \) (see (3.55)), from Theorem E we obtain that,

\[
- u_k(z) = \ln \frac{e^{-u_k(0)}}{1 + \|\alpha_k\|^2(x_k,l) e^{-u_k(0)}|z|^2} + O(1) \quad \text{in} \quad B_r. \quad (3.89)
\]

In particular, such an estimate allows us to compare the blow-up rates at different blow-up points as follows:

\[
\frac{1}{C} \leq \frac{e^{-u_k(x_k,h)}\|\alpha_k\|^2(x_k,h)}{e^{-u_k(x_k,l)}\|\alpha_k\|^2(x_k,l)} \leq C \quad \text{for all} \quad h, l = 1, \ldots, m
\]

for suitable \( C \geq 1 \).

At this point we may continue the proof of Theorem [10] and we can take advantage of the information about \( \xi_k \) provided by Theorem E to estimate the integral:

\[
\int_{B_r} e^{\xi_k}(\ast_E^{-1} \hat{\alpha}_k, \ast_E^{-1} \alpha) dA = \int_{B_r} e^{\xi_k(z)} \left( \Pi_{j=1}^{s} (z - p_{j,k})^{n_j} \right) \psi_k(z) \tilde{\varphi}(z) \lambda^{-1}(z) dz d\bar{z}.
\]

To this purpose we see that, as \( k \longrightarrow \infty \),

\[
\varepsilon_k := |p_{s_1,k}| \longrightarrow 0, \quad \lambda_k^2 := e^{\xi_k(0)} + 2 \ln \varepsilon_k + \ln(W_k(0)) \longrightarrow \infty,
\]

\[
\varepsilon_{j,k} := \frac{\varepsilon_k}{|p_{j,k}|} \longrightarrow L_j \geq 0 \quad \text{for all} \quad j = 1, \ldots, s_1. \quad (3.90)
\]

Hence \( t_k^2 = \frac{W_k(0)}{8} e^{\xi_k(0)} = \frac{1}{8} \left( \frac{\Lambda_k}{\varepsilon_k} \right)^2 \longrightarrow \infty, \quad \text{as} \quad k \longrightarrow \infty, \quad \text{with}
\]

\[
\frac{W_k(0)}{8} = (\Pi_{j=1}^{s} |p_{j,k}|^{2n_j}) |\psi_k(0)|^2 = ||\hat{\alpha}_k||^2(x_k,l) = e^{-s_k ||\alpha_k||^2(x_k,l)} > 0. \quad (3.91)
\]
Therefore, for \( R > 0 \) large, we have

\[
\left| \int_{B_r(z)} \left( \Pi_{j=1}^s (z - p_{j,k})^{n_j} \psi_k(z) \bar{\varphi}(z) \lambda^{-1}(z) \frac{i}{2} dz \right) \right|
\leq C \left( \int_{B_r(z)} e^{\xi_k(z)} \right)^{1/2} \left( \int_{B_r(z)} e^{\xi_k(z)} W_k(z) \right)^{1/2}
\leq C \left( \int_{B_r(z)} e^{\xi_k(z)} \right)^{1/2} \leq \frac{C}{(W_k(0))^{1/2}} \leq \frac{1}{\|\alpha_k\|(x_{k,l})} \frac{C}{R}.
\]

While, for \( \hat{p}_{j,k} := \frac{p_{j,k}}{|p_{j,k}|} \rightarrow \hat{p}_j \) with \( |\hat{p}_j| = 1, j = 1, \ldots, s \); in view of \( 3.90 \)-\( 3.91 \) we have:

\[
\int_{\{ |w| \leq R \}} \frac{e^{\xi_k(z)}(\Pi_{j=1}^s (z - p_{j,k})^{n_j})}{(1 + |w|^2)^2} \psi_k(z) \bar{\varphi}(z) \lambda^{-1}(z) \frac{i}{2} dw \wedge d\bar{w} + o(1)
\]

\[
= \frac{8}{W_k(0)} \int_{\{ |w| \leq R \}} \frac{(\Pi_{j=1}^s (\xi_{j,k} w - \hat{p}_{j,k})^{n_j})}{|\psi_k(0)| (1 + |w|^2)^2} \frac{i}{2} dw \wedge d\bar{w} + o(1)
\]

\[
= \frac{1}{\|\alpha_k\|(x_{k,l})} \left[ \int_{\{ |w| \leq R \}} \frac{\pi \Pi_{j=1}^s (-\hat{p}_{j,k})^{n_j} \psi_k(0) |\varphi(0)|}{|\varphi(0)|} \bar{\varphi}(0) + O\left( \frac{1}{R^2} \right) + o(1) \right],
\]

where \( o(1) \rightarrow 0 \) uniformly in \( R \), as \( k \rightarrow +\infty \). So, by letting \( R \rightarrow +\infty \), and combining the above estimates, we conclude that the analogous of \( 3.79 \) holds also when \( q_l \in S \cap Z_0 \). In other words, in terms of \( u_k \), we have established that

\[
\int_{B_r(x_{k,l})} e^{-u_k} e^{\xi_k} \frac{1}{|E^{-1} \alpha_k|} \frac{1}{|E^{-1} \alpha_k|} dA = e^{-\frac{1}{2}} \int_{B_r(x_{k,l})} e^{\xi_k} (E^{-1} \alpha_k) \alpha_k dA
\]

\[
= \frac{1}{\|\alpha_k\|(x_{k,l})} (H_l \varphi_l + o(1)), \quad \forall \ l = 1, \ldots, m
\]

for suitable \( H_l, \varphi_l \in C, H_l \neq 0 \) and \( |\varphi_l| = \|\alpha\|(q_l) \). Consequently, from \( 3.78 \) we conclude that

\[
\int_X \beta_0 \wedge \alpha = \sum_{l=1}^m \frac{1}{\|\alpha_k\|(x_{k,l})} (H_l \varphi_l + o(1)) + o(1), \quad \text{as} \ k \rightarrow +\infty. \quad (3.92)
\]

In particular, as above, by taking \( q_l \in S \cap Z_0 \) (so \( \alpha_0(q_l) = 0 \)) and a holomorphic quadratic differential \( \alpha \in C_2(X) \) such that \( \|\alpha\|(q_l) \neq 0 \), but \( \|\alpha\|(q_l) = 0 \) for all \( q \in S \setminus \{ q_l \} \), from \( 3.78 \) we find:

\[
\int_X \beta_0 \wedge \alpha = \frac{1}{\|\alpha_k\|(x_{k,l})} (H_l \varphi_l + o(1)) + o(1) \text{ with } |H_l \varphi_l| = |H_l\|\alpha\|(q_l) > 0.
\]

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As a consequence, we deduce that \( \| \hat{\alpha}_k \| (x_{k,l}) \leq C \), and since
\[
\| \hat{\alpha}_k \| (x_{k,l}) \rightarrow \| \hat{\alpha}_0 \| (q) = 0, \quad \text{as } k \rightarrow \infty,
\]
we find that,
\[
e^{-s_k} \leq C \| \hat{\alpha}_k \| (x_{k,l}) \rightarrow 0, \text{ i.e. } s_k \rightarrow \infty, \quad \text{as } k \rightarrow \infty,
\]
as claimed.

**Claim:**
\[
\min \{ \| \alpha_k \| (x_{k,l}) : l = 1, \ldots, m \} \leq C \tag{3.93}
\]
Argue by contradiction, and assume that, (along a subsequence)
\[
\| \alpha_k \| (x_{k,l}) \rightarrow +\infty, \quad \text{as } k \rightarrow +\infty, \text{ for every } l = 1, \ldots, m.
\]
In case \( q_l \in S \cap Z_0 \) then,
\[
\int_{B_r(x_{k,l})} e^{-u_k} = e^{-s_k} \int_{B_r(x_{k,l})} e^{\xi_k} \\
= \frac{1}{\| \alpha_k \|^2 (x_{k,l})} \int_{\{ |z| \leq \frac{1}{s_k} \}} \frac{1}{(1 + |z|^2)^2} dz d\overline{z} \tag{3.94}
\]
\[
\leq \frac{C}{\| \alpha_k \|^2 (x_{k,l})} \rightarrow 0, \quad \text{as } k \rightarrow +\infty.
\]
On the other hand, in case \( q_l \in S \setminus Z_0 \) \( \neq 0 \) then we know that, \( q_l \notin Z \) and so \( \| \hat{\alpha}_k \| \) is bounded below away from zero in a small neighbourhood of \( q_l \). As a consequence, for \( r > 0 \) sufficiently small, we find \( \int_{B_r(x_{k,l})} e^{\xi_k} \leq C \) and we obtain:
\[
\int_{B_r(x_{k,l})} e^{-u_k} = e^{-s_k} \int_{B_r(x_{k,l})} e^{\xi_k} \leq Ce^{-s_k} \rightarrow 0, \quad \text{as } k \rightarrow \infty.
\]
Since we know also that,
\[
e^{-u_k} = e^{\xi_k - s_k} \leq Ce^{-s_k} \quad \text{in } X \setminus \bigcup_{l=1}^m B_r(x_{k,l}), \tag{3.95}
\]
then we may conclude that, \( \int_X e^{-u_k} \rightarrow 0, \text{ as } k \rightarrow +\infty \) in contradiction to part (v) of Lemma 3.7, and (3.93) is established.

Since for \( q_l \in S \setminus Z_0 \) (i.e. \( \| \hat{\alpha}_0 \| (q_l) > 0 \)) we have: \( \| \alpha_k \|^2 (x_{k,l}) = O(e^{s_k}) \rightarrow +\infty \), then the condition \( \| \alpha_k \| (x_{k,l}) \leq C \) implies that necessarily, \( q_l \in S \cap Z_0 \).

Without loss of generality and after relabelling (along a subsequence), we can assume that
\[
q_1 \in S \cap Z_0 \text{ and } \mu_{k,1} = \| \alpha_k \|^2 (x_{k,1}) = \min_{l=1, \ldots, m} \| \alpha_k \|^2 (x_{k,l}) \leq C. \tag{3.96}
\]
As above, we can use such information in (3.92) with \( \alpha \in C_2(X) \) vanishing in \( S \setminus \{q_1\} \), but not in \( q_1 \), to obtain:

\[
\int_X \beta_0 \wedge \alpha = \frac{1}{\sqrt{\mu_{k,1}}} (H_1 \tilde{\varphi}_1 + o(1)), \quad \text{as} \ k \to +\infty,
\]

with \( H_1 \neq 0 \) and \( |\varphi_1| = \|\alpha\|(q_1) > 0 \). So, by (3.96), we derive first that, \( \int_X \beta_0 \wedge \alpha \neq 0 \), and then we obtain also that, \( \mu_{k,1} \) us uniformly bounded below away from zero. In fact, more generally, we conclude:

\[
\mu_{k,l} = \|\alpha_k\|^2(x_{k,l}) \geq c > 0, \quad \forall \ k \in \mathbb{N} \quad \text{and} \quad \forall \ l \in \{1, \ldots, m\}.
\]

Next, we define the set of indices:

\[
J = \{ l \in \{1, \ldots, m\} : \lim_{k \to +\infty} \sup \mu_{k,l} < \infty \},
\]

so that \( 1 \in J \). Furthermore, if \( l \in J \) then necessarily \( q_l \in S \cap Z_0 \) and (along a subsequence) we can summarize the properties of the elements of \( J \) as follows:

\[
\mu_{k,l} \to \mu_l > 0 \quad \text{and} \quad q_l \in S \cap Z_0, \quad \forall \ l \in J \neq \emptyset.
\]

On the contrary,

if \( \{1, \ldots, m\} \setminus J \neq \emptyset \), then \( \mu_{k,l} \to +\infty \) for \( l \in \{1, \ldots, m\} \setminus J \).

Therefore, (3.92) can be expressed as follows:

\[
\int_X \beta_0 \wedge \alpha = \left( \sum_{l \in J} \frac{1}{\|\alpha_k\|(x_{k,l})} H_l \tilde{\varphi}_l \right) + o(1), \quad \text{as} \ k \to +\infty,
\]

and we easily derive that,

if \( \alpha \in C_2(X) \) and \( \|\alpha\|(q_l) = 0, \forall \ l \in J \) then \( \int_X \beta_0 \wedge \alpha = 0 \),

and also (ii) is established.

At this point we can combine the estimate (3.95) together with the integral estimate (3.94) over the ball \( B_r(x_{k,l}) \) when \( l \in \{1, \ldots, m\} \setminus J \), and the estimate (3.89) around \( x_{k,l} \) when \( l \in J \), to deduce that, \( e^{-u_k} \to \pi \sum_{j \in J} \frac{1}{\mu_1} \delta_{q_l} \), weakly in the sense of measure, as claimed in (i).

Finally, for \( r > 0 \) sufficiently small, we have:

\[
\int_X |\nabla w_k|^2 = \int_X |\nabla \xi_k|^2 = \sum_{l=1}^m \int_{B_r(q_l)} |\nabla \xi_k|^2 + O(1),
\]

and by means of the gradient estimate (3.88) in Theorem 3, together with (3.59), (3.86) and (3.91) we find:

\[
\int_{B_r(x_{k,l})} |\nabla \xi_k|^2 = 16\pi (d_k - \log \|\alpha_k\|^2(x_{k,l})) + O(1).
\]
On the other hand, for $l \in J$ we know that $\|\alpha_k\|^2(x_{k,l}) = O(1)$, and we conclude:

$$c_k = D_{t_k}(u_k, \eta_k) = \frac{1}{4} \int_X |\nabla w_k|^2 - 4\pi(g - 1) d_k + O(1)$$

$$= -4\pi \left((g - 1 - m) d_k + \sum_{l=1}^m \log(\|\alpha_k\|^2(x_{k,l}))\right) + O(1),$$

and also (3.77) is established with $d_k \to \infty$, as $k \to \infty$.

It is reasonable to expect that the set of indices $J$ in Theorem 10 either is a "singleton" or it covers the full set of indices. In either cases we get a "cleaner" version of Theorem 10 as already expressed by Theorem 4 of the introduction, in case $S$ is a "singleton". On the other hand we have:

**Corollary 3.2.** Let $S$ be the (non-empty) blow-up set of $\xi_k$ in (3.58) and assume $J = \{1, \ldots, m\}$ in Theorem 10. Then $S \subset Z_0$ and there holds:

(i) $\mu_{k,l} \xrightarrow{k \to \infty} \mu_l > 0$, $\forall l = 1, \ldots, m$;

(ii) $e^{-u_k} \to \pi \sum_{l=1}^m \frac{1}{\mu_l} \delta_{q_l}$, as $k \to +\infty$, weakly in the sense of measures;

(iii) $\int_X \beta_0 \wedge \alpha = 0$, $\alpha \in Q_2[q_1, \ldots, q_m]$;

(iv) $c_k = -4\pi(g - 1 - m) d_k + O(1)$, with $d_k = \int_X u_k \to +\infty$, as $k \to +\infty$.

Thus, under the assumptions of Corollary 3.2 we can still conclude, as before, that, when $\rho(\beta) = 4\pi(g - 1)$ (i.e. $m = g - 1$) then $D_0$ is bounded from below.

At this point, Theorem 4, Theorem 5 and Theorem 6 can be easily derived.

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