BLOWING UP DETERMINANTAL SPACE CURVES
AND VARIETIES OF CODIM 2 (*)

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Introduction.

Let \( k \) be an algebraically closed field with \( \text{char} \, k = 0 \) and let \( C \subseteq \mathbb{P}^3 = \mathbb{P}^3_k \) be a curve (i.e. a 1-dimensional, smooth, irreducible scheme).

We want to study the scheme \( X_C \) which is the blow-up of \( \mathbb{P}^3 \) along \( C \). Let \( E \) be the exceptional divisor of the blow-up, and \( H \) the strict transform of a generic hyperplane, then \( \text{Pic} \, X_C = \mathbb{Z} < H, E > \), and the kind of questions we would like to address are:

1) For which \( p \) is the linear system \( |pH - E| \) very ample on \( X_C \)?

2) When is \( |pH - E| \) normally generated?

3) Let \( \mathcal{O}_{X_C}(pH - E) \) be generated by its global sections (g.b.g.s. for short), then it defines a morphism \( \phi_p : X_C \to \mathbb{P}^N \), where \( N + 1 = \dim \mathcal{H}^0(\mathcal{O}_{X_C}(pH - E)) \). What can we say about the ideal of \( Y_{C,p} = \phi_p(X_C) \)? More specifically, we try to get information about the generators and the resolution of its ideal from the data known about \( C \).

Note that by \( |pH - E| \) normally generated we mean that \( \mathcal{O}_{X_C}(pH - E) \) is very ample and \( Y_{C,p} \) is arithmetically Cohen-Macaulay (a.C.M. for short), i.e. its coordinate ring is Cohen-Macaulay.

We will restrict ourselves to the case when \( C \) is projectively normal. Let \( I_C \subseteq \mathcal{O}_R = k[w_0, w_1, w_2, w_3] \) be its homogeneous ideal; by the Hilbert-Burch Theorem we know that \( I_C \) is generated by the maximal minors of a \( \rho \times (\rho + 1) \) matrix:

\[
M = \begin{pmatrix}
M_{11} & M_{12} & \cdots & M_{1,\rho+1} \\
M_{21} & M_{22} & \cdots & M_{2,\rho+1} \\
\vdots & \vdots & \ddots & \vdots \\
M_{\rho1} & M_{\rho2} & \cdots & M_{\rho,\rho+1}
\end{pmatrix}
\]

where the \( M_{ij} \)'s are forms in \( R \) and \( \deg M_{ij} = \deg M_{i1} + \deg M_{1j} - \deg M_{11} \). The degree matrix, \( \partial M \), is determined by the degrees of a minimal set of generators of \( I_C \) or, equivalently in this case, by the graded Betti numbers of a minimal resolution of \( I_C \).

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A resolution of the ideal sheaf $I_C$ is of the type:

$$0 \to \bigoplus_{j=1}^n \mathcal{O}_{P^3}(-n_j) \to \bigoplus_{i=1}^r \mathcal{O}_{P^3}(-d_i) \to I_C \to 0.$$ 

A first answer to question 1) above is given by [Co, Theorem 1]: let $V \subseteq P^n$ be a smooth irreducible projective scheme and $\lambda$ such that $V$ is scheme-theoretically generated in degrees $\leq \lambda$, then (with obvious notation) $|pH - E|$ is very ample on the blow up $X_V$ of $P^n$ along $V$ for all $p \geq \lambda + 1$.

In particular, let $V$ be a.C.M. and consider the invariant

$$\sigma = \sigma(V) = \min\{ t| \Delta^{n-1} H(V,t) = 0 \},$$

where $H(V,t)$ is the Hilbert function of $V$, $\Delta^i H(V,t) = \Delta^{i-1} H(V,t) - \Delta^{i-1} H(V,t-1)$ and $\Delta^0 H(V,t) = H(V,t)$.

It is well known that the homogeneous ideal of $V$, $I_V$, is generated in degrees $\leq \sigma$ (see e.g. [L.1, Theorem 2]), hence $|pH - E|$ is always very ample for $p \geq \sigma + 1$.

Several examples and known facts lead to the following conjecture:

**Conjecture:** Let $C \subseteq P^3$ be a projectively normal curve. Then $|\sigma H - E|$ is very ample on $X_C$ if and only if $C$ has no $\sigma$-secant lines. Moreover, if this is the case, $|\sigma H - E|$ is normally generated.

Note that $\mathcal{O}_{X_C}(\sigma H - E)$ is generated by its global section since $I_C$ is generated in degrees $\leq \sigma$.

The bound $p \geq \sigma$ is sharp with no other specification on $C$, since $|\sigma H - E|$ is not very ample when $C$ has a $\sigma$-secant $L$ (the divisors in $|\sigma H - E|$ will not intersect the strict transform of $L$). Examples of curves $C$ possessing $\sigma$-secants are well known:

- rational normal cubics curves ($\sigma = 2$), the most trivial example;
- projectively normal sextic curves of genus 3 (here $\sigma = 3$ and they have infinitely many 3-secants, see e.g. [R]);
- quintic curves of genus two of type (3,2) on a quadric surface ($\sigma = 3$).

The whole problem can be generalized in a very natural way to the blow-up, $X_V$, of $P^n$ along a codimension 2 (smooth, irreducible) variety $V \subseteq P^n$, $n \geq 3$. If we require that $V$ is a.C.M. we still have that the ideal $I_V$ of $V$ is generated in degrees $\leq \sigma$ and, by Hilbert-Burch theorem, determinantal.

Notice that the analog of the Conjecture above for points in $P^2$ is true (see [D-G]), but we will not consider this case here, since it has been studied by many authors (e.g. see the surveys [Gi.2], [Gi.3]).

Note also that, when $\rho \geq 2$, the locus $\Sigma$ where the rank of the matrix $M$ drops twice is singular for $V$ and its codimension is $\leq 6$, hence, for $n \geq 6$, $\Sigma$ is non-empty and $V$ has to be singular.

Thus, for $n \geq 6$, only $\rho = 1$ is allowed in order to have $V$ smooth, i.e. $V$ has to be a complete intersection, so its ideal is quite simple and if it is generated by two forms of degrees $d_1, d_2$, say with $d_1 \geq d_2$, we have that $|pH - E|$ is very ample if $p \geq d_2 + 1$ (e.g. by [Co]). We will not treat that case here, see e.g. [S-T-V] and [G-G-H] for the case $n = 2$.

The plan of the paper is the following:

§1: In the general case, we show that $|\sigma H - E|$ is very ample on $X_V - E$ (i.e. $(I_V)_\sigma$ gives a very ample linear system on $P^n - V$).
§2: We prove our Conjecture when the entries of the matrix $M$ are linear forms. In this case a quite complete description of the ideal of $Y_{C,\sigma}$ (which turns out to be determinantal itself) and of its resolution can be given.

Furthermore, we extend our result to the case of codim 2 subschemes of $\mathbb{P}^n$, $n = 4, 5$, which do not contain lines and we give numerical conditions that describe when the generic determinantal scheme of the type under consideration does contain lines or possess $\sigma$-secants.

§3: For a given degree $d \geq 3$, we will consider projectively normal curves $C$ of degree $d$ with a "generic" resolution (these curves have minimal genus for that degree).

In this case we are able to find the ideal generation of $Y_{C,\sigma}$.

§4: We give some examples of the construction seen above in the case of Fano varieties of dimensions 3, 4, 5.

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1. Very ampleness on $\mathbb{P}^n - V$.

Let $V \subseteq \mathbb{P}^n$ be a smooth, irreducible, a.C.M. scheme of codimension 2 and let $\sigma$, $X_V$ be as in the introduction. The first step in studying the very ampleness of $|\sigma H - E|$ is to show that $\phi_{\sigma}$ is generically 1:1; more precisely we will show:

**Proposition 1.1:** If there are no $\sigma$-secant lines to $V$, the linear system $|\sigma H - E|$ separates points and tangent vectors on $X_V - E$.

Note that this is equivalent to saying that the sections of $H^0(\mathbb{P}^n, \mathcal{I}_V(\sigma)) \cong (I_V)_\sigma$ separate points and tangent vectors on $\mathbb{P}^n - V$.

**Proof:** What we have to show is that given any 0-dimensional scheme $T \subseteq \mathbb{P}^n - V$ of degree 2, we have that the restriction map:

\[ H^0(\mathbb{P}^n, \mathcal{I}_V(\sigma)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_T(\sigma)) \]

is surjective.

Let $L$ be the line defined by $T$; we will be done if we show that the restriction of $I_{\sigma} = (I_V)_\sigma$ to $L$ defines a very ample linear system on $L - (V \cap L)$ if and only if $L$ is not a $\sigma$-secant for $V$.

Let $I_L$ be the homogeneous ideal of $L$ and consider the exact sequence:

\[ 0 \rightarrow (I_V \cap I_L)_\sigma \rightarrow (I_V)_\sigma \rightarrow (I_{V \cap L, L})_\sigma \alpha \]

Since $V \cap L$ is given by $k$ points (not necessarily distinct) on $L$, and $0 \leq k \leq \sigma - 1$, we have $\dim(I_{V \cap L, L})_\sigma = \sigma - k + 1 \geq 2$, hence it defines, out of $V \cap L$, a $g^2_{\sigma - k}$ on $L \cong \mathbb{P}^1$, which is very ample on $L - (V \cap L)$ because it does not have base points (out of $V \cap L$) since $I_{V \cap L, L}$ is generated in degree $k < \sigma$.

So we will be done if $\alpha$ is surjective.

**Fact:** Let $\Pi$ be a generic plane containing $L$ and let $Z = V \cap \Pi$, then $Z$ is a 0-dimensional scheme of degree $d = \deg V$ in $\Pi \cong \mathbb{P}^2$ and the homogeneous ideal of $Z$ in $\Pi$ is determinantal (with the same degree matrix as $V$).
In order to check that such II exists one has to show that there are \( n - 2 \) hyperplanes \( H_1, ..., H_{n-2} \) such that \( V_i = V \cap H_1 \cap ... \cap H_i \) has dimension \( n - 2 - i, \ i = 1, ..., n - 2 \) (then we will choose \( \Pi = H_1 \cap ... \cap H_{n-2} \)).

Consider the linear system \( W \subseteq H^0(V, O_V(1)) \) which is cut on \( V \) by the hyperplanes through \( L_i \); since \( \dim(L \cap V) = 0 \), \( W \) has no fixed components of dimension \( \geq 1 \). Thus, if we start with any \( H_1 \) through \( L \), and \( D_1 = V \cap H_1 \in W \), not all \( D_2 \in W \) will contain a component of \( D_1 \), hence \( D_2 \in W \) can be chosen such that \( \dim(D_1 \cap D_2) = n - 4 \) (\( D_1 \) and \( D_1 \cap D_2 \) are of pure dimension, since \( V \) is a.C.M. and we cut each time with a non-zero divisor).

Iterating this procedure, at each cut the dimension drops by \( 1 \) until \( D_1 \cap ... \cap D_{n-2} \) has dimension zero, as required.

We have another exact sequence:

\[
0 \to (I_Z \cap I_L)_\sigma \to (I_Z)_\sigma \to (I_{Z \cap L,L})_\sigma \to 0
\]

and we will be done if \( \alpha' \) is surjective, i.e. if

\[
\dim(I_Z \cap I_L)_\sigma = \dim(I_Z)_\sigma - \dim(I_{Z \cap L,L})_\sigma = \left(\frac{\sigma + 2}{2}\right) - d - (\sigma + 1 - k) = \left(\frac{\sigma + 1}{2}\right) - d + k.
\]

Let \( Z' = Z - (Z \cap L) \) as schemes (i.e. \( Z' \) is the subscheme of \( \Pi \) defined by \( I_Z : I_L \)); note that

\[
\dim(I_Z \cap I_L)_\sigma = \dim(I_{Z'})_{\sigma - 1},
\]

since all the forms in \( (I_Z \cap I_L)_\sigma \) are the product of a linear form (defining \( L \) on \( \Pi \)) and a form of degree \( \sigma - 1 \) containing \( Z' \). Thus all we need to show is that:

\[
\dim(I_{Z'})_{\sigma - 1} = \left(\frac{\sigma + 1}{2}\right) - d + k
\]

i.e. that \( Z' \) imposes independent conditions to the curves of degree \( \sigma - 1 \) (in fact \( \deg Z' = d - k \)), but this is trivial, since \( Z' \subseteq Z \), and \( Z \) already does (it follows from the definition of \( \sigma \)).

\[\square\]

Remark 1.2:

Note that the above proof works on every line \( L \subseteq \mathbb{P}^n \), \( L \not\subseteq V \), hence it also shows that \( |\sigma H - E| \) separates points and tangent vectors lying on the strict transform of such a line.

Now let \( \pi : X_V \to \mathbb{P}^n \) be the blowing up, \( P \in V \), and \( \pi^{-1}(P) = F \subseteq E \) an \( (n - 2) \)-dimensional linear space in the ruling of \( E \).

Proposition 1.3: The linear system \( |\sigma H - E| \) is very ample on \( F \).

Proof: This is quite an elementary fact, see e.g. [Co, §1]: we have to show that points \( P \in F \) and tangent vectors \( t \in T_P(F) \) can be separated by divisors \( D \in |\sigma H - E| \). The passage of \( D \) through \( P \) and its tangency to \( t \) are equivalent to the fact that \( \pi(D) \) has certain tangent vectors at \( \pi(P) \), and, by [Co, Lemma 1.5], for every hyperplane \( \Lambda \in T_{\pi(P)}(\mathbb{P}^n) \), one can find \( D \in |\sigma H - E| \) such that \( T_{\pi(P)}(\pi(D)) = \Lambda \), thus one can separate tangent vectors at \( \pi(P) \).

\[\square\]

Remark 1.4:
Note that the above proof works also to show that $\forall P \in E$, vectors $t \in T_P(X_V)$ can be separated by divisors $D \in |\sigma H - E|$.

2. Varieties defined by matrices of linear forms.

Let $V$ be as in §1 and let its defining matrix $M$ be such that all its entries are linear forms. In this case we have $\rho = \sigma$ and it is quite easy to check that:

**Proposition 2.1:** The variety $Y_{V,\sigma} = \phi_{\sigma}(X_V)$ is a.C.M.

**Sketch of proof:** Let $M = (L_{ij})$, where $L_{ij} = \sum_{k=0}^{n} \delta_{ij}^k w_k$. The homogeneous ideal of $Y_{V,\sigma}$, $I_Y \subseteq k[x_1, \ldots, x_{\sigma+1}]$, is generated by the maximal minors of a $(\sigma \times \sigma)$-matrix $N$ of linear forms in the $x_i$'s, and since $\text{ht} I = \text{cod} Y_{V,\sigma} = \sigma - n$, $I_Y$ is perfect and the matrix $N$ has entries $N_{ik} = \sum_{j=1}^{\sigma+1} \delta_{ij}^k x_j$ (e.g. see [E-N], [R], [Gi.1]).

Now it is quite immediate to give a resolution of the ideal $I_Y$:

**Corollary 2.2:** A minimal resolution of $I_Y$ is given by:

$$
0 \to \oplus_{(\sigma)} S(-\sigma) \to \cdots \to \oplus_{(\sigma)} S(-n-i) \to \cdots \to \oplus_{(\sigma)} S(-(n+1)) \to I_Y \to 0.
$$

where $i = 1, \ldots, \sigma - n$ and $S = k[x_0, \ldots, x_{\sigma}]$ is the coordinate ring of $\mathbb{P}^\sigma$.

**Proof:** From the proof of 2.1, we have that $I_Y$ is generated by the maximal minors of the matrix $N$, so the resolution above is given by an Eagon-Northcott complex (see [E-N]).

Now let us go back to the case of curves in $\mathbb{P}^3$: in particular we will consider the case when the curve $C$ is defined by a matrix $M$ of linear forms; in this case we are able to prove the Conjecture, namely we have the following:

**Theorem 2.3:** Let $C$ be a projectively normal space curve and let $I_C$ be generated by the maximal minors of a matrix of linear forms. Then $|\sigma H - E|$ is very ample on $X_C$ if and only if $C$ does not have any $\sigma$-secant.

**Proof:** We only have to show that the absence of $\sigma$-secants implies the very ampleness of $|\sigma H - E|$, since we already noticed in the introduction that this is a necessary condition for very ampleness.

In the case when $C$ is defined by a matrix $M$ of linear forms in $k[w_0, \ldots, w_3]$ we know that $M$ is a $\sigma \times (\sigma + 1)$-matrix, that $d = \deg C = (\sigma + 1)$ and the genus of $C$ is $g = 2(\sigma + 1) - (\sigma + 1) + 1$ (e.g. see [R]).

We will indicate:

$$
M = \begin{pmatrix}
L_{11} & L_{12} & \cdots & L_{1\sigma+1} \\
L_{21} & L_{22} & \cdots & L_{2\sigma+1} \\
\vdots & \vdots & \ddots & \vdots \\
L_{\sigma 1} & L_{\sigma 2} & \cdots & L_{\sigma \sigma+1}
\end{pmatrix}
\quad \text{where} \quad L_{ij} = \sum_{k=0}^{3} \delta_{ij}^k w_k.
$$

Let $F_\beta = (-1)^{\beta} F'_\beta$, where $F'_\beta$ is the minor of $M$ obtained by erasing its $\beta$th column and consider the map $\phi : \mathbb{P}^3 - C \to \mathbb{P}^\sigma$ given by $\phi(P) = [F_1(P) : \ldots : F_{\sigma+1}(P)]$ (we have that $\phi_{\sigma} = \pi \circ \phi$ on $X_C - E$).
Note that we surely have $\sigma \geq 3$, otherwise $C$ would possess 1- or 2-secants.

We have to show that $|\sigma H - E|$ is very ample, i.e. that given any 0-dimensional scheme $T \subseteq X_C$, with $\deg T = 2$, the restriction map:

$$H^0(X_C, \mathcal{O}_{X_C}(\sigma H - E)) \to H^0(X_C, \mathcal{O}_T(\sigma H - E)) = H^0(X_C, \mathcal{O}_T)$$

is surjective.

We are already done if either:
- $T \subseteq F \subseteq E$, where $F$ is a fiber of $\pi$, by Proposition 1.3;
- $T \subseteq X_C - E$, by Proposition 1.1;
- $\text{Supp } T = P \in E$, by Remark 1.4.

Hence we are left with the following possibilities:

Case 1): $T = Q_1 \cup Q_2 \subseteq E$, $Q_1 \neq Q_2$, and, if $P_i = \pi(Q_i)$, $P_1 \neq P_2$;

Case 2): $T = Q_1 \cup Q_2$, $Q_1 \in E$, $Q_2 \notin E$.

Case 1): There exist vectors $t_i \in T_{P_i}(\mathbb{P}^3) - T_{P_i}(C)$, $i = 1, 2$ such that $\forall D \in |\sigma H - E|$, $Q_i \in D$ if and only if $t_i \in T_{P_i}(\pi(D))$; see e.g. \cite{Co}, §1.

Let $L_i \subseteq \mathbb{P}^3$ be the line determined by $t_i$; if $L_1 = L_2 = L$, then $T$ is contained in the strict transform of $L$, and we are done by Remark 1.2.

Let $L_1 \neq L_2$, and suppose that $L_1 \cap L_2 = \emptyset$, then, with a linear change of coordinates, we can assume that $P_1 = (1 : 0 : 0 : 0)$, $P_2 = (0 : 0 : 0 : 1)$, $L_1 = \{w_2 = w_3 = 0\}$, $L_2 = \{w_0 = w_1 = 0\}$.

What we have to check is that for surfaces $S = \{f = 0\}$, with $f \in (I_C)_\sigma$, the condition:

a) $t_1 \in T_{P_1}(S)$

does not imply the condition:

b) $t_2 \in T_{P_2}(S)$.

Fix the basis $\{F_1, ..., F_{\sigma+1}\}$ for $(I_C)_\sigma$, so we can write any $f \in (I_C)_\sigma$ as:

$$f = \lambda_1 F_1 + \lambda_2 F_2 + \ldots + \lambda_{\sigma+1} F_{\sigma+1} = \det \begin{pmatrix} \lambda_1 & M \\ \lambda_2 & \ldots & \lambda_{\sigma+1} \end{pmatrix}$$

and note that condition a) determines a $\sigma$-dimensional vector space $V_1 \subseteq (I_C)_\sigma$.

Let:

$$M_k = \begin{pmatrix} \delta^{k}_{11} & \delta^{k}_{12} & \ldots & \delta^{k}_{1\sigma+1} \\ \delta^{k}_{21} & \delta^{k}_{22} & \ldots & \delta^{k}_{2\sigma+1} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{k}_{\sigma1} & \delta^{k}_{\sigma2} & \ldots & \delta^{k}_{\sigma\sigma+1} \end{pmatrix}$$

It is well-known and easy to check that the rows of $M_0$ give elements $f = \delta^{0}_{11} F_1 + \ldots + \delta^{0}_{1\sigma+1} F_{\sigma+1} \in V_1$, since $f = \det \begin{pmatrix} \delta^{0}_{11} & M \\ \delta^{0}_{21} & \ldots & \delta^{0}_{2\sigma+1} \end{pmatrix}$, and this matrix has rank $\sigma - 2$ at $P_1$ (rank $M_0 = \text{rank}M(P_1) = \sigma - 1$, since $P_1 \in C$ and $C$ is smooth).

In an analogous way, b) determines a $\sigma$-dimensional subspace $V_2 \subseteq (I_C)_\sigma$, which contains the vectors given by the rows, $\delta^{k}_{ij}$, of $M_2$.

There exists an element $a = (a_1, ..., a_{\sigma+1}) \in k^{\sigma+1}$ such that the set $B = \{a^3, \ldots, a^3, a\}$ is a (redundant) system of generators of $V_2$.

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If \( a \Rightarrow b \), then \( \mathbf{V}_1 \subseteq \mathbf{V}_2 \), and we have that all vectors \( \mathbf{d}^0 \) are linear combinations of elements of \( \mathcal{B} \), i.e.

\[ \forall i = 1, \ldots, \sigma \text{ there exists } (\alpha_1^i, \ldots, \alpha_{\sigma}^i, \alpha_i^i) \in k^{\sigma+1} \text{ such that:} \]

\[ \mathbf{d}^0 = \sum_{k=1}^\sigma \alpha_k^i \mathbf{d}_k^3 + \alpha_i^i \mathbf{d}_i. \]

Now consider the line \( L = \{ w_1 = w_2 = 0 \} \): we have that \( M|_L = w_0 M_0 + w_3 M_3 \). Since \( \text{rank} M_3 = \sigma - 1 \), there will be two independent linear combinations of the columns of \( M_3 \) which give the zero-vector:

\[ \beta_1 \begin{pmatrix} \delta_{11}^3 \\ \vdots \\ \delta_{\sigma 1}^3 \end{pmatrix} + \ldots + \beta_{\sigma+1} \begin{pmatrix} \delta_{1\sigma+1}^3 \\ \vdots \\ \delta_{\sigma 3}^3 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\sigma+1} \beta_j \delta_{j 1}^3 \\ \vdots \\ \sum_{j=1}^{\sigma+1} \beta_j \delta_{j \sigma}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}; \]

\[ \gamma_1 \begin{pmatrix} \delta_{11}^3 \\ \vdots \\ \delta_{\sigma 1}^3 \end{pmatrix} + \ldots + \gamma_{\sigma+1} \begin{pmatrix} \delta_{1\sigma+1}^3 \\ \vdots \\ \delta_{\sigma 3}^3 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\sigma+1} \gamma_j \delta_{j 1}^3 \\ \vdots \\ \sum_{j=1}^{\sigma+1} \gamma_j \delta_{j \sigma}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \]

If we use \((\beta_1, \ldots, \beta_{\sigma+1})\) as coefficients for a linear combination of the columns of \( M|_L \), we get:

\[ \beta_1 \begin{pmatrix} \delta_{11}^0 w_0 + \delta_{11}^3 w_3 \\ \vdots \\ \delta_{\sigma 1}^0 w_0 + \delta_{\sigma 1}^3 w_3 \end{pmatrix} + \ldots + \beta_{\sigma+1} \begin{pmatrix} \delta_{1\sigma+1}^0 w_0 + \delta_{1\sigma+1}^3 w_3 \\ \vdots \\ \delta_{\sigma \sigma+1}^0 w_0 + \delta_{\sigma \sigma+1}^3 w_3 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\sigma+1} \beta_j \delta_{j 1}^0 \\ \vdots \\ \sum_{j=1}^{\sigma+1} \beta_j \delta_{j \sigma}^0 \end{pmatrix}. \]

By (1), we have: \( \delta_{ij}^0 = \sum_{k=1}^{\sigma} \alpha_k^i \delta_{kj}^3 + \alpha^i a_j \), hence:

\[ w_0 \begin{pmatrix} \sum_{j=1}^{\sigma+1} \sum_{k=1}^{\sigma} \beta_j (\alpha_k^i \delta_{kj}^3 + \alpha^i a_j) \\ \vdots \\ \sum_{j=1}^{\sigma+1} \sum_{k=1}^{\sigma} \beta_j (\alpha_k^i \delta_{kj}^3 + \alpha^i a_j) \end{pmatrix} = w_0 \begin{pmatrix} \sum_{k=1}^{\sigma} \alpha_k^i (\sum_{j=1}^{\sigma+1} \beta_j \delta_{kj}^3) + \alpha^i \sum_{j=1}^{\sigma+1} \beta_j a_j \\ \vdots \\ \sum_{k=1}^{\sigma} \alpha_k^i (\sum_{j=1}^{\sigma+1} \beta_j \delta_{kj}^3) + \alpha^i \sum_{j=1}^{\sigma+1} \beta_j a_j \end{pmatrix}. \]

which, since \( \sum_{j=1}^{\sigma+1} \beta_j \delta_{kj}^3 = 0 \), yields:

\[ (\sum_{j=1}^{\sigma+1} \beta_j a_j) w_0 \begin{pmatrix} \alpha_1^i \\ \vdots \\ \alpha^i \end{pmatrix}. \]

In an analogous way one gets:

\[ w_0 \begin{pmatrix} \sum_{j=1}^{\sigma+1} \sum_{k=1}^{\sigma} \gamma_j (\alpha_k^i \delta_{kj}^3 + \alpha^i a_j \gamma_j) \\ \vdots \\ \sum_{j=1}^{\sigma+1} \sum_{k=1}^{\sigma} \gamma_j (\alpha_k^i \delta_{kj}^3 + \alpha^i a_j \gamma_j) \end{pmatrix} = (\sum_{j=1}^{\sigma+1} \gamma_j a_j) w_0 \begin{pmatrix} \alpha_1^i \\ \vdots \\ \alpha^i \end{pmatrix}. \]

Since these two columns are multiples of each other, by elementary columns operations we can replace a column in \( M|_L \), say the first one, by a column of zeros.

Hence either the rank of \( M \) drops at each point of \( L \), i.e. \( L \subseteq C \), or all the minors of \( M|_L \) are zero except \( F_1|_L \), i.e. \( C \cap L \) consists of \( \sigma \) points (the zeros of \( F_1|_L \)).

Since both cases cannot occur, we are done.

In the case when \( L_1 \cap L_2 \neq \emptyset \), they are contained in a plane \( \Gamma \), say \( \Gamma = \{ w_3 = 0 \} \). Then the procedure is exactly the same, simply choose on \( \Gamma \): \( P_1 = (1:0:0), P_2 = (0:0:1), L_1 = \{ w_2 = 0 \}, L_2 = \{ w_0 = 0 \}, \)

\( L = \{ w_1 = 0 \} \).

Case 2). This case is quite similar to the previous one. Let \( P_1 = \pi(Q_1) \), so \( P_1 \in C, P_2 \notin C \), and let \( t_1 \in T_{P_1}(P^3) - T_{P_1}(C) \) be as in the previous case.

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As before, we have to check that, for \( S = \{ f = 0 \}, f \in (I_C)_\sigma \), the condition

a) \( t_1 \in T_{P_1}(S) \)
does not imply the condition:
b) \( P_2 \in S \).

Choose for \( P_1, P_2, L_1 \) the same coordinates as before. Since b) gives a subspace of \((I_C)_\sigma\) defined by \( \{ \delta_{11}^3, \ldots, \delta_{\sigma+1}^3 \} \) (this time they are independent since \( P_2 \notin C \), i.e. \( \text{rk}(M_3) = \sigma \), if a) \( \Rightarrow \) b), we will again have that all the rows in \( M_0 \) are linear combination of the rows in \( M_3 \), and we can conclude as before (here we just have \( a = 0 \)). □

It is possible to extend the result above to the case of varieties of codimension 2 which do not contain lines.

**Proposition 2.4:** Let \( V \subseteq \mathbb{P}^n \) be a smooth a.C.M. variety of codim 2 and let \( I_V \) be generated by the maximal minors of a matrix of linear forms. Then \( |σH − E| \) is very ample on \( X_V \) if \( V \) does not contain any line and it has no \( σ \)-secant line.

The proof of Prop. 2.4 follows the same lines as that of Thm. 2.3; the hypothesis that \( V \) does not contain lines is needed to exclude the possibility that the rank of \( M \) drops at every point of \( L \) (see Case 1 in the proof of 2.3). □

Now, the next problems that are quite natural to consider for these varieties are:
1) when does \( V \), defined by a generic matrix of linear forms, contain lines?
2) when does such a \( V \) have \( σ \)-secants?

Note that here "generic" matrix means that the coefficients of its entries are generic in \( k^{σ(σ+1)(n+1)} \).

The answer to these questions is given by the next proposition and it was probably classically known. We give a proof here, for lack of reference (notice that we consider only the case \( σ \geq n + 1 \), since for \( σ \leq n \) \( V \) trivially has \( σ \)-secants).

**Proposition 2.5:** Let \( M \) be a generic \( σ \times (σ + 1) \)-matrix, \( σ \geq n + 1 \), whose entries are linear forms in \( k[w_0, w_1, w_2, \ldots, w_n] \), \( n \geq 3 \). Then \( M \) defines an irreducible a.C.M. variety \( V \) in \( \mathbb{P}^n \) and

i) \( V \) contains a line if and only if \( σ \leq 2n − 5 \).

ii) \( V \) has \( σ \)-secants if and only if \( σ \leq 2n − 2 \).

**Proof:** For the fact that a generic matrix defines an irreducible 2-codimensional variety see e.g. [S] (recall, see the introduction, that \( V \) will be smooth if and only if \( n \leq σ \)).

As before, let

\[
M = \begin{pmatrix} L_{11} & L_{12} & \ldots & L_{1σ+1} \\
L_{21} & L_{22} & \ldots & L_{2σ+1} \\
\vdots & \vdots & \ddots & \vdots \\
L_{σ1} & L_{σ2} & \ldots & L_{σσ+1} \\
\end{pmatrix}
\]

where \( L_{ij} = \sum_{k=0}^{n} δ_{ij}^k w_k \).

For each \( z = [z_1 : \ldots : z_σ] \), let us consider the \( σ + 1 \) hyperplanes:

(1) \( \{ z_1 L_{1j} + z_2 L_{2j} + \ldots + z_σ L_{σj} = 0 \}, \quad j = 1, \ldots, σ + 1 \).
Then $V$ can be viewed as the locus of meets of the $\sigma + 1$ hyperplanes above, as $\z_1$ varies in $\mathbb{P}^{\sigma - 1}$ (this is classically known as the \textit{projective generation} of $V$).

Proof of i):

For a given $\z \in \mathbb{P}^{\sigma - 1}$, let us denote by $T_\z \subset V$ the linear space of the solutions of the system (in the $w_k$'s) defined by (†), whose matrix of coefficients $Z$ is an $(n+1) \times (\sigma+1)$-matrix of linear forms in $k[z_1, \ldots, z_\sigma]$:

$$Z_{kj} = \sum_{i=1}^\sigma \delta_{ij}^k z_i.$$

We will have that $T_\z$ contains a line if and only if the rank of $Z$ is $\leq n - 1$. $Z$ is a generic matrix since $M$ is (the coefficients that show up are the same), hence the codimension of the locus $\Lambda$ where the rank of $Z$ is $\leq n - 1$ is $(\sigma + 1 - n + 1)(n + 1 - n + 1) = 2\sigma - 2n + 4$. Thus $\Lambda \subset \mathbb{P}^{\sigma - 1}$ is not empty if and only if $2\sigma - 2n + 4 \leq \sigma - 1$, i.e. whenever $\sigma \leq 2n - 5$.

Now let $L$ be a line in $V$; if we show that $L$ is contained in some $T_\z$ we are done, but this is obvious since $L \subseteq V$, so the rank of $M|L$ drops, i.e. its rows are linearly dependent, therefore there is a solution $\z$ for the system (†), hence $L \subseteq T_\z$.

Proof of ii):

Every family of hyperplanes given by $z_1 L_{1j} + z_2 L_{2j} + \ldots + z_\sigma L_{\sigma j} = 0$ has a "center", given by $\{L_{1j} = L_{2j} = \ldots = L_{\sigma j} = 0\}$. Via a linear combination of the columns of $M$, one can consider

$$T_\y = \{ \sum_{j=1}^{\sigma+1} y_j L_{1j} = \sum_{j=1}^{\sigma+1} y_j L_{2j} = \ldots = \sum_{j=1}^{\sigma+1} y_j L_{\sigma j} = 0 \}$$

as one of the "centers" of (†), for every choice of $\y \in \mathbb{P}^\sigma$.

If we consider the linear system defined by $T_\y$ in the $w_k$'s, the entries of its $(\sigma \times (n + 1))$-matrix of coefficients $N$ are given by the forms (in $k[y_1, \ldots, y_{\sigma+1}]$):

$$N_{ik} = \sum_{j=1}^{\sigma+1} \delta_{ij}^k y_j$$

Now, for a given $\y$, if the matrix $N(\y)$ has rank $n + 1$, then $T_\y$ is empty; if $\text{rk}N(\y) = n$, then $T_\y$ is a point; while $T_\y$ is a line for $\text{rk}N(\y) = n - 1$.

In the last case, the line $T_\y$ intersects $V$ (see e.g. $[\mathbb{R}]$) in a subvariety of $T_\y \cong \mathbb{P}^1$ defined by a $(\sigma \times \sigma)$-matrix of linear forms, i.e. in $\sigma$ points (with multiplicities, i.e. in a divisor of degree $\sigma$ in $\mathbb{P}^1$).

Since $M$ was generic, so is $N$ (the coefficients of the forms are the same), hence the scheme $\Gamma \subseteq \mathbb{P}^\sigma$, defined as the rank $n - 1$ locus of $N$, has codimension $(\sigma - (n - 1)) \times (n + 1 - (n - 1)) = 2(\sigma - n + 1) = 2(\sigma - n + 1)$ in $\mathbb{P}^\sigma$, so it is not empty for $\sigma \leq 2n - 2$, as we wanted.

We are left with checking that when $\sigma \geq 2n - 1$, there are no $\sigma$-secants to $V$.

We will be done if we show that, when $V$ possesses a $\sigma$-secant $L$, $L$ can be viewed as before, i.e. as the "center" of a family of hyperplanes which gives the projective generation of $V$.

Let $L = \{w_2 = w_3 = \ldots = w_n = 0\}$; then on $L$ we have that $M|L$ defines the homogeneous ideal, in $R' = k[w_0, w_1]$, of $\sigma$ points, i.e. that its maximal minors $F_1, \ldots, F_{\sigma+1}$ (where $F_j$ is the minor with sign obtained by erasing the $j^{\text{th}}$ column) can be viewed as multiples of one of them (which is not identically zero), say $F_1$. 

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This implies that the columns of the matrix \( M|_L \), are linearly dependent, hence there is a linear combination of them, say with coefficients \( y_1, \ldots, y_{\sigma+1} \), which gives a zero column.

Then the same linear combination on \( M \) gives a form in \( k[w_2, w_3, \ldots, w_n] \), i.e. we have that:

\[
\left\{ \sum_{j=1}^{\sigma+1} y_j L_{1j} = \sum_{j=1}^{\sigma+1} y_j L_{2j} = \ldots = \sum_{j=1}^{\sigma+1} y_j L_{\sigma j} = 0 \right\}
\]

has solution on all of \( L = \{ w_2 = w_3 = \ldots = w_n = 0 \} \). But the matrix of this linear system is what we have called \( N(y) \), so it must have rank \( \leq n - 1 \), which is possible only for \( \sigma \leq 2n - 2 \), as we have observed before.

\[ \square \]

Let us give an example of what happens for the first values of \( \sigma \), when \( V = C \) is a curve:
- when \( \sigma = 2 \), \( C = C_0^2 \) is a rational normal curve and, for every \( y \in \mathbb{P}^2 \), \( T_y \) is a secant line of \( C \).
- when \( \sigma = 3 \), \( C = C_3^2 \) is a sestic curve of genus three, and \( \Gamma \cong \mathbb{P}^2 \) gives a family of trisecants of \( C \).
- when \( \sigma = 4 \), \( \Gamma \subset \mathbb{P}^4 \) is a set of 20 points, corresponding to 20 4-secants of \( C = C_{11}^3 \).

¿From the above results, it follows the analog of the Conjecture for a ”generic” smooth \( V \) of codimension 2:

**Corollary 2.6:** Let \( M \) be a generic \( \sigma \times (\sigma + 1) \)-matrix of linear forms in \( k[w_0, \ldots, w_n] \), with \( 1 < n < 6 \), and let \( V \subset \mathbb{P}^n \) be the smooth a.C.M. variety of codim 2 defined by \( I_V \), the ideal generated by the maximal minors of \( M \). Then \( |\sigma H - E| \) is very ample on \( X_V \) if and only if \( \sigma > 2n - 2 \).

### 3. Generic curves of minimal genus.

In this section we will consider projectively normal curves \( C \subset \mathbb{P}^3 \) such that they are ”generic for their degree” in the sense that if \( \deg C = s \), the ideal sheaf of \( C \) has the same kind of resolution (in the sense that it has the same graded Betti numbers) as a generic set of \( s \) points in \( \mathbb{P}^2 \). Which are the Betti numbers of this ”generic resolution” (in the sense of [L.2] ) will be specified below (see comments before Prop. 3.3).

We will say that such a \( C \) is **Betti minimal**, B-minimal for short, and note that its Hilbert-Burch matrix has entries of minimal degree. As we will see below, such curves have minimal genus among projectively normal curve of their degree.

Since a generic hyperplane section of \( C \) has the same graded Betti numbers as \( C \), by cutting with a generic plane in order to understand the Hilbert-Burch matrix of a B-minimal \( C \), we can reduce ourselves to look at a (reduced) set of points \( Z \in \mathbb{P}^2 \). The requirement on the resolution of \( Z \) implies that the degree matrix of the ideal \( I_Z \) (which is the same as the degree matrix of \( I_C \)) is made by 1’s and 2’s and the Hilbert function of \( Z \) is maximal, i.e. \( H(Z, t) = \min \{ \binom{t+2}{2}, s \} \) (see [C-G-O] , [Gi-Lo] ).

The Hilbert function of \( Z \) is related to the genus of \( C \) by the following well-known fact (which we prove here for lack of reference):

**Proposition 3.1:** Let \( C \subset \mathbb{P}^3 \) be projectively normal, and let \( Z \) be a generic plane section of \( C \). Let \( \deg C = s \); then the genus of \( C \) is:

\[ g = \sum_{t \geq 1} s - H(Z, t). \]

**Proof:** We have \( g = h^2(\mathbb{P}^3, \mathcal{I}_C) \), and \( s - H(Z, t) = h^1(\mathbb{P}^2, \mathcal{I}_Z(t)) \). ¿From the resolution of \( C \):

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Corollary 3.2: Any B-minimal curve has minimal genus among the projectively normal curves of its degree.

Now let us describe more precisely the degree matrix of B-minimal curves; let \( s = \deg C = \deg Z \), \( d = \min\{ t| s \leq \binom{t+2}{2} \} \) and \( s = \binom{d+1}{2} + k \), with \( 1 \leq k \leq d+1 \) (note that we have \( s = \sigma(I_C) = d+1 \)).

The ideal \( I_C \) will be generated by \( d-k+1 \) forms of degree \( d \) and \( h \) forms of degree \( d+1 \), where \( h \) is either 0 or \( 2k-d \), according to whether \( d \geq 2k \) or not.

Since given any B-minimal \( C \) the integers \( d, k \) are determined by its degree \( s \), we will also speak of "B-minimal curve of type \( d, k \)".

Denote by \( F_1, ..., F_{d-k+1} \) the generators of degree \( d \) and by \( G_1, ..., G_{2k-d} \) those of degree \( d+1 \) (when present).

We have that the Hilbert-Burch matrix is a \( \rho \times (\rho + 1) \) matrix, where:

\[
\rho = \begin{cases} 
  k & \text{if } d \leq 2k \\
  d-k & \text{if } d \geq 2k
\end{cases}
\]

In the case \( d \leq 2k \), the matrix \( M \) is:

\[
M = \begin{pmatrix}
L_{11} & L_{12} & \cdots & L_{1d-k} & Q_{11} & \cdots & Q_{1d-k+1} \\
L_{21} & L_{22} & \cdots & L_{2d-k} & Q_{21} & \cdots & Q_{2d-k+1} \\
& \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
L_{k1} & L_{k2} & \cdots & L_{k2d-k} & Q_{k1} & \cdots & Q_{kd-k+1}
\end{pmatrix}
\]
where the $L_{ij}$'s are linear forms and the $Q_{il}$'s are forms of degree 2. The $F_j$'s are the algebraic minors obtained by deleting the $j^{th}$ column, while the $G_i$'s are obtained by deleting the $i^{th}$ column.

In this case the resolution of $I_C$ is:

$$0 \to \oplus^k \mathcal{O}_\mathbb{P}^3(-d-2) \to (\oplus^{2k-d} \mathcal{O}_\mathbb{P}^3(-d-1)) \oplus (\oplus^{d-k+1} \mathcal{O}_\mathbb{P}^3(-d)) \to I_C \to 0.$$

In the other case $(d \geq 2k)$ the matrix is:

$$M = \begin{pmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1d-k+1} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{k1} & Q_{k2} & \cdots & Q_{kd-k+1} \\
L_{11} & L_{12} & \cdots & L_{1d-k+1} \\
\vdots & \vdots & \ddots & \vdots \\
L_{d-2k1} & L_{d-2k2} & \cdots & L_{d-2kd-k+1}
\end{pmatrix},$$

Hence the resolution of $I_C$ is:

$$0 \to (\oplus^k \mathcal{O}_\mathbb{P}^3(-d-2)) \oplus (\oplus^{d-2k} \mathcal{O}_\mathbb{P}^3(-d-1)) \to (\oplus^{d-k+1} \mathcal{O}_\mathbb{P}^3(-d)) \to I_C \to 0.$$

In this case we have that the $F_j$ are the maximal algebraic minors of $M$, and the ideal $I_C$ is generated in degree $d$.

¿From what we have observed, it easily follows that our conjecture is trivially true for B-minimal curves of type $d,k$ with $d \geq 2k$.

**Proposition 3.3:** Let $C$ be B-minimal of type $d,k$ with $d \geq 2k$. Then $|\sigma H - E|$ is very ample on $X_C$.

**Proof:** The Proposition follows immediately from [Co] , Theorem 1, quoted in the introduction, since $I_C$ is generated in degree $d = \sigma - 1$ (notice that for this reason $C$ cannot possess $\sigma$-secants).

Let us notice that, by what we have seen in the previous section, for any $C$ of type $(*)$, the linear system $|\sigma H - E|$ always defines a morphism $\phi_\sigma : X_C \to \mathbb{P}^N$, where $N+1 = h^0(X_C, \mathcal{O}_{X_C}(\sigma H - E)) = \dim(I_C)_\sigma$, whose image is a rational threefold (by Prop.1.1, $\phi_\sigma$ is generically 1:1). We have:

**Theorem 3.4:** Let $C \subseteq \mathbb{P}^3$ be a projectively normal curve which is B-minimal of type $d,k$. Then there exist two matrices $B,X$ of linear forms (in the coordinates ring $S$ of $\mathbb{P}^N$) such that: $X$ is a $4 \times (d-k+1)$-matrix, $B$ is a $k \times 4$-matrix, and the homogeneous ideal $I$ of $Y_{C,\sigma} = \phi_\sigma(X_C)$ in $S$ is generated by the $(2 \times 2)$-minors of $X$, the entries of $B \cdot X$ and the $(4 \times 4)$-minors of $B$. Moreover, $Y_{C,\sigma}$ is a.C.M.

**Sketch of proof:** The proof is very similar to the one given in §4 of [Gi-Lo] , for points in $\mathbb{P}^2$, which essentially relies upon the structure of the matrix $M$ and its genericity, hence it easily extends to this case. We refer to that paper for full details, here we simply explain the main steps of the procedure:

1) Construction of a homogeneous ideal $I$ as described in the statement, such that $Y_{C,\sigma} \subseteq Z(I)$.
2) Prove that $I$ is prime and perfect.
3) Show that $Y_{C,\sigma} = Z(I)$.
Step 1 - The idea for the construction of \( I \) is the fact that its elements can be obtained from syzygies of \( I_C \) whose components belong to \( I_C \) itself; so we have to show that we can find the appropriate syzygies which will give us a set of generators for \( I \).

First of all, recall that we denoted by \( S = k[x,y] \) the coordinate ring of \( \mathbb{P}^N \), where \( x = (x_{ij}) \), \( h = 0, \ldots, 4; j = 1, \ldots, d - k + 1 \), and by \( R = k[w] \) the coordinate ring of \( \mathbb{P}^3 \). Let \( R' \) be the graded subring of \( R \) defined by \( R' = \oplus kR'_h \), where \( R'_h = R_k(d+1) \). Now define \( \psi : k[x,y] \rightarrow k[w] \) by \( \psi(x_{ij}) = (w_hF_j) \), \( \psi(y_{ij}) = G_t \), which is graded (i.e. homogeneous of degree 0). Let \( N' = 3d - 3k + p + 3 \). When \( d \geq 2k \), we have that \( N' = N \) and \( I_Y = \text{Ker}\psi \). When \( d > 2k \), we can derive, from the matrix \( M \), \( d - 2k \) linearly independent linear forms \( H_1, \ldots, H_{d-2k} \in \text{Ker}\psi \) so that \( I_Y \) may be identified with \( \langle \overline{M}_{13}, \overline{M}_{d-2k} \rangle \).

The first obvious generators of \( I \) will all be the elements \( x_{h'j'}x_{hj} - x_{h'j}x_{hj'} \), which are in \( \text{Ker}\psi \) (they correspond to the trivial syzygies \( w_hF_jw_hF_j - w_hF_jw_hF_j \)), and which can be viewed as the \((2 \times 2)\)-minors of the matrix:

\[
X = \begin{pmatrix}
x_{01} & x_{02} & \cdots & x_{0d-k+1} \\
x_{11} & x_{12} & \cdots & x_{1d-k+1} \\
\vdots & \vdots & \ddots & \vdots \\
x_{31} & x_{32} & \cdots & x_{3d-k+1}
\end{pmatrix}.
\]

Now assume \( d \leq 2k \) and consider the matrix \( M_u \), \( u = 1, \ldots, k \), obtained by repeating the \( u \)-th row in \( M \):

\[
M_u = \begin{pmatrix}
L_{u1} & L_{u2} & \cdots & L_{u2-k-d} & Q_{u1} & \cdots & Q_{ud-k+1}
\end{pmatrix}.
\]

let \( L_{ui} = \sum_{i=0}^{3} \delta_{ui}^i w_i \), and \( Q_{uj} = \sum_{i=0}^{3} \sum_{h=1}^{d-k+1} \beta_{hi}^j w_h w_i \); by expanding \( \text{det} M_u \) with respect to the last row we get:

\[
0 = \sum_{i=1}^{2k-d} L_{ui}G_i + \sum_{j=1}^{d-k+1} Q_{uj}F_j = \sum_{i=1}^{2k-d} \left( \sum_{i=0}^{3} \delta_{ui}^i w_i \right) G_i + \sum_{j=1}^{d-k+1} \left( \sum_{i=0}^{3} \sum_{h=0}^{d-k+1} \beta_{hi}^j w_h w_i \right) F_j;
\]

and so, by multiplying by \( F_{\nu} \), we get \( \forall \nu = 1, \ldots, d - k + 1 \):

\[
0 = F_{\nu} \text{det} M_u = \sum_{i=0}^{3} \left( \sum_{j=1}^{d-k+1} \left( \sum_{i=0}^{3} \sum_{h=0}^{d-k+1} \beta_{hi}^j \psi(x_{ij}) \right) \psi(x_{i\nu}) \right) w_{ii};
\]

which can be written as:

\[
0 = \psi \left( \sum_{i=0}^{3} x_{i\nu}B_{ui} \right),
\]

where

\[
B_{ui} = \sum_{l=1}^{2k-d} \delta_{ui}^l y_l + \sum_{j=1}^{d-k+1} \sum_{h=0}^{d-k+1} \beta_{ui}^{h,j} x_{hj}.
\]

The degree 2 forms \( \sum_{i=1}^{3} x_{i\nu}B_{ui} \) in \( \text{Ker}\psi \) will be added as generators of \( I \) and can be viewed as the entries of the product matrix \( B \cdot X \), where \( B \) is the \((4 \times k)\)-matrix of linear forms: \( B = (B_{ui})_{ui} \).

Furthermore, denote \( C_{ui} = \psi(B_{ui}) \); then the image under \( \psi \) of the maximal minors of \( B \) are precisely the maximal minors of the matrix \( C = (C_{ui})_{ui} \). Now, for each \( u = 1, \ldots, k \) and \( i = 0, \ldots, 3 \), consider the matrix

\[
D_{ui} = \begin{pmatrix}
\delta_{ui}^0 & \cdots & \delta_{ui}^{2k-d} & M_{\sum_{h=0}^{3} \beta_{ui}^{h1} w_h} & \cdots & M_{\sum_{h=0}^{3} \beta_{ui}^{hd-k+1} w_h}
\end{pmatrix}.
\]

Then, it turns out that \( \text{det} D_{ui} = C_{ui} \) and that, \( \forall (a_0 : \ldots : a_3) \in \mathbb{P}^3 \),

\[
0 = \text{det} M_u(a_0 : \ldots : a_3) = \sum_{i=0}^{3} a_i \text{det} D_{ui}(a_0 : \ldots : a_3) = \sum_{i=0}^{3} a_i C_{ui}(a_0 : \ldots : a_3).
\]

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Therefore the rank of $C$ is not maximal, and so all the maximal minors of $B$ belong to $\text{Ker} \psi$. We add these forms of degree 4 to the generators of $I$.

The case $d > 2k$ is alike, but the matrix $M_u$, obtained by repeating the $u^{th}$ row of $M$, in this case looks like

$$M_u = \left( Q_{u1} \ M \ldots \ Q_{ud-k+1} \right).$$

Note that, in this case, there are no $G_l$, hence no $y_l$ will show up in the resulting matrix $B$.

Also, in this case, $I$ will be defined as the ideal generated by the residue classes of the $2 \times 2$ minors of $X$, of the entries of $B \cdot X$, and of the $4 \times 4$ minors of $B$, modulo $(H_1, \ldots, H_{d-2k})$.

**Step 2** - A genericity argument and a Theorem by Huneke ([Hu, Thm. 60], see also [Gi-Lo, Thm. 4.1]) allow us to prove that $I$ is prime and perfect, just as in [Gi-Lo, Thm. 4.2].

**Step 3** - After the necessary changes, the proof of [Gi-Lo, Prop. 4.4] works throughout.

\[\square\]

**Remark 1.** When $k = d + 1$, $M$ is a matrix of linear forms and in this case only the matrix $B$ will appear in the construction of the ideal $I$ of $\phi \sigma(C)$, which will be determinantal (the situation will be the one described in Corollary 2.2, with $k = \rho = \sigma$).

**Remark 2.** The threefold $Y_{C,\sigma}$ considered in 3.4 need not be smooth: there are many cases when $\phi \sigma$ is not very ample (e.g. see the Remark in the next section).

There is another case in which it is not hard to compute the ideal generation of the embedded threefold (this time via the linear system $|((\sigma+1)H - E)|$):

**Proposition 3.5:** Let $C \subseteq \mathbb{P}^3$ be a projectively normal curve whose Hilbert-Burch matrix is made of linear forms and let $\deg C = \left(\begin{smallmatrix}d+1 \\ 2\end{smallmatrix}\right)$ (hence $\sigma(C) = d$). Then the homogeneous ideal of $Y_{C,\sigma+1} \subseteq \mathbb{P}^{3d+3}$ is generated by the $(2 \times 2)$-minors of a $4 \times (d+1)$-matrix $X$ of linear forms (in the coordinate ring of $\mathbb{P}^{3d+3}$).

The proof works as in the previous theorem (where one could consider $k = 0$): this time only the matrix $X$ is to be considered (see [Ge-Gi] for a description of this construction in the case of points in $\mathbb{P}^2$). In order to check that $Y_{C,\sigma+1} \subseteq \mathbb{P}^N$, with $N = 3d + 3$, consider that $\deg C = \left(\begin{smallmatrix}d+1 \\ 2\end{smallmatrix}\right)$, and $g(C) = \left(\begin{smallmatrix}d+1 \\ 2\end{smallmatrix}\right) \left(\begin{smallmatrix}2d-2 \\ 3\end{smallmatrix}\right) + 1$ (see Prop. 3.1), hence:

$$H(C, d+1) = \left(\begin{smallmatrix}d+1 \\ 2\end{smallmatrix}\right) (d+1) - g(C) + 1$$

implies

$$N + 1 = \left(\begin{smallmatrix}d+1+3 \\ 3\end{smallmatrix}\right) - H(C, d+1) = \left(\begin{smallmatrix}d+4 \\ 3\end{smallmatrix}\right) - \left(\begin{smallmatrix}d+1 \\ 2\end{smallmatrix}\right) (d+1) + g(C) - 1 + 1 = 3d + 4.$$  

\[\square\]

Notice that in this case also the resolution of $I_{Y_{C,\sigma+1}}$ can be computed via a Lascoux complex (see [La] and also [PW]).

Finally, we can observe that it is not hard to extend the results of 3.4 and 3.5 to the case of 2-codimensional (smooth, irreducible, a.C.M.) subschemes $V \subseteq \mathbb{P}^n$, $n = 4, 5$. Namely, let $V$ be as above, and
also B-minimal (with an obvious extension of the definition given for curves in \( \mathbb{P}^3 \)). Let \( \deg V = s = \binom{d+1}{2} + k \), \( 0 \leq k \leq d + 1 \), in analogy of what we did before. Then, if \( Y_{V,d+1} \) is the image of \( X_V \) via the linear system \( |(d + 1)H - E| \), we have:

**Proposition 3.6:** With the above notation and hypotheses, we have that the homogeneous ideal of \( Y_{V,d+1} \subseteq \mathbb{P}^N \) is generated by the \((2 \times 2)\) minors of a \( k \times (n + 1)\)-matrix of linear forms \( X \), by the \((n + 1) \times (n + 1)\) minors of a \( k \times (n + 1)\)-matrix \( B \) of linear forms and by the entries of \( B \cdot X \). Moreover, \( Y_{V,d+1} \) is a.C.M.

The proof of 3.6 works as the ones of 3.4 and 3.5.

4. Examples: Some Fano varieties and their ideal generation.

**Example 1** Let \( C = C_7^5 \subseteq \mathbb{P}^3 \) be a projectively normal curve of degree 7 and genus 5. \( C \) is B-minimal of type 3,1 and its Hilbert-Burch matrix has degrees:

\[
\begin{pmatrix}
2 & 2 & 2 \\
1 & 1 & 1
\end{pmatrix}.
\]

Hence \( I_C \) is generated by cubics, and in this case \( \sigma = 4 \), so the linear system \( |4H - E| \) is very ample on \( X_C \); notice that \( -K_{X,C} = 4H - E \), hence \( X_C \) is a Fano threefold of index 1.

The image of \( X_C \) is a \( Y_{C,4} \subseteq \mathbb{P}^{10} \), and a curve-section of it can be viewed as the image of a curve \( C' \subseteq \mathbb{P}^3 \) which is the residual intersection, with respect to \( C \), of two quartic surfaces. Such curve has degree 9 and its genus is given by the following formula, where \( a, b \) are the degrees of two surfaces whose intersection is formed (scheme theoretically) by \( C \cup C' \) (see e.g. [P-S] ) :

\[
g(C) - g(C') = \left( \frac{a + b}{2} - 2 \right) (\deg C - \deg C')
\]

hence \( g(C') = 9 \).

So \( Y_{C,4} \) has degree \( 2g(C') - 2 = 16 \) and sectional genus 9.

By Theorem 3.4, the homogeneous ideal of \( Y_{C,4} \) will be generated by quadratic forms, given by the \( 2 \times 2 \) minors of a \( (4 \times 3)\)-matrix \( X \) of linear forms and by the entries of \( B \cdot X \), where \( B \) is a \( (1 \times 4)\)-matrix of linear forms.

**Remark.** Notice that \( C = C_7^5 \) is the only B-minimal curve with \( \sigma = 4 \) for which \( |4H - E| \) is very ample on \( X_C \); the other B-minimal curves with \( \sigma = 4 \) are in fact: \( C_8^8 \), \( C_9^9 \) and \( C_{11}^{10} \), whose degree matrices are, respectively:

\[
\begin{pmatrix}
1 & 2 & 2 \\
1 & 2 & 2
\end{pmatrix}; \quad \begin{pmatrix}
1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2
\end{pmatrix}; \quad \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

In the first case it is trivial that the two linear form in the first column of the matrix define a 4-secant line; in the second it can be seen that any smooth cubic surface containing the curve has six lines (one part of its ”double-six” among its 27 lines) which are 4-secants to the curve; the last case has been considered in Proposition 2.5 (there are twenty 4-secant lines).
Notice also that in these cases $|4H - E|$ is not even ample (any of its multiples will have zero intersection with the 4-secants), hence those $X_C$'s are not Fano.

**Example 2** Of course one can get a Fano threefold also when $\sigma(C) < 4$; in particular let us consider $C = C_3^6$: in this case $I_C$ is generated by the $3 \times 3$-minors of a $3 \times 4$-matrix of linear forms and $\sigma(C) = 3$, hence we are in the case of Proposition 3.5: the Fano threefold $Y_{C, 4} \subseteq \mathbb{P}^{12}$ has degree 20 and sectional genus 11 (two quartics through C have residual intersection in a $C_{11}^{10}$) and its homogeneous ideal is generated by the $2 \times 2$ minors of a $(4 \times 4)$-matrix of linear forms.

**Example 3** Working as in Example 1, it is possible to find Fano varieties of dimension 4 and 5. In $\mathbb{P}^4$, consider the generic determinantal surfaces of degree 11 and 12, whose Hilbert-Burch matrices have, respectively, degrees:

$$
\begin{pmatrix}
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 2
\end{pmatrix}.
$$

Those surfaces are generated in degree 4, hence the (anticanonical) linear system $|5H - E|$ is very ample on the blow-up $X_V$, and the generation of the ideals of the embedded 4-folds is given by Proposition 3.6.

Similarly, in $\mathbb{P}^5$, we have that we can use the two determinantal threefold (of degrees, respectively, 16 and 17) whose Hilbert-Burch matrices have entries, respectively, of the following degrees:

$$
\begin{pmatrix}
2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1
\end{pmatrix}
$$

to get Fano 5-fold whose ideal generation is again described by Prop. 3.6.

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