A strong form of Plessner’s theorem

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Abstract

Let $f$ be a holomorphic, or even meromorphic, function on the unit disc. Plessner’s theorem then says that, for almost every boundary point $\zeta$, either (i) $f$ has a finite nontangential limit at $\zeta$, or (ii) the image $f(S)$ of any Stolz angle $S$ at $\zeta$ is dense in the complex plane. This paper shows that statement (ii) can be replaced by a much stronger assertion. This new theorem and its analogue for harmonic functions on halfspaces also strengthen classical results of Spencer, Stein and Carleson.

1 Introduction

Let $D$ be the unit disc of the complex plane, $T$ be the unit circle, and $\lambda_n$ denote Lebesgue measure on $\mathbb{R}^n$ ($n \geq 1$), where we identify $\mathbb{C}$ with $\mathbb{R}^2$, and $T$ with $[0, 2\pi)$, in the usual way. By a Stolz angle at a point $\zeta$ of $T$ we mean an open triangular subset of $D$ that has a vertex at $\zeta$ and is symmetric about the diameter of $D$ through $\zeta$. A fundamental result concerning the boundary behaviour of holomorphic functions is as follows (see the original paper [15] or any of the books [4], [16], [9]).

Theorem A (Plessner) Let $f$ be a holomorphic function on $D$. Then, for $\lambda_1$-almost every point $\zeta$ of $T$, either

(i) $f$ has a finite nontangential limit at $\zeta$, or

(ii) $f(S) = \mathbb{C}$ for every Stolz angle $S$ at $\zeta$.

Although this result describes a stark dichotomy in boundary behaviour, it has been a long-standing open question whether condition (ii) can be significantly strengthened. For example, Collingwood, one of the authors of the standard text [4] on cluster sets, asked over 50 years ago whether the statement that $f(S) = \mathbb{C}$ can be replaced by the much stronger assertion that $\lambda_2(\mathbb{C}\setminus f(S)) = 0$ (Problem 5.20 in [10] or [11]; see also Problem 5.57 in [11]). Below we give a different substantial improvement of Plessner’s theorem. We denote the circle $\{z \in \mathbb{C} : |z - w| = r\}$ by $C_{w,r}$.

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Theorem 1 Let \( f \) be a holomorphic function on \( \mathbb{D} \). Then, for \( \lambda_1 \)-almost every point \( \zeta \) of \( \mathbb{T} \), either

(i) \( f \) has a finite nontangential limit at \( \zeta \), or

(ii) for every Stolz angle \( S \) at \( \zeta \),

\[
\int_{S \cap f^{-1}(C_{w,r})} |f'(z)| \, |dz| = \infty
\]

for \( \lambda_3 \)-almost every \((w, r) \in \mathbb{C} \times (0, \infty)\).

The integral in (1) measures the total arc length of the image of \( S \cap f^{-1}(C_{w,r}) \) under \( f \), taking account of multiplicities. Thus, although this image is contained in the circle \( C_{w,r} \), condition (ii) makes the striking assertion that its length, counting multiplicities, is infinite for almost every choice of \((w, r)\).

Plessner’s theorem holds more generally for meromorphic functions \( f \) on \( \mathbb{D} \), and the same is true of Theorem 1. We will explain at the end of the proof of Theorem 1 how the argument can be adapted to cover meromorphic functions as well.

We next recall a further classical result concerning the boundary behaviour of holomorphic functions (Theorem 5 in [17]; see also Theorem X.1.3 in [9], and p. 364 of the survey article [19] for its wider significance).

Theorem B (Spencer) Let \( f \) be a holomorphic function on \( \mathbb{D} \). Then, for \( \lambda_1 \)-almost every point \( \zeta \) of \( \mathbb{T} \), either

(i) \( f \) has a finite nontangential limit at \( \zeta \), or

(ii) for every Stolz angle \( S \) at \( \zeta \),

\[
\int_{S} \frac{|f'|^2}{1 + |f|^2} \, d\lambda_2 = \infty.
\]

If \( f \) and \( S \) are as above, then the co-area formula (see Section 3.4.3 of [5], or Section 1.2.4 of [13]) tells us that

\[
\int_{S \cap \{a < |f-w| < b\}} |f|^2 \, d\lambda_2 = \int_{(a,b)} \int_{S \cap \{|f-w|=t\}} |f'(z)| \, |dz| \, d\lambda_1(t)
\]

whenever \( w \in \mathbb{C} \) and \( 0 \leq a < b \). Hence Theorem 1 is also much stronger than Spencer’s result. In particular, condition (ii) of Theorem 1 clearly implies that, for every Stolz angle \( S \),

\[
\int_{S \cap \{|f-w| < r\}} |f|^2 \, d\lambda_2 = \infty \quad (w \in \mathbb{C}, r > 0).
\]

(This last integral measures the total area of the image of \( S \cap \{|f-w| < r\} \) under \( f \), counting multiplicities.)
Theorem 1 and its generalization to meromorphic functions, will be proved in the next section, after which we will briefly discuss its application to the theory of universal series. The final section of the paper presents an analogue of Theorem 1 for harmonic functions on halfspaces, which strengthens well known results of Carleson and Stein.

2 Proof of Theorem 1

We define nontangential approach regions at points \( \zeta \) of \( T \) by

\[
S(\zeta, \delta) = \left\{ z \in \mathbb{D} : \sqrt{1 - \delta^2}|z - \zeta| < 1 - |z| < \delta \right\} \quad (0 < \delta < 1).
\]

Let \( f \) be a holomorphic function on \( \mathbb{D} \). Then it is well known that the set of points in \( T \) at which \( f \) has a finite nontangential limit is a Borel subset of \( T \). It follows easily from the lemma below that the same can also be said of the set of points \( \zeta \) in \( T \) for which condition (ii) of Theorem 1 holds. Let

\[
L_j(\zeta, w, r) = \int_{S(\zeta, j^{-1}) \cap f^{-1}(C_{w, r})} |f'(z)| \, |dz| \quad (\zeta \in T, w \in \mathbb{C}, r > 0) \quad (2)
\]

for each \( j \in \{2, 3, \ldots\} \).

**Lemma 2** The function \( L_j \) is Borel measurable on \( T \times \mathbb{C} \times (0, \infty) \) for each \( j \in \{2, 3, \ldots\} \).

**Proof.** We dismiss the trivial case where \( f \) is constant and so \( L_j \equiv 0 \). Let

\[
Z = \{ z \in \mathbb{D} : f'(z) = 0 \}
\]

and then let

\[
L_j^{(m)}(\zeta, w, r) = \int_{S(\zeta, j^{-1}) \cap A_m \cap f^{-1}(C_{w, r})} |f'(z)| \, |dz| \quad (\zeta \in T, w \in \mathbb{C}, r > 0).
\]

Since \( L_j(\zeta, w, r) = \lim_{m \to \infty} L_j^{(m)}(\zeta, w, r) \), it will be enough to show that \( L_j^{(m)} \) is Borel measurable on \( T \times \mathbb{C} \times (0, \infty) \) for any \( m \).

We now fix both \( j \) and \( m \). For any open set \( U \) such that \( \overline{U} \subset \mathbb{D} \setminus Z \), we define

\[
T_U(\zeta, w, r) = \int_{S(\zeta, j^{-1}) \cap A_m \cap f^{-1}(C_{w, r})} |f'(z)| \, |dz|.
\]

Let \( W \) be an open set satisfying \( \overline{W} \subset S(\zeta, j^{-1}) \cap A_m \cap U \). If \((\zeta_k, w_k, r_k) \to (\zeta, w, r)\) in \( T \times \mathbb{C} \times (0, \infty) \), then there exists \( k_0 \) such that

\[
\overline{W} \subset S(\zeta, j^{-1}) \cap A_m \cap U \quad (k \geq k_0).
\]
Further, if \(f|_U\) is injective, then \(\int_{W \cap f^{-1}(C_{w,r})} |f'(z)||dz|\) is the total arc length of \(f(W) \cap C_{w,r}\), and so

\[
\liminf_{k \to \infty} T_U(\zeta_k, w_k, r_k) \geq \liminf_{k \to \infty} \int_{W \cap f^{-1}(C_{w_k,r_k})} |f'(z)||dz| \geq \int_{W \cap f^{-1}(C_{w,r})} |f'(z)||dz|.
\]

On enlarging \(W\) we see that

\[
\liminf_{k \to \infty} T_U(\zeta_k, w_k, r_k) \geq T_U(\zeta, w, r),
\]

whence \(T_U\) is lower semicontinuous on \(T \times C \times (0, \infty)\).

Now let \(U = U_1 \cup U_2\), where \(U_1, U_2\) are open sets such that \(\overline{U}_i \subset \mathbb{D} \setminus Z\) and \(f|_{U_i}\) is injective for each \(i\). Then

\[
T_U(\zeta, w, r) = T_{U_1}(\zeta, w, r) + T_{U_2}(\zeta, w, r) - T_{U_1 \cap U_2}(\zeta, w, r),
\]

so \(T_U\) is Borel measurable. Similarly, \(T_U\) is Borel measurable when \(U\) is any finite union of such open sets \(U_i\).

Since \(f\) is locally injective on \(\mathbb{D} \setminus Z\), we may, by compactness, choose open sets \(U_1, ..., U_l\) such that \(\overline{U}_i \subset \mathbb{D} \setminus Z\) and \(f|_{U_i}\) is injective for each \(i\), and also \(A_m \subset U\), where \(U = \bigcup_{i=1}^{l} U_i\). Then \(L_j^{(m)}(\zeta, w, r) = T_U(\zeta, w, r)\) and so, by the previous paragraph, \(L_j^{(m)}\) is Borel measurable on \(T \times C \times (0, \infty)\), as required.

**Proof of Theorem 1** Let \(f\) be a holomorphic function on \(\mathbb{D}\). We may assume that \(f\) is nonconstant. The above lemma tells us that the function \(L_j\) defined by (2) is Borel measurable on \(T \times C \times (0, \infty)\) for each \(j\). Further, the sequence \((L_j)\) is decreasing. The set of all points \(\zeta\) in \(T\) satisfying condition (ii) of Theorem 1 is given by \(\bigcap_j E_j\), where

\[
E_j = \left\{ \zeta \in T : \int_{C \times (0, \infty)} \chi_{\{L_j < \infty\}}(\zeta, w, r)\, d\lambda_3(w, r) = 0 \right\}, \quad (j \geq 2),
\]

and \((E_j)\) is a decreasing sequence. The sets \(E_j\) are Borel because the functions \(L_j\) are Borel. Let \(F\) be the Borel set of all points in \(T\) at which \(f\) has a finite nontangential limit. Theorem 1 will follow if we can show that \(\lambda_1(T \setminus (E_j \cup F)) = 0\) for each \(j\). We now suppose, for the sake of contradiction, that there exists \(j\) such that \(\lambda_1(B_1) > 0\), where \(B_1 = T \setminus (E_j \cup F)\). We proceed below with this particular choice of \(j\).

For each \(\zeta \in B_1\) we know that

\[
\int_{C \times (0, \infty)} \chi_{\{L_j < \infty\}}(\zeta, w, r)\, d\lambda_3(w, r) > 0.
\]
Hence we can choose $\rho > 0$ large enough to ensure that $\lambda_1(B_2) > 0$, where

$$B_2 = \left\{ \zeta \in B_1 : \int_{\mathbb{C} \times (0, \infty)} \chi_{\{L_j \leq \rho\}}(\zeta, w, r) \, d\lambda_3(w, r) > 0 \right\}.$$  

Let

$$A(w, r) = \{ \zeta \in B_2 : L_j(\zeta, w, r) \leq \rho \} \quad (w \in \mathbb{C}, r > 0).$$

Then

$$\int_{\mathbb{C} \times (0, \infty)} \lambda_1(A(w, r)) \, d\lambda_3(w, r) = \int_{\mathbb{C} \times (0, \infty)} \int_{B_2} \chi_{\{L_j \leq \rho\}}(\zeta, w, r) \, d\lambda_1(\zeta) \, d\lambda_3(w, r)$$

by Tonelli’s theorem and the choice of $B_2$. In particular, there exist $w_0 \in \mathbb{C}$ and $r_0 > 0$ such that $\lambda_1(A(w_0, r_0)) > 0$. For each $\zeta \in A(w_0, r_0)$ we know that

$$\int_{S(\zeta, j^{-1}) \cap f^{-1}(C_{w_0, r_0})} |f'(z)| \, |dz| \leq \rho. \quad (3)$$

We define arcs in $\mathbb{T}$ by

$$I(z, \delta) = \{ \eta \in \mathbb{T} : z \in S(\eta, \delta) \} \quad (0 < \delta < 1, 1 - \delta < |z| < 1).$$

Thus $\zeta \in I(z, \delta)$ whenever $z \in S(\zeta, \delta)$, and

$$\frac{\lambda_1(I(z, \delta))}{1 - |z|} \to \frac{2\delta}{\sqrt{1 - \delta^2}} \quad (|z| \to 1-). \quad (4)$$

It follows from the Lebesgue density theorem that, for $\lambda_1$-almost every point $\zeta$ of $A(w_0, r_0)$,

$$\liminf_{|z| \to 1-} \frac{\lambda_1(I(z, j^{-1}) \cap A(w_0, r_0))}{\lambda_1(I(z, j^{-1}))} \geq \frac{1}{2}$$

and so

$$\liminf_{|z| \to 1-} \frac{\lambda_1(I(z, j^{-1}) \cap A(w_0, r_0))}{1 - |z|} \geq \frac{j^{-1}}{\sqrt{1 - j^{-2}}} = \frac{1}{\sqrt{j^2 - 1}}$$

by (4). Since $\lambda_1(A(w_0, r_0)) > 0$, we can choose $\varepsilon \in (0, j^{-1})$ and a subset $A_0$ of $A(w_0, r_0)$ such that $\lambda_1(A_0) > 0$ and

$$\lambda_1(I(z, j^{-1}) \cap A(w_0, r_0)) \geq j^{-1}(1 - |z|) \quad (1 - \varepsilon < |z| < 1, z \in \Omega). \quad (5)$$
where 
\[ \Omega = \bigcup_{\zeta \in A_0} S(\zeta, j^{-1}). \]

We obtain a measure on \( \mathbb{D} \) by defining 
\[ \mu = \Delta \log^+ \frac{|f - w_0|}{r_0} \]
in the sense of distributions. It has support in the set \( I = \{|f - w_0| = r_0\} \).

Also, \( \| \nabla \log |f - w_0| \| = |f'| / |f - w_0| \) wherever \( f \neq w_0 \), and \( \nabla \log |f - w_0| \) is normal to \( I \) on the set \( \{z \in I : f'(z) \neq 0\} \). Given any test function \( \phi \in C_0^\infty(\mathbb{D}) \), we can choose \( s \in (0, 1) \) so that the support of \( \phi \) is contained in \( \{z : |z| < s\} \), and apply Green’s identity on the open set 
\[ U = \{z : |z| < s \text{ and } |f(z) - w_0| > r_0\} \]
to see that 
\[
\left( \Delta \log^+ \frac{|f - w_0|}{r_0} \right)(\phi) = \int_U \left( \log^+ \frac{|f - w_0|}{r_0} \right) \Delta \phi \, d\lambda_2
= \int_{\{|f-w_0|=r_0\}} \phi(z) \| \nabla \log |f - w_0| (z) \| |dz|
= \frac{1}{r_0} \int_{\{|f-w_0|=r_0\}} \phi(z) |f'(z)| |dz|.
\]

Hence
\[
\mu(J) = \frac{1}{r_0} \int_{J \cap f^{-1}(C_{w_0, r_0})} |f'(z)| |dz| \quad \text{for any open set } J \subset \mathbb{D}. \tag{6}
\]

Further, by \( \mathcal{L} \),
\[
\int_{S(\zeta, j^{-1})} d\mu = \frac{1}{r_0} \int_{S(\zeta, j^{-1}) \cap f^{-1}(C_{w_0, r_0})} |f'(z)| |dz| \leq \frac{\rho}{r_0} \quad (\zeta \in A(w_0, r_0)).
\]

We now see that
\[
\frac{\rho}{r_0} \lambda_1(A(w_0, r_0)) \geq \int_{A(w_0, r_0)} \int_{S(\zeta, j^{-1})} d\mu(\xi) \lambda_1(\zeta)
= \int_{A(w_0, r_0)} \int_{\Omega} \chi_{S(\zeta, j^{-1})}(\xi) \, d\mu(\xi) \lambda_1(\zeta)
= \int_{\Omega} \int_{A(w_0, r_0)} \chi_{S(\zeta, j^{-1})}(\xi) \, d\lambda_1(\zeta) \, d\mu(\xi)
= \int_{\Omega} \lambda_1(I(\xi, j^{-1}) \cap A(w_0, r_0)) \, d\mu(\xi)
\geq j^{-1} \int_{\Omega \cap \{1-\varepsilon < |\xi| < 1\}} (1 - |\xi|) \, d\mu(\xi),
\]

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by (5). In particular, this last integral is finite, so we can define the Green potential

\[ u(z) = \int_{\Omega} G_D(z, \xi) \, d\mu(\xi) \quad (z \in \mathbb{D}), \]

where \( G_D(\cdot, \cdot) \) denotes the Green function for the unit disc (see Theorem 4.2.5(ii) of [1]).

We know (see Corollary 4.3.3 and Theorems 4.3.8(i) and 4.3.5 of [1]) that the function

\[ h(z) = \log^+ \frac{|f(z) - w_0|}{r_0} + \frac{u(z)}{2\pi} \quad (z \in \mathbb{D}) \quad (7) \]

is harmonic on \( \Omega \), and clearly \( h > 0 \). We can now combine a standard conformal mapping argument (see the proof of Theorem VI.2.2 in [9]) with Fatou’s theorem for positive harmonic functions on \( \mathbb{D} \) (Theorem 4.6.7 in [1]) to see that \( h \) has a finite nontangential limit at \( \lambda_1 \)-almost every point of \( A_0 \).

Since \( h \geq \log(|f - w_0|/r_0) \), it follows that \( f \) is nontangentially bounded \( \lambda_1 \)-almost everywhere on \( A_0 \), and so \( f \) has finite nontangential limits \( \lambda_1 \)-almost everywhere on \( A_0 \) by a theorem of Privalov (Theorem VI.2.2 of [9]). However,

\[ A_0 \subset A(w_0, r_0) \subset B_2 \subset B_1 = \mathbb{T}\backslash(E_j \cup F) \subset \mathbb{T}\backslash F, \]

so \( f \) cannot have a finite nontangential limit at any point of \( A_0 \), and \( \lambda_1(A_0) > 0 \). The assumption that \( \lambda_1(B_1) > 0 \) has thus led to a contradiction. We conclude that \( \lambda_1(\mathbb{T}\backslash(E_j \cup F)) = 0 \) for every \( j \), and so the theorem is established.

\[ \square \]

Remark Let \( f \) be a holomorphic function on \( \mathbb{D} \), and \( Y \) be a countable subset of \( \mathbb{C} \times (0, \infty) \). Then, for \( \lambda_1 \)-almost every point \( \zeta \) of \( \mathbb{T} \), either (i) \( f \) has a finite nontangential limit at \( \zeta \), or (ii) for every Stolz angle \( S \) at \( \zeta \) and every pair \( (w, r) \) in \( Y \) equation (1) holds. To see this, it is enough to consider the case where \( Y \) is a singleton \( \{(w_0, r_0)\} \). We can then follow the outline of the proof of Theorem [1] provided that we now define

\[ E_j = \{ \zeta \in \mathbb{T} : L_j(\zeta, w_0, r_0) = \infty \} \]

and choose \( \rho > 0 \) large enough so that \( \lambda_1(A(w_0, r_0)) > 0 \), where

\[ A(w_0, r_0) = \{ \zeta \in \mathbb{T}\backslash(E_j \cup F) : L_j(\zeta, w_0, r_0) \leq \rho \}. \]

This variant of Theorem [1] also implies Plessner’s theorem, since we may choose \( Y \) to be dense in \( \mathbb{C} \times (0, \infty) \).

Extension to meromorphic functions. Now suppose that \( f \) is merely meromorphic on \( \mathbb{D} \). We note that Lemma [2] extends easily to this case, and we follow the proof of Theorem [1] as far as the sentence containing (6). Then

\[ \Delta \log^+ \frac{|f - w_0|}{r_0} = \mu - 2\pi \mu_1, \]

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where $\mu$ again satisfies (6) and $\mu_1$ is the measure which counts the poles of $f$ according to multiplicity. The function $h$ in (7) now satisfies $\Delta h = -2\pi \mu_1$ on $\Omega$, and so is superharmonic there. The conformal mapping argument that we used previously thus yields a positive superharmonic function $v$ on $\mathbb{D}$ such that $(-\Delta v)/(2\pi)$ is a sum of Dirac measures. By the Riesz decomposition theorem (Theorem 4.4.1 of [1]) and Theorem 4.2.5(ii) of [1], the discrete measure $-\Delta v$ satisfies the Blaschke condition on $\mathbb{D}$ and $v$ has the form $v = h_1 - \log |B|$, where $h_1$ is a positive harmonic function on $\mathbb{D}$ and $B$ is a Blaschke product associated with $(-\Delta v)/(2\pi)$. Hence $v$ has a finite nontangential limit $\lambda_1$-almost every point of $A_0$. The proof concludes as before.

3 Application to universal series

A power series $\sum a_n z^n$ with radius of convergence 1 is said to belong to the collection $\mathcal{U}$ if, for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement and every continuous function $g : K \to \mathbb{C}$ that is holomorphic on $K^c$, there is a subsequence $(m_k)$ of the natural numbers such that $\sum_{n=0}^{m_k} a_n z^n \to g(z)$ uniformly on $K$. Members of $\mathcal{U}$ are called universal Taylor series, and such functions have been widely studied in recent years. Nestoridis [14] showed that this universal approximation property is a generic feature of holomorphic functions on the unit disc; more precisely, he proved that $\mathcal{U}$ is a dense $G_\delta$ subset of the space of all holomorphic functions on $\mathbb{D}$ endowed with the topology of local uniform convergence.

In Theorem 2 of [6] nontrivial potential theoretic arguments were used to show that universal Taylor series cannot have nontangential limits at a boundary set of positive measure, and so must satisfy condition (ii) of Plessner’s theorem at $\lambda_1$-almost every point $\zeta$ of $\mathbb{T}$. If we substitute Theorem 1 of this paper for Plessner’s theorem in the proof given in [6], we immediately obtain the following improvement. It implies, in particular, that a generic property of holomorphic functions on $\mathbb{D}$ is that condition (ii) of Theorem 1 holds for almost every $\zeta$ in $\mathbb{T}$.

**Corollary 3** If $f \in \mathcal{U}$, then for $\lambda_1$-almost every point $\zeta$ of $\mathbb{T}$ equation (7) holds for every Stolz angle $S$ at $\zeta$ and $\lambda_3$-almost every $(w, r) \in \mathbb{C} \times (0, \infty)$.

The same improvement may be made to Corollary 2 of [7], which generalizes Theorem 2 of [6], and to Corollary 5 of [8], which establishes the corresponding boundary behaviour of universal Dirichlet series (in this case we would use the obvious analogue of Theorem 1 for holomorphic functions in a halfplane).
4 Boundary behaviour of harmonic functions

We will now present an analogue of Theorem 1 for harmonic functions on the halfspace $H = \{(x_1, ..., x_N) \in \mathbb{R}^N : x_N > 0\}$, where $N \geq 2$. By a Stolz domain at $y \in \partial H$ we mean a truncated cone in $H$ that meets $\partial H$ precisely at its vertex $y$ and with its axis normal to $\partial H$. We will consider $\lambda_{N-1}$ as a measure on $\partial H$ by identifying this set with $\mathbb{R}^{N-1}$ in the obvious way.

If $h$ is a harmonic function on $H$, then a result of Carleson [3] is equivalent to saying that, for $\lambda_{N-1}$-almost every point $y$ of $\partial H$, either (i) $h$ has a finite nontangential limit at $y$, or (ii) $h(S) = \mathbb{R}$ for every Stolz domain $S$ at $y$. Another well-known result (Theorem 1 of [18]) may be formulated as follows.

**Theorem C (Stein)** Let $h$ be a harmonic function on $H$. Then, for $\lambda_{N-1}$-almost every point $y$ of $\partial H$, either

(i) $h$ has a finite nontangential limit at $y$, or

(ii) for every Stolz domain $S$ at $y$,

$$\int_S x_2^{N-N} \|\nabla h(x)\|^2 \, d\lambda_N(x) = \infty.$$ 

Let $\sigma$ denote surface area measure on level sets of (nonconstant) harmonic functions. As before, the co-area formula shows that, for $h$ and $S$ as above,

$$\int_{S \cap \{a < h < b\}} x_2^{N-N} \|\nabla h(x)\|^2 \, d\lambda_N(x) = \int_{(a,b)} \int_{S \cap \{h = t\}} x_2^{N-N} \|\nabla h(x)\| \, d\sigma(x) \, d\lambda_1(t)$$

whenever $a < b$. Hence the following analogue of Theorem 1 simultaneously strengthens the results of both Carleson and Stein.

**Theorem 4** Let $h$ be a harmonic function on $H$. Then, for $\lambda_{N-1}$-almost every point $y$ of $\partial H$, either

(i) $h$ has a finite nontangential limit at $y$, or

(ii) for every Stolz domain $S$ at $y$,

$$\int_{S \cap h^{-1}(\{t\})} x_2^{N-N} \|\nabla h(x)\| \, d\sigma(x) = \infty$$

for $\lambda_1$-almost every $t \in \mathbb{R}$.

**Outline proof.** The proof of Theorem 4 is similar in pattern to that of Theorem 1 and simpler in some respects. We will therefore only give an outline as a guide to the reader, and indicate a few points of difference. Let

$$S_N(y, \delta) = \{x \in H : \sqrt{1 - \delta^2} \|x - y\| < x_N < \delta\} \quad (y \in \partial H, 0 < \delta < 1).$$
For each \( j \geq 2 \) we define the Borel function
\[
M_j(y, t) = \int_{S_N(y, j^{-1} \cap h^{-1}(t))} x_N^{2-N} \|\nabla h(x)\| \, d\sigma(x) \quad (y \in \partial \mathbb{H}, t \in \mathbb{R}).
\]
The implicit function theorem allows us to express the set \( h^{-1}(\{t\}) \) locally on \( \mathbb{H} \setminus \{\nabla h = 0\} \) in the form
\[
x_p = g(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_N, t)
\]
for some \( p \in \{1, \ldots, N\} \), where \( g \) is continuously differentiable and \( p \) is chosen so that \( \partial h/\partial x_p \neq 0 \). We can then argue as in Lemma 2 to see that \( M_j \) is Borel measurable on \( \partial \mathbb{H} \times \mathbb{R} \). Thus the sets
\[
E_j = \left\{ y \in \partial \mathbb{H} : \int_{\mathbb{R}} \chi_{(M_j < \infty)}(y, t) \, d\lambda_1(t) = 0 \right\} \quad (j \geq 2)
\]
are Borel. The set \( F \) of all boundary points at which \( h \) has a finite non-tangential limit is also Borel. We suppose that there exists \( j \) such that \( \lambda_{N-1}(B_1) > 0 \), where \( B_1 = \partial \mathbb{H} \setminus (E_j \cup F) \), and seek a contradiction.

Let \( \rho > 0 \) be large enough so that \( \lambda_{N-1}(B_2) > 0 \), where
\[
B_2 = \left\{ y \in B_1 : \int_{\mathbb{R}} \chi_{(M_j \leq \rho)}(y, t) \, d\lambda_1(t) > 0 \right\},
\]
and define
\[
A(t) = \{ y \in B_2 : M_j(y, t) \leq \rho \} \quad (t \in \mathbb{R}).
\]
Then
\[
\int_{\mathbb{R}} \lambda_{N-1}(A(t)) \, d\lambda_1(t) = \int_{\mathbb{R}} \int_{B_2} \chi_{(M_j \leq \rho)}(y, t) \, d\lambda_{N-1}(y) \, d\lambda_1(t)
\]
\[
= \int_{B_2} \int_{\mathbb{R}} \chi_{(M_j \leq \rho)}(y, t) \, d\lambda_1(t) \, d\lambda_{N-1}(y) > 0,
\]
so \( \lambda_{N-1}(A(t_0)) > 0 \) for some \( t_0 \). We choose a bounded subset \( A_1(t_0) \) of \( A(t_0) \) such that \( \lambda_{N-1}(A_1(t_0)) > 0 \). For each \( y \in A_1(t_0) \) we know that
\[
\int_{S_N(y, j^{-1} \cap h^{-1}(t_0))} x_N^{2-N} \|\nabla h(x)\| \, d\sigma(x) \leq \rho. \quad (8)
\]
If we define
\[
I_N(x, \delta) = \{ y \in \partial \mathbb{H} : x \in S_N(y, \delta) \} \quad (0 < \delta < 1, 0 < x_N < \delta),
\]
then \( \lambda_{N-1} \left( I_N(x, j^{-1}) \right) \) is proportional to \( x_N^{N-1} \) and, as before, there is a positive constant \( C(N, j) \) such that
\[
\liminf_{x_N \to 0^+} \frac{\lambda_{N-1} \left( I_N(x, j^{-1}) \cap A_1(t_0) \right)}{x_N^{N-1}} \geq C(N, j)
\]

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for $\lambda_{N-1}$-almost every point $y$ of $A_1(t_0)$. We can choose $\varepsilon \in (0,j^{-1})$ and $A_0 \subset A_1(t_0)$ such that $\lambda_{N-1}(A_0) > 0$ and

$$
\lambda_{N-1} \left( I_N(x,j^{-1}) \cap A_1(t_0) \right) \geq \frac{C(N,j)}{2} x_N^{N-1} \quad (0 < x_N < \varepsilon, x \in \Omega),
$$

(9)

where

$$
\Omega = \bigcup_{y \in A_0} S_N(y,j^{-1}).
$$

We obtain a measure on $\mathbb{H}$ by defining $\mu = \Delta(h-t_0)^+$ in the sense of distributions, and then use Green’s identity to see that

$$
\mu(J) = \int_{J \cap h^{-1}(t_0)} \|\nabla h(x)\| \ d\sigma(x) \quad \text{for any open set } J \subset \mathbb{H}.
$$

(When $N \geq 3$ the set where $\nabla h = 0$ need no longer be discrete, but it is contained in an $(N-2)$-dimensional manifold: see pp. 716, 717 of [12].)

From (8) and (9) we see that

$$
\rho \lambda_{N-1}(A_1(t_0)) \geq \int_{A_1(t_0)} \int_{S_N(y,j^{-1})} x_N^{2-N} \ d\mu(x) d\lambda_{N-1}(y)
$$

$$
\geq \int_{\Omega} x_N^{2-N} \int_{A_1(t_0)} \chi_{S_N(y,j^{-1})}(x) \ d\lambda_{N-1}(y) d\mu(x)
$$

$$
\geq \frac{C(N,j)}{2} \int_{\Omega} x_N \ d\mu(x),
$$

and so we can form the Green potential $u$ in $\mathbb{H}$ of the measure $\mu|_{\Omega}$ (see Theorem 4.2.5(iii) in [1]).

As before (by Corollary 4.3.3 and Theorems 4.3.8(i) and 4.3.5 of [1]), the function

$$
(h-t_0)^+ + \frac{u}{\sigma_N \max\{1, N-2\}}
$$

where $\sigma_N$ denotes the surface area of the unit sphere in $\mathbb{R}^N$, is positive and harmonic on $\Omega$. Although the conformal mapping argument that we used previously is no longer available, it remains true that any positive harmonic function on $\Omega$ has a finite nontangential limit at $\lambda_{N-1}$-almost every point of $A_0$, by a result of Brelot and Doob (Théorème 10 of [2]). Hence we can proceed, as in the proof of Theorem 1, to obtain a contradiction.

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