q-Quaternions and q-deformed su(2) instantons

Gaetano Fiore,

Dip. di Matematica e Applicazioni, Università “Federico II”
V. Claudio 21, 80125 Napoli, Italy
I.N.F.N., Sezione di Napoli,
Complesso MSA, V. Cintia, 80126 Napoli, Italy

Abstract

We construct (anti)instanton solutions of a would-be q-deformed su(2) Yang-Mills theory on the quantum Euclidean space $R_q^4$ [the $SO_q(4)$-covariant noncommutative space] by reinterpreting the function algebra on the latter as a q-quaternion bialgebra. Since the (anti)selfduality equations are covariant under the quantum group of deformed rotations, translations and scale change, by applying the latter we can generate new solutions from the one centered at the origin and with unit size. We also construct multi-instanton solutions. As they depend on noncommuting parameters playing the roles of ‘sizes’ and ‘coordinates of the centers’ of the instantons, this indicates that the moduli space of a complete theory should be a noncommutative manifold. Similarly, gauge transformations should be allowed to depend on additional noncommutative parameters.
1 Introduction

The search for instantonic solutions has become a central point of investigation of Yang-Mills gauge theories on noncommutative manifolds after the discovery [38] that deforming $\mathbb{R}^4$ into the Moyal-Weyl noncommutative Euclidean space regularizes the zero-size singularities of the instanton moduli space (see also [47]). Other noncommutative geometries have been considered, mostly deformations [11, 5, 12, 31] of the sphere $S^4$, because the latter, as a compactification of $\mathbb{R}^4$, provides a better framework to display the topological properties of the instanton bundles. It is therefore tempting to investigate this issue also on another available deformation of $\mathbb{R}^4$, the Faddeev-Reshetikhin-Takhtadjan noncommutative Euclidean space $\mathbb{R}^4_q$ covariant under $SO_q(4)$ [15].

At least in our opinion, there is still no fully satisfactory formulation of gauge field theory on quantum group covariant noncommutative spaces (shortly: quantum spaces) like $\mathbb{R}^4_q$ (see e.g. [28] for an attempt). One main reason is the lack of a proper (i.e. cyclic) trace to define gauge-invariant observables (action, etc). Another one is the $\star$-structure of the differential calculus, which for real $q$ is problematic. Probably a satisfactory formulation will be possible within a generalization of the standard framework of noncommutative geometry [9] where gauge transformations, gauge potentials, and the corresponding field strengths will depend not only on coordinate, but also on derivatives (as suggested e.g. in [13, 1]) and/or possibly on additional noncommuting parameters (see section 6 below). Here we leave these issues aside and just ask for nontrivial differential 2-forms solutions of the deformed (anti)selfduality equations: results in this direction might contribute to suggest more general formulations of gauge theories on noncommutative manifolds that include quantum spaces.

As known, the search and classification [3] of Yang-Mills instantons on $\mathbb{R}^4$ is greatly simplified when the latter is endowed with the structure of a quaternion algebra $\mathbb{H}$. Therefore, following the undeformed case, we first (section 2) introduce a notion of a $q$-quaternion as a $2 \times 2$ matrix which can be factorized as the product of the defining matrix of $SU_q(2)$ by an element of a semigroup isomorphic to the semigroup $\mathbb{R}_{\geq}$ of nonnegative real numbers, and reformulate the algebra $\mathcal{A}$ of functions on $\mathbb{R}^4_q$ as a $\star$-bialgebra $C(\mathbb{H}_q)$. The bialgebra structure encodes the property that the product of two quaternions is a quaternion and is inherited from the bialgebra of $2 \times 2$ quantum matrices [14, 16, 54, 15] (therefore it differs from the proposal in [36]). We shall give more details and further developments in Ref. [23]. It also turns out that our $\star$-algebra $\mathcal{A}$ and the $C^*$-algebra of functions on the quantum 4-sphere of Ref. [12] are made to be isomorphic (as $\star$-algebras) if they are slightly extended so as to contain suitable rational functions of their respective central elements; therefore that noncommutative sphere can be regarded as a compactification of $\mathcal{A}$. In section 3 we reformulate in $q$-quaternion language the $SO_q(4)$-covariant
differential calculus [this turns out to coincide with the bicovariant differential calculus on $M_q(2), GL_q(2)$, and after imposing the unit $q$-determinant condition with the Woronowicz 4D-bicovariant differential calculus on $C(SU_q(2))$, the $SO_q(4)$-covariant $q$-epsilon tensor and Hodge map on $\Omega^*(R^4_q)$. In section 4 we recall some basic notions about the standard framework for gauge theories on noncommutative spaces, pointing out where it doesn’t fit the present model, and we formulate (anti)selfduality equations. In section 5 we find a large family of solutions $A$ of the (anti)self-duality equations in the form of $2 \times 2$ matrices both in the “regular” and in the “singular gauge”. There is a larger indeterminacy than in the undeformed theory because we are not yet able to formulate and impose the correct antihermiticity condition on the gauge potential. Among the solutions there are some distinguished choices that closely resemble (in $q$-quaternion language) their undeformed counterparts (instantons and anti-instantons) in $su(2)$ Yang-Mills theory on $R^4$. The (still missing) complete gauge theory might however be a deformed $u(2)$ rather than $su(2)$ Yang-Mills theory. We also make contact with the today standard formulation of gauge theory on noncommutative spaces based on the identification of vector bundles on the latter with projective modules over $A$ by constructing in $q$-quaternion language the hermitean projector associated to the $q$-deformed instanton projective module, and we find that it coincides (for a specific choice of the instanton size parameter) with the one found in Ref. As in the undeformed (and in the Nekrasov-Schwarz) case, applying the quantum group $SO_q(4)$ of $q$-deformed dilatations and the braided group of $q$-deformed translations one finds gauge inequivalent solutions. The difference is however that a dependence on additional noncommutative parameters is introduced: this global gauge transformation depends on the noncommuting “coordinates of the center” of the (anti)instanton. Finally (section 7), we find first $n$-instantons solutions in the “singular” gauge for any integer $n$; the construction procedure is not yet the deformed analog of the general ADHM one, but rather of the procedure initiated in and developed in, which reduces to the determination of a suitable scalar potential, expressed in quaternion language. Then for $n = 1, 2$ we transform the singular solutions into “regular” solutions by “singular gauge transformations”, as in the undeformed case (of course the $n = 1$ regular instanton solution is again the one found in section 4). The solutions are parametrized by noncommuting parameters playing the role of “sizes” and “coordinates of the centers” of the (anti)instantons. This indicates that the moduli space of a complete theory will be a noncommutative manifold.
2 Promoting $C\left(\mathbb{R}_q^4\right)$ to the $q$-quaternion bi- (or Hopf) algebra $C\left(\mathbb{H}_q\right)$

We start by recalling how the (undeformed) quaternion $\star$-algebra $\mathbb{H}$ can be formulated in terms of $2 \times 2$ matrices: any $X \in \mathbb{H}$ is given by

$$X = x_1 + x_2 i + x_3 j + x_4 k,$$

with $x \in \mathbb{R}^4$ and imaginary $i, j, k$ fulfilling

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1.$$

One refers to $x_1$ and to the following three terms as to the ‘real’ and ‘imaginary’ part of $X$ respectively. Replacing $i, j, k$ by Pauli matrices times the imaginary unit $i$ we can associate to $X$ a matrix

$$X \leftrightarrow x \equiv \begin{pmatrix} x_1 + x_4 i & x_3 + x_2 i \\ -x_3 + x_2 i & x_1 - x_4 i \end{pmatrix} =: \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

(where $\alpha, \gamma \in \mathbb{C}$). The quaternionic product becomes represented by matrix multiplication, and the quaternionic conjugation becomes represented by hermitean conjugation of the matrix $x$. Therefore $\mathbb{H}$ can be seen also as the subalgebra of $M_2(\mathbb{C})$ consisting of all complex $2 \times 2$ matrices of this form. Since the determinant of any $x$ is nonnegative,

$$|x|^2 \equiv \det(x) = |a|^2 + |\gamma|^2 \geq 0,$$

any $x$ can be factorized in the form

$$x = T|x|,$$

where $T \in SU(2)$ and $|x|$ belongs to the semigroup $\mathbb{R}_{\geq}$ of nonnegative real numbers. Hence any $x$ belongs also to the semigroup $SU(2) \times \mathbb{R}_{\geq}$.

We $q$-deform this just replacing $SU(2)$ by $SU_q(2)$ in the dual picture of the algebra of functions of the matrix elements of $x$. In other words, we define a $q$-quaternion just as one introduces the defining matrix of $SU_q(2)$ [52, 54], but without imposing the unit $q$-determinant condition. For $q \in \mathbb{R} \setminus \{0\}$ consider the unital associative $\star$-algebra $A \equiv C(\mathbb{H}_q)$ generated by elements $\alpha, \gamma, \alpha^*, \gamma^*$ fulfilling the commutation relations

$$\begin{align*}
\alpha \gamma &= q \gamma \alpha, & \alpha \gamma^* &= q \gamma^* \alpha, & \gamma \alpha^* &= qa^* \gamma, \\
\gamma^* \alpha^* &= qa^* \gamma^*, & [\alpha, \alpha^*] &= (1-q^2)\gamma \gamma^* & [\gamma^*, \gamma] &= 0.
\end{align*}$$

(2.1)

Introducing the matrix

$$x \equiv \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} := \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

(2.2)
we can rewrite these commutation relations as
\[ \hat{R} x_1 x_2 = x_1 x_2 \hat{R} \] (2.3)
and the conjugation relations as
\[ x^{\alpha \beta \ast} = \epsilon^{\beta \gamma} x^{\delta \gamma} \epsilon_{\delta \alpha}, \] i.e.
\[ x^\dagger = \bar{x} \] where \( \bar{a} := \epsilon a^T \epsilon^{-1} \) \( \forall a \in M_2. \) (2.4)
Here as usual \( x_1 \equiv x \otimes \mathbb{C} \bar{I}_2, \) \( x_2 \equiv \bar{I}_2 \otimes \mathbb{C} x \) (\( I_2 \) is the \( 2 \times 2 \) unit matrix), \( \hat{R} \) is the braid matrix of \( M_q(2), \) \( GL_q(2) \) and \( SU_q(2) \)
\[ \hat{R}_{\gamma \delta}^{\alpha \beta} = q^{\delta \gamma} \epsilon_{\delta \beta}^\gamma + \epsilon^{\alpha \beta} \epsilon_{\gamma \delta}, \] (2.5)
and \( \epsilon \) is the corresponding completely \( q \)-antisymmetric tensor
\[ \epsilon \equiv (\epsilon_{\alpha \beta}) := \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \] \[ \epsilon^{-1} \equiv (\epsilon^{\alpha \beta}) = -q^{-1}(\epsilon_{\alpha \beta}). \] (2.6)
So \( A := C(H_q) \) can be naturally endowed with a \( \ast \)-bialgebra structure (we are not excluding \( 0_2 \) from the spectrum of \( x \)), more precisely the above real section of the bialgebra \( C(M_q(2)) \) of \( 2 \times 2 \) quantum matrices \([14, 16, 54, 15]\).
In the sequel we shall write the corresponding coproduct
\[ \Delta(x^{\alpha \gamma}) = x^{\alpha \beta} \otimes x^{\beta \gamma} \] (2.7)
in the more compact matrix product form
\[ \Delta(x) = ax \] (2.8)
where we have renamed \( x \otimes 1 \rightarrow a, \) \( 1 \otimes x \rightarrow x. \) Since the coproduct is a \( \ast \)-algebra map, \( \Delta(x), \) or equivalently the matrix product \( ax \) of any two matrices \( a, x \) with mutually commuting entries and fulfilling \([2.3, 2.4]\), again fulfills the latter. Therefore we shall call any such matrix \( x \) a \( q \)-quaternion, and \( A := C(H_q) \) the \( q \)-quaternion bialgebra. Note that, according to this definition, the unit matrix is a \( q \)-quaternion. Note that \( I_2 \) is a \( q \)-quaternion, and \( x \) is a \( q \)-quaternion iff \( -x \) is.
As well-known, the so-called ‘\( q \)-determinant’ of \( x \)
\[ |x|^2 \equiv \det_q(x) := x^{11} x^{22} - q x^{12} x^{21} = \alpha^* \alpha + \gamma^* \gamma \]
\[ = \frac{1}{1+q^2} x^{\alpha \alpha'} x^{\beta \beta'} \epsilon_{\alpha \beta} \epsilon_{\alpha' \beta'}, \] (2.9)
is central, manifestly nonnegative-definite and group-like. Therefore at representation level it will have zero eigenvalue iff \( x \) has \( 0_2 \) eigenvalue matrix. Replacing \([2.5]\) in \([2.3]\) we find that the latter is equivalent to
\[ x \bar{x} = \bar{x} x = |x|^2 I_2. \] (2.10)
If we extend \( C(H_q) \) by the new (central, positive-definite and group-like) generator \( |x|^{-1} \) (this will exclude \( x = 0_2 \) from the spectrum), the
matrix $x$ becomes invertible and we obtain even a Hopf $\star$-algebra with antipode $S$ defined by

\[ Sx = x^{-1} = \frac{x}{|x|^2}, \quad S|x|^{-1} = |x|. \quad (2.11) \]

The matrix elements of $T := \frac{x}{|x|}$ fulfill the ‘RTT’ \cite{15} relations \cite{23} and

\[ T^\dagger = T^{-1} = T, \quad \det_q(T) = 1, \quad (2.12) \]

namely generate $C(SU_q(2))$ \cite{52, 54} as a quotient subalgebra. Therefore the $x^{\alpha\alpha'}$ generate the (Hopf) $\star$-algebra $C(SU_q(2) \times GL^+(1))$ of functions on the “quantum group $SU_q(2) \times GL^+(1)$ of nonvanishing $q$-quaternions” [a real section of the Hopf algebra $C(GL_q^+(2))$, in analogy with the $q = 1$ case.

In view of the construction of instanton solutions we also extend $\mathcal{A} = C(H_q)$ by adding as generators

\[ \frac{1}{1 + |x|^2}, \quad \rho \in \mathbb{R}^+. \]

**Identifying $H_q$ as $\mathbb{R}^4_q$**

One can easily verify that as a $\star$-algebra $\mathcal{A} := C(H_q)$ coincides with the algebra of functions on the $SO_q(4)$-covariant quantum Euclidean Space $\mathbb{R}^4_q$ of \cite{15}. We identify the present $qx^{11}, x^{12}, -qx^{21}, x^{22}$ with the generators $x^1, x^2, x^3, x^4$ of \cite{15} (in their original indices convention) or with the generators $x^{-2}, x^{-1}, x^1, x^2$ in the convention of Ref. \cite{39} (which has been heavily used by the author of the present work). We shall denote by $B \equiv (B_{\alpha\alpha'})$ this (diagonal and invertible) matrix entering the linear transformation $x^a = B^a_{\alpha\alpha'} x^{\alpha\alpha'}$. The braid matrix of $SO_q(4)$ is obtained as

\[ \hat{R} \equiv (\hat{R}_{cd}) = q^{-1}B(\hat{\mathcal{R}} \otimes \hat{C})\hat{R}B^{-1} \quad (2.13) \]

(recall that the tensor product of two braid matrices is again a braid matrix), where $\hat{B}_{\alpha\beta\alpha'\beta'}^{ab} := B^a_{\alpha\alpha'} B^b_{\beta\beta'}$. Its decomposition

\[ \hat{R} = q\mathcal{P}_3 - q^{-1}\mathcal{P}_4 + q^{-3}\mathcal{P}_5 \quad (2.14) \]

in orthogonal projectors follows from that of the braid matrix of $M_q(2), GL_q(2), SU_q(2)$,

\[ \hat{R} = q\mathcal{P}_3 - q^{-1}\mathcal{P}_3, \quad (2.15) \]
since $P := B(\mathcal{P} \otimes C \mathcal{P}^\prime)B^{-1}$ is a projector whenever $\mathcal{P}, \mathcal{P}^\prime$ are. In fact,

\begin{align*}
P_s &= B(\mathcal{P}_s \otimes C \mathcal{P}_s)B^{-1}, & \mathcal{P}_s &= B(\mathcal{P} \otimes C \mathcal{P}_s)B^{-1}, \\
P_a &= B(\mathcal{P}_a \otimes C \mathcal{P}_a)B^{-1}, & \mathcal{P}_a^\prime &= B(\mathcal{P}_a \otimes C \mathcal{P}_s)B^{-1}, \\
P_t &= \mathcal{P}_a + \mathcal{P}_a^\prime.
\end{align*}

(2.17)

$\mathcal{P}_s, \mathcal{P}_a, \mathcal{P}_a^\prime$ are respectively $GL_q(2)$-covariant deformations of the symmetric and antisymmetric projectors, and have dimension 3, 1. They can be expressed in terms of the $q$-deformed $\epsilon$-tensor by

\begin{align*}
P_{a\gamma\delta}^{\alpha\beta} &= -\frac{\epsilon_{\alpha\beta}^\gamma\delta}{q + q^{-1}}, & P_{s\gamma\delta}^{\alpha\beta} &= \delta_{\gamma\delta}^\alpha\beta + \frac{\epsilon_{\alpha\beta}^\gamma\delta}{q + q^{-1}}. \\
& & (2.18)
\end{align*}

$\mathcal{P}, \mathcal{P}_a, \mathcal{P}_t$ are $SO_q(4)$-covariant deformations of the symmetric trace-free, antisymmetric and trace projectors respectively; as we shall see $\mathcal{P}_a, \mathcal{P}_a^\prime$ are projectors respectively on the selfdual and antiselfdual 2-forms subspaces. By (2.17) the $\mathcal{P}_s, \mathcal{P}_a, \mathcal{P}_t, \mathcal{P}_a^\prime, \mathcal{P}_a$ respectively have dimensions 9, 3, 3, 6, 1, and

\begin{equation}
R_{ijkl} = (g^{sm}g^{sm})^{-1}g^{ij}g_{kl} = \frac{1}{(q + q^{-1})^2}g^{ij}g_{kl} 
\end{equation}

(2.19)

where the $4 \times 4$ matrix $g_{ab}$ (denoted as $C_{ab}$ in [15]) is given by

\begin{equation}
g_{ab} = B^{-1}_{a\alpha\alpha'}B^{-1}_{b\beta\beta'}\epsilon_{\alpha\beta}^\alpha\beta'; \tag{2.20}
\end{equation}

it is the $SO_q(4)$-isotropic 2-tensor, deformation of the ordinary Euclidean metric, and “Killing form” of $U_q so(4)$. Recalling that $\tilde{R}' = \tilde{R}$ one immediately checks that the commutation relations become

\begin{equation}
\mathcal{P}_{a\gamma\delta}^{ij}x^kx^l = 0 
\end{equation}

(2.21)

as in the definition [15] of the quantum Euclidean space.

The commutation relations and the $\star$-structure are covariant under, i.e. preserved by, matrix multiplication

\[ x \rightarrow a x b \]

by the defining matrices $a, b$ of two copies $SU_q(2), SU_q(2)'$ of the special unitary quantum group, or of two copies $\mathbb{H}_q, \mathbb{H}_q'$ of the quaternion quantum group, respectively, whose entries commute with each other and with the entries of $x$. In other words they are covariant under the (mixed left-right)

*The orthonormality relations for the $P_{\mu}$, with $\mu = s, a$,

\begin{align*}
P_{\mu}P_{\nu} &= P_{\mu}\delta_{\mu\nu}, & \sum_{\mu} P_{\mu} &= I. \\
& & (2.16)
\end{align*}

trivially imply the orthogonality relations for the $P_{\mu}$, with $\mu = s, a, a', t$. 7
coactions of $SU_q(2) \otimes SU_q(2)' = Spin_q(4)$ and of $H_q \otimes H_q'$. This follows from the fact that the twofold coproduct $\Delta^{(2)}(x) = axb$,

$$\Delta^{(2)}(x^{\alpha\alpha'}) = a^{\alpha\beta}b^{\beta\alpha'} \otimes x^{\beta\beta'}, \quad \text{i.e.} \quad x \mapsto x \Delta_L x,$$

(2.22)
is a $\star$-homomorphism, or equivalently both the left coaction $x \mapsto a x$ and the right one $x \mapsto x b$ are.

Upon applying the linear transformation $B$ (2.22) takes the form

$$\Delta_L(x^i) = T^i_j \otimes x^j, \quad T^i_j := B^i_{\alpha\alpha'}a^{\alpha\beta}b^{\beta\alpha'}B^{-1\beta\beta'}_j.$$

(2.23)

Relation (2.24) follows from (2.13) and (2.3) for both $a$ and $b$. Relation (2.24) follows from (2.4) for both $a$ and $b$. $g_{ij}T^i_j T^j_{i'} = g_{jj'}$ follows from (2.9), (2.20) when $\|a\|b\| = 1$. The analogy with the case $q = 1$ would be complete if one were able to further extend the action of $\overline{ISO_q(4)}$ into that of a quantum conformal group. This is out of the scope of this work and will hopefully treated elsewhere [23]. A quantum deformation of the Universal Enveloping Algebra (U.E.A.) of the conformal group having the U.E.A. of the $q$-deformed Poincaré group [40] as a closed subalgebra was already constructed in [29].
Comparison and links with other formulations

A matrix version of the 4-dim quantum Euclidean space (with no interpretation in terms of $q$-deformed quaternions) was proposed also in [36]. However, the $\star$-relations and the $SO_q(4)$-coaction are different, i.e. cannot be put both in the form (2.1), (2.22), even by a relabelling of the generators.

As a $\star$-algebra, our $A$ slightly differs from the one of the quantum 4-sphere $S^4_q$ proposed in [12] (which was introduced as a ‘suspension’ of the algebra of the quantum 3-sphere $S^3_q$), in the sense that a slight extension $A^{ext}$ of $A$ by some rational functions of $|x|$ contains that algebra as a $\star$-subalgebra. Define

$$\alpha' = \sqrt{2} \alpha^* \frac{2}{1+2|x|^2} e^{ia}, \quad \alpha'^* = \sqrt{2} \alpha \frac{2}{1+2|x|^2} e^{-ia},$$

$$\beta' = \sqrt{2} \gamma^* \frac{2}{1+2|x|^2} e^{ib}, \quad \beta'^* = \sqrt{2} \gamma \frac{2}{1+2|x|^2} e^{-ib},$$

$$z = \frac{1-2|x|^2}{1+2|x|^2}$$

where $\alpha, \gamma, \alpha^*, \gamma^*$ fulfill (2.1) and $e^{ia}, e^{ib} \in U(1)$ are possible phase factors. Then $\alpha', \beta', z$ fulfill the defining relation (1) of the $C^*$-algebra considered in Ref. [12] (where these elements are respectively denoted as $\alpha, \beta, z$), in particular

$$\alpha'\alpha'^* + \beta'\beta'^* + z^2 = 1,$$

(2.26)

and the invertible function $z(|x|)$ spans $[-1,1]$, i.e. all the spectrum of $z$ except the eigenvalue $z = 1$, as $|x|$ spans all its spectrum $[0,\infty]$. Viceversa, starting from the latter and enlarging it so that it contains the element $(1+z)/2(1-z) = |x|$, then inverting the above formulae one obtains elements $\alpha, \gamma, \alpha^* \gamma^*$ fulfilling our defining relations (2.1).

The redefinitions (2.25) have exactly the form of a stereographic projection of $\mathbb{R}^4$ on a sphere $S^4$ of unit radius (recall that $x \cdot x = 2|x|^2$): $S^4$ is the sphere centered at the origin and $\mathbb{R}^4$ the subspace $z = 0$ immersing both in a $\mathbb{R}^5$ with coordinates defined by $X \equiv (\text{Re}(\alpha'), \text{Im}(\alpha'), \text{Re}(\beta'), \text{Im}(\beta'), z)$. In the commutative theory adjoining the missing point $X = (0,0,0,0,1)$ of $S^4$ amounts to adding to $\mathbb{R}^4$ the point at infinity, i.e. to compactifying $\mathbb{R}^4$ to $S^4$. We can thus regard the transition from our algebra to the one considered in Ref. [12] as a compactification of $\mathbb{R}^4_q$ into their $S^4_q$.

3 The $SO_q(4)$-covariant differential calculi

The $SO_q(4)$-covariant differential calculus [6] ($d, \Omega^*$) on $\mathbb{R}^4_q \sim H_q$ is obtained imposing covariant homogeneous bilinear commutation relations (3.1) between the $x^i$ and the differentials $\xi^i := dx^i$. Partial derivatives
are introduced through the decomposition \( d = \xi^\alpha \partial_\alpha = \xi^{\alpha\prime} \partial_{\alpha\prime} \) of the \( (SO_q(4)\text{-invariant}) \) exterior derivative. All other commutation relations are derived by consistency with nilpotency and the Leibniz rule. Besides \( 2.21 \), or equivalently \( 2.3 \), we have

\[
x^h \xi^i = q \hat{R}_{jh}^i \xi^j = 0 \quad \Leftrightarrow \quad x^{\alpha\prime} \xi^{\beta\prime} = \hat{R}_{\alpha\prime\beta\prime}^{\gamma\delta} \xi^{\gamma} x^{\delta},
\]

\[
\partial_{i} = (2.21), \quad \text{or equivalently } 2.3, \quad \text{we have}
\]

\[
(\partial_{i} + \hat{R}_{ih}^j \xi^j = 0 \quad \Leftrightarrow \quad \partial_{\alpha\prime} x^{\beta\prime} = \hat{R}_{\alpha\prime\beta\prime}^{\gamma\delta} \xi^{\gamma} \delta_{\gamma\delta},
\]

\[
\partial_{i} x^j = (3.4) \quad \text{and } \partial_{i} \partial_{j} = (3.5) \quad \text{it follows}
\]

\[
\partial \hat{\Omega} \equiv (3.3) \quad \text{is} \quad \text{harmonic, as in the undeformed case. There exists a special combination } V \text{ of } 1, x \cdot \partial, \Box \text{ which is unitary and fulfills}
\]

\[
V x^i = q x^i V, \quad V \partial^i = q^{-1} \partial^i V, \quad V \xi^i = \xi^i V.
\]

We add as new generator its "inverse square root", a unitary element \( \lambda \) such that \( \lambda^2 V = V \lambda^2 = 1 \) and

\[
\lambda x^i = q^{-1} x^i \lambda, \quad \lambda \partial^i = q \partial^i \lambda, \quad \lambda \xi^i = \xi^i \lambda.
\]

We introduce the following unital associative algebras:
We shall denote by $\bigwedge^*$ (exterior algebra, or algebra of exterior forms) the $\sharp$-graded algebra generated by the $\xi^i$, where the grading $\sharp$ is the degree in $\xi^i$; any component $\bigwedge^p$ having $\sharp = p$ carries a corepresentation of $SO_q(4)$ and has the same dimension $\binom{4}{p}$ as in the $q = 1$ case. In particular, up to a factor there exists a unique 4-form which we shall denote as $d^4x$. $\bigwedge^p$ is irreducible if $p \neq 2$, and, as we shall see, splits into a selfdual and an antiselfdual part if $p = 2$, exactly as in the $q = 1$ case.

We shall denote by $D\mathcal{C}^*$ (“differential calculus algebra”) the $\natural$-graded algebra generated by $x^i, \xi^i, \partial_i$. Elements of $D\mathcal{C}^0$ are differential-operator-valued $p$-forms.

We shall denote by $\Omega^* \equiv (\text{algebra of differential forms})$ the $\natural$-graded subalgebra generated by the $\xi^i, x^i$. By definition $\Omega^0 = \mathcal{A}$ itself, and both $\Omega^*$ and $\Omega^p$ are $\mathcal{A}$-bimodules. Also, we shall denote by $\Omega^*_S$ the subalgebra and $C(SU_q(2))$-bimodule generated by $T_{\alpha\alpha}', dT_{\alpha\alpha}'$ (this is still 4-dim!), and by $\tilde{\Omega}^*$ the extension of $\Omega^*$ with the unitary generators $\lambda^\pm_1$ obeying (3.11).

We shall denote by $\mathcal{H}$ (Heisenberg algebra) the subalgebra generated by the $x^i, \partial_i$. By definition, $D\mathcal{C}^0 = \mathcal{H}$, and both $D\mathcal{C}^*$ and $D\mathcal{C}^p$ are $\mathcal{H}$-bimodules.

Remark 1. The whole set of commutation relations (2.3, (3.1-3.5) is [7] in fact invariant under the replacement $x_{\alpha\alpha}'/|x|^2q^2(1 - q^2) \to \partial_{\alpha\alpha}'$ (this is an algebra homomorphism).

As a corollary, on $\Omega^*$ one can realize the action of the exterior derivative as the (graded) commutator
\[ d\omega_p = [-\theta, \omega_p] := -\theta \omega_p + (-)^p \omega_p \theta, \quad \omega_p \in \Omega^p \] (3.12)
with the special $SO_q(4)$-invariant 1-form [8] [48] (the ‘Dirac Operator’, in Connes’ [9] parlance)
\[ \theta := (d|x|^2)|x|^{-2} \frac{1}{q^2 - 1} \xi^{\alpha\alpha'} x^{\beta\beta'} |x|^2 \epsilon_{\alpha\beta} \epsilon_{\alpha'\beta'}. \] (3.13)
\[ \theta \text{ is closed:} \quad d\theta = 0, \quad \theta^2 = 0. \] (3.14)
Applying $d$ to (2.10) we find
\[ x\xi + \xi x = (q^2 - 1)\theta |x|^2 I_2, \quad \bar{x}\xi + \xi \bar{x} = (q^2 - 1)\theta |x|^2 I_2. \] (3.15)
Relation (3.11) implies $|x|^2 \xi^i = q^2 \xi^i |x|^2$, which we generalize as usual to
\[ |x| \xi^i = q \xi^i |x|, \quad \Rightarrow \quad |x| \theta = q \theta |x|. \] (3.16)
By a straightforward computation one also finds
\[ dT^{\alpha\alpha'} = q^{-1} \xi^{\alpha\alpha'} \frac{1}{|x|} + (q^{-1} - 1)\theta T^{\alpha\alpha'}. \] (3.17)
By (2.22) the 1-form-valued $2 \times 2$ matrices
\[
(dT)^T, \quad (\bar{d}T)T
\] (3.18)
are manifestly invariant under respectively the right and left coaction of the Hopf algebra $SU_q(2)$, or equivalently the $SU_q(2)'$ and the $SU_q(2)$ part of $SO_q(4)$ coaction. Setting $Q := -\epsilon^{-1} \epsilon^T$ one finds
\[
\text{tr}[Q(dT)\bar{T}] = \text{tr}[Q^{-1}(d\bar{T})T] = (q-1)(q^{-2})\theta;
\]
only in the $q \to 1$ limit these traces vanish. That’s why for generic $q \neq 1$ the four matrix elements of either $(dT)^T$ or $(d\bar{T})T$ are independent, and make up alternative bases for both $\Omega^*_N$ and $\Omega^*$. Actually, one can check (we will give details in [18]) that $(d, \Omega^*)$ coincides with the bicovariant differential calculus on $M_q(2), GL_q(2)$ [44, 46], and $(d, \Omega^*_N)$ coincides with the Woronowicz 4D- bicovariant one [53, 43] on $C(SU_q(2))$. One major problem in the present $q \in \mathbb{R} \setminus \{0\}$ case is that the calculus is not real: there is no $*$-structure such that $d(f^*) = (df)^*$, nor is there a $*$-structure $*: \Omega^* \to \Omega^*$. Formally, a $*$-structure would map the commutation relations of $(d, \Omega^*)$ into the ones of $(\hat{d}, \hat{\Omega}^*)$, and conversely. At least, there is a $*$-structure [41]
\[
*: DC^* \to DC^*
\]
having the desired commutative limit (the $*$-structure of the De Rham calculus on $\mathbb{R}^4$), but a rather nonlinear character (incidentally, the latter has been recently [17] recast in a much more suggestive form), in other words objects of the second calculus can be realized nonlinearly in terms of objects of the first (and conversely).

One could introduce a simpler $[SO_q(4)\text{-covariant}]$ $*$-structure
\[
*: \hat{\Omega}^* \to \hat{\Omega}^*. \quad (3.19)
\]
It would be compactly summarized in the formula
\[
\theta^* = -q\lambda^{-2}\theta, \quad (3.20)
\]
and would coincide with the one suggested as a side-remark in formula (7.2) of [24]. But this would not be useful for the present purposes, because its $q \to 1$ limit is not the $*$-structure of the De Rham calculus on $\mathbb{R}^4$ unless in the commutative limit some coordinates $x^a$ vanish (instead of becoming cartesian coordinates).

\[\text{Eq. (3.20) is equivalent to } (d|x|^2)^* = -q^{-1}\lambda^{-2}d|x|^2. \] In the limit $q \to 1 \lambda \to 1$, so that $(d|x|^2)^* = -d|x|^2$, i.e. $d|x|^2$ is purely imaginary, rather than real! A short computation also shows that in this limit $(\xi T(\xi)T)T \in \Omega^{2'}$, in other words $^*\to$ maps selfdual into antiselfdual 2-forms (and conversely), instead of preserving the chirality!
3.1 Hodge operator and (anti)selfdual 2-forms

The Hodge map is a $SO_q(4)$-covariant, $A$-bilinear map $\ast : \tilde{\Omega}^p \rightarrow \tilde{\Omega}^{4-p}$ such that $\ast^2 = \text{id}$, defined by

$$\ast \varepsilon^{i_1 \ldots i_p} = c_p \varepsilon^{i_{p+1} \ldots i_{4}} \varepsilon_{i_1 \ldots i_{p+1} i_1 \ldots i_p} \lambda^{2p-4},$$

where $\varepsilon^{hijk} = q$-epilson tensor \cite{21, 19, 37, 20} and $c_p$ are suitable normalization factors \cite{21, 19, 37, 20}. Actually this extends \cite{20} to a $H$-bilinear map $\ast : DC^p \rightarrow DC^{4-p}$ with the same features. For $p = 2$ $\lambda$-powers disappear and one even gets a map $\ast : \Omega^2 \rightarrow \Omega^2$ defined by

$$\ast \xi^i \xi^j = \frac{1}{[2]_q} \xi^h \xi^k \xi_{kh} \epsilon_{ij} \omega_{ji}, \quad (3.21)$$

where $[2]_q = q + q^{-1}$. By an explicit calculation one finds that this amounts to

$$\ast \xi^i \xi^j = (P_a - P_a') \xi^i \xi^j \xi^k, \quad (3.22)$$

with $P_a, P_a'$ defined in (2.17). $\Lambda^2$ splits into the direct sum

$$\Lambda^2 = \tilde{\Lambda}^2 \oplus \tilde{\Lambda}^2' = P_a \Lambda^2 \oplus P_a' \Lambda^2 \quad (3.23)$$

of the eigenspaces $\tilde{\Lambda}^2, \tilde{\Lambda}^2'$ of $\ast$ with eigenvalues $1, -1$ (the “subspaces of selfdual and antiselfdual exterior forms” respectively), which carry the (3,1)- and (1,3)-dimensional corepresentation of $SU_q(2) \times SU'_q(2)$. By (2.17), (3.2) and (2.18) $\tilde{\Lambda}^2, \tilde{\Lambda}^2'$ are respectively spanned by

$$f^{\alpha \beta} := P_{\gamma \delta} \varepsilon_{\gamma \delta} \xi^{\gamma \delta} \xi^\alpha \xi^\beta = (\xi \xi^T)^{\alpha \beta} \quad (3.24)$$

and their antiselfdual partner

$$f'^{\alpha' \beta'} := P_{\gamma' \delta'} \varepsilon_{\gamma' \delta'} \xi^{\gamma' \delta'} \xi^\alpha \xi^{\beta'} = (\xi^T \xi)^{\alpha' \beta'} \quad (3.24')$$

As expected, only three out of the four matrix elements $f^{\alpha \beta}$ (resp. $f'^{\alpha' \beta'}$) are independent, as (3.2) implies $\epsilon_{\alpha \beta} f^{\alpha \beta} = 0 = \epsilon_{\alpha' \beta'} f'^{\alpha' \beta'}$. As a basis we can alternatively use also the matrix elements of $\xi \xi$ (resp. $\xi^T \xi$), because

$$(\xi \xi)^{\alpha \beta} = f^{\alpha \gamma} \epsilon^{\beta \gamma}, \quad (\xi^T \xi)^{\alpha' \beta'} = f'^{\alpha' \gamma'} \epsilon^{\beta' \gamma'} \quad (3.25)$$

From the decomposition $P_a = P_a + P_a'$ one easily finds

$$\xi^{\alpha \alpha'} \xi^{\beta \beta'} = -\frac{1}{q + q^{-1}} [f^{\alpha \beta} \epsilon^{\alpha' \beta'} + \epsilon^{\alpha \beta} f'^{\alpha' \beta'}] \quad (3.26)$$

Using relations (3.2) and (2.5) one easily derives the following relations

$$x^{\alpha \alpha'} f^{\beta \gamma} = q(\hat{R}_{12} \hat{R}_{23})^{\alpha \beta \gamma \delta} f^{\lambda \mu \nu \sigma} \epsilon_{\mu \nu \sigma}, \quad (3.27)$$

$$\partial^{\alpha \alpha'} f^{\beta \gamma} = q^{-1}(\hat{R}_{12} \hat{R}_{23})^{\alpha \beta \gamma \delta} f^{\lambda \mu \nu \sigma} \epsilon_{\mu \nu \sigma}. \quad (3.28)$$
The second is obtained from the first by applying □ and recalling (3.8) (or, alternatively, Remark 1). As done in (3.24'), in the sequel we shall usually label a formula regarding antiselfdual 2-forms by adding a prime to the label of its selfdual counterpart, and possibly omit it, whenever it can be obtained from the latter by the obvious replacements. As another example, the analog of (3.27) reads

$$x^{\alpha\alpha'} f^{\beta\gamma'} = q(R_{12}R_{23})^\alpha_{\lambda\mu} x^{\lambda\mu} x^{\alpha'^{}}.$$  (3.27)

From the previous three formulae and (3.28)' it follows that $$\Omega^2$$ (resp. $$\tilde{\Omega}^2$$) splits into the direct sum of $$A$$- (resp. $$H$$-) bimodules

$$\Omega^2 = \tilde{\Omega}^2 + \tilde{\Omega}^2'$$

(resp. $$\tilde{\Omega}^2 = \tilde{\Omega}^2 + \tilde{\Omega}^2'$$) (3.29)

of the eigenspaces of $$\ast$$ with eigenvalues 1, -1 respectively, whose elements we shall call as usual “self-dual and anti-self-dual 2-forms”.

**Proposition 1** For any $$\omega_2 \in \tilde{\Omega}^2$$, $$\omega'_2 \in \tilde{\Omega}^2'$$, (resp. $$\omega_2 \in \tilde{\Omega}^2$$, $$\omega'_2 \in \tilde{\Omega}^2'$$)

$$\omega_2 \omega'_2 = \omega'_2 \omega_2 = 0.$$  (3.30)

**Proof** Since $$\tilde{\Omega}^2, \tilde{\Omega}^2'$$ are $$A$$-bimodules (resp. $$\tilde{\Omega}^2, \tilde{\Omega}^2'$$ are $$H$$-bimodules) to prove (3.30) it is sufficient to prove

$$f^{\alpha\beta} f^{\gamma\delta'} = 0, \quad f^{\gamma\delta'} f^{\alpha\beta} = 0.$$

By construction the lhs's belong to the (3, 3)-dimensional (irreducible) corepresentation of $$SU_q(2) \times SU'_q(2)$$; at the same time, being 4-forms, they must be proportional to the invariant 4-form $$d^4 x$$, i.e. belong to the (1, 1)-dimensional corepresentation. Therefore they have to vanish. ∎

The 2-forms $$(\xi \bar{\xi})^{\alpha\beta}, (\bar{\xi} \xi)^{\alpha'\beta'}$$ are exact. One can find 1-form-valued matrices $$a, a'$$ such that

$$da = \xi \bar{\xi}, \quad da' = \bar{\xi} \xi.$$  (3.31)

Clearly, they are defined up to $$d$$-exact terms. Among the simplest choices we have

$$\hat{a} := -\xi \bar{\xi}, \quad \hat{a}' := -\bar{\xi} \xi.$$  (3.32)

They have the following commutation relations with the coordinates:

$$x^{\alpha\alpha'} (\hat{a} \epsilon)^{\beta\gamma} = (q R_{12} R_{23}^{-1})_{\lambda\mu}^{\alpha} (\hat{a} \epsilon)^{\lambda\mu} x^{\alpha'^{}}.$$  (and similarly for $$\hat{a}'$$). The four matrix elements of $$\hat{a}$$ are all independent and make up an alternative basis for $$\Omega^1$$; they belong to the (3, 1) + (1, 1)-dimensional (reducible) corepresentation of $$SU_q(2) \times SU'_q(2)$$. (And similarly for $$\hat{a}'$$). These properties remain true for any combination

$$a_\kappa := \hat{a} + \kappa \theta |x|^2 I_2.$$  (3.33)
with complex \( \kappa \neq \kappa_0 := q^2(q^2 - 1)/(q^2 + 1) \), whereas there are only three independent

\[
a_{\kappa_0}^{\alpha \beta} = P_{\gamma \delta}^{\alpha \lambda} (\xi \epsilon x^T) \gamma^\delta \epsilon^{\beta \lambda},
\]

(3.34)

because \( a_{\kappa_0}^{\alpha \beta} (\epsilon \epsilon^T)_{\beta \alpha} = 0 \); the latter belong to the (3,1) irreducible corepresentation of \( SU_q(2) \times SU'_q(2) \). There is no other matrix \( a \) with the latter property. In the \( q = 1 \) limit (3.34) becomes the familiar

\[
a_{\kappa_0}^{\alpha \beta} = -\left( (\xi \epsilon - 1) x^T \right)^{(\alpha \lambda)} \epsilon^{\lambda \beta} = - \{ \text{Im}(\xi \bar{x}) \}^{\alpha \beta},
\]

where \((\alpha \lambda)\) denotes symmetrization w.r.t. \( \alpha \lambda \) and \( \text{Im} \) the imaginary part.

From (3.15), (3.16), (3.12), (3.14) we easily derive

\[
a_{\kappa} a_{\kappa} = (1 - \kappa) [\xi \bar{x} + (1 - q^2) \theta \bar{x}] |x|^2 = q^2 (1 - \kappa) [\xi \bar{x} + (1 - q^{-2}) a_{\kappa} \theta] |x|^2
\]

(3.35)

An analogous statement holds for their primed counterparts. By straightforward calculations one also finds

\[
T a_{\kappa} T = -q^{-1} (1 + \kappa) [d' + \kappa' \theta] |x|^2 I_2 = -q^{-1} (1 + \kappa) a'_{\kappa'}
\]

(3.36)

where \( \kappa' := q^2/(1 + \kappa) - 1 \). Looking for a \( \kappa \) such that \( \kappa' = \kappa \) we find two solutions \( \kappa_{\pm} = 1 \pm q \), which yield the simple changes

\[
T a_{\kappa_{\pm}} T = \mp a'_{\kappa_{\pm}}
\]

(3.37)

under the ‘similarity’ transformation \( T \); it is immediate to check that

\[
a_{\kappa_{\pm}} = -q (dT) T |x|^2,
\]

which has a well defined limit as \( q \rightarrow 1 \), whereas in the same limit \( a_{\kappa_{-}} \) diverges. Since \((\xi \bar{x})^{\alpha' \beta'} \in \Omega^{2^l} \), which is an \( A \)-bimodule, we also find

\[
T \xi \bar{x} T = q^{-2} \bar{x} \xi \bar{x} = -\xi \bar{x} \xi \bar{x} \bar{x} - q^{-2} (1 - q^{-2}) \theta |x|^2 \xi \bar{x} \bar{x} \bar{x} q^{-2} + (1 - q^{-2}) \theta |x|^2 \xi \bar{x} \bar{x} q^{-2} + (q^{-2} - 1) \theta \bar{x} \bar{x} q^{-2} + (q^{-2} - q^{-2}) \bar{x} \bar{x} \bar{x} q^{-2}
\]

(3.38)

4 Looking for a suitable noncommutative gauge theory framework

We recall some minimal common elements in the formulations of gauge theories on commutative as well as noncommutative spaces [9, 32] (see also [30, 26]). We denote by \( \mathcal{A} \) the ‘\( \ast \)-algebra of functions on the noncommutative space’ under consideration, by \((d, \Omega^*)\) a differential calculus on \( \mathcal{A} \), real in the sense that \( d(f^*) = (df)^* \). In \( U(n) \) gauge theory the
gauge transformations $U$ are unitary $A$-valued $n \times n$ unitary matrices, $U \in M_n(A) \equiv M_n(C) \otimes C A$,

$$U^{-1} = U^\dagger, \quad U \in U_n.$$  \hspace{1cm} (4.1)

Gauge potentials are antihermitean $n \times n$ 1-form-valued matrices $A \equiv (A^\alpha_\beta)$, $A \in M_n(\Omega^1) \equiv M_n(C) \otimes C \Omega^1$. The case $n = 1$ corresponds to electromagnetism. The covariant derivative $D : M_n(\Omega^p) \rightarrow M_n(\Omega^{p+1})$ is defined as usual by

$$D\omega_p := d\omega_p + [A, \omega_p],$$  \hspace{1cm} (4.2)

and is therefore hermitean, $D(f^\dagger) = (Df)^\dagger$. The associated field strength $F \in M_n(\Omega^2)$ is defined as usual by

$$F := dA + AA.$$  \hspace{1cm} (4.3)

At the right-hand side the product $AA$ is both a (row by column) matrix product and a wedge product. It is automatically hermitean. As in commutative geometry, it is immediate to prove that $F$ satisfies the Bianchi identity

$$DF = 0.$$  \hspace{1cm} (4.4)

The Yang-Mills equation reads as usual

$$D^* F = 0.$$  \hspace{1cm} (4.5)

If the exterior derivative can be realized as the graded commutator [3,12] with a special 1-form $-\theta$, then introducing the 1-form-valued matrix $B := -\theta I_n + A$ one finds that

$$F = BB, \quad D = [B, \cdot]$$  \hspace{1cm} (4.6)

and Bianchi identity is now even more trivial. In Connes’ noncommutative geometry $-\theta$ is the so-called ‘Dirac operator’, which has to fulfill more stringent requirements [9].

In commutative geometry the so-called Serre-Swan theorem [49, 10] states that vector bundles over a compact manifold coincide with finitely generated projective modules $E$ over $A$. The gauge connection $A$ of a gauge group (fiber bundle) acting on a vector bundle is expressed in terms of the corresponding projector $P$. Therefore the projectors characterizing the projective modules can be used to completely determine the connections. In Connes’ standard approach [9] to noncommutative geometry the finitely generated projective modules are the primary objects to define and develop the gauge theory. The topological properties of the connections can be classified in terms of topological invariants (Chern numbers), and the latter can be computed directly in terms of characters of the projectors (Chern-Connes characters).
Because of the Bianchi identity, in a 4D Riemannian geometry endowed with a (involutive) Hodge map $\ast$ the YM equation is automatically satisfied by a solution of the (anti)self-duality equations

\begin{align*}
\ast F &= F & \text{self-duality}, \\
\ast F &= -F & \text{anti-self-duality}.
\end{align*}

(4.7)

If $\Omega^2$ splits as in (3.29) then $F$ is uniquely decomposed in a selfdual and an antiselfdual part

\[ F = F^+ + F^- . \]

(4.8)

Under a gauge transformation $U$

\[ A \to A^U = U^{-1}(AU + dU), \quad \Leftrightarrow \quad B \to B^U = U^{-1}BU \]

(4.9)

implying as usual

\[ F \to F^U = U^{-1}FU. \]

(4.10)

The Bianchi identity, the Yang-Mills equation, the (anti)self-duality equations, the splitting (4.8), the flatness condition $F = 0$ are preserved by gauge transformations. As usual, $A = U^{-1}dU$ implies $F = 0$.

Up to normalization factors, the gauge invariant action $S$ and the ‘Pontryagin index’, or ‘second Chern number’, $Q$ (a topological invariant) are defined by

\begin{align*}
S &= \text{Tr}(F^*F), \\
Q &= \text{Tr}(FF),
\end{align*}

(4.11, 4.12)

where Tr stands for a positive-definite trace (as such, it has to fulfill the cyclic property) combining the $n \times n$-matrix trace with the integral over the noncommutative manifold. If integration $\int$ fulfills itself the cyclic property then this is obtained by simply choosing $\text{Tr} = \int \text{tr}$, where $\text{tr}$ stands for the ordinary matrix trace. $S$ is automatically nonnegative.

$Q$ can be computed in terms of the second Chern-Connes character of the projector $P$ associated to the connection $A$ when Connes’ formulation of noncommutative geometry applies.

If, as in the case under discussion, (3.30) holds, $S, Q$ split into the sum, difference of the two nonnegative contributions

\begin{align*}
S &= \text{Tr}(F^+ *F^+) + \text{Tr}(F^- *F^-), \\
Q &= \text{Tr}(F^+ *F^+) - \text{Tr}(F^- *F^-).
\end{align*}

(4.13, 4.14)

As in the commutative case, these relations imply $S \geq |Q|$.

In the present $\mathcal{A} \equiv C(R^4_q) = C(H_q)$ case the above scheme is not fully applicable because of two main problems:

1. Integration over $R^4_q$ fulfills a deformed cyclic property [18].
2. As already recalled, \( d(f^*) \neq (df)^* \) and there is no \(*\)-structure \( \star : \Omega^* \to \Omega^* \), but only a \(*\)-structure \( \star : \mathcal{DC}^* \to \mathcal{DC}^* \), with a rather nonlinear character.

A solution to both problems might be obtained
1. allowing for \( \mathcal{DC}^1 \)-valued \( A \Rightarrow \mathcal{DC}^2 \)-valued \( F \)'s), and/or
2. defining a cyclic trace \( \text{Tr} \) by \( \text{Tr}(\omega) := \int \text{tr}(W^{(1)}\omega W^{(2)}) \), with some suitable positive definite \( W^{(1)} \otimes W^{(2)} \in M_n(\mathcal{H}) \otimes M_n(\mathcal{H}) \) (in Sweedler notation with suppressed sum symbol). (A \( W \in M_n(\mathcal{H}) \) is a pseudodifferential-operator-valued \( n \times n \) matrix).

This hope is based on our results \[17\]: 1) the Hodge map \( \star \) is not only \( \mathcal{A}\)-bilinear, but fully \( \mathcal{H}\)-bilinear; 2) the \(*\)-structure \( \star : \mathcal{DC}^* \to \mathcal{DC}^* \) can been recast in a much more suggestive form involving only a similarity transformation with the realization as pseudodifferential operators of the ribbon element \( \tilde{w} \) and of the “vector field generators” \( \tilde{Z}^i \) of the central extension of \( U_q\text{so}(4) \) with dilatations; 3) \( d \) and the exterior coderivative \( \delta := -\ast d\ast \) become conjugated of each other

\[
(\alpha_p, d\beta_{p-1}) = (\delta\alpha_p, \beta_{p-1}), \quad (d\beta_{p-1}, \alpha_p) = (\beta_{p-1}, \delta\alpha_p)
\]

if one defines

\[
(\alpha_p, \beta_p) = \int_{\mathbb{R}_q^4} \alpha^*_p \tilde{w'}^{1/2} \ast \beta_p
\]

where \( \tilde{w}' \) is the realization of \( \tilde{w} \) as a pseudodifferential operator.

5 \( q \)-deformed \( su(2) \) instanton

We look for \( A \in M_2(\Omega^1) \) solutions of the (anti)self-duality equations (4.7) virtually yielding a finite action functional (4.11). Among them we expect deformations of the (multi)instanton solutions of \( su(2) \) Yang-Mills theory on the “commutative” \( \mathbb{R}^4 \). We first recall the instanton solution of Belavin et al. \[4\], which we write down both in \( \text{t'} \) Hooft \[50\] and in ADHM \[3\] quaternion notation:

\[
A = dx^i \sigma^a \eta^a_j x^j \frac{1}{\rho^2 + r^2/2},
\]

\[
= - \text{Im} \left\{ \xi \frac{\bar{x}}{|x|^2} \right\} \frac{1}{1 + \rho^2 |x|^2} \frac{1}{\rho^2 + r^2/2}
\]

\[
= -(dT)T \frac{1}{1 + \rho^2 |x|^2} \quad \text{(5.1)}
\]

\[
F = \xi \bar{\xi} \rho^2 \frac{1}{(|x|^2 + \rho^2)^2}.
\]
Here \( r^2 := x \cdot x = 2|x|^2 \), \( \sigma^a \) are the Pauli matrices, \( \eta_{ij}^a \) are the so-called t’ Hooft \( \eta \)-symbols, \( \rho \) is the size of the instanton (here centered at the origin). The third equality is based on the identity
\[
\xi \frac{\bar{x}}{|x|^2} = (dT)\bar{T} + I_2 \frac{d|x|^2}{2|x|^2}
\]
and the observation that the first and second term at the rhs are respectively antihermitean and hermitean, i.e. the imaginary and the real part of the quaternion.

In terms of the modified gauge potential \( B := A - \theta I_2 \) a natural Ansatz for the deformed instanton solution in the ‘regular gauge’ is (in matrix notation)
\[
B = \xi \frac{\bar{x}}{|x|^2} l + \theta I_2 m,
\]
where \( l, m \) are functions of \( x \) only through \( |x| \). For any \( f(x) \) we shall denote \( f_q(x) := f(qx) \). Using (3.16), (3.14), (3.15), (3.12), (2.10) we find
\[
F = B^2 = \xi \frac{\bar{x}}{|x|^2} l \xi \frac{\bar{x}}{|x|^2} l + \xi \frac{\bar{x}}{|x|^2} l \theta m + \theta m \xi \frac{\bar{x}}{|x|^2} l + \theta m \theta m
\]
\[
= \xi \bar{x} \xi \bar{x} l \frac{q^{-2}}{|x|^4} + \xi \bar{x} \theta l q \frac{q^{-2}}{|x|^2} + \theta \xi \bar{x} m q l \frac{1}{|x|^2} + \theta^2 m q m
\]
\[
= \xi \left[ -\bar{x} + (q^2 - 1) \theta |x|^2 \right] \bar{x} l q \frac{q^{-2}}{|x|^4} + \xi \left[ \bar{x} + \theta \bar{x} \right] l q m q^{-2} \frac{1}{|x|^2} - \xi \bar{x} m q l \frac{1}{|x|^2}
\]
\[
= \xi \bar{x} (m - l) q \frac{q^{-2}}{|x|^2} + \xi \bar{x} l \left[ (q^2 - 1) l q + l q m - q^2 m q l \right] q^{-2} \frac{1}{|x|^2}.
\]
A sufficient condition for \( F \) to be selfdual is that the expression in the square bracket vanishes. Setting \( h := m/l \) this amounts to the equation \( q^2 h q - h = (q^2 - 1) \), which is solved by
\[
m = \left[ 1 + \rho^2 \frac{1}{|x|^2} \right] l,
\]
where \( \rho^2 \) is a constant, or might be a further generator of the algebra, commuting with \( \theta \). Replacing in the expression for \( A, F \), we find a family of solutions
\[
A_l = \xi \frac{\bar{x}}{|x|^2} l + \theta I_2 \left\{ 1 + \left[ 1 + \rho^2 \frac{1}{|x|^2} \right] l \right\}
\]
\[
= q(dT)\bar{T} l + \theta I_2 \left\{ 1 + \left[ q + \rho^2 \frac{1}{|x|^2} \right] l \right\},
\]
\[
F_l = \xi \frac{1}{|x|^2} \rho^2 \frac{q^2}{|x|^2} l q,
\]
parametrized by the function \( l \). This large (compared to the undeformed case) freedom in the choice of the solution is due to the fact that we have not yet imposed in \( A \) the antihermiticity condition. Actually, we don’t
know yet what the ‘right’ antihermiticity condition is: in fact, for no \( l \) is \( A \) antihermitean w.r.t. the \( \star \)-structure \([4.11]\) mentioned in section 3. In any case, one should check that for the final \( A \) the resulting \( F \) decreases faster than \( |x|^{-2} \) at infinity, so that the resulting action functional \([4.11]\) is finite.

The second term in \([5.3]_1\) is proportional to \( d|x|^2 \); in the commutative limit \( q = 1 \) it is a connection associated to the noncompact factor \( GL^+(1) \) of \( \Pi \). In this limit the antihermiticity condition on \( A \) amounts to the vanishing of this term and completely determines the solution. It factors \( GL^+(1) \) out of the gauge group to leave a pure \( su(2) \) gauge theory. In this case the associated gauge theory would necessarily be a deformed \( u(2) \) one.

For the moment we cannot solve the ambiguity, and content ourselves with writing the solution for a couple of selected choices of \( l \). If we choose \( l \) so that the second term in \([5.3]_1\) vanishes and set \( \rho^2 = \bar{\rho}^2 q^{-1} \) we obtain

\[
A = -(dT)T \frac{1}{1 + \rho^2 |x|^2},
\]

\[
F = q^{-1} \xi \bar{\xi} \frac{1}{q^2 |x|^2 + \rho^2} \frac{1}{|x|^2 + \rho^2}.
\]

This has manifestly the desired \( q \to 1 \) limit \([5.1]\). The second choice,

\[
l = -\frac{1 + q^2}{1 + q^4 \frac{1}{1 + q^2 \rho^2}}
\]

\[
\rho^2 := \frac{1 + q^2}{1 + q^4 \rho^2},
\]

is designed in order that \( A \) is proportional to the \( a_{\alpha_0} \) of \([3.34]\), so that \( A^{\alpha \beta} \) span the (3,1) dimensional, irreducible corepresentation of \( SU_q(2) \times SU_q'(2) \). The result is:

\[
\tilde{A} = -\frac{1 + q^2}{1 + q^4 \rho^2} a_{\alpha_0} |x|^2 + \rho^2
\]

\[
\tilde{F} = \frac{1 + q^4 \rho^2}{1 + q^4 \rho^2} \xi \bar{\xi} \frac{1}{|x|^2 + \rho^2} \frac{1}{|x|^2 + \rho^2}.
\]

This also has the desired \( q \to 1 \) limit \([5.1]\). If \( \rho^2 \neq 0 \), in both cases \( FF \) is regular everywhere and decreases as \( 1/|x|^8 \) as \( x \to \infty \), therefore it virtually will yield finite action \( S \) and Pontryagin index \( Q \) upon integration.

As in the undeformed case, to make the determination of multi-instanton solutions easier it is useful to go to the “singular gauge”. Note that as in the \( q = 1 \) case \( T = x/|x| \) is unitary and formally not continuous at \( x = 0 \), so it can play the role of a ‘singular gauge transformation’. In fact \( A \) can be obtained through the gauge transformation \( A = T(A T + dT) \) from the “singular” gauge potential

\[
\hat{A} = TdT \frac{1}{1 + |x|^2 \rho^2}.
\]
\[
\hat{F} = Tq^{-1}\bar{\xi}\bar{\xi} - \frac{1}{q+1}\xi_{\alpha\alpha'}x^{\beta\beta'} - \frac{1}{1 + |x|^2/\rho^2} \tag{5.7}
\]

which is the analog of the instanton solution in the “singular gauge” found by ’t Hooft in [50]. By singular gauge potential it is meant that it has a pole in \(|x| = 0\). More generally, the generic solution (5.3) can be obtained through the gauge transformation \(\hat{A}_l = T(\hat{A}_l T + dT)\) from a singular solution \(\hat{A}_l\). The latter can be obtained also by starting from an Ansatz like \(\hat{B} = \bar{\xi}\bar{\xi}_l + \theta I_2 \tilde{m}\), instead of (5.2), and imposing that the \(\bar{\xi}\bar{\xi}\) and the \(\bar{\xi}\theta x\) term in \(\hat{F} = \hat{B}^2\) appear in a combination proportional to (3.38).

A straightforward computation by means of (3.9) shows that \(\hat{A}\) can be expressed also in the form

\[
\hat{A} = (\hat{D}\phi)\phi^{-1}, \tag{5.9}
\]

where \(\hat{D}\) is the first-order-differential-operator-valued \(2 \times 2\) matrix obtained from the expression in the square bracket in (5.7) by the replacement

\[
x^{\alpha\alpha'}/|x|^2 \rightarrow q^4 \partial^{\alpha\alpha'}, \nonumber
\]

\[
\hat{D} := q^3\bar{\xi}\partial - \frac{q}{q+1}dI_2, \quad \tag{5.10}
\]

(for simplicity we are here assuming that \(\rho^2\) commutes with \(\xi\partial\)) and \(\phi\) is the harmonic potential

\[
\phi := 1 + \rho^2 \frac{1}{|x|^2}, \quad \Box \phi = 0.
\]

This is the analog of what happens in the classical case.

The \textbf{anti-instanton solution} is obtained just by converting unbarred into barred matrices, and conversely, as in the \(q = 1\) case. For instance, from (5.4) we obtain the anti-instanton solution in the regular gauge

\[
A' = -(dT)T \frac{1}{1 + \rho^2/|x|^2}, \tag{5.11}
\]

and for the one in the singular gauge \(\hat{A}' = (\hat{D}'\phi)\phi^{-1}\), where

\[
\hat{D}' := q^3\bar{\xi}\partial - \frac{q}{q+1}dI_2. \tag{5.12}
\]
Recovering the instanton projective module of [12]

In commutative geometry the instanton projective module $E$ over $A$ and the associated gauge connection can be most easily obtained using the quaternion formalism, in the way described e.g. in Ref. [2]. $H \sim \mathbb{R}^4$ can be compactified as $P_1(H) \sim S^4$. Let $(w, x) \in H^2$ be homogeneous coordinates of the latter, and choose $w = I_2$ on the chart $H \sim \mathbb{R}^4$. The element $u \in H^2$ defined by

$$u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I_2 \\ \rho x/|x|^2 \end{pmatrix} \left(1 + \frac{\rho^2}{|x|^2}\right)^{-1/2}$$  \hspace{1cm} (5.13)

fulfills $u^\dagger u = I_2 1$, and the $4 \times 2 A$-valued matrix $u$ has only three independent components. Therefore the $4 \times 4 A$-valued matrix

$$\mathcal{P} := uu^\dagger = \begin{pmatrix} I_2 \\ \rho \bar{x}/|x|^2 \\ \rho^2/|x|^2 I_2 \end{pmatrix} \left(1 + \frac{\rho^2}{|x|^2}\right)^{-1/2}$$  \hspace{1cm} (5.14)

is a self-adjoint three-dimensional projector. It is the projector associated in the Serre-Swan theorem correspondence to the gauge connection (5.6), by the formula $\hat{A} = u^\dagger du$. The associated projective module $\mathcal{E}$ is embedded in the free module $A^{16}$ seen as $M_4(A)$, and is obtained from the latter as $\mathcal{E} = \mathcal{P} M_4(A)$.

In the present $q$-deformed setting we immediately check that the element $u \in H^2_q$ defined by (5.13) fulfills $u^\dagger u = I_2 1$ again, so that the $4 \times 2 A$-valued matrix $\mathcal{P}$ defined by (5.14) is hermitean and idempotent, and has only 3 independent components. Therefore, it defines the ‘instanton projective module’ $\mathcal{E} = \mathcal{P} M_4(A)$ also in the $q$-deformed case. One can easily verify that $\mathcal{P}$ reduces to the hermitean idempotent $e$ of [12] if one chooses the instanton size as $\rho = 1/\sqrt{2}$ and performs the change of generators (2.25). Therefore, interpreting the model [12] as a compactification to $S_4^4$ of ours, we can use all the results [12] about the Chern-Connes classes of $e$.

Unfortunately in the $q$-deformed case it is no more true that $\hat{A} = u^\dagger du$, essentially because the $|x|$-dependent global factor multiplying the matrix at the rhs(5.14) does not commute with the 1-forms of the present calculus ($|x|\xi^i = q\xi^i|x|$).

6 Changing the size and shifting the center of the (anti)instanton

Applying the $\tilde{SO}_q(4)$ coaction (2.22) to $|x|^2, \xi \bar{x}, \xi x$ and using (2.10) we obtain

$$\Delta_L (|x|^2) = |x|^2|c|^2, $$
\[
\Delta_L(\xi x) = |c|^2 a \xi x a^{-1}, \quad \Delta_L(\xi x) = |c|^2 b \xi x b^{-1},
\]
\[
\Delta_L(\xi x) = |c|^2 a \xi x a^{-1}, \quad \Delta_L(\xi x) = |c|^2 b \xi x b^{-1},
\]
where \(|c|^2 := |a|^2 |b|^2\). The result is the same also if we consider \(|c|^2\) as an independent parameter and choose \(a, b\) with unit \(q\)-determinant \(|a| = |b| = 1\). If we apply \(\Delta_L\) to the instanton gauge potentials \([5.3]\) we thus find
\[
\Delta_L(A_l(\xi, x)) = a A_l(\xi|c|, x|c|) a^{-1}
\]
\[
\Delta_L(F_l(\xi, x)) = a F_l(\xi|c|, x|c|) a^{-1}.
\]
In particular, on the gauge potential \([5.4]\)
\[
\Delta_L(A) = -a (dT) T^{-1} a^{-1}
\]
\[
\Delta_L(F) = a \xi \frac{1}{q^2 |x|^2 + \rho^2} q^{-1} \rho^2 \frac{1}{|x|^2 + \rho^2} a^{-1}
\]
where we have set \(\rho^2 := \rho^2 |c|^{-2}\). These gauge potentials are again solutions of the self-duality equation, since the latter is covariant under the \(SO_q(4)\) coaction. The result of the \(SO_q(4)\) coaction \(||a| = |b| = 1\) can be reabsorbed into a (global) gauge transformation \([4.9]\), with \(U = a^{-1}\) (and similarly \(U = b^{-1}\) for the anti-instanton gauge potentials), i.e. is a gauge equivalent solution. Note that we are thus introducing gauge transformations depending on the additional noncommuting parameters \(a, b\). A full \(\widetilde{SO_q(4)}\) coaction \(||c| \neq 1\) instead involves also a change of the size of the instanton, and gives an inequivalent solution. We can thus obtain any size starting from the instanton with unit size.

Having built an (anti)instanton “centered at the origin” with arbitrary size one would like first to translate the center to another point \(y\), then to construct \(n\)-instanton solutions “centered at points \(y\_\mu\), \(\mu = 1, 2, \ldots, n\). The appropriate framework is to replace tensor products \(\otimes\) by braided tensor products \(\otimes\) and apply the braided coaddition \([34]\) to the covectors \(x\). This gives new (i.e. gauge inequivalent) solutions. The braided coaddition \([34]\) of the coordinates \(x\) reads
\[
\Delta(x) = x \otimes 1 + 1 \otimes x \equiv x - y,
\]
where we have renamed \(x := x \otimes 1\), \(y := -1 \otimes x\). It follows
\[
P_{h_k}^{ij} y^h y^k = 0 \quad \iff \quad y \bar{y} = \bar{y} y = I_2 |y|^2
\]
Out of the two possible braidings we choose the following one:
\[
y^h x^i = q \tilde{R}_{jk}^{hi} x^j y^k \quad \iff \quad y^{\alpha \gamma} x^{\beta \gamma'} = \tilde{R}_{\alpha \delta}^{\alpha \gamma} \tilde{R}_{\gamma' \delta}^{\beta \gamma'},
\]
\[
\partial_i y^j = q \tilde{R}_{jk}^{hi} y^k \partial_h \quad \iff \quad \partial_{\alpha \gamma} y^{\beta \gamma'} = \tilde{R}_{\alpha \delta}^{\beta \gamma} \tilde{R}_{\gamma' \delta}^{\alpha \gamma'},
\]
\[
y^h \xi^i = q \tilde{R}_{jk}^{hi} \xi^j y^k \quad \iff \quad y^{\alpha \gamma} \xi^{\beta \gamma'} = \tilde{R}_{\alpha \delta}^{\beta \gamma} \tilde{R}_{\gamma' \delta}^{\alpha \gamma'},
\]

23
(the commutation relations between $y$ and $\xi$ are determined up to a ‘conformal factor’; we have fixed the latter in such a way that they look exactly as the commutations relations between $x$ and $\xi$). As a result,

\begin{align*}
  d y &= y d, \\ y \theta &= \theta y, \\
  (x - y)^i \xi^i &= q R^{ij}_{hk} s^h (x - y)^k, \\
  \partial (x - y)^i &= \delta^i_j + q R^{jk}_{ih} (x - y)^k \partial_h,
\end{align*}

in other words, the differential calculus is invariant under the replacement $x \to x - y$ (i.e. under $A$). This implies that under this replacement solutions go into solutions. Therefore the instanton solution with “shifted” center $y$ will read in the regular gauge

\begin{align*}
  A &= -d \left[ \frac{(x - y)}{|x - y|} \left( \frac{|x - y|}{|x - y|} \right)^{1 + \rho^2 / |x - y|^2} \right] \\
  F &= q^{-1} \xi \xi \left( \frac{|x - y|}{|x - y|} \right)^{2 + \rho^2 / |x - y|^2}.
\end{align*}

and in the singular gauge

\begin{align*}
  \hat{A} &= (\hat{D} \phi) \phi^{-1}, \\
  \hat{F} &= q^{-1} \xi \xi \left( \frac{|x - y|}{|x - y|} \right)^{2 + \rho^2 / |x - y|^2}.
\end{align*}

We conclude this section by sketching how one obtains the ‘infinitesimal’ version of (6.1), (6.9), i.e. transformations of the solutions under the action of the cross-product $F^{\prime} \triangleright U_q \mathfrak{so}(4)$ [10] [33] [11] [22] (i.e. the U.E.A. of the Euclidean quantum group extended with dilatations), where $F'$ is the subalgebra of $\mathcal{H}$ generated by the $\Delta_{\alpha \alpha'}$. As known, the (right) action $\triangleright$ of the dual Hopf algebra $H'$ of a Hopf algebra $H$ can be obtained from the (left) coaction $\Delta_L (v) = v(1) \otimes v(2)$ of the latter (in Sweedler notation) by the rule $v \triangleright h' = \langle v(1), h' \rangle v(2)$ (here $\langle , , \rangle$ denotes the pairing between $H, H'$). For $H' = U_q su(2) \otimes U_q' su(2)$ one finds in particular

\begin{equation}
  v^{\alpha \alpha'} \triangleright g g' = [\tau(g) v \tau(g')]^{\alpha \alpha'} = \tau^{\alpha \beta}_\beta(g) v^{\beta \beta'} \tau^{\beta'}_{\alpha'}(g'),
\end{equation}

where $v = x, \partial, \xi, g \in U_q su(2), g' \in U'_q su(2)$ $[g g' = g' g$ in $U_q su(2) \otimes U_q' su(2)]$, and $\tau$ is the fundamental 2-dim representation of $U_q su(2)$. One finds the following transformation of the instanton solution $A_l$ under $q$-rotations:

\begin{align*}
  A_l \triangleright g g' &= \varepsilon(g') A_l, \\
  A_l \triangleright g &= \tau(g(1)) A_l \tau(S g(2))
\end{align*}

where $\varepsilon, S$ denotes the counit, antipode of $U_q su(2), U_q' su(2)$ and we have used Sweedler notation (with suppressed summation index) for the coproduct $\Delta(g) = g(1) \otimes g(2)$. The transformation law for the antiinstanton solution is obtained exchanging $g$ with $g'$.

\footnote{On the FRT generators $L^\pm_\delta$ of $U_q su(2)$ one has $\tau^{\alpha}_\beta(L^\pm_\delta) = \hat{R}^{\pm 17 \alpha}_\beta$.}
In section 3 we have introduced partial derivatives \( \partial_i \) acting from the left, as conventional. This means that the deformed Leibniz rule takes the form \( \partial_{\alpha \alpha'} (f f') = \partial_{\alpha \alpha'} (f) f' + O_{\alpha \alpha'}^{\gamma \gamma'} (f) \partial_{\gamma \gamma'} (f') \), with a suitable linear operator \( O_{\alpha \alpha'}^{\gamma \gamma'} \). The generators of infinitesimal translations in the right action \( \xi \) are instead derivatives \( \partial_{\alpha \alpha'} \) acting from the right, i.e. \( f f' \partial_{\alpha \alpha'} = f (f' \partial_{\alpha \alpha'}) + (f \partial_{\gamma \gamma'}) O_{\alpha \alpha'}^{\gamma \gamma'} (f') \). The quickest way to determine their action on a function (or differential from) form \( \omega \) is to recall [35] that this is determined by the equation

\[
\omega (x - y) = \omega (x) - (\omega (x) \xi) y^{\gamma \gamma'} + O (y^2)
\]

namely is the coefficient of the term of degree 1 in \(- y^{\gamma \gamma'}\) in the expansion of \( \omega (x - y) \) in powers of \( y^{\gamma \gamma'} \) (put on the right of all \( \xi, x \)'s). One thus easily finds, for instance,

\[
A^{\alpha \beta} \xi_{\gamma \gamma'} = q^{-2} \left[ A^{\alpha \beta} \frac{x^{\lambda \lambda'}}{\rho^2 + |x|^2} + \varepsilon^{\lambda \lambda'} \frac{\delta_{\alpha \beta}}{2 (\mu^2 + \rho^2)} \right] \xi_{\lambda \lambda'} \xi_{\gamma \gamma'} - \xi_{\alpha \gamma'} \xi_{\beta \gamma'} \xi_{\gamma \gamma'} \xi_{\gamma \gamma'} \frac{\rho^2 + \mu^2}{\rho^2 + \mu^2}
\]

(6.13)

on the instanton gauge potentials (5.3). The \( \partial_{\alpha \alpha'} \) can be easily realized as elements of \( F' \simeq U_q so(4) \), or also of the Heisenberg algebra \( \mathcal{H} \).

7 Multi-instanton solutions

On the basis of the latter and of the \( q = 1 \) results [50][51], we first look for \( n \)-instanton solutions of the self-duality equation in the “singular gauge” in the form (5.9). Beside the coordinates \( x^i \equiv - y^i_0 \) we introduce \( n \) other coordinates \( y^i_\mu, \mu = 1, 2, ..., n \) generating as many \( \hat{R}^4_q \) and braided to each other:

\[
P_{A^{ij} k} y^j_\mu y^k_\mu = 0 \quad \iff \quad y_\mu \bar{y}_\mu = \bar{y}_\mu y_\mu = I_2 |y_\mu|^2
\]

(7.1)

\[
y^i_\nu y^j_\mu = q \hat{R}^{hi}_{jk} y^j_\mu y^k_\nu \quad \iff \quad y^{i \alpha \beta} y^{j \beta \gamma} = \hat{R}^{i \gamma \delta}_{j \beta \delta} y^{j \gamma \gamma'} y^{j \gamma \delta'}
\]

with \( \mu < \nu \). We shall call \( A_n \) the larger algebra generated by the \( y^i_\mu \)'s and by parameters \( \rho^i_\mu, \mu = 1, ..., n \) fulfilling the commutation relations

\[
\rho^2_\mu \rho^2_\nu = q^2 \rho^2_\mu \rho^2_\nu, \quad \nu < \mu,
\]

(7.2)

\[
\rho^2_\mu y^i_\nu - y^i_\mu \rho^2_\nu = \begin{cases} q^{-2} & \nu < \mu, \\ 1 & \nu \geq \mu. \end{cases}
\]

(7.3)

We shall also enlarge \( A_n \) to the extended Heisenberg algebra \( \mathcal{H}_n \) and extended algebra of differential forms \( \Omega^* (\mathcal{A}_n) \) by adding as generators the \( \partial_i \) and the \( \xi^i \) respectively, and to the extended differential calculus algebra \( \mathcal{D} (\mathcal{A}_n) \) by adding as generators both the \( \xi^i, \partial_i \), with cross commutation
relations
\[ \rho_\mu^2 \xi^i = \xi^i \rho_\mu^2, \quad \partial_i \rho_\mu^2 = \rho_\mu^2 \partial_i, \quad (7.4) \]
\[ \partial_i y_j^\mu = q \hat{R}^j_{ik} y_k^\mu \partial_h \quad \Leftrightarrow \quad \partial_{\alpha \gamma} y_\mu^{\beta \gamma'} = \hat{R}^\beta_{\alpha \gamma'} \hat{R}^\gamma_{\gamma' \beta} y_\mu^{\gamma'} \partial_\beta', \quad (7.5) \]
\[ y_k^h \xi^i = q \hat{R}^h_{jk} \xi^j y_k^\mu \quad \Leftrightarrow \quad y_\mu^{\alpha \gamma} \xi^{\beta \gamma'} = \hat{R}^\alpha_{\gamma \delta} \hat{R}^{\gamma'}_{\delta \gamma} \xi^{\beta \gamma'} y_\mu^{\delta \gamma'}, \quad (7.6) \]

Note that the first relations, together with the decomposition \( d = \xi^i \partial_i \), imply
\[ d \rho_\mu^2 = \rho_\mu^2 d. \quad (7.7) \]

Also, from these relations it is evident that \( \hat{\Omega}^2 (A_n), \hat{\Omega}'^2 (A_n) \) are \( A_n \)-bimodules (resp. \( \hat{DC}^2 (A_n), \hat{DC}'^2 (A_n) \) are \( H_n \)-bimodules).

In the sequel we shall introduce the short-hand notation
\[ z^i_\mu := x^i - v^i_\mu, \quad v^i_\mu := \sum_{\nu=1}^{\mu} y^i_\nu, \quad \mu = 1, 2, ..., n; \]

\( v^i_\mu \) will play the role of coordinates of the center of the \( \mu \)-th instanton. It is easy to check from (7.1) that these new \( n \) sets of variables generate as many copies of the quantum Euclidean space \( \mathbb{R}_{\hbar}^4 \), namely
\[ P_{\alpha \beta} z^i_\mu z^j_\mu = 0, \quad \Leftrightarrow \quad z^i_\mu z^i_\mu = |z^i_\mu|^2 I_2 \quad (7.8) \]
and together with \( x^i \) make up an alternative Poincaré-Birkhoff-Witt basis of the algebra \( A_n \), (i.e. ordered monomials in these variables make up a basis of the vector space underlying \( A_n \)). Moreover, differentiating \( z^i_\mu \) and commuting it with \( \xi^j \) is like differentiating and commuting \( x^j \):
\[ \partial_i z^j_\mu = \delta^j_i + q \hat{R}^j_{ik} z^k_\mu \partial_h, \quad (3.4) \]
\[ z^j_\mu \xi^i = q \hat{R}^j_{ik} \xi^j \xi^i. \quad (3.1) \]

Therefore for any \( \mu = 1, 2, ..., n \) the replacement \( x \rightarrow z^i_\mu \) in any true relation involving \( x, \partial, \xi \) will generate a new true relation, which we shall label by adding the subscript \( \mu \) to the original one, as we have just done.

The solution \( \phi \) searched for (5.9) is of the form
\[ \phi \equiv \phi_n = 1 + \rho_1^2 \frac{1}{|x - y_1|^2} + \rho_2^2 \frac{1}{|x - y_1 - y_2|^2} + ... + \rho_n^2 \frac{1}{|x - y_1 - ... - y_n|^2} \quad (7.9) \]
or, more compactly,
\[ \phi_n = 1 + \sum_{\mu=1}^{n} \rho_\mu^2 \frac{1}{|z^i_\mu|^2}, \]

namely a scalar “function” of the coordinates \( x^i \), of the instanton “sizes” \( \rho_\mu \) and of the “coordinates of their centers”. For this to be allowed we have further enlarged \( A_n, \Omega^*(A_n), H_n, DC(A_n) \) to extended algebras
\(A^e_n, \Omega^e(A^e_n)H^e_n, \mathcal{D}C(A^e_n)\) by adding as generators inverse elements \(1/|z_\mu|\), but we also add the inverses \(1/\phi_m\), together with corresponding commutation relations (see the appendix) consistent with the ones given so far.

By Remark 1 and relation (3.10), \(\phi\) is harmonic, exactly as in the classical case. In the appendix we prove more:

**Lemma 1** Denoting \(\phi_q(\{z_i\}) := \phi(\{qz_i\})\),

\[
\Box \phi \sim \bar{\partial}\partial \phi = \partial \bar{\partial} \phi = 0 \quad (i.e. \phi \text{ is harmonic}), \tag{7.10}
\]

\(\phi \xi^i = \xi^i \phi_q\), \tag{7.11}

\([\phi, (\partial_i \phi)] = 0 = [\phi, (\partial_h \partial_i \phi)]\), \tag{7.12}

\(\mathcal{P}_{h_{ij}}(\partial_h \phi)(\partial^i \phi) = 0\), \tag{7.13}

\((d \phi)(d \phi) = 0\). \tag{7.14}

We are now ready to prove

**Theorem 1** \(\hat{A} = (\hat{\mathcal{D}} \phi)\phi^{-1}\) with \(\phi\) defined in (7.9) fulfills the selfduality equation (4.7).1.

**Proof** We denote \(n_q := 1 + q + \ldots + q^{n-1}\). We find

\[
d \xi = -\xi d = \frac{-q^2}{1 + q^2} [\epsilon^{-1} (\xi \xi \bar{\theta}) + \xi \xi \bar{\theta}], \tag{7.15}
\]

\[\hat{\mathcal{D}} \xi = q^2 \bar{\xi} \bar{\partial} \xi - \frac{q}{q + 1} d \xi - q \bar{\xi} \bar{\partial} \xi - \frac{q^{-1} \delta_q}{2q} d \xi, \tag{7.16}\]

\[d \bar{\xi} \partial \phi = -\bar{\xi} d \partial \phi \tag{7.11}\]

Moreover,

\[
(\xi \partial \phi)(d \phi) = -d [(\xi \partial \phi) \phi] + (d \xi \partial \phi) \phi \tag{7.12}
\]

\[
= -d [\phi_q^{-1}(\bar{\xi} \partial \phi)] + (d \bar{\xi} \partial \phi) \phi \tag{7.12}
\]

\[
= -d (\phi_q^{-1})(\xi \partial \phi) - \phi_q^{-1}(d \xi \partial \phi) + (d \bar{\xi} \partial \phi) \phi \tag{7.12}
\]

\[
= -d (\bar{\xi} \partial \phi)(\phi - \phi_q). \tag{7.18}
\]

Therefore

\[
\hat{F} = d \left[ (\hat{\mathcal{D}} \phi)\phi^{-1} + (\hat{\mathcal{D}} \phi)\phi^{-1}(\hat{\mathcal{D}} \phi)\phi^{-1} \right]
\]

\[
= (d \hat{\mathcal{D}} \phi)(d \phi)\phi^{-1} + (\hat{\mathcal{D}} \phi)\phi^{-1}(d \phi)\phi^{-1} + (\hat{\mathcal{D}} \phi)\phi^{-1}(\hat{\mathcal{D}} \phi)\phi^{-1}
\]

\[
= (d \hat{\mathcal{D}} \phi)\phi^{-1} + (\hat{\mathcal{D}} \phi) \left[ (\hat{\mathcal{D}} + d) \phi \right] \phi^{-1} \phi_q^{-1}
\]

\[
= (q^2 d \xi \partial \phi) \phi^{-1} + (\hat{\mathcal{D}} \phi) \left[ (q^2 \hat{\xi} \partial + \frac{1}{q + 1} d) \phi \right] \phi^{-1} \phi_q^{-1}
\]

27
an element of our algebra is: 1. analytic in $z$ strength. Here we address this issue semi-heuristically. We shall say that singularity is only due to the choice of a singular gauge and can be removed if this is a selfdual matrix, $\hat{F} \in M_2 (\hat{\Omega}^2 (A_n))$, because $\xi \bar{\xi}$ is.

Formally, as $x \to \infty$ also $z_\mu \to \infty$, $\phi \to 1$, and a simple inspection shows that $\hat{A} \to 0$ as $1/|x|^3$, $\hat{F} \to 0$ as $1/|x|$, exactly as in the case $q = 1$. Therefore $\hat{F} \hat{F}$ decreases fast enough at infinity for integrals like $\int \text{tr}(\hat{F} \hat{F})$ to be well defined at infinity.

On the other hand, as $z_\mu \to 0$ the function $\phi$ and therefore the gauge potential $\hat{A}$ are singular, i.e. formally diverge. We don’t know yet whether the singularity will cause problems also in a proper functional-analytical treatment (this requires analyzing representations of the algebra). If this is the case then, as in the undeformed theory, the question arises if this singularity is only due to the choice of a singular gauge and can be removed by performing a suitable gauge transformation, or it really affects the field strength. Here we address this issue semi-heuristically. We shall say that an element of our algebra is: 1. analytic in $z_\mu$ if its power expansion has no poles in $z_\mu$, i.e. does not depend on $1/|z_\mu|$; 2. regular in $z_\mu$ if it formally keeps finite as $z_\mu \to 0$, i.e. in its power expansion the dependence on $1/|z_\mu|$ occurs only through $z_\mu/|z_\mu|$. Since such dependences might change upon changing the order in which the variables $z_1, z_2, \ldots, z_n$, and possible extra variables $1/|z_1 - z_2|, 1/|z_1 - z_3|, \ldots$ (if necessary), are displayed, these conditions have to be met for any order. In appendix A.3 we show that performing the “singular gauge transformation” $U_2$ defined by

$$U_2 \equiv U_2(z_1, z_2) := \frac{z_1 \ y_2 \ z_2}{|z_1| \ |y_2| \ |z_2|} \quad (7.19)$$

on $\hat{A}_2$ we obtain a 2-istanton solution

$$A_2 = U_2^{-1} \left( \hat{A} U_2 + dU_2 \right) \quad (7.20)$$

analytic in both $z_1, z_2$; the corresponding selfdual field strength will be analytic as well. The form of $U_2$ exactly mimics the undeformed one of Ref. [25] [42]. Of course, for this to make sense, we have to further enlarge

\[ (q^3 d\xi \partial \phi) \phi^{-1} + \left[ q^3 \left( \hat{D} \xi \phi \right) \partial \phi + \frac{1}{q+1} \left( \hat{D} \phi \partial \phi \right) \phi^{-1} \right] \phi^{-1} \phi_q^{-1} \]

\[ (q^3 d\xi \partial \phi) \phi^{-1} + \left\{ q \left( \hat{D} \xi \phi \right) \left( \partial \phi \right) + \frac{q^3}{q+1} \left( \xi \partial \phi \phi \right) \phi^{-1} \right\} \phi^{-1} \phi_q^{-1} \]

\[ (q^3 d\xi \partial \phi) \phi^{-1} - \left\{ \left[ q^2 \xi \partial \phi + \frac{3}{2} d\xi \phi \right] \phi^{-1} \phi_q^{-1} \right\} \phi^{-1} \phi_q^{-1} \]

\[ q^3 (d\xi \partial \phi) \left[ \phi^{-1} + \frac{1}{2q} (\phi_q^{-1} - \phi^{-1}) \right] - \left[ q^2 \xi \partial \phi + (q^2 + 1)(d\xi \partial \phi) \right] \phi_q^{-1} \phi^{-1} \phi_q^{-1} \]

\[ -\frac{q^5}{4q} \left[ -q^2 \xi \partial \phi \right] \left[ \phi^{-1} + \phi_q^{-1} \right] + q^2 \xi \partial \phi \phi^{-1} \phi_q^{-1} \phi^{-1} \phi_q^{-1} ; \]
the algebras adding as a generator $1/|y_2|$ with consistent commutation relations; this is done in the subsection A.1. By generalization of the undeformed results [25, 42], we are led to the

**Conjecture.** Performing the singular gauge transformation $U_n$ recursively defined by $U_0 = 1_2$ and

$$U_n = U_n(z_1, ..., z_n) := U_{n-1}(z_1, ..., z_{n-1})U_{n-1}^{-1}(y)\frac{\bar{z}_{n}}{|z_n|}, \quad (7.21)$$

with $U_m(y)$ the function of $y_1, ..., y_m$ only defined by $U_m(y) := U_m(z_1 - z_n, ..., z_{n-1} - z_n)$, we finally obtain a regular $n$-istanton solution

$$A \equiv A_n = U_n^{-1}(\hat{A}U_n + dU_n) \quad (7.22)$$

and a corresponding regular selfdual field strength, for any $n$.

Results for the $n$-antiistanton solutions are obtained by the already mentioned replacements. In particular, the singular ones $\hat{A}$ are simply obtained replacing $D$ with $D'$ in (5.9).

### A Appendix

#### A.1 Additional relations for the extended algebra

Let $z := x - y$, where $y$ is defined as in section 6. Let $a \cdot b := a^{\alpha\beta}b^{\beta\gamma}\epsilon_{\alpha\beta\epsilon\gamma\gamma'}$. The following relations are consequences of the commutation relations for the generators $x^i, y^i, z^i, \rho_x$ or are (the only) consistent extensions of these consequences to the square root, inverse, and inverse square root of $|z|^2, |x|^2, |y|^2$ having the desired, commutative $q \to 1$ limit.

$$C\xi^i = q^2\xi^iC, \quad \text{for } C = |x|^2, |y|^2, x \cdot y, |z|^2 \quad (A.1)$$

$$\gamma\xi^i = q^{-1}\xi^i\gamma, \quad \text{for } \gamma = \frac{1}{|x|}, \frac{1}{|y|}, \frac{1}{|z|} \quad (A.2)$$

$$y^i|x|^2 = q^2|x|^2 y^i, \quad x^i|y|^2 = q^{-2}|y|^2 x^i, \quad x \cdot y = q^2 y \cdot x, \quad (A.3)$$

$$y^i|x|^\pm1 = q^\mp1|x|^\pm1 y^i, \quad x^i|y|^\pm1 = q^\pm1|y|^\pm1 x^i, \quad \frac{1}{|y|} \frac{1}{|x|} = \frac{q}{|x|} \frac{1}{|y|} \quad (A.4)$$

$$[y^i, y \cdot x] = |y|^2 x^i(1 - q^{-2}), \quad [x^i, y \cdot x] = -y^i|x|^2(1 - q^{-2}) \quad (A.5)$$

$$z^j \frac{x^j}{|x|^2} = q^{-1}R_{jk}^{ij} \frac{x^h}{|x|^2} z^k + (1 - q^{-2})g^{ij} \quad (A.6)$$

$$z^\bar{x} \frac{\bar{x}}{|x|^2} = -q^{-2} \frac{x}{|x|^2} \bar{z} + q^{-4} \frac{x}{|x|^2} \cdot z + (1 - q^{-4}) \quad (A.7)$$
\[ \frac{x^i}{|x|} z^2 = q^2 |z|^2 \frac{x^i}{|x|^2} (1 - q^2) z^i, \quad (A.8) \]

\[ \frac{1}{|z|^2} \frac{x^i}{|x|^2} = \frac{x^i}{|x|^2} q^2 (1 - q^2) z^i, \quad (A.9) \]

\[ \frac{x^i}{|x|^2} |z| = q |z| \frac{x^i}{|x|^2} (1 - q) z^i, \quad (A.10) \]

\[ \frac{1}{|z|} \frac{x^i}{|x|^2} = \frac{x^i}{|x|^2} q (1 - q) \frac{z^i}{|z|^2}, \quad \frac{q}{|z|} \frac{y^j}{|y|^2} = \frac{y^j}{|y|^2} \frac{1}{|z|} (1 - q) \frac{z^i}{|z|^2}, \quad (A.11) \]

\[ |z|^2 \frac{1}{|x|^2} = |x|^2 [q^4 |z|^2 + (1 - q^2) x \cdot z], \quad (A.12) \]

\[ |z|^2 \frac{1}{|x|^2} = q^4 |x|^2 |z|^2 + (1 - q^2) \left[ q^4 \frac{x}{|x|^2} \cdot z + (1 - q^{-2}) \right] \quad (A.13) \]

\[ \frac{1}{|z|} \frac{1}{|x|^2} = \frac{q^2}{|x|^2} \frac{1}{|z|} + (q^{-1} - 1) \frac{x}{|x|^2} \frac{z}{|z|^2} \quad (A.14) \]

\[ \left[ \xi^h \frac{x^i}{|x|^2}, z \right]^2 = (1 - q^2) \xi^h z^i, \quad (A.15) \]

\[ \left[ \xi^h \frac{x^i}{|x|^2}, \frac{1}{|z|} \right] = (1 - q^{-1}) \xi^h z^i, \quad (A.16) \]

\[ [T \partial_z, \frac{1}{|z|}] = (1 - q^{-1}) T_z a \partial_z \frac{1}{|z|}, \quad (A.17) \]

\[ \frac{|x|}{|y|} \text{ commutes with } x^i, y^j, z^i, |x|, |y|, |z| \quad (A.18) \]

For instance, relations (A.2) are postulated by consistency with (A.1). Relation (A.7) follows from (A.4), (2.21), (2.19), (2.14), (6.5). Eq. (A.8), (A.9) follow from the preceding ones. Relations (A.10), (A.11) are postulated by consistency with (A.7), (A.8). Eq. (A.15) follows from (A.8), (A.9) and (A.1). Relation (A.16) follows from (A.10), (A.11) and (2). Eq. (A.17), where we have set \( T_z := z/|z| \), is a particular consequence of (A.16). Eq. (A.18) follows from (7.3).

From (A.4) it also follows

\[ \frac{1}{|x|} z = z \frac{q}{|x|} + (1 - q) \frac{x}{|x|}, \quad \frac{1}{|y|} z = z \frac{q^{-1}}{|y|} + (q^{-1} - 1) \frac{y}{|y|}, \quad (A.19) \]

\[ |x| \frac{1}{|y|} x = q^{-1} x |x| \frac{1}{|y|}, \quad |x| \frac{1}{|y|} y = q^{-1} y |x| \frac{1}{|y|}, \]

\[ |x| \frac{1}{|y|} z = z |x| \frac{q^{-1}}{|y|}, \quad |x| \frac{1}{|y|} z = z |x| \frac{1}{|y|}, \quad \left[ |x| \frac{1}{|y|}, \frac{z}{|z|} \right] = 0. \quad (A.20) \]
Lemma 2

\[
\begin{align*}
\bar{z} \frac{\bar{y}}{|y|} \frac{x}{|x|} &= \frac{x}{|x|} \frac{\bar{y}}{|y|} \bar{z}, \\
\bar{z} \frac{x}{|x|} \frac{\bar{y}}{|y|} &= \frac{\bar{y}}{|y|} \frac{x}{|x|} \bar{z}, \\
\bar{z} \frac{x}{|x|} \frac{\bar{y}}{|y|} &= \frac{\bar{y}}{|y|} \frac{x}{|x|} \bar{z}, \\
\bar{z} \frac{x}{|x|} \frac{\bar{y}}{|y|} &= \frac{\bar{y}}{|y|} \frac{x}{|x|} \bar{z}.
\end{align*}
\]

(A.21)

Proof

We use (A.20) and (A.8)

\[
\bar{z} \frac{\bar{y}}{|y|} \frac{x}{|x|} = \frac{x}{|x|} \frac{\bar{y}}{|y|} \bar{z},
\]

(A.22)

whereas

\[
\bar{z} \frac{x}{|x|} \frac{\bar{y}}{|y|} = \frac{\bar{y}}{|y|} \frac{x}{|x|} \bar{z}.
\]

(A.23)

Proposition 2

\[
\begin{align*}
\left| z \right|^2, \frac{\bar{y}}{|y|} \frac{x}{|x|} &= \left| z \right|^2, \frac{\bar{y}}{|y|} \frac{x}{|x|} = \left| z \right|^2, \frac{\bar{y}}{|y|} \frac{x}{|x|} = \left| z \right|^2, \frac{\bar{y}}{|y|} \frac{x}{|x|} = 0,
\end{align*}
\]

(A.24)

\[
\begin{align*}
\left| z \right|^\pm 1, \frac{\bar{y}}{|y|} \frac{x}{|x|} &= \left| z \right|^\pm 1, \frac{\bar{y}}{|y|} \frac{x}{|x|} = \left| z \right|^\pm 1, \frac{\bar{y}}{|y|} \frac{x}{|x|} = \left| z \right|^\pm 1, \frac{\bar{y}}{|y|} \frac{x}{|x|} = 0,
\end{align*}
\]

(A.25)

\[
\begin{align*}
U_2(x, z) := \left| z \right| \frac{\bar{y}}{|y|} \frac{x}{|x|} &= \left| z \right| \frac{\bar{y}}{|y|} \frac{x}{|x|} = \left( \frac{\bar{z}}{|z|^2} - \frac{\bar{x}}{|x|^2} \right) \frac{\bar{y}}{|y|} \frac{x}{|x|} \left| z \right|, \\
U_2^{-1}(x, z) &= \left| z \right| \frac{\bar{y}}{|y|} \frac{x}{|x|} = \left( \frac{\bar{z}}{|z|^2} - \frac{\bar{x}}{|x|^2} \right) \frac{\bar{y}}{|y|} \frac{x}{|x|} \left| z \right|,
\end{align*}
\]

(A.26)

(A.27)

(A.28)

Proof

Eq. (A.24) are direct consequences of \( \left| z \right|^2 = \bar{z} z = z \bar{z} \) and of the relations in the lemma. (A.25) are derived by consistency with (A.24). The first equality in (A.26) is a direct consequence of (A.21), and of (A.25); the second equality is a consequence of (A.23), (A.10), (A.20). Eq. (A.28) follows from (A.25) and \( [z, |z|] = [\bar{z}, |z|] = 0 \).
Relations (3.5), (7.6), (3.1), (6.5) respectively imply
\[ \xi \bar{\partial} + q^2 \partial \xi = q^{-2} dI_2, \quad \bar{\xi} \partial + q^2 \bar{\partial} \xi = q^{-2} dI_2, \] (A.29)
\[ \xi \bar{y} + y \bar{\xi} = q^{-2} (\xi \cdot y) I_2, \quad \bar{\xi} y + \bar{y} \xi = q^{-2} (\xi \cdot y) I_2, \] (A.30)
\[ \xi \bar{z} + z \bar{\xi} = q^{-2} (\xi \cdot z) I_2, \quad \bar{\xi} z + \bar{z} \xi = q^{-2} (\xi \cdot z) I_2, \] (A.31)
\[ x \bar{y} + y \bar{x} = q^{-2} (x \cdot y) I_2, \quad \bar{x} y + \bar{y} x = q^{-2} (x \cdot y) I_2, \] (A.32)

A quick way to prove these relations is to note that they can be obtained from (3.15) by the following replacements: \( x/|x|^2 q^2 (1-q^2) \ra \partial \) (see Remark 1), \( x \ra y, x \ra z, x \ra y \) and \( \xi \ra x \), respectively.

**Remark 2.** By (A.19), reordering \( 1/|x|, 1/|y| \) w.r.t. \( x^i, z^i \) does not introduce additional powers of \( 1/|x|, 1/|y|, 1/|z| \). Consequently, for any \( f(x,z) \) analytic w.r.t. \( x, z, f1/|x| = (1/|x|)g, f1/|y| = (1/|y|)h \), with \( g(x,z), h(x,z) \) analytic functions w.r.t. \( x, z \). By (A.14), reordering \( 1/|z| \) w.r.t. \( 1/|x|^2 \) does not introduce additional powers of \( 1/|x| \).

**Remark 3.** Any relation (...) or Remark proved/postulated so far in this appendix is mapped into a new true/consistent one, which we shall label as (...)\( _\mu \) or (...)\( _{\mu, \nu} \) according to the cases, by the replacements \( x \ra z_\mu, \rho_x \ra \rho_\mu, y \ra \sum_{\lambda=\mu+1}^{\nu} y_\lambda, z \ra z_\nu, \rho_z \ra \rho_\nu \) with \( \nu > \mu \).

### A.2 Proof of Lemma 1

Relation (7.10) is a straightforward consequence of (7.4)\( _\mu \) and of (3.10)\( _\mu, \mu = 1, 2, ..., n \). Relation (7.11) is a straightforward consequence of (7.4)\( _1 \) and of (A.2)\( _\mu \). To prove (7.12), (7.13) we first state the following relations:

\[
\frac{1}{|z_\nu|^2} y_\mu^i = \frac{q^2}{|z_\nu|^2} y_\mu^i, \quad \frac{\rho_\nu^2}{|z_\nu|^2} y_\mu^i = \frac{q^2}{|z_\nu|^2} y_\mu^i, \quad \left[ \frac{\rho_\nu^2}{|z_\nu|^2}, \frac{\rho_\mu^2}{|z_\mu|^2} \right] = 0, \quad \nu < \mu \] (A.33)

\[
\left[ \frac{\rho_\nu^2}{|z_\nu|^2}, \frac{\rho_\mu^2}{|z_\mu|^2} \right] = 0, \quad \nu \leq \mu \] (A.34)

\[
\left[ \frac{\rho_\nu^2}{|z_\nu|^2}, \frac{\rho_\mu^2}{|z_\mu|^2} \right] = (1 - q^2) \frac{z_\mu^k}{|z_\mu|^4} \rho_\mu^2 \frac{\rho_\mu^2}{|z_\mu|^2}, \quad \nu > \mu \] (A.35)

\[
P_{\mu}^{ij} z_\mu^i k \frac{1}{|z_\mu|^2} = -P_{\mu}^{ij} z_\mu^i k \frac{1}{|z_\mu|^2}, \quad \nu > \mu \] (A.36)

\[
P_{\mu}^{ij} z_\mu^i k \frac{1}{|z_\mu|^2} = -q^2 P_{\mu}^{ij} z_\mu^i k \frac{1}{|z_\mu|^2}, \quad \nu > \mu \] (A.37)

Relations (A.33) follow from (A.4)\( _{\mu, \nu} \) and (7.3). The first two relations (A.34) follow from (A.33), the third from (7.3). (A.35) is an immediate
As a consequence we have also (A.34). Relation (A.36) is a consequence of (A.34) and (A.37), a consequence of (A.34) and (A.37). Relation (A.36) is a consequence of (A.34) and (A.37), a consequence of (A.37), (7.8), (7.1), and (2.14), A.38 a consequence of (A.37), (7.8), (A.8) and 

To prove (7.12) we proceed as follows:

\[ \phi(\partial^k \phi) = -q^{-4} \left[ 1 + \sum_{\mu = 0}^{n} \rho^2_{\mu} \sum_{\nu = 0}^{n} \frac{z^k_{\nu}}{|z_{\nu}|^2} \right] \sum_{\nu = 0}^{n} \frac{z^k_{\nu}}{|z_{\nu}|^4} \rho^2_{\nu} \]

\[ = (\partial^k \phi) - q^{-4} \left[ \sum_{\mu = 0}^{n} \frac{\rho^2_{\mu}}{|z_{\mu}|^2} \frac{z^k_{\mu}}{|z_{\mu}|^2} \sum_{\nu > \mu}^{n} \frac{\rho^2_{\nu}}{|z_{\nu}|^2} \frac{z^k_{\nu}}{|z_{\nu}|^2} \right] + q^2 \sum_{\mu, \nu = 0}^{n} \frac{z^k_{\mu}}{|z_{\mu}|^2} \frac{\rho^2_{\nu}}{|z_{\nu}|^2} \left[ \sum_{\mu = 0}^{n} \frac{\rho^2_{\mu}}{|z_{\mu}|^2} \right] \]

\[ = (\partial^k \phi) - q^{-4} \sum_{\nu = 0}^{n} \frac{z^k_{\nu}}{|z_{\nu}|^2} \left[ \sum_{\mu = 0}^{n} \frac{\rho^2_{\mu}}{|z_{\mu}|^2} \right] = (\partial^k \phi) \phi \]

(in the fourth equality we have used the fact that the second term in the inner bracket is proportional to and therefore can be put together with the second in the square bracket). Similar is the proof of (7.12). As a consequence we have also \[ \phi^{-1}, (\partial^k \phi) \] = 0 and, by the replacement \( z_{\mu} \rightarrow q z_{\mu}, \phi^{-1}, (\partial^k \phi) = 0 \). We now prove (7.13)

\[ P_{A_{hk}} (\partial^k \phi) (\partial^k \phi) \]

\[ \sim P_{A_{hk}} \left[ \sum_{\mu = 0}^{n} \frac{z^h_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} \frac{z^k_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} + \sum_{\mu, \nu = 0}^{n} \frac{z^h_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} \frac{z^k_{\nu} \rho^2_{\nu}}{|z_{\nu}|^2} + \sum_{\mu, \nu = 0}^{n} \frac{z^h_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} \frac{z^k_{\nu} \rho^2_{\nu}}{|z_{\nu}|^2} \right] \]

\[ = P_{A_{hk}} \left[ \sum_{\mu = 0}^{n} \frac{z^h_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} |z^k_{\mu}|^2 + \sum_{\mu, \nu = 0}^{n} \frac{z^h_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} \frac{z^k_{\nu} \rho^2_{\nu}}{|z_{\nu}|^2} + \sum_{\mu, \nu = 0}^{n} \frac{z^h_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} \frac{z^k_{\nu} \rho^2_{\nu}}{|z_{\nu}|^2} \right] \]

The first term in the square bracket vanishes because of (7.8), whereas, because of (A.34), (A.36), the other two give, as claimed,

\[ P_{A_{hk}} (\partial^k \phi) (\partial^k \phi) \sim P_{A_{hk}} \sum_{\mu, \nu = 0}^{n} \left[ \frac{1}{|z_{\mu}|^2} \frac{z^h_{\mu} \rho^2_{\mu}}{|z_{\mu}|^2} \frac{z^k_{\nu} \rho^2_{\nu}}{|z_{\nu}|^2} + \frac{z^h_{\nu} \rho^2_{\nu}}{|z_{\nu}|^2} \left( \frac{z^k_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} \frac{\rho^2_{\nu}}{|z_{\nu}|^2} + (1 - q^2) \frac{z^k_{\mu} \rho^2_{\mu}}{|z_{\mu}|^4} \frac{\rho^2_{\nu}}{|z_{\nu}|^2} \right) \right] \]
A.3 Proof of the analyticity of \( A \)

In the last but one equality we have used (A.38), (7.8). Finally, (A.11)

\[
(d\phi)(d\phi) = (d\phi)(\xi^i \partial_i \phi) = (d\xi^i g^j)(\partial_i \phi) = -q^{-2} \xi^i (d\phi)(\partial_i \phi)
\]

\[
= -q^{-2} \xi^i \xi^j (\partial_j \phi)(\partial_i \phi) = -q^{-2} \xi^i \xi^k \partial^j \partial^m \phi_{ij} \partial^m \phi_{ij}
\]

\[
= -q^{-2} \xi^i \xi^k g^j g^l \phi_{ij} \partial^m \phi_{ij}
\]

proves (7.14). In the last but one equality we have used the property (see e.g. [15, 19]) \([P_A, P(g \otimes Cg)] = 0\), where \( P \) denotes the permutation matrix.

A.3 Proof of the analyticity of \( A_2 \)

For any quaternion \( w \) let \( V(w) := w/|w| \). As a consequence, \( V^{-1}(w) := \overline{w}/|w| \). So \( T = V(x) \). We shall use also the shorter notation \( T_n := V(z_n) \).

Having defined \( U_2 \) as in (7.19), we find

\[
U_2^{-1}(dU_2) = T_2 V^{-1}(y_2) T_1 (d\overline{T}_1) V(y_2) \overline{T}_2 + T_2 (d\overline{T}_2).
\]

(A.39)

From the definition (7.39) it follows, for both \( \mu = 1, 2 \),

\[
\phi_2^{-1} = \frac{|z_\mu|^2}{\rho_\mu^2} f_\mu, \text{ where } f_\mu \text{ is analytic in } z_\mu.
\]

(A.40)

Using properties (A.35), (A.34) we immediately find for \( m = 1, 2 \)

\[
[\phi_m, T_2] = 0, \quad [\phi_1, \phi_2] = 0.
\]

(A.41)

whereas we find, as consequences of (A.18), (A.25), (7.3)

\[
[\phi_m, T_1 V^{-1}(y_2)] = [\phi_m, V(y_2) \overline{T}_1] = [\phi_m, U_2] = 0.
\]

(A.42)

Moreover, by straightforward calculations,

\[
(\hat{\mathcal{D}} \phi_2) = -(d\overline{T}_1) T_1 \frac{\rho_1^2}{|z_1|^2} - (d\overline{T}_2) T_2 \frac{\rho_2^2}{|z_2|^2}.
\]

(A.43)

We first show that \( A_2 \) is an analytic function of \( z_1 \):
Finally, the term $T z \phi$ As

Looking at (3.17) we see that $T_2^2 \left[ \hat{(A^2)} \right] V(y_2) T_2^2 \left( 1 + \frac{1}{T_2^2} \right) \frac{f_1}{T_2^2} + T_2(dT_2)$

$= U_2^{-1}(dT_1)V(y_2)T_2^2 \left( \phi_2 - \frac{\rho_2}{|z_1|^2} \right) \phi_2^{-1}$

$= -U_2^{-1}(dT_2)T_2 U_2 \phi_2^{-1} \frac{\rho_2^2}{|z_2|^2} + T_2(dT_2)$

$= T_2^2 V^{-1}(y_2)T_1(dT_1) V(y_2) T_2^2 \left( 1 + \frac{1}{T_2^2} \right) \frac{|z_1|^2}{\rho_1} \frac{f_1}{T_2^2} + T_2(dT_2)$

$= T_2^2 V^{-1}(y_2)T_1(dT_1) \frac{|z_1|^2}{\rho_1} V(y_2) T_2^2 \left( 1 + \frac{1}{T_2^2} \right) f_1$

$= -q^{-1} \frac{|z_1|^2}{\rho_1} V^{-1}(y_2)T_2(dT_2) V(y_2) \frac{z_1}{\rho_1} f_1 \frac{\rho_2^2}{|z_2|^2} + T_2(dT_2)$.

As $\phi_1$ does not depend on $z_2$ and $T_2(dT_2)|z_2|^2, f_2$ are analytic in $z_2$, the
first term is. On the other hand, the second term is manifestly analytic in $z_2$. Now, by a further gauge transformation $\tilde{U} := V(y_2)T_1$,

$$A_2 = \tilde{U}^{-1}A\tilde{T}_2\tilde{U} + T_1(dT_1).$$

$\tilde{U}$ is an analytic function of $z_2$, therefore by Remark 2 the first term remains analytic in $z_2$ (however we fix to order the variables $z_1, z_2, 1/|y_2|$); the second term is even independent of $z_2$, so $A_2$ is analytic in $z_2$.

References

[1] P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml, J. Wess, “Twisted Gauge Theories”, *Lett. Math. Phys.* 78 (2006), 61-71

[2] M. F. Atiyah, *Geometry on Yang-Mills fields Lezioni Fermiane*, Scuola Normale Superiore di Pisa (1979).

[3] M. F. Atiyah, N. J. Hitchin, V. G. Drinfel’d, Yu. I. Manin, *Construction of instantons, Phys. Lett.* 65A (1978), 185-187.

[4] A. A. Belavin, A. M. Polyakov, A. S. Schwarz and Yu. S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations, Phys. Lett.* 59B (1975), 85-87.

[5] F. Bonechi, N. Ciccoli, M. Tarlini, “Noncommutative instantons on the 4-sphere from quantum groups” *Commun. Math. Phys.* 226 (2002), 419-432.

[6] U. Carow-Watamura, M. Schlieker, S. Watamura, “$SO_q(N)$ covariant differential calculus on quantum space and quantum deformation of Schroedinger equation”, *Z.Phys.* C 49 (1991), 439-446.

[7] B. L. Cerchiai, G. Fiore, J. Madore, “Geometrical Tools for Quantum Euclidean Spaces”, *Commun. Math. Phys.* 217 (2001), 521-554.

[8] C. S. Chu, P. M. Ho, B. Zumino, “Some complex quantum manifolds and their geometry”, *Quantum Fields and Quantum Space Time* (Cargese, 1996), 281–322, NATO Adv. Sci. Inst. Ser. B Phys. 364, Plenum, New York, 1997 (Editors: G.’t Hooft, A.Jaffe, G.Mack, P. Mitter, R. Stora).

[9] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[10] A. Connes, *Non-commutative Geometry and Physics*, Les Houches, Session LVII, Elsevier Science B. V., 1994.

[11] A. Connes, G. Landi, “Noncommutative manifolds, the instanton algebra and isospectral deformations” *Commun. Math. Phys.* 221 (2001), 141-159.

[12] “Ludwik Dabrowski, Giovanni Landi, Tetsuya Masuda, Instantons on the Quantum 4-Spheres $S^4_\theta$”, *Commun. Math. Phys.* 221 (2001), 161-168.
[13] M. Dimitrijevic, F. Meyer, L. Möller, J. Wess, “Gauge theories on the kappa-Minkowski spacetime”, Eur. Phys. J. C36 (2004), 117-126.

[14] V. Drinfeld, “Quantum groups,” in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[15] L. D. Faddeev, N. Y. Reshetikhin, L. A. Takhtadjan, “Quantization of Lie groups and Lie algebras”, Algebra i Analiz 1 (1989), 178-206, translated from the Russian in Leningrad Math. J. 1 (1990), 193-225.

[16] L.D. Faddeev, L. A. Takhtadjan, “Liouville model on the lattice”, Lecture Notes in Physics 246 (1989), Springer, New York, 1986, pp. 166-179.

[17] G. Fiore, “On the hermiticity of q-differential operators and forms on the quantum Euclidean spaces $\mathbb{R}^N_q$”, Rev. Math. Phys. 18 (2006), 79-117.

[18] G. Fiore, “$q$-Deformed quaternions and $su(2)$ instantons”, J. Phys.: Conference Series 53 (2006), 885-899. Proceedings of ”Noncommutative Geometry in Field and String Theories”, Satellite Workshop of ”CORFU Summer Institute 2005”.

[19] G. Fiore, “$q$-Euclidean Covariant Quantum Mechanics on $\mathbb{R}^N_q$: Isotropic Harmonic Oscillator and Free particle” (PhD Thesis), SISSA-ISAS (1994).

[20] G. Fiore, “Quantum group covariant (anti)symmetrizers, $\varepsilon$-tensors, vielbein, Hodge map and Laplacian”, J. Phys. A: Math. Gen. 37 (2004), 9175-9193.

[21] G. Fiore, “Quantum Groups $SO_q(N), Sp_q(n)$ have q-Determinants, too”, J. Phys. A: Math Gen. 27 (1994), 3795-3802.

[22] G. Fiore, “The Euclidean Hopf algebra $U_q(\mathfrak{e}^N)$ and its fundamental Hilbert space representations”, J. Math. Phys. 36 (1995), 4363-4405.

[23] G. Fiore, “On q-quaternions”, in preparation.

[24] G. Fiore, J. Madore “The geometry of the quantum Euclidean space” J. Geom. Phys. 33 (2000), 257-287.

[25] J. J. Giambiaggi, K. D. Rothe “Regular N-istanton fields and singular gauge transformations” Nucl. Phys. B129 (1977), 111-124.

[26] J. M. Gracia-Bondía, J. C. Varilly, H. Figueroa, Elements of Noncommutative Geometry, Birkhäuser Boston, Inc., Boston, MA, 2001.

[27] R. Jackiw, C. Rebbi, “Conformal properties of a Yang-Mills pseudoparticle”, Phys. Rev. D14 (1976), 517-523. Phys. Rev. D14 (1976), 518-523.

[28] B. Jurco, L. Möller, S. Schraml, P. Schupp, J. Wess, “Construction of non-Abelian gauge-theories on noncommutative spaces”, Eur. Phys. J. C21 (2001), 383-388.
[29] T. Kobayashi, T. Uematsu, “$q$-deformed conformal and Poincaré algebras on quantum 4-spinors”, Z. Phys. C58 (1993), 559-566.

[30] G. Landi, An Introduction to Noncommutative Spaces and Their Geometries, Springer-Verlag, Berlin, 1997.

[31] G. Landi, W. van Suijlekom, “Noncommutative instantons from twisted conformal symmetries”, Comm. Math. Phys. 271 (2007), 591–634.

[32] J. Madore, “An introduction to noncommutative differential geometry and its physical applications”. Second edition. London Mathematical Society Lecture Note Series, 257. Cambridge University Press, Cambridge, 1999.

[33] S. Majid, “Braided Momentum Structure of the q-Poincaré Group”, J. Math. Phys. 34 (1993) 2045-2058.

[34] For a review see for instance: S. Majid, Foundations of Quantum Groups, Cambridge Univ. Press (1995); and references therein.

[35] S. Majid, “Free Braided Differential Calculus, Braided Binomial Theorem and the Braided Exponential Map”, J.Math.Phys. 34 (1993), 4843-4856.

[36] S. Majid, $q$-Euclidean space and quantum group wick rotation by twisting, J. Math. Phys. 35 (1994), 5025-5034.

[37] S. Majid, “$q$-Epsilon tensor for quantum and braided spaces” J. Math. Phys. 36 (1995), 1991-2007.

[38] N. Nekrasov, A. Schwarz, “Instantons on noncommutative $\mathbb{R}^4$, and $(2,0)$ superconformal six dimensional theory” Commun.Math.Phys. 198 (1998), 689-703.

[39] O. Ogievetsky “Differential operators on quantum spaces for $GL_q(n)$ and $SO_q(n)$” Lett. Math. Phys. 24 (1992), 245-255.

[40] O. Ogievetsky, WB Schmidke, J. Wess, B. Zumino, “$q$-Deformed Poincaré algebra”, Commun. Math. Phys. 150 (1992), 495-518.

[41] O. Ogievetsky, B. Zumino “Reality in the Differential calculus on the $q$-Euclidean Spaces”, Lett. Math. Phys. 25 (1992), 121-130.

[42] D. I. Olive, S. Sciuto, R. J. Crewther, “Instantons in field theory”, Riv. Nuovo Cim. 2 (1979), 1-117.

[43] P. Podlés, S. L. Woronowicz, “Quantum deformation of Lorentz group”, Commun. Math. Phys. 130 (1990), 381-431.

[44] A. Schirrmacher, “Remarks on the use of $R$-matrices”, in Quantum groups and related topics (Wrocław, 1991), 55–65,Math. Phys. Stud. 13, Kluwer Acad. Publ., Dordrecht, 1992. A. Sudbery, Math. Proc. Cambridge Philos. Soc. 114 (1993), 111-130; Phys. Lett. B 284 (1992), 61-65; Phys. Lett. B 291 (1992), 519-519.
[45] M. Schlieker, W. Weich and R. Weixler, “Inhomogeneous quantum groups”, \textit{Z. Phys. C} \textbf{53} (1992), 79-82;

[46] P. Schupp, P. Watts, B. Zumino, “Differential Geometry on Linear Quantum Groups”, \textit{Lett. Math. Phys.} \textbf{25} (1992), 139-148.

[47] N. Seiberg, E. Witten, “String Theory and Noncommutative Geometry” \textit{JHEP} \textbf{9909} (1999) 032 (93 pp.).

[48] H. Steinacker, “Integration on quantum Euclidean space and sphere in $N$ dimensions”, \textit{J. Math Phys.} \textbf{37} (1996), 4738-4749.

[49] R. G. Swan, “Vector Bundles and Projective Modules”, \textit{Trans. Am. Math. Soc.} \textbf{105} (1962), 264-277.

[50] G. ‘t Hooft, “Computation of the Quantum Effect Due to a Four-dimensional Pseudoparticle” \textit{Phys. Rev.} \textbf{D14} (1976), 3432-3450.

[51] F. Wilczek, in ‘Quark confinement and field theory’, Ed. D. Stump and D. Weingarten, John Wiley and Sons, New York (1977). E. Corrigan, D. B. Fairlie, \textit{Phys. Lett.} \textbf{67B} (1977), 69-71. R. Jackiw, C. Nohl, C. Rebbi \textit{Phys. Rev.} \textbf{D15} (1977), 1642-1646.

[52] S. L. Woronowicz, “Compact Matrix Pseudogroups”, \textit{Commun. Math. Phys.} \textbf{111} (1987) 613-665.

[53] S. L. Woronowicz, “Differential calculus on compact matrix pseudogroups (Quantum Groups),” \textit{Commun. Math. Phys.} \textbf{122} (1989) 125-170.

[54] S. L. Woronowicz, “Twisted $SU(2)$ group, an example of a noncommutative differential calculus,” \textit{Publ. RIMS, Kyoto Univ.} \textbf{23} (1987) 117.