Inverse spectral problem for normal matrices and a generalization of the Gauss-Lucas theorem

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Abstract. We establish an analog of the Cauchy-Poincare separation theorem for normal matrices in terms of majorization. Moreover, we present a solution to the inverse spectral problem (Borg-type result). Using this result we essentially generalize and extend the known Gauss–Lucas theorem about the location of the roots of a complex polynomial and of its derivative. The last result is applied to prove the well-known old conjectures of de Bruijn-Springer and Schoenberg.

1. Introduction

Let $A = A^*$ be a selfadjoint $n \times n$-matrix, $A_{n-1}$ its principal $(n - 1) \times (n - 1)$ submatrix, obtained by deleting the last row and column. According to the Cauchy–Poincare interlacing theorem their spectra $\sigma(A) = \{\lambda_j\}_{1}^{n}$ and $\sigma(A_{n-1}) = \{\mu_j\}_{1}^{n-1}$ separate each other, that is

\[ \lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n \]  

(1.1)

It is known (see [16, 23]) that the converse is also true, that is for any two sequences $\{\lambda_j\}_{1}^{n}$ and $\{\mu_j\}_{1}^{n-1}$ of real numbers, satisfying (1.1), there exists (nonunique) $n \times n$ selfadjoint matrix $A$ such that $\sigma(A) = \{\lambda_j\}_{1}^{n}$ and $\sigma(A_{n-1}) = \{\mu_j\}_{1}^{n-1}$. We say that such a matrix $A = A^*$ solves the inverse spectral problem for these sequences.

The result of Hochstadt, see [15] (an analog the well known Borg uniqueness result for Sturm-Liouville equation) claims that there exists the unique Jacobi (tridiagonal) selfadjoint matrix $A$ solving the inverse problem.

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In the present paper we generalize both the Cauchy-Poincare separation theorem and the Hochstadt theorem to the case of normal matrices.

It is obvious that an analogous result can not hold literally for a normal matrix $A$, first because the eigenvalues are not real and second, a principal submatrix $A_{n-1}$ is normal only in trivial cases (see [10] and Lemma 4.9).

Our first main result is Theorem 3.7. It provides necessary and sufficient geometric conditions for the sequences $\{\lambda_j\}_{n}^{1}$ and $\{\mu_j\}_{n-1}^{1}$ to be the spectrum of a normal matrix $A$ and its submatrix $A_{n-1}$ respectively.

In order to formulate these geometrical conditions we introduce (in Section 2) several concepts of majorization for sequences of vectors from $\mathbb{R}^n$, being natural generalizations of the classical ones and coinciding with those for $n = 1$.

Note however, that the sufficient part of the above mentioned conditions can be expressed analytically without majorization and reads as follows

$$c_k := \frac{\prod_{j=1}^{n-1}(\mu_j - \lambda_k)}{\prod_{1 \leq j \leq n, j \neq k}(\lambda_j - \lambda_k)} \geq 0, \quad k \in \{1, \ldots, n\}. \quad (1.2)$$

The second main topic of our paper is concentrated around the known Gauss-Lucas theorem [28]. According to this theorem the roots $\{\mu_k\}_{1}^{n-1}$ of the derivative $p'$ of any complex polynomial $p(\in \mathbb{C}[z])$ of degree $n$ lie in the convex hull of the roots $\{\lambda_j\}_{1}^{n}$ of the polynomial $p$.

At a first view this topic is rather far from the above one. Nevertheless the following proposition establishes a "bridge" between the two main topics (parts) of the paper.

**Proposition 1.1.** Let $p(z)$ be a polynomial of degree $n$ with zeros $\{\lambda_j\}_{1}^{n}$ and $\{\mu_j\}_{1}^{n-1}$ the zeros of its derivative. Then there exists a (nonunique) normal matrix $A \in M_n(\mathbb{C})$ such that $\sigma(A) = \{\lambda_j\}_{1}^{n}$ and $\sigma(A_{n-1}) = \{\mu_j\}_{1}^{n-1}$.

The proof is immediately implied by (1.2) since now $c_k = 1$, $k \leq n$.

Combining Proposition 1.1 with Theorem 3.7 (our solution to inverse problem) and setting $\mu_n := (\sum_{j=1}^{n} \lambda_j)/n$, we immediately arrive (see Proposition 1.2) at the following result: let $\mu := \{\mu_j\}_{1}^{n}$, $\lambda := \{\lambda_j\}_{1}^{n}$. There exists a doubly stochastic $n \times n$-matrix $S$ such that $\mu = S\lambda$.

This result essentially improves the Gauss-Lucas theorem. Proposition 1.1 allows us to apply linear algebra techniques to the investigation of the location of the zeros of a polynomial and of its derivative. For example, using the exterior algebra techniques we obtain a generalization
of the Gauss-Lucas theorem for the products of roots (see Proposition 4.2).

Note, that while the second topic has attracted a lot of attention during two last decades (see [2], [5], [7], [26], [27], [31]), our approach seems to be new and perspective. In particular, in the framework of this approach we get simple (and short) solutions to the old problems of de Bruijn-Springer [6] and Schoenberg [33].

Let us briefly describe these problems, fixing the above notations.

In 1948 de Bruijn and Springer [6] conjectured that the following inequality holds for any convex function $f : \mathbb{C} \to \mathbb{R}$:

$$1 \leq 1 \frac{1}{n-1} \sum_{j=1}^{n-1} f(\mu_j) \leq \frac{1}{n} \sum_{j=1}^{n} f(\lambda_j).$$

They succeeded in proving this inequality for a class of convex functions. We provide a proof of this conjecture by showing that a bistochastic matrix $S$ in the above mentioned representation $\mu = S\lambda$ can be choosen in such a way that all entries in the last row equal $1/n$.

Moreover, we prove (Theorem 4.6) that the following inequality is valid for any $k \in \{1, \ldots, n - 1\}$

$$\frac{1}{(n-1)k!} \sum_{\pi=1}^{n-1} f\left(\prod_{j=1}^{k} (\mu_{\pi_j} - \alpha)\right) \leq \frac{1}{k!} \sum_{\pi=1}^{n} f\left(\prod_{j=1}^{k} (\lambda_{\pi_j} - \alpha)\right).$$

(1.3)

In 1986 Schoenberg [33] (see also [7]) conjectured that if $\sum_{j=1}^{n} \lambda_j = 0$, then

$$n \sum_{j=1}^{n-1} |\mu_j|^2 \leq (n - 2) \sum_{j=1}^{n} |\lambda_j|^2$$

and the equality holds if and only if all numbers $\lambda_j$ lie on the same line.

We establish this inequality in Proposition 1.1 (see Theorem 4.10).

Let us briefly sketch the contents of the paper.

In section 2 we introduce two new notions of majorization for sequences of vectors with nonequal numbers of entries and establish some simple properties of those. We also study a connection between different concepts of majorization and show that (on the contrary to the scalar case), they are not equivalent. In particular, this provides a negative answer to the question from the book of Marshall and Olkin [24], p.433.

In section 3 we establish an analog of the Cauchy-Poincare separation theorem for normal matrices (Theorem 3.7). As a corollary, we get
an analogous result for "noncommutative" convex combinations of normal matrices (Corollary 3.11). Moreover, we solve an inverse Borg-type problem, generalizing the result of Hochstadt [15].

In Section 4 we essentially generalize and extend the known Gauss–Lucas theorem based on our solution to inverse problem for normal matrices. Finally, we apply this result in order to obtain complete solutions to the de Bruijn-Springer and Schoenberg conjectures.

The preliminary version of our paper has already been published as a preprint [20]. The main results of the paper have been announced without proofs in [21].

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2. Majorization

2.1. Two definitions of majorization.

We start with several known definitions (see [14], [23], [24]).

Notation 2.0.1. Let $X$ be a subset of $\mathbb{R}^m$. Denote by $\text{conv}(X)$ the convex hull of $X$, i.e. the smallest convex set, containing $X$.

If $X$ is convex, let $\text{Ext}X$ denote the (nonempty) set of its extreme points.

Next, $\text{CV}(Y)$ stands for the set of all convex functions on a convex set $Y$.

As usual, denote by $C := A \circ B$ the Schur (element-wise) product of two $n \times n$-matrices $A = (a_{ij})$ and $B = (b_{ij}) : (c_{ij}) := (a_{ij}b_{ij})$.

Definition 2.1. a) A matrix $A \in M_n(\mathbb{R})$ is called bistochastic (doubly stochastic) iff all its entries are nonnegative and the sum of the elements in each row and each column equals one.

We denote the set of all bistochastic matrices by $\Omega_n \subset M_n(\mathbb{R})$.

b) A matrix $A \in M_n(\mathbb{R})$ is called unitary-stochastic (orthostochastic) if there exists a unitary (orthogonal) matrix $U \in M_n(\mathbb{C})(U \in M_n(\mathbb{R}))$ such that $A = U \circ \bar{U}$.

The set of all unitary stochastic matrices is denoted by $\Omega_n^u$.

Remark 2.1. The set of all bistochastic matrices is convex and contains all transposition matrices.
The known Theorem of Birkhoff states that the set \( \text{Ext}(\Omega_n) \) coincides with the set of all transposition matrices, and thus by the Krein-Mil’man theorem \( \Omega_n \) is the convex hull of all transposition matrices \([14], [23], [24]\).

Not that each unitary stochastic \( n \times n \)-matrix is bistochastic, i.e. \( \Omega_n^u \subset \Omega_n \). But converse is not true: not every bistochastic matrix is unitary-stochastic \([23], [24]\).

**Definition 2.2.** Let \( x = \{x_k\}_{k=1}^l \) and \( y = \{y_k\}_{k=1}^m \) be two sequences of vectors in \( \mathbb{R}^n \) and \( l \leq m \). Suppose that the following conditions are fulfilled:

\[
\text{conv}(x_{i_1}, \, i_1 = 1, l) \subset \text{conv}(y_i : \, i = 1, m),
\]

\[
\text{conv}(x_{i_1} + \ldots + x_{i_k} : \, 1 \leq i_1 < \ldots < i_k \leq l) \subset \text{conv}(y_{i_1} + \ldots + y_{i_k} : \, 1 \leq i_1 < \ldots < i_k \leq m)
\]

\[
x_1 + \ldots + x_l \in \text{conv}(y_{i_1} + \ldots + y_{i_l} : \, 1 \leq i_1 < \ldots < i_l \leq m)
\]

Then we say that the sequence \( \{y_k\} \) majorates the sequence \( \{x_k\} \) and write \( x \prec y \);

**Remark 2.2.** If \( l = m \), the last condition turns into

\[
x_1 + \ldots + x_m = y_1 + \ldots + y_m.
\]

**Definition 2.3.** Let \( x := \{x_k\}_{k=1}^l \) and \( y := \{y_k\}_{k=1}^m \) be two sets of vectors \( x_k, y_k \in \mathbb{R}^n \).

We say that \( x \) is bistochastically majorated by \( y \) and write \( x \prec_{uds} y \), if there exist vectors \( x_{i_{l+1}}, \ldots, x_m \in \mathbb{R}^n \) and a bistochastic matrix \( S \in \Omega_m \), such that \( \overline{x} := \{x_k\}_{k=1}^m = (I_n \otimes S)y \).

If \( S \) can be chosen to be unitary stochastic, we write \( x \prec_{uds} y \).

Next we compare these definitions to the classical ones.

For this purpose we recall the notion of majorization (\([14], [23], [24]\)) for sequences of real numbers (the case \( n = 1 \)).

**Definition 2.4.** Let there be given two real sequences \( \alpha := \{\alpha_k\}_{1}^m \) and \( \beta := \{\beta_k\}_{1}^m \). Let also \( \hat{\alpha} \) and \( \hat{\beta} \) be these sequences, reordered to be decreasing. If

\[
\hat{\beta}_1 + \ldots + \hat{\beta}_j \leq \hat{\alpha}_1 + \ldots + \hat{\alpha}_j, \quad j \in \{1, \ldots, m\}
\]

then it is usually written \( \beta \ll \alpha \).
If, moreover,
\[ \sum_{k=1}^{m} \alpha_k = \sum_{k=1}^{m} \beta_k, \]
then the sequence \( \beta \) is said to be majorized by \( \alpha \) which is denoted by \( \beta \prec \alpha \).

The following famous theorem due to Weyl, Birkhoff and Hardy-Littlewood-Polya (see [13, 14, 23, 24]), explains the connection between two different definitions of majorization in the scalar case \((n = 1)\).

**Theorem 2.5.** Let \( \alpha, \beta \in \mathbb{R}^m \) be to real sequences. Then the following are equivalent
1) \( \beta \prec \alpha \).
2) The following inclusion holds true:
\[ \beta \in \text{conv}(\{ A\alpha : A \text{ is a permutation matrix} \}) = \text{conv}(\{\{\alpha_{i_1}, \ldots, \alpha_{i_m}\} : \{i_1, \ldots, i_m\} \text{ is a permutation of the set } \{1, \ldots, m\}\}) \]
(2.3)
3) There exists a bistochastic matrix \( S \in M_m(\mathbb{R}) \), such that \( \beta = S\alpha \).
   It fact, the matrix \( S \) can be chosen to be orthostochastic.
4) The inequality
\[ \sum_{i=1}^{m} f(\beta_i) \leq \sum_{i=1}^{m} f(\alpha_i) \]
holds for any convex function \( f \) on \( \mathbb{R} \).
5) The inequality
\[ f(\beta_1, \ldots, \beta_m) \leq f(\alpha_1, \ldots, \alpha_m) \]
holds for any convex function \( f \) in \( \mathbb{R}^m \) which is symmetric, that is, invariant under any permutation of the coordinates.

**Remark 2.3.** Note that in the case \( n = 1 \) and \( l = m \) definitions [2.2] and [2.3] are equivalent to the above Definition [2.4] of majorization for real sequences. In fact, we have
\[ \beta \prec \alpha \iff \beta = S\alpha \iff (-\beta) \prec (-\alpha). \]
(this fact very easy to check explicitly). Now, the convex hull of a set of real numbers is the closed interval between the minimal to the maximal numbers. Thus Definition [2.2] is a natural generalization of the standard one in \( \mathbb{R}^1 \).

It is very easy to see that the following proposition is valid
Proposition 2.6. If \( x = \{x_k\} \prec_{ds} y = \{y_k\} \), then \( x \prec y \).

One can suppose, that a complete analog of Theorem 2.5 is valid, that is the partial orders \( \prec \) and \( \prec_{ds} \) are equivalent. But the following example shows, that it is not the case.

Example 2.7. Let \( n = 2 \) and \( m = 4 \). Set
\[
x = \{x_1, \ldots, x_4\} = \{(12, 12), (12, 12), (5, 3), (3, 5)\},
y = \{y_1, \ldots, y_4\} = \{(8, 16), (16, 8), (0, 0), (8, 8)\}
\]
(2.6)

It is easy to check by hand, that \( x \prec y \). At the same time it is easy to see that the vector \( x_1 = (12, 12) \) can be uniquely expressed as a convex combination of \( y_k \)-s: \( x_1 = 1/2(y_1 + y_2) \). Suppose now, that there exists a bistochastic matrix \( S \), such that \( x = Sy \). Then \( S \) has the form
\[
S = \begin{pmatrix}
1/2 & 1/2 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
0 & 0 & s_{33} & s_{34} \\
0 & 0 & s_{43} & s_{44}
\end{pmatrix},
\]
which is impossible, since \( x_3, x_4 \) do not belong to the convex hull of \( y_3, y_4 \).

Remark 2.4. On the p. 433 of [24] Marshall and Olkin, mention the condition 2) of Proposition 2.10 as the weakest possible notion of majorization. By virtue of Proposition 2.10 it is equivalent to \( \prec \) for \( l = m \). Furthermore they say that the relation between \( \prec \) and \( \prec_{ds} \) is not clear. Example 2.7 provides a negative answer to this question.

We note the following simple

Proposition 2.8. a) Let \( \{y_k\}_{k=1}^m \) be such, that \( \text{conv}(\{y_k\}_{k=1}^m) \) is affine isomorphic to the standard simplex
\[
\Sigma_{m-1} := \{(t_1, \ldots, t_m) \in \mathbb{R}^m : t_k \geq 0, \sum_k t_k = 1\}.
\]
Then \( x = \{x_k\} \prec y = \{y_k\} \) if and only if \( x \prec_{ds} y \).

b) for \( m = 3 \) the orders \( \prec \) and \( \prec_{ds} \) are equivalent.

Proof. a) Under the assumption a) each \( x_k \) can be uniquely represented as a convex combination of \( y_k \):
\[
x_k = \sum_{i=1}^m s_{ki}y_i, \quad k \in \{1, \ldots, m\}
\]
with $\sum_i s_{ki} = 1$. Now, by definition, $x \prec y$ yields

$$(1/m) \sum_k y_k = (1/m) \sum_k x_k = (1/m) \left( \sum_{i=1}^m \left( \sum_{k=1}^m s_{ki} \right) y_i \right).$$

Defining

$$\beta_i = 1/m \left( \sum_{k=1}^m s_{ki} \right)$$

we get $\beta_i = 1/m$ for all $i$, since the expression via extreme points is unique. Thus the matrix $S = (s_{ki})_{m \times m}$ is the required bistochastic matrix.

b) $m = 3$. If the endpoints of $y_1, y_2, y_3$ in $\mathbb{R}^n$ do not lie on the same line, then a) applies. If they do, then shifting them all by the same vector so that the line becomes passing through the origin we reduce the problem to the case $n = 1$, contained in Theorem 2.5.

**Remark** 2.5. It is clear, that if $\{y_k\}$ are linearly independent, then they satisfy the hypothesis of a).

It is also interesting to note that in the case a) only the first condition $x_k \in \text{conv}\{y_k\}_{k=1}^m$ and the last one $\sum x_k = \sum y_k$ are sufficient for the existence of a bistochastic matrix.

### 2.2. The set of extreme points of the set Maj(y).

Let $n = 1$ and $x = \{x_k\}_{k=1}^m$, $y = \{y_k\}_{k=1}^m \in \mathbb{R}^m$. It is known (see [22]), that in this case ($n = 1$) the set of extreme points of the set $\text{Maj}(y) := \{x : x \prec y\}$ is

$$\text{Ext}(\text{Maj}(y)) = \{Py : P \in \Omega_m, P \text{ is a permutation matrix}\}.$$  \hspace{1cm} (2.7)

The following statement easily follows from the scalar case.

**Proposition 2.9.** Let $x = \{x_k\}_{k=1}^m$, $y = \{y_k\}_{k=1}^m$ be two sequences of vectors in $\mathbb{R}^n$ and $\text{Maj}(y) := \{x : x \prec y\}$. Then

$$\text{Ext}(\text{Maj}(y)) \supset \{(I_n \otimes P)y : P \text{ is a permutation matrix}\}.$$  \hspace{1cm} (2.8)

The following questions naturally arise in this connection:

**Questions:**

1) Find some additional geometric conditions, such that together with (2.1) they imply $x \prec_{ds} y$.

2) What are the extreme points of the set Maj(y)?

3) Under which conditions on the sequence $y = \{y_k\}$ the sets $\{x : x \prec y\}$ and $\{x : x \prec_{ds} y\}$ coincide?

Let $CVS(\mathbb{R}^n)$ be the closed in the point-wise convergence topology cone in $CV(\mathbb{R}^n)$, generated by the set of convex functions

$$\{f(\langle x, y \rangle) : f \in CV(\mathbb{R}), y \in \mathbb{R}^n\}.$$  \hspace{1cm} (2.8)
The class $CVS(\mathbb{R}^n)$ naturally arises in the following proposition being a partial generalization of Theorem 2.5.

**Proposition 2.10.** Let $x := \{x_j\}_1^l$, $y := \{y_k\}_1^m$ be systems of vectors from $\mathbb{R}^n$. The following conditions are equivalent

1) $x \prec y$;
2) for any vector $h \in \mathbb{R}^n$,
\[
\langle x_1, h \rangle, \ldots, \langle x_l, h \rangle \prec \langle y_1, h \rangle, \ldots, \langle y_m, h \rangle;
\]
3) the inequality
\[
\sum_{i=1}^l f(x_i) \leq \sum_{i=1}^m f(y_i) \quad (2.9)
\]
holds true for any nonnegative $f \in CVS(\mathbb{R}^n)$ when $l < m$ and for any $f \in CVS(\mathbb{R}^n)$ if $l = m$.

**Proof.** 1) $\iff$ 2). A vector $x \in \mathbb{R}^n$ lies in a convex set $Y \subset \mathbb{R}^n$ iff its projection to any line lies in the projection of $Y$ onto the same line. Thus
\[
x_{i_1} + \ldots + x_{i_k} \in \text{conv}(\{y_{j_1} + \ldots + y_{j_k}\})
\]
iff
\[
\langle x_{i_1}, h \rangle + \ldots + \langle x_{i_k}, h \rangle \in \text{conv}(\{\langle y_{j_1}, h \rangle + \ldots + \langle y_{j_k}, h \rangle\})
\]
for any $h \in \mathbb{R}^n$. By Remark 2.3 this is equivalent to 2).

2) $\iff$ 3). By a result of Fisher and Holbrook [11], $\{\langle x_k, h \rangle\}_1^l \prec \{\langle y_k, h \rangle\}_1^m$ if and only if
\[
\sum_{k=1}^l f(\langle x_k, h \rangle) \leq \sum_{k=1}^m f(\langle y_k, h \rangle)
\]
for any nonnegative function $f \in CV(\mathbb{R})$. If $l = m$, we have an equality in (2.9) for any linear function. And any convex function on the line is a sum of a linear function and a nonnegative convex function. This immediately yields the required. \qed

We mention also the following beautiful result, being another partial generalization of the Hardy-Littlewood-Polya theorem. It is due to Sherman [32] for $l = m$ and to Fisher and Holbrook [11] for $l < m$:

**Theorem 2.11.** [32], [11] Let $x := \{x_j\}_1^l$ and $y := \{y_k\}_1^m$ be systems of vectors in $\mathbb{R}^n$. Then $x \prec_d y$ if and only if the inequality
\[
\sum_{i=1}^l f(x_i) \leq \sum_{i=1}^m f(y_i) \quad (2.10)
\]
is valid for any nonnegative \( f \in \text{CV}(\mathbb{C}) \) if \( l < m \) and for any \( f \in \text{CV}(\mathbb{C}) \) if \( l = m \).

The proof is completely different from the scalar case.

Note, that combining Proposition 2.10, Theorem 2.11 and Example 2.7 we arrive at the relation \( \text{CVS}(\mathbb{R}^n) \neq \text{CV}(\mathbb{R}^n) \). Note also that using our Example 2.7 F. V. Petrov has constructed a simple explicit counterexample of a function \( f \in \text{CV}(\mathbb{R}^n) \setminus \text{CVS}(\mathbb{R}^n) \).

To finish the section, we mention the following elegant result due to F. Petrov.

**Proposition 2.12.** 1) \( \{x_k\}_{k=1}^l \prec \{y_k\}_{k=1}^m \) if and only if the sum of any \( s \) of \( x_i \)'s, \( 1 \leq s \leq l \) is a linear combination of \( z_j \)'s with coefficients between 0 and 1 and the sum of coefficients equal \( s \);

2) \( \{x_k\}_{k=1}^l \prec \{y_k\}_{k=1}^m \) if and only if there exist vectors \( x_{l+1}, \ldots, x_m \in \mathbb{R}^n \) such that

\[
\{x_k\}_{k=1}^m \prec \{y_k\}_{k=1}^m.
\]

**Proof.** 1) easily follows from Proposition 2.10. 2).

2) It is easy to see from 1) that if \( \{x_k\}_{k=1}^l \prec \{z_k\}_{k=1}^m \), then \( \{x_k\}_{k=1}^{l+1} \prec \{z_k\}_{k=1}^m \), where \( x_{l+1} = \frac{\sum_{i=1}^m z_i - \sum_{i=1}^l x_i}{m-l} \). \( \square \)

3. Inverse problem and interlacing theorem for normal matrices

In this section we will use the partial orders \( \prec \) and \( \prec_{ds} \) for vectors with complex entries. In this case we identify \( \mathbb{C} \) with \( \mathbb{R}^2 \) and so Definition 2.2 does not change.

Let \( A_{i_1, \ldots, j_k} \) denote the submatrix of \( A \) with the \( i_1, \ldots, i_k \) and columns \( j_1, \ldots, j_k \). We denote for the brevity \( A_{n-1} := A_{1 \ldots n-1} \).

3.1. Preliminary solution to the inverse problem by two spectra.

**Proposition 3.1.** Let \( \{\lambda_k\}_{k=1}^n \) and \( \{\mu_j\}_{j=1}^{n-1} \) be two sequences of complex numbers. Then the system of inequalities:

\[
\prod_{j=1}^{n-1} (\mu_j - \lambda_k) / \prod_{1 \leq j \leq n, j \neq k} (\lambda_j - \lambda_k) \geq 0, \quad k \in \{1, \ldots, n\}
\]

(3.1)

is valid if and only if there exists a normal matrix \( \tilde{A} \) with the spectrum \( \sigma(\tilde{A}) = \{\lambda_1, \ldots, \lambda_n\} \) such that the spectrum of \( \tilde{A} = A_{n-1} \) is \( \sigma(\tilde{A}) = \{\mu_1, \ldots, \mu_{n-1}\} \).
Proof. Necessity. Let $A$ be a normal matrix with the spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ such that $\sigma(\tilde{A}) = \{\mu_1, \ldots, \mu_{n-1}\}$. Let $e = (0, \ldots, 0, 1)$. Consider the function:

$$\Delta(\lambda) := (A - \lambda)^{-1} e, e = \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k - \lambda} = \frac{\det(\tilde{A} - \lambda)}{\det(A - \lambda)}$$

(3.2)

where $x_k$ are the coordinates of $e$ in the orthonormal basis of eigenvectors of $A$. Clearly, the poles of $\Delta(\lambda)$ are in the spectrum of $A$ and the residues in these poles are equal to $x_k^2$ and hence are nonnegative. But, by (3.2) these residues equal the numbers (3.1).

Sufficiency. Let (3.1) be fulfilled. Consider the function

$$\Delta(\lambda) := \prod_{j=1}^{n-1} (\mu_j - \lambda) \prod_{k=1}^{n} (\lambda_k - \lambda)$$

(3.3)

By (3.1), the residues of $\Delta(\lambda)$ in its poles $\lambda_k$ are nonnegative and hence equal $x_k^2$ for some real numbers $x_k$. Clearly, we have

$$\Delta(\lambda) = \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k - \lambda}$$

(3.4)

and

$$\sum_{k=1}^{n} x_k^2 = \lim_{\lambda \to \infty} -\lambda \Delta(\lambda) = 1.$$ 

Therefore, considering the diagonal matrix $A = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ and writing it down in an orthonormal basis with the last vector $e_n = (x_1, \ldots, x_n)$ we get the required normal matrix. \qed

Remark 3.1. Note, that the poles of the function $\Delta(\lambda)$ (see (3.2)) are simple (i.e. of multiplicity one), since the matrix $A$ is (unitary) diagonalizable. Therefore it easily follows from (3.2), as well as from general dimension arguments, that if $A$ has a $k$-multiple eigenvalue $\lambda_0$, then $\lambda_0$ is an eigenvalue of $A_{n-1}$ of multiplicity at least $k - 1$. Note, however, that $A_{n-1}$ is normal only in very special cases (see Lemma 4.9). Moreover, it may even happen that $A_{n-1}$ is not of simple structure, that is it may be nondiagonalizable.

Corollary 3.2. Let two systems of complex numbers $\{\mu_j\}_{j=1}^{n-1}$ and $\{\lambda_j\}_{j=1}^{n}$ satisfy

$$\Delta(\lambda) := \frac{\prod_{j=1}^{n-1} (\mu_j - \lambda)}{\prod_{j=1}^{n} (\lambda_j - \lambda)} = \sum_{j=1}^{n} \frac{|x_j|^2}{\lambda_j - \lambda}$$

(3.5)
with some complex numbers \( \{x_j\}_{j=1}^n \). Then for any unitary matrix \( U = (u_{ij})_{n,j=1}^n \) with the last row \( (u_{n1}, \ldots, u_{nm}) = (x_1, \ldots, x_n) \), the matrix \( A := U \text{diag}\{\lambda_j\}_1^n U^* \) satisfies the hypothesis of Proposition 3.1.

3.2. Quasi-Jacobi normal matrices and an analog of the Hochstadt theorem.

It is known and easy to see that any selfadjoint matrix (bounded operator) is unitary equivalent to a selfadjoint tridiagonal (Jacobi) matrix.

Here we find an analog of such a form for a normal matrix and apply it in order to obtain an analog of the Hochstadt result \([15]\) on the unique recovery of a Jacobi matrix from two spectra.

**Proposition 3.3.** Every normal \( m \times m \) matrix \( A \) is unitary equivalent to a direct sum of normal matrices \( A_i, i = 1, \ldots, k \) satisfying \((A_i)_{j,k} = 0 \) for \( k \geq j + 2 \) and \((A_i)_{j,j+1} \neq 0 \). Moreover, \( A \) has simple spectrum iff it is unitary equivalent to only one such matrix.

**Proof.** The proof is very simple and standard. It is clear that it suffices to consider only the case of simple spectrum.

In this case, taking any cyclic vector \( x \), we get that \( \{A^jx\}_{j=0}^{m-1} \) forms a basis in \( \mathbb{C}^m \). After the Gram-Schmidt procedure we arrive at the required basis. \( \square \)

**Definition 3.4.** A matrix \( A \) is called quasi-Jacobi if it satisfies the hypothesis of Proposition 3.1.

Thus, quasi-Jacobi form is in a sense a normal form for a normal matrix. Now we can complement Proposition 3.1 with a uniqueness result.

**Theorem 3.5.** For any two systems of complex numbers \( \{\lambda_j\}_1^n \) and \( \{\mu_j\}_1^{n-1} \), satisfying (3.1) there exists a unique normal quasi-Jacobi matrix \( A \) such that \( \sigma(A) = \{\lambda_j\}_1^n \) and \( \sigma(A_{n-1}) = \{\mu_j\}_1^{n-1} \).

**Proof.** Writing the function \( \Delta(\lambda) \) from (3.3) in the form

\[
\Delta(\lambda) = \int \frac{d\mu(z)}{\lambda - z}
\]

with \( d\mu = \sum_{k=1}^n x_k^2 \delta_{\lambda_k} \) we can introduce the orthogonal polynomials with respect to measure just like in the Jacobi case (see, e.g. \([1]\) and Gesztesy and Simon \([12]\)) and then, following the same lines as in \([12]\), we get the result. \( \square \)

**Remark 3.2.** The function \( \Delta(\lambda) \) is an analog of the Weyl M-function in this case (see \([12], [17]\)). Note that our proof of Proposition 3.1 is similar to that proposed in \([12], [17]\).
3.3. The set of all possible diagonals in the "unitary" orbit of a normal matrix.

The criterion (3.1) of Proposition 3.1 is trivial and provides no information on the geometry of the sequences \( \{ \lambda_k \}_{k=1}^{n} \) and \( \{ \mu_j \}_{j=1}^{n-1} \). It is even unclear how far can \( \mu_j \) lie from \( \lambda_k \). Therefore it would be desirable to have a more "geometric" answer, being an analog of the Poincare Theorem.

In this subsection we start with an arbitrary normal matrix \( A \) and give a (rather trivial) description of the set of diagonals of its "unitary" orbit \( \{ UAU^* : U \in M_n(\mathbb{C}), U^*U = I \} \). In the next section we apply this result to complete solution to the inverse spectral problem for a normal matrix.

**Proposition 3.6.** Let \( A \in M_n(\mathbb{C}) \) be a normal matrix with the spectrum \( (\lambda_1, \ldots, \lambda_n) \). Then

a) \((a_{11}, \ldots, a_{nn}) \prec_{uds} (\lambda_1, \ldots, \lambda_n)\);

b) there exists an orthonormal basis \( \{ e_i \}_{i=1}^{n} \) such that \( (Ae_i, e_i) = \alpha_i \) iff there exists a unitary stochastic matrix \( O \) such that \( \text{col}(\alpha_1, \ldots, \alpha_n) = O \text{col}(\lambda_1, \ldots, \lambda_n) \).

c) if \( A \) is selfadjoint, then the set of all possible diagonals (in all orthonormal bases) is convex;

d) the set of all possible diagonals (in all orthonormal bases) of a fixed normal matrix \( A \) is not necessarily convex.

**Proof.** The validity of a) and b) is obvious. It is also clear that all permutations of the set \( (\lambda_1, \ldots, \lambda_n) \) are realized by diagonals. Thus, if the set of diagonals were convex, it would contain all vectors of the form \( S \text{col}(\lambda_1, \ldots, \lambda_n) \) with a bistochastic \( S \).

c) is due to Horn \[16\].

d) take

\[
S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\]

It is known (see \[23\]) that \( S \) is not unitary-stochastic. Set \( (\lambda_1, \lambda_2, \lambda_3) = (1, i, 0) \) and

\[
S(\lambda_1, \lambda_2, \lambda_3) = (\alpha_1, \alpha_2, \alpha_3)^t = (1 + i, 1, i)^t.
\]

If there exists a unitary-stochastic matrix \( O \), such that

\[
O(\lambda_1, \lambda_2, \lambda_3)^t = (\alpha_1, \alpha_2, \alpha_3)^t,
\]

then it immediately yields \( O = S \). \( \Box \)

**Remark 3.3.** The result of c) is a due to Horn \[23, 24, 14, 16\]. All other statements are folklore.
3.4. Analog of the Cauchy-Poincare interlacing theorem and a solution to the inverse spectral problem for normal matrices.

Now we are ready to state the main result of the section. Define for any vector \( \{\lambda_j\}_1^m \in \mathbb{C}^m \) the vector

\[
C_k(\{\lambda_j\}_1^m) := \{\lambda_{i_1} \cdots \lambda_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq m} \in C_k^{(m)}.
\]

(3.6)

**Theorem 3.7.** Let \( \{\lambda_1, \ldots, \lambda_n\} \) and \( \{\mu_1, \ldots, \mu_{n-1}\} \) be two systems of complex numbers. Then for the existence of a normal matrix \( A \) such that \( \sigma(A) = \{\lambda_1, \ldots, \lambda_n\} \) and \( \sigma(A_{1,n-1}) = \{\mu_1, \ldots, \mu_{n-1}\} \) it is necessary that the condition

\[
C_k(\{\mu_j - \alpha\}_1^{n-1}) \prec_{uds} C_k(\{\lambda_j - \alpha\}_1^n)
\]

(3.7)

be fulfilled for any complex number \( \alpha \in \mathbb{C} \) and any \( k \in \{1, \ldots, n-1\} \) and sufficient that it be fulfilled for \( k = n-1 \) and all \( \alpha \in \{\lambda_1, \ldots, \lambda_n\} \).

**Proof.** a) **Sufficiency.** Let \( \alpha = \lambda_k \). (3.7) for \( k = n-1 \) reads

\[
\prod_{i=1}^{n-1} (\mu_i - \lambda_k) \in \text{conv} \left( \prod_{1 \leq i \leq n, i \neq l} \{\lambda_i - \lambda_k\} : l = 1, n \right)
\]

\[= \text{conv} \left( 0; \prod_{1 \leq i \leq n, i \neq k} \{\lambda_i - \lambda_k\} \right)
\]

(3.8)

Hence (3.1) is valid for all \( k \). Proposition 3.4 yields the required.

b) **Necessity.** Let \( A \) be a normal matrix with the spectrum \( \sigma(A) = \{\lambda_1, \ldots, \lambda_n\} \) and \( \sigma(A_{1,n-1}) = \{\mu_1, \ldots, \mu_{n-1}\} \). Let us prove (3.7). By the Shur Theorem [23, 16] there exists a unitary matrix \( V_1 \in M_{n-1}(\mathbb{C}) \) such that the matrix \( V_1^* A_{1,n-1} V_1 \) is upper triangular. Therefore, considering the matrix \( U_1 := V_1 \oplus 1 \in M_n(\mathbb{C}) \) we get the normal matrix \( B := U_1^* A U_1 \) with the same spectrum as \( A \), \( \sigma(B) = \sigma(A) \), but the \( \mu_j \)-s are on the diagonal. Therefore we can take \( B \) instead of \( A \). Proposition 3.6 implies \( \{\mu_j\}_1^{n-1} \prec_{uds} \{\lambda_j\}_1^n \).

Take an arbitrary \( \alpha \in \mathbb{C} \). The matrix \( B - \alpha I \) is also normal. Let us consider its exterior power \( C_k(B - \alpha I) := \wedge^k(B - \alpha I) \) acting on the space \( H := \bigwedge_{k \text{ times}} \). Then this matrix is also normal with the spectrum

\[
\sigma(C_k(B - \alpha I)) = \{(\lambda_{i_1} - \alpha) \cdots (\lambda_{i_k} - \alpha)\}_{1 \leq i_1 < \cdots < i_k \leq n}.
\]

The diagonal elements of \( C_k(B - \alpha I) \) are the \( k \times k \) principal minors of \( B - \alpha I \). Since \( B_{n-1} - \alpha I_{n-1} \) is upper triangular, then the \( k \times k \) principal
minors of $B_{n-1} - \alpha I$ equal
\[(\mu_{i_1} - \alpha) \ldots (\mu_{i_k} - \alpha), 1 \leq i_1 < \ldots < i_k \leq n - 1.\]
Thus Proposition 3.6 a) implies the required.

\[\square\]

**Remark 3.4.** It is easy to see from the proof that the unitary stochastic matrices in (3.7) are independent of $\alpha$.

**Corollary 3.8.** There exists a normal matrix $A$ such that $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(A_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$ if and only if
\[\prod_{k=1}^{n-1} (\lambda_j - \mu_k) \in \text{conv}(0; p'(\lambda_j)) \text{ for all } j \in \{1, \ldots, n\}\]
where $p(\lambda) = \prod_{k=1}^{n} (\lambda - \lambda_k)$

**Example 3.9.** In the case $n = 3$ the orders $\prec$ and $\prec_{ds}$ are equivalent. Therefore in this case conditions (3.7) take a specially simple form:
\[
\begin{align*}
\mu_1, \mu_2 &\in \text{conv}\{\lambda_j\}_{j=1}^{3}, \\
\mu_1 + \mu_2 &\in \text{conv}\{\lambda_1 + \lambda_2, \lambda_2 + \lambda_3, \lambda_1 + \lambda_3\}, \\
\mu_1\mu_2 - \alpha(\mu_1 + \mu_2) &\in \text{conv}\{\lambda_k\lambda_p - \alpha(\lambda_k + \lambda_p)\}_{1 \leq k < p \leq 3}, \alpha \in \mathbb{C}.
\end{align*}
\]

An immediate consequence is:

**Corollary 3.10.** Let $A$ be a normal matrix with the spectrum $\{\lambda_1, \ldots, \lambda_n\}$ and $P$ an $m(\leq n)$-dimensional orthoprojection in $\mathbb{C}^n$. Let also $B := PA[P\mathbb{C}^n$ and $\sigma(B) := (\mu_1, \ldots, \mu_m)$. Then
\[C_k(\{\mu_j - \alpha\}_{j=1}^{m}) \prec_{uds} C_k(\{\lambda_j - \alpha\}_{j=1}^{n})\] (3.9)
for all $k, 1 \leq k \leq m$.

**Corollary 3.11.** Let $A_i \in M_{n_i}(\mathbb{C}), i = 1, \ldots, p$ be a $p$-tuple of normal matrices, and $\sigma(A_i) = \{\lambda_k^i\}_{k=1}^{n_i}$ their spectra. Let $S_i$ be $m \times n_i$ matrices, $i = 1, p$ such that
\[\sum_{i=1}^{p} S_i^* S_i = I_m\] (3.10)
is the identity operator in $\mathbb{C}^m$. Consider
\[B = \sum_{i=1}^{p} S_i^* A_i S_i\] (3.11)
and let $\sigma(B) = (\mu_1, \ldots, \mu_m)$. Set $n = n_1 + \ldots + n_p$ and
\[(\lambda_1, \ldots, \lambda_n) = (\lambda_1^1, \ldots, \lambda_1^{n_1}, \lambda_2^1, \ldots, \lambda_2^{n_2}, \ldots, \lambda_p^1, \ldots, \lambda_p^{n_p}).\] (3.12)
Then the systems of numbers $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_m)$ satisfy conditions (3.9).
Proof. Consider the normal matrix \( A := \bigoplus_{i=1}^l A_i \). Condition (3.10) means that the operator
\[
V := \begin{pmatrix} S_1 \\ \vdots \\ S_n \end{pmatrix} : \mathbb{R}^m \rightarrow \mathbb{R}^n
\]
is an isometry and hence the operator
\[
B = V^* AV
\]
is unitary equivalent to \( PA [PC^n] \) where \( P \) is the projector onto the image of \( V \). Corollary 3.10 completes the proof. \( \square \)

Remark 3.5. There is another way to implement the trick of going from Corollary 3.10 to Corollary 3.11. See [18], [19].

4. Location of roots of a polynomial and of its derivative

4.1. Generalization of the Gauss-Lucas Theorem.

Recall the known Gauss-Lucas theorem

Theorem 4.1. The roots of the derivative \( p' \) of a polynomial \( p \in C[z] \) lie in the convex hull of the roots of \( p \).

Numerous papers are devoted to different generalizations and improvements of this results (see e.g. [8], [2], [31]).

In what follows we denote by \( p \in C[z] \) a polynomial of degree \( n \) with complex coefficients. Let also \( \{ \lambda_j \}_1^n \) be the roots of \( p \) and \( \{ \mu_k \}_1^{n-1} \) the roots of its derivative \( p' \). We set additionally \( \mu_n := (\sum_1^n \lambda_j)/n \). Then the Gauss-Lucas theorem reads as follows: there exists a stochastic (by rows) matrix \( S' \) such that \( \mu = S' \lambda \), where \( \mu := \{ \mu_k \}_1^n \) and \( \lambda := \{ \lambda_j \}_1^n \).

The following result, being a corollary of Theorem 3.7, improves the Gauss-Lucas theorem.

Proposition 4.2. Let \( p(\in C[z]) \) be a degree \( n \) complex polynomial with roots \( \{ \lambda_j \}_1^n \) and \( \{ \mu_k \}_1^{n-1} \) the roots of its derivative \( p' \). Then the sequences \( \{ \lambda_j \}_1^n \) and \( \{ \mu_k \}_1^{n-1} \) satisfy (3.7) for \( k \in \{ 1, \ldots, n - 1 \} \). In particular, there exists a matrix \( S \in \Omega_n \) such that \( \mu = S \lambda \), that is, the vector \( \mu \) is bistochastically majorized by the vector \( \lambda \).

Proof. It follows from Corollary 3.2 and the obvious identity
\[
p'(z)/p(z) = \sum 1/(z - \lambda_j) \quad (4.1)
\]
that there exists a normal matrix \( A \) such that \( \sigma(A) = \{ \lambda_j \}_1^n \) and \( \sigma(A_{n-1}) = \{ \mu_j \}_1^{n-1} \). One completes the proof by applying Theorem 3.7. \( \square \)
Note that this corollary does not give a complete information on the location. In particular, it does not anyhow explain the identities:

\[
\frac{1}{C_{n-1}^k} \sum_{1 \leq i_1 < \ldots < i_k \leq n-1} \prod_{j=1}^{k} (\mu_{ij} - \alpha) = \frac{1}{C_{n}^k} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} (\lambda_{ij} - \alpha) \quad (4.2)
\]

which mean that even the products of the roots are equally distributed.

It turns out that in this case we can obtain a more complete information on the matrices \( S \in \Omega_n^a \), realizing majorization.

**Theorem 4.3.** Let \( k \in \{1, \ldots, n-1\} \), \( a := (n-1)_k \), \( b := (n)_k \). Then there exists a matrix \( S_k = (s_{ijk}) \in \Omega_n^a \) such that \( \sum_{i=a+1}^{b} s_{ijk} = k/n \) for all \( j \in \{1, \ldots, b\} \) and \( C_k(\{\mu_j - \alpha\}_1^{n-1}) = P S_k C_k(\{\lambda_j - \alpha\}_1^n) \) (see (3.6)), where \( P : \mathbb{C}^b \to \mathbb{C}^a \) is the natural orthoprojection.

In particular, if \( k = 1 \) then there exists \( S_1 \in \Omega_n^a \) such that \( s_{n1j} = 1/n, j \in \{1, \ldots, n\} \) and \( \{\mu_j\}_1^n = S_1 \{\lambda_j\}_1^n \).

**Proof.** Set \( D := (\lambda_1, \ldots, \lambda_n) \). By Corollary 3.2 and (4.1) we have, that for any unitary matrix \( V \) with the last row consisting of \( 1/\sqrt{n} \), the normal matrix \( A := V D V^* \) solves the inverse problem for a pair of sequences \( \{\lambda_j\}_1^n \) and \( \{\mu_j\}_1^{n-1} \), that is \( \sigma(A) = \{\lambda_j\}_1^n \) and \( \sigma(A_{n-1}) = \{\mu_j\}_1^{n-1} \). Let, further \( U_1 \) be the same unitary matrix as in the proof of Theorem 3.7. Then \( U := U_1 V (\in M_n(\mathbb{C})) \) is a unitary matrix with the last row consisting of \( 1/\sqrt{n} \). Moreover, it follows from the proof of Theorem 3.7 that \( \{\mu_j\}_1^n = S_1 \{\lambda_j\}_1^n \), where \( S_1 := U \circ \bar{U} (\in \Omega_n^a) \) is the unitary stochastic matrix with the last row consisting of \( 1/n \).

Therefore, passing to the exterior powers as in the proof of Theorem 3.7 we get that the unitary stochastic matrix

\[
S_k := C_k(U) \circ C_k(\bar{U}), \quad k \in \{1, \ldots, n\}
\]

realizes the unitary stochastic majorization of the systems of numbers \( C_k(\{\mu_j\}_1^{n-1}) \) and \( C_k(\{\lambda_j\}_1^n) \). We prove that \( S_k \) has additional properties

\[
\sum_{1 \leq i_1 < \ldots < i_k \leq n-1} \left| U_{i_1, \ldots, i_k, n} \right|^2 = \frac{n-k}{n}. \quad (4.3)
\]

Because of the symmetry, the same identity is certainly valid for all other choices of \( k \) columns.

Expanding each minor with respect to the last row we get

\[
U_{i_1, \ldots, i_k, n} = \frac{1}{\sqrt{n}} \sum_{l=1}^{k} (-1)^{l+1} U_{i_1, \ldots, i_{l-1}, i_{l+1}, \ldots, i_k, n} \quad (4.4)
\]
and thus
\[ |U_{i_1, \ldots, i_{k-1}, n}^{1, \ldots, k}|^2 = \frac{1}{n} \sum_{p,q=1}^{k} U_{i_1, \ldots, i_{k-1}}^{1, \ldots, \hat{p}, k} U_{i_1, \ldots, i_{k-1}}^{1, \ldots, \hat{q}, k} \] (4.5)

Consider now the matrix \( V := U_{i_1, \ldots, n-1}^{1, \ldots, k} \). Then, since \( U \) is unitary, we get
\[ T := V^* V = \{t_{ij}\}_{i,j=1}^{k} \]
with
\[ t_{ij} = \begin{cases} \frac{n-1}{n}, & i = j \\ \frac{1}{n}, & i \neq j \end{cases} \]
Therefore we get
\[ C_{k-1}(V)^* \cdot C_{k-1}(V) = C_{k-1}(T). \]
Because of (4.5), one easily sees that the required sum in (4.3) is a linear combination of the elements of the matrix \( C_{k-1}(V)^* \cdot C_{k-1}(V) \). Thus this sum depends only on the matrix \( T \), is independent of the choice of \( U \) and is the same for all other choices of \( k \) columns.

But by (1.2), these sums can be nothing but \((n - k)/n\). The proof is complete. \( \square \)

**Corollary 4.4.** Let \( U \) be an \( n \times n \) unitary matrix with \( u_{nj} = 1/\sqrt{n} \). Let \( C_k(U) \) be its \( k \)-th exterior power. Then for \( k \leq n \) \( S_k := C_k(U) \circ C_k(\overline{U}) \) is a unitary stochastic matrix, satisfying
\[ \sum_{i=1}^{\binom{n-1}{k}} s_{ij} = \frac{n-k}{n} \]
for all \( j \in \{1, \ldots, \binom{n}{k}\} \).

**4.2. Conjecture of de Bruijn-Springer and its generalization.**

In 1947 de Bruijn and Springer [6] conjectured that the inequality
\[ \frac{1}{n-1} \sum_{j=1}^{n-1} f(\mu_j) \leq \frac{1}{n} \sum_{j=1}^{n} f(\lambda_j) \] (4.6)
holds for any convex function \( f : \mathbb{C} \to \mathbb{R} \).

In order to prove this conjecture as well as its generalization we need the following simple lemma.
Lemma 4.5. Let \( \{x_j\}_1^k, \{y_j\}_1^n \) be two sequences of vectors from \( \mathbb{R}^m \). Suppose that there exists a matrix \( S = (s_{ij}) \in \mathbb{R}^{k \times n} \) with non-negative entries and such that \( \{x_j\}_1^k = (S \otimes I_m) \{y_j\}_1^n \) and

\[
\sum_{j=1}^n s_{ij} = 1, \quad i \in \{1, \ldots, k\} \quad \text{and} \quad \sum_{i=1}^k s_{ij} = \frac{k}{n}, \quad j \in \{1, \ldots, n\}. \tag{4.7}
\]

Then the inequality

\[
\frac{1}{k} \sum_{j=1}^k f(x_j) \leq \frac{1}{n} \sum_{j=1}^n f(y_j) \tag{4.8}
\]

holds true for any function \( f \in CV(\mathbb{R}^m) \).

Proof. One obtains the proof by combining the Jensen inequality with relations (4.7). \( \square \)

The following result contains in particular a positive solution to the conjecture of de Bruijn and Springer \( [6] \).

Theorem 4.6. The following inequality holds true for any convex function \( f : \mathbb{C} \to \mathbb{R} \) and any \( k, 1 \leq k \leq n \):

\[
\frac{1}{\binom{n-1}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} f \left( \prod_{j=1}^k (\mu_{i_j} - \alpha) \right) \leq \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f \left( \prod_{j=1}^k (\lambda_{i_j} - \alpha) \right). \tag{4.9}
\]

Proof. The inequality immediately follows by combining Theorem 4.3 with Lemma 4.5. \( \square \)

Remark 4.1. In the case \( k = 1 \) inequality (4.9) coincides with inequality (4.6), that is with the de Bruijn–Springer conjecture \( [6] \).

Remark 4.2. According to the result of Sherman \( [32] \), the existence of a \( k \times n \) matrix \( S \), satisfying the hypothesis of Lemma 4.5 is actually equivalent to the validity of inequality (4.8) for each function \( f \in CV(\mathbb{R}^m) \).

4.3. The Schoenberg conjecture.

Now we are ready to prove the famous Schoenberg conjecture \( [33], [7] \).

We will need two Lemmas. The first one is known \( [23] \), but we present it with a proof for the reader’s convenience.
Lemma 4.7. Any matrix $A = (a_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$ with spectrum $\sigma(A) = \{\lambda_j\}_{j=1}^n$ satisfies the inequality

$$\sum_{j=1}^n |\lambda_j|^2 \leq \|A\|_2^2 = \sum_{i,j=1}^n |a_{ij}|^2 \quad (4.10)$$

and the equality holds if and only if $A$ is normal.

Proof. The inequality (4.10) is known. It is clear that the equality holds true for a normal matrix.

Conversely, let $A$ satisfy the equality. By the Schur theorem $A$ is unitary equivalent to an upper triangular matrix with $\lambda_j$-s on the diagonal. Since $\|A\|_2$ is unitary invariant, this matrix will be diagonal, that is $A$ is normal. \(\square\)

Lemma 4.8. Let $\varepsilon = e^{2\pi i/n}$ and $U = n^{-1/2}(\varepsilon^{k(j-1)})_{k,j=1}^n$. Let also $\sum_{j=1}^n \lambda_j = 0$. Define

$$r(z) := \sum_{j=1}^n \lambda_j z^{j-1}$$

and

$$A := U \text{diag}(\lambda_j)_{j=1}^n U^* = \frac{1}{n}(r(\varepsilon^{k-j}))_{k,j=1}^n =: (a_{ij})_{i,j=1}^n.$$

Then $A$ is a normal matrix with spectrum $\sigma(A) = \{\lambda_j\}_{j=1}^n$ and $\sigma(A_{n-1}) = \{\mu_j\}_{j=1}^{n-1}$. Moreover the following identity holds true

$$n\|A_{n-1}\|_2^2 = (n-2)\|A\|_2^2.$$

Proof. The first statement follows from Corollary 3.21 since the last row of $U$ consists of $\frac{1}{\sqrt{n}}$. It remains to prove the last identity.

It is easy to see that $a_{jj} = 0$ and $a_{nj} = a_{n-k,j-k}$ for $k < j$ and $a_{nj} = a_{k-j,k}$ for $k > j$. Therefore the required identity takes the form

$$\sum_{j=1}^n |\lambda_j|^2 = n \sum_{j=1}^{n-1} |a_{nj}|^2.$$

But

$$n \sum_{j=1}^{n-1} |a_{nj}|^2 = \frac{(n-1)}{n} \sum_{j=1}^n |\lambda_j|^2 + \frac{1}{n} \sum_{i<j} \Re(\lambda_i \bar{\lambda}_j c_{ij})$$

where

$$c_{ij} = \sum_{k=1}^{n-1} \varepsilon^{(i-j)(n-k)} = -1$$
for all $1 \leq i \neq j \leq n$. But we have

$$\left| \sum_{j=1}^{n} \lambda_j \right|^2 = 0 \iff 2 \sum_{i<j} \Re(\lambda_i \overline{\lambda_j}) = -\sum_{j=1}^{n} |\lambda_j|^2.$$ 

This completes the proof. \(\square\)

The next lemma is due to Fan Ky and Pall \([10]\).

**Lemma 4.9.** Let $A$ be a normal matrix such that its submatrix $A_{n-1}$ is also normal and $A \neq A_{n-1} \oplus a_m$. Then all the eigenvalues of $A$ lie on the same line.

The following result has been conjectured by Schoenberg \([33]\) (see also \([7]\)).

**Theorem 4.10.** Let $\sum_{j=1}^{n} \lambda_j = 0$. Then

$$n \sum_{j=1}^{n-1} |\mu_j|^2 \leq (n - 2) \sum_{j=1}^{n} |\lambda_j|^2$$

and the equality holds if and only if all the numbers $\lambda_j$ lie on the same line.

**Proof.** Combining Lemma 4.7 with Lemma 4.8 we get

$$n \sum_{j=1}^{n-1} |\mu_j|^2 \leq n \|A_{n-1}\|_2^2 = (n - 2) \|A\|_2^2 = (n - 2) \sum_{j=1}^{n} |\lambda_j|^2. \quad (4.11)$$

Moreover, by Lemma 4.7 identity \((4.11)\) holds if and only if $A_{n-1}$ is normal. On the other hand, by Lemma 4.9 this is possible if and only if all $\lambda_j$-s lie on the same line. \(\square\)

**4.4. The Mason-Shapiro Polynomials.** In \([25]\) Gisli Masson and Boris Shapiro initiated study of a class of differential operators $T_Q$ defined as follows: let $Q$ be a degree $k$ monic polynomial. Then $T_Q$ is defined via

$$T_Q : f \rightarrow (Qf)^{(k)}.$$ 

They have shown that for each $m$ there exists a unique polynomial eigenfunction $p_m$ of $T_Q$ of degree $m$. Moreover,

$$T_Q p_m = \lambda_{m,k} p_m \quad (4.12)$$

and $\lambda_{m,k}$ depends only on $k, m$, namely

$$\lambda_{m,k} = (m + 1)(m + 2) \cdots (m + k).$$

One of the results of \([25]\) is the following interesting analog of the Gauss-Lucas Theorem:
Theorem 4.11. The zeros of $p_m$ are contained in the convex hull of the set of zeros of $Q$ for each $m$.

The authors have also made a number of beautiful conjectures about the asymptotic distribution of the zeros of $p_m$-s, recently proved in [3].

We strengthen Theorem 4.11 in the following way:

**Theorem 4.12.** Let $\{z_j\}_{j=1}^k$ be the zeros of $Q$ and $\{w_j\}_{1}^m$ the zeros of $p_m$. Then

(i) there exists an $m \times k$ matrix $S$ with $\sum_{j=1}^k s_{ij} = 1$ for all $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^m s_{ij} = m/k$ for all $j \in \{1, \ldots, k\}$ and such that

\[ \{w_j\}_{1}^m = S\{z_j\}_{1}^k, \quad (4.13) \]

(ii) for any convex function $f : \mathbb{C} \to \mathbb{R}$:

\[ \frac{\sum_{j=1}^m f(w_j)}{m} \leq \sum_{j=1}^k \frac{f(z_j)}{k}. \quad (4.14) \]

Proof. Applying Theorem 4.11 $k$ times to the polynomial $Qp_m$ and its consequence derivatives we arrive at the representation

\[ \{w_j\}_{1}^m = \tilde{S}(\text{col}(\{w_j\}_{1}^m, \{z_j\}_{1}^k)) \quad (4.15) \]

with $m \times (m + k)$ matrix $\tilde{S} = (s'_{ij})$ being a product of the $k$ corresponding matrices and satisfying

\[ \sum_{j=1}^k s'_{ij} = 1, \quad i \in \{1, \ldots, m\} \quad \text{and} \quad \sum_{i=1}^m s'_{ij} = \frac{m}{m + k}, \quad j \in \{1, \ldots, m + k\}. \]

Now applying Lemma 4.5 we arrive at the inequality

\[ \frac{1}{m} \sum_{j=1}^m f(w_j) \leq \frac{1}{k + m} \left( \sum_{j=1}^k f(z_j) + \sum_{j=1}^m f(w_j) \right) \]

which yields (4.14).

It is not difficult to construct the matrix $S$ satisfying (4.13) and the other required properties by resolving the last identity for $w_j$-s in (4.15). But its existence is immediately implied by (4.14) due to the result of Sherman [32] (see Theorem 2.11).

Remark 4.3. In the case of real numbers $w_j$ and $z_j$ inequality (4.14) has been mentioned (without proof) by Harold Shapiro [30].

**Final remarks.** 1) Let $A \in M_n(\mathbb{C})$ be a normal matrix, $e \in \mathbb{C}^n$ a vector and $P$ the orthoprojection onto the orthogonal complement of $e$. It is not difficult to construct examples of a nondiagonalizable $A_e :=$
PA[P^n]. It would be interesting to investigate the Jordan structure and other similarity (or unitary) invariants of the operator \( A_+ \).

2) We do not know whether the relations (3.9) are also sufficient.

3) The most famous unsolved conjectures connected with the Gauss-Lucas Theorem are the conjectures of Sendov and Smale (see [31] for a survey on this topic).

4) The relative location of the zeros of \( p(z) \), \( p'(z) \) and \( p''(z) \) may be very nontrivial even in the case of real roots [29].

5) Some very interesting relations between the zeros of polynomials, their derivatives and majorization are studied in a recent paper [4] by J. Borcea and B. Shapiro. They formulate many interesting open problems.

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