Maxwell-$CP(2)$ vortices in the presence of magnetic impurities

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We consider a Maxwell-$CP(2)$ model extended to include a magnetic impurity. We focus our attention on the time-independent configurations with radial symmetry, from which we minimize the corresponding energy by following the Bogomol'nyi-Prasad-Sommerfield (BPS) prescription. We use the general first-order expressions in order to introduce modified scenarios in which the impurity plays a relevant role. We then solve the effective first-order equations numerically by means of a finite-difference scheme, from which we comment on the main changes on the shape of the final solutions caused by the presence of a localised impurity. We also discuss the limit when the impurity becomes a delta function.

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I. INTRODUCTION

In the context of classical field models, topologically nontrivial configurations are commonly described as the time-independent solutions inherent to highly nonlinear field models in the presence of a self-interaction potential supporting the occurrence of a spontaneous symmetry breaking [1]. These solutions are expected to be formed during a phase transition, their stability being therefore guaranteed by topological arguments.

Also, under very special circumstances, the aforesaid configurations can be obtained by solving a particular set of two coupled first-order differential equations, instead of the second-order Euler-Lagrange equations. These first-order equations usually appear as a consequence of the minimization of the effective energy functional, and also provide a well-defined lower bound for the energy of the resulting first-order solutions [2]. Here, it is instructive to point out that the first-order results can alternatively be constructed via the study of the conservation of the corresponding energy-momentum tensor [3] or the implementation of the so-called On-Shell Method [4].

In this context, vortices stand for rotationally symmetric structures which were first obtained in a planar Maxwell-Higgs scenario [5]. In the beginning of the 90’s, first-order vortices were also found in both Chern-Simons-Higgs [6] and Maxwell-Chern-Simons-Higgs [7] theories.

More recently, time-independent vortices were also studied in connection to a Maxwell-$CP(2)$ model via the second-order Euler-Lagrange equations [8], from which some of us have verified the existence of their first-order counterparts [9]. Rotationally symmetric configurations were also considered in the context of both the Chern-Simons-$CP(2)$ [10] and the Maxwell-Chern-Simons-$CP(2)$ [11] theories, these investigations being motivated by a close phenomenological relation between the gauged $CP(N)$ models and four-dimensional Yang-Mills theories.

Another interesting issue concerning the first-order vortices is whether they can be obtained from a gauged model extended to include the effects of the presence of both electric or magnetic impurities. This idea was first considered by Tong and Wong in [12]. Their paper clarifies how the presence of an impurity affects the moduli space inherent to a Maxwell-Higgs type model. Moreover, existence theorems for both vortices and anti-vortices in the presence of magnetic impurities were studied in [13]. In addition, the first-order vortices coming from a Chern-Simons-Higgs model extended to include impurities was considered in [14]. In Ref. [15], some of us investigated the dynamics of Maxwell-Higgs vortices in the presence of a localised magnetic impurity at critical coupling. The numerical results confirmed the existence of a moduli space of vortex solutions.

We now go a little bit further by investigating whether first-order time-independent vortices can be obtained from a Maxwell-$CP(2)$ model extended to include the presence of a magnetic impurity.

In order to present our results, this manuscript is organized as follows: in the next Section, we define the overall gauged model and our main conventions. We then focus our attention on those field configurations with radial symmetry, and minimize the effective energy according the standard BPS idea. As a consequence, we obtain the general form of the first-order equations to be satisfied by the BPS fields, together with a lower bound for the value of the energy itself. In Section III, we show how to use the expressions introduced previously in order to define effective first-order scenarios, including the corresponding potential which engenders self-duality (i.e. the construction of the overall first-order framework). We solve the resulting first-order equations numerically by means of a finite-difference algorithm, from which we depict the numerical profiles for the relevant fields. We also comment on the main changes on the shape of the final solutions caused by the presence of a localised magnetic impurity, and the limit when the impurity approaches a $\delta$-function. Finally, we present our conclusions and perspectives in Section IV.
In this work, we adopt \(\eta^{\mu\nu} = (+ - - -)\) as the metric signature of the flat spacetime, together with the natural units system, for the sake of simplicity.

II. \(CP(2)\) VORTICES

We begin this Section by defining the model we will investigate. It consists of a gauged \(CP(2)\) theory extended to include an additional term which represents the presence of a magnetic impurity. The resulting Lagrange density is then given by

\[
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (P_{ab}D_\mu \varphi_b)^* P_{ac}D_\mu \varphi_c - U(\varphi_3, \Delta) + \Delta B, \tag{1}
\]

where the complex fields \(\varphi_a\) are homogeneous coordinates of \(\mathbb{C}P^2\), such that \((\varphi_1, \varphi_2, \varphi_3)\) and \((\lambda \varphi_1, \lambda \varphi_2, \lambda \varphi_3)\) denote the same point in \(\mathbb{C}P^2\) for \(\lambda \neq 0\). We impose the normalisation condition \(\varphi_3^* \varphi_3 = \hbar\) so that \(\hbar \equiv 0\) is excluded. \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) stands for usual field strength tensor and \(P_{ab} = \delta_{ab} - \hbar^{-1} \varphi_a \varphi_b^*\) is the projection operator which ensures that the kinetic term is well defined on \(CP(2)\). Moreover, \(D_\mu \varphi_a = \partial_\mu \varphi_a - i\varphi_a A_\mu\varphi_b\) is the corresponding covariant derivative and \(Q_{ab}\) stands for a real charge matrix (diagonal and traceless). As usual, the Einstein summation convention is implied, and Greek indices represent space-time coordinates, while the Latin indices label the complex components of the scalar \(CP(2)\) sector.

In the absence of impurities (\(\Delta = 0\)) we only consider potentials of the form \(U = U(\varphi_3)\) in (1). This is well-defined on \(CP^2\) since we have fixed the normalisation, and \(U\) only depends on the modulus of \(\varphi_3\). In the presence of impurities we will allow the potential \(U\) to depend on \(\varphi_3\) and \(\Delta\). The final term in (1) couples the magnetic field \(B\) to a function \(\Delta\). This function \(\Delta = \Delta(r)\) is supposed to depend on the radial coordinate \(r\) only, therefore breaking the translational invariance of the overall theory. This is not a problem at all: the model (1) can be thought as an effective one. The point here is that function \(\Delta\) can be understood as representing the presence of a magnetic impurity, see the arguments in [12].

Variation of the Lagrange density (1) with respect to the electric potential \(A^0\) leads to the Gauss law. The presence of the term \(\Delta B\) does not change the structure of that expression, from which we conclude that the Gauss law for time-independent fields coming from (1) is the very same one already obtained in the context of the usual Maxwell-\(CP(2)\) model without magnetic impurity, see the Eq. (2) in Ref. [9] and the discussion therein. The point to be highlighted here is that such time-independent law is identically solved by \(A^0 = 0\), which can be therefore chosen as a legitimate gauge condition (the temporal gauge). Hence, we conclude that the time-independent solutions inherent to the model (1) present no electric field and charge density.

We then focus our attention on those time-independent configurations with no electric field via the rotationally symmetric map, i.e.

\[
A_i = -A_i = -\frac{1}{gr} \epsilon^{ij} n^j A(r), \tag{2}
\]

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix} = \hbar \begin{pmatrix}
\sin m \theta \\ 
\sin m \theta \\
\sin 2m \theta
\end{pmatrix}, \tag{3}
\]

where \(\epsilon^{ij}\) stands for the two-dimensional Levi-Civita’s symbol (with \(\epsilon^{12} = +1\) and \(n^j = (\cos \theta, \sin \theta)\) represents the unit vector. Here, \(m_1, m_2\) and \(m_3 \in \mathbb{Z}\) are the winding numbers (vorticities) of the corresponding fields, with both the profile functions \(\alpha = \alpha(r)\) and \(\beta = \beta(r)\) depending a priori on the radial coordinate \(r\).

Due to the rotational symmetry of (2) and (3) regular configurations with no singularities can be attained via those profile functions satisfying

\[
\alpha(r = 0) = 0 \quad \text{and} \quad A(r = 0) = 0. \tag{4}
\]

In addition, regarding the combination between the charge matrix \(Q_{ab}\) and the vorticities, we consider \(m_1 = -m_2 = m \quad \text{(with } m \in \mathbb{Z})\), \(m_3 = 0\) and

\[
Q_{ab} = \frac{1}{2} \text{diag}(1, -1, 0). \tag{5}
\]

Hence, the profile function \(\beta(r)\) respects the very same Euler-Lagrange equation already obtained in the context of the Maxwell-\(CP(2)\) scenario in the absence of magnetic impurity, i.e.

\[
\frac{d^2 \beta}{dr^2} + \left(\frac{1}{r} + 2 \frac{d^2}{dr} \cot \alpha\right) \frac{d\beta}{dr} = \frac{\sin^2 \alpha \sin(4\beta)}{r^2} \left(m - \frac{A}{2}\right)^2, \tag{6}
\]

whose simplest solutions are

\[
\beta(r) = \beta_1 = \frac{\pi}{4} + \frac{\pi}{2} k \quad \text{or} \quad \beta(r) = \beta_2 = \frac{\pi}{2} k, \tag{7}
\]

with \(k \in \mathbb{Z}\).

However, it is also known that these two a priori different solutions for \(\beta(r)\) are phenomenologically equivalent given that the first-order results for \(\beta(r) = \beta_2\) can be obtained directly from those ones for \(\beta(r) = \beta_1\) via the redefinitions \(\alpha \to 2\alpha\) and \(h \to h/4\). Therefore, we will mainly focus on the case \(\beta(r) = \beta_1\).

We now look for the first-order framework inherent to the extended model (1) by means of the standard BPS prescription, i.e. via the minimization of the total energy of the effective scenario, the starting-point being the expression for the corresponding energy distribution. In view of the rotationally symmetric map (2) and (3), and given all the conventions introduced above, the time-independent energy density can be written as

\[
\varepsilon = \frac{1}{2} B^2 + U(\alpha) - \Delta B + \hbar \left[\left(\frac{d^2}{dr}\right)^2 + \frac{1}{r^2} \left(\frac{A}{2} - m\right)^2 \sin^2 \alpha \right]. \tag{8}
\]
After some algebra, the expression for the energy density can be rewritten in the form
\[
\varepsilon = \varepsilon_{bps} + \frac{1}{2} \left( B \mp \sqrt{2U} \right)^2 + h \left( \frac{d\alpha}{dr} + \frac{\sin \alpha}{r} \left( \frac{A}{2} - m \right) \right)^2,
\]
where
\[
\varepsilon_{bps} = \mp \frac{1}{gr} \left[ \left( \sqrt{2U} \mp \Delta \right) \frac{d(A - 2m)}{dr} \right] + gh(A - 2m) \frac{d}{dr}(\cos \alpha),
\]
with
\[
B(r) = -\frac{1}{gr} \frac{dA}{dr},
\]
i.e. the rotationally symmetric expression for the magnetic field.

In order to complete the implementation of the BPS prescription, it is necessary to rewrite (10) as a total derivative with respect to the radial coordinate \( r \). Therefore, we impose the differential constraint
\[
\frac{d}{dr} \left( \sqrt{2U} \mp \Delta \right) = gh \frac{d}{dr}(\cos \alpha),
\]
from which the expression for \( \varepsilon_{bps} \) reduces to
\[
\varepsilon_{bps} = \mp \frac{1}{gr} \frac{d}{dr} \left[ (A - 2m) \left( \sqrt{2U} \mp \Delta \right) \right].
\]

Now, in view of (9) and (13), the expression for the total energy of the rotationally symmetric configurations can be promptly written as
\[
\frac{E}{2\pi} = \frac{E_{bps}}{2\pi} + \frac{1}{2} \int \left( B \mp \sqrt{2U} \right)^2 r dr + h \int \left( \frac{d\alpha}{dr} + \frac{\sin \alpha}{r} \left( \frac{A}{2} - m \right) \right)^2 r dr,
\]
where
\[
E_{bps} = 2\pi \int r \varepsilon_{bps} dr,
\]
the upper (lower) sign holds for negative (positive) values of the winding number \( m \).

Therefore, in view of (14) and (16), we can write the general expression
\[
\frac{E}{2\pi} = \frac{E_{bps}}{2\pi} + \frac{1}{2} \int \left( B \mp \sqrt{2U} \right)^2 r dr + h \int \left( \frac{d\alpha}{dr} + \frac{\sin \alpha}{r} \left( \frac{A}{2} - m \right) \right)^2 r dr \geq \frac{2}{g} |m| \left( \sqrt{2U_0} \mp \Delta_0 \right),
\]
from which one clearly sees that when the fields satisfy
\[
B = \pm \sqrt{2U},
\]
they support the existence of time-independent rotationally symmetric structures whose total energy is equal to
\[
E = E_{bps} = \frac{4\pi}{g} |m| \left( \sqrt{2U_0} \mp \Delta_0 \right),
\]
therefore saturating the lower bound defined in (16).

It is important to emphasize that such a first-order construction only holds due to the constraint (12) (which, as we show below, allows us to determine the self-dual potential itself). In the next Section, we also demonstrate that the conditions which were imposed on both \( \sqrt{2U} \mp \Delta \) and \( \sqrt{2U_0} \mp \Delta_0 \) in order to allow the calculation of the result in (16) (i.e. \( \sqrt{2U} \mp \Delta \rightarrow 0 \), with \( \sqrt{2U_0} \mp \Delta_0 \) finite and positive) are effectively satisfied in a rather natural way.

### III. THE FIRST-ORDER SCENARIO AND ITS NUMERICAL SOLUTIONS

We now demonstrate how the first-order expressions we have introduced in the previous Section generate well-defined BPS scenarios. Here, we proceed as follows: firstly, we choose one particular solution for \( \beta(r) \) between the ones in (7), from which we solve the differential constraint (12) in order to obtain the functional form of the relation \( \sqrt{2U} \mp \Delta \) (or, in other words, the relation between the self-dual potential and the magnetic impurity). We then use this result to calculate the value of the Bogomol’nyi bound explicitly (i.e. the total energy of the first-order solutions). A posteriori, we write down the effective first-order equations themselves and, by supposing a localized magnetic impurity, we solve these equations numerically by means of a finite-difference scheme, from which we identify and comment the main effects which appear on the shape of the resulting vortices due to the presence of the magnetic impurity.
Equation (12) can be integrated to obtain
\[ \sqrt{2U} \mp \Delta = gh \cos \alpha, \tag{21} \]
where we have used \( C = 0 \) for the integration constant.

In view of (21), we conclude that the presence of the magnetic impurity into the original Lagrange density (1) requires an adjust on the potential in order to support the existence of first-order configurations, the potential itself being given by
\[ U(\alpha, \Delta) = \frac{g^2h^2}{2} \left( \cos \alpha \pm \frac{1}{gh} \Delta \right)^2, \tag{22} \]
which can also be written as a function of \( \phi_3 \), i.e.
\[ U(\phi_3, \Delta) = \frac{g^2h}{2} \left( \phi_3 \pm \frac{1}{\sqrt{gh}} \Delta \right)^2, \tag{23} \]
which spontaneously breaks the original \( SU(3) \) symmetry, as expected.

A similar calculation for \( \beta(r) = \beta_2 \) gives the potential
\[ U(\alpha, \Delta) = \frac{g^2h^2}{8} \left( \cos^2 \alpha \pm \frac{2}{gh} \Delta - \frac{1}{2} \right)^2. \tag{24} \]
which can also be obtained directly from \( U(\alpha, \Delta) \) in (22) by the substitutions \( h \to h/4 \) and \( \alpha \to 2\alpha \). Hence, both choices are in equivalent, even in the presence of impurities. Also note that potential (24) can be written as a function of \( |\phi_3| \), i.e.
\[ U(|\phi_3|, \Delta) = \frac{g^2}{8} \left( \frac{h}{2} \mp \frac{\Delta}{g} - |\phi_3|^2 \right)^2. \tag{25} \]
which is manifestly compatible with our choice of inhomogeneous coordinates on \( CP^2 \). However, (22) is simpler expression than (24). Therefore, we will from now on only consider the case \( \beta(r) = \beta_1 \).

Now, given the results above, the expression (8) for the energy density can be rewritten in the form
\[ \varepsilon = \frac{1}{2} B^2 + \frac{g^2h^2}{2} \left( \cos \alpha \pm \frac{1}{gh} \Delta \right)^2 - \Delta B \]
\[ + h \left[ \left( \frac{d\alpha}{dr} \right)^2 + \frac{1}{r^2} \left( \frac{A}{2} - m \right)^2 \right] \sin^2 \alpha, \tag{26} \]
via which one concludes that, in the presence of a localized magnetic impurity (i.e. for \( \Delta_{\infty} \equiv \Delta (r \to \infty) \to 0 \)), the finite-energy condition \( \varepsilon (r \to \infty) \to 0 \) is attained by those profile functions which satisfy
\[ \alpha (r \to \infty) \to \frac{\pi}{2} \quad \text{and} \quad A (r \to \infty) \to 2m, \tag{27} \]
which are the very same asymptotic boundary conditions which appear in the standard Maxwell-\(-CP^2(2)\) case without impurity (i.e. the presence of a localized impurity does not change the behavior of the first-order fields in the limit \( r \to \infty \)).

In view of the discussion above and given that \( \alpha (r = 0) = 0 \) (according the conditions in (4)), the expression in (21) gives
\[ \sqrt{2U_0} \mp \Delta_0 = gh, \tag{28} \]
from which we calculate the total energy of the resulting first-order configurations explicitly as
\[ E = E_{bps} = 4\pi h |m|, \tag{29} \]
which is quantized according the winding number \( m \), see the expression (20). In this case, we highlight that the energy of the BPS fields is not affected by the presence of the magnetic impurity.

Moreover, given the self-dual potential (22), the first-order equations (18) and (19) can be written, respectively, in the form
\[ \frac{1}{r} \frac{dA}{dr} = \mp g^2h \left( \cos \alpha \pm \frac{1}{gh} \Delta \right), \tag{30} \]
\[ \frac{d\alpha}{dr} = \pm \sin \alpha \left( \frac{A}{2} - m \right), \tag{31} \]
where we have used (11) for the rotationally symmetric magnetic field.

In what follows, we implement
\[ \Delta (r) = c \exp (-dr^2) \tag{32} \]
for the localized magnetic impurity. Here, both \( c \) and \( d \) \( \in \mathbb{R} \), with \( d > 0 \). In this case, the parameters \( c \) and \( d \) control the height and width of the impurity, respectively.

In view of (32), the first-order equations (30) and (31) reduce to
\[ \frac{1}{r} \frac{dA}{dr} = \mp g^2h \left( \cos \alpha \pm \frac{c}{gh} \exp (-dr^2) \right), \tag{33} \]
\[ \frac{d\alpha}{dr} = \pm \sin \alpha \left( \frac{A}{2} - m \right), \tag{34} \]
which must be solved according the boundary conditions (4) and (27).

It is worthwhile to point out that, given the impurity (32), the first-order potential (22) can be written as
\[ U = \frac{g^2h^2}{2} \left( \cos \alpha \pm \frac{c}{gh} \exp (-dr^2) \right)^2, \tag{35} \]
in which the upper (lower) sign holds for negative (positive) values of \( m \) (the vorticity). In this work, we mainly consider the case of \( d = 1 \) and different values for \( c \); as we explain later below, this case engenders an interesting modification on the form of the effective potential which
FIG. 1: Numerical solutions to $\alpha(r)$ coming from (33) and (34) in the presence of (4) and (27). Here, we have fixed $g = h = 1$, $m = 1$ (lower signs in the first-order expressions) and $d = 1$. This Figure shows the profiles for $c = -5$ (dashed orange line), $c = -4$ (dashed red line), $c = -2$ (dashed blue line), $c = 0$ (usual solution, no impurity, solid black line), $c = +2$ (solid blue line), $c = +4$ (solid red line).

can be understood as the origin of the changes on the shape of the resulting first-order vortices.

We fix $g = h = 1$, for simplicity. Also, we mainly consider $m = 1$ (i.e. the lower signs in the first-order expressions) and $d = 1$ (a fixed value for the width of the impurity). We then study the resulting first-order equations by means of a finite-difference algorithm for different values of $c$ (the height of the impurity), from which we depict the numerical solutions to the profile functions $\alpha(r)$ and $A(r)$, the magnetic field $B(r)$ and the energy distribution $\varepsilon_{bps}$.

The figures 1 and 2 show the numerical profiles to $\alpha(r)$ and $A(r)$, for $c = -5$ (dashed orange line), $c = -4$ (dashed red line), $c = -2$ (dashed blue line), $c = 0$ (usual solution, no impurity, solid black line), $c = +2$ (solid blue line) and $c = +4$ (solid red line). The behavior of these fields at the boundaries are constrained by the boundary conditions. On the other hand, we see that, for negative values of $c$, the resulting profiles lose the original monotonicity. The profiles for $\alpha$ and $A$ have a global maximum for $c = -4$ and $c = -5$. For $c = 2$ and $c = 4$ the profile $A$ has a minimum.

It is interesting to study how these solutions depend on the radial coordinate $r$ as they approach the boundaries. So, we now perform the linearisation of the first-order equations for both $r = 0$ and $r \to \infty$. Then, near the origin, we approximate the profile fields via

$$\alpha(r) \approx \delta \alpha(r),$$

where both $\delta \alpha(r)$ and $\delta A(r)$ are supposed to be small fluctuations around the boundary values $\alpha(r = 0) = 0$ and $A(r = 0) = 0$. Now, in view of (36) and (37), we can derive the linearised version of the first-order equations (in which we use $\exp(-dr^2) \approx 1 - dr^2$ for the magnetic impurity), i.e.

$$\frac{d}{dr} \delta \alpha = m \frac{\delta \alpha}{r},$$

$$\frac{1}{r} \frac{d}{dr} \delta A = g^2 h \left( 1 - \frac{c}{gh} \right) + gcd^2 r^2,$$

whose solutions are (here, $C_0 > 0$ is a real constant)

$$\delta \alpha(r) = C_0 r^m,$$

$$\delta A(r) = \frac{g^2 h}{2} \left( 1 - \frac{c}{gh} \right) r^2 + \frac{gcd}{4} r^4,$$

via which we also obtain the approximate solutions for the profile functions near the origin, i.e.

$$\alpha(r) \approx C_0 r^m,$$

$$A(r) \approx \frac{g^2 h}{2} \left( 1 - \frac{c}{gh} \right) r^2 + \frac{gcd}{4} r^4,$$

which promptly recover the usual results for $c = 0$ (i.e. in the absence of the magnetic impurity).
shows the numerical solutions to the energy
exactly.

The linearised first-order equations
approximately approach the origin. However, the impurity (via its height parameter
c) controls the sign of the magnetic sector as this
represents the masses of the scalar and gauge bosons, respectively (in the context of a first-order framework, the
relation $M_\alpha/M_A = 1$ typically defines the self-dual limit).

Therefore, the expressions (49) and (50) reveal that, in
the presence of a localised magnetic impurity, the resulting
first-order vortices mimic the standard asymptotic
behavior, i.e. a localised impurity does not affect the
way the fields behave in the asymptotic region.

We now return to the analysis of the numerical pro-
files themselves. In Figure 3, we depict the numerical
solutions to the magnetic sector $B(r)$. In this case, the
results suggest that the parameter $c$ (the height of the
impurity) induces the occurrence of an inversion on the
sign of the magnetic field as it approaches the origin.
This phenomenon can be understood as follows: given the
rotationally expression $B(r) = -A'/gr$ for the mag-
netic field (here, prime denotes derivative with respect
to the radial coordinate $r$), the approximate result (43)
can be used to obtain the corresponding one for the
$B(r)$ near the origin, i.e.

$$B(r) \approx B_0 - cdr^2,$$

in which we have identified

$$B_0 = B(r = 0) = c - gh$$

as the value of the magnetic field at $r = 0$.

The expression for $B_0$ above clarifies how the sign of
the magnetic field near the origin depends on the value of $c$: for $c = gh$, the magnetic field at the origin vanishes
($B_0 = 0$), while for $c > gh$ ($c < gh$), we find that $B_0 > 0$
($B_0 < 0$). Hence, we have established the way the pa-
parameter $c$ controls the sign of the magnetic sector as this
field approaches the origin. In addition, the interested
reader can easily verify that our calculations match the
numerical results in Figure 3 exactly.

Figure 4 shows the numerical solutions to the energy
distribution $\varepsilon(r)$ of the first-order vortices, from which
we note that $c$ seems to control also the sign of $\varepsilon(r)$ as it
approaches $r = 0$. In particular, those positive values of $c$
induce the appearance of a particular region in space in
which the energy density assumes negative values. This
can be seen as consequence of the fact that, given the

\[
\delta A(r) = 2M_A C_\infty r e^{-M_A r},
\]

where $C_\infty > 0$ stands for a positive real constant. The
approximate expressions for the profile functions $\alpha(r)$
and $A(r)$ in the asymptotic limit $r \to \infty$ then read

\[
\alpha(r) \approx \frac{\pi}{2} - C_\infty e^{-M_\alpha r},
\]

\[
A(r) \approx 2m - 2M_A C_\infty r e^{-M_A r},
\]

where

\[
M_\alpha = M_A = g \sqrt{\frac{h}{2}}
\]

represent the masses of the scalar and gauge bosons, re-
respectively (in the context of a first-order framework, the
relation $M_\alpha/M_A = 1$ typically defines the self-dual limit).

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\delta A(r) = 2M_A C_\infty r e^{-M_A r},
\]

where $C_\infty > 0$ stands for a positive real constant. The
approximate expressions for the profile functions $\alpha(r)$
and $A(r)$ in the asymptotic limit $r \to \infty$ then read

\[
\alpha(r) \approx \frac{\pi}{2} - C_\infty e^{-M_\alpha r},
\]

\[
A(r) \approx 2m - 2M_A C_\infty r e^{-M_A r},
\]

where

\[
M_\alpha = M_A = g \sqrt{\frac{h}{2}}
\]

represent the masses of the scalar and gauge bosons, re-
respectively (in the context of a first-order framework, the
relation $M_\alpha/M_A = 1$ typically defines the self-dual limit).

Therefore, the expressions (49) and (50) reveal that, in
the presence of a localised magnetic impurity, the resulting
first-order vortices mimic the standard asymptotic
behavior, i.e. a localised impurity does not affect the
way the fields behave in the asymptotic region.

We now return to the analysis of the numerical pro-
files themselves. In Figure 3, we depict the numerical
solutions to the magnetic sector $B(r)$. In this case, the
results suggest that the parameter $c$ (the height of the
impurity) induces the occurrence of an inversion on the
sign of the magnetic field as it approaches the origin.
This phenomenon can be understood as follows: given the
rotationally expression $B(r) = -A'/gr$ for the mag-
netic field (here, prime denotes derivative with respect
to the radial coordinate $r$), the approximate result (43)
can be used to obtain the corresponding one for the
$B(r)$ near the origin, i.e.

\[
B(r) \approx B_0 - cdr^2,
\]

in which we have identified

\[
B_0 = B(r = 0) = c - gh
\]
Conventions as in the Fig. 3. In this case, sufficiently large values of \( c \) lead us to \( \varepsilon (r = 0) \approx -c \), see the Eq. (56).

The inversion on the sign of \( \varepsilon (r) \) can be explained in the very same way as before, i.e. given the solutions (42) and (43), we can approximate the energy distribution near the origin by

\[
\varepsilon(r) \approx \varepsilon_0 + (C_0^2 B_0 + c d) gh r^2 = 2 C_0^2 - gB_0 \varepsilon_0 \approx -ghc_{\text{bps}}(r) \text{.}
\]

with \( B_0 = B(r = 0) \) still given by (53).

Moreover, for \( m = 1 \) (the case which we are mainly considering in this manuscript), the approximate expression for the energy distribution reduces to

\[
\varepsilon(r) \approx \varepsilon_0 + (C_0^2 B_0 + c d) gh r^2 = 2 C_0^2 - gB_0 \varepsilon_0 \approx -ghc_{\text{bps}}(r) \text{.}
\]

where we have introduced

\[
\varepsilon_0 = \varepsilon (r = 0) = (2C_0^2 - gB_0) h
\]

as the value of the energy density at \( r = 0 \).

We note that for \( c = gh \) (i.e. \( B_0 = 0 \)), the energy density at the origin is \( \varepsilon_0 = 2C_0^2 h > 0 \). On the other hand, sufficiently large values of \( c \) lead to \( B_0 \approx c \), from which we approximate (56) to \( \varepsilon_0 \approx -ghc_{\text{bps}} \). In this case, we point out in a qualitative way how \( c \) controls the sign of \( \varepsilon_0 = \varepsilon (r = 0) \): for \( c \to +\infty \) (\( c \to -\infty \)), the value of energy density near the origin decreases (increases) linearly.

We end this Section by clarifying that, as we have calculated above, a localised impurity does not alter the way the fields behave in the asymptotic region. For completeness, we point out that the approximate solutions for both the magnetic field and the energy density in the asymptotic limit can be promptly verified to be

\[
B(r) \approx -ghC_\infty e^{-MAr},
\]

and

\[
\varepsilon(r) \approx 2gh^2 C_\infty e^{-2MAr},
\]

which vanish in the limit \( r \to \infty \) in order to satisfy the finite-energy requirement, as expected.

In figure 5 we show the potential \( U(\alpha, \Delta) \) as a function of \( r \), using the corresponding profile \( \alpha(r) \) and the impurity \( \Delta(r) \) in (32). From the asymptotic we have \( U(0) = \frac{1}{2}(c - 1)^2 \). For \( c > 0 \) there is an additional local minimum. For \( c < 0 \) the value of the potential \( U \) is monotonically increasing. The behaviour of \( U(r) \) for \( c = -4 \) is very similar to that of \( c = -2 \), but the corresponding values are much larger. Therefore, we only display \( c = -2 \) in Fig. 5.
FIG. 6: Numerical solutions to the $\alpha(r)$ for the vacuum configuration $m = 0$ with impurity (32) with black line for $d = 1$ ($c = 4$), blue line for $d = 2$ ($c = 8$), red line for $d = 4$ ($c = 16$), and orange line for $d = 256$ ($c = 1024$). The red dotted line corresponds to $\alpha(r)$ for the $m = 1$ configuration with no impurity.

Finally, we discuss impurities for different values of $d$. We are particularly interested in the limit $d \to \infty$ and $|c| \to \infty$, where the ratio $c/d$ remains fixed. This limit corresponds to a $\delta$ function of strength $4\pi$. For Ginzburg-Landau vortices at critical coupling a $\delta$-function impurity with zero topological charge behaves the same as a charge one vortex at critical coupling [15]. Surprisingly, this remains true for axially symmetric configurations away from critical coupling [16]. In figure 6 we show the profile function $\alpha(r)$ for $d = 1$, 2, 4, and 256. As $d$ increases, the shape function approaches that of vortex with $m = 1$. Figure 7 shows the gauge field $A(r)$ for $d = 1$, 2, 4 and 256. As $d$ increases the minimum of the gauge field $A$ tends to $-2$ and moves towards the origin. This can be compared to the gauge field of a vortex with $m = 1$ shifted by $-2$. Hence, we showed numerically that $\delta$ function impurities “behave” like vortices, also in this more complicated model. While the profile function remains smooth, the gauge function $A(r)$ becomes singular in the limit $d \to \infty$ and develops a jump at the origin.

IV. FINAL COMMENTS AND PERSPECTIVES

We have investigated the rotationally symmetric first-order configurations inherent to a gauged Maxwell-$CP(2)$ theory extended to include a term which represents a magnetic impurity.

We have presented the basic conventions and also defined the effective scenario, from which we have proceeded the minimization of the corresponding energy following the well-established BPS prescription. We have clarified how this minimization depends on the introduction of a differential constraint whose solution is the self-dual potential itself.

As a consequence of the aforementioned procedure, we have obtained the first-order BPS equations themselves and also the lower bound (the Bogomol’nyi one) to be imposed on the total energy of the rotationally symmetric configurations.

The point to be raised here is that, despite its influence on the first-order equations and on the self-dual potential itself, the impurity does not change the effective value of the total energy of the resulting first-order configurations, which remains quantized according the vorticity $m$.

We have considered a localized impurity whose general form is controlled by two real parameters $c$ (its height) and $d$ (its width). We have then solved the resulting first-order equations numerically by means of a finite-difference scheme for a fixed $d$ and different values of $c$, from which we have obtained the numerical profiles for all the fields of the model. In particular, we have implemented the linearization procedure in order to determine the way those field solutions depend on the radial coordinate $r$ as they approach the boundaries, from which we have clarified how the values of $c$ control the inversion on the signs of the magnetic field and the energy distribution as these quantities approach the origin.

It is important to highlight that the effects which we
have identified in this work also appear in the context of the first-order Maxwell-Higgs model in the presence of a magnetic impurity. Such a correspondence reinforces the coherence and validity of our results.

We note that the dramatic changes which we have reported on this work can be thought as being caused by the modification on the form of the effective potential due to the presence of the impurity, see the expression (35) and figure 5.

Vortex and impurities interactions have been observed in various different physical systems such as condensed matter [17], Bose-Einstein condensates [18] and neutron stars [19]. The dynamics of such systems has been explored for example in [20, 21]. A systematic way of introducing impurities to BPS solitons systems so that half of the BPS equations are preserved has recently been developed in [22, 23]. This allows the detailed investigation of, for example, a kink scattering off a kink trapped by an impurity, see also [24] for an earlier study. The corresponding relativistic dynamics of Ginzburg-Landau vortices with magnetic impurities has been studied in detail in [16]. It is an open question, how Manton’s Schrödinger-Chern-Simons model [25] and its interesting vortex dynamics [26, 27] is affected by impurities.

The configurations which we have obtained in this work can be used as the initial ones during the numerical study of the dynamical interaction between the Maxwell-CP(2) vortices and a localized impurity. Also, the idea regarding the impurity can be applied to the first-order Chern-Simons-CP(2) scenario, from which we expect the occurrence of interesting changes on the shape of the electric field. These issues are currently under investigation, and we hope relevant results to be presented in a future contribution.

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