A look at representations of $SL_2(\mathbb{F}_q)$ through the lens of size

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Abstract
How to study a nice function on the real line? The physically motivated Fourier theory technique of harmonic analysis is to expand the function in the basis of exponentials and study the meaningful terms in the expansion. Now, suppose the function lives on a finite non-commutative group $G$, and is invariant under conjugation. There is a well-known analog of Fourier analysis, using the irreducible characters of $G$. This can be applied to many functions that express interesting properties of $G$. To study these functions one wants to know how the different characters contribute to the sum? In this note we describe the $G = SL_2(\mathbb{F}_q)$ case of the theory we have been developing in recent years which attempts to give a fairly general answer to the above question for finite classical groups. The irreducible representations of $SL_2(\mathbb{F}_q)$ are “well known” for a very long time (Frobenius in Sitzber Preuss Akad Wiss 985–1021, 1896; Jordan in Am J Math 29:387–405, 1907; Schur in Journal für die reine und angewandte Mathematik 132:85–137, 1907) and are a prototype example in many introductory courses on the subject. We are happy that we can say something new about them. In particular, it turns out that the representations that were considered as “anomalous” in the “old” point of view (known as the “philosophy of cusp forms”) are the building blocks of the current approach.

Keywords  Size · Rank · Harmonic analysis · Character ratios · Eta correspondence

To Joe Wolf mazal tov ad me’ah v’esrim

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1 Introduction

An important invariant attached to any finite group $G$ is its representation ring \([12] \]
\[ R(G) = \mathbb{Z}[\hat{G}], \]
i.e., the ring generated from the set $\hat{G}$ of isomorphism classes of irreducible representations (irreps) using the operations of addition and multiplication given, respectively, by direct sum $\oplus$ and tensor product $\otimes$.

The main goal of this note is to advertise, using the special linear group over the finite field $\mathbb{F}_q$ with $q$ odd elements $SL_2(\mathbb{F}_q)$, our recent discovery \([12–14,21,22] \] that in the case that $G$ is a finite classical group the ring $R(G)$ has a natural “tensor rank” filtration that encodes the analytic properties of the representations. In particular, moving to the associated graded pieces, each member $\rho$ of the unitary dual $\hat{G}$ gets a well defined non-negative integer called tensor rank and denoted $\text{rank}_\otimes(\rho)$. This integer seems—see Figure 1 for illustration\(^1\) in the case of $G = SL_2(\mathbb{F}_q)$—to be intimately related to analytic properties of $\rho$ such as its dimension and size of character values.

In the rest of the introduction we discuss two aspects of the tensor rank filtration. The first is the fact that this is the first point of view that makes the degenerate discrete and principal series representations of $SL_2(\mathbb{F}_q)$ seem significant rather than anomalous. The second aspect we discuss, through a specific example for $SL_2(\mathbb{F}_q)$, is the potential usefulness of the tensor rank filtration to the harmonic analysis of the finite classical groups.

1.1 Degenerate discrete and principal series representations

Let $G = G(\mathbb{F}_q)$ be the group of $\mathbb{F}_q$-rational points of a reductive algebraic group $G$ defined over $\mathbb{F}_q$. The Gelfand–Harish-Chandra “philosophy of cusp forms” \([6,15] \] asserts that each member $\rho$ of $\hat{G}$ can be realized, in some effective manner, inside the induction $\text{Ind}_{P}^{G}(\sigma)$ from a cuspidal representation (i.e., one that is not induced from a smaller parabolic subgroup) of a parabolic subgroup $P$ of $G$

\[ \hat{G} = \left\{ \begin{array}{l} \rho \in \text{Ind}_{P}^{G}(\sigma), \\ G \supset P \text{ - parabolic}, \\ \sigma \text{ cuspidal repn of } P \end{array} \right\}. \]

The above philosophy is a central one and leads to important developments in representation theory of reductive groups over local and finite fields, in particular the

\(^1\) The numerics in this note were generated using the computer algebra system Magma.
work of Deligne–Lusztig [2] on the construction of representations of finite reductive groups and Lusztig’s striking achievement: The classification [29] of these representations.

Let us see how this philosophy manifests itself in the cases when $G$ is $GL_2(\mathbb{F}_q)$ or $SL_2(\mathbb{F}_q)$—see Table 1 (let us ignore for this discussion the one dimensional representations). In both cases around 50% of the irreps are cuspidals, i.e., $P = G$, and form the so called discrete series, and 50% are the so called principal series, i.e., induced from the Borel subgroup $P = B$ of upper triangular matrices in $G$. Moreover, it is well known, and not so difficult to show [4], that the discrete and principal series representations of $SL_2(\mathbb{F}_q)$ can be obtained by restriction from the corresponding irreps of $GL_2(\mathbb{F}_q)$. These restrictions are typically irreducible, with only two exceptions: One discrete series representation of dimension $q - 1$ and one principal series representation of dimension $q + 1$ split into two pieces. Hence one has—see Table 1—two discrete and two principal series representations of dimensions $\frac{q-1}{2}$ and $\frac{q+1}{2}$, respectively. These representations are called in the literature “degen-

\begin{table}[h]
\centering
\begin{tabular}{|l|l|l|}
\hline
$G$ & $\text{Discrete series (P=G)}$ & $\text{Principal series (P=B)}$ \\
\hline
$GL_2(\mathbb{F}_q)$ & $\dim(\rho) = q - 1$ & $\dim(\rho) = q + 1$ or $q$ (one irrep) \\
\hline
$SL_2(\mathbb{F}_q)$ & $\begin{cases}
q - 1 \\
\frac{q - 1}{2}
\end{cases}$ & $\begin{cases}
q + 1 \\
\frac{q + 1}{2}
\end{cases}$ \\
\hline
\end{tabular}
\caption{Rough description of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$}
\end{table}
erate discrete series” and “degenerate principal series” and are usually treated as anomalous.

In this note we show that the tensor rank filtration proposes a quantitative measurement that, in particular, demonstrates that the degenerate representations are significant rather than anomalous. In fact, in some formal sense these irreps are the “atoms” of the all theory of “size” of representations we are describing.

1.2 Harmonic analysis on a finite group

Suppose we want to study a class function \( \mathcal{N} \), defined on a finite group \( G \):

\[
\mathcal{N} : G \to \mathbb{C}, \quad \mathcal{N}(hgh^{-1}) = \mathcal{N}(g),
\]

for every \( g, h \in G \).

Many interesting examples of \( \mathcal{N} \) present similar manner discussed below.

1.2.1 The problem of harmonic analysis on a finite group

Class functions can, in principle, be investigated using the representation theory of \( G \): it is a basic fact [34] that one can expand them as a linear combination of the irreducible characters of \( G \). This is the harmonic analysis approach for studying class functions on \( G \). In particular, this technique can be applied to many functions that express interesting properties of \( G \) [3,9,27,28,31,35,36]. In that latter case the expansion is in many cases of the form

\[
\mathcal{N} = \sum_{\rho \in \hat{G}} \frac{\chi_{\rho}(g)}{\dim(\rho)} \cdot \rho,
\]

where \( \frac{\chi_{\rho}(g)}{\dim(\rho)} \) stands for some relevant expression in terms of the character ratios \( \frac{\chi_{\rho}(g)}{\dim(\rho)} \) of the irreps of \( G \).

The above discussion suggests the following:

**Problem (Main problem of harmonic analysis on \( G \)).** Estimate the character ratios

\[
\frac{\chi_{\rho}(g)}{\dim(\rho)}, \quad \rho \in \hat{G}, \quad g \in G,
\]

and possible relations among them.

1.2.2 Example: the commutator mapping on \( SL_2(\mathbb{F}_q) \)

Consider the group \( G = SL_2(\mathbb{F}_q) \) and the commutator mapping

\[
[\cdot, \cdot] : G \times G \to G, \quad [x, y] = xyx^{-1}y^{-1}.
\]

For an element \( g \in G \) let us denote by \([\cdot, \cdot]_g\) the set \( \{(x, y) \in G \times G; [x, y] = g\} \) called the fiber over \( g \) of the commutator mapping. We are interested in the distribution of
Table 2 Values of $\mathcal{N}(g)$ for various $g$, $q$.

| $q$ (below) vs. $g$ (right) | $I$ | $-I$ | $\begin{pmatrix} \pm 1 & 1, \varepsilon \\ 0 & \pm 1 \end{pmatrix}$ | $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ | $\begin{pmatrix} x \varepsilon y \\ y \varepsilon x \end{pmatrix}$ |
|-----------------------------|-----|------|-------------------------------------------------|---------------------------------|---------------------------------|
| 101                         | 105 | 1    | 0.98                                           | 0.97                            | 0.97                            |
| 197                         | 201 | 1    | 0.99                                           | 1.01                            | 0.98                            |
| $q$                         | $q + 4$ | 1    | $1 + O(1/q)$                                   | $1 + O(1/q)$                    | $1 + O(1/q)$                    |

the number of elements $\#([1], g)$ when $g$ runs in the set $G \setminus \{\pm I\}$. It is natural [35] to make a suitable normalization and study the class function

$$\mathcal{N}(g) = \#([1], g)/\#G.$$  \hspace{1cm} (1.2)

To see how this function behaves over the various conjugacy classes of $G$, let us recall [4] that $G$ has $q + 4$ conjugacy classes with standard representatives

$$I, -I, \begin{pmatrix} \pm 1 & 1, \varepsilon \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} x \varepsilon y \\ y \varepsilon x \end{pmatrix},$$  \hspace{1cm} (1.3)

where $\varepsilon$ is a non-square \(^2\), $a \neq \pm 1$, $y \neq 0$, and $x^2 - \varepsilon y^2 = 1$.

Using numerics—see Table 2 for illustration—one can suspect the following to be true:

**Theorem (Uniformity of the commutator map).** We have\(^3\)

$$\mathcal{N}(g) = 1 + O(1/q), \ g \neq \pm I.$$  \hspace{1cm} (1.4)

The relevant formula enabling the representation theoretic study of the function $\mathcal{N}$ is due to Frobenius [5]

$$\mathcal{N}(g) = \sum_{\rho \in \hat{G}} \frac{\chi_{\rho}(g)}{\dim(\rho)}.$$  \hspace{1cm} (1.5)

Looking back on (1.4), and taking into account the trivial representation in (1.5), we see that to verify the uniformity statement above we need to

**Goal:** Explore the cancellation

$$\sum_{1 \neq \rho \in \hat{G}} \frac{\chi_{\rho}(g)}{\dim(\rho)} = O(1/q), \ g \neq \pm I.$$  \hspace{1cm} (1.6)

In fact, it is possible (see [8]) to verify (1.6) invoking available character tables of $G$ [4,5,25,32]. There are two drawbacks to this approach. First, it will prevent us

\(^2\) For the rest of this note $\varepsilon$ stands for a non-square element of $F_q$.

\(^3\) For numerical sequences $a(q), b(q)$, we write $a(q) = O(b(q))$ if there is $C$ s.t. $|a(q)| \leq C \cdot |b(q)|$ for all sufficiently large $q$.

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Table 3  Order of magnitude of $\chi_{\rho}(g)/\dim(\rho)$ for $\rho$’s irreps of $G = SL_2(\mathbb{F}_q)$

| $\rho$ (below) vs. $g$ (right) | $\left(\pm 1, \varepsilon \right)$ | $\left(a \ 0 \ a^{-1}\right)$ | $\left(x \ y \ x \right)$ |
|---------------------------------|-------------------------------------|---------------------------------|---------------------------|
| 2 irreps $\rho$ of dim. $q^{-1}/2$ | 1/$\sqrt{q}$ | 0 | 1/q |
| 2 irreps $\rho$ of dim. $q+1/2$ | 1/$\sqrt{q}$ | 1/q | 0 |
| 1 irrep $\rho$ of dim. $q$ | 0 | 1/q | 1/q |
| $q-1$ irreps $\rho$ of dim. $q-1$ | 1/q | 0 | 1/q |
| $q-3$ irreps $\rho$ of dim. $q+1$ | 1/q | 1/q | 0 |

from telling the story we are trying to make. Second, and more seriously, it is not really feasible to have explicit character tables for more complicated groups, e.g., the high rank symplectic groups. This is a family that contains $SL_2(\mathbb{F}_q) = Sp_2(\mathbb{F}_q)$ as a member and for which one might expect (see the numerics in [12]) similar behavior of the relevant Frobenius sum (1.6). So we would like to have a different approach to achieve the goal (1.6). In this note we demonstrate, in the case of $SL_2(\mathbb{F}_q)$, how the tensor rank filtration might assists with that task.

2 Character ratios and tensor rank

A priori, it is possible that the sum (1.6) is small as it is because each of the (around) $q$ terms there is so small. This is not the case.

2.1 Character ratios

A quick numerical experiment with Magma tells us (see Table 3 for the outcome) that indeed the sum (1.6) is small due to cancellations. For example, in the last two rows of the second column of Table 3 there are around $q$ character ratios each one of order of magnitude $1/q$. Moreover, looking again on the second and third rows of the second column, one hope for two “sub-cancellations” in the sum (1.6), one between the character ratios of the irreps of dimensions $q\pm1/2$, and another one between the character ratios of the other irreps.

We will show that indeed this what is going on.

2.2 Tensor rank of a representation

The different orders of magnitude of the character ratios discussed above—see also Figure 1 for illustration—is a manifestation of a particular invariant—called tensor rank—that can be attached to any representation of $G$ and should be regarded in a formal sense as its “size”.

Let $k$ be a non-negative integer and consider the collection $F^k = F^k R(G)$ of representations in $R(G)$ that appear inside sums (with integer coefficients) of $\ell$-fold,
\( \ell \leq k \), tensor product of the four irreps of \( G \) of dimensions \( q^{\pm 1} \). One has \( F^k \subset F^{k+1} \), \( F^i F^j \subset F^{i+j} \) and \( \cup_k F^k = R(G) \), so the \( F^k \)'s form a filtration of \( R(G) \) that we call tensor rank filtration. In fact,

**Proposition 2.1** We have \( F^2 = R(G) \).

Proposition 2.1 is verified using explicit computations in Sect. 4.3—see Sect. 4.3.3.

Now we can pass to the associated graded pieces \( \operatorname{Gr}^k = F^k / F^{k-1} \) and in particular obtain a well defined notion of tensor rank for representations [13]. Concretely,

**Definition 2.2** (**Tensor rank**) Let \( \rho \) be a representation of \( G \). We say that \( \rho \) has tensor rank \( k \), denoted \( \operatorname{rank}_\otimes(\rho) = k \), if \( \rho \in F^k \setminus F^{k-1} \).

Let us denote by \( \hat{G}_k \) the family of irreps of tensor rank \( k \). Combining Proposition 2.1 and the information appearing in Table 1 we see that \( \hat{G}_0 \) consists of the trivial representation, \( \hat{G}_1 \) is the collection of irreps of dimensions \( q^{\pm 1/2} \), and finally \( \hat{G}_2 \) is the family of irreps of dimensions \( q \pm 1 \) and \( q \).

Going back to the problem of exploring the cancellation in the sum (1.6) our idea is to split it over the tensor ranks

\[
\sum_{1 \neq \rho \in \hat{G}} \frac{\chi_\rho(g)}{\dim(\rho)} = \sum_{\rho \in \hat{G}_1} \frac{\chi_\rho(g)}{\dim(\rho)} + \sum_{\rho \in \hat{G}_2} \frac{\chi_\rho(g)}{\dim(\rho)},
\]

and to witness the cancellation in each partial sum. Of course, for this we will need more information on the irreps of each given tensor rank.

### 3 The Heisenberg and oscillator representations

Where do the tensor rank one irreps of \( G = SL_2(\mathbb{F}_q) \) come from? A. Weil shed some light on this in [39]. They can be found by considering the Heisenberg group.

#### 3.1 The Heisenberg group

We consider the vector space \( W = \mathbb{F}_q^{2n} \) of column vectors of length \( 2n \) with entries in \( \mathbb{F}_q \). We equip \( W \) with the non-degenerate skew-symmetric bilinear form

\[
\langle w, w' \rangle = w^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} w',
\]

where \( w^t \) is the row vector transpose of \( w \), and \( I \) is the \( n \times n \) identity matrix. The pair \((W, \langle , \rangle)\) is called symplectic vector space.

The Heisenberg group attached to \((W, \langle , \rangle)\) is a two-step nilpotent group that can be realized by the set

\[
H = W \times \mathbb{F}_q,
\]
with the group law

\[(w, z) \cdot (w', z') = (w + w', z + z' + \frac{1}{2}\langle w, w' \rangle).\]

In particular, the center \(Z\) of the Heisenberg group

\[Z = \{(0, z); \ z \in F_q = F_q,\]

is equal to its commutator subgroup.

### 3.2 Representations of the Heisenberg group

We would like to describe the irreps of the Heisenberg group.

Take an irreducible representation \(\pi\) of \(H\). Then, by Schur’s lemma [34], the center \(Z\) will act by scalars

\[\pi(z) = \psi_\pi(z)I, \ z \in Z,\]

where \(I\) is the identity operator on the representation space of \(\pi\), and \(\psi_\pi \in \hat{Z}\) is a character of \(Z\), called the central character of \(\pi\). If \(\psi_\pi = 1\), then \(\pi\) factors through \(H/Z \cong W\), which is abelian, so \(\pi\) is itself a character of \(W\). The case of non-trivial central character is described by the following celebrated theorem [30]:

**Theorem 3.1** (Stone–von Neumann–Mackey) Up to equivalence, there is a unique irreducible representation \(\pi_\psi\) with given non-trivial central character \(\psi\) in \(\hat{Z} \smallsetminus \{1\}\).

We will call the (isomorphism class of the) representation \(\pi_\psi\) the Heisenberg representation associated to the central character \(\psi\).

**Remark 3.2** (Realization) There are many ways to realize (i.e., to write explicit formulas for) \(\pi_\psi\) [7,10,11,16,39]. In particular, it can be constructed as induced representation from any character extending \(\psi\) to any maximal abelian subgroup of \(H\). To have a concrete one, note that the inverse image in \(H\) of any maximal subspace of \(W\) on which \(\langle \cdot, \cdot \rangle\) is identically zero (a.k.a. Lagrangian) will be a maximal abelian subgroup for which we can naturally extend the character \(\psi\). For example, consider the Lagrangian

\[X = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\} \subset W, \tag{3.1} \]

and the associated maximal abelian subgroup \(\tilde{X}\) with character \(\tilde{\psi}\) on it, given by

\[\tilde{X} = X \times F_q, \ \tilde{\psi}(x, z) = \psi(z).\]
Then we have the explicit realization of $\pi_\psi$, given by the action of $H$, by right translations, on the space

$$\text{Ind}_H^H(\tilde{\psi}) = \{ f : H \to \mathbb{C}; \ f(\tilde{x}h) = \tilde{\psi}(\tilde{x})f(h), \ \tilde{x} \in \tilde{\mathcal{X}}, \ h \in H \}. \tag{3.2}$$

In particular, we have $\dim(\pi_\psi) = q^n$.

### 3.3 The oscillator representation

Consider the group $Sp(W)$ of automorphism of $W$ that preserve the form $(\cdot, \cdot)$. The action of $Sp(W)$ on $W$ lifts to an action on $H$ by automorphisms leaving the center point-wise fixed. The precise formula is $g(w, z) = (gw, z), \ g \in Sp(W)$. It follows from the Stone–von Neumann–Mackey theorem, that the induced action of $Sp(W)$ on the set $\text{Irr}(H)$ will leave fixed each isomorphism class $\pi_\psi, \psi \in \hat{\mathbb{Z}} \setminus \{1\}$. This means that, if we fix a vector space $\mathcal{H}_\psi$ realizing $\pi_\psi$, then for each $g$ in $Sp(W)$ there is an operator $\omega_\psi(g)$ that acts on space $\mathcal{H}_\psi$ and satisfies the equation

$$\omega_\psi(g)\pi_\psi(h)\omega_\psi(g)^{-1} = \pi_\psi(g(h)). \tag{3.3}$$

Note that, by Schur’s lemma, the operator $\omega_\psi(g)$ is defined by (3.3) up to scalar multiples. This implies that for any $g, g' \in Sp(W)$ we have $\omega_\psi(g)\omega_\psi(g') = c(g, g')\omega_\psi(gg')$, where $c(g, g')$ is an appropriate complex number of absolute value 1. It is well known (see [7,10,11] for explicit formulas) that over finite fields of odd characteristic the mapping $\omega_\psi$ can be defined so that $c(g, g') = 1$ for every $g, g \in Sp(W)$, i.e., $\omega_\psi$ defines a representation of $Sp(W)$. We summarize:

**Theorem 3.3** (Oscillator representation) There exists a representation

$$\omega_\psi : Sp(W) \rightarrow GL(\mathcal{H}), \tag{3.4}$$

that satisfies the identity (3.3).

We will call $\omega_\psi$ the oscillator representation. This is a name that was given to this representation in [16] due to its origin in physics [33,37]. Another popular name for $\omega_\psi$ is the Segal–Shale–Weil or just Weil representation, following the influential paper [39].

**Remark 3.4** (Schrödinger model) We would like to have some useful formulas for $\omega_\psi$. We consider the Lagrangian decomposition $W = X \oplus Y$, where $X$ is the first $n$ coordinates Lagrangian (3.1) and $Y$ is the second $n$ coordinates Lagrangian defined similarly. In terms of this decomposition, the space (3.2) is naturally identified with

$$L^2(Y) = \text{functions on } Y. \tag{3.5}$$

---

4 This representation is unique except the case $n = 2$ and $q = 3$, where there is a canonical one [10,11].
On the space (3.5) we realize the representation ωψ. This realization is sometime called the Schrödinger model [19]. In particular, in that model for every \( f \in L^2(Y) \) we have [7,12,39]

\[
\begin{align*}
(A) \quad & \left[ \omega_\psi \left( \begin{array}{cc} I & A \\ 0 & I \end{array} \right) f \right](y) = \psi(\frac{1}{2}A(y, y))f(y), \text{ where } A : Y \to X \text{ is symmetric}; \\
(B) \quad & \left[ \omega_\psi \left( \begin{array}{cc} 0 & B \\ -B^{-1} & 0 \end{array} \right) f \right](y) = \frac{1}{\gamma(B, \psi)} \sum_{y' \in Y} \psi(B(y, y'))f(y'), \text{ where } B : Y \sim X \text{ is symmetric, and } \gamma(B, \psi) = \sum_{y \in Y} \psi(-\frac{1}{2}B(y, y)) \text{ the quadratic Gauss sum [24]}; \\
(C) \quad & \left[ \omega_\psi \left( \begin{array}{cc} tC^{-1} & 0 \\ 0 & C \end{array} \right) f \right](y) = \left( \frac{\det(C)}{q} \right)f(C^{-1}y), \\
& \quad \text{where } C \in GL(Y), tC^{-1} \in GL(X) \text{ its transpose inverse, and } \left( \frac{\cdot}{q} \right) \text{ is the Legendre symbol.}
\end{align*}
\]

It turns out that the isomorphism class of \( \omega_\psi \) does depends on the central character \( \psi \) in \( \hat{\mathbb{Z}} \smallsetminus \{1\} \). However, this dependence is weak. The following result [7,16,17] indicates that there are only two possible oscillator representations. For a character \( \psi \) in \( \hat{\mathbb{Z}} \smallsetminus \{1\} \) denote by \( \psi_a, a \in \mathbb{F}_q^* \), the character \( \psi_a(z) = \psi(az) \).

**Proposition 3.5** We have \( \omega_\psi \simeq \omega_{\psi'} \) iff \( \psi' = \psi_{a^2} \) for some \( a \in \mathbb{F}_q^* \).

### 3.4 The representations of tensor rank one

The oscillator representations are slightly reducible. Indeed, for any two representations \( \varrho, \varrho' \) of any group one has the intertwining number \( (\varrho, \varrho') = \dim \text{Hom}(\varrho, \varrho') \).

The following is well known [7,16,20].

**Proposition 3.6** We have \( (\omega_\psi, \omega_\psi) = 2 \).

For the sake of completeness, we give a proof of Proposition 3.6 in Appendix C.1.

The meaning of Proposition 3.6 is that \( \omega_\psi \) has two irreducible pieces. They can be computed explicitly as follows. The center \( Z(Sp(W)) = \{ \pm I \} \) acts on the representation \( \omega_\psi \), inducing the decomposition into irreps

\[
\omega_\psi = \omega_{\psi,1} \oplus \omega_{\psi,sgn}, \tag{3.6}
\]

where \( \omega_{\psi,1} \) is the subspace of vectors on which \( Z(Sp(W)) \) acts trivially, and \( \omega_{\psi,sgn} \) is the subspace of vectors on which \( Z(Sp(W)) \) acts via the sign character. Moreover, using the description for the action of \(-I\) given in Remark 3.4 (C), it is easy to verify that for \( n \) odd

\[
\dim(\omega_{\psi,1}) = \begin{cases} \frac{q^n+1}{2}, & \text{if } q \equiv 1 \text{ mod } 4; \\ \frac{q^n-1}{2}, & \text{if } q \equiv 3 \text{ mod } 4, \end{cases} \quad \text{and } \dim(\omega_{\psi,sgn}) = \begin{cases} \frac{q^n-1}{2}, & \text{if } q \equiv 1 \text{ mod } 4; \\ \frac{q^n+1}{2}, & \text{if } q \equiv 3 \text{ mod } 4, \end{cases}
\]

\( \text{For } a \in \mathbb{F}_q^* \) the Legendre symbol \( \left( \frac{a}{q} \right) = +1 \) or \(-1\), according to \( a \) being a square or not, respectively.
and for \( n \) even

\[
\dim(\omega_{\psi,1}) = \frac{q^n + 1}{2}, \quad \text{and} \quad \dim(\omega_{\psi,\text{sgn}}) = \frac{q^n - 1}{2}.
\]

Now, substituting \( n = 1 \) we have \( Sp(W) = SL_2(\mathbb{F}_q) \) and using the two possible oscillator representations (see Proposition 3.5) we obtain all the irreps of tensor rank one. These have dimensions \( \frac{q^\pm 1}{2} \). They are exactly the representations that appear “anomalous” in the philosophy of cusp forms—see Sect. 1.1.

Next, we want information on the tensor rank two irreps.

4 The eta correspondence

How to get information on the tensor rank two irreps of \( G = SL_2(\mathbb{F}_q) \)? This section will include an answer to this question. In fact, we introduce a systematic method called \( \eta \)-correspondence that works uniformly for all tensor ranks \( k = 0, 1, 2 \), generalizes the construction of tensor rank one irreps described in Sect. 3, and exists for all classical groups over finite and local fields \([12,13,21,22]\). The eta correspondence is related, but different, to the theta correspondence in the representation theory of classical groups over local fields \([18]\).

4.1 The \((O_{k\pm}, SL_2(\mathbb{F}_q))\) dual pair

Consider the vector space \( U = \mathbb{F}_q^k \) and let \( \beta \) be an inner product (i.e., a non-degenerate symmetric bilinear form) on \( U \). The pair \((U, \beta)\) is called a quadratic space. In this note we will be interested only in the cases \( k = 0, 1, 2 \). In these cases we have the following examples:

- \( k = 0 \): The zero space \( U = 0 \) with \( \beta = 0 \).
- \( k = 1 \): The line \( U = \mathbb{F}_q \) with the form \( \beta_1^+(x) = x^2 \) or with the form \( \beta_1^-(x) = \varepsilon x^2 \), where \( \varepsilon \) is a non-square in \( \mathbb{F}_q \).
- \( k = 2 \): The plane \( U = \mathbb{F}_q^2 \) with
  - the form \( \beta_2^+(x, y) = x^2 - y^2 \). In this case—called hyperbolic plane—i.e., there are two lines on which the form is identically zero.
  - the form \( \beta_2^-(x, y) = x^2 - \varepsilon y^2 \), where \( \varepsilon \) is a non-square in \( \mathbb{F}_q \). In this case—called anisotropic plane—there is no line on which the form is identically zero.

It is well known \([26]\) that up to isometry there are only two quadratic spaces of dimension \( k \neq 0 \) over \( \mathbb{F}_q \). In particular, for \( k = 0, 1, 2 \), the above list of examples is exhaustive.

Finally, for a quadratic space \((U, \beta)\) we denote by \( O_\beta \) the associated isometry group and call it the orthogonal group. We might also denote this group by \( O_{k+} \) or \( O_{k-} \) according as the form \( \beta \) is, respectively, \( \beta^+ \) or \( \beta^- \), or by \( O_{k\pm} \) to mean either one of these groups.
Now, let us take the plane $V = \mathbb{F}_q^2$ with the symplectic form

$$\{v, v'\}_V = v' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v'.$$

and consider the vector space $U \otimes V$—the tensor product of $U$ and $V$ [1]. It has a natural structure of a symplectic space, with the symplectic form given by $\beta \otimes \langle \cdot, \cdot \rangle_{V}$.

The groups $O_\beta$ and $G = SL_2(\mathbb{F}_q)$ act on $U \otimes V$ via their actions on the first and second factors, respectively,

$$O_\beta \curvearrowright U \otimes V \curvearrowleft G.$$

Both actions preserve the form $\beta \otimes \langle \cdot, \cdot \rangle_{V}$, and moreover the action of $O_\beta$ commutes with that of $G$, and vice versa. In particular, we have a map

$$O_\beta \times G \longrightarrow Sp(U \otimes V),$$

that embeds each of the two factors $O_\beta$ and $G$ in $Sp(U \otimes V)$, and the two images form a pair of commuting subgroups. In fact, each is the full centralizer of the other inside $Sp(U \otimes V)$. Thus, the pair $(O_\beta, G)$ forms what has been called in [16] a dual pair of subgroups of $Sp(U \otimes V)$.

### 4.2 The eta correspondence

Let us fix for the rest of this paper a non-trivial character $\psi$ of $\mathbb{F}_q$. Consider the symplectic space $W = U \otimes V$ with the form $\langle \cdot, \cdot \rangle = \beta \otimes \langle \cdot, \cdot \rangle_{V}$ described in Sect. 4.1 above. In this setting, we have the oscillator representation $\omega_{U \otimes V} = \omega_{U \otimes V, \psi}$ of $Sp(U \otimes V)$ given by (3.4).

Note that for $k = 0$, $\omega_{U \otimes V}$ is the trivial representation, and for $k = 1$ it is, respectively, the oscillator representation $\omega_{\psi^+}$ or $\omega_{\psi^-}$ of $G = SL_2(\mathbb{F}_q)$, according as the form $\beta$ is $\beta_1^+$ or $\beta_1^-$.  

**Proposition 4.1** Assume that $\dim(U) = 2$. As a representation of $G = SL_2(\mathbb{F}_q)$, in case the form $\beta$ is $\beta_2^+$, we have

$$\omega_{U \otimes V|G} = \begin{cases} \omega_{\psi^+} \otimes \omega_{\psi^+} & \text{if } q \equiv 1 \text{ mod } 4; \\ \omega_{\psi^+} \otimes \omega_{\psi^-} & \text{if } q \equiv 3 \text{ mod } 4, \end{cases}$$

and in case $\beta$ is $\beta_2^-$, we have

$$\omega_{U \otimes V|G} = \begin{cases} \omega_{\psi^+} \otimes \omega_{\psi^-} & \text{if } q \equiv 1 \text{ mod } 4; \\ \omega_{\psi^+} \otimes \omega_{\psi^+} & \text{if } q \equiv 3 \text{ mod } 4. \end{cases}$$

For a proof of Proposition 4.1 see Appendix C.2.

For the rest of this note, if not otherwise stated, we assume that the dimension of $U$ is $k = 0, 1$ or 2. Note that, by Proposition 4.1, every representation of $G$ that appears
in \( \omega_{U \otimes V} \) is of tensor rank \( \leq k \). Let us denote by \( \omega_{U \otimes V,k} \) the \( k \)-spectrum of \( G \) in \( \omega_{U \otimes V} \), i.e., the subspace of \( \omega_{U \otimes V} \) consisting of the isotypic components of tensor rank \( k \) irreps of \( G \). The group algebras \( A_{O_\beta} = \mathbb{C}[\omega_{U \otimes V}(O_\beta)] \), \( A_G = \mathbb{C}[\omega_{U \otimes V}(G)] \) act on \( \omega_{U \otimes V,k} \). Let us denote by \( \hat{G}_k \) the irreps of \( G \) of tensor rank \( k \). The following is a key result:

**Theorem 4.2** (\( \eta \)-correspondence) We have

1. The algebras \( A_{O_\beta} \) and \( A_G \) are each other’s commutant in \( \text{End}(\omega_{U \otimes V,k}) \). In particular, the double commutant theorem (see Appendix B) implies that, we have an injection

\[
\tau \mapsto \eta(\tau), \tag{4.4}
\]

from a subset of \( \hat{O}_\beta \) to \( \hat{G}_k \).

2. Using the two possible \( \beta \)s we obtain all of \( \hat{G}_k \), i.e.,

\[
\eta(O_{k+}) \cup \eta(O_{k-}) = \hat{G}_k,
\]

where \( \eta(O_{k\pm}) \) denotes the image of the mapping (4.4).

We call (4.4) the eta correspondence.

In [12,13], following initial ideas from [16], we introduced techniques proving Theorem 4.2 in generality for any dual pair when one of the members is moving in a Witt tower.

In this note we offer more elementary and explicit verification.

### 4.3 Explicit description of the eta correspondence

We are interested in getting information on the restriction of the oscillator representation \( \omega_{U \otimes V} \) of \( \text{Sp}(U \otimes V) \) to \( G = \text{Sp}(V) = \text{SL}_2(F_q) \). Consider the intertwining number \( (\omega_{U \otimes V}, \omega_{U \otimes V})_G = \dim \text{End}_G(\omega_{U \otimes V}) \). In case of \( \dim(U) = k = 0, 1 \), we know (see Proposition 3.6 for \( k = 1 \)) that this number is 1 and 2, respectively.

**Proposition 4.3** Suppose \( \dim(U) = 2 \). Then,

\[
(\omega_{U \otimes V}, \omega_{U \otimes V})_G = 2q + 1. \tag{4.5}
\]

For a proof of Proposition 4.3 see Appendix C.3.

Now, we consider the restriction, i.e., the pullback via the map (4.1), of \( \omega_{U \otimes V} \) to the product \( O_\beta \times G \). We decompose this restriction into isotypic components for \( O_\beta \)

\[
\omega_{U \otimes V}|_{O_\beta \times G} = \sum_{\tau \in \hat{O}_\beta} \tau \otimes \Theta(\tau), \tag{4.6}
\]

where \( \Theta(\tau) \) is a representation of \( G \).

We want now to make the decomposition (4.6) explicit. In most cases \( \Theta(\tau) \) is equal to \( \eta(\tau) \), however, there are cases when \( \eta(\tau) \) is a proper sub-representation, and cases when \( \Theta(\tau) \) does not contribute to the \( k \)-spectrum at all.
Table 4  The irreps of $O_{2\pm}$

| dim | $\widetilde{O}_{2\pm}$ | $\hat{\tau}_{\lambda_2}^\pm$, sgn, $\tau_{\lambda_2}^\pm$ | $\tau_{\lambda} = Ind_{SO_{2\pm}^+}(\lambda)$, $\lambda \neq \lambda^{-1}$ | $\frac{q+5}{2}$ irreps |
|-----|-----------------|---------------|-----------------|-----------------|
| 1   | $\widetilde{O}_{2+}$ | 1, sgn, $\tau_{\lambda_2}^+$ | $\tau_{\lambda} = Ind_{SO_{2+}^+}(\lambda)$, $\lambda \neq \lambda^{-1}$ | $\frac{q+5}{2}$ irreps |
| 2   | $\widetilde{O}_{2-}$ | 1, sgn, $\tau_{\mu_2}^+$ | $\tau_{\mu} = Ind_{SO_{2-}^-}(\mu)$, $\mu \neq \mu^{-1}$ | $\frac{q+7}{2}$ irreps |

In case dim$(U) = 0$ we make the convention $O_0 = \{1\}$ and so $\tau = 1$ and $\Theta(1) = 1$ the trivial representation.

4.3.1 The $O_{1\pm}$-$G$ decomposition

Let us denote by $\omega_{\psi \pm}$ the two oscillator representations $\omega_U \otimes V$ of $SL_2(F_q) = Sp(U \otimes V)$ in the cases that $U$ is equipped with the forms $\beta_{1\pm}$, respectively. The orthogonal groups $O_{1\pm} = \{\pm 1\}$ act on the representations $\omega_{\psi \pm}$, respectively. In these cases the isotypic components in (4.6) are $1 \otimes \omega_{\psi \pm, 1}$ and $sgn \otimes \omega_{\psi \pm, sgn}$, where $1, sgn$, are the trivial and sign representations of $O_{1\pm}$, and $\omega_{\psi \pm, 1}, \omega_{\psi \pm, sgn}$, are the four irreps that we discussed in Sect. 3.4. In particular, in these cases $\eta(\tau) = \Theta(\tau)$, and we verified Theorem 4.2 explicitly.

4.3.2 The $O_{2\pm}$-$G$ decomposition

We first recall some information on the orthogonal groups $O_{2\pm}$ and their irreps, and then we describe the decomposition (4.6) associated with these groups.

The irreps of $O_{2\pm}$

The groups $O_{2\pm}$ fit into a short exact sequence

$$1 \to SO_{2\pm} \to O_{2\pm} \to \{\pm 1\} \to 1,$$

where the third morphism is det, and $SO_{2\pm}$ stand for the special orthogonal groups. It might be convenient for us to choose a splitting of (4.7), i.e., to consider a reflection

$$r \in O_{2\pm} \setminus SO_{2\pm}, \quad r^2 = 1.$$  

Then $O_{2\pm} = SO_{2\pm} \rtimes < r >$ and $r sr = s^{-1}$ for every $s \in SO_{2\pm}$. Moreover, the groups $SO_{2\pm}$ are cyclic with $SO_{2+} \simeq F_q^*$ and $SO_{2-} \simeq \{\xi \in F_q^2; \text{Norm}(\xi) = 1\}$ - the norm one elements in a quadratic extension of $F_q$. In particular, $\#(SO_{2+}) = q - 1$ and $\#(SO_{2-}) = q + 1$.

Let us recall the irreps of $O_{2+}$—see Table 4 for a summary. There are $\frac{q+5}{2}$ of them and they can be realized as follows. For every character $\lambda$ of $SO_{2+}$ we consider the induced representation

$$\tau_{\lambda} = Ind_{SO_{2+}^+}(\lambda) = \{ f : O_{2+} \to \mathbb{C}; \quad f(sh) = \lambda(s)f(h), \quad \text{for} \quad s \in SO_{2+}, \quad h \in O_{2+} \}.$$  

Note that
• dim(Ind^{O_{2^+}}_{SO_{2^+}}(\lambda)) = 2 for each \lambda.
• r (4.8) induces an isomorphism Ind^{O_{2^+}}_{SO_{2^+}}(\lambda) \cong Ind^{O_{2^+}}_{SO_{2^+}}(\lambda^{-1}), f \mapsto f_r, where f_r(h) = f(\lambda h).
• Ind^{O_{2^+}}_{SO_{2^+}}(\lambda)|_{SO_{2^+}} = \lambda \oplus \lambda^{-1}.

In particular, the \tau_\lambda's with \lambda \neq \lambda^{-1} give the \frac{q-3}{2} irreps of dimension 2. In addition, there are 4 irreps of dimension 1: The trivial representation 1, the sign character sgn = \det, and the two components \tau_{\lambda_2} of Ind^{O_{2^+}}_{SO_{2^+}}(\lambda_2) on which the reflection (4.8) acts by +1 or -1, respectively, where \lambda_2 is the unique non-trivial character of SO_{2^+} such that \lambda_2^2 = 1.

The \frac{q+7}{2} irreps of O_{2^-} are described in the same manner starting with \frac{q-1}{2} of them of dimension 2, induced from characters \mu, \mu \neq \mu^{-1}, of SO_{2^-}, etc.—see Table 4 for the outcome in this case.

The decomposition

We need to describe (4.6). Let us first treat the O_{2^-} case—see Table 5 (bottom) for a summary.

**Proposition 4.4 (The \omega_U \otimes V|O_{2^-} \times G decomposition)** Every representation \tau \in \widehat{O_{2^-}}, \tau \neq sgn, appears in \omega_U \otimes V|O_{2^-}, and in that range the corresponding representations \Theta(\tau) of G (4.6) are irreducible and pairwise non-isomorphic. In particular, Part (1) of Theorem 4.2 holds true. Moreover, for the description of \Theta(\tau) and \eta(\tau) we have

1. The representation \Theta(1) is of tensor rank 2 and dimension q. We denote it by \eta(1) or by \hat{St} since it is known as the Steinberg representation.
2. The representations \Theta(\tau_\mu), \mu \neq \mu^{-1}, are of tensor rank 2 and dimension q - 1. We denote them by \eta(\tau_\mu).
3. The representations \Theta(\tau_{\pm \lambda_2}) are the two tensor rank 1 irreps of dimension \frac{q-1}{2}.

For a proof of Proposition 4.4 see Appendix C.4.

Parts (1) and (2) of Proposition 4.4 give an explicit description of the eta correspondence (4.4).

Next, we treat the O_{2^+} case—see Table 5 (top) for a summary.

**Proposition 4.5 (The \omega_U \otimes V|O_{2^+} \times G decomposition)** Every representation \tau \in \widehat{O_{2^+}} appears in \omega_U \otimes V|O_{2^+}. Moreover, for the description of \Theta(\tau) and \eta(\tau), we have

1. The representation \Theta(1) = 1 \oplus \hat{St}, where \hat{St} is the Steinberg representation of tensor rank 2 and dimension q. We denote it by \eta(1). In addition, \Theta(\text{sgn}) = 1.
2. The representations \Theta(\tau_\lambda), \lambda \neq \lambda^{-1}, are irreducible, pairwise non-isomorphic, of tensor rank 2 and dimension q + 1. We denote them by \eta(\tau_\lambda).
3. The representations \Theta(\tau_{\pm \lambda_2}) are the two tensor rank 1 irreps of dimension \frac{q+1}{2}.

In particular, note that on the tensor rank 2 spectrum (see Sect. 4.2), i.e., on the subspace \omega_U \otimes V, Parts (1) and (2) give the conclusion of Part (1) of Theorem 4.2.

---

6 The notation \tau_{\pm \lambda_2} depends on r. If \tau' is another reflection, then \tau' = sr for unique s \in SO_{2^\pm}.
Table 5 Explicit description of $\omega_{U \otimes V | O_{2}\pm \times G}$

| $\tau$ | $\Theta(\tau)$ | $\dim \Theta(\tau)$ | $\eta(\tau)$ | $\dim \eta(\tau)$ | # |
|--------|----------------|----------------------|------------|----------------|---|
| 1      | $1 \oplus St$  | $q + 1$              | $St$       | $q$            | 1 |
| $sgn$  | 1              | 1                     | $\times$  | $\times$       | $\times$ |
| $\tau_{\pm}^{\pm}$ | $\Theta(\tau_{\pm}^{\pm})$ | $\frac{q+1}{2}$ | $\times$ | $\times$       | $\times$ |
| $\tau_{\lambda}, \lambda \neq \lambda^{-1}$ | $\eta(\tau_{\lambda})$ | $q + 1$ | $\eta(\tau_{\lambda})$ | $q + 1$ | $\frac{q-3}{2}$ |

$\omega_{U \otimes V | O_{2\pm} \times G}$

| $\tau$ | $\Theta(\tau)$ | $\dim \Theta(\tau)$ | $\eta(\tau)$ | $\dim \eta(\tau)$ | # |
|--------|----------------|----------------------|------------|----------------|---|
| 1      | $St$           | $q$                  | $St$       | $q$            | 1 |
| $sgn$  | $\times$      | $\times$             | $\times$  | $\times$       | $\times$ |
| $\tau_{\mu}^{\mu}$ | $\Theta(\tau_{\mu}^{\mu})$ | $\frac{q-1}{2}$ | $\times$ | $\times$       | $\times$ |
| $\tau_{\mu}, \mu \neq \mu^{-1}$ | $\eta(\tau_{\mu})$ | $q - 1$ | $\eta(\tau_{\mu})$ | $q - 1$ | $\frac{q-1}{2}$ |

For a proof of Proposition 4.5 see Appendix C.4.

Parts (1) and (2) of Proposition 4.5 give an explicit description of the eta correspondence (4.4).

**Remark 4.6 (Double commutant property)** Note the difference between the two decompositions appearing in Propositions 4.4 and 4.5. In the first case $O_{2-}$ and $G$ generate each other’s commutant when acting on the space $\omega_{U \otimes V}$. This is not what is happening in the second case of $O_{2+}$ and $G$. However, they do generate each other’s commutant—see Table 5 (top)—after taking away from $\omega_{U \otimes V}$ the isotypic component of the trivial representation of $G$.

4.3.3 Summary

The $\eta$-correspondence that was described explicitly in this section, gives a way to realize $q + 4$ irreps of $G$. It does it for each tensor rank $k = 0, 1, 2$ separately. The realization is inside specific oscillator representations, and in each one is described in term of irreps of corresponding orthogonal groups. In particular, $\eta(\hat{O}_0) = \hat{G}_0$, $\eta(\hat{O}_{1+}) \cup \eta(\hat{O}_{1-}) = \hat{G}_1$, and $\eta(\hat{O}_{2+}) \cup \eta(\hat{O}_{2-}) = \hat{G}_2$, verifying Part (2) of Theorem 4.2. Finally, by counting $\#(\hat{G}) = q + 4$ so $\hat{G}_0 \cup \hat{G}_1 \cup \hat{G}_2 = \hat{G}$, which verifies Proposition 2.1.

5 Application to uniformity of the commutator map

We go back to the problem of the uniformity of the commutator map $[\cdot, \cdot] : G \times G \to G$ described in Sect. 1.2.2 for $G = SL_2(\mathbb{F}_q)$.
5.1 Statement

We considered the function \( N \) on \( G \) given by (1.2) \( N(g) = \#([, ]_g)/\#G \), i.e., the normalized cardinality of the fiber over \( g \) of the map \([, ]\).

We wanted to show that,

**Theorem 5.1** For \( \pm I \neq g \in G \) we have

\[
N(g) = 1 + O(1/q). \tag{5.1}
\]

5.2 Verifying cancellations in Frobenius’s character sum

Recall that Frobenius’s Formula (1.5) gives a representation theoretic interpretation of \( N \) as a character sum

\[
N(g) = 1 + \sum_{1 \neq \rho \in \hat{G}} \frac{\chi_\rho(g)}{\dim(\rho)}. \tag{5.2}
\]

In Sect. 2 we looked on (5.1) and (5.2) and since—see Table 3—around \( q \) of the summands in (5.2) are of size \( \approx 1/q \) we understood that we need to verify cancellations in the sum on the right-hand side of (5.2). Moreover, we noticed that for some elements \( g \in G \), the size of the summands in (5.2) seem—see Table 3—larger for irreps of tensor rank 1 then for these of tensor rank 2. Hence, at the end of Sect. 2 we proposed to verify the cancellations in (5.2) by splitting the sum into two partial sums, one over \( \hat{G}_1 = \) tensor rank one irreps and one over \( \hat{G}_2 = \) tensor rank two irreps

\[
\sum_{1 \neq \rho \in \hat{G}} \frac{\chi_\rho(g)}{\dim(\rho)} = \sum_{\rho \in \hat{G}_1} \frac{\chi_\rho(g)}{\dim(\rho)} + \sum_{\rho \in \hat{G}_2} \frac{\chi_\rho(g)}{\dim(\rho)},
\]

and explore cancellations within each sub-sum.

The numerics appearing in Table 6 suggest that the above idea might work. Indeed, using the information on \( \hat{G}_1 \) and \( \hat{G}_2 \) supplied by the \( \eta \)-correspondence, we can show that

| Table 6 | Partial sums over irreps of tensor rank one and two for \( SL_2(\mathbb{F}_{101}) \) |
|---------|--------------------------------------------------|
| Conjugacy class \( g \) | \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) | Other |
| \( S_1(g) = \sum_{\rho \in SL_2(\mathbb{F}_{101})_1} \frac{\chi_\rho(g)}{\dim(\rho)} \) | 0.004 \( \approx 4/101 \) | 0.04 \( \approx 4/101 \) |
| \( S_2(g) = \sum_{\rho \in SL_2(\mathbb{F}_{101})_2} \frac{\chi_\rho(g)}{\dim(\rho)} \) | 0.02 \( \approx 2/101 \) | 0.01 \( \approx 1/101 \) |

\( \hat{G} \) Springer
Proposition 5.2 Let $\pm I \neq g \in G$. We have

$$S_1(g) = \sum_{\rho \in \hat{G}_1} \frac{\chi_\rho(g)}{\dim(\rho)} = \left\{ \begin{array}{ll} O(1/q^2), & \text{if } g \sim \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right); \\ O(1/q), & \text{other}. \end{array} \right.$$ \hspace{1cm} (5.3)$$

and

$$S_2(g) = \sum_{\rho \in \hat{G}_2} \frac{\chi_\rho(g)}{\dim(\rho)} = O(1/q).$$ \hspace{1cm} (5.4)$$

Proposition 5.2 implies Theorem 5.1.

5.3 Proof of Proposition 5.2

The $\eta$-correspondence enables to relate the sums (5.3) and (5.4) that we want to estimate, to some other sums that we can actually compute using only two inputs: The character of the oscillator representation, for which we have explicit formulas \cite{7,10,16,17,23,38}; and the character of the Steinberg representation, for which a simple description is known \cite{4}.

Let us denote by $\chi_{\omega \psi_{\pm}}$ the characters of the two oscillator representations of $G = SL_2(\mathbb{F}_q)$ associated with a character $1 \neq \psi$ of $\mathbb{F}_q$ (see Sect. 4.3.1). The formulas (see loc. cit.) in particular give:

Fact 5.3 For $I \neq g \in G$ we have

1. The oscillator characters satisfies,

$$\chi_{\omega \psi_{\pm}}(g) = \left\{ \begin{array}{ll} \pm \gamma(\psi), & \text{if } g \sim \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right); \\ \mp \gamma(\psi), & \text{if } g \sim \left( \begin{array}{cc} 1 & \epsilon \\ 0 & 1 \end{array} \right); \\ \left( -\frac{\det(g-I)}{q} \right), & \text{other}, \end{array} \right.$$ \hspace{1cm} (5.5)$$

where $\gamma(\psi) = \sum_{z \in \mathbb{F}_q} \psi \left( \frac{1}{2} z^2 \right)$ is the usual quadratic Gauss sum \cite{24}, $\left( \frac{z}{q} \right)$ is the Legendre symbol, and $\sim$ means in the same conjugacy class.

2. The Steinberg Character satisfies,

$$\chi_{St}(g) = \#(L^g) - 1,$$ \hspace{1cm} (5.6)$$

where $L^g$ is the collection of lines in the plane $V$ that are fixed by $g$.

The tensor rank one sum

First, we estimate the sum $S_1(g)$ (5.3) for $g \neq \pm I$.

Let us write $\chi_{\omega \psi_{\pm}} = \chi_{++} + \chi_{+-}$, where $\chi_{++}$ and $\chi_{+-}$ denote the characters of the components of $\omega \psi_{\pm}$ of dimensions $\frac{q+1}{2}$ and $\frac{q-1}{2}$, respectively, and likewise $\chi_{\omega \psi_{-}} = \chi_{--} + \chi_{-+}$.
We have
\[ S_1 = \frac{X_+ -}{q^{-1} + \frac{1}{2}} + \frac{X_- -}{q^{-1} + \frac{1}{2}} + \frac{X_+ +}{q + 1 + \frac{1}{2}} + \frac{X_- +}{q + 1 + \frac{1}{2}} \]
\[ = \left( \frac{1}{q^{-1} + \frac{1}{2}} - \frac{1}{q + 1 + \frac{1}{2}} \right) (X_+ - + X_- -) + \frac{1}{q + 1} (X_+ + + X_- + + X_- + + X_- -). \quad (5.7) \]

But, Formula (5.5) implies that
\[ \chi_+ + (g) + \chi_- (g) + \chi_+ (g) + \chi_- (g) = \chi_{\omega \psi} (g) + \chi_{\omega \psi} (g) \]
\[ = \begin{cases} 0, & \text{if } g \sim \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right); \\ \pm 2, & \text{other}, \end{cases} \quad (5.8) \]

and, moreover, from the definition of \( \chi_+ \) and \( \chi_- \) and Formula (5.5), we get
\[ \chi_+ (g) + \chi_- (g) = \begin{cases} -1, & \text{if } g \sim \left( \begin{smallmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{smallmatrix} \right); \\ |.| \leq 2, & \text{other}. \end{cases} \quad (5.9) \]

So overall, by a combination (5.7), (5.8), and (5.9), we have
\[ |S_1 (g)| = \begin{cases} 4 q^{-2}, & \text{if } g \sim \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right); \\ 4 q^{-1} + |\delta|, & \text{other}, \end{cases} \]

where \(|\delta| \leq \frac{8}{q^{-1}}\).

This completes a quantitative form of (5.3).

**The tensor rank two sum**

Next, we estimate the sum \( S_2 (g) \) for \( g \neq \pm 1 \). It has a similar structure to \( S_1 (g) \).

Using the explicit description of the \( \eta \)-correspondence—see Table 5—we have
\[ S_2 = \sum_{\tau_\mu \in O_{2-}, \mu \neq \mu^{-1}} \frac{X_\eta (\tau_\mu)}{q - 1} + \sum_{\tau_\lambda \in O_{2+}, \lambda \neq \lambda^{-1}} \frac{X_\eta (\tau_\lambda)}{q + 1} + \chi_{St} \]
\[ = \left( \frac{1}{q^{-1} + \frac{1}{2}} - \frac{1}{q + 1 + \frac{1}{2}} \right) \sum_{\tau_\mu \in O_{2-}, \mu \neq \mu^{-1}} X_\eta (\tau_\mu) \]
\[ + \frac{1}{q + 1} \left( \sum_{\tau_\mu \in O_{2-}, \mu \neq \mu^{-1}} X_\eta (\tau_\mu) + \sum_{\tau_\lambda \in O_{2+}, \lambda \neq \lambda^{-1}} X_\eta (\tau_\lambda) + \chi_{St} \right) + \left( \frac{1}{q} - \frac{1}{q + 1} \right) \chi_{St}. \quad (5.10) \]

We would like to estimate the various terms in (5.10).
Consider the sums $\mathcal{X}_2^-$ and $\mathcal{X}_2^+$ above. Denote by $U_2^-$ and $U_2^+$, respectively, the anisotropic and hyperbolic planes, and by $\chi_{\omega U_2^\pm \otimes V}$, respectively, the characters of the oscillator representations of the groups $Sp(U_2^\pm \otimes V)$ (see Sect. 4.2). Then,

$$2\mathcal{X}_2^-(g) + \chi_{+-}(g) + \chi_{--}(g) + \chi_{si}(g) + 2\mathcal{X}_2^+(g) + \chi_{++}(g) + \chi_{-+}(g) + \chi_{si}(g) = \chi_{\omega U_2^\pm \otimes V}(I \otimes g),$$

where the first equality follows from the explicit descriptions of $\omega U \otimes V|O_2^\pm \times G$ appearing in Table 5, and the second equality follows from a combination of Proposition 4.1 and Formula (5.5). So, together with (5.8) we get,

$$\mathcal{X}_2^-(g) + \mathcal{X}_2^+(g) + \chi_{si}(g) = \begin{cases} -1, & \text{if } g \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \\ \pm 2, & \text{other}, \end{cases}$$

(5.11)

Next, concerning the term $\mathcal{X}_2^-$ in (5.10). From the explicit description of $\omega U \otimes V|O_2^- \times G$ appearing in Table 5, we obtain

$$\mathcal{X}_2^-(g) = \frac{\chi_{\omega U_2^- \otimes V}(I \otimes g) - \chi_{+-}(g) - \chi_{--}(g) - \chi_{si}(g)}{2}.$$ 

(5.12)

So, using Formulas (4.3), (5.5), and (5.9), (5.6), inserted in (5.12), we get

$$\mathcal{X}_2^-(g) = \begin{cases} -\frac{q-1}{|\cdot|}, & \text{if } g \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \\ |\cdot| \leq 2, & \text{other.} \end{cases}$$

(5.13)

Combining (5.10), (5.6), (5.11), and (5.13), we have for every $\pm I \neq g \in G$,

$$|S_2(g)| \leq \frac{2}{q+1} + \delta,$$

where $|\delta| \leq \frac{5}{q^2-1}$.

This is a quantitative form of (5.4).

This completes effective proofs (with explicit bounds) for Proposition 5.2 and Theorem 5.1.

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Appendix A: The Weyl transform

Let us denote by $\mathcal{H}_\psi$ a vector space supporting the Heisenberg representation $\pi_\psi$ of the Heisenberg group $H = H(W)$ defined in Sect. 3.2. Various calculations done in this note involve the space $\text{End}(\mathcal{H}_\psi)$ of all linear transformations on $\mathcal{H}_\psi$. Interestingly, this operator space has an invariant description [10,16] that was discovered by Weyl in the context of quantum mechanics [40].

A.1. Definition of the Weyl transform

Consider the vector space $L^2(W)$ of complex valued functions on the symplectic space $W$. Then, we have a canonical isomorphism, called the Weyl transform,

$$\mathcal{W} : \text{End}(\mathcal{H}_\psi) \rightarrow L^2(W)$$

$$\mathcal{W}(T)(w) = \frac{1}{\dim(\mathcal{H}_\psi)} \text{trace}(T \circ \pi_\psi(-w)), \tag{A.1}$$

for every $T \in \text{End}(\mathcal{H}_\psi), w \in W$.

Here is a useful application of the Weyl transform (A.1).

A.2. Application for intertwining numbers

We describe application for the computation of intertwining numbers for restrictions of the oscillator representation $\omega_\psi$ to subgroups $K < \text{Sp}(W)$. Note that the identification $\mathcal{W}$ intertwines the conjugation action of $\text{Sp}(W)$ on $\text{End}(\mathcal{H}_\psi)$ with its permutation action on $L^2(W)$. Consider the intertwining number $(\omega_\psi, \omega_\psi)_K = \dim \text{End}_K(\mathcal{H}_\psi)$, i.e., the dimension of the space of operators on $\mathcal{H}_\psi$ that commute with the action of $K$. Denote by $W/K$ the set of orbits for the action of $K$ on $W$. It follows

Corollary A.1 We have $(\omega_\psi, \omega_\psi)_K = \#(W/K)$.

Appendix B: The double commutant theorem

In Sect. 4.2 we used the double commutant theorem [41] to define the $\eta$-correspondence.

Formulation

For the convenience of the reader, here is the statement.

Theorem B.1 (Double commutant theorem) Let $\mathcal{H}$ be a finite dimensional vector space. Let $\mathcal{A}, \mathcal{A}' \subset \text{End}(\mathcal{H})$ be two sub-algebras, such that

1. The algebra $\mathcal{A}$ acts semi-simply on $\mathcal{H}$.
2. Each of $\mathcal{A}$ and $\mathcal{A}'$ is the full commutant of the other in $\text{End}(\mathcal{H})$. 

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Then $A'$ acts semi-simply on $\mathcal{H}$, and as a representation of $A \otimes A'$ we have

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i \otimes \mathcal{H}'_i,$$

(B.1)

where $\mathcal{H}_i$ are all the irreducible representations of $A$, and $\mathcal{H}'_i$ are all the irreducible representations of $A'$. In particular, we have a bijection between irreducible representations of $A$ and $A'$, and moreover, every isotypic component for $A$ is an irreducible representation of $A \otimes A'$.

On the other hand, if $A$, $A'$ commute and as a representation of $A \otimes A'$ the decomposition (B.1) holds, then each of $A$ and $A'$ is the full commutant of the other in $\text{End}(\mathcal{H})$.

Appendix C: Proofs

C.1. Proof of Proposition 3.6

Proof Take $K = Sp(W)$ in Corollary A.1 and note $#(W/Sp(W)) = 2$, as claimed.

C.2. Proof of Proposition 4.1

Proof Let us denote by $\omega(W, \langle, \rangle, \psi)$ the oscillator representation associated with a symplectic vector space $(W, \langle, \rangle)$ and a central character $\psi$. It is well known that $\omega(W, a \langle, \rangle, \psi) \simeq \omega(W, \langle, \rangle, \psi_a)$, and $\omega((W_1, \langle, \rangle_1) \oplus (W_2, \langle, \rangle_2), \psi) \simeq \omega(W_1, \langle, \rangle_1, \psi) \otimes \omega(W_2, \langle, \rangle_2, \psi)$, for every $a \in \mathbb{F}_q^*$, and two symplectic spaces $(W_i, \langle, \rangle_i)$, $i = 1, 2$.

Next, in the case $U$ endowed with the form $\beta_2^+$, we can identify $(U \otimes V, \beta_2^+ \otimes \langle, \rangle_V)$, as a $G = Sp(V)$-space, with $(V, \langle, \rangle_V) \oplus (V, -\langle, \rangle_V)$, and in the case $U$ is with form $\beta_2^-$, we can identify $(U \otimes V, \beta_2^- \otimes \langle, \rangle_V)$, as a $G = Sp(V)$-space, with $(V, \langle, \rangle_V) \oplus (V, -\varepsilon \langle, \rangle_V)$ where $\varepsilon \in \mathbb{F}_q^*$ a non-square.

Now, combining the above functorial properties with Proposition 3.5, we get identities (4.2) and (4.3), as claimed.

C.3. Proof of Proposition 4.3

Proof We take $W = U \otimes V$ and $G = Sp(V) < Sp(W)$ in Corollary A.1, and compute $#(W/G)$. We identify the action of $G$ on $W$ with its diagonal action on $V \times V$, and find that the orbits are

- For each $0 \neq a \in \mathbb{F}_q^*$ the orbit $\{(u, v) \in V \times V; \langle u, v \rangle_V = a\}$.
- For each $0 \neq b \in \mathbb{F}_q^*$ the orbit $\{(u, v) \in (V \setminus 0) \times (V \setminus 0); u = bv\}$.
- The orbits $\{(0, v); 0 \neq v \in V\}, \{(v, 0); 0 \neq v \in V\}, \{(0, 0)\}$.

So, overall we have $2q + 1$ orbits, as claimed.

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C.4. Proof of Propositions 4.4 and 4.5

Proof We give a proof that works more or less in a uniform manner for both groups $O_{2\pm}$.

The verification will use a realization of $\omega_{U \otimes V}$ which is convenient for the groups $O_{2\pm}$. Indeed, $\omega_{U \otimes V}$ can be realized on the space $L^2(U)$ in such a way that for $O_{2\pm}$ it is its permutation representation. To arrive to this realization, write $V = F_q \oplus F_q$ and consider the associated Lagrangian decomposition $U \otimes V = (U \otimes F_q) \oplus (U \otimes F_q) = U \oplus U$. Then we have the Schrödinger model (see Remark 3.4) realizing $\omega_{U \otimes V}$ as described above.

Recall that by Witt’s theorem [26] the groups $O_{2\pm}$ act transitively on the each of the sets $U_a = \{u \in U; \; \beta_{2\pm}(u) = a\}$ of vectors with a given value $a \in F_q$ for the inner product. To see what representations of $O_{2\pm}$ appear in $L^2(U)$ we decompose

$$L^2(U) = \bigoplus_{a \in F_q} L^2(U_a),$$

and study the spaces $L^2(U_a)$.

Let us start with the $q - 1$ non-isotropic orbits, i.e., $U_a, a \neq 0$. The stabilizer subgroup of a non-isotropic vector $u \in U_a$ is the orthogonal group of the orthogonal complement of the line spanned by $u$, i.e., the group generated by the reflection $r_u$ with respect to that line. In particular, the groups $SO_{2\pm}$ act simply transitively on $U_a, a \neq 0$.

It follows that the $\det = \text{sgn}$ character can not appear in $L^2(U_a), a \neq 0$, since it would have to be represented by the constant function 1 by transitivity of $SO_{2\pm}$, but this transforms by the identity character of $O_{2\pm}$.

For the characters of $O_{2\pm}$ that are non-trivial on $SO_{2\pm}$, one will take value 1 on a given reflection, and one will take value $-1$. Clearly, only the one that takes value 1 on the reflection $r_u$ that stabilizes the vector $u \in U_a$ can live on the orbit. That means that each of these characters live on half of the orbits of non-isotropic vectors.

Finally, it is easy to see that each of the two-dimensional irreps of $O_{2\pm}$ appear once in $L^2(U_a), a \neq 0$.

For the double commutant property for $O_{2-}$, we know from the above analysis that every irrep but $\det = \text{sgn}$ appears in $L^2(U)$, so by Plancherel [34] it gives #($O_{2-}$) $- 1 = 2(q + 1) - 1 = 2q + 1$ linearly independent operators in the the commutant $\text{End}_G(\omega_{U \otimes V})$, so by (4.5) it gives the all commutant.

In summary, the isotypic components in $\omega_{U \otimes V}$ of irreps of $O_{2-}$ are

- $1 \otimes \Theta(1), \Theta(1) \in \hat{G}$ with $\dim \Theta(1) = q$ (note that $U_0 = \{0\}$ for $\beta_{2-}$), so $\Theta(1) = St$.
- $\tau_{\mu} \otimes \Theta(\tau_{\mu}), \mu \neq \mu^{-1}, \Theta(\tau_{\mu}) \in \hat{G}$ pairwise non-isomorphic with $\dim \Theta(\tau_{\mu}) = q - 1$.
- $\tau_{\mu_2} \otimes \Theta(\tau_{\mu_2}), \Theta(\tau_{\mu_2}) \in \hat{G}$, two non-isomorphic irreps with $\dim \Theta(\tau_{\mu_2}) = q - 1$. This completes the Proof of Proposition 4.4.

Consider the group $O_{2+}$, note that it acts simply transitively on the non-zero isotropic vectors, so the representation of $O_{2+}$ on $U_0$ is the direct sum of the triv-
ial representation and the regular representation. So $L^2(U_0)$ contains the $\det = sgn$ representation once, the trivial representation twice, the two characters of $O_{2+}$ which are non-trivial on $SO_{2+}$ each once, and each of the two-dimensional irreps of $O_{2+}$ twice.

For the double commutant property for $O_{2+}$, we have all irreps appearing, but they only give $\#(O_{2+}) = 2(q - 1)$ operators, so by (4.5) we are missing 3.

We will show that

Lemma C.1 The isotypic component in $\omega_{U \otimes V}$ of the trivial representation of $G$ is two-dimensional on which $O_{2+}$ acts by $\text{sgn} \oplus 1$.

This means that $O_{2+}$ only supplies two of the four operators in the commuting algebra for this $G$ component. So if we leave out the trivial representation of $G$, then we need only $2q + 1 - 4 = 2q - 3$ operators. On the other hand, from Lemma C.1 and the analysis we have already done above we see that, on the complement of the trivial representation of $G$ we get all the representations of $O_{2+}$ except $\text{sgn}$, so this supplies $2(q - 1) - 1 = 2q - 3$ operators, which is the needed number.

In summary, this will show that the isotypic components in $\omega_{U \otimes V}$ of irreps of $O_{2+}$ are

- $1 \otimes (1 \oplus St)$.
- $\text{sgn} \otimes 1$.
- $\tau_\lambda \otimes \Theta(\tau_\lambda), \lambda \neq \lambda^{-1}, \Theta(\tau_\lambda) \in \hat{G}$ pairwise non-isomorphic with $\dim \Theta(\tau_\lambda) = q + 1$.
- $\tau^\pm_{\lambda_2} \otimes \Theta(\tau^\pm_{\lambda_2}), \Theta(\tau^\pm_{\lambda_2}) \in \hat{G}$, two non-isomorphic irreps with $\dim \Theta(\tau^\pm_{\lambda_2}) = \frac{q + 1}{2}$.

To prove Lemma C.1 it is enough, due to our knowledge on the size of the commutant, to show that $\text{sgn} \oplus 1$ sits inside the $G$-invariant subspace of $\omega_{U \otimes V}$. To show this, we will use another realization of $\omega_{U \otimes V}$, that helps to visualize the action of $G$. Decompose $U$ into a direct sum $U = \ell \oplus \ell^*$ of isotropic lines. This induces the Lagrangian decomposition $U \otimes V = (\ell \otimes V) \oplus (\ell^* \otimes V)$, that we can further identify with $V \oplus V$. In particular, we can realize $\omega_{U \otimes V}$ on the space $L^2(V)$, where

1. The action of $G = Sp(V)$ is its permutation representation.
2. The action of $SO_{2+}$ is identified with the scaling action of $\mathbb{F}_{q^*}$.
3. The action of the reflection $r : \ell \to \ell^*$ is via the symplectic Fourier transform $F$ given by
   \[ F[f](v) = \frac{1}{q} \sum_{v' \in V} \psi(\langle v, v' \rangle) f(v'), \]
   for every $f \in L^2(V), v \in V$.

   Now, denote by $\delta_0$ the Dirac delta function at the origin of $V$, and by $1_V$ the constant function $1_V(v) = 1$ for every $v \in V$. Consider the functions $f_\pm = \delta_0 \pm \frac{1}{q}1_V \in L^2(V)$. Then, $f_\pm$ are $G$-invariant and $F(f_\pm) = \pm f_\pm$. This completes the proof of Lemma C.1 and Proposition 4.5.
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