A Note on the Size of the Largest Ball Inside a Convex Polytope

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Abstract. Let $m > 1$ be an integer, $B_m$ the set of all unit vectors of $\mathbb{R}^m$ pointing in the direction of a nonzero integer vector of the cube $[-1, 1]^m$. Denote by $s_m$ the radius of the largest ball contained in the convex hull of $B_m$. We determine the exact value of $s_m$ and obtain the asymptotic equality $s_m \sim \frac{2}{\sqrt{\log m}}$.

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§1. INTRODUCTION

Let $m \geq 2$ be an integer, and consider the sets

$$A_m = \{-1, 0, 1\}^m \setminus \{\vec{0}\}, \quad \text{and} \quad B_m = \left\{ \frac{v}{||v||} \mid v \in A_m \right\}.$$

Let $C_m$ be the convex hull of $B_m$, and $s_m$ the radius of the largest ball contained in $C_m$. (Due to the apparent symmetries of $C_m$, such a largest ball is necessarily centered at the origin.) In the paper [B-M-S(2005)] (dealing with rotation numbers/vectors of billiards) we needed sharp lower and upper estimates for the extremal radius $s_m$. Here we determine the exact value of $s_m$ which, of course, implies such estimates.
Theorem.

\[
s_m = \left( \sum_{k=1}^{m} \frac{1}{(\sqrt{k} + \sqrt{k-1})^2} \right)^{-1/2}
\]

The Theorem implies that

\[
\frac{1}{4} \log m < s_m^{-2} < \frac{1}{4} \log m + \frac{5}{4}.
\]

As an immediate corollary, the quantity \( s_m \) is asymptotically equal to \( \frac{2}{\sqrt{\log m}} \).

§2. Proof of the Theorem

The proof will be split into a few lemmas. The first one of them is a trivial observation.

Lemma 1. The set of vertices \( B_m \) of the convex polytope \( C_m \), and hence \( C_m \) itself, is invariant under the action of the full isometry group \( G \) of the cube \([-1,1]^m\). (The group \( G \) is generated by all permutations of the coordinates in \( \mathbb{R}^m \), and by all reflections across the coordinate hyperplanes.) \( \square \)

We will use the notation \( v_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} e_i \) \((k = 1, \ldots, m)\) for some specific vertices of \( C_m \). (Here \( e_i \) stands for the \( i \)-th standard unit vector of \( \mathbb{R}^m \).)

Lemma 2. The simplex \( S \), spanned by the linearly independent vectors \( v_k \) \((k = 1, \ldots, m)\) as vertices, is a face of the polytope \( C_m \) whose outer normal vector is \( u = (u_1, \ldots, u_m) \) with the coordinates \( u_i = \sqrt{i} - \sqrt{i-1} \).

Proof. Consider the scalar product function \( \langle v, u \rangle \) \((v \in B_m)\) restricted to the set \( B_m \) of vertices of the polytope \( C_m \). Elementary inspection shows that this scalar product function can only attain its maximum value at the vertices \( v_k \), and actually,

\[
\langle v_k, u \rangle = 1
\]

for each \( k = 1, \ldots, m \). This proves all claims of the lemma. \( \square \)

Lemma 3. For any face \( F \) of the polytope \( C_m \) there exists a congruence \( g \in G \) such that \( g(F) = S \).
Proof. Fix a non-zero vector $w = (w_1, \ldots, w_m)$ whose ray $R(w) = \{ \lambda w \mid \lambda \geq 0 \}$ intersects the interior of the face $F$. By selecting $w$ in a generic manner, we can assume that the absolute values $|w_i|$ of its coordinates are distinct and all different from zero. Therefore, by applying a suitable element $g \in G$, we can even assume that

$$w_1 > w_2 > \cdots > w_m > 0. \quad (2)$$

We claim that $g(F) = S$. Indeed, by (2) we have the linear expansion

$$w = \sum_{k=1}^{m} \sqrt{k}(w_k - w_{k+1})v_k.$$

of $w$ in the basis $\{v_1, \ldots, v_m\}$ with positive coefficients. (With the natural convention $w_{m+1} = 0$.) This proves that some positive multiple of $w$ is a convex linear combination of the vertices of $S$ with non-zero coefficients, so the face $g(F)$ shares an interior point with $S$. □

It follows from the previous lemma that the radius $s_m$ of the inscribed sphere is actually the distance between $S$ and the origin. However, this distance is equal to $s_m = \langle u, e_1 \rangle / ||u|| = 1 / ||u||$ by (1). It is clear that

$$||u||^2 = \sum_{k=1}^{m} \frac{1}{k^2}.$$

finishing the proof of our theorem. □

Define $R_m = \sum_{k=1}^{m} \frac{1}{k}$. For the asymptotic value of $s_m$ we use the elementary fact that $\log m < R_m < \log m + 1$.

$$\frac{1}{4} \log m < \sum_{k=1}^{m} \frac{1}{4k} < \sum_{k=1}^{m} \frac{1}{k^2} = ||u||^2$$

$$< 1 + \sum_{k=2}^{m} \frac{1}{4(k-1)} < 1 + \frac{1}{4} (\log m + 1) = 1 + \frac{1}{4} \log m + \frac{5}{4}.$$

Remark 1. Let $K$ be the convex cone generated by the vectors $v_k$, $k = 1, \ldots, m$. The meaning of Lemma 3 is that the cones $g(K)$ ($g \in G$) form a triangulation of the space $\mathbb{R}^m$. As a matter of fact, the intersections of the cones $g(K)$ with the standard $(m-1)$-simplex

$$S_{m-1} = \left\{ x \in \mathbb{R}^m \mid \sum_{i=1}^{m} x_i = 1, x_i \geq 0 \text{ for all } i \right\}$$

form the baricentric subdivision of $S_{m-1}$. 
Remark 2. The following natural question has been considered in several papers, for instance in [B-F(1988)] and [B-W(2003)]. What is the maximal radius \( r(m, N) \) of the inscribed ball of the convex hull of \( N \) points chosen from the unit ball of \( \mathbb{R}^m \)? In our case \( N = 3^m - 1 \) and one may wonder how close \( s_m \) and \( B_m \) are to the maximal radius and best arrangement. It turns out that they are very far: it follows from the results of [B-F(1988)] and [B-W(2003)] that, in the given range \( N = 3^m - 1 \),

\[
r(m, N) = \left( \frac{8}{9} \right)^{1/2} (1 + o(1))
\]

as \( m \to \infty \). So the optimal radius is much larger than \( s_m \). This also shows that, as expected, \( B_m \) is far from being distributed uniformly on the unit sphere.

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