On decay of almost periodic viscosity solutions to Hamilton-Jacobi equations

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Abstract

We establish that a viscosity solution to a multidimensional Hamilton-Jacobi equation with a convex non-degenerate Hamiltonian and Bohr almost periodic initial data decays to its infimum as time $t \to +\infty$.

1 Introduction

In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0, +\infty)$, we consider the Cauchy problem for a first order Hamilton-Jacobi equation

$$u_t + H(\nabla_x u) = 0 \quad (1.1)$$

with merely continuous Hamiltonian function $H(v) \in C(\mathbb{R}^n)$, and with initial condition

$$u(0, x) = u_0(x) \in BUC(\mathbb{R}^n), \quad (1.2)$$

where $BUC(\mathbb{R}^n)$ denotes the Banach space of bounded uniformly continuous functions on $\mathbb{R}^n$ equipped with the uniform norm $\|u\|_\infty = \sup |u(x)|$.

We recall the notions of superdifferential $D^+ u$ and subdifferential $D^- u$ of a continuous function $u(t, x) \in C(\Pi)$:

$$D^+ u(t_0, x_0) = \{ (s, v) = \nabla \varphi(t_0, x_0) \mid \varphi(t, x) \in C^1(\Pi), \quad (t_0, x_0) \text{ is a point of local maximum of } u - \varphi \},$$

$$D^- u(t_0, x_0) = \{ (s, v) = \nabla \varphi(t_0, x_0) \mid \varphi(t, x) \in C^1(\Pi), \quad (t_0, x_0) \text{ is a point of local minimum of } u - \varphi \}.$$ 

Let us denote by $BUC_{loc}(\bar{\Pi})$ the space of continuous functions on $\bar{\Pi} = C^1 \Pi = [0, +\infty) \times \mathbb{R}^n$, which are bounded and uniformly continuous in any layer $[0, T) \times \mathbb{R}^n$, $T > 0$. Recall the notion of viscosity solution of (1.1), (1.2).
Definition 1. A function $u(t, x) \in BUC_{loc}(\Pi)$ is called a viscosity sub-solution (vsubs. for short) of problem (1.1), (1.2) if $u(0, x) \leq u_0(x)$ and $s + H(v) \leq 0$ for all $(s, v) \in D^+ u(t, x), (t, x) \in \Pi$.

A function $u(t, x) \in BUC_{loc}(\Pi)$ is called a viscosity supersolution (v.supers.) of problem (1.1), (1.2) if $u(0, x) \geq u_0(x)$ and $s + H(v) \geq 0$ for all $(s, v) \in D^- u(t, x), (t, x) \in \Pi$.

Finally, $u(t, x) \in BUC_{loc}(\Pi)$ is called a viscosity solution (v.s.) of (1.1), (1.2) if it is a v.subs. and a v.supers. of this problem simultaneously.

The theory of v.s. was developed in [2, 3]. This theory extended the earlier results of S.N. Kruzhkov [5, 6].

It is known that for each $u_0(x) \in BUC(\mathbb{R}^n)$ there exists a unique v.s. of problem (1.1), (1.2). The uniqueness readily follows from the more general comparison principle.

Theorem 1 (see [3]). Let $u_1(t, x), u_2(t, x) \in BUC_{loc}(\Pi)$ be a v.subs. and a v.supers. of (1.1), (1.2) with initial data $u_{10}(x), u_{20}(x)$, respectively. Assume that $u_{10}(x) \leq u_{20}(x) \forall x \in \mathbb{R}^n$. Then $u_1(t, x) \leq u_2(t, x) \forall (t, x) \in \Pi$.

Corollary 1. Let $u_1(t, x), u_2(t, x) \in BUC_{loc}(\Pi)$ be v.s. of (1.1), (1.2) with initial data $u_{10}(x), u_{20}(x)$, respectively. Then for all $t > 0$

$$\inf(u_{10}(x) - u_{20}(x)) \leq u_1(t, x) - u_2(t, x) \leq \sup(u_{10}(x) - u_{20}(x)).$$

In particular, $\|u_1 - u_2\|_\infty \leq \|u_{10} - u_{20}\|_\infty$.

Proof. Let

$$a = \inf(u_{10}(x) - u_{20}(x)), \quad b = \sup(u_{10}(x) - u_{20}(x)).$$

Obviously, the functions $a + u_2(t, x), b + u_2(t, x)$ a v.s. of (1.1), (1.2) with initial data $a + u_{20}(x), b + u_{20}(x)$, respectively. Since $a + u_{20}(x) \leq u_{10}(x) \leq b + u_{20}(x)$, then by Theorem 1 $a + u_2(t, x) \leq u_1(t, x) \leq b + u_2(t, x) \forall (t, x) \in \Pi$, which completes the proof.

In the case when $H(0) = 0$ constants are v.s. of (1.1). By Corollary 1 with $u_1 = u, u_2 = 0$ we find that a v.s. $u = u(t, x)$ is bounded, namely $\|u\|_\infty \leq \|u_0\|_\infty$. Notice that the requirement $H(0) = 0$ does not reduce the generality because we can make the change $\tilde{u} \rightarrow u + H(0)t$, which provides a v.s. $\tilde{u}$ of the problem $\tilde{u}_t + H(\nabla_x \tilde{u}) - \tilde{H}(0) = 0, \tilde{u}(0, x) = u_0(x)$, with new hamiltonian $\tilde{H}(v) = H(v) - H(0)$ satisfying the desired condition $\tilde{H}(0) = 0$. 


We are going to study long time decay property of v.s. to the problem \((\ref{1.1}), (\ref{1.2})\) with almost periodic initial data. Recall that the space \(AP(\mathbb{R}^n)\) of Bohr (or uniform) almost periodic functions is a closure of trigonometric polynomials, i.e. finite sums \(\sum a_\lambda e^{2\pi i \lambda \cdot x}\), in the space \(BUC(\mathbb{R}^n)\) (by \(\cdot \) we denote the inner product in \(\mathbb{R}^n\)). It is clear that \(AP(\mathbb{R}^n)\) contains continuous periodic functions (with arbitrary lattice of periods). Let \(C_R\) be the cube \[\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x|_\infty = \max_{i=1,\ldots,n} |x_i| \leq R/2 \}, \quad R > 0.\]

It is known (see for instance \cite{8} ) that for each function \(u \in AP(\mathbb{R}^n)\) there exists the mean value \[\bar{u} = \int_{\mathbb{R}^n} u(x) dx = \lim_{R \to +\infty} R^{-n} \int_{C_R} u(x) dx\]

and, more generally, the Bohr-Fourier coefficients \[a_\lambda = \int_{\mathbb{R}^n} u(x) e^{-2\pi i \lambda \cdot x} dx, \quad \lambda \in \mathbb{R}^n.\]

The set \[Sp(u) = \{ \lambda \in \mathbb{R}^n \mid a_\lambda \neq 0 \}\]

is called the spectrum of an almost periodic function \(u(x)\). It is known \cite{8}, that the spectrum \(Sp(u)\) is at most countable.

Now we assume that the initial function \(u_0(x) \in AP(\mathbb{R}^n)\). Let \(M_0\) be the smallest additive subgroup of \(\mathbb{R}^n\) containing \(Sp(u_0)\). Notice that in the case when \(u_0\) is a continuous periodic function \(M_0\) coincides with the dual lattice to the lattice of periods.

We are going to find an exact condition on the hamiltonian \(H(v)\) for the fulfillment of the decay property \[u(t, x) \Rightarrow c = \text{const} \quad \text{as} \quad t \to +\infty. \quad (1.3)\]

We assume that \(H(0) = 0\) and introduce the following non-degeneracy condition of \(H(v)\) at point \(v = 0\) in “resonant” directions \(\xi \in M_0\)

\[
\forall \xi \in M_0, \xi \neq 0 \quad \text{the functions} \quad s \to H(s\xi)
\]

are not linear in any interval \(|s| < \delta, \delta > 0. \quad (1.4)\]

Notice that the similar condition (non-linearity of resonant components of the flux vector \(f(u)\)) is known to be an exact condition for decay of almost
periodic entropy solutions to scalar conservation laws \( u_t + \text{div}_x f(u) = 0 \), see [11] and, in the periodic case, [4, 9, 10].

Let us demonstrate that requirement (1.4) is necessary for the decay of v.s. of (1.1), (1.2) such that \( Sp(u_0) \subset M_0 \). In fact, if (1.4) fails then there exist a nonzero vector \( \xi \in M_0 \) and a positive \( \delta > 0 \) such that \( H(s\xi) = \alpha s \) for \( |s| \leq \delta \) for some \( \alpha \in \mathbb{R} \). Then the function \( u(t,x) = \frac{\delta}{2\pi} \sin(2\pi(\xi \cdot x - \alpha t)) \) is a classical (and, therefore, a v.s.) of (1.1), (1.2) with initial function \( u_0(x) = \frac{\delta}{2\pi} \sin(2\pi \xi \cdot x) \) because

\[
\nabla_x u(t,x) = s\xi, \quad u_t(t,x) = -s\alpha, \quad s = \delta \cos(2\pi(\xi \cdot x - \alpha t)) \in [-\delta,\delta],
\]

so that \( u_t + H(\nabla_x u) = 0 \). Obviously, \( u_0(x) \) is a periodic function and

\[
Sp(u_0) = \{ k\xi \mid k \in \mathbb{Z}, k \neq 0 \} \subset M_0.
\]

But the decay property is evidently violated.

In the case \( n = 1 \) condition (1.4) reads that \( H(v) \) is not linear in any vicinity of zero. In recent preprint [12] this condition was shown to be sufficient for the decay property.

In this paper we prove the similar result in arbitrary dimension \( n \geq 1 \) but for a convex hamiltonian. Our main results is the following

**Theorem 2.** Assume that the hamiltonian \( H(v) \) is convex, \( H(0) = 0 \), and condition (1.4) is satisfied. Then the decay property (1.3) holds. Moreover, the limit constant \( c = \inf u_0(x) \).

Notice that in the case of strictly convex hamiltonian condition (1.4) is always satisfied. In this case the statement of Theorem 2 follows from the general results of [5, Theorem 8], [6, Theorem 6], for arbitrary \( u_0(x) \in BUC(\mathbb{R}^n) \).

We also remark that in the case of arbitrary \( H(0) \) decay property (1.3) should be revised as

\[
u(t,x) + H(0)t \Rightarrow c = \text{const} \quad \text{as } t \to +\infty.\]

2 The case of periodic initial function

In the case of convex hamiltonian the unique v.s. \( u(t,x) \) of (1.1), (1.2) can be found by the known Hopf-Lax-Oleinik formula [6, 11]

\[
u(t,x) = \min_{y \in \mathbb{R}^n}[u_0(y) + tH^*((x-y)/t)],
\]

(2.1)
where
\[ H^*(p) = \sup_{v \in \mathbb{R}^n} [p \cdot v - H(v)] \quad (2.2) \]
is the Legendre transform of \( H \). To simplify the proofs, we will assume in addition that the following coercivity condition is satisfied (in Remark 1 below we demonstrate how to avoid this condition)
\[ H(v)/|v| \to +\infty \quad \text{as} \quad v \to \infty \quad (2.3) \]
(here and in the sequel we denote by \( |v| \) the Euclidean norm of a finite-dimensional vector \( v \)). Under this condition the Legendre conjugate function \( H^*(p) \) is everywhere defined convex function satisfying the coercivity condition:
\[ H^*(p)/|p| \to +\infty \quad \text{as} \quad p \to \infty . \]

It is known that for every \( v_0 \in \mathbb{R}^n \) the sub-differential \( D^-H(v_0) \) of a convex function \( H(v) \) on \( \mathbb{R}^n \) coincides with the set
\[ \partial H(v_0) = \{ p \in \mathbb{R}^n \mid H(v) - H(v_0) \geq p \cdot (v - v_0) \}, \]
which is a nonempty convex compact in \( \mathbb{R}^n \). By the assumption \( H(0) = 0 \) the conjugate function \( H^*(p) \geq 0 \) and \( H^*(p_0) = 0 \) if and only if \( p_0 \in \partial H(0) \). We fix such \( p_0 \) and introduce the convex set \( \partial H^*(p_0) \). Since \( 0 = H^*(p_0) = \min H(p) \) then \( 0 \in \partial H^*(p_0) \). By the duality,
\[ H(v) = H^{**}(v) = \max_{p \in \mathbb{R}^n} [p \cdot v - H^*(p)]. \]
As readily follows from this relation,
\[ H(v) = p_0 \cdot v \quad \forall v \in \partial H^*(p_0). \quad (2.4) \]

We fix \( \varepsilon > 0 \) and introduce the polar set
\[ G = (\partial H^*(p_0))' = \{ p \in \mathbb{R}^n \mid p \cdot v \leq \varepsilon \forall v \in \partial H^*(p_0) \}. \quad (2.5) \]
Then \( G \) is a closed convex set and, by the bipolar theorem \[ \text{Theorem 14.5}],
\[ \partial H^*(p_0) = G' = \{ v \in \mathbb{R}^n \mid p \cdot v \leq \varepsilon \forall p \in G \}. \quad (2.6) \]
Let \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) be the torus, \( \text{pr} : \mathbb{R}^n \to \mathbb{T}^n \) be the natural projection.

**Proposition 1.** Let \( G \subset \mathbb{R}^n \) be a convex set such that \( \text{pr}(G) \) is not dense in \( \mathbb{T}^n \). Then there exists \( \xi \in \mathbb{Z}^n \) such that the linear functional \( p \to \xi \cdot p \) is bounded on \( G \).
Proof. Without loss of generality we may suppose that $G$ is a closed convex set and that $0 \in G$. We define the dimension $\dim A$ of any set $A \subset \mathbb{R}^n$ as the dimension of its linear span. Let $m(G)$ be the maximal of such integer $m$ that there exists a convex cone $C \subset G$ of dimension $m$. Since the trivial cone $\{0\} \subset G$, then $0 \leq m(G) \leq n$. We will prove our statement by induction in the value $k = n - m(G)$. If $k = 0$ then there exists a cone $C \subset G$ of full dimension $n$. This implies that $C$ contains some ball $B_\delta(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| \leq \delta \}$. Obviously, $B_{r\delta}(rx_0) = rB_\delta(x_0) \subset C$ for any $r > 0$. In particular, the set $G \supset C$ contains balls of arbitrary radius. This implies that $G$ contains a cube $K$ with side greater than 1. Since $\text{pr}(K) = \mathbb{T}^n$, we conclude that $\text{pr}(G) = \mathbb{T}^n$. Hence in the case $k = 0$ the assumptions that $\text{pr}(G)$ is not dense in $\mathbb{T}^n$ cannot be satisfied and our assertion is true.

Now assume that $k > 0$ and the statement of our proposition holds for all sets $G$ such that $n - m(G) < k$. We have to prove our statement in the case $n - m(G) = k$. Let $C \subset G$ be a cone of maximal dimension $n - k$, $L$ be the linear span of $C$. By our assumption $K = \text{Cl} \text{pr}(C)$ is a proper compact subset of $\mathbb{T}^n$. Introduce the set

$$M = \{x \in \mathbb{R}^n \mid \text{pr}(\lambda x) + y \in K \ \forall y \in K, \ \lambda \geq 0 \}.$$ 

Here $+$ is a standard group operation on the torus $\mathbb{T}^n$. Since

$$\text{pr}(\lambda(c_1x_1 + c_2x_2)) + y = \text{pr}(\lambda c_1x_1) + (\text{pr}(\lambda c_2x_2) + y) \in K$$

for all $y \in K$, then $c_1x_1 + c_2x_2 \in M$ whenever $x_1, x_2 \in M, c_1, c_2 \geq 0$. This means that $M$ is a convex cone. Let us demonstrate that $-x \in M$ for each $x \in M$. For that we define the standard metric $d$ on $\mathbb{T}^n$,

$$d(y_1, y_2) = \min\{ |x_1 - x_2| \mid y_i = \text{pr}(x_i), i = 1, 2 \},$$

and introduce for $x_0 \in \mathbb{R}^n$, $y \in K$ the function $f(s) = d(\text{pr}(sx_0) + y, K) = \min_{z \in K} d(\text{pr}(sx_0) + y, z), s \in \mathbb{R}$. It is clear that $f(s)$ is an almost periodic function (as a composition of the continuous periodic function $g(x) = d(\text{pr}(x) + y, K)$ and the linear embedding $s \rightarrow sx_0$). If $x_0 \in M$ then $f(s) = 0$ for all $s \geq 0$. Since $f(s)$ is an almost periodic function, the latter is possible only if $f \equiv 0$. In particular, $\text{pr}(sx_0) + y \in K$ for all $s \leq 0$ and $y \in K$, that is, $-x_0 \in M$. Hence, $M$ is a symmetric convex cone, i.e., it is a linear space. If $x \in C$, $y = \text{pr}(z), z \in C$, then $\text{pr}(\lambda x) + y = \text{pr}(\lambda x + z) \in \text{pr}(C)$. Thus, $\text{pr}(\lambda x) + y \in \text{pr}(C)$ whenever $y \in \text{pr}(C), \lambda \geq 0$. By the continuity of the
group operation we find that $\text{pr}(\lambda x) + y \in K$ for all $y \in K = \text{Cl} \text{pr}(C)$, $\lambda \geq 0$, that is, $x \in M$. We conclude that $C \subset M$. Since $M$ is a linear subspace, it follows that $L \subset M$. Notice that $0 \in K$, $\text{pr}(M) = \text{pr}(M) + 0 \subset K \neq \mathbb{T}^n$. Let $S = \text{Cl} \text{pr}(M) \subset K$. Then $S$ is a proper closed subgroup of $\mathbb{T}^n$. By Pontryagin duality [13], there exists a character $\chi(y) = e^{2\pi i \xi \cdot y}$, $\xi \in \mathbb{Z}^n$, $\xi \neq 0$, such that $\chi(y)$ $= 1$ on $S$. It follows that $\xi \cdot x = 0$ for all $x \in M$, and in particular, $\xi \in L^\perp$. If the linear functional $x \rightarrow \xi \cdot x$ is bounded on $G$, then the desired statement is proved. Assuming the contrary, that is, this functional is not bounded on $G$, we can find the sequence $p_r \in G$, $r \in \mathbb{N}$, such that $\xi \cdot p_r \rightarrow \infty$ as $r \rightarrow \infty$. We introduce the new convex set $G' = \text{Cl} (L + G)$ and remark that

$$\text{pr}(L + G) = \text{pr}(L) + \text{pr}(G) \subset K + \text{pr}(G) = \text{Cl} \text{pr}(C) + \text{pr}(G) \subset \text{Cl} \text{pr}(C + G) \subset \text{Cl} \text{pr}(G).$$

We utilized that $C + G \subset G$. To prove this inclusion, we fix $x \in C$, $p \in G$ and observe that

$$x + p = \lim_{r \rightarrow \infty} \left( x + \frac{r - 1}{r} p \right),$$

while $x + \frac{r - 1}{r} p = \frac{1}{r} rx + \frac{r - 1}{r} p \in G$ for all $r > 1$ by the convexity of $G$ (notice that $rx \in C \subset G$). Since $G$ is closed, relation (2.8) implies that $x + p \in G$ for each $x \in C$, $p \in G$, as was to be proved.

By (2.7) $\text{Cl} \text{pr}(G') = \text{Cl} \text{pr}(G)$ is a proper subset of $\mathbb{T}^n$. Let $q_r \in L$ be orthogonal projection of $p_r$, so that $p_r - q_r \perp L$. Since $\xi \cdot (p_r - q_r) = \xi \cdot p_r \rightarrow \infty$ as $r \rightarrow \infty$, we have $\alpha_r = \| p_r - q_r \| \rightarrow +\infty$ as $r \rightarrow \infty$. Since $0, p_r - q_r \in G'$ then $\lambda(\alpha_r)^{-1}(p_r - q_r) \in G'$ if $\alpha_r > \lambda \geq 0$. Passing to a subsequence if necessary, we can suppose that the sequence of unite vectors $(\alpha_r)^{-1}(p_r - q_r) \rightarrow h$ as $r \rightarrow \infty$. Evidently, $|h| = 1$ and $h \perp L$. Besides, $\lambda h = \lim_{r \rightarrow \infty} \lambda(\alpha_r)^{-1}(p_r - q_r) \in G'$ because the set $G'$ is closed. We find that the cone $C' = L + \{ \lambda h \mid \lambda \geq 0 \} \subset G'$ while $\dim C' = m(G) + 1$. We see that $n - m(G') < k$. By the induction hypothesis there exists a vector $\xi \in \mathbb{Z}^n$, $\xi \neq 0$ such that the corresponding linear functional $x \rightarrow \xi \cdot x$ is bounded on $G'$ and therefore also on $G \subset G'$. We prove the assertion of our proposition for the case $n - m(G) = k$. By the principle of mathematical induction, this completes the proof. \hfill \Box

**Corollary 2.** Assume that the following non-degeneracy condition holds:

$$\forall \xi \in \mathbb{Z}^n, \xi \neq 0 \text{ the functions } s \rightarrow H(s\xi)$$

are not linear in any vicinity of zero. 

(2.9)
Then for each $\varepsilon > 0$ the set $\text{pr}(G)$ is dense in $\mathbb{T}^n$, where the convex set $G$ is given by (2.3).

**Proof.** Assuming the contrary and applying Proposition 1 we can find $\xi \in \mathbb{Z}^n$, $\xi \neq 0$, and a positive constant $c$ such that $|\xi \cdot p| \leq c$ for all $p \in G$. In view of (2.6) this implies that $s\xi \in G' = \partial H^*(p_0)$ for $|s| < \delta = \varepsilon/c$. By (2.4) we find that the function $H(s\xi) = sp_0 \cdot \xi$ is linear on the interval $|s| < \delta$. But this contradicts to our assumption. Thus, $\text{pr}(G)$ is dense in $\mathbb{T}^n$. The proof is complete.

Let $u_0(x) \in C(\mathbb{T}^n)$ be a periodic function (with the standard lattice of periods $\mathbb{Z}^n$), and $u = u(t,x)$ be the unique v.s. of the problem (1.1), (1.2). Observe that the group $M_0$ coincides with $\mathbb{Z}^n$, and condition (1.4) reduces to (2.9). Now, we are ready to prove our main result.

**Theorem 3.** Under non-degeneracy condition (2.9) a v.s. $u(t,x)$ of (1.1), (1.2) satisfies the following decay property:

$$u(t,x) \Rightarrow c = \min u_0(x) \quad \text{as } t \to +\infty.$$ (2.10)

**Proof.** We fix $p_0 \in \partial H(0)$, $\varepsilon > 0$ and consider the corresponding set $G$ introduced in (2.3). Since $u_0(y)$ is a uniformly continuous function on the compact $\mathbb{T}^n$, there exists such $\delta > 0$ that

$$|u_0(y_1) - u_0(y_2)| < \varepsilon \quad \forall y_1, y_2 \in \mathbb{T}^n, d(y_1, y_2) \leq \delta.$$ (2.11)

By Corollary 2 the set $\text{pr}(G)$ is dense in $\mathbb{T}^n$. Therefore, there exists a finite $\delta$-net $Y = \{y_1, \ldots, y_m\} \subset \text{pr}(G)$. We choose $q_k \in G$ such that $y_k = \text{pr}(q_k)$, $k = 1, \ldots, m$. Let a point $y_s \in \mathbb{T}^n$ be such that $u_0(y_s) = c = \min u_0(y)$. Since $Y$ is a $\delta$-net in $\mathbb{T}^n$, then for each $(t,x) \in \Pi$ there exists such $k \in \{1, \ldots, m\}$ that

$$d(\text{pr}(x - p_0 t - q_k), y_s) = d(\text{pr}(x - p_0 t - y_s, y_k) \leq \delta.$$ (2.11)

In view of (2.11), $u_0(x - p_0 t - q_k) < u_0(y_s) + \varepsilon = c + \varepsilon$. From (2.11) it follows that

$$u(t,x) \leq u_0(x - p_0 t - q_k) + tH^*((x - (x - p_0 t - q_k))/t) =$$

$$u_0(x - p_0 t - q_k) + tH^*(p_0 + q_k/t) < c + \varepsilon + tH^*(p_0 + q_k/t).$$ (2.12)

Notice also that in view of Corollary 2 with $u_1 = u(t,x)$, $u_2 \equiv 0$, $u(t,x) \geq c$ for all $(t,x) \in \Pi$. From this inequality and (2.12) it now follows that

$$c \leq u(t,x) < c + \varepsilon + \alpha(t).$$ (2.13)
where
\[ \alpha(t) = \max_{k=1,\ldots,m} tH^*(p_0 + q_k/t). \]
Since \( H^*(p) \) is a convex function and \( H^*(p_0) = 0 \), there exist limits
\[ \lim_{t \to +\infty} tH^*(p_0 + q_k/t), \]
which coincide with directional derivatives \( D_{q_k}H^*(p_0) \). It is known (see [14])
that \( D_{q_k}H^*(p_0) = \max_{v \in \partial H^*(p_0)} q_k \cdot v \).

Thus, \( \lim_{t \to +\infty} \|u(t, \cdot) - c\|_\infty \leq 2\varepsilon \).

To complete the proof it only remains to notice that \( \varepsilon > 0 \) is arbitrary. \( \square \)

**Remark 1.** The statement of Theorem 1.2 remains valid for arbitrary convex
hamiltonian \( H(v) \), which may not satisfy the coercivity condition (2.3).

Assume first that the initial function \( u_0(x) \) is Lipschitz:
\[ |u_0(x) - u_0(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^n, \]
\( L > 0 \) is a Lipschitz constant. By Corollary 1 we have
\[ |u(t, x + h) - u(t, x)| \leq \sup |u_0(x + h) - u_0(x)| \leq L|h| \quad \forall x, h \in \mathbb{R}^n, t > 0. \]

Thus, the functions \( u(t, \cdot) \) satisfy the Lipschitz condition with the constant \( L \). Therefore, the generalized gradient \( \nabla_x u \in L^\infty(\Pi, \mathbb{R}^n) \), \( \|\nabla_x u\|_\infty \leq L \).

This readily implies that \( |v| \leq L \) whenever \( (s, v) \in D^\pm u(t, x), (t, x) \in \Pi \).

We see that the behavior of \( H(v) \) for \( |v| > L \) does not matter and we can
always improve the convex hamiltonian \( H(v) \) in the domain \( |v| > L \) in such a
way that the corrected hamiltonian satisfies the coercivity assumption. It is
clear that the non-degeneracy condition (2.9) remains valid. By Theorem 3
we conclude that decay property (2.10) holds.

In the general case we construct the sequence \( u_{0k} \in C(T^n), k \in \mathbb{N} \), of
periodic Lipschitz functions such that \( u_{0k} \to u_0 \) as \( k \to \infty \) in \( C(T^n) \). Let
$u_k = u_k(t, x)$ be a v.s. of (1.1), (1.2) with initial data $u_{0k}$. Then, taking into account Corollary 1, we find that as $k \to \infty$

$$u_k(t, x) \to u(t, x), \quad c_k = \min u_{0k}(y) \to c = \min u_0(y).$$  \hspace{1cm} (2.14)

As was already proved, for each $k \in \mathbb{N}$

$$u_k(t, \cdot) \to c_k \quad \text{as} \quad t \to +\infty.$$  

In view of (2.14) we can pass to the limit as $k \to \infty$ in the above relation and derive the desired result (2.10).

We underline that condition (2.9) can be satisfied even for a hamiltonian linear on each of two half-spaces with a common boundary hyper-space.

Example 1. Let $p \in \mathbb{R}^n$ be a nonzero vector. We consider the equation

$$u_t + |\partial_p u| = 0,$$

where $\partial_p u = p \cdot \nabla x u$ is the directional derivative of $u$. Obviously, the hamiltonian $H(v) = |p \cdot v|$ satisfies (2.9) if and only if $p \cdot \xi \neq 0$ for every $\xi \in \mathbb{Z}^n$, $\xi \neq 0$. This means that the coordinates $p_j, j = 1, \ldots, n,$ of the vector $p$ are linearly independent over the field $\mathbb{Q}$ of rationals.

3 The case of almost periodic initial data

In this section we prove Theorem 2 in the general case $u_0(x) \in AP(\mathbb{R}^n)$.

We will need some simple general properties of v.s. collected in the following lemma.

Lemma 1. (i) If $u(t, x)$ is a v.s. of (1.1), (1.2), then $v = -u(t, x)$ is a v.s. to the problem

$$v_t - H(-\nabla x v) = 0, \quad v(0, x) = -u_0(x);$$

(ii) Let $y = Ax$ be a non-degenerate linear operator on $\mathbb{R}^n$, $v_0(y) \in BUC(\mathbb{R}^n), v(t, y) \in BUC_{loc}(\Pi)$. Then the function $u(t, x) = v(t, Ax)$ is a v.s. of (1.1), (1.2) with initial data $u_0(x) = v_0(Ax)$ if and only if $v(t, y)$ is a v.s. of the problem

$$v_t + H(A^*\nabla y v) = 0, \quad v(0, y) = v_0(y),$$

10
where $A^*$ is the conjugate operator;

(iii) Let $H(p, q) \in C(\mathbb{R}^n \times \mathbb{R}^m)$. We consider the equation

$$U_t + H(\nabla_x U, \nabla_y U) = 0$$

(3.1)

in the half-space $\{ (t, x, y) \mid t > 0, x \in \mathbb{R}^n, y \in \mathbb{R}^m \}$. Then $U(t, x, y) = u(t, y)$ is a non-depending on $x$ v.s. of (3.1) if and only if $u(t, y)$ is a v.s. of the reduced equation

$$u_t + H(0, \nabla_y u) = 0, \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m.$$

Proof. (i) As is easy to verify, $(s, w) \in D^\pm v(t_0, x_0)$ if and only if $(-s, -w) \in D^\mp u(t_0, x_0)$. Since $u(t, x)$ is a v.s. of (1.1) we obtain that, respectively, $\pm (-s + H(-w)) \geq 0$, i.e., $\mp (s - H(-w)) \geq 0$. By the definition, this means that $v(t, x)$ is a v.s. of the equation $v_t - H(-\nabla_x v) = 0$. Since the initial condition $v(0, x) = -u_0(x)$ is evident, this completes the proof of (i).

Assertion (ii) follows from the fact that $(t_0, x_0)$ is a point of local maximum (minimum) of $u(t, x) - \psi(t, Ax)$, with $\psi(t, y) \in C^1(\Pi)$, if and only if $(t_0, Ax_0)$ is a point of local maximum (minimum) of $v(t, y) - \psi(t, y)$ and from the classical identity $A^* \nabla_y \psi(t, y) = \nabla_x \psi(t, Ax)$, $y = Ax$.

Finally, assertion (iii) readily follows from the evident equalities

$$D^\pm U(t, x, y) = \{ (s, 0, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \mid (s, v) \in D^\pm u(t, y) \}.$$
Proof. Let \( u(t, x, y) \) be a v.s. of \((3.2), (3.3)\), and \( y_0 \in \mathbb{R}^m \). We assume that \( \varphi(t, x) \in C^1(\Pi) \) and \((t_0, x_0) \in \Pi \) is a point of local maximum of \( u^{y_0} - \varphi \). Moreover, replacing \( \varphi \) by \( \varphi(t, x) + (t-t_0)^2 + |x-x_0|^2 + u(t_0, x_0, y_0) - \varphi(t_0, x_0) \), we can suppose, without loss of generality, that \((t_0, x_0) \in \Pi \) is a point of strict local maximum of \( u^{y_0} - \varphi \), and that in this point \( u^{y_0}(t_0, x_0) - \varphi(t_0, x_0) = 0 \). Therefore, there exists \( c > 0 \) such that

\[
\varphi(t, x) - u(t, x, y_0) > c \quad \forall (t, x) \in \Pi, \quad (t-t_0)^2 + |x-x_0|^2 = r^2,
\]

for some \( r \in (0, t_0) \). By the continuity there exists \( h > 0 \) such that \( \varphi(t, x) - u(t, x, y) > c/2 \) for all \((t, x, y) \in \mathbb{R}^n \times \mathbb{R}^m \), \( (t-t_0)^2 + |x-x_0|^2 = r^2, |y-y_0| \leq h \). We can choose such \( C_0 > 0 \) that

\[
C_0 h^2 - c > \max\{ u(t, x, y) - \varphi(t, x) \mid (t-t_0)^2 + |x-x_0|^2 \leq r^2, |y-y_0| = h \}.
\]

Then for each natural \( k > C_0 \) the function \( p_k(t, x, y) = \varphi(t, x) + k|y-y_0|^2 \in C^1(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m) \) and satisfies the property

\[
p_k(t, x, y) - u(t, x, y) > c/2 > 0 = p_k(y_0, x_0, y_0) - u(t_0, x_0, y_0)
\]

\( \forall (t, x, y) \in \partial V_{r,h} \), where we denote by \( V_{r,h} \) the domain

\[
V_{r,h} = \{ (t, x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid (t-t_0)^2 + |x-x_0|^2 < r^2, |y-y_0| < h \}.
\]

In view of \((3.4)\) the point \((t_k, x_k, y_k)\) such that

\[
p_k(t_k, x_k, y_k) - u(t_k, x_k) = \min_{(t, x, y) \in \Pi \cap \partial V_{r,h}} (p_k(t, x, y) - u(t, x, y))
\]

lies in \( V_{r,h} \) and, therefore, it is a point of local maximum of the difference \( u(t, x, y) - p_k(t, x, y) \). Since \( \nabla p_k(t, x, y) = (\partial_t \varphi(t, x), \nabla_x \varphi(t, x), 2k(y-y_0)) \), then by the definition of v.s. of \((3.2)\)

\[
\partial_t \varphi(t_k, x_k) + H(\nabla_x \varphi(t_k, x_k)) \leq 0.
\]

Since \( \min_{(t, x, y) \in \Pi \cap \partial V_{r,h}} (p_k(t, x, y) - u(t, x, y)) \leq p_k(t_0, x_0, y_0) - u(t_0, x_0, y_0) = 0 \), then \( k|y_k - y_0|^2 \leq m = \max_{(t, x, y) \in \Pi \cap \partial V_{r,h}} (u(t, x, y) - \varphi(t, x)) \). In particular \( y_k \to y_0 \) as \( k \to \infty \). Taking into account that \((t_0, x_0) \) is a point of strict local maximum of \( u(t, x, y_0) - \varphi(t, x) \), we derive that \((t_k, x_k) \to (t_0, x_0) \) as \( k \to \infty \). Therefore, it follows from \((3.5)\) in the limit as \( k \to \infty \) that

\[
\partial_t \varphi(t_0, x_0) + H(\nabla_x \varphi(t_0, x_0)) \leq 0.
\]
This means that \( u(t, x, y_0) \) is a v.subs. of (1.1). By the similar reasons we obtain that
\[
\partial_t \varphi(t_0, x_0) + H(\nabla_x \varphi(t_0, x_0)) \geq 0
\]
whenever \((t_0, x_0)\) is a point of strict local minimum of \( u(t, x, y_0) - \varphi(t, x) \), where \( \varphi(t, x) \in C^1(\Pi) \), that is, \( u(t, x, y_0) \) is a v.supers. of (1.1). Thus, \( u(t, x, y_0) \) is a v.s. of (1.1) for each \( y_0 \in \mathbb{R}^m \).

Conversely, assume that \( u^y(t, x) \) is a v.s. of (1.1) for every \( y \in \mathbb{R}^m \). Suppose that \( \varphi(t, x, y) \in C^1(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m) \) and that \((t_0, x_0, y_0)\) is a point of local maximum (minimum) of \( u(t, x, y) - \varphi(t, x, y) \). Then the point \((t_0, x_0)\) is in \( \Pi \) is a point of local maximum (minimum) of \( u^{y_0}(t, x) - \varphi(t, x, y_0) \). Since \( u^{y_0} \) is a v.s. of (1.1) then \( \varphi(t_0, x_0, y_0) + H(\nabla_x \varphi(t_0, x_0, y_0)) \leq 0 \) (respectively, \( \varphi(t_0, x_0, y_0) + H(\nabla_x \varphi(t_0, x_0, y_0)) \geq 0 \)). Hence, \( u(t, x, y) \) is a v.s. of (1.2). To complete the proof it only remains to notice that initial condition (3.3) is satisfied if and only if \( u^y(t, x) \) satisfies (1.2) with initial data \( u_0^y \) for all \( y \in \mathbb{R}^m \).

Now, we can extend the statement of Lemma 1(ii) to the case of arbitrary linear maps.

**Proposition 2.** Let \( \Lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear map, and \( v(t, y) \) be a v.s. to the problem
\[
v_t + H(\Lambda^* \nabla_y v) = 0, \quad v(0, y) = v_0(y)
\]
in the half-space \((t, y) \in \mathbb{R}_+ \times \mathbb{R}^m \). Then \( u(t, x) = v(t, \Lambda x) \) is a v.s. of original problem (1.1), (1.2) with initial function \( u_0(x) = v_0(\Lambda x) \).

**Proof.** We introduce the invertible linear operator \( \tilde{\Lambda} \) on the extended space \( \mathbb{R}^{n+m} \), defined by the equality \( \tilde{\Lambda}(x, z) = (x, z + \Lambda x) \). Since \( \tilde{\Lambda}^*(x, y, y) = (x + \Lambda^* y, y) \), equation (3.6) can be rewritten in the form
\[
v_t + H(\tilde{\Lambda}^*(0, \nabla_y v)) = 0,
\]
where \( H(p, q) = H(p), \; p \in \mathbb{R}^n, \; q \in \mathbb{R}^m \). By Lemma 1(iii) the function \( v = v(t, y) \) is a v.s. of equation
\[
v_t + H(\tilde{\Lambda}^*(\nabla_x v, \nabla_y v)) = 0
\]
in the extended domain \((t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \). Then, by Lemma 1(ii) the function \( u(t, x, z) = v(t, z + \Lambda x) \) is a v.s. of (1.1) considered in the extended domain \((t, x, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \). Applying Lemma 2 we conclude that
$u^z(t,x) = u(t,x,z)$ is a v.s. of (1.1) for all $z \in \mathbb{R}^m$. Taking $z = 0$ we find that $u(t,x) = v(t,\Lambda x)$ is a v.s. of (1.1). It is clear that $u(0,x) = v_0(\Lambda x) = u_0(x)$, that is, $u(t,x)$ is a v.s. of original problem (1.1), (1.2).

Now we are ready to prove our main Theorem 2.

**Proof of Theorem 2.** We first assume that the initial function is a trigonometric polynomial $u_0(x) = \sum_{\lambda \in S} a_\lambda e^{2\pi i \lambda \cdot x}$. Here $S = Sp(u_0) \subset \mathbb{R}^n$ is a finite set. Then the subgroup $M_0$ is a finite generated torsion-free abelian group and therefore it is a free abelian group of finite rank (see [7]). Hence, there is a basis $\lambda_j \in M_0$, $j = 1, \ldots, m$, so that every element $\lambda \in M_0$ can be uniquely represented as $\lambda = \lambda(\bar{k}) = \sum_{j=1}^m k_j \lambda_j$, $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$. In particular, we can represent the initial function as

$$u_0(x) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \sum_{j=1}^m k_j \lambda_j \cdot x},$$

where $J = \{ \bar{k} \in \mathbb{Z}^m \mid \lambda(\bar{k}) \in S \}$ is a finite set. By this representation $u_0(x) = v_0(y(x))$, where

$$v_0(y) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \bar{k} \cdot y}$$

is a periodic function on $\mathbb{R}^m$ with the standard lattice of periods $\mathbb{Z}^m$ while $y = \Lambda x$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by the equalities $y_j = \lambda_j \cdot x$, $j = 1, \ldots, m$. We consider the Hamilton-Jacobi equation (3.6). Let $v(t,y)$ be a v.s. of the Cauchy problem for equation (3.6) with initial function $v_0(y)$. Then by Proposition 2 we have the identity $u(t,x) = v(t,\Lambda x)$. Let us verify that the hamiltonian $\tilde{H}(w) = H(\Lambda^* w)$ of equation (3.6) satisfies condition (2.9). Indeed,

$$\tilde{H}(s\xi) = H(s\Lambda^* \xi) = H(s\lambda),$$

where $\lambda = \Lambda^* \xi = \sum_{j=1}^m \xi_j \lambda_j \in M_0$ for each $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{Z}^m$. Since $\lambda_j$, $j = 1, \ldots, m$, is a basis, $\lambda = \Lambda^* \xi \neq 0$ if $\mathbb{Z}^m \ni \xi \neq 0$. By assumption (1.4) for every $\lambda \in M_0$, $\lambda \neq 0$, the function $s \to H(s\lambda)$ is not linear in any vicinity of zero. In view of (3.7) the convex hamiltonian $\tilde{H}(w)$ satisfies the non-degeneracy requirement (2.9) (with dimension $m$ instead of $n$). By Theorem 3

$$v(t,y) \Rightarrow c = \min v_0(y)$$
as $t \to +\infty$. Since $u_0(x) = v_0(\Lambda x)$, $u(t, x) = v(t, \Lambda x)$, the latter relation reduces to the following one

$$u(t, x) \Rightarrow c = \min v_0(y) \quad \text{as } t \to +\infty.$$  

Observe, that the set $\text{pr}(\Lambda(\mathbb{R}^n))$ is dense in $\mathbb{T}^m$ (in particular, this follows from Proposition 1) while $v_0(y) \in C(\mathbb{T}^m)$. Therefore, $c = \min v_0(y) = \inf u_0(\Lambda x) = \inf u_0(x)$, which completes the proof in the case when $u_0(x)$ is a trigonometric polynomial.

The general case of arbitrary $u_0 \in \text{AP}(\mathbb{R}^n)$ will be treated by approximation arguments. There exists a sequence of trigonometric polynomials $u_{0m}(x)$, $m \in \mathbb{N}$, such that $\text{Sp}(u_{0m}) \subset M_0$ and $u_{0m} \Rightarrow u_0$ as $m \to \infty$. For instance, we can choose $u_{0m}$ as the sequence of Bochner-Fejér trigonometric polynomials, see [8]. Let $u_m(t, x)$ be a v.s. of (1.1), (1.2) with initial data $u_{0m}$. By Corollary 1

$$\|u_m - u\|_\infty \leq \|u_{0m} - u_0\|_\infty \to 0 \quad \text{as } t \to \infty.$$  

As we have already established in the first part of the proof, v.s. $u_m$ satisfy the decay property

$$u_m(t, x) \Rightarrow c_m = \inf u_{0m}(x). \quad (3.8)$$  

Since $u_m \Rightarrow u$, $c_m \to c = \inf u_0(x)$ as $m \to \infty$, then the assertion of Theorem 2 follows from (3.8) in the limit as $m \to \infty$. \hfill \Box

**Remark 2.** In the case of concave hamiltonian $H(v)$

$$u(t, \cdot) \Rightarrow c = \sup u_0(x) \quad \text{as } t \to +\infty. \quad (3.9)$$  

Indeed, by Lemma 1(i) the function $w = -u(t, x)$ is a v.s. of the problem

$$w_t - H(-w_x) = 0, \quad w(0, x) = -u_0(-x),$$

with the convex hamiltonian $-H(-w)$. By Theorem 2

$$w(t, x) = -u(t, x) \Rightarrow \inf -u_0(x) = -\sup u_0(x) \quad \text{as } t \to +\infty,$$

which reduces to (3.9).

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