NULL CURVES IN $\mathbb{C}^3$ AND CALABI-YAU CONJECTURES

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ABSTRACT. For any open orientable surface $M$ and convex domain $\Omega \subset \mathbb{C}^3$, there exists a Riemann surface $N$ homeomorphic to $M$ and a complete proper null curve $F : N \to \Omega$. This result follows from a general existence theorem with many applications. Among them, the followings:

- For any convex domain $\Omega$ in $\mathbb{C}^3$ there exist a Riemann surface $N$ homeomorphic to $M$ and a complete proper holomorphic immersion $F : N \to \Omega$. Furthermore, if $D \subset \mathbb{R}^2$ is a convex domain and $\Omega$ is the solid right cylinder $\{x \in \mathbb{C}^2 \mid \text{Re}(x) \in D\}$, then $F$ can be chosen so that $\text{Re}(F) : N \to D$ is proper.
- There exists a Riemann surface $N$ homeomorphic to $M$ and a complete bounded holomorphic null immersion $F : N \to \text{SL}(2, \mathbb{C})$.
- There exists a complete bounded CMC-1 immersion $X : M \to \mathbb{H}^3$.
- For any convex domain $\Omega \subset \mathbb{R}^3$ there exists a complete proper minimal immersion $(X_i)_{i=1,2,3} : M \to \Omega$ with vanishing flux. Furthermore, if $D \subset \mathbb{R}^2$ is a convex domain and $\Omega = \{(x_i)_{i=1,2,3} \in \mathbb{R}^3 \mid (x_1, x_2) \in D\}$, then $X$ can be chosen so that $(X_1, X_2) : M \to D$ is proper.

Any of the above surfaces can be chosen with hyperbolic conformal structure.

1. INTRODUCTION

Calabi [Ca] asked whether or not there exists complete minimal surfaces in a bounded domain, or more generally, with bounded projection into a straight line. The first answer to Calabi’s question was given by Jorge and Xavier [JX], who exhibited complete non flat minimal discs in a slab of $\mathbb{R}^3$. Later, Yau [Ya1, Ya2] revisited these conjectures and opened new lines for research. It was Nadirashvili [Nad] who developed a powerful technique for constructing complete bounded minimal surfaces in $\mathbb{R}^3$. His examples are minimal discs with never vanishing Gaussian curvature, providing counterexamples to classical Hadamard’s conjecture for negatively curved surfaces as well. Nadirashvili’s ideas have been a fountain of insight that has strongly influenced the global theory of minimal surfaces along the last decade. López, Martín and Morales [LMM] added handles to Nadirashvili’s surfaces, and some years later Alarcón, Ferrer, Martín and Morales [MM1, MM2, AFM] constructed proper complete minimal surfaces in smooth domains of $\mathbb{R}^3$, under some restrictions on the topology of the surfaces and the geometry of the domains. Recently, Ferrer, Martín and Meeks [FMM] have given a complete solution to the proper Calabi-Yau problem for minimal surfaces of arbitrary topology in both convex and smooth bounded domains of $\mathbb{R}^3$, even with disjoint limit sets for distinct ends.

The embedded Calabi-Yau problem for minimal surfaces has radically different nature. Colding and Minicozzi [CM] have proved that any complete embedded minimal surface in $\mathbb{R}^3$ with finite topology is proper in $\mathbb{R}^3$, and Meeks, Pérez and Ros [MPR] have extended this result to the family of minimal surfaces with finite genus and countably many ends.

Calabi-Yau and Hadamard’s conjectures are closely related and make sense for a wide range of surfaces and ambient manifolds. This paper is devoted to the existence of complete proper null curves in convex domains of $\mathbb{C}^3$. Given an open Riemann surface $N$, a map $F = (F_j)_{j=1,2,3} : N \to \mathbb{C}^3$ is said to be a null curve in $\mathbb{C}^3$ if $F$ is a holomorphic immersion and $\sum_{j=1}^3 |dF_j|^2 = 0$. The Riemannian metric on $N$ induced by the Euclidean metric in $\mathbb{C}^3$ is given by $ds_N^2 := \sum_{j=1}^3 |dF_j|^2$. Throughout this paper we adopt

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column notation for both vectors and matrices of linear transformations in $C^3$, and make the convention
\[ C^3 \ni (z_1, z_2, z_3)^T \equiv (\text{Re}(z_1), \text{Re}(z_2), \text{Re}(z_3), \text{Im}(z_1), \text{Im}(z_2), \text{Im}(z_3))^T \in \mathbb{R}^6, \]
where as usual $(\cdot)^T$ means “transpose”.

Calabi-Yau problem for null curves in $C^3$ is interesting by itself, but also because it provides a key for solving other Calabi-Yau conjectures, namely, for null curves in $SL(2, C)$, holomorphic immersed curves in $C^2$, CMC-1 surfaces (or Bryant surfaces) in $H^3$ and, of course, minimal surfaces in $R^3$.

Jones [Jo] constructed a complete bounded holomorphic immersion of the unit disc $D$ in $C^2$, a complete bounded holomorphic embedding of $D$ in $C^3$, and a complete proper holomorphic immersion of $D$ in the unit ball of $C^4$. As a consequence, he produced complete bounded null curves in $C^n$ for any $n \geq 4$. On the other hand, Bourgain [Bo] showed that there are no complete bounded null curves in $C^2$.

However, the Calabi-Yau problem for null curves in $C^3$ has remained open for a long time. Recently, Martín, Umehara and Yamada [MUY1] have given examples of complete simply connected bounded null curves in $C^3$. Their method consists of producing complete bounded null curves in $SL(2, C)$ applying Nadirashvili’s technique to CMC-1 surfaces theory in $H^3$, and then use a correspondence between null curves in $SL(2, C)$ and $C^3$ (see (1.1)).

Our Main Theorem below (see Theorem 3.2) represents a wide generalization of all these results, and as we will see later, has many interesting consequences. For a rigorous statement, the following notations are required. If $\rho = \{\rho_i\}_{1 \leq i \leq n} \subseteq \{1, \ldots, 6\}$ is a strictly increasing sequence, $n \geq 1$, and $\rho^* = \{\rho_i^*\}_{1 \leq i \leq n-1}$ is the (possibly void) complementary one in $\{1, \ldots, 6\}$, we set $R^\rho = \{(x_i)_{1 \leq i \leq 6} \in \mathbb{R}^6 \mid x_j = 0 \ \forall j \in \rho^*\}$ and label $\Pi_{\rho} : \mathbb{R}^6 \to \mathbb{R}^\rho$ as the corresponding Euclidean orthogonal projection. The sequence $\rho$ is said to be wide if $n \geq 2$ and $\rho \neq \{2j-1, 2j\}, j = 1, 2, 3$. Given $\Omega \subset \mathbb{R}^6$, we call $C_\rho(\Omega)$ as the cylinder $\{x \in \mathbb{R}^6 \mid \Pi_{\rho}(x) \in \Omega\}$. When $\rho^* = \emptyset$ then $R^\rho = \mathbb{R}^6$, $\Pi_\rho = \text{Id}_{R^6}$ and $C_\rho(\Omega) = \Omega$, and we make the conventions $R^\emptyset = \{\emptyset\}$ and $\Pi_{\rho} \equiv \emptyset$.

**Main Theorem.** Let $\rho$ be a wide sequence in $\{1, \ldots, 6\}$, and let $\Omega$ be a convex domain in $\mathbb{R}^\rho$ (possibly all $\mathbb{R}^6$).

Then for any open orientable surface $M$ there exists a hyperbolic Riemann surface $N$ homeomorphic to $M$ and a complete null curve $F : N \to C_\rho(\Omega)$ such that $\Pi_{\rho} \circ F : N \to \Omega$ is proper.

It is classically known that any open hyperbolic Riemann surface $M$ carries neither proper holomorphic functions $f : M \to \mathbb{C}$ nor proper harmonic functions $h : M \to \mathbb{R}$. Therefore, Main Theorem does not hold if $\rho$ is not wide and $\Omega = \mathbb{R}^\rho$, and in this sense is sharp (see Remark 3.4).

To prove this theorem, we have developed an original technique that differs substantially from those of Nadirashvili, Martin-Morales and Martin-Umehara-Yamada. Our arguments rely only on the geometry of $C^3$, and the involved approximation results for null curves are of **extrinsic** nature. Roughly speaking, the null curve $F$ in the theorem is obtained by deforming recursively a sequence of compact null curves in $C_\rho(\Omega)$. Unlike previous methods, during the deformation we have direct control over the immersion itself instead of over its derivatives (or Weierstrass data). Furthermore, completeness and properness can be achieved at the same time in the process and checked extrinsically as well.

Different choices of sequence $\rho$ and convex domain $\Omega \subset \mathbb{R}^\rho$ generate a list of suggestive corollaries. The most straightforward is the following one:

**Corollary I (Calabi-Yau problem in $C^3$).** For any open orientable surface $M$ and any convex domain $\Omega$ in $C^3$, there exist a Riemann surface $N$ homeomorphic to $M$ and a proper complete null curve $F : N \to \Omega$.

In particular, this answers affirmatively the Problem 1 in [MUY1]: “Are there complete null curves properly immersed in the unit ball of $C^3$?”. A partial result in the line of Corollary I can also be found in [AFL].

Denote by $\langle \cdot , \cdot \rangle$ the Hermitian inner product in $SL(2, C)$ given by $\langle A,B \rangle = \text{trace}(A \cdot B^T)$, $A,B \in SL(2, C)$. A map $Z : N \to SL(2, C)$ is said to be a null curve in $SL(2, C)$ if $Z$ is a holomorphic immersion and $\det(dZ) = 0$. The Riemannian metric on $N$ induced by $\langle \cdot , \cdot \rangle$ is given by $d^2Z = [dZ,dZ]$. The
following correspondence is a biholomorphism preserving null curves (see [MUY1]):

\[ T : \mathbb{C}^3 - \{z_3 = 0\} \rightarrow \{(a_{ij}) \in \text{SL}(2,\mathbb{C}) \mid a_{11} \neq 0\}, \quad T((z_i)_{i=1,2,3}) = \frac{1}{z_3} \left( \frac{1}{z_1 - iz_2} \frac{z_1 + iz_2}{z_1^2 + z_2^2 + z_3^2} \right) \]

The transformation \( T \) provides complete null curves in \( \text{SL}(2,\mathbb{C}) \) when applied to complete bounded null curves in \( \mathbb{C}^3 - \{|z_3| > 1\} \). Unfortunately, proper examples in \( \text{SL}(2,\mathbb{C}) \) can not be constructed by this procedure. Taking into account Corollary I, we get that:

**Corollary II** [Calabi-Yau problem in \( \text{SL}(2,\mathbb{C}) \)]. For any open orientable surface \( M \), there exists a Riemann surface \( N \) homeomorphic to \( M \) and a complete bounded null curve \( F : N \rightarrow \text{SL}(2,\mathbb{C}) \).

Let \( \mathbb{H}^3 = \{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + 1 = x_0^2, x_0 > 0 \} \) be the hyperboloid model of the 3-dimensional hyperbolic space. We call \( \langle \cdot, \cdot \rangle_0 \) as the hyperbolic metric in \( \mathbb{H}^3 \) induced by the 4-dimensional Lorentz-Minkowski space \( \mathbb{L}^4 \) of signature \((- , +, +, +)\). Up to the canonical identification

\[ (x_0, x_1, x_2, x_3) \equiv \left( \begin{array}{c} x_0 + x_3 \\ x_1 - ix_2 \\ x_1 + ix_2 \\ x_0 - x_3 \end{array} \right), \]

\( \mathbb{H}^3 = \{ A \cdot A^T \mid A \in \text{SL}(2,\mathbb{C}) \} \). With this language, Bryant's projection \( B : \text{SL}(2,\mathbb{C}) \rightarrow \mathbb{H}^3, B(A) = A \cdot A^T \), maps null curves in \( \text{SL}(2,\mathbb{C}) \) into conformal immersions of mean curvature \( H = 1 \) in \( \mathbb{H}^3 \). Furthermore, if \( Z : N \rightarrow \text{SL}(2,\mathbb{C}) \) is a null curve then the pull back \( (Z \cdot Z^T)^{*} \langle \cdot, \cdot \rangle_0 \) coincides with \( \frac{1}{2} d\bar{z}_2^2 \) (see [Br,UY] for a good setting).

The family of complete CMC-1 surfaces in \( \mathbb{H}^3 \) with finite topology is very vast, see [RUY,PP] for a good reference. For the arbitrary topology case, there is no general existence result available known to the authors. Regarding Calabi-Yau questions, only recently were simply connected complete bounded Bryant surfaces constructed [MUY1]. However, applying Bryant's projection \( B \) to the complete bounded null curves of Corollary II we infer that:

**Corollary III** [Calabi-Yau problem in \( \mathbb{H}^3 \)]. For any open orientable surface \( M \), there exists a complete bounded CMC-1 immersion \( X : M \rightarrow \mathbb{H}^3 \).

Martín, Umehara and Yamada [MUY2] extended Jones' existence result [Jo] to complete bounded complex submanifolds with arbitrary finite genus and finitely many ends in \( \mathbb{C}^2 \). On the other hand, the existence of proper holomorphic immersions in \( \mathbb{C}^2 \) with arbitrary topological type is well known [Bi,Nar,Re,AL]. From Main Theorem it follows considerably more:

**Corollary IV** [Calabi-Yau problem in \( \mathbb{C}^2 \)]. For any open orientable surface \( M \) and any convex domain \( \Omega \) in \( \mathbb{C}^2 \) (possibly \( \Omega = \mathbb{C}^2 \)), there exist a Riemann surface \( N \) homeomorphic to \( M \) and a complete proper holomorphic immersion \( F : N \rightarrow \Omega \). Furthermore, if \( D \subset \mathbb{R}^2 \) is a convex domain and \( \Omega \) is the solid right cylinder \( \{ x \in \mathbb{C}^2 \mid \text{Re}(x) \subset D \} \), then \( F \) can be chosen so that \( \text{Re}(F) : N \rightarrow D \) is proper.

The real part of a null curve in \( \mathbb{C}^3 \) is a minimal immersion in \( \mathbb{R}^3 \) with vanishing flux, that is to say, such that the integral of the conormal vector to the immersion along any arc-length parameterized closed curve in the surface vanishes. As a consequence of Main Theorem,

**Corollary V** [Calabi-Yau problem in \( \mathbb{R}^3 \)]. For any open orientable surface \( M \) the following assertions hold:

(i) For any convex domain \( \Omega \subset \mathbb{R}^2 \), there exists a complete proper minimal immersion \( X : M \rightarrow \Omega \) with vanishing flux.

(ii) For any convex domain \( D \subset \mathbb{R}^2 \), there exists a complete minimal immersion \( X = (X_i)_{i=1,2,3} : M \rightarrow \mathbb{R}^3 \) with vanishing flux such that \( (X_1, X_2)(M) \subset D \) and \( (X_1, X_2) : N \rightarrow D \) is proper.

(iii) There exists a bounded complete flux vanishing minimal immersion \( X : M \rightarrow \mathbb{R}^3 \) such that all its associate immersions are bounded.
Although certainly Corollary V-(i) is strongly related with Ferrer-Martín-Meeks theorem [FMM], these results do not imply each other. Recently, the authors [AL] have constructed minimal surfaces with arbitrary conformal structure properly projecting into $\mathbb{R}^2$, answering a question posed by Schoen and Yau (see [SY, AG] for a good setting). Corollary V-(ii) shows that the analogous result for convex domains of $\mathbb{R}^2$ holds as well.

Finally, we remark that all the open Riemann surfaces involved in the above corollaries are of hyperbolic conformal type.

2. Preliminaries

As usual, we denote by $\| \cdot \|$ as the Euclidean norm in $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and for any compact topological space $X$ and continuous map $f : X \to \mathbb{K}^n$ we call $\|f\| = \max\{\|f(p)\| \mid p \in X\}$ as the maximum norm of $f$ on $X$.

Given a $n$-dimensional topological manifold $M$, we denote $\partial M$ as the $(n - 1)$-dimensional topological manifold determined by its boundary points. For any $A \subset M$, $A^\circ$ and $\overline{A}$ will denote the interior and the closure of $A$ in $M$, respectively. Open connected subsets of $M - \partial(M)$ will be called domains, and those proper topological subspaces of $M$ being $n$-dimensional manifolds with boundary are said to be regions. If $M$ is a topological surface, $M$ is said to be open if it is non-compact and $\partial M = \emptyset$.

An open Riemann surface is said to be hyperbolic if it carries non constant negative subharmonic functions.

**Remark 2.1.** Throughout this paper $\mathcal{N}$ will denote a fixed but arbitrary open hyperbolic Riemann surface.

A Jordan arc in $\mathcal{N}$ is said to be analytical if it is contained in an open analytical Jordan arc in $\mathcal{N}$.

A subset $A \subset \mathcal{N}$ is said to be incompressible if the inclusion map $i_A : A \hookrightarrow \mathcal{N}$ induces a group monomorphism $(i_A)_* : \mathcal{H}_1(A, \mathbb{Z}) \hookrightarrow \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$. In this case we identify the groups $\mathcal{H}_1(A, \mathbb{Z})$ and $(i_A)_*(\mathcal{H}_1(A, \mathbb{Z})) \subset \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$ via $(i_A)_*$ and consider $\mathcal{H}_1(A, \mathbb{Z}) \subset \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$.

Two incompressible subsets $A_1, A_2 \subset \mathcal{N}$ are said to be isotopic if $\mathcal{H}_1(A_1, \mathbb{Z}) = \mathcal{H}_1(A_2, \mathbb{Z})$. Two incompressible subsets $A_1, A_2 \subset \mathcal{N}$ are said to be homeomorphically isotopic if there exists a homeomorphism $\sigma : A_1 \to A_2$ such that $\sigma_* = \text{Id}_{\mathcal{H}_1(A_1, \mathbb{Z})}$, where $\sigma_*$ is the induced group morphism on homology. In this case $\sigma$ is said to be an isotopical homeomorphism. Two incompressible domains with finite topology (or two compact regions) in $\mathcal{N}$ are isotopic if and only if they are homeomorphically isotopic.

**Definition 2.2 (Admissible set).** A compact subset $S \subset \mathcal{N}$ is said to be admissible if and only if:

- $M_S := \overline{S} \setminus \mathcal{N}$ is a finite collection of pairwise disjoint compact regions in $\mathcal{N}$ with $C^0$ boundary,
- $C_S := \mathcal{S} - M_S$ consists of a finite collection of pairwise disjoint analytical Jordan arcs,
- any component $\alpha$ of $C_S$ with an endpoint $P \in M_S$ admits an analytical extension $\tilde{\alpha}$ in $\mathcal{N}$ such that the unique component of $\tilde{\alpha} - \alpha$ with endpoint $P$ lies in $M_S$, and
- $S$ is incompressible.

If $S \subset \mathcal{N}$ is a compact subset satisfying the first three items in the above definition, then $S$ is admissible if and only if $\mathcal{N} - S$ has no bounded components in $\mathcal{N}$ (by definition, a connected component $V$ of $\mathcal{N} - S$ is said to be bounded if $\overline{V}$ is compact).

Let $W$ be an incompressible domain of finite topology in $\mathcal{N}$, and let $A$ be either a compact region or an admissible subset in $\mathcal{N}$. $W$ is said to be a tubular neighborhood of $A$ if $A \subset W$ and $A$ is isotopic to $W$. In other words, if $A \subset W$ and $W - A$ consists of a finite collection of pairwise disjoint open annuli.

For any subset $A \subset \mathcal{N}$, we denote by $\mathcal{F}_0(A)$ as the space of continuous functions $f : A \to \mathbb{C}$ which are holomorphic on an open neighborhood of $A$. Likewise, $\mathcal{F}_0^A(A)$ will denote the space of continuous functions $f : A \to \mathbb{C}$ being holomorphic on $A^\circ$. Call $\Omega_0(A)$ as the space of holomorphic 1-forms on an open neighborhood of $A$, and label $\Omega_0^A(A)$ as the space of complex 1-forms $\theta$ of type $(1,0)$ that are continuous on $A$ and holomorphic on $A^\circ$. As usual, a 1-form $\theta$ on $A$ is said to be of type $(1,0)$ if for any conformal chart $(U, z)$ in $\mathcal{N}$, $\theta|_{U \cap A} = h(z)dz$ for some function $h : U \cap S \to \mathbb{C}$. 
Let $S$ be an admissible subset of $N$. A function $f \in F_0^3(S)$ is said to be smooth if $f|_{M_\alpha}$ admits a smooth extension $f_\theta$ to a domain $W$ containing $M_\alpha$, and for any component $\alpha$ of $C_\infty$ and any open analytical Jordan arc $\beta$ in $N$ containing $\alpha$, $f$ admits a smooth extension $f_\beta$ to $\beta$ satisfying that $f_\beta|_{W \cap \beta} = f_\theta|_{W \cap \beta}$. Likewise, a 1-form $\theta \in \Omega^1(S)$ is said to be smooth if, for any closed conformal disk $(U, z)$ on $N$ such that $S \cap U$ is admissible, $\theta|dz$ is smooth in the previous sense. Given a smooth function $f \in F_0^3(S)$, we set $df \in \Omega^1(S)$ as the smooth 1-form given by $df|_{M_\alpha} = d(f|_{M_\alpha})$ and $df|_{W \setminus U} = (f \circ \alpha)'(x)dz|_{W \cap U}$, where $(U, z = x + iy)$ is a conformal chart on $N$ such that $\alpha \cap U = \{z \in \mathbb{C} \mid z = z(U)\}$. A smooth 1-form $\theta \in \Omega^1(S)$ is said to be exact if $\theta = df$ for some smooth function $f \in F_0^3(S)$, or equivalently if $\int_U \theta = 0$ for all $U \in \mathcal{U}(S, Z)$.

Let $W \subset N$ be a domain containing $S$. We shall say that a function $f \in F_0^3(S)$ can be uniformly approximated on $S$ by functions in $F_0(W)$ if there exists $\{f_n\}_{n \in \mathbb{N}} \subset F_0(W)$ such that $\{\|f_n - f\|\}_{n \in \mathbb{N}} \to 0$ uniformly on $S$. Likewise, a 1-form $\theta \in \Omega^1(S)$ can be uniformly approximated on $S$ by 1-forms in $\Omega^1(W)$ if there exists $\{\theta_n\}_{n \in \mathbb{N}} \subset \Omega^1(W)$ such that $\{\|\theta_n - \theta\|\}_{n \in \mathbb{N}} \to 0$ uniformly on $S \cap W$. Finally, we say that a function $f \in F_0^3(S)$ can be uniformly $C^1$-approximated on $S$ by functions in $F_0(W)$ if there exists $\{f_n\}_{n \in \mathbb{N}} \subset F_0(W)$ such that $\{\|f_n - f\|\}_{n \in \mathbb{N}} \to 0$ and $\{\|df_n - df\|\}_{n \in \mathbb{N}} \to 0$ uniformly on $S$. In a similar way we define the uniform approximation (and $C^1$-approximation) on $S$ of maps in $F_0^3(S)^3 \equiv \{(f_j)_{j=1,2,3} : S \to C^3 | f_j \in F_0^3(S), j = 1, 2, 3\}$ by functions in $F_0(W)^3$.

2.1. Null Curves in $C^3$. Given $A \subset C^3$, we set span$_R(A) = \{\sum_{j=1}^n v_j \mid v_j \in \mathbb{R}, v_j \in A, n \in \mathbb{N}\}$ and span$_C(A) = \text{span}_R(A) + \text{span}_C(A)$, where $J : C^3 \to C^3$, $J(v) = iv$, is the usual complex structure. If $V \subset C^3$ is a real subspace, the complex subspace $V_C := V \cap J(V)$ is said to be the kernel complex of $V$.

Label $\langle , \rangle : C^3 \times C^3 \to C$, $\langle u, v \rangle := u^T v$, as the usual Hermitian inner product in $C^3$. Given $A \subset C^3$ we denote by $\langle A \rangle^\perp = \{v \in C^3 \mid \langle u, v \rangle = 0 \forall u \in A\}$. We also write $(,) = \Re(\langle , \rangle)$ as the Euclidean scalar product of $C^3 \equiv \mathbb{R}^6$, and call $(A)^\perp = \{v \in C^3 \mid \langle u, v \rangle = 0 \forall u \in A\}$, $A \subset C^3$.

Finally, we set $\langle , \rangle : C^3 \times C^3 \to C$ as the complex symmetric bilinear 1-form given by $\langle u, v \rangle := u^T v$, $v$, and $\langle u, v \rangle^\perp = \{v \in C^3 \mid \langle u, v \rangle = 0\}$. Notice that $\langle u, v \rangle^\perp = \langle u \rangle^\perp \subset (u)^\perp$ for all $u \in C^3$ (the equality holds iff $u = 0$). A vector $u \in C^3 - \{0\}$ is said to be null if $\langle u, u \rangle = 0$. We label $\Theta = \{u \in C^3 - \{0\} \mid u \text{ is null}\}$.

Remark 2.3. $\Theta = \{(\frac{1}{2}z(1 - w^2), \frac{1}{2}z(1 + w^2), zw) \mid w, z \in C, z \neq 0\}$. As a consequence, $\Theta$ is a complex conical submanifold of $C^3$ not contained in a finite union of real (or complex) hyperplanes of $C^3$.

A basis $\{u_1, u_2, u_3\}$ of $C^3$ is said to be $\langle , \rangle$-conjugate if $\langle u_j, u_k \rangle = \delta_{jk}, j, k \in \{1, 2, 3\}$. Likewise we define the notion of $\langle , \rangle$-conjugate basis of a complex subspace $U$, provided that $\langle , \rangle \mid_{U \times U}$ is not a degenerate complex bilinear form.

A real or complex hyperplane $V \subset C^3$ is said to be $\langle , \rangle$-conjugate if $\langle , \rangle \mid_{V_C \times V_C}$ is a degenerate complex bilinear 1-form, that is to say, if $V_C = \langle u \rangle^\perp$ for some null vector $u$. For instance, if $H = \langle v \rangle^\perp$, $v \neq 0$, then $H_C = \langle T \rangle^\perp$ and $H$ is $\langle , \rangle$-conjugate if and only if $v$ is null. If $v$ is not null, there exists a $\langle , \rangle$-conjugate basis $\{u_1, u_2, u_3\}$ of $C^3$ so that $u_3 = T$ and span$_C(\{u_1, u_2\}) = H_C$.

We denote $O(3, C)$ as the complex orthogonal group $\{A \in M_3(C) \mid A^T \cdot A = I_3\}$, i.e., the group of matrices whose column vectors determine a $\langle , \rangle$-conjugate basis of $C^3$. As usual, we also call $A : C^3 \to C^3$ as the complex linear transformation induced by $A \in O(3, C)$.

It is clear that $A(\Theta) = \Theta$ for all $A \in O(3, C)$. Since $A(\langle v \rangle^\perp) = \langle A \cdot v \rangle^\perp$ for any $A \in O(3, C)$ and $v \in C^3 - \{0\}$, the $\langle , \rangle$-degeneracy of real (or complex) hyperplanes is preserved by complex orthogonal transformations.

Definition 2.4. A domain $\Omega \subset C^3$ with non empty $\partial \Omega$ is said to be $\langle , \rangle$-regular if it is regular (i.e., with smooth $\partial \Omega$) and the real tangent space $T_p \partial \Omega$ is not $\langle , \rangle$-degenerate for almost every $p \in \partial \Omega$.

The $\langle , \rangle$-regularity of domains is preserved by complex orthogonal transformations.
Let $M$ be an open Riemann surface. Using the above language, a holomorphic map $F : M \to \mathbb{C}^3$ is a null curve iff $\langle \Phi, \Phi \rangle = 0$ and $\langle \Phi, \Phi \rangle$ never vanishes on $M$, where $\Phi = dF$. Conversely, given an exact holomorphic vectorial 1-form $\Phi$ on $M$ satisfying that $\langle \Phi, \Phi \rangle = 0$ and $\langle \Phi, \Phi \rangle$ never vanishes on $M$, the map $F : M \to \mathbb{C}^3, F(p) = \int^p \Phi$, defines a null curve in $\mathbb{C}^3$. In this case we call $\Phi$ as the Weierstrass representation of $F$.

If $F : M \to \mathbb{C}^3$ is a null curve, the pull back metric $ds_F^2 := F^* \langle , , \rangle$ on $M$ coincides with $\langle \Phi, \Phi \rangle = \langle \Phi, \Phi \rangle$. Given two subsets $V_1, V_2 \subset M$, we denote by $\text{dist}_{(M,F)}(V_1, V_2)$ the intrinsic distance between $V_1$ and $V_2$ with respect to the metric $ds_F^2$.

**Remark 2.5.** Let $F : M \to \mathbb{C}^3$ be a null curve and $A = (a_{jk})_{jk=1,2,3} \in \mathcal{O}(3, \mathbb{C})$. Then $A \circ F : M \to \mathbb{C}^3$ is also a null curve and $ds_F^2 \geq \frac{1}{|A|^2} ds_{A \circ F}^2$, where $\|A\| = (\sum_{jk} |a_{jk}|^2)^{1/2}$.

The following definitions deal with the notion for null curve on admissible subsets.

**Definition 2.6.** Given a proper subset $M \subset N$, we denote by $\mathcal{N}(M)$ the space of maps $X : M \to \mathbb{C}^3$ extending as a null curve to an open neighborhood of $M$ in $N$.

**Definition 2.7.** Let $S \subset N$ be an admissible subset. A smooth map $F : S \to \mathbb{C}^3$ is said to be a generalized null curve in $\mathbb{C}^3$ if it satisfies the following properties:

- $F|_{M_S} \in \mathcal{N}(M_S)$,
- $\langle dF, dF \rangle = 0$ on $S$,
- $\langle dF, dF \rangle$ never vanishes on $S$.

The following Lemma will be required to approximate generalized null curves by null curves defined on larger domains.

**Lemma 2.8.** Let $W \subset N$ be an incompressible domain of finite topology, and let $S$ be a connected admissible compact set contained in $W$ and isotopic to $W$. Let $F = (F_j)_{j=1,2,3} : S \to \mathbb{C}^3$ be a generalized null curve.

Then $F$ can be uniformly $C^1$-approximated on $S$ by a sequence $(H_n = (H_{jn}(x,3,2,3)_{n \in \mathbb{N}}$ in $\mathcal{N}(W)$. In addition, we can choose $H_{3,n} = F_3$ for all $n \in \mathbb{N}$ provided that $F_3 \in \mathcal{F}_0(W)$ and $dF_3$ never vanishes on $C_3$.

**Proof.** Use the Approximation Lemma in [AL] for $\Phi = dF$ to get a sequence of exact vectorial 1-forms $\{\Phi_n\}_{n \in \mathbb{N}} \subset \mathcal{O}_n(W)^3$ converging to $dF$ uniformly on $S$. Since $S$ is isotopic to $W$, $H_n := F(p) + \int_{F_0} \Phi_n$ is well defined on $W$ for all $n \in \mathbb{N}$, where $F_0$ is any point in $M_0$. $\{H_n\}_{n \in \mathbb{N}}$ solves the lemma.

### 2.2. Convex Domains

Throught this section, $\mathcal{D}$ will denote a regular convex domain of $\mathbb{R}^n$, $\mathcal{D} \neq \mathbb{R}^n$, $n \geq 2$.

Recall that $\mathcal{D} \cap (p + T_p \partial \mathcal{D}) = \emptyset$ for all $p \in \partial \mathcal{D}$, where $T_p \partial \mathcal{D}$ denotes the tangent space of $\partial \mathcal{D}$ at $p$. Therefore $\overline{\mathcal{D}} = \bigcap_{p \in \partial \mathcal{D}} H_p$, where $H_p$ is the closed half space bounded by $p + T_p \partial \mathcal{D}$ and containing $\mathcal{D}$, $p \in \partial \mathcal{D}$.

Let $\nu_{\mathcal{D}} : \partial \mathcal{D} \to S^{n-1}$ be the outward pointing unit normal of $\partial \mathcal{D}$. Given $p \in \partial \mathcal{D}$ and $v \in T_p \partial \mathcal{D} \cap S^{n-1}$, we denote by $\kappa_{\mathcal{D}}(p, v)$ the normal curvature at $p$ in the direction of $v$ with respect to $-\nu_{\mathcal{D}}$, obviously non negative. In particular, the principal curvatures of $\partial \mathcal{D}$ at $p$ with respect to $-\nu_{\mathcal{D}}$ are non negative. We let $\kappa(p) \geq 0$ denote the maximum of these principal curvatures at $p \in \partial \mathcal{D}$, and write

$$\kappa(\mathcal{D}) := \sup\{\kappa(p) \mid p \in \partial \mathcal{D}\} \in [0, +\infty].$$

If $p \in \partial \mathcal{D}$, $v \in S^{n-1} \cap T_p \partial \mathcal{D}$ and $\kappa_{\mathcal{D}}(p, v) > 0$, basic convex geometry gives that $\lim_{\lambda \to -\infty} \text{dist}(p + \lambda v, \partial \mathcal{D}) = +\infty$, where as usual dist() means Euclidean distance. The domain $\mathcal{D}$ is said to be strictly convex if $\kappa_{\mathcal{D}}(p, v) > 0$ for all $p \in \partial \mathcal{D}$ and $v \in S^{n-1} \cap T_p \partial \mathcal{D}$.

For any $t \in (-\frac{1}{\kappa(\mathcal{D})}, +\infty)$ let $\mathcal{D}_t$ denote the convex domain in $\mathbb{R}^n$ bounded by $\partial \mathcal{D}_t = \{p + t \cdot \nu_{\mathcal{D}}(p) \mid p \in \partial \mathcal{D}\}$ and such that $\mathcal{D} \subset \mathcal{D}_t$ if $t \geq 0$, and $\mathcal{D}_t \subset \mathcal{D}$ if $t \leq 0$. We have made the conventions
$-\frac{1}{\kappa(D)} = -\infty$ and $-\frac{1}{\kappa(D)} = 0$ provided that $\kappa(D) = 0$ and $\kappa(D) = +\infty$, respectively. We label $D_{-1/\kappa(D)}$ as the closed subset $\bigcap_{D_{-1/\kappa(D)}} D$. Note that $\partial D$ is a regular (convex) hypersurface $\forall t \in (-\frac{1}{\kappa(D)}, +\infty)$.

Set $\pi_D : \mathbb{R}^n - D_{-1/\kappa(D)} \to \partial D$ as the normal projection given by $\pi_D(p + t\nu_D(p)) = p$, and keep denoting by $\nu_D$ the extended normal map $\nu_D = \pi_D : \mathbb{R}^n - D_{-1/\kappa(D)} \to S^{n-1}$.

A vector $v \in \mathbb{R}^n - \{0\}$ is said to be a **escaping vector** in $D$ if $D$ contains no half lines parallel to $v$, or equivalently, if $\lim\sup_{R \to +\infty} \text{dist}(p + \lambda v, D) = +\infty$ for all $p \in \mathbb{R}^n$. In this case $\lim\inf_{R \to +\infty} \text{dist}(p + \lambda v, D) > 0$ for all $p \in \mathbb{R}^n$. The set $\mathcal{E}_D$ of escaping vectors in $D$ is empty if and only if $D$ contains a half space, and otherwise it is the complement in $\mathbb{R}^n$ of a double cone with vertex $0$ and base a compact convex subset of $S^{n-1}$. If $p \in \partial D$ and $v \in T_p \partial D - \mathcal{E}_D$ then $\partial D \cap T_p \partial D$ contains a half line parallel to $v$ and with initial point $p$, whereas $v \in \mathcal{E}_D \cap T_p \partial D$ implies that $\partial D \cap \{p + \lambda v \mid \lambda \in \mathbb{R}\}$ is a compact segment containing $p$. If $D$ is strictly convex then $T_p \partial D - \{0\} \subset \mathcal{E}_D$, $\partial D \cap \{p + T_p \partial D\} = \{p\}$ and $\lim_{T_p \partial D \to +\infty} \text{dist}(p + v, \partial D) = +\infty$ for all $p \in \partial D$.

Assume that $\kappa(D) < +\infty$ and take $r \in (0, 1/\kappa(D))$ and $p \in D - \overline{D}_r$. Consider $\delta \in (0, r)$ and a neighborhood $U_p$ of $p$ so that $U_p \subset D - \overline{D}_{r-\delta}$ and $\text{diam}(U_p) < \delta$. Then it is straightforward to check that

$$\partial D - \{0\} \cap (q_1 + T_{\pi_D(q_1)} \partial D) = \emptyset \text{ for all } q_1, q_2 \in U_p.$$

**Remark 2.9.** Let $D$ and $\hat{D}$ be two regular convex domains in $\mathbb{R}^n$ with $\kappa(D) < +\infty$ and $\overline{D} \subset \hat{D}$. Consider $p \in \partial D$, and take $r \in (0, 1/\kappa(D))$ and $q \in \partial D - r$ with $\pi_D(q) = p$. Basic trigonometry gives that $\text{dist}(q, (q + T_p \partial D - r) \cap \partial \hat{D}) \geq \frac{d_r^2 + 2 - \frac{d_r^2}{\kappa(D)}}{1 - \frac{\kappa(D)}{r}}$, where $d_r = \text{dist}(D - r, \partial \hat{D})$. Since $T_q \partial D - r = T_p \partial D$, $d_r \geq d_0 + r$ and $\kappa(D - r) = \frac{\kappa(D)}{1 - \frac{\kappa(D)}{r}}$, we infer that

$$\text{dist}(q, (q + T_p \partial D) \cap \partial \hat{D}) \geq \sqrt{d_0^2 + 2 - \frac{d_0^2}{\kappa(D)}}.$$

Given two compact subsets $C, D \subset \mathbb{R}^n$, the Hausdorff distance between $C$ and $D$ is defined by

$$\delta^H(C, D) = \max \left\{ \sup_{x \in C} \inf_{y \in D} ||x - y||, \sup_{y \in D} \inf_{x \in C} ||x - y|| \right\}.$$

A sequence $\{K_n\}_{n \in \mathbb{N}}$ of (possibly unbounded) closed subsets of $\mathbb{R}^n$ is said to be convergent in the Hausdorff topology to a closed subset $K_0$ of $\mathbb{R}^n$ if $\{K_n \cap B\}_{n \in \mathbb{N}} \rightarrow K_0 \cap B$ in the Hausdorff distance for any closed ball $B \subset \mathbb{R}^n$. If $K_j \subset K_{j+1} \subset K_0 \forall j \in \mathbb{N}$ and $\{K_j\}_{j \in \mathbb{N}} \rightarrow K_0$ in the Hausdorff topology, we simply write $\{K_j\}_{j \in \mathbb{N}} \rightarrow K_0$. Likewise we put $\{K_j\}_{j \in \mathbb{N}} \rightarrow K_0$ provided that $K_0 \subset K_{j+1} \subset K_j \forall j \in \mathbb{N}$ and $\{K_j\}_{j \in \mathbb{N}} \rightarrow K_0$ in the Hausdorff topology.

The following theorem follows from classical Minkowski’s Theorem [Mi] (see also [MY]):

**Theorem 2.10.** Let $C$ be a (possibly neither bounded nor regular) convex domain of $\mathbb{R}^n$.

Then there exists a sequence $\{C_k\}_{k \in \mathbb{N}}$ of bounded strictly convex analytic domains in $\mathbb{R}^n$ with $\overline{C_k} \subset \mathbb{C}$.

In addition $C$ is bounded, then there exists a sequence $\{D_k\}_{k \in \mathbb{N}}$ of bounded strictly convex analytic domains in $\mathbb{R}^n$ with $\overline{D_k} \subset \mathbb{C}$.

Recall that a convex domain $D$ is said to be analytic if $\partial D$ is an analytical hypersurface of $\mathbb{R}^n$.

**Definition 2.11.** Let $C$ be a (possibly neither bounded nor regular) convex domain in $\mathbb{R}^n$. A sequence $\{C_k\}_{k \in \mathbb{N}}$ of convex domains in $\mathbb{R}^n$ is said to be proper in $C$ if $C_k$ is bounded regular strictly convex for all $k$, $\{\overline{C_k}\}_{k \in \mathbb{N}} \rightarrow C$ in the Hausdorff topology and $\sum_{k \in \mathbb{N}} \sqrt{\text{dist}(C_k, \partial C)} = +\infty$.

**Lemma 2.12.** Any convex domain in $\mathbb{R}^n$ admits a proper sequence of convex domains.
Proof. Let $C$ be a convex domain in $\mathbb{R}^n$. By Theorem 2.10, there exists a sequence $\{C_j\}_{j \in \mathbb{N}}$ of bounded strictly convex analytic domains in $\mathbb{R}^n$ with $\{\overline{C_j}\}_{j \in \mathbb{N}} \not\subset \mathcal{C}$. For the sake of simplicity write $d_j = \text{dist}(C_j, \partial C_{j+1})$ and $\kappa_j = \kappa(C_j)$ for all $j$.

Recall that $\frac{\kappa_j}{2\pi} \sum_{a=1}^{m_i} \frac{1}{a} = 1$, and for each $j \in \mathbb{N}$ fix $m_j \in \mathbb{N}$ large enough so that $\frac{\kappa_j}{2\pi} \sum_{a=1}^{m_i} \frac{1}{a} \geq 1$. Call $d_{a,j} = d_{\frac{\kappa_j}{2\pi} \sum_{a=1}^{m_i} \frac{1}{a}}$, set $C_{a,j} = (C_j)_{d_{a,j}}$, $a = 1, \ldots, m_j$, and make the convention $C_{0,j} = C_j$. It is clear that $d_{a,j} < d_j$, $C_{a,j}$ is analytical and strictly convex, $C_{a,j} \subset C_{a+1,j} \subset C_{a+1} \subset C_{j+1}$, and $\text{dist}(C_{a,j}, \partial C_{a+1,j}) = \frac{6d_{a,j}}{\pi (a+1)^2}$ for all $a = 0, \ldots, m_j - 1$. Furthermore, $\kappa(C_{a,j}) = \kappa_j \frac{1}{1 + \frac{d_{a,j}}{\kappa_j}} \leq \kappa_j$ for all $a = 0, \ldots, m_j - 1$.

Let $\{D_k\}_{k \in \mathbb{N}}$ denote the enumeration of $\{C_{a,j} \mid a = 0, \ldots, m_j\}$ so that $\overline{D_k} \subset D_{k+1}$ for all $k$. Since

$$\sum_{k \in \mathbb{N}} \sqrt{\frac{\text{dist}(D_k, \partial D_{k+1})}{\kappa(D_k)}} \geq \sum_{j \in \mathbb{N}} \left( \sum_{a=0}^{m_j-1} \sqrt{\frac{\text{dist}(C_{a,j}, \partial C_{a+1,j})}{\kappa(C_{a,j})}} \right) = +\infty,$$

$\{D_k\}_{k \in \mathbb{N}}$ is proper in $C$ and we are done. \hfill $\Box$

### 2.2.1. Convex domains in $\mathbb{C}^3$

Given a regular convex domain $D \subset \mathbb{C}^3$ and a point $p \in \mathbb{C}^3 - D - 1/\kappa(D)$, we call $\Theta_D(p)$ as the space $\langle \nu_D(p) \rangle^\perp \cap \Theta \subset T_{\pi_D(p)} \partial D$.

**Definition 2.13.** A regular convex domain $D \subset \mathbb{C}^3 \equiv \mathbb{R}^6$ is said to be null strictly convex if $\mathcal{E}_D \cap \Theta_D(p) \not\subset \emptyset \not\subset \forall p \in \partial D$.

This occurs, for instance, if for any $p \in \partial D$ there is $v \in \Theta_D(p) \cap S^5$ such that $\kappa(p, v) > 0$.

**Claim 2.14.** If $A \in \mathcal{O}(3, \mathbb{C})$, $p \in \partial D$ and $v \in \Theta_A(\partial D) \cap \mathcal{E}_D$ then $A(v) \in \Theta_A(\partial D)(A(p)) \cap \mathcal{E}_A(\partial D)$. In particular, the null strictly convexity is preserved by complex orthogonal transformations.

**Proof.** It is clear that $A(p) \in \partial A(D)$ and $\|A(\nu_D(p))\| = \|\nu_D(A(p))\| = 0 = A(\nu_D(p))$. Therefore, $\langle \nu_A(\partial D)(A(p)) \rangle^\perp = A(\langle \nu_D(p) \rangle^\perp)$ and $A(v) \in \Theta_A(\partial D)(A(p))$. Since $\text{dist}(p,q) \leq \|A^{-1}\| \text{dist}(A(p), A(q))$, then $A(v) \in \mathcal{E}_A(\partial D)$ as well and we are done. \hfill $\Box$

It is interesting to notice that a strictly increasing sequence $p = \{p_i\}_{1 \leq i \leq n} \subset \{1, \ldots, 6\}$ is wide if and only if dim$_{\mathbb{C}}(\operatorname{span}_C(\mathbb{R}^6)) \geq 2$ (see Section 1 for notations).

**Proposition 2.15.** Let $p$ be a wide sequence in $\{1, \ldots, 6\}$, and let $\Omega \subset \mathbb{R}^6$ be a regular strictly convex domain with $\kappa(\Omega) < +\infty$. Then $\mathcal{C}_p(\Omega)$ is $\langle , , \rangle$-regular, null strictly convex and $\kappa(\mathcal{C}_p(\Omega)) < +\infty$.

**Proof.** Label $n$ as the length of $p$. Take $p \in \partial C_p(\Omega)$, and observe that $\Pi_p(\nu_p(\Omega)(p)) = \nu_p(\Pi_p(p))$ and $\Pi_p(\nu_p(\Omega)(p)) = 0$. Call $S_p^u = S_p \cap \mathbb{R}^6$ and $\Theta_p = \partial \cap S_p^u$.

Let us observe that $\Theta_p$ is an analytical submanifold of $S_p^u$ of dimension $n - 1$ and so has measure zero in $S_p^u$. Indeed, reason by contradiction and suppose that $\Theta_p = S_p^u$, that is to say, $\mathbb{R}^6 \subset \Theta$. Then $\text{span}_{\mathbb{C}}(\mathbb{R}^6) \subset \Theta$ as well, and so $\Theta$ contains a complex hyperplane of $\mathbb{C}^3$ (recall that $p$ is wide), a contradiction. The strictly convexity of $\Omega$ implies that $\nu_p : \partial \Omega \rightarrow S_p^u$ is an injective local diffeomorphism, hence $\nu_p |_{\mathcal{C}_p(\Omega)(p)}$ is null for almost every $p \in \mathcal{C}_p(\Omega)$, proving the $\langle , , \rangle$-regularity.

For the null strictly convexity, let us show first that $\Pi_p(\Theta_p(\mathcal{C}_p(\Omega)(p))) \not\subset \{0\}$ for all $p \in \partial C_p(\Omega)$. If $\nu_p(\mathcal{C}_p(\Omega)(p))$ is null then $\Theta_p(\mathcal{C}_p(\Omega)(p)) \not\subset \emptyset$ and $\Pi_p(\mathcal{C}_p(\Omega)(p)) = 0$. Assume now that $\nu_p(\mathcal{C}_p(\Omega)(p))$ is not null, and reasoning by contradiction suppose that $\Theta_p(\mathcal{C}_p(\Omega)(p)) \subset \mathbb{R}^6$. Since $\Theta_p(\mathcal{C}_p(\Omega)(p))$ contains two $\mathbb{C}$-linearly independent null vectors then dim$_{\mathbb{C}}(\mathbb{R}^6)_{\mathbb{C}} \geq 2$, contradicting that $p$ is wide.
To finish, take any \( v \in \Theta_{C_p}(\Omega) \) such that \( \Pi_p(v) \neq \emptyset \). By the strictly convexity of \( \Omega \) we have that \( \Pi_p(v) \in E_\Omega \), and so \( v \in E_{C_p}(\Omega) \). Since \( \kappa(C_p(\Omega)) = \kappa(\Omega) < +\infty \), we are done. \( \square \)

Remark 2.16. If \( \rho = \{2j - 1, 2j\} \) for some \( j \in \{1, 2, 3\} \) and \( \Omega \subset \mathbb{R}^p \) is a regular convex domain, then \( C_p(\Omega) \) is not null strictly convex. Moreover, both open slabs and half spaces in \( \mathbb{C}^3 \equiv \mathbb{R}^3 \) are not null strictly convex as well.

3. The Main Theorem

The following Lemma, which will be proved later in Section 4, is the kernel of the proof of our main Theorem.

Lemma 3.1. Let \( D \) be a \( \leq, \geq \)-regular and null strictly convex domain in \( \mathbb{C}^3 \) with \( \kappa(D) < +\infty \), and consider \( \rho \in (0, 1/\kappa(D)) \). Let \( M \) be an incompressible compact region in \( N \), \( p_1 \in M^0 \) and \( F \in N(M) \) satisfying that:

\[
(3.1) \quad F(\partial(M)) \subset D - \overline{D}_{-r}.
\]

Then, for any regular convex domain \( \mathcal{D} \) and \( \epsilon > 0 \) such that \( \mathcal{D} \subset \mathcal{D}_{-\epsilon} \subset \mathcal{D} \subset \mathcal{D}_{1/\epsilon} \), there exist an incompressible compact region \( \tilde{M} \subset N \) and \( \tilde{F} \in N(M) \) satisfying that:

(i) \( M \subset \tilde{M}^0 \) and \( \tilde{M} \) is isotopic to \( M \).
(ii) \( \| F - \tilde{F} \| < \epsilon \) on \( M \).
(iii) \( \tilde{F}(\partial(M)) \subset \mathcal{D} - \overline{\mathcal{D}}_{-\epsilon} \).
(iv) \( \tilde{F}(\tilde{M} - \tilde{M}^0) \subset \mathcal{D} - \overline{\mathcal{D}}_{-r} \).
(v) \( \text{dist}_{(M, F)}(p_1, \partial(M)) > \text{dist}_{(M, F)}(p_0, \partial(M)) + \sqrt{\text{dist}(\partial D, \partial \mathcal{D})} \), where \( d = \text{dist}(\partial D, \partial \mathcal{D}) \).

We are now ready to state and prove our main theorem.

Theorem 3.2. Let \( \rho \subset \{1, \ldots, 6\} \) be a wide sequence, and let \( \Omega \) be a (possibly neither bounded nor regular) convex domain in \( \mathbb{R}^p \). Let \( M \subset N \) be an incompressible compact region in \( N \), and consider a null curve \( X \in N(M) \) satisfying that:

\[
(3.2) \quad \Pi_p(X(\partial(M))) \subset \Lambda - \overline{\Lambda}_{-r},
\]

where \( \Lambda \) is a bounded regular strictly convex domain in \( \mathbb{R}^p \) so that \( \Lambda \subset \Omega \) and \( r \in (0, 1/\kappa(\Lambda)) \).

Then, for any \( \xi > 0 \) there exist an open domain \( N \subset N \) and a null curve \( Y : N \rightarrow \mathbb{C}^3 \) satisfying that

(a) \( M \subset N \), \( N \) is incompressible and \( N \) is isotopic to \( N \),
(b) \( \| Y - X \| \leq \xi \) on \( M \),
(c) \( Y \) is complete,
(d) \( \Pi_p \circ Y : N \rightarrow \Omega \) is proper and \( \Pi_p(Y(M) - M^0) \subset \Omega - \overline{\Omega}_{-r} \).

Proof. Consider an exhaustion \( \{M_j\}_{j \in \mathbb{N}} \) of \( N \) by incompressible compact regions so that \( M^0_j \) is a tubular neighborhood of \( M_j \), and \( M_{j-1} \subset M^0_j \) and the Euler characteristic \( \chi(M_j - M^0_{j-1}) \in \{-1, 0\} \) for all \( j \geq 2 \).

Let \( \{\Omega_j\}_{j \in \mathbb{N}} \) be a proper sequence in \( \Omega \) of convex domains (see Lemma 2.12), and label \( d_j = \text{dist}(\Omega_j, \Omega_{j+1}) \) and \( \kappa_j = \kappa(\Omega_j) \). Without loss of generality, we can suppose that \( \Lambda \subset \Omega_j \). Call \( \Omega_0 = \Lambda \) and \( \mathcal{D}_j = C_p(\Omega_j) \) for all \( j \geq 0 \). Notice that \( \mathcal{D} \) is \( \leq, \geq \)-regular, null strictly convex and \( \kappa(\mathcal{D}) < +\infty \), \( j \geq 0 \) (see Proposition 2.15).

Without loss of generality, we will assume that \( \xi \) is small enough so that \( \epsilon_0 := r - \xi > 0 \) and

\[
\Pi_p(X(\partial(M))) \subset \Lambda - \overline{\Lambda}_{-\epsilon_0}.
\]

Consider a decreasing sequence \( \{\epsilon_j\}_{j \in \mathbb{N}} \) such that \( \epsilon_j \in (0, \xi/2^{j+1}) \) for all \( j \), \( \overline{\mathcal{D}}_{\epsilon_0} \subset \mathcal{D} - \epsilon_0 \subset \mathcal{D} \subset \mathcal{D}_{1/\epsilon_0} \), and \( \mathcal{D}^{-1} \subset \mathcal{D} - \epsilon_1 \subset \mathcal{D}_{1/\epsilon_1} \subset \mathcal{D} \subset \mathcal{D}^{-1} \subset \mathcal{D}_{1/\epsilon_j} \). Fix \( Q_0 \in M^0 \) and label \( N_0 = M \), \( \epsilon_0 = \text{dist}_{M_0} \) and \( \epsilon_0 = X_0 = X \).

Claim 3.3. There exists a sequence \( \{(N_j, \epsilon_j, X_j)\}_{j \in \mathbb{N}} \) where \( N_j \subset N \) is an incompressible compact region isotopic to \( M_j \), \( \epsilon_j : N_j \rightarrow M_j \) is an isotopic homeomorphism and \( X_j \in N(N_j) \) for all \( j \), satisfying that:
Lemma 3.1 for the data

Proof. The sequence will be constructed recursively. Choose \((N_1, X_1)\) as the couple \((\tilde{M}, \tilde{F})\) arising from Lemma 3.1 for the data \((D, r, M, F, \tilde{D}, e, P_0) = (D^0, e_0, N_0, X_0, D^1, e_1, Q_0)\). Note that \(N_0 \subset N_1^0\) and that \(N_1, M_1\) and \(N_0\) are homeomorphic. Choosing \(c_1 : N_1 \rightarrow M_1\) any homeomorphism satisfying \(\sigma_1|N_0 = \text{Id}_{N_0}\), all the above items hold for \(j = 1\) (item (Vj) makes no sense).

Assume that we have already constructed \(X_1, \ldots, X_{j-1}\) satisfying the required properties, and let us construct \(X_j\). Let us distinguish two cases:

- \(\chi(M_i - M_{i-1}^0) = 0\). As before, we take \((N_j, X_j)\) as the couple \((\tilde{M}, \tilde{F})\) arising from Lemma 3.1 for the data \((D, r, M, F, \tilde{D}, e, P_0) = (D^{i-1}, e_{j-1}, N_{j-1}, X_{j-1}, D^1, e_j, Q_0)\). As \(N_j, N_{j-1}\) and \(M_j\) are homeomorphic and \(N_j - 1 \subset N_{j-1}^0\), we can choose \(\sigma_j : N_j \rightarrow M_j\) any homeomorphism so that \(\sigma_j|N_{j-1} = \sigma_{j-1}\). Properties (Ij), (IIj), (IIIj) and (IVj) are straightforward, whereas (Vj) follows from Lemma 3.1-(v) and (Vj-1).

- \(\chi(M_j - M_{j-1}^0) = -1\). Consider a closed curve \(\hat{a} \in \mathcal{H}_1(M_j, \mathbb{Z}) - \mathcal{H}_1(M_{j-1}, \mathbb{Z})\) contained in \(M_j^0\) and intersecting \(M_j - M_{j-1}\) in a Jordan arc \(a\) with endpoints in \(\partial(M_{j-1})\) and otherwise disjoint from \(M_{j-1}\). As \(M_j\) is incompressible then \(\mathcal{H}_1(M_j, \mathbb{Z}) = \mathcal{H}_1(M_{j-1} \cup a, \mathbb{Z})\) and \(N_j \rightarrow (M_{j-1} \cup a)\) has no bounded components. Take a Jordan arc \(\gamma \subset N - N_{j-1}^0\) and an isotopic homeomorphism \(\zeta : N_{j-1} \cap \gamma \rightarrow M_{j-1} \cup a\) so that \(\zeta|N_{j-1} = \zeta_j - 1\) and \(\zeta(\gamma) = a\). Note that \(N_j \rightarrow (N_{j-1} \cup \gamma)\) has no bounded components as well, hence without loss of generality we can assume that \(S := N_{j-1} \cup \gamma\) is admissible.

Let \(M_{j-1}' \subset M_j^0\) be any compact region isotopic to \(M_j\) and containing \(M_{j-1} \cup a\) in its interior. Construct a generalized null curve \(Z : S \rightarrow \mathbb{C}^3\) with \(Z|_{N_{j-1}} = X_{j-1}\) and \(Z(\gamma) \subset D^{i-1} - (\overline{D^{i-1}} - e_{j-1})\) (this is possible by (IIIj-1)). Applying Lemma 2.8 to \(S, Z\) and any tubular neighborhood \(W\) of \(S\), we get a compact region \(N_{j-1}'\) and a null curve \(X_{j-1}' \subset N(N_{j-1}')\) such that

- \(S \subset (N_{j-1}')^0\) and \(N_{j-1}'\) is isotopic to \(W\),
- \(\|X_{j-1}' - X_{j-1}\| < e_j/2\) on \(N_{j-1}'\),
- \(X_{j-1}'(\partial(N_{j-1}')) \subset D^{i-1} - (\overline{D^{i-1}} - e_{j-1})\),
- \(X_{j-1}'(N_{j-1}' - N_{j-1}^0) \subset D^{i-1} - (\overline{D^{i-1}} - e_{j-1})\),
- \(\|\partial(N_{j-1}') - \partial(N_{j-1})\| = \|X_{j-1}' - X_{j-1}\|\) so \(\zeta_{j-1}'\).

Fix an isotopic homeomorphism \(\sigma_{j-1}' : N_{j-1}' \rightarrow M_{j-1}'\) with \(\sigma_{j-1}'|N_{j-1} = \zeta\). To finish, notice that \(\chi(M_j - (M_{j-1}')^0) = 0\), set \((N_j, X_j)\) as the couple \((\tilde{M}, \tilde{F})\) arising from Lemma 3.1 for the data \((D, r, M, F, \tilde{D}, e, P_0) = (D^{i-1}, e_{j-1}, N_{j-1}', X_{j-1}', D^1, e_j/2, Q_0)\), and take \(\sigma_j : N_j \rightarrow M_j\) any homeomorphic extension of \(\sigma_{j-1}'\).

Label \(N = \bigcup_{j \in \mathbb{N}} N_j\) and set \(\sigma : N \rightarrow N\), \(\sigma|_{N_j} = \sigma_j\). Since \((M_j)_{j \in \mathbb{N}}\) is an exhaustion of \(N\) by incompressible compact regions and \(\sigma_j\) is an isotopic homeomorphism for all \(j\), then \(\sigma\) is an isotopic homeomorphism as well and item (a) holds.

By items (IIj), \(j \in \mathbb{N}\), the sequence \(\{X_j\}_{j \in \mathbb{N}}\) uniformly converges on compact subsets of \(N\) to a holomorphic map \(Y : N \rightarrow \mathbb{C}^3\) such that \(\langle dY, dY \rangle = 0\) and \(\|Y - X\| \leq \xi\) on \(M\), which corresponds to (b). Let us check that \(Y\) is an immersion. Indeed, note that the Weierstrass data of \(X_n\) converge to the ones of \(Y\). By Hurwitz’s Theorem, either \(\langle dY, dY \rangle = 0\) never vanishes on \(N\) or \(dY = 0\). However, the fact that \(\|Y - X\| \leq \xi\) on \(M\) prevents \(Y(N)\) to be a point provided that \(\xi\) is taken small enough from the beginning.
The completeness of $Y$ follows from $(V_j)$, $j \in \mathbb{N}$, and the fact that the series $\sum_{i \geq 1} \frac{1}{\sqrt{d_i}}$ is divergent (recall that $\{\Omega_j\} \subset \mathbb{N}$ is proper). Finally, let us check (d). Since $(\Pi_k), k > j$, we get that $\|Y - X_j\| \leq \frac{\xi}{2}$ on $\Omega_j$ and from (III) that $Y(\partial \Omega_j) \subset (\Omega_j)_{\frac{1}{2}-\frac{\xi}{2}}$. Thus $\Pi_p(Y(\partial \Omega_j)) \subset (\Omega_j)_{\frac{1}{2}-\frac{\xi}{2}} - (\Omega_j)_{\frac{1}{2}-\frac{\xi}{2}}$, and so by the maximum principle $\Pi_p(Y(\partial \Omega_j)) \subset (\Omega_j)_{\frac{1}{2}-\frac{\xi}{2}}$ for all $j$. Therefore, $\Pi_p(Y(\Omega_j)) \subset \Omega$, hence $\Pi_p(Y(\Omega_j)) \subset \Omega$ again by the maximum principle. From (IV) we deduce that $\Pi_p(Y(\Omega_j)) \subset \Omega - (\Omega_{j-1})_{\frac{1}{2}-\frac{\xi}{2}}$ for all $j \geq 1$, proving that $\Pi_p \circ Y : N \to \Omega$ is proper. Since $\Lambda \subset \Omega_j$ and $\frac{1}{\sqrt{d_i}} \xi \leq \frac{\xi}{2}$ then $\Lambda_{\frac{1}{2}-\frac{\xi}{2}} \subset (\Omega_j)_{\frac{1}{2}-\frac{\xi}{2}}$, and so $\Pi_p(Y(\Omega_j)) \subset \Omega - \Lambda_{\frac{1}{2}-\frac{\xi}{2}}$ for all $j \geq 1$, which proves (d) and the theorem.

Remark 3.4. The hypothesis that $\rho$ is wide and Proposition 2.15 allow us to use Lemma 3.1 during the proof of Theorem 3.2 (see Remark 2.16). Hence, they play a crucial role in this setting.

Moreover, recall that $\mathcal{N}$ is hyperbolic, then so is $N$.

Complete null curves in $\mathbb{C}^3$ project on complete holomorphic immersions in $\mathbb{C}^2$ and complete minimal immersions in $\mathbb{R}^3$. From Theorem 3.2 we infer that:

**Corollary 3.5.** The following assertions hold:

(a) For any convex domain $\Omega$ in $\mathbb{C}^3$ there exist a domain $N \subset \mathcal{N}$ homeomorphic to $\mathbb{N}$ and a complete proper null curve $F : N \to \Omega$.

(b) For any convex domain $\Omega$ in $\mathbb{R}^3$ there exist a domain $N \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ and a complete proper minimal immersion $X : N \to \Omega$ with vanishing flux.

(c) For any convex domain $\Omega$ in $\mathbb{C}^2$ there exist a domain $N \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ and a complete proper holomorphic immersion $F : N \to \Omega$.

(d) For any convex domain $\Omega$ in $\mathbb{R}^3$ there exist a domain $N \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ and a complete holomorphic immersion $F : N \to \mathbb{C}^2$ such that $\Re(F)(N) \subset \Omega$ and $\Re(F) : N \to \Omega$ is proper.

(e) For any convex domain $\Omega$ in $\mathbb{R}^3$ there exist a domain $N \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ and a conformal complete minimal immersion $X = (X_j)_{j=1,2,3} : N \to \mathbb{R}^3$ with vanishing flux such that $(X_1, X_2)(N) \subset \Omega$ and $(X_1, X_2) : N \to \Omega$ is proper.

**Proof.** Consider a bounded regular strictly convex domain $\Lambda$ with $\overline{\Lambda} \subset \Omega$, and fix $r \in (0, 1/\kappa(\Lambda))$ and $\xi > 0$. Set $\rho = \{1, 2, 3, 4, 5, 6\}, \{1, 3, 5\}, \{1, 2, 3, 4\}, \{1, 3\}$ and $\{1, 3\}$ in item (a), (b), (c), (d) and (e), respectively. In each case $\Omega \subset \mathbb{R}^d$ and we label $D = C_\mu(\Lambda)$.

Let $H : C \to \mathbb{C}^3$ be the properly embedded null curve given by $H(z) = (iz, z, \sqrt{z})^T$, and note that $M := H^{-1}(D_{r/2})$ is a a closed disc with $H(\partial (M)) \subset D - (D_{r/2})$. Without loss of generality we can assume that $M \subset \mathcal{N}$.

Let $Y : N \to \mathbb{C}^3$ be the null curve arising from Theorem 3.2 for the data $p, \Omega, \Lambda, r, M, X = H|_M$ and $\xi$. The immersion $F = \Pi_p \circ Y$ solves items (a), (b) and (c). For items (d) and (e) choose $F = \Pi_p \circ Y$, where $\hat{\rho} = \{1, 2, 3, 4\}$ and $\hat{\rho} = \{1, 3, 5\}$, respectively.

A null curve $Z : M \to \text{SL}(2, \mathbb{C})$ is bounded and complete if an only if so is its Bryant’s projection $B(Z) = Z \cdot Z^T$. Furthermore, if $F : M \to \mathbb{C}^3$ is a complete bounded null curve such that $F(M) \cap \{z_3 = 0\} = \emptyset$, then $T \circ F : M \to \text{SL}(2, \mathbb{C})$ is a complete bounded null curve as well (see (1.1) and [MUYI]).

**Corollary 3.6.** The following assertions hold:

- There exist a domain $N \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ and a complete bounded null curve $Z : N \to \text{SL}(2, \mathbb{C})$.
- There exist a domain $N \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ and a conformal complete bounded CMC-1 immersion $X : N \to \text{IH}^3$.

**Proof.** Use Corollary 3.5-(a) for an Euclidean ball $\Omega$ in $\mathbb{C}^3$ whose closure is disjoint from $\{z_3 = 0\}$, and take into account the transformations $B$ and $T$. 

\[ \Box \]
4. Proof of Lemma 3.1

Recall that Remark 2.9 gives that \( \text{dist}(p, \partial D \cap (p + (\nu_D(p))^\perp)) \geq \sqrt{d^2 + 2 \frac{d}{k(D)}} \) for all \( p \in D - \overline{D}_r \), where \( d = \text{dist}(\partial D, \hat{D}) \). Take \( \varepsilon_1 \in (0, \varepsilon) \) small enough so that \( \text{dist}(p, \partial D \cap (p + (\nu_D(p))^\perp)) > \sqrt{d^2 + 2 \frac{d}{k(D)}} + \varepsilon_1 \).

Then, by a continuity argument and equation (2.1), there exists an open neighborhood \( V_p \) of \( p \) in \( D - \overline{D}_r \) such that

\[
\hat{V}_p \cap \overline{D}_r = \emptyset \quad \text{and} \quad \text{dist}(q, \partial D \cap \hat{V}_p) > \sqrt{d^2 + 2 \frac{d}{k(D)}} + \varepsilon_1 \quad \text{for all} \quad q \in V_p,
\]

where \( \hat{V}_p = U_{(q_1, q_2) \in V_p \times V_p} (q_1 + (\nu_D(q_2))^\perp) \).

Label \( \Theta^* = \Theta \cap S^5 \), and consider the continuous function \( f : S^5 \times \Theta^* \to [-1, 1], f(\sigma, \theta) = (\sigma, \theta) \).

Remark 2.3 implies that \( \mu^* := \max\{ f(\sigma, \theta) \mid \theta \in \Theta^* \} > 0 \) for all \( \sigma \in S^5 \). Label \( \mu = \min\{ \mu^* \mid \sigma \in S^5 \} > 0 \), and for each \( p \in D - \overline{D}_r \) choose \( \theta_p \in \Theta^* \) and an open neighborhood \( U_p \) of \( p \) in \( D - \overline{D}_r \) so that

\[
f(\nu_D(q), \theta_p) > \mu/2 \quad \text{for all} \quad q \in U_p.
\]

Set \( W_p = V_p \cap U_p \subset D - \overline{D}_r \) for all \( p \in D - \overline{D}_r \) and call \( \mathcal{W} = \{ W_p \mid p \in D - \overline{D}_r \} \). Write \( a_1, \ldots, a_k \) as the connected components of \( \partial(M) \). For each \( m \in \mathbb{N} \) let \( \mathbb{Z}_m = \{0, 1, \ldots, m-1\} \) denote the additive cyclic group of integers modulus \( m \). From (3.1), \( \mathcal{W} \) is an open covering of \( \partial(M) \), and so there exist \( m \in \mathbb{N}, m \geq 3, \) and a collection \( \{ a_i \mid (i, j) \in \{1, \ldots, k\} \times \mathbb{Z}_m \} \) such that for any \( i \in \{1, \ldots, k\} : \)

- \( \bigcup_{j=1}^m a_i(j) = a_i \),
- \( a_{ij} \) and \( a_{ij+1} \) have a common endpoint \( Q_{ij} \) and are otherwise disjoint for all \( j \in \mathbb{Z}_m \),
- \( F(a_i \cup a_{ij+1}) \subset W_{ij} \subset \mathcal{W} \) for all \( j \in \mathbb{Z}_m \).

As \( D \) is \( <, > \)-regular, then we can find \( p_{ij} \in W_{ij-1} \cap W_{ij} \) such that \( (e_{ij})^\perp \) is not \( <, > \)-degenerate, where \( e_{ij} = \nu_D(p_{ij}), j \in \mathbb{Z}_m \) (see Figure 1).

![Figure 1. The sets \( W_{ij} \)]

Label \( w_{ij} = \overrightarrow{e_{ij}} / \langle \overrightarrow{e_{ij}}, \overrightarrow{e_{ij}} \rangle_{\mathbb{R}} \), for all \( i, j \) and notice that \( \langle e_{ij} \rangle^\perp = \langle w_{ij} \rangle^\perp \). Since \( w_{ij} \) is not null, we can take \( u_{ij}, v_{ij} \in \langle w_{ij} \rangle^\perp \) such that \( \{u_{ij}, v_{ij}, w_{ij}\} \) is \( <, > \)-conjugate in \( \mathbb{C}^3 \). Denote \( A_{ij} \) as the complex orthogonal matrix \( (u_{ij}, v_{ij}, w_{ij})^{-1} \), for all \( i \in \{1, \ldots, k\} \) and \( j \in \mathbb{Z}_m \).

Since \( W_{ij} \in \mathcal{W}, (2.4) \) yields \( \theta_{ij} \in \Theta^* \) so that \( f(\nu_D(q), \theta_{ij}) > \mu/2 \) for all \( q \in W_{ij} \). In particular,

\[
(4.3) \quad \langle e_{ij}, \theta_{ij} \rangle \text{ and } 
\langle e_{ij+1}, \theta_{ij} \rangle \text{ are both positive, } i = 1, \ldots, k, j \in \mathbb{Z}_m.
\]

Let \( \mathcal{W} \) be a tubular neighborhood of \( M \) and let \( C_1, \ldots, C_k \) denote the finite collection of open annuli in \( \mathcal{W} - M \), where up to relabeling \( a_i \subset \partial C_i, i = 1, \ldots, k \).
Let \( \{r_{ij} \subset W \mid i = 1, \ldots, k, j \in \mathbb{Z}_m \} \) be a collection of pairwise disjoint analytical Jordan arcs in \( W \) such that \( r_{ij} \) has initial point \( Q_{ij} \) and \( r_{ij} - \{Q_{ij}\} \subset C_i \) for all \( i \) and \( j \). Label \( T_{ij} \) as the final point of \( r_{ij} \), and split \( r_{ij} \) into two subarcs \( s_{ij} \) and \( t_{ij} \), where \( Q_{ij} \in s_{ij} \) and \( T_{ij} \in t_{ij} \). In addition, choose these arcs so that \( S = M \cup (\bigcup_{i,j} r_{ij}) \) is admissible.

**Claim 4.1.** There exists \( H \in \mathbb{N}(W) \) such that for any \( (i, j) \in \{1, \ldots, k\} \times \mathbb{Z}_m \):

(a1) \( H(s_{ij-1} \cup a_{ij} \cup s_{ij} \cup a_{ij+1} \cup s_{ij+1}) \subset W_{ij} \)

(a2) if \( j \subset s_{ij} \) is a Borel measurable subset, then

\[
\min\{L((A_{ij} \cdot H|_{|S_j})_3), L((A_{ij+1} \cdot H|_{|S_j})_3)\} + \\
\min\{L((A_{ij} \cdot H|_{|S_j})_3), L((A_{ij+1} \cdot H|_{|S_j})_3)\} > \max\{\|A_{ij}\|, \|A_{ij+1}\|\}\left(\frac{d}{\kappa(D)} + \epsilon_1\right),
\]

where \((\cdot)_3\) means third (complex) coordinate and \( L \) Euclidean length in \( C \).

(a3) \((H(P) - H(Q_{ij}), e_{ij}), (H(P) - H(Q_{ij}), e_{ij+1}) > 0 \) for all \( P \in t_{ij} \) and \((H(T_{ij}) + (e_{ij})^\perp) \cup (H(T_{ij}) + (e_{ij+1})^\perp) \subset \bigcup_{i} \)

(a4) \(|H - F| < \epsilon_1 / (1 + km) \) on \( M \).

**Proof.** Let \( r_{ij}(u), u \in [0, 1] \), be a smooth parameterization of \( r_{ij} \) so that \( r_{ij}([0, 1/2]) = s_{ij} \) and \( r_{ij}([1/2, 1]) = t_{ij} \). Let \( \lambda_{ij} \in \mathbb{R} \) be a null vector so that \( (A_{ij}(\lambda_{ij}))_3, (A_{ij+1}(\lambda_{ij}))_3 \neq 0 \) (see Remark 2.3) and the segment \( \{\lambda_{ij}(s) := F(Q_{ij}) + s\lambda_{ij} \mid s \in [0, 1]\} \) lies in the interior of \( W_{ij-1} \cap W_{ij} \cap W_{ij+1} \), and set \( \Lambda_{ij}(s) = \Lambda_{ij}(1 - s), s \in [0, 1] \). Take \( N \in \mathbb{N} \) large enough so that

\[
2N \min\{||A_{ij}(\lambda_{ij})_3||, ||A_{ij+1}(\lambda_{ij})_3||\} > \max\{||A_{ij}||, ||A_{ij+1}||\}\left(\frac{d}{\kappa(D)} + \epsilon_1\right)
\]

and

\[
||F(Q_{ij}) + N\theta_{ij} + (e_{ij})^\perp \cup (F(Q_{ij}) + N\theta_{ij} + (e_{ij+1})^\perp) || \subset \bigcup_{i}.
\]

For (4.5) take into account (4.3). Set \( d_{ij} : [0, 1] \rightarrow C^3 \),

- \( d_{ij}(u) = \Lambda_{ij}(8Nu + b - 1) \) if \( u \in \left[\frac{1}{bN}, \frac{1}{bN} + 1\right] \) and \( b \in \{1, \ldots, 2N\} \) is odd,
- \( d_{ij}(u) = \Lambda_{ij}(8Nu + b - 1) \) if \( u \in \left[\frac{1}{bN}, \frac{1}{bN} + 1\right] \) and \( b \in \{1, \ldots, 2N\} \) is even,
- \( d_{ij}(u) = F(Q_{ij}) + \frac{\epsilon_{ij}(4u - 1)}{\theta_{ij}} \) if \( u \in \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right] \), where \( \epsilon_{ij} > 0 \) is small enough so that \( d_{ij}(\left[\frac{1}{4}, \frac{1}{2}\right]) \subset W_{ij-1} \cap W_{ij} \cap W_{ij+1} \) and
- \( d_{ij}(u) = F(Q_{ij}) + \left(\epsilon_{ij} + N(2u - 1)\right)\theta_{ij} \) if \( u \in [1/2, 1] \).

Notice that the curves \( d_{ij} \) are continuous, weakly differentiable and satisfy that \(<d_{ij}'(u), d_{ij}'(u) > = 0 \).

Up to replacing \( H|_{r_{ij}} \) for \( d_{ij} \) for all \( i, j \), items (a1), (a2) and (a3) formally hold. To finish, approximate \( d_{ij} \) by a smooth curve \( c_{ij} \), matching smoothly with \( F \) at \( Q_{ij} \) and so that the map \( \tilde{H} : S \rightarrow C^3 \) given by \( \tilde{H}|_{M} = F, \tilde{H}|_{r_{ij}(u)} = c_{ij}(u) \) for all \( u \in [0, 1], i \) and \( j \), is a generalized null curve satisfying all the above items. Indeed, if \( c_{ij} \) is chosen close enough to \( d_{ij} \), (a1) is obvious, (a2) follows from (4.4), and (a3) is an elementary consequence of (4.5). The Claim follows by a direct application of Lemma 2.8 to \( S, \tilde{H} \) and \( W \).

Let \( M_0 \) be a compact region such that \( S - (\bigcup_{i,j} \{T_{ij}\}) \subset M_0^o = M_0 \subset M_0 \subset W_i \cup \{T_{ij}\} \subset \partial M_0 \) and \( M_0 - M^o \) consists of \( k \) pairwise disjoint compact annuli \( \hat{C}_1, \ldots, \hat{C}_k \), where \( \hat{C}_i \subset \mathbb{C} \). Label \( \Omega_{ij} \) as the closed disc in \( M_0 - M^o \) bounded by \( a_{ij}, r_{ij-1}, r_{ij} \) and a piece, named \( \rho_{ij} \) of \( \partial \hat{C}_i \) connecting \( T_{ij-1} \) and \( T_{ij} \).

Consider compact neighborhoods (with the topology of a closed disc) \( \tilde{a}_{ij}, \tilde{b}_{ij} \) and \( \tilde{I}_{ij} \) in \( M_0 - M^o \) of \( a_{ij}, \rho_{ij} \) and \( t_{ij} \), respectively, \( i = 1, \ldots, k, j \in \mathbb{Z}_m \), and satisfying the following properties (see Figure 2):

(b1) \( \tilde{a}_{ij} \cap (\tilde{I}_{ij-1} \cup \tilde{I}_{ij}) = \emptyset, \tilde{a}_{ij} \cap (\tilde{b}_{ij} \cup \tilde{I}_{ij}) = \emptyset, a \neq j - 1, j, (\tilde{b}_{ij-1} \cup \tilde{I}_{ij-1}) \cap (\tilde{b}_{ij} \cup \tilde{I}_{ij}) = \emptyset, \)
(b2) \( K_{ij} := \overline{\Omega_{ij} - (\tilde{s}_{ij-1} \cup \tilde{\alpha}_{ij} \cup \tilde{\beta}_{ij} \cup \tilde{t}_{ij})} \) is a compact disc and \( K_{ij} \cap \beta_{ij} \) is a Jordan arc disjoint from \( \{T_{ij-1}, T_{ij}\} \).

(b3) \( H(\tilde{s}_{ij-1} \cup \tilde{\alpha}_{ij} \cup \tilde{s}_{ij} \cup \tilde{\alpha}_{ij+1} \cup \tilde{s}_{ij+1}) \subset W_{ij} \).

(b4) if \( J \subset \tilde{s}_{ij} \) is an arc connecting \( \tilde{\alpha}_{ij} \cup \tilde{\alpha}_{ij+1} \) and \( \tilde{t}_{ij}, J_1 = J \cap \Omega_{ij} \) and \( J_2 = J \cap \Omega_{ij+1} \), then

\[
L((A_{ij} \cdot H)_{J_1}) + L((A_{ij+1} \cdot H)_{J_2}) > \max\{\|A_{ij}\|, \|A_{ij+1}\|\} \left( \sqrt[\kappa(D)]{d} + \epsilon_1 \right),
\]

(b5) \( (H(P) - H(Q_{ij-1}, e_{ij})) > 0 \forall P \in \tilde{t}_{ij-1}, (H(P) - H(Q_{ij}, e_{ij})) > 0 \forall P \in \tilde{t}_{ij}, \) and \( (H(Q) + (e_{ij})^\perp) \cap \overline{D_{1/\epsilon_1}} = \emptyset \forall Q \in (\tilde{t}_{ij-1} \cup \tilde{t}_{ij}) \cap \partial M_0 \).

These choices are possible due to properties (a1), (a2) and (a3) and a continuity argument.

![Figure 2. \( M_0 \)]

Let \( \eta : \{1, \ldots, km\} \to \{1, \ldots, k\} \times \mathbb{Z}_m \) be the bijection \( \eta(n) = (E(\frac{n-1}{m}) + 1, n - 1) \), where \( E(\cdot) \) means integer part.

Claim 4.2. There exists a sequence \( H_0, H_1, \ldots, H_{km} \) of null curves in \( N(W) \) satisfying the following properties:

(c1\(_n\)) \( H_n(s_{\eta(j)-0,1} \cup \tilde{\alpha}_{\eta(j)} \cup \tilde{s}_{\eta(j)-0,1} \cup \tilde{\alpha}_{\eta(j)} + (0,1) \cup \tilde{s}_{\eta(j)} + (0,1)) \subset W_{\eta(j)}, \forall j \in \{1, \ldots, km\} \).

(c2\(_n\)) \( H_n((\tilde{s}_{\eta(j)-0,1} \cup \tilde{\alpha}_{\eta(j)} \cup \tilde{s}_{\eta(j)} + (0,1)) \cap \Omega_{\eta(j)} \subset W_{\eta(j)} + \|e_{\eta(j)}\|, \forall j \in \{1, \ldots, n\}, n \geq 1 \).

(c3\(_n\)) \( H_n((\tilde{s}_{\eta(j)-0,1} \cup \tilde{\alpha}_{\eta(j)} \cup \tilde{s}_{\eta(j)} + (0,1)) \cap \Omega_{\eta(j)} \subset W_{\eta(j)}, \forall j \in \{n + 1, \ldots, km\}, \).

(c4\(_n\)) if \( J \subset \tilde{s}_{\eta(j)} \) is an arc connecting \( \tilde{\alpha}_{\eta(j)} \cup \tilde{s}_{\eta(j)} + (0,1) \) and \( \tilde{t}_{\eta(j)}, J_1 = J \cap \Omega_{\eta(j)} \) and \( J_2 = J \cap \Omega_{\eta(j) + (0,1)}, \) then

\[
L((A_{\eta(j)} \cdot H_n)_{J_1}) + L((A_{\eta(j) + (0,1)} \cdot H_n)_{J_2}) > \max\{\|A_{\eta(j)}\|, \|A_{\eta(j) + (0,1)}\|\} \left( \sqrt[\kappa(D)]{d} + \epsilon_1 \right),
\]

\( \forall j \in \{1, \ldots, km\}, \)

(c5\(_n\)) \( \langle H_n(P) - H_n(Q_{\eta(j)}, e_{\eta(j)}) \rangle > 0 \forall P \in \tilde{t}_{\eta(j)} \cap \Omega_{\eta(j)}, \forall j \in \{1, \ldots, km\}, \)

(c6\(_n\)) \( \langle H_n(P) - H_n(Q_{\eta(j)}, e_{\eta(j)}) \rangle > 0 \forall P \in \tilde{t}_{\eta(j) - (0,1)} \cap \Omega_{\eta(j)}, \forall j \in \{1, \ldots, km\}, \)

(c7\(_n\)) \( (H_n(Q) + \|e_{\eta(j)}\|) \cap \overline{D_{1/\epsilon_1}} = \emptyset \forall Q \in (\tilde{t}_{\eta(j)} \cup \tilde{t}_{\eta(j) - (0,1)} \cap \partial M_0 \cap \Omega_{\eta(j)}, \forall j \in \{1, \ldots, km\}, \)

(c8\(_n\)) \( H_n(K_{\eta(j)}) \cap \overline{D_{1/\epsilon_1}} = \emptyset, \forall j \in \{1, \ldots, n\}, n \geq 1, \) and

(c9\(_n\)) \( \|H_n - H_{n-1}\| < \epsilon_1/(km + 1) \) on \( M_0 - \Omega_{\eta(n)}, n \geq 1. \)

Proof. Take \( H_0 := H. \) Notice that (c1\(_n\)), (c3\(_n\)), (c4\(_n\)), (c5\(_n\)), (c6\(_n\)) and (c7\(_n\)) follow from (b3), (b4) and (b5), whereas the remaining properties make no sense for \( n = 0. \)

Reason by induction and assume we have constructed \( H_0, \ldots, H_{n-1} \) satisfying all the above properties. Let us construct \( H_n. \)

Label \( G = A_{\eta(n)} \cdot H_{n-1} \in N(W) \) and write \( G = (G_1, G_2, G_3)^T. \)
Let $\gamma$ denote a Jordan arc in $\mathcal{A}_{\eta(n)}$ disjoint from $(\mathcal{S}_{\eta(n)}-(0,1)) \cup \mathcal{I}_{\eta(n)}-(0,1)) \cup (\mathcal{S}_{\eta(n)} \cup \mathcal{I}_{\eta(n)})$, with initial point $\Gamma_1 \in a_{\eta(n)}$ and final point $\Gamma_2 \in \partial K_{\eta(n)}$ and otherwise disjoint from $\partial K_{\eta(n)}$. Choose $\gamma$ so that $dC_3|_{\eta}$ never vanishes. Denote $S_\eta$ as the closure of $(M_0 - \Omega_{\eta(n)}) \cup \gamma \cup K_{\eta(n)}$, and without loss of generality assume that $S_\eta$ is admissible.

Recall that $A_{\eta(n)} \in \mathcal{O}(3, \mathbb{C})$ implies that $A_{\eta(n)}(\langle e_{\eta(n)} \rangle) = \langle u(A(D)A_{\eta(n)}(\pi(D)(\eta(n)))) \rangle$, and from the definition of $A_{\eta(n)}, A_{\eta(n)}(\langle e_{\eta(n)} \rangle) = \{(z_1, z_2, 0) \in \mathbb{C}^3 | z_1, z_2 \in \mathbb{C} \}$. Since $A_{\eta(n)}(D)$ is null strictly convex (see Claim 2.14) and $A_{\eta(n)}(\overline{D}_1/\epsilon_1) \in (A_{\eta(n)}(D))_{|A_{\eta(n)}(D)/\epsilon_1}$, there exists $\xi \in \mathcal{E}_{A_{\eta(n)}(D)} \cap \Theta \cap A_{\eta(n)}(\langle e_{\eta(n)} \rangle) = \langle 1 \rangle$ such that

\[(4.6) \quad \text{dist}(G(\Gamma_1) + \zeta, A_{\eta(n)}(\overline{D}_1/\epsilon_1)) > \text{diam}(G(\gamma \cup K_{\eta(n)})), \]

where $\text{diam}(\cdot)$ means Euclidean diameter in $\mathbb{C}^3$.

Let $\gamma(u), u \in [0, 1]$, be a smooth parameterization of $\gamma$ with $\gamma(0) = \Gamma_1$. Label $\tau_j = \gamma([0, 1])$ and consider the parameterization $\tau_j(u) = \gamma(u/j), u \in [0, 1]$. Write $G_{3,j}(u) = G_3(\tau_j(u)), u \in [0, 1]$, and notice that $dG_{3,j}(0) = \frac{dG_{3,j}(0)}{du}$ for all $j \in \mathbb{N}$.

Since $e_{\eta(n)}$ is not null and $A_{\eta(n)} \in \mathcal{O}(3, \mathbb{C})$, there is a null vector $\zeta^* \in A_{\eta(n)}(\langle e_{\eta(n)} \rangle)$ so that $\{\zeta, \zeta^*\}$ is a basis of $A_{\eta(n)}(\langle e_{\eta(n)} \rangle)$ and $\langle \zeta, \zeta^* \rangle 
eq 0$.

Set $\zeta_j = \zeta - \frac{(dG_{3,j}(0))}{2} \zeta^* \in A_{\eta(n)}(\langle e_{\eta(n)} \rangle)$, $j \in \mathbb{N}$, and observe that $\lim_{j \to \infty} \zeta_j = \zeta$ and $\langle \zeta_j, \zeta^* \rangle = -\frac{(dG_{3,j}(0))}{2}$. For all $j$. Furthermore, by (4.6) we can also assume that $\text{dist}(G(\Gamma_1) + \zeta_j, A_{\eta(n)}(\overline{D}_1/\epsilon_1)) > \text{diam}(G(\gamma \cup K_{\eta(n)}))$ for all $j \in \mathbb{N}$.

Set $h_j : [0, 1] \to \mathbb{C}^3, h_j(u) = \Gamma_1 + \frac{G_{3,j}(u)}{2} \zeta_j + (0, 0, G_{3,j}(u) - G_3(\Gamma_1))$, and notice that $\langle h_j(u), h_j(u) \rangle > 0$ and $\langle h_j(u), h_j(u) \rangle \gg 0$ never vanishes on $[0, 1], j \in \mathbb{N}$. Up to choosing a suitable branch of $\langle \zeta_j, \zeta_j \rangle > 1/2$, the sequence $\{h_j\}_{j \in \mathbb{N}}$ converges uniformly on $[0, 1]$ to $h_\infty : [0, 1] \to \mathbb{C}^3, h_\infty(u) = u \zeta + G(\Gamma_1)$. As $\text{dist}(h_\infty(1), A_{\eta(n)}(\overline{D}_1/\epsilon_1)) > \text{diam}(G(\gamma \cup K_{\eta(n)}))$ (see equation (4.6)), then there exists $j_0 \in \mathbb{N}$ such that

\[(4.7) \quad \text{dist}(h_{j_0}(1), A_{\eta(n)}(\overline{D}_1/\epsilon_1)) > \text{diam}(G(\gamma \cup K_{\eta(n)})). \]

Set $\hat{h} : \tau_{j_0} \to \mathbb{C}^3, \hat{h}(P) = h_{j_0}(u(P))$, where $u(P) \in [0, 1]$ is the only value for which $\tau_{j_0}(u(P)) = P$. Let $\hat{\mathcal{G}} = (\hat{G}_1, \hat{G}_2, \hat{G}_3) : S_n \to \mathbb{C}^3$ denote the continuous map given by:

$\hat{\mathcal{G}}_{\mid M_0 - \Omega_{\eta(n)}} = G_{\mid M_0 - \Omega_{\eta(n)}}$,
$\hat{\mathcal{G}}_{\mid \tau_{j_0}} = \hat{h}$,
$\hat{\mathcal{G}}_{\mid (\gamma - \tau_{j_0}) \cup K_{\eta(n)}} = G_{\mid (\gamma - \tau_{j_0}) \cup K_{\eta(n)}} - G(\tau_{j_0}(1)) + \hat{h}(\tau_{j_0}(1))$.

Notice that $\hat{\mathcal{G}}_3 = G_3|_{S_n}$. The equation $\langle d\hat{\mathcal{G}}, d\hat{\mathcal{G}} \rangle = 0$ formally holds except at the points $\Gamma_1$ and $\tau_{j_0}(1)$ where smoothness could fail. Up to smooth approximation, $\hat{\mathcal{G}}$ is a generalized null curve satisfying that

$\hat{\mathcal{G}}_{\mid M_0 - \Omega_{\eta(n)}} = G_{\mid M_0 - \Omega_{\eta(n)}}$,
$\hat{\mathcal{G}}_3 = G_3|_{S_n}$,
$\hat{\mathcal{G}}(K_{\eta(n)}) \cap A_{\eta(n)}(\overline{D}_1/\epsilon_1) = \emptyset$.

Take $\epsilon_0 > 0$ to be specified later. Applying Lemma 2.8 to $S_n, W$ and $\hat{\mathcal{G}}$, we get a null curve $Z = (Z_1, Z_2, Z_3)^T \in \mathbb{N}(W)$ such that

\[(4.8) \quad \|Z - A_{\eta(n)} \cdot H_{n-1}\| < \epsilon_0 \|M_0 - \Omega_{\eta(n)}\|, Z_3 = (A_{\eta(n)} \cdot H_{n-1}Z_3), Z(K_{\eta(n)}) \cap A_{\eta(n)}(\overline{D}_1/\epsilon_1) = \emptyset. \]

Define $H_n := A_{\eta(n)}^{-1} \cdot Z \in \mathbb{N}(W)$. From (4.8), $H_n$ trivially satisfies that:

\[(d1) \quad \|H_n - H_{n-1}\| < \epsilon_0 \|A_{\eta(n)}^{-1}\| \text{ on } M_0 - \Omega_{\eta(n)}, \]
\[(d2) \quad \langle H_n - H_{n-1}, e_{\eta(n)} \rangle \gg 0 \text{ (see the definition of } A_{\eta(n)}), \]
\[(d3) \quad H_n(K_{\eta(n)}) \cap (\overline{D}_1/\epsilon_1) = \emptyset. \]
Let us check that \( H_n \) fulfills properties (c1) to (c9) provided that \( \epsilon_0 \) is chosen small enough. Item (c1\(_{n-1}\)) and (d1) imply (c1\(_n\)) for small enough \( \epsilon_0 \). Item (c2\(_n\)) follows from (d1) and (c2\(_{n-1}\)) for \( j < n \) provided that \( \epsilon_0 \) is small enough, and from (d2) and (c5\(_{n-1}\)) for \( j = n \). Item (c5\(_{n-1}\)) and (d1) also give (c3\(_n\)) for small enough \( \epsilon_0 \). Observe that (d1) shows that \( \| A\_n(j) \cdot H_n - A\_n(j) \cdot H_{n-1} \| < \epsilon_0 \| A\_n^{-1} \| \|A\_n(j)\| \) and \( \| A\_n(j) + (0,1) \cdot H_n - A\_n(j) + (0,1) \cdot H_{n-1} \| < \epsilon_0 \) \( \| \tilde{\gamma} \_n \| \) on \( \Omega\_n \) for \( j \neq n \), whereas (d2) gives that \( (A\_n(j) \cdot H_n) = (A\_n(j) \cdot H_{n-1}) \) on \( \Omega\_n \). Then, taking into account (c4\(_{n-1}\)) we get (c4\(_n\)) provided that \( \epsilon_0 \) is small enough.

To prove (c5\(_n\)) (respectively, (c6\(_n\), (c7\(_n\))) we distinguish cases. If \( j \neq n \), use (d1) and (c5\(_{n-1}\)) (respectively, (c6\(_{n-1}\), (c7\(_{n-1}\))) for small enough \( \epsilon_0 \), whereas for \( j = n \) we use (d2) and (c5\(_{n-1}\)) (respectively, (c6\(_{n-1}\), (c7\(_{n-1}\))). Item (c8\(_n\)) follows from (c8\(_{n-1}\)) and (d1) and (d3) for \( \epsilon_0 \) small enough. Finally, (c9\(_n\)) is an immediate consequence of (d1) for \( \epsilon_0 < \frac{1}{\| A\_n^{-1} \|} \).

The proof of Claim 4.2 is done. \( \square \)

Notice that \( H\_km(\partial M \cup (\bigcup_{j=1}^{km} s\_n(j))) \subset D - \overline{D} - r \) and \( H\_km(\partial M \cup (\bigcup_{j=1}^{km} K\_n(j))) \cap \overline{D} / \epsilon_1 = \emptyset \). Indeed, use (c1\(_{km}\)) and that \( \bigcup_{j=1}^{km} W\_n(j) \subset D - \overline{D} - r \) for the first assertion, and (c7\(_{km}\)) and (c8\(_{km}\)) for the second one. Call \( \hat{M}_0 \) as the connected component of \( H\_km^{-1}(\overline{D}) \) containing \( M \), and observe that

\[
H\_km(\partial \hat{M}_0) \subset \partial \hat{D},
\]

and by the convex hull property \( \hat{M}_0 - M^s \) consists of \( k \) pairwise disjoint closed annuli.

Consider the null curve \( \hat{f}_0 := H\_km|\hat{M}_0 \in N(\hat{M}_0) \), and observe that \( \hat{f}_0 \) satisfies the following properties:

(P1) \( \| \hat{f}_0 - F \| < \epsilon_1 \) on \( M \). Use (a4) and (c9\(_n\)), \( n = 1, \ldots, km \).

(P2) \( \hat{f}_0(\hat{M}_0 - M^s) \subset \overline{D} - \overline{D} - r \). It suffices to check that \( \hat{f}_0(\hat{M}_0 - M^s) \cap \overline{D} - r = \emptyset \). Let \( P \in \hat{M}_0 - M^s \). Taking into account (4.10), there are three possible locations for the point \( P \):

- \( P \in \hat{f}_0(\hat{M}_0) \cap \Omega\_n \) for some \( n \in \{1, \ldots, km\} \). Then, combining (c2\(_{km}\)) and (4.1), we get \( \hat{f}_0(P) \notin \overline{D} - r \).
- \( P \in \hat{f}_0(\hat{M}_0) \cap \Omega\_n \) for some \( n \in \{1, \ldots, km\} \). Use that \( \hat{f}_0(Q\_n(n)) \in W\_n(n) \) (see (c1\(_{km}\))), \( \hat{f}_0(Q\_n(n)) + \epsilon_1 \hat{\gamma\_n(n)} \) \( \cap \overline{D} - r = \emptyset \) (see (4.1)) and \( \| \hat{f}_0(P) - \hat{f}_0(Q\_n(n)) \| \hat{\epsilon\_n(n)} > 0 \) (see (c5\(_{km}\))), and recall that \( \hat{\gamma\_n(n)} \) is the outward pointing normal to \( \partial \hat{D} \) at \( \pi\_D(P\_n(n)) \).
- \( P \in \hat{f}_0(\hat{M}_0 - M^s) \cap \Omega\_n \). Use that \( \hat{f}_0(Q\_n(n)) \in W\_n(n) \) and argue as above but using (c6\(_{km}\)) instead of (c5\(_{km}\)).

(P3) \( \text{dist}(\hat{M}_0, \hat{f}_0(\hat{M}_0)) > \text{dist}(\gamma, \gamma) \). Taking into account (P1), it suffices to prove that \( \text{dist}(\hat{M}_0, \hat{f}_0(\hat{M}_0)) > \sqrt{\text{dist}(\gamma)} + \epsilon_1 \). Let \( P \in \hat{M}_0 \) and \( \gamma \subset \hat{M}_0 - M^s \) be a connected curve connecting \( \partial M \) and \( P \). There are two possible locations for the point \( P \):

- If \( P \in \hat{f}_0(\hat{M}_0) \) (see Figure 3) for some \( n \in \{1, \ldots, km\} \), then by (4.10) there exists a sub-arc \( \hat{\gamma} \subset \gamma \) connecting \( \tilde{\gamma\_n(n)} \cup \tilde{\gamma\_n(n)} + (0,1) \) and \( \hat{f}_0(\hat{M}_0) \). Thus, Remark 2.5 and (c4\(_{km}\)) give that \( \sqrt{\text{dist}(\gamma)} + \epsilon_1 < L(\hat{f}_0(\hat{M}_0)) \). Suppose now that \( P \in \Omega\_n \cap (\tilde{\gamma\_n(n)} + (0,1) \cup \tilde{\gamma\_n(n)} \cup \tilde{\gamma\_n(n)}) \) (see Figure 4) for some \( n \in \{1, \ldots, km\} \).

By the preceding case, we can restrict ourselves to the case \( \gamma \cap (\bigcup_{j=1}^{km} \tilde{\gamma\_n(n)}) = \emptyset \). By (4.10) again, \( P \notin \tilde{\gamma\_n(n)} + (0,1) \cup \tilde{\gamma\_n(n)} \cup \tilde{\gamma\_n(n)} \) and therefore there exists a sub-arc \( \hat{\gamma} \subset \gamma \) contained in \( \Omega\_n \cap (\tilde{\gamma\_n(n)} + (0,1) \cup \tilde{\gamma\_n(n)} \cup \tilde{\gamma\_n(n)}) \) connecting a point \( Q \in \tilde{\gamma\_n(n)} + (0,1) \cup \tilde{\gamma\_n(n)} \cup \tilde{\gamma\_n(n)} \) and \( P \). Since (c1\(_{km}\)), \( \hat{f}_0(Q) \in W\_n(n) \), and (c2\(_{km}\)) and (4.9) give \( \| \hat{f}_0(\hat{M}_0) - \partial \hat{D} \cap W\_n(n) \| < \epsilon_1 \). Therefore (4.1) implies that \( \sqrt{\text{dist}(\gamma)} + \epsilon_1 < \text{dist}(\hat{f}_0(\hat{M}_0)) \leq L(\hat{f}_0(\hat{M}_0)) \).

Define \( \hat{f} := \hat{f}_0|\hat{M} \in N(\hat{M}) \), where \( \hat{M} \subset \hat{M}_0 \) is close enough to \( \hat{M}_0 \) so that \( \hat{f}(\hat{M}) \subset D - \overline{D} - \epsilon_1 \) and \( \hat{f} \) satisfies (P1), (P2) and (P3). The region \( \hat{M} \) and the null curve \( \hat{f} \) solves the Lemma.
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FIGURE 3. \( P \in \tilde{t}_{\eta(n)} \).

FIGURE 4. \( P \in \Omega_{\eta(n)} \cap (\tilde{s}_{\eta(n)} - (0,1) \cup \tilde{\alpha}_{\eta(n)} \cup \tilde{s}_{\eta(n)}). \)

REFERENCES

[AFL] A. Alarcón, I. Fernández and F.J. López, Complete minimal surfaces and harmonic functions. Preprint (arXiv:0910.4284).

[AFM] A. Alarcón, L. Ferrer and F. Martín, Density theorems for complete minimal surfaces in $\mathbb{R}^3$. Geom. Funct. Anal. 18 (2008), 1-49.

[AG] A. Alarcón and J.A. Gálvez, Proper harmonic maps from hyperbolic Riemann surfaces into the Euclidean plane. Preprint (arXiv:0906.2638).

[AL] A. Alarcón and F.J. López, Minimal surfaces in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$. Preprint (arXiv:0910.4124).

[Bi] E. Bishop, Mappings of partially analytic spaces. Amer. J. Math. 83 (1961), 209-242.

[Bo] J. Bourgain, On the radial variation of bounded analytic functions on the disc. Duke Math. J. 69 (1993), 671-682.

[Br] R. Bryant, Surfaces of mean curvature one in hyperbolic space. Théorie des variétés minimales et applications (Palaiseau, 1983-1984), Astérisque 154-155 (1987), 321-347.

[Ca] E. Calabi, Problems in differential geometry. Ed. S. Kobayashi and J. Ells, Jr., Proceedings of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965. Nippon Hyoronsha Co., Ltd., Tokyo (1966) 170.

[CM] T.H. Colding and W.P. Minicozzi II, The calabi-yau conjectures for embedded surfaces. Ann. of Math. (2) 167 (2008), 211-243.

[FMM] L. Ferrer, F. Martín and W.H. Meeks III, Existence of proper minimal surfaces of arbitrary topological type. Preprint (arXiv:0903.4194).

[Jo] P. W. Jones, A complete bounded complex submanifold of $\mathbb{C}^3$. Proc. Amer. Math. Soc. 76 (1979), 305-306.

[JX] L.P.M. Jorge and F. Xavier, A complete minimal surface in $\mathbb{R}^3$ between two parallel planes. Ann. of Math. (2) 112 (1980), 203-206.

[LMM] F.J. López, F. Martín and S. Morales, Adding handles to Nadirashvili's surfaces. J. Differential Geom. 60 (2002), 155-173.

[MM1] F. Martin and S. Morales, Complete proper minimal surfaces in convex bodies of $\mathbb{R}^3$. Duke Math. J. 128 (2005), 559-593.

[MM2] F. Martin and S. Morales, Complete proper minimal surfaces in convex bodies of $\mathbb{R}^3$. II. The behavior of the limit set. Comment. Math. Helv. 81 (2006), 699-725.

[MUY1] F. Martin, M. Umehara and K. Yamada, Complete bounded null curves immersed in $\mathbb{C}^3$ and $\text{SL}(2, \mathbb{C})$. Calc. Var. Partial Differential Equations 36 (2009), 119-139.

[MUY2] F. Martin, M. Umehara and K. Yamada, Complete bounded holomorphic curves immersed in $\mathbb{C}^2$ with arbitrary genus. Proc. Amer. Math. Soc. 137 (2009), 3437-3450.

[MPR] W.H. Meeks III, J. Pérez and A. Ros, The embedded calabi-yau conjectures for finite genus. Preprint.

[MY] W.H. Meeks III and S.-T. Yau, The classical Plateau problem and the topology of three-dimensional manifolds. The embedding of the solution given by Douglas Morrey and an analytic proof of Dehn's lemma. Topology 21 (1982), 409-442.
[Mi] H. Minkowski, *Volumen und Oberfläche*. Math. Ann. 57 (1903), 447-495.

[Nad] N. Nadirashvili, *Hadamard’s and Calabi-Yau’s conjectures on negatively curved and minimal surfaces*. Invent. Math. 126 (1996), 457-465.

[Nar] R. Narasimhan, *Imbedding of holomorphically complete complex spaces*. Amer. J. Math. 82 (1960), 917-934.

[PP] F. Pacard and F.A.A. Pimentel, *Attaching handles to constant-mean-curvature-1 surfaces in hyperbolic 3-space*. J. Inst. Math. Jussieu 3 (2004), 421–459.

[Re] R. Remmert, *Sur les espaces analytiques holomorphiquement séparables et holomorphiquement convexes*. C. R. Acad. Sci. Paris 243 (1956), 118-121.

[RYU] W. Rossman, M. Umehara and K. Yamada, *Mean curvature 1 surfaces in hyperbolic 3-space with low total curvature, I*. Adv. Stud. Pure Math. 34 (2002), 245–253.

[SY] R. Schoen and S.T. Yau, *Lectures on Harmonic Maps*. Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[UY] M. Umehara and K. Yamada, *Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space*. Ann. of Math. (2) 137 (1993), 611-638.

[Ya1] S.-T. Yau, *Problems section. Seminar on Differential Geometry*. Ann. of Math. Stud. 102, Princeton Univ. Press, Princeton, N.J., 1982, 669-706.

[Ya2] S.-T. Yau, *Review of geometry and analysis*. Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, 353-401.

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