Kobayashi-Hitchin correspondence for analytically stable bundles

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Abstract

We prove the existence of a Hermitian-Einstein metric on holomorphic vector bundles with a Hermitian metric satisfying the analytic stability condition, under some assumption for the underlying Kähler manifolds. We also study the curvature decay of the Hermitian-Einstein metrics. It is useful for the study of the classification of instantons and monopoles on the quotient of 4-dimensional Euclidean space by some types of closed subgroups. We also explain examples of doubly periodic monopoles corresponding to some algebraic data.

Keywords: Hermitian-Einstein metric, instanton, monopole, analytically stable bundle, Kobayashi-Hitchin correspondence.

MSC: 53C07, 14D21.

1 Introduction

Let $Y$ be a Kähler manifold equipped with a Kähler form $\omega_Y$. Let $(E, \partial E)$ be a holomorphic vector bundle on $Y$. Let $A^{p,q}(E)$ denote the space of $C^\infty$-sections of $E \otimes \Omega^{p,q}$. Let $\Lambda : A^{p,q}(E) \to A^{p-1,q-1}(E)$ denote the operator obtained as the adjoint of the multiplication of $\omega$. (See [14].)

Let $h$ be a Hermitian metric of $E$. Let $F(h) \in A^{1,1}(\text{End}(E))$ denote the curvature of the Chern connection of $E$ which is a unique unitary connection determined by $\partial E$ and $h$. The metric $h$ is called a Hermitian-Einstein metric if the following condition is satisfied:

$$\Lambda F(h) = \frac{\text{tr} \Lambda F(h)}{\text{rank } E} \text{id}_E.$$ 

If the base space is a smooth projective manifold, or more generally a compact Kähler manifold, according to the celebrated theorem of Donaldson [8] and Uhlenbeck-Yau [26], $(E, \partial E)$ has a Hermitian-Einstein metric if and only if $(E, \partial E)$ is stable with respect to the Kähler form $\omega$.

In the fundamental work [24], Simpson generalized the theorem to several directions. He introduced the concept of Hermitian-Einstein metrics for Higgs bundles. Under some assumption for the base Kähler manifold which are not necessarily compact, he introduced the analytic stability condition for Higgs bundles $(E, \overline{\partial} E, \theta)$ equipped with a Hermitian metric $h_0$. Then, he proved that if $(E, \overline{\partial} E, \theta)$ is analytically stable, then $(E, \overline{\partial} E, \theta)$ has a Hermitian-Einstein metric $h$ such that (i) $h$ and $h_0$ are mutually bounded, (ii) $\det(h) = \det(h_0)$, (iii) $(\overline{\partial} + \theta)(hh_0^{-1})$ is $L^2$. (See [24 Theorem 1].) The result of Simpson has been quite useful in the study of the Kobayashi-Hitchin correspondences for tame and wild harmonic bundles on projective manifolds in [15, 17, 18].

As mentioned, in the theorem of Simpson, the base Kähler manifold should satisfy some conditions. For instance, the volume should be finite. (See [24 §2 for more details.) In the study of tame and wild harmonic bundles on projective manifolds, it is not so restrictive. Indeed, because the condition for pluri-harmonic metrics depends only on the complex structure of the base space, the role of Kähler metrics is rather auxiliary, and we may choose an appropriate Kähler metric satisfying the conditions.

However, the condition for Hermitian-Einstein metrics depend on the Kähler metrics. There are many natural non-compact Kähler manifolds such that we cannot directly apply the theorem of Simpson to the construction of Hermitian-Einstein metrics for Higgs bundles on the spaces. For example, we may mention the quotient space $\mathbb{C}^2/\Gamma$, where $\Gamma \simeq \mathbb{Z}^a$ ($a < 4$) because the volume of such spaces are infinite.

There are many interesting studies of the Hermitian-Einstein metrics on vector bundles over Kähler manifolds with infinite volume. For example, see [11, 13, 16, 21, 22, 23]. We may also mention the construction of monopoles.
on $\mathbb{R}^3$ in [13] as a related work. However, the author does not find any systematic study on the relation between the existence of a Hermitian-Einstein metric and the analytic stability condition in a generalized context.

In this paper, we introduce a weaker condition for non-compact Kähler manifolds (Assumption 2.1), and we study the existence of Hermitian-Einstein metrics for analytically stable holomorphic vector bundles. Suppose that $(X, g_X)$ is a Kähler manifold satisfying the condition in Assumption 2.1. Let $(E, \overline{\partial} E)$ be a holomorphic vector bundle on $X$ with a Hermitian metric $h_0$ satisfying the analytic stability condition. (See Definition 2.2.) Then, we prove that $(E, \overline{\partial} E)$ has a Hermitian-Einstein metric $h$ such that (i) $\det(h) = \det(h_0)$, (ii) $h$ and $h_0$ are mutually bounded, (iii) $\overline{\partial} E(h h_0^{-1})$ is $L^2$ (Theorem 2.3). We also study the curvature decay (Propositions 2.6, 2.8), and the uniqueness of such metrics (Proposition 2.11). The Higgs case will be studied elsewhere.

Because $\mathbb{R} \times T^3$ and $\mathbb{R}^2 \times T^2$ satisfy the condition for the underlying Kähler manifolds as explained in [3], where $T^j$ denotes a $j$-dimensional real torus, Theorem 2.5 is useful in the study of instantons and monopoles on $\mathbb{C}^2 / \Gamma$ for some types of $\Gamma$. Indeed, the author has already applied it to the study of monopoles with Dirac type singularity on $S^1 \times \mathbb{R}^2$ in [20]. We also explain a way to construct examples of doubly periodic monopoles from some algebraic data in [3, 5, 6, 9].

For the proof of Theorem 2.5, instead of the method of the heat equation, we apply the deep result of Donaldson on the Dirichlet problem for Hermitian-Einstein metrics [9]. Suppose that $(E, \overline{\partial} E, h_0)$ is analytically stable. For simplicity, let us consider the case $\text{Tr } F(h_0) = 0$. We take a sequence of closed submanifolds with boundary $X_i \subset X$ ($i = 1, 2, \ldots$) such that $\bigcup X_i = X$. According to [3], we have a Hermitian metric $h_i$ of $E|_{X_i}$ satisfying $\Lambda F(h_i) = 0$ and $h_i|_{\partial X_i} = h_0|_{\partial X_i}$. Note that we have $\det(h_i) = \det(h_0|_{X_i})$. It is natural to ask the convergence of the sequence $\{h_i\}$ on any compact subset, which should have the desired property. We give a useful argument to obtain the convergence from a boundedness of the sequence $h_i$ in [22]. Hence, our issue is to obtain a $C^0$-bound from the analytic stability condition. We consider the Donaldson functional $M(h^{(1)}, h^{(2)})$ on each $X_i$. We always have the inequality $M(h_{0|X_i}, h_i) \leq 0$. It allows us to deduce the desired $C^0$-bound (Proposition 2.22), for which we essentially apply the argument of Simpson in the proof of [24, Proposition 5.3] by adjusting to the condition in Assumption 2.1.

Remark 1.1 In the proof of [3, Theorem 0.12, Proposition 5.12], a generalization of the theorem of Simpson [24, Theorem 1] is mentioned on the basis of [2]. (See Remark 2.1 below.) It is not clear how it is related with our result in this paper. Anyway, the author thinks it useful to give a general statement with a proof for further studies on similar issues. (See [20], for example.)

Remark 1.2 We may expect that the results in this paper might be useful for the construction of instantons on ALG-spaces and ALH-spaces. It is not clear if the analytic stability condition is useful or not in the case of ALF-spaces. For example, in the construction of monopoles on $\mathbb{R}^3$ in [13], it seems that the analytic stability condition has no role apparently. We should note that Bando [1] gave the construction of instantons on ALE spaces.

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2 Kobayashi-Hitchin correspondence for analytically stable bundles

2.1 Assumption on the base space

Let $G$ be a compact Lie group. Let $(X, g_X)$ be an $n$-dimensional connected Kähler manifold. Suppose that $(X, g_X)$ is equipped with a left $G$-action $\kappa$ in the following sense.

- $\kappa : G \times X \to X$ is a $C^\infty$-map satisfying $\kappa(a, \kappa(b, x)) = \kappa(ab, x)$ and $\kappa(1, x) = x$. Set $\kappa(a, x) := \kappa_a(x)$.
- $\kappa_a : X \to X$ is holomorphic for each $a \in G$.
- $\kappa_a g_X = g_x$.

Let $\omega_X$ denote the Kähler form, and let $\text{dvol}_X$ denote the volume form associated to $g_X$. We shall often denote $\int_X f \, \text{dvol}_X$ by $\int_X f$. We set $\Delta_X := -\sqrt{-1}\Lambda\partial_X\overline{\partial}_X$.

**Assumption 2.1** We are given a $G$-invariant function $\varphi : X \to [0, \infty[$ with $\int_X \varphi < \infty$, and positive constants $C_i$ $(i = 1, 2)$, such that the following holds.

- Let $f : X \to [0, \infty[$ be a bounded function such that $\Delta_X f \leq B \varphi$ for a positive number $B$ as a distribution. Then, we have
  \[
  \sup_{P \in X} f(P) \leq C_1 B + C_2 \int_X f \varphi.
  \]
  Moreover, if the bounded function $f$ satisfies the stronger condition $\Delta_X f \leq 0$ on $X$, we have $\Delta_X f = 0$.

See [3] for examples satisfying the assumption.

2.2 Analytic stability condition for $G$-equivariant bundles

Let $(E, \overline{\partial}_E)$ be a holomorphic vector bundle on $X$ with a Hermitian metric $h_0$. It is assumed to be $G$-equivariant in the following sense.

- We are given $C^\infty$-isomorphisms $\Theta : \kappa^{-1}(E) \simeq p_2^{-1}(E)$, here $p_2$ is the projection $G \times X \to X$. Let $\Theta_a$ denote the restriction of $\Theta$ to $\{a\} \times X$.
- We have $\Theta_{a_1 \cdot a_2} = \Theta_{a_2} \circ \kappa_{a_2}^{-1}(\Theta_{a_1})$.
- $\Theta_a$ gives a holomorphic isomorphism $\kappa_a^{-1}(E, \overline{\partial}_E) \simeq (E, \overline{\partial}_E)$, and an isometry $\kappa_a^{-1}(E, h_0) \simeq (E, h_0)$.

Let $F(h_0)$ denote the curvature of the Chern connection of $(E, \overline{\partial}_E, h_0)$. We assume the following.

- We have $B > 0$ such that $|\Lambda F(h_0)|_{h_0} \leq B \varphi$.

As in [24], we set
\[
\deg(E, h_0) := \sqrt{-1} \int_X \text{Tr}(\Lambda F(h_0)).
\]

Let $V$ be any $\mathcal{O}_X$-submodule of $E$ which is saturated, i.e., $E/V$ is torsion-free. We have a closed complex analytic subset $Z(V) \subset X$ such that (i) dim $Z(V) \leq \dim X - 2$, (ii) $V_{|X \setminus Z(V)}$ is a locally free $\mathcal{O}$-module. We have the induced metric $h_{0,V}$ of $V_{|X \setminus Z(V)}$. We set
\[
\deg(V, h_0) := \sqrt{-1} \int_{X \setminus Z(V)} \text{Tr}(\Lambda F(h_{0,V})).
\]

The integral is well defined in $\mathbb{R} \cup \{-\infty\}$ by the Chern-Weil formula:
\[
\deg(V, h_0) = \sqrt{-1} \int_{X \setminus Z(V)} \text{Tr}(\pi_V \Lambda F(h_0)) - \int_{X \setminus Z(V)} |\overline{\partial}_E \pi_V|^2_{h_0, g_X}.
\]

Here, $\pi_V$ denote the orthogonal projection of $E_{|X \setminus Z(V)}$ to $V_{|X \setminus Z(V)}$, and $|\cdot|_{h_0, g_X}$ denote the norm induced by $h_0$ and $g_X$.

We say that an $\mathcal{O}_X$-submodule $V$ of $E$ is $G$-equivariant if $\Theta_g(\kappa_g^* V) = V$ for any $g \in G$. 

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Let Chern-Weil formula, we have and satisfies flat with respect to the Chern connections of \((E, \overline{\partial}_E)\).

If the equality holds, we have Proposition 2.4 Let \((E, \overline{\partial}_E, h_0)\) is analytically polystable if it is analytically semistable with respect to the \(G\)-action, and \(G\)-equivariantly isomorphic to the direct sum \(\bigoplus (E_i, \overline{\partial}_{E_i}, h_{0,i})\), where each \((E_i, \overline{\partial}_{E_i}, h_{0,i})\) is analytically stable with respect to the \(G\)-action.

2.3 Hermitian-Einstein metrics

Let \((E, \overline{\partial}_E)\) be a \(G\)-equivariant holomorphic vector bundle on \(X\) with a \(G\)-invariant Hermitian metric \(h\). Let \(F(h)^{\perp}\) denote the trace free part of \(F(h)\). We say that \(h\) is a Hermitian-Einstein metric if \(\Lambda F(h)^{\perp} = 0\), i.e., \(\Lambda F(h) = \text{Tr} \Lambda (F(h)) i d_E / \text{rank} E\).

Lemma 2.3 Suppose that \(h\) is a Hermitian-Einstein metric of \((E, \overline{\partial}_E)\), and that \(|\text{Tr} \Lambda F(h)| < B\varphi\) for some \(B > 0\). Then, \((E, \overline{\partial}_E, h)\) is analytically polystable with respect to the \(G\)-action.

Proof Let \(V\) be any saturated \(G\)-equivariant \(O_X\)-submodule of \(E\) such that \(0 < \text{rank} V < \text{rank} E\). By the Chern-Weil formula, we have

\[
\frac{\deg(V, h_0)}{\text{rank} V} < \frac{\deg(E, h_0)}{\text{rank} E}
\]

holds for any saturated \(G\)-equivariant \(O_X\)-submodule \(V\) of \(E\) such that \(0 < \text{rank} V < \text{rank} E\). We say that \((E, \overline{\partial}_E, h_0)\) is analytically semistable with respect to the \(G\)-action if

\[
\frac{\deg(V, h_0)}{\text{rank} V} \leq \frac{\deg(E, h_0)}{\text{rank} E}
\]

holds for any saturated \(G\)-equivariant \(O_X\)-submodule \(V\) of \(E\). We say that \((E, \overline{\partial}_E, h_0)\) is analytically polystable if \(\text{deg}(V, h_0) / \text{rank} V < \text{deg}(E, h_0) / \text{rank} E\).

Definition 2.2 We say that \((E, \overline{\partial}_E, h_0)\) is analytically stable with respect to the \(G\)-action if

\[
\frac{\deg(V, h_0)}{\text{rank} V} < \frac{\deg(E, h_0)}{\text{rank} E}
\]

holds for any saturated \(G\)-equivariant \(O_X\)-submodule \(V\) of \(E\) such that \(0 < \text{rank} V < \text{rank} E\). We say that \((E, \overline{\partial}_E, h_0)\) is analytically semistable with respect to the \(G\)-action if

\[
\frac{\deg(V, h_0)}{\text{rank} V} \leq \frac{\deg(E, h_0)}{\text{rank} E}
\]

holds for any saturated \(G\)-equivariant \(O_X\)-submodule \(V\) of \(E\). We say that \((E, \overline{\partial}_E, h_0)\) is analytically polystable if it is analytically semistable with respect to the \(G\)-action, and \(G\)-equivariantly isomorphic to the direct sum \(\bigoplus (E_i, \overline{\partial}_{E_i}, h_{0,i})\), where each \((E_i, \overline{\partial}_{E_i}, h_{0,i})\) is analytically stable with respect to the \(G\)-action.

Proposition 2.4 Let \(h_1\) and \(h_2\) be \(G\)-equivariant Hermitian-Einstein metrics of \((E, \overline{\partial}_E)\) satisfying the following.

- \(h_1\) and \(h_2\) are mutually bounded.
- \(\Lambda F(h_1) = \Lambda F(h_2)\).

Then, we have a \(G\)-invariant holomorphic decomposition \(E = \bigoplus_{i=1}^m E_i\) and a tuple \((c_1, \ldots, c_m) \in \mathbb{R}_{>0}^m\) such that the following holds:

- The decomposition \(E = \bigoplus_{i=1}^m E_i\) is orthogonal with respect to both \(h_i\) \((i = 1, 2)\).
- \(h_{1|E_i} = c_i \cdot h_{2|E_i}\).

Proof Let \(b\) be the endomorphism of \(E\) determined by \(h_1 = h_2 b\), which is \(G\)-invariant. According to [24, Lemma 3.1], we have

\[
\Delta_X \text{Tr}(b) = -|\overline{\partial}_E(b)b^{-1/2}|_{h_2, g_X}^2 \leq 0.
\]

By the assumption on \(X\), we obtain that \(\Delta_X \text{Tr}(b) = 0\). Hence, we obtain \(|\overline{\partial}_E(b)b^{-1/2}|_{h_2, g_X} = 0\). It implies that \(\overline{\partial}_E(b) = 0\). Because \(b\) is self-adjoint with respect to \(h_i\) \((i = 1, 2)\), we also have \(\partial_{E, h_i}(b) = 0\), i.e., \(b\) is flat with respect to the Chern connections of \((E, \overline{\partial}_E, h_i)\). In particular, the eigenvalues of \(b\) are constant. Let \(E = \bigoplus_{i=1}^m E_i\) denote the eigen decomposition of \(b\). It satisfies the condition desired in the lemma.
2.4 Statements

2.4.1 Existence of Hermitian-Einstein metrics and the analytic stability condition

Let \((E,\overline{\partial}_E)\) be a \(G\)-equivariant holomorphic vector bundle on \(X\). Let \(h_0\) be a \(G\)-invariant Hermitian metric of \(E\) such that \(|AF(h_0)|_{h_0} \leq B\varphi\) for \(B > 0\). We shall prove the following theorem in \(\S 2.6\).

**Theorem 2.5** If \((E,\overline{\partial}_E; h_0)\) is analytically stable with respect to the \(G\)-action, there exists a \(G\)-invariant Hermitian-Einstein metric \(h\) of \((E,\overline{\partial}_E)\) satisfying the following conditions.

- \(\det(h) = \det(h_0)\).
- Let \(b\) be the endomorphism of \(E\) determined by \(h = h_0b\). Then, \(|b|_{h_0}\) and \(|b^{-1}|_{h_0}\) are bounded, and \(\int_X |\overline{\partial} b|^2_{h_0,g_X} < \infty\).

2.4.2 Complement on curvature decay

Let \(Y\) be a \(G\)-invariant closed end of \(X\). We impose the following condition on \((Y, g_X|_Y)\)

- The curvature of \(g_X|_Y\) is bounded.
- We have \(r_0 > 0\) and a compact subset \(Y_0 \subset Y\) such that the injective radius of any point \(P \in Y \setminus Y_0\) in \(Y\) is larger than \(r_0\).

Let \((E,\overline{\partial}_E, h_0)\) be as in Theorem 2.5. We shall prove the following proposition in \(\S 2.8\).

**Proposition 2.6** We impose the following condition on \((E,\overline{\partial}_E, h_0)|_Y\).

- For any \(\epsilon > 0\), we have a compact subset \(K(\epsilon)\) of \(Y\) such that \(|F(h_0)^\perp|_{h_0,g_X} \leq \epsilon\) and \(|\overline{\partial}_E \Lambda F(h_0)^\perp|_{h_0,g_X} \leq \epsilon\) on \(Y \setminus K(\epsilon)\). Note that it also implies that \(|\overline{\partial}_{E,h_0} \Lambda F(h_0)^\perp|_{h_0,g_X} \leq \epsilon\) on \(Y \setminus K(\epsilon)\).

Let \(h\) be the \(G\)-invariant Hermitian-Einstein metric of \((E,\overline{\partial}_E)\) as in Theorem 2.5. Then, for any \(\epsilon > 0\), there exists a compact subset \(K(\epsilon) \subset Y\) such that \(|F(h)|^\perp|_{h,g_X} \leq \epsilon\) on \(Y \setminus K(\epsilon)\).

The following proposition is similar and proved in \(\S 2.8\).

**Proposition 2.7** We impose the following condition on \((E,\overline{\partial}_E, h_0)|_Y\).

- We have \(C > 0\) such that \(|F(h_0)^\perp|_{h_0,g_X} \leq C\) and \(|\overline{\partial}_E \Lambda F(h_0)^\perp|_{h_0,g_X} \leq C\) on \(Y\). Note that it also implies that \(|\overline{\partial}_{E,h_0} \Lambda F(h_0)^\perp|_{h_0,g_X} \leq C\) on \(Y\).

Let \(h\) be the \(G\)-invariant Hermitian-Einstein metric of \((E,\overline{\partial}_E)\) as in Theorem 2.5. Then, we have \(C_1 > 0\) such that \(|F(h)|^\perp|_{h,g_X} \leq C_1\) on \(Y\).

2.4.3 Complement on the \(L^2\)-property of the curvature

We give some sufficient conditions for the metric \(h\) in Theorem 2.5 to be \(L^2\). We shall prove the following proposition in \(\S 2.9\).

**Proposition 2.8** Let \((E,\overline{\partial}_E, h_0)\) be as in Theorem 2.5. We assume \(\int_X |F(h_0)|^2_{h_0,g_X} < \infty\). Moreover, we assume that there exists a \(G\)-invariant exhaustion function \(\phi\) on \(X\) satisfying the following conditions.

- \(\overline{\partial} \phi\) is bounded.
- \(\lim_{t \to \infty} \frac{1}{t} \int_{\{\phi \leq t\}} |F(h_0)|^\perp \cdot \overline{\partial} \phi|_{h_0,g_X} = 0\).
- \(\lim_{t \to \infty} \frac{1}{t} \left( \int_{\{\phi \leq t\}} |F(h_0)|^\perp \cdot \overline{\partial} \phi|_{h_0,g_X}^2 \right)^{1/2} = 0\).
Let $h$ be the $G$-invariant Hermitian-Einstein metric of $(E, \overline{\partial} E)$ as in Theorem 2.5. Then, we have
\[ \int_X \text{Tr}((F(h)\perp)^2) \cdot \omega_X^{n-2} = \int_X \text{Tr}((F(h_0)\perp)^2) \cdot \omega_X^{n-2}. \] (1)

In particular, $F(h)\perp$ is $L^2$.

**Corollary 2.9** Suppose that there exists a $G$-invariant exhaustion function $\phi$ on $X$ such that (i) $\partial \overline{\partial} \phi$ is $L^2$ and bounded, (ii) $\partial \phi$ is bounded. Let $(E, \overline{\partial} E)$ be a holomorphic vector bundle on $X$ with a Hermitian metric $h_0$ such that (a) $|\Lambda F(h_0)| \leq B \phi$ for some $B > 0$, (b) $\text{tr} F(h_0) = 0$, (c) $F(h_0)$ is $L^2$. If $(E, \overline{\partial} E, h_0)$ is analytically stable with respect to the $G$-action, then we have a $G$-invariant Hermitian-Einstein metric $h$ of $E$ such that (i) $h$ and $h_0$ are mutually bounded, (ii) $\det(h) = \det(h_0)$, (iii) $\Lambda F(h) = 0$, (iv) $F(h)$ and $\overline{\partial} E(hh_0^{-1})$ are $L^2$. Moreover, the equality \[(1)\] holds.

**Proof** Let $h$ be the $G$-invariant Hermitian-Einstein metric for $(E, \overline{\partial} E)$ in Theorem 2.5. Because $\text{tr} F(h)$ and $\det(h) = \det(h_0)$, we have $\text{tr} F(h) = 0$. In particular, we have $\Lambda F(h) = 0$. Because the assumptions in Proposition 2.8 are satisfied for $(E, \overline{\partial} E, h_0)$ with $\phi$, we obtain that $F(h)$ is $L^2$.

**Remark 2.10** Proposition 2.8 and Corollary 2.9 are variants of [24] Proposition 3.5, Lemma 7.4.

### 2.4.4 Uniqueness

Let us give a sufficient condition for the uniqueness of metrics with the properties in Theorem 2.5. We shall prove the following proposition in [24, Proposition 5.1].

**Proposition 2.11** Suppose that we have an exhaustion function $\phi_1 : X \to \mathbb{R}_{>0}$ such that $\overline{\partial} \log \phi_1$ is $L^2$ on $X$. Let $(E, \overline{\partial} E, h_0)$ be an analytically stable bundle on $X$ with respect to the $G$-action as in [2.4.1]. Suppose that $h_i$ ($i = 1, 2$) are $G$-invariant Hermitian-Einstein metrics such that (i) $\det(h_i) = \det(h_0)$, (ii) $h_i$ and $h_0$ are mutually bounded. Then, $h_1 = h_2$.

**Remark 2.12** It would be instructive to state explicitly that a Hermitian-Einstein metric in [24, Theorem 1] is unique without any additional assumption. Indeed, suppose that there exist Hermitian-Einstein metrics $h_i$ ($i = 1, 2$) under the assumption in [24], such that (i) $h_i$ are mutually bounded with $h_0$, (ii) $\det(h_i) = \det(h_0)$. By an argument in Proposition 2.4, we obtain the holomorphic decomposition $(E, \overline{\partial} E) = \bigoplus_{i=1}^{n} (E_i, \overline{\partial} E_i)$, which are orthogonal with respect to both $h_i$. For $a = (a_1, \ldots, a_m) \in \mathbb{R}^m_{>0}$ with $\prod a_i^\text{rank} E_i = 1$, we consider the automorphism $F_a = \bigoplus_{i=1}^{n} a_i \text{id}_{E_i}$ of $E$. Let $s_a$ be determined by $F_a h_1 = h_0 e^{s_a}$. According to [24, Proposition 5.3], we have positive constants $C_i$ ($i = 1, 2$) such that $\sup_X |s_a|_{h_0} \leq C_1 + C_2 M(h_0, F_a h_1)$. Here, $M(h_0, F_a h_1)$ denotes the Donaldson functional in [24, §5]. By [24, Proposition 5.1], we have $M(h_0, F_a h_1) = M(h_0, h_1) + M(h_1, F_a h_1)$. Because $F_a h_1$ are Hermitian-Einstein with $\det(F_a h_1) = \det(h_1)$, we have $\partial \overline{\partial} (F_a h_1) = 0$ as observed in the proof of Proposition 2.3. Hence we have $M(h_1, F_a h_1) = 0$ by the definition of Donaldson functional in [24, §5]. We obtain $\sup_X |s_a|_{h_0} \leq C_1 + C_2 M(h_0, h_1)$ for any $a$. If $m \geq 2$, we can take a sequence $a_i = (a_{i,1}, \ldots, a_{i,m})$ such that $a_{i,1} \to \infty$, which implies $\sup_X |s_{a_i}|_{h_0} \to \infty$. Hence, we have $m = 1$.

### 2.5 Review of the Dirichlet Problem for Hermitian-Einstein metrics

Let $(Z, g_Z)$ be a connected compact Kähler manifold with a non-empty smooth boundary $\partial Z$ equipped with an action of a compact Lie group $G$. For simplicity, we assume that $Z$ is embedded to a Kähler manifold $(Z', g_{Z'})$ equipped with a $G$-action such that $Z \setminus \partial Z$ is a $G$-invariant open subset of $Z'$.

Let $(E', \overline{\partial} E')$ be a $G$-equivariant holomorphic vector bundle on a neighbourhood of $Z$. Let $(E, \overline{\partial} E)$ be the restriction of $(E', \overline{\partial} E')$ to $Z$. We say that a Hermitian metric $h$ of $E$ is $C^\infty$ if for each $k \in \mathbb{Z}_{>0}$ there exists a $C^k$-metric $h'_k$ of $E'$ such that $h'_k|_Z = h'$. A section $s$ of $E$ is called $C^\infty$ if for each $k \in \mathbb{Z}_{>0}$ there exists a $C^k$-section $s'_k$ of $E'$ such that $s'_k|_Z = s$. In the following, Hermitian metrics and sections are $C^\infty$ unless otherwise specified.

Let us recall an important theorem of Donaldson.
Proposition 2.13 (Donaldson [9]) Let $h_{E;\partial Z}$ be a $G$-invariant $C^\infty$-Hermitian metric of $E_{\partial Z}$. Then, there exists a unique $G$-invariant $C^\infty$-Hermitian metric $h_E$ of $E$ satisfying the following condition.

- $\Lambda F(h_E) = 0$.
- $h_{E;\partial Z} = h_{E,\partial Z}$.

Note that the $G$-invariance of $h_E$ follows from the uniqueness.

Let $h_0$ be any $G$-equivariant $C^\infty$-Hermitian metric of $(E,\overline{\partial}_E)$.

Corollary 2.14 We have the $G$-invariant Hermitian-Einstein metric $h_E$ of $E$ such that (i) $\det(h_E) = \det(h_0)$ on $Z$, (ii) $h_{E;\partial Z} = h_{0;\partial Z}$.

Proof We have the $G$-invariant Hermitian-Einstein metric $h_1$ of $E$ such that (i) $\Lambda F(h_1) = 0$, (ii) $h_{1;\partial Z} = h_{0;\partial Z}$. We have the function $a$ determined by $\det(h_0) = \det(h_1) \cdot a$. Set $h_E := h_1 e^{-a/rank E}$. Then, $h_E$ satisfies the desired condition.

2.5.1 Complement for the Donaldson functional

Let us recall the construction of the Donaldson functional in this context by following [24].

Preliminary Let $U$ be any finite dimensional complex vector space with a Hermitian metric $h_U$. Let $b$ be an endomorphism of $U$ which is self-adjoint with respect to $h_U$. Let $e_1,\ldots,e_r$ be an orthonormal base of $U$ such that $b(e_i) = \lambda_i e_i$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any function. Then, $f(b)$ denote the endomorphism of $U$ determined by $f(b)(e_i) = f(\lambda_i)e_i$. It is self-adjoint with respect to $h_U$. Let $\Phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be any function. Let $e_1^\vee,\ldots,e_r^\vee$ denote the dual base of $U^\vee$. Then, we have the endomorphism $\Phi(b)$ of $End(E)$ determined as $\Phi(b)(e^\vee_i \otimes e_j) := \Phi(\lambda_i,\lambda_j)e^\vee_i \otimes e_j$. The endomorphism $\Phi(b)$ is self-adjoint with respect to the induced metric on $End(U)$. If $f$ (resp. $\Phi$) is positive valued, then $f(b)$ (resp. $\Phi(b)$) is positive definite.

Let $E$ be a vector bundle on $Z$ with a $C^\infty$-Hermitian metric $h_0$. Suppose that we are given a $C^\infty$-section $b$ of $End(E)$ which is self-adjoint with respect to $h_0$. For any $C^\infty$-function $f : \mathbb{R} \rightarrow \mathbb{R}$, we have a $C^\infty$-section $f(b)$ of $End(E)$ which is self-adjoint with respect to $h_0$. For any $C^\infty$-function $\Phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, we obtain a $C^\infty$-endomorphism $\Phi(b)$ of $End(E)$, which is self-adjoint with respect to $h_0$.

Donaldson functional Let $(E,\overline{\partial}_E)$ be a holomorphic vector bundle on $Z$ as above. Let $h_0$ be a $C^\infty$-Hermitian metric of $E$. Let $\mathcal{P}$ denote the space of $C^\infty$-Hermitian metrics of $E$ such that $h_{0;\partial Z} = h_{0;\partial Z}$.

We take $h_1, h_2 \in \mathcal{P}$. Let $s$ be the endomorphism of $E$ which is self-adjoint with respect to $h_i$ $(i = 1,2)$ determined by the condition $h_2 = h_1 e_s$. By the construction, we have $s_{\partial Z} = 0$. We set

$$\Psi(\lambda_1,\lambda_2) := \frac{e^{\lambda_2 - \lambda_1} - (\lambda_2 - \lambda_1) - 1}{(\lambda_2 - \lambda_1)^2}.$$  

Let $\bar{h}_1$ be the metric on $End(E) \otimes \Omega^{0,1}$ induced by $h_1$ and $g_X$. Then, we put

$$M(h_1, h_2) := \sqrt{-1} \int_Z Tr(sA\Phi(h_1)) + \int_Z \bar{h}_1(\Psi(s)(\overline{\partial}_E s), \overline{\partial}_E s).$$

Proposition 2.15 For any $h_i \in \mathcal{P}$ $(i = 1,2,3)$, we have $M(h_1, h_3) = M(h_1, h_2) + M(h_2, h_3)$.

Proof We just apply the argument of Donaldson [7, 8] and Simpson [24] under the Dirichlet condition. We give only an outline.

We take a sufficiently large $p > 0$. Let $\mathcal{P}$ denote the space of $L^p_Z$-Hermitian metrics $h$ of $E$ such that $h_{\partial Z} = h_{0;\partial Z}$. It is naturally a Banach manifold. (See [12] for Sobolev spaces on manifolds with boundary.) For each $h \in \mathcal{P}$, the tangent space of $\mathcal{P}$ at $h$ is naturally identified with the space $S_h$ of $L^p_Z$-sections of $End(E)$ which are self-adjoint with respect to $h$ such that $a_{\partial Z} = 0$. Let $U_h := \{a \in S_h \mid sup |a|^2 \leq 1/2 \}$. We have the map $\Upsilon_h : U_h \rightarrow \mathcal{P}$ given by $\Upsilon_h(b) = h \cdot (id + b)$, which gives an isomorphism of $U_h$ and a neighbourhood of $h$ in $\mathcal{P}$.  

\[7\]
For each $h \in \mathcal{P}$, we have the linear map $\Phi_h : S_h \to \mathbb{R}$ given by

$$
\Phi_h(b) = \sqrt{-1} \int_Z \text{Tr}(b\Lambda F(h)).
$$

Thus, we obtain a differential 1-form $\Phi$ on $\mathcal{P}$.

**Lemma 2.16** $\Phi$ is a closed 1-form.

**Proof** It is enough to prove $(d\Phi)_{h_1} = 0$ at each $h_1 \in \mathcal{P}$. We take $u, v \in S_{h_1}$. They naturally gives a vector field on the linear space $S_{h_1}$. By $\Upsilon_{h_1}$, they induce vector fields on $\Upsilon_{h_1}(U_{h_1})$, which are denoted by $\underline{v}$ and $\underline{u}$. We have $[\underline{u}, \underline{v}] = 0$. For $b \in U_{h_1}$, we have $\underline{w}_{h_1, (id+b)} = (id+b)^{-1}u \in S_{h_1, (id+b)}$. We have

$$
\Phi_{h_1}(\underline{w}_{h_1, (id+\epsilon v)}(\underline{w}_{h_1, (id+\epsilon v)})) = \sqrt{-1} \int_Z \Lambda \text{Tr}\left((id+\epsilon v)^{-1}u F(h_1(id+\epsilon v))\right).
$$

Hence, we have

$$
\Phi_{h_1}(\underline{w}_{h_1, (id+\epsilon v)}) - \Phi_{h_1}(\underline{w}_{h_1}) = \epsilon \sqrt{-1} \int_Z \Lambda \text{Tr}\left((-\epsilon u)F(h_1) + u \frac{\partial}{\partial h_1}v + O(\epsilon)\right).
$$

We obtain the following:

$$
\left(\underline{w}\Phi_{h_1}(\underline{w}_{h})\right)_{h=h_1} - \left(\underline{w}\Phi_{h_1}(\underline{w}_{h})\right)_{h=h_1} = \sqrt{-1} \int_Z \Lambda \text{Tr}\left((-\epsilon u + \epsilon v)F(h_1) + u \frac{\partial}{\partial h_1}v - v \frac{\partial}{\partial h_1}u\right). \tag{2}
$$

Note the relation $(\partial\partial_{h_1} + \partial_{h_1}\partial) v = [F(h_1), v]$. Because $u|_{\partial Z} = v|_{\partial Z} = 0$, we obtain the following by the Stokes formula:

$$
\int_Z \Lambda \text{Tr}(u \frac{\partial}{\partial h_1}v - v \frac{\partial}{\partial h_1}u) = \int_Z \Lambda \text{Tr}(u(F(h_1)v - vF(h_1))).
$$

Hence, the right hand side of (2) is 0. Thus, we obtain the claim of the lemma. 

For $h_1, h_2 \in \mathcal{P}$, we take a path $\gamma$ from $h_1$ to $h_2$ in $\mathcal{P}$, and we set $M^D(h_1, h_2) := \int_{\gamma} \Phi$. We clearly have $M^D(h_1, h_3) = M^D(h_1, h_2) + M^D(h_2, h_3)$ for $h_i \in \mathcal{P}$ $(i = 1, 2, 3)$. For $s \in S_{h_1}$, we have

$$
\frac{d^2}{dt^2} M^D(h_1, h_1 e^{ts})_{t=t_0} = \left(\frac{d}{dt} M^D(h_1, h_1 e^{ts}, h_1 e^{t_1+s})_{t=0}\right)_{t_1=t_0} = \sqrt{-1} \int_Z \Lambda \text{Tr}(s \frac{\partial}{\partial h_1}e^{t_0+s}). \tag{3}
$$

As given in the proof of [24, Proposition 5.1], we have the following:

$$
\frac{d^2}{dt^2} M(h_1, h_1 e^{ts})_{t=t_0} = \int_Z \bar{\Upsilon}_1(t_0) \left(s \frac{\partial}{\partial h_1}e^{t_0+s}, \frac{\partial}{\partial s}\right).
$$

(4)

Here, $\Psi_{1,t_0}(y_1, y_2) = e^{t_0(y_2-y_1)}$. Hence, the right hand side of (4) is

$$
- \sqrt{-1} \int_Z \Lambda \text{Tr}\left(\frac{\partial}{\partial h_1}e^{t_0+s}\right). \tag{5}
$$

Because $s|_{\partial Z} = 0$, we obtain the following equality from (3), (5) and the Stokes formula:

$$
\frac{d^2}{dt^2} M^D(h_1, h_1 e^{ts}) = \frac{d^2}{dt^2} M(h_1, h_1 e^{ts}).
$$

We clearly have $M(h_1, h_1 e^{ts})_{t=0} = 0 = M^D(h_1, h_1 e^{ts})_{t=0}$ and

$$
\frac{d}{dt} M(h_1, h_1 e^{ts})_{t=0} = \sqrt{-1} \int_Z \text{Tr}(s \Lambda F(h_1)) = \frac{d}{dt} M^D(h_1, h_1 e^{ts})_{t=0}.
$$
Hence, we have \( M(h_1, h_1 e^t) = M^D(h_1, h_1 e^t) \). In particular, we obtain \( M(h_1, h_3) = M(h_1, h_2) + M(h_2, h_3) \) for any \( h_i \in \mathcal{P} \ (i = 1, 2, 3) \).

Let us consider the heat equation associated to the Hermitian-Einstein condition:

\[
h_t^{-1} \frac{dh_t}{dt} = -\sqrt{-1} \Lambda F(h_t).
\]

(6)

According to Simpson [24], we have a unique solution \( h_t \) of the heat equation satisfying \( h_t|_{t=0} = h_0 \) and \( h_t|_{t=0} \cap \partial Z = h_0 \cap \partial Z \).

**Lemma 2.17** \( M(h_0, h_t) \) is \( C^1 \) with respect to \( t \), and we have

\[
\frac{d}{dt} M(h_0, h_t) = -\int_Z |\Lambda F(h_t)|^2_{h_t}.
\]

**Proof** We give only an indication by following the argument of Simpson in [24] Lemma 7.1] closely. Indeed, because the base space is compact, the argument is easier.

By Proposition 2.15, it is enough to prove the equality at \( t = 0 \). Let \( s_t \) be determined by \( h_t = h_0 e^{s_t} \). By the heat equation, we have \( \lim_{t \to 0} \frac{1}{t} s_t = -\sqrt{-1} \Lambda F(h_0) \), and hence

\[
\lim_{t \to 0} \frac{1}{t} \int_Z \text{Tr}(s_t \Lambda F(h_t)) = \int_Z \text{Tr}(\Lambda F(h_0)) = -\int_Z |\Lambda F(h_0)|^2_{h_0}.
\]

Set \( b_t := e^{s_t} \). We have \( b_t > 1 \) such that \( |b_t - id|_{h_0} \leq 1/2 \) for any \( t < t_0 \). Then, \( s_t = \sum_l (-1)^{l-1} j^{-1}(b_t - id)^j \).

We have \( C_1 > 0 \) such that \( |\bar{\partial} s_t|_{h_0} \leq C_1 |\bar{\partial} b_t|_{h_0} \) for \( 0 \leq t \leq t_0 \).

We have \( C_2 > 0 \) such that \( h_0(\Psi(s_t)\bar{\partial} E s_t, \bar{\partial} E s_t) \leq C_2 |\bar{\partial} E(b_t) b_t^{-1/2}|^2_{h_0} \) for any \( 0 \leq t \leq t_0 \). Hence, it is enough to prove

\[
\lim_{t \to 0} \frac{1}{t} \int_Z |\bar{\partial} E(b_t) b_t^{-1/2}|^2_{h_0} = 0.
\]

(7)

Because \( \text{Tr} \Lambda F(h_t) = \text{Tr} \Lambda F(h_0) \), we have the following equality:

\[
\Delta \text{Tr}(b_t) = \sqrt{-1} \text{Tr} \left( (b_t - id)(\Lambda F(h_t) - \Lambda F(h_0)) \right) - |\bar{\partial} E(b_t) b_t^{-1/2}|^2_{h_0}.
\]

Because \( \text{det}(b_t) = 1 \), we have \( \text{Tr}(b_t) \geq \text{rank} E \) on \( Z \). We also have \( \text{Tr}(b_t) = \text{rank} E \) on the boundary \( \partial Z \). We obtain \( \partial_{\nu} \text{Tr}(b_t) \leq 0 \) at \( \partial Z \), where \( \partial_{\nu} \) denote the outer normal vector field of \( \partial Z \subset Z \). Hence, by the Stokes formula (see Lemma 2.19 below, for example), we have

\[
\int_Z \Delta \text{Tr}(b_t) = -\int_{\partial Z} \partial_{\nu} \text{Tr}(b_t) \geq 0.
\]

Hence, we obtain

\[
\int_Z |\bar{\partial} E(b_t) b_t^{-1/2}|^2_{h_0} \leq \int_Z \left| (b_t - id) |(\Lambda F(h_t) - \Lambda F(h_0))|_{h_0}
\]

Because \( t^{-1}(b_t - id) \) is uniformly bounded, we obtain (7).

**Lemma 2.18** Let \( h_E \) be the Hermitian-Einstein metric of \((E, \bar{\partial} E)\) such that \( \text{det}(h_E) = \text{det}(h_0) \) and \( h_E|_{\partial Z} = h_0|_{\partial Z} \). Then, we have \( M(h_0, h_E) \leq 0 \).

**Proof** Let \( h_t \ (t \geq 0) \) be the solution of the heat equation \((5) \) with the initial value \( h_0 \). By Lemma 2.17, \( M(h_0, h_t) \) is non-increasing. Because \( M(h_0, h_0) = 0 \), we have \( M(h_0, h_t) \leq 0 \) for any \( t \). According to Donaldson [9], a subsequence \( \{ h_t \} \) converges to \( h_E \) in \( C^\infty \) as \( t_1 \to \infty \). Hence, we obtain \( M(h_0, h_E) \leq 0 \).
2.5.2 Stokes formula (Appendix)

Let \( \partial_\nu \) denote the outer normal vector bundle at \( \partial Z \). We recall the following general formula.

**Lemma 2.19** We have the following equality for \( C^\infty \)-functions \( f \) and \( \varphi \) on \( Z \):

\[
\int_Z f \Delta \varphi \text{dvol}_Z = \int_Z \varphi \Delta f \text{dvol}_Z - \int_{\partial Z} f \partial_\nu \varphi \text{dvol}_{\partial Z} + \int_{\partial Z} \varphi \partial_\nu f \text{dvol}_{\partial Z}.
\]  \( (8) \)

**Proof** Let \( * \) denote the Hodge star operator of the Kähler manifold \((Z, g_Z)\). We have the following:

\[
\int_Z f \Delta \varphi \text{dvol}_Z = \int_Z f (\ast (d \varphi)) \text{dvol}_Z = - \int_Z f \cdot (d \ast \varphi).
\]

It is equal to

\[
- \int d(f \cdot (d \varphi)) + \int d f \cdot (d \varphi) = - \int d(f \cdot (d \varphi)) + \int df \cdot (d \varphi)
\]

\[
= - \int d(f \cdot (d \varphi)) + \int d(\varphi \cdot (df)) - \int \varphi d(df).
\]  \( (9) \)

Hence, we have the following equality:

\[
\int f \Delta \varphi \text{dvol}_Z = \int \varphi \Delta f \text{dvol}_Z - \int d(f \cdot (d \varphi)) + \int d(\varphi \cdot (df))
\]

We have the following general formula for \( C^\infty \)-functions \( \psi \) and \( g \) on \( Z \):

\[
\int_Z d(\psi \ast dg) = \int_{\partial Z} \psi \ast dg \mid_{\partial Z} = \int_{\partial Z} \psi \partial_\nu (g) \text{dvol}_{\partial Z}.
\]

Thus, we obtain \( \blacksquare \).

2.6 Sequence of Hermitian-Einstein metrics and \( C^0 \)-bound

We return to the setting in \( [24] \). If \( X \) is compact, the claim of Theorem \( 2.5 \) is contained in the result of Simpson \( [24] \) Theorem 1]. (If \( X \) is compact and \( G \) is trivial, it is a result of Donaldson \( [8] \) and Uhlenbeck-Yau \( [26] \).) Hence, we shall prove Theorem \( 2.5 \) under the assumption where \( X \) is non-compact. Recall that \( X \) is assumed to be connected.

We give a proof of the following elementary lemma in \( [2.6.3] \).

**Lemma 2.20** We have a \( G \)-invariant \( C^\infty \)-function \( f \) on \( X \) which is exhaustive.

**Lemma 2.21** We have an increasing sequence of compact subsets \( X_i \) (\( i = 1, 2, \ldots \)) in \( X \) with \( \bigcup X_i = X \) satisfying the following condition.

- Each \( X_i \) is a \( G \)-invariant submanifold with non-empty smooth boundary \( \partial X_i \) such that \( X_i \setminus \partial X_i \) is an open subset of \( X \). Moreover each connected component of \( X_i \) has non-empty boundary.

**Proof** We take a \( G \)-invariant \( C^\infty \)-function \( f : X \rightarrow \mathbb{R}_{\geq 0} \) which is exhaustive. We can take a sequence \( a_i \rightarrow \infty \) such that each \( a_i \) is not a critical value of \( f \). It is enough to put \( X_i := f^{-1}(\{ t \in \mathbb{R} \mid t \leq a_i \}) \). Note that if the boundary of a connected component of \( X_i \) is empty, the component should be equal to \( X \) because \( X \) is assumed to be connected. But, it contradicts with our assumption that \( X \) is non-compact. \( \blacksquare \)

We take a sequence of compact subsets \( X_i \subset X \) as in Lemma \( 2.21 \). We set \( (E_i, \overline{\partial}_E, h_{0,i}) := (E, \overline{\partial}_E, h_0) \mid_{X_i} \).

According to the theorem of Donaldson (Corollary \( 2.14 \)), we have a unique \( G \)-invariant Hermitian-Einstein metric \( h_i \) of \( (E_i, \overline{\partial}_E) \) such that \( h_{ij} \mid_{X_i} = h_{0,i} \mid_{X_i} \), and \( \det(h_i) = \det(h_{0,i}) \). Let \( s_i \) be the endomorphism of \( E_i \) determined by \( h_i = h_{0,i} e^{s_i} \). Note that \( \text{Tr}(s_i) = 0 \).

We shall prove the following, which is the counterpart of \( [24] \) Proposition 5.3.

**Proposition 2.22** Suppose that \( (E, \overline{\partial}_E, h_0) \) is analytically stable. Then, we have a positive constant \( C_1 \) such that \( \sup X_i |s_i|_{h_{0,i}} \leq C_1 \) for any \( i \).

We give only an outline of the proof.
2.6.1 Comparison of the sup-norms and the $L^1$-norms

Set $r := \text{rank } E$. According to [24, Lemma 3.1], we have the following inequality on $X_i$:

$$\Delta_X (\log (\text{Tr}(e^s)) \cdot r) \leq |AF(h_{0,i})|_{h_{0,i}}.$$  

We extend $\log (\text{Tr}(e^s)/r)$ and $|AF(h_{0,i})|_{h_{0,i}}$ to the functions $\log (\text{Tr}(e^s)/r)^\sim$ and $|AF(h_{0,i})|_{h_{0,i}}^\sim$ on $X$ by setting $0$ outside $X_i$.

**Lemma 2.23** We have the following inequality as distributions on $X$:

$$\Delta_X (\log (\text{Tr}(e^s)/r))^\sim \leq |AF(h_{0,i})|_{h_{0,i}}^\sim.$$  

**Proof**  On $\partial X_i$, we have $\log (\text{Tr}(e^s)/r) = 0$. Let $\partial_{\nu,i}$ be the outer normal vector field at $\partial X_i$. Then, we have

$$\partial_{\nu,i} (\log (\text{Tr}(e^s)/r)) \leq 0.$$  

Let $\phi$ be any $\mathbb{R}_{\geq 0}$-valued test function on $X$. We have

$$\int_{X_i} \log (\text{Tr}(e^s)/r) \cdot \Delta_X (\phi|_{X_i}) = \int_{X_i} \Delta_X (\log (\text{Tr}(e^s)/r) \cdot \phi - \int_{\partial X_i} \log (\text{Tr}(e^s)/r) \cdot \partial_{\nu,i} \phi + \int_{\partial X_i} \phi \cdot \partial_{\nu,i} \log (\text{Tr}(e^s)/r). \quad (10)$$  

We have $\int_{\partial X_i} \log (\text{Tr}(e^s)/r) \cdot \partial_{\nu,i} \phi = 0$ and $\int_{\partial X_i} \phi \cdot \partial_{\nu,i} \log (\text{Tr}(e^s)/r) \leq 0$. Hence, we have

$$\int_{X_i} \log (\text{Tr}(e^s)/r) \cdot \Delta_X (\phi|_{X_i}) \leq \int_{X_i} \Delta_X (\log (\text{Tr}(e^s)/r) \cdot \phi \leq \int_{X_i} |AF(h_{0,i})|_{h_{0,i}} \cdot \phi.$$  

It implies the claim of the lemma. 

By the assumption on $(X, g_X)$, we have a constant $C_{10}, C_{11} > 0$ such that the following holds for any $i$:

$$\sup_{X_i} \log (\text{Tr}(e^s)/r) \leq C_{10} + C_{11} \int_{X_i} \log (\text{Tr}(e^s)/r) \cdot \varphi|_{X_i}.$$  

Hence, we have positive constants $C_{12}, C_{13}$ such that the following holds for any $i$:

$$\sup_{X_i} |s_i|_{h_0} \leq r^{1/2} \sup_{X_i} \log (\text{Tr}(e^s)) \leq r^{1/2} \left( \log r + C_{10} + C_{11} \int_{X_i} \log (\text{Tr}(e^s)/r) \cdot \varphi|_{X_i} \right)$$

$$\leq C_{12} + C_{13} \int_{X_i} |s_i|_{h_0} \cdot \varphi|_{X_i}. \quad (11)$$

2.6.2 Proof of Proposition 2.22

Suppose that we have a sequence $s_i$ such that $\sup_{X_i} |s_i|_{h_0} \to \infty \ (i \to \infty)$. By the estimate (11), we have

$$\ell_i := \int_{X_i} |s_i|_{h_0} \varphi \to \infty. \quad \text{We set } u_i := \ell_i^{-1} s_i. \quad \text{They are } G\text{-invariant sections of } \text{End}(E_i) \text{ on } X_i \text{ which are self-adjoint with respect to } h_i.$$  

**Lemma 2.24** After going to a subsequence, we have the following.

- We have a $G$-invariant $L^2$-section $u_\infty$ of $\text{End}(E)$ on $X$ such that $\{u_i\}$ is weakly convergent to $u_\infty$ in $L^2_1$ on any compact subset of $X$. 

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• Let \( \Phi : \mathbb{R} \times \mathbb{R} \to [0, \infty] \) be a \( C^\infty \)-function such that \( \Phi(y_1, y_2) < (y_1 - y_2)^{-1} \) if \( y_1 > y_2 \). Then, the following holds:

\[
\sqrt{-1} \int_X \text{Tr}(u_\infty \Lambda F(h_0)) + \int_X \overline{h}_0(\Phi(u_\infty)(\overline{\partial}_E u_\infty), \overline{\partial}_E u_\infty) \leq 0. \tag{12}
\]

(See [25, Lemma 5.4] for the notation \( \Phi(u_\infty)(\overline{\partial}_E u_\infty) \). Here, \( \overline{h}_0 \) denotes the metric of \( \text{End}(E) \otimes \Omega^{2,0} \) induced by \( h_0 \) and \( g_X \).

**Proof** We closely follow the argument of [24, Lemma 5.4]. By Lemma 2.18 we have \( M(h_{0,i}, h_i) \leq 0 \), and hence

\[
\sqrt{-1} \int_{X_i} \text{Tr}(u_i \Lambda F(h_0)) + \int_{X_i} \overline{h}_0(\ell_i \Psi(\ell_i u_i)(\overline{\partial}_E u_i), \overline{\partial}_E u_i) \leq 0.
\]

Note that we have \( C_1 > 0 \) such that \( \sup |u_i| < C_1 \) for any \( i \). We remark that \( \ell_i \Psi(\ell_i \lambda_1, \ell_i \lambda_2) \) is monotonously increasing for \( \ell \), and convergent to \( (\lambda_1 - \lambda_2)^{-1} \) (if \( \lambda_1 > \lambda_2 \)) or \( \infty \) (if \( \lambda_1 \leq \lambda_2 \)). Hence, we have

\[
\sqrt{-1} \int_{X_i} \text{Tr}(u_i \Lambda F(h_0)) + \int_{X_i} \overline{h}_0(\Phi(u_i)(\overline{\partial}_E u_i), \overline{\partial}_E u_i) \leq 0. \tag{13}
\]

We have \( C_2 > 0 \) such that \( \int_{X_i} |u_i \cdot \Lambda F(h_0)|_{h_{0,i}} \leq C_2 \) for any \( i \). By (13), we have \( C_3 > 0 \) such that

\[
\int_{X_i} |\overline{\partial}_E u_i|_{h_0}^2 < C_3.
\]

Hence, for any compact subset \( K \) of \( X \), we obtain that \( u_i \) are bounded in \( L^2_1 \) on \( K \). By going to a subsequence, we may assume that \( \{u_i\} \) is weakly convergent in \( L^2_1 \) on any compact subset of \( X \). Let \( u_\infty \) denote the weak limit. Because \( |u_i|_{h_{0,i}} \leq C_1 \), we have \( |u_\infty|_{h_0} \leq C_1 \). Because \( u_i \) are \( G \)-invariant, \( u_\infty \) is also \( G \)-invariant.

For any compact subset \( Z \) of \( X \), we have \( \int_Z |u_i|_{h_{0,i}} \varphi \to \int_Z |u_\infty|_{h_0} \varphi \). Because \( |u_i|_{h_0} \leq C_1 \) for any \( i \), we have a compact subset \( Z \) such that \( \int_{X_i \setminus Z} |u_i|_{h_{0,i}} \varphi \leq 1/2 \) for any \( i \). Then, we obtain that \( \int_Z |u_i|_{h_{0,i}} \varphi \geq 1/2 \), which implies \( \int_Z |u_\infty|_{h_0} \varphi \geq 1/2 \). In particular, \( u_\infty \neq 0 \).

For any compact subset \( Z \) of \( X \), we have the convergence

\[
\lim_{i \to \infty} \int_Z \text{Tr}(u_i \Lambda F(h_0)) = \int_Z \text{Tr}(u_\infty \Lambda F(h_0)).
\]

Note that \( |u_\infty|_{h_0} \leq C_1 \) and \( |u_i|_{h_{0,i}} \leq C_1 \) for any \( i \). Hence, for any \( \epsilon > 0 \), we have a compact subset \( Z \) such that

\[
\int_{X_i \setminus Z} \left| \text{Tr}(u_i \Lambda F(h_0)) \right| < \epsilon, \quad \int_{X \setminus Z} \left| \text{Tr}(u_\infty \Lambda F(h_0)) \right| < \epsilon.
\]

Hence, we have the convergence

\[
\lim_{i \to \infty} \int_{X_i} \text{Tr}(u_i \Lambda F(h_0)) = \int_X \text{Tr}(u_\infty \Lambda F(h_0)). \tag{14}
\]

Take any \( \epsilon > 0 \). Because (14), we have \( i_1 \) such that the following holds for any \( i \geq i_1 \):

\[
\sqrt{-1} \int_X \text{Tr}(u_\infty \Lambda F(h_0)) + \int_X \overline{h}_0(\Phi^{1/2}(u_i)(\overline{\partial}_E u_i))_{h_{0,i}}^2 \leq \epsilon.
\]

If \( i \leq j \), we have \( \int_{X_i} \left| \Phi^{1/2}(u_j)(\overline{\partial}_E u_j) \right|_{h_{0,j}}^2 \leq \int_{X_j} \left| \Phi^{1/2}(u_j)(\overline{\partial}_E u_j) \right|_{h_{0,j}}^2 \). Hence, we have the following for any \( i < j \):

\[
\sqrt{-1} \int_X \text{Tr}(u_\infty \Lambda F(h_0)) + \int_X \overline{h}_0(\Phi^{1/2}(u_j)(\overline{\partial}_E u_j))_{h_{0,j}}^2 \leq \epsilon.
\]

On the compact space \( X_i \), by applying the argument of Simpson in the proof of [24, Lemma 5.4], we obtain

\[
\sqrt{-1} \int_X \text{Tr}(u_\infty \Lambda F(h_0)) + \int_X \overline{h}_0(\Phi^{1/2}(u_\infty)(\overline{\partial}_E u_\infty))_{h_0}^2 \leq 2\epsilon.
\]
Hence, we obtain
\[ \sqrt{\frac{1}{n}} \int_X \text{Tr}(u_\infty \Lambda F(h_0)) + \int_X |\Phi^{1/2}(u_\infty)(\partial u_\infty)|^2_{h_0} \leq 2\epsilon. \]
Because this holds for any \( \epsilon > 0 \), we obtain the desired inequality \([12]\).

We obtain the following lemma by the argument of Simpson \([24]\), Lemma 5.5, Lemma 5.6.

**Lemma 2.25**

- The eigenvalues of \( u_\infty \) are constant. We denote them by \( \lambda_1, \lambda_2, \ldots, \lambda_{\text{rank } E} \).
- Let \( \Phi : \mathbb{R} \times \mathbb{R} \to ]0, \infty[ \) be a \( C^\infty \)-function such that \( \Phi(\lambda_i, \lambda_j) = 0 \) if \( \lambda_i > \lambda_j \). Then, \( \Phi(u_\infty)(\partial u_\infty) = 0 \).

Let \( \gamma \) be an open interval between the eigenvalues. We take a \( C^\infty \)-function \( p_\gamma : \mathbb{R} \to ]0, \infty[ \) such that \( p_\gamma(\lambda_i) = 1 \) if \( \lambda_i < \gamma \) and \( p_\gamma(\lambda_i) = 0 \) if \( \lambda_i > \gamma \). Set \( \pi_\gamma := p_\gamma(u_\infty) \). By the construction, we have \( \pi_\gamma^2 = \pi_\gamma \) and \( \pi_\gamma h_0 = \pi_\gamma \). Moreover \( \pi_\gamma \) is \( G \)-invariant.

Because \( \partial \pi_\gamma = dp_\gamma(u_\infty)(\partial u_\infty) \), \( \pi_\gamma \) is a locally \( L^2 \)-section of \( \text{End}(E) \), where
\[ dp_\gamma(y_1, y_2) := (y_1 - y_2)^{-1}(p_\gamma(y_1) - p_\gamma(y_2)). \]
We also have \( \int |\partial \pi_\gamma|_{h_0}^2 < \infty \). By setting \( \Phi_\gamma := (1 - p_\gamma(y_2))dp_\gamma(y_1, y_2) \). Because \( \Phi_\gamma(u_\infty)(\partial u_\infty) = (1 - \pi_\gamma)(\partial \pi_\gamma) \), we have \( (1 - \pi_\gamma)(\partial \pi_\gamma) = 0 \) by Lemma 2.25.

According to \([20]\), \( \pi_\gamma \) determines a saturated \( \mathcal{O}_X \)-submodule \( V_\gamma \) of \( E \) such that \( \pi_\gamma \) is the orthogonal projection of \( E \) onto \( V_\gamma \) outside \( Z(V_\gamma) \). Because \( \pi_\gamma \) is \( G \)-invariant, we obtain that \( V_\gamma \) is also \( G \)-invariant.

The rest of the proof of Proposition 2.22 is completely the same as the proof of \([21]\), Proposition 5.3, which we do not repeat.

### 2.6.3 Appendix: Proof of Lemma 2.20

We take an increasing sequence of compact subsets \( K_i \subset X \) \((i = 1, 2, \ldots)\) such that \( \bigcup K_i = X \). We set \( \tilde{K}_i := \bigcup_{g \in G} \kappa_g(K_i) \). Because \( \tilde{K}_i \) is the image of the map \( G \times K_i \to X \), \( \tilde{K}_i \) is compact and contains \( K_i \). We take a relatively compact open neighbourhood \( U_1 \) of \( \tilde{K}_i \cup K_2 \). Let \( \overline{U}_1 \) be the closure of \( U_1 \) in \( X \), which is compact. We set \( \tilde{K}_2 := \bigcup_{g \in G} \kappa_g(\overline{U}_1) \). Inductively, we construct a sequence of compact subsets \( \tilde{K}_i \subset X \) \((i = 1, 2, \ldots)\) as follows. Suppose that we have already constructed \( \tilde{K}_i \). We take a relatively compact open neighbourhood \( U_i \) of \( \tilde{K}_i \cup K_{i+1} \). Let \( \overline{U}_i \) be the closure of \( U_i \) in \( X \), which is compact. We set \( \tilde{K}_{i+1} := \bigcup_{g \in G} \kappa_g(\overline{U}_i) \). Thus, we obtain an increasing sequence of \( G \)-invariant compact subsets \( \tilde{K}_i \subset X \) \((i = 1, 2, \ldots)\) such that \( \tilde{K}_{i+1} \) contains an open neighbourhood of \( \tilde{K}_i \) and that \( \bigcup \tilde{K}_i = X \). Let \( L_i \) denote the closure of \( X \setminus \tilde{K}_i \).

We can take a \( C^\infty \)-function \( \phi_i : X \to [0, t \leq 1] \) such that \( \phi_i(P) = 0 \) \((P \in \tilde{K}_{i-1})\) and \( \phi_i(P) = 1 \) \((P \in L_i)\). We set \( \tilde{\phi}_i := \int_G \kappa_g^*(\phi_i) \), where the integral is taken with respect to the normalized bi-invariant measure of \( G \). Then, \( \tilde{\phi}_i \) are \( C^\infty \)-functions such that \( \tilde{\phi}_i(P) = 0 \) \((P \in \tilde{K}_{i-1})\) and \( \tilde{\phi}_i(P) = 1 \) \((P \in L_i)\).

We set \( f := \sum_{i=1}^\infty \tilde{\phi}_i \). On \( K_{i+1} \cap L_i \), we have \( \tilde{\phi}_i = 0 \) \((j \geq i + 1)\) and \( \tilde{\phi}_j = 1 \) \((j < i)\). Hence, we have \( i \leq f \leq i + 1 \) on \( K_{i+1} \cap A_i \). Hence, \( f \) is a function as desired in Lemma 2.20.

### 2.7 Proof of Theorem 2.5

By Proposition 2.22, we have \( \sup_X |s_i|_{h_0, i} < C_1 \). Set \( b_i := e^{\nu i} \) on \( X_i \). We have \( C_20 > 0 \) such that
\[ \sup_{X_i} (|b_i|_{h_0, i} + |b_i^{-1}|_{h_0, i}) < C_20. \]

By \([21]\), Lemma 3.1, we have the following equality on \( X_i \):
\[ -\sqrt{-1} \Lambda \partial E_i, h_{0, i}, \overline{\partial} E_i(b_i) = -b_i \sqrt{-1} \Lambda F(h_0, i)^{-1} + \sqrt{-1} \Lambda \partial E_i(b_i)b_i^{-1} \partial E_i, h_{0, i}(b_i). \]
Hence, we have the following on $X_i$:

$$\Delta_{X_i} \operatorname{Tr}(b_i) = -\operatorname{Tr}(b_i \sqrt{-1} A F(h_{0,i})^i) - |\overline{\partial}_{E_i}(b_i) b_i^{-1/2}|_{h_{0,i}}^2.$$ 

We have $\operatorname{Tr}(b_i) - \text{rank } E \geq 0$ on $X_i$ and $\operatorname{Tr}(b_i) - \text{rank } E = 0$ on $\partial X_i$. By the Stokes formula and the inequality $\partial_{v,i} \operatorname{Tr}(b_i) \leq 0$ on $\partial X_i$, we also have the following:

$$\int_{X_i} \Delta_{X_i}(\operatorname{Tr}(b_i) - \text{rank } E) = -\int_{\partial X_i} \partial_{v,i}(\text{Tr}(b_i) - \text{rank } E) \geq 0.$$

Hence, we obtain the following:

$$0 \leq -\int_{X_i} \operatorname{Tr}(b_i \sqrt{-1} A F(h_{0,i})^i) - \int_{X_i} |\overline{\partial}_{E_i}(b_i) b_i^{-1/2}|_{h_{0,i}}^2.$$ 

We have positive constants $C_{21}, C_{22}$ such that the following holds for any $i$:

$$\int_{X_i} |\overline{\partial}_{E_i}(b_i) b_i^{-1/2}|_{h_{0,i}}^2 \leq C_{21} \int_{X_i} |A F(h_0)|_{h_0} \leq C_{22}.$$

Hence, we have $C_{23} > 0$ such that the following holds for any $i$:

$$\int_{X_i} |\overline{\partial}_i|_{h_{0,i}}^2 \leq C_{23} \quad (15)$$

Let us prove that the $C^1$-norm of $b_i$ are locally bounded by using the argument in the proof of [22, Theorem 2.10]. Let $P$ be any point of $X$. Take a holomorphic coordinate neighbourhood $(X_P; z_1, \ldots, z_n)$ around $P$ in $X$. We have $i(P)$ such that $X_P \subset X_i$ for any $i \geq i(P)$. We have a relatively compact neighbourhood $X_P$ of $P$ in $X_P$.

**Lemma 2.26** $|\overline{\partial}_i|_{h_{0,i}}$ are bounded on $X_P$.

**Proof** We use the metric $g_P = \sum dz_i d\overline{z}_i$ of $X_P$. We may assume to have a frame $v_1, \ldots, v_r$ of $E_{|X_P}$. Let $h_P$ be the metric of $E_{|X_P}$ determined as $h_P(v_i, v_i) = 1$ and $h_P(v_i, v_j) = 0$ ($i \neq j$). Let $b'_i$ be the automorphism of $E_{|X_P}$ determined by $h_{i|X_P} = h_P b'_i$. Let $h_P$ be the metric of $\text{End}(E) \otimes \Omega^{p-q}$ on $X_P$ induced by $h_P$ and $g_P$. It is enough to prove that $|\overline{\partial}_{E_i}(b'_i)|_{h_P}$ are bounded on $X_P$. As proved in [21 §2.3], the function $\|(b'_i)^{-1} \overline{\partial}_{E_i} b'_i\|^2_{h_P}$ is subharmonic. From (15), we have $\int_{X_P} \|(b'_i)^{-1} \overline{\partial}_{E_i} b'_i\|^2_{h_P} < C_P$ for some $C_P > 0$. Hence, we obtain the boundedness of $\|(b'_i)^{-1} \overline{\partial}_{E_i} b'_i\|^2_{h_P}$ on $X_P$.

By using the equation for Hermitian-Einstein metrics, we obtain that the sequence $\{b_i\}$ is bounded in $L^2$ for any $p \geq 1$ on any compact subset $K$ of $X$. Going to a subsequence, we may assume that $\{b_i\}$ is weakly convergent in $L^p$ for any $p \geq 1$ on any compact subset of $X$. Then, we have the limit $\overline{h} = \lim h_i$, which is a Hermitian-Einstein metric of $(E, \overline{\partial}_E)$ with the desired property. Thus, we obtain Theorem 2.5.

### 2.8 Proof of Proposition 2.6 and Proposition 2.7

#### 2.8.1 Kodaira identity

Let $(Z, g)$ be a Kähler manifold. Let $(V, \overline{\partial}_V)$ be any holomorphic vector bundle on $Z$ with a Hermitian metric $h$. Let $\Lambda^{p,q}(V)$ denote the space of $C^\infty$-sections of $V \otimes \Omega^{p,q}$. Let $\partial_{V,h} + \overline{\partial}_V$ denote the Chern connection of $V$. The differential operators $A^{r,s}(V) \rightarrow A^{r,s+1}(V)$ induced by $\partial_{V,h}$ are also denoted by $\partial_{V,h}$.

Take a non-negative integer $p$. We set $\tilde{V} := V \otimes \Omega^{p,0}_Z$. It is naturally a holomorphic vector bundle, and equipped with the induced Hermitian metric $\tilde{h}$. Let $\partial_{\tilde{V},\tilde{h}} + \overline{\partial}_{\tilde{V}}$ denote the Chern connection of $(\tilde{V}, \overline{\partial}_{\tilde{V}}, \tilde{h})$. The differential operators $A^{r,s}(\tilde{V}) \rightarrow A^{r,s+1}(\tilde{V})$ are also denoted by $\partial_{\tilde{V},\tilde{h}}$.

For any $\tau \in A^{p,1}(V) = A^{0,1}(\tilde{V})$, we obtain the following elements:
Lemma 2.27  For any $\tau \in A^{p,1}(V) = A^{0,1}(\tilde{V})$, we have the following equality in $A^{0,0}(V) = A^{0,0}(\tilde{V})$:

$$-\sqrt{-1}\Lambda \partial \tilde{V}^h \tau (\tau) + (-1)^{p+1}\sqrt{-1}\partial \tilde{V}^h \Lambda \tau + (-1)^p \sqrt{-1}\partial \tilde{V}^h (\tau) = 0.$$ 

Proof  It is enough to prove the equality at each point $P \in Z$. We take a holomorphic coordinate $(z_1, \ldots, z_n)$ around $P$ such that (i) $z_i(P) = 0$, (ii) $g = \sum dz_i \bar{dz}_i + O(\sum |z|^2)$. For any tuple $I = (i_1, \ldots, i_r) \in \{1, \ldots, n\}^r$, we set $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_r}$.

We have the expression

$$\tau = \frac{1}{p!} \sum_{|I| = p} \sum_j \tau_{I,j} dz_I d\bar{z}_j.$$ 

We set $\eta := \partial \tilde{V}^h \tau$. We have the expression

$$\eta = \frac{1}{(p + 1)!} \sum_{|I| = p+1} \sum_j \eta_{I,j} dz_I d\bar{z}_j.$$ 

We assume that $\tau_{I,j} = \text{sign}(\sigma) \sigma_{I,j}$ (resp. $\eta_{I,j} = \text{sign}(\sigma) \eta_{I,j}$) for any $\sigma \in \mathcal{S}_p$ (resp. $\sigma \in \mathcal{S}_{p+1}$). We have the relation

$$\frac{1}{p!} \sum_k \sum_{|I| = p} \nabla_k \tau_{I,j} dz_k dz_I d\bar{z}_j = \frac{1}{(p + 1)!} \sum_{|I| = p+1} \sum_j \eta_{I,j} dz_I d\bar{z}_j.$$ 

We have the following relation for any $(a_0, \ldots, a_p) \in \{1, \ldots, n\}^{p+1}$:

$$\sum_{i=0}^p \nabla_{a_i} \tau_{a_0, a_1, \ldots, a_i-1, a_i+1, \ldots, a_p} dz_{a_0} dz_{a_0} \cdots dz_{a_i-1} dz_{a_i+1} \cdots dz_{a_p} d\bar{z}_j = \eta_{(a_0, \ldots, a_p), j} dz_{a_0} \cdots dz_{a_p} d\bar{z}_j.$$ 

Hence, we have the following for any $(a_1, \ldots, a_p) \in \{1, \ldots, n\}^p$ with $a_i \neq j$:

$$\eta_{(a_1, \ldots, a_p), j} = \nabla_j \tau_{a_1, \ldots, a_p} + \sum_{i=1}^p (-1)^i \nabla_{a_i} \tau_{a_1, \ldots, a_i-1, a_i+1, \ldots, a_p}.$$ 

(16)

We have the following expression:

$$\partial \tilde{V}^h \tau = \frac{1}{p!} \sum_{k, j, l} \nabla_k \tau_{I,j} \otimes dz_I d\bar{z}_j.$$ 

Hence, we have the following:

$$-\sqrt{-1}\Lambda \partial \tilde{V}^h \tau = -\frac{1}{p!} \sum_{j, l} \nabla_j \tau_{I,j} dz_I + A_0$$ 

$$= -\frac{1}{p!} \sum_{j} \sum_{|I| = p} \nabla_j \tau_{I,j} dz_I - \frac{1}{(p - 1)!} \sum_{j} \sum_{|K| = p-1} \nabla_j \tau_{(j, K)} dz_j d\bar{z}_K + A_0.$$ 

(17)

Here, $A_0(0, \ldots, 0) = 0$. We have the following:

$$\Lambda \eta = \frac{1}{p!} (-1)^{p+1} \sqrt{-1} \sum_{j} \sum_{|I| = p} \eta_{(j, I), j} dz_I + A_1.$$ 

(18)
Here, $A_1 = O\left(\sum |z_i|^2\right)$. (See [14] §3.2, for example.) We also have
\[
\Lambda \tau = \frac{1}{(p-1)!} (-1)^p \sqrt{-1} \sum_j \sum_{|K|=p-1} \tau_{(j,K)} \partial_j dz_K + A_2.
\]
Here, $A_2 = O\left(\sum |z_i|^2\right)$. Hence, we have
\[
\partial_{\nu,h} \Lambda \tau = \frac{1}{(p-1)!} (-1)^p \sqrt{-1} \sum_k \sum_{|K|=p-1} \sum_{j,k \notin K} \nabla_k \tau_{(j,K)} \partial_j dz_k \cdot dz_K + A_3
\]
\[
= \frac{1}{(p-1)!} (-1)^p \sqrt{-1} \sum_k \sum_{j,k \notin K} \sum_{|K|=p-1} \nabla_j \tau_{(j,K)} \partial_j dz_j \cdot dz_K
\]
\[
+ \frac{1}{(p-1)!} (-1)^p \sqrt{-1} \sum_k \sum_{j,k \notin K} \sum_{|K|=p-1} \nabla_k \tau_{(j,K)} \partial_j dz_k \cdot dz_K + A_3. \quad (19)
\]
Here, $A_3(0,\ldots,0) = 0$. By [16], [17] and [19], we obtain the following:
\[- \sqrt{-1} \Lambda \partial_{\nu,h} \tau + (-1)^{p+1} \sqrt{-1} \partial_{\nu,h} \Lambda \tau =
\]
\[- \frac{1}{p!} \sum_j \sum_{|I|=p} \sum_{j \notin I} \nabla_j \tau_{I,j} \partial_j dz_I + \frac{1}{(p-1)!} \sum_k \sum_{j \notin k} \sum_{|K|=p-1} \nabla_k \tau_{(j,K)} \partial_j dz_k \cdot dz_K + A_4
\]
\[
= - \frac{1}{p!} \sum_j \sum_{|I|=p} \sum_{j \notin I} \eta_{(j,I),j} \partial_j dz_I + A_4. \quad (20)
\]
Here, $A_4(0,\ldots,0) = 0$. By [18], we obtain that $- \sqrt{-1} \Lambda \partial_{\nu,h} \tau + (-1)^{p+1} \sqrt{-1} \partial_{\nu,h} \Lambda \tau + (-1)^p \sqrt{-1} \Lambda \eta$ is 0 at $P$. Thus, we obtain the claim of Lemma 2.27.

2.8.2 Some computations

Let $(E, \overline{\partial}_E, h_0)$ be as in Theorem 2.5. We have the Hermitian-Einstein metric $h$ of $(E, \overline{\partial}_E)$ as in Theorem 2.5. Let $b_1$ be determined by $h_0 = h b_1$. Note that $b_1 = b^{-1}$. Set $\bar{E} := E \otimes \Omega^{1,0}$. Let $\bar{h}$ denote the Hermitian metric of $\bar{E}$ induced by $h$ and $g_X$.

**Lemma 2.28** We have the following equality:
\[
\Delta_{\bar{E},\bar{h}}(b_1^{-1} \partial_{E,h} b_1) = - \sqrt{-1} \partial_{E,h_0} \Lambda F(h_0)^\perp + \sqrt{-1} [b_1^{-1} \partial_{E,h} b_1, \Lambda F(h_0)^\perp] - \sqrt{-1} \Lambda [b_1^{-1} \partial_{E,h} b_1, F(h_0)^\perp].
\]

**Proof** By applying Lemma 2.27 to $\overline{\partial}_E(\bar{b}_1^{-1} \partial_{E,h} b_1)$, we obtain the following equality:
\[- \Lambda \partial_{\bar{E},\bar{h}} \overline{\partial}_E(\bar{b}_1^{-1} \partial_{E,h} b_1) = \partial_{E,h}(- \Lambda \overline{\partial}_E(\bar{b}_1^{-1} \partial_{E,h} b_1)) + \sqrt{-1} \Lambda \partial_{E,h} \overline{\partial}_E(\bar{b}_1^{-1} \partial_{E,h} b_1).
\]
Because $h_0 = h b_1$, we have $\partial_{E,h_0} = \partial_{E,h} + b_1^{-1} \partial_{E,h} b_1$ and $F(h_0) - F(h) = \overline{\partial}_E(\bar{b}_1^{-1} \partial_{E,h} b_1)$. By the Hermitian-Einstein condition and $\det(h) = \det(h_0)$, we have $\Lambda(F(h_0) - F(h)) = \Lambda F(h_0)^\perp$, and hence
\[
\partial_{E,h}(- \Lambda \overline{\partial}_E(\bar{b}_1^{-1} \partial_{E,h} b_1)) = - \Lambda (\partial_{E,h_0} - b_1^{-1} \partial_{E,h} b_1) \Lambda F(h_0)^\perp
\]
\[
= - \Lambda \partial_{E,h_0} \Lambda F(h_0)^\perp + \sqrt{-1} [b_1^{-1} \partial_{E,h} b_1, \Lambda F(h_0)^\perp]. \quad (21)
\]
We have $\partial_{E,h_0} F(h_0) = 0$ and $\partial_{E,h} F(h) = 0$. Hence, we have the following:
\[
\sqrt{-1} \Lambda \partial_{E,h} \overline{\partial}_E(\bar{b}_1^{-1} \partial_{E,h} b_1) = \sqrt{-1} \Lambda \partial_{E,h} (F(h_0) - F(h)) = \sqrt{-1} \Lambda (\partial_{E,h_0} - b_1^{-1} \partial_{E,h} b_1) F(h_0)
\]
\[
= - \sqrt{-1} \Lambda [b_1^{-1} \partial_{E,h} b_1, F(h_0)] = - \sqrt{-1} \Lambda [b_1^{-1} \partial_{E,h} b_1, F(h_0)^\perp]. \quad (22)
\]
Then, we obtain the claim of the lemma.
Lemma 2.29 We have the following inequality:

\[ \Delta_X |b_1^{-1}\partial_{E,h} b_1|^2_h \leq 2 \text{Re} \tilde{h}(\Delta_{E,h}(b_1^{-1}\partial_{E,h} b_1), b_1^{-1}\partial_{E,h} b_1). \]

Proof We have the following equality:

\[ -\sqrt{-1} \Lambda \tilde{d} \tilde{h}(b_1^{-1}\partial_{E,h} b_1, b_1^{-1}\partial_{E,h} b_1) = 2 \text{Re} \tilde{h}(\Delta_{E,h}(b_1^{-1}\partial_{E,h} b_1), b_1^{-1}\partial_{E,h} b_1) \]

\[ + \sqrt{-1} \tilde{d}(\Delta_{E,h}(b_1^{-1}\partial_{E,h} b_1, \tilde{F}(b_1^{-1}\partial_{E,h} b_1)) - \sqrt{-1} \tilde{d}(\Delta_{E,h}(b_1^{-1}\partial_{E,h} b_1), \partial_{E,h}(b_1^{-1}\partial_{E,h} b_1)). \] (23)

Then, we obtain the claim of the lemma.

2.8.3 Proof of Proposition 2.6

Let \( Y \) be an end of \( X \) as in Proposition 2.6. We know that \( h \) and \( h_0 \) are mutually bounded. Let \( b \) be determined by \( h = h_0b \) as in Theorem 2.3. Because

\[ b_1^{-1}\partial_{E,h} b_1 = \partial_{E,h_0} - b^{-1}\partial_{E,h_0} b, \]

we have \( \int_Y |b_1^{-1}\partial_{E,h} b_1|^2_h < \infty \). For any \( \epsilon > 0 \), we have a compact subset \( K_\epsilon \subset Y \) such that the following holds.

- \( \int_{Y \setminus K_\epsilon} |b_1^{-1}\partial_{E,h} b_1|^2_h < \epsilon. \)
- \( |\partial_{E,h_0} A F(h_0)^{\perp}|_h < \epsilon \) on \( Y \setminus K_\epsilon. \)
- \( |F(h_0)|^2_h \leq \epsilon \) on \( Y \setminus K_\epsilon. \)

By Lemma 2.28 and Lemma 2.29 we have \( C_1 > 0 \), which is independent of \( (\epsilon, K_\epsilon) \), such that the following holds on \( Y \setminus K_\epsilon: \)

\[ \Delta_X |b_1^{-1}\partial_{E,h} b_1|^2_h \leq C_1 \epsilon |b_1^{-1}\partial_{E,h} b_1|^2_h + 2C_1 \epsilon |b_1^{-1}\partial_{E,h} b_1|^2_h \leq \frac{C_1 \epsilon}{2} + \frac{5C_1 \epsilon}{2} |b_1^{-1}\partial_{E,h} b_1|^2_h. \]

Hence, we have a constant \( C_2 > 0 \) which is independent of \( (\epsilon, K_\epsilon) \), such that the following holds on \( Y \setminus K_\epsilon: \)

\[ (\Delta_X - C_2 \epsilon) |b_1^{-1}\partial_{E,h} b_1|^2_h \leq C_2 \epsilon. \]

By using [11] Theorem 9.20, we obtain the following.

Lemma 2.30 For any \( \epsilon > 0 \), we have a compact subset \( K_\epsilon' \subset Y \) such that the following holds on \( Y \setminus K_\epsilon': \)

\[ \sup_{Y \setminus K_\epsilon'} |b_1^{-1}\partial_{E,h} b_1|^2_h \leq \epsilon. \]

By [23], for any \( \epsilon > 0 \), we have a compact subset \( K_\epsilon'' \subset Y \) such that \( |b_1^{-1}\partial_{E,h_0} b|^2_{h_0} \leq \epsilon \) on \( Y \setminus K_\epsilon''. \) According to [23] Lemma 3.1, we have the following relation:

\[ \Delta_{E,h_0}(b) = -b \sqrt{-1} \Lambda F(h_0)^{\perp} + \sqrt{-1} \tilde{d}(E)(b) b^{-1}\partial_{E,h_0} b. \] (25)

Hence, for any \( \epsilon > 0 \), we have a compact subset \( K^{(3)}_\epsilon \subset Y \) such that \( |\Delta_{E,h_0}(b)|_{h_0} \leq \epsilon \) on \( Y \setminus K^{(3)}_\epsilon. \)

Take a large \( p \). We obtain that the \( L^2_E \)-norm of \( b - \text{id} \) on the disc with radius \( r_0 \) centered at \( P \in Y \) goes to \( 0 \) as \( P \) goes to \( \infty \). By using [23], we obtain that the \( L^2_E \)-norm of \( b - \text{id} \) on the disc with radius \( r_0 \) centered at \( P \in Y \) goes to \( 0 \) as \( P \) goes to \( \infty \). Then, we obtain the claim of Proposition 2.6.
2.8.4 Proof of Proposition 2.7

We use the notation in 2.8.3. We have $C_{10} > 0$ such that the following holds:

- $\int_Y |b^{-1}_1 \partial E, b_{b_1}|_h^2 \leq C_{10}$.
- $|\partial E, b_{h_0} AF(h_0)_{L^2}^1|_h^2 \leq C_{10}$ and $|F(h_0)_{L^2}^1|_h^2 \leq C_{10}$ hold on $Y$.

We have $C_{11} > 0$ such that the following holds on $Y$:

$$\Delta_X |b^{-1}_1 \partial E, b_{b_1}|_h^2 \leq C_{11} |b^{-1}_1 \partial E, b_{b_1}|_h^2 + 2C_{11} |b^{-1}_1 \partial E, b_{b_1}|_h^2 \leq \frac{C_{11}}{2} + \frac{5C_{11}}{2} |b^{-1}_1 \partial E, b_{b_1}|_h^2.$$  

Again, by using [11] Theorem 9.20], we obtain that there exists a constant $C_{12} > 0$ such that

$$\sup_Y |b^{-1}_1 \partial E, b_{b_1}|_h^2 \leq C_{12}.$$  

Hence, we have $C_{13} > 0$ such that $|b^{-1}_1 \partial E, b_{b_1}|_h^2 \leq C_{13}$ on $Y$. By using [25], we obtain that there exists a compact subset $K \subset Y$ such that the $L^p_{\bullet}$-norms of $b - \text{id}$ on the disc with radius $r_0$ centered at $P \in Y \setminus K$ are bounded. Hence we obtain the boundedness of $b$. Thus, the proof of Proposition 2.4 is completed. 

2.9 Proof of Proposition 2.8

We closely follow the argument of Simpson in the proof of [24] Proposition 3.5, Lemma 7.4. We shall prove the following inequalities:

$$\int_X \text{Tr} (F(h_1)_{L^2}^1 \omega_{X}^{\dim X - 2}) \leq \int_X \text{Tr} (F(h_0)_{L^2}^1 \omega_{X}^{\dim X - 2}), \quad (26)$$

$$\int_X \text{Tr} (F(h_0)_{L^2}^1 \omega_{X}^{\dim X - 2}) \leq \int_X \text{Tr} (F(h_1)_{L^2}^1 \omega_{X}^{\dim X - 2}). \quad (27)$$

We may assume that $X_i = \{ P \in X | \phi(P) \leq a \}$ for a sequence $a_i > 0$. On $X_i$, we set $f_i := 1 - a_i^{-1} \phi$. We have $f_i > 0$ on $X_i \setminus \partial X_i$, and $f_i = 0$ on $\partial X_i$. Let $\omega_{X_i}$ denote the Kähler form of $X_i$. In the following, when we are given a Hermitian metric $k$ on $E$, the induced metrics $E \otimes \Omega^{p,q}$ are also denoted by $k$ for simplicity of the description.

2.9.1 Proof of (26)

Let $P_i$ denote the space of $C^\infty$ Hermitian metrics $k$ of $E_i$ such that $k_{|\partial X_i} = h_{0|\partial X_i}$. Take any $k_1, k_2 \in P_i$. We have the endomorphism $s$ of $\text{End}(E_i)$ determined by $k_2 = k_1 \cdot e^s$. We put $\tau_i := -2\sqrt{-1} \partial \bar{\partial} f_i = 2\sqrt{-1} a_i^{-1} \partial \bar{\partial} \phi$ on $X_i$. Let $n := \dim X_i$. We set:

$$M_i(k_1, k_2) := \sqrt{-1} \int_{X_i} \text{Tr}(s F(k_1)) \tau_i \omega_{X_i}^{n-2} - \sqrt{-1} \int_{X_i} \text{Tr}(\Psi(s)(\bar{s} \partial k_i)) \tau_i \omega_{X_i}^{n-2}.$$  

As in [24] Lemma 7.2], we have the following equality:

$$M_i(k_1, k_2) = \int_{X_i} f_i (\text{Tr}(F(k_1)^2) - \text{Tr}(F(k_2)^2)) \omega_{X_i}^{n-2}.$$  

Let $h_{0,i}$, $h_i$ and $s_i$ be as in (26).

**Lemma 2.31** We have a constant $C_1 > 0$ such that the following holds for any $i$:

$$M_i(h_{0,i}, h_i) \geq -\frac{C_1}{a_i} \int_{X_i} |(F(h_{0,i})_{L^2}^1 \cdot \partial \bar{\partial} \phi)|_{X_i} h_{0,i} - \frac{C_1}{a_i} \int_{X_i} |\bar{\partial} E_{s_i}|_{h_{0,i}}^2. \quad (28)$$
Proof We consider $M_i(h_{0,i}, h_{0,i}e^{y_i})$ for $0 \leq y \leq 1$. We have

$$
\frac{d^2}{dy^2} M_i(h_{0,i}, h_{0,i}e^{y_i}) = \sqrt{-1} \int_{X_i} \text{Tr}(s_i \delta \partial h_{0,i} e^{y_i}(s_i)) \tau_i \omega_{X_i}^{n-2} = -\sqrt{-1} \int_{X_i} \text{Tr}(\bar{\partial} E_i(s_i) \cdot \delta h_{0,i} e^{y_i}(s_i)) \tau_i \omega_{X_i}^{n-2}.
$$

Take any point $P \in X_i \setminus \partial X_i$. We take a holomorphic coordinate $(z_1, \ldots, z_n)$ around $P$ such that (i) $z_i(P) = 0$, (ii) $g_{X_i} = (\sum d\bar{z}_i dz_i)^{p}$, (iii) $\tau_{p} = \sum b_i (dz_i d\bar{z}_i)^p$ for some $(b_1, \ldots, b_n) \in \mathbb{C}^n$. We have $C_2 > 0$ independent of $i$ and $P$ such that $|b_p| < C_2/a_i$. We have the expression

$$
\bar{\partial} E_i(s_i) = \sum A_p d\bar{z}_p, \quad \partial h_{0,i} e^{y_i}(s_i) = \sum (A_p)_{h_{0,i}} e^{y_i}, dz_p.
$$

At $P$, we have

$$
\text{Tr}(\bar{\partial} E_i(s_i) \delta h_{0,i} e^{y_i}(s_i)) \cdot \tau_i \omega_{X_i}^{n-2} = \sum_{p \neq q} [A_p]_{h_{0,i} e^{y_i}}^2 \cdot b_q (dz_p d\bar{z}_p)(dz_q d\bar{z}_q) \omega_{X_i}^{n-2} = C_3 \sum_{p \neq q} [A_p]_{h_{0,i} e^{y_i}}^2 \cdot b_q \cdot \omega_{X_i}^2.
$$

Here, $C_3$ depends only on the dimension $n$. Because $0 \leq y \leq 1$, and $h_0$ and $h$ are mutually bounded on $X$, we have positive constants $C_5$ and $C_6$, which are independent of $i$ and $P$, such that the following holds:

$$
\left| \sum_{p \neq q} [A_p]_{h_{0,i} e^{y_i}}^2 \cdot b_q \right| \leq \frac{C_5}{a_i} \left| \bar{\partial} E_i(s_i) \right|_{h_{0,i} e^{y_i}}^2 \leq \frac{C_5}{a_i} \left( \left| \bar{\partial} E_i(s_i) \right|_{h_{0,i}}^2 + \left| \bar{\partial} E_i(s_i) \right|_{h_{0,i}}^2 \right) \leq \frac{C_6}{a_i} \left| \bar{\partial} E_i(s_i) \right|_{h_{0,i}}^2.
$$

Hence, we obtain the following:

$$
\frac{d^2}{dy^2} M_i(h_{0,i}, h_{0,i}e^{y_i}) \geq -\frac{C_6}{a_i} \int_{X_i} \left| \bar{\partial} E_i(s_i) \right|_{h_{0,i}}^2.
$$

Because $\text{Tr}(s_i) = 0$, we have

$$
\frac{d}{dy} M_i(h_{0,i}, h_{0,i}e^{y_i})|_{y=0} = \sqrt{-1} \int_{X_i} \text{Tr}(s_i F(h_{0,i})) \tau_i \omega_{X_i}^{n-2} = \sqrt{-1} \int_{X_i} \text{Tr}(s_i F(h_{0,i})) \tau_i \omega_{X_i}^{n-2}.
$$

Hence, we have a positive constant $C_7$, which is independent of $i$, such that the following holds:

$$
\frac{d}{dy} M_i(h_{0,i}, h_{0,i}e^{y_i}) \geq -\frac{C_7}{a_i} \int_{X_i} \left| s_i \right|_{h_{0,i}} \cdot |F(h_{0,i})|_{h_{0,i}}^2 \cdot \partial \bar{\partial} \phi |_{h_{0,i}} - \frac{C_7}{a_i} \int_{X_i} \left| \bar{\partial} E_i(s_i) \right|_{h_{0,i}}^2.
$$

Then, we obtain the inequality \((28)\).

Hence, we have a positive constant $C_{10}$, $C_{11}$ which is independent of $i$ such that the following holds:

$$
\int_{X_i} f_i \text{Tr}(F(h_i)^2) \leq \int_{X_i} f_i \text{Tr}(F(h_{0,i})^2) + \frac{C_{10}}{a_i} \int_{X_i} |F(h_{0,i})|_{h_{0,i}}^2 \cdot \partial \bar{\partial} \phi |_{h_{0,i}}.
$$

Because $\text{Tr} F(h_{0,i}) = \text{Tr} F(h_i)$ on $X_i$, we have

$$
\int_{X_i} \text{Tr}(f_i(F(h_i)^2) \leq \int_{X_i} \text{Tr}(f_i(F(h_{0,i})^2) + \frac{C_{10}}{a_i} \int_{X_i} |F(h_{0,i})|_{h_{0,i}}^2 \cdot \partial \bar{\partial} \phi |_{h_{0,i}}.
$$

By the Hermitian-Einstein condition and the non-negativity $f_i \geq 0$, we have the following for any $i \leq j$:

$$
\int_{X_i} f_j \text{Tr}((F(h_j)^2) \leq \int_{X_j} f_j \text{Tr}((F(h_j)^2) \leq \int_{X_j} f_j \text{Tr}((F(h_j)^2) + \frac{C_{10}}{a_j} \int_{X_j} |F(h_{0,j})|_{h_{0,j}}^2 \cdot \partial \bar{\partial} \phi |_{h_{0,j}}.
$$

(29)
We have the convergences
\[
\lim_{j \to \infty} \int_{X_j} f_j \text{Tr}
\left( (F(h_j)^{\perp})^2 \right) = \int_{X_i} \text{Tr}
\left( (F(h)^{\perp})^2 \right),
\]
\[
\lim_{j \to \infty} \int_{X_j} f_j \text{Tr}
\left( (F(h_{0, j})^{\perp})^2 \right) = \int_{X_i} \text{Tr}
\left( (F(h_0)^{\perp})^2 \right).
\]
Hence, we obtain
\[
\int_{X_i} \text{Tr}
\left( (F(h)^{\perp})^2 \right) \leq \int_{X_i} \text{Tr}
\left( (F(h_0)^{\perp})^2 \right).
\]
By taking the limit \(i \to \infty\), we obtain (26).

### 2.9.2 Proof of (27)

We extend the function \(f_i\) on \(X_i\) to the function \(\tilde{f}_i\) on \(X\) by setting 0 outside \(X_i\). We obtain the current \(\tilde{\tau}_i := -2\sqrt{-1}\partial\overline{\partial} f_i\) on \(X\).

Let \(k_1, k_2\) be \(C^\infty\) Hermitian metrics of \(E\). We have the endomorphism \(s\) of \(E\) determined by \(k_2 = k_1 e^s\). We set
\[
\tilde{M}_i(k_1, k_2) := \sqrt{-1} \int_{X} \text{Tr}
\left( sF(k_1) \right) \tilde{\tau}_i \omega_X^{n-2} - \sqrt{-1} \int_{X} \text{Tr}
\left( \Psi(s) \partial S \cdot \partial k_i s \right) \cdot \tilde{\tau}_i \omega_X^{n-2}.
\]
As proved in [24] Lemma 7.2, we have
\[
\tilde{M}_i(k_1, k_2) = \int_{X} \tilde{f}_i \text{Tr}
\left( (F(k_1)^{\perp})^2 - (F(k_2)^{\perp})^2 \right) \omega_X^{n-2}.
\]

**Lemma 2.32** We have a positive constant \(C_{20}\) such that the following holds for any \(i\):
\[
\tilde{M}_i(h_0, h) \leq \frac{C_{20}}{a_i} \left( \int_{X_i} |F(h_0)^{\perp} \cdot \partial \phi|^2_{h_0} \right)^{1/2} \left( \int_{X_i} |\overline{\partial} g s|^2_{h_0} \right)^{1/2} + \frac{C_{20}}{a_i} \int_{X} |\overline{\partial} g s|^2_{h_0}.
\]
Here, \(s\) is determined by \(h = h_0 e^s\).

**Proof** Recall \(\text{Tr}(s) = 0\). By definition, we have the following:
\[
\left| \int_{X} \text{Tr}(sF(h_0)) \tilde{\tau}_i \omega_X^{n-2} \right| = \left| \int_{X} \text{Tr}(sF(h_0)^{\perp}) \tilde{\tau}_i \omega_X^{n-2} \right| = 2 \left| \int_{X} \overline{\partial} g \text{Tr}(sF(h_0)^{\perp})(1 - a_i^{-1} \phi) \omega_X^{n-2} \right|
\]
\[
= 2 \left| \int_{X} \text{Tr}(\overline{\partial} g \cdot F(h_0)^{\perp} \cdot \partial \phi) \omega_X^{n-2} \right|.
\]
Hence, we have
\[
\left| \int_{X} \text{Tr}(sF(h_0)) \tilde{\tau}_i \omega_X^{n-2} \right| \leq 2 \left( \int_{X_i} |\overline{\partial} g s|^2_{h_0} \right)^{1/2} \left( \int_{X_i} |F(h_0)^{\perp} \partial \phi|^2_{h_0} \right)^{1/2}.
\]

By definition, we have
\[
- \sqrt{-1} \int_{X} \text{Tr}(\Psi(s) \overline{\partial} g h_0 s) \tilde{\tau}_i \omega_X^{n-2} = - \sqrt{-1} \int_{X_i} - \overline{\partial} g \text{Tr}(\Psi(s) \overline{\partial} g h_0 s)(-2\sqrt{-1}f_i) \omega_X^{n-2}
\]
\[
= \sqrt{-1} \int_{\partial X_i} \text{Tr}(\Psi(s) \overline{\partial} g h_0 s)(-2\sqrt{-1}(\overline{\partial} f_i)) \omega_X^{n-2} - \sqrt{-1} \int_{X_i} \text{Tr}(\Psi(s) \overline{\partial} g h_0 s)(-2\sqrt{-1}(\overline{\partial} f_i)) \omega_X^{n-2}.
\]

**Lemma 2.33** We have
\[
\sqrt{-1} \int_{\partial X_i} \text{Tr}(\Psi(s) \overline{\partial} g h_0 s)(-2\sqrt{-1}(\overline{\partial} f_i)) \omega_X^{n-2} \leq 0.
\]
Proof It is enough to prove that the integrand is positive at each $P \in \partial X_i$. Note that $X_i = \{ f_i \geq 0 \} = \{ \phi - a_i \leq 0 \}$. We take a holomorphic coordinate neighbourhood $(z_1, \ldots, z_n) = (x_1 + \sqrt{-1}y_1, \ldots, x_n + \sqrt{-1}y_n)$ around $P$ such that (i) $z_i(P) = 0$, (ii) $g_{X_i} = \sum dz_p \, d\bar{z}_p$, (iii) $(d\phi)_P = b \, dx_1$ for $b > 0$. We may also assume the following:

$$\sqrt{-1} \text{Tr}(\Psi(s)\bar{\partial}s \partial h_0 s) = - \sum c_p \sqrt{-1}dz_p \, d\bar{z}_p$$

for some $c_p \geq 0$ ($p = 1, \ldots, n$). We have $(-2\sqrt{-1}(\bar{\partial}f_i))_P = -a_i^{-1}(-2\sqrt{-1}(\bar{\partial}\phi))_P = a_i^{-1}b(\sqrt{-1}dz_1)_P$. As a form on the tangent space $T_P \partial X_i$, we have

$$\left(\sqrt{-1} \text{Tr}(\Psi(s)\bar{\partial}s \partial h_0 s)(-2\sqrt{-1}(\bar{\partial}f_i))_P\omega^{n-2}\right)_{P} = - \sum_{p=2}^{n} c_p \sqrt{-1}dz_p \, d\bar{z}_p \left( a_i^{-1}b \, dy_1 \right)_P \cdot \left( \sum_{i=2}^{n} \frac{\sqrt{-1}}{2} \, dz \, d\bar{z}_i \right)^{n-2}.$$  (33)

Then, the claim of the lemma follows.

Because $s$ and $\partial \bar{\partial} \phi$ are bounded, we have a constant $C_{21} > 0$ such that the following holds for any $i$:

$$\left| \sqrt{-1} \int_{X_i} \text{Tr}(\Psi(s)\bar{\partial}s \partial h_0 s)(-2\sqrt{-1}(\bar{\partial}f_i))_P \omega^{n-2} \right| \leq \frac{C_{21}}{a_i} \int_{X_i} |\partial s|_{h_0}^2.$$  (30)

Hence, we obtain

$$\sqrt{-1} \int_{X} \text{Tr}(\Psi(s)\bar{\partial}s \partial h_0 s)\tau_i \omega^{n-2} \leq \frac{C_{21}}{a_i} \int_{X} |\partial s|_{h_0}^2.$$  (30)

Thus, we obtain (30). We have positive constants $C_{30}$ and $C_{31}$ such that the following holds for any $i$:

$$\int_{X} f_i \text{Tr}((F(h_0)^{1/2})^2) \leq \int_{X} f_i \text{Tr}((F(h_0)^{1/2})^2) + \frac{C_{30}}{a_i} + \frac{C_{31}}{a_i} \left( \int_{X} |F(h_0)^{1/2} \partial \phi|_{h_0}^2 \right)^{1/2}.$$  (30)

By the theorem of Lebesgue, we have the following convergence:

$$\lim_{i \to \infty} \int_{X} f_i \text{Tr}((F(h_0)^{1/2})^2) = \int_{X} \text{Tr}((F(h_0)^{1/2})^2).$$

Note that $Tr((F(h_0)^{1/2})^2)$ is positive by the Hermitian-Einstein condition, and $f_i$ is monotonously increasing for $i$. Hence, we have the following convergence:

$$\lim_{i \to \infty} \int_{X} f_i \text{Tr}((F(h_0)^{1/2})^2) = \int_{X} \text{Tr}((F(h_0)^{1/2})^2).$$

Hence, we obtain (27). Thus the proof of Proposition 2.8 is completed.

2.10 Proof of Proposition 2.11

Let $h_1$ and $h_2$ be metrics as in Proposition 2.11. For the proof of the proposition, we may assume that $h_1$ is as in Theorem 2.8. Namely, let $b_1$ be determined by $h_1 = h_0 b_1$, then $\bar{\partial} b_1$ is $L^2$.

By Proposition 2.9 we have the decomposition $(E, \bar{\partial} E) = \bigoplus_{j=1}^{m}(E_j, \bar{\partial} E_j)$ such that (i) the decomposition is orthogonal with respect to $h_i$ ($i = 1, 2$), (ii) $h_{2|E_j} = a_j \cdot h_{1|E_j}$ for some $a_j > 0$. Let $\pi_i$ denote the projection onto $E_i$ with respect to the decomposition. Let $\pi_i^\dagger_{h_0}$ denote the adjoint of $\pi_i$ with respect to $h_0$. Note that $\pi_i$ are bounded with respect to $h_0$, because $h_0$ and $h_i$ are mutually bounded.

Lemma 2.34 $\partial_{E,h_0} \pi_i$ and $\bar{\partial}_{E} \pi_i^\dagger_{h_0}$ are $L^2$.  

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Proof Because the holomorphic decomposition $E = \bigoplus E_i$ is orthogonal with respect to $h_1$, we have $\partial_{E_i} \pi_i = 0$. We have $\partial_{E_0} h_1 = \partial_{E_0} h_0 + b_1^{-1} \partial_{E_0} b_1$. Because $\partial_{E_0} h_0 \pi_{i,h_0} = -|b_1^{-1} \partial_{E_0} b_1, \pi_{i,h_0}|$, we obtain that $\partial_{E_0} h_0 \pi_{i,h_0}$ is $L^2$. We also obtain that $\overline{\partial} E \pi_{i,h_0}$ are $L^2$.

We consider the Hermitian metric $h_3$ obtained as the direct sum of $h_{0|E}$. 

Lemma 2.35 $h_3$ and $h_0$ are mutually bounded.

Proof Because $h_0$ and $h_1$ are mutually bounded, $h_{0|E}$ and $h_{1|E}$ are mutually bounded. Because $h_1 = \bigoplus h_{1|E}$, we obtain that $h_1$ and $h_3$ are mutually bounded. Then, we obtain the claim of the lemma.

Let $b_3$ be determined by $h_3 = h_0 b_3$. We have 

$$b_3 = \sum_{j=1}^{m} \pi_j \circ \pi_j.$$ 

Lemma 2.36 $b_3^{-1} \overline{\partial} E b_3$ and $b_3^{-1} \partial_{E_0} b_3$ are $L^2$ with respect to $h_0$ and $g_X$.

Proof By Lemma 2.34 $\overline{\partial} E b_3$ and $\partial_{E_0} b_3$ are $L^2$ with respect to $h_0$ and $g_X$. By Lemma 2.35 $b_3$ and $b_3^{-1}$ is also bounded with respect to $h_0$. Then, we obtain the claim of the lemma.

Lemma 2.37 $\Lambda \overline{\partial} \text{Tr}(b_3^{-1} \partial_{E_0} b_3)$ is $L^1$.

Proof We have $\overline{\partial} E (b_3^{-1} \partial_{E_0} b_3) = -b_3^{-1} (\partial_{E_0} b_3) \cdot b_3^{-1} \partial_{E_0} b_3 + b_3^{-1} \overline{\partial} E \partial_{E_0} b_3$. We also have the following:

$$\overline{\partial} E \partial_{E_0} b_3 = \sum_{j=1}^{m} \overline{\partial} E \pi_{j,h_0} \circ \partial_{E_0,h_0} \pi_j + \sum_{j=1}^{m} \pi_{j,h_0} \circ [F(h_0), \pi_j].$$

By the assumption $|\Lambda F(h_0)|_{h_0} \leq B_{\varphi}$, $|\Lambda F(h_0)|_{h_0}$ is $L^1$. Then, we obtain the claim of the lemma.

Lemma 2.38 We have $\int_{X} \text{Tr}(\overline{\partial} E (b_3^{-1} \partial_{E_0} b_3)) \omega_X^{\dim X-1} = 0$.

Proof We take a $C^\infty$-function $\rho : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ such that $\rho(t) = 0$ if $t \geq 2$ and $\rho(t) = 1$ if $t \leq 1$. We set $\chi_N := \rho(N^{-1} \phi_1)$ on $X$. Because $\text{Tr}(\overline{\partial} E (b_3^{-1} \partial_{E_0} b_3)) \omega_X^{\dim X-1}$ is $L^1$, it is enough to prove

$$\lim_{N \rightarrow \infty} \int_{X} \chi_N \cdot \text{Tr}(\overline{\partial} E (b_3^{-1} \partial_{E_0} b_3)) \omega_X^{\dim X-1} = 0. \quad (34)$$

We have the following:

$$\int_{X} \chi_N \cdot \text{Tr}(\overline{\partial} E (b_3^{-1} \partial_{E_0} b_3)) \omega_X^{\dim X-1} = -\int_{X} \partial_{E_0} \chi_N \cdot \text{Tr}(b_3^{-1} \partial_{E_0} b_3) \omega_X^{\dim X-1}$$

$$= -\int_{X} \rho'(N^{-1} \phi_1) N^{-1} \overline{\partial} \phi_1 \cdot \text{Tr}(b_3^{-1} \partial_{E_0} b_3) \omega_X^{\dim X-1} \quad (35)$$

Note that if $\rho'(N^{-1} \phi_1) \neq 0$, we have $N \leq \phi_1 \leq 2N$. Hence, we have $|N^{-1} \overline{\partial} \phi_1|_{g_X} \leq 2 |\phi_1^{-1} \overline{\partial} \phi_1|_{g_X} = 2 |\overline{\partial} \log \phi_1|_{g_X}$. Then, we obtain (34) by the theorem of Lebesgue.

Because $\text{Tr} F(h_3) = \text{Tr} F(h_0) + \Lambda \text{Tr}(b_3^{-1} \partial_{E_0} b_3)$, we obtain

$$\int_{X} \Lambda \text{Tr} F(h_0) = \int_{X} \Lambda \text{Tr} F(h_3) = \sum_{i=1}^{m} \int_{X} \Lambda \text{Tr} F(h_{0|E_i}).$$

We also have $\text{rank } E = \sum_{i=0}^{m} \text{rank } E_i$. Then, it is standard that there exists $i_0$ such that $\deg(E, h_0)/\text{rank } E \leq \deg(E_{i_0}, h_0)/\text{rank } E_{i_0}$. By the analytic stability of $(E, \overline{\partial} E, h_0)$, we obtain that $m = 1$. It implies $h_1 = h_2$. 

3 Examples

3.1 Preliminary

3.1.1 Ahlfors type lemma

Take $R > 0$ and $C_0 > 0$. We use the standard coordinate $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We set $r(x) := \sqrt{\sum_{i=1}^{n} x_i^2}$. We set $U(R) := \{x \in \mathbb{R}^n \mid r(x) \geq R\}$. We set $\Delta := -\sum \partial_i^2$.

Let $C_0$ be a positive constant. Let $g : U(R) \rightarrow \mathbb{R}_{\geq 0}$ be a function such that

\[ \Delta g \leq -C_0 g. \]

We also assume that $g = O(r(x)^N)$ for some $N > 0$.

**Lemma 3.1** We have $\epsilon_1 > 0$, depending only on $C_0$, such that $g(x) = O\left(\exp(-\epsilon_1 r(x))\right)$.

**Proof** For any $a \in \mathbb{R}$, we have

\[ -(\partial^2 + (n-1)r^{-1}\partial_r) \exp(ar) = (-a^2 - (n-1)r^{-1}a) \exp(ar). \]

Hence, we have $\epsilon_1 > 0$ and $R_1 \geq R$ such that the following holds on $\{x \in \mathbb{R}^n \mid r(x) \geq R_1\}$:

\[ \Delta \exp(-\epsilon_1 r(x)) \geq -C_0 \exp(-\epsilon_1 r(x)), \quad \Delta \exp(\epsilon_1 r(x)) \geq -C_0 \exp(\epsilon_1 r(x)). \]

We take $C_1 > 0$ such that $g(x) < C_1 \exp(-\epsilon_1 r(x))$ on $\{r(x) = R_1\}$. For any $\delta > 0$, we set

\[ F_\delta(x) := C_1 \exp(-\epsilon_1 r(x)) + \delta \exp(\epsilon_1 r(x)). \]

We have $g(x) < F_\delta(x)$ on $\{r(x) = R_1\}$. We also have $\Delta(g - F_\delta) \leq C_0(g - F_\delta)$. We set

\[ Z(\delta) := \{x \in \mathbb{R}^n \mid r(x) \geq R_1, \ g(x) > F_\delta(x)\}. \]

Because $g(x) = O(r(x)^N)$ and $g(x) < F_\delta(x)$ on $\{r(x) = R_1\}$, $Z(\delta)$ is relatively compact in $\{r(x) > R_1\}$. On $Z(\delta)$, we have

\[ \Delta(g - F_\delta) < 0. \]

On $\partial Z(\delta)$, we have $g - F_\delta = 0$. Hence, if $Z(\delta) \neq \emptyset$, we obtain $g - F_\delta \leq 0$ on $Z(\delta)$, which contradicts with the construction of $Z(\delta)$. Hence, we have $Z(\delta) = \emptyset$. Namely, we have $g \leq F_\delta$ on $\{r(x) > R_1\}$ for any $\delta > 0$. We obtain that $g \leq C_1 \exp(-\epsilon_1 r(x))$.

3.1.2 An estimate on $\mathbb{R}$

Let $\varphi$ be a positive $C^\infty$-function on $\mathbb{R}$ such that $\varphi(t) = e^{-\delta |t|}$ ($|t| \geq 1$).

**Lemma 3.2** We have $C_i (i = 0, 1)$ such that the following holds.

- Let $g$ be a bounded function $\mathbb{R} \rightarrow [0, \infty[ \ such \ that \ -\partial^2 g \leq B \varphi$ for $B > 0$. Then, $\sup g \leq C_0 B + C_1 \int \varphi g$.

**Proof** On $t \geq 1$, we set $F := g + \delta^2 B e^{-\delta t}$. Then, we have $\partial^2 F \geq 0$ on $t \geq 1$. Because $F$ is bounded, we obtain that $\partial_t F \leq 0$ on $t \geq 1$, and hence $F(t) \leq F(1)$ ($t \geq 1$). It implies that $g(t) \leq g(1) + \delta^2 B e^{-\delta t}$ for $t \geq 1$. Similarly, we obtain that $g(t) \leq g(-1) + \delta^2 B e^{-\delta t}$ for $t \leq -1$. Because $-\partial^2 g \leq B \varphi$, we have $C_i (i = 2, 3)$ such that $g(t) \leq C_2 B + C_3 \int_2^t 2g \varphi$ for $-1 \leq t \leq 1$. Then, we obtain the claim of the lemma.
3.1.3 Inequality for distributions

Let $U$ be a neighbourhood of $(0,0,0)$ in $\mathbb{R}^3$. Set $\tilde{U} := S^1 \times U$ and $W := S^1 \times \{(0,0,0)\}$. We set $\tilde{U}^* := \tilde{U} \setminus W$. We regard $S^1 = \mathbb{R}/\mathbb{Z}$. Let $t$ be the standard coordinate of $\mathbb{R}$, which induces local coordinates on $S^1$. Let $(x,y,z)$ be the standard coordinate of $\mathbb{R}^3$. We set $\Delta := -(\partial_x^2 + \partial_y^2 + \partial_z^2)$. Let $\varphi$ be a bounded positive function on $\tilde{U}$. It is implicitly implied in the proof of Proposition 2.2 of \cite{23}.

**Lemma 3.3** Let $g : \tilde{U}^* \to \mathbb{R}_{\geq 0}$ be a bounded function such that $\Delta g \leq \varphi$ on $\tilde{U}^*$. Then, we have $\Delta g \leq \varphi$ as distributions on $\tilde{U}$.

**Proof** In the proof of \cite{23} Proposition 2.2, it is studied in the case where $\tilde{W}$ is a complex submanifold. The argument can work even in the case of real submanifold whose codimension is larger than 2.

3.2 Instantons

3.2.1 Doubly periodic instantons

Let $\Gamma$ be a lattice in $\mathbb{C}$. Let us consider the action of $\Gamma$ on $\mathbb{C} \times \mathbb{C}$ given by $\chi(z,w) = (z + \chi, w)$. Set $X := (\mathbb{C} \times \mathbb{C}) / \Gamma$ with the Kähler metric $g_X := dz \, d\bar{z} + dw \, d\bar{w}$. We set $\varphi := (1 + |w|^2)^{-\delta}$ for a $\delta > 0$.

**Proposition 3.4** $(X,g_X,\varphi)$ satisfies the assumption in \cite{23}.

**Proof** Let us regard $\mathbb{C}$ with the metric $\varphi \, dw \, d\bar{w}$ as a Kähler manifold. The Laplacian is given by $-\varphi^{-1} \partial_w \partial_{\bar{w}}$. Recall the following, which is a special case of \cite{23} Proposition 2.4.

**Lemma 3.5** There exists an increasing function $a_1 : [0, \infty[ \to [0, \infty[$ with $a_1(0) = 0$ and $a_1(x) = x$ ($x \geq 1$), such that if $g$ is a bounded function $\mathbb{C} \to [0, \infty[$ satisfying $-\varphi^{-1} \partial_w \partial_{\bar{w}} g \leq B$, then

$$
\sup g \leq a_1 \left( \int_{\mathbb{C}} g \varphi \, dw \, d\bar{w} \right)
$$

holds, where $C_1(B)$ is a positive constant depending on $B$. Moreover, if $-\partial_w \partial_{\bar{w}} f \leq 0$ then $f$ is constant.

Let $f$ be an $\mathbb{R}_{\geq 0}$-valued bounded function on $X$ such that $-(\partial_z \partial_{\bar{z}} + \partial_w \partial_{\bar{w}}) f \leq B \varphi$ for $B > 0$. Set $T := \mathbb{C} / \Gamma$. For any $w \in \mathbb{C}$, we set $F(w) := \int_{T \times \{w\}} f$. Then, we have $-\partial_w \partial_{\bar{w}} F(w) \leq B \varphi$. By Lemma 3.5 we obtain

$$
\sup_{w \in \mathbb{C}} F(w) \leq a_1 \left( \int_{\mathbb{C}} \varphi \, dw \, d\bar{w} \right).
$$

For any $w_0 \in \mathbb{C}$ and $r_0 > 0$, let $B_{r_0}(w_0) := \{w \in \mathbb{C} \mid |w - w_0| < r_0\}$. We have $\Delta_X f \leq B \varphi$ for $B > 0$ on $T \times B_{r_0}(w_0)$. We also have

$$
\int_{T \times B_{r_0}(w_0)} f \leq \int_{B_{r_0}(w_0)} F \leq a_1 \left( \int_{T \times \mathbb{C}} \varphi \, dw \, d\bar{w} \right).
$$

Then, we have positive constant $C_i$ ($i = 2,3$) such that

$$
\sup_{w \in B_1(w_0)} f(w) \leq C_2 B + C_3 C_1(B) a_1 \left( \int_{T \times \mathbb{C}} \varphi \, dw \, d\bar{w} \right).
$$

Suppose that $f$ satisfies the stronger condition $-(\partial_z \partial_{\bar{z}} + \partial_w \partial_{\bar{w}}) f \leq 0$ on $X$. Let us prove that $f$ is constant. We set $F(w) := \int_{T \times \{w\}} f$ as above. Then, $F$ is a bounded function on $\mathbb{C}$ satisfying $-\partial_w \partial_{\bar{w}} F \leq 0$. Hence, by Lemma 3.5, $F$ is constant.

We have the following:

$$
-(\partial_z \partial_{\bar{z}} + \partial_w \partial_{\bar{w}}) |f|^2 = -2(\partial_z \partial_{\bar{z}} + \partial_w \partial_{\bar{w}}) f \cdot f - |\partial_z f|^2 - |\partial_{\bar{z}} f|^2 - |\partial_w f|^2 - |\partial_{\bar{w}} f|^2 \leq -|\partial_w f|^2.
$$
By using the Fourier expansion in the $T$-direction, we have the decomposition $f = f_0 + f_1$, where $f_0$ is constant in the $T$-direction, and $\int_{T} f_1 = 0$. We have $\int_{T} (f_0 + f_1)^2 = \int_{T} f_0^2 + 2 \int_{T} f_0 f_1 + \int_{T} f_1^2$. Because $F(w) = \int_{T} f = \int_{T} f_0$ is constant, $f_0$ and $\int_{T} f_1^2$ are constant. We have $C > 0$ such that

$$\int_{T} |\partial_{w} f|^2 = \int_{T} |\partial_{w} f_1|^2 \geq C \int_{T} f_1^2.$$ 

Hence, we obtain the following inequality on $\mathbb{C}_w$:

$$-\partial_{w} \partial_{\overline{w}} \int_{T} |f_1|^2 \leq -C \int_{T} |f_1|^2.$$ 

Because $\int_{T} |f_1|^2$ is bounded, it follows from Lemma 3.1 that $\int_{T} |f_1|^2 = O(\exp(-c|w|))$ for some $c > 0$ as $|w| \to \infty$. Because $\int_{T} |f_1|^2$ is subharmonic, we obtain that $\int_{T} |f_1|^2 \leq 0$, and hence $f_1 = 0$. It implies that $f$ is constant.

Take $\lambda \in \mathbb{C}$. We have the complex structure $(\xi, \eta) = (z + \lambda \overline{w}, w - \lambda \overline{z})$ on $\mathbb{R}^4 = \mathbb{C}_z \times \mathbb{C}_w$, which induces a complex structure on $X$. The complex manifold is denoted by $X^\lambda$. Let $(E, \nabla_E)$ be a holomorphic vector bundle on $X^\lambda$ with a Hermitian metric $h_0$ such that (a) $|\Lambda F(h_0)|_{h_0} \leq B\varphi$ for $B > 0$, (b) $tr F(h_0) = 0$, (c) $|F(h_0)|_{h_0, g_X}$ is $L^2$.

**Corollary 3.6** If $(E, \nabla_E, h_0)$ is analytically stable, then we have a Hermitian metric $h$ of $E$ such that (i) $h$ and $h_0$ are mutually bounded, (ii) $\det(h) = \det(h_0)$, (iii) $\Lambda F(h) = 0$, (iv) $\nabla_E(h h_0^{-1})$ and $F(h)|_{h, g_X}$ are $L^2$. Moreover, the equality (1) holds. If $h'$ is a Hermitian metric of $E$ satisfying the conditions (i), (ii), (iii), then $h' = h$.

**Proof** We take a positive $C^\infty$-function $\phi$ on $X$ such that $\phi(z, w) = \log |w|$ ($|w| > 1$). Note that $w = (1 + |\lambda|^2)^{-1}(\eta + \lambda \overline{z})$. On $\{ |w| > 1 \}$, we have the following equality with respect to the complex structure of $X^\lambda$:

$$\overline{\partial}_\phi = \frac{1}{1 + |\lambda|^2} \left( \frac{dn}{w} + \overline{\lambda} \frac{d\xi}{w} \right) = O(|w|^{-1}), \quad \overline{\partial}_\phi = \frac{1}{(1 + |\lambda|^2)^2} \left( -\lambda \frac{d\overline{\xi}}{w^2} + \lambda \frac{d\overline{\xi}}{w^2} \right) = O(|w|^{-2}).$$

Hence, $\overline{\partial}_\phi$ is bounded, and $\overline{\partial}_\phi$ is $L^2$ and bounded on $X^\lambda$. Moreover, $\overline{\partial}_\phi \log \phi = O((\log |w|)^{-1}|w|^{-1})$ is $L^2$. Then, the claim follows from Theorem 2.6, Corollary 2.9, and Proposition 2.11.

**Remark 3.7** In [3, Theorem 0.12, Proposition 5.12], in the rank 2 case, Biquard and Jardim studied the correspondence between instantons on $X^0$ with quadratic curvature decay and parabolic bundles on the natural compactification of $X^0$. (See [3, Theorem 0.12] for more details.) In the proof, they seem to mention a generalization of [24, Theorem 1] in the setting where the volume of the base Kähler manifold is infinite, on the basis of [2]. Indeed, Biquard kindly replied to the author that the volume finiteness condition is not essential in his argument in [2]. It is not clear to the author how their generalization is related to Corollary 3.6.

The author studied the correspondence between $L^2$-instantons on $X^0$ and polystable filtered bundles with degree 0 on the natural compactification of $X^0$ in [19], as a generalization of [3, Theorem 0.12]. For the construction of $L^2$-instantons from stable filtered bundles with degree 0, we used the Nahm transforms between $L^2$-instantons on $X^0$ and wild harmonic bundles on the dual torus $\mathbb{C}/\Gamma$. Corollary 3.6 should allow us to construct $L^2$-instantons from stable filtered bundles with degree 0 more directly.

We plan to study the correspondence between $L^2$-instantons on $X^\lambda$ and filtered bundles on the natural compactification of $X^\lambda$ for general $\lambda$ by using Corollary 3.6.

### 3.2.2 Spatially periodic instantons

Let $\Gamma$ be a lattice in $\mathbb{R}^3$. We consider $X := (\mathbb{R}^3/\Gamma) \times \mathbb{R}$ with the Euclidean metric $g_X$ for which $\mathbb{R}^3/\Gamma$ and $\mathbb{R}$ are orthogonal. We take a complex structure $\mathbb{R}^4 \simeq \mathbb{C}^2$ for which the multiplication of $\sqrt{-1}$ is an isometry. It induces a complex structure on $X$. Let $t$ be the standard coordinate on $\mathbb{R}$. Let $\varphi_0 : \mathbb{R} \to [0, \infty[$ be a $C^\infty$-function on $\mathbb{R}$ such that $\varphi_0(t) = e^{-|t|/\delta}$ ($|t| > 1$). Let $\varphi$ be the $C^\infty$-function $X \to [0, \infty[$ obtained as the pull back of $\varphi_0$ by the projection $X = (\mathbb{R}^3/\Gamma) \times \mathbb{R} \to \mathbb{R}$. 

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Proposition 3.8 \((X,g_X,\varphi)\) satisfies the assumption in [21].

**Proof** Let \(f\) be a bounded function on \(X\) such that \(\Delta(f) \leq B\varphi\) for \(B > 0\). We consider the function \(F(t) := \int_{T \times X} f\), which satisfies \(-\partial_t^2 F(t) \leq B\varphi\). By Lemma 5.2 we obtain that \(\sup F(t) \leq C_1 B + C_2 \int_{X} F\varphi\). Then, we obtain the estimate \(\sup f(t) \leq C_3 B + C_4 \int_{X} F\varphi\) as in the case of Proposition 3.8. Suppose that \(f\) satisfies the stronger condition \(\Delta(f) \leq 0\) on \(X\). We set \(F := \int_{T^2 \times X} f\) as above. We obtain \(-\partial_t^2 F \leq 0\). Because \(F : \mathbb{R} \rightarrow [0,\infty]\) is bounded and convex from below, we obtain that \(F\) is constant. Then, we obtain that \(f\) is also constant by using the argument in the proof of Proposition 3.3.

Let \((E,\overline{\partial}_E)\) be a holomorphic vector bundle on \(X\) with a Hermitian metric \(h_0\) such that (a) \(|A F(h_0)|_{h_0} \leq B\varphi\) for \(B > 0\), (b) \(\text{tr} F(h_0) = 0\), (c) \(F(h_0) \) is \(L^2\).

**Corollary 3.9** If \((E,\overline{\partial}_E, h_0)\) is analytically stable, then we have a Hermitian metric \(h\) of \(E\) such that (i) \(h\) and \(h_0\) are mutually bounded, (ii) \(\det(h) = \det(h_0)\), (iii) \(A F(h) = 0\), (iv) \(|\overline{\partial}_E(\Lambda h^{-1})|\) and \(F(h)\) are \(L^2\). Moreover, the equality (1) holds. If \(h'\) is a Hermitian metric of \(E\) satisfying the conditions (i), (ii) and (iii), then \(h = h'\).

**Proof** Let \(\phi(t) = \log|t|\) if \(|t| > 1\). Let \(\phi\) be the \(C^\infty\)-function on \(X\) obtained as the pull back of \(\phi\) by the projection \(X \rightarrow \mathbb{R}\). We can take a complex coordinate \((z, w)\) such that \(\text{Re}(z) = t\) and \(g_X = dz \overline{dz} + dw \overline{dw}\). On \(\{t > 1\}\), we have \(\partial \phi = -t^{-1}dz - \overline{w} dw\). Let \(\partial \phi = -t^{-1}dz - \overline{w} dw\). We also have \(\partial \log \phi = -t^{-1}(\log t) t^{-1}dz\). Hence, \(\partial \phi\) and \(\overline{\partial} \phi\) are bounded, and \(\partial \log \phi\) and \(\overline{\partial} \phi\) are \(L^2\). Then, the claim follows from Theorem 2.9, Corollary 2.10 and Proposition 2.11.

**Remark 3.10** Spatially periodic instantons were studied by Charbonneau in [4], and more recently by Yoshino [27]. It is natural to expect to have a correspondence between \(L^2\)-instantons on \(X\) and filtered bundles on the natural compactification of \(X\) depending on the holomorphic structure, for which Corollary 3.9 will be useful.

3.3 Monopoles

3.3.1 Periodic monopoles

For any \(T > 0\), we set \(S^1_T := \mathbb{R}/T\mathbb{Z}\). If \(T = 1\), we denote it by \(S^1\). Take a finite subset \(Z \subset S^1_T \times \mathbb{C}_w\). Let \(\pi : S^1 \times S^1_T \times \mathbb{C}_w \rightarrow S^1 \times \mathbb{C}_w\) denote the projection \(\pi(t_1, t_2, w) = (t_2, w)\). We regard \(S^1 \times S^1_T\) as the quotient space \(\mathbb{C}_z/(\mathbb{Z} + \sqrt{-1}\mathbb{T})\) by \((t_1, t_2) \rightarrow z = t_1 + \sqrt{-1}t_2\). We set \(X := (S^1 \times S^1_T \times \mathbb{C}_w) \setminus \pi^{-1}(Z)\) with the Kähler metric \(g_X := dz \overline{dz} + dw \overline{dw}\). Set \(\varphi(z, w) := (1 + |w|^2)^{-\delta - 1}\) for \(\delta > 0\).

**Proposition 3.11** \((X, g_X, \varphi)\) satisfies the assumption in [21].

**Proof** Let \(f\) be an \(\mathbb{R}_{\geq 0}\)-valued function such that \(\Delta_X f \leq B\varphi\) on \(X\) for \(B \geq 0\). As remarked in Lemma 3.3, the inequality holds on \(S^1 \times S^1_T \times \mathbb{C}\) as distributions. Then, we obtain the claim from Proposition 3.3.

Note that \((X, g_X)\) is a hyper-Kähler manifold. We consider the complex manifold \(X^\lambda\) corresponding to the twistor parameter \(\lambda \in \mathbb{C}\). Indeed, we regard \(X\) as the quotient of an open subset in \(\mathbb{C}^2 = \{(z, w)\) \subset \mathbb{R}^4\). Local holomorphic coordinates for \(X^\lambda\) are given by \((\xi, \eta) = (z + \lambda w, w - \lambda \overline{w})\).

We have the natural \(S^1\)-action on \(S^1 \times S^1_T \times \mathbb{C}_w\) given by \(s_1(z, w) = (z + s_1, w)\). Let \((E, \overline{\partial}_E)\) be an \(S^1\)-equivariant holomorphic vector bundle on \(X^\lambda\). Let \(h_0\) be an \(S^1\)-invariant Hermitian metric of \(E\) such that (a) \(|A F(h_0)|_{h_0} \leq B\varphi\) for \(B > 0\), (b) \(\text{tr} F(h_0) = 0\).

**Corollary 3.12** Suppose that \((E, \overline{\partial}_E, h_0)\) is analytically stable with respect to the \(S^1\)-action.

- We have a unique \(S^1\)-invariant Hermitian metric \(h\) of \(E\) such that (i) \(h\) and \(h_0\) are mutually bounded, (ii) \(\det(h) = \det(h_0)\), (iii) \(A F(h) = 0\), (iv) \(|\overline{\partial}_E(\Lambda h)^{-1}\) is \(L^2\). If \(h'\) is a Hermitian metric satisfying the conditions (i), (ii) and (iii), then \(h' = h\).

- If \((E, \overline{\partial}_E, h_0)\) satisfies the additional condition (c) \(F(h_0)^+ \rightarrow 0\) and \(\overline{\partial}_E A F(h_0)^+ \rightarrow 0\) as \(|w| \rightarrow \infty\), then \(h\) satisfies the condition (e) \(F(h) \rightarrow 0\) as \(|w| \rightarrow \infty\).
Proof We have \( R > 0 \) such that \( Z \) is contained in \( S^1 \times \{|w| < R\} \). For each \( P \in Z \), let \( r_P : (S^1 \times \mathbb{C}_w) \rightarrow \mathbb{R}_{\geq 0} \) be the distance function from \( P \). We take a positive \( C^\infty \)-function \( \phi_1 \) on \( X \) such that \( \phi_1(z, w) = \log |w| \) if \(|w| > R\), and \( \phi_1 = -\log(r_P \circ \pi) \) around \( \pi^{-1}(P) \). Let us consider \( \partial \log \phi_1 \) on \( X^\lambda \setminus \pi^{-1}(P) \) for each \( P \in Z \). We have already observed that the restriction to \( \{|w| > R\} \) is \( L^2 \). Around \( P \), we have
\[
\partial \log \phi_1 = \phi_1^{-1} \cdot (r_P \circ \pi)^{-1} \cdot \partial (r_P \circ \pi) = O(\phi_1^{-1} \cdot (r_P \circ \pi)^{-1}).
\]
Hence, \( \partial \log \phi_1 \) is \( L^2 \) around \( P \). Then, the claim follows from Theorem 2.5, Proposition 2.6, and Proposition 2.11.

Remark 3.13 Note that \( S^1 \)-equivariant instantons on \( X^\lambda \) are equivalent to monopoles on \( (S^1 \times \mathbb{C}_w) \setminus Z \). In [20], by using Corollary 3.12, we study the correspondence of monopoles on \( (S^1 \times \mathbb{C}_w) \setminus Z \) of GCK type and filtered mini-holomorphic bundles on the compactification of \( (S^1 \times \mathbb{C}_w) \setminus Z \) depending on \( \lambda \). The latter objects are regarded as difference modules with parabolic structure.

Remark 3.14 In [5], Charbonneau and Hurtubise efficiently applied the theorem of Simpson for the construction of Hermitian-Einstein monopoles with Dirac type singularity on the product of a circle and a Riemann surface.

3.3.2 Doubly periodic monopoles

Let \( T^2 \) be the quotient space \( \mathbb{R}^2/\Gamma \) for a lattice \( \Gamma \subset \mathbb{R}^2 \). Take a finite subset \( Z \subset T^2 \times \mathbb{R} \). Let \( \pi : S^1 \times T^2 \times \mathbb{R} \rightarrow T^2 \times \mathbb{R} \) be the projection. We regard \( S^1 \times T^2 \) as the quotient of \( \mathbb{R}^3 \). We set \( X := (S^1 \times T^2 \times \mathbb{R}) \setminus \pi^{-1}(Z) \) with the Euclidean metric \( g_X \) as in (3.22). We take an \( \mathbb{R} \)-isomorphism \( \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \simeq \mathbb{C}^2 \) for which the multiplication of \( \sqrt{-1} \) is an isometry with respect to \( g_X \). It induces a complex structure on \( X \). Let \( \varphi \) be a \( C^\infty \)-function on \( X \) as in (3.22).

Proposition 3.15 \((X, g_X, \varphi)\) satisfies the assumption in (2.1).

Proof It follows from Proposition 3.8 as in the case of Proposition 3.11.

Let \((E, \overline{\partial}_E)\) be an \( S^1 \)-equivariant holomorphic vector bundle on \( X^\lambda \) with an \( S^1 \)-invariant Hermitian metric \( h_0 \). Suppose that (a) \( |Af(h_0)|_h \leq B \varphi \) for some \( B > 0 \), (b) \( \text{tr} F(h_0) = 0 \). By an argument similar to the proof of Corollary 3.12 by using Proposition 2.7 instead of Proposition 2.6, we obtain the following.

Corollary 3.16 Suppose that \((E, \overline{\partial}_E, h_0)\) on \( X^\lambda \) is analytically stable with respect to the \( S^1 \)-action.

- We have a unique \( S^1 \)-invariant Hermitian metric of \( E \) such that (i) \( h \) and \( h_0 \) are mutually bounded, (ii) \( \det(h) = \det(h_0) \), (iii) \( Af(h) = 0 \), (iv) \( \overline{\partial}_E(h h_0^{-1}) \) is \( L^2 \), If \( h' \) is an \( S^1 \)-invariant Hermitian metric satisfying the conditions (i), (ii) and (iii), then \( h' = h \).
- If \( h_0 \) satisfies the additional condition (c) \( F(h_0)^+ \) and \( \overline{\partial}_E AF(h_0)^- \) are bounded on \( \{|t| > R\} \) for some \( R \), then \( h \) satisfies the condition (c) \( F(h)^+ \) is bounded on \( \{|t| > R\} \).

Remark 3.17 \( S^1 \)-equivariant instantons on \( X \) are equivalent to monopoles on \((T^2 \times \mathbb{R}) \setminus Z\). We plan to study the correspondence between monopoles on \((T^2 \times \mathbb{R}) \setminus Z\) and filtered mini-holomorphic bundles on the compactifications depending on complex structures, for which Corollary 3.16 would be useful. For appropriate complex structures, the latter objects are regarded as \( q \)-difference modules with parabolic structure.

3.3.3 Doubly periodic monopole corresponding to some concrete algebraic data

To show that Proposition 3.15 (and hence Theorem 2.5) is already useful for the construction of doubly periodic monopoles, let us explain the existence of doubly periodic monopoles corresponding to some algebraic data. There are many similar constructions.

For simplicity, let us consider the case where \( T^2 \) is isometric to the product \( \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \). We may identify \( S^1 \times T^2 \times \mathbb{R} \) with the Euclidean metric and \((\mathbb{C}_z/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})) \times \mathbb{C}_w^* \) with the metric \( dz d\sigma + |w|^{-2} dw d\tau \). We regard it as the quotient of \( \mathbb{C} \times \mathbb{C}^* \) by the \((\mathbb{Z} + \sqrt{-1}\mathbb{Z})\)-action given as \( \tilde{\kappa}_\chi(z, w) = (z + \chi, w) \ (\chi \in \mathbb{Z} + \sqrt{-1}\mathbb{Z}) \).
We set $T_z := \mathbb{C}_z/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$ and $\Gamma := \mathbb{Z} + \sqrt{-1}\mathbb{Z}$. We define the $\mathbb{R}$-action on $\mathbb{C}_z \times \mathbb{C}_w^*$ by $\rho_\alpha(z, w) = (z + \alpha, w)$. The induced $S^1$-action on $T_z \times \mathbb{C}_w^*$ is also denoted by $\rho_\alpha$.

Take $\alpha \in \mathbb{C}^*$. Let $L$ be the line bundle on $\mathbb{C}_z \times \mathbb{C}_w^*$ with a frame $e$. We define the $\Gamma$-action on $L$ by $\tilde{\kappa}_1^*\rho_\alpha = e$ and $\tilde{\kappa}_1^*(\alpha e) = \alpha e$. We also define the $\mathbb{R}$-action on $L$ by $\rho_\alpha^* = e$. We obtain the $S^1$-equivariant holomorphic bundle $\mathcal{L}$ on $S^1 \times \mathbb{C}_w^*$ induced by $L$. Let $h_{L, \alpha}$ be the metric of $L$ given by

$$h_{L, \alpha}(e, e) = |\alpha|^{2 \Im(z)}.$$ 

Then, we have $\tilde{\kappa}_1^*h_{L, \alpha} = h_{L, \alpha}$ and $\rho_\alpha^*h_{L, \alpha} = h_{L, \alpha}$. The curvature of $(L, \tilde{\mathcal{D}}_L, h_{L, \alpha})$ is $\mathcal{D}\mathcal{D}\log |\alpha|^{2 \Im(z)} = 0$. Hence, $(L, \tilde{\mathcal{D}}_L, h_{L, \alpha}$) gives an $(S^1 \times \Gamma)$-equivariant unitary flat line bundle on $\mathbb{C}_z \times \mathbb{C}_w$. It induces an $S^1$-equivariant instanton $(L, \tilde{\mathcal{D}}_L, h_{L, \alpha})$ on $T_z \times \mathbb{C}_w$, i.e., a monopole on $S^1 \times \mathbb{C}_w$.

Let $V$ be a holomorphic vector bundle on $\mathbb{C}_z \times \mathbb{C}_w$ of rank 2 equipped with a holomorphic frame $e_1, e_2$. We define the $\Gamma$-action on $V$ as follows:

$$\tilde{\kappa}_1^*(e_1, e_2) = (e_1, e_2), \quad \tilde{\kappa}_1^*(\alpha e_1, e_2) = (e_1, e_2) \left( \begin{array}{cc} w & a \\ a & \bar{w}^{-1} \end{array} \right).$$

We also define the $\mathbb{R}$-action on $V$ by $\rho_\alpha^*(e_1, e_2) = (e_1, e_2)$. We obtain an $S^1$-equivariant holomorphic bundle $\mathcal{Y}$ on $T_z \times \mathbb{C}_w^*$.

Set $L := \det(V)$, which is equipped with the frame $e := e_1 \wedge e_2$. We have $\tilde{\kappa}_1^*(e) = e$, $\tilde{\kappa}_1^*(\alpha e) = (1 - a^2)e$ and $\rho_\alpha^*(e) = e$. Note that $(L, \tilde{\mathcal{D}}_L, h_{L, \alpha})$ is an $S^1$-equivariant unitary flat line bundle, as explained above. We have the induced $S^1$-equivariant instanton $(L, \tilde{\mathcal{D}}_L, h_{L, \alpha})$ on $T_z \times \mathbb{C}_w$.

Let us prove that we have a family of $S^1$-invariant Hermitian metrics $\tilde{h}_V^{(c_0, c_\infty)}$ ($(c_0, c_\infty) \in \mathbb{R}^2$) of $\mathcal{Y}$ such that (i) $\Lambda F(\tilde{h}_V^{(c_0, c_\infty)}) = 0$, (ii) $\det(\tilde{h}_V^{(c_0, c_\infty)}) = h_{L, \alpha}$.

As a preliminary, we construct $S^1$-invariant metrics $\tilde{h}_V^{(c_0, c_\infty)}$ of $\mathcal{Y}$ which satisfies the ASD-equation outside a compact subset. Note that the roots of the polynomial $T^2 - (w + w^{-1})T + 1 - a^2$ are

$$\frac{1}{2}(w + w^{-1} \pm \sqrt{(w + w^{-1})^2 - 4(1 - a^2)}).$$

Let $U_\infty$ be a small neighbourhood of $\infty$ in $\mathbb{P}^1_w$. Set $U_\infty^* := U_\infty \setminus \{\infty\}$. On $U_\infty$, let $\delta_\infty(w^{-1})$ be the branch of $\sqrt{1 + (-2 + 4a^2)w^{-2} + w^{-4}}$ such that $\delta_\infty(0) = 1$. We set

$$\beta_1(w^{-1}) = \frac{w}{2}(1 + w^{-2} + \delta_\infty(w^{-1})).$$

It is a root of $T^2 - (w + w^{-1})T + 1 - a^2$. The other root is

$$\beta_2(w^{-1}) = \frac{w}{2}(1 + w^{-2} - \delta_\infty(w^{-1})) = \beta_1(w^{-1})^{-1}(1 - a^2).$$

We define a holomorphic frame $v_1, v_2$ of $V$ on $\mathbb{C}_z \times U_\infty^*$ as follows:

$$v_i := \alpha e_1 + (\beta_i(w^{-1}) - w)e_2 \quad (i = 1, 2).$$

Then, we have $\tilde{\kappa}_{n_1 + n_2 \sqrt{-1}}(v_i) = \beta_i(w^{-1})^{n_2}v_i$ and $\rho_\alpha^*(v_i) = v_i (i = 1, 2)$. We have

$$v_1 \wedge v_2 = a(\beta_2(w^{-1}) - \beta_1(w^{-1}))e = -aw\delta_\infty(w^{-1})e.$$

Take any $c_\infty \in \mathbb{R}$. We define the metric $\tilde{h}_V^{(c_\infty)}$ of $V_{\mathbb{C}_z \times U_\infty^*}$ as follows:

$$\tilde{h}_V^{(c_\infty)}(v_1, v_1) = |w|^{c_\infty}|\beta_1(w^{-1})|^{2 \Im(z)}|aw\delta_\infty|^2, \quad \tilde{h}_V^{(c_\infty)}(v_2, v_2) = |w|^{-c_\infty}|\beta_2(w^{-1})|^{2 \Im(z)}, \quad \tilde{h}_V^{(c_\infty)}(v_1, v_2) = 0.$$
Then, the metric $h_{0,\mathcal{U}_w}^{(c_0)}$ is invariant under the $\Gamma$-action and the $\mathbb{R}$-action. The induced metric on $L|_{\mathbb{C}_z \times U_{\infty}^*}$ is equal to $h_{L,1-a^2|\mathbb{C}_z \times U_{\infty}^*}$. We can easily see the following for $i = 1, 2$:

$$\partial_{z} \partial_{\bar{z}} \log h_{0,\mathcal{U}_w}^{(c_0)}(v_i, v_i) = w \partial_{w} \overline{\partial_{w}} \log h_{0,\mathcal{U}_w}^{(c_0)}(v_i, v_i) = 0.$$  

We also have

$$\partial_{w} \overline{\partial_{w}} \log h_{0,\mathcal{U}_w}^{(c_0)}(v_i, v_i) = \frac{1}{2 \sqrt{-1}} \frac{w \partial_{w} \overline{\partial_{w}} \log h_{0,\mathcal{U}_w}^{(c_0)}}{\beta_1(w^{-1})}, \quad \partial_{w} \overline{\partial_{w}} \log h_{0,\mathcal{U}_w}^{(c_0)}(v_i, v_i) = \frac{\sqrt{-1} w \partial_{w} \overline{\partial_{w}} \log h_{0,\mathcal{U}_w}^{(c_0)}}{\beta_1(w^{-1})}.$$  

Hence, we have $\Lambda F(h_{0,\mathcal{U}_w}^{(c_0)}) = 0$, and $F(h_{0,\mathcal{U}_w}^{(c_0)})$ is bounded.

We make a similar construction around $w = 0$. Let $U_0$ be a small neighbourhood of 0 in $\mathbb{C}_w$. We set $U_0^*: = U_0 \setminus \{0\}$. On $U_0$, let $\delta_0(w)$ be the branch of $\sqrt{1 + (-2 + 4a^2)w^2 + w^4}$ such that $\delta_0(0) = 1$. We set

$$\gamma_1(w) = \frac{w^{-1}}{2}(1 + w^2 + \delta_0(w)).$$  

It is a root of $T^2 - (w + w^{-1})T + 1 - a^2$. The other root is

$$\gamma_2(w) = \frac{w^{-1}}{2}(1 + w^2 - \delta_0(w)) = \gamma_1(w)^{-1}(1 - a^2).$$  

We define a holomorphic frame $u_1, u_2$ of $V|_{\mathbb{C}_z \times U_{w}}$ as follows:

$$u_i = ae_1 + (\gamma_i(w) - w)e_2 \quad (i = 1, 2).$$  

Then, we have $\bar{\kappa}_{n+1} + \overline{\partial_{w}}(u_i) = \gamma_i(w)^{n2}u_i$ and $\rho^*_\mathbb{C}_z(u_i) = u_i$ for $i = 1, 2$. We have $u_1 \wedge u_2 = -aw\delta_0 e$.

Take any $c_0 \in \mathbb{R}$. We define the metric $h_{0,\mathcal{U}^*_w}^{(c_0)}$ of $V|_{\mathbb{C}_z \times U_{w}}^*$ as follows:

$$h_{0,\mathcal{U}^*_w}^{(c_0)}(u_1, u_1) = |w|^{-c_0}|\gamma_1(w)|^{2\text{Im}(z)}|a w^{-1}\delta_0(w)|^{2}, \quad h_{0,\mathcal{U}^*_w}^{(c_0)}(u_2, u_2) = |w|^{c_0}|\gamma_2(w)|^{2\text{Im}(z)}, \quad h_{0,\mathcal{U}^*_w}^{(c_0)}(u_1, u_2) = 0.$$  

Then, the metric $h_{0,\mathcal{U}^*_w}^{(c_0)}$ is invariant under the action of $\mathbb{R}$ and $\Gamma$. The induced metric on $L|_{\mathbb{C}_z \times U_{w}}^*$ is equal to $h_{L,1-a^2|\mathbb{C}_z \times U_{w}}^*$. As in the case of $h_{0,\mathcal{U}^*_w}^{(c_0)}$, we have $\Lambda F(h_{0,\mathcal{U}^*_w}^{(c_0)}) = 0$, and $F(h_{0,\mathcal{U}^*_w}^{(c_0)})$ is bounded.

We take a Hermitian metric $h_{0,\mathcal{V}}^{(c_0, c_{\infty})}$ of $V$ such that (i) it is invariant under the actions of $\mathbb{R}$ and $\Gamma$, (ii) the induced metric on $L$ is equal to $h_{L,1-a^2}$, (iii) the restriction to $\mathbb{C}_z \times U_{\infty}^*$ is equal to $h_{0,\mathcal{U}_w}^{(c_0)}$, (iv) the restriction to $\mathbb{C}_z \times U_{w}^*$ is equal to $h_{0,\mathcal{U}^*_w}^{(c_0)}$.

We obtain the $S^1$-invariant Hermitian metric $h_{0,\mathcal{V}}^{(c_0, c_{\infty})}$ of $V$ induced by $h_{0,\mathcal{V}}^{(c_0, c_{\infty})}$. By the construction, we have $\det(h_{0,\mathcal{V}}^{(c_0, c_{\infty})}) = h_{L,1-a^2}$. We also have $\Lambda F(h_{0,\mathcal{V}}^{(c_0, c_{\infty})}) = 0$ on $T_z \times (U_{w}^* \sqcup U_{\infty}^*)$, and $F(h_{0,\mathcal{V}}^{(c_0, c_{\infty})})$ is bounded on $S^1 \times \mathbb{C}_w$.

**Proposition 3.18** For each $(c_0, c_{\infty}) \in \mathbb{R}^2$, we have a unique $S^1$-invariant Hermitian metric $h_{0,\mathcal{V}}^{(c_0, c_{\infty})}$ of $V$ such that (i) $h_{0,\mathcal{V}}^{(c_0, c_{\infty})}$ and $h_{0,\mathcal{U}_w}^{(c_0)}$ are mutually bounded, (ii) $\det(h_{0,\mathcal{V}}^{(c_0, c_{\infty})}) = h_{L,1-a^2}$, (iii) $\Lambda F(h_{0,\mathcal{V}}^{(c_0, c_{\infty})}) = 0$, (iv) $\overline{\partial_{E}}(h_{0,\mathcal{V}}^{(c_0, c_{\infty})})$ is $L^2$, (v) $F(h_{0,\mathcal{V}}^{(c_0, c_{\infty})})$ is bounded.

**Proof** According to Corollary 3.16, it is enough to prove that $(\mathcal{V}, \overline{\partial_{E}}, h_{0,\mathcal{V}}^{(c_0, c_{\infty})})$ is analytically stable. Let us check a stronger condition that a non-trivial $S^1$-equivariant holomorphic subbundle of $\mathcal{V}$ is 0 or $\mathcal{V}$.

Let $\mathcal{V}'$ be the product bundle on $\mathbb{R} \times \mathbb{C}_w$ with a frame $e_1', e_2'$. We define the action of $\mathbb{Z}$ on $\mathcal{V}'$ by

$$\kappa^*_w(e_1', e_2') = (e_1', e_2') \left( \begin{array}{cc} w & a \\ a & w^{-1} \end{array} \right).$$  

We obtain the bundle $\mathcal{V}'$ on $S^1 \times \mathbb{C}_w$.  

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We define the differential operators $\partial_{\nu',t}$ and $\partial_{\nu',\pi}$ on $\mathcal{V}'$ as follows, for any $f \in C^\infty(\mathbb{R} \times \mathbb{C}^*_w)$:

$$\partial_{\nu',t}(fe_i') = \partial_t(f)e_i', \quad \partial_{\nu',\pi}(fe_i') = \partial_{\pi}(f)e_i'.$$

It induces differential operators $\partial_{\nu',t}$ and $\partial_{\nu',\pi}$ on $\mathcal{V}'$.

The restriction $\mathcal{V}'|_{(0)\times \mathbb{C}^*_w}$ and $\mathcal{V}'|_{(1)\times \mathbb{C}^*_w}$ are holomorphic vector bundles by $\partial_{\nu',\pi}$. We have the isomorphism $\Phi : \mathcal{V}'|_{(0)\times \mathbb{C}^*_w} \cong \mathcal{V}'|_{(1)\times \mathbb{C}^*_w}$ given by the parallel transport along the paths $(t,w)$ $(0 \leq t \leq 1)$ for any $w \in \mathbb{C}^*$ with respect to $\partial_{\nu',t}$. Because $\partial_{\nu',t}$ and $\partial_{\nu',\pi}$ are commutative, $\Phi$ is holomorphic. It induces the monodromy automorphism $\Phi$ on $\mathcal{V}'|_{(0)\times \mathbb{C}^*_w}$.

**Lemma 3.19** Suppose that $\mathcal{V}'_1$ is a subbundle of $\mathcal{V}'$ such that the following holds:

- If $s$ is a local section of $\mathcal{V}'_1$, then $\partial_{\nu',t}s$ and $\partial_{\nu',\pi}s$ are also local sections of $\mathcal{V}'_1$.

Then, $\mathcal{V}'_1$ is $\mathcal{V}'$ or $0$.

**Proof** By the condition, $\mathcal{V}'|_{(0)\times \mathbb{C}^*_w}$ is a holomorphic subbundle of $\mathcal{V}'|_{(0)\times \mathbb{C}^*_w}$ such that $\Phi(\mathcal{V}'|_{(0)\times \mathbb{C}^*_w}) = \mathcal{V}'|_{(1)\times \mathbb{C}^*_w}$. Because the eigenvalues of $\Phi$ are not single-valued, we can conclude that $\mathcal{V}'|_{(1)\times \mathbb{C}^*_w}$ is $\mathcal{V}'|_{(0)\times \mathbb{C}^*_w}$ or $0$. Then, we obtain the claim of Lemma 3.19.

Let $p : \mathbb{C}^*_w \times \mathbb{C}^*_w \longrightarrow \mathbb{R} \times \mathbb{C}^*_w$ be given by $p(z,w) = (\text{Im}(z),w)$. We have the $C^\infty$-isomorphism $V \cong p^{-1}(V')$ by identifying $e_i$ and $p^{-1}(e_i')$. By the construction, for any section $s$ of $\mathcal{V}'$, we have

$$\partial_{\nu,\pi}p^{-1}(s) = \frac{-1}{2}p^{-1}(\partial_{\nu',t}s), \quad \partial_{\nu,\pi}p^{-1}(s) = p^{-1}(\partial_{\nu',\pi}s).$$

(36)

Let $p_1 : T_z \times \mathbb{C}^*_w \longrightarrow S^1 \times \mathbb{C}^*_w$ be the projection induced by $p$. The isomorphism $V \cong p^{-1}(V')$ induces an isomorphism $V \cong p_1^{-1}(V')$.

Let $\mathcal{V}'_1 \subset \mathcal{V}$ be an $S^1$-invariant holomorphic subbundle. It induces a $C^\infty$-subbundle $\mathcal{V}'_1 \subset \mathcal{V}'$. Moreover, for any section $s$ of $\mathcal{V}'_1$, $\partial_{\nu',t}s$ and $\partial_{\nu',\pi}s$ are also sections of $\mathcal{V}'_1$, which we can observe by (36). Hence, by Lemma 3.19 we obtain that $\mathcal{V}'_1$ is $\mathcal{V}'$ or $0$. It implies that $\mathcal{V}_1$ is $\mathcal{V}$ or $0$. Thus, Proposition 3.18 is proved.

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