BOUNDS ON THE CROSSCAP NUMBER OF TORUS KNOTS

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Abstract. For a torus knot $K$, we bound the crosscap number $c(K)$ in terms of the genus $g(K)$ and crossing number $n(K)$: $c(K) \leq \lfloor (g(K) + 9)/6 \rfloor$ and $c(K) \leq \lfloor (n(K) + 16)/12 \rfloor$. The $(6n-2,3)$ torus knots show that these bounds are sharp.

1. Introduction

In 1978, Clark [C] defined the crosscap number $c(K)$ of the knot $K$ to be the minimal genus of all non-orientable surfaces which span the knot and gave an upper bound for this number in terms of $g(K)$, the genus: $c(K) \leq g(K) + 1$. The obvious next question is if there is some way of bounding the genus in terms of the crosscap number. Or, as Adams [A] asked, is there some family of knots for which the difference $|g(K) - c(K)|$ increases without bound?

The torus knots are a natural target for this question since Teragaito [T] has recently classified their crosscap numbers. We soon noticed that the $(2n+1,2)$ torus knots provide an answer to Adams’s question. The genus of such a knot is $n$ while the crosscap number is one. (Indeed, it’s quite easy to see that these knots span Möbius bands. Take a strip of paper and give it $2n + 1$ half twists before joining the ends. The band’s edge will be a $(2n+1,2)$ torus knot.) Thus, for the $(2n+1,2)$ torus knots the difference $g(K) - c(K)$ is $n - 1$ and this difference increases without bound as $n$ approaches infinity.

This example suggests that $g(K) + 1$ is a rather poor estimate for crosscap number if we restrict attention to the class of torus knots. Indeed, we have the following:

**Theorem 1.** For a torus knot $K$, $c(K) \leq \lfloor (g(K) + 9)/6 \rfloor$.

Here, $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

Since the crossing number of a torus knot is roughly twice the genus, we can also improve the bound on crosscap number in terms of the crossing number $n(K)$.

**Theorem 2.** For a torus knot $K$, $c(K) \leq \lfloor (n(K) + 16)/12 \rfloor$.

Compare this with Murakami and Yasuhara’s [MY] general result that $c(K) \leq \lfloor n(K)/2 \rfloor$.

The $(6n-2,3)$ torus knots show that the inequalities in our two theorems are sharp. These knots have genus $6n - 3$ and crossing number $12n - 4$. Thus, both $(g(K) + 9)/6$ and $(n(K) + 16)/12$ yield the crosscap number $n + 1$ for these knots.

1991 Mathematics Subject Classification. Primary 57M25.
Key words and phrases. crosscap number, non-orientable genus, genus, crossing number, torus knot.

The second author is an undergraduate student who was supervised by the first author during an REU held at CSU, Chico in the summer of 2004 and funded by NSF REU Award 0354174.
Our arguments make use of Teragaito's classification of the crosscap number of torus knots. He shows that the crosscap number of a \((p, q)\) torus knot is given by summing certain coefficients in the continued fraction expansion of \(p/q\), \(q/p\), \((pq + 1)/p^2\), or \((pq - 1)/p^2\). Our main tool is an observation about the sums of continued fraction coefficients: if \(p > q > 0\) are relatively prime integers, then the sum of the coefficients in the continued fraction expansion of \(p/q\) is at most \(p\). This provides a bound on the crosscap number that we can in turn compare to the genus or crossing number.

The paper is organised as follows. After this introduction, we give a brief overview of Teragaito's classification in Section 2 as well as some basic results about continued fractions. In Section 3, we apply these techniques to prove Theorem 2. In Section 4, we outline the proof of Theorem 1.

2. Continued Fractions and Teragaito's classification

In this section, we give an overview of Teragaito's classification of the crosscap number of torus knots as well as some basic facts about continued fractions that will prove useful in the sequel. Recall that a positive rational number \(r\) can be represented uniquely by a simple continued fraction

\[
r = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}} = [a_0, a_1, \ldots, a_n]
\]

where each \(a_i, 1 \leq i \leq n\), is a positive integer and \(a_n > 1\). We will write \(r = [a_0, a_1, \ldots, a_n]\). If \(\frac{p}{q} = [a_0, a_1, \ldots, a_n]\), then the function \(N(p, q)\) defined by Bredon and Wood is given as follows. We first sum the \(a_i\) in order, beginning with \(a_0\), and skipping the succeeding \(a_i\) whenever the partial sum becomes even. We then halve the total. For example, \(8/3 = [2, 1, 2]\), so \(N(8, 3) = (2 + 2)/2 = 2\). Since \(34/49 = [0, 1, 2, 3, 1, 3]\), we have \(N(34, 49) = (0 + 2 + 1 + 3)/2 = 3\).

For a \((p, q)\) torus knot \(K\) with \(p, q > 0\), we will say \(K\) is odd (respectively, even) if \(pq\) is odd (respectively, even). For the statement of Teragaito's theorem, we will assume \(p > q\) if \(K\) is odd and \(p\) is even if \(K\) is even.

**Theorem 3** (Theorem 1 of [T]). Let \(K\) be the non-trivial torus knot of type \((p, q)\), where \(p, q > 0\).

1. If \(K\) is even, then \(c(K) = N(p, q)\).
2. If \(K\) is odd, then \(c(K) = \min\{N(pq - 1, p^2), N(pq + 1, p^2)\}\)

In light of Theorem 3, the following observation about continued fractions will be useful in bounding the crosscap numbers of torus knots. We omit the straightforward proof by induction.

**Lemma 4.** Let \(p > q > 0\) be relatively prime and let \(p/q = [a_0, a_1, \ldots, a_n]\) be a simple continued fraction. Then \(\sum_{i=0}^{n} a_i \leq p\).

Finally, we will make use of a lemma that relates the continued fractions of \((pq + 1)/p^2\) and \(q/p\). For this lemma, assume \(p > q > 1\).
Lemma 5 (Lemma 9 of [1]). If \( q/p = [a_0, a_1, a_2, \ldots, a_n] \), then

\[
\frac{(pq - 1)}{p^2} = \begin{cases} 
[a_0, a_1, a_2, \ldots, a_n-1, a_n + 1, a_n - 1, \\
a_{n-1}, a_{n-2}, \ldots, a_2, a_1] & \text{if } n \text{ is odd,} \\
a_0, a_1, a_2, \ldots, a_n-1, a_n - 1, a_n + 1, \\
a_{n-1}, a_{n-2}, \ldots, a_2, a_1] & \text{if } n \text{ is even,}
\end{cases}
\]

and

\[
\frac{(pq + 1)}{p^2} = \begin{cases} 
[a_0, a_1, a_2, \ldots, a_n-1, a_n - 1, a_n + 1, \\
a_{n-1}, a_{n-2}, \ldots, a_2, a_1] & \text{if } n \text{ is odd,} \\
a_0, a_1, a_2, \ldots, a_n-1, a_n + 1, a_n - 1, \\
a_{n-1}, a_{n-2}, \ldots, a_2, a_1] & \text{if } n \text{ is even.}
\end{cases}
\]

Note that, since \( p > q \), \( a_0 = 0 \) in the lemma. Also, if \( a_1 = 1 \), then \([a_0, \ldots, a_3, a_2, a_1]\) should be replaced by \([a_0, \ldots, a_3, a_2 + 1]\).

3. Proof of Theorem 2

In this section, we will prove

Theorem 2. For a torus knot \( K \), \( c(K) \leq ⌊(n(K) + 16)/12⌋ \).

Proof: By definition [C], the crosscap number of the unknot is zero and the theorem holds in this case. So, let \( K \) be a \((p,q)\) torus knot where \( p > q > 1 \) are relatively prime integers. The proof breaks into three cases according to whether \( q \) is even, \( p \) is even, or both are odd.

3.1. \( q \) is even. Let us assume \( q \) is even. By the Euclidean algorithm, there are unique positive integers \( m \) and \( k \) with \( p = qm - k \) and \( k < q \). Since \( p > q \), we have \( m > 1 \). The crosscap number of \( K \), \( N(p,q) \), is determined by the continued fraction

\[
\frac{q}{p} = \frac{q}{qm - k} = 0 + \frac{1}{(m-1) + \frac{1}{\frac{1}{q}}}
\]

In calculating \( N(p,q) \) we would skip \( m - 1 \) (since the first partial sum \( a_0 = 0 \) is even) and add certain of the coefficients in the continued fraction of \( q/(q-k) \). By Lemma 3, the sum of all the coefficients in \( q/(q-k) \) is bounded by \( q \). For \( N(p,q) \), the sum is halved, so we have \( c(K) = N(p,q) \leq q/2 \).

Since \( q < p \), the crossing number of \( K \) is \( n(K) = p(q-1) = (qm-k)(q-1) \). In order to prove the theorem in this case, it’s enough to show that

\[
\frac{q}{2} \leq \frac{(qm-k)(q-1)+16}{12}.
\]

(1)

Our strategy is to use induction on \( m \) and \( k \). In the \( m \) inductive step, \( m \) will increase by one, while for the \( k \) induction, we’ll decrement by one at each step. We have already mentioned that \( m > 1 \) and, since \( k < q \), we begin our induction with the case \( m = 2 \) and \( k = q - 1 \). Then Equation 1 becomes \( q^2 - 6q + 15 \geq 0 \) which is true for all integers \( q \). So the theorem is proved in this case. If \( m \) is increased by 1, then the right hand side of Equation 1 is increased by \( q(q - 1)/12 \) while the left hand side is unchanged. Since \( q > 1 \), the equation will still hold if \( m \) is increased by 1. For the \( k \) induction, if \( k \) is decreased by 1, the right hand side of Equation 1 is increased by \( (q - 1)/12 \) while the left hand side is unchanged. By induction, Equation 1 holds for all \( m > 1 \) and all \( 0 \leq k < q \). This proves the theorem in the case \( q \) is even.
3.2. \textbf{p is even.} Let us assume \(p\) is even. In this case \(c(K) = N(p,q)\). We can write \(p = 2qm - k\) for some positive integers \(m\) and \(k\) with \(k < 2q\). If \(k = q\), then \(q \mid p\) contradicting our assumption that \(p\) and \(q\) are relatively prime. We have two subcases depending on whether \(k < q\) or \(k > q\).

Suppose \(k < q\). The continued fraction for \(N(p,q)\) is

\[
p/q = (2qm - k)/q = 2m - 1 + \frac{1}{q-k}\
\]

so that \(c(K) = N(p,q) \leq (2m-1+q)/2\) (using Lemma \ref{lem:continued-fraction}). On the other hand, the crossing number is \(n(K) = p(q-1) = (2qm-k)(q-1)\), so the theorem can be proved by verifying

\[
\frac{2m-1+q}{2} \leq \frac{(2qm-k)(q-1) + 16}{12}.
\]

Again, we will use induction on \(m\) and \(k\). If \(m = 1\) and \(k = q-1\), the inequality becomes \((q-3)^2 \geq 0\). Note that, as \(p\) is even, \(q\) is odd. We were already assuming \(q > 1\), so we must have \(q \geq 3\). Thus, Equation \(\ref{eq:case1}\) holds in the case \(m = 1, k = q-1\). If \(m\) is increased by 1, the left hand side of the inequality is increased by 1 while the right is increased by \(2q(q-1)/12\). Since \(q \geq 3\), we have \(2q(q-1)/12 \geq 1\) and the inductive step for \(m\) is proved. If \(k\) is decreased by 1, the left hand side is unchanged while the right hand side increases by \((q-1)/12\). Thus, Equation \(\ref{eq:case1}\) holds for all \(m \geq 1\) and \(k < q\) and the theorem is proved in the case where \(k < q\).

If \(k > q\), we have

\[
p/q = (2qm - k)/q = 2m - 2 + \frac{1}{q-k}\
\]

so that \(c(K) = N(p,q) \leq m - 1 + q/2\). In this case we must verify the inequality

\[
m - 1 + \frac{q}{2} \leq \frac{(2qm-k)(q-1) + 16}{12}.
\]

If \(m = 1\), then \(p = 2qm - k = 2q - k < q\) which contradicts our assumption that \(p > q\). Therefore, the base step for the induction is \(m = 2\) and \(k = 2q - 1\). With these values, Equation \(\ref{eq:case1}\) becomes \(2q^2 - 7q + 3 \geq 0\) and this inequality holds for all \(q \geq 3\). The induction for \(m\) and \(k\) is similar to the previous subcase and, thus, the theorem is proved for all even \(p\).

3.3. \textbf{pq odd.} Suppose both \(p\) and \(q\) are odd, and let \(p = qm - k\) where \(m\) and \(k\) are positive integers with \(k < q\). Since \(p > q\), we have \(m > 1\). In this case \(c(K)\) is determined by the continued fractions of \((pq \pm 1)/p^2\) and, by Lemma \ref{lem:continued-fraction}, these are related to \(q/p\). Now,

\[
q/p = q/(qm - k) = 0 + \frac{1}{m-1 + \frac{1}{a_2 + \cdots}}.
\]

So, if we write \(q/p = [a_0, a_1, \ldots, a_n]\) as in Lemma \ref{lem:continued-fraction}, then \(a_0 = 0, a_1 = m-1,\) and \(q/(q-k) = [a_2, a_3, \ldots, a_n]\). Moreover, by Lemma \ref{lem:continued-fraction}, \(\sum_{i=2}^{\infty} a_i \leq q\).

For odd knots, \(c(K) = \min\{N(pq-1, p^2), N(pq+1, p^2)\}\). By Lemma \ref{lem:continued-fraction} the continued fractions for \((pq \pm 1)/p^2\) both begin \([0, m-1, a_2, \ldots]\). So in calculating \(N(pq \pm 1, p^2)\) we omit the \(m-1\) coefficient and the summation effectively begins with \(a_2\). Moreover, by the lemma, whichever of \((pq+1)/p^2\) and \((pq-1)/p^2\) is used (and whether or not \(n\) is even), the sum of the coefficients beginning with \(a_2\) is
2 \sum_{i=2}^{n} a_i + a_1. \text{ Thus, } c(K) \leq \sum_{i=2}^{n} a_i + a_1/2 \leq q + (m - 1)/2. \text{ Since } c(K) \text{ is an integer, we have } c(K) \leq \lfloor q + (m - 1)/2 \rfloor.

The crossing number in this case is again \( n(K) = p(q - 1) \); so we can prove the theorem by verifying the inequality:

\[
|q + \frac{m - 1}{2}| \leq \frac{(qm - k)(q - 1) + 16}{12}.
\]

In fact, this inequality does not hold for all choices of \( q, m, \) and \( k \). Let us begin by delineating the cases where it does hold.

Since \( p \) and \( q \) are both odd, we will need to carry out two induction arguments, one for the case where \( m \) is odd and \( k \) is even and one with the opposite parities. The base case for \( m \) even is \( m = 2, k = q - 2 \). In this case Equation 4 becomes \( q^2 - 11q + 14 \geq 0 \) which is valid for all \( q > 9 \). The base case for \( m \) odd is \( m = 3, k = q - 1 \). Here the inequality becomes \( 2q^2 - 13q + 3 \geq 0 \) which is valid for all \( q \geq 7 \). Since \( q \) is odd, the smallest \( q \) for which both cases apply is \( q = 11 \). Let us show the induction for \( q \geq 11 \) and then examine smaller values of \( q \) individually.

**Case 1** \( q = 11 \). We've established that the base cases both hold if \( q = 11 \). We're left to verify the inductive steps. Note that in both inductions \( m \) will be increased by 2 at each step and \( k \) will be decreased by 2. If \( m \) is increased by 2, the left hand side of Equation 4 is increased by 1 while the right hand side increases by \( q(q - 1)/6 \). Since \( q \geq 11 \), the \( m \) inductive step preserves the inequality. (Indeed, this inductive step will be valid so long as \( q \geq 3 \).) If \( k \) is decreased by 2, the right hand side is increased by \( (q - 1)/6 \) and the left hand side is unchanged. Thus, the theorem is proved in the case \( q = 11 \).

**Case 2** \( q = 9 \). Since the \( m = 2 \) base case is problematic, let's instead begin the even \( m \) induction with \( m = 4 \) and \( k = q - 2 \). Then Equation 4 becomes \( 3q^2 - 13q + 2 \geq 0 \) which is true for all \( q \geq 5 \). As noted above, the \( m = 3 \) base case is valid, and the inductive arguments also go through when \( q = 9 \). So, in order to complete this case, we must address the knots that have \( m = 2 \). That is, we must verify the theorem for the knots \((17, 9), (13, 9), \) and \((11, 9)\). (Note that 15 and 9 are not relatively prime.) The crossing numbers \( n(K) \) of these knots are, respectively, 136, 104, and 88. The crosscap numbers \( c(K) \) are 5, 4, and 5. Thus, the theorem holds in these cases as well and is proved for the case \( q = 9 \).

**Case 3** \( q = 7 \). Our arguments above show that the theorem holds for \( q = 7 \) provided \( m \geq 3 \). Again, we can verify the knots with \( m = 2, (13, 7), (11, 7), \) and \((9, 7)\), directly. The crossing numbers for these knots are 78, 66, and 54 while the crosscap numbers are 4, 3, and 4. So these knots also satisfy the theorem.

**Case 4** \( q = 5 \). When \( q = 5 \), the \( m = 3 \) base case is no longer valid. However, Equation 4 is satisfied if we take \( q = 5, m = 5 \) and \( k = q - 1 = 4 \). Thus, induction arguments will take care of all cases where \( m \geq 4 \). We are left to investigate \( m = 2 \) and \( m = 3 \). That is, we are left with the knots \((13, 5), (11, 5), (9, 5), \) and \((7, 5)\). The crossing numbers are 52, 44, 36, and 28 while the crosscap numbers are all 3. This completes the argument in the case \( q = 5 \).

**Case 5** \( q = 3 \). If \( q = 3 \), we can explicitly calculate the continued fraction \( q/p \). Since \( p \) is odd and relatively prime to 3, \( p \) is of the form \( 6m + 1 \) or \( 6m - 1 \).

If \( p = 6m + 1 \), then \( q/p = [0, 2m, 3] \). As Teragaito shows, since \( 3x \equiv -1 \mod p \) has the even solution \( x = 2m \), the crosscap number is \( c(K) = N(pq - 1, p^2) \). By Lemma 5, \( (pq - 1)/p^2 = [0, 2m, 2, 4, 2m] \). Therefore, \( c(K) = (0 + 2 + 2m)/2 = m + 1 \). On the other hand, the crossing number is \( n(K) = p(q - 1) = 2(6m + 1) \). Thus,
\[ ((n(K) + 16)/12) = ((12m + 18)/12) = m + 1 = c(K) \] and the theorem holds in this case.

Finally, suppose \( p = 6m - 1 \) and \( q = 3 \). Then, \( q/p = \lfloor 0, 2m - 1, 1, 2 \rfloor \). Since \( 3x \equiv -1 \mod p \) has the odd solution \( x = 4m - 1 \), the crosscap number is \( N(pq + 1, p^2) \) (see [1]). By Lemma 4 \( (pq + 1)/p^2 = \lfloor 0, 2m - 1, 1, 3, 1, 2m - 1 \rfloor \) if \( m \neq 1 \) and \( [0, 1, 1, 3, 2] \) when \( m = 1 \). Thus, \( c(K) = (1 + 1 + 1 + 2m)/(2m - 1) = m + 1 \). Now, the crossing number is \( p(q - 1) = 2(6m - 1) \). Thus, \( ((n(K) + 16)/12) = ((12m + 14)/12) = m + 1 = c(K) \) and the theorem holds in this case as well.

Thus, we have proved Theorem 2 when \( pq \) is odd. This completes the proof of the theorem. \( \square \)

4. Proof of Theorem 1

In this section we prove

**Theorem 1.** For a torus knot \( K \), \( c(K) \leq \lfloor (g(K) + 9)/6 \rfloor \).

**Proof:** The argument is very similar to that used in proving Theorem 2. The theorem holds for the trivial knot, so we will assume \( p > q > 1 \) are relatively prime and \( K \) is the \((p, q)\) torus knot. We have three cases depending on whether \( q \) is even, \( p \) is even, or \( pq \) is odd.

Let’s assume \( q \) is even. Then as in the previous section, we have \( c(K) \leq q/2 \). Writing \( p = qm - k \) with \( m > 1 \) and \( k < q \) positive and using \( g(K) = (p-1)(q-1)/2 \), we can prove the theorem by verifying

\[
\frac{q}{2} \leq \frac{(qm - k - 1)(q - 1)/2 + 9}{6}
\]

If \( m = 2 \) and \( k = q - 1 \), the inequality becomes \( q^2 - 7q + 18 \geq 0 \) which is true for all integers \( q \). If \( m \) is increased by one or \( k \) is decreased by one, the right hand side increases while the left hand side is unchanged. Thus the inequality holds for all \( m > 2 \) and all positive \( k < q \). This proves the theorem in the case \( q \) is even.

Next, assume \( p \) is even and write \( p = 2qm - k \) with \( m \) and \( k < 2q \) positive. We have two subcases: \( k < q \) and \( k > q \). Suppose \( k < q \). Then, as in the previous section, \( c(K) \leq (2m - 1 + q)/2 \) and we can prove the theorem by verifying

\[
\frac{2m - 1 + q}{2} \leq \frac{(2qm - k - 1)(q - 1)/2 + 9}{6}
\]

If \( m = 1 \) and \( k = q - 1 \), the inequality becomes \((q - 3)(q - 4) \geq 0 \) which is true for all integers \( q \). Increasing \( m \) by one will increase the left hand side by one and the right hand side by \( q(q - 1)/6 \). So, this inductive step will go through for all \( q \geq 3 \). If \( k \) is decreased by one, the left hand side is unchanged and the right hand side increases so the \( k \) induction step will also preserve the inequality. So the theorem is proved when \( k < q \).

Now suppose \( k > q \) and \( p = 2qm - k \) is even. As in the previous section, we have \( c(K) \leq m - 1 + q/2 \) so it will be enough to show

\[
\frac{m - 1 + q}{2} \leq \frac{(2qm - k - 1)(q - 1)/2 + 9}{6}
\]

If \( m = 1 \), since \( k < q \), then \( p = 2qm - k \) will be less than \( q \) contradicting an earlier assumption. So \( m \geq 2 \). Substituting \( m = 2 \) and \( k = 2q - 1 \) into Equation 4 results in the inequality \((q - 1)(q - 3) \geq 0 \) which is true since \( q \geq 3 \). Again, the \( m \) and \( k \)
inductive steps preserve the inequality and the theorem is proved in the case $k < q$ as well.

Finally, we have the case where $pq$ is odd. As in the previous section, we can show $c(K) \leq \lfloor q + (m - 1)/2 \rfloor$. It's enough to verify

$$
\lfloor q + \frac{m - 1}{2} \rfloor \leq \frac{(qm - k - 1)(q - 1)/2 + 9}{6}.
$$

If $m = 2$ and $k = q - 2$ we have $q^2 - 12q + 17 \geq 0$ which is valid for $q \geq 11$. If $m = 3$ and $k = q - 1$, we have $2q^2 - 14q + 6 \geq 0$ which is valid for $q \geq 7$. So we will look at the induction when $q \geq 11$ and then take smaller values of $q$ in turn.

**Case 1** $q \geq 11$. If $q \geq 11$, both base cases hold, so it's enough to check that the inductive steps preserve the inequality. If $m$ is increased by 2, the left side of Equation 8 goes up by 1 while the right hand side increases by $q(q - 1)/6$. Thus, the induction will work as long as $q \geq 3$. If $k$ is decremented by 2, the left hand side of the inequality is unchanged while the right increases by $(q - 1)/6$. Therefore, the theorem is proved when $q \geq 11$.

**Case 2** $q = 9$. If $m = 4$ and $k = q - 2$, Equation 8 becomes $3q^2 - 14q + 5 \geq 0$ which is valid for $q \geq 5$. So, induction arguments show the theorem holds for $m \geq 3$. For $m = 2$, we have the knots $(17, 9), (13, 9)$, and $(11, 9)$ of genus 72, 48, and 40 respectively. As these knots have crosscap number 5, 4, and 5, the theorem holds for these knots and, therefore, for all knots with $q = 9$.

**Case 3** $q = 7$. As above, induction will take care of all cases where $m \geq 3$. For $m = 2$, we have the knots $(13, 7), (11, 7)$, and $(9, 7)$ of genus 36, 30, and 24 respectively. Since the crosscap numbers are 4, 3, and 4, we have verified the theorem in the case $q = 7$.

**Case 4** $q = 5$. If $m = 5$ and $k = q - 1$, Equation 8 becomes $4q^2 - 16q - 6 \geq 0$ which is valid when $q = 5$. So, induction takes care of the cases where $m \geq 4$ and we’re left with the knots $(13, 5), (11, 5), (9, 5)$, and $(7, 5)$ of genus 24, 20, 16, and 12 respectively. These all have crosscap number 3, so the theorem holds in this case as well.

**Case 5** $q = 3$. If $p = 6m + 1$, then $c(K) = m + 1, g(K) = 6m$, and the theorem holds. If $p = 6m - 1$, then $c(K) = m + 1, g(K) = 6m - 2$, and the theorem holds.

This completes the proof of Theorem 1. □

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