THE INTERACTION FORCE BETWEEN ROTATING BLACK HOLES AT EQUILIBRIUM *

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Abstract

We study the previously constructed Riemann problem whose solutions correspond to equilibrium configurations of black holes. We evaluate the metric coefficients at the symmetry axis and the interaction force between the black holes.

1 Introduction

In the previous work [1], we studied the axially symmetric stationary vacuum solution of Einstein equations describing the equilibrium configuration of rotating black holes. The equilibrium configuration was understood to mean a stationary solution possessing a disconnected event horizon. All such solutions satisfy a certain boundary-value problem for a system of elliptic nonlinear equations that can be easily obtained from [2], where the regularity conditions for the symmetry axis and the event horizon were first formulated.

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Our investigation showed that this boundary-value problem reduces to a matrix Riemann problem with a rational conjugation matrix. We also noted that solutions corresponding to stationary equilibrium configuration of black holes are likely to develop a conical singularity on the symmetry axis, which is the reason why the black hole do not fall on each other. We note that for rotating black holes, the presence of conical singularity for arbitrary values of the solution parameters is not quite obvious from the physical standpoint, because it can be assumed that the rotation of black holes leads to repulsion forces that can compensate the gravitational attraction.

The conical singularity itself allows us to introduce the notion of the interaction force between the black holes. We consider this in more detail in the next section. The analytic expression for the interaction force between Schwarzschild black holes has long been known \[3\]. The main result of the present work is our generalization of this to the case of rotating black holes.

## 2 The boundary-value problem

In cylindrical coordinates, the axially symmetric metric is

\[
ds^2 = -V dt^2 + 2 W dt d\phi + X d\phi^2 + \frac{X}{\rho^2} e^\beta (d\rho^2 + dz^2),
\]

where the metric coefficient depend only on \( \rho \) and \( z \). The Einstein equations are then divided into two system of nonlinear equations, the first of which can be brought to the form

\[
(\rho g_{\rho \rho}^{-1})_{\rho} + (\rho g_{\rho z}^{-1})_{z} = 0, \quad g = \begin{pmatrix} -V & W \\ W & X \end{pmatrix}, \quad \det g = -\rho^2,
\]

and the second one to the form

\[
(\ln(\frac{X}{\rho^2} e^\beta))_\zeta = \frac{i}{2\rho} \left(1 + \text{tr}(\rho g_{\zeta}^{-1} g^{-1})^2\right), \quad \partial_\zeta = \frac{1}{2}(\partial_z - i \partial_\rho).
\]

Let the event horizon have \( N \) disconnected components, let \( z_1, \ldots, z_{2N} \) be the \( z \)-coordinates of the intersection points of the event horizon and the symmetry axis, and let \( \Omega_i \) be the angular velocity of the \( i \)th black hole. In the neighborhood of the \( i \)th black hole, we choose the coordinate system

\[
\rho^2 = (\lambda^2 - m_i^2)(1 - \mu^2), \quad m_i = \frac{z_{2i} - z_{2i-1}}{2},
\]
\[ z - \frac{z_{2i} + z_{2i-1}}{2} = \lambda \mu, \quad |\mu| \leq 1, \]
in which the regularity condition of the symmetry axis and the event horizon can be written as \[ \Omega_i \]
\begin{align*}
\begin{pmatrix}
1 & \Omega_i \\
0 & 1
\end{pmatrix} g
\begin{pmatrix}
1 & 0 \\
\Omega_i & 1
\end{pmatrix}
= \begin{pmatrix}
(\lambda^2 - m_i^2)\hat{V}(\lambda, \mu) & \rho^2 \hat{W}(\lambda, \mu) \\
\rho^2 \hat{W}(\lambda, \mu) & (1 - \mu^2)\hat{X}(\lambda, \mu)
\end{pmatrix}, \quad (2.3)
\end{align*}
where \( \hat{X} \) and \( \hat{V} \) are smooth functions not equal to zero. Using (2.3), we can easily obtain the boundary conditions for the system (2.1)
\begin{align*}
\rho g, \rho g^{-1} &= \begin{pmatrix} 0 & O(1) \\ 0 & 2 \end{pmatrix}, \quad \rho \to 0, \quad z \in \Gamma, \quad (2.4a) \\
\hat{\Omega}_i \rho g, \rho g^{-1} \hat{\Omega}_i^{-1} &= \begin{pmatrix} 2 & 0 \\ O(1) & 0 \end{pmatrix}, \quad \rho \to 0, \quad z \in I_i, \quad \hat{\Omega}_i = \begin{pmatrix} \Omega_i & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.4b) \\
\rho g, \rho g^{-1} &= O(1), \quad \rho \to 0, \quad z \in R. \quad (2.4c)
\end{align*}
where \( \Gamma \) is the symmetry axis consisting of \( N + 1 \) connected components,
\[ \Gamma = \bigcup \Gamma_j = R \setminus \bigcup I_i \quad (j = 1, \ldots, N + 1, \ i = 1, \ldots, N), \quad I_i = (z_{2i-1}, z_{2i}). \]
The symbol \( O(1) \) denotes uniformly bounded functions on the corresponding interval. We impose the condition at infinity
\[ W = \rho^2 O\left(\frac{1}{\rho^3}\right), \quad X = \rho^2 (1 + O\left(\frac{1}{\rho}\right)), \quad r = \sqrt{\rho^2 + z^2}. \quad (2.5) \]
Boundary conditions (2.4) and (2.5) are sufficient to construct solutions of the first group of Einstein equations (2.1).

We now consider the properties of solutions of system (2.2). Using (2.3), we can easily verify that for \( \rho = 0 \) and \( z \in \Gamma \), we have
\[ \partial_z \beta = 0, \quad \beta|_{\Gamma_i} = b_i, \quad (2.6) \]
where \( b_i \) are some constants. The condition for the solution to be regular on the symmetry axis \( \left( \frac{X^a X_a}{4X} \to 1 \text{ on the symmetry axis} \right) \) is satisfied if and only if Eq. (2.3) is supplemented by the additional requirement that
\[ b_i = 0. \quad (2.7) \]
For asymptotically flat solutions (those for which $\beta \to 0$ as $r \to \infty$) condition (2.7) is automatically satisfied on the first and the last components of the symmetry axis but cannot be satisfied on the remaining components (see [4, 5, 6, 7]). In the general case, the last statement is not yet proved. In this work, we evaluate the constant $b_i$, but the corresponding analytic expression is unfortunately too complicated to analyze the solvability of Eq.(2.7).

The $b_i$ parameters also have their own interpretation: the quantity

$$ F_i = \frac{1}{4}(e^{-b_i/2} - 1) \quad (2.8) $$

can be considered as the interaction force between the black holes. Here and what follows, we assume that $b_1 = b_{N+1} = 0$. We now comment on the origin of this interpretation. If condition (2.7) is not satisfied, the curvature tensor becomes singular in the neighborhood of the corresponding symmetry axis components. We assume that the singular points are filled with ”matter”, which is precisely what prevents the black hole falling on each other. The tension in $z$ direction per unit surface is $T(e_z, e_z)$ where $T$ is the energy-momentum tensor and $e_z$ is the normalized vector parallel to $\partial/\partial z$. It is natural to define the interaction force of the black holes as the integral

$$ \int_{S_\varepsilon} T(e_z, e_z)ds, $$

where $S_\varepsilon$ is the 2-surface coordinatized by $\rho\phi$(with $\rho \leq \varepsilon$) and $ds$ is the corresponding area element. Evaluating this integral, we obtain Eq. (2.8) [4].

To conclude this section, we give a brief derivation of representation for $b_i$ [4, 5]. We bring Eq.(2.2) to the form

$$ \partial_\rho \beta = \frac{\rho}{2}\left(\frac{\partial_\rho \ln X}{\rho^2}\right)^2 - \left(\partial_z \ln \frac{X}{\rho^2}\right)^2 + \frac{(\partial_\rho Y)^2 - (\partial_z Y)^2}{X^2}, \quad (2.9a) $$

$$ \partial_z \beta = \partial_z \ln X \left(\rho \partial_\rho \ln X - 2\right) + \frac{\rho \partial_z Y \partial_\rho Y}{X^2}, \quad (2.9b) $$

where

$$ dY = \frac{1}{\rho} \ast (X dW - W dX), \quad \ast d\rho = dz, \ast dz = -d\rho. $$

4
Let \( C_\varepsilon \) be a curve connecting \( \Gamma_{i+1} \) and \( \Gamma_i \):

\[
\rho = \varepsilon \sin \tau, \quad z - \frac{z_{2i} + z_{2i-1}}{2} = -\sqrt{\varepsilon^2 + m_i^2} \cos \tau, \quad 0 \leq \tau \leq \pi.
\]

Then

\[
b_{i+1} - b_i = \int_{C_\varepsilon} d\beta. \tag{2.10}
\]

The left-hand side of (2.10) is independent of \( \varepsilon \). Taking \( \varepsilon \) to zero and using (2.3) and (2.9), we obtain that

\[
b_{i+1} - b_i = -2 \left( \ln \hat{X}(z_{2i}) - \ln \hat{X}(z_{2i-1}) \right), \tag{2.11}
\]

where \( \hat{X}(z_{2i}) = \hat{X}(m_i, 1) \) and \( \hat{X}(z_{2i-1}) = \hat{X}(m_i, -1) \). In the next section, we use Eq. (2.11) to evaluate \( b_i \).

### 3 The Riemann problem

We showed in [1] that for every solution of boundary-value problem (2.4), (2.5) there exists a piecewise analytic matrix \( \chi(\omega) \) satisfying the conjugation condition on the imaginary axis

\[
\chi_-(\omega) = \chi_+(\omega) \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} T(k) \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix}, \quad k = z + (\omega^2 - \rho^2)/2\omega \tag{3.1a}
\]

and normalized at infinity by

\[
\chi(\omega) \to I, \quad \omega \to \infty. \tag{3.1b}
\]

The rational matrix \( T(k) \) is defined as

\[
T(k) = \hat{D}_{N+1} T_N \hat{D}_N \ldots T_1 \hat{D}_1, \tag{3.2}
\]

where

\[
T_j = \begin{pmatrix} 1 - \frac{2M_j}{k-z_{2j-1}} & -4M_j\Omega_j \\ \frac{2L_j}{(k-z_{2j})(k-z_{2j-1})} & 1 + \frac{2M_j}{k-z_{2j}} \end{pmatrix}, \quad \hat{D}_j = \begin{pmatrix} 1 & -D_j \\ 0 & 1 \end{pmatrix}. \tag{3.3}
\]

The \( M_j, L_j \) parameters have the physical interpretation of the respective full mass and angular momentum of \( j \)th black hole and are related by

\[
M_j = m_j + 2\Omega_j L_j, \quad 2m_j = z_{2j} - z_{2j-1}. \tag{3.4}
\]
An additional requirement that the matrix $T(k)$ be symmetric leads to $2N + 1$ nonlinear algebraic equations on the parameters $D_j, L_j$ and $\Omega_j$, which determine $D_j$ and $L_j$ as functions of $\Omega_j$ and $z_j$. In particular, 

$$
\sum_{j=1}^{N+1} D_j + \sum_{j=1}^N 4\Omega_j M_j = 0. 
$$

(3.5)

Let $d_1 = D_1, d_{i+1} = d_i + D_{i+1} + 4M_i\Omega_i$ and 

$$
t_{2j−1}(k) = C_{2j−1}^{-1} \left( \begin{array}{cc}
\frac{1}{2M_j(k−z_{2j−1})} & 0 \\
1 & 1
\end{array} \right) C_{2j−1}, \quad t_{2j}(k) = C_{2j}^{-1} \left( \begin{array}{cc}
1 & \frac{\Omega_j}{2(k−z_{2j})} \\
0 & 1
\end{array} \right) C_{2j},
$$

where 

$$
C_{2j−1} = \left( \begin{array}{cc}
1 & -d_j \\
0 & 1
\end{array} \right), \quad C_{2j} = \left( \begin{array}{cc}
0 & \Omega_j \\
-1/\Omega_j & 4M_j
\end{array} \right) C_{2j−1}.
$$

Then applying (3.5) and easily verified identity 

$$
T_j \left( \begin{array}{cc}
1 & -d_j \\
0 & 1
\end{array} \right) t_{2j−1}^{-1} t_{2j}^{-1} = \left( \begin{array}{cc}
1 & -d_j - 4M_j\Omega_j \\
0 & 1
\end{array} \right),
$$

we derive that 

$$
T(k) = t_{2N}(k)t_{2N−1}(k) \ldots t_2(k)t_1(k).
$$

(3.6)

We now note that the matrix 

$$
\left( \begin{array}{cc}
1 & 0 \\
0 & \omega
\end{array} \right) t_{2j}(k)t_{2j−1}(k) \left( \begin{array}{cc}
1 & 0 \\
0 & 1/\omega
\end{array} \right)
$$

has no singularity at $\omega = 0$ and tends to identity matrix as $\omega \to \infty$, 

$$
\left( \begin{array}{cc}
1 & 0 \\
0 & \omega
\end{array} \right) t_{2j}(k)t_{2j−1}(k) \left( \begin{array}{cc}
1 & 0 \\
0 & 1/\omega
\end{array} \right) \bigg|_{\omega=0} = \left( \begin{array}{cc}
1 & -8M_j(2M_j\Omega_j + d_j)/\rho^2 \\
0 & 1
\end{array} \right).
$$

(3.7)

Therefore, the same properties are shared by the matrix 

$$
\left( \begin{array}{cc}
1 & 0 \\
0 & \omega
\end{array} \right) T(k) \left( \begin{array}{cc}
1 & 0 \\
0 & 1/\omega
\end{array} \right),
$$

and Riemann problem (3.1) is correctly posed for any $D_j, L_j, \Omega_j$ satisfying (3.5).
The only singularities of the conjugation matrix are simple poles at points

\[ \omega_i^\pm = (z_i - z) \pm \sqrt{(z_i - z)^2 + \rho^2}, \quad \omega_i^- = -\rho^2/\omega_i^+. \]

It now follows from Eq.(3.1) that \( \chi_\pm(\omega) \) are rational functions with simple poles at the points \( \omega_i^\pm (\omega_i^+ > 0, \omega_i^- < 0) \),

\[ \chi_\pm(\omega) = I + \sum_{j=1}^{2N} \frac{A_j^\pm}{\omega - \omega_j^\pm}, \quad (3.8) \]

with \( A_j^\pm \) being independent of \( \omega \). It follows from the unimodularity of \( T(k) \) that \( \det \chi_\pm(\omega) = 1 \), and \( \det A_j^\pm = 0 \). Let \( \xi_j \) be the eigenvector of \( A_j^- \); then

\[ A_j^- \xi_j = 0, \quad A_j^- = a_j \xi_j^\sigma, \quad \xi_j^\sigma = (-\xi_j^2, \xi_j^1), \quad (3.9) \]

with \( a_j \) being a column vector. We further note that \( \chi_-(\omega) \) is unimodular if and only if there exists a constant \( v_i \) such that

\[ a_i = v_i \lim_{\omega \to \omega_i^-} \chi_-(\omega) \xi_i, \quad (3.10) \]

Equation (3.9) and (3.10) yield a system of linear equations for \( a_i \),

\[ a_i = v_i \xi_i + v_i \sum_{i \neq k} \frac{\xi_k^\sigma \xi_i}{\omega_i^- - \omega_k^-} a_k. \quad (3.10a) \]

The parameters \( v_i \) and \( \xi_i \) can be easily determined from conjugation matrix. We define the matrix \( G_i(k) = t_{i-1}(k) \ldots t_1(k) \) and the vectors

\[ c_{2j} = C_{2j}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c_{2j-1} = C_{2j-1}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Then

\[ \xi_i = \begin{pmatrix} 1 & 0 \\ 0 & \omega_i^- \end{pmatrix} G_i^{-1}(z_i) c_i, \quad (3.11) \]

and also

\[ v_i = \frac{\omega_i^- v_i^0}{u_i \omega_i^- - v_i^0 \xi_i^\sigma \xi_i^0 \xi_i}, \quad (3.12) \]
where

\[ v^0_{2j} = \frac{\Omega_j}{\omega^2_{2j} - \omega^2_{2j}}, \quad v^0_{2j-1} = -\frac{1}{\Omega_j} \frac{1}{\omega^2_{2j-1} - \omega^2_{2j-1}} \]

and

\[ u_i = 1 + (\omega_i - \omega_i^+)v^0_i c^\sigma_i \partial_k G_i(z_i) G_i^{-1}(z_i) c_i. \]

We also note that

\[ \xi^\sigma_i = \omega_i^+ c^\sigma_i G_i(z_i) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1/\omega_i^- \end{array} \right) \]

The approach adopted here to solve the Riemann problem is similar to one used in [8]. As is clear from what follows, to construct an exact solution to nonlinear system (2.1) we can take \( T \) to be any symmetric matrix. This method for deriving exact solutions is evidently different from the method used in [9, 10]; however, it does not lead to new solutions of (2.1). In particular, the solution that we investigate here is in the class of \( 2N \)-solitons in Minkowski space background [10].

We now show how the solution of (2.1) can be reconstructed from the solution of Riemann problem (3.1). Let

\[ D_1 = \partial_z - \frac{2\omega^2}{\omega^2 + \rho^2} \partial_\omega, \quad D_2 = \partial_\rho + \frac{2\omega \rho}{\omega^2 + \rho^2} \partial_\omega. \]

Because \( D_1 k = D_2 k = 0 \), it follows from (3.1) that "logarithmic" derivatives \( D\Psi \Psi^{-1} \) of the functions

\[ \Psi(\omega) = \chi(\omega) \left( \begin{array}{cc} 1 & 0 \\ 0 & \omega \end{array} \right) \]

have no other singularities in addition to the simple poles at \( \pm i\rho \). Therefore,

\[ D_1 \Psi_-(\omega) \Psi^{-1}_-(\omega) = -i\rho \partial_\omega \Psi_-(i\rho) \Psi^{-1}_-(i\rho) + i\rho \partial_\omega \Psi_-(i\rho) \Psi^{-1}_-(i\rho) \]

\[ D_2 \Psi_-(\omega) \Psi^{-1}_-(\omega) = \frac{\rho \partial_\omega \Psi_-(i\rho) \Psi^{-1}_-(i\rho)}{\omega - i\rho} = \frac{\rho \partial_\omega \Psi_-(i\rho) \Psi^{-1}_-(i\rho)}{\omega + i\rho} \]

We now take into account that \( T(\bar{\omega}) = T(\omega) \) and define the real matrix

\[ g = -\chi_-(0) \left( \begin{array}{cc} 1 & 0 \\ 0 & -\rho^2 \end{array} \right). \]
From (3.13) with $\omega = 0$, one obtains
\begin{align*}
\partial_\zeta g g^{-1} = \partial_\omega \Psi_- (i\rho) \Psi_-^{-1} (i\rho), \\
\bar{\partial}_\zeta g g^{-1} = \partial_\omega \Psi_- (-i\rho) \Psi_-^{-1} (-i\rho).
\end{align*}
(3.15)

The compatibility condition for Eqs.(3.13) is given by system (2.1), and therefore, Eq.(3.14) is a solution of (2.1).

From here on, we assume that the parameters $d_i$ and $M_i$ are determined from the requirement that $T(k)$ be symmetric. Then
\begin{align}
\sum_{i=1}^{N} M_i (d_i + 2\Omega_i M_i) = 0, \quad (\lim_{k \to \infty} k (T_{21}(k) - T_{12}(k)) = 0)
\end{align}
(3.16)
and $\chi_- (0) = \chi_+ (0)$ (see (3.7)). Using the last identity and the uniqueness of the solution to the Riemann problem, we obtain the reduced problem
\begin{align}
\chi_- (\omega) = -g \bar{\chi}_+^{-1} (-\rho^2 / \omega) \begin{pmatrix} 1 & 0 \\ 0 & -1/\rho^2 \end{pmatrix} \left( \bar{T} (-\rho^2 / \omega) = T (\omega) \right),
\end{align}
(3.17)
where $\bar{T}$ denotes the transposed matrix. Taking $\omega$ either to zero or to infinity in (3.17), we see that $g$ is a symmetric matrix.

We now investigate the properties of solutions of the Riemann problem as $\rho \to 0$. Let $z \in \Gamma_{m+1}$, then
\begin{align}
\omega_i^+ \to 2(z_i - z), \quad \omega_i^- \to 0, \quad i \geq 2m + 1, \quad \rho \to 0 & \quad \text{(3.18a)} \\
\omega_i^+ \to 0, \quad \omega_i^- \to 2(z_i - z), \quad i \leq 2m, \quad \rho \to 0 & \quad \text{(3.18b)}
\end{align}

Using the formula $\omega_i^+ = -\rho^2 / \omega_i^-$, we can easily verify that all the coefficients of linear system (3.10) also well defined for $\rho = 0$, and it is therefore natural to assume the existence of the limit $\lim_{\rho \to 0} a_i$. This assumption ensures the existence of all the limits used in what follows.

Introduce the matrix
\begin{align}
\chi_1 (\omega) = \chi_- (\omega) \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} G_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix} = \chi_+ (\omega) \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} Q_2^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix}
& = I + \sum_{i=1}^{2m} \frac{B_i}{\omega - \omega_i^+} + \sum_{i=2m+1}^{2N} \frac{B_i}{\omega - \omega_i^-},
\end{align}
(3.19)
where
\begin{align}
Q_i (k) = t_{2N} (k) \ldots t_i (k), \quad T (k) = Q_i (k) G_i (k).
\end{align}
(3.20)
We can see from (3.18) that
\[
\lim_{\rho \to 0} \chi_1(\omega) = I + \frac{B}{\omega}. \tag{3.21}
\]

Further, let
\[
\chi_2(\omega) = \chi_+(\omega) \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \tilde{G}_{2m+1} \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix}
\]
In view of the identities
\[
\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \tilde{t}_{2j-1} \tilde{t}_{2j} \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix} \bigg|_{\omega=0} = I,
\]
\[
\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \tilde{t}_{2j-1} \tilde{t}_{2j} \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix} \bigg|_{\omega=\infty} = \begin{pmatrix} 1 & 8M_j(d_j + 2M_j \Omega_j) \\ 0 & 1 \end{pmatrix},
\]
we have
\[
\chi_+(0) = \chi_2(0), \quad \chi_2(\infty) = \begin{pmatrix} 1 & 8 \sum_{j=1}^{m} M_j(d_j + 2M_j \Omega_j) \\ 0 & 1 \end{pmatrix}. \tag{3.22}
\]

In terms of \(\chi_1\) and \(\chi_2\), Riemann problem (3.1) can be rewritten as
\[
\chi_2(\omega) = \chi_1(\omega) \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} Q_{2m+1}^{-1} \tilde{G}_{2m+1} \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix}, \tag{3.23}
\]
and reduced problem (3.17) as
\[
\chi_2(\omega) = -g\tilde{\chi}_1^{-1}(-\rho^2/\omega) \begin{pmatrix} 1 & 0 \\ 0 & -1/\rho^2 \end{pmatrix} = B^0 + \sum_{i=1}^{2m} \frac{B_i^2}{\omega - \omega_i^2} + \sum_{i=2m+1}^{2N} \frac{B_i^2}{\omega - \omega_i^2}. \tag{3.24}
\]

Taking \(\rho \to 0\) in (3.23) and taking into account (3.21), we see that
\[
\chi_2(\omega) = B^0 + \sum_{i=1}^{2N} \frac{B_i^2}{\omega - 2(z_i - z)} = (I + \frac{B}{\omega}) \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \gamma_{m+1}^{-1}(\omega) \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix}, \tag{3.25}
\]
where
\[
\gamma_{m+1}(\omega) = Q_{2m+1}^{-1} (z + \omega/2) \tilde{G}_{2m+1} (z + \omega/2). \tag{3.26}
\]
The function \(\chi_2(\omega)\) is nonsingular at \(\omega = 0\). This is possible only if
\[
B = \begin{pmatrix} 0 & -\frac{\gamma_{m+1}^{-1}(0)}{\gamma_{m+1}^{-1}(0)} \\ 0 & 0 \end{pmatrix}.
\]
Finally,

\[
\chi_-(0) = \left(\frac{1}{\gamma_{m+1}^{22}(0)} \left(\gamma_{m+1}^{22}(0)\partial_\omega \gamma_{m+1}^{12}(0) - \gamma_{m+1}^{12}(0)\partial_\omega \gamma_{m+1}^{22}(0)\right) / \gamma_{m+1}^{22}(0)\right)
\]

Recall that \(\chi_-(0) = \chi_+ (0) = \chi_2(0)\). We now note that \(\gamma_{m+1}^{22}(0)\) is a symmetric real matrix with the unit determinant and also that \(\gamma_{m+1}^{22}(0) \to 1\) as \(z \to \infty\); therefore,

\[
\gamma_{m+1}^{22}(0) > 0.
\]

(3.27)

Since

\[
\left.\frac{\hat{X}}{\lambda^2 - m_i^2}\right|_{r_{i+1}} = \gamma_{i+1}^{22}(0, z), \quad \left.\frac{\hat{X}}{\lambda^2 - m_i^2}\right|_{r_i} = \gamma_i^{22}(0, z),
\]

and \(\lambda^2 - m_i^2 \to (z - z_2i)(z - z_{2i-1})\) as \(\rho \to 0\), we find

\[
\hat{X}(z_{2i}) = m_i \text{res}_{z_{2i}} \gamma_{i+1}^{22}(0, z), \quad \hat{X}(z_{2i-1}) = -m_i \text{res}_{z_{2i-1}} \gamma_i^{22}(0, z).
\]

(3.28)

It follows from (3.27) that

\[
\text{res}_{z_{2i}} \gamma_{i+1}^{22}(0, z) > 0, \quad \text{res}_{z_{2i-1}} \gamma_i^{22}(0, z) < 0.
\]

Equations (2.11) and (3.28) give the sought-for analytic expression for the constants \(b_i\) and the interaction force (2.8) of black holes.

4 Interaction force between two black holes

For two black holes, the symmetry axis has three connected components. We normalize \(\beta\) such that \(b_1 = b_3 = 0\), then

\[
F = \frac{1}{4} \left(\frac{\hat{X}(z_2)}{X(z_1)} - 1\right),
\]

(4.1)

and

\[
\hat{X}(z_2) = m_1 \text{res}_{z_2} \gamma_{2}^{22}(0, z), \quad \hat{X}(z_1) = -m_1 \text{res}_{z_1} \gamma_{1}^{22}(0, z).
\]

(4.2)

Expression (4.1) depends on \(d_i, \Omega_i, L_i, M_i\) and \(z_i\). These parameters are not independent and must satisfy the system of nonlinear algebraic equations

\[
\text{res}_{z_i} T_{12}(k) = \text{res}_{z_i} T_{21}(k), \quad M_i = m_i + 2\Omega_i L_i.
\]

(4.3)
It can be easily seen from (3.16) that
\[
d_1 = -2M_1\Omega_1 + M_2d, \quad d_2 = -2M_2\Omega_2 - M_1d.
\] (4.4)

Using (4.3) and (4.4), we can eliminate parameters \(d_1, d_2, \Omega_1\) and \(\Omega_2\) from the expression for the interaction force (4.1). Unfortunately, the formula thus obtained is extremely cumbersome. The situation is greatly simplified in one particular case, however. Namely, let the black holes rotate in opposite directions and have the same total and irreducible masses:
\[
M_1 = M_2 = M, \quad m_1 = m_2 = m, \quad \Omega_1 = -\Omega_2 = \Omega.
\]

We set
\[
z_1 = -m_1, \quad z_2 = m_1, \quad z_3 = R - m_2, \quad z_4 = R + m_2.
\] (4.5)

Then
\[
L = 2M^2(m + M)\frac{R + 2M}{R - 2M}\Omega, \quad M = m + 2\Omega L, \quad L_1 = -L_2 = L, \quad d = 0 \quad (4.6)
\]
and
\[
F = \frac{M^2}{R^2 - 4M^2}.
\] (4.7)

By definition, \(R \geq 2m\). Using (4.6), we can easily show that \(2M \leq R\) (where the equality takes place only in the limiting case of \(R = 2m\)); therefore, \(F > 0\) and tends to infinity as \(R \to 2m\). We also observe that the interaction force in the case under consideration is exactly the same as for two Schwarzschild black holes with identical masses (equal to \(M\)). The interaction force of two nonrotating black holes (\(\Omega_1 = \Omega_2 = 0\)) is given by
\[
F = \frac{m_1m_2}{R^2 - (m_1 + m_2)^2}.
\] (4.8)

It is easy to see that in the cases we have considered, \(F\) tends to the Newtonian limit as \(R \to \infty\). This result can be generalized; the estimate
\[
F = \frac{M_1M_2}{R^2} + O\left(\frac{1}{R^4}\right)
\] (4.9)
holds as \(R \to \infty\). To derive (4.9), it suffices to use the approximate solution of (4.3):
\[
d = -\frac{2L_2M_1 + 2L_1M_2}{M_1M_2} \frac{1}{R^2} + 4\left(L_1 + L_2 + \frac{2L_1M_2^2 + 2L_2M_1^2}{M_1M_2}\right) \frac{1}{R^3} + O\left(\frac{1}{R^4}\right).
\]
\[ \Omega_i = \frac{L_i a_i}{2 M_i^2 (M_i + m_i)} \]

and

\[ a_1 = 1 - \frac{4 M_2}{R} + \frac{8 M_2^2}{R^2} + O\left(\frac{1}{R^3}\right), \quad a_2 = 1 - \frac{4 M_1}{R} + \frac{8 M_1^2}{R^2} + O\left(\frac{1}{R^3}\right). \]

We now discuss the general properties of constraints (4.3) on the angular velocities and angular momenta. We especially note that the equality \( L_2 = \Omega_2 = 0 \) cannot be satisfied for any values of the remaining parameters with \( L_1 \neq 0 \). This means that if \( L_2 = 0 \) and the second black hole does not contribute to the total angular momentum, an approaching observer would still see this black hole rotating with a certain angular velocity. We consider this behavior quite natural. Indeed, the notion of the angular velocity is related to the behavior of geodesics in the neighborhood of black hole: every test particle approaching the black hole is involved in its rotation, and the angular velocity of this rotation becomes equal to the angular velocity of the black hole when the particle reaches the event horizon. That the angular velocity is nonzero is therefore determined by the fact that the first black hole involves the second one in its rotation. This explanation corresponds to the behavior of \( \Omega_2 \) at large distances: if \( L_2 = 0 \) then \( \Omega_2 = O(1/R^3) \).

We return to the exact solution (4.6). The foregoing discussion allows us to take the angular momenta of each black hole as the independent parameters. We then have \( \Omega \to 0 \) and \( M \to m \) as \( R \to 2m \) which means that as black holes approach each other, they loose the full mass and "slow down" each others rotation.

Evidently, the realistic solution for the configuration of two black holes cannot be stationary. As we have seen however, the stationary solutions admits a sufficiently realistic physical interpretation and demonstrate, at least qualitatively, the effects caused by the interaction of black holes. In our opinion, this is because the Einstein equations determine not only the gravitational field but also the law of motion of material bodies [11].

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