Structural tensors of generalized Kenmotsu manifolds

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Abstract. The paper introduces the Kirichenko structural tensors of generalized Kenmotsu manifolds, called the first, second, and third structural tensors. The properties of structural tensors of generalized Kenmotsu manifolds are studied, analytical expressions of structural tensors are obtained, and covariant differentials of structural tensors are calculated.

1. Introduction

Let M be a connected smooth manifold of dimension $(2n+1)$, $C^\infty(M)$ - the algebra of smooth functions on M, $\mathfrak{X}(M) – C^\infty$-module of smooth vector fields on the manifold M, $d$ - the operator of external differentiation. If a Riemannian metric $\langle \cdot, \cdot \rangle$ is given on M, then the corresponding Riemannian connection is denoted by $\nabla$. We assume that all manifolds, tensor fields (tensors), and other objects are smooth of class $C^\infty$.

Definition 1.1 [1] An almost contact metric structure on a manifold M is the set $\{\xi, \eta, g, \phi\}$ of tensor fields on M, where $g=\langle \cdot, \cdot \rangle$ is a (pseudo) Riemannian metric, $\phi$ is a tensor of type $(1,0)$ called the structural endomorphism, $\xi$ is a vector field called characteristic, $\eta$ is a differential 1-form called the contact form of the structure. Wherein:

1. $\phi(\xi)=0$; 2. $\eta(\xi)=0$; 3. $\eta(\xi)=1$; 4. $\phi^2=\eta \otimes \xi$; 5. $\langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y) ; X, Y \in \mathfrak{X}(M)$ (1)

Today, active research is being carried out on the geometry of almost contact metric structures on manifolds. One of the most pressing issues in this section of geometry is the question of studying individual classes of almost contact metric manifolds. In 1972, Kenmotsu [2] introduced a class of almost contact metric structures characterized by the identity $\nabla_X(\phi)Y = \langle \phi X, Y \rangle \xi - \eta(\phi X)X, Y \in \mathfrak{X}(M)$. Polarizing the identity characterizing Kenmotsu manifolds, S. Umnova [3] identified in her dissertation a class of almost contact metric manifolds, which is a generalization of Kenmotsu manifolds and called a class of generalized (in short, GK-) Kenmotsu manifolds. It was proved in [3] that generalized Kenmotsu manifolds of constant curvature are Kenmotsu manifolds of constant curvature $-1$.

Generalized Kenmotsu manifolds were studied in [4-8]. In this paper, we continue the study of generalized Kenmotsu manifolds and investigate the properties of structural tensors of this class of manifolds.

This paper is organized as follows. In Section 2, we give the preliminary information needed in the subsequent presentation, construct the space of the adjoint G-structure, give the first group of structural equations of an almost contact metric manifold, introduce the Kirichenko structural tensors of an almost contact metric structure, and present their properties. In Section 3, we give a definition of generalized Kenmotsu manifolds, give a complete group of structural equations, and give definitions...
of special generalized Kenmotsu manifolds of the first and second kind. In Section 4, using the identity recovery procedure [9-10], we obtain an analytic expression of the first and second structural tensors of generalized Kenmotsu manifolds, and also give the properties of these tensors and the third structural tensor. In addition, in this section, we calculate the covariant derivative of the first and second structural tensors and present the identities that the tensors satisfy. Thee is a list of references at the end of the paper.

2. Preliminary Information

Let M be a smooth manifold, dimensions $2n+1$, $X(M) - C^\infty$-module of smooth vector fields on the manifold M. Further, all manifolds, tensor fields, and other objects are assumed to be smooth of $C^\infty$ class.

Definition 2.1. [9,10] An almost contact structure on a manifold M is a triple $(\eta, \xi, \Phi)$ of tensor fields on this manifold, where $\eta$ is a differential 1-form called a contact form of a structure, $\xi$ is a vector field called a characteristic field, $\Phi$ is an endomorphism of the module $X(M)$ called structural endomorphism. Wherein

1) $\eta(\xi) = 1$; 2) $\eta \circ \Phi = 0$; 3) $\Phi(\xi) = 0$; 4) $\Phi^2 = - \text{id} + \eta \otimes \xi$  

(2)

If, in addition, such a Riemannian structure $g = \langle \cdot, \cdot \rangle$ is fixed on M that

$\langle \Phi X, \Phi Y \rangle = \eta(X) \eta(Y)$, $X, Y \in X(M)$  

(3)

tetrad $(\eta, \xi, \Phi, g = \langle \cdot, \cdot \rangle)$ is called an almost contact metric (in short, AC-) structure. A manifold on which an almost contact (metric) structure is fixed is called an almost contact (metric (in short, AC-)) manifold.

The skew-symmetric tensor $\Omega(X, Y) = \langle X, \Phi Y \rangle$, $X, Y \in X(M)$ is called the fundamental form of the AC-structure [9,10].

Let $(\eta, \xi, \Phi, g)$ be an almost contact metric structure on the manifold $M^{2n+1}$. In the module $X(M)$, two mutually complementary projectors $m = \eta \otimes \xi$ and $\text{id} = - \Phi^2$ are internally defined [10]; thus, $X(M) = L \oplus M$, where $L = \text{Im} (\Phi) = \text{ker} \eta$ is the so-called contact (or first fundamental) distribution, $\text{dim} L = 2n, M = \text{Im} (\Phi) = \text{ker} \eta = L (\xi)$ - the linear span of the structural vector or the so-called second fundamental distribution (moreover, $L$ and $m$ are projectors on the submodules $L$, $M$, respectively) [9, 10]. Obviously, the distributions of $L$ and $M$ are invariant with respect to $\Phi$ and are mutually orthogonal. It is also obvious that $\Phi^2 = - \text{id}$, $(\Phi X, \Phi Y) = \langle X, Y \rangle$, $X, Y \in X(M)$, where $\Phi = \Phi | L$. Therefore, $(\Phi_p, \text{id} | L_p)$ is a Hermitian structure on the space $L_p$.

The complexification $X(M)^C$ of the module $X(M)$ splits into the direct sum $X(M)^C = D_\Phi^1 \oplus D_\Phi^1 \oplus D_\Phi^1$ of proper subspaces of the structural endomorphism $\Phi$ corresponding to the eigenvalues $\sqrt{-1}, \sqrt{-1}$ and 0, respectively. Moreover, the projectors on the terms of this direct sum are, respectively, endomorphisms [17, 18]

$$\pi = \sigma_0 | L = \frac{1}{2} (\Phi^2 + \sqrt{-1} \Phi), \quad \text{where} \quad \sigma_0 | L = \frac{1}{2} (\Phi^2 + \sqrt{-1} \Phi),$$

(4)

where $\sigma = \frac{1}{2} (\text{id} + \sqrt{-1} \Phi), \quad \bar{\sigma} = \frac{1}{2} (\text{id} - \sqrt{-1} \Phi)$.

The expressions $\sigma_p: L_p \rightarrow D_\Phi^1$ and $\bar{\sigma}_p: L_p \rightarrow D_\Phi^1$ are an isomorphism and anti-isomorphism, respectively, of Hermitian spaces. Therefore, to each point $p \in M^{2n+1}$, we can add a family of frames of the space $T_p(M)^C$ of the form $(p, e_0, e_1, \ldots, e_{2n}, e_{2n+1}, \ldots, e_{2n})$, where $e_a = \sqrt{2} \sigma_p (e_a), \quad e_{2n+1} = \sqrt{2} \bar{\sigma}_p (e_a), \quad e_0 = \xi_p$; where
The set \( \{e_a\} \) is the orthonormal basis of the Hermitian space \( L_P \). Such a frame is called an A-frame [10]. It is easy to see that the matrices of the components of the tensors \( \Phi_n \) and \( g_p \) in the A-frame have the form, respectively:

\[
\left( \Phi^i_j \right) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \sqrt{-1}I_n & 0 \\
0 & 0 & -\sqrt{-1}I_n
\end{pmatrix},
\left( g_{ij} \right) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & I_n \\
0 & I_n & 0
\end{pmatrix}
\]  

(4)

where \( I_n \) – an identity matrix of order \( n \). It is well known [17, 18] that the set of such frames defines a G-structure on \( M \) with the structural group \( \{1\} \times \text{U}(n) \) represented by matrices of the form:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & A & 0 \\
0 & 0 & A
\end{pmatrix}
\]  

where \( A \in \text{U}(n) \). This G-structure is called adjoint [9, 10].

Let \( \left( M^{2n+1}, \Phi_i \xi_n, g=\langle \cdot, \cdot \rangle \right) \) be the almost contact metric manifold. We agree that throughout the work, unless otherwise indicated, the indices \( i,j,k,l, \ldots \) run through the values from 1 to \( 2n \), the indices \( a,b,c,d, \ldots \) - the values from 1 to \( n \), and assume that \( \hat{a}=a, \hat{n}=n, \hat{0}=0 \). Let \( (U,\varphi) \) be a local map on the manifold \( M \). According to the Main tensor analysis theorem [9], defining the structural endomorphism \( \Phi \) and the Riemannian structure \( g=\langle \cdot, \cdot \rangle \) on the manifold \( M \) induces the setting on the total space \( BM \) of the bundle of frames over \( M \) of a system of functions \( \left\{ \Phi^i_j \right\}, \left\{ g_{ij} \right\} \) satisfying in the coordinate neighborhood \( W=\pi^{-1}(U) \subset BM \) a system of differential equations of the form:

\[
d\Phi^i_j + \Phi^k_j \theta_{ij}^k - \Phi^i_j \theta_{ij}^k = \omega_{ijk}, \quad dg_{ij} - g_{kj} \theta^k_j - g_{ik} \theta^k_j = \omega_{ijk} \omega^k
\]  

(5)

where \( \left\{ \omega^i \right\}, \left\{ \theta^i_j \right\} \) – the components of the displacement forms and the Riemannian connection \( \nabla \), respectively. \( \Phi^i_{j,k}, g_{ij,k} \) are the components of the covariant differential of the tensors \( \Phi \) and \( g \) in this connection, respectively. Moreover, by the definition of the Riemannian connection, \( \nabla g=0 \) and, therefore,

\[
g_{ij,k} = 0
\]  

(6)

Relations (2.4) on the space of the adjoint G-structure are written in the form [9, 10]

\[
\Phi^a_{b,k} = 0, \quad \Phi^\delta_{b,k} = 0, \quad \Phi^\alpha_{0,k} = 0
\]

\[
\theta^\alpha_b = -\frac{\sqrt{-1}}{2} \Phi^\alpha_{b,k} \omega^k, \quad \theta^\delta_b = -\frac{\sqrt{-1}}{2} \Phi^\delta_{b,k} \omega^k
\]

\[
\theta^\alpha_0 = -\sqrt{-1} \Phi^\alpha_{0,k} \omega^k, \quad \theta^\delta_0 = -\sqrt{-1} \Phi^\delta_{0,k} \omega^k
\]

\[
\theta^\alpha_a = -\sqrt{-1} \Phi^\alpha_{a,k} \omega^k, \quad \theta^\delta_a = -\sqrt{-1} \Phi^\delta_{a,k} \omega^k
\]

\[
\theta^\alpha_j + \theta^\delta_j = 0, \quad \theta^\alpha_0 = 0
\]  

(7)

In addition, we note that since the corresponding forms and tensors are real \( \overline{\omega^i} = \omega^i, \overline{\theta^\alpha_j} = \theta^\alpha_j, \overline{\nabla \Phi^i_{j,k}} = \nabla \Phi^i_{j,k} \), where \( t \to \overline{t} \) is the complex conjugation operator.

The first group of structural equations of the Riemannian connection \( d\omega^i = -\theta^\alpha_j \omega^j \) on the space of the adjoint G-structure of an almost contact metric manifold can be written in the following form called the first group of structural equations of an almost contact metric manifold [9, 10]:
\[ \text{d}\omega = C_{ab}\omega^a\Lambda^b + C^{ab}\omega_a\Lambda^b + C^{ab}\omega_b\Lambda^a + C_{ab}\omega^a\Lambda^b \]

\[ \text{d}\omega^a = -\delta^a_b\Lambda^b + B^{ab}\omega_c\Lambda^a + B^{ab}\omega_a\Lambda^b = B^a_b\omega^b \]

\[ \text{d}\omega = \delta^a_b\Lambda^b + B^{bc}\omega_c\Lambda^a + B^{bc}\omega_a\Lambda^b = B^a_b\omega^b \]

where \( \omega^0 = \pi^* (\eta) \); \( \pi \) – natural projection of the space of the adjoint \( G \)-structure onto the manifold \( M \). \( \omega_i = g_{ij}\omega^j \),

\[ B^{ab} = \frac{\sqrt{-1}}{2} \Phi_{b,c}^a, \quad B^{ac} = \frac{\sqrt{-1}}{2} \Phi_{b,c}^a, \quad B^{abc} = \frac{\sqrt{-1}}{2} \Phi_{b,c}^a \]

\[ B_{abc} = \frac{\sqrt{-1}}{2} \Phi_{b,c}^a, \quad B_{ab} = \sqrt{-1} \Phi_{b,d}^a, \quad B_{b} = \sqrt{-1} \Phi_{b,d}^a \]

\[ B^{ab} = \sqrt{-1} \left( \Phi_{0,5}^a - \frac{1}{2} \Phi_{0,6}^a \right), \quad B_{ab} = \sqrt{-1} \left( \Phi_{0,5}^a + \frac{1}{2} \Phi_{0,6}^a \right) \]

\[ C^{ab} = \sqrt{-1} \Phi_{b,c}^a, \quad C_{ab} = \sqrt{-1} \Phi_{b,c}^a, \quad C^{b} = \sqrt{-1} \left( \Phi_{a,6}^b + \Phi_{a,5}^b \right) = B^b a - B^b a \]

We introduce the notation \[ \Phi_{k,l,} = \Phi_{k,l,} \]

\[ C^{abc} = \frac{\sqrt{-1}}{2} \Phi_{b,c}^a, \quad C_{abc} = \frac{\sqrt{-1}}{2} \Phi_{b,c}^a, \quad F^{ab} = -\sqrt{-1} \Phi_{a,b}^0, \quad F_{ab} = -\sqrt{-1} \Phi_{a,b}^0 \]

Let’s consider the following families of functions on the space of an adjoint \( G \)-structure [9]:

1. \( B = \{ B^i_j \}; B_{bc}^{ab} = B_{bc}^{ab} = B_{bc}^c; \) all other components of family \( B \) are zero;
2. \( C = \{ C_{ij} \}; C_{bc}^{ab} = C_{bc}^{ab}; \) all other components of family \( C \) are zero;
3. \( D = \{ D^i_j \}; D_{bc}^{ab} = D_{bc}^{ab}; \) all other components of family \( D \) are zero;
4. \( E = \{ E^i_j \}; E_{bc}^{ab} = E_{bc}^{ab}; \) all other components of family \( E \) are zero;
5. \( F = \{ F^i_j \}; F_{bc}^{ab} = F_{bc}^{ab}; \) all other components of family \( F \) are zero;
6. \( G = \{ G_i \}; G_{bc}^{ab} = G_{bc}^{ab}; \) all other components of family \( G \) are zero.

These systems of functions define tensors of the corresponding types on the manifold \( M \), which are called the first, second, ..., sixth structural tensors of the AC-structure, respectively. The following holds:

Proposition [9] 2.1. The structural tensors of the AC-structure have the following properties:

1. \( \Phi \circ B(X,Y) = B(\Phi X,Y) = B(X,\Phi Y) \);
2. \( \langle B(X,Y), Z \rangle + \langle Y, B(X,Z) \rangle = 0 \);
3. \( \Phi \circ C(X,Y) = C(\Phi X,Y) = C(X,\Phi Y) \);
4. \( \langle C(X,Y), Z \rangle + \langle Y, C(X,Z) \rangle = 0 \);
5. \( \Phi \circ D = D \circ \Phi \);
6. \( \Phi \circ E = E \circ \Phi \);
7. \( \Phi \circ F = F \circ \Phi \);
8. \( G \in L \); where \( \langle X,Y \rangle = \langle X,Y \rangle + \sqrt{-1} \langle X,\Phi Y \rangle \), \( X, Y, \in X(M) \).

3. Generalized Kenmotsu manifolds
Let \((M^{2n+1}, \Phi, \varphi, g=\{\cdot, \cdot\})\) be an almost contact metric manifold.

Definition 3.1. [3,4] The class of almost contact metric manifolds characterized by the identity

\[
\nabla_X(\Phi)Y + \nabla_Y(\Phi)X - \eta(Y)\Phi X - \eta(X)\Phi Y; X, Y \in \text{X}(M)
\]

is called generalized Kenmotsu manifolds (in short, GK-manifolds).

Note that in literature, this class of manifolds is called the nearly Kenmotsu class of manifolds, and others. We will call these manifolds, as in [3], generalized Kenmotsu manifolds, and shortly -GK-manifolds.

The following theorem holds.

Theorem 3.1. [5] The full group of structural equations of GK-manifolds on the space of the adjoint G-structure has the form:

1) \(d\omega = F_{ab} \omega^a \omega^b + F_{ab} \omega_a \omega_b;\)
2) \(d\omega^a = -\omega^a \Lambda \omega + C_{abc} \omega_b \omega_c - \frac{1}{2} F_{abc} \omega_a \omega_b + \delta^a_0 \omega_0 \omega_0;\)
3) \(d\omega_0 = \theta_0 \Lambda \omega + C_{abc} \omega^b \Lambda \omega^c - \frac{1}{2} F_{abc} \omega^0 \omega^0 + \theta_0 \omega_0 \omega_0;\)
4) \(d\theta_0 = \theta_0 \Lambda \omega + \left(\frac{3}{2} F_{abc} \omega^0 \right) \omega_a \omega_b \omega_c + \left(\frac{3}{2} F_{abc} \omega^0 \right) \omega_a \omega_b \omega_c + \left(\frac{3}{2} F_{abc} \omega^0 \right) \omega_a \omega_b \omega_c;\)
5) \(dC_{abc} + C_{dbc} \delta^d_0 + C_{adc} \delta^d_0 + C_{abd} \delta^d_0 = C_{ade} \omega^d - 2 \delta^d_0 F_{abc} \omega^d - C_{abc};\)
6) \(dC_{a0c} + C_{a0c} \delta^d_0 + C_{a0c} \delta^d_0 = C_{a0c} \omega^0 - 2 \delta^0_0 F_{abc} \omega^0 + C_{abc};\)
7) \(dF_{ab} + F_{ab} \delta^c_0 = -2 F_{ab};\)
8) \(dA^{ad} + A^{ab} \delta^b_0 + A^{ab} \delta^b_0 = A^{ad} - A^{ad};\)
9) \(A^{ad} + A^{ad} \delta^b_0 + A^{ad} \delta^b_0 = A^{ad} - A^{ad};\)

where

\[
C_{[abc]} = C_{abc}, \quad C_{[abc]} = C_{abc}, \quad F_{[abc]} = F_{abc}, \quad F_{[abc]} = F_{abc}, \quad F_{[abc]} = F_{abc}, \quad F_{[abc]} = F_{abc}, \quad F_{[abc]} = F_{abc}, \quad F_{[abc]} = F_{abc}.
\]
1) since $\xi^a=\xi_a=0$, then $C(\xi,X)=C_{abc}^bX^b\xi^c+\delta^{abc}\xi^cX_a=0$. Similarly, 
$C(X,\xi)=C_{abc}X^b\xi^c+\delta^{abc}\xi^cX_a=0$.

2) Due to (3.3), we have $C(X,Y)=C_{abc}X^bY^c\xi^a+\delta^{abc}X_aY_a\xi^a=-C(X,Y)$.

3) $C(\Phi X,Y)=C_{abc}(\Phi X)^bY^c\xi^a+\delta^{abc}(\Phi X)_aY_a\xi^a=-\Phi^bC(\Phi X,Y)$.

Using the properties of tensor $C$, we find an explicit analytic expression for this tensor. By definition, we have: 
$$[C(\xi,\xi)]^a_b\xi_a=\frac{1}{2}\Phi^a_b\xi_a=\frac{1}{2}(\nabla_{\xi}^a(\Phi)e_b)^a\xi_a=\frac{1}{2}(\Phi_\xi\nabla_{\xi}^a(\Phi)e_b)^a\xi_a,$$
i.e. 
$$[C(\xi,\xi)]^a_b\xi_a=\frac{1}{2}(\Phi_\xi\nabla_{\xi}^a(\Phi)e_b)^a\xi_a.$$ Since the vectors $\{\xi_a\}$ form the basis of the subspace 
$$\left(D_\Phi^{-1}\right)_p,$$ 
the submodules $D_\Phi^{-1}$ and $D_\Phi^0$ are endomorphisms $\pi=\sigma_\xi=\frac{1}{2}(\Phi^2+\sqrt{-1}\Phi)$, $\bar{\pi}=\bar{\sigma}_\xi=\frac{1}{2}(\Phi^2+\sqrt{-1}\Phi)$, then the equality written above can be rewritten in the form 
$$\left(\Phi^2+\sqrt{-1}\Phi\right)_pC\left(-\Phi^2X+\sqrt{-1}\Phi X+\Phi^2\nabla_{\Phi^2Y}(\Phi)\Phi X+\Phi^2\nabla_{\Phi^2Y}(\Phi)\Phi X+\Phi^2\nabla_{\Phi^2Y}(\Phi)\Phi^2X\right).$$

Opening the brackets and simplifying, we get 
$$C(X,\Phi Y)=\frac{1}{8}\left\{(-\Phi_\xi\nabla_{\Phi^2Y}(\Phi)\Phi^2X+\Phi_\xi\nabla_{\Phi^2Y}(\Phi)\Phi X+\Phi^2\nabla_{\Phi^2Y}(\Phi)\Phi X+\Phi^2\nabla_{\Phi^2Y}(\Phi)\Phi^2X\right\} \quad (14)$$

The above procedure is called the identity recovery procedure [9,10].

Note. Formula (4.2) was first obtained in [11].

For GK-manifolds, the equality $\Phi^a_{\xi,\xi}=0$ holds, i.e. $\Phi^a_{\xi,\xi}e_a=(\nabla_{\xi}^a(\Phi)e_b)^a\xi_a=0$. Since the vectors $\{\xi_a\}$ form the basis of the subspace 
$$\left(D_\Phi^{-1}\right)_p,$$ vectors $\{\xi_a\}$ form the basis of the subspace 
$$\left(D_\Phi^{-1}\right)_p,$$ and the projectors of the module $X(M)^C$ onto the submodules $D_\Phi^{-1}$ and $D_\Phi^0$ are endomorphisms $\pi=\sigma_\xi=\frac{1}{2}(\Phi^2+\sqrt{-1}\Phi)$, $\bar{\pi}=\bar{\sigma}_\xi=\frac{1}{2}(\Phi^2+\sqrt{-1}\Phi)$, then the equality written above can be rewritten in the form 
$$\left(\Phi^2+\sqrt{-1}\Phi\right)_p\Phi\nabla_{\Phi^2Y}(\Phi)\Phi X+\Phi_\xi\nabla_{\Phi^2Y}(\Phi)\Phi X+\Phi_\xi\nabla_{\Phi^2Y}(\Phi)\Phi X=0; \forall X,Y\in X(M).$$

Opening the brackets and simplifying, we get 
$$\Phi_\xi\nabla_{\Phi^2Y}(\Phi)\Phi^2X+\Phi_\xi\nabla_{\Phi^2Y}(\Phi)\Phi X+\Phi^2\nabla_{\Phi^2Y}(\Phi)\Phi X+\Phi^2\nabla_{\Phi^2Y}(\Phi)\Phi X=0; \forall X,Y\in X(M) \quad (15)$$
In view of the obtained equality (15), relation (14) takes the form:

\[
C(X,Y) = \frac{1}{4} \left\{ \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 (X) \right\} = \frac{1}{4} \left\{ \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 (X) \right\}; \forall X,Y \in X(M) \tag{16}
\]

Thus, the theorem is proved.

Theorem 4.2. The structural tensor of the GK-structure is calculated by the formula

\[
\left\{ \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 (X) \right\} - \left\{ \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 (X) \right\} - \left\{ \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 (X) \right\} = \left\{ \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 (X) \right\}; \forall X,Y \in X(M)
\]

Let's consider a family of functions on the space of an adjoint G-structure \( F = \{ F_i \}; F_{a,b} = F_{a,b}; \) all other components of the family \( F \) are zero. These systems of functions define a tensor of type (1) on the manifold \( M \), which we call the second structural tensor of the GK-structure.

Theorem 4.3. The second structural tensor of the GK-structure has the properties:

1) \( F(\xi) = 0 \); 2) \( F(X,Y) = \Phi(Y,F(X)) \); 3) \( \Phi \circ F = - F \circ \Phi \); 4) \( \eta F(X) = 0; \forall X,Y \in X(M) \tag{17} \)

Proof.

1) Since in the \( A \)-frame \( \xi^0 = 1, \xi^i = 0 \), then \( F(\xi) = F_i = F_{i}^{-1}\xi^i + F_{ab}\xi^a\xi^b = 0 \).

2) Due to (2.3), we have \( (\Phi \circ F)(X,Y) = \Phi \circ F(Y,X) = \Phi \circ F(X,Y) \).

3) \( \left\{ \Phi \circ F \right\}(X) = \Phi \circ F_{a} = \Phi_{a} \circ F_{a} = \Phi_{a} \circ F_{a} \).

4) Since in the \( A \)-frame \( \eta_{1} = 1, \eta_{a} = 0 \), then \( \Phi \circ F(X) = \Phi(X) = \Phi_{a} \circ F_{a} = \Phi_{a} \circ F_{a} \).

Theorem 4.4. The second structural tensor of the GK-structure has the following expressions:

\[
F(X) = \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 (X) = \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 (X); \forall X,Y \in X(M) \tag{18}
\]

Proof. Using the properties of tensor \( F \), we find an explicit analytic expression for this tensor. By definition, we have:

\[
\left[ F(\xi) \right]^{a} \xi_{a} = - \Phi_{0}^{a}_{c} \xi_{c} = \Phi_{a} \circ \xi_{a} = \Phi_{a} \circ \xi_{a} = \left( \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 \right) \xi_{a}; \text{ i.e.}
\]

\[
\left[ F(\xi) \right]^{a} \xi_{a} = \Phi_{a} \circ \xi_{a} = \left( \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 \right) \xi_{a}.
\]

Since the vectors \( \{ \xi_{a} \} \) form the basis of the subspace \( D_{\Phi} \), vectors \( \{ \xi_{a} \} \) form the basis of the subspace \( D_{\Phi}^{1} \), and the projectors of the module \( X(M) \) onto the submodules \( D_{\Phi}^{1} \) and \( D_{\Phi}^{0} \) are endomorphisms \( \pi_{1} = \Phi_{0}^{a}_{c} \xi_{c} = - \Phi_{a} \circ \xi_{a} = \left( \Phi \circ \nabla_{\Phi} \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 \right) \xi_{a} \), then the equality written above can be rewritten in the form:

\[
\left( \Phi^{2} + \sqrt{-1} \Phi \right) \circ F = - \left( \Phi^{2} + \sqrt{-1} \Phi \right) \circ F = \left( \Phi^{2} + \sqrt{-1} \Phi \right) \circ F = \left( \Phi^{2} + \sqrt{-1} \Phi \right) \circ F; \forall X \in X(M).
\]

Opening the brackets and simplifying, we get
The last equality can be written as:

$$F(X) = \frac{1}{2} \{\Phi \circ \nabla_{\Phi^2 \phi X}(\Phi) \xi \circ \Phi^2 \circ \nabla_{\Phi X}(\Phi) \xi \}; \forall X \in \mathbb{X}(M)$$

(19)

Applying the identity recovery procedure to equalities $B^b_{\alpha} = \sqrt{-1} \Phi^a_{\alpha} = \Phi^a_{a}$, we get

$$E(X) = \frac{1}{2} \{\Phi \circ \nabla_{\Phi^2 \phi X}(\Phi) \xi \circ \Phi^2 \circ \nabla_{\Phi X}(\Phi) \xi \} = \Phi^2 X; \forall X \in \mathbb{X}(M)$$

(21)

or

$$\Phi \circ \nabla_{\Phi}(\Phi) \xi + \nabla_{\Phi X}(\Phi) \xi = 2\Phi X; \forall X \in \mathbb{X}(M)$$

(22)

And the theorem is follows from (19) – (22).

Since $C$ is a tensor of type (2), by the Main theorem of tensor analysis, we have:

$$dC_{jk} + C_{jk} \theta^l_{ik} \eta_{kl} - C_{ik} \theta^l_{jl} \eta_{kl} = C_{ik} \theta^l_{jl}$$

(23)

where $\{C^i_{jk, l}\}$ – a system of functions serving on the space of the bundle of all frames as components of the covariant differential of the structural tensor $C$. Writing this equality on the space of the adjoint $G$-structure, we obtain:

$$dC_{ab} + C_{ab} \theta^c_{bc} - C_{bc} \theta^a_{ac} = C_{bc} \theta^a_{ac}$$

(24)

And the remaining components are zero.

So, the theorem is proved.

Theorem 4.5. The components of the covariant differential of the structural tensor $C$ of a GK-manifold on the space of an adjoint G-structure have the form:

1) $C_{ab,c} = C_{abc}$; 2) $C_{ab,c} = C_{abc}$; 3) $C_{bc,d} = C_{ab} d_{hc}$; 4) $C_{bc,d} = C_{ab} d_{hc}$; 5) $C_{bc,d} = C_{ab} d_{hc}$; 6) $C_{bc,d} = C_{ab} d_{hc}$; 7) $C_{bc,d} = C_{ab} d_{hc}$; 8) $C_{bc,d} = C_{ab} d_{hc}$; 9) $C_{bc,d} = C_{ab} d_{hc}$; 10) $C_{bc,d} = C_{ab} d_{hc}$; 11) $C_{bc,d} = C_{ab} d_{hc}$; 12) $C_{bc,d} = C_{ab} d_{hc}$; 13) $C_{bc,d} = C_{ab} d_{hc}$; 14) $C_{bc,d} = C_{ab} d_{hc}$; 15) $C_{bc,d} = C_{ab} d_{hc}$; 16) $C_{bc,d} = C_{ab} d_{hc}$; 17) $C_{bc,d} = C_{ab} d_{hc}$.

(25)

And the remaining components are zero.

Let’s consider the expression $C(X,M) \times \mathbb{X}(M) \times \mathbb{X}(M) \to \mathbb{X}(M)$ given by the formula

$$C(X,Y,Z) = C_{ab} h_{cd} X^a Y^d Z_b e_a + C_{ab} h_{cd} X_c Y_d Z^a e_a$$

(26)

**Theorem 4.6.** For a GK-manifold, the following relations hold:

1) $C(X,Y,Z) = C(X,Y,Z)$; 2) $C(X,Y,Z) = C(X,Y,Z)$; 3) $C(X,Y,Z) = C(X,Y,Z)$; 4) $C(X,Y,Z) = C(X,Y,Z)$; 5) $C(X,Y,Z) = C(X,Y,Z)$; 6) $C(X,Y,Z) = C(X,Y,Z)$; 7) $C(X,Y,Z) = C(X,Y,Z)$; 8) $C(X,Y,Z) = C(X,Y,Z)$; 9) $C(X,Y,Z) = C(X,Y,Z)$; 10) $C(X,Y,Z) = C(X,Y,Z)$; 11) $C(X,Y,Z) = C(X,Y,Z)$; 12) $C(X,Y,Z) = C(X,Y,Z)$; 13) $C(X,Y,Z) = C(X,Y,Z)$; 14) $C(X,Y,Z) = C(X,Y,Z)$; 15) $C(X,Y,Z) = C(X,Y,Z)$; 16) $C(X,Y,Z) = C(X,Y,Z)$; 17) $C(X,Y,Z) = C(X,Y,Z)$.

(27)

**Proof.** The proof is carried out by direct calculation. For example:

$$C(X_1+X_2,Y,Z) = C_{ab} h_{cd}(X_1+X_2) Y^d Z_b e_a + C_{ab} h_{cd}(X_1+X_2) Y_d Z^a e_a$$

The remaining properties are proved similarly.
We introduce the endomorphism defined in the A-frame by the matrix \( \mathbf{C}_a^b = (\mathbf{C}_{a}^{bc}) \). This endomorphism is Hermitian conjugate, which means that we are diagonalized in a suitable A-frame, i.e. in this frame

\[
\mathbf{C}_a^b = \mathbf{C}_b^a \delta_a^b \quad (27)
\]

where \( \{ \mathbf{C}_a^b \} \) – eigenvalues of this endomorphism. Moreover, the Hermitian form \( \mathbf{C}(\mathbf{X}, \mathbf{Y}) = \mathbf{C}_a^b \mathbf{X}_a \mathbf{Y}_b \) corresponding to this endomorphism is positively semidefinite, since \( \mathbf{C}(\mathbf{X}, \mathbf{X}) = \mathbf{C}_a^b \mathbf{X}_a \mathbf{X}_b = \mathbf{C}_{a}^{bc} \mathbf{C}_{db} \mathbf{X}_a \mathbf{X}_d = \sum_{c,d} \mathbf{C}_{cd} \mathbf{X}_a \mathbf{X}_d \geq 0 \). Therefore, \( C_a \geq 0, a=1,2,\ldots,n \). If \( C_a = 0 \), a GK-manifold is an SGK-manifold of the first kind. And since, according to [5], a GK-manifold is an SGK-manifold of the first kind only in dimension 5, then if \( C_a = 0 \), then the GK-manifold is a five-dimensional SGK-manifold of the first kind.

We consider some properties of tensor \( \mathbf{V} \).

Applying the procedure for the restoration of identity [9,10] to relations (20), we obtain the following theorem.

Theorem 4.7. Tensor \( \mathbf{V} \) has the following properties:

\[
1) \quad \nabla_2 (C)(\xi, \mathbf{X}) = \nabla_2 (C)(\xi, \mathbf{X}) = 0; \quad 2) \quad \nabla_2 (C)(\xi, \Phi^2 \mathbf{Y}) = \nabla_2 (C)(\xi, \Phi^2 \mathbf{Y}) = \nabla_2 (C)(\xi, \Phi^2 \mathbf{Y}) = \mathbf{C}(\mathbf{X}, \mathbf{Y}) ;
\]

\[
3) \quad \nabla_2 (C)(\mathbf{X}, \mathbf{Y}) = \nabla_2 (C)(\Phi^2 \mathbf{Y}, \mathbf{Y}) = -2 \mathbf{C}(\mathbf{X}, \mathbf{Y}) ;
\]

\[
4) \quad \nabla_2 (C)(\mathbf{Y}, \mathbf{Y}) = \nabla_2 (C)(\Phi^2 \mathbf{Y}, \mathbf{Y}) = -2 \mathbf{C}(\mathbf{X}, \mathbf{Y}) ;
\]

\[
5) \quad \nabla_2 (C)(\Phi^2 \mathbf{Y}, \mathbf{Y}) = -2 \mathbf{C}(\mathbf{X}, \mathbf{Y}) ;
\]

\[
6) \quad \nabla_2 (C)(\Phi^2 \mathbf{Y}, \mathbf{Y}) = -2 \mathbf{C}(\mathbf{X}, \mathbf{Y}) ;
\]

\[
7) \quad \nabla_2 (C)(\Phi^2 \mathbf{Y}, \mathbf{Y}) = -2 \mathbf{C}(\mathbf{X}, \mathbf{Y}) ;
\]

\[
8) \quad \nabla_2 (C)(\Phi^2 \mathbf{Y}, \mathbf{Y}) = -2 \mathbf{C}(\mathbf{X}, \mathbf{Y}) ;
\]

\[
9) \quad \nabla_2 (C)(\Phi^2 \mathbf{Y}, \mathbf{Y}) = -2 \mathbf{C}(\mathbf{X}, \mathbf{Y}) ;
\]

\[
10) \quad \nabla_2 (C)(\Phi^2 \mathbf{Y}, \mathbf{Y}) = -2 \mathbf{C}(\mathbf{X}, \mathbf{Y}) .
\]

Therefore, if \( \mathbf{F} \) is a tensor of type \((1,1)\), by the Main theorem of tensor analysis, we have:

\[
d \mathbf{F}^k_j + F^k_{ij} F^i_j = F^j_{ik} \delta^k_l = F_{j,k} \delta^k_l \quad (29)
\]

where \( \{ F^i_{j,k} \} \) – a system of functions serving on the space of the bundle of all frames as components of the covariant differential of the structural tensor \( \mathbf{F} \). Writing this equality on the space of the adjoint G-structure, we obtain:

\[
1) \quad F^0_{ab} = F_{ab}; \quad 2) \quad F^0_{ab} = -F_{ac} F_{cb}; \quad 3) \quad F^0_{ab} = -F_{ac} F_{cb}; \quad 4) \quad F^0_{ab} = -F_{ac} F_{cb}; \quad 5) \quad F^0_{ab} = -F_{ac} F_{cb}; \quad 6) \quad F^0_{ab} = -F_{ac} F_{cb}; \quad 7) \quad F^0_{ab} = -2 F_{ab}; \quad 8) \quad F^0_{ab} = -F_{ab}; \quad 9) \quad F^0_{ab} = -F_{ab}; \quad 10) \quad F^0_{ab} = -2 F_{ab} .
\]

And the remaining components are zero.

We introduce the endomorphism \( \mathbf{F} \) defined in the A-frame by the matrix \((\mathbf{F}_a^b) = (\mathbf{F}_{a}^{bc}) \). This endomorphism is Hermitian conjugate, which means that we are diagonalized in a suitable A-frame, i.e. in this frame

\[
\mathbf{F}_a^b = \mathbf{F}_b^a \delta_a^b , \quad (31)
\]

where \( \{ \mathbf{F}_a \} \) – eigenvalues of this endomorphism. Moreover, the Hermitian form \( \mathbf{F}(\mathbf{X}, \mathbf{Y}) = \mathbf{F}_a^b \mathbf{X}_a \mathbf{Y}_b \) corresponding to this endomorphism is positively semidefinite, since \( \mathbf{F}(\mathbf{X}, \mathbf{X}) = \mathbf{F}_a^b \mathbf{X}_a \mathbf{X}_b = \mathbf{F}_{a}^{bc} \mathbf{F}_{cb} \mathbf{X}_a \mathbf{X}_d = \sum_{c,d} \mathbf{F}_{cd} \mathbf{X}_a \mathbf{X}_d \geq 0 \). Therefore, \( F_a \geq 0, a=1,2,\ldots,n \). If \( F_a = 0 \), then the manifold is an SGK-manifold of Kenmotsu of the second kind.

From (3.2: 9) by the Main theorem of the tensor analysis it follows that the functions of families \( \{ \mathbf{A}_{a}^{cd} \} \) on the space of the adjoint G-structure symmetric in the upper and lower indices form a pure tensor on \( M^{2n+1} \), called the tensor of \( \Phi \)-holomorphic sectional curvature [3,5]. This tensor defines the expression \( A: \mathbf{X}(\mathbf{M}) \times \mathbf{X}(\mathbf{M}) \times \mathbf{X}(\mathbf{M}) \rightarrow \mathbf{X}(\mathbf{M}) \), which is given by the relation

\[
A(\Phi \mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}_{a}^{cd} \mathbf{X}^a \mathbf{Y}^b \mathbf{Z}^d \mathbf{c} + \mathbf{A}_{a}^{cd} \mathbf{X}^a \mathbf{Y}_b \mathbf{Z}^d \mathbf{c} .
\]

It is easy to verify by direct calculation that the tensor of the \( \Phi \)-holomorphic sectional curvature has the following properties:

1) \( A(\Phi \mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\Phi \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{Z}, \mathbf{Y}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}) = \mathbf{A}(\mathbf{X}, \mathbf{Z}) = \mathbf{A}(\mathbf{Z}, \mathbf{X}) ;
\]

2) \( \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) ;
\]

3) \( \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) ;
\]
Indeed,
\[
A(\Phi X, Y, Z) = A_{\alpha\beta}^{cd}(\Phi X)^b Y^b Z_c^d + A_{\alpha\beta}^{cd}(\Phi X)_a Y_b Z_c^d = \sqrt{-1} T_{\alpha\beta}^{cd} X^a Y^b Z_c^d \sqrt{-1} T_{\alpha\beta}^{cd} X_b Z_c^d = A_{\alpha\beta}^{cd} X^a (\Phi Y)^b Z_c^d + A_{\alpha\beta}^{cd} X_a (\Phi Y)_b Z_c^d = A(X, \Phi Y, Z)
\]

Similarly,
\[
A(\Phi X, Y, Z) = A_{\alpha\beta}^{cd}(\Phi X)^b Y^b Z_c^d + A_{\alpha\beta}^{cd}(\Phi X)_a Y_b Z_c^d = \sqrt{-1} T_{\alpha\beta}^{cd} X^a Y^b Z_c^d \sqrt{-1} T_{\alpha\beta}^{cd} X_a Y_b Z_c^d = -A_{\alpha\beta}^{cd} X^a (\Phi Y)^b Z_c^d - A_{\alpha\beta}^{cd} X_a (\Phi Y)_b Z_c^d = -A(X, \Phi Y, Z)
\]

Property 5) follows from the symmetry of the tensor $A_{\alpha\beta}^{cd}$ with respect to the lower pair of indices. Symmetry in the upper pair of indices implies property 6).

Thus, the theorem is proved.

Theorem 4.8. The tensor of the $\Phi$-holomorphic sectional curvature of a GK-manifold has the properties:

1) $A(\Phi X, Y, Z) = A(X, \Phi Y, Z) = A(X, Y, \Phi Z) = -A(Z, X, Y)$; 2) $A(Z, X, \phi^i Y) = A(Z, X, Y)$; 3) $\eta A(X, Y, Z) = 0$;

4) $A(\xi, Y, Z) = A(X, \xi, Z) = A(X, Y, \xi) = 0$; 5) $A(X, Y, Z) = A(Y, X, Z)$;

6) $A(X, Y, Z, W) = A(X, Y, W, Z); \ \forall X, Y, Z, W \in X(M)$

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