EFFECTIVE UPPER BOUNDS ON THE NUMBER OF RESONANCES IN POTENTIAL SCATTERING

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Abstract. We prove upper bounds on the number of resonances and eigenvalues of Schrödinger operators $-\Delta + V$ with complex-valued potentials, where $d \geq 3$ is odd. The novel feature of our upper bounds is that they are effective, in the sense that they only depend on an exponentially weighted norm of $V$. Our main focus is on potentials in the Lorentz space $L^{(d+1)/2,1/2}$, but we also obtain new results for compactly supported or pointwise decaying potentials. The main technical innovation, possibly of independent interest, are singular value estimates for Fourier-extension type operators. The obtained upper bounds not only recover several known results in a unified way, they also provide new bounds for potentials which are not amenable to previous methods.

1. Introduction

1.1. Counting resonances. Assume that $d \geq 3$ is odd and that $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ is a bounded, compactly supported potential, possibly complex-valued. The resolvent

$$R_V(\lambda) := (-\Delta + V - \lambda^2)^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad \text{Im } \lambda \gg 1,$$

extends to a meromorphic family (see [DZ19, Thm. 3.8])

$$R_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}^d) \to L^2_{\text{loc}}(\mathbb{R}^d), \quad \lambda \in \mathbb{C}.$$ 

Scattering resonances are defined as the poles of this meromorphic continuation. Eigenvalues $z = \lambda^2$ correspond to resonances in the upper half plane. Let $n_V(r)$ denote the number of resonances (counted with multiplicity) with absolute value at most $r$,

$$n_V(r) := \#\{\text{resonances } \lambda : |\lambda| \leq r\}.$$ 

The first polynomial bound $n_V(r) \leq C_V r^{d+1}$ was proved by Melrose [McS83]. Zworski [Zwo89a] proved the sharp upper bound

$$n_V(r) \leq C_V r^d, \quad r \geq 1,$$ 

see also [DZ19] Thm. 3.27] for a textbook presentation which uses a substantial simplification of the argument due to Vodev [Vod92]. Christiansen and Hislop [CH05, CH10] proved that (1) is optimal for (Baire) generic complex or real-valued potentials in the sense that $\limsup_{r \to \infty} \log n(r)/\log r = d$. An example of Christiansen shows that there are complex-valued potentials with no resonances.
The question of whether (1) is optimal for arbitrary real-valued $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ is still open. Asymptotics for $n_V(r)$ are known in $d = 1$ [Zwo87, Fro97, Reg58, Sim00] and for certain radial potentials [Zwo89b, Ste06], or generic (in the sense of pluripotential theory) non-radial potentials supported in a ball in $d \geq 3$ [DV14]. For potentials decaying like $\exp(-|x|^{1+\varepsilon})$, Sá Barreto and Zworski [SBZ95] proved

$$n_V(r) \leq C_V r^{d(1+1/\varepsilon)}, \quad r \geq 1.$$  \hspace{1cm} (2)

For super-exponentially decaying potentials, Froese [Fro98] established an upper bound on $n_V(r)$ in terms of the growth of the Fourier transform of $V$. Returning to the compactly supported case, the estimate (1) admits far reaching generalizations to operators other than $-\Delta + V$, e.g. metric perturbations, obstacle scattering and scattering by finite-volume surfaces. The appropriate abstract framework is called black box scattering. We will not discuss this further but refer to the recent book [DZ19] and survey article [Zwo17] for more details and references. We also consider the semiclassical Schrödinger operator $-\hbar^2 \Delta + V$. It is clear that $n_V(r, h)$, the number of resonances $\leq r$, is equal to $n_{V/h^2}(r/h)$. The semiclassical analogue of (1) (see [DZ19, Thm. 4.13]) is

$$n_V(r, h) \leq C_V h^{-d}r^d, \quad r \geq 1, \quad h \in (0, 1].$$  \hspace{1cm} (3)

### 1.2. Effective bounds

The previously discussed upper bounds are meaningful for a fixed potential. Our main aim is to prove effective upper bounds that makes the dependence of $C_V$ on $V$ explicit. Korotyaev [Kor16, Theorem 1.1] proved an effective upper bound in $d = 1$, a simplified version of which (without explicit constants) being

$$n_V(r) \lesssim r a + \ln(2 + r) + \frac{Q}{1 + r}, \quad r > 0,$$  \hspace{1cm} (4)

provided $\text{supp}(V) \subset [-a, a]$ and $Q := \int_{\mathbb{R}} (1 + |x|)|V(x)|dx < \infty$. Our contribution is a far-reaching generalization of (1) to higher dimensions and to larger potential classes (Lorentz space, compactly supported or pointwise decaying). For the sake of exposition, we temporarily continue to assume that $V$ is bounded and compactly supported (so that resonances are defined as above), but our estimates will be uniform in the respective potential class. We postpone the definition of resonances for non compactly supported potentials to Section 1.3. The appearance of exponentially weighted norms is natural in view of the structure of the Schwartz kernel of the free resolvent (see e.g. [DZ19, 3.1.16]). For the definition of the Lorentz norm $\|V\|_{(d+1)/2,1/2}$, the reader is referred to Section 5.1

**Theorem 1.1.** Suppose $d \geq 3$ is odd and that $V$ is bounded and compactly supported with $\text{supp} V \subset B(0, R)$. For every $\varepsilon > 0$ there exists a constant $C_1 = C_1(\varepsilon)$ (independent of $V$ and $R$), such that every resonance $\lambda$ satisfies

$$|\lambda| \leq C_1 e^{2(1+\varepsilon)(\text{Im} \lambda)^{-1}} V^{(d+1)/2}, \quad |\lambda| \leq C_1 R(\lambda R)^{-\varepsilon} e^{2(1+\varepsilon)(\text{Im} \lambda)^{-1}} R\|V\|_{\infty}, \quad |\lambda| \leq C_1(\lambda)^{\varepsilon} e^{2(1+\varepsilon)(\text{Im} \lambda)^{-1}} |\lambda|^{1+\varepsilon} V^{(d+1)/2},$$

\hspace{1cm} (5)  \hspace{1cm} (6)  \hspace{1cm} (7)
where $(\text{Im} \, \lambda)_- := \max(0, -\text{Im} \, \lambda)$ and $\langle \lambda \rangle := 2 + |\lambda|$. Moreover, there exists an absolute constant $C_2 \geq 2$ such that the number of resonances $|\lambda| \leq r$ is bounded by

$$n_V(r) \leq C_1 \ln(C_1 C_2 r^{-(d+2)(d+1)/2}) e^{2(1+\epsilon)r|\lambda|V/(d+1)/2)}$$

if $r \geq C_2 \|V\|^{(d+1)/2}$.

$$n_V(r) \leq C_1 \ln(C_1 C_2 r^{-1}(rR)^{e^{2(1+\epsilon)rR}}V/\|V\|_{\infty}^d$$

if $r \geq C_2 \|V\|_{\infty}^{1/2}$,

$$n_V(r) \leq C_1 \ln(C_1 C_2 r^{-(1+\epsilon)r|\lambda|V/\|V\|_{\infty}^d$$

if $r \geq C_2 \|V\|_{\infty}^{1/2}$.

A more refined version of Theorem 1.1 is stated in Section 7 (Theorem 7.1). The refinement has several aspects, among others:

1. The dependence of $C_1$ on $\epsilon$ is explicit;
2. The upper bound is valid for all $r > 0$ (at the expense of having additional terms on the right hand side). We reiterate that our bounds are uniform, not just asymptotic for large $r$. For this reason, we include terms like $r^{-1}$ in (9), which would be negligible in the limit $r \to \infty$. They also play a role in the semiclassical bound and ensure that the argument of the logarithms in (8), (9) are dimensionless, thus respecting dilation symmetry (scaling). The symmetry is broken in (10) since we have fixed a scale by using $\langle x \rangle$ in the norm. Theorem 7.1 will involve an arbitrary scale $R > 0$, which restores the symmetry and makes the similarity to the compactly supported case more obvious.
3. The upper bound holds for the number of resonances in a larger disk $D(\lambda_0, |\lambda_0| + r)$, where $\lambda_0 \in \mathbb{R}_+$ (linearly) depends on a parameter $A$ that may be tuned. In particular, by choosing $A \gg r \gg 1$, one gets a bound on the number of resonances close to the real axis.
4. There are additional parameters that may be tuned to change the exponential weights.

Remark 1.2. The bounds (8), (9), (10) correctly reproduce the asymptotic results discussed before. More precisely, straightforward calculations reveal the following (see Appendix A for proofs):

(i) (8), (9), (10) all imply Zworski’s bound (4).
(ii) (8), (10) both imply the bound (2) of Sá Barreto and Zworski.
(iii) (9), (10) both imply the semiclassical bound (4).

The theorem also yields genuinely new bounds that cannot be obtained from any previous results in the literature.

Example 1.3. Assume $V = \sum_{j=1}^{\infty} H_j \chi_{\Omega_j}$, where $H_j \in \mathbb{C}$ and $\Omega_j \subset \mathbb{R}^d$ are mutually disjoint bounded measurable sets. Assume that $L_j := \text{dist}(\Omega_j, \bigcup_{i \neq j} \Omega_i \setminus \Omega_j)$ is increasing, $\sum_{j} \exp(-\eta L_j) < \infty$ for every $\eta > 0$, and $\lim_{j \to \infty} L_j^{-1} \text{diam}(\Omega_j) = 0$. Potentials of this type were called sparse in [Cue22]. For simplicity, assume that $|H_j| = 1$, $L_j = j$ and $\Omega_j$ are balls with $\Omega_j \exp(Mj) < \infty$ for all $M > 0$. In particular, $V$ need not decay at infinity. However, (8) implies that $n_V(r)$ is finite for all $r > 0$ (see Section A for details).

1.3. Meromorphic continuation. By a simple density argument, the result of Theorem 1.1 can be extended to non compactly supported potentials as long as
the respective norms are finite. In order to define resonances in this case we need the following results (see Section 5 for the proofs).

**Proposition 1.4.** Let \( d \geq 3 \) be odd, \( \gamma > 0 \). If \( V \in e^{-2\gamma |\cdot|} L^{(d+1)/2,1/2}(\mathbb{R}^d) \), then \( R_V \) has a meromorphic continuation to \( \text{Im} \lambda > -\gamma \).

**Proposition 1.5.** Let \( d \geq 3 \) be odd, \( \gamma, \epsilon > 0 \). If \( V \in e^{-2\gamma |\cdot|} |\cdot|^{-1-\epsilon} L^\infty(\mathbb{R}^d) \), then \( R_V \) has a meromorphic continuation to \( \text{Im} \lambda > -\gamma \).

1.4. **Resonances in the upper half plane.** The techniques used to prove resonance bounds are close to those used to prove eigenvalue bounds for Schrödinger operators with complex potentials. We refer for instance to [Cue22] for references on this subject. Eigenvalues \( z = \lambda^2 \) are of course just resonances in the upper half-plane \( \text{Im} \lambda \geq 0 \). If \( V \) is real-valued, then (e.g. by the Cwikel–Lieb–Rozenbljum bound if \( d \geq 3 \)) there are only finitely many negative eigenvalues. If \( V \) is complex-valued, Frank [Fra11] proved that all eigenvalues are contained in the disk

\[
|z|^{p-d/2} \lesssim \|V\|_p^p
\]

for \( d \geq 2 \) and \( d/2 < p \leq (d+1)/2 \). Bögli and the author [BC22] produced a counterexample that shows (11) is false for \( p > (d+1)/2 \), thus disproving a conjecture of Laptev and Safronov [LS09]. A consequence of this counterexample is that there exist potentials with arbitrary small \( L^p \) and \( L^\infty \) norm whose eigenvalues accumulate to every point of the essential spectrum \([0, \infty)\). Compared to bounds on single eigenvalues, much less is known about the distribution and especially about the number of eigenvalues, i.e. the quantity

\[
n^+_V := \# \{ \text{resonances } \lambda : \text{Im} \lambda \geq 0 \}.
\]

Frank, Laptev and Safronov [FLS16] proved that, when \( d \geq 3 \) is odd, then

\[
n^+_V \lesssim \gamma^{-2} \|e^{2\gamma |\cdot|} V\|^{d+1}_{(d+1)/2}
\]

for any \( \gamma > 0 \) (with implicit constant independent of \( \gamma \)). The techniques developed here can be used to obtain an alternative (and sometimes sharper) bound. To set a benchmark, we test (12) on bounded potentials supported in \( B(0, R) \), where \( R \) is large and \( \|V\|_\infty \leq 1 \). Then (12) implies \( n^+_V \lesssim R^{d+2} \). Our method yields the improved bound \( n^+_V \lesssim R^{2d} \). Indeed, by [AAD01] (or by (59) with \( k = 1 \) and the Birman–Schwinger principle), all resonances in the upper half plane (eigenvalues) must lie in a half-disk \( |\lambda| \leq C_0 R \), \( \text{Im} \lambda \geq 0 \). Hence, choosing \( r = C_0 R \) in Theorem 1.1, it follows from (9) that \( n^+_V \lesssim R^{2d} \) for all \( R > \max(1, C_2/C_0) \).

1.5. **Overview of the proof.** The proof of (1) starts by identifying resonances with a subset of the zeros of a certain determinant, see [DZ19] for a textbook presentation. The determinant there is

\[
det(I - (VR_0(\lambda)\rho)^{d+1}), \quad \lambda \in \mathbb{C},
\]

where \( \rho \in C^\infty_0(\mathbb{R}^d) \) is equal to 1 on \( \text{supp} \ R \). Recall that, if \( K \) is a trace class operator, the Fredholm determinant of \( I - K \) is defined by

\[
det(I - K) := \prod_{k \in \mathbb{N}} (1 - \lambda_k(K)),
\]
where \( \lambda_j(K) \) are the eigenvalues of \( K \), repeated according to multiplicities. The power \( p = d + 1 \) in (13) is convenient, but any integer \( \alpha > d/2 \) could be chosen. We will work with a different Fredholm determinant,

\[
H_\alpha(\lambda) := \det(I - (-BS(\lambda))^\alpha), \quad \lambda \in \mathbb{C},
\]

where \( \alpha > d/2 \) is an integer, defined in terms of the Birman-Schwinger operator

\[
BS(\lambda) := |V|^{1/2}R_0(\lambda)V^{1/2}.
\]

Then the poles of the resolvent \( R_V \) are among the zeros of \( H_\alpha \) (this follows from the resolvent identity (23)). Let \( n_{H_\alpha}(r) \) be the number of zeros (counting multiplicities) of \( H_\alpha \) in the disk \( D(0, r) \). The definition of the multiplicity \( m_R(\lambda) \) of a resonance \( \lambda \) is subtle, and we refer the reader to [DZ19, Sect. 3.2]. However, for the purpose of proving upper bounds, it suffices to know that \( m_R(\lambda) \) is less or equal than the multiplicity of \( \lambda \) as a zero of \( H_\alpha \), see [DZ19, Thm. 3.26] (The proof there works for any integer \( \alpha > d/2 \)). In particular, this implies that \( n_V(r) \leq n_{H_\alpha}(r) \).

In order to ensure that \( H_\alpha \) is an entire function we need to restrict to bounded, compactly supported potentials; in the general case, the estimates on \( BS(\lambda) \) do not allow us to exclude a singularity at the origin (see e.g. (21)). Froese [Fro98] also used a Birman–Schwinger argument, but he considers a different determinant (the determinant of the scattering matrix). Our definition is closer to [FLS16] and other works concerned with the distribution of eigenvalues for Schrödinger operators with complex potentials, e.g. [BGK09, DHK09, Fra18, FS17, LS09]. However, a major difference to these works is that we do not use regularized determinants. Zworski [Zwo89a] observed that, except when \( d = 3 \), these grow too fast as \( \text{Im} \lambda \to -\infty \) (also see the discussion before [DZ19, Thm. 3.27]), and thus they become less useful for the purpose of counting resonances (in the lower half plane).

It will be convenient to consider the regularized counting functions

\[
N_V(r) := \int_0^r \frac{n_V(t)}{t} \, dt, \quad N_{H_\alpha}(r) := \int_0^r \frac{n_{H_\alpha}(t)}{t} \, dt,
\]

for which we also have \( N_V(r) \leq N_{H_\alpha}(r) \). Moreover, for any \( s > 1 \), we have that \( n_V(r) \leq (\ln s)^{-1}N_V(sr) \) (see e.g. [Fro97]). It will therefore be sufficient to prove upper bounds on \( N_{H_\alpha}(sr) \) for all \( r > 0 \) and some fixed \( s > 1 \). This is achieved by means of Jensen’s formula,

\[
N_{H_\alpha}(sr) = \frac{1}{2\pi} \int_0^{2\pi} \ln |H_\alpha(sr e^{i\theta})| \, d\theta - \ln |H_\alpha(0)|. \tag{14}
\]

Using Weyl’s inequality between eigenvalues and singular values one can show that (see Lemma 6.1)

\[
\ln |H_\alpha(\lambda)| \lesssim \sum_{k \in \mathbb{N}} \ln(1 + |s_k(BS(\lambda))|\alpha),
\]

which gives control over the first term in (14), provided we can prove good upper bounds on the singular values \( s_k(BS(\lambda)) \). However, the second term is a nuisance.
since the upper bounds for the quantity $-\ln |H_\alpha(\lambda)|$ will blow up at $\lambda = 0$. To circumvent this problem we will consider a different function

$$F_\alpha(k) := \frac{H_\alpha(\lambda_0 + k)}{H_\alpha(\lambda_0)}, \quad k \in \mathbb{C},$$

(15) that is normalized at zero, i.e. $F_\alpha(0) = 1$, and that counts resonances in a larger disk $D(\lambda_0, |\lambda_0| + sr)$, with $\lambda_0 \in i\mathbb{R}_+$ (see Figure 1). Choosing $|\lambda_0|$ sufficiently large, one can show that (see Lemma 6.2)

$$-\ln |H_\alpha(\lambda_0)| \lesssim \sum_{k \in \mathbb{N}} |s_k(BS(\lambda_0))|^\alpha.$$

Applying Jensen’s formula to $F_\alpha$ then yields

$$n_V(r) \ln s \lesssim \max_{\lambda \in \partial D(\lambda_0, |\lambda_0| + sr)} \sum_{k \in \mathbb{N}} \ln(1 + |s_k(BS(\lambda))|^\alpha) + \sum_{k \in \mathbb{N}} |s_k(BS(\lambda_0))|^\alpha, \quad (16)$$

The maximum will be estimated separately for the part of the boundary in the upper and in the lower half plane. In the upper half plane, we will prove polynomial bounds of the form

$$s_k(BS(\lambda)) \lesssim M_+(\lambda)k^{-1/\beta_+}, \quad \Im \lambda \geq 0,$$

(17) where $M_+(\lambda)$ is one of the norms of $V$ appearing in Theorem 1.1 and $\beta_+ > 0$. The singular value estimates for $\Im \lambda < 0$ will be based on Stone’s formula (see e.g. [DZ19] (3.1.19)) in the form

$$BS(\lambda) - BS(-\lambda) = a_d\lambda^{d-2}|V|^{1/2}\partial\varepsilon(\lambda)(\overline{E(\lambda)}^*)V^{1/2}, \quad \Im \lambda < 0,$$

(18) where $\varepsilon(\lambda)f(x) := \mathcal{F}(fdS)(-\lambda x)$ denotes the (analytically continued) Fourier extension operator (see Section 2.3). We will prove exponential bounds of the form

$$s_k(|V|^{1/2}\partial\varepsilon(\lambda)\overline{E(\lambda)}^*|V|^{1/2}) \lesssim M_-(\lambda)\exp(-ck^{1/\beta_-}), \quad \lambda \in \mathbb{C},$$

(19) where $M_-(\lambda)$ is of the same type as $M_+(\lambda)$ and $c, \beta_- > 0$. For instance, an admissible choice would be

$$M_+(\lambda) = |\lambda|^{\frac{2}{2d+1}||V||^{(d+1)/2}},$$

$$M_-(\lambda) = |\lambda|^{-\frac{d+1}{2d+1}||e^{2(1/c)||\varepsilon(\lambda)||}|V||^{(d+1)/2}}$$

(20) and $\beta_+ = d + 1, \beta_- = d - 1$ (see Proposition 5.1). Using (17), (18), (19), we can estimate (16), leading to an upper bound

$$n_V(r) \lesssim (\ln s)^{-1}(\tilde{n}_+(sr) + \tilde{n}_-(sr) + \tilde{n}_0),$$

where $\tilde{n}_+(sr), \tilde{n}_-(sr)$ denote the contributions of the first term in (16), corresponding to the upper and lower half plane, respectively, and $\tilde{n}_0$ denotes the contribution of the second term in (16). Of course, these quantities also depend on $V$, but only through one of its norms. For large $r$, the dominant term is $n_-(r)$, and this leads to the bounds (8), (9), (10) stated in Theorem 1.1. The resonance-free regions can be determined by the Birman–Schwinger principle, i.e. the fact that $\lambda^2$ is an eigenvalue of $-\Delta + V$ if and only if $-1$ is an eigenvalue of $BS(\lambda)$. This implies
that \( \|BS(\lambda)\| \geq 1 \) for any resonance, which yields (4), (9), (7). Since the operator norm equals the first singular value, these will therefore also follow from the singular value bounds.

1.6. Technical novelties. The main technical novelty in this work compared to the textbook result [DZ19] is the uniform control of the singular values (17), (19) in terms of the potential. By keeping track of the constants in the presentation of [DZ19] one could in principle also obtain effective estimates, in terms of \( \|V\|_\infty \) and the size of the support of \( V \). However, compared to (4) (the compactly supported case), this bound would involve \( R^d \|V\|_\infty \) as opposed to \( R \|V\|_\infty \). This is because instead of the trivial bound for the Fourier extension operator (taking absolute values inside the integral) we use a new energy estimate (Lemma 3.1). The latter can be viewed as a generalization of the classical Agmon–Hörmander bound [AH76] to “complex energies”. Similar results were obtained by Burq [Bur02] and Gannot [Gan15], but our estimate is scale-invariant and is nearly optimal in the exponential weight. The singular value estimates for compactly supported potentials can be summed dyadically to yield the results for pointwise decaying potentials.

The main advancement in this article is that we allow potentials with only average decay (the Lorentz case). To the best of our knowledge, our bounds on the number of resonances and on resonance-free regions are the first that involve only a (weighted) Lorentz norm. The corresponding singular value estimates are related to a result of Frank–Laptev–Safronov [FLS16], who proved

\[
\|BS(\lambda)\|_{\mathcal{G}^{d+1}} \lesssim |\lambda|^{-\frac{d}{d+1}} e^{2C_d (\text{Im} \lambda) - |\cdot|} \|V\|_{(d+1)/2}, \quad \lambda \in \mathbb{C}, \tag{21}
\]

for some \( C_d > 1 \). The case \( \text{Im} \lambda \geq 0 \) was established by Frank–Sabin [FS17].

In view of the exponential growth of the free resolvent, the weight cannot be smaller than \( e^{2(\text{Im} \lambda) - |\cdot|} \). By Stone’s formula, (21) implies a polynomial bound on the singular values of the operator on the left of (19) (involving the Fourier extension operator \( E(\lambda) \)). If \( \lambda \) were real, then this would be a \( TT^* \) operator, and its singular values would coincide with those of \( T^*T \). The latter is an operator on the unit sphere, and one expects its singular values to decay exponentially (if \( V \) has sufficient decay). If \( \lambda \) is complex, then \( E(\lambda)E(\bar{\lambda})^* \) is not a \( TT^* \) operator, but it still has a translation-invariant kernel. However, to recover the exponential decay of singular values, one has to look at \( E(\lambda)E(\lambda)^* \), which is not controlled by a translation-invariant kernel. Hence, the (by now) standard methods of proving Schatten norm estimates for such operators (either complex interpolation [FS17] or the multilinear Hardy–Littlewood–Sobolev inequality [FLLS16]) are not readily applicable. To overcome this difficulty, we adapt Bourgain’s proof of the Stein–Tomas inequality [Bou91] to the trace ideal setting. It turns out that this method is robust enough to handle the case of complex \( \lambda \). Moreover, it yields estimates that are stronger than (21) in two important ways:

1. We obtain exponential decay of singular values in (19) and
2. The weight \( e^{2(1+\epsilon)(\text{Im} \lambda) - |\cdot|} \) in (20) is essentially optimal (possibly up to an \( \epsilon \) loss).

However, our method only gives a restricted weak type inequality. This means two things: First, the Schatten space \( \mathcal{G}^{d+1} \) has to be replaced by the weak Schatten
space $S^{d+1,w}$. Second, the $L^{(d+1)/2}$ of the potential has to be replaced by the $L^{(d+1)/2,1/2}$ Lorentz norm.

In addition to the technical innovations discussed above, there are two additional new features worth highlighting. The first is a stationary phase estimate for certain oscillatory integrals with complex-valued phase functions. Although such integrals are covered e.g. in Hörmander’s treatise of stationary phase [Hor90], our phase functions are more special and allow for a more elementary proof (without reference to the Malgrange preparation theorem). The second new feature is a structure formula for $E(\lambda)(I - \epsilon^2 \Delta_S)$, where $\Delta_S$ the Laplace–Beltrami operator on the unit sphere $S^{d-1}$. This is needed in order to reduce the exponential decay estimate (19) to Weyl’s asymptotics. The trick is not new (see [DZ19] and references there) but the argument is much more delicate since we take cancellations of the phase into account. Again, if one only uses size estimates (taking absolute values inside the integral), this step becomes almost trivial.

1.7. Outline of the paper. In Section 2, we briefly review some standard results about singular values of compact operators, the particular resolvent identity used in this paper, and the Fourier extension operator $E(\lambda)$. In Section 3, we we prove a generalization of the Agmon–Hörmander bound for $E(\lambda)$ to complex $\lambda$. In Section 4, we prove a similar generalization of the Stein–Tomas bound. Section 5 contains the key estimates for the singular values of the Birman–Schwinger operator for the potential classes discussed in the introduction (Lorentz, compactly supported, poitwise decaying). In Section 6 we discuss Fredholm determinants and make the arguments in the introduction more precise. In Section 7, we prove very detailed effective bounds for the number of resonances, Theorem 7.1, and we show that Theorem 1.1 follows from this. In Appendix A, we give the proof of a certain ‘structure formula’ for commuting $E(\lambda)$ with powers of the Laplacian on the unit sphere. In Appendix B, we provide additional details of the proof of Theorem 7.1. In Appendix C, we prove Remark 1.2 and Example 1.3.

Notation. We write $A \lesssim B$ or $A = O(1)B$ for two non-negative quantities $A, B$ to indicate that there is a constant $C > 0$ such that $A \leq CB$. To highlight the dependence of the constant on a parameter $\epsilon$, the notation $A \lesssim_{\epsilon} B$ is sometimes used. The dependence on fixed parameters like the dimension $d$ is usually omitted. The notation $A \asymp B$ means $A \lesssim B \lesssim A$, and $A \ll B$ means that $A \leq cB$ for some sufficiently small constant $c > 0$. If $T : X \to Y$ is a bounded linear operator between two Hilbert spaces $X$ and $Y$, we denote its operator norm by $\|T\|_{X \to Y}$, or just by $\|T\|$ if the spaces are clear from the context. The indicator function of a set $\Omega \subset \mathbb{R}^d$ is denoted by $1_{\Omega}$. We define $\langle x \rangle := 2 + |x|$. The natural numbers are denoted by $\mathbb{N} = \{1, 2, \ldots\}$. The upper/lower half plane is denoted by $\mathbb{C}^{\pm} = \{ \lambda \in \mathbb{C} : \pm \text{Im} \lambda > 0 \}$. We also use the following abbreviations, especially in Sections 5 and 6,

$$c_\delta := \delta^{-\frac{(d-1)^2}{d+1}}, \quad \rho_\delta(x, \lambda) := e^{-\sqrt{1+\delta^2} |\text{Im} \lambda||x|}, \quad \rho_{\delta,\epsilon}(x, \lambda) := e^{-(\sqrt{1+\delta}\epsilon |\text{Im} \lambda|+\epsilon |\lambda|) |x|}.$$

Acknowledgments. The author thanks Alexei Stepanenko and Konstantin Merz for useful discussions, and Maciej Zworski for valuable feedback on an earlier version of the article. He also thanks the anonymous reviewers for their careful
2. Preliminaries

2.1. Singular values and Schatten spaces. We denote by $\mathcal{S}^p$ the Schatten class of order $p \in (0, \infty)$ over the Hilbert space $L^2(\mathbb{R}^d)$ and by $\| \cdot \|_{\mathcal{S}^p}$ the corresponding Schatten norm

$$
\|K\|_{\mathcal{S}^p} := \left( \sum_{k \in \mathbb{N}} s_k(K)^p \right)^{\frac{1}{p}},
$$

where $(s_k(K))_{k \in \mathbb{N}}$ denotes the sequence of singular values of the compact operator $K$, that is, the square roots of the eigenvalues of $K^*K$, in nonincreasing order and repeated according to multiplicities (note that [DZ19] use a different convention, where the enumeration starts at $k = 0$). The weak Schatten class $\mathcal{S}^p, w$ of order $p \in (0, \infty)$ is the set of all compact operators on $L^2(\mathbb{R}^d)$ for which the quasinorm

$$
\|K\|_{\mathcal{S}^p, w} := \sup_{N \in \mathbb{N}} N^{\frac{1}{p}} - 1 \sum_{k=1}^{N} s_k(K),
$$

is finite. For $p > 1$, an equivalent norm is given by

$$
\|K\|_{\mathcal{S}^p, w} := \sup_{N \in \mathbb{N}} N^{\frac{1}{p}} - 1 \sum_{k=1}^{N} s_k(K),
$$

see e.g. [BS87, Ch. 11, Sect. 6]. We will use the following standard facts for singular values repeatedly throughout the article (see e.g. [Sim05]). If $A, B$ are compact operators, then for $j, k \geq 0$,

$$
s_{j+k+1}(AB) \leq s_{j+1}(A)s_{k+1}(B),
$$

$$
s_{j+k+1}(A + B) \leq s_{j+1}(A) + s_{k+1}(B). \tag{22}
$$

If $A$ is compact and $B$ is bounded, then

$$
s_k(AB), s_k(BA) \leq s_k(A)\|B\|.
$$

Moreover, we have the elementary equality $s_k(K) = s_k(K^*)$.

2.2. Resolvent identity. If $V$ is bounded, then, by iterating the second resolvent identity,

$$
R_V(\lambda) = R_0(\lambda) - R_0(\lambda)V R_V(\lambda), \quad \text{Im } \lambda \gg 1,
$$

one obtains

$$
R_V(\lambda) = R_0(\lambda) - R_0(\lambda)V^{1/2}(I + BS(\lambda))^{-1}V^{1/2}R_0(\lambda). \tag{23}
$$

This formula is valid for $\text{Im } \lambda \gg 1$ since $\|BS(\lambda)\| \leq |\text{Im } (\lambda^2)|^{-1} \|V\|_{\infty}$. If $V$ is unbounded, then one can use (23) as the definition of $R_V(\lambda)$, $\text{Im } \lambda \gg 1$. This is a classical construction going back to Kato [Kat66], see also [GLMZ05] for abstract results in the nonselfadjoint setting.
2.3. Fourier extension operator. Consider the (analytically continued) Fourier extension operator
\[ \mathcal{E}(\lambda) : L^1(\mathbb{S}^{d-1}) \to L^\infty_{\text{loc}}(\mathbb{R}^d), \]
\[ \mathcal{E}(\lambda)g(x) = \int_{\mathbb{S}^{d-1}} e^{i\lambda x \cdot \xi} g(\xi) dS(\xi), \quad x \in \mathbb{R}^d, \lambda \in \mathbb{C}, \]
where \(dS\) denotes induced Lebesgue measure on the unit sphere \(\mathbb{S}^{d-1}\). It is immediate from the triangle inequality that \(\mathcal{E}(\lambda)\) is a bounded operator. By Cauchy–Schwarz, it also follows that \(\mathcal{E}(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2_{\text{loc}}(\mathbb{R}^d)\) is bounded. For \(\lambda \geq 0\), the following estimates hold for any \(g \in C^\infty(\mathbb{S}^{d-1})\):
\[ \|\mathcal{E}(\lambda)g\|_{L^2(\mathbb{S}^{d-1})} \leq \lambda^{\frac{d(d-1)}{2}} \|g\|_{L^2(\mathbb{S}^{d-1})}, \]  
\[ \|\mathcal{E}(\lambda)g\|_{L^2(B(0,R))} \leq R^\frac{d}{2} \lambda^{-\frac{d-1}{2}} \|g\|_{L^2(\mathbb{S}^{d-1})}. \]  
These estimates follow by rescaling from \(\lambda = 1\), in which case \(24\) is known as the Stein–Tomas estimate \([\text{Ste93, Prop. IX.2.1}]\) and \(25\) as the Agmon–Hörmander estimate \([\text{AH76, Thm. 2.1}]\).

3. Agmon–Hörmander estimate

In this section we prove a generalization of the Agmon–Hörmander bound \(25\). For later purposes it will be convenient to consider the slightly more general operators
\[ \mathcal{E}_a(\lambda)g(x) := \int_{\mathbb{S}^{d-1}} e^{i\lambda x \cdot \xi} a(x, \xi) g(\xi) dS(\xi), \quad x \in \mathbb{R}^d, \lambda \in \mathbb{C}, \]
where \(a(x, \xi)\) is bounded and smooth in the \(\xi\)-variable. We are looking for uniform estimates for \(\mathcal{E}_a\) over a family of functions \(a\) in the unit ball of \(L^\infty(\mathbb{R}^d ; C^N(\mathbb{S}^{d-1}))\):
\[ \|a\|_{L^\infty(\mathbb{R}^d ; C^N(\mathbb{S}^{d-1}))} \leq 1. \]  
Here \(N\) is a sufficiently large (depending only on the dimension) but fixed integer. The following proposition is an analogue of the Agmon–Hörmander bound \(25\) for \(\text{Im } \lambda < 0\) and is one of the main technical ingredients. It improves upon \([\text{Bur02, (2.5)}]\) and \([\text{Gan15, Lemma 1.1}]\).

**Proposition 3.1.** Let \(d \geq 2\), \(\lambda \in \mathbb{C}\). For each \(R > 0\), \(\delta \in (0, 1]\), we have
\[ \|\rho_\delta \mathcal{E}_a(\lambda)g\|_{L^2(B(0,R))} \leq R^\frac{d}{2} |\delta\lambda|^{-\frac{d-1}{2}} \|g\|_{L^2(\mathbb{S}^{d-1})}, \]  
uniformly subject to \(20\), where \(\rho_\delta(x, \lambda) := e^{-\sqrt{1+\delta} |\text{Im } \lambda||x||}. \)

For the proof we will need the following lemma; for \(\text{Im } \lambda = 0\) it is an immediate consequence of Plancherel’s theorem.

**Lemma 3.2.** Let \(n \geq 1\), \(\lambda \in \mathbb{C}\), \(\epsilon > 0\). If \(f\) is supported in \(B(0, \epsilon/2) \subset \mathbb{R}^n\), then
\[ \|e^{-\epsilon|\text{Im } \lambda||x||} \hat{f}(\lambda \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim |\lambda|^{-\frac{d}{2}} \|f\|_{L^2(\mathbb{R}^n)}. \]  

Proof. 1. By scaling, we may assume without loss of generality that $\epsilon = 1$. Indeed, if $f$ is supported in $B(0, \epsilon / 2)$, then $g(\xi) := f(\epsilon \xi)$ is supported in $B(0, 1 / 2)$, and if (28) holds with $g$ in place of $f$ and with $\epsilon = 1$, then

$$\epsilon^{-n} \| e^{-|\text{Im} \lambda|} \cdot \hat{f}(\lambda / \epsilon \cdot) \|_{L^2(\mathbb{R}^n)} \lesssim |\lambda|^{-\frac{n}{2}} \epsilon^{-\frac{N}{2}} \| f \|_{L^2(\mathbb{R}^n)}.$$  

Changing variables $\lambda = \epsilon \hat{\lambda}$ yields (28) with $\hat{\lambda}$ in place of $\lambda$. By the monotone convergence theorem it suffices to prove the estimate for $\epsilon = 1$ over a ball $B(0, R)$ with arbitrary $R > 0$. By a change of scale $x = R y$, the inequality for $R = 1$ and $\lambda$ replaced by $\lambda R$ yields the general case $R > 0$.

2. Let $\chi_1, \chi_2$ be non-negative bump functions such that $\chi_1 \geq 1_{B(0,1)}$, $\text{supp}(\chi_1) \subset B(0, 2)$ and $\chi_2 \geq 1_{B(0,1/2)}$, $\text{supp}(\chi_2) \subset B(0, 3/4)$. Because of the support assumption on $f$, we then have

$$\| e^{-|\text{Im} \lambda|} \cdot \hat{f}(\lambda) \|_{L^2(B(0,1))}^2 \leq \| \chi_1 e^{-|\text{Im} \lambda|} \cdot \hat{\chi}_2 f(\lambda) \|_{L^2(\mathbb{R}^n)}.$$  

Since $\chi_2$ is smooth and supported in $B(0, 3/4)$, it follows that $\hat{\chi}_2$ is an entire function, satisfying

$$|\hat{\chi}_2(z)| \leq C_N (1 + |z|)^{-N} e^{\frac{2}{3} |\text{Im} z|}$$  

for all $z \in \mathbb{C}^n$ and all $N > 0$, see e.g. [Hor90] 7.3. Thus,

$$|\hat{\chi}_2(f(x))| = | \int_{\mathbb{R}^n} \hat{\chi}_2(\lambda x - y) \hat{f}(y) dy | \leq C_N e^{\frac{2}{3} |\text{Im} \lambda| |x|} \left( \int (1 + |y| + |(\text{Im} \lambda) x|)^{-N} dy \right)^{\frac{1}{2}} \| f \|_{L^2},$$  

where we used Cauchy–Schwarz and Plancherel and changed variables $y \rightarrow y - (\text{Re} \lambda) x$. This yields

$$\| \chi_1 e^{-|\text{Im} \lambda|} \cdot \hat{\chi}_2 f(\lambda) \|_{L^2}^2 \leq C_N \| f \|_{L^2}^2 \int (1 + |y| + |(\text{Im} \lambda) x|)^{-N} \chi_1(x)^2 dy dx.$$  

Since $\chi_1$ has compact support, this is bounded by $\| f \|_{L^2}^2$, which is the desired bound if $|\lambda| \leq 1$.

3. If $|\lambda| > 1$, we distinguish between $|\text{Im} \lambda| \geq |\text{Re} \lambda|$ or $|\text{Im} \lambda| < |\text{Re} \lambda|$. In the first case, we change the order of integration in (30) (Fubini–Tonelli) and change variables $x \rightarrow |\text{Im} \lambda|^{-1} x$ to obtain an upper bound of $|\lambda|^{-n} \| f \|_{L^2}^2$, which is the desired bound in this case. If $|\text{Im} \lambda| < |\text{Re} \lambda|$, instead of using Cauchy–Schwarz in (28), we only estimate

$$|\hat{\chi}_2 f(\lambda x)| \leq C_N e^{\frac{2}{3} |\text{Im} \lambda| |x|} \int (1 + |(\text{Re} \lambda) x - y|)^{-N} |\hat{f}(y)| dy$$  

and use

$$(1 + s)^{-N} \lesssim 1 \{ s \leq 1 \} + \sum_{j=0}^{\infty} 2^{-Nj} 1 \{ 2^j-1 \leq s \leq 2^j \}.$$  

(31)
for \( s = |\text{Re} \lambda x - y| \). We use Cauchy–Schwarz for each term separately,

\[
\int 1\{ |(\text{Re} \lambda) x - y| \leq 1 \}|\hat{f}(y)|dy \lesssim \left( \int |(\text{Re} \lambda) x - y| \leq 1 |\hat{f}(y)|^2 dy \right)^{\frac{1}{2}},
\]

\[
\int 1\{ 2^{j-1} \leq |(\text{Re} \lambda) x - y| \leq 2^j \}|\hat{f}(y)|dy \lesssim 2^{\delta j} \left( \int 2^{j-1} \leq |(\text{Re} \lambda) x - y| \leq 2^j |\hat{f}(y)|^2 dy \right)^{\frac{1}{2}}.
\]

Then, for \( N > n \), (31) implies

\[
\| \chi_1 e^{-|\text{Im} \lambda||y|} |\lambda_2 f(\lambda)| \|_{L^2}^2 \lesssim N \int \int |(\text{Re} \lambda) x - y| \leq 1 |\hat{f}(y)|^2 dy dx
\]

\[
+ \sum_{j=0}^{\infty} 2^{n-2^j} \left( \int 2^{j-1} \leq |(\text{Re} \lambda) x - y| \leq 2^j |\hat{f}(y)|^2 dy \right)^{\frac{1}{2}} \int dx
\]

\[
\lesssim \int \int |(\text{Re} \lambda) x - y| \leq 1 |\hat{f}(y)|^2 dy dx + \sum_{j=0}^{\infty} 2^{n+1} j^{-2 N j} \int \int 2^{j-1} \leq |(\text{Re} \lambda) x - y| \leq 2^j |\hat{f}(y)|^2 dy dx,
\]

where we used Cauchy–Schwarz for the sum in the last line. By Fubini–Tonelli,

\[
\int \int |(\text{Re} \lambda) x - y| \leq 1 |\hat{f}(y)|^2 dy dx \lesssim |\text{Re} \lambda|^{-n} \int |\hat{f}(y)|^2 dy,
\]

\[
\int \int 2^{j-1} \leq |(\text{Re} \lambda) x - y| \leq 2^j |\hat{f}(y)|^2 dy dx \lesssim 2^{j} |\text{Re} \lambda|^{-n} \int |\hat{f}(y)|^2 dy.
\]

The resulting geometric series is summable for \( 2N > 2n + 1 \), and Plancherel yields the desired bound. \( \square \)

Proof of Proposition 3.1. We first give the proof for \( \mathcal{E}(\lambda) \). By a standard partition of unity argument, we may replace \( \mathcal{E}(\lambda) \) by (with abuse of notation)

\[
\mathcal{E}_1(\lambda) g(x) := \int_{\mathbb{R}^{d-1}} e^{i \lambda (x' \cdot \xi' + x_1 \psi(\xi'))} \chi(\xi') g(\xi') d\xi',
\]

where \( x = (x_1, x') \), \( \xi = (\xi_1, \xi') \), \( \psi(\xi') = \sqrt{1 - |\xi'|^2} \) and \( \chi \) is a smooth function that has support in a fixed small neighborhood of the origin. Since this neighborhood can be covered by approximately \( \delta^{-\frac{d+1}{2}} \) finitely overlapping balls of diameter \( C\delta^\frac{d}{2} \), it suffices to prove

\[
\| \rho_\delta \mathcal{E}_1(\lambda) g \|_{L^2(B(0, R))} \lesssim R^{\frac{d}{2}} |\lambda|^{-\frac{d+1}{2}} \| g \|_{L^2(S^{d-1})},
\]

whenever \( g \) is supported in \( B(0, C\delta^\frac{d}{2}) \). We freeze the \( x_1 \) variable and denote the operator acting on the remaining variables \( x' = (x_2, \ldots, x_d) \) by \( \mathcal{E}_{x_1}(\lambda) \). From the elementary estimate \( 2|x_1||x'| \leq \delta x_1^2 + \delta^{-1} |x'|^2 \) it follows that

\[
(|x_1| + |x'|)^2 \leq (1 + \delta) x_1^2 + (1 + \delta^{-1}) |x'|^2.
\]
Rescaling $x_1 \to \frac{x_1}{\sqrt{1+\delta}}, \ x' \to \frac{x'}{\sqrt{1+\delta}}$ and multiplying the above inequality by $(1+\delta)$ we obtain

$$\sqrt{1+\delta}|(x_1, x')| \geq |x_1| + \epsilon|x'| \quad \text{with} \quad \epsilon := \sqrt{(1+\delta)/(1+\delta^{-1})}.$$  

Using this, one observes that

$$|e^{-\sqrt{1+\delta} \Im \lambda ||(x_1,x')||} \mathcal{E}_{x_1}(\lambda) g(x')| \leq e^{-\epsilon |\Im\lambda| |x'|} |g_{\lambda,x_1}(-\lambda x')|,$$

where $g_{\lambda,x_1}(\xi') := e^{-|\Im\lambda||x_1|}e^{i\lambda x_1 \psi(\xi')} \chi(\xi')g(\xi')$. Since $\frac{1}{4} \delta^2 \leq \frac{1}{2} \epsilon$, Lemma 3.2 implies

$$\|e^{-\sqrt{1+\delta} \Im \lambda ||(x_1,x')||} \mathcal{E}_{x_1}(\lambda) g\|_{L^2(\mathbb{R}^d-1)} \lesssim |\lambda|^{-\frac{d-1}{2}} \|g\|_{L^2(\mathbb{R}^d-1)},$$

whenever $\text{supp}(\chi) \subset B(0, \frac{1}{\delta^2})$, were we used that $|e^{i\lambda x_1 \psi(\xi')}| \leq e^{i|\Im\lambda||x_1|}$. Squaring and integrating over $|x_1| \leq R$ yields (33).

2. To prove this for $\mathcal{E}_a$ we assume first that $a(x, \xi)$ factors as a product, $a(x, \xi) = b(x)c(\xi)$. Then the previous argument can be used without modification. In the general case, we expand $\chi a$ in a Fourier series,

$$\chi(\xi')a(x, (\psi(\xi'), \xi')) = \sum_{j' \in \delta^d \mathbb{Z}^{d-1}} a_{j'}(x)e^{ij'\xi'}.$$

By a scaling argument, we may assume without loss of generality that $\delta = 1$. Then, by (26), we have $|a_{j'}(x)| \lesssim (1 + |j'|)^{-N}$ for any $N > 0$. The result for the general case then follows from the factored case and the triangle inequality. □

Remark 3.3. From the proof of Proposition 3.7 it transpires that if we replace the Euclidean norm $| \cdot |$ by the equivalent norm

$$|x|_0 := \max_{i = 1, \ldots, d} \{|x_i| + |x'_i|\},$$

where $x'_i \in \mathbb{R}^{d-1}$ is the vector $x$ with the $i$-th component omitted, then we have

$$\|e^{-|\Im\lambda||x|} \mathcal{E}_a(\lambda) g\|_{L^2(\mathbb{R}^d)} \lesssim R^2 |\lambda|^{-\frac{d-2}{4}} \|g\|_{L^2(\mathbb{R}^d-1)}.$$  

4. Stein–Tomas estimate

Here we state a generalization of the Stein–Tomas bound (24).

Proposition 4.1. Let $d \geq 2$, $\lambda \in \mathbb{C}$. Then for each $\delta \in (0, 1)$,

$$\|\rho_\delta \mathcal{E}_a(\lambda) g\|_{L^{2(d+1)}(\mathbb{R}^d)} \lesssim \delta^{-\frac{(d-1)^2}{4(d+1)}} \lambda^{-\frac{(d-1)}{2(d+1)}} \|g\|_{L^2(\mathbb{R}^d-1)},$$

uniformly subject to (26).

We omit the proof since the result is a consequence of the stronger singular value estimate (36). The key ingredient in the proof of the latter will be the following stationary phase estimate.

Lemma 4.2. Let $\lambda \in \mathbb{C}$. The Schwartz kernel $K_\lambda$ of $\mathcal{E}_a(\lambda)\mathcal{E}_a(\lambda)^*$ satisfies

$$|K_\lambda(x-y)| \lesssim e^{i\Im\lambda|x-y|(1 + |\lambda||x-y|)^{-\frac{d-1}{4}}} \quad (34)$$
for \( x, y \in \mathbb{R}^d \). Moreover, the kernel \( \tilde{K}_\lambda \) of \( \mathcal{E}_a(\lambda)\mathcal{E}_a(\lambda)^* \) satisfies
\[
|\tilde{K}_\lambda(x, y)| \lesssim e^{4N} (x, y)^1 (1 + |\lambda| |x - y| + |\lambda| |x + y|)^{-\frac{d+1}{2}}.
\]
Both (33), (34) are uniform over \( a \) in the unit ball of \( L^\infty(\mathbb{R}^d; C^N(S^{d-1})) \).

**Proof.** 1. We start with the proof of (34). As in the proof of Theorem 3.1 it suffices to consider the case \( a = 1 \) and to replace \( E(\lambda) \) by \( \mathcal{E} \). Note that \( \psi(0) = 0 \), \( \nabla \psi(0) = 0 \). The kernel \( K_{1,\lambda}(x - y) \) of \( \mathcal{E}_1(\lambda)\mathcal{E}_1(\lambda)^* \) is given by
\[
K_{1,\lambda}(z) := \int_{\mathbb{R}^d} e^{\lambda z} \phi(\omega, \xi') \chi_1(\xi')^2 d\xi', \quad \omega := z/|z| \in S^{d-1},
\]
where \( \phi(\omega, \xi') := \omega \cdot (\psi(\xi'), \xi') \). For \( \lambda > 0 \), this is essentially the Fourier transform of the surface measure of \( \lambda S^d \), and the proof of (34) is classical, see e.g. [Ste93 VIII.3.1.1]. The proof for \( \lambda \in \mathbb{C} \) is a simple modification, but we shall provide some details. The phase \( \phi \) is stationary at the points \( (\pm e_1, 0) \), i.e. \( \nabla \phi(\pm e_1, 0) = 0 \), where \( e_1 = (1, 0, \ldots, 0) \). Since
\[
\det [D^2\phi(\omega, \xi')] = \omega_t d^{-1} \det [D^2\psi(\xi')] \neq 0
\]
at \( (\omega, \xi') = (\pm e_1, 0) \), the implicit function theorem implies that, for \( \omega \) in a small neighborhood \( \mathcal{N}_\pm \) of \( \pm e_1 \), there is a unique solution \( \xi'_\pm = \xi'_\pm(\omega) \) of the critical point equation \( \nabla \phi(\omega, \xi') = 0 \) (assuming, as we may, that the support of \( \chi_1 \) is sufficiently small). Moreover, if \( \mathcal{N}_\pm \) are sufficiently small, then (37) still holds at \( (\omega, \xi'_\pm(\omega)) \).

2. We will only consider the two cases \( \omega \in \mathcal{N}_+ \) and \( \omega \in S^{d-1} \setminus (\mathcal{N}_+ \cup \mathcal{N}_-) \). The proof for \( \omega \in \mathcal{N}_- \) is the same as for \( \omega \in \mathcal{N}_+ \). If \( \omega \in S^{d-1} \setminus (\mathcal{N}_+ \cup \mathcal{N}_-) \), then \( \phi \) has no stationary point (if the support of \( \chi \) is chosen sufficiently small) since \( \nabla \phi(\omega, 0) = \omega' \) and \( |\omega'| \geq c > 0 \). In this case, integration by parts yields
\[
|K_{1,\lambda}(z)| \lesssim e^{4N} (1 + |\lambda| |z|)^{-N}
\]
for any \( N > 0 \), where we used that \( |\phi(\omega, \xi')| \leq 1 \) and thus \( e^{4N} |\phi(\omega, \xi')| \leq e^{4N} |\phi(\omega, \xi')| \). If \( \omega \in \mathcal{N}_+ \), we Taylor expand \( \phi \) around the critical point,
\[
\phi(\omega, \xi') = \phi(\omega, \xi'(\omega)) + \frac{1}{2} \langle \eta', D_{\xi'}^2 \phi(\omega, \xi'(\omega)) \eta' \rangle + O(|\eta'|^3),
\]
where \( \xi'(\omega) = \xi'_0(\omega) \) and \( \eta' = \xi' - \xi'(\omega) \) (note that the prime does not stand for a derivative here). Then (37) implies that \( |\nabla_\xi \phi(\omega, \xi')| \geq |\eta'| \) on \( \text{supp}(\chi_1) \). We split the integral in (30) into two parts, \( K_{1,\lambda}(z) = I + II \), where
\[
I := \int_{\mathbb{R}^{d-1}} e^{4N} (\phi(\omega, \xi'))^2 \rho(|\lambda z|^{1/2}(\xi' - \xi'(\omega))) d\xi',
\]
and \( \rho \) is a bump function that localizes near the origin. A simple base times height estimate then shows that \( |I| \lesssim e^{4N} |\lambda z|^{-d-1}/2 \). For the second integral
\[
II = \int_{\mathbb{R}^{d-1}} e^{4N} (\phi(\omega, \xi'))^2 (1 - \rho(|\lambda z|^{1/2}\eta')) d\xi' \quad (\eta' = \xi' - \xi'(\omega)),
\]
integration by parts yields, for any integer \( N > d - 1 \) (similarly as in the proof of \([MS13\text{ Lemma 4.15}]\),
\[
|I| \lesssim e^{\text{Im }\lambda|z|}|z|^{-N} \int_{|\eta'|>|z|^{-1/2}} (|\eta'|^{N/2}|\eta'|^{-N} + |\eta'|^{-2N})d\eta' \lesssim e^{\text{Im }\lambda|z|}|z|^{-\frac{d-1}{2}}.
\]
This proves \( |K_{1,\lambda}(z)| \lesssim e^{\text{Im }\lambda|z|}(1+|z|)^{-\frac{d-1}{2}} \) for \( |z| \geq 1 \). The case \( |z| < 1 \) is of course trivial.

3. The proof of (35) is similar. Instead of (36), we need to consider
\[
\tilde{K}_{1,\lambda}(x, y) := \int_{\mathbb{R}^{d-1}} e^{i(\lambda x - \overline{\lambda y})\cdot(\psi(\xi'), \xi')} \chi(\lambda')^2d\xi'.
\]
Note, however, that this is no longer a convolution kernel. We can write the phase as \( |\lambda x - \overline{\lambda y}|\phi(\omega, \xi') \), with the same function \( \phi(\omega, \xi') = \omega \cdot (\psi(\xi'), \xi') \) as before, and with \( \omega := \frac{\lambda x - \overline{\lambda y}}{|\lambda x - \overline{\lambda y}|} \in \mathbb{C}^d \), \( |\omega| = 1 \). If \( |\omega'| \ll 1 \), then \( |\nabla \xi' \phi(\omega, \xi')| \gtrsim 1 \) if the support of \( \chi \) is sufficiently small, which we may assume. Integration by parts and the identity
\[
\lambda x - \overline{\lambda y} = \text{Re }\lambda(x - y) + i\text{Im }\lambda(x + y)
\]
then yield
\[
|\tilde{K}_{1,\lambda}(x, y)| \lesssim N \ e^{\text{Im }\lambda|z|+y}(1+|\text{Re }\lambda||x-y| + |\text{Im }\lambda||x+y|)^{-N}.
\]
If \( |\omega'| \ll 1 \), then \( |\omega_1| \gtrsim 1 \). Since \( \psi(\xi') = \sqrt{1-\xi' \cdot \xi'} \) has an analytic continuation to a neighborhood of the origin in \( \mathbb{C}^{d-1} \) and \( \psi(\xi') = 1 - \frac{1}{2} \xi' \cdot \xi' + O(|\xi'|^3) \), there is a unique critical point of \( \phi(\omega, \cdot) \) near the origin, given by \( \xi(\omega) = \frac{\omega'}{\omega^2}(1+|\xi'|^2) \in \mathbb{C}^{d-1} \).

Since \( D_{\xi'}^2 \phi(\omega, \xi') = -\omega_1^2 I + O(|\xi'|^2) \) is nondegenerate there, (35) again implies
\[
|\nabla \xi' \phi(\omega, \xi')| \gtrsim |\xi' - \xi'(\omega)|.
\]
Splitting \( \tilde{K}_{1,\lambda}(z) = I + II \), where now
\[
I := \int_{\mathbb{R}^{d-1}} e^{i(\lambda x - \overline{\lambda y})\cdot(\psi(\xi'), \xi')} \chi(\lambda')^2 \rho(|\lambda x - \overline{\lambda y}|^{1/2}(\xi' - \text{Re }\xi'(\omega)))d\xi',
\]
and repeating the estimates used for (39), (40), we obtain (35). \( \square \)

5. **Singular value estimates**

5.1. **Lorentz space potentials.** Assume that \( V \in e^{-2\gamma |.|}L^{\frac{d+1}{2}}(\mathbb{R}^d) \) for some \( \gamma > 0 \). Here, \( \| \cdot \|_{p,q} \) denotes the quasinorm
\[
\|V\|_{p,q} := p^{\frac{1}{q}} \left( \int_0^\infty \alpha^q \|\{x \in \mathbb{R}^d : |V(x)| > \alpha\}\|_p \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}}
\]
in the Lorentz space \( L^{p,q}(\mathbb{R}^d) \), \( p, q \in (0, \infty) \). Note that \( L^{p,q}(\mathbb{R}^d) = L^p(\mathbb{R}^d) \) and \( L^{p,q}(\mathbb{R}^d) \subset L^{p,d}(\mathbb{R}^d) \) for \( q < p \).

**Proposition 5.1.** Suppose \( d \geq 3 \) is odd. Then
\[
s_k(BS(\lambda)) \lesssim k^{-\frac{d+1}{2}} |\lambda|^{-\frac{d+1}{2}}\|V\|_{(d+1)/2}, \quad \text{Im }\lambda \geq 0, \tag{41}
\]
and for any \( \delta \in (0, 1] \), \( \gamma \geq \sqrt{1+\delta|\text{Im }\lambda|} \),
\[
s_k(BS(\lambda)) \lesssim \delta^{-\frac{d+1}{4}} k^{-\frac{d+1}{2}} |\lambda|^{-\frac{d+1}{2}}\|e^{2\gamma |.|}V\|_{(d+1)/2,1/2}, \quad \text{Im }\lambda < 0. \tag{42}
\]
Moreover, if \( \delta, \epsilon \in (0, 1], \gamma \geq \sqrt{1 + \delta |\text{Im} \lambda| + \epsilon |\lambda|} \), then
\[
s_k(BS(\lambda) - BS(-\lambda)) \lesssim |\lambda|^{-\frac{d-1}{2+\gamma}} \delta \exp \left( -\epsilon k \frac{1}{|\lambda|} \right) \| e^{2\gamma |V|} \|_{(d+1)/2,1/4} \tag{43}
\]
for \( \lambda \in \mathbb{C} \).

**Corollary 5.2.** If \( \gamma > 0 \), \( V \in e^{-2\gamma |V|} L^{(d+1)/2,1/2}(\mathbb{R}^d) \) and \( \text{Im} \lambda > -\gamma \), then we have that \( BS(\lambda) \in \mathcal{S}^{d+1,w} \). In particular, \( BS(\lambda) \) is compact.

**Proof.** For \( \text{Im} \lambda \geq 0 \) the claim follows from (41). If \( -\gamma < \text{Im} \lambda < 0 \), then there exists \( \delta > 0 \) such that \( \gamma \geq \sqrt{1 + \delta |\text{Im} \lambda|} \), and the claim follows from (12).

**Proof of Proposition 1.4.** In view of the resolvent identity (23) and since \( BS(\lambda) \) is compact, this follows from a routine application of the meromorphic Fredholm theorem \[\text{DZ19}, \text{Thm. C.8}]. \]

**Lemma 5.3.** Let \( d \geq 2 \). Then for all \( \lambda \in \mathbb{C}, \delta \in (0, 1], \)
\[
s_k(W_1 \rho_3 \mathcal{E}_a(\lambda) \mathcal{E}_a(\Lambda)^* \rho_3 W_2) \lesssim c_3 k^{-\frac{d-1}{d+1}} \| W_1 \|_{d+1,1} \| W_2 \|_{d+1,1}, \tag{44}
\]
uniformly over \( a \in L^\infty(\mathbb{R}^d; C^N(S_{\xi}^{d-1})) \) in the unit ball. Here,
\[
c_3 := \delta^{-\frac{(d-1)^2}{d+1}}, \quad \rho_3(x, \lambda) := e^{-\sqrt{1+\delta |\text{Im} \lambda|} |\xi|}.
\]

**Proof.** 1. By scaling, we may assume that \( |\lambda| = 1 \). Then (44) would follow from the restricted weak type inequality
\[
s_k(1_{\Omega_1} \rho_3 \mathcal{E}_a(\lambda) \mathcal{E}_a(\Lambda)^* \rho_3 1_{\Omega_2}) \lesssim c_3 k^{-\frac{1}{d+1}} \| \Omega_1 \|_{d+1,1} \| \Omega_2 \|_{d+1,1}, \tag{45}
\]
where \( \Omega_1, \Omega_2 \subset \mathbb{R}^d \) are measurable sets. To see that (45) implies (44), one observes that, by the layer cake representation of \( |W_1|, |W_2| \),
\[
|W_j| = \int_0^\infty 1_{\{ |W_j| > \alpha_j \}} d\alpha_j,
\]
the triangle inequality for the weak Schatten norm \( \| \cdot \|_{\mathcal{S}^{d+1,w}} \) (see Section 2 for the definition and the discussion that this is a norm) and the definition of the Lorentz norm \( \| \cdot \|_{d+1,1} \),
\[
\| W_1 \rho_3 \mathcal{E}_a(\lambda) \mathcal{E}_a(\Lambda)^* \rho_3 W_2 \|_{\mathcal{S}^{d+1,w}}
\leq \int_0^\infty \int_0^\infty \| 1 \{ |W_1| > \alpha_1 \} \rho_3 \mathcal{E}_a(\lambda) \mathcal{E}_a(\Lambda)^* \rho_3 1 \{ |W_2| > \alpha_2 \} \|_{\mathcal{S}^{d+1,w}} d\alpha_1 d\alpha_2
\lesssim c_3 k^{-\frac{1}{d+1}} \int_0^\infty \int_0^\infty \| 1 \{ |W_1| > \alpha_1 \} \|_{d+1,1} \| |W_2| > \alpha_2 \|_{d+1,1} \|_{d+1,1} d\alpha_1 d\alpha_2
\leq c_3 k^{-\frac{1}{d+1}} \| W_1 \|_{d+1,1} \| W_2 \|_{d+1,1}.
\]
Then (44) follows from the equivalence of \( \| \cdot \|_{\mathcal{S}^{d+1,w}} \) and \( \| \cdot \|_{\mathcal{S}^{d+1,w}} \).
2. Let \( \{Q_\alpha\}_{\alpha \in \mathbb{Z}^d} \) be a partition of \( \mathbb{R}^d \) into cubes of sidelength \( R \) and denote by \( \chi_\alpha \) the corresponding characteristic functions. Let
\[
\Sigma_\geq := \sum_{|\alpha - \beta| > 1} \chi_\alpha \rho_\delta \mathcal{E}_\alpha(\lambda) \mathcal{E}_\beta^*(\lambda) \rho_\delta \chi_\beta,
\]
\[
\Sigma_\leq := \sum_{|\alpha - \beta| \leq 1} \chi_\alpha \rho_\delta \mathcal{E}_\alpha(\lambda) \mathcal{E}_\beta^*(\lambda) \rho_\delta \chi_\beta.
\]

Then we have
\[
s_k(1_{\Omega_1} \rho_\delta \mathcal{E}_\alpha(\lambda) \mathcal{E}_\beta^*(\lambda) \rho_\delta 1_{\Omega_2}) \leq s_k(1_{\Omega_1} \Sigma_\geq 1_{\Omega_2}) + \|1_{\Omega_1} \Sigma_\leq 1_{\Omega_2}\|. \tag{46}
\]

To estimate the second term we use Proposition 3.3.4 (observing that the assumptions and the right hand side of (27) are symmetric under complex conjugation), which gives
\[
\|\chi_\alpha \rho_\delta \mathcal{E}_\alpha(\lambda)\|, \|\mathcal{E}_\alpha(\lambda)^* \rho_\delta \chi_\beta\| \lesssim \delta^{-\frac{d-1}{2}} R^{\frac{1}{2}}. \tag{47}
\]

Combining (47) with the triangle inequality and Cauchy–Schwarz, we have that for any \( f, g \in L^2(\mathbb{R}^d) \),
\[
|\langle f, \Sigma \rangle| \leq \sum_{|\alpha - \beta| \leq 1} \|\mathcal{E}_\alpha(\lambda)^* \rho_\delta \chi_\alpha f\| \|\mathcal{E}_\beta(\lambda)^* \rho_\delta \chi_\beta g\|
\leq \left( \sum_{\alpha} \|\mathcal{E}_\alpha(\lambda)^* \rho_\delta \chi_\alpha f\|^2 \right)^{1/2} \left( \sum_{\alpha} \|\mathcal{E}_\beta(\lambda)^* \rho_\delta \chi_\beta g\|^2 \right)^{1/2}
\leq R \delta^{-(d-1)} \left( \sum_{\alpha} \|\mathcal{E}_\alpha f\|^2 \right)^{1/2} \left( \sum_{\alpha} \|\mathcal{E}_\beta g\|^2 \right)^{1/2} = R \delta^{-(d-1)} \|f\| \|g\|,
\]

which implies
\[
\|\Sigma\| \lesssim R \delta^{-(d-1)}. \tag{48}
\]

3. To estimate \( s_k(1_{\Omega_1} \Sigma_\geq 1_{\Omega_2}) \) we use Markov’s inequality
\[
|\{s_k(1_{\Omega_1} \Sigma_\geq 1_{\Omega_2}) > \alpha\}| \leq \alpha^{-2} \sum_{n \in \mathbb{N}} s_n(1_{\Omega_1} \Sigma_\geq 1_{\Omega_2})^2 = \alpha^{-2} \|1_{\Omega_1} \Sigma_\geq 1_{\Omega_2}\|^2. \tag{49}
\]

The Hilbert–Schmidt norm is estimated using (51), which yields
\[
\|1_{\Omega_1} \Sigma_\geq 1_{\Omega_2}\|^2 \lesssim R^{-\frac{d-1}{2}}. \tag{50}
\]

Here we used that \( e^{it \mathcal{E}(\lambda)[|x - y| - \sqrt{|1 + \delta \langle x, y \rangle}|]} \leq 1 \) and \( |x - y| > R \) on the support of \( \Sigma_\geq \). Thus we have
\[
\|1_{\Omega_1} \Sigma_\geq 1_{\Omega_2}\|^2 \lesssim R^{-(d-1)} \int_{\Omega_2} \int_{\Omega_1} \left( \sum_{|\alpha - \beta| > 1} \chi_\alpha(x) \chi_\beta(y) \right)^2 dx dy.
\]

Since the cubes are almost disjoint (they intersect at most in a set of measure zero), we have
\[
\|1_{\Omega_1} \Sigma_\geq 1_{\Omega_2}\|^2 \lesssim R^{-(d-1)} \sum_{\alpha, \beta} |\Omega_1 \cap Q_\alpha| |\Omega_2 \cap Q_\beta| \leq R^{-(d-1)} |\Omega_1||\Omega_2|. \tag{51}
\]
4. Together, (49) and (51) imply
\[ s_k(1_{\Omega_2} \Sigma_{\geq 1} 1_{\Omega_2}) \lesssim k^{-\frac{d}{2}} R^{-\frac{d-1}{2}} |\Omega_1|^\frac{1}{2} |\Omega_2|^\frac{1}{2}. \] (52)

Combining (48), (38), (32), we get
\[ s_k(1_{\Omega_2} \rho_2 \mathcal{E}_k(\lambda) \mathcal{E}_a(\lambda) \rho_2 1_{\Omega_2}) \lesssim \delta^{-(d-1)} R + k^{-\frac{d}{2}} R^{-\frac{d-1}{2}} |\Omega_1|^\frac{1}{2} |\Omega_2|^\frac{1}{2}. \]

Optimizing over \( R \), i.e. taking \( R = \delta^{\frac{d(d-1)}{2(d+1)}} k^{-\frac{1}{d+1}} |\Omega_1|^{\frac{1}{d+1}} |\Omega_2|^{\frac{1}{d+1}} \), yields (45). \( \Box \)

We will also need the following variant of (44).

**Lemma 5.4.** Suppose \( d \geq 2 \). Then for all \( \lambda \in \mathbb{C} \), \( \delta \in (0,1] \),
\[ s_k(W \rho_3 \mathcal{E}_a(\lambda)) \lesssim c_\delta^{1/2} k^{-\frac{1}{d+1}} |\lambda|^{-\frac{d(d-1)}{2(d+1)}} \|W\|_{d+1,1}, \] (53)
uniformly over \( a \) in the unit ball of \( L^\infty(\mathbb{R}^d; C^N(\mathbb{S}^{d-1})) \).

**Proof.** If \( \lambda \in \mathbb{R} \), then (53) follows from (11) by a \( TT^* \) argument. If \( \lambda \notin \mathbb{R} \), we distinguish the cases \( |\Re \lambda| \gtrsim |\lambda| \) and \( |\Re \lambda| \ll |\lambda| \). By scaling, we may again assume \( |\lambda| = 1 \), so that we consider the cases \( |\Re \lambda| \gtrsim 1 \) and \( |\Im \lambda| \gtrsim 1 \). It suffices to prove
\[ s_k(1_{\Omega_2} \rho_3 \mathcal{E}_a(\lambda) \mathcal{E}_a(\lambda)^* \rho_3 1_{\Omega_2}) \lesssim c_\delta |\Omega_1|^{\frac{1}{d+1}} |\Omega_2|^{\frac{1}{d+1}}, \]
This is the same as (45), but without the complex conjugate. If \( |\Re \lambda| \gtrsim 1 \), the proof is exactly the same as that of Lemma 5.3 except that we now use (35) to get (55). If \( |\Im \lambda| \gtrsim 1 \), we replace \( \alpha - \beta \) by \( \alpha + \beta \) in the definition of \( \Sigma_{\geq}, \Sigma_{\leq} \). Then (35) again yields (55), and the estimate (48) for \( \Sigma_{\leq} \) is unchanged, due to the first bound in (47). \( \Box \)

**Lemma 5.5.** Suppose \( d \geq 2 \). Then there exists \( c > 0 \) such that for all \( \lambda \in \mathbb{C} \) and \( \delta, \epsilon \in (0,1] \),
\[ s_k(W \rho_{\delta, \epsilon} \mathcal{E}(\lambda)) \lesssim c_\delta^{1/2} \exp \left( -c_\delta k \frac{1}{d+1} \right) |\lambda|^{-\frac{d(d-1)}{d+1}} \|W\|_{d+1,1}, \]
where
\[ \rho_{\delta, \epsilon}(x, \lambda) := e^{-(\sqrt{1+\delta} |\Im \lambda| + \epsilon |\lambda|)|x|}. \]

**Proof.** For any \( l \in \mathbb{N} \) we have
\[ s_k(W \rho_{\delta, \epsilon} \mathcal{E}(\lambda)) \leq \|W \rho_{\delta, \epsilon} \mathcal{E}(\lambda)(I - \epsilon^2 \Delta_S)^l\|s_k((I - \epsilon^2 \Delta_S)^{-l}) \] (54)
where \( \Delta_S \) is the Laplace–Beltrami operator on the sphere \( S = \mathbb{S}^{d-1} \). By Weyl’s asymptotics,
\[ s_k((I - \epsilon^2 \Delta_S)^{-l}) \leq (C_{\delta} k \frac{1}{d+1})^{-2l}. \] (55)

To estimate the operator norm in (54), we use the **structure formula**
\[ \mathcal{E}(\lambda)(I - \epsilon^2 \Delta_S)^l = \mathcal{E}_a(\lambda), \] (56)
where \( a_l \) satisfies
\[ \sum_{|\beta| \leq N} |\partial_x^\beta a_l(x, \xi)| \leq C_{\delta} a_l(2l) ! \exp(\epsilon |\lambda x|), \quad x \in \mathbb{R}^d, \quad \xi \in S \] (57)
for any $N \in \mathbb{N}_0$. The proof of (56), (57) is postponed to Section A. Lemma 5.4 (with $k = 1$) then implies

$$
\|W_{\rho,\epsilon}\mathcal{E}_0(\lambda)\| \lesssim_c 1/3 \mathcal{C}_{N,d}^2 (2l)^{2l} |\lambda|^{- \frac{d(d-1)}{2(d+l)}} \|W\|_{d+1,1},
$$

(58)

where we also used that $(2l)! \leq (2l)^{2l}$ and $s_1(K) = \|K\|$. Combining (51), (55), (56), (58), we obtain

$$
s_k(W_{\rho,\epsilon}\mathcal{E}(\lambda)) \lesssim_c 1/3 (2l)^{2l} \left( C_w C_{N,d}^{-1} k^{\frac{d+1}{d}} \right) ^{-2l} |\lambda|^{- \frac{d(d-1)}{2(d+l)}} \|W\|_{d+1,1}.
$$

Since $\inf_{l \in \mathbb{N}}(2l)^{2l} M^{-2l} \lesssim \exp(-cM)$ for any $M > 0$ and some $c > 0$, the claim follows.

Proof of Proposition 5.7. The estimate (41) for $\Im \lambda \geq 0$ follows from (21). For $\Im \lambda < 0$ we use Stone’s formula (18). In view of (41) for $\Im \lambda \geq 0$, (42) follows from

$$
s_k(|V|^{1/2} \rho,\mathcal{E}(\lambda)\mathcal{E}(\lambda)^* \rho,\mathcal{E}(\lambda)|V|^{1/2}) \lesssim c_3 k^{- \frac{d-1}{d+1}} |\lambda|^{- \frac{d(d-1)}{2(d+l)}} \|V\|_{d+1/2},
$$

which is a consequence of the more general bound in Lemma 5.3. Similarly, (43) follows from

$$
s_k(|V|^{1/2} \rho,\mathcal{E}(\lambda)\mathcal{E}(\lambda)^* \rho,\mathcal{E}(\lambda)|V|^{1/2}) \lesssim c_4 k^{- \frac{d-1}{d+1}} |\lambda|^{- \frac{d(d-1)}{2(d+l)}} \exp(-ck\frac{1}{d+1}) \|V\|_{d+1/2},
$$

which is a consequence of Lemma 5.5.

5.2. Compactly supported potentials.

Proposition 5.6. Suppose $d \geq 3$ is odd. Let $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ be supported in a ball $B_R = B(0, R)$. Then

$$
s_k(BS(\lambda)) \lesssim (R|\lambda| + k^{\frac{d}{2}})^{-1} \|V\|_\infty R^2, \quad \text{Im } \lambda \geq 0,
$$

(59)

and for $\lambda$ in the exterior of a fixed cone,

$$
s_k(BS(\lambda)) \lesssim (k^{\frac{d}{2}} R^{-2} + |\lambda|^2)^{-1} \|V\|_\infty, \quad \text{Im } \lambda \gtrsim |\lambda|.
$$

(60)

In particular, for any $\theta \in [0, 1]$ and $\text{Im } \lambda \gtrsim |\lambda|,

$$
s_k(BS(\lambda)) \lesssim k^{- \frac{2d}{d+1}} R^{2d} |\lambda|^{-2(1-\theta)} \|V\|_\infty.
$$

(61)

If $\nu \geq 0$, $\theta \in [0, 1]$, then for $\text{Im } \lambda \geq 0$,

$$
s_k(BS(\lambda)) \lesssim (k^{- \frac{2d}{d+1}} |\lambda R| \nu \ln(\lambda R) R |\lambda|^{-1} + k^{- \frac{2d}{d+1}} R^{2d} |\lambda|^{-2(1-\theta)}) \|V\|_\infty,
$$

(62)

and for any $\delta \in (0, 1]$, $\text{Im } \lambda < 0$,

$$
s_k(BS(\lambda) - BS(-\lambda)) \lesssim k^{- \frac{2d}{d+1}} \delta^{1-d} |\lambda R| \nu \ln(\lambda R) R |\lambda|^{-1} \|\rho,\mathcal{E}_0^{-2} V\|_\infty.
$$

(63)

Moreover, for any $\delta, \epsilon \in (0, 1]$, $\text{Im } \lambda < 0$,

$$
s_k(BS(\lambda) - BS(-\lambda)) \lesssim \delta^{1-d} R |\lambda|^{-1} \|\rho,\mathcal{E}_0^{-2} V\|_\infty \exp(-ck\frac{1}{d+1}).
$$

(64)

All implicit constants depend only on $d, \theta, \nu$. 

**Proof.** 1. The proof strategy is similar to that of Proposition [5.1]. We first note that
\[(k^{2}R^{-2} + \lambda^{2})^{-1} \leq k^{-d}R^{2\theta}|\lambda|^{-2(1-\theta)}\]
for any \(\theta \in [0,1]\). Hence (51) follows from (60). For the proof of (62) it will be convenient to split \(R_{0}(\lambda) = R_{0}(\lambda)^{\text{high}} + R_{0}(\lambda)^{\text{low}}\), where \(R_{0}(\lambda)^{\text{low}}\) has Fourier multiplier \((\xi^{2} - \lambda^{2})^{-1}1\chi(\xi/|\lambda|)\) and \(\chi\) is a smooth, radial function on \(\mathbb{R}^{d}\) that equals 1 on \(B(0,2)\) and is supported on \(B(0,4)\). In view of Stone’s formula \(15\) and since \(|V| \leq \|V\|_{\infty}1_{B_{R}}\), it will then suffice to prove
\[s_{k}(1_{B_{R}}R_{0}(\lambda)1_{B_{R}}) \lesssim (R|\lambda| + k^{\frac{1}{2}})^{-1}R^{2}, \quad \text{Im} \lambda \geq 0, \quad (65)\]
\[s_{k}(1_{B_{R}}R_{0}(\lambda)1_{B_{R}}) \lesssim (k^{2}R^{-2} + \lambda^{2})^{-1}, \quad \text{Im} \lambda \geq |\lambda|, \quad (66)\]
\[s_{k}(1_{B_{R}}R_{0}(\lambda)^{\text{high}}1_{B_{R}}) \lesssim (k^{2}R^{-2} + \lambda^{2})^{-1}, \quad \text{Im} \lambda \geq 0, \quad (67)\]
\[s_{k}(1_{B_{R}}R_{0}(\lambda)^{\text{low}}1_{B_{R}}) \lesssim k^{-\frac{d}{2}+d}R|\lambda|^{-1}(\lambda R)^{d} \ln(\lambda R), \quad \text{Im} \lambda \geq 0, \quad (68)\]
\[s_{k}(1_{B_{R}}\rho_{3}E(\lambda)) \lesssim k^{-\frac{d}{2}+d}R\delta|\lambda|^{-d+1}\exp\left(-ck^{d}R\right), \quad \lambda \in \mathbb{C}, \quad (69)\]
\[s_{k}(1_{B_{R}}\rho_{6}E(\lambda)) \lesssim R\delta|\lambda|^{-d+1}\exp\left(-ck^{d}R\right), \quad \lambda \in \mathbb{C}. \quad (70)\]

2. We first prove (65). We start with the following resolvent estimate, see e.g. [DA20] (3.3):
\[|\lambda||\|R_{0}(\lambda)f||_{\hat{Y}} + \|\nabla R_{0}(\lambda)f||_{\hat{Y}} \lesssim \|f||_{\hat{Y}^{*}}, \quad \text{Im} \lambda \geq 0, \quad (71)\]
where the space \(\hat{Y}\) is defined through the norm
\[\|f\|_{\hat{Y}}^{2} := \sup_{R>0} \frac{1}{R} \int_{|x| \leq R} |f(x)|^{2}dx, \]
and the (pre-) dual \(\hat{Y}^{*}\) can be equipped with the equivalent norm
\[\|f\|_{\hat{Y}^{*}} = \sum_{j \in \mathbb{Z}} 2^{j/2} \|f\|_{L^{2}(2^{j}|x| \in 2^{j+1})}. \]
Note that \(\hat{Y}^{*}\) can be viewed as a homogeneous version of the “B-space” of Agmon and Hörmander [AH76]. We have
\[s_{k}(1_{B_{R}}R_{0}(\lambda)1_{B_{R}}) \leq s_{k}((|\lambda|^{2} - \Delta_{T_{R}})^{-\frac{1}{2}})(|\lambda|^{2} - \Delta_{T_{R}})^{-\frac{1}{2}}1_{B_{R}}R_{0}(\lambda)1_{B_{R}}\|, \]
where \(T_{R} := \mathbb{R}^{d}/(2R\mathbb{Z})^{d}\) is the \(d\)-dimensional torus of sidelenth \(2R\). By Weyl’s asymptotics,
\[s_{k}((|\lambda|^{2} - \Delta_{T_{R}})^{-\frac{1}{2}}) \lesssim (|\lambda|^{2} + R^{-2}k^{\frac{1}{2}})^{-\frac{1}{2}}. \]
Using (71), we have
\[\|(|\lambda|^{2} - \Delta_{T_{R}})^{-\frac{1}{2}}1_{B_{R}}R_{0}(\lambda)1_{B_{R}}\| \lesssim |\lambda|\|1_{B_{R}}R_{0}(\lambda)1_{B_{R}}\| + \|1_{B_{R}}\nabla R_{0}(\lambda)1_{B_{R}}\| + R^{-1}\|1_{B_{2R}}\nabla R_{0}(\lambda)1_{B_{2R}}\| \lesssim R. \]
In the first inequality we used that $\|(|\lambda|^2 - \Delta_{\text{eff}})\varphi\|_s$ is equivalent to $|\lambda|\|\cdot\| + \|\nabla\cdot\|$ and inserted a smooth cutoff function $\chi_R = \chi(\cdot/R)$, with $1_{B_R} \leq \chi \leq 1_{B_{2R}}$. Combining the last two displays proves (65) and hence (59).

3. Next, we prove (66), (67), (68). Since the Fourier multiplier of $R_0(\lambda)^{\text{high}}$ is bounded by $(\xi^2 + \lambda^2)^{-1}$, Weyl asymptotics for $-\Delta$ on $B_R$ yield (67). The same estimate holds for $R_0(\lambda)$ instead of $R_0(\lambda)^{\text{high}}$ if $\text{Im} \lambda \gtrsim |\lambda|$, which proves (66).

We will momentarily show that the estimate (68) for $R_0(\lambda)^{\text{low}}$ would follow from

$$s_k(1_{B_R}\mathcal{E}(\lambda)) \lesssim k^{-\frac{\nu}{d-1}} R^\nu |\lambda|^{-\frac{d-1}{2}} (\lambda R)\varphi^-, \quad \text{Im} \lambda \geq 0.$$  

(72)

We defer the proof of (72) to Lemma 5.7 below. By scaling, we may assume that $\text{Re} \lambda = 1$. Moreover, since we have already proved (66), we may assume $0 \leq \text{Im} \lambda < 1$. Using polar coordinates, we have

$$R_0(1 \pm i \text{Im} \lambda)^{\text{low}} f = (2\pi)^{-d} \int_0^4 \chi(r)\mathcal{E}(r)\hat{\phi}(r)^* f \, dr.$$  

It would then follow from (72) that

$$\|1_{B_R}R_0(1 \pm i \text{Im} \lambda)^{\text{low}} 1_{B_R}\|_{\psi_{\frac{d-1}{2}}} \lesssim R(1 + R)^\nu \int_0^4 |r - 1|^{-1} \, dr.$$  

Unfortunately, the integral is divergent. The redeeming feature is that the spatial localization to $B_R$ smooths out the integrand on the $1/R$ scale, i.e. $|r - 1|$ may be replaced by $|r - 1| + 1/R$. To make this rigorous, one observes that, by the convolution theorem,

$$1_{B_R}m(D)1_{B_R} = 1_{B_R}m_R(D)1_{B_R},$$

whenever $m(D)$ is a Fourier multiplier, $m_R := \varphi * m$, $\varphi_R(\xi) := R^d \varphi(|\xi|)$ and $\varphi$ is a Schwartz function such that $\hat{\varphi} = 1$ on $B(0,2)$. The previous argument thus yields (see [CM22] Sect. 6.2 for more details)

$$\|1_{B_R}R_0(1 \pm 0)^{\text{low}} 1_{B_R}\|_{\psi_{\frac{d-1}{2}}} \lesssim R(1 + R)^\nu \int_0^4 (|r - 1| + 1/R)^{-1} \, dr \lesssim R(1/R)^\nu \ln(R),$$  

which completes the proof of (68).

4. We defer the proof of (69) to Lemma 5.7 below and continue with the proof of (70). As in the proof of Lemma 5.5 we estimate

$$s_k(1_{B_R}\rho_s\mathcal{E}(\lambda)) \lesssim \|1_{B_R}\rho_s\mathcal{E}(\lambda)(I - \epsilon^2\Delta S)^{\frac{1}{2}}\| s_k((I - \epsilon^2\Delta S)^{-\frac{1}{2}}).$$

The same argument as there then shows that (70) is a consequence of the norm bound (27) and Weyl’s asymptotics (59). □

**Lemma 5.7.** Suppose $d \geq 2$, $R > 0$, $\nu \geq 0$. Then (72) holds. Moreover, (69) holds for any $\delta \in (0,1]$.

**Proof.** The proof is again similar to that of Lemma 5.5. The argument is simpler since $\nu$ is fixed; however, $\nu$ is not necessarily an integer. We can still write

$$s_k(1_{B_R}\mathcal{E}(\lambda)) \leq \|1_{B_R}\mathcal{E}(\lambda)(I - \Delta S)^{\frac{1}{2}}\| s_k((I - \Delta S)^{-\frac{1}{2}}),$$  

(73)

$$s_k(1_{B_R}\rho_s\mathcal{E}(\lambda)) \leq \|1_{B_R}\rho_s\mathcal{E}(\lambda)(I - \Delta S)^{\frac{1}{2}}\| s_k((I - \Delta S)^{-\frac{1}{2}}),$$
where \((I - \Delta_S)^c\) is defined in terms of the eigenfunction expansion of \(-\Delta_S\). If \(\nu/4\) is an integer, then integration by parts together with (25) yields
\[
\|1_{B_R} e(\lambda)(I - \Delta_S)^{\frac{\nu}{2}}\| \lesssim R^\frac{\nu}{4}|\lambda|^{-\frac{\nu}{4}}(\lambda R)^\frac{\nu}{4}, \quad \lambda > 0,
\]
which in combination with (73) and Weyl’s asymptotics (59) yields (72) in this case. For non integer \(\nu/4\), the result follows by complex interpolation applied to the analytic family of operators \(\kappa \mapsto (\lambda R)^{-2\kappa}(I - \Delta_S)^{\kappa}\). The proof of (69) is analogous, but uses (27) instead of (25).

5.3. **Pointwise decaying potentials.** We will use the following abbreviations:
\[
v_{p,R} := \sup_{x \in \mathbb{R}^d} (1 + |x|/R)^{\theta}|V(x)|, \quad v_{p,R,\gamma} := \sup_{x \in \mathbb{R}^d} e^{2\gamma |x|} (1 + |x|/R)^{\theta}|V(x)|.
\]

**Proposition 5.8.** Suppose \(d \geq 3\) is odd, \(\theta \in [0, 1]\), \(\rho, \gamma > 0\). If \(\rho > 2\theta\), then for \(\lambda\) in the exterior of a fixed cone \(\text{Im } \lambda \gtrsim |\lambda|\),
\[
s_k(BS(\lambda)) \lesssim k^{-\frac{\nu}{2\theta}} R^{2\theta} |\lambda|^{-2(1-\theta)} v_{p,R}.
\]

If \(\nu \geq 0\), \(\kappa > 0\) and \(\rho > \max(1 + \nu + \kappa, 2\theta)\), then for \(\text{Im } \lambda \geq 0\),
\[
s_k(BS(\lambda)) \lesssim (k^{\frac{\nu}{2\theta}} |\lambda R|^{\nu + \kappa} R|\lambda|^{-1} + k^{\nu/2} R^{2\theta} |\lambda|^{-2(1-\theta)}) v_{p,R}.
\]

Moreover, for \(\nu, \kappa, \rho\) as above, \(\delta \in (0, 1]\), \(\gamma \geq \sqrt{1 + \delta |\text{Im } \lambda|}, \text{Im } \lambda < 0\), we have
\[
s_k(BS(\lambda)) \lesssim \delta^{1-d} k^{-\frac{\nu}{2\theta}} (\lambda R)^{\nu + \kappa} R|\lambda|^{-1} v_{p,R,\gamma} + k^{\nu/2} R^{2\theta} |\lambda|^{-2(1-\theta)} v_{p,R}.
\]

and for \(\delta, \epsilon \in (0, 1]\), \(\gamma \geq \sqrt{1 + \delta |\text{Im } \lambda| + \epsilon |\lambda|}, \text{Im } \lambda < 0\), we have
\[
s_k(BS(\lambda) - BS(-\lambda)) \lesssim \delta^{1-d} R|\lambda|^{-1} v_{p,R,\gamma} \exp\left(-\alpha k^{\frac{\nu}{2\theta}}\right).
\]

All implicit constant depend only on \(d, \theta, \rho, \nu, \kappa\).

**Proof.** The results follow from Proposition 5.6 by dyadic summation. More precisely, we have \(V = \sum_{j \in \mathbb{N}_0} V_j\) where \(V_j = V 1_{A_j}\) is supported in the dyadic shell \(A_j = \{2^j R \leq |x| \leq 2^{j+1} R\}, j \in \mathbb{N}_0\). Then (61) implies
\[
\|BS(\lambda)\|_{\mathcal{S}_{\mathcal{F}, \psi}} = \|R_0(\lambda)^{\frac{\nu}{2}} V R_0(\lambda)^{\frac{\nu}{2}}\|_{\mathcal{S}_{\mathcal{F}, \psi}} \leq \sum_j \|R_0(\lambda)^{\frac{\nu}{2}} V_j R_0(\lambda)^{\frac{\nu}{2}}\|_{\mathcal{S}_{\mathcal{F}, \psi}} = \sum_j \|V_j R_0(\lambda) V_j R_0(\lambda)^{\frac{\nu}{2}}\|_{\mathcal{S}_{\mathcal{F}, \psi}} \lesssim |\lambda|^{2(1-\theta)} R^{2\theta} v_{p,R} \sum_j 2^{(2\theta - \rho)j}.
\]

Since \(\rho > 2\theta\), this yields (74). Similarly, dyadic summation of (62), (63), using \(\text{Im } (\lambda R) \lesssim |\lambda R|\), yields (75), (76). To prove (64), we observe that
\[
\|e^{2(\sqrt{|\text{Im } \lambda| + |\lambda|}) |V| L^{\infty}(2^{-j} R \leq |x| \leq 2^j R)} \| \leq v_{p,R,\gamma} 2^{-\rho j}.
\]

Thus, dyadic summation of (64) yields (77).

**Corollary 5.9.** Assume that \(\theta \in [1/2, 1]\), \(\rho > 2\theta\) and \(v_{p,R,\gamma} < \infty\). Then we have \(BS(\lambda) \in \mathcal{S}^{\alpha}\) for \(\text{Im } \lambda \geq -\gamma\) and \(\alpha > \max(\frac{\nu}{4}, \frac{\nu + \kappa}{\theta})\). In particular, \(BS(\lambda)\) is compact.
Proof. Assume Im $\lambda \geq 0$ first. Let $\kappa > 0$ be such that 
$$\nu := \rho - 1 - 2\kappa > 0$$ 
and $\alpha > \max\left(\frac{1}{\rho}, \frac{1}{2\theta}\right)$. Since $\rho > \max(1 + \nu + \kappa, 2\theta)$, (75) implies that $BS(\lambda) \in \mathcal{S}^\alpha$. If $0 > \text{Im} \lambda > -\gamma$, then there exists $\delta > 0$ such that $\gamma \geq \sqrt{1 + \delta |\text{Im} \lambda|}$, and the claim follows from (76) in the same way. \hfill $\square$

The proof of Proposition 1.5 then follows from the same routine argument as that of Proposition 1.4.

6. Fredholm determinant

We place $D(0, r)$ into a larger disk $D(\lambda_0, |\lambda_0| + r)$, where $\lambda_0 \in \mathbb{iR}_+$ will be suitably chosen, and consider the function $F_\alpha$ defined in (15). We recall the definitions of $H_\alpha(\lambda)$ and $F_\alpha(\lambda)$:

$$H_\alpha(\lambda) := \det(I - (-BS(\lambda))^{\alpha}), \quad F_\alpha(k) := \frac{H_\alpha(\lambda_0 + k)}{H_\alpha(\lambda_0)}, \quad \lambda, k \in \mathbb{C}.$$ 

We also recall the definition of the ‘averaged counting functions’

$$N_V(r) := \int_0^r \frac{n_V(t)}{t} dt, \quad N_{H_\alpha}(r) := \int_0^r \frac{n_{H_\alpha}(t)}{t} dt, \quad N_{F_\alpha}(r) := \int_0^r \frac{n_{F_\alpha}(t)}{t} dt,$$

where $n_V(t)$ denotes the number of resonances of $-\Delta + V$ and $n_{H_\alpha}, n_{F_\alpha}$ denote the number of zeros (counting multiplicity) of the analytic functions $H_\alpha, F_\alpha$, respectively, in the disk $D(0, r)$. Finally, we recall the estimates $N_V(r) \leq N_{H_\alpha}(r)$ and $n_V(r) \leq (\ln s)^{-1} N_V(sr)$, for any fixed $s > 1$. Hence, we have

$$n_V(r) \ln s \leq N_V(sr) \leq N_{H_\alpha}(sr) \leq N_{F_\alpha}(|\lambda_0| + sr),$$

where the last inequality follows from the triangle inequality. Indeed, if $\lambda \in D(0, sr)$ is a zero of $H_\alpha$, then $k = \lambda - \lambda_0$ is a zero of $F_\alpha$ and $|k| < |\lambda_0| + sr$. By Jensen’s formula, since $F_\alpha(0) = 1$, we have

$$N_{F_\alpha}(|\lambda_0| + sr) = \frac{1}{2\pi} \int_0^{2\pi} \ln |F_\alpha((|\lambda_0| + sr)e^{i\theta})| d\theta \leq \max_{|k| = |\lambda_0| + sr} \ln |F_\alpha(k)|.$$ 

Figure 1. The disk $D(0, r)$ inside the larger disk $D(\lambda_0, |\lambda_0| + r)$. 


By definition of $F_\alpha$, 
\[
\max_{|k|=|\lambda_0|+sr} \ln |F_\alpha(k)| = \max_{|k|=|\lambda_0|+sr} \ln |H_\alpha(\lambda_0 + k)| - \ln |H_\alpha(\lambda_0)| \\
= \max_{\lambda \in \partial D(\lambda_0, |\lambda_0|+sr)} \ln |H_\alpha(\lambda)| - \ln |H_\alpha(\lambda_0)|
\]

To estimate this we set 
\[
S_+ := S_+(sr, \lambda_0) := \{\lambda \in \mathbb{C} : \text{Im} \lambda \geq 0, |\lambda| \geq P\}, \\
S_- := S_-(sr, \lambda_0) := \{\lambda \in \mathbb{C} : -sr \leq \text{Im} \lambda < 0, sr \leq |\lambda| \leq P\},
\]
where $P := P(sr, \lambda_0) := \sqrt{2|\lambda_0|sr + (sr)^2}$. Elementary geometry (see Figure 1) reveals that 
\[
\partial D(\lambda_0, |\lambda_0| + sr) \subset S_+(sr, \lambda_0) \cup S_-(sr, \lambda_0).
\]

In summary, we have 
\[
N_V(sr) \leq \max_{\lambda \in S_+ \cup S_-} \ln |H_\alpha(\lambda)| - \ln |H_\alpha(\lambda_0)|, 
\tag{78}
\]
where we recall that $S_\pm$ depend on $sr$ and $\lambda_0$, and $\lambda_0 \in \mathbb{iR}^+$ is to be determined. By the Weyl inequality \cite[5.8]{DZ19}, for any trace class operator $A$, 
\[
\ln |\det(I - A)| \leq \sum_{k \in \mathbb{N}} \ln(1 + s_k(A)) \leq \sum_{k \in \mathbb{N}} s_k(A), 
\tag{79}
\]
where we used $\ln(1 + x) \leq x$ for $x \geq 0$ in the second inequality. We use \tag{79} with $A = (-BS(\lambda))^\alpha$. By sparsifying the sum 
\[
\sum_{k \in \mathbb{N}} = \sum_{k \in \mathbb{N}} + \sum_{k \in \mathbb{N}+1} + \ldots + \sum_{k \in \mathbb{N}(N-1)+1}
\]
and using the monotonicity of the singular values, it is sufficient to estimate the last sum in the above display, which can be written as 
\[
\sum_{j=0}^{\infty} \ln(1 + s_{\alpha j+1}((-BS(\lambda))^{\alpha})).
\]
Repeated application of \tag{22} shows that 
\[
s_{\alpha j+1}((-BS(\lambda))^{\alpha}) \leq [s_{j+1}(BS(\lambda))]^{\alpha}.
\]
We thus obtain the following estimate on the first term of \tag{78}.

**Lemma 6.1.** Assume that $\alpha > d/2$ is an integer. Then for $\lambda \in S_+ \cup S_-$ we have 
\[
\ln |H_\alpha(\lambda)| \leq \alpha \sum_{k \in \mathbb{N}} \ln(1 + [s_k(BS(\lambda))]^{\alpha}) \leq \alpha \|BS(\lambda)\|^{\alpha}. 
\tag{80}
\]

We also need an estimate for the second term in \tag{78}. To this end we use \tag{79} with $A = (I - K)^{-1}K$ and $K = (-BS(\lambda_0))^\alpha$. In view of the identity $(I - K)^{-1} = I + A$, this gives 
\[
|\det((I - K)^{-1})| \leq \prod_{k \in \mathbb{N}} (1 + \|I - K\|^{-1} s_k(K)).
\]
Assuming that $\|K\| \leq 1/2$, we could estimate $\|(I - K)^{-1}\| \leq 2$. Then, using \(\det((I - K)^{-1}) = (\det(I - K))^{-1}\), we would have
\[- \ln |H_\alpha(\lambda_0)| \leq \sum_{k \in \mathbb{N}} \ln(1 + 2s_k(K)).\]

The argument leading to (80) then yields the following estimate.

**Lemma 6.2.** Assume that $\alpha > d/2$ is an integer and that $\|BS(\lambda_0)\| \leq 1/2$. Then
\[- \ln |H_\alpha(\lambda_0)| \leq \alpha \sum_{k \in \mathbb{N}} \ln(1 + 2[s_k(\lambda_0)]\alpha) \leq 2\alpha \|BS(\lambda_0)\|_{\infty}. \quad (81)\]

Since $\alpha > 1$, the assumption of Lemma 6.2 would be satisfied if $\|BS(\lambda_0)\| \leq 1/2$. In view of the estimates (21), (60) and (74) (with $k = 1$ and $\theta = 0$) this will be achieved if we set
\[2\lambda_0 := \begin{cases} 
\text{i}A\|V\|^{(d+1)/2} & \text{(Lorentz space),} \\
iA\|V\|^{2}_{\infty} & \text{(compactly supported),} \\
iA\varphi_{\rho,R} & \text{(pointwise decaying).} 
\end{cases} \quad (82)\]

where $A$ is a sufficiently large dimensionless constant. Here $\rho > 1$ and $R > 0$ are fixed. In conclusion, we state the bounds that will be needed in the proof of Theorem 7.1. By (78), (80), (81) we have
\[n_{\nu}(r) \ln s \leq N_{\nu}(sr) \lesssim \alpha(\tilde{n}_{+}(sr) + \tilde{n}_{-}(sr) + \tilde{n}_0) \quad (83)\]

where (fixing an integer $\alpha > d/2$ and $\lambda_0$ as in (82))
\[\tilde{n}_{\pm}(r) := \max_{\lambda \in S_{\pm}(r, \lambda_0)} \sum_{k \in \mathbb{N}} \ln(1 + [s_k(\lambda)]\alpha), \quad \tilde{n}_0 := \sum_{k \in \mathbb{N}} \ln(1 + [s_k(\lambda_0)]\alpha). \quad (84)\]

## 7. Proof of Theorem 7.1

We will prove the following stronger versions of Theorem 7.1.

**Theorem 7.1.** Suppose $d \geq 3$ is odd, $r, \gamma > 0$ and $\delta, \epsilon \in (0, 1]$. In the compactly supported case, assume that $\text{supp} V \subset B(0, R)$. In the pointwise decaying and compactly supported case, we also fix $\theta \in [1/2, 1]$, $\nu \geq 0$, $\kappa > 0$, and $\rho > \max(1 + \nu + \kappa, 2\theta)$ (the latter is only needed in the pointwise decaying case).

(a) Then every resonance $\lambda \in \mathbb{C}$ satisfies
\[|\lambda| \lesssim_{\nu, \kappa, \rho} \begin{cases} 
\delta^{-(d-1)/2} \|e^{2\sqrt{V + \delta}(|\lambda_\nu|)}V\|_{(d+1)/2,2,1/2} (\text{Lorentz space},) \\
\delta^{-(d-1)/2} \|\lambda R\|^{\kappa}e^{2\sqrt{V + \delta}(|\lambda_\nu|)}R \|V\|_{\infty} (\text{compactly supported},) \\
\delta^{-(d-1)/2} \|\lambda R\|^{\kappa}e^{2\sqrt{V + \delta}(|\lambda_\nu|)}(\text{pointwise decaying}). 
\end{cases} \]

(b) Moreover, there exists $A_0 > 0$, depending only on $d$, such that if $A \geq A_0$ and $s > 1$, then $n_{\nu}(r) \lesssim_{\nu, \kappa, \rho} (\ln s)^{-1}(n_{+}(sr) + n_{-}(sr)) + n_0$ (the same upper
Remark 7.3. For definiteness, but any
More precisely, we have (see Appendix B for the proof)
Hence, there exist
Remark 7.2. For \( \text{Im} \lambda \geq 0 \), the exponential weight and the negative power of \( \delta \) in (a) can be omitted. Moreover, the (second) Lorentz exponent can be omitted. More precisely, we have (see Appendix [B] for the proof)
\[
|\lambda| \lesssim_{\kappa, r} \begin{cases}
\|V\|_{(d+1)/2}^{(d+1)/2} & (\text{Lorentz space}), \\
R(\lambda R)^{\kappa} \|V\|_{\infty} & (\text{compactly supported}), \\
R(\lambda R)^{\kappa} v_{\rho, R} & (\text{pointwise decaying}).
\end{cases}
\]  
(85)
Remark 7.3. For the compactly supported and pointwise decaying case, the term \( \langle rR + AR \rangle^{\nu + \kappa} \) in \( n_+(r) \) can be replaced by \( \langle rR \rangle^{\nu + \kappa} \) if \( \nu + \kappa \leq 1 \), see Appendix B.2.2 for details.

Proof of Theorem 7.1 assuming Theorem 7.2. We only give details for the number of resonances; the bounds (53), (63), (77) are obvious. To avoid confusion, we rename \( \epsilon \) in Theorem 1.1 to \( \epsilon_1 \).

Consider first the Lorentz case. Let \( A = A_0 \) in Theorem 7.1. The lower bound on \( r \) in (8) then implies that
\[
\gamma(sr) \leq (1 + \delta)^{1/2} sr + cr(s^2 + sA_0C_2^{-1})^{1/2}.
\]
Hence, there exist \( \delta = \delta(\epsilon_1) > 0, \epsilon = \epsilon(\epsilon_1) > 0 \) and \( s = s(\epsilon_1) > 1 \) such that \( \gamma(sr) \leq (1 + \epsilon_1)r \). It only remains to observe that \( n_+(r), n_0 \lesssim n_-(r) \). This is clear since \( n_+(r), n_0 \lesssim 1 \) (for \( n_+(r) \) this again follows from the lower bound on \( r \) in (8)).

In the compactly supported case we take \( A = Cr\|V\|_{\infty}^{-1/2} \) with some constant \( C \geq A_0/C_2 \). Then again there exist \( \delta > 0, \epsilon > 0, s > 1 \) such that \( \gamma(sr) \leq (1 + \epsilon_1)r \). Moreover, since \( A \gtrsim 1 \), we have \( n_0(r) \lesssim n_+(r) \). In the following, we fix \( \theta = 1/2 \) for definiteness, but any \( \theta \in [1/2, 1] \) would work. The lower bound on \( r \) in (8)
implies that $\|V\|_\infty \lesssim r^2$, so that
\[
n_+(r) \lesssim (rR)^{\nu+\kappa} (Rr)^{\frac{d-1}{r}} + (Rr)^d \lesssim \langle Rr \rangle^d,
\]
where we fixed $\nu = d$ in the last inequality (any $\nu > d - 1$ would work) and chose $\kappa$ small enough. If $Rr < 1$, then clearly $n_+(r) \lesssim 1 \lesssim n_-(r)$, where the second inequality holds in view of the final comment in Remark 1.2. If $Rr \geq 1$, we first observe that (6), with $\epsilon_1/2$ in place of $\epsilon$, yields
\[
1 \leq C_1 (\epsilon_1/2)^{\langle rR \rangle^\epsilon} e^{\epsilon r^2} \parallel V \parallel_\infty.
\]
Together with $\langle rR \rangle^\epsilon \lesssim \epsilon, \epsilon_1 e^{\epsilon_1/2}$, this implies $n_- (r) \gtrsim \epsilon_1 (Rr)^d \gtrsim n_+(r)$.

The proof in the pointwise decaying case is similar; here we select $\kappa > 0$ and $\nu > d - 1$ such that $1 + \nu + \kappa \leq d + \epsilon_1$. $\square$

Before giving the proof of Theorem 7.1 it is useful to record the following elementary estimates.

Lemma 7.4. i) For any $M > 0$, $\alpha > \beta > 0$, we have
\[
\sum_{k \in \mathbb{N}} \ln(1 + (Mk^{-1/\beta})^\alpha) \leq C_{\alpha,\beta} M^\beta.
\]
(86)

ii) For any $M > 2$, $\alpha, \beta, \epsilon > 0$, we have
\[
\sum_{k \in \mathbb{N}} \ln(1 + (M \exp(-\epsilon k^{1/\beta}))^\alpha) \leq C_{\alpha,\beta} \epsilon^{-\beta} \ln M)^{\beta+1}.
\]
(87)

Proof. Without loss of generality we may assume that $\alpha = 1$, otherwise we replace $M, \beta, \epsilon$ by $M^\alpha, \beta/\alpha$ in i) and $M, \epsilon$ by $M^\alpha, \alpha \epsilon$ in ii).

i) Set $\mu_k := Mk^{-1/\beta}$. The estimate (86) is obvious if $M \leq 1$ since then
\[
\sum_{k \leq M} \ln(1 + \mu_k) \leq M \sum_{k} \ln(1 + \mu_k) \leq M \leq M^\beta,
\]
where we used that $\beta < 1$. If $M > 1$, then we have
\[
\sum_{k > M^\beta} \ln(1 + \mu_k) \leq \sum_{k > M^\beta} Mk^{-1/\beta} \lesssim M(M^\beta)^{1-1/\beta} = M^\beta,
\]
\[
\sum_{k \leq M^\beta} \ln(1 + \mu_k) \lesssim \ln(1 + M) + \int_1^{M^\beta} \ln(1 + Mx^{-1/\beta})dx
\]
\[
\lesssim \ln(1 + M) + M^\beta \int_1^{M} \ln(1 + y)y^{-1/\beta}dy \lesssim M^\beta.
\]

ii) Set $\mu_k := M \exp(-\epsilon k^{1/\beta})$. For any $K \in \mathbb{N}$ we have
\[
\sum_k \ln(1 + \mu_k) \leq \sum_{k \leq K} \ln(1 + \mu_k) + \sum_{k > K} \mu_k,
\]
(88)
where we again used $\ln(1 + \mu_k) \leq \mu_k$ in the second sum. Estimating the latter by an integral, we have
\[
\sum_{k > K} \mu_k \leq M \int_K^\infty \exp(-\epsilon y^{1/\beta})dy \lesssim M \epsilon^{-\beta} \exp\left(\frac{1}{2} \epsilon K^{\frac{1}{\beta}}\right)
\]
(89)
On the other hand, since $\mu_k$ is decreasing,

$$\sum_{k \leq K} \ln(1 + \mu_k) \leq K \ln M. \quad (90)$$

Combining (88), (89), (90), we have

$$\sum_{k} \ln(1 + \mu_k) \lesssim K \ln M + M \epsilon^{-\beta} \exp \left( -\epsilon K^\frac{1}{\beta} \right)$$

for any $K \in \mathbb{N}$. We choose $K^\frac{1}{\beta} = (2/\epsilon) \ln(CM)$, where $C > 1$ is such that $K \in \mathbb{N}$. This yields

$$\sum_{k} \ln(1 + \mu_k) \lesssim \epsilon^{-\beta} \ln(CM) \beta \ln M + \epsilon^{-\beta} \frac{M}{1 + CM} \lesssim \epsilon^{-\beta} \ln M^{\beta+1}.$$

□

Proof of Theorem 7.1. We only provide the main arguments here, details are postponed to Appendix B.

(a) This a consequence of the Birman–Schwinger principle: If $\lambda$ is a resonance, then $\|BS(\lambda)\| \geq 1$. Using the bounds (41), (42) with $k = 1$ yields an upper bound on $s_1(\lambda) = \|BS(\lambda)\|$ in the Lorentz case. In the compactly supported case we use (62), (63) (with $\theta = 1/2$, $\nu = 0$). In the pointwise decaying case we use (75), (76) (with the same choice of $\theta, \nu$).

(b) The starting point is (83), (84), i.e. the estimate on $n_V(r)$ in terms of the singular values of $BS(\lambda)$. We point out that the constant $A$ in the theorem is the same constant that appear in (82). In the Lorentz space case we take $\alpha = d + 1$, in the other two cases we take $\alpha > \max(d-1, d\varepsilon)$. The exact choice of $\alpha$ is not important, but the constants will depend on $\alpha$. Our choice guarantees that the sums in the last display converge and are bounded by the expressions in the theorem. This is a consequence of the singular value estimates in Section 5 and Lemma 7.4. Indeed, the bounds for $n_+(r)$ follow immediately from (11), (42), (75) and (80), recalling that $|\lambda| \geq r$ for $\lambda \in S_+$ (in the compactly supported and pointwise decaying case, an additional argument is needed for $\nu + \kappa > 1$, see Appendix B). Those for $n_0(r)$ follow from (11), (61), (74) and (80) and the definition of $\lambda_0$, see (82). Those for $n_-(r)$ follow from (43), (61), (77) and (87), with

$$M = C \begin{cases} |\lambda|^{-\frac{d+1}{2}} \delta^{-1} |e^{2\gamma(sr)}||V||_{(d+1)/2, 1/2}, \\ \delta^{1-d} R|\lambda|^{-1} ||\rho_{\delta, \epsilon} V||_{\infty}, \\ \delta^{-1} R|\lambda|^{-1} v_{\rho, R, \gamma(sr)} \end{cases}$$

and $\beta = d - 1$. From the upper bound on $|\lambda|$ in part (a) it follows that there is a $C > 0$ such that $M \geq 2$; we choose $C$ in such a way. We also use the fact that $\text{Im } \lambda \geq -sr$, $|\lambda| \leq \sqrt{2|\lambda_0||sr + (sr)^2|}$ for $\lambda \in S_{\rho_0}$, which implies that $||\rho_{\delta, \epsilon} V||_{\infty} \leq e^{2\gamma(sr)R|V||_{\infty}}$. □
APPENDIX A. PROOF OF THE STRUCTURE FORMULA

We give a proof of (56), (57).

Proof. 1. Integration by parts yields
\[
E(\lambda)(I - \epsilon^2 \Delta S)^l g(x) = \int_S [(I - \epsilon^2 \Delta S)^l e^{i\lambda x \cdot \xi}] g(\xi) dS(\xi),
\]
which shows that
\[
E(\lambda)(I - \epsilon^2 \Delta S)^l = E_{a_l}(\lambda), \quad \text{with} \quad a_l(x, \xi) = e^{-i\lambda x \cdot \xi} (I - \epsilon^2 \Delta S)^l e^{i\lambda x \cdot \xi}.
\]
We will prove the bound (57) for \( N = 0 \), the other cases being similar. Let \( v \in \mathbb{C}^d \) and let \( \phi_v(\xi) := v \cdot \xi / |\xi|, \xi \in \mathbb{R}^d \setminus \{0\} \), so that \( e^{\phi_v(\xi)} \) is the degree zero homogeneous extension of \( e^{i\lambda x \cdot \xi} \) for \( v = i\lambda x \). By a well-known formula for the spherical Laplacian,
\[
\Delta_S e^{i\lambda x \cdot \xi} = \Delta e^{\phi_v(\xi)}, \quad \xi \in S,
\]
where \( \Delta = \sum_{j=1}^d \partial_{\xi_j} \) is the Euclidean Laplacian. It thus suffices to prove that
\[
|e^{-\phi_v(\xi)} (I - \epsilon^2 \Delta)^l e^{\phi_v(\xi)}| \leq C^2 l! \exp(\epsilon v), \quad v \in \mathbb{C}^d, \quad \xi \in \mathbb{R}^d \setminus \{0\}.
\]
2. By the multinomial formula, the left hand side of the last display consists of \( (d + 1)^l \) terms of the form \( e^{-\phi_v(\xi)} (\epsilon \partial)^{\alpha} e^{\phi_v(\xi)} \) with \( |\alpha| \leq 2l \). Note that the factor \( (d + 1)^l \) can be absorbed into the constant in (57). Hence, we have reduced the task to proving that for \( |\alpha| \leq 2l \), the function
\[
a_\alpha(v, \xi) = e^{-\phi_v(\xi)} \partial^\alpha e^{\phi_v(\xi)}
\]
satisfies
\[
|e^{\alpha|} a_\alpha (v, \xi)| \leq C|\alpha| |\alpha|! \exp(\epsilon v), \quad v \in \mathbb{C}^d, \quad \xi \in \mathbb{R}^d \setminus \{0\}.
\]
Since this is a local statement, it suffices to prove it for some fixed \( \xi \in S \). Since the upper bound in (57) is invariant under rotations, we may assume without loss of generality that \( \xi = e_1 \).

3. It follows from the multivariate Faà di Bruno formula (see [CS96, Corollary 2.10] for details) that
\[
a_\alpha (v, \xi) = \sum_{r=1}^{\alpha} \sum_{p(\alpha, r)} \alpha! \prod_{j=1}^{\alpha} \frac{[\partial^{\lambda_j} \phi_v(\xi)]_{k_j} (k_j!)^{\lambda_j}}{(k_j!)^{\lambda_j}} k_i \lambda_i = \alpha,
\]
where \( p(\alpha, r) \) is a set of \((k_1, \ldots, k_{|\alpha|}; \lambda_1, \ldots, \lambda_{|\alpha|})\), \( k_i \in \mathbb{N}_0, \lambda_i \in \mathbb{N}^d_0 \), satisfying the constraints
\[
\sum_{i=1}^{\alpha} k_i = r, \quad \sum_{i=1}^{\alpha} k_i \lambda_i = \alpha.
\]
Since \( \phi_v \) is homogeneous of degree one in \( v \) and since \( \epsilon \leq 1 \), it follows from (91) and the first identity in (92) that, with \( \omega := v/|v| \),

\[
|\epsilon^{\alpha} a_\alpha(v, \xi)| = \left| \sum_{r=1}^{\alpha} |\epsilon v|^r \sum_{p(\alpha, r)} \alpha! \prod_{j=1}^{\alpha} \left| \epsilon \partial^{\lambda_j} \phi_\omega(\xi)^{k_j} \right| (k_j!)(\lambda_j!)^{k_j} \right|
\leq \exp(|\epsilon v|) \sum_{p(\alpha, r)} r! \alpha! \prod_{j=1}^{\alpha} \left| \partial^{\lambda_j} \phi_\omega(\xi)^{k_j} \right| (k_j!)(\lambda_j!)^{k_j}.
\]

4. It remains to show that

\[
\sum_{p(\alpha, r)} r! \alpha! \prod_{j=1}^{\alpha} \left| \partial^{\lambda_j} \phi_\omega(\xi)^{k_j} \right| (k_j!)(\lambda_j!)^{k_j} \leq C^{\alpha+1} \alpha!.
\]  

(93)

Here and in the following, \( C \) denotes a generic constant (only depending on the dimension \( d \)) that is allows to change from line to line. To prove this we use that \( \phi_\omega(\xi) \) is real-analytic at \( \xi = e_1 \), which means that we have derivative bounds

\[
|\partial^{\lambda} \phi_\omega(\xi)| \leq C^{\lambda+1} \lambda!
\]

(94)

for \( \xi \) in an open ball around \( e_1 \). Since \( \omega \) ranges over the unit sphere, the constant can be taken uniform in \( \omega \). Using (94) in (93) together with the second identity in (92) yields

\[
\sum_{p(\alpha, r)} r! \alpha! \prod_{j=1}^{\alpha} \left| \partial^{\lambda_j} \phi_\omega(\xi)^{k_j} \right| (k_j!)(\lambda_j!)^{k_j} \leq C^{\alpha+1} \alpha! \sum_{p(\alpha, r)} \prod_{j=1}^{\alpha} \frac{1}{k_j!}.
\]

The last term can be evaluated to (see [CS96, page 515])

\[
r! \sum_{p(\alpha, r)} \prod_{j=1}^{\alpha} \frac{1}{k_j!} = \binom{|\alpha| - 1}{r - 1},
\]

and we estimate

\[
\binom{|\alpha| - 1}{r - 1} \leq \sum_{r=2}^{|\alpha|} \binom{|\alpha| - 1}{r - 1} = 2^{|\alpha| - 1}.
\]

This proves (93).

\[\square\]

**Appendix B. Details of the proof of Theorem 7.1**

**B.1. Details for part (a).** We first recall the Birman–Schwinger principle: If \( \text{Im} \lambda \in \mathbb{C} \) is a resonance of \(-\Delta + V\), then

\[
1 \leq \| \text{BS}(\lambda) \| = s_1(\text{BS}(\lambda)).
\]
B.1.1. Lorentz case: We start with the easier bound (85) for $\text{Im} \lambda \geq 0$. By (41) with $k = 1$, we have

$$1 \leq \| BS(\lambda) \| \lesssim |\lambda|^{-\frac{d}{2}} \| V \|_{(d+1)/2} \implies |\lambda| \lesssim \| V \|_{(d+1)/2}^{d+1/2}$$

For $\text{Im} \lambda < 0$, we use (42) with $k = 1$,

$$\| BS(\lambda) \| \lesssim \delta^{-\frac{d-1}{2}} |\lambda|^{-\frac{d}{2}} e^{2\sqrt{\frac{1}{1+\frac{1}{d}}}} |\lambda|^{-2} \| e_{(d+1)/2} \| V \|_{(d+1)/2},$$

which leads to the stated bound in Theorem 7.1 (a) in the Lorentz case. Note that the bound there is stated for all $\lambda \in \mathbb{C}$, which is still true since the bound for $\text{Im} \lambda \geq 0$ is clearly dominated by that for $\text{Im} \lambda < 0$. This remark also applies to the compactly supported and pointwise decaying case, and we will not repeat it.

B.1.2. Compactly supported case: For $\text{Im} \lambda \geq 0$, we use (62) with $k = 1$:

$$\| BS(\lambda) \| \lesssim \langle \lambda R \rangle^{\nu+\kappa} \ln(\lambda R) R |\lambda|^{-1} + R^{2\theta} |\lambda|^{-2(1-\theta)} \| V \|_{\infty}.$$  

Choosing $\theta = \frac{1}{2}$, $\nu = 0$, the first term dominates the second, and we get

$$\| BS(\lambda) \| \lesssim \ln(\lambda R) R |\lambda|^{-1} \| V \|_{\infty}. \quad (95)$$

Since $\ln(\lambda R) \lesssim \kappa \langle \lambda R \rangle^\kappa$ for any $\kappa > 0$, we get the stated bound in (85).

For $\text{Im} \lambda < 0$, we use (63) with $k = 1$:

$$\| BS(\lambda) - BS(-\lambda) \| \lesssim \delta^{-d} (\lambda R)^{\nu} \ln(\lambda R) R |\lambda|^{-1} \| v_{p,R} \| \sqrt{1+\frac{1}{d}} |\text{Im} \lambda|^{1/2} R^{2\theta} |\lambda|^{-2(1-\theta)} \| v_{p,R} \|.$$  

Since the upper bound (95) for $\text{Im} \lambda \geq 0$ is dominated by the right hand side in the last display, an application of the triangle inequality for the operator norm $\| \cdot \|$ yields the same upper bound for $\| BS(\lambda) \|$. This gives the stated bound in Theorem 7.1 (a) in the compactly supported case.

B.1.3. Pointwise decaying case: For $\text{Im} \lambda \geq 0$, we use (76) with $k = 1$:

$$\| BS(\lambda) \| \lesssim (\lambda R)^{\nu+\kappa} R |\lambda|^{-1} + R^{2\theta} |\lambda|^{-2(1-\theta)} v_{p,R}.$$  

Again, choosing $\theta = \frac{1}{2}$, $\nu = 0$ and observing that the first term dominates the second, we get

$$\| BS(\lambda) \| \lesssim (\lambda R)^{\kappa} R |\lambda|^{-1} v_{p,R},$$  

which gives the stated bound in (85).

For $\text{Im} \lambda < 0$, we use (79) with $k = 1$:

$$\| BS(\lambda) \| \lesssim \delta^{-d} (\lambda R)^{\nu+\kappa} R |\lambda|^{-1} v_{p,R,\sqrt{1+\frac{1}{d}} |\text{Im} \lambda|} + R^{2\theta} |\lambda|^{-2(1-\theta)} v_{p,R}.$$  

Choosing $\theta = \frac{1}{2}$, $\nu = 0$, the first term again dominates the second, and we get

$$\| BS(\lambda) \| \lesssim \delta^{-d} (\lambda R)^{\kappa} R |\lambda|^{-1} v_{p,R,\sqrt{1+\frac{1}{d}} |\text{Im} \lambda|}.$$  

This gives the stated bound in Theorem 7.1 (a) in the pointwise decaying case.
B.2. Details for part (b). Recall (83), (84):

\[ n_V(r) \lesssim (\ln s)^{-1}(\tilde{n}_+(sr) + \tilde{n}_-(sr) + \tilde{n}_0), \]

\[ \tilde{n}_\pm(r) := \max_{\lambda \in S_\pm(r, \lambda_0)} \sum_{k \in \mathbb{N}} \ln(1 + [s_k(BS(\lambda))]^\alpha), \quad \tilde{n}_0 := \sum_{k \in \mathbb{N}} \ln(1 + [s_k(BS(\lambda_0))]^\alpha), \]

where we omitted the dependence of the implicit constant on the (fixed) integer \( \alpha > d/2 \). We will show that, if \( n_\pm(r) \), \( n_0 \) denote the quantities appearing in the statement of Theorem 7.1 then

\[ \tilde{n}_+(r) \lesssim n_+(r), \quad \tilde{n}_-(r) \lesssim n_+(r) + n_-(r), \quad \tilde{n}_0 \lesssim n_0. \]

B.2.1. Lorentz case: Using (83), (84) and the fact that \( x \mapsto \ln(1 + x^\alpha) \) is monotonically increasing on \((0, \infty)\), we get

\[ \tilde{n}_+(r) \lesssim \max_{\lambda \in S_+(r, \lambda_0)} \sum_{k \in \mathbb{N}} \ln(1 + [k^{-\frac{d}{d+1}} |\lambda|^{-\frac{2}{d+1}} \|V\|_{(d+1)/2}]^\alpha). \]

Lemma 7.4 i) applied with \( M = |\lambda|^{-\frac{2}{d+1}} \|V\|_{(d+1)/2} \) and \( \beta = d + 1 \) yields

\[ \tilde{n}_+(r) \lesssim \max_{\lambda \in S_+(r, \lambda_0)} |\lambda|^{-2} \|V\|_{(d+1)/2} \leq r^{-2} \|V\|_{(d+1)/2} = n_+(r), \]

where, in the last inequality, we used that \( |\lambda| \geq r \) for \( \lambda \in S_+(r, \lambda_0) \).

To estimate \( \tilde{n}_+(r) \) we first sparsify the sum over \( k \in \mathbb{N} \), by splitting it into a sum over even and odd \( k \) (a similar argument was used in Section 6). By monotonicity of the singular values, it suffices to consider one of these, say the even ones. Using \( s_{2j+1}(A + B) \leq s_{j+1}(A) + s_{j+1}(B) \), which follows from (22), we get

\[ \sum_{k \in 2\mathbb{N}} \ln(1 + [s_k(BS(\lambda))]^\alpha) \lesssim \sum_{k \in \mathbb{N}} \ln(1 + [2s_k(BS(-\lambda))]^\alpha) \]

\[ + \sum_{k \in \mathbb{N}} \ln(1 + [2s_k((BS(\lambda) - BS(-\lambda))]^\alpha). \]

Here we again used the monotonicity of \( x \mapsto \ln(1 + x^\alpha) \), as well as its consequence

\[ \ln(1 + (x + y)^\alpha) \leq \ln(1 + 2^\alpha x^\alpha + 2^\alpha y^\alpha) \leq \ln((1 + 2^\alpha x^\alpha)(1 + 2^\alpha y^\alpha)) \]

\[ = \ln(1 + 2^\alpha x^\alpha) + \ln(1 + 2^\alpha y^\alpha). \]

Taking the maximum over \( \lambda \in S_+(r, \lambda_0) \), the first sum on the right hand side of (96) is estimated by \( n_+(r) \), so it remains to show that the second sum is dominated by \( n_-(r) \). By 43, for \( \lambda \in S_+(r, \lambda_0) \), this sum is bounded by

\[ \sum_{k \in \mathbb{N}} \ln(1 + [2^\alpha |\lambda|^{-\frac{2}{d+1}} \delta^{-\frac{(d+1)^2}{2}} \exp\left(-\alpha k^{\frac{1}{d+1}}\right) \|e^{2\gamma(r)}|V|_{(d+1)/2, 2/1, 1}^\alpha]. \]

Lemma 7.4 ii), with \( M = 2^\alpha |\lambda|^{-\frac{2}{d+1}} \delta^{-\frac{(d+1)^2}{2}} \|e^{2\gamma(r)}|V|_{(d+1)/2, 2/1, 1}^\alpha \) and \( \beta = d - 1 \), shows that this is dominated by \( n_-(r) \).
The bound for $\tilde{n}_0$ is similar to that for $\tilde{n}_+$. We again use (11), but this time with $\lambda_0$ instead of $\lambda$, resulting in

$$\tilde{n}_0 \lesssim \sum_k \ln(1 + [k^{-\frac{1}{\lambda_0}}(A/2)]^{-\frac{1}{\lambda_0}}),$$

where we used the definition of $\lambda_0$, see (82). Lemma (7.4) applied with $M = (A/2)^{\frac{1}{2d+1}}\|V\|_{(d+1)/2}$ and $\beta = d + 1$ yields $\tilde{n}_0 \lesssim A^{-2} = n_0$.

B.2.2. Compact supported and decaying case. The argument is exactly the same as for the Lorentz case, with the appropriate replacement of the singular value bounds (the relevant bounds are cited in the proof of Theorem 7.1). The only difference is the appearance of a factor $\langle \lambda R \rangle^{\nu+\kappa}$ with a positive power of $\lambda$. But for $\lambda \in S_\pm(r, \lambda_0)$, we have a prior only a lower bound $|\lambda| \geq r$, so it is not immediately clear how to estimate the maximum over such $\lambda$. This factor appears in the expression $\langle \lambda R \rangle^{\nu+\kappa} |\lambda|^{-1} \|V\|_\infty$ (or $v_{\rho, R}$). If $\nu + \kappa \leq 1$, then we can estimate

$$\langle \lambda R \rangle^{\nu+\kappa} |\lambda|^{-1} \lesssim (\lambda R)^{\nu+\kappa} R^{-1}$$

for $|\lambda| \geq r$. Indeed, if $|\lambda| R \leq 2$, then this clearly holds. If $|\lambda| R > 2$, then the left hand side is bounded by $|\lambda|^{\nu+\kappa-1} R^{\nu+\kappa+1}$. Since $\nu + \kappa \leq 1$, the claim follows. However, if $\nu + \kappa > 1$, then the lower bound $|\lambda| \geq r$ is not enough, and we need to use the stronger fact that $\lambda$ lies in the intersection of the upper half plane with $\partial D(\lambda_0, |\lambda_0| + r)$, which implies that $|\lambda| \leq 2|\lambda_0| + r$ (see Figure 1 in Section 6). This leads to the bound for $n_+(r)$ stated in Theorem 7.1.

APPENDIX C. PROOF OF REMARK 1.2 AND EXAMPLE 1.3
C.1. Proof of Remark 1.2

(i) Assume that $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$, supp($V$) $\subset B(0, R)$. It suffices to show that the right hand sides of (3), (9), (10) are all bounded by $C_{\nu} r^d$ for $r \geq 1$. This is obvious for (9), (10). To estimate the Lorentz norm in (3) we use

$$|\{x : e^{2\gamma|x|}|V(x)| > \alpha\}| \leq |B(0, R)| \{\alpha < e^{2\gamma R} \|V\|_\infty\}$$

to find that (with $\gamma = 1 + \epsilon$)

$$\|e^{2(1+\epsilon)r} |V|\|_{(d+1)/2} \lesssim \left(\int_0^\infty \alpha^{-\frac{d}{2}} |\{x : e^{2\gamma|x|}|V(x)| > \alpha\}| \frac{d\alpha}{\alpha} \right)^{2} \lesssim R^{d+1} e^{2\gamma R} \|V\|_\infty.$$

(ii) Without loss of generality we assume that $|V(x)| \leq \exp(-|x|^{1+\epsilon})$. We show that the right hand sides of (3), (10) are bounded by $C_{\nu} r^{d(1+1/\epsilon)}$ for $r \geq 1$. In the pointwise decaying case (10) this follows from the fact that

$$|V(x)| \leq C_{\rho, \gamma}(1 + |x|)^{-\rho} \exp(4^{1/\epsilon} \gamma^{1+1/\epsilon}) e^{-2\gamma |x|}$$

(97)

for every $\rho, \gamma > 0$. Indeed, if $|x|^{\epsilon} \geq 4\gamma$, then

$$|V(x)| \leq e^{-\frac{d}{2} |x|^{1+\epsilon}} e^{-2\gamma |x|} \leq C_{\rho, e}(1 + |x|)^{-\rho} e^{-2\gamma |x|},$$

and if $|x|^{\epsilon} < 4\gamma$, then $e^{-\gamma |x|} > \exp(-4^{1/\epsilon} \gamma^{1+1/\epsilon})$, so that

$$|V(x)| \leq C_{\rho, e}(1 + |x|)^{-\rho} < C_{\rho, e}(1 + |x|)^{-\rho} \exp(4^{1/\epsilon} \gamma^{1+1/\epsilon}) e^{-2\gamma |x|}.$$
In the Lorentz case \( \mathbb{S} \), let
\[
U_\alpha := \{ x : e^{2\gamma|x|} |V(x)| > \alpha \}, \quad U_{\alpha, <} := U_\alpha \cap B(0, (4\gamma)^{1/2}).
\]
Then for \( x \in U_\alpha \setminus U_{\alpha, <} \), we have \( 2\gamma|x| - |x|^{1+\epsilon} \leq -\frac{1}{2} |x|^{1+\epsilon} \), which implies
\[
|U_\alpha \setminus U_{\alpha, <}| \lesssim (\ln(1/\alpha))^{\frac{d}{\gamma}} 1_{\alpha < 1}.
\]
On the other hand, we have
\[
|U_{\alpha, <}| \lesssim \gamma^d 1_{\alpha < \alpha_{\text{max}}},
\]
where \( \alpha_{\text{max}} = \max_{x \in \mathbb{R}^d} e^{2\gamma|x|} |V(x)| \leq \exp(C\epsilon_1^{1+1/\epsilon}) \), in view of \( [97] \). Combining the last two displays, we have
\[
\|e^{2(1+\epsilon)r|V|}\|_{(d+1)/2, 1/2} \lesssim \left( \int_0^\infty \alpha^{-\frac{d}{\gamma}} |U_\alpha|^{\frac{d}{\gamma}} d\alpha \right)^{2}
\lesssim \left( \int_1^\infty \alpha^{-\frac{d}{\gamma}} [\ln(1/\alpha)]^{\frac{d}{\gamma}} d\alpha \right)^{2} + \gamma^d \alpha_{\text{max}} \leq \exp(C\epsilon_1^{1+1/\epsilon}),
\]
where we used that the integral involving the logarithm is \( O(1) \).

(iii) Assume again that \( V \in L_{\text{comp}}(\mathbb{R}^d) \), \( \supp(V) \subset B(0, R) \). Without loss of generality we may assume that \( R > 2 \) and \( \|V\|_\infty \leq C_2^{-2} \). Since \( n_V(r, h) = n_V/\sqrt{2}(r/h) \), it is easy to see that \( [9] \) implies \( n_V(r, h) \lesssim |rh^{-1}R|^d \) for \( r \geq 1 \), \( h \in (0, 1) \). The same argument works for (III). It does not work for (III) since the condition \( r \geq C_2 \|V\|_{(d+1)/2} \) is not invariant under rescaling \( r \to r/h, V \to V/h^2 \).

C.2. Details of Example 1.3. Since \( V \) is sparse, we have \( L_j(1 + o(1)) \geq |x| \) for \( x \in \Omega_j \) \( [Cue22, (16)] \). Therefore,
\[
\|e^{2\gamma|x|}\|_{(d+1)/2, 1/2} \leq \| \sum_j e^{O(1)\gamma L_j} |H_j| 1_{\Omega_j} \|_{(d+1)/2, 1/2}.
\]
We assume \( |H_j| = 1, L_j = j, |\Omega_j| \leq C_M \exp(-Mj) \) for all \( M > 0 \). Then
\[
|\{ x : \sum_j e^{O(1)\gamma L_j} |H_j| 1_{\Omega_j}(x) > \alpha \}| \leq \sum_{j > c \ln \alpha/\gamma} |\Omega_j| \lesssim C_M \alpha^{-cM/\gamma}
\]
for all \( \alpha > 1 \), some \( c > 0 \), and zero else. Thus,
\[
\|e^{2\gamma|x|}\|_{(d+1)/2, 1/2} \lesssim \int_1^\infty (C_M \alpha^{-cM/\gamma})^{\frac{d}{\gamma}} d\alpha \lesssim C_M^\frac{2}{\gamma} (c'M/\gamma - 1)^{-1}
\]
for \( \gamma < c'M \) and some \( c' > 0 \). Thus, \( [8] \) yields
\[
n_V(r) \lesssim \left[ \ln \left( O(1)(C_M/r)^{\frac{d}{\gamma}} \right) \right]^{\frac{d}{\gamma}} \left( \frac{r}{M - O(1)r} \right)^{\frac{d}{\gamma}}
\]
for \( 1 \leq r \ll M \). Since \( M \) can be taken arbitrarily large, it follows that \( n_V(r) < \infty \) for all \( r > 0 \).
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