Perturbative String Dynamics Near the Photon Sphere

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(March 28, 2022)

Abstract

String dynamics near the photon sphere in Schwarzschild spacetime is considered on the basis of a perturbative approach with respect to a rescaled string tension as a small parameter. The perturbative string solution in the zeroth and first approximation is presented. The perturbative solution corresponds to a small deformation of the photon sphere in Schwarzschild spacetime.

PACS number(s): 04.70.Bw, 04.70.Dy, 11.25.-w, 11.25 Mj

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I. INTRODUCTION

The classical evolution of strings in curved backgrounds is described by a complicated system of second-order non-linear coupled partial differential equations which is integrable only for some special configurations [1]. A vast simplification of the equations of motion arises when one neglects string tension and considers the null (tensionless) strings [2]. Their equations of motion are null geodesic equations of General Relativity appended by an additional ‘stringy’ constraint. The exact null string configurations were studied in Schwarzschild and cosmological spacetimes recently [3–6]. In particular in Ref. [3] the solution for a null string moving along the photon sphere of the Schwarzschild spacetime was presented.

However, the important physical information about tensile string dynamics can be obtained from studying the approximate solutions of its equations of motion as it was proposed in Ref. [7]. In Ref. [8,9] it was considered the realisation of a perturbative scheme which was based on the assumption of a small value of a rescaled string tension parameter $\frac{\gamma}{\alpha'} \ll 1$.

The objective of this paper is to apply the expansion scheme as proposed in [9] for studying string dynamics in Schwarzschild spacetime and compare the results with a qualitative picture given in [3].

II. RESCALED TENSION AS A PERTURBATION

In this Section we shortly discuss the main points of the perturbation scheme of Ref. [9]. The basic idea is to use for the string action a generalization of the action given for massless point particle Ref. [12]. We assume a perturbative parameter $\varepsilon \equiv \frac{\gamma}{\alpha'} \ll 1$ with $\gamma$ a constant and $\alpha'$ the inverse string tension parameter. It was shown in Ref. [9] that for $\varepsilon \ll 1$ the solutions for the string equations of motion are approximated by the solutions of a null geodesic equation with an additional constraint.

$^1$Suggestion to consider string tension as a small parameter was also made in Ref. [10]. However, in contradiction to Ref. [8,9] the authors of Ref. [10,11] considered a perturbative scheme with null strings as zeroth approximation.
the case of small $\varepsilon \ll 1$ one can introduce a macroscopic ‘slow’ worldsheet time parameter

$$T = \varepsilon \tau,$$

(1)

where $\tau$ is the proper string time parameter. On the scale $T$ the string oscillations can be considered as perturbations with respect to the translational motion of the string points and described in the form of asymptotic expansion

$$X^\mu(T, \sigma) = \varphi^\mu(T) + \varepsilon \psi^\mu(T, \sigma) + \varepsilon^2 \chi^\mu(T, \sigma) + \ldots,$$

(2)

with $\sigma$ being a spacelike worldsheet string coordinate and $\mu, \nu, \rho, \kappa = 0, 1, 2, 3$. After introducing the expansion (2) the perturbative equations of motion and constraints in the first approximation have the form (3)

$$(D^2_T - \partial^2_\sigma) \psi^\mu + R^\mu_{\nu\rho\kappa}(\varphi) \varphi^\nu_{,T} \varphi^\rho_{,T} \psi^\kappa = 0,$$

$$(\varphi_{\mu,T} D^T_T \psi^\mu) = 0,$$

$$(\varphi_{\mu,T} \psi^\mu) = 0,$$

(3)

where $D^T_T \psi^\mu = \psi^\mu_{,T} + \varphi^\nu_{,T} \Gamma^\mu_{\nu\rho}(\varphi) \psi^\rho$, $R^\mu_{\nu\rho\kappa}$ is the Riemann tensor and $(\ldots)_T = \partial / \partial T$. First of Eqs.(3) is of the form of the geodesic deviation equation with an additional term $\partial^2_\sigma \psi^\mu$ describing the elastic string force. The zeroth order equations for $\varphi^\mu(T)$ are geodesic equations for a massless particle in a given curved space, i.e.,

$${D_T} \varphi^\mu_{,T} = 0,$$

$$\left( \varphi^\mu_{,T} \varphi_{,T} \right) = 0.$$

(4)

III. STRING DYNAMICS IN PERTURBATIVE APPROACH

Referring to the results of Ref. [3] we want to use the perturbative scheme of Section II for the discussion of string dynamics in Schwarzschild spacetime with the line element given by (M is the Schwarzschild mass with the dimension of (length)$^{-1}$, $\hbar = c = 1$ and $G$ is Newton constant with the dimension of (length)$^{2}$)
\[ ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2GM}{r}} - r^2 \left(d\theta^2 - \sin^2 \theta d\phi^2\right), \]

so the set of spacetime coordinates is \( X^\mu = (t, r, \theta, \phi). \)

Our suggestion is to choose \( M^{-2} \) to play the role of the constant \( \gamma \) in the rescaled tension parameter

\[ \varepsilon = \frac{M^{-2}}{\alpha'}, \]

and the perturbative approach can be applied for the case when \( M^{-2} \ll \alpha' \).

In order to solve the perturbative equations (3) for the first approximation functions \( \psi^\mu \) we need the solution for the zeroth approximation functions \( \varphi^\mu \). The general solution of the geodesic equations (4) for a massless particle in Schwarzschild spacetime is well-known [13]. Here we want to apply a particular form of the solution of the geodesic motion which describes a massless particle moving on the photon sphere in Schwarzschild spacetime in the form [3]

\[ \varphi^0(T) = 3ET, \quad \varphi^1(T) = 3GM, \quad \varphi^2(T) = \pm \frac{E\tau}{\sqrt{3GM}} + \theta_0, \quad \varphi^3(T) = \phi_0, \]

with \( E, \theta_0, \phi_0 \) constant. It is important that (7) automatically satisfies the constraint (4).

Substituting (7) into the equations for the first approximation functions (3) and making use of the background metric components (5) we get the equations of motion for \( \psi^\mu \)

\[ \psi^0_{,TT} - \psi^0_{,\sigma\sigma} + \frac{2E}{GM} \psi^1_{,T} = 0, \]

\[ \psi^1_{,TT} - \psi^1_{,\sigma\sigma} = 0, \]

\[ \psi^2_{,TT} - \psi^2_{,\sigma\sigma} = 0, \]

\[ \psi^3_{,TT} - \psi^3_{,\sigma\sigma} = 0, \]

and the constraints are [8]

\[ \psi^0_{,\sigma} \mp 3\sqrt{3}GM \psi^2_{,\sigma} = 0, \]

\[ \psi^0_{,T} \mp 3\sqrt{3}GM \psi^2_{,T} = 0. \]
The constraints (9) can be integrated to give only one condition

$$\psi^0 = \mp 3\sqrt{3}GM\psi^2,$$  \hspace{1cm} (10)

and the set of equations (8) reads

$$\psi^0_{, TT} - \psi^0_{, \sigma \sigma} = 0,$$

$$\psi^1_{, TT} - \psi^1_{, \sigma \sigma} - \frac{E^2}{3G^2M^2}\psi^1 = 0,$$  \hspace{1cm} (11)

$$\psi^2_{, TT} - \psi^2_{, \sigma \sigma} = 0,$$

$$\psi^3_{, TT} - \psi^3_{, \sigma \sigma} = 0.$$

After inspecting Eq.(9) we can easily notice that Eqs.(11) result in three two-dimensional wave equations for $\psi^0$, $\psi^2$, and $\psi^3$ which refer to the perturbations in the components $t$, $\theta$ and $\phi$ of the metric (5) respectively. The solutions for these components are

$$\psi^0 = \sum_{k=-\infty}^{\infty} \left( \alpha^0_k e^{ik(\sigma - T)} + \beta^0_k e^{-ik(\sigma - T)} \right),$$

$$\psi^0 = \sum_{k=-\infty}^{\infty} \left( \alpha^2_k e^{ik(\sigma - T)} + \beta^2_k e^{-ik(\sigma - T)} \right),$$  \hspace{1cm} (12)

$$\psi^0 = \sum_{k=-\infty}^{\infty} \left( \alpha^3_k e^{ik(\sigma - T)} + \beta^3_k e^{-ik(\sigma - T)} \right),$$

which describe the small string oscillations with frequencies $k = 1, 2 \ldots$ on the macroscopic scale $T$. The emergence of these oscillations is a pure consequence of admitting the non-zero (but very weak) string tension. These oscillations are the oscillations which take place on the surface of the photon sphere and they do not lead to any deformations of this sphere. On the other hand, the equation for the radial correction $\psi^1$ in (11) has the following solution

$$\psi^1 = A^1 \sin \left( \frac{E}{\sqrt{3}GM} \sigma \right) + B^1 \cos \left( \frac{E}{\sqrt{3}GM} \sigma \right),$$  \hspace{1cm} (13)

with $A^1, B^1$ constant. The solution (13) must subject the periodicity condition

$$\psi(0) = \psi(2\pi).$$  \hspace{1cm} (14)

As a consequence of (14) we get the ‘quantization’ of the parameter $E$
\[ E = \sqrt{3}GMn \quad (n = 0, \pm 1, \pm 2, \ldots), \] 

and the final solution for \( \psi^1 \) is

\[ \psi^1 = A^1 \sin n\sigma + B^1 \cos n\sigma. \] 

### IV. CONCLUSION

From (16) we can conclude that opposite to the case of oscillations for the components \( \psi^0, \psi^2 \) and \( \psi^3 \) as described by (12) the solution (16) corresponds to a small static deformation of the photon sphere \( r = 3GM + \varepsilon(A^1 \sin n\sigma + B^1 \cos n\sigma) \) and \( \varepsilon \ll 1. \)

This result confirms the qualitative statement of Ref. [3] namely that it is impossible to have the tensile strings with a constant value of the radial coordinate \( r \) in Schwarzschild spacetime. Besides, in both zeroth and first order approximations of the discussed perturbative scheme only one constraint exists. We note this is in agreement with the discussion of Ref. [3] concerning the absence of the additional integral of motion for tensile strings in Schwarzschild spacetime which suggests their chaotic behaviour.

The successful application of the perturbative scheme of Ref. [9] in this paper to the qualitative discussion of [3] for the particular zeroth order solution (a particle on the photon sphere) in Schwarzschild spacetime appears to give hope that the different exact solutions for the particle motion in Schwarzschild spacetime [13] can be considered as zeroth order approximations for the perturbative description of the tensile string dynamics in this spacetime.

### V. ACKNOWLEDGMENTS

MPD was supported by the Polish Research Committee (KBN) grant No 2 PO3B 196 10. AAZ acknowledges the hospitality of the Institute of Physics, University of Szczecin.
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