Discrete Galerkin Method for Fractional Integro-Differential Equations

P. Mokhtary
Department of Mathematics, Faculty of basic Sciences
Sahand University of Technology, Tabriz, Iran
E-Mails: mokhtary.payam@gmail.com mokhtary@sut.ac.ir

January 13, 2015

Abstract

In this paper, we develop a fully discrete Galerkin method for solving initial value fractional integro-differential equations (FIDEs). We consider Generalized Jacobi polynomials (GJPs) with indexes corresponding to the number of homogeneous initial conditions as natural basis functions for the approximate solution. The fractional derivatives are used in the Caputo sense. The numerical solvability of algebraic system obtained from implementation of proposed method for a special case of FIDEs is investigated. We also provide a suitable convergence analysis to approximate solutions under a more general regularity assumption on the exact solution.

Subject Classification: 34A08; 65L60

Keywords: Fractional integro-differential equation (FIDE), Galerkin Method, Generalized Jacobi Polynomials (GJPs), Caputo derivative.

1 Introduction

In this paper, we provide a convergent numerical scheme for solving FIDE

\[
\begin{cases}
D^q u(x) = p(x,u(x) + f(x) + \lambda \int_0^x K(x,t)u(t)dt, & x \in \Omega = [0,1], \\
u(0) = 0,
\end{cases}
\]

where \( q \in \mathbb{R}^+ \cap (0,1) \). The symbol \( \mathbb{R}^+ \) is the collection of all positive real numbers. \( p(x) \) and \( f(x) \) are given continuous functions and \( K(x,t) \) is a given sufficiently smooth kernel function, \( u(x) \) is the unknown function.

Note that the condition \( u(0) = 0 \) is not restrictive, due to the fact that (1) with nonhomogeneous initial condition \( u(0) = d \), \( d \neq 0 \) can be converted to the following homogeneous FIDE

\[
\begin{cases}
D^q \tilde{u}(x) = p(x,\tilde{u}(x) + \bar{f}(x) + \lambda \int_0^x K(x,t)\tilde{u}(t)dt, & x \in \Omega = [0,1], \\
\tilde{u}(0) = 0,
\end{cases}
\]

by the simple transformation \( \tilde{u}(x) = u(x) - d \), where \( \bar{f}(x) = f(x) + d \left( p(x) + \lambda \int_0^x K(x,t)dt \right) \).
Such kind of equations arising in the mathematical modeling of various physical phenomena, such as heat conduction, materials with memory, combined conduction, convection and radiation problems ([3], [5], [26], [27]).

\[ D^q u(x) = \mathcal{I}^{1-q} u'(x), \]  

where

\[ \mathcal{I}^\mu u(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-s)^{\mu-1} u(s)ds, \]  

is the fractional integral operator from order \( \mu \). \( \Gamma(\mu) \) is the well known Gamma function. The following relation holds ([8]):

\[ \mathcal{I}^q (D^q u(x)) = u(x) - u(0). \]  

From the relation above, it is easy to check that (1) is equivalent with the following weakly singular Volterra integral equation

\[ u(x) = g(x) + \lambda \int_0^x K(x, t)u(t)dt. \]  

Here \( g(x) = \mathcal{I}^q f(x) \) and \( K(x, t) = \frac{(x-t)^{q-1}}{\Gamma(q)} p(t) + \int_t^x \frac{(x-s)^{q-1}}{\Gamma(q)} K(s, t)ds. \) From the well known existence and uniqueness Theorems ([4], [32]), it can be concluded that if the following conditions are fulfilled

- \( f(x) \in C^l(\Omega), \quad l \geq 1 \)
- \( p(x) \in C^l(\Omega), \quad l \geq 1 \)
- \( K(x, t) \in C^l(D), \quad D = \{(x, t); 0 \leq t \leq x \leq 1\}, \quad l \geq 1 \)
- \( K(x, x) \neq 0, \)

the regularity of the unique solution \( u(x) \) of (5) and also (1) is described by

\[ u(x) = \sum_{(j,k)} \gamma_{j,k} x^j x^{kq} + U_l(x; q) \in C^l(0, 1) \bigcap C(\Omega), \quad \text{with} \quad |u'(x)| \leq C_q x^{q-1}, \]  

where the coefficients \( \gamma_{j,k} \) are some constants, \( U_l(\cdot; q) \in C^l(\Omega) \) and \( (j, k) := \{(j, k) : j, k \in \mathbb{N}_0, j + kq < l\} \). Here \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), where the symbol \( \mathbb{N} \) denotes the collection of all natural numbers. Thus, we must expect the first derivative of the solution to have a discontinuity at the origin. More precisely, if the given functions \( g(x), p(x) \) and \( K(x, t) \) are real analytic in their domains then it can be concluded that there is a function \( U = U(z_1, z_2) \) real and analytic at \( (0, 0) \), so that solutions of (5) and also (1) can be written as \( u(x) = U(x, x^q) \) ([4], [32]).

Recently, several numerical methods for the numerical solution of FIDE’s have been proposed. In [25], fractional differential transform method was developed to solve FIDE’s with nonlocal boundary conditions. In [30], Rawashdeh studied the numerical solution of FIDE’s by polynomial spline functions. In [2], an analytical solution for a class of FIDE’s was proposed. Adomian decomposition method to solve nonlinear FIDE’s was proposed in [21]. In [33], authors solved fractional nonlinear Volterra integro differential equations using the second kind Chebyshev wavelets. In [18], Taylor
expansion approach was presented for solving a class of linear FIDE’s including those of Fredholm and Volterra types. In [22], authors were solved FIDE’s by adopting Hybrid Collocation method to an equivalent integral equation of convolution type. In [21], Chebyshev Pseudospectral method was implemented to solve linear and nonlinear system of FIDE’s. In [23], authors proposed an analyzed spectral Jacobi Collocation method for the numerical solution of general linear FIDE’s. In [10], authors applied Collocation method to solve the nonlinear FIDE’s. In [24], Mokhtary and Ghoreishi, proved the $L^2$ convergence of Legendre Tau method for the numerical solution of nonlinear FIDE’s.

Many of the techniques mentioned above or have not proper convergence analysis or if any, very restrictive conditions including smoothness of the exact solution are assumed. In this paper we will consider non smooth solutions of (1). In this case although the discrete Galerkin method can be implemented directly but this method leads to very poor numerical results. Thus it is necessary to introduce a regularization procedure that allows us to improve the smoothness of the given functions and then to approximate the solution with a satisfactory order of convergence. To this end, we propose a regularization process which the original equation (1) will be changed into a new equation which possesses a more regularity properties by taking a suitable coordinate transformation. Our logic in choosing proper transformation is based upon the formal asymptotic expansion of the exact solution in (6). Consider (1), using the variable transformation

$$x = v^\frac{1}{q}, \quad v = x^q, \quad t = w^\frac{1}{q}, \quad w = t^q,$$

we can change (1) to the following equation

$$\mathcal{M}^q \bar{u}(v) = \bar{p}(v)\bar{u}(v) + \bar{f}(v) + \lambda \int_0^v \bar{K}(v, w)\bar{u}(w)dw,$$

where

$$\bar{p}(v) = p(v^\frac{1}{q}), \quad \bar{f}(v) = f(v^\frac{1}{q}), \quad \bar{K}(v, w) = \frac{1}{q}K(v^\frac{1}{q}, w^\frac{1}{q}).$$

$$\mathcal{M}^q \bar{u}(v) = \frac{1}{\Gamma(1-q)} \int_0^v (v^\frac{1}{q} - w^\frac{1}{q})^{-q} \bar{u}'(w)dw.$$

From (6), the exact solution $\bar{u}(v)$ can be written as $\bar{u}(v) = u(v^\frac{1}{q}) = \sum_{(j,k)} \gamma_{j,k} v^{\frac{1}{q}+k} + U_t(v^{\frac{1}{q}}; q)$. It can be easily seen that $\bar{u}'(v) \in C(\Omega)$. It is trivial that for $q = \frac{1}{n}$, $n \in \mathbb{N}$, the unknown function $\bar{u}(v)$ will be in the form

$$\bar{u}(v) = u(v^n) = \sum_{(j,k)} \gamma_{j,k} v^{n+j+k} + U_t(v^n; q), \quad n \in \mathbb{N},$$

which is infinitely smooth. Then we can deduce that the solution $\bar{u}(v)$ of the new equation (8) possesses better regularity and discrete Galerkin theory can be applied conveniently to obtain high order accuracy.

In the sequel, we introduce the discrete Galerkin solution $\bar{u}_N(v)$ based upon GJPs to (8). Since the exact solutions of (1) can be written as $u(x) = \bar{u}(v)$ then we define $u_N(x) = \bar{u}_N(v)$, $x, v \in \Omega$ as the approximate solution of (1).

Spectral Galerkin method is one of the weighted residual methods (WRM), in which approximations are defined in terms of truncated series expansions, such that residual which should be exactly equal
to zero, is forced to be zero only in an approximate sense. It is well known that, in this method, the expansion functions must satisfy in the supplementary conditions. The two main characteristics behind the approach are that, first it reduces the given problems to those of solving a system of algebraic equations, and in general converges exponentially and almost always supplies the most terse representation of a smooth solution([6], [15], [31]).

In this article, we use shifted GJPs on Ω, which are mutually orthogonal with respect to the shifted weight function $\delta_{\alpha,\beta}(v) = (2 - 2v)^\alpha (2v)^\beta$ on Ω where $\alpha, \beta$ belong to one of the following index sets

$N_1 = \{(\alpha, \beta); \alpha, \beta \leq -1, \alpha, \beta \in \mathbb{Z}\}$,  
$N_2 = \{(\alpha, \beta); \alpha \leq -1, \beta > -1, \alpha \in \mathbb{Z}, \beta \in \mathbb{R}\}$,  
$N_3 = \{(\alpha, \beta); \alpha > -1, \beta \leq -1, \alpha \in \mathbb{R}, \beta \in \mathbb{Z}\}$,  
$N_4 = \{(\alpha, \beta); \alpha, \beta > -1, \alpha, \beta \in \mathbb{R}\}$,

where the symbol $\mathbb{Z}$ is the collection of all integer numbers. The main advantage of GJPs is that these polynomials, with indexes corresponding to the number of homogeneous initial conditions in a given FIDE, are the natural basis functions to the Galerkin approximation of this problem([12], [13]).

The organization of this paper is as follows: we begin by reviewing some preliminaries which are required for establishing our results in Section 2. In Section 3, we introduce the discrete Galerkin method based on the GJPs and its application to [3]. Numerical solvability of the algebraic system obtained from discrete Galerkin discretization of a special case of [3] with $0 < q < \frac{1}{2}$ and $\bar{p}(v) = 1$ based on GJPs is given in Section 4. Convergence analysis of the proposed scheme is provided in Section 5. Numerical experiments are carried out in Section 6.

2 Preliminaries and Notations

In this section, we review the basic definitions and properties that are required in the sequel.

Defining weighted inner product

$$\left(u_1, u_2\right)_{\alpha,\beta} = \int_{\Omega} u_1(v)u_2(v)\delta_{\alpha,\beta}(v)dv,$$

and discrete Jacobi-Gauss inner product

$$\left(u_1, u_2\right)_{N,\alpha,\beta} = \sum_{k=0}^{N} u_1(v_k^{\alpha,\beta})u_2(v_k^{\alpha,\beta})\delta_k^{\alpha,\beta},$$

we recall the following norms over Ω

$$\|u\|_{\alpha,\beta}^2 = \left(u, u\right)_{\alpha,\beta}, \quad \|u\|_{N,\alpha,\beta}^2 = \left(u, u\right)_{N,\alpha,\beta}, \quad \|u\|_{\infty} = \sup_{v \in \Omega} |u(v)|.$$

Here, $v_k^{\alpha,\beta}$ and $\delta_k^{\alpha,\beta}$ are the shifted Jacobi Gauss quadrature nodal points on Ω and corresponding weights respectively.

The non-uniformly Jacobi-weighted Sobolev space denotes by $B_{\alpha,\beta}^k(\Omega)$ and defines as follows

$$B_{\alpha,\beta}^k(\Omega) = \{u : \|u^{(s)}\|_{\alpha+s,\beta+s} < \infty; \ 0 \leq s \leq k\},$$

equipped with the norm and semi-norm

$$\|u\|_{\alpha,\beta,k}^2 = \sum_{s=0}^{k} \|u^{(s)}\|_{\alpha+s,\beta+s}^2, \quad |u|_{\alpha,\beta,k} = \|u^{(k)}\|_{\alpha+k,\beta+k}.$$
The space $B^k_{\alpha,\beta}(\Omega)$ distinguishes itself from the usual weighted Sobolev space $H^k_{\alpha,\beta}(\Omega)$ by involving different weight functions for derivatives of different orders. The usual weighted Sobolev space $H^k_{\alpha,\beta}(\Omega)$ is defined as

$$H^k_{\alpha,\beta}(\Omega) = \{ u : \|u^{(s)}\|_{\alpha,\beta} < \infty; \ 0 \leq s \leq k \},$$

equipped with the norm

$$\|u\|_{H^k_{\alpha,\beta}(\Omega)}^2 = \sum_{s=0}^k \|u^{(s)}\|_{\alpha,\beta}^2.$$

We denote the shifted GJPs on $\Omega$ by $G^0_n(\alpha,\beta)(v)$ and define as

$$G^0_n(\alpha,\beta)(v) = \begin{cases} (2 - 2v)^{-\alpha}(2v)^{-\beta}J_{n-n_0}^{\alpha,-\beta}(v), & (\alpha, \beta) \in \mathcal{N}_1, \ n_0 = -\alpha - \beta, \\ (2 - 2v)^{-\alpha}J_{n-n_0}^{\alpha,\beta}(v), & (\alpha, \beta) \in \mathcal{N}_2, \ n_0 = -\alpha, \\ (2v)^{-\beta}J_{n-n_0}^{\alpha,\beta}(v), & (\alpha, \beta) \in \mathcal{N}_3, \ n_0 = -\beta, \\ J_{n-n_0}^{\alpha,\beta}(v), & (\alpha, \beta) \in \mathcal{N}_4, \ n_0 = 0, \end{cases}$$

where $J_{n}^{\alpha,\beta}(v)$ is the classical shifted Jacobi polynomials on $\Omega$; see [31]. An important fact is that the shifted GJPs $\{G^0_n(\alpha,\beta)(v); n \geq 1\}$ form a complete orthogonal system in $L^2_{\alpha,\beta}(\Omega)$; see (12) [13]. To present a Galerkin solution for (8) it is fundamental that the basis functions in the approximate solution satisfy in the homogeneous initial condition. To this end, since $G^0_{n-1}(0) = 0, \ n \geq 1$, then we can consider $\{G^0_{n-1}(v), \ n \geq 1\}$ as suitable basis functions to the Galerkin solution of (8).

From (10) and the following formula [9]

$$J_i^{\alpha,\beta}(v) = \sum_{k=0}^i (-1)^{i-k} \frac{\Gamma(i + \beta + 1)\Gamma(i + k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1)\Gamma(i + \alpha + \beta + 1)(i-k)!} v^k, \ \alpha, \beta \in \mathcal{N}_4,$$

we can obtain the following explicit formula for $G^0_{i-1}(v)$

$$G^0_{i-1}(v) = (2v)J_{i-1}^{0,1}(v) = 2 \sum_{k=0}^{i-1} (-1)^{i-1-k} \frac{(i+k)!}{(k+1)!(i-1-k)!} v^{k+1}, \ \ i \geq 1. \ (11)$$

For any continuous function $Z(v)$ on $\Omega$, we define the Legendre Gauss interpolation operator $\mathcal{I}_N$, as

$$\mathcal{I}_N Z(v) = \sum_{s=0}^N \left( Z, J_s^{0,0}(v) \right)_{N,0,0} J_s^{0,0}(v). \ (12)$$

Let $\mathcal{P}_N$ be the space of all algebraic polynomials of degree up to $N$. We introduce Legendre projection $\Pi_N : L^2(\Omega) \rightarrow \mathcal{P}_N$ which is a mapping such that for any $Z(v) \in L^2(\Omega)$,

$$\left( Z - \Pi_N Z, \phi \right)_{0,0} = 0, \ \ \forall \phi \in \mathcal{P}_N. \ (13)$$
3  Discrete Galerkin Approach

In this section, we present the numerical solution of (8) by using the discrete Galerkin method based on GJPs.

Let
\[ \tilde{u}_N(v) = \sum_{i=1}^{N} b_i G_i^{0,-1}(v), \]
\[ (14) \]
be the Galerkin solution of (8). It is trivial that \( \tilde{u}_N(0) = 0 \).

Galerkin formulation of (8) is to find \( \tilde{u}_N(v) \), such that
\[
\left( M^i \tilde{u}_N, G_i^{0,-1} \right)_{0,-1} = \left( p(v) \tilde{u}_N(v), G_i^{0,-1} \right)_{0,-1} + \left( f(v), G_i^{0,-1} \right)_{0,-1} + \lambda \left( K(\tilde{u}_N), G_i^{0,-1} \right)_{0,-1}, \quad i = 1, 2, ..., N
\]
\[ (15) \]
where \( K(\tilde{u}_N) = \int_0^1 \bar{K}(v, w) \tilde{u}_N(w) dw \).

Applying transformation \( w(\theta) = v\theta, \ \theta \in \Omega \) we get
\[
K(\tilde{u}_N) = K_\theta(\tilde{u}_N) = v \int_0^1 \bar{K}(v, w(\theta)) \tilde{u}_N(w(\theta)) d\theta.
\]
\[ (16) \]
Substituting (10) in (15) yields
\[
\left( M^i \tilde{u}_N, G_i^{0,-1} \right)_{0,-1} = \left( p(v) \tilde{u}_N(v), G_i^{0,-1} \right)_{0,-1} + \left( f(v), G_i^{0,-1} \right)_{0,-1} + \lambda \left( K_\theta(\tilde{u}_N), G_i^{0,-1} \right)_{0,-1},
\]
\[ i = 1, 2, ..., N. \]
\[ (17) \]
Inserting (14) in (17) we get
\[
\sum_{j=1}^{N} b_j \left\{ \left( M^j G_j^{0,-1}(v), G_i^{0,-1} \right)_{0,-1} - \left( p(v) G_j^{0,-1}(v), G_i^{0,-1} \right)_{0,-1} - \lambda \left( K_\theta(G_j^{0,-1}), G_i^{0,-1} \right)_{0,-1} \right\}
\]
\[
= \left( f(v), G_i^{0,-1} \right)_{0,-1}, \quad i = 1, 2, ..., N.
\]
\[ (18) \]
Following the relation \( G_i^{0,-1}(v) = G_i^{0,1}(v) \), we can rewrite (18) as
\[
\sum_{j=1}^{N} b_j \left\{ \left( M^j G_j^{0,-1}(v), G_i^{0,1} \right)_{0,0} - \left( p(v) G_j^{0,-1}(v), G_i^{0,1} \right)_{0,0} - \lambda \left( K_\theta(G_j^{0,-1}), G_i^{0,1} \right)_{0,0} \right\}
\]
\[
= \left( f(v), G_i^{0,1} \right)_{0,0}, \quad i = 1, 2, ..., N.
\]
\[ (19) \]
Now, we try to find an explicit form for $\mathcal{M}^qG_{j}^{0,-1}$. To this end, using (11) we have

$$\mathcal{M}^qG_{j}^{0,-1}(v) = \frac{2}{\Gamma(1-q)} \sum_{k=0}^{j-1} (-1)^{j-1-k} \frac{(j+k)!}{(k+1)!k!(j-1-k)!} \int_0^v \left( v^{\frac{1}{q}} - w^{\frac{1}{q}} \right)^{-q}(w^{k+1})'dw$$  \hspace{1cm} (20)$$

Applying the relation \[11\]

$$\int_0^v \left( v^{\frac{1}{q}} - w^{\frac{1}{q}} \right)^{-q} w^k dw = \left( \frac{q\pi \csc (\pi q)\Gamma(q+gk)}{\Gamma(q)\Gamma(1+gq)} \right) v^k, \ k \geq 0,$$

in (20) we can obtain the following explicit formula for $\mathcal{M}^qG_{j}^{0,-1}$:

$$\mathcal{M}^qG_{j}^{0,-1}(v) = \frac{2}{\Gamma(1-q)} \sum_{k=0}^{j-1} (-1)^{j-1-k} \frac{(j+k)!}{(k+1)!k!(j-1-k)!} \left( \frac{q\pi \csc (\pi q)\Gamma(q+gk)}{\Gamma(q)\Gamma(1+gq)} \right) v^k =: \Psi_{j,q}(v),$$  \hspace{1cm} (21)$$

Substituting (21) in (19) we obtain

$$\sum_{j=1}^{N} b_j \left\{ \left( \Psi_{j,q}(v), G_{i-1}^{0,1} \right)_{0,0} - \left( \tilde{p}(v)G_{j}^{0,-1}(v), G_{i-1}^{0,1} \right)_{0,0} - \lambda \left( \mathcal{K}_0(G_{j}^{0,-1}), G_{i-1}^{0,1} \right)_{0,0} \right\}$$

$$= \left( \tilde{f}(v), G_{i-1}^{0,1} \right)_{0,0}, \ i = 1, 2, ..., N.$$  \hspace{1cm} (22)$$

In this position, we approximate the integral terms of (22) using $(N + 1)$–point Legendre Gauss quadrature formula. Our discrete Galerkin method is to seek

$$\tilde{u}_N(v) = \sum_{i=1}^{N} a_i G_{i}^{0,-1}(v),$$  \hspace{1cm} (23)$$

such that coefficients $\{a_j\}_{j=1}^{N}$ satisfies in the following algebraic system of linear equations

$$\sum_{j=1}^{N} a_j \left\{ \left( \Psi_{j,q}(v), G_{i-1}^{0,1} \right)_{0,0} - \left( \tilde{p}(v)G_{j}^{0,-1}(v), G_{i-1}^{0,1} \right)_{N,0,0} - \lambda \left( \mathcal{K}_{N,0}(G_{j}^{0,-1}), G_{i-1}^{0,1} \right)_{N,0,0} \right\}$$

$$= \left( \tilde{f}(v), G_{i-1}^{0,1} \right)_{N,0,0}, \ i = 1, 2, ..., N.$$  \hspace{1cm} (24)$$

where

$$\mathcal{K}_{N,0}(G_{j}^{0,-1}) = v \sum_{k=0}^{N} \tilde{K}(v, w(\theta_k))G_{j}^{0,-1}(w(\theta_k))\delta_k.$$  \hspace{1cm} (25)$$

Here $\{\theta_k\}_{k=0}^{N}$ and $\{\delta_k\}_{k=0}^{N}$ are the shifted Legendre Gauss quadrature points on $\Omega$ and corresponding weights respectively. Note that, from (21) we can see that $\Psi_{j,q}(v)$ is a polynomial from degree at most $N$, then we have $\left( \Psi_{j,q}(v), G_{i-1}^{0,1} \right)_{N,0,0} = \left( \Psi_{j,q}(v), G_{i-1}^{0,1} \right)_{0,0}$. It is trivial that the solution of (24) gives us unknown coefficients $\{a_i\}_{i=1}^{N}$ in (23).
4 Existence and Uniqueness Theorem for Discrete Galerkin Algebraic System

The main object of this section is providing an existence and uniqueness Theorem for a special case of the discrete Galerkin algebraic system of equations (24) with $\bar{p}(v) = 1$ and $0 < q < \frac{1}{2}$. Throughout the paper, $C_i$ will denote a generic positive constant that is independent on $N$.

First, we give some preliminaries which will be used in the sequel.

**Definition 4.1** Let $\mathcal{X}, \mathcal{Y}$ be normed spaces. A linear operator $\mathcal{A} : \mathcal{X} \rightarrow \mathcal{Y}$ is compact if the set $\{\mathcal{A}x | \|x\|_\mathcal{X} \leq 1\}$ has compact closure in $\mathcal{Y}$.

**Theorem 4.2** [16, 17, 34] Assume $\mathcal{X}, \mathcal{Y}$ be two banach spaces. Let

$$v = \mathcal{A}v + f,$$  \hspace{1cm} (26)

be a linear operator equation where $\mathcal{A} : \mathcal{X} \rightarrow \mathcal{Y}$ is a linear continuous operator, and the operator $I - \mathcal{A}$ is continuously invertible. As an approximation solution of (26) we consider the equation

$$v_N = \mathcal{A}_N v_N + \mathcal{B}_N f,$$  \hspace{1cm} (27)

which can be rewritten as

$$v_N = \tilde{\mathcal{B}}_N \mathcal{A}v_N + \mathcal{S}_N v_N + \mathcal{B}_N f,$$  \hspace{1cm} (28)

where $\mathcal{A}_N$ is a linear continuous operator in a closed subspace $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$. $\mathcal{B}_N : \mathcal{Y} \rightarrow \tilde{\mathcal{Y}}$ are linear continuous projection operators and $\mathcal{S}_N = \mathcal{A}_N - \tilde{\mathcal{B}}_N \mathcal{A}$ is a linear operator in $\tilde{\mathcal{Y}}$. If the following conditions are fulfilled

- for any $Z \in \tilde{\mathcal{Y}}$ we have $\|\mathcal{S}_N Z\| \rightarrow 0$ as $N \rightarrow \infty$
- $\|\mathcal{A} - \tilde{\mathcal{B}}_N \mathcal{A}\| \rightarrow 0$ as $N \rightarrow \infty$
- $\|f - \mathcal{B}_N f\| \rightarrow 0$ as $N \rightarrow \infty$

then (28) possesses a uniquely solution $v_N \in \tilde{\mathcal{Y}}$, for a sufficiently large $N$.

**Lemma 4.3** [1] Let $\mathcal{X}, \mathcal{Y}$ be banach spaces and $\tilde{\mathcal{Y}}$ be a subspace of $\mathcal{Y}$. Let $\tilde{\mathcal{B}}_N : \mathcal{Y} \rightarrow \tilde{\mathcal{Y}}$ be a family of linear continuous projection operators with

$$\tilde{\mathcal{B}}_N y \rightarrow y \text{ as } N \rightarrow \infty, \ y \in \mathcal{Y}.$$

Assume that linear operator $\mathcal{A} : \mathcal{X} \rightarrow \mathcal{Y}$ be compact. Then

$$\|\mathcal{A} - \tilde{\mathcal{B}}_N \mathcal{A}\| \rightarrow 0 \text{ as } N \rightarrow \infty.$$

**Lemma 4.4** (Interpolation error bound[31]) Let $I_N Z$ be the interpolation polynomial approximation of the function $Z(v)$ defined in (12). For any $Z(v) \in B_{0,0}^k(\Omega)$ with $k \geq 1$, we have

$$\|Z - I_N Z\|_{0,0} \leq CN^{-k}|Z|_{0,0,k}.$$
Lemma 4.5 \[ \] For every bounded function \( Z(v) \), there exists a constant \( C \) independent of \( Z \) such that
\[
\sup_N \| \mathcal{I}_N Z \|_{0,0} \leq C \sup_v |Z(v)|.
\]

Lemma 4.6 (Legendre Gauss quadrature error bound) If \( Z(v) \in B^k_{0,0}(\Omega) \) for some \( k \geq 1 \) and \( \Phi \in \mathcal{P}_N \), then for the Legendre Gauss integration we have
\[
\left| (Z, \Phi)_{0,0} - (Z, \Phi)_{N,0,0} \right| \leq C N^{-k} \| Z \|_{0,0,k} \| \Phi \|_{0,0}.
\]

Now we intend to prove existence and uniqueness Theorem for a special case of the discrete Galerkin system (24) with \( \bar{p}(v) = 1 \) and \( 0 < q < \frac{1}{2} \).

Theorem 4.7 (Existence and Uniqueness) Let \( 0 < q < \frac{1}{2} \) and \( \bar{p}(v) = 1 \). If (8) has a uniquely solution \( \bar{u}(v) \) then the linear discrete Galerkin system (24) has a uniquely solution \( \bar{u}_N(v) \in \mathcal{P}_N \) for sufficiently large \( N \).

Proof: Our strategy in proof is based on two steps. First, we try to represent (24) in the operator form (28). Then by applying Theorem 4.2 to operator form obtained in the first step the desired result have been concluded.

Step 1: In this step, we show that the discrete Galerkin system (24) can be written in the operator form (28). To this end, consider (8) and define
\[
\mathcal{R}_N(v) = \mathcal{M}^q \bar{u}_N(v) - \bar{u}_N(v) - \lambda \mathcal{K}_{N,\theta}(\bar{u}_N) - \bar{f}(v).
\]

According to the proposed method, we have
\[
\left( \mathcal{R}_N(v), G_{i-1}^{0,1}(v) \right)_{N,0,0} = 0, \quad i = 1, 2, ..., N.
\]

From interpolation and Legendre Gauss quadrature properties, we can write
\[
\left( \mathcal{R}_N(v), G_{i-1}^{0,1}(v) \right)_{N,0,0} = \left( \mathcal{I}_N(\mathcal{R}_N), G_{i-1}^{0,1}(v) \right)_{N,0,0} = \left( \mathcal{I}_N(\mathcal{R}_N), G_{i-1}^{0,1}(v) \right)_{0,0} = 0, \quad i = 1, 2, ..., N.
\]

(29)

Since \( \mathcal{I}_N(\mathcal{R}_N(v)) \) is a polynomial, it can be represented by a linear orthogonal polynomial expansion based on \( \{ G_i^{0,-1}(v) \}_{i=0}^{N} \), as
\[
\mathcal{I}_N(\mathcal{R}_N) = \sum_{i=1}^{N} \frac{\mathcal{I}_N(\mathcal{R}_N), G_i^{0,-1}}{\| G_i^{0,-1} \|_{0,-1}} G_i^{0,-1}(v) = \sum_{i=1}^{N} \frac{\mathcal{I}_N(\mathcal{R}_N), G_i^{0,1}}{\| G_i^{0,-1} \|_{0,-1}} G_i^{0,-1}(v).
\]

(30)

Using the relations (29) and (30) yields \( \mathcal{I}_N(\mathcal{R}_N) = 0 \). Thus
\[
\mathcal{I}_N \left( \mathcal{M}^q \bar{u}_N - \bar{u}_N(v) - \lambda \mathcal{K}_{N,\theta}(\bar{u}_N) - \bar{f} \right) = 0,
\]
which can be rewritten as
\[ \tilde{u}_N(v) = \mathcal{I}_N \left( \mathcal{M}^q - \lambda \mathcal{K}_{N,\theta} \right) \tilde{u}_N - \mathcal{I}_N \tilde{f} = \mathcal{I}_N \mathcal{T}_N \tilde{u}_N - \mathcal{I}_N \tilde{f}, \tag{31} \]
and thereby
\[ \tilde{u}_N(v) = \Pi_N \mathcal{T} \tilde{u}_N + S_N \tilde{u}_N - \mathcal{I}_N \tilde{f}, \tag{32} \]
where \( \mathcal{I}_N \) and \( \Pi_N \) are defined in (12) and (13) respectively and
\[
\begin{align*}
\mathcal{T} &= \mathcal{M}^q - \lambda \mathcal{K}_\theta, \\
\mathcal{T}_N &= \mathcal{M}^q - \lambda \mathcal{K}_{N,\theta}, \\
S_N &= \mathcal{I}_N \mathcal{T}_N - \Pi_N \mathcal{T}.
\end{align*}
\]

Since (32) is the form in which the discrete Galerkin method is implemented, as it leads directly to the equivalent linear system (24). It can be easily check that (32) can be considered in the operator form (25) by assuming
\[
\begin{align*}
\mathcal{X} &= H^1_{0,0}(\Omega), & \mathcal{Y} &= L^2(\Omega), & \mathcal{Y} &= \mathcal{P}_N, \\
v_N &= \tilde{u}_N, & \mathcal{B}_N &= \Pi_N, & \mathcal{B}_N &= \mathcal{I}_N, \\
\mathcal{A} &= \mathcal{T}, & \mathcal{A}_N &= \mathcal{I}_N \mathcal{T}_N,
\end{align*}
\tag{33}
\]
which can be completed the desired result of step 1.

Step 2: In this step we intend to apply Theorem 4.2 with the assumptions (33) to prove the Theorem. To this end, following Theorem 4.2 we must show that
\[
\begin{align*}
1) & \text{ for any } Z \in \mathcal{P}_N \text{ we have } \|S_N Z\|_{0,0} = \|\mathcal{I}_N \mathcal{T}_N Z - \Pi_N \mathcal{T} Z\|_{0,0} \to 0 \text{ as } N \to \infty, \\
2) & \|\mathcal{T} - \Pi_N \mathcal{T}\|_{0,0} \to 0 \text{ as } N \to \infty, \\
3) & \|\tilde{f} - \mathcal{I}_N \tilde{f}\|_{0,0} \to 0 \text{ as } N \to \infty. \tag{34}
\end{align*}
\]

First, we prove the first condition in (34). For this, we can write
\[
\|S_N Z\|_{0,0} = \left\| \mathcal{I}_N \left( \mathcal{T}_N - \Pi_N \mathcal{T} \right) Z \right\|_{0,0} = \|\mathcal{I}_N (\mathcal{T}_N - \mathcal{T}) Z\|_{0,0} + \left\| \mathcal{I}_N - \Pi_N \right\| \mathcal{T} Z\|_{0,0}. \tag{35}
\]
Since \( \mathcal{M}^q Z \in \mathcal{P}_N \) for any \( Z \in \mathcal{P}_N \) then
\[
\|S_N Z\|_{0,0} \leq \lambda \left( \left\| \mathcal{I}_N (\mathcal{K}_{N,\theta} - \mathcal{K}_\theta) Z\right\|_{0,0} + \left\| \mathcal{I}_N - \Pi_N \right\| \mathcal{K}_\theta Z\|_{0,0} \right). \tag{36}
\]
Using Lemmas 4.5 and 4.6 and the relations (16) and (24) we can obtain
\[
\|\mathcal{I}_N (\mathcal{K}_{N,\theta} - \mathcal{K}_\theta) z\|_{0,0} \leq \sup_{v \in \Omega} |\mathcal{K}_\theta(z(v)) - \mathcal{K}_{N,\theta}(z(v))| \leq C_1 N^{-k_1} \sup_{v \in \Omega} \left( \|\tilde{K}(v, w(\theta))\|_{0,0,k_1} \|\mathcal{K}(w(\theta))\|_{0,0} \right), \tag{37}
\]
where norms \( \|\tilde{K}(v, w(\theta))\|_{0,0,k_1} \) and \( \|\mathcal{K}(w(\theta))\|_{0,0} \) are applied with respect to the variable \( \theta \).

According to Lemma 4.6 we have
\[
\left| \left( \mathcal{K}_\theta, J^0_s \right)_{N,0,0} - \left( \mathcal{K}_\theta, J^0_s \right)_{0,0} \right| \leq C N^{-k_2} \|\mathcal{K}_\theta\|_{0,0,k_2} \|J^0_s\|_{0,0}, \quad s = 0, 1, ..., N, \tag{38}
\]
where norm $\|K_\theta\|_{0,0,k_2}$ is applied with respect to the variable $v$. Using (38) and the relations (32), (33) we can yields
\[ \left\| \left( I_N - \Pi_N \right) K_\theta Z \right\|_{0,0} \to 0 \quad \text{as} \quad N \to \infty. \tag{39} \]

Substituting (37) and (39) in (36) we can conclude the first condition in (34). Applying Lemma 4.6 and Lemma 5.1 gives us the third condition in (34). To complete the proof, it is sufficient that we prove the second condition in (34). Applying Lemma 4.3 with the assumptions (33). Since \[ \frac{1}{v^q - w^q} \to 0 \quad \text{as} \quad N \to \infty \] for $y \in L^2(\Omega)$ (see [72]), the second condition in (34) can be achieved by proving compactness of the operator $\mathcal{T}$. Since $k(v,w)$ is a continuous kernel, the operator $\mathcal{T}$ will be compact if $\mathcal{M}^q(\bar{u}) : H^1_{0,0}(\Omega) \to L^2(\Omega)$ be a compact operator.

For $\bar{u} \in H^1_{0,0}(\Omega)$, using Cauchy-Schwartz inequality we have
\[ \mathcal{M}^q \bar{u} \in L^2(\Omega) \quad \text{and} \quad \| \mathcal{M}^q \| = \sup \left\{ \frac{\| \mathcal{M}^q \bar{u} \|_{0,0}}{\| \bar{u} \|_{H^1_{0,0}(\Omega)}} : \bar{u} \neq 0, \bar{u} \in H^1_{0,0}(\Omega) \right\} \leq M < \infty. \]

Therefore $\mathcal{M}^q : H^1_{0,0}(\Omega) \to L^2(\Omega)$ defined by (8), is a bounded operator. If we proceed same as Theorem 3.4 in [29], we can conclude compactness of the operator $\mathcal{M}^q(\bar{u})$. Then the operator $\mathcal{T}$ is a compact operator and from Lemma 4.3 we have \[ \left\| \mathcal{T} - \Pi_N \mathcal{T} \right\|_{0,0} \to 0 \quad \text{as} \quad N \to \infty. \]

In this position, all conditions in (34) that are required to deduce existence and uniqueness of solutions of the discrete Galerkin system (24) have been proved and then the proof is completed. \( \Box \)

### 5 Convergence analysis

In this section, we will try to provide a reliable convergence analysis which theoretically justify convergence of the proposed discrete Galerkin method for the numerical solution of a special case of (5) with $\bar{p}(v) = 1$.

In the sequel, our discussion is based on these Lemmas:

**Lemma 5.1** [31] For any $Z \in B^1_{0,0}(\Omega)$, we have
\[ \left\| I_N Z \right\|_{0,0} \leq C \left( \| Z \|_{0,0} + N^{-1} \| Z' \|_{1,1} \right). \]

**Lemma 5.2** The fractional integral operator $I^\mu$ is bounded in $L^2(\Omega)$ and
\[ \left\| I^\mu Z \right\|_{0,0} \leq C \| Z \|_{0,0}, \quad Z \in L^2(\Omega). \]
Lemma 5.3 \[\text{(Gronwall inequality)}\]
Assume that $Z(v)$ is a non-negative, locally integrable function defined on $\Omega$ which satisfies

$$Z(v) \leq b(v) + B \int_0^v (v - w)^\alpha w^\beta Z(w)dw, \quad w \in \Omega,$$

where $\alpha, \beta > -1$, $b(v) \geq 0$ and $B \geq 0$. Then, there exist a constant $C$ such that

$$Z(v) \leq b(v) + C \int_0^v (v - w)^\alpha w^\beta b(w)dw, \quad w \in \Omega.$$ 

Now, we state and prove the main result of this section regarding the error analysis of the proposed method for the numerical solution of a special case of (8) with $\bar{p}(v) = 1$.

Theorem 5.4 \(\text{(Convergence)}\)
Let $u(x)$ and $\bar{u}(v)$ are the exact solutions of the equations (1) and (8) respectively that is related by $u(x) = \bar{u}(v)$. Assume that $u_N(x) = \bar{u}_N(v)$ be the approximate solution of (1), where $\bar{u}_N(v)$ is the discrete Galerkin solution of the transformed equation (8). If the following conditions are fulfilled

1. $\bar{f}(v) \in B_{k_1,0}^k(\Omega)$ for $k_1 \geq 1$,
2. $K(\bar{u}) \in B_{k_2,0}^k(\Omega)$ for $k_2 \geq 1$,

then for sufficiently large $N$ we have

$$\|e_N(u)\|_{0,0} \leq CN^{-\xi} \left( |\bar{f}|_{0,0,k_1} + |K(\bar{u})|_{0,0,k_2} \right)$$

where $\xi = \min\{k_1, k_2\}$ and $e_N(u) = u(x) - u_N(x)$ denotes the error function.

Proof:
Since $M^q\bar{u}_N$ is a polynomial from degree of at most $N$ then we can rewrite (31) as

$$M^q\bar{u}_N - \bar{u}_N(v) = I_N \left( \lambda K_{N,\theta}(\bar{u}_N) \right) = I_N \bar{f}. \quad (40)$$

Subtracting (8) from (40) and some simple manipulations we can obtain

$$M^q\bar{e}_N = e_N(v) + e_{I_N}(\bar{f}) + \lambda K(\bar{e}_N) + \lambda e_{I_N}(K(\bar{u}_N)) + \lambda I_N \left( K_{\theta}(\bar{u}_N) - K_{N,\theta}(\bar{u}_N) \right), \quad (41)$$

where $e_{I_N}(g) = g - I_N(g)$ denotes the interpolating error function and $\bar{e}_N(v) = \bar{u}(v) - \bar{u}_N(v)$ is the error function of approximating (8) using discrete Galerkin solution $\bar{u}_N(v)$.

Applying the transformation (7) to the operator $\mathcal{I}^qu$ we get

$$M^q_1\bar{u} = \frac{1}{\Gamma(q)} \int_0^v \left( u^{\frac{1}{q}} - w^{\frac{1}{q}} \right)^{q-1} \bar{u}(w) w^{\frac{1}{q} - 1} dw.$$ 

Following (41) it is easy to check that

$$M^q_1(M^q\bar{u}(v)) = \bar{u}(v) - \bar{u}(0). \quad (42)$$
Applying the operator $\mathcal{M}_1^q$ on the both sides of (41) and using (42) we can yield

$$|\bar{e}_N(v)| \leq \mathcal{M}_1^q(|\bar{e}_N| + |\lambda \mathcal{K}(\bar{e}_N)|) + |\mathcal{M}_1^q\left(e_{N}(\bar{f}) + \lambda e_{N}(\mathcal{K}(\bar{u}_N)) + \lambda \mathcal{I}_N\left(\mathcal{K}_\theta(\bar{u}_N) - \mathcal{K}_{N,\theta}(\bar{u}_N)\right)\right)|.$$  (43)

Since

$$\mathcal{M}_1^q \int_0^v \bar{K}(v,w)\bar{e}_N(w)dw = \frac{1}{\Gamma(q)} \int_0^v \int_0^w (v^{\frac{1}{q}} - w^{\frac{1}{q}})^{q-1} \frac{w^{\frac{1}{q}-1}}{q} |\bar{K}(w,s)||\bar{e}_N(s)dsdw$$

$$= \frac{1}{\Gamma(q)} \int_0^v \int_s^v (v^{\frac{1}{q}} - w^{\frac{1}{q}})^{q-1} \frac{w^{\frac{1}{q}-1}}{q} |\bar{K}(w,s)||\bar{e}_N(s)dsdw$$

$$= \frac{1}{\Gamma(q)} \int_0^v |\bar{K}_1(v,s)||\bar{e}_N(s)ds \leq \frac{\|\bar{K}_1(v,s)\|_\infty}{\Gamma(q)} \int_0^v |\bar{e}_N(s)ds,$$  (44)

then (43) can be rewritten as

$$|\bar{e}_N(v)| \leq \frac{1}{\Gamma(q)} \left(\int_0^v (v^{\frac{1}{q}} - w^{\frac{1}{q}})^{q-1} \frac{w^{\frac{1}{q}-1}}{q} |\bar{e}_N(w)|dw + |\lambda||\bar{K}_1(v,s)\|_\infty \int_0^v |\bar{e}_N(w)|dw\right)$$

$$+ \left|\mathcal{M}_1^q\left(e_{N}(\bar{f}) + \lambda e_{N}(\mathcal{K}(\bar{u}_N)) + \lambda \mathcal{I}_N\left(\mathcal{K}_\theta(\bar{u}_N) - \mathcal{K}_{N,\theta}(\bar{u}_N)\right)\right)\right|$$

$$\leq \tilde{C} \left(\int_0^v (v^{\frac{1}{q}} - w^{\frac{1}{q}})^{q-1} |\bar{e}_N(w)|dw\right)$$

$$+ \left|\mathcal{M}_1^q\left(e_{N}(\bar{f}) + \lambda e_{N}(\mathcal{K}(\bar{u}_N)) + \lambda \mathcal{I}_N\left(\mathcal{K}_\theta(\bar{u}_N) - \mathcal{K}_{N,\theta}(\bar{u}_N)\right)\right)\right|,$$  (45)

where

$$\bar{K}_1(v,s) = \int_s^v (v^{\frac{1}{q}} - w^{\frac{1}{q}})^{q-1} \frac{w^{\frac{1}{q}-1}}{q} |\bar{K}(w,s)|dw, \quad \tilde{C} = \max \left\{\frac{1}{q\Gamma(q)}, \frac{|\lambda||\bar{K}_1(v,s)||_\infty}{\Gamma(q)}\right\} \|1+(v^{\frac{1}{q}} - w^{\frac{1}{q}})^{1-q}\|_\infty.$$

Using Gronwall inequality(Lemma 5.3) in (45) yields

$$\|\bar{e}_N\|_{0,\frac{1}{q}-1} \leq C_1 \left(\|\mathcal{M}_1^q\left(e_{N}(\bar{f}) + \lambda e_{N}(\mathcal{K}(\bar{u}_N)) + \lambda \mathcal{I}_N\left(\mathcal{K}_\theta(\bar{u}_N) - \mathcal{K}_{N,\theta}(\bar{u}_N)\right)\right)\|_{0,\frac{1}{q}-1}\right).$$  (46)

It can be checked that by applying the transformation (6) we obtain

$$\|\bar{e}_N(v)\|_{0,\frac{1}{q}-1}^2 = q^{2\frac{1}{q}-1}\|e_N(x)\|_{0,0}^2.$$  (47)

On the other hand, by applying (6) and Lemma 5.2 we can get

$$\|\mathcal{M}_1^q \tilde{Z}(v)\|_{0,\frac{1}{q}-1}^2 = q^{2\frac{1}{q}-1}\|\mathcal{I}^q Z(x)\|_{0,0}^2 \leq C q^{2\frac{1}{q}-1}\|Z(x)\|_{0,0}^2 = C \|\tilde{Z}(v)\|_{0,\frac{1}{q}-1}^2,$$  (48)
where \( Z(x) \) is a given function and \( Z(v) = z(v^{\frac{1}{q}}) \). Using the relations \((47)\) and \((48)\) in \((46)\) we can obtain
\[
\sqrt{q^{2\frac{1}{q}-1}}e_N \|_{0,0} \leq C_2 \|e_{I_N}(\tilde{f}) + \lambda e_{I_N}(\mathcal{K}(\tilde{u}_N)) + \lambda I_N(\mathcal{K}_\theta(\tilde{u}_N) - \mathcal{K}_{N,\theta}(\tilde{u}_N))\|_{0,\frac{1}{q}-1}
\]
\[
\leq C_2 \left( \|e_{I_N}(\tilde{f})\|_{0,\frac{1}{q}-1} + |\lambda| \|e_{I_N}(\mathcal{K}(\tilde{u}_N))\|_{0,\frac{1}{q}-1} + |\lambda| \|I_N(\mathcal{K}_\theta(\tilde{u}_N) - \mathcal{K}_{N,\theta}(\tilde{u}_N))\|_{0,\frac{1}{q}-1} \right).
\]

Since \( \delta^{0,\frac{1}{q}-1} \leq \delta^{0,0} \) then we have \( \|e_N\|_{0,0} \leq \|e_N\|_{0,0} \) and thereby \((49)\) can be written as
\[
\|e_N\|_{0,0} \leq C_3 \left( \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \right),
\]
where
\[
\mathcal{L}_1 = \|e_{I_N}(\tilde{f})\|_{0,0}, \quad \mathcal{L}_2 = \|e_{I_N}(\mathcal{K}(\tilde{u}_N))\|_{0,0}, \quad \mathcal{L}_3 = \|I_N(\mathcal{K}_\theta(\tilde{u}_N) - \mathcal{K}_{N,\theta}(\tilde{u}_N))\|_{0,0}.
\]

Now, it is sufficient to find suitable upper bounds to the above norms. According to Lemma \(4.4\) we have
\[
\mathcal{L}_1 = \|e_{I_N}(\tilde{f})\|_{0,0} \leq C_3 N^{-k_1} |\tilde{f}|_{0,0,k_1}, \quad \text{(51)}
\]
\[
\mathcal{L}_2 = \|e_{I_N}(\mathcal{K}(\tilde{u}_N))\|_{0,0} \leq C_4 N^{-k_2} \|\mathcal{K}(\bar{u} - \bar{e}_N(u))\|_{0,0,k_2}.
\]

Using Lemmas \(4.5\) and \(4.6\) we can obtain
\[
\mathcal{L}_3 = \|I_N(\mathcal{K}_\theta(\tilde{u}_N) - \mathcal{K}_{N,\theta}(\tilde{u}_N))\|_{0,0} \leq \sup_{v \in \Omega} |\mathcal{K}_\theta(\tilde{u}_N) - \mathcal{K}_{N,\theta}(\tilde{u}_N)|
\]
\[
\leq C_7 N^{-k_3} \sup_{v \in \Omega} \left( \|\tilde{K}(v, w(\theta))\|_{0,0,k_3} \|v \tilde{u}_N(w(\theta))\|_{0,0} \right)
\]
\[
\leq C_7 N^{-k_3} \left( \sup_{v \in \Omega} \|\tilde{K}(v, w(\theta))\|_{0,0,k_3} \right) \|\tilde{u}_N(w)\|_{0,0}
\]
\[
\leq C_7 N^{-k_3} \left( \sup_{v \in \Omega} \|\tilde{K}(v, w(\theta))\|_{0,0,k_3} \right) \left( \|\tilde{u}\|_{0,0} + \|\bar{e}_N\|_{0,0} \right), \quad \text{(52)}
\]

where norms \( \|\tilde{K}(v, w(\theta))\|_{0,0,k_3} \) and \( \|v \tilde{u}_N(w(\theta))\|_{0,0} \) are applied with respect to the variable \( \theta \).

For sufficiently large \( N \) the desired result can be concluded by inserting \((51)\) and \((52)\) in \((50)\). \(\square\)

6 Numerical Results

In this section we apply a program written in Mathematica to a numerical example to demonstrate the accuracy of the proposed method and effectiveness of applying GJPs. the ”Numerical Error” always refers to the \( L^2 \)-norm of the obtained error function.

Example 6.1 Consider the following FIDE
\[
D^{\frac{1}{2}} u(x) = u(x) + f(x) + \frac{1}{2} \int_0^x \sqrt{xt} u(t) dt, \quad u(0) = 0,
\]

\[
14
\]
with the exact solution $u(x) = \frac{\sin x}{\sqrt{x}}$ and

$$f(x) = -x + \sqrt{x} \left( \sqrt{x} \cos x + \sqrt{\pi} \left( J_0 \left( \frac{x}{2} \right) \cos \frac{x}{2} - J_1 \left( \frac{x}{2} \right) \sin \frac{x}{2} \right) \right) - 2 \sin x,$$

Here $J_n(z)$ gives the Bessel function of the first kind.

This example has a singularity at the origin, i.e.,

$$u'(x) \sim \frac{1}{\sqrt{x}}.$$

In the theory presented in the previous section, our main concern is the regularity of the transformed solution. For the present problem by applying coordinate transformation $x = v^2$, the infinitely smooth solution

$$\bar{u}(v) = u(v^2) = \frac{\sin v^2}{v},$$

is obtained. The main purpose is to check the convergence behavior of the numerical solutions with respect to the approximation degree $N$. Numerical results obtained are given in the Table 1 and Figure 1. As expected, the errors show an exponential decay, since in this semi-log representation(Figure 1) one observes that the error variations are essentially linear versus the degrees of approximation.

Table 1: The numerical errors of example 6.1

| $N$ | Numerical Errors |
|-----|------------------|
| 2   | $7.86 \times 10^{-3}$ |
| 4   | $4.71 \times 10^{-5}$ |
| 6   | $2.15 \times 10^{-6}$ |
| 8   | $1.05 \times 10^{-8}$ |
| 10  | $4.42 \times 10^{-10}$ |
| 12  | $2.05 \times 10^{-12}$ |
| 14  | $2.64 \times 10^{-14}$ |
| 16  | $1.91 \times 10^{-16}$ |
Figure 1: The numerical errors of example 6.1 with various values of N.

References

[1] K. E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge, (1997).

[2] F. Awawdeh, E. A. Rawashdeh, H. M. Jaradat, Analytic solution of fractional integro-differential equations, Ann. Univ. Craiova math. Comput. Sci. Ser. 38 (2011), 1-10.

[3] R. L. Bagley, P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheol., 27 (1983), 201-210.

[4] H. Brunner, Collocation Methods for Volterra and Related Functional Equations, Cambridge University Press, Cambridge, (2004).

[5] M. Caputo, Linear models of dissipation whose Q is almost frequency independent II, Geophys. J. Roy. Astron. Soc., 13 (1967), 529-539.

[6] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, Spectral Methods, Fundamentals in Single Domains, Springer-Verlag, Berlin, (2006).

[7] Y. Chen Y., T. Tang, Convergence analysis of the Jacobi spectral collocation methods for volterra integral equations with a weakly singular kernel, Math. Comput. 79 (2010), 147-167.

[8] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag Berlin, (2010).

[9] E. H. Doha, A. H. Bhrawy, S. S. Ezz-Eldien, A new Jacobi operational matrix: an application for solving fractional differential equations, Appl. Math. Model., 36 (2012), 4931-4943.

[10] M. R. Eslahchi, M. Dehghan, M. Parvizi, Application of the Collocation method for solving nonlinear fractional integro-differential equations, J. Comput. Appl. Math., 257 (2014), 105-128.

[11] F. Ghoreishi, P. Mokhtary, Spectreal Collocation method for multi-order fractional differential equations, Int. J. Comput. Math., 11 (2014), no. 5, 23pp.
[12] Ben-Yu Guo, Jie Shen, Li-Lian Wang, Optimal Spectral-Galerkin methods using Generalized Jacobi polynomials, *J. Sci. Comput.*, **27**(2006), 305-322.

[13] Ben-Yu Guo, Jie Shen, Li-Lian Wang, Generalized Jacobi polynomials/functions and their applications, *Appl. Numer. Math.*, **59**(2009), 1011-1028.

[14] L. Huang, X. F. Li, Y. L. Zhao, X. Y. Duan, Approximate solution of fractional integro-differential equations by Taylor expansion method, *Comput. Math. Appl.*, **62**(2011), 1127-1134.

[15] J. S. Hesthaven, S. Gottlieb, D. Gottlieb, *Spectral Methods for Time-Dependent Problems*, Cambridge University Press, (2007).

[16] L. V. Kantrovich, Functional analysis and applied mathematics, *Usp. Mat. Nauk*, **3** (1984), 89-185.

[17] L. V. Kantrovich, G. P. Akilov, *Functional Analysis in Normed Spaces*(Funktsional’nyi analiz v normirovannykh prostranstvakh), Fizmatgiz, Moscow, (1959).

[18] L. Huang, X. F. Li, Y. L. Zhao, X. Y. Duan, Approximate solution of fractional integro-differential equations by Taylor expansion method, *Comput. Math. Appl.*, **62**(2011), 1127-1134.

[19] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, (2006).

[20] M. M. Khader, N. H. Sweilam, On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method, *Appl. Math. Model.*, **27**(2013), no. 24, 819-9828.

[21] R. C. Mittal, R. Nigam, Solution of fractional calculus and fractional integro-differential equations by Adomian decomposition method, *Int. J. Appl. Math. Mech.*, **4**(2008), no. 4, 87-94.

[22] Xiaohua Ma, C. Huang, Numerical solution of fractional integro-differential equations by Hybrid Collocation method, *Appl. Math. Comput.*, **219**(2013), 6750-6760.

[23] Xiaohua Ma, C. Huang, Spectral Collocation method for linear fractional integro-differential equations, *Appl. Math. Model.* **38** (2014), no. 4, 1434-1448.

[24] P. Mokhtar, F. Ghoreishi, The $L^2$-convergence of the Legendre spectral Tau matrix formulation for nonlinear fractional integro differential equations, *Numer. Algorithms*, **58**(2011), 475-496.

[25] D. Nazari, S. Shahmorad, Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions, *J. Comput. Appl. Math.*, **234**(2010), 883-891.

[26] K. B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York, (1974).

[27] W. E. Olmstead, R. A. Handelsman, Diffusion in a semi-infinite region with nonlinear surface dissipation, *SIAM Rev.* **18**(1976), 275-291.

[28] I. Podlubny, *Fractional Differential Equations*, Academic Press, (1999).

[29] D. Porter, David S. G. Stirling, *Integral Equations, A Practical Treatment, from Spectral Theory to Applications*, Cambridge University Press, New York, (1990).

[30] E. A. Rawashdeh, Numerical solution of fractional integro-differential equations by Collocation method, *Appl. Math. Comput.*, **176**(2006), 1-6.
[31] Jie Shen, Tao Tang, Li-Lian Wang, *Spectral Methods, Algorithms, Analysis and Applications*, Springer, (2011).

[32] Li Xianjuan, T. Tang, Convergence analysis of Jacobi spectral Collocation methods for Abel-Volterra integral equations of second kind, *Front. Math. China*, 7(2012), no. 1, 69-84.

[33] Li Zhu, Qibin Fan, Numerical solution of nonlinear fractional order Volterra integro differential equations by SCW, *Commun. Nonlinear Sci. Numer. Simul.*, 18 (2013), no. 15, 1203-1213.

[34] G. M. Vainikko, Galerkin’s perturbation method and the general theory of approximate methods for non-linear equations, *USSR Computational Mathematics and Mathematical Physics*, 7(1967), no. 4, 1-41.