THE BLOCH-OKOUNKOV CORRELATION FUNCTIONS AT HIGHER LEVELS

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Abstract. We establish an explicit formula for the \( n \)-point correlation functions in the sense of Bloch-Okounkov for the irreducible representations of \( \widehat{\mathfrak{gl}}_\infty \) and \( \mathcal{W}_{1+\infty} \) of arbitrary positive integral levels.

1. Introduction

The representation theories of the infinite-dimensional Lie algebras \( \widehat{\mathfrak{gl}}_\infty \) and \( \mathcal{W}_{1+\infty} \) have been well developed. Among the most interesting representations are the integrable highest weight modules of a positive integral level (cf. \[1, 3, 6, 7\] and the references therein). In \[2\], Bloch and Okounkov formulated certain \( n \)-point correlation functions on the fermionic Fock space and established a beautiful closed formula for them in terms of theta functions. In light of free field realizations of \( \widehat{\mathfrak{gl}}_\infty \) and \( \mathcal{W}_{1+\infty} \), their work amounts to the study of certain trace functions on the irreducible modules of \( \widehat{\mathfrak{gl}}_\infty \) or \( \mathcal{W}_{1+\infty} \) of level 1, which can be regarded as a character formula of these modules involving all elements in the infinite-dimensional Cartan subalgebras of \( \widehat{\mathfrak{gl}}_\infty \) or \( \mathcal{W}_{1+\infty} \).

The aim of this article is to formulate and establish an explicit formula for the \( n \)-point correlation function for an arbitrary integrable highest weight module of \( \widehat{\mathfrak{gl}}_\infty \) or \( \mathcal{W}_{1+\infty} \) of any positive integral level \( \ell \in \mathbb{N} \). Our main result is Theorem 2.2, which gives an elegant simple formula for these correlation functions essentially as the product of \( \ell \) copies of the “normalized” Bloch-Okounkov \( n \)-point function of level 1 and the \( q \)-dimension formula of the corresponding integrable module of level \( \ell \). The main tool used here is a duality between finite-dimensional representations of \( \text{GL}(\ell) \) and the integrable modules of \( \widehat{\mathfrak{gl}}_\infty \) or \( \mathcal{W}_{1+\infty} \) of level \( \ell \) (cf. \[4, 6, 11\]). This duality is an infinite-dimensional generalization of the Howe duality \[5\]. Using the Bloch-Okounkov formula Theorem 2.2 is then derived by exploiting several combinatorial consequences of this duality. We also present another formula for these \( n \)-point correlation functions of higher levels involving inverse Kostka numbers. As an immediate consequence some combinatorial identities are obtained by comparing these two formulas.

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Besides the obvious representation-theoretic motivations, the interest in these $n$-point functions is stimulated by the fact that Bloch-Okounkov’s correlation functions of level 1 also have connections to combinatorics of partitions (cf. [2, 10]) and geometry (such as Gromov-Witten theory of an elliptic curve by Okounkov-Pandharipande, equivariant intersections on Hilbert schemes by Li-Qin-Wang). It would be interesting to see whether these $n$-point functions at higher levels afford similar geometric applications. We also refer to [9, 12] for a twisted analog of [2] and interesting connections to vertex operators.

The paper is organized as follows. In Section 2 we introduce the necessary notation and formulate the problem of $n$-point correlation functions of higher levels. We also recall the Bloch-Okounkov formula and conclude the section by stating our main theorem. Section 3 is divided into three subsections and its main purpose is to provide a proof of the main theorem. In Section 4 we derive another formula for these $n$-point correlation functions of higher levels involving inverse Kostka numbers and from it some combinatorial identities.

2. FORMULATION AND STATEMENT OF THE MAIN THEOREM

Recall that any finite-dimensional irreducible rational representation of the general linear group $GL(\ell)$ is obtained by taking the tensor product of an integral power of the determinant representation with an irreducible polynomial representation. Thus, the irreducible rational representations of $GL(\ell)$ are parameterized by generalized partitions of length $\ell$, where by a generalized partition $\lambda$ of length $\ell$ we mean a non-increasing sequence of integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.

A generalized partition $\lambda$ of length $\ell$ we mean a non-increasing sequence of integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Given such a $\lambda$, we denote the corresponding irreducible $GL(\ell)$-module by $V_\lambda$.

Let $gl_\infty$ denote the Lie algebra spanned by matrices of the form $(a_{ij})_{i,j \in \mathbb{Z}}$ with $a_{ij} = 0$, for $|i - j| >> 0$, that is, $gl_\infty$ is the Lie algebra of infinite matrices with finitely many non-zero diagonals. The Lie algebra $gl_\infty = \sum_{j \in \mathbb{Z}} (gl_\infty)_j$ is $\mathbb{Z}$-graded by setting $\deg(E_{ij}) = j - i$, where $E_{ij}$ denotes the matrix with 1 in the $i$th row and $j$th column and zero elsewhere. Furthermore it has a non-trivial 2-cocycle $\alpha$ defined by $\alpha(A, B) = \text{Tr}([J, A]B)$, which gives rise to a central extension $\hat{gl}_\infty$. Here $J = \sum_{i \leq 0} E_{ii}$. We shall denote by $K$ the central element of this central extension. The Lie algebra $\hat{gl}_\infty$ inherits from $gl_\infty$ the $\mathbb{Z}$-gradation $\hat{gl}_\infty = \sum_{j \in \mathbb{Z}} (\hat{gl}_\infty)_j$ with $\deg(K) = 0$, and its Cartan subalgebra is $(\hat{gl}_\infty)_0 = (gl_\infty)_0 + CK$.

Thus any element $\Lambda$ in the restricted dual $(\hat{gl}_\infty)_0^*$ gives rise to a highest weight irreducible $\hat{gl}_\infty$-module $L(\hat{gl}_\infty, \Lambda)$ of highest weight $\Lambda$.

The fundamental weights of $\hat{gl}_\infty$ are given as follows. For $i > 0$ we set $\Lambda_i(K) = 1$ and

$$\Lambda_i(E_{jj}) = \begin{cases} 1, & \text{if } 0 < j \leq i, \\ 0, & \text{otherwise}. \end{cases}$$
Now if \( i < 0 \) we set \( \Lambda_i(K) = 1 \) and
\[
\Lambda_i(E_{jj}) = \begin{cases} 
-1, & \text{if } i < j \leq 0, \\
0, & \text{otherwise}.
\end{cases}
\]
Finally we define \( \Lambda_0 \) by \( \Lambda_0(K) = 1 \) and \( \Lambda_0(E_{jj}) = 0, \) for all \( j \in \mathbb{Z}. \)

Given a positive integer \( \ell \) and a generalized partition \( \lambda = (\lambda_1, \cdots, \lambda_\ell) \) of length \( \ell, \) we denote by \( \Lambda(\lambda) \) the highest weight \( \Lambda_{\lambda_1} + \cdots + \Lambda_{\lambda_\ell}. \) The highest weight modules \( L(\hat{\mathfrak{gl}}_\infty, \Lambda(\lambda)) \) with highest weight vector \( v_{\Lambda(\lambda)} \) of level \( \ell \) are the so-called integrable modules, which are arguably the most interesting \( \hat{\mathfrak{gl}}_\infty \)-modules.

We define
\[
T(t) = \sum_{k \in \mathbb{Z}} t^{k - \frac{1}{2}} E_{kk} + \frac{1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} K,
\]
where \( t \) is an indeterminate.

Extending the definition of Bloch and Okounkov \cite{BlochOkounkov} of level 1, we define the level \( \ell \) \( n \)-point correlation function on \( L(\hat{\mathfrak{gl}}_\infty, \Lambda(\lambda)) \) by
\[
F^\ell_{\lambda}(q; t_1, \cdots, t_n) := \text{Tr}_{L(\hat{\mathfrak{gl}}_\infty, \Lambda(\lambda))}(q^H T(t_1) T(t_2) \cdots T(t_n)).
\]
Here \( H \) is the energy operator characterized by
\[
\begin{align*}
H \cdot v_{\Lambda(\lambda)} &= (\lambda^2/2) \cdot v_{\Lambda(\lambda)}, \\
[H, E_{ij}] &= (i - j) E_{ij},
\end{align*}
\]
where by definition
\[
\lambda^2 := \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_\ell^2.
\]
The (2.2) is introduced for convenience later on (see (3.3) below).

A remarkable closed formula for the \( n \)-point function at level 1 has been established in \cite{BlochOkounkov}. Before stating it we need some additional notation.

Let \( q \) denote a formal parameter. Let \( \Theta(t) \) denote the following theta function
\[
\Theta(t) = \Theta_{11}(t; q) := \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+\frac{1}{2})^2} t^{n+\frac{1}{2}}.
\]
It is agreed that \( 1/(-k)! = 0 \) for \( k > 0 \) below. Let
\[
\Theta^{(k)}(t) := (t \frac{d}{dt})^k \Theta(t),
\]
and let \( \varphi(q) \) denote the Euler product
\[
\varphi(q) = \prod_{j=1}^\infty (1 - q^j).
\]
Theorem 2.1. (Bloch-Okounkov [2]) The correlation function $F_{(0)}^1(q; t_1, \cdots, t_n)$ of level 1 equals
\[
\frac{1}{\varphi(q)} \cdot \sum_{\sigma \in S_n} \frac{\det \left( \frac{\Theta^{(j-i+1)(t_{\sigma(1)} \cdots t_{\sigma(n-j)})}}{(j-i+1)!} \right)_{i,j=1}^n}{\Theta(t_{\sigma(1)})\Theta(t_{\sigma(1)}t_{\sigma(2)} \cdots \Theta(t_{\sigma(1)}t_{\sigma(2)} \cdots t_{\sigma(n)})).}
\] (2.3)

The expression (2.3) will be denoted by $F_{BO}(q; t_1, \cdots, t_n)$. It is implicitly given in [10] that, for $k \in \mathbb{Z}$,
\[
F_{(k)}^1(q; t_1, \cdots, t_n) = q^{k^2}(t_1 \cdots t_n)^k F_{BO}(q; t_1, \cdots, t_n).
\]

Our main result is the following formula for the Bloch-Okounkov $n$-point correlation functions at level $\ell$. The proof will be given in Section 3.

Theorem 2.2. The $n$-point correlation function of level $\ell$ associated to the generalized partition $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ is given by
\[
F_{\lambda}^\ell(q; t_1, \cdots, t_n) = q^{\lambda^2} (t_1 t_2 \cdots t_n)^{\lambda_1} \prod_{1 \leq i < j \leq \ell} \left( 1 - q^{\lambda_i - \lambda_j + j - i} \right) \varphi(q)^{\ell} \times
\left( \sum_{\sigma \in S_n} \frac{\det \left( \frac{\Theta^{(j-i+1)(t_{\sigma(1)} \cdots t_{\sigma(n-j)})}}{(j-i+1)!} \right)_{i,j=1}^n}{\Theta(t_{\sigma(1)})\Theta(t_{\sigma(1)}t_{\sigma(2)} \cdots \Theta(t_{\sigma(1)}t_{\sigma(2)} \cdots t_{\sigma(n)})}. \right) ^\ell.
\]

Remark 2.1. The $W_{1+\infty}$ algebra $\hat{\mathcal{D}}$ is the central extension of the Lie algebra $\mathcal{D}$ of differential operators on the circle. Write $D = t \frac{d}{dt}$ and elements in $\mathcal{D}$ can be written as a linear combination of $t^k f_k(D)$, $k \in \mathbb{Z}$, where the $f_k$'s are polynomials in one variable. Setting the degrees of $t^{-k}, D$ and $C$ to $k, 0$ and $0$, respectively defines a $\mathbb{Z}$-grading of $\hat{\mathcal{D}}$. A Lie algebra homomorphism $\phi$ of $\hat{\mathcal{D}}$ into $\hat{\mathfrak{gl}}_\infty$, preserving the $\mathbb{Z}$-gradation, is given by
\[
\phi \left( t^k f(D) \right) = \sum_{j \in \mathbb{Z}} f(-j) E_{j-k,j}.
\]

The pullback of the integrable module $L(\hat{\mathfrak{g}}_\infty, \Lambda)$ via $\phi$ remains an irreducible $\hat{\mathcal{D}}$-module, and the most interesting $\hat{\mathcal{D}}$-modules are obtained in this way or its variant (cf. [3, 4]). In this way, the $n$-point functions introduced above can be regarded as $n$-point functions for irreducible $\hat{\mathcal{D}}$-modules of level $\ell$. 

3. Proof of the main theorem

3.1. The \((\text{GL}(\ell), \hat{\text{gl}}_{\infty})\)-Duality on \(\mathfrak{g}^\ell\). Let \(z\) be a formal indeterminate and, for \(1 \leq i \leq \ell\), let

\[
\begin{align*}
\psi^+ i(z) &= \sum_{r \in \frac{1}{2} + \mathbb{Z}} \psi^+_r z^{-r - \frac{1}{2}}, \\
\psi^- i(z) &= \sum_{r \in \frac{1}{2} + \mathbb{Z}} \psi^- r z^{-r - \frac{1}{2}}
\end{align*}
\]

denote \(\ell\) pairs of free fermionic fields whose non-trivial (super)commutation relations are given by:

\[
[\psi^+_i, \psi^-_j] = \delta_{ij} \delta_{r-s} I, \quad 1 \leq i, j \leq \ell.
\]

Let \(\mathfrak{g}^\ell\) denote the Fock space associated to these \(\ell\) pairs of fermions with vacuum vector \(|0\rangle\) satisfying

\[
\psi^\pm r |0\rangle = 0, \quad 1 \leq i \leq \ell, \quad r > 0.
\]

In terms of the generating function \(E(z, w) = \sum_{i,j \in \mathbb{Z}} E_{ij} z^i w^j\), we have a free field realization for \(\hat{\text{gl}}_{\infty}\) of level \(\ell\) (i.e. the central element \(K\) acts as \(\ell \cdot I\)) on \(\mathfrak{g}^\ell\) given by:

\[
E(z, w) = \sum_{i=1}^{\ell} :\psi^+_i(z)\psi^- i(w):.
\]

The normal ordering \(:\) here and further is defined by moving the annihilation operators to the right (up to a sign).

Put

\[
e_{ij} = \sum_{r \in \frac{1}{2} + \mathbb{Z}} :\psi^+_r \psi^-_r :; \quad 1 \leq i, j \leq \ell.
\]

Then the \(e_{ij}\)'s define a representation of the general linear Lie algebra \(\text{gl}(\ell)\) on \(\mathfrak{g}^\ell\), which lifts to an action of the Lie group \(\text{GL}(\ell)\). In particular as a \(\text{GL}(\ell)\)-module \(\mathfrak{g}^\ell\) is completely reducible.

**Proposition 3.1.** \([4]\) (cf. \([11]\)) The pair \((\text{GL}(\ell), \hat{\text{gl}}_{\infty})\) on \(\mathfrak{g}^\ell\) forms a dual reductive pair in the sense of Howe. Furthermore, as a \(\text{GL}(\ell) \times \hat{\text{gl}}_{\infty}\)-module, \(\mathfrak{g}^\ell\) is multiplicity-free and decomposes into

\[
\mathfrak{g}^\ell = \bigoplus_{\lambda} V_\ell^\lambda \otimes L(\hat{\text{gl}}_{\infty}, \Lambda(\lambda)),
\]

where \(\lambda\) is summed over all generalized partitions of length \(\ell\).

**Remark 3.1.** As explained in \([9]\), the duality \((3.1)\) extends to a \((\text{GL}(\ell), W_{1+\infty})\) duality via the homomorphism in Remark 2.1.
The energy operator $H$ on $\mathcal{F}^\ell$ can be realized as
\begin{equation}
H = \sum_{i=1}^{\ell} \sum_{r \in \frac{1}{2} \mathbb{Z} + \mathbb{Z}} r : \psi_r^{+i} \psi_r^{-i} :,
\end{equation}
which may be regarded as the zero-mode of a Virasoro field.

Similar to the definition of monomial symmetric polynomials and Schur polynomials, one may define the monomial symmetric (Laurent) polynomial $m_\lambda$ and Schur (Laurent) polynomial $s_\lambda$ associated to a generalized partition $\lambda$. The $s_\lambda(z_1, \ldots, z_\ell)$'s, where the $\lambda$'s are generalized partitions of length $\ell$, form a linear basis for $\mathbb{C}[z_1^{\pm 1}, \ldots, z_\ell^{\pm 1}] S_\ell$. Similar statement is true for the $m_\lambda(z_1, \ldots, z_\ell)$'s.

**Corollary 3.1.** We have the following combinatorial identity.
\begin{equation}
\prod_{i=1}^{\ell} \prod_{r \in \frac{1}{2} \mathbb{Z} + \mathbb{Z}} (1 + q^r z_i)(1 + q^r z_i^{-1}) = \sum_\lambda s_\lambda(z_1, \cdots, z_\ell) q^{\lambda^2} \prod_{1 \leq i < j \leq \ell} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{\varphi(q)^\ell},
\end{equation}
where $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ is summed over all generalized partitions of length $\ell$.

**Proof.** We apply the trace of the operator $q^H z_1^{e_{11}} z_2^{e_{22}} \cdots z_\ell^{e_{\ell\ell}}$ to both sides of the duality (3.2). Applying it to the left-hand side of (3.2) gives rise to
\begin{equation}
\text{Tr}_{\mathcal{F}^\ell}(q^H z_1^{e_{11}} z_2^{e_{22}} \cdots z_\ell^{e_{\ell\ell}}) = \prod_{i=1}^{\ell} \prod_{r \in \frac{1}{2} \mathbb{Z} + \mathbb{Z}} (1 + q^r z_i)(1 + q^r z_i^{-1}).
\end{equation}
On the other hand applying the operator $q^H z_1^{e_{11}} z_2^{e_{22}} \cdots z_\ell^{e_{\ell\ell}}$ to the right-hand side of (3.2) we see that $z_1^{e_{11}} z_2^{e_{22}} \cdots z_\ell^{e_{\ell\ell}}$ acts on the first tensor factor, while $q^H$ acts on the second tensor factor. Now the $q$-dimension formula for the $\hat{\mathfrak{gl}}_\infty$-module $L(\hat{\mathfrak{gl}}_\infty, \Lambda(\lambda))$ is well-known to be (cf. e.g. [3])
\begin{equation}
\text{Tr}_{L(\hat{\mathfrak{gl}}_\infty, \Lambda(\lambda))} q^H = \frac{q^{\lambda^2} \prod_{1 \leq i < j \leq \ell} (1 - q^{\lambda_i - \lambda_j + j - i})}{\varphi(q)^\ell},
\end{equation}
while $s_\lambda(z_1, \ldots, z_\ell)$ is the character of $V^\lambda_\ell$. Thus the trace operator when applied to the right-hand side of (3.2) gives rise to the right-hand side of (3.4). \qed

### 3.2. Traces of operators on $\mathcal{F}^\ell$.

It follows from (3.1) that the operator $T(t)$ in (2.10) acting on $\mathcal{F}^\ell$ is given by the formula
\begin{equation}
T(t) = \sum_{i=1}^{\ell} \sum_{r \in \frac{1}{2} \mathbb{Z}} t^r \psi_r^{+i} \psi_r^{-i}.
\end{equation}
Let $S$ be the shift operator on $\mathfrak{f}^1$, which is uniquely determined by the relations

\[ S^{-1}\psi^\pm r S = \psi^\pm r \pm \frac{1}{2}, \quad r \in \frac{1}{2} + \mathbb{Z}, \]
\[ S|0\rangle = \psi^+_\frac{1}{2}|0\rangle. \]

For $\ell = 1$, $e_{11} \in \text{GL}(1)$ coincides with the standard charge operator $C$ on $\mathfrak{f}^1$. Introduce

\[ F(z, q; t_1, \cdots, t_n) = \text{Tr}_{\mathfrak{f}^1}(z^{e_{11}} q^H T(t_1) \cdots T(t_n)). \]

We have the following lemma (compare with [10]).

**Lemma 3.1.** We have

\[ F(z, q; t_1, \cdots, t_n) = F_{BO}(q; t_1, \cdots, t_n) \sum_{k \in \mathbb{Z}} (zt_1 t_2 \cdots t_n)^k q^\frac{k^2}{2}. \]

**Proof.** Proposition 3.1 for $\ell = 1$ is simply the boson-fermion correspondence:

\[ \mathfrak{f}^1 = \sum_{k \in \mathbb{Z}} V_k^1 \otimes L(\hat{\mathfrak{gl}}_\infty, \Lambda(k)), \]

where $V_k^1$ is the 1-dimensional module over GL(1) of (highest) weight $k$. The space $L(\hat{\mathfrak{gl}}_\infty, \Lambda(k))$ is exactly the charge $k$ subspace $\mathfrak{f}^1(k)$ of $\mathfrak{f}^1$. The following commutation relations are standard (cf. e.g. [10]):

\[ S^k \mathfrak{f}^1(0) = \mathfrak{f}^1(k), \quad S^{-1}T(t)S = t T(t) \]
\[ S^{-1}CS = C + 1, \quad S^{-1}HS = H + C + \frac{1}{2}. \]

Using these relations, we compute that

\[ F(z, q; t_1, \cdots, t_n) = \sum_{k \in \mathbb{Z}} \text{Tr}_{\mathfrak{f}^1(k)}(z^C q^H T(t_1) \cdots T(t_n)) \]
\[ = \sum_{k \in \mathbb{Z}} \text{Tr}_{\mathfrak{f}^1(0)}(z^{-k} q^H T(t_1) \cdots T(t_n) S^k) \]
\[ = \sum_{k \in \mathbb{Z}} \text{Tr}_{\mathfrak{f}^1(0)}(z^{C+k} q^H t^2 + kC T(t_1) \cdots T(t_n))(t_1 \cdots t_n)^k \]
\[ = \sum_{k \in \mathbb{Z}} \text{Tr}_{\mathfrak{f}^1(0)}((zt_1 \cdots t_n)^k q^H t^2 + kC T(t_1) \cdots T(t_n)) \]
\[ = \sum_{k \in \mathbb{Z}} (zt_1 \cdots t_n)^k q^\frac{k^2}{2} \text{Tr}_{\mathfrak{f}^1(0)}(q^H T(t_1) \cdots T(t_n)) \]
\[ = F_{BO}(q; t_1, \cdots, t_n) \sum_{k \in \mathbb{Z}} (zt_1 \cdots t_n)^k q^\frac{k^2}{2}. \]

This finishes the proof. \qed
Lemma 3.2. We have the following combinatorial identity:

\[
\prod_{i=1}^{\ell} F(z_i, q; t_1, \cdots, t_n) = \sum_{\lambda} s_{\lambda}(z_1, \cdots, z_\ell) F^\ell_{\lambda}(q; t_1, \cdots, t_n),
\]

where \( \lambda \) above is summed over all generalized partitions of length \( \ell \).

Proof. The lemma follows from the computation of the trace

\[
\text{Tr}_{\mathfrak{g}^\ell}(z_1^{\epsilon_1} \cdots z_\ell^{\epsilon_\ell} q^H T(t_1) \cdots T(t_n))
\]

in two different ways. Note that all the operators involved here commute with one another.

First note that \( \mathfrak{g}^\ell \) is the tensor product of the \( \ell \) Fock spaces associated to each of the pairs of fermions \( \psi^{\pm i}(z) \), \( 1 \leq i \leq \ell \). The operators \( q^H \) and \( T(t_i) \) act diagonally on this tensor product, while \( z^{\epsilon_i} \) acts nontrivially only on the \( i \)-th tensor factor. These considerations and the definition of \( F \) give rise to the left-hand side of (3.5).

On the other hand, we may compute the trace by using the Howe duality (3.2). The \( z_1^{\epsilon_1} \cdots z_\ell^{\epsilon_\ell} \) acts on the first tensor factor \( V_{\ell, \lambda} \) only and \( q^H T(t_1) \cdots T(t_n) \) acts only on the second factor \( L(\hat{\mathfrak{gl}}_{\infty}, \Lambda(\lambda)) \). Noting that \( \text{Tr}(z_1^{\epsilon_1} \cdots z_\ell^{\epsilon_\ell}) \) on \( V_{\ell, \lambda} \) is exactly \( s_{\lambda}(z_1, \cdots, z_\ell) \), we obtain the right-hand side of (3.5). \( \square \)

3.3. Completion of the proof of Theorem 2.2. It follows from Lemma 3.1 that

\[
\prod_{i=1}^{\ell} F(z_i, q; t_1, \cdots, t_n) = F_{BO}(q; t_1, \cdots, t_n)^\ell \prod_{i=1}^{\ell} \sum_{\beta \in \mathbb{Z}} (z_i t_1 \cdots t_n)^{k_i} q^{k_i^2}.
\]

Using the celebrated Jacobi triple product identity

\[
\prod_{j=1}^{\infty} (1 - q^j)(1 + x_i q^{j-\frac{1}{2}})(1 + x_i^{-1} q^{j-\frac{1}{2}}) = \sum_{k \in \mathbb{Z}} x_i^k q^{k^2 / 2},
\]

we can rewrite (3.6), with \( x_i = z_i t_1 \cdots t_n \), as

\[
\prod_{i=1}^{\ell} F(z_i, q; t_1, \cdots, t_n) = F_{BO}(q; t_1, \cdots, t_n)^\ell \varphi(q)^\ell \prod_{i=1}^{\ell} \prod_{j=1}^{\infty} (1 + (z_i t_1 \cdots t_n) q^{j-\frac{1}{2}})(1 + (z_i t_1 \cdots t_n)^{-1} q^{j-\frac{1}{2}}).
\]
Using Corollary 3.1 now with $z_1 t_1 \cdots t_n$ replacing $z_i$, we thus obtain

$$\prod_{i=1}^{\ell} F(z_i, q; t_1, \cdots, t_n) =$$

$$\sum_{\lambda} s_{\lambda}(z_1 t_1 \cdots t_n, \cdots, z_\ell t_1 \cdots t_n) F_{BO}(q; t_1, \cdots, t_n)^\ell \cdot q^{\frac{\lambda^2}{2}} \prod_{1 \leq i < j \leq \ell} \left(1 - q^{\lambda_i - \lambda_j + j - i}\right) =$$

$$\sum_{\lambda} s_{\lambda}(z_1, \cdots, z_\ell)(t_1 \cdots t_n)^\ell F_{BO}(q; t_1, \cdots, t_n)^\ell \cdot q^{\frac{\lambda^2}{2}} \prod_{1 \leq i < j \leq \ell} \left(1 - q^{\lambda_i - \lambda_j + j - i}\right).$$

The last identity follows from the homogeneity of the Schur polynomials. Noting that the Schur functions associated to generalized partitions are linearly independent, we can now complete the proof of Theorem 2.2 by comparing the coefficient of $s_{\lambda}(z_1, \cdots, z_\ell)$ in (3.7) with the one in (3.5). \hfill $\Box$

**Corollary 3.2.** The $n$-point correlation function of the vacuum module of $\hat{\mathfrak{gl}}_\infty$ (i.e. associated to the trivial partition $(0)$) at level $\ell$ is given by

$$F_{(0)}^\ell(q; t_1, \cdots, t_n) = F_{BO}(q; t_1, \cdots, t_n)^\ell \prod_{1 \leq i < j \leq \ell} \left(1 - q^{j-i}\right).$$

The next corollary follows from Corollary 3.1 and the Jacobi triple product identity.

**Corollary 3.3.** We have the following combinatorial identity:

$$\prod_{i=1}^{\ell} \sum_{k_i \in \mathbb{Z}} z_i^k q^{\frac{k^2}{2}} = \sum_{\lambda} s_{\lambda}(z_1, \cdots, z_\ell) q^{\frac{\lambda^2}{2}} \prod_{1 \leq i < j \leq \ell} \left(1 - q^{\lambda_i - \lambda_j + j - i}\right),$$

where $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ is summed over all generalized partitions of length $\ell$.

**Remark 3.2.** In [11] several other types of Howe duality, realized on some Fock spaces, were established. They typically involve pairs consisting of a finite-dimensional Lie group of classical type and an infinite-dimensional Lie algebra, which is a classical Lie subalgebra of $\hat{\mathfrak{gl}}_\infty$. We expect that the duality method used in this paper can be extended to compute the analogous $n$-point functions for the irreducible modules of these Lie algebras (cf. [12] for results in a very special case). This direction will be pursued elsewhere.

4. **Another formula for the $n$-point functions**

Recall that the Kostka number $K_{\lambda \mu}$ (cf. [8]) associated to two partitions $\lambda$ and $\mu$ is determined by

$$s_{\lambda} = \sum_{\mu} K_{\lambda \mu} m_{\mu}.$$
Since the monomial symmetric Laurent polynomials and the Schur Laurent functions corresponding to generalized partitions are linear bases for $\mathbb{C}[z_1^{\pm 1}, \ldots, z_\ell^{\pm \ell}]$, we can define the Kostka number $K_{\lambda\mu}$ by means of (4.1), for generalized partitions $\lambda$ and $\mu$.

Let $r$ be a non-negative integer such that $\lambda + (r\ell)$ and $\mu + (r\ell)$ are partitions, where $\lambda + (r\ell)$ denotes $(\lambda_1 + r, \ldots, \lambda_\ell + r)$. One clearly has

$$m_{\lambda+(r\ell)} = (z_1 \cdots z_\ell)^r m_\lambda, \quad s_{\lambda+(r\ell)} = (z_1 \cdots z_\ell)^r s_\lambda,$$

(4.2)

In particular $K_{\lambda\mu}$ is zero unless both $\lambda$ and $\mu$ have the same size.

**Theorem 4.1.** The $n$-point correlation function associated to the generalized partition $\lambda$ of level $\ell$ is given by

$$(4.3) \quad F^\ell_\lambda(q; t_1, \cdots, t_n) = (t_1 t_2 \cdots t_n)^{|\lambda|} F_{BO}(q; t_1, \cdots, t_n)^\ell \sum_\mu q^{\frac{\mu^2}{2}} K_{\mu \lambda}^{(-1)},$$

where the summation above is over all generalized partitions $\mu$ of length $\ell$, and $(K_{\mu \lambda}^{(-1)})$ denotes the inverse Kostka matrix.

**Proof.** It follows from Lemma 3.1 that

$$(4.4) \quad \prod_{i=1}^{\ell} F(z_i, q; t_1, \cdots, t_n) = F_{BO}(q; t_1, \cdots, t_n)^\ell \sum_{(k_1, \cdots, k_\ell) \in \mathbb{Z}^\ell} (t_1 \cdots t_n)^{k_1+\cdots+k_\ell} z_1^{k_1} \cdots z_\ell^{k_\ell} q^{\sum_{i=1}^{\ell} \frac{k_i^2}{2}}.$$

We may rewrite

$$(4.5) \quad \sum_{(k_1, \cdots, k_\ell) \in \mathbb{Z}^\ell} (t_1 \cdots t_n)^{k_1+\cdots+k_\ell} z_1^{k_1} \cdots z_\ell^{k_\ell} q^{\frac{k^2}{2}} = \sum_\mu (t_1 \cdots t_n)^{|\mu|} m_\mu(z_1, \ldots, z_\ell) q^{\frac{\mu^2}{2}},$$

summed over the generalized partition $\mu$ of length $\ell$.

By recalling the definition of the Kostka numbers $K_{\lambda\mu}$ (4.1), we may rewrite the right-hand side of (4.3) as

$$ (4.6) \quad \sum_\mu (t_1 \cdots t_n)^{|\mu|} m_\mu(z_1, \ldots, z_\ell) q^{\frac{\mu^2}{2}} = \sum_{\lambda,\mu,|\lambda|=|\mu|} (t_1 \cdots t_n)^{|\mu|} q^{\frac{\mu^2}{2}} K_{\mu \lambda}^{(-1)} s_\lambda(z_1, \ldots, z_\ell).$$
By combining (4.4), (4.5) and (4.6), we arrive at

\[
(4.7) \quad \prod_{i=1}^{\ell} F(z_i, q; t_1, \cdots, t_n) = \sum_{\lambda} (t_1 \cdots t_n)^{|\lambda|} F_{BO}(q; t_1, \cdots, t_n)^{\ell} \left( \sum_{\mu: |\mu|=|\lambda|} q^{\frac{r^2}{2}} K_{\mu\lambda}^{(-1)} \right) s_\lambda(z_1, \cdots, z_\ell).
\]

Now the Schur functions \(s_\lambda(z_1, \cdots, z_\ell)\)'s are linearly independent. Hence the theorem follows from comparing (3.5) with (4.7). \(\Box\)

From Theorem 2.2 and Theorem 4.1 we obtain the following.

**Corollary 4.1.** For a generalized partition \(\lambda = (\lambda_1, \cdots, \lambda_\ell)\) of length \(\ell\) we have

\[
(4.8) \quad \sum_{\mu} q^{\frac{r^2}{2}} K_{\mu\lambda}^{(-1)} = q^{\frac{r^2}{2}} \prod_{1 \leq i < j \leq \ell} \left( 1 - q^{\lambda_i - \lambda_j + j - i} \right),
\]

where the summation \(\mu\) is over generalized partitions of the same size as \(\lambda\).

In the case of ordinary partitions Corollary 4.1 may be restated as follows.

**Corollary 4.2.** Given a partition \(\lambda\) we have the following combinatorial identity:

\[
(4.9) \quad \sum_{\mu} K_{\mu\lambda}^{(-1)} q^{\frac{r^2}{2}} + \sum_{r=1}^{\ell-1} \sum_{l(\mu) < \ell} K_{\mu,\lambda+\langle r \rangle}^{(-1)} q^{\sum_{i=1}^{r} (\mu_i - r)^2} = q^{\frac{r^2}{2}} \prod_{1 \leq i < j \leq \ell} \left( 1 - q^{\lambda_i - \lambda_j + j - i} \right),
\]

where the \(\mu\)'s are partitions and \(l(\mu)\) denotes the length of \(\mu\). In particular if \(\lambda\) is the trivial partition, we obtain

\[
\sum_{r=0}^{\ell-1} \sum_{l(\mu) < \ell} K_{\mu,\lambda+\langle r \rangle}^{(-1)} q^{\sum_{i=1}^{r} (\mu_i - r)^2} = \prod_{1 \leq i < j \leq \ell} (1 - q^{i-j}).
\]

**Proof.** The proof amounts to showing that the left-hand side of (4.8) equals the left-hand side of (4.9). This follows from the combinatorial formula (2) on page 107 in [8] for \(K_{\mu\lambda}^{(-1)}\) and (4.2) (note that we have interchanged the roles of \(\lambda\) and \(\mu\) from loc. cit.). We remark that an ingredient in the proof is the fact that \(K_{\mu\lambda} = 0\) for \(\lambda_\ell \geq \ell\) and \(l(\mu) < \ell\), which is a consequence of the combinatorial formula for \(K_{\mu\lambda}^{(-1)}\) in loc. cit.. \(\Box\)

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