Convexity of Čebyšev Sets

in Hilbert Spaces

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Abstract. The aim of this paper is state of conditions that ensure the convexity of a Čebyšev sets in Hilbert spaces.

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1. Introduction

The approximation theory is one of the important branch of functional analysis that Čebyšev originated it in nineteenth century. But, convexity of Čebyšev sets is one of the basic problems in this theory. In a finite dimensional smooth normed linear space a Čebyšev set is convex [2]. Also, every boundedly compact Čebyšev set in a smooth Banach space is convex [3,7] and in a Banach space which is uniformly smooth, each approximately compact Čebyšev set is convex [4]. In addition, in a strongly smooth space, every Čebyšev set with continuous metric projection is convex [5,6]. Regarding convexity of Čebyšev sets, there are still several open problems. It is a wellknown problem whether a Čebyšev set in a Hilbert space must be convex. Of course, in a finite-dimensional Hilbert space, every Čebyšev set is convex. For inverse, we know that every closed convex set in a strictly convex reflexive Banach space and in particular Hilbert space is Čebyšev. However, this problem that whether every Čebyšev set in a strictly convex reflexive Banach space is convex is still open.
2. Basic definitions and Preliminaries

In this section we collect some elementary facts which will help us to establish our main results.

Definition 2.1. Let \((X, \| \cdot \|)\) be a real normed linear space, \(x \in X\) and \(X^*\) be its dual space. For a nonempty subset \(K\) in \(X\), the distance of \(x\) from \(K\) is defined as \(d_K(x) = \inf \{ \| x - v \| ; v \in K \}\). The \(K\) is said to be a Čebyševo set if, each point in \(X\) has a unique best approximation in \(K\). In other words, for every \(x \in X\), there exist a unique \(v \in K\) such that \(\| x - v \| = d_K(x)\). (This concept was introduced by S. B. Stechkin in honour of the founder of best approximation theory, Čebyšev). The metric projection is given by \(P_K(x) = \{ v \in K ; \| x - v \| = d_K(x) \}\) which consists of the closest points in \(K\) to \(x\). The \(P_K\) is said to be continuous if, \(P_K(x)\) is a singleton for each \(x \in X \setminus K\) and it is sequently continuous.

Definition 2.2. A norm \(\| \cdot \|\) on \(X\) is said to be Kadec if, each weakly convergent sequence \((x_n)_{n=1}^{\infty}\) in \(X\) with the weak limit \(x \in X\) converges in norm to \(x\) whenever \(\| x_n \| \to \| x \|\) as \(n \to \infty\).

Definition 2.3. The space \(X\) is said to be strictly convex if, \(x = y\) whenever \(x, y \in S(X)\) and \(\frac{x + y}{2} \in S(X)\), where \(S(X) = \{ x \in X ; \| x \| = 1 \}\).

Remark 2.4. Every Hilbert space is strictly convex. Hence the dual of each Hilbert space is strictly convex.

Related to the notion of strict convexity, is the notion of smoothness.

Definition 2.5. For each \(x \in X\) the element \(x^* \in S(X^*)\) satisfying \(\| x \| = \langle x^*, x \rangle\) is called the support functional corresponding to \(x\) and \(X\) is smooth in a non-zero \(x \in X\) if, the support functional corresponding to \(x\) is unique.

Of course, the Hahn-Banach extension theorem, ensures the existence of at least one such support functional.

Smoothness and strict convexity are not quite dual properties. There are examples of smooth spaces whose duals fail to be strictly convex.

Theorem 2.6. Each Hilbert space is smooth.

Example 2.7. The space \(\mathbb{R}^2\) with Euclidian norm,

\[ \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2} \]

is a smooth and strictly convex space.

Theorem 2.8. If \(X\) be a reflexive and smooth space, then the dual space \(X^*\) is strictly convex.

Definition 2.9. The space \(X\) is uniformly convex if, for every sequences \((x_n)_{n=1}^{\infty}\) and \((y_n)_{n=1}^{\infty}\) we have, \(\lim_{n \to \infty} \| x_n - y_n \| = 0\) whenever, \(\lim_{n \to \infty} \| x_n + y_n \| = 2\).

Theorem 2.10. Every uniformly convex space, is strictly convex.
Convexity of Čebyšev sets in Hilbert spaces

**Remark 2.11.** [1] The inverse of theorem (2.10) is not true, necessary. For example, define a norm $||.||$ on $C[0, 1]$ by $||x||^2 = ||x||^2 + ||x||^2$, where $||.||_{\infty}$ and $||.||_2$, denote the norms of $C[0, 1]$ and $L^2[0, 1]$, respectively. Then $||.||_2$ is strictly convex but, not uniformly convex on $C[0, 1]$.

**Theorem 2.12.** [1] Each Hilbert space, is uniformly convex.

**Theorem 2.13.** [1] Every uniformly convex Banach space, is reflexive.

**Theorem 2.14.** [1] The norm of every uniformly convex space, is Kadec.

### 3. Main Results

In this section, we state the conditions that ensure the convexity of a Čebyšev sets in Hilbert spaces

**Theorem 3.1.** Let $K$ be a weakly closed set in a reflexive space $X$ with Kadec norm. Then the metric projection $P_K$ is continuous.

*Proof.* Let $x \in X \setminus K$, $v \in P_K(x)$ and suppose $(x_n)_{n=1}^{\infty} \subseteq X$, $(v_n)_{n=1}^{\infty} \subseteq P_K(x_n)$ such that $x_n \to x$ in norm. It is sufficient show that $P_K(x)$ is a singleton and $v_n \to v$ in norm. Since that $d_K$ is continuous, we have:

$$d_K(x) = ||x - v|| \leq ||v_n - v|| \leq ||x_n - v_n|| + ||x_n - x|| = d_K(x_n) + ||x_n - x|| \to d_K(x)$$

So, $\lim_{n \to \infty} ||v_n - x|| = ||x - v||$ and hence $(v_n)_{n=1}^{\infty}$ is bounded. Thus $(v_n)_{n=1}^{\infty}$ is contained in an set $A$ such that $A$ is weakly closed and boundedly in norm. Since $X$ is reflexive, the set $A$ is weakly compact. Hence there exists a weakly convergent subsequence $(v_{n_k})_{k=1}^{\infty}$ of $(v_n)_{n=1}^{\infty}$ whose weak limit $v_0$ lies in $A$: Such an $v_0$ must be in $K$. Note that the norm on a normed space is lower semicontinuous for the weak topology. Then

$$||x - v|| = d_K(x) \leq ||x - v_0|| \leq \liminf_{k \to \infty} ||v_{n_k} - x|| = d_k(x) = ||x - v||$$

This implies $v_0 = v$ and so $P_K(x)$ is a singleton. The $(x - v_{n_k})_{k=1}^{\infty}$ is weakly converges to $x - v$ and satisfies $\lim_{k \to \infty} ||v_{n_k} - x|| = ||x - v||$. Since the norm on $X$ is Kadec, the sequence $(x - v_{n_k})_{k=1}^{\infty}$ is normly convergent to $(x - v)$. Therefore, $(v_{n_k})_{k=1}^{\infty}$ converges to $v$ in norm and consequently $(v_n)_{n=1}^{\infty}$ converges to $v$ in norm. This proves that $P_K$ is continuous.

Now by the theorems (2.12), (2.13), (3.1), we have:

**Corollary 3.2.** Let $K$ be a weakly closed set in a uniformly convex Banach space $X$. Then the metric projection $P_K$ is continuous.

**Theorem 3.3.** [5,6] Every Čebyšev set $K$ with continuous metric projection $P_K$, in a Banach space $X$ with strictly convex dual $X^*$, is convex.

Now by remark (2.4) and the previous theorem, we have:

**Corollary 3.4.** Every Čebyšev set with continuous metric projection, in a Hilbert space is convex.
Now by the theorems (3.3), (2.12), (2.8) and corollary (3.2), we have:

**Theorem 3.5.** Every weakly closed Čebyšev set in a smooth uniformly convex Banach space, is convex.

Finally, by the theorems (2.6), (2.11) and the previous theorem, we have:

**Corollary 3.6.** Every weakly closed Čebyšev set in a Hilbert space, is convex.

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