Ultraslow Wave Nuclear Burning of Uranium-Plutonium Fissile Medium on Epithermal Neutrons

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Abstract

For a fissile medium, originally consisting of uranium-238, the investigation of fulfillment of the wave burning criterion in a wide range of neutron energies is conducted for the first time, and a possibility of wave nuclear burning not only in the region of fast neutrons, but also for cold, epithermal and resonance ones is discovered for the first time.

For the first time the results of the investigation of the Feoktistov criterion fulfillment for a fissile medium, originally consisting of uranium-238 dioxide with enrichments 4.38\%, 2.00\%, 1.00\%, 0.71\% and 0.50\% with respect to uranium-235, in the region of neutron energies 0.015÷10.00 eV are presented. These results indicate a possibility of ultraslow wave neutron-nuclear burning mode realization in the uranium-plutonium media, originally (before the wave initiation by external neutron source) having enrichments with respect to uranium-235, corresponding to the subcritical state, in the regions of cold, thermal, epithermal and resonance neutrons.

In order to validate the conclusions, based on the slow wave neutron-nuclear burning criterion fulfillment depending on the neutron energy, the numerical modeling of ultraslow wave neutron-nuclear burning of a natural uranium in the epithermal region of neutron energies (0.1÷7.0 eV) was conducted for the first time. The presented simulated results indicate the realization of the ultraslow wave neutron-nuclear burning of the natural uranium for the epithermal neutrons.

Introduction

Nowadays the development of the theory of the wave reactors with internal safety (reactors of Feoktistov type) \([1–4]\) as well as of the natural georeactor \([3–5]\) is topical.

In \([1]\) by an example of originally engineering uranium, irradiated by an external neutron source, for the arising uranium-plutonium fissile medium (the fertile nuclide \(^{238}\text{U}\) and the fissile nuclide \(^{239}\text{Pu}\)) L.P. Feoktistov proposed a criterion (a condition), the fulfillment of which in the neutron multiplication medium leads to a stationary wave of slow nuclear burning formation. The Feoktistov criterion consists in the condition that the equilibrium concentration of a fissile

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nuclide (for the considered neutron multiplication medium) must exceed its critical concentration. For the uranium-plutonium chain of nuclear reactions considered in [1], by equilibrium and critical concentrations of the active component $^{239}\text{Pu}$ we mean the concentrations, for which the rate of the active component nuclei creation is equal to the rate of their disappearance during nuclear reactions, and the rate of neutron birth is equal to the rate of their absorption, respectively. For the particular case considered in [1], this criterion reads: $N_{\text{equil}}^{239}\text{Pu} > N_{\text{crit}}^{239}\text{Pu}$. Since the computer modeling of the neutron multiplication medium kinetics is a very complicated problem, requiring large computational burden, the check of the Feoktistov criterion becomes the only way of the preliminary search for nuclide composition of the neutron multiplication medium and the external parameters, for which the realization of a slow nuclear burning wave is possible. However, conducting of such a search is even more complicated by the fact that the equilibrium and critical concentrations of the fissile nuclide, present in the Feoktistov criterion, are functions of the energy spectrum of neutrons, which changes with the nuclide composition and the external parameters such as the temperature, the pressure and geometry of a fissile medium.

1 Akhiezer wave of nuclear burning for the thermal neutrons

Although in [2] the other problem was solved and investigated, it is of undoubtful interest for our subject and makes a certain contribution into theoretical generalization of the possible neutron multiplicative burning processes and substantiation of the possibility of the wave burning realization in the thermal region of neutron energies in principle.

A chain nuclear reaction on thermal neutrons in the large neutron-multiplication system in the form of a cylinder or a parallelepiped of big length (one of the geometric parameters must exceed two others considerably) was considered in the one-group diffusion-age approximation in [2]. The chain nuclear reaction was initiated by the external neutron source, which was set by its neutron density flux on the cylinder butt (the problem II in [2]).

At the same time in [2] the thermal neutron density $n(\vec{r},t)$ satisfies the following diffusion integro-differential equation, which can be derived from the effective one-group equation in the diffusion-age approximation [6–8], taking into account that the so-called age of thermal neutrons $\tau$ is proportional to the mean value of the squared neutron shift during the deceleration process from the point of fast neutron birth to the point of thermal neutron birth (see e.g. [7–9]):

$$\frac{\partial n(\vec{r},t)}{\partial t} = D \Delta n(\vec{r},t) - \frac{1}{r_e} n(\vec{r},t) + \frac{k_\infty}{\pi^{3/2}r_0^3\tau_c} \int_V n(\vec{r}',t) \exp \left( -\frac{|\vec{r} - \vec{r}'|^2}{r_0^2} \right) dV'$$

(1)

where $D$ is the diffusion coefficient for thermal neutrons, $r_c$ is their lifetime with respect to capture, $k_\infty$ is the neutron infinite multiplication factor, $r_0$ is the mean neutron moderation length.

The form of the integrand in (1) supposes that all dimensions of a neutron-multiplication system are considerably larger than the neutron moderation length $r_0$. Let us note that although integration in (1) is performed over the volume occupied by the neutron-multiplication medium, in fact, because of the smallness of the mean neutron moderation length $r_0$ in comparison with typical dimensions of the system, it covers only a small neighbourhood of the point $\vec{r}'$ with the diameter of the order of $r_0$. This allows to extend the limits of integration in (1), containing the Gaussian function, to infinity with exponential accuracy.

As of the neutron multiplication factor, it was set in [2] in the form of the known four factor formula for thermal reactors (see e.g. [7–9]) and neglecting the fast multiplication factor.
\( \varepsilon \), which is close to unity for thermal reactors \((\varepsilon \cong 1.00 \div 1.03)\):

\[
k_\infty = \nu \varepsilon \theta_f \phi \approx \nu \theta_f \phi
\]

where \( \nu \) is the average number of neutrons, generated under capture of one neutron by the fuel, \( \theta_f \) is a probability of thermal neutron absorption by an uranium nucleus, \( \phi \) is the probability function for the resonance neutron absorption by uranium-238 nuclei.

However, in order to simplify the problem, only the kinetic equation for neutrons was considered, and the kinetic equations for densities of nuclides in the multiplication fissile medium were not taken into account. Otherwise one had to solve the system of kinetic equations with nonlinear feedbacks (an example of such kinetic system of 20 equations is presented below in section 5, as well as e.g. in \([1, 3–5, 10]\)).

Really, the neutron multiplication factor in \([2]\) was set at the level of the expression (2), although, as is known (see e.g. \([7–9]\)), each of the factors present in (2) depends on the composition of the fuel fissile medium and its construction, for example, \( \nu \) is set by the following expression:

\[
\nu = \frac{\sum_i \nu_i \sigma_f^i N_i}{\sum_i \sigma_a^i N_i}
\]

where \( \nu_i \) is the average number of neutrons, generated by the fission of one nucleus of the \( i^{th} \) fissile nuclide, \( \sigma_f^i \) is the fission cross-section of the \( i^{th} \) nuclide, \( \sigma_a^i \) is the neutron absorption cross-section of the \( i^{th} \) nuclide, \( N_i \) is the density of the \( i^{th} \) nuclide nuclei.

Thus, in \([2]\) the change of the fissile medium composition was not taken into account, and the process of neutron multiplication was described at the level of the neutron multiplication factor (the expressions (1) and (2)), which is known to depend on the nuclide composition of the fissile medium and cross-sections of neutron-nuclear reactions (the expression (3)). The cross-sections, in their turn, depend on the neutron energy and can change their values considerably (see e.g. figures 1 and 2). Therefore, under such a simplified description, the dependences on the fissile medium composition and the neutron energy were at the back of the multiplication factor. Naturally, this imposed corresponding restrictions on the solutions obtained and conclusions made in \([2]\).

The solution of the kinetic equation for the density of thermal neutrons (1) was obtained in \([2]\) for a semi-infinite parallelepiped \((0 \leq z \leq +\infty)\) of square section \((0 \leq x, y \leq a)\), on the butt of which \((z = 0)\) either the flux \( j_0(x, y, t) \), or the neutron density \( n_0(x, y, t) \) was set, corresponding to presence of the external neutron source at the butt (the problem II in \([2]\)). The investigation of its behavior for asymptotics with respect to time and parallelepiped length was also carried out for two particular cases of the above-critical and subcritical states of the neutron multiplication fissile medium.

Let us note that the choice of geometry of a fissile medium in the form of a cylinder or a parallelepiped of large length was made in \([2]\), because in this case the above-critical or subcritical states practically do not depend on the length, i.e. one can neglect the influence of butts, and the ratio of the surface area to the volume remains practically constant with the length change.

Following \([2]\), if the system is surrounded by vacuum, and its dimensions are large in comparison with the neutrons free path, the boundary condition at the external lateral surface \((x = 0 \div a; y = 0 \div a)\) consists in the equality of the neutron density to zero. Thus, we look for the solutions of (1) in the following form:

\[
n(r, t) = n(z, t) \sin(\pi x/a) \sin(\pi y/a)
\]
Let us note that, of course, the zero neutron flux density should be set on the extrapolated lateral surface of the system, but this inaccuracy in [2] may be disregarded, since it can be easily corrected.

Substituting (4) into the equation (1) we get the following equation for the function \( n(z, t) \):

\[
\frac{\partial n(z, t)}{\partial t} = D \frac{\partial^2 n(z, t)}{\partial z^2} - 2\frac{\pi^2 D}{a^2} n(z, t) - \frac{1}{\tau_c} n(z, t) + \frac{k_\infty}{\pi^{1/2} r_0 \tau_c} e^{-\frac{\pi^2 z^2}{a^2}} \int_0^\infty n(z', t) \exp \left( -\frac{(z - z')^2}{r_0^2} \right) dz' \tag{5}
\]

Since the diffusion flux reads \( j_z = -D \frac{\partial n(z, t)}{\partial z} \), the boundary and initial conditions for the problem II in [2] have the form:

\[
\left. n(z, t) \right|_{t=0} = 0, \quad \left. n(z, t) \right|_{z=0} = n_0(t) \quad \text{or} \quad \left. \frac{\partial n(z, t)}{\partial z} \right|_{z=0} = -\frac{j_0(t)}{D} \tag{6}
\]

At the same time the boundary conditions on the butt of the cylinder were as follows:

\[
j_0(x, y, t) = j_0(t) \sin(\pi x/a) \sin(\pi y/a); \quad n_0(x, y, t) = n_0(t) \sin(\pi x/a) \sin(\pi y/a) \tag{7}
\]

The solution of the equation (5) with the initial and boundary conditions (6) and (7) (the problem II in [2]) with the help of the direct and inverse Fourier transforms was obtained in [2] in the following form:

\[
n(z, t) = \frac{1}{\sqrt{\pi D^*}} \int_0^t j_0(t - \mu) \exp \left( \frac{A^* \mu - z^2}{4D^* \mu} \right) \frac{d\mu}{\sqrt{\mu}}, \tag{8}
\]

where

\[
D^* = D + \frac{1}{4} k_\infty \frac{r_0^2}{\tau_c} \tag{9}
\]

is called the effective diffusion coefficient [6],

\[
A^* = k_\infty - 1 - \frac{2\pi^2}{a^2} D^* \tau_c \tag{10}
\]

is the effective neutron multiplication factor.

In [2] the obtained solution (8) was studied assuming that the neutron flux is constant at the boundary of the right-angled cylinder for \( z = 0 \), i.e. it does not depend on time. In this case the solution (8) is simplified and has the following form:

\[
n(z, t) = \frac{j_0}{\sqrt{\pi D^*}} \int_0^t \frac{\exp \left( \frac{A^* \mu - z^2}{4D^* \mu} \right)}{\sqrt{\mu}} d\mu = \frac{j_0}{\sqrt{\pi D^*}} J \tag{11}
\]

where \( J \) represents a designation for the integral in (11).

Authors of [2] were first of all interested in the above-critical mode \( A^* > 0 \), and it was shown that in this case one can speak about some velocity of slow nuclear burning. In order to find this velocity, asymptotics of the expression (11) for were considered for \( z = vt, v = \text{const}, z, t \to \infty \), i.e. in observation points moving along the body axis with one or another constant velocity \( v \). The asymptotics of \( n(z, t) \) are easily reproduced, if one transforms the integral \( J \) in (11) via the substitution of variables to the form of the standard Laplace integral \( \int_a^b \varphi(u) \exp(\lambda u) du \), where \( \lambda \) is a large positive parameter, and \( u \) is a real number [2].
As shown in [2], if one introduces the quantities

\[ v_0 = 2 \sqrt{\frac{A^* D^*}{\tau_c}} \]  
\[ (12) \]

and

\[ L_0 = 2 \sqrt{\frac{D^* \tau_c}{A^*}} = \frac{4D^*}{v_0} \]  
\[ (13) \]

having dimensionalities of velocity and length, then, if the quantity \( 2z/L_0 \) in the exponential function in the integral \( J \), transformed to the form of the standard Laplace integral, is a large parameter (\( z/L_0 \gg 1 \)), integrating \( J \) by parts, we find the main term of asymptotics of the integral \( J \), and, accordingly, the asymptotics of the expression (11) for the neutron density \( n(z,t) \) for \( z = vt \to \infty, v = const \) in the following form:

\[ n(z,t) \approx j_0 \frac{L_0}{2\sqrt{\pi D^*}} \frac{L_0}{z} \exp \left[ \frac{z}{L_0} \left( \frac{v_0}{v} - \frac{v}{v_0} \right) \right] \varphi \left( \frac{1}{2} \left( \frac{v_0}{v} - \frac{v}{v_0} \right) \right) = \]

\[ = \frac{4j_0}{\sqrt{\pi D^*}} \frac{L_0}{z} \exp \left[ \frac{z}{L_0} \left( \frac{v_0}{v} - \frac{v}{v_0} \right) \right] \frac{v_0^{1/2}}{v^2 + v_0^2} \]  
\[ (14) \]

As it is seen from (14), for all constant velocities \( v < v_0 \) the quantity \( n(z,t) \) exponentially grows up with the increase of the distance \( z \) from the cylinder butt (or with time, since \( z = vt \)). At the same time in the case \( v > v_0 \), on the contrary, the quantity \( n(z,t) \) exponentially decreases with the increase of the distance \( z \). If the velocity \( v \) is equal to the velocity \( v_0 \), then an abrupt change of the asymptotics type of \( n(z,t) \) occurs: instead of being exponential, it becomes power and decreases very slowly with distance:

\[ n(z,t) \approx j_0 \sqrt{\frac{\pi}{v_0}} \frac{L_0}{\sqrt{z}} \]  
\[ (15) \]

Using the expression (14), one can find the velocity of the constant thermal neutron density propagation at large distances from the cylinder butt. Supposing \( n(z,t) = \tilde{n} \), we seek for the solution for the velocity \( z/t \) in the following form:

\[ \frac{z}{t} = v_0 + \varepsilon(z), \quad \frac{\varepsilon(z)}{v_0} \ll 1 \]  
\[ (16) \]

As a result, we get

\[ \frac{\varepsilon(z)}{v_0} \approx \frac{1}{2} \frac{L_0}{z} \ln \left( \frac{j_0}{\sqrt{\pi}v_0} \sqrt{\frac{L_0}{z}} \right), \quad \frac{z}{L_0} \gg 1 \]  
\[ (17) \]

Differentiating (16) with respect to time \( t \), we find the following expression for the instantaneous velocity \( dz/dt \):

\[ \frac{dz}{dt} = \frac{v_0 + \varepsilon(z)}{1 - \frac{\varepsilon(z)}{v_0}} \approx \frac{v_0}{4z} \left( 1 - \frac{L_0}{4z} \right) \]  
\[ (18) \]

Thus, the instantaneous velocity of the constant thermal neutron density propagation in the first order of \( L_0/z \) does not depend on the neutron density \( \tilde{n} \) and at large distances from the cylinder butt tends asymptotically to \( v_0 \) (the velocity of slow nuclear burning in a fissile medium). Really, in the case of usual slow burning, when a certain temperature is reached as a result of the reaction, the slow burning velocity is proportional to \( \sqrt{\lambda/\tau} \), where \( \lambda \) is the
thermal diffusivity, and \( \tau \) is the characteristic time of the reaction \([11, 12]\). In the considered case it is a question of achievement of some fixed neutron density at the given point, caused by multiplication as well as by neutron diffusion. According to the expression (12) for \( v_0 \), the lifetime of a neutron \( \tau_c \) plays the role of a characteristic time of the reaction, while the geometric mean from the effective diffusion coefficient \( D^* \) (9) and the effective multiplication coefficient \( A^* \) (10) plays the role of the transport factor.

In [2] the asymptotics of the neutron density \( n(z, t) \) were also considered in two important cases (of large distances and large time lapses), and the following asymptotic expressions were obtained:

- in the case of growing \( z \) and fixed \( t \), i.e. when \( z \to \infty, t = \text{const} \) and \( v/v_0 \gg 1 \) (\( v \neq \text{const} \))

\[
n(z, t) \approx \frac{4 j_0 \sqrt{D^* t^3}}{z^2} e^{A^* t/\tau_c} e^{-z^2/(4D^* t)}, \quad \frac{z^2}{4D^* t} \gg 1; \tag{19}
\]

- in the case of growing \( t \) and fixed \( z \), i.e. when \( t \to \infty, z = \text{const} \) and \( v/v_0 \ll 1 \)

\[
n(z, t) \approx \frac{2 j_0}{\sqrt{\pi v_0}} \frac{1}{\sqrt{A^* \tau_c}} e^{A^* t/\tau_c}, \quad A^* \frac{1}{\tau_c} \gg 1, \quad v = \frac{z}{t}. \tag{20}
\]

The first of the considered cases corresponds to the "instantaneous" picture of neutron density distribution over the whole cylinder length, and the second one describes the density evolution at each given fixed point \( z \). From the expressions for asymptotics (19) and (20) it is clear that at large distances the quantity \( n(z, t) \) decreases exponentially with distance, and for large lapses of time for the above-critical mode (\( A^* > 0 \)) it grows up exponentially with time because of multiplication. It is also clear that for \( A^* > 0 \) the neutron density grows up with time unlimitedly, and the quantity \( \frac{D^*}{|A^*|} \) is a typical time. This means that in this case the chain reaction of spontaneous fissions under impossibility of neutron and energy removal leads to a system explosion. As noted in [2], it is necessary, however, to keep in mind that the exponential growth of \( n(z, t) \) is related essentially to linearity of the used approximation, and if the nonlinear terms with respect to \( n(z, t) \) are taken into account, the growth intensity decreases. The authors of [2] supposed that the neutron density \( n(z, t) \) is a finite function, and the velocity of propagation of the burning front is the same as in the linear approximation.

In the case of the subcritical mode \( A^* < 0 \), when the number of neutrons, being born during the nuclei fission, is not enough for maintaining the spontaneous chain nuclear reaction, the exponents in the expressions (19) and (20) are negative and the neutron density decreases with time as well as with the increase of the distance from the cylinder butt. In this case a characteristic length of neutron propagation may be considered. The quantity \( \sqrt{D^* \tau_c/|A^*|} \) represents such length, according to [2]. The most important conclusion for us, made in [2] as a result of the analysis of behaviour for asymptotics of the solution (11) for the neutron density \( n(z, t) \) and asymptotics with respect to time and the cylinder length (19) and (20), consists in the fact that for the case of the above-critical state of the neutron-multiplication fissile medium, a possible existence of the wave burning, propagating with the constant velocity and practically unchangeable amplitude along the cylinder axis and representing superposition of processes of multiplication and diffusion of neutrons, is demonstrated, and in the case of the subcritical state of the neutron-multiplication fissile medium, the neutron process decays. It is also important, as it is shown in [2], that the wave burning velocity is defined by the formula similar to usual slow burning \([11, 12]\). The estimate of this wave burning velocity, presented in [2], amounts to \( \sim 100 \) sm/s. Let us also note that the analysis of the asymptotics behaviour with respect to time and the cylinder length showed that for the case of the above-critical state
of the neutron-multiplication fissile medium, for the values of velocities of the chain process propagation bigger than the aforesaid wave process velocity, the chain process run-away, i.e. the explosion, begins, and for the values of velocities of the chain process propagation, smaller than the aforesaid wave process velocity, the chain process decays. In [2] conditions and criteria, under which the wave burning mode, that may be called an above-critical Akhiezer wave of slow nuclear burning, can be realized, are absolutely not investigated, and the only condition, designated in this paper, is the initial above-critical state of the fissile medium for the whole cylinder. Of course, it is also very interesting to investigate the features of this wave kinetics, its stability, the degree of burn-up for fuel medium nuclides, etc.

Thus in [2] the uranium-plutonium [1–5] and thorium-uranium (see e.g. [2–5, 10]) chains of nuclear reactions, underlying slow wave burning (the slow Feoktistov wave), on which the whole conception of wave reactors of new generation is based and due to which these reactors have the property of internal safety and allow to use the originally unenriched or slightly enriched media of uranium-238 or thorium-232 as fuel, were not considered. For slow wave nuclear burning, being the subject of the given paper, it is essential that the multiplication fissile uranium-plutonium or thorium-uranium (or even thorium-uranium-plutonium) medium was initially in the subcritical state, and as it follows from the results of [2], in this case the wave chain process is not discovered.

Let us also note that the velocity of the slow Feoktistov wave is several orders of magnitude smaller than the velocity of the Akhiezer wave, and therefore the Feoktistov wave may be called a wave of ultraslow neutron-nuclear burning. Besides, the burning region in the Feoktistov wave passes temporarily into the above-critical state as a result of accumulation of the fissile nuclide (plutonium-239 or uranium-233 for the thorium-uranium cycle, or both simultaneously), and after that the total process of multiplication and diffusion of neutrons, apparently, becomes similar to the chain neutron process, considered in [2], however, here the differences are also noticeable, since in the case of the Feoktistov wave this supercriticality is local, and for the above-critical Akhiezer wave supercriticality is set initially for the whole cylinder of the fissile medium.

It is natural to note that Feoktistov and Akhiezer criteria of the wave burning realization, as it follows from the present paper, differ essentially, that also indicates that these wave processes are different.

In connection with the aforesaid it is interesting to conduct the following computer modelling experiment: in the cylindrical uranium (or uranium-plutonium) neutron-multiplication medium, being in the above-critical state with respect to uranium-235 (or plutonium-239), with the help of the permanent external neutron source to initiate the Akhiezer burning wave (the wave of uranium-235 burning), after run of which along the cylinder by the condition of non-hundred-percent burn-up of uranium-238 (we make this stipulation because the kinetics of the Akhiezer wave is not studied for the time being) after some time, typical for lighting of the Feoktistov wave, the external neutron source will initiate the Feoktistov wave, which will also propagate along the cylinder. Of course, conditions of such a computer experiment must be agreed with criteria of existence of these wave processes, and first of all with respect to neutron energies. In this connection let us note that from physical considerations it is clear that the Akhiezer wave by the corresponding initial enrichment with respect to the fissile nuclide, ensuring the above-critical state of the fissile medium, can be realized not only for thermal neutrons, as considered in [2], but also for other neutron energies, e.g. for fast neutrons. This is also confirmed by the fact that the used in [2] equation for thermal neutrons in the one-group diffusion-age approximation can be also generalized for other neutron energies (see e.g. [8, 13–15]).

A similar computer experiment may be conducted also for thorium-uranium or thorium-uranium-plutonium fissile media.)
2 Fulfillment of Feoktistov’s criterion for uranium-plutonium neutron multiplication medium and the neutron energy of 0.1 eV ÷ 1 MeV

In [1] for the uranium-plutonium medium under a number of simplifications of the kinetic system of equations for the considered process (the one-dimensional medium and the fixed neutron energy (the one-group approximation) are considered, the neutron diffusion is not taken into account, the kinetic equation for plutonium-239 is written assuming that uranium-238 turns directly into plutonium-239 with some typical time of the β-transition \( \tau_\beta \), delayed neutrons and the fissile medium temperature are not taken into account) the following expressions for the equilibrium concentration \( N_{eq}^{Pu} \) of the fissile nuclide \( ^{239}Pu \) and its critical concentration \( N_{crit}^{Pu} \) are obtained:

\[
N_{eq}^{Pu}(E_n) \approx \frac{\sigma_c^8(E_n)}{\sigma_c^{Pu}(E_n) + \sigma_f^{Pu}(E_n)} N^8 = \frac{\sigma_c^8(E_n)}{\sigma_a^{Pu}(E_n)} N^8 \tag{21}
\]

\[
N_{crit}^{Pu}(E_n) \approx \frac{\sum_{i \neq Pu} \sigma_i^8(E_n) N^i - \sum_{i \neq Pu} \nu_i \sigma_i^8(E_n) N^8}{\left(\nu_{Pu} - 1\right) \sigma_f^{Pu}(E_n) - \sigma_c^{Pu}(E_n)} \tag{22}
\]

where \( \sigma_c^i, \sigma_f^i, \sigma_a^i \) are micro-cross-sections of the neutron radiative capture reactions, fission and neutron absorption, respectively, for the \( i \)th nuclide of the fissile medium; \( \tau_\beta \) is a typical time for two β-decays, transforming \(^{239}U\) (arising under the radiation capture of neutrons by uranium \(^{238}\)) into \(^{239}Np\), and \(^{239}Np\) into \(^{239}Pu\); \( \nu_i \) and \( \nu_{Pu} \) represent average numbers of neutrons being born as a result of fission of one nucleus of the one \( i \)th nuclide and \(^{239}Pu\), respectively.

In the figures 1 and 2 the dependences of the cross-sections of nuclear reactions of fission and radiative capture on the energy of neutrons are given. In these figures the energy of neutrons ranges from 10 \(^{-5}\) eV to 10 \(^7\) eV.

According to the relationships (21) and (22), we have calculated the equilibrium \( N_{eq}^{Pu} \) and critical \( N_{crit}^{Pu} \) concentrations of \(^{239}Pu\) for the uranium-plutonium fissile medium. The results of the calculations are presented in figures 3 – 5.

The analysis of the presented in figures 3 – 5 results allows to draw a conclusion that there exist several regions of energies of neutrons 0.015 ÷ 0.05 eV (Fig. 3), 0.6 ÷ 6 eV (Fig. 3), 90 ÷ 300 eV (Fig. 4) and 0.24 ÷ 1 MeV (Fig. 5), where the Feoktistov criterion \( N_{eq}^{239}Pu > N_{crit}^{239}Pu \) holds true, i.e. in these regions of energies realization of modes of wave neutron-nuclear burning is possible.

Thus, in contrast to the conclusion about a possibility of wave neutron-nuclear burning only in the region of fast neutrons, drawn in [1] and based on estimates of the equilibrium \( N_{eq}^{Pu} \) and critical \( N_{crit}^{Pu} \) concentrations of \(^{239}Pu\) only for two values of the neutron energy (thermal 0.025 eV, fast 1 MeV), in the present paper we find out the fulfillment of the Feoktistov criterion and, consequently, a possibility of realization of modes of neutron-nuclear burning also in the region of cold, epithermal and resonance neutron energies.

Let us consider the thermal region of neutron energies. Existence of regions of neutron-nuclear burning for the neutron energies 0.015 ÷ 0.05 eV and 0.6 ÷ 6 eV, in contrast to the region of energies 0.05 ÷ 0.6 eV, where the Feoktistov criterion does not hold true (see Fig. 3) and, consequently, the mode of neutron-nuclear burning is not realized, can be explained by the presence of the resonance on the curve of the \(^{239}Pu\) radiation capture cross-section dependence on the neutron energy in the range of energies 0.05 ÷ 0.6 eV (Fig. 2) and the analytical form of the expressions (21) and (22) for the equilibrium and critical concentrations of \(^{239}Pu\). Indeed, the cross-section of the neutron radiative capture for \(^{239}Pu\) is a positive term in the denominator of
Figure 1: The dependences of fission cross-sections on the energy of neutrons for $^{239}Pu$, $^{235}U$ and $^{238}U$, taken from the database ENDF/B-VII.0.

the expression (21) for the equilibrium concentration of $^{239}Pu$, that leads to the abrupt decrease of the $^{239}Pu$ equilibrium concentration in the resonance energy range 0.05÷0.6 eV on the curve of the $^{239}Pu$ radiation capture cross-section dependence. In contrast to the expression (21), the neutron radiation capture cross-section for $^{239}Pu$ is a negative term in the denominator of the expression (22) for the critical concentration of $^{239}Pu$, which leads to the abrupt increase of the $^{239}Pu$ critical concentration in the resonance energy range 0.05÷0.6 eV on the curve of the $^{239}Pu$ radiation capture cross-section dependence.

In the region of resonance neutron energies 90÷300 eV the visual analysis of the results, presented in Fig. 4, discovers the region of possible realization of the neutron-nuclear burning mode. Let us also note that for this resonance energy region it is possible to make more correct estimation of the equilibrium and critical concentrations of $^{239}Pu$ as a result of averaging over the neutron energy spectrum.

At the same time in the region of fast neutrons 0.24÷1 MeV (Fig. 5), similar to [1], a possibility of the neutron-nuclear burning mode is confirmed.

3 Generalized Feoktistov’s criterion for Uranium-Plutonium neutron multiplication medium

Let us note that in (21) and (22) the arguments, on which the concentrations of the nuclides must depend, are not shown explicitly, since we consider kinetics of the system of neutrons and nuclides, and the arguments, on which the cross-sections depend, are also not shown, and therefore one has the impression that $N_{eq}^{Pu}$ and $N_{crit}^{Pu}$ are constants. However, this simplification
Figure 2: The dependences of radiation capture cross-sections on the energy of neutrons for $^{239}\text{Pu}$, $^{235}\text{U}$ and $^{238}\text{U}$, taken from the database ENDF/B-VII.0.

has reasonable explanation. In [1] the idea of a possibility of the nuclear burning wave existence itself was grounded, and at least approximate estimates, confirming this at least for particular cases of equilibrium $N_{\text{eq}}^{\text{Pu}}$ and critical $N_{\text{crit}}^{\text{Pu}}$ concentrations of $^{239}\text{Pu}$ were necessary. The author of [1], apparently, reasoned in the following way: since the concentration of uranium-238 in the local region of the fissile medium for the system considered in [1], being the closest to the external neutron source, only decreases from the initial concentration 100\% with the lapse of time, the maximum value of the estimate of the equilibrium concentration of plutonium-239, according to (21) (where the cross-sections are constants for the fixed neutron energy), must be obtained exactly for this initial maximum concentration of uranium-238. At the same time the concentration of plutonium-239 in the same local region grows from zero to its maximum value with the lapse of time. As mentioned above, according to the Feoktistov criterion, the relationship $N_{\text{eq}}^{^{239}\text{Pu}} > N_{\text{crit}}^{^{239}\text{Pu}}$ must hold true for the appearance of the nuclear burning wave, therefore, if this relationship does not hold true for the initial concentration of uranium-238, it will not hold true according to (21) later as well, and in such a system a nuclear burning wave should not exist. Obviously, therefore in [1] with the help of the expressions (21) and (22) for the fixed concentration of uranium-238, being equal to 100\% (engineering uranium), for two fixed neutron energies (1 MeV for fast and 0.025 eV for thermal neutrons) estimations were made, demonstrating the fulfilment of the Feoktistov criterion for fast neutrons and, consequently, also a possibility of existence of a nuclear burning wave for a fast uranium-plutonium reactor and its impossibility for a slow uranium-plutonium reactor.

However, as said above, everything is much more complicated in reality.

Taking into account the real neutron spectrum in the fissile medium and introducing the
Figure 3: The simulation of dependences of equilibrium $N_{eq}^{Pu}$ and critical $N_{crit}^{Pu}$ densities of $^{239}Pu$ on the energy of neutrons, ranging from 0.015 eV to 10.05 eV.

probability function of the neutron distribution over the energy spectrum $\rho(r, E_n, t)$, for this function we can write down the formula

$$\rho(\vec{r}, E_n, t) = \frac{\Phi(\vec{r}, E_n, t)}{\int \Phi(\vec{r}, E_n, t) dE_n} = \frac{\Phi(\vec{r}, E_n, t)}{\Phi(\vec{r}, t)}, \quad (23)$$

where the total neutron density flux reads $\Phi(\vec{r}, t) = \int \Phi(\vec{r}, E_n, t) dE_n$.

Further, if one takes into account the three-dimensional geometry, the dependence of cross-sections on the neutron energy $E_n$ and the temperature of the fissile medium $T$ (taking into account the temperature influence is obligatory, since the wave nuclear burning modes with burning down to 50% [1–5] can be realized) by preservation of the simplifying assumptions (as in [1], the kinetic equation for plutonium-239 is written assuming that uranium-238 turns directly into plutonium-239 with some typical transition time, the delayed neutrons are disregarded), which allow to preserve the general form of the expressions (21) and (22), we can rewrite them in the new form. In order to do it, we have to return to the balance equations for plutonium-239 and neutrons, similar to the same equations in [1], and from which in [1] the expressions (21) and (22) were obtained:

$$\frac{\partial N_{Pu}(\vec{r}, T, E_n, t)}{\partial t} \approx \Phi(\vec{r}, E_n, T, t) \left[ \sigma_a^8 N^8(\vec{r}, T, t) - \sigma_a^{Pu}(E_n, T) N_{Pu}(\vec{r}, T, t) \right] \quad (24)$$

and

$$\frac{\partial n(\vec{r}, T, E_n, t)}{\partial t} \approx \Phi(\vec{r}, E_n, T, t) \left[ \sum_i \nu_i \sigma_{f}^i(E_n, T) N^i(\vec{r}, T, t) - \sum_i \sigma_{a}^i(E_n, T) N^i(\vec{r}, T, t) \right] \quad (25)$$

where $n(\vec{r}, E_n, T, t)$ and $N_{Pu}(\vec{r}, T, E_n, t)$ represent phase concentrations of neutrons and plutonium, respectively, while $\Phi(\vec{r}, E_n, T, t)$ is the phase neutron density flux.

Integrating left and right hand sides of the expressions (24) and (25) over the neutron energy and dividing them by the total neutron density flux $\Phi(\vec{r}, t)$, for the total plutonium concentration $N_{Pu}(\vec{r}, T, t)$ and the total neutron density $n(\vec{r}, T, t)$, taking into account (23), we obtain the following expressions:
Figure 4: The simulation of dependences of equilibrium $N_{eq}^{Pu}$ and critical $N_{crit}^{Pu}$ densities of $^{239}Pu$ on the energy of neutrons, ranging from 10 eV to 1 keV.

Figure 5: The simulation of dependences of equilibrium $N_{eq}^{Pu}$ and critical $N_{crit}^{Pu}$ densities of $^{239}Pu$ on the energy of neutrons, ranging from 1 keV to 1 MeV.

\[
\frac{\partial N_{eq}^{Pu}(\vec{r}, T, t)}{\partial t} \approx \Phi(\vec{r}, T, t) \left[ \bar{\sigma}_8^8 N^8(\vec{r}, T, t) - \bar{\sigma}_a^{Pu}(T) N_{eq}^{Pu}(\vec{r}, T, t) \right]
\]

(26)

and

\[
\frac{\partial n(\vec{r}, T, t)}{\partial t} \approx \Phi(\vec{r}, T, t) \left[ \sum_i \nu_i \bar{\sigma}_j^i(T) N^i(\vec{r}, T, t) - \sum_i \bar{\sigma}_a^i(T) N^i(\vec{r}, T, t) \right]
\]

(27)

where $\bar{\sigma}_j^i(\vec{r}, T, t) = \int \sigma_j^i(E_n, T) \rho(\vec{r}, E_n, t) dE_n$ are the cross-sections of the $j^{th}$ nuclear reaction for the $i^{th}$ nuclide of the fissile medium averaged over the neutron energy spectrum.

Further, from (26) and (27), equating the derivatives to zero, we obtain the following expressions for the equilibrium concentration $\bar{N}_{eq}^{Pu}$ (of course, it is not an equilibrium, but some stationary concentration, but we use this term after Feoktistov) of the fissile nuclide $^{239}Pu$ and its critical concentration $\bar{N}_{crit}^{Pu}$:
\[ \tilde{N}_{eq}^{Pu}(\vec{r}, T) \approx \frac{\tilde{\sigma}_a^8(T)}{\bar{\sigma}_{Pu}^a(T)} N^8(\vec{r}, T) \]  
(28)

and

\[ \tilde{N}_{crit}^{Pu}(\vec{r}, T) \approx \sum_{i \neq Pu} \tilde{\sigma}_a^i(T) N^i(\vec{r}, T) - \sum_{i \neq Pu} \nu_i \tilde{\sigma}_f^i(T) N^i(\vec{r}, T) \]  
(29)

Thus, keeping the physical meaning of the Feoktistov criterion, we will approximate to the more realistic analysis of a possibility of a nuclear burning wave realization in the uranium-plutonium fissile medium, if we rely on the relationship

\[ \tilde{N}_{eq}^{239Pu} > \tilde{N}_{crit}^{239Pu} \]  
(30)

where \( \tilde{N}_{eq}^{239Pu} \) and \( \tilde{N}_{crit}^{239Pu} \) are set by the expressions (28) and (29).

For the calculation of the cross-sections of the \( j^{th} \) nuclear reaction for the \( i^{th} \) nuclide of the fissile medium \( \tilde{\sigma}_j^i(\vec{r}, T, t) \) averaged over the neutron energy spectrum, present in the expressions (28) and (29), on which, consequently, the fulfilment of the Feoktistov criterion (30) depends, one needs to know (be able to calculate) the energy spectrum of the slowing-down neutrons and the dependences of the nuclear reactions cross-sections on the neutron energy and the temperature of the fissile medium (the Doppler effect).

In order to calculate the cross-sections averaged over the neutron energy spectrum we used the spectrum of thermal neutrons of the WWER reactor [16].

![Energy spectrum of neutrons in the WWER](image)

Figure 6: Energy spectrum of neutrons in the WWER [16] (1 – spectrum of neutrons; 2 – Maxwellian distribution; \( T \) is a transit time (in microseconds) for some standard distance; \( E \) is the energy of neutrons, corresponding to this transit time; \( n(v) \) is the density of neutrons with the velocity \( v \)).

The equilibrium and critical concentrations of \( ^{239}Pu \) were calculated with the help of the expressions (28) and (29), into which the cross-sections of nuclear reactions averaged over the neutron spectrum of the WWER enter. The following estimates of the averaged cross-sections of fission and radiation capture for uranium-238 and plutonium-239 were obtained:

\[ \bar{\sigma}_{Pu}^c = 339.10 b, \quad \bar{\sigma}_{Pu}^f = 553.60 b, \quad \bar{\sigma}_{238}^c = 255.33 b, \quad \bar{\sigma}_{238}^f = 0.00 b. \]  
(31)
Using the obtained estimates for the averaged cross-sections for nuclides of uranium-238 and plutonium-239 according to the expressions (8) and (9) we calculated the equilibrium and critical concentrations of $^{239}Pu$ and obtained the following values:

$$\tilde{N}_{eq}^{Pu} \approx 1.302 \cdot 10^{22} \text{cm}^{-3} \quad \text{and} \quad \tilde{N}_{crit}^{Pu} \approx 1.618 \cdot 10^{22} \text{cm}^{-3}. \quad (32)$$

The obtained estimates show that the Feoktistov’s criterion does not hold true. This can be explained by the fact that the majority of neutrons of thermal WWER reactor, according to Fig. 6, corresponds to the energy range $0.05 \div 0.6 \text{ eV}$, coinciding with the resonance energy range on the curve of the $^{239}Pu$ radiative capture cross-section dependence (Fig. 2), that, as we saw in the previous section, leads to the decrease of the equilibrium concentration of $^{239}Pu$ and to the increase of the critical one.

Also we calculated the values of averaged cross-sections of fission and radiation capture for uranium-238 and plutonium-239 as well as the equilibrium and critical concentrations of $^{239}Pu$ (the expressions (28) and (29)) for the neutron energy ranges $0.015 \div 0.05 \text{ eV}$ and $0.60 \div 6.00 \text{ eV}$ of the WWER reactor spectrum, in which, as we saw in section 2 (see Fig. 3), in concordance with the obtained earlier estimates the Feoktistov criterion holds true. The following estimates were obtained:

- for the neutron energy range $0.015 \div 0.06 \text{ eV}$
  $$\tilde{N}_{eq}^{Pu} \approx 1.152 \cdot 10^{20} \text{cm}^{-3} \quad \text{and} \quad \tilde{N}_{crit}^{Pu} \approx 1.045 \cdot 10^{20} \text{cm}^{-3};$$

- for the neutron energy range $0.60 \div 6.00 \text{ eV}$
  $$\tilde{N}_{eq}^{Pu} \approx 1.632 \cdot 10^{21} \text{cm}^{-3} \quad \text{and} \quad \tilde{N}_{crit}^{Pu} \approx 1.269 \cdot 10^{21} \text{cm}^{-3};$$

As it is seen from the obtained values for the equilibrium and critical concentrations of $^{239}Pu$, in these regions of energies of the WWER reactor spectrum the generalized over the spectrum criterion of realization of the neutron-nuclear wave burning mode holds true.

4 Fulfilment of Feoktistov’s criterion for fissile medium, originally consisting of uranium-238 dioxide with enrichments $4.38\%$, $2.00\%$, $1.00\%$, $0.71\%$ and $0.50\%$ with respect to uranium-235, for the region of neutron energies $0.015 \div 10.00 \text{ eV}$

Existence of three fissile nuclides of uranium-235, plutonium-239 and uranium-233, two latter of which are fissile nuclides primarily for ultraslow neutron-nuclear burning in uranium-plutonium and thorium-uranium nuclear reactions chains, respectively, allows to consider the realization of ultraslow neutron-nuclear burning modes more complicated with respect to the initial composition of the fissile media [4], than those, which were considered in the papers by Feoktistov and Teller. For example, one may consider a fissile medium originally consisting of uranium-238 and uranium-235 with different enrichments with respect to uranium-235, that corresponds to most widespread nuclear fuel of modern nuclear reactors, a fissile medium originally consisting of uranium-238 and plutonium-239 with different enrichments with respect to plutonium-239, the fissile medium originally consisting of thorium-232 and uranium-233 with different enrichments with respect to uranium-233, as well as a fissile medium originally consisting of any...
combination of those. It is clear that such broadening of possible compositions of fissile media enables to control the possible modes of ultraslow neutron-nuclear burning.

For the fissile medium consisting of uranium-238 dioxide with enrichments 4.38%, 2.00%, 1.00%, 0.71% (natural uranium) and 0.50% with respect to uranium-235, for the neutron energy region $0.015 \div 10.00$ eV according to the expressions (21) and (22) we calculated the equilibrium and critical concentrations of $^{239}Pu$. The results of calculations are presented in Fig. 7.

The results presented in Fig. 7 indicate that, e.g. for the fissile medium, consisting of uranium-238 dioxide with the enrichment 4.38% with respect to uranium-235, the critical concentration of plutonium-239 exceeds zero only for the neutron energy range $1.8 \div 10.0$ eV (for the neutron energies less than 1.8 eV the uranium fissile medium with the enrichment 4.38%
with respect to uranium-235 is already in the above-critical state; it is well known that the natural uranium under the thermal energy of neutrons 0.025 eV has the criticality factor 1.32 (see e.g. [17]), and therefore the critical concentration of plutonium-239 in this neutron energy region, calculated by the formula (22), is negative and is not presented in Fig. 7. Practically in the whole this region of neutron energies 1.8÷6.0 eV the Feoktistov’s criterion holds true, and so does the generalized over the neutron spectrum criterion. Thus, if one forms such composition, structure and geometry of an active zone of a nuclear reactor that the neutron spectrum corresponds mainly to this neutron energy region, then realization of such wave nuclear reactor is possible.

Similar conclusions are true also for all other dependences, presented in Fig. 7.

Let us also note that the equilibrium concentration, calculated by means of the approximate expression (21), does not take into account the possibility of the initial enrichment of the uranium-238 dioxide by uranium-235 and therefore does not change under different enrichments with respect to uranium-235.

It is important to note that the results presented in Fig. 7 demonstrate the division of the considered neutron energy region into energy regions, where the Akhiezer mode of slow burning is realized (the regions, where the estimate of the critical concentration of plutonium-239 is negative, e.g. as already mentioned above for natural uranium (the enrichment 0.71% with respect to uranium-235), the region of neutron energies less than 1.8 eV) and energy regions, where the Feoktistov mode of ultraslow burning is realized (the regions, where the estimate of the critical concentration of plutonium-239 is positive and the Feoktistov criterion holds true, e.g. as already mentioned above for the natural uranium, the region of neutron energies less than 1.8÷6.0 eV) for the given compositions of the fissile medium. In Fig. 7 those regions, where the modes of slow Akhiezer burning and ultraslow Feoktistov burning are realized, are coloured differently for clearness.

Thus, the general criterion of the wave neutron-nuclear burning modes realization (both Akhiezer and Feoktistov waves) could be formulated in the following way:

• if the neutron-multiplication fissile medium is originally (before the action of the external source of neutrons) in the above-critical state, then under the action of the external source of neutrons the Akhiezer wave of slow burning is realized in it;

• if the neutron-multiplication fissile medium is originally in the critical state and if the Feoktistov criterion holds true, then under the action of the external source of neutrons the Feoktistov wave of ultraslow burning is realized in it.

The analysis of the results presented in Fig. 7 also allows to conclude that when the enrichment of uranium-238 dioxide decreases from 4.38% to 0.50% with respect to uranium-235, the broadening of the neutron energy region, where the critical concentration of plutonium-239 exceeds zero, to thermal and even cold neutron energies happens, and in all these regions the Feoktistov criterion \(N_{eq}^{239}\text{Pu} > N_{crit}^{239}\text{Pu} \) and the generalized over the neutron spectrum criterion \(\tilde{N}_{eq}^{239}\text{Pu} > \tilde{N}_{crit}^{239}\text{Pu} \) (see (30)) hold true, i.e. realization of the slow wave neutron-nuclear burning mode is possible.

For practical realization of wave reactors it is important to note that, as it follows from Fig. 7, in natural uranium dioxide (the enrichment 0.71% with respect to uranium-235) the criteria of the slow wave neutron-nuclear burning realization hold true practically for the whole regions of thermal and epithermal neutrons.

Thus, in the present paper it is concluded for the first time that creating a thermal-epithermal wave nuclear reactor with natural uranium in its various forms as a fuel is possible, which is substantiated by the calculations results.
Indeed, all publications have been discussing only different variants of wave fast neutron reactors so far. In the closing paragraph of [1] a supposition about a possibility in principle of wave burning of plutonium-239 in a heavy-water thermal natural uranium reactor is made. However, this supposition is wrong. Really, as it follows from the results presented above (Fig. 7) and their analysis, in order for the wave burning in the thermal region of neutron energies to exist, the neutron energy region, where the critical concentration of plutonium-239 is positive, should exist. And we know that natural uranium fuel (the enrichment 0.71% with respect to uranium-235) exactly in a heavy-water reactor already has supercriticality (as is known, the criticality factor for natural uranium for thermal neutrons amounts to 1.32 (see e.g. [17]) and already exceeds 1.00), that ensures its advantages over light-water reactors. Thus, as it follows from the above-stated in this section, in a heavy-water natural uranium reactor there is no region, where the critical plutonium concentration is positive, and consequently, the mode of slow wave neutron-nuclear burning of plutonium-239 cannot be realized. The same is true for the natural uranium reactors with a gas coolant. In these reactors slow wave nuclear burning is possible for fuel with so lesser enrichment with respect to uranium-235, that in the thermal energy region the plutonium-239 criticality region exists (the estimate of the critical concentration of plutonium-239 is positive). For example, uranium-238 dioxide with the enrichment 0.50% with respect to uranium-235 (see Fig. 11) or with even lesser enrichment, or engineering uranium, or spent nuclear fuel, satisfying this condition, will do.

And light-water thermal reactors with natural uranium fuel will right do for slow wave burning of plutonium-239, since natural uranium in them, as is well known, is in the subcritical state, and exactly because of this the additional enrichment of fuel 2.0%÷3.5% with respect to uranium-235 is required for such reactors operation.

5 Modeling of neutron-nuclear burning of natural uranium for the epithermal region of neutron energies

In order to confirm the validity of the afore-cited estimates and conclusions, based on the analysis of the slow wave neutron-nuclear burning criterion fulfilment depending on the neutron energy, the numerical modelling of neutron-nuclear burning in natural uranium within the epithermal region of neutron energy (0.1÷7.0 eV) was carried out.

We consider a semispace with respect to the coordinate \( x \), filled with natural uranium (99.28% of uranium-238 and 0.72% of uranium-235), which is lighted from the open surface by the neutron source. For simplicity the diffusion one-group approximation is considered (the neutron energy 1 eV). Uranium-238, absorbing a neutron, turns into uranium-239, which then as a result of two \( \beta \)-decays with a typical time of the \( \beta \)-decay \( \tau_\beta \sim 3 \) days turns into fission-active isotope of plutonium-239. As shown above in Section 4, in such a medium a slow neutron-fission wave of plutonium-239 burning can arise.

Taking into account the delayed neutrons, kinetics of such a wave is described by a system of 20 partial differential equations with nonlinear feedbacks concerning 20 functions \( n(x,t) \), \( N_5(x,t) \), \( N_6(x,t) \), \( N_8(x,t) \), \( N_{Pu}(x,t) \), \( \hat{N}_i(Pu)(x,t) \), \( \hat{N}_i(5)(x,t) \), \( \bar{N}_i(Pu)(x,t) \), \( \bar{N}_i(5)(x,t) \) of two variables \( x \) and \( t \), which can be written down in the following form.

First let us write the kinetic equation for the density of neutrons:

\[
\frac{\partial n(x,t)}{\partial t} = D \Delta n(x,t) + q(x,t),
\]

where the source volume density \( q(x,t) \) reads.
\[
q(x, t) = [\nu^{(Pu)}(1 - p^{(Pu)}) - 1] \cdot n(x, t) \cdot V_n \cdot \sigma_f^{Pu} \cdot N_{Pu}(x, t) + \\
+ [\nu^{(5)}(1 - p^{(5)}) - 1] \cdot n(x, t) \cdot V_n \cdot \sigma_5^F \cdot N_5(x, t) + \ln 2 \cdot \sum_{i=1}^{6} \left[ \frac{N_i^{(Pu)}}{T_i^{(Pu)}/2} + \frac{N_i^{(5)}}{T_i^{(5)}/2} \right] - \\
- n(x, t) \cdot V_n \cdot \left[ \sum_{5,8,9,Pu} \sigma_c^i \cdot N_i(x, t) + \sum_{i=1}^{6} \left[ \sigma_c^{i(Pu)} \cdot \tilde{N}_i^{(Pu)}(x, t) + \sigma_c^{i(5)} \cdot \tilde{N}_i^{(5)}(x, t) \right] \right] - \\
- n(x, t) \cdot V_n \cdot \left[ \sigma_c^{eff(Pu)}(x, t) \tilde{N}^{(Pu)}(x, t) + \sigma_c^{eff(5)} \tilde{N}^{(5)} + \sigma_c^{eff} \tilde{n} N(x, t) \right],
\]

where \( n(x, t) \) is the density of neutrons; \( D \) is the neutron diffusion coefficient; \( V_n \) is the neutron velocity (\( E = 3 \) eV, the one-group approximation); \( \nu^{(Pu)} \) and \( \nu^{(5)} \) represent mean numbers of instantaneous neutrons per one fission event for \( ^{239}Pu \) and \( ^{235}U \), respectively; \( N_5, N_8, N_9, N_{Pu} \) are densities of \( ^{235}U, ^{238}U, ^{239}U, ^{239}Pu \), respectively; \( \tilde{N}_i^{(Pu)} \) and \( \tilde{N}_i^{(5)} \) are densities of surplus neutron fragments of fission of nuclei \( ^{239}Pu \) and \( ^{235}U \), respectively; \( \tilde{N}_i^{(Pu)} \) and \( \tilde{N}_i^{(5)} \) are densities of all other fragments of fission of nuclei \( ^{239}Pu \) and \( ^{235}U \), respectively; \( \tilde{N}(x, t) \) is the density of accumulated nuclei of “slags”; \( \sigma_c \) and \( \sigma_f \) are micro-cross-sections of neutron radiative capture and nucleus fission reactions; the parameters \( p_i \) \( (p = \sum_{i=1}^{6} p_i) \) and \( T_i^{1/2} \), characterizing groups of delayed neutrons for main fuel fissile nuclides are known and given e.g. in [17, 18]. Let us note that while deriving the equation for \( q(x, t) \) for taking into account the delayed neutrons, the Akhiezer-Pomeranchuk method [19] was used.

The last terms in square brackets in the right hand side of (34) were set according to the method of averaged effective cross-section for “slags” [17], e.g. for fission fragments of nuclei:

\[
n(x, t)V_n \sum_{i=fission \ fragments} \sigma_c^i \tilde{N}_i(x, t) = n(x, t)V_n \sigma_c^{eff} \tilde{N}(x, t),
\]

where \( \sigma_c^{eff} \) is some effective micro-cross-section of the radiation capture of neutrons for fragments.

The kinetic equations for \( \tilde{N}^{(Pu)}(x, t) \) and \( \tilde{N}^{(5)}(x, t) \) have the following form:

\[
\frac{\partial \tilde{N}^{(Pu)}(x, t)}{\partial t} = 2 \left( 1 - \sum_{i=1}^{6} p_i^{(Pu)} \right) \cdot n(x, t) \cdot V_n \cdot \sigma_f^{Pu} \cdot N_{Pu}(x, t) - V_n \cdot n(x, t) \cdot \sigma_c^{eff(Pu)} \cdot \tilde{N}^{(Pu)}(x, t)
\]

and

\[
\frac{\partial \tilde{N}^{(5)}(x, t)}{\partial t} = 2 \left( 1 - \sum_{i=1}^{6} p_i^{(5)} \right) \cdot n(x, t) \cdot V_n \cdot \sigma_5^F \cdot N_5(x, t) - V_n \cdot n(x, t) \cdot \sigma_c^{eff(5)} \cdot \tilde{N}^{(5)}(x, t)
\]

Consequently, we obtain the following system of 20 kinetic equations:

\[
\frac{\partial m(x, t)}{\partial t} = D \Delta n(x, t) + q(x, t),
\]

where \( q(x, t) \) is given by the expression (34);

\[
\frac{\partial N_8(x, t)}{\partial t} = -V_n n(x, t) \sigma_c^8 N_8(x, t);
\]
\[
\frac{\partial N_9(x,t)}{\partial t} = V_n n(x,t) \left[ \sigma_e^8 N_8(x,t) - \sigma_e^9 N_9(x,t) \right] - \frac{1}{\tau_\beta} N_9(x,t); \quad (40)
\]

\[
\frac{\partial N_{Pu}(x,t)}{\partial t} = \frac{1}{\tau_\beta} N_9(x,t) - V_n n(x,t) (\sigma_f^{Pu} + \sigma_e^{Pu}) N_{Pu}(x,t); \quad (41)
\]

\[
\frac{\partial N_5(x,t)}{\partial t} = -V_n n(x,t) (\sigma_f^5 + \sigma_e^5) N_5(x,t); \quad (42)
\]

\[
\frac{\partial \tilde{N}_i^{(Pu)}(x,t)}{\partial t} = p_i^{(Pu)} \cdot V_n \cdot n(x,t) \cdot \sigma_f^{Pu} \cdot N_{Pu}(x,t) - \frac{\ln 2 \cdot \tilde{N}_i^{(Pu)}(x,t)}{T_{1/2}^{(Pu)}} - V_n \cdot n(x,t) \cdot \sigma_c^{eff(Pu)} \cdot \tilde{N}_i^{(Pu)}(x,t), \quad i = 1, 6; \quad (43)
\]

\[
\frac{\partial \tilde{N}_i^{(5)}(x,t)}{\partial t} = p_i^{(5)} \cdot V_n \cdot n(x,t) \cdot \sigma_f^5 \cdot N_5(x,t) - \frac{\ln 2 \cdot \tilde{N}_i^{(5)}(x,t)}{T_{1/2}^{(5)}} - V_n \cdot n(x,t) \cdot \sigma_c^{eff(5)} \cdot \tilde{N}_i^{(5)}(x,t), \quad i = 1, 6; \quad (44)
\]

\[
\frac{\partial \tilde{N}^{(Pu)}(x,t)}{\partial t} = 2 \left( 1 - \sum_{i=1}^{6} p_i^{(Pu)} \right) \cdot V_n \cdot n(x,t) \cdot \sigma_f^{Pu} \cdot N_{Pu}(x,t) - V_n \cdot n(x,t) \cdot \sigma_c^{eff(Pu)} \cdot \tilde{N}^{(Pu)}(x,t); \quad (45)
\]

\[
\frac{\partial \tilde{N}^{(5)}(x,t)}{\partial t} = 2 \left( 1 - \sum_{i=1}^{6} p_i^{(5)} \right) \cdot V_n \cdot n(x,t) \cdot \sigma_f^5 \cdot N_5(x,t) - V_n \cdot n(x,t) \cdot \sigma_c^{eff(5)} \cdot \tilde{N}^{(5)}(x,t); \quad (46)
\]

\[
\bar{N}(x,t) = V_n n(x,t) \left[ \sigma_e^9 N_9(x,t) + \sigma_e^{Pu} N_{Pu}(x,t) + \sigma_e^5 N_5(x,t) + \sum_{i=1}^{6} \left( \sigma_c^{eff(Pu)} \tilde{N}_i^{(Pu)}(x,t) + \sigma_c^{eff(5)} \tilde{N}_i^{(5)}(x,t) \right) + \sigma_c^{eff(Pu)} \tilde{N}^{(Pu)} + \sigma_c^{eff(5)} \tilde{N}^{(5)}(x,t) \right] + \sum_{i=1}^{6} \left( \frac{\tilde{N}_i^{(Pu)}}{T_{1/2}^{(Pu)}} + \frac{\tilde{N}_i^{(5)}}{T_{1/2}^{(5)}} \right) \ln 2; \quad (47)
\]

where \(\bar{N}(x,t)\) is the total number of “slagging” nuclei, while \(\tau_\beta\) is the lifetime of the nucleus with respect to the \(\beta\)-decay.

The boundary conditions:

\[
n(x,t)|_{x=0} = \frac{\Phi_0}{V_n} \quad \text{and} \quad n(x,t)|_{x=l} = 0; \quad (48)
\]

where \(\Phi_0\) is the flux density of neutrons, created by a plane diffusion source of neutrons, situated at the border \(x = 0\); \(l\) is the length of a block of natural uranium, set while modelling.

The initial conditions:
\begin{align*}
n(x, t)_{|x=0, t=0} &= \frac{\Phi_0}{V_n}, \quad \text{and} \quad n(x, t)_{|x\neq0, t=0} = 0; \\
N_8(x, t)_{|t=0} &= 0.9928 \cdot \frac{\rho_8 N_A}{\mu_8} \approx 0.9928 \cdot \frac{19}{238} N_A \quad \text{and} \quad N_5(x, t)_{|t=0} \approx 0.7200 \cdot \frac{19}{238} N_A, \quad (50)
\end{align*}
where \( \rho_8 \) is the mass density \((g/cm^3)\) of uranium-238, \( \mu_8 \) is the molar mass \((g \cdot \text{mole}^{-1})\) of uranium-238, \( N_A \) is the Avogadro number;

\begin{align*}
N_9(x, t)_{|t=0} &= 0, \quad N_{Pu}(x, t)_{|t=0} = 0, \quad \tilde{N}_i^{(Pu)}(x, t)_{|t=0} = 0, \\
\tilde{N}_i^{(5)}(x, t)_{|t=0} &= 0, \quad \tilde{N}_i^{(Pu)}(x, t)_{|t=0} = 0, \quad \tilde{N}_i^{(5)}(x, t)_{|t=0} = 0. \quad (51)
\end{align*}

The numerical solution of the system of equations (38) – (47) with the boundary and initial conditions (48) – (51) was performed in Mathematica 8.

For optimization of the process of the numerical solution of the system of equations we passed to dimensionless quantities according to the following relationships:

\begin{align*}
n(x, t) = \frac{\Phi_0}{V_n} n^*(x, t), \quad N(x, t) = \frac{\rho_8 N_A}{\mu_8} N^*(x, t). \quad (52)
\end{align*}

The model calculations were carried out for several variants of setting the constant coefficients of differential equations. Below in the present paper the results of two model calculations are presented.

For the first calculation the following numerical values were set for the constant coefficients of the differential equations:
\[ D = 2.0 \cdot 10^4 \text{ cm}^2/\text{s}; \quad V_n = 1.0 \cdot 10^6 \text{ cm/s}; \quad \Phi_0 = 1.0 \cdot 10^{23} \text{ cm}^{-2}\text{s}^{-1}; \quad \tau_\beta \sim 3.3 \text{ days}; \]
\[ \nu^{(\text{Pu})} = 2.90; \quad \nu^{(\text{5})} = 2.41; \]
\[ \sigma_f^{\text{Pu}} = 477.04 \cdot 10^{-24} \text{ cm}^2; \quad \sigma_f^{\text{5}} = 286.15 \cdot 10^{-24} \text{ cm}^2; \quad \sigma_f^{\text{8}} = 252.50 \cdot 10^{-24} \text{ cm}^2; \]
\[ \sigma_f^{\text{5}} = 136.43 \cdot 10^{-24} \text{ cm}^2; \quad \sigma_f^{\text{5}} = 57.61 \cdot 10^{-24} \text{ cm}^2; \quad \sigma_f^{\text{9}} = 4.80 \cdot 10^{-24} \text{ cm}^2; \]
\[ \sigma_c^{\text{eff}(\text{Pu})} = 1.10 \cdot 10^{-24} \text{ cm}^2; \quad \sigma_c^{\text{eff}(\text{Pu})} = 1.00 \cdot 10^{-24} \text{ cm}^2, \quad i = 1.6; \]
\[ \sigma_c^{\text{eff}(\text{5})} = 1.10 \cdot 10^{-24} \text{ cm}^2; \quad \sigma_c^{\text{eff}(\text{5})} = 1.00 \cdot 10^{-24} \text{ cm}^2, \quad i = 1.6; \]

\[ T^{(\text{Pu})}_1 = 56.28 \text{ s}; \quad T^{(\text{Pu})}_2 = 23.04 \text{ s}; \quad T^{(\text{Pu})}_3 = 5.60 \text{ s}; \]
\[ T^{(\text{Pu})}_4 = 2.13 \text{ s}; \quad T^{(\text{Pu})}_5 = 0.62 \text{ s}; \quad T^{(\text{Pu})}_6 = 0.26 \text{ s}; \]
\[ p^{(\text{Pu})}_1 = 0.0072 \cdot 10^{-3}; \quad p^{(\text{Pu})}_2 = 0.626 \cdot 10^{-3}; \quad p^{(\text{Pu})}_3 = 0.444 \cdot 10^{-3}; \]
\[ p^{(\text{Pu})}_4 = 0.685 \cdot 10^{-3}; \quad p^{(\text{Pu})}_5 = 0.180 \cdot 10^{-3}; \quad p^{(\text{Pu})}_6 = 0.093 \cdot 10^{-3}; \]
\[ p^{(\text{Pu})} = \sum_{i=1}^{6} p^{(\text{Pu})}_i = 0.0021; \quad \sigma_c^{\text{eff}} = 1.10 \cdot 10^{-24} \text{ cm}^2, \]

\[ T^{(5)}_1 = 55.72 \text{ s}; \quad T^{(5)}_2 = 22.72 \text{ s}; \quad T^{(5)}_3 = 6.22 \text{ s}; \]
\[ T^{(5)}_4 = 2.30 \text{ s}; \quad T^{(5)}_5 = 0.61 \text{ s}; \quad T^{(5)}_6 = 0.23 \text{ s}; \]
\[ p^{(5)}_1 = 0.210 \cdot 10^{-3}; \quad p^{(5)}_2 = 1.400 \cdot 10^{-3}; \quad p^{(5)}_3 = 1.260 \cdot 10^{-3}; \]
\[ p^{(5)}_4 = 2.520 \cdot 10^{-3}; \quad p^{(5)}_5 = 0.740 \cdot 10^{-3}; \quad p^{(5)}_6 = 0.27 \cdot 10^{-3}; \]
\[ p^{(5)} = \sum_{i=1}^{6} p^{(5)}_i = 0.0064; \quad l = 100 \text{ cm}; \quad (53) \]

Let us note that the aforesaid cross-sections for the neutron-nuclear reactions for nuclides were set by their values averaged over the epithermal region of neutron energies (0.1÷7.0 eV).

During the calculation, the results of which are presented below in figures 8–12, the length of the fissile medium, where the wave of neutron-nuclear burning propagates, amounts to 100 cm, the full time of modelling is \( t = 30 \text{ min} \), the temporal step is \( \Delta t = 10 \text{ s} \), the spatial coordinate step is \( \Delta x = 0.01 \text{ cm} \).

Of course, one would like to carry out the calculation for considerably longer computer experiment, to have the temporal step \( \Delta t \approx 10^{-5} \div 10^{-7} \text{ s} \) and to consider the external source of neutrons with the smaller flux density, but the authors of the paper, while choosing the aforesaid parameters for the calculation, were confined to their available computational resources.

The results of numerical modelling of the wave neutron-nuclear burning in natural uranium within the epithermal region of neutron energies (0.1÷7.0 eV) presented in figures 8–12 indicate the realization of such a mode. Indeed in Fig. 12 we can clearly see the wave burning of plutonium-239. At the same time, according to Fig. 9 and Fig. 10, uranium-238 and uranium-235 burn down practically completely. It should be noted that the results for the neutron density kinetics presented in Fig. 8 do not demonstrate the neutron wave, in contrast to our results (e.g. [3–5, 10]) published earlier, for neutron-nuclear burning of uranium-238 for fast neutrons (with the energy of the order of 1 MeV). The authors explain this by the following fact. Since the system of differential equations was solved numerically for dimensionless (according to the relationships (51)) variables, and while making them dimensionless, the density of neutrons was divided by the flux density of the external source, which was set by a specially overrated
value $\Phi_0 = 1.0 \cdot 10^{23} \text{ cm}^{-2} \text{s}^{-1}$ for the purpose of reducing the computation time, the difference between the scales of the external source flux density and the flux density of neutrons in the region of nuclear burning in the steady-state regime of wave burning does not allow to see the neutron wave. It is also possible that the wave burning of plutonium-239 is not visible in Fig. 8 for the density of neutrons, since burning of uranium-235 is superimposed on it. Really, the results of kinetics modelling for the uranium-235 nuclei density, presented in Fig. 10, show that uranium-235 burns down practically completely, and its concentration, being equal to 0.7%, is almost three orders bigger than the amplitude of the steady wave concentration of plutonium-239, which according to Fig. 12 equals 0.001%.

Let us emphasize that according to the results presented in Fig. 12, the wave of slow neutron-nuclear burning of plutonium-239 has been practically formed during the time of modelling, being equal to 18 minutes (the time of the wave lighting). The shorter time of the wave lighting in comparison with the results of modelling of slow wave burning for fast neutrons (see e.g. [3–5, 10]) is explained by the value of the cross-section for the radiation capture of neutrons for uranium-238, set for the epithermal region amounting to 252.5 barn (see (52)), being two orders higher than the corresponding value for fast neutrons. It should be noted that since the cross-section for the radiation capture of neutrons with energy 7 eV for uranium-238 has a resonance $\sim 10^4 \text{ barn}$ (see Fig. 2), the displacement of the maximum of the neutron energy spectrum closer to 7 eV can reduce the lighting time or the neutron flux density, created by the external (lighting) source by two more orders.

For the second calculation the same (aforesaid) constant coefficients of differential equations were set, as for the first calculation, with the exception of the effective cross-sections of neutron radiation capture reactions for fragments and slags. In this calculation the effective cross-
sections of the neutron radiation capture reactions for fragments and slags were increased by one order in comparison with the first calculation and had the following values:

\[ \sigma^\text{eff}(\text{Pu}) = 10.1 \cdot 10^{-24} \text{ cm}^2; \quad \sigma^\text{eff}(5) = 10.1 \cdot 10^{-24} \text{ cm}^2; \quad \sigma^\text{eff} = 10.1 \cdot 10^{-24} \text{ cm}^2; \]  

(54)

These values were set according to the data of the base ENDF/B-VII.0 for the cross-sections of the neutron radiation capture for products of the fissile uranium-plutonium medium for the considered epithermal region of neutron energies.

The length of the fissile medium, where the wave of neutron-nuclear burning propagates, is equal to 1000 cm, the full time of modelling is \( t = 60 \text{ days} \), the temporal step is \( \Delta t = 300 \text{ s} \), the spatial coordinate step is \( \Delta x = 0.001 \text{ cm} \).

The results of the second calculation are presented below in figures 13-18.

The presented results of the second numerical modelling of the wave neutron-nuclear burning of a natural uranium in the epithermal region of neutron energies (0.1 ÷ 7.0 eV) also indicate the realization of such a mode. For example, in Fig. 17 we can see the wave burning of plutonium-239. The time of the steady autowave burning establishing for plutonium is 45 ÷ 48 days, and the velocity of the steady wave burning of plutonium in this case equals \( u_{\text{umer}} \approx 1.85 \cdot 10^{-5} \text{ cm/s} \) (see Fig. 17), being two orders smaller than the velocity of the wave burning of plutonium for the first numerical calculation (see Fig. 12). Thus, the increase of the radiation capture effective cross-section for slags by one order in comparison with the first calculation has led to decrease of the plutonium burning wave velocity by two orders. At the same time the maximum plutonium concentration in the wave grew up to 15%.
Figure 10: Kinetics of the uranium-235 nuclei density under wave neutron-nuclear burning of natural uranium. The dependence of the dimensionless density of uranium-235 nuclei on the spatial coordinate $N^*_5(x)$ for the moments of time (a) $t = 6 \text{ min}$; (b) $t = 12 \text{ min}$; (c) $t = 18 \text{ min}$; (d) $t = 24 \text{ min}$.

6 Estimate of the slow neutron-nuclear burning rate for the thermal region of neutron energies

According to the theory of a soliton-like neutron wave of slow nuclear burning, developed on basis of the theory of quantum chaos in [10], the velocities of neutron-nuclear burning must satisfy the Wigner distribution. The phase velocity $u$ of the soliton-like neutron wave of nuclear burning is determined by the following approximate equality:

$$\Lambda(a_*) = \frac{u \tau_\beta}{2L} \approx \left(\frac{8}{3\sqrt{\pi}}\right)^6 a_*^4 \exp\left(-\frac{64}{9\pi}a_*^2\right), \quad a_*^2 = \frac{\pi^2}{4} \cdot \frac{N_{Pu}^{eq} - N_{Pu}^{crit}}{N_{Pu}^{eq}} , \quad (55)$$

where $\Lambda(a_*)$ is a dimensionless invariant, depending on the parameter $a_*$; $N_{Pu}^{eq}$ and $N_{Pu}^{crit}$ are the equilibrium and critical concentrations of $^{239}Pu$, $L$ is the mean free path of neutrons, $\tau_\beta$ is the delay time, connected with birth of an active (fissile) isotope and equal to the effective period of the $\beta$-decay of compound nuclei in the Feoktistov uranium-plutonium cycle.

For the purpose of checking the velocity of slow neutron-nuclear burning of natural uranium in the epithermal region of energies of neutrons $(1.0 \div 7.0 \text{ eV})$, obtained during numerical modelling, the corresponding equilibrium and critical concentrations for plutonium-239 were calculated according to the expressions (28) and (29), and with their help, using the relationship (55), the estimates of the parameter $a_*^2$ and the invariant $\Lambda(a_*)$ were made.

In order to calculate the equilibrium and critical concentrations for plutonium-239 according to the expressions (28) and (29), for the epithermal region of neutron energies $(1.0 \div 7.0 \text{ eV})$ we
preliminarily calculated the cross-sections of neutron-nuclear reactions averaged over the neutron energy spectrum, present in the expressions (28) and (29). The averaging of the neutron-nuclear reactions cross-sections in the epithermal region of neutron energies (1.0÷7.0 eV) was carried out over the neutron spectrum, obtained from the spectrum of WWER neutrons, presented in Fig. 6, by such its displacement to the epithermal region that the maximum of the neutrons spectrum corresponded to their energy of 3 eV. The following values were obtained:

\[
\begin{align*}
\bar{\sigma}^{Pu}_{c} &= 5.43 \; b, \; \bar{\sigma}^{Pu}_{f} = 26.05 \; b, \; \bar{\sigma}^{238}_{c} = 251.68 \; b, \; \bar{\sigma}^{238}_{f} = 0.0001 \; b, \\
\bar{\sigma}^{235}_{c} &= 49.41 \; b, \; \bar{\sigma}^{235}_{f} = 60.19 \; b, \; \bar{\sigma}^{239}_{c} = 4.68 \; b, \; \bar{\sigma}^{239}_{f} = 1.64 \; b, \\
\tilde{N}_{eq}^{Pu} &\approx 3.852 \cdot 10^{23} \; 1/cm^3; \; \text{and} \; \tilde{N}_{crit}^{Pu} \approx 4.097 \cdot 10^{22} \; 1/cm^3; \\
a_* &= 0.5418 \; \text{and} \; \Lambda(a_*) = 0.5144.
\end{align*}
\]  

The obtained estimates of the parameter \(a_*\) and the invariant \(\Lambda(a_*)\) for slow neutron-nuclear burning of the natural uranium in the epithermal region of neutron energies (1.0÷7.0 eV) are presented in Fig. 13.

At the same time the estimate of the phase velocity of neutron-nuclear burning of natural uranium in the epithermal region of neutron energies, obtained by means of the numerical modelling results presented in section 5 (see e.g. Fig. 12) is approximately equal to

\[
\begin{align*}
u_{\text{numer}} &\approx 1 \; \text{cm}/(12 \cdot 60 \; \text{s}) \approx 1 \cdot 10^{-3} \; \text{cm/s}.
\end{align*}
\]
Figure 12: Kinetics of the plutonium-239 nuclei density under wave neutron-nuclear burning of natural uranium. The dependence of the dimensionless density of plutonium-239 nuclei on the spatial coordinate $N_{Pu}^*$ for the moments of time (a) $t = 6$ min; (b) $t = 12$ min; (c) $t = 18$ min.

The mean free path for neutrons of the indicated epithermal energy region is equal to

$$L = \frac{1}{\Sigma_a} = \frac{1}{\bar{\sigma}^a N_a(t = 0)} \approx \frac{1}{4.68 \cdot 10^{-24} \, cm^2 \cdot 0.48 \cdot 10^{23} \, cm^{-3}} \approx 4.45 \, cm. \quad (58)$$

Here it should be noted that in the previous expression (57) for the estimation of the averaged free path for neutrons we used the value of the radiation capture cross-section for uranium-239 instead of the radiation capture cross-section for uranium-238, since under the set neutron flux density uranium-238 changes to uranium-239 in a very short time as a result of the neutron radiative capture reaction.

Thus, on basis of the numerical modeling results we get the following estimate of the invariant of nuclear burning:

$$\frac{u_{\text{numer}}}{L} \approx \frac{1.00 \cdot 10^{-3} \, cm/s}{4.45 \, cm} \approx 2.25 \cdot 10^{-4} \, s^{-1} \quad (59)$$

The estimate of $u/L$ can be also obtained from the expression (30) if the values of the invariant $\Lambda$ presented in (56) and $\tau_\beta$ are known. Indeed, according to (55) we obtain

$$\frac{u}{L} = \frac{2 \cdot \Lambda}{\tau_\beta} \approx \frac{2 \cdot 0.51}{2.85 \cdot 10^5 \, s} \approx 3.6 \cdot 10^{-4} \, s^{-1}. \quad (60)$$

The comparison of the values (59) and (60) allows to draw a conclusion about their good agreement. The difference between the value of the invariant (59), obtained from the results of numerical modelling, and the theoretical estimate (60) can be caused by the fact that we used the one-group diffusion approximation of the neutron transport theory for the numerical modelling. Owing to this, we set the values of the neutron-nuclear reactions cross-sections (53), obtained by averaging over the range of epithermal energies ($0.1 \div 7.0$ eV) for the input data without taking into account the form of the neutron energy spectrum. We set the values corresponding to the energy of 1 eV for the diffusion coefficient and the neutron velocity.
Figure 13: Kinetics of neutrons under wave neutron-nuclear burning of natural uranium. The dependence of the dimensionless density of neutrons on the spatial coordinate $n^*(x)$ for (a) $t = 12$ days; (b) $t = 24$ days; (c) $t = 36$ days; (d) $t = 48$ days; (e) $t = 60$ days).

It should be noted that the decrease of the plutonium wave burning velocity in the second modelling calculation in comparison with the first one, caused by the increase of the effective cross-section of the neutron radiation capture for fragments of fission and “slags” by an order of magnitude, agrees well with the expression for the phase velocity of wave burning set by the Wigner distribution (55) and the expressions (28) and (29) (or the similar ones (21) and (22)) for the equilibrium and critical plutonium concentrations. The increase of the effective cross-section of neutron radiation capture for fragments of fission and “slags”, according to the expression (28), does not change the estimate for the equilibrium plutonium concentration and, according to the expression (29), leads to the increase of the first term in the numerator of the expression (29), leading to the increase of the estimate for the critical plutonium concentration. The increase of the critical plutonium concentration for the constant equilibrium plutonium concentration and under fulfilment of the burning condition $\tilde{N}_{eq}^{239Pu} > \tilde{N}_{crit}^{239Pu}$, according to the expression (55) for the parameter $a_\alpha$, leads to its increase, since the numerator grows up while the denominator goes down, and such parameter increase causes the decrease of the phase velocity of plutonium wave burning also according to the expression (55).
Figure 14: Kinetics of the uranium-238 nuclei density under wave neutron-nuclear burning of natural uranium. The dependence of the dimensionless uranium-238 nuclei density on the spatial coordinate $N_8^*(x)$ for (a) $t = 12$ days; (b) $t = 24$ days; (c) $t = 36$ days; (d) $t = 48$ days; (e) $t = 60$ days).

Conclusions

A general criterion of the wave modes of neutron-nuclear burning realization for both Akhiezer and Feoktistov waves is formulated for the first time.

The investigation of the wave burning criterion fulfilment for a fissile medium originally consisting of uranium-238 in a wide range of neutron energies is conducted for the first time. The possibility of the wave nuclear burning not only in the region of fast neutrons, but also for cold, epithermal and resonance ones is also discovered for the first time.

The results of the investigation of the Feoktistov criterion fulfilment for a fissile medium, originally consisting of uranium-238 dioxide with enrichments 4.38%, 2.00%, 1.00%, 0.71% and 0.50% with respect to uranium-235, in the region of neutron energies $0.015 \div 10.00$ eV are presented for the first time. These results indicate a possibility of the ultraslow wave neutron-nuclear burning of the uranium-plutonium media, originally (before the wave mode initiation with some external neutron source) having enrichments with respect to uranium-235, corresponding to the subcritical state, in the regions of cold, thermal, epithermal and resonance neutrons.
Figure 15: Kinetics of the uranium-235 nuclei density under wave neutron-nuclear burning of natural uranium. The dependence of the dimensionless uranium-235 nuclei density on the spatial coordinate $N_5^*(t, x)$ for (a) $t = 12 \text{ days}$; (b) $t = 24 \text{ days}$; (c) $t = 36 \text{ days}$; (d) $t = 48 \text{ days}$; (e) $t = 60 \text{ days}$.

In order to confirm the validity of the conclusions based on the analysis of the slow wave neutron-nuclear burning criterion fulfilment depending on the neutron energy, the numerical modelling of the ultraslow wave neutron-nuclear burning in the natural uranium within the epithermal region of neutron energies ($0.1 \div 7.0 \text{ eV}$) was conducted for the first time. The presented results of the numerical modelling of such conditions indicate the realization of such a mode.

A conclusion about the possibility of creating the thermal-epithermal wave nuclear reactor in which a natural uranium in its various forms can be used as fuel, substantiated by the calculations results, is made for the first time. We also make a conclusion that light-water thermal reactors with natural uranium fuel are appropriate for slow wave burning of plutonium-239. In a heavy-water natural uranium reactor there is no region, where the critical plutonium concentration is positive and, consequently, the mode of slow wave neutron-nuclear burning (the Feoktistov wave) cannot be realized. The same is true for the natural uranium reactors with a gas coolant. In these reactors slow wave nuclear burning is possible for fuel with so small enrichment with respect to uranium-235, that in the thermal energy region the plutonium-239 criticality region exists. For example, engineering uranium or already spent nuclear fuel with appropriate burn-up will do.

Thus we make the conclusion about a possibility of creation of wave nuclear reactors on cold,
Figure 16: Kinetics of the uranium-239 nuclei density under wave neutron-nuclear burning of natural uranium. The dependence of the dimensionless uranium-239 nuclei density on the spatial coordinate $N_9^*(x)$ for (a) $t = 12$ days; (b) $t = 24$ days; (c) $t = 36$ days; (d) $t = 48$ days; (e) $t = 60$ days).

thermal, epithermal and resonance neutrons, and not only on fast neutrons for the first time, which is substantiated by the corresponding calculations. It is extremely important for a number of reasons, the major ones of which are the following. First, the problem of materials radiation resistance, being topical for fast wave reactors [20], is resolved automatically, since the neutron energy is smaller by approximately six orders (from $\sim 1$ MeV to $\sim 1$ eV) and, consequently, the integral fluence on the material of the fuel element walls also much smaller. However, we would like to draw attention to the fact that, as it follows from the results presented in Fig. 5, the wave burning criterion holds true not only for fast reactors with the hard spectrum (the average neutron energy $\sim 1$ MeV), but also for fast reactors with the softened neutron spectrum (the average neutron energy less then $\sim 100$ keV, and a “tail” of the neutron spectrum with the energy less than 10 keV is rather large, see e.g. [17]). And the solution of the problem of the materials radiation resistance, essential for fast wave reactors with the hard neutron spectrum, requires creation of new structural materials for the fuel element walls, withstanding the radiation load of 500 dpa (displacement per atom), whereas the materials operating nowadays withstand $\sim 100$ dpa (see [20]), i.e. the increase of radiation resistance in five times is necessary. Therefore, perhaps, it will be possible to solve the existing problem of radiation resistance of materials for fast wave reactors with the hard neutron spectrum by creating the fast wave reactors with
Figure 17: Kinetics of the plutonium-239 nuclei density under wave neutron-nuclear burning of natural uranium. The dependence of the dimensionless plutonium-239 nuclei density on the spatial coordinate $N_{Pu}^*(x)$ for (a) $t = 12$ days; (b) $t = 24$ days; (c) $t = 36$ days; (d) $t = 48$ days; (e) $t = 60$ days.

The softened neutron spectrum, since the transition to such reactors reduces the radiation load by an order. Let us also note that the prohibition on usage of steel as a constructional material of the active zone and necessity of use the expensive aluminium and zirconium alloys instead, which takes place for usual thermal reactors operating in the aforesaid ranges of neutron energies, do not exist for the wave reactors, since there is no need to preserve the neutrons for maintaining the chain reaction in wave reactors. Consequently, a possibility to exclude the danger of zirconium-steam reaction with the subsequent explosion of the hydrogen mixture [15, 21, 22] appears. Second, the wave reactors on resonance neutrons can act as the transmutators of the nuclides most dangerous for biosphere, being fragments of the fuel nuclides fissions (the so-called biocompatibility of the nuclear reactor of new generation). Third, the realization of the reactors on cold neutrons simplifies their radiation protection seriously because of small penetrability of the neutrons with these energies, which in combination with internal safety of the wave reactors can ensure their wide spread adoption. Fourth, a possibility of the wave reactor realization in the epithermal neutron region (the maximum of neutron distribution of the Maxwellian type in the region from 3 eV to 7 eV) attracts profound interest, since the requirements for the flux density of the external neutron source, ensuring the lighting of the nuclear burning, decrease sharply in this case (approximately by three-four orders). This is
Figure 18: Kinetics of the “slags” nuclei density under wave neutron-nuclear burning of natural uranium. The dependence of the dimensionless “slags” nuclei density on the spatial coordinate $\overline{N}(x)$ for (a) $t = 12$ days; (b) $t = 24$ days; (c) $t = 36$ days; (d) $t = 48$ days; (e) $t = 60$ days).

also true for the lighting times, because of the presence of the radiation capture cross-section maximum for uranium-238 of the order of 10000 barn for the neutron energy 7 eV. A source with natural radioactivity of neutrons with the energy 7 eV could be such a source, as well as the accelerator of charged particles (e.g. protons or electrons), which create mainly neutrons with the energy 7 eV by interaction with the target nuclei as a result of the nuclear reaction.

Let us note that wave nuclear burning in the epithermal region of neutron energies is of serious interest also for investigation of the possible burning modes of the wave georeactor [5], since the epithermal region of neutron energies with the distribution maximum for neutrons of Maxwellian type in the region $\sim 1$ eV can be considered as a region of thermalized neutrons in the fissile uranium-plutonium medium, being in a state with the temperature $\sim 5000$ K, which corresponds to the temperature of nuclear burning of the georeactor at the interface of the solid and liquid Earth cores.
Figure 19: The theoretical dependence (the solid line) and the simulation (dots) for phase velocities of neutron-nuclear burning $\Lambda(a_*) = u\tau_\beta/2L$ on the parameter $a_*$, presented in [10] and supplemented with an estimate, obtained in the present paper for slow wave burning of natural uranium in the epithermal region of the neutrons energy (1.0÷7.0 eV).

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