JONES-WASSERMANN SUBFACTORS FOR MODULAR TENSOR CATEGORIES

ZHENGWEI LIU AND FENG XU

Abstract. The representation theory of a conformal net is a unitary modular tensor category. It is captured by the bimodule category of the Jones-Wassermann subfactor. In this paper, we construct multi-interval Jones-Wassermann subfactors for unitary modular tensor categories. We prove that these subfactors are self-dual. It generalizes and categorifies the self-duality of finite abelian groups and we call it modular self-duality.

1. Introduction

Subfactor theory provides an entry point into a world of mathematics and physics containing large parts of conformal field theory, quantum algebras and low dimensional topology (cf. Jon90 and references therein). In Jon V. Jones has devised a renormalization program based on planar algebras as an attempt to show that all finite depth subfactors are related to CFT, i.e., the double of a finite depth subfactor should be related to CFT.

More generally, the program is the following: given a unitary modular tensor category (MTC) C, (cf. Tur94), can we construct a CFT whose representation category is isomorphic to C? We shall call such a program “reconstruction program”, analogue to a similar program in higher dimensions by Doplicher-Roberts (cf. DR89).

Given a rational conformal net A, and let I be a union of n > 1 disconnected intervals. The Jones-Wassermann subfactor is the subfactor A(I) ⊂ A(I′) (LR95, Was98, Xu00, KLM01). This subfactor is related to permutation orbifold and a simple application of orbifold theory shows that the Jones-Wassermann subfactor is self-dual, see Remark 6.16 and KLX05.

If the reconstruction program works, then for any MTC C we can find a rational conformal net A such that the category of representations of A is isomorphic to C, it will follow that there are self-dual Jones-Wassermann subfactors for each integer m > 1. Hence a positive solution to reconstruction program would imply that we can construct self-dual Jones-Wassermann subfactors for each integer m > 1 associated with any unitary MTC C. This is the motivation for our paper.

Our main result gives a construction of self-dual Jones-Wassermann subfactors for each integer m > 1 associated with any unitary MTC C. The main difficulty is the proof of the self-duality for Jones-Wassermann subfactors for MTC. The proof of the self-duality essentially requires the modularity of C, so we call it the modular self-duality. We believe that our construction will shed new light on the reconstruction program.

We construct the “m-interval” Jones-Wassermann subfactor associated with a unitary MTC C by a Frobenius algebra γm in Cm, the mth tensor power of C, although there is no notion of intervals in modular tensor categories. We give an explicit formula for the objects and morphisms of these Frobenius algebras. When m = 2, the Jones-Wassermann subfactor defines the quantum double of C (Dri86, Ocn91, Pop94, LR95, Müg03).
The bimodule category of a subfactor is described by a subfactor planar algebra [Jon98]. The $n$-box space of the planar algebra of the $m$-interval Jones-Wassermann subfactor for $\mathcal{C}$ is given by the vector space $\text{hom}_{\mathcal{C}-m}(1, \gamma_m^n)$. It turns out to be natural to represent these vectors by a 3D picture. This representation identifies $\text{hom}_{\mathcal{C}-m}(1, \gamma_m^n)$ as a configuration space $Conf_{n,m}$ on a 2D $n \times m$ lattice. Therefore the configuration space $\{Conf_{n,m}\}_{m,n \in \mathbb{N}}$ unifies the Jones-Wassermann subfactors for all $m \geq 1$. It is a natural candidate for the configuration space of a 2D lattice model that can be used in the reconstruction program.

Moreover, we show that planar tangles can act on $\{Conf_{n,m}\}_{m,n \in \mathbb{N}}$ in two different directions independently. In one direction $m$ is fixed. These actions are the usual ones in the planar algebra of the $m$-interval Jones-Wassermann subfactor. In the other direction $n$ is fixed. These actions relate the Jones-Wassermann subfactor with different intervals which have not been studied before.

The bi-directional actions of planar tangles are compatible with the geometric actions on the 2D lattices. We call such family of vector spaces a bi-planar algebra. It is a new subject in subfactor theory and it adds one additional dimension to the theory of planar algebras.

This 3D representation also leads to the discovery of a new symmetry between $m$ and $n$, although the meaning of the actions of planar tangles in the two directions are completely different. It will be interesting to understand these additional symmetries in conformal field theory.

When $\mathcal{C}$ is the representation category of a finite abelian group $G$, the configuration space $Conf(\mathcal{C})_{2,2}$ becomes $L^2(G)$. Moreover, the modular self-duality coincides with the self-duality of $G$. The proof of the self-duality of $G$ uses the discrete Fourier transform on $G$. We construct the string Fourier transform (SFT) on the configuration space $\text{hom}_{\mathcal{C}-m}(1, \gamma_m^n)$ to prove the modular self-duality. From this point of view, the modular self-duality and the SFT generalize and categorify the self-duality and the Fourier transform of finite abelian groups.

Moreover, the SFT on $Conf(\mathcal{C})_{2,2}$ is the same as the modular $S$-matrix of $\mathcal{C}$. Therefore we can study the Fourier analysis of the $S$-matrix on $Conf(\mathcal{C})_{2,2}$. It fits into the recent progress about the Fourier analysis on subfactors [Lin16, JLW16, LW17, JLW].

The modular self-duality has been used in the quon 3D language for quantum information [LWJ], where the vector space $\text{hom}_{\mathcal{C}-2}(1, \gamma_2^2)$ is considered as the 1-quon space. A combination of these two works leads to further applications in the study of MTC.

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2. Configuration spaces

2.1. Modular tensor categories. We refer the readers to [Lin94] for basic definitions about modular tensor categories. Suppose $\mathcal{C}$ is a unitary modular tensor category. Let $\text{Irr}$ be the set of simple objects of $\mathcal{C}$ and the unit is denoted by $1$. For an object $X$, its dual object is denoted by $\overline{X}$. Its quantum dimension is $d(X)$. Let $\mu = \sum_{X \in \text{Irr}} d(X)^2$ be the global dimension of $\mathcal{C}$.

The modular conjugation $\theta_\mathcal{C}$ on $\mathcal{C}$ is a horizontal reflection. We have that $\theta_\mathcal{C}(X) = \overline{X}$. Moreover, for objects $X$, $Y$, $Z$ in $\mathcal{C}$, $\theta_\mathcal{C} : \text{hom}(X \otimes Y, Z) \rightarrow \text{hom}(Y \otimes \overline{X}, Z)$ is an anti-linear algebroid isomorphism. The adjoint operator $\star$ on $\mathcal{C}$ is a vertical reflection. We have that $X^\star = X$. Moreover, $\star : \text{hom}(X \otimes Y, Z) \rightarrow \text{hom}(Z, X \otimes Y)$ is an anti-linear algebroid anti-isomorphism. The contragredient map $\rho_\pi$ on $\mathcal{C}$ is a rotation by $\pi$. We have that $\rho(X) = \overline{X}$. Moreover, $\rho : \text{hom}(X \otimes Y, Z) \rightarrow \text{hom}(Z, Y \otimes X)$ is a natural isomorphism.
hom(\mathcal{Z}, \mathcal{Y} \otimes \mathcal{X}) is a linear algebroid anti-isomorphism. Furthermore
\[ \theta_{\mathcal{C}} = \rho_{\pi} \circ \ast. \]

We can identify the morphism spaces \( \text{hom}(\mathcal{Z}, \mathcal{X} \otimes \mathcal{Y}) \) and \( \text{hom}(1, \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}) \) as follows: For a morphism \( \alpha \in \text{hom}(1, \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}) \), we obtain a morphism \( \tilde{\alpha} = (1 \otimes 1 \otimes \phi_{\mathcal{Z} \otimes \mathcal{Z}})(\alpha \otimes 1) \) in \( \text{hom}(\mathcal{Z}, \mathcal{X} \otimes \mathcal{Y}) \), where \( \phi_{\mathcal{Z} \otimes \mathcal{Z}} \in \text{hom}(1, \mathcal{Z} \otimes \mathcal{Z}) \) is the duality map.

**Notation 2.1** (Frobenius reciprocity). Diagrammatically we represent \( \tilde{\alpha} \) as

\[ \tilde{\alpha} := \alpha \]

**Notation 2.2.** For an object \( X \) in \( \mathcal{C} \), we denote an ortho-normal-basis of \( \text{hom}_{\mathcal{C}}(1, X) \) by \( \text{ONB}_{\mathcal{C}}(X) \), or \( \text{ONB}(X) \) for short. We denote an ortho-normal-basis of \( \text{hom}_{\mathcal{C}}(X, 1) \) by \( \text{ONB}^*_{\mathcal{C}}(X) \), or \( \text{ONB}^*(X) \) for short.

For two objects \( X \) and \( Y \), we have the resolution of the identity:
\[ 1_X \otimes 1_Y = \sum_{Z \in \text{Irr}, \alpha \in \text{ONB}_{\mathcal{C}}(X \otimes Y \otimes Z)} d(Z) \alpha \]

\[ \alpha \]

(2)

**2.2. Configuration spaces.** Now let us define the configuration space on a finite 2D-lattice with the target space \( \mathcal{C} \). Each configuration has three parts: \( Z \)-, \( X \)-, \( Y \)- configurations.

We use \( \text{Grid}(n, m) \) to represent the grid \( \mathbb{Z}_n \times \mathbb{Z}_m \times \{\pm 1\} \). We allocate the vertices of the grid at \((i, j, \pm 1), 0 \leq i \leq m - 1 \text{ and } 1 \leq j \leq n \) in the 3D space which are indicated by the bullets in Fig. 1.

To simplify the notations, we draw pictures for \( n = 4, m = 3 \). The reader can figure out the general case.

![Figure 1. Grid(n,m) for n = 4, m = 3.](image)

For the lattice \( \text{Lat} = \mathbb{Z}_n \times \mathbb{Z}_m \), a \( Z \)-configuration is a map from the sites of the lattice to simple objects in \( \mathcal{C} \). We denote the simple object at the site \((i, j)\) as \( X_{i,j} \). We denote this \( Z \)-configuration by \( X_{i,j} \) and represent it in the 3D space by assigning the object \( X_{i,j} \) to the line from \((i, j, 1)\) to \((i, j, -1)\) as in Fig. 2.
We denote $X_{\vec{i},\vec{j}} = X_{1,j} \otimes \cdots \otimes X_{n,j}$ and $X_{i,\vec{j}} = X_{i,0} \otimes \cdots \otimes X_{i,m-1}$. Moreover,

$$d(X_{\vec{i},\vec{j}}) := \prod_{1 \leq i \leq n, 0 \leq j \leq m-1} d(X_{i,j}),$$

$$d(X_{i,\vec{j}}) := \prod_{1 \leq i \leq n} d(X_{i,j}),$$

$$d(X_{\vec{i},\vec{j}}) := \prod_{0 \leq j \leq m-1} d(X_{i,j}).$$

An $X$-configuration with boundary $X_{\vec{i},\vec{j}}$ is a morphism $a_j$ in $\text{hom}(1, X_{\vec{i},\vec{j}})$. We denote the boundary by $X(a_j) := X_{\vec{i},\vec{j}}$. A $Y$-configuration with boundary $X_{i,\vec{j}}$ is a morphism $b_i$ in $\text{hom}(X_{i,\vec{j}}, 1)$. We denote the boundary by $X(b_i) := X_{i,\vec{j}}$. We represent them in the 3D space in Fig. 3. Moreover, we call $a_{\vec{j}} = a_0 \otimes \cdots \otimes a_m$ an $X$-configuration with boundary $X_{\vec{i},\vec{j}}$ and $b_{\vec{i}} = b_1 \otimes \cdots \otimes b_n$ a $Y$-configuration with boundary $X_{\vec{i},\vec{j}}$.

We call $a_{\vec{j}} \otimes b_{\vec{i}}$ a configuration with boundary $X_{\vec{i},\vec{j}}$, denoted by $X(a_{\vec{j}} \otimes b_{\vec{i}}) := X_{\vec{i},\vec{j}}$. We represent it in the 3D space as in Fig. 4. We define the configuration space on the $n \times m$ 2D-lattice $\text{Lat}$ to be the Hilbert space

$$\text{Conf}(\text{Lat}) = \text{Conf}(\mathcal{C})_{m,n} := \bigoplus_{X_{\vec{i},\vec{j}} \in \text{Irr}_{nm}} \left( \bigotimes_{j=0}^{m-1} \text{hom}(1, X_{\vec{i},\vec{j}}) \otimes \bigotimes_{i=1}^{n} \text{hom}(X_{i,\vec{j}}, 1) \right),$$

where each hom space is considered as a Hilbert space. We simply use the notation $\sum a_{\vec{j}} \otimes b_{\vec{i}}$ to represent an element in $\text{Conf}(\text{Lat})$. 
2.3. Duality. When we consider Lat = \( \mathbb{Z}_n \times \mathbb{Z}_m \) as a lattice on a torus, its dual lattice \( \text{Lat}' \) is also \( \mathbb{Z}_n \times \mathbb{Z}_m \) and the configuration space on the dual lattice is \( \text{Conf}(\mathcal{C})_{m,n} \). We allocate the vertices of the corresponding Grid\((n,m)\) at \((i+1/2, j-1/2, \pm 1)\), \(0 \leq i \leq m-1\) and \(1 \leq j \leq n\) in the 3D space.

We define a bilinear form \( LL \) on the configuration spaces of the lattice and the dual lattice \( \text{Conf}(\text{Lat}) \otimes \text{Conf}(\text{Lat}') = \text{Conf}(\mathcal{C})_{m,n} \otimes \text{Conf}(\mathcal{C})_{m,n} \). For \( a_j \otimes b_i \) with boundary \( X_{i,j} \) in \( \text{Conf}(\text{Lat}) \), and \( a'_j \otimes b'_i \) with boundary \( X'_{i,j} \) in \( \text{Conf}(\text{Lat}') \), the bilinear form \( LL \) is defined as

\[
LL(a_j \otimes b_i, a'_j \otimes b'_i) = \mu^{(1-n)(m-1)} \sqrt{d(X_{i,j})d(X'_{i,j})}
\]

When \( m = 0 \) or \( n = 0 \), we define the configuration space as the ground field. We define \( LL \) as the multiplication of the two scalars.

**Theorem 2.3.** The configurations spaces of the lattice and the dual lattice are dual to each other. Precisely the map from \( \text{Conf}(\text{Lat}) \) to the dual space of \( \text{Conf}(\text{Lat}') \) induced by \( LL(-,-) \) is an isometry.

We first prove the case for \( m = n = 2 \). We prove the general case by a bi-induction in the rest of the paper. The order of the proofs is shown at the end of this Section.

**Proof for the case \( m = n = 2 \):** When \( m = n = 2 \), the diagram in Equation (3) becomes the Hopf link and \( LL \) defines the \( S \) matrix of \( \mathcal{C} \).
By the modularity of \( \mathcal{C} \), the map induced by \( LL \) is an isometry. \( \square \)

**Proposition 2.4.** Suppose \( V \) is a Hilbert space and \( \{ \alpha_i \} \) is an ONB. Let \( V' \) be the dual space of \( V \). For \( f \in V' \), a linear functional on \( V \),

\[
    r(f) = \sum_i \overline{f(\alpha_i)} \alpha_i
\]

is independent of the choice of the basis.

**Proof.** It follows directly from definition. \( \square \)

The map \( r : V^* \to V \) is an anti-isometry which is well-known as the Riesz representation. Therefore we obtain an anti-isometry \( D : \text{Conf}(\text{Lat}') \to \text{Conf}(\text{Lat}) \) that we call the duality map:

**Definition 2.5 (duality maps).** We define

\[
    D_+(x) = \sum_{x' \in B'} LL(x, x') x',
\]

\[
    D_-(x') = \sum_{x \in B} LL(x, x') x,
\]

where \( B \) is an ONB of \( \text{Conf}(\text{Lat}) \) and \( B' \) is an ONB of \( \text{Conf}(\text{Lat}') \).

Therefore Theorem 2.3 is equivalent to the following Proposition.

**Proposition 2.6.** The map \( D_+ \) is an anti-linear isometry from \( \text{Conf}(\text{Lat}) \) to \( \text{Conf}(\text{Lat}') \), and \( D_- \) is its inverse.

**Definition 2.7.** We use \( 1_{n,m} \) to denote the trivial configuration whose \( Z \)-, \( X \)-, \( Y \)-configurations are all 1. We define

\[
    \mu_{n,m} : = D_-(1_{n,m}).
\]

**Definition 2.8.** We define \( L \) as a linear functional on \( \text{Conf}(\text{Lat}) \) as

\[
    L(x) = LL(x, 1_{n,m}).
\]

Then

\[
    L(a_j \otimes b_l) = \mu^{(1-n)(m-1)} \sqrt{d(X_{i,j})} a_0 b_1 b_2 b_3 b_4.
\]

(6)
and

$$\mu_{n,m} = \sum_{\alpha \in B} L(\alpha) \alpha,$$

(7)

where $B$ be is an ONB of $Conf(Lat)$.

In §3, we study the actions of rotations and reflections in $X$- and $Y$-directions on the lattices and the induced actions the configuration spaces. In §4, §5, §6 we fix $m$ and study the structure of the configuration space for different $n$. We prove that these configuration space admit the action of planar tangles (or operas) in the $X$-direction:

**Theorem 2.9.** For each $m \geq 1$, $\{S_n = Conf(C_{m,n})\}_{n \in \mathbb{N}}$ is an unshaded subfactor planar algebra.

This defines the self-dual $m$-interval Jones-Wassermann subfactor. It is proved in Theorems 4.11 and 6.13. Moreover, the duality map defines the string Fourier transform (SFT) of the unshaded planar algebra.

**Remark 2.10.** If we fix $n$, instead of $m$, then all the results also work. So we also have the action of planar tangles on the configuration spaces in the $Y$-direction. Therefore the configuration spaces $\{Conf(C_{m,n})\}_{m,n \in \mathbb{N}}$ admit the action of planar tangles in two different directions.

**Proposition 2.11.** Let $B$ be an ONB of $\text{hom}(\gamma, 1)$ whose elements are $Y$-configurations. Let $1_\gamma$ be the canonical inclusion from 1 to $\gamma$ and $b_1, b_2 \in \text{hom}(\gamma, 1)$. Then

$$\delta^{-2} \sum_{b' \in B} d(X(b')) = \langle 1^*_\gamma, b_1 \rangle \langle 1^*_\gamma, b_2 \rangle$$

**Proof.** Without loss of generality, we assume that $b_1$ and $b_2$ are unit vectors. Note that if $X(b_1) \neq X(\theta_1(b_2))$, then both sides are zero. We assume that $X(b_2) = X(\theta_1(b_1))$.

If a $Y$-configuration $b$ in $\text{hom}(\gamma, 1)$ is a unit vector, then

$$\text{dim hom}_{\mathbb{C}^m}(1, X(b) \otimes X(\theta(b))) = 1.$$ 

So there is only one $X$-configuration with boundary $X(b) \otimes X(\theta(b))$ up to a scalar. Let $a_b$ be the the canonical inclusion from 1 to $X(b) \otimes X(\theta(b))$ in $\mathbb{C}^m$. Let $C' = \{a_j \otimes b'_i\}$ be an ONB of $Conf(Lat')$. 

Applying Theorem 2.3 for $n = 2$, we have that
\[
\langle 1^*, b_1 \rangle \langle 1^*, b_2 \rangle = \langle 1_{m,2}, a_{b_1} \otimes (b_1 \otimes b_2) \rangle,
\]
\[
= \sum_{a_{b_1}' \otimes b_1' \in C'} LL(1_{m,2}, a_{b_1}' \otimes (b_1' \otimes b_2)) LL(a_{b_1} \otimes (b_1 \otimes b_2), a_{b_1}' \otimes (b_1' \otimes b_2))
\]
\[
= \sum_{b_1' \in B} LL(1_{m,2}, a_{b_1}' \otimes (b_1' \otimes \theta_1(b_1'))) LL(a_{b_1} \otimes (b_1 \otimes b_2), a_{b_1}' \otimes (b_1' \otimes \theta_1(b_1')))
\]
\[
= \delta^{-2} \sqrt{d(X(b_1))} \sum_{b_1' \in B} d(X(b_1'))
\]

If $X(b_1) \neq 1$, then both sides are zero. If $X(b_1) = 1$, then $d(X(b_1)) = 1$ and the statement holds. □

If we switch $n$ and $m$ in Proposition 2.11, then we have obtained the following equivalent result:

**Proposition 2.12.** Take $\tilde{\mathcal{X}} = \bigoplus_{X \in \text{Irr}} X$ and $1_{\tilde{\mathcal{X}}}$ to be the conical inclusion from 1 to $\tilde{\mathcal{X}}$. Then
\[
\mu^{1-n} \sum_{X_j \in \text{Irr}} d(X_j) \sum_{\alpha \in \text{ONB}(X_j)} = 1_{\tilde{\mathcal{X}}} 1_{\tilde{\mathcal{X}}} 1_{\tilde{\mathcal{X}}}
\]

**Proof.** Taking inner product of both sides with an element in $\text{hom}(\tilde{\mathcal{X}}^n, \tilde{\mathcal{X}}^n)$, the statement follows from Proposition 2.11. □

**Remark 2.13.** When $m = n = 2$, Equation (8) is the killing relation.

We prove Proposition 2.11 and 2.12 using Theorem 2.3. Proposition 2.12 will be used in Lemma 6.8. Then we prove Theorem 2.9 and Theorem 2.3. We prove Theorems 2.3 and 2.9 in the following order:

1. Theorem 2.3 for $m = 2$, $n = 2$;
2. → Theorem 2.9 for $m = 2$;
3. → Theorem 2.3 for $m = 2$, $n \geq 1$;
4. ↔ Theorem 2.3 for $m \geq 1$, $n = 2$;
5. → Theorem 2.9 for $m \geq 1$;
6. → Theorem 2.3 for $m \geq 1$, $n \geq 1$.

(When $m = 1$, the configuration space $\text{Conf}(\mathcal{C})_{m,n}$ is $\mathbb{C}$. The theorems are obvious.)
3. Actions on configuration spaces

3.1. Automorphisms on the lattice. Note that the lattice $\mathbb{Z}_m \times \mathbb{Z}_n$ is invariant under the following actions:

- The clockwise $2\pi/n$ rotation around the $Y$-direction $\rho_1$: $(i, j) \rightarrow (i - 1, j)$.
- The reflection in the $X$-direction $\theta_1$: $(i, j) \rightarrow (n + 1 - i, j)$.
- The clockwise $2\pi/m$ rotation around the $X$-direction $\rho_2$: $(i, j) \rightarrow (i, j + 1)$.
- The reflection in the $Y$-direction $\theta_2$: $(i, j) \rightarrow (i, m - 1 - j)$.

Now let us define the induced action on the configuration space $Conf(C_n,m)$.

For $k = 1, 2$, the induced actions on the $Z$-configurations are

\[
\rho_k(X)_{i,j} = X_{\rho_k^{-1}(i,j)},
\theta_k(X)_{i,j} = \theta(X_{\theta_k^{-1}(i,j)}).
\]

For an $X$-configuration $a_k$, we define

\[
\rho_1(a_j) = X_{2,j} X_{3,j} X_{4,j} X_{1,j},
\theta_1(a_j) = \theta(a_j),
\rho_2(a_j) = a_j,
\theta_2(a_j) = X_{1,j} X_{2,j} X_{3,j} X_{4,j}.
\]

For a $Y$-configuration $b_i$, we define
\[ \rho_1(b_i) = b_i, \]
\[ \theta_1(b_i) = X_{i,0}, \]
\[ \rho_2(b_i) = X_{i,2}, \]
\[ \theta_2(b_i) = \theta^\prime(b_i). \]

**Definition 3.1.** For a configuration \( a_j \otimes b_i \), we define
\[
\begin{align*}
\rho_1(a_j \otimes b_i) &= (\rho_1(a_0) \otimes \cdots \otimes \rho_1(a_{m-1}) \otimes (b_2 \otimes \cdots \otimes b_n \otimes b_1), \\
\theta_1(a_j \otimes b_i) &= (\theta_1(a_0) \otimes \cdots \otimes \theta_1(a_{m-1}) \otimes (\theta_1(b_n) \otimes \cdots \otimes \theta_1(b_1)), \\
\rho_2(a_j \otimes b_i) &= (a_{m-1} \otimes a_1 \otimes \cdots \otimes a_{m-2} \otimes (\rho_2(b_1) \otimes \cdots \otimes \rho_2(b_n)), \\
\theta_2(a_j \otimes b_i) &= (\theta_2(a_{m-1}) \otimes \cdots \otimes \theta_2(a_0) \otimes (\theta_2(b_1) \otimes \cdots \otimes \theta_2(b_n)).
\end{align*}
\]

The actions on the configuration space are defined by their linear or anti-linear extensions.

Note that this definition coincide with the geometric actions on the configurations in Fig. 4. Therefore their relations also hold on the configuration space.

**Proposition 3.2.** On the configuration space, \( \rho_1 \) and \( \theta_1 \) commute with \( \rho_2 \) and \( \theta_2 \), and for \( k = 1, 2 \),
\[
\begin{align*}
\rho_k \theta_k &= \theta_k \rho_k, \quad \rho_k^m = 1, \quad \theta_k^2 = 1.
\end{align*}
\]

3.2. **Automorphisms on the dual pair of lattices.** Similarly we also define the four actions on the dual lattice \( \text{Lat}' \) and the configuration space \( \text{Conf}(\text{Lat}') \).

**Proposition 3.3.** For \( x \in \text{Conf}(\text{Lat}) \) and \( x' \in \text{Conf}(\text{Lat}') \), we have that
\[
\begin{align*}
LL(x, x') &= LL(\rho_k(x), \rho_k(x')), \quad k = 1, 2 \\
LL(x, x') &= LL(\theta_1(x), \rho_1(x')), \\
LL(x, x') &= LL(\rho_1 \theta_2(x), \theta_2(x')).
\end{align*}
\]
Proof. It is enough to prove the case $x = a_j \otimes b_i$, $x' = a'_j \otimes b'_i$.

Recall that $LL$ is defined by a closed diagram in the 3D space as shown in Equation (3). Applying the rotation on the 3D diagram, we obtain Equation (9).

If we consider the 3D diagram as an element in $C$, then we have that

$$LL(a_j \otimes b_i, a'_j \otimes b'_i) = \mu \frac{(1-n)(m-1)}{4} \sqrt{d(X_{i,j}^-)d(X_{i,j}^+)}$$

If $x = 1_{n,m}$ in Proposition 3.3, we obtain that

$$= LL(\theta_1(x), \rho \theta_1(x'))$$

The proof of Equation (11) is similar.

By definitions, $\rho_k, \theta_k, k = 1, 2$, preserve $1_{n,m}$. Take $x' = 1_{n,m}$ in Proposition 3.3 we obtain that
**Proposition 3.4.** For any \( x \) in \( \text{Conf}(\text{Lat}) \), \( k = 1, 2 \),
\[
L(\rho_k(x)) = L(x),
\]
\[
L(\theta_k(x)) = \overline{L(x)}.
\]

**Proposition 3.5.** The four actions \( \rho_k, \theta_k, k = 1, 2 \), preserve \( \mu_{n,m} \).

**Proof.** The statement follows from Proposition 3.4 and the definition of \( \mu_{n,m} \) in Equation (7).

\[\square\]

4. Jones-Wassermann Subfactors for MTC

4.1. Identification. Suppose \( \mathcal{C} \) is a unitary modular tensor category. Let \( \mathcal{C}^m \) be the \( m \)th tensor power of \( \mathcal{C} \). Let \( \text{Irr}_m \) be the set of simple objects of \( \mathcal{C}^m \). We can represent a simple object in \( \text{Irr}_m \) as \( \vec{X} := X_0 \otimes \cdots \otimes X_{m-1} \) for some simple objects \( X_j \) in \( \mathcal{C} \). Let \( d(\vec{X}) \) be the quantum dimension of the object \( \vec{X} \). Then \( d(\vec{X}) = \prod_{j=0}^{m-1} d(X_j) \).

Take
\[ \gamma = \gamma_m = \bigoplus_{\vec{X}} N_{\vec{X}} \vec{X}, \] (12)
where \( N_{\vec{X}} = \dim \text{hom}_\mathcal{C}(\vec{X}, 1) \). Recall that \( \mu \) is the global dimension of \( \mathcal{C} \).

**Proposition 4.1.** For \( m \geq 1 \), \( d(\gamma) = \mu^{m-1} \).

**Proof.** It is obvious for \( m = 1 \). When \( m \geq 2 \),
\[
d(\gamma) = \sum_{\vec{X} \in \text{Irr}_m} N_{\vec{X}} d(\vec{X})
\]
\[= \sum_{\vec{X} \in \text{Irr}_m, Y \in \text{Irr}_1} \dim \text{hom}_\mathcal{C}(\vec{X}, Y) d(\vec{X}) d(Y) \quad \text{by Frobenius reciprocity}
\]
\[= \sum_{\vec{X} \in \text{Irr}_m-1} d(\vec{X})^2
\]
\[= \mu^{m-1}.
\]
\[\square\]

**Notation 4.2.** Take \( \delta = \mu \frac{m-1}{2} \).

For each \( \vec{X} \), let \( ONB^*_{\mathcal{C}}(\vec{X}) \) be an ONB of \( \text{hom}_\mathcal{C}(\vec{X}, 1) \). Then we can use the basis to represent the multiplicity of \( \vec{X} \) in \( \gamma \).
\[
\gamma = \bigoplus_{\vec{X} \in \text{Irr}_m, b \in \text{ONB}^*_{\mathcal{C}}(\vec{X})} \vec{X}(b),
\] (13)
where \( \vec{X}(b) = \vec{X} \).

The representation is covariant with respect to the choice of the ONB: For an object \( Y \) in \( \mathcal{C}^m \) and a morphism \( y \in \text{hom}_\mathcal{C}(Y, N_{\vec{X}} \vec{X}) \), we take two ONB \( B(1), B(2) \) of \( \text{hom}_\mathcal{C}(\vec{X}, 1) \). Then we obtain
two representations
\[ y = \bigoplus_{b_1 \in B(1)} y(b_1), \quad y(b_1) \in \text{hom}(Y, \bar{X}(b_1)), \]
\[ y = \bigoplus_{b_2 \in B(2)} y(b_2), \quad y(b_2) \in \text{hom}(Y, \bar{X}(b_2)). \]

The representation is covariant means that
\[ y(b) = \sum_{b'} b' \cdot y(b'), \]
where
\[ \text{hom}_{C^m}(1, \gamma^n) = \bigoplus_{X_{i,j} \in \text{Irr}^{nm}} \bigoplus_{b_i \in \text{ONB}^*(X_{i,j})} \text{hom}_{\mathcal{E}^m}(1, X_{i,j}(b_i)). \]

We call it the \( n \)-box space of \( \gamma \). For \( a_j \in \text{hom}_{\mathcal{E}}(1, X_{i,j}), 0 \leq j \leq m - 1, \) we have \( a_j(b_i) \in \text{hom}_{\mathcal{E}^m}(1, X_{i,j}(b_i)). \)

**Definition 4.3.** We define a map \( \Phi: \text{hom}_{\mathcal{E}^m}(1, \gamma^n) \rightarrow \text{Conf}(\mathcal{E})_{n,m} \) as a linear extension of
\[ \Phi(a_j(b_i)) = a_j \otimes b_i. \]

The definition is independent of the choice of the ONB \( b_i \), since the representation is covariant. Moreover,
\[ \langle a_j(b_i), c_j(d_i) \rangle = \langle a_j, c_j \rangle(b_i, d_i) = \langle a_j \otimes b_i, c_j \otimes d_i \rangle. \]

So \( \Phi \) is an isometry. Therefore we can identify the vectors in the two Hilbert spaces \( \text{hom}_{\mathcal{E}^m}(1, \gamma^n) \) and \( \text{Conf}(\mathcal{E})_{n,m} \). We simply use the notation \( \sum a_j(b_i) \) to represent an element in \( \text{hom}_{\mathcal{E}^m}(1, \gamma^n) \).

**Definition 4.4.** Induced by the isometry \( \Phi \), the four actions \( \rho_k, \theta_k \), \( k = 1, 2 \), and the contractions \( \wedge_k, k \geq 0 \), are also defined on \( \text{hom}(1, \gamma^n) \), still denoted by \( \rho_k, \theta_k \).

Recall that the multiplicity of \( X_{-j} \) in \( \gamma \) is represented by \( b \) in \( \text{ONB}^*(X_{-j}) \). We need an anti-isometric involution on \( \text{ONB}^*(X_{-j}) \) to specify the dual of \( X_{-j}(b) \). To be compatible with the geometric interpretation of the configuration in the 3D space, we define the dual by \( \theta_1 \):

**Definition 4.5.** For an object \( X_{-j}(b) \), we define its dual object as \( \overline{X_{-j}(b)} \).

Note that \( \theta_1^2(b) = b \), thus \( \overline{\overline{X_{-j}(b)}} = X_{-j}(b) \). By Frobenius reciprocity, the modular conjugation on \( \mathcal{E}^m \) is given by \( \theta_1 \).

The element in \( \text{hom}_{\mathcal{E}^m}(1, \gamma^n) \) is usually represented by a diagram on the 2D plane. To be compatible with the isometry \( \Phi \), we simplify the 3D pictures for configurations by their projections on the plane \( Y = 0 \) as follows:

The configuration in Fig. 4 is simplified as the following notation:
Induced by the isometry $\Phi$, $L$ becomes a linear functional on $\text{hom}(1, \gamma^n)$,

$$L(a_j(b_i)) := L(a_j \otimes b_i) = \delta^{1-n} \sqrt{d(X_{i,j})},$$

where we simplify the diagram in Equation (8) by its projection on the plane $Y = 0$.

4.2. Contractions. The multiplication on $\mathcal{C}$ defines a map from $\text{hom}(\gamma^n, \gamma^k) \otimes \text{hom}(\gamma^k, \gamma^l)$ to $\text{hom}(\gamma^n, \gamma^l)$. Applying Frobenius reciprocity, we obtain a contraction $\land_k : \text{hom}(1, \gamma^{n+k}) \otimes \text{hom}(1, \gamma^{k+l}) \to \text{hom}(1, \gamma^{n+l})$. Then $\land_k$ is also defined on the configuration spaces induced by $\Phi_i$. We give the definition in detail here.

Remark 4.6. The notation $\land_k$ comes from the graded multiplication in [GJS10].

Suppose $X_{i,j}, Y_{i,j}, Z_{i,j}$ are $Z$-configurations of size $n \times m$, $\ell \times m$, $k \times m$. For $X$-configurations $a_j \in \text{hom}(1, X_{i,j} \otimes Z_{i,j})$ and $c_j : \text{hom}(1, \theta_1(Z_{i,j}) \otimes Y_{i,j})$, we define the $k$-string contraction, $k \geq 0$, as

$$a_j \land_k c_j := \begin{array}{c|c|c|c|c}
X_{1,j} & \cdots & X_{n,j} & Z_1 & \cdots & Z_k & Z_1 & \cdots & Y_{1,j} & \cdots & Y_{\ell,j}
\end{array},$$

Moreover, $a_j \land_k c_j := (a_1 \land c_1) \otimes \cdots \otimes (a_n \land c_n)$. Suppose $b_i \in \text{hom}(1, X_{i,j} \otimes Z_{i,j})$ and $d_i : \text{hom}(1, \theta_1(Z_{i,j}) \otimes Y_{i,j})$ are $Y$-configurations. We define the $k$-string contraction $\land_k$ on the configurations $a_j \otimes b_i$ and $c_j \otimes d_i$ as

$$(a_j \otimes b_i) \land_k (c_j \otimes d_i) = \prod_{s=1}^{k} (\theta_1(b_{k+1-s}), b_{n+s})(a_j \land_k c_j) \otimes \bigotimes_{i=1}^{n} b_i \otimes \bigotimes_{i=k+1}^{k+\ell} d_i).$$

We define $\land_k : \text{Conf}(\mathcal{C})_{n+k,m} \otimes \text{Conf}(\mathcal{C})_{k+\ell,m} \to \text{Conf}(\mathcal{C})_{k+\ell,m}$ by a linear extension on the configuration spaces.

Therefore we can identify $\text{Conf}(\mathcal{C})_{k+\ell,m}$ as operators from $\text{Conf}(\mathcal{C})_{k,m}$ to $\text{Conf}(\mathcal{C})_{\ell,m}$ corresponding to Frobenius reciprocity. Moreover, the composition of these operators is associative.

Proposition 4.7. Recall that $\rho_2$ and $\theta_2$ are actions in the $Y$-directions. They commute with the contraction $\land_k$ in the $X$-direction.

Proof. Recall that $\rho_2$ is an isometry and it commutes with $\theta_1$ by Proposition 3.2 so

$$\rho_2(a_j \otimes b_i) \land_k \rho_2(c_j \otimes d_i)$$

$$= \prod_{s=1}^{k} (\theta_1(\rho_2(d_{k+1-s}), b_{n+s})(\rho_2(a_j) \land_k \rho_2(c_j)) \otimes \bigotimes_{i=1}^{n} \rho_2(b_i) \otimes \bigotimes_{i=k+1}^{k+\ell} \rho_2(d_i))$$

$$= \prod_{s=1}^{k} (\theta_1(\rho_2(d_{k+1-s}), b_{n+s})\rho_2(a_j \land_k c_j) \otimes \bigotimes_{i=1}^{n} \rho_2(b_i) \otimes \bigotimes_{i=k+1}^{k+\ell} \rho_2(d_i))$$

$$= \rho_2((a_j \otimes b_i) \land_k (c_j \otimes d_i)).$$

The proof for $\theta_2$ is similar. \qed


Lemma 4.8. When $k = 1$, we have

$$\sum_{b \in ONB(Z_j)}$$

Proof. We apply Equation (2), the resolution of identity in $C$, to $1_{Z_j}$ in the first diagram. Only the component equivalent to 1 remains non-zero, since the first diagram has no boundary on the left. This component gives the second diagram.

4.3. Frobenius algebras.

Definition 4.9. We define $\mu_n = \Phi^{-1}(\mu_{n,m})$, for $n \geq 1$, and $\mu_0 = 1$. Then

$$\mu_n = \sum_{\alpha \in B} L(\alpha)\alpha,$$

where $B$ be is an ONB of $\text{hom}(1, \gamma^n)$.

Moreover, $\mu_1$ is the canonical inclusion from 1 to $\gamma$, and $\mu_2$ is the canonical inclusion from 1 to $\gamma \otimes \gamma$ which defines the dual of objects. By Proposition 3.5, for $k = 1, 2$,

$$\rho_k(\mu_n) = \mu_n,$$

$$\theta_k(\mu_n) = \mu_n.$$ (14)

Let us prove that $\mu_3$ defines a Frobenius algebra.

Lemma 4.10. For $n, \ell \geq 1$,

$$\mu_n \land_1 \mu_\ell = \delta^{-1} \mu_{n+\ell-2}.$$ (15)

Proof. Suppose $X_{i,j}, Y_{i,j}, Z_j$ are $Z$-configurations of size $n \times m, \ell \times m, 1 \times m$. Note that

$$\{ \sqrt{d(Z_j)a_j \land_1 c_j} | a_j \in ONB(X_{i,j} \otimes Z_j), c_j \in ONB(Z_j \otimes Y_{i,j}) \}$$

defines an ONB($X_{i,j} \otimes Y_{i,j}$). Take $b_i = b_1 \otimes \cdots \otimes b_n$ and $d_i = d_1 \otimes \cdots \otimes d_\ell$, where $b_i \in ONB(X_{i,j})$ and $d_i \in ONB(Y_{i,j})$. By Lemma 4.8, we have

$$\delta \sum_{b \in ONB^{*}(Z_j)} L(\sqrt{d(Z_j)a_j \land_1 c_j})a_j \land_1 c_j (b \otimes d_i)$$

$$= \delta \left( \sum_{b \in ONB^{*}(Z_j)} L(\sqrt{d(Z_j)a_j \land_1 c_j})a_j \land_1 c_j (b \otimes b) \right) \land_1 \left( \sum_{b \in ONB^{*}(Z_j)} L(\sqrt{d(Z_j)a_j \land_1 c_j})a_j \land_1 c_j (b \otimes d_i) \right).$$
Sum over $a_j(b_i)$, $c_j(d_i)$, we have

$$\mu_{n+\ell-2} = \delta\mu_n \wedge_1 \mu_\ell.$$ 

**Theorem 4.11.** By Frobenius reciprocity, we can identify $\mu_3$ as a morphism in $\text{hom}(\gamma, \gamma^2)$ or $\text{hom}(\gamma^2, \gamma)$. Then $(\gamma, \mu_3)$ is a Frobenius algebra in $\mathcal{C}^m$.

**Proof.** It follows from Equations (14), (15) and Lemma 4.10.

We call the subfactor associated with the Frobenius algebra $(\gamma, \mu_3)$ the $m$-interval Jones-Wassermann subfactor. The modularity is not used in the construction of the Jones-Wassermann subfactors, but it is crucial in the proof of the self-duality of the Jones-Wassermann subfactor. The formula of $(\gamma, \mu_3)$ in terms of the 3D configuration is intuitive in the proof of self-duality.

We can also derive this Frobenius algebra through the tensor functor $\text{Fun}$ from $\mathcal{C}^m$ to $\mathcal{C}$. (1) The adjoint functor of $\text{Fun}$ sends the trivial Frobenius algebra in $\mathcal{C}$ to a Frobenius algebra in $\mathcal{C}^m$ which is our $(\gamma, \mu_3)$. (2) The functor $\text{Fun}$ defines an inclusion from $\text{hom}_{\mathcal{C}^m}(\tilde{X}^{mn})$ to $\text{hom}_{\mathcal{C}}(\tilde{X}^{mn})$. The inductive limit of this inclusion for $n \to \infty$ defines a subfactor, which was studied by Erlijman and Wenzl in [EW07]. The corresponding Frobenius algebra is $(\gamma, \mu_3)$.

The Frobenius algebra $(\gamma, \mu_3)$ defines a $\gamma - \gamma$ bimodule category induced by $\mathcal{C}^m$. It is a unitary fusion category, called the dual of $\mathcal{C}^m$ with respect to $(\gamma, \mu_3)$. When $m = 2$, the dual of $\mathcal{C}^2$ is known as the quantum double of $\mathcal{C}$. For a general $m$, we call the dual of $\mathcal{C}^m$ with respect to the Frobenius algebra $(\gamma, \mu_3)$ the quantum $m$-party, or quantum multiparty, of $\mathcal{C}$.

## 5. The string Fourier transform on planar algebras

Once we obtain a Frobenius algebra $(\gamma, \mu_3)$, we can define a subfactor planar algebra $\mathcal{S} = \{\mathcal{S}_{n,\pm}\}_{n \in \mathbb{N}}$, such that $\mathcal{S}_{n,+} = \text{hom}(1, \gamma^n)$. This is a part of Theorem 2.3. We show that the planar algebra is unshaded by constructing a planar algebraic $^*$-isomorphism from $\mathcal{S}_{n,-}$ to $\mathcal{S}_{n,+}$ in §6.

The modular conjugation $\theta_1$ defines the involution $^*$ of the subfactor planar algebra $\mathcal{S}$. In the planar algebra $\mathcal{S}_{n,+}$, the element $\delta_n \mu_3$ is represented by

![Diagram $\delta_n \mu_3$](image)

where the diagram has $2n$ boundary points at the bottom.

**Remark 5.1.** Convention: We omit the $\$$ sign of the planar diagram if it is on the left.

The action of any planar tangle on $\mathcal{S}_{n,+}$ is a composition of the following 6 elementary ones, for $n, \ell \geq 0$:

- The rotation $\rho : \mathcal{S}_{n,+} \to \mathcal{S}_{n,+}$,
• The wedge product $\wedge : \mathcal{I}_{n,+} \otimes \mathcal{I}_{l,+} \to \mathcal{I}_{n,+}$,

\[
x \wedge y = \begin{array}{c}
\vdots \\
\vdots \\
\end{array}
\begin{array}{c}
x \\
\vdots \\
y \\
\vdots \\
\end{array}
\]

• The inclusion $t_0 : \mathcal{I}_{n,\pm} \to \mathcal{I}_{n+1,\pm}$,

\[
t_0(x) = \delta^{-1/2} \begin{array}{c}
\vdots \\
\vdots \\
\end{array}
\begin{array}{c}
x \\
\vdots \\
\end{array}
\]

• The contraction $\phi_0 : \mathcal{I}_{n+1,\pm} \to \mathcal{I}_{n,\pm}$,

\[
\phi_0(x) = \delta^{-1/2} \begin{array}{c}
\vdots \\
\vdots \\
\end{array}
\begin{array}{c}
x \\
\vdots \\
\end{array}
\]

• The inclusion $t_1 : \mathcal{I}_{n,\pm} \to \mathcal{I}_{n+1,\pm}$,

\[
t_1(x) = \delta^{-1/2} \begin{array}{c}
\vdots \\
\vdots \\
\end{array}
\begin{array}{c}
x \\
\vdots \\
\end{array}
\]

• The contraction $\phi_1 : \mathcal{I}_{n+1,\pm} \to \mathcal{I}_{n,\pm}$,

\[
\phi_1(x) = \delta^{-1/2} \begin{array}{c}
\vdots \\
\vdots \\
\end{array}
\begin{array}{c}
x \\
\vdots \\
\end{array}
\]

The first two are isometries. The last four are partial isometries. These actions except the rotation can be written as contractions:

\[
t_0(x) = \mu_1 \wedge x \\
\phi_0(x) = \mu_1 \wedge_1 x \\
t_1(x) = \delta \mu_3 \wedge_1 x \\
\phi_1(x) = \delta \mu_3 \wedge_2 x.
\]

Moreover, $\phi_k$ is the adjoint operator of $t_k$:

**Proposition 5.2.** For $x \in \mathcal{I}_{n,\pm}, y \in \mathcal{I}_{n+2,\pm}, k = 0, 1$, we have

\[
< t_k(x), y >= < x, \phi_k(y) >.
\]

As a subfactor planar algebra, we have the involution on $\mathcal{I}_{n,+}$ defined by the reflection $\theta_1$ which is an anti-isometry.

We have also these actions on the configuration space in the Y-direction. In particular, the rotation $\rho_2$ and the reflection $\theta_2$ preserves the size $m$, and they are defined on $\mathcal{I}_{n,+} = \text{hom}(1, \gamma^n)$.

**Theorem 5.3.** The action $\rho_2$ on $\mathcal{I}$ is a planar algebra $\ast$-isomorphism. The action $\theta_2$ on $\mathcal{I}$ is an anti-linear planar algebra $\ast$-isomorphism.

**Proof.** By Propositions 3.2, $\rho_2$ and $\theta_2$ commute with $\rho$ and $\theta_1$. By Propositions 3.5 and 4.7, we have that $\rho_2$ and $\theta_2$ commute with $\wedge, t_k$ and $\phi_k$, for $k = 1, 2$. Therefore they are (anti-linear) planar algebraic $\ast$-isomorphisms. □
Similarly we have the 6+1 elementary actions on $\mathcal{I}_{n,\pm}$ by switching the shading. The string Fourier transform (SFT) $\mathcal{F}_n : \mathcal{I}_{n,\pm} \to \mathcal{I}_{n,\mp}$ is an isometry given by a clockwise one-string rotation. Applying the SFT, we can represent the element in $\mathcal{I}_{n,\mp}$ by $\mathcal{F}_n(x)$ for $x$ in $\mathcal{I}_{n,\pm}$ and derive the six elementary actions on $\mathcal{I}_{n,\pm}$ by actions on $\mathcal{I}_{n,\mp}$.

For an element $x \in \mathcal{I}_{n,\pm}$, its SFT $\mathcal{F}_n(x) \in \mathcal{I}_{n,\mp}$ is given by

$$\mathcal{F}(x) = \begin{array}{c}
\vspace{0.5cm}
\end{array}$$

(16)

Then $\rho = \mathcal{F}^2$. Moreover, $\iota_k := \mathcal{F}^{-k}\iota_0 \mathcal{F}^k$, $1 \leq k \leq 2n$, is adding a cap before the $k^{th}$ boundary points, and $\phi_k := \mathcal{F}^{-k}\phi_0 \mathcal{F}^k$, $1 \leq k \leq 2n$, is a contraction between the $k+1^{th}$ and $k+2^{th}$ boundary points.

**Notation 5.4.** By the spherical property, we define $\phi_1$ on $\mathcal{I}_{1,\pm}$ by $\phi_0$.

For $x \in \mathcal{I}_{n,+}, y \in \mathcal{I}_{n',+}$, we define $x \star y \in \mathcal{I}_{n+n',+}$ as

$$x \star y = \begin{array}{c}
\vspace{0.5cm}
\end{array}$$

(17)

Then

$$\rho \mathcal{F}(x) = \mathcal{F} \rho(x),$$

$$\mathcal{F}(x) \wedge \mathcal{F}(y) = \mathcal{F}(x \star y),$$

$$\phi_k \mathcal{F}(x) = \mathcal{F} \phi_{k+1}(x),$$

$$\iota_{k+1} \mathcal{F}(x) = \mathcal{F} \iota_k(x),$$

$$\theta_1(\mathcal{F}(x)) = \mathcal{F} \theta^{-1}_1(x).$$

Recall that $\mathcal{I}_{n,+} = \text{hom}(1, \gamma^n)$, thus $\star : \text{hom}(1, \gamma^n) \otimes \text{hom}(1, \gamma^{n'}) \to \text{hom}(1, \gamma^{n+n'})$ is also defined. From the planar algebra $\mathcal{S}$ to category $\mathcal{C}^m$, the shaded strip becomes a $\gamma$-string. Then Equation (17) becomes

$$x \star y = \delta^2$$

(17)
6. Modular self-duality

6.1. The self-duality of Jones-Wassermann subfactors. Suppose that \( \mathcal{I} \) is the subfactor planar algebras of the Jones-Wassermann subfactor for a unitary modular tensor category \( \mathcal{C} \). In this section, we construct a planar algebraic *-isomorphism from \( \mathcal{I}_{n,-} \) to \( \mathcal{I}_{n,+} \). Then the subfactor planar algebra \( \mathcal{I} \) is unshaded. Equivalently the Jones-Wassermann subfactor is self-dual.

Induced by \( \Phi \), we define \( LL \) on \( \text{hom}(1, \gamma^n) \otimes \text{hom}(1, \gamma^n) \). Moreover, we use the following notation to simplify the diagram in Equation (3):

\[
LL(a_j(b_i), a'_j(b'_i)) = LL(a_j \otimes b_i, a'_j \otimes b'_i) = \delta^{1-n} \sqrt{d(X_{ij})d(X'_{ij})}
\]

\[
(\theta_2(b)) = (\theta_1(b_1) \cdots \theta_1(b_4))
\]

\[
(23)
\]

Notation 6.1. We use \( A_n \) to denote an ONB of \( \text{hom}(\mathcal{C}, (\gamma^n, 1)) \). We use \( B \) to denote an ONB of \( \text{hom}(\gamma, 1) \).

Definition 6.2 (string Fourier transform). We represent elements in \( \mathcal{I}_{n,-} \) as \( F(x) \) for \( x \in \mathcal{I}_{n,+} \). We define \( \Psi : \mathcal{I}_{n,-} \to \mathcal{I}_{n,+} \), \( n \geq 0 \),

\[
\Psi(F(x)) = \sum_{x' \in B_n} LL(x, \theta_{2}(x'))x'.
\]

(24)

When \( n = 0 \), \( \Psi \) maps 1 to 1. When \( n = 1 \), \( \Psi \) maps the canonical inclusion from 1 to \( \gamma \) in \( \mathcal{I}_{1,-} \) to the canonical inclusion in \( \mathcal{I}_{1,+} \). Let us prove that \( \Psi \) commutes with the 6+1 elementary actions, so \( \Psi \) is a planar algebraic *-isomorphism from \( \mathcal{I}_{n,-} \) to \( \mathcal{I}_{n,+} \). Then \( \mathcal{I} \) becomes an unshaded planar algebra. Moreover, the map \( \Psi_F \) in Equation (24) defines the SFT on the unshaded planar algebra \( \mathcal{I}_n \).

When \( m = n = 2 \), \( \gamma = \bigoplus_{X \in \text{Irr}(\mathcal{C})} X \otimes \overline{X} \). The vectors \( \{v_X\}_{X \in \text{Irr}(\mathcal{C})} \) form an ONB of \( \text{hom}_{\mathcal{C}}(1, \gamma^2) \), where \( v_X \) is the canonical inclusion from 1 to \( (X \otimes X) \otimes (X \otimes X) \) in \( \mathcal{C}^2 \). We call the ONB \( \{v_X\}_{X \in \text{Irr}(\mathcal{C})} \) a standard basis of \( \text{hom}_{\mathcal{C}}(1, \gamma^2) \).

The vector \( v_X \) is independent of the choice of the representative of \( X \otimes \overline{X} \) in \( \gamma \). For convenience, we take \( b^* \) to be the canonical inclusion from 1 to \( X \otimes \overline{X} \) to indicate the multiplicity of \( X \otimes \overline{X} \) in \( \gamma \), then \( \overline{(X \otimes X)(b)} = (X \otimes X)(\theta_1(b)) \).

The following result is a consequence of the modular self-duality and our definition of the SFT.

Proposition 6.3. The SFT on the standard basis of \( \text{hom}_{\mathcal{C}}(1, \gamma^2) \) is the same as the modular \( S \) matrix of the MTC \( \mathcal{C} \): for any \( X, X' \in \text{Irr} \mathcal{C} \),

\[
(\Psi_F(v_X), v_{X'}) = S_{XX'}.
\]

(25)
Proof. By Definition 6.2, the matrix units of $\Psi F$ on the basis $\{v_X\}_{X \in Irr(\varphi)}$ is
\[
\langle \Psi F(v_X), v_{X'} \rangle = LL(x, \theta_2(x')) = \mu^{-\frac{1}{2}}X X' = S_{XX'} \tag{27}
\]
\[
= \mu^{-\frac{1}{2}} X X' = S_{XX'} \tag{28}
\]

Proposition 6.4. For $x \in hom(1, \gamma^n)$, $n \geq 1$,
\[
\Psi(\Psi F(\rho(x))) = \rho \Psi(\Psi F(x)),
\]
\[
\Psi(\Psi F(\rho^{-1}\theta_1(x))) = \theta_1(\Psi(\Psi F(x))).
\]
Proof. By Propositions 3.2 and 3.3 we have
\[
\Psi(\Psi F(\rho(x))) = \sum_{x' \in A_n} LL(\rho(x), \theta_2(x'))x' \tag{26}
\]
\[
= \sum_{x' \in A_n} LL(x, \rho^{-1}\theta_2(x'))x' \tag{27}
\]
\[
= \sum_{x' \in A_n} LL(x, \rho^{-1}\theta_2(x'))x' \tag{28}
\]
\[
= \sum_{x' \in A_n} LL(x, \theta_2\rho^{-1}(x'))x' \tag{29}
\]
\[
= \rho \Psi(\Psi F(x)). \tag{30}
\]

Similarly we have $\Psi(\Psi F(\rho^{-1}\theta_1(x))) = \theta_1(\Psi(\Psi F(x)))$. \hfill \Box

Lemma 6.5. For $x, x' \in \text{hom}(1, \gamma^n)$, $y, y' \in \text{hom}(1, \gamma^\ell)$, $n, \ell \geq 1$,
\[
LL(x \ast y, x' \wedge y') = LL(x, x')LL(y, y').
\]
Proof. Take $x = a_j(b_1)$, $x' = a'_i(b'_1)$, $y = c_j(d_1)$ and $y' = c'_i(d'_1)$. Note that the boundary of a $Y$-configuration $b$ is a $Z$-configuration, denoted by $\bar{X}(b)$. It represents a simple sub object of $\gamma$ in $\mathcal{C}^m$. For $Y$-configurations $b, d \in B$, we define $A_{b,d}$ to be an ONB of $\text{hom}(1, X(b) \otimes X(\theta_1(b_1)) \otimes X(d) \otimes X(\theta_1(d_1)))$, a sub space of $\text{hom}(1, \gamma^2)$. Then
\[
LL(\vec{x} \star \vec{y}, \vec{x}' \land \vec{y}')
\]
\[
= \sum_{b, d \in B_1, \alpha \in A_{b, d}} \delta^{1-n-\ell} \delta^3 \sqrt{\frac{d(b) d(d) d(b_j) d(b'_j) d(d_j) d(d'_j)}{d(b_1) d(d_1)}} L(\alpha)
\]
\[
= \sum_{b, d \in B_1, \alpha \in A_{b, d}} \delta^{3-n-\ell} \sqrt{\frac{d(b) d(d) d(b_j) d(b'_j) d(d_j) d(d'_j)}{d(b_1) d(d_1)}} L(\alpha)
\]
\[
= \sum_{b, d \in B_1, \alpha \in A_{b, d}} \delta^4 \frac{1}{d(b_1) d(d_1)} |L(\alpha)|^2 LL(x, x') LL(y, y')
\]
\[
= \sum_{d \in B_1} \delta^{n-2} d(d) LL(x, x') LL(y, y') \quad \text{by Lemma 4.8 and Equation (2)}
\]
\[
= LL(x, x') LL(y, y') \quad \text{by Proposition 4.1}
\]

By the linearity, the equation holds for any \(x\) and \(x'\). (Here we give the pictures for \(n = \ell = 3\). One can figure out the general case.)

**Proposition 6.6.** For \(x \in \text{hom}(1, \gamma^n)\), \(y \in \text{hom}(1, \gamma^{n'})\), \(n, n' \geq 1\),

\[
\Psi(\vec{\mathfrak{S}}(x \star y)) = \Psi(\vec{\mathfrak{S}}(x)) \wedge \Psi(\vec{\mathfrak{S}}(y)).
\]
**Proof.** By Lemma 6.5

\[
\Psi(\mathcal{F}(x \ast y)) = \sum_{x', y' \in B, x', y' \in B} LL(\bar{x} \ast \bar{y}, \bar{x'} \otimes \bar{y'}) x' \otimes y' \\
= \sum_{x', y' \in B} LL(\bar{x}, \bar{x'}) LL(\bar{y}, \bar{y'}) x' \otimes y'
\]

\[
= \Psi(\mathcal{F}(x)) \wedge \Psi(\mathcal{F}(y)).
\]

\[\square\]

**Lemma 6.7.** For \(x \in \text{hom}(1, \gamma^n)\) and \(x' \in \text{hom}(1, \gamma^{n-1})\), \(n \geq 1\),

\[
LL(\phi_1(x), x') = LL(x, \iota_0(x')).
\]

**Proof.** When \(n = 1\), the statement is obvious.

When \(n \geq 2\), suppose \(x = a\vec{j}(b\vec{i})\) and \(x' = a'\vec{j}(b'\vec{i})\). For \(Y\)-configurations \(b \in B\), we define \(A_b\) to be an ONB of \(\text{hom}_{\varphi^1}(1, X(b) \otimes X(\theta_1(b_2)) \otimes X(\theta_1(b_1)))\), a sub space of \(\text{hom}_{\varphi}(1, \gamma^3)\). Then by Lemma 4.8 and Equation (2), we have

\[
\begin{align*}
LL(\phi_1(x), x') &= \sum_{b \in B_1, \alpha \in A_b} \delta^{1-(n-1)} \delta\delta^{-2} d(b) \sqrt{d(b_j)} d(b'_j) \\
&= \delta^{1-n} \sqrt{d(b_j)} \sqrt{d(b'_j)} \\
&= LL(x, \iota_0(x')).
\end{align*}
\]

The general case follows from the linearity. \[\square\]

From the proof of Lemma 6.7, we see that the contraction \(\phi_1\) on the configuration space is contracting the \(Z\)-configurations \(X_{1, j}\) and \(X_{2, j}\). The diagrammatic representation of the contracted
configuration is given by
\[
\delta^{1-n} \sqrt{d(X_{i,j})}
\]
(37)

Lemma 6.8. For \( x \in \text{hom}(1, \gamma^n) \) and \( x' \in \text{hom}(1, \gamma^{n-1}) \), \( n \geq 2 \),
\[
LL(\phi_2(x), x') = LL(x, \iota_1(x'))
\]

Proof. For \( b', b' \in B \), take \( A_{b', b'} \) to be an ONB of \( \text{hom}(1, X(b') \otimes X(b') \otimes X(\theta_1(b'))) \). Then by Lemma 4.8 Equation (2) and Proposition 2.12, we have
\[
LL(\phi_2(x), x') = \delta^{1-n} \delta \delta^{-2} d(b')^d(b') \sqrt{d(b_j^d)}
\]
(38)

\[
= \sum_{b'_{b'}, b'_{b'} \in B} \delta^{1-n} \delta \delta^{-2} d(b')^d(b') \sqrt{d(b_j^d)}
\]
(39)

\[
= \sum_{b'_{b'} \in B} \delta^{-n} d(b') \sqrt{d(b_j^d)}
\]
(40)

\[
= \delta_{b_1, b_2} \delta^2 \delta^{-n}
\]
(41)

\[
= LL(\phi_2(x), x')
\]
(42)

\[
= LL(\phi_2(x), x')
\]
(43)

\[
= LL(\phi_2(x), x')
\]
(44)

\[
= LL(\phi_2(x), x')
\]
(45)
Lemma 6.9. Suppose $H_1$ and $H_2$ are Hilbert spaces, and $T$ is an operator from $H_1$ to $H_2$. If $x \perp T(H_1)$ in $H_2$, then $T^*x = 0$.

Proof. If $x \perp T(H_1)$ in $H_2$, i.e., $<x, Ty> = 0, \forall y \in H_2$, then $<T^*x, y> = 0$. Thus $T^*x = 0$. \qed

Proposition 6.10. For $0 \leq k \leq 2n - 2$, $x \in \text{hom}(1, \gamma^n)$,

$$
\Psi(\mathfrak{F}(\phi_{k+1}(x))) = \phi_k \Psi(\mathfrak{F}(x)).
$$

Proof. For $k = 0, 1$,

$$
\Psi(\mathfrak{F}(\phi_{k+1}(x))) = \sum_{x' \in B_{n-1}} LL(\phi_{k+1}(x), \theta_2(x'))x' \\
= \sum_{x' \in B_{n-1}} LL(x, \iota_k \theta_2(x'))x' \quad \text{by Lemmas 6.7, 6.8}
$$

$$
= \sum_{x' \in B_{n-1}} LL(x, \theta_2 \iota_k (x'))x' \quad \text{by Proposition 3.2}
$$

$$
= \sum_{x'' \in \iota_k(B_{n-1})} LL(x, \theta_2(x'')) \phi_k(x'') \quad \text{by Proposition 5.2 and Lemma 6.9}
$$

$$
= \phi_k \Psi(\mathfrak{F}(x)).
$$

The general case follows from Proposition 6.4. \qed

Proposition 6.11. The map $\Psi : \mathcal{I}_{n,-} \to \mathcal{I}_{n,+}$ is an isometry.

Proof. It is true for $n = 0, 1$ by definition. When $n \geq 2$, for $x, y$ in $\mathcal{I}_{n,+}$,

$$
(\Psi \mathfrak{F}(x), \Psi \mathfrak{F}(y)) = \delta^{n/2} \phi_0 \phi_1 \cdots \phi_{2n-1}(\Psi \mathfrak{F}(x) \wedge \Psi \mathfrak{F}(y)) \\
= \delta^{n/2} \phi_0 \phi_2 \cdots \phi_{2n}(x \ast y) \quad \text{by Propositions 6.6, 6.10}
$$

$$
= \delta^{n/2} \phi_0 \phi_2 \cdots \phi_{2n}(x \ast y) \\
= \langle x, y \rangle \\
= \langle \mathfrak{F}(x), \mathfrak{F}(y) \rangle
$$

\qed

Proposition 6.12. For $0 \leq k \leq 2n - 2$, $x \in \text{hom}(1, \gamma^n)$,

$$
\Psi(\mathfrak{F}(\iota_k(x))) = \iota_{k+1} \Psi(\mathfrak{F}(x)).
$$
Proof. By Propositions 6.11, 5.2 and 6.10 we have
\[ \langle \Psi(\mathcal{F}(x)), y \rangle. \]
\[ = \langle \iota(x), \Psi(\mathcal{F}(y)) \rangle. \]
\[ = \langle x, \phi_k \Psi(\mathcal{F}(y)) \rangle. \]
\[ = \langle x, \Psi(\mathcal{F}(\phi_k+1(y))) \rangle. \]
\[ = \langle \iota_{k+1} \Psi(\mathcal{F}(x)), y \rangle. \]
Therefore \( \Psi(\mathcal{F}(\iota_k(x))) = \iota_{k+1} \Psi(\mathcal{F}(x)). \)

\[ \square \]

Theorem 6.13. The map \( \Psi \) is a planar algebraic *-isomorphism from \( \mathcal{J}_{n-} \) to \( \mathcal{J}_{n+} \). Therefore, the \( m \)-interval Jones-Wassermann subfactor is self-dual for any \( m \geq 1 \).

Proof. We write an elements in \( \mathcal{J}_{n-} \) as \( x' = \mathcal{F}(x), y' = \mathcal{F}(y) \), for \( x, y \in \mathcal{J}_{n+} \).

By Equation (18) and Proposition 6.4, \( \Psi(\rho(x')) = \Psi(\mathcal{F}(\rho(x))) = \rho \Psi(x') \).

By Equation (19) and Proposition 6.6, \( \Psi(x' \wedge y') = \Psi(F(x \ast y)) = \Psi(x') \wedge \Psi(y') \).

By Equation (20) and Proposition 6.10, \( \Psi(\phi_k(x')) = \Psi(\mathcal{F}(\phi_k+1(x))) = \phi_k \Psi(x') \).

By Equation (21) and Proposition 6.12, \( \Psi(\iota_k(x')) = \Psi(\mathcal{F}(\iota_{k+1}(x))) = \iota_k \Psi(x') \).

By Equation (22) and Proposition 6.4, \( \Psi(\theta(x')) = \Psi(\mathcal{F}(\rho^{-1}(x)) = \theta_1(\Psi(x')). \)

That means \( \Psi \) commutes with the \( 6 \! + \! 1 \) elementary actions of planar algebras. So \( \Psi \) is a planar algebraic *-isomorphism.

\[ \square \]

Remark 6.14. The modularity is essential in the proof of the self-duality of Jones-Wassermann subfactors for the unitary MTC \( \mathcal{C} \), so we call this property the modular self-duality of the MTC.

Remark 6.15. Recall that \( \rho_2 \) is a planar algebraic *-isomorphism of \( \mathcal{J}_{n+} \) with periodicity \( m \), then for each \( k \in \mathbb{Z}_m \), \( \Psi(\rho^k_2) \) is a planar algebraic *-isomorphism from \( \mathcal{J}_{n-} \) to \( \mathcal{J}_{n+} \). Therefore there are \( k \) different ways to lift the shading of \( \mathcal{J}_{n,\pm} \). Each choice defines an unshaded subfactor planar algebra.

Remark 6.16. From orbifold theory it is easy to see that the Jones-Wassermann subfactors for \( n \) disjoint intervals are isomorphic to its dual as subfactors. Here is a proof using orbifold theory: the dual of \( \pi_{1,0,1,...,n-1} \) is \( \pi_{1,-1,n-2,...,0} \), but \( \{ n-1, n-2, ..., 0 \} \) is conjugate to \( \{ 0, 1, ..., n-1 \} \) in \( S_n \) via \( g(i) = n - i - 1, i = 0, 1, ..., n - 1 \), hence \( \pi_{1,0,1,...,n-1} \simeq g \pi_{1,-1,n-2,...,0} g^{-1} \). We refer the readers to [KLX05] for details.

If we take \( \mathcal{C} \) to be the unitary modular tensor category, such that its fusion ring is the cyclic group \( \mathbb{Z}_d \) and its \( S \) matrix is the discrete Fourier transform of \( \mathbb{Z}_d \), then two-box space of the two-interval Jones-Wassermann subfactor is isomorphic to \( L^2(\mathbb{Z}_d) \). It is known that the usual multiplication and coproduct on the 2-box space in subfactor theory coincide with the multiplication and convolution on \( L^2(\mathbb{Z}_d) \). In addition, we have shown that the SFT is the \( S \) matrix which becomes the usual discrete Fourier transform. The modular self-duality reduces to the self-duality of \( \mathbb{Z}_d \) on \( L^2(\mathbb{Z}_d) \). Therefore the modular self-duality generalize and categorify the self-duality of finite abelian groups.

6.2. Actions of planar tangles on the configuration space. Motivated by the Jones-Wassermann subfactor, we obtain actions of planar tangles on the configuration spaces \( \{ \text{Conf}_{n,m} \}_{n,m \in \mathbb{N}} \) in both \( X \) - and \( Y \) -directions. Moreover, these actions coincide with the geometric action on the lattices: The contraction tangle \( \phi_1 \) corresponds contractions of lattices to as shown in Equation (37). The correspondence for the other \( 6+1 \) elementary tangles are more straightforward. Thus the actions of planar tangles in two different directions commute. We call the (Hilbert) space \( \{ \text{Conf}_{n,m} \}_{n,m \in \mathbb{N}} \)
equipped with such commutative actions of bidirectional planar tangles a bi-planar algebra which we will study in the future.

Note that
\[
\mathcal{D}_+(x) = \sum_{x' \in B} \overline{L(x, x')} x' = \theta_2 \Psi(\tilde{\gamma}(x)).
\] (46)

Since \(\theta_2\) is anti-isometry, we obtain Theorem 2.3 from Proposition 6.11.

Moreover, \(\theta_2\) commute with the action of planar tangles, we have the following result corresponding to Propositions 6.4, 6.6, 6.10, 6.12:

**Proposition 6.17.** For \(x \in \text{Conf}(\mathcal{C})_{n,m}, y \in \text{Conf}(\mathcal{C})_{t,m},\)
\[
\mathcal{D}_+ \rho(x) = \rho \mathcal{D}_+(x),
\]
\[
\mathcal{D}_+ \rho^{-1} \theta_1(x) = \theta_1 \mathcal{D}_+(x),
\]
\[
\mathcal{D}_+(x \ast y) = \mathcal{D}_+(x) \wedge \mathcal{D}_+(y),
\]
\[
\mathcal{D}_+ \phi_{k+1}(x) = \phi_k \mathcal{D}_+(x),
\]
\[
\mathcal{D}_+ \iota_k(x) = \iota_{k+1} \mathcal{D}_+(x).
\]

**References**

[DR89] S. Doplicher and J. Roberts, *A new duality theory for compact groups*, Invent. Math. 98 (1989), no. 1, 157–218.

[Dri86] V. G. Drinfeld, *Quantum groups*, Zapiski Nauchnykh Seminarov POMI 155 (1986), 18–49.

[EW07] J. Erljman and H. Wenzl, *Subfactors from braided C* tensor categories*, Pacific Journal of Mathematics 231 (2007), no. 2, 361–399.

[GLS10] A. Guionnet, V. F. R. Jones, and D. Shlyakhtenko, *Random matrices, free probability, planar algebras and subfactors*, Quanta of maths: Non-commutative Geometry Conference in Honor of Alain Connes, Clay Math. Proc. 11 (2010), 201–240.

[LW16] Z. Liu, and J. Wu *The noncommutative Fourier transform: a survey*, Acta Mathematica Sinica, Chinese Series, 60, (2017), no. 1, 81–96.

[JLCW] C. Jiang, Z. Liu, and J. Wu, *Extremal pairs of young’s inequality and renormalization maps*.

[Jon98] V. F. R. Jones, *Some unitary representations of thompson’s groups F and T*, [http://arxiv.org/abs/1412.7740](http://arxiv.org/abs/1412.7740).

[Jon00] V. F. R. Jones, *von Neumann algebras in mathematics and physics*, Proc. Internat. Congress Math. Kyoto (1990), 121–138.

[KLM01] Y. Kawahigashi, R. Longo, and M. Müger, *Multi-interval subfactors and modularity of representations in conformal field theory*, Commun. Math. Phys. 219 (2001), no. 3, 631–669.

[KLX05] V. Kac, R. Longo, and F. Xu, *Solitons in affine and permutation orbifolds*, Commun. Math. Phys. 253 (2005), no. 3, 723–764.

[Liu16] Z. Liu, *Exchange relation planar algebras of small rank*, Trans. AMS 368 (2016), 8303–8348, DOI: [http://dx.doi.org/10.1090/tran/6882](http://dx.doi.org/10.1090/tran/6882).

[LR95] R. Longo and K-H Rehren, *Nets of subfactors*, Reviews in Mathematical Physics 7 (1995), no. 04, 567–597.

[LWX3] Z. Liu, A. Wozniakowski, and A. Jaffe, *The 3D quon language for quantum information*, [http://arxiv.org/abs/1612.02630v2](http://arxiv.org/abs/1612.02630v2).

[Müg93] M. Müger, *From subfactors to categories and topology II: The quantum double of tensor categories and subfactors*, Journal of Pure and Applied Algebra 180 (2003), no. 1, 159–219.

[Ono91] K. Oono, *Quantum symmetry, differential geometry of finite graphs, and classification of subfactors*, Univ. of Tokyo Seminary Notes, 1991.

[Pop94] S. Popa, *Classification of amenable subfactors of type II*, Acta Math. 172 (1994), 352–445.

[Tur94] VG. Turaev, *Quantum invariants of knots and 3-mansifolds*, Walter de Gruyter, 1994.

[Wat98] A. Wassermann, *Operator algebras and conformal field theory III. fusion of positive energy representations of \(\text{SU}(N)\) using bounded operators*, Invent. Math. 133 (1998), no. 3, 467–538.

[Xu00] F. Xu, *Jones-wassermann subfactors for disconnected intervals*, Communications in Contemporary Mathematics 2 (2000), no. 03, 307–347.
Department of Mathematics and Department of Physics, Harvard University
E-mail address: zhengweiliu@fas.harvard.edu

Department of Mathematics, University of California, Riverside
E-mail address: xufeng@math.ucr.edu