Combinatorial representations of Coxeter groups over a field of two elements

Hau-wen Huang† Chih-wen Weng‡

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Abstract

Let $W$ denote a simply-laced Coxeter group with $n$ generators. We construct an $n$-dimensional representation $\phi$ of $W$ over the finite field $\mathbb{F}_2$ of two elements. The action of $\phi(W)$ on $\mathbb{F}_2^n$ by left multiplication is corresponding to a combinatorial structure extracted and generalized from Vogan diagrams. In each case $W$ of types A, D and E, we determine the orbits of $\mathbb{F}_2^n$ under the action of $\phi(W)$, and find that the kernel of $\phi$ is the center $Z(W)$ of $W$.

Keywords: Coxeter groups; Dynkin diagrams; Group representations; Vogan diagrams.

1 Introduction

A simply-laced Coxeter group is a group $W_S(m)$ with a finite set of generators $S \subseteq W_S(m)$ subject only to relations

$$(ss')^{m(s,s')} = 1,$$

where $m(s, s) = 1$ and $m(s, s') = m(s', s) \in \{2, 3\}$ for $s \neq s'$ in $S$. When $m$ is specified, we write $W_S$ for $W_S(m)$, and if both $S$ and $m$ are specified, we
write $W$ for $W_S(m)$. A Coxeter graph $S$ represents a simply-laced Coxeter group $W_S(m)$, and vice versa. The vertex set of this graph is $S$, and there is an edge joining two vertices $s$ and $s'$ whenever $m(s,s') = 3$. There are Coxeter groups which are not simple-laced. In this article we always assume simple-laced property in a Coxeter group to make the corresponding Coxeter graph $S$ a simple graph.

We shall investigate a kind of flipping puzzle, which is also studied in [10, 11], associated with a given Coxeter graph $S$. The configuration of the flipping puzzle is $S$, together with an assignment of a unique state, white or black, on each vertex of $S$. A move in the puzzle is to select a vertex $s$ which has black state, and then flip the state of each neighbor of $s$. When $S$ is one of the Dynkin diagrams described in Figure 1, the configuration above is essentially a Vogan diagram with identity involution, which was first defined in [9], in a more general way, as a combinatorial object representing the real form of the corresponding complex simple Lie algebra and a system of choices. See also [1, 2, 5].

\[ A_n(n \geq 1) \]

\[ D_n(n \geq 4) \]

\[ E_6 \]

\[ E_7 \]

\[ E_8 \]

\textbf{Figure 1:} Simply-laced Dynkin diagrams.

We fix a simply-laced Coxeter group $W$ and its Coxeter graph $S$, where
Let $F_2$ denote the finite field of two elements 0 and 1. In Section 2, we use the column vector set $F_2^S = F_2^n$ to describe the set of configurations in the flipping puzzle associated with $S$ by setting that $\ell_s = 1$ iff the configuration $\ell \in F_2^n$ has black state in the vertex $s$. For each vertex $s \in S$, we find a way to associate the move with selecting vertex $s$ as an $n \times n$ invertible matrix $s$ over $F_2$. This $s$ acts on a configuration $\ell \in F_2^n$ by left multiplication to become a new configuration $s\ell$ which has the desired property as stated in the definition of the flipping puzzle when $\ell_s = 1$. Unlike in the definition, our move $s$ does not select configuration $\ell$, but if a configuration has white state in $s$, it makes no effect; i.e. if $\ell_s = 0$ then $s\ell = \ell$.

Let $GL_n(F_2)$ denote the set of $n \times n$ invertible matrices over $F_2$ and let $W$ denote the subgroup of $GL_n(F_2)$ generated by the moves $s$ for $s \in S$. We refer $W$ to a flipping group of $S$. In Section 3, we find that the canonical map $\phi : W \to GL_n(F_2)$, lifted from $\phi(s) = s$ for $s \in S$, is a homomorphism with $\phi(W) = W$. Due to its origination, we refer such a map to the Vogan representation of $W$. Then we find that the flipping group $W$ has trivial center in Section 4. In Sections 5, 6 and 7, we assume $W$ to be $A_n$, $D_n$ and $E_n$ respectively. By using the finiteness of $W$, we can determine the size of the corresponding flipping group $W$. We find that the kernel of the Vogan representation of $W$ is the center $Z(W)$ of $W$ when $W$ is finite.

In the flipping puzzle on a Coxeter graph $S$, two configurations are said to be equivalent if one can be obtained from the other by a sequence of moves. Let $\mathcal{P}$ denote the partition of configurations (i.e. $F_2^n$) according to the above equivalent relation. As a byproduct of our work, we solve the flipping puzzle associated with $S$ when $S$ is each of $A_n$, $D_n$ and $E_n$ by determining $\mathcal{P}$. Note that when $S$ is a tree, a generalization of Dynkin diagrams, some partial results on $\mathcal{P}$ are obtained in [11] and [10].

## 2 Flipping groups

Throughout this article, $W$ will be a simply-laced Coxeter group with corresponding Coxeter graph $S$ of $n$ elements and edge set $R = \{ss' \mid m(s, s') = 3\}$. We shall construct a matrix group associated with the flipping puzzle on the Coxeter graph $S$. Let $\text{Mat}_n(F_2)$ denote the set of $n \times n$ matrices over $F_2$ with rows and columns indexed by $S$. Let $F_2^n$ denote the set of $n$-dimensional
column vectors over $F_2$ indexed by $S$. For $s \in S$, let $\tilde{s}$ denote the characteristic vector of $s$ in $F_2^n$; that is $\tilde{s} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t$, where 1 is in the position corresponding to $s$.

**Definition 2.1.** For $s \in S$, we associate a matrix $s \in \text{Mat}_n(F_2)$, denoted by the bold type of $s$, as

$$s_{uv} = \begin{cases} 1, & \text{if } u = v, \text{ or } v = s \text{ and } uv \in R; \\ 0, & \text{else}, \end{cases}$$

where $u, v \in S$.

The following is a reformulating of Definition 2.1.

**Lemma 2.2.** For $s, v \in S$,

$$s v = \begin{cases} \tilde{v}, & \text{if } v \neq s; \\ \tilde{v} + \sum_{uv \in R} \tilde{u} & \text{if } v = s. \end{cases}$$

□

The flipping puzzle associated with $S$, which is described in the introduction, is now restated as follows. A configuration is simply a vector $\ell \in F_2^n$, where $\ell_s = 1$ (resp. $\ell_s = 0$) means that the vertex $s \in S$ has black state (resp. white state). In this setting, if $\ell_s = 1$ then $s \ell$ is the new configuration after the move to select the vertex $s$. Note that if $\ell_s = 0$, we have $s \ell = \ell$ from Lemma 2.2, so we can view the action of $s$ on $\ell$ as a feigning move on $\ell$ which is not originally defined as a move in the flipping puzzle. The following lemma is immediate from this combinatorial realization.

**Lemma 2.3.** For $s \in S$, $s$ is an involution; that is $s^2 = I$, the identity matrix. □

From Lemma 2.3, $s$ is invertible, so we can give the following definition.

**Definition 2.4.** Let $W$ denote the subgroup of $\text{GL}_n(F_2)$ generated by the set $\{s \mid s \in S\}$. $W$ is referring to the flipping group of $S$. 

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3 Coxeter groups and their combinatorial representations

Let $W$ denote a simply-laced Coxeter group. Recall that an $n$-dimensional representation of $W$ over $F_2$ is a homomorphism of $W$ into $GL_n(F_2)$. It is notorious difficult in the study of groups only defined by generators and relations. Hence the representation theory of Coxeter groups plays an important role in the study. In [7, Section 5.3], Humphreys gives "geometric representations" of Coxeter groups and use these representations to show that the finite Coxeter groups are essentially those associated with Dynkin diagrams. In this section we shall show that the flipping groups defined in the last section give "combinatorial representations" of simply-laced Coxeter groups. First we need a lemma.

Lemma 3.1. Let $W$ denote a simply-laced Coxeter group with Coxeter graph $S$. For $s \in S$, set $E_s \in \text{Mat}_n(F_2)$ by

$$E_s \tilde{v} = \begin{cases} 0, & \text{if } v \neq s; \\ \sum_{uv \in R} \tilde{u}, & \text{if } v = s \end{cases} \quad \text{for } v \in S. \quad (3.1)$$

Then with referring to the notation in Definition 2.1, the following (i)-(iii) hold.

(i) $s = I + E_s$ for $s \in S$

(ii) $E_s E_s = 0$, if $s' s / \in R$.

(iii) If $s_i s_{i-1} \in R$ for $i = 1, 2, \ldots, t$, then

$$E_{s_t} E_{s_{t-1}} \cdots E_{s_0} = \begin{cases} E_{s_0}, & \text{if } s_t = s_0; \\ E_{s_t} E_{s_0}, & \text{if } s_t s_0 \in R. \end{cases}$$

Proof. (i) is immediate from Lemma 2.2. Note that $E_s E_s \tilde{v} = 0$ by (3.1) for any $v, s, s' \in S$ with $s' s / \in R$, and hence we have (ii). (iii) follows from the same reason as in (ii) by applying the product of matrices in either side of the equation to $\tilde{v}$ and obtaining the desired equality in each case. \qed

Theorem 3.2. Let $W$ denote a simply-laced Coxeter group with Coxeter graph $S$. Let $W$ denote the flipping group of $S$. Then there exists a surjective homomorphism $\phi : W \to W$ such that $\phi(s) = s$ for $s \in S$. In particular, $\phi$ is a representation of $W$ over $F_2$. 

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Proof. We have seen $s^2 = I$ for $s \in S$. It remains to show $(ss')^2 = I$ if $s \neq s'$ and $ss' \not\in R$, and to show $(ss')^3 = I$ if $ss' \in R$. For $s, s' \in S$,

$$ss' = (I + E_s)(I + E_{s'}) = I + E_s + E_{s'} + E_s E_{s'}$$

by Lemma 3.1(i). In the case $s \neq s'$ and $ss' \not\in R$,

$$(ss')^2 = (I + E_s + E_{s'})(I + E_s + E_{s'}) = I + 2E_s + 2E_{s'}$$

by Lemma 3.1(ii). In the case $ss' \in R$,

$$(ss')^2 = (I + E_s + E_{s'} + E_s E_{s'})(I + E_s + E_{s'} + E_s E_{s'}) = I$$

by Lemma 3.1(iii).

\[ \square \]

**Definition 3.3.** The representation $\phi$ defined in Theorem 3.2 is called the **Vogan representation** of $W$.

Suppose $J \subseteq S$. Let $W_J$ denote the subgroup of $W$ generated by the set $\{s \mid s \in J\}$ and $W_J$ denote simply-laced Coxeter group with the set $J$ of generators with the function $m \upharpoonright J \times J$, the restriction of $m$ to $J \times J$. Note that $W_J$ is isomorphic to the subgroup of $W$ generated by the set $\{s \mid s \in J\}$ [7, Section 5.5]. Hence we use the same symbol $W_J$ to express these two isomorphic groups. It makes no confused if the place that $W_J$ appears is also considered. For example, the first $W_J$ in (iii) of the following lemma is in the first meaning and the remaining two $W_J$ are in the second meaning. Note that $W_J$, which is different to $W_J$, is the flipping group on $J$. Let $G[J]$ denote the submatrix of $G \in \text{Mat}_n(F_2)$ with rows and columns indexed by $J$, and $W_J[J] := \{G[J] \mid G \in W_J\}$.
Lemma 3.4. Suppose $J \subseteq S$. The following (i)-(iii) hold.

(i) $W_J[J] = W_J$.

(ii) The map $\psi : W_J \to W_J$, defined by $\psi(G) = G[J]$ for $G \in W_J$, is a surjective homomorphism.

(iii) Let $\phi$ and $\phi'$ denote the Vogan representations of $W_S$ and $W_J$ respectively. Then $\phi' = \psi \circ \phi \upharpoonright W_J$. In particular, $\operatorname{Ker} \phi \upharpoonright W_J \subseteq \operatorname{Ker} \phi'$.

Proof. By Definition 2.1, $s_{uv} = 0$ for $s, u \in J$ and $v \in S - J$. By this, each matrix $G \in W_J$ has the form

$$G = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

if indices in $J$ are placed in the beginning of rows and columns, where $A$ is a $|J| \times |J|$ matrix, $B$ is an $(n - |J|) \times |J|$ matrix, $C$ is an $(n - |J|) \times (n - |J|)$ matrix, and $0$ is a $|J| \times (n - |J|)$ zero matrix. Then (i) and (ii) follow from the following matrix product rule in block form:

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} A' & 0 \\ B' & C' \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ BA' + CB' & CC' \end{pmatrix}.$$

Since $\psi \circ \phi(s) = s[J] = \phi'(s)$ by (i) for all $s \in J$, we see $\phi' = \psi \circ \phi \upharpoonright W_J$, and this implies $\operatorname{Ker} \phi \upharpoonright W_J \subseteq \operatorname{Ker} \phi'$.

\[\square\]

4 The center of a flipping group

As we shall see in Proposition 6.10 that the Coxeter group of type $D_n$ has nontrivial center when $n$ is even. In this section, we show that the center $Z(W)$ of any flipping group $W$ of a Coxeter graph $S$ is trivial. Therefore, the center $Z(W)$ of any Coxeter group $W$ is contained in the kernel of the Vogan representation of $W$. Recall that a Coxeter graph $S$ is disconnected if there is a partition of $S = S' \cup S''$ with $S', S'' \neq \emptyset$ and there is no edge $uv \in R$ with $u \in S'$ and $v \in S''$. In this case the Coxeter group $W$ is isomorphic to the direct product $W' \times W''$ of the Coxeter groups $W' = W_{S'}$ and $W'' = W_{S''}$. $S$ is connected if $S$ is not disconnected.

Proposition 4.1. Let $W$ denote a simple-laced Coxeter group with Coxeter graph $S$. Then the center $Z(W)$ of the flipping group $W$ of $S$ is trivial.
Proof. It suffices to assume that $S$ is connected with at least two vertices. Let $Z$ be an element in the center of $W$ and let $u,v$ be two distinct elements in $S$. We show that $Z_vu = 0$ to conclude $Z = I$. Suppose $Z_vu = 1$. On the one hand $vZ_vu \neq Z_vu$ since $Z_vu$ has 1 in the $v$th position. On the other hand, $vZ_vu = Zv\tilde{u} = Z\tilde{u}$ since $v\tilde{u} = \tilde{u}$. Hence we have a contradiction.

From the above Proposition 4.1 we immediately have the following corollary.

**Corollary 4.2.** Let $W$ denote a simply-laced Coxeter group. Then the center $Z(W)$ is contained in the kernel of the Vogan representation of $W$. \qed

## 5 Coxeter groups of type $A_n$

Recall that the Vogan representation $\phi$ of $W$ is *faithful* whenever $\phi$ is injective. Also $\phi$ is *irreducible* if there is no subspace $V \subseteq F_2^n$, $V \neq 0, F_2^n$, such that $\phi(W)V \subseteq V$. For $a \in F_2^n$, the subset of $F_2^n$ consisting of all elements $Ga$ with $G \in \phi(W)$ is called the orbit of $F_2^n$ containing $a$ under the action of $\phi(W)$.

In this section we assume that $W$ is of type $A_n$ with the Coxeter graph $S$ as shown in Fig. 1, and determine the orbits of $F_2^n$ under the action of $\phi(W)$. We also show that the kernel of the Vogan representation $\phi$ of $W$ is the center $Z(W)$ of $W$ and determine the reducibility of $\phi$. The trivial case is given in the following.

**Proposition 5.1.** Let $W$ be a Coxeter group of type $A_1$ with the Vogan representation $\phi$. Then the orbits of $F_2$ are $\{0\}, \{1\}$ under the action of $\phi(W)$, $\text{Ker } \phi = \{1, s_1\} = W = Z(W)$, and $\phi$ is irreducible.

*Proof.* This follows from that $W = \{1, s_1\}$ and $\phi(W)$ is a trivial group. \qed

In the remaining of this section, we always assume $n \geq 2$. Set

$$\overline{1} = \tilde{s}_1, \quad \overline{i + 1} = s_is_{i-1} \cdots s_1 \overline{1} \quad \text{for } 1 \leq i \leq n. \quad (5.1)$$

Note that

$$\overline{i} = \tilde{s}_{i-1} + \tilde{s}_i \quad \text{for } 2 \leq i \leq n, \quad (5.2)$$

and

$$\overline{n + 1} = \tilde{s}_n = \overline{1} + \overline{2} + \cdots + \overline{n}. \quad (5.3)$$
Set \( \Delta = \Delta(A_n) := \{1, 2, \ldots, n\} \). Note that \( \Delta \) is a basis of \( F_2^n \). We refer \( \Delta \) to a simple basis of \( F_2^n \). For \( a \in F_2^n \), let \( \Delta(a) \) denote the subset of \( \Delta \) consisting of all the elements appeared in the expression of \( a \) as a linear combination of elements in \( \Delta \). The weight of an element \( a \in F_2^n \) is \( wt(a) := |\Delta(a)| \). For example, \( \Delta(n + 1) = \Delta \) and \( wt(n + 1) = n \).

**Lemma 5.2.** \( s_i \overline{i} = \overline{i + 1}, s_i \overline{i + 1} = \overline{i} \) and \( s_i \) fixes other vectors in \( \{1, 2, \ldots, n + 1\} \) − \( \{i, i + 1\} \) for \( 1 \leq i \leq n \).

**Proof.** This is immediate by applying Lemma 2.2, (5.1) and (5.2).

Let \( S_{n+1} \) denote the group of permutations on \( \{1, 2, \ldots, n + 1\} \). By Lemma 5.2, we can give the following definition.

**Definition 5.3.** Let \( \alpha : W \to S_{n+1} \) be the homomorphism defined by

\[
\alpha(G) \overline{j} = G \overline{j}
\]

for each \( 1 \leq j \leq n + 1 \) and \( G \in W \).

Note that \( \alpha(s_i) \) is the transposition \( \overline{i, i + 1} \) in \( S_{n+1} \) for each \( 1 \leq i \leq n \).

**Lemma 5.4.** \( \alpha \) is an isomorphism from \( W \) onto \( S_{n+1} \).

**Proof.** \( \alpha \) is surjective since the transpositions \( \alpha(s_1), \alpha(s_2), \ldots, \alpha(s_n) \) generate \( S_{n+1} \). Since \( \Delta \cup \{n + 1\} \) spans \( F_2^n \), \( \alpha \) is injective.

The next proposition determines the orbits of \( F_2^n \) under the action of \( W \).

**Proposition 5.5.** For \( 0 \leq i \leq \lfloor \frac{n+1}{2} \rfloor \),

\[
O_i = \{a \in F_2^n \mid wt(a) = i \text{ or } n + 1 - i\}
\]

is an orbit of \( F_2^n \) under the action of \( W \), where \( \lfloor t \rfloor \) is the largest integer less than or equal to \( t \).

**Proof.** Suppose \( a \in F_2^n \) with \( wt(a) = i \). Observe that from Lemma 5.4 and (5.3),

\[
\Delta(Ga) = \begin{cases} 
\alpha(G) \Delta(a), & \text{if } n + 1 \notin \alpha(G) \Delta(a) ; \\
\Delta - \alpha(G) \Delta(a), & \text{if } n + 1 \in \alpha(G) \Delta(a)
\end{cases}
\]

for \( G \in W \). The proposition follows from this observation because the subgroup of \( \alpha(W) = S_{n+1} \) generated by the transpositions \( \alpha(s_1), \alpha(s_2), \ldots, \alpha(s_{n-1}) \) acts transitively on the fixed size subsets of \( \Delta \), and \( s_n \overline{n} = \overline{1} + \overline{2} + \cdots + \overline{n} \) by Lemma 5.2 and (5.3).
In the following propositions, we study the reducibility of $\phi$ and Ker $\phi$.

**Proposition 5.6.** The Vogan representation $\phi$ of $W$ is irreducible if and only if $n$ is even.

**Proof.** Let $V$ denote a nontrivial proper subspace of $F_2^n$ such that $\phi(W)V \subseteq V$. Referring to Proposition 5.5, note that

$$V = \bigcup_{i \in J} O_i$$

(5.4)

for some proper subset $J \subseteq \{0, 1, \ldots, \frac{n+1}{2}\}$ with $J \neq \{0\}$. Note that the set in the right side of (5.4) to be closed under addition is when it is the set of even weight vectors, and this occurs if and only if $n$ is odd. □

**Proposition 5.7.** The Vogan representation $\phi$ of $W$ is faithful. In particular, Ker $\phi = Z(W)$ is the trivial group.

**Proof.** The first statement follows from that Proposition 5.4 and $W$ is isomorphic to $S_{n+1}$ [7, p41]. The second follows from the first and Corollary 4.2. □

### 6 Coxeter groups of type $D_n$

Fix an integer $n \geq 4$. Let $W$ denote the Coxeter group of type $D_n$ with the Coxeter graph in Fig. 1. Let $\phi$ denote the Vogan representation of $W$, and $W = \phi(W)$ be the flipping group of $S$. Set

$$\overline{1} = \overline{s_1}, \ i + \overline{1} = s_is_{i-1} \cdots s_1\overline{1} \text{ for } 1 \leq i \leq n - 1, \text{ and } n + \overline{1} = \overline{s_n}. \quad (6.1)$$

Note that

$$\overline{i} = \overline{s}_{i-1} + \overline{s}_i \quad \text{for } 2 \leq i \leq n - 2,$$

$$\overline{n - 1} = \overline{s}_{n-2} + \overline{s}_{n-1} + \overline{s}_n, \quad (6.2)$$

and

$$\overline{n} = \overline{s}_{n-1} + \overline{s}_n = \overline{1} + \overline{2} + \cdots + \overline{n - 1} + \overline{n - 1}. \quad (6.3)$$

Set $\Delta = \Delta(D_n) := \{\overline{1}, \overline{2}, \ldots, \overline{n-1}, \overline{n+1}\}$ to be the simple basis of $F_2^n$ in the case of type $D_n$. Set $\Delta(a)$ and $wt(a)$ as before for $a \in F_2^n$. For example, $\Delta(\overline{n}) = \Delta - \{\overline{n+1}\}$ by (6.3), and $wt(\overline{n}) = n - 1$. 10
Lemma 6.1. The following (i),(ii) hold.

(i) For each $1 \leq i \leq n - 1$, $s_i \overline{i} = \overline{i + 1}$, $s_i \overline{i + 1} = \overline{i}$, and $s_i \overline{j} = \overline{j}$ for $j \in \{1, 2, \ldots, n + 1\} - \{i, i + 1\}$.

(ii) $s_n \overline{n} = \pi$, $s_n \overline{n - 1} = \overline{n - 1}$, $s_n \overline{n + 1} = \overline{n - 1} + \pi + n + 1$, and $s_n \overline{j} = \overline{j}$ for $j \in \{1, 2, \ldots, n - 2\}$.

In particular, $n + 1 \in \Delta(G_n + 1)$ and $G(\{1, 2, \ldots, n\}) \subseteq \{1, 2, \ldots, n\}$ for all $G \in W$.

Proof. This follows immediately from Lemma 2.2, (6.1) and (6.2).

Definition 6.2. Let $\beta : W \to S_n$ denote the homomorphism defined by $\beta(G)(j) = G_j$ for $1 \leq j \leq n$ and $G \in W$.

In fact, $\beta$ is surjective since the $n - 1$ transpositions $\beta(s_1)$, $\beta(s_2)$, \ldots, $\beta(s_{n-1})$ generate $S_n$. Let $Z$ denote the additive group of $(n - 1)$-dimensional subspace of $F_2^n$ spanned by the set $\{1, 2, \ldots, n - 1\}$. Note that $a \in Z$ iff $n + 1 \not\in \Delta(a)$ for $a \in F_2^n$. By Lemma 6.1 and (6.3), $Z$ is closed under the left multiplication of matrices in $W$.

Proposition 6.3. The Vogan representation $\phi$ of $W$ is not irreducible. In particular $\phi(W)Z \subseteq Z$.

Hence $Z$ is a disjoint union of orbits of $F_2^n$ under the action of $W$. Note that $F_2^n - Z$ is also a disjoint union of orbits of $F_2^n$ under the action of $W$. The following proposition determines all the orbits of $F_2^n$ under the action of $W$.

Proposition 6.4. The following are orbits of $F_2^n$ under the action of $W$.

$$
O_i = \{a \in Z \mid \text{wt}(a) = i \text{ or } n - i\},
$$

$$
\Omega_o = \{a \in F_2^n - Z \mid \text{wt}(a) \equiv 1 \text{ or } n - 1 \pmod 2\},
$$

$$
\Omega_e = \{a \in F_2^n - Z \mid \text{wt}(a) \equiv 0 \text{ or } n \pmod 2\},
$$

where $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. In particular $\Omega_o = \Omega_e = F_2^n - Z$ is an orbit when $n$ is odd.
Proof. The proof is similar to the proof of Proposition 5.5. The reason that $O_i$ is an orbit follows from two facts: (i) $\beta(s_1), \beta(s_2), \ldots, \beta(s_{n-2})$ generate the subgroup $S_{n-1}$ of $S_n$ consisting of permutations on $\Delta - \{n+1\}$ and $S_{n-1}$ acts transitively on fixed size subsets of $\Delta - \{n+1\}$, and (ii)

$$s_{n-1}n-1 = s_n n-1 = \overline{n-1} = \overline{1+2+\cdots+n-1}$$

by Lemma 6.1(i),(ii) and (6.3). The reason that $\Omega_o$ and $\Omega_e$ are orbits follows from an additional fact that

$$wt(s_n(n+1)) = wt(n-1 + \overline{n+n+1}) = wt(\overline{1+2+\cdots+n-2+n+1}) = n-1.$$  

\[\square\]

We study the structure of $W$.

**Definition 6.5.** Let $\gamma : W \to \text{Aut}(Z)$ denote the homomorphism from $W$ into the group $\text{Aut}(Z)$ of automorphisms of $Z$ such that

$$\gamma(G)(u) = Gu$$

for $G \in W$ and $u \in Z$.

**Lemma 6.6.** There exists a unique homomorphism $\theta : S_n \to \text{Aut}(Z)$ such that $\gamma = \theta \circ \beta$.

**Proof.** Since $\beta$ is surjective, it suffices to show that the kernel of $\beta$ is contained in the kernel of $\gamma$. Suppose $G \in \text{Ker} \beta$. Then $G^i = 1$ for $1 \leq i \leq n$; in particular $G$ fixes each element in the basis $\Delta - \{n+1\}$ of $Z$. Thus $G \in \text{Ker} \gamma$. \[\square\]

Let $Z \rtimes_\theta S_n$ denote the group of *external semidirect product* of $Z$ and $S_n$ with respect to $\theta$[8, p.155]; i.e. $Z \rtimes_\theta S_n$ is the set $Z \times S_n$ with the following product rule:

$$(u, \sigma)(v, \tau) = (u + \theta(\sigma)(v), \sigma \tau),$$

where $u, v \in Z$ and $\sigma, \tau \in S_n$. Note that $\overline{n+1 + G\overline{n+1}} \in Z$ for any $G \in W$ by Lemma 6.1.

**Definition 6.7.** Let $\delta : W \to Z \rtimes_\theta S_n$ denote the map defined by

$$\delta(G) = (\overline{n+1 + G\overline{n+1}}, \beta(G))$$

for any $G \in W$.  

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Lemma 6.8. $\delta$ is an injective homomorphism of $W$ into $Z \rtimes \theta S_n$.

Proof. For $G, H \in W$,

$$\delta(G)\delta(H) = \frac{(n+1 + Gn + 1, \beta(G))(n + 1 + Hn + 1, \beta(H))}{(n + 1 + Gn + 1 + \theta(\beta(G))(n + 1 + Hn + 1), \beta(G)\beta(H))}$$

$$= \frac{(n + 1 + G(n + 1) + G(n + 1) + Hn + 1), \beta(G)\beta(H))}{(n + 1 + G + Hn + 1, \beta(G))}$$

$$= \frac{(n + 1 + GHn + 1, \beta(GH))}{\delta(GH)}.$$

This shows that $\delta$ is a homomorphism. $\delta$ is injective since if $n + 1 + Gn + 1 = 0$ and $G \in \text{Ker } \beta$, then $G$ fixes all vectors in $\Delta$, so $G$ is the identity matrix. \qed

Note that $Z = \frac{n + 1}{\omega}$ if $n$ is odd, and $Z = (\frac{n + 1}{\omega} \cup \frac{n + 1}{\omega})$ if $n$ is even.

Lemma 6.9. $\delta(W) = (\frac{n + 1}{\omega}) \rtimes \theta S_n$. In particular $\delta(W) = Z \rtimes \theta S_n$ if $n$ is odd; $\delta(W)$ has index 2 in $Z \rtimes \theta S_n$ if $n$ is even.

Proof. Note that $\delta(s_1), \delta(s_2), \ldots, \delta(s_{n-2}), \delta(s_{n-1})$ generate $0 \rtimes \theta S_n$. Since $\omega$ is an orbit containing $n + 1$, we have $\delta(W) = (n + 1 + \omega) \rtimes \theta S_n$. The second part follows from Proposition 6.4. \qed

Proposition 6.10. The Vogan representation $\phi$ of $W$ is faithful when $n$ is odd; $\text{Ker } \phi$ has order 2 when $n$ is even. Moreover, $\text{Ker } \phi$ is the center $Z(W)$ of $W$.

Proof. Note that $W$ is isomorphic to the semidirect product $Z \rtimes S_n$ of $Z$ and $S_n$ [7, p.42]. By Lemma 6.9, $\phi$ is faithful when $n$ is odd, and $\text{Ker } \phi$ has order 2 when $n$ is even. From Corollary 4.2, $Z(W) \subseteq \text{Ker } \phi$, and from the fact that a normal subgroup of order 2 is contained in the center, we have $\text{Ker } \phi \subseteq Z(W)$.

\section{Coxeter groups of type $E_n$}

Fix an integer $n \geq 6$. Let $W$ denote the Coxeter group of type $E_n$ with the Coxeter graph $S$ in Fig. 2. In this section we shall determine the orbits of $F_2^n$ under the action of the flipping group $W$ of $S$. Restricting the attention
to the case $n = 6, 7$ or $8$ in which $W$ is finite, we show that the kernel of the Vogan representation $\phi$ of $W$ is the center $Z(W)$ of $W$.

Set $\mathcal{I} = \bar{s}_1, \bar{i+1} = s_is_{i-1} \cdots s_1i$ for $1 \leq i \leq n - 1$ and $n + \bar{1} = \bar{s}_n$. Note that

$$\begin{align*}
\bar{i} &= \bar{s}_i + \bar{s}_{i-1} \quad \text{for } 2 \leq i \leq n-3, \\
\bar{n-2} &= \bar{s}_{n-3} + \bar{s}_{n-2} + \bar{s}_n, \\
\bar{n-1} &= \bar{s}_{n-2} + \bar{s}_{n-1} + \bar{s}_n, \\
\bar{n} &= \bar{s}_{n-1} + \bar{s}_n.
\end{align*}$$

Set $\Delta = \Delta(E_n) := \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ to be the simple basis of $F_2^n$ in this case. Observe that

$$\bar{n+1} = \bar{1} + \bar{2} + \cdots + \bar{n}. \quad (7.2)$$

Set $\Delta(a)$ and $\text{wt}(a)$ as before for $a \in F_2^n$. For example, $\Delta(n+1) = \Delta$ and $\text{wt}(n+1) = n$.

**Lemma 7.1.** The following (i),(ii) hold.

(i) For each $1 \leq i \leq n - 1$, $s_is_{i+1} = i + 1$, $s_i\bar{i} + \bar{1} = \bar{i}$, and

$$s_i\bar{j} = \bar{j} \quad \text{for } \bar{j} \in \{\bar{1}, \bar{2}, \ldots, \bar{n+1}\} - \{\bar{i}, \bar{i+1}\}.$$

(ii) $s_{n+1} = n - 2 + n - 1 + n, s_{n+1} = n - 2 + n - 1 + n + 1, s_{n+1} = n - 2 + n + 1, s_{n+1} = n - 1 + n + 1$, and

$$s_i\bar{j} = \bar{j} \quad \text{for } 1 \leq j \leq n - 3.$$
Proof. This is immediate by applying Lemma 2.2 and (7.1).  

Let \( S_n \) denote the group of permutations on \( \Delta = \{1, 2, \ldots, n\} \). Set \( T := \{s_1, s_2, \ldots, s_{n-1}\} \). Recall that \( W_T \) is the subgroup of \( W \) generated by \( \{s \mid s \in T\} \). By Lemma 7.1, we find that the set \( \Delta \) is closed under the left multiplication of elements in \( W_T \).

**Definition 7.2.** Let \( \varepsilon : W_T \to S_n \) denote the homomorphism satisfying

\[
\varepsilon(G)(j) = G_j
\]

for \( 1 \leq j \leq n \) and \( G \in W_T \).

In fact, \( \varepsilon \) is an isomorphism since \( \Delta \) is a spanning set and the \( n - 1 \) transpositions \( \varepsilon(s_1), \varepsilon(s_2), \ldots, \varepsilon(s_{n-1}) \) generate \( S_n \).

**Proposition 7.3.** The following are orbits of \( F_2^n \) under the action of \( W \).

\[
O_0 = \{0\},
\]

\[
O_1 = \{a \in F_2^n \mid a \neq 0, \text{wt}(a) \equiv 1 \text{ or } n - 2 \pmod{4}\}, \quad (7.3)
\]

\[
O_2 = \{a \in F_2^n \mid a \neq 0, \text{wt}(a) \equiv 2 \text{ or } n - 3 \pmod{4}\},
\]

\[
O_3 = \{a \in F_2^n \mid a \neq 0, \text{wt}(a) \equiv 3 \text{ or } n \pmod{4}\},
\]

\[
O_4 = \{a \in F_2^n \mid a \neq 0, \text{wt}(a) \equiv 0 \text{ or } n - 1 \pmod{4}\}.
\]

In particular \( O_1 = O_3 \) when \( n \equiv 1 \pmod{4} \), \( O_1 = O_4 \) and \( O_2 = O_3 \) when \( n \equiv 2 \pmod{4} \), \( O_2 = O_4 \) when \( n \equiv 3 \pmod{4} \), and \( O_1 = O_2 \) and \( O_3 = O_4 \) when \( n \equiv 0 \pmod{4} \).

**Proof.** It is clear that \( O_0 \) is an orbit. There are four cases to put nonzero vectors \( a, b \) in an orbit. (a) \( \text{wt}(a) = \text{wt}(b) \): This is because \( \varepsilon(W_T) = S_n \) acts transitively on the fixed size subsets of \( \Delta \); (b) \( \text{wt}(b) = n + 3 - \text{wt}(a) \), or \( n - 1 - \text{wt}(a) \): This is from (a) and the observation that

\[
\text{wt}(s_na) = \begin{cases} 
  n + 3 - \text{wt}(a), & \text{if } |\Delta(a) \cap \{n, n-1, n-2\}| = 3; \\
  n - 1 - \text{wt}(a), & \text{if } |\Delta(a) \cap \{n, n-1, n-2\}| = 1; \\
  \text{w(a),} & \text{else}
\end{cases}
\]

by Lemma 7.1(ii) and (7.2); (c) \( \text{wt}(a) = \text{wt}(b) - 4 \): This is by applying the first case of (7.4) and then applying the second case of (7.4); and (d) \( \text{wt}(a) = \text{wt}(b) + 4 \): This is by applying the second case of (7.4) and then the first case of (7.4). The proposition follows from the above cases (a)-(d).  

\[\square\]
Remark 7.4. With reference to Proposition 7.3, for each orbit \( O \) of \( F_2^n \) with \( O \neq O_0 \) there is \( 1 \leq i \leq n \) such that \( \tilde{s}_i \in O \). For example \( \tilde{s}_i \in O_i \) for \( i = 1, 2, 3 \) and \( \tilde{s}_{n-1} \in O_4 \).

Similar to case of \( A_n \), we determine the reducibility of \( \phi \) from Proposition 7.3 immediately.

Proposition 7.5. The Vogan representation \( \phi \) is irreducible if and only if \( n \) is even. \( \square \)

Recall that for \( a \in F_2^n \), the isotropy group of \( a \) in \( W \) is \( \{ G \in W \mid Ga = a \} \), and the cardinality of the orbit of \( a \) is equal to the index of the isotropy group of \( a \).

Corollary 7.6. For \( J := \{ s_2, s_3, \ldots, s_n \} \), the number \( |W_J||O_1| \) divides \( |W| \), where

\[
|O_1| = \begin{cases} 
2^{n-1} - (-1)^\frac{n-2}{4}2^{\frac{n-2}{2}}, & \text{if } n \equiv 0 \pmod{4}, \\
2^{n-1}, & \text{if } n \equiv 1 \pmod{4}, \\
2^{n-1} + (-1)^\frac{n-2}{4}2^{\frac{n-2}{2}} - 1, & \text{if } n \equiv 2 \pmod{4}, \\
2^{n-2} + (-1)^\frac{n-3}{4}2^{\frac{n-3}{2}}, & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

(7.5)

Proof. Since \( W_J \) is a subgroup of the isotropy group of \( \Gamma \), the number \( |W_J||O_1| \) divides \( |W| \). Note that by (7.3)

\[
|O_1| = \begin{cases} 
\sum_{k=1, 2 \pmod{4}} \binom{n}{k}, & \text{if } n \equiv 0 \pmod{4}, \\
\sum_{k=1 \pmod{2}} \binom{n}{k}, & \text{if } n \equiv 1 \pmod{4}, \\
\sum_{k=0, 3 \pmod{4}} \binom{n}{k}, & \text{if } n \equiv 2 \pmod{4}, \\
\sum_{k=1 \pmod{4}} \binom{n}{k}, & \text{if } n \equiv 3 \pmod{4},
\end{cases}
\]

where \( \binom{n}{k} \) is the binomial coefficient. From this, we routinely prove (7.5) by induction on \( n \). \( \square \)

We need to quote a lemma.

Lemma 7.7. ([4, Lemma 10.2.11]) If \( W \) is of type \( E_7 \) or \( E_8 \) then \( Z(W) = \{1, w_0\} \), where \( w_0 \) is the longest element of \( W \). \( \square \)
Recall that \( T = \{ s_1, s_2, \ldots, s_{n-1} \} \) and \( J = \{ s_2, s_3, \ldots, s_n \} \) when we indicate that the Coxeter group \( W \) is of type \( E_n \).

**Proposition 7.8.** The Vogan representation \( \phi \) of \( W \) is faithful if \( W \) is of type \( E_6 \), and \( |\text{Ker} \phi| = 2 \) if \( W \) is of type \( E_7 \). Moreover, \( \text{Ker} \phi = Z(W) \) if \( W \) is of type \( E_6 \) or \( E_7 \).

**Proof.** Suppose \( W \) is of type \( E_6 \). With referring to Corollary 7.6, we have \( |O_1| = 27 \). By Lemma 3.4(iii) and Proposition 6.10 (the case \( D_5 \)), we know \( |W_J| = 2^4 \cdot 5! \), where \( J \) is of type \( D_5 \). Since \( |W_J| |O_1| \) divides \( |W| \), we have \( |W| \geq 2^4 \cdot 5! \cdot 27 = 2^7 \cdot 3^4 \cdot 5 \). Since \( |W| = 2^6 \cdot 3^4 \cdot 5 \) [7, p.44], \( W \) is isomorphic to \( W \) and \( \text{Ker} \phi \) is trivial. By this and Corollary 4.2, \( Z(W) \) is trivial.

Suppose \( W \) is of type \( E_7 \). From Corollary 4.2 and Lemma 7.7, \( |\text{Ker} \phi| \geq 2 \). Since \( |W| = 2^6 \cdot 3^4 \cdot 5 \cdot 7 \) [7, p.44], we see that \( |W| \leq 2^6 \cdot 3^4 \cdot 7 \). On the other hand, according to a similar counting argument as above, we have \( |O_1| = 28 \), \( |W_J| = 2^7 \cdot 3^5 \), where \( J \) is of type \( E_6 \), and hence \( |W| \geq 2^6 \cdot 3^4 \cdot 7 \). Thus, \( |W| = 2^6 \cdot 3^4 \cdot 7 \) and \( |Z(W)| = |\text{Ker} \phi| = 2 \).

We now go to the last case \( W \) of type \( E_8 \). Note that \( J = \{ s_2, s_3, \ldots, s_8 \} \) is of type \( E_7 \) and \( T \cap J = \{ s_2, s_3, \ldots, s_7 \} \) is of type \( A_6 \). We need more information of the nontrivial element \( w_0 \) in the center \( Z(W_J) \) of \( W_J \). It is quite complicate to describe \( w_0 \) directly as a product of elements in \( J \). We borrow two notations to describe \( w_0 \). Let \( \phi \) denote the Vogan representation of \( W \). Note that \( \phi \uparrow W_J \cap J \) is an isomorphism of \( W_J \cap J \) onto \( W_J \) by Lemma 3.4(ii) and Proposition 5.7. Also \( \epsilon \uparrow W_J : W_J \rightarrow S_7 \) is an isomorphism, where \( \epsilon \) is as in Definition 7.2 and \( S_7 \) is the group of permutations on \( \{ 2, 3, \ldots, 8 \} \).

The expression of \( w_0 \) is as follows.

\[
\begin{align*}
w_0 &= \phi^{-1}(\epsilon^{-1}((2, 3, 5, 7, 4, 6, 5)))s_8\phi^{-1}(\epsilon^{-1}((7, 8)(4, 7)(3, 6)))s_8 \\
&\quad \phi^{-1}(\epsilon^{-1}((1, 8)(3, 7)(2, 6)))s_8\phi^{-1}(\epsilon^{-1}((5, 8)(4, 7)))s_8 \quad (7.6) \\
&\quad \phi^{-1}(\epsilon^{-1}((3, 7)(2, 6)))s_8.
\end{align*}
\]

It is routine to check that the above \( w_0 \) maps to \( -I \) by the faithful representation defined in [3, Proposition 8] with \( c = 0 \) or in [6, p. 291] to conclude \( w_0 \) is in the center of \( W_J \) and indeed is the longest element of \( W_J \) by [3, Proposition 21]. Thus, we have the following lemma.

**Lemma 7.9.** Let \( W \) be of type \( E_8 \) with the Vogan representation \( \phi \) and \( w_0 \in Z(W_J) \) be not identity. Then \( \phi(w_0) \) is

\[
\begin{align*}
&\epsilon^{-1}((2, 3, 5, 7, 4, 6, 5)))s_8\epsilon^{-1}((7, 8)(4, 7)(3, 6))s_8\epsilon^{-1}((1, 8)(3, 7)(2, 6))s_8 \\
&\times \epsilon^{-1}((5, 8)(4, 7)))s_8\epsilon^{-1}((3, 7)(2, 6))s_8.
\end{align*}
\]
Note that $W_J$ is not isomorphic to its flipping group $W_J$ by Proposition 7.8. The following lemma claims that $W_J$ is isomorphic to the subgroup $W_J$ of $W$.

**Lemma 7.10.** Let $W$ be of type $E_8$ with the Vogan representation $\phi$. Then the restriction $\phi \mid W_J$ of $\phi$ to $J$ is injective.

**Proof.** Let $\phi' : W_J \to W_J$ denote the Vogan representation of $W_J$. From Lemma 3.4(iii) and Proposition 7.8, we see that $\text{Ker} \phi \mid W_J \subseteq \text{Ker} \phi' = \{1, w_0\}$, where $w_0$ is given in (7.6). To prove that $\text{Ker} \phi \mid W_J$ is trivial, it suffices to show that $\phi(w_0) \neq I$. This follows from the computation

$$\phi(w_0)\mathfrak{S} = \mathfrak{T} + \mathfrak{S}$$

by applying the expression $\phi(w_0)$ in Lemma 7.9 to $\mathfrak{S}$ and using Lemma 7.1 and (7.2) for $n = 8$ to simplify. 

There is a similar result about $W$ of type $E_8$.

**Proposition 7.11.** If $W$ is of type $E_8$ then $\text{Ker} \phi$ has order 2. Moreover, $\text{Ker} \phi = Z(W)$.

**Proof.** We have $|O_1| = 2^3 \cdot 3 \cdot 5$ from (7.5), $|W_J| = |W_J| = 2^{10}3^15 \cdot 7$ from Lemma 7.10 and $|W| = 2^{14}3^55^27$[7, p.44]. Therefore, as the proof of Proposition 7.8, $\text{Ker} \phi$ has order 2 and $\text{Ker} \phi = Z(W)$. 

## 8 Concluding remarks

We list the main results of this article as follows.
Dynkin diagram reducibility of $\phi$ & $|\text{Ker } \phi|$ \\
$A_n$ & $\phi$ is irr. iff $n = 1$ or $n$ is even. & $\begin{cases} 2, & \text{if } n = 1, \\ 1, & \text{else.} \end{cases}$ \\
$D_n$ & $\phi$ is not irr. & $\begin{cases} 2, & \text{if } n \text{ is even,} \\ 1, & \text{else.} \end{cases}$ \\
$E_6$ & $\phi$ is irr. & 1 \\
$E_7$ & $\phi$ is not irr. & 2 \\
$E_8$ & $\phi$ is irr. & 2 \\

Table 1: The reducibility and the kernel of a Vogan representation $\phi$.

Coxeter graph orbits

| $A_n$ | $O_i = \{a \in F_{2}^n \mid wt(a) = i \text{ or } n+1-i\} (0 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor)$. |
|-------|---------------------------------------------------------------------------------------------------------------|
| $D_n$ | $\begin{align*}
O_i &= \{a \in Z \mid wt(a) = i \text{ or } n-i\} (0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor), \\
\Omega_o &= \{a \in F_{2}^n - Z \mid wt(a) \equiv 1 \text{ or } n-1 \pmod{2}\}, \\
\Omega_e &= \{a \in F_{2}^n - Z \mid wt(a) \equiv 0 \text{ or } n \pmod{2}\}, \\
\Omega_o &= \Omega_e = F_{2}^n - Z \text{ when } n \text{ is odd.}
\end{align*}$ |
| $E_n$ | $\begin{align*}
O_0 &= \{0\}, \\
O_1 &= \{a \in F_{2}^n \mid a \neq 0, wt(a) \equiv 1 \text{ or } n-2 \pmod{4}\}, \\
O_2 &= \{a \in F_{2}^n \mid a \neq 0, wt(a) \equiv 2 \text{ or } n-3 \pmod{4}\}, \\
O_3 &= \{a \in F_{2}^n \mid a \neq 0, wt(a) \equiv 3 \text{ or } n \pmod{4}\}, \\
O_4 &= \{a \in F_{2}^n \mid a \neq 0, wt(a) \equiv 0 \text{ or } n-1 \pmod{4}\}. \\
O_1 &= O_3 \text{ when } n \equiv 1 \pmod{4}, \\
O_1 &= O_4 \text{ and } O_2 = O_3 \text{ when } n \equiv 2 \pmod{4}, \\
O_2 &= O_4 \text{ when } n \equiv 3 \pmod{4}, \\
O_1 &= O_2 \text{ and } O_3 = O_4 \text{ when } n \equiv 0 \pmod{4}.
\end{align*}$ |

Table 2: The orbits of $F_{2}^n$ under the action of the flipping group of a Coxeter graph $S$.

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Hau-wen Huang
Department of Applied Mathematics
National Chiao Tung University
1001 Ta Hsueh Road
Hsinchu, Taiwan 30050, R.O.C.
Email: poker80@msn.com
Fax: +886-3-5724679