HEAT KERNEL BOUNDS, ANCIENT $\kappa$ SOLUTIONS AND THE POINCARÉ CONJECTURE

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Abstract. We establish certain Gaussian type upper bound for the heat kernel of the conjugate heat equation associated with 3 dimensional ancient $\kappa$ solutions to the Ricci flow.

As an application, using the $W$ entropy associated with the heat kernel, we give a different and much shorter proof of Perelman’s classification of backward limits of these ancient solutions. The method is partly motivated by [CS] and [S]. The current paper or [CL] combined with [ChZ] and [Z2] lead to a simplified proof of the Poincaré conjecture without using reduced distance and reduced volume.

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1. Introduction

The main goal of the paper is to establish certain Gaussian type upper bound for the heat kernel (fundamental solutions) of the conjugate heat equation associated with 3 dimensional ancient $\kappa$ solutions to the Ricci flow. Heat kernel estimates have been an active area of research. When coupled with Ricci flow, various estimates can be found in [C], [P1] Section 9, [N], and [Z1]. For example, in Section 9 of [P1] Perelman proved a lower bound for the fundamental solution of the conjugate heat equation for general Ricci flow. So far an upper bound corresponding to this lower bound has been missing. Our result is a progress in this direction when the Ricci flow is a 3 dimensional ancient $\kappa$ solution.

One motivation of the work is that it induces a simpler proof of the Poincaré conjecture. The most difficult analytical parts of the proof can now be treated by one unifying theme: Perelman’s $W$ entropy and related (log) Sobolev inequalities and heat kernel estimates. Let us explain the point in more detail. From Perelman’s original papers [P1], [P2], [P3] and the works by Cao and Zhu [CZ], Kleiner and Lott [KL] and Morgan and Tian [MT], and Tao [T2], [T], it is clear that the bulk of the proof of the Poincaré conjecture is consisted of two items. One is the proof of local non-collapsing with or without surgeries, and the other is the classification of backward limits of ancient $\kappa$ solutions. After these are
done, one can show that regions where the Ricci flow is close to forming singularity have simple topological structure, i.e. canonical neighborhoods. Then one proceeds to prove that the singular region can be removed by finite number of surgeries in finite time. When the initial manifold is simply connected, the Ricci flow becomes extinct in finite time [P3] (see also [CM]). Thus the manifold is diffeomorphic to $S^3$, as conjectured by Poincaré.

Besides the results and techniques by R. Hamilton, the main new tools Perelman used in carrying out the proof are several monotone quantities along Ricci flow. These include the $W$ entropy, reduced volume and the associated reduced distance. In [P1], Perelman first used his $W$ entropy to prove local non-collapsing for smooth Ricci flows. However he then turned to the reduced volume (distance) to prove the classification and non-collapsing with surgeries. The $W$ entropy is not used anymore. The reduced distance, not being smooth or positive in general, is one of the causes of the complexity of the original proof.

It turns out that the $W$ entropy is just the formula in a log Sobolev inequality (c.f. [Gr] in the fixed metric case) and the monotonicity of the $W$ entropy implies certain uniform Sobolev inequalities along the Ricci flow. Using this idea and being inspired by the last section of [P2] and [KL], we proved in [Z2] a stronger local non-collapsing result for Ricci flow with surgeries. The proof, without using reduced distance or volume, is short and seems more accessible. It also strengthens and clarifies the original result by doing analysis at one time level each time, thus avoiding the complication associated with surgeries. In the wake of this development, it would be desirable that the classification mentioned above can also be done by using the $W$ entropy alone. Such a view was also expressed in [T] e.g.

As one application of the main result of the paper, using the $W$ entropy associated with the heat kernel, we give a different and much shorter proof of Perelman’s classification of backward limits of these ancient solutions. Thus, the current paper together with [Z2] and [ChZ] (see explanation 4 paragraphs below) lead to a simplified proof of the Poincaré conjecture. Of course we still follow the framework by Perelman. However, much of the highly intensive analysis involving reduced distance and volume is now replaced by the study of the $W$ entropy and the related uniform Sobolev inequalities and heat kernel estimates. Sobolev inequalities and heat kernels are familiar to many mathematicians. Therefore the current proof is more accessible to a wider audience. Besides, due to the relative simplicity, we hope the current technique can lead to better understanding of other problems for Ricci flow.

We should mention that the reduced distance and volume are still needed for the proof of the geometrization conjecture. Specifically, they are needed, but only in the proof of Perelman’s no local collapsing Theorem II with surgeries.

Let us outline the proof. In the next section we prove Theorem 1.1 concerning the bounds for the heat kernel of the conjugate heat equation. The proof follows the framework in section 5 of [Z1]. There an upper bound in the case of Ricci flow with nonnegative Ricci curvature was given. In the current situation, the ancient $\kappa$ solutions provide better control on curvature and volume. These allow us to find a better Gaussian upper bound for the heat kernel. These bounds can be regarded as generalization of the heat kernel bounds of Li and Yau [LY] in the fixed metric case.

Using this heat kernel bound, in Section 3 we show that the $W$ entropy associated with the heat kernel is uniformly bounded from below after certain scaling. After this done, we use Perelman’s monotonicity formula for the $W$ entropy to prove the backward limit is a
shrinking gradient Ricci soliton. This part of the arguments resembles that in the paper
[10] and [8] where forward convergence results for normalized Ricci flow were proven.
Finally one needs to prove universal non-collapsing for ancient $\kappa$ solutions without
reduced distance or volume. But this is already done in [ChZ], even in certain more
general 4 dimensional situation. We will just describe their proof.

Now let us introduce the definitions and notations in order to present our result
precisely. $M$ denotes a complete compact, or noncompact Riemannian manifold, unless stated
otherwise; $g, R_{ij}$ (or $Ric$) will be the metric and Ricci curvature; $\nabla, \Delta$ the corresponding
gradient and Laplace-Beltrami operator; $c$ with or without index denote generic positive
constant that may change from line to line. If the metric $g(t)$ evolves with time, then
d($x, y, t$) will denote the corresponding distance function; $dg(x, t)$ or $dg(t)$ denote the vol-
ume element under $g(t)$; We will use $B(x, r; t)$ to denote the geodesic ball centered at $x$
with radius $r$ under the metric $g(t)$; $|B(x, r; t)|_s$ to denote the volume of $B(x, r; t)$ un-
der the metric $g(s)$. We will still use $\nabla, \Delta$ to denote the corresponding gradient and
Laplace-Beltrami operator for $g(t)$, without mentioning the time $t$, when no confusion
arises.

We use the following concept of ancient $\kappa$ solutions according to Perelman.

**Definition 1.1.** A solution to the Ricci flow $\partial_t g = -2\text{Ric}$ is an ancient $\kappa$ solution if it
satisfies the following properties.
1. It is complete (compact or noncompact) and defined on an ancient time interval
$(-\infty, T_0], T_0 > 0$.
2. It has nonnegative curvature operator and bounded curvature at each time level.
3. It is $\kappa$ noncollapsed on all scales for some positive constant $\kappa$. i.e.
Suppose that $x_0 \in M$, $t_0 \in (-\infty, T_0]$. Let $P(x_0, t_0, r, -r^2)$ be the parabolic ball
$$\{(x, t) \mid d(x, x_0, t) < r, \quad t_0 - r^2 < t < t_0\}.$$ 
Then $M$ is $\kappa$ non-collapsed at $(x_0, t_0)$ at scale $r$ if $|Rm| \leq r^{-2}$ on $P(x_0, t_0, r, -r^2)$ and
vol($B(x_0, t_0, r)$) $\geq \kappa r^3$.

For convenience, we take the final time $T_0$ of the ancient solution to be 0 throughout
the paper. The conjugate heat equation is

\begin{equation}
\Delta u - Ru - \partial_t u = 0.
\end{equation}

Here and always $\tau = -t$. $\Delta$ and $R$ are the Laplace-Beltrami operator and the scalar
curvature with respect to $g(t)$. This equation, coupled with the initial value $u_{\tau=0} = u_0$ is
well posed if $M$ is compact or the curvature is bounded, and if $u_0$ is bounded [4].

We use $G = G(x, \tau; x_0, \tau_0)$ to denote the heat kernel (fundamental solution) of (1.1).

Here $\tau > \tau_0$ and $x, x_0 \in M$. Existence of $G$ was established in [G]. The main result of the paper is

**Theorem 1.1.** (i) Let $(M, g(t))$ be a $n$ dimensional ancient $\kappa$ solution of the Ricci flow.
Suppose also that $R(x, t) \leq \frac{D_0}{1 + |t|}$ for some $D_0 > 0$ and for $t \in [-T, 0]$. Here $T$ is any
positive number or $T = \infty$. Then exist positive numbers $a$ and $b$ depending only on $n$, $\kappa$ and
$D_0$ such that the following holds.

For all $x, x_0 \in M$,

$$G(x, \tau; x_0, \tau_0) \leq \frac{a}{|B(x, \sqrt{\tau - \tau_0}, t_0)|_{t_0}} e^{-bd^2(x, x_0, t_0)/(\tau - \tau_0)},$$

where $\tau = -t$, $\tau_0 = -t_0$, $\tau > \tau_0$ and $t \in [-T, 0]$.

(ii). In particular, if $R(x, t) \leq \frac{D_0}{1 + |t|}$ for all $t \leq 0$, namely $(M, g(t))$ is a Type I ancient solution, there exist positive numbers $a_1$ and $b_1$ depending only on $\kappa$ and $D_0$ such that the following holds. For all $x, x_0 \in M$, and all $\tau = -t > 0$,

$$\frac{1}{a_1 \tau^{n/2}} e^{-d^2(x, x_0, t)/(b_1 \tau)} \leq G(x, \tau; x_0, \tau/2) \leq \frac{a_1}{\tau^{n/2}} e^{-b_1 d^2(x, x_0, t)/\tau}.$$

Remark. The full Gaussian lower bound in part (ii) of the theorem is not needed for the application in Section 3. One only needs the lower bound for one point in the ball $B(x_0, \sqrt{b|t|}, t)$ for some $b > 1$, which is a simple consequence of the upper bound.

The Gaussian upper and lower bounds seem to be of interest that is independent of the Poicaré conjecture. For instance, Perelman [P1] used heat kernel bounds to prove his pseudo locality theorem. In Section 9 of the same paper, a lower bound for the heat kernel was proven. However the upper bound is missing. In this sense, this paper is not just a reproof of a known result.

2. Proof of Theorem 1.1: the heat kernel bounds

We divide the proof into three steps. The first two are for part (i) of the theorem. We always assume that all the time variables involved are not smaller than $-T$, so that the condition $R(\cdot, t) \leq \frac{D_0}{1 + |t|}$ holds. As mentioned in the introduction, the proof follows the framework of Theorem 5.2 in [Z1], where certain upper bound for $G$ under Ricci flow with nonnegative Ricci curvature was derived. Comparing with that case, we have two new ingredients coming from ancient $\kappa$ solutions. One is the non-collapsing condition on all scales. The other is the bound on the scalar curvature. These allow us to prove a better bound. During the proof, there will be overlaps with [Z1]. They are here so that the paper is self contained. Without loss of generality we assume $\tau_0 = 0$ in $G(x, \tau; x_0, \tau_0)$. It is convenient to work with the reversed time $\tau$. Note that the Ricci flow is a backward flow with respect to $\tau$ and the conjugate heat equation is a forward heat equation with a potential term.

Step 1.

Since $\text{Ricci} \geq 0$, it is well known (see Theorem 3.7 [Heb1] e.g.) the following Sobolev inequality holds: Let $B(x, r, t)$ be a proper subdomain for $(M, g(t))$. For all $v \in W^{1,2}(B(x, r, t))$, there exists $c_n > 0$ depending only on the dimension $n$ such that

$$\left(\int v^{2n/(n-2)} d\sigma(t)\right)^{(n-2)/n} \leq \frac{c_n r^2}{|B(x, r, t)|^{1/2}} \int \left[|\nabla v|^2 + r^{-2} v^2 \right] d\sigma(t).$$

For our purpose, we only need to take $r = c \sqrt{|t|}$, for $c < 1$. By the assumption that $R(x, t) \leq \frac{D_0}{1 + |t|}$ and the $\kappa$ non-collapsing property, we have

$$|B(x, \sqrt{|t|}, t)|_r \geq \kappa D_0^{-n}|t|^{n/2}.$$

Therefore the above Sobolev inequality becomes

$$\left(\int v^{2n/(n-2)} d\sigma(t)\right)^{(n-2)/n} \leq \frac{c_n D_0^2}{\kappa^{2/n}} \int \left[|\nabla v|^2 + |t|^{-1} v^2 \right] d\sigma(t)$$

for all $v \in W^{1,2}(B(x, \sqrt{|t|}, t))$.  


Before moving forward, we would like to clarify a technical point in the definition of Perelman’s $\kappa$ non-collapsing as given in Definition 1.1. The issue is whether the metric balls $B(x, r, t)$ in the definition are required to be a proper subdomain of the manifold $M$. When $M$ is noncompact, $B(x, r, t)$ is always a proper subdomain so this issue is mute. Now one assumes that $M$ is compact. Without requiring $B(x, r, t)$ being a proper subdomain, if $r$ is larger than the diameter of $M$, then $B(x, r, t)$ is the whole manifold. In this case $|B(x, r, t)|_t$ can not be greater than $\kappa r^n$ for large $r$. So to be $\kappa$ non-collapsed, at some point in the parabolic ball $|Rm|$ is greater than $1/r^2$. In other words, if $|Rm| \leq 1/r^2$ in the parabolic ball, then the volume of the manifold is at least $\kappa r^n$. If the Ricci curvature is nonnegative, then by standard volume comparison theorem, the diameter of the manifold at time $t$ is at least $\kappa r$.

In this paper, we take this explanation for Perelman’s $\kappa$ non-collapsing, i.e $B(x, r, t)$ in the definition of $\kappa$ ancient solutions is not required to be a proper subdomain. This seems to be the prevailing view in the literature. That is why the Sobolev imbedding 2.2 holds without requiring that $B(x, \sqrt{|t|}, t)$ is a proper subdomain of $M$. A natural question is: what happens when $B(x, r, t)$ is implicitly assumed as a proper subdomain in the definition of $\kappa$ solutions? Then we have to make this extra assumption throughout. However either way does not affect the application for the Poincaré conjecture in the next section. The reason is compact ancient solutions are already taken care of. See the beginning of the proof of Theorem 3.1.

Next we show that, under the assumptions of the theorem, $(M, g(t))$ possess a space time doubling property: the distance between two points at times $t_1$ and $t_2$ are comparable if $t_1$ and $t_2$ are comparable. The proof is very simple. Given $x_1, x_2 \in M$, let $r$ be a shortest geodesic connecting the two. Then

$$\partial_t d(x_1, x_2, t) = -\int_r \text{Ric}(\partial_r, \partial_r)ds.$$ 

Since the sectional curvature is nonnegative, it holds

$$|\text{Ric}(x, t)| \leq cR(x, t) \leq \frac{cD_0}{1 + |t|}.$$ 

Therefore

$$-\frac{cD_0}{1 + |t|}d(x_1, x_2, t) \leq \partial_t d(x_1, x_2, t) \leq 0.$$ 

After integration, we arrive at:

$$(|t_1|/|t_2|)^{cD_0} \leq d(x_1, x_2, t_1)/d(x_1, x_2, t_2) \leq 1$$

for all $t_2 < t_1 < 0$. Note that the above inequality is of local nature. If the distance is not smooth, then one can just shift one point, say $x_1$, slightly and then obtain the same integral inequality by taking limits.

Similarly, we have

$$0 \geq \partial_t \int_{B(x, \sqrt{|t_1|}, t_1)} dg(t) = -\int_{B(x, \sqrt{|t_1|}, t_1)} \text{R}(y, t)dg(t) \geq -\frac{D_0}{1 + |t|} \int_{B(x, \sqrt{|t_1|}, t_1)} dg(t).$$ 

Upon integration, we know that the volume of the balls

$$(2.3) \quad |B(x, \sqrt{|t_3|}, t_4)|_{t_5}$$

are all comparable for $t_3, t_4, t_5 \in [t_2, t_1]$, provided that $t_1$ and $t_2$ are comparable.
Let $u$ be a positive solution to (1.1) in the region
\[ Q_{στ}(x, τ) \equiv \{(y, s) \mid y \in M, τ - (στ)^2 \leq s \leq τ, d(y, x, -s) \leq στ\}. \]
Here $r = \sqrt{t}/8 > 0, 2 \geq σ \geq 1$. Given any $p \geq 1$, it is clear that
\[ (2.5) \quad Δu^p - pRu^p - \partial_t u^p \geq 0. \]

Let $φ : [0, ∞) → [0, 1]$ be a smooth function such that $|φ'| ≤ 2/((σ - 1)r), φ' \leq 0$, $φ(ρ) = 1$ when $0 \leq ρ \leq r$, $φ(ρ) = 0$ when $ρ \geq στ$. Let $η : [0, ∞) → [0, 1]$ be a smooth function such that $|η'| ≤ 2/((σ - 1)r)^2$, $η' ≥ 0$, $η ≥ 0$, $φ(s) = 1$ when $τ - r^2 \leq s ≤ τ$, $φ(s) = 0$ when $s ≤ τ - (στ)^2$. Define a cut-off function $ψ = φ(d(x, y, -s))η(s)$.

Writing $w = u^p$ and using $wψ^2$ as a test function on (2.5), we deduce
\[ (2.6) \quad \int \nabla(wψ^2)\nabla wdgd(y, -s)ds + p \int Rw^2ψ^2dg(y, -s)ds \leq - \int (∂_sw)wψ^2dg(y, -s)ds. \]

By direct calculation
\[ \int \nabla(wψ^2)\nabla wdgd(y, -s)ds = \int |∇(wψ)|^2dg(y, -s)ds - \int |∇ψ|^2w^2dg(y, -s)ds. \]

Next we estimate the righthand side of (2.6).
\[ - \int (∂_sw)wψ^2dg(y, -s)ds \]
\[ = \int w^2ψ\partial_swψdg(y, -s)ds + \frac{1}{2} \int (wψ)^2Rdg(y, -s)ds - \frac{1}{2} \int (wψ)^2dg(y, -τ). \]

Observe that
\[ ∂_sw = η(s)φ'(d(y, x, -s))\partial_s(d(x, y, -s)) + φ(d(y, x, -s))η'(s) \leq φ(d(y, x, -s))η'(s). \]
This is so because $φ' ≤ 0$ and $∂_sd(y, x, -s) ≥ 0$ under the Ricci flow with nonnegative Ricci curvature. Hence
\[ (2.7) \quad - \int (∂_sw)wψ^2dg(y, -s)ds \]
\[ \leq \int w^2ψφ(d(y, x, -s))η'(s)dg(y, -s)ds + \frac{1}{2} \int (wψ)^2Rdg(y, -s)ds - \frac{1}{2} \int (wψ)^2dg(y, -τ). \]

Combing (2.6) with (2.7), we obtain, in view of $p ≥ 1$ and $R ≥ 0$,
\[ (2.8) \quad \int |∇(wψ)|^2dg(y, -s)ds + \frac{1}{2} \int (wψ)^2dg(y, -τ) \leq \frac{c}{(σ - 1)^2r^2} \int Q_{στ}(x, τ) w^2dg(y, -s)ds. \]

By Hölder’s inequality
\[ (2.9) \quad \int ψw^{2(1+2/n)}dg(y, -s) ≤ \left( \int (ψw)^{2n/(n-2)}dg(y, -s) \right)^{(n-2)/n} \left( \int (ψw)^2dg(y, -s) \right)^{2/n}. \]

By the $κ$ non-collapsing assumption, $|B(x, \sqrt{t}/8, t)| ≥ κc2r^n$. Since $M$ has nonnegative Ricci curvature, the diameter of $M$ at time $t$ is a least a constant multiple of $c\sqrt{|t|}$ for some $c = c_n > 0$. Recall that $r = \sqrt{|t|}/8$. Therefore by the distance doubling property
\( B(x, \sigma r, -s) \) is a proper sub-domain of \( M \), \( s \in [\tau - (\sigma r)^2, \tau] \). Here we just take the number 8 for simplicity. If it is not large enough, we just replace it by a sufficiently large number \( D \) and consider \( r = \sqrt{|t|}/D \) instead. By the Sobolev inequality (2.2), it holds
\[
\left( \int (\psi w)^{2n/(n-2)} \, dg(y, -s) \right)^{(n-2)/n} \leq c(\kappa, D_0) \int |\nabla (\psi w)|^2 + r^{-2} (\psi w)^2 \, dg(y, -s),
\]
for \( s \in [t - (\sigma r)^2, t] \). Substituting this and (2.8) to (2.9), we arrive at the estimate
\[
\int_{Q_{r/2}(x, \tau)} w^{2\theta} \, dg(y, -s) \, ds \leq c(\kappa, D_0) \left( \frac{1}{(\sigma - 1)^2 r^2} \int_{Q_{sr}(x, \tau)} w^2 \, dg(y, -s) \, ds \right)^\theta,
\]
with \( \theta = 1 + (2/n) \). Now we apply the above inequality repeatedly with the parameters \( \sigma_0 = 2, \sigma_i = 2 - \Sigma_{j=1}^i 2^{-j} \) and \( p = \theta^i \). This shows a \( L^2 \) mean value inequality
\[
(2.10) \quad \sup_{Q_{r/2}(x, \tau)} u^2 \leq \frac{c(\kappa, D_0)}{r^{n+2}} \int_{Q_r(x, \tau)} u^2 \, dg(y, -s) \, ds.
\]
This inequality clearly also holds if one replaces \( r \) by any positive number \( r' < r \) since \( |B(x, r', t)| \geq k c_n B(x, r, t) |(r'/r)^n | c r^m \) by the doubling condition for manifolds with nonnegative Ricci curvature. Then one can just rerun the above Moser’s iteration.

From here, by a generic trick of Li and Schoen [LS], applicable here since it uses only the doubling property of the metric balls, we arrive at the \( L^1 \) mean value inequality
\[
\sup_{Q_{r/2}(x, \tau)} u \leq \frac{c(\kappa, D_0)}{r^{n+2}} \int_{Q_r(x, \tau)} u d g(z, -s) \, ds.
\]
We remark that the doubling constant is uniform since the metrics have nonnegative Ricci curvature.

Now we take \( u(x, \tau) = G(x, \tau; x_0, 0) \). Note that \( \int_M u(z, s) d g(z, -s) = 1 \) and \( r = \sqrt{|t|} \).
\[
(2.11) \quad G(x, \tau; x_0, 0) \leq \frac{c(\kappa, D_0)}{|t|^{n/2}}.
\]

**step 2.** proof of the Gaussian upper bound.

We begin by using a modified version of the exponential weight method due to Davies [Da]. Pick a point \( x_0 \in M \), a number \( \lambda < 0 \) and a function \( f \in C_0^\infty(M, g(0)) \). Consider the functions \( F \) and \( u \) defined by
\[
(2.12) \quad F(x, \tau) \equiv e^{\lambda d(x, x_0, t)} u(x, \tau) \equiv e^{\lambda d(x, x_0, t)} \int G(x, \tau; y, 0) e^{-\lambda d(y, x_0, 0)} f(y) \, dg(y, 0).
\]
Here and always \( \tau = -t \). It is clear that \( u \) is a solution of (1.1). By direct computation,
\[
\begin{align*}
\partial_\tau \int F^2(x, \tau) \, dg(x, t) &= \partial_\tau \int e^{2\lambda d(x, x_0, t)} u^2(x, \tau) \, dg(x, t) \\
&= 2\lambda \int e^{2\lambda d(x, x_0, t)} \partial_\tau d(x, x_0, t) u^2(x, \tau) \, dg(x, t) + \int e^{2\lambda d(x, x_0, t)} u^2(x, \tau) R(x, t) \, dg(x, t) \\
&\quad + 2 \int e^{2\lambda d(x, x_0, t)} [\Delta u - R(x, t) u(x, \tau)] u(x, \tau) \, dg(x, t).
\end{align*}
\]
By the assumption that $Ricci \geq 0$ and $\lambda < 0$, the above shows

$$\partial \tau \int F^2(x, \tau) dg(x, t) \leq 2 \int e^{2\lambda d(x,x_0,t)} u \Delta u(x, \tau) dg(x, t).$$

Using integration by parts, we turn the above inequality into

$$\partial \tau \int F^2(x, \tau) dg(x, t) \leq -4\lambda \int e^{2\lambda d(x,x_0,t)} u \nabla d(x, x_0, t) \nabla u dg(x, t) - 2 \int e^{2\lambda d(x,x_0,t)} |\nabla u|^2 dg(x, t).$$

Observe also

$$\int |\nabla F(x, \tau)|^2 dg(x, t) = \int |\nabla (e^{\lambda d(x,x_0,t)} u(x, \tau))|^2 dg(x, t)$$

$$= \int e^{2\lambda d(x,x_0,t)} |\nabla u|^2 dg(x, t) + 2\lambda \int e^{2\lambda d(x,x_0,t)} u \nabla d(x, x_0, t) \nabla u dg(x, t)$$

$$+ \lambda^2 \int e^{2\lambda d(x,x_0,t)} |\nabla d|^2 u^2 dg(x, t).$$

Combining the last two expressions, we deduce

$$\partial \tau \int F^2(x, \tau) dg(x, t) \leq -2 \int |\nabla F(x, \tau)|^2 dg(x, t) + \lambda^2 \int e^{2\lambda d(x,x_0,t)} |\nabla d|^2 u^2 dg(x, t).$$

By the definition of $F$ and $u$, this shows

$$\partial \tau \int F^2(x, \tau) dg(x, t) \leq \lambda^2 \int F(x, \tau)^2 dg(x, t).$$

Upon integration, we derive the following $L^2$ estimate

$$(2.13) \quad \int F^2(x, \tau) dg(x, t) \leq e^{\lambda^2 \tau} \int F^2(x, 0) dg(x, 0) = e^{\lambda^2 \tau} \int f(x)^2 dg(x, 0).$$

Recall that $u$ is a solution to (1.11). Therefore, by the mean value inequality (2.11), the following holds

$$u(x, \tau) \leq \frac{c(\kappa, D_0)}{\tau^{1+n/2}} \int_{\tau/2}^{\tau} \int_{B(x, \sqrt{|\tau|/2}, -s)} u^2(z, s) dg(z, -s) ds.$$

By the definition of $F$ and $u$, it follows that

$$u(x, \tau)^2 \leq \frac{c(\kappa, D_0)}{\tau^{1+n/2}} \int_{\tau/2}^{\tau} \int_{B(x, \sqrt{|\tau|/2}, -s)} e^{-2\lambda d(z,x_0,-s)} F^2(z, s) dg(z, -s) ds.$$

In particular, this holds for $x = x_0$. In this case, for $z \in B(x_0, \sqrt{|\tau|/2}, -s)$, there holds $d(z, x_0, -s) \leq \sqrt{|\tau|}/2$. Therefore, by the assumption that $\lambda < 0$,

$$u(x_0, \tau)^2 \leq \frac{c(\kappa, D_0)}{\tau^{1+n/2}} e^{-\lambda \sqrt{2|\tau|}} \int_{\tau/2}^{\tau} \int_{B(x_0, \sqrt{|\tau|/2}, -s)} F^2(z, s) dg(z, -s) ds.$$

This combined with (2.13) shows that

$$u(x_0, \tau) \leq \frac{c(\kappa, D_0)}{\tau^{n/2}} e^{\lambda^2 \tau - \lambda \sqrt{2|\tau|}} \int f(y)^2 dg(y, 0).$$
This is a solution to the conjugate of the conjugate equation (1.1), i.e.

\[
\left( \int G(x_0, \tau; z, 0) e^{-\lambda d(z, x_0, 0)} f(z) dg(z, 0) \right)^2 \leq \frac{c(\kappa, D_0)}{\tau^{n/2}} e^{\lambda^2 \tau - \lambda \sqrt{2|\tau|}} \int f(y)^2 dg(y, 0).
\]

Now, we fix \(y_0\) such that \(d(y_0, x_0, 0)^2 \geq 4t\). Then it is clear that, by \(\lambda < 0\) and the triangle inequality,

\[-\lambda d(z, x_0, 0) \geq -\frac{\lambda}{2} d(x_0, y_0, 0)\]

when \(d(z, y_0, 0) \leq \sqrt{|\tau|}\). In this case, the above integral inequality implies

\[
\left( \int_{B(y_0, \sqrt{\tau}]} G(x_0, \tau; z, 0) f(z) dg(z, 0) \right)^2 \leq \frac{c(\kappa, D_0) e^{\lambda d(z, y_0, 0)} + \lambda^2 \tau - \lambda \sqrt{2|\tau|}}{\tau^{n/2}} \int f(y)^2 dg(y, 0).
\]

Note that this inequality hold for all \(-T \leq t < 0\) and \(\lambda < 0\). For an arbitrarily fixed \(t \in [-T, 0]\), we take

\[
\lambda = -\frac{d(x_0, y_0, 0)}{\beta \tau}
\]

with \(\beta > 0\) sufficiently large. Since \(f\) is arbitrary, this shows, for some \(b > 0\),

\[
\int_{B(y_0, \sqrt{\tau}]} G^2(x_0, \tau; z, 0) dg(z, 0) \leq \frac{c(\kappa, D_0) e^{-b d(x_0, y_0, 0)^2/\tau}}{\tau^{n/2}}.
\]

Hence, there exists \(z_0 \in B(y_0, \sqrt{\tau}])\) such that

\[
G^2(x_0, \tau; z_0, 0) \leq \frac{c(\kappa, D_0)}{\tau^{n/2} |B(x_0, \sqrt{\tau}])|_0} e^{-b d(x_0, y_0, 0)^2/\tau}.
\]

In order to get the upper bound for all points, let us consider the function

\[v = v(z, l) \equiv G(x_0, \tau; z, l)\]

This is a solution to the conjugate of the conjugate equation (1.1), i.e.

\[
\Delta_z G(x, \tau; z; l) + \partial_l G(x, \tau; z, l) = 0, \quad \partial_l g = 2 \text{Ric}.
\]

Therefore, we can use Theorem 3.3 in [Z1], after a reversal in time. Note this theorem was stated only for compact manifolds. However, as remarked there, it is valid in he noncompact case whenever the maximum principle for the heat equation holds. Since the proof is quite short, we will present it in the appendix. It is just a simple generalization of Hamilton’s first result in [H] to the Ricci flow case. Consequently, for \(\delta > 0, C > 0\),

\[
G(x_0, \tau; y_0, 0) \leq CG^{1/(1+\delta)}(x_0, \tau, z_0, 0) M^{\delta/(1+\delta)},
\]

where \(M = \sup_{[0, \tau/2]} G(x_0, \tau, \cdot, \cdot)\). By Step1, there exists a constant \(c(\kappa, D_0) > 0\), such that

\[
M \leq \frac{c(\kappa, D_0)}{\tau^{n/2}}.
\]

Consequently

\[
G^2(x_0, \tau; y_0, 0) \leq \frac{c(\kappa, D_0)}{\tau^{n/2} |B(x_0, \sqrt{\tau}])|_0} e^{-b d(x_0, y_0, 0)^2/\tau} \cdot \frac{c(\kappa, D_0)}{|B(x_0, \sqrt{\tau}])|_0} e^{-b d(x_0, y_0, 0)^2/\tau}.
\]

The last step holds since the Ricci curvature is nonnegative.

Since \(x_0\) and \(y_0\) are arbitrary, the proof of part (i) is done.
step 3
In this step, we prove the upper and lower bound for $G(x, \tau; x_0, \tau/2)$ in the case of type I ancient solution. The upper bound is already proven in view the distance and volume comparison result (2.3), (2.4) and the fact that $|B(x, \sqrt{t}, t)| \geq c(\kappa, D_0)|t|^{n/2}$. So we just need to prove the lower bound.

For a number $\beta > 0$ to be fixed later, the upper bound implies
\[
\int_{B(x_0, \sqrt{\beta|t|}, t)} G^2(x, \tau; x_0, \tau/2) dg(x, t)
\geq \frac{1}{|B(x_0, \sqrt{\beta|t|}, t)|} \left( \int_{B(x_0, \sqrt{\beta|t|}, t)} G(x, \tau; x_0, \tau/2) dg(x, t) \right)^2
= \frac{1}{|B(x_0, \sqrt{\beta|t|}, t)|} \left( 1 - \int_{B(x_0, \sqrt{\beta|t|}, t)} c(\kappa, D_0) e^{-b d(x_0, y_0,t)^2/t} dg(x, t) \right)^2
\geq \frac{1}{|B(x_0, \sqrt{\beta|t|}, t)|} \left( 1 - \int_{B(x_0, \sqrt{\beta|t|}, t)} c(\kappa, D_0) e^{-b d(x_0, y_0,t)^2/t} dg(x, t) \right)^2
\]

Since the Ricci curvature is nonnegative, one can use the volume doubling property to compute that
\[
\int_{B(x_0, \sqrt{\beta|t|}, t)} c(\kappa, D_0) e^{-b d(x_0, y_0,t)^2/t} dg(x, t) \leq 1/2
\]
provided that $\beta$ is sufficiently large. Here we stress that all constants are independent of $t$. Since $|B(x_0, \sqrt{\beta|t|}, t)| \leq c_n(\beta|t|)^{n/2}$ by standard volume comparison theorem, this shows
\[
\int_{B(x_0, \sqrt{\beta|t|}, t)} G^2(x, \tau; x_0, \tau/2) dg(x, t) \geq \frac{c(\kappa, D_0)}{|t|^{n/2}}.
\]

Hence there exists $x_1 \in B(x_0, \sqrt{\beta|t|}, t)$ such that
\[
G(x_1, \tau; x_0, \tau/2) \geq \frac{c(\kappa, D_0)}{|t|^{n/2}}.
\]

For applications in Section 3, this lower bound is already sufficient.

An inspection of the proof shows that actually for any $\lambda \in [3/4, 4]$, it holds, for some $x_{\lambda} \in B(x_0, \sqrt{\beta|t|}, t)$,
\[
G(x_{\lambda}, \lambda \tau; x_0, \tau/2) \geq \frac{c(\kappa, D_0)}{|t|^{n/2}}.
\]

It is well known that such a lower bound implies the full Gaussian lower bound if one has a suitable Harnack inequality. Such Harnack inequality already exists. For the heat kernel, it is in Section 9 of [21]. For all positive solutions it is in Corollary 2.1 (a) in [KZ] and [CH]. Applying Corollary 2.1 (a) in [KZ], we get
\[
G(x_{3/4}, \frac{3}{4} \tau; x_0, \tau/2) \leq G(x, \tau; x_0, \tau/2) \left( \frac{\tau}{\tau_{3/4}} \right)^n \exp \left[ t_0^1 \frac{[4|\gamma'(s)|^2 + (\tau/4)^2 R]}{2(\tau/4)} ds \right],
\]
where $\gamma$ is a smooth curve on $\mathbf{M}$ such that $\gamma(0) = x_{3/4}$ and $\gamma(1) = x$. Also $|\gamma'(s)|^2 = g_{-1}(\gamma'(s), \gamma'(s))$, and $l = 3\tau/4 + s\tau/4$. 
This inequality together with the decay property of $R$ and compatibility of distances to conclude

$$G(x,\tau;x_0,\tau/2) \geq \frac{c(\kappa,D_0)}{|t|^{n/2}} e^{-b_1d(x,x_0,t)^2/\tau}.$$ 

This finishes the proof of the theorem. □

### 3. Applications to Ancient Solutions and the Poincaré Conjecture

In this section we use Theorem 1.1 to give a different proof for Perelman’s classification result of backward limits of ancient $\kappa$ solutions.

**Theorem 3.1. (Perelman)** Let $g(\cdot,t)$ with $t \in (-\infty,0]$ be a nonflat, 3 dimensional ancient $\kappa$ solution for some $\kappa > 0$. Then there exist sequences of points $\{q_k\} \subset M$ and times $t_k \to -\infty$, $k = 1, 2, \ldots$, such that the scaled metrics $g_k(x,s) \equiv R(q_k,t_k)g(x,t_k + sR^{-1}(q_k,t_k))$ around $q_k$ converge to a nonflat gradient shrinking soliton in $C^\infty_{\text{loc}}$ topology.

**Proof.**

We divide the proof into several cases.

Case 1 is when the section curvature is zero somewhere and $M$ is noncompact. Then Hamilton’s strong maximum principle for tensors show that $M = M_2 \times \mathbb{R}^1$ where $M_2$ is a 2 dimensional, nonflat ancient $\kappa$ solution. According to Hamilton, $M_2$ is either $S^2$ or $\mathbb{RP}^2$. So the theorem is already proven in this case. This case can also be covered in Case 4 below together.

Case 2 is when the section curvature is zero somewhere and $M$ is compact. Then, again using maximum principle, Hamilton (see Theorem 6.64 in [CLN] e.g) showed that $M$ is the metric quotient of $\mathbb{R}^3$ with the flat metric or that of $S^2 \times \mathbb{R}^1$. So the theorem is also proven in this case.

Case 3 is when the sectional curvature is positive everywhere and $M$ is a type II ancient solution. i.e. $\sup_{t<0} |t| R(\cdot,t) = \infty$.

In this case Hamilton [H2] showed by a scaling argument and his matrix maximum principle that the backward limit is a steady gradient soliton. See also Theorem 9.29 in [CLN], in which a proof is given for the non-compact case. However the compact case can be proven in the same way with the $\kappa$ non-collapsing assumption. So one can take a scaling limit to a shrinking gradient soliton. See Theorem 9.66 in [CLN] e.g. If the ancient solution arises from the blow up of finite time type II singularity, then Hamilton [H2] even proved that $M$ is a steady gradient soliton. If $M$ is compact, then it is well known that $M$ is an Einstein manifold. Since the curvature is positive, $M$ has to be $S^3$.

So there is only one case left.

Case 4: $M$ has positive sectional curvature and is of type I ancient solution.

If $M$ is compact, N. Sesum already proved the theorem in this case [S]. Actually she proved a stronger result, namely, $M$ is a shrinking gradient soliton. See also p 302 [CZ] and the work of X.D. Cao [Cx].

So we will assume that $M$ is noncompact and of type I for the rest of the proof. In fact our proof works in both compact and noncompact cases.

By the $k$ noncollapsing assumption and the bound $R(\cdot,t) \leq \frac{D_0}{1+|t|}$, we can find a sequence $\tau_k \to \infty$ such that the following holds:
the pointed manifolds \((M, g_k, y_k)\) with the metric
\[
g_k = \tau_k^{-1}g(\cdot, -s\tau_k)
\]
converge, in \(C^\infty_0\) sense, to a pointed manifold \((M_\infty, g_\infty(\cdot, s), y_\infty)\). Here \(s > 0\).

We aim to prove that \(g_\infty\) is a gradient, shrinking Ricci soliton. Note that we are scaling by \(\tau_k^{-1}\). By the upper and lower bound on the scalar curvature, this scaling is equivalent to scaling by the scalar curvatures. We define, for \(x \in M\) and \(s \geq 1\), the functions
\[
    u_k = u_k(x, s) = \tau_k^{n/2}G(x, s\tau_k; x_0, 0).
\]
Here \(G\) is the heat kernel of the conjugate heat equation and \(x_0\) is a fixed point. We choose \(y_k = x_0\) in the scaled of metrics above. By Theorem 1.1 (actually (2.11)), we know that \(u_k(x, s) \leq U_0\) uniformly for all \(k = 1, 2, \ldots, x \in M\) and \(s\) in a compact interval. Here \(U_0\) is a positive constant. Note that \(u_k\) is a positive solution of the conjugate heat equation under the metric on \((M, g(s))\) i.e.
\[
    \Delta g_k u_k - R g_k u_k - \partial_s u_k = 0.
\]

We have seen that \(u_k\) and \(R g_k\) are uniformly bounded on compact intervals of \(s\) in \((0, \infty)\), and also the Ricci curvature is nonnegative and the curvature tensors are uniformly bounded. The standard parabolic theory shows that \(u_k\) is Hölder continuous uniformly with respect to \(g_k\). Hence we can extract a subsequence, still called \(\{u_k\}\), which converges in \(C^\alpha_{loc}\) sense, modulo diffeomorphism, to a \(C^\alpha_{loc}\) function \(u_\infty\) on \((M_\infty, g_\infty(s), y_\infty)\).

Using integration by parts, it is easy to see that \(u_\infty\) is a weak solution of the conjugate heat equation on \((M_\infty, g_\infty(s))\), i.e.
\[
    \int \int (u_\infty \Delta \phi - \partial_s u_\infty \phi + u_\infty \partial \phi) \, dg_\infty(s) \, ds = 0
\]
for all \(\phi \in C^\infty_0(M_\infty \times (-\infty, 0])\).

By standard parabolic theory, the function \(u_\infty\), being bounded on compact time intervals, is a smooth solution of the conjugate heat equation on \((M_\infty, g_\infty(s), y_\infty)\). We need to show that \(u_\infty\) is not zero. One can even show that it is actually the fundamental solution of the conjugate heat equation with pole at \(y_\infty\) (the image of the same \(x_0\) in the limiting manifold). Let \(u = u(x, \tau) = G(x, \tau; x_0, 0)\). We claim that for a constant \(a > 0\) and all \(\tau \geq 1\),
\[
u(x_0, \tau) \geq \frac{a}{\tau}.
\]

Here is the proof. Define \(f\) by
\[
(4\pi \tau)^{-n/2}e^{-f} = u.
\]
By Corollary 9.4 in [P], which is a consequence of his differential Harnack inequality for fundamental solutions, we have, for \(\tau = -t\),
\[
-\partial_t f(x_0, t) \leq \frac{1}{2} R(x_0, t) \leq \frac{1}{2\tau} f(x_0, t).
\]
Since \(R(x_0, t) \leq c/\tau\), we can integrate the above from \(\tau = 1\) to get
\[
f(x_0, \tau) \leq c + \frac{f(x_0, 1)}{\tau} \leq C.
\]
Here we have use the fact that \(f(x_0, 1)\) is bounded, by the standard short time bounds for \(G = G(x_0, 1; x_0, 0)\). This proves the claim. By definition of \(u_k\) as a scaling of \(u\),
we know that \( u_k(x_0, s) \geq b > 0 \) for \( s \in [1, 4] \). Here \( b \) is independent of \( k \). Therefore \( u_\infty(x_0, s) \geq b > 0 \). The maximum principle shows \( u_\infty \) is positive everywhere.

Let us recall that Perelman’s \( W \) entropy for each \( u_k \) is

\[
W_k(s) = W(g_k, u_k, s) = \int [s(|\nabla f_k|^2 + R_k) + f_k - n] u_k dg_k(s)
\]

where \( f_k \) is determined by the relation

\[
(4\pi s)^{-n/2} e^{-f_k} = u_k;
\]

and \( R_k \) is the scalar curvature under \( g_k \). By the uniform upper bound for \( u_k \), we know that there exist \( c_0 > 0 \) such that

\[
f_k = -\ln u_k - \frac{n}{2} \ln(4\pi s) \geq -c_0
\]

for all \( k = 1, 2, \ldots \) and \( s \in [1, 3] \). Here the choice of this interval for \( s \) is just for convenience. Any finite time interval also works. Since \( M \) is noncompact, one needs to justify the integral in \( W_k(s) \) is finite. For fixed \( k \), \( u_k \) has a generic Gaussian upper and lower bound with coefficients depending on \( \tau_k \) and curvature tensor and their derivatives, as shown in [G]. The manifold has nonnegative Ricci curvature and bounded curvature. So the term \( f_k u_k \) which is essentially \(-u_k \ln u_k \) is integrable. The term \(|\nabla f_k|^2 u_k = |\nabla u_k|^2 / u_k \) which is integrable by Theorem 3.3 in [Z1], given in the appendix. These together imply that \( W_k(s) \) is well defined.

Since \( \int_M u_k dg_k = 1 \), we know that

\[
(3.1) \quad W_k(s) \geq -c_0 - n
\]

for all \( k = 1, 2, \ldots \) and \( s \in [1, 3] \).

There is an alternative proof of the lower bound for \( W_k \). Actually \( W_k(s) \) is uniformly bounded from below if \( u_k \) is replaced by any \( v \in W^{1, 2} \) such that \( \|v\|_2 = 1 \). This can be seen since \((M, g_k(s), g_k), s \in [1, 3] \) has uniformly bounded curvature operator and are \( \kappa \) noncollapsed. Therefore, a uniform Sobolev inequality holds, which implies the lower bound of \( W_k(s) \). The later is nothing but a lower bound on the best constants of log Sobolev inequalities.

By scaling it is easy to see that

\[
W_k(s) = W(g, u, s\tau_k),
\]

where \( u = u(x, l) = G(x, l, x_0, 0) \). According to [P1],

\[
(3.2) \quad \frac{dW_k(s)}{ds} = -2s \int |\text{Ric}_{g_k} + \text{Hess}_{g_k} f_k - \frac{1}{2s} g_k|^2 u_k dg_k(s) \leq 0.
\]

Note that the integral on the right hand side is finite by a similar argument as in the case of \( W_k(s) \). So, for a fixed \( s \), \( W_k(s) = W(g, u, s\tau_k) \) is a non-increasing function of \( k \). Using the lower bound on \( W_k(s) \), we can find a function \( W_\infty(s) \) such that

\[
\lim_{k \to \infty} W_k(s) = \lim_{k \to \infty} W(g, u, s\tau_k) = W_\infty(s).
\]

Now we pick \( s_0 \in [1, 2] \). Clearly we can find a subsequence \( \{\tau_{n_k}\} \), tending to infinity, such that

\[
W(g, u, s_0\tau_{n_k}) \geq W(g, u, (s_0 + 1)\tau_{n_k}) \geq W(g, u, s_0\tau_{n_{k+1}}).
\]
Since
\[
\lim_{k \to \infty} W(g, u, s_0 \tau_{\nu_k}) = \lim_{k \to \infty} W(g, u, s_0 \tau_{\nu_{k+1}}) = W_\infty(s_0),
\]
we know that
\[
\lim_{k \to \infty} [W(g, u, s_0 \tau_{\nu_k}) - W(g, u, (s_0 + 1) \tau_{\nu_k})] = 0.
\]
That is
\[
\lim_{k \to \infty} [W_{\nu_k}(s_0) - W_{\nu_k}(s_0 + 1)] = 0.
\]
Integrating (3.2) from \(s_0\) to \(s_0 + 1\), we use the above to conclude that
\[
\lim_{k \to \infty} \int_{s_0}^{s_0+1} \int s |Ric_{\nu_k} + Hess_{\nu_k} f_{\nu_k} - \frac{1}{2s} g_{\nu_k}|^2 u_{\nu_k} dg_{\nu_k}(s) ds = 0.
\]
Therefore
\[
Ric_\infty + Hess_\infty f_\infty - \frac{1}{2s} g_\infty = 0.
\]
So the backward limit is a gradient shrinking Ricci soliton.

Finally we need to show the soliton is non-flat. We can assume the original ancient solution is not a gradient shrinking soliton. Otherwise there is nothing to prove. Hence, we know that \(W_k(s) < W_k(0) = W_0 = 0\) where \(W_0\) is the Euclidean \(W\) entropy with respect to the standard Gaussian. Hence \(W_\infty(s) \leq W_k(s) < W_0\). If the gradient shrinking soliton \(g_\infty\) were flat, it is known to be \(\mathbb{R}^3\). Hence \(W_\infty(s) = W_0\), a contradiction. □

Remark. Case 4 with positive curvature tensor can also be dealt with by the method in [CL]. There Chow and Lu actually constructed an embedded region of the flow, which is close to \(S^2 \times \mathbb{R}\). They even do not need to assume the soliton is \(\kappa\) non-collapsed on all scales. In fact, there does not exist type I, noncompact, \(\kappa\) ancient solution with positive curvature tensor, after all. This is due to Perelman’s classification of backward limits.

Also the on diagonal lower bound of the fundamental solution \(G\) in the middle of the proof can be extended to full lower bound by the theorem in the appendix. But we do not need it here.

In the last part of the section, we discuss the ramification of the above method to the proof of the Poincaré conjecture. After the classification of the backward limits and \(\kappa\) non-collapsing with surgeries, the only part of Perelman’s proof of the Poincaré conjecture that requires the reduced distance and volume is the universal non-collapsing of ancient \(\kappa\) solutions. Interestingly, a different proof of this fact already exists in Section 3.2 of the paper of Chen and Zhu [ChZ], where certain more general 4 dimensional result is proven (see the paragraph after the proof of Proposition 3.4 there). In the 3 dimensional case, the proof looks longer than Perelman’s original proof. However it is basically a reshuffling of certain arguments suggested by Perelman, all which are needed to prove the canonical neighborhood property for ancient \(\kappa\) solutions. In this sense, the proof of the universal noncollapsing is a by product of canonical neighborhood property for ancient \(\kappa\) solutions. Indeed, the canonical neighborhood property for ancient \(\kappa\) solutions can be proven exactly the same way without the universal noncollapsing property, except that the constants in the property depend on the noncollapsing constant \(\kappa\). But this is enough to show that after a conformal change of metric using the scalar curvature function, the ancient solution is \(\epsilon\) close to model manifolds which are universal noncollapsed. Therefore the former is also universal noncollapsed.
Let us state the result and sketch the proof.

**Proposition 3.1.** (Perelman) There exists a positive constant $\kappa_0$ with the following property. Suppose we have a non-flat, 3 dimensional ancient $\kappa$ solution arising from finite time singularity of a Ricci flow, for some $\kappa > 0$. Then either the solution is $\kappa_0$ non-collapsed on all scales or it is a metric quotient of the round 3 sphere.

*Proof.* (sketched as a special case of Chen and Zhu’s proof in Section 3.2, the statement after Proposition 3.4 [ChZ])

Note we use an extra assumption that $\kappa$ solution is arising from finite time singularity of a Ricci flow. This will make the proof more transparent since type II $\kappa$ solution in this case is just steady gradient Ricci soliton as proven by Hamilton [H2].

If the three dimensional $M$ is compact, then they are explicitly known to be gradient solitons as mentioned in Cases 1-4 in the proof of the previous theorem. Anyway they are not needed in singularity analysis leading to the Poincaré conjecture. So we just need to prove that noncompact 3 dimensional $\kappa$ ancient solutions are universal non-collapsed on all scales. The proof is divided into 3 steps.

**step 1.** one proves the compactness of ancient $\kappa$ solutions with any fixed $\kappa > 0$. i.e.

The set of nonflat 3 dimensional ancient $\kappa$ solutions, for any fixed $\kappa > 0$, is compact modulo scaling in the following sense: for any sequence of such solutions and marking points in space time $(x_k, 0)$ with $R(x_k, 0) = 1$, one can extract a $C^\infty_{\text{loc}}$ converging subsequence whose limit is also an ancient $\kappa$ solution.

The proof is identical to that in [P1], the Theorem in Section 11.7. Note that no universal non-collapsing is needed here. This actually is the original order of proof by Perelman.

**step 2.** One proves certain elliptic type estimates for the scalar curvature.

There exist a positive constant $\eta$ and a positive increasing function $w : [0, \infty) \to (0, \infty)$ with the following property. Let $(M, g_{ij}(t))$, $-\infty < t \leq 0$ is a 3 dimensional ancient $\kappa$ solution for a fixed $\kappa > 0$. Then

(i) for every $x, y \in M$ and $t \in (-\infty, 0]$, there holds

$$R(x, t) \leq R(y, t) w(R(y, t)d^2(x, y, t));$$

(ii) for all $x \in M$ and $t \in (-\infty, 0]$, there hold

$$|\nabla R| \leq \eta R^{3/2}(x, t), \quad |\partial_t R(x, t) | \leq \eta R^2(x, t).$$

(iii) Suppose for some $(y, t_0)$ in space time and a constant $\zeta > 0$ there holds

$$\frac{|B(y, R(y, t_0)^{-1/2}, t_0)|_{t_0}}{R(y, t_0)^{-3/2}} \geq \zeta.$$

Then there exist a positive functions $w$ depending only on $\zeta$ such that, for all $x \in M$,

$$R(x, t_0) \leq R(y, t_0) w(R(y, t_0)d^2(x, y, t_0)).$$

The proof of statements (i) and (ii) is almost a carbon copy of Theorem 6.4.3 in [CZ] (3 d case) or Proposition 3.3 (4 d case) in [ChZ], or the corresponding results in [KL] and [MT]. The one difference is that one uses $\kappa$ non-collapsing assumption instead the universal non-collapsing that is being proved. Therefore the constant $\eta$ and the function $w$
may depend on $\kappa$. Part (iii) is the remark after Proposition 3.3 (4 d case) in [ChZ], which includes the 3 dimension case as a special situation. Its proof is a moderate refinement of that of statement (i), by keeping a careful track of constants.

**step 3.** For any point $(x, t)$, one shows that either it is a center of the $\epsilon$ neck, or it lies in a compact manifold with boundary, called $M_\epsilon$. After scaling by scalar curvature at one of its boundary points, this manifold is $\epsilon$ close to a compact manifold of finite diameter and whose scalar curvature is bounded between two positive constants which are independent of the noncollapsing constant $\kappa$. This step follows Proposition 3.4 in [ChZ] which is a 4 dimensional result that includes the 3 dimension one as a special case. They use a blow up argument, taking advantage of the property that a boundary point of $M_\epsilon$ is the centered of a ball which is $2\epsilon$ close to that of $S^2 \times \mathbb{R}$ after scaling. Then they use (iii) in step 2 to obtain the bounds on scalar curvature. The bounds depend only on the noncollapsing constant of $S^2 \times \mathbb{R}$.

This means that after scaling by scalar curvature, every point on the ancient solution has a ball of fixed diameter that is $\epsilon$ close to a model manifold which is universal non-collapsed. Therefore ancient $\kappa$ solutions is also universal non-collapsed.

Let us close by presenting the flow chart of a simplified proof of the Poincaré conjecture without reduced distance or volume.

Step 1. $W$ entropy and its monotonicity ([P1]). See also [Cetc], [CZ], [KL], [MT].

Step 2. Local non-collapsing result via Step 1 ([P1]). See also [Cetc], [CZ], [KL], [MT].

Step 3. getting ancient $\kappa$ solutions by blowing up of singularity using Step 2 and Hamilton’s compactness theorem ([P1]). See also [Cetc], [CZ], [KL], [MT].

Step 4. (i) showing the backward limits of ancient $\kappa$ solutions are gradient shrinking solitons. Earlier work of Hamilton [H2] for type II case and [CL] or this paper for type I case.

(ii) universal non-collapsing of ancient $\kappa$ solutions. Section 3.2 of [ChZ].

(iii) curvature and volume estimates for ancient solutions ([P1]). See also [Cetc], [CZ], [KL], [MT].

Step 5. classification of gradient shrinking solitons. [P1]. See also [Cetc], [CZ], [KL], [MT].

Step 6. canonical neighborhood property [P1]. That is: regions of high scalar curvature resemble the ancient solution after appropriate scaling.

See also [Cetc], [CZ], [KL], [MT].

Step 7. surgery procedure, including properties of the standard solution [P2]. See also [CZ], [KL], [MT].

Step 8. local $\kappa$ non-collapsing with surgeries [Z2].

Step 9. canonical neighborhood property with surgeries [P2]. See also [CZ], [KL], [MT].

Step 10. existence of Ricci flow with surgeries, i.e. proving there are finitely many surgeries within finite time. [P2]. See also [CZ], [KL], [MT].

Step 11. Finite time extinction of Ricci flow on simply connected manifolds [P3]. See also [CM] and [MT].

4. Appendix

Here we state and prove Theorem 3.3 in [Z1], which was used at the end of Step 2 in the proof of Theorem [L1]. See also [CH].
Theorem 4.1. Let $M$ be a compact or complete noncompact Riemannian manifold with bounded curvature and equipped with a family of Riemannian metric evolving under the forward Ricci flow $\partial_t g = -2Ric$ with $t \in [0, T]$. Suppose $u$ is any positive solution to $\Delta u - \partial_t u = 0$ in $M \times [0, T]$. Then, it holds

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq \sqrt{\frac{1}{t} \log \frac{M}{u(x,t)}}$$

for $M = \sup_{M \times [0,T]} u$ and $(x,t) \in M \times [0,T]$.

Moreover, the following interpolation inequality holds for any $\delta > 0$, $x, y \in M$ and $0 < t \leq T$:

$$u(y,t) \leq c_1 u(x,t)^{1/(1+\delta)} M^{\delta/(1+\delta)} e^{c_2 d(x,y,t)^2/t}.$$

Here $c_1, c_2$ are positive constants depending only on $\delta$.

Proof

This is almost the same as that of Theorem 1.1 in [H]. By direct calculation

$$\Delta (u \log \frac{M}{u}) - \partial_t (u \log \frac{M}{u}) = -\frac{\|\nabla u\|^2}{u},$$

$$(\Delta - \partial_t) (\frac{\|\nabla u\|^2}{u}) = 2 \frac{\partial_i \partial_j u - \partial_i u \partial_j u}{u} \geq 0.$$

The first inequality follows immediately from the maximum principle since $t \frac{\|\nabla u\|^2}{u} - u \log \frac{M}{u}$ is a sub-solution of the heat equation.

To prove the second inequality, we set $l(x,t) = \log (M/u(x,t))$.

Then the first inequality implies

$$|\nabla \sqrt{l(x,t)}| \leq 1/\sqrt{t}.$$

Fixing two points $x$ and $y$, we can integrate along a geodesic to reach

$$\sqrt{\log (M/u(x,t))} \leq \sqrt{\log (M/u(y,t))} + \frac{d(x,y,t)}{\sqrt{t}}.$$

The result follows by squaring both sides. □

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