Octonionic representations of Clifford algebras and triality

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The theory of representations of Clifford algebras is extended to employ the division algebra of the octonions or Cayley numbers. In particular, questions that arise from the non-associativity and non-commutativity of this division algebra are answered. Octonionic representations for Clifford algebras lead to a notion of octonionic spinors and are used to give octonionic representations of the respective orthogonal groups. Finally, the triality automorphisms are shown to exhibit a manifest \( \Sigma_3 \times SO(8) \) structure in this framework.

I. INTRODUCTION

The existence of classical supersymmetric string theories in \((n + 1, 1)\) dimensions has been linked to the existence of the normed division algebras \(K_n\), where \(K_n = \mathbb{R}, \mathbb{C}, \mathbb{H}, \) and \(\mathbb{O}\) for \(n = 1, 2, 4,\) and \(8\) are the algebras of the reals, complexes, quaternions, and octonions. One reason for this correspondence is the isomorphism \(sl(2, K_n) \cong so(n + 1, 1)\) on the Lie algebra level [3]. However, because of the non-associativity of the octonions, the extension of this result to finite Lorentz transformations, i.e., on the Lie group level, for \(n = 8\) has posed a problem until recently [4,5]. Nevertheless, octonionic spinors based on \(sl(2, \mathbb{O})\) have been used successfully as a tool to solve and parametrize classical solutions of the superstring and superparticle [5–7].

Another link between octonions and supersymmetric theories is given by the triality automorphisms of \(SO(8)\), which interchange the spaces of vectors, even spinors, and odd spinors. These automorphisms are constructed using the Chevalley algebra, which combines these three spaces into a single 24-dimensional algebra. Our formulation of the Chevalley algebra in terms of \(3 \times 3\) octonionic hermitian matrices naturally extends to the exceptional Jordan algebra. A variety of articles connect this algebra to theories of the superstring, the superparticle, and supergravity [10,11].

Division algebras are also used in the spirit of GUTs to provide a group structure that contains the known interactions [12].

The contribution of this paper is to bring these many isolated observations together and place them on the foundation of the theory of Clifford algebras. Our framework allows an elegant unified derivation of all the previous results about orthogonal groups. The octonionic triality automorphisms, for example, are completely symmetric with respect to the spaces of vectors, even spinors, and odd spinors, as they should be. We explain new features and properties of octonionic representations of Clifford algebras related to the possible choices of different octonionic multiplication rules. We also find that not all of the common constructions from complex representations have exact analogues for octonionic representations because of the non-commutativity of the octonions. For example, the octonionic analogue of the charge conjugation operation involves the opposite octonionic algebra, without which the transformation behavior is inconsistent. However, the extra structure of two distinguished octonionic algebras may turn out to be a feature of our formalism rather than a bug.

In a previous article [4] a demonstration of the construction of \(SO(7), SO(8), SO(9,1),\) and \(G_2\) is given, which illustrates how the octonionic algebra works explicitly. However, in this article, we only use the general algebraic properties of the octonions, rather than rely on explicit computations involving a specific multiplication rule. This approach is taken to highlight the central role of the alternativity of the octonions in the development of our formalism. In essence, we suggest the division algebra of the octonions not as an afterthought, but as a starting point for incorporating Lorentzian symmetry and supersymmetry in supersymmetrical theories. This principle is brought to fruition in a fully octonionic description of the triality automorphisms of the Chevalley algebra.

The content of this article is organized as follows: First we give a thorough introduction to composition algebras and the division algebra of the octonions. In particular, we devote a large part of section 1 to the investigation...
of the relationship amongst different multiplication tables of the octonions. In section II we state basic concepts about Clifford algebras and their representations. We characterize the Clifford group and the orthogonal group of a vector space with a metric by generating sets. This approach turns out to be better adapted to octonionic representations than the usual Lie algebra one. Then we introduce the octonionic representation of the Clifford algebra in 8-dimensional Euclidean space in section IV. In section V the reductions to 7 and 6 dimensions and the extension to 9+1 dimensions are discussed. In section VI, we introduce an octonionic description of the Chevalley algebra and show that the triality symmetry is inherent in the octonionic description. Then, in section VII, we briefly explain how our results with regard to sets of finite generators of Lie groups are related to the usual description in terms of infinitesimal generators of the corresponding Lie algebra. Section VIII discusses our results.

II. THE DIVISION ALGEBRA OF THE OCTONIONS

This section lays the first part of the foundation for octonionic representations of Clifford algebras, namely it introduces the octonionic algebra. The first subsection deals with some general properties of composition algebras. A subsection introducing our convention for octonions follows. We then turn our attention to the relationship among different multiplication tables for the octonions and introduce the opposite octonionic algebra. For further information and omitted proofs see [13,14,3]. A less rigorous approach is taken in [4].

A. Composition algebras

An algebra \( \mathfrak{A} \) over a field \( F \) is a vector space over \( F \) with a multiplication that is distributive and \( F \)-linear:

\[
\begin{align*}
(x + y)z &= xz + yz, \\
(xy)z &= x(yz),
\end{align*}
\]

\( \forall x, y, z \in \mathfrak{A}, \) \( \forall f \in F. \) (1)

\( \mathfrak{A} \) is also assumed to have a multiplicative identity \( 1_\mathfrak{A} \).

A composition algebra \( \mathfrak{A} \) over a field \( F \) is defined to be an algebra equipped with a non-degenerate symmetric \( F \)-bilinear form,

\[
\langle \cdot, \cdot \rangle : \mathfrak{A} \times \mathfrak{A} \to F
\]

\( (x, y) \mapsto \langle x, y \rangle , \) (3)

with the special property that it gives rise to a quadratic norm form which is compatible with multiplication in the algebra:

\[
|\cdot|^2 : \mathfrak{A} \to F
\]

\( x \mapsto |x|^2 := \langle x, x \rangle , \) (4)

\[
|xy|^2 = |x|^2 |y|^2 \quad \forall x, y \in \mathfrak{A}.
\]

(In the case of the octonions (3) is known as the eight-squares theorem, i.e., a sum of eight squares is the product of two sums of eight squares, and many applications rely on this identity.) Two main consequences can be derived (see [3]) from this essential property of composition algebras. Firstly, these algebras exhibit a weak form of associativity:

\[
\begin{align*}
x(xy) &= (xx)y, \\
(yx)x &= y(xx)
\end{align*}
\]

\( \forall x, y \in \mathfrak{A}. \) (6)

Defining the associator as a measure of the deviation from associativity via

\[
[x, y, z] := x(yz) - (xy)z, \quad x, y, z \in \mathfrak{A},
\]

(7)

then (3) implies

\[
[x, x, y] = [y, x, x] = 0 \quad \forall x, y \in \mathfrak{A}
\]

(8)
Since the associator is linear in its arguments, we can put (15), (18), and (19) together:
\[
[x, y, z] = -[x, z, y] = -[y, x, z] \quad \forall x, y, z \in \mathfrak{A},
\]  
\[i.e., the associator is an alternating function of its arguments. This weak form of associativity is also called\textit{ alternativity}. (15) and (19) are equivalent, if the characteristic \(\chi(F)\) of \(F\) does not equal 2, which is assumed from now on. As shown in [13], alternativity implies the so-called Moufang [15] identities,
\[
\begin{align*}
(xy(x))z &= x(y(xz)) \quad \forall x, y, z \in \mathfrak{A}, \\
z(xy) &= ((zx)y)x \quad \forall x, y, z \in \mathfrak{A}, \\
x(yz)x &= (xy)(zx)
\end{align*}
\]  
which will turn out to be useful later on.

Secondly, composition algebras are endowed with an involutory antiautomorphism \(\ast\):
\[
\ast : \mathfrak{A} \to \mathfrak{A} \quad x \mapsto x^\ast := 2 \langle 1, x \rangle - x, \quad \forall x, y \in \mathfrak{A}.
\]  
(Obviously, we view \(F\) as embedded in the algebra \(\mathfrak{A}\) via \( F \cong F1_\mathfrak{A} \subseteq \mathfrak{A} \), in particular \(1_\mathfrak{A} = 1_F = 1\). With this identification and (13), multiplication with an element of \(F\) is commutative, i.e., \(F \subseteq Z\), where \(Z\) is the center of \(\mathfrak{A}\).) We observe that \(\ast\) is linear and fixes \(F\). (Note that \(\langle 1, 1 \rangle = 1\), since \(\langle x, x \rangle = \langle x, x \rangle \langle 1, 1 \rangle \quad \forall x \in \mathfrak{A}\).) This antiautomorphism can be shown to provide a way to express the quadratic form \(|\cdot|^2\):
\[
x^x = x^\ast x = |x|^2 \quad \forall x \in \mathfrak{A}.
\]  
So all elements of \(\mathfrak{A}\) satisfy a quadratic equation over \(F\):
\[
x^2 - 2\langle 1, x \rangle x + |x|^2 = 0 \quad \forall x \in \mathfrak{A}.
\]  
Polarizing (12) results in an expression for the bilinear form:
\[
\langle x, y \rangle = \frac{1}{2}(x y^\ast + y x^\ast) \quad \forall x, y \in \mathfrak{A}.
\]  
We determine inverses:
\[
x^{-1} = \frac{x^\ast}{|x|^2} \quad \forall x \in \mathfrak{A}, |x|^2 \neq 0.
\]  
However, in order to solve a linear equation \(ax = b\), we need \(a^{-1}(ax) = x\). To see that we do indeed have associativity in this case, we need the following relationship,
\[
6[x, y, z] = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \quad \forall x, y, z \in \mathfrak{A},
\]  
which is defined as usual. So for \(\chi(F) \neq 2, 3\), we see that products with elements in \(Z\) are associative:
\[
x \in Z \iff [x, y] = 0 \quad \forall y \in \mathfrak{A} \iff [x, y, z] = 0 \quad \forall y, z \in \mathfrak{A}.
\]  
Since the associator is linear in its arguments, we can put (13), (1), and (18) together:
\[
[x^{-1}, x, y] = \frac{[x^\ast, x, y]}{|x|^2} = \frac{2\langle 1, x \rangle [1, x, y] - [x, x, y]}{|x|^2} = 0 \quad \forall x, y \in \mathfrak{A}, |x|^2 \neq 0.
\]  
Finally, we observe more general consequences of (11) and (18):
\[
[x^\ast, y] = -[x, y] = [x, y]^\ast \quad \forall x, y \in \mathfrak{A}
\]  
and
\[
[x^\ast, y, z] = -[x, y, z] = [x, y, z]^\ast \quad \forall x, y, z \in \mathfrak{A},
\]  
which imply that both commutators and associators have vanishing inner products with 1:
\[
\langle 1, [x, y] \rangle = \langle 1, [x, y, z] \rangle = 0 \quad \forall x, y, z \in \mathfrak{A}.
\]  
We will now turn to the specific composition algebra of the octonions.
B. Octonions

According to a theorem by Hurwitz [3], which relies heavily on (13) there are only four composition algebras over the reals with a positive definite bilinear form, namely the reals, \( \mathbb{R} \); the complexes, \( \mathbb{C} \); the quaternions, \( \mathbb{H} \) [17]; and the octonions or Cayley numbers, \( \mathbb{O} \) [18]. Their dimensions as vector spaces over \( \mathbb{R} \) are 1, 2, 4, and 8. Since the norm is positive definite, there exist inverses for all elements except 0 in these algebras. Therefore, they are also called normed division algebras.

For specific calculations the following concrete form of \( \mathbb{O} \) is useful. \( \mathbb{O} \cong \mathbb{R}^8 \) as a normed vector space. Fortunately, it is always possible to choose an orthonormal basis \( \{i_0, i_1, \ldots, i_7\} \) which induces a particularly simple multiplication table for the basis elements such as the one given by the following triples:

\[
\begin{align*}
i_0 &= 1, \\
i_a^2 &= -1 \quad (1 \leq a \leq 7), \\
i_a i_b = i_c &= -i_b i_a \quad \text{and cyclic for } (a,b,c) \in P = \{(1,2,3),(1,4,5),(1,6,7),(2,6,4),(2,5,7),(3,4,7),(3,5,6)\}.
\end{align*}
\]

The algorithm to obtain such a basis is similar to the Gram-Schmidt procedure [19] with additional requirements about products of the basis elements (see [4]).

Working over the field of real numbers, the following definitions of real and imaginary parts are customary:

\[
\begin{align*}
\text{Re}\ x &= \langle 1, x \rangle = \frac{1}{2}(x + x^*) \in \mathbb{R}, \\
\text{Im}\ x &= x - \langle 1, x \rangle = \frac{1}{2}(x - x^*) \in \mathbb{R}^\perp.
\end{align*}
\]

Also \( i_0 \) is called the real unit and the other basis elements are called imaginary units,

\[
\text{Re}\ i_0 = i_0, \quad \text{Im}\ i_a = i_a \quad (1 \leq a \leq 7).
\]

In analogy to \( \mathbb{C} \) and \( \mathbb{H} \), the antiautomorphism \( \cdot^* \) is called “octonionic conjugation”. It also changes the sign of the imaginary part. With these conventions (22) reads

\[
\text{Re}\ [x,y] = \text{Re}\ [x,y,z] = 0 \quad \forall x,y,z \in \mathfrak{A}.
\]

C. Multiplication tables

The question of possible multiplication tables arises, for example, when one reads another article on octonions, which, of course, uses a different one from the one given in (23). Usually it is remarked, that all 480 possible ones are equivalent, i.e., given an octonionic algebra with a multiplication table and any other valid multiplication table one can choose a basis such that the multiplication follows the new table in this basis. One may also take the point of view, that there exist different octonionic algebras, i.e., octonionic algebras with different multiplication tables. With this interpretation the previous statement means that all these octonionic algebras are isomorphic. However, this fact does not imply that a physical theory might not make use of more than one multiplication table at any given time. In this section we extend the ideas of Coxeter [20], giving a detailed description of how the various multiplication tables are related to each other. A new result, which emerges from our description, is that two classes of multiplication tables can be identified, namely the class corresponding to a given algebra and the one corresponding to its opposite algebra. In a physical theory, the distinction between these two classes becomes important when parity is not a good symmetry, i.e., in a chiral theory.

The set \( P \) in (23) can be taken to represent a labeling of the projective plane \( \mathbb{Z}_2 P^2 \) over the field with two elements \( \mathbb{Z}_2 = GF(2) = \{0,1\} \) (see Fig. 1). Before we explain this correspondence, we introduce the basic properties of \( \mathbb{Z}_2 P^2 \). (Readers who are not familiar with projective geometry may consult [21].) This plane contains as points the one-dimensional linear subspaces of \( (\mathbb{Z}_2)^3 \). Given a basis of \( (\mathbb{Z}_2)^3 \) these subspaces are

\[
\begin{align*}
p_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & p_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & p_3 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & p_4 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
p_5 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, & p_6 &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, & p_7 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\end{align*}
\]

(27)
FIG. 1. The projective plane $\mathbb{Z}_2 P^2$ representing a multiplication table for the octonions.

(Since these linear subspaces contain only one non-zero element, we will drop the angle brackets and identify the points with the non-zero elements of $(\mathbb{Z}_2)^3$.) The lines $l_1, l_2, \ldots, l_7$ of the plane are the two-dimensional linear subspaces of $(\mathbb{Z}_2)^3$, which can also be described by their normal vectors $n_1, n_2, \ldots, n_7$, i.e., the dual vectors that annihilate the subspaces:

\[
\begin{align*}
n_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
n_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
n_3 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
n_4 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
n_5 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},
n_6 &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},
n_7 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\end{align*}
\]  

(28)

So there are also seven lines in $\mathbb{Z}_2 P^2$. The geometry of the plane is then defined by the incidence of points and lines, where

\[
p_j \text{ and } l_k \text{ are incident } \iff p_j \subset l_k \iff n_k^T p_j \equiv 0 \pmod{2},
\]  

(29)

for example, $p_3$, $p_5$, and $p_6$ are incident with $l_7$.

We are now in a position to specify the previously mentioned correspondence between $\mathbb{Z}_2 P^2$ and $P$. $P$ contains seven triples formed out of seven labels. The labels represent points and the triples represent lines containing the three points given by the labels, i.e., a label and a triple are incident, if and only if the label is part of the triple. Cyclic permutations of a triple change neither the multiplication table nor the geometry of the plane. However, $P$ does define an orientation on each line, since a transposition in a triple would change the multiplication table. This notion of orientation on the lines, is represented by arrows in Fig. 1. So we can read the multiplication table off the triangle. If we follow a line connecting two labels in direction of the arrow we obtain the product, for example, $i_3 i_4 = i_7$. When moving opposite to the direction of the arrow we pick up a minus sign, $i_4 i_2 = -i_6$. (Note that in projective geometry the ends of the lines are connected, i.e., lines are topologically circles, $S^1$.

What are possible transformations of the multiplication table $P$ and how do they correspond to transformations of the projective plane $\mathbb{Z}_2 P^2$? Looking at Fig. 1, we see that there are three ways to change the picture:

(i) We may relabel the corners, leaving the arrows unchanged.

(ii) The labels may be kept fixed while some or all arrows are reversed.

(iii) Minus signs may be attached to the labels, i.e., we change part of (28) to read $i_a i_b = -i_c = -i_b i_a$ and cyclic for $(a,b,c) \in P$.

The sign change of a label in type (iii) is equivalent to reversing the orientation of the three lines through that point and therefore is included in the transformations of type (ii). For the second kind of transformation, we have to make
sure, that the multiplication table so obtained satisfies alternativity, for it to define another octonionic algebra. One can show that given the arbitrary orientation of four lines including all seven points, the orientations of the remaining three lines are determined by alternativity. (Note that there is only one case to consider. Among the four lines there are necessarily three which have one point in common. Two of those together with the fourth one fix one of the remaining orientations.) This in turn implies that elementary transformations of type (ii) change the orientation of three lines which have one point in common. So the transformations of type (ii) and type (iii) are equivalent. Since four arrows can be chosen freely, we obtain sixteen as the number of possible configurations of arrows, i.e., the number of distinct multiplication tables that can be reached this way, namely: 1 original configuration with no changes, 7 with the orientation of three lines through one point reversed, 7 with the orientation of four lines avoiding one point reversed, and 1 with the orientation of all lines reversed.

In order to discuss these transformations further, we will introduce some notation. (Before developing this framework, I verified most of these results using the computer algebra package Maple. So the reader who is not algebraically inclined may take this proof by exhaustion as sufficient. For a basic reference on group theory see [22].) We denote an octonionic algebra given by an orthonormal basis of $\mathbb{R}^8$ and a set $P$ of the type given in (24) by $O_P$, and the set made up of all such octonionic algebras by $O := \{ \text{all possible } P : O_P \}$. “All possible $P$” means those that induce a multiplication table satisfying alternativity. So $O$ can be viewed as the set of possible multiplication tables.

We now consider the group action of $T = T_1 * T_2$, the free product of transformations of type (i) and (ii), on $O$:

$$T \times O \rightarrow O$$

$$(t, O_P) \mapsto O_{t(P)}.$$ (30)

Thus each $t \in T$ induces an isomorphism $O_P \rightarrow O_{t(P)}$. The group of transformations $T_1$ of type (i), i.e., the relabelings of the corners, is of course the permutation group on seven letters, $\Sigma_7$, acting in the obvious way. We identify the group $T_2$ of transformations of type (ii) as $(\mathbb{Z}_2)^7$, with the 7 generators acting as the elementary transformations reversing the orientation of the three lines through one point. Earlier we saw that the orbits of an element of $O$ under the action of this group are of size 16: $|\text{Orb}_{(\mathbb{Z}_2)^7}(O_P)| = 16$. In order to determine the orbits of $\Sigma_7$ we first consider its subgroup $H$ which acts as the group of projective linear transformations on $\mathbb{Z}_2 P^2$ labeled as in Fig. 1, i.e., we let $H$ act on one specific $O_P \in O$, namely with $P$ as in (23). $H \cong \text{PGL}(3, \mathbb{Z}_2) \cong GL(3, \mathbb{Z}_2)$ is generated by the permutations $(1243675)$ and $(125)(374)$. $H$ is in fact simple, of Lie-type, of order 168 = $2^3 \cdot 3 \cdot 7$, and denoted by $A_2(2)$ (see [23]). Since elements of $H$ as projective linear transformation do not change the geometry of $\mathbb{Z}_2 P^2$, they can only reverse the orientations of lines, i.e., $\text{Orb}_H(O_P) \subseteq \text{Orb}_{(\mathbb{Z}_2)^7}(O_P)$. Hence, we have $|\text{Orb}_{H \ast (\mathbb{Z}_2)^7}(O_P)| = 16$. Thus the index of the stabilizing subgroup of $H$ has to divide 16:

$$|H : \text{Stab}_H(O_P)| = |\text{Orb}_H(O_P)| \big| 16.$$ (31)

Since the action of $H$ is not trivial and $H$ being simple of order 168 cannot have subgroups of index 2 or 4, we conclude $|\text{Orb}_H(O_P)| = 8$. To determine $|\text{Orb}_{\Sigma_7}(O_P)|$ we need to consider the cosets of $H$ in $\Sigma_7$. There are $|\Sigma_7 : H| = 30$ of them corresponding to distinct geometries of $\mathbb{Z}_2 P^2$, i.e., the incidence of lines and points is different for different cosets. Therefore, there are 30 distinct classes of multiplication tables, with members of one class related by a projective linear transformation. So it follows

$$|\text{Orb}_{\Sigma_7}(O_P)| = 30 \cdot 8 = 240,$$

$$|\text{Orb}_T(O_P)| = 30 \cdot 16 = 480.$$ (32)

So relabelings of the corners reach only half of the possible multiplication tables, which is a consequence of the fact that projective linear transformations reach only half of the possible configurations of arrows. Why is this so and what are the possible implications? To answer these questions we need to understand how elements of $H$ change orientations of lines. We can decompose the action of elements of $H$ into one part that permutes the lines and another one that reverses the orientation of certain lines in the image. An element $t_1 \in H$ of odd order $p$ may only change the orientation of an even number of lines. For $t_1^p = 1$ has to act trivially on $P$, and the changes of orientation add up modulo 2. However, $H$ is generated by elements of odd order, so all of its elements change only the orientation of an even number of lines. To obtain the full orbit we may add just one element $\zeta \in T_2$ that changes the orientation of an odd number of lines. A particularly good choice for $\zeta$ is the product of all generators, i.e., the one corresponding to reversing all seven lines (or attaching minus signs to all labels when viewed as type (iii) transformation). Obviously, $t_1 \zeta(P) = \zeta t_1(P) \ \forall t_1 \in T_1$, so that we may form the direct product $T_1' = T_1 \times \{1, \zeta\}$ and $\text{Orb}_{T_1'}(O_P) = \text{Orb}_{T_1}(O_P)$. Note that $\zeta$ corresponds to the operation of octonionic conjugation, so that the isomorphism given by $\zeta$ is illustrated by the following diagram:
\[ \mathbb{O}_P \times \mathbb{O}_P \rightarrow \mathbb{O}_P \]
\[ (a, b) \mapsto ab \]
\[ \zeta \times \zeta \quad \searrow \quad \downarrow \zeta \quad . \]
\[ \mathbb{O}_{\zeta(P)} \times \mathbb{O}_{\zeta(P)} \rightarrow \mathbb{O}_{\zeta(P)} \]
\[ (a^*, b^*) = (a', b') \mapsto (ab)^* = b^* a^* = b' a' . \]

(33)

Therefore, \( \mathbb{O}_{\zeta(P)} \) is the opposite algebra of \( \mathbb{O}_P \), i.e., the algebra obtained by reversing the order of all products. So for octonionic algebras, there is an isomorphism of an algebra and its opposite algebra given by octonionic conjugation, besides the natural anti-isomorphism given by identification. What are the consequences of these results for a physical theory? Usually, the physical theory will contain a vector space of dimension 8, for which we want to introduce an octonionic description. This description, however, should be invariant under the appropriate symmetry group, most commonly, \( SO(8) \). The multiplication table changes in a more general way under \( SO(8) \). The product of two basis elements will turn out to be a linear combination of all basis elements, but the relabelings given by \( \Sigma_{7} \) are certainly a subgroup contained in \( SO(8) \). Moreover, \( \zeta \notin SO(8) \), which implies that the most general multiplication tables with respect to an orthonormal basis split in two classes with \( SO(8) \) acting transitively on each class, but only \( SO(8) \times \{1, \zeta\} \cong O(8) \) acting transitively on all of them. In fact we will find it useful to consider two algebra structures, namely \( \mathbb{O} \) and its opposite \( \mathbb{O}_{\text{opp}} \), on the same \( \mathbb{R}^8 \) to describe the spinors of opposite chirality.

In a recent article, Cederwall & Preitschopf \[24\] introduce an “X-product” on \( \mathbb{O} \) via
\[ a \odot_X b := (a X)(X^* b), \quad a, b, X \in \mathbb{O}, \ X X^* = 1, \]

(34)

which is just the original product for \( X = 1 \). As \( X \) becomes different from 1, the multiplication table for this product changes continuously in a way related to the \( SO(8) \) transformations that leave 1 fixed. This changing product appears naturally when the basis of a spinor space is changed, see section [VF].

III. CLIFFORD ALGEBRAS AND THEIR REPRESENTATIONS

The second building block for octonionic representations of Clifford algebras is presented in this section. First we define an abstract Clifford algebra and observe some of its basic properties. Then we consider the Clifford group which gives us the action of the orthogonal groups on vectors and spinors. In our approach to the Clifford group in this second subsection we also introduce the key idea of characterizing groups by finite generators. The third subsection states the necessary facts about representations of Clifford algebras, i.e., how we can find matrix algebras to describe Clifford algebras. For further reference and proofs that are left out see \[25,26,27\]. We only consider the real or complex field, i.e., \( F = \mathbb{R}, \mathbb{C} \), in this section, even though some of the statements generalize to other fields, in particular of characteristic different from 2.

A. Clifford algebras

The tensor algebra \( T(V) \) of a vector space \( V \) of dimension \( n \) over a field \( F \) is the free associative algebra over \( V \):

(All the products in this section are associative.)

\[ T(V) := \bigoplus_{k=0}^{\infty} (V)^k, \]

(35)

where

\[ (V)^k = V \otimes V \otimes \cdots \otimes V, \quad k > 0, \quad (V)^0 = F. \]

(36)

The identity element is \( 1 \in F \) and \( F \) lies in the center of \( T(V) \). Given a metric \( g \) on \( V \), i.e., \( g \) is a non-degenerate symmetric bilinear form on \( V \), the Clifford algebra \( Cl(V,g) \) is defined to be

\[ Cl(V,g) := T(V)/I(g), \]

(37)

where
\(I(g) = \langle u \otimes u - g(u, u) : u \in V \rangle\) (38)

is the two-sided ideal generated by all expressions of the form \(u \otimes u - g(u, u)\). If \(V\) is unambiguously defined from the context, we simply write \(\text{Cl}(g)\). We denote multiplication in \(\text{Cl}(g)\) by

\[u \vee v := \pi^{-1}(u) \otimes \pi^{-1}(v) + I(g) \quad \forall \, u, v \in \text{Cl}(g),\] (39)

where \(\pi\) is the canonical projection:

\[\pi : \mathcal{T}(V) \to \text{Cl}(g), \quad u \mapsto u + I(g)\] (40)

and \(\pi^{-1}(u)\) is any preimage of \(u\). Since \(\pi\) restricted to \(F \oplus V\) is injective, we identify this space with its embedding in \(\text{Cl}(g)\).

From a more practical perspective a Clifford product is just a tensor product with the additional rule that

\[u \vee u = g(u, u) \quad \forall \, u \in V.\] (41)

As a consequence elements of \(V \subseteq \text{Cl}(g)\) anticommute up to an element of \(F\):

\[\{u, v\} := u \vee v + v \vee u = 2g(u, v) \quad \forall \, u, v \in V\] (42)

or in terms of an orthonormal basis \(\{e_1, e_2, \ldots, e_n\}\)

\[\{e_i, e_j\} := 2g(e_i, e_j) = \begin{cases} \pm 2, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases} \quad (1 \leq i, j \leq n).\] (43)

Based on these relationships, we find a basis for \(\text{Cl}(g)\) as a vector space,

\[\{e_{a_1} \vee e_{a_2} \vee \cdots \vee e_{a_k} : 0 \leq k \leq n, 1 \leq a_1 < a_2 < \cdots < a_k\},\] (44)

which shows that

\[\text{dim} \text{Cl}(g) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.\] (45)

The product \(\eta = e_1 \vee e_2 \vee \cdots \vee e_n\) is called the volume form and has the special property

\[\eta \vee u = (-1)^{n+1} u \vee \eta \quad \forall \, u \in V.\] (46)

So for odd \(n\), \(\eta\) lies in the center \(Z\) of \(\text{Cl}(g)\). In fact

\[Z = \begin{cases} F, & \text{for } n \text{ even} \\ F \oplus F\eta, & \text{for } n \text{ odd} \end{cases}.\] (47)

There are two involutions on \(\mathcal{T}(V)\) given by the obvious extensions of the following maps on \(V \subseteq \mathcal{T}(V)\): the main automorphism \(\alpha\),

\[\alpha|_V : V \to V \quad u \mapsto -u,\] (48)

and the main antiautomorphism \(\beta\),

\[\beta|_V : V \to V \quad u \mapsto u \quad \beta(u \otimes v) = v \otimes u \quad \forall \, u, v \in V.\] (49)

Since \(I(g)\) is invariant under \(\alpha\) and \(\beta\), we obtain maps on the quotient \(\text{Cl}(g)\). The main antiautomorphism can also be understood as an isomorphism between \(\text{Cl}(g)\) and its opposite algebra \((\text{Cl}(g))_{\text{opp}}\)
The even and odd part of the Clifford algebra are defined to be
\[ \alpha \]
The main automorphism \( \beta \) is generated by a primitive idempotent \( Q \in C(l(g)) \), i.e.,
\[ Q^2 = Q, \quad \not\exists Q_1, Q_2 : Q_1^2 = Q_1 \neq 0, \quad Q_2^2 = Q_2 \neq 0, \quad Q = Q_1 + Q_2. \]
(These characterizations of minimal left ideals rely on the fact that Clifford algebras over \( \mathbb{R} \) and \( \mathbb{C} \) are semisimple, see Section III C.) If the primitive idempotent is even, the spinor space \( S \) decomposes into the spaces of even and odd Weyl spinors:
\[ S = S_0 \oplus S_1, \quad S_k = P_k S = C(l(k)) \vee Q, \quad (k = 0, 1). \]
Different names for these spaces are used within the mathematical physics community. \( S \) is also called the space of Dirac spinors and \( S_0 \) and \( S_1 \) are called semi-spinor spaces. Sometimes, elements of \( S \) are called bi-spinors and elements of \( S_0 \) and \( S_1 \) are just called even and odd spinors.

For a mixed primitive idempotent \( Q \) there may still be a Weyl decomposition (see (86)), but it is not compatible with the \( \mathbb{Z}_2 \) grading on \( C(l(g)) \):
\[ S = C(l(g)) \vee Q = C(l_0(g)) \vee Q = C(l_1(g)) \vee Q. \]

For odd \( n \), \( S \) is also called the space of Pauli spinors or semi-spinors. If only the double \( 2S := S \oplus S \) carries a faithful representation of \( C(l(g)) \) (see (90)), then some authors refer only to \( 2S \) as the space of spinors.

**B. The Clifford group**

The connection of the symmetry group of the metric, i.e., the orthogonal group, with the Clifford algebra is made in this subsection via the Clifford group \( \Gamma(g) \) (see (53), (54), and (55)). We use a non-standard definition of the Clifford group in terms of a set of finite generators. We were led to this approach because octonionic representations are naturally implemented in this way. However, we feel this characterization of the Clifford group is simpler in many applications. By relating both definitions to the orthogonal group we show that they are essentially equivalent (see (77), (88), and (90)).

We define the Clifford group \( \Gamma(g) \) to be the group generated by the vectors of non-zero norm, i.e.,
\[ \Gamma(g) := \{ u \in V : u^2 = g(u, u) \neq 0 \}. \] (58)

As we will see, this definition is almost equivalent to the usual one,
\[ \Gamma'(g) := \{ u \in Cl(g) : u \text{ invertible, } u \cdot x \cdot u^{-1} \in V \ \forall x \in V \} \supseteq \Gamma(g). \] (59)

Considering \( u \in \Gamma(g) \cap V \) and any \( x \in V \) we see that
\[
\begin{align*}
  u \cdot x \cdot u^{-1} &= u \cdot x \cdot \frac{g(u, u)}{g(x, u)} = \frac{g(x, u)}{g(x, u)}(-x \cdot u + 2g(x, u)) \cdot u \\
  &= -x + 2\frac{g(x, u)}{g(x, u)}u \in V.
\end{align*}
\] (60)

Therefore, \( \Gamma'(g) \supseteq \Gamma(g) \) indeed, and in particular \( \Gamma'(g) \cap V = \Gamma(g) \cap V \). In fact, the definition of \( \Gamma'(g) \) implies that \( \Gamma(g) \) is stable under conjugation in \( \Gamma'(g) \), i.e., \( \Gamma(g) \) is a normal subgroup of \( \Gamma'(g) \). We will investigate the structure of the Clifford group on the basis of this group action of \( \Gamma'(g) \) on \( V \):
\[
\phi' : \Gamma'(g) \times V \rightarrow V \\
(u, x) \mapsto \phi'_u(x) := u \cdot x \cdot u^{-1}.
\] (61)

Dropping all the primes we have the obvious restriction
\[
\phi : \Gamma(g) \times V \rightarrow V \\
(u, x) \mapsto \phi_u(x) := u \cdot x \cdot u^{-1}.
\] (62)

(We will not explicitly give the unprimed analogues of expressions below.) Of course, these actions can be extended to give inner automorphisms of \( Cl(g) \). According to \( \Phi \), the action of \( u \in V \cap \Gamma'(g) \) is just a reflection of \( x \) at the hyperplane orthogonal to \( u \) composed with an inversion of the whole space. In particular \( \phi'_u(x) \in V \) and \( \phi'_u \) is an isometry:
\[
g(\phi'_u(x), \phi'_u(x)) = \phi'_u(x) \cdot \phi'_u(x) = (\frac{1}{u})^2 u \cdot x \cdot u^{-1} \cdot u \cdot x \cdot u^{-1} \\
= u \cdot x \cdot u^{-1} = g(x, x).
\] (63)

So \( \Phi' \) (resp. \( \Phi \)) gives a homomorphism \( \Phi' \) (resp. \( \Phi \)) of \( \Gamma'(g) \) (resp. \( \Gamma(g) \)) to the group of isometries or orthogonal transformations \( O(g) \) of \( V \):
\[
\Phi' : \Gamma'(g) \rightarrow O(g) \\
\eta \mapsto \phi'_{\eta} : V \rightarrow V \\
x \mapsto \phi'_{\eta}(x) = u \cdot x \cdot u^{-1}
\] (64)

To compare \( \Gamma'(g) \) (resp. \( \Gamma(g) \)) with \( O(g) \) we need to know the range and the kernel of \( \Phi' \) (resp. \( \Phi \)). Since the reflections at hyperplanes generate all orthogonal transformations \( \Phi' \) (resp. \( \Phi \)) is onto, if we can find a preimage of the inversion \( x \mapsto -x \). Because of \( \Phi \), \( \eta \in \Gamma(g) \subseteq \Gamma' \) does the job for even \( n \). For odd \( n \), there is no element of \( Cl(g) \) that anticommutes with all \( x \in V \). So there is no preimage of the inversion, which leaves us with \( SO(g) \) as the range. The kernel coincides with the part of the center, that lies in the Clifford group. Thus we have according to the homomorphism theorems
\[
\begin{align*}
\Gamma(g)/\mathbb{F}^* &\cong O(g) \cong \Gamma'(g)/\mathbb{F}^* & \text{(for even } n \text{)} \tag{65} \\
\Gamma(g)/\mathbb{F}^*(\eta) &\cong SO(g) \cong \Gamma'(g)/\mathbb{F}^* & \text{(for odd } n \text{)}, \tag{66}
\end{align*}
\]

where \( \mathbb{F}^* = \mathbb{F} \setminus \{0\}, \langle \eta \rangle \) is the group generated by \( \eta \), and \( \mathbb{F}^* := \Gamma'(g) \cap \mathbb{Z} \) is the invertible part of the center. So the Clifford group is isomorphic to the orthogonal (resp. simple orthogonal) group up to a subgroup of the center \( \mathbb{Z} \). Therefore,
\[
\begin{align*}
\Gamma(g) &\cong \Gamma'(g) & \text{(for even } n \text{)} \tag{67} \\
\Gamma(g) \times \mathbb{F}^*/\mathbb{F}^*(\eta) &\cong \Gamma'(g) & \text{(for odd } n \text{)}, \tag{68}
\end{align*}
\]

So for even \( n \) both definitions \( \text{(58)} \) and \( \text{(59)} \) of the Clifford group are equivalent. For odd \( n \) they differ by inhomogeneous elements of the invertible part of the center \( \mathbb{F}^* \). For our purposes it will be sufficient to consider the Clifford group \( \Gamma(g) \) as defined in \( \text{(58)} \) only.
For both even and odd $n$, we obtain a homomorphism from the even Clifford group $\Gamma_0(g)$,
\[ \Gamma_0(g) := \Gamma(g) \cap C\ell_0(g) = P_0\Gamma(g) = \Gamma'(g) \cap C\ell_0(g) = P_0\Gamma'(g), \]
onto $SO(g)$:
\[ \Gamma_0(g)/\mathbb{Z} \cong SO(g). \]
(69)

The even Clifford group is generated by pairs of vectors with non-zero norm:
\[ \Gamma_0(g) = \langle u \vee v : u, v \in V, g(u, u) \neq 0 \neq g(v, v) \rangle. \]
(70)

In fact one of the vectors may be fixed,
\[ \Gamma_0(g) = \langle u \vee v : u \in V, g(u, u) \neq 0 \rangle, \text{ for some } w \in V, g(w, w) \neq 0, \]
(71)
since any product of two vectors $u, v$ can be written as a product of two pairs that contain $w$: $u \vee v = (u \vee w) \vee (v' \vee w)$, where $v' = \frac{1}{g(w, w)}w \vee v \vee w^{-1} \in V$.

We also have an action $\psi$ of $\Gamma(g)$ on the Clifford algebra $C\ell(g)$ and in particular on any of its minimal left ideals, a space of spinors $S$:
\[ \psi : \Gamma(g) \times S \to S \\
(u, s) \mapsto \psi_u(s) := u \vee s. \]
(73)

So we have two actions of the Clifford group $\Gamma(g)$ and its subgroup $\Gamma_0(g)$, the action $\phi$ on vectors and the action $\psi$ on spinors. These actions give rise to the so-called vector and spinor representations of the simple orthogonal group via the isomorphism (70). All octonionic representations of orthogonal groups in sections IV and V are based on this relationship. In physics particles are understood in terms of representations of groups describing the symmetries in the physical theory, in particular the Lorentz group. Therefore these representations of the orthogonal group are important, because they determine how physical fields transform.

The way in which $\mathbb{F}^*$ should be divided out in (70) is obvious for the vector representation, since $\mathbb{F}^*$ is the kernel of $\Phi$. For the spinor representation, requiring the invariance of the spinor bilinear form (see section III D) determines how to divide out scalars (see (101) and (101)). Actually, this leads to a homomorphism of $\Gamma_0(g)$ onto the universal covering group of $SO(g)$, which is also called $Spin(g)$. We will take $SO(g)$ to be the appropriate group depending on the context and not make a distinction in notation between $SO(g)$ and $Spin(g)$.

### C. Representations of Clifford algebras

In this subsection we describe how we can get a matrix algebra that is isomorphic to a Clifford algebra. In a sense this is the analogue to [17], where we gave an explicit form of the octonions, which implemented their abstract properties. We start out by introducing some definitions concerning representations in general. Algebras are assumed to be finite dimensional and contain a unit element. (For a general reference for representation theory see, e.g., [28].)

A representation $\gamma$ of an algebra $\mathfrak{A}$ over a field $\mathbb{F}$ in a vector space $W$ is a homomorphism
\[ \gamma : \mathfrak{A} \to \text{End}_{\mathbb{F}}(W) \\
a \mapsto \gamma(a) : W \to W \\
w \mapsto \gamma(a)w, \]
(74)
i.e.,
\[ \gamma(a \vee b) = \gamma(a)\gamma(b) \]
\[ \gamma(a + b) = \gamma(a) + \gamma(b) \]
\[ \forall a, b \in \mathfrak{A}, \]
(75)

where we denote multiplication in $\mathfrak{A}$ by $\vee$ even though $\mathfrak{A}$ is not necessarily a Clifford algebra. Given a basis of $W$, $\gamma(a)$ as an endomorphism of $W$ may be understood as an $l \times l$-matrix, where $l = \dim W$ is called the dimension of the representation. The representation is called faithful, if $\gamma$ is injective. $R$ is an invariant subspace of $\gamma$, if $\gamma(a)R \subseteq R \forall a \in \mathfrak{A}$. The representation $\gamma$ is called irreducible, if there are no invariant subspaces of $\gamma$ other than $W \neq \{0\}$ and $\{0\}$. A reducible representation $\gamma$ may be reduced to a representation $\gamma_R$ on an invariant subspace $R$,
i.e., \( \mathfrak{A} \hookrightarrow \text{End}_F(R) \) requiring \( \gamma_R(a)w = \gamma(a)w \) \( \forall w \in R, \ a \in \mathfrak{A} \). An algebra is called simple, if it allows a faithful and irreducible representation. An algebra is called semisimple if it is a direct sum of simple algebras.

Since a left ideal \( J \) of \( \mathfrak{A} \) is by definition stable under left multiplication,

\[
\mathfrak{A} \cdot J \subseteq J, \tag{76}
\]

and since \( J \) is a vector space, we have a natural representation \( \lambda_J \) of \( \mathfrak{A} \) on \( J \). (Again given a basis \( \{b_1, b_2, \ldots, b_l\} \) we have a representation in terms of matrices: \( a \cdot b_i = \lambda_J(a)b_i \).) Taking \( J = \mathfrak{A} \) we obtain the so-called left regular representation, which is faithful. If \( J \) is a minimal left ideal, then the representation on it is irreducible, since invariant subspaces would correspond to proper subspaces of \( J \) which are left ideals and contradict the minimality of \( J \).

If the algebra \( \mathfrak{A} \) is semisimple then the converse is also true, i.e., any irreducible representation can be written as a \( \lambda_J \) for some minimal left ideal \( J \): In this case an irreducible representation \( \gamma \) of \( \mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_k \) is an irreducible representation of one of the simple components, say \( \mathfrak{A}_k \). So a minimal ideal \( L \) of \( \gamma(\mathfrak{A}) \) can be lifted to a minimal ideal of \( J \subseteq \mathfrak{A}_j \), such that \( \gamma(J) = L \). Then the following diagram commutes,

\[
\begin{array}{c}
\mathfrak{A} \\
\lambda_J \downarrow \\
\text{End}(J)
\end{array} \xleftarrow{\gamma} \begin{array}{c}
\gamma(\mathfrak{A}) \subseteq \text{End}(W) \\
\gamma(J) = L \\
\text{End}(L)
\end{array}. \tag{77}
\]

Since the maps \( \lambda_L \) and \( \gamma(J) = L \) are isomorphisms, there is an isomorphism relating \( W \) and \( J \) as vector spaces,

\[
F : W \to J, \tag{78}
\]

such that

\[
\gamma(a) \circ F = F \circ \lambda_J(a) \quad \forall a \in \mathfrak{A}. \tag{79}
\]

\( F \) is said to intertwine the representations \( \gamma \) and \( \lambda_J \):

\[
\begin{array}{c}
W \\
F \downarrow \\
J
\end{array} \xleftarrow{\gamma(a)} \begin{array}{c}
W \\
F \downarrow \\
J
\end{array} \xrightarrow{\lambda_J(a)} \begin{array}{c}
W \\
F \downarrow \\
J
\end{array}. \tag{80}
\]

Representations related in this way are called equivalent. In terms of their matrix form, equivalent representations are related by a basis transformation. This observation also shows that for a simple algebra all irreducible representations are equivalent to \( \lambda_I \) and therefore equivalent to each other.

As is shown in the references given (see in particular [3, 27]), Clifford algebras over \( \mathbb{R} \) and \( \mathbb{C} \) are simple or semisimple. Therefore, there is an equivalent definition for spinors in terms of representations of \( Cl(g) \), i.e., a spinor space \( S \) can be defined to be the carrier space of an irreducible representation of \( Cl(g) \).

In order to find a concrete representation, we must still find a primitive idempotent \( Q \) that generates a minimal left ideal \( J \) and observe how the basis elements of \( Cl(g) \) act on it. Actually, we will give a procedure to construct a representation that does not use a primitive idempotent explicitly. For this purpose we define the signature of a metric for the case \( F = \mathbb{R} \). We say that \( g \) has the signature \( p, q \) (written \( g_{p,q} \)), where \( \dim V = p + q = n \), if there is an orthonormal basis \( \{e_1, e_2, \ldots, e_n\} \) of \( V \), such that

\[
g_{ij} := g(e_i, e_j) = \begin{cases} 
0, & \text{for } i \neq j \\
1, & \text{for } i = j \leq p \\
-1, & \text{for } i = j > p
\end{cases}. \tag{81}
\]

We write \( Cl(p,q) \) and \( \gamma_{p,q} \) to denote \( Cl(g_{p,q}) \) and one of its representations. It is particularly simple to give a procedure that produces a representation of \( Cl(m,m) \), i.e., in the case of a so-called neutral space. The procedure starts by “guessing” a representation \( \gamma_{1,1} \) for \( Cl(1,1) \):

\[
\gamma_{1,1}(e_1) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: \sigma \quad \text{and} \quad \gamma_{1,1}(e_2) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: \epsilon
\]

\[
\implies \gamma_{1,1}(e_1 \circ e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: \tau \quad \text{and} \quad \gamma_{1,1}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1. \tag{82}
\]
If $A$ is a representation of a group, it follows by complex conjugation that representations $\gamma$ are necessarily equivalent to a real one. To examine this issue we define the complex conjugate $p, q$ of the decomposition of the Clifford algebra in its even and odd part. Let, for example, $Q$ be an irreducible if $\gamma$ is faithful and irreducible. The representation $\gamma$ to a representation $\gamma'$ of $C(I(V', g'))$ with $\dim V' = 2m + 2$:

$$\gamma'(e'_i) = \sigma \otimes \gamma_{p,q}(e_i) \quad (1 \leq i \leq 2m), \quad \gamma'(e'_{2m+1}) = \sigma \otimes \gamma_{p,q}(\eta), \quad \gamma'(e'_{2m+2}) = \epsilon \otimes \gamma_{p,q}(1).$$

(Of course, there are other extensions using the same building blocks.) It is easy to check that $\gamma'$ is faithful and irreducible if $\gamma_{p,q}$ was. The signature of the resulting metric $g'$ depends on the value of $\gamma_{p,q}(\eta)^2 = (-1)^\nu(\nu-1)1$.

For even $\frac{p}{2}$, $\gamma_{p,q}(\eta)$ has eigenvalues $+1$ and $-1$ and we have Weyl projections $P_\pm$:

$$P_\pm := \frac{1}{2}(1 \pm \eta).$$

One of these projectors can be decomposed to give an even primitive idempotent $Q$. A representation such as the one given, where $\gamma_{p,q}(\eta) = \left(\begin{smallmatrix} 1_{m \times m} & \epsilon \otimes \gamma_{p,q}(\eta) \\ 0 & -1_{m \times m} \end{smallmatrix}\right)$ is called a Weyl representation, since the Weyl projections $P_\pm$ take a simple form. Due to the property (46) of $\eta$,

$$P_\pm a = a_0 P_\pm + a_1 P_\mp,$$

where $a_0$ and $a_1$ are the even and odd part of $a$. Since either $P_+$ or $P_-$ annihilates the even primitive idempotent $Q$, we indeed get projections onto the spaces of even and odd Weyl spinors. Let, for example, $P_+ \lhd Q = Q$ and $P_- \lhd Q = 0$, then for $s = a \lhd Q = a_0 \lhd Q + a_1 \lhd Q \in S_0 \oplus S_1$, $a = a_0 + a_1$ as before,

$$P_+ \lhd s = P_+ \lhd a \lhd Q = a_0 \lhd P_+ \lhd Q + a_1 \lhd P_- \lhd Q = a_0 \lhd Q \in S_0,$n

$$P_- \lhd s = P_- \lhd a \lhd Q = a_0 \lhd P_- \lhd Q + a_1 \lhd P_+ \lhd Q = a_1 \lhd Q \in S_1.$$ (88)

If we choose a mixed primitive idempotent $Q$, then we get a different decomposition $S = P_+ S \oplus P_- S$ unrelated to the decomposition of the Clifford algebra in its even and odd part.

The procedure continues for even $n$ and $\nu \neq 0$. In this case we can get a complex representation of the same dimension $l = 2m$ by complexifying and transforming the metric to obtain a neutral space. (We can change the sign of $g(e_j, e_j)$ for given $j$ using the transformation $e_k \mapsto \begin{cases} e_k, & \text{for } k \neq j \\ e_j, & \text{for } k = j \end{cases}$. Thus we may choose a basis to obtain a form $S$ of the metric with any $p, q$ where $p + q = n$.) This complex representation is faithful and irreducible but not necessarily equivalent to a real one. To examine this issue we define the complex conjugate $\gamma^*$ of a representation $\gamma : \mathfrak{A} \to \text{End}_C(W)$ by

$$\gamma^* : \mathfrak{A} \to \text{End}_C(W^*) \quad \gamma^* a = (\gamma(a))^*.$$

If $\mathfrak{A}$ is simple then $\gamma$ and $\gamma^*$ are equivalent, i.e., there exists a linear map $C : W \to W^*$ intertwining these two representations:

$$\gamma^* a \circ C = C \circ \gamma(a) \quad \forall a \in \mathfrak{A}.$$ (90)

It follows by complex conjugation that
whence by Schur’s Lemma $C^* \circ C$ is proportional to the identity. Since $C^* \circ C$ has a real eigenvalue, $C$ can be normalized to satisfy

$$C^* \circ C = \pm 1.$$  

(92)

If and only if $C^* \circ C = +1$, then we can find a basis transformation to make $\gamma_{p,q}$ real. This is the case for $\nu \equiv 0, 2 \pmod{8}$. In practice, we relate $W$ and $W^*$ by complex conjugation in the obvious way. $C$ is found by imposing (84) for $a \in \{e_1, e_2, \ldots, e_n\}$. (Following the procedure given above, any of the matrices $\gamma(e_k)$ is either real or purely imaginary, so that $C$ either commutes or anticommutes with it.) The new basis is a basis of eigenvectors for $C$, which is invariant under $s \mapsto s_C := (Cs)^*$. $(s_C$ is essentially the charge conjugate spinor for $s$.) For the cases $\nu \equiv 0, 6 \pmod{8}$ we can make a similar transformation to make $\gamma_{p,q}$ purely imaginary. These real (resp. purely imaginary) representations are known as Majorana representations of the first (resp. second) kind. Of course, even for $\nu \equiv 1$ (mod 8) we can find an irreducible real representation of higher dimension, namely $l = 2m+1$, by letting $I \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1$ and $i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \epsilon$ in an irreducible complex representation.

D. Bilinear forms on spinors

Physical observables are tensors, which in terms of the Clifford algebra transform under the orthogonal group like $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, while spinors transform like $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For this reason it seems likely that a bilinear form on spinors may provide observables based on spinors. The algebraic approach uses the fact that for $u \in \Gamma(g)$ its inverse $u^{-1}$ is proportional to $\beta(u)$. Therefore, up to a normalization $s_\beta(s')$ transforms under the tensorial action of $\Gamma(g)$. A decomposition in terms of a basis of the Clifford algebra gives the tensorial pieces of certain rank. In terms of representations we construct a bilinear form on spinors considering induced representations of the opposite Clifford algebra. Given a representation $\gamma: \mathfrak{A} \to \text{End}_F(W)$ there is an induced representation $\gamma^T$, its “transpose”:

$$\gamma^T: \mathfrak{A}_{\text{opp}} \to \text{End}_F(W^T)$$

$$a_{\text{opp}} \mapsto (\gamma^T)(a_{\text{opp}}) := (\gamma(a))^T : W^T \to W^T$$

$$w^T \mapsto \gamma^T(a_{\text{opp}})(w^T) = w^T \gamma(a_{\text{opp}}).$$

This is indeed a representation since

$$\gamma^T(a_{\text{opp}} \cdot b_{\text{opp}}) = \gamma((b_{\text{opp}} \cdot a_{\text{opp}}))^T = \gamma(b)\gamma(a)^T$$

$$= \gamma^T(a_{\text{opp}})\gamma^T(b_{\text{opp}}) \quad \forall a_{\text{opp}}, b_{\text{opp}} \in \mathfrak{A}_{\text{opp}}.$$  

(96)
As we pointed out in (50), the main antiautomorphism $\beta$ can be viewed as connecting the algebra $\mathfrak{A}$ and its opposite $\mathfrak{A}_{opp}$, so that we may obtain another induced representation $\hat{\gamma}$ for $\mathfrak{A}$ by

$$\hat{\gamma}(a) := \gamma^T(\beta(a)) = (\gamma(\beta(a)))^T \quad (a \in \mathfrak{A}),$$

(97)

where we interpret $\beta$ first $\mathfrak{A} \overset{\beta}{\longrightarrow} \mathfrak{A}_{opp}$ as in [51] and then as an antiautomorphism $\mathfrak{A} \overset{\beta}{\longrightarrow} \mathfrak{A}$ on $\mathfrak{A}$.

Since a bilinear form on spinors can be understood as a linear transformation $B : W \rightarrow W^T$, we take $B$ to be a map that intertwines the representations $\gamma$ and $\hat{\gamma}$. Such a map exists if the representation $\gamma$ is irreducible, whence $\hat{\gamma}$ is also irreducible. In this case $B$ is defined up to a constant by

$$B \circ \gamma(a) = \hat{\gamma}(a) \circ B \quad \forall a \in \mathfrak{A} \quad \iff \quad B\gamma(e_k) = (\gamma(e_k))^T B \quad \forall k \in \{1, \ldots, n\}.$$  

(98)

We understand $B$ as a bilinear form on $W$:

$$B : W \times W \rightarrow \mathbb{F}$$

$$s, s' \mapsto B(s, s') := (B(s))(s') = \pi s' = s^T B s'$$

(99)

both as a map and as its matrix form. $\pi := B(s) = s^T B$ is the adjoint to $s$ with respect to $B$. Indeed, $B(s, s')$ transforms like a scalar (compare (73)):

$$B(s, s') \overset{\psi_s}{\rightarrow} B(u \triangledown s, u \triangledown s') = s^T \gamma(u)^T B(u)s' = s^T \hat{\gamma}(\beta(u)) B(u)s'$$

$$= s^T B^\gamma(\beta(u)) \gamma(u)s' = [\beta(u) \triangledown u] s^T B s',$$

(100)

if $u = u_1 \triangledown \cdots \triangledown u_k \in \Gamma(g)$ and $u_1, \ldots, u_k \in V$ such that

$$\beta(u) \triangledown u = g(u_1, u_1) \cdots g(u_k, u_k) = 1.$$  

(101)

For $x \in V$, $x \triangledown s' \overset{\psi_s}{\rightarrow} u \triangledown x \triangledown s' = (u \triangledown x \triangledown u^{-1}) \triangledown (u \triangledown s')$, hence $B(s, x \triangledown s')$ also transforms like a scalar. Therefore, a vector $y$ is given by

$$y_k = B(s, e_k \triangledown s') = s^T B\gamma(e_k)s' \quad (1 \leq k \leq n).$$

(102)

In a similar way, a tensor $Y$ of rank $r$ may be formed:

$$Y_{k_1 \cdots k_r} = B(s, e_{k_1} \triangledown \cdots \triangledown e_{k_r} \triangledown s') = s^T B\gamma(e_{k_1}) \cdots \gamma(e_{k_r})s' \quad (1 \leq k_1, \ldots, k_r \leq n).$$

(103)

Another bilinear form $E$ may be obtained by replacing the main antiautomorphism $\beta$ with $\alpha \circ \beta$ which, of course, is an antiautomorphism also. So $E$ is determined up to a constant by

$$E \circ \gamma(a) = \hat{\gamma}(\alpha(a)) \circ E \quad \forall a \in \mathfrak{A} \quad \iff \quad E\gamma(e_k) = -(\gamma(e_k))^T E \quad \forall k \in \{1, \ldots, n\};$$

(104)

therefore, for even $n$

$$E = B\gamma(\eta).$$

(105)

The condition (101) changes to

$$(\alpha \circ \beta)(u) \triangledown u = (-1)^kg(u_1, u_1) \cdots g(u_k, u_k) = 1,$$

(106)

which reduces to the previous condition for $u \in \Gamma_0(g)$. So both bilinear forms are invariant under the action of normalized elements of $\Gamma_0(g)$.

Both of these bilinear forms may be combined with $C$ to give a sesquilinear form $A : W \rightarrow W^\dagger$ on $W$. We only consider the combination $A := B^* \circ C$ here:

$$A \circ \gamma(a) = B^* \circ C \circ \gamma(a) = B^* \circ \gamma^*(a) \circ C = \gamma^*(\beta(a))^T \circ B^* \circ C$$

$$= \gamma^T(\beta(a)) \circ A \quad \forall a \in \mathfrak{A},$$

(107)

$$\iff \quad A\gamma(e_k) = \gamma^T(e_k)A \quad \forall k \in \{1, \ldots, n\},$$

By a similar argument as in (91),

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\[(A^{-1})^\dagger \circ A \circ \gamma(a) = (A^{-1})^\dagger \circ \gamma((\beta(a)) \circ (A^\dagger \circ (A^{-1})^\dagger) \circ A \]
\[= (A^{-1})^\dagger \circ (A \circ \gamma((\beta(a))) \circ (A^{-1})^\dagger) \circ A \]
\[= (A^{-1})^\dagger \circ (\gamma((\beta \circ \beta)(a)) \circ A) \circ (A^{-1})^\dagger \circ A \]
\[= ((A^{-1})^\dagger \circ A^\dagger) \circ \gamma(a) \circ A^\dagger \circ A \]
\[= \gamma(a) \circ A^\dagger \circ A \quad \forall \ a \in \mathbb{A}, \]

we conclude by Schur’s Lemma that we can normalize \( A \) to satisfy
\[(A^{-1})^\dagger \circ A = I. \] (109)

Therefore, \( A \) may be assumed to be hermitian. Of course, \( A \) like \( B \) may be used to define a spinor adjoint \( \mathcal{F} := A(s) = s^\dagger A \) and to construct tensors of various rank as sesquilinear forms of spinors. Which one of these forms is chosen depends on the signature and the physical theory.

In all of our derivations involving \( C, B, E, \) and \( A \), we relied on certain properties of matrix multiplication over the field \( \mathbb{C} \) (resp. \( \mathbb{R} \)), namely the fact that transposition is an anti-automorphism and complex conjugation is an automorphism of matrix multiplication. We are about to replace \( \mathcal{F} \) by \( \mathcal{O} \). Since octonionic multiplication is not commutative and octonionic conjugation has become an anti-automorphism, the only remaining anti-automorphism of octonionic matrix multiplication is hermitian conjugation. Due to the non-associativity of the octonions even the carrier space \( W \) is no longer a vector space, but an “octonionic module”. It is surprising but true that there are natural resolutions for these difficulties as we show in the following section \( \text{IV} \).

### IV. An Octonionic Representation of \( \text{Cl}(8,0) \)

In this section we will put the results of sections \( \text{I} \) and \( \text{II} \) to work and examine the features of octonionic representations of Clifford algebras, considering the example of \( \text{Cl}(8,0) \). So \( V = \mathbb{R}^8 \) with a positive definite norm. Let \( \{e_0, e_1, \ldots, e_7\} \) be an orthonormal basis of \( V \). Note that we choose indices ranging from 0 to 7 in this section. The octonionic algebra \( \mathcal{O} \) is assumed to be given with basis \( \{i_0, i_1, \ldots, i_7\} \) obeying the multiplication table \( \text{[3]} \). However, the properties
\[i_0 = 1, \]
\[i_a^2 = 1 \quad (1 \leq a \leq 7), \]
\[i_ai_b = -i_bi_a \quad (1 \leq a < b \leq 7). \] (110)

rather than the particular multiplication rule, i.e., the particular set \( P \) of triples, will be relevant. Furthermore, we identify \( V \) with \( \mathcal{O} \) as vector spaces by \( x^k e_k \mapsto x^k i_k \).

#### A. The representation

An octonionic representation \( \gamma_{8,0} : \text{Cl}(8,0) \to M_2(\mathcal{O}) \) is given by
\[\gamma_{8,0}(e_k) := \begin{pmatrix} 0 & i_k \\ i_k^* & 0 \end{pmatrix} =: \Gamma_k \quad (0 \leq k \leq 7) \] (111)
\[\Leftrightarrow \quad \gamma_{8,0}(x) := \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} = x^k \Gamma_k =: \# \quad (x \in V). \] (112)

The carrier space \( W \) of the representation is understood to be \( \mathcal{O}^2 \), i.e., the set of columns of two octonions, with \( \gamma_{8,0}(x) \) acting on it by left multiplication. Therefore, octonionic matrix products are interpreted as being associated to the right and acting on \( W \), i.e., octonionic matrix multiplication is understood to be composition of left multiplication onto \( W \). For example, if we want to verify that \( \text{[112]} \) is a representation, then checking that
\[\# \# = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} = \begin{pmatrix} xx^* & 0 \\ 0 & xx^* \end{pmatrix} = |x|^2 1 = g(x,x)1 \quad \forall x \in V \] (113)
in accordance with \( \text{[83]} \) is not sufficient. This relationship has to hold even when acting on an element \( w = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \in W \):
This completes the proof that \( \gamma \) (Note that \( \Gamma_0 \neq 0 \) than multiplication, we have an octonionic Weyl representation: the sign difference corresponds to the two classes of multiplication tables. Since \( \Gamma_0 \) Which sign is true depends on the specific multiplication rule. With our convention the plus sign applies. In fact, the Weyl projections take the form \( \{1\} \in W \) can be mapped to any \( w \in W \): \[
(\psi - \psi_0 I) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & w \end{pmatrix} \begin{pmatrix} 0 & w \end{pmatrix} = \begin{pmatrix} 0 & w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (w_0, w_1), \]
(115)
Second, we will show that any non-zero \( w \in W \) can be mapped to \( \{1\} \), using the Weyl projections \( P_{\pm} \). If this is so, then there are no non-trivial invariant subspaces of the representation \( \gamma_{8,0} \).
Since \( \{1\} \) holds for the volume element \( \eta \), we have for \( \Gamma_9 := \gamma_{8,0}(\eta) = \psi_0 \eta_1 \ldots \eta_7 \)
\[
\Gamma_9 x = -x^*(i_0(i_1(i_2(...(i_7x)...))) = (w_0, w_1), \]
(116)
hence
\[
i_0(i_1(i_2(...(i_7x)...))) = x(i_0(i_1(i_2(...(i_6i_7)...)))) \quad \forall x \in \mathbb{O}. \]
(117)
Since \( \Gamma_9^2 = 1 \), \( \Gamma_9 \) has eigenvalues \( \pm 1 \), whence we can find solutions to the equation
\[
\Gamma_9 w = \pm w \iff \begin{pmatrix} i_0(i_1(i_2(...(i_7w_0)...)) \end{pmatrix} = \begin{pmatrix} -w_0(i_0(i_1(i_2(...(i_6i_7)...))) \end{pmatrix} = \pm \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}. \]
(118)
Since a non-trivial solution exists,
\[
i_0(i_1(i_2(...(i_7x)...))) = \pm x \quad \forall x \in \mathbb{O}. \]
(119)
Which sign is true depends on the specific multiplication rule. With our convention the plus sign applies. In fact, the sign difference corresponds to the two classes of multiplication tables. Since \( \Gamma_9 \) is defined by its action under left multiplication, we have an octonionic Weyl representation:
\[
\Gamma_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
(120)
The Weyl projections take the form
\[
P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]
(121)
For any \( 0 \neq w \in W \), at least one of \( P_+ w \) or \( P_- w \) does not vanish. If \( P_+ w \neq 0 \), then
\[
\psi_0^{-1} I P_+ w = \begin{pmatrix} 0 & w_0 \\ (w_0^{-1})^* & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]
(122)
(Note that \( I = \Gamma_0 \) corresponds to a vector \( e_0 \in V \subseteq Cl(8,0) \) and is to be distinguished from the identity \( \gamma(1) = 1 \).)
If \( P_- w \neq 0 \), then
\[
\psi_1^{-1} P_- w = \begin{pmatrix} 0 & w_0 \\ (w_0^{-1})^* & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]
(123)
This completes the proof that \( \gamma_{8,0} \) is irreducible. Since \( Cl(8,0) \) is simple, it does not contain any two-sided ideals other than \( \{0\} \) and itself, which are also the only candidates for the kernel of any representation of \( Cl(8,0) \). Therefore, \( \gamma_{8,0} \)
Apart from the scalar, we form tensors as spinor bilinears as in (103):

the Clifford algebra. So if the representation obtained for the Clifford group is faithful, then so is the representation for the Clifford algebra.

In this article we have chosen to rely only on the algebraic properties of the octonions, rather than using the correspondence to a real representation. However, for completeness, we give the matrices corresponding to left multiplication with respect to our convention:

\[
\begin{align*}
\Gamma_0 &= \sigma \otimes 1 \otimes 1 \otimes 1, \\
\Gamma_1 &= -\epsilon \otimes 1 \otimes 1 \otimes \epsilon, \\
\Gamma_2 &= -\epsilon \otimes \tau \otimes \epsilon \otimes \tau, \\
\Gamma_3 &= -\epsilon \otimes 1 \otimes \epsilon \otimes \sigma, \\
\Gamma_4 &= -\epsilon \otimes \epsilon \otimes 1 \otimes \tau, \\
\Gamma_5 &= -\epsilon \otimes \epsilon \otimes \tau \otimes \sigma, \\
\Gamma_6 &= -\epsilon \otimes \sigma \otimes \epsilon \otimes \tau, \\
\Gamma_7 &= -\epsilon \otimes \epsilon \otimes \sigma \otimes \sigma.
\end{align*}
\] (124)

Since we have an irreducible representation, we may identify the carrier space \(W\) with the space of spinors. So for now we consider elements of \(\mathbb{O}^2\) as octonionic spinors. Later in section IV E we will add a subtle twist to this understanding.

**B. The hermitian conjugate representation and spinor covariants**

Since octonionic conjugation is an antiautomorphism of \(\mathbb{O}\), the octonionic conjugate of the product of two matrices is not the product of the octonionic conjugates. Matrix transposition requires a commutative multiplication to be an antiautomorphism. Thus only hermitian conjugation, which combines both operations, remains as an antiautomorphism of \(M_2(\mathbb{O})\). More precisely, for products of three matrices we need to keep the grouping of the product the same, i.e., under hermitian conjugation left multiplication by a matrix goes to right multiplication by its hermitian conjugate and vice versa. So we can define \(\tilde{\gamma}_{8,0} : Cl(8,0) \rightarrow (M_2(\mathbb{O}))^\dagger\) by

\[
\tilde{\gamma}_{8,0}(a) := (\gamma_{8,0}(\beta(a)))^\dagger \quad (a \in Cl(8,0)).
\] (125)

This representation acts on the set \(W^\dagger = (\mathbb{O}^2)^\dagger\) of rows of two octonions by right multiplication. It is also faithful and irreducible and therefore equivalent to \(\gamma_{8,0}\). The isomorphism \(A\) intertwining \(\gamma_{8,0}\) and \(\tilde{\gamma}_{8,0}\) is given by

\[
A : \begin{pmatrix} w_0 & w_1 \end{pmatrix} \mapsto \begin{pmatrix} w_0^* \end{pmatrix}, \quad w = \begin{pmatrix} w_0 & w_1 \end{pmatrix}.
\] (126)

Its matrix form is just the identity,

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\] (127)

which is verified,

\[
A \circ \gamma_{8,0}(a) = \tilde{\gamma}_{8,0}(a) \circ A \quad \forall a \in Cl(8,0) \quad \iff \quad A \gamma_{8,0}(x) = (\gamma_{8,0}(x))^\dagger \circ A \quad \forall x \in V,
\] (128)

considering \(\Gamma_k = (\Gamma_k)^\dagger\) \((0 \leq k \leq 7)\).

From \(A\) we obtain a hermitian form on \(W\):

\[
A : W \times W \to \mathbb{R} \quad (w, z) \mapsto A(w, z) := (A(w))(z) = \text{Re}(w^\dagger A z) = \text{Re}(w_0^* w_1^T) \left( \begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix} \right) = \text{Re}(w_0^* z_0 + w_1^* z_1).
\] (129)

The designation “hermitian” is somewhat misleading, since the octonionic representation \(\gamma_{8,0}\) is Majorana, i.e., essentially real, which is also the reason for taking the real part above. So the spinor adjoint is given by

\[
\bar{w} := A(w) = w^\dagger A = w^\dagger \quad (w \in W).
\] (130)

Apart from the scalar, we form tensors as spinor bilinears as in (103):
\[ Y_{k_1\ldots k_r} := \text{Re} \, \overline{\mathbf{w}} \Gamma_{k_1} \ldots \Gamma_{k_r} z. \] 

Since the real part of an associator vanishes (20) and \( A \) is real, we may associate the matrices sandwiched between the two spinors differently:

\[
\text{Re} \, \overline{\mathbf{w}} \Gamma_{k_1} \ldots \Gamma_{k_r} z = \text{Re} \left( w^A \right) \left[ \Gamma_{k_1} \ldots (\Gamma_{k_r} z) \ldots \right] = \text{Re} \left[ (w^A) \Gamma_{k_1} \ldots (\Gamma_{k_r} z) \ldots \right] = \text{Re} \left[ w^A (\Gamma_{k_1} \ldots (\Gamma_{k_r} z) \ldots) \right] = \text{Re} \left[ (w^A \Gamma_{k_1}) \Gamma_{k_2} \ldots (\Gamma_{k_r} z) \ldots \right] = \text{Re} \Gamma_{k_1} \overline{w} (\Gamma_{k_2} \ldots (\Gamma_{k_r} z) \ldots). \]

Since the real part of a commutator vanishes also, we may cyclically permute, if a trace is included

\[
\text{Re} \, \overline{\mathbf{w}} \Gamma_{k_1} \ldots \Gamma_{k_r} z = \text{Re} \text{tr} \left( \overline{w} (\Gamma_{k_1} \ldots (\Gamma_{k_r} z) \ldots) \right) = \text{Re} \text{tr} \left( (\Gamma_{k_1} \ldots (\Gamma_{k_r} z) \ldots) \overline{w} \right) \Gamma_{k_1} \right] \right). \]

For the vector covariant, we have a particular expression

\[
y_k := \text{Re} \, \overline{\mathbf{w}} \Gamma_k z = \text{Re} \left( w_0^* w_1^* \right) \left( \begin{array}{cc} 0 & i_k \\
1_k & 0 \end{array} \right) \left( \begin{array}{c} z_0 \\
1_1 \end{array} \right) = \text{Re} \left( w_0^* i_k z_0 + w_1^* i_k z_0 \right) = \text{Re} \left( i_k z_0 w_0^* + z_0 w_1^* i_k \right) = \left( w_0 z_1^* + z_0 w_1^* \right),
\]

where we used once for part of the expression that the real part does not change under octonionic conjugation. So we can express the \( k \)-th component of \( y \) by the \( k \)-th component of an octonionic product, which allows us to write \( \hat{y} \) without the use of the matrix representations of the basis elements:

\[
\hat{y} = \left( \begin{array}{cc} 0 & y \\
y^* & 0 \end{array} \right) = \Gamma_k \text{Re} \, \overline{\mathbf{w}} \Gamma_k \overline{z}
\]

\[
\hat{y} = \left( \begin{array}{cc} 0 & w_0 z_1^* + z_0 w_1^* \\
0 & w_0 z_1^* + z_0 w_1^* \end{array} \right).
\]

C. Orthogonal transformations

From section II B we know the action of the Clifford group on vectors (12) and spinors (13). The condition (101) shows how to divide out \( \mathbb{R}^* \) to obtain the orthogonal group. So elements of \( V \) satisfying

\[
\beta(u) \vee u = 1 \iff u \vee u = g(u, u) = |u|^2 = 1
\]

generate the orthogonal transformations via

\[
\hat{x}' = (\gamma \circ \phi_u)(x) = \check{y} \hat{y} = \left( \begin{array}{c} u^* x u^* \\
0 \end{array} \right),
\]

\[
w' = \psi_u(w) = \check{y} w = \left( \begin{array}{c} w u_1 \\
u^* u_0 \end{array} \right).
\]

The Moufang (11) identities ensure that (137) is unambiguous and even holds under the action of left multiplication, which can be seen in the example, \( (x \vee w)' = x' \vee w' \):

\[
\check{y} w = \left( \begin{array}{cc} 0 & u^* x u^* \\
0 & u^* u_0 \end{array} \right) \left( \begin{array}{c} w u_1 \\
u^* u_0 \end{array} \right) = \left( \begin{array}{c} (u^* x u^*) (w u_1) \\
(u^* u_0) (w u_0) \end{array} \right) = \left( \begin{array}{c} u^* (x (u^* u) w_1) \\
u^* (u (u^* u_0) w_0) \end{array} \right) = |u|^2 \left( \begin{array}{c} u^* (x w_1) \\
u^* (x w_0) \end{array} \right) = \check{y} \hat{y} (w) = (\check{y} w)',
\]

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The third Moufang identity guarantees that the vector covariant \([135]\) of two spinors transform correctly:

\[
\begin{align*}
\hat{y}' &= \left( \begin{array}{cc} 0 & uy^* \\ wyu & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & u(w_0z_1^* + z_0w_1^*)u^* \\ u^*(w_0z_1^* + z_0w_1^*)u & 0 \end{array} \right) \\
&= \left( \begin{array}{cc} u^*(w_0z_1^* + z_0w_1^*)u & 0 \\ u^*(w_0z_1^* + z_0w_1^*)u^* & 0 \end{array} \right) \\
&= \left( \begin{array}{cc} (u^*w_0)(z_1^*u^*) + (u^*z_0)(w_1^*u^*)^* & 0 \\ 0 & 0 \end{array} \right) \\
&= \Gamma^k \text{Re} \overline{w} \Gamma_k z'.
\end{align*}
\]

According to \([72]\), simple orthogonal transformations are generated by pairs \((u, v) \in V \times V\), where we take \(v = e_0\) fixed and \(|u|^2 = 1\):

\[
\begin{align*}
\hat{y}' &= (\gamma_{8,0} \circ \phi_{(u,v)})(x) = \#y \#y \#y = \left( \begin{array}{c} 0 \\ u^*x^*u^* \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ (uxu)^* \\ 0 \end{array} \right), \\
w' &= \psi_{(u,v)}(w) = \#w = \left( \begin{array}{c} uw_0 \\ u^*w_1 \end{array} \right).
\end{align*}
\]

Choosing the fixed vector to be \(e_0\) allows significant simplification, since its representation \(\Gamma_0\) is real. How to construct any orthogonal transformation from these generators is thoroughly explained in \([4]\). These transformation properties imply that the definition of the spinor covariants in section \([5, 6]\) is consistent. For example,

\[
\text{Re} \overline{w} \hat{y}' \hat{y}' = \text{Re} \overline{\#w} \#y \#y \#y \#w z = \text{Re} w^\dagger \# I^\dagger A \# I^\dagger z = \text{Re} w^\dagger A I \# y \# y \# y \# y \# w \# z = \text{Re} \bar{w} \bar{y} z.
\]

### D. Related representations using the opposite octonionic algebra \(\mathcal{O}_{\text{opp}}\)

As pointed out in section \([5, 6]\), transposition and octonionic conjugation are not (anti-)automorphisms of octonionic matrix multiplication. However, we can find (anti-)isomorphisms to matrix algebras by using the opposite octonionic algebra \(\mathcal{O}_{\text{opp}}\). We define the octonionic conjugate representation \(\gamma^*\) of an octonionic representation \(\gamma : \mathcal{A} \to M_l(\mathcal{O})\) by

\[
\gamma^* : \mathcal{A} \to M_l(\mathcal{O}_{\text{opp}}), \quad a \mapsto \gamma^*(a) = (\gamma(a))^*_{\text{opp}}.
\]

Octonionic products are now to be evaluated in the opposite algebra as is indicated in the following examples. First we consider the action of \(\gamma_{8,0}(x)\) for \(x \in V\) on an element \(w^*\) of the carrier space \(W_{\text{opp}}^*\)

\[
\gamma_{8,0}(x)w_{\text{opp}} = \left( \begin{array}{c} 0 \\ x^* \\ 0 \end{array} \right)^* \left( \begin{array}{c} w_0 \\ w_1 \end{array} \right)^*_{\text{opp}} = \left( \begin{array}{c} 0 \\ x^* \\ 0 \end{array} \right) \left( \begin{array}{c} w_0^* \\ w_1^* \end{array} \right)_{\text{opp}}
\]

\[
= \left( \begin{array}{c} w_1^*x^* \\ w_0^*x \end{array} \right)
\]

\[
= (\gamma_{8,0}(x)w)^* = \left( \begin{array}{c} xw_1 \\ x^*w_0 \end{array} \right)^* = \left( \begin{array}{c} (xw_1)^* \\ (x^*w_0)^* \end{array} \right)
\]

So in this representation the action on the carrier space is effectively right multiplication by octonions.

We check that \(\gamma_{8,0}^*\) is indeed a representation. Let \(u, v \in V\), then

\[
\gamma_{8,0}^*(u)\gamma_{8,0}^*(v) = \left( \begin{array}{c} 0 \\ u^* \\ 0 \end{array} \right)^*_{\text{opp}} \left( \begin{array}{c} 0 \\ v^* \\ 0 \end{array} \right)^*_{\text{opp}} = \left( \begin{array}{c} 0 \\ u^* \\ 0 \end{array} \right) \left( \begin{array}{c} v^* \\ 0 \\ 0 \end{array} \right)_{\text{opp}}
\]

\[
= \left( \begin{array}{c} vu^* \\ 0 \\ 0 \end{array} \right)
\]

\[
= \gamma_{8,0}(u \lor v) = (\gamma_{8,0}(u \lor v))^* = (\#y)^*
\]

\[
= \left( \begin{array}{c} uw^* \\ 0 \\ u^*v \end{array} \right)^*.
\]
In both cases the subscript “opp” indicates that the remaining products are to be done in the opposite octonionic algebra. However the final result is to be interpreted as an element of $W^*_\text{opp}$ (resp. $M_2(\mathbb{O}_\text{opp}^*)$).

Since
\[ \Gamma_0 \Gamma_k = (\Gamma_k)^* \Gamma_0 \quad (0 \leq k \leq 7), \] (147)

we define the map
\[ C : W \to W^*_\text{opp} \]
\[ w \mapsto C(w) := \Gamma_0 w^*_\text{opp}. \] (148)

This map gives rise to an operation on $W$ which is analogous to charge conjugation:
\[ w_C := C(w)^* = \Gamma_0 w^* = \begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix} \in W. \] (149)

Let us examine how $w_C$ transforms under an orthogonal transformation:
\[ (w_C)' = \Psi w_C = \begin{pmatrix} 0 \\ u \end{pmatrix} \begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]
\[ \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} = \begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}. \]
\[ \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix}, \]
\[ \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}, \]
\[ \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}. \]
\[ \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}, \]
\[ \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}. \]
\[ \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}, \]
\[ \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}. \]

So the map $C$ almost intertwines $\gamma^*_{8,0}$ and $\gamma_{8,0}$, except that the opposite octonionic algebra has to be included explicitly:
\[ (w_C)' \neq (C(w'))^* = \left[ \Gamma_0 \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \right]^* = \left[ \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix} \right]^*. \] (151)

Of course, octonionic conjugation intertwines $\gamma^*_{8,0}$ and $\gamma_{8,0}$, but there is no octonionic linear transformation that does the job.

Related to matrix transposition we obtain another representation $\tilde{\gamma}$ involving $\mathbb{O}_\text{opp}$:
\[ \tilde{\gamma} : \mathfrak{A} \to M_2^T(\mathbb{O}_\text{opp}) \]
\[ a \mapsto \tilde{\gamma}(a) := (\gamma(\beta(a)))^T_{\text{opp}} : W^*_\text{opp} \to W^*_\text{opp} \]
\[ w^T \mapsto \tilde{\gamma}(a)(w^T) = (w^T \gamma(\beta(a))) = (\gamma^T(\beta(a))) w_{\text{opp}}^T. \] (152)

The verification of $\tilde{\gamma}(a \cdot b) = \tilde{\gamma}(a) \tilde{\gamma}(b)$ is another exercise in applying opposite algebras:
\[ \tilde{\gamma}(a \cdot b) = (\gamma(\beta(a \cdot b)))^T = (\gamma(\beta(b) \cdot \beta(a)))^T = (\gamma(\beta(b)) \gamma(\beta(a)))^T = (\gamma(\beta(a)))^T (\gamma(\beta(b)))^T_{\text{opp}} = \tilde{\gamma}(a) \tilde{\gamma}(b). \] (153)

The map that almost intertwines $\gamma^*_{8,0}$ and $\gamma_{8,0}$ is
\[ B : W \to W^*_\text{opp} \]
\[ w \mapsto B(w) := w_{\text{opp}}^T \Gamma_0, \] (154)

since
\[ \Gamma_0 \Gamma_k = (\Gamma_k)^T \Gamma_0 \quad (0 \leq k \leq 7). \] (155)
We have seen that the non-commutativity of the octonions has important consequences for representations that are related by octonionic conjugation and matrix transposition. The natural space for these representations to act on involves the opposite octonionic algebra, which prevents us from finding intertwining maps. Therefore special care should be taken when octonionic conjugation or matrix transposition is part of a manipulation involving octonionic spinors. However, this additional freedom of choosing different multiplication rules for different representations and carrier spaces may turn out to be advantageous in applications. In the following section we will observe how more general changes of multiplication rules further increase the flexibility of an octonionic representation.

### E. Octonionic spinors as elements of minimal left ideals

In this section we take a different perspective on octonionic spinors, regarding them as elements of a minimal left ideal which is generated by a certain primitive idempotent. The choice of an idempotent will turn out to be equivalent to the choice of a basis of the carrier space of the representation, which may be understood as a change of the multiplication rule of the octonions. An immediate application of the ideas presented here can be found in [29].

In a real or complex representation $\gamma : \mathfrak{A} \to \text{End}_F(W,W)$ of dimension $l$ an idempotent is given by an $l \times l$-matrix $Q$ satisfying the minimal polynomial $Q(Q - 1) = 0$. Therefore, $Q$ can be diagonalized with eigenvalues 0 and 1. If the representation is onto and the idempotent is primitive, then $Q$ is of rank 1 and there is a transformation such that $Q$ takes the form

$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \quad (156)$$

So for a surjective representation a primitive idempotent is represented by a matrix of the form

$$Q = q p^T, \quad p^T q = 1 \quad (q, p \in W). \quad (157)$$

The minimal left ideal $J = \mathfrak{A} \cdot Q$ generated by $Q$ in this representation consists of matrices with linearly dependent columns. Therefore, the action of the Clifford algebra on the minimal left ideal $J$ is determined by $q$. So the relevant choices of primitive idempotents are given by the choices for $q$. The choice of a basis for $J$ is still arbitrary at this point. For the octonionic case, however, there is a connection between the choice of $q$ and a multiplication rule.

In terms of the octonionic representation $\gamma_{8,0}$ we have $q = (q_0, q_1) \in \mathbb{O}^2$. For $q$ to correspond to an even primitive idempotent $Q$, one of its components has to vanish. (Note that even elements of the Clifford algebra are represented by diagonal matrices, whereas for odd elements the matrices have vanishing diagonal components.) We may also normalize $q$. So let $q = \left(\frac{q_0}{q_1}\right)$ with $|q|^2 = 1$. (A vanishing upper component leads to similar results.) A natural parametrization of the spinor space $J$ is given by

$$s := (s_1 + s_0 \Gamma_0) q = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} s_0 q_0 + s_1 q_1 \rho \\ s_0 q_1 - s_1 q_0 \rho \end{pmatrix}. \quad (158)$$

We interpret the octonions $s_0$ and $s_1$ as the new labels or components for the spinor $s$. The choice of this parametrization is natural, since it is up to octonionic conjugation the only one that involves only one left multiplication by an octonion. How does the Clifford algebra act in terms of the new spinor components? For $x \in V$

$$s' = \# s = \begin{pmatrix} x(s_1^* \rho) \\ x^* (s_0 \rho) \end{pmatrix} = \begin{pmatrix} s_0^* \rho \\ s_1^* \rho \end{pmatrix} \quad (159)$$

which leads to two other versions of the “$X$-product” [3] with $X = \rho$:

$$s_0' = [x(s_1^* \rho)] \rho^* = [(x \rho^*)(s_1^* \rho)] \rho^* = [(x \rho^*)(s_1^* \rho)] \rho^* = (x \rho)(s_1^* \rho),$$

$$s_1' = \rho [(\rho^* s_0^* \rho)x] = \rho [(\rho^* s_0)(\rho^* x)] = \rho [(\rho^* s_0)(\rho^* x)] = (s_0 \rho)(\rho^* x) = s_0^* \rho x. \quad (160)$$
where the fourth equality uses (10). Octonionic conjugation is also an antiautomorphism of the “$X$-product”, which gives the transformation behavior of $s_1^*$ as new spinor component.

$$s_1^{*} = s_1^{*'} = (s_0^* \circ \rho)^* = x^* \circ s_0,$$

where the fourth equality uses (10). Octonionic conjugation is also an antiautomorphism of the “$\rho$-product”. We confirm this result for the scalar formed out of two spinors (compare (129)):

$$\text{Re} \gamma s' = \text{Re} \left( \rho^* s_0^* \rho^* s_1 \right) = \text{Re} \left[ (\rho^* s_0^*) (s_0^* \rho^* s_1) + (\rho^* s_0^*) (s_0^* \rho^* s_1^*) \right]$$

$$= \text{Re} \left[ (s_0^* \rho^* s_1^*)^{(s_0^* \rho^* s_1^*)} + (s_0^* \rho^* s_1^*)^{(s_0^* \rho^* s_1^*)} \right] = \text{Re} \left( s_0^* \circ s_1^* \circ s_1^* \right)$$

as well as the vector (compare (133))

$$\Gamma^k \text{Re} \gamma \Gamma_k s' = \left( \begin{array}{cc} 0 & (s_0^* \rho^* s_1) + (s_0^* \rho^* s_1)^* \\ 0 & s_0^* \circ s_1^* + s_0^* \circ s_1 \\ \end{array} \right).$$

Of course, orthogonal transformations, as described in section IV C, also induce a change of basis on the spinor space. The corresponding change of the octonionic multiplication rule is more complex since the real part is no longer fixed (compare section II C).

V. OTHER OCTONIONIC REPRESENTATIONS

In this section, we point out the constructions of octonionic representations related to $\gamma_{8,0}$. We follow the program outlined in section II C. First we shrink the representation of $\mathcal{C}l(8,0)$ to obtain one of $\mathcal{C}l_0(8,0) \cong \mathcal{C}l(0,7)$ and further of $\mathcal{C}l(0,6)$. Then we look at the extension to a representation of $\mathcal{C}l(9,1)$, which is of particular importance, since it applies to superstring and superparticle models.

A. $\mathcal{C}l_0(8,0)$ and $\mathcal{C}l(0,7)$

Restricting the representation $\gamma_{8,0}$ to $\mathcal{C}l_0(8,0) \cong \mathcal{C}l_0(0,8)$ produces a faithful representation with the generators

$$\Gamma_0 \Gamma_k = \gamma_{8,0}(e_0 \circ e_k) = \left( \begin{array}{cc} i_k & 0 \\ 0 & i^*_k \end{array} \right) = \left( \begin{array}{cc} i_k & 0 \\ 0 & -i_k \end{array} \right) \quad (1 \leq k \leq 7).$$

So $\mathcal{C}l_0(8,0)$ is represented by diagonal matrices, i.e., this representation decomposes into two irreducible representations given by the two elements on the diagonal. By the isomorphism $\mathcal{C}l_0(8,0) \cong \mathcal{C}l(0,7)$ (James), these two are also irreducible representations $\gamma_{0,7}^+ : \mathcal{C}l(0,7) \rightarrow \mathcal{M}_1(\mathbb{O}) = \mathbb{O}$,

$$\gamma_{0,7}^+(e_k) := \pm i_k \quad (1 \leq k \leq 7)$$

$$\Leftrightarrow \gamma_{0,7}^+(x) := \pm x = \pm \text{Im} x \quad (x \in V = \mathbb{R}^7).$$

So we identify $V = \mathbb{R}^7$ with the purely imaginary subspace of the octonions $\text{Im} \mathbb{O}$. A faithful representation of $\mathcal{C}l(0,7)$ is found by letting $\gamma_{0,7}(e_k) = \Gamma_0 \Gamma_k$ in (164):

$$\gamma_{0,7} := \gamma_{0,7}^+ \oplus \gamma_{0,7}^- \quad \Leftrightarrow \gamma_{0,7}(a) = \left( \begin{array}{cc} \gamma_{0,7}^+(a) & 0 \\ 0 & \gamma_{0,7}^-(a) \end{array} \right).$$
A hermitian form $A^\prime : \mathbb{O}^\pm \to \mathbb{O}^{\pm \dagger}$ on the carrier space of an irreducible representation is given by

$$A^\prime(w) := w^* \quad (168)$$

with the property

$$A^\prime_{\gamma_{0,7}}(e_k) = -\gamma_{0,7}(e_k)A' = -(\gamma_{0,7}(e_k))^*(1 \leq k \leq 7). \quad (169)$$

Thus the form $A'$ intertwines $\gamma_{0,7}^\pm$ and $\gamma_{0,7}^\pm \circ \alpha \circ \beta$:

$$A' \circ \gamma_{0,7}^\pm(a) = (\gamma_{0,7}^\pm((\alpha \circ \beta)(a)))^\dagger \circ A' \quad (a \in Cl(0,7)). \quad (170)$$

There is no sesquilinear form satisfying (107) on a carrier space of the irreducible representation. However, one can intertwine $\gamma_{0,7}^\pm$ and $\gamma_{0,7}^\pm$ to obtain such a form on the carrier space $2\mathbb{O} = \mathbb{O}^+ \oplus \mathbb{O}^-$ of the faithful representation that swaps the two copies $\mathbb{O}^+$ and $\mathbb{O}^-$ of $\mathbb{O}$ since

$$\gamma_{0,7}^\pm(a) = (\gamma_{0,7}^\pm(\beta(a)))^* \quad (a \in Cl(0,7)). \quad (171)$$

A, defined by

$$A(w^+ \oplus w^-) := w^-^* \oplus w^+^* = \overline{\psi} \iff A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma, \quad (172)$$

satisfies

$$A \circ \gamma_{0,7}(a) = \gamma_{0,7}^\dagger(\beta(a)) \circ A \quad (a \in Cl(0,7))$$

$$\iff A\gamma_{0,7}(e_k) = \gamma_{0,7}^\dagger(e_k)A \quad (1 \leq k \leq 7). \quad (173)$$

Simple orthogonal transformations are generated by unit vectors $u \in \text{Im } \mathbb{O}$, $|u|^2 = -u^2 = 1$ via

$$x' = (\gamma_{0,7}^\pm \circ \phi_u)(x) = (\pm u)x(\pm u)^{-1} = uux^* = -uxu, \quad (174)$$

$$w^\pm' = \psi_u(w) = \pm uw. \quad (175)$$

Since the real part of $u$ vanishes, $u^{-1} = -u$. Therefore, the transformations have the same form as (137) and (138) up to signs and the Moufang identities ensure the compatibility of the spinor and vector transformations as before. As is seen from (12), improper rotations, for example, inversion of $\mathbb{R}^7$, $x \mapsto -x = x^*$, is not described by the action of the Clifford group for odd $n$. In fact, inversion is equivalent to octonionic conjugation or switching from $\gamma_{0,7}^\pm$ to $\gamma_{0,7}^\mp$. In order to implement inversion we need to use the faithful representation:

$$\hat{x}' = -e\hat{x}e = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\hat{x}, \quad (176)$$

$$w' = \begin{pmatrix} w^+ & w^- \end{pmatrix} = e\psi(w) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w^+ \\ w^- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} w^- \\ -w^+ \end{pmatrix}. \quad (176)$$

The transformation preserves scalars:

$$\overline{\psi} \hat{x} = w^\dagger \sigma(-e)(-e)\hat{x} \epsilon = (w^\dagger \epsilon \sigma)(-e)\hat{x} \epsilon = \overline{\psi} \hat{x}' \epsilon. \quad (177)$$

**B. $Cl_0(0,7)$ and $Cl(0,6)$**

Shrinking a representation of $Cl(0,7)$ further leads to the smallest Clifford algebra that has the octonions as a natural carrier space for a representation. Both irreducible representations $\gamma_{0,7}^\pm$ and $\gamma_{0,7}^\pm$ agree on the even Clifford algebra $Cl_0(0,7) \cong Cl_0(7,0)$. Their restriction is an irreducible representation given by the generators

$$\gamma_{0,7}^\pm(e_k \vee e_7) = i_k i_7 \quad (1 \leq k \leq 6). \quad (178)$$
which act by successive left multiplication on the carrier space \( W = \mathbb{O} \). Again by the isomorphism \( Cl_0(0, 7) \cong Cl(0, 6) \) [33], we obtain a faithful and irreducible representation of \( Cl(0, 6) \), \( \gamma_{0, 6} : Cl(0, 6) \rightarrow M_1(\mathbb{O}) = \mathbb{O} \),
\[
\gamma_{0, 6}(e_k) := i_k i_7 \quad (1 \leq k \leq 6),
\]
\[
\iff \quad \gamma_{0, 6}^\pm (x) := xi_7 \quad (x \in V = \mathbb{R}^6).
\]

\( V = \mathbb{R}^6 \) is identified with the imaginary subspace of \( \mathbb{O} \) with vanishing 7-component, \( \{x \in \text{Im} \mathbb{O} : x^7 = 0\} \). The volume form \( \eta \) is represented by
\[
\gamma_{0, 6}(\eta) = \gamma_{0, 6}(e_1 \vee e_2 \vee \cdots \vee e_6) = i_1 i_7 i_2 i_7 \cdots i_6 i_7 = -i_1 i_2 \cdots i_6 = i_7,
\]
according to (119). A hermitian form \( A' : \mathbb{O} \rightarrow \mathbb{O} \) is given by
\[
A'(w) := w^*.
\]

Orthogonal transformations are generated by unit vectors \( u \in \mathbb{R}^6 \), \( |u|^2 = -u^2 = 1 \) via
\[
x' = (\gamma_{0, 6} \circ \phi_u)(x) = (u i_7 x i_7) u \quad (183)
\]
\[
w' = \psi_u(w) = u (i_7 w) \quad (184)
\]

Since these transformations have the same structure as the simple orthogonal transformations for \( V = \mathbb{R}^6 \), the Moufang identities ensure their compatibility and their validity under the interpretation of left multiplication. Since \( \gamma_{0, 6} \) is faithful and irreducible and \( Cl(0, 6) \) is a 2^6-dimensional algebra, we conclude from this section that left multiplication by octonions generates a 64-dimensional algebra isomorphic to \( M_8(\mathbb{R}) \).

**C. \( Cl(9, 1) \)**

In this section we will give a little more detail because of the frequent use of \( Cl(9, 1) \) in supersymmetric models. Starting from \( Cl(8, 0) \), we do a Cartan extension [84] to obtain a representation of \( Cl(9, 1) \), \( \gamma_{9, 1} : Cl(9, 1) \rightarrow M_4(\mathbb{O}) \), given by the generators
\[
\gamma_{9, 1}(e_k) := \sigma \otimes \gamma_{8, 0}(e_k) = \left( \begin{array}{cc} 0 & \Gamma_k \\ \Gamma_k & 0 \end{array} \right) \quad (0 \leq k \leq 7),
\]
\[
\gamma_{9, 1}(e_8) := \sigma \otimes \gamma_{8, 0}(\eta) = \left( \begin{array}{cc} 0 & \tau \\ \tau & 0 \end{array} \right),
\]
\[
\gamma_{9, 1}(e_{-1}) := -\epsilon \otimes \gamma_{8, 0}(1) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),
\]

or equivalently by
\[
\gamma_{9, 1}(x) := \hat{x} = x^\mu \gamma_\mu = \left( \begin{array}{cc} 0 & X \\ X & 0 \end{array} \right)
\]
\[
= \left( \begin{array}{cc} 0 & (x^+ x, x^x x^-) \\ (-x^- x^x, x^+ x^-) & 0 \end{array} \right),
\]

(186)

where we defined
\[
X := x^\mu \Gamma_\mu = \left( \begin{array}{cc} x^+ x, x^x x^- \end{array} \right), \quad \Gamma_8 := \tau, \quad \Gamma_{-1} := 1, \quad x_\pm := x_{-1} \pm x_8,
\]
\[
\check{X} := x^\mu \check{\Gamma}_\mu = \left( \begin{array}{cc} x^- x, x^x x^- \end{array} \right), \quad \check{\Gamma}_\mu := \left\{ \frac{\Gamma_\mu}{(0 \leq \mu \leq 8)}, \frac{\Gamma_{-1}}{(\mu = -1)} \right\},
\]
\[
\gamma_\mu := \gamma_{9, 1}(e_\mu) = \left( \begin{array}{cc} 0 & \Gamma_\mu \\ \Gamma_\mu & 0 \end{array} \right) \quad (-1 \leq \mu \leq 8).
\]

(187)
(Labeling the basis elements of $V = \mathbb{R}^{10}$ by indices ranging from $-1$ to $8$, allows us to keep the notation we developed for $\gamma_{8,0}$.) The representation $\gamma_{9,1}$ is Weyl, since the volume element $\eta = e_{-1} \vee e_0 \vee \ldots \vee e_8$ is represented by

$$\gamma_{9,1}(\eta) = \tau \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma_1 \gamma_0 \ldots \gamma_8 =: \gamma_{11}. \quad (188)$$

The Weyl projections take the form

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (189)$$

We denote an element $w \in W = \mathbb{O}^4$ of the carrier space by its Weyl projections

$$w_\pm := P_\pm w \in \mathbb{O}^2, \quad (190)$$

where we discard the two vanishing components of $w_\pm$. The identity

$$# = x^\mu x_\mu 1 \iff \begin{pmatrix} \tilde{X} \tilde{X} & 0 \\ 0 & XX \end{pmatrix} = x^\mu x_\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (191)$$

holds under left multiplication because of the alternative property, since only one full octonion $x$ and its conjugate are contained in $X$ and $\tilde{X}$. Noting that

$$\tilde{X} = X - (\text{tr}(X)) 1, \quad (192)$$

it follows that

$$X\tilde{X} = X^2 - (\text{tr}(X)) X = \tilde{X}X = - \det X 1 = x^\mu x_\mu 1, \quad (193)$$

since the characteristic polynomial for a hermitian $2 \times 2$-matrix $A$ is $p_A(\lambda) = \lambda^2 - \text{tr}(A) \lambda + \det A$. Polarizing, we get

$$2x_\mu y^\mu 1 = X\tilde{X} + Y\tilde{X} = \tilde{X}Y + Y\tilde{X} \iff 2g_{\mu\nu} 1 = \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = \tilde{\Gamma}_\mu \Gamma_\nu + \tilde{\Gamma}_\nu \Gamma_\mu. \quad (194)$$

To extract components, we have the familiar formulas involving traces:

$$x_\mu = \frac{1}{4} \text{Re tr}(\eta \gamma_\mu) = \frac{1}{4} \text{Re tr} \left( X\tilde{\Gamma}_\mu + \tilde{X}\Gamma_\mu \right) = \frac{1}{2} \text{Re tr} \left( X\tilde{\Gamma}_\mu \right) = \frac{1}{2} \text{Re tr} \left( \tilde{X}\Gamma_\mu \right). \quad (195)$$

Considering

$$\gamma_\mu = \begin{cases} \gamma_\mu, & (\mu \neq -1) \\ -\gamma_\mu, & (\mu = -1) \end{cases}, \quad (196)$$

a hermitian form $A$ is given by

$$A(w) := w^\dagger A = w^\dagger \gamma_{11} \gamma_{-1} = \begin{pmatrix} w_+^\dagger & w_-^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} w_-^\dagger & w_+^\dagger \end{pmatrix} =: \overline{w}. \quad (197)$$

So the scalar covariant formed out of $w, z \in W$ is

$$A(w, z) = \text{Re} \overline{w} z = \text{Re} \begin{pmatrix} w_-^\dagger & w_+^\dagger \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \text{Re} \left( w_-^\dagger z_+ + w_+^\dagger z_- \right), \quad (198)$$

which only involves terms combining spinors of opposite chirality. For the vector covariant $y$, we obtain
\[ y_\mu := \text{Re} \bar{\gamma}_\mu z \]
\[ = \text{Re} \left[ \begin{pmatrix} w_+^1 & w_+^2 \\ 0 & \Gamma_\mu \end{pmatrix} \right] \begin{pmatrix} \Gamma_\mu \\ \mu \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} \]
\[ = \text{Re} (w_+^1 \tilde{\Gamma}_\mu z_+ + w_+^2 \Gamma_\mu z_-) = \text{Re} \left( z_+ w_+^1 \tilde{\Gamma}_\mu + z_- w_+^2 \Gamma_\mu \right) \]
\[ = \frac{1}{2} \left[ \text{Re} \left( z_+ w_+^1 \tilde{\Gamma}_\mu + z_- w_+^2 \Gamma_\mu \right) + \text{Re} \left( (z_+ w_+^1 \tilde{\Gamma}_\mu + z_- w_+^2 \Gamma_\mu)^\dagger \right) \right] \]
\[ = \frac{1}{2} \text{Re} \left( [z_+ w_+^1 + w_+ z_+^\dagger] \tilde{\Gamma}_\mu + [z_- w_+^2 + w_+ z_-^\dagger] \Gamma_\mu \right) \]
\[ = \frac{1}{2} \text{Re} \left( [z_+ w_+^1 + w_+ z_+^\dagger] \tilde{\Gamma}_\mu + [z_- w_+^2 + w_+ z_-^\dagger] \Gamma_\mu \right). \]

So the vector covariant is formed of combinations of spinors of the same chirality. Since the hermitian matrix $Y$ is completely determined by the components according to (199) and the terms in square brackets are hermitian, we can give a formula analogous to (193):

\[ \hat{y} = \begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix} := \gamma^k \text{Re} \bar{\gamma}_k z \]
\[ = \begin{pmatrix} \tilde{z}_+ w_+^1 + \tilde{w}_+^2 z_+^\dagger \\ \tilde{z}_- w_+^1 + \tilde{w}_+^2 z_-^\dagger \end{pmatrix} \begin{pmatrix} \Gamma_\mu \\ \mu \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} \]
\[ = \begin{pmatrix} \tilde{z}_+ w_+^1 + \tilde{w}_+^2 z_+^\dagger \\ \tilde{z}_- w_+^1 + \tilde{w}_+^2 z_-^\dagger \end{pmatrix}. \]

Proper Lorentz transformations are generated by pairs of timelike (resp. spacelike) unit vectors $u, v \in V$, i.e., $u_\mu u^\mu = \mp 1 = v_\mu v^\mu$. We choose $v = e_{-1}$ fixed

\[ \hat{y}' = \hat{y} \gamma_{-1} \hat{y} = \begin{pmatrix} 0 & UXU \\ UXU & 0 \end{pmatrix}, \]
\[ w' = \hat{y} \gamma_{-1} w = \begin{pmatrix} -Uw_+ \\ \tilde{Uw}_- \end{pmatrix}. \]

The correct transformation behavior of spinors and vectors is ensured by the Moufang identities as in the 8-dimensional case, since $\hat{y}$ contains additional real parameters but only one full octonion. This form of proper Lorentz transformations makes the isomorphism $SL(2, \mathbb{O}) \cong SO(9,1)$ as Lie groups precise.

Since for $C := \gamma_{-1} \gamma_{0} \gamma_{8} = -\epsilon \otimes \epsilon$

\[ C \gamma_\mu = \gamma_\mu C \quad (1 \leq \mu \leq 8), \]

a “charge conjugation” operation is given by

\[ w_C := C(w)^* = -\epsilon \otimes \epsilon w^* = \begin{pmatrix} \epsilon w_-^* \\ -\epsilon w_+^* \end{pmatrix}, \]

which must involve the opposite octonionic algebra as it was pointed in (150) and (151). This transition to the opposite algebra for spinors with opposite chirality may be useful in theories with $N > 1$ supersymmetry.

Of course, we may iterate the process of shrinking and extending of a representation with $\gamma_{0,1}$ as a starting point. We can shrink it to obtain representations of $Cl_0(9,1) \cong Cl(9,0) \cong Cl(1,8)$ and from there to $Cl_0(9,0) \cong Cl(0,8)$ and $Cl_0(1,8) \cong Cl(1,7)$. Also an extension to a representation of $Cl(10,2)$ is possible.

VI. AN OCTONIONIC DESCRIPTION OF THE CHEVALLEY ALGEBRA AND TRIALITY

The triality automorphisms of the Chevalley algebra are well known and have been discussed in detail before, even in an octonionic formulation [11]. However, in our opinion, the following treatment based on the preparatory work of section V adds another unique and very transparent perspective with regard to this topic.

In the case of 8 euclidean dimensions we are in a special situation; the spaces of vectors, $V$, even spinors, $S_0$, and odd spinors, $S_1$, have the same dimension, namely 8. This allows the construction of the triality maps that interchange the transformation behavior of these three spaces. We define the Chevalley algebra $A := V \oplus S_0 \oplus S_1$ to be the direct
sum of these three spaces. This definition automatically provides a vector space structure for $\mathcal{A}$. Furthermore, $\mathcal{A}$ inherits an $SO(8)$-invariant bilinear form $\mathbf{B} = 2g \otimes 2A$ from the metric $g$ on the vector space and the hermitian form $A$ on $S = S_0 \oplus S_1$. (For notational convenience later on, we put in a factor of 2 in the definition of $\mathbf{B}$.) For $a = a_v \oplus a_0 \oplus a_1$, $b = b_v \oplus b_0 \oplus b_1 \in \mathcal{A}$, we obtain

$$\mathbf{B}(a, b) = 2g(a_v, b_v) + 2A\left(\begin{pmatrix} a_0 & a_1^* \\ a_v^* & a_1 \end{pmatrix}, \begin{pmatrix} b_0 & b_1^* \\ b_v^* & b_1 \end{pmatrix}\right) = 2\text{Re}(a_v b_v^* + a_0^* b_0 + a_1^* b_1),$$

(204)

where we used the parametrization of the spinor components introduced in section IV.C. (204) confirms that $A$ decomposes and is a real symmetric bilinear form on the 16 real spinor components. The $SO(8)$-invariance of $\mathbf{B}$ is clear using the results of section IV.C. Furthermore, we observed in (143) that the expression

$$\mathbf{T}'(a) := \text{Re} \eta_{\mathbb{O}}^* \mathbf{a}_0 = \text{Re} \left[ \begin{pmatrix} 0 & a_1 \\ a_v & 0 \end{pmatrix} \begin{pmatrix} 0 & a_v^* \\ a_0 & 0 \end{pmatrix} \right] = \text{Re} a_1 a_v a_0$$

(205)

is $SO(8)$-invariant. (Note that, we also redefined our basis of $V$ by octonionic conjugation for symmetry reasons, which will become relevant below.) By polarization, we define a $SO(8)$-invariant symmetric trilinear form on $\mathcal{A}$, which we denote by $\mathbf{T}$:

$$\mathbf{T}(a, b, c) := \text{Re}(a_1 b_v c_0 + a_0 c_v b_0 + b_1 a_v c_0 + b_0 a_v c_0 + c_1 a_v b_0 + c_0 a_v b_0 + c_1 b_v a_0) \quad (a, b, c \in \mathcal{A}).$$

(206)

The Chevalley product “$\circ_A$” is then implicitly defined to satisfy the following condition connecting $\mathbf{B}$ and $\mathbf{T}$:

$$\mathbf{B}(a \circ_A b, c) = \mathbf{T}(a, b, c) \quad \forall a, b, c \in \mathcal{A}.$$  

(207)

The Chevalley product is obviously symmetric and $SO(8)$-invariant.

In this setting the triality maps are just automorphisms of the Chevalley algebra, which interchange $V$, $S_0$, and $S_1$. But before we describe the triality maps, we will take advantage of the octonionic formalism and rewrite the bilinear and trilinear forms, $\mathbf{B}$ and $\mathbf{T}$, and the Chevalley product by representing elements of the Chevalley algebra by octonionic hermitian $3 \times 3$-matrices with vanishing diagonal elements,

$$a = \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_v^* & a_1 & 0 \end{pmatrix} = \begin{pmatrix} a_s^* & a_s \\ a_s & 0 \end{pmatrix} \in \mathcal{A},$$

(208)

where $a_s = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = a_v \oplus a_1 \in S$. Then the bilinear form $\mathbf{B}$ is given by

$$\mathbf{B}(a, b) = \frac{1}{2} \text{tr} (ab + ba) = \text{tr} (a \circ b)$$

$$= \frac{1}{2} \text{tr} \left( \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_v^* & a_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & b_v^* & b_0 \\ b_v & 0 & b_1^* \\ b_v^* & b_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b_v^* & b_0 \\ b_v & 0 & b_1^* \\ b_v^* & b_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_v^* & a_1 & 0 \end{pmatrix} \right)$$

$$= \frac{1}{2} \text{tr} \left( \begin{pmatrix} a_s^* b_v + a_0 b_0^* \\ a_v^* b_v^* + a_1^* b_1 \\ a_v b_0 \end{pmatrix} \begin{pmatrix} a_0^* b_v^* + a_1^* b_1 \\ a_1 b_v \\ a_0 b_0 + a_1 b_1 \end{pmatrix} \right)$$

$$+ \frac{1}{2} \text{tr} \left( \begin{pmatrix} b_v^* a_v + b_0 a_0^* \\ b_1 a_v \\ b_v a_v^* \\ b_v^* a_v \end{pmatrix} \begin{pmatrix} b_v^* a_0 + b_1 a_1 \\ b_v^* a_0 \\ b_v a_v^* \\ b_v^* a_v \end{pmatrix} \right)$$

$$= \frac{1}{2} \left[ (a_s^* b_v + b_s^* a_v + a_0^* b_0^* + b_v^* a_v) + (a_0^* b_0^* + b_0^* a_0 + a_s b_0 + b_v a_v) \right.$$  

$$+ \left. (a_1^* b_1 + b_s^* a_1 + a_1) \right] = 2 \text{Re}(a_v b_v^* + a_0^* b_0 + a_1^* b_1),$$

(209)

where “$\circ$” denotes the symmetrized matrix product

$$a \circ b := \frac{1}{2} (ab + ba).$$

(210)

In fact, the symmetrized product is the Jordan product and the matrices that we are dealing with are a subset of the exceptional Jordan algebra of $3 \times 3$ octonionic hermitian matrices $\mathbb{O}$.  

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For the trilinear form $\mathbf{T}$ we find
\[
\mathbf{T}(a, b, c) = \text{tr} \left( (a \circ b) \circ c \right) = \frac{1}{4} \left( (a_0 b_1 c_v + a_v a_0 b_1 + b_1^* a_0^* a_v + c_v^* b_1^* a_0^* + (b_0 a_1 a_v + c_v b_0 a_1 \\
+ a_1^* b_0^* c_v + c_v^* a_1^* b_0^*) + (a_v b_0 a_1 + c_v a_0 b_0 + b_0^* a_v^* c_v + c_v^* b_0^* a_v^*) \\
+ (b_0 a_0 c_v + a_v b_v a_0 + a_0^* b_v^* a_v^* + c_v a_0^* b_v^*) + (a_1 b_v a_v + c_v b_0 a_v + b_0^* a_v^* c_v + c_v^* b_0^* a_v^*) \right) \\
= \text{Re} \left( (b_v c_v a_0 + a_v c_v b_0 + c_v a_0 b_0 + a_0^* b_v^* a_v^* + b_0^* a_v^* c_v + c_v^* b_0^* a_v^*) \right)
\]

It follows from (207), (209), and (211) that the Chevalley product “$\circ_A$” is given by the off-diagonal elements of the symmetrized matrix product “$\circ$”,
\[
(a \circ_A b) = \mathbf{B}(a \circ_A b, c) = \mathbf{T}(a, b, c) = \text{tr} \left( (a \circ b) \circ c \right)
\]
where the subscript “$A$” on a matrix denotes the matrix with erased diagonal elements, i.e.,
\[
(a \circ b)_A := \begin{pmatrix}
0 & a_0 b_1 + b_0 a_1 \\
0 & a_v b_0 + b_v a_0 \\
a_1 b_v + b_1 a_v & a_v b_0 + b_v a_0 \\
a_0 b_1 + b_0 a_1 & a_v b_0 + b_v a_0 & 0
\end{pmatrix}
\]

(Note that only the off diagonal elements of $a \circ b$ contribute to the last term of (213)). Traditionally the Chevalley product is written in terms of Clifford products, which we combine into the 3 × 3-matrix
\[
a \circ_A b = \begin{pmatrix}
\Gamma_k \Gamma_1 b_v & a_v b_s + b_v^* a_s \\
\Gamma_1 \Gamma_0 b_s & 0
\end{pmatrix}
\]
What we have done is to utilize the Jordan product and project onto the Chevalley algebra. Since both $\mathbf{B}$ and $\mathbf{T}$ are expressed entirely in terms of the Jordan product, automorphisms of the Jordan product, that map the Chevalley algebra onto itself, will also be automorphisms of the Chevalley algebra. We have already encountered one such automorphism, namely the orthogonal transformation corresponding to a generator $p_v \in V$ with $|p_v|^2 = 1$, which is written in matrix form
\[
\tau_{p_v}(a) := \begin{pmatrix}
0 & 0 & 0 \\
p_v & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & a_v^* & a_0 \\
0 & a_0 & a_v \\
a_v & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & p_v \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a_v^* & a_0 \\
0 & a_0 & a_v \\
a_v & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
p_v^* a_v^* b_v^* & a_v^* b_s + b_v^* a_s \\
0 & 0 & 0
\end{pmatrix}
\]

This first triality map combines the vector action and spinor action of the Clifford group (see section [[113]]). The action of the generator $p_v$ is a reflection at a hyperplane orthogonal to $p_v$ combined with an inversion of the whole space. This transformation is an improper rotation and interchanges even and odd spinors:
\[
\tau_{p_v}(a_v) = p_v a_v^* p_v \in V, \\
\tau_{p_v}(a_0) = (p_0 a_0)^* \in S_1, \\
\tau_{p_v}(a_1) = (a_1 p_v)^* \in S_0.
\]

Using the Moufang identities, it is easy to check that $\tau_{p_v}$ is indeed an automorphism of $A$ of order 2, i.e., $\tau_{p_v}^2 = 1$. Composing an even number of maps $\tau_{p_v}$ with different parameters $p_v$, we generate the simple orthogonal group $SO(8)$ as is seen in (141) and (132). From the form of (210), it is obvious that there are two more families of automorphisms of $A$ of order 2, parametrized by an even spinor variable $p_0$ and an odd spinor variable $p_1$ with $|p_0|^2 = 1 = |p_1|^2$:
\[
\tau_{p_0}(a) := \begin{pmatrix}
0 & 0 & p_0 \\
p_0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a_v^* & a_0 \\
0 & a_0 & a_v \\
a_v & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & p_0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a_v^* & a_0 \\
0 & a_0 & a_v \\
a_v & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
p_0^* a_v^* b_v^* & a_v^* b_s + b_v^* a_s \\
0 & 0 & 0
\end{pmatrix}
\]
and
\[
\tau_{p_1}(a) := \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & p_1 \\
0 & p_1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a_v^* & a_0 \\
0 & a_0 & a_v \\
a_v & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & p_1 \\
0 & p_1 & 0
\end{pmatrix}
= \begin{pmatrix}
p_v^* a_v^* b_v^* & a_v^* b_s + b_v^* a_s \\
0 & 0 & 0
\end{pmatrix}
\]
For these two families of maps, the matrix formalism shows the clear parallel structure to the maps \( \tau_{p_v} \). Traditionally expressions in terms of both Clifford products and the spinor bilinear form are used for the maps \( \tau_{p_v} \) and \( \tau_{p_1} \), which obscures this symmetry, because in \( \tau_{p_v} \) only Clifford products are used. These two families preserve one of the spinor spaces and interchange the other one with \( V \):

\[
\begin{align*}
\tau_{p_0}(a_v) &= (a_v p_0)^* \in S_1, \\
\tau_{p_0}(a_0) &= p_0 a^*_0 p_0 \in S_0, \\
\tau_{p_0}(a_1) &= (p_0 a_1)^* \in V,
\end{align*}
\]

(220)

and

\[
\begin{align*}
\tau_{p_1}(a_v) &= (p_1 a_v)^* \in S_0, \\
\tau_{p_1}(a_0) &= (a_0 p_1)^* \in V, \\
\tau_{p_1}(a_1) &= p_1 a^*_1 p_1 \in S_1,
\end{align*}
\]

(221)

By combining two triality maps with the same octonionic parameter \( p_v = p = p_0 \) from different families, we obtain an automorphisms \( \Xi_p \) of order 3:

\[
\Xi_p(a) = (0 \ 1 \ 0) \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_0^* & a_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & p^* \\ 1 & 0 \ 0 \end{pmatrix} (a \in A),
\]

(222)

hence

\[
\begin{align*}
\Xi_p(a_v) &= p^* a_v \in S_0, \\
\Xi_p(a_0) &= p a_0 p \in S_1, \\
\Xi_p(a_1) &= p^* a_1 \in V.
\end{align*}
\]

(223)

As is seen from their matrix forms, \( \tau_{p_v=p} \) and \( \Xi_p \) generate \( \Sigma_3 \), the permutation group on three letters. (In particular for \( p = 1 \), this is easy to verify.) We observed before that the maps \( \tau_{p_v} \) generate \( O(8) \), so that the triality maps, we have found so far, have a group structure isomorphic to \( \Sigma_3 \times SO(8) \). It is known (see [8]) that this is the full automorphism group of the Chevalley algebra, which is also the automorphism group of \( SO(8) \). This concludes our demonstration of triality.

VII. FINITE VS. INFINITESIMAL GENERATORS

In this article we characterize orthogonal groups in terms of a set of finite generators. This approach is not as widely used as the description in terms of infinitesimal generators, i.e., the Lie algebra of the group. In this section we compare the two approaches.

If we want to compare two Lie groups given by infinitesimal generators we know how to proceed [32]. We determine their Lie algebra by working out the commutators of the generators. We then determine their structure constants and identify the Lie algebra. For semi-simple Lie algebras the Cartan-Weyl normalization provides a unique identification. We may also use a Lie algebra homomorphism and determine its image and kernel to relate the two groups in question. Whether the homomorphism is surjective and injective can often be determined by counting the dimension of the Lie algebras involved. Having identified the Lie algebra we have full knowledge of the local structure of the Lie group. From this information we can construct the simply connected universal covering group, which has this local structure. However, the Lie group we are trying to characterize may be neither connected nor simply connected. So in order to compare two groups we need to have some global information about them in addition to the infinitesimal generators.

In section [11] we compared two groups given by finite generators, namely the orthogonal group generated by reflections on hyperplanes and the Clifford group generated by non-null vectors of the Clifford algebra. The relationship was established considering a group homomorphism. The homomorphism is surjective if the generators lie in the image. This is the analogue to counting the dimension of the Lie algebras. Determining the kernel, which has to be a normal subgroup, completes the comparison. The advantage of finite generators is the global information that they carry. Having found an isomorphism based on the finite generators, we know that the groups have the same global structure.

Even though the two descriptions have different features, they are closely related. The exponential map provides a means to parametrize a neighborhood of the identity element of the group. This coordinate chart can be translated by
The finite generators that determine the groups considered in this article are elements of a topological manifold of dimension less than the dimension of the group. For example, the octonions that generate $SO(8)$ \cite{11} are elements of the octonionic unit sphere, $S^7$. Translating a disk centered at a point $p \in S^7$ by $p^{-1} \in S^7$, we obtain a submanifold of the group containing the identity. (A generating set of a group is always assumed to contain inverses of every element.) This submanifold is of lower dimension than the Lie group, so its tangent space at the identity is only a linear subspace of the Lie algebra. In most of our examples it is sufficient to consider the translation of a sufficient number of disks contained in the generating set to obtain linear subspaces that span the Lie algebra. Otherwise the process continues by taking products of elements of two disks around $p_1$ and $p_2$ in the generating set and translating these products by $(p_1 p_2)^{-1}$ to the identity. An example of this latter construction is the $S^8$ generating $SO(8)$ described in \cite{3}. In this way infinitesimal generators can be found starting from finite ones.

There is also a formal construction of the entire group; namely, the group is given by the set of equivalence classes of finite sequences of generators. The group product of two elements $[g_1], [g_2]$ is just the class of the juxtaposition $[g_1 g_2]$ of two representatives. For the octonionic description we need to do this decomposition into generators to find spinor and vector transformations that are consistent. For example, if a vector given by $x \in \mathbb{O}$ transforms by $x \mapsto u x u^*$, which is an $SO(8)$ transformation, we need to re-express $uxu^*$ as $v_1 (v_2 (\ldots (v_k x v_k) \ldots ) v_2 ) v_1$ with $|v_1|^2 = |v_2|^2 = \cdots = |v_k|^2$ in order to determine the corresponding spinor transformation $w \mapsto v_1 (v_2 (\ldots (v_k w) \ldots ))$. In general, octonionic transformations, because of their non-associativity, involve this nesting of multiplications. Therefore the octonionic description of Lie groups in terms of generators is the natural one. Octonionic descriptions of Lie algebras, which are also possible, have the disadvantage that the exponential map no longer works because of the non-associativity. So this avenue does not provide a construction of finite group elements.

**VIII. CONCLUSION**

We have demonstrated that the abstract octonionic algebra is a suitable structure to represent Clifford algebras in certain dimensions. We obtained most of our results from the basic property of composition algebras, which is the norm compatibility of multiplication, and its consequence alternativity. The alternative property, in particular in the form of the Moufang identities, was found to be responsible for ensuring the correct transformation behavior of octonionic spinors and for ensuring the consistency of the representation in terms of left multiplication by octonionic matrices. The choice of a multiplication rule for the octonions, in particular, the modified “X-product”, was found to be related to coordinate transformations or a change of basis of the spinor space. The opposite octonionic algebra was shown to be connected to an analogue of the charge conjugate representation. The Clifford group and its action on vectors and spinors led to octonionic representations of orthogonal groups in corresponding dimensions. The natural octonionic description of these groups is in terms of generating sets of the Lie group rather than in terms of generators of the Lie algebra. This is due to the nested structure which is necessary to accommodate the non-associativity of the octonions.

The usefulness of this tool of octonionic representations was evident in the presentation of the triality automorphisms of the Chevalley algebra. This presentation unequivocally showed that the spaces of vectors and even and odd spinors are interchangeable in this case. We expect that a similar, fully octonionic treatment of supersymmetrical theories will make their symmetries more transparent. In fact, we have successfully applied the methods of this article to the CBS-superparticle \cite{29}. We hope to be able to find a parallel treatment of the Green-Schwarz superstring.

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