Is there paradox with infinite space?

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Abstract

We argue that an infinite universe should not necessarily be avoided on philosophical grounds. Paradoxes of repeating behaviour in the infinite, or eternal inflationary, universe can be alleviated by a realistic definition of differing lives: not simply permutations of various quantum states. The super-exponential growth in the rules of a cellular automata is used as an example of surpassing the holography bound. We also critically question the notion that our universe could simply be a simulation in somebody else's computer.

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Introduction

There is much discussion about the topology of the universe, most recently a closed compact Dodecahedral shaped universe has been claimed to be consistent with the latest WMAP data [1]. Previously, there was interest in compact open universes - for general reviews on topology of the universe see [2].

For a while in cosmology there has been a prejudice against infinite universes since they apparently give the paradoxical situation that there must also be other identical planets and life to ours. Assuming the Copernican principle that our observable part of the universe is not unusual, this is possible in a open or flat FRW universe [3] or if an infinite number of finite closed universes are created [4] as in the eternal chaotic universe scenario [5]. One might first wonder if similar life, beyond the particle horizon in our universe, is more disconcerting than identical life in other totally disconnected universes. For this reason many cosmologists prefer the idea of a bound universe to try to prevent such worrisome “clones of themselves” living elsewhere in the universe. As we later remark this would not anyway be a satisfactory situation; it implies that all possible human endeavor is finite but by restricting to a subset we never have to be aware of this limitation. The only real consideration is that we should be in a low entropy state as any activity will cause entropy increase. But while in such a state we are effectively isolated thermodynamically from the rest of the universe.

Although, we have doubts as to whether certain eternal universe schemes are viable we wish to argue that this sort of infinity is not so frightening as often suspected. Because our environment at our instigation can be extremely complex there seems little chance of a life actually repeating the same historic path, despite an apparent finite number of possible quantum states. We shall try and argue for a no-repeat condition for life in the universe. This is independent of global thermodynamical properties of the universe. We shall use a single cellular automata to illustrate how the holography bound can be circumvented. Although the finite number of states of a cellular automata means that it alone is not sufficient to totally prevent a no-repeat condition, it does illustrate limitations in how the holography principle has been interpreted.

Some related argument can be made against the notion that we are merely the result of somebody’s computer simulation [6]. Such a computer simulation also has problems with maintaining the notion of free will for life within the simulation.
Although this debate might seem rather abstract spurious philosophical notions seem to have been suggested from such examples. The infinite universe is said to include somewhere “every possible action you choose NOT to perform” [7]. Likewise in the computer simulation our actions are merely to “amuse others” and so in either case, we are not really under any compunction to modify our behaviour [8].

**State counting is insufficient**

It is easy to formulate a puzzle with more permutations than particles in the universe $\sim 10^{80}$. For example a travelling salesman with $N$ places to visit has $N!$ possible routes. So $N \sim 100$ routes can surpass this $\sim 10^{80}$ figure. One can say that the complexity of our environment is not limited by this particle number figure, since anyway why should the actual number of particles vastly distant from us limit what is or isn’t possible for us on earth.

Now if we consider the number of possible quantum states within the observable universe, which is the essence of the repeat argument[4,9], we obtain a much larger figure $X \sim 10^{10^{120}}$. This is obtained by applying a holography type principle to the universe. According to Rucker [10], who argues against this sort of reasoning, E. Wette first obtained $\sim 10^{10^{10}}$ possible space-time states. The actual figure is not crucial for our argument. Now is this figure anymore relevant than the previous number in limiting the complexity of our lives? Although, a game like Chess or Go does not have this many game permutations [11] I again can surpass this figure by considering a more elaborate puzzle: actually by means of cellular automata - see [12] for a full description.

For a block cellular automata with blocks of size $n$ and allowing $k$ colours there are $k^{nk^n}$ rules or “games to play”, before we worry about the number of ways of playing each game [12]. So for say size 1000 blocks and ten colours I again can surpass this large figure apparently imposed by the holography argument. The large number of quantum states does not constrain the extra patterns that we might explore even for relatively few atoms involved in such a cellular automata. Note also that actual information is equal to the logarithm, to base 2, of the number of possible quantum states. So the visible universe can only store $\sim 10^{120}$ bits of information with this many quantum states. This fact will later be useful in understanding the actual relevance of the holography bound.

What possible maximum number of rules, say $Y$, that could be explored on a physically realistic computers is unknown but will in effect be almost infinite, in comparison with the earlier apparent large number of quantum
states i.e. $Y >> X$. Such complexity is probably existing already in nature, for example patterns on cone shells [12].

But is this sufficient complexity to remedy recurrence occurring? After all without an infinite grid the number of permutations $Y$ is still finite and repeats would still eventually occur. But by surpassing the holography bound there is now no known reason to believe the number of possibilities is actually finite. So the resolution of the problem is currently unknown. However, I think there is strong evidence to suppose that the quantum universe should not be simpler than a classical one. Recall, if I considered the number of possible football games and there was no limit to how accurately I could measure the position of the players and ball I would obtain $\aleph_1$ possible games. The same cardinality as all the points within a 4-dimensional box, whether finite or infinite sized - see e.g. [10]. So in such a classical universe any clones do not play exactly the same game regardless of whether the universe is finite or infinite. Ordinary quantum mechanics does not alter this since the underlying modes are still continuous, although such modes are often made of discrete quanta. In many ways the quantum version is more complicated since Uncertainty Principles are only limits on complementary measurements while from Bell’s inequality type arguments we know there is not even an “underlying reality of the universe” - see e.g. [13]. Indeed, quantum mechanics allows randomness to remain even when a measurement has been made. This process itself is rather analogue in that continuous differing directions can be chosen for the measurement of say the polarization of photons. As emphasized recently by Hawking [14] in his so called top-down approach to cosmology, there is no observer independent history to the universe. Rather our measurements determine what histories have dominated in a Feynman path integral for the state of the universe.

Whether quantum gravity will eventually show that space comes in packets or is continuous should not significantly detract from the above reasoning unless quantum mechanics is itself simplified by the ultimate quantum gravity theory. Presently there still seems many uncertainties in reconciling quantum mechanical notions with discreteness at the Planck scale cf. information loss [15]. Incidentally if there is such discreteness and space-time comes in packets then the infinite universe has at most size $\aleph_0$ packets.

In conclusion, the actual complexity of our environment although not precisely known is not simply limited by the number of particle states within the universe.

**Does Eternal inflation give infinite space?**
We have previously given a number of possible reasons why the eternal inflationary mechanism might not hold [16,17]. These include a) The adaption of the Hawking radiation calculation, derived from Black hole physics, to de Sitter space not being entirely correct—see [18] for some differences between the two cases. This might alter the fluctuation result required for galaxy formation. b) Black holes or other topological defects being produced that reduce the surface gravity and so the perturbation strength. Especially since the Planck scale might be altered in Brane models. c) Higher dimensions becoming relevant that likewise reduce the surface gravity of de Sitter space. In Brane models the continual presence of a fixed higher dimensional bulk AdS space could constrain the process cf.[19]. Large gravitational wave fluctuations from presently inflating regions might take “short cuts” through the bulk space and over produce matter perturbations in our region of the Brane, or give a large dark radiation term. d) The required superposition of quantum modes being prevented by a decoherence mechanism cf.[13]. e) The quantum fluctuations of matter not automatically being transferred to the geometric left hand side of Einstein’s equation. f) The imposition of the 2nd law of thermodynamics preventing decrease in entropy within any horizon volumes—related to the decoherence mechanism if many other particle modes are present. Other issues have been emphasized by Turok [20].

We will anyway for now allow the eternal mechanism the benefit of the doubt on these sort of issues. However more seriously in this regard some geodesic incompleteness results for expanding universes suggests that inflation has only been occurring for a finite time [21]. So if the initial domain was a closed compact space only finite numbers of new closed universes can be obtained. It follows from only so many new Hubble sized domains per unit time being formed. Actually even with infinite time being available only $\aleph_0$ universes would be formed. Note that this argument is similar to one for particle creation numbers in a steady state universe [10]. There is also a “new inflationary” eternal scheme [22] where the universe remains globally de Sitter but fluctuations cause local FRW universes to condense out. Again the finiteness or not depends on the geometry of the background which is an ab initio input.

There is yet another variation on this scheme, involving quantum tunnelling, where open universes can occur [23]. From within such a universe the universe appears infinite in size like an open FRW universe. However, there is some slight of hand with this argument as observers outside this bubble would still think it is of finite but growing size. Also since the time
outside the bubble is limited by the geodesic incompleteness results, and so cannot be taken to infinity, the full Penrose diagram of the open universe is not strictly valid - see Fig.(1). We therefore do not think that the eternal universe mechanism can actually give an infinite total size universe if the original domain was finite - this claim suffers from a lack of gauge invariance. It cannot actually change finite to infinite because of the geodesic incompleteness restriction. One might start with an initially infinite domain but then the eternal inflationary mechanism doesn’t per se cause this infinity. But again for the sake of argument lets still assume open FRW universe can be formed.

**Real life complexity**

The archetypal infinite universe is still then just the flat or open FRW universe case [3]; also present in the pre-big bang model if the initial state
is infinite [24]. By means of a single cellular automata we have given an example that allows more rules to explore than are apparently allowed by an holography argument for the number of quantum states. This alone pushes repeat lives vastly further away than previously expected by quantum state considerations. If cellular automata are a realistic representation of nature we can speculate that in our actual lives we are, at least, playing many cellular automata in parallel. Even this is probably a vast underestimate since quantum mechanics allows “non-local” effects whereas a cellular automata has only nearest neighbour rules. We therefore suspect that the actual value of \( Y \) will approach \( \aleph_0 \). If so then unless the size of the universe was some higher order infinity repeats would be absent.

In conclusion, it seem that even if possible identical lifeforms are present in the infinite universe it can be argued that the complexity of the games we play or simply “our lives” will fortunately not be being repeated. In practice we play many complex games simultaneously and sequentially so the total number of possible permutations astronomically grows towards what can only be approximated by countable infinity \( \aleph_0 \). There now seems little difference between an infinite or finite universe in this regard. A definite proof will require a more complete understanding of what possibly limits the mathematical complexity we can interact with, and exactly how future quantum gravity phenomena should be included.

**Simulation of universe**

Another extreme type of cosmology is that we are simply being simulated in a computer in somebody else’s universe [6]. This is just the latest version of a long standing idea in cosmology [25].

Bostrom [6] is concerned that if humans don’t become extinct they could start simulating life on their machines and contribute excessively to the total number of humans who have ever “lived”. We then are most likely if we are a “typical observer” to be actually one of the lives in the machine.

This problem firstly concerns deep philosophical issues some of which are closely related to those considered by Wittgenstein [26,27]. For example, if nothing around us is then actually real; it is only some sort of “activity in a structure” pretending to be just like our universe, how can we then know anything concretely about the true universe? This argument is related to some in Wittgenstein’s *On Certainty* [26] and concerns the difficulty of the skeptic once all ground rules are done away with. Of course the simulation exponents want to take notions gleaned from within our “false” universe to understand the real universe - but this is a philosophical impasse.
Let us think also about simulation. We do not have remotely realistic computers to do this but we do have some understanding of the “computer” in our heads which can simulate or imagine different events. Now a dream in my mind is a simulation, but note that the characters in my dream do not have true existence as far I am aware. There is therefore no essential need for the “people” within this future simulation to actually exist in the sense of self-conscious awareness. What the advocates of this scheme are suggesting is therefore a more extreme “mind creation” device not a simple simulation.

The connection between a brain state and an actual understanding is also more involved than suggested. On a practical ground there are estimated to be roughly $\sim 10^{10^{10}}$ possible patterns within the brain [28,29]. The notion of keeping track of these fragile states is not compatible with quantum mechanics, since observation alone would alter the states.

However, Wittgenstein’s “Philosophical investigations”[27] suggests that without “direct contact” you cannot gain true understanding in your mind. So these created minds must at some stage have interacted and learnt with real things, that have some similarity to those in our world. There is a similar reasoned example by Putnam [30] that if we were simply “brains in a vat” being fed impulses we couldn’t refer outside the simulation. We could never even imagine that we might be all just brains in a vat. The proponents of the scheme don’t object to the universes being similar to some extent, but we will later show that this seems incompatible with other considerations. Although Wittgenstein is not the last word in philosophy: the notion that our understanding, or theirs, is strongly grounded in your present world would place great difficulty in anybody actually simulating or understanding what was happening in a totally different world.

Returning to actual computers. If one takes a cellular automata [12] then properties like self reproduction can occur. However this sort of scheme doesn’t seem consistent with our notions of free will. If we instead have free will the simulation cannot follow a deterministic procedure since it would need to know our choices beforehand cf. Mackay’s argument [31,32] against a pre-determined universe. This seems especially true in quantum mechanics where even just the possibility of measurement can be important cf. delayed choice experiments using photons gravitationally lensed from across the universe—see e.g.[13].

For cellular automata, as mentioned by Barrow [29,32], a threshold of complexity occurs once self-reference is possible. At this stage a classical paradox, like “the Liar’s”, can be described within the system and Gödel
like incompleteness is present. Barrow [33] has suggested errors in the computer programming could be apparent in our universe since Gödel incompleteness has been shown to correspond to not knowing the shortest program to produce a task. However, the physical universe might not require such an elaborate algebra that contains incompleteness [32].

But if we consider the mathematical world this problem must be apparent. In a simulation nothing is inherent and so everything has to be put there explicitly. For example, there is an interesting problem, beyond current computing capacity, involving the sign of $\text{Li}(n) - \pi(n)$ where $\text{Li}(n)$ is the logarithmic integral function and $\pi(n)$ the number of prime numbers less than $n$. The value changes sign repeatedly for large arbitrary numbers -see e.g. [34]. Storing its actual behaviour seems effectively impossible in any physical system. But if the computer doing the simulation only works with approximate algorithms we should be able to overwhelm it, if only by chance we happen upon an improved algorithm. This is unlike the physical world which might have completeness.

One might also worry about uncomputable problems in our universe and how they can be taken care of. For a really astronomical number consider the Busy Beaver problem with a 12 state Turing machine [35] and games that might be formulated!

There are also results [36] that a computer within our universe cannot process information as fast as the universe, or solve every problem we might consider. So if this result can be extended and applied to the universe in which the simulation of our universe occurs we either a) have a much less sophisticated universe than they do or b) the evolution speed has been altered to prevent this limitation appearing. The constants of nature would apparently differ in our universe from theirs (we must be running slower). A slower evolving simulation of an inferior universe would not be very interesting to the viewers.

We should also point out that since the computer cannot totally predict its own universe it is still rather unprotected. Since the motivation of simulated universes is to argue against Doomsday extinction, this computer still seems exposed to unforeseen danger cf.[6]. Especially if there is some sort of infinite regress of simulations within simulations. A “power failure” in the primary universe would take them all out. One would anyway expect some sort of degradation due to simulation i.e. entropy rapidly building up in the initial actually physical universe.

There must also be some constraint preventing us developing a more pow-
erful computer than that doing the simulation - a rather spurious imposition.

If the universe is really made up of quantum gravity packets of space cf. loop quantum gravity [37], then these cannot be compressed further. It violates the quantum rules to define them more sharply. Either a) laws of physics are different within the simulating machine or b) only a subset of the states we think are in the visible universe are being modelled. Neither is philosophically appealing.

We can see this problem more directly: recall the universe is made up of \( X \) quantum states which can code information only of maximum \( \log X \), see e.g. [38]. But if this is a simulation within some “computer ” it would take at least \( 2^X \) distinct states to simply note these states within the computer. The information within the computer is therefore greater than that present or ever achievable in the universe. The simulation is actually massively inefficient. Again one gets the conclusion that only a vastly inferior, by a two logarithms amount, universe can be simulated. This is also related to the underlying randomness inherent in quantum mechanics needing instead to be accurately described within the simulating computer, i.e. the so-called “hidden variables” are un-hidden in the simulator cf.[13]

Conclusions

By describing a cellular automata with \( Y \) possibilities we have obtained a stronger constraint on the distance that you might encounter clones of oneself having identical lives. But once the holography argument is surpassed we see little evidence to further restrain the complexity that restricts our lives: on the contrary quantum mechanics should actually mean the universe is more complicated than a purely classical universe. This might be too subjective for some people since it depends on us defining structures and games that we interact with to distinguish different lives.

Recall the archetypal issue is if all possible molecules for life like DNA have finite variations then repeats happen. Incidentally, we suspect the way the instructions are implemented allows vastly more variations depending on chemical concentrations, background radiation etc, than often supposed. Even then if clones do exist their differing food sources, environment etc. will cause some divergences in their behaviour. However, our main point is that our games or “culture” is so potentially elaborate that no repeats in life history are realistically possible. Since genetic twins occur anyway upon earth, and human cloning is nearly possible, this seems the only realistic form of non-repetition we could achieve. An implication of this argument is that whether the universe is finite or infinite becomes less consequential for the
actual complexity of our lives. Only a bigger still infinity e.g. \( \aleph_1 \) of universes, that is not produced by inflationary expansion, might allow one to claim repetition is necessarily occurring. This expanded complexity of our world can be contrasted with the philosophy outlined by Wolfram [12] that the universal cellular automata has the same level of complexity as the whole universe. Although it might take unreasonable amounts of runtime to calculate the consequences. Note that unlike Wolfram we are not advocating here that cellular automata necessarily describe all reality. We have doubts that, for example, the “non-local” effects on quantum mechanics could adequately be modelled, although some quantum like automata (complementarity games) have been proposed [39].

Of course other reasons might still prefer the finite case, e.g. having finite action or simplifying the boundary term for a “quantum creation” calculation. One such finite model is the emergent universe [40], an update of the Eddington-Lemaître universe. We would just add that one motivation for this model, that it maximizes the entropy does not seem correct. The later inflationary state itself allows a higher entropy, and a Black hole with the same mass produces even more [41,17].

We also argued against the notion that our universe could simply be the inside of a computer existing within a larger universe. Whether the hypothetical quantum computer alters any of these arguments is uncertain, but the results of Wolpert [36], for example, already assume arbitrary fast processing. The quantum computer is anyway likely to be limited by quantum gravity cf.[42]

Let us return to the original motivation for the finite universe. The argument is that there are only say \( N \sim 10^{10^{120}} \) different possible universes of the present visible horizon size. So if we took a random selection of \( N \) closed universes of similar size to ours, or if our universe has \( N \) times more space then we should expect repeats to start happening. Many of the properties of these universes will be constrained by the Laws of Physics, and initial conditions at the start of the universe. So many fewer actual alterations are allowed by our own volition.

If the observable universe is growing many more regions are coming into causal contact so allowing large extra numbers of possible quantum states. This increase is likely to dominate over any possible changes i.e. like moving objects, we might make. So this sort of reasoning actually ignores the effects of human or any other life forms intervention - unless they start to develop galactic scale alterations. The \( N \) possible universes are therefore effectively
independent of life which is only a miniscule perturbation to the actual allowed states. So any actual universe, say \( n \), out of the \( N \) possibilities is equally useful for our use. Human life which is so paramount to us is just treated as extreme fine structure in this quantum state number argument [4,9]. But by the same token it cannot then be applied to proscribe life if this is what we are actually interested in cf.[7]. If the universe is actually accelerating then the argument can be replaced that we should have only negligible effect upon the growing entropy of the universe \( S = \log W \). In fact the actual entropy of the visible universe is only \( \sim 10^{90} \) -see for example [38,41], considerably smaller than that given by the quantum state calculation \( \sim 10^{120} \) which is the maximum possible entropy allowed in the visible universe. It is this low entropy that allows us to “play games” without restricting their complexity.

If thermodynamical equilibrium was achieved then the behaviour would then start to simply ergodically repeat states. So the repeat arguments seem actually relevant to a system in thermodynamic equilibrium where life anyway is not possible. It is then erroneous to try and conclude things about life in this case. We have assumed a Copernican type principle so that the Laws of Physics and initial conditions are closely similar. Other work has considered the possibility of allowing an ensemble of different Laws or Mathematics which would allow an even greater variety of possibilities -see also [43,44].

It seems therefore that as long as our games have negligible effect upon the universe it doesn’t matter whether we are in a finite or infinite universe. Whether other life forms have exactly the same culture, music, sport etc. as us depends on other arguments about how many varieties of such things might be conceivable. But they are in the finer details of the universe. Ultimately, the global topology will determine whether “games” can continue to be played. For example if the universe continues accelerating it will reach a stage of heat death with entropy maximized [17]. But again remaining in a low entropy state and not the topology of the global universe is the dominant factor.

In conclusion, we don’t feel the need to presently exclude infinite universes necessarily for unwanted philosophical reasons. Or that infinite production of inflationary universes would mean “anything in life” must happen somewhere. Admittedly this argument might be superseded by future results in quantum gravity but not by the holography principle per se. This is fortunate since the universe regardless of size or state, leaves us with a blank
canvas upon which to impose our own, to us important, values. In a sense we are disconnected: the infinite universe isn’t some maelstrom dragging everything, including life, in an ergodic fashion like atoms in a gas.

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\(^1\)There is a similar analogy that gravitationally bound systems can drop out of the expansionary global behaviour of the universe.
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