A Comment on the Infra-Red Problem in the AdS/CFT Correspondence

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Abstract. In this note we report on some recent progress in proving the AdS/CFT correspondence for quantum fields using rigorously defined Euclidean path integrals. We also comment on the infra-red problem in the AdS/CFT correspondence and argue that it is different from the usual IR problem in constructive quantum field theory. To illustrate this, a triviality proof based on hypercontractivity estimates is given for the case of an ultraviolet regularized potential of type $\phi^4$. We also give a brief discussion on possible renormalization strategies and the specific problems that arise in this context.

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1. Introduction

Often, the AdS/CFT correspondence between string theory or some other theory including quantized gravity on bulk AdS and super-symmetric Yang-Mills theory on its conformal boundary [12, 17] is formulated in terms of Euclidean path integrals. In the absence of mathematically rigorous approaches to path integrals of string type (see however [1]) or even gravity, it seems to be reasonable to use the well-established theory of constructive quantum field theory (QFT) [5] as a testing lab for some aspects of the more complex original AdS/CFT conjecture. That such simplified versions of the AdS/CFT correspondence are in fact possible was already noted by Witten [17] (see also [8]) and further elaborated by [4]. In [6] we give a mathematically rigorous version of the latter work (in [9] one finds some related ideas), leaving however the infra-red (IR) problem open. In this note we come back to the IR problem and we show how the difference between the IR problem in the AdS/CFT correspondence as compared with the usual IR problem in constructive QFT leads to somewhat unexpected results.
The authors would like to underline that, in contrast to [6], the present article is rather focused on ideas and thus leaves space for the interpretation of the validity of the results. We will comment on that in several places.

The article is organized as follows: In the following section we introduce the mathematical framework of AdS/CFT correspondence and define rigorous probabilistic path integrals on AdS. In Section 3 we recall the main results from [4, 6], i.e. that the generating functional that is obtained from imposing certain boundary conditions at the conformal boundary (which is the way generating functionals are defined in string theory) can in fact be written as a usual generating functional of some other field theory. From the latter form it is then easy to extract structural properties, e.g. reflection positivity of the functional, in the usual way. Somewhat unexpectedly, it is not clear whether a functional integral can be associated to the boundary theories. These statements hold for all sorts of interactions with a IR-cut-off. In Section 4 the IR-problem in this version of the AdS/CFT correspondence is discussed on a heuristic level. We also sketch the proof of triviality of the generating functional of the conformally invariant theory on the conformal boundary of AdS for the case of an UV-regularized \( \phi^4 \) interaction. We briefly survey strategies that might be candidates to overcome the triviality obstacle at a non-rigorous level and we comment on specific problems with such strategies. The final section gives some preliminary conclusions and an outlook on open research problems in understanding further the mathematical basis of AdS/CFT.

2. Functional integrals on AdS

Let us consider the \( d+2 \) dimensional ambient space \( \mathbb{R}^{d,2} = \mathbb{R}^{d+2} \) with inner product of signature \((-,+,...,+,-)\), i.e. \( \zeta^2 = -\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{d+1}^2 - \zeta_{d+2}^2 \) where \( \zeta \in \mathbb{R}^{d,2} \). Then the submanifold defined by \( \{ \zeta \in \mathbb{R}^{d,2} : \zeta^2 = -1 \} \) is a \( d+1 \) dimensional Lorentz manifold with metric induced by the ambient metric. It is called the \( d+1 \) dimensional Anti de Sitter (AdS) space. Formal Wick rotation \( \zeta_1 \rightarrow i\zeta_1 \) converts the ambient space into the space \( \mathbb{R}^{d+1,1} \) with signature \((+,...,+,-)\). Under Wick rotation, the AdS space is converted to the Hyperbolic space \( \mathbb{H}^{d+1} : \{ \zeta \in \mathbb{R}^{d+1} : \zeta^2 = -1, \zeta^d > 0 \} \), which is a Riemannian submanifold of the ambient \( d+2 \) dimensional Minkowski space. We call \( \mathbb{H}^{d+1} \) the Euclidean AdS space.

It has been established with full mathematical rigor that Euclidean random fields that fulfill the axioms of invariance, ergodicity and reflection positivity give rise, via an Osterwalder–Schrader reconstruction theorem, to local quantum field theories on the universal covering of the relativistic AdS, cf. [3] [10] justifying the above sketched formal Wick rotation. Hence a constructive approach with reflection positive Euclidean functional integrals is viable.

It is convenient to work in the so called half-space model of Euclidean AdS (henceforth the word Euclidean will be dropped). This coordinate system is obtained via the change of variables \( \zeta_i = x_i/z, \) \( i = 1,...,d, \) \( \zeta_{d+1} = -(z^2 + x^2 - 1)/2z, \) \( \zeta_{d+2} = (z^2 + x^2 + 1)/2z \) which maps \( \mathbb{R}^{d+1} = \{(z,x) \in \mathbb{R}^{d+1} : z > 0 \} \) to \( \mathbb{H}^{d+1} \).
We will use the notation \( x \) for \((z, x_1, \ldots, x_d) \in \mathbb{R}^{d+1}_+\). The metric on \( \mathbb{R}^{d+1}_+ \) is given by \( g = (dz^2 + dx_1^2 + \cdots + dx_d^2)/z^2 \) which implies that the canonical volume form is \( dz g = z^{d-1} dz \wedge dx_1 \wedge \cdots \wedge dx_d \). The conformal boundary of \( \mathbb{H}^{d+1} \) then is the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) with metric \( ds^2 = dx_1^2 + \cdots + dx_d^2 \) which is obtained via the limit \( z \to 0 \) and a conformal transformation of the AdS metric. Of course, the upshot of the AdS/CFT correspondence is that the action of the Lorentz group on the AdS space \( \mathbb{H}^{d+1} \) gives rise to an action of the conformal group transformations on the conformal boundary. One thus expects an AdS symmetric QFT (or string/quantum gravity... theory) on the bulk \( \mathbb{H}^{d+1} \) to give, if properly restricted to the conformal boundary, a conformally invariant theory on \( \mathbb{R}^d \).

We will now make this precise. On the hyperbolic space \( \mathbb{H}^{d+1} \) one has two invariant Green’s functions (“bulk-to-bulk propagators”) for the operator \(-\Delta_g + m^2\), with \( \Delta_g \) the Laplacian and \( m^2 \) a real number suitably bounded from below, that differ by scaling properties towards the conformal boundary

\[
G_{\pm}(z, x; z', x') = \gamma_{\pm} (2u)^{-\Delta_{\pm}} F(\Delta_{\pm}, \Delta_{\pm} + \frac{1-d}{2}; 2\Delta_{\pm} + 1 - d; -2u^{-1})
\]

(2.1)

Here \( F \) is the hypergeometric function,

\[
u = \frac{(z-z')^2 + (x-x')^2}{dz^2 + (x-x')^2}, \quad \Delta_{\pm} = \frac{d}{2} \mp \sqrt{d^2 + 4m^2}
\]

\[\gamma_{\pm} = \frac{\Gamma(\Delta_\pm)}{2\pi^{d/2} \Gamma(\Delta_\pm + 1 - d/2)} \] [6]. Taking pointwise scaling limits for \( z \to 0 \) in one or two of the arguments, the bulk-to-boundary and boundary-to-boundary propagators are obtained

\[
H_{\pm}(z, x; x') = \lim_{z' \to 0} z'^{-\Delta_{\pm}} G_{\pm}(z, x; z', x') = \gamma_{\pm} \left( \frac{z}{z^2 + (x-x')^2} \right)^{\Delta_{\pm}}
\]

(2.2)

and

\[
\alpha_{\pm}(x, x') = \lim_{z' \to 0} z^{-\Delta_{\pm}} H_{\pm}(z, x; x') = \gamma_{\pm} (x - x')^{-2\Delta_{\pm}}.
\]

(2.3)

If (2.2) or (2.3) do not define locally integrable functions, the expressions on the right hand side are defined via analytic continuation in the weights \( \Delta_{\pm} \). An important relation between \( G_+, G_-, H_+ \) and \( \alpha_- \) is the covariance splitting formula for \( G_- \) given by

\[
G_-(x, x') = G_+(x, x') + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H_+(x, y) c^2 \alpha_-(y, y') H_+(y', y') dy dy',
\]

(2.4)

with \( c = 2\nu \).

We now pass on to the description of mathematically well-defined functional integrals. Let \( \mathcal{D} = \mathcal{D}(\mathbb{H}^{d+1}, \mathbb{R}) \) be the infinitely differentiable, compactly supported functions on \( \mathbb{H}^{d+1} \) endowed with the topology of compact convergence. The propagator \( G_+ \) is the resolvent function to the Laplacian \( \Delta_g \) with Dirichlet boundary conditions at conformal infinity, from which it follows that \( G_+ \) is stochastically positive,

\[
\langle f, f \rangle_{-1} = G_+(f, f) = \int_{\mathbb{H}^{d+1} \times \mathbb{H}^{d+1}} G_+(x, x') f(x) f(x') \, dy \, dy' \geq 0
\]

\( \forall f \in \mathcal{D} \), and reflection positive as long as \( m^2 > -\frac{d^2}{4} \). The latter value is determined by the lower bound of the spectrum of \( \Delta_g \) on \( \mathbb{H}^{d+1} \). In explicit, if
The reflection \( \Theta \) is the reflection in \( x_1 \)-direction, then for any \( f \in \mathcal{D}_+ = \{ h \in \mathcal{D} : h(z) = 0 \text{ if } x_1 \leq 0 \} \) we have

\[
\int_{\mathbb{H}^{d+1} \times \mathbb{H}^{d+1}} G_+(z, z') f_\theta(z') d\theta_2 dz' \geq 0,
\]

cf. [5]. Here, \( f_\theta(z) = f(u^{-1} z) \) for \( u \in \text{Iso}(\mathbb{H}^{d+1}) \).

Consequently, via application of Minlos theorem, there exists a unique probability measure \( \mu_{G_+} \) on the measurable space \( (\mathcal{D}', \mathcal{B}) \), where \( \mathcal{D}' \) is the topological dual space of \( \mathcal{D} \) and \( \mathcal{B} \) the associated Borel sigma algebra, such that \( \int_{\mathcal{D}'} e^{\langle \phi, f \rangle} d\mu_{G_+}(\phi) = e^{\frac{1}{2} \| f \|_2} \). By setting \( \varphi(f)(\phi) = \phi(f) \) we define the canonical random field associated with \( \mu_{G_+} \), i.e. a random variable valued distribution. In the following we omit the distinction between \( \varphi \) and \( \phi \) and write \( \phi \) for both.

Let \( \mathcal{B}_\Lambda, \Lambda \subseteq \mathbb{H}^{d+1} \) be the smallest sigma algebra generated by the functions \( \mathcal{D}' \ni \phi \rightarrow \langle \phi, f \rangle \), \( \sup f \in \Lambda \) and \( M(\Lambda) \) be the functions that are \( \mathcal{B}_\Lambda \)-measurable. We use the special abbreviations \( \mathcal{B}_+ = \mathcal{B}_\Lambda(\mathbb{Z} \subseteq \mathbb{H}^{d+1}: x_1 > 0) \) and \( M_+ = M(\mathcal{B}_+) \). Then \( \mu_{G_+} \) is reflection positive, i.e.

\[
\int_{\mathcal{D}'} \Theta F(\phi) F(\phi) d\mu_{G_+}(\phi) \geq 0, \forall F \in M_+. \tag{2.5}
\]

The reflection \( \Theta F(\phi) \) is defined as \( F(\varphi_\theta) \) with \( \langle \phi_\theta, f \rangle = \langle \phi, f \rangle \) \( \forall \phi \in \mathcal{D}' \), \( f \in \mathcal{D} \) and \( u \in \text{Iso}(\mathbb{H}^{d+1}) \). \( \langle \phi, f \rangle \) is the duality between \( \mathcal{D}' \) and \( \mathcal{D} \) induced by the \( L^2(\mathbb{H}^{d+1}, d\mu_{\mathbb{H}_+}) \) inner product.

Let \( \{ V_\Lambda \} : \mathcal{D}' \rightarrow \mathbb{R} \) be a set of interaction potentials indexed by the net of bounded, measurable subsets \( \Lambda \) in \( \mathbb{H}^{d+1} \). In particular these sets have finite volume \( |\Lambda| = \int_{\mathbb{H}_+} d\mu_{\mathbb{H}_+} \). We require that the following conditions hold:

- (i) Integrability: \( e^{-V_\Lambda} \in L^1(\mathcal{D}', d\mu_{G_+}) \) \( \forall \Lambda \);
- (ii) Locality: \( V_\Lambda \in M(\mathcal{B}_\Lambda) \);
- (iii) Invariance: \( V_\Lambda(\phi_\theta) = V_{\Lambda^{-1}\Lambda}(\phi) \) \( \mu_{G_+} \)-a.s.\;
- (iv) Additivity: \( V_\Lambda + V_{\Lambda'} = V_{\Lambda \cup \Lambda'} \) for \( \Lambda \cup \Lambda' = \emptyset \);
- (v) Non-degeneracy: \( V_\Lambda = 0 \) \( \mu_{G_+} \)-a.s. if \( |\Lambda| = 0 \).

Then, using (i), we obtain a family of interacting measures on \( (\mathcal{D}', \mathcal{B}) \), indexed by the net \( \{ \Lambda \} \), by setting \( d\mu_{G_+ + \Lambda} = e^{-V_\Lambda} d\mu_{G_+} / Z_\Lambda \) with \( Z_\Lambda = \int_{\mathcal{D}'} e^{-V_\Lambda} d\mu_{G_+} \). Furthermore, using (ii)-(v) we get whenever \( \theta \Lambda = \Lambda \)

\[
\int_{\mathcal{D}'} \Theta F F d\mu_{G_+ + \Lambda} = \frac{1}{Z_\Lambda} \int_{\mathcal{D}'} \Theta \left( F e^{-V_\Lambda} \right) \left( F e^{-V_{\Lambda'}} \right) d\mu_{G_+} \geq 0, \forall F \in M_+, \tag{2.6}
\]

where \( \Lambda_+ = \Lambda \cap \{ z \in \mathbb{H}^{d+1} : x_1 > 0 \} \). Hence reflection positivity is preserved under the perturbation. Furthermore, from the invariance of \( \mu_{G_+} \) under \( \text{Iso}(\mathbb{H}^{d+1}) \) we get that \( u \ast \mu_{G_+ + \Lambda} = \mu_{G_+ + u \Lambda} \). Here \( u \in \text{Iso}(\mathbb{H}^{d+1}) \) induces an action on \( \mathcal{D}' \) via \( \phi \rightarrow \phi_\theta \) and \( u \ast \) is the pushforward under this action. Consequently, if the limit (in distribution) \( \mu_{G_+ + \mathbb{H}^{d+1}} = \lim_{\Lambda \rightarrow \mathbb{H}^{d+1}} \mu_{G_+ + \Lambda} \) exists and is unique, the limiting measure is invariant under \( \text{Iso}(\mathbb{H}^{d+1}) \) and reflection positive. Invariance follows from the equivalence of the nets \( \{ \Lambda \} \) and \( \{ u \Lambda \} \) and the postulated uniqueness of the limit over the net \( \{ \Lambda \} \).
Let us next consider functional integrals associated with the Green’s function $G_-$. In the case when $2
u < d$ ($\Leftrightarrow m^2 < 0$) we get that $\alpha_-$ is stochastically positive since $\alpha_-(\hat{f}, f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \alpha_-(x, x') \hat{f}(x) f(x') \, dx \, dx' = C_{-\nu} \int_{\mathbb{R}^d} |k|^{-2\nu} |\hat{f}(k)|^2 \, dk \geq 0$. $\hat{f}$ denotes the Fourier transform of $f$. For simplicity, we sometimes assume that $V$ can be defined.)

To simplify the model and for the sake of concreteness we will consider a regularized effective potential for that field obtained via integrating out the remaining degrees of freedom (leaving open the question how such an “integral” can be defined). The reflection positivity of $G_-$ does not follow from the reflection positivity of $G_+$ and $\alpha_-$ due to the non-local effect of $H_+$. We will however not need it here. We thus conclude that for $\text{sup} \, \text{spec}(\Delta_g) < m^2 < 0$ a unique probability measure $\mu_{G_-}$ on $(\mathcal{D}', \mathcal{B})$ with Laplace transform $\int_{\mathcal{D}'} e^{\langle \phi, f \rangle} \, d\mu_{G_-}(\phi) = e^{\frac{1}{2} \langle f, f \rangle_{-1,-}}$ exists. Here $\langle f, f \rangle_{-1,-} = G_-(f, f)$. The perturbation of $\mu_{G_-}$ with an interaction can now be discussed in analogy with the above case – where however the reflection positivity for the perturbed measure remains open, as reflection positivity of the free measure does not necessarily hold.

3. Two Generating Functionals

On the string theory side of the AdS/CFT correspondence, generating functionals for the boundary theory are calculated fixing boundary conditions at the conformal boundary (so called Dirichlet boundary conditions). Little is known about the mathematical properties of such kinds of generating functionals. E.g. their stochastic and reflection positivity is far from obvious, leaving the linkage to path integrals and relativistic physics open. It was noticed by Dütsch and Rehren [4] that such kinds of generating functionals can however be re-written in terms of ordinary generating functionals, from which the structural properties can be read off in the usual way. These ideas in [6] have been made fully rigorous in the context of constructive QFT. We will now briefly review these results.

The generating functional $Z(f)/Z(0)$, $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, the space of Schwartz functions, in the AdS/CFT correspondence from a string theoretic point of view can be described as follows: Let $\phi$ be some scalar quantum field that is included in the theory (e.g. the dilaton field) and let $V_\Lambda$ be the (IR and eventually UV-regularized) effective potential for that field obtained via integrating out the remaining degrees of freedom (leaving open the question how such an “integral” can be defined). To simplify the model and for the sake of concreteness we will sometimes assume that $V_\Lambda$ is of polynomial type. Formally,

$$Z(f) = \int_{\phi_0 = \phi|_{\partial \mathbb{R}^{d+1}}} e^{-S_0(\phi) - V_\Lambda(\phi)} \, d\phi = \int \delta(\phi_0 - f) e^{-S_0(\phi) - V_\Lambda(\phi)} \, d\phi \quad (3.1)$$

where $S_0 = |\nabla \phi|^2 + m^2 \phi^2$, $\phi_0 = \phi|_{\partial \mathbb{R}^{d+1}}$ are suitably rescaled boundary values of the field $\phi$ and $d\phi$ is the heuristic flat measure on the space of all field configurations. The first step in making this formal expression rigorous is to replace
$e^{-S_0(\phi)} \, d\phi$ with a well-defined probabilistic path integral. It turns out that $d\mu_{G_-}(\phi)$ is the right candidate and hence for the moment restriction to $m^2 < 0$ is necessary.

In a second step we have to make sense out of the boundary condition $\phi_0 = f$ or the functional delta distribution on the boundary values of the field, respectively. Using the covariance splitting formula (2.4) we obtain the splitting $\phi_-(x) = \phi_+(x) + \int_{\mathbb{R}^d} H_+(x',x)\phi_-(x') \, dx'$, where $\phi_\pm$ are the canonical random fields associated with $G_{\pm}$ and $\phi_{\alpha_-}$ is the canonical random field associated to the functional measure $\mu_{\alpha_-}$, i.e. the Gaussian measure with generating functional $e^{\frac{i}{2} \mu_-(f,f)}$ living on the conformal boundary of $\mathbb{H}^{d+1}$.

The following step is to construct a finite dimensional approximation $\psi_{\alpha_-}$ of the boundary field $\phi_{\alpha_-}$ by projecting it via a basis expansion to $\mathbb{R}^n$. Thereafter, one can implement the delta distribution as a delta distribution on $\mathbb{R}^n$. Finally one can remove the finite dimensional approximation via a limit $n \to \infty$. It turns out that this limit exists and is unique up to a diverging multiplicative constant. This constant however drops out in the quotient $Z(f)/Z(0)$. With the projection to the first $n$ terms of the basis expansion denoted by $p_n$ and $\eta$ a linear mapping from this space to $\mathbb{R}^n$ we get

$$C_{A_-} \int_{\mathbb{R}^n} \int_{\mathcal{D}'} \delta(\psi_{\alpha_-} - \eta p_n f) e^{-V_\Lambda(\phi_+ + cH_+(\eta^{-1}\psi_{\alpha_-}))} d\mu_{G_+}(\phi_+) e^{-\frac{i}{2}(\psi_{\alpha_-},A_- \psi_{\alpha_-})} d\psi_{\alpha_-} = C_{A_-} e^{-\frac{i}{2}(f,p_n \alpha - p_n \eta)^{-1} f} \int_{\mathcal{D}'} e^{-V_\Lambda(\phi_+ + cH_+(p_n f))} d\mu_{G_+}(\phi_+) =: Z_n(f),$$

(3.2)

where $A_- := (\eta p_n \alpha - p_n \eta)^{-1}$ and $C_{A_-} = \frac{i \det A_-}{(2\pi)^{\frac{d}{2}}}$. One can then show that

$$Z(f)/Z(0) := \lim_{n \to \infty} Z_n(f)/Z_n(0) = e^{-\frac{i}{2}(f,\alpha^{-1} f)} \int_{\mathcal{D}'} e^{-V_\Lambda(\phi_+ + cH_+(f))} d\mu_{G_+}(\phi_+) \int_{\mathcal{D}'} e^{-V_\Lambda(\phi_+)} d\mu_{G_+}(\phi_+)$$

(3.3)

converges under rather weak continuity requirements on $V_\Lambda$ that are fulfilled e.g. for UV-regularized potentials in arbitrary dimension and for $P(\phi)_2$ potentials without UV cut-offs in $d+1 = 2$. Obviously, the limit does not depend on the details of the finite dimensional approximation. For the details we refer to [6]. We now realize that the right hand side of (3.3) also makes sense for $m^2 \geq 0$ and we adopt (3.3) as a definition of (3.1). At this point one would like to associate a boundary field theory to the generating functional $\mathcal{C}(f) = Z(f)/Z(0)$. In order to obtain a functional integral associated to $\mathcal{C} : \mathcal{S} = \mathcal{S}(\mathbb{R}^d,\mathbb{R}) \to \mathbb{R}$ we require that $\mathcal{C}$ is continuous wrt the Schwartz topology, normalized, $\mathcal{C}(0) = 1$ and stochastically positive, $\sum_{l=1}^n \bar{z}_j z_l \mathcal{C}(f_j + f_l) \geq 0 \ \forall \ n \in \mathbb{N}, f_j \in \mathcal{S}, z_j \in \mathcal{C}$. Furthermore, in order to have a well defined passage from Euclidean time to real time QFT one requires reflection positivity $\sum_{l=1}^n \bar{z}_j z_l \mathcal{C}(f_j, \theta + f_l) \geq 0 \ \forall \ n \in \mathbb{N}, f_j \in \mathcal{S}_+, z_j \in \mathcal{C}$. Here $\mathcal{S}_+ = \{ f \in \mathcal{S} : \text{supp} f \subseteq \{ x \in \mathbb{R}^d : x_1 > 0 \} \}$. Finally, the theory obtained at the boundary should be conformally invariant, provided the IR cut-off $\Lambda$ is removed from $V_\Lambda$ via taking the limit of the generating functionals wrt the net $\{ \Lambda \}$. 

Gottschalk and Thaler
It has been pointed out in \[1\] that an alternative representation of the functional \( \langle 3.3 \rangle \) answers a number of the questions raised above. Let \( \phi(x) = \phi(z, x) \) be the canonical random field associated with the measure \( \mu_{G_+} \). The idea is to smear \( \phi(z, x) \) in the \( x \)-variable with a test function \( f \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}) \) and then scale \( z \to 0 \). In the light of \( \langle 2.1 \rangle \), one has to multiply \( \phi(z, f) = (\phi, \delta_z \otimes f) \) with a factor \( z^{-\Delta_+} \) in order to obtain a finite result in the limit. We set

\[
Y_z(f) = \int_{\mathcal{D}'} e^{\langle \phi, z^{-\Delta_+} \delta_z \otimes f \rangle} e^{-V_\Lambda(\phi)} \, d\mu_{G_+}(\phi). \tag{3.4}
\]

Clearly, under the conditions on \( V_\Lambda \) given in the preceding section and for \( \Lambda = \theta \Lambda \), \( Y_z(f)/Y_z(0) \) converges to a continuous, normalized, stochastically positive and reflection positive generating functional for all \( z > 0 \). Using the fact that \( G_+ (\delta_z \otimes f) \) is in the Cameron-Martin space of the measure \( \mu_{G_+} \), one gets with \( f_z = z^{-\Delta_+} \delta_z \otimes f \), cf. \[6\],

\[
Y_z(f)/Y_z(0) = e^{\frac{1}{2} G_+ (f_z, f_z)} \int_{\mathcal{D}'} e^{-V_\Lambda(\phi + G_+ f_z)} \, d\mu_{G_+}(\phi)/Y_z(0). \tag{3.5}
\]

We now want to take the limit \( z \to 0 \). Using \( \langle 2.2 \rangle \) one can show under rather weak continuity requirements on \( V_\Lambda \) that the functional integral on the rhs of \( \langle 3.5 \rangle \) converges to \( \int_{\mathcal{D}'} e^{-V_\Lambda(\phi + H_+ f)} \, d\mu_{G_+}(\phi) \). The prefactor however diverges. The reason is that the limit in \( \langle 2.3 \rangle \) is only a pointwise limit for \( x \neq x' \) and not a limit in the sense of tempered distributions. One can however show that \[6\]

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \alpha_+(x, y) f(x) f(y) \, dx \, dy = \lim_{z \to 0} z^{-2\Delta_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_+(z, x; z, y) f(x) f(y) \, dx \, dy - \frac{1}{(2\pi)^2} \left( \sqrt{\nu + \frac{1}{2}} \right) 2^{1-\nu} \sum_{j=0}^{|\nu|} z^{-2(\nu-j)} (-1)^j a_j \int_{\mathbb{R}^d} |\hat{f}(k)|^2 |k|^{2j} \, dk.
\]

\[
= \lim_{z \to 0} z^{-2\Delta_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_+(z, x; z, y) f(x) f(y) \, dx \, dy - \text{Corr}(z) f, f. \tag{3.6}
\]

Here \( a_j = \int_{-\infty}^{\infty} (\int_0^1 \cos(\omega t) (1 - t^2)^{\nu - \frac{1}{2}} dt) \omega^{2(\nu-j)-1} \, d\omega \). Thus, the right hand side of \( \langle 3.5 \rangle \) multiplied with \( e^{-\frac{1}{2} \text{Corr}(z) f, f} \) converges and we obtain the limiting functional

\[
\hat{C}(f) = \lim_{z \to 0} e^{-\frac{1}{2} \text{Corr}(z) f, f} \left( Y_z(f)/Y_z(0) \right) = \lim_{z \to 0} e^{\frac{1}{2} \text{Corr}(z) f, f - \text{Corr}(z) f, f} \int_{\mathcal{D}'} e^{-V_\Lambda(\phi + G_+ f_z)} \, d\mu_{G_+}(\phi)/Y_z(0) = e^{\frac{1}{2} \alpha_+(f, f)} \int_{\mathcal{D}'} e^{-V_\Lambda(\phi + H_+ f)} \, d\mu_{G_+}(\phi) / \int_{\mathcal{D}'} e^{-V_\Lambda(\phi)} \, d\mu_{G_+}(\phi). \tag{3.7}
\]

This, together with \( \alpha_- = -c^2 \alpha_+ \), establishes the crucial identity \[1, 6\]

\[
C(f) = \hat{C}(\epsilon f), \quad \forall f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}). \tag{3.8}
\]
Let us now investigate the structural properties of the generating functional $C : S \to \mathbb{R}$. If there were not the correction factor $(\text{Corr}(z)f, f)$, $C$ would be stochastically positive and reflection positive as the limit of functionals with that property, since we can combine (3.7) and (3.8) for a representation of $C$. However, due to the signs in (3.6) $S \ni f \mapsto e^{-\frac{1}{2}(\text{Corr}(z)f, f)} \in \mathbb{R}$ is not stochastically positive and consequently the stochastic positivity of $e^{-\frac{1}{2}(\text{Corr}(z)f, f)} (Y_z(f)/Y_z(0))$ is at least unclear. Hence we do not have any reason to believe that the limiting functional $C$ is stochastically positive and can be associated with a probabilistic functional integral. An exception is the case where $V_\Lambda \equiv 0$ where we can dwell on the fact that $S \ni f \mapsto e^{\frac{1}{2}\alpha + (f, f)} \in \mathbb{R}$ is manifestly stochastically positive since $\hat{\alpha}(k) = C - \nu^2 |k|^2 \nu \in \mathbb{R}$ with $C - \nu^2 > 0$. It is therefore questionable if one can use the AdS/CFT correspondence to generate conformally invariant models in statistical mechanics.

We next investigate the question of reflection positivity. Since the correlation length of the distributional kernels of $\text{Corr}(z)$ is zero, we get that $(\text{Corr}(z)(f_j, \theta + f_l), (f_j, \theta + f_l)) = (\text{Corr}(z)(f_j, \theta) + (\text{Corr}(z)f_j, f_l) = (\text{Corr}(z)f_j, f_l) + (\text{Corr}(z)f_l, f_l)$ for $f_j \in S_+$. Consequently, $\forall f_j \in S_+, z_1, \ldots, z_n \in \mathbb{C}$ and $\Lambda$ such that $\theta \Lambda = \Lambda$ we get

$$
\sum_{j,l=1}^n C(f_j, \theta + f_l) \bar{z}_j z_l = \lim_{z \to 0} \sum_{j,l=1}^n (Y_z(cf_j, \theta + cf_l)/Y_z(0)) \bar{z}_j' z_l' \geq 0 \quad (3.9)
$$

with $z_j' = z_j e^{-\frac{1}{2}(\text{Corr}(z)cf_j, cf_j)}$. For a proof that the reflection positivity of generating functionals implies the reflection positivity of Schwinger functions [5] also in the absence of stochastic positivity, cf. [7]. As in [4, 6, 16], we thus come to the conclusion that the crucial property for the existence of a relativistic theory is preserved in the AdS/CFT correspondence.

Finally we address the invariance properties of the limiting generating functional $C$. For being the generating functional of a CFT, we require invariance under conformal transformations, i.e. $C(f) = C(u^{-1}f) \forall f \in S$ where $u$ is an element of the conformal group on $\mathbb{R}^d$ and

$$
\lambda_u(x) = \left| \det \left( \frac{\partial u(x)}{\partial x} \right) \right|^{-\Delta_u/d}. \quad (3.10)
$$

Certainly, as long as an interaction with IR cut-off is included in the definition of $C = C_\Lambda$, conformal invariance can not hold. Using the identification of $\text{Iso}(\mathbb{H}^{d+1})$ and the conformal group on $\mathbb{R}^d$, we get that $H_+$ intertwines the respective representations on function spaces, i.e. [6]

$$
H_+(u(z, x); x') = \left| \det \left( \frac{\partial u^{-1}(x')}{\partial x'} \right) \right|^{\frac{\Delta_u}{2}} H_+(z, x; u^{-1}(x')). \quad (3.11)
$$
Combining this, the conformal invariance of $\alpha_+$ under the given representation of the conformal group and (3.3), we obtain
\[ C(\lambda^{-1} u) = C_0(f) \quad \forall f \in \mathcal{S}. \quad (3.12) \]
Hence, if the generating functionals $\{C_\Lambda\}$ have a unique limit $C$ wrt the net $\{\Lambda\}$, then $C$ is reflection positive and conformally invariant and hence is the generating functional of a boundary CFT.

4. The Infra-Red Problem and Triviality

In this section we investigate the net limit of $\{C_\Lambda\}$ which is needed to establish the full AdS/CFT correspondence. This problem has been left open in [6] and we will show that this kind of IR problem behaves somewhat wired.

The reason is the following: When we identified the generating functionals $C_\Lambda$ and $\tilde{C}_\Lambda$, we have seen from the latter functional that it originated from a usual QFT generating functional with $z^{-\Delta+\delta_2} \otimes f$ giving rise to a source term which needs to be considered in the limit $z \to 0$. As (3.6) shows, this source term corresponds to an interaction of an “exterior field” with the quantum field $\phi$ which, already for the free field, has zero expectation but infinite fluctuations in the limit $z \to 0$. Without any correction term, this would have led to a generating functional which converges to zero for any $f \neq 0$. We already then needed an ultra-local correction term to deal with the prescribed infinite energy fluctuations.

If we now switch on the interaction, a shift term $H_+ f$ in the bulk theory is generated, cf. (3.7). If we e.g. restrict to polynomial interactions, this shift leads to re-defined $f$-dependent couplings that diverge towards the conformal boundary. This again leads to an infinite energy transfer and it is probable that this infinite amount of energy plays havoc with the generating functional. Here we will show that in some situations this indeed happens.

Let us first investigate the behavior of the shift $H_+ f$ towards the conformal boundary. Let $f \in \mathcal{S}$ be such that $f(0) \neq 0$. Choosing spherical coordinates, we denote by $f_{\text{rad}}(r)$ the integral of $f(x)$ over the angular coordinates. We get from (2.2) via a change of coordinates
\[ H_+ f(z,0) = \gamma_+ z^{-\Delta_+ + d} \int_0^\infty \left( \frac{1}{1+r^2} \right)^{\Delta_+} f_{\text{rad}}(zr) r^{d-1} dr \quad (4.1) \]
and we see that the integral on the rhs converges to $f(0) \int_0^\infty \left( \frac{1}{1+r^2} \right)^{\Delta_+} r^{d-1} dr = f(0) \times \Gamma(\Delta_+ - d/2)\Gamma(d/2)/2\Gamma(\Delta_+)$, hence $H_+ f(z,x) \sim z^{-\Delta_+ + d}$ if $f(x) \neq 0$ by translation invariance.

Let us now work with the generating functional as defined by (3.7). The prefactor on the rhs is independent of $\Lambda$, hence we have to investigate the behavior of
\[ C'(f) = \frac{\int_{D'} e^{-V_\Lambda(f+H_+f)} d\mu_{G_+}(\phi)}{\int_{D'} e^{-V_\Lambda(f)} d\mu_{G_+}(\phi)}. \quad (4.2) \]
We restrict ourselves to the simplest possible case - an ultra-violet regularized $\phi^4$ potential in arbitrary dimensions $d + 1$

$$V_\lambda(\phi) = \lambda \int_\Lambda : \phi_\kappa^4 : (x) \, d_\nu x$$

(4.3)

where $\phi_\kappa$ denotes the random field $\phi$ with UV-cut off $\kappa$. Due to this cut-off, the locality axiom in Section 2 will in general be violated. This however does not matter in the following discussion. We furthermore require that the locality axiom in Section 2 will in general be violated.

Taking the expected value of the shifted potential wrt $\mu_{G_+}$, one obtains $\lambda \times \int_\Lambda (H_+ f)^4 d_\nu x$ which in the light of (4.1) clearly diverges as $\Lambda \to \infty$ whenever $f \neq 0$.

Let us now focus on a specific class of cut-offs of the form $\Lambda(z_0) = \Lambda(z_0, l) = [z_0, A] \times [-l, l]^d$ where we keep $l > 0, A > 0$ arbitrarily large but fixed. Let $V(z_0, f)(\phi) = V_{\Lambda(z_0)}(\phi + H_+ f)$. Since $d_\nu x = z^{-d-1}dzdx$ we obtain the scaling of the expected shifted interaction energy

$$E(z_0, f) = \mathbb{E}[V(z_0, f)] = \lambda \int_{[z_0, A]} \int_{[-l, l]^d} (H_+ f)^4 (z, x) \, dxz^{-d-1}dz$$

$$\sim z_0^{-d-4(\Delta_+ - d)} \text{ as } z_0 \to 0.$$  

(4.5)

Let us next investigate the fluctuations in the shifted energy as $z_0 \to 0$. Denoting the standard deviation of $V(z_0, f)$ with $\sigma(z_0, f)$, we obtain using (4.5) and $\mathbb{E}[: \phi_\kappa^a : (x) : \phi_\kappa^b : (y) :] = a! \delta_{a,b} G_+^a (x, y)^b$, $a, b \in \mathbb{N}$,

$$\sigma(z_0, f) = \left[ 24 \int_{\Lambda(z_0)^2} G_+ (x, y)^4 d_\nu x d_\nu y \right. \right.$$  

$$+ 96 \int_{\Lambda(z_0)^2} (H_+ f(x)H_+ f(y)G_+ (x, y))^3 d_\nu x d_\nu y$$

$$+ 72 \int_{\Lambda(z_0)^2} (H_+ f(x))^2 (H_+ f(y))^2 G_+ (x, y)^3 d_\nu x d_\nu y$$

$$+ 16 \int_{\Lambda(z_0)^2} (H_+ f(x))^3 (H_+ f(y))^3 G_+ (x, y)^4 d_\nu x d_\nu y \right]^{1/2}$$

$$\sim z_0^{-d-3(\Delta_+ - d)} \text{ or slower as } z_0 \to 0,$$

where we took the factors $G_+^a$ out of the integral and replaced them with a majorizing constant in order to obtain an upper bound on the scaling. Apparently, the
quotient \( \gamma(z_0, f) = 2\sigma(z_0, f)/E(z_0, f) \sim z_0^{\Delta + d} \) scales down to zero if \( m^2 > 0 \). Using the Chebyshev inequality \( \mu_{G_+}(\|V(z_0, f) - E(z_0, f)\| \leq E(z_0, f)/2) \leq \gamma(z_0, f)^2 \) we see from this that \( V(z_0, f) \to \infty \mu_{G_+} \text{-a.s.} \).

To determine the behavior of \( C_{\Lambda(z_0)}(f) \) for \( f \neq 0 \) we however need an argument based on the hypercontractivity estimate \( \|F\|_p \leq (p-1)^{\alpha/2} \|F\|_2 \forall F \) that are in the \( L^p(D', B, \mu_{G_+}) \)-closure of the span of Wick monomials: \( \phi(f_1) \cdots \phi(f_s) \): with \( s \leq n \). Applying this to \( V(z_0, f) = V_{\Lambda(z_0)}(\phi + H_+, f) \) with \( n = 4 \) one obtains

\[
\mu_{G_+}(V(z_0, f) \leq \frac{E(z_0, f)}{2}) \leq \mu_{G_+}(\|V(z_0, f) - E(z_0, f)\| \geq \frac{E(z_0, f)}{2}) \leq \frac{2^p}{E(z_0, f)^p} \|V(z_0, f) - E(z_0, f)\|_p^p \leq \frac{2^p}{E(z_0, f)^p} (p-1)^{2p} \|V(z_0, f) - E(z_0, f)\|_2^p = \gamma(z_0, f)^p (p-1)^{2p}. \tag{4.7}
\]

The next step is to optimize this estimate wrt \( p \) for \( z_0 \to 0 \). Equivalently, one can ask for the minimum of the logarithm of the rhs wrt to \( p \). Taking the \( p \)-derivative of this expression and setting it zero yields \( 0 = p \log \gamma(z_0, f) + \frac{2p(z_0)}{p(z_0)-1} + 2 \log(p(z_0) - 1) \) with \( p(z_0) \) the optimal \( p \). Apparently \( p(z_0) \to \infty \) as \( z_0 \to 0 \) and thus \( 2p(z_0)/(p(z_0) - 1) \to 2 \), hence \( p(z_0) \) scales as

\[
p(z_0) \sim e^{-1} \times \gamma(z_0, f)^{-1/2} \sim Ce^{-1} \times z_0^{-(\Delta+d)/2}. \tag{4.8}
\]

Combining (4.7) and (4.8) yields

\[
\mu_{G_+}(V(z_0, f) < \frac{E(z_0, f)}{2}) \leq \gamma(z_0, f)^e^{-1} \times \gamma(z_0, f)^{-1/2} (e^{-1} \times \gamma(z_0, f)^{-1/2} - 1)^{2e^{-1} \times \gamma(z_0, f)^{-1/2}} \sim e^{-2e^{-1} \times \gamma(z_0, f)^{-1/2}} \sim e^{-2Ce^{-1} \times z_0^{(d-\Delta+d)/2}}. \tag{4.9}
\]

We have thus seen that the portion of the probability space where \( V(z_0, f) \) does not get large as \( z_0 \to 0 \) has a rapidly falling probability. We need an estimate that controls the negative values on this exceptional set. The ultra-violet cut-off implies: \( \varphi_0^k \): \( \varphi(x) \geq -Bc_2^k \), \( B \) independent of \( \kappa \), \( c_\kappa = \sup_x \mu_G[G_k(x, \varphi)] \), \( \mu_{G_+} \text{-a.s.} \), which provides us with a pointwise lower bound for \( V(z_0, f) \) that is depending on \( z_0 \) as

\[
V(z_0, f) \geq -\lambda B c_2^k |\Lambda(z_0)| = -[\lambda B c_2^k (2l)^d] \times (z_0^{-d} - A^{-d})/d \mu_{G_+} \text{-a.s.} \tag{4.10}
\]

Combination of (4.9) and (4.10) gives for \( z_0 \) sufficiently small

\[
\mathbb{E} \left[ e^{-V(z_0, f)} \right] \leq e^{-\frac{1}{2}E(z_0, f)} + e^{[\lambda B c_2^k (2l)^d] \times (z_0^{-d} - A^{-d})/d - 2Ce^{-1} \times z_0^{(d-\Delta+d)/2}} \to 0, \tag{4.11}
\]
if $\Delta_+ > 3d \iff m^2 > 6d^2$. Furthermore, by Jensen’s inequality and $\mathbb{E}[V(z_0, 0)] = 0$,
\[ \mathbb{E}[e^{-V(z_0, 0)}] \geq e^{-\mathbb{E}[V(z_0, 0)]} = 1, \tag{4.12} \]
which implies that for $m^2$ sufficiently large
\[ C'_{\Lambda(z_0)}(f) = \frac{\mathbb{E}[e^{-V(z_0, f)}]}{\mathbb{E}[e^{-V(z_0, 0)}]} \to 0 \text{ as } z_0 \to 0, \tag{4.13} \]
We have thus obtained the following result:

**Theorem 4.1.** If the generating functional $C(f) = \lim_{\Lambda} C_{\Lambda}(f)$ exists for the UV-regularized $\phi^4$ interaction and is unique (as required in order to obtain conformal invariance from AdS-invariance) it is also trivial ($C(f) = 0$ if $f \neq 0$) provided $m^2 \geq 6d^2$.

The above triviality result relies on three crucial assumptions.

(i) The potential is quartic, cf. (4.3);
(ii) There is a UV-cut-off;
(iii) The mass is sufficiently large.

In order to assess the relevance of the triviality result for the general case, let us give some short comments on the role of each of these assumptions:

(i) At the cost of a more restrictive mass bound, assumption (i) can easily be relaxed from quartic to polynomial interactions. For non-polynomial interactions, however, the hypercontractivity estimate can not be used. This might be of relevance, if we consider $V$ as an effective potential, which in general will be non-polynomial.

(ii) The fact that there is a UV-cut-off enters our triviality argument via (4.10). When removing the UV-cut-off at least in dimension $d+1 = 2$, we therefore have to modify the triviality argument. It turns out that the bound obtained from the hypercontractivity estimate for the UV-problem is not good enough to reproduce the above argument. It seems to be necessary to combine UV and IR-hypercontractivity bounds in a single estimate in order to obtain triviality without cut-offs in $d + 1$ dimensions. We will come back to this point elsewhere.

(iii) The mass bound to us rather seems to be a technical consequence of the methods used and not so much a true necessity for the onset of triviality. Different methods, e.g. based on decoupling via Dirichlet- and Neumann boundary conditions on a partition of $H^{d+1}$ e.g. combined with large deviation methods might very well lead to less restrictive mass bounds or eliminate them completely.

On a heuristic level, the problem that expectation and variance of the shifted potential and the non shifted potential will have different scalings under the limit $\Lambda \nearrow \mathbb{H}^{d+1}$ prevails for a large class of polynomial and non-polynomial interactions with and without cut-offs. Thus, in our eyes, the three assumptions (i)–(iii) are not essential but rather technical. The result above therefore should be taken rather as an example of what can happen in the AdS/CFT correspondence than a definite mathematical statement. Of course, at the present and very preliminary state of the affair, everybody is free to think differently.
5. Conclusions and Outlook

In this section we give an essentially non-technical discussion on repair strategies that would cure the obstacle of triviality.

(i) coupling constant renormalization: The simplest way to deal with the divergences in the potential energy $V(z_0, f)$ would be to make $\lambda$ a $z_0$-dependent quantity. In fact, a naive guess at the scaling behavior suggests that $\lambda(z_0) \sim z_0^{d+4(\Delta_+ - d)}$ would compensate for the increase in the expected value of the interaction energy $\mathbb{E}[V(z_0, f)] = \lambda C \int f^4 dx$ converges to a constant with $C = (\gamma^+ \Gamma(\Delta_+ - d/2)\Gamma(d/2)/2\Gamma(\Delta_+))^4$, cf. (4.1) and the paragraph thereafter. Furthermore, one can expect that the subleading terms ($j = 1 \ldots 4$ in (4.4)) converge to zero and do not affect the generating functional. It thus seems reasonable that with this renormalization the generating functional gives in the limit $z_0 \to 0$

$$C(f) = e^{\frac{1}{2} \alpha_+ (f, f) - \lambda C \int f^4 dx}$$

which is reflection positive as a limit of reflection positive functionals (it is manifestly not stochastically positive for all $\lambda > 0$ and hence gives a nice illustration for the destruction of stochastic positivity due to the correction term in (3.6) and (3.7)). The problem with this functional however is that the additional term in the interaction is an ultra local term and hence does not influence the corresponding real time CFT – which is a free theory determined by the analytic continuation of $\alpha_+$. Hence this sort of renormalization only trades in another kind of triviality for the triviality observed in Section 4.

(ii) bulk counterterms: Such terms can simply be added to the (formal) Lagrangian. The problem to use this method in the AdS/CFT correspondence is twofold: Firstly, the infra-red divergences that are occurring in $V(z_0, f)$ are $f$-dependent. If we however want to cure them with $f$-dependent counterterms, the renormalization description of $C_{\Lambda(z_0)}$ becomes $f$-dependent. Bulk counterterms however only preserve the structural properties of stochastic and reflection positivity, if the same renormalization prescription is chosen for all $f$. Hence, $f$-dependent counterterms would lead to a limiting functional, for which it is not known, whether it is reflection positive or not. The situation is worsened from the observation that, unlike in other IR problems, in the AdS/CFT correspondence the divergences in the nominator and denominator scale differently - as seen in our triviality result. This means for bulk counterterms, that, if they are working out fine for the nominator, they probably create new divergences in the denominator. Different renormalizations for the potential in the nominator and in the denominator in the limit might lead to a non normalizable vacuum for the boundary theory, which does not make sense.

(iii) boundary counterterms: The problems described above for bulk counterterms also have to be taken into account for boundary counterterms. Furthermore, while bulk counterterms, at least if they are not $f$-dependent, do not spoil
the conformal invariance of the boundary theory, boundary counterterms theoretically might do so. Hence one needs a separate argument to show that they don’t. But there is still another problem with boundary counterterms. We have seen that we cannot take it for granted that a limiting functional measure exists for the boundary theory. But if the boundary theory is not described by a functional integral $\mu_{bd}$, it is not clear how to define boundary counterterms on a mathematical basis: recall that a counterterm (at a finite value of the cut-off $z_0$ is defined by $d\mu_{bd, ren, z_0}(\varphi) = e^{-L_{ren}(z_0, \varphi)} d\mu_{bd}(\varphi)/\int_{D(\mathbb{R}^d)} e^{-L_{ren}(z_0, \varphi')} d\mu_{bd}(\varphi')$ and it is not obvious how this can be defined if $\mu_{bd}$ is not a measure.

(iv) giving up generating functionals: The triviality result of Section 4 relied on the scaling behavior of the expected value of $V(z_0, f)$ under the limit $z_0 \to 0$. This expected value can be associated with the Witten graph $\otimes$ which gives rise to the first order contribution to the four point function $\int_{\mathbb{R}^{d+1}} \prod_{l=1}^4 H_+(z, f_l) d^d z$ which is converging as long as $\text{supp} f_j \cap \text{supp} f_l = \emptyset$ if $j \neq l$, cf. [2.2] and [4.1] (see also [11] for concrete calculations). One may thus hope that the triviality result of Section 4 is an artefact of using generating functionals which makes it necessary to evaluate Schwinger functions at unphysical coinciding points. A reasonable approach to the infra-red problem in AdS/CFT would thus be to use (3.7) to define reflection positive Schwinger functions with cut off and then remove the cut-off for the Schwinger functions at physical (non coinciding) points, only. This might then work out without further renormalization along the lines of [5], as divergences might only occur on the diagonal. If this is true, triviality does only occur on the level of generating functionals – which are reminiscent of the Laplace transform of a functional measure for the boundary theory that might not exist in the present context.

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