The Radius Rigidity Theorem for Manifolds of Positive Curvature

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Abstract

Recall that the radius of a compact metric space \((X, \text{dist})\) is given by
\[
\text{rad} X = \min_{x \in X} \max_{y \in X} \text{dist}(x, y).
\]
In this paper we generalize Berger’s \(\frac{1}{4}\)-pinched rigidity theorem and show that a closed, simply connected, Riemannian manifold with sectional curvature \(\geq 1\) and radius \(\geq \frac{\pi}{2}\) is either homeomorphic to the sphere or isometric to a compact rank one symmetric space.

The classical sphere theorem states that a complete, simply connected Riemannian \(n\)-manifold with positive, strictly \(1/4\)-pinched sectional curvature is homeomorphic to \(S^n\) ([Ber1], [K], and [Rch]). The weakly \(1/4\)-pinched case is covered by

**Berger’s Rigidity Theorem** Let \(M\) be a complete, simply connected Riemannian \(n\)-manifold with sectional curvature, \(1 \leq \sec M \leq 4\). Then either

(i) \(M\) is homeomorphic to \(S^n\), or

(ii) \(M\) is isometric to a compact rank one symmetric space.

([Ber2])

The hypotheses of Berger’s Theorem imply (with a lot of work) that the injectivity radius of \(M\) satisfies \(\text{inj} M \geq \frac{\pi}{2}\) ([CG2] or [KS]). The diameter

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therefore, also satisfies \( \text{diam } M \geq \pi/2 \), and the class of complete Riemannian manifolds with

\[
\sec \geq 1 \text{ and } \text{diam } \geq \pi/2
\]

contains Berger’s class. The former class is in fact, much vaster, since it contains, for example, metrics with arbitrarily small volume (see [Ber3] and Example 2.4 in [GP1]).

On the other hand, the set of smooth manifolds admitting metrics satisfying (*) is nearly the same as for Berger’s class. Indeed, in [GG1] Gromoll and Grove extended Berger’s Rigidity Theorem and the Diameter Sphere Theorem ([GS]) in proving the

**Diameter Rigidity Theorem:** Let \( M \) be a complete, simply connected Riemannian \( n \)-manifold with sectional curvature \( \sec M \geq 1 \) and diameter \( \text{diam } M \geq \pi/2 \). Then either

(i) \( M \) is homeomorphic to \( S^n \),

(ii) \( M \) is isometric to a compact rank one symmetric space, or

(iii) \( M \) has the cohomology algebra of the Cayley Plane, \( \text{CaP}^2 \).

An open question regarding this theorem is whether the possibility (iii) can be removed from the conclusion. This seems to be a very difficult problem; however, there is a natural hypothesis that falls between those of the two rigidity theorems. Observe that the hypothesis \( \text{inj } M \geq \pi/2 \) (which is satisfied by Berger’s class) implies that given any point \( x \in M \), there is a point \( y \in M \) so that \( \text{dist}(x,y) \geq \pi/2 \). This later condition can be expressed succinctly in terms of a well known metric invariant called the radius.

**Definition 1 (Radius)** Let \((X, \text{dist})\) be a compact metric space. The radius of \( X \) is given by,

\[
\text{rad } X = \min_{x \in X} \max_{y \in X} \text{dist}(x,y).
\]

(The concept of radius was invented in [SY]. The name radius was first used in [GP2].)

Clearly \( \text{inj } M \geq \pi/2 \Rightarrow \text{rad } M \geq \pi/2 \Rightarrow \text{diam } M \geq \pi/2 \), suggesting the following generalization of Berger’s Rigidity Theorem.
Radius Rigidity Theorem: Let $M$ be a complete, simply connected Riemannian $n$-manifold with sectional curvature $\sec M \geq 1$ and radius $\rad M \geq \pi/2$. Then either

(i) $M$ is homeomorphic to $S^n$, or
(ii) $M$ is isometric to a compact rank one symmetric space.

A crucial step in the proof of the Diameter Rigidity Theorem is to show that if $M$ is not homeomorphic to $S^n$, then there are certain points $x$ whose unit tangent sphere is mapped via $v \mapsto \exp_x \frac{\pi}{2} v$ onto the cut locus of $x$, and that this map is a Riemannian submersion with connected fibers. Since the unit tangent sphere is isometric to the unit sphere $S^n \subset \mathbb{R}^{n+1}$, the classification theorem from [GG2] can be invoked. It states that up to isometric equivalence the only Riemannian submersions of Euclidean spheres (with connected fibers) are the Hopf fibrations, except possibly for fibrations of the 15-sphere by homotopy 7-spheres. It was shown in [GG1] that if the exception could be removed from the submersion theorem in [GG2], then (iii) can be removed from the statement of the Diameter Rigidity Theorem (see Remark 4.4 in [GG1]). Although we have not been able to remove the exception from the submersion classification, we have proved the following.

Main Lemma 2 Let $S^n(r)$ denote $\{ v \in \mathbb{R}^{n+1} \mid \|v\| = r \}$. Let $\Pi : S^{15}(1) \to V$ be a Riemannian submersion with connected, 7-dimensional fibers, and let $G$ be the set of points $v \in V$ so that $\Pi^{-1}(v)$ is totally geodesic.

Then either $G$ is discrete or $G$ is a totally geodesic and isometrically embedded copy of $S^l(\frac{1}{2})$ for some $1 \leq l \leq 8$.

Moreover, in case $G$ is discrete, either

$$\dist(x, y) = \frac{k\pi}{q}$$

for some $k, q \in \mathbb{N}$ so that $q \equiv 0 \mod 4$, or

$$\dist(x, y) = \frac{\pi}{2}$$

for all $x, y \in G$.

The Riemannian manifolds with

$\sec M \geq 1$, $\diam M \geq \frac{\pi}{2}$, and nontrivial fundamental group \textit{(***)}
were completely classified in [GG1]. Naturally, the class with $\sec M \geq 1$, $\rad M \geq \frac{\pi}{2}$, and nontrivial fundamental group is contained in (**). It is not difficult to prove that this containment is proper.

**Theorem 4** Let $M$ be a closed, Riemannian $n$-manifold with sectional curvature $\sec M \geq 1$, radius $\rad M \geq \frac{\pi}{2}$, and nontrivial fundamental group $\Gamma$. Then either

(i) The universal cover $\tilde{M}$ of $M$ is isometric to $S^n(1)$, and every orbit of the action of $\Gamma$ is contained in a proper invariant totally geodesic subsphere, or

(ii) For some $d \geq 2$, $M$ is isometric to the $\mathbb{Z}_2$-quotient of $CP^{2d-1}$ given by the involution

$$[z_1, z_2, \ldots, z_{2d}] \mapsto [\bar{z}_{d+1}, \ldots, \bar{z}_{2d}, -\bar{z}_1, \ldots, -\bar{z}_d]$$

in homogeneous coordinates on $CP^{2d-1}$.

Moreover, all such spaces have $\sec M \geq 1$ and $\rad M \geq \frac{\pi}{2}$.

Recall ([W]) that a representation $\rho : \Gamma \rightarrow O(n+1)$ is called fixed point free if and only if $S^n(1)/\rho(\Gamma)$ is a space form.

The actions of the groups in (i) are necessarily reducible; however, it is not immediately apparent (at least to the author) exactly which (reducible) space forms satisfy the conclusion of (i). As a partial answer we will prove

**Theorem 5** Let $\rho : \Gamma \rightarrow O(n+1)$ be a fixed point free representation that decomposes as a direct sum

$$\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

of $k \geq 2$ irreducible representations.

(i) A necessary condition for $S^n(1)/\rho(\Gamma)$ to have radius $= \frac{\pi}{2}$ is that $\rho_i$ be equivalent to $\rho_j$ for some $i \neq j$.

(ii) In case $\Gamma$ is abelian (i) is also a sufficient condition.

(iii) If $\Gamma$ is not abelian and $\tilde{\rho} : \Gamma \rightarrow O(d)$ is a fixed point free, irreducible representation, then $\rad S^{2d-1}(1)_{(\tilde{\rho} \oplus \rho)(\Gamma)} < \frac{\pi}{2}$. 

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(iv) If \( \text{rad } S^n(1)/\rho(\Gamma) = \frac{\pi}{2} \) and \( \sigma : \Gamma \to O(d) \) is another fixed point free representation of \( \Gamma \), then \( \text{rad } S^{n+d(1)}(\rho \circ \sigma)(\Gamma) = \frac{\pi}{2} \).

(v) There is a \( k_0 \) (depending on \( \Gamma \)) so that if \( k \geq k_0 \), then \( \text{rad } S^n(1)/\rho(\Gamma) = \frac{\pi}{2} \).

Given a smooth manifold \( M \), the tangent and unit tangent bundles of \( M \) will be denoted by \( TM \) and \( SM \) respectively. If \( V \subset M \) is a smooth submanifold, then the normal bundle of \( V \) in \( M \) will be denoted by \( NV \). When there is no possibility of confusion we denote \( S^n(1) \) by \( S^n \).

For simplicity we abbreviate compact rank one symmetric space as \( \text{CROSS} \). All geodesics will be parametrized by arc length on \([0, \cdot]\) unless otherwise indicated.

The remainder of the paper is divided into four sections and an appendix. The first two sections contain the proof of the main lemma and a review of certain material from \([GG1]\). The Radius Rigidity Theorem is proved in section 3, and Theorems 4 and 5 are proven in section 4. In the appendix, we give the proof of an inequality that is used in the proof of the main lemma.

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1 Reflecting Good Points

First we review some basic facts about Riemannian submersions. Recall that if \( \pi : M \to B \) is a Riemannian submersion, then the vectors tangent to the fibers are called vertical vectors and the vectors perpendicular to the fibers are called horizontal vectors. We denote these two subbundles of \( TM \) by \( VM \) and \( HM \) respectively.

The fundamental tensors of a submersion were defined in \([O]\) as follows.
For arbitrary vector fields $E$ and $F$ on $M$ the tensor $T$ is defined by
\[ T_E F = (\nabla_{E^v} F^v)^h + (\nabla_{E^v} F^h)^v, \]
where the superscripts $h$ and $v$ denote the horizontal and vertical parts of the vectors in question. Note that the first summand is the second fundamental form of a fiber applied to $E^v$ and $F^v$, and the second term is the shape operator of a fiber applied to $E^v$ and $F^h$.

The other fundamental tensor, $A$, is obtained by dualizing $T$, that is, by switching all horizontal and vertical parts in the definition of $T$. Thus
\[ A_E F = (\nabla_{E^h} F^h)^v + (\nabla_{E^h} F^v)^h. \]

It is shown by O’Neill in [O], that all of the sectional curvatures of $M$ can be written in terms of $A$, $T$, $\nabla A$, $\nabla T$, the sectional curvatures of $B$, and the intrinsic sectional curvatures of the fibers. In particular, he proves that if $X$ and $Y$ are orthonormal horizontal vector fields and $V$ is a unit vertical field, then

**Horizontal Curvature Equation**
\[ K(X, Y) = K(d\pi X, d\pi Y) - 3\|A_X Y\|^2, \quad \text{and} \quad (6) \]

**Vertizontal Curvature Equation**
\[ K(X, V) = \langle(\nabla_X T)V, X \rangle + \|A_X V\|^2 - \|T_V X\|^2. \quad (7) \]

We refer the reader to [O] for the statements and proofs of the basic facts about $T$ and $A$ and other basic facts and definitions about Riemannian submersions that we will use freely and without further mention.

Now we begin the proof of the main lemma. Let $\Pi$, $V$, and $G$ be as in Main Lemma 2. We will call the members of $G$ “good points”.

**Lemma 8**

(i) If $x \in G$, then there is a unique point $a(x) \in V$ at maximal distance from $x$, $\text{dist}(x, a(x)) = \frac{\pi}{2}$, and $a(x)$ is also in $G$.

(ii) $V$ is Wiedersehen at $x$ and $a(x)$, i.e. the cut locus of $x$ is $a(x)$, and the cut locus of $a(x)$ is $x$. Furthermore, the fibers of $\Pi$ are invariant under the antipodal map, and every geodesic in $V$ is periodic with period $\pi$. 

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Remark: Gromoll and Grove have proven independently that the fibers of any Riemannian submersion with connected fibers of the 15-sphere are invariant under the antipodal map (even ones with $G = \emptyset$) ([GG3]).

Proof: First we review the notion of Holonomy Displacement ([H], p. 238, also [GG2], p. 150). Let $\gamma$ be a geodesic in $V$. If we consider all of the horizontal lifts of $\gamma$ to $S^{15}$, then we obtain a family of diffeomorphisms, $\psi_{s,t} : \Pi^{-1}(\gamma(s)) \rightarrow \Pi^{-1}(\gamma(t))$ given by $\psi_{s,t}(z) = \gamma_z(t)$, where $\gamma_z$ denotes the unique horizontal lift of $\gamma$ with $\gamma_z(s) = z$.

Now suppose that $F_x \equiv \Pi^{-1}(x)$ is totally geodesic. Then all horizontal geodesics emanating from $F_x$ are in a totally geodesic 7-sphere $F_{a(x)}$ at time $\pi/2$. Hence $F_{a(x)}$ is also a fiber of $\Pi$, and $\Pi(F_{a(x)})$ is the desired point $a(x)$. This proves (i).

Since every horizontal geodesic emanating from $F_x$ reaches $F_{a(x)}$ at time $\pi/2$, every geodesic emanating from $x$ reaches $a(x)$ at time $\pi/2$, and hence is minimal up to time $\pi/2$. Thus $V$ is Wiedersehen at $x$ and by symmetry at $a(x)$.

It follows that reflection in $x$ is a homeomorphism of $V$. Hence reflection in $F_x$ is an isometry of $S^{15}$ that maps fibers to fibers. Similarly, reflection in $F_{a(x)}$ maps fibers to fibers. But the composition of the two reflections is the antipodal map, $a$, of $S^{15}$. So if we knew that the composition of reflection in $x$ with reflection in $a(x)$ were the identity map of $V$, then we know that the fibers are invariant under the antipodal map.

To establish this, let $r_x$, $r_{a(x)}$, $r_{F_x}$ and $r_{F_{a(x)}}$ be the four reflections. Note that $r_{F_x} \circ r_{a(x)} \circ r_x = a \circ a = id_{S^{15}}$. So $r_x \circ r_{a(x)} \circ r_{a(x)} \circ r_x = id_{V}$. The differential $d(r_x \circ r_{a(x)})_{a(x)}$ is therefore a linear isometry of $T_{a(x)} V$ whose square is the identity, and hence $T_{a(x)} V$ has a basis of eigenvectors for $d(r_x \circ r_{a(x)})_{a(x)}$ with corresponding eigenvalues of either 1 or $-1$. Suppose $v$ is an eigenvector whose eigenvalue is $-1$. Then $-v = d(r_x \circ r_{a(x)})_{a(x)} v = d(r_x)_{a(x)} v - v$. This implies that the reflection isometry $r_x$ fixes the geodesic $\gamma_{-v} : t \mapsto \exp_{a(x)} (-tv)$, which is absurd, since $\gamma_{-v}(\frac{\pi}{2}) = x$. So the only possible eigenvalue is 1, and we can conclude that the fibers are indeed invariant under the antipodal map.

The invariance of the fibers under the antipodal map implies immediately that every geodesic in $V$ is periodic with period $\pi$. □

We saw in the proof above that reflection in a totally geodesic fiber is an isometry of $S^{15}$ that preserves the fibers of $\Pi$. By using this fact over and over again, we can prove
Lemma 9 Let $x, a(x) \in V$ be good points at maximal distance. Let $z \in V \setminus \{x, a(x)\}$ be another good point, and let $\gamma : [0, \infty) \to V$ be the unique geodesic that passes through $x, z$, and then $a(x)$ so that $\gamma(0) = x$. Then $\gamma(k \cdot \text{dist}(x, z))$ is a good point for all integers $k$. In particular, if $\text{dist}(x, z)$ is an irrational multiple of $\pi$, then all points along $\gamma$ are good.

If $G$ has an accumulation point, then using Lemmas 8 and 9 and the fact that $G$ is closed, we see that $G$ contains the image of an entire periodic geodesic of length $\pi$. Thus in the indiscrete case it is enough to prove the following corollary of the main lemma.

Corollary 10 If $V_l \subset G_V$ is totally geodesic in $V$ and isometric to $S^{l(\frac{1}{2})}$ for some $l \geq 1$, and if there is a good point $x \in V \setminus V_l$, then there is a totally geodesic set of good points, $V_{l+1} \subset V$, that contains $V_l \cup \{x\}$ and is isometric to $S^{l+1}(\frac{1}{2})$.

We will focus on the proof of (10) for nearly all of the remainder of this section.

Note that $x \in G_V \setminus V_l$ implies $a(x) \in G_V \setminus V_l$. So by Lemma 8, $\text{dist}(x, \cdot)$ is smooth along $V_l$. If $\text{dist}(x, \cdot)|_{V_l}$ is not constant, there is therefore some open set $O \subset V_l$ for which the subset $I_O = \{q \in O \mid \text{dist}(x, q) \text{ is an irrational multiple of } \pi\}$ is a dense $G_\delta$. By Lemma 4, the geodesics between $x$ and the points in $I(O)$ consist of good points. By continuity, then, all geodesics between $x$ and points of $O$ consist of good points. Since $x$ is a good point and $a(x) \notin V_l$, there is a unique minimal geodesic between $x$ and every point in $O$. Let $C(O)$ denote the union of these geodesics. Then $C(O) \setminus \{x, a(x)\}$ is a smooth, $(l + 1)$-dimensional submanifold of $V$ composed entirely of good points.

Now consider a point $y \in O$. Let $V^{l+1}$ be the image of the set of geodesics emanating from $y$ which are initially tangent to $C(0)$. It follows from Lemma 8 that $V^{l+1}$ consists entirely of good points, and by Lemma 8 it is homeomorphic to $S^{l+1}$. To help understand the infinitesimal geometry of $V^{l+1}$ we prove the following.

Lemma 11 If $\gamma : [0, \pi] \to V$ is a geodesic whose image consists entirely of good points, then all radial sectional curvatures of $V$ along $\gamma$ are constant and equal to $4$. 
Proof: Let \( \tilde{\gamma} \) be a horizontal lift of \( \gamma \) to \( S^{15} \). Let \( X \) and \( \tilde{X} \) denote the tangent fields to \( \gamma \) and \( \tilde{\gamma} \) respectively, and let \( V \) be a vertical unit field along \( \tilde{\gamma} \) so that \( (\nabla_X V)^v = 0 \). Then, using the equation for the vertizontal curvatures, we find that \( K(\tilde{X}, V) \) along \( \tilde{\gamma} \) is,

\[
1 \equiv K(\tilde{X}, V) = \langle (\nabla_{\tilde{X}} T) V, \tilde{X} \rangle + \|A_{\tilde{X}} V\|^2 - \|T_V \tilde{X}\|^2 = \|A_{\tilde{X}} V\|^2.
\]

(12)

It follows from (12) that the map \( v \mapsto A_{\tilde{X}} v \) from \( VS^{15} \) to \( HS^{15} \cap \tilde{X} \perp \) is bijective. (13)

Combining this and (12) we can show that \( \|A_{\tilde{X}} y\| \equiv 1 \) for all unit vectors \( y \in HS^{15} \cap \tilde{X} \perp \). (14)

Indeed, if \( \langle A_{\tilde{X}} y, v \rangle \) were bigger than 1 for some unit vector \( v \in VS^{15} \), then we would have \( |\langle A_{\tilde{X}} y, v \rangle| = \|v, A_{\tilde{X}} y\| > 1 \), contrary to (12). On the other hand, by (12) and (13), \( y = A_{\tilde{X}} v \) for some unit vector \( v \in VS^{15} \), thus \( |\langle A_{\tilde{X}} y, v \rangle| = \|y, A_{\tilde{X}} v\| = 1 \), and \( \|A_{\tilde{X}} y\| \geq 1 \) as well.

Let \( Y \) be a unit field along \( \gamma \) that is perpendicular to \( X \), and let \( \tilde{Y} \) denote the horizontal lift of \( Y \). Then it follows from (14) that

\[
K(X, Y) = K(\tilde{X}, \tilde{Y}) + 3\|A_{\tilde{X}} \tilde{Y}\|^2 = 1 + 3 \times 1 = 4.
\]

□

The proof of Corollary 10 in case \( dist(x, \cdot)|_{V^i} \) is not constant is completed by applying the following result to \( V^{i+1} \).

Lemma 15 Let \( P \subset T_y V \) be a \( k \)-dimensional subspace of a tangent fiber of \( V \) such that \( W \equiv \exp_y P \) is a subset of \( G_V \). Then \( W \) is totally geodesic and isometric to \( S^k(\frac{1}{2}) \).

Proof: Let \( \iota : T_y V \longrightarrow T_z S^k(\frac{1}{2}) \) be a linear isometry. It follows from Lemma 11 that the differentials of \( \exp_z \circ \iota \circ \exp_y^{-1} \) are isometries along \( V^{i+1} \), and it follows from Lemma 8 that \( \exp_z \circ \iota \circ \exp_y^{-1} \) is an embedding whose image is isometric to \( S^k(\frac{1}{2}) \). □

To prove Corollary 11 it remains to consider the case when the restriction of \( dist(x, \cdot) \) to \( V^i \) is constant. In this case it turns out that

\[
dist(x, \cdot)|_{V^i} \equiv \pi/4.
\]

(16)
To see this, first note that for \( z \in V^l \), \( \pi/2 = \text{dist}(z,a(z)) \leq \text{dist}(z,x) + \text{dist}(x,a(z)) = 2\text{dist}(z,x) \). On the other hand \( d_{\text{Haus}}(S^{15}, \pi^{-1}(V^l)) \leq \pi/4 \), so \( \text{dist}(x, V^l) \leq \pi/4 \).

By combining (16) with Lemma 8, we see that all geodesics from \( x \) that pass through \( V^l \) go through good points at times 0, \( \pi/4 \), \( \pi/2 \), and \( 3\pi/4 \). For the rest of this section we will study geodesics with this property. Our goal will be to prove that all of the points along such a geodesic are good. The first step is to estimate the average Ricci curvature along \( \gamma \).

**Lemma 17** Let \( \gamma : [0, \pi] \longrightarrow V \) be a geodesic so that the points \( \gamma(0), \gamma(\pi/4), \gamma(\pi/2), \) and \( \gamma(3\pi/4) \) are good, and let \( \{E_i\}_{i=1}^7 \) be an orthonormal collection of parallel unit normal fields along \( \gamma \). Then

\[
\int_0^\pi \text{Ric}(\dot{\gamma}, \dot{\gamma}) \leq 28\pi, \quad \text{and} \\
\int_0^\pi \sum_{i=1}^7 \|A_\gamma \tilde{E}_i\|^2 \leq 7\pi,
\]

where \( \tilde{\gamma} \) and \( \tilde{E}_i \) denote the horizontal lifts of \( \dot{\gamma} \) and \( E_i \) respectively.

**Proof:** Inequality (19) is clearly a consequence of (18), the equation for horizontal curvatures, and the fact the the curvature of \( S^{15} \) is identically 1. So it suffices to prove (18).

By Lemma 8 there are no conjugate points along \( \gamma \) prior to time \( \pi/2 \). Using this and a “Bonnet-Meyers” type of argument we can show,

\[
\int_0^{\pi/2} K(\dot{\gamma}, E_i) \sin^2 2t \leq 4 \int_0^{\pi/2} \sin^2 2t
\]

for all \( i = 1, \ldots, 7 \).

Suppose (21) is false. Then for some \( l < \frac{\pi}{2} \) and \( i = 1, \ldots, 7 \),

\[
\int_0^l \sin^2 \frac{\pi t}{l} K(\dot{\gamma}, E_i) > \frac{\pi^2}{l^2} \int_0^l \sin^2 \frac{\pi t}{l}.
\]

Now set \( W_i = \sin(\frac{\pi t}{l})E_i(t) \), and compute the index:

\[
I(W_i, W_i) = \int_0^l \langle \nabla_\gamma \sin(\frac{\pi t}{l})E_i, \nabla_\gamma \sin(\frac{\pi t}{l})E_i \rangle - \sin^2(\frac{\pi t}{l})\langle R(E_i, \gamma)\dot{\gamma}, E_i \rangle dt =
\]
\[ \int_0^l -\langle \sin\left(\frac{\pi t}{l}\right)E_i, \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} \sin\left(\frac{\pi t}{l}\right)E_i \rangle - \sin^2\left(\frac{\pi t}{l}\right)\langle R(E_i, \dot{\gamma})\dot{\gamma}, E_i \rangle dt = \int_0^l \frac{\pi^2}{l^2} \sin^2\left(\frac{\pi t}{l}\right)\langle E_i, E_i \rangle - \sin^2\left(\frac{\pi t}{l}\right)\langle R(E_i, \dot{\gamma})\dot{\gamma}, E_i \rangle dt = \int_0^l \sin^2\left(\frac{\pi t}{l}\right)\left( \frac{\pi^2}{l^2} - \langle R(E_i, \dot{\gamma})\dot{\gamma}, E_i \rangle \right) dt \]

If (24) holds for some \( i = 1, \ldots, 7 \), it follows that \( I(W_i, W_i) < 0 \), implying that \( \gamma|_{[0,l]} \) is not minimal, a contradiction.

Applying the same argument to \( \gamma|_{[\pi/4,3\pi/4]} \), \( \gamma|_{[\pi/2,\pi]} \), and \( \gamma|_{[3\pi/4,5\pi/4]} \) shows

\[ \int_{\pi/4}^{3\pi/4} K(\dot{\gamma}, E_i) \cos^2 2t \leq 4 \int_{\pi/4}^{3\pi/4} \cos^2 2t, \]

\[ \int_{\pi/2}^{\pi} K(\dot{\gamma}, E_i) \sin^2 2t \leq 4 \int_{\pi/2}^{\pi} \sin^2 2t, \quad \text{and} \]

\[ \int_{3\pi/4}^{5\pi/4} K(\dot{\gamma}, E_i) \cos^2 2t \leq 4 \int_{3\pi/4}^{5\pi/4} \cos^2 2t \]

for all \( i = 1, \ldots, 7 \). Combining these with (20) and using the fact that \( \gamma \) is periodic with period \( \pi \) yields (18). \( \square \)

From (19) we get

**Lemma 22** Let \( \gamma \) be as in Lemma 17. Let \( \{v_i\}_{i=1}^7 \) be an orthonormal basis for \( V S^{15}\gamma(0) \), and let \( V_i \) be an extension of \( v_i \) to a vertical field such that \( (\nabla_{\dot{\gamma}} V_i)^n = 0 \). Then

\[ \int_\gamma \Sigma_{i=1}^7 \|A_{\dot{\gamma}} V_i\|^2 \leq 7\pi. \] (23)

The verification of Lemma 22 is a lengthly but rather routine exercise in linear algebra, so we defer it to the appendix.

The proof of Corollary 10 is completed by combining Lemmas 11 and 15 and Equation 16 with the following result.

**Lemma 24** If \( \gamma : [0, \pi] \to V \) is a geodesic so that the fibers \( p^{-1}(\gamma(0)) \), \( p^{-1}(\gamma(\pi/4)) \), \( p^{-1}(\gamma(\pi/2)) \), and \( p^{-1}(\gamma(3\pi/4)) \) are totally geodesic, then all of the fibers \( p^{-1}(\gamma(t)) \) are totally geodesic.
Proof: Let \( \{V_i\}_{i=1}^7 \) be as in the statement of (22). Averaging vertical curvatures along \( \tilde{\gamma} \) we find
\[
\int_0^\pi \Sigma_{i=1}^7 \langle R(V_i, \dot{\tilde{\gamma}}), \dot{\tilde{\gamma}}, V_i \rangle = \int_0^\pi \Sigma_{i=1}^7 \dot{\tilde{\gamma}} \langle T_{V_i} V_i, \dot{\tilde{\gamma}} \rangle + \|A_{\tilde{\gamma}} V_i\|^2 - \|T_{V_i} \dot{\tilde{\gamma}}\|^2 dt \tag{25}
\]
The first term on the right is equal to
\[
\Sigma_{i=1}^7 \langle T_{V_i} V_i, \dot{\tilde{\gamma}} \rangle |_0^\pi.
\]
All of these terms are zero, since \( \gamma(0) = \gamma(\pi) \) is a good point. Thus (25) becomes
\[
7\pi = \int_\gamma \Sigma_{i=1}^7 \langle R(V_i, \dot{\gamma}), \dot{\gamma}, V_i \rangle = \int_\gamma \Sigma_{i=1}^7 \|A_{\dot{\gamma}} V_i\|^2 - \|T_{V_i} \dot{\gamma}\|^2 dt.
\]
Combining this with (23) shows
\[
\int_\gamma \Sigma_{i=1}^7 \|T_{V_i} \dot{\gamma}\|^2 dt = 0.
\]
Thus \( T_{V_i} \dot{\gamma} \equiv 0 \), and hence
\[
T_v \dot{\gamma} = 0 \text{ for all } v \in VS^{15}|_{\dot{\gamma}}. \tag{26}
\]

It turns out that the condition (26) implies that all of the Holonomy Displacement maps for \( \gamma \) are isometries. This and the fact that the fiber \( \pi^{-1}(\gamma(0)) \) is totally geodesic yields the conclusion of Lemma 24.

So it remains to see that the above Holonomy Displacement maps, \( \psi_{s,t} \), are isometries. Consider a curve \( c : [0, l] \rightarrow \Pi^{-1}(\gamma(t_0)) \) with \( \|\dot{c}\| \equiv 1 \) and the variation \( W(s, t) \) of \( c \) that is given by \( W(s, t) = \psi_{t_0, t}(c(s)) \). The variation field of \( W \) along \( c \) is the horizontal lift of \( \dot{\gamma}(t_0) \). Denote it by \( \tilde{X} \). By the first variation formula, we have
\[
\frac{d}{dt} \text{Length}[W(\cdot, t)] = \langle \tilde{X}, \dot{c} \rangle^0_t - \int_0^t \langle \tilde{X}, \nabla_v \dot{c} \rangle = \int_0^t \langle T_{\dot{c}} \tilde{X}, \dot{c} \rangle = 0.
\]
The second equality is due to the properties of \( T \) and the facts that \( \tilde{X} \) is horizontal and \( \dot{c} \) is vertical. The last equality follows from (26). \( \square \)

To complete the proof of the main lemma, we note that if \( G \) is discrete, then by Lemmas 8, 9, and 24, the equations in (3) hold.
2 Review of the Diameter Rigidity Theorem

If $M$ satisfies the hypotheses of the Radius Rigidity Theorem, then $M$ also satisfies the hypotheses of the Diameter Rigidity Theorem, so the only way $M$ can fail to satisfy the conclusion of the Radius Rigidity Theorem is if it has the cohomology algebra of $CaP^2$. We assume throughout sections 2 and 3 that $\sec M \geq 1$, $\text{Rad} M \geq \pi/2$, $\pi_1(M) = \{e\}$, and $H^*(M) \cong H^*(CaP^2)$, and we attempt to show that $M$ is isometric to $CaP^2$.

By the Diameter Sphere Theorem ([GS]), $\text{diam } M = \pi/2$. We would like to focus on this property for awhile; so let $N$ be a Riemannian $n$-manifold with $\sec N \geq 1$, $\pi_1(N) \cong \{e\}$, and $\text{diam } N = \pi/2$, that is not homeomorphic to $S^n$. Many basic aspects of the geometry of $N$ can be described in terms of so called dual sets ([GG1]). (Cf also [Sa], [Sh], and [SS].)

Definition 27 (Dual Sets) For any subset $B \subset N$, the dual set of $B$ is,

$$B' = \{x \in N \mid \text{dist}(x, B) = \pi/2\}$$

The following properties of dual sets were observed in [GG1] (cf also [Sa], [Sh], and [SS]).

(i) $B'$ is totally $\pi$-convex, that is, any geodesic of length strictly less than $\pi$ whose end points lie in $B'$ lies entirely in $B'$.

(ii) $B \subset B''$.

(iii) If $A \subset B$, then $A' \supset B'$.

(iv) $B' = B''$.

It follows from (i) and [CG1] that $B$ is a topological manifold with (possibly empty) boundary and smooth, totally geodesic interior. If we start with a set $B$ so that $B' \neq \emptyset$ and set $A = B'$, then $A = (A')'$. Thus

$$A = \{x \in N \mid \text{dist}(x, A') = \pi/2\}$$

and $A' = \{x \in N \mid \text{dist}(x, A) = \pi/2\}$, and $A$ and $A'$ are called a dual pair.

The proof in [GG1] proceeds from this point to use comparison theory and other geometric and topological tools to argue that the geometry of $N$ is more and more like the geometry of a CROSS. For example, it is shown
that $\partial A = \partial A' = \emptyset$, that $\text{cutlocus}(A) = A'$ and $\text{cutlocus}(A') = A$, and that for any $p \in A$ the map $\Pi_p : U N A_p \to A'$ from the unit normal sphere to $A$ at $p$ to $A'$ given by $\Pi_p(u) = \exp(\frac{\pi}{2} u)$ is a Riemannian submersion with connected fibers. This allows them to apply the classification theorem in [GG2] and conclude that $\Pi_p$ is isometrically equivalent to a Hopf fibration (except possibly if the fibers are 7-dimensional). The proof is completed with further comparison arguments. The exception to the conclusion is accounted for by the fact that the classification in [GG2] is not quite complete. It leaves open the possibility of nonstandard Riemannian submersions of the 15-sphere by homotopy 7-spheres. On the other hand, using arguments from [GG1] it is easy to prove that this is the only possible obstruction.

**Proposition 28** If $N$ has a dual pair $(A, A')$ such that one of the submersions $\Pi_p$ is isometrically equivalent to a Hopf fibration, then $N$ is isometric to a CROSS.

**Proof:** Say $p \in A$, and $\Pi_p$ is isometrically equivalent to a Hopf fibration. It was shown in [GG1] (p. 236) that it is enough to find a dual pair $\{q\}, \{q'\}$, where $\{q\}$ is a singleton and $\Pi_q$ is isometrically equivalent to a Hopf fibration. So we may assume that $A \neq \{p\}$. By the Diameter Rigidity Theorem, we may assume that $N$ has the properties of the possibly exceptional manifold $M$, on page 13, $\sec N \geq 1$, $\text{Rad } N \geq \pi/2$, $\pi_1(N) = \{e\}$, and $H^*(N) \cong H^*(\text{CaP}^2)$. So we can refer to $N$ as $M$. We also know that $A'$ is isometric to a CROSS, $P^m(K)$. It was observed in [GG1] (p. 236) that the dual set $B$ (in $A'$) of any singleton $\{x\} \subset A'$ is isometric to $P^{m-1}(K)$, and that the double dual of $\{x\}$ (in $M$) is again $\{x\}$. It follows from the convexity properties of $A'$ that the fibers of the submersion $SA'_x \to B$ are also fibers of the submersion $\Pi_x : SM_x \to \{x\}'$, and it follows from our simplifying assumptions that the dimension of these fibers is $< 7$. Therefore the submersion $\Pi_x$ is equivalent to a Hopf fibration and $M$ is isometric to a CROSS. □

We now restrict our attention to the possibly exceptional manifold $M$.

**Proposition 29**

(i) The set of dual pairs is a covering of $M$.

(ii) Every dual pair consists of a singleton and a set that is homeomorphic to $S^8$.

(iii) If $(p, V)$ and $(q, W)$ are distinct dual pairs, then $V \cap W$ is a point.
Proof: (i) is an immediate consequence of properties (ii) and (iv) of dual sets and the fact that \( \text{rad } M = \pi/2 \).

To prove (ii) first note that if \((p,V)\) is a dual pair and \(V\) is not 8-dimensional, then the Riemannian submersion \( \Pi_p : SM_p \rightarrow V \) is isometrically equivalent to a Hopf fibration; so \( M \) is isometric to a CROSS. If \( V \) is 8-dimensional, then the fibers of \( \Pi_p \) are homotopy 7-spheres (see Theorem 5.1 in [Br]). It follows from the long exact homotopy sequence of the fibration \( \Pi_p \) that \( V \) is a homotopy 8-sphere, and hence a topological 8-sphere. Finally, if there is a dual pair \((A,A')\) so that neither \(A\) nor \(A'\) is a point, then \(1 \leq \dim NS_p \leq 14\), and the submersion \( \Pi_p \) is equivalent to a Hopf fibration.

To prove (iii) observe that since \( \sec M > 0 \) and \( \dim V + \dim W = \dim M \), a Synge Theorem type of argument shows that \( V \cap W \neq \emptyset \) (see [F] and also Proposition 1.4 in [GG1]). Next observe that \( V \cap W = \{p,q\}' \), so \((V \cap W, (V \cap W)')\) is a dual pair. By (ii), one of these dual sets is a point. Since \( p,q \in (V \cap W)' \), we conclude that \((V \cap W)\) is a point. \(\square\)

If \((p,V)\) is a dual pair, then we will (optimistically) refer to \(V\) as a Cayley line. This name is partially justified by the fact that once we have proven that \(M\) is isometric to \(CaP^2\) we will know that all of these \(V\)'s are isometric to \(CaP^1\).

### 3 Intersecting Cayley Lines

In this section we prove the Radius Rigidity Theorem.

If \((p,V)\) is a dual pair, then we have seen that it is enough to show that the submersion \( \Pi_p : S_p \rightarrow V \) is isometrically equivalent to the Hopf fibration \( S^7 \hookrightarrow S^{15} \rightarrow S^8 \). This holds if its fibers are totally geodesic (see [Rj]). Roughly speaking, the strategy of our proof is to find dual pairs \((p,V)\) so that \( \Pi_p \) contains more and more totally geodesic fibers. Our method for finding totally geodesic fibers will be to find more and more “good points” in \(M\).

**Definition 30 (Good Point)** If \((p,V)\) is a dual pair, then we shall call a point \(x \in V\) good if and only if \(\Pi_p^{-1}(x)\) is totally geodesic. A point \(m \in M\) will be called good if and only if \(m \in G_W\) for some Cayley line \(W \subset M\). The sets of good points in \(V\) and \(M\) will be denoted by \(G_V\) and \(G_M\) respectively.
The fact that \( G_M \) is rather large is a consequence of Proposition 29 and the next result.

**Proposition 31** Let \((p,V)\) be a dual pair.

(i) A point \( x \in V \) is good if and only if there is a Cayley line \( W \) so that \( V \cap W = \{x\} \).

(ii) \( G_M \) is closed. In fact, \( m \in G_M \) if and only if there are points \( x, y \in M \) so that \( \text{dist}(x,m) = \text{dist}(m,y) = \text{dist}(x,y) = \frac{\pi}{2} \).

**Remark:** Gromoll and Grove proved independently that \( G_M \neq \emptyset \) ([GG3]).

**Proof:** If there is a dual pair \((z,W)\) so that \( W \cap V = \{x\} \), then \( p \) and \( z \) are distinct points in \( \{x\}' \), so \( \{x\}' \) is a Cayley line and hence intersects \( V \) in a single point \( y \). Since \( p \) and \( x \) are distinct points of \( \{y\}' \), \( \{y\}' \) is a Cayley line. It follows that the set of minimal geodesics from \( p \) to \( x \) is contained in \( \{y\}' \). Thus \( \Pi^{-1}_p(x) \) is contained in the unit tangent sphere \( S\{y\}'_p \) to \( \{y\}' \) at \( p \). But since both of these sets are homotopy 7-spheres they must coincide. compact

Since \( S\{y\}'_p \) is totally geodesic in \( SM_p \), \( \Pi^{-1}_p(x) \) is as well. This proves the “if” part of (i).

On the other hand, if \( x \in G_V \), then by Lemma 3 there is a unique point \( a(x) \in V \) so that \( \text{dist}(x,a(x)) = \frac{\pi}{2} \). Since \( x, p \in \{a(x)\}' \), \( \{a(x)\}' \) is a Cayley line. By Proposition 29, \( x = \{a(x)\}' \cap V \). This proves the “only if” part of (i). Since \( \text{dist}(x,a(x)) = \text{dist}(a(x),p) = \text{dist}(p,x) = \frac{\pi}{2} \) it also proves the “only if” part of (ii).

To prove the “if” part of (ii) note that \( x, y \in \{m\}', m, y \in \{x\}' \), and \( m, x \in \{y\}' \). So \( \{m\}', \{x\}' \), and \( \{y\}' \) are all Cayley lines, and \( x \), for example, is good since \( x = \{m\}' \cap \{y\}' \). □

The Radius Rigidity Theorem would follow if we could show that there is a Cayley line \( V \) so that every point in \( V \) is good. We will do this by finding Cayley lines with good points in sets that are isometric to spheres of constant curvature 4 of progressively higher and higher dimension. Since each point in \( M \) lies on at least one Cayley line, we can certainly find a countably infinite family of Cayley lines \( \{V_i\}_{i=1}^\infty \). Next we observe that there is a Cayley line \( W \) so that

\[
\text{card} \ \{W \cap V_i\}_{i=1}^\infty
\]

is infinite. Indeed if \( \text{card} \ \{W \cap V_i\}_{i=1}^\infty \) were finite, then there would be an infinite set \( \{V_{i_j}\}_{j=1}^\infty \) and a point \( x \) so that \( W \cap V_{i_j} = \{x\} \) for all \( j \). But then
\( \{x\}' \) is a Cayley line and the points \( \{x\}' \cap V_i \) must all be distinct. So we can find a Cayley line (let’s call it \( V \)) with infinitely many distinct good points. It follows that the set of good points in \( V \) contains an accumulation point and hence, using Lemma 9 and the fact that \( G_V \) is closed, the image of an entire geodesic.

To prove the Radius Rigidity Theorem we argue by contradiction. It follows from the main lemma that the set of good points in each Cayley line is either discrete, an entire geodesic, or a sphere of constant curvature 4. Let \( V \) be a Cayley line whose set of good points has maximal dimension, \( d \). We’ve seen that \( d \geq 1 \), and if the Radius Rigidity Theorem were false, then we would know \( d \leq 7 \). Consider the configuration \( C \) consisting of all Cayley lines of the following types:

**type 1** \( V \),

**type 2** All the Cayley lines between the good points of \( V \) and \( V' \),

**type 3** For each \( W \) of type 2 we also include all of the Cayley lines between each of the good points of \( W \) and \( W' \) that are neither of type 1 nor of type 2.

We point out that if \( W \) is of type 2, then \( W' \) is a good point of \( V \), and if \( U \) is a line of type 3 between a good point of \( W \) and \( W' \), then \( U' \) is a good point of \( W \).

Suppose we could find a Cayley line \( Z \) that is not included in the configuration above. Then either,

\[
Z \cap V \notin G_V \text{ or } \quad Z \cap V \in G_V
\]

But neither of these is possible. The first can not occur because \( G_V \) consists of all of the good points of \( V \). On the other hand, if \( Z \cap V \in G_V \), then \( (Z \cap V)' \) is a line of type 2, and \( Z \) is a Cayley line between \( (Z \cap V)' \) and \( Z \cap V \), implying that \( Z \) is of type 3, a contradiction. Therefore,

\[
C \text{ contains all of the lines of } M. \tag{32}
\]

Next we prove
Lemma 33 We may assume that \( \dim G_U = d \) for every line \( U \) in the configuration.

Proof: We prove this in a step by step manner.

We know that there is at least one line of type 3, since otherwise the configuration would only be 8-dimensional in a neighborhood of a bad point of \( V \) and hence would not cover \( M \). Since all of the lines of type 2 intersect at \( V' \), they must intersect at distinct points of each line of type 3. Thus \( \dim G_U \geq 1 \) for all lines of type 3, and if \( U \) is a line of type 3, then there are infinitely many lines between \( U \) and \( U' \in W_0 \), where \( W_0 \) is a line of type 2. Since all of these lines intersect at \( U' \) they must intersect each line of type 2 (other than \( W_0 \)) in infinitely many places. Therefore \( \dim G_W \geq 1 \) for all lines \( W \neq W_0 \) of type 2. Since the set of all good points in \( M \) is closed, \( \dim G_{W_0} \geq 1 \) as well.

For each point \( v \in G_V \), the set \( L_v = \{ \text{lines } U \text{ in the configuration } | \ v \in U \} \) can be topologized by declaring that it is homeomorphic to \( G_{\{v\}} \). We will show that for each \( v \in G_V \), \( L_v = \{ \text{lines } U \text{ in } L_v \ | \ \dim G_U = d \} \) is both closed and open. Since \( V \in L_v^d \) and \( \cup_{v \in G_v} L_v = C \), it will follow that \( \dim G_U = d \) for every line in the configuration. Let \( \{U_i\} \) be a sequence in \( L_v^d \) which is converging to a line \( U \) in \( L_v \). Then by passing to a subsequence if necessary, we may assume that \( \{G_{U_i}\} \) converges (in the classical Hausdorff topology) to some subset \( G \) of \( U \) (cf Theorem 4.2 in [Mi]). By the main lemma, \( G \) is isometric to \( S^d(\frac{1}{2}) \), and by Proposition [31] \( G \subset G_U \). In fact \( G = G_U \) by the maximality of \( d \). So \( L_v^d \) is closed. To see that it is open, let \( U \in L_v^d \) and let \( W \in L_v \) be close to \( U \). Consider the set \( L(U, U') \) of lines between \( U \) and \( U' \). Each \( u \in G_U \) is on exactly one line \( Z_u \in L(U, U') \), and the map \( G_U \rightarrow G_W \) given by \( u \mapsto Z_u \cap W \) preserves distances up to small additive error. It follows from the main lemma and the maximality of \( d \) that if \( W \) was originally chosen to be sufficiently close to \( U \), then \( G_W \) is isometric to \( S^d(\frac{1}{2}) \). □

Consider the following subset of \( TM \).

\[
TC|_V(\pi/2) = \{v \in TM|_{G_v} \ | \ \|v\| \leq \pi/2 \text{ and } v \text{ is tangent to a line in the configuration}\}.
\]

If the Radius Rigidity Theorem is false, then \( \exp : TC|_V(\frac{\pi}{2}) \rightarrow C \) is a surjective Lipschitz map, and the set of points in \( M \) whose inverse image is a singleton is an open and dense set. Indeed, \( \exp \) is surjective since the
configuration has to cover \( M \), and \( \exp \) has unique preimages on \( M \setminus G_M \). The set \( M \setminus G_M \) is open and dense, since \( G_M \) consists of points of the form \( U' \) where \( U \) is a line in the configuration, and the points of this form all lie in proper subspaces of lines of type 2.

The fact that \( \exp \) is surjective and Lipschitz yields a contradiction in case \( d \leq 3 \) since it implies that \( \dim_{\text{Haus}} M \leq \dim_{\text{Haus}} TC|_V(\pi/2) \leq 3+3+8 = 14 \).

The case \( 5 \leq d \leq 7 \) is also easy to eliminate since in this case \( \dim TC|_V(\varpi/2) \geq 5 + 5 + 8 > 16 = \dim M \). So it is impossible for \( \exp|_{TC|_V} \) to have unique preimages on an open dense set.

The case \( d = 4 \) is also not possible, but it is much harder to rule out. We will get a contradiction in this case by showing that there is \((S^4 \times S^8)\)-bundle \( E \) over \( S^4 \) and a degree 1 map from \( E \) to \( M \). To see that this is a contradiction, note that a spectral sequence argument shows that if \( E \) is any \((S^4 \times S^8)\)-bundle over \( S^4 \), then

\[
H^*(E) \cong H^*(S^4 \times S^8 \times S^4). \tag{33}
\]

Since \( H^*(M) \cong H^*(CaP^2) \), the existence of a degree 1 map \( E \to M \) implies that the fundamental cohomology class in \( E \) has a square root, and a simple algebraic argument combined with (33) shows that it does not.

**Proposition 34** If \( d = 4 \), then \( G_M \) is a totally geodesic submanifold of \( M \) that is isometric to \( HP^2 \) with its canonical metric with \( 1 \leq \sec HP^2 \leq 4 \).

**Proof:** For any line \( U \) in \( C \) we can let \( U \) play the role of \( V \) and define a configuration \( C_U \) consisting of lines of type \( 1_U, 2_U, \) and \( 3_U \) in a way analogous to what we did on page 17. Of course assertion (32) is valid for each \( C_U \), and for each such configuration \( C_U, G_M = \cup_W \) a line of type \( 2_U G_W \), since otherwise there would be a line not included in \( C_U \).

Now let \( u \) and \( w \) be two points in \( G_M \). Since \( w \) must lie on a line of type \( 2_{(u)} \), there is a Cayley line \( Z \) containing \( u \) and \( w \). Since \( G_Z \) is isometric to \( S^4(\varpi/2) \), we can find a geodesic in \( G_Z \) between \( u \) and \( w \). Using Lemma 8 and the fact that \( Z \) is totally \( \pi \)-convex we see that if \( \text{dist}(u, w) < \varpi \) then the geodesic constructed above is the unique geodesic in \( M \) between \( u \) and \( w \). This shows that \( G_M \) is totally \( \varpi \)-convex and hence, by [CG1], a topological manifold with boundary and smooth, totally geodesic interior. But the above construction also indicates that every geodesic in \( G_M \) can be indefinitely prolonged (to a geodesic in \( G_M \)). Therefore \( \partial G_M = \emptyset \). Thus \( G_M \) with its intrinsic metric is
a Riemannian manifold with sectional curvature $\geq 1$ and diameter $= \frac{\pi}{2}$. The proposition follows by analyzing the structure of the dual sets in $G_M$ and applying the classification theorem in $[GG1]$. □

To construct $E$, first let $E' = \{ v \in TC_V(\frac{\pi}{2}) \mid \|v\| = 1 \}$. Let $p_{E'} : E' \to G_V$ be the restriction of the projection map of $TC_V(\frac{\pi}{2})$. Given any $v \in G_V$, $p_{E'}^{-1}(v) = \Pi^{-1}_v(G(v)')$, where $\Pi : SM_v \to \{ v \}'$ is the Riemannian submersion discovered by Gromoll and Grove. Since $G(v)'$ is contractible in $\{ v \}'$, $\Pi_v|_{\Pi^{-1}_v(G(v)')}$ is trivial. This shows that that $E'$ is an $S^7 \times S^4$-bundle over $G_V$. The desired bundle $E$ will be obtained by suspending the “$S^7$ parts” of the fibers of $E'$. To help see that this can be done we prove

**Lemma 35** There is a bundle $S^4 \hookrightarrow Q \xrightarrow{p_Q} G_V$ and a bundle $S^7 \hookrightarrow E' \xrightarrow{p_{E',Q}} Q$ so that $p_{E'} = p_Q \circ p_{E',Q}$.

**Proof:** Let $P = \{ v \in NG_V \subset TG_M \mid \|v\| \leq \frac{\pi}{4} \}$, and let $Q$ be the double of $P$ (cf [Mu]). For convenience we distinguish between the two copies $P_1$ and $P_2$ of $P$ in $Q$ by setting

$$\|v\|_Q = \begin{cases} \|v\|_{P_1} & \text{if } v \in P_1 \\ -\|v\|_{P_2} + \frac{\pi}{2} & \text{if } v \in P_2 \end{cases}$$

For $i = 1, 2$, let $p_{P_i} : P_i \to G_V$ be the the projection map of $P_i$. By setting

$$p_Q(v) = \begin{cases} a \circ p_{P_1}(v) & \text{if } v \in P_1 \\ a \circ p_{P_2}(v) & \text{if } v \in P_2 \end{cases}$$

we see that $Q$ is an $S^4$-bundle over $G_V$. (Here $a$ is the antipodal map of $G_V$.)

We can even define an exponential map $\exp_Q : Q \to M$ by setting

$$\exp_Q v = \begin{cases} \exp_P \|v\|_Q \ x_v & \text{if } \|v\|_Q \neq 0, \frac{\pi}{2} \\ p_{P_1}(v) & \text{if } \|v\|_Q = 0 \\ V' & \text{if } \|v\|_Q = \frac{\pi}{2} \end{cases}.$$ 

Using the definition of $\| \cdot \|_Q$, Lemma [3], and the definition of double ([Mu]), it is easy to check that $\exp_Q$ is well defined even when $\|v\|_Q = \frac{\pi}{4}$ and smooth even when $\|v\|_Q = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

Let $p_{E',Q} : E' \to Q$ be the map such that $p_{E',Q}(u) = x$ if and only if $u \in T_xU$ for the Cayley line $U$ and $v \in G_V$, $x \in p_{Q}^{-1}(v)$, and $\exp_Q x = U \cap \{ v \}'$. 20
$p_{E',Q}$ is smooth on $p_{E',Q}^{-1}(\exp_Q^{-1}(G_M \setminus V'))$ since on this set it is the composition of the smooth map $\exp_Q^{-1}$ with the map $E' \to M$ given by $e \mapsto \exp_{\frac{\pi}{2}e}$. It is also clear that

(i) $p_Q \circ p_{E',Q} = p_{E'}$, and

(ii) the restriction of $p_{E',Q}$ to a fiber $p_{E'}^{-1}(v)$ of $p_{E'}^{-1}$ is $\exp_Q^{-1} \circ \Pi_v$.

Since $p_{E',Q}|_{\exp_Q^{-1}(G_M \setminus V')}^{-1}$ is smooth, it follows from (i) and (ii) that it is a submersion. Given any point $x \in p_{Q}^{-1}(v) \subset Q$, $p_{E',Q}(x) = E' \cap T U_v$ where $U$ is the Cayley line between $\exp_Q x$ and $v$. So $E'|_{p_{E',Q}^{-1}(\exp_Q^{-1}(G_M \setminus V'))}$ is an $S^7$ bundle with projection map $p_{E',Q}$.

It remains to find bundle charts for $p_{E',Q}$ about points in $\exp_Q^{-1}(V')$. Let $\Phi$ be the geodesic flow of $M$. Observe that

$$p_{E',Q}^{-1}(\exp_Q^{-1}(V')) = \{ u \in E' \mid u \text{ is tangent to a line of type } 2 \}$$

and that the map $p_{E',Q}^{-1}(\exp_Q^{-1}(V')) \to \Pi_{V'}^{-1}(G_V)$ given by $u \mapsto \Phi^u(x)$ is a diffeomorphism which commutes with the obvious projection maps onto $G_V$. This shows that $E'|_{p_{E',Q}^{-1}(\exp_Q^{-1}(V'))}$ is a trivial $S^7$ bundle over $G_V$. The 0-section $s_0$ of $P_2$ provides a way of identifying $\exp_Q^{-1}(V')$ with $G_V$, and $p_{E',Q}|_{p_{E',Q}^{-1}(\exp_Q^{-1}(V'))} = s_0 \circ a \circ p_{E'}|_{p_{E',Q}^{-1}(\exp_Q^{-1}(V'))}$; so there is a diffeomorphism

$$\psi : E'|_{p_{E',Q}^{-1}(\exp_Q^{-1}(V'))} \to \exp_Q^{-1}(V') \times S^7$$

which commutes with the projections onto $\exp_Q^{-1}(V')$.

It follows from (ii) that the restriction of $p_{E',Q}$ to any fiber of $p_{E'}$ is a fiber bundle with fiber $S^7$. In fact, given any fixed $v \in G_V$, we can extend $\psi|_{p_{E',Q}^{-1}(s_0 \circ a(v))}$ to a chart

$$p_{E',Q}^{-1}(P_2 \cap p_{Q}^{-1}(v)) \xrightarrow{\psi_v} (P_2 \cap p_{Q}^{-1}(v)) \times S^7.$$ 

by using the holonomy displacement maps of $\Pi_v$ given by the radial geodesics in $\{ v \}'$ emanating from $V'$. Clearly the $\psi_v$'s vary continuously with $v$. So given any open disk $U \subset G_V$ we get a bundle chart

$$p_{E',Q}^{-1}(P_2 \cap p_{Q}^{-1}(U)) \xrightarrow{\psi_U} P_2 \cap p_{Q}^{-1}(U) \times S^7.$$

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by setting $\psi_U(x) = \psi_{p_{E'}}(x)$. □

Let $E$ be the $S^8$-bundle over $Q$ obtained by suspending the fibers of $S^7 \hookrightarrow E' \overset{p_{E'}}{\to} Q$. We think of $E$ as the quotient space obtained from $E' \times [0, \frac{\pi}{2}]$ by the equivalence relation

$$(e_1, t) \sim (e_2, s) \quad \text{only if } t = s.$$  

$$(e_1, 0) \sim (e_2, 0) \quad \text{if and only if } p_{E', Q}(e_1) = p_{E', Q}(e_2)$$  

$$(e_1, \frac{\pi}{2}) \sim (e_2, \frac{\pi}{2}) \quad \text{if and only if } p_{E', Q}(e_1) = p_{E', Q}(e_2)$$  

for $0 < t < \frac{\pi}{2}, (e_1, t) \sim (e_2, t)$ only if $e_1 = e_2$

Since $E$ is an $S^8$-bundle over $Q$ and $Q$ is an $S^4$-bundle over $G_V$, $E$ is an $S^4 \times S^8$-bundle over $G_V$. We get the desired map $\psi : E \to M$ by setting $\psi[(e, t)] = \exp(te)$. (Here $[(e, t)]$ denotes the equivalence class of $(e, t)$ in $E$.) That $\psi$ is well defined follows from Lemma 8. That $\psi$ is degree 1 follows from the properties of $\exp : TC|\nu(\frac{\pi}{2}) \to M$.

4 The Nonsimply Connected Case

Let $M$ satisfy the hypotheses of Theorem 4. As we indicated in the introduction the classification theorem in [GG1] applies to $M$. In particular, we know that $M$ is either isometric to a space form or the quotient of $CP^{2k-1}$ in Theorem (ii).

Suppose $M$ is a space form, $O$ is an orbit of the action of $\Gamma$ on $S^n$, and $p : S^n \to M$ is the universal covering map. Then since $rad M = \frac{\pi}{2}$, we can find a dual pair $(A, A')$ in $M$ with $p(O) \in A$. Since $A$ is totally $\pi$-convex so is $\tilde{A} = p^{-1}(A)$. Since $\partial A = \emptyset$ (2.5, 3.4, and 3.5 in [GG1]), $\partial \tilde{A} = \emptyset$. $\tilde{A}$ is therefore, a $\Gamma$-invariant, great subsphere of $S^n$ that contains $O$.

On the other hand, if $S^n/\Gamma$ is a space form and an orbit $O$ of $\Gamma$ is contained in a proper, invariant, totally geodesic subsphere, $S^k$, then $A(S^k) = \{x \in S^n \mid dist(x, S^k) = \frac{\pi}{2}\}$ is also invariant. Hence $dist(p(O), p(A(S^k))) = \frac{\pi}{2}$, and $rad S^n/\Gamma = \frac{\pi}{2}$, if every orbit of $\Gamma$ is contained in a proper, invariant, great subsphere.

To complete the proof of Theorem it remains to show that the space in (ii) has radius $= \frac{\pi}{2}$. The orbit of an arbitrary point for the corresponding action on $S^{4k-1}$ is

$$(z_1, z_2, \ldots, z_{2d}) \mapsto (\overline{z}_{d+1}, \ldots, \overline{z}_{2d}, -\overline{z}_1, \ldots, -\overline{z}_d) \mapsto (-z_1, \ldots, -z_d, -z_{d+1}, \ldots, -z_{2d}) \mapsto$$
Thus each orbit (in $S^{4d-1}$) is contained in an invariant geodesic that is parallel to the fibers of the Hopf fibration $S^1 \hookrightarrow S^{4d-1} \to CP^{2d-1}$. It follows that each orbit in $CP^{2d-1}$ is contained in an invariant geodesic. If $\gamma : [0, \pi] \to CP^{2d-1}$ is an invariant geodesic, then $(image(\gamma))'$ is also invariant (and $\neq \emptyset$ since $d \geq 2$). So the radius of the quotient is $\frac{\pi}{2}$.

Now we focus on the proof of (iii).

Proof of (i): Let $\rho : \Gamma \to O(n + 1)$ be a fixed point free representation that respects an orthogonal splitting $V_1 \oplus V_2 \oplus \cdots \oplus V_k$ of $\mathbb{R}^{n+1}$ so that $\rho|_{V_i}$ is irreducible for all $i$. It follows from Theorem 7.2.18 in [W] that $dim V_i = dim V_j(\equiv d)$ for all $i, j$. Suppose $rad S^n/\rho(\Gamma) = \frac{\pi}{2}$. Then using (4, i) we can find a proper, invariant subspace that is not a direct sum of $V_i$’s and then an irreducible invariant subspace $W$ that is distinct from all of the $V_i$’s. The orthogonal projections $p_i : W \to V_i$ are all $\rho$-equivariant, so by Schur’s Lemma, they are either zero maps or isomorphisms. If they are all zero maps, then we have $W \subset (V_1 \oplus V_2 \oplus \cdots \oplus V_k)^\perp$, which is impossible. So at least one of the projections (say $p_1$) is an isomorphism. If all of the other $p_i$’s are zero maps, then we have $W \subset (V_2 \oplus \cdots \oplus V_k)^\perp$, which would imply that $W = V_1$, also impossible. So at least one other projection (say $p_2$) is an isomorphism. Thus $\rho|_{W}$ is linearly (and hence orthogonally by Lemma 4.7.1 in [W]) equivalent to both $\rho|_{V_1}$ and $\rho|_{V_2}$.

Proof of (ii): By (iv) it suffices to consider the case $k = 2$. In this case the action of $\rho(\Gamma)$ on $S^3$ is orthogonally equivalent to a subaction $\bar{\rho}(\Gamma)$ of the Hopf action, and the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ induces a Riemannian submersion $S^1 \hookrightarrow S^3/\bar{\rho}(\Gamma) \to S^2(\frac{1}{2})$. Thus $rad S^3/\rho(\Gamma) = rad S^3/\bar{\rho}(\Gamma) \geq rad S^2(\frac{1}{2}) = \frac{\pi}{2}$.

Proof of (iii): Suppose there are points $u, v \in S^{d-1}$ and members $g_1, g_2, \ldots, g_d, g_{d+1}$ of $\Gamma$ so that $\{g_1(u), g_2(u), \ldots, g_d(u)\}$ and $\{g_1(v), g_2(v), \ldots, g_d(v)\}$ are linearly independent and such that the sets of coefficients $\{a_i\}, \{b_i\}$ so that

$$a_1g_1(u) + \cdots + a_dg_d(u) = g_{d+1}(u) \quad \text{and}$$

$$b_1g_1(v) + \cdots + b_dg_d(v) = g_{d+1}(v)$$

are distinct. It follows that the vector $\frac{1}{\sqrt{2}}(g_{d+1}(u), g_{d+1}(v))$ in $\mathbb{R}^{2d}$ is not in the span of $\{\frac{1}{\sqrt{2}}g_1(u, v), \frac{1}{\sqrt{2}}g_2(u, v), \ldots, \frac{1}{\sqrt{2}}g_d(u, v)\}$. On the other hand, $\{\frac{1}{\sqrt{2}}g_1(u, v), \frac{1}{\sqrt{2}}g_2(u, v), \ldots, \frac{1}{\sqrt{2}}g_d(u, v)\}$ is linearly independent since its pro-
jection onto \( \mathbb{R}^d \times \{0\} \) is \( \{ \frac{1}{\sqrt{2}}g_1u, \frac{1}{\sqrt{2}}g_2u, \ldots, \frac{1}{\sqrt{2}}g_du \} \). Therefore the set
\[
\left\{ \frac{1}{\sqrt{2}}g_1(u,v), \frac{1}{\sqrt{2}}g_2(u,v), \ldots, \frac{1}{\sqrt{2}}g_d(u,v), \frac{1}{\sqrt{2}}g_{d+1}(u,v) \right\}
\]
is linearly independent, and we are be done by Theorem 4(i). But according to [W], the image of every irreducible representation of a fixed point free, nonabelian group \( \Gamma \) contains matrices of the form
\[
A = \begin{pmatrix}
e^{\frac{2\pi ik}{m}} & e^{\frac{2\pi kr}{m}} & & \\
e^{\frac{2\pi ik}{m}} & e^{\frac{2\pi kr}{m}} & & \\
& & \ddots & \\
& & & e^{\frac{2\pi ikd-1}{m}}
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
& \ddots & \cdots & \\
& \cdots & \ddots & \\
0 & & \cdots & 1
\end{pmatrix}
\]
(35)
where \( d, k, l, m, n' \) and \( r \) are as in Theorem 5.5.6 in [W] (and we have used complex coordinates). So it suffices to set \( u = (1,0,\ldots,0) \), \( v = (0,1,0,\ldots,0) \), \( g_1 = Id, g_2 = A, g_3 = BA, g_4 = ABA, \ldots, g_{d+1} = (BA)^{\frac{d}{2}} \). (Note that \( d \) is even.) (To quickly see that there are matrices of the form (35) in the image of every irreducible representation of a fixed point free, nonabelian group, note that such matrices are in the image of every such representation of a so called “group of type 1” (Theorem 5.5.6 and 5.5.10 in [W]) and that other nonabelian fixed point free groups contain groups of type 1 as subgroups ([W] pages 204-208).) □

Proof of (iv): View \( S^{n+d}(1) \) as the join \( S^n(1) \star S^{d-1}(1) \), and view \( \rho \oplus \sigma \) as the join of \( \rho \) and \( \sigma \). Then every orbit of \( \rho \oplus \sigma \) is contained in the join of an orbit of \( \rho \) with an orbit of \( \sigma \). Since the orbits of \( \rho \) are all contained in proper invariant totally geodesic subspheres of \( S^n \), the orbits of \( \rho \oplus \sigma \) are contained in the joins of proper great subspheres of \( S^n \) with \( S^{d-1} \). □

Proof of (v): For example, if \( k \) is so large that the order of \( \Gamma \) is less than \( n + 1 \), then every orbit is automatically contained in an invariant subspace. □

Appendix

Here we prove Lemma 22. Let \( UVS^{15} \) and \( UHS^{15} \) be the subbundles of \( VS^{15} \) and \( HS^{15} \), respectively, that consist of unit vectors. Set \( X = \hat{\gamma} \).
Let \( \{ E_i \}_{i=1}^7 \) and \( \{ \tilde{E}_i \}_{i=1}^7 \) be as in the statement of Lemma 17, and define \( A^*_X : VS^{15}_{\tilde{\gamma}(t)} \longrightarrow HS^{15}_{\tilde{\gamma}(t)} \) by \( A^*_X v = A_X v \).

**Proposition 36** Suppose \( K_v \oplus L_v \) and \( K_h \oplus L_h \) are orthogonal splittings of \( VS^{15}_{\tilde{\gamma}(t)} \) and \( HS^{15}_{\tilde{\gamma}(t)} \) respectively so that \( A^*_X K_v \subset K_h \), \( A^*_X L_v \subset L_h \), \( A_X K_h \subset K_v \), and \( A_X L_h \subset L_v \). Let \( v \in L_v \cap UVS^{15}_{\tilde{\gamma}(t)} \) be such that \( \| A_X v \| = \max_{w \in L_v \cap UVS^{15}_{\tilde{\gamma}(t)}} \| A_X w \| \), and suppose that \( A_X v = \lambda y \), where \( y \) is a unit vector and \( \lambda > 0 \). Then

(i) \( A_X y = -\lambda v \),

(ii) \( \| A_X y \| = \max_{z \in L_h \cap UHS^{15}_{\tilde{\gamma}(t)}} \| A_X z \| \),

(iii) \( A^*_X (y^\perp) \subset y^\perp \), and

(iv) \( A_X (y^\perp \cap HS^{15}) \subset v^\perp \).

**Proof:** If (i) is false, then \( A_X y = \lambda w \) for some unit vector \( w \in L_v \) that is different from \( v \) and \( -v \). Note that

\[
|\lambda_w| = |\lambda_w \langle w, w \rangle| > |\lambda w \langle w, v \rangle| = |\langle A_X y, v \rangle| = |\langle y, A_X v \rangle| = |\lambda|,
\]

and \( \langle A_X w, y \rangle = -\langle w, A_X y \rangle = -\lambda w \), which contradicts the maximality of \( \| A_X v \| \).

If (ii) is false, then there is a horizontal unit vector \( y_1 \in L_h \) different from \( y \) and \( -y \), a unit vector \( w \in L_v \) and a \( \lambda_1 > \lambda \) so that \( A_X y_1 = \lambda_1 w \). But then \( \langle A_X w, y_1 \rangle = -\langle w, A_X y_1 \rangle = -\lambda_1 \), and this contradicts the maximality of \( \| A_X v \| \).

If \( w \) is a vertical vector in \( v^\perp \), then \( \langle A_X w, y \rangle = -\langle w, A_X y \rangle = 0 \), by part (i).

Similarly, if \( z \) is a horizontal vector in \( y^\perp \), then \( \langle A_X z, v \rangle = -\langle z, A_X v \rangle = 0 \).

Using Proposition 36 we can inductively construct an orthonormal basis for \( V S^{15}_{\tilde{\gamma}(t)} \), \( \{ u_1, \ldots, u_k, u_{k+1}, \ldots, u_7 \} \) with the following properties:

(i) \( \{ u_1, \ldots, u_k \} \) is a basis for \( \ker A^*_X \),

(ii) \( \{ u_{k+1}, \ldots, u_7 \} \) satisfies \( \langle A_X u_i, A_X u_j \rangle = 0 \), for \( i \neq j \), and
(iii) for $i = k + 1, \ldots, 7$, $A_X u_i = \lambda_i y_i$ for some unit vector $y_i$ and some $\lambda_i > 0$, and $A_X y_i = -\lambda_i u_i$.

To construct $\{u_{k+1}, \ldots, u_7\}$ choose $u_{k+1}$ so that $\|A_X u_{k+1}\| = \max_{w \in UVS_{15}(t)} \|A_X w\|$ and in general choose $u_i$ so that $\|A_X u_i\| = \max_{w \in (\text{span}\{u_1, \ldots, u_k, u_{k+1}, \ldots, u_{i-1}\})^\perp \cap UVS_{15}(t)} \|A_X w\|$.

Let $\{y_1, \ldots, y_7\}$ be a completion of $\{y_{k+1}, \ldots, y_7\}$ to an orthonormal basis for $H_{S^15\bar{(t)}}^1 \cap x^\perp$. Note that

$$Ric(d\Pi(X), d\Pi(X)) = \sum_{i=1}^7 K(d\Pi(X), E_i) = \sum_{i=1}^7 K(d\Pi(X), d\Pi(y_i))$$

(37)

Using the Horizontal Curvature Equation, (37) becomes

$$\sum_{i=1}^7 K(X, \tilde{E}_i) + 3\|A_X \tilde{E}_i\|^2 = \sum_{i=1}^7 K(X, y_i) + 3\|A_X y_i\|^2.$$  

(38)

And since the sectional curvature of $S^15$ is constant, (38) becomes

$$\sum_{i=1}^7 \|A_X \tilde{E}_i\|^2 = \sum_{i=1}^7 \|A_X y_i\|^2.$$  

(39)

Using (39) and the properties of the $u_i$’s we see that

$$\sum_{i=1}^7 \|A_X \tilde{E}_i(t)\|^2 = \sum_{i=1}^7 \|A_X u_i\|^2.$$  

(40)

For our given orthonormal basis $\{V_1(t), \ldots, V_7(t)\}$ for $VS_{15\bar{(t)}}^1$ we have $A_X V_j(t) = \sum_i A_X (V_j(t), u_i) u_i$. Therefore

$$\sum_j \|A_X V_j(t)\|^2 = \sum_j \sum_i (V_j(t), u_i)^2 \|A_X u_i\|^2 = \sum_i \sum_j (V_j(t), u_i)^2 \|A_X u_i\|^2 = \sum_{i=1}^7 \|A_X u_i\|^2.$$  

Combining this with (39) and (40) completes the proof of (22).

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