TASI 2011: Four Lectures on TeV Scale Extra Dimensions

Eduardo Pontón

Department of Physics
538 W 120th street
Columbia University, New York, NY 10027

Abstract

Compact spatial dimension at the TeV scale remain an intriguing possibility that is currently being tested at the LHC. We give an introductory review of extra-dimensional models and ideas, from a phenomenological perspective, but emphasizing the appropriate theoretical tools. We emphasize the power and limitations of such constructions, and give a self-contained account of the methods necessary to understand the associated physics. We also review a number of examples that illustrate how extra-dimensional ideas can shed light on open questions in the Standard Model. An introduction to holography is provided. These are the notes of my TASI 2011 Lectures on Extra Dimensions.
0 Introduction

It is illustrative to consider the first *modern* attempt at entertaining seriously the possibility that our universe could have more than $3 + 1$ dimensions. In 1921 [1], not long after the development of General Relativity (GR), the German mathematician and physicist Theodor Kaluza extended GR to $4 + 1$ dimensions. He noticed that “splitting” one of the spatial dimensions from the rest made it apparent that the structure of electromagnetism was contained within GR. With the introduction of “curled up”, or *compact*, dimensions by the Swedish physicist Oskar Klein in 1926 [3], the possibility of electromagnetism simply being part of $4 + 1$-dimensional general relativity, with one of the spatial dimensions compactified, seemed to offer a beautiful path towards the unification of the known forces at the time. This is known as the Kaluza-Klein theory. The idea is fairly simple, even if profound, and is illustrated in the figure to the right. The vertical dimension stands for the $3 + 1$ (infinitely large) dimensions we are familiar with. The fifth dimension, on the other hand, is *finite*, being compactified on a circle of radius $R$. Note that our world corresponds to the “surface of the cylinder”: there is nothing physical inside or outside of it. When fields propagate on such a manifold, the fifth component of the momentum is quantized in units of $1/R$, the *compactification scale*. Due to the associated gradients, exciting such field configurations requires energies of that order, and in fact appear to 4D observers as *massive 4-dimensional fields excitations*. At energies low compared to the compactification scale, the relevant field configurations are constant in the 5th dimension, and are known as *0-modes*. The 5D metric tensor of the Kaluza-Klein (KK) theory takes the schematic form

$$G_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} & \bar{A}_\mu \\ \bar{A}_\mu^T & g_{55} \end{pmatrix}$$

where we show the expected spin components under the 4D Lorentz group. It turns out to be

$$G_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} & \bar{A}_\mu \\ \bar{A}_\mu^T & g_{55} \end{pmatrix}$$

Under 4D

Lorentz group

“spin-2” $\bar{g}_{\mu\nu}$

+ “spin-1” $\bar{A}_\mu$

+ “spin-0” $g_{55}$

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*Strictly speaking, some of the main ideas had already been introduced in 1914 by the Finish physicist Gunnar Nordström [2]. He observed that extending the Maxwell theory to 5D, written in terms of a 5-vector $A_M$, contained the 4-vector $A_\mu$—to be identified with the electromagnetic vector potential—plus a 4D scalar obeying the field equations for his own proposed *scalar theory of gravity*. I emphasize here the Kaluza-Klein theory both because it is based on general relativity, now known to be the correct description of 4D gravity, as well as because it allows us to neatly introduce concepts such as that of a *radion field*. 
more convenient to parametrize the 0-modes as
\[ ds^2 = \phi^{-1/3} (g_{\mu\nu} - \phi A_\mu A_\nu) \, dx^\mu \, dx^\nu - 2\phi^{2/3} A_\mu \, dx^\mu \, dy - \phi^{2/3} \, dy^2 , \]  
where \( y \equiv x^5 \), and all fields are taken to be \( y \)-independent. Replacing this ansatz in the 5D Einstein-Hilbert action gives
\[ -\frac{1}{2} M_5^3 \int d^5x \sqrt{G} \, R_5[G] = -\frac{1}{2} M_P^2 \int d^4x \sqrt{g} \left[ R_4[g] + \frac{1}{4} \phi \, F_{\mu\nu} F^{\mu\nu} - \frac{1}{6} \frac{\partial_\mu \phi \partial^\mu \phi}{\phi^2} \right] , \]
where, on the r.h.s. all the contractions are done with \( g^{\mu\nu}(x) \), and \( M_P^2 = (2\pi R) M_5^3 \) is to be identified with the 4D reduced Planck mass. The appearance of the \( F_{\mu\nu} F^{\mu\nu} \) structure can be understood by realizing that under an infinitesimal general coordinate transformation \( \delta G_{MN} = \xi_{M;N} + \xi_{N;M} \), and when the infinitesimal parameter \( \xi(x) \) is \( y \)-independent, one has
\[ \delta A_\mu = (\partial_\mu \xi^\rho) A_\rho + \xi^\rho (\partial_\rho A_\mu) + \partial_\mu \xi_5 . \]

The first term accounts for the transformation of the 4-vector index, the second for the argument of the field \( A_\mu(x) \), and the last one is recognized as a \( U(1) \) gauge transformation, which ensures that \( A_\mu \) must appear as part of \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). After the field redefinition \( \phi \rightarrow e^{-\sqrt{6} \phi / M_P} \), the scalar kinetic term becomes canonical, and \( \phi \) acquires the couplings of a dilaton field. Thus, at low energies the Kaluza-Klein theory describes 4D gravity + a \( U(1) \) gauge theory + a real scalar field with dilaton couplings. The latter can also be thought as parameterizing small oscillations in the size of the extra dimension, and is often called a radion field.

From a modern perspective we may draw some lessons:

- The principle of general coordinate invariance in 4D is well-tested experimentally, in the form of General Relativity (the natural framework to describe massless spin-2 particles, in quantum mechanical language). In our applications, the assumption that the full spacetime is also described by GR gives a set of well-defined rules to write down extra-dimensional models.

- The KK theory provides an intriguing way to achieve the unification of apparently disparate forces, by obtaining a \( U(1) \) field from a fifth dimension. In fact, non-abelian fields can arise from spacetimes of higher dimensionality, when appropriately compactified. Unfortunately, describing what we know today about the Standard Model (SM) cannot be done within the Kaluza-Klein framework [4]. The problem arises from the fact that the SM fermions are chiral under the SM gauge group, while Kaluza-Klein theories give rise to a vector-like fermion spectrum.
In modern (early 21st century) proposals, the previous difficulty is overcome by considering compactifications with some sort of (mild) singularities. To illustrate what we mean, consider the case of two extra-dimensions. A simple way to compactify them is the torus compactification, where adjacent sides of a rectangular region are identified, as shown in Fig. 1. This is a straightforward generalization of the 5D Kaluza-Klein theory to 6D, and gives rise necessarily to a vector-like theory at low energies, that is not suitable for embedding the SM.

A simple modification is achieved with the Chiral Square compactification, where adjacent, rather than opposite sides are identified, as illustrated in Fig. 2. In this case, the compact space has three conical singularities. The boundary conditions that codify the chiral square compactification are

\[
\begin{align*}
\Phi(y,0) &= e^{in\pi/2}\Phi(0,y), \\
\Phi(y,\pi R) &= e^{in\pi/2}\Phi(\pi R, y), \\
\partial_5 \Phi|_{(x^4,x^5)=(y,0)} &= -e^{in\pi/2} \partial_4 \Phi|_{(x^4,x^5)=(0,y)}, \\
\partial_5 \Phi|_{(x^4,x^5)=(y,\pi R)} &= -e^{in\pi/2} \partial_4 \Phi|_{(x^4,x^5)=(\pi R,y)},
\end{align*}
\]

which say that, up to a possible phase \(e^{in\pi/2}\), the field and its derivatives are continuous across the glued boundaries. The integer \(n = 0, 1, 2, 3\) labels the consistent possibilities. As it turns out, fermions propagating on the chiral square are necessarily chiral at low energies,
hence the name of the compactification, which has been used to describe possible physics beyond the SM. In the body of these lectures, we will consider an even simpler possibility: the *interval compactification* in 5 dimensions.²

- Extra-dimensional models beg questions such as: what fixes the size of the extra dimensions? (Or, in more technical language, what sets the mass for the radion field?) What about their “shape”? Or for that matter, what selects how many dimensions are compact/“infinite”?

Compared to the old Kaluza-Klein proposal, our aim here is at the same time more and less ambitious. It is less ambitious in that we do not attempt to achieve a unification of forces through the geometric structure. We will simply assume the presence of higher-dimensional fields of various spins, apart from the higher-dimensional metric. The “chiral compactifications” will typically get rid of the \( U(1) \) Kaluza-Klein gauge field, but not of the radion mode. In addition, we will be content with assuming the number of compact dimensions (together with the familiar 3+1 non-compact dimensions), and also the type of compactification. Our guide will be phenomenological. At the same time we are more ambitious in that these extra-dimensional models are built to be consistent with all aspects of the SM. Also, at least in principle, many of these extensions can be tested or ruled out at the TeV scale.

A very important question concerns the size of the compact extra dimensions, if they exist. Within string theory it may be natural for this size to be of the order of the Planck or string scales. If so, such dimensions will not be probed directly, although even very small extra dimensions can have observable indirect effects. Another possibility is that the compactification scale may be related in some deep way to the weak scale, perhaps even playing a role in the breaking of the electroweak symmetry (EWSB). This type of new physics might very well be testable during the coming decade.

As already mentioned, realistic extra-dimensional scenarios possess certain singularities, or defects, often also called *branes* (loosely borrowing stringy terminology). Such defects can support localized fields of various spins, while other fields may propagate in the bulk of the space. The situation is illustrated in Fig. 3. The possibility of localizing fields at different places of the extra dimensional real estate marks an important difference with 4-dimensional model building. The use of locality in the extra dimensions has been successfully used in a number of applications. This also means that specifying the localization (or not) of the various fields in the theory becomes part of the *definition* of the model, beyond the specification of the field content and relevant symmetries.

The above brings an additional issue: the cosmological constant. Presumably, the gravita-

²However, 6D spaces have intriguing properties not present in 5D, that we will not have space to explore.
Figure 3: General extra-dimensional setup, with defects of reduced dimensionality. These may be thought to have a thickness much smaller than the typical size of the compact space, denoted by $L$. Some fields, denoted here by $Q$ and $X$, may be restricted to propagate on such “branes”, while other may propagate in the bulk of the space.

The only constraint we have on $\Lambda_5$ is that the effective 4D cosmological constant should be very small:

$$M_{Pl}^2 \Lambda_4 \sim (\text{contribution from gravit. background}) + (\text{contribution from } \Lambda_5) + (\text{contribution from other sources, e.g. Casimir energies}) \lesssim \mathcal{O}(10^{-3} \text{ eV})^4.$$

One may envision two extremes here:

1. The various contributions have a natural size of order $M_5^4$, but they cancel out almost precisely. This cancellation is just the well-known cosmological constant problem, which so far seems to necessitate an extraordinary fine-tuning (even if the final understanding is based on “anthropic considerations”).

2. Perhaps, for some unspecified reason, $\Lambda_5$ is very suppressed so that the extra dimensions are essentially flat. Note that the “contribution from other sources” can be expected to contain...
EWSB effects, or the effects from QCD chiral symmetry breaking, as well as the quantum contribution from the compactification, itself of order $1/R^4$ (often called Casimir energy). Thus, fine-tuning is still necessary, beyond the suppression of $\Lambda_5$. Nevertheless, flat extra dimensions illustrate an interesting benchmark from the point of view of phenomenology.

Naturally, one can also consider scenarios where $\Lambda_5$ takes an intermediate value (perhaps related to the weak scale?). In any case, from our point of view, $\Lambda_5$ is a free parameter, and it is important to be aware that it can have a significant impact on the phenomenology. This is the basis for the classification between flat versus warped extra dimensions.

In these lectures, we explore various aspects of extra-dimensional model building. In the first lecture, we discuss a number of general features that should be confronted when considering field theories in more than four dimensions. The student may want to have a cursory look at this short lecture, and come back to it later on. In the second lecture, we develop the basic language required to discuss extra-dimensional theories and their phenomenology. We restrict to the case of one compact extra spatial dimension. This simplest case already contains most of the main features and it pays to get fully familiar with it. In addition, many models of physical interest fall in this category. We derive our results in a general background, specializing to particular cases only when necessary. In the third lecture, we review a number of such models, without intending to be exhaustive. Nevertheless, the student should be able to get a good exposure to the possibilities that arise in extra-dimensional frameworks. In Fig. 4 we show a schematic guide to a few popular extra-dimensional scenarios and some of their prominent features. We will only touch on a subset of these, in particular only those with extra dimensions at the TeV scale. Finally, in Lecture 4 we give an elementary introduction to holography.

There have already been a number of excellent TASI lectures on Extra Dimensions from a phenomenological point of view. The student interested in the subject is encouraged to look at all of these since they cover different aspects and present different points of view:

- C. Csáki, “TASI lectures on extra dimensions and branes,” hep-ph/0404096
- R. Sundrum, “Tasi 2004 lectures: To the fifth dimension and back,” hep-th/0508134
- C. Csáki, J. Hubisz and P. Meade, “TASI lectures on electroweak symmetry breaking from extra dimensions,” hep-ph/0510275
- G. D. Kribs, “TASI 2004 lectures on the phenomenology of extra dimensions,” hep-ph/0605325
- B. A. Dobrescu, “Particle physics in extra dimensions,” FERMILAB-CONF-08-703-T.
- H. -C. Cheng, “2009 TASI Lecture – Introduction to Extra Dimensions,” arXiv:1003.1162
- T. Gherghetta, “TASI Lectures on a Holographic View of Beyond the Standard Model Physics,” arXiv:1008.2570
Figure 4: A guide to *representative* extra-dimensional scenarios (not intended to be complete!).
Lecture 1: Quantum Field Theory in More than 4D

Recall that our expectation is that any Quantum Field Theory (QFT) should be regarded as an effective low-energy description. Even in a “renormalizable” theory such as the Standard Model, higher-dimension operators are most likely present, although they may be suppressed by a very high scale. In our discussion of extra-dimensional physics it will be useful to have in mind the classic example of an effective theory, the Fermi theory of $\beta$-decay:

$$L_{\text{Fermi}} = G_F \bar{\psi}_1 \gamma^\mu \psi_2 \bar{\psi}_3 \gamma^\mu \psi_4,$$

where $G_F \sim \frac{1}{v^2}$. The mass scale associated with the Fermi constant implies that Eq. (5) could have been valid at most up to a cutoff of order $4\pi v$. As it happens, the Fermi theory breaks down at the somewhat lower scale associated with the weak gauge bosons, due to the perturbative nature of the UV completion (i.e. the electroweak theory). Generic higher-dimensional theories are very similar to the Fermi theory in this respect. In other words, extra-dimensional models contain an intrinsic UV cutoff, that forces us to treat them with the methods and spirit appropriate to Effective Field Theories (EFT’s): i) processes with characteristic energies parametrically lower than the cutoff can be reliably described, ii) there are UV dominated observables displaying a power-law sensitivity to the unspecified UV completion, iii) IR-dominated observables become particularly interesting.

Thus, one should keep in mind that these models may have limitations in terms of calculability. The basic reason for the above arises from dimensional analysis. Consider a generic field theory in $D$-dimensions that describes interacting fields of various spins,

$$L_D \sim \int d^D x \left\{ -\frac{1}{4g_D^2} F_{MN} F^{MN} + \bar{\Psi} i \mathcal{D} \Psi + |D_M \Phi|^2 + \cdots \right\},$$

where $D_M = \partial_M + iA_M$ implies the mass dimension $[A_M] = 1$, i.e. $[F_{MN}] = 2$ and $g_D^2 = 4 - D$. In particular, the gauge coupling has inverse mass dimension for $D > 4$, which introduces an intrinsic scale into the theory. Similarly, since fermions and scalars have $[\Psi] = \frac{1}{2}(D - 1)$ and $[\Phi] = \frac{1}{2}(D - 2)$, respectively, if we add a Yukawa interaction of the type $L_{\text{Yuk}} = y_D \Phi \bar{\Psi} \Psi$, it follows that $[y_D] = D - \frac{1}{2}(D - 2) - (D - 1) = 2 - \frac{1}{2}D$ has also inverse mass dimensions for $D > 4$.

**Exercise:** What is the mass dimension of a scalar quartic coupling?

**Exercise:** We have mentioned the possibility of localizing fields on 4D subspaces, e.g.

$$L_5 \supset \delta(y - y_0) \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + i \bar{\phi} \mathcal{D} \phi + \cdots \right\},$$

[one can also consider $\partial_5$ derivatives, although these require some care in the interpretation]. Work out the mass dimension for

a) a Yukawa coupling between a bulk fermion $\Psi$ and a localized scalar $\phi$.

b) a Yukawa coupling between a localized fermion $\psi$ and a bulk scalar $\Phi$.

c) Explore various combinations of bulk/localized scalar self-interactions.
The point of these observations is that most interesting (from our low-energy 4D point of view) interactions become non-renormalizable when considering $D > 4$. To better appreciate the meaning of this, consider the 1-loop self-renormalization of a bulk Yukawa interaction:

\[ (iy_D)^3 \int \frac{d^D p}{(2\pi)^D} \frac{i}{p - M_\Psi} \frac{i}{p - M_\Psi} \frac{i}{p^2 - M_\Psi^2} \]

\[ \sim \frac{y_D^3}{(2\pi)^D} \Omega_D \int_0^\Lambda p^{D-2} \frac{dp}{2} \frac{1}{p^2} \frac{1}{p^2} \]

\[ \sim \frac{\Omega_D}{2(2\pi)^D} \frac{y_D^3}{D - 4} \Lambda^{D-4} , \tag{7} \]

where $\Omega_D \equiv \int d\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the $D$-dimensional solid angle. A few comments are in order:

i) We have assumed that $D > 4$, so that the integral is dominated by the region near the cutoff $\Lambda$, as indicated in the second line. We also take the fermion and scalar masses to be small compared to $\Lambda$.

ii) One obtains a $D$-dimensional loop factor

\[ \frac{1}{l_D} \equiv \frac{\Omega_D}{2(2\pi)^D} = \frac{1}{(4\pi)^{D/2}\Gamma(D/2)} = \left\{ \begin{array}{ll} \frac{1}{16\pi^2} & \text{for } D = 4 \\ \frac{1}{24\pi^4} & \text{for } D = 5 \end{array} \right. , \tag{8} \]

where we exhibit the 4D case to make contact with a familiar result, as well as the 5D case that will be of use in the following.

iii) It may be illuminating to consider the regulator dependence of the above result, which was obtained by imposing a hard-momentum cutoff. Let us consider again the momentum integral, taking for concreteness $D = 5$ and $M_\Psi = 0$. In Euclidean space, we have

\[ \int_0^\Lambda p^3 dp^2 \frac{1}{p^2} \frac{1}{p^2 + M_\Phi^2} = 2\Lambda - 2M_\Phi \tan^{-1} \left( \frac{\Lambda}{M_\Phi} \right) \]

\[ = 2\Lambda - \pi M_\Phi \left[ 1 + O \left( \frac{M_\Phi}{\Lambda} \right) \right] , \]

where we show also the subleading term in the second line. If we were to regulate the momentum integral with a Pauli-Villars (anticommuting) scalar, we would then get

\[ \int_0^\infty p^3 dp^2 \frac{1}{p^2} \left[ \frac{1}{p^2 + M_\Phi^2} - \frac{1}{p^2 + \Lambda_{PV}^2} \right] = \pi (\Lambda_{PV} - M_\Phi) . \]

The fact that, for odd $D$, the integrand is not an analytic function of $p^2$ (depends on $|p|$ through the measure) can lead to the appearance of factors of $\pi$ on top of those in the
loop factor, Eq. (8). Nevertheless, here we see that the connection between the momentum
cutoff and the Pauli-Villars mass is given by $2\Lambda \leftrightarrow \pi \Lambda_{PV}$, which shows that they differ
by an order one factor. For the most part, we will use a momentum cutoff to estimate
UV-dominated contributions, but should remember that there is some uncertainty as to
where to “expect” the new physics to come in.

iv) Since $[y_D] = 2 - \frac{1}{2}D$, we will often write it in terms of the cutoff scale, $y_D \equiv c_y \Lambda^{2-\frac{4}{D}}$, for
some dimensionless $c_y$. Note that the dimensions of Eq. (7) come from $y_D^3 \Lambda^{D-4} = c_y^3 \Lambda^{2-\frac{1}{2}D}$,
which coincide with those of $y_D$, as required by dimensional analysis (remember that the
amputated diagram corresponds to a Yukawa interaction in the quantum effective action).

v) We have assumed above that we are working in a flat and infinite $D$-dimensional space. We
will comment further on this later on, but suffice it to say here that if the cutoff length is
small compared to the “size” of the extra-dimensional space, as well as the curvature radius
(if in curved spacetime), it should be intuitively clear that the above estimates continue to
hold. Only the IR (i.e. truly calculable) contributions depend on such details.

The above considerations are useful because

- They exhibit the *power-law* dependence on the UV cutoff, generally present in
  $D > 4$ (for $D = 4$, we would recover $\frac{1}{D-4} \Lambda^{D-4} \rightarrow \ln \Lambda$). This signals that certain
  quantities are, strictly speaking, incalculable.
- They also allow to define a reasonable criterion for how large the cutoff of a higher-
  dimensional theory can be, in terms of the (dimensionful) couplings defining it.

As we will see, the above does not mean that we cannot use extra-dimensional models to un-
derstand physics questions or even make predictions. It only means that we should regard them
as an effective description valid over a finite energy regime, with uncertainties pertaining to the
actual (unknown) UV completion. We therefore turn to making the second bullet above more
precise.

**Naive Dimensional Analysis (NDA)**

Recall from 4D experience that, after renormalization, the effects of higher-dimension operators
on physical observables scale like powers of $E/\Lambda$, where $E$ characterizes the energy scale of
the process under consideration and $\Lambda$ characterizes the mass scale of the coefficient of the higher-
dimension operator. Similarly, in $D$-dimensions, cutoff-dependent effects, as estimated in Eq. (7)
are absorbed, through the process of renormalization, into the experimentally measured values
of the couplings of the theory (a Yukawa coupling in our example). As in “non-renormalizable”
4D theories, an infinite tower of interactions is induced in this manner. Nevertheless, at energies
low compared to the cutoff, only a finite number of them is relevant, at least if we are content
with a finite precision. Only when the characteristic energy approaches the cutoff, does the full
set of higher-dimension operators become important, and therefore the EFT loses predictive
power. This is not surprising, as the details of the UV completion should start to become crucial.
The above can also be related to the breakdown of the loop expansion, i.e. when diagrams at all
loop orders contribute equally (and are also comparable to the “tree-level” effects). This implies
a breakdown of the perturbative expansion, and amounts to the onset of “strong dynamics” (or
more generally, to the appearance of new degrees of freedom at or below \( \Lambda \). There are therefore two aspects to our discussion:

1. Operators of arbitrary dimension can contribute equally at tree-level.

2. The quantum (i.e. loop) contributions associated with a given interaction can all contribute equally (and at the same order as tree-level).

Let us start by considering loop effects at \( E \sim \Lambda \), again considering a Yukawa interaction for definiteness. Note that now we regard the coupling \( y_D \) as the renormalized Yukawa interaction, so that we are talking about a result that is finite (i.e. cutoff-independent, other than through the value of \( y_D \)). Nevertheless, when setting the external \( E \sim \Lambda \gg M_\Psi, M_\Phi \), our estimate proceeds as in Eq. (7), except that now the \( \Lambda \)-dependence is understood as coming from the external momenta. Thus, we find that the renormalized three-point function evaluated at external momenta of order \( \Lambda \) takes the form

\[
y_D \left\{ 1 + O(1) \times \frac{1}{l_D^2} y_D^2 \Lambda^{D-4} + \text{higher-order loops} \right\} \sim y_D \left\{ 1 + \frac{c_y^2}{l_D} + \cdots \right\}. \tag{9}
\]

This suggests that the loop expansion breaks down when \( c_y^2 \sim l_D \). In other words, the cutoff in the above sense is estimated as

\[
\Lambda \sim \left[ \frac{c_y^{\text{NDA}}}{y_D} \right]^{\frac{2}{4-D}} \equiv \left[ \frac{l_D^{1/2}}{y_D} \right]^{\frac{2}{4-D}}, \tag{10}
\]

where we defined the NDA value \( c_y^{\text{NDA}} = l_D^{1/2} \) based on the above considerations.

**Exercise:** Estimate by power counting, as above, some two-loop contributions to the 3-point correlator above, and show that they are also as large as the tree-level and one-loop contributions, for \( c_y = c_y^{\text{NDA}} \).

Thus, the previous “NDA rule” provides a reasonable criterion for the largest the cutoff \( \Lambda \) can be. We note that this criterion also addresses the first bullet above regarding the interplay of several higher-dimension operators: if we estimate their coefficients by NDA (say, by quick 1-loop estimates), then they also give similar tree-level contributions to a given observable when \( E \sim \Lambda \). This will be illustrated in the following

**Exercise:** consider the operator \( y_D^{(2)} (\Box \Phi) \bar{\Psi} \Psi \), where all fields propagate in the bulk.

a) What is the mass dimension of \( y_D^{(2)} \)?

b) Writing \( y_D^{(2)} \) in units of \( \Lambda \), where \( \Lambda \) is defined by \( y_D \) above, estimate the NDA value of \( y_D^{(2)} \) from a 1-loop computation.

c) Show that the tree-level contribution of \( y_D^{(2)} \) to the \( \langle \Phi \Psi \bar{\Psi} \rangle \) 3-point function is as large as \( y_D \), for \( E \sim \Lambda \).

Thus, the NDA couplings give an estimate for the “boundary” between perturbativity and non-perturbativity. Before giving an efficient prescription for obtaining the NDA estimates, it will be useful to look at two more examples.
Gauge Self-energy and Gauge Coupling

As was already noted, the gauge couplings become dimensionful for \( D \neq 4 \). Consider the 1-loop contribution of a fermion to the gauge self-energy:

\[
\mu \nu \quad q \quad p \quad \sim \quad (ig_D)^2 \int \frac{d^Dp}{(2\pi)^D} \text{Tr} \left( \Gamma^\mu \frac{i}{p - M_\Psi} \Gamma^\nu \frac{i}{(p + q) - M_\Psi} \right). 
\]

Superficially, the integral diverges like \( \Lambda^{D-2} \). However, gauge invariance requires that it be proportional to \( q^2 \eta_{\mu\nu} - q_\mu q_\nu \). As is well-known, to obtain such a result it is necessary to use a gauge-invariant regulator, such as Pauli-Villars regularization.\(^3\) Nevertheless, for power-counting arguments it is sufficient to extract the above transverse projector, and estimate the remaining integral with a straightforward momentum cutoff:

\[
\mu \nu \quad q \quad \sim \quad (q^2 \eta_{\mu\nu} - q_\mu q_\nu) \frac{g_D^2}{l_D} \int^{\Lambda} p^{D-2} dp^2 \frac{1}{p^4} \sim \quad (q^2 \eta_{\mu\nu} - q_\mu q_\nu) \frac{g_D^2}{l_D} \frac{1}{D - 4} \Lambda^{D-4}, \quad (11)
\]

where we omitted the factors from the trace (which count the fermionic degrees of freedom). Adding this to the tree-level contribution, which is simply \( q^2 \eta_{\mu\nu} - q_\mu q_\nu \), we get the correction factor

\[
1 + \mathcal{O}(1) \times \frac{1}{l_D} g_D^2 \Lambda^{D-4} + \cdots \sim \quad 1 + \frac{c_2^2}{l_D} + \cdots, \quad (12)
\]

where we wrote \( g_D = c_g/\Lambda^{\frac{1}{2}D-2} \). Thus, the NDA value is \( c_g^{\text{NDA}} = l_D^{1/2} \).

**Exercise:** For non-abelian groups there are several states in a given irreducible representation, leading to a counting factor \( \text{Tr}(T^A T^B) = d_F \delta^{AB} \) [in the standard physics normalization, \( d_F = \frac{1}{2} \) in the fundamental rep.]. By considering the self-energy due to the gauge self-interactions, show that the correction goes like \( N_c c_g^2 / l_D \), where \( N_c \) is the “number of colors”. Hence a somewhat better estimate is \( c_g^{\text{NDA}} = (l_D / N_c)^{1/2} \).

As we will see, the connection between the (dimensionful) 5D gauge coupling and the (dimensionless) 4D gauge coupling takes the form

\[
g_4^2 = \frac{g_5^2}{L} = \frac{c_g^2}{\Lambda L}, \quad \text{(tree-level matching)} \quad (13)
\]

\(^3\)Although dimensional regularization is very convenient in practical computations, by defining the power-law divergences to zero, it obscures the physical points that interest us here.
where $L$ is the size of the compactified fifth dimension. Since for the SM gauge couplings [e.g. $g_s$ for $SU(3)_C$], $g_4^2 \sim O(1)$, we find based on NDA\footnote{One may want to write $\Lambda L \sim l_5/\pi N_c$ anticipating numerical factors in the direction suggested by the Pauli-Villars regularization.}:

$$\Lambda L \sim \frac{l_5}{N_c},$$

(14)

where $\Lambda$ is interpreted as the cutoff associated to the 5D $SU(3)_C$ gauge interactions.

**Important lesson:** the cutoff $\Lambda$ cannot be too far above $1/L$, of order the compactification scale (in flat space).

**Exercise:**

a) Using that, in flat space, $L \sim \pi R$, where $1/R \equiv m_c$ characterizes the scale of the new 5D physics (beyond the SM), estimate $L R \sim \text{number of KK-modes below the cutoff}$.

b) Repeat the exercise for arbitrary $D$.

**Localized Interactions**

We have already mentioned that realistic extra-dimensional models contain special subspaces (defects) on which fields, or interactions that may involve both bulk and localized fields, are localized. The presence of the defect implies that momentum is not conserved in certain directions (due to the breaking of translational invariance by the defect). For instance, consider the operator in 5D:

$$\mathcal{L}_5 \supset \delta(y - y_0) \tilde{y}_5 \Phi \overline{\Psi} \Psi,$$

(15)

where $[\tilde{y}_5] = -3/2$. The presence of the $\delta$-function implies that when going to momentum space, the $dy$ integration is trivial and cannot be used to recover the representation of the momentum-conserving $\delta$-function along the fifth dimension. Thus, some of the $dp_5$ integrals coming from the $\int d^5p \ldots$ representation of the propagators cannot be “eliminated” and remain as part of the loop integration. For example, taking vanishing external momenta,

$$\int \frac{d^5p}{(2\pi)^5} \frac{dp_5}{(2\pi)} \frac{i}{p - M_\Psi} \frac{i}{p' - M_\Psi} \frac{i}{p'' - M_\Phi}.$$

**Exercise:** if $q_1, q_2$ and $q_3$ denote the external momenta, what relation should their fifth components satisfy, using only momentum conservation at the two $y_5$ (bulk) vertices? Based on the above, which operator is being renormalized by the above diagram? Double-check your conclusion by computing the mass dimension of the above diagram, and comparing against those of $y_5$ and $\tilde{y}_5$. 


We can now estimate the diagram by power counting:

$$\frac{1}{2\pi} \frac{1}{l_5} \tilde{y}_5 y_5^2 \Lambda^2 \sim y_5 \times \frac{1}{2\pi} \frac{1}{l_5} \frac{\tilde{c}_y}{\Lambda^{3/2}} \frac{c_y}{\Lambda^{1/2}} \Lambda^2 .$$

Plugging in $c_y = c_y^{\text{NDA}} = \sqrt{l_5}$, and applying the NDA rules to this diagram, we conclude that

$$\frac{1}{2\pi} \frac{\tilde{c}_y^{\text{NDA}}}{\sqrt{l_5}} \sim 1 \implies \tilde{c}_y^{\text{NDA}} = 4\sqrt{6} \pi^{5/2} \sim 1.1 \times l_4 ,$$

where $l_4 = 16\pi^2$. The fact that a 4D loop factor is recovered is not a coincidence.

**NDA Made Trivial**

As we have seen, the game is to use standard dimensional analysis by expressing all quantities in units of $\Lambda$, as well as keeping track of the explicit $\pi$-factors from the phase space integrations (the "naive" part), and maybe also an even more naive estimate to count the number of states.

Recall that the loop expansion (with an additional $\frac{1}{16\pi^2}$ loop factor per loop in 4D) follows from keeping track of $\hbar$, which appears as an overall factor in front of the action in the path-integral formulation of QFT. The same argument suggests that we write

$$\mathcal{L}_D = \frac{N}{l_D} \mathcal{L}_{\text{Bulk}}(A_M, \Phi, \Psi) + \frac{N}{l_4} \delta^{(D-4)}(y - y_0) \mathcal{L}_{\text{Brane}}(\Phi, \phi, \ldots) ,$$

where all the couplings in $\mathcal{L}_{\text{Bulk}}$ and $\mathcal{L}_{\text{Brane}}$ are expressed in units of $\Lambda$ with coefficients of order one (except for well-understood combinatoric factors when several identical fields are involved). For instance,

$$\mathcal{L}_D = \frac{N}{l_D} \left\{ -\frac{1}{4} \Lambda^{D-4} F_{MN} F^{MN} + \overline{\Psi} \not{D} \Psi + |D_M \Phi|^2 + \Lambda^{2-D} \Phi \not{\partial} \Phi \Psi + \ldots \right\}
+ \frac{N}{l_4} \delta^{(D-4)}(y - y_0) \left\{ |D_\mu \phi|^2 + \ldots + \Lambda^{4-D} \phi \not{\partial} \phi \Psi + \Lambda^{6-D} \phi \not{\partial} \phi \Psi + \ldots \right\} .$$

If there are self-interactions for a real scalar, we would write e.g. $\frac{1}{n!} \Lambda^k \phi^n$, where $k$ depends on whether this is a bulk or localized interaction/field.

The idea behind Eq. (18), borrowed from the semi-classical expansion, is that the explicit factors in front of $\mathcal{L}_{\text{Bulk}}$ and $\mathcal{L}_{\text{Brane}}$ will precisely cancel the loop factors appearing at each order in the loop expansion, thus realizing the “strong coupling” regime discussed above. Of course, it is possible that some of the operators have smaller coefficients than above. The aim here is simply to characterize the maximum size of those coefficients.\footnote{If we include in Eq. (18) an overall factor of $1/\epsilon$, for $\epsilon < 1$, we can characterize a situation where each additional loop brings a suppression of $\epsilon$ in processes with $E \sim \Lambda$.} Regarding the factor of $N$, the cancellation need not work perfectly since the dependence on the specific field content of the theory is more complicated than can be parameterized above. Nevertheless, the factors of $N$ give us a rough idea of how the multiplicity of states affects the results.
Note that we chose to write a 4D defect simply described by a \( \delta \)-function, and that we write a 4D loop factor in front (the generalization to defects of other dimensionalities is straightforward). The 4D loop factor is easily understood for processes that involve only localized fields and interactions (works identically to \( 1/l_\text{D} \) in front of \( \mathcal{L}_{\text{Bulk}} \)). As we will see, it also reproduces our less trivial result involving both bulk and localized interactions (and works more generally).

In order to use the NDA parametrization given in Eq. (18), we first normalize the fields canonically:

\[
\Psi \rightarrow \left( \frac{l_\text{D}}{N} \right)^{1/2} \Psi,
\]

\[
\Phi \rightarrow \left( \frac{l_\text{D}}{N} \right)^{1/2} \Phi,
\]

\[
\phi \rightarrow \left( \frac{l_4}{N} \right)^{1/2} \phi,
\]

Although one can also normalize canonically the gauge field, it is more useful here to keep conventions with \( D_M = \partial_M + iA_M \), so that \([A_M] = 1\) and the gauge coupling can be read from the coefficient of the gauge kinetic term, as in Eq. (6). Then the Lagrangian in Eq. (19) becomes

\[
\mathcal{L}_D = -\frac{1}{4} \frac{N}{l_\text{D}} \Lambda^{D-4} F_{MN} F^{MN} + i \overline{\Psi} D \Psi + |D_M \Phi|^2 + \left( \frac{l_\text{D}}{N} \right)^{1/2} \Lambda^{2-\frac{D}{2}} \overline{\Phi} \Phi \Psi + \cdots
\]

\[
+ \delta^{(D-4)}(y - y_0) \left\{ |D_M \phi|^2 + \left( \frac{l_4}{N} \right)^{1/2} \frac{l_\text{D}}{l_4} \Lambda^{4-D} \overline{\phi} \phi \Psi + \frac{l_\text{D}}{l_4} \left( \frac{l_\text{D}}{N} \right)^{1/2} \Lambda^{6-\frac{3D}{2}} \overline{\phi} \phi \Psi + \cdots \right\},
\]

from which we can immediately read

\[
y_\text{D}^2 = \frac{l_\text{D}/N}{\Lambda^{D-4}}, \quad y_D = \left( \frac{l_\text{D}}{N} \right)^{1/2} \Lambda^{2-\frac{D}{2}},
\]

\[
y_D' = \left( \frac{l_4}{N} \right)^{1/2} \frac{l_\text{D}}{N} \Lambda^{4-D}, \quad \bar{y}_D = \frac{l_\text{D}}{l_4} \left( \frac{l_\text{D}}{N} \right)^{1/2} \Lambda^{6-\frac{3D}{2}}.
\]

**Exercise:**

a) Compare to the previous results and convince yourself that this works.

b) Choose a 1-loop example that renormalizes \( y_D' \), and check the above NDA estimate.

**What have we achieved?**

We have emphasized that higher-dimensional theories have an intrinsic cutoff \( \Lambda \), and that some observables are very sensitive to the physics at and above \( \Lambda \), and are therefore incalculable within the extra-dimensional model. One should simply include in the theory the corresponding local operators, with coefficients *to be determined by experiment*. In addition, the simple considerations described above allow us to
• Estimate the largest that the cutoff can be (in a given model).

• Estimate the maximum (and perhaps, expected) size of the coefficients of the tower of operators consistent with the assumed symmetries.

The above situation is not intrinsically different from other 4D effective theories, but it is important to keep in mind the limitations imposed by the existence of a cutoff relatively close to the scale of the new extra-dimensional physics. In many models of interest, the SM interactions are promoted to higher-dimensional bulk interactions. To estimate the cutoff, we can then focus on the strongest of these, namely the $SU(3)_C$ and top Yukawa interactions. In flat 5D space, we then expect that the number of KK-modes below the cutoff, $N_{KK} \lesssim 24\pi / N_c \sim 8\pi < O(30)$, where we used the estimate suggested by a Pauli-Villars regularization (see footnote 4). We will see later that this is reduced in the presence of 5D curvature (even in flat space, other considerations based on unitarity can put more stringent bounds, see [7]).

**Extra Dimensions and the Hierarchy Problem**

The previous discussion also highlights a possible application to particle physics. With a weakly coupled SM(-like) Higgs boson, one faces a potential naturalness problem, associated with the quadratic sensitivity of the Higgs mass parameter to ultra-short distance physics. This suggests that there should exist a physical cutoff at or near the TeV scale, such that the Higgs quadratic divergences can be under theoretical control, and hence the EW scale can be understood without fine-tuned cancellations within the (unknown) high-energy contributions.

Having extra-dimensions at the TeV scale, provides a strong rationale for why a cutoff should exist near that scale. Although this observation by itself does not fully address the Higgs hierarchy problem, it does provide a possible starting point for a potentially deeper, and perhaps more satisfactory, understanding of the naturalness issue. Possible additional features one would like to have in this context are:

1. A mechanism that selects the compactification scale to be near the TeV scale. If one regards other higher scales (such as the Planck scale) as fundamental, one would like to “generate” the compactification scale dynamically.

2. A dynamical connection between the compactification scale/extra-dimensional physics, and the breaking of the EW symmetry.

Note, however, that even in the absence of such theoretical “goodies”, if an extra-dimensional structure at the TeV scale was discovered experimentally, we would be forced to accept the existence of a cutoff not far above. In this sense, the big Planck/weak scale hierarchy problem of the 4D SM would become less urgent to fully solve: the priority would be to understand the potential “little hierarchy” between the weak scale and the cutoff associated with the extra-dimensional physics.

In the above sense, generic extra-dimensional models (not necessarily addressing points 1. and 2. above) can be regarded as motivated by the hierarchy problem, provided the compactification scale is near the weak scale, similar to the SUSY solution to the hierarchy problem when the superpartner masses are around a TeV (although, of course, there are significant differences...
between the two approaches). This provides our motivation for focusing on TeV scale extra dimensions in these lectures.

**Additional comments**

Our NDA discussion was framed in the language of uncompactified extra-dimensional field theories, even though we know that any extra-dimension would have to be compactified. Technically, the distinction is contained in the replacement (in flat 5D, for illustration)

$$\int \frac{dp_5}{2\pi} \rightarrow \frac{1}{2\pi R} \sum_n ,$$

where $\Delta p_5 = 1/R$. Since the compact extra dimension is different from the four non-compact ones, strictly speaking we cannot assume spherical symmetry as we did when we computed $\Omega_D$, the $D$-dimensional solid angle. Nonetheless, our interest was mainly in the UV contributions (which is what NDA is useful for). Thus, to the extent that $\Lambda R \gg 1$, the discrete sums can be approximated by integrals, and we conclude that, up to $O(1)$ factors, our discussion carries over to the compact case without change.

We further assumed that the background was exactly flat. Non-zero curvature is an intrinsic feature of many extra-dimensional models. Nevertheless, again the fact that we were interested in ultra-short distance physics allows us to extend our discussion to warped backgrounds. All we need is that the curvature $k$ be small compared to the cutoff $\Lambda$. In this case it is intuitively clear (and can be checked in explicit examples) that curvature effects play a negligible role in the NDA discussion. Note that the hierarchy $k \ll \Lambda$ is typically a tacit assumption, or else we would have little control over the gravitational background.

The most important difference between flat and warped backgrounds is the relation between the compactification scale, $m_{KK}$, and the “size” of the extra dimension, e.g. as measured with a meter stick (if we could do that):

- **Flat space:** $m_{KK} = 1/R = \frac{\pi}{L}$,
- **Warped space:** $m_{KK} \sim ke^{-kL}$ if $kL \gg 1$.

In both cases, as argued above, one has that $\Lambda L \sim l_5/(\pi N_c)$, but the number of KK modes below $\Lambda$ read

- **Flat space:** $N_{KK} = \Lambda R = \frac{l_5}{\pi^2 N_c}$,
- **Warped space:** $N_{KK} = \frac{\Lambda}{k} = \frac{1}{kL} \frac{l_5}{\pi N_c}$.

The fact that in the warped case the number of KK modes below the cutoff is given by $\Lambda/k$ will become clear when we have developed the necessary technology. The point here is that the suppression factor from $1/kL$ can be significant, and therefore one expects far fewer KK modes before reaching the strong-coupling regime in warped scenarios than in flat space models.
2 Lecture 2: The Tools of the Trade

In this lecture we will collect a number of basic results that will allow us to discuss specific extra-dimensional models. It will be worthwhile doing it in some generality, that can be specialized according to the cases of interest.

We have already mentioned that the compactification of the extra dimensions must be accompanied by certain “singularities” or “defects”. This is essential to obtain models that are chiral at low energies (e.g. the LH and RH electrons have different quantum numbers). A useful construction is called a “Field theory orbifold”. However, in 5D, a more straightforward and somewhat more general prescription is to compactify on an interval, i.e.

\[
\begin{align*}
    ds^2 &= g_{MN} dx^M dx^N \\
    &= e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2 ,
\end{align*}
\]

where \( y \in [0, L] \), and \( A(y) \) is arbitrary at this point (although we can always rescale the \( x^\mu \) to require \( A(0) = 0 \), which we shall do). Thus, one assumes that the extra dimension is a simply connected compact space. Our ansatz for the line element is the most general one exhibiting 4D Lorentz invariance. The “flat 4D sections” correspond to the vanishing of the effective 4D cosmological constant. The \( y \)-coordinate is chosen here as the proper distance along the extra dimension. If \( A(y) = \text{const} \) then the 5D spacetime is flat; otherwise we say it is warped.

2.1 Boundary Conditions

An important point to realize is that when the spacetime has a boundary at a finite distance, the theory is not fully defined until one specifies the boundary conditions (b.c.’s). In infinite spacetime, the issue of boundary conditions is often implicit because there is a natural choice: the fields should vanish at infinity “sufficiently fast”. This corresponds to the physical notion that we are dealing with local physics, and the same prescription will be used for the 4 non-compact dimensions in our case. However, there can be several boundary conditions at \( y = 0 \) and \( y = L \) that are physically acceptable. Nevertheless, not quite anything is allowed: the boundary conditions should reflect the fact that nothing escapes the interval, just because nothing is supposed to exist outside.

A general prescription for finding the b.c.’s (in the above sense) was described in detail in the “TASI Lectures on EWSB and Extra Dimensions” by Csáki, Hubisz and Meade. We will briefly review the idea, and refer the student to those lectures for further details.

For simplicity, consider a real scalar field:

\[
S = \int d^5 x \sqrt{g} \left\{ \frac{1}{2} g^{MN} \partial_M \Phi \partial_N \Phi - V(\Phi) \right\} ,
\]

where it is assumed that the metric background [i.e. the function \( A(y) \)] is specified. Under a variation \( \delta \Phi \), one finds

\[
\delta S = \delta S_V + \delta S_S ,
\]
where
\[
\delta S_V = - \int d^5x \sqrt{g} \left\{ \frac{1}{\sqrt{g}} \partial_M \left[ \sqrt{g} g^{MN} \partial_N \Phi \right] + \frac{\partial V}{\partial \Phi} \right\} \delta \Phi ,
\] (26)
is a volume term, and
\[
\delta S_S = \int d^4x \sqrt{g} g^{5N} \partial_N \Phi \cdot \delta \Phi \bigg|_{y=L},
\] (27)
is a surface term arising from the integration by parts. We assumed that \( \Phi \to 0 \) as \( x^\mu \to \pm \infty \).

We recognize \( \delta S = 0 \) as the variational principle that allow us to derive the equations of motion (EOM) for \( \Phi \), provided the b.c.’s are such that the surface term vanishes.

The idea then is to impose, for arbitrary \( \delta \Phi \),
\[
\delta S_V = 0 \quad \& \quad \delta S_S = 0.
\] (28)
The first condition leads to the EOM
\[
\frac{1}{\sqrt{g}} \partial_M \left[ \sqrt{g} g^{MN} \partial_N \Phi \right] + V'(\Phi) = 0,
\] (29)
which in our background reads
\[
e^{2A} \partial_\mu \partial^\mu \Phi - e^{4A} \partial_y \left[ e^{-4A} \partial_y \Phi \right] + V' = 0.
\] (30)
Note that here \( \partial_\mu \partial^\mu \) is understood to be contracted with the Minkowski metric. The second condition in Eq. (28) defines the allowed set of b.c.’s (in our diagonal background):
\[
- \partial_y \Phi \delta \Phi \bigg|_{y=L} + \partial_y \Phi \delta \Phi \bigg|_{y=0} = 0.
\] (31)
One way to satisfy this condition is to impose periodicity: \( \Phi(L) = \Phi(0) \) and \( \Phi'(L) = \Phi'(0) \). However, this defines the compactification on a circle, which is a smooth manifold and leads to a vector-like low-energy theory, as we will soon see when we describe bulk fermions. We therefore discard this option as uninteresting for our phenomenological applications.

Other simple possibilities are to impose Neumann or Dirichlet boundary conditions:
\[
\text{Neumann (N):} \quad \partial_y \Phi = 0,
\]
\[
\text{Dirichlet (D):} \quad \Phi = 0,
\]
in various combinations, that we denote by
\[
(+, +) = (N \text{ at } y = 0, N \text{ at } y = L),
\]
\[
(+, -) = (N \text{ at } y = 0, D \text{ at } y = L),
\]
\[
(-, +) = (D \text{ at } y = 0, N \text{ at } y = L),
\]
\[
(-, -) = (D \text{ at } y = 0, D \text{ at } y = L).
\]
The upshot, from a technical point of view, is that
• We can freely integrate by parts and discard the surface terms in all our manipulations.
• The EOM with the allowed b.c.’s defines a self-adjoint eigenvalue problem:
   a) Real eigenvalues
   b) Orthogonal eigenfunctions

These properties turn out to be extremely convenient in the manipulations and physical interpretation that follow. From a physics perspective, the above properties allow us to define conserved charges, i.e. no flow outside $[0, L]$. In this sense, there is really nothing outside the interval!

Nonetheless, N or D boundary conditions are not the most general ones. Sometimes one finds mixed b.c.’s, e.g.

$$\partial_y \Phi |_{y=0} - m \Phi |_{y=0} = 0 .$$

Typically, this indicates that there is a source on the boundary. For instance, adding a boundary term to the action

$$\Delta S_{\text{boundary}} = - \int d^4 x \sqrt{\bar{g}} \frac{1}{2} m \Phi^2 |_{y=0}$$

leads to a contribution to $\delta S$ that results in the mixed N/D boundary condition above. Note that we use the induced metric, denoted by $\bar{g}_{\mu\nu}$, to write the boundary terms. In Section 4.1 we provide expressions for the case with the most general quadratic boundary terms. What matters is that, as long as the b.c.’s can be derived from the variational procedure described above, we are guaranteed to have the mathematical properties that lead to sensible physics.

### 2.2 Fermions in 5D

The description of 5D fermions proceeds as follows. First, we need five anticommuting Dirac $\Gamma$-matrices, for which we can take

$$\Gamma^A = (\gamma^\alpha, -i\gamma_5), \quad \alpha = 0, 1, 2, 3$$

where, in the Weyl representation,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (1_{2\times2}, \bar{\sigma}), \quad \bar{\sigma} = (1_{2\times2}, -\bar{\sigma})$$

$$\gamma_5 = \begin{pmatrix} -1_{2\times2} & 0 \\ 0 & 1_{2\times2} \end{pmatrix},$$

which obey $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$. Here $\bar{\sigma}$ stands for the three Pauli matrices, while $\gamma_5$ is the 4D chirality operator. It is also useful to define the projectors on 4D chirality, $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$.

One also needs the fünfbein, $e^A_M$ defined by

$$g_{MN} = e^A_M e^B_N \eta_{AB},$$

21
and the inverse fünfbein $e^M_A$ obeying $e^M_A e_M^B = \delta^B_A$. In our background, the latter is
\[ e^\mu_\alpha = e^{+A(y)} \delta^\mu_\alpha : \quad e^5_\alpha = +1 , \]
with all other components vanishing.

In general backgrounds one also needs a spin connection to define the covariant derivative w.r.t. both general coordinate and local Lorentz transformations. In our case, the covariant derivative, $D_M = \partial_\mu + \frac{i}{8} \omega_{MAB} [\Gamma^A, \Gamma^B]$, takes the form
\[ D_\mu = \partial_\mu - \frac{i}{2} e^{-A} A' \gamma_\mu \gamma_5 , \quad D_5 = \partial_5 . \]

The (explicitly hermitian) action for a free 5D fermion reads
\[ S_\Psi = \int d^5 x \sqrt{-g} \left\{ \frac{i}{2} \Psi^M A^M D_M \Psi - \frac{i}{2} (D_M \Psi)^\dagger \Gamma^0 e^M_A \Gamma^A \Psi - M \overline{\Psi} \right\} . \]

The determination of the allowed b.c.’s proceeds via the variational principle described in the scalar case. Here the most direct procedure is to replace first the metric background, noting that the spin connection cancels out in the action and we might as well replace $D_M \rightarrow \partial_M$:
\[ S_\Psi = \int d^5 x e^{-3A} \left\{ i \gamma^\mu \partial_\mu \Psi + \frac{1}{2} e^{-A} \left[ \overline{\Psi} \gamma_5 \partial_5 \Psi - (\partial_5 \overline{\Psi}) \gamma_5 \Psi \right] - e^{-A} M \overline{\Psi} \right\} , \]
where we integrated by parts along $x^\mu$ (but not $y$) and dropped the corresponding terms, assuming that the field vanishes at infinity. Now we can proceed with the variation w.r.t. $\Psi$. For instance, under $\delta \Psi$:
\[ \delta S_\Psi^V = \int d^5 x e^{-3A} \delta \overline{\Psi} \left\{ i \gamma^\mu \partial_\mu \Psi + e^{-A} \gamma_5 \partial_5 \Psi - \frac{1}{2} A' e^{-A} \gamma_5 \Psi - e^{-A} M \Psi \right\} , \]
while the integration by parts along $x^5 = y$ generates the surface term
\[ \delta S_\Psi^S = - \int d^4 x e^{-4A} \delta \overline{\Psi} \gamma_5 \Psi \bigg|_{y=0}^{y=L} . \]

Our prescription of requiring $\delta S_\Psi^V = \delta S_\Psi^S = 0$ under any $\delta \overline{\Psi}$ leads to the EOM
\[ \left\{ i e^A \gamma^\mu \partial_\mu + \left( \partial_5 - \frac{1}{2} A' \right) \gamma_5 - M \right\} \Psi = 0 , \]
and to the requirement
\[ - \delta \overline{\Psi} \gamma_5 \Psi \bigg|_{y=L} + \delta \overline{\Psi} \gamma_5 \Psi \bigg|_{y=0} = 0 . \]

It is useful to express this in terms of $\Psi_{L,R} \equiv P_{L,R} \Psi$:
\[ \delta \overline{\Psi}_L \Psi_R - \delta \overline{\Psi}_R \Psi_L \bigg|_{y=0}^{y=L} = 0 . \]

We can now be more explicit about the relation between chirality and compactification:

---

\[ \text{For completeness, recall that } \omega^A_{AB} = e^A_N (\partial_M e^N_B + e^S_B \Gamma^N_M ) , \text{ where the Christoffel symbols are } \Gamma^K_M_N = \frac{1}{2} g^{KL} \left\{ \partial_N g_{LM} + \partial_M g_{LN} - \partial_L g_{MN} \right\} \text{. Here the only non-vanishing components are } \omega^a_5 = -\omega^5_a = \partial_5 (e^{-A}) \delta^a_5 . \]
i) If we were to impose periodic b.c.’s (compactification on a circle)

\[
\Psi_L|_{y=L} = \Psi_L|_{y=0} \quad \& \quad \Psi_R|_{y=L} = \Psi_R|_{y=0},
\]

there would be nothing that distinguishes between the two chiralities, L and R.

ii) If, on the other hand, we treat the two boundaries in our general condition \((44)\) separately (interval compactification), we see that we can use

\[
\Psi_L| = 0 \quad \text{or} \quad \Psi_R| = 0.
\]

Either one of these (with the four possible combinations at the two boundaries) is sufficient to ensure \(\delta S^S_{\Psi} = 0\). In fact, we do not have the right to impose Dirichlet b.c.’s conditions on both chiralities at a given boundary!

Indeed, recalling the EOM \((42)\), which can be split into L and R as

\[
i e^{A_\gamma^\mu} \partial_\mu \Psi_L + \left[ \left( \partial_5 - \frac{1}{2} A' \right) - M \right] \Psi_L = 0,
\]

\[
i e^{A_\gamma^\mu} \partial_\mu \Psi_R + \left[ - \left( \partial_5 - \frac{1}{2} A' \right) - M \right] \Psi_L = 0,
\]

we see that, if \(\Psi_L| = 0\), then the first equation implies

\[
\partial_5 \Psi_R| = \left( \frac{1}{2} A' + M \right) \Psi_R|.
\]

If we were to also impose \(\Psi_R| = 0\), we would automatically have \(\partial_5 \Psi_R| = 0\), and then –from the second equation– we would find \(\partial_5 \Psi_L| = 0\). But then the only allowed solution of the system of two first-order differential equations (in \(y\)) is \(\Psi \equiv 0\).

Thus, we settle on two interesting possibilities:

\((-) \equiv \Psi_L| = 0, \quad \text{hence} \quad \partial_5 \Psi_R| = \left( \frac{1}{2} A' + M \right) \Psi_R|,
\]

\[\text{or} \]

\((+) \equiv \Psi_R| = 0, \quad \text{hence} \quad \partial_5 \Psi_L| = \left( \frac{1}{2} A' - M \right) \Psi_L|.
\]

[Note the sign flip on the mass term]. This is at the basis of the advertised result: compactification on an interval necessarily leads to boundary conditions that distinguish L from R, which will allow us to easily embed the SM structure.

Similar to the scalar case, specifying the boundary conditions on both boundaries leads to four possibilities that we label as

\((+,+) \quad (+,-) \quad (-,+) \quad (-,-)\).

Also, as in the scalar case, one can generalize these boundary conditions by including localized terms in the action (via a \(\delta\)-function, i.e. a boundary term), which contribute directly to \(\delta S^S_{\Psi}\).
2.3 5D Gauge Fields

The case of gauge fields, which will be the last case we will review here, involves a new ingredient, the need for gauge fixing:

\[
S_A = \int d^5x \sqrt{g} \left\{ -\frac{1}{4} g^{MN} g^{KL} F_{MK} F_{NL} \right\} + S_{GF},
\]

(50)

where the gauge-fixing term \( S_{GF} \) will be specified shortly. In this section we will be interested in the quadratic part of the action, so that there is no distinction between the abelian and non-abelian cases, and we can take \( F_{MN} = \partial_M A_N - \partial_N A_M \). To motivate the choice of \( S_{GF} \) let us expand the gauge kinetic term in our background, Eq. (23):

\[
\int d^5x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} e^{-2A} \partial_\mu A_5 \partial^\mu A_5 - \partial_5 \left[ e^{-2A} A_5 \right] \partial_\mu A^\mu + \frac{1}{2} e^{-2A} \partial_5 A_\mu \partial_5 A^\mu \right\},
\]

(51)

where all the contractions are now done with the Minkowski metric. We also integrated by parts in both \( x^\mu \) and \( y \) and dropped the terms at infinity, as usual. However, we must keep track of the generated surface term in the \( y \) direction:

\[
S_S = \int d^4x \left. e^{-2A} A_5 \partial_\mu A^\mu \right|_{y=0}.
\]

(52)

We see that there is a bulk term mixing the \( A_\mu \) and \( A_5 \) components, which also results in mixing in the EOM. The gauge-fixing action is chosen so as to cancel this term \[9\], specifically:

\[
S_{GF} = \int d^5x - \frac{1}{2\xi} \left\{ \partial_\mu A^\mu - \xi \partial_5 \left[ e^{-2A} A_5 \right] \right\}^2,
\]

(53)

where \( \xi \) is an arbitrary gauge-fixing parameter. The total gauge action then becomes

\[
S_A = \int d^5x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \right\} \left\{ \partial_\mu A^\mu - \xi \partial_5 \left[ e^{-2A} A_5 \right] \right\}^2\left\{ -\frac{1}{2} \right\} e^{-2A} \partial_\mu A_5 \partial^\mu A_5 - \frac{1}{2} \xi \left( \partial_5 \left[ e^{-2A} A_5 \right] \right)^2,
\]

(54)

together with the surface term, Eq. (53). We may now apply the variational procedure. Under \( \delta A_\mu \):

\[
\delta S_A^V = \int d^5x \delta A_\mu \left\{ \left[ \eta^{\mu\nu} \Box - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] \right\} A_\nu,
\]

\[
\delta S_A^S = \int d^4x e^{-2A} \delta A_\mu \left\{ \partial_5 A^\mu - \partial^\mu A_5 \right\} \left|_{y=L} \right.,
\]

(53)

where \( \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu \), and we took into account the variation of Eq. (53).
Under $\delta A_5$:

\[
\delta S_A^V = \int d^5 x \, e^{-2A} \delta A_5 \left\{ -\Box + \xi \partial_5^2 e^{-2A} \right\} A_5,
\]

\[
\delta S_A^S = -\int d^4 x \, e^{-2A} \delta A_5 \left\{ \xi \partial_5 \left[ e^{-2A} A_5 \right] + \partial_\mu A^\mu \right\} \bigg|_{y=0}.
\]

Therefore, the EOM are:

\[
0 = \left[ \eta^{\mu\nu} \Box - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A_\nu - \partial_5 \left[ e^{-2A} \partial_5 A^\mu \right], \tag{55}
\]

\[
0 = \Box A_5 - \xi \partial_5^2 \left[ e^{-2A} A_5 \right]. \tag{56}
\]

Focusing on interval boundary conditions, we also identify two interesting possibilities:

\[(+) \equiv A_5 \bigg|_{y} = 0, \text{ hence } \partial_5 A_\mu \bigg|_{y} = 0, \]

\[(-) \equiv A_\mu \bigg|_{y} = 0, \text{ hence } \partial_5 \left[ e^{-2A} A_5 \right] \bigg|_{y} = 0. \]

Similarly to the case of fermions, we see that the b.c.’s for $A_\mu$ and $A_5$ are correlated. Again, there are four cases that we label by

\[(+,+) \quad (+,-) \quad (-,+) \quad (-,-), \]

where the first (second) entry refers to the b.c. at $y = 0$ ($y = L$).

Additional boundary terms in the action translate into a modification of the boundary conditions above, and are read with the same methods. One interesting case corresponds to localized gauge-kinetic terms, and we refer the interested reader to [10, 11, 12] for further details (see also Section 4.2.2 in the fourth lecture). We should also comment that the Faddeev-Popov procedure, when applied to our gauge-fixing choice, leads to ghost fields. However, we will not need them in these lectures.

**Exercise:**

a) Show in full detail how the $(+)$ and $(-)$ b.c.’s arise from $\delta S_A^S = 0$. In particular, argue for the correlation between the b.c.’s of $A_\mu$ and $A_5$.

b) Are there other types of b.c.’s consistent with $\delta S_A^S = 0$?

**Exercise:** It is possible that the bulk gauge field is coupled to a “Higgs field” that gets a non-zero vev. This can lead to an effective $y$-dependent mass for the gauge field, $M_A^2(y)$. How is the above formalism modified?

As we will see in the next lecture

- The SM gauge fields arise from $(+,+)$ b.c.’s.
- Models where the Higgs is interpreted as an extra-dimensional polarization of higher-dimensional gauge fields use $(-,-)$ b.c.’s.
- $(+,-)$ and $(-,+)$ b.c.’s show up in models with custodial symmetries, among others.
2.4 The Metric Background

So far, we have kept the metric background arbitrary, imposing only that it respects 4D Lorentz invariance. However, the choice of background is not only necessary to solve explicitly the previous EOM; it also largely determines the resulting phenomenology.

Presumably, the background is a solution to the higher-dimensional Einstein equations:\(^7\) There may exist (scalar) fields sourcing these equations, e.g.

\[ S = \int_M d^5x \sqrt{g} \left[ -\frac{1}{2} M_5^3 \mathcal{R}_5 + \frac{1}{2} \nabla_M \Phi_i \nabla^M \Phi_i - V(\Phi_i) \right] + \int_{\partial M} d^4x \sqrt{\bar{g}} L_4(\Phi_i) , \]  

(57)

where \( M_5 \) is the (reduced) 5D Planck mass, \( \mathcal{R}_5 \) is the 5D Ricci scalar, and the last term allows for operators localized on the boundary.\(^8\)

Solving the EOM that follow from the above action is, in general, quite hard. However, it turns out that there is a class of potentials that is amenable to fully analytic solutions to the coupled metric/scalar system. Before presenting the method, let us make a few remarks.

Although the class of potentials that allows for closed solutions is rather special (and not necessarily stable under radiative corrections), we can mention a number of reasons for why it is worthwhile to study these cases:

1. Having explicit solutions for varied (even if special) potentials allows to develop an intuition for the effects associated with backreaction.
2. In many cases, the physical properties of interest can be seen not to depend on the special relations that make the problem analytically soluble. Thus, the analytic solutions can be taken as a convenient way to understand the relevant physics.
3. Models based on this method have been recently proposed.
4. “Traditional” backgrounds, such as flat space or AdS\(_5\) are particular examples of this scheme. Although the machinery we will present is overkill for these simple cases, it allows for a unified treatment, and generalizations.

2.4.1 The “Superpotential” Method

This is a supergravity-inspired approach that was applied to extra-dimensional scenarios in \(^13\) and in a more general form in \(^14\). We assume a single scalar field in 5D (or, more generally, in codimension one). The class of potentials of interest here are those that can be written as

\[ V(\Phi) = \frac{1}{8} \left( \frac{\partial W(\Phi)}{\partial \Phi} \right)^2 - \frac{1}{6 M_5^2} W(\Phi)^2 , \]  

(58)

\(^7\)As long as we assume that the extra-dimensions share the properties of the dimensions we have observed, at least at short distances.

\(^8\) We also assume the presence of the Gibbons-Hawking term, \( \int_{\partial M} d^4x \sqrt{\bar{g}} M_5^3 K \), where \( K \) is the trace of the extrinsic curvature and \( \bar{g}_{\mu\nu} \) is the induced metric. In our coordinates, we have \( K_{\mu\nu} = \frac{1}{4} \partial_{\mu} \Phi \partial_{\nu} \Phi \), which implies that \( K = \Phi^\mu K_{\mu\nu} = 4A' \) in our background. This term is important when performing the variation of the action w.r.t. \( A(y) \) in order to determine the appropriate boundary conditions governing the gravity sector.
for some “superpotential” $W(\Phi)$. We emphasize that, in spite of the language, there is no assumption that supersymmetry is involved here.

The 5D EOM for the metric/scalar system are, in the bulk

$$4(A')^2 - A'' = -\frac{2}{3M_5^3} V ,$$

$$\langle A' \rangle^2 = \frac{1}{12M_5^3} (\Phi')^2 - \frac{1}{6M_5^3} V ,$$

$$\phi'' - 4A' \phi' = \frac{\partial V}{\partial \phi} ,$$

where we assumed the line element Eq. (23), and that the scalar vev $\langle \Phi \rangle = \phi(y)$ is $x^\mu$ independent (as required by 4D Lorentz invariance). The solutions with vanishing effective 4D Cosmological Constant (C.C.) (i.e. flat 4D sections, as in our ansatz) lead to a particularly simple solution when $V$ is given by Eq. (58). Indeed, it is straightforward to check that if

$$\phi'(y) = \frac{1}{2} \frac{\partial W(\phi)}{\partial \phi} ,$$

$$A'(y) = \frac{1}{6M_5^3} W(\phi(y)) ,$$

then Eqs. (59) are fulfilled. The above represents a huge simplification: given $W(\phi)$, Eq. (60) is a first-order ODE for $\phi$ that we can always solve by separation of variables. Having an explicit solution for $\phi(y)$, Eq. (61) has an explicit function of $y$ on the r.h.s., and can be integrated immediately to find $A(y)$. Thus, the problem has been reduced to quadrature.

We have described the bulk solution, but still need to specify the b.c.’s that allow such a solution. This will require specifying appropriate sources at the boundary $\partial M$. Indeed, Eqs. (60) and (61) imply

$$\phi'|_0 = \frac{1}{2} \frac{\partial W}{\partial \phi} , \quad A'|_0 = \frac{1}{6M_5^3} W|_0 .$$

We want to check that these conditions can be derived from a variational principle. Assuming that $L_4$ in Eq. (57) does not involve $y$ derivatives (e.g. if it is a pure potential term), varying the action with respect to both $\Phi$ and $A$, and requiring that the surface terms obtained from the integrations by parts vanish, leads to

$$\mp \Phi'|_{0,L} = \frac{\partial L_4}{\partial \Phi}|_{0,L} , \quad \mp A'|_{0,L} = \frac{1}{3M_5^3} L_4|_{0,L} .$$

9The method can be generalized to cases where the 4D geometry is de Sitter or Anti-de Sitter [4].

10Here it is important to include the terms arising from the variation of the Gibbons-Hawking term (see footnote 8). This generates a boundary term proportional to $\delta A'$ that cancels a similar term from the variation of the Einstein-Hilbert action, and leaves behind a term proportional to $\delta A$ that results in the l.h.s. of the second equation in (63).
We see that Eqs. (62) follow from Eqs. (63) when

\[ \mp L_4(\Phi) \bigg|_{0,L} = \frac{1}{2} W(\bar{\phi}_{0,L}) + \frac{1}{2} W'(\bar{\phi}_{0,L})(\Phi - \bar{\phi}_{0,L}) \mp \Delta L_4(\Phi), \]  

(64)

where \( \bar{\phi}_{0,L} \) are constants to be interpreted as the boundary field values

\[ \Phi(y = 0) = \bar{\phi}_0, \quad \Phi(y = L) = \bar{\phi}_L, \]  

(65)

and \( \Delta L_4 \) is arbitrary, except for the requirement that \( L_4(\Phi = \bar{\phi}_{0,L}) = 0 \). This freedom is often used to write a stiff potential that enforces Eqs. (65) in a simple way.

**Exercise:** Check the above statements in detail.

### 2.4.2 Application: Radion Stabilization

Generically, non-trivial scalar profiles lead to *radion stabilization*, i.e. to a preferred value for the size of the extra dimension, \( L \). This can be transparently seen from the above formalism. Assume, for simplicity, that \( \Delta L_4 \big|_{y=0,L} \) are stiff potentials that essentially fix the scalar boundary values to \( \bar{\phi}_0 \) and \( \bar{\phi}_L \). For instance,

\[ \Delta L_4(\Phi) \bigg|_{0,L} = -\gamma (\Phi^2 - \bar{\phi}_{0,L}^2)^2, \quad \text{for } \gamma \to \infty. \]  

(66)

The argument is illustrated in Fig. 5: basically, given the bulk solution and \( \bar{\phi}_0 \) and \( \bar{\phi}_L \), only special values of \( L \) lead to consistency between the bulk solution and the boundary conditions. Thus, a particular size for the extra dimension is selected *dynamically*.

Let us add a few comments:

- Depending on the bulk solution (i.e. the superpotential \( W \)) and the choice of the parameters \( \bar{\phi}_0 \) and \( \bar{\phi}_L \), it may happen that there is no “intersection” in Fig. 5, for example, if \( |\bar{\phi}_L - \bar{\phi}_0| > \max_y \phi(y) - \min_y \phi(y) \), where \( \phi(y) \) is the bulk profile. In such a case, one of the assumptions must break down, most likely the flat 4D sections ansatz. There may be solutions for the same theory which are de Sitter or Anti-de Sitter, for fixed \( y \).

- Somewhat related to the above, if the boundary Lagrangians \( L_4 \) do not have the special form given above (which require tuning), then what happens in general is that the solutions have non-vanishing 4D curvature.

- Assuming that the solution with 4D flat sections is allowed, what happens if we displace “by hand” the branes from their preferred separation? This is a dynamical, time-dependent situation that goes beyond our ansatz. If the static solution is stable, then the system will oscillate about the preferred brane separation. This motion corresponds to “radion excitations” and will show up as a particle with properties rather similar to the Higgs boson!

\[ ^{11} \] This happens here at tree-level, but it is possible to imagine stabilization mechanisms that rely on quantum effects, see e.g. [11, 15].
Figure 5: Illustration of “radion stabilization” as arising from the interplay between the bulk scalar solution and the boundary conditions. We may assume, by a shift of the $y$ coordinate, that $\langle \Phi(0) \rangle = \bar{\phi}_0$. The size of the extra dimension is then determined by $\langle \Phi(L) \rangle = \bar{\phi}_L$.

The issue of stability can be investigated on a case by case basis by analyzing small oscillations of the scalar modes in the gravity/bulk scalar system, to check that there are no negative mass eigenvalues (tachyonic modes). This also allows to determine the radion mass by the KK-decomposition procedure to be presented below.

- The stabilization of the radion by non-trivial bulk profiles was first studied in [16] in the limit that the scalar backreaction is small. In this limit the stabilization can be understood from a 4D effective potential, and the curvature at the minimum of this potential gives the radion mass. This is just a different language to study the same physics as above. It allows for more general potentials than the “superpotential approach”, but the parameters must be chosen so that the backreaction on the metric is a small effect.

- If $W$ is odd in $\Phi$, the resulting backgrounds have a KK-parity symmetry, as in Universal Extra Dimensional models (UED’s) – to be discussed in the next lecture – but allowing for non-trivial warping (see [17]).

### 2.4.3 Simple Limits

Two simple backgrounds may be easily obtained in the framework of the superpotential approach.

**Flat space:** Taking $W = 0$ (one may just remove $\Phi$ since it plays no role), one has $A'(y) = 0$, hence $A(y) = \text{constant}$. By rescaling $x^\mu$, we simply have $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu - dy^2$, i.e. flat space.

**AdS$_5$:** Removing again $\Phi$ and taking $W = 6M_5^3k$, for some constant $k$ (with mass dimension one), we have $A'(y) = k$, hence $A(y) = ky + \text{constant}$. By rescaling $x^\mu$, we can again set the constant to zero, and obtain

$$ds^2 = e^{-2ky} \eta_{\mu\nu}dx^\mu dx^\nu - dy^2,$$  \hspace{1cm} (67)
which can be recognized as Anti-de Sitter space with curvature $k$ (see Fig. 6). In this case, the required boundary Lagrangians read

$$\mathcal{L}_{4,0,L} = \pm 6M_5^3k \equiv T_{0,L},$$

(68)

while the bulk potential is $V = -6M_5^3k^2 \equiv \frac{1}{2}M_5^3\Lambda_5 < 0$. This reproduces the two fine-tunings found in $[18]^{12}$

$$T_0 = -T_L = -\frac{M_5^3\Lambda_5}{2k}.$$  

(69)

In this limit the radion has no potential (i.e. the size of the extra dimension is not stabilized), which accounts for one of the fine-tunings. The second tuning amounts to setting the 4D C.C. to zero, and thus is tied to the C.C. problem. Although this second fine-tuning is always present, the first one disappears once the radion is stabilized in the more general setup involving the scalar field.

### 2.5 The Kaluza-Klein Decomposition

A central concept in extra-dimensional physics is that the theory of a bulk field propagating in a compact extra dimension can be rewritten as a 4D theory involving an infinite number of 4D fields. For example, for a free scalar

$$\int d^5x \sqrt{g} \left\{ \frac{1}{2} \partial_M \Phi \partial^M \Phi - \frac{1}{2} M^2 \Phi^2 \right\} = \sum_n \int d^4x \frac{1}{2} \left\{ \partial_\mu \phi_n \partial^\mu \phi_n - m_n^2 \phi_n^2 \right\},$$

(70)

\[\text{Note that my } M_5^3 \text{ is their } 4M^3, \text{ and my } \frac{1}{2}M_5^3\Lambda_5 \text{ is their } \Lambda.\]
where we assumed 4D flat sections. The $\phi_n(x^\mu)$ are known as Kaluza-Klein (KK) modes. The information about the background (and $M$) is fully contained in the KK spectrum, $m_n$.

The above is just a generalization of the Fourier-mode decomposition, written here as

$$
\Phi(x^\mu, y) = \frac{e^{A(y)}}{\sqrt{L}} \sum_n \phi_n(x^\mu) f_n(y),
$$

(71)

where $\{f_n\}$ is any set of complete functions that allow us to expand an arbitrary $y$-dependence, with $x^\mu$-dependent coefficients, $\phi_n(x^\mu)$. We will see soon how to choose this set in the most convenient way. They will be called "KK wavefunctions". Note that in our expansion, Eq. (71), we pulled out an explicit factor of $e^{A(y)}$, which gives the $f_n$'s so defined a rather direct physical interpretation in terms of localization properties of the KK-modes along the extra dimension. Hence I will call them "physical wavefunctions" (This will become clearer in subsequent sections.) In addition, we also wrote a factor of $1/\sqrt{L}$ in Eq. (71). This allows us to display more transparently the different dimensions of the fields involved: $\Phi$ has mass dimension $3/2$, while the $\phi_n$ have mass dimension 1, as appropriate to their 4D interpretation. Our conventions will always be that the KK wavefunctions $f_n$ are dimensionless.

To understand how to choose the set $f_n$, recall the scalar EOM derived in Eq. (30), which reads here

$$
e^{2A} \partial_\mu \partial^\mu \Phi - e^{4A} \partial_y [e^{-4A} \partial_y \Phi] + M^2 \Phi = 0.
$$

(72)

The interpretation of $\phi_n$ as a (free) 4D scalar field of mass $m_n$ means that

$$
\partial_\mu \partial^\mu \phi_n + m_n^2 \phi_n = 0.
$$

(73)

If we take a special field configuration with a single $n$, $\Phi(x^\mu, y) = \frac{e^{A(y)}}{\sqrt{L}} \phi_n(x^\mu) f_n(y)$, and replace in Eq. (72) using Eq. (73), we get

$$
- e^{2A} m_n^2 f_n - e^{3A} \partial_y [e^{-4A} \partial_y (e^A f_n)] + M^2 f_n = 0,
$$

(74)

or more explicitly

$$
f''_n - 2A' f'_n + [A'' - 3(A')^2 - M^2 + e^{2A} m_n^2] f_n = 0.
$$

(75)

Here is where our discussion of boundary conditions pays off, for if we multiply by $e^{-2A} f_m$ and subtract the same expression with $m \leftrightarrow n$, we find

$$
\partial_y \left \{ e^{-2A} (f_m f'_n - f_n f'_m) \right \} + (m_n^2 - m_m^2) f_m f_n = 0.
$$

(76)

If we integrate $\int_0^L dy$, the first term gives rise to surface terms of exactly the form found when applying the variational procedure. Since these are required to vanish, we find

$$
(m_n^2 - m_m^2) \int_0^L dy f_m f_n = 0,
$$

(77)
i.e. we have derived an orthogonality relation for $m_n^2 \neq m_m^2$.\footnote{The same line of reasoning can be used to show that for two solutions, $h_1$ and $h_2$, of Eq. (75), one has $e^{-2A(y)} W(y) = e^{-2A(y')} W(y')$, where $W(y) = h_1(y)h_2(y) - h_2(y)h_1(y)$ is the Wronskian.} In fact, the differential equation for $f_n$, Eq. (71), together with the boundary conditions derived from the variational principle, define a self-adjoint problem that guarantees not only the orthogonality of the solutions, but also that the $\{f_n\}$ form a complete set that we can use to “Fourier” expand the $y$-dependence of any field configuration $\Phi(x^\mu, y)$.

Choosing the normalization
\[
\frac{1}{L} \int_0^L dy f_m f_n = \delta_{mn}, \quad (78)
\]
and replacing the KK expansion, Eq. (77) in the free bulk scalar action, it is straightforward to obtain the advertised result, Eq. (70). Of course, this 4D action just embodies the EOM for the KK modes, Eq. (73). The KK masses, $m_n^2$, are found by solving the ODE for the $f_n$, Eq. (75), and requiring that the desired b.c.’s be satisfied.

Note the philosophy here: the KK decomposition is defined in terms of the free action, while interactions are to be treated perturbatively after inserting the previous KK decomposition. For instance, for a $\frac{1}{3!}\lambda_5 \Phi^3$ interaction, we get in the equivalent KK theory:
\[
\mathcal{L}_{\text{int}} = \int_0^L dy \sqrt{g} \frac{1}{3!}\lambda_5 \Phi^3 = \sum_{m,n,r} \frac{1}{3!} \lambda_{mnr} \phi_m \phi_n \phi_r, \quad (79)
\]
where
\[
\lambda_{mnr} = \frac{\lambda_5}{L^{3/2}} e^{-A(L)} \int_0^L dy e^{-[A(y)-A(L)]} f_m(y) f_n(y) f_r(y). \quad (80)
\]
We have written the factors of $e^{\pm A(L)}$ for later physical interpretation. These overlap integrals carry information about the relative localization of the KK-modes.

**Exercise:** What are the mass dimensions of $\lambda_5$ and $\lambda_{mnr}$?

### 2.5.1 KK Decomposition for Fermions

We now quickly summarize the KK decomposition for a 5D fermion field. As in the scalar case, the KK wavefunctions are chosen to obey the classical (free) EOM, which together with the allowed b.c.’s guarantees that they form a complete, orthogonal set. We write now
\[
\Psi_{L,R}(x^\mu, y) = \frac{e^{\frac{2}{3}A(y)}}{\sqrt{L}} \sum_n \psi^n_{L,R}(x^\mu) f^n_{L,R}(y), \quad (81)
\]
where
\[
\left( \partial_y + M - \frac{1}{2} A' \right) f^n_L = m_n e^A f^n_R, \quad (82)
\]
\[
\left( \partial_y - M - \frac{1}{2} A' \right) f^n_R = -m_n e^A f^n_L, \quad (83)
\]
and
\[ \frac{1}{L} \int_{0}^{L} dy f_{L,R}^{m} f_{L,R}^{n} = \delta_{mn}, \] (84)

Note the factor \( e^{\frac{3}{2}A(y)} \) in Eq. (81) that defines the \( f_{L,R}^{n}(y) \) as physical wavefunctions, and that we allow for different KK expansions for the LH and RH chiralities, in accord with the fact that these are distinguished by the b.c.’s. The two types of boundary conditions discussed in Section 2.2 read
\[ (\cdot) : \partial_{y} f_{R}^{n} - \left( M + \frac{1}{2} A' \right) f_{R}^{n} \bigg|_{y} = 0 \quad \& \quad f_{L}^{n} \bigg|_{y} = 0, \] (85)

or
\[ (+) : \partial_{y} f_{L}^{n} + \left( M - \frac{1}{2} A' \right) f_{L}^{n} \bigg|_{y} = 0 \quad \& \quad f_{R}^{n} \bigg|_{y} = 0, \]

while the KK theory simply reads
\[ S_{\Psi} = \sum_{n} \int d^{4}x \bar{\psi}^{n} (i\gamma^{\mu} \partial_{\mu} - m_{n}) \psi^{n}. \] (86)

**Exercise:** Check the above results. Derive also decoupled 2nd order ODE’s for the massive modes \( f_{L}^{n} \) and \( f_{R}^{n} \).

### 0-modes

The KK fermion equations, together with the b.c.’s (85), have the very interesting property that they always allow a massless mode. Indeed, setting \( m_{n=0} = 0 \), the first order ODE’s, Eqs. (82) and (83) decouple and can be integrated immediately to give:
\[ f_{L,R}^{0}(y) = N_{L,R} \exp \left( \frac{1}{2} A(y) \mp \int_{0}^{y} dz \, M(z) \right), \] (87)

where the \( N_{L,R} \) are fixed by the normalization (84), and we allowed for a \( y \)-dependent mass, as would arise from a Yukawa coupling to a bulk scalar that gets a non-trivial vev profile:
\[ -y_{S} \Phi \overline{\Psi} \Psi \rightarrow -y_{S} \phi(y) \overline{\Psi} \Psi \equiv -M(y) \overline{\Psi} \Psi. \] (88)

Note also that the \((+,+)\) b.c.’s allow for \( f_{L}^{0} \) but imply \( N_{R} = 0 \), while the \((-,-)\) b.c.’s allow for \( f_{R}^{0} \) but imply \( N_{L} = 0 \). Thus, the 0-mode level is indeed chiral. Presumably, the SM fermions could be identified with such 0-modes. Note that the above KK-masses are related to the compactification of the extra dimension, but that there can be additional contributions to the physical masses arising from EWSB. Thus, in realistic models, the “0-modes” may not be strictly massless. Nevertheless, since typically the masses associated to EWSB are small compared to the compactification scale, it is customary to continue referring to these would-be zero-modes simply as 0-modes.
A special case arises when
\[ M(y) \approx \pm c A'(y) \]
for some dimensionless “c-parameter”. In such a case we have
\[ f^0_{L,R}(y) \approx f^0_c(y) \equiv N_0 e^{-(c-\frac{1}{2})A(y)} , \]
for some dimensionless “c-parameter”. In such a case we have
\[ f^0_{L,R}(y) \approx f^0_c(y) \equiv N_0 e^{-(c-\frac{1}{2})A(y)} , \]
independently of chirality [L for (+, +), R for (−, −)]. The localization properties can then be conveniently described by c. For instance, in the case of AdS$_5$, with $A' = k$, c is just the Dirac mass $M$ in units of the curvature scale $k$. However, it is worth pointing out that the relation \[(89)\] can arise in more general settings. We illustrate the localization of the fermion 0-modes in Fig. 7.

2.5.2 KK Decomposition for Gauge Fields

The 5D gauge fields are written as [here $A_M(x^\mu, y)$ has mass dimension 3/2, see Eq. (50)]
\[ A_{\mu,5}(x^\mu, y) = \frac{1}{\sqrt{L}} \sum_n A_{\mu,5}^n(x^\mu)f^n_{A,5}(y) , \]
where
\[ \partial_y \left[ e^{-2A}\partial_y f^n_A \right] + m_n^2 f^n_A = 0 , \]
while the appropriate equation for $f^n_5$ is
\[ \partial_y^2 \left( e^{-2A}f^n_5 \right) + m_n^2 f^n_5 = 0 . \]
Note that a solution to Eq. (92) with $m_n^2 \neq 0$ immediately gives a solution to Eq. (93) as $f^n_5 = \frac{1}{m_n^2}\partial_y f^n_A$. As we further comment below, this relation is not accidental.
The KK wavefunctions are normalized as expected:

\[ \frac{1}{L} \int_0^L dy f_{A,5}^m f_{A,5}^n = \delta_{mn} . \]  

(94)

For \((+,-)\) b.c.’s, i.e. \(\partial_y f_A^n = f_5^n y = 0\) on both branes, one finds a spin-1 zero-mode with \(f_A^0(y) = 1\) and \(f_5^0(y) = 0\). Presumably, such a state could be identified with one of the observed SM gauge bosons. Notice that, unlike the case of fermion 0-modes (which are controlled by the 5D Dirac mass), there are no free parameters to control the localization of the gauge zero-mode. The fact that the 0-mode is flat in our physical normalization is closely related to gauge invariance, and the necessary universality of the gauge interactions. If the gauge 0-mode was not flat, the non-trivial localizations of fermions or scalars would allow us to adjust their gauge couplings in an arbitrary way. As it happens, the overlap integrals arising from the gauge vertices \(\overline{\Psi} A \Psi, \partial_\mu \Phi^\dagger A^\mu \Phi\) and \(\Phi^\dagger A_\mu A^\mu \Phi\), after replacing the KK-mode decompositions with the gauge 0-mode, always reduce to the fermion or scalar orthonormality conditions, so that the corresponding 4D gauge interactions are indeed universal. In particular, taking \(f_A^0(y) = 1\), one finds the tree-level matching relation [see also Eq. (128) later on]

\[ g_4^2 = \frac{g_5^2}{L} . \]  

(95)

For \((-,-)\) b.c.’s, i.e. \(\partial_y (e^{-2A} f_5^n) = f_A^n y = 0\) on both branes, there is no spin-1 zero-mode, but there is a spin-0 zero-mode with \(f_5^0(y) \propto e^{2A(y)}\). In certain constructions, such scalars can be identified with the SM Higgs field. Notice again that, given the background, the profile of such a 4D scalar 0-mode is fixed (no adjustable parameters to control its localization).

**Comment:** the massive \(A^a_5\), with \(f_A^n = \frac{1}{m_n} \partial_y f_{A,5}^n\), provide the longitudinal polarizations of the associated massive gauge fields. What happens is that the compactification breaks the gauge invariance associated to the \(n\)-th KK-level spontaneously, and there is a Higgs mechanism at work at each KK-level. Thus, the physical massive KK spectrum consists only of massive spin-1 fields (with three physical polarizations each), and there are no massive physical scalars. Such massive gauge fields can, and do have non-trivial wavefunction profiles, and their interactions with fermion or scalar pairs can depend on the details of those fields. The spontaneous breaking of the \(n\)-th KK level gauge invariance leaves no obvious trace of the underlying 5D gauge invariance, although it is still manifested in a less trivial manner, e.g. in the form of sum rules.

**Exercise:** Establish as many of the the facts stated in this section as possible.

### 2.5.3 Special Cases

Here we collect the KK decompositions for the two simple backgrounds discussed in Section 2.4.3.

**Flat space:** \(A(y) = 0\)

This case is particularly simple as far as KK decompositions: all KK equations reduce to the Simple Harmonic Oscillator.

\[ f''_n + (m_n^2 - M^2) f_n = 0 . \]  

(96)
so that the $f_n$’s are just sines and cosines.

The most widely studied application is to UED’s where all fields obey $(+, +)$ or $(-, -)$ boundary conditions. In addition, the 5D fermion Dirac masses are assumed to vanish. In such a case:

$$f_n^{(+, +)} = \begin{cases} 1 & \text{for } n = 0 \\ \sqrt{2} \cos m_n y & \text{for } n \neq 0 \end{cases},$$

$$f_n^{(-, -)} = \sqrt{2} \sin m_n y,$$

where $m_n = \frac{\pi n}{L} \equiv \frac{n}{R}$ for fermions or gauge fields, while $m_n^2 = M^2 + (n/R)^2$ for the Higgs field.

**Exercise:** If the 5D fermion Dirac mass does not vanish, the KK wavefunctions are linear superpositions of sines and cosines, and the KK masses are not as above. Work out this case in detail.

**AdS$_5$ (or RS):** $A(y) = k y$

This case can also be treated analytically, since the wave equations reduce to Bessel equations.

**Scalars:** writing the bulk mass as $M^2 = (\alpha^2 - 4)k^2$, one finds

$$f_n = N_n e^{ky} \left\{ J_\alpha \left( \frac{m_n}{k} e^{ky} \right) + b_n Y_\alpha \left( \frac{m_n}{k} e^{ky} \right) \right\},$$

where $b_n$ depends on the type of boundary conditions. It can be easily worked out in each case.

**Fermions:** writing the bulk mass as $M = \pm ck$, where the $+$ sign applies to $(+, +)$ b.c.’s while the $-$ sign applies to $(-, -)$ b.c.’s, one finds

$$f_{n L,R}^n = N_n e^{ky} \left\{ J_{c \pm \frac{1}{2}} \left( \frac{m_n}{k} e^{ky} \right) + b_n Y_{c \pm \frac{1}{2}} \left( \frac{m_n}{k} e^{ky} \right) \right\}.$$

Due to our sign convention for the Dirac mass in terms of $c$, this expression applies to both types of boundary conditions (at the massive KK-level). The constants $N_n$ and $b_n$ are the same for both chiralities.

**Gauge:** (unbroken case)

$$f_A^n = N_n e^{ky} \left\{ J_1 \left( \frac{m_n}{k} e^{ky} \right) + b_n Y_1 \left( \frac{m_n}{k} e^{ky} \right) \right\}.$$

**Comments:**

- The various solutions differ in the index of the Bessel functions, as well as in the mass spectrum, which is obtained by matching the b.c.’s according to the case.
- Up to an overall phase, the gauge case coincides exactly with the $(+, +)$ LH fermion case with $c = 1/2$ (or the $(-, -)$ RH fermion case with $c = -1/2$). This limit would apply to gauginos in models with unbroken $N = 1$ SUSY.
- The expression for possible 0-modes were given explicitly in the previous section.
3 Lecture 3: Concrete Models

In this lecture, we will review a number of models that illustrate various aspects of extra-dimensional phenomenology. The results summarized in the previous lecture constitute the basic language that is required. Fig. 4 serves as a rough guide to the organization of this lecture.

As already mentioned, a given model is defined by i) specifying the gauge and global symmetries (as in 4D models), and ii) specifying the field content, including which fields propagate in the bulk and which are localized on the boundaries (physically, over a distance smaller than the inverse cutoff, Λ). In general, the Lagrangian will contain a number of parameters without a 4D counterpart. An important example is provided by the Dirac masses that control the localization of the fermion 0-modes. The scale of the new KK resonances is also typically a free parameter, one of central importance for collider phenomenology.

The procedure to establish the connection to experiment is in principle straightforward:

• Perform the KK decomposition for all the bulk fields. The details depend on the metric background, which should be specified consistently. Nevertheless, as we will see, there are a number of qualitative features that hold in large (and interesting) classes of backgrounds. The data we obtain in this way are spectra (up to an overall scale) and wavefunctions that determine the couplings among the KK modes (including the 0-modes). In general, these will depend on several microscopic model parameters.

• Check for consistency with existing data, for instance
  * Electroweak precision constraints (EWPT).
  * Flavor/CP constraints.
  * Direct collider bounds.

In other words, check that the properties of the 0-mode sector are sufficiently similar to the SM, and that the new physics could have escaped direct detection in high-energy (or other) processes. Potential deviations from the SM at low energies are associated with the heavy modes, and can be grouped in two classes:

1. **Tree-level effects**: if present, these tend to imply stringent constraints.

2. **Loop effects**: these encode the quantum aspects of the extra-dimensional model. Here one has to be careful due to the possible sensitivity to physics at the cutoff scale. Sometimes, the best one can do is estimate the expected size of such effects. However, it may happen that the quantity in question, within a given model, is dominated by the physics around the compactification scale, not the cutoff scale. In such cases, a more detailed computation is warranted. One should note that it may happen that the cutoff sensitivity enters at a higher loop-order (for instance, at 2-loops but not at 1-loop). Then we may regard the quantity as effectively calculable.

The upshot of this analysis is typically a lower bound on the compactification scale, as a function of other microscopic model parameters.

• At this point, one is ready to explore the collider phenomenology and investigate how to test the predictions of the model. This typically depends on a finite (not too large) number of particles beyond the SM.
Nowadays, it is not too difficult to implement the relevant part of the KK Lagrangian (just a 4D QFT) into programs such as LanHEP [19] or FeynRules [20] that accept a Lagrangian written in a form closely related to the one we are used to, compute the corresponding Feynman rules, and generate code appropriate for a number of event generators. These can then be further processed to reach detector-level objects. Needless to say, it is important to have a solid understanding of the physics to validate the output of such codes.

• One might also be interested in other applications. Apart from flavor or CP signals, a subject that is often relevant relates to Dark Matter (DM) and the associated cosmology, as well as expectations for direct and indirect DM detection.

Our aim in this lecture is not to be exhaustive, but rather to provide an educated feeling for the possibilities opened up to model building by the potential existence of compact extra dimensions. We would like to emphasize the most important physics aspects and highlight the features that are common/different in broad classes of scenarios. We will start with the simplest models, and slowly add structure that permits addressing issues, such as those discussed in the first lecture.

### 3.1 Universal Extra Dimensions (UED’s)

Perhaps the most straightforward idea is to promote the SM (gauge group and field content) to $1 + (3 + n)$ Minkowski spacetime. Remarkably, the resulting scenarios can have interesting and non-trivial features.

In a first approximation, UEDs are defined by:

1) The extra dimensions are assumed to be exactly flat.

2) All SM fields are promoted to higher-dimensional (bulk) fields.

As we will see, in practice, UED scenarios involve additional assumptions to be specified shortly. We pointed out in the previous lecture that the KK wavefunctions in this case are sines/cosines. We reproduce them here for the case of 5D, to which we shall be mostly confined, for easy reference:

$$f_n^{(+,+)} = \begin{cases} 
1 & \text{for } n = 0 \\
\sqrt{2} \cos \frac{ny}{R} & \text{for } n \neq 0
\end{cases} \quad \text{or} \quad f_n^{(-,-)} = \sqrt{2} \sin \frac{ny}{R} \quad (102)$$

obeying the orthonormality conditions

$$\frac{1}{\pi R} \int_0^{\pi R} dy f_m^{(+,+)} f_n^{(+,+)} = \delta_{mn} \quad \text{and} \quad \frac{1}{\pi R} \int_0^{\pi R} dy f_m^{(-,-)} f_n^{(-,-)} = \delta_{mn} \quad (103)$$

---

14We do not show EWSB effects here, although they can play an important role in certain aspects of the phenomenology.
However, sines/cosines as above are very special in that they obey additional “selection rules”, involving integrals of three or more wavefunctions. For instance, we have

$$\frac{1}{\pi R} \int_0^{\pi R} dy f_{n_1}^{(+,+)} f_{n_2}^{(+,+)} f_{n_3}^{(+,+)} = \frac{1}{\sqrt{2\pi}} \left\{ \frac{\sin(n_1 + n_2 + n_3)\pi}{n_1 + n_2 + n_3} + \frac{\sin(n_1 + n_2 - n_3)\pi}{n_1 + n_2 - n_3} \right. \right.$$  

$$+ \left. \frac{\sin(-n_1 - n_2 + n_3)\pi}{n_1 - n_2 + n_3} + \frac{\sin(-n_1 - n_2 - n_3)\pi}{n_1 - n_2 - n_3} \right\} , (104)$$

and, since $n_1$, $n_2$ and $n_3$ are integers, one can see that the integral vanishes unless $n_1 \pm n_2 \pm n_3 = 0$ for some choice of $\pm$ signs.

**Exercise:** Convince yourself that the above is true (within this example). It may be useful to think in terms of the momentum components contained in the trigonometric functions.

In fact, the above selection rule can be generalized to

$$n_1 \pm n_2 \pm \cdots \pm n_N = 0 , \quad (105)$$

for any $N$-point function, provided there is an even number of $f_n^{(-,-)}$ insertions. As it turns out this is the case for all vertices associated with SM interactions; for instance

- Yukawa couplings:
  
  $$H \overline{Q} U = \sum_{(+,+)(+,-)(+,-)} H \overline{Q}_L U_R + \sum_{(+,+)(+,-)(+,-)} H \overline{Q}_R U_L \quad .$$

- $A_5$ is $(-,-)$ but always comes with another $(-,-)$ field, e.g.
  
  $$\overline{Q}_L A_5 Q_L = \sum_{(+,+)(-,-)(-,-)} \overline{Q}_L A_5 Q_R - \sum_{(+,+)(-,-)(-,-)} \overline{Q}_R A_5 Q_L .$$

Note that, in UED models, the $A_5$’s for $SU(3)_C \times SU(2)_L \times U(1)_Y$ and the “wrong chirality fermions” are the only $(-,-)$ fields.

The above selection rules have important consequences, e.g.

- $0 \quad \overline{0} \quad 0 \quad 1$ is allowed, but $0 \quad \overline{0} \quad 0,1 \quad 1$ is not allowed.

Thus, at tree-level, the new states can only be pair produced. However, at loop level, the selection rules allow

$$0 \quad \overline{0} \quad 0 \quad 1 \quad 2 \quad \propto \frac{y g^2}{16 \pi^2} \ln \Lambda .$$

39
Figure 8: The Kaluza-Klein parity in UED models arises from a geometrical reflection symmetry. We show the 5D and 6D cases. Sometimes these geometries are described as $S^1/Z_2$ and $T^2/Z_4$ (field theory) orbifolds, respectively. Certain points, marked here with large dots, are sometimes called “fixed points”. The $S^1/Z_2$ orbifold corresponds to the interval compactification, while the $T^2/Z_4$ corresponds to the Chiral Square compactification (see Fig. 2).

Thus, a second-level KK mode can be singly produced. The fact that the effect is (logarithmically) divergent means that there must exist a local 5D operator associated to this process. On the other hand, all bulk operators lead to the tree-level selection rule above, which is explicitly violated by a 0–0–2 process. The resolution is that the effect corresponds to an operator localized on the boundaries, with the same coefficient on both:

$$\mathcal{L}_5 \supset [\delta(y) + \delta(y - L)] \mathcal{O} .$$ (106)

**Exercise:** Show, by taking $\mathcal{O} = \Phi^3$, and replacing the UED KK expansions, that 0–0–2 processes exist, but not 0–0–3 ones. Note the importance of the equality of the coefficients on both boundaries.

**KK-Parity:** The bulk theory above has an exact discrete symmetry under which

$$\phi_n \mapsto (-1)^n \phi_n \quad \text{where } \phi = A_\mu, \psi, H .$$ (107)

This symmetry, by treating different KK modes differently, must be a spacetime symmetry. In fact, it follows from invariance under a reflection about the middle of the interval, as shown in Fig. 8. This important property can be generalized to more than 5D, as illustrated also in the figure.

Now we are ready to state the remaining assumptions that are part of the definition of UED models (as understood in the literature).

1) The reflection symmetry about $y = L/2$ is imposed, i.e. boundary operators are written symmetrically about the center. In more detail, the theory is invariant under

$$A_\mu(x, y) \mapsto A_\mu(x, L - y) \quad \Psi(x, y) \mapsto \pm \gamma_5 \Psi(x, L - y)$$

$$A_5(x, y) \mapsto -A_5(x, L - y) \quad H(x, y) \mapsto H(x, L - y) .$$
4) It is assumed that the size of the coefficients of the localized operators is of 1-loop order. Such an assumption is technically natural, i.e. they can be this small without tuned cancellations (as estimated by NDA).

The above have important consequences

- KK-parity is an *exact* symmetry. Although this is an assumption, it is non-trivial that it can be imposed.
- The single production of even KK-number states is 1-loop suppressed (either because it corresponds to a bona fide loop effect arising from bulk interactions, or because it arises at tree-level from a localized operator, but with a small coefficient of 1-loop order). *Odd KK-number states cannot be singly produced.*
- The corrections to EW observables arise at 1-loop order (or have 1-loop size). Thus, the new physics could appear at the few hundred GeV scale. For a light Higgs, such EW precision bounds can push the KK scale to several hundred GeV.
- The flavor structure of the bulk Lagrangian is as in the SM, including the GIM mechanism. However, higher-dimension operators, such as four-fermion interactions, can be dangerous due to the low suppression scale. The UV completion above the cutoff $\Lambda$ must have special flavor properties, if the compactification scale is as low as allowed by EWPT. In any case, one should remember that UED models do not pretend to provide a theory of flavor.
- Localized operators (localized kinetic terms, in particular) induce mass splittings within a given KK level. In the absence of these, all KK fields at the $n = 1$ level would have masses given by $1/R$ (up to typically fairly small EWSB effects). The actual mass splittings are relatively small due to assumption 4). This approximate degeneracy also plays an important role in the phenomenology of UED scenarios.
- The lightest $n = 1$ state is exactly stable (i.e. the lightest state carrying the non-trivial KK-parity charge). This leads to a (WIMP) dark matter candidate.

**Minimal UED’s: *(mUED)***

Often one makes an additional assumption: that the localized operators are generated *entirely* by loops of bulk operators. This allows to predict the spectrum, up to the overall scale. One finds that

- Strongly-interacting particles are the heaviest (receiving a 20-30% upward mass-shift from 1-loop diagrams).
- Weakly-interacting particles having $SU(2)_L$ interactions receive a 5-10% mass correction.
- Particles with only $U(1)_Y$ quantum numbers are the lightest.
- The lightest KK particle (the LKP) is $B^{(1)}_\mu$, the first KK resonance of the hypercharge gauge boson. It receives a small and negative 1-loop mass correction.

The latter is a rather interesting DM candidate, and can provide the observed DM relic density when its mass is in the several hundred GeV range 21 22. In addition, since $B^{(1)}_\mu$ couples to hypercharge, it has a large annihilation cross section into (RH) lepton pairs, leading to a
characteristic positron signal \cite{23}. However, one should keep in mind that mild departures from \textit{mUED} can lead to a significantly different phenomenology (see e.g. \cite{24}).

### 3.2 Warped Extra Dimensions

We now turn our attention to cases where the spacetime curvature is important (though still with vanishing 4D C.C.) Much of the literature has focused on the AdS\textsubscript{5} background. However, the central features are more general –as long as we are in the strong warping regime– and we will consider special backgrounds only when necessary. We have in mind the line element of Eq. \((23)\) where \(A(y)\) is a monotonically increasing function of \(y\).\footnote{The function \(A(y)\) is expected to be convex, \(A'' \geq 0\). If a minimum is reached within the region of physical interest, one can treat the regions to the left and to the right of this minimum separately, and then join them smoothly.} This covers a relatively large class of backgrounds, including those that arise from simple choices of \(W\) in the \textit{superpotential approach} described in Section 2.4.

What we mean by \textit{strong warping} corresponds to

\[
A(L) \gg 1 ,
\]

where we assume, without loss of generality, that \(A(0) = 0\). In this case, the scale of the KK resonances is given by

\[
\tilde{k}_{\text{eff}} \equiv A'(L) e^{-A(L)} ,
\]

that is, a measure of the \textit{warped-down curvature at} \(y = L\). One refers to \(y = 0\) as the UV boundary (brane), and to \(y = L\) as the IR boundary (brane). Under the assumption of strong warping, the solutions to the various equations presented in the previous lecture have the feature that the massive KK wavefunctions are IR localized, as shown in Fig. 9 for the gauge and fermion cases. Qualitatively, the generic \textit{strong warping} solutions are similar to the explicit Bessel function solutions appropriate to the AdS\textsubscript{5} background. The spectrum, in units of \(\tilde{k}_{\text{eff}}\) is also very similar to the AdS\textsubscript{5} spectrum in units of \(\tilde{k} = k e^{-kL}\). Thus, the intuition gained from the AdS\textsubscript{5} limit is of wider applicability.

The above feature of the KK wavefunctions has important consequences. Consider a fairly generic interaction with the \textit{schematic} structure

\[
\mathcal{L}_{\text{int}} \supset \lambda_5 \Phi^N \Psi^M \left( e^{\mu} \gamma^{\alpha} \partial_{\mu} \right)^S \left( g^{\beta \nu} A_{\beta} \partial_{\nu} \right)^K ,
\]

involving bulk scalar, fermion and gauge fields, and with a number of derivatives. Some of the latter may be contracted through the \textit{fünfbein}, while others are contracted via the metric. In Eq. (110), the derivatives should be understood to act on the fields in various ways. For simplicity, we focus on \(\partial_{\mu}\) derivatives, but our conclusion does not change when \(\partial_5\) derivatives are included. We could similarly include \textit{brane-localized} fields.
Figure 9: Example of gauge (left panel) and fermion (right panel) KK modes in strongly warped backgrounds. The wavefunctions are IR localized, but note the behavior near the UV brane.

The 5D coupling has mass dimension $[\lambda_5] = 5 - N \times \frac{3}{2} - M \times 2 - K \times \frac{3}{2} - (S + K)$. Replacing our KK decompositions, Eqs. (71), (81) and (91), we get interactions among KK fields (schematically)

$$
\lambda_{n_1 \cdots n_N m_1 \cdots m_M k_1 \cdots k_K} \phi_{n_1} \cdots \phi_{n_N} \psi_{m_1} \cdots \psi_{m_M} A_{k_1} \cdots A_{k_K} \phi^{S+K},
$$

(111)

where

$$
\lambda_{n_1 \cdots n_N m_1 \cdots m_M k_1 \cdots k_K} = \frac{\lambda_5}{L^{(N+M+K)/2}} e^{-[\lambda_4] A(L)} \int_0^L dy e^{-[\lambda_4] (A(y) - A(L))} f_{n_1}(y) \cdots f_{k_K}(y)
$$

has mass dimension

$$
[\lambda_4] = 4 - N \times 1 - M \times \frac{3}{2} - K \times 1 - (S + K).
$$

(112)

We now use the fact that the (massive) KK wavefunctions are localized within a distance of order $k_{\text{eff}}^{-1} \equiv [A'(L)]^{-1}$ from the IR brane. This means that the support of the integral is dominated by the region $[L - k_{\text{eff}}^{-1}, L]$, and that in this region the exponential in the integrand above is of “order one”. In addition, due to the normalization of the physical wavefunctions, Eq. (78), we can estimate

$$
f_{n}(L) \sim \sqrt{k_{\text{eff}} L},
$$

(113)

and, therefore, the 4D coupling is of order

$$
\frac{\lambda_5}{L^{(N+M+K)/2}} e^{-[\lambda_4] A(L)} \frac{1}{k_{\text{eff}}^{(N+M+K)/2}} \sim \frac{\lambda_5 k_{\text{eff}}^{-(N+M+K)/2}}{\text{has mass dimension } [\lambda_4]} e^{-[\lambda_4] A(L)}.
$$

Thus, we learn that, quite generically, the scale of the effective 4D coupling is red-shifted precisely by its mass dimension. This is how strongly warped scenarios generate exponentially small scales.
starting from "fundamental" ones. The most famous application is to the generation of the Planck/weak scale hierarchy, through the redshift of the Higgs mass parameter.

**Comment:** Above we simply estimated the size of the overlap integral based on the generic localization of the KK wavefunctions within a distance $k_{\text{eff}}^{-1}$ from $y = L$. In practice, the overlap integral may be further suppressed due to destructive interference (from the oscillatory behavior). Such an interference will be more pronounced when states with very different masses are involved. Even more importantly, 0-modes need not be IR-localized, as already illustrated in Fig. 7. Thus, integrals involving such modes may be exponentially suppressed due to the wide separation of the relevant wavefunctions. These additional suppressions from the details of the wavefunctions, which appear on top of the redshift dictated by dimensional analysis, are usefully described as the result of physical localization effects. It is in this sense that our conventions for the physical wavefunctions, introduced in the previous lecture, factor out the universal effects of the geometry (background) from the geography of localization that characterizes a given model. It will be useful to remember that gauge 0-modes are flat, fermion 0-modes have profiles as in Eqs. (87) or (90), and scalar 0-modes arising from 5D gauge fields obeying $(-, -)$ b.c.’s have $f_0^G(y) \propto e^{-2A(y)}$. These properties can have interesting phenomenological applications, to be discussed below.

### 3.3 Sample Scenarios

We are now ready to present a few representative scenarios based on warped extra dimensions.

#### 3.3.1 The Original Version: RS1

All the SM fields, except for gravity, are assumed to be $\delta$-function localized on the IR brane. This model was proposed as a solution to the Planck/weak scale hierarchy problem [18], and as an alternative to supersymmetry:

- The background is taken as AdS$_5$ so that $A(y) = ky$ (see Section 2.5.3 in the previous lecture).
- The warped-down curvature scale is taken as $k_{\text{eff}} \equiv k \sim \text{TeV}$. The Higgs mass parameter, and hence EWSB, is then naturally at the TeV scale.
- The graviton 0-mode (i.e. massless spin-2 fluctuations in the KK spectrum of the 5D metric) is given by

$$ds^2 = e^{-2ky} [\eta_{\mu\nu} + h_{\mu\nu}(x)] dx^\mu dx^\nu - dy^2,$$

i.e. it shares the same profile as the background. In our KK-decomposition, we would factor out the $e^{-2ky}$ and define the graviton physical wavefunction as $f^G_0(y) = 1$. As for the spin-1 case, the flatness of this wavefunction is closely connected to the necessary universality of the gravitational interactions, here following from general covariance. The couplings of the graviton are always set by $M_P^2 = (1 - e^{-2kL}) M_5^3/k$, independently of the localization properties of the gravitating fields (see below).
In this model one expects massive spin-2 resonances that may be narrow or wide, depending on the size of the curvature relative to the 5D Planck scale. It should be emphasized that 4-fermion operators, which are necessarily localized on the IR brane, are only suppressed by $O(\text{TeV})$. Thus, as in UED models, the UV completion must have a special flavor structure to be consistent with existing flavor (and CP) constraints. In general, one would expect these models to suffer from severe FCNC and CP problems, which has motivated the variants discussed next.

### 3.3.2 The SM in a Warped Background

It turns out that giving up the assumption that all the SM fields are localized on the IR brane opens up a way to address the flavor problem of RS1, without giving up the RS solution to the hierarchy problem. The essential feature to be preserved is that the Higgs (0-mode) be localized within a distance of order $k_{\text{eff}}^{-1}$ from $y = L$, as should be clear from our general discussion in Section 3.2 regarding how the warped down scale is generated. It is, however, not necessary that the SM fields be localized as in the RS1 model, and in these scenarios one assumes that all fields can propagate in the bulk.

**Gravitational Interactions:**

We will not review here the appropriate gauge fixing necessary to perform the KK decomposition of the gravity sector. The KK expansion for the spin-2 modes takes the form

$$G_{\mu\nu}(x,y) = e^{-2A(y)} \sum_n g_{\mu\nu}^n(x)f^n_G(y),$$

(115)

where all quantities are *dimensionless*. If we restrict to a 0-mode background,

$$ds^2 = e^{-2A(y)}g_{\mu\nu}(x)dx^\mu dx^\nu - dy^2,$$

(116)

we have that $\mathcal{R}_5[G] = e^{2A}\mathcal{R}_4[g]$, so that the 5D Einstein-Hilbert action takes the form

$$S_{\text{EH}} = -\frac{1}{2}\int d^5x \sqrt{G} M_5^3 \mathcal{R}_5[G]$$

$$= -\frac{1}{2}\int d^4x \sqrt{g} M_5^3 \int_0^L dy e^{-4A} e^{2A} \mathcal{R}_4[g],$$

(117)

which identifies the 4D Planck mass as

$$M_P^2 = M_5^3 \int_0^L dy e^{-2A(y)}.$$  

(118)

In the *strong warping* limit the integrand has support over a distance $1/k_{\text{UV}} \ll L$ from $y = 0$, where $k_{\text{UV}} = A'(0)$ measures the curvature in the vicinity of the UV brane. We therefore have

$$M_P^2 \sim \frac{M_5^3}{k_{\text{UV}}},$$

(119)

Sometimes one assumes that the Higgs field is $\delta$-function localized, but even this is not necessary: the localization need be only of order $k_{\text{eff}}^{-1} \ll L$. 

45
which has no powers of the warp factor. Thus, we may take $M_5 \sim k_{UV} \sim M_P$, while still having at our disposal the much smaller scale in Eq. (109).

**Exercise:** Use NDA to show that the cutoff associated with the 5D gravitational interactions is given by $\Lambda^2_5 \sim M_5^2/l_5$, where $l_5 = 24\pi^3$ is the 5D loop factor. How does this cutoff compare to the one associated with the $SU(3)_C$ interactions?

**Flavor Anarchy:**

The freedom to localize the fermion 0-modes offers an appealing understanding of the observed fermion mass hierarchies (and quark mixing angles). Recall that the fermion masses span about six orders of magnitude, from $y_e \sim 10^{-6}$ to $y_t \sim 1$. The neutrinos may be associated with even smaller Yukawa couplings.

In 5D, a bulk Yukawa operator leads to

$$y_5 H \Psi_{1L} \Psi_{2L} + \text{h.c.} \mapsto y_4 H^0 \tilde{\psi}_{1L} \psi_{2R}^0 + \text{h.c.} + \text{“KK-modes”},$$

where

$$y_4 = \left( \frac{y_5}{\sqrt{L}} \right) \frac{1}{L} \int_0^L dy f_H^0 f_{\psi_1}^0 f_{\psi_2}^0,$$

$$\sim \left( \frac{y_5}{\sqrt{L}} \right) \frac{1}{k_H L} \sqrt{k_H L} f_{\psi_1}^0 (L) f_{\psi_2}^0 (L),$$

and in the second line we assumed that the Higgs 0-mode is localized within a distance $k_H^{-1}$ of the IR brane, hence $f_H^0 (L) \sim \sqrt{k_H L}$. Thus, the 4D Yukawa couplings depend not only on the 5D Yukawa couplings, but also on the fermion localization. The assumption of flavor anarchy in this context\(^{17}\) corresponds to taking the 5D Yukawa matrices to be generic matrices with all entries of the same order, i.e. no special flavor structure. The hierarchical structure of the observed fermion masses (and mixing angles) would be a consequence of the fermion geography, as illustrated in Fig. 10. The first two generation quarks, plus the RH bottom, are localized close to the UV brane, so that their overlap with the Higgs field is small. The top quark, on the contrary, is localized near the IR brane, and its overlap with the Higgs wavefunction is of order one so that the top mass is naturally comparable to the electroweak vev. The overlap of the Higgs profile with the gauge bosons ($Z$ and $W$) is only mildly suppressed by a volume factor. Thus, these can also get EWSB masses of order the electroweak vev. For clarity in the figure, we have oversimplified by not distinguishing between chiralities and the different light fermions. For instance, typically the RH top is more localized towards the IR brane than the LH top\(^{18}\). The LH bottom quark is then also somewhat localized towards the IR brane, but the UV localization of the RH bottom can result in a suppressed (4D) bottom Yukawa coupling. The upshot is that the observed structure of fermion masses follows from underlying $c$-parameters in the vicinity of

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\(^{17}\)The underlying idea of anarchy was first discussed in the 4D context\(^{25, 26}\).

\(^{18}\)Although scenarios with the LH top closer to the IR brane than the RH top can be motivated in certain scenarios.
Figure 10: Illustration of the “flavor anarchy” setup. The light fermions (including $b_R$) are localized towards the UV brane, while the top is localized near the IR brane. The overlaps with the Higgs wavefunction can naturally lead to exponential Yukawa hierarchies.

1/2, in the language of Fig. 7. As a bonus, higher-dimension operators, such as 5D four-fermion interactions involving the light families, get also a suppression (beyond the standard redshift) that amounts to an effective scale well-above the TeV. This dramatically improves the tension with precision flavor constraints.

One should point out that there are tree-level FCNC effects arising from the fact that the light families are not localized in exactly the same way. This means that there is some flavor-dependence in the couplings of the fermions to KK gluons:

$$g_c = \left( \frac{g_5}{\sqrt{L}} \right) \frac{1}{L} \int_0^L dy \left[ f_0^0(y) \right]^2 f_{G'}(y),$$

(122)

where $f_0^0(y)$ are the fermion 0-mode wavefunctions given in Eq. (90) and $f_{G'}(y)$ is the first KK-gluon wavefunction (see the blue curve in the left panel of Fig. 9). Although these couplings are diagonal in the fermion gauge eigenbasis, the fact that they are fermion-dependent (through the $c$-dependence), means that in the mass eigenbasis, off diagonal couplings are induced. For instance,

$$d \rightarrow K^0 - \bar{K}^0 \text{ oscillations} : \Delta m_K^2.$$

However, as indicated in the left panel of Fig. 9, the KK-gluon wavefunction has the property that it is almost flat, except very near the IR brane. Thus, when $f_0^0$ is UV-localized, we have

$$\frac{1}{L} \int_0^L dy \left[ f_0^0(y) \right]^2 f_{G'}(y) \approx \frac{f_{G'}(0)}{L} \int_0^L dy \left[ f_0^0(y) \right]^2 = f_{G'}(0),$$

(123)
i.e. the $c$-dependence is very weak. This leads to suppressed flavor-changing KK-gluon vertices, and is known as the RS-GIM mechanism.

**Exercise:**
- Check in the AdS$_5$ limit, using the explicit expressions for the KK-gluon wavefunction, that the profile has the properties stated above. Evaluate also the overlap integral to get a feeling for the $c$-dependence.
- Argue, from the gauge KK equation, that the previous property holds more generally in strongly warped backgrounds.

The above is remarkable: the anarchy assumption amounts to a “theory of flavor” with new physics at the TeV scale; and yet the scenario can be roughly consistent with FCNC constraints. (Naively, such flavor constraints would put bounds on the new physics scale of about $10^3$ TeV or higher.) At the same time, the strongest constraints do arise from the above KK-gluon exchange. When analyzed in detail, one finds that some CP-violating phases have to be somewhat suppressed. In spite of this, the above structure remains as one of the striking properties of the bulk RS scenario.

**Electroweak Constraints:**
As already mentioned, one has to check that the SM properties are not distorted too much by the new physics (i.e. by the KK-modes). Given that these are decoupling effects, this can be used to set a lower bound on the KK scale. The most sensitive and robust constraints are derived from the EW precision measurements, mostly done at LEP/SLD and the Tevatron. We will present a simple formalism to perform such an analysis in the next lecture, but will discuss here the physics in the context of several examples.

Generically, the largest deviations from SM properties arise from the following observation: when the 0-mode Higgs, that by assumption has an IR-brane localized profile (perhaps $\delta$-function), acquires a vacuum expectation value, it adds a $y$-dependent mass to the gauge EOM. The gauge spectrum is shifted (by a small amount if $v \ll \tilde{k}_{\text{eff}}$), and in particular no longer admits an exactly massless solution. The would-be 0-mode wavefunction is modified near the IR brane, as illustrated in Fig. [11], so that the $y$-gradient near this brane accounts for the non-zero mass, which would be identified with $M_Z$ or $M_W$. The dominant source of constraints arises from the requirement to fulfill the relation $M_W^2 \approx M_Z^2 \cos^2 \theta_w$, which is observed to hold to excellent accuracy and suggests the existence of a custodial symmetry. More precisely, within the SM, if one neglects the hypercharge and Yukawa interactions one can show that the model has an exact global symmetry arising as

$$SU(2)_{\text{gauge}} \times SU(2)_{\text{global}} \underset{\text{EWSB}}{\rightarrow} SU(2)_{\text{Diag}} \equiv SU(2)_{\text{custodial}} .$$

(124)

Technically, the simplest way to see this is to rewrite the SM (with $g' = y_i = 0$) in terms of a

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19These solutions correspond to linear superpositions of the “unperturbed” KK modes described in the previous sections (i.e. with $v = 0$). Thus, one can understand the non-trivial effects on the EW precision observables as arising from mixing of the 0-modes with their massive KK-towers.
Figure 11: Effect of a $y$-dependent Higgs vev on the gauge 0-mode. The deformation of the wavefunction affects the $Z$ and $W$, but not the gluons or the photon.

Higgs $SU(2)_L \times SU(2)_R$ “bidoublet”

$$\Phi = \begin{pmatrix} H^0 & H^+ \\ -H^- & H^0 \end{pmatrix} \begin{pmatrix} SU(2)_L \\ SU(2)_R \end{pmatrix}$$

which implies that its vev, $\langle \Phi \rangle = v \times 1_{2 \times 2}$ preserves the diagonal subgroup of $SU(2)^{\text{gauge}}_L \times SU(2)^{\text{global}}_R$. Since the three $W$ gauge bosons transform as a triplet under $SU(2)^{\text{gauge}}_L$ (and singlets under $SU(2)^{\text{global}}_R$), hence like a triplet of $SU(2)^{\text{custodial}}$, the charged and neutral $W$ components are forced to have the same mass, even after EWSB. Turning on the small $g'$ induces mixing in the neutral gauge sector (between $W^3_\mu$ and the hypercharge gauge boson), and hence a relatively small mass splitting between the $Z$ and $W^\pm$ masses, $M^2_W = M^2_Z \cos^2 \theta_w$, while the custodial-breaking effects from the Yukawa couplings are even smaller, as they enter at loop level (dominantly from the top Yukawa).

What happens when we lift the SM to a warped extra dimension? Since the Lagrangian has the same form as the SM, it is still the case that for $g' = 0$ and in the absence of Yukawa couplings, there is a custodial symmetry. Again, there are tree-level violations proportional to $g'$, and loop-level corrections controlled by the top Yukawa. The main difference is that the hypercharge KK-modes couple with an enhanced coupling to the Higgs

$$g_{B^n} = \left( \frac{g_s}{\sqrt{L}} \right) \frac{1}{L} \int_0^L dy \left[ f_H^0(y) \right]^2 f_{B^n}(y) \sim g' \frac{1}{k_{\text{eff}} L} (k_{\text{eff}} L)^{3/2},$$

where we estimate the integral based on by now familiar arguments (assuming, for simplicity, that all wavefunctions are localized within a distance $1/k_{\text{eff}}$ of $y = L$). Thus, $g_{B^n} \sim \sqrt{k_{\text{eff}} L} g' \gg g'$, since solving the hierarchy problem is related to $k_{\text{eff}}L \gg 1$. We then see that the custodial violations due to the hypercharge KK-tower are enhanced compared to that of the hypercharge 0-mode. The way to suppress them is by increasing the KK-masses, i.e. the overall KK scale.
\( k_{\text{eff}} \). Indeed, in the limit that \( \tilde{v} \ll k_{\text{eff}} \), where we use the notation \( \tilde{v} \) to emphasize that this is the *warped-down* vev, to be identified with the weak scale, the effect corresponds to

\[
\langle H \rangle \propto g'^4 \left( k_{\text{eff}} L \right) \frac{\tilde{v}^2}{k_{\text{eff}}^2} \tilde{v}^2.
\]  

where the internal (magenta) line denotes the exchange of massive KK modes. Similarly, the couplings of the top KK-tower to the Higgs are enhanced by \( \sqrt{k_{\text{eff}}} L \) compared to the top Yukawa, and lead to

\[
W_\mu^{(0)} W_\nu^{(0)} \propto y_t^4 (k_{\text{eff}} L)^4 \frac{\tilde{v}^2}{k_{\text{eff}}^2} \tilde{v}^2 \times \text{loop integral}.
\]

The integral is actually UV sensitive, and the best one can do at this point is to estimate its size (e.g. using the NDA philosophy).

The actual analysis shows that in the anarchic RS model (based on AdS\(_5\) and with a \( \delta\)-function localized Higgs), the first gauge KK resonance should obey

\[
m_{\text{KK}} \gtrsim 13 \ (8) \ \text{TeV} \quad \text{for} \quad m_h = 120 \ (500) \ \text{GeV} \quad \text{at 95\% C.L.}
\]

The bounds can be somewhat relaxed by allowing the Higgs to propagate in the bulk (though still localized within \( k_{\text{eff}}^{-1} \) of \( y = L \)), but they are still significant. There are a number of approaches one can take to alleviate these bounds and have the chance of producing the new physics at the LHC. Some involve going beyond the minimal 5D field content, but not all of them.

### 3.3.3 Relaxing the Bounds from EW Constraints

As noted above, the severity of the bounds on the KK scale from EW precision constraints is directly tied to the strength of the couplings of the (localized) Higgs and the gauge/top KK towers. The first class of proposals aim at reducing these couplings, i.e. reduce the relevant overlap integrals.

1) **Brane-localized Kinetic Terms:**

We have mentioned several times the possibility of writing boundary terms (in fact, we have argued that these must be written since they are induced radiatively). The simplest example corresponds to brane kinetic terms (BKT’s):

\[
\mathcal{L}_5 = \delta(y) \left\{ -\frac{1}{4} r_{\text{UV}} F_{\mu\nu} F^{\mu\nu} + \alpha_{\text{UV}} \overline{\Psi} i \gamma^\mu \partial_\mu \Psi \right\} + \delta(y - L) \left\{ -\frac{1}{4} r_{\text{IR}} F_{\mu\nu} F^{\mu\nu} + \alpha_{\text{IR}} \overline{\Psi} i \gamma^\mu \partial_\mu \Psi \right\},
\]
where the BKT coefficients, \( r_i \) and \( \alpha_i \), have mass dimension \(-1\), and we illustrate the idea with \( \partial_\mu \) derivatives (\( \partial_y \) derivatives require more care). These terms can be included in a straightforward manner when performing the KK decomposition, as a modification of the b.c.'s (see previous lecture, and also Section 4.2.2 in the next one). They also modify the orthonormality relations, for instance for the gauge KK modes, Eq. (94), to:

\[
\frac{1}{L} \int_0^L f^n_A(y) f^m_A(y) \, dy + \frac{r_{\text{UV}}}{L} f^n_A(0) f^m_A(0) + \frac{r_{\text{IR}}}{L} f^n_A(L) f^m_A(L) = \delta_{mn},
\]

(127)

and a similar modification for the fermion wavefunctions with \( r_i \rightarrow \alpha_i \). The effect of the BKT’s is to repel the wavefunctions from the respective brane (for positive \( r_i, \alpha_i \)). In Fig. 12, we illustrate the effect of IR BKT’s. Due to the normalization condition, the (absolute) value of the wavefunction in the UV region increases, and therefore also the couplings to UV localized fields. We can see in the figure how the overlap with the Higgs profile would be reduced compared to the case of vanishing BKT’s. In addition, for fixed \( \tilde{k}_{\text{eff}} \), the masses of the KK modes are lowered. However, for these effects to be phenomenologically significant, the size of the BKT coefficients has to be larger than those induced radiatively.

Let us also mention here that an important effect of the BKT’s is to modify the relation between the 5D and 4D gauge couplings, Eq. (95):

\[
\frac{1}{g_4^2} = \frac{1}{g_5^2} \left[ L + r_{\text{UV}} + r_{\text{IR}} \right].
\]

(128)

Sometimes this is called loop-level matching, especially when the BKT’s are assumed to arise dominantly from radiative corrections (in which case \( r_{\text{UV}} \gg r_{\text{IR}} \)).
2) Modifications of the Gravitational Background:

A different avenue to relax the bounds on the KK scale is to explore non-trivial deviations from the AdS$_5$ background, as proposed in [27]. For instance, including a bulk scalar with a potential generated by the superpotential

$$W(\Phi) = 6k \left(1 + e^{\nu\Phi/\sqrt{6}}\right),$$

the bulk solutions found from the superpotential method described in the previous lecture give

$$\langle \Phi \rangle = -\sqrt{6} \nu \ln \left[\nu^2 k (y_s - y)\right],$$

$$A(y) = ky - \frac{1}{\nu^2} \ln \left(1 - \frac{y}{y_s}\right),$$

which are controlled by two parameters: $\nu$ and $y_s$. These bulk solutions exhibit a singularity at $y = y_s$, which is chosen to lie beyond $y = L$, but in its vicinity. In the left panel of Fig. 13, we show both the scalar profile and the metric background function $A(y)$. As explained in the previous lecture, the relative position of the branes is determined by the scalar boundary values (b.c.’s). Note that, except close to the IR brane, $A(y)$ is approximately linear in $y$, which corresponds to the AdS$_5$ limit. The nearby presence of the singularity (even if outside the physical region) modifies the background around the IR brane, which in turn affects the various wavefunctions through their wave-equations. In order to discuss a bulk Higgs, one chooses a coupling to the stabilizing scalar as

$$\mathcal{L} = M(\Phi)^2 H^1 H,$$

where $M(\Phi)^2 = ak[ak - (2/3)W(\Phi)]$ for some dimensionless $a > 2$. The physical Higgs wavefunction (lightest mode) is approximately given by

$$f_0^h(y) = N_h e^{-A(y)} e^{aky}, \quad \frac{1}{L} \int_0^L dy \left[f_0^h(y)\right]^2 = 1.$$

The form of the above expression is similar to what one would find in an AdS$_5$ background, except that here the function $A(y)$ is not simply linear in $y$ near the IR brane, where the Higgs is localized. We illustrate the result in the right panel of Fig. 13 which illustrates how the overlap integral between the Higgs profile and the gauge KK modes can be suppressed by the “repulsion” of $f_0^h(y)$ from the IR brane. The gauge KK-wavefunctions are also affected by the deformed background, being attracted towards the IR brane. As a result, one can have gauge KK resonances as light as $\sim 1$ TeV, consistent with EWPT.

The proposals 1) and 2) above do not require an extension of the 5D field content to lower the bound on the KK scale (apart from the stabilizing scalar). The second class of proposals, to be discussed next, focuses on protecting certain observables by the imposition of symmetries, which require an extension of the field content.
Figure 13: Left panel: scalar and metric background profiles for the superpotential of Eq. (129). We use arbitrary units on the vertical axis to show both profiles together. Right panel: associated deformation of the Higgs profile, compared to the first gauge KK mode.

### 3.3.4 Bulk Models with Custodial Symmetries

The idea here\cite{28} is to force the custodial symmetry to be as exact as possible by gauging it! Thus, the SM gauge group is extended to

\[
SU(2)_L^{\text{gauge}} \times SU(2)_R^{\text{gauge}} \times U(1)_X^{\text{gauge}}, \quad (X = B - L). \tag{134}
\]

The hypercharge generator is identified with \(Y = X + T_3^R\). This gauge group is broken by boundary conditions, so that only the SM subgroup has associated massless gauge bosons (before EWSB). Specifically, one breaks the gauge symmetry down to \(SU(2)_L \times U(1)_Y\) on the UV brane:

\[
W_{R}^{1,2},
Z' = \frac{1}{\sqrt{g_R^2 + g_X^2}} \{g_R W_{R}^3 - g_X X\} \quad (-, +)
W_{L}^{1,2,3},
B_{\mu} = \frac{1}{\sqrt{g_R^2 + g_X^2}} \{g_X W_{R}^3 + g_R X\} \quad (+, +)
\]

Note that the LR symmetry remains unbroken on the IR brane, where the strongly interacting states are localized. Thus, the custodial symmetry is exact on this boundary, even for \(g' \neq 0\). Schematically, the \(T\)-parameter receives contributions from the massive \(W_L\) and \(W_R\) fields, leading to a good degree of cancellation, since at the massive level the differences between \(L\) and
R are small, their wavefunctions being suppressed near the UV brane:

\[ W_L \] 

\[ - W_R \]

To be more precise, the leading \( k_{\text{eff}}L \) enhanced term shown in Eq. (126) cancels out. Of course, at the 0-mode level, the custodial symmetry is broken, exactly as in the SM. There are corresponding cancellations in loop diagrams: the bulk gauge symmetry requires top partners in order to span full bulk gauge multiplets, and these ensure that the loop diagrams are finite. The finite effect can still play a role in constraining the model, and should be investigated on a case by case basis, but the loop suppression factor brings it to the right level to not be the overwhelming single source of constraints. Typically, the dominant source of constraints in custodial models arises from tree-level vertex diagrams, such as

\[ W^{(0)}_\mu \]

\[ \psi^{(0)} \]

which, in anarchic scenarios, are nearly universal for the light fermions and have no \( k_{\text{eff}}L \) enhancement. In this case, they can be thought as a contribution to the S-parameter.

A special case occurs for the well-measured \( Zb_L \bar{b}_L \) coupling. This is because the \((t_L, b_L)\) \( SU(2)_L \) doublet cannot be localized too far from the IR brane or else the top mass would be predicted to be suppressed compared to the weak scale. This leads to a non-universal vertex correction in the \( Zb_L \bar{b}_L \) coupling that can place significant bounds. Interestingly, this vertex is protected when \[ g_L = g_R \] and \[ T^3_R(b_L) = T^3_L(b_L) = -\frac{1}{2} \].

Due to the connection to the underlying custodial structure, Eq. (134), this is known as a custodial protection of \( Zb_L \bar{b}_L \). Realizing the above quantum number assignments requires the embedding of the \((t_L, b_L)\) \( SU(2)_L \) doublet into a \( SU(2)_L \times SU(2)_R \) bidoublet with \( X = 2/3 \):

\[ Q_L = \begin{pmatrix} t_L (+,+) & \chi^{5/3}_L (-,+) \\ b_L (+,+)& \chi^{2/3}_L (-,+) \end{pmatrix} \]

Here we also exhibit the boundary conditions for the LH components of the fermion bi-doublet, \( Q \), which are chosen to preserve the full \( SU(2)_L \times SU(2)_R \) on the IR brane, but to lift the
unwanted fermion 0-modes by the choice on the UV. The b.c.’s on the RH chiralities can be read from the above, following the general rules explained in the previous lecture. The superscripts on the new 5D fermions, \( \chi^{5/3}_{L} \) and \( \chi^{2/3}_{L} \), indicate their electric charges, where \( Q = T^3_L + T^3_R + X \).

The new charge-2/3 tower, with \( T^3_L(\chi^{2/3}_L) = -T^3_L(t_L) \), is responsible for the required cancellations in the fermion loop contributions to the T-parameter. Interestingly, a prediction of the setup is the presence of fermion resonances with an exotic electric charge, ±5/3. A global fit to the EW precision data shows that [30], under the assumption of flavor anarchy,

\[
m_{KK} \gtrsim 4 \text{ TeV} \quad \text{at } 95\% \text{ C.L.}
\]

However, if one relaxes the anarchy assumption, one can have

\[
m_{KK} \gtrsim 2 \text{ TeV} \quad \text{at } 95\% \text{ C.L.}
\]

The original RS flavor problem can still be addressed through a version of the General Minimal Flavor Violation ansatz: in this extra-dimensional realization, the SM flavor symmetries are gauged, but broken by b.c.’s on the UV brane, in analogy to the breaking of the LR symmetry described above.

### 3.3.5 Models of Dynamical EWSB

So far, we have treated the Higgs as a “fundamental” field, while EWSB is described—in analogy with the SM—by writing an appropriate phenomenological Higgs potential (with the important difference that the warping provides the TeV scale). However, it is possible to go further, and use the extra-dimensional structure to construct models where the EW symmetry is broken dynamically. Here we mention two examples:

- **Condensation**: no Higgs scalar is introduced. The strong interactions associated with the KK gluons may trigger the condensation of fermion 0-mode pairs, if these are localized sufficiently close to the IR brane. If the fermion bilinear has non-trivial EW quantum numbers, EWSB ensues [31, 32, 33]. A relatively light scalar bound state of these fermions unitarizes \( W_L W_L \rightarrow W_L W_L \) scattering.

- **Gauge-Higgs Unification (GHU)**: in these models there are no 5D scalars, but 4D scalars arise from the extra-dimensional polarizations of gauge fields obeying \((-,-)\) b.c.’s. The 5D gauge symmetry forbids a tree-level (5D) potential for these modes, but the compactification leads to a calculable (i.e. UV insensitive) potential [34]. The interplay between the top Yukawa and weak gauge interactions generates a non-vanishing expectation value for these scalars, which if appropriately charged under the EW group induce EWSB.

Let us only mention a few aspects of one of the most studied GHU scenarios [35]. One needs a gauge group large enough to contain the SM group plus additional gauge fields that can give rise to the 4D scalar 0-modes with the quantum numbers of the Higgs. One may also want to embed the custodial symmetry in the model. The smallest group that satisfies these requirements is

\[
SO(5) \times U(1)_{B-L} \leftarrow SU(2)_L \times U(1)_Y \quad \text{on the UV brane}
\]

\[
SU(2)_L \times SU(2)_R \times U(1)_{B-L} \quad \text{on the IR brane}
\]
where the breaking occurs by b.c.’s. The Higgs scalars are embedded as

\[ A_5^{\hat{a}} T^{\hat{a}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0_{4 \times 4} & H^+ \\ H^0 & -H^- \\ -H^0 & -H^0 \end{pmatrix}, \]  

(136)

where \( \hat{a} \) runs over the \( SO(5)/SO(4) \) generators. The embedding of the fermion sector is somewhat complicated and there are several possible realizations. Thus, we will be content with describing the results in general terms. The computation of the Higgs potential was illustrated, in flat space, in the TASI 2005 Lecture on “To the 5th Dimension and Back”, by R. Sundrum [36]. An elegant computation of the computation for arbitrary warp factor was given in [37]. The object of interest is the Coleman-Weinberg potential, which for a real bulk scalar with non-trivial \( SO(5) \times U(1)_{B-L} \) quantum numbers takes the form

\[ V = \frac{1}{2} \sum_n \int \frac{d^4p}{(2\pi)^4} \ln \left( p^2 + m_n^2 \right). \]  

(137)

Here \( m_n \) is the spectrum obtained by the scalar KK-decomposition in the presence of the Higgs background, Eq. (136), and contains the information about the metric background as well as the imposed boundary conditions. The contributions from fields of other spins are obtained by using the corresponding spectrum and multiplying by the number of 4D massive degrees of freedom (2 for complex scalars, 4 for fermions, 3 for real gauge bosons, 5 for the graviton), and including a \((-1)\) for the fermionic contributions.

The above Coleman-Weinberg potential can be manipulated into a convergent integral

\[ V = \sum_{r=\text{fields}} \frac{N_r}{16\pi^2} \int_0^\infty dp p^3 \ln \left[ 1 + F_r(-p^2) \sin^2 \left( \frac{\lambda_r h}{f_h} \right) \right], \]  

(138)

where \( \lambda_r \) is a representation-dependent, order one constant,

\[ f_h^2 = \frac{1}{g_5^2 \int_0^L dy e^{-2A(y)}} \]  

(139)

is the Higgs decay constant, and \( h \) is the Higgs vev. The dependence on the latter enters only through the explicit sine in Eq. (138). The form factor \( F_r(-p^2) \) contains the model information. Falkowski found simple approximations that capture the physics quite well [37]

\[ \begin{align*}
\text{Gauge:} & \quad F_{(1)}(-p^2) \approx \frac{g^2 f_h^2}{M_{\text{KK}}^2 \sinh^2 (p/M_{\text{KK}})}, \\
\text{Fermion:} & \quad F_{(1/2)}(-p^2) \approx \frac{y^2 f_h^2}{M_{\text{KK}}^2 \sinh^2 (p/M_{\text{KK}})},
\end{align*} \]  

(140)
Figure 14: Radiatively induced Higgs potential in GHU models, illustrating the interplay between the gauge (stabilizing) and fermion (destabilizing) effects. The black curve displays a non-trivial minimum in the angular variable $h/f_h$.

where $g$ and $y$ are the 0-mode gauge and Yukawa couplings, respectively, and

$$M_{KK} \equiv \frac{1}{\int_0^L dy \, e^{-A(y)}}. \quad (141)$$

In Fig. 14 we illustrate how the gauge and Yukawa contributions can generate a non-vanishing minimum for $h$. This minimum can lie at $h \ll f_h$, which corresponds to a limit where the Higgs has SM-like properties. We refer the reader to the literature for further details of GHU scenarios, and close here our (non-exhaustive) survey of extra-dimensional models with phenomenological applications at the weak scale.
4 Lecture 4: Introduction to Holography

In the previous lectures we have analyzed compact extra-dimensional theories by rewriting them in terms of Kaluza-Klein modes. This language is of direct interest for the collider phenomenology, since it is these (mass eigenstates) that would be produced in high-energy collisions.

However, the KK decomposition is not the only rewriting that is useful. In this lecture, we review the holographic approach, which in certain cases provides the most direct way of deriving properties of the theory. One such application is to the analysis of EW constraints. In addition, holography gives rise to a dual 4D description of extra-dimensional models that is quite interesting in its own right.

4.1 The Holographic Approach: Scalar Case

It will be useful to illustrate the idea, and its connection to the KK picture, in the simplest case of a real bulk scalar, propagating in our generic background, Eq. (23). First, let us generalize slightly our previous free scalar action to

\[ S = \int d^5x \sqrt{g} \left\{ \frac{1}{2} \partial_M \Phi \partial^M \Phi - \frac{1}{2} M(y)^2 \Phi^2 \right\} + \int_{UV} d^4x \sqrt{g} L_0(\Phi) + \int_{IR} d^4x \sqrt{\bar{g}} L_L(\Phi), \]  

by allowing a potentially \( y \)-dependent mass, as well as (quadratic) boundary terms.\(^{20}\) The latter will typically translate into a certain form of the boundary conditions, as dictated by the variational principle discussed in Lecture 2. For instance, consider

\[ L_i(\Phi) = \frac{1}{2} r_i \bar{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} M_i \Phi^2, \]  

where \( \bar{g}_{\mu\nu} \) denotes the induced metric, and \( r_i \) and \( M_i \) have mass dimension \(-1\) and \(1\), respectively (recall that \( [\Phi] = 3/2 \)). If we write, for convenience\(^{21}\)

\[ M_0 = - (\alpha - 2) k_{UV} + m_{UV}, \]  

\[ M_L = + (\alpha - 2) k_{IR} + m_{IR}, \]  

where \( k_{UV} \equiv A'(0) \) and \( k_{IR} \equiv A'(L) \), the associated b.c.’s read

\[ \partial_y \Phi \bigg|_0 = - \left[ (\alpha - 2) + b_{UV} \left( \frac{i \partial_\mu}{k_{UV}} \right) \right] k_{UV} \Phi \bigg|_0, \]  

\[ \partial_y \Phi \bigg|_L = - \left[ (\alpha - 2) - b_{IR} \left( \frac{e^{A(L)}}{k_{IR}} i \partial_\mu \right) \right] k_{IR} \Phi \bigg|_L, \]  

\(^{20}\)The philosophy here is the same as in the KK approach: use the free theory to define a useful set of variables, and then treat possible interactions perturbatively, after rewriting them in terms of the “free variables”.

\(^{21}\)This is particularly useful when \( k_{UV} \sim k_{IR} \equiv k_{eff} \), and \( M^2 = (\alpha^2 - 4) k_{eff}^2 \).
where we defined, for \( i = UV, IR \),

\[
    b_i(z) = \hat{r}_i z^2 - \hat{m}_i, \quad \hat{r}_i \equiv r_i k_i, \quad \hat{m}_i \equiv \frac{m_i}{k_i}.
\] (148)

It is understood here that when \( z \propto \partial_\mu \), then \( z^2 \propto \partial_\mu \partial_\nu \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \), i.e. the Lorentz contraction is done with the Minkowski metric. Our sign conventions are such that positive \( r_i \) (\( m_i \)) decrease (increase) the KK masses, as could be expected from physical considerations.

**Exercise:** Convince yourself that by sending \( \hat{m}_i \to \infty \) one recovers the Dirichlet b.c.’s on the respective boundary.

In the holographic approach, the UV and IR boundaries are treated asymmetrically. In particular, special importance is given to the UV brane (where \( A(0) = 0 \)). The scalar theory above is subjected to a two-step process. In the first step, while the b.c. on the IR brane is imposed as in the KK analysis, one requires that on the UV brane the field attains a prescribed value, \( \phi_0(x) \):

\[
    \Phi(x, y = 0) = \phi_0(x) \quad \text{4D mom. space} \quad \implies \quad \tilde{\Phi}(p, y = 0) = \tilde{\phi}_0(p),
\] (149)

where the tildes on the fields indicate a 4D Fourier transform. To keep track of the connection to our KK analysis, we can factor out one power of the warp factor [as we did to define the physical wavefunctions in Eq. (71)]

\[
    \tilde{\Phi}(p, y) = e^{A(y)} \tilde{\phi}(p, y),
\] (150)

so that the EOM and IR b.c. read [c.f. Eq. (75)]:

\[
    0 = \partial_y^2 \tilde{\phi} - 2A' \partial_y \tilde{\phi} + \left[ A'' - 3(A')^2 - M^2 + e^{2A} p^2 \right] \tilde{\phi},
\] (151)

\[
    \partial_y \tilde{\phi} \bigg|_L = - \left[ (\alpha - 1) - b_{IR} \left( \frac{p e^{A(L)}}{k_{IR}} \right) \right] k_{IR} \tilde{\phi} \bigg|_L.
\] (152)

Note that compared to the KK equations we have \( m_n \to p \). If we replace such a solution back into the action and integrate by parts, we find that

1. The bulk part is proportional to the EOM, Eq. (151).
2. The surface term generated at \( y = L \) from the integration by parts combines with \( \mathcal{L}_L(\Phi) \) to reproduce the IR b.c., Eq. (152).

It follows that these two contributions vanish, and the action evaluated on the above solution is determined by \( \mathcal{L}_0(\Phi) \) and the generated surface term at \( y = 0 \):

\[
    S[\phi_0] = \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2} \tilde{\Phi} \partial_y \tilde{\Phi} + \mathcal{L}_0(\tilde{\Phi}) \right\}_0
\]

\[
    = \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2} \tilde{\phi} \partial_y \tilde{\phi} + \frac{1}{2} k_{UV} \tilde{\phi}^2 + \mathcal{L}_0(\tilde{\phi}) \right\}_0.
\] (153)
Note that this action depends only on $\tilde{\phi}_0$, the UV boundary field, that will be called the *holographic field*. In particular, $\partial_y \tilde{\phi} \big|_0$ is fixed by the holographic field and the IR boundary condition.

**Exercise:** Derive the previous result, Eq. (153).

At this point it is convenient to define a *holographic profile*, $K(p,y)$, with $\tilde{\phi}_0$ factored out:

$$
\tilde{\phi}(p,y) = \tilde{\phi}_0(p) \cdot K(p,y),
$$

where $K(p,y)$ obeys the same EOM and IR b.c. as $\tilde{\phi}$, Eqs. (151) and (152), and in addition

$$
K(p,y=0) = 1.
$$

In terms of this profile, we get the *holographic action*

$$
S[\phi_0] = \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{1}{2} \tilde{\phi}_0 \partial_y K(p,y) \big|_0 + \frac{1}{2} k_{UV} \tilde{\phi}_0^2 + L_0(\tilde{\phi}_0) \right\},
$$

where $\tilde{\phi}_0^2$ is understood as $\tilde{\phi}_0(p)\tilde{\phi}_0(-p)$. In the second step of the holographic approach, one requires that $S[\phi_0]$ be stationary against variations in $\phi_0$:

$$
\delta S[\phi_0] = \int \frac{d^4 p}{(2\pi)^4} \left\{ \tilde{\phi}_0 \partial_y K(p,y) \big|_0 + k_{UV} \tilde{\phi}_0 + \frac{\partial L_0}{\partial \tilde{\phi}_0} \right\} \delta \tilde{\phi}_0
$$

$$
= 0 \quad \text{for any } \delta \tilde{\phi}_0.
$$

It follows that

$$
\partial_y \tilde{\phi} \big|_0 = -k_{UV} \tilde{\phi} - \frac{\partial L_0}{\partial \tilde{\phi}_0} \big|_0.
$$

For $L_0$ as given in Eq. (143), we get

$$
\partial_y \tilde{\phi} \big|_0 = -\left[ (\alpha - 1) k_{UV} + r_{UV} p^2 - m_{UV} \right] \tilde{\phi} \big|_0,
$$

which matches the UV b.c. Eq. (146) after recalling Eq. (150).

**The lesson is that:**

1. Considering classical solutions that take a *fixed* field value at $y = 0$, and
2. Then allowing the fixed value to vary arbitrarily (i.e. integrating over it in the path-integral sense),

is completely equivalent to performing a KK analysis. This observation is, in a sense, remarkable. We have that the 4D holographic theory, Eq. (156), treated with the usual tools (i.e. deriving the EOM by the principle of least action w.r.t. $\phi_0$), contains all the information of the 5D theory.
The latter describes an infinite number of 4D degrees of freedom (the KK modes), yet we seem to have a single $\phi_0(x)$ in the holographic action.

On closer inspection, we see that $S[\phi_0]$ is highly non-local, i.e. the momentum dependence of $\partial_y K(p, y)|_0$ can be non-analytic in $p$, as dictated by the 5D EOM. What happens is that

$$\Pi \tilde{\phi}_0 \equiv \left\{ \partial_y K(p, y)|_0 + k_{UV} \right\} \tilde{\phi}_0 + \frac{\partial L_0}{\partial \tilde{\phi}_0}$$

vanishes for an infinite number of values of $p$. When $L_0$ is quadratic, these solutions coincide with the KK masses: $p^2 = m_n^2$.

To exhibit this in more detail, consider the Neumann propagator, $G^N(p; y, y')$, satisfying

$$\left\{ \partial_y^2 - 2A' \partial_y + \left[ A'' - 3(A')^2 - M^2 + e^{2A} p^2 \right] \right\} G^N(p; y, y') = e^{2A} \delta(y - y') ,$$

as well as the boundary conditions

$$\left. \partial_y \left[ (\alpha - 1) + b_{UV} \left( \frac{p}{k_{UV}} \right) \right] k_{UV} \right|_{y=0} = 0 ,$$

$$\left. \partial_y \left[ (\alpha - 1) - b_{IR} \left( \frac{p e^{A(L)}}{k_{IR}} \right) \right] k_{IR} \right|_{y=L} = 0 .$$

To find an expression for $G^N(p; y, y')$ it is convenient to use the holographic profile $K(p, y)$, together with a second function that also obeys the scalar EOM, Eq. (151), but satisfies the b.c.’s $S(p, y = 0) = 0$, $\partial_y S(p, y)|_{y=0} = 1$.

The Neumann propagator can then be expressed as

$$G^N(p; y, y') = \frac{1}{\Pi(p)} K(p, y) K(p, y') - S(p, y_<) K(p, y_>) ,$$

where $\Pi$ is defined by Eq. (160) and $y_> = \max\{y, y'\}$, $y_< = \min\{y, y'\}$.

**Exercise:**

a) Check that the above expression for $G^N(p; y, y')$ indeed satisfies Eqs. (161)–(163).

b) Show that $G^D(p; y, y') \equiv -S(p, y_<) K(p, y_>)$ is the Dirichlet propagator satisfying Eq. (161) and the IR b.c., Eq. (163), but obeying the UV b.c. $G^D(p; y = 0, y') = 0$.

It follows that

$$G^N(p; 0, 0) = \frac{1}{\Pi(p)} ,$$
and therefore that the holographic action, Eq. (156), can be written as (recall we are assuming here that $L_0$ is quadratic in the field)

$$S[\phi_0] = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2} \tilde{\phi}_0 \Pi \tilde{\phi}_0 = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2} \tilde{\phi}_0 \left[ G^N(p; 0, 0) \right]^{-1} \tilde{\phi}_0 .$$  

(168)

On the other hand, if we define the KK wavefunctions by applying the variational procedure reviewed in Lecture 2 to the scalar action, Eq. (142), namely

$$\left\{ \frac{d^2}{dy^2} - 2A' \frac{d}{dy} + \left[ A'' - 3(A')^2 - M^2 + e^{2A} m^2_n \right] \right\} f_n(y) = 0 , \quad (169)$$

$$\left\{ \frac{d}{dy} + \left[ (\alpha - 1) + b_{UV} \left( \frac{m_n}{k_{UV}} \right) \right] k_{UV} \right\} f_n(y) \bigg|_{y=0} = 0 , \quad (170)$$

$$\left\{ \frac{d}{dy} + \left[ (\alpha - 1) - b_{IR} \left( \frac{m_n c^{A(L)}}{k_{IR}} \right) \right] k_{IR} \right\} f_n(y) \bigg|_{y=L} = 0 , \quad (171)$$

where the $f_n$ are normalized according to \[\text{see also Eq. (127)}\]

$$\frac{1}{L} \int_0^L f_n(y) f_m(y) dy + \frac{r_{UV}}{L} f_n(0) f_m(0) + \frac{r_{IR}}{L} f_n(L) f_m(L) = \delta_{mn} , \quad (172)$$

it is known on general grounds that the (Neumann) Green function can be represented as

$$G^N(p; y, y') = \frac{1}{L} \sum_n \frac{f_n(y) f_n(y')}{p^2 - m^2_n} . \quad (173)$$

It is then clear from Eq. (167) that the zeros of $\Pi$ coincide with the poles of $G^N(p; 0, 0)$, i.e. the KK masses $p^2 = m^2_n$, as we wanted to show.

**Comments:** The *holographic profile*, $K(p, y)$, can be written as

$$K(p, y) = \Pi(p) G^N(p; 0, y) = \frac{G^N(p; 0, y)}{G^N(p; 0, 0)} , \quad (174)$$

which we will call the *amputated boundary-to-bulk propagator*, where amputation here means dividing by the boundary-to-boundary propagator, $G^N(p; 0, 0)$.

Note also that, for $p \sim m_n$, the holographic Lagrangian reads

$$\mathcal{L}_{\text{hol}} \sim \frac{L}{2} \frac{1}{[f_n(0)]^2} \tilde{\phi}_0 (p^2 - m^2_n) \tilde{\phi}_0 + \text{suppressed} , \quad (175)$$

and, therefore, the wavefunctions evaluated at $y = 0$ can be read from the coefficient of the “kinetic term” in the region dominated by the corresponding KK mode. Normalizing canonically, these wavefunctions will control the interaction terms in the more general interacting theory.
4.2 Explicit Expressions for Wavefunctions, Propagators, etc.

It is possible to write completely general and fairly explicit formulas for all the central objects in our discussion. In this section we provide such expressions, and also give the formulas in the AdS$_5$ limit, for easy reference.

4.2.1 General Case

Assume that $h_1(p, y)$ and $h_2(p, y)$ are two independent solutions to the bulk EOM, Eq. (151). In particular, their Wronskian is non-vanishing (see also footnote 13 in Section 2.5 of Lecture 2):

$$ W(y) = h_1 \partial_y h_2 - h_2 \partial_y h_1 \neq 0 . $$

(176)

The brane-localized terms, which determine the b.c.’s, enter only through

$$ \tilde{h}^\text{UV}_i(p) \equiv \left\{ \partial_y h_i(p, y) + \left[ (\alpha - 1) + b_\text{UV} \left( \frac{p}{k_\text{UV}} \right) \right] k_\text{UV} h_i(p, y) \right\} \bigg|_{y=0} , $$

(177)

$$ \tilde{h}^\text{IR}_i(p) \equiv \left\{ \partial_y h_i(p, y) + \left[ (\alpha - 1) - b_\text{IR} \left( \frac{p e^{A(L)}}{k_\text{IR}} \right) \right] k_\text{IR} h_i(p, y) \right\} \bigg|_{y=L} , $$

(178)

for $i = 1, 2$, where $b_\text{UV}(z)$ and $b_\text{IR}(z)$ are as in Eq. (148). We also define the auxiliary functions

$$ f^\text{UV}(p, y) = \tilde{h}^\text{UV}_2(p) h_1(p, y) - \tilde{h}^\text{UV}_1(p) h_2(p, y) , $$

(179)

$$ f^\text{IR}(p, y) = \tilde{h}^\text{IR}_2(p) h_1(p, y) - \tilde{h}^\text{IR}_1(p) h_2(p, y) , $$

(180)

which are solutions to the EOM, obeying UV / IR (+) b.c.’s, respectively. Imposing the (+) type b.c. on the other brane for either of these functions results, up to normalization, in the KK wavefunctions, $f_n(p) \propto f^\text{UV}(m_n, y) \propto f^\text{IR}(m_n, y)$, where the KK masses are solutions, $p = m_n$, of

$$ D_N(p) \equiv \tilde{h}^\text{UV}_1(p) \tilde{h}^\text{IR}_2(p) - \tilde{h}^\text{UV}_2(p) \tilde{h}^\text{IR}_1(p) = 0 . $$

(181)

If the KK problem required Dirichlet b.c.’s on the UV brane (but (+) on the IR brane), the KK wavefunctions would be given by $f_n(p) \propto f^\text{IR}(m_n, y)$ with the spectrum now determined by

$$ D_D(p)/k_\text{UV} \equiv f^\text{IR}(p, 0) = h_1(p, 0) \tilde{h}^\text{IR}_2(p) - h_2(p, 0) \tilde{h}^\text{IR}_1(p) = 0 . $$

(182)

Note also that $W(0) = f^\text{UV}(p, 0)$ and $W(L) = f^\text{IR}(p, L)$. In terms of the above we have that

$$ K(p, y) = \frac{f^\text{IR}(p, y)}{f^\text{IR}(p, 0)} , $$

(183)

$$ S(p, y) = \frac{1}{W(0)} \left[ h_1(p, 0) h_2(p, y) - h_2(p, 0) h_1(p, y) \right] , $$

(184)

$$ G^N(p; y, y') = \frac{f^\text{UV}(p, y')}{{W(0) D_N(p)}} . $$

(185)

In particular, $G^N(p; 0, 0) = D_D(p)/[k_\text{UV} D_N(p)]$, a relation of central importance in the holographic interpretation to be presented in Section 4.4. The student should have no problem checking the above formulas.
4.2.2 The AdS$_5$ Limit

Here we collect the expressions for the wavefunctions and propagators in AdS$_5$. The formulas below are also useful for fields of other spins. For fermions with a LH (RH) 0-mode, one should identify $\alpha = c + 1/2$ ($\alpha = c - 1/2$), and set $m_{\text{UV}} = m_{\text{IR}} = 0$. For the spin-1 case, one should take $\alpha = 1$. Up to respective factors of $k$ and $k \, e^{kL}$, the functions of Eqs. (177) and (178) read:

$$J^\text{UV}_\alpha(z) = zJ_{\alpha-1}(z) + b_{\text{UV}}(z) J_\alpha(z) \ ,$$  

$$J^\text{IR}_\alpha(z) = zJ_{\alpha-1}(z) - b_{\text{IR}}(z) J_\alpha(z) \ ,$$

with analogous definitions for $\tilde{Y}^\text{UV}_\alpha(z)$ and $\tilde{Y}^\text{IR}_\alpha(z)$. In terms of these, the KK wavefunctions, obeying Eqs. (169) and (171) are

$$f_\alpha(y) = N_n e^{ky} \left[ J_\alpha \left( \frac{m_n}{k} e^{ky} \right) + B^\text{IR}_\alpha \left( m_n \right) Y_\alpha \left( \frac{m_n}{k} e^{ky} \right) \right] \ ,$$

where $N_n$ is determined by the normalization condition, Eq. (78), and

$$D^\text{IR}_\alpha(p) = -\frac{\tilde{J}^\text{IR}_\alpha \left( \frac{p}{k} e^{kL} \right)}{Y^\text{IR}_\alpha \left( \frac{p}{k} e^{kL} \right)} \ .$$

The UV b.c., Eq. (170), determines the KK masses from

$$D_N(m_n) \equiv \tilde{J}^\text{UV}_\alpha \left( \frac{m_n}{k} \right) \tilde{Y}^\text{IR}_\alpha \left( \frac{m_n}{k} e^{kL} \right) - \tilde{Y}^\text{UV}_\alpha \left( \frac{m_n}{k} \right) \tilde{J}^\text{IR}_\alpha \left( \frac{m_n}{k} e^{kL} \right) = 0 \ .$$

If instead we were to impose Dirichlet b.c.’s on the UV brane, the spectrum would be given by

$$D_D(m_n) \equiv J_\alpha \left( \frac{m_n}{k} \right) \tilde{Y}^\text{IR}_\alpha \left( \frac{m_n}{k} e^{kL} \right) - Y_\alpha \left( \frac{m_n}{k} \right) \tilde{J}^\text{IR}_\alpha \left( \frac{m_n}{k} e^{kL} \right) = 0 \ .$$

The holographic profile $K(p, y)$ and the function $S(p, y)$ take the form:

$$K(p, y) = e^{ky} \left[ \frac{J_\alpha \left( \frac{p}{k} e^{ky} \right) + B^\text{IR}_\alpha(p) Y_\alpha \left( \frac{p}{k} e^{ky} \right)}{J_\alpha \left( \frac{p}{k} \right) + B^\text{IR}_\alpha(p) Y_\alpha \left( \frac{p}{k} \right)} \right] \ ,$$

$$S(p, y) = -\frac{\pi}{2k} e^{ky} \left[ Y_\alpha \left( \frac{p}{k} \right) J_\alpha \left( \frac{p}{k} e^{ky} \right) - J_\alpha \left( \frac{p}{k} \right) Y_\alpha \left( \frac{p}{k} e^{ky} \right) \right] \ .$$

Then the Neumann propagator is given by

$$G^N(p; y, y') = \frac{\pi e^{k(y+y')}}{2k D_N(p)} \left[ \tilde{Y}^\text{UV}_\alpha \left( \frac{p}{k} \right) J_\alpha \left( \frac{p}{k} e^{ky} \right) - \tilde{J}^\text{UV}_\alpha \left( \frac{p}{k} \right) Y_\alpha \left( \frac{p}{k} e^{ky} \right) \right] \times \left[ \tilde{Y}^\text{IR}_\alpha \left( \frac{p}{k} e^{kL} \right) J_\alpha \left( \frac{p}{k} e^{ky} \right) - \tilde{J}^\text{IR}_\alpha \left( \frac{p}{k} e^{kL} \right) Y_\alpha \left( \frac{p}{k} e^{ky} \right) \right] \ ,$$

while the Dirichlet propagator reads:

$$G^D(p; y, y') = \frac{\pi e^{k(y+y')}}{2k D_D(p)} \left[ Y_\alpha \left( \frac{p}{k} \right) J_\alpha \left( \frac{p}{k} e^{ky} \right) - J_\alpha \left( \frac{p}{k} \right) Y_\alpha \left( \frac{p}{k} e^{ky} \right) \right] \times \left[ \tilde{Y}^\text{IR}_\alpha \left( \frac{p}{k} e^{kL} \right) J_\alpha \left( \frac{p}{k} e^{ky} \right) - \tilde{J}^\text{IR}_\alpha \left( \frac{p}{k} e^{kL} \right) Y_\alpha \left( \frac{p}{k} e^{ky} \right) \right] \ .$$

One can easily check that the above expressions obey the EOM and all the relevant b.c.’s.
4.3 Application: EW Constraints in Anarchic Models

The holographic rewriting of the 5D theory allows for a simple derivation of the expressions for the oblique parameters that control some of the most important corrections to the EW observables in models with flavor anarchy (i.e. with the light families localized near the UV brane). Non-universal corrections, such as $\delta g_{ZbL\bar{L}}$, require more work.

Consider a bulk $SU(2)_L \times SU(2)_R \times U(1)_X$ custodial model. It will be easy to specialize the results to models with only the SM gauge group by setting to zero the extra gauge bosons/propagators. Our starting action is then

$$S_{\text{Cust. gauge}} = \int \frac{d^4p}{(2\pi)^4} \int_0^L dy \sqrt{g} \left\{ -\frac{1}{4g_{5L}^2} (L_{MN}^a)^2 - \frac{1}{4g_{5R}^2} (R_{MN}^a)^2 - \frac{1}{4g_{5X}^2} (X_{MN}^a)^2 + \frac{1}{2} v(y)^2 (L_{M}^a - R_{M}^a)^2 \right\} ,$$

where $L_{MN}^a$, $R_{MN}^a$ and $X_{MN}^a$ are the field strengths for $SU(2)_L$, $SU(2)_R$, and $U(1)_X$, respectively, while $g_{5L}$, $g_{5R}$ and $g_{5X}$ are the corresponding gauge couplings. Note also that we allow for a $y$-dependent vev, as would arise from a bulk Higgs with a non-trivial profile.

**Exercise:** Show that the $(L - R)^2$ structure for the gauge mass term arises when the Higgs is a bidoublet of $SU(2)_L \times SU(2)_R$.

It is useful to rewrite the above action in the “$V - A$” basis:

$$V_M^a = \frac{g_{5R}}{g_{5L}} L_M^a + \frac{g_{5L}}{g_{5R}} R_M^a , \quad A_M^a = L_M^a - R_M^a ,$$

so that

$$S_{\text{Cust. gauge}} = \int \frac{d^4p}{(2\pi)^4} \int_0^L dy \sqrt{g} \left\{ -\frac{1}{4g_{5Z}^2} [(V_{MN}^a)^2 + (A_{MN}^a)^2] - \frac{1}{4g_{5X}^2} (X_{MN}^a)^2 + \frac{1}{2} v(z)^2 A_M^a \right\} ,$$

where $g_{5Z}^2 \equiv g_{5L}^2 + g_{5R}^2$, and $V_{MN}^a$ and $A_{MN}^a$ are the field strengths associated with $V_M$ and $A_M$, respectively. We can now rewrite this action holographically by defining the holographic profiles $K_A$, $K_V$ and $K_X$, satisfying [see also Eq. (92)]

$$\partial_y \left[ e^{-2A} \partial_y K_A \right] - \left[ M_A(y)^2 - p^2 \right] K_A = 0 ,$$

$$\partial_y \left[ e^{-2A} \partial_y K_{V,X} \right] + p^2 K_{V,X} = 0 ,$$

where $M_A(y)^2 = g_{5Z}^2 v(y)^2$. For simplicity, we assume that the possible quadratic IR-localized terms vanish, so that the $K_i$ simply obey

$$K_i(p, y = 0) = 1 ,$$

$$\partial_y K_i(p, y)|_{y=L} = 0 .$$
for $i = A, V, X$. In these models all fields obey Neumann (+) b.c.’s at $y = L$ so that the custodial symmetry is well-preserved in the IR region. On the UV brane, we imagine there is physics that “lifts” the unwanted gauge 0-modes:

$$ R^b \equiv W^b_R , \quad \text{for } b = 1, 2 $$

$$ Z' = \frac{1}{\sqrt{g_{5R}^2 + g_{5X}^2}} (R^3 - X) , $$

by adding a UV-localized Lagrangian

$$ L_0 = \frac{1}{2} M^2_R \left[ (W^1_R)^2 + (W^2_R)^2 \right] + \frac{1}{2} M^2_{Z'} (Z')^2 , $$

and sending $M^2_R, M^2_{Z'} \to \infty$. This imposes effectively Dirichlet (−) b.c.’s on the UV brane, so that there are no corresponding holographic fields (the boundary values vanish).

We are therefore left with the SM fields, related to $A, V$ and $X$ by

$$ A^a = W^a_L - \delta^{a3} B , \quad V^a = \frac{g_{5R}}{g_{5L}} W^a_L + \delta^{a3} \frac{g_{5L}}{g_{5R}} B , \quad X = B , $$

where we switched to the more familiar notation $W^a_L \equiv L^a$. We then have

$$ S[\bar{W}_L, \bar{B}] \overset{\text{Hol}}{=} \int \frac{d^4p}{(2\pi)^4} \left\{ -\frac{1}{2} P^{\mu\nu} \left\{ A^a_\mu \Pi_A(p^2) A^a_\nu + V^a_\mu \Pi_V(p^2) V^a_\nu + X_\mu \Pi_X(p^2) X_\nu \right\} \right\} $$

$$ = \int \frac{d^4p}{(2\pi)^4} (-P^{\mu\nu}) \left\{ \bar{W}^+_\mu \Pi_+ \bar{W}^-_\nu + \frac{1}{2} \bar{W}^3_\mu \Pi_{BB} \bar{W}^3_\nu + \frac{1}{2} B_\mu \Pi_{BB} B_\nu + \bar{W}^3_{3B} \Pi_{3B} B_\nu \right\} , $$

where $P^{\mu\nu} = \eta^{\mu\nu} - p^\mu p^\nu / p^2$ is the transverse projector, and we use here a bar to denote the boundary values (i.e. the holographic fields). We also defined $\bar{W}^\pm \equiv (W^1 \mp i W^2) / \sqrt{2}$. To obtain the second line, we used Eq. (205) applied to the holographic fields. In doing so, we identify

$$ \Pi_{aa} = \Pi_A + \frac{g_{5R}^2}{g_{5L}^2} \Pi_V , \quad a = 1, 2, 3 $$

$$ \Pi_{BB} = \Pi_A + \frac{g_{5L}^2}{g_{5R}^2} \Pi_V + \Pi_X , $$

$$ \Pi_{3B} = -\Pi_A + \Pi_V , $$

where

$$ \Pi_{A,V} = \frac{1}{g_{5Z}^2} \partial_y K_{A,V} \bigg|_{y = 0} , \quad \Pi_X = \frac{1}{g_{5X}^2} \partial_y K_X \bigg|_{y = 0} , $$

are the boundary propagators, which take into account the gauge couplings in front of the gauge kinetic terms in Eq. (198).
UV-localized Fermions

Up to this point we have not mentioned anything about fermions. A holographic approach for fermions has been discussed in [39]. If these are UV-localized their couplings to gauge bosons are nearly independent of the precise fermion localization profiles. In this case, the couplings of the fermionic holographic fields take the universal form

$$\mathcal{L}_{\text{Univ}} = \bar{W}_\mu J^\mu_L + \bar{B}_\mu J^\mu_Y,$$  \hspace{1cm} (211)

where

$$J^\mu_L \equiv \sum_\psi \bar{\psi} \gamma^\mu T^a_L \psi, \hspace{1cm} J^\mu_Y \equiv \sum_\psi \bar{\psi} \gamma^\mu Y \psi.$$  \hspace{1cm} (212)

The holographic procedure then automatically gives an effective 4D theory in oblique form.

When the theory has a mass gap, so that

$$\Pi_i(p^2) = \Pi_i(0) + p^2 \Pi'_i(0) + \frac{1}{2} p^4 \Pi''_i(0) + \cdots$$  \hspace{1cm} (213)

have an analytic (Taylor) low-energy expansion, the effects of the new physics (contained in the $\Pi$'s) can be parameterized in terms of the following [40]:

$$\hat{T} = 1 - \frac{\Pi_{33}(0)}{\Pi_{+-}(0)}, \hspace{1cm} W = \frac{1}{2} g^2 M_W^2 \Pi''_{33}(0),$$

$$\hat{S} = g^2 \Pi'_{3B}(0), \hspace{1cm} Y = W,$$  \hspace{1cm} (214, 215)

where

$$\frac{1}{g^2} = \Pi'_{11}(0), \hspace{1cm} \frac{1}{g'^2} = \Pi'_{BB}(0), \hspace{1cm} -\frac{M_W^2}{g^2} = \Pi_{+-}(0).$$  \hspace{1cm} (216)

The first two parameters are related to the Peskin-Takeuchi $S$ and $T$ [41]

$$\alpha S = 4 \hat{S}_W \hat{S}, \hspace{1cm} \alpha T = \hat{T}.$$  \hspace{1cm} (217)

Given a specific background, one can solve for the holographic profiles to find the $\Pi_i(p^2)$, and use the previous general formulas to compare to the EW data.

As an example, for the AdS$_5$ background, and assuming that the Higgs is $\delta$-function IR-localized, one finds in the non-custodial case (just getting rid of the ($-$, $+$) fields above completely)

$$\hat{T}_{UV} = \frac{g^{'2}}{2 g^2} \left( \frac{M_W}{k} \right)^2 kL, \hspace{1cm} W_{UV} = \frac{1}{4 k L} \left( \frac{M_W}{k} \right)^2,$$  \hspace{1cm} (218)

$$\hat{S}_{UV} = \frac{1}{2} \left( \frac{M_W}{k} \right)^2, \hspace{1cm} Y_{UV} = W_{UV},$$  \hspace{1cm} (219)

67
where the UV subscripts remind us that the fermions are assumed to be UV localized. One should note the pattern of \( kL \) enhancements and suppressions. In the presence of the custodial structure, one finds \( \hat{T}_{UV} = 0 \), while the other remain unchanged.

**Arbitrary Fermion Localization**

It is possible to extend the holographic approach to cases where the fermion 0-modes are arbitrarily localized. We illustrate the idea in a toy model with a single gauge field coupled to a (single) fermion current. The new effects can be expressed in the following diagrammatic form:

**Vertex corrections:**

\[
\gamma = \frac{1}{G^N(p; 0, 0)} \int_0^L dy \left[ f^0_\psi(y) \right]^2 G^N(p; 0, y) = \int_0^L dy \left[ f^0_\psi(y) \right]^2 K(p, y),
\]

where, for simplicity, we omit the spinor indices provided by the Dirac \( \gamma \). We see how we obtain the overlap integral of the gauge and fermion wavefunctions, which we call \( \bar{g}(p) \). Note that it depends on the 4D momentum, and also on the fermion type through its wavefunction profile [recall the \( c \)-parameters introduced in Eqs. (89) and (90)].

**Comment:** at zero momentum, \( K(p = 0, y) \) is precisely the profile for the gauge 0-mode, so the above integral matches exactly our KK notion of couplings as overlap integrals. Also, if the gauge field is massless, one has \( \bar{g}(p = 0) = 1 \). For a full comparison with our KK results, recall that here the gauge field is not canonically normalized, but rather has a factor \( L / \tilde{g}_5^2 \) in front of the kinetic term.

**4-fermion interactions:**

\[
\gamma' = \int_0^L dy dy' \left[ f^0_\psi(y) \right]^2 \left[ f^0_\psi(y') \right]^2 G^N(p; y, y') = G^N(p; 0, 0) \bar{g}(p)^2 + \frac{g_5^2}{L} \gamma_{\psi \psi'},
\]

where again we omitted the spinor structure and a Lorentz \( \eta_{\gamma \gamma'} \). In the second line we used Eqs. (166)-(167):

\[
G^N(p, y, y') = G^N(p, 0, 0)K(p, y)K(p, y') + G^D(p, y, y'),
\]
where \(G_D(p; y, y')\) is the Dirichlet propagator [see the exercise after Eq. (166)], but note that we include a factor \(g_5^2\) in the definition of both \(G_D\) and \(G_N\), due to the non-canonical gauge normalization of this section. See also Eq. (210). In addition, we defined

\[
\gamma_{\psi\psi'} \equiv \frac{1}{g_5^2L} \int_0^L dy dy' \left[ f_\psi^0(y) \right]^2 G_D(p; y, y') \left[ f_\psi^0(y') \right]^2 ,
\]

(221)

which has mass dimension \(-2\). In this definition, the explicit factor of \(1/g_5^2\) simply cancels the corresponding factor in \(G_D(p; y, y')\). Note also that our wavefunctions, which are dimensionless, differ by a factor of \(\sqrt{L}\) (and powers of the warp factor) from those used in Ref. [42].

The above effects can be encoded in the 4D effective Lagrangian:

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} (-P^{\mu\nu}) \tilde{A}_\mu \left[ G^N(p; 0, 0) \right]^{-1} \tilde{A}_\nu + \tilde{g}(p) \tilde{A}_\mu J^\mu + \frac{1}{2} \frac{g_5^2}{L} \gamma_{\psi\psi'} J^\mu J^\mu .
\]

(222)

\[
\frac{\mathcal{O}_{4\text{fermions}}}{\mathcal{O}_{4\text{fermions}}} = \tilde{g}, \quad \frac{\mathcal{O}_{4\text{fermions}}}{\mathcal{O}_{4\text{fermions}}} = \frac{g_5^2}{L} \gamma_{\psi\psi'}
\]

so that the net 4-fermion operator in the effective theory is

\[
\frac{\mathcal{O}_{4\text{fermions}}}{\mathcal{O}_{4\text{fermions}}} + \frac{\mathcal{O}_{4\text{fermions}}}{\mathcal{O}_{4\text{fermions}}} = \tilde{g}(p) G^N(p; 0, 0) \tilde{g}(p) + \frac{g_5^2}{L} \gamma_{\psi\psi'} ,
\]

thus matching the full theory result above. It is then straightforward to put these ingredients together in the context of a full \(SU(2)_L \times SU(2)_R \times U(1)_X\) model to include the cases where some fermions are not UV localized. The most notable example arises from the third generation \((t_L, b_L)\) \(SU(2)_L\) doublet, which can lead to important non-universal corrections to the coupling of the LH bottom to the \(Z\) gauge boson.

**Comments:**

- The propagators used in this section have mass dimension \(-2\), unlike those introduced in the generic discussion based on a bulk scalar field of Section 4.1 which had 5D normalization. The difference in dimensionality is provided by the factors of \(g_5^2\) already mentioned above.

- In our diagrammatic arguments, for simplicity we omitted the factors of \(i\) in the Feynman rules, as well as the spinor/Lorentz structure, but they can be easily restored.

- The above 4D effective Lagrangian is not in the oblique form even when the fermion profiles are universal, so that \(J^\mu = \sum_\psi \tilde{\psi}\gamma^\mu T_\psi\) and \(\tilde{g}(p)\) is independent of the fermion type. This is because the factor in front of \(A_\mu J^\mu\) is not 1, and also because there are 4-fermion interactions. However, when the fermion profiles are universal, one can redefine the holographic gauge fields to put the effective Lagrangian in the oblique form. For an explicit example, in the \(SU(2)_L \times SU(2)_R \times U(1)_X\) case, the student is referred to [42].

- An alternative way of obtaining the same result is to integrate out the heavy KK modes, and match onto the SM + dimension-6 operators. The above reference also reviews the
method in detail. Here we rather quote, for reference, the elegant expressions obtained in \[43, 44\].

\[
\alpha T = s_W^2 M_Z^2 L \int_0^L dy e^{2A(y)} \left[ \Omega_\psi - \Omega_h \right]^2 ,
\]

\[
\alpha S = 8 s_W^2 M_Z^2 L \int_0^L dy e^{2A(y)} \left[ \Omega_\psi - \frac{y}{L} \right] \left[ \Omega_\psi - \Omega_h \right] ,
\]

\[
Y = W = c_W^2 M_Z^2 L \int_0^L dy e^{2A(y)} \left[ \Omega_\psi - \frac{y}{L} \right]^2 ,
\]

where

\[
\Omega_{\psi,h}(y) \equiv \frac{1}{L} \int_0^y dy' \left[ f^0_{\psi,h}(y') \right]^2 ,
\]

which obeys \( \Omega_{\psi,h}(L) = 1 \) due to the normalization of the wavefunctions. These expressions hold for any background and any Higgs vev profile. They only assume that the \( f^0_{\psi} \)'s lead to universal shifts (so that the oblique parameter analysis applies), and that \( v \ll \tilde{k}_{\text{eff}} \), the scale of new physics.

- We also provide general expressions that allow to compute the most important non-oblique effect in anarchic scenarios with custodial symmetry:

\[
\delta g_{Z\bar{b}bL} = \frac{g^2 v^2}{2 c_W^2} \left\{ \left[ g_L^2 T_L^3 - g_R^2 T_R^3 \right] \beta_Q^D + \left[ g_L^2 T_L^3 - g_R^2 T_R^3 \right] \left( \beta_Q^N - \beta_{UV}^N - \beta_Q^D \right) \right\} ,
\]

where \( T_L^3 = \frac{1}{2} \) and \( T_R^3 \) are the \( SU(2)_L \times SU(2)_R \) quantum numbers for the LH bottom (also \( Y = \frac{1}{6} \)), and we defined the mass-dimension \(-2\) quantities

\[
\beta_{\psi}^{N,D} = \frac{1}{g_L^2 L} \int_0^L dy dy' \left[ f^0_{\psi}(y) \right]^2 \tilde{G}^{N,D}(p = 0; y, y') \left[ f^0_{h}(y') \right]^2 ,
\]

with \( \beta_Q^N = \beta_{\psi}^{N,D} , \beta_{UV}^N = \beta_{\text{light fermions}}^N \). In Eq. (228), the tilde on the propagator indicates that the 0-mode needs to be subtracted (for the Neumann b.c.’s).

When \( g_R = g_L \) and \( T_L^3 = T_R^3 \) (custodial protection of \( \delta g_{Z\bar{b}bL} \)), the above simplifies to

\[
\delta g_{Z\bar{b}bL} = -\frac{g^2 v^2}{2 c_W^2} \left[ \frac{1}{2} g^2 + \frac{1}{6} g^2 Y \right] \left( \beta_Q^N - \beta_{UV}^N - \beta_Q^D \right) .
\]

It is also worth noting the alternate forms for the \( \beta \) coefficients:

\[
\beta_{\psi}^N = \frac{L}{2} \int_0^L dy e^{2A(y)} \left[ \Omega_\psi - \frac{y}{L} \right] \left[ \Omega_h - \frac{y}{L} \right] ,
\]

\[
\beta_{\psi}^D = \frac{L}{2} \int_0^L dy e^{2A(y)} \Omega_\psi \Omega_h .
\]
4.4 The 4D Dual Interpretation

After this excursion into the depths of EW precision constraints, let us go back to the holographic rewriting of the 5D theory. We will offer some elementary remarks on how the 4D dual interpretation arises. For further details, I refer the student to T. Gherghetta’s 2010 TASI Lectures on “A Holographic view and BSM physics”.

We will illustrate the main points in the context of the scalar example (in sufficient generality to make it easily applicable to other spins). We found that, as a functional of the 5D scalar boundary value, the holographic action reads

\[ S[\phi_0] = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2} \tilde{\phi}_0 \Pi \phi_0 , \] (232)

where \( \Pi(p) = [G^N(p; 0, 0)]^{-1} \) is the inverse boundary-to-boundary propagator. This propagator obeys b.c.’s on the UV brane that take into account possible UV localized terms. For instance, if we write a UV localized mass term, and then take it to infinity, we get \( G^N \rightarrow G^D \), the propagator obeying Dirichlet b.c.’s on the UV brane.

The holographic procedure of replacing the classical bulk EOM, for prescribed \( \phi_0 \), back into the 5D action corresponds to the leading-order in the evaluation of the (Euclidean) path integral

\[ Z_h[\phi_0] = C \int_{\Phi(y=0)=\phi_0} D\Phi e^{-S_{\text{Bulk}}[\Phi]} \approx e^{-S[\phi_0]} , \] (233)

where \( C \) is a normalization constant chosen so that \( Z_h[0] = 1 \), and the second equality follows from the evaluation of the integral with the method of steepest descent that leads to the semiclassical expansion. Since \( S[\phi_0] \) is a 4D scalar action, we may regard \( Z_h[\phi_0] \) as the generating functional for correlators in some (not explicitly specified) 4D theory, with \( \phi_0 \) acting as an external source. More precisely, \( \phi_0 \) is interpreted as the source for some operator \( O \) in the dual theory:

\[ Z[\phi_0] = C' \int D\chi e^{-S_{\text{Dual}}[\chi]} + \int d^4x \phi_0 O , \] (234)

where we denote the dynamical variables in the dual theory by \( \chi \), and the operator \( O \) is built out of the \( \chi \)’s. The \( O \)-correlators are given by

\[ \langle O_1 \cdots O_n \rangle = \delta \frac{\delta}{\delta \phi_0(x_1)} \cdots \delta \frac{\delta}{\delta \phi_0(x_n)} Z[\phi_0] \Big|_{\phi_0=0} . \] (235)

In general, for a given 5D theory, we do not explicitly know its dual, \( S_{\text{Dual}}[\chi] \), but by identifying \( Z[\phi_0] = Z_h[\phi_0] \) we can find the correlators for \( O \) from the holographic action arising from the 5D theory. The connected Green functions are then

\[ \langle O_1 \cdots O_n \rangle^c = -\delta \frac{\delta}{\delta \phi_0(x_1)} \cdots \delta \frac{\delta}{\delta \phi_0(x_n)} S[\phi_0] \Big|_{\phi_0=0} , \] (236)

\(^{22}\text{Recall that, in coordinate space, } [\phi_0] = \frac{3}{2} , \text{ not } [\phi_0] = 1.\)
where we used the semi-classical leading order (or tree-level) term.

In particular, the two-point function (which is the only non-vanishing one if we use the free bulk scalar action) is given by

\[ \langle O(p)O(-p) \rangle = -\Pi(p) = -\left[ G^N(p; 0, 0) \right]^{-1}. \] (237)

Thus the zeros of \( G^N(p; 0, 0) \) give the masses of the states in the dual theory that can be created by \( O \).

In general, loops involving the \( \chi \) fields will induce terms that depend on \( \phi_0 \), giving rise to a contribution to the effective dual action of the form (we use here \( L \) to match dimensions)

\[ \Delta S_{\text{Dual}} = \int d^4x \left\{ \frac{L}{2} \nabla_\mu \phi_0 \nabla^\mu \phi_0 + \frac{L}{2} \Delta m^2 \phi_0^2 + \cdots \right\}. \] (238)

On the holographic side, these are matched by the terms we already wrote:

\[ S_0 = \int d^4x L_0(\Phi) = \int d^4x \left\{ \frac{1}{2} r_0 \partial_\mu \phi_0 \partial^\mu \phi_0 + \frac{1}{2} M_0 \phi_0^2 \right\}. \] (239)

We again see that such boundary terms are not really optional: they are generated by quantum effects. If we integrate, in the path-integral sense, over \( \phi_0 \) (as we know we have to do in order to recover the correct b.c.’s on the UV brane), the above terms imply that \( \phi_0 \) is a dynamical field. Then the term \( \int d^4x \phi_0 O \) corresponds to a coupling of the dual d.o.f. to \( \phi_0 \), which should be thought of as a field external to the dual theory. The physical states of the theory should then correspond to admixtures between the dual fields (also called composite states) and the elementary field \( \phi_0 \). We can, nevertheless, consider a limit where such a mixing is absent. If the localized mass \( M_0 \) is very large, the fluctuations in \( \phi_0(x) \) will be very suppressed. In the limit \( M_0 \to \infty \), the EOM will force \( \phi_0 \) to vanish, so that the dual theory contains no \( \phi_0 O \) term when on-shell. Since in this limit the holographic theory is equivalent to a KK theory obeying Dirichlet b.c.’s on the UV brane, we can expect that the Dirichlet KK states map exactly into the pure dual states, without mixing with \( \phi_0 \).

We can make the above more precise by recalling our general expressions for the Neumann propagator, given in Section 4.2.1, and in particular, the result mentioned after Eq. (185):

\[ G^N(p; 0, 0) = \frac{D_D(p)}{k_{\text{UV}} D_N(p)}, \] (240)

where \( D_D(m_n) = 0 \) and \( D_N(m_n) = 0 \) determine the KK spectrum for (−) and (+) b.c.’s on the UV brane, respectively. Thus, we see that the poles of \( \langle O(p)O(-p) \rangle = -\Pi(p) \) indeed coincide with the Dirichlet KK masses. In summary, we learn that

\[
\text{Dirichlet KK states} \quad \leftrightarrow \quad \text{pure dual states}
\]

There is also a sense in which the associated KK wavefunctions, obeying Dirichlet b.c.’s on the UV brane, characterize the properties of pure dual states. The student is referred to Ref. [45] for full details.
Figure 15: Detail of the inverse brane-to-brane Neumann propagator (for concreteness, in the AdS$_5$ case). The poles occur at the Dirichlet KK masses, which can be interpreted as the masses of the dual states. The zeros, i.e. the poles of the Neumann propagator, are the generic KK masses (red dot). For large UV localized mass $M_0$ (equivalently $m_{\text{UV}}$) the two coincide. The mass splitting is interpreted in the dual theory as arising from mixing between the elementary $\phi_0$ and the dual states $\chi$. In the AdS$_5$ background, one can further interpret the dual states as bound states in a CFT where the conformal symmetry has been spontaneously broken (by the IR boundary). The radion/dilaton is the (pseudo) Nambu-Goldstone mode corresponding to the conformal breaking. A stabilizing scalar field would correspond to explicit breaking of the conformal symmetry, as is the Planck mass associated to the UV brane.

For finite $M_0$, the KK spectrum [i.e. the poles of $G^N(p; 0, 0)$] differs from the Dirichlet spectrum [i.e. the poles of the Dirichlet propagator, $G^D(p; 0, 0)$, which coincide with the zeros of $G^N(p; 0, 0)$]. These "mass shifts" in the KK theory (illustrated in Fig. 15) can be interpreted as due to kinetic and mass mixing between $\phi_0$ and the $\chi$ states in the dual theory. In Ref. [46], such a picture has been worked out in full detail in the AdS$_5$ limit, and assuming that $r_0 = M_0 = 0$ (so that $\Phi$ obeys standard Neumann b.c.'s, $\partial_y \Phi |_{y=0} = 0$ on the UV brane) by diagonalizing the Dirichlet basis to the Neumann basis in the presence of mixing.

Therefore, one obtains a generic interpretation of the KK states as mixed dual/elementary states:

$$|\phi_n\rangle = \sin \theta_n |\phi_0\rangle + \cos \theta_n |\chi\rangle , \quad (241)$$

where the $|\phi_n\rangle$ are the generic KK states, which are mass eigenstates, and are the ones we would see in experiments. Here we used a schematic notation, as if this was a two-state system, which it is not, but should be sufficient to illustrate the general idea (from the comments above, we can think of the $\chi$ as the Dirichlet KK states). The point of physics is that the massive KK states of the theory ($n \neq 0$) have $\sin \theta_n \ll 1$, $\cos \theta_n \approx 1$, so that they are mostly composite. For
the 0-modes the elementary/composite content depends on the localization:

UV localization \[\rightarrow\] mostly elementary \[\sin \theta_0 \approx 1, \cos \theta_0 \ll 1\],

IR localization \[\rightarrow\] mostly composite \[\sin \theta_0 \ll 1, \cos \theta_0 \approx 1\],

Approximately flat \[\rightarrow\] largely elementary, but some compositeness \[\cos \theta_0 \sim 1/\sqrt{k_{\text{eff}}L}\].

One should emphasize that this holographic language simply provides a way to look at the physics obtained from the more simple-minded Kaluza-Klein approach (after all, the KK states are the physical mass eigenstates). In fact, the way we have described it here (which is well-defined operationally) the whole re-interpretation amounts to a definition, since we do not have independent access to the dual theory \[S_{\text{Dual}}[\chi]\]. Nevertheless, in concrete examples like \(\text{AdS}_5\), the detailed properties of the various propagators allow for non-trivial checks that the interpretation makes physical sense. Of course, the \(\text{AdS}_5\) limit is closely tied to the setting on which the holographic duality was first proposed \[47\], and on which the method is inspired \[48\]. In this case, both theories are known and justify the name duality, which indicates a strong/weak coupling mapping:

\[
\begin{array}{ccc}
\text{Type IIB String Theory} & \xrightarrow{\text{Dual}} & 4D \mathcal{N} = 4 \text{ SU}(N) \\
on \text{AdS}_5 \times S^5 & & \text{super Yang-Mills}
\end{array}
\]

Many non-trivial checks (computations on both sides) strongly suggest that the conjectured duality is indeed true.

One simple map is between the isometries of \(\text{AdS}_5\) and the (super) conformality of \(\mathcal{N} = 4\) SYM. Thus, one expects, more generally, that \(\text{AdS}_5\) backgrounds map onto conformal theories (where the conformal symmetry is explicitly broken by the UV brane, and spontaneously broken by the IR brane). This is why in many applications one thinks of a CFT dual. For further aspects of the dictionary see \[49, 50\].

Let us conclude this lecture by using the holographic interpretation to understand some of the results we have derived in the KK language:

- The Higgs field, being IR localized, is almost pure CFT composite. Since the compositeness scale is of order \(\tilde{k}_{\text{eff}} \ll M_P\), we can understand why \(v_{\text{EW}} \ll M_P\) (up to little hierarchies).
- Gauge fields, being flat, have a non-negligible CFT admixture. Thus their properties can receive sizable corrections from the strongly coupled CFT sector (the \(k_{\text{eff}}L\) enhancements we have found).
- When light fields are UV localized, they are mostly elementary. The \(S\)-parameter involves both these light fermions and the Higgs: it can be important, but not as much as the \(T\)-parameter which involves only the Higgs (and gauge bosons). In custodial models, the custodial symmetry of the CFT sector ensures that its contribution to the \(T\) parameter vanishes.
• The top quark, as well as the LH bottom quark, have a sizable CFT content, and their properties can receive sizable corrections (e.g. $\delta g Z\bar{b}_Lb_L$). Since both chiralities of the top quark are mostly composite, its mixing with the CFT sector leads to a large top mass, comparable to the EWSB vev, which is also connected to the CFT strong dynamics. The bottom quark can be lighter, since only the LH component has a composite nature, but the RH bottom is mostly elementary.

• **Custodial Models:**

Consider the boundary value of a gauge field $\bar{A}^a_\mu(x)$. If it is to be interpreted as a dynamical field coupled to a dual 4D theory

$$\mathcal{L}_{\text{Dual}} \supset \bar{A}^a_\mu J^{\mu a}, \quad (242)$$

the CFT currents $J^{\mu a}$ better be conserved. Thus, the dual theory must possess a global symmetry that is weakly gauged by $\bar{A}^a_\mu(x)$. If the $\bar{A}^a_\mu(x)$ obey Dirichlet boundary conditions on the UV brane (add a mass, then send it to infinity) then the CFT is left just with the global symmetry. This leads to the important mapping

$$\text{5D gauge symmetries} \quad \leftrightarrow \quad \text{(weakly gauged) CFT global symmetries}$$

depending on whether the gauge field obeys Dirichlet b.c.’s on the UV, or not.

In the custodial models we have a bulk

$$SU(2)_L \times SU(2)_R \times U(1)_X \quad \text{(243)}$$

gauge symmetry, but all non-SM gauge fields obey Dirichlet b.c.’s on the UV brane. Thus, they correspond to global symmetries of the CFT:

$$[SU(2)_L \times SU(2)_R \times U(1)_X]^\text{global} \supset \text{weakly gauged } SU(2)_L \times U(1)_Y.$$ \quad (244)

Similarly to the SM, it is this global symmetry that protects the relation $M_W^2 \approx M_Z^2 \cos^2 \theta_w$.

Recall that the SM gauge fields (0-modes) are mostly elementary, hence not quite part of the CFT. In particular, $g'$ breaks the custodial symmetry, much as in the SM.

**Exercise:** What about the loop effects associated with the top tower?

Have a productive extra-dimensional journey!

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