LOCAL WELL-POSEDNESS FOR THE INHOMOGENEOUS NONLINEAR SCHröDINGER EQUATION

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ABSTRACT. We consider the Cauchy problem for the inhomogeneous nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u = \frac{\mu}{|x|} - b|u|^{\alpha} u, \quad u(0) \in H^s(\mathbb{R}^N), \ N \geq 1, \ \mu \in \mathbb{C}, \ b > 0 \text{ and } \alpha > 0. \]

Only partial results are known for the local existence in the subcritical case \( \alpha < \frac{(4 - 2b)(N - 2s)}{N} \) and much more less in the critical case \( \alpha = \frac{(4 - 2b)(N - 2s)}{N} \). In this paper, we develop a local well-posedness theory for the both cases. In particular, we establish new results for the continuous dependence and for the unconditional uniqueness. Our approach provides simple proofs and allows us to obtain lower bounds of the blowup rate and of the life span. The Lorentz spaces and the Strichartz estimates play important roles in our argument. In particular this enables us to reach the critical case and to unify results for \( b = 0 \) and \( b > 0 \).

1. Introduction. In this paper we consider the Cauchy problem for the inhomogeneous nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u = \mu|x|^{-b}|u|^\alpha u, \quad u(0,.) = u_0, \]

where \( u_0 \) is a complex valued function, \( u_0 \in H^s(\mathbb{R}^N), \ s \geq 0, \ u = u(t, x) \in \mathbb{C}, \ t \in \mathbb{R}, \ x \in \mathbb{R}^N \setminus \{0\}, \) and

\[ N \geq 1, \ \mu \in \mathbb{C}, \ 0 < b < 2, \ \alpha > 0. \]

Equation (1) for \( b = 0 \) corresponds to the standard nonlinear Schrödinger equation. For \( b > 0 \), (1) plays a crucial rule as a limiting case in the study of the nonlinear inhomogenous Schrödinger equation

\[ i\partial_t u + \Delta u = \mu K(x)|u|^\alpha u, \]

with regular potential \( K(x) \sim |x|^{-b} \) as \( |x| \to \infty \). Equation (2) appears in diverse branches of physics such as in nonlinear optics [16, Section 6]. Other equations with the same type of singularity in (1) are also treated, such as heat equation. See for example [41] and references therein.

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In this article, we deal with the case $b > 0$. The difficulty in the study of (1) comes from the singularity of the potential at the origin. We are mainly interested in the local well-posedness problem for (1). More precisely, we shall study the local existence, the unconditional uniqueness and the continuous dependence on initial data.

There are vast amount results on local existence for the standard nonlinear Schrödinger equation ($b = 0$). See, among many, [18, 44, 7, 25] and references therein. While, only a few works are interested in $b > 0$. To our best knowledge, the first result on the local well-posedness in $H^1(\mathbb{R}^N)$ for $\mu \in \mathbb{R}$, has been done in [17]. This was proved using an abstract theory developed in [5]. Recently, for $\mu \in \mathbb{C}$ and under some conditions on $b$ and $s$, the author of [20] has established the local well-posedness for the critical case $\alpha = (4 - 2b)/(N - 2s)$ is an open problem. Recently, we have learned in [29, 30] that the local existence for this case is established using Strichartz estimates in some weighted Lebesgue spaces. Their results are given for $N \geq 3$, $0 \leq s < 1/3$ or $s = 1$ and under some restrictive hypotheses on $\alpha$ and $b$. See also [9] for radially symmetric initial data in $H^1(\mathbb{R}^3)$. It seems that the spaces used in [20, 13, 29, 30] are not the natural one to give the complete local well-posedness theory. In fact, no blowup criterium is stated for the critical case. Also, neither continuous dependence in $H^s$ in the standard sense (see [6] for $b = 0$) nor unconditional uniqueness are given even for the subcritical case. See [29, Theorem 1.1] and [20, Theorem 1.4] for continuous dependence results in weaker sense.

The main goal of this paper is to give a new strategy in order to develop the well-posedness theory for the problem (1) in adequate spaces. Our approach is to consider Strichartz estimates in Lorentz spaces. The solutions that we construct belong to the spaces considered in [20, 13, 29, 30]. Moreover, we aim to build a local theory as the one developed for the case $b = 0$. See [7] and references therein. Our framework spaces are also well adapted to establish qualitative properties of solutions, such as global existence for oscillatory initial data and the scattering theory. This is done in [1, 2].

The first main purpose is to study the local well-posedness. To give our result we need the following definition.

**Definition 1.1 (Admissible pair).** We say that $(r, p)$ is an admissible pair if it satisfies

\[ \frac{2}{r} + \frac{N}{p} = \frac{N}{2} \tag{3} \]

and

\[ 2 \leq p \leq \frac{2N}{N - 2}, \quad N \geq 3, \quad (2 \leq p < \infty \text{ if } N = 2, \quad 2 \leq p \leq \infty \text{ if } N = 1). \]

In this paper, only the case $p < \infty$ will be considered.

Let us introduce the following particular admissible pair $(\gamma, \rho)$ defined by

\[ \gamma = \frac{4(\alpha + 2)}{\alpha(N - 2s) + 2b}, \quad \rho = \frac{N(\alpha + 2)}{N + \alpha s - b}. \tag{4} \]
As is standard practice, we study the problem (1) via its integral version
\[ u(t) = e^{it\Delta}u_0 - i\mu \int_0^t e^{i(t-\sigma)\Delta}(|x|^{-\beta}|u|^\alpha u(\sigma))d\sigma, \] (5)
where \( e^{it\Delta} \) is the free Schrödinger group. Let
\[ BR(H^s) := \{ f \in H^s(\mathbb{R}^N); \| f \|_{H^s(\mathbb{R}^N)} \leq R \}, \]
be the closed ball of the Sobolev space \( H^s(\mathbb{R}^N) \), centered at the origin with radius \( R > 0 \). We denote by
\[ W^{s,p}_q(\mathbb{R}^N) := (I - \Delta)^{-s/2}L^{p,q}(\mathbb{R}^N), \]
the Sobolev-Lorentz space, where \( L^{p,q}(\mathbb{R}^N) \) is the Lorentz space. We recall the definition of these spaces in Section 2. We have obtained the following result.

**Theorem 1.2** \((H^s \text{ local well-posedness})\). Let \( N \geq 1, \mu \in \mathbb{C}, 0 \leq s \leq 1 \) and \( s < N/2 \). Assume that \( 0 < b < \min(2, N - 2s) \) and
\[ 0 < \alpha \leq \frac{4 - 2b}{N - 2s}. \]
Let \( \gamma \) and \( \rho \) be given by (4). Then for every \( u_0 \in H^s(\mathbb{R}^N) \) there exist \( T_{\text{max}}(u_0) > 0 \) and a unique solution \( u \in C([0, T_{\text{max}}(u_0)), H^s(\mathbb{R}^N)) \cap L^r_{\text{loc}}([0, T_{\text{max}}(u_0)) \), \( W_{2}^{s,p}(\mathbb{R}^N) \) of (5). Moreover, we have the following properties.

(i) \( u \) is unique in \( L^r(0, T; W_{2}^{s,p}(\mathbb{R}^N)) \); for every \( 0 < T < T_{\text{max}}(u_0) \).

(ii) \( u \in L^r(0, T; W_{2}^{s,p}(\mathbb{R}^N)) \) for every \( 0 < T < T_{\text{max}}(u_0) \) and every admissible pair \( (r, p) \).

(iii) Let \( R > 0 \) and \( K \) be a compact of \( H^s(\mathbb{R}^N) \). If \( \alpha < (4 - 2b)/(N - 2s) \) (respectively, \( \alpha = (4 - 2b)/(N - 2s) \)), then there exists \( T = T(R) > 0 \) (respectively \( T = T(K) > 0 \)), such that for all \( u_0 \in BR(H^s) \) (respectively, \( u_0 \in K \)), (5) has a unique solution \( u \) in \( C([0, T), H^s(\mathbb{R}^N)) \cap L^r(0, T; W_{2}^{s,p}(\mathbb{R}^N)) \). Moreover for every admissible pair \( (r, p) \), there exists a constant \( C > 0 \) such that
\[ \|u - v\|_{L^r(0, T; L^p_{\text{loc}}(\mathbb{R}^N))} \leq C\|\varphi - \psi\|_{H^s(\mathbb{R}^N)}, \] (6)
for all \( \varphi, \psi \in BR(H^s) \) (respectively \( \varphi, \psi \in K \)), where \( u \) and \( v \) are the solutions of (5) with initial data respectively \( \varphi \) and \( \psi \).

(iv) (Blow-up criterion) If \( T_{\text{max}}(u_0) < \infty \), then \( \|u\|_{L^\infty(0, T_{\text{max}}(u_0); W_{2}^{s,p}(\mathbb{R}^N))} = \infty \).

(v) (Global existence for small data) If \( \alpha = (4 - 2b)/(N - 2s) \) and \( \|(-\Delta)^{s/2}u_0\|_{L^2(\mathbb{R}^N)} \) is sufficiently small, then \( T_{\text{max}}(u_0) = \infty \) and \( u \in L^r(0, \infty; W_{2}^{s,p}(\mathbb{R}^N)) \) for every admissible pair \( (r, p) \). Moreover, the global solution \( u \) scatters in \( H^s(\mathbb{R}^N) \), as \( t \to \infty \), to a solution of the linear Schrödinger equation (1) with \( \mu = 0 \).

(vi) (Lower estimate of the blow-up rate) If \( \alpha < (4 - 2b)/(N - 2s) \) and \( T_{\text{max}}(u_0) < \infty \), then there exists a constant \( C > 0 \) such that for \( 0 < t < T_{\text{max}}(u_0) \),
\[ \|u(t)\|_{\dot{H}^s(\mathbb{R}^N)} \geq C(T_{\text{max}}(u_0) - t)^{-\frac{4 - 2b - (N - 2s)\alpha}{4\alpha}}. \]
In particular,
\[ \lim_{t \to T_{\text{max}}(u_0)} \|u(t)\|_{\dot{H}^s(\mathbb{R}^N)} = \infty. \]

(vii) (Lower estimate of the life-span) Let \( \lambda > 0 \) and \( \alpha < (4 - 2b)/(N - 2s) \). Then there exists a constant \( C = C(\|u_0\|_{\dot{H}^s(\mathbb{R}^N)}) > 0 \) such that
\[ T_{\text{max}}(\lambda u_0) \geq C\lambda^{-\frac{4\alpha}{4 - 2b - (N - 2s)\alpha}}. \]
(viii) (Solution of the differential equation) If $N \geq 3$ and $s = 1$, then $u \in C^1([0, T_{\text{max}}(u_0)), H^{-1}(\mathbb{R}^N))$ and it satisfies the differential equation (1).

**Remark 1.**

1) The upper bound $\alpha \leq (4 - 2b)/(N - 2s)$ in Theorem 1.2 for which a solution exists with initial value in $H^s(\mathbb{R}^N)$ can be explained by a scaling argument. In fact, the scaling critical exponent $s_c$ is given by $s_c := \frac{N}{2} - \frac{2 - b}{\alpha}$. To see this, we observe that the transformation $\lambda^{\frac{2-b}{\alpha}} u(\lambda^2 t, \lambda x)$, which leaves invariant the set of solutions to (1), acts on initial values as $\lambda^{\frac{2-b}{\alpha}} u_0(\lambda \cdot)$. Furthermore,

$$\|\lambda^{\frac{2-b}{\alpha}} u_0(\lambda)\|_{H^s(\mathbb{R}^N)} = \lambda^{\frac{2-b}{\alpha} - \frac{N}{2}} \|u_0\|_{H^s(\mathbb{R}^N)} = \lambda^{s - s_c} \|u_0\|_{H^s(\mathbb{R}^N)}, \quad \forall \lambda > 0.$$ 

In other words, the homogeneous Sobolev norm of $H^s(\mathbb{R}^N)$ is invariant under the action of $\lambda^{\frac{2-b}{\alpha}} u_0(\lambda \cdot)$ precisely if $s = s_c$. Theorem 1.2 states, as expected, that solution exists if the initial value is in $H^s(\mathbb{R}^N)$ with $s \geq s_c$, i.e. $\alpha \leq (4 - 2b)/(N - 2s)$.

2) The similar statements of Theorem 1.2 hold also for negative time.

3) The previous result and its proof remain valid for the case $b = 0$. In this context, Theorem 1.2 unifies the results for $b = 0$ and $b > 0$. In particular we find the lower estimates of the blowup rate and the life span, well known for the case of the standard nonlinear Schrödinger equation. See [7, Theorem 1.1 (vii), inequality (1.8), p. 809], [7, Inequality (4.7), p. 825] and [43].

4) Due to the Strichartz estimates in Lorentz spaces ([26]), we obtain more precise regularity results with respect to the known ones. Precisely, we show that, for any admissible pair $(r, p)$, the $H^s-$solutions belong to $L_{\text{loc}}^r([0, T_{\text{max}}(u_0)), W^{s,p}_2(\mathbb{R}^N))$ which is a strict subspace of $L_{\text{loc}}^r([0, T_{\text{max}}(u_0)), W^{s,p}(\mathbb{R}^N))$. See [20, 13].

5) The local well-posedness result in the critical case $\alpha = (4 - 2b)/(N - 2s)$ is completely new for $N = 1, N = 2$ and $N \geq 3$ with $1/3 < s < 1$. For the other cases, the solution constructed here is more regular than the one given in [29, 30]. This follows by using the Hölder inequality in the Lorentz spaces.

6) For the subcritical case $\alpha < (4 - 2b)/(N - 2s)$, we give a different and simple proof compared with the ones in [20, 13]. Moreover, we establish more precise properties of solutions. In particular, uniqueness in a larger space, blowup criterium, blowup rate and life span.

7) The previous Theorem is satisfied if we replace $|x|^{-b} |u|^\alpha u$ in (1) by $g(x) F(u)$ with $g \in L^{N/b, \infty}(\mathbb{R}^N)$, $(-\Delta)^{s/2} g \in L^{N/(b+s), \infty}(\mathbb{R}^N)$ and $F$ is as (for example) in [5, 25]. We present our results for $|x|^{-b} |u|^\alpha u$ so not to complicate unnecessarily the proofs and deviate from the main ideas.

8) Theorem 1.2 guarantees the local well-posedness in $H^1(\mathbb{R}^3)$ for $0 < b < 1$. The case $b \geq 1$, will be done in [2]. Also, the study of the local well-posedness for $s > 1$ or $s \geq N/2$ will be considered elsewhere.

In the previous theorem the uniqueness is given under auxiliary conditions. Our second aim is to prove an unconditional uniqueness result. We have obtained the following.

**Theorem 1.3** (Unconditional Uniqueness). Assume that one of the following holds

(i) $N \geq 1$, $s \geq N/2$, $0 < b < \min(2, N)$, $\alpha > 0$.

(ii) $N = 1$, $0 \leq s < 1/2$, $0 < b < (1 + 2s)/2$, $0 < \alpha < (1 + 2s - 2b)/(1 - 2s)$.

(iii) $N = 2$, $0 \leq s < 1$, $0 < b < 1 + s$, $0 < \alpha < (1 + s - b)/(1 - s)$.
Let $u$ with initial value $u_0 \in H^s(\mathbb{R}^N)$ and $T > 0$. Then (5) has at most one solution in $C([0,T], H^s(\mathbb{R}^N))$.

Remark 2. 
1) The previous theorem is completely new for $b > 0$ and it extends the known results for $b = 0$. See [18, 24, 25, 5, 35].
2) The unconditional uniqueness for the case $N \geq 3$, $0 < s < N/2$, $0 < b < 2$, $\alpha = (4-2b)/(N-2s)$.

Our third aim is to study the continuous dependence of solutions with respect to initial data. Let us first recall the known results for the case $b = 0$. For the subcritical case the continuous dependence is proved in [44] for $s = 0$ and in [24] for $s = 1$. Similar results are established for the critical case in [7] if $s = 0$ and in [7, 38, 27, 28] if $s = 1$. The case $0 < s < \min(1, N/2)$ is studied in [6]. The case $b > 0$ we are not aware of any previous continuous dependence result. Our main result is the following.

Theorem 1.4 (Continuous dependence in $H^s$). Let $N \geq 1$, $\mu \in \mathbb{C}$, $0 \leq s \leq 1$ and $s < N/2$. Assume that $0 < b < \min(2, N - 2s)$ and

$$0 < \alpha \leq \frac{4 - 2b}{N - 2s}.$$ 

Let $\varphi \in H^s(\mathbb{R}^N)$ and $u \in C\left([0, T_{\max}(\varphi)], H^s(\mathbb{R}^N)\right)$ be the maximal solution of (1) with initial value $\varphi$. Then we have the following.

(i) If $s = 0$, $s > 0$ and $\alpha > 1$, or $\alpha = s = 1$, then for all $0 < T < T_{\max}(\varphi)$ there exists $\delta_0 > 0$ such that if $\psi \in H^s(\mathbb{R}^N)$ satisfying $\|\varphi - \psi\|_{H^s(\mathbb{R}^N)} < \delta_0$, we have $T < T_{\max}(\psi)$ and for any admissible pair $(r, p)$, the corresponding maximal solution $v$ verifies

$$\|u - v\|_{L^r([0,T], W^{s,p}_x(\mathbb{R}^N))} \leq C\|\varphi - \psi\|_{H^s(\mathbb{R}^N)},$$

(ii) If $s = 1$, $\alpha = \min((4-2b)/(N-2), 1)$ and if $\varphi_k \to \varphi$ in $H^s(\mathbb{R}^N)$, then for all $0 < T < T_{\max}(\varphi)$ the exists $k_0$ such that $T < T_{\max}(\varphi_k)$ for all $k > k_0$ and $u_k \to u$ in $L^r([0,T], W^{1,p}_x(\mathbb{R}^N))$ for all admissible pair $(r, p)$, where $u_k$ is the solution of (1) with initial data $\varphi_k$. In particular, $u_k \to u$ in $C\left([0, T], H^1(\mathbb{R}^N)\right)$.

(iii) If $s = 1$, $\alpha = (4-2b)/(N-2) < 1$, and $\mu \in \mathbb{R}$, then, with the notations of (ii), $u_k \to u$ in $L^r\left([0,T], H^1(\mathbb{R}^N)\right)$, for every $2 < r < \infty$. If we assume further $\mu > 0$ then the conclusion of (ii) holds.

Remark 3. The case where $b > 0$, $\alpha \leq 1$ and $0 < s < \min(1, N/2)$, is an open problem. See [6], for $b = 0$.

Let us now give some ideas about the proofs. The proof of Theorem 1.2 is done via a contraction mapping argument. Since we are proving local existence in the Sobolev spaces $H^s(\mathbb{R}^N)$, we need, as in many works, Strichartz estimates in spaces such as $L^r(0, T, W^{s,p}_x(\mathbb{R}^N))$. Unfortunately, the singular potential does not belong to any Lebesgue spaces, and so the nonlinear part $|x|^{-\mu}|u|^\alpha u$ do not operate well on the Sobolev spaces. Thus, we consider Sobolev-Lorentz spaces $W^{s,p}_q(\mathbb{R}^N)$ and
we use the Strichartz estimates of type $L^r(0, T, L^{p,2}(\mathbb{R}^N))$, established in [26]. The admissible pair $(\gamma, \rho)$ given by (4) will play an essential role in our estimates. The value of $\rho$ allows the map $u \mapsto |x|^{-b}|u|^\alpha u$ to apply the homogeneous Sobolev-Lorentz space $W_2^{s,\rho}(\mathbb{R}^N)$ into $W_2^{s,\rho'}(\mathbb{R}^N)$. To estimate the nonlinear part for $s \in (0, 1)$, we use the Leibnitz fractional rule ([12]) and we establish a fractional chain rule in Lorentz spaces. This is done by adapting the known results in Lebesgue spaces ([10]).

To prove Theorem 1.3, we use some argument of [5] and the inhomogeneous Strichartz estimates for non admissible pairs established in [37]. For this, we construct particular non admissible pairs. See Proposition 7 below.

Our proof of Theorem 1.4 uses some argument of [7, 24, 6] with the Strichartz estimates in Lorentz spaces. We also use the fractional Leibnitz and chain rule in Lorentz spaces.

The rest of this paper is organized as follows. In Section 2, we recall some basic notion on Lorentz and Sobolev-Lorentz spaces. We also recall the Leibnitz fractional rule and the Strichartz estimates. Then, we establish the fractional chain rule in these spaces. Section 3 contains the proof of the local well-posedness, that is Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.3. Finally in Section 5, we give the proof of Theorem 1.4. In the sequel, a functional space on $\mathbb{R}^N$, $X(\mathbb{R}^N)$ will be denoted simply by $X$.

The notation $A \lesssim B$ for positive numbers $A$ and $B$ means that there exists a positive constant $C$ such that $A \leq CB$. If $A \lesssim B$ and $B \approx A$, we write $A \sim B$. We denote $a_+ := \max(0, a)$ for a real number $a$. Also, $C$ will denotes a constant which may be different at different places.

2. Preliminaries. In this section we recall some basic functional spaces and give the Leibnitz and the chain rules. We also recall the Strichartz estimates in the Lorentz spaces.

2.1. Useful functional spaces. In this subsection we recall some basic facts about the Lorentz spaces which are relevant to our study. We refer the reader to [4, 19, 31, 32, 23, 36, 39, 40] and references therein for more properties and information.

Let $f$ be a measurable function on $\mathbb{R}^N$. We define on $(0, \infty)$ the distribution function of $f$, which we denote $d_f$, by:

$$d_f(\lambda) = \left| \{ x \in \mathbb{R}^N, \ |f(x)| > \lambda \} \right|.$$ 

Here $|A|$ is the Lebesgue measure of a subset $A$ of $\mathbb{R}^N$. We then define on $(0, \infty)$, the decreasing rearrangement function of $f$, which we denote $f^*$, by:

$$f^*(s) = \inf \{ \lambda > 0, \ d_f(\lambda) \leq s \}.$$ 

It is clear that $d_f$ and $f^*$ are non-negative and non-increasing functions. We give the definition of the Lorentz spaces.

**Definition 2.1.** Let $p \in (0, \infty)$, $q \in (0, \infty]$. The Lorentz space $L^{p,q}(\mathbb{R}^N)$ is defined by

$$L^{p,q}(\mathbb{R}^N) = \{ f \text{ measurable, } \|f\|_{L^{p,q}} < \infty \},$$

where

$$\|f\|_{L^{p,q}} = \begin{cases} \left( \frac{q}{p} \int_0^\infty \left( s^{1/p} f^*(s) \right)^q \frac{1}{s} ds \right)^{1/q}, & p \in (0, \infty), \ q \in (0, \infty), \\ \sup_{s > 0} s^{1/p} f^*(s), & p \in (0, \infty), \ q = \infty \end{cases}.$$
We have $L^p = L^p$ and by convention $L^\infty \cap L^\infty = L^\infty$. The quantity $\| \cdot \|_{L^{p,q}}$ induces a natural topology to the Lorentz space $L^{p,q}$. The quantity $\| \cdot \|_{L^{p,q}}$ is a quasi–norm in $L^{p,q}$, because the triangular inequality is satisfied up to a constant. So $L^{p,q}$ are quasi–normed spaces. Also, $L^{p,q}$ are quasi-Banach spaces.

For any fixed $p$, the Lorentz spaces $L^{p,q}$ increase as the exponent $q$ increases. That is, for $0 < p < \infty$ and $0 < q < r \leq \infty$, $L^{p,q} \subset L^{p,r}$ and there exists $C = C(p,q,r) > 0$ such that

$$\| f \|_{L^{p,r}} \leq C \| f \|_{L^{p,q}}.$$  \hfill (7)

Also, for any $0 < p, \mu < \infty$ and any $0 < q \leq \infty$,\
$$\| |f|^{\mu} \|_{L^{p,q}} = \| f \|_{L^{p,\mu q}}^{\mu}.$$  \hfill (8)

For $1 < p < \infty$ and $1 \leq q \leq \infty$, $L^{p,q}$ can be normed to become Banach spaces. To introduce a norm, we consider

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad t > 0$$

and define

$$\| f \|_{L^{p,q}} = \begin{cases} \left( \frac{2}{p} \int_0^\infty \left( s^{1/p} f^{**}(s) \right)^q \frac{1}{s} ds \right)^{1/q}, & p \in (1, \infty), \quad q \in [1, \infty), \\ \sup_{s > 0} s^{1/p} f^{**}(s), & p \in (1, \infty), \quad q = \infty. \end{cases}$$

$L^{p,q}$, $1 < p < \infty$, $1 \leq q \leq \infty$, endowed with the norm $\| f \|_{L^{p,q}}$ are Banach spaces. The following inequalities hold

$$\| f \|_{L^{p,q}} \leq \| f \|_{L^{p,q}}^* \leq \frac{p}{p-1} \| f \|_{L^{p,q}}.$$ 

In some sense $\| f \|_{L^{p,q}}$ is equivalent to $\| f \|_{L^{p,q}}^*$ for $1 < p < \infty$ and $1 \leq q < \infty$. It will also be sufficient to estimate $\| f \|_{L^{p,q}}$ instead of $\| f \|_{L^{p,q}}^*$ since the last one is more difficult to manipulate. The function $|x|^{-\gamma} \in L^{p,q}$ if and only if $p = N/\gamma$, $q = \infty$. Hence $|x|^{-\gamma} \in L^{N,\infty}$, and does not belong to any Lebesgue space. We recall the following generalized Hölder’s inequality. See [34, Theorems 3.4, 3.5, p. 141]. See also [31, Proposition 2.3, p. 19].

**Proposition 1** (Generalized Hölder’s Inequality ([34])). Let $1 < p_1, p_2 < \infty$, $1 \leq q_1, q_2 \leq \infty$. Let $f \in L^{p_1,q_1}$, $g \in L^{p_2,q_2}$. Then we have the following:

(i) If $\frac{1}{p_1} + \frac{1}{p_2} < 1$ then $fg \in L^{p,q}$ where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad q \geq 1 \text{ is any number (}1 \leq q \leq \infty) \text{ such that}$$

$$\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}.$$ 

Moreover, there exists a constant $C = C(p,q,p_1,p_2,q_1,q_2) > 0$ such that

$$\| fg \|_{L^{p,q}} \leq C \| f \|_{L^{p_1,q_1}} \| g \|_{L^{p_2,q_2}}.$$ 

(ii) If $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and $\frac{1}{q_1} + \frac{1}{q_2} \geq 1$ then $fg \in L^1$. Moreover, there exists a constant $C = C(p_1, p_2, q_1, q_2) > 0$ such that

$$\| fg \|_1 \leq C \| f \|_{L^{p_1,q_1}} \| g \|_{L^{p_2,q_2}}.$$
(iii) If \( p_2 = q_2 = \infty \) then \( fg \in L^{p_1,q_1} \) and there exists a constant \( C > 0 \) such that
\[
\|fg\|_{L^{p_1,q_1}} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{\infty}.
\]

We have the following interpolation inequality in Lorentz spaces
\[
\|f\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}}^{\theta}\|f\|_{L^{p_2,q_2}}^{1-\theta},
\]
where \( 1 < p, p_1, p_2 < \infty, 1 \leq q, q_1, q_2 \leq \infty, \theta \in (0,1) \) and
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} \leq \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.
\]

For \( s \geq 0, 1 < p < \infty, 1 \leq q \leq \infty, \) we define the Sobolev-Lorentz spaces (See [21, page 571]) as follows
\[
W^{s,p}_q(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N), (I-\Delta)^{s/2}f \in L^{p,q} \},
\]
\[
\dot{W}^{s,p}_q(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N), (-\Delta)^{s/2}f \in L^{p,q} \},
\]
where
\[
(I-\Delta)^{s/2}f = \mathcal{F}^{-1}\left((1 + |\xi|^2)^{s/2}\mathcal{F}(f)\right),
\]
\[
(-\Delta)^{s/2}f = \mathcal{F}^{-1}\left(|\xi|^s\mathcal{F}(f)\right),
\]
with \( \mathcal{F}(f) \) is the Fourier transform of \( f \) and \( \mathcal{F}^{-1} \) is the inverse Fourier transform. The spaces \( W^{s,p}_q \) and \( \dot{W}^{s,p}_q \), endowed respectively with the norms
\[
\|f\|_{W^{s,p}_q} = \|f\|_{L^{p,q}} + \|(-\Delta)^{s/2}f\|_{L^{p,q}},
\]
\[
\|f\|_{\dot{W}^{s,p}_q} = \|(-\Delta)^{s/2}f\|_{L^{p,q}},
\]
are Banach spaces.

We recall the homogenous Sobolev-Lorentz embedding (See [31, Theorem 2.4 (iii), p. 20]) : \( \dot{W}^{s,p}_q \hookrightarrow L^{\tilde{p},\tilde{q}}, \) where \( 1 < \tilde{p} < \infty, 1 \leq \tilde{q} \leq \infty, 0 < s < \frac{N}{\tilde{p}} \) and
\[
\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s}{N}.
\]
That is there exists a constant \( C > 0 \) such that
\[
\|f\|_{L^{\tilde{p},\tilde{q}}} \leq C\|(-\Delta)^{s/2}f\|_{L^{p,q}}, \quad f \in \dot{W}^{s,p}_q.
\]

By the well known Sobolev embedding \( H^s \hookrightarrow L^p \) and interpolation, we have the following
\[
H^s \hookrightarrow L^p, \quad s \geq 0, \quad \frac{1}{2} - \frac{s}{N} \leq \frac{1}{p} \leq \frac{1}{2}, \quad p < \infty.
\]

We recall the Gagliardo-Nirenberg inequality in the Lorentz spaces
\[
\|f\|_{L^{p,q}} \leq C\|(-\Delta)^{s/2}f\|_{L^{p_1,q_1}}^{\theta}\|f\|_{L^{p_2,q_2}}^{1-\theta},
\]
where \( 1 < p, p_2 < \infty, 1 < q, q_1, q_2 < \infty, 0 < s < N, 1 < p_1 < N/s, 0 < \theta < 1, \) and
\[
\frac{1}{p} = \frac{\theta}{p_1} - \frac{\theta s}{N} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} \leq \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.
\]
See [21, Theorem 2.1, p. 571].

We will use the following lemma.

**Lemma 2.2.** Let \( 0 \leq b < N, \) \( p > \frac{N}{N-b} \) and \( q \geq 1. \) Then the multiplication operator
\[
|.|^{-b} : L^{p,q} \to L^{p_1,q_1}
\]
is continuous, where \( \frac{1}{p_1} = \frac{1}{p} + \frac{b}{N} \) and \( q_1 \geq q. \)
Lemma 2.3. (Fractional Leibniz Rule in Lorentz spaces, [12, Theorem 6.1]) Let $s \in [0,1]$ and $1 < p$, $p_1$, $p_2$, $p_3$, $p_4 < \infty$, $1 \leq q$, $q_1$, $q_2$, $q_3$, $q_4 \leq \infty$ be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.$$ 

Then,

$$\|(-\Delta)^{s/2}(fg)\|_{L^{p,q}} \lesssim \|(-\Delta)^{s/2}f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}} + \|f\|_{L^{p_3,q_3}}\|(-\Delta)^{s/2}g\|_{L^{p_4,q_4}}.$$ 

We will prove the following chain rule in Lorentz spaces, which is already known in Lebesgue spaces. See [10].

Lemma 2.4. (Fractional Chain Rule in Lorentz spaces) Let $s \in [0,1]$, $F \in C^1(\mathbb{C}, \mathbb{C})$ and $1 < p$, $p_1$, $p_2 < \infty$, $1 \leq q$, $q_1$, $q_2 < \infty$ be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1}.$$ 

Then,

$$\|(-\Delta)^{s/2}F(f)\|_{L^{p,q}} \lesssim \|F'(f)\|_{L^{p_1,q_1}}\|(-\Delta)^{s/2}f\|_{L^{p_2,q_2}}.$$ 

In order to prove the previous lemma, let us consider $\eta \in C_0^\infty(\mathbb{R}^N)$, nonnegative function supported in $\{1/2 < |\xi| < 2\}$ and satisfying

$$\sum_{|\xi| < 2} \eta(2j\xi) = 1, \ \xi \in \mathbb{R}^N \setminus \{0\}.$$ 

We define the Fourier multiplier operators

$$Q_j(f) = \eta(2^{-j} \cdot) \ast f, \ f \in S'(\mathbb{R}^N), \ j \in \mathbb{Z}.$$ 

We need the following result.

Lemma 2.5. Let $1 < p < \infty$ and $1 \leq q < \infty$. Consider the square-function operator

$$S(f)(x) = \left(\sum_{j \in \mathbb{Z}} |Q_j(f)(x)|^2\right)^{1/2}, \ x \in \mathbb{R}^N.$$ 

Then there exist positive constants $C_1$, $C_2$ such that

$$C_1\|f\|_{L^{p,q}} \leq \|S(f)\|_{L^{p,q}} \leq C_2\|f\|_{L^{p,q}}. \quad (12)$$

Proof. By [15, Theorem 8.6, p. 162], there exists $C > 0$, such that $\|S(f)\|_{L^p} \leq C\|f\|_{L^p}$. Then the right-hand side of (12) follows by interpolation. The left-hand side of (12) can be obtained by duality as for the Lebesgue spaces. In fact, from the identity

$$\int_{\mathbb{R}^N} fg = \int_{\mathbb{R}^N N} \sum_{j \in \mathbb{Z}} Q_j(f)\overline{Q_j(g)},$$
the Hölder inequality in Lorentz spaces and the characterization of the dual of Lorentz spaces \( L^{p,q} \) with \( q < \infty \), we have

\[
\|f\|_{L^{p,q}} = \sup \left\{ \left\| \int_{R^N} f(y) dy \right\|_{L^{p',q'}} : \|g\|_{L^{p',q'}} \leq 1 \right\}
\]

\[
= \sup \left\{ \left\| \int_{R^N} \sum_{j \in Z} Q_j(f) \mathcal{Q}_j(g) \right\|_{L^{p,q}} : \|g\|_{L^{p',q'}} \leq 1 \right\}
\]

\[
\leq C \sup \left\{ \left\| \left( \sum_{j \in Z} |Q_j(f)|^2 \right)^{1/2} \right\|_{L^{p,q}} \left\| \left( \sum_{j \in Z} |Q_j(g)|^2 \right)^{1/2} \right\|_{L^{p',q'}} : \|g\|_{L^{p',q'}} \leq 1 \right\}
\]

\[
\leq C \left( \sum_{j \in Z} |Q_j(f)|^2 \right)^{1/2} \left( \sum_{j \in Z} |Q_j(g)|^2 \right)^{1/2}.
\]

This completes the proof. \( \square \)

**Proof of Lemma 2.4.** The result holds obviously for \( s = 0 \) or \( s = 1 \). Let us consider the Hardy-Littlewood maximal function

\[
M(f)(x) = \sup_{r > 0} \frac{1}{|B_r|} \int_{B_r} |f(x - y)| dy,
\]

where \( f \in L^1_{loc}(\mathbb{R}^N) \) and \( B_r = \{ x \in \mathbb{R}^N, |x| < r \} \). It is known that \( M : L^p \to L^p \) is continuous for \( 1 < p < \infty \). See \([15, \text{Theorem 2.5, p. 31}]\). Then by interpolation, \( M : L^{p,q} \to L^{p,q} \) is continuous for \( 1 < p < \infty, 1 \leq q \leq \infty \). See \([4]\). Let

\[
\Phi : L^p(\ell^2) \to L^p(\ell^2), (h_k) \mapsto (Mh_k).
\]

By \([3, 11]\), \( \Phi \) is continuous. Then by interpolation \( \Phi : L^{p,q}(\ell^2) \to L^{p,q}(\ell^2), 1 < p < \infty 1 \leq q \leq \infty \) is also continuous. By Lemma 2.5 and the fact that

\[
\left( \sum_{j \in Z} \left| Q_j \left( (-\Delta)^{s/2} g \right) \right|^2 \right)^{1/2} \sim \left( \sum_{j \in Z} 2^{js} \left| \mathcal{Q}_j (g) \right|^2 \right)^{1/2}, \quad (13)
\]

where

\[
\mathcal{Q}_j (f) = \tilde{\eta}(2^{-j} \cdot) * f, \quad f \in S'((\mathbb{R}^N), \ j \in \mathbb{Z},
\]

for \( \tilde{\eta} \in C_0^\infty (\{ 1/2 < |\xi| < 2 \}) \) satisfying \( \tilde{\eta} \cdot \eta = \eta \), we have

\[
\left\| (-\Delta)^{s/2} F(f) \right\|_{L^{p,q}} \lesssim \left\| S((-\Delta)^{s/2} F(f)) \right\|_{L^{p,q}} \lesssim \left\| \left( \sum_{j \in Z} 2^{2js} \left| \mathcal{Q}_j (F(f)) \right|^2 \right) \right\|_{L^{p,q}}^{1/2}.
\]

By \([10, \text{Proof of Proposition 3.1, p. 93}]\), we have

\[
\left( \sum_{j \in Z} 2^{2js} \left| \mathcal{Q}_j (F(f)) \right|^2 \right)^{1/2} \lesssim M(F'(f)) \left( \sum_{j \in Z} 2^{2js} \left[ M^2 (\mathcal{Q}_j (f)) \right]^2 \right)^{1/2},
\]

where \( M^2 = M \circ M \). Combining the last inequality with the Hölder inequality in Lorentz spaces, the continuity of \( M \) and \( \Phi \circ \Phi \), equivalence (13) together with
Lemma 2.5, we get
\[
\|(-\Delta)^{s/2} F(f)\|_{L^p,q} \lesssim \| M(F'(f)) \left( \sum_{j \in \mathbb{Z}} 2^{2js} \left[ M^2(\tilde{Q}_j(f)) \right]^2 \right)^{1/2} \|_{L^{p,q}} \\
\lesssim \| M(F'(f))\|_{L^{p_1,q_1}} \left\| \left( \sum_{j \in \mathbb{Z}} 2^{2js} \left[ M^2(\tilde{Q}_j(f)) \right]^2 \right)^{1/2} \right\|_{L^{p_2,q_2}} \\
\lesssim \| F'(f)\|_{L^{p_1,q_1}} \left\| \left( \sum_{j \in \mathbb{Z}} 2^{2js} |\tilde{Q}_j(f)|^2 \right)^{1/2} \right\|_{L^{p_2,q_2}} \\
\lesssim \| F'(f)\|_{L^{p_1,q_1}} \left\| \left( \sum_{j \in \mathbb{Z}} |Q_j((-\Delta)^{s/2} f)|^2 \right)^{1/2} \right\|_{L^{p_2,q_2}} \\
\lesssim \| F'(f)\|_{L^{p_1,q_1}} \left\| (-\Delta)^{s/2} f \right\|_{L^{p_2,q_2}}.
\]

This achieves the proof of Lemma 2.4.

2.3. Strichartz estimates. The following result is a key tool for our work. It is established in [26, Theorem 10.1] (see also [37], Theorem 1.2 page 828) with the notation of [26] we take
\[
\sigma = N/2, \quad H = B_0 = L^2, \quad B_1 = L^1, \quad \theta = \frac{4}{rN} = 1 - \frac{2}{p},
\]
B_θ = L^{p',2}. (Note that from (3) \( r \geq \frac{4}{N} \) hence \( \theta \leq 1 \).)

**Proposition 2.** [26, Theorem 10.1] (i) Let \((r, p)\) be an admissible pair. Then there exists a constant \(C > 0\), such that
\[
\left\| e^{it\Delta} \varphi \right\|_{L^r(\mathbb{R}, L^p, \mathbb{R}^2)} \leq C\| \varphi \|_{L^2},
\]
for every \( \varphi \in L^2 \).

(ii) Let \((r_i, p_i); i = 1, 2\) be two admissible pairs. Then there exists a constant \(C > 0\), such that
\[
\left\| \int_0^t e^{i(t-s)\Delta} f(s, .)ds \right\|_{L^{r_1}(\mathbb{R}, L^{p_1,2})} \leq C\| f \|_{L^{r_2}(\mathbb{R}, L^{p_2,2})},
\]
for every \( f \in L^{r_2}(\mathbb{R}, L^{p_2,2}) \).

The following proposition will be used to prove the unconditional uniqueness. We first give the following definition.

**Definition 2.6.** We say that a pair \((r, p)\) is acceptable if either
\[
\frac{1}{r} < \frac{N}{2} - \frac{N}{p}, \quad 1 \leq r < \infty, \quad 2 < p < \infty
\]
or \( (r, p) = (\infty, 2) \).

**Proposition 3.** ([37, Theorem 1.4, p. 828]) Let \( N \geq 3 \) be a positive integer. Let \((r, p), (\tilde{r}, \tilde{p})\) be two real acceptable pairs satisfying the scaling condition
\[
\frac{1}{r} + \frac{1}{\tilde{r}} = \frac{N}{2} \left( 1 - \frac{1}{p} - \frac{1}{\tilde{p}} \right),
\]
and
and
\[
\frac{1}{r} + \frac{1}{\tilde{r}} < 1, \quad \frac{N-2}{p} \leq \frac{N}{\tilde{p}}, \quad \frac{N-2}{\tilde{p}} \leq \frac{N}{p}.
\]

Then there exists a constant \( C > 0 \), such that
\[
\left\| \int_0^t e^{i(t-s)\Delta} f(s, .) ds \right\|_{L^r(R, L^p, 2)} \leq C \| f \|_{L^\tilde{r}'(R, L^\tilde{p}', 2)} ,
\]
for every \( f \in L^\tilde{r}'(R, L^\tilde{p}', 2) \).

**Remark 4.** The nonhomogeneous Strichartz estimates (15) and (16) remain true if we replace \( R \) (in time) by an interval of \( \mathbb{R} \) (bounded or not).

3. **Local well-posedness.** In this section we will prove the local well-posedness for the problem (1). We will use the particular admissible pair \((\gamma, \rho)\) defined by (4).

It is clear that for \( b = 0 \), the previous pair \((\gamma, \rho)\) coincides with those introduced in \([7]\). We first have the following.

**Proposition 4.** Suppose that \( N \geq 1 \), \( 0 \leq s < N/2 \), \( 0 < b < \min(2, N-2s) \) and \( 0 < \alpha \leq (4-2b)/(N-2s) \). Let \( \gamma \) and \( \rho \) be given by (4). Set
\[
\frac{1}{\rho_1} = \frac{1}{\rho} - \frac{b}{N},
\]
and
\[
\delta = 1 - \frac{\alpha(N-2s)}{4} - \frac{b}{2}.
\]

Then we have the following.

(i) \( 2 < \rho < \frac{2N}{(N-2)_+} \), \((\gamma, \rho)\) is an admissible pair;

(ii) \( \rho < \frac{2N}{N-2s} \) and \( \rho < \rho^* < \infty \) if \( s > 0 \),

(iii) \( \frac{1}{\rho_1} = \frac{\rho^*}{\rho} + \frac{1}{\rho} \),

(iv) \( \delta = 1 - \frac{\alpha + 2}{\gamma} \geq 0 \) and
\[
\delta = 0 \text{ if and only if } \alpha = \frac{4-2b}{N-2s}.
\]

**Proof.** (i) The fact that \( \rho > 2 \) is equivalent to \( N(\alpha + 2) > 2(N + \alpha s - b) \), which follows from the fact that \( 2s - N < 0 < b \). If \( N \geq 3 \), the inequality \( \rho < \frac{2N}{(N-2)_+} \) is equivalent to \( \alpha(N-2-2s) < 4 - 2b \), which is true since \( \alpha(N-2-2s) \leq 4 - 2b \). The cases \( N = 1 \) and \( N = 2 \) are obvious. The fact that \((\gamma, \rho)\) is an admissible pair follows by simple calculation.

(ii) We note that
\[
\rho = \frac{N s}{\alpha + \frac{N-2}{s} < \frac{N}{s}}, \quad \text{if } \frac{N-b}{s} > 2.
\]

(iii) This property follows by the definition of \( \rho \). Note that we have
\[
\frac{\alpha}{\rho^*} = 1 - \frac{2}{\rho} - \frac{b}{N},
\]
and
\[
\frac{1}{\rho_1} = 1 - \frac{1}{\rho} - \frac{b}{N}.
\]
(iv) The equality follows by definition of $\gamma$. $\delta \geq 0$ is equivalent to $\gamma \geq \alpha + 2$. By an obvious calculation we have

$$\gamma \geq \alpha + 2 \text{ if and only if } \alpha \leq \frac{4 - 2b}{N - 2s}.$$ 

This completes the proof of the proposition. \hfill \Box

In the sequel, we denote

$$\mathcal{N}(u) = i\mu \int_0^t e^{i(t-\sigma)\Delta}(|x|^{-b}|u|^\alpha u(\sigma))d\sigma,$$

and

$$\mathcal{F}(u) = e^{i\mu\Delta}u_0 - \mathcal{N}(u).$$

To prove Theorem 1.2, we need the following proposition.

**Proposition 5.** Assume the conditions on $N, b, \alpha, s$ of Theorem 1.2. Let $T > 0$, $(\gamma, \rho)$ be given by (4) and $u, v \in L^\gamma(0, T, W_2^{s, \rho})$. Let $\delta$ be given by (19) and $(r, p)$ be any admissible pair. Then

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^r(0, T, L^p)} \lesssim T^\delta \left(\|u\|_{L^\gamma(0, T, W_2^{s, \rho})} + \|v\|_{L^\gamma(0, T, W_2^{s, \rho}} \right) \times \|u - v\|_{L^\gamma(0, T, L^p)},$$

and

$$\|\mathcal{N}(u)\|_{L^r(0, T, W_2^{s, \rho})} \lesssim T^\delta \|u\|_{L^\gamma(0, T, W_2^{s, \rho})}^{\alpha+1}.$$  

**Proof.** We first prove (21). From Proposition 4, Proposition 2 (ii) and Lemma 2.2, we have

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^r(0, T, L^p)} \lesssim \|x|^{-b}|u|^\alpha u - |v|^\alpha v\|_{L^\gamma(0, T, L^{p,2})} \lesssim \|u|^\alpha u - |v|^\alpha v\|_{L^\gamma(0, T, L^{p,2})},$$

where $\rho_1$ is defined by (17). By the well known inequality

$$\|u|^\alpha u - |v|^\alpha v\| \lesssim (|u|^\alpha + |v|^\alpha) |u - v|,$$

and using the Hölder inequality and the embedding of the Lorentz spaces with respect to the second index, we obtain

$$\|u|^\alpha u - |v|^\alpha v\|_{L^{p,2}} \lesssim \left(\|u\|_{L^\gamma(0, T, W_2^{s, \rho})}^{\alpha + \frac{2(\alpha+1)}{\alpha}} + \|v\|_{L^\gamma(0, T, W_2^{s, \rho})}^{\alpha + \frac{2(\alpha+1)}{\alpha}}\right) \|u - v\|_{L^{p,2}},$$

$$\lesssim \left(\|u\|_{L^{p,2}}^{\alpha} + \|v\|_{L^{p,2}}^{\alpha}\right) \|u - v\|_{L^{p,2}},$$

where $\rho^*$ is given by (18). By the Sobolev-Lorentz homogenous embedding (10), with $p = \rho$ and $\rho^* = \rho^*$ we get from the last inequality that

$$\|u|^\alpha u - |v|^\alpha v\|_{L^{p,2}} \leq C \left((\|(-\Delta)^{s/2} u\|_{L^{p,2}}^{\alpha} + \|(-\Delta)^{s/2} v\|_{L^{p,2}}^{\alpha}) \|u - v\|_{L^{p,2}} \right),$$

Combining (23), (24) and using the Hölder inequality in time, we obtain

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^r(0, T, L^p)} \leq \|1_{[0,T]}\|_{L^{1/\delta}(0, T)} \left((\|(-\Delta)^{s/2} u\|_{L^\gamma(0, T, L^p)}^{\alpha} + \|(-\Delta)^{s/2} v\|_{L^\gamma(0, T, L^p)}^{\alpha}) \|u - v\|_{L^\gamma(0, T, L^p)} \right).$$
where \( 1_{[0,T]} \) is the characteristic function of \([0,T]\) and \( \delta \) is given by (19). Then we conclude that
\[
\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^r(0,T,L^{p,2})} \leq C T^{\delta} \left( \|(-\Delta)^{s/2} u\|_{L^\gamma(0,T,L^{p,2})} + \|(-\Delta)^{s/2} v\|_{L^\gamma(0,T,L^{p,2})} \right) \|u - v\|_{L^\gamma(0,T,L^{p,2})}.
\]

We now turn to prove (22). By the Strichartz estimates, we have
\[
\|(-\Delta)^{s/2} \mathcal{N}(u)\|_{L^r(0,T,L^{p,2})} \lesssim \|\mathcal{N}(u)\|_{L^r(0,T,L^{p,2})} \lesssim \|(-\Delta)^{s/2}(|x|^{-b}|u|^\alpha u)\|_{L^r(0,T,L^{p,2})}.
\]

By Lemma 2.3 and the fact that
\[
\frac{\alpha + 1}{\rho^*} = \frac{1}{\rho'} - \frac{b + s}{N},
\]
we have
\[
\|(-\Delta)^{s/2}(|x|^{-b}|u|^\alpha u)\|_{L^{p,2}} \lesssim \|(-\Delta)^{s/2}(|x|^{-b}|u|^\alpha u)\|_{L^{p,2}} + \|(-\Delta)^{s/2}(|u|^\alpha u)\|_{L^{p,2}} \lesssim \|u\|_{L^{p,2}}^{\alpha + 1} + \|(-\Delta)^{s/2}(|u|^\alpha u)\|_{L^{p,2}} \lesssim I + II.
\]

For the first term, by the Sobolev-Lorentz embedding, we have
\[
I = \|u\|_{L^{p,2}}^{\alpha + 1} \lesssim \|(-\Delta)^{s/2} u\|_{L^{p,2}}^{\alpha + 1}.
\]

For the second term, by Lemma 2.4 and the Sobolev-Lorentz embedding, we have
\[
II \lesssim \|u\|_{L^{p,2}}^{\alpha + 1} \lesssim \|(-\Delta)^{s/2} u\|_{L^{p,2}}^{\alpha + 1} \lesssim \|(-\Delta)^{s/2} u\|_{L^{p,2}}^{\alpha + 1}.
\]

Then
\[
\|(-\Delta)^{s/2}(|x|^{-b}|u|^\alpha u)\|_{L^{p,2}} \lesssim \|(-\Delta)^{s/2} u\|_{L^{p,2}}^{\alpha + 1}.
\]

Therefore we conclude, by Hölder’s inequality in time, that
\[
\|(-\Delta)^{s/2} \mathcal{N}(u)\|_{L^r(0,T,L^{p,2})} \leq C T^{\frac{\alpha(N-2)}{4} - \frac{s}{2}} \|(-\Delta)^{s/2} u\|_{L^\gamma(0,T,L^{p,2})}.
\]

This completes the proof of the proposition. \(\square\)

To prove our result for the critical case we need the following proposition.

**Proposition 6.** Let \( s \geq 0 \) and \( \mathcal{K} \) be a compact of \( H^s \). Then for any admissible pair \((r,p)\), with \( r < \infty \);
\[
\lim_{T \to 0} \sup_{\varphi \in \mathcal{K}} \|e^{i\Delta \varphi}\|_{L^r(0,T,W^{s,p}_2)} = 0.
\]

**Proof.** By the Strichartz estimate, we have
\[
\lim_{T \to 0} \|e^{i\Delta \varphi}\|_{L^r(0,T,W^{s,p}_2)} = 0,
\]
for any \( \varphi \in \mathcal{K} \).
Let $\varepsilon > 0$. Since $\mathcal{K}$ is compact, there exist $\varphi_1, \cdots, \varphi_k \in \mathcal{K}$ such that $\mathcal{K} \subset \bigcup_{j=1}^{k} B_{H^s}(\varphi_j, \varepsilon/2)$. We have, for any $j \in \{1, \cdots, k\}$, there exists $\eta_j > 0$ such that if $0 < T < \eta_j$, then
\[
\|e^{t\Delta} \varphi_j\|_{L^r(0,T,W^s_2)} < \varepsilon/2.
\] (25)
Put $\eta = \min_{1 \leq j \leq k}(\eta_j)$. Let $0 < T < \eta$ and $\varphi \in \mathcal{K}$. Then there exists $j_0 \in \{1, \cdots, k\}$ such that $\varphi \in B_{H^s}(\varphi_{j_0}, \varepsilon/2)$. Using (25) and the Strichartz estimate, we deduce that
\[
\|e^{t\Delta}\varphi\|_{L^r(0,T,W^s_2)} \lesssim \|e^{t\Delta}(\varphi - \varphi_{j_0})\|_{L^r(0,T,W^s_2)} + \|e^{t\Delta}\varphi_{j_0}\|_{L^r(0,T,W^s_2)} \lesssim \|\varphi - \varphi_{j_0}\|_{H^s} + \varepsilon/2 \leq \varepsilon.
\]
This completes the proof of the proposition. \hfill \Box

Let us now prove Theorem 1.2.

**Proof of Theorem 1.2.** We first establish the existence then we prove the various properties of solutions.

**Existence part:** We will prove the existence result via the Banach fixed point theorem. For $M > 0, T > 0$, we consider the space
\[
X(M, T) := X = \{u \in L^\gamma(0, T, W^s_2); \|u\|_{L^\gamma(0, T, W^s_2)} \leq M\},
\]
endowed with the metric $d(u, v) = \|u - v\|_{L^\gamma(0, T, L^p)}$. The space $L^\gamma(0, T, W^s_2)$ is reflexive. Indeed, because $1 < \rho < \infty$, the Lorentz space $L^{\rho,2}$ is uniformly convex. See [22]. Using the fact that the norm in $W^s_2$ is defined by $\|u\|_{W^s_2} = \|(-\Delta)^{s/2} u\|_{L^{2,2}}$, that is it involves a uniformly convex norm, then $W^s_2$ is also uniformly convex, hence reflexive. It follows by the reflexivity of $L^\gamma(0, T, W^s_2)$ that its closed ball of radius $M$ is weakly compact. Hence $(X, d)$ is a complete metric space.

**First case.** $\alpha < (4 - 2b)/(N - 2s)$. Let $R > 0$ and $u_0 \in B_R(H^s)$. Let us show that $\mathcal{F}$ defined in (20) applies the space $X$ into itself for suitable $M > 0$ and $T > 0$. Since $(\gamma, \rho)$ is an admissible pair, so from Proposition 2 Part (i) we have
\[
\|e^{it\Delta}u_0\|_{L^\gamma(0, T, L^{p,2})} \leq C\|u_0\|_{L^2}
\]
and also
\[
\|((-\Delta)^{s/2} e^{-it\Delta} u_0)\|_{L^\gamma(0, T, L^{p,2})} \leq C\|u_0\|_{H^s}.
\]
Applying Proposition 5 with $(r, p) = (\gamma, \rho)$, we deduce that, if $u \in X(M, T)$ then $N(u) \in L^\gamma(0, T, W^{s,2}_2)$ and we obtain the following estimate
\[
\|\mathcal{F}(u)\|_{L^\gamma(0, T, W^s_2)} \leq C\|u_0\|_{H^s} + 2CT\delta M^{\alpha + 1}.
\]
Now we will prove the contraction of $\mathcal{F}$ for some $M$ and $T$. We have
\[
\|\mathcal{F}(u) - \mathcal{F}(v)\|_{L^\gamma(0, T, L^{p,2})} = \|N(u) - N(v)\|_{L^\gamma(0, T, L^{p,2})}.
\]
By (22) of Proposition 5, we have that
\[
\|\mathcal{F}(u) - \mathcal{F}(v)\|_{L^\gamma(0, T, L^{p,2})} \leq 2CT\delta M^{\alpha}\|u - v\|_{L^\gamma(0, T, L^{p,2})},
\]
for any $u, v \in X(M, T)$. We choose $M > 0$ and $T > 0$ such that
\[
C\|u_0\|_{H^s} + 2CT\delta M^{\alpha + 1} \leq M
\]
and
\[
2CT\delta M^{\alpha} < 1.
\] (26)
Let $M > 0$ be such that $M > 2CR$. Then there exists $T = T(R) > 0$ satisfying (26) and (27). By this choice, $\mathcal{F}$ has a fixed point in the space $X$. This proves the local existence of solution to the problem (5) for the subcritical case $\alpha < (4 - 2b)/(N - 2s)$.

**Second case.** $\alpha = (4 - 2b)/(N - 2s)$. In this case $\delta = 0$ and we argue as follows. Let $K$ be a compact of $H^s$ and $u_0 \in K$. As above, we have that
\[ \|\mathcal{F}(u)\|_{L^\gamma(0,T,W^{s,p}_2)} \leq \|e^{it\Delta}u_0\|_{L^\gamma(0,T,W^{s,p}_2)} + 2CM^{\alpha+1}. \]
Set $\eta_0 = \frac{1}{2}(\frac{1}{2\theta})^{1/\alpha}$ and let $0 < \eta < \eta_0$. By Proposition 6, there exists $T = T(K) > 0$ such that
\[ \|e^{it\Delta}u_0\|_{L^\gamma(0,T,W^{s,p}_2)} \leq \eta. \]
We choose $M = 2\eta$. Then we can verify that (27) is satisfied with $\delta = 0$ and the inequality
\[ \|e^{it\Delta}u_0\|_{L^\gamma(0,T,W^{s,p}_2)} + 2CM^{\alpha+1} \leq M \]
holds. So the fixed point argument on $X(M, T)$ with the above values of $M$ and $T$ works and gives a unique solution $u \in X(M, T)$ of (5). Moreover, for any admissible pair $(r, p)$, $u \in L^r(0, T; W^{s,p}_2)$ and we have
\[ \|u\|_{L^\gamma(0,T;W^{s,p}_2)} \leq 2\eta, \]
and there exists $C = C(r, p) > 0$ with
\[ \|u\|_{L^r(0,T;W^{s,p}_2)} \leq C\eta. \]
The estimate (6) follows by Proposition 5.

Similarly as in [8, Proposition 4.2, p. 824], we have that $u \in C([0, T], H^s) \cap L^r(0, T; W^{s,p}_2)$ for any admissible pair $(r, p)$. Also, uniqueness holds in $L^\gamma(0, T; W^{s,p}_2)$, for any $T > 0$. Thus we may define the maximal solution $u \in C([0, T_{\max}(u_0)), H^s) \cap L^\gamma([0, T_{\max}(u_0)), W^{s,p}_2)$. The proof of (iv) follows by classical arguments. See for example [8]. So, it remains to prove (v)-(viii).

**Properties of solutions:**

**(v)** We now prove the global existence for small data in the critical case. We suppose that $\alpha = (4 - 2b)/(N - 2s)$. In this case inequalities (26)-(27) read
\[ C\|u_0\|_{H^s} + 2CM^{\alpha+1} \leq M, \]
\[ 2CM^\alpha < 1, \]
where there is no dependence on $T$. We choose $M > 0$ such that $2CM^\alpha < 1$ and $u_0$ such that $\|u_0\|_{H^s} \leq M (1 - 2CM^\alpha) C^{-1}$. Then the fixed point argument in this case works in $X(\infty, M)$. This implies that for such initial data, $T_{\max}(u_0) = \infty$ and $u \in L^\gamma(0, \infty; W^{s,p}_2)$. By Proposition 5, $u \in L^r(0, \infty; W^{s,p}_2)$ for any admissible pair $(r, p)$.

Let us now show that the solution scatters. Write
\[ u(t) = e^{it\Delta}u_0 - i\mu \int_0^t e^{i(t-\sigma)\Delta}(|x|^{-b}|u|^\alpha u(\sigma))d\sigma. \]
Set
\[ z(t) := e^{-it\Delta}u(t) = u_0 - i\mu \int_0^t e^{-i\sigma\Delta}(|x|^{-b}|u|^\alpha u(\sigma))d\sigma. \]
For $0 < t < \tau$, we have
\[ z(t) - z(\tau) = -i\mu \int_\tau^t e^{-i\sigma\Delta}(|x|^{-b}|u|^\alpha u(\sigma))d\sigma. \]
By the Strichartz estimates and similar calculations as in the proof of (22), we get
\[ \|z(t) - z(\tau)\|_{H^s} \leq |\mu| \left\| \int_R e^{-i\sigma \Delta} [1_{(t,\tau)}(-\Delta)^{s/2}|x|^{-\beta}|u|^\alpha u] d\sigma \right\|_{L^2} \]
\[ \leq C \left\| 1_{(t,\tau)}(-\Delta)^{s/2}|x|^{-\beta}|u|^\alpha u \right\|_{L^\infty((R,L^{\infty})^2)} \]
\[ \leq C \|(-\Delta)^{s/2}u\|_{L^\infty((\tau,T,\tau,\tau)^2)} \to 0 \]
as \( t, \tau \to \infty \), since \( u \in L^\infty(0,\infty;W^s_{1,\infty}) \). Clearly, the previous calculations hold for \( s = 0 \). Thus \( \|z(t) - z(\tau)\|_{H^s} \to 0 \) as \( t, \tau \to \infty \). Then there exists \( \varphi^+ \in H^s \), such that \( e^{-it\Delta}u(t) \to \varphi^+ \) as \( t \to \infty \), in \( H^s \).

(vi) We now give the proof of the lower estimate for the blowup rate. In fact, the fixed point argument used to prove local existence to the nonlinear integral equation (5) yields lower estimate for the blowup rate. We apply an idea which was first used in [45, Section 4 and Remark 6 (2)]. See also [33, Proposition 5.3, p. 901], where this idea was formulated more clearly.

Let \( \alpha < (4 - 2b)/(N - 2s) \). Let \( u_0 \in H^s \) be such that \( T_{\text{max}}(u_0) < \infty \) and \( u \in C([0,T_{\text{max}}(u_0)),H^s) \) be the maximal solution of (5). Fix \( t_0 \in (0,T_{\text{max}}(u_0)) \) and let
\[ w(t) = u(t + t_0), \quad t \in (0,T_{\text{max}}(u_0) - t_0) . \]
We have,
\[ C\|w(0)\|_{H^s} + CM^{\alpha + 1}(T_{\text{max}}(u_0) - t_0)^{1 - \frac{\alpha(N - 2s)}{4} - \frac{h}{2}} > M, \quad \forall M > 0 . \quad (31) \]
In fact, otherwise there exists \( M > 0 \) such that
\[ C\|w(0)\|_{H^s} + C(T_{\text{max}}(u_0) - t_0)^{1 - \frac{\alpha(N - 2s)}{4} - \frac{h}{2}} M^{\alpha + 1} \leq M, \]
and \( w \) will be defined on \([0,T_{\text{max}}(u_0) - t_0] \). In particular, \( u(T_{\text{max}}(u_0)) \) is well defined. This leads to a contradiction. Hence (31) is verified for any \( t \in (0,T_{\text{max}}(u_0)) \) and for all \( M > 0 \). Let
\[ M = 2C\|u(t)\|_{H^s} . \]
From (31) we have
\[ C^{\alpha + 1} 2^\alpha + 1\|u(t)\|_{H^s}^{\alpha}(T_{\text{max}}(u_0) - t)^{1 - \frac{\alpha(N - 2s)}{4} - \frac{h}{2}} > 1 . \]
Hence
\[ \|u(t)\|_{H^s} \geq C(T_{\text{max}}(u_0) - t)^{\frac{N - 2s}{4} - \frac{2h}{2\alpha}} . \]
In particular, \( \lim_{t \to T_{\text{max}}(u_0)} \|u(t)\|_{H^s} = \infty \). This completes the proof of (vi).

(vii) Let us now prove the lower bound estimate of the life-span. We aim to obtain lower estimates for \( T_{\text{max}}(\lambda u_0) \), for all \( \lambda > 0 \). As for the proof of (vi), we show that the fixed point argument used to prove local existence to the nonlinear integral equation (5) yields lower estimate for the life-span. We apply an idea used in [42, Theorem 2.6 (ii), p. 161]. See also [43], where this idea is developed more clearly. Let \( \lambda > 0 \) and \( \alpha < (4 - 2b)/(N - 2s) \). Let \( \varphi \in H^s \) and \( u \in C([0,T_{\text{max}}(\lambda \varphi));H^s) \) be the maximal solution of (5) on \([0,T_{\text{max}}(\lambda \varphi)) \) with \( u_0 = \lambda \varphi \). By the inequality (26), for \( T > 0, M > C\|\varphi\|_{H^s} \) and
\[ C\|\varphi\|_{H^s} + CT^{1 - \frac{\alpha(N - 2s)}{4} - \frac{h}{2}} M^{\alpha + 1} \leq M \]
we have that the solution $u$ is defined on $[0, T]$. Then for $T = T_{\text{max}}(\varphi)$ we should have
\[ C\|\varphi\|_{H^s} + C\left(T_{\text{max}}(\varphi)\right)^{1 - \frac{\alpha(N-2\alpha)}{4} - \frac{b}{2}}M^{\alpha+1} > M, \]
for all $M > C\|\varphi\|_{H^s}$. That is, it must be
\[ \lambda C\|\varphi\|_{H^s} + C\left(T_{\text{max}}(\lambda\varphi)\right)^{1 - \frac{\alpha(N-2\alpha)}{4} - \frac{b}{2}}M^{\alpha+1} > M, \]
for all $M > \lambda C\|\varphi\|_{H^s}$. If we set $M = 2\lambda C\|\varphi\|_{H^s}$ we obtain from the last inequality that
\[ 2^{\alpha+1}C\alpha T_{\text{max}}(\lambda\varphi)^{1 - \frac{\alpha(N-2\alpha)}{4} - \frac{b}{2}}\lambda^\alpha > 1. \]
Then there exists $C = C(N, b, \alpha, s, \|\varphi\|_{H^s}) > 0$ such that
\[ T_{\text{max}}(\lambda\varphi) \geq C\lambda^{-\frac{2\alpha}{N-2\alpha} - \frac{N-2\alpha}{4}}. \]
This completes the proof of (vii).

(viii) We now prove that $u \in C^{1}([0, T_{\text{max}}(u_0)], H^{-1})$. We have that $u \in C([0, T_{\text{max}}(u_0)], H^1)$ then $\Delta u \in C([0, T_{\text{max}}(u_0)], H^{-1})$. Using
\[ u(t) = e^{it\Delta}u_0 - i\mu e^{it\Delta} \int_0^t e^{-i\sigma\Delta}(|x|^{-b}|u|^\alpha u) d\sigma, \]
and $\Delta e^{it\Delta}u_0 \in C(\mathbb{R}, H^{-1})$, it follows that $\Delta \left(e^{it\Delta} \int_0^t e^{-i\sigma\Delta}(|x|^{-b}|u|^\alpha u) d\sigma\right) \in C([0, T_{\text{max}}(u_0)], H^{-1})$. That is $\Delta \left(\int_0^t e^{-i\sigma\Delta}(|x|^{-b}|u|^\alpha u) d\sigma\right) \in C([0, T_{\text{max}}(u_0)], H^{-1})$. Since $e^{it\Delta}u_0 \in C^{1}(\mathbb{R}, H^{-1})$, it remains to show that $|x|^{-b}|u|^\alpha u \in C([0, T_{\text{max}}(u_0)], H^{-1})$. We have $N \geq 3$ and $\alpha \leq (4 - 2b)/(N - 2)$, then
\[ \frac{2N}{(N-2)(\alpha+1) + 2b} > 1. \]
We distinguish two cases.

First case. \( \frac{2N}{(N-2)(\alpha+1) + 2b} > 2. \) In this case $b < 1$. Hence $b < \frac{N}{2}$ and
\[ 2 \leq \frac{2N(\alpha+1)}{N - 2b} < \frac{2N}{N-2}. \]
We know that $u \in C([0, T_{\text{max}}(u_0)], H^1)$. By Sobolev-Lorentz embedding, it follows that
\[ u \in C([0, T_{\text{max}}(u_0)], L^{\frac{2N(\alpha+1)}{N - 2b}}). \]
That is, $|u|^\alpha u \in C([0, T_{\text{max}}(u_0)], L^{\frac{2N}{N - 2b}} \supset C([0, T_{\text{max}}(u_0)], L^{\frac{2N}{N - 2b}})$. By Lemma 2.2, we get $|x|^{-b}|u|^\alpha u \in C([0, T_{\text{max}}(u_0)], L^2) \subset C([0, T_{\text{max}}(u_0)], H^{-1})$.

Second case. \( \frac{2N}{(N-2)(\alpha+1) + 2b} \leq 2. \) Since $\alpha \leq (4 - 2b)/(N - 2)$ then
\[ \frac{2N}{(N-2)(\alpha+1) + 2b} \geq \frac{2N}{N+2} \]
and so $L^{\frac{2N}{N+2}} \leftrightarrow H^{-1}$. We have $u \in C([0, T_{\text{max}}(u_0)], H^1)$, then $u \in C\left(0, T_{\text{max}}(u_0)\right), L^{\frac{2N}{N+2}}$, that is $|u|^\alpha u \in C\left(0, T_{\text{max}}(u_0)\right), L^{\frac{2N}{N+2}}$. Therefore, By Lemma 2.2 we get $|x|^{-b}|u|^\alpha u \in C\left(0, T_{\text{max}}(u_0)\right), L^{\frac{2N}{N+2}}$, which gives that $|x|^{-b}|u|^\alpha u \in C\left(0, T_{\text{max}}(u_0)\right), H^{-1})$. This completes the proof of (vii) and that of Theorem 1.2. □
4. Unconditional uniqueness. Let us now prove the unconditional uniqueness in $C([0, T], H^s)$, for $s \geq 0$. To prove Theorem 1.3, we first establish the following.

Proposition 7. Assume (ii), (iii) or (iv) of Theorem 1.3. Then there exist two pairs $(r, p)$ and $(r_1, p_1)$ such that :

1) If $N = 1, 2$, $(r, p)$ and $(r_1, p_1)$ are admissible pairs.
2) If $N \geq 3$, $(r, p)$ and $(r_1, p_1)$ satisfy the hypotheses of Proposition 3.
3) $\frac{1}{p_2} := \frac{1}{p_1} - bN \in (0, 1)$, $\frac{\alpha}{p_3} := \frac{1}{p_2} - \frac{1}{p} \in (0, \min(1, \alpha))$.
4) $H^s \hookrightarrow L^{p_2}$.
5) $H^s \hookrightarrow L^{p_3, 2}$.
6) $r > r'_1$.

The proof will be done via three lemmas.

Lemma 4.1. Assume the hypotheses of Theorem 1.3. Then for $s > 0$, there exists $p$ satisfying the following.

If $N = 1$,
$$\max \left( \frac{(1-2s)^+}{2}, \frac{1}{2} - \frac{\alpha}{2} - b \right) < \frac{1}{p} < \min \left( \frac{1}{2}, 1 - b - \frac{\alpha(1-2s)^+}{2} \right).$$

If $N = 2$,
$$\max \left( \frac{(1-s)^+}{2}, \frac{1}{2} - \alpha - b \right) < \frac{1}{p} < \min \left( \frac{1}{2}, 1 - b - \frac{\alpha(1-s)^+}{2} \right).$$

If $N \geq 3$,
$$\max \left( \frac{N-2}{2N}, \frac{(N-2)^+}{2N}, \frac{1}{2} - \alpha - \frac{b}{N} \frac{N-2}{2(N-1)} - \frac{\alpha(N-2)}{4(N-1)} - \frac{b(N-2)}{2N(N-1)} \right) < \frac{1}{p}$$
and
$$\frac{1}{p} < \min \left( \frac{1}{2}, \frac{N}{2(N-1)} - \frac{b}{2(N-1)} - \frac{\alpha(N-2s)^+}{4(N-1)} \right).$$

Proof. If $s \geq N/2$ the existence of $p$ satisfying the above conditions is obvious.

If $N = 1$ and $0 \leq s < 1/2$, we see that $p$ exists if and only if $\frac{1-2s}{2} < 1 - b - \frac{\alpha(1-2s)}{2}$, which is satisfied by the conditions on $b$ and $\alpha$.

If $N = 2$ and $0 \leq s < 1$, we see that $p$ exists if and only if $\frac{1-s}{2} < 1 - b - \frac{\alpha(1-s)}{2}$, which is satisfied by the conditions on $b$ and $\alpha$.

If $N \geq 3$ and $0 < s < N/2$, we see that $p$ exists if and only if
$$\frac{(N-2s)^+}{2N} < \frac{N}{2(N-1)} - \frac{b}{2(N-1)} - \frac{\alpha(N-2s)}{4(N-1)}.$$ This is satisfied by the conditions on $b$ and for $\alpha < \frac{(2+4s-2b-4s/N)(N-2s)}{(N-2s)}$.

Lemma 4.2. Assume the hypotheses of Theorem 1.3. Let $p$ be a real number constructed in Lemma 4.1. Then for $s > 0$, there exists $p_1$ satisfying
$$\max \left( \frac{(N-2)^+}{pN}, \frac{1}{p} - \frac{1}{2} - \frac{\alpha}{N} - \frac{b}{N} - \frac{1}{p} \right) < \frac{1}{p_1} < \min \left( \frac{1}{2}, 1 - \frac{1}{p} - \frac{b}{N} - \frac{\alpha(N-2s)^+}{2N} \right).$$

Proof. If $s \geq N/2$ the proof is obvious. If $0 < s < N/2$, then $p_1$ exists if and only if $p$ satisfies the conditions of Lemma 4.1 and
$$1 - \frac{2}{N} - \frac{1}{p} < 1 - \frac{1}{p} - b - \frac{\alpha(N-2s)}{2N}.$$ This is realized since $\alpha < \frac{(4-2b)(N-2s)}{(N-2s)}$. □
Lemma 4.3. Assume the hypotheses of Theorem 1.3 with $N \geq 3$. Let $p$ be a real number constructed in Lemma 4.1 and $p_1$ be a real number constructed in Lemma 4.2. Then there exist real numbers $r, r_1$ such that

$$0 < \frac{1}{r} < \frac{N}{2} - \frac{N}{p}, \quad 0 < \frac{1}{r_1} < \frac{N}{2} - \frac{N}{p_1},$$

$$\frac{1}{r_1} + \frac{1}{r} = N \left(1 - \frac{1}{p} - \frac{1}{p_1}\right) < 1.$$ 

Proof. We choose $r$ such that

$$\max\left(0, -\frac{N}{2p} + \frac{N}{2p_1}\right) < \frac{1}{r} < \min\left(\frac{N}{2} - \frac{N}{p}, \frac{N}{2} - \frac{N}{2p} - \frac{N}{2p_1}\right).$$

This is possible since $p, p_1 > 2$ and $\frac{1}{p_1} < 1 - \frac{1}{p}$. Then we take $r_1$ such that

$$\frac{1}{r_1} + \frac{1}{r} = N \left(1 - \frac{1}{p} - \frac{1}{p_1}\right).$$

The last assertion follows by the condition on $p_1$ in Lemma 4.2 if $s > 0$. This completes the proof of the lemma.

We now give the proof of Proposition 7.

Proof of Proposition 7. For $s > 0$ we take $p, p_1$ satisfying the conditions of Lemmas 4.1 and 4.2.

1) For $N = 1, 2$ we take $r, r_1$ such that $(r, p), (r_1, p_1)$ are admissible pairs. This is possible by the conditions on $p$ and $p_1$.

2) For $N \geq 3$, one can verify that

$$\frac{N - 2}{p} < \frac{N - 2}{p_1} < \frac{N}{p},$$

and by Lemma 4.3, we have that $(r, p), (r_1, p_1)$ are acceptable pairs and satisfies the conditions of Proposition 3.

3) We have

$$\frac{1}{p_2} = 1 - \frac{1}{p_1} - \frac{b}{N}, \quad \frac{\alpha}{p_3} = 1 - \frac{1}{p_1} - \frac{b}{N} - \frac{1}{p}.$$

Then assertion 2) follows by the conditions on $p, p_1$.

4) This follows by the conditions on $p$ since we have that $\frac{(N - 2s)_+}{2N} < \frac{1}{p}$.

5) By the conditions on $p, p_1$ we have that

$$\frac{\alpha(N - 2s)_+}{2N} < 1 - \frac{1}{p_1} - \frac{b}{N} - \frac{1}{p} < \frac{\alpha}{2},$$

which gives the desired result.

6) If $N = 1, 2$ the pairs $(r, p)$ and $(r_1, p_1)$ are admissible so $\frac{1}{r_1} + \frac{1}{r} < 1$. If $N \geq 3$, the property follows by Lemma 4.3.

For $s = 0$, we take $p = 2, \frac{1}{p_1} = \frac{1}{2} - \frac{b}{N} - \frac{1}{2}, r = \infty$ and $\frac{1}{r_1} = \frac{b}{2} + \frac{N\alpha}{4}$. In this case the properties are obviously satisfied. This completes the proof of the proposition.

We now give the proof of Theorem 1.3.
Proof of Theorem 1.3. (i)-(iv). Let \( u, v \in C([0, T], H^s) \) be two solutions of (5). We have

\[
u(t) - v(t) = -i\mu \int_0^t e^{i(t-s)\Delta}(|x|^{-b}(|u|^\alpha u(s) - |v|^\alpha v(s)))ds
\]

Let \( T_1 \in (0, T] \) and set \( J = [0, T_1] \). Using the inhomogeneous Strichartz estimate for a pairs \((r, p), (r_1, p_1)\) satisfying the previous Proposition, we get

\[
\|u - v\|_{L^r(J, L^p J^)} \leq C\|x|^{-b}(|u|^\alpha u - |v|^\alpha v)\|_{L^r(J, L^p J^)}.
\]

By Lemma 2.2, we have

\[
\|x|^{-b}(|u|^\alpha u - |v|^\alpha v)\|_{L^r(J)} \leq C\|\|u\|_{H^s}^\alpha + \|v\|_{H^s}^\alpha\|_{L^p(J)}^{\frac{2(\alpha+1)}{\alpha}}\|u - v\|_{L^p(J)}.
\]

Then we deduce that

\[
\|u - v\|_{L^r(J, L^p J^)} \leq C(\|u\|_{L^\infty(J, L^p J^)} + \|u\|_{L^\infty(J, L^p J^)} )\|u - v\|_{L^r(J, L^p J^)}
\]

\[
\leq C(\|u\|_{L^\infty(J, H^s)} + \|u\|_{L^\infty(J, H^s)} )\|u - v\|_{L^r(J, L^p J^)}.
\]

Put \( C_1 = \|u\|_{L^\infty([0, T], H^s)} + \|v\|_{L^\infty([0, T], H^s)} \). Using Hölder inequality in time, we obtain

\[
\|u - v\|_{L^r(J, L^p J^)} \leq C_1 T_1^{\frac{2}{r} - \frac{1}{r}}\|u - v\|_{L^r(J, L^p J^)}.
\]

Since \( \frac{2}{r} - \frac{1}{r} > 0 \), then choosing \( T_1 \) small, we obtain \( u = v \) on \([0, T]\). By a Boot-strap argument we conclude that \( u = v \) on \([0, T]\).

(v). \( N \geq 3, 1 \leq s < N/2 \) and \( \alpha = (4 - 2b) / (N - 2s) \). Note that in this case by Lemma 4.2 we can only take the unique value \( \frac{1}{p_1} = 1 - \frac{1}{p} - \frac{2}{N} \). Hence \( \frac{1}{r_1} - \frac{1}{r} = 0 \). So we can not conclude using the previous calculations. We apply some argument of [5] done for the case \( b = 0 \).

Let \( I = [0, T], 0 < T < \infty, M > 0 \) and set

\[
f^M = 1_{\{|u| + |v| > M\}}(|u|^{\frac{4-2b}{N-2s}} u - |v|^{\frac{4-2b}{N-2s}} v),
\]

\[
f_M = 1_{\{|u| + |v| \leq M\}}(|u|^{\frac{4-2b}{N-2s}} u - |v|^{\frac{4-2b}{N-2s}} v).
\]

It is easy to see that

\[
|f_M| \leq CM \frac{4-2b}{N-2s} |u - v| \tag{32}
\]

and

\[
|f^M| \leq C1_{\{|u| + |v| > M\}} (|u| + |v|) \frac{4-2b}{N-2s} |u - v|. \tag{33}
\]

Assume \( b < 2 \). Let \( p_1 \) and \( p_3 \) be such that

\[
\max \left( \frac{N - 2b}{2N}, \frac{N - 2b}{2N} \right) \leq \frac{1}{p_1} = \min \left( \frac{1}{2}, \frac{N + 2 - 2b}{2N} \right).
\]

\[
\frac{1}{p_3} = \frac{1}{p_1} - \frac{b}{N} = 1 - \frac{1}{p_1} - \frac{b}{N}.
\]

It is clear that

\[
2 \leq p_1, p_3 \leq \frac{2N \alpha}{N - 2}.
\]
Consider \( r_1 \) and \( r_3 \) such that the pairs \((r_1, p_1)\) and \((r_3, p_3)\) are admissible. Let \( \tau \in (0, T) \). Using Strichartz estimates we have
\[
\|u - v\|_{L^3(0, \tau, L^{p_3, 2})} + \|u - v\|_{L^2\left(0, \tau, \frac{3N}{N - 2s}\right)} \leq C\|f\|_{L^r(0, \tau, L^{p_1, 2})} + \|f\|_{L^2(0, \tau, L^{p_1, 2})},
\]
where \( C \) is a constant.

On one hand, by Lemma 2.2, we have
\[
\|\|x\|^{-b}f\|_{L^{r_1}(0, \tau, L^{p_3, 2})} \leq C\|f\|_{L^{r_1}(0, \tau, L^{p_3, 2})} \leq C\left(\|u - v\|_{L^{r_1}(0, \tau, L^{p_3, 2})}\right),
\]
On the other hand, we have
\[
\|\|x\|^{-b}f\|_{L^{r_1}(0, \tau, L^{p_3, 2})} \leq C\|\|x\|^{-b}1_{\{|u| + |v| > M\}}\|u\|^\alpha + |v|^\alpha\|L^{N/2, \infty}_{\tau} \|u - v\|_{L^{\frac{2N}{N - 2s}, 2}} \leq C\|1_{\{|u| + |v| > M\}}\|u\|^\alpha + |v|^\alpha\|L^{N/2, \infty}_{\tau} \|u - v\|_{L^{\frac{2N}{N - 2s}, 2}}.
\]
That is, since \( \alpha = (4 - 2b)/(N - 2s) \), we get
\[
\|\|x\|^{-b}f\|_{L^2\left(0, \tau, \frac{3N}{N - 2s}\right)} \leq C\|1_{\{|u| + |v| > M\}}\|u\|^\alpha + |v|^\alpha\|L^{\frac{2N}{N - 2s}, \infty}_{\tau} \|u - v\|_{L^{\frac{3N}{N - 2s}, 2}} \times
\]
\[
\|u - v\|_{L^2\left(0, \tau, \frac{3N}{N - 2s}\right)} \leq C\|1_{\{|u| + |v| > M\}}\|u\|^\alpha + |v|^\alpha\|L^{\frac{2N}{N - 2s}, \infty}_{\tau} \|u - v\|_{L^{\frac{3N}{N - 2s}, 2}} \times
\]
\[
\|u - v\|_{L^2\left(0, \tau, \frac{3N}{N - 2s}\right)}.
\]
We deduce that
\[
\|u - v\|_{L^3(0, \tau, L^{p_3, 2})} + \|u - v\|_{L^2(0, \tau, L^{\frac{3N}{N - 2s}, 2})} \leq C\|1_{\{|u| + |v| > M\}}\|u\|^\alpha + |v|^\alpha\|L^{\frac{2N}{N - 2s}, \infty}_{\tau} \|u - v\|_{L^{\frac{3N}{N - 2s}, 2}} + CM\|u - v\|_{L^{\frac{3N}{N - 2s}, 2}}.
\]
Arguing as in [5, p. 87], we have \(\|1_{\{|u| + |v| > M\}}\|u\|^\alpha + |v|^\alpha\|L^{\frac{2N}{N - 2s}, \infty}_{\tau} \|u - v\|_{L^{\frac{3N}{N - 2s}, 2}} \rightarrow 0\), when \(M\) tends to \(\infty\). So for \(M\) sufficiently large we obtain
\[
\|u - v\|_{L^3(0, \tau, L^{p_3, 2})} + \|u - v\|_{L^2(0, \tau, L^{\frac{3N}{N - 2s}, 2})} \leq \varepsilon\|u - v\|_{L^2(0, \tau, L^{\frac{3N}{N - 2s}, 2})} + CM\|u - v\|_{L^{\frac{3N}{N - 2s}, 2}}.
\]
for \(\varepsilon > 0\) small enough. We conclude that
\[
\|u - v\|_{L^3(0, \tau, L^{p_3, 2})} \leq C\|u - v\|_{L^{\frac{3N}{N - 2s}, 2}},
\]
for every \(\tau \in (0, T)\). One can verify that \(r_3 > r_1\), so by [5, Lemma 4.2.2, page 85] we deduce that \(u = v\). This achieves the proof of Theorem 1.3.

5. Continuous dependence. In this section we prove the continuous dependence of solutions with respect to the initial data, that is we give the proof of Theorem 1.4.
Proof of Theorem 1.4. Proof of (i). We consider two cases.

Case 1. \( \alpha < (4 - 2b)/(N - 2s) \). In this case we can choose \( T \) so that the fixed point argument used to construct the solutions can be done for any initial values \( \psi \) having a norm less than \( 2 \| \varphi \|_{H^s} \). Then for sufficiently small \( \delta_0 \), if \( \| \varphi - \psi \|_{H^s} < \delta_0 \), we have \( T < T_{\text{max}}(\psi) \) and

\[
\| u \|_{L^\gamma(0,T,W^\rho_{2,s})}, \| v \|_{L^\gamma(0,T,W^\rho_{2,s})} \leq 4\| \varphi \|_{H^s}.
\]

Now by Proposition 5 and (21), we obtain

\[
\| u - v \|_{L^\gamma(0,T,L^p;2)} \lesssim \| \varphi - \psi \|_{L^2} + T^{1-\frac{\alpha+2}{2}} \left[ \| u \|_{L^\gamma(0,T,W^\rho_{2,s})} \right] \times \| u - v \|_{L^\gamma(0,T,L^p;2)},
\]

for any admissible pair \((r,p)\). We apply this estimate for \((r,p) = (\gamma,\rho)\) and choosing \( T \) small we get

\[
\| u - v \|_{L^\gamma(0,T,L^p;2)} \lesssim \| \varphi - \psi \|_{L^2},
\]

combined with the previous inequality gives

\[
\| u - v \|_{L^\gamma(0,T,L^p;2)} \lesssim \| \varphi - \psi \|_{L^2}.
\]

Since \( T \) depends only on \( \| \varphi \|_{H^s} \), then by a classical continuation argument we obtain (i) for \( s = 0 \).

To prove this result for \( s > 0 \) we need to control the \((-\Delta)^{s/2}\)-term. We write,

\[
(-\Delta)^{s/2}u(t) = (-\Delta)^{s/2}v(t) + e^{it\Delta} \left[ (-\Delta)^{s/2}\varphi - (-\Delta)^{s/2}\psi \right] + (-i)\mu \int_0^t e^{i(t-\sigma)\Delta} \left[ (-\Delta)^{s/2} \left( |x|^{-b} |u|^{\alpha}u(\sigma) - |v|^{\alpha}v(\sigma) \right) \right] d\sigma.
\]

Let \((r,p)\) be an admissible pair. By the homogeneous Strichartz estimate, we have

\[
\| I \|_{L^\gamma(0,T,L^p;2)} \lesssim \| (-\Delta)^{s/2}\varphi - (-\Delta)^{s/2}\psi \|_{L^2} \lesssim \| \varphi - \psi \|_{H^s}.
\]

By the inhomogeneous Strichartz estimate and the Leibnitz fractional rule, Lemma 2.3, we have

\[
\| II \|_{L^\gamma(0,T,L^p;2)} \lesssim \| (-\Delta)^{s/2} \left( |x|^{-b} |u|^{\alpha}u - |v|^{\alpha}v \right) \|_{L^\gamma(0,T,L^p;2)} + \| |x|^{-b} \|_{L^\frac{p}{p^*}\infty} \| (-\Delta)^{s/2} \left( |u|^{\alpha}u - |v|^{\alpha}v \right) \|_{L^\gamma(0,T,L^{p^*}1;2)}
\]

\[
= (III) + (IV),
\]

where \( p_1 \) and \( p^* \) are given respectively by (17) and (18). Using the Hölder inequality in time and space, we have

\[
(III) \lesssim \| |x|^{-b-s} \|_{L^\frac{p^*}{p^*}2\infty} \| 1_{[0,T]} \|_{L^{1/s}} \times \left[ \| u \|_{L^\gamma(0,T,L^{p^*}\infty)} + \| v \|_{L^\gamma(0,T,L^{p^*}\infty)} \right] \| u - v \|_{L^\gamma(0,T,L^{p^*}2)},
\]

where \( \delta \) is given by (19). By the Sobolev-Lorentz embedding, we have

\[
(III) \lesssim T^\delta \left[ \| u \|_{L^\gamma(0,T,W^\rho_{2,s})} + \| v \|_{L^\gamma(0,T,W^\rho_{2,s})} \right] \| u - v \|_{L^\gamma(0,T,W^\rho_{2,s})}.
\]
For the term (IV) we separate the cases $\alpha > 1$ and $\alpha = 1$. We begin by the case $\alpha > 1$. Set $F(z) = |z|^\alpha \cdot z$. We have

$$(-\Delta)^{s/2}(F(u) - F(v)) = \int_0^1 (-\Delta)^{s/2}(F'(t u + (1 - t)v)(u - v))dt.$$

By the Leibnitz fractional rule (Lemma 2.3) we have

$$\|(-\Delta)^{s/2}(F'(t u + (1 - t)v)(u - v))\|_{L^{\gamma'}(0, T, L^{\rho_2})} \lesssim \|(-\Delta)^{s/2}(u - v)\|_{L^\gamma(0, T, L^{\rho_1})} \|F'(t u + (1 - t)v)\|_{L^{\gamma_2}(0, T, L^{\rho_2'}/s, \infty)} + \|(u - v)\|_{L^{\gamma}(0, T, L^{\rho_3', s\alpha_1})} \|(-\Delta)^{s/2}(F'(t u + (1 - t)v))\|_{L^{\gamma_2}(0, T, L^{\rho_2'/s, \frac{2s}{\gamma_1}})} \lesssim (IV)_1 + (IV)_2,$$

where

$$\frac{1}{\gamma_2} \equiv \frac{1}{\gamma} - \frac{1}{\gamma_1}, \frac{1}{\rho_2} \equiv \frac{1}{\rho_1} - \frac{1}{\rho_3'}.$$

Using that $|F'(t u + (1 - t)v)| \lesssim |u|^\alpha + |v|^\alpha$, we have

$$(IV)_1 \lesssim \|1_{[0, T]} L^{1/s}(\|u\|_{L^\gamma(0, T, L^{\rho_3, 2})} + \|v\|_{L^\gamma(0, T, L^{\rho_3})})\|u - v\|_{L^{\gamma}(0, T, W^{s\alpha, \rho}_2)} \lesssim T^\delta \left[\|u\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)} + \|v\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)}\right]\|u - v\|_{L^{\gamma}(0, T, W^{s\alpha, \rho}_2)}.$$

Since $\alpha > 1$, then $F \in C^2(\mathbb{C}, \mathbb{C})$. Using the chain rule (Lemma 2.4) we have

$$\|(-\Delta)^{s/2}(F'(t u + (1 - t)v))\|_{L^{\gamma_2}(0, T, L^{\rho_2'/s, \frac{2s}{\gamma_1}})} \lesssim \|(-\Delta)^{s/2}(t u + (1 - t)v)\|_{L^{\gamma}(0, T, L^{\rho_3, 4s\alpha})} \|F''(t u + (1 - t)v)\|_{L^{\gamma_2}(0, T, L^{\rho_2'/s, \frac{2s}{\gamma_1}})}.$$

Since $|F''(t u + (1 - t)v)| \lesssim |u|^{\alpha - 1} + |v|^{\alpha - 1}$, then

$$(IV)_2 \lesssim \|1_{[0, T]} L^{1/s}(\|u\|_{L^{\gamma}(0, T, W^{s\alpha, \rho}_2)} + \|v\|_{L^{\gamma}(0, T, W^{s\alpha, \rho}_2)}) \times \left(\|u\|_{L^{\gamma}(0, T, L^{\rho_3, 2})} + \|v\|_{L^{\gamma}(0, T, L^{\rho_3, 2})}\right)\|u - v\|_{L^{\gamma}(0, T, L^{\rho_3, 2})} \lesssim T^\delta \left[\|u\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)} + \|v\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)}\right]\|u - v\|_{L^{\gamma}(0, T, W^{s\alpha, \rho}_2)}.$$

We finally get that

$$(VI) \lesssim T^\delta \left[\|u\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)} + \|v\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)}\right]\|u - v\|_{L^{\gamma}(0, T, W^{s\alpha, \rho}_2)}.$$

If $\alpha = 1$ then we take only $s = 1$. In this case we estimate the term (IV) as follows. We have the following inequality

$$\|\nabla(|u| |v|)\| \lesssim |u| \|\nabla u - \nabla v\| + |\nabla v| |u - v|.$$

As above, we have

$$(IV) = \|\nabla(|x|^{-\alpha})\|_{L^{\gamma}(0, T, L^{\rho_3, 2})} \|\nabla(|u| |v|)\|_{L^\gamma(0, T, L^{\rho_3, 2})} \lesssim \|1_{[0, T]} L^{1/s}(\|u\|_{L^{\gamma}(0, T, L^{\rho_3, 2})} \|\nabla u - \nabla v\|_{L^\gamma(0, T, L^{\rho_3, 2})} + \|\nabla v\|_{L^{\gamma}(0, T, L^{\rho_3, 2})})\|u - v\|_{L^\gamma(0, T, L^{\rho_3, 2})} \lesssim T^\delta \left[\|u\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)} + \|v\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)}\right]\|u - v\|_{L^\gamma(0, T, W^{s\alpha, \rho}_2)}.$$
By the estimates of (I)–(IV), we conclude that
\[ \|u - v\|_{L^r(0,T,\dot{W}^s_{2,r})} \lesssim \|\varphi - \psi\|_{H^s} + T^3 \left[ \|u\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}^\alpha + \|v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}^\alpha \right] \times \|u - v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}. \]

This gives
\[ \|u - v\|_{L^r(0,T,\dot{W}^s_{2,r})} \lesssim \|\varphi - \psi\|_{H^s} + T^3 \|u - v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}. \]

Hence for \((r, p) = (\gamma, \rho)\) and \(T\) sufficiently small, we derive the inequality
\[ \|u - v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} \lesssim \|\varphi - \psi\|_{H^s}, \]
which still verified for any admissible pair \((r, p)\). Combining this estimate with (35) we complete the proof of (i) for the subcritical case.

**Case 2.** \(\alpha = (4 - 2b)/(N - 2s)\). Let \(\eta_0\) be as in the proof of Theorem 1.2. Let \(0 < \tilde{\eta} < \eta_0\) to be fixed later. Let \(K\) be a compact in \(H^s\). Choose \(T > 0\) such that
\[ \sup_{\varphi \in K} \|e^{i\Delta \varphi}\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} \leq \tilde{\eta}/2. \]

Let \(\varphi \in K\). By the Strichartz inequality, we have
\[ \|e^{i\Delta \varphi}\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} \leq \|e^{i\Delta \varphi}\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} + \|e^{i\Delta (\varphi - \psi)}\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} \leq \|e^{i\Delta \varphi}\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} + C(\gamma, \rho)\|\varphi - \psi\|_{H^s}, \]
for any \(\psi \in H^s\). Set \(\eta = \tilde{\eta}/(2C(\gamma, \rho))\). Then for \(\psi \in B(\varphi, \eta)\), we have
\[ \|e^{i\Delta \varphi}\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} \leq \tilde{\eta}. \]

By the local well-posedness we have that \(T < T_{\text{max}}(\varphi), T < T_{\text{max}}(\psi)\) and
\[ \|u\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}, \|v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} \leq 2\tilde{\eta}. \]

For the critical case, (34) and (36) become
\[ \|u - v\|_{L^r(0,T,L^2)} \lesssim \|\varphi - \psi\|_{L^2} + \left[ \|u\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}^\alpha + \|v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}^\alpha \right] \times \|u - v\|_{L^\gamma(0,T,L^2)} \]
and
\[ \|u - v\|_{L^r(0,T,\dot{W}^s_{2,r})} \lesssim \|\varphi - \psi\|_{H^s} + \left[ \|u\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}^\alpha + \|v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}^\alpha \right] \times \|u - v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})}. \]

Taking \((r, p) = (\gamma, \rho)\) and \(\tilde{\eta}\) sufficiently small we get
\[ \|u - v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} \lesssim \|\varphi - \psi\|_{H^s}, \]
which still verified for any admissible pair \((r, p)\). We conclude that for any compact \(K\) in \(H^s\) and for any admissible pair \((r, p)\) there exist \(\eta > 0, C > 0, T > 0\) such that for all \(\varphi \in K\) and \(\psi \in B(\varphi, \eta)\) we have
\[ \|u - v\|_{L^\gamma(0,T,\dot{W}^s_{2,r})} \leq C\|\varphi - \psi\|_{H^s}. \]

To conclude, we have to show that the previous inequality holds for any \(T < T_{\text{max}}(\varphi)\). So, for \(T > 0\) we consider the property:
\[ (P_T) : \exists \eta_1, \text{ such that for any admissible pair } (r, p), \exists C_1(r, p) > 0, \text{ for which we have } \forall \psi \in B(\varphi, \eta_1), T_{\text{max}}(\psi) > T, \text{ and } (37) \text{ holds with } C = C_1. \]

Let
\[ \bar{T} := \sup \{0 < T < T_{\text{max}}(\varphi), \text{ such that } (P_T) \text{ holds} \}. \]
By the previous argument we have $T > T_0$. We will show that $\tilde{T} = T_{\max}(\varphi)$. Assume that $T < T_{\max}(\varphi)$. Since $u \in C([0, \tilde{T}], H^s)$, so $K_0 := \cup_{t \in [0, \tilde{T}]} \{u(t)\}$ is a compact in $H^s$. Let $(r, p)$ be an admissible pair. By the previous argument, there exist $\eta_0$, $T_0 > 0$ and $C_0 > 0$ such that (37) holds for any $\varphi \in K_0$ and $\psi \in B(\varphi, \eta_0)$.

Let $0 < \tau < \tilde{T}$, $\tau + T_0 > \tilde{T}$. Set $\eta_2 = \min(\eta_1, \eta_0/C_1(\infty, 2))$ and consider $\psi \in B(\varphi, \eta_2)$. By the definition of $\tilde{T}$ we have
\[
\|u(\tau) - v(\tau)\|_{H^s} \leq \|u - v\|_{L^\infty(0, \tau, H^s)} \leq C_1(\infty, 2)\|\varphi - \psi\|_{H^s} \leq \eta_0.
\]
Then $v(\tau) \in B(u(\tau), \eta_0)$. Since $u(\tau) \in K_0$, then for $u_t = u(\tau + \cdot)$ and $v_t(\tau + \cdot)$, we have $T_{\max}(u(\tau)) > T_0$, $T_{\max}(v(\tau)) > T_0$ and
\[
\|u - v\|_{L^r(\tau, \tau + T_0, W^{s, p})} = \|u_\tau - v_\tau\|_{L^r(0, T_0, W^{s, p})} \leq C_0\|u(\tau) - v(\tau)\|_{H^s} \leq C_0C_1(\infty, 2)\|\varphi - \psi\|_{H^s}.
\]
Hence the property $(P_{\tau + T_0})$ is verified. This contradicts the definition of $\tilde{T}$ and concludes the proof of Part (i).

**Proof of (ii).** We take $T > 0$ as given in the corresponding case for (i). We keep the same estimates for (I) and (II). For (III) we argue as follows.
\[
\|III\|_{L^r(0, T, L^{p, 2})} \lesssim \|1_{[0, T]}\|_{L^{1/\alpha}}\|u\|_{L^\gamma(0, T, L^{p, 2})}^\alpha \|
abla u - \nabla u_k\|_{L^\gamma(0, T, L^{p, 2})} + \|
abla u_k\|_{L^\gamma(0, T, L^{p, 2})}\|u - u_k\|_{L^\gamma(0, T, L^{p, 2})}^\alpha \\
\lesssim T^\delta \|
abla u\|_{L^\gamma(0, T, L^{p, 2})}^\alpha \|
abla u - \nabla u_k\|_{L^\gamma(0, T, L^{p, 2})} + \|
abla u_k\|_{L^\gamma(0, T, L^{p, 2})}\|u - u_k\|_{L^\gamma(0, T, L^{p, 2})}^\alpha.
\]
It remains to prove that
\[
\lim_{k \to \infty} \|u - u_k\|_{L^\gamma(0, T, L^{p, 2})} = 0.
\]
Set
\[
f(t) = \|u(t) - u_k(t)\|_{L^{p, 2}}^\alpha.
\]
Let $\max(2, \alpha\gamma_2) < \gamma_3 < \gamma$. We claim that
\[
\lim_{k \to \infty} \|u - u_k\|_{L^{\gamma_3}(0, T, L^{p, 2})} = 0.
\]
In fact, let $\rho_3$ be such that $(\gamma_3, \rho_3)$ is an admissible pair. By the Gagliardo-Nirenberg inequality, we have
\[
\|u - u_k\|_{L^{\gamma_3}(0, T, L^{p, 2})} \leq \|u - u_k\|_{L^{\gamma_3}(0, T, L^{p_3, 2})} \|
abla u - \nabla u_k\|_{L^{\gamma_3}(0, T, L^{p_3, 2})}^\theta,
\]
where
\[
\theta = N \left( \frac{1}{\rho^*} - \frac{1}{\rho_3} + \frac{1}{N} \right) \in (0, 1).
\]
From (35), which still holds for $\alpha < 1$, we have that
\[
\lim_{k \to \infty} \|u - u_k\|_{L^{\gamma_3}(0, T, L^{p_3, 2})} = 0.
\]
By the fixed point procedure, $\|
abla u - \nabla u_k\|_{L^{\gamma_3}(0, T, L^{p_3, 2})}$ is bounded for sufficiently large $k$. This proves the claim.

Hence,
\[
\|u - u_k\|_{L^{\gamma_3}(0, T, L^{p, 2/\alpha})} = \int_0^T f(t) dt \leq T^{1 - \frac{\gamma_3}{\alpha}} \|u - u_k\|_{L^{\gamma_3}(0, T, L^{p_3, 2})} \to 0,
\]
as \( k \to \infty \). The proof is achieved by classical continuation argument.

**Proof of (iii).** We have \( \alpha = (4 - 2b)/(N - 2) \), so \( \gamma_2 = \gamma \). Hence the previous argument fails. We use similar argument as in [7]. Arguing as in the beginning of the part (i) case 2, we know that for any compact \( K \subset H^1 \) there exists \( T > 0 \) such that for any \( \varphi \in K, T < T_{\max}(\varphi) \) and if \( \varphi_k \to \varphi \) in \( H^1 \), then for sufficiently large \( k \), \( T < T_{\max}(\varphi_k) \), \( u_k \to u \) in \( L'(\[0, T], L^{p, 2}) \) and \( (u_k) \) is bounded in \( L'(\[0, T], W^{1, p}_1) \), for every admissible pair \((r, p)\).

Let \((r, p)\) be an admissible pair such that \( 2 < p < N, p < 2N/(N - 2) \). By the Gagliardo-Nirenberg inequality, we have that
\[
\|u_k - u\|_{L^{\frac{2N}{N-2}, 2}} \leq C\|\nabla u_k - \nabla u\|_{L^{p, 2}}\|u_k - u\|_{L^{p, 2}}^{1-\theta},
\]
where \( 0 < \theta < 1 \) satisfies
\[
\frac{N - 2}{2N} = \frac{1}{p} - \frac{\theta}{N}.
\]
Taking the \( L' \)-norm of the last inequality in time, we deduce that \( u_k \to u \) in \( L'(\[0, T], L^{\frac{2N}{N-2}, 2}) \), for \( p \) sufficiently close to 2 hence for \( r < \infty \) arbitrarily large.

Let us now show that the operator
\[
| \cdot |^{-b/(\alpha + 2)} : H^1 \to L^{\alpha + 2}, \ f \mapsto | \cdot |^{-b/(\alpha + 2)} f,
\]
is continuous. In fact, using Hölder’s inequality in Lorentz spaces, we have
\[
\| | \cdot |^{-b} f \|_{L^{\alpha + 2}} \leq C\| | \cdot |^{-b} \|_{L^{\frac{\infty}{\alpha + 2}}} \| f \|_{L^{\frac{\alpha + 2}{\alpha - 2}}},
\]
for any \( f \in H^1 \), where we have used \( \frac{N(\alpha + 2)}{N - b} = \frac{2N}{N - 2} \). Hence
\[
\| | \cdot |^{-b/(\alpha + 2)} f \|_{L^{\alpha + 2}} \leq C\| f \|_{H^1}.
\]
Let us define the energy
\[
E(u)(t) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2} + \frac{\mu}{\alpha + 2} \| | \cdot |^{-b} u(t) \|_{L^{\alpha + 2}}^2.
\]
By the above calculations, \( E(u) \) is well defined. We now consider separately the cases \( \mu \in \mathbb{R} \) and \( \mu > 0 \).

**Case 1.** \( \mu \in \mathbb{R} \). Since \( \varphi_k \to \varphi \) in \( H^1 \), then by the continuity of the above operator, \( E(\varphi_k) \to E(\varphi) \) and by conservation of energy (because \( \mu \in \mathbb{R} \) ) \( E(u_k)(t) \to E(u)(t) \). So \( \|\nabla u_k(t)\|_{L^2} \to \|\nabla u(t)\|_{L^2} \) in \( L'(\[0, T]) \). On the other hand since \( u_k \to u \) in \( C(\[0, T], L^2) \) then for all \( t \in \[0, T] \),
\[
u_k(t) \to u(t) \text{ in } L^2.
\]
Also, by the boundedness in \( C(\[0, T], H^1) \) we deduce that \( u_k(t) \to u(t) \) weakly in \( H^1 \). Then one conclude that \( u_n \to u \) in \( L'(\[0, T], H^1) \). The rest of the proof follows similarly as in [5, Step 6, p. 108].

**Case 2.** \( \mu > 0 \). We have to show that \( u_k \to u \) in \( C(\[0, T], H^1) \). We argue by contradiction. Then there exist \( \varepsilon > 0 \) and a sequence \( t_k \to t \in \[0, T] \) such that \( \|u_k(t_k) - u(t_k)\|_{H^1} \geq 2\varepsilon \) for all \( k \). Since \( u \in C(\[0, T], H^1) \), we have \( u(t_k) \to u(t) \) in \( H^1 \). Then \( \|u_k(t_k) - u(t)\|_{H^1} \geq \varepsilon \), for large \( k \). We know that \( u_k(t_k) \to u(t) \) in \( H^1 \), so,
by the continuity of the above operator, we have \( | \cdot |^{-b/(\alpha+2)} u_k(t_k) \to | \cdot |^{-b/(\alpha+2)} u(t) \) weakly in \( L^{\alpha+2} \). Since \( \mu > 0 \), then by the conservation of the energy, we deduce that

\[
\frac{1}{2} \| \nabla u_k(t_k) \|_{L^2}^2 \to \frac{1}{2} \| \nabla u(t) \|_{L^2}^2
\]

and

\[
\| | \cdot |^{-b/(\alpha+2)} u_k(t_k) \|_{L^{\alpha+2}} \to \| | \cdot |^{-b/(\alpha+2)} u(t) \|_{L^{\alpha+2}}.
\]

Combining (38) (which is valid for \( t \) replaced by \( t_k \) in the first term) and (39), we get that \( u_k(t_k) \to u(t) \) strongly in \( H^1 \). This leads to a contradiction. Then \( u_k \to u \) in \( C(\{0, T\}, H^1) \).

We have now shown for both cases the existence of \( 0 < T < T_{\max}(\varphi) \) such that the convergence in (iii) holds. To prove that this convergence is satisfied for any \( 0 < T < T_{\max}(\varphi) \), we consider the following property: \( (Q_T) \): If \( \varphi_k \to \varphi \) there exists \( k_0 > 0 \) such that if \( k > k_0 \) then \( T_{\max}(\varphi_k) > T \) and \( u_k \to u \) in \( L^\infty(0, T, H^1) \).

Replacing the property \( (P_T) \) by \( (Q_T) \) and adapting the continuation argument of the part (i) case 2, we can show that Property \( (Q_T) \) holds for any \( T < T_{\max}(\varphi) \). This finishes the proof of Theorem 1.4.

\[ \square \]

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