Applications of chiral supersymmetry for spin fields in self-dual backgrounds

L. Fehér*, P. A. Horváthy† and L. O’Raifeartaigh

Dublin Institute for Advanced Studies, Ireland

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Abstract

Due to chiral supersymmetry the (nonzero mode) spectral and symmetry properties of a 4-dimensional, self-dual Dirac-Yang-Mills operator $\mathbf{D}$ can be recovered from those of the corresponding scalar Laplacian $D^2$. It is shown that a similar result holds for higher spins, and that in the 4-vector case the supersymmetric partners are $-D^2\mathbf{I}_4$ and the fluctuation operator. The reduction to $D^2$ is used to simplify previous analyses of the (nonzero mode) spectrum of $\mathbf{D}$ and of the fluctuations for a BPS-monopole, and to explain the Kepler and $\text{su}(2/2)$ (super)symmetries of a system studied recently by D’Hoker and Vinet.

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1 Introduction

It is known that the Dirac-Yang-Mills (DYM) operator $\mathbf{D} = \gamma^\mu D_\mu$, $D_\mu = \partial_\mu - iA_\mu$, is chiral-supersymmetric in any even dimensions \cite{1}, and it is easy to verify that for a 4-dimensional, self-dual gauge potential $A_\mu$, the squared operator $\mathbf{D}^2$ reduces to the scalar Laplacian $D^2$ (times the unit matrix) on one of the supersymmetric sectors. The reduction of $\mathbf{D}^2$ to $D^2$ has been used in constructing propagators in instanton background \cite{2}. The purpose of the present paper is to show how the reduction can be used to simplify and clarify some recent work on the discrete spectra and symmetries of self-dual DYM operators. We first show how the discrete eigenvector system (apart from the zero modes) and the symmetries of $\mathbf{D}$ may be reconstructed from those of $D^2$, and then use $D^2$ to

(i) simplify and generalize to any isospin the previous analyses \cite{3,4} of the (nonzero mode) spectrum of the DYM operator of a Bogomolny-Prasad-Sommerfield (BPS) monopole \cite{5} (the zero-modes have been completely analyzed in Refs. \cite{6} and \cite{7}),

(ii) explain the remarkable but complicated ‘dynamical’ Kepler and supersymmetries of the self-dual $\mathbf{D}^2$ found recently by D’Hoker and Vinet \cite{8},

(iii) show how the DYM supersymmetry has an immediate generalization from Dirac-spinor $(m = 0)$ to $D^{(1/2,m)} \oplus D^{(m,1/2)}$ wave functions for any $m = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (note that $D^{(p,q)}$ denotes the representation of $\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2)$, which is obtained by taking the tensor

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*On leave from Bolyai Institute, University of Szeged. Present address: Research Institute for Particle and Nuclear Physics, Budapest, Hungary. e-mail: lfeher-at-rmki.kfki.hu

†Department de Mathématiques Université d’Avignon. Present address: Laboratoire de Mathématiques et de Physique Théorique, Université de Tours, France. e-mail: horvathy-at-lmpt.univ-tours.fr
product of the angular momentum $p$ and $q$ representations of the left and right SU(2) subgroups, respectively, and that in the 4-vector case ($m = \frac{1}{2}$) the SUSY partners are $-D^2 \mathds{1}_4$ and the operator on the left-hand side of the fluctuation equation \[9\]
\[
- D^2 \delta_{\mu\nu} + 2i F_{\mu\nu} = \omega^2 (\delta A^\mu), \quad F_{\mu\nu} \equiv i [D_\mu, D_\nu].
\] (1.1)
The latter result throws a new light on the well-known fact \[10, 9, 3, 4\] that the $\omega \neq 0$ eigenfluctuations can be expressed in terms of the eigenfunctions of either $\hat{\mathcal{D}}$ or $D^2$.

The starting point is the usual 4-dimensional Euclidean DYM operator in the chiral (supersymmetric) basis, namely,
\[
Q = \begin{bmatrix} 0 & S \\ S^\dagger & 0 \end{bmatrix} \equiv i \hat{\mathcal{D}} = \begin{bmatrix} 0 & \sigma^k D_k - i D_4 \\ -\sigma^k D_k - i D_4 & 0 \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix},
\] (1.2)
from which one has, for the squared operator,
\[
Q^2 = \begin{bmatrix} SS^\dagger & 0 \\ 0 & S^\dagger S \end{bmatrix} = -\hat{\mathcal{D}}^2 = \begin{bmatrix} -D^2 - \sigma^k (B_k + E_k) & 0 \\ 0 & -D^2 - \sigma^k (B_k - E_k) \end{bmatrix},
\] (1.3)
the $\sigma$’s being the three Pauli matrices. From \[1, 3\] it is evident that for self-dual gauge fields ($E_k = B_k$) the operator $\hat{\mathcal{D}}^2$ reduces to $D^2$ on one of the supersymmetric sectors.

First we wish to show that, at least for the discrete spectrum, the eigenvector system and the symmetries of $\hat{\mathcal{D}}$ can be reconstructed from those of $D^2$. For this one notes that if, for a general QMSUSY system, the zero eigenmodes of $Q$ are excluded from the Hilbert space, the operator
\[
U = S \frac{1}{\sqrt{SS^\dagger}}
\] (1.4)
becomes unitary (at least on the bound state sector we are interested in), and intertwines the supersymmetric partners of $Q^2$, i.e.,
\[
SS^\dagger = U (S^\dagger S) U^\dagger.
\] (1.5)
It follows that if $\psi$ is an eigenvector of $S^\dagger S$ with eigenvalue $\omega^2 > 0$, then $U \psi$ is an eigenvector of $SS^\dagger$ with the same eigenvalue and
\[
\psi_{\pm \omega} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} U \psi \\ \pm \psi \end{pmatrix}
\] (1.6)
are eigenvectors of $Q$ with eigenvalues $\pm \omega$ ($\omega > 0$), respectively. Similarly, if $G$ is a symmetry group of $S^\dagger S$ then $UGU^\dagger$ is a symmetry group of $SS^\dagger$ and the diagonal subgroup of $(UGU^\dagger) \times G$ is a symmetry group of $Q$. Applying these results to the case $Q = i \hat{\mathcal{D}}$ for a self-dual $A_\mu$, one sees at once that the discrete eigenvector system and the symmetries of $\hat{\mathcal{D}}$ can be reconstructed from those of $D^2$. Examples of the symmetry reconstruction will be given in Sec. 3.

2 Nonzero mode spectrum for the BPS monopole

Let us first apply the reduction of $\hat{\mathcal{D}}^2$ to $D^2$ to the study of the bound state (nonzero mode) spectrum of $\hat{\mathcal{D}}$ for the gauge field of an SU(2) (isospin), static, spherically symmetric BPS-monopole. This is the special case in which, in the usual three-dimensional notation, the self-dual gauge potential $A_\mu$ is
\[
A_4^a = \Phi_4 = -\frac{x^a}{r} H, \quad A_i^a = \epsilon_{aik} \frac{x^k}{r^2} (1 - K), \quad H = r \coth r - 1, \quad K = \frac{r}{\sinh r},
\] (2.1)
(\(A^2_3\) being identified physically as a Higgs field \(\Phi^a\), and correspondingly \(E_k \rightarrow D_k \Phi\)). The bound state problem in this background has been extensively investigated before [3] for an isospin 1 particle on the basis of the Jackiw-Rebbi [3] angular momentum analysis of \(\Phi\) (see also ref. [4]). If we exclude the zero modes then, according to the arguments given above, it is sufficient to study the scalar Laplacian \(D^2\), and for the gauge potential (2.1), this Laplacian (for \(x^4\)-independent wave-functions) is easily computed to be

\[
- D^2 = - \Delta_r + \frac{1}{r^2} \left\{ \Lambda^2 + K(K - 2)T^2 - 2KI_0 + \left[ H^2 - (1 - K)^2 \right]Q_{em}^2 \right\}, \tag{2.2}
\]

where \(\Delta_r\) is the usual radial Laplacian, \(\Lambda^i\) is the total angular momentum,

\[
\Lambda^i = L^i + T^i, \tag{2.3}
\]

\(L^i\) and \(T^i\) are the orbital angular momentum and isospin \(\text{su}(2)\) generators, respectively, and \(I_0\) and \(Q_{em}\) are the isospin-orbit coupling and the electric charge operators defined as

\[
I_0 = L_k T^k, \quad \text{and} \quad Q_{em} = - \frac{x^a}{r} T_a \left( = \Phi^a \right). \tag{2.4}
\]

The operator \(D^2\) acts on the function space \(\mathcal{H} = L^2(\text{d}r \text{d}\theta) \oplus L^2(S^2) \oplus \mathbb{C}^{2t+1}\) and since it commutes with the total isospin \(T^2 = \sum_i (T^i)^2\) and the total angular momentum vector \(\Lambda^k\), it can be studied for each set of eigenvalues \(\{t(t+1), \lambda(\lambda+1), \lambda_3\}\) of the operators \(\{T^2, \Lambda^2, \Lambda_3\}\) separately. This labeling of the nonradial part of \(\mathcal{H}\) leaves a \((2t + 1)\)-dimensional degeneracy, which can be removed in a natural way by labeling with the \((2t + 1)\)-eigenvalues \(-t \leq q \leq t\) of \(Q_{em}\). Since \(Q_{em}\) does not commute with \(D^2\) the various \(Q_{em}\) eigenstates are coupled, but because \(D^2\) (and \(\Lambda^i\)) commute with the space-parity operator \(P\), it is possible to decompose the \((2t + 1)\)-states into two decoupled sets of opposite parity. Because \(P\) and \(Q_{em}\) anticommute, the eigenstates of \(P\) are simply the sums and differences of the eigenstates of \(Q_{em}\) for fixed \(q^2\) and the subspaces of definite parity are (in general) dimension \((t + \frac{1}{2})\) for \(t = \text{half-odd-integer}\), and \(t\) and \((t + 1)\) for \(t = \text{integer}\). Since the angular operators \(I_0\) and \(Q_{em}\) which occur in \(D^2\) do not commute, and are irreducible on the parity eigenspaces for fixed \((t, \lambda, \lambda_3)\), the radial equations within each parity sector cannot be further decoupled.

For example, for the case \(t = 1\), to which the problem of nonzero mode fluctuations around the monopole can be reduced (see Sec. 4), one sees that for each value of \(\lambda, \lambda_3\) \((\lambda = 1, 2, \ldots)\) there are one uncoupled and two coupled radial equations, and that the operator \(-D^2\) takes the form

\[
\begin{bmatrix}
- \Delta_r + \frac{\lambda(\lambda + 1)}{r^2} + \frac{H^2 + K^2 - 1}{r^2}
\end{bmatrix} \tag{2.5}
\]

and

\[
\begin{bmatrix}
- \Delta_r + \frac{\lambda(\lambda + 1)}{r^2} + \frac{1}{r^2} \begin{pmatrix}
2K^2 & 0 \\
0 & (H^2 + K^2 - 1)
\end{pmatrix} - \frac{2K \sqrt{\lambda(\lambda + 1)}}{r^2} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\end{bmatrix} \tag{2.6}
\]

in the respective parity sectors. (Note that \(Q_{em}^2\) is also diagonal in (2.5)-(2.6)). For the lowest angular momentum \(\lambda = 0\) only the upper \((Q_{em} = 0)\) half of (2.6) survives.

The main question concerning the operator (2.2) is whether it has a discrete spectrum, i.e., admits bound states, and since the functions \(H(r)\) and \(K(r)\) are well-behaved at the origin, this question can be investigated by concentrating on the asymptotic \((r \rightarrow \infty)\) form of \(-D^2\), which is easily seen to be

\[
- D^2 \left|_{r \rightarrow \infty} \right. = \left[ - \Delta_r + \frac{\Lambda^2}{r^2} - 2 \frac{Q_{em}}{r} + Q_{em}^2 \right]. \tag{2.7}
\]
In this limit the operator $I_0$ drops out, the $(2t+1)$ equations for fixed $(t,\lambda,\lambda_3)$ completely decouple, and one obtains a set of Schrödinger operators with Coulomb potentials, and a free Schrödinger operator for $Q_{em} = 0$. Thus if (2.7) were the true operator, there would be a discrete Coulomb spectrum (for $0 < E < Q_{em}^2$) for each nonzero value $q$ of $Q_{em}$. For the true operator (2.2), one would therefore expect to have bound states in all irreducible sectors except those which contain the neutral ($Q_{em} = 0$) states. Thus for $t = \text{half-odd integer}$ one would expect to have bound states in both parity sectors, and for $t = \text{integer}$, one would expect to have them only in the $t$-dimensional sector (in which $Q_{em}^2 \neq 0$). Furthermore, since the true operator (2.2) differs appreciably from (2.7) only in the core region ($r \leq 1$), one would expect the true bound state spectra to be close to the Coulomb spectra for all but the lowest eigenvalues of the angular momentum. These expectations have been confirmed [11] numerically for $t = \frac{1}{2}$ and $t = 1$. Concentrating on the $t = 1$ case, we note that the bound states of (2.5) with $\lambda = 1,2,\ldots$ yield via (1.6) bound states of $i\not\!D$ with $j = \lambda \pm \frac{1}{2} = \frac{1}{2},\frac{3}{2},\ldots$, where the quantum number $j$ refers to the angular momentum operator $J^i = \Lambda^i + \sigma^j/2$ containing also the spin. In fact, the $j = \lambda - \frac{1}{2}$ series of bound states constructed in this way belongs to the $F_{\pm,\lambda}^{\pm}$ Jackiw-Rebbi sector [3] and reproduce the bound states found in Ref. [3]. The $j = \lambda + \frac{1}{2}$ series which is in the $F_{\pm,\lambda}^{\pm}$ Jackiw-Rebbi sector went apparently unnoticed in Ref. [3].

3 A point-like self-dual monopole

Let us consider the case where the asymptotic form

$$A_4^a = \Phi^a = -\frac{x^a}{r} \left(1 - \frac{1}{r} \right), \quad A_i^a = \epsilon_{akr} \frac{x^k}{r^2}$$

(3.1)

of the gauge potential (2.1) is taken as the gauge potential for all $r \neq 0$. (In spite of the singularity at $r = 0$ the operator $i\not\!D$ is self-adjoint for (3.1) just as for the Coulomb potential.) For this potential $-D^2$ is given (for all $r \neq 0$) by the expression (2.7), where

$$\Lambda^i = \epsilon_{ijk} x^j (-i\not\!D^k) - Q_{em} \frac{x^i}{r}$$

(3.2)

is now the angular momentum operator for a Wu-Yang monopole. In this case the electric charge $Q_{em}$ commutes with $D^2$ and we restrict ourselves to one of its eigensubspaces with a nonzero eigenvalue $q$. The $\Lambda^i$ then become the angular momentum of a charged particle in a Dirac monopole background, and the operator (2.7) itself becomes the Hamiltonian considered about 20 years ago by Zwanziger [12] and by McIntosh and Cisneros [13] (where a term $q^2/r^2$, which comes from the Higgs field $\Phi^a$ in the above derivation, was introduced ‘by hand’ in order to change a dyon-background potential to a Coulomb one). In Refs. [12] and [13] it was shown that the Hamiltonian (2.7) has an $o(4)/o(3,1)$ Kepler type symmetry, but one generated not by the usual $H$-atom generators but by $\Lambda^i$ and the Runge-Lenz vector

$$K^i = -\frac{i}{2} \epsilon_{ijk} (D^j \Lambda^k - \Lambda^j D^k) - q^2 \frac{x^i}{r}.$$  

(3.3)

The interesting feature of the potential (3.1) is that the operator $-D^2$ for this potential (and for fixed $q \neq 0$) is just the $4 \times 4$ supersymmetric Hamiltonian studied by D’Hoker and Vinet [8], and one of its chiral projections is the $g = 4$ Pauli Hamiltonian studied by these authors. Once this is realized, the reason for the value 4 of the gyromagnetic ratio, and for the symmetries of the Pauli Hamiltonian, becomes clear: the value 4 comes from the fact that for self-dual fields $\sigma^k (\not\!E_k + B_k) \to 2 \sigma^k B_k$ in $SS^\dagger$, the ‘extra’ $su(2)$ is really the SUSY partner of the trivial spin $su(2)$ symmetry of $S^\dagger S = -D^2 1_2$ present in any self-dual background, and the
Kepler symmetry is simply the SUSY partner of the Kepler symmetry of the Hamiltonian \((2.7)\). In fact the symmetry generators in the lower sector are \(\sigma^j/2\) and \((\Lambda^1 \mathbf{1}_2, K^i \mathbf{1}_2)\), or equivalently \((J^i, \sigma^j/2, K^i \mathbf{1}_2)\), where \(J^i = \Lambda^1 \mathbf{1}_2 + \sigma^j/2\) is invariant with respect to the SUSY transformation \(U\), and (on the subspaces \(SS^\dagger = \omega^2 > 0\)) the SUSY partners of \(\sigma^j/2\) and \(K^i \mathbf{1}_2\) are calculated to be
\[
\Omega^i \equiv U \left( \frac{1}{2} \sigma^j \right) U^\dagger = \frac{1}{\omega^2} \left[ \frac{1}{2} (D^2 - D_0^2) \sigma^j - D_4 (\epsilon^{ijk} D_j \sigma_k) - (\sigma^k D_k) D^i \right],
\]
and
\[
U \left( K^i \mathbf{1}_2 \right) U^\dagger = K^i \mathbf{1}_2 - i \epsilon^{ijk} D^j \sigma^k + \left( \frac{q}{r} - \frac{q}{2} \right) \sigma^i - (\sigma^k B_k) x^i + q \Omega^i,
\]
and it is easy to verify that the D’Hoker-Vinet generators are linear combinations of these operators and \(J^i\). Note however that what D’Hoker-Vinet call the ‘Runge-Lenz’ vector is the transform of \(K^i \mathbf{1}_2 - q \sigma^j/2\), and not of \(K^i \mathbf{1}_2\). We should like to emphasize that the operator \(\Omega^j\) \((3.3)\) commutes with \(SS^\dagger\) in any self-dual background, and, moreover, the direct sum operator \(\Sigma^i = \Omega^i + \sigma^i/2\) commutes with the Dirac operator \(\slashed{D}\) itself. We shall consider the D’Hoker-Vinet \(su(2,2)\) supersymmetry in Sec. 5.

4 Fluctuations and generalized DYM supersymmetry

Let us next consider the question of (nonzero mode) fluctuations around a self-dual background. The fluctuation eigenmodes are \([2]\) transverse \(D_\mu (\delta A_\mu) = 0\) solutions of Eq. \((1.1)\), and what we wish to show is that, just like the DYM eigenfunctions, the eigenfunctions \(\delta A_\mu\) of \((1.1)\) (with \(\omega^2 \neq 0\)) are supersymmetric partners of the eigenfunctions of the scalar Laplacian \(D^2\). We shall show this by first generalizing the DYM supersymmetry \((1.2), (1.3)\) from Dirac spinors \((m = 0)\) to \(D^{(1/2,m)} \oplus D^{(m,1/2)}\) wave-functions for any \(m = 0, 1/2, 1, 3/2, \ldots\), then specializing the result to the 4-vector case \((m = 1/2)\).

The generalization is made by replacing the Dirac spinor representation space of \(SO(4), D^{(1/2,0)} \oplus D^{(0,1/2)}\), by the representation space \(D^{(1/2,m)} \oplus D^{(m,1/2)}\) and the operator \(S\) in \((1.2)\) by \(S : D^{(m,1/2)} \rightarrow D^{(1/2,m)}\) defined by
\[
S^{A,k'}_{n,B'} \equiv D_\mu (\sigma_\mu)_{B'}^{A'} \delta^k_n, \quad \mu = 1, \ldots, 4, \quad (\sigma_4)_{B'}^{A'} = -i \sigma_{B'}^{A'}, \quad (4.1)
\]
where \(A, B' = 1, 2\) and \(n, k' = 1, \ldots, 2m + 1\) are the usual left and right spinor indices for \(SO(4) \simeq SU(2) \times SU(2)\). Since \(S\) is given by a trivial matrix on the \(m\)-part of the representation space, the chiral projections of \(Q^2\) are similar to those in Eq. \((1.3)\), namely,
\[
(SS^\dagger)^{A,k'}_{B,n'} = -D^2 \delta^A_{B'} \delta^{k'}_{n'} - (B^i + E^i \sigma_4)_{B'}^{A'} \delta^{k'}_{n'}, \quad (4.2)
\]
\[
(S^\dagger S)^{k,A'}_{n,B'} = -D^2 \delta^k_n \delta^{A'}_{B'} - (B^i - E^i \sigma_4)_{B'}^{A'} \delta^k_n. \quad (4.3)
\]

In particular, in a self-dual background the spin couplings cancel in \(S^\dagger S\), so \(S^\dagger S\) becomes \(-D^2 \mathbf{1}_{4m+2}\), i.e., it essentially reduces to the scalar Laplacian \(D^2\). (Note that the Pauli matrices play completely different roles in Eqs. \((4.1)\) and \((1.2), (1.3)\), being intertwining operators in \((4.1)\) and group operators in \((1.2), (1.3)\).) For \(m = 1/2, D^{(1/2,m)}\) and \(D^{(m,1/2)}\) both become the 4-vector representations of \(SO(4)\), and in the usual vector basis \(S\) and \(SS^\dagger\) take the form
\[
S_{\mu \nu} = -iD_k (\eta^k_\mu)_{\nu} - iD_{4 \mu \nu}, \quad (4.4)
\]
\[
(SS^\dagger)_{\mu \nu} = -D^2 \delta_{\mu \nu} + i(B_k + E_k) (\eta^k_\mu)_{\nu}, \quad (4.5)
\]
respectively, where the \((\eta^k_\mu)\) are the usual ‘t Hooft matrices \([14]\), \((\eta^k_\mu)_{ij} = \epsilon_{ijk}, (\eta^k_\mu)_{jk} = \delta_{ij}\).

The important point for our considerations is that in a self-dual background the operator \((4.5)\)
is identical to the fluctuation operator \([-D^2\delta_{\mu\nu} + 2iF_{\mu\nu}\)] appearing on the left-hand side of (1.1). Hence for a self-dual background the fluctuation operator is the SUSY partner of \(-D^2\mathbb{1}_4\) and the formula
\[
(\delta A_{\mu})^{(e)} = \frac{i}{\omega}S_{\mu\nu}\phi, \quad \text{where} \quad -D^2\phi = \omega^2\phi, \quad \omega^2 \neq 0,
(4.6)
\]
provides us with four orthogonal fluctuation eigenmodes. (From the explicit expressions of the ’t Hooft matrices one sees that \((\delta A_{\mu})^{(e)}\) is a gauge mode, and the other three satisfy the transversality condition.) Formula (4.4) is known \([9, 10]\), but we thought it worth pointing out that it is a manifestation of the purely vectorial supersymmetry (4.4)-(4.5) and that the latter is just the \(m = \frac{1}{2}\) special case in the series of supersymmetries (4.2)-(4.3). Finally, we remark that for the BPS-monopole gauge potential (2.1) formula (4.6) produces bound vector-Higgs fluctuations with \(j = \lambda, \lambda \pm 1\) form any bound state of (2.5) with angular momentum \(\lambda = 1, 2, \ldots\), but only the \(j = \lambda - 1\) series seems to have been noticed in Ref. [3].

5 An su\((n/n)\) superalgebra

Since for self-dual gauge fields the operator \(S^\dagger S\) of (4.3) reduces to \(-D^2\mathbb{1}_n\), where \(n = 2(2m+1)\), it has an obvious su\((n)\) symmetry, and we wish to show that the generators of \(\Sigma^i\) of this su\((n)\) induce an su\((n/n)\) superalgebra which commutes with \(Q^2\). For this we first construct superpartner generators \(\Sigma^i_+ = USU^\dagger\) (given explicitly for \(\text{su}(2)\) in (3.4)) which, together with the \(\Sigma^i_-\), span an su\((n)\oplus\text{su}(n)\) Lie algebra that commutes with \(Q^2\). Note that since the \(\Sigma^i_-\) generate the defining representation of their respective su\((n)\)'s, they, and hence the diagonal generators \(\Sigma^i = \Sigma^i_+ + \Sigma^i_-\), close (modulo central terms) with respect to commutation and anticommutation:
\[
[\Sigma^i, \Sigma^j] = if_{ijk}\Sigma^k, \quad \{\Sigma^i, \Sigma^j\} = id_{ijk}\Sigma^k + c\delta_{ij}\mathbb{1}_{2n},
(5.1)
\]
where the \(f\)'s and \(d\)'s are structure constants in a trace-orthogonal basis. We then introduce the odd (i.e. anticommuting with \(\gamma_5\)) Hermitian quantities
\[
Q_1 \equiv Q, \quad Q_2 \equiv i\gamma_5Q_1, \quad \text{and} \quad Q^i_+ \equiv Q_\alpha\Sigma^i, \quad \alpha = 1, 2, \quad \gamma_5 = \begin{pmatrix} \mathbb{1}_n & -\mathbb{1}_n \\ -\mathbb{1}_n & \mathbb{1}_n \end{pmatrix},
(5.2)
\]
which are su\((n)\oplus\text{su}(n)\) scalars and vectors, respectively, and commute with \(Q^2\). Because of (5.1) the anti-commutators of the operators (5.2),
\[
\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}Q^2
\]
\[
\{Q_\alpha, Q^i_\beta\} = 2Q^2\delta_{\alpha\beta}\Sigma^i
\]
\[
\{Q_i^+, Q^j_+\} = Q^2\delta_{\alpha\beta}\{\Sigma^i, \Sigma^j\} - Q^2\epsilon_{\alpha\beta}[\Sigma^i, \Sigma^j](i\gamma_5), \quad (\alpha, \beta = 1, 2)
(5.3)
\]
are linear combinations of the \(\Sigma^i_+\) (times \(Q^2\)) and a central term. Thus, for each fixed nonzero eigenvalue of \(Q^2\), the even operators \(\Sigma^i_+\) and the odd ones (5.2) generate a superalgebra [15], and since there are \(2n^2\) linearly independent hermitian quantities in (5.2) it is clear that this superalgebra is su\((n/n)\).

Note that if \(S^\dagger S\) allows for an additional symmetry operator \(K\) which commutes with the \(\Sigma^i_-\), then the operator \(UKU^\dagger \oplus K\) will commute not only with \(Q\), but also with the whole su\((n/n)\) superalgebra constructed above.

For the Dirac Laplacian \(\not{D}^2\) (1.3) the superalgebra su\((n/n)\) reduces to su\((2/2)\), and it is easy to identify the su\((2/2)\) superalgebra found by D’Hoker and Vinet [8] as this superalgebra, in the special case when the self-dual gauge potential is (3.1). For the latter gauge potential
the operator $S^\dagger S$ admits also the Kepler symmetry discussed in Sec. 3 and its generators $\Lambda^i$ commute with $\Sigma^i = \sigma^i/2$. Therefore $\Lambda^i$ and $K^i$ give rise to symmetries of $\mathcal{H}$ which commute with the whole $\text{su}(2/2)$ superalgebra, as a special case of the situation mentioned in the previous paragraph. Thus, finally, the full symmetry algebra of $\mathcal{H}^2$ for the gauge-potential (3.1) is $\text{su}(2/2) \oplus \text{o}(4)$ for the bound states and $\text{su}(2/2) \oplus \text{o}(3,1)$ for the scattering states.

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