Implementation of a spatial two-dimensional quantum random walk with tunable decoherence

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We put forward a new, versatile and highly-scalable experimental setup for the realization of discrete two-dimensional quantum random walks with a single-qubit coin and tunable degree of decoherence. The proposed scheme makes use of a small number of simple optical components arranged in a multi-path Mach-Zehnder-like configuration, where a weak coherent state is injected. Environmental effects (decoherence) are generated by a spatial light modulator, which introduces pure dephasing in the transverse spatial plane, perpendicular to the direction of propagation of the light beam. By controlling the characteristics of this dephasing, one can explore a great variety of scenarios of quantum random walks: pure quantum evolution (ballistic spread), fast fluctuating environment leading to a diffusive classical random walk, and static disorder resulting in the observation of Anderson localization.

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I. INTRODUCTION

The underlying principles of many quantum information protocols can be traced back to the concept of quantum random walk (QRW), which since its first description have become a fundamental paradigm in quantum science \cite{a1, a2}. The idea of QRWs was originally conceived by Aharonov et al. \cite{a3} as an extension of the well-known classical random walk (CRW) \cite{a4}. The main distinguishing feature of a QRW, compare to a CRW, is the possibility of interference between the multiple paths that can be simultaneously traversed by a quantum walker, enabling thus a faster spreading of the uncertainty of location of the walker than in the classical case \cite{a5,a6}.

The temporal evolution of a quantum system, such as a QRW, depends on the presence, and specific characteristics, of the environmental effects (decoherence) which can modify it \cite{a7}. In most cases, the influence of decoherence during the evolution of a quantum walker transforms an originally pure state into a mixed state, lowering the uncertainty about the location of the walker as it propagates. In the limiting case, when all cross-interference terms between different lattice sites are completely erased, the state of pure diffusive classical propagation is reached \cite{a8}.

QRWs have been theoretically explored for the case of one-dimensional lattices \cite{a9,a10}, and experimentally implemented by means of different physical platforms, such as photon-based systems \cite{a11,a12}, optical lattices \cite{a13} and waveguide arrays \cite{a14}. Also, QRWs have been implemented using trapped ions \cite{a15} and nuclear magnetic resonance systems \cite{a16}.

Although the implementation of one-dimensional QRWs has showed to be useful when describing several quantum information systems, there is a great interest in expanding the concept to multidimensional lattices. Along these lines, two-dimensional QRWs provide a powerful tool for modeling complex quantum information and energy transport systems \cite{a17,a18}. Notwithstanding, their realization represents a challenge because of the need of a four-level coin operation \cite{a19,a20}. One way to overcome this drawback is to make use of different degrees of freedom of photons, such as polarization and orbital angular momentum, as it has been shown in \cite{a18}. Another approach is to mimic the two-dimensional QRWs evolution by performing two subsequent one-dimensional QRWs \cite{a21,a22}.

Here, we make use of the latter approach to put forward an experimental setup for the realization of two-dimensional QRWs. We include the environmental effects (decoherence) as pure dephasing by means of the introduction of random phase patterns, generated by a spatial light modulator (SLM), which can be different from site to site (spatial disorder). By controlling the degree of decoherence, we study the transition from the quantum ballistic spreading to the diffusive classical walk. Also, by adding static disorder, we show the possibility of observing Anderson localization \cite{a23}.

Importantly, our proposal provides a versatile, highly-scalable experimental setup, which may be used as a tool for understanding quantum processes whose underlying physics can be somehow traced to the concept of random walks, such as energy transport in photosynthetic light-harvesting complexes \cite{a24,a25} and material band gap structures \cite{a26}.

The structure of the article is as follows. In Sec. II, we introduce the QRW model system, including dephasing, considered here. The proposed experimental setup is described in Sec. III. Numerical results are discussed in Sec. IV. Finally, we summarize our results in the conclusion.
II. A TWO-DIMENSIONAL QUANTUM RANDOM WALK WITH DEPHASING

A typical discrete quantum random walk comprises two operations: a coin-tossing operation and a shift operation. Here, the coin-tossing operation is performed in the Hilbert space $\mathcal{H}_p$ spanned by vectors $\{|H\rangle, |V\rangle\}$, corresponding to the photon polarization. The random walk is performed in the Hilbert space $\mathcal{H}_X \otimes \mathcal{H}_Y$, corresponding to the position of the photon in the transverse plane, spanned by vectors $\{|i,j\rangle\} (i,j \text{ integers})$, which indicate sites $(i,j)$ in the transverse plane $(i,j = \ldots -2, -1, 0, 1, 2\ldots)$. The global quantum system thus evolve in the Hilbert space

$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_p.$$  \hfill (1)

The state of the system is described by the density matrix $\hat{\rho}^{(n)}$, which is transformed to a new density matrix each step $n$ via the map

$$\hat{\rho}^{(n+1)} = \hat{S}_Y \hat{H} \hat{S}_X \hat{H} \hat{\rho}^{(n)} \hat{H} \hat{S}_X \hat{H} \hat{S}_Y \hat{\rho}^{(n)} \hat{S}_Y \hat{H} \hat{S}_X \hat{H} \hat{\rho}^{(n)} \hat{S}_Y \hat{H} \hat{S}_X \hat{H} \hat{\rho}^{(n)}.$$  \hfill (2)

$\hat{H}$ denotes the Hadamard operator

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$  \hfill (3)

which acts on the polarization degree of freedom. The operators $\hat{S}_X$ and $\hat{S}_Y$, which describe the walker’s shift in the transverse dimensions $x$ and $y$, independently, read as

$$\hat{S}_X = \sum_{i,j} |i-1,j,H\rangle \langle i,j,H| + |i+1,j,V\rangle \langle i,j,V|,$$  \hfill (4)

and

$$\hat{S}_Y = \sum_{i,j} |i,j-1,H\rangle \langle i,j,H| + |i,j+1,V\rangle \langle i,j,V|.$$  \hfill (5)

The coupling of the quantum walker with the environment is described by pure dephasing $[31]$. The form of the unitary dephasing operator considered here can be written as

$$\hat{\rho}^{(n)} = \sum_{ij} e^{-i \phi_{ij}^{(n)} \sigma_z} |i,j\rangle \langle i,j|,$$  \hfill (6)

where $\phi_{ij}^{(n)}$ is a random phase matrix and $\sigma_z$ is the Pauli operator. Inspection of Eq. 6 shows that $\phi_{ij}^{(n)}$ represents a newly introduced phase difference between the horizontal and vertical polarizations at each site. Concerning this, we will consider three physically relevant scenarios, that can be easily implemented in the set-up proposed here (see Sec. III). In the general case, the phase differences $\phi_{ij}^{(n)}$ are independent random variables, but with the same probability distribution. Moreover, the ensemble of phase differences $\phi_{ij}^{(n)}$ can change from step $n$ to step $n+1$. In the following, we will refer to this case as a QWR influenced by dynamical spatial disorder.

The easiest probability distribution that we can consider is an uniform probability distribution. If phases can be chosen arbitrarily between the extreme values $-\zeta$ and $\zeta$, there is a constant probability $1/(2\zeta)$ to obtain any phase in this interval. $\zeta = \pi$ is the maximal phase which we can have between the two orthogonal polarizations. $\zeta = 0$ corresponds to the absence of any spatial disorder. If phases do not change during propagation, even though they might differ from site to site, i.e. $\phi_{ij}^{(n)} = \phi_{ij}^{(n+1)}$, then we have static spatial disorder. Finally, if all phase differences are the same for all sites, but they can still change from one step to the following, we have dynamical dephasing without spatial disorder.

The probability of detecting a photon in the site $(i, j)$ is

$$p^{(n)}(i,j) = \langle i,j| Tr_p[\hat{\rho}^{(n)}]|i,j\rangle,$$  \hfill (7)

where the density matrix that describes the whole system is traced out over the polarization degree of freedom ($Tr_p$).

The spreading of the uncertainty of photon location is characterized by dependence of the variance on the step index $n$

$$V^{(n)} = \sum_{i,j} p^{(n)}(i,j) |r_{ij} - \mu|^2.$$  \hfill (8)

where $r_{ij} = (i,j)$ represents the lattice site with indexes $(i,j)$ and $\mu$ is the mean position, i.e., $\mu = \sum_{i,j} p^{(n)}(i,j) r_{ij}$. 

![General scheme for the implementation of a two-dimensional random walk with decoherence](image.png)
where Gaussian beam of 2 mm beam waist, corresponding to a space due to diffraction. For instance, by making use of a Gaussian beam profile of the pulse has to be carefully cho-
the setup in one cycle. The transverse size of the Gaussian beam profile of the pulse has to be carefully cho-
another FL) then serve to return all beams back to the original direction of propagation, so that all beams remain in the same position in the traverse plane at all time.

IV. NUMERICAL RESULTS

A. Quantum random walk

Let us consider first the case when the SLM does not introduce any phase shift ($\phi_{ij}^{(n)} = 0$ for all $(i,j)$). This corresponds to the case $\zeta = 0$. Figure 2 shows the probability distribution function $p^{(n)}(i,j)$ for (a) $n = 10$ and (b) $n = 20$ steps. In both cases, the distribution shows a symmetrical shape around the lines $x = 0$ and $y = 0$, with four groups of peaks located along the $x$ and $y$ axes. The resulting symmetry comes from the specific initial quantum state chosen in Eq. (9). When the number of steps is increased, the peaks move further away from the central site $(0,0)$. The shapes obtained in Fig. 3 correspond to the probability distributions of a two-dimensional Grover walk [19, 25]. The walker propagates with ballistic speed, characterized by a quadratic dependence of the variance $V^{(n)}$ with the step index, i.e., $V^{(n)} \approx n^2$. This case is shown in Figs. 4(a) and 4(b), corresponding to the case with $\zeta = 0$.

B. Quantum random walk affected by dephasing

The dephasing effect introduced by the SLM allows to induce a transition from the quantum to the classical random walk via two mechanisms. First, as shown in Fig. 4(a), by means of dynamical spatial disorder. The phase-matrix $\phi_{ij}^{(n)}$ shows independent and randomly chosen values for each site, and it is refreshed each step. The case $\zeta = 0$ corresponds to the QRW with no dephasing. Increasing the amount of disorder, characterized by a corresponding increase of the parameter $\zeta$, reduce the spreading of $V^{(n)}$ as can be seen in Fig. 4(a). In the limiting case, which is reached for $\zeta = \pi$, the observed dependence ($\sim n^2$) of variance $V^{(n)}$ is a direct indication of the transition to the classical regime of random walks.

The classical limit can also be reached by means of dynamical dephasing without spatial disorder, as it is shown in Fig. 4(b). The reduction of the uncertainty of
FIG. 5. (Color online) Observation of the spatial Anderson localization. a) Example of a matrix $\phi^{(n)}_{ij}$ for $\zeta = \pi$ that leads to the Anderson localization. b) Corresponding probability distribution $p^{(n)}(i,j)$ after 20 steps. c) Averaged probability distribution over 500 realizations. d) Cuts of the data shown in Fig. 5(c) along the X and Y axes, to highlight this feature.

the photon position is less dramatic than in the case with dynamical spatial disorder. For the dynamical spatial disorder, $V^{(n)} \sim 51.25$ for $\zeta = \pi$ after 20 steps. On the contrary, for dynamical dephasing without spatial disorder, we have $V^{(n)} \sim 98.45$ under the same conditions. Indeed, the $n$-dependence of the typical deviation, characteristic of the classical regime, it is not yet reached after 20 steps, as is readily observed in Fig. 4(b).

C. Anderson localization

In the context of our discussion, the Anderson localization is the reduction of spreading of the uncertainty of the photon position [26]. We will demonstrate that this effect can also be observed in the set-up considered here. In [33], Anderson localization was observed in the transverse plane of a light beam passing through a crystal with random static fluctuations of the index of refraction. Since the randomness in the index of refraction is affecting only the phase of the propagating beam, it is possible to imitate these phase-fluctuations with a SLM, under the consideration of static spatial disorder, since Anderson localization does not appear with dynamical spatial disorder.

In Fig. 5(a) we present a typical profile of the phase-matrix $\phi^{(n)}_{ij}$, independent of $n$, which leads to beam localization. Fig. 5(b) shows the corresponding probability distribution of the photon position for this specific phase profile. Notice that it contains a strong peak located in the middle of the lattice. The presence of Anderson localization is confirmed in Fig. 5(c), where we show the averaged probability distribution function for $\zeta = \pi$, exhibiting an exponential suppression of probabilities for sites distant from the center. For the sake of clarity, we also plotted in Fig. 5(d) two cuts of the averaged probability distribution along the X and Y axes, to highlight this feature.

V. CONCLUSION

We have put forward a new, highly scalable and easily implemented experimental configuration to observe spatial two-dimensional random walks under a great variety of circumstances, by means of the implementation of two consecutive one-dimensional random walks. The proposal makes use of only a small amount of simple optics components, and allows us to simulate many different quantum systems and protocols based on the quantum random walk concept. Additionally, by carefully controlling the amount and type of disorder present in the system, we have shown the effects of different environmental effects: dynamical spatial disorder, dynamical dephasing without spatial disorder and static spatial disorder. The last case drove us to the observation of Anderson localization. The control of environmental effects is paramount importance in nearly all quantum systems. In some cases, it is even crucial to understand the dynamics experimentally observe. For instance, in light harvesting complexes
the interplay between coherent evolution and noise is a critical ingredient, which allows to obtain a high efficiency energy transport.

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