SUPERCONDUCTORS WITH MESOSCOPIC PHASE SEPARATION

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A model of superconductivity is proposed taking into account repulsive particle interaction, mesoscopic phase separation and softening of crystalline lattice. These features are typical of many high-temperature superconductors. The main results obtained for the model are: (i) phase separation is possible only if repulsive forces play a significant role; (ii) the critical temperature as a function of the superconducting phase fraction can have non-monotonic behaviour; (iii) superconductivity is possible in heterophase systems even when it would be forbidden in pure samples. These results are in agreement with experiments.

I. INTRODUCTION

In the standard theory of superconductivity one employs a reduced Hamiltonian involving only the attractive part of an effective particle interaction, responsible for the existence of superconductivity, and omitting the repulsive part of the interaction as irrelevant. This approach was adapted in the original paper of Bardeen, Cooper and Schrieffer [1]. The consideration of the Coulomb interaction leads to a renormalization of the coupling constant [1,2], which can be effectively included in the formula for the superconducting critical temperature [3]. In usual metals the averaged Coulomb interaction is small as compared to the coupling parameter mediated by phonon exchange. A detailed discussion of reasons why the repulsive interaction may be neglected has been given in a review by Legget [4].

These arguments concern the direct influence of repulsive interactions. However, they can affect superconductivity indirectly. An obvious example is that all interactions are included in the definition of the electronic energy bands and the corresponding density of states [5,6]. An interaction that is not directly responsible for the pairing mechanism, can, nevertheless, drastically strengthen the order by providing a high density of states at the Fermi surface, thus ensuring a large value of the effective coupling parameter, which, in turn, leads to a higher superconducting transition temperature. Such an increase of the density of states can occur, for instance, when a van Hove singularity is located almost precisely at the Fermi level [7].

It is necessary to keep in mind that it is Coulomb interactions, and not pairing interactions that define the space structure of matter. The electrical properties change drastically from one structure to another [5-8]. Moreover, the stability or instability of a crystalline (and chemical) structure is closely connected with superconductivity. This was noticed long before the discovery of the high-temperature superconductors. For example, Testardi [9] analysed a large number of superconducting A-15 compounds and concluded that superconductivity and structural instability are really related: the higher the averaged Coulomb interaction is small as compared to the coupling parameter mediated by phonon exchange. A detailed discussion of reasons why the repulsive interaction may be neglected has been given in a review by Legget [4].

The main result of structural instabilities is an essential softening of the lattice. This softening is manifested in various precursor effects occurring near the superconducting transition temperature, such as anomalies in elastic moduli, in strain parameters, and in sound velocity [9,11]. The softening of phonon modes can be observed by infrared, Raman, and neutron scattering experiments [11]. Nuclear gamma - resonance studies [12,13] also show that a dramatic lattice softening occurs at $T_c$ for high-temperature superconductors, the most noticeable result of which is an anomalous sagging of the Mössbauer factor.

The review of experimental data also provides evidence that the lattice softening and the increase in $T_c$ are both associated with the formation of phase precursor clusters with structural (spatial or chemical) composition different from the basic component [9,11]; the linear size of the clusters is of the order $10 - 100 \AA$. Such mixed structures exist in many ceramic superconductors, as has been observed by using pulsed-neutron scattering, synchrotron x-ray powder diffraction, nuclear magnetic resonance, nuclear quadrupole resonance, and nuclear gamma resonance
For instance, the low-temperature orthorhombic and high-temperature tetragonal structures coexist in oxide superconductors in a region around $T_c$ [15-17]. A more detailed discussion of these and other experiments can be found in a recent review [18]. The relation between the existence of soft vibrational modes and the appearance of clusters of a mesoscopic size has been also proved by molecular dynamics computer simulation for metastable glassy models [19,20].

The clusters of one phase structure inside another, occurring near a phase transition point, are randomly distributed in space, and often, being unstable, change with time. To emphasize the randomness of their distribution, they can be called heterostructural, or heterophase fluctuations, and to stress their mesoscopic sizes, they can be named mesoscopic structural fluctuations [18].

Since structural characteristics are intimately related to electronic and conducting properties, the existence of structural fluctuations implies the appearance of spatial fluctuations in the superconducting properties. Mesoscopic structural fluctuations are accompanied by large energy fluctuations which favour a stochastic separation of a sample into superconducting and normal regions [18,21]. There is clear experimental evidence that the superconducting cuprates display a nanoscale phase separation into insulating and superconducting nanodomains [7]. This is in agreement with some simple models [7,22-24] predicting that such a phase separation can be thermodynamically profitable. Recent experiments with high-temperature superconductors confirm that only part of a given sample - often less than a few percent - is in a superconducting phase (see [25-27] and references therein).

In the present paper we suggest a way of taking into account the three mutually interrelated factors: Coulomb interaction, phase separation, and lattice softening; all of which factors are very important in high-temperature superconductors. To clearly emphasize the influence of these factors as such we shall use the standard approximations accepted for superconductors. We give a detailed analysis of the superconducting critical temperature as a function of parameters related to the attracting and repulsive interactions and to the superconducting phase fraction.

Everywhere below we use the system of units with $\hbar \equiv k_B \equiv 1$.

II. PHASE SEPARATION

In order to take into account phase separation, we must admit such a possibility from the beginning, and then the corresponding phase probabilities must be defined in a self-consistent way. Assume that a phase separation has occurred so that our sample consists of regions occupied by two different phases numbered by the index $\nu = 1, 2$; the phase regions being randomly distributed in space. Assign the index $\nu = 1$ to the superconducting phase, and $\nu = 2$, to the normal phase. For each phase distribution, or phase configuration, the sample is nonuniform, which greatly complicates the consideration. However, assuming that each phase distribution is random, we may average observable quantities over these phase configurations. As a result, we come to a system described by a renormalized Hamiltonian that allows usual techniques supplemented with additional equations for phase probabilities and conditions of stability. The procedure of averaging over configurations has been expounded, in all necessary detail, in earlier papers [28-31] and discussed in a recent review [18]. Therefore, we shall not repeat it here, especially since the mathematical foundation is quite long and tedious even though the final result is rather simple.

The averaging over phase configurations, in the case of a random mixture of two phases, leads to the definition of a renormalized Hamiltonian

$$\bar{H} = H_1 \oplus H_2,$$

which is a direct sum of terms

$$H_\nu = w_\nu H^{\text{kin}}_\nu + w^2_\nu H^{\text{int}}_\nu,$$

in which $H^{\text{kin}}_\nu$ is a kinetic-energy operator, including external fields, if any, $H^{\text{int}}_\nu$ is an interaction-energy operator. The renormalizing factor $w_\nu$ is the geometric probability of the $\nu$-phase, that is, the ratio of the average volume occupied by the $\nu$-phase to the total volume of the sample. The Hamiltonian (2) is defined on the space

$$\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2,$$

which is a tensor product of Hilbert spaces, where $\mathcal{V}_\nu$ is the space of states typical of the $\nu$-phase. When $\nu = 1$ corresponds to a superconducting phase, and $\nu = 2$, to a normal phase, the state-spaces are chosen so that the so-called anomalous averages calculated with the states from $\mathcal{V}_1$ are not identically zero, while those calculated with the states from $\mathcal{V}_2$ are zero. In other words, the gaps related to the anomalous averages satisfy the condition

$$\Delta_1 \neq 0, \quad \Delta_2 = 0.$$
This condition can be interpreted in several ways. For example, we can always posit that each Hilbert space of microstates, $\mathcal{Y}_\nu$, is restricted to functions which are invariant under a symmetry group yielding desired properties for the order parameters [32]. Then, the order parameters are just the gaps in (4), and the corresponding microstates can, for example, be chosen as the BCS wave functions [1,33,34] for the superconducting phase and the usual Slater determinants for the normal phase. Another way to interpret the choice of the order parameters (4) is to connect them, and the values of the anomalous averages, with the off-diagonal long-range order of reduced density matrices [35]. The largest eigenvalues of the latter are also known [35-37] to be directly related to this off-diagonal order. A more detailed discussion of these relations can be found in reviews [38,39]. Such relations between microscopic and macroscopic characteristics can be described in a general way by introducing the notion of order indices [40,41].

If $\hat{\rho}^N_n$ is an $n$-particle reduced density matrix for a system of $N$ particles, then the order indices are defined as

$$
\alpha_n \equiv \lim_{N \to \infty} \frac{\log \|\hat{\rho}^N_n\|}{\log N}.
$$

It has been shown [41] that the index $\alpha_n$ satisfies the restriction

$$
0 \leq \alpha_n \leq n \quad (0 \leq n \leq N).
$$

Different thermodynamic phases correspond to different sets of order indices, so that a $\nu$-phase is characterized by a set $\{\alpha_n(\nu)\}$. Necessary and sufficient conditions that the phase with $\nu = 1$ is superconducting and the phase with $\nu = 2$ is normal are given by

$$
\alpha^{(1)}_{2n} = n, \quad \alpha^{(2)}_{2n} < n. \quad (5)
$$

Thus, there are several well-developed methods for defining thermodynamic phases. What we still need to define is how to find the geometric probabilities $w_\nu$ of the corresponding phases when considering a mixture described by the Hamiltonian (1).

According to the general procedure [18], the phase probabilities are given by the minimization of the thermodynamic potential

$$
f = -\frac{T}{N} \ln \text{Tr} e^{-\beta \hat{H}} \quad (\beta T = 1) \quad (6)
$$

under the condition

$$
w_1 + w_2 = 1, \quad 0 \leq w_\nu \leq 1, \quad (7)
$$

where $T$ is temperature. The normalization condition (7) can be taken into account explicitly by using the notation

$$
w_1 \equiv w \quad w_2 \equiv 1 - w. \quad (8)
$$

Then $w$ satisfies the equations

$$
\frac{\partial f}{\partial w} = 0, \quad \frac{\partial^2 f}{\partial w^2} > 0. \quad (9)
$$

The first of the equations in (9) is equivalent to

$$
\langle \frac{\partial \hat{H}}{\partial w} \rangle = 0. \quad (10)
$$

Substituting the Hamiltonian (1) and introducing the notation

$$
K_\nu \equiv \langle H^{\text{kin}}_\nu \rangle, \quad U_\nu \equiv \langle H^{\text{int}}_\nu \rangle, \quad (11)
$$

we obtain the equation

$$
w = \frac{2U_2 + K_2 - K_1}{2(U_1 + U_2)} \quad (12)
$$

for the probability of the superconducting phase. In case of thermodynamic phases with equal average densities, the probabilities $w_\nu$ coincide with the phase concentrations defined as the ratios of the number of particles in each phase.
to the total number of particles. The average density of the superconducting and of the normal phase are practically the same. Therefore we may speak of (12) as of the superconducting phase concentration.

The inequality in (9) shows when the phase separation in a sample is thermodynamically advantageous, as compared to a pure superconducting system. Taking account of (6), we find

\[
\left( \frac{\partial^2 \tilde{H}}{\partial w^2} \right) > \beta \left( \frac{\partial \tilde{H}}{\partial w} \right)^2.
\]  

(13)

In accordance with (7) and (12), we have

\[-2U_1 \leq K_1 - K_2 \leq 2U_2.\]  

(14)

Substituting the Hamiltonian \(\tilde{H}\), defined in (1) and (2), into the left-hand side of (13), we get

\[\langle \frac{\partial^2 \tilde{H}}{\partial w^2} \rangle = 2(U_1 + U_2).\]

Using this and noticing that the right-hand side of (13) is always non-negative, we obtain the inequality

\[U_1 + U_2 > 0.\]  

(15)

Thus, phase separation is possible only when there are sufficiently strong repulsive interactions in the system for (15) to be satisfied. In other words, inequality (15) is a necessary condition for phase separation. This general result is in agreement with Hubbard-model calculations for the copper-oxide superconductors [42]. According to these it is necessary to include a sufficiently strong nearest-neighbour Coulomb repulsion in order to produce phase separation.

A physical system will remain in a mixed state, with phase separation, as long as this is thermodynamically favourable compared with pure phases. The boundary, in the space of thermodynamic parameters, between mixed and pure phases is the set of points at which a necessary condition for phase stability is violated, - one of the conditions (13) or (14), for example. Experiment supports the usual assumption that transformation of a pure phase into a mixed phase begins with the appearance of nuclei of a competing phase within the pure phase. This process can be called nucleation and the point in the phase diagram at which this occurs, a nucleation point. At such a point, some thermodynamic and dynamic characteristics may display a singularity. This may occur, for example, in the density-density response function which is determined by the Fourier transform of the second-order density matrix. The behaviour of the thermodynamic and dynamic characteristics in the vicinity of such a phase transition is a separate problem which does not fall within the purview of the present paper. Rather, it is our object to delineate features which we regard as basic to any reasonable model of superconductors which exhibit mesoscopic phase separation.

### III. SPECIFICATION OF HAMILTONIAN

Let us now specify the Hamiltonian, defined in (1) and (2), keeping in mind that \(H_1\) corresponds to the superconducting phase; and \(H_2\), to the normal phase. This implies that the corresponding order parameters and order indices must satisfy conditions (4) and (5), respectively. These conditions can also be formulated for anomalous averages defined through the field operators, \(\psi_{\nu s}(\vec{r})\), where \(s\) denotes spin and \(\vec{r} \in \mathbb{R}^3\).

The anomalous averages

\[\langle \psi_{1s}(\vec{r})\psi_{1s}(\vec{r}') \rangle \neq 0\]  

(16)

for the first representation are not zero, while those for the second representation vanish,

\[\langle \psi_{2s}(\vec{r})\psi_{2s}(\vec{r}') \rangle \equiv 0.\]  

(17)

As is obvious, the conditions (16) and (17) are directly related to the corresponding properties of symmetry of microscopic states [4,18] or to the values (5) of the order indices [40,41].

The kinetic part of Hamiltonian (2) can be written in the form

\[H_{\nu}^{\text{kin}} = \int \sum_s \psi_{\nu s}(\vec{r}) \left[ K_{\nu}(\vec{r}, \vec{r}') - \mu \delta(\vec{r} - \vec{r}') \right] \psi_{\nu s}(\vec{r}') d\vec{r} dl \vec{r}',\]  

(18)
in which \( K_{\nu}(\vec{r}, \vec{r}') \) is a kinetic kernel including external scalar fields, if any, and \( \mu \) is the chemical potential. Take for the interaction part of (2) a general expression
\[
H_{\nu}^{\text{int}} = \frac{1}{2} \int \sum_{ss'} \psi^{\dagger}_{\nu s}(\vec{r}_1)\psi^{\dagger}_{\nu s'}(\vec{r}_2)V_{\nu}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4)\psi_{\nu s}(\vec{r}_3)\psi_{\nu s'}(\vec{r}_4) d\vec{r}_1 d\vec{r}_2 d\vec{r}_3 d\vec{r}_4,
\]
where \( V(\ldots) \) is an interacting vertex including all effective interactions, repulsive and attractive, direct and indirect. For this moment, we do not need to specify these interactions. However, as an example, we can think of direct interaction as a repulsive Coulomb force taking screening into account, and that indirect interactions are mediated by an exchange of phonons or other bosons.

In order not to complicate our discussion with the consequences of a variety of other known factors on the properties of superconductors, we limit ourselves to isotropic matter. Thus, we leave aside anisotropy effects and the related van Hove singularities [7]. For an isotropic system we can use the expansion
\[
\psi_{\nu s}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{k} c_{\nu s}(\vec{r}) e^{i\vec{k} \cdot \vec{r}},
\]
in which \( V \) is the total volume of the system.

In what follows, to simplify the notation, we shall accept the convention
\[
c_{1s}(\vec{k}) \equiv c_s(\vec{k}).
\]
Then, we shall deal mainly with expressions corresponding to the superconducting phase, since an identical treatment can be given for the normal phase. The difference between the phases can be taken into account at the final stage by invoking conditions (16) and (17).

Substituting (20) in (18), and using the property
\[
\frac{1}{V} \int e^{i(\vec{k} - \vec{k}') \cdot \vec{r}} d\vec{r} = \delta_{kk'} \equiv \begin{cases} 1, & k = k' \\ 0, & k \neq k' \end{cases},
\]
we have
\[
H_{1}^{\text{kin}} = \sum_{k,k',s} \left[ T_{1}(\vec{k}, \vec{k'}) - \mu \delta_{kk'} \right] c_{1s}(\vec{k})c_{s}(\vec{k'}),
\]
where the convention (21) is used and
\[
T_{\nu}(\vec{k}, \vec{k'}) \equiv \frac{1}{V} \int e^{-i\vec{k} \cdot \vec{r}} K_{\nu}(\vec{r}, \vec{r'}) e^{i\vec{k'} \cdot \vec{r'}} d\vec{r} d\vec{r'}
\]
is the transport matrix. For (19) we get
\[
H_{\nu}^{\text{int}} = \frac{1}{2V} \sum_{k_1, \ldots, k_4} \sum_{ss'} \Gamma_{1}(k_1, k_2, k_3, k_4)c_{1s}(k_1)c_{s}(k_2)c_{s}(k_3)c_{s}(k_4)
\]
with the vertex
\[
\Gamma_{1}(k_1, k_2, k_3, k_4) \equiv \frac{1}{V} \int V_{1}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) \times \\
\times \exp \left\{ -i k_1 r_1 -i k_2 r_2 +i k_3 r_3 +i k_4 r_4 \right\} d\vec{r}_1 d\vec{r}_2 d\vec{r}_3 d\vec{r}_4.
\]
approach as in the Blatt quasi-chemical equilibrium theory [47]. Or we may use the fundamental ansatz for the many-body wave function yielding what is called [38,39] the antisymmetrized geminal power. All these approaches are equivalent to one another [4,34,48,49]. To our mind, the main common idea lying behind these approaches can, in second-quantization language, be expressed by the operator approximation

\[ c_1^c c_2^c c_3 c_4 \approx c_1^c c_4 (c_2^c c_3) + (c_1^c c_4) c_2^c c_3 - (c_1^c c_4) (c_2^c c_3) - \\
- c_1^c c_3 (c_2^c c_4) + (c_1^c c_3) c_2^c c_4 + (c_1^c c_3) (c_2^c c_4) + \\
+ c_4^c c_1^c (c_2^c c_3) + (c_4^c c_1^c) c_2^c c_3 - (c_4^c c_1^c) (c_2^c c_3), \]  

(26)

where, for compactness, we write \( c_n \) instead of \( c_{s_n}(\vec{k}) \). The right-hand side of (26) may be called the operator antisymmetrized geminal power. Averaging (26), we obtain the familiar approximation for the correlation function

\[ \langle c_1^c c_2^c c_3 c_4 \rangle \approx \langle c_1^c c_4 \rangle \langle c_2^c c_3 \rangle - \langle c_1^c c_3 \rangle \langle c_2^c c_4 \rangle + \langle c_1^c c_4 \rangle \langle c_2^c c_3 \rangle \]

(27)

of the Hartree-Fock-Bogolubov form.

Applying (26) to (24), we present the latter as a sum

\[ H_1^{int} = H_1^{nor} + H_1^{sup} + B_1, \]  

(28)

in which the first term

\[ H_1^{nor} = \frac{1}{V} \sum_{k_1 \ldots k_4} \sum_{ss'} \Gamma(k_1, k_2, k_3, k_4) \times \\
\times \left[ c_4^c(n_1) c_s(n_2) \langle k_4 | c_s' \rangle (k_3) - c_1^c(n_1) c_s' \langle k_3 | c_s \rangle (k_4) \right] \]

(29)

has the normal Hartree-Fock form with

\[ \Gamma_a \left( k_1, k_2, k_3, k_4 \right) = \frac{1}{2} \left[ \Gamma_a(k_1, k_2, k_3, k_4) + \Gamma_a(k_2, k_1, k_3, k_4) \right]. \]

(30)

The second term in (28), that is,

\[ H_1^{sup} = \frac{1}{2V} \sum_{k_1 \ldots k_4} \sum_{ss'} \Gamma(k_1, k_2, k_3, k_4) \times \\
\times \left[ c_1^c(n_1) c_s' \langle k_2 | c_s \rangle (k_3) + c_s \langle k_1 | c_s' \rangle (k_2) c_s' \langle k_3 | c_s \rangle (k_4) \right], \]

(31)

contains anomalous averages and is, thus, responsible for superconductivity. The last term in (28) is

\[ B_1 = \frac{1}{2} \sum_{k_1 \ldots k_4} \sum_{ss'} \Gamma(k_1, k_2, k_3, k_4) \times \\
\times \left[ \langle c_1^c(n_1) c_s' \rangle (k_2) - \langle c_1^c(n_1) \rangle (k_2) - \langle c_s' \rangle (k_3) - \langle c_s \rangle (k_4) \right]. \]

(32)

The Hamiltonian defined by (1), together with (2), (18) and (19), describes a system in which the total momentum and spin are conserved. Therefore

\[ \langle c_s^c(n) c_s' \rangle = \delta_{kk'} \delta_{ss'} \langle c_s^c(n) \rangle c_s(n), \]

\[ \langle c_s^c(n) c_s' \rangle = \delta_{-kk'} \delta_{-ss'} \langle c_s^c(n) \rangle c_s(-n). \]

(33)
If we denote the up and down spins as $s = \uparrow, \downarrow$, then in (33) for $s = \uparrow$ the notation $-s$ means $\downarrow$, and for $s = \downarrow, -s$ means $\uparrow$. For the normal average introduce the notation
\[ n_1(\vec{k}) \equiv \sum_s n_{1s}(\vec{k}); \quad n_{1s}(\vec{k}) = \langle c_s^\dagger(\vec{k}) c_s(\vec{k}) \rangle, \]
(34)
and for the anomalous average,
\[ \sigma_1(\vec{k}) \equiv \langle c_{-s}(-\vec{k}) c_s(\vec{k}) \rangle. \]
(35)

The function $n_\nu(\vec{k})$ is a momentum distribution of particles. Since the transport matrix (23) and the vertex (25) do not depend on spin, we have
\[ n_\nu(\vec{k}) = 2n_{\nu s}(\vec{k}). \]
(36)

The normal term (29), with the use of (33) and (34), can be written as
\[ H_1^{\text{nor}} = \sum_{kk',s} M_1(\vec{k}, \vec{k}'; \vec{k}, \vec{k}') c_s^\dagger(\vec{k}) c_s(\vec{k}'), \]
(37)
where $M_\nu$ is a mass operator with kernel
\[ M_\nu(\vec{k}, \vec{k}') = \frac{1}{V} \sum_p \left[ \Gamma_\nu(\vec{k}, \vec{p}, \vec{p}, \vec{k}') - \frac{1}{2} \Gamma_\nu(\vec{k}, \vec{p}, \vec{k}', \vec{p}) \right] n_\nu(\vec{p}). \]
(38)

By using (33) and (35), the superconducting term (31) becomes
\[ H_1^{\text{sup}} = \frac{1}{2} \sum_{kk',\vec{p}} \Gamma_1(\vec{k}, \vec{k}', -\vec{p}, \vec{p}) \times \left[ c_s^\dagger(\vec{k}) c_{-s}^\dagger(\vec{k}') \sigma_1(\vec{p}) + c_s(\vec{k}) c_{-s}(\vec{k}') \sigma_1(\vec{p}) \right], \]
(39)

And the scalar term (32) simplifies to
\[ B_1 = -\frac{1}{2} \sum_k M_1(\vec{k}, \vec{k}) n_1(\vec{k}) - \frac{1}{V} \sum_{kp} \Gamma_1(\vec{k}, -\vec{k}, -\vec{p}, \vec{p}) \sigma_1(\vec{k}) \sigma_1(\vec{p}). \]
(40)

Notice that the necessity of an accurate and detailed analysis of all transformations presented here is dictated by our main goal which is to consider the role of all terms of the Hamiltonian not omitting any of them. As we have already shown and shall see in what follows, all these terms, repulsive as well as attractive, play a crucial role for superconductors with phase separation.

### IV. GAP EQUATION

The sole approximation we have invoked so far now is the operator geminal power form (26), which is equivalent to the Hartree - Fock - Bogolubov approximation or to the antisymmetrized geminal power approach. The structure of the resulting Hamiltonian is still too complex to permit specific conclusions. Actually, a group theoretical analysis by Ozaki [50] of the Hartree - Fock - Bogolubov approximation, enumerated 73 possible ordered states, including 26 superconducting states! The latter include non-Cooper pairing, when an electron pair has nonzero momentum, and various superconducting states coexisting with other nonsuperconducting orders.

To make the problem tractable, we can resort to an approximation that restricts the space of microscopic states to a subspace satisfying some additional constraints [48,51,52]. In our case a natural such constraint is to consider
only those quantum states that conserve momenta, that is, when the momentum conservation occurs not only on the average, as in (33), but for all operator combinations:

\[ c_s^\dagger(\mathbf{k}) c_s(\mathbf{k}) = \delta_{kk'} c_s^\dagger(\mathbf{k}) c_s(\mathbf{k}) , \]

\[ c_s^\dagger(\mathbf{k}) c_{s'}^\dagger(\mathbf{k}) = \delta_{kk'} c_s^\dagger(\mathbf{k}) c_{s'}(\mathbf{k}) . \]  

(41)

Such a restricted space consists of BCS wave functions that are a particular kind of antisymmetric geminal power functions [34,53].

The restriction (41) makes it possible to greatly simplify all formulas. To this end, let us introduce the single-particle spectrum

\[ \varepsilon\nu(\mathbf{k}) \equiv T\nu(\mathbf{k}, \mathbf{k}) , \]

(42)

the diagonal mass-operator

\[ M\nu(\mathbf{k}) \equiv M\nu(\mathbf{k}, \mathbf{k}) \]

(43)

and an effective interaction

\[ J\nu(\mathbf{k}, \mathbf{p}) \equiv \Gamma\nu(\mathbf{k}, -\mathbf{k}, -\mathbf{p}, \mathbf{p}) . \]  

(44)

Taking account of (41) and (42) - (44) we obtain for the kinetic term (22)

\[ H_1^{\text{kin}} = \sum_{k,s} \left[ \varepsilon_{1}(\mathbf{k}) - \mu \right] c_s^\dagger(\mathbf{k}) c_s(\mathbf{k}) , \]

(45)

for the normal term (37)

\[ H_1^{\text{nor}} = \sum_{k,s} M_{1}(\mathbf{k}) c_s^\dagger(\mathbf{k}) c_s(\mathbf{k}) , \]

(46)

and for the superconducting term (39)

\[ H_1^{\text{sup}} = \frac{1}{2V} \sum_{kp,s} J_1(\mathbf{k}, \mathbf{p}) \left[ c_s^\dagger(\mathbf{k}) c_{s'}^\dagger(-\mathbf{k}) \sigma_1(\mathbf{p}) + c_s(\mathbf{k}) c_{-s}(\mathbf{k}) \sigma_1(\mathbf{p}) \right] . \]  

(47)

The scalar term (40) becomes

\[ B\nu = -\frac{1}{2} \sum_{k} M_{\nu}(\mathbf{k}) n_{\nu}(\mathbf{k}) \frac{1}{V} \sum_{kp} J_{\nu}(\mathbf{k}, \mathbf{p}) \sigma_\nu(\mathbf{k}) \sigma_\nu(\mathbf{p}) . \]  

(48)

Finally, collecting all these terms, for the Hamiltonian of the superconducting phase, given in (2), we obtain the expression

\[ H_1 = w_1 \sum_{k} \sum_{s} \omega_{1}(\mathbf{k}) c_s^\dagger(\mathbf{k}) c_s(\mathbf{k}) + w_2^2 H_1^{\text{sup}} + w_1^2 B_1 , \]

(49)

in which \( H_1^{\text{sup}} \) and \( B_1 \) are defined by (47) and (48), and

\[ \omega_{\nu}(\mathbf{k}) \equiv \varepsilon_{\nu}(\mathbf{k}) - \mu + w_{\nu} M_{\nu}(\mathbf{k}) \]

(50)

plays the role of an effective spectrum, renormalized, as compared to the single-particle spectrum (42), by the presence of the mass operator (43).

The Hamiltonian (49) can be diagonalized by the Bogolubov canonical transformation
in which both $c_s$ and $a_s$ satisfy the Fermi commutation relations. Diagonalization is achieved with

$$|u(k)|^2 = \frac{1}{2} \left| 1 + \frac{\omega_1(k)}{E_1(k)} \right|, \quad |v(k)|^2 = \frac{1}{2} \left| 1 - \frac{\omega_1(k)}{E_1(k)} \right|$$

leading to the Hamiltonian

$$H_1 = w_1 \sum_{k,s} E_1(k) \frac{1}{2} a_s \rightarrow k a_s \rightarrow k + w_1 C_1,$$

where the quasiparticle spectrum

$$E_1^2(k) \equiv \Delta_1^2(k) + \omega_1^2(k)$$

contains the gap

$$\Delta_1(k) \equiv -\frac{w_1}{V} \sum_p J_1(k, \rightarrow p) \sigma_1(\rightarrow p).$$

For the scalar part of (52) we have

$$c_\nu \equiv \sum_k \left[ \omega_\nu(k) - E_\nu(k) + \Delta_\nu(k) \sigma_\nu(k) - \frac{w_\nu}{2} M_\nu(k) n_\nu(k) \right].$$

With the diagonal Hamiltonian (52) it is straightforward to calculate the momentum distribution (34),

$$n_1(k) = 1 - \frac{\omega_1(k)}{E_1(k)} \tanh \frac{w_1 E_1(k)}{2T},$$

and the anomalous averages (35),

$$\sigma_1(k) = \frac{\Delta_1(k)}{2E_1(k)} \tanh \frac{w_1 E_1(k)}{2T}.$$

So, for the gap (54) we obtain the equation

$$\Delta_1(k) = -\frac{w_1}{V} \sum_p J_1(k, \rightarrow p) \frac{\Delta_1(\rightarrow p)}{2E_1(\rightarrow p)} \tanh \frac{w_1 E_1(\rightarrow p)}{2T}.$$

Let us emphasize that in the case of a superconductor with phase separation, with which we are dealing, all formulas obtained above involve the phase probabilities $w_\nu$ in an intricate way. And it is very important where and how the latter enter into the expressions. Even though the technical approximations we have adopted are standard, it is essential to accurately trace the role of the phase probabilities since it is these which distinguish the heterophase from pure superconductor. Also, we have to carefully take into account all interactions since, as has been shown, the presence of repulsive interactions is decisive for the phase separation itself.

For the superconducting phase we need a nontrivial solution of (58). For the normal phase according to condition (17) instead of (56) we have

$$n_2(k) = 1 - \tanh \frac{w_2 \omega_2(k)}{2T} = \frac{2}{\exp[w_2 \omega_2(k)/T] + 1},$$

and instead of (57) and (58) we have

$$\sigma_2(k) = 0, \quad \Delta_2(k) = 0,$$

with the effective spectrum, $\omega_2(k)$, given by (50).
V. INTERACTION POTENTIALS

It follows from the previous Section that all characteristics of the system can be calculated provided we know the mass operator (43) and the effective interaction (44). These quantities are not independent of each other since they are both defined, by means of (38) and (44), through the vertex (25). If the real-space vertex entering into the hamiltonian (19), describes an interaction between particles, which is, as usual, invariant with respect to permutation of particles and space reflections, then the momentum-space vertex (25) has the properties [2]:

\[
\Gamma^\nu(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = \Gamma^\nu(\vec{k}_2, \vec{k}_1, \vec{k}_4, \vec{k}_3) = \Gamma^\nu(-\vec{k}_1, -\vec{k}_2, -\vec{k}_3, -\vec{k}_4) = \Gamma^\nu(\vec{k}_1, -\vec{k}_3, -\vec{k}_2, \vec{k}_4) = \Gamma^\nu(\vec{k}_3, \vec{k}_4, \vec{k}_1, \vec{k}_2).
\]

With these properties, the symmetrized vertex (30) coincides with (25),

\[
\tilde{\Gamma}^\nu(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = \Gamma^\nu(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4).
\]

It then follows from (38),(43) and (44) that

\[
M^\nu(\vec{k}) = \frac{1}{V} \sum_p n^\nu(\vec{p}) \left[ J^\nu(\vec{k}, \vec{p}) - \frac{1}{2} J^\nu(\vec{p}, \vec{k}) \right].
\]

(61)

Thus, the mass operator (43) is completely defined by the effective interaction (44) by the relation (61).

As we have emphasized, we have to keep in the Hamiltonian (19) all interactions, direct and indirect. Therefore, the vertex (25) can be written as a sum \( \Gamma^\nu = \Gamma^\nu_{\text{dir}} + \Gamma^\nu_{\text{ind}} \) of direct and indirect interactions. Hence, the effective interaction (44) is a sum

\[
J^\nu(\vec{k}, \vec{p}) = J^\nu_{\text{dir}}(\vec{k}, \vec{p}) + J^\nu_{\text{ind}}(\vec{k}, \vec{p})
\]

(62)

of direct and indirect interactions.

In principle, we could continue our analysis employing only the occurrence of the separation (62) of the effective interaction into two terms, without specifying their nature. For example, as interacting particles we could think of electrons or of atoms of \(^3\text{He}\) or of nucleons, taking for the real-space direct interactions the Coulomb, Lennard-Jones or Yukawa potentials, respectively. The indirect interaction might be assumed to be mediated by a boson-exchange mechanism involving phonons or excitons or something else.

However, we prefer to be more concrete, so that in what follows, we shall think of particles as electrons whose direct interaction is described by a screened Coulomb potential and indirect interaction induced by phonon exchange. Generally, the screened Coulomb potential depends on the properties of a phase inside which electrons interact, since the screening is described by an inverse dielectric function reflecting the features of matter [54,55]. The simplest form of screening is obtained by using the Thomas-Fermi approximation which is equivalent to the static approximation for the Lindhard dielectric function [54,55]. Then, in real space the screened Coulomb interaction of particles immersed in the \(\nu\)-phase has the form

\[
\Phi^\nu(r) = \frac{e^2}{r} e^{-\kappa^\nu r},
\]

(63)
in which \(e\) is the charge, \(\kappa^{-1}\) is a screening radius with

\[
\kappa^2 = \frac{4}{a_B} \left( \frac{3}{\pi} n^\nu \right)^{1/3} \left( a_B \equiv \frac{1}{m e^2} \right),
\]

where \(m\) is the electron mass and \(n^\nu\), the electron density in the \(\nu\)-phase. In momentum space, (63) yields

\[
v^\nu(k) = \frac{4\pi e^2}{k^2 + \kappa^2}.
\]

(64)
Recall that the particular form (64) is not of great importance for our argument: for example, we could take for \( v_\nu(k) \) the more general expression \((4\pi/k^2)e^{-1}(k,\omega_{\nu}^{\text{exc}}(k))\), in which \( \varepsilon_\nu(k,\omega) \) is a dielectric function and \( \omega_{\nu}^{\text{exc}}(k) \), a spectrum of elementary excitations in the \( \nu \)-phase. Also, for the dielectric function we could invoke a more refined approximation [55-60]. Here we have chosen the simple form (64) in order to illustrate how the screening can depend on the phase properties. In any case, whether \( v_\nu(k) \) is given by (64) or by a more complicated expression, the part of (62) related to the direct interaction is

\[
J^{\text{dir}}_\nu(\rightarrow k) = \int \Phi_\nu(r) e^{-i(\rightarrow k - \rightarrow p) \cdot r} d \rightarrow r = v_\nu \left( | \rightarrow k - \rightarrow p | \right). \tag{65}
\]

For the effective indirect interaction we take one mediated by phonon exchange in the form [55]

\[
J^{\text{ind}}_\nu(\rightarrow k, \rightarrow p) = -\frac{|\alpha_\nu|^2}{\omega_{\nu0}^2 - [\omega_\nu(k) - \omega_\nu(\rightarrow p)]^2}, \tag{66}
\]

where \( \omega_{\nu0} \) is a characteristic phonon frequency, \( \alpha_\nu \) is the electron - phonon coupling taking account of screening,

\[
\alpha_\nu = -\frac{4\pi i Z e^2}{k_{\nu F}(1 + n^2/\rho_{\nu M}) \left( \rho_{\nu M} \right)^{1/2}},
\]

\( Z \) and \( M \) are an ion charge and mass, respectively, \( \rho_{\nu} \) is the density of ions in the \( \nu \)-phase, and \( k_{\nu F} \approx (3\pi^2 n_{\nu})^{1/3} \) is the Fermi momentum of an electron.

The presentation (62) of the effective interaction as a sum of direct and indirect interactions leads to a similar decomposition

\[
M_\nu(\rightarrow k) = M^{\text{dir}}_\nu(\rightarrow k) + M^{\text{ind}}_\nu(\rightarrow k). \tag{67}
\]

of the mass operator (61). The first term in (67) is related to (65) giving

\[
M^{\text{dir}}_\nu(\rightarrow k) = n_\nu v_\nu(0) - \frac{1}{2V} \sum_\nu \sum_\nu n_\nu(\rightarrow p) v_\nu(\rightarrow k - \rightarrow p), \tag{68}
\]

while the second term in (67) is expressed through (66) yielding

\[
M^{\text{ind}}_\nu(\rightarrow k) = -n_\nu \frac{|\alpha_\nu|^2}{\omega_{\nu0}} - \frac{1}{2V} \sum_\nu \sum_\nu n_\nu(\rightarrow p) J^{\text{ind}}_\nu(\rightarrow k, \rightarrow p), \tag{69}
\]

where

\[
n_\nu \equiv \frac{1}{V} \sum_\nu n_\nu(\rightarrow p) \tag{70}
\]

is the electron density in the \( \nu \)-phase.

The chemical potential, as a function of temperature and density, is defined by the formula

\[
N = -\left( \frac{\partial F}{\partial \mu} \right) = \sum_\nu \int \psi_\nu^\dagger(r) \psi_\nu(r) d \rightarrow r \tag{71}
\]

for the total number of electrons in the system. Eq.(71), with notation (34) and (36), gives

\[
N = \sum_\nu \left[ w_1 n_1(\rightarrow p) + w_2 n_2(\rightarrow p) \right]. \tag{72}
\]

Using (70), we can reduce (72) to the equation

\[
n \equiv \frac{N}{V} = w_1 n_1 + w_2 n_2 \tag{73}
\]
connecting the average electron density in the system with the electron densities in each of the phases which the system is composed of and with the corresponding phase probabilities. Note that when the electron densities in both phases coincide with each other, that is when \( n_\nu = n \), then (73) reduces to (7).

Considering the gap equation (58) we pass to the thermodynamic limit with the standard replacement \( \frac{1}{V} \sum_p \to \int \frac{d^3 p}{(2\pi)^3} \). The main contribution to the resulting integral in the right-hand side of (58) comes from momenta close to the Fermi surface which is defined by the condition

\[
\omega_1(\vec{k}_F) = 0. \tag{74}
\]

Keeping this in mind, we can rewrite the gap equation (58) as

\[
\Delta_1(\vec{k}) = \left( \frac{|\alpha_1|^2}{\omega_{10}^2 - \omega_1^2(k)} - \bar{v}_1(k) \right) \times \int \frac{\Delta_1(\vec{p})}{\sqrt{\Delta_1^2(\vec{p}) + \omega_1^2(\vec{p})}} \tanh \frac{w_1 \sqrt{\Delta_1^2(\vec{p}) + \omega_1^2(\vec{p})}}{2T} \frac{d\vec{p}}{(2\pi)^3}, \tag{75}
\]

where

\[
\bar{v}_1(k) = \lim_{p \to \vec{k}_F} \frac{1}{4\pi} \int v_1(|\vec{k} - \vec{p}|)d\Omega(\vec{p}) = \frac{\pi e^2}{kk_F} \ln \left| \frac{(k + k_F)^2 + \kappa_1^2}{(k - k_F)^2 + \kappa_1^2} \right| \tag{76}
\]

is the screened Coulomb interaction averaged over spherical angles.

A necessary condition that (75) has a nonzero solution for the gap \( \Delta_1(\vec{p}) \) is

\[
\frac{|\alpha_1|^2}{\omega_{10}^2 - \omega_1^2(k)} - \bar{v}_1(k) > 0, \tag{77}
\]

which occurs in the vicinity of the Fermi surface so that \( \omega_1^2(k) < \omega_{10}^2 \). Actually, inequality (77) has almost the same form as the BCS criterion for superconductivity [1]. However in our case the quantities entering into (77) depend on the superconducting phase probability. This dependence is essential and as we show below can dramatically change the characteristics of superconductors with phase separation making the existence of superconductivity possible even for values of the parameters which in a pure sample would imply the absence of superconductivity.

Equation (75) can be simplified by considering the value of the gap at the Fermi surface, i.e.

\[
\Delta \equiv \lim_{k \to k_F} \Delta_1(\vec{k}), \tag{78}
\]

and by introducing the level density per spin,

\[
N_1(\omega) \equiv \frac{1}{(2\pi)^3} \int \frac{dS(\omega)}{|\nabla_\omega \omega_1(k)|}, \tag{79}
\]

where the integration is over the surface given by the equation \( \omega_1(\vec{k}) = \omega \). Then (75) yields

\[
\int_0^{\omega_{10}} \frac{\lambda_{eff}}{\sqrt{\Delta^2 + \omega^2}} \tanh \frac{w \sqrt{\Delta^2 + \omega^2}}{2T} d\omega = 1, \tag{80}
\]

where the effective coupling parameter is

\[
\lambda_{eff} \equiv w N_1(0) \left[ \frac{|\alpha_1|^2}{\omega_{10}^2} - \bar{v}_1(k_F) \right], \tag{81}
\]

and

\[
\omega_1(\vec{k}_F) = 0. \tag{82}
\]
$N_1(0)$ being the level density (78) at the Fermi surface defined by (74).

The criterion (77) implies that for superconductivity the effective coupling parameter (80) is positive. For metals one has $\kappa_1 \sim k_F$ and $\bar{v}_1 (k_F) \sim \pi e^2/k_F^2$. Therefore in this case we have the estimate

$$\frac{\bar{v}_1 (k_F)}{|\alpha_1|^2/\omega_{10}^2} \sim \frac{\omega_{10}^2}{\Omega_p^2}, \quad (\Omega_p^2 \equiv \frac{4\pi M_1 Z^2 e^2}{M}),$$

in which $\Omega_p$ is the ion plasma frequency. The condition $\lambda_{eff} > 0$ means that $\omega_{10} < \Omega_p$. The latter inequality makes it possible to understand why structural instabilities and the related softening of the lattice, typical of high - temperature superconductors, favour the appearance of superconductivity. Lattice softening is associated with a decrease in the characteristic phonon frequency $\omega_{10}$, this makes it easy to satisfy the condition $\omega_{10} < \Omega_p$ and favours the onset of superconductivity. However, the latter occurs not in the whole volume of a sample, but only in those parts of it that are occupied by the superconducting phase. This is why lattice softening, enhancement of superconductivity and phase separation are phenomena that are intimately related to one another. This conclusion is supported as well by the consideration of s Hubbard - type model for which it has been found [42] that superconducting correlations are enhanced in the phase - separation regions.

**VI. CRITICAL TEMPERATURE**

The superconducting phase transition occurs at the critical temperature $T_c$ where $\Delta = 0$. The equation for $T_c$ follows from (79):

$$\lambda_{eff} \int_0^{\omega_{10}} \frac{1}{\omega} \tanh \frac{\omega}{2T_c} d\omega = 1. \quad (81)$$

Recall that here $w(T_c)$ is the geometric probability of the superconducting phase, that is, the relative part of volume occupied by this phase. The upper limit in the integral (81) is the characteristic phonon frequency exhibiting softening because of structural fluctuations and the related phase separation [18,61]. The simultaneous occurrence of lattice softening and phase separation can be clearly observed by using Mössbauer spectroscopy [62,63]. In high - temperature superconductors these effects lead to anomalous saggings of the Mössbauer factor at critical temperature [12,13,64]. The softening of phonon frequencies at $T_c$ can also be observed by other experimental methods such as infrared, Raman and neutron scattering [11]. The softening of the characteristic frequency $\omega_{10}$, as was shown in [61,63], can be expressed by the relation

$$\omega_{10} = w^\varphi \omega_0, \quad (82)$$

in which $\omega_0$ is a characteristic phonon frequency of a pure superconductor, without phase separation, and the parameter $\varphi$ measures the intensity of softening. Weak softening corresponds to $\varphi = 1/2$, moderate, to $\varphi = 1$, and the strong softening, to $\varphi = 3/2$.

For superconductors with phase separation the dependence of the critical temperature $T_c$ on the superconducting - phase probability $w$ is given by an intricate relation through (80) - (82). If one interprets the onset of superconductivity as the condensation of Cooper pairs, then one can call $w$ the concentration of superconducting condensates. No matter what it is called, the main point is that this quantity can be varied in high - temperature superconductors by varying their chemical structure, for example by doping. The experimentally measured dependence of $T_c$ on $w$ exhibits, for some high - temperature superconductors, a quite unusual nonmonotonic behaviour, as is reviewed in refs.[65,66]. For this reason it is especially interesting to study the dependence of $T_c$ on $w$.

Formula (81) allows us to obtain two asymptotic expressions. One limit corresponds to the case

$$T_c \ll \frac{\omega_0}{\pi} w^{1+\varphi}, \quad \lambda_{eff} \ll 1, \quad (83)$$

when

$$T_c \simeq 1.134 w^{1+\varphi} \omega_0 \exp \left( -\frac{1}{\lambda_{eff}} \right). \quad (84)$$

Note, that if we consider $\omega_{eff} = w^{1+\varphi} \omega_0$ and $\lambda_{eff}$ as fitting parameters, then expression (84), as is known [67,68], describes the majority of low - temperature superconductors quite well.
Another limiting case, opposite to (83), is when
\[ T_c \gg \omega_0 \pi w^{1+\varphi}, \quad \lambda_{\text{eff}} \gg 1. \] (85)

Then (81) gives
\[ T_c \simeq \frac{1}{2} w^{1+\varphi} \lambda_{\text{eff}} \omega_0. \] (86)

There is a temptation to treat (85) as the strong coupling limit. In doing this, we should be very cautious and not forget that \( \lambda_{\text{eff}} \), given by (80), is an effective coupling parameter. So, it may happen that \( \lambda_{\text{eff}} \) is large, even if the bare coupling constant, \( \lambda \) is not. If \( \lambda \gg 1 \), then by a standard argument Eliashberg equations imply that \( T_c \sim \sqrt{\lambda} \), although the dependence \( T_c \sim \lambda \), as is stated in [69], is also consistent with the strong coupling limit of these equations.

The involvement of the superconducting phase concentration in the definition of the effective coupling parameter (80) makes the applicability of such simple asymptotic expressions, as (84) and (86), to high-temperature superconductors with phase separation quite limited.

We therefore attempt an accurate analysis of the dependence of \( T_c \) on \( w \). First, we have to remember that the level density (78) also depends on \( w \) through the effective spectrum (50) renormalized by the mass operator (67). Consider the isotropic case leaving aside the very interesting, but separate problem of van Hove singularities [7]. Then, differentiating the effective spectrum (50), we find
\[ \lim_{k \to k_F} \left| \nabla_{k \omega_1}(k) \right| = \varepsilon'_{F} + wM'_F, \] (87)

where
\[ \varepsilon'_{F} \equiv \lim_{k \to k_F} \frac{\partial}{\partial k} \varepsilon_1(k), \quad M'_F \equiv \lim_{k \to k_F} \frac{\partial}{\partial k} M_1(k). \]

The level density (78) at the Fermi surface becomes
\[ N_1(0) = \frac{N_1^*(0)}{|1 + \gamma w|}, \] (88)

where we use the notation
\[ N_1^*(0) = \frac{k_F^2}{2\pi^2|\varepsilon'_{F}|}, \quad \gamma \equiv \frac{M'_F}{\varepsilon'_{F}}. \] (89)

To estimate the value of \( M'_F \), take into account that \( n_1 \approx k_F^3/3\pi^2 \) and \( \kappa_1 \approx k_F \), then \( M'_F \approx 4e^2/3\pi \). For a parabolic zone \( \varepsilon'_{F} \approx k_F/m \), so that for the parameter \( \gamma \) in (89) we get \( \gamma \sim a_e/\pi^2a_B \), where \( a_e \) is the average distance between electrons, and \( a_B \) is the Bohr radius. In good conductors \( a_e \sim a_B \), whence \( \gamma \sim 0.1 \). This means that in good metals the renormalization of the spectrum, due to the mass operator, is quite weak.

In bad conductors with low electron density one has \( a_e \gg a_B \). Therefore \( \gamma \) may equal 1 or more. This makes the role of the mass operator in renormalizing the electron spectrum and consequently in influencing the level density (88) very important. Such a situation is directly related to high-temperature superconductors which are, as is known, bad conductors having a low density of carriers.

To distinguish the effects due to phase separation, we introduce the quantities
\[ \lambda^* \equiv N_1^*(0) \frac{|\alpha_1|^2}{\omega_0^2}, \quad \mu^* \equiv N_1^*(0) \tilde{v}_1(k_F) \] (90)

which do not depend on the phase probability \( w \). The first quantity in (90) is the electron-phonon coupling constant for a pure system without phase separation. The second quantity in (90) is the average intensity of screened Coulomb interaction.

With notation (90), for the effective coupling parameter (80) we obtain
\[ \lambda_{\text{eff}} = \frac{\lambda^* - \mu^* w^{2\varphi}}{w^{2\varphi-1}(1 + \gamma w)}. \] (91)
The criterion of superconductivity means that $\lambda_{eff} > 0$, which leads because of (91) to

$$\lambda^* - \mu^* w^{2\varphi} > 0.$$  

This inequality, in the case of pure superconductor, when $w \equiv 1$, reduces to the standard condition $\lambda^* > \mu^* \gamma_w$ usually valid for low-temperature superconductors. When the phase separation occurs in a superconductor, then the condition $\lambda_{eff} > 0$ is easier to satisfy. Really, when $0 < w < 1$, then the inequality $\lambda^* > \mu^* w^{2\varphi}$ can be true even if $\lambda^* < \mu^*$. In this way, phase separation favours the appearance of superconductivity. A sample with phase separation can become superconductive even if a similar sample, without phase separation, cannot have superconducting properties.

The superconducting critical temperature is given by (81) which becomes

$$\frac{\lambda^* - \mu^* w^{2\varphi}}{w^{2\varphi - 1}|1 + \gamma w|} \int_0^{w^\omega_0} \frac{1}{\omega} \tanh \frac{w \omega}{2T_c} d\omega = 1.$$  

(92)

As we see, it is not easy to analyse precisely the influence of phase separation on the critical temperature, that is, to solve explicitly for the dependence of $T_c$ on the phase probability $w$. To slightly simplify the situation, we may notice that since $\varepsilon_F \approx k_F/m$, $N_0^*(0) \approx (mk_F/2\pi^2)$ and $v_1 (k_F) \approx \pi e^2/k_F^2$, then $\mu^* \approx \gamma$. Therefore, for simplicity, we put $\gamma = \mu^*$.

Of course, such a slight simplification does not help much, and to proceed further in defining the dependence of $T_c$ on $w$ we have to resort to numerical analysis of (92). To this end, it is convenient to introduce the dimensionless critical temperature

$$t_c \equiv \frac{T_c}{\omega_0}.$$  

Now we can reorganize (92) in the form

$$\frac{\lambda^* - \mu^* w^{2\varphi}}{w^{2\varphi - 1}(1 + \mu^* w)} \int_0^1 \tanh \left( \frac{w^{1+\varphi}}{2t_c} \right) \frac{dx}{x} = 1.$$  

(93)

From (93) we can immediately conclude that the behaviour of $t_c = t_c(w)$ can be, in general, nonmonotonic, since there are two points at which $t_c(w)$ tends to zero. The first point corresponds to $w \to 0$. Then from (93) we obtain

$$t_c \simeq \frac{1}{2} w^{2-\varphi} \frac{\lambda^* - \mu^* w^{2\varphi}}{1 + \mu^* w}.$$  

(94)

The second case, when $t_c \to 0$, is when $w \to (\lambda^*/\mu^*)^{1/2 \varphi}$, then

$$t_c \simeq 1.134 w^{1+\varphi} \exp \left\{ - \frac{w^{2\varphi - 1}(1 + \mu^* w)}{\lambda^* - \mu^* w^{2\varphi}} \right\}.$$  

(95)

Recall that the dependence of $t_c$ on $w$ is interesting to analyse because the superconducting phase concentration $w$ can be measured and can be varied in experiments by changing the chemical structure of materials, for example, by doping [6,11,65,66].

We made a detailed analysis of the function $t_c(w)$ by solving the equation (93) numerically. Graphs of the resulting functions are presented in figs.1-12. We did not try to fit any particular experimental situation. Rather we wanted to understand the whole picture as to how the behaviour of $t_c(w)$ changes qualitatively with the change of parameters $\varphi, \mu^*$ and $\lambda^*$.

We were pleasantly surprised by the wide variety of curves which were obtained. It is apparent from the accompanying graphs that by the choice of the corresponding parameters it will be possible to obtain a reasonably good fit to any experimental curve.

Figs.1-4 correspond to the case of weak softening; figs.5-8, to that of moderate softening; and figs.9-12, to the case of strong softening. With the increase of the Coulomb parameter the function $t_c(w)$ really becomes nonmonotonic. The behaviour of $t_c(w)$ plotted in figs.3,4,7 and 8 has a striking similarity to the corresponding experimental curves for high-temperature cuprate superconductors (see [6,11,65,70] and references therein). The coincidence becomes practically complete if we redraw our figures in the relative coordinates $\tilde{T} \equiv t_c/t_c^{max}$, $\sigma \equiv w/w^{max}$, as in [65,66], where the point $(t_c^{max}, w^{max})$ denotes the point of a maximum of the considered curve.
It is clear from the preceding analysis, as illustrated in our graphs, that, for fixed $\varphi$, $\mu$ and $\lambda$, the value of the reduced critical temperature, $T_c$, depends crucially on the parameter $w$. The operational meaning of this parameter is rather different according as the the system is 1) stable, or 2) metastable.

Notice that while $w$ occurs explicitly on the left hand side of equation (12) it also occurs implicitly in a complicated manner in the right hand side as a result of the renormalization of the Hamiltonian (1). Thus for a stable system satisfying conditions (9) and (13), we can think of $w$ as determined self-consistently by equation (12) once all characteristics of the system, - such as chemical composition, particle masses, interaction potentials, temperature, density and pressure - are given. Note that all of these characteristics are necessary for $w$ to be uniquely determined. In particular, the total role of interactions must be taken into account. In contrast to this, following common practice, in our model the gap equation (79) and consequently equation (93) for the critical temperature, take into account only interactions specified on the Fermi surface. Essential to the argument of the present paper is the conviction that it would not be reasonable to try to treat superconductivity in heterophase system with a model in which $w$ is determined by parameters defined merely on the Fermi surface. For instance, parameters characterizing the ground state would be indispensable. The equations for phase probabilities always contain more characteristic constants than the equations for an order parameter [18]. It is therefore permissible to contemplate the possibility of holding some parameters, such as $\lambda^*$ and $\mu^*$, fixed while $w$ varies as a result of changing other parameters such as the chemical composition.

On the other hand, for a metastable system for which (13) does not hold even though thermal and mechanical stability are preserved, the fraction $w$ is not necessarily determined by equation (12), but might be arbitrary depending on the preparation of the sample. This possibility should not be overlooked since many high-$T_c$ superconductors are metastable.

VII. CONCLUSION

We have developed an approach, for describing superconductors, taking into account three mutually interrelated factors: (i) repulsive interactions, (ii) lattice softening and (iii) phase separation. We have deliberately limited ourselves to the use of only commonly accepted approximations. This was in order to emphasize that our results are not artifacts of some technical tricks, but the direct consequences of the physical reasons offered. The main results can be summarized as follows:

(i) A necessary condition for phase separation in a superconductor is the presence of repulsive interactions.

(ii) Phase separation favours superconductivity making it possible in a heterophase sample even if it were impossible in a pure sample.

(iii) The superconducting critical temperature as a function of the relative concentration of the superconducting phase can display the nonmonotonic behaviour typical of high-temperature cuprate superconductors.

It should be a straightforward matter to adapt the basic approach of this paper to other models of superconductors by using alternative approximation methods.

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Figure Captions

Fig.1.
The superconducting critical temperature as a function of the superconducting phase concentration for the parameters \( \varphi = 0.5, \mu^* = 0.1 \) and \( \lambda^* = 1 \) (lower curve), \( \lambda^* = 5 \) (middle curve) and \( \lambda^* = 10 \) (upper curve).

Fig.2.
The same as in fig.1, but for the parameters \( \varphi = 0.5, \mu^* = 1 \), and \( \lambda^* = 1 \) (lower curve), \( \lambda^* = 5 \) (middle curve) and \( \lambda^* = 10 \) (upper curve).

Fig.3.
The same as in fig.1 for the parameters \( \varphi = 0.5, \mu^* = 5 \), and \( \lambda^* = 1 \) (hardly visible lower curve), \( \lambda^* = 5 \) (middle curve) and \( \lambda^* = 10 \) (upper curve).

Fig.4.
The same as in fig.1 for the parameters \( \varphi = 0.5, \mu^* = 10 \), and \( \lambda^* = 5 \) (lower curve) and \( \lambda^* = 10 \) (upper curve). The curve corresponding to \( \lambda^* = 1 \) in this case is invisible.
Fig. 5. The same as in fig.1 for the parameters $\varphi = 1$, $\mu^* = 0.1$, and $\lambda^* = 1$ (lower curve), $\lambda^* = 5$ (middle curve) and $\lambda^* = 10$ (upper curve).

Fig. 6. The same as in fig.1 for the parameters $\varphi = 1$, $\mu^* = 1$, and $\lambda^* = 1$ (lower curve), $\lambda^* = 5$ (middle curve) and $\lambda^* = 10$ (upper curve).

Fig. 7. The same for the parameters $\varphi = 1$, $\mu^* = 5$, and $\lambda^* = 1$ (lower curve), $\lambda^* = 5$ (middle curve) and $\lambda^* = 10$ (upper curve).

Fig. 8. The same for the parameters $\varphi = 1$, $\mu^* = 10$, and $\lambda^* = 1$ (lower curve), $\lambda^* = 5$ (middle curve) and $\lambda^* = 10$ (upper curve).

Fig. 9. The same for the parameters $\varphi = 1.5$, $\mu^* = 0.1$, and $\lambda^* = 1$ (lower curve), $\lambda^* = 5$ (middle curve) and $\lambda^* = 10$ (upper curve).

Fig. 10. The same for the parameters $\varphi = 1.5$, $\mu^* = 1$, and $\lambda^* = 1$ (lower curve), $\lambda^* = 5$ (middle curve) and $\lambda^* = 10$ (upper curve).

Fig. 11. The same for the parameters $\varphi = 1.5$, $\mu^* = 5$, and $\lambda^* = 1$ (lower curve), $\lambda^* = 5$ (middle curve) and $\lambda^* = 10$ (upper curve).

Fig. 12. The same for the parameters $\varphi = 1.5$, $\mu^* = 10$, and $\lambda^* = 1$ (lower curve), $\lambda^* = 5$ (middle curve) and $\lambda^* = 10$ (upper curve).