Abstract

We consider extremal black hole attractors (both BPS and non-BPS) for $\mathcal{N} = 3$ and $\mathcal{N} = 5$ supergravity in $d = 4$ space-time dimensions.

Attractors for matter-coupled $\mathcal{N} = 3$ theory are similar to attractors in $\mathcal{N} = 2$ supergravity minimally coupled to Abelian vector multiplets.

On the other hand, $\mathcal{N} = 5$ attractors are similar to attractors in $\mathcal{N} = 4$ pure supergravity, and in such theories only $\frac{1}{2}$-BPS non-degenerate solutions exist.

All the above mentioned theories have a simple interpretation in the first order (fake supergravity) formalism. Furthermore, such theories do not have a $d = 5$ uplift.

Finally we comment on the “duality” relations among the attractor solutions of $\mathcal{N} \geq 2$ supergravities sharing the same full bosonic sector.
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1 Introduction

The Attractor Mechanism [1]–[4] is an important dynamical phenomenon in the theory of gravitational objects, which naturally appears in modern theories of gravity, such as supergravity, superstrings [5]–[8] or M-theory [9, 10].

Even if such a phenomenon was originally shown to occur for \( \frac{1}{\text{BPS}} \) extremal black holes (BHs) in \( \mathcal{N} = 2 \), \( d = 4 \) ungauged supergravity coupled to Abelian vector multiplets, it has a more general validity, since it may take place also for non-BPS extremal BHs, irrespective of whether the underlying gravitational theory is endowed with local supersymmetry or not [11]–[66] (for further developments, see also e.g. [67]–[70]). Moreover, such a phenomenon also exists in higher space-time dimensions for black \( p \)-branes coupled to scalar fields, provided certain constraints are met.

In theories with \( \mathcal{N} > 2 \) local supersymmetry, extremal BH attractors with regular horizon geometry and non-vanishing classical Bekenstein-Hawking entropy exhibit a new feature. This indicates that the Hessian matrix of a suitably defined effective BH potential \( V_{BH} \) may present, in contrast with the \( \mathcal{N} = 2 \) case, “flat” directions even for \( (\frac{1}{2} \text{-BPS}) \) configurations. This is for instance the case when the scalar manifold is a locally symmetric space, as it holds for all \( \mathcal{N} > 2 \), \( d = 4 \) supergravities. A general analysis of extremal BH attractors in \( \mathcal{N} = 2 \) symmetric special Kähler geometry was performed in [25], and the related moduli spaces were discovered and classified in [41]. For \( \mathcal{N} > 2 \) supergravities similar results were obtained in [71] and [56]. In [38] it was further observed that “flat” directions for \( \mathcal{N} > 2 \) (both \( \frac{1}{2} \text{-BPS} \) and non-BPS) attractors, as well as for \( \mathcal{N} = 2 \) non-BPS attractors, are closely related to the fact that in \( \mathcal{N} = 2 \) ungauged supergravity the hypermultiplets’ scalars do not participate in the Attractor Mechanism. As a consequence, the moduli space of \( \frac{1}{2} \text{-BPS} \) attractors in \( \mathcal{N} > 2 \) supergravities is a quaternionic manifold, spanned by the left-over would-be hypermultiplets’ scalar degrees of freedom in the supersymmetry reduction of the original theory down to \( \mathcal{N} = 2 \).

The corresponding orbits of electric and magnetic BH charges, supporting the critical points of \( V_{BH} \) which determine the attractor scalar configurations on the BH event horizon, have also been classified in [25] and [56]. The non-compactness of the stabilizer of such orbits (with the only exception of \( \mathcal{N} = 2 \) BPS orbits) is responsible for the existence of “flat” directions of the BH potential at its corresponding critical points.

Most of the supergravities based on symmetric scalar manifolds have the property that the classical BH entropy, as given by the Bekenstein-Hawking entropy-area formula [72], is expressed in terms of the square root of the absolute value of a (\( n \text{ unique} \)) invariant \( I_4 \) of the relevant representation of the \( U \)-duality group. Such an invariant is quartic in electric and magnetic BH charges:

\[
S_{BH} = \frac{A}{4} = \pi V_{BH}\big|_{\partial V_{BH}=0} = \pi \sqrt{|I_4|}. \tag{1.1}
\]

\( I_4 \), which is moduli-independent, can also be written in terms of the “dressed” charges, i.e. in terms of the moduli-dependent central charge matrix and matter charges, in a (\( n \text{ unique} \)) combination, such that the overall dependence on moduli drops out. However, a peculiar class of \( d = 4 \) supergravities exists, such that the unique \( U \)-duality invariant (and thus moduli-independent) combination of “dressed” charges turns out to be a perfect square of a quadratic expression in the skew-eigenvalues of the relevant central charge matrix. Namely, this holds for pure \( \mathcal{N} = 4 \) [73] and \( \mathcal{N} = 5 \) [74] supergravity.

Furthermore, another class of \( d = 4 \) theories exists, so that the unique \( U \)-invariant \( I_2 \) is quadratic in BH charges, yielding:

\[
S_{BH} = \frac{A}{4} = \pi V_{BH}\big|_{\partial V_{BH}=0} = \pi |I_2|. \tag{1.2}
\]

\( I_2 \) is also given by a quadratic expression in terms of the “dressed” charges. Such a class of theories is given by \( \mathcal{N} = 2 \) supergravity minimally coupled to Abelian vector multiplets [75], and by \( \mathcal{N} = 3 \) supergravity coupled to matter (Abelian vector) multiplets [76].

For both the peculiar class of theories admitting \( I_4 \) as a perfect square of the skew-eigenvalues of the central charge matrix and the supergravities admitting \( I_2 \), there is a very simple alternative expression for the classical Bekenstein-Hawking entropy in terms of a (square) effective horizon radius \( R_{BH} \). This turns out to be moduli-independent, and dependent only on the set of magnetic and electric BH charges,
shortly indicated as \((p,q)\). The formula for the entropy of the extremal BH in these cases reads:

\[
S_{BH} = \frac{A}{4} = \pi V_{BH} \mid \partial V_{BH} = 0 \equiv \pi R_H^2 (p, q) = \begin{cases} 
\frac{\pi \sqrt{|I_4|}}{\pi \langle I_2 \rangle} 
\text{or} \n\frac{\pi \langle I_2 \rangle}{\pi \langle Z \rangle}
\end{cases}
\]

\[
= \pi \left[ r_H^2 (\varphi_\infty, p, q) - \frac{1}{2} G_{ab} (\varphi_\infty) \Sigma^a (\varphi_\infty, p, q) \Sigma^b (\varphi_\infty, p, q) \right].
\]

\((1.3)\)

\(r_H\) is the radius of the unique \((event)\) horizon of the extremal BH, \(\Sigma^a\) denoting the set of \textit{scalar charges} asymptotically associated to the scalar field \(\varphi^a\), and \(G_{ab}\) is the covariant metric tensor of the scalar manifold (in the real parametrization). Notice that the first line of Eq. \((1.3)\) only contains the definition of \(R_H^2\) itself, whereas the second line of the very same Eq. expresses it through a \textit{moduli-independent} combination of \textit{moduli-dependent} quantities, holding only for the aforementioned \(d = 4\) supergravity theories.

Actually, \(R_H^2\) can be expressed as a suitable integral in terms of a \textit{square effective radius} \(R^2\), as follows:

\[
R_H^2 = R^2 (r, \varphi_\infty, p, q) \bigg|_{r=r_H (\varphi_\infty, p, q)}
\]

\[
R^2 (r, \varphi_\infty, p, q) \equiv r^2 - \frac{1}{2} G_{ab} (\varphi_\infty) \Sigma^a (\varphi_\infty, p, q) \Sigma^b (\varphi_\infty, p, q) \leq r^2,
\]

\((1.5)\)

where \(r\) is the usual radial coordinate, and in the last inequality the positive definiteness of \(G_{ab}\) was exploited.

It is worth pointing out that the second line of Eq. \((1.3)\) and Eqs. \((1.4)\) and \((1.5)\) are generalizations of the formulae holding in the so-called \textit{Maxwell-Einstein-axion-dilaton system} (similarly also in the \textit{non-extremal case}, see \textit{e.g.}, [77] and [78]; see also the treatment, and in particular Eqs. \((2.7)\), \((2.8)\) and \((2.15)\), of [63]). Notice that in the first of Refs. [77] the variable \(R\) is named \textit{physical radial coordinate} (\textit{see e.g.} Eq. \((72)\) therein). Clearly, as the second line of Eq. \((1.3)\), also the definition \((1.5)\) of \(R^2 (r, \varphi_\infty, p, q)\) holds only for the aforementioned \(d = 4\) supergravity theories.

Within the \textit{first order (fake supergravity) formalism} [79], recently used to describe non-BPS attractor flows of \(d = 4\) extremal BHs [35, 40], the quantities appearing in the second line of Eq. \((1.3)\) can easily be expressed in terms of a \textit{real “fake superpotential”} \(W (\varphi, p, q)\) as follows (\textit{see Eqs. \((2.5)\) and \((2.6)\) below, respectively)}:

\[
r_H (\varphi_\infty, p, q) = W (\varphi_\infty, p, q);
\]

\((1.6)\)

\[
\Sigma^a (\varphi_\infty, p, q) = 2 G^{ab} (\varphi_\infty) (\partial_b W) (\varphi_\infty, p, q).
\]

\((1.7)\)

An explicit expression for \(W\) can be given for the supergravity theories mentioned above [40]. It is worth noticing here that for \(\frac{1}{8}\)-BPS \textit{non-degenerate} attractor flows, it simply holds \(W (\varphi, p, q) = |Z| (\varphi, p, q)\), where \(|Z|\) is the biggest absolute value of the skew-eigenvalues of the \textit{central charge matrix} \(Z_{AB}\), saturating the BPS bound [80].

Eq. \((1.3)\) would seem to yield a \textit{moduli-dependent} expression for \(R_H^2\), but, as we prove explicitly in the present paper, for the class of \(d = 4\) ungauged supergravities under consideration it just transpires that the dependence on moduli drops out in the combination \(r_H^2 - \frac{1}{2} G_{ab} \Sigma^a \Sigma^b\), when Eqs. \((1.6)\) and \((1.7)\) are taken into account. Summarizing, such a phenomenon happens in the following theories:

- \(\mathcal{N} = 2\) supergravity \textit{minimally coupled} to Abelian vector multiplets [75], whose scalar manifold is endowed with a symmetric special Kähler geometry, with the completely symmetric rank-3 tensor \(C_{ijk} = 0\), and with \(U\)-invariant \textit{quadratic} in BH charges. For such a theory, in [63] Eq. \((1.3)\) has been proved to hold for both \(\frac{1}{2}\)-BPS and non-BPS \((Z = 0)\) attractor flows

- \(\mathcal{N} = 3\) supergravity coupled to matter (Abelian vector) multiplets [76], with \(U\)-invariant \textit{quadratic} in BH charges

- \(\mathcal{N} = 4\) \textit{pure} supergravity [73], with \(U\)-invariant \textit{quartic} in BH charges
\( \mathcal{N} = 5 \) supergravity [74], with \( U \)-invariant \textit{quartic} in BH charges.

It is worth pointing out that \( \mathcal{N} = 2 \) supergravity \textit{minimally coupled} to one Abelian vector multiplet, corresponding to \( (U(1))^3 \to (U(1))^2 \) gauge truncation of \( \mathcal{N} = 4 \) \textit{pure} supergravity, is nothing but the so-called \textit{Maxwell-Einstein-axion-dilaton system}, studied in [77, 78] and recently discussed in [71] and in [63]. As stated above, the formula (1.3) indeed holds true (actually, with suitable changes, also in the \textit{non-extremal} case).

Furthermore, it is interesting to notice that \textit{all} the above mentioned theories are \textit{all} the \( \mathcal{N} \geq 2 \), \( d = 4 \) supergravities based on \textit{symmetric} scalar manifolds which do \textit{not} admit an uplift\(^1\) to \( d = 5 \) space-time dimensions [81].

The present paper is organized as follows.

In Sect. 2 we briefly introduce the fundamentals of the \textit{first order (fake supergravity)} formalism for the \textit{non-degenerate} attractor flows (both BPS and non-BPS) of extremal BHs in \( d = 4 \) space-time dimensions.

Sect. 3 is thus devoted to a detailed study of \( \mathcal{N} = 2 \), \( d = 4 \) supergravity \textit{minimally coupled} to Abelian vector multiplets. In Subsect. 3.1 the related \textit{Attractor Equations} are explicitly solved, for both the classes of \textit{non-degenerate} critical points of \( V_{BH} \): the \( \frac{1}{2} \)-BPS one (Subsubsect. 3.1.1) and the non-BPS \( Z = 0 \) one, this latter with related moduli space (Subsubsect. 3.1.2). By exploiting the \textit{first order (fake supergravity) formalism}, in Subsects. 3.2 and 3.3 the \textit{ADM} (Arnowitt-Deser-Misner) mass \( M_{ADM} \) [82], covariant scalar charges \( \Sigma \), and \textit{(square) effective horizon radius} \( R^2_H \) are explicitly computed respectively for \( \frac{1}{2} \)-BPS and non-BPS \( Z = 0 \) attractor flows, proving that the second line of Eq. (1.3) holds true. This latter result, already proved in [63], generalizes the findings of [77, 78], also holding in the \textit{non-extremal} case.

Sect. 4 deals with \( \mathcal{N} = 3 \), \( d = 4 \) supergravity coupled to matter (Abelian vector) multiplets. In Subsect. 4.1 the related \textit{Attractor Equations} are explicitly solved, for both the classes of \textit{non-degenerate} critical points of \( V_{BH} \): the \( \frac{1}{2} \)-BPS one (Subsect. 4.1.1) and the non-BPS \( Z_{AB} = 0 \) one (Subsect. 4.1.2), both with related moduli space. Once again, by using the \textit{first order (fake supergravity) formalism}, in Subsects. 4.2 and 4.3 the \textit{ADM mass} \( M_{ADM} \), covariant scalar charges \( \Sigma \) and \textit{(square) effective horizon radius} \( R^2_H \) respectively for \( \frac{1}{2} \)-BPS and non-BPS \( Z_{AB} = 0 \) attractor flows are explicitly computed, proving that the second line of Eq. (1.3) holds true also for such a theory. This, as the \textit{minimally coupled} \( \mathcal{N} = 2 \) supergravity, has a \textit{unique} \( U \)-invariant \textit{quadratic} in BH charges.

Comments on the invariance properties of BH entropy in \textit{minimally coupled} \( \mathcal{N} = 2 \), as well as in \textit{matter-coupled} \( \mathcal{N} = 3 \), \textit{ungauged} \( d = 4 \) supergravity are given in Sect. 5.

Next, Sect. 6 deals with \( \mathcal{N} = 5 \), \( d = 4 \) supergravity, which does \textit{not} allow for matter coupling and whose field content thus only consists of the gravity multiplet (\textit{pure} theory). In Subsect. 6.1 the related \textit{Attractor Equations} are explicitly solved for the unique class of \textit{non-degenerate} critical points of \( V_{BH} \), namely the \( \frac{1}{2} \)-BPS one. In Subsect. 6.1.1 such a class is studied, along with the related moduli space and Bekenstein-Hawking classical BH entropy [72]. This latter is proportional to the unique \( U \)-invariant \( T_4 \) of \( \mathcal{N} = 5 \) supergravity, whose \textit{quartic} expression in terms of the BH charges is also explicitly derived. Through the formal machinery presented in Sect. 2, in Subsect. 6.2 the \textit{ADM mass} \( M_{ADM} \), covariant scalar charges \( \Sigma \) and \textit{(square) effective horizon radius} \( R^2_H \) for the \( \frac{1}{2} \)-BPS attractor flow are explicitly given, proving that (the second line of) Eq. (1.3) holds true also for such a theory. This is somewhat surprising because, as mentioned above, \( \mathcal{N} = 5 \) supergravity, in contrast to the \textit{minimally coupled} \( \mathcal{N} = 2 \) and \textit{matter-coupled} \( \mathcal{N} = 3 \) cases, has a \textit{unique} \( U \)-invariant \textit{quartic}, rather than \textit{quadratic}, in BH charges.

So, in Sect. 7 the extremal BH attractors in \( \mathcal{N} = 4 \), \( d = 4 \) \textit{pure} supergravity are revisited. In Subsect. 7.1 the resolution of the corresponding \textit{Attractor Equation} [71] is reviewed for the unique class of \textit{non-degenerate} critical points of \( V_{BH} \), namely the \( \frac{1}{2} \)-BPS one. Its corresponding Bekenstein-Hawking classical BH entropy [72] is given by the unique \( U \)-invariant \( T_4 \) of \( \mathcal{N} = 4 \) \textit{pure} supergravity, which is also reported. By using the formulæ of Sect. 2, in Subsect. 7.2 the \textit{ADM mass} \( M_{ADM} \), covariant axion-dilaton charge \( \Sigma \) and \textit{square effective horizon radius} \( R^2_H \) for the \( \frac{1}{2} \)-BPS attractor flow are explicitly given, proving that the second line of Eq. (1.3) holds true also for such a theory. Also such a result is rather surprising, for the same reason mentioned above: \textit{pure} \( \mathcal{N} = 4 \) supergravity, as \( \mathcal{N} = 5 \) theory and

\(^{1}\)Throughout all the treatment of the present paper, by “uplift to \( d = 5 \)” we mean the dimensional uplift to a \( d = 5 \) Poincaré supergravity theory, having the same massless degrees of freedom of the original \( d = 4 \) supergravity.
in contrast to the minimally coupled $\mathcal{N} = 2$ and matter-coupled $\mathcal{N} = 3$ cases, has a unique $U$-invariant quartic, rather than quadratic, in BH charges.

However, as pointed out in the Introduction, pure $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravities are peculiar theories, because their unique (moduli-independent) $U$-duality invariant, quartic in BH charges, when expressed as a (unique) combination of “dressed” (moduli-dependent) charges, turns out to be a perfect square of a quadratic expression in the skew-eigenvalues $Z_1$ and $Z_2$ of the relevant central charge matrix. Such a key feature is studied in Sect. 8.

In Sect. 9 we consider all ungauged $\mathcal{N} \geq 2, d = 4$ supergravities sharing the same bosonic sector, and thus with the same number of fermion fields, but with different supersymmetric completions. Beside the well-known case of the “duality” between $\mathcal{N} = 2$ and $\mathcal{N} = 4$ (matter-coupled) and $\mathcal{N} = 6$ (pure) supergravity (see [25] and Refs. therein), other two cases exist, namely:

- the “duality” exhibited by $\mathcal{N} = 2$ supergravity minimally coupled to 3 Abelian vector multiplets, and $\mathcal{N} = 3$ supergravity coupled to 1 matter (Abelian vector) multiplet
- the “duality” between $\mathcal{N} = 2$ supergravity coupled to 6 Abelian vector multiplets, with scalar manifold given by the symmetric reducible special Kähler manifold $SU(1,1)/U(1) \times SO(2) \times SO(6)^2$, and $\mathcal{N} = 4$ supergravity coupled to 2 matter (Abelian vector) multiplets.

It is here worth commenting that such “dualities” constitute evidence against the conventional wisdom that bosonic interacting theories have an unique supersymmetric extension. The sharing of the same bosonic backgrounds with different supersymmetric completions implies its “dual” interpretation with respect to the supersymmetry-preserving properties. Consistent with local supersymmetry, the number of fermion fields is the same in both theories, but with different spin/field contents, simply related by the interchange among spin-$\frac{1}{2}$ (gaugino) and spin-$\frac{3}{2}$ (gravitino) fields.

Sect. 10 contains some comments, outlook and directions for further developments.

Finally, Appendix I concludes the paper. It presents $\mathcal{N} = 4, d = 4$ ungauged supergravity coupled to 1 matter (Abelian vector) multiplet (upliftable to the $\mathcal{N} = 4, d = 5$ pure theory). This constitutes a counterexample of a theory with unique (moduli-independent) $U$-duality invariant quartic in BH charges which, when expressed as a combination of “dressed” (moduli-dependent) charges, does not turn out to be a perfect square of a quadratic expression in the skew-eigenvalues of the central charge matrix and in the matter charge(s). As a consequence, the explicit expression of $R_H^2$ given by (the second line of) Eq. (1.3) does not hold for such a theory, as well as for all other $d = 4$ (ungauged) supergravities not explicitly mentioned above and in the treatment given below.

## 2 Fake Supergravity Formalism and Effective Horizon Radius

for $d = 4$ Extremal Black Holes

We recall some facts about the first order (fake supergravity) formalism [79] for static, spherically symmetric, asymptotically flat dyonic extremal (i.e. with $c = 0$) BHs in $d = 4$, introduced in [35] and [40] (see also [63]).

Let us start with the general formula for the (positive definite) BH effective potential of $d = 4$ supergravities:

$$V_{BH} = \frac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^I,$$

(2.1)

where $Z_{AB} = Z_{\{AB\}}$ ($A, B = 1, \ldots, \mathcal{N}$) is the central charge matrix, and $Z_I$ ($I = 1, \ldots, n$) are the matter charges, $n \in \mathbb{N}$ being the number of matter multiplets (if any) coupled to the gravity multiplet. Equivalently, in the first order formalism (see Eq. (2.23) of [35]):

$$V_{BH} = W^2 + 4G^{ij} (\partial_i W) \bar{\partial}_j W = W^2 + 4G^{ij} (\nabla_i W) \bar{\nabla}_j W,$$

(2.2)

where $W$ is the moduli-dependent so-called first order fake superpotential, and $\nabla$ denotes the relevant covariant differential operator.

An alternative expression for $V_{BH}$ can be given as follows (see Eq. (5.7) of [22]):

$$V_{BH} = e^G \left[ 1 + G^{ij} (\partial_i G) \bar{\partial}_j \bar{G} \right] = e^G \left[ 1 + G^{ij} (\nabla_i G) \bar{\nabla}_j \bar{G} \right],$$

(2.3)
where now
\[ W \equiv e^{G/2}. \] (2.4)

By recalling Eq. (65) of [63] and Eqs. (84) and (114) of [63] (which in turn can be traced back to Eq. (29) of [40]), in the same framework the covariant scalar charges and the squared ADM mass [82] can respectively be written as follows:\(^2\)

\[ \Sigma_i = 2 \lim_{\tau \to 0^+} \nabla_i W = 2 \lim_{\tau \to 0^+} \partial_i W; \] (2.5)

\[ M^2_{ADM} = r_H^2 = \lim_{\tau \to 0^+} \left[ V_{BH} - 4G^{-1}(\partial_i W) \overline{\partial_j W} \right] = \lim_{\tau \to 0^+} W^2, \] (2.6)

where \( \tau \equiv (r_H - r)^{-1} \). Hence, one can introduce the (square) effective horizon radius (recall the notation \( R_{+,c=0} = R_{-,c=0} \equiv R_H \); see the treatment of [63]):

\[ R_H^2 = \lim_{\tau \to -\infty} V_{BH} = V_{BH}|_{\partial V_{BH}=0, V_{BH} \neq 0} = \lim_{\tau \to -\infty} W^2 = W^2|_{\partial W=0, W \neq 0} = \]

\[ = \frac{A_{eff}(p,q)}{4\pi} = \frac{S_{BH}(p,q)}{\pi}, \] (2.7)

where \((p,q)\) denotes the set of magnetic and electric BH charges, \( A_{eff} \) (simply named \( A \) in the Introduction) is the effective area of the BH (i.e. the area of the surface pertaining to \( R_H \)), \( S_{BH} \) is the classical BH entropy, and the Bekenstein-Hawking entropy-area formula [72] has been used.

Whenever allowed by the symmetric nature of the scalar manifold, \( R_H^2 \) can thus be expressed in terms of a suitable power of the (generally unique) invariant of the relevant representation of the \( U-duality \) group \( G \), determining the symplectic embedding of the vector field strengths. In \( d = 4 \) \( S_{BH} \) is homogeneous of degree two in \((p,q)\), and only two possibilities arise:

\[ R_H^2 = |\mathcal{I}_2(p,q)|, \text{ or } R_H^2 = \sqrt{|\mathcal{I}_4(p,q)|}, \] (2.8)

where \( \mathcal{I}_2 \) and \( \mathcal{I}_4 \) respectively denote \( U \)-invariants quadratic and quartic in BH charges.

By exploiting the \( \tau \)-monotonicity of \( W \) (which is indeed an example of \( C \)-function [17] for extremal BHs) [40]:

\[ \frac{dW(z(\tau), \overline{z}(\tau); p,q)}{d\tau} \geq 0, \] (2.9)

the following inequality (holding for \( c = 0 \)) can be obtained [63]:

\[ M^2_{ADM}(z_{\infty}, \overline{z}_{\infty}, p,q) = \lim_{\tau \to 0^+} \left[ V_{BH} - 4G^{-1}(\partial_i W) \overline{\partial_j W} \right] = \]

\[ = \lim_{\tau \to 0^+} W^2 \equiv r_H^2 (z_{\infty}, \overline{z}_{\infty}, p,q) \geq R_H^2 (p,q) = \lim_{\tau \to -\infty} W^2 = \lim_{\tau \to -\infty} V_{BH}, \] (2.10)

where the radius \( r_H \) of the BH event horizon was introduced. More concisely:

\[ r_H^2 (z_{\infty}, \overline{z}_{\infty}, p,q) \geq R_H^2 (p,q), \quad \forall (z_{\infty}, \overline{z}_{\infty}) \in \mathcal{M}_{\infty}, \] (2.11)

holding in the whole asymptotical scalar manifold \( \mathcal{M}_{\infty} \).

In the minimally matter-coupled \( N = 2, d = 4 \) supergravity based on the sequence of symmetric special Kähler manifolds (complex Grassmannians) \( SU(1,n)/SU(n) \times SU(1) \) [75] (see also the treatment of [63]), as well as in \( N = 3 \), pure \( N = 4 \) and \( N = 5, d = 4 \) supergravity, it is possible to specialize further the inequality (2.11). Indeed, for such theories it holds that (recall Eq. (1.3), as well as Eqs. (1.4) and (1.5))

\[ R_H^2 (p,q) \equiv \frac{S_{BH}(p,q)}{\pi} = \]

\[ = r_H^2 (z_{\infty}, \overline{z}_{\infty}, p,q) - G_{ij} \overline{\partial_j W} = r_H^2 (z_{\infty}, \overline{z}_{\infty}, p,q) - 4 \lim_{\tau \to 0^+} G_{ij} (\partial_i W) \overline{\partial_j W}, \] (2.12)

\(^2\)Here and in all our analysis we assume all functions of moduli to be sufficiently regular, in order to allow one to perform smoothly the radial asymptotical (\( \tau \to 0^+ \)) and near horizon (\( \tau \to -\infty \)) limits.
where in the last step Eq. (2.5) was used. Eq. (2.12), clearly yielding the inequality (2.11) by the presence of non-vanishing scalar charges and the (strict) positive definiteness of $G_{ij}$, is nothing but a many-moduli generalization of the formula holding for the so-called (axion-)dilaton extremal BH [77].

The crucial feature, expressed by Eq. (2.12) and shared by the aforementioned supergravities, is the disappearance of the dependence on the asymptotical moduli $(z_\infty, \tau_\infty)$ in the combination of quantities $r_H^2 - G_{ij}\Sigma_i^\tau\Sigma_j^r$, which separately do depend on moduli$^3$.

As a generalization of the formula holding (also in the non-extremal case) in the Maxwell-axion-dilaton supergravity (see e.g. [77, 78], and also [63]), in [63] Eq. (2.12) was proved to hold in the extremal case for the whole sequence of $\mathcal{N} = 2$, $d = 4$ supergravity minimally coupled to Abelian vector multiplets [75], in terms of the (unique) invariant $\mathcal{I}_2$ of the $U$-duality group $G = SU(1, n)$, which is quadratic in charges:

$$R_H^2 (p, q) = r_H^2 (z_\infty, \tau_\infty, p, q) - 4 \lim_{\tau \to 0} G^{\overline{j}j} (\partial_\tau W) \overline{\partial_j} W = |\mathcal{I}_2 (p, q)|.$$  

(2.13)

We will report such results in Subsects. 3.2 and 3.3.

Thence, by exploiting the first order formalism for $d = 4$ extremal BHs outlined above, we will show that the same happens for the following $d = 4$ supergravities:

- $\mathcal{N} = 3$ (matter-coupled) [76], as intuitively expected by the strict similarity with the so-called minimally coupled $\mathcal{N} = 2$ theory (Subsects. 4.2 and 4.3);
- $\mathcal{N} = 5$ [74], with $|\mathcal{I}_2|$ replaced by $\sqrt{|\mathcal{I}_4|}$ (Subsect. 6.2);
- pure $\mathcal{N} = 4$ [73], with $|\mathcal{I}_2|$ replaced by $\sqrt{|\mathcal{I}_4|}$ (Subsect. 7.2).

Let us here note that while $\mathcal{N} = 5$ theory cannot be coupled to matter, in the case $\mathcal{N} = 4$ matter coupling is allowed, but Eq. (2.12) holds only in $\mathcal{N} = 4$ pure supergravity. Having a(n unique) $U$-invariant $\mathcal{I}_4$ quartic in charges, the aforementioned $\mathcal{N} = 4$ and $\mathcal{N} = 5$ theories are pretty different from the minimally coupled $\mathcal{N} = 2$ and $\mathcal{N} = 3$, $d = 4$ supergravity, as we will point out in the treatment below.

It is here worth pointing out that in the non-extremal case (i.e. $c \neq 0$) the expression generalizing Eq. (2.12), namely

$$R_H^2 (z_\infty, \tau_\infty, p, q) = \frac{S_{BH,c \neq 0} (z_\infty, \tau_\infty, p, q)}{\pi} = R_H^2 (z_\infty, \tau_\infty, p, q) =$$

$$= r_H^2 (z_\infty, \tau_\infty, p, q) - G_{ij}\Sigma_i^\tau\Sigma_j^r$$  

(2.14)

can be only guessed, but at present cannot be rigorously proved. Indeed, for static, spherically symmetric, asymptotically flat dyonic non-extremal BHs a first order formalism is currently unavailable, so there is no way to compute the scalar charges (beside the direct integration of the Eqs. of motion of the scalars, as far as we know at present feasible only for the (axion-)dilaton BH [77], and - partially - for $stu$ model [54]).

---

$^3$The BH event horizon, which is the unique geometrical horizon for extremal BHs (in which $c = 0 \not\iff r_+ = r_H \equiv r_+$; for a recent treatment, see e.g. [63]). It depends on the dyonic BH charges $p^A$ and $q_A$, but, in presence of non-vanishing scalar charges, also on the asymptotical scalar fields $(z_\infty, \tau_\infty)$.

In order to make contact with the Attractor Mechanism, and thus to characterize $r_H$ as the fixed point of the scalar radial dynamics (in the considered static, spherically symmetric and asymptotically flat extremal BH background), one has to evaluate $r_H$ at the peculiar geometrical locus in the (asymptotical) moduli space defined by the (non-degenerate) criticality condition of $V_{BH}$ (i.e. by $\partial V_{BH} = 0$, with $V_{BH} |_{\partial V_{BH} = 0} \neq 0$). Eq. (2.5) yields that

$$\Sigma^i (z_H (p, q), \tau_H (p, q), p, q) = 0 \ \forall i,$$

where $(z_H (p, q), \tau_H (p, q))$ are defined by

$$[\partial V_{BH} (z, \tau, p, q)]_{(z, \tau) = (z_H (p, q), \tau_H (p, q))} \equiv 0.$$

Thus, Eqs. (2.6) and (2.7) (or Eq. (2.12)) consistently yield that

$$r_H (z_H (p, q), \tau_H (p, q), p, q) = R_H (p, q).$$
3 $\mathcal{N} = 2$ Minimally Coupled Supergravity

We consider $\mathcal{N} = 2$, $d = 4$ ungauged supergravity minimally coupled (mc) [75] to $n_V$ Abelian vector multiplets, in the case in which the scalar manifold is given by the sequence of homogeneous symmetric rank-1 special Kähler manifolds

$$
\mathcal{M}_{N=2,mc,n} = \frac{G_{N=2,mc,n}}{H_{N=2,mc,n}} = \frac{SU(1,n)}{SU(n) \times U(1)}, \quad \text{dim}_{\mathbb{R}} = 2n, \quad n = n_V \in \mathbb{N}.
$$

The $1+n$ vector field strengths and their duals, as well as their asymptotical fluxes, sit in the fundamental representation of the $U$-duality group $G_{N=2,mc,n} = SU(1,n)$, in turn embedded in the symplectic group $Sp(2+2n,\mathbb{R})$.

The general analysis of the Attractor Equations, BH charge orbits and attractor moduli spaces of such a theory has been performed in [25] and [41].

By fixing the Kähler gauge such that $X^0 = 1$ and in a suitable system of local symplectic special coordinates, the geometry of $\mathcal{M}_{N=2,mc,n}$ is determined by the holomorphic prepotential function:

$$
\mathcal{F}(z) \equiv -\frac{i}{2} \left[ 1 - (z^i) \right]^2.
$$

The Kähler potential of $\mathcal{M}_{N=2,mc,n}$ can be computed to be ($\Lambda = 0, 1, ..., n$ throughout all the present Section, and $|z|^2 \equiv \sum_{i=1}^{n_V} |z^i|^2$)

$$
K(z,\bar{z}) = -\log \left[ i \left( \bar{X}^i F_{A} - X^A \bar{F}_{i} \right) \right] = -\log \left[ 2 \left( 1 - |z|^2 \right) \right],
$$

yielding the metric constraint $1 - |z|^2 > 0$, and the covariant and contravariant metric tensors to be respectively $(G_{ij}(z,\bar{z}) G^{ij}(z,\bar{z}) = \delta_i^j)$:

$$
G_{ij}(z,\bar{z}) = \frac{(1 - |z|^2)^2 \delta_{ij} + \overline{z}^i z^j}{(1 - |z|^2)^2} = 2e^K \delta_{ij} + 4e^{2K} z^i \overline{z}^j;
$$

$$
G^{ij}(z,\bar{z}) = \frac{(1 - |z|^2) \left( \delta_{ij} - z^i \overline{z}^j \right)}{2} = \frac{1}{2} e^{-K} \left( \delta_{ij} - z^i \overline{z}^j \right);
$$

From its very definition (see e.g. [84], and Refs. therein), the covariantly holomorphic $\mathcal{N} = 2$, $d = 4$ central charge function can be computed to be

$$
Z = e^{K/2} W = e^{K/2} \left[ q_0 + ip^0 + (q_i - ip^i) z^i \right] = \frac{1}{\sqrt{2} \sqrt{1 - |z|^2}} \left[ q_0 + ip^0 + (q_i - ip^i) z^i \right],
$$

where $W$ is the $\mathcal{N} = 2$, $d = 4$ superpotential (also named holomorphic central charge function, with Kähler weights $(2,0)$).

On the other hand, the so-called matter charges read

$$
Z_i \equiv D_i Z = \partial_i Z + \frac{1}{2} (\partial_i K) Z = e^{K/2} [\partial_i W + (\partial_i K) W] = \frac{1}{\sqrt{2} (1 - |z|^2)^{1/2}} \left[ (q_i - ip^i) (1 - |z|^2) + (q_0 + ip^0) \overline{z} + (q_j - ip^j) z^j \overline{z} \right].
$$

Here, $D$ denotes the $U(1)$-Kähler and $H_{N=2,mc,n}$-covariant differential operator. Due to the global vanishing of the $C_{ijkl}$-tensor of special Kähler geometry, there are only two ($U(1)$-Kähler-) and $H_{N=2,mc,n}$-invariants, namely

$$
\alpha_1 \equiv |Z|;
$$

$$
\alpha_2 \equiv \sqrt{Z_i \overline{Z}^i} = \sqrt{G_i^j Z_i \overline{Z}^j} = \sqrt{G_i^j (D_i Z) \overline{D}^j \overline{Z}},
$$

\footnote{In all our analysis we consider the (semi)classical limit of continuous (unquantized), large BH charges.}
both (homogeneous) of degree 1 in BH charges \((p,q)\) (in particular, square roots of quantities quadratic in \((p,q)\)). By a suitable rotation of \(U(n)\), the vector \(Z_i\) of matter charges can be chosen real and pointing in a given direction, e.g.

\[
Z_i = \sqrt{Z_j Z^j} \delta_{i1} = \alpha_2 \delta_{i1}.
\]  

As recalled at the start of the next Subsection, only \((\frac{1}{2}, \frac{1}{2})\)-BPS and non-BPS \((Z = 0)\) attractor flows are non-degenerate (i.e. corresponding to large BHs, see below) [25], and the corresponding (squared) first order fake superpotentials are ([40]; recall Eq. (4.3) and (4.5), respectively)

\[
W_{\frac{1}{2}, \text{BPS}}^2 \propto |Z|^2 = \alpha_1^2 = \\
= \frac{[\bar{q}_0 + i \bar{p}^0 + (q_i - ip^i) z^i] [\bar{q}_0 + i \bar{p}^0 + (q_j + ip^j) \bar{z}^j]}{2 (1 - |z|^2)};
\]

(3.10)

\[
W_{\text{non-BPS},Z=0}^2 = G^{\bar{7}} (D_i Z) \bar{D}_j \bar{Z} = \alpha_2^2 = \\
= \frac{1}{2 (1 - |z|^2)^2} \left( \delta^{\bar{7}} \bar{z} - z \bar{\delta}^7 \right) \\
\cdot \left[ (q_i - ip^i)(1 - |z|^2) + (q_0 + ip^0) \bar{z}^i + (q_r - ip^r) z^r \bar{z}^i \right],
\]

(3.11)

\[
(3.12)
\]

where use of Eqs. (3.6) and (3.7) was made.

### 3.1 Attractor Equations and their Solutions

The **BH effective potential** can be written as

\[
V_{\text{BH}} = |Z|^2 + G^{\bar{7}} (D_i Z) \bar{D}_j \bar{Z} = \alpha_1^2 + \alpha_2^2.
\]  

(3.13)

The \(N = 2, d = 4\) Attractor Eqs. in the case of minimal coupling to Abelian vector multiplets are nothing but the criticality conditions for such an \(H_{N=2,mc,n}\)-invariant (and Kähler-gauge-invariant) quantity. Such criticality conditions are satisfied for two classes of critical points:

- \((\frac{1}{2}, \frac{1}{2})\)-BPS:
  \[
  D_i Z = 0 \ \forall i = 1, \ldots, n \Leftrightarrow \alpha_2 = 0, \ Z \neq 0;
  \]
  
  (3.14)

- non-supersymmetric (non-BPS with \(Z = 0\)) :
  \[
  D_i Z \neq 0 \ (\text{at least for some } i), \ Z = 0 \Leftrightarrow \alpha_1 = 0.
  \]
  
  (3.15)

It is worth counting here the degrees of freedom related to Eqs. (3.14) and (3.15). The \((\frac{1}{2}, \frac{1}{2})\)-BPS criticality conditions (3.14) are \(n\) complex independent ones, thus all scalars are stabilized by such conditions. On the other hand, there is only one complex non-BPS \(Z = 0\) criticality condition (3.15). This fact paves the way to the possibility to have a moduli space of non-BPS \(Z = 0\) attractors, spanned by the \(n - 1\) complex scalars unstabilized by Eq. (3.15); this actually holds true [41], as it will be explicitly found below for the first time (see Subsection 3.1.2).

#### 3.1.1 \((\frac{1}{2}, \frac{1}{2})\)-BPS Attractors

An algebraic, equivalent approach to the direct resolution of the \(n\) complex \((\frac{1}{2}, \frac{1}{2})\)-BPS criticality conditions (3.14) is based on the resolution of the special Kähler geometry identities evaluated along the geometrical
locus in $\mathcal{M}_{N=2,\text{quad}, n}$ defined by the constraints (3.14). By following such an approach, the electric and magnetic BH charges are constrained as follows [2]:

$$
\begin{align*}
\begin{cases}
p^\Lambda = ie^{K/2}(ZX^\Lambda - Z\bar{X}^\Lambda); \\
q_\Lambda = ie^{K/2}(ZF_\Lambda - Z\bar{F}_\Lambda).
\end{cases}
\end{align*}
$$

(3.16)

Summing such two sets of symplectic-covariant Eqs., one gets

$$X^\Lambda q_\Sigma - p^\Lambda F_\Sigma = ie^{K/2} Z(\bar{Z}^\Lambda F_\Sigma - X^\Lambda \bar{F}_\Sigma),$$

(3.17)

in which the scalars $z^i$ and $\bar{z}^j$ and the central charge function $Z$ are understood to be evaluated at the BH horizon. Thence, we can proceed to solve for the scalars, stabilized at the BH horizon in terms of the BH charges; by rewriting Eqs. (3.17) in components, one achieves the following result:

$$
\begin{align*}
(\Lambda, \Sigma) &= (0, 0) : q_0 + ip^0 = 2 e^{K/2} Z; \\
(\Lambda, \Sigma) &= (0, i) : q_i - iz^i p^0 = - e^{K/2} Z(z^i + \bar{z}^i); \\
(\Lambda, \Sigma) &= (i, 0) : z^i q_0 + ip^i = e^{K/2} Z(z^i + \bar{z}^i); \\
(\Lambda, \Sigma) &= (i, i) : z^i (q_i - ip^i) = - 2 e^{K/2} Z |z|^2.
\end{align*}
$$

(3.18)

The decoupling of such $2n_V + 2$ real algebraic Eqs. in terms of the $n_V$ complex unknowns $z^i$ (the two additional real degrees of freedom residing in the homogeneity of degree 1 of the system (3.18) in BH charges) allows for an effortless resolution, yielding the following explicit expression of the $n$ complex scalars determining the $\frac{1}{2}$-BPS attractor scalar horizon configurations:

$$z^i_{BPS} = - \frac{(q_i + ip^i)}{q_0 - ip^0}, \quad \forall i = 1, \ldots, n.$$

(3.19)

Notice that all $n_V$ complex scalars $z^i$ are stabilized in terms of the BH charges, and thus, as well known, no classical moduli space for the $\frac{1}{2}$-BPS attractors exists at all. By recalling Eqs. (1.2) and (3.13), and plugging Eqs. (3.19) into Eqs. (3.6) and (3.7), one obtains that

$$
\frac{S_{BH,BPS}}{\pi} = \frac{A_{H,BPS}}{4} = V_{BH}|_{BPS} = \alpha_{1,BPS}^2 = |Z|_{BPS}^2 = \mathcal{I}_2 > 0,
$$

(3.20)

where $\mathcal{I}_2$ is the (unique) invariant of the fundamental/anti-fundamental $(1 + n, 1 + n)$ representation of the $U$-duality group $G_{N=2,mc,n}$ (not irreducible with respect to $G_{N=2,mc,n}$ itself), quadratic in BH charges (see Eq. (5.1) below):

$$
\mathcal{I}_2 = \frac{1}{2} \left[ q^2_0 - q^2_i + (p^0)^2 - (p^i)^2 \right] = \frac{1}{2} (q^2 + p^2),
$$

(3.21)

where $q^2 = \eta^{\Lambda\Sigma} q_\Lambda q_\Sigma$ and $p^2 = \eta^{\Lambda\Sigma} p^\Lambda p^\Sigma$, $\eta^{\Lambda\Sigma} = \eta_{\Lambda\Sigma}$ being the $(1 + n)$-dim. Lorentzian metric with signature $(+, - , \ldots, -)$ (see the discussion in Sect. 5). In terms of the dressed charges $Z$ and $D_i Z$, the (only apparently moduli-dependent) expression of $\mathcal{I}_2$ reads (see e.g. [25]):

$$
\mathcal{I}_2 = |Z|^2 - G^{ij} (D_i Z) \bar{D}_j Z = \alpha_1^2 - \alpha_2^2.
$$

(3.22)

It can be explicitly checked that $z^i_{BPS}$ given by Eqs. (3.19) satisfy the metric constraint $1 - |z^i|^2 > 0$ (yielded by Eq. (3.3)).

It is well known that $\frac{1}{2}$-BPS critical points of $V_{BH}$ are stable, at least as far as the metric $G_T$ of the special Kähler scalar manifold is positive definite (at such points); indeed, the $2n_V \times 2n_V$ (covariant)
Hessian matrix $\mathcal{H}^{V_{BH}}_{BPS}$ of $V_{BH}$ at its $\frac{1}{2}$-BPS critical points has rank $2n_V$, and it reads [4]:

$$\mathcal{H}^{V_{BH}}_{BPS} = 2|Z|_{BPS}^2 \begin{pmatrix} 0 & G_{ij} \\ G_{ji} & 0 \end{pmatrix}_{BPS} = \left[ q_0^2 - q_k^2 + (p^0)^2 - (p^k)^2 \right].$$

$$\begin{pmatrix} 0 & e^{K_{BPS}} \tilde{\delta}_{ij} + e^{2K_{BPS}} \tilde{\tau}_{i,BPS} \tilde{\tau}_{j,BPS} \\ e^{K_{BPS}} \tilde{\delta}_{ji} + e^{2K_{BPS}} \tilde{\tau}_{j,BPS} \tilde{\tau}_{i,BPS} & 0 \end{pmatrix} \right),$$

where use of the stabilization Eqs. (3.19) was made.

### 3.1.2 Non-BPS ($Z = 0$) Attractors and their Moduli Space

As yielded by Eqs. (3.15), non-BPS ($Z = 0$) attractor solutions are given by $D_i Z \neq 0$ for at least some $i \in \{1, \ldots, n_V\}$, and by the vanishing of the central charge $Z$, which in the considered theory reads as follows (within the metric constraint $1 - |z|^2 > 0$; recall Eq. (3.6)):

$$Z = 0 \Leftrightarrow q_0 + i p^0 = - (q_i - i p^i) z^i_{\text{non-BPS}}. \quad (3.24)$$

As noticed above, this is one complex Eq. in terms of $n_V$ complex unknowns $z^i$, thus at most only one of them will be stabilized in terms of the BH charges. Indeed, one can choose, without any loss of generality, to solve Eq. (3.24) for $z^1$, getting ($i = 2, \ldots, n_V$):

$$z^1_{\text{non-BPS}} = - \frac{(q_i - i p^i) z^{\hat{i}}_{\text{non-BPS}}}{q_i - i p^i} - \frac{q_0 + i p^0}{q_i - i p^i}. \quad (3.25)$$

The remaining scalars $z^i$ are not stabilized at the considered (non-degenerate) non-BPS $Z = 0$ critical points of $V_{BH}$. As known from group-theoretical arguments (see Table 3 of [41]), such scalars span a moduli space given by the rank-1 symmetric special Kähler manifold

$$\mathcal{M}_{N=2,mc,n,\text{non-BPS}} = \frac{SU (1,n-1)}{SU (n-1) \times U (1)} = \mathcal{M}_{N=2,mc,n-1,\text{dim}_R = 2 (n-1)} \quad (3.26)$$

The unique element of the sequence $\mathcal{M}_{N=2,mc,n}, n \in \mathbb{N}$, in which the non-BPS $Z = 0$ attractors have no associated moduli space is the $n = 1$ case (the so-called $t^2$ model), in which all non-degenerate critical points of $V_{BH}$ are stable, with no “flat” directions at all.

The existence of $n - 1$ “flat” directions at all orders in the (covariant) differentiation of $V_{BH}$ at its non-degenerate non-BPS $Z = 0$ critical points in the considered theory can be realized also by the following argument.

Firstly, it can be explicitly computed that the application of an odd number of covariant differential operators on $V_{BH}$ always yields a vanishing result (here the tilded indices can be either holomorphic or anti-holomorphic; $m \in \mathbb{N}$ throughout):

$$(D_{i_1} D_{i_2} \ldots D_{i_{2m-1}} V_{BH})_{\text{non-BPS}} = 0. \quad (3.27)$$

Thence, the $2n_V \times 2n_V$ (covariant) Hessian matrix $\mathcal{H}^{V_{BH}}_{\text{non-BPS}}$ of $V_{BH}$ at its non-BPS $Z = 0$ critical points can be computed to be:

$$\mathcal{H}^{V_{BH}}_{\text{non-BPS}} = 2\begin{pmatrix} 0 & (D_i Z) \overline{T}_j Z \\ (D_j Z) \overline{T}_i Z & 0 \end{pmatrix}_{\text{non-BPS}}, \quad (3.28)$$

and it has thus rank 2, with 2 strictly positive and $2n_V - 2$ vanishing real eigenvalues (massless “Hessian modes”).
In order to investigate the persistence of such \(2n_V - 2\) massless “Hessian modes” to higher order in the covariant differentiation of \(V_{BH}\), one can define a “putative” mass matrix \(\mathcal{H}^{V_{BH}}_{m=0}\) for scalars, such that \(\mathcal{H}^{V_{BH}}_{m=0} = \mathcal{H}^{V_{BH}}\) (covariant Hessian matrix of \(V_{BH}\)), in the following way:

\[
\mathcal{H}^{V_{BH}}_{m=0} = \left(D_i D_j D_{i_1} \ldots D_{i_{2m}} V_{BH}\right) \tilde{Z}^{i_1} \ldots \tilde{Z}^{i_{2m}},
\]

where \(\tilde{Z}\) denotes the relevant contravariant matter charge. It can be thus calculated that \(\mathcal{H}^{V_{BH}}_{m=0-\text{BPS}} = 2^{2m} (V_{BH,\text{non-BPS}})^m \mathcal{H}^{V_{BH}}_{\text{non-BPS}}\).

Therefore, regardless of \(m\) the “putative” mass matrix \(\mathcal{H}^{V_{BH}}_{m=0}\) has rank 2, with 2 strictly positive and \(2n_V - 2\) vanishing real eigenvalues, and these latter thus span a moduli space.

By recalling Eq. (3.7) and plugging Eq. (3.24) into the matter charges \(D_i Z\), one obtains:

\[
D_i Z |_{\text{non-BPS}} = \frac{q_i - i p_i}{\sqrt{2} \sqrt{1 - |z|^{2}_{\text{non-BPS}}}}.
\]

By such a result, \(\mathcal{H}^{V_{BH}}_{\text{non-BPS}}\) can be rewritten as follows:

\[
\mathcal{H}^{V_{BH}}_{\text{non-BPS}} = \frac{1}{1 - |z|^{2}_{\text{non-BPS}}} \begin{pmatrix} 0 & (q_i - i p_i)(q_j + i p_j) \\ (q_i - i p_i)(q_j + i p_j) & 0 \end{pmatrix}.
\]

Due to Eqs. (3.24) and (3.25), in general the matter charges \(D_i Z\) and \(\mathcal{H}^{V_{BH}}\) are not stabilized in terms of the BH charges at the considered non-BPS \(Z = 0\) critical points, but nevertheless this does not affect the moduli-independence of the BH entropy. Indeed, by recalling Eqs. (1.2) and (3.13), and plugging Eqs. (3.15) and (3.24) into Eqs. (3.6) and (3.7), one obtains that

\[
\frac{S_{BH,\text{non-BPS}}}{\pi} = \frac{A_{H,\text{non-BPS}}}{4} = V_{BH,\text{non-BPS}} = \alpha_{2,\text{non-BPS}}^{2} =
\]

\[
\left[G^\mathcal{O} (D_i Z) \Pi_{\mathcal{O}} \Pi_{\mathcal{O}}\right]_{\text{non-BPS}} = \left[G^\mathcal{O} (\partial Z) \Pi_{\mathcal{O}} \Pi_{\mathcal{O}}\right]_{\text{non-BPS}} =
\]

\[
= - I_2 > 0,
\]

where \(I_2\) is the (unique) quadratic \(G_{N=2,mc,n}\)-invariant given by Eqs. (3.21) and (3.22).

Thus, in \(N = 2, d = 4\) supergravity minimally coupled to Abelian vector multiplets, the BH charges supporting non-degenerate critical points of \(V_{BH}\) are split in two branches: the \((\frac{1}{2})\)-BPS one, defined by \(I_2 > 0\), and the non-BPS \((Z = 0)\) one, corresponding to \(I_2 < 0\).

### 3.2 Black Hole Parameters for \(\frac{1}{2}\)-BPS Flow

By using the explicit expressions of \(W_{BPS}^{2}\) given by Eq. (6.32), using the differential relations of special Kähler geometry of \(\mathcal{M}_{N=2,mc,n}\) (see e.g. [84], and Refs. therein), and exploiting the first order (fake supergravity) formalism discussed in Sect. 2, one respectively obtains the following expressions of the (square) ADM mass, covariant scalar charges and (square) effective horizon radius for the \(\frac{1}{2}\)-BPS
attractor flow\(^5\) [63]:
\[
\begin{align*}
 r_{H,BPS}^2 (z_\infty, \varpi_\infty, p, q) &= M_{ADM,BPS}^2 (z_\infty, \varpi_\infty, p, q) = W_{BPS}^2 (z_\infty, \varpi_\infty, p, q) = \\
 &= \lim_{\tau \to 0^-} |Z|^2 (z (\tau), \varpi (\tau), p, q) = \\
 &= \frac{[q_0 + ip^0 + (q_i - ip^i) z^i_\infty]}{2 (1 - |z_\infty|^2)} \left[ q_0 - ip^0 + (q_j + ip^j) \varpi^j_\infty \right]; \tag{3.34}
\end{align*}
\]
\[
\Sigma_{i,BPS} (z_\infty, \varpi_\infty, p, q) = 2 \lim_{\tau \to 0^-} \left( \partial_i W_{BPS} \right) (z (\tau), \varpi (\tau), p, q) = \\
= \frac{1}{M_{ADM,BPS} (z_\infty, \varpi_\infty, p, q)} \lim_{\tau \to 0^-} (\overline{Z} D_i Z) (z (\tau), \varpi (\tau), p, q) = \\
= \frac{1}{\sqrt{2} (1 - |z_\infty|^2)^{3/2}} \frac{q_0 - ip^0 + (q_j + ip^j) \varpi^j_\infty}{q_0 + ip^0 + (q_k - ip^k) z^k_\infty} \\
\cdot \left[ (q_i - ip^i) (1 - |z_\infty|^2) + (q_0 + ip^0) \varpi^0_\infty + (q_r - ip^r) z^r_\infty \varpi^r_\infty \right]; \tag{3.35}
\]
\[
R_{H,BPS}^2 = \lim_{\tau \to 0^-} \left[ W_{BPS}^2 (z (\tau), \varpi (\tau), p, q) + \\
-4G \overline{G} (z (\tau), \varpi (\tau)) (\partial_i W_{BPS}) (z (\tau), \varpi (\tau), p, q) \right] = \\
= \mathcal{I}_2 (p, q) = V_{BH,BPS} = \frac{S_{BH,BPS} (p, q)}{\pi}. \tag{3.36}
\]

Eq. (3.36) proves Eq. (2.13) for the \(1/2\)-BPS attractor flow of the \(N = 2, d = 4\) supergravity minimally coupled to \(n = n_V\) Abelian vector multiplets.

Notice that in the extremality regime \(c = 0\) the effective horizon radius \(R_H\), and thus \(A_H\) and the Bekenstein-Hawking entropy \(S_{BH}\) are independent on the particular vacuum or ground state of the considered theory, i.e. on \(\left(z^i_\infty, \varpi^i_\infty\right)\), but rather they depend only on the electric and magnetic charges \(q_A\) and \(p_A\), which are conserved due to the overall \((U(1))^{n+1}\) gauge-invariance. The independence on \(\left(z^i_\infty, \varpi^i_\infty\right)\) is of crucial importance for the consistency of the microscopic state counting interpretation of \(S_{BH}\), as well as for the overall consistency of the macroscopic thermodynamic picture of the BH. However, it is worth recalling that the ADM mass \(M_{ADM}\) generally does depend on \(\left(z^i_\infty, \varpi^i_\infty\right)\) also in the extremal case, as yielded by Eq. (3.34) for the considered \(1/2\)-BPS attractor flow.

Furthermore, Eq. (3.34) yields that the \(1/2\)-BPS attractor flow of the \(N = 2, d = 4\) supergravity minimally coupled to \(n = n_V\) Abelian vector multiplets does not saturate the marginal stability bound (see [83] and [54]).

\(^5\)Throughout the whole paper, for all the considered functions \(f (z, \varpi, p, q)\) we assume:
\[
( f (z, \varpi, p, q) )_\infty \equiv \lim_{\tau \to 0^-} f (z (\tau), \varpi (\tau), p, q) = f (z_\infty, \varpi_\infty, p, q).
\]

Furthermore, we assume \(f (z, \varpi, p, q)\) to be smooth enough to split the asymptotical limit of a product into the product of the asymptotical limits of the factors.
3.3 Black Hole Parameters for Non-BPS (Z = 0) Flow

By using the explicit expressions of $\mathcal{W}_{\text{non-BPS}}^2$ given by Eq. (3.12), using the differential relations of special Kähler geometry of $\mathcal{M}_{N=2,m,n}$ (see e.g. [84] and Refs. therein), and exploiting the first order (fake supergravity) formalism discussed in Sect. 2, one respectively obtains the following expressions of the (square) ADM mass, covariant scalar charges and (square) effective horizon radius for the non-BPS $Z = 0$ attractor flow [63]:

$$
\begin{align*}
\sqrt{2}^2 H_{\text{non-BPS}} (z_\infty, \overline{z}_\infty, p, q) &= M_{\text{ADM,non-BPS}}^2 (z_\infty, \overline{z}_\infty, p, q) = \mathcal{W}_{\text{non-BPS}}^2 (z_\infty, \overline{z}_\infty, p, q) =
\lim_{\tau \to 0} \left[ G(D, Z \overline{D}) \overline{D} \right] (z(\tau), \overline{z}(\tau), p, q) = \\
&= \frac{1}{2 \left( 1 - |z_\infty|^2 \right)^2} \left( \partial^\tau - z_\infty \partial^\tau \right) \cdot \\
&\quad \cdot \left[ (q_i - ip^i)(1 - |z_\infty|^2) + (q_0 + ip^0)z_\infty\overline{z}_\infty + (q_r - ip^r)z_\infty \overline{z}_\infty \right] \cdot \\
&\quad \cdot \left[ (q_j + ip^j)(1 - |z_\infty|^2) + (q_0 - ip^0)z_\infty \overline{z}_\infty + (q_n + ip^n)z_\infty \overline{z}_\infty \right] ;
\end{align*}
(3.37)

$$
\begin{align*}
\Sigma_{\text{non-BPS}} (z_\infty, \overline{z}_\infty, p, q) &= 2 \lim_{\tau \to 0} (\partial_\tau \mathcal{W}_{\text{non-BPS}}) (z(\tau), \overline{z}(\tau), p, q) = \\
&= \frac{1}{M_{\text{ADM,non-BPS}} (z_\infty, \overline{z}_\infty, p, q)} \lim_{\tau \to 0} (Z D, Z) (z(\tau), \overline{z}(\tau), p, q) = \\
&= \frac{1}{\sqrt{2}} \left( 1 - |z_\infty|^2 \right)^{-1/2} \cdot \\
&\quad \cdot \left[ (q_0 - ip^0 + (q_j + ip^j)z_\infty \overline{z}_\infty) \right] \cdot \\
&\quad \cdot \left[ (q_i - ip^i)(1 - |z_\infty|^2) + (q_0 + ip^0)z_\infty \overline{z}_\infty + (q_n - ip^n)z_\infty \overline{z}_\infty \right] \cdot \\
&\quad \cdot \left[ (q_j + ip^j)(1 - |z_\infty|^2) + (q_0 - ip^0)z_\infty \overline{z}_\infty + (q_n + ip^n)z_\infty \overline{z}_\infty \right] ;
\end{align*}
(3.38)

$$
\begin{align*}
R_{\text{H,non-BPS}}^2 &= \lim_{\tau \to 0} \left[ \mathcal{W}_{\text{non-BPS}}^2 (z(\tau), \overline{z}(\tau), p, q) + \\
&\quad -4G(D, Z \overline{D}) (\partial_\tau \mathcal{W}_{\text{non-BPS}}) (z(\tau), \overline{z}(\tau), p, q) \right] = \\
&= -\mathcal{T}_2 (p, q) = V_{\text{BH,non-BPS}} = \frac{S_{\text{BH,non-BPS}}(p, q)}{\pi}.
\end{align*}
(3.39)

Eq. (3.39) proves Eq. (2.13) for the non-BPS $Z = 0$ attractor flow of the the $N = 2, d = 4$ supergravity minimally coupled to $n = n_Y$ Abelian vector multiplets. The considerations made at the end of Subsect. 3.2 hold also for the considered attractor flow.
It is worth noticing out that Eqs. (3.36) and (3.39) are consistent, because, as pointed out above, the \((\frac{1}{2})\)-BPS- and non-BPS \((Z = 0)\)- supporting BH charge configurations in the considered theory are respectively defined by the quadratic constraints \(I_2(p, q) > 0\) and \(I_2(p, q) < 0\).

As yielded by Eqs. (3.35) and (3.38) for both non-degenerate attractor flows of the considered theory it holds the following relation among scalar charges and ADM mass:

\[
\Sigma_i = \frac{1}{M_{ADM}} \lim_{\tau \to 0^-} D_i (|Z|^2). \tag{3.40}
\]

Furthermore, Eq. (3.37) yields that the non-BPS \(Z = 0\) attractor flow of the \(\mathcal{N} = 2, d = 4\) supergravity minimally coupled to \(n \equiv n_V\) Abelian vector multiplets does not saturate the marginal stability bound (see [83] and [54]).

As it will be proved in the next Sections, for all non-degenerate attractor flows of the considered \(d = 4\) supergravities the marginal stability bound is not saturated. A more detailed discussion of such an issue falls beyond the scope of the present investigation, and it will be given elsewhere [85].

4 \(\mathcal{N} = 3\) Supergravity

The (Kähler) scalar manifold is [76]

\[
\mathcal{M}_{\mathcal{N}=3,n} = \frac{G_{\mathcal{N}=3,n}}{H_{\mathcal{N}=3,n}} = \frac{SU(3,n)}{SU(3) \times SU(n) \times U(1)}, \quad \text{dim}_R = 6n. \tag{4.1}
\]

The \(3 + n\) vector field strengths and their duals, as well as their asymptotical fluxes, sit in the fundamental \(3 + n\) representation of the \(U\)-duality group \(G_{\mathcal{N}=3,n} = SU(3,n)\), in turn embedded in the symplectic group \(Sp(6 + 2n, \mathbb{R})\).

\(Z_{AB} = Z_{[AB]} (A,B = 1,2,3 = \mathcal{N})\) is the central charge matrix, and \(Z_I (I = 1, ..., n)\) are the matter charges, where \(n \in \mathbb{N}\) is the number of matter (Abelian vector) multiplets coupled to the gravity multiplet. By a suitable transformation of the \(R\)-symmetry \(U(3)\), \(Z_{AB}\) can be skew-diagonalized by putting it in the normal form (see e.g. [40] and Refs. therein):

\[
Z_{AB} = \begin{pmatrix} Z_1 & X \\ 0 & 0 \end{pmatrix}, \tag{4.2}
\]

where \(\epsilon\) is the \(2 \times 2\) symplectic metric, and \(Z_1 \in \mathbb{R}^+\) is the unique \(\mathcal{N} = 3\) (moduli-dependent) skew-eigenvalue, which can be expressed in term of the unique \(U(3)\)(and also \(H_{\mathcal{N}=3,n} = U(3) \times U(n)\))-invariant as follows:

\[
Z_1 = \sqrt{\frac{1}{2} \sqrt{Z_{AB}Z^{AB}}}. \tag{4.3}
\]

On the other hand, by a suitable rotation of \(U(n)\), the vector \(Z_I\) of matter charges can be chosen real and pointing in a given direction, e.g.

\[
Z_I = \rho \delta_{I1}, \tag{4.4}
\]

where \(\rho\) can be expressed the unique \(U(n)\)(and also \(H_{\mathcal{N}=3,n} = U(3) \times U(n)\))-invariant as:

\[
\rho = \sqrt{Z^I Z_I}. \tag{4.5}
\]

The simplest holomorphic parametrization of \(\mathcal{M}_{\mathcal{N}=3,n}\) can be written in terms of the \((3 + n) \times (3 + n)\) coset representative [86, 87]

\[
L = \begin{pmatrix} \sqrt{1 + XX^\dagger} & X \\ X^\dagger & \sqrt{1 + X^\dagger X} \end{pmatrix}, \tag{4.6}
\]

where \(X\) is a complex \(n \times 3\) matrix in the bi-fundamental of \(SU(3) \times SU(n) = H_{\mathcal{N}=3,n} \setminus U(1)\), whose component are nothing but the \(3n\) complex scalars \(z^I (i = 1, ..., 3n)\) spanning \(\mathcal{M}_{\mathcal{N}=3,n}\). The embedding of \(G_{\mathcal{N}=3,n}\) into the symplectic group \(Sp(6 + 2n, \mathbb{R})\):

\[
SU(3,n) \rightarrow Sp(6 + 2n, \mathbb{R}) \quad , \quad g \equiv L(z) \rightarrow S(g) \equiv S(L(z)), \tag{4.7}
\]
is determined by the $(6 + 2n) \times (6 + 2n)$ matrix

\[
S(g) = \left( \begin{array}{cc} \phi_0 & \overline{\phi}_1 \\ \phi_1 & \overline{\phi}_0 \end{array} \right) \in SU(3, n) \subset Sp(6 + 2n, \mathbb{R}),
\]  

(4.8)
such that the $(3 + n) \times (3 + n)$ sub-blocks $\phi_0$ and $\phi_1$ satisfy the relations

\[
\phi_0^\dagger \phi_0 - \phi_1^\dagger \phi_1 = 1, \quad \phi_0^\dagger \phi_1 - \phi_1^\dagger \phi_0 = 0.
\]  

(4.9)

Let us here recall that in the Gaillard-Zumino formalism [88], the vector kinetic matrix can be written as ($\Lambda = 1, 2, 3, 4, ..., 3 + n$ throughout all the present Section)

\[
N_{\Lambda \Sigma} = (\phi_0^\dagger + \phi_1^\dagger)^{-1}(\phi_0^\dagger - \phi_1^\dagger).
\]  

(4.10)

The embedding $SU(3, n) \rightarrow Sp(6 + 2n, \mathbb{R})$ is determined once $S$ is written as a functions of $X(z)$, namely

\[
S(X) = \begin{pmatrix} \sqrt{1 + XX^\dagger} & 0 & 0 & X \\ 0 & \sqrt{1 + X^\dagger X} & X^\dagger & 0 \\ 0 & X^\dagger & \sqrt{1 + XX^\dagger} & 0 \\ X^\dagger & 0 & 0 & \sqrt{1 + X^\dagger X} \end{pmatrix},
\]  

(4.11)

that is

\[
\phi_1 = \begin{pmatrix} \sqrt{1 + XX^\dagger} \\ 0 \\ X^\dagger \\ 0 \end{pmatrix};
\]  

(4.12)

\[
\phi_0 = \sqrt{1 + \phi_1 \phi_1} \begin{pmatrix} \sqrt{1 + XX^\dagger} & 0 \\ 0 & \sqrt{1 + X^\dagger X} \end{pmatrix}.
\]  

(4.13)

The vector kinetic matrix $N_{\Lambda \Sigma}$ can be written in terms of the $(3 + n) \times (3 + n)$ symplectic sections (and their inverse) as follows (see e.g. [84], and Refs. therein):

\[
N_{\Lambda \Sigma} = h_{\Omega \Lambda} (f^{-1})^\Omega_{\Sigma}.
\]  

(4.14)

The explicit dependence of the symplectic sections on the sub-blocks of $S(X)$ is simply

\[
h_{\Lambda \Sigma} = -\frac{i}{\sqrt{2}} (\phi_0 - \phi_1), \quad f^\Lambda_{\Sigma} = \frac{1}{\sqrt{2}} (\phi_0 + \phi_1),
\]  

(4.15)

whereas in terms of the matrix $X(z)$ they read

\[
h_{\Lambda \Sigma} = -\frac{i}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + XX^\dagger} & -X^\dagger \\ -X & \sqrt{1 + X^\dagger X} \end{pmatrix} \equiv (h_{\Lambda|AB}, \overline{h}_{\Lambda|I}) \equiv h;
\]  

(4.16)

\[
f^\Lambda_{\Sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + XX^\dagger} & \overline{X} \\ X^\dagger & \sqrt{1 + X^\dagger X} \end{pmatrix} \equiv (f^\Lambda_{AB}, f^\Lambda_I) \equiv f.
\]  

(4.17)

By rewriting Eqs. (4.9) in terms of the symplectic sections, one finds [89, 90]

\[
i (f^\dagger h - h^\dagger f) = 1;
\]  

(4.18)

\[
h^T f - hf^T = 0.
\]  

(4.19)

The central charge matrix $Z_{AB}$ and the matter charges $Z_I$ are respectively defined as the integral over the 2-sphere at infinity $S^2_\infty$ of the dressed graviphoton and matter field strengths [89, 90, 91]:

\[
Z_{AB} = -\int_{S^2_\infty} T_{AB} = -\int_{S^2_\infty} T_{\Lambda\Lambda}^\Lambda = f_{AB}^\Lambda q_{\Lambda} - h_{\Lambda|AB} p^\Lambda;
\]  

(4.20)

\[
Z_I = -\int_{S^2_\infty} T_I = -\int_{S^2_\infty} T_I^\Lambda = f_I^\Lambda q_{\Lambda} - h_{\Lambda|I} p^\Lambda.
\]  

(4.21)
Using the explicit expression for the symplectic sections given in Eqs. (4.16) and (4.17), one obtains

\[
Z_{AB} = \frac{1}{\sqrt{2}} \left[ (\sqrt{1+XX^\dagger})^{C}_{\ AB} (q_C + ip^C) + XX^\dagger (q_i - ip^i) \right]; \quad (4.22)
\]

\[
Z_I = \frac{1}{\sqrt{2}} \left[ (X^\dagger)^A (q_A - ip^A) + (\sqrt{1+XTX^\dagger})^i (q_i + ip^i) \right]. \quad (4.23)
\]

As recalled at the start of the next Subsection, only \((\frac{1}{2})\)BPS and non-BPS \((Z_{AB} = 0)\) attractor flows are non-degenerate (i.e. corresponding to large BHs), and the corresponding (square) first order fake superpotentials are \(([40]; \text{recall Eq. (4.3) and (4.5), respectively})\)

\[
W^2_{(\frac{1}{2})\text{BPS}} = \frac{1}{2} Z_{AB} Z^{AB} = Z_1^2 =
\]

\[
= \frac{1}{2} \left[ \left( q_C - ip^C \right) \left( \sqrt{1+XX^\dagger} \right)^C (q_i + ip^i) (X^\dagger)^i \right].
\]

\[
= \frac{1}{2} \left[ (1 + XX^\dagger)^{AB} (q_A - ip^A)(q_B + ip^B) +
\right.
\]

\[
\left. + (\sqrt{1+XX^\dagger} X)^{A\ d} (q_i + ip^i)(q_A + ip^A) +
\right.
\]

\[
\left. + (X^\dagger \sqrt{1+XX^\dagger} X^\dagger)^{B\ j} (q_A - ip^A)(q_j - ip^j) +
\right.
\]

\[
\left. + (X^\dagger X)^{k\ i} (q_i + ip^i)(q_k - ip^k) \right];
\]

\[
(4.24)
\]

\[
W^2_{\text{non-BPS}, (Z_{AB} = 0)} = Z_1 Z^I = \rho^2 =
\]

\[
= \frac{1}{2} \left[ \left( q_D + ip^D \right) X^D (q_i - ip^i) \left( \sqrt{1+XTX^\dagger} \right)^i \right].
\]

\[
= \frac{1}{2} \left[ (X^\dagger)^C \left( q_C - ip^C \right) + \left( \sqrt{1+XTX^\dagger} \right)^i (q_i + ip^i) \right].
\]

\[
= \frac{1}{2} \left[ (XX^\dagger)^{CD} (q_C + ip^C)(q_D - ip^D) +
\right.
\]

\[
\left. + (\sqrt{1+XX^\dagger} X)^{A\ d} (q_i - ip^i)(q_C - ip^C) +
\right.
\]

\[
\left. + (X^\dagger \sqrt{1+XX^\dagger} X^\dagger)^{B\ j} (q_D + ip^D)(q_i + ip^i) +
\right.
\]

\[
\left. + (1 + XX^\dagger)^{k\ i} (q_i - ip^i)(q_k + ip^k) \right].
\]

\[
(4.25)
\]

where Eqs. (4.22) and (4.23) were used. Notice that, since all the contractions of \(SU(3)\) and \(SU(n)\) indices of electric and magnetic BH charges are uniquely defined with respect to the row or columns of the matrix \(X\), every transposition index has been suppressed in Eqs. (4.24) and (4.25).

By introducing the complexified graviphoton and matter BH charges respectively as follows:

\[
Q_C = q_C + ip^C;
\]

\[
Q_i = q_i + ip^i,
\]

\[
(4.26)
\]

\[
(4.27)
\]
Eqs. (4.24) and (4.25) can be rewritten as follows:

\[
W_{BP S}^2 = \frac{1}{2} \left[ (1 + XX^\dagger)^{AB}Q_A Q_B + (\sqrt{1 + XX^\dagger} X)^{Ai}Q_i Q_A + (X^\dagger X)^{k\ell}Q_k Q_{\ell} \right]; \tag{4.28}
\]

\[
W_{\text{non-BPS}}^2 = \frac{1}{2} \left[ (XX^\dagger)^{AB}Q_A Q_B + (\sqrt{1 + X^\dagger X} X^\dagger)^{iA}Q_i Q_A + (X^\dagger X)^{k\ell}Q_k Q_{\ell} + (1 + X^\dagger X)^{kl}Q_k Q_{\ell} \right]. \tag{4.29}
\]

4.1 Attractor Equations and their Solutions

The BH effective potential can be written as

\[
V_{BH} = \frac{1}{2} Z_{AB} \overline{Z}^{AB} + Z_I \overline{Z}^I = \rho^2. \tag{4.30}
\]

The \( N = 3, d = 4 \) Attractor Eqs. are nothing but the criticality conditions for such an \( H_{N=3,n} \)-invariant (and Kähler-gauge-invariant) quantity. Such criticality conditions are satisfied for two classes of critical points:

- \((\frac{1}{4})\)-BPS:

\[
\begin{align*}
Z_I &= 0 \forall I = 1, \ldots, n \Leftrightarrow \rho = 0; \\
Z_{AB} &\neq 0; \tag{4.31}
\end{align*}
\]

- non-supersymmetric (non-BPS with \( Z_{AB} = 0 \)):

\[
\begin{align*}
Z_I &\neq 0 \text{ (at least for some } I); \\
Z_{AB} &= 0 \Leftrightarrow Z_I = 0. \tag{4.32}
\end{align*}
\]

It is worth counting here the degrees of freedom related to Eqs. (4.31) and (4.32).

The \( \frac{1}{4} \)-BPS criticality conditions (4.31) are \( n \) complex independent ones, thus a moduli space of \( \frac{1}{4} \)-BPS attractors, spanned by the \( 2n \) complex scalars unstabilized by Eq. (4.31) might - and actually does [25] - exist.

Furthermore, there are only three complex non-BPS \( Z = 0 \) criticality conditions (4.32). This fact paves the way to the possibility to have a moduli space of non-BPS \( (Z_{AB} = 0) \) attractors, spanned by the \( 3(n-1) \) complex scalars unstabilized by Eq. (4.32); this actually holds true [41], as it will be explicitly found below (see Subsection 4.1.2).

4.1.1 \( \frac{1}{4} \)-BPS Attractors and their Moduli Space

By inserting the \((\frac{1}{4})\)-BPS criticality conditions (4.31) into the expression (4.23) of the matter charges \( Z_I \) and recalling the definitions (4.26) and (4.27) of the complexified BH charges, one obtains

\[
(X^\dagger)^{Ai}Q_A = -\left( \sqrt{1 + X X^\dagger} \right)^{iA}Q_i, \forall I = 1, \ldots, n. \tag{4.33}
\]

By plugging such an expression and its Hermitian conjugate into Eq. (4.28), and using the identity (holding for any matrix \( A \))

\[
A^\dagger \sqrt{1 + AA^\dagger} = \sqrt{1 + A^\dagger A} A^\dagger \tag{4.34}
\]

and its Hermitian conjugate

\[
\sqrt{1 + AA^\dagger} A = A \sqrt{1 + A^\dagger A}, \tag{4.35}
\]

one obtains that
where in the last step we exploited the Hermiticity of \( XX^\dagger \), yielding that \((\cdot, \cdot)_{XX^\dagger}\) denotes the \( XX^\dagger\)-dependent square norm of complexified BH charges

\[
(X X^\dagger)^{AB} Q_A Q_B = (Q, Q)_{XX^\dagger} = (\overline{Q}, Q)_{XX^\dagger} = (XX^\dagger)^{AB} \overline{Q}_A Q_B .
\]

By recalling Eqs. (1.2) and (4.30), and using Eq. (4.36), one achieves the following result:

\[
\frac{S_{BH,BPS}}{\pi} = \frac{A_{H,BPS}}{4} = V_{BH}|_{BPS} = Z_{1,BPS}^2 = \frac{1}{2} (Z_{AB} \overline{Z}^{AB})_{BPS} = \frac{1}{2} (Q_A \overline{Q}_A - Q_i \overline{Q}_i) = \mathcal{I}_2 > 0.
\]

Here \( \mathcal{I}_2 \) denotes the (unique) invariant of the fundamental/anti-fundamental \((3 + n, n + 3)\) representation of the \(U\)-duality group \( G_{N=3,n} \) (not irreducible with respect to \( G_{N=3,n} \) itself), quadratic in BH charges (see Eq. (5.1) below):

\[
\mathcal{I}_2 = \frac{1}{2} \left[ q_A^2 - q_i^2 + (p^A)^2 - (p^i)^2 \right] = \frac{1}{2} (q^2 + p^2),
\]

where \( q^2 \equiv \eta^{AB} q_A q_B \) and \( p^2 \equiv \eta^{AB} p_A p_B \), \( \eta^{AB} = \eta_{AB} \) being the \((3 + n)\)-dim. Lorentzian metric with signature \((3, n)\) (see the discussion in Sect. 5). In terms of the dressed charges \( Z_{AB} \) and \( Z_i \), the (only apparently moduli-dependent) expression of \( \mathcal{I}_2 \) reads (see e.g. [91, 89, 90]):

\[
\mathcal{I}_2 = \frac{1}{2} Z_{AB} \overline{Z}^{AB} - Z_i \overline{Z}^i = Z_i^2 - \rho^2.
\]

As mentioned above, the \( \frac{1}{3} \)-BPS criticality conditions (4.31) or (4.33) are a set of \( n \) complex equations, which thus does not stabilize all the \( 3n \) complex scalar fields \( z_i \) in terms of the BH electric and magnetic charges. In [56] the residual \( 2n \) unstabilized scalars have been shown to span the \( \frac{1}{3} \)-BPS moduli space

\[
\mathcal{M}_{N=3,n,BPS} = \frac{SU(2,n)}{SU(2) \times SU(n) \times U(1)}, \quad \text{dim}_R = 4n.
\]

### 4.1.2 Non-BPS (\( Z_{AB} = 0 \)) Attractors and their Moduli Space

By inserting the non-BPS (\( Z_{AB} = 0 \)) criticality conditions (4.32) into the expression (4.22) of the central charge matrix \( Z_{AB} \) and recalling the definitions (4.26) and (4.27) of the complexified BH charges, one obtains

\[
\left( \sqrt{1 + XX^\dagger} \right)^{C}_{AB} Q_C = - (X^\dagger)^{AB} \overline{Q}_i, \quad \forall A, B = 1, 2, 3.
\]

By plugging such an expression and its Hermitian conjugate into Eq. (4.29), and using the identities (4.34) and (4.35), one obtains that
\[
2 \left(ZI\bar{Z}'\right)_{\text{non-BPS}} = (XX')^{AB} Q_A \bar{Q}_B + (X' \sqrt{1 + XX'})^i A_i Q_i \bar{Q}_A + \\
+ (\sqrt{1 + XX'} X)^B j Q_B Q_j + (1 + X' X)^k i \bar{Q}_k Q_l = \\
= (XX')^{AB} Q_A \bar{Q}_B - (1 + XX')^{AB} Q_A \bar{Q}_B + \\
- (X' X)^i j \bar{Q}_i Q_j + (1 + X' X)^k i \bar{Q}_k Q_l = \\
= Q_i \bar{Q}_i - Q_A \bar{Q}_A.
\]

(4.43)

where once again Eq. (4.37) was used.

By recalling Eqs. (1.2) and (4.30), and using Eq. (4.43), one achieves the following result:

\[
\frac{S_{BH,\text{non-BPS}}}{\pi} = \frac{A_{H,\text{non-BPS}}}{4} = V_{BH,\text{non-BPS}} = \rho_{\text{non-BPS}}^2 = \left(ZI\bar{Z}'\right)_{\text{non-BPS}} = \\
= -\frac{1}{2} (Q_A \bar{Q}_A - Q_i \bar{Q}_i) = -\mathcal{I}_2 > 0,
\]

(4.44)

where \(\mathcal{I}_2\) is the quadratic \(G_{N=3,n}\)-invariant, given by Eqs. (4.39) and (4.40).

As mentioned above, the non-BPS criticality conditions (4.32) or (4.42) are a set of 3 complex equations, which thus does not stabilize all the 3n complex scalar fields \(z^i\) in terms of the BH electric and magnetic charges. In [56] the residual 3 \((n-1)\) unstabilized scalars have been shown to span the non-BPS \((Z_{AB} = 0)\) moduli space

\[
\mathcal{M}_{N=3,n,\text{non-BPS}} = \frac{SU(3, n-1)}{SU(3) \times SU(n-1) \times U(1)} = \mathcal{M}_{N=3,n-1}, \ \text{dim}_R = 6(n-1).
\]

(4.45)

Thus, as it holds in symmetric \(N = 2, d = 4\) supergravity minimally coupled to \(n_U = n\) Abelian vector multiplets, also in the considered \(N = 3, d = 4\) supergravity the BH charges supporting non-degenerate critical points of \(V_{BH}\) are split in two branches: the \((\frac{1}{3})\)-BPS one, defined by \(\mathcal{I}_2 > 0\), and the non-BPS \((Z_{AB} = 0)\) one, corresponding to \(\mathcal{I}_2 < 0\).

### 4.2 Black Hole Parameters for \(\frac{1}{3}\)-BPS Flow

By using the Maurer-Cartan Eqs. of \(N = 3, d = 4\) supergravity (see e.g. [91, 89, 90]), one gets [40]

\[
\partial_t Z_1 = \partial_t W_{BPS} = \frac{1}{2 \sqrt{2}} P_{IAB,1} Z^{\prime AB} = \frac{1}{4Z_1} P_{IAB,1} Z^{\prime AB} = \frac{1}{4Z_1} P_{IAB,1} Z^{\prime AB},
\]

(4.46)

where \(P_{IAB,1} dz^i\) is the holomorphic Vielbein of \(\mathcal{M}_{N=3,n}\). Here, \(\nabla\) denotes the \(U(1)\)-Kähler and \(H_{N=3,n}\)-covariant differential operator.

Thus, by using the explicit expressions of \(W_{BPS}^2\) given by Eq. (4.28), using the Maurer-Cartan Eqs. of \(N = 3, d = 4\) supergravity (see e.g. [91, 89, 90]), and exploiting the first order (fake supergravity) formalism discussed in Sect. 2, one respectively obtains the following expressions of the (square) ADM mass, covariant scalar charges and (square) effective horizon radius for the \(\frac{1}{3}\)-BPS attractor flow:

\[
\begin{align*}
\tau^2_{H,BPS}(z_\infty, \bar{z}_\infty, p, q) &= M_{A,ADM,BPS}^2 (z_\infty, \bar{z}_\infty, p, q) = W_{BPS}^2 (z_\infty, \bar{z}_\infty, p, q) = \\
&= \frac{1}{2} \lim_{\tau \to 0^+} \left( Z_{AB} Z^{\prime AB} \right) (z(\tau), \bar{z}(\tau), p, q) = \\
&= \frac{1}{2} \left[ (1 + X_\infty X_\infty)^{AB} \bar{Q}_A Q_B + (\sqrt{1 + X_\infty X_\infty} X_\infty) \bar{A}_i Q_i \bar{Q}_A + \\
&\quad + (\sqrt{1 + X_\infty X_\infty})^j B \bar{Q}_B \bar{Q}_j + (X_\infty X_\infty)^k i \bar{Q}_k Q_l \right] \tag{4.47}
\end{align*}
\]
\[
\Sigma_{AB} (z, \bar{z}, p, q) = 2 \lim_{\tau \to 0^+} (\partial_i W_{BPS}) (z (\tau), \bar{z} (\tau), p, q) = \\
= \frac{1}{\sqrt{2}} \left[ \frac{P_{IAB,1} \bar{z} \bar{Z}^{AB}}{\sqrt{Z_{CD} Z^{CD}}} \right]_{\infty} = \frac{1}{2} \left[ \frac{1}{Z} P_{IAB,1} \bar{Z} \bar{Z}^{AB} \right]_{\infty} = \\
= \frac{1}{2 M_{ADM,BPS} (z, \bar{z}, p, q)} \left( P_{IAB,1} \bar{Z} \bar{Z}^{AB} \right)_{\infty} 
\]

\[
R_{H,BPS}^2 = \lim_{\tau \to 0^+} \left[ \begin{array}{c}
W_{BPS}^2 (z (\tau), \bar{z} (\tau), p, q) + \\
-4 G^7 (z (\tau), \bar{z} (\tau)) (\partial_i W_{BPS}) (z (\tau), \bar{z} (\tau), p, q) \cdot \\
\end{array} \right] 
\]

\[
= \mathcal{I}_2 (p, q) = V_{BH,BPS} = \frac{S_{BH,BPS} (p, q)}{\pi}, 
\]

where

\[
X_\infty = \lim_{\tau \to 0^-} X (\tau) .
\]

Throughout all the treatment, the subscript "\( \infty \)" indicates the evaluation at the scalars at radial infinity \( z^\infty \).

Eq. (4.49) proves Eq. (2.13) for the \( \frac{1}{3} \)-BPS attractor flow of the considered \( \mathcal{N} = 3, d = 4 \) supergravity. Such a result was obtained by using Eq. (4.46) and computing that

\[
G^7 P_{IAB,1} T_{JEF} \bar{Z} \bar{Z}^{AB} Z^{EF} = \delta_{IJ} (\delta_{AE} \delta_{BF} - \delta_{AF} \delta_{BE}) \]

was exploited.

The considerations made at the end of Subsect. 3.2 hold also for the considered attractor flow.

As pointed out above, the same also holds for the \( \frac{1}{3} \)-BPS attractor flow of) \( \mathcal{N} = 2, d = 4 \) supergravity minimally coupled to Abelian vector multiplets (see Eq. (150) of [63]), in which the (unique) invariant of the \( U \)-duality group \( SU \) \( (1, n) \) is quadratic in BH electric and magnetic charges. Such a similarity is ultimately due to the fact that \( SU \) \( (m, n) \) is endowed with a pseudo-Hermitian quadratic form built out of the fundamental \( m + n \) and antifundamental \( m + n \) representations.

Furthermore, Eq. (4.47) yields that the \( \frac{1}{3} \)-BPS attractor flow of the \( \mathcal{N} = 3, d = 4 \) supergravity does not saturate the marginal stability bound (see [83] and [54]; see also the discussion at the end of Subsect. 3.3).

### 4.3 Black Hole Parameters for Non-BPS (\( Z_{AB} = 0 \)) Flow

By using the Maurer-Cartan Eqs. of \( \mathcal{N} = 3, d = 4 \) supergravity (see e.g. [91, 89, 90]), one gets [40]

\[
\partial_i \rho = \partial_i W_{non-BPS} = \frac{1}{4} P_{IAB,1} \bar{z} \bar{Z}^{AB} \frac{1}{\sqrt{Z_J Z^J}} = \frac{P_{IAB,1} \bar{Z} \bar{Z}^{AB}}{4 \rho} .
\]

Thus, by using the explicit expressions of \( W_{non-BPS}^2 \) given by Eq. (4.29), using the Maurer-Cartan Eqs. of \( \mathcal{N} = 3, d = 4 \) supergravity (see e.g. [91, 89, 90]), and exploiting the first order (fake supergravity)
formalism discussed in Sect. 2, one respectively obtains the following expressions of the (square) ADM mass, covariant scalar charges and (square) effective horizon radius for the non-BPS attractor flow:

\[
r^2_{H,\text{non-BPS}} (z_\infty, \bar{z}_\infty, p, q) = M^2_{\text{ADM,non-BPS}} (z_\infty, \bar{z}_\infty, p, q) = \mathcal{W}^2_{\text{non-BPS}} (z_\infty, \bar{z}_\infty, p, q) = \mathcal{W}^2_{\text{non-BPS}} (z_\infty, \bar{z}_\infty, p, q) = \\
= \lim_{\tau \to 0^-} \left( Z_i Z^i \right) (z (\tau), \bar{z} (\tau), p, q) = \\
= \frac{1}{2} \left[ (X_\infty X_\infty)^{AB} Q_A \bar{Q}_B + (\sqrt{1 + X_\infty^1 X_\infty^{1'}})^{iA} \bar{Q}_A + + (X_\infty \sqrt{1 + X_\infty^1 X_\infty^{1'}})^{AB} j Q_B j + (1 + X_\infty^1 X_\infty^{1'}) Q_f \bar{Q}_f \right] ;
\]

(4.54)

\[
\Sigma_{i,\text{non-BPS}} (z_\infty, \bar{z}_\infty, p, q) = 2 \lim_{\tau \to 0^-} (\partial_i \mathcal{W}_{\text{non-BPS}}) (z (\tau), \bar{z} (\tau), p, q) = \\
= \frac{1}{2} \left[ \frac{P_{iAB,i} Z^i Z^{AB}}{Z_i Z^i} \right] = \frac{1}{2} \left[ \frac{P_{iAB,i} Z^i Z^{AB}}{\rho} \right] = \\
= \frac{1}{2M_{\text{ADM,non-BPS}}} \left( Z_i Z^i \right) \mathcal{W}^{AB} (\rho) ;
\]

(4.55)

\[
R^2_{H,\text{non-BPS}} = \lim_{\tau \to 0^-} \left[ \mathcal{W}^2_{\text{non-BPS}} (z (\tau), \bar{z} (\tau), p, q) + - 4G^{TA} (z (\tau), \bar{z} (\tau)) (\partial_i \mathcal{W}_{\text{non-BPS}}) (z (\tau), \bar{z} (\tau), p, q) \right] = \\
= - T_2 (p, q) = V_{BH,\text{non-BPS}} = \frac{S_{BH,\text{non-BPS}} (p, q)}{\pi} .
\]

(4.56)

Eq. (4.49) proves Eq. (2.13) for the non-BPS ($Z_{AB} = 0$) attractor flow of the considered $\mathcal{N} = 3, d = 4$ supergravity. Such a result was obtained by using Eq. (4.53) and computing that

\[
4G^{TA} (\partial_i \mathcal{W}_{\text{non-BPS}}) Z_i \mathcal{W}_{\text{non-BPS}} = 4G^{TA} (\partial_i \rho) Z_i \rho = \\
= \frac{1}{4} \delta_{IK} (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}) \frac{Z^{AB} Z^{CD} Z^{IJ} Z^K}{Z_i Z^i} = \\
= \frac{1}{2} Z_{AB} Z^{AB} = 2 \tau^2 .
\]

(4.57)

The considerations made at the end of Subsect. 3.2 hold also for the considered attractor flow.

It is worth noticing out that Eqs. (4.49) and (4.56) are consistent, because, as pointed out above, the ($\frac{1}{3}$-BPS) and non-BPS ($Z_{AB} = 0$)- supporting BH charge configurations in the considered theory are respectively defined by the quadratic constraints $T_2 (p, q) > 0$ and $T_2 (p, q) < 0$.

As yielded by Eqs. (4.48) and (4.55) for both non-degenerate attractor flows of the considered theory it holds the following relation among scalar charges and ADM mass:

\[
\Sigma_i = \frac{1}{2M_{\text{ADM}}} \lim_{\tau \to 0^-} P_{iAB,i} Z^{AB} .
\]

(4.58)

Furthermore, Eq. (4.54) yields that the non-BPS $Z_{AB} = 0$ attractor flow of the $\mathcal{N} = 3, d = 4$ supergravity does not saturate the marginal stability bound (see [83] and [54]; see also the discussion at the end of Subsect. 3.3).
5 Black Hole Entropy in Minimally Coupled $\mathcal{N} = 2$ and $\mathcal{N} = 3$ Supergravity

It is here worth remarking that the classical Bekenstein-Hawking [72] $d = 4$ BH entropy $S_{BH}$ for minimally coupled $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supergravity is given by the following $SU(m,n)$-invariant expression:

$$S_{BH} = \frac{1}{2} |q^2 + p^2|,$$

(5.1)

where $q^2 = \eta^{\Lambda\Sigma} q_{\Lambda} q_{\Sigma}$ and $p^2 = \eta_{\Lambda\Sigma} p^{\Lambda} p^{\Sigma}$, $\eta^{\Lambda\Sigma} = \eta_{\Lambda\Sigma}$ being the Lorentzian metric with signature $(m,n)$. As said above, $\mathcal{N} = 2$ is obtained by putting $m = 1$, whereas $\mathcal{N} = 3$ is given by $m = 3$ (see Eqs. (3.21) and (4.39) above, respectively). Thus, in Eq. (5.1) the positive signature pertains to the graviphoton charges, while the negative signature corresponds to the charges given by the fluxes of the vector field strengths from the matter multiplets.

The supersymmetry-preserving features of the attractor solution depend on the sign of $q^2 + p^2$. The limit case $q^2 + p^2 = 0$ corresponds to the so-called small BHs (which however, in the case $\mathcal{N} = 3$, do not enjoy an enhancement of supersymmetry, contrarily to what usually happens in $\mathcal{N} \geq 4$, $d = 4$ supergravities; see e.g. the discussion in [71]).

By setting $n = 0$ in $\mathcal{N} = 3$, $d = 4$ supergravity (with resulting $U$-duality $U(3)$ which, due to the absence of scalars, coincides with the $\mathcal{N} = 3$ $\mathcal{R}$-symmetry $U(3)$ [92]), one gets

$$S_{BH} = \frac{1}{2} \left[ q_1^2 + q_2^2 + q_3^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 \right],$$

(5.2)

which is nothing but the sum of the entropies of three extremal (and thus BPS; see e.g. the discussion in [63]) Reissner-Nordstrom BHs, without any interference terms. Such a result can be simply understood by recalling that the generalization of the Maxwell electric-magnetic duality $U(1)$ to the case of $n$ Abelian gauge fields is $U(n)$ [88], and that the expression in the right-hand side of Eq. (5.2) is the unique possible $U(3)$-invariant combination of charges.

Moreover, it is here worth noticing that $\mathcal{N} = 3$, $d = 4$ supergravity is the only $\mathcal{N} > 2$ supergravity in which the gravity multiplet does not contain any scalar field at all, analogously to what happens in the case $\mathcal{N} = 2$. Thus, in minimally coupled $\mathcal{N} = 2^d$ and $\mathcal{N} = 3$, $d = 4$ supergravity the pure supergravity theory, obtained by setting $n = 0$, is scalarless, with the $U$-duality coinciding with the $\mathcal{R}$-symmetry [92].

This does not happen for all other $\mathcal{N} > 2$ theories. For instance, the $\mathcal{N} = 4$, $d = 4$ gravity multiplet does contain one complex scalar field (usually named axion-dilaton) and six Abelian vectors; thus, the pure $\mathcal{N} = 4$ theory, obtained by setting $n = 0$ (see Eqs. (A.1) and (7.1)), is not scalarless. By further truncating four vectors out (i.e. by performing a $(U(1))^6 \to (U(1))^2$ gauge truncation) and analyzing the bosonic field content, one gets the bosonic sector of $\mathcal{N} = 2$, $d = 4$ supergravity minimally coupled to one vector multiplet, the so-called axion-dilaton supergravity (for a discussion of the invariance properties of the classical BH entropy in such cases, see e.g. [63] and Refs. therein).

6 $\mathcal{N} = 5$ Supergravity

The (special Kähler) scalar manifold is [74]

$$\mathcal{M}_{\mathcal{N}=5} = \frac{G_{\mathcal{N}=5}}{H_{\mathcal{N}=5}} = \frac{SU(1,5)}{SU(5) \times U(1)}, \quad \text{dim}_R = 10.$$

(6.1)

As previously mentioned, no matter coupling is allowed (pure supergravity).

The 10 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the three-fold antisymmetric irreps. 20 of the $U$-duality group $G_{\mathcal{N}=5} = SU(1,5)$ (or equivalently of the compact form $SU(6)c$), and not in its fundamental repr. 6.

$Z_{AB} = Z_{\{AB\}}$, $A, B = 1, 2, 3, 4, 5 = \mathcal{N}$ is the central charge matrix. By means of a suitable transformation of the $\mathcal{R}$-symmetry $H_{\mathcal{N}=5} = U(5)$, $Z_{AB}$ can be skew-diagonalized by putting it in the normal

[6] Let us notice also that $\mathcal{N} = 2$ minimally coupled theory is the only (symmetric) $\mathcal{N} = 2$, $d = 4$ supergravity which yields the pure $\mathcal{N} = 2$ supergravity simply by setting $n = 0$. 

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form (see e.g. [40] and Refs. therein):

\[ Z_{AB} = \begin{pmatrix} Z_1 & Z_2 \\ Z_2^\dagger & Z_1^\dagger \end{pmatrix}, \]  

(6.2)

where \( Z_1, Z_2 \in \mathbb{R}_0^+ \) are the \( N = 5 \) (moduli-dependent) skew-eigenvalues, which can be ordered as \( Z_1 \geq Z_2 \) without any loss of generality (up to renamings; see e.g. [40]), and can be expressed as follows (see also the treatment of [93]):

\[
\begin{align*}
Z_1 &= \frac{1}{\sqrt{2}} \sqrt{I_1 + \sqrt{2}I_2 - I_1^2}; \\
Z_2 &= \frac{1}{\sqrt{2}} \sqrt{I_1 - \sqrt{2}I_2 - I_1^2};
\end{align*}
\]

(6.3)

where

\[
\begin{align*}
I_1 &= \frac{1}{2} Z_{AB} Z^{AB}, \\
I_2 &= \frac{1}{2} Z_{AB} Z^{BC} Z^{CD} Z^{DA}
\end{align*}
\]

(6.4)

are the two unique (moduli-dependent) \( H_N=5 \)-invariants.

From the Lagrangian density of \( N = 5, d = 4 \) supergravity [74], the vector kinetic terms read as follows:

\[ \mathcal{L}_{\text{vec}}^{\text{kin}} = -\frac{1}{4} \sqrt{-g} (S^{ijkl}) F^{ij \mu \nu} F_{\mu \nu} + \text{h.c.}; \]  

(6.6)

here and below \( i = 1, ..., 5 \) and antisymmetrization of couple of \( i \)-indices is understood. In the Gaillard-Zumino formalism, it also holds (see Eq. (36) of [71])

\[ \mathcal{L}_{\text{vec}}^{\text{kin}} = \sqrt{-g} \frac{i}{4} N_{ijkl} F^{ij \mu \nu} F_{\mu \nu} + \text{h.c.}. \]  

(6.7)

By comparing Eqs. (6.6) and (6.7), can write the \( 10 \times 10 \) vector kinetic matrix \( N_{ijkl} \) as follows:

\[ N_{ijkl} = i (S^{ijkl} - \frac{1}{2} \delta^{ijkl}) . \]

(6.8)

The matrix \( S \) satisfies the relation

\[ (\delta^{ijkl} - S^{ijkl}) S^{klmn} = \delta^{ij mn}, \]

(6.9)

and, in a suitable choice of the scalar fields, it holds that

\[ S^{ijkl} = -\frac{1}{2} \epsilon^{ijklm} z_m. \]

(6.10)

Thus, one finds (square brackets denote antisymmetrization of enclosed indices throughout)

\[ S^{ijkl} = \frac{1}{1 - (z_n)^2} \left[ \delta^{ij} - \frac{1}{2} \epsilon^{ijklm} z_m - 2 \delta^{[i} z_{j]} z_{l]} \right], \]

(6.11)

where the last term is normalized as

\[ \delta^{[i} z_{j]} z_{l]} = \frac{1}{4} (\delta^{i} z_{j} z_{l} \pm \text{permutations}...). \]

(6.12)

Consequently, one achieves the following expression of the vector kinetic matrix:

\[ N_{ijkl} = \frac{\alpha}{1 - (z_m)^2} \left( \frac{1}{2} \left[ 1 + (z_n)^2 \right] \delta^{ij} - \frac{1}{2} \epsilon^{ijklm} z_p - 2 \delta^{[i} z_{j]} z_{l]} \right), \]

(6.13)
where $\alpha$ is a factor to be determined by the relations satisfied by the symplectic sections $h$ and $f$ defined in the last steps of Eqs. (4.16) and (4.17) above.

The supersymmetry transformation of the vector field can be written as [94]
\[
\delta A^\mu_{ij} = 2 f^i j |AB \psi^A_{\alpha} + 2 f^i j A^B \psi^A_{\mu} \epsilon^B.
\] (6.14)

Such a formula is equivalent to the following expression [74] (see also [95]):
\[
\delta A^\mu_{ij} = \left( S^{ij|kl} - \delta^{ij|kl} \right) (C^{-1})_{kl} AB \left( e^C_\gamma \chi_{ABC} + 2 \sqrt{2} e^A_{AB} \psi^B \right),
\] (6.15)

where
\[
C_{ij}^{kl} = \frac{1}{e_1} \delta^{ij} - \frac{e_2}{e_1} \delta^{[i|z_j]_{kl}};
\] (6.16)
\[
(\text{C}^{-1})_{ij}^{kl} = \left( e_1 \delta^{kl} + 2 e_2 \delta^{[i|z_j]_{kl}} \right),
\] (6.17)
and
\[
e_1^2 = \frac{1}{1 - |z|^2};
\] (6.18)
\[
e_2 = \frac{1}{|z|^2} (1 - e_1).
\] (6.19)

By comparing Eqs. (6.14) and (6.15), one obtains that
\[
f^{ij}_{AB} = \left( e_1 \delta^{ij}_{AB} + e_1 \epsilon^j A^m z_m + 2 e_2 \delta^{[i|z_j]_{AB} z_l} \right).
\] (6.20)

The symplectic section $h$ is
\[
h_{ij|AB} = N'_{ij|mn} f^{mn}_{AB},
\] (6.21)
and thus it explicitly reads
\[
\hat{h}_{ij|AB} = \frac{\alpha}{1 - (z)^2} \left\{ \frac{1}{e_1} \left( 1 - (z_{i|j})^2 \right) \delta_{kl} \epsilon^{ijk|m} z_m - 2 \delta_{[i|z_j]_{kl}} \right\}.
\] (6.22)

Next, we check the above results and determine the overall numerical factor $\alpha$ of the matrix $N$, by expliciting writing down the identities (4.18) and (4.19) in the considered framework.

By recalling that
\[
\frac{1}{4} e^{ijkl} \epsilon^{CDL}_{kl} z_k z_l = (z_m)^2 \delta^{AB}_{CD} - 2 \delta_{[A|C]_{D} z_j z_l};
\] (6.23)
\[
\delta^{[A|z_j]_{B} z_l} \delta^{[C|z_j]_{D} z_l} = \frac{1}{2} (z_m)^2 \delta^{[A|C]_{z_j} z_l} z_B,
\] (6.24)
the identity (4.19) is easily verified, because
\[
(f^T h)_{CD}^{AB} = (h^T f)^{AB}_{CD} = \alpha \left\{ \frac{e_1^2}{2} \delta_{CD}^{ij} + \frac{e_1^2}{2} \epsilon_{ijABk} z_k + e_2 \delta_{[i}^A [z]_{j]} \right\}.
\]

Consequently, the identity (4.18) is satisfied iff
\[
\alpha = \frac{1}{2} \left( e_1 + e_2 + e_2 \right) \delta_{[i}^A [z]_{j]} + 2 e_2 \delta_{[i}^A [z]_{j]} + e_2 \delta_{[i}^A [z]_{j]} + \frac{e_1 e_2 + e_2}{2} \Re \left( \delta_{[i}^A [z]_{j]} \right).
\]

In order to check the identity (4.18) and determine \( \alpha \), we compute
\[
f^i h = f_{ij}^{AB} h_{ij|CD} = \alpha \left\{ \frac{1}{2} \delta_{CD}^{ij} + \frac{e_1^2}{4} (\epsilon_{ijABk} z_k - \epsilon_{ijABk} z_k) - \left( e_1^2 + e_2 + e_2 \right) \delta_{[i}^A [z]_{j]} \right\} = \alpha \left\{ \frac{1}{2} \delta_{CD}^{ij} + \frac{e_1^2}{2} \Re \left( \epsilon_{ijABk} z_k \right) \right\}.
\]

where we used the facts that \( |z|^2 e_2 = 1 - e_1 \), and \( e_1^2 + e_1 e_2 + e_2 = 0 \), directly following from the definitions (6.18) and (6.19).

By an analogous calculation, one achieves the following result:
\[
h^i f = (h_{ij|CD})^i j^{AB} = \alpha \left[ \frac{1}{2} \delta_{CD}^{ij} + \frac{e_1^2}{2} \Re \left( \epsilon_{ijABk} z_k \right) \right].
\]

Consequently, the identity (4.18) is satisfied iff \( \alpha = -i \). By substituting such a value into Eq. (6.13), one obtains the following expression of the vector kinetic matrix of \( \mathcal{N} = 5, d = 4 \) ungauged supergravity:
\[
\mathcal{N}_{ijk|l} = -\frac{i}{1 - (z_m)^2} \left( \frac{1}{2} \left[ (z_n)^2 \right] + \frac{1}{2} \epsilon_{ijklp} z_p - 2 \delta_{[i}^A [z]_{j]} \right).
\]

From the general definition\(^7\) (4.20), the central charge matrix \( Z_{AB} \) is defined as the integral over the 2-sphere at infinity \( S^2_\infty \) of the dressed graviphoton field strength \([89, 90, 91]\):
\[
Z_{AB} = -\int_{S^2_\infty} T_{AB} = -\int_{S^2_\infty} T_{AB}^\Lambda = f_{ij}^{AB} q_{ij} - h_{ij|AB} \Lambda = f_{ij}^{AB} q_{ij} - h_{ij|AB} \Lambda.
\]

\(^7\)We rescale the symplectic section \( f \) by a factor \( \frac{1}{2} \), and the kinetic matrix \( \mathcal{N} \) correspondingly by a factor 2. The definition of the symplectic section \( h \) through Eq. (6.21) and the identities (4.18) and (4.19) are left unchanged. Such a redefinition is performed in order to avoid an unsuitable rescaling of the magnetic charges, and thus to define the complexified BH charges as in Eq. (6.30).
By recalling Eqs. (6.20) and (6.22) with $\alpha = -i$, and, analogously to definitions (4.26) and (4.27), by introducing the complexified BH charges as

$$Q^{ij} = \frac{1}{2} (q_{ij} + ip^{ij}),$$  \hspace{1cm} (6.30)

one gets

$$Z_{AB}(z, \bar{z}, q, p) = e_1 Q^{AB} + \frac{e_1}{2} \epsilon^{ABijkl} Q_{ij} q_k - 2e_2 z^{[A} Q^{B]C} z^C. \hspace{1cm} (6.31)$$

As recalled at the start of the next Subsection, only $(\frac{1}{5})$-BPS attractor flow is non-degenerate (i.e. corresponding to large BHs; see e.g. the discussion in [71]), and the corresponding (squared) first order fake superpotential is ( [40]; recall Eqs. (6.3)-(6.5))

$$W^2_{(\frac{1}{5})BPS} = \frac{1}{2} \left[ \frac{1}{2} Z_{AB} \bar{Z}^{AB} + \sqrt{Z_{AB} \bar{Z}^{BC} Z_{CD} \bar{Z}^{DA}} - \frac{1}{4} \left( Z_{AB} \bar{Z}^{AB} \right)^2 \right] = \frac{1}{2} \left[ I_1 + \sqrt{2I_2 - I_1^2} \right] = Z_1^2. \hspace{1cm} (6.32)$$

6.1 Attractor Equations and their Solutions

Due to the absence of matter charges, the BH effective potential can be written as

$$V_{BH} = \frac{1}{2} Z_{AB} \bar{Z}^{AB} = Z_1^2 + Z_2^2 = I_1. \hspace{1cm} (6.33)$$

The $\mathcal{N} = 5$, $d = 4$ Attractor Eqs. are nothing but the criticality conditions for the $\mathcal{H}_{\mathcal{N}=5}$-invariant $I_1$. By using the Maurer-Cartan Eqs. of $\mathcal{N} = 5$, $d = 4$ supergravity (see e.g. [91, 89, 90]), such Attractor Eqs. can be written as follows:

$$\epsilon^{ABCDE} Z_{AB} Z_{CD} = 0; \hspace{1cm} (6.34)$$

$$\epsilon^{ABCDE} \bar{Z}^{AB} \bar{Z}^{CD} = 0, \hspace{1cm} (6.35)$$

where the two lines are reciprocally complex conjugated. They can also be written more explicitly, by using the expression of central charge matrix $Z_{AB}$ given by Eq. (6.31). For instance, Eqs. (6.34) can be rewritten in the following way:

$$0 = \epsilon^{ABCDE} Z_{AB} Z_{CD} =$$

$$= e_1^2 \epsilon^{ABCDE} Q^{AB} Q^{CD} + e_1^2 (\epsilon^{ABCDE} Q^{AB} Q^{CD} z_i) z_E + 8e_1^2 (Q^{AB} Q^{AB} z_E +$$

$$+ 16e_1^2 z_i Q^{AC} z_E - 4e_1 e_2 \epsilon^{ABCDE} Q^{AC} z C z_i +$$

$$- 8e_1 e_2 (z_i Q^{CD} z C) z_E - 16e_1 e_2 |z|^2 z_j Q^{CD} z E). \hspace{1cm} (6.36)$$

The criticality conditions (6.35) and (6.36) are satisfied for a unique class of critical points (for further elucidation, see e.g. the treatment in [71]):

- $(\frac{1}{5})$BPS:
  $$Z_2 = 0, \ Z_1 > 0. \hspace{1cm} (6.37)$$

It is worth counting here the degrees of freedom related to Eqs. (6.34) and (6.35), or equivalently to the unique $(\frac{1}{5})$-BPS solution given by Eq. (6.37). Eqs. (6.34) and (6.35) are 10 real equations, but actually only 6 real among them are independent. Thus a moduli space of $(\frac{1}{5})$-BPS attractors, spanned by the 2 complex scalars unstabilized by Eq. (6.37), might - and actually does [56] - exist. Such a counting of “flat” directions of $V_{BH}$ at its $(\frac{1}{5})$-BPS critical points was given in terms of the left-over would-be $\mathcal{N} = 2$ hyperscalars’ degrees of freedom in the $\mathcal{N} = 5 \rightarrow \mathcal{N} = 2$ supersymmetry reduction in [91].
6.1.1 Entropy of $\frac{1}{4}$-BPS Attractors and their Moduli Space

By recalling Eqs. (1.1) and (6.33), and using Eq. (6.37), one achieves the following result:

$$\frac{S_{B H,B P S}}{\pi} = \frac{A_{H,B P S}}{4} = V_{B H}|_{B P S} = Z_{1,B P S} = \sqrt{I_4}. \quad (6.38)$$

$I_4$ here denotes the (unique) invariant of the three-fold antisymmetric $20$ representation of the $U$-duality group $G_{N=5}$. Such a representation is symplectic, containing the singlet $1_a$ in the tensor product $20 \times 20$ [96], and it is thus irreducible with respect to both $G_{N=5}$ and $Sp(20, \mathbb{R})$. $I_4$ is quartic in BH charges.

In terms of the dressed charges, i.e. of the central charge matrix $Z_{A B}$, $I_4$ can be written as follows:

$$I_4 = Z_{A B} \bar{Z}^{B C} Z_{C D} \bar{Z}^{D A} - \frac{1}{4} \left( Z_{A B} \bar{Z}^{A B} \right)^2 = 2 I_2 - I_4^2 = \text{Tr}(A^2) - \frac{1}{4} \left( \text{Tr}(A) \right)^2. \quad (6.39)$$

where the (moduli-dependent) matrix $A_A^B = Z_{A C} \bar{Z}^{B C}$ and the related quantities [71]

$$\text{Tr}(A) = Z_{A B} \bar{Z}^{A B} = 2 I_1 = 2 V_{B H}; \quad (6.40)$$

$$\text{Tr}(A^2) = Z_{A B} \bar{Z}^{B C} Z_{C D} \bar{Z}^{D A} = 2 I_2, \quad (6.41)$$

were introduced, $I_1$ and $I_2$ being the two unique (moduli-dependent) $H_{N=5}$-invariants, respectively defined by Eqs. (6.4) and (6.5).

The $U$-invariant $I_4$, introduced in Eq. (6.38) and expressed in terms of dressed charges by Eq. (6.39), is the unique (moduli-independent) independent $H_{N=5}$-invariant combination of (moduli-dependent) $H_{N=5}$-invariant quantities (see e.g. [91], and [71], as well as Refs. therein). Its quarticity in BH charges is ultimately due to the symplectic nature of the $20$ of $SU(1, 5)$ (irreducible with respect to both $SU(1, 5)$ and $Sp(20, \mathbb{R})$), i.e. to the fact that the tensor product $20 \times 20$ contains an antisymmetric singlet $1_a$ [96], thus yielding a vanishing quartic invariant of $SU(1, 5)$.

Thus, the expression (6.39) is moduli-dependent only apparently. Being moduli-independent, $I_4$ can actually be written only in terms of the BH charges. In order to determine such an expression, one can use the fact that, in the considered coordinate parametrization, the origin $O$ of $\mathcal{M}_{N=5}$ (determined by $z^i = 0 \forall i = 1, ..., 5$) is the invariant point under the action of $H_{N=5} = U(5)$ (see e.g. Eq. (2.17) of [74]). Thus, the explicit dependence of $I_4$ on BH charges is obtained simply by computing $Z_{A B}|_O$ and using Eq. (6.39). By recalling Eq. (6.31) and Eqs. (6.40) and (6.41), one obtains:

$$I_4 = Q_{A B} Q^{B C} Q_{C D} Q^{D A} - \frac{1}{4} (Q_{A B} Q^{A B})^2 =$$

$$= \frac{1}{24} \left( q_{A B} - i p^{A B} \right) \left( q_{B C} + i p^{B C} \right) \left( q_{C D} + i p^{C D} \right) \left( q_{D A} + i p^{D A} \right) +$$

$$- \frac{1}{24} \left[ (q_{A B} - i p^{A B}) (q_{A B} + i p^{A B}) \right]^2, \quad (6.42)$$

where the complexified BH charges $Q^{ij}$ defined by Eq. (6.30) have been used, with $Q_{A B} \equiv Q^{A B}$. Eq. (6.42) is manifestly $H_{N=5}$-invariant, the $10 + 10$ real BH charges being arranged in the (reciprocally conjugated) complex rank-2 tensors $Q^{ij}$ and $Q_{ij}$ in the two-fold antisymmetric $10$ and $10'$ of $SU(5)$, or equivalently in the real rank-2 tensors $q_{ij}$ and $p^{ij}$ in a pair of two-fold antisymmetric $10$ of $SO(5)$.

Clearly, the BH electric and magnetic BH charges, being the asymptotical fluxes of the vector field strengths and of their duals, can actually be arranged to sit in the three-fold antisymmetric, symplectic representation $20$ of the $U$-duality group $G_{N=5} = SU(1, 5)$. The embedding of the two-fold antisymmetric $10^{(i)}$ of $SU(5)$ into the three-fold antisymmetric $20$ of $SU(1, 5)$ is given by the formula

$$(a, b, c, d, e = 1, ..., 5)$$

$$\epsilon^{a b c d e} = \frac{1}{3!} \epsilon^{a b c d e} t_{d e}, \quad (6.43)$$

$$t_{d e} =$$
or, more precisely (recall $A, B, C = 1, \ldots, 5$, and moreover $\hat{A}, \hat{B}, \hat{C} = 1, \ldots, 6$, here and below):

$$Q_{\hat{A}\hat{B}\hat{C}} \equiv q_{\hat{A}\hat{B}\hat{C}} + ip\hat{A}\hat{B}\hat{C} = (Q_{ABC}, Q_{AB})$$

(6.44)

$$Q_{AB} \equiv \frac{1}{2} (q_{AB} + ip^{AB})$$

(6.45)

$$Q_{6AB} \equiv q_{6AB} + ip^{6AB}$$

(6.46)

Eq. (6.44) describes the splitting of the $20$ of $SU(1,5)$ into the $10$ and $10'$ of $SU(5)$, whose embedding is determined by Eqs. (6.45) and (6.46), respectively. As given by Eq. (6.46), by identifying $t_{de} \equiv t_{de6}$, Eq. (6.43) can be seen as part of a self-reality condition (admitting solutions in $SU(1,5)$, but not in $SU(6)$).

Thus, it can be easily shown that Eq. (6.42) can be recast in a manifestly $U = G_{N=5}$-invariant way, with BH charges in the three-fold antisymmetric $20$ of $SU(1,5)$, as follows:

$$\mathcal{I}_4 = \frac{1}{3 \cdot 2^5} \epsilon^{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}} \epsilon^\hat{A} \hat{A} \hat{B} \hat{C} \hat{D} \hat{E} Q_{\hat{A}\hat{B}\hat{C}} Q_{\hat{A}\hat{B}\hat{C}'} Q_{\hat{A}\hat{B}\hat{C}'} Q_{\hat{A}\hat{B}\hat{C}''} Q_{\hat{A}\hat{B}\hat{C}''} = \frac{1}{3 \cdot 2^5} \epsilon^{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}} \epsilon^\hat{A} \hat{A} \hat{B} \hat{C} \hat{D} \hat{E} Q_{\hat{A}\hat{B}\hat{C}} Q_{\hat{A}\hat{B}\hat{C}'} Q_{\hat{A}\hat{B}\hat{C}'} Q_{\hat{A}\hat{B}\hat{C}''} Q_{\hat{A}\hat{B}\hat{C}''}$$

(6.47)

In order to prove such a formula, let us explicit the entries “6” in Eq. (6.47), obtaining

$$\mathcal{I}_4 = \frac{1}{3 \cdot 2^5} \left[ 9 \epsilon_{A'B'C'C''} \epsilon_{6B'C''} Q_{6BC} Q_{AB'C'} Q_{6B'C''} Q_{A'B''C''} + 12 \epsilon_{ABC6'C''} \epsilon_{6B'C''} Q_{ABC} Q_{6B'C'} Q_{6B'C''} Q_{A''B''C''} + 6 \epsilon_{ABC'B''} \epsilon_{6B'C''} Q_{ABC} Q_{6B'C'} Q_{6B'C''} Q_{A''B''C''} + 4 \epsilon_{ABC'B''} \epsilon_{6B'C''} Q_{ABC} Q_{6B'C'} Q_{6B'C''} Q_{A''B''C''} + \epsilon_{ABC'B''} \epsilon_{6B'C''} Q_{ABC} Q_{6B'C'} Q_{6B'C''} Q_{A''B''C''} \right].$$

(6.49)

By using the embedding Eqs. (6.45) and (6.46), it is immediate to check that Eq. (6.49) yields Eq. (6.42).

As mentioned above, only 6 out of the 10 real $\frac{1}{2}$-BPS criticality conditions (6.34) and (6.35) are actually independent. Thus, they do not stabilize all the 5 complex scalar fields $z^i$ in terms of the BH electric and magnetic scalar charges, but only 3 of them. In [56] the residual 2 unstabilized complex scalar degrees of freedom have been shown to span the $\frac{1}{4}$-BPS moduli space

$$\mathcal{M}_{N=5,BPS} = \frac{SU(2,1)}{SU(2) \times SU(1)} = \mathcal{M}_{N=3,n=1,BPS}, \text{ dim}_R = 4.$$ 

(6.50)

### 6.2 Black Hole Parameters for $\frac{1}{5}$-BPS Flow

By using the Maurer-Cartan Eqs. of $N = 5$, $d = 4$ supergravity (see e.g. [91, 89, 90]), one gets [40]

$$\partial_t Z_1 = \partial_t \mathcal{W}_{BPS} = \frac{P}{\sqrt{2}} \left[ \frac{1}{2} Z_{AB} Z^{AB} - \frac{1}{2} Z_{AB} Z^{BC} Z_{CD} Z^{DA} - \frac{1}{4} \left( Z_{AB} Z^{AB} \right)^2 \right] = P_1 Z_2,$$

(6.51)

where $P = P_{1234} P_{ABCD} = P_{ABCD,dz^i} = \epsilon_{ABCDDE} p^E$ being the holomorphic Vielbein of $\mathcal{M}_{N=5}$. Here, $\nabla$ denotes the $U(1)$-Kähler and $H_{N=5}$-covariant differential operator.
Thus, by using the explicit expressions of \( W_{\text{BPS}}^2 \) given by Eq. (6.32), using the Maurer-Cartan Eqs. of \( \mathcal{N} = 5, d = 4 \) supergravity (see e.g. [91, 89, 90]), and exploiting the first order (fake supergravity) formalism discussed in Sect. 2, one respectively obtains the following expressions of the (square) ADM mass, covariant scalar charges and (square) effective horizon radius for the \( \frac{1}{2} \)-BPS attractor flow:

\[
\begin{align*}
 r_{H,\text{BPS}}^2 (z_\infty, \tau_\infty, p, q) &= M_{\text{ADM, BPS}}^2 (z_\infty, \tau_\infty, p, q) = W_{\text{BPS}}^2 (z_\infty, \tau_\infty, p, q) = \\
 &= \frac{1}{2} \lim_{\tau \to 0^+} \left[ \frac{1}{2} Z_{AB} Z^{AB} + \sqrt{Z_{AB} Z^{BC} Z_{CD} Z^{DA}} - \frac{1}{4} \left( Z_{AB} Z^{AB} \right)^2 \right] = \\
 &= Z_4^2 \bigg|_{\infty};
\end{align*}
\]

(6.52)

\[
\begin{align*}
 \Sigma_{i,\text{BPS}} (z_\infty, \tau_\infty, p, q) &= 2 \lim_{\tau \to 0^+} (\partial_\tau W_{\text{BPS}}) (z(\tau), \tau(\tau), p, q) = \\
 &= \sqrt{2} \lim_{\tau \to 0^+} \left[ P_i \sqrt{\frac{1}{2} Z_{AB} Z^{AB} - \sqrt{Z_{AB} Z^{BC} Z_{CD} Z^{DA}} - \frac{1}{4} \left( Z_{AB} Z^{AB} \right)^2} \right] = \\
 &= 2 (P_i, Z_4)_{\infty};
\end{align*}
\]

(6.53)

\[
R_{H,\text{BPS}}^2 = W_{\text{BPS}}^2 (z_\infty, \tau_\infty, p, q) + \\
-4G^5 (z_\infty, \tau_\infty) (\partial_\tau W_{\text{BPS}}) (z_\infty, \tau_\infty, p, q) (\partial_\tau W_{\text{BPS}}) (z_\infty, \tau_\infty, p, q) = \\
= \sqrt{Z_{AB} Z^{BC} Z_{CD} Z^{DA}} - \frac{1}{4} \left( Z_{AB} Z^{AB} \right)^2 = \\
= \sqrt{2} Z_4^2 = 2 (P_i, Z_4)_{\infty} = \sqrt{I_4 (p, q)} > 0.
\]

(6.54)

Eq. (6.54) proves Eq. (2.13) for the \( \frac{1}{2} \)-BPS attractor flow of the considered \( \mathcal{N} = 5, d = 4 \) supergravity. Such a result was obtained by using Eq. (6.51) and computing that

\[
4G^5 (\partial_\tau W_{\text{BPS}}) (\partial_\tau W_{\text{BPS}}) = 4G^5 (\partial_\tau Z_4) (\partial_\tau Z_4) = \\
= 2G^5 P_i \overline{P}^j \left[ \frac{1}{2} Z_{AB} Z^{AB} - \sqrt{Z_{AB} Z^{BC} Z_{CD} Z^{DA}} - \frac{1}{4} \left( Z_{AB} Z^{AB} \right)^2 \right] = \\
= \frac{1}{2} \left[ \frac{1}{2} Z_{AB} Z^{AB} - \sqrt{Z_{AB} Z^{BC} Z_{CD} Z^{DA}} - \frac{1}{4} \left( Z_{AB} Z^{AB} \right)^2 \right] = Z_4^2,
\]

(6.55)

where the relation

\[
4G^5 P_i \overline{P}^j = 1
\]

(6.56)

was used.

The considerations made at the end of Subsect. 3.2 hold also for the considered attractor flow.

It is worth noticing out that Eq. (6.54) is consistent, because, as pointed out above, the \( \frac{1}{2} \)-BPS-supporting BH charge configurations in the considered theory is defined by the quartic constraints \( I_4 (p, q) > 0 \).

Furthermore, Eq. (6.52) yields that the \( \frac{1}{2} \)-BPS attractor flow of \( \mathcal{N} = 5, d = 4 \) supergravity does not saturate the marginal stability bound (see [83] and [54]; see also the discussion at the end of Subsect. 3.3).
7 \( \mathcal{N} = 4 \) Pure Supergravity Revisited

The treatment of \( \mathcal{N} = 4, d = 4 \) pure supergravity is pretty similar to the one given for \( \mathcal{N} = 5, d = 4 \) supergravity in Sect. 6.

The (special Kähler) scalar manifold is [73]

\[
\mathcal{M}_{\mathcal{N}=4,\text{pure}} = \frac{G_{\mathcal{N}=4,\text{pure}}}{H_{\mathcal{N}=4,\text{pure}}} = SU(1,1) \times SU(4) = SU(1,1) \times SU(4), \quad \text{dim}_\mathbb{R} = 2, \quad (7.1)
\]

spanned by the complex scalar

\[
s \equiv a + ie^{-2\varphi}, \quad a, \varphi \in \mathbb{R}, \quad (7.2)
\]

where \( a \) and \( \varphi \) are usually named axion and dilaton, respectively.

The 6 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the bifundamental irrepr. (2, 6) of the \( \mathcal{U}(1) \) duality group \( G_{\mathcal{N}=4,\text{pure}} = SU(1,1) \times SO(6) \sim SU(1,1) \times SU(4) \).

\( Z_{AB} = Z_{[AB]}, \quad A, B = 1, 2, 3, 4 = N \) is the central charge matrix. By means of a suitable transformation of the \( R \)-symmetry \( H_{\mathcal{N}=4,\text{pure}} = U(1) \times SO(6) \sim U(1) \times SU(4) \), \( Z_{AB} \) can be skew-diagonalized by putting it in the normal form (see e.g. [40] and Refs. therein):

\[
Z_{AB} = \begin{pmatrix} Z_1 & \epsilon \\ \epsilon & Z_2 \end{pmatrix}, \quad (7.3)
\]

where \( Z_1, Z_2 \in \mathbb{R}_+^3 \) are the \( N = 4 \) (moduli-dependent) skew-eigenvalues, which can be ordered as \( Z_1 \geq Z_2 \) without any loss of generality (up to renamings; see e.g. [40]), and can be formally expressed by the very same Eqs. (6.3)-(6.5), where now \( I_1 \) and \( I_2 \) are the two unique (moduli-dependent) \( H_{\mathcal{N}=4,\text{pure}} \)-invariants.

The symplectic sections read as follows (\( \Lambda \equiv [AB] = 1, \ldots, 6 \) throughout) [89, 90, 71]

\[
f^A_{AB} = e^b \delta^A_{AB}, \quad \delta^A_{AB} = (ae^\varphi + i e^{-\varphi}) \delta^A_{AB}, \quad (7.4)
\]

such that the kinetic vector matrix is given by (recall Eq. (4.14))

\[
N_{\Lambda\Sigma} = (hf^{-1})_{\Lambda\Sigma} = s\delta_{\Lambda\Sigma}. \quad (7.5)
\]

By the general definition (4.20), the central charge matrix is given by

\[
Z_{AB} = f^A_{AB} p^A - h_{[AB]} p^A = e^b \delta^A_{AB} q_A - se^\varphi \delta^A_{AB} p^A = -e^\varphi (s p_{AB} - q_{AB}). \quad (7.6)
\]

Such an explicit expression allows one to elaborate Eqs. (6.3)-(6.5) further, obtaining

\[
I_1 = \frac{1}{2} Z_{AB} \overline{Z}^A_{BC} = Z_1^2 + Z_2^2 = \frac{1}{2} e^{2\varphi} (s p_{AB} - q_{AB}) (\overline{s} p^A_{BC} - q^A_{BC}) = \frac{1}{2} e^{2\varphi} (s p_{AB} - q_{AB}) (\overline{s} p^A_{BC} - q^A_{BC}) =
\]

\[
= (e^{2\varphi} a^2 + e^{-2\varphi}) p^2 + e^{2\varphi} q^2 - 2ae^{2\varphi} p \cdot q; \quad (7.7)
\]

\[
I_2 = \frac{1}{2} Z_{AB} \overline{Z}^A_{CD} Z^D = Z_1^2 + Z_2^2 = \frac{1}{2} e^{2\varphi} (s p_{AB} - q_{AB}) (\overline{s} p^A_{BC} - q^A_{BC}) (s p_{CD} - q_{CD}) (\overline{s} p^D - q^D), \quad (7.8)
\]

where \( p^2 \equiv (p^1)^2 + \ldots + (p^6)^2, \quad q^2 \equiv (q_1)^2 + \ldots + (q_6)^2, \quad \text{and} \quad p \cdot q \equiv p^A q_A \) (see Eq. (7.1) of [63], fixing a typo in Eq. (225) of [71]).

Only \( \left(\frac{1}{2}\right) \)-BPS attractor flow is non-degenerate (i.e. corresponding to large BHs; see e.g. the \( n = 0 \) limit of the discussion in [71]), and the corresponding (squared) first order fake superpotential is identical to the one of the \( \left(\frac{1}{2}\right) \)-BPS attractor flow in \( \mathcal{N} = 5, d = 4 \) supergravity [40], given by Eq. (6.32) above,
which in the considered framework can be further elaborated as follows:

\[
W_{(\frac{1}{4})\text{BPS}}^2 = \frac{1}{2} \left[ I_1 + \sqrt{2I_2 - I_1^2} \right] = Z_1^2 = \\
= \frac{e^{2\varphi}}{4} \left[ (sp_{AB} - q_{AB}) (s p^{AB} - q^{AB}) + \\
+ 4 (sp_{AB} - q_{AB}) (s p^{BC} - q^{BC}) (sp_{CD} - q_{CD}) (s p^{DA} - q^{DA}) + \\
- [s p_{AB} - q_{AB}] (s p^{AB} - q^{AB}) \right]^2.
\]

(7.9)

7.1 Attractor Equations and their Solutions: 
\(\frac{1}{4}\)-BPS Attractors and their Entropy

Due to the absence of matter charges, the BH effective potential \(V_{BH}\) reads

\[
V_{BH} = \frac{1}{2} Z_{AB} Z^{AB} = Z_1^2 + Z_2^2 = I_1 = \\
= (e^{2\varphi} a^2 + e^{-2\varphi}) p^2 + e^{2\varphi} q^2 - 2ae^{2\varphi} p \cdot q,
\]

(7.10)

where in the second line we recalled Eq. (7.7) (see also the treatments of [71] and [63]). The complex Attractor Eq. of \(N = 4, d = 4\) pure supergravity is nothing but the criticality condition for the \(H_{N=4,\text{pure}}\)-invariant \(I_1\). By using the relevant Maurer-Cartan Eqs. (see e.g. [91, 89, 90]), such a complex Attractor Eq. can be written as follows (\(|\varphi| < \infty\)):

\[
\epsilon_{ABCD} Z^{AB} \overline{Z}^{CD} \Leftrightarrow \epsilon_{ABCD} (s p^{AB} - q^{AB})^2 (s p^{CD} - q^{CD}) = 0.
\]

(7.11)

An equivalent set of two real Eqs. is given by the system

\[
\frac{\partial V_{BH}}{\partial a} = 0, \quad \frac{\partial V_{BH}}{\partial \varphi} = 0,
\]

(7.12)

with \(V_{BH}\) given by Eqs. (7.7) or (7.10), yielding Eqs. (7.2)-(7.3) of [63].

Thus, the criticality conditions (7.11), or equivalently (7.12), are satisfied for a unique class of critical points:

\(\frac{1}{4}\text{-BPS}:

\[
Z_2 = 0, \quad Z_1 > 0,
\]

(7.13)

yielding Eqs. (7.2)-(7.3) of [63]. Eqs. (7.12) are 2 real Eqs. in 2 real unknowns, namely the axion \(a\) and the dilaton \(\varphi\), which both are stabilized solely in terms of the magnetic and electric BH charges. Thus no moduli space of \(\frac{1}{4}\)-BPS attractors in \(N = 4, d = 4\) pure supergravity exists at all [71, 56, 63].

By recalling Eqs. (1.1) and (7.10), and using Eq. (7.13), one achieves the following result [97]:

\[
\frac{S_{BH,\text{BPS}}}{\pi} = A_{H,\text{BPS}} = V_{BH}|_{\text{BPS}} = Z_1^2 = \sqrt{I_4}.
\]

(7.14)

\(I_4\) denotes the (unique) invariant of the bi-fundamental representation \((2,6)\) of the \(U\)-duality group \(G_{N=4,\text{pure}}\). Such a representation is symplectic, containing the singlet \(1_n\) in the tensor product \((2,6) \times (2,6)\) [96] (and thus yielding a vanishing quadratic \(U\)-invariant); consequently, it is irreducible with respect to both \(G_{N=4,\text{pure}}\) and \(Sp(12, \mathbb{R})\). \(I_4\) is quartic in BH charges (see Eq. (7.4), and the related discussion, of [63]) [97]:

\[
I_4 = 4 \left[ p^2 q^2 - (p \cdot q)^2 \right].
\]

(7.15)

\(^8\)This fact has been observed above also for \(N = 5, d = 4\) supergravity.

Even though in general the (unique) quartic invariant of a (semi-simple) Lie group can be built from a non-symplectic representation, this never happens for the \(U\)-duality groups of \(N = 2\) symmetric and \(N > 2, d = 4\) theories. Thus, for all such supergravities having a (unique) \(U\)-invariant quartic in BH charges, the relevant representation of the \(U\)-duality group is symplectic (irreducible to both \(U\)-duality and relevant symplectic group).

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In terms of the *dressed* charges, *i.e.* of the central charge matrix $Z_{AB}$, $I_4$ is formally given by the very same Eqs. (6.39)-(6.41). $I_4$ is the *unique* (moduli-independent) independent $G_{N=4, \text{pure}}$-invariant combination of (moduli-dependent) $H_{N=4, \text{pure}}$-invariant quantities (see *e.g.* the $n = 0$ limit of the discussion in [91] and [71], and Refs. therein).

It is worth pointing out that only when $\Lambda = 1, 2$ (corresponding to the truncation $(U(1))^6 \to (U(1))^2$ of the gauge group) $I_4$ is a perfect square, thus reproducing the quadratic invariant $I_2$ of the (1-modulus, $n = 1$ element of the) minimally coupled $\mathcal{N} = 2, d = 4$ sequence, given by Eq. (3.21) (see *e.g.* [25] and [63], and Refs. therein).

7.2 Black Hole Parameters for $\mathcal{I}_4$-BPS Flow

By using the *Maurer-Cartan* Eqs. of $\mathcal{N} = 4, d = 4$ pure supergravity (see *e.g.* [91, 89, 90]), one gets [40]

$$
\partial_s Z_1 = \partial_s W_{BPS} = \frac{P_s}{\sqrt{2v}} \left[ \frac{1}{2} Z_{AB} \overrightarrow{Z}^{AB} - \sqrt{Z_{AB} \overrightarrow{Z}^{BC} Z_{CD} \overrightarrow{Z}^{DA} - \frac{1}{4} \left( Z_{AB} \overrightarrow{Z}^{AB} \right)^2} \right] = P_s Z_2,
$$

(7.16)

where $P = P_s ds$ is the holomorphic Vielbein of $\mathcal{M}_{N=5}$. Here, $\nabla$ denotes the $U(1)$-Kähler and $H_{N=4, \text{pure}}$-covariant differential operator.

Thus, by using the explicit expressions of $W_{BPS}^2$ given by Eq. (7.9), using the *Maurer-Cartan* Eqs. of $\mathcal{N} = 5, d = 4$ supergravity (see *e.g.* [91, 89, 90]), and exploiting the first order (fake supergravity) formalism discussed in Sect. 2, one respectively obtains the following expressions of the (square) ADM mass, axion-dilaton charge and (square) effective horizon radius for the $\mathcal{I}_4$-BPS attractor flow:

$$
r_{H, BPS}^2 (s_\infty, \overline{s}_\infty, p, q) = M_{ADM, BPS}^2 (s_\infty, \overline{s}_\infty, p, q) = W_{BPS}^2 (s_\infty, \overline{s}_\infty, p, q) =
$$

$$
= \frac{1}{2} \lim_{\tau \to 0^-} \left[ \frac{1}{2} Z_{AB} \overrightarrow{Z}^{AB} + \sqrt{Z_{AB} \overrightarrow{Z}^{BC} Z_{CD} \overrightarrow{Z}^{DA} - \frac{1}{4} \left( Z_{AB} \overrightarrow{Z}^{AB} \right)^2} \right] =
$$

$$
= Z_1^2 \big|_\infty = \frac{i}{2 (s_\infty - \overline{s}_\infty)},
$$

(7.17)

$$
\Sigma_{BPS} (s_\infty, \overline{s}_\infty, p, q) = 2 \lim_{\tau \to 0^-} (\partial_s W_{BPS}) (s (\tau), \overline{s} (\tau), p, q) =
$$

$$
= \sqrt{2} \left[ P_s \sqrt{\frac{1}{2} Z_{AB} \overrightarrow{Z}^{AB} - \sqrt{Z_{AB} \overrightarrow{Z}^{BC} Z_{CD} \overrightarrow{Z}^{DA} - \frac{1}{4} \left( Z_{AB} \overrightarrow{Z}^{AB} \right)^2}} \right] =
$$

$$
= 2 (P_s Z_2)_\infty,
$$

(7.18)
where, with suitable changes, the matrix $A_A^B$ and related quantities are defined by Eqs. (6.40) and (6.41).

Eq. (7.19) proves Eq. (2.13) for the $\frac{1}{4}$-BPS attractor flow of $\mathcal{N} = 4$, $d = 4$ pure supergravity. Such a result was obtained by using Eq. (7.16) and computing that

$$4G^{\sigma \tau} (\partial_{\sigma} W_{BPS}) \partial_{\tau} W_{BPS} = 4G^{\sigma \tau} (\partial_{\sigma} Z_1) \partial_{\tau} Z_1 =$$

$$= 2G^{\sigma \tau} P_{s} \sqrt{1 - \sqrt{1 - (Z_{AB} Z_{AB})^2}} =$$

$$= \frac{1}{1} = 1$$

(7.21)

where the relation

$$4G^{\sigma \tau} P_{s} \sqrt{1} = 1$$

was used.

From its very definition, by using Eqs. (7.9) or (7.17), the axion-dilaton charge $\Sigma_{s,BPS}$ can be explicitly computed as follows:

$$\Sigma_{s,BPS} = \frac{1}{M_{ADM,BPS}} (\partial_{s} W_{BPS})_\infty = \frac{1}{M_{ADM,BPS}}.$$

Furthermore, from the definition (7.2) and Eq. (7.18), it follows that

$$\Sigma_{s,BPS} \equiv 2 \lim_{\tau \to 0} (\partial_{s} W_{BPS}) = \Sigma_{s,BPS} + \frac{i}{2} e^{2\varphi} \Sigma_{q,BPS},$$

(7.23)
where
\[
\Sigma_{a,BPS} = 2 \lim_{\tau \to 0^-} (\partial_\tau W_{BPS}) = Re (\Sigma_{s,BPS}); \tag{7.24}
\]
\[
\Sigma_{\varphi,BPS} = 2 \lim_{\tau \to 0^-} (\partial_\tau W_{BPS}) = 2 e^{-2\varphi} Im (\Sigma_{s,BPS}) = -i (s_\infty - \overline{s}_\infty) Im (\Sigma_{s,BPS}) \tag{7.25}
\]
respectively are the axionic and dilatonic charges pertaining to the $1\over 4$-BPS attractor flow.

The considerations made at the end of Subsect. 3.2 hold also for the considered attractor flow.

It is worth noticing out that Eq. (7.19) is consistent, because, as pointed out above, the $1\over 4$-BPS-supporting BH charge configurations in the considered theory is defined by the quartic constraints $I_4 (p,q) > 0$.

Furthermore, Eq. (7.17) yields that the $1\over 4$-BPS attractor flow of $\mathcal{N} = 4$, $d = 4$ pure supergravity does not saturate the marginal stability bound (see [83] and [54]; see also the discussion at the end of Subsect. 3.3).

8 Peculiarity of Pure $\mathcal{N} = 4$ and $\mathcal{N} = 5$ Supergravity

By exploiting the first order (fake supergravity) formalism discussed in Sect. 2, the expression of the squared effective horizon radius (in the extremal case $c = 0$) $R^2_H$ given by Eq. (2.12) has been shown to hold for the following $d = 4$ supergravity theories:

- **minimally coupled $\mathcal{N} = 2$** theory, whose scalar manifold is given by the sequence $\mathbb{SU}(1,n) / \mathbb{SU}(n) \times U(1)$ ([75]), also named multi-dilaton system in [63]; see Subsects. 3.2 and 3.3);
- $\mathcal{N} = 3$ ([76], see Subsects. 4.2 and 4.3);
- $\mathcal{N} = 5$ ([74], see Subsect. 6.2);
- $\mathcal{N} = 4$ pure ([73], see Subsect. 7.2).

Such theories differ by a crucial fact: whereas the $U$-invariant of minimally coupled $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supergravity is quadratic, the $U$-invariant of $\mathcal{N} = 5$ and pure $\mathcal{N} = 4$ theories is quartic in BH charges.

Thus, among all $d = 4$ supergravities with $U$-invariant quartic in BH charges, the cases $\mathcal{N} = 5$ and pure $\mathcal{N} = 4$ turn out to be peculiar ones.

Such a peculiarity can be traced back to the form of their Attractor Eqs., which are structurally identical to the ones of the minimally coupled $\mathcal{N} = 2$ and $\mathcal{N} = 3$ cases (see e.g. the treatments in [71] and [40]), and actually also to the very structure of $W^2_{BPS}$, as given by Eqs. (6.32) and (7.9), respectively.

This is ultimately due to a remarkable property, expressed by the last two lines of Eqs. (6.54) and (7.19): the (unique) invariant $I_4 (p,q)$ of $G_{N=5}$ and $G_{N=4,\text{pure}}$, which is quartic in the electric and magnetic BH charges $(p,q)$, is a perfect square of a quadratic expression when written in terms of the moduli-dependent skew-eigenvalues $Z_1$ and $Z_2$:
\[
I_4 (p,q) = Z_{AB} Z_{BC} Z_{CD} Z_{DA} - \frac{1}{4} \left( Z_{AB} Z_{AB} \right)^2 = Tr (A^2) - \frac{1}{4} (Tr (A))^2 = (Z_1^2 - Z_2^2)^2. \tag{8.1}
\]

Such a result, which is true in the whole scalar manifolds $\mathcal{M}_{N=5}$ and $\mathcal{M}_{N=4,\text{pure}}$, does not generally hold for all other $\mathcal{N} > 2$, $d = 4$ supergravities with (unique) quartic $U$-invariant, i.e. for $\mathcal{N} = 4$ matter-coupled and $\mathcal{N} = 6,8$ theories, as well as for $\mathcal{N} = 2$ supergravity whose scalar manifold does not belong to the aforementioned sequence of complex Grassmannians $\mathbb{SU}(1,n) / \mathbb{SU}(n) \times U(1)$.

This allows one to state that the relation (in the extremal case $c = 0$) between the square effective horizon radius $R^2_H$ and the square BH event horizon radius $r^2_H$ for the non-degenerate attractor flows of such supergravities, if any, is structurally different from the one given by Eq. (2.12). Of course, in such theories one can still construct the quantity $r^2_H (z_\infty, \overline{z}_\infty, p,q) - G_{\mathcal{N}} \Sigma^2 \Sigma^\dagger$ (eventually within a real parametrization of the scalar fields), but, also in the extremal case, it will be moduli-dependent, thus not determining $R^2_H (p,q)$.
Orbit | $\mathcal{N} = 2$ minimally coupled, $n_V = 3$ | $\mathcal{N} = 3$, $m = 1$
---|---|---
$\frac{SU(1,3)}{SU(3)}$ | $\mathcal{O}_{1/2}^{-\text{BPS}}$, no mod. space, $\mathcal{I}_{2,N=2} > 0$ | $\mathcal{O}_{\text{non-BPS},Z_{AB}=0}$, no mod. space, $\mathcal{I}_{2,N=3} < 0$
$\frac{SU(1,3)}{SU(1,2)}$ | $\mathcal{O}_{\text{non-BPS},Z=0}$, mod. space $= \frac{SU(1,2)}{SU(2) \times U(1)}$, $\mathcal{I}_{2,N=2} > 0$ | $\mathcal{O}_{1/2}^{-\text{BPS}}$, mod. space $= \frac{SU(1,2)}{SU(2) \times U(1)}$, $\mathcal{I}_{2,N=3} > 0$

Table 1: $\mathcal{N}$-dependent BPS-interpretations of the classes of non-degenerate orbits of the symmetric special Kähler manifold $\frac{SU(1,3)}{SU(3) \times U(1)}$.

9 $\mathcal{N} \geq 2$ Supergravities with the same Bosonic Sector and “Dualities”

In the present Section we consider $\mathcal{N} \geq 2$, $d = 4$ supergravities sharing the same bosonic sector, and thus with the same number of fermion fields, but with different supersymmetric completions.

I)
- $\mathcal{N} = 2$ (matter-coupled) magic supergravity based on the degree 3 complex Jordan algebra $J_3^\mathbb{H}$;
- $\mathcal{N} = 6$ supergravity.

The scalar manifold of both such theories (upliftable to $d = 5$) is $\frac{SO^*(12)}{SU(6) \times U(1)}$ (rank-3 homogeneous symmetric special Kähler space). In both theories the 16 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the left-handed spinor repr. 32 of the $U$-duality group $SO^*(12)$, which is symplectic, containing the symmetric singlet $1_a$ in the tensor product $32 \times 32$, and thus irreducible with respect to both $SO^*(12)$ and $Sp(32, \mathbb{R})$. For a discussion of the spin/field content, see e.g. [89, 90].

The correspondences among the various classes of non-degenerate extremal BH attractors of such two theories have been studied in [25] (see e.g. Table 9 therein).

II)
- $\mathcal{N} = 2$ supergravity minimally coupled to $n = n_V = 3$ Abelian vector multiplets;
- $\mathcal{N} = 3$ supergravity coupled to $m = 1$ matter (Abelian vector) multiplet.

All such theories (matter-coupled, with quadratic $U$-invariant, and not upliftable to $d = 5$) share the same scalar manifold, namely the rank-1 symmetric special Kähler space $\frac{SU(1,3)}{SU(3) \times U(1)}$. Furthermore, in both such theories the 4 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the fundamental 4 repr. of the $U$-duality group $SU(3,1)$ (not irreducible with respect to $SU(3,1)$ itself, but only with respect to $Sp(8, \mathbb{R})$).

By (local) supersymmetry, the number of fermion fields is the same in the three theories, namely there are 8 bosons and 8 fermions, but with different relevant spin/field contents:

$$\mathcal{N} = 2 \text{ minimally coupled, } n_V = 3 : \quad [1 \, (2), \, 2 \, (\frac{1}{2}), \, 1 \, (1)], \, 3 \, [1 \, (1), \, 2 \, (\frac{1}{2}), \, 1 \, C \, (0)];$$

$$\mathcal{N} = 3, \, m = 1 : \quad [1 \, (2), \, 3 \, (\frac{1}{2}), \, 3 \, (1), \, 1 \, (\frac{1}{2})], \, 1 \, [1 \, (1), \, 4 \, (\frac{1}{2}), \, 3 \, C \, (0)].$$

(9.1)

From this it follows that one can switch between such two theories by transforming 1 gravitino in 1 gaugino, and vice versa.

---

9 The relation between $\mathcal{N} = 1$ and $\mathcal{N} = 2$, $d = 4$ supergravities and their attractor solutions is discussed in [36].
\[ N = 2, n_V = 7 \]
\[ N = 4, n = 2 \]

| Orbit \( \frac{SU(1,1) \times SO(2,6)}{SO(2) \times SO(6)} \) | \( N = 2, n_V = 7 \) | \( N = 4, n = 2 \) |
| --- | --- | --- |
| \( O_{BPS}^+ \)\( I_{4,N=2} > 0 \) | \( O_{BPS}^+ \)\( I_{4,N=4} > 0 \) | \( O_{BPS}^+ \)\( I_{4,N=2} > 0 \) | \( O_{BPS}^+ \)\( I_{4,N=4} > 0 \) | \( O_{BPS}^+ \)\( I_{4,N=2} > 0 \) | \( O_{BPS}^+ \)\( I_{4,N=4} > 0 \) |
| \( O_{nBPS,Z=0} \) mod. space \( \frac{SO(2,4)}{SO(2) \times SO(4)} \) | \( O_{nBPS,Z=0} \) mod. space \( \frac{SO(2,4)}{SO(2) \times SO(4)} \) | \( O_{nBPS,Z\neq 0} \) mod. space \( \frac{SO(1,1) \times SO(1,1)}{SO(1) \times SO(1)} \) | \( O_{nBPS,Z\neq 0} \) mod. space \( \frac{SO(1,1) \times SO(1,1)}{SO(1) \times SO(1)} \) |

Table 2: \( N \)-dependent BPS-interpretations of the classes of non-degenerate orbits of the reducible symmetric special Kähler manifold \( SU(1,1) / U(1) \times SO(2,6) / SO(2) \times SO(6) \). The structure of the “duality” is analogous to the one pertaining to the manifold \( SU(1,1) / U(1) \times SO(2,6) / SO(2) \times SO(6) \) (see point I above, as well as Table 9 of [25]).

The relation among the various classes of non-degenerate extremal BH attractors of such three theories [71, 56] is given in Table 1.

When switching between \( N = 2 \) and \( N = 3 \), the flip in sign of the quadratic \( U \)-invariant \( I_2 = q^2 + p^2 \) can be understood by recalling that \( q^2 = \eta_{A\Sigma} q_A q_{\Sigma} \) and \( p^2 = \eta_{A\Sigma} p_A p_{\Sigma} \), with \( \eta_{A\Sigma} = \eta_{\Lambda\Sigma} = diag (1, -1, -1, -1) \) in the case \( N = 2 \), and \( \eta_{A\Sigma} = \eta_{\Lambda\Sigma} = diag (1, 1, 1, -1) \) in the case \( N = 3 \) (recall Eq. (5.1)). It is here worth pointing out once again that the positive signature pertains to the graviphoton charges, while the negative signature corresponds to the charges given by the asymptotical fluxes of the vector field strengths from the matter multiplets (see also the discussion in Sect. 5). As yielded by Table 1, the supersymmetry-preserving features of the attractor solutions depend on the sign of \( I_2 \).

III)

- \( N = 2 \) supergravity coupled to \( n_V = n + 1 = 7 \) Abelian vector multiplets, with scalar manifold \( SU(1,1) / U(1) \times SO(2,6) / SO(2) \times SO(6) \) (shortly named “cubic”, \( n_V = 7 \) in Eq. (9.2));

- \( N = 4 \) supergravity coupled to \( n_m = 2 \) matter (Abelian vector) multiplets.

The scalar manifold of both such theories (upliftable to \( d = 5 \)) is \( SU(1,1) / U(1) \times SO(2,6) / SO(2) \times SO(6) \) (homogeneous symmetric reducible special Kähler, with rank 3). In both theories the 8 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the (spinor/doublet)-vector (bi-fundamental) repr. \( (2,8) \) of the \( U \)-duality group \( SU(1,1) \times SO(2,6) \), which is symplectic, containing the antisymmetric singlet \( 1_a \) in the tensor product \( (2,8) \times (2,8) \), and thus irreducible with respect to both \( SU(1,1) \times SO(2,6) \) and \( Sp(16,\mathbb{R}) \).

Notice that, due to the isomorphism \( so(6,2) \sim so^*(8) \) (see e.g. [86]), the “dual” supersymmetric interpretation of the scalar manifold \( SU(1,1) / U(1) \times SO(2,6) / SO(2) \times SO(6) \) can be considered, disregarding the axion-dilaton sector \( SU(1,1) / U(1) \times SO(2,6) / SO(2) \times SO(6) \), as a “subduality” of the “duality” discussed in point I.

By (local) supersymmetry, the number of fermion fields is the same in the three theories, namely there are 16 bosons and 16 fermions, but with different relevant spin/field contents:

\[ N = 2 \ “cubic” \] \[ n_V = 7 \] \[ N = 2 \ "cubic", n_V = 7 \] \[ [1 \ (2), 2 \ (\frac{3}{2}), 1 \ (1)], 7 \ [1 \ (1), 2 \ (\frac{3}{2}), 1_c \ (0)] \]

\[ N = 4, n_m = 2 \] \[ [1 \ (2), 4 \ (\frac{3}{2}), 6 \ (1), 4 \ (\frac{3}{2}), 1_c \ (0)], 2 \ [1 \ (1), 4 \ (\frac{3}{2}), 3_c \ (0)] \]

(9.2)

From this it follows that one can switch between such two theories by transforming 2 gravitinos in 2 gauginos, and vice versa.
The correspondences among the various classes of non-degenerate extremal BH attractors of such two theories have been studied in [25, 41, 71, 56], and it is given in Table 2.

As yielded by the comparison of Table 9 of [25] and Table 2, such a “duality” is pretty similar to the “duality” between $\mathcal{N} = 2 J^3_3$ and $\mathcal{N} = 6$ considered at point I, also because the sign of the quartic $U$-invariant is unchanged by the “duality” relation (this is also consistent with the “subduality” relation mentioned above). In this sense, it differs from the “duality” between $\mathcal{N} = 2$ minimally coupled, $n_{V} = 3$ and $\mathcal{N} = 3$, $m = 1$ considered at point II, because in both such theories the $U$-invariant is quadratic, and its sign is flipped by the “duality” relation (see Table 1).

Points I-III present evidences against the conventional wisdom that interacting bosonic field theories have a unique supersymmetric extension. The sharing of the same bosonic backgrounds with different supersymmetric completions implies the “dual” interpretation with respect to the supersymmetry-preserving properties of non-degenerate extremal BH attractor solutions (see respectively Table 9 of [25], Table 1 and Table 2).

10 Conclusion

In the present investigation we have considered the class of $\mathcal{N} \geq 2$, $d = 4$ ungauged supergravity theories which do not have a counterpart in $d = 5$ space-time dimensions. For such theories, the extremal BH parameters, namely, the ADM mass $M_{\text{ADM}}$, the scalar charges $\Sigma^a$ and the effective horizon radius $R_H$, pertaining to non-degenerate attractor flows have a simple formulation in the first order (fake supergravity) formalism.

All such theories share the property that an effective radial variable $R$ can be defined, such that the effective BH (horizon) area $A$ is simply given by the surface of a sphere of radius $R_H(p,q)$, where $R_H(p,q)$ is the moduli-independent effective horizon radius of the extremal BH.

For $\mathcal{N} = 2$ this holds for supergravity minimally coupled to Abelian vector multiplets, but it does not hold for more general matter couplings, such as symmetric spaces coming from degree three Jordan Algebras [81] with cubic holomorphic prepotential. In this respect, minimally coupled $\mathcal{N} = 2$ supergravity can be considered as a multi-dilaton system in that it generalizes the Maxwell-Einstein-axion-dilaton system, studied in Refs. [77] and [78] (see also the recent treatment in [63]). Matter-coupled supergravity with $\mathcal{N} = 3$ shares similar properties. Both such theories exhibit two class of non-degenerate attractors (BPS and non-BPS), and a Bekenstein-Hawking classical BH entropy quadratic in the electric and magnetic BH charges. Furthermore, non-BPS ($Z = 0$) $\mathcal{N} = 2$ attractors and $\mathcal{N} = 3$ (both $\frac{1}{3}$)BPS and non-BPS ($Z_{AB} = 0$) attractors yield a related moduli space of solutions.

Pure $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravities have also the same formula yielding to define $R_H(p,q)$, in spite of the fact that the classical BH entropy is not quadratic, but rather the square root of a quartic expression, in terms of the BH charges. Such theories have only BPS attractors, and for $\mathcal{N} = 5$ a residual moduli space of solutions also exists.

It would be interesting to extend the notion of effective radius and of fake supergravity formalism to other $d = 4$ theories, such as $\mathcal{N} = 2$ not minimally coupled to Abelian vector multiplets, matter-coupled $\mathcal{N} = 4$, $\mathcal{N} = 6$ and $\mathcal{N} = 8$. In these cases, different formula should occur to determine the BH parameters, such as ADM mass $M_{\text{ADM}}$ and scalar charges $\Sigma^a$, in terms of the geometry of the underlying (asymptotical) scalar manifold.

Finally, one may wonder about a stringy realization of the theories discussed in the present paper, and their extremal BH states. At the string tree level, the massless spectrum of $\mathcal{N} = 3$ and $\mathcal{N} = 5$, $d = 4$ supergravity can be obtained via asymmetric orbifolds of Type II superstrings [100]–[103], or by orientifolds [104]. Furthermore, it should be remarked that in string theory the Attractor Mechanism is essentially a non-perturbative phenomenon, either because it fixes the dilaton, or because it involves non-perturbative string states made out of D-brane bound states [105].

It is worth pointing out once again that the present analysis only covers non-degenerate extremal BH attractors, determining “large” BH horizon geometries with non-vanishing classical effective BH horizon area in the field theory limit. For degenerate extremal BHs, having “small” BH horizon geometries with vanishing classical effective BH horizon area, a departure from the Einsteinnian approximation, including higher curvature terms in the gravity sector, is at least required [106]. Such corrections in supersymmetric theories of gravity have been considered in [11] and [107]–[109].
Acknowledgments

The authors gratefully acknowledge L. Andrianopoli, M. Bianchi, A. Ceresole, R. D’Auria, F. Morales, M. Trigiante and A. Yeranyan for enlightening and fruitful discussions.

We also would like to thank Mrs. Helen Webster for careful proofreading.

A. M. would like to thank the Department of Physics, Theory Unit Group at CERN, where part of this work was done, for kind hospitality and stimulating environment.

The work of S.F. has been supported in part by the European Community Human Potential Programme under contract MRTN-CT-2004-503369 “Constituents, Fundamental Forces and Symmetries of the Universe” in association with LNF-INFN and in part by DOE grant DE-FG03-91ER40662, Task C.

The work of A.M. has been supported by Museo Storico della Fisica e Centro Studi e Ricerche “Enrico Fermi”, Rome, Italy, in association with INFN-LNF.
A Appendix I

A Counterexample : \( \mathcal{N} = 4 \) Matter-Coupled Supergravity

As an example of \( d = 4 \) supergravity in which \( \mathcal{I}_4 \) is not a perfect square of an expression quadratic in all the relevant geometrical quantities (such as the skew-eigenvalues of the central charge matrix and the matter charges), let us consider the \( \mathcal{N} = 4, d = 4 \) matter-coupled theory [98, 99].

The real scalar manifold is

\[
\mathcal{M}_{\mathcal{N}=4} = \frac{G_{\mathcal{N}=4}}{H_{\mathcal{N}=4}} = \frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6) \times SO(n)}, \quad \dim_R = 6n + 2. \tag{A.1}
\]

The \( 6 + n \) vector field strengths and their duals, as well as their asymptotical fluxes, sit in the bi-fundamental irrepr. \((2,6+n))\ of the U-duality group \( G_{\mathcal{N}=4} = SU(1,1) \times SO(6,n)\).

\( Z_{AB} = Z_{[AB]}, A,B = 1,2,3,4 = \mathcal{N} \) is the central charge matrix. As in the pure supergravity treated in Sect. 7 (obtained by setting \( n = 0 \) in Eq. (A.1)), by means of a suitable transformation of the \( \mathcal{R}\)-symmetry \( U(1) \times SO(6) \sim U(1) \times SU(4)\), \( Z_{AB} \) can be skew-diagonalized by putting it in the normal form (see e.g. [40]) and Refs. therein):

\[
Z_{AB} = \begin{pmatrix} Z_1 \epsilon & \cdots & Z_n \epsilon \end{pmatrix}, \tag{A.2}
\]

where \( Z_1, Z_2 \in \mathbb{R}_0^+ \) are the \( \mathcal{N} = 4 \) (moduli-dependent) skew-eigenvalues, which can be ordered as \( Z_1 \geq Z_2 \) without any loss of generality (up to renamings; see e.g. [40]), and can be formally expressed by the very same Eqs. (6.3)-(6.5), where \( I_1 \) and \( I_2 \) are still the two (moduli-dependent) \( H_{\mathcal{N}=4,pure}(\text{and also } H_{\mathcal{N}=4})\)-invariants.

In this case, similarly to \( \mathcal{N} = 3 \) supergravity, also the matter charges \( Z_I (I = 1,\ldots,n) \) enter the game, \( n \in \mathbb{N} \) denoting the number of matter multiplets coupled to the gravity multiplet. By a suitable rotation of \( SO(n) \), the vector \( Z_I \) can be reduced in such a way that only one (strictly positive) real and one complex matter charge are non-vanishing [40]:

\[
Z_I = \begin{pmatrix} Z_1 = \rho_1, Z_2 = \rho_2 e^{i\theta}, Z_3 = 0 \end{pmatrix}, \quad \rho_1, \rho_2 \in \mathbb{R}_0^+, \theta \in [0,2\pi), \hat{T} = 3,\ldots,n. \tag{A.3}
\]

Thus, one can introduce the (moduli-dependent, unique) \( SO(n)\)-(and also \( H_{\mathcal{N}=4}\))-invariants

\[
I_3 = \|Z_I Z^I\| = \rho_1^2 + \rho_2^2; \tag{A.4}
\]

\[
I_4 = Re\{Z_I Z^I\} = \rho_1^2 + \rho_2^2 \cos(2\theta). \tag{A.5}
\]

Now, there are only three (moduli-dependent) \( SO(6,n)\)-invariants, reading as follows [91]:

\[
I_1 = I_1 - I_3 = 2Z_1^2 + 2Z_2^2 - \rho_1^2 - \rho_2^2; \tag{A.6}
\]

\[
I_2 = \frac{1}{4} \epsilon^{ABCD} Z_{AB} Z_{CD} - Z_1^2 Z_2^I = 2Z_1 Z_2 - \rho_1^2 - 2\rho_2 e^{-2i\theta}; \tag{A.7}
\]

\[
I_3 = \|Z_I Z^I\| = 2Z_1 Z_2 - \rho_1^2 - \rho_2^2 e^{2i\theta}. \tag{A.8}
\]

The quartic \( G_{\mathcal{N}=4}\)-invariant \( \mathcal{I}_4 \) of \( \mathcal{N} = 4, d = 4 \) supergravity is the following unique (moduli-dependent) \( G_{\mathcal{N}=4}\)-invariant combination of \( I_1, I_2 \) and \( I_3 \) [91]:

\[
\mathcal{I}_4 \equiv I_1^2 - I_2 I_3 = I_1^2 - |I_2|^2 =
\]

\[
= (Z_1^2 - Z_2^2)^2 + (Z_1 Z^I)^2 - 2(Z_1^2 + Z_2^2)Z_1 Z^I + \]

\[
+ 2Z_1 Z_2 \left( Z_1 Z^I + Z_1 Z^I \right) - |Z_1 Z^I|^2 = \tag{A.9}
\]

\[
= (Z_1^2 - Z_2^2)^2 + (\rho_1^2 + \rho_2^2)^2 - 2(Z_1^2 + Z_2^2) (\rho_1^2 + \rho_2^2) + \]

\[
+ 4Z_1 Z_2 \left[ \rho_1^2 + \rho_2^2 \cos(2\theta) \right] - \left[ \rho_1^4 + \rho_1^4 + 2\rho_1^2 \rho_2^2 \cos(2\theta) \right].
\]
On the other hand, in terms of the BH charges \((q,p)\), \(I_4\) reads as follows (recall Eq. (7.15)):

\[
I_4 = 4 \left[ p^2 q^2 - (p \cdot q)^2 \right],
\]

where \(p^2 = p^\Lambda p^\Sigma \eta_{\Lambda \Sigma}, q^2 = q_M q_N \eta^{MN}\), with \(\Lambda\) ranging \(1, \ldots, n+6\), and the scalar product \(\cdot\) is defined by \(\eta_{\Lambda \Sigma} = \eta^{\Lambda \Sigma}\), the Lorentzian metric with signature \((n,6)\) (see [63] and Refs. therein).

Looking at Eq. (A.9), it is easy to realize that \(I_4\) is a non-trivial perfect square of a function of degree 2 of \(Z_1, Z_2, \rho_1, \rho_2\) and \(\theta\) only in the pure supergravity theory (obtained by setting \(n = 0\)), i.e. only in the case \(\rho_1 = \rho_2 = 0\). In such a limit, Eq. (A.9) consistently reduces to Eq. (8.1).

As an example, we can workout the case \(n = 1\) (which uplifts to pure \(N = 4, d = 5\) supergravity). In this case, only a (strictly positive) real matter charge \(Z_1 = \rho_1 \in \mathbb{R}^+\) is present, and the quartic invariant \(I_4\) acquires the following form:

\[
I_4 = (Z_1^2 - Z_2^2)^2 - 2(Z_1^2 + Z_2^2)Z_1^2 + 4Z_1 Z_2 Z_1^2 = (Z_1 - Z_2)^2 \left[ (Z_1 + Z_2)^2 - 2Z_1^2 \right] = (Z_1 - Z_2)^2 \left( Z_1 + Z_2 + \sqrt{2}\rho_1 \right) \left( Z_1 + Z_2 - \sqrt{2}\rho_1 \right),
\]

which is not a non-trivial perfect square of \(Z_1, Z_2\) and \(\rho_1\).
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