SIGN CHANGES IN MERTENS’ FIRST AND SECOND THEOREMS

JEFFREY P.S. LAY

Abstract. We show that the functions \( \sum_{p \leq x} \frac{\log p}{p} - \log x - E \) and \( \sum_{p \leq x} \frac{1}{p} - \log \log x - B \) change sign infinitely often, and that under certain assumptions, they exhibit a strong bias towards positive values. These results build on recent work of Diamond and Pintz [DP09] and Lamzouri [Lam] concerning oscillation of Mertens’ product formula, and answer a question posed by Rosser and Schoenfeld [RS62].

1. Introduction

Mertens’ first two theorems concerning the density of the primes can be stated as the asymptotic formulae (see [Dus99, Thms 11 & 12] for explicit bounds)

\[
M_1(x) := \sum_{p \leq x} \frac{\log p}{p} - \log x - E = O\left( \frac{1}{\log x} \right),
\]

\[
M_2(x) := \sum_{p \leq x} \frac{1}{p} - \log \log x - B = O\left( \frac{1}{\log^2 x} \right),
\]

as \( x \to \infty \), where, writing \( C_0 \) for Euler’s constant,

\[
E := -C_0 - \sum_{k=2}^{\infty} \sum_p \frac{\log p}{p^k} = -1.332 \ldots , \quad B := C_0 - \sum_{k=2}^{\infty} \sum_p \frac{1}{kp^k} = 0.261 \ldots .
\]

Concerning the signs of \( M_1(x) \) and \( M_2(x) \), calculations by Rosser and Schoenfeld [RS62, Thms 20 & 21] show that \( M_1(x) > 0 \) and \( M_2(x) > 0 \) for all \( 1 < x < 10^8 \), and they questioned, by analogy with Littlewood’s famous result on \( \pi(x) - \text{li}(x) \), whether both inequalities fail for arbitrarily large \( x \).

Diamond and Pintz [DP09] established oscillation in Mertens’ product formula, answering an analogous question of Rosser and Schoenfeld. Precisely, they showed that the function

\[
\sqrt{x} \left( \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} - e^{C_0 \log x} \right)
\]

attains arbitrarily large positive and negative values as \( x \to \infty \). Motivated by their work, we prove that both \( M_1(x) \) and \( M_2(x) \) change sign infinitely often. Moreover, we provide estimates regarding the growth of their oscillations.

Theorem 1. For each \( i \in \{1, 2\} \) the following assertion holds: There exists a function \( f_i(x) \) going to infinity as \( x \to \infty \) such that

\[
\liminf_{x \to \infty} \frac{\sqrt{x \log^{i-1} x}}{f_i(x)} M_i(x) < -1, \quad \limsup_{x \to \infty} \frac{\sqrt{x \log^{i-1} x}}{f_i(x)} M_i(x) > 1.
\]

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Remark 1. Our methods show that if the Riemann Hypothesis (RH) is true, we may take $f_i(x) = c_i \log \log \log x$ for some fixed $c_i > 0$ (cf. §4 in [DP09]). Estimating the growth of these functions unconditionally appears to remain a formidable problem.

Remark 2. The main result in this paper is the case $i = 1$ of Theorem 1 since oscillation of $M_2(x)$ follows as a simple corollary from oscillation of Mertens’ product formula [DP09, Thm 1.1], in view of the asymptotic

$$- \log \left( 1 - \frac{1}{p} \right) = \sum_{k=1}^{\infty} \frac{1}{kp^k} = \frac{1}{p} + O \left( \frac{1}{p^2} \right).$$

We refer the reader to §5 for exact details.

Due to the nature of oscillation theorems, it is convenient to break the proof of the case $i = 1$ of Theorem 1 into two cases, the first in which RH is assumed to fail, and the second in which it is assumed to hold. We tackle these individual cases in §3 and §4, respectively.

We investigate in §6 why the functions $M_1(x)$ and $M_2(x)$ are biased towards positive values, explaining the observations of Rosser and Schoenfeld. In general, we say that $f(x)$ is biased towards values in $S \subset \mathbb{R}$ if $\delta(\{x : f(x) \in S\}) > 1/2$ for an appropriate notion of density $\delta$. It turns out (see, for example, [Win41]) that the logarithmic density is the appropriate density to use for oscillation theorems; suffice it to say, the usual density does not exist. We recall its definition: for any $S \subset \mathbb{R}$, we define

$$\underline{\delta}(S) := \liminf_{X \to \infty} \frac{1}{\log X} \int_{t \in S \cap [2, X]} \frac{dt}{t}, \quad \overline{\delta}(S) := \limsup_{X \to \infty} \frac{1}{\log X} \int_{t \in S \cap [2, X]} \frac{dt}{t}.$$

If $\underline{\delta}(S) = \overline{\delta}(S)$, we call the resultant quantity the logarithmic density of $S$, and denote it by $\delta(S)$.

We also recall the following conjecture concerning the vertical distribution of the non-trivial zeroes of the Riemann zeta-function $\zeta(s)$.

Conjecture 1 (Linear Independence Hypothesis (LI)). The set of positive ordinates of the non-trivial zeroes of $\zeta(s)$ is linearly independent over $\mathbb{Q}$.

This conjecture encapsulates the widely-held belief that there should not exist any algebraic relations between the non-trivial zeroes of $\zeta(s)$; it also implies that all such zeroes are simple. Analogous statements are expected to hold for generalised $L$-functions.

Rubinstein and Sarnak [RS94] showed that under the assumption of both RH and LI, we have

$$\delta(1) := \delta(\{x \geq 2 : \pi(x) > \text{li}(x)\}) = 0.00000026 \ldots;$$

thus, the difference $\pi(x) - \text{li}(x)$ is highly biased towards negative values. Lamzouri [Lam] recently studied the bias in Mertens’ product formula using the framework developed by Rubinstein and Sarnak. He determined that under the assumptions of RH and LI, we have

$$\delta \left( \left\{ x \geq 2 : \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} > e^\gamma \log x \right\} \right) = 1 - \delta(1) = 0.99999973 \ldots.$$

We shall prove that $M_1(x)$ and $M_2(x)$ are both biased towards positive values with logarithmic density $1 - \delta(1)$.

Theorem 2. For each $i \in \{1, 2\}$ the following assertion holds: Let $W_i$ denote the set of real numbers $x \geq 2$ such that $M_i(x) > 0$. Then, assuming RH, we have $0 < \underline{\delta}(W_i) \leq \overline{\delta}(W_i) < 1$. If in addition to RH we assume LI, then in fact $\delta(W_i) = 1 - \delta(1)$.
We shall see that the case \( i = 2 \) follows immediately from the work of Lamzouri \cite{Lam}, owing to the almost identical behaviour between \( M_2(x) \) and the logarithmic form of Mertens' product formula. A full proof will be given for the case \( i = 1 \); we follow the argument given in \cite{Lam}.

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2. Notation

As usual, we write \( f(x) = O(g(x)) \) or, equivalently, \( f(x) \ll g(x) \), if \( |f(x)| \leq cg(x) \) is satisfied for some \( c > 0 \) and all sufficiently large \( x \). For oscillation estimates, we say that \( f(x) = \Omega_{\pm}(g(x)) \) if there exists \( c' \geq 0 \) such that both \( \liminf_{x \to \infty} f(x)/g(x) < -c' \) and \( \limsup_{x \to \infty} f(x)/g(x) > c' \) hold.

For a complex variable \( s = \sigma + it \), \( \Re s \) and \( \Im s \) will denote, respectively, the real and imaginary parts of \( s \). The letter \( p \) will always represent a prime number, and we use \( \rho = \beta + i\gamma \) to denote a non-trivial zero of \( \zeta(s) \).

Finally, since we will be using some probability theory, we write \( \mathbb{P} \) for probability and \( \mathbb{E} \) for expectation.

3. Oscillation of \( M_1(x) \): the non-RH case

The first step is to replace the terms in \( M_1(x) \) involving sums over primes with an appropriate Stieltjes integral.

**Lemma 1.** We have

\[
M_1(x) = \int_1^x \frac{d\psi(t)}{t} - \log x + C_0 + O\left(\frac{1}{\sqrt{x}}\right),
\]

where, as usual, \( \psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{p^k \leq x} \log p \).

**Proof.** We observe that

\[
\int_1^x \frac{d\psi(t)}{t} = \sum_{n \leq x} \frac{\Lambda(n)}{n} = \sum_{p \leq x} \frac{\log p}{p} + \sum_{\substack{p^k \leq x \\kappa \geq 2}} \frac{\log p}{p^k}
\]

\[
= \sum_{p \leq x} \frac{\log p}{p} + \sum_{\kappa = 2}^{\infty} \sum_{p} \frac{\log p}{p^\kappa} - \left( \sum_{p \leq x} \frac{\log p}{p^2} + \sum_{\substack{p^k \leq x \\kappa \geq 2}} \frac{\log p}{p^k} \right),
\]

so it remains to estimate the term in brackets. Using the well-known estimate \( \theta(x) := \sum_{p \leq x} \log p \ll x \), we see that

\[
\sum_{\substack{p^k \leq x \\kappa \geq 2}} \frac{\log p}{p^k} \ll \sum_{p \leq x} \frac{\log p}{p^2} = \int_x^\infty \frac{d\theta(t)}{t^2} \ll \frac{1}{x} + \int_x^\infty \frac{\theta(t)}{t^3} \, dt \ll \frac{1}{x^2}.
\]

For the remaining sum, we use the estimate \( \pi(x) \ll x/\log x \) to obtain

\[
\sum_{\substack{p^k \leq x \\kappa \geq 2}} \frac{\log p}{p^k} \ll \sum_{\sqrt{x} < p \leq x} \frac{\log p}{p^2} + \sum_{p \leq \sqrt{x}} \frac{\log x}{x} \ll \frac{1}{\sqrt{x}}.
\]
We conclude that
\[
\sum_{p \leq x} \frac{\log p}{p} + \sum_{k=2}^{\infty} \sum_{p} \frac{\log p}{p^k} = \int_{1}^{x} \frac{d\psi(t)}{t} + O\left(\frac{1}{\sqrt{x}}\right),
\]
from which the result follows. \(
\square
\)

Now set
\[
U(x) := \int_{1}^{x} \frac{d\psi(t)}{t} - \log x + C_0, \quad \mathcal{V}(x) := \frac{1}{\sqrt{x}}.
\]
The theorem clearly follows if we can show that for any fixed \(K \in \mathbb{R}\), the function \(U(x) + KV(x)\) changes sign infinitely often. This is achieved through an application of the following famous theorem of Landau (see [Ing32, Thm H]).

**Theorem 3** (Landau’s oscillation theorem). Suppose \(f(x)\) is of constant sign for all sufficiently large \(x\). Then the real point \(s = \sigma_0\) of the line of convergence of the Dirichlet integral
\[
\int_{1}^{x} x^{-s} f(x) \, dx
\]
is a singularity of the function represented by the integral.

This approach naturally leads to a consideration of the Mellin transforms of \(U(x)\) and \(\mathcal{V}(x)\). Recall the well-known identity [Ing32, Eq. (17)]
\[
-\frac{\zeta'(s)}{\zeta(s)} = s \int_{1}^{\infty} x^{-s-1} \psi(x) \, dx = \int_{1}^{\infty} x^{-s} \, d\psi(x), \quad \Re s > 1,
\]
whence it follows from a change of variables and integration by parts that
\[
-\frac{\zeta'(s+1)}{\zeta(s)} = \int_{1}^{\infty} x^{-s} \frac{d\psi(x)}{x} = s \int_{1}^{\infty} x^{-s-1} \int_{1}^{x} \frac{d\psi(t)}{t} \, dx, \quad \Re s > 0.
\]
Moreover, we have from elementary means
\[
\int_{1}^{\infty} x^{-s-1} \log x \, dx = \frac{1}{s^2}, \quad \Re s > 0,
\]
and
\[
\int_{1}^{\infty} x^{-s-1} \, dx = \frac{1}{s}, \quad \Re s > 0.
\]
These give us the Mellin transforms
\[
\tilde{U}(s) := \int_{1}^{\infty} x^{-s-1} U(x) \, dx = -\frac{1}{s} \frac{\zeta'(s+1)}{\zeta(s)} - \frac{1}{s} \frac{1}{2} + \frac{1}{s} C_0, \quad \Re s > 0,
\]
and, replacing \(s\) with \(s + 1/2\) in (3),
\[
\tilde{V}(s) := \int_{1}^{\infty} x^{-s-1} \mathcal{V}(x) \, dx = \int_{1}^{\infty} x^{-(s+1/2)-1} \, dx = \frac{2}{1+2s}, \quad \Re s > -\frac{1}{2}.
\]
We now investigate the analytic behaviour of the point of convergence of the Mellin transform \(\tilde{U}(s) + K\tilde{V}(s)\).

Consider the explicit formula [MV07, Cor. 10.14]
\[
-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2} C_0 + 1 - \log 2\pi + \frac{1}{s-1} + \frac{1}{2} \Gamma'\left(\frac{s}{2} + 1\right) - \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),
\]
whence
\[ -\frac{1}{s} \zeta'(s + 1) = \frac{1}{s} \left( \frac{1}{2} C_0 + 1 - \log 2\pi \right) + \frac{1}{s^2} \]
\[ + \frac{1}{2s} \frac{\Gamma'}{\Gamma} \left( \frac{s+1}{2} + 1 \right) - \frac{1}{s} \sum_{\rho} \left( \frac{1}{s+1-\rho} + \frac{1}{\rho} \right). \]

It remains to classify the simple poles of the last two terms on the right-hand side of (5).

**Lemma 2.** We have
\[ -\frac{1}{s} \sum_{\rho} \left( \frac{1}{s+1-\rho} + \frac{1}{\rho} \right) = \frac{1}{s} \left( -C_0 - 2 + \log 4\pi \right) + F(s), \]
where \( F(s) \) is some function regular for \( \Re s > 0 \).

**Proof.** This follows from the identity \[\text{[MV07, Eq. (10.30)]}\]
\[ -\sum_{\rho} \left( \frac{1}{s+1-\rho} + \frac{1}{\rho} \right) = -C_0 - 2 + \log 4\pi, \]
whence
\[ \text{Res}_{s=0} \left[ -\frac{1}{s} \sum_{\rho} \left( \frac{1}{s+1-\rho} + \frac{1}{\rho} \right) \right] = -C_0 - 2 + \log 4\pi. \]

**Lemma 3.** We have
\[ \frac{1}{2s} \frac{\Gamma'}{\Gamma} \left( \frac{s+1}{2} + 1 \right) = \frac{1}{s} \left( -\frac{1}{2} C_0 - \log 2 + 1 \right) + G(s), \]
where \( G(s) \) is some function regular for \( \Re s > 0 \).

**Proof.** Logarithmically differentiating Legendre’s duplication formula \[\text{[MV07, Eq. (C.9)]}\] yields the functional equation
\[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + s \right) = -\frac{\Gamma'}{\Gamma} (s) - 2 \log 2 + 2 \frac{\Gamma'}{\Gamma} (2s). \]

Using the fact that for \( n \in \mathbb{N} \) we have \( \Gamma'(n+1) = n! \times (-C_0 + \sum_{k=1}^{n} 1/k) \), and recalling the identity \( -\Gamma'(1) = C_0 \), we deduce that
\[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + 1 \right) = -\frac{\Gamma'}{\Gamma} (1) - 2 \log 2 + 2 \frac{\Gamma'}{\Gamma} (2) = -C_0 - 2 \log 2 + 2, \]
whence
\[ \text{Res}_{s=0} \left[ \frac{1}{2s} \frac{\Gamma'}{\Gamma} \left( \frac{s+1}{2} + 1 \right) \right] = -\frac{1}{2} C_0 - \log 2 + 1. \]

Combining Lemmas 2 and 3 with equation (5) gives us the formula
\[ -\frac{1}{s} \zeta'(s + 1) = -\frac{1}{s} C_0 + \frac{1}{s^2} + F(s) + G(s), \]
whence
\[ \hat{U}(s) + K \hat{V}(s) = F(s) + G(s) + K \frac{2}{1 + 2s}. \]
To conclude the proof of the theorem, fix $K$ (positive or negative) and suppose RH is false. Then $\zeta'(s+1)/\zeta(s+1)$ has a singularity at a complex point $s_0$ with $\Re s_0 > -1/2$, so the abscissa of convergence of the Mellin transform

$$\hat{U}(s) + K\hat{V}(s) = -\frac{1}{s} \zeta'(s+1) - \frac{1}{s^2} + \frac{1}{s} C_0 + K \frac{2}{1 + 2s}$$

is at least $-1/2$. But (8) shows that the possible singularity at $s = 0$ is removable, so we conclude that the point of convergence of the Mellin transform is a regular point. It follows from Theorem 3 that $U(x) + KV(x)$ changes sign infinitely often.

4. Oscillation of $M_1(x)$: the RH case

We start with a formula that relates $M_1(x)$ to the error term in the prime number theorem.

**Lemma 4.** We have (unconditionally)

$$M_1(x) = \frac{\psi(x) - x}{x} - \int_x^\infty \frac{\psi(t) - t}{t^2} dt + O\left(\frac{1}{\sqrt{x}}\right). \quad (9)$$

**Proof.** Integration by parts yields

$$\int_1^x \frac{d\psi(t)}{t} = \frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)}{t^2} dt$$

$$= \log x + \frac{\psi(x) - x}{x} + 1 + \int_1^x \frac{\psi(t) - t}{t^2} dt$$

$$= \log x + \frac{\psi(x) - x}{x} + 1 + \int_1^\infty \frac{\psi(t) - t}{t^2} dt - \int_x^\infty \frac{\psi(t) - t}{t^2} dt,$$

making use of the fact that $\lim_{x \to \infty} \int_1^x (\psi(t) - t)/t^2 dt \ll 1$, which can be seen via a simple application of the prime number theorem. We conclude from (1) that

$$M_1(x) - C_0 - \left(1 + \int_1^\infty \frac{\psi(t) - t}{t^2} dt\right) = \frac{\psi(x) - x}{x} - \int_x^\infty \frac{\psi(t) - t}{t^2} dt + O\left(\frac{1}{\sqrt{x}}\right).$$

It remains to assign an explicit value to the constant term $1 + \int_1^\infty (\psi(t) - t)/t^2 dt$. We first observe that

$$1 + \int_1^\infty \frac{\psi(t) - t}{t^2} dt = 1 + \lim_{s \to 1} \int_1^\infty t^{-s-1}(\psi(t) - t) dt,$$

which, appealing to (2) and (3), is equal to

$$1 + \lim_{s \to 1} \left[-\frac{1}{s} \zeta'(s) - \frac{1}{s - 1}\right] = \lim_{s \to 1} \left[-\frac{1}{s} \zeta'(s) + \frac{1}{s} - \frac{1}{s - 1}\right] = \lim_{s \to 1} \left[-\frac{1}{s} \zeta'(s) - \frac{1}{s(s - 1)}\right].$$

We now apply (4), (6), and (7) to deduce that the limit attains the value $-C_0$, as desired. □

Using the famous oscillation result [Ing32 Thm 34] of $\psi(x) - x$, we obtain

$$\frac{\psi(x) - x}{x} = \Omega_{\pm} \left(\frac{\log \log x}{\sqrt{x}}\right), \quad (10)$$

which immediately gives the desired estimate for the first term on the right-hand side of (9). The theorem follows if we can show that the integral is sufficiently small.

To achieve this, we invoke the following powerful result of Cramér [Cra21 Thm IV] concerning the average order of the error term in the prime number theorem. This enables us to save a logarithmic factor that we would otherwise have to deal with using point-wise estimates.
**Theorem 4** (Cramér). If RH is true, then
\[
\frac{1}{x} \int_{1}^{x} \frac{\psi(t) - t}{\sqrt{t}} \, dt \ll 1.
\]

From this, we see that
\[
\frac{1}{\sqrt{2x}} \int_{x}^{2x} |\psi(t) - t| \, dt + \int_{1}^{x} \frac{\psi(t) - t}{\sqrt{t}} \, dt \leq \int_{1}^{2x} \frac{\psi(t) - t}{\sqrt{t}} \, dt \ll x,
\]
whence
\[
(11) \quad \int_{x}^{2x} |\psi(t) - t| \, dt \ll x \sqrt{x}.
\]

The strategy is to use the estimate (11) to bound the integral in (9) using dyadic interval estimates.

Using (11), we have for all non-negative integers \(k\)
\[
\int_{2^k x}^{2^{k+1} x} \frac{\psi(t) - t}{t^2} \, dt \leq \frac{1}{2^{2k+1} x^2} \int_{2^k x}^{2^{k+1} x} |\psi(t) - t| \, dt \ll \left( \frac{1}{\sqrt{2}} \right)^k \frac{1}{x}.
\]

Thus, we obtain the estimate
\[
\int_{x}^{\infty} \frac{\psi(t) - t}{t^2} = \sum_{k=0}^{\infty} \int_{2^k x}^{2^{k+1} x} \frac{\psi(t) - t}{t^2} \, dt \ll \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^k \ll \frac{1}{\sqrt{x}}.
\]

We see that the integral is smaller than the oscillation term (11) when \(x \to \infty\), so we conclude from (11) that
\[
M_1(x) = \Omega_{\pm} \left( \frac{\log \log \log x}{\sqrt{x}} \right).
\]

### 5. Oscillation of \(M_2(x)\)

The main result of Diamond and Pintz [DP09] towards establishing sign changes of \(M_3(x)\) is the following oscillation estimate:

**Theorem 5** (Diamond and Pintz). There exists a function \(f_3(x)\) going to infinity as \(x \to \infty\) such that
\[
- \sum_{p \leq x} \log \left( 1 - \frac{1}{p} \right) - \log \log x - C_0 = \Omega_{\pm} \left( \frac{f_3(x)}{\sqrt{x} \log x} \right).
\]

In particular, we may take \(f_3(x) = \log \log \log x\) assuming the truth of RH.

Thus, the case \(i = 2\) of Theorem 1 follows immediately upon showing

**Lemma 5.** We have
\[
(12) \quad M_2(x) = - \sum_{p \leq x} \log \left( 1 - \frac{1}{p} \right) - \log \log x - C_0 + O \left( \frac{1}{x} \right).
\]

**Proof.** Taking the Taylor expansion of the logarithmic term yields
\[
- \sum_{p \leq x} \log \left( 1 - \frac{1}{p} \right) = \sum_{p \leq x} \sum_{k=1}^{\infty} \frac{1}{kp^k} = \sum_{p \leq x} \frac{1}{p} + \sum_{k=2}^{\infty} \sum_{p \leq x} \frac{1}{kp^k} - \sum_{p > x \ k \geq 2} \frac{1}{kp^k},
\]
so it remains to show that the last sum is $O(x^{-1})$. But this follows readily from the generous estimate

$$
\sum_{\substack{p > x \\ k \geq 2}} \frac{1}{kp^k} \ll \sum_{p > x} \frac{1}{p^2} \ll \int_x^\infty \frac{dt}{t^2} \ll \frac{1}{x}.
$$

\[ \square \]

6. Investigating the bias

We begin by listing some of the main results in [Lam].

**Proposition 1** (Corollary 2.2 in [Lam]). Assuming RH, we have

$$
\sqrt{x} \log x \left( -\sum_{p \leq x} \log \left( 1 - \frac{1}{p} \right) - \log \log x - C_0 \right)
= 1 + 2\Re \sum_{0 < \gamma \leq T} \frac{x^{i\gamma}}{-\frac{1}{2} + i\gamma} + O\left( \frac{\sqrt{x} \log^2(xT)}{T} + \frac{1}{\log x} \right).
$$



**Proposition 2** (See §4 in [Lam]). Let $\tilde{W}$ denote the set of real numbers $x \geq 2$ such that $\prod_{p \leq x} (1 - 1/p)^{-1} > e^{\gamma} \log x$, and let $\tilde{Z}$ denote the random variable

$$
\tilde{Z} := 1 + 2\Re \sum_{\gamma > 0} \frac{\tilde{X}(\gamma)}{\sqrt{\frac{1}{4} + \gamma^2}},
$$

where $\tilde{X}(\gamma)$ is a sequence of independent random variables indexed by the positive imaginary parts of the non-trivial zeroes of $\zeta(s)$. Then, assuming RH and LI, we have

$$
\delta(\tilde{W}) = \mathbb{P}[\tilde{Z} > 0] = 1 - \delta(1).
$$

In fact, we can see straight away why the case $i = 2$ of Theorem 2 follows from the work of Lamzouri. Combining (12) and (13), we deduce that

$$
\sqrt{x} \log x \left( \sum_{p \leq x} \frac{1}{p} - \log \log x - B \right)
= 1 + 2\Re \sum_{0 < \gamma \leq T} \frac{x^{i\gamma}}{-\frac{1}{2} + i\gamma} + O\left( \frac{\sqrt{x} \log^2(xT)}{T} + \frac{1}{\log x} \right),
$$

so the explicit formula for $M_2(x)$ in terms of the non-trivial zeroes of the Riemann zeta-function is identical to that of Mertens’ product formula (13) (up to small error). The rest of this section is thus devoted to proving the case $i = 1$: we give full details of this proof, which follows the method of Lamzouri.

Recall that our goal is to measure the logarithmic density of the set

$$
\mathcal{W}_1 = \left\{ x \geq 2 : \sum_{p \leq x} \frac{\log p}{p} > \log x + E \right\}.
$$

To achieve this, define

$$
\mathcal{E}(x) := \sqrt{x} \left( \sum_{p \leq x} \frac{\log p}{p} - \log x - E \right) = \sqrt{x} M_1(x),
$$
and note that $x \in W_i$ if, and only if, $E(x) > 0$. Our main result is the following formula that explicitly relates $E(x)$ to the non-trivial zeroes of $\zeta(s)$.

**Proposition 3.** For all $x, T \geq 5$ we have

$$E(x) = 1 - \sum_{|\gamma| \leq T} \frac{x^{\rho - 1/2}}{\rho - 1} + O\left(\frac{\sqrt{x} \log^2(xT)}{T} + \frac{1}{\log x}\right).$$

**Proof.** Recall from the proof of Lemma [1] that

$$\sum_{p \leq x} \frac{\log p}{p} + \sum_{k=2}^{\infty} \sum_{p \leq x} \frac{\log p}{p^k} = \sum_{n \leq x} \frac{\Lambda(n)}{n} + \sum_{p \leq x, p^k > x} \frac{\log p}{p^k} + O\left(\frac{1}{x}\right).$$

We require a sharp estimate for the last sum on the right-hand side. First note that

$$\sum_{p \leq x, p^k > x} \frac{\log p}{p^k} = \sum_{\sqrt{2} \leq p \leq x} \frac{\log p}{p^2} + O\left(\frac{\log x}{x^{2/3}}\right),$$

where the contribution from prime powers $p^k$ with $k \geq 3$ was estimated trivially. For the sum over squares of primes, it suffices to use the classical prime number theorem estimate

$$\theta(x) = x + O\left(x \exp\left(-c \sqrt{\log x}\right)\right)$$

to obtain

$$\sum_{\sqrt{2} \leq p \leq x} \frac{\log p}{p^2} = \int_{\sqrt{2}}^{x} \frac{d\theta(t)}{t^2} = \frac{1}{\sqrt{x}} + O\left(\frac{1}{\sqrt{x} \log x}\right),$$

where the last error term was chosen for convenience. Combining the above estimates, we conclude that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \lim_{\alpha \to 1^+} \left(-\frac{\zeta'}{\zeta}(\alpha) + \frac{x^{1-\alpha}}{1-\alpha} - \sum_{|\gamma| \leq T} \frac{x^{\rho - 1}}{\rho - 1} + O\left(\frac{\log x}{x} + \frac{\log^2 x}{T} + \frac{\log^2 T}{T \log^2 x}\right).$$

We now introduce an explicit formula for the weighted sum of the von Mangoldt function. Lamzouri [Lam, Lem. 2.4] showed that for $\alpha > 1$ and $x, T \geq 5$, we have

$$\sum_{n \leq x} \frac{\Lambda(n)}{n^\alpha} = -\frac{\zeta'}{\zeta}(\alpha) + \frac{x^{1-\alpha}}{1-\alpha} - \sum_{|\gamma| \leq T} \frac{x^{\rho - 1}}{\rho - 1} + O\left(x^{-\alpha} \log x + \frac{x^{1-\alpha}}{T} \left(4^\alpha + \log^2 x + \frac{\log^2 T}{\log x}\right) + \frac{1}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\alpha+1/\log x}}\right).$$

Since

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+1/\log x}} = -\frac{\zeta'}{\zeta}(1 + \frac{1}{\log x}) = \log x + O(1), \quad x \to \infty,$$

we therefore obtain, taking the limit $\alpha \to 1^+$ in [10],

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \lim_{\alpha \to 1^+} \left(-\frac{\zeta'}{\zeta}(\alpha) + \frac{x^{1-\alpha}}{1-\alpha} - \sum_{|\gamma| \leq T} \frac{x^{\rho - 1}}{\rho - 1} + O\left(\frac{\log x}{x} + \frac{\log^2 x}{T} + \frac{\log^2 T}{T \log^2 x}\right).$$
To evaluate the limit term in (17), we compute the Laurent series
\[
\frac{x^{1-\alpha}}{1-\alpha} = \sum_{k=-1}^{\infty} \frac{(1-\alpha)^k \log^{k+1} x}{(k+1)!} = \frac{1}{1-\alpha} + \log x + \frac{1}{2} (1-\alpha) \log^2 x + \cdots ,
\]
which, together with equations (4), (6), and (7), gives us
\[
\lim_{\alpha \to 1^+} \left( \frac{\zeta'\zeta(\alpha)}{\alpha-1} + x^{1-\alpha} \right) = \frac{1}{2} C_0 + 1 - \log x + \frac{1}{2} \pi + \frac{\log^2 x}{T} + \cdots
\]
and multiplying through by \(\sqrt{x}\) gives the result.

Remark 3. It is immediately clear from (14) that the constant 1 is responsible for the positive bias of \(M_1(x)\).

Lemma 6. Assuming RH, we have
\[
E(x) = 1 - 2\Re \sum_{0<\gamma \leq T} \frac{x^{i\gamma}}{\frac{1}{2} + i\gamma} + O\left( \frac{\log^2(xT)}{T} + \frac{1}{\log x} \right),
\]
and, in particular,
\[
E(x) = -2 \sum_{0<\gamma \leq T} \frac{\sin(\gamma \log x)}{\gamma} + O\left( 1 + \frac{\sqrt{x} \log^2(xT)}{T} \right).
\]

Proof. Equation (18) follows immediately upon writing \(\rho = 1/2 + i\gamma\) in (14). To deduce (19), we combine (18) with the observation that
\[
\left| \sum_{0<\gamma \leq T} \frac{x^{i\gamma}}{\frac{1}{2} + i\gamma} - \sum_{0<\gamma \leq T} \frac{x^{i\gamma}}{i\gamma} \right| \ll \sum_{0<\gamma \leq T} \frac{1}{\gamma^2} \ll 1,
\]
where convergence of the last sum follows from the Riemann–von-Mangoldt formula.

The existence of the upper and lower logarithmic densities is due to the following result from Section 2.2 of [RS94].

Proposition 4 (Rubinstein and Sarnak). There exists absolute positive constants \(a_1\) and \(a_2\) such that for all \(\lambda \gg 1\) and \(Y\) sufficiently large,
\[
\frac{1}{Y} \text{meas} \left\{ y \in [2,Y] : \sum_{0<\gamma \leq e^y} \frac{\sin(\gamma y)}{\gamma} > \lambda \right\} \geq \frac{a_1}{\exp \left( \exp(a_2\lambda) \right)},
\]
and
\[ \frac{1}{Y} \text{meas} \left\{ y \in [2, Y] : \sum_{0 < \gamma \leq e^y} \frac{\sin(\gamma y)}{\gamma} < -\lambda \right\} \geq \frac{a_1}{\exp\left(\exp(a_2 \lambda)\right)}. \]

We are now ready to prove the first assertion of the theorem; henceforth, assume RH. Substituting \( y = \log x \) into (19) gives us
\[ E(e^y) = -2 \sum_{0 < \gamma \leq T} \frac{\sin(\gamma y)}{\gamma} + O\left(1 + \frac{e^{y/2}(y + \log T)^2}{T}\right), \]
whence we deduce that for all sufficiently large \( Y \), there exists \( A > 0 \) such that for all \( 2 \leq y \leq Y \),
\[ -2 \left( \sum_{0 < \gamma \leq e^y} \frac{\sin(\gamma y)}{\gamma} + A \right) < E(e^y) < -2 \left( \sum_{0 < \gamma \leq e^y} \frac{\sin(\gamma y)}{\gamma} - A \right). \]
Using this, we see that \( \sum_{0 < \gamma \leq e^y} \frac{\sin(\gamma y)}{\gamma} < -A \) implies \( E(e^y) > 0 \). It follows from Proposition 4 that
\[ \frac{1}{\log x} \int_{x \in W_1 \cap [2, x]} \frac{dt}{t} = \frac{1}{Y} \text{meas} \left\{ y \in [\log 2, Y] : E(e^y) > 0 \right\} \]
\[ \geq \frac{1}{Y} \text{meas} \left\{ y \in [2, Y] : \sum_{0 < \gamma \leq e^y} \frac{\sin(\gamma y)}{\gamma} < -A \right\} \]
\[ \geq \frac{1}{2} \frac{a_1}{\exp\left(\exp(a_2 A)\right)}, \]
say, if \( Y \) is large enough. Hence, we deduce that
\[ \delta(W_1) \geq \frac{1}{2} \frac{a_1}{\exp\left(\exp(a_2 A)\right)} > 0. \]
By a similar argument, we see that \( E(e^y) > 0 \) implies \( \sum_{0 < \gamma \leq e^y} \frac{\sin(\gamma y)}{\gamma} < A \), whence
\[ \frac{1}{\log x} \int_{x \in W_1 \cap [2, x]} \frac{dt}{t} \leq \frac{1}{Y} \text{meas} \left\{ y \in [2, Y] : \sum_{0 < \gamma \leq e^y} \frac{\sin(\gamma y)}{\gamma} < A \right\} + O\left(\frac{1}{Y}\right) \]
\[ \leq 1 - \frac{1}{2} \frac{a_1}{\exp\left(\exp(a_2 A)\right)}, \]
say, from which we conclude that \( \delta(W_1) < 1 \).

It remains to prove that under the additional assumption of LI, the quantities \( \hat{\delta}(W_1) \) and \( \bar{\delta}(W_1) \) coincide and attain the value \( 1 - \delta(1) \).

**Proposition 5.** Assuming RH, there exists a probability measure \( \mu_\mathcal{E} \) on \( \mathbb{R} \) such that for all bounded continuous functions \( u : \mathbb{R} \to \mathbb{R} \), we have
\[ \lim_{x \to \infty} \frac{1}{\log x} \int_2^x u(E(t)) \frac{dt}{t} = \int_{-\infty}^\infty u(t) \, d\mu_\mathcal{E}. \]
If in addition to RH we assume LI, then we have the following explicit formula for the Fourier transform of \( \mu_E \):

\[
\hat{\mu}_E(t) = \int_{-\infty}^{\infty} e^{-it} \, d\mu_E = e^{-it} \prod_{\gamma > 0} J_0 \left( \frac{2t}{\sqrt{\frac{1}{4} + \gamma^2}} \right),
\]

where \( J_0(t) := \sum_{k=0}^{\infty} (-1)^k (k!)^{-2} (t/2)^{2k} \) is the Bessel function of the first kind of order zero.

**Proof.** Set \( y = \log x \) in the explicit formula (18), and let \( \nu(y,T) := e^{y/2} \frac{(y + \log T)^2}{T} + 1/y \) denote the error term. A simple calculation shows that

\[
\lim_{Y \to \infty} \frac{1}{Y} \int_{\log Y}^{Y} |\nu(y,e^Y)|^2 \, dy = 0;
\]

thus, the mean square of the error is uniformly small. It follows from the work of Rubinstein and Sarnak [RS94] and Akbary, Ng, and Shahabi [ANS14, Thm 1.2] that \( E(x) \) is a \( B_2 \)-almost periodic function and thus possesses a limiting distribution (20). In particular, the Fourier transform (21) was deduced from [ANS14, Thm 1.9]. \( \square \)

Note that under LI, the quantities \( x^{i\gamma} \) appearing in equation (18) can be viewed as points uniformly distributed on the unit circle. This leads to the following statistical characterisation of the measure \( \mu_E \).

**Lemma 7.** Assume RH and LI. Let \( X(\gamma) \) denote a sequence of random variables indexed by the positive ordinates of the non-trivial zeroes of \( \zeta(s) \), and distributed uniformly on the unit circle. Then \( \mu_E \) is the distribution of the random variable

\[
Z := 1 - 2\Re \sum_{\gamma > 0} \frac{X(\gamma)}{\sqrt{\frac{1}{4} + \gamma^2}}.
\]

**Proof.** We see from the definition of \( Z \) that

\[
\mathbb{E}[e^{-itZ}] = e^{-it} \prod_{\gamma > 0} \mathbb{E} \left[ \exp \left( i \frac{2t}{\sqrt{\frac{1}{4} + \gamma^2}} \Re X(\gamma) \right) \right].
\]

However, we note that for a random variable \( X \) uniformly distributed on the unit circle,

\[
\mathbb{E}[e^{it\Re X}] = \frac{1}{2\pi} \int_0^{2\pi} e^{it\cos \theta} \, d\theta = J_0(t),
\]

making use of the integral representation of the Bessel function. Hence, the right-hand side of (22) is equal to

\[
e^{-it} \prod_{\gamma > 0} J_0 \left( \frac{2t}{\sqrt{\frac{1}{4} + \gamma^2}} \right),
\]

so we conclude that \( \mathbb{E}[e^{-itZ}] = \hat{\mu}_E(t) \) by (21). \( \square \)

Now observe that \( Z \) and \( \tilde{Z} \) have the same distribution, in view of the fact that the \( X(\gamma_n) \) are symmetric random variables. Using Proposition 2 this implies

\[
\mathbb{P}[Z > 0] = \mathbb{P}[\tilde{Z} > 0] = 1 - \delta(1),
\]

so the second assertion of the theorem follows upon showing

**Lemma 8.** Assuming RH and LI, we have \( \delta(W_1) = \mathbb{P}[Z > 0] \).

Proof. Since $Z$ is the sum of continuous random variables, it follows from Lemma 7 that $\mu_E$ is an absolutely continuous probability distribution. Let $\epsilon > 0$, and let $u_1(x)$ and $u_2(x)$ be continuous functions such that
\[ u_1(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ \epsilon \in [0,1] & \text{if } x \in (-\epsilon,0), \\ 0 & \text{otherwise}, \end{cases} \]
\[ u_2(x) = \begin{cases} 1 & \text{if } x \geq \epsilon, \\ \epsilon \in [0,1] & \text{if } x \in (0,\epsilon), \\ 0 & \text{otherwise}. \end{cases} \]
It follows from Proposition 5 and Lemma 7 that
\[ \delta(W_1) \leq \lim_{x \to \infty} \frac{1}{\log x} \int_2^x u_1(E(t)) \frac{dt}{t} = \int_{-\infty}^{\infty} u_1(t) \, d\mu_E \leq \mu_E(-\epsilon,\infty) = \mathbb{P}[Z > 0] + O(\epsilon), \]
and, using a similar argument,
\[ \delta(W_1) \geq \lim_{x \to \infty} \frac{1}{\log x} \int_2^x u_2(E(t)) \frac{dt}{t} = \int_{-\infty}^{\infty} u_2(t) \, d\mu_E \geq \mu_E(\epsilon,\infty) = \mathbb{P}[Z > 0] + O(\epsilon). \]
The result follows on taking $\epsilon \to 0$. \qed

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Mathematical Sciences Institute, The Australian National University, Canberra ACT 2601, Australia

E-mail address: jeffrey.lay@anu.edu.au