Results on some partition functions arising from certain relations involving the Rogers-Ramanujan continued fractions

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Abstract. Relations involving the Rogers-Ramanujan continued fractions $R(q)$, $R(q^3)$, and $R(q^4)$ are used to find new generating functions and congruences modulo 5 and 25 for 3-core, 4-core, 4-regular, and colored partition functions.

Key Words: Generating function, congruence, $t$-core, $\ell$-regular partition, colored partition, continued fraction.

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1. Introduction

A partition of a positive integer $n$ is a finite non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$, called parts of $\lambda$, such that

$$\sum_{j=1}^{k} \lambda_j = n.$$ 

Let $Q(n)$ denote the number of partitions of $n$ into distinct parts (equivalently, by Euler’s theorem, into odd parts). In [2] Baruah and Begum found the exact generating functions for $Q(5n+1)$, $Q(25n+1)$, $Q(125n+26)$, and a few congruences modulo 5 and 25. They used Ramanujan’s theta function identities and some identities for the Rogers-Ramanujan continued fraction. In particular, certain relations between the continued fractions $R(q)$ and $R(q^2)$ were employed to establish their results, where $R(q) = q^{1/5}/R(q)$ with $R(q)$ being the famous Rogers-Ramanujan continued fraction usually given by

$$R(q) := \frac{q^{1/5}}{1 + q^{1/5}} \frac{q}{1 + q} \frac{q^2}{1 + q^2} \frac{q^3}{1 + q^3} \cdots = q^{1/5} \frac{(q; q^5)_{\infty}(q^4; q^5)_{\infty}}{(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}},$$

where for any complex number $a$ and $|q| < 1$, the standard $q$-product $(a; q)_{\infty}$ is defined by

$$(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k),$$

and the product representation of $R(q)$ is due to both Rogers and Ramanujan (See [5] pp. 158–160). The technique in [2] was further used effectively in [3, 4, 7, 8].

In this work, we explore relations involving the Rogers-Ramanujan continued fraction $R(q)$ with those of $R(q^3)$ and $R(q^4)$ to deduce some new generating functions and congruences modulo 5 and 25 for certain partition functions mentioned in the following.

The Ferrers-Young diagram of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is an array of left-aligned nodes with $\lambda_i$ nodes in the $i$-th row. If $\lambda'_j$ denotes the number of nodes in column $j$, then the hook number of the node $(i, j)$ is defined by $H(i, j) := \lambda_i + \lambda'_j - i - j + 1$. A partition of $n$ is called a $t$-core of $n$ if none of the hook numbers is a multiple of $t$. For example, the
Ferrers-Young diagram of the partition \( \lambda = (5, 2, 1) \) is given by

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & & & & & \\
\end{array}
\]

The nodes \((1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), \) and \((3,1)\) have hook numbers 7, 5, 3, 2, 1, 3, 1, and 1, respectively. Since none of these is a multiple of 4, so \( \lambda \) is a 4-core. Obviously, it is a \( t \)-core for \( t \geq 8 \). Let \( a_t(n) \) denote the number of partitions of \( n \) that are \( t \)-cores. It is well-known that the generating function for \( a_t(n) \) is given by

\[
\sum_{n=0}^{\infty} a_t(n)q^n = \frac{E_t}{E_1},
\]

where here and throughout the sequel, for a positive integer \( n \),

\[
E_n := (q^n; q^n)_{\infty}.
\]

Next, a partition of a positive integer \( n \) is said to be \( \ell \)-regular if none of its parts is divisible by \( \ell \). If \( b_\ell(n) \) denotes the number of \( \ell \)-regular partitions of \( n \) with \( b_\ell(0) = 1 \), then the generating function for \( b_\ell(n) \) is given by

\[
\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{E_\ell}{E_1}.
\]

Finally, for an integer \( r > 1 \), let \( p_{[1^k, k]}(n) \) denote the number of partitions of \( n \) in which multiples of \( r \) appear in \( k+1 \) colors and the rest of the parts in \( k \) colors. The generating function for \( p_{[1^k, k]}(n) \) is given by

\[
\sum_{n=0}^{\infty} p_{[1^k, k]}(n)q^n = \frac{1}{E_k E_r^k}.
\]

We are now in a position to state our results. The results from Theorem 1.1 through Theorem 1.8 are obtained by using identities involving \( R(q) \) and \( R(q^3) \).

**Theorem 1.1.** For any integer \( n \geq 0 \), we have

\[
\sum_{n=0}^{\infty} a_3(5n + 3)q^n = q \frac{E_3}{E_5}^{15}.
\] (1.1)

With the aid of (1.1), we find the following corollary.

**Corollary 1.2.** For any integer \( n \geq 0 \), \( k > 0 \), and \( r = 0, 2, 3 \) and 4, we have

\[
a_3(25n + 5r + 3) = 0
\] (1.2)

and

\[
a_3 \left( 5^{2k}n + \frac{5^{2k} - 1}{3} \right) = a_3(n).
\] (1.3)

We note that (1.3) is only a special case of a more general result by Hirschhorn and Sellers [12, Corollary 8].
Theorem 1.3. For any integer $n \geq 0$, we have

\[
\sum_{n=0}^{\infty} p_{[13]}(5n + 1)q^n = \frac{E_5^5}{E_1^5E_{15}} + 10q\frac{E_5^{10}}{E_1^4E_3} + q^2\frac{E_5^5}{E_1^3E_5} + 45q^3\frac{E_5^5}{E_1^2E_3} - 90q^5\frac{E_5^{10}}{E_1^3E_5}. \tag{1.4}
\]

\[
\sum_{n=0}^{\infty} p_{[123]}(5n + 2)q^n = 5 \left( \frac{E_5^{10}}{E_1^{12}E_{15}} + 20q\frac{E_5^{15}}{E_1^{13}E_3} + q^2 \left( 80\frac{E_5^{20}}{E_1^{14}E_{15}^2} + 12\frac{E_5^4}{E_1^6E_5^2} \right) \right) + q^3 \left( 306\frac{E_5^4}{E_1^6E_3} - 36\frac{E_5^2}{E_1^7E_3} \right) + q^4 \left( \frac{E_5^{10}}{E_1^{12}E_3} + 540\frac{E_5^{15}}{E_1^{13}E_3} \right) + q^5 \left( 306\frac{E_5^4}{E_1^6E_3} + 324\frac{E_5^9}{E_1^7E_3} \right) + 4745q^6\frac{E_5^{10}}{E_1^{12}E_3} - 180q^7\frac{E_5^{15}}{E_1^6E_3} \right) - 4860q^8\frac{E_5^{15}}{E_1^6E_3} + 6480q^{10}\frac{E_5^{15}}{E_1^6E_3} \right) \tag{1.5}
\]

and

\[
\sum_{n=0}^{\infty} p_{[133]}(5n + 3)q^n = 25\frac{E_5^{15}}{E_1^{18}E_{15}} + q \left( 216\frac{E_5^{35}}{E_1^{22}E_3E_{15}} + 234\frac{E_5^{20}}{E_1^{19}E_3E_5^2} + 234\frac{E_5^{19}}{E_1^{19}E_3E_5^2} + 81\frac{E_3^3}{E_1^3E_5^3} \right) + q^2 \left( 2976\frac{E_5^{25}}{E_1^{20}E_3E_{15}} + 2049\frac{E_5^{24}}{E_1^{19}E_3E_5} + 27\frac{E_5^{10}}{E_1^{19}E_3E_5} + 3528\frac{E_3^3}{E_1^3E_5^3} \right) + q^3 \left( 13224\frac{E_5^{30}}{E_1^{21}E_3E_{15}} + 38858\frac{E_5^{15}}{E_1^{18}E_3E_5} + 27\frac{E_5^{15}}{E_1^{18}E_3E_5} + 3528\frac{E_3^3}{E_1^3E_5^3} \right) + q^4 \left( 109101\frac{E_5^{20}}{E_1^{19}E_3E_5^2} + 2226\frac{E_5^{20}}{E_1^{19}E_3E_5^2} + 3528\frac{E_5^{15}}{E_1^{18}E_3E_5^2} - 3\frac{E_5^{10}}{E_1^{19}E_3E_5} \right) + q^5 \left( 270086\frac{E_5^{25}}{E_1^{20}E_3E_{15}} + 18018\frac{E_5^{10}}{E_1^{19}E_3E_5^2} + 13855\frac{E_5^{10}}{E_1^{19}E_3E_5^2} - 2002\frac{E_5^{15}}{E_1^{18}E_3E_5^2} \right) + q^6 \left( 1270766\frac{E_5^{15}}{E_1^{18}E_3E_{15}} - 5571\frac{E_5^{10}E_5^{14}}{E_1^{18}E_3E_5} + 25\frac{E_5^{15}}{E_1^{18}E_3E_5} \right) + q^7 \left( 441524\frac{E_5^{20}}{E_1^{19}E_3E_5^2} + 10854\frac{E_5^{15}}{E_1^{19}E_3E_5^2} - 54675\frac{E_5^{14}E_5^{15}}{E_1^{18}E_3E_5^2} + 38858\frac{E_5^{15}}{E_1^{18}E_3E_5} \right) + q^8 \left( 1270766\frac{E_5^{15}}{E_1^{18}E_3E_{15}} + 496539\frac{E_5^{10}E_5^{14}}{E_1^{18}E_3E_5} + 25\frac{E_5^{15}}{E_1^{18}E_3E_5} \right) + q^9 \left( 1210417\frac{E_5^{15}}{E_1^{18}E_3E_{15}} - 2106\frac{E_5^{20}}{E_1^{18}E_3E_5} + 18954\frac{E_5^{19}}{E_1^{18}E_3E_5} \right) + q^{10} \left( 1800306\frac{E_5^{19}E_5^{15}}{E_1^{16}E_3E_{15}} - 981909\frac{E_5^{20}E_5^{15}}{E_1^{11}E_3E_5} \right) - 3973716q^{11}\frac{E_5^{10}E_5^{15}}{E_1^{11}E_3E_5} \right) \tag{1.5}
\]
We derive the following congruences from (1.4).

**Corollary 1.4.** For any integer \( n \geq 0 \), we have
\[
p_{\[1^33\]}(25n + 21) \equiv 0 \pmod{5} \tag{1.7}
\]
and
\[
p_{\[1^33\]}(625n + 521) \equiv 0 \pmod{5^2}. \tag{1.8}
\]

Congruence (1.7) was found earlier by Ahmed, Baruah, and Dastidar [1] whereas (1.8) seems to be new. Computational evidences indicate that there might exist congruences modulo higher powers of 5 similar to (1.7) and (1.8). To that end, we pose the following conjecture.

**Conjecture 1.5.** For any integer \( n \geq 0 \) and \( k > 0 \), we have
\[
p_{\[1^33\]} \left( 5^{2k} n + \frac{5^{2k+1} + 1}{6} \right) \equiv 0 \pmod{5^k}. \tag{1.5}
\]

The above infinite family of congruences is analogous to the one for \( p_{\[1^22\]} \) discovered independently by Chan and Toh [6] and Xiong [13].

We derive the next corollary from (1.5).

**Corollary 1.6.** For any integer \( n \geq 0 \) and \( k > 0 \), we have
\[
p_{\[1^32\]}(5n + 2) \equiv 0 \pmod{5} \tag{1.9}
\]
and
\[
p_{\[1^32\]} \left( 5^{2k} n + \frac{2 \times 5^{2k} + 1}{3} \right) \equiv p_{\[1^32\]}(25n + 17) \pmod{5^2}. \tag{1.10}
\]

**Remark 1.7.** It is proved in Section 3 that
\[
p_{\[1^33\]}(5n + 3) \equiv 0 \pmod{5}. \tag{1.11}
\]
We could not transform the generating function (1.6) of \( p_{\[1^33\]}(5n + 3) \) effectively to a form similar to (1.5) which could have immediately implied the above congruence.

Next, Zhang and Shi [14] studied the sixth order mock theta function \( \beta(q) \), defined by
\[
\beta(q) := \sum_{n=0}^{\infty} \frac{q^{3n^2+3n+1}}{(q; q^3)_{n+1}(q^2; q^3)_{n+1}} =: \sum_{n=0}^{\infty} p_\beta(n) q^n.
\]

They proved that
\[
\sum_{n=0}^{\infty} p_\beta(3n + 1) q^n = \frac{E_3^3}{E_1^2}.
\]
and

\[ \sum_{n=0}^{\infty} p_\beta(9n+5)q^n = 3 \frac{E_3^6}{E_1^6}. \]

They also found some congruences for \( p_\beta(n) \) modulo 3, 5, and 7. In particular, they proved the following three congruences by using elementary techniques.

**Theorem 1.8.** For any integer \( n \geq 0 \), we have

\[ p_\beta(15n + 7) \equiv 0 \pmod{5}, \quad (1.12) \]

\[ p_\beta(45n + 23) \equiv 0 \pmod{15} \quad (1.13) \]

and

\[ p_\beta(45n + 41) \equiv 0 \pmod{15}. \quad (1.14) \]

In this paper, we present alternative proofs of the above congruences by using relations between \( R(q) \) and \( R(q^3) \).

Finally, we present the following two new results on \( b_4(n) \) and \( a_4(n) \) that are obtained by using identities involving \( R(q) \) and \( R(q^4) \).

**Theorem 1.9.** For any integer \( n \geq 0 \), we have

\[ \sum_{n=0}^{\infty} b_4(5n+3)q^n = 3 \frac{E_2^3 E_{10}^6}{E_1^3 E_4 E_{20}^2} + q \frac{E_3^3 E_5^3 E_{10}^{10}}{E_2^6 E_4^2 E_{10}^4} + 4q^2 \frac{E_4^3 E_{10}^6 E_{20}^3}{E_5^6 E_4^2}. \quad (1.15) \]

**Theorem 1.10.** For any integer \( n \geq 0 \), we have

\[
\sum_{n=0}^{\infty} a_4(5n)q^n = \frac{E_4^4 E_{10}^{10}}{E_1^2 E_2^8 E_5^5 E_{16}^{20}} - q \left( 3 \frac{E_3^3 E_{10}^{15}}{E_1^3 E_2^6 E_{20}^2} - 4 \frac{E_3^3 E_{10}^{30}}{E_1^3 E_2^6 E_5^{10} E_{11}^{20}} \right) \\
- q^2 \left( 12 \frac{E_3^3 E_{20}^{20}}{E_1^3 E_4 E_5 E_{20}^6} + 20 \frac{E_4 E_5 E_{10}^{10} E_{10}^{10}}{E_2^6 E_2 E_{20}^2} - 24 \frac{E_3^3 E_{35}^3}{E_1^3 E_2^5 E_5^{10} E_{11}^{20}} \right) \\
- q^3 \left( 27 \frac{E_3^3 E_{6}^6}{E_1^2} + 60 \frac{E_4 E_{10}^{10}}{E_1^2 E_4 E_{20}^2} - 196 \frac{E_3^3 E_{25}^5}{E_1^3 E_2^{10} E_5^{10} E_{20}^6} \right) \\
- q^4 \left( 83 \frac{E_3^3 E_{24}^4}{E_1^6} - 456 \frac{E_4 E_{10}^{15}}{E_1^2 E_4 E_5 E_{20}^6} \right) + q^5 \left( 296 \frac{E_{10}^{10} E_2^4}{E_1^5 E_2^2} + 96 \frac{E_4 E_{20}^{20}}{E_1^5 E_2^5 E_5 E_{20}^6} \right) \\
+ q^6 \left( 128 \frac{E_2 E_5^5 E_{20}^{20}}{E_1^6 E_4 E_{10}^{10} E_5^2} + 592 \frac{E_{10}^{10} E_2^4}{E_1^2 E_4^2 E_5^2} \right) + 512q^7 \frac{E_9^{20}}{E_5^2}. \quad (1.16) \]

This work is organized as follows. In Section 2, we present some preliminary lemmas. In Section 3, we prove our results in Theorem 1.9 – Theorem 1.8. In the final section we prove Theorem 1.9 and Theorem 1.10.
2. Preliminary lemmas

The first lemma comprises of the well-known 5-dissections of $E_1$ and $1/E_1$.

**Lemma 2.1.** We have

$$E_1 = E_{25} \left( R(q^5) - q - \frac{q^2}{R(q^5)} \right)$$

and

$$\frac{1}{E_1} = \frac{E_{25}}{E_5} \left( R(q^5)^4 + qR(q^5)^3 + 2q^2R(q^5)^2 + 3q^3R(q^5) + 5q^4 - 3\frac{q^5}{R(q^5)} \right) + 2\frac{q^6}{R(q^5)^2} - \frac{q^7}{R(q^5)^3} + \frac{q^8}{R(q^5)^4}.$$  

**Proof.** See [5, Chapter 7, pp. 161–165]. □

In the next lemma, we present two useful relations among $R(q)$, $R(q^2)$, and $E_n$.

**Lemma 2.2.** We have

$$A = A(q) := R(q)^5 - \frac{q^2}{R(q^5)} = 11q + \frac{E_1^6}{E_5^6}$$

and

$$B = B(q) := \frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2} = 4qE_1^5E_{10}^5E_2^5E_5^5.$$  

**Proof.** See [5] Chapter 7, p. 164] and [2] Lemma 2.2.1]. □

Some relations among $R(q)$, $R(q^3)$, and $E_n$ are stated in the following lemma.

**Lemma 2.3.** We have

$$C_1 := \frac{R(q)^3}{R(q^3)} + \frac{R(q^3)}{R(q)^3} = 2 + 9q^2 \frac{E_1 E_{15}^5}{E_3 E_5^5},$$

$$C_2 := R(q)R(q^3)^3 + \frac{q^4}{R(q)R(q^3)^3} = \frac{E_2 E_5^5}{E_1 E_{15}^5} - 2q^2$$

and

$$C_3 := R(q)R(q^3)^2 + \frac{R(q)}{R(q^3)^2} + q^2 \frac{R(q)}{R(q^3)^2} - \frac{q^2}{R(q^3)} = 3q.$$  

**Proof.** See [9] Theorem 5.1] and [11] p. 194]. □

Our next lemma provides a relation among $R(q)$, $R(q^4)$, and $E_n$.

**Lemma 2.4.** We have

$$D_1 := R(q)R(q^4) + \frac{q^2}{R(q)R(q^4)} = 2q + \frac{E_1 E_4 E_{10}^{10}}{E_2^5 E_5^5 E_{20}^5}.$$  

(2.7)
Proof. From [10, Theorem 3.3(iii)], we have
\[
R(q)R(q^4) + \frac{q^2}{R(q)R(q^4)} = 3q + \frac{\psi^2(-q)}{\psi^2(-q^5)},
\]
where
\[
\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}.
\]
From [5, p. 6 and p. 11], we have
\[
\psi(q) = \frac{E_2}{E_1}.
\]
Now, from [11, Chapter 34, p. 313], we recall that
\[
\psi^2(q) - q\psi^2(q^5) = \frac{E_2E_5^3}{E_1E_{10}}.
\]
Replacing \(q\) by \(-q\) in the above, we have
\[
\psi^2(-q) + q\psi^2(-q^5) = \frac{E_2E_{-5}^3}{E_{-1}E_{10}},
\]
where for a positive odd integer \(n\), \(E_{-n} := (-q^n; -q^n)_\infty\). By elementary \(q\)-product manipulation, it follows that
\[
E_{-n} = \frac{E_{-2n}^3}{E_nE_{4n}}.
\]
Dividing (2.10) by \(\psi^2(-q^5)\) and then employing (2.9) and (2.11), we obtain
\[
\frac{\psi^2(-q)}{\psi^2(-q^5)} + q = \frac{E_1E_{10}E_{14}}{E_2^3E_5^5E_{20}}.
\]
Identity (2.7) now follows readily from (2.8) and (2.12). □

Our final lemma of this section states a relation among \(R(q)\), \(R(q^2)\), \(R(q^4)\), and \(E_n\).

Lemma 2.5. We have
\[
F := \frac{R(q)^2R(q^2)}{R(q^4)} + \frac{R(q^4)}{R(q)^2R(q^2)} = 2 + 4q^2\frac{E_2E_{20}^5}{E_4E_{10}^5}.
\]

Proof. See [10, Theorem 3.6(ii), Lemma 1.1, p. 185]. □

We end this section by defining an extraction operator. For a power series \(\sum_{n=0}^{\infty} A(n)q^n\) and \(r = 0, 1, 2, 3\) and 4, we define the operator \([q^{5n+r}]\) by
\[
[q^{5n+r}] \left\{ \sum_{n=0}^{\infty} A(n)q^n \right\} = \sum_{n=0}^{\infty} A(5n + r)q^n.
\]
3. Proofs of (1.1)-(1.4) using identities satisfied by $R(q)$ and $R(q^3)$

Proof of Theorem 1.1. We have

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{E_3^3}{E_1}.$$ 

Employing Lemma 2.1 in the above and then applying $[q^{5n+3}]$, we find that

$$\sum_{n=0}^{\infty} a_3(5n+3)q^n = \frac{E_5^5 E_5^5}{E_1^6} \left( 3 \left( \frac{R(q)R(q^3)^3}{R(q)} \right) + q \left( \frac{R(q)^3 R(q)}{R(q)^4} - q^2 \frac{R(q^3)^3}{R(q^3)^3} \right) \right) - 3 \left( \frac{R(q)^4 R(q^3)^2}{R(q)^4 R(q^3)^2} - 3q \left( \frac{R(q)^3}{R(q)} - q^2 \frac{R(q)}{R(q)} \right) \right) + 25q^2.$$ (3.1)

Now, using Lemma 2.2 and Lemma 2.3 we have

$$C_4 := \frac{R(q)^3}{R(q)} - q^2 \frac{R(q)^4}{R(q^3)^3} = A - C_3(C_1 + 1) = \frac{E_1^6}{E_5^6} + 2q - 27q^3 \frac{E_5^5 E_5}{E_3 E_5^2},$$ (3.2)

and

$$R(q)^4 R(q^3)^2 + q^4 \frac{R(q)^4 R(q^3)^2}{R(q)^4 R(q^3)^2} = C_2 - (C_1 - 2)q^2 + C_3 \left( \frac{R(q)^2 R(q^3)}{R(q)^2} - q^2 \frac{R(q^3)}{R(q^3)} \right).$$

Subtracting $3q \left( \frac{R(q^3)^2}{R(q)} - q^2 \frac{R(q)}{R(q^3)} \right)$ from both sides of the last identity and then employing Lemma 2.3 we obtain

$$R(q)^4 R(q^3)^2 + q^4 \frac{R(q)^4 R(q^3)^2}{R(q)^4 R(q^3)^2} - 3q \left( \frac{R(q^3)^2}{R(q)} - q^2 \frac{R(q)}{R(q^3)} \right) = \frac{E_3 E_5^5}{E_1 E_5} + 7q^2 - 9q^4 \frac{E_1 E_5^5}{E_3 E_5^2}.$$ 

Employing the above identity, (2.5), and (3.2) in (3.1), we arrive at (1.1).

Proof of Corollary 1.2. The identity (1.1) can be written as

$$\sum_{n=0}^{\infty} a_3(5n+3)q^n = \sum_{n=0}^{\infty} a_3(n)q^{5n+1}.$$ 

Equating the coefficients of $q^{5n+r}$ for $r = 0, 2, 3$ and 4 from both sides of the above, we arrive at (1.2). On the other hand, equating the coefficients of $q^{5n+1}$, we have

$$a_3(25n + 8) = a_3(n)$$ from which (1.3) follows by induction.

Proof of Theorem 1.3. First, we prove (1.4). We have

$$\sum_{n=0}^{\infty} p_{[1;3]}(n)q^n = \frac{1}{E_1 E_3}.$$
Employing (2.2) in the above and then applying \([q^{5n+1}]\), we find that

\[
\sum_{n=0}^{\infty} p_{[1^{1}3^{1}]}(5n+1)q^n = \frac{E_5^3}{E_1^3} \left( 25q^3 + 6q^2 \left( R(q)^2 R(q^3) + q^2 \frac{R(q)}{R(q)^2} - \frac{q^2}{R(q)^2 R(q^3)} \right) \right.
\]

\[
+ 2q \left( R(q)^4 R(q^3)^2 + q^4 \frac{R(q)^2}{R(q)^2} + \frac{R(q)^3}{R(q)^2} + \frac{q^4}{R(q)^4 R(q^3)^2} \right)
\]

\[
- 3q \left( q^2 \frac{R(q)^3}{R(q^3)} - R(q) R(q^3)^3 - \frac{q^4}{R(q) R(q^3)^3} + q^2 \frac{R(q^3)}{R(q^3)} \right)
\]

\[
+ \left( R(q)^3 R(q^3)^4 - \frac{q^6}{R(q)^3 R(q^3)^4} + \frac{R(q)^3}{R(q)^4} - q^2 \frac{R(q^3)}{R(q^3)} \right) \right). \tag{3.3}
\]

Now, with the aid of Lemma 2.2 and Lemma 2.3 we have

\[
C_5 := R(q)^3 R(q^3)^4 - \frac{q^6}{R(q)^3 R(q^3)^4} = A(q^3) + (C_2 - q^2)C_3
\]

\[
= \frac{E_3^6}{E_1^6} + 3q \frac{E_5^6 E_3}{E_1^6 E_{15}^3} + 2q^3 \tag{3.4}
\]

and

\[
C_6 := R(q)^4 R(q^3)^2 + q^4 \frac{R(q)^2}{R(q^3)^4} + \frac{R(q)^3}{R(q)^2} + \frac{q^4}{R(q)^4 R(q^3)^2} = C_1 C_2
\]

\[
= 2 \frac{E_3 E_5^6}{E_1 E_{15}^6} + 5q^2 - 18q^4 \frac{E_1 E_{15}^6}{E_3 E_5^6}. \tag{3.5}
\]

Using Lemma 2.3 (3.2), (3.3), and (3.5) in (3.3), we arrive at (1.4).

Next, we prove (1.5). We have

\[
\sum_{n=0}^{\infty} p_{[1^{2}3^{2}]}(n)q^n = \frac{1}{E_1^2 E_3^2}.
\]

Employing (2.2) in the above and then applying \([q^{5n+2}]\), we find that

\[
\sum_{n=0}^{\infty} p_{[1^{2}3^{2}]}(5n+2)q^n = \frac{E_5^{10} E_{15}^{10}}{E_1^{12} E_3^{12}} \left( \left( R(q)^6 R(q^3)^8 + q^4 \frac{R(q)^6}{R(q)^8} + q^8 \frac{R(q)^8}{R(q)^6} + \frac{q^{12}}{R(q)^6 R(q^3)^8} \right) \right.
\]

\[
+ 2q \left( R(q)^7 R(q^3)^6 + q^2 \frac{R(q)^7}{R(q)^6} - q^8 \frac{R(q)^6}{R(q)^7} - \frac{q^{10}}{R(q)^7 R(q^3)^6} \right)
\]

\[
+ 4q \left( R(q) R(q^3)^8 - q^4 \frac{R(q)^8}{R(q^3)} + q^6 \frac{R(q)^3}{R(q)^8} - \frac{q^{10}}{R(q) R(q^3)^8} \right) \right). \tag{3.3}
\]
and

\[ 8 \left( R(q)^4 R(q^3)^7 - q^4 \frac{R(q^3)^4}{R(q)^7} + q^6 \frac{R(q)^7}{R(q^3)^4} - \frac{q^{10}}{R(q)^4 R(q^3)^7} \right) \]

\[ + 4 q^2 \left( R(q)^8 R(q^3)^4 + \frac{R(q^3)^8}{R(q)^4} + q^8 \frac{R(q)^4}{R(q^3)^8} + \frac{q^{10}}{R(q)^8 R(q^3)^4} \right) \]

\[ - 8 q^2 \left( \frac{R(q^3)^7}{R(q)} - q^2 R(q)^7 R(q^3) - \frac{q^6}{R(q)^7 R(q^3)} + q^8 \frac{R(q)}{R(q^3)^7} \right) \]

\[ + 20 q^2 \left( R(q)^5 R(q^3)^5 - q^2 R(q)^5 R(q^3)^5 + q^6 \frac{R(q)^5}{R(q^3)^5} + \frac{q^{10}}{R(q)^5 R(q^3)^5} \right) \]

\[ + 27 q^2 \left( R(q)^2 R(q^3)^6 + q^4 \frac{R(q)^6}{R(q)^2} + q^4 \frac{R(q)^3}{R(q)^2} + \frac{q^{10}}{R(q)^2 R(q^3)^6} \right) \]

\[ + 16 q^2 \left( R(q)^6 R(q^3)^3 - \frac{R(q^3)^6}{R(q)^3} + q^6 \frac{R(q)^3}{R(q^3)^6} - \frac{q^{10}}{R(q)^6 R(q^3)^3} \right) \]

\[ + 30 q^2 \left( R(q^3)^5 - \frac{q^6}{R(q^3)^5} + q^2 R(q)^5 - \frac{q^{10}}{R(q)^5} \right) \]

\[ + 64 q^2 \left( R(q)^3 R(q^3)^4 + q^2 \frac{R(q)^3}{R(q)^4} - q^4 \frac{R(q)^4}{R(q^3)^4} - \frac{q^6}{R(q)^3 R(q^3)^4} \right) \]

\[ + 64 q^4 \left( R(q) R(q^3)^3 - q^2 \frac{R(q^3)^3}{R(q^3)} - q^2 \frac{R(q^3)}{R(q^3)^3} + \frac{q^4}{R(q) R(q^3)^3} \right) \]

\[ + 108 q^4 \left( \frac{R(q)^4 R(q^3)^2}{R(q)} + q^4 \frac{R(q)^2}{R(q)^4} + q^4 \frac{R(q^3)^4}{R(q^2)} + \frac{q^8}{R(q)^4 R(q^3)^2} \right) \]

\[ + 108 q^5 \left( \frac{R(q)^2 R(q^3)}{R(q)} - \frac{R(q)^2}{R(q^2)} + q^2 \frac{R(q)}{R(q^2)} - \frac{q^2}{R(q)^2 R(q^3)} + 45 q^6 \right). \]  

By Lemma 2.2 and Lemma 2.3 we find that

\[ C_7 := \frac{R(q^3)^7}{R(q)} - q^2 R(q)^7 R(q^3) - \frac{q^6}{R(q)^7 R(q^3)} + q^8 \frac{R(q)}{R(q^3)^7} = C_4 C_5, \]  

\[ C_8 := R(q) R(q^3)^8 - q^4 \frac{R(q)^8}{R(q^3)^8} + q^6 \frac{R(q)^3}{R(q^3)^8} - \frac{q^{10}}{R(q) R(q^3)^8} = A(q^3) C_2 + q^4 (C_3 - AC_1), \]

\[ C_9 := R(q)^4 R(q^3)^7 - q^4 \frac{R(q)^3}{R(q)^7} + q^6 \frac{R(q)^7}{R(q^3)^7} - \frac{q^{10}}{R(q)^4 R(q^3)^7} = C_2 C_5 - q^4 (C_1 C_4 + C_3) \]

and

\[ C_{10} := R(q)^7 R(q^3)^6 + q^2 \frac{R(q)^3}{R(q)^6} - q^8 \frac{R(q)^6}{R(q^3)^6} - \frac{q^{10}}{R(q)^7 R(q^3)^6} \]

\[ = C_6 (C_5 - C_8 + q^2 C_4) - q^2 A(q^3) + q^4 (C_3 - A). \]  

Note that each of \( C_7 - C_{10} \) can be expressed in terms of \( E_n \)'s. Using (2.3), Lemma 2.3 (3.2), (3.4), (3.5), and (3.7)–(3.10) in (3.6), we arrive at (3.6).
Finally, we sketch the proof of (1.6). We have
\[
\sum_{n=0}^{\infty} p_{[1^33]}(n)q^n = \frac{1}{E_1E_3^3}.
\]
As in the previous cases, we employ (2.2) in the above and then apply \([q^{5n+3}]\) to deduce an identity similar to (3.6). The resulting identity can be shown to be equivalent to (1.6) with the help of Lemma 2.3, (2.3), (2.4)–(2.6), (3.2), and (3.4). □

**Proof of Corollary 1.4.** By the binomial theorem, we note that, for any positive integer \(k\),
\[
E_5^k \equiv E_{5k} \pmod{5}.
\]
Therefore, from (1.4), we have
\[
\sum_{n=0}^{\infty} p_{[1^33]}(5n+1)q^n \equiv \frac{E_5^4}{E_1E_5^4} + q^2 \frac{E_5^4}{E_3E_5} \pmod{5}.
\]
Employing (2.2) in the above and then applying \([q^{5n+4}]\), we find that
\[
p_{[1^33]}(25n+21) \equiv 0 \pmod{5},
\]
which is (1.7).

Now, we prove (1.8). Again from (1.4), we have
\[
\sum_{n=0}^{\infty} p_{[1^33]}(5n+1)q^n \equiv \frac{E_5^5}{E_1^6E_5} + 10q \frac{E_5^9}{E_1^2E_5^4} + q^2 \frac{E_5^9}{E_3^2E_5^3} + 45q^3 \frac{E_3^4E_5^4}{E_1E_3}
- 90q^5 \frac{E_5^9}{E_3^2E_5} \pmod{25}.
\]
Next, we employ (2.2) followed by the extraction operator \([q^{5n+4}]\) and (3.11) to each term of the right side of the above to find the following identities.
\[
[q^{5n+4}] \left\{ \frac{E_5^5}{E_1^6E_5} \right\} = \frac{5}{E_3} \left( 63 \frac{E_5^6}{E_1^6} + 52 \times 5^3 q \frac{E_5^{12}}{E_1^{13}} + 63 \times 5^5 q^2 \frac{E_5^{18}}{E_1^{19}} 
+ 6 \times 5^8 q^3 \frac{E_5^{24}}{E_1^{25}} + 5^3 q \frac{E_5^{30}}{E_1^{31}} \right)
\equiv 5 \times 63 \frac{E_3^1E_5^4}{E_3} \pmod{25}, \tag{3.13}
\]
\[
[q^{5n+4}] \left\{ 10q \frac{E_5^9}{E_1^2E_5^4} \right\} = [q^{5n+3}] \left\{ 10 \frac{E_5^9}{E_1^2E_5^4} \right\}
= 10 \frac{E_1^9}{E_3} \cdot \frac{E_1^{10}}{E_1^{12}} \left( 15q + 10 \left( R(q)^5 - \frac{q^2}{R(q)^5} \right) \right)
\equiv 0 \pmod{25}, \tag{3.14}
\]
\[
[q^{5n+4}] \left\{ q^2 \frac{E_5^{15}}{E_3^5 E_5^5} \right\} = \frac{5}{E_1} \left( 63 \frac{E_5^{12}}{E_3^3} + 52 \times 5^3 q \frac{E_5^{10}}{E_3^5} + 63 \times 5^5 q^2 \frac{E_5^{18}}{E_3^9} + 6 \times 5^8 q^3 \frac{E_5^{24}}{E_3^{15}} + 5^{10} q^4 \frac{E_5^{30}}{E_3^{21}} \right)
\equiv 5 \times 63 q^2 \frac{E_5^6}{E_1 E_3^3}
\equiv 5 \times 63 q^2 \frac{E_3^4 E_5^4}{E_1} \pmod{25},
\tag{3.15}
\]

and
\[
[q^{5n+4}] \left\{ 45q^3 \frac{E_3^4 E_5^4}{E_1} \right\} = [q^{5n+1}] \left\{ 45 \frac{E_5^4 E_3^4}{E_1 E_3} \right\}
\equiv 45 E_3^4 \left( \frac{E_5^4}{E_1 E_3} + q^2 \frac{E_5^4}{E_3 E_5} \right)
\equiv 45 \left( \frac{E_3^4 E_5^4 E_3^4}{E_1 E_3} + q^2 \frac{E_3^4 E_3^4 E_5^4}{E_5} \right)
\equiv 45 \left( \frac{E_3^4 E_3^4}{E_3} + q^2 \frac{E_3^4 E_5^4}{E_1} \right) \pmod{25}
\tag{3.16}
\]

and
\[
[q^{5n+4}] \left\{ 90q^5 \frac{E_5^9}{E_3^2 E_5} \right\} = 90 \frac{E_5^{10}}{E_1 E_3^5} \left( 15q^5 + 10q^2 \left( R(q^3)^5 - \frac{q^6}{R(q^3)^5} \right) \right)
\equiv 0 \pmod{25}.
\tag{3.17}
\]

Using (3.13)–(3.17) in (3.12), we arrive at
\[
\sum_{n=0}^{\infty} p_{[1^{3}]}(25n + 21) q^n \equiv 10 \left( \frac{E_3^4 E_5^4}{E_3} + q^2 \frac{E_3^4 E_5^4}{E_1} \right) \pmod{25}.
\]

Employing Lemma 2.1 in the above and then applying \([q^{5n}]\), we obtain
\[
\sum_{n=0}^{\infty} p_{[1^{3}]}(125n + 21) q^n
\equiv 10 \left( \frac{E_3^4 E_5^4 E_5^5}{E_3^6 E_5^5} \left( R(q)^3 R(q^3)^4 - \frac{q^6}{R(q^3)^4} \frac{R(q)}{R(q)^3} + q^3 \left( 25 - 3 \left( \frac{R(q)^3}{R(q)^3} + \frac{R(q^3)}{R(q)} \right) \right) \right) \right)
- q^5 \frac{E_3^4 E_5^5 E_5^5}{E_1^6} \left( 3 \left( R(q)^4 R(q^3)^2 + q^4 \frac{R(q)^2}{R(q^3)^2} + \frac{R(q^3)}{R(q)^2} \right) \right)
- 3 \left( R(q) R(q^3)^3 + \frac{q^4}{R(q) R(q^3)^3} \right) + q^5 \left( \frac{R(q^3)}{R(q)^3} - q^2 \frac{R(q)}{R(q^3)^2} \right)
+ 3q \left( R(q)^2 R(q^3) - \frac{R(q)^3}{R(q)} + q^2 \frac{R(q)}{R(q^3)^2} \frac{R(q^3)}{R(q)} \right) \pmod{25}.
\]
Applying Lemma 2.3, (3.2), (3.4), and (3.5) in the above, we find that

\[ \sum_{n=0}^{\infty} p_{[1^{13}]}(125n + 21)q^n \equiv 10 \left( \frac{E_5}{E_1 E_{15}} + q^2 \frac{E_5}{E_3 E_5} \right) \pmod{25}. \]

Once again employing (2.2) in the above and then applying \([q^{5n+4}]\), we arrive at (1.8).

**Proof of Corollary 1.6.** By (1.5), it is obvious that (1.9) is true. It remains to prove (1.10). Using (3.11) in (1.5), we find that

\[ \sum_{n=0}^{\infty} p_{[1^{23}]}(5n + 2)q^n \equiv 5 \frac{E_5}{E_1^2 E_{15}^2} + 10q^2 \frac{E_5^3 E_3}{E_1 E_3} + 5q^4 \frac{E_5^8}{E_3^2 E_5^2} \pmod{25}. \]  

(3.18)

From (3.14) and (3.17), we have

\[ [q^{5n+3}] \left\{ \frac{1}{E_7} \right\} \equiv 0 \pmod{5} \]

and

\[ [q^{5n+4}] \left\{ \frac{1}{E_3} \right\} \equiv 0 \pmod{5}. \]

Employing (2.2) in (3.18), applying \([q^{5n+3}]\) and then with an aid from the above congruences, we obtain

\[ \sum_{n=0}^{\infty} p_{[1^{23}]}(25n + 17)q^n \equiv 10 \frac{E_5^5}{E_1^2 E_3^3} \left( R(q)^3 R(q)^4 + q^2 \frac{R(q)^3}{R(q)^4} - q^4 \frac{R(q)^4}{R(q)^3} - \frac{q^6}{R(q)^3 R(q)^4} \right) \]

\[ + 2q \left( R(q)^4 R(q)^3 - q^2 \frac{R(q)^3}{R(q)^4} - q^4 \frac{R(q)^4}{R(q)^3} + \frac{q^6}{R(q)^3 R(q)^4} \right) \]

\[ + 3q \left( R(q) R(q)^3 - q^2 \frac{R(q)^3}{R(q)^4} - q^4 \frac{R(q)^4}{R(q)^3} + \frac{q^6}{R(q)^3 R(q)^4} \right) \]

\[ + 6q^2 \left( R(q)^2 R(q)^3 - q^2 \frac{R(q)^3}{R(q)^4} - q^4 \frac{R(q)^4}{R(q)^3} + \frac{q^6}{R(q)^3 R(q)^4} \right) \pmod{25}, \]

which by Lemma 2.3, (3.2), (3.4), and (3.5) reduces to

\[ \sum_{n=0}^{\infty} p_{[1^{23}]}(25n + 17)q^n \equiv 10 \left( E_1^2 E_3^3 E_5^4 + q^2 E_1^2 E_3 E_{15}^4 \right) \pmod{25}. \]  

(3.19)

We again employ (2.1) in the above and then apply \([q^{5n+1}]\) and Lemma 2.3 to obtain

\[ \sum_{n=0}^{\infty} p_{[1^{23}]}(125n + 42)q^n \equiv 20 \left( 4 \frac{E_5^8}{E_1^2 E_{15}^2} + 3q^2 \frac{E_5^3 E_{15}}{E_1 E_3} + 4q^4 \frac{E_5^8}{E_3^2 E_5^2} \right) \pmod{25}. \]
The procedure from (3.18) to (3.19) can be repeated in the above, to arrive at
\[ \sum_{n=0}^{\infty} p_{[1^23^2]}(625n + 417)q^n \equiv 10 \left( E_1^2 E_3^3 \frac{E_5}{E_{15}} + q^2 E_1^3 E_3^2 \frac{E_{15}^4}{E_5} \right) \pmod{25}. \] (3.20)

From (3.19) and (3.20), it follows that
\[ \sum_{n=0}^{\infty} p_{[1^23^2]}(625n + 417) \equiv p_{[1^23^2]}(25n + 17) \pmod{25}, \]
which by induction gives (1.10).

\[ \square \]

**Proof of (1.11).** Using (2.1), (2.6), and (3.11), we can easily see that
\[ \sum_{n=0}^{\infty} p_{[1^33^3]}(n)q^n = \frac{E_3}{E_5} \equiv E_2 \pmod{5} \]
and
\[ [q^{5n+3}] \{ E_1^2 E_2^2 \} \equiv 0 \pmod{5}. \]

Congruence (1.11) follows immediately from the above.

\[ \square \]

**Proof of Theorem 1.8.** We recall from Section 1 that
\[ \sum_{n=0}^{\infty} p_\beta(9n + 5)q^n = 3 \frac{E_3}{E_1}. \]

Employing Lemma 2.1 in the above and then applying \([q^{5n+2}]\), we find that
\[ \sum_{n=0}^{\infty} p_\beta(15n + 7)q^n = 5 \frac{E_1^{10} E_{15}^3}{E_{12}^4} \left( 4qC_2 + 10q^2 A + 4q^2 C_4 - 3q^3 C_1^2 + 21q^3 + R(q)^6 R(q^3)^3 - \frac{q^6}{R(q)^6 R(q^3)^3} \right. \]
\[ \left. - 12q \left( R(q)^4 R(q^3)^2 + \frac{q^4}{R(q)^4 R(q^3)^2} \right) + 12q^2 \left( \frac{R(q^3)^2}{R(q)} - q^2 \frac{R(q)}{R(q^3)^2} \right) \right) \pmod{5}. \] (3.21)

Congruence (1.12) follows immediately from the above.

Now, we prove the remaining two congruences. From Section 1, we also recall that
\[ \sum_{n=0}^{\infty} p_\beta(9n + 5)q^n = 3 \frac{E_6}{E_5}. \]

Employing (3.11) in the above, we have
\[ \sum_{n=0}^{\infty} p_\beta(9n + 5)q^n \equiv 3 \frac{E_{15} E_3}{E_5} \pmod{5}, \]
which, by (2.1), can be rewritten as
\[ \sum_{n=0}^{\infty} p_\beta(9n + 5)q^n \equiv 3 \frac{E_{15} E_{15}}{E_5} \left( R(q^{15}) - q^3 + \frac{q^6}{R(q^{15})} \right) \pmod{5}. \]
Applying \([q^{5n+2}]\) and \([q^{5n+4}]\) in the above, we obtain (1.13) and (1.14), respectively.

\[\square\]

**Remark 3.1.** We could not effectively transform \((R(q^3)^2/R(q) - q^2 R(q)/R(q^3)^2)\) in (3.21) into an expression involving only \(E_n\)’s, which could have helped in reducing the right side of (3.21) in terms of \(E_n\)’s with an aid from (2.6). The transformed equivalent form of (3.21) might have lead to congruences modulo higher powers of 5, including the following congruences conjectured by Zhang and Shi [14, Conjecture 6].

\[p_\beta(3 \cdot 5^2 n + 22) \equiv p_\beta(3 \cdot 5^2 n + 52) \equiv p_\beta(3 \cdot 5^2 n + 67) \equiv 0 \pmod{5^2},\]

\[p_\beta(3 \cdot 5^4 n + 547) \equiv p_\beta(3 \cdot 5^4 n + 1297) \equiv p_\beta(3 \cdot 5^4 n + 1672) \equiv 0 \pmod{5^3} .\]

4. Proofs of Theorems 1.9–1.10 using identities for \(R(q)\) and \(R(q^4)\)

**Proof of Theorem 1.9.** We have

\[
\sum_{n=0}^{\infty} b_4(n) q^n = \frac{E_4}{E_1}.
\]

Employing Lemma 2.1 in the above and then applying \([q^{5n+3}]\), we find that

\[
\sum_{n=0}^{\infty} b_4(5n + 3) q^n = \frac{E_5^5 E_{20}^5}{E_1^6} \left( 3 \left( R(q) R(q^4) + \frac{q^2}{R(q) R(q^4)} \right) - q \left( 5 \left( \frac{R(q)^4}{R(q)} - \frac{R(q^4)}{R(q^4)} \right) \right) \right).
\]  

Now, from Lemma 2.2 and Lemma 2.5 it follows that

\[
D_2 := \frac{R(q)^4}{R(q^4)} - \frac{R(q^4)}{R(q^4)} = BF + B(q^2)
\]

\[
= \frac{E_1 E_5^5}{E_2 E_5^5} + 4q^2 \frac{E_2 E_5^5}{E_4 E_10^5} + 16q^3 \frac{E_1 E_5^5}{E_4 E_5^5}. \tag{4.2}
\]

Using (2.7) and (4.2) in (4.1), we find that

\[
\sum_{n=0}^{\infty} b_4(5n + 3) q^n
\]

\[
= 3 \frac{E_4 E_{10}^5}{E_1^5 E_2^4 E_4} + q \frac{E_5 E_{20}^5}{E_1^6} - 8q^2 \frac{E_5 E_{20}^5}{E_1^6 E_2} - 4q^3 \frac{E_2 E_5 E_{20}^5}{E_1^6 E_4 E_{10}^5} - 16q^4 \frac{E_{20}^5}{E_1^7 E_4}
\]

\[
= \left( \frac{E_5^5}{E_2^4 E_{20}^5} - 4q^2 \frac{E_5^5}{E_2^5 E_4} \right) \left( 3 \frac{E_2 E_5 E_{10}^5}{E_1^7 E_{10}^5} + q \frac{E_2 E_5 E_{20}^5}{E_1^7 E_4} + 4q^2 \frac{E_2 E_{20}^5}{E_1^7 E_2} \right). \tag{4.3}
\]

From [2, Lemma 2.2.2], we recall that

\[
\frac{E_5^5}{E_1^5 E_{10}^5} = \frac{E_5}{E_2^5} + 4q \frac{E_{10}^2}{E_1^5 E_2}.
\]

Replacing \(q\) by \(q^2\) in the above and then using the resulting identity in (4.3), we arrive at (1.15).

\[\square\]
Proof of Theorem 1.10. We have
\[ \sum_{n=0}^{\infty} a_4(n)q^n = \frac{E_4^1}{E_1}. \]

Employing Lemma 2.1 in the above and then applying \([q^{5n}]\), we find that
\[ \sum_{n=0}^{\infty} a_4(5n)q^n \]
\[ = \frac{E_5^5 E_2^1}{E_1^6} \left( R(q)^4 R(q^4)^4 + \frac{q^8}{R(q)^4 R(q^4)^4} + q \left( -4 \left( R(q)^3 R(q^4)^3 + \frac{q^6}{R(q)^3 R(q^4)^3} \right) \right) \right. \\
- 3 \left( \frac{R(q)^4}{R(q)} - q^6 \frac{R(q)}{R(q^4)^2} \right) + q^2 \left( 4 \left( R(q)^2 R(q^4)^2 + \frac{q^4}{R(q)^2 R(q^4)^2} \right) \right) \right.
\]
\[ - 8 \left( \frac{R(q)^3}{R(q)^2} - \frac{R(q)^2}{R(q^4)^3} \right) + q^4 \left( 24 \left( R(q) R(q^4) + \frac{q^2}{R(q) R(q^4)} \right) \right) \right.
\]
\[ - 2 \left( \frac{R(q)^4}{R(q)^3} - q^2 \frac{R(q)^3}{R(q^4)^2} \right) + q^4 \left( -25 - 8 \left( \frac{R(q)^4}{R(q^4)} - \frac{R(q^4)}{R(q^4)^4} \right) \right). \]  \tag{4.4}

It follows from Lemma 2.2 and Lemma 2.4 that
\[ D_3 := \frac{R(q)^4}{R(q)^3} - q^2 \frac{R(q)^3}{R(q^4)^2} = A - D_1 D_2, \]  \tag{4.5}
\[ D_4 := \frac{R(q)^4}{R(q)^2} - q^4 \frac{R(q)^2}{R(q^4)^3} = D_1 D_3 + q^2 D_2, \]  \tag{4.6}
\[ D_5 := \frac{R(q)}{R(q)} - q^6 \frac{R(q)}{R(q^4)^4} = D_1 D_4 - q^2 D_3, \]  \tag{4.7}
\[ D_6 := R(q)^2 R(q^4)^2 + \frac{q^4}{R(q)^2 R(q^4)^2} = D_1^2 - 2q^2, \]  \tag{4.8}
\[ D_7 := R(q)^3 R(q^4)^3 + \frac{q^6}{R(q)^3 R(q^4)^3} = D_1^3 - 3q^2 D_1 \]  \tag{4.9}
and
\[ D_8 := R(q)^4 R(q^4)^4 + \frac{q^8}{R(q)^4 R(q^4)^4} = D_1^2 - 2q^4. \]  \tag{4.10}

Using (2.7), (4.2), and (4.5)–(4.10) in (4.4), we arrive at (1.16). \qed

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Partition identities arising from the Rogers-Ramanujan continued fractions \( R(q), R(q^3), \) and \( R(q^4) \)

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