Uniform Decay for a Viscoelastic Wave Equation with Density and Time-Varying Delay in $\mathbb{R}^n$

Salah Zitouni$^a$, Khaled Zennir$^b$, Lamine Bouzettouta$^c$

$^a$Department of mathematics, University 20 Août 1955- Skikda, 21000, Algeria
$^b$Department of Mathematics, College of Sciences and Arts, Al-Ras, Qassim University, Kingdom of Saudi Arabia
$^c$Laboratory LAMAHIS, Department of mathematics, University 20 Août 1955- Skikda, 21000, Algeria.

Abstract. A linear viscoelastic wave equation with density and a time-varying delay term in the internal feedback is considered. Under suitable assumptions on the relaxation function, we establish a decay result of solution for by using energy perturbation method in the space $\mathbb{R}^n$ ($n > 2$). We extend a recent result in Feng [10].

1. Introduction and position of problem

It is well known that the PDEs with time delay have been much studied during the last years and their results is by now rather developed especially in the varying delay case, see [1], [7]–[9], [16]–[18], [21], and so on. In the classical theory of delayed wave equations, several main parts are joined in a fruitful way, it is very remarkable that the damped wave equation with varying delays occupies a similar position and arise in many applied problems.

In this paper, we consider the following wave equation with a time-varying delay term in the internal feedback:

$$\begin{cases}
\mathbf{u}_{tt} - \phi(x) \left( \Delta_x u - \int_0^t g(t-s) \Delta_x u(s) ds \right) + \mu_1 \mathbf{u}_t + \mu_2 \mathbf{u}_t(x, t-\tau(t)) = 0, \\
\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \mathbf{u}_t(x, 0) = \mathbf{u}_1(x), \quad x \in \mathbb{R}^n, \\
\mathbf{u}_t(x, t) = f_0(x, t), \quad x \in \mathbb{R}^n, \quad t \in (-\tau(0), 0),
\end{cases}$$

where $\mathbf{u}_0(x)$, $\mathbf{u}_1(x)$, and $f_0(x, t-\tau(0))$ are given initial data and the function $g$ is the relaxation function. The function $\phi(x) := (\rho(x))^{-1}$ is the speed of sound at the point $x \in \mathbb{R}^n$ and the function $\rho(x)$ is the density. The constants $\mu_1$ and $\mu_2$ are two real numbers and the function $\tau(t)$ is the varying delay term.

We assume, on the time-delay functions, that there exist positive constants $\underline{\tau}_0$ and $\overline{\tau}$ such that

$$0 < \underline{\tau}_0 \leq \tau(t) \leq \overline{\tau}, \quad \forall t > 0,$$

2010 Mathematics Subject Classification. Primary 35B35; Secondary 35L05

Keywords. Wave equation, varying delay term, exponential stability.

Received: 10 May 2017; Accepted: 14 August 2017

Communicated by Marko Nedeljkov

Email addresses: zitsala@yahoo.fr (Salah Zitouni), k.Zennir@qu.edu.sa (Khaled Zennir), lami_750000@yahoo.fr (Lamine Bouzettouta)
Moreover, we assume
\[ \tau \in W^{2,\infty}([0, T]), \quad \forall T > 0, \quad (3) \]
\[ \tau' (t) \leq d < 1, \quad \forall t > 0, \quad (4) \]
where \( d \) is the positive constant.

The relaxation function \( g \) satisfies the following assumptions:

1. \( g : \mathbb{R} \rightarrow \mathbb{R} \) is a \( C^1 \) function satisfying
   \[ g(0) > 0, \quad 1 - \int_0^\infty g(s) ds = l > 0. \quad (5) \]

2. There exists a non-increasing differentiable function \( \zeta : \mathbb{R} \rightarrow \mathbb{R} \) such that
   \[ \int_0^\infty \zeta(s) ds = +\infty, \quad g'(t) \leq -\zeta(t) g(t), \quad \text{for } t \geq 0. \quad (6) \]

The modified energy functional associate with problem (1) is given by
\[ E(t) = \frac{1}{2} \| u(t) \|_{L^2}^2 + \frac{1}{2} \left( 1 - \int_0^t g(s) ds \right) \| \nabla_x u(t) \|_{L^2}^2 \]
\[ + \int_0^t (g \circ \nabla_x u)(s) + \frac{\xi}{2} \int_{t-\tau(t)}^t \int_{\mathbb{R}^n} \rho (x) e^{l(t-s)} u_s (x, s) dx ds. \quad (7) \]
where and \( \xi > 0 \) will be chosen later, and the constant \( \lambda > 0 \), see [19], satisfies
\[ \lambda < \frac{1}{\tau_1} \left| \log \left| \frac{\mu_1}{\sqrt{1 - d}} \right| \right. \]
and
\[ (g \circ \nabla_x u)(t) = \int_0^t g(t-s) \| \nabla_x u(t) - \nabla_x u(s) \|_{L^2}^2 ds. \quad (8) \]

For \( \tau (t) = \tau_0 \), system (1) has been investigated recently by many authors, where they showed the well-posedness and stabilities results in bounded/unbounded domains (see [1], [2], [3], [5], [7], and so on). Concerning the existence and uniqueness result, we refer the reader to read the existing works which is not our aim interesting here (see [10], Theorem 3.1). In the present work, we extend the result in [10] to time-varying delay.

The plan of the paper is as follows. The first section is devoted to introduce the problem. In Section 2, we give some preliminaries and our main results. In Section 3, we shall prove the stability of energy to the problem.

2. Preliminaries and main result

As in [11], [22], we introduce the weighted spaces \( D^{1,2} (\mathbb{R}^n) \) and \( L^p_p (\mathbb{R}^n) \) for our system. First we assume the density \( \rho (x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies the following conditions.

(A) \( \rho (x) > 0, \rho \in C^{0,\gamma} (\mathbb{R}^n) \) with \( \gamma \in (0,1) \) and \( \rho \in L^2 (\mathbb{R}^n) \cap L^\infty (\mathbb{R}^n) \). Now we define the weighted spaces
\[ D^{1,2} (\mathbb{R}^n) \text{ and } L^p_p (\mathbb{R}^n), \quad (1 < p < \infty). \]

1. The space \( D^{1,2} (\mathbb{R}^n) \) is defined to be the closure of \( C_0^\infty (\mathbb{R}^n) \) functions with respect to which norm
\[ D^{1,2} (\mathbb{R}^n) = \left\{ u \in L^{\frac{2n}{n-2}} (\mathbb{R}^n) : \nabla_x u \in L^2 (\mathbb{R}^n) \right\}, \]
equipped with the norm $\|u\|_{L^2_p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx$.

(2) We introduce the weighted space $L^2_p(\mathbb{R}^n)$ to be defined the closure of $C^\infty_0(\mathbb{R}^n)$ functions with respect to the inner product

$$(u, v)_{L^2_p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho |u|^p \, dx,$$

and we know that $L^2_p(\mathbb{R}^n)$ is a separable Hilbert space and $\|u\|_{L^2_p(\mathbb{R}^n)} = (u, u)_{L^2_p(\mathbb{R}^n)}$.

(3) If $u$ is a measurable function on $\mathbb{R}^n$, we define

$$\|u\|_{L^p_u(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \rho |u|^p \, dx \right)^{\frac{1}{p}}, \quad \text{for} \ 1 < p < \infty,$$

and let $L^p_u(\mathbb{R}^n)$ consist of all $u$ for which $\|u\|_{L^p_u(\mathbb{R}^n)} < \infty$.

We have the following Lemma.

**Lemma 2.1.** [10], [11],[23] Assume the function $\rho$ satisfies (A), then for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$

$$\|u\|_{L^2_u(\mathbb{R}^n)} \leq \|\rho\|_{L^1(\mathbb{R}^n)} \|\nabla u\| \quad \text{with} \quad s = \frac{2n}{2n - qn + 2q} \quad \text{and} \quad 2 \leq q \leq \frac{2n}{n - 2}. \quad (1)$$

**Remark 2.2.** For $q = 2$, we have

$$\|u\|_{L^2_u(\mathbb{R}^n)} \leq \|\rho\|_{L^1(\mathbb{R}^n)} \|\nabla u\|. \quad (2)$$

If $\rho \in L^2(\mathbb{R}^n)$, we have

$$\|u\|_{L^2_u(\mathbb{R}^n)} \leq c \|\nabla u\|, \quad (3)$$

where $c, > 0$ is a constant.

The main result of the present work is to establish a general decay rate of the energy, which is given by the following theorem.

**Theorem 2.3.** Assume the assumptions (G1)-(G2) and $|\mu_2| < \sqrt{1-d}\mu_1$ hold. Let $U(0) = (u_0, u_1) \in \mathcal{D}^{1,2}(\mathbb{R}^n) \times L^2_u(\mathbb{R}^n)$ and $f_0(x, t) \in L^2_u(\mathbb{R}^n \times (-\tau(0), 0))$, then there exist two constants $\beta > 0$ and $\gamma > 0$ such that the energy $E(t)$ defined by (7) satisfies

$$E(t) \leq \beta e^{-\gamma \int_0^t \xi(t) \, ds}, \quad \forall t \geq 0. \quad (4)$$

for all fixed $t_0 > 0$.

### 3. Proof of stability result

In this section, we show that problem (1), is uniformly exponentially stable using the multiplier technique. To achieve our goal, we need the following lemmas.

**Lemma 3.1.** Under the assumptions of Theorem 2.3, the modified energy functional defined by (7) satisfies for any $t \geq 0$,

$$E'(t) \leq \left( \frac{|\mu_2|}{2 \sqrt{1-d}} - \mu_1 + \frac{\xi}{2} \right) \|u(t)\|_{L^2_u(\mathbb{R}^n)}^2$$

$$+ \left( \frac{|\mu_2|}{2 \sqrt{1-d}} - \frac{\xi}{2} e^{-\lambda t} (1 - d) \right) \int_{\mathbb{R}^n} \rho(x) u_2(t - \tau(t)) \, dx + \frac{1}{2} (q' \circ \nabla u)(t)$$

$$- \frac{1}{2} \rho(t) \|\nabla u(t)\|_2^2 - \frac{\xi \lambda}{2} \int_{t-\tau(t)}^t \int_{\mathbb{R}^n} \rho(x) e^{-\lambda(t-s)} u_2^2(x,s) \, dx \, ds. \quad (1)$$
Proof. Taking derivative of $E(t)$, we have

$$E'(t) = \int_{\mathbb{R}^n} \rho(x) u_t u_t dx - \frac{1}{2} g(t) \|\nabla_x u\|^2 + \left(1 - \int_0^t g(s) ds\right) \int_{\mathbb{R}^n} \nabla_x u \nabla_x u dx$$

$$+ \frac{1}{2} (g' \circ \nabla_x u) + \int_0^t g(t-s) \int_{\mathbb{R}^n} (\nabla_x u(t) - \nabla_x u(s)) \nabla_x u(t) dx ds$$

$$- \lambda \frac{\xi}{2} \int_{\mathbb{R}^n} \int_{I-t}^t \rho(x) e^{\lambda(t-s)} u_x^2(x, s) ds dx$$

$$- \frac{\xi}{2} \frac{1}{\tau'} (t) e^{-\tau(t)} \int_{\mathbb{R}^n} \rho(x) u_t^2(x, t-\tau(t)) dx$$

By using equation (1) and integration by parts, we can easily get

$$E'(t) = \frac{1}{2} (g' \circ \nabla_x u) - \frac{1}{2} g(t) \|\nabla_x u\|^2 - \mu_1 \int_{\mathbb{R}^n} \rho(x) u_t^2 dx + \frac{\xi}{2} \|u_t\|^2_{L^2}$$

$$- \mu_2 \int_{\mathbb{R}^n} \rho(x) u_t u_t(x, t-\tau(t)) dx$$

$$- \frac{\xi}{2} (1 - \tau'(t)) e^{-\tau(t)} \int_{\mathbb{R}^n} \rho(x) u_t^2(x, t-\tau(t)) dx$$

$$- \lambda \frac{\xi}{2} \int_{\mathbb{R}^n} \int_{I-t}^t \rho(x) e^{\lambda(t-s)} u_x^2(x, s) ds dx.$$

By using Young’s inequality, we can get

$$- \mu_2 \int_{\mathbb{R}^n} \rho u_t(t) \cdot u_t(t-\tau(t)) dx \leq \frac{|\mu_2|}{2} \|u_t\|^2_{L^2} + \frac{|\mu_2|}{2} \sqrt{1-d} \int_{\mathbb{R}^n} \rho u_t^2(t-\tau(t)) dx,$$

which gives us (1). The proof is complete. □

Lemma 3.2. Under the assumptions of Theorem 2.3, let $(u, u_t)$ be the solution of problem (1). The functional $F_1(t)$ defined by

$$F_1(t) = \int_{\mathbb{R}^n} \rho u_t dx$$

(2)

satisfies that there exist a positive constants $\kappa_1$, $\kappa_2$ and $\kappa_3$ such that for any $t > 0$,

$$F_1(t) \leq -\frac{1}{2} \|\nabla_x u(t)\|^2 + \kappa_1 \|u_t(t)\|_{L^2}^2 + \kappa_2 \int_{\mathbb{R}^n} \rho u_t^2(x, t-\tau(t)) dx + \kappa_3 (g \circ \nabla_x u)(t)$$

(3)
By using Young’s and Hölder’s inequalities, we arrive at for any \( \varepsilon > 0 \)

\[
\mathcal{K}_1(t) = \int_{\mathbb{R}^n} \rho u_1^2 dx + \int_{\mathbb{R}^n} u \Delta u dx - \int_{\mathbb{R}^n} u \int_0^t g(t-s) \Delta u(s) ds dx \\
- \mu_1 \int_{\mathbb{R}^n} \rho \mu u_t dx - \mu_2 \int_{\mathbb{R}^n} \rho \mu u_t (x, t - \tau(t)) dx \\
= \int_{\mathbb{R}^n} \rho u_1^2 dx - \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u (t) \int_0^t g(t-s) \nabla u(s) ds dx \\
- \mu_1 \int_{\mathbb{R}^n} \rho \mu u_t dx - \mu_2 \int_{\mathbb{R}^n} \rho \mu u_t (x, t - \tau(t)) dx \\
= \int_{\mathbb{R}^n} \rho u_1^2 dx - \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u (t) \int_0^t g(t-s) |\nabla u(s) - \nabla u (t)| ds dx \\
- \mu_1 \int_{\mathbb{R}^n} \rho \mu u_t dx - \mu_2 \int_{\mathbb{R}^n} \rho \mu u_t (x, t - \tau(t)) dx \\
- \mu_2 \int_{\mathbb{R}^n} \rho \mu u_t (x, t - \tau(t)) dx \\
\tag{4}
\]

By using Young’s and Hölder’s inequalities, we arrive at for any \( \varepsilon > 0 \)

\[
\int_{\mathbb{R}^n} \nabla u (t) \int_0^t g(t-s) (\nabla u(s) - \nabla u (t)) ds dx \\
\leq \varepsilon \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{4\varepsilon} \left( \int_0^t g(s) ds \right) (g \circ \nabla u) \\
\leq \varepsilon |\nabla u|^2 + \frac{1}{4\varepsilon} (g \circ \nabla u). \\
\tag{5}
\]

By using the same calculations and (3) we have for any \( \varepsilon > 0 \)

\[
- \mu_1 \int_{\mathbb{R}^n} \rho \mu u_t dx \leq \mu_1 \varepsilon \| \nabla u \|^2 + \frac{\mu_1}{4\varepsilon} \| u_t \|^2 \\
- \mu_2 \int_{\mathbb{R}^n} \rho \mu u_t (x, t - \tau(t)) dx \leq \mu_2 \varepsilon \| \nabla u \|^2 + \frac{\mu_2}{4\varepsilon} \int_{\mathbb{R}^n} u_t^2 (x, t - \tau(t)) dx \\
\tag{6,7}
\]

Inserting (5)-(7) into (4), using Assumption (G1) and taking \( \varepsilon > 0 \) small enough, we can get (3) with

\[
\kappa_1 = 1 + \frac{\mu_1}{4\varepsilon}, \quad \kappa_2 = \frac{\mu_2}{4\varepsilon}, \quad \kappa_3 = \frac{1-l}{4\varepsilon}.
\]

The existence of viscoelastic term forces us to introduce the next Lemma.

**Lemma 3.3.** Under the assumptions of Theorem 2.3, let \((u, u_t)\) be the solution of problem (1). The functional \( F_2 (t) \) defined by

\[
\mathcal{F}_2(t) = \int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s) (u(t) - u(s)) ds dx \\
\tag{8}
\]

satisfies that there exists a positive constant \( \kappa_4 \) such that for any \( \delta > 0 \)

\[
\mathcal{F}_2(t) \leq \left( 2\delta - \int_0^t g(s) ds \right) \| u_t(t) \|^2 + \left( \delta + 2\delta (1-l)^2 \right) \| \nabla u(t) \|^2 + \kappa_4 \left( g \circ \nabla u \right) (t) \\
- \frac{g(0)\kappa_2^2}{4\delta} \left( g' \circ \nabla u \right) (t) + \delta \int_{\mathbb{R}^n} \rho u_t^2 (x, t - \tau(t)) dx \\
\tag{9}
\]
Proof. We derive \( \overline{\mathcal{N}}_2(t) \) and use (1) to obtain

\[
\overline{\mathcal{N}}_2(t) = - \int_{\mathbb{R}^n} \Delta_x u \int_0^t g(t-s) (u(t) - u(s)) \, ds \, dx
+ \int_{\mathbb{R}^n} \left( \int_0^t g(s) \Delta_x u(s) \, ds \right) \left( \int_0^t g(t-s) (u(t) - u(s)) \, ds \right) \, dx
+ \mu_1 \int_{\mathbb{R}^n} \rho u_i \int_0^t g(t-s) (u(t) - u(s)) \, ds \, dx
+ \mu_2 \int_{\mathbb{R}^n} \rho u_i (x, t - \tau(t)) \int_0^t g(s) \, ds \, dx
- \int_{\mathbb{R}^n} \rho u_i \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, dx - \int_0^t g(s) \, ds \, ||u_i||^2_{L^2} \tag{10}
\]

Using integration by parts, Young’s inequality and Hölder’s inequality, we have for any \( \delta > 0 \),

\[
\left| - \int_{\mathbb{R}^n} \Delta_x u \int_0^t g(t-s) (\nabla_x u(s) - \nabla_x u(t)) \, ds \, dx \right|
\leq \delta ||\nabla_x u||^2 + \frac{1 - l}{4 \delta} (g \circ \nabla_x u)
\tag{11}
\]

\[
= \left| \int_{\mathbb{R}^n} \left( \int_0^t g(t-s) (u(t) - u(s)) \, ds \right) \left( \int_0^t g(t-s) \Delta_x u(s) \, ds \right) \, dx \right|
\]

\[
\leq \delta \int_{\mathbb{R}^n} \left( \int_0^t g(t-s) \nabla_x u(s) \, ds \right)^2 \, dx + \frac{1}{4 \delta} \int_{\mathbb{R}^n} \left( \int_0^t g(t-s) (\nabla_x u(t) - \nabla_x u(s)) \, ds \right)^2 \, dx
\]

\[
\leq 2 \delta \int_{\mathbb{R}^n} \left[ \left( \int_0^t g(t-s) (\nabla_x u(t) - \nabla_x u(s)) \, ds \right)^2 + \int_0^t g(t-s) \nabla_x u(s) \, ds \right]^2 \, dx
+ \frac{1}{4 \delta} \int_{\mathbb{R}^n} \left( \int_0^t g(t-s) (\nabla_x u(t) - \nabla_x u(s)) \, ds \right)^2 \, dx
\]

\[
\leq \left( 2 \delta + \frac{1}{4 \delta} \right) \left( \int_0^t g(s) \, ds \right) (g \circ \nabla_x u) + 2 \delta \left( \int_0^t g(s) \, ds \right)^2 ||\nabla_x u||^2
\]

\[
\leq \left( 2 \delta + \frac{1}{4 \delta} \right) (1 - l) (g \circ \nabla_x u) + 2 \delta (1 - l)^2 ||\nabla_x u||^2 \tag{12}
\]

\[
\left| \mu_1 \int_{\mathbb{R}^n} \rho u_i \int_0^t g(t-s) (u(t) - u(s)) \, ds \, dx \right|
\leq \delta ||u_i||^2_{L^2} + \frac{\epsilon^2}{4 \delta} (g \circ \nabla_x u)
\tag{13}
\]

\[
\left| \mu_2 \int_{\mathbb{R}^n} \rho u_i (x, t - \tau(t)) \int_0^t g(s) \, ds \, dx \right|
\leq \delta \int_{\mathbb{R}^n} \rho u_i^2 (x, t - \tau(t)) \, dx + \frac{\epsilon^2}{4 \delta} (g \circ \nabla_x u)
\]
Combining (11)-(14) with (10), we can obtain (9) with
\[
\kappa_4 = (1 - l) \left[ \left( 2\delta + \frac{1}{4\delta} \right) + \frac{c^2}{4\delta} \right]
\]
(14)
The proof of the Lemma is complete.

Define the Lyapunov functional
\[
\mathcal{U}(t) = E(t) + N_1 \mathcal{\Phi}_1(t) + N_2 \mathcal{\Phi}_2(t)
\]
(15)
where, \(N_1\) and \(N_2\) are positive constants that will be fixed later.

**Lemma 3.4.** For \(N_1 > 0\) and \(N_2 > 0\) small enough, we have
\[
\frac{1}{2} E(t) \leq \mathcal{U}(t) \leq 2E(t)
\]
(16)

**Proof.** By using H"{o}lder's inequality, Young's inequality and making use of the above Lemmas, and (3) we obtain for any \(\delta > 0\)
\[
|\mathcal{U}(t) - E(t)| \leq N_1 \int_{\mathbb{R}^n} |\rho u_t dx| + N_2 \int_{\mathbb{R}^n} |\rho u_t \int_0^t g(t - s) (u(t) - u(s)) ds dx|
\leq N_1 \left( \delta \|u_t\|_{L^2}^2 + \frac{c^2}{4\delta} \|\nabla_x u\|_{L^2}^2 \right) + N_2 \left( \delta \|u_t\|_{L^2}^2 + \frac{c^2}{4\delta} (1 - l) (g \circ \nabla_x u) \right)
\leq \delta (N_1 + N_2) \|u_t\|_{L^2}^2 + \frac{N_1 c^2}{4\delta} \|\nabla_x u\|_{L^2}^2 + \frac{N_2 c^2}{4\delta} (1 - l) (g \circ \nabla_x u)
\]
which implies us there exists a positive constant \(\epsilon > 0\) such that
\[
|\mathcal{U}(t) - E(t)| \leq \epsilon E(t),
\]
(17)
or
\[
(1 - \epsilon) E(t) \leq \mathcal{U}(t) \leq (1 + \epsilon) E(t)
\]
(18)
when we choose \(\epsilon_1 > 0\) and \(\epsilon_2 > 0\) small enough. The proof is complete.

**Proof of Theorem 2.3.** For any fixed \(t_0 > 0\), we know that for any \(t \geq t_0\),
\[
\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds := g_0.
\]
(19)
Now we derive (15) and using (9), (3) and (1)

\[
\Psi'(t) \leq \left( \frac{|\mu_2|}{2 \sqrt{1-d}} - \frac{\mu_1 + \frac{\xi}{2} + N_1 \kappa_1 + N_2 (2 \delta - g_0)}{2} \right) \|\mu_1\|_{1,2}^2
\]

\[
+ \left( \frac{1}{2} - N_2 \frac{g(0)c_2^2}{4\delta} \right) (g' \circ V_\delta u) + \left( N_2 \left( \delta + 2 \delta (1 - \eta) \right) - N_1 \frac{l}{2} \right) \|\nabla_3 u\|^2
\]

\[
+ \left( \frac{|\mu_2|}{2} \sqrt{1-d} - \frac{\xi}{2} e^{-\lambda \tau} (1-d) + N_1 \kappa_2 + N_2 \delta \right) \int_{\mathbb{R}^l} \rho(x) u_0^2 (x,t - \tau(t)) \, dx
\]

\[
- \frac{\lambda \xi}{2} \int_{\mathbb{R}^l} \int_{\tau(t)}^\infty \rho(x) e^{\lambda(t-s)} u_0^2 (x,s) \, ds \, dx,
\]

\[
+ (N_1 \kappa_3 + N_2 \kappa_4) (g \circ V_\delta u).
\]

(20)

We can easily get that \( e^{\lambda t} \) goes to 1 as \( \lambda \to 0^+ \). Noting the continuity of the set of real numbers, we can take \( \lambda \) so small that there exists a positive constant \( \xi \) such that

\[
\frac{e^{\lambda t} |\mu_2|}{\sqrt{1-d}} < \xi < \mu_1.
\]

(21)

From (21) we infer that

\[
\frac{|\mu_2|}{2 \sqrt{1-d}} - \frac{\mu_1 + \xi}{2} < 0,
\]

(22)

and

\[
\frac{|\mu_2|}{2} \sqrt{1-d} - \frac{\xi}{2} e^{-\lambda \tau} (1-d) < 0.
\]

(23)

We can choose \( 0 < \delta < \frac{\phi}{\xi} \) such that \( (2 \delta - g_0) < 0 \). For any fixed \( \delta < 0 \), we at last choose \( N_2 \) and \( N_1 \) small enough so that

\[
N_2 < \min \left( \frac{2 \delta}{g(0)c_2^2}, \frac{1}{\delta} \left( \frac{|\mu_2|}{2} \sqrt{1-d} + \frac{\xi}{2} e^{-\lambda \tau} (1-d) \right) \right),
\]

(24)

and

\[
\frac{2N_2}{l} \left( \delta + 2 \delta (1 - \eta)^2 \right) < N_1 < \min \left( \frac{N_2}{\kappa_1} (g_0 - 2 \delta), \frac{|\mu_2|}{2} \sqrt{1-d} + \frac{\xi}{2} e^{-\lambda \tau} (1-d) - \frac{N_2 \delta}{\kappa_2} \right),
\]

(25)

which gives us

\[
\frac{1}{2} - N_2 \frac{g(0)c_2^2}{4d} > 0, \quad \frac{|\mu_2|}{2} \sqrt{1-d} - \frac{\xi}{2} e^{-\lambda \tau} (1-d) + N_2 \delta < 0,
\]

(26)

\[
N_1 \kappa_1 + N_2 (2 \delta - g_0) < 0, \quad N_2 \left( \delta + 2 \delta (1 - \eta)^2 \right) - N_1 \frac{l}{2} < 0,
\]

(27)

and

\[
\frac{|\mu_2|}{2} \sqrt{1-d} - \frac{\xi}{2} e^{-\lambda \tau} (1-d) + N_1 \kappa_2 + N_2 \delta < 0.
\]

(28)

At this point it follows that there exist two positive constants \( \gamma_1 \) and \( \gamma_2 \) such that for any \( t \geq t_0 \),

\[
\Psi'(t) \leq -\gamma_1 E(t) + \gamma_2 (g \circ V_\delta u).
\]

(29)
We multiply (29) by $\zeta'(t)$ which is $\zeta''(t) \leq 0$ and from (G2), we use
$$
\zeta(t)(g \circ \nabla u) \leq -(g' \circ \nabla u) \leq -2E(t)
$$
we obtain
$$
\zeta(t) \zeta''(t) \leq -\gamma_1 \zeta(t) E(t) + \gamma_2 \zeta(t)(g \circ \nabla u) \\
\leq -\gamma_1 \zeta(t) E(t) - 2\gamma_2 E'(t).
$$
which implies
$$
\zeta(t) \zeta''(t) + 2\gamma_2 E'(t) \leq -\gamma_1 \zeta(t) E(t).
$$
We note $E(t)$ such that
$$
E(t) = \zeta(t) \zeta'(t) + 2\gamma_2 E'(t),
$$
then $E(t)$ is equivalent to the modified energy $E(t)$ by using (20), which implies there exist two positive constants $\beta_1$ and $\beta_2$ such that
$$
\beta_1 E(t) \leq E(t) \leq \beta_2 E(t).
$$
By using (31) and (32), we infer that for any $t \geq t_0$,
$$
E'(t) \leq -\gamma_1 \zeta(t) E(t) \leq -\frac{\gamma_1}{\beta_2} \zeta(t) E(t),
$$
we get
$$
E(t) \leq E(t_0) e^{-\frac{\gamma_1}{\beta_2} \int_{t_0}^{t} \zeta(s) ds},
$$
which implies
$$
E(t) \leq \frac{\beta_2}{\beta_1} E(t_0) e^{-\frac{\gamma_1}{\beta_2} \int_{t_0}^{t} \zeta(s) ds}.
$$
By renaming the constants, and by the continuity and boundedness of $E(t)$. This completes the proof of Theorem 2.3.

Acknowledgments.
The authors wish to thank deeply the anonymous referee for his/her useful remarks and his/her careful reading of the proofs presented in this paper.

References
[1] C. Abdallah, P. Dorato, J. Benitez-Read, and R. Byrne, Delayed positive feedback can stabilize oscillatory system, Proceedings of the 1993 American Control Conference, pp. 3106–3107, San Francisco, CA, USA, 1993.
[2] A. Benaissa and S. A. Messaoudi, Global existence and energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks, J. Math. Phys., 53(2012), 123514, 19pp.
[3] A. Benaissa, A. Benguessoum and S. A. Messaoudi, Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the nonlinear internal feedback, Int. J. Dyna. Syst. Differ. Equa., 5(1)(2014), 1-26.
[4] A. Beniami, A. Benaissa and Kh. Zennir, Polynomial Decay of Solutions to the Cauchy Problem for a Petrovsky-Petrovsky System in $\mathbb{R}^n$, Acta. Appl. Math. 146 (2016), pp. 67-79.
[5] Q. Dai and Z. Yang, Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay, Z. Angew. Math. Phys., 65(2014), 885-903.
[6] M. Daoulatli, I. Lasiecka and D. Toundykov, Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth restrictions, Discrete Conti. Dyna. Syst., 2(2009), 67-95.
[7] R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, SIAM J. Control Optim., 26 (1988):697–713.

[8] R. Datko, Two questions concerning the boundary control of certain elastic systems, J. Differential Equations, 92 (1) (1991):27-44.

[9] R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, SIAM J. Control Optim., 24(1), (1986):152-156.

[10] B. Feng, General decay for a viscoelastic wave equation with density and time delay term in $\mathbb{R}^n$, Taiwanese J. Math. in appear (2017).

[11] N. I. Karachalios and N. M. Stavrakakis, Existence of global attractor for semilinear dissipative wave equations on $\mathbb{R}^n$, J. Differential Equations 157 (1999) 183-205.

[12] W. J. Liu, General decay of the solution for viscoelastic wave equation with a time-varying delay term in the internal feedback. J. Math. Phys., 54(2013), 043504.

[13] W. J. Liu, General decay rate estimate for the energy of a weak viscoelastic equation with an internal time-varying delay term, Taiwanese J. Math., 17(2013), 2101-2115.

[14] G. Liu and H. Zhang, Well-posedness for a class of wave equation with past history and a delay, Z. Angew. Math. Phys., 67(2016), DOI 10.1007/s00033-015-0593-z.

[15] P. Martinez, A new method to obtain decay rate estimates for dissipative systems, ESAIM Control Optim. Calc. Var. 4(1999)419-444.

[16] S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim., 45(5), (2006): 1561-1585.

[17] S. Nicaise, J. Valein and E. Fridman, Stability of the heat and of the wave equations with boundary time-varying delays, Discrete Contin. Dyn. Syst. Ser. S, 2, (2009): 599–611.

[18] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, Differential Integral Equations 21:9-10 (2008), 935–958.

[19] S. Nicaise and C. Pignotti, Interior feedback stabilization of wave equations with time dependent delay, Electron. J. Differ. Equ., 2011(41)(2011), 1-20.

[20] C. Pignotti, Stability for second-order evolution equations with memory and switching time-delay. J. Dyn. Diff. Equat., (2016), DOI 10.1007/s10884-016-9545-3

[21] I. H. Suh and Z. Bien, Use of time delay action in the controller design, IEEE Trans. Autom. Control 25:3 (1980), 600–603.

[22] Kh. Zennir, General decay of solutions for damped wave equation of Kirchhoff type with density in $\mathbb{R}^n$. Ann Univ Ferrara, 61, (2015) 381-394.

[23] S. Zitouni and Kh. Zennir, On the existence and decay of solution for viscoelastic wave equation with nonlinear source in weighted spaces, Rend. Circ. Mat. Palermo, II. Ser, 2016, DOI 10.1007/s12215-016-0257-7.