Cosmology with primordial black holes

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Abstract. We study the effect of accretion of radiation in the early universe on primordial black holes in various modified gravity theories. We consider separately, a braneworld model, Brans-Dicke theory, and a more general scalar-tensor model. The rate of growth of a primordial black hole due to accretion of radiation in all these modified gravity models is smaller than the rate of growth of the cosmological horizon, thus making available sufficient radiation density for the black hole to accrete causally. We show that accretion of radiation overrides the effect of Hawking evaporation during the radiation dominated era. The subsequent evaporation of the black holes in later eras could further be modified due to either the modified braneworld geometry, or the variable gravitational coupling in scalar-tensor theories, and they could survive up to much longer times compared to the case of standard cosmology.

1. Introduction
The cosmological evolution of primordial black holes is an interesting subject that has been studied by a number of researchers over the last few decades. Black holes could be formed in the early universe through several mechanisms [1] and in a wide mass range. Their impact on cosmological evolution could be diverse, and is yet to be investigated in full details for different models of black hole formation and cosmological evolution. Among the notable cosmological consequences of surviving primordial black holes are the possibility of generation of the baryon asymmetry in the universe [2], [3], and their viability as candidates of cold dark matter [4], [5]. Primordial black holes could also have interesting astrophysical consequences, such as the seeding of supermassive black holes [6]. The most relevant issue concerning primordial black holes is their longevity which depends crucially on the effectiveness of various accretion processes.

In this article we provide a brief description of the evolution of primordial black holes in different alternate gravity scenarios [7]. We consider a braneworld model [8], the Brans-Dicke theory [9], and a more general scalar-tensor model [10]. The evolution of primordial black holes in the RS-II braneworld model leads to their growth due to accretion in the early radiation dominated high energy era [11] The possibility of black hole solutions in BD theory was first proposed by Hawking [12]. Due to the variable gravitational constant, the rate of energy loss and gain of primordial black holes through evaporation and accretion of radiation, respectively, gets modified [13]. Using scalar-tensor gravity theories Barrow and Carr [14] have studied PBH evaporation during various eras. It has been recently observed that in the context of generalised Brans-Dicke theory, inclusion of the effect of accretion leads to the prolongation of PBH lifetimes [15]. In all the considered alternate gravity models we show that accretion of radiation in the early radiation dominated universe is a dominant process overcoming the effects of Hawking radiation, enabling the black holes to survive for much longer times. A significant number of
primordial black holes in these scenarios could survive up to present times and contribute to the dark matter density of the universe.

2. Cosmological evolution of primordial black holes

Consider a population of primordial black holes in the radiation dominated era of the universe that could be formed by various mechanisms [1] operative in the post-inflationary stage of the universe. To keep the analysis simple, we do not consider an initial mass distribution of black holes, but assume that all the black holes are formed at time $t_0$ with mass $M_0$. The time evolution of the FRW scale factor $a(t)$ of the universe is governed by the Friedmann equation with a dominating radiation component and also a matter component provided by the black holes.

The mass of a black hole will change with time due to two factors. First, the process of Hawking radiation causes a loss of mass. Secondly, accretion of the surrounding matter by the black hole leads to an increase in its mass. The rate of change of mass of a single black hole can be written as

$$\dot{M} = 4\pi R_{BH}^2 \left(-\sigma T_{BH}^4 + f \rho_R \right)$$

(1)

where $R_{BH}$ is the Schwarzschild radius, $\sigma$ is the Stefan-Boltzmann constant, $T_{BH}$ is the Hawking temperature, and $f$ is an $O(1)$ accretion efficiency factor. Relativistic particles and photons accreted by the black hole have density $\rho_R = \frac{\pi g T^4}{270}$ where $g$ is the number of particle species.

The energy density of the relativistic matter $\rho_R$ will evolve both due to the expansion of the universe, as well as because of the absorption or emission by the black holes. The time-variation of $\rho_R$ is thus given by

$$\frac{d}{dt}(\rho_R(t)a^4(t)) = -\dot{M} N_{BH} a(t)$$

(2)

where $N_{BH}$ denotes the number of primordial black holes present. When the right-hand side of Eq.(1) vanishes, it yields the usual result $\rho_R \propto a^{-4}$. This happens when either no black holes are formed, i.e., $N_{BH} = 0$, or when the individual black hole masses do not change with the cosmic time.

The equations governing the cosmological evolution of the universe beyond the time $t_0$ are given by [16] the hubble expansion

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3M_4^2} \left(\rho_R + \frac{M\beta\rho(t_0)}{M_0} \frac{a^3(t_0)}{a^3} \right)$$

(3)

the evolution of radiation density

$$\rho_R = -4\rho_R \frac{\dot{a}}{a} - \frac{\dot{M} \beta \rho(t_0)}{M_0} \frac{a^3(t_0)}{a^3}$$

(4)

and the rate of change of black hole mass Eq.(1). The above coupled equations have to be integrated for the specific gravity model under consideration. The modified geometry for the black holes alters the evaporation and accretion rates. Further, the cosmological evolution of the scale factor could also be altered in a model-dependent way, as we would discuss later in context of a braneworld model, and also in context of scalar-tensor models.

3. Survival of primordial black holes in the braneworld scenario

The cosmology of the RS-II model [8] entails a modified high energy phase in the early radiation dominated era of the universe during which the right hand side of the Einstein equation contains terms that are quadratic in the brane energy momentum tensor [17]. Other modifications include
the so-called “dark-radiation” term which is given by the projection of the bulk Weyl tensor. Transition to the standard radiation dominated era takes place when $t \geq t_c = l/2$. Such a modified high energy evolution has rich consequences for the physics of the early universe[17]). In particular, the inflationary scenario is altered, allowing the possibility of steep inflaton potentials to accomplish the desired features [18], [19]. Assuming a radiation dominated equation of state, the solutions for the Friedmann equation lead to an evolution such that for times earlier that $t$ much later than $t_c$, i.e., $t \leq t_c$ (or $\rho \geq \lambda$), one has the non-standard high energy regime during which the radiation density and the scale factor evolve as $\rho_R = \frac{3M_t^2}{32\pi^2 l_t^4}$, and $a = a_0 (t/t_0)^{1/4}$, respectively. But, for times much later than $t_c$, i.e., $t > t_c$ (or $\rho << \lambda$), one gets back the standard radiation dominated cosmological evolution given by $\rho_R = \frac{3M_t^2}{32\pi^2 l_t^4}$, and $a = a_0 (t/(t_0 t_c)^1)^{1/2}$.

The formation and evolution of black holes is an interesting and complex issue of investigation in braneworld cosmology [7]. Horizon sized density perturbations in the high energy phase could lead to the formation of black holes on the brane by the process of gravitational collapse. The geometry of such black holes is inherently 5-dimensional [20] with an altered mass-radius relationship given by [21]

$$r_{BH} = \left(\frac{8}{3\pi}\right)^{1/2} \left(\frac{t}{t_4}\right)^{1/2} \left(\frac{M}{M_4}\right)^{1/2} l_4$$

(5)

Taking into account these effects of accretion and evaporation together, the rate of change of mass $M$ of a braneworld black hole is given by [11]

$$\dot{M} = \frac{AM_t^2}{Mt_c} + \frac{BM}{t}$$

(6)

where $A \simeq (3/(16)^3 \pi)$ and $B \simeq (2f/\pi)$. The exact solution for the black hole rate equation was obtained earlier [11]. It was shown that a black hole grows in size by the accretion of radiation during the high energy radiation dominated era, with its mass given by [22], [11]

$$\frac{M(t)}{M_0} \simeq \left(\frac{t}{t_0}\right)^B$$

(7)

The growth of mass decreases in the low energy radiation dominated regime, and subsequently, in the matter dominated era there is a lack of sufficient radiation density for the black hole to accrete. Hence, the black hole mass grows to a maximum value $M_{max}$ at time $t_f$ before evaporation starts dominating. The expression for the lifetime $t_{end}$ of a black hole in this scenario is given by [11]

$$\frac{t_{end}}{t_4} \simeq \frac{4}{A}\left(2\sqrt{2}\right)^B \left(\frac{M_0}{M_4}\right)^{2-B} \frac{t_c}{t_4} \left(\frac{t_4}{t_c t_4}\right)^B$$

(8)

Depending upon the values of the parameters $t_c$ and $l$, one could obtain several examples of primordial black holes surviving till cosmologically interesting eras [11]. For example, a black hole formed with an initial mass $M_0 = 10^8 M_4 \simeq 10^3 g$, survives up to the present era if one chooses $(l/l_4) \simeq 10^{30}$. If the black holes are produced with an initial mass spectrum, then one would have evaporating black holes at different eras. Observational constraints impacting different cosmological impose restrictions on the initial mass spectrum of braneworld black holes. Clancy et al [23] have shown how standard constraints are modified in the case of braneworld cosmology. There have been further studies on the impact of braneworld primordial black holes on the high energy diffuse photon background [24] and cosmic ray antiprotons [25]. It has been also shown how the high energy era dynamics is conducive to the formation of black hole binaries [26].
4. Primordial black holes in scalar-tensor theories

Scalar-tensor models (or Generalized Jordan-Brans-Dicke (JBD) models) are obtained in the low energy limit of higher dimensional theories. The JBD [9] theory is one of the earliest and most well-motivated alternatives of the Einstein theory of gravitation. The value of the gravitational "constant" is set by the inverse of a time-dependent classical scalar field with a coupling parameter \( \omega \). General relativity is recovered in the limit of \( \omega \to \infty \). Solar system observations impose lower bounds on \( \omega \) [27]. Higher dimensional models after compactification of the extra dimensions yield variants of scalar-tensor models in which the the scalar field coupling \( \omega \) may become dynamical.

4.1. Brans-Dicke theory

For a spatially flat \((k = 0)\) FRW universe with scale factor \( a \), the Einstein equations and the equation of motion for the JBD field \( \Phi \) take the form

\[
\frac{a^2}{a^2} + \frac{\dot{a}}{a} \Phi - \frac{\omega \Phi^2}{6 \Phi} = \frac{8\pi \rho}{3\Phi} \tag{9}
\]
\[
2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a} \Phi + \frac{\omega \Phi^2}{2 \Phi} + \frac{\Phi}{\Phi} = -\frac{8\pi p}{\Phi} \tag{10}
\]
\[
\frac{\dot{\Phi}}{8\pi} + \frac{5\dot{\Phi}}{8\Phi} = \rho - 3p + \frac{\rho - p}{2\omega + 3}. \tag{11}
\]

The energy conservation equation is given by

\[
\dot{\rho} + 3(\gamma + 1)H\rho = 0. \tag{12}
\]

Here we have used the perfect fluid equation of state \( p = \gamma \rho \).

Barrow and Carr [14] have obtained the following solutions for \( a \) and \( G \) for different eras, as

\[
a(t) \propto \begin{cases} 
    t^{(1-\sqrt{n})/3} & (t < t_1) \\
    t^{1/2} & (t_1 < t < t_e) \\
    t^{(2-n)/3} & (t > t_e)
\end{cases} \tag{13}
\]

and

\[
G(t) = \begin{cases} 
    G_0 \left( \frac{t}{t_e} \right)^{\sqrt{n}} \left( \frac{t}{t_e} \right)^n & (t < t_1) \\
    G_0 \left( \frac{\omega}{t_e} \right)^n & (t_1 < t < t_e) \\
    G_0 \left( \frac{\omega}{t} \right)^n & (t > t_e)
\end{cases} \tag{14}
\]

where the radiation dominated era lasts from \( t_1 \) to \( t_e \), \( G_0 \) is the present value of \( G \), \( t_0 \) the present time \( n \) is a parameter which related to \( \omega \) by \( n = \frac{2}{4\pi\sqrt{3}\omega} \). Since observations[30] require that \( \omega \) be large (\( \omega \geq 10^4 \)), \( n \) is very small (\( n \leq 0.00007 \)).

Barrow and Carr [14] have considered only evaporation of the primodial black holes due to Hawking radiation. If we consider accretion which is effective in radiation dominated era, then the primordial black holes take more time to evaporate. Here we have studied how accretion changes the life time of the primordial black holes. If we consider both evaporation and accretion simultaneously, then the rate at which the primordial black hole mass changes is given by

\[
\dot{M}_{BH} = 6fG \left( \frac{\dot{a}}{a} \right)^2 M^2 - \frac{a_H}{256\pi^3} \frac{1}{G^2 M^2}. \tag{15}
\]

This equation can not be solved exactly. But we can very well approximate it during different regimes when either accretion or evaporation is the dominant process.
For $t < t_1$ :
The Hawking evaporation rate for this era is given by
\[
\dot{M}_{\text{evap}} = -\alpha \frac{t_e}{t_0} \left( \frac{1}{t_1} \right)^{2\sqrt{\pi}} \left( \frac{t^2\sqrt{\pi}}{M^2} \right) \tag{16}
\]
where $\alpha = \frac{a_M}{2\sqrt{4\pi\omega_0^2}}$. Integrating the above equation, we get
\[
M^3 = M_i^3 + 3\alpha \left( 1 + 2\sqrt{n} \right)^{-1} \left( \frac{t_e}{t_0} \right)^{2n} \left( \frac{t_1}{t_0} \right)^{-2\sqrt{\pi}} \left( t^2\sqrt{\pi} + 1 - t^2\sqrt{\pi} + 1 \right) \tag{17}
\]
where $M_i$ is the black hole mass at its formation time $t_i \simeq G^{-1}(t_i)t_i$. The black hole evaporation time is given by
\[
\tau = \left[ (3\alpha)^{-1}(1 + 2\sqrt{n}) \left( \frac{t_0}{t_e} \right)^{2n} M_i^3 t_1^{2\sqrt{n}} + t_i^{1+2\sqrt{n}} \right]^\frac{1}{1+2\sqrt{n}} \tag{18}
\]

For $t_1 < t < t_e$ :
In the radiation dominated era we consider two possibilities like: PBHs created before $t_1$ and those created after $t_1$.

**CASE-I** ($t_1 < t_1$) :
Again considering both evaporation and accretion, we can write
\[
\dot{M}_{BH} = \left[ -\alpha \left( \frac{1}{t_1} \right)^{2\sqrt{\pi}} \left( \frac{t_e}{t_0} \right)^{2n} \left( \frac{t^2\sqrt{\pi}}{M^2} \right) + \frac{3}{2} f G_0 \left( \frac{t_0}{t_e} \right)^n \left( \frac{M^2}{t^2} \right) - \alpha \left( \frac{t_e}{t_0} \right)^{2n} \left( \frac{1}{M^2} \right) \right] . \tag{19}
\]
The first term of r.h.s. is integrated over the time period $t_i$ to $t_1$ and last two terms of r.h.s. are integrated over the time period $t_1$ to $t$. PBHs formed at time $t_i < t_1$ evaporate out in the radiation dominated era [13].

**CASE-II** ($t_1 > t_1$) :
Taking both accretion and evaporation into account, we can write
\[
\dot{M}_{BH} = \frac{3}{2} f G_0 \left( \frac{t_0}{t_e} \right)^n \left( \frac{M^2}{t^2} \right) - \alpha \left( \frac{t_e}{t_0} \right)^{2n} \left( \frac{1}{M^2} \right) . \tag{20}
\]
In the radiation dominated era for a particular primordial black hole, accretion is dominant upto a value of $t$ at which accretion equals evaporation ($t_c$) and after that evaporation dominates over accretion. Taking this into account we have considered only accretion upto $t_c$ and only evaporation after that. For $t = t_c$, the magnitude of the accretion term is equal to the magnitude of the evaporation term. So we have
\[
\frac{3}{2} f G_0 \left( \frac{t_0}{t_c} \right)^n \left( \frac{M^2}{t^2} \right) = \alpha \left( \frac{t_c}{t_0} \right)^{2n} \left( \frac{1}{M^2} \right) . \tag{21}
\]
which gives
\[
M_{\text{max}} = \left( \frac{A}{f} \right)^\frac{1}{4} \times \left( t_c \right)^{\frac{3}{2}} \tag{22}
\]
where $A = \frac{2}{3} G^{-1} \alpha \left( \frac{t_c}{t_0} \right)^{3n}$ and $M_{\text{max}} = M(t_c)$. But, from the PBH accretion equation, we have
\[
M_{\text{max}} = M_i \left[ 1 + \frac{3}{2} f \left( \frac{t_i}{t_c} - 1 \right) \right]^{-1} . \tag{23}
\]
Equating the above two equations, we get

\[ t_c^{1/2} = \left( \frac{f}{A} \right)^{1/4} \times \frac{M_i}{1 - \frac{3}{2}f} \, . \]

(24)

and hence

\[ M_{\text{max}} = \frac{M_i}{1 - \frac{3}{2}f} \, . \]

(25)

This leads to the condition \( f < \frac{2}{3} \).

Since evaporation dominates over accretion from \( t_c \), we get

\[ M = M_{\text{max}} \left[ 1 + 3\alpha \left( \frac{t_0}{t_0} \right)^{2n} \left( \frac{t_0}{M_{\text{max}}} \right) \left( 1 - \left( \frac{t}{t_c} \right) \right) \right]^{\frac{1}{3}} \, . \]

(26)

So, the lifetime for these PBHs is given by

\[ t_{\text{evap}} = t_c \left[ 1 + (3\alpha)^{-1} \left( \frac{t_0}{t_0} \right)^{2n} \left( \frac{M_{\text{max}}^3}{t_{\text{crit}}} \right) \right] \, . \]

(27)

For the PBHs which do not evaporate out till the end of radiation dominated era, one has

\[ M = M_e \left[ 1 + 3\alpha (2n + 1) \left( \frac{t_0}{t_0} \right)^{2n} \left( \frac{t_0}{M_e} \right) \left( 1 - \left( \frac{t}{t_e} \right)^{2n+1} \right) \right]^{\frac{1}{3}} \, . \]

(28)

where \( M_e = M(t_e) \). Hence, the lifetime becomes

\[ t_{\text{evap}} = t_e \left[ 1 + (3\alpha)^{-1} (2n + 1) \left( \frac{t_0}{t_0} \right)^{2n} \left( \frac{M_e^3}{t_e} \right)^{\frac{1}{2n+1}} \right] \, . \]

(29)

which leads to

\[ M_e = M_{\text{max}} \left[ 1 + 3\alpha \left( \frac{t_e}{t_0} \right)^{2n} \left( \frac{t_e}{M_{\text{max}}^3} \right) \left( 1 - \left( \frac{t}{t_e} \right)^{2n+1} \right) \right]^{\frac{1}{3}} \, . \]

(30)

From the above expression we can calculate the initial masses of the primordial black holes which are evaporating today for different values of the accretion efficiency \( f \). The results are presented in Ref. [13].

For \( t > t_e \):

The primordial black holes which are created before radiation dominated era are completely evaporated during radiation dominated era. So for \( t > t_e \), only those primordial black holes, which are created after \( t_1 \), exist. Considering both accretion and evaporation into account, we can write

\[ \dot{M}_{BH} = \left\{ \frac{3}{2} f G_0 \left( \frac{t_0}{t_e} \right)^n \left( \frac{M^2}{t_e^2} \right) - \alpha \left( \frac{t_0}{t_0} \right)^{2n} \frac{1}{M^2} \right\} - \alpha \left( \frac{t_0}{t_0} \right)^{2n} \left( \frac{t_0^2}{M^2} \right) \, . \]

(31)

The first two term of r.h.s. are integrated over the period \( t_i \) to \( t_e \) and the last term is integrated over period \( t_e \) to \( t \). The above equation can be integrated numerically to yield the lifetimes of PBHs formed at various initial times.
4.2. Scalar-tensor cosmology

Let us now consider a more general scalar-tensor model of gravity. A generalised JBD action is of the form

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [\phi R - \frac{\omega(\phi)}{\phi} (\partial\mu \phi)^2] + S_{\text{matter}} \tag{32}$$

where $S_{\text{matter}}$ corresponds to the action of the relativistic fluid of particles in the early universe. The Friedman equation and the equation of motion for the JBD field $\phi$ admit solutions of the type $[10]$

$$a(t) = a_i \left( \frac{t}{t_i} \right)^{\frac{3}{\omega + 6}} \tag{33}$$

$$\dot{\phi}(t) = \dot{\phi}_i \left( \frac{t}{t_i} \right)^{-\frac{3}{\omega + 6}} \tag{34}$$

$$2\omega(t) + 3 = (2\omega_i + 3) \left( \frac{t}{t_i} \right)^{\frac{2\omega_i + 3}{\omega_i + 6}} \tag{35}$$

where the subscript $i$ indicates the initial values of the variables. In order to minimize the departure from the standard radiation dominated evolution, we choose $|\omega| \ll 1$. Note that the solution for the JBD field $\phi$ indicating the strengthening of gravity with the increase of cosmic time is a generic feature of several generalised JBD or scalar tensor models $[28]$. 

We now consider the evolution of PBHs in the cosmological background governed by the above solutions. The rates of accretion and evaporation by a JBD black hole can be obtained by identifying $r_{BH} = 2M/\phi$ as the black hole radius, and $T = \phi/(8\pi M)$ as the Hawking temperature. The complete evolution for the PBH is described by $[15]$

$$\dot{M} = -\frac{A}{M^2 t} + \frac{BM^2}{t} \tag{36}$$

For PBHs with initial mass $M_i < (A/B)^{1/4}$, the rate of evaporation exceeds that of accretion. Hence, a PBH with initial mass $M_i < (A/B)^{1/4}$ evaporates out quickly. But PBHs with initial mass $M_i > (A/B)^{1/4}$ experience growth, reaching a maximum mass given by $[15]$

$$\frac{M_{\text{max}}}{M_i} \approx \frac{1}{1 - BM_i \ln(t_{eq}/t_i)} \tag{37}$$

The lifetime for the PBHs $t_{\text{evap}}$ is given by

$$t_{\text{evap}} = t_{eq} \left[ 1 - \frac{M_i^3}{3 \left(1 - BM_i \ln(t_{eq}/t_i)\right)^3 \left(A_t M_{pl}^4\right)} \right]^{-3/5} \tag{38}$$

The overall lifetime of a JBD PBH has an interesting comparison to a PBH lifetime in standard cosmology. A JBD PBH with initial mass $M_i < (A/B)^{1/4}$, evaporates out quicker than a PBH in standard cosmology, being unable to accrete in the radiation dominated era. However, when the initial mass exceeds $(A/B)^{1/4}$, accretion proceeds effectively and dominates over evaporation in the radiation dominated era increasing the PBH mass at the time of matter-radiation equality. Certain interesting examples of evolving PBHs in JBD cosmology have been observed $[15]$. A PBH which forms at time $t_{EW}$ with an initial mass of the order of the cosmological horizon mass at $t_{EW}$ satisfies the condition $M_i > (A/B)^{1/4}$, enabling it to survive beyond the present era ($t_{\text{evap}} > t_{\text{now}}$), with the PBH thus being a component of cold dark matter.
5. Conclusions
We have discussed the evolution of primordial black holes in separate scenarios of modified gravity leading to different cosmological evolutions in the early universe. The modified high energy behaviour of the expansion of the Hubble volume in the braneworld scenario makes it feasible for accretion by the primordial black holes of the surrounding radiation to take place. This feature is primarily responsible for the mass growth and prolonged survival [11], [22] of the braneworld black holes. In Brans-Dicke theory, the evolution of primordial black holes due to accretion and evaporation is modified by the variable gravitational constant. This leads to the increase of black hole lifetimes, and consequently the modification of several observational constraints from their evaporation [13]. In the context of scalar-tensor cosmology, we have obtained a cut-off value for the initial mass which decides whether the PBHs could survive today. An interesting consequence of this analysis is that no PBHs evaporate during the matter dominated [15] era, and hence there is no impact on the cosmic microwave background radiation due to evaporating PBHs.

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