Fourth derivative gravity in the auxiliary fields representation and application to the black hole stability

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Abstract. We consider an auxiliary fields formulation for the general fourth-order gravity on an arbitrary curved background. The case of a Ricci-flat background is elaborated in full details and it is shown that there is an equivalence with the standard metric formulation. At the same time, using auxiliary fields helps to make perturbations to look simpler and the results more clear. As an application we reconsider the linear perturbations for the classical Schwarzschild solution. We also briefly discuss the relation to the effect of massive unphysical ghosts in the theory.

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1 Introduction

The long-standing interest to higher derivative gravity theories is based on their critical importance for quantum extensions of gravity. In this case the general fourth order gravity represents a minimal UV completion in the vacuum (purely metric) sector of the theory. Such a theory is renormalizable at both semiclassical [1] (see also [2, 3] for the introduction) and quantum [4] levels. At the same time the higher derivative quantum gravity can not be considered as a completely consistent theory, because its spectrum of states includes unphysical massive ghost
- the spin-2 state with negative kinetic energy. Starting from 70-ies there were several different proposals on how to solve the problem of massive ghosts [4, 5, 6, 7], mainly based on the quantum field theory methods, but all of them did not lead to conclusive results [8]. Very recently one of us proposed a more simple classical way of dealing with ghosts [9]. The new approach to the problem is based on the study of stability of the classical solutions in the low-energy sector of the theory, and in this respect it is similar to the consideration based on the effective quantum field theory ideas [10].

The linear stability of the given solution of some differential equation is usually sufficient for the stability at any perturbative level. Therefore, in order to observe the effect of the spin-2 ghosts one can simply look at the asymptotic behaviour of the gravitational waves on the given background. In the case of cosmological solutions discussed in [9], we could observe that the ghost does not produce instability below the threshold, which approximately corresponds to the Planck scale of frequencies. Only starting from the Planck scale one can observe an explosive-type dynamics of the gravitational waves. The situation is completely different for tachyons, in this case an instability takes place independent on the energy scale. This important difference was recently discussed in [11] and [12].

The stability of the cosmological background due to the huge mass of the ghost [9] is in accordance with the previously known result for the particular deSitter background [13, 14, 15]. In these papers the anomaly-induced semiclassical contributions were taken into account, but finally were shown to be irrelevant [16]. It would be definitely interesting to consider other physically relevant background solutions, in particular for the Schwarzschild metric. During some time the unique study of the stability of the $d = 4$ classical Schwarzschild solution in the presence of higher derivative gravitational terms was the work by Whitt [17]. The main result is that the solution is stable in the case of fourth derivative gravity. Recently there was another paper by Myung, [18], which argued in favor of the opposite result, by using the analogy with the well-known Gregory-Laflamme instabilities [19] (see also [20]). The difference between [17] and [18] does not concern the equations for perturbations, but only their solutions.

In this short report we discuss the procedure of separating Ricci-flat and higher-derivative perturbations of [17]. For this end we employ the method of auxiliary fields, as it was described in a recent paper [21] (see also [22] for standard reviews and further references). In our opinion, the possibility to present the general fourth derivative gravity in the second-order form by means of tensor auxiliary fields has some independent interest. On the top of that we apply this new description to the analysis of metric perturbations on the background of Schwarzschild solution.

2 Standard approach to fourth-derivative perturbations

In this section we summarize the Whitt approach [17], which will be verified in the next section by using auxiliary fields. According to [17], it is possible to explore the stability by means of the second-order equations for the perturbation of the Ricci tensor, instead of the fourth order equations in the metric perturbations. This approach helps to simplify the standard treatment of the stability problem in the theory of the fourth order in derivatives.
Let us consider the gravitational action of the fourth order theory,

\[ S[g_{\mu\nu}] = \int d^4x \sqrt{-g} \left( -\frac{1}{\kappa^2} R - \frac{\omega}{3\lambda} R^2 - \frac{1}{2\lambda} C^2_{\mu\nu\alpha\beta} \right), \]  

where \( C_{\mu\nu\alpha\beta} \) is the Weyl tensor, \( C^2_{\mu\nu\alpha\beta} = E + 2(R^2_{\mu\nu} - 1/3 R^2) \). In this expression we use notations which are typical in quantum gravity, in particular the transition to the notations of Whitt [17] can be performed as follows:

\[ 16\pi G = \kappa^2, \quad \frac{1}{\lambda} = \frac{\beta}{16\pi G}, \quad \frac{1 - \omega}{3\lambda} = \frac{\alpha}{16\pi G}. \]  

Furthermore, we disregard the Gauss-Bonnet term with the integrand \( E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2 \), since it is not relevant at the classical level.

The variation of the action (1) with respect to the metric yields

\[
\frac{1}{\kappa^2} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + g_{\mu\nu} \left[ \frac{1 - \omega}{6\lambda} R^2 - \frac{1}{2\lambda} R^2_{\rho\sigma} + \frac{4\omega - 1}{6\lambda} \Box R \right] - \frac{2\omega + 1}{3\lambda} \nabla_\mu \nabla_\nu R + \frac{1}{\lambda} \Box R_{\mu\nu} + \frac{2}{\lambda} R^\rho_\sigma R_{\rho\mu\sigma\nu} + \frac{2(\omega - 1)}{3\lambda} R R_{\mu\nu} = 0.
\]

Taking trace we arrive at

\[
\frac{1}{\kappa^2} R + \frac{2\omega}{\lambda} \Box R = 0.
\]

According to this equation, the scalar curvature obeys the covariant minimal Klein-Gordon equation. Furthermore, in case of \( \omega = 0 \) the last formula boils down to the non-dynamical constraint on the scalar curvature \( R = 0 \). The choice \( \omega = 0 \) corresponds to the case of a pure Weyl-squared higher derivative part in Eq. (1). The higher derivative part of the action possess local conformal invariance, the symmetry which is violated only by the Einstein-Hilbert term.

Equation (3) shows that any vacuum solution of Einstein equations is also a solution for the theory of the fourth order (1). For the perturbations this may be not true, but in the case when the background metric satisfies vacuum Einstein equations, they can be classified into the two distinct classes:

- The perturbations which satisfy Einstein’s equations in vacuum, \( R_{\mu\nu}(g_{\alpha\beta} + h_{\alpha\beta}) = 0 \).
- The ones which do not satisfy these equations.

In the first case we have exactly the same problem as in Einstein’s theory because

\[
R_{\mu\nu}(g_{\alpha\beta} + h_{\alpha\beta}) = R_{\mu\nu}(g_{\alpha\beta}) + R^{(1)}_{\mu\nu}(h_{\alpha\beta}) = R^{(1)}_{\mu\nu}(h_{\alpha\beta}) = 0,
\]

where \( R^{(1)}_{\mu\nu} \) is the first order expansion of the Ricci tensor, which will be specified below.

This ensures, for example, the stability of the Schwarzschild metric, as it was discussed in [23, 24, 25]. In the second case one can consider the perturbation of the Ricci tensor,

\[
R'_{\mu\nu} = R_{\mu\nu} + R^{(1)}_{\mu\nu}.
\]
Substituting (6) into (3) and remembering that $R_{\mu \nu} (g_{\alpha \beta}) = 0$, we obtain the equation for the linear perturbations of the Ricci tensor

$$
\frac{1}{\kappa^2} \left[ R_{\mu \nu}^{(1)} - \frac{1}{2} g_{\mu \nu} R^{(1)} \right] + \frac{4 \omega - 1}{6 \lambda} g_{\mu \nu} \Box R^{(1)} - \frac{2 \omega + 1}{6 \lambda} \nabla_\mu \nabla_\nu R^{(1)} + \frac{1}{\chi} \Box R_{\mu \nu}^{(1)} + \frac{2}{\lambda} R^{(1) \rho \sigma} R_{\mu \sigma \nu \rho} = 0 .
$$

(7)

where we use notation $R^{(1)}_{\mu \nu} = g_{\mu \nu} R^{(1)}_{\mu \nu}$. Let us note that in the Ricci-flat case $R^{(1)}_{\mu \nu}$ is also the first order term in the expansion of the curvature of the second type.

Taking the trace of (7) we get

$$
\frac{1}{\kappa^2} R^{(1)} + \frac{2 \omega}{\lambda} \Box R^{(1)} = 0 ,
$$

(8)

which is the linearized form of Eq. (4). For $\omega = 0$ this gives the constraint $R^{(1)} = 0$.

In what follows we shall limit our analysis to this particular case. So eq. (7) becomes

$$
\Box R_{\mu \nu}^{(1)} + 2 R_{\tau \mu \lambda \nu} R^{(1) \tau \lambda} + \frac{\lambda}{\kappa^2} R_{\mu \nu}^{(1)} = 0 .
$$

(9)

Eq. (9) determines the dynamics of small gravitational perturbations. It turns out that the stability of the background Schwarzschild metrics in the fourth order theory in $d = 4$ is defined by the same equations as in the black string case in $d = 5$ space-time dimensions [19] and as in the bi-metric theory of gravity [20]. As it was noted in [18], in these cases one meets a well-known Gregory-Laflamme instability [19].

3 Using auxiliary fields

Let us consider the action with auxiliary fields $\phi_{\mu \nu}$,

$$
S_2 = \int d^4 x \sqrt{-g} \left\{ - \frac{1}{\kappa^2} R + \phi_{\mu \nu} R^{\mu \nu} + \xi \frac{1}{2} \phi_{\mu \nu} (A^{-1})^{\mu \nu, \alpha \beta} \phi_{\alpha \beta} \right\} ,
$$

(10)

where $A^{\mu \nu, \alpha \beta}$ is an invertible $c$-number operator depending only on the metric. The variation of this action with respect to the field $\phi_{\mu \nu}$ gives

$$
\phi_{\alpha \beta} = - \frac{1}{\xi} A_{\alpha \beta, \mu \nu} R^{\mu \nu} .
$$

(11)

After using the last relation, the action $S_2$ on-shell becomes

$$
S_2 = \int d^4 x \sqrt{-g} \left( - \frac{1}{\kappa^2} R - \frac{1}{2 \xi} R^{\mu \nu} A_{\mu \nu, \alpha \beta} R^{\alpha \beta} \right) .
$$

(12)

Comparing (1) with (12) we obtain

$$
\xi = \frac{\lambda}{2} , \quad A_{\mu \nu, \alpha \beta} = \delta_{\mu \nu, \alpha \beta} - 7 g_{\mu \nu} g_{\alpha \beta} ,
$$

(13)

1For different values of $\omega$, the trace free part and the trace part of $R_{\mu \nu}^{(1)}$ do not decouple leading to a fourth order equation for $\tilde{R}_{\mu \nu}^{(1)}$, where $\tilde{R}_{\mu \nu}^{(1)} = R_{\mu \nu}^{(1)} - \frac{1}{2} g_{\mu \nu} R^{(1)}$. 

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where \( \gamma = (1 - \omega)/3 \). Here we use the standard DeWitt notation
\[
\delta_{\mu\nu,\alpha\beta} = \frac{1}{2}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}).
\] (14)

The operator defined by (13) does not have an inverse if \( \gamma = 1/4 \), that is for \( \omega = 1/4 \). Let us simply assume \( \gamma \neq 1/4 \). Then the inverse operator can be easily obtained with
\[
A_{\mu\nu,\alpha\beta}^{-1} A^{\alpha\beta,\tau\lambda} = \delta_{\mu\nu,\tau\lambda},
\] (15)
and one can check that
\[
A_{\mu\nu,\alpha\beta}^{-1} = \delta_{\mu\nu,\alpha\beta} - \theta g_{\mu\nu}g_{\alpha\beta},
\] (16)
where
\[
\theta = \frac{1 - \omega}{1 - 4\omega}.
\] (17)

In this way we arrive at the explicit form of Eq. (10),
\[
S_2 = \int d^4x \sqrt{-g} \left\{ -\frac{1}{\kappa^2} R + \phi_{\mu\nu}R^{\mu\nu} + \frac{\lambda}{4} \phi_{\mu\nu}(\delta^{\mu\nu,\alpha\beta} - \theta g^{\mu\nu}g^{\alpha\beta})\phi_{\alpha\beta} \right\}.
\] (18)

It is natural to check whether the auxiliary field \( \phi_{\alpha\beta} \) satisfied some constraints. First of all, without losing generality we can assume that \( \phi_{\alpha\beta} \) is a symmetric space-time tensor. Furthermore, one can use Eqs. (11) and (13) to derive the Bianchi identity for this field in the form
\[
\nabla_{\mu}\phi^\mu_{\nu} = \left( \frac{1}{2} - \theta \right) \nabla_{\nu}\phi, \quad \text{where} \quad \phi = \phi^{\mu\nu}g_{\mu\nu}.
\] (19)

There is a complicated question on whether the auxiliary field \( \phi_{\alpha\beta} \) should satisfy the constraint (19) at the quantum level, but since in the present work our attention is restricted to the purely classical case, this relation should be respected.

The action (18) describes a dynamical theory of the two symmetric tensor fields \( g_{\mu\nu} \) and \( \phi_{\mu\nu} \). It is important to note that the auxiliary field is dynamical, because variation with respect to the metric produce second derivatives of \( \phi_{\mu\nu} \) in the equations of motion. The action (18) is second-derivative, but it is dynamically equivalent to the original fourth-derivative action (1). This equivalence means one can map any solution of one theory to some solution of another one and vice versa. Indeed, this is true for both background and perturbed solutions, so the action (18) can be useful to explore the perturbations.

In order to illustrate how it works, let us expand this action up to the second order
\[
S_2 = S_2^{(0)} + S_2^{(1)} + S_2^{(2)}
\] (20)
in the perturbations of both metric and auxiliary field. The expansions of the fields are defined as
\[
g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu},
\] (21)
\[
\phi_{\mu\nu} \rightarrow \bar{\phi}_{\mu\nu} = \phi_{\mu\nu} + \psi_{\mu\nu},
\] (22)

\(^2\)In the special case \( \gamma = 1/4 \) the formulation via auxiliary tensor field is possible, but this field must be traceless. It is not difficult to elaborate this version, but there is no much practical interest to do it.
In what follows we shall also use notations for the traces $h = h_{\mu}^{\mu} = h_{\mu\nu}g^{\mu\nu}$ and $\psi = \psi_{\mu}^{\mu}$. One can easily check that the first terms in the expansions $R_{\mu\nu}^{(1)}$ and $\psi_{\mu\nu}$ satisfy the same Bianchi identities as the background quantities,

$$\nabla_{\mu}R_{\nu\lambda}^{(1)\mu} = \frac{1}{2} \nabla_{\nu}R_{\lambda\mu}^{(1)} \quad \text{and} \quad \nabla_{\mu}\psi_{\nu}^{\mu} = \left(\frac{1}{2} - \theta\right) \nabla_{\nu}\psi. \quad (23)$$

Other relevant quantities are expanded to the second order as follows:

$$g^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_{\lambda}^{\nu} + \ldots,$$

$$\sqrt{-g} = \sqrt{-g}\left(1 + \frac{1}{2}h - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta} + \frac{1}{8}h^{2} + \ldots\right).$$

Furthermore, the first-order expansion for the Ricci tensor is

$$R_{\mu\nu}^{(1)} = -\frac{1}{2} \Box h_{\mu\nu} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} h + \frac{1}{2} \nabla_{\lambda} \nabla_{\mu} h_{\nu}^{\lambda} + \frac{1}{2} \nabla_{\lambda} \nabla_{\nu} h_{\mu}^{\lambda} \quad (24)$$

and the second-order term is

$$R_{\mu\nu}^{(2)} = \frac{1}{2} \eta^{\lambda}(\nabla_{\tau} \nabla_{\lambda} h_{\mu\nu} + \nabla_{\mu} \nabla_{\nu} h_{\tau\lambda} - \nabla_{\lambda} \nabla_{\nu} h_{\tau\mu} - \nabla_{\lambda} \nabla_{\mu} h_{\tau\nu})$$

$$+ \frac{1}{2} (\nabla_{\tau} h_{\tau\lambda} - \frac{1}{2} \nabla^{\lambda} h) (\nabla_{\lambda} h_{\mu\nu} - \nabla_{\mu} h_{\lambda\nu} - \nabla_{\nu} h_{\lambda\mu})$$

$$+ \frac{1}{4} \nabla_{\mu} h_{\tau\lambda} \nabla_{\nu} h_{\tau\rho} + \frac{1}{2} \nabla_{\tau} h_{\lambda\mu} \nabla^{\tau} h_{\nu}^{\lambda} - \frac{1}{2} \nabla_{\tau} h_{\mu}^{\lambda} \nabla_{\lambda} h_{\nu}^{\tau} \quad (25)$$

The Ricci scalar in the first order is

$$R^{(1)} = \nabla_{\mu} \nabla_{\nu} h^{\mu\nu} - \Box h - R_{\mu\nu} h^{\mu\nu}, \quad (26)$$

and in the second-order we meet

$$R^{(2)} = h^{\mu\nu}(\Box h_{\mu\nu} + \nabla_{\mu} \nabla_{\nu} h - \nabla_{\mu} \nabla_{\nu} h_{\lambda}^{\lambda} - \nabla_{\lambda} \nabla_{\nu} h_{\mu}^{\lambda}) + \nabla_{\mu} h \nabla_{\nu} h^{\mu\nu} - \nabla_{\mu} h^{\mu\nu} \nabla_{\lambda} h_{\nu}^{\lambda}$$

$$- \frac{1}{4} \nabla_{\lambda} h \nabla_{\mu} h_{\mu}^{\lambda} - \frac{1}{4} \nabla_{\lambda} h \nabla_{\nu} h_{\nu}^{\lambda} - \frac{1}{2} \nabla_{\lambda} h_{\mu\nu} \nabla_{\mu} h_{\nu}^{\lambda} + R_{\mu\nu} h_{\mu\lambda} h_{\nu}^{\lambda} \quad (27)$$

Replacing these formulas into the action $S_{2}$ (18), we obtain the first-order expansion

$$S_{2}^{(1)} = \int d^{4}x \sqrt{-g} \left\{ R_{\mu\nu}^{\mu\nu} \psi_{\mu\nu} + \frac{h}{2} \left[-\frac{1}{\kappa^{2}} R + \phi_{\mu\nu} R_{\mu\nu}^{\mu\nu} + \frac{\lambda}{4} (\phi_{\mu\nu} \phi^{\mu\nu} - \theta \phi^{2}) \right] \right.$$ 

$$- 2 \phi_{\mu\alpha} R_{\nu}^{\alpha\mu} h_{\mu\nu} - \frac{\lambda}{2} \phi_{\mu\alpha} \phi_{\nu}^{\alpha\mu} h^{\mu\nu} + \frac{\lambda}{2} \theta \phi_{\mu\nu} h^{\mu\nu} + \frac{\lambda}{2} (\phi_{\mu\nu} \psi_{\mu\nu} - \theta \psi)$$

$$- \frac{1}{\kappa^{2}} (\nabla_{\mu} \nabla_{\nu} h_{\mu\nu} - \Box h - R_{\mu\nu} h_{\mu\nu}) + \phi_{\mu\nu} (\nabla_{\lambda} \nabla_{\mu} h_{\nu}^{\lambda} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} h - \frac{1}{2} \Box h_{\mu\nu}) \right\}. \quad (28)$$
Finally, the second-order term is

\[ S_2^{(2)} = \int d^4x \sqrt{-g} \left\{ \frac{\lambda}{4} \phi_{\mu\nu} \phi^{\mu\nu} - \theta \phi^2 - 2\psi_{\mu}^{\nu} R_{\mu\nu} h^{\beta\nu} - \lambda \phi_{\mu\nu} \psi^{\mu\nu} + \phi^{\mu\nu} R_{\mu\nu} \right\} + \frac{\lambda}{2} \left( \phi \phi_{\alpha\beta} h^{\alpha\beta} + \psi \phi_{\alpha\beta} h^{\alpha\beta} \right) + \frac{h}{2} \left( \psi^{\mu\nu} R_{\mu\nu} + \frac{\lambda}{2} \left( \phi^{\mu\nu} \psi_{\mu\nu} - \theta \phi \psi \right) + \phi^{\mu\nu} R_{\mu\nu}^{(1)} \right) + \left( \psi_{\mu\nu} - 2\phi_{\beta}^{\nu} h^{\beta\nu} \right) R_{\mu\nu}^{(1)} + \left( \phi_{\alpha\beta} h^{\mu\nu} h^{\nu\beta} + 2\psi_{\beta}^{\nu} h^{\beta\nu} \lambda - \phi_{\beta}^{\nu} h^{\nu\beta} h \right) R_{\mu\nu} - \frac{\lambda}{4} h (\phi_{\mu\nu} \phi_{\beta}^{\mu} h^{\nu\beta} - \theta \phi_{\alpha\beta} h^{\alpha\beta}) - \frac{1}{\kappa^2} \left[ R_{\mu\nu}^{(2)} + \frac{1}{2} h R_{\mu\nu}^{(1)} \right] + \frac{\phi_{\mu\nu} \phi_{\lambda\beta}}{4} \left[ \frac{\lambda}{4} h_{\mu\nu} h^{\nu\beta} \right. \right.
\left. + \frac{\lambda}{2} g_{\mu\lambda} h^{\nu\lambda} h^{\beta} - \frac{\lambda \theta}{4} h_{\mu\nu} h^{\alpha\beta} - \frac{\lambda \theta}{2} g^{\mu\nu} h^{\alpha\lambda} h_{\lambda\beta} \right] + \left( \frac{1}{8} h^2 - \frac{1}{4} h_{\alpha\beta} h^{\alpha\beta} \right) \left[ \phi_{\mu\nu} R_{\mu\nu} + \frac{\lambda}{4} \left( \phi_{\mu\nu} \phi^{\mu\nu} - \theta \phi^2 \right) - \frac{1}{\kappa^2} R \right]. \]  

(29)

The expression (29) contains the complete equations for the perturbations. However, for the sake of simplicity we assume that the background metric satisfies Einstein equations in vacuum, such that we take \( R_{\mu\nu} = 0 \) and \( \phi_{\mu\nu} = 0 \). After discarding surface terms, we obtain the simplified expression

\[ S_2^{(2)} = \int d^4x \sqrt{-g} \left\{ \frac{\lambda}{4} \psi_{\alpha\beta} (\delta_{\mu\nu} h^{\alpha\beta} - \theta g^{\mu\nu} g^{\alpha\beta}) \right\} \]

\[ + \psi_{\mu\nu} \left[ - \frac{1}{2} h_{\mu\nu} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} h + \nabla_{\mu} \nabla_{\nu} h^{\beta} - R_{\mu\nu\beta} h^{\alpha\beta} \right] \]

\[ + \frac{1}{2 \kappa^2} h^{\mu\nu} \left[ \frac{1}{2} g_{\mu\nu} \nabla^{2} - \nabla_{\mu} \nabla_{\nu} h^{\beta} \right. \]  

\[ \left. - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} h^{\beta} - \frac{1}{2} \nabla_{\mu} h_{\nu\beta} - g_{\mu\nu} \nabla_{\beta} h^{\alpha\beta} - R_{\mu\nu\beta} h^{\alpha\beta} \right]. \]  

(30)

The equations that determine the dynamics of the perturbations have the form

\[ - \frac{\lambda}{2} (\delta_{\mu\nu} h^{\alpha\beta} - \theta g_{\mu\nu} g^{\alpha\beta}) \psi^{\alpha\beta} = R_{\mu\nu}^{(1)} \]  

(31)

and

\[ \frac{1}{\kappa^2} h^{\mu\nu} \right. \]  

\[ \left. \nabla_{\lambda} \psi^{\lambda} \nabla_{\mu} \nabla_{\nu} \right. \]

\[ + \nabla_{\mu} \nabla_{\lambda} \psi^{\lambda} \nabla_{\nu} \psi^{\mu} - g_{\mu\nu} \nabla_{\alpha} \nabla_{\beta} \psi^{\alpha\beta} - 2R_{\mu\nu\beta} \psi^{\alpha\beta} = 0. \]  

(32)

By means of relations (23) and (26) the last equation can be presented in the form

\[ \frac{1}{\kappa^2} \left[ R_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu} R^{(1)} \right] = \frac{1}{2} \nabla^{2} \psi_{\mu\nu} + R_{\mu\nu\lambda\beta} \psi^{\alpha\beta} + \left( \theta - \frac{1}{2} \right) \left( \nabla_{\mu} \nabla_{\nu} \psi - \frac{1}{2} g_{\mu\nu} \nabla^{2} \psi \right). \]  

(33)

Inverting Eq. (31) and inserting it into Eq. (33) one can reproduce Eq. (7). For the conformal fourth-derivative case, when \( \omega = 0 \), the equation is

\[ \nabla^{2} \psi_{\mu\nu} + R_{\mu\nu\lambda\beta}^{(1)} \psi^{\alpha\beta} + \left( \theta - \frac{1}{2} \right) \left( \nabla_{\mu} \nabla_{\nu} \psi - \frac{1}{2} g_{\mu\nu} \nabla^{2} \psi \right) = 0. \]  

(34)

where we used the constraint \( R_{\mu\nu}^{(1)} = 0 \) (consequently \( \psi = 0 \) too). Furthermore, one can impose the \( TT \)-gauge on the metric fluctuations,

\[ h = 0, \quad \nabla_{\mu} h^{\mu\nu} = 0, \]  

(35)
so that Eq. (24) which relates metric and Ricci fluctuations becomes

$$\Box h_{\mu\nu} + 2R_{\tau\mu\lambda\nu} h^{\tau\lambda} + 2R^{(1)}_{\mu\nu} = 0 ,$$

(36)

or, in terms of the auxiliary field,

$$\Box h_{\mu\nu} + 2R_{\tau\mu\lambda\nu} h^{\tau\lambda} = \lambda \psi_{\mu\nu} .$$

(37)

In the last formula we took into account the constraint $\psi = 0$ which holds in the case $\omega = 0$. The same constraint, together with the TT-gauge (35), reduce the Eq. (32) to the relation

$$\Box h_{\mu\nu} + 2R_{\mu\alpha\nu\beta} h^{\alpha\beta} = -\kappa^2 \left( \Box \psi_{\mu\nu} + 2R_{\mu\alpha\nu\beta} \psi^{\alpha\beta} \right).$$

(38)

Taken together, Eqs. (37) and (38) lead to the massive equation for the auxiliary field alone,

$$\Box \psi_{\mu\nu} + 2R_{\mu\alpha\nu\beta} \psi^{\alpha\beta} + \frac{\lambda}{\kappa^2} \psi_{\mu\nu} = 0 .$$

(39)

4 Brief discussion

In the previous section we have shown that the two descriptions of perturbations are indeed equivalent. Let us see how one can apply the auxiliary fields description for the black-hole background case.

Starting from equations (34) and (36) one can define a new field

$$\sigma_{\mu\nu} = \frac{1}{\kappa^2} h_{\mu\nu} + \psi_{\mu\nu} = \frac{1}{\kappa^2} h_{\mu\nu} - \frac{2}{\lambda} R^{(1)}_{\mu\nu} ,$$

(40)

which satisfies the massless equation (on a Ricci-flat background)

$$\Box \sigma_{\mu\nu} + 2R_{\mu\alpha\nu\beta} \sigma^{\alpha\beta} = 0 .$$

(41)

This is exactly the equation for the Ricci-flat perturbations in Einstein theory, which were explored by Regge, Wheeler et al in [23, 24, 25]. It is known to have no unstable modes. Therefore, as far as unstable modes are concerned we can set $\sigma_{\mu\nu} = 0$, i.e. $\psi_{\mu\nu} = -h_{\mu\nu}/\kappa^2$, making Eq. (36) to take the same form as Eq. (34).

In order to establish the relation between massive ghosts and instabilities, one has to note that the ghost with the mass of the Planck order of magnitude does not produce instabilities in flat space or in a very weak gravitational field [11, 10]. The practical consequence for the black hole case should be a selection rule for the boundary conditions of the perturbations at space infinity. Neglecting the oscillatory part, we can impose the requirement for those perturbations which are related to the ghost instabilities that $\psi_{\mu\nu}$, $h_{\mu\nu}$ vanish for large values of $r$, where the background solution approaches Minkowski space. Only those solutions for the perturbations should be permitted, if we intend to estimate the effect of massive ghost on the instabilities on the black-hole background.
The stability of the perturbations is completely defined by eq. (34). This equation is known to possess a spherically symmetric unstable mode provided

\[ 0 < \frac{1}{\sqrt{\beta}} = \frac{\sqrt{\lambda}}{\kappa} < \frac{O(1)}{r_S}, \]

where \( r_S \) is Schwarzschild radius [19]. Such a mode grows exponentially in time as \( e^{\Omega t}, \Omega > 0 \), spatially vanishes at infinity and is regular at the future horizon. This mode is present in both \( R^{(1)}_{\mu \nu} \) (as also noted in [18]) and \( h_{\mu \nu} \).

The Gregory-Laflamme unstable mode exponentially decays in the radial direction, leading to

\[ (h_{\mu \nu}, \psi_{\mu \nu}) \sim e^{\Omega t - r\sqrt{\Omega^2 + \lambda/\kappa^2}}. \]  

Eq. (42) tells us that in any given finite space point at some instant of time the perturbations become large and then show an unrestricted growth. According to our previous considerations, one can consider this as an indication that the Gregory-Laflamme instability is not related to the presence of massive ghosts in the spectrum. Finally, we note that the end-state of this instability is thought to be a black hole with a massive graviton halo \(^3\).

5 Conclusions

We presented a new form of the action and of equations of motion for the higher derivative gravity, which uses auxiliary fields on an arbitrary curved space-time background. Previously, the same representation has been known in flat space [27] and for the special de Sitter background [28]. The main result is the eq. (29) from which one can easily extract the full set of equations in the physically interesting cases or approximations. For the sake of simplicity, we elaborate only the perturbations for the Ricci-flat background and arrive the coupled set of equations for metric and auxiliary field perturbations.

As an application we consider linear perturbations around the static spherically symmetric black hole. The use of auxiliary tensor field enables one to distinguish the perturbations related to the massive unphysical ghost which is present in the spectrum of the theory. As a result we can see that these perturbations can not be identified with the Gregory-Laflamme instability.

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\(^3\)In the context of massive gravity a numerical analysis [26] showed that for graviton mass \( \sim 1/r_S \) likely candidates are black holes with massive graviton halo, but none have been found for smaller masses.
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