Black hole quantum spectrum

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Abstract

Introducing a black hole (BH) effective temperature, which takes into
account both the non-strictly thermal character of Hawking radiation and
the countable behavior of emissions of subsequent Hawking quanta, we re-
cently re-analysed BH quasi-normal modes (QNMs) and interpreted them
naturally in terms of quantum levels. In this work we improve such an
analysis removing some approximations that have been implicitly used in
our previous works and obtaining the corrected expressions for the formu-
las of the horizon’s area quantization and the number of quanta of area
and hence also for Bekenstein-Hawking entropy, its sub-leading corrections
and the number of micro-states, i.e. quantities which are fundamental to
realize the underlying quantum gravity theory, like functions of the QNMs
quantum “overtone” number $n$ and, in turn, of the BH quantum excited
level. An approximation concerning the maximum value of $n$ is also cor-
rected. On the other hand, our previous results were strictly corrected
only for scalar and gravitational perturbations. Here we show that the
discussion holds also for vector perturbations.

The analysis is totally consistent with the general conviction that BHs
result in highly excited states representing both the “hydrogen atom” and
the “quasi-thermal emission” in quantum gravity. Our BH model is some-
what similar to the semi-classical Bohr’s model of the structure of a hy-
drogen atom.

The thermal approximation of previous results in the literature is con-
sistent with the results in this paper. In principle, such results could also
have important implications for the BH information paradox.

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1 Introduction

The non-strictly thermal spectrum by Parikh and Wilczek \cite{1,2} of Hawking radiation \cite{3} implies that emissions of subsequent Hawking quanta are countable \cite{4, 5, 17, 20, 21, 22, 23, 24}, and, in turn, generates a natural correspondence between Hawking radiation and BH QNMs \cite{4, 5, 17}, permitting to naturally interpret QNMs as quantum levels \cite{4, 5, 17}. In fact, Parikh and Wilczek \cite{1, 2} obtained their important result in the tunnelling framework, which has been an elegant and largely used approach to obtain Hawking radiation in recent years, see for example \cite{1, 2, 19, 28, 29, 30, 31} and refs. within. In the tunnelling framework, Hawking’s mechanism of particles creation by BHs \cite{3} can be described as tunnelling arising from vacuum fluctuations near the BH horizon \cite{1, 2, 19, 28, 29, 30, 31}. If a virtual particle pair is created just inside the horizon, the virtual particle with positive energy can tunnel out. Then, it materializes outside the BH as a real particle. Analogously, if a virtual particle pair is created just outside the horizon, the particle with negative energy can tunnel inwards. In both of the situations, the BH absorbs the particle with negative energy. The absorptions of such particles with negative energy are exactly the perturbations generating the QNMs. The result will be that the BH mass decreases and the particle with positive energy propagates towards infinity. Thus, subsequent emissions of quanta appear as Hawking radiation. QNM can be naturally considered in terms of quantum levels if one interprets the absolute value of a QNM frequency as the total energy emitted at that level \cite{4, 5, 17}. In other words, QNMs frequencies are the eigenvalues of the system. The Hawking quanta are then interpreted as the “jumps” among the levels. This key point agrees with the idea that, in an underlying quantum gravity theory, BHs result in highly excited states.

We recently used this important issue to re-analyse the spectrum of BH QNMs through the introduction of a BH effective temperature \cite{4, 5}. In our analysis the formula of the horizon’s area quantization and the number of quanta of area resulted to be functions of the quantum QNMs “overtone” number \(n\) \cite{4, 5}, i.e. of the BH quantum level. Consequently, Bekenstein-Hawking entropy, its sub-leading corrections and the number of micro-states resulted functions of \(n\) too \cite{4, 5}.

Here the analysis is improved, removing some approximations that have been implicitly used in previous works \cite{4, 5} and obtaining the corrected expressions for the cited formulas like functions of \(n\). An approximation concerning the maximum value of \(n\) is also corrected. As our previous results \cite{4, 5} were strictly corrected only for scalar and gravitational perturbations, in this work we show that the analysis holds also for vector perturbations.

We shortly discuss potential important implications also for the BH information paradox \cite{18}.

The analysis in this paper is totally consistent with the general conviction that BHs result in highly excited states representing both the “hydrogen atom” and the “quasi-thermal emission” in quantum gravity. Previous results in the literature \cite{6, 14, 15}, obtained in strictly thermal approximation, are consistent
with the results in this paper.

At the present time, we do not yet have a full theory of quantum gravity. Thus, we have to content with the semi-classical approximation. In fact, as for large $n$ Bohr’s correspondence principle \[25, 26, 27\] holds, such a semi-classical description is adequate. In this framework, our BH model is somewhat similar to the semi-classical Bohr’s model of the structure of a hydrogen atom \[26, 27\]. In our BH model, during a quantum jump a discrete amount of energy is radiated and for large values of the principal quantum number $n$ the analysis becomes independent from the other quantum numbers. In a certain sense, QNMs represent the "electron" which jumps from a level to another one and the absolute values of the QNMs frequencies represent the energy "shells". In Bohr’s model \[26, 27\], electrons can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation with an energy difference of the levels according to the Planck relation $E = hf$, where $h$ is the Planck constant and $f$ the transition frequency. In our BH model, QNMs can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels. On the other hand, Bohr’s model is an approximated model of the hydrogen atom with respect to the valence shell atom model of full quantum mechanics. In the same way, our model should be an approximated model of the emitting BH with respect to the definitive, but at the present time unknown, model of full quantum gravity theory.

2 Quasi-normal modes in non-strictly thermal approximation

Working with $G = c = k_B = \hbar = \frac{1}{4\pi\alpha_0} = 1$ (Planck units), in strictly thermal approximation the probability of emission of Hawking quanta is \[1, 2, 3\]
\[
\Gamma \sim \exp(-\frac{\omega}{T_H}),
\]
where $T_H = \frac{1}{8\pi M}$ is the Hawking temperature and $\omega$ the energy-frequency of the emitted radiation.

The important correction, due to the BH varying geometry yields \[1, 2\]
\[
\Gamma \sim \exp[-\frac{\omega}{T_H}(1 - \frac{\omega}{2M})].
\]

This result takes into account the BH back reaction and adds the term $\frac{\omega}{2M}$ like correction \[1, 2\]. In a recent paper \[19\] we have improved the tunnelling picture in \[1, 2\]. In fact, we have shown that the probability of emission \[2\] is indeed associated to the two distributions \[19\]
\[
<n>_{\text{boson}} = \frac{1}{\exp[-4\pi n (M - \omega) \omega] - 1}, \quad <n>_{\text{fermion}} = \frac{1}{\exp[-4\pi n (M - \omega) \omega] + 1},
\]
for bosons and fermions respectively, which are non strictly thermal.
As in various fields of physics and astrophysics the deviation of the spectrum of an emitting body from the strict thermality is taken into account by introducing an effective temperature (i.e. the temperature of a black body emitting the same total amount of radiation) we introduced the effective temperature in BH physics too [4, 5].

\[
T_E(\omega) \equiv \frac{2M}{2M - \omega}T_H = \frac{1}{4\pi(2M - \omega)}. \tag{4}
\]

One sees that \(T_E\) depends on the frequency of the emitted radiation. Therefore, one can rewrite eq. (2) in Boltzmann-like form as [4, 5]

\[
\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp\left(-\frac{\omega}{T_E(\omega)}\right), \tag{5}
\]

where \(\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}\) and \(\exp[-\beta_E(\omega)\omega]\) is the effective Boltzmann factor appropriate for a BH with inverse effective temperature \(T_E(\omega)\) [4, 5]. The deviation of the BH radiation spectrum from the strict thermality is given by the ratio \(\frac{T_E(\omega)}{T_H} = \frac{2M}{2M - \omega}\) [4, 5]. The introduction of \(T_E(\omega)\) permits the introduction of the effective mass and of the effective horizon [4, 5]

\[M_E \equiv M - \frac{\omega}{2}, \ r_E \equiv 2M_E \tag{6}\]

of the BH during the emission of the particle, i.e. during the BH contraction phase [4, 5]. They are average values of the mass and the horizon before and after the emission [4, 5].

We have shown that the correction to the thermal spectrum is also very important for the physical interpretation of BH QNMs, which, in turn, is very important to realize the underlying quantum gravity theory as BHs represent theoretical laboratories for developing quantum gravity and BH QNMs are the best candidates like quantum levels [4, 5, 6].

QNMs are radial spin-\(j\) perturbations (\(j = 0, 1, 2\) for scalar, vector and gravitational perturbation respectively) of the Schwarzschild background usually labelled as \(\omega_{nl}\), being \(l\) the angular momentum quantum number [4, 5, 6, 7]. For each \(l \geq 2\) for BH perturbations, there is a countable sequence of QNMs, labelled by the “overtone” number \(n\) (\(n = 1, 2, \ldots\)) [4, 5, 7]. For large \(n\) the QNMs of the Schwarzschild BH become independent of \(l\). In strictly thermal approximation and for scalar and gravitational perturbations their frequencies are given by [4, 5, 6, 7]

\[
\omega_n = \ln 3 \times T_H + 2\pi i(n + \frac{1}{2}) \times T_H + O(n^{-\frac{3}{2}}) = \\
= \frac{\ln 3}{8\pi M} + \frac{2\pi i}{8\pi M}(n + \frac{1}{2}) + O(n^{-\frac{3}{2}}). \tag{7}
\]

The introduction of the effective temperature \(T_E(\omega)\) is useful also to analyse BH QNMs [4, 5]. An important issue is that eq. (4) is an approximation as it assumes that the BH radiation spectrum is strictly thermal. If one wants to take into account the deviation from the thermal spectrum the Hawking temperature \(T_H\) must be replaced by the effective temperature \(T_E\) in eq. (7) [4, 5]. Thus, for scalar and gravitational perturbations the correct expression for
the Schwarzschild BH QNMs, which takes into account the non-strict thermality of the spectrum is \[4, 5\]

\[
\omega_n = \ln 3 \times T_E(\omega_n) + 2\pi i(n + \frac{1}{2}) \times T_E(\omega_n) + \mathcal{O}(n^{-\frac{1}{2}}) = \\
= \frac{\ln 3}{4\pi[2M - (\omega_n)]} + \frac{2\pi i}{4\pi[2M - (\omega_n)]}(n + \frac{1}{2}) + \mathcal{O}(n^{-\frac{1}{2}}) \simeq \frac{2\pi i n}{4\pi[2M - (\omega_n)]},
\]

where \((\omega_0)_n \equiv |\omega_n|\). We derived eq. (8) in \[4, 5\]. A more rigorous derivation of it can be found in detail in the Appendix of this paper. In that Appendix we also show that the behavior

\[
\omega_n \simeq \frac{2\pi i n}{4\pi[2M - (\omega_0)_n]}
\]

also holds for \(j = 1\) (vector perturbations) and this is in full agreement with Bohr’s correspondence principle \[25, 26, 27\] which states that “transition frequencies at large quantum numbers should equal classical oscillation frequencies” \[10\].

The physical solution for \((\omega_0)_n\) in eq. \(8\) is \[4, 5\]

\[
(\omega_0)_n = M - \sqrt{M^2 - \frac{1}{4\pi}[(\ln 3)^2 + 4\pi^2(n + \frac{1}{2})^2]} \simeq M - \sqrt{M^2 - \frac{1}{2}(n + \frac{1}{2})},
\]

where the term \(M - \sqrt{M^2 - \frac{1}{2}(n + \frac{1}{2})}\) represent the solution of eq. \(9\), which in turn holds for \(j = 0, 1, 2\).

### 3 A note on black hole’s remnants

As \((\omega_0)_n\) is interpreted like the total energy emitted at level \(n\) \[4, 5\], one needs also

\[
M^2 - \frac{1}{4\pi}[(\ln 3)^2 + 4\pi^2(n + \frac{1}{2})^2] \simeq M^2 - \frac{1}{2}(n + \frac{1}{2}) \geq 0
\]

in eq. \(10\). In fact, BHs cannot emit more energy than their total mass. The expression \(11\) is solved giving a maximum value for the overtone number \(n\) \[4\]

\[
n \leq n_{\text{max}} = 2\pi^2 \left(\sqrt{16M^4 - \left(\frac{\ln 3}{\pi}\right)^2} - 1\right) \simeq 2\pi^2\pi(4M^2 - 1),
\]

which corresponds to \((\omega_0)_{n_{\text{max}}} = M\). Thus, the countable sequence of QNMs for emitted energies cannot be infinity although \(n\) can be extremely large \[4\]. On the other hand, we recall that, by using the Generalized Uncertainty Principle, Adler, Chen and Santiago \[5\] have shown that the total BH evaporation is prevented in exactly the same way that the Uncertainty Principle prevents the hydrogen atom from total collapse. In fact, the collapse is prevented, not by symmetry, but by dynamics, as the Planck distance and the Planck mass are approached \[5\]. That important result implies that eq. \(11\) has to be slightly modified, becoming (the Planck mass is equal to 1 in Planck units)
By solving eq. (13) one gets a different value of the maximum value for the overtone number \( n \)

\[
n \leq n_{\text{max}} = 2\pi^2 \left( \sqrt{16(M^2 - 1)^2 - \left( \frac{\ln 3}{\pi} \right)^2} - 1 \right) \approx 2\pi^2(4M^2 - 5). \tag{14}
\]

The result in eq. (14) improves the one of eq. (12) that we originally derived in [4].

### 4 Quasi-normal modes like natural quantum levels

Bekenstein [9] has shown that the Schwarzschild BH area quantum is \( \Delta A = 8\pi \) (the Planck length \( l_p = 1.616 \times 10^{-33} \text{ cm} \) is equal to one in Planck units). By analysing Schwarzschild BH QNs, Hod found a different numerical coefficient [10]. Hod’s work was refined by Maggiore [6], who re-obtained the original result by Bekenstein. In [4, 5] we further improved the result by Maggiore taking into account the deviation from the strictly thermal feature. In fact, in [4, 5] we used eq. (8) instead of eq. (7) [4, 5]. From eq. (10) one gets that an emission involving \( n \) and \( n - 1 \) gives a variation of energy [4, 5]

\[
\Delta M_n = (\omega_0)_{n-1} - (\omega_0)_n = -f_n(M, n) \tag{15}
\]

where one defines [4, 5]

\[
f_n(M, n) \equiv \frac{1}{4\pi} \sqrt{(\ln 3)^2 + 4\pi^2(n - \frac{1}{2})^2} - \frac{1}{4\pi} \sqrt{(\ln 3)^2 + 4\pi^2(n + \frac{1}{2})^2} \approx \frac{1}{4\pi} \sqrt{M^2 - \frac{1}{2}(n - \frac{1}{2})} - \frac{1}{4\pi} \sqrt{M^2 - \frac{1}{2}(n + \frac{1}{2})} \tag{16}
\]

Together with collaborators, we have recently shown that the results (15) (16) also hold for Kerr BHs in the case \( M^2 \gg J \), where \( J \) is the angular momentum of the BH [17].

Recalling that, as for the Schwarzschild BH the horizon area \( A \) is related to the mass through the relation \( A = 16\pi M^2 \), a variation \( \Delta M \) in the mass generates a variation

\[
\Delta A = 32\pi M \Delta M \tag{17}
\]

going with the area. Combining eqs. (17) and (15) one gets [4, 5]

\[
\Delta A = 32\pi M \Delta M_n = -32\pi M \times f_n(M, n). \tag{18}
\]
As we consider large $n$, eq. (16) is well approximated by $f_n(M, n) \approx \frac{1}{4\pi}$ and eq. (18) becomes $\Delta A \approx -8\pi$ which is the original result by Bekenstein for the area quantization (a part a sign because we consider an emission instead of an absorption). Then, only when $n$ is enough large the levels are approximately equally spaced \[4, 5\]. Instead, for smaller $n$ there are deviations. One assumes that, for large $n$, the horizon area is quantized \[4, 5, 6\] with a quantum $|\Delta A|$. The total horizon area must be $A = N|\Delta A|$ where the integer $N$ is the number of quanta of area. One gets \[4, 5\]

$$N = \frac{A}{|\Delta A|} = \frac{16\pi M^2}{32\pi M \cdot f_n(M, n)} = \frac{M}{2f_n(M, n)}. \quad (19)$$

This permits to write the famous formula of Bekenstein-Hawking entropy \[3, 9\] like a function of the quantum overtone number $n$ \[4, 5\]

$$S_{BH} = \frac{A}{4} = 8\pi NM \cdot f_n(M, n). \quad (20)$$

As we consider large $n$, the approximation $f_n(M, n) \approx \frac{1}{4\pi}$ permits to re-obtain the standard result \[6, 11, 12, 13\]

$$S_{BH} \to 2\pi N, \quad (21)$$

like good approximation. Recent results show that that BH entropy contains three parts which are important to realize the underlying quantum gravity theory. They are the Bekenstein-Hawking entropy and two sub-leading corrections: the logarithmic term and the inverse area term \[14, 15\]

$$S_{total} = S_{BH} - \ln S_{BH} + \frac{3}{2A}. \quad (22)$$

Thus, eq. (20) permits to re-write eq. (22) like \[4, 5\]

$$S_{total} = 8\pi NM \cdot f_n(M, n) - \ln [8\pi NM \cdot f_n(M, n)] + \frac{3}{64\pi NM \cdot f_n(M, n)} \quad (23)$$

that, as one considers large $n$, is well approximated by \[4, 5\]

$$S_{total} \simeq 2\pi N - \ln (2\pi N) + \frac{3}{16\pi N}. \quad (24)$$

Thus, at a level $n - 1$, the BH has a number of micro-states \[4, 5\]

$$g(N) \propto \exp \left\{ 8\pi NM \cdot f_n(M, n) - \ln [8\pi NM \cdot f_n(M, n)] + \frac{3}{64\pi NM \cdot f_n(M, n)} \right\}, \quad (25)$$

that, for large $n$, is well approximated by \[4, 5\]

$$g(N) \propto \exp \left\{ 2\pi N - \ln (2\pi N) + \frac{3}{16\pi N} \right\}. \quad (26)$$

Eqs. (24), (25) and (26) are in agreement with previous literature \[6, 14, 15\], in which the strictly thermal approximation were used.

Actually, in previous discussion, and hence also in \[4, 5\], we used an implicit simplification. Now, we improve the analysis by removing such a simplification and by giving the correct results.

In fact, we note that, after an high number of emissions (and potential absorptions as the BH can capture neighboring particles), the BH mass changes from $M$ to
\[ M_{n-1} \equiv M - (\omega_0)_{n-1}, \quad (27) \]

where \((\omega_0)_{n-1}\) is the total energy emitted by the BH at that time, and the BH is excited at a level \(n - 1\). In the transition from the state with \(n - 1\) to the state with \(n\) the BH mass changes again from \(M_{n-1}\) to

\[ M_n \equiv M - (\omega_0)_{n-1} + \triangle M_n, \quad (28) \]

which, by using eq. (16), becomes

\[ M_n = M - (\omega_0)_{n-1} - f_n(M, n) = \]
\[ = M - (\omega_0)_{n-1} + (\omega_0)_{n-1} - (\omega_0)_n = M - (\omega_0)_n. \quad (29) \]

Now, the BH is excited at a level \(n\). By considering eq. (10), eqs. (27) and (29) read

\[ M_{n-1} = \sqrt{M^2 - \frac{1}{4\pi} \sqrt{(\ln 3)^2 + 4\pi^2(n - \frac{1}{2})^2}} \approx \sqrt{M^2 - \frac{1}{2}(n - \frac{1}{2})}, \quad (30) \]

and

\[ M_n = \sqrt{M^2 - \frac{1}{4\pi} \sqrt{(\ln 3)^2 + 4\pi^2(n + \frac{1}{2})^2}} \approx \sqrt{M^2 - \frac{1}{2}(n + \frac{1}{2})}. \quad (31) \]

For extremely large \(n\) the condition \(M_n \approx M\), that we implicitly used in previous discussion, does not hold because the BH has emitted a large amount of mass. This implies that, if one uses eqs. (10) and eq. (30), eq. (18) has to be correctly rewritten as

\[ \triangle A_{n-1} \equiv -32\pi M_{n-1} \triangle M_n = -32\pi M_{n-1} \times f_n(M, n) \quad (32) \]

This equation should give the area quantum of an excited BH for an emission from the level \(n - 1\) to the level \(n\) in function of the quantum number \(n\) and of the initial BH mass. Actually, there is a problem in eq. (32). In fact, an absorption from the level \(n\) to the level \(n - 1\) is now possible, with an absorbed energy \([5]\)

\[ (\omega_0)_n - (\omega_0)_{n-1} = f_n(M, n) = -\triangle M_n. \quad (33) \]

In that case, the quantum of area should be

\[ \triangle A_n \equiv -32\pi M_n \triangle M_n = 32\pi M_n \times f_n(M, n), \quad (34) \]

and the absolute value of the area quantum for an absorption from the level \(n\) to the level \(n - 1\) is different from the absolute value of the area quantum for an emission from the level \(n - 1\) to the level \(n\) because \(M_{n-1} \neq M_n\). Clearly, one
indeed expects the area spectrum to be the same for absorption and emission. This inconsistency is solved if, once again, one considers the effective mass which correspond to the transitions between the two levels \( n \) and \( n - 1 \), which is the same for emission and absorption

\[
M_{E(n, n-1)} \equiv \frac{1}{2} (M_{n-1} + M_n) = \\
= \frac{1}{2} \left( \sqrt{M^2 - \frac{1}{4\pi} \sqrt{(\ln 3)^2 + 4\pi^2(n - \frac{1}{2})^2}} + \sqrt{M^2 - \frac{1}{4\pi} \sqrt{(\ln 3)^2 + 4\pi^2(n + \frac{1}{2})^2}} \right) \approx \\
\approx \frac{1}{2} \left( \sqrt{M^2 - \frac{1}{2}(n - \frac{1}{2})} + \sqrt{M^2 + \frac{1}{2}(n - \frac{1}{2})} \right) \tag{35}
\]

Replacing \( M_{n-1} \) with \( M_{E(n, n-1)} \) in eq. \( 32 \) and \( M_n \) with \( M_{E(n, n-1)} \) in eq. \( 34 \) we obtain

\[
\Delta A_{n-1} \equiv 32\pi M_{E(n, n-1)} \Delta M_n = -32\pi M_{E(n, n-1)} \times f_n(M, n) \quad \text{emission}
\]

\[
\Delta A_n \equiv -32\pi M_{E(n, n-1)} \Delta M_n = 32\pi M_{E(n, n-1)} \times f_n(M, n) \quad \text{absorption}
\]

and now one gets \( \alpha = |\Delta A_n| = |\Delta A_{n-1}| \). By using eqs. \( 10 \) and \( 20 \) one finds

\[
\alpha = |\Delta A_n| = |\Delta A_{n-1}|
\]

\[
= 4 \left( \sqrt{(\ln 3)^2 + 4\pi^2(n + \frac{1}{2})^2} - \sqrt{(\ln 3)^2 + 4\pi^2(n - \frac{1}{2})^2} \right) \approx 8\pi. \tag{37}
\]

Hence, the introduction of the effective temperature and of the effective mass does not degrade the importance of the Hawking temperature. Indeed, the effective temperature and the effective mass are introduced because the values of the Hawking temperature and of the mass change with discrete behavior in time. Thus, it is not clear which value of the Hawking temperature and of the mass have to be associated to the emission (or to the absorption) of the particle. Has one to consider the values of the Hawking temperature and of the mass before the emission (absorption) or the values after the emission (absorption)? The answer is that one must consider intermediate values, the effective temperature and the effective mass. In a certain sense, they represent the values of the BH temperature and BH mass during the emission.

We note that, as we consider large \( n \), the result \( 37 \) is \( \approx 8\pi \), which also holds for vector perturbations. Thus, one can take the result \( 37 \) as the quantization of the area of the horizon of a Schwarzschild BH. Again, the original famous result by Bekenstein \( 9 \) is a good approximation and a confirmation of the correctness of the current analysis. This is also in full agreement with Bohr’s correspondence principle \( 25, 26, 27 \).

In previous analysis and in \( 4, 5 \), the simplification \((\omega_0)_{n-1} \ll M \) has been implicitly used, i.e. the energy associated to the QNM is much less than the
original mass-energy of the BH. Clearly, in that case the correction given by eq. (32) results non-essential, as one can neglect the difference between the initial BH mass $M$ and the mass of the excited BH $M_{n-1}$, but it becomes very important when $M_n \simeq M$ does not hold, i.e. for very highly excited BHs. In that case, for example in the latest stages of the BH evaporation (but before arriving at the Planck scale, where our semi-classical approximation breaks down and a full quantum gravity theory is needed), it could be $(\omega_0)_n \lesssim M$, and further corrections on previous formulas are needed. Putting $A_{n-1} \equiv 16\pi M_{n-1}^2$ and $A_n \equiv 16\pi M_n^2$, the formulas of the number of quanta of area and of the Bekenstein-Hawking entropy become

$$N_{n-1} = \frac{A_{n-1}}{|\Delta A_{n-1}|} = \frac{16\pi M_{n-1}^2}{32\pi M_{E(n, n-1)} \cdot f_n(M, n)} = \frac{M_{n-1}^2}{2M_{E(n, n-1)} \cdot f_n(M, n)}$$

(38)

before the emission, and

$$N_n = \frac{A_n}{|\Delta A_n|} = \frac{16\pi M_n^2}{M_{E(n, n-1)} \cdot f_n(M, n)} = \frac{M_n^2}{2M_{E(n, n-1)} \cdot f_n(M, n)}$$

(39)

after the emission respectively. One can easily check that

$$N_n - N_{n-1} = \frac{M_n^2 - M_{n-1}^2}{2M_{E(n, n-1)} \cdot f_n(M, n)} = \frac{f_n(M, n)}{2M_{E(n, n-1)} \cdot f_n(M, n)} (M_{n-1} + M_n) = 1$$

(40)

as one expects. Hence, the formulas of the Bekenstein-Hawking entropy read

$$(S_{BH})_{n-1} = 4\pi M_{E(n, n-1)} \cdot f_n(M, n) + M_{n-1}$$

$$= 4\pi \left( M_n^2 - \frac{1}{4\pi} \sqrt{\ln 3^2 + 4\pi^2(n - \frac{1}{2})^2} \right)$$

(41)

before the emission and

$$(S_{BH})_n = 4\pi M_n \cdot f_n(M, n)$$

$$= 4\pi \left( M_n^2 - \frac{1}{4\pi} \sqrt{\ln 3^2 + 4\pi^2(n + \frac{1}{2})^2} \right)$$

(42)

after the emission respectively. Notice that, as $n \gg 1$, one obtains

$$(S_{BH})_n \simeq (S_{BH})_{n-1} \simeq 4\pi \left( M_n^2 - \frac{n}{2} \right).$$

(43)

Again, formula (43) works for all $j = 0, 1, 2$. The formulas of the total entropy that takes into account the sub-leading corrections to Bekenstein-Hawking entropy become

$$(S_{total})_{n-1} = 8\pi N_{n-1} M_{n-1} \cdot f_n(M, n)$$

$$- \ln [8\pi N_{n-1} M_{n-1} \cdot f_n(M, n)] + \frac{3}{64\pi N_{n-1} M_{n-1} \cdot f_n(M, n)}$$

(44)

before the emission, and

$$(S_{total})_n = 8\pi N_n M_n \cdot f_n(M, n)$$

$$- \ln [8\pi N_n M_n \cdot f_n(M, n)] + \frac{3}{64\pi N_n M_n \cdot f_n(M, n)}$$

(45)
after the emission, respectively.

Hence, at level $n-1$ the BH has a number of micro-states

$$g(N_{n-1}) \propto \exp\{8\pi N_{n-1} M_{n-1} \cdot f_n(M, n) + - \ln [8\pi N_{n-1} M_{n-1} \cdot f_n(M, n)] + \frac{3}{64\pi N_{n-1} M_{n-1} \cdot f_n(M, n)}\}$$

and, at level $n$, after the emission, the number of micro-states is

$$g(N_n) \propto \exp\{8\pi N_n M_n \cdot f_n(M, n) + - \ln [8\pi N_n M_n \cdot f_n(M, n)] + \frac{3}{64\pi N_n M_n \cdot f_n(M, n)}\}$$

All these corrections, which represent the correct formulas of an excited BH for a transition between the levels $n-1$ and $n$ in function of the quantum number $n$, result very important for very highly excited BHs, when $n$ becomes extremely large and $M_n \simeq M$ does not hold (in particular in the last stages of the BH evaporation, before arriving at the Planck scale, when $(\omega_0)_n \lesssim M$).

Instead, when $(\omega_0)_n \ll M$, formulas (18), (19), (23) and (25), are a good approximation. Such formulas are a better approximation with respect to formulas (21), (24) and (26) which were used in previous results in the literature [6, 14, 15]. In those works the strictly thermal approximation was used. We also see that, for $(\omega_0)_{n-1} \ll M$, $M_n \simeq M_{n-1} \simeq M$, and eqs. (18), (19), (23) and (25), are easily recovered.

5 Final discussion and conclusion remarks

We explain the way in which our BH model works. Let us consider a BH original mass $M$. After an high number of emissions (and potential absorptions as the BH can capture neighboring particles), the BH will be at an excited level $n-1$ and its mass will be $M_{n-1} \equiv M - (\omega_0)_{n-1}$ where $(\omega_0)_{n-1}$ is the absolute value of the frequency of the QNM associated to the excited level $n-1$. $(\omega_0)_{n-1}$ is also the total energy emitted at that time. The BH can further emit an energy to jump to the subsequent level: $\Delta M_n = (\omega_0)_{n-1} - (\omega_0)_n$. Now, the BH at an excited level $n$ and the BH mass will be

$$M_n \equiv M - (\omega_0)_{n-1} + \Delta M_{n-1} =$$

$$= M - (\omega_0)_{n-1} + (\omega_0)_{n-1} - (\omega_0)_n = M - (\omega_0)_n.$$  

The BH can, in principle, return to the level $n-1$ by absorbing an energy $-\Delta M_n = (\omega_0)_n - (\omega_0)_{n-1}$. We have also shown that the quantum of area is the same for both absorption and emission, given by eq. (37), as one expects.

There are three different physical situations for excited BHs ($n \gg 1$):

1. $n$ is large, but not enough large. It is also $(\omega_0)_n \ll M_n \simeq M$ and one can use eqs. (18), (20), (23) and (25) which results a better approximation than eqs. (21), (24) and (26) which were used in previous literature in strictly thermal approximation [6, 14, 15].
2. $n$ is very much larger than in point 1, but before arriving at the Planck scale. In that case, it can be $(\omega_0)_n \lesssim M$, while $M_n \approx M$ does not hold and one must use the eqs. (37), and from (41) to (47).

3. At the Planck scale $n$ is larger also than in point 2, we need a full theory of quantum gravity.

In summary, in this paper we analyzed BH QNMs in terms of quantum levels following the idea that, in an underlying quantum gravity theory, BHs result in highly excited states. By using the concept of effective temperature, we took into account the important issue that QNMs spectrum is not strictly thermal.

The obtained results improve our previous analysis in [4, 5] because here we removed some approximations that we implicitly used in [4, 5]. The results look particularly intriguing as important modifies on BH quantum physics have been realized. In fact, we found the correct formulas of the horizon’s area quantization and of the number of quanta of area like functions of the quantum “overtone” number $n$. Consequently, we also found the correct expressions of Bekenstein-Hawking entropy, its sub-leading corrections and the number of micro-states, i.e. quantities which are fundamental to realize the underlying quantum gravity theory, like functions of the quantum “overtone” number $n$. In other words, the cited important quantities result to depend on the excited BH quantum state. An approximation concerning the maximum value of $n$ has been also corrected. As our previous results in [4, 5] were strictly corrected only for scalar and gravitational perturbations, the results of this work have shown that the analysis holds also for vector perturbations.

We stress that the analysis is totally consistent with the general conviction that BHs result in highly excited states representing both the “hydrogen atom” and the “quasi-thermal emission” in quantum gravity and that previous results in the literature, where the thermal approximation has been used [6, 14, 15], are consistent with the results in this paper. As at the present time we do not yet have a full theory of quantum gravity. Then, we must be content with the semi-classical approximation. As for large $n$ Bohr’s correspondence principle holds [23, 28, 29], such a semi-classical description is adequate. Thus, we can consider an intriguing analogy in which our BH model is somewhat similar to the semi-classical Bohr’s model of the structure of a hydrogen atom [26, 27]. In our BH model, during a quantum jump a discrete amount of energy is indeed radiated and for large values of the principal quantum number $n$ the analysis becomes independent from the other quantum numbers. In a certain sense, QNMs represent the "electron" which jumps from a level to another one and the absolute values of the QNMs frequencies represent the energy "shells". In Bohr’s model [26, 27], electrons can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation with an energy difference of the levels according to the Planck relation $E = hf$, where $h$ is the Planck constant and $f$ the transition frequency. In our BH model, QNMs can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels according to eqs. (15) and (16).
On the other hand, Bohr’s model is an approximated model of the hydrogen atom with respect to the valence shell atom model of full quantum mechanics. In the same way, our BH model should be an approximated model of the emitting BH with respect to the definitive, but at the present time unknown, model of full quantum gravity theory.

It is also important to emphasize that the results in this paper could have important implications for the BH information paradox. The more important arguments that information can be lost during BH evaporation rely indeed on the assumption of strict thermal behavior of the spectrum [18]. On the other hand, the results in this paper show that BHs seem to be well defined quantum mechanical systems, which have ordered, discrete quantum spectra. This important point is surely consistent with the unitarity of the underlying quantum gravity theory and endorses the idea that information should come out in BH evaporation [20, 21, 22, 23, 24, 32, 33].

6 Acknowledgements

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Appendix. Derivation of the quasi-normal modes equation in non-strictly thermal approximation

Being frequencies of the radial spin−$j = 0, 1, 2$ perturbations $\phi$ of the Schwarzschild space-time, QNMs are governed by the Schrodinger-like equation [4, 5, 16]

$$\left(-\frac{\partial^2}{\partial r^2} + V(r) - \omega^2\right)\phi. \quad (49)$$

In strictly thermal approximation one introduces the Regge-Wheeler potential

$$V(r) = V[x(r)] = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + 2\frac{(1-j^2)M}{r^3}\right). \quad (50)$$

We recall that $j = 0, 1, 2$ for scalar, vector and gravitational perturbation respectively.

The relation between the Regge-Wheeler “tortoise” coordinate $x$ and the radial coordinate $r$ is [4, 5, 16]

$$x = r + 2M \ln \left(\frac{r}{2M} - 1\right) \quad (51)$$

$$\frac{\partial}{\partial x} = (1 - \frac{2M}{r}) \frac{\partial}{\partial r}.$$

These states are analogous to quasi-stationary states in quantum mechanics [16]. Thus, their frequency is allowed to be complex [16]. They must have purely outgoing boundary conditions both at the horizon ($r = 2M$) and in the asymptotic region ($r \to \infty$) [16]

$$\phi(x) \sim c_\pm \exp(\mp i\omega x) \quad \text{for} \quad x = \pm \infty. \quad (52)$$
Considering the non-strictly thermal behavior of BHs, one substitutes the original BHs $M$ in eqs. (49) and (50) with the effective mass of the contracting BH defined in eq. [4, 5]. Hence, eqs. (50) and (51) are replaced by the effective equations [4, 5]

$$V(x) = V[x(r)] = \left(1 - \frac{2M_E}{r}\right)\left(\frac{l(l+1)}{r^2} + 2\left(1 - \frac{j^2}{r^3}\right)M\right)$$  \hspace{1cm} (53)

and

$$x = r + 2M_E \ln \left(\frac{r}{2M_E} - 1\right)$$  \hspace{1cm} (54)

In order to streamline the formulas, here we also set

$$2M_E = r_E \equiv 1 \quad \text{and} \quad m \equiv n + 1.$$  \hspace{1cm} (55)

As the Planck mass $m_p$ is equal to 1 in Planck units one rewrites (8) as

$$\omega_0 m = \frac{m_p^2}{m_p^2} \left(\frac{3}{4\pi} + \frac{i}{2}(m - \frac{1}{2}) + O(m^{-\frac{3}{2}}), \quad \text{for} \quad m \gg 1,\right.$$

where now $m_p \neq 1$. Putting

$$\tilde{\omega}_m \equiv \frac{\omega_m}{m_p^2};$$  \hspace{1cm} (57)

eqs. (8), (49), (53) and (54) become

$$\tilde{\omega}_m = \left(\frac{3}{4\pi} + \frac{i}{2}(m - \frac{1}{2}) + O(m^{-\frac{3}{2}}), \quad \text{for} \quad m \gg 1,\right.$$

$$\left(-\frac{\partial^2}{\partial x^2} + V(x) - \tilde{\omega}^2\right)\phi,$$

$$V(x) = V[x(r)] = \left(1 - \frac{1}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{3(1 - j^2)}{r^3}\right)$$  \hspace{1cm} (60)

and

$$\frac{\partial}{\partial x} = \left(1 - \frac{1}{r}\right)\frac{\partial}{\partial r}$$  \hspace{1cm} (61)

respectively. Although $M_E$ and $r_E$ (and consequently the tortoise coordinate and the Regge-Wheeler potential) are frequency dependent, eq. (55) translates such a frequency dependence into a continually rescaled mass unit in the following discussion. We will show at the end of this Appendix that such a rescaling is extremely slow and always included within a factor 2. Thus, it does not influence the following analysis.

We emphasize that here after we closely follow ref. [16]. The solution of eq. (54) can be expanded as [16].
\[ \phi(r) = \left( \frac{r - 1}{r^2} \right)^{i\tilde{\omega}} \exp[-i\tilde{\omega}(r-1)] \sum_m a_m \left( \frac{r - 1}{r} \right)^m. \] (62)

The pre-factor has to satisfy the boundary conditions (62) both at the effective horizon \((r = 1)\) and in the asymptotic region \((r = \infty)\) [16]:

1. One needs \(\exp[-i\tilde{\omega}(r - 1)]\) for the correct leading evolution at \(r \to \infty\)
2. \((r - 1)^{i\tilde{\omega}}\) fixes the evolution at \(r \to 1^+\)
3. \(\left( \frac{1}{r} \right)^{i\tilde{\omega}}\) arranges the sub-leading evolution at \(r \to \infty\) which arises from the logarithmic term in eq. (61).

The power series (62) converges for \(\frac{1}{2} < r \leq \infty\) assuming that the boundary conditions at \(r = \infty\) are preserved [16]. On the other hand, eq. (49) is equivalent to the recursion relation [16]

\[ c_0(m, \tilde{\omega})a_m + c_1(m, \tilde{\omega})a_m - 1 + c_2(m, \tilde{\omega})a_m - 2 = 0. \] (63)

One can extract the coefficients \(c_k(m, \tilde{\omega})\) from eq. (63) and rewrite them in a more convenient way [16]:

\[ c_0(m, \tilde{\omega}) = m(m + 2i\tilde{\omega}) \] (64)

\[ c_1(m, \tilde{\omega}) = -2(m + 2i\tilde{\omega} - \frac{1}{2})^2 + j^2 - \frac{1}{2} \] (65)

\[ c_2(m, \tilde{\omega}) = (m + 2i\tilde{\omega} - 1)^2 - j^2. \] (66)

We see that, except for \(c_0\), the coefficients \(c_k\) depend on \(m, \tilde{\omega}\) only through their combination \(m + 2i\tilde{\omega}\). The preservation of the boundary conditions at \(r = 1\) is guaranteed by the initial conditions for the recursion relation [16]. Such conditions are \(a_0 = 1\) (in general any non-zero constant that we set equal to the unity for the sake of simplicity), and \(a_{-1} = 0\) (that, together with eq. (63) implies \(a_{-m} = 0\) for all positive \(m\)) [15]. One also defines [16]

\[ R_m = -\frac{a_m}{a_{m-1}}, \] (67)

where we choose the minus sign in agreement with [16]. By using eq. (63), one gets [16]

\[ c_1(m, \tilde{\omega}) - c_0(m, \tilde{\omega})R_m = \frac{c_2(m, \tilde{\omega})}{R_{m-1}} \] (68)

which can be rewritten as

\[ R_{m-1} = \frac{c_2(m, \tilde{\omega})}{c_1(m, \tilde{\omega}) - c_0(m, \tilde{\omega})R_m}. \] (69)
Therefore, one can write $R_m$ in terms of a continued fraction. The condition $a_{-1} = 0$ becomes

$$R_0 = \infty \rightarrow c_1(1, \tilde{\omega}) - c_0(1, \tilde{\omega})R_1 = 0. \quad (70)$$

As eq. (62) converges at $r = \infty$, there is a particular asymptotic form of $R_m$ for large $m$ (with $|R_m| < 1$ for very large $m$) [16]. Thus, the boundary conditions require eq. (70). One can write down eq. (70) in terms of continued fractions

$$0 = c_1(1, \tilde{\omega}) - c_0(1, \tilde{\omega})\frac{c_2(2, \tilde{\omega})}{c_1(2, \tilde{\omega}) - c_0(2, \tilde{\omega})\ldots \ldots}, \quad (71)$$

which is a condition for the existence of the quasi-normal modes.

As $R_m \rightarrow -1$ for large $m$ and the changes of $R_m$ slow down for large $|\tilde{\omega}|$, assuming that $R_m$ changes adiabatically is an excellent approximation and one gets

$$\frac{R_m}{R_{m-1}} \simeq 1 + O(\tilde{\omega}^{-\frac{3}{2}}). \quad (72)$$

This approximation works for both $Re (m + 2i\tilde{\omega}) > 0$, when one computes $R_m$ recursively from $R_{\infty} = -1$, and for $Re (m + 2i\tilde{\omega}) < 0$, when one starts to compute from $R_0 = \infty$ [16]. If one inserts $R_{m-1} = R_m$ in eq. (68), one gets a quadratic equation, having the solutions (at the leading terms for large $m$) [16]

$$R_m^\pm = \frac{-(m + 2i\tilde{\omega}) \pm \sqrt{2i\omega(m + 2i\tilde{\omega})}}{m} + O(m^{-\frac{3}{2}}). \quad (73)$$

The approximated solution (73) can be carefully checked by using Mathematica [16]. The issue that $R_m$ must satisfy eq. (73) for one of the signs is a necessary condition [16]. One needs a more deep discussion in order to see if that condition is also sufficient [16]. When $Re (m + 2i\tilde{\omega}) < 0$, the sign arises from the condition $R_1$ is small. Two terms in eq. (73) are approximately deleted. The sign for $Re (m + 2i\tilde{\omega}) > 0$ arises from the condition $|R_m| < 1$ for very large $m$ [16].

When $|\tilde{\omega}|$ is very large (but minor than the total mass of the black hole), one chooses an integer $N$ such that [16]

$$1 \ll N \ll |\tilde{\omega}|. \quad (74)$$

For the values of $N$ in eq. (74), eq. (73) can be used to determine $R_{[-2i\omega]}$ for only the second term in the RHS of eq. (73) results to be relevant (the symbol for the integer part $[-2i\omega]$ represents an arbitrary integer differing from $-2i\omega$ by a number much smaller than $N$ which is assumed to be even) [16].

Such a relevant term in eq. (73) implies [16]

$$R_{[-2i\omega]+x} \propto \pm \frac{i\sqrt{x}}{\sqrt{-2i\omega}} \text{ for } 1 \ll x \ll |\tilde{\omega}|, \quad (75)$$

while the first term in eq. (73) is $\propto \frac{1}{x}$ and results subleading [16]. Neglecting such a first term is equivalent to neglecting $c_1(m, \tilde{\omega})$ in the original equation.
In fact, this term is irrelevant for all the $m$ for large $|\tilde{\omega}|$, with the possible exclusion of some purely imaginary frequencies where $c_0(m)$ or $c_2(m)$ vanish \[16\]. The ratio $R_{[-2i\tilde{\omega}]_{\pm}N}$ is computed from eq. \[73\] like the ratio of $\sqrt{x}$ for $x = \pm N$ \[10\]

$$\frac{R_{[-2i\tilde{\omega}]_{\pm}N}}{R_{[-2i\tilde{\omega}]_{-}N}} = \pm i + O(\tilde{\omega}^{-\frac{1}{2}}). \tag{76}$$

The assumptions that permitted to obtain eq. \[73\] break down when $|m+2i\tilde{\omega}| \sim 1$ \[10\]. In that case, the coefficients $c_0(m)$, $c_1(m)$, $c_2(m)$ contain terms of order 1 that cannot be neglected \[10\]. They also strongly depend on $m$. The adiabatic approximation breaks down in this region and the quantities $R_{[-2i\tilde{\omega}]_{\pm}N}$ have to be related through the original continued fraction. Below, we will calculate the continued fraction exactly in the limit of very large $|\tilde{\omega}|$. The continued fraction will give the same result of eq. \[76\] like the adiabatic argument. In fact, the two solutions will eventually “connect” \[16\]. Such a connection will release a non-trivial constraint on $\tilde{\omega}$.

As $R_{[-2i\tilde{\omega}]_{-}x} \propto \frac{1}{\sqrt{\tilde{\omega}}}$, $c_1(m)$ in the denominator of eq. \[63\] is negligible when compared to the other term (which results $\propto \sqrt{\tilde{\omega}}$) \[10\]. By fixing $N$, one understands that the effect of $c_1(m)$ in eq. \[63\] vanishes for large $|\tilde{\omega}|$ \[10\]. An exception should appear when $c_0(m)$ and/or $c_2(m) \rightarrow 0$ for some $m$, but we will show that this exception cannot occur when $Re(\tilde{\omega}) \neq 0$.

We note that the orbital angular momentum $l$ is irrelevant because $c_1(m)$ does not affect the asymptotic frequencies \[16\]. This is not surprising as it is in agreement with Bohr correspondence principle \[10\] \[25\] \[26\] \[27\]. One can replace the factor $m$ in eq. \[53\] with $[-2i\tilde{\omega}]$. In fact, $|\tilde{\omega}|$ becomes extremely large when one studies only a relatively small neighborhood of $m \sim [-2i\tilde{\omega}]$ \[16\]. We also see that the continued fraction is simplified into an ordinary fraction. If one inserts eq. \[63\] recursively into itself $2N$ times, one gets \[10\]

$$R_{[-2i\tilde{\omega}]_{-}N} = \prod_{k=1}^{N} \left( \frac{c_2([-2i\tilde{\omega}] - N + 2k - 1)c_0([-2i\tilde{\omega}] - N + 2k)}{c_0([-2i\tilde{\omega}] - N + 2k - 1)c_2([-2i\tilde{\omega}] - N + 2k)} \right) R_{[-2i\tilde{\omega}]_{+}N}. \tag{77}$$

The dependence of the $c_k(m)$ on the frequency is suppressed. Thus, one can combine eqs. \[76\] and \[77\] to eliminate the other $R_m$. We require that the generic solution \[73\], which holds almost everywhere, is “patched” with the solution \[77\], which holds for $m \sim [-2i\tilde{\omega}]$ and for $|\tilde{\omega}|$ extremely large \[10\].

One can express the products of $c_0(m)$ and $c_2(m)$ in terms of the gamma function $\Gamma$, which is an extension of the factorial function to real and complex numbers \[16\]. As $c_2(m)$ is bilinear, the four factors of eq. \[77\] lead to a product of 6 factors, i.e. $(2+1+1+2)$. Each of those equals a ratio of two $\Gamma$ functions. In other words, one gets a ratio of twelve $\Gamma$ functions \[10\].

One can write down the resulting condition in terms of a shifted frequency defined by $-2if \equiv -2i\tilde{\omega} - [-2i\tilde{\omega}]$ \[10\].
\[ \pm i = \frac{\Gamma\left(N + \frac{1}{2}\right)\Gamma\left(N + \frac{1}{2} + j\right)\Gamma\left(-N + \frac{1}{2}\right)}{\Gamma\left(\frac{3N+1}{2}\right)\Gamma\left(\frac{3N+1}{2} + j\right)\Gamma\left(-N + \frac{1}{2}\right)}. \]  

(78)

Considering eq. (66), the factors \( m \) result cancelled. Six of the \( \Gamma \) functions in eq. (78) show an argument with a huge negative real part. Hence, they can be converted into \( \Gamma \) of positive numbers by using the formula \[ \Gamma(x) = \frac{\pi}{\sin(\pi x) \Gamma(1 - x)}. \]  

(79)

Thus, the \( \pi \) factors cancel like the Stirling approximations for the \( \Gamma \) functions which have a huge positive argument \[ \Gamma(m + 1) \approx \sqrt{2\pi m} \left(\frac{m}{e}\right)^m, \]  

(80)

while the factors with \( \sin(x) \) survive. The necessary condition for regular frequencies for large \( m \) (the frequencies for which the analysis is valid) will return on this point later reads \[ \pm 1 = \frac{\sin[\pi(if + \frac{1}{2})] \sin[\pi(if + \frac{1}{2})] \sin[\pi(if + \frac{1}{2})]}{\sin[\pi(i\omega + \frac{1}{2})] \sin[\pi(i\omega + \frac{1}{2})] \sin[\pi(i\omega + \frac{1}{2})]}. \]  

(81)

Choosing \( N \in 4\mathbb{Z} \), one erases \( N \) from the arguments of the trigonometric functions. One also replaces \( f \) by \( \tilde{\omega} \) again as the functions in eq. (81) are periodic with the right periodicity and the number \([-2i\tilde{\omega}]\) can be chosen even. We can use the terms \( \frac{1}{2} \) in the denominator in order to convert the sin functions into cos. Multiplying eq. (81) by the denominator of the RHS and expanding the sin functions in terms of the exponentials (one has to be careful about the signs) the result is (\( \epsilon(y) \) is the sign function) \[ \exp\left[\epsilon Re(\omega) \cdot 4\pi \tilde{\omega}\right] = -1 - 2\cos(\pi j). \]  

(82)

Hence, for scalar or gravitational perturbations the allowed frequencies are \[ \tilde{\omega}_m = \frac{i}{2}(m - \frac{1}{2}) \pm \frac{\ln 3}{4\pi} + O(m^{-\frac{1}{2}}), \]  

(83)

while for vector perturbations one gets \[ \tilde{\omega}_m = \frac{im}{2} + O(m^{-\frac{1}{2}}). \]  

(84)

We note that, as we are in the large \( m \) regime, one gets \( \tilde{\omega}_m \approx \frac{im}{2} \) independent of \( j \). This implies the correctness of eq. (9) and the important issue that the quantum of area (37) is an intrinsic property of Schwarzschild BHs. Again, this is in full agreement with Bohr correspondence principle [10, 25, 26, 27].
In order to finalize the analysis one has to resolve the question marks concerning the special frequencies where the used approximation, which neglects $c_1(m)$, breaks down [10].

The first step is to argue that the “regular” solutions must exist [16]. In fact, if one can relate the remainders $R_{[-2i\tilde{\omega}]\pm N}$ by using the continued fraction, where $c_1(m)$ can be neglected, one can also extrapolate them to eq. (73) [16].

From the boundary conditions $R_0 = \infty$, $R_\infty = -1^+$, one sees that a specific sign of the square root in eq. (73) has to be separately chosen for $m < [-2i\tilde{\omega}]$ and $m > [-2i\tilde{\omega}]$ [16]. But one finds that the signs agree with the signs of $\pm i$ that automatically lead to the solutions [16].

The condition for $\tilde{\omega}$ is both necessary and sufficient [16]. This kind of solutions are only the $\ln(3)$ solutions from eq. (58). The existence of those solutions is guaranteed [16].

As one needs to find all irregular solutions, let us recall two useful points [16].

1. The continued fractions of eq. (71) depend on the coefficients $c_0(m)$ and $c_2(m+1)$ only through their product $c_0(m)c_2(m+1)$ [16].

2. If one finds zeroes in eq. (77) exclusively in the numerator or in the denominator, the ratio $R_{[-2i\tilde{\omega}]\pm N}$ can be only either zero, or infinite. When one takes into account $c_1(m)$, “zero” or “infinity” results to be replaced by a negative or positive power of $|\tilde{\omega}|$, respectively [16].

Point 2. means that one can obtain irregular solutions only by finding $\omega$ such that eq. (77) becomes an indeterminate form $\frac{0}{0}$ [16].

As $c_0(m)$ can be null at most for one value of $m$, one finds that there is at least one value of $m$ where $c_2(m)$ vanishes [15]. Thus, eq. (69) implies that one between $2(i\tilde{\omega} + 1)$ and $2(i\tilde{\omega} - 1)$ (maybe both) must be integer [10]. As the two conditions are equivalent, both numbers $2(i\tilde{\omega} \pm 1)$ must be integers to give a chance to exist to the quasi-normal frequency [10]. Thus, the two numbers differ by an even number. Then, both the vanishing factors of $c_2(m)$ must appear in the numerator of eq. (77), or, alternatively, both must appear in the denominator of such an equation [10]. One can assume, for example, that they appear in the denominator without loss of generality [10]. Thus, one finds the indeterminate form only if the vanishing $c_0(m)$ appears in the numerator [10]. Clearly, $2(i\tilde{\omega} \pm 1)$ and $2i\tilde{\omega}$ are different modulo two. Thus, the effect of $c_1(m)$ gives the desired result, confirming that the regular states of eq. (83) are the only solutions [10]. By using eqs. (57) and (55) one easily returns to Planck units and obtains eq. (8).

Now, we show that the continually rescaled mass unit in the above discussion, which is due to the frequency dependence of $M_E$ and $r_E$, did not influence the analysis. We note that, although $\tilde{\omega}$ in the analysis can be very large because of definition (57), $\omega$ must instead be always minor than the BH initial mass as BHs cannot emit more energy than their total mass. Inserting this constrain in eq. (6) we obtain the range of permitted values of $M_E(\omega_n)$ as
\[
\frac{M}{2} \leq M_E(|\omega_n|) \leq M. \tag{85}
\]

Thus, setting \(2M_E(|\omega_n|) = r_E(|\omega_n|) ÷ 1(\omega_n)\) one sees that the range of permitted values of the continually rescaled mass unit is always included within a factor 2. On the other hand, we recall that the countable sequence of QNMs is very large, see Section 3 and [4]. Thus, the mass unit’s rescaling is extremely slow. Hence, one can easily check, by reviewing the above discussion step by step, that the continually rescaled mass unit did not influence the analysis.

Another argument which remarks the correctness of the analysis in this Appendix is the following. One can choose to consider \(M_E\) as being constant within the range [5]. In that case, it is easy to show that such an approximation is indeed very good. In fact, eq. (85) implies that the range of permitted values of \(T_E(|\omega_n|)\) is

\[
T_H = T_E(0) \leq T_E(|\omega_n|) \leq 2T_H = T_E(|\omega_{n_{\text{max}}}|), \tag{86}
\]

where \(T_H\) is the initial BH Hawking temperature. Therefore, if one fixes \(M_E = \frac{M}{2}\) in the analysis, the approximate result is

\[
\omega_n \simeq 2\pi n \times 2T_H. \tag{87}
\]

On the other hand, if one fixes \(M_E = M\) (thermal approximation), the approximate result is

\[
\omega_n \simeq 2\pi n \times T_H. \tag{88}
\]

As both the approximate results in correspondence of the extreme values in the range [5] have the same order of magnitude, fixing \(2M_E = r_E ÷ 1\) does not change the order of magnitude of the final (approximated) result with respect to the exact result. In particular, if we set \(T_E = \frac{1}{2}T_H\) the uncertainty in the final result is 0.33, while in the result of the thermal approximation [88] the uncertainty is 2. Thus, even considering \(M_E\) as constant, our result is more precise than the thermal approximation of previous literature and the order of magnitude of the total emitted energies [10] is correct.

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