From Seiberg-Witten invariants to topological Green-Schwarz string

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Abstract

In this note we describe the physics of equivalence of the Seiberg-Witten invariants of 4-manifolds and certain Gromov-Witten invariants defined by pseudo-holomorphic curves. We show that physics of the pseudo-holomorphic curves should be governed by the N=2 Green-Schwarz string.

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1 Introduction

In the recent paper [1] Taubes announced the equivalence of the Seiberg-Witten invariants [2] of symplectic 4-manifolds [3, 4] and certain Gromov-Witten invariants defined by pseudo-holomorphic curves [5, 6, 7]. Besides its main application to topology the result has also interesting physical aspects. It sounds like the long standing expectations that in some cases gauge field configurations should be well approximated by strings. In our case the string theory would be the topological theory of pseudo-holomorphic curves. Of course the problem lies far apart from the $1/N_c$ expansion [8, 9] because there is nothing like $N_c$ here. It shall appear that the localization is strictly related to properties of topological field theory. Anyway the mechanism of localization has an interesting physical background and moreover it shows relations to the N=2 Green-Schwarz string [11]. These are the reasons why we have decided to base our discussion on physical N=2 SUSY theories which by “twisting” are related to topological field theories [10].

We start with the N=2 SUSY SU(2) Yang-Mills theory in $R^4$ and show that after inclusion of Feyet-Iliopoulos (FI) term it localizes on pseudo-holomorphic curves. In the next section we show the relation to the N=2 Green-Schwarz string.

2 From Seiberg-Witten theory to pseudo-holomorphic curves

We start from the N=2 SUSY SU(2) Yang-Mills theory which is a basis for the Seiberg-Witten theory of invariants of 4-manifolds [2]. We recall that the moduli space of the N=2 SUSY SU(2) Yang-Mills theory [12] contains two singular points. At these points the low energy effective theory is N=2 SUSY U(1) theory coupled to an additional massless matter (monopoles or dyons) in the form of the N=2 hypermultiplet. Hence the low energy fields are: the vector multiplet (gauge field $A_\mu$, SU$_I$(2) doublet of fermions $\lambda_\pm$, scalar $\phi$ plus SU$_I$(2) triplet of auxiliary fields $\bar{Y}_I$ and the hypermultiplet (SU$_I$(2) doublet of scalars $M^i$, two fermions $\psi_\pm$). We shall write down only the relevant

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2I would like to thank R.Stora for seeding an uneasiness into my naive belief in “twisting”.

3Our conventions are: SU$_I$(2) spinors doublets respect the so-called symplectic Majorana condition $\psi^*_\pm = 1^T (i\sigma^2)(\psi^I_\pm)^*$, $i,j = 1,2$, where $\psi^I_\pm$ are spinors of SU$_\pm$(2) and the Lorenz group is SO(4) = SU$_+$ (2) x SU$-$ (2)/Z2. Also $\sigma^\mu = (1,i\vec{\tau})$, $\bar{\sigma}^\mu = (1,-i\vec{\tau})$ and $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \bar{\sigma}^\nu\sigma^\mu)$, $\bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \sigma^\nu\bar{\sigma}^\mu)$. 

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terms of the low energy Lagrangian. Thus we drop fermions and the boson $\phi$ which has trivial vacuum.

$$L = \frac{1}{4} F^{\mu \nu} - \frac{1}{2}(\nabla)\phi^2 + D_\mu \bar{M}^i D^\mu M^i - \bar{Y} \bar{M}^i \tau^{ij} M^j$$  \hspace{1cm} (1)

Because of the future application to topological field theory we have written the Lagrangian (1) in the Euclidean space.

It is known that N=2 SUSY admits also Feyet-Iliopoulos (FI) term of the form

$$-\bar{Y} \bar{\omega}$$  \hspace{1cm} (2)

The term goes with SU$_I$(2) triplet of coupling constants $\bar{\omega}$ of the dimension of mass squared. The auxiliaries are members of the vector multiplet so FI term does not break U(1) explicitly but can break it spontaneously. It can also spontaneously break SUSY.

Let us examine the classical vacuum of the model. After solution of the equation of motions for auxiliaries from (1) plus (2) we get that the potential is

$$V = \frac{1}{2}(\bar{M}^i \tau^{ij} M^j - \bar{\omega})^2.$$  \hspace{1cm} (3)

thus the vacuum respects $\bar{Y} = \bar{\omega} - \bar{M}^i \tau^{ij} M^j = 0$. This breaks the U(1) completely.

In this case standard arguments lead to the conclusion that there exist strings as topological solitons. The width of the string is of the order $1/|\omega|$. In the limit $|\omega| \to \infty$ all classical field configurations are localized along these strings.$^4$

The supersymmetry transformations in the nontrivial (bosonic) background are as follows:

$$\delta \lambda_+^i = i(\bar{\tau}^{ij} \bar{Y} + F_{+\mu \nu} \bar{\bar{\sigma}}^{\mu \nu} \delta^{ij}) \xi_+^j$$  \hspace{1cm} (4)

$$\delta \lambda_-^i = i(\bar{\tau}^{ij} \bar{Y} + F_{-\mu \nu} \bar{\bar{\sigma}}^{\mu \nu} \delta^{ij}) \xi_-^j$$  \hspace{1cm} (5)

$$\delta \psi_+ = -i \bar{\sigma}^\mu D_\mu M^i \epsilon^{ij} \xi_-^j$$  \hspace{1cm} (6)

$$\delta \psi_- = -i \sigma^\mu D_\mu M^i \epsilon^{ij} \xi_+^j$$  \hspace{1cm} (7)

where $F_\pm = \frac{1}{2}(F \pm *F)$. Vanishing of (4,7) for some parameters $\xi^i$ yields Bogomolny type conditions. This kind of conditions has been recently thoroughly studied within

$^4$Of course in the case of Euclidean theory instead of static strings we shall have 2d surfaces as topological solitons. In the course of the paper we have decided to keep the physical terminology hoping that it will not lead to confusions. Moreover we shall deal only with SUSY transformations generated by $\xi_+$ because we want to twist SU$_+(2)$ subgroup of the (Euclidean Lorentz) group O(4).
the context of the so-called extremal p-branes for string inspired Lagrangians [13]. In the context of topological field theory the standard choice for a constant spinor is \( \xi^+_i = \epsilon^{i\dot{\alpha}} \eta \) with real \( \eta \) (\( \dot{\alpha} \) is the index of spinor representation of \( SU_+(2) \)). Then the equation of motions following from vanishing of (4,7) read

\[
0 = F_{+\mu\nu} + Y_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3
\]

\[
0 = \sigma_{\alpha\beta}^\mu D_{\mu} M^{\dot{\beta}}
\]

(8)

where \( Y_{\mu\nu} = \omega_{\mu\nu} - M \bar{\sigma}_{\mu\nu} M \) is self dual tensor defined by \( Y^{0i} = Y^i/2 \). These are famous Seiberg-Witten equations perturbed by the Feyet-Iliopoulos term \( \bar{\omega} \). This perturbation has been considered previously by many authors [2, 14, 15].

Moreover the configurations respecting (8) break only half (two out of four) of the supersymmetries \( \xi^+_i \). One is \( \xi^{i\dot{\alpha}} = \epsilon^{i\dot{\alpha}} \eta \). In order to see the another one let us denote the operator standing on the r.h.s. of (4) by \( \Theta = (F_{+\mu\nu} \bar{\sigma}^{\mu\nu} \otimes 1 + \bar{Y} 1 \otimes \bar{\tau}) \). The operator is hermitian and traceless. It also anticommutes with the symplectic Majorana operator \( i\tau^2 \otimes i\tau^2 \) (see the last footnote). It follows that zero eigenvalues go in pairs. Thus there must be two of them and we can choose them to be the eigenvalues of \( i\tau^2 \otimes i\tau^2 \). One of them is just \( \xi^{i\dot{\alpha}} = \epsilon^{i\dot{\alpha}} \eta \). The second SUSY transformation (7) does not give any new condition. To show it we apply the chiral Dirac operator \( \bar{\sigma}^\mu D_{\mu} \) on the r.h.s. of (7).

\[
\bar{\sigma}^\mu D_{\mu} \delta \psi_\chi = [D^2 - F_{\mu\nu} \bar{\sigma}^{\mu\nu}] M^i \epsilon^{ij} \xi^+_j
\]

(9)

By the formula (4), vanishing of \( \delta \lambda^+_i \) transforms this to

\[
[D^2 M^i - \bar{Y} \bar{\tau}^{ik} M^k] \epsilon^{ij} \xi^+_j
\]

(10)

The expression in square brackets is just the equation of motion for the scalar \( M \) following from (8) or (4). Thus \( \bar{\sigma}^\mu D_{\mu} \delta \psi_\chi = 0 \). We expect that in generic case this implies \( \delta \psi_\chi = 0 \). We conclude that out of four components of SUSY generators two of them are preserved by the background which respect (8).

The two broken SUSY generators have nice physical interpretation [16]. They are Goldstone fermions of partially broken supersymmetry and they become fermionic coordinates of the soliton. As we shall see the effective description of the solitonic string in the zero width limit will be given by the Green-Schwarz action. This string has two physical space-time spinors. As we shall see these spinors are exactly the two
SUSY generators that have been broken. This will be one of the arguments for the identification of the Green-Schwarz string as the string describing solutions of (8) in the limit $|\omega| \to \infty$.

Here we shall recall some properties of the physical stringy solitons in order to build-up an intuition what happens in the Euclidean case. We notice that the solitons are BPS states thus if they are parallel they do not interact. This means that multiple string configurations are also solutions of the equation of motions. For the detailed discussion of this point see [17]. The width of a single string is of the order $1/\sqrt{|\omega|}$ and shrinks to zero when $|\omega| \to \infty$. This limit in quantum field theory (1) should be somehow ill defined as we deal with UV region of non asymptotically free field theory. This is probably the source of troubles of the Green-Schwarz string. The troubles vanish when one goes to topological field theory. Thus we expect no troubles with the $|\omega| \to \infty$ limit. In this limit the topological configurations solving (8) should be localized on a 2d surfaces and moreover whenever the notion of being “parallel” can be extended to curved 4-manifolds we should expect multi-soliton solutions. The last situation can occur for the surfaces being tori.

Now we shall derive the equations determining the surface on which solutions to (8) localizes when $|\omega| \to \infty$. The position of the string is determined by the equation $M^i(x) = 0$. We dwell for a moment on the physical, static configurations. Thus we pick one of the coordinate (say $x^0$) to be the time and choose $A_0 = 0$ gauge. Then Eqs.(8) have the form

$$\vec{B} = -\frac{1}{2}(\vec{\omega} + \vec{M}\vec{\tau}M)$$  \hspace{1cm} (11)

With the appropriate boundary conditions at infinity the direction of the string (i.e. the line $X^m(\sigma)$, which respects $M^i(\vec{x}|_{\vec{x}} = \vec{X}(\sigma) = 0)$ is given by $\vec{\omega}$ i.e. the string equation is $\partial_\sigma \vec{X} = \vec{\omega}$ where $\sigma$ parameterize the string. Fields $M^i$, $\vec{A}$ have non-trivial behavior on the surface perpendicular to $\vec{\omega}$. Now we go the general case. Let the surface of zero locus of $M^i$ be given by an immersion $X: \Sigma \to M = R^4$. Thus if we pull back first of the Eqs.8 on $X$ we get

$$F_{\mu\nu}t^\mu_+ = \omega^{\mu\nu}t^\nu_+ = const \neq 0$$  \hspace{1cm} (12)

where $t^\mu_+ = t^{\mu\nu} + *t^{\mu\nu}$ and $t^{\mu\nu} \equiv e^{ab}\partial_a X^\nu \partial_b X^\nu / \sqrt{g}$ ($a, b$ indexes coordinates on $\Sigma$, $g = det(g_{ab})$, $g_{ab} = \partial_a X^\nu \partial_b X^\nu$ is the induced metric). The form $\omega$ determines an
almost complex structure \( J_\omega \) on our space-time \( M \) as well as on the pull-back tangent bundle \( X_*TM \). Using \( J_\omega \) one can decompose the space of self-dual forms as follows

\[
X_*\Lambda^2_+ M = X_*\Lambda^{(1,1)} \oplus X_*\Lambda^{(2,0)} \oplus X_*\Lambda^{(0,2)}.
\] (13)

Moreover we choose a hermitian metric on \( M \). In this metric all components of the above sum are orthogonal to each other. The equation (12) implies

\[
t^{\mu\nu}_+ = \frac{\omega^{\mu\nu}}{|\omega|} + \ldots
\] (14)

where the dots denotes the undetermined contribution of the \( X_*\Lambda^{(2,0)} \) and \( X_*\Lambda^{(0,2)} \) forms. Now we try to match with the just discussed static case. We pick up the gauge \( X^0 \propto t \). In this gauge \( \vec{X} \) are functions of \( \sigma \) only. Now comparing \( \partial_\sigma \vec{X} = \vec{\omega} \) with (14) one sees that the dots in the latter do not contribute. This fixes (14) (without dots) as the equation for the surface on which the solutions of the deformed Seiberg-Witten equations localizes in the limit \( |\omega| \to \infty \). The equations (14) are, in fact, equations for pseudo-holomorphic curves. We can see it multiplying both sides of (14) by \( \partial_\nu X^\nu \).

Using the definition of the induced metric \( g_{ab} = \partial_a X^\nu \partial_b X^\nu \) we get

\[
\frac{\omega^{\mu\nu}}{|\omega|} \partial_\sigma X^\nu = \epsilon_a^b \frac{\partial_\nu X^\mu}{\sqrt{g}}
\] (15)

Pseudo-holomorphic curves defines the so-called Gromov-Witten invariants.

These simple arguments do not give the precise relation between the Seiberg-Witten theory and the Gromov-Witten theory but show the physics of the localization. For the mathematical description we refer the reader to the original papers of Taubes [1]. Nevertheless one obtains a simple understanding of certain properties of the relation e.g. the necessity for multiple-covered tori or the use of disconnected surfaces. The physical approach presented above allows also to derive the string theory relevant for this problem. We shall dwell on this problem in the next section.

3 Green-Schwarz string

As we have showed in the previous section, the Seiberg-Witten theory (8), in the limit \( |\omega| \to \infty \), localizes on the pseudo-holomorphic curves (14). Here we would like to concentrate on the physical string theory which is directly related to the previously
studied field theory. We shall show that the string theory is familiar \( N=2 \) Green-Schwarz string \([11]\). Of course the model is quantum mechanically ill-defined for 4d space-times what is presumably reflection of the discussed UV sickness of the \( N=2 \) SUSY U(1) field theory. As we are interested in topological field theory this limit should not produce any problems.

The are several arguments leading to the \( N=2 \) Green-Schwarz string. First of all the string has massless spectrum exactly the same as that of the \( N=2 \) SUSY U(1) coupled to one \( N=2 \) hypermultiplet. Moreover as we shall show below the spinors of the \( N=2 \) Green-Schwarz string are the Goldstone modes of the partially broken supersymmetry of the field theory. Together with the supersymmetry of the theory this uniquely identifies the \( N=2 \) Green-Schwarz string as the effective theory describing the solitonic string of the previous section.

To match with field theory description of the previous section we must slightly modify the original formulation of the Green-Schwarz string. The modification reveals the \( SU_I(2) \) structure of the theory. One should expect this because the equation of string world-sheet surface \([14]\) depends explicitly on \( \vec{\omega} \). The \( N=2 \) Green-Schwarz string consists of the Nambu-Goto term and the Wess-Zumino term \( L_{WZ} \):

\[
S = \int d^2z \left\{ \frac{1}{2} \sqrt{g} + L_{WZ} \right\}
\]

where \( g_{a\bar{b}} = \Pi^a\Pi^\bar{b}, \Pi^a = \partial_a X^\mu - i(\bar{\theta}_+ \sigma^\mu \partial_\mu \theta_- + \bar{\theta}_- \sigma^\mu \partial_\mu \theta_+) \) \((i = 1, 2)\). The famous Wess-Zumino term is responsible for the existence of the local \( \kappa \)-symmetry. The two terms transform differently under \( SU_I(2) \) : the kinetic term is a singlet, the Wess-Zumino term is a component of a triplet. The easiest way to produce the Wess-Zumino term is to construct closed 3-forms in the target superspace. It appears that they form a \( SU_I(2) \) triplet

\[
\vec{\Omega} = \Pi^\mu d\bar{\theta}_+^i \sigma^\mu d\theta_-^j \vec{\tau}^{ij}
\]

The forms \( \vec{\Omega} \) are exact thus we define triplet of the Wess-Zumino 2-forms \( \vec{\Omega}_{WZ} \) by

\[
\vec{\Omega} = d\vec{\Omega}_{WZ},
\]

The proposed Wess-Zumino term is:

\[
L_{WZ} = \vec{\Omega}_{WZ} \vec{y}
\]

\[
= -\frac{1}{2} \epsilon^{ab} [\partial_a X^\mu - i(\bar{\theta}_+ \sigma^\mu \partial_\mu \theta_- + \bar{\theta}_- \sigma^\mu \partial_\mu \theta_+)] (\bar{\theta}^k_+ \sigma^\mu \partial_\mu \theta_- - \partial_\mu \theta^k_+ \sigma^\mu \theta_-) \vec{\tau}^{kl} \vec{y}
\]
where \(|\vec{y}| = 1\). The vector \(\vec{y}\) will be determined by comparison with the field theory of the previous section. The standard Green-Schwarz string action is recovered for \(\vec{g}_{gs} = (0, 0, 1)\). The global SUSY transformation is not changed by this modification,

\[
\delta X^\mu = -i(\bar{\theta}_i^+ \bar{\sigma}^{\mu} \xi_i^+ + \bar{\theta}_i^- \sigma^{\mu} \xi_i^-) \tag{19}
\]

\[
\delta \theta_i^+ = \xi_i^+ \tag{20}
\]

\[
\delta \theta_i^- = \xi_i^- \tag{21}
\]

but the \(\kappa\)-symmetry is modified:

\[
\delta X^\mu = i(\bar{\theta}_+^i \sigma^{\mu} \delta \theta_i^+ + \bar{\theta}_-^i \sigma^{\mu} \delta \theta_i^-) \tag{22}
\]

\[
\delta \theta_i^+ = i(\vec{y} \tau^{ij} + \delta^{ij} \Gamma_+) \kappa_i^+ \tag{23}
\]

\[
\delta \theta_i^- = i(\vec{y} \tau^{ij} + \delta^{ij} \Gamma_-) \kappa_i^- \tag{24}
\]

and similarly for \(\delta \theta_i^\pm\). In the above

\[
\Gamma_+ = \frac{\epsilon^{cd} \Pi_c^\mu \Pi_d^\nu \bar{\sigma}^{\mu\nu}}{\sqrt{g}} \tag{25}
\]

We note here that \(\bar{\sigma}^{\mu\nu}\) is self-dual so it picks up only the self-dual part of the tensor on the r.h.s. of (25). For \(\theta_\pm = 0\) we have \(\Gamma_+ = \frac{1}{2} t^{\mu\nu} \bar{\sigma}^{\mu\nu}\). This will be crucial for comparison with field theory.

Now we connect this picture with the previous field theory considerations. We recall that the center of the solitonic string is defined by \(M^i = 0\), so at this point the first Seiberg-Witten equation gives \(F_{+\mu\nu} = \omega_{\mu\nu}\) thus, by (24), also \(F_{+\mu\nu}/|\omega| = \frac{1}{2} t_{+\mu\nu}\). Inserting this back to (24) and identifying \(|\omega|\vec{y} = \vec{\omega}|_{M^i = 0} = \vec{\omega}\) we realize that the latter coincides with field theory transformation (11) taken at the string world-sheet. This means that the spinor which can be gauged away by (24) is exactly the field theory spinor of preserved SUSY transformation (4). This is in perfect agreement with physical intuition that only broken generators of a symmetry constitute the physical degrees of the string [16].

Thus the theory (16) with the Wess-Zumino term given by (19) and \(\vec{y} = \vec{\omega}/|vom|\) is the seeking effective string theory describing \(N=2\) SUSY \(U(1)\) fields theory string.

There are several things which should be farther clarified and worked out. Among them there is the topological field theory behind the above \(N=2\) Green-Schwarz string and its comparison with the theory of pseudo-holomorphic curves (14). We shall devote our future publication to these subjects.
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