Approximate solutions in scalar and fermionic theories within 
the exact renormalization group approach

Jordi Comellas †
Departament d’Estructura i Constituents de la Matèria, Facultat de Física
Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain

Yuri Kubyshin ‡§
Departamento de Física Teórica C-XI, Universidad Autónoma de Madrid
Cantoblanco, 28049 Madrid, Spain

and

Enrique Moreno ¶
Department of Physics, City College of New York
New York, NY 10031, U.S.A.

January 18, 1996

Abstract

We give a review of the exact renormalization group (ERG) approach and illustrate its applications in scalar and fermionic theories. The derivative expansion and approximations based on the derivative expansion with further truncation in the number of fields (mixed approximation) are discussed. We analyse the mixed approximation for a three-dimensional scalar theory and show that it is less effective than the pure derivative expansion. For pure fermionic theories analytical solutions for the pure derivative expansion and mixed approximation in the limit $N \to \infty$, where $N$ is the number of fermionic species, are found. For finite $N$ a few series of fixed point solutions with their anomalous dimensions and critical exponents are computed numerically. We argue that one of the fermionic solutions can be identified with that of Dashen and Frishman, whereas the others seem to be new ones. The issues of spurious solutions and scheme dependence of the results are discussed.
1 Introduction

One of the methods, which, in principle, is capable of handling non-perturbatively many problems of quantum field theory (such as calculation of exact renormalization flows, explicit derivation of low-energy effective Lagrangians from high-energy ones, extension of the c-theorem to higher dimensions, etc.) is the exact renormalization group (ERG, hereafter). Originally developed by Wilson in his seminal articles in the early seventies [1] (see Ref. [2] for a classical review), it has recently attracted much attention.

In this article we are going to consider a version of the ERG approach based on a functional differential equation. It determines how the action changes as momentum modes are integrated out in lowering down a characteristic momentum scale (cutoff) Λ while keeping the $S$-matrix elements invariant. This is the so-called ERG equation. Using it for the most general action, consistent with the symmetries of the model under consideration, the complete set of β-functions (or flow equations) can be derived, and from them the location of fixed points, critical exponents, behaviour of the renormalization flows, low-energy effective action, etc. can be computed in principle. However, with known techniques it is not possible to handle such general actions for physically interesting models. One needs to choose an approximation which, on one hand, reduces sufficiently the complexity of the problem and, on the other hand, captures the essential features of the physical system under consideration. Approximations most commonly used in the literature are the local potential approximation [3] - [7], polynomial approximation for the effective potential, which can be regarded as further simplification of the local potential approximation, [3], [4] and the derivative expansion [10] - [13]. Discussion of their main features and some results obtained by these techniques are given in Sect. 2.3. For calculations we will use the derivative expansion in the case of fermions and the derivative expansion combined with expansion in powers of fields (mixed approximation) for fermionic and scalar theories.

In this article we will consider the Polchinski type ERG equation [15] with its derivation being discussed along the lines of Ref. [16]. It is worth mentioning that there are other types of ERG equations used in quantum field theory calculations. Among them is the one of Ref. [3], based on the sharp cutoff realization of the Wilson idea by Wegner and Houghton [17], the average effective action equation [18], a version of the ERG for the effective action with a sharp cutoff of Ref. [19], to name a few.

Although firstly used for studies of critical phenomena in condensed matter problems [20], now the scope of the ERG approach covers many other areas [21], including field theory, as it was demonstrated by Polchinski in an elegant paper where he proved the perturbative renormalizability of $\lambda\phi^4$ in a quite simple way [15]. Most of the work was devoted to scalar theories, both to their general studies [15], [22] and to computation of fixed points and the critical exponents [3] - [6], [8] - [9], [11] - [14]. Much work has been done on gauge theories, see Refs. [23] for examples. However, a satisfactory formulation of the approach is lacking in this case, the main difficulty being to maintain the gauge invariance along the renormalization flow of the regularized action. It is worth mentioning studies of phase transitions at finite temperature [24], of bound states [25], [26] and in relativistic cosmology [27] within the ERG approach. Investigations of theories with fermions, which have been done so far, are very restricted. In [28] the ERG approach was used for proofs of the renormalizability and convergence of the perturbative expansion of the two-dimensional Gross-Neveu model. Discussions of physical effects in many of the articles are limited to perturbative expansions [29] or models with fermions interacting with scalars either through Yukawa coupling [29] or through coupling of the fermion bilinears to a scalar potential and then studying the fixed points and renormalization flows for this potential [30], [31]. The ERG formalism for pure fermionic theories, in which fermionic degrees of freedom are treated directly and non-perturbatively, has been developed recently in Ref. [32]. Fixed point solutions and the critical exponents were also analysed there.

The purpose of this article is to give an account of basic ideas of the approach and main steps of the derivation of the ERG equation (for the scalar and fermionic theories) and to describe main techniques used for non-perturbative calculations of fixed points and critical exponents. We will also give a review of main results of such calculations and illustrate the application of the techniques in two concrete models. One of them is a three-dimensional model of one-component scalar field with $Z_2$-symmetry. It possesses the non-trivial fixed point (Wilson fixed point), and was analysed by many authors (see [8], [5], [3] - [5], [11] - [14]). The other example is a two-dimensional Gross-Neveu type theory which seems to be a perfect site for such studies: its phase diagram is non-trivial and, on the other hand, the two-dimensional
Clifford algebra is the simplest one and generated by the Pauli matrices. In the analysis of this model we will follow mainly Ref. [32], but also some new results on the pure derivative expansion will be presented.

We would like to mention that pure fermionic theories, analysed in this article, are widely used in description of fundamental interactions. There are some interesting phenomenological models of this kind, like the celebrated Fermi theory of weak interactions [33], models of resonance physics [34] based on extensions of the Nambu-Jona-Lasinio action [35], models explaining the symmetry-breaking sector of the electroweak theory, especially in connection with technicolor theories [36], etc. Also we would like to remind that fermions are not always easily manageable by non-perturbative methods, e.g. in lattice formulation. On the contrary, as we will show, there are no principal difficulties in treating fermions within the ERG approach. Moreover, once truncated, the ERG equations for fermions and bosons look very similarly, thus making possible applications of techniques, already known in bosonic theories, in the case of fermions.

Moreover, the two-dimensional Gross-Neveu model [37], which contains \( N \) species of fermions transforming under a global representation of the unitary group \( U(N) \), is interesting in its own. It is asymptotically free and renormalizable within perturbation theory and also within the \( 1/N \) expansion. However, none of these approximations is capable to find any non-trivial fixed point for \( d = 2 \). An interesting modification leads to the so-called chiral Gross-Neveu model [37], which is chosen to have the additional symmetry of the \( U(1) \) chiral group. As in the previous case no fixed points, besides the Gaussian one, can be found within the \( 1/N \) approximation. However, using non-perturbative methods based on current algebra and conformal techniques Dashen and Frishman in Ref. [38] found two critical curves in the space of couplings for which the theory is scale invariant. One of the lines corresponds to the abelian Thirring model, whereas the other one is truly non-trivial and does not pass through the origin. A very remarkable fact is that this result is exact and is not given by a zero of a \( \beta \)-function, neither in the perturbation theory nor in the large \( N \) expansion. For this two continuum sets of critical theories the value of the coupling constant associated to the \( SU(N) \) degrees of freedom is fixed to be equal to zero or \( 4\pi/(N+1) \) respectively, while the coupling associated to the abelian degrees of freedom is arbitrary. More recently, using bosonization, current algebra and conformal techniques, other non-trivial fixed points in two-dimensional fermionic models were found [39].

The article is organized as follows. In Sect. 2 we derive the ERG equation for scalar theories and study the mixed approximation for a three-dimensional model. Results of computation of the fixed points and critical exponents are compared with those obtained by the pure derivative expansion and the polynomial approximation in earlier works. In Sect. 3 we derive the ERG equation for pure fermionic theories for arbitrary number of spacetime dimensions. Sect. 4 is devoted to the construction of the action. The chiral Gross-Neveu type model, we are going to study, is defined through its symmetries, and the approximations, we are going to use, are explained. In Sect. 5 we work out the pure derivative expansion and obtain analytically two fixed point solutions. The calculation of the \( \beta \)-functions, fixed points and critical exponents within the mixed approximation in the fermionic model is covered by Sect. 6. Sect. 7 contains discussion of the results and some conclusions.

## 2 Scalar theory

In this section we explain the main points of the derivation of the ERG equation for the scalar case following the lines of Ref. [40]. We also present an example of calculation of the fixed point and the critical exponents in a scalar \( Z_2 \)-symmetric model in three (Euclidean) dimensions in order to illustrate how the whole machinery works.

As usual in this type of problems, there are three important steps. The first one is to derive an ERG equation which governs the behavior of the action as we integrate out modes, that is, how the action of our effective theory has to be modified when we vary the characteristic scale (cutoff) \( \Lambda \). This is achieved by requiring the independence of the S-matrix elements on \( \Lambda \). However, it is more convenient to impose a stronger condition, namely the independence of Green’s functions on \( \Lambda \). The second step is to choose the appropriate “space of local interactions”. This amounts to define properly the theory and a sensible approximation of the general action. The last step is to compute the \( \beta \)-functions, which characterize the RG flow of the theory, and to obtain physically relevant information from it. We may be interested in global properties of the flow, like the number of the fixed points it contains, or just local ones, like the
behavior of the flow in the vicinity of them. The latter is characterized by the critical exponents, which are, moreover, universal quantities.

### 2.1 ERG equation

To derive an ERG equation we impose the condition that the generating functional of Green’s functions

$$ Z[J] = \int D\phi \exp \left( -S[\phi; \Lambda] + \int_p J_p Q_{\Lambda}^{-1} \phi_{-p} + f_\Lambda \right) $$

is independent of some cutoff \( \Lambda \) that sets the scale of the theory. Here \( J_p \) is an external source, \( Q_{\Lambda} \) a regulating function and \( f_\Lambda \) is a c-number quantity.

The action is arbitrarily divided into a “kinetic term” and an interaction part as

$$ S[\phi; \Lambda] \equiv \frac{1}{2} \int_p \phi_{p} \phi_{-p} P_{\Lambda}^{-1} (p^2) + S_{\text{int}} [\phi; \Lambda], $$

where \( P_{\Lambda} \) stands for the regulated free propagator

$$ P_{\Lambda} (p^2) = (2\pi)^d \frac{K(z)}{p^2} $$

with \( K(z) \) being an arbitrary (but fixed) cutoff function which vanishes faster than any polynomial when \( p^2 \to \infty \) and satisfies \( K(0) = 1 \).

Using the path integral identity

$$ \int D\phi \frac{\delta}{\delta \phi_p} \left( \frac{1}{2} \frac{\delta}{\delta \phi_{-p}} + P_{\Lambda}^{-1} \phi_p - Q_{\Lambda}^{-1} J_p \right) e^{-\frac{1}{2} S[\phi; \Lambda]} + \int_p J_p Q_{\Lambda}^{-1} \phi_{-p} + f_\Lambda = 0, $$

we can write the condition of the independence of the generating functional on the cutoff \( \Lambda \) as an evolution equation for the total action \( S[\phi, \Lambda] \)

$$ \left\langle \dot{S} \right\rangle = \left\langle \frac{1}{2} \int_p \dot{P}_{\Lambda} (p^2) \left( \frac{\delta S}{\delta \phi_p} \frac{\delta S}{\delta \phi_{-p}} - \frac{\delta^2 S}{\delta \phi_p \delta \phi_{-p}} \right) - \int_p \frac{\delta S}{\delta \phi_p} \dot{P}_{\Lambda} P_{\Lambda}^{-1} \phi_p \right\rangle (3) $$

provided

$$ f_\Lambda = \int dt \left\{ \int_p \dot{Q}^{-2} (p^2) \frac{\dot{P}_{\Lambda} (p^2)}{P_{\Lambda} (p^2)} J_p J_{-p} - \int_p \frac{\dot{P}_{\Lambda} (p^2)}{P_{\Lambda} (p^2)} \delta (0) \right\}, $$

where \( Q_{\Lambda} (p^2) = P_{\Lambda} (p^2) \dot{Q} (p^2) \) and \( \dot{Q} (p^2) \) satisfies \( \dot{Q} (p^2) = (\eta/2) \dot{Q} (p^2) \). (We have defined \( \dot{\phi} \equiv \left( \eta/2 \right) \dot{\phi} \), and the derivatives with respect to \( \Lambda \) should be understood as acting only on the coefficients of the action, not on the fields. This convention will be used throughout this paper.

Note that even though Eq. (3) depends on the source \( J_p \) through the Boltzmann weight, the argument of the v.e.v.’s does not. So Eq. (3) is valid for arbitrary \( J_p \) if and only if the arguments of the r.h.s and l.h.s vacuum expectation values are equal. So from now on we will drop the v.e.v brackets.

To complete the renormalization group transformation we should add the canonical rescalings of the couplings. The easiest way to proceed is to re-write Eq. (3) in terms of dimensionless quantities by extracting the appropriate powers of \( \Lambda \) carrying the canonical scale dependencies. We get

$$ \dot{S} = \int_p (2\pi)^d K' (p^2) \left( \frac{\delta S}{\delta \phi_p} \frac{\delta S}{\delta \phi_{-p}} - \frac{\delta^2 S}{\delta \phi_p \delta \phi_{-p}} \right) + dS $$

where all quantities should be understood as dimensionless. We have defined \( K'(z) \equiv dK(z)/dz \) and the prime in \( \delta / \delta p \) means that the derivative does not act on the momentum conservation delta functions in \( S \). This is our final ERG equation for a scalar theory. It is exact, and being supplied with the initial condition \( S |_{t=0} \) defines completely the renormalization group flow of the action.
2.2 The action

The second step is to define the theory, i.e. to specify the space-time on which the model is defined, its fields and its symmetries. Then, after characterizing the model we are instructed to take an action which is the most general one consistent with the stated symmetries. In our example this would be a power-like interaction with arbitrary number of derivatives invariant under space rotations and discrete $Z_2$-symmetry. The next step would be to use our equation to obtain the exact renormalization flows of our theory.

This program is, however, as nice as unrealistic. It is obviously impossible to deal with such a large action. And even in the case the $\beta$-functions were computed somehow, the infinite set of equations would be intractable. Consequently, one has to develop a reasonable approximation.

In the fermionic case we will use a mixed approximation consisting in expansion in derivatives and truncation in the number of fields. This will also illustrate the calculation of fixed points and their critical exponents within the ERG approach. A short review of other approximations used in the literature and discussion of their effectiveness is given at the end of the next subsection. Namely, we consider a scalar field theory on the Euclidean space with $Z_2$ symmetry and we keep all local terms up to six fields and two derivatives. The number of the derivatives is chosen to be the minimum one which allows, in principle, a non-vanishing anomalous dimension. Therefore, we take the following initial interaction action:

\[
S_{\text{int}} = (2\pi)^{-d} \int \sum_{p} a_1 \phi_p \phi_{-p} + (2\pi)^{-3d} \int_{p_1, \ldots, p_4} (a_2 + b_2 p_1^2) \phi_{p_1} \cdots \phi_{p_4} \delta(\Sigma p_i)
\]

\[
+ (2\pi)^{-5d} \int_{p_1, \ldots, p_6} (a_3 + b_3 p_1^2) \phi_{p_1} \cdots \phi_{p_6} \delta(\Sigma p_i)
\]

where $a_1, a_2, a_3, b_2$ and $b_3$ are real coupling constants.

2.3 Fixed points and critical exponents

It is a simple exercise to substitute the above action into Eq. (4) and obtain after a bit of algebra the complete set of $\beta$-functions within this approximation. The system of $\beta$-functions will depend on the particular shape of the regulating function $K(x)$ through four real quantities:

\[
\alpha = \int_p K'(p^2), \quad \beta = \int_p p^2 K'(p^2), \quad \gamma = K'(0), \quad \delta = K''(0).
\]

However it can be proved that, within our approximation, the real dependence on the scheme is given not by four but only by two independent parameters. This simplification can be made explicit after the following redefinition:

\[
a_1 \to \frac{1}{\gamma} a_1, \quad a_2 \to \frac{1}{\alpha \gamma} a_2, \quad a_3 \to \frac{1}{\alpha \gamma^2} a_3, \quad b_2 \to \frac{\delta}{\alpha \gamma^2} b_2, \quad b_3 \to \frac{\delta}{\alpha \gamma^2} b_3.
\]

So, in these new variables the set of $\beta$-functions reads,

\[
0 = \eta - y \left(12b_2 - 8a_1^2\right)
\]

\[
a_1' = (2 + \eta) a_1 - 12a_2 - 6b_2/x + 4a_1^2
\]

\[
a_2' = (4 - d + 2\eta) a_2 - 30a_3 - 10b_3/x + 16a_1 a_2
\]

\[
a_3' = (6 - 2d + 3\eta) a_3 + 24a_1 a_3 + 16a_2^2
\]

\[
b_2' = (2 - d + 2\eta) b_2 + 16a_1 a_2 + 16a_1 b_2 - 20b_3
\]

\[
b_3' = (4 - 2d + 3\eta) b_3 + \frac{192}{5} a_2 b_2 + 24a_1 b_3 + 24a_1 a_3 + \frac{144}{5} a_2^2,
\]

where $y \equiv \delta/\gamma^2$ and $x \equiv \alpha \gamma / \beta \delta$. 
The value of the coupling constants at fixed points will be sensitive to \( x \) and \( y \). The critical exponents in principle are independent on them. However, the exponents depend on the scheme because of the truncation, as one can immediately realize noting that the anomalous dimension \( \eta \) (which in the vicinity of a fixed point becomes one of the critical exponents) depends almost linearly on \( y \) (some non-linear dependence enters through \( a_1 \) and \( b_2 \)). A similar pattern of scheme dependence was found in Ref. \[14\].

To be concrete, let us consider briefly the numerical results that can be obtained from Eq. (7) in \( d = 3 \). To this end we find first the fixed points of the theory: the points in the space of couplings for which all the \( \beta \)-functions vanish. That is, we solve the following system of equations:

\[
\begin{align*}
0 &= (2 + \eta) a_1 - 12 a_2 - 6 b_2 \frac{x}{x} + 4 a_1^2 \\
0 &= (1 + 2\eta) a_2 - 30 a_3 - 10 b_3 \frac{x}{x} + 16 a_1 a_2 \\
0 &= 3\eta a_3 + 24 a_1 a_3 + 16 a_2^2 \\
0 &= (-1 + 2\eta) b_2 + 16 a_1 a_2 + 16 a_1 b_2 - 20 b_3 \\
0 &= (-2 + 3\eta) b_3 + \frac{192}{5} a_2 b_2 + 24 a_1 b_3 + 24 a_1 a_3 + \frac{144}{5} a_2^2.
\end{align*}
\]

(8)

where \( \eta \) is defined in Eq. (1).

By resolving the second, third and fifth equations in (8) with respect to \( a_2 \), \( a_3 \) and \( b_3 \) respectively one can easily reduce this system to a pair of non-linear coupled algebraic equations for \( a_1 \) and \( b_2 \). These remaining equations have to be solved numerically. The trivial point, where all the couplings are zero, is always a solution and corresponds to the Gaussian fixed point. As it is known, in the model under consideration there is also one non-trivial fixed point, the Wilson fixed point. The critical behavior of the model at this point belongs to the same universality class as the Ising model in \( d = 3 \). The system of equations (8) has many other zeroes besides the two corresponding to the physical fixed points. The appearance of spurious solutions is a common feature of the polynomial approximation, and as we will discuss later, is very difficult to handle (see Refs. [6], [9]). However, analysing successive approximations it is possible to identify the physical solution among the fictitious ones: the latter do not stabilize as we increase the order of the approximation. In our simple scalar model we can manage high orders of the polynomial expansion, so we can easily find the true fixed point. This is not the case for fermionic systems where a six-degree polynomial in fields contains more than hundred terms within the approximation with two derivatives. We will discuss these issues more extensively later in this article.

Having found the true fixed point, the next step is to study the asymptotic behavior of the coupling constants flow in the vicinity of it, governed by the critical exponents. They can be computed by linearizing the RG transformations near the fixed point. That is, if \( g_i \) is a generic coupling constant, then its variation in the vicinity of a fixed point \( g^* = (g_1^*, g_2^*, \ldots, g_i^*, \ldots) \) is approximated by \( \delta g_i = \dot{g}_i = R_{ij}(\omega) \delta g_j \), where \( g_i(t) = g_i^* + \delta g_i(t) \) and \( R_{ij} \) is the matrix \( \partial \delta g_i / \partial g_j \). The eigenvalues of \( R_{ij} \omega^* \) can be identified with critical exponents. The biggest one, \( \lambda_1 \), is the inverse of the exponent \( \nu \), which governs the correlation length in the critical domains, and the second one \( \lambda_2 \), is minus the exponent \( \omega \) associated to the slope of the \( \phi^4 \) operator \( \beta \)-function at the critical point.

For our system (4) the matrix of linear deviations reads:

\[
\begin{pmatrix}
2 - \eta + 8 a_1 & -12 & 0 & -6/x & 0 \\
16 a_2 & 1 - 2\eta + 16 a_1 & -30 & 0 & -10/x \\
24 a_3 & 32 a_2 & -3\eta + 24 a_1 & 0 & 0 \\
16(a_2 + b_2) & 16 a_1 & 0 & -1 - 2\eta + 16 a_1 & -20 \\
24(a_3 + b_3) & \frac{288}{5} a_2 + \frac{144}{5} b_2 & 24 a_1 & 192/5 a_2 & -2 - 3\eta + 24 a_1
\end{pmatrix}
\]

where \( \eta \) takes the value given in the first line of Eq. (7) and all the variables are evaluated at the fixed point.

It is a simple exercise to carry out the above program in our example. The numerical analysis of the solutions of (8) shows that the anomalous dimension \( \eta \) is almost linear in \( y \) (for fixed \( x \)). On the contrary, for any fixed \( y \) the function \( \eta(x) \) has a maximum at some value \( x^*(y) \). Consequently we used this minimal sensitivity criterion \[40\] to fix the parameter \( x \) at \( x = x^*(y) \). Unfortunately, due to the
monotonous dependence of the solution on \( y \) we are unable to set it by a similar prescription. A more
careful analysis requires the study of the dependence of the critical exponents on this last parameter,
and, perhaps, following this direction one could fix it by applying the principle of minimal sensitivity.
We made this study for the critical exponents \( \nu \) and \( \omega \) but with our working precision we could not find
any noticeable oscillation. This, perhaps indicates the poorness of the approximation.

Within the range \( 0.1 \leq y \leq 1 \) we computed the anomalous dimension and the first two critical
exponents (\( \nu \) and \( \omega \)). Considering the scantiness of our truncation the results for the anomalous dimension
and the critical exponents \( \nu \) and \( \omega \) are quite good. As we mentioned above the values of \( \eta \) as a function of
\( y \) are monotonous ranging from 0.016 when \( y = 0.1 \) to 0.1 when \( y = 1 \). Notice that this range includes the “known” value of the anomalous dimension (\( \eta = 0.035 \pm 0.005 \)). Within our accuracy, the value of \( \nu \)
in this range is constant,

\[
\nu = 0.53 \pm 0.015
\]  

(9)

and \( \omega \) ranges between

\[
0.86 \leq \omega \leq 0.96.
\]  

(10)

The known values for \( \nu \) and \( \omega \) are, respectively, \( \nu = 0.635 \pm 0.005 \) and \( \omega = 0.8 \pm 0.05 \) [11].

Now let us discuss other approximations and compare with our results. As it was already mentioned
in the Introduction, one of the simplest approximations is the local potential approximation in which the
interaction Lagrangian just reduces to the effective potential [3] (see Refs. [4] - [7] for further applications).
In this case the flow equation is a partial differential equation, and the fixed point equation is a non-linear ordinary differential equation. Though this equation has a family of solutions only one of them
is finite for every value of the field and we immediately recognize it as the Wilson fixed point For this
approximation the anomalous dimension is zero and the values of the critical exponents depend on the
particular ERG equation used. Using a sharp cutoff ERG equation for the effective potential the authors
of Ref. [3] obtained \( \nu = 0.687, w = 0.595 \). In Ref. [11] these critical exponents were calculated from the
flow equation for the Legendre effective action with a power-like additive cutoff. The results there are
\( \nu = 0.66, w = 0.628 \). With a Polchinski type equation within the same approximation the authors of
Ref. [12] obtained \( \nu = 0.649 \) and \( w = 0.66 \).

Further simplification of the local potential approximation can be achieved by representing the effective
potential as a polynomial in the field [8], [9]. In this case the differential fixed point equation transforms
into a system of the algebraic equations, but many spurious solutions appear. The presence of such
fictitious solutions is thus the characteristic feature of the approximations based on truncations in the
number of fields. Analyzing successive orders of the approximation it is possible in some cases to recognize
the true fixed point solution, at least for not very low dimensions (for example for \( d > 8/3 \)). Another and
more serious problem is the convergence of the polynomial approximation as the order of the polynomial
increases. Indeed there are some evidence that the polynomial approximation does not converge, at least
for the sharp cutoff approximation [8], even though the low orders give reasonably good numerical results
for the critical exponents. Thus, for \( d = 3 \) the polynomial approximation of the Wilson fixed point with
7 terms gives \( \nu = 0.657 \) and \( w = 0.705 \) [8].

In general better numerical results are obtained in the framework of the derivative expansion [10] -
[12], where the interaction Lagrangian is expanded in powers of momenta. Since we are interested in the
low-energy properties of the model, contributions of terms with high powers of momenta are expected to
be sub-leading. No general solid results on the convergence of this expansion have been obtained so far
(see, however, a discussion in Ref. [13]). The local potential approximation is thus the leading order of
the derivative expansion. When the next-to-leading order (i.e. terms with two derivatives) is taken into
account the numerical results improve in general. Thus, for the ERG equation of Ref. [11] \( \eta = 0.054, 
\nu = 0.618 \) and \( w = 0.897 \). For the Polchinski type equation the critical exponents turn out to be scheme
dependent, as in our example above. In [14] it was shown that for a certain reasonable variation of the
scheme parameters the ranges of variation of the critical exponents are the following: \( 0.019 < \eta < 0.056, 
0.616 < \nu < 0.637 \) and \( 0.70 < w < 0.85 \).

We see that the mixed approximation with only 6 terms, studied in this article, works rather well
comparing to other methods. Its obvious advantage is simplicity. However, it possesses the drawbacks of
both the polynomial approximation and the derivative expansion. We expect that corrections due
to higher orders in the action should improve the results [11] and [12]. In principle, they also can be
improved by choosing the best suited renormalization scheme, as it has been done in Ref. [14].
3 ERG equation for fermionic theories

In this section we derive the ERG equation for a pure fermionic field theory on the Euclidean space of dimension $d$. The equation will be very similar to that for bosons.

The action is splitted, as usual,

$$S = S_{kin} + S_{int},$$

where $S_{int}$ is an arbitrary function of the fields and derivatives or momenta (if we work in the momentum space) and $S_{kin}$ is a regulated version of the usual kinetic term,

$$S_{kin} = \int_p \overline{\psi}_p P^{-1}_\Lambda (p) \psi_p,$$

where $P_\Lambda$ is now the matrix

$$P_\Lambda (p) = (2\pi)^d K (\frac{p^2}{\Lambda^2}) i \hat{p}.$$

$K(z)$ is a cutoff function with the same properties as that for bosons.

The starting point of the derivation is now the relation

$$\left< \left( \frac{\delta}{\delta \overline{\psi}_p} - \overline{\psi}_p P^{-1}_\Lambda + \overline{\chi}_p Q^{-1}_\Lambda \right) \dot{P}_\Lambda^{-1} \left( \frac{\delta}{\delta \overline{\psi}_p} + P^{-1}_\Lambda \psi_p - Q^{-1}_\Lambda \chi_p \right) \right> = - \left< \text{tr} \left( P^{-1}_\Lambda \dot{P}_\Lambda \right) \delta (0) \right> - \left< \left( \overline{\psi}_p P^{-1}_\Lambda - \overline{\chi}_p Q^{-1}_\Lambda \right) \dot{P}_\Lambda \left( P^{-1}_\Lambda \psi_p - Q^{-1}_\Lambda \chi_p \right) \right>,$$

where the trace is taken over the spinor indices. This is the counterpart of Eq. (2) of the previous section and, as there, it can be used to identify the rate of change of the kinetic term.

Imposing the independence of the generating functional $Z = \langle 1 \rangle$ on the scale $\Lambda$, we come, after a bit of algebra, to the relation

$$\left< \dot{S}_{int} \right> = - \int_p \left< \frac{\eta}{2} \overline{\psi}_p P^{-1}_\Lambda \psi_p + \overline{\psi}_p \dot{P}^{-1}_\Lambda \psi_p \right> - \frac{\eta}{2} \int_p \left< \overline{\psi}_p \frac{\delta S_{int}}{\delta \overline{\psi}_p} + \psi_p \frac{\delta S_{int}}{\delta \psi_p} \right> + \int_p \left< \frac{\eta}{2} \overline{\chi}_p Q^{-1}_\Lambda \psi_p + \frac{\eta}{2} \overline{\psi}_p Q^{-1}_\Lambda \chi_p + \overline{\chi}_p \dot{Q}^{-1}_\Lambda \psi_p + \overline{\psi}_p \dot{Q}^{-1}_\Lambda \chi_p \right> + \left< \dot{f}_\Lambda \right>,$$

where the anomalous dimension is similarly defined by

$$\psi_p = \frac{\eta}{2} \psi_p, \quad \overline{\psi}_p = \frac{\eta}{2} \overline{\psi}_p.$$

We can write an equation for $\left< \dot{S} \right>$ by just adding the contribution of the kinetic term. It will be satisfied if

$$\dot{S} = \int_p \left( \frac{\delta S}{\delta \overline{\psi}_p} \dot{P}_\Lambda (p) \frac{\delta S}{\delta \psi_p} - \frac{\delta}{\delta \overline{\psi}_p} \dot{P}_\Lambda (p) \frac{\delta S}{\delta \psi_p} \right) + \int_p \left[ \frac{\delta S}{\delta \psi_p} \left( \dot{P}_\Lambda (p) P^{-1}_\Lambda (p) \psi_p - \overline{\psi}_p \left( P^{-1}_\Lambda (p) \dot{P}_\Lambda (p) \right) \frac{\delta S}{\delta \psi_p} \right] \right].$$

Similarly, the equations for the terms containing $\chi_p$ and $\overline{\chi}_p$ and the term $f_\Lambda$ without the sources and the fields are

$$Q_\Lambda (p) = P_\Lambda (p) \dot{Q} (p^2), \quad f_\Lambda = - \int dt \int_p \dot{Q}^{-2} (p^2) \overline{\chi}_p \dot{P}_\Lambda^{-1} (p) \chi_p,$$
with $\tilde{Q}(p^2)$ being a scalar function which evolves according to the equation $\dot{\tilde{Q}}(p^2) = (\eta/2)\tilde{Q}(p^2)$.

Finally, after performing the pertinent rescalings, we arrive at the equation we will use in the sequel,

$$
\dot{S} = \int_p 2(2\pi)^d K'(p^2) \left( \frac{\delta S}{\delta \bar{\psi}_p} \frac{\delta S}{\delta \bar{\psi}_p} - \frac{\delta}{\delta \bar{\psi}_p} i \gamma^\mu \frac{\delta S}{\delta \bar{\psi}_p} \right) + dS + \int_p \left( \frac{1}{2} \frac{d}{2} - \frac{1}{2} - 2p^2 K'(p^2) \psi_p \frac{\delta S}{\delta \bar{\psi}_p} \right) \left( \psi_p \frac{\delta S}{\delta \bar{\psi}_p} + \bar{\psi}_p \frac{\delta S}{\delta \bar{\psi}_p} \right) - \int_p \left( \bar{\psi}_p \gamma^\mu \frac{\partial}{\partial p^\mu} \frac{\delta S}{\delta \bar{\psi}_p} + \psi_p \gamma^\mu \frac{\partial}{\partial p^\mu} \frac{\delta S}{\delta \bar{\psi}_p} \right).
$$

Notice that, once the above ERG equation is derived, it would be easy to obtain a similar one for a model involving also other fields and interactions just by combining the manipulations here with those of the previous section. (The resemblance of Eq. (11) and the Polchinski type equation for scalar theories is pretty evident.)

As a final comment about our equation let us comment that we present it on Euclidean space as it is customary in the field. For our purposes, however, there is nothing special about the Euclidean formulation, as finally what one obtains is just a set of relations among coupling constants. In fact, we have also derived the counterpart of Eq. (11) for Minkowski space. It is not so nice because of the presence of an extra imaginary unit coming from the functional derivatives of the Minkowskian “Boltzmann” factor $e^{i\phi}$ in the second term. Nevertheless, with this equation we have computed the $\beta$-functions for a simplified action (one without operators with six fields) in much the same way we will explain later for Euclidean space: they are finite, real and consistent with the desired symmetries, as they should be. We have not proceeded further, but the parallelism between them and their Euclidean counterparts strongly supports the common lore that both should contain the same physical information and that the choice of space is much a matter of taste. Nevertheless, it would probably be nice to afford a complete calculation in Minkowski space.

4 The fermionic action

We turn now to consider spin 1/2 fields. We want to apply the machinery to a concrete model, which we define in this section through its symmetries, then discuss sensible truncations and, finally, comment on the prescriptions which we have actually used to build its action in a systematic way.

We choose a general theory with $N$ spin-1/2 fields on the two-dimensional Euclidean space which obeys the discrete symmetries of parity, charge conjugation and, to obtain reflection positive Green functions, reflection hermiticity (see Ref. [42] for a precise definition of them) in addition to the invariance under the standard Euclidean transformations. Further we will impose the invariance under the chiral symmetry transformations of $U(N)_R \times U(N)_L$.

The next step is to choose an appropriate truncation. A reasonable idea is to use a kind of the derivative expansion, which was proved to be quite efficient in bosonic theories [10] - [14]. However, unlike in the scalar case, the approximation without derivatives (effective potential) does not work and the leading order should include terms both without derivatives and with one derivative. The reason is simple: Eq. (11) contains, due to the sum over polarizations, the factor $\bar{\psi}$ in the Kadanoff terms, while a similar equation for bosons does not (see, for instance, Eq. (4)). Another significant difference is that in the scalar case a general potential contains an infinite number of independent operators (powers of fields), whereas for finite $N$ a general function of fermionic variables with fixed number of derivatives has in any case a finite number of terms due to the Grassmann nature of the field.

However, for a given but large enough $N$ (the case which we will be mainly interested in) the number of different structures grows fast as the number of derivatives increases. In practice, the approximation is intractable already in the order with 3 derivatives unless the degree of the polynomial of the fields is also restricted. We consider two approximations here. The first one is the pure derivative expansion in next-to-leading order (with all terms without derivatives and with one derivative). The second approximation is the mixed one where the number of derivatives and also the power of fields are truncated. This allows for non-vanishing anomalous dimension.
The final preparatory step is to write down the action. To construct it systematically we consider each symmetry in turn and derive the restrictions it imposes. For the second approximation, we will neither comment the details of the construction, nor write down the whole action, which in this case consists of 107 independent operators. The interested reader is referred to [12] for a thorough discussion. Here as an example we present only terms \( S^{(s,q)} \) (\( s \) stands for the number of fermionic fields and \( q \) for the number of derivatives) of the action with 2 and 4 fields. In the momentum representation they read:

\[
\begin{align*}
S^{(2,1)} &= p_{12}^{-j} i V_{12}^{j}, \\
S^{(4,0)} &= g_1 (S_{12} S_{34} - P_{12} P_{34}) + g_2 V_{12}^{j} V_{34}^{j}, \\
S^{(4,2)} &= \{ m_1 p_{12}^{+2} + m_2 p_{12}^{-} \cdot p_{34}^{-} + m_3 p_{12}^{-2} \} \times (S_{12} S_{34} - P_{12} P_{34}) \\
&+ (r_1 p_{12}^{+2} + r_2 p_{12}^{-} \cdot p_{34}^{-} + r_3 p_{12}^{-2}) \cdot V_{12}^{j} V_{34}^{j} + \{ s_1 p_{12}^{+j+k} + s_2 p_{12}^{-j-k} + s_3 p_{12}^{-j} p_{34}^{-k} \} \\
&+ s_4 p_{12}^{-j-k} \} \times V_{12}^{j} V_{34}^{k} + t p_{12}^{+j+k} e^{jk} (S_{12} S_{34} - P_{12} P_{34}),
\end{align*}
\]

(12)

where we introduced the following notations:

\[
S_{12} \equiv \overline{\psi} (p_1) \gamma^a \psi (p_2), \quad P_{12} \equiv \overline{\psi} (p_1) \gamma^{ij} \psi (p_2), \quad V_{12}^{j} \equiv \overline{\psi} (p_1) \gamma^j \psi (p_2)
\]

(14)

(the flavour indices are summed up) and \( p_{kl}^\pm \equiv (p_k \pm p_l)^j \). \( S^{(2,1)} \) is of course the kinetic term.

In the first case, however, the action is written as an infinite series in increasing powers of derivatives with coefficients being arbitrary functions of scalar operators without derivatives built out of the fermions. We work in the \( N \rightarrow \infty \) limit.

From the discussion above we see that there are two independent scalar operators without derivatives, which in the coordinate representation are (compare with Eq. [14])

\[
R(x) = \overline{\psi} (x) \gamma^a \psi (x) \overline{\psi} (x) \gamma^b \psi (x)
\]

(15)

and

\[
U(x) = \overline{\psi} (x) \gamma^a \psi (x) \overline{\psi} (x) \gamma^b \psi (x) - \overline{\psi} (x) \gamma_a \psi (x) \overline{\psi} (x) \gamma_b \psi (x).
\]

Then, the most general action in our approximation can be written as

\[
S = \int d^2 x \left\{ A(R(x), U(x)) \right. \\
+ \overline{\psi} (x) \gamma^a \partial_\nu \psi (x) B(R(x), U(x)) \\
+ \overline{\psi} (x) \gamma^a \partial_\nu \psi (x) \overline{\psi} (x) \gamma^b \psi (x) C(R(x), U(x)) \\
+ \left. \overline{\psi} (x) \gamma^a \partial_\nu \psi (x) \overline{\psi} (x) \gamma^b \psi (x) - \overline{\psi} (x) \gamma_a \partial_\nu \psi (x) \overline{\psi} (x) \gamma_b \psi (x) - \overline{\psi} (x) \gamma_a \psi (x) \overline{\psi} (x) \gamma_b \psi (x) \right. \\
+ \left. \overline{\psi} (x) \gamma^a \psi (x) \gamma^b \psi (x) \gamma_c \partial_\nu \psi (x) \overline{\psi} (x) \gamma_d \psi (x) \right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. 
\]
characterized by the arbitrary continuous parameter $g$ point for any value of model the sector, corresponding to this term, decouples from the rest of the model and is at the fixed $= 0$ for this fixed point. Note that Eq. (18) is valid for any anomalous dimension the solution (18). Because of the too low order of our approximation, we do not think that this fixed is given by $A$ gives rise to the $gr$ is completely determined by the function $A$ $t$ $\alpha$ and $\gamma$ are scheme parameters defined by Eq. (3). As before the dot means the derivative with respect to the flow parameter $t$. We see that the function $E(r, u)$ appears only in the last equation and is completely determined by the function $A(r, u)$.

There is only one solution regular in $r$ and $u$ at $r = u = 0$. It is $u$-independent and is given by the following expressions for the generating functions:

$$
\begin{align*}
A(r, u) &= -2\alpha N + gr, \quad B(r, u) = 1 + \gamma g^2 r, \quad C(r, u) = -2\gamma g^2, \quad D(r, u) = 0, \quad E(r, u) = 0,
\end{align*}
$$

where the normalization of the kinetic term has been taken into account. The anomalous dimension $\eta = 0$ for this fixed point. Note that Eq. (13) is valid for any $N$. This solution is polynomial in $r$ and is characterized by the arbitrary continuous parameter $g$. Since $r$ corresponds to the operator $[13]$ in the original action, the term $gr$ in the expression for $A(r, u)$ corresponds to the operator $gV_{12}V_{13}$, which gives rise to the $U(1)$ Thirring like term. This agrees with the known fact that in the chiral Gross-Neveu model the sector, corresponding to this term, decouples from the rest of the model and is at the fixed point for any value of $g$. However, the coupling constant, corresponding to the $SU(N)$ sector, is zero for the solution (18). Because of the too low order of our approximation, we do not think that this fixed point can be reliably identified with any of the known fixed points of the Gross-Neveu model.

Let us mention for completeness that the system (17) has another formal fixed point solution. The anomalous dimension $\eta = 0$ again, and the generating functions do not depend on $r$. The expression for $A$ is given by

$$
A(u) = A_0 + G_0 \int_{u_0}^{u} \frac{du'}{1 - 2G_0\alpha\gamma N \ln(u'/u_0)},
$$

where $A_0$, $G_0$ and $u_0$ are integration constants (only two of them are independent). One can see that derivatives of $A$ are not regular at $u = 0$. Other generating functions for this solution are expressed through similar integrals. It is not clear for us whether such formal solution has any physical meaning.

### 6 Mixed approximation for fermions

In this section we will consider the approximation including the terms with up to three derivatives and up to six field operators. Simplification due to the truncation in the number of fields allows us to increase the number of the derivatives in the approximation, that in its turn amounts to non-zero anomalous dimension, as we will explain shortly. Having constructed the corresponding action, as it was explained above, the next step is to substitute it into the ERG equation (11) and to compute the $\beta$-functions or flow equations of our model within the chosen approximation.

One comment is in order here. In the actual calculation of the $\beta$-functions we used an extended action which was larger than the one discussed in the previous section in order to have some extra check of the equations. Namely, we considered an action written in a basis of functionals with two fermions and one and three derivatives, four fermions with zero and two derivatives and six fermions with one and three derivatives which were restricted by $U(N)$ symmetry, parity and Euclidean invariance only (that is, we imposed neither reflection hermiticity, nor charge conjugation, nor chirality). We then projected the space of functionals generated by this basis onto the subspace of functionals invariant under all symmetries of the theory and its direct complement. The required flow equations were obtained after we performed the first projection, while the projection onto the complementary subspace provided us a consistency check of the calculation. The latter defined a set of null equations, or identities, which had to be satisfied.
along the renormalization flow. Indeed, since the initial action at \( t = 0 \) is symmetric, appearance of any non-zero non-symmetric term at \( t > 0 \) would indicate an anomaly, which, as we have argued, does not appear.

### 6.1 Generalities

The next step is to find the fixed point solutions, that is, the sets of values of the coupling constants which make all the \( \beta \)-functions to vanish. They correspond to stationary points of the RG flows, thus providing us with the first information of how the phase diagram of the system looks like.

\( \dot{S} = 0 \) is equivalent to the system of 106 non-linear algebraic equations. A simplification comes from the observation that all the coupling constants of operators with six fields must enter linearly. The reason is that the only source of non-linearity in Eq. (11) is the first term of its r.h.s., and neither \( \eta \) equations, plus an equation for the anomalous dimension with respect to the couplings of six-field operators we reduce the system to a set of only 13 non-linear equations, plus an equation for the anomalous dimension \( \eta \) that satisfies

\[
\eta = 4\alpha[-m_1 + m_2 + m_3 + s_1 + s_2 + s_3 + s_4 + t - 2N(r_2 + 2s_3 + s_4)].
\]

Here \( N \) is the number of flavours; \( \alpha, \beta, \gamma, \delta \) are scheme dependent parameters defined by the same formulas as [3]; the rest of coefficients of (19) are different coupling constants of the four-field functionals of our action, see Eq. [4].

The appearance of the above dependence just reflects the freedom in choosing the renormalization scheme, or the cutoff function in our case. Furthermore, although the \( \beta \)-functions contain four scheme parameters, we will see that, as in the scalar field case, the fixed point solution and critical exponents depend only on two combinations of them.

We found the fixed point numerically for finite \( N \), whereas for \( N \to \infty \) we could solve it analytically in the leading order of the \( 1/N \) expansion. Results of both calculations are presented below.

After the fixed points are found, the behaviour of the theory near each of them is controlled by the critical exponents. One of them is already known once we solved our system of equations: it is the anomalous dimension at the fixed point. The rest are found by linearizing the ERG flow equations near a chosen fixed point \( g^* \) in the same way as it was explained for the bosonic case in Sect. 2.3. The eigenvalues of \( R_{ij}|g^* = (\partial y_i/\partial y_j)|g^* \) are identified with the critical exponents. They can be viewed as the anomalous dimensions of the corresponding operators in the vicinity of the fixed point.

Finally, let us turn again to the issue of scheme dependence. As we have already mentioned the values of the coupling constants at a fixed point are scheme dependent, thus reflecting that they are not universal quantities. Critical exponents, on the other hand, are universal and should be scheme independent. Nonetheless, due to the truncation, the scheme dependence will inevitably appear. We will proceed in a common way [4] and try to find a scheme where the effect of the dependence will be the least. To this end, we will apply the principle of minimal sensitivity in order to fix the scheme dependence.

### 6.2 \( N \to \infty \)

Now we are going to define the large \( N \) expansion in our model which allows to obtain analytical results in the leading order in \( 1/N \). To this end we substitute each coupling constant \( g_i \) by \( N^{z_i}g_i \) and study the limit \( N \to \infty \) keeping \( g_i \) fixed. \( z_i \) can be any real number, but for the sake of simplicity we restrict ourselves to integer values.

We are looking for the sets \( \{z_i\} \) for which the \( \beta \)-functions for redefined couplings are finite and, if possible, non-trivial in the \( N \to \infty \) limit. With the above requirements, we found two different patterns of the \( 1/N \) expansion, which lead to inequivalent results. We will label them by I and II, and discuss in turn.

The Type I solution is obtained by taking \( z_i = -1 \), where \( i \) runs over all four-fermion couplings. With this definition, the anomalous dimension vanishes in leading order in \( 1/N \) and at the fixed point the coupling constant \( g_1 \) of the operator \( (S_{12}S_{34} - P_{12}P_{34}) \) in \( S^{(4,0)} \), Eq. [12], is fixed,

\[
g_1 = -1/ (\alpha \gamma)
\]
whereas the coupling $g_2$ of the operator $V_{12}^2 V_{14}^2$, giving rise to the $U(1)$ Thirring excitations, is arbitrary.

The characteristic polynomial $P(\lambda)$, associated to the matrix $R_{ij}|_{\sigma^*}$, can be computed analytically:

$$P(\lambda) = \lambda^2 (\lambda + 2)^{12} (\lambda + 4)^{83} (\lambda + 6) (\lambda^2 + 6\lambda - 8) \times (-\lambda^3 - 12\lambda^4 + (8w - 44)\lambda^3 + (64w - 16)\lambda^2 + (32w + 64)\lambda - (128w + 256)),$$

where $w = \beta\delta/(\alpha\gamma)$ and $z = \delta/\gamma^2$. The critical exponents can be read from $P(\lambda)$. There are 100 scheme independent eigenvalues, most of them coinciding with the canonical values $0, -2, -4$ and $-6$. The non-trivial ones are $-3 + \sqrt{17} = 1.1231...$ and $-3 - \sqrt{17} = -7.1231...$. The rest of the eigenvalues are given by the roots of the polynomial

$$Q(\lambda) = -\lambda^5 - 12\lambda^4 + (8w - 44)\lambda^3 + (64w - 16)\lambda^2 + (32w + 64)\lambda - (128w + 256)$$

and are $w$-dependent. If $w < 0$, that includes, for instance, the case of the exponential cutoff function $K(z) = e^{-\alpha z}$, the most relevant critical exponent is $\lambda_1 = 1.1231...$

The fixed point solution contains two free continuous parameters $g_2$ and $m_1$. This is the expected result for the chiral Gross-Neveu type model because $U(1)$ Thirring like excitations (which in our action are controlled by $g_2$) decouple from the rest and this subsystem is conformal invariant (i.e. it is at the fixed point) for any value of $g_2$. For the $SU(N)$ part there exists a discrete set of fixed points, the fixed point of Dashen and Frishman being one of them. It is reached when the constant $q_1$ is of the order $1/N$, as in our case. So we can expect that our solution is the fixed point of Ref. [38]. However, the values of the anomalous dimension do not match. For the cited fixed point it is non-vanishing in leading order in $1/N$ and not zero as we have found. This discrepancy with the exact result of Dashen and Frishman could be caused by the low accuracy of our truncation. We cannot reject, however, the possibility of having found a different fixed point (see Ref. [39]).

For the Type II solution we let a combination of coupling constants to be of order 1, instead of $O(N^{-1})$. We do not know if this possibility is an accident of the truncation or if it would survive in a complete calculation. The most significant feature of this solution is its non-zero anomalous dimension,

$$\eta = \frac{4}{3} \frac{\beta\gamma}{6\alpha} = \frac{4}{3} \frac{1}{6} \frac{w}{z}.$$  

Unlike the previous case, we could not find the exact analytical expression for the characteristic polynomial. However, by computing numerically the eigenvalues for different values of $z$ and $w$ we could deduce some exact results. It turned out that none of the critical exponents coincide with their canonical counterparts. Moreover, most of them are functions of the combination $\frac{w}{z}$. Thus, there are 82 eigenvalues $\lambda = -\frac{w}{z}$, 8 eigenvalues equal to $\frac{2}{3}(1 - \frac{w}{z})$ and 4 equal to $2 - \frac{w}{z}$. The remaining ones are functions with more complicated dependence on $w$ and $z$ then the ratio $w/z$ (and even a few have a non-vanishing imaginary part, which is not unusual in approximations based on truncations). We studied numerically the most relevant critical exponent, which turned out to belong to the class with non-simple dependence on $w$ and $z$, for a wide range of values of the scheme parameters. As it also happens in the scalar case, for any value of $z$ this exponent always has a minimum at some $w = w^\ast$. Such behaviour suggests to use the minimal sensitivity criterion to fix the parameter $w$ to its critical value $w^\ast$. Unfortunately, due to the monotonous dependence of the exponent on the parameter $z$ in the range analysed, we were unable to fix it by a similar prescription. We show in Table 1 $\lambda_1^\ast(z) = \lambda_1(w^\ast, z)$ and $w^\ast$ for some values of $z$ and the corresponding anomalous dimension.

| $z$ | $\lambda_1^\ast$ | $w^\ast$ | $\eta$ |
|-----|-----------------|----------|-------|
| 0.1 | 2.258           | 0.122    | 1.130 |
| 0.5 | 2.239           | 0.616    | 1.128 |
| 1.0 | 2.217           | 1.250    | 1.125 |
| 2.0 | 2.175           | 2.610    | 1.116 |

Table 1: Local minimum $\lambda_1^\ast$ of $\lambda_1(w, z)$, the most relevant critical exponent, for different values of $z$. $w^\ast$ is the value of $w$ at which the minimum is reached and $\eta$ is the corresponding value of the anomalous dimension.
6.3 Finite $N$

For finite number of flavours we could not find fixed point solutions analytically, so we studied the zeroes of the $\beta$-functions numerically. In general the number of different solutions of a system of coupled non-linear equations is not known a priori, and common programs for root-finding (such as the FindRoot routine of Mathematica) do not give all its solutions. Nevertheless, after some experience was acquired and relying on the results for high $N$ we could choose a reasonable range of values of the coupling constants and examine it minutely in searching for the fixed point solutions.

A more serious problem is to discriminate between zeros, which correspond to real fixed point solution, and spurious roots, which are artefacts of the truncation. This problem, which appears in the bosonic case too, is perhaps the Achilles’ heel of the approximations based on truncations [6], [9]. We present the class of solutions of which we are more confident. These are mainly the ones which asymptotically match with some solution clearly identified in the framework of the large $N$ expansion.

To begin with, let us select a particular scheme and find the solution for different values of $N$. We take $w = -2$ and $z = 0.5$, corresponding to the exponential regulating function $K(x) = e^{-x^2}$. We will analyse the dependence on these parameters later on.

There is a sequence of fixed point solutions for various (integer) $N$ which asymptotically approaches the type I solution of the $N \to \infty$ limit discussed above. For this sequence $N\eta$ increases with $N$ and tends to $4.87...$, while the most relevant critical exponent $\lambda_1$ decreases with $N$ and asymptotically approaches the value $1.1231...$, in a full agreement with our previous results on the $1/N$ expansion. The second eigenvalue was found to be complex, that is apparently an artefact of the approximation used. Another mismatch is that the type of the solutions of the sequence does not fit the type of the $N \to \infty$ fixed point. As we wrote above, the latter is the two-parameter continuous line of solutions parametrized by $g_2$, which is related to the Thirring $U(1)$ sector, and $m_1$, whereas the solutions for finite $N$ are isolated. Most likely, this qualitative change of the type of the solution occurs when we go from infinite $N$ to, though large, but finite $N$ within our approximation. However, we have not studied this phenomenon in detail. We present in Fig. 1 the plots of $N\eta$ and $\lambda_1$ as functions of $N$.

![Figure 1](image)

Figure 1: $N \cdot \eta$ (solid line) and $\lambda_1$ (dashed line) as functions of $N$. The corresponding fixed point solution matches the Type I solution of the large $N$ limit.

Let us discuss now the dependence of the solutions on the renormalization scheme. The parameter $z$ enters the flow equations only through the anomalous dimension and, for this reason, the dependence of the fixed points solutions on $z$ is rather simple: for instance, it is almost linear for $\eta$. The dependence on $w$ is more complicated, and we studied the behaviour of $\eta$ and $\lambda_1$ under the change of $w$ for fixed $N$ and $z$. In this analysis, as in the scalar case, we looked for some non-linear $w$-dependence so that we could invoke the principle of minimal sensitivity to fix the value of this parameter and reduce the scheme.
dependence. To this end, we take $z = 0.5$ and $N = 1000$. The curve $\eta$ vs. $w$ is monotonous and decreases with $w$, while the first eigenvalue $\lambda_1$ reaches its minimum value $\lambda_1 = 1.12511$ at $w = -45$, which increases as we lower $N$: it is equal to 1.1273 for $N = 500$ (and the minimum is reached at $w = -23$), 1.146 for $N = 200$ (at $w = -10$), 1.519 for $N = 100$ (at $w = -8$), 1.695 for $N = 10$ (at $w = -2.4$) and finally, 2.560 for $N = 3$ (at $w = -0.5$).

For a sequence of fixed point solutions which match the Type II solution the situation is different. For $N$ large enough, (say $N \geq 1000$), the numerical solutions are in good agreement with the infinite $N$ analytical result (for example the value of $\eta = 1.99$ for $z = 0.5$ and $w = -2$, this is to be compared with the exact result $\eta = 2$ for $N \to \infty$). As we lower $N$ the values of the anomalous dimension $\eta$ and the most relevant eigenvalue $\lambda_1$ decrease and near $N = 142$ (actually at $N = 142.8$ if we let $N$ to take non-integer values) the solution ceases to exist and joins another branch of fixed point solutions. For fixed points belonging to the second branch $\eta$ and $\lambda_1$ remain finite as $N \to \infty$. However, we found, that some couplings did not behave as powers of $N$ in this limit and, therefore, this solution cannot be associated with a fixed point in the $N \to \infty$ limit in the sense stated previously. At the joining point $\eta = 1.88$ and $\lambda_1 = 5.80$. We show in Fig. 2 the curves $\eta(N)$ and $\lambda_1(N)$. Further studies should be carried out in order to understand the structure of these solutions better.

![Figure 2: $\eta$ (solid line) and $\lambda_1$ (dashed line) as functions of $N$ for $z = 0.5$ and $w = -2$. In both curves the upper branch corresponds to the solution which matches the Type II solution of the large $N$ limit.](image_url)

There are also other solutions, which were found for low $N$. For some of them either $\eta$ or $\lambda_1$ have minima in $w$, but in other cases both curves are monotonous in $w$. They also display various behaviours as $N \to \infty$.

Finally we will consider the special case $N = 1$. Fierz transformations relate $U(N)$-covariant local operators (like $\bar{\psi}^a(p_1) \psi^b(p_2)$) to $U(N)$ scalars (like $\bar{\psi}^a(p_1) \psi^a(p_2)$). So, for $N = 1$ the Fierz transformations establish additional relations between the available operators, which give relations between the coupling constants that allow us to reduce the system of the flow equations considerably.

After a bit of algebra and discarding the trivial solution we finally get an equation for $\eta$,

\[
0 = -120w^2z + 288wz^2 + \eta(13w^2 - 132wz + 210w^2z + 288z^2 - 720wz^2) + \eta^2(99wz - 90w^2z - 432z^2 + 594wz^2) + \eta^3(162z^2 - 162wz^2).
\]

As in the previous analysis we had to choose some particular scheme, i.e. fix $w$ and $z$, and solve the
equation numerically. Unfortunately, the system of fixed point equations is not underdetermined and there is no room for a free parameter in the fixed point solution as it is the case in the Thirring model. In fact such property takes place in the previous order approximation (terms with less than three derivatives), however \( \eta \) vanishes identically there. This is in a complete agreement with the first solution of the pure derivative expansion considered in Sect. 5. The reason why this property is lost in the approximation with three derivatives is unclear for us. Probably, one has to go to the next order of the approximation to understand this issue.

We solved the equation numerically for different values of \( w \) and \( z \). As in previous examples the fixed point solutions are almost linear in the parameter \( z \), their behaviour with \( w \) is more complicated. However, in the range of values studied, we did not find any non-monotonous dependence on \( w \) either of the critical couplings or of the anomalous dimension. We present in Table 2 some of the results for \( \eta \).


dd

| \( z \) = 0.1 | \( w = -0.5 \) | \( w = -1.0 \) | \( w = -2.0 \) |
|-------|-------|-------|-------|
| 1.763 | 3.691 | 6.316 | 11.747 |
| 1.418 | 1.790 | 1.418 | 3.388 |
| 1.376 | 1.559 | 1.811 | 2.349 |
| 1.354 | 1.445 | 1.569 | 1.834 |

dd

Table 2: Values of \( \eta \) for \( N = 1 \) for different values of the scheme parameters \( z \) and \( w \).

7 Discussion and conclusions

In this article we gave an account of main ideas and techniques of the ERG approach. We presented steps of the derivation of the ERG equation, which plays the crucial role, for scalar and fermionic theories. We discussed main approximations, which have been developed so far, for solving it and showed how some characteristics of models, like fixed points and critical exponents, can be computed. We also reviewed briefly some non-perturbative results, obtained within this approach for the scalar \( Z_2 \)-symmetric model in three dimensions and for the two-dimensional Gross-Neveu type model. In the latter case the results are quite recent. As we argue above, within the \( 1/N \) expansion the Type I fixed point solution is an excellent candidate for the Dashen-Frishman fixed point, whereas the other one presents evidences to be a new fixed point, with quite intricate properties, not discussed previously in the literature.

We saw that in scalar theories truncation in the power of fields (like in the polynomial approximation of the local effective potential or in the mixed approximation) leads to numerous spurious solutions with the subsequent problem of identification of the true physical one, especially when the order of the polynomial is high enough. However, if the pure derivative expansion is used, the spurious solutions and the problem of identification do not appear. We demonstrated that in the case of fermionic theories the situation is similar. One should only keep in mind that for finite \( N \), due to the Grassmannian nature of the fields, terms with \( l \) derivatives do not contain more than \( k_{\text{max}}(l) = 4N + 2[l/2] \) spinors (here \([x]\) stands for the integer part of \( x \)). Thus, each term of this expansion is a polynomial in fermionic fields. When we use the mixed approximation, i.e. we restrict the action to the number of fields less than \( k_{\text{max}}(l) \) for a given \( l \), as it was the case in most of the examples considered in Sect. 6, we observe the appearance of many spurious solutions. When we carry out the pure derivative expansion, either in a general form (see Sect. 5) or by taking into account all possible operators with powers till \( k_{\text{max}}(l) \) in fields, like in the example with \( N = 1 \) in Sect. 6.3 \((l \leq 3 \) there), we see that no spurious solutions appear. Thus, for \( N = 1 \) there are only three fixed point solutions, besides the trivial one, two of them having complex coupling constants, thus being rejected at once. It would be interesting to perform a pure derivative expansion for, say, \( N = 2 \) and check if the above features hold.

All approximations which include derivatives contain scheme parameters, which cannot be eliminated by redefinition of the coupling constants. Such scheme dependence does not appear, however, in the lowest approximations without derivatives. This is clearly illustrated by our scalar example in Sect. 2.3. For the approximation without derivatives, i.e. when \( b_2 \) and \( b_3 \) are zero, \( \eta = 0 \) and the equation (\[ \]) is not present, no scheme parameters appear in the first three equations of the system (\[ \]). When the terms with two derivatives are taken into consideration, the fixed point solution and critical exponents depend on
the scheme parameters $x$ and $y$. This dependence on the regulating (cutoff) function is analogous to the renormalization scheme dependence in the perturbation theory calculations. For observable quantities, like critical exponents, this dependence is unphysical and is an artefact of the approximation. We used the principal of minimal sensitivity to "minimize" their dependence on the scheme.

An interesting feature is the seemingly good results for the large $N$ limit in the fermionic model. At the computational level this is related to the fact that at the leading order of the $1/N$ expansion the system of the flow equations simplifies dramatically and no room is left for spurious solutions. The same simplification occurs in scalar models as well \[17\], \[5\]. It would be useful to understand deeper reasons for this. Apparently, improvement of results in the large $N$ limit is a quite general feature of the ERG approach.

Recent developments of the ERG formalism allow us to conclude that it can be regarded as a reliable non-perturbative method of studies of physical models. Calculational techniques, developed in scalar theories, were proved to be effective enough in calculations of the numerical characteristics of the models and analysis of renormalization flows. In fermionic theories much less work has been done so far. It would be interesting to probe similar calculational techniques in higher approximations, than discussed here, or in other physical models. For example, the ERG formalism should be extended to higher dimensions. The equation \[11\] is prepared for that. The number of spinor structures will increase in this case and, therefore, one will have to handle more terms in the action. However, we guess that, due to the greater complexity, two derivatives may be sufficient to obtain interesting results, or at least, according to the standard rule that quantum effects become less important when the dimension is increased, we hope that the results will be more transparent within the same approximation.

We did not discuss gauge theories in this article. This is because of lacking of a satisfactory ERG formalism for the moment. The main problem is to maintain the gauge invariance of the ERG equation and the renormalization flow, which is obviously broken when cutoff functions of the type \[4\] are introduced in order to set the physical scale. Development of an effective formalism for gauge theories seems to be the most challenging problem in the ERG approach nowadays.

**Acknowledgements**

We are sincerely indebted to J.I. Latorre, who encouraged us in this work and provided us fruitful suggestions during its completion. Discussions with A.A. Andrianov and D. Espriu are also acknowledged. Yu. K. and E. M. thank the Department ECM of the University of Barcelona for warm hospitality and friendly atmosphere during their stay there. Yu. K. also would like to thank the Department of Theoretical Physics of the UAM for hospitality. This work was supported in part by funds provided by CICYT under contract AEN95-0590 and by the M.E.C. grants SAB94-0087 and SAB95-0224.

**References**

[1] K.G. Wilson, Phys. Rev. B4 (1971) 3174; 3184.
[2] K.G. Wilson and J. Kogut, Phys. Rep. 12 (1974) 75.
[3] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B270 (1986) 687.
[4] J.F. Nicoll, T.S. Chang and H.E. Stanley, Phys. Rev. Lett. 33 (1974) 540; J.F. Nicoll, T.S. Chang and H.E. Stanley, Phys. Lett. A57 (1976) 7; V.I. Tokar, Phys. Lett. A104 (1984) 135; G. Felder, Comm. Math. Phys. 111 (1987) 101; P. Hasenfratz and J. Nager, Z. Phys. C37 (1988) 477; C. Bagnuls and C. Bervillier, Phys. Rev. B41 (1990) 402; C. Wetterich, Nucl. Phys. B352 (1991) 529; A.E. Filipov and S.A. Breus, Phys. Lett. A158 (1991) 300; M. Alford, Phys. Lett. B336 (1994) 237; U. Ellwanger, Z. Phys. C62 (1994) 503; K. Halpern and K. Huang, Phys. Rev. Lett. 74 (1995) 3526; S. Bornholdt, N. Tetradis and C. Wetterich, Phys. Lett. B348 (1995) 89.
[5] N. Tetradis and D.F. Litim, “Analytical Solutions of Exact Renormalization Group Equations”, hep-th/9512073, 1995.
[6] T.R. Morris, Phys. Lett. B334 (1994) 355.
[7] M. Maggiore, Z. Phys. C41 (1989) 687.
[8] A. Margaritis, G. Ódor and A. Patkós, Z. Phys. C39 (1988) 109.
[9] P.E. Haagensen, Y. Kubyshin, J.I. Latorre and E. Moreno, Phys. Lett. B323 (1994) 330.
[10] G.R. Golner, Phys. Rev. B33 (1986) 7863; A.E. Filippov and A.V. Radievsky, Phys. Lett. A169 (1992) 195; S.-B. Liao and J. Polonyi, Ann. Phys. 222 (1993) 122; Phys. Rev. D51 (1995) 4474; N. Tetradis and C. Wetterich, Nucl. Phys. B422 (1994) 541.
[11] T.R. Morris, Phys. Lett. B329 (1994) 241.
[12] T.R. Morris, Phys. Lett. B345 (1995) 139; Nucl. Phys. B (Proc. Suppl.) 42 (1995) 811.
[13] T.R. Morris, “Momentum Scale Expansion of Sharp Cutoff Flow Equations”, hep-th/9508017, 1995.
[14] R.D. Ball, P.E. Haagensen, J.I. Latorre and E. Moreno, Phys. Lett. B347 (1995) 80.
[15] J. Polchinski, Nucl. Phys. B231 (1984) 269.
[16] R.D. Ball and R.S. Thorne, Ann. Phys. 236 (1994) 117.
[17] F.J. Wegner and A. Houghton, Phys. Lett. A8 (1972) 401.
[18] C. Wetterich, Nucl. Phys. 408 (1993) 91; Phys. Lett. B301 (1993) 90.
[19] T.R. Morris, Int. Jour. Mod. Phys. A9 (1994) 2411.
[20] K.G. Wilson, Phys. Rev. Lett. 28 (1972) 548; M.E. Fisher and K.G. Wilson, Phys. Rev. Lett. 28 (1972) 240; M.E. Fisher, Rev. Mod. Phys. 46 (1974) 597.
[21] T.S. Chang, D.D. Vvedensky and J.F. Nicoll, Phys. Rep. 217 (1992) 279.
[22] G. Keller and C. Kopper, Comm. Math. Phys. 148 (1992) 445; 153 (1993) 245; M. Bonini, M. D’Attanasio and G. Marchesini, Nucl. Phys. B409 (1993) 441.
[23] B.J. Warr, Ann. Phys. 183 (1988) 1, 59; G. Keller and C. Kopper, Phys. Lett. B273 (1991) 323; C. Becchi, “On the construction of renormalized quantum field theory using renormalization group techniques”, in M. Bonini, G. Marchesini and E. Onofri, editors, Elementary particles, Field theory and Statistical mechanics, Parma University, Parma, 1993; M. Bonini, M. D’Attanasio and G. Marchesini, Nucl. Phys. B421 (1994) 429; B437 (1995) 163; Phys. Lett. B346 (1995) 87; M. Reuter and C. Wetterich, Nucl. Phys. B417 (1994) 181; B427 (1994) 291; U. Ellwanger, Phys. Lett. B335 (1994) 364; T.R. Morris, Phys. Lett. B357 (1995) 225; “Two Phases for Compact $U(1)$ Pure Gauge Theory in Three Dimensions”, hep-th/9505003, 1995; U. Ellwanger, M. Hirsch and A. Weber, “Flow equations for the relevant part of the pure Yang-Mills action”, hep-th/9506019, 1995; S.-B. Liao, “Operator cutoff regularization and renormalization group in Yang-Mills theory”, hep-th/9511046, 1995; M. Reuter, “Effective Average Action of Chern-Simons Field Theory”, hep-th/9511128, 1995.
[24] N. Tetradis and C. Wetterich, Nucl. Phys. B398 (1993) 659; Int. Jour. Mod. Phys. A9 (1994) 4029; M. Reuter, N. Tetradis and C. Wetterich, Nucl. Phys. B401 (1993) 567; M. Alford and J. March-Russell, Nucl. Phys. B417 (1994) 527; S.-B. Liao, J. Polonyi and D. Xu, Phys. Rev. D51 (1995) 748; S.-B. Liao and M. Strickland, “Renormalization group approach to field theory at finite temperature”, hep-th/9506137, 1995; S. Bornholdt and N. Tetradis, “High temperature phase transition in two scalar theories”, hep-ph/9503282, 1995.
[25] E. Ellwanger, Z. Phys. C58 (1993) 619; C62(1994) 503.
[26] E. Ellwanger and C. Wetterich, Nucl. Phys. B423 (1994) 137.
[27] M. Carfora and K. Piotrkowska, Phys. Rev. D52 (1995) 4393.
[28] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. 54 (1985) 2191; Comm. Math. Phys. 102 (1985) 1.

[29] M. Bonini, M. D’Attanasio and G. Marchesini, Nucl. Phys. B418 (1994) 81; Phys. Lett. B329 (1994) 249.

[30] C. Wetterich, Z. Phys. C48 (1990) 693;
   S. Bornholdt and C. Wetterich, Z. Phys. C58 (1993) 585;
   U. Ellwanger and L. Vergara, Nucl. Phys. B398 (1993) 52;
   T.E. Clark, B. Haeri, S.T. Love, W.T.A. ter Veldhuis and M.A. Walker Phys. Rev. D50 (1994) 606.

[31] T.E. Clark and S.T. Love, Phys. Lett. B344 (1995) 266;
   D.U. Jungnickel and C. Wetterich, “Effective Action for the Chiral Quark Meson Model”, \texttt{hep-ph/9505267}, 1995.

[32] J. Comellas, Yu. Kubyshin and E. Moreno, ”Exact renormalization group study of fermionic theories”, \texttt{hep-th/9512086}, 1995.

[33] E. Fermi, Z. Phys. 88 (1934) 161.

[34] J. Bijnens, C. Bruno and E. de Rafael, Nucl. Phys. B390 (1993) 501;
   A.A. Andrianov and V.A. Andrianov, Theor. Math. Phys. 94 (1993) 3.

[35] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345.

[36] V.A. Miransky, M. Tanabashi and K. Yamawaki, Phys. Lett. B221 (1989) 177; Mod. Phys. Lett. A4 (1989) 1043;
   W.A. Bardeen, C.T. Hill and M. Lindner, Phys. Rev. D41 (1990) 1647.

[37] D. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.

[38] R. Dashen and Y. Frishman, Phys. Rev. D11 (1975) 2781.

[39] P.K. Mitter and P.H. Weisz, Phys. Rev. D8 (1973) 4410;
   E. Moreno and F. Schaposnik, Int. Jour. Mod. Phys. A4 (1989) 2827;
   C. Hull and O.A. Soloviev, “Conformal Points and Duality of Non-Abelian Thirring Models and Interacting WZNW Models”, \texttt{hep-th/9503021}, 1995.

[40] P.M. Stevenson, Phys. Rev. D23 (1981) 2916.

[41] C. Itzikson and J.M. Drouffe, Statistical field theory, vol. I, Cambridge Monographs on Mathematical Physics, 1989.

[42] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford Science Publications, Oxford, 1989.