Spacelike slices from globally well-behaved simultaneity connections

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As shown by the development of Special Relativity the simultaneity concept should be related to that of reference frame. Poincaré proposed to define the simultaneity of two events by means of light signals following what is nowadays known as the Einstein simultaneity convention. The need of a simultaneity definition is present also in general relativity and in curved spacetimes in order to provide the observers with a coordinate time.

It is recognized that the old Einstein simultaneity convention is nothing but a connection on a suitable trivial bundle that defines the reference frame. Unfortunately, it has a non-vanishing holonomy in curved and even in flat spacetimes a fact that makes it almost useless. We point out the advantage of local simultaneity conventions showing that they are represented by local simultaneity connections. Among them there is one, uniquely determined by the reference frame, which is particularly well-behaved globally.

1 Introduction

At the beginning of the last century telegraphers were used to synchronize distant clocks according to a procedure that took into account the finite velocity of propagation of the telegraphic signal [2]. This procedure, later considered by Poincaré [13, 14] for the propagation of light signals, has become known as the Einstein synchronization (simultaneity) convention. Consider two distant clocks A and B at rest in an inertial frame. The Einstein convention consists in the following steps: An observer at A (observer A for short) sends a light beam from A to B where it is reflected back to A. Using its clock the observer at A measures the round-trip time $\Delta \tau$ of the light beam, and the instant of departure $\tau_{Ai}$ while the observer at B measures, with its own clock, the time of reflection $\tau_{Br}$ that later is communicated to A. A then moves the hands of its own clock forward of a quantity $\delta = \tau_{Br} - \Delta / 2 - \tau_{Ai}$ in such way that if the entire process would have been done since the beginning with the new setting A would have found $\delta = 0$. The two clocks can be considered synchronized if this procedure, no matter how many times repeated, gives always $\delta = 0$. The experience tells us that in fact, for clocks that have been initially synchronized, if $\delta = 0$ at one time then $\delta = 0$ at any later time. We recall that two clocks are synchronized if their rate agree when moved in the same space point for the comparison. From now on clocks will be assumed synchronized. Should we expect that if A and B are synchronized and B and C are synchronized then A and C are synchronized? That is, should we expect that the Einstein synchronization convention in an inertial frame is transitive? The answer is affirmative. It can be shown [16, 11] that the transitiveness of the Einstein convention follows only from the observable fact that the speed of light over a closed path is a constant independent of the path (here the light beam moves along the closed path by means of suitable mirrors). It can therefore be safely applied to synchronize all the clocks in space. With a lattice of clocks the observers in the inertial
frame are therefore able to construct a coordinate time i.e. they are able to assign to each event its coordinate time by simply looking at the nearest clock when the event happens. The spacetime is thus foliated by simultaneity slices of constant coordinate time, and the existence of such slices is ultimately due to the transitiveness of the Einstein simultaneity convention.

2 The general relativistic picture

The concepts and results of the previous section are, unfortunately, only approximate since global inertial reference systems do not exist. General relativity suggests that these results could hold locally but not globally. Indeed it is easy to show that the Einstein convention is not transitive, not even in flat spacetime, if one considers rotating reference frames. First of all one needs a concept for reference frame in general relativity since the simultaneity concept should be related with that of frame. Roughly speaking a reference frame is a collection of objects (the points of the frame) moving together as a single object. Mathematically the reference frame is therefore a congruence of timelike curves \[u(x)\] on the curved spacetime \(M\), that is, a reference frame is determined by a normalized field \(u(x)\).

The integral lines of this field are the worldlines of the points at rest in the frame. If \(S\) is the manifold of integral wordlines we have (at least locally) a projection \(\pi: M \to S\) that associate to each event the worldline passing through it or, which is the same, a point of space. Now, imagine applying the Einstein convention on the neighborhood of an observer \(A\) moving along the congruence flow. Since in special relativity the Einstein simultaneity convention determines as simultaneity planes those perpendicular to the observer worldline the same should be true locally in general relativity. Indeed, let \(H_u(x)\) be the plane orthogonal to \(u(x)\) at \(x\). This is the plane of simultaneity of an observer moving with the flow, and thus having 4-velocity \(u\), in the sense that the events simultaneous to \(x\) according to the Einstein convention and with respect to the frame determined by \(u(x)\) lie near the exponential map at \(x\) of \(H_u(x)\).

Now, notice that a distribution of planes in a fibration \(\pi: M \to S\) determines a splitting of the tangent space \(TM_x\) and therefore defines a generalized connection. The conclusion is that the Einstein simultaneity convention is in fact a connection \[\mathbf{9}\] in the sense of generalized gauge theories \[\mathbf{8, 12}\]. Generalized gauge theories have been studied extensively in the past since they have a structure as rich as usual gauge theories without requiring a principal bundle (that is the fibration is not necessarily generated by the action of a group).

It its interesting to look at the meaning of the horizontal lift in this gauge theory of simultaneity \[\mathbf{4}\]. Consider a succession of observers at rest in the frame. Let \(A\) be the first observer and let observers \(B, C, D\ldots\) lie around a closed curve \(\gamma\) on \(S\). In other words let the observers be disposed in circle and let them synchronize their clocks with that of the observer at their own right-hand side. Starting from observer \(A\) the observers synchronize their clocks and thus implicitly define a simultaneity convention which take as simultaneous those events that correspond to the same clock reading, no matter where placed along the curve. This 'simultaneous' events stay in a line which is the horizontal lift of \(\gamma\) on \(S\). The horizontal lift corresponds therefore to the operation of pointwise synchronization along the base curve (see figure \[\mathbf{4}\]). Unfortunately, in presence of a non-vanishing holonomy the lifted curve does not close. This means that the simultaneity considered is not transitive along the closed curve and therefore it fails to determine a spacetime foliation and hence a time coordinate. We shall see, as a particular case of our study below, that the Einstein simultaneity convention (connection) has a non vanishing curvature (holonomy) whenever the vorticity vector \(w^\eta = \frac{1}{2}h_u^\eta_\gamma u_\gamma u_\alpha\gamma \) differs from zero. Here \(h_u^\eta_\gamma = \delta^\eta_\nu - u^\nu u_\nu\) is the projector on the horizontal space \(H_u\) (we use the timelike convention \(\eta_{00} = 1\)). Now, this vector field differs from zero even in flat spacetime if a rotating congruence is considered. The
impossibility of determining a global simultaneity definition using the Einstein convention in a rotating frame is well known even experimentally. Indeed, the non-vanishing holonomy implies that two beams of light moving over a closed curve $\gamma$ in opposite directions close the curve in different times. This is the celebrated Sagnac effect \cite{15, 1}. These difficulties involving the Einstein simultaneity convention and the practical need of a coordinate time over rotating frames like the earth naturally arise the question as to whether some better simultaneity convention could be found. The next section presents some results obtained in \cite{10}. We refer the reader to that work for the detailed proofs.

3 Alternative local simultaneity connections

It is important to distinguish between local and non-local simultaneity conventions. Non-local simultaneity conventions are those that in order to construct a global spacetime foliation make use of global information. This could be information on the global shape and metric of the spacetime manifold (say the Schwarzschild metric) or on the motion of distant objects (say the satellites used in the GPS). Local simultaneity conventions are those conventions that hopefully determine a global spacetime foliation starting from local information only. They apply as rules between neighboring observers exactly as the Einstein convention does. The single observer at rest in the frame does not need global information on the frame or on the spacetime metric structure. These are the most conceptually easier simultaneity conventions, but despite of this, conventions of this kind were almost overlooked in previous literature with the notable exception of the Einstein convention. Mathematically a local simultaneity convention is, analogously to the Einstein case, a connection over the frame bundle, that is a splitting of the tangent space in vertical and horizontal. This can be determined in a number of ways. Here we shall identify the connection with a 1-form $\omega$ normalized so that $\omega(u) = 1$ and such that $\omega^\mu$ is a timelike vector (Llosa and Soler would call our connection $\omega$ a normalized time scale \cite{5}). This last condition is imposed since the horizontal space $H_\omega(x)$ at $x$ is identified with the ker of $\omega$ at $x$, and the previous condition assures that it is spacelike, a minimal requirement for a simultaneity plane. This defines a general simultaneity connection. However, we are interested in local simultaneity connections. This requirement imposes some more conditions on the shape of $\omega$ and its meaning should be briefly discussed first. We have already said that a local convention should use only local information that the generic observer at rest should find with experiments in its local comoving laboratory. This information may consist in data on the motion of the congruence as the 4-velocity $u^\mu$, the vorticity vector $w^\mu$, the acceleration $a^\mu = u_\mu;\nu u^{\nu}$, the shear
tensor, the expansion and in data on the spacetime metric structure such as the Riemann
tensor. A local simultaneity connection (convention) is therefore a connection \( \omega \) that can be
contructed from local tensors. Exotic tensors can in principle enter the contruction, however,
this is unlikely since they should have a clear operational meaning. That is, it should be clear
what the observer should do in order to measure their components in a suitable base. It is
convenient to introduce the vector product between the vorticity vector and the acceleration
\( m_{\alpha} = \epsilon_{\alpha \beta \gamma \delta} a^{\beta} u^{\gamma} w^{\delta} \) and limit for the moment our analysis only to those spacetime regions
where \( m_{\alpha} \neq 0 \). We also define \( a^{2} = -a^{\mu} a_{\mu} \), \( w^{2} = -w^{\mu} w_{\mu} \) and \( m^{2} = -m^{\mu} m_{\mu} = a^{2} w^{2} \sin^{2} \theta \)
where \( \theta \) is the angle between the vorticity vector and the acceleration for a n observer mov-
ing at speed \( u \). Since \( u^{\mu} \), \( a^{\mu} \), \( w^{\mu} \) and \( m^{\mu} \) are linearly independent any local simultaneity
convention takes the form
\[
\omega_{\alpha} = u_{\alpha} + \psi^{m}(x)m_{\alpha} + \psi^{a}(x)a_{\alpha} + \psi^{w}(x)w_{\alpha}, \tag{1}
\]
for suitable functions \( \psi^{m}, \psi^{a}, \psi^{w} \). From the definition of local simultaneity convention it
follows moreover that \( \psi^{m}, \psi^{a}, \psi^{w} \), depend on the acceleration \( a \) the vorticity \( w \), the angle
\( \theta \), and possibly on other scalars (note that in a stationary frame the possibilities are re-
duced since the shear and the expansion vanish). Note that if the \( \psi \) functions are small
the simultaneity connection may be considered as a perturbation of Einstein’s for which
\( \omega_{\alpha} = u_{\alpha} \).

Let us come to the curvature. Here, for short, we identify it with the vector (for the
relations between different definitions see [10])
\[
v_{\eta} = \frac{1}{2} h_{\rho \sigma \eta} \epsilon^{\rho \beta \alpha \gamma} \omega_{\beta \gamma} \omega_{\alpha \gamma} \tag{2}
\]
It is a kind of generalized vorticity vector for the present non-time orthogonal connection.
Taking into account the Frobenius condition of integrability, \( \omega \wedge d \omega = 0 \), it is not difficult
to show that the equation \( v^{\mu} = 0 \) implies that the distribution of planes \( H_{\omega} \) in the ker of \( \omega \)
is integrable.

Since in the Einstein case the curvature coincides with the vorticity vector we might
look for new connections that reduce to Einstein’s if \( w^{\mu} = 0 \). Hence the additional term
\( \psi^{m}(x)m_{\alpha} + \psi^{a}(x)a_{\alpha} + \psi^{w}(x)w_{\alpha} \) in the expression for \( \omega \) should vanish whenever the vorticity
vanish. The idea is that of looking for some functions \( \psi \) that tilting suitably the simultaneity
planes make them integrable (see figure 2). However, notice that since the functions \( \psi \) depend
on \( x \) only indirectly through the dependence on some measurable scalars we are actually
looking for a local rule of tilting. In practice, in order to apply the simultaneity convention
\( \omega \), the observers at rest in the frame apply the same procedure using light rays of the
Einstein convention. This time, however, they slightly modify the coefficient \( \delta \). Rewrite the
connection \( \omega_{\mu} \) as \( \omega_{\mu} = u + \phi n_{\mu} \) where \( n \) is a spacelike normalized vector. Since the functions
\( \psi^{m}, \psi^{a} \) and \( \psi^{w} \) are measurable so are \( \phi(x) \) and \( n(x) \). Let us return to two neighboring
observers A and B and consider a light beam sent from A to B and then reflected back

Fig. 2. Can the horizontal planes of the Einstein simultaneity convention be slightly tilted according
to a local rule so as to obtain a new integrable distribution of planes?
to A. Let $\alpha$ be the angle between the direction $AB$ and $n$. With the previous notation, in order to apply the convention $\omega$ the observer $A$ should move forward the hands of its clock of a quantity (for simplicity we are considering here the case in which there is no redshift between $A$ and $B$; this formula can, however, be generalized to include the redshift)$\delta_{\omega} = \tau_{Br} - \frac{1}{2} - \tau_{Ai} + \phi \Delta/2 \cos \theta = \delta_{u} + \frac{1}{2} \phi \Delta \cos \theta$ with $\delta_{u} \equiv \delta$.

Let us now try to determine the most convenient local simultaneity connection. We make some simplifying assumptions

(a) The frame is generated by a Killing vector field $k$.
(b) The functions $\psi^{m}, \psi^{w}$ and $\psi^{a}$ are constructed from the observable quantities $a, w$ and $\theta$ (or equivalently $a, w$ and $m$ with $m = aw \sin \theta$).
(c) The curvature $\nu^{m}$ of $\omega_{\mu}$ is proportional to the Riemann tensor (through contraction with a suitable tensor).

The first two conditions are natural simplifications that allow us to tackle the problem while keeping the calculations at a reasonable size. The last one is imposed since the requirement $\nu^{m} = 0$ would be too restrictive and no simultaneity connection satisfying that requirement would be eventually found (note that the condition $\nu^{m} = 0$ implies condition (c), thus if a connection that satisfies $\nu^{m} = 0$ exists, it should be found between those that we selected imposing (c)). With our condition (c), at least in the weak field limit, the distribution of horizontal planes becomes integrable providing a useful definition of simultaneity.

The following theorem holds

**Theorem 1.** In a stationary spacetime let $k$ be a timelike Killing vector field and set $u = k/\sqrt{k \cdot k}$. Let $U$ be the open set $U = \{x : m(x) > 0 \text{ and } a(x) \neq w(x)\}$. Consider in $U$ the connection

$$\omega_{\alpha} = u_{\alpha} + \psi^{m}(x)m_{\alpha} + \psi^{a}(x)a_{\alpha} + \psi^{w}(x)w_{\alpha.}$$

Let $\psi^{m}, \psi^{a}, \psi^{w},$ be $C^{3}$ functions dependent only on $a, w$ and $\theta$. Then, regardless of the stationary spacetime considered, the connection is timelike in $U$ (and hence it is a simultaneity connection in $U$) and has a curvature proportional to the Riemann tensor in $U$ only if

$$\psi^{m} = \frac{a^{2} + w^{2} - \sqrt{(a^{2} + w^{2})^{2} - 4m^{2}}}{2m^{2}}.$$  \hspace{1cm} (4)

**4 Conclusions**

Thanks to the previous theorem the simultaneity connection

$$\bar{\omega}_{\alpha} = u_{\alpha} + \frac{a^{2} + w^{2} - \sqrt{(a^{2} + w^{2})^{2} - 4m^{2}}}{2m^{2}} m_{\alpha},$$  \hspace{1cm} (5)

that we call $\bar{C}$-simultaneity, is particularly well behaved in those spacetime regions where the Riemann tensor is sufficiently weak. It can be shown [10] that it can be extended by continuity to the set $C = A - B$ where, $A = \{x : a^{2} + w^{2} > 0\}, B = \{x : a = w \neq 0 \text{ and } \theta = \pi/2\}$, by defining, $\bar{\omega}_{\alpha} = u_{\alpha}$, in those points where $m = 0$.

The $\bar{C}$-simultaneity follows almost uniquely from the geometrical requirements discussed above. While the observers have the freedom to choose a local simultaneity convention, the requirement of being almost integrable in the weak field limit imposes strong constraints on its actual expression. The geometry tells the observers what simultaneity convention is better to use in practice and remarkably the simultaneity convention that turns out is not Einstein’s but rather $\bar{C}$-simultaneity. This convention is integrable in Minkowski spacetime contrary to the Einstein convention which is not integrable, for instance, in the case of the
rotating platform. This integrability can also be verified studying the more complicated Killing vector field that gives rise to the pseudocylindric coordinates of Letaw and Pfautsch \[4, 5\].

We believe that our approach shares a number of features that makes it preferable over other approaches to simultaneity. Indeed it is coordinate independent, it has a clear operational meaning, and the locality property makes it independent of the global information that one may or may not have (e.g. it does not depend on the model of the earth geoid). Finally, it does not require particular spacetime symmetries apart from that of stationarity (e.g. the spherical symmetry is not required). Our findings seems to be useful for all those synchronization approaches that try to adapt a global coordinate time to a spacetime network of observers (e.g. computers connected over the earth surface) without privileging a particular vertex in the network and without making use of elements outside the network (e.g. the satellites in the GPS).

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