The first occurrence of a number in Gijswijt’s sequence

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Abstract

This article builds on a previous article on curling numbers by van de Bult et al. There, sequences $A^{(m)}$ were defined for $m \in \mathbb{Z}_{\geq 1}$ which we call the level-$m$ Gijswijt sequences. Van de Bult et al. proved that these sequences contain all integers $n$ that are at least $m$. They also made a conjectural estimate for the position of the first $n$ in Gijswijt’s sequence $A^{(1)}$. The main result of this article is an expression for the exact position of the first $n$ in the level-$m$ Gijswijt sequence, thereby proving the estimate by van de Bult et al. Along the way, we give a rigorous definition for constants $\epsilon_m$ for $m \geq 1$ which occurred in their paper. Our expression also uses constants $\nu_m$ for $m \geq 1$. We prove that the irrationality measures of both $\epsilon_m$ and $\nu_m$ are at least $m + 1$, for all $m \geq 1$. Additionally, we prove the estimate of van de Bult et al. for the first 5 in Gijswijt’s sequence. We also prove that each integer $n$ has an asymptotic density in $A^{(m)}$, and that $A^{(m)}$ has a mean value.

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1 Introduction

Van de Bult et al. [1, p. 2] defined the curling number of a string as follows. For a finite nonempty string $S$ (over an arbitrary alphabet), the curling number of $S$ is the largest integer $k$ such that

$$S = X Y Y \ldots Y$$

$k$ times $Y$

for some strings $X$ and $Y$, where $Y$ must be nonempty. This number is denoted by $C(S)$. For example, we have $C(112112223) = 1$ and $C(222322232223) = 3$, the latter since 2223 is repeated 3 times. Then van de Bult et al.[1, p. 2] defined the infinite string $A$ as follows:

$$A(1) := 1, \quad A(n + 1) := C((A(1), A(2), \ldots, A(n)))$$

for all $n \geq 1$.

This sequence is now called Gijswijt’s sequence (see entry A090822 in the OEIS [8]). It starts with

$$A = 11211223112112223112112223211211222311211222322232223321 \ldots$$

The first 4 occurs at position 220, but the first 5 is conjectured to occur at about position $10^{10^{23}}$ [1, p. 22]. Van de Bult et al. [1, p. 12] proved that all positive integers occur in Gijswijt’s sequence. In order to prove this, van de Bult et al. had to define a family of sequences that generalizes Gijswijt’s sequence. We shall call these the level-$m$ Gijswijt sequences. These sequences, denoted by $A^{(m)}$, are given by

$$A^{(m)}(1) := m, \quad A^{(m)}(n + 1) := \max(m, C((A(1), A(2), \ldots, A(n))))$$

for all $n \geq 1$.

Note that $A^{(1)} = A$. The level-2 and level-3 Gijswijt sequences (see entries A091787 and A091799 in the OEIS [8]) start as follows:

$$A^{(2)} = 222322232223222322232223222322233342 \ldots$$

$$A^{(3)} = 3333433343334333433343334333433343334443 \ldots$$

It turned out [1, Section 3] that the level-$m$ Gijswijt sequences are builded from finite strings $B_t^{(m)}$, $S_t^{(m)}$ (the $S_t^{(m)}$ are called glue strings). They satisfy the following relations:

$$B_1^{(m)} = m$$

$$B_t^{(m)} = B_t^{(m)} B_t^{(m)} \ldots B_t^{(m)} S_t^{(m)}$$

$m+1$ times $B_t^{(m)}$ for all $t, m \geq 1$ (1)

$$A^{(m)} = \lim_{t \to \infty} B_t^{(m)}$$

for all $m \geq 1$

$$A^{(m+1)} = S_1^{(m)} S_2^{(m)} S_3^{(m)} \ldots$$
The limit in the second line means that every $B_t^{(m)}$ string is a prefix of $A^{(m)}$. This structure implies that every element of $A^{(m+1)}$ occurs in $A^{(m)}$ also. Indeed, if $x$ is in $A^{(m+1)}$, then there is a $t$ such that $x$ is in $S_t^{(m)}$. Therefore, $x$ occurs in $B_t^{(m)}$, and hence in $A^{(m)}$. Now from induction we see that for $n \geq m$, every element of $A^{(n)}$ occurs in $A^{(m)}$. Since $A^{(n)}$ starts with $n$, we conclude that $A^{(m)}$ contains all integers $n \geq m$.

Additionally, $T_t^{(m)}$ strings were defined as follows:

$$T_t^{(m)} := S_1^{(m)} S_2^{(m)} \ldots S_{t-1}^{(m)}. \quad (2)$$

Now it follows that

$$A^{(m+1)} = \lim_{t \to \infty} T_t^{(m)}.$$ 

From induction on $t$, using Equation 1, it follows also that $T_t^{(m)}$ is a suffix of $B_t^{(m)}$ for all $t, m \geq 1$.

The lengths of $B_t^{(m)}, S_t^{(m)}, T_t^{(m)}$ are denoted by respectively $\beta^{(m)}(t), \sigma^{(m)}(t), \tau^{(m)}(t)$.

After proving that each number $n \geq m$ occurs in $A^{(m)}$, van de Bult et al. [1, Section 4] made a number of conjectural estimates in order to find the approximate location of the first occurrence of a number $n$ in the level-$m$ Gijswijt sequences, and specifically in Gijswijt’s sequence $A = A^{(1)}$. Conjecture 4.4 of van de Bult et al. states that for $n \geq 5$, the first occurrence of $n$ in Gijswijt’s sequence is about position

$$2^{2^{n-1}} \quad (3)$$

No specific meaning was assigned to the term ‘about’), but Expression 3 indicates the slow growth of the sequence.

In this article, we shall provide an expression for the exact position of the first $n$ in the level-$m$ Gijswijt sequence. Before we give the expression, we propose the following notation:

**Definition 1.1.** For $n \geq m \geq 1$, define $\phi^{(m)}(n)$ as the smallest number such that $A^{(m)}(\phi^{(m)}(n)) = n$.

Now the first values of $\phi^{(1)}(n)$ are 1, 3, 9, 220, . . . . This is Entry [A091409](https://oeis.org/A091409) in the OEIS [8].

For the next definition we need a lemma:

**Lemma 1.2.** For all integers $m, n$ with $n \geq m \geq 1$, we have the following.

(a) There is a unique number $t^{(m)}(n)$ such that $\phi^{(m)}(n) = \beta^{(m)}(t^{(m)}(n))$.

(b) For this number $t^{(m)}(n)$, we have $T_{t^{(m)}(n)}^{(m)} = B_{t^{(m)}(n)}^{(m+1)}$. 

4
Proof. (a) For \( n = m \), we have \( \phi^{(m)}(m) = 1 = \beta^{(m)}(1) \), so we can define \( t^{(m)}(m) := 1 \).

Now suppose that \( n > m \). Then let \( t \) be minimal such that \( n \) is in \( B_t^{(m)} \). Since \( n \) is not an element of \( B_{t-1}^{(m)} \), the first \( n \) in \( A^{(m)} \) must be in the glue string \( S_{t-1}^{(m)} \). But \( C(A^{(m)}[1, \phi^{(m)}(n)]) = 1 \). Therefore, \( A^{(m)}(\phi^{(m)}(n) + 1) = m \). Since \( m \not\in S_{t-1}^{(m)} \), it follows that the first \( n \) is the last element of \( B_t^{(m)} \). Therefore, \( \phi^{(m)}(n) = \beta^{(m)}(t) \).

Uniqueness follows from the definition of \( \phi^{(m)}(n) \).

(b) Since \( T_{t^{(m)}(n)}^{(m)} \) is a suffix of \( B_{t^{(m)}(n)}^{(m)} \), it follows that the last element of \( T_{t^{(m)}(n)}^{(m)} \) is \( n \) and that \( T_{t^{(m)}(n)}^{(m)} \) has no other elements \( n \). Also, \( T_{t^{(m)}(n)}^{(m)} \) is a prefix of \( A^{(m+1)} \). So it follows by the definition of \( \phi^{(m+1)}(n) \) that \( T_{t^{(m)}(n)}^{(m)} = A^{(m+1)}[1, \phi^{(m+1)}(n)] \), which equals \( A^{(m+1)}[1, \beta^{(m+1)}(t^{(m+1)}(n)) \) by Part (a). This equals \( B_{t^{(m+1)}(n)}^{(m+1)} \).

Hence we can make the following definition:

**Definition 1.3.** For all integers \( m, n \) with \( n \geq m \), define \( t^{(m)}(n) \) as the integer such that \( \phi^{(m)}(n) = \beta^{(m)}(t^{(m)}(n)) \) and \( T_{t^{(m)}(n)}^{(m)} = B_{t^{(m)}(n)}^{(m+1)} \).

The values of \( t^{(1)} \) are given in Entry \( \text{A357064} \) of the OEIS [8].

Let us consider a few examples.

**Example 1.4.** For all \( m \geq 1 \), we have the following.

- \( t^{(m)}(m) = 1 \) since \( B_{1}^{(m)} = m \).
- \( t^{(m)}(m + 1) = 2 \) since \( B_{2}^{(m)} = m^{m+1}(m + 1) \).
- For \( n = m + 2 \), observe that \( \sigma^{(m)}(x) = 1 \) for \( x = 1, 2, \ldots, m \) and \( \sigma^{(m)}(m + 1) = 3 \) by Equation 35 from van de Bult et al. [1]. Therefore, \( \tau^{(m)}(m + 2) = m + 3 \), hence \( T_{m+2}^{(m)} = A^{(m+1)}[1, m + 3] = (m + 1)^{m+2}(m + 2) \). It follows that \( t^{(m)}(m + 2) = m + 2 \).
- From the bottom of page 21 of van de Bult et al. [1, p. 21], we know that \( \phi^{(m)}(m) = 1, \phi^{(m)}(m + 1) = m + 2 \), and \( \phi^{(m)}(m + 2) = \frac{(m+1)^{m+2}+2m-1}{m} \).

In our new terminology, Van de Bult et al. [1, p. 22] obtained the values \( t^{(2)}(5) = 80 \), \( \phi^{(2)}(5) = 7709040388415370160829246932345692180 \), and the estimate \( \phi^{(1)}(5) \approx 10^{10^{23}} \).

Conjecture 4.4 of van de Bult et al. [1] becomes

\[
\phi^{(1)}(n) \approx 2^{2^{2^{\ldots^{n-1}}}}
\]

for all \( n \geq 5 \) (no specific meaning was assigned to \( \approx \)).
Table 1: Values of $B_t^{(m)}$

| $m \setminus t$ | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|
| 1               | 1 | 112| 112112223| 1121122231121122232 |
| 2               | 2 | 2223| 2223223223222332223223223223223223223334 |
| 3               | 3 | 33334| (33334)44| (33334333433343334444)44 |
| 4               | 4 | 444445| (444445)55| ((444445)55)55 |

Table 2: Values of $S_t^{(m)}$

| $m \setminus t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|---|---|---|---|---|---|---|
| 1               | 2 | 223| 22322322322232223223223223223223223223334 | 2 |
| 2               | 3 | 334| 3 | 3 | 334 | 3 |
| 3               | 4 | 4 | 445 | 4 | 4 | 4 |
| 4               | 5 | 5 | 5 | 5 | 556 | 5 | 5 |

We can now state our expression, which we shall prove in Theorem 7.11. For all integers $m, n$ with $1 \leq m \leq n - 2$ and $(m, n) \neq (1, 3)$, we have:

$$
\phi^{(m)}(n) = [1 - \epsilon_m + \epsilon_m \cdot (m + 1)\lceil \nu_{m:(m+1)}[\nu_{m+1:(m+2)}...[\nu_{n-2:(n-1)}]...]\rceil].
$$

This formula is not an estimate, but the exact position of the first $n$ in the level-$m$ Gijswijt sequence.

Here the constants $\epsilon_m$ for $m \geq 1$ are as mentioned in van de Bult et al. [1, p. 19], but the constants $\nu_m$ for $m \geq 1$ are not mentioned earlier in the literature; we shall define them in Definition 6.1. Note that substituting $m = 1$ gives us the expression

$$
[1 - \epsilon_1 + \epsilon_1 \cdot 2^{[\nu_{1:2}[\nu_{2:3}...[\nu_{n-2:(n-1)}]...]}],
$$

which looks similar to Expression 3.

Since Expression 4 contains constants $\epsilon_m$ and $\nu_m$, it is not entirely a direct formula. While we have found a direct formula for $\phi^{(m)}(m + 3)$, it is too ugly to publish here. However, we shall see that the constants $\epsilon_m$ and $\nu_m$ are close to 1. Therefore, Expression 4 shows how slow the sequences are growing. We provide algorithms to approximate values of $\epsilon_m$ and $\nu_m$ at [4] and [6]. Specific values can be found in Tables 8 and 11.

1.1 Tables

We have included tables of the first values of $B_t^{(m)}, S_t^{(m)}, T_t^{(m)}, \beta^{(m)}(t), \sigma^{(m)}(t), \tau^{(m)}(t)$:
| $m \backslash t$ | 1 | 2 | 3 | 4 | 5 | 6 |
|--------------|---|---|---|---|---|---|
| 1            | $\varepsilon$ | 2 | 2223 | 22232 | 22232223222332 | 222322232223322232 |
| 2            | $\varepsilon$ | 3 | 33 | 33334 | 333343 | 3333433 |
| 3            | $\varepsilon$ | 4 | 44 | 444 | 444445 | 4444454 |
| 4            | $\varepsilon$ | 5 | 55 | 555 | 5555 | 5555556 |

Table 3: Values of $T_t^{(m)}$

| $m \backslash t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|--------------|---|---|---|---|---|---|---|---|
| 1            | 1 | 3 | 9 | 19 | 47 | 98 | 220 | 441 |
| 2            | 1 | 4 | 13 | 42 | 127 | 382 | 1149 | 3448 |
| 3            | 1 | 5 | 21 | 85 | 343 | 1373 | 5493 | 21973 |
| 4            | 1 | 6 | 31 | 156 | 781 | 3908 | 19541 | 97706 |
| 5            | 1 | 7 | 43 | 259 | 1555 | 9331 | 55989 | 335935 |

Table 4: Values of $\beta^{(m)}(t)$. Compare with Entries A091411 and A357063 in the OEIS [8]. We shall use these values later in the article.

| $m \backslash t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|--------------|---|---|---|---|---|---|---|---|---|
| 1            | 1 | 3 | 1 | 9 | 4 | 24 | 1 | 3 | 1 |
| 2            | 1 | 3 | 3 | 1 | 1 | 3 | 1 | 1 | 9 |
| 3            | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 |
| 4            | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 5: Values of $\sigma^{(m)}(t)$ (compare with Entries A091579 and A091840 in the OEIS [8])

| $m \backslash t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------------|---|---|---|---|---|---|---|---|---|----|
| 1            | 0 | 1 | 1 | 1 | 5 | 14 | 18 | 42 | 43 | 46 |
| 2            | 0 | 1 | 2 | 5 | 6 | 7 | 10 | 11 | 12 | 21 |
| 3            | 0 | 1 | 2 | 3 | 6 | 7 | 8 | 9 | 12 | 13 |
| 4            | 0 | 1 | 2 | 3 | 4 | 7 | 8 | 9 | 10 | 11 |

Table 6: Values of $\tau^{(m)}(t)$ (compare with Entry A092432 in the OEIS [8])
1.2 Notation and definitions

When \( f : A \to B \) is a function, the image of \( f \) is defined as \( \text{Im}(f) := \{ f(a) \mid a \in A \} \).

For \( z \in \mathbb{Z} \), we define the set \( \mathbb{Z}_{\geq z} := \{ z, z + 1, z + 2, \ldots \} \). For integers \( m, n \in \mathbb{Z} \) such that \( m \leq n \), we denote the interval \( \{ m, m + 1, \ldots, n \} \) by \([m, n]\). For (half) open intervals \((m, n], [m, n), (m, n)\), the elements \( m, n \) or both are deleted from the interval. We shall use \([m, m - 1]\) or \((m, m)\) to denote the empty interval.

Occasionally, we shall use \([a, b], (a, b), [a, b), (a, b)\) to denote intervals of real numbers instead of intervals of integers. This shall be clear from the context.

1.2.1 String definitions

We will use a number of definitions on strings (i.e., alphabet, prefix, suffix, proper prefix, proper suffix, concatenation) that can be found in the book of Lothaire [3]. However, we shall use string instead of word.

We say that \( x \) is an element of a string \( S \) if there is an \( i \) such that \( S(i) = x \).

Usually when we denote a string, we shall omit the commas and brackets. For example, \( 1222 \) means \((1, 2, 2, 2)\). For \( n \in \mathbb{Z}_{\geq 1}, \Omega^* \) is the set of all strings over \( \Omega \), including infinite strings and the empty string \( \varepsilon \). For \( \Omega = \mathbb{Z}_{\geq m} \), we write \( (\mathbb{Z}_{\geq m})^* \) simply as \( \mathbb{Z}_{\geq m}^* \). The length of a string \( S \), which is an element of \( \mathbb{Z}_{\geq 0} \cup \{ \infty \} \), is denoted by \( l(S) \).

For a string \( S \) and \( m, n \in \mathbb{Z}_{\geq 1} \) such that \( m \leq n \leq l(S) \), we define \( S[m, n] \) as the string \((S(m), S(m + 1), \ldots, S(n))\). We shall occasionally use \( S[m, m - 1] \) to denote the empty string \( \varepsilon \); this shall be useful when we want to speak about a string \( S[m, n] \) for which we do not know whether it is empty or not. Also, for an infinite string \( S \) and \( m \in \mathbb{Z}_{\geq 1} \) we define \( S[m, \infty] \) as the infinite string \((S(m), S(m + 1), S(m + 2), \ldots)\). The strings \( S[m, n] \) are called substrings of \( S \).

We say that a string \( S \) contains a string \( T \) if \( T \) occurs as a substring in \( S \). For two infinite strings \( S, T \), we say that \( S \) is a subsequence of \( T \) if \( S \) can be obtained by deleting elements of \( T \). Note the difference between ‘substring’ and ‘subsequence’: for a substring, no elements in between are to be deleted.

For a finite string \( S \) and an integer \( k \geq 1 \), we denote \( S S \ldots S \) by \( S^k \). We say that a string of the form \( X^2 \) is a square, and a string of the form \( X^3 \) is a cube.

For strings \( S_1, S_2, S_3 \ldots \) such that \( S_n \) is a proper prefix of \( S_{n+1} \) for all \( n \geq 1 \), we define the limit \( \lim_{n \to \infty} S_n \) as the unique infinite string \( S \) such that \( S_n \) is a prefix of \( S \) for all \( n \geq 1 \).

We say that a finite string \( Y \) is a conjugate of \( X \) if there are (possibly empty) strings \( A, B \) such that \( X = AB \) and \( Y = BA \). We shall use that a conjugate of a power of \( X \) is also a power of a conjugate of \( X \), and vice versa.

For a string \( S \) and a number \( n \), let \#(n \in S) be the number of occurrences of \( n \) in \( S \).
1.2.2 Other definitions

For integers \( n \geq 1 \) and \( m \geq 2 \), we define the \( m \)-adic order \( \text{ord}_m(n) \) as the largest integer \( z \) such that \( m^z \) divides \( n \). Now \( \text{ord}_m(n) \in \mathbb{Z}_{\geq 0} \). For \( m \geq 2 \), we define the \( m \)-th order ruler sequence as the infinite string \( r_m := (\text{ord}_m(1), \text{ord}_m(2), \text{ord}_m(3), \ldots) \). The sequences \( r_2 - r_8 \) can be found in Entries A007814, A007949, A235127, A112765, A122841, A214411, 244413 of the OEIS [8].

Also, we can write \( n \) in base \( m \); this means that there are unique integers \( a_1, \ldots, a_x \in \{0, 1, \ldots, m - 1\} \) with \( a_x \neq 0 \) such that \( n = a_x \cdot m^{x-1} + \cdots + a_1 \cdot m^0 \). We shall denote the largest digit of \( n \) in base \( m \) by \( \maxdigit_m(n) \), i.e., we have \( \maxdigit_m(n) := \max(a_1, \ldots, a_x) \).

Furthermore, we define transformations \( \chi^{(m)} \) of \( \mathbb{Z}_{\geq 1} \) as follows: when \( n = a_x \cdot m^{x-1} + \cdots + a_1 \cdot m^0 \) in base \( m \), we define \( \chi^{(m)}(n) := a_x \cdot (m + 1)^{x-1} + \cdots + a_1 \cdot (m + 1)^0 \). The functions \( \chi^{(2)} - \chi^{(10)} \) are given in Entries A005836, A023717, A020654, A037465, A020657, A037474, A037477, A007095, A171397 of the OEIS [8].

We shall use the following asymptotic relations. For two sequences \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) of real numbers, we say that \( a_n \sim b_n \) for \( n \to \infty \) if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \). We shall use \( a_n \approx b_n \) only for heuristic means, so the symbol \( \approx \) has no exact meaning.

For a subset \( S \subset \mathbb{Z}_{\geq 1} \), we say that \( S \) has asymptotic density equal to \( \delta \) if

\[
\lim_{N \to \infty} \frac{|S \cap [1, N]|}{N} = \delta.
\]

Consider a function \( f : \mathbb{Z}_{\geq 1} \to \mathbb{R} \). If the limit

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(i)
\]

exists, it is called the mean value of \( f \).

The irrationality measure \( \mu(x) \) of a real number \( x \) is defined as follows. Let \( S \) be the set of real numbers \( \mu \) such that the equation \( 0 < |x - \frac{p}{q}| < \frac{1}{q^\mu} \) has infinitely many solutions in integers \( p, q \) with \( q > 0 \). Then \( \mu(x) \) is the supremum \( \sup(S) \).

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2 Overview

The bulk of this article (Sections 3–7) is dedicated to proving an expression for the exact location of the first n in the level-m Gijswijt sequence (Theorem 7.11). No such an expression was previously known. The expression contains constants εm and νm. We shall prove that the irrationality measures of εm and νm are at least m + 1 (Theorems 6.8, 6.21 and 7.13).

We start Section 3 by observing a ruler sequence pattern in the sequences A(m). Specifically, we define π strings in Definition 3.2:

\[ π^{(m)}(a, b) := \prod_{i=a}^{b} m \cdot T^{(m)}_{\ord_{m+1}(i)+1} \]

We prove in Equation 5 that

\[ A^{(m)} = π^{(m)}(1, \infty). \]

Now the strings π(m+1)(1, x) are exactly the prefixes of A(m+1) with curling number at most m + 1. On the other hand, the strings T(m) t are exactly the prefixes of A(m+1) with curling number at most m (see Lemma 3.1). Therefore, every string T(m) t can be written as π(m+1)(1, x) for some x. To find the exact relation, we need more information on the curling numbers of the π strings. Therefore, we continue Section 3 by studying suffixes of strings π(n)(1, x) of the form X^l.

For this, we reduce the problem to an equality in a ruler sequence, and solve the problem there. The case of cube and higher power suffixes (Corollary 3.9), turns out to be a lot easier than square suffixes (Lemma 4.5 and the subsequent discussion). Therefore, we can make the following definition:

**Definition** (see Definition 4.6). For all m, t ≥ 1, let \( \nu_m(t) \) be the unique number in \( \mathbb{Z}_{≥0} \) such that

\[ T^{(m)}_t = π^{(m+1)}(1, \chi^{(m+1)}(\nu_m(t))). \]

Here \( \chi^{(m)}(a) \) is defined as the number \( \sum_i a_i \cdot (m + 1)^{i-1} \), where \( a = \sum_i a_i \cdot m^{i-1} \) is the m-ary expansion of \( a \).

Now the function \( \nu_m \) contains, in some way, the information how \( A^{(m)} \) can be reconstructed from \( A^{(m+1)} \); \( \nu_m \) shows us how the \( T^{(m)}_t \) are located in \( A^{(m+1)} \). Then we know that \( S^{(m)}_t \) is the string such that \( T^{(m)}_{t+1} = T^{(m)}_t S^{(m)}_t \), and finally we obtain the \( B^{(m)}_t \) strings by starting with \( B^{(m)}_1 = m \) and then iteratively using that \( B^{(m)}_{t+1} = (B^{(m)}_t)^{m+1} \cdot S^{(m)}_t \). Now \( A^{(m)} = \lim_{t \to \infty} B^{(m)}_t \).
After listing some basic properties of $\iota_m$, we define $\iota_m^{-1}(p)$ in Definition 4.9 as the largest integer such that $\iota_m(\iota_m^{-1}(p)) \leq p$.

Section 5 explores a further connection between the iota functions. We start by defining the level-$m$ expansion of a number $a$:

**Definition** (see Definition 5.1). Let $m \geq 1$ and $a \geq 1$. Let $n$ be the last element of $\pi^{(m+1)}(1, \chi^{(m+1)}(a))$. Define the numbers $a_{m+1}, a_{m+2}, \ldots, a_n$ recursively in the following way:

- Take $a_{m+1} := a$;
- When $a_l$ is constructed for $l < n$, define $a_{l+1} := \iota_l(\text{ord}_l(a_l) + 1)$.

The sequence $a_{m+1}, a_{m+2}, \ldots, a_n$ is called the **level-m expansion** of $a$.

The goal for defining the level-$m$ expansion, is to study the string $\pi^{(m+1)}(1, \chi^{(m+1)}(a))$. It turns out (see Lemma 5.2 (b)) that for all $l$, $\pi^{(l)}(1, \chi^{(l)}(a_l))$ is the longest suffix of $\pi^{(m+1)}(1, \chi^{(m+1)}(a))$ that contains only elements in $\mathbb{Z}_{\geq 1}$.

The most important level-$m$ expansion is that of $a := \iota_m(\ell^{(m)}(n))$. We prove in Proposition 5.3 that the expansion consists of $\iota_m(\ell^{(m)}(n)), \iota_{m+1}(\ell^{(m+1)}(n)), \ldots, \iota_{n-1}(\ell^{(n-1)}(n))$. This implies the following equation (see Equation 9):

$$\ell^{(m)}(n) = \iota_m^{-1}((m + 1)^{\ell^{(m+1)}(n)-1})$$

This is the first ingredient of our expression for the first $n$ in $A^{(m)}$ (Theorem 7.11).

Using Corollary 3.9, we prove Theorem 5.4. This theorem states that for $m \geq 2$, a number $a$ is in the image of $\iota_m$ if and only if its level-$m$ expansion satisfies certain properties. For $m = 1$, we use Lemma 3.7 to prove a similar statement in Theorem 5.10.

In Subsection 5.1, we discuss an implementation of Theorems 5.4 and 5.10 in Python.

In Section 6, we prove that for all $m \geq 1$, $\frac{\pi}{\iota_m(x)}$ converges as $x \to \infty$. We define $\nu_m$ as the limit of this expression (see Definition 6.1). We approximate the constants $\nu_m$ by rationals to prove that their irrationality measures are at least $m + 1$. For this, we have to make a distinction between the cases $m \geq 2$ and $m = 1$. The case $m \geq 2$ (Theorem 6.8) is a lot easier than $m = 1$ (Theorem 6.21). This is a consequence of the fact that Corollary 3.9 is easier than Lemma 3.7.

For $m \geq 2$, we first prove that there exists a set $V_m$ such that $x$ is in the image of $\iota_m$ if and only if $\text{ord}_{m+1}(x) \in V_m$ (see Definition 6.3). For the proof we use Theorem 5.4. Next, we show in Proposition 6.5 (b) that

$$\nu_m = \frac{m}{m + 1} \cdot \sum_{\nu \in V_m} \frac{1}{(m + 1)^\nu}.$$ 

We establish patterns in the sets $V_m$ in Lemma 6.7. Those patterns allow us to prove in Theorem 6.8 that $\mu(\nu_m) \geq m + 1$. Now by Roth’s theorem, $\nu_m$ is transcendental.

For $m = 1$, things get more complicated. Instead of one set $V_m$, we define two sets $Q, R$ in Definition 6.10, where $R$ consists of pairs of integers. In Corollary 6.11, which follows from Theorem 5.10, we show how these sets are related to the image of $\iota_1$. 
We prove in Proposition 6.13 (c) that
\[
\nu_1 = \sum_{(a,b) \in R} \frac{1}{2^{b+1}}.
\]
In Lemma 6.14, we establish patterns in Q and R. Finally, we prove in Theorem 6.21 that \( \mu(\nu_1) \geq 2 \), which implies that \( \nu_1 \) is irrational.

A number of the estimates for \( \nu_m \) that we encounter in this section, are also useful for proving Theorem 7.11. For some values of \( y \), it turns out that \( \nu_m^{-1}(y) - 1 = [\nu_m \cdot y] \) (see Propositions 6.5 (c) and 6.18). This is the second ingredient of the proof of Theorem 7.11. We use our results until now to obtain an initial tower of exponents in Theorem 6.20.

We finish Section 6 with approximations for \( \nu_1, \nu_2, \ldots, \nu_{10} \) in Table 8. We also give the first elements of \( V_m \), \( Q \) and \( R \) in Tables 9 and 10.

Section 7 analyses the asymptotic behavior of \( \beta^{(m)}(n) \). In Theorem 7.2 we prove, using the iota functions, that for all \( m \geq 1 \) there is a constant \( \epsilon_m \) such that \( \beta^{(m)}(n) \sim \epsilon_m \cdot (m+1)^{n-1} \) as \( n \to \infty \). The constants \( \epsilon_m \) are mentioned in van de Bult et al. [1, p. 19]. In Corollary 7.3, we deduce the equation
\[
\epsilon_m = 1 + \sum_{i=1}^{\infty} \frac{\sigma^{(m)}(i)}{(m+1)^i}.
\]
After establishing patterns in the values of \( \sigma^{(m)} \), we prove estimates for \( \epsilon_m \) in Lemma 7.7.

We use this to prove Proposition 7.9, which expresses \( \beta^{(m)}(t^{(m)}(n)) \) in \( t^{(m)}(n) \). This is the final ingredient for Theorem 7.11. Note that the formula for \( \phi^{(m)}(n) \) in Theorem 7.11 does not help much with calculating specific values of \( \phi^{(m)}(n) \). However, these values increase so rapidly that even \( \phi^{(1)}(5) \) can not be written down. Nevertheless does Theorem 7.11 give insight in the growth rate of the level-\( m \) Gijswijt sequences, and the theorem shows to what extent Conjecture 4.4 from van de Bult et al. [1] holds.

Again using Lemma 7.7, we prove in Theorem 7.13 that \( \epsilon_m \) has irrationality measure at least \( m+1 \). It follows that \( \epsilon_1 \) is irrational, and for \( m \geq 2 \), Roth’s theorem tells us that \( \epsilon_m \) is transcendental.

We end Section 7 with approximations for \( \epsilon_m \) for \( m = 1, 2, \ldots, 10 \) in Table 11.

In Section 8, we prove some further results on the level-\( m \) Gijswijt sequences. The three subsections are independent of each other.

We start in Subsection 8.1, by proving a observation from van de Bult et al. [1, pp. 14–19]. It says that for \( m \geq 1 \), the sequence
\[
\sigma^{(m)}(1), \sigma^{(m)}(2), \sigma^{(m)}(3), \ldots
\]
is ‘almost’ equal to a sequence \( \rho \circ r_{m+1} \), for some function \( \rho \). To prove this, we will not only look at the lengths \( \sigma^{(m)}(t) \) of the glue strings \( S_t^{(m)} \), but we shall also consider the glue strings \( S_t^{(m)} \) themselves. We define strings \( P_t^{(m)} := mB_1^{(m)}S_1^{(m)}B_2^{(m)}S_2^{(m)} \ldots B_{t-1}^{(m)}S_{t-1}^{(m)} \) in Definition 8.1, and it turns out in Theorem 8.3 that for all \( t \geq 1, m \geq 1 \), there is an integer \( u \geq 1 \) such that either \( S_t^{(m)} = P_u^{(m+1)} \) or \( S_t^{(m)} = P_u^{(m+1)} \cdot (m+1) \).
From these insights, Conjectures 4.1 and 4.2 from van de Bult et al. [1] follow as a corollary.

In Subsection 8.2, we calculate the position of the first 5 in Gijswijt’s sequence. The position can be found (expressed in $\epsilon_1$) in Equation 35. A similar estimate for this position was already predicted at Page 23 of van de Bult et al. [1]: the actual value is only 2 or 3 smaller! We also calculate the position of the first substring 44.

In Subsection 8.3, we use estimates from Section 7 to prove that the set $\{p \in \mathbb{N} : A^{(m)}(p) = n\}$ has an asymptotic density for all $m,n$. For Gijswijt’s sequence, this was conjectured in a comment at Entry A090822 of the OEIS [8]. We also prove that $A^{(m)}$ has a mean value for all $m$.

In Section 9, we state some open questions.


3 Square and cube substrings

In this section, we consider substrings $X^k$ of the level-$m$ Gijswijt sequences, for $k \geq 2$. Lemma 3.7 is the main result of this section, which gives a partial classification of the substrings $X^2$ in $A(m)$. To arrive at that result, we will need to consider the level-$m$ Gijswijt sequences from a new perspective. We do this by making a new definition, of $\pi$ strings (Definition 3.2).

We start with some basic observations:

Since $B_t^{(m)}$ is a prefix of $A^{(m)}$ and since $S_t^{(m)}$ and $T_t^{(m)}$ are substrings of $A^{(m+1)}$ for all $m, t \geq 1$, it follows that $B_t^{(m)} \in \mathbb{Z}_{\geq m}$ and $S_t^{(m)}, T_t^{(m)} \in \mathbb{Z}_{\geq m+1}$ for all $m, t \geq 1$. All these strings are finite, and except for the strings $T_1^{(m)}$ all these strings are nonempty.

Van de Bult et al. [1] only defined the curling number of a finite nonempty string; we additionally define $C(\varepsilon) := 1$. We shall often use the following: if $S$ is a string with suffix $T$, then $C(S) \geq C(T)$.

**Lemma 3.1.** For all $m \geq 1$, the strings $T_t^{(m)}$ are exactly the prefixes of $A^{(m+1)}$ with curling number at most $m$.

**Proof.** In van de Bult et al. [1] at pages 12-13, an element $A^{(m)}(i)$ of $A^{(m)}$ is called promoted if $C(A^{(m)}[1, i-1]) < m$. Therefore, an element $A^{(m+1)}(i)$ is promoted if $C(A^{(m+1)}[1, i-1]) \leq m$. Now the first two lines of the Sketch of the proof of Theorem 3.5 in van de Bult et al. [1] state the following:

'By Theorem 3.1, $A^{(m+1)} = S_1^{(m)} S_2^{(m)} \cdots$. The promoted elements $a^{(m+1)}(i_1), a^{(m+1)}(i_2), \ldots$ of $A^{(m+1)}$ are precisely the initial elements of the $S_k^{(m)}$.'

Now since $T_t^{(m)} = S_1^{(m)} \cdots S_{t-1}^{(m)}$ for all $t \geq 1$, the lemma follows. \hfill $\square$

We define the $\pi$ strings as follows:

**Definition 3.2.** For $m, \alpha \geq 1$, and $\beta \in \mathbb{Z}_{\geq \alpha-1} \cup \{\infty\}$, define

$$\pi^{(m)}(\alpha, \beta) := \prod_{i=\alpha}^{\beta} m \cdot T_{\text{ord}_{m+1}(i)+1}^{(m)}.$$ 

We shall see shortly that $\pi^{(m)}(1, \infty) = A^{(m)}$ for all $m$. Note that the indices of the $T$ strings follows the pattern of the ruler sequence $r_{m+1}$. We will now prove some properties of the ruler sequences. We shall use the elementary fact that for integers $m \geq 2$, $a \geq 1$, and $b \geq 1$, $\text{ord}_m(a) > \text{ord}_m(b)$ implies $\text{ord}_m(a + b) = \text{ord}_m(b)$.

**Lemma 3.3.** Let $m \geq 2$ be an integer.

(a) For integers $a \geq 0, b \geq 0$ with $a \leq b$ and a (possibly empty) string $A$, suppose that $AaAb$ is a substring of $r_m$. Then there is an integer $\mu \in \{1, 2, \ldots, \lfloor \frac{m}{y} \rfloor\}$ such that $l(A) + 1 = \mu \cdot m^y$, where $y$ is the maximal element of $Aa$. 

14
(b) For integers \( a \geq 0, b \geq 0 \) with \( a > b \) and \( a \) (possibly empty) string \( A \), suppose that \( AaAb \) is a substring of \( r_m \), i.e., we have \( r_m[\alpha, \alpha + 2p - 1] = AaAb \) for some integers \( \alpha, p \geq 1 \). Then there exist integers \( \lambda \geq 1 \) and \( \mu \in \{1, 2, \ldots, m - 1\} \) such that \( \alpha = \lambda \cdot m^b + \mu \cdot m^b + 1 \) and \( p = \mu \cdot m^b \). Moreover, we have \( \alpha > 1 \) and \( r_m(\alpha - 1) = b \).

(c) Let \( a \geq 1 \) be an integer with \( m \)-ary expansion \( a = a_x \cdot m^{x-1} + a_{x-1} \cdot m^{x-2} + \cdots + a_1 \cdot m^0 \). Then the following holds:
\[
r_m[1, a] = (r_m[1, m^{x-1}])^{a_x} \cdot (r_m[1, m^{x-2}])^{a_{x-1}} \cdots (r_m[1, m^0])^{a_1}.
\]

Proof.

(a) Take \( \alpha \geq 1, p \geq 1 \) such that \( AaAb = r_m(\alpha, \alpha + 2p - 1) \), and define \( y \) as the maximal element of \( Aa \). Now there is an integer \( i \in \{0, \ldots, p - 1\} \) such that \( r_m(\alpha + i) = y \) and \( r_m(\alpha + p + i) \geq y \). Therefore, \( m^y \mid p \), so there is an integer \( \mu \geq 1 \) such that \( p = \mu \cdot m^y \).

Suppose that \( \mu > \frac{m}{2} \). Then it follows that \( l([\alpha, \alpha + 2p - 2]) = 2p - 1 \geq 2 \cdot \left(\frac{m+1}{2} \right) \cdot m^y - 1 = m^y + m^y - 1 \geq m^y+1 \). Therefore, there is an integer \( i \) in \([\alpha, \alpha + 2p - 2]\) such that \( i \equiv 0 \mod m^y+1 \). But then we have \( r_m(i) \geq y+1 \). That is a contradiction with the definition of \( y \). Therefore, \( \mu \leq \frac{m}{2} \).

(b) We have \( m^b \mid a + 2p - 1 \) and \( m^b \mid a + p - 1 \), hence \( m^b \mid p \). Also, \( m^{b+1} \nmid a + 2p - 1 \), but \( m^{b+1} \mid m^a \mid a + p - 1 \), hence \( m^{b+1} \nmid p \). Therefore, we have \( b = \text{ord}_m(p) \). Now since \( m^{b+1} \mid a + p - 1 \) and \( m^b \mid p \), there are integers \( \lambda \geq 1 \) and \( \mu \geq 1 \) such that \( \alpha + p - 1 = \lambda \cdot m^{b+1} \) and \( p = \mu \cdot m^b \).

Now we shall prove that \( \mu \leq m - 1 \). For this, suppose that \( \mu \geq m \). Then we have \( p \geq m^{b+1} \). Therefore, there is a number \( j \in [\alpha+p, \alpha+2p-1] \) such that \( r_m(j) \geq b+1 \).

Now \( j \neq \alpha + 2p - 1 \) since \( r_m(\alpha + 2p - 1) = a \), hence \( r_m(j - p) = r_m(j) \geq b + 1 \) (using that \( j \) is in the second copy of \( A \)). From this it follows that \( m^{b+1} \mid p \). But that is a contradiction with what we saw earlier. So indeed, \( \mu \leq m - 1 \).

Now \( \alpha \geq m^{b+1} - (m-1) \cdot m^b + 1 = m^b + 1 > 1 \), and it follows from \( \text{ord}_m(\alpha - 1) = \text{ord}_m(\lambda \cdot m^{b+1} - \mu \cdot m^b) = b \) that \( r_m(\alpha - 1) = b \).

(c) For all \( i \in \{1, 2, \ldots, x\} \), we have
\[
  r_m[a_x \cdot m^{x-1} + a_{x-1} \cdot m^{x-2} + \cdots + a_{i+1} \cdot m^i + 1, a_x \cdot m^{x-1} + a_{x-1} \cdot m^{x-2} + \cdots + a_{i+1} \cdot m^i + a_i \cdot m^{i-1}]
  = r_m[1, a_i \cdot m^{i-1}] = (r_m[1, m^{i-1}])^{a_i}.
\]

The lemma follows by concatenating those strings in reversed order: first \( i = x \), then \( i = x - 1 \), and so on.
We can apply Lemma 3.3 (c) to the $\pi$ strings to obtain the following:

**Lemma 3.4.** For all integers $m \geq 1$ and $a \geq 1$, we have

$$
\pi^{(m)}(1, a) = (B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1},
$$

where $a_x \ldots a_1$ is the $(m + 1)$-ary expansion of $a$.

**Proof.** It follows by Lemma 3.3 (c) combined with Definition 3.2 that

$$
\pi^{(m)}(1, a) = (\pi^{(m)}(1, (m + 1)^{x-1}))^{a_x} \cdots (\pi^{(m)}(1, (m + 1)^0))^{a_1}.
$$

So it remains to prove that $\pi^{(m)}(1, (m + 1)^{x-1}) = B_x^{(m)}$. We do this by induction on $x$. For $x = 1$, both strings equal $m$. Now suppose that it holds for $x = k$. Then we have

$$
\pi^{(m)}(1, (m + 1)^k) = (\pi^{(m)}(1, (m + 1)^{k-1}))^{m+1} \cdot S_k^{(m)} \quad \text{by Definition 3.2}
$$

$$
= (B_k^{(m)})^{m+1} \cdot S_k^{(m)} \quad \text{by induction hypothesis}
$$

$$
= B_k^{(m+1)}.
$$

This completes the induction step. \qed

Now since $A^{(m)} = \lim_{t \to \infty} B_t^{(m)}$ and $\pi^{(m)}(1, \infty) = \lim_{x \to \infty} \pi^{(m)}(1, (m + 1)^{x-1})$, we obtain:

$$
A^{(m)} = \pi^{(m)}(1, \infty) = \prod_{i=1}^{\infty} m \cdot T_{\text{ord}_{m+1}(i)+1}^{(m)} \text{ for all } m \geq 1. \quad (5)
$$

Therefore, the strings $\pi^{(m)}(1, x)$ are exactly the prefixes of $A^{(m)}$ which are followed by an $m$. Since we will focus on suffixes of the $\pi$ strings, we will need the following lemma:

**Lemma 3.5.** Let $m, x \geq 1$. Let $S$ be a nonempty suffix of $\pi^{(m)}(1, x)$. Let $n$ be the smallest element of $S$. Then there is a $y$ such that $S$ is also a suffix of $\pi^{(n)}(1, y)$.

**Proof.** We prove by induction on $l \leq n$ that $S$ is a suffix of $\pi^{(l)}(1, x_l)$ for some $x_l$. For $l = m$, we know that it holds with $x_m := x$. Now suppose that it is true for $l = k$, with $k < n$. Then it follows since $k \notin S$ that $S$ is also a suffix of $T_{\text{ord}_{k+1}(x_k)+1}^{(k)}$. By Lemma 3.1, we have $C(T_{\text{ord}_{k+1}(x_k)+1}^{(k)}) \leq k$. Now it follows that the prefix $T_{\text{ord}_{k+1}(x_k)+1}^{(k)}$ of $A^{(k+1)}$ is followed by a $k + 1$. So by definition of the $\pi^{(m+1)}$ strings, there is a number $x_{k+1}$ such that $T_{\text{ord}_{k+1}(x_k)+1}^{(k+1)} = \pi^{(k+1)}(1, x_{k+1})$. This completes the induction step. \qed

**Remark 3.6.** The construction of $x_m, \ldots, x_n$ used in the proof of this lemma, is similar to Definition 5.1.

Now that we have established some basic properties of the $\pi$ strings, we shall research square suffixes of strings $\pi^{(m)}(1, x)$. The reason that we are interested in suffixes of $\pi$ strings, shall become clear in the next section, where we shall compare $T$ strings to $\pi$ strings.
Lemma 3.7. Let \( n, x \geq 1 \). Suppose that \( \pi^{(n)}(1, x) = QX^2 \) for a string \( X \) with \( n \in X \). Then one of the following holds:

(a) There are integers \( t \geq 1 \) and \( \mu \in \{1, 2, \ldots, \lfloor \frac{n+1}{2} \rfloor \} \) such that \( X \) is a conjugate of \( (B_t^{(n)})^\mu \).

(b) There are integers \( t \geq 2, \ u > t, \mu \in \{1, 2, \ldots, n\} \), and \( \rho \geq 1 \) such that \( n + 1 \nmid \rho \), and there is a nonempty suffix \( S \) of \( T_t^{(n)} \) for which \( T_t^{(n)} \cdot S = T_u^{(n)} \), such that

\[
QX^2 = \pi^{(n)}(1, \rho \cdot (n + 1)^{u-1} + \mu \cdot (n + 1)^{t-1}) = Q \cdot S \cdot \pi^{(n)}(\rho \cdot (n + 1)^{u-1} - \mu \cdot (n + 1)^{t-1} + 1, \rho \cdot (n + 1)^{u-1} + \mu \cdot (n + 1)^{t-1})
\]

Proof. Let \( t := \text{ord}_{n+1}(x) + 1 \). Let \( \alpha \geq 1 \) be the integer such that \( \alpha - 1 \) is the number of elements \( n \) in \( Q \), and let \( p \geq 1 \) be the number of elements \( n \) of \( X \). By looking at the elements \( n \) in \( QX^2 \) we see that \( Q \) is a prefix of \( \pi^{(n)}(1, \alpha - 1) \), and \( \pi^{(n)}(1, \alpha + 2p - 2) \) is a proper prefix of \( QX^2 \), and \( QX^2 \) is a prefix of \( \pi^{(n)}(1, \alpha + 2p - 1) \). Consider the following picture:

\[
X^2 = S \cdot nY_0 \cdot nY_{\alpha+1} \ldots nY_{\alpha+p-2} \cdot nT_t^{(n)}S \cdot nY_{\alpha+p} \cdot nY_{\alpha+p+1} \ldots nY_{\alpha+2p-2} \cdot nT_t^{(n)}
\]

where

- \( Y_i := T_{\text{ord}_{n+1}(i)+1}^{(n)} \) for all \( i \)
- \( S \) is a suffix of \( Y_{\alpha-1} \), or is empty if \( \alpha = 1 \)
- \( T_t^{(n)}S = Y_{\alpha+p-1} \)

We obtain for all \( i \in [\alpha, \alpha+p-2] \) that \( n \cdot T_{\text{ord}_{n+1}(i)+1}^{(n)} = n \cdot T_{\text{ord}_{n+1}(i+p)+1}^{(n)} \), hence \( \text{ord}_{n+1}(i) = \text{ord}_{n+1}(i + p) \). Therefore, \( r_{n+1}[\alpha, \alpha+p-2] = r_{n+1}[\alpha + p, \alpha + 2p - 2] \), where \( r_{n+1} \) is the ruler sequence of order \( n + 1 \). We distinguish two cases.

- Suppose that \( r_{n+1}(\alpha + p - 1) \leq r_{n+1}(\alpha + 2p - 1) \). Then it follows from Lemma 3.3 (a) that there is an integer \( \mu \in \{1, 2, \ldots, \lfloor \frac{n+1}{2} \rfloor \} \) such that \( p = \mu \cdot (n + 1)^y \), where \( y \) is the maximal element of \( r_{n+1}[\alpha, \alpha + p - 1] \). Also, it follows from the picture that \( \pi^{(n)}(\alpha, \alpha + p - 1) \) is a conjugate of \( X \). Furthermore, \( r_{n+1}[\alpha, \alpha + p - 1] \) (which has length \( \mu \cdot (n + 1)^y \) and maximal element \( y \)) is a conjugate of \( r_{n+1}[1, \mu \cdot (n + 1)^y] \), hence \( \pi^{(n)}(\alpha, \alpha + p - 1) \) is a conjugate of \( \pi^{(n)}(1, \mu \cdot (n + 1)^y) \). Since conjugation defines an equivalence relation (see Lothaire [3, p. 4]), we conclude that \( X \) is a conjugate of \( \pi^{(n)}(1, \mu \cdot (n + 1)^y) \), which equals \( (B_{y+1}^{(n)})^\mu \) by Lemma 3.4.
Now suppose that \( r_{n+1}(\alpha + p - 1) > r_{n+1}(\alpha + 2p - 1) \). Then it follows from Lemma 3.3 (b) that there exist integers \( \lambda \geq 1 \) and \( \mu \in \{1, 2, \ldots, n\} \) such that
\[
\alpha = \lambda \cdot (n+1)^{b+1} - \mu \cdot (n+1)^b + 1 + p = \mu \cdot (n+1)^b, \quad \text{where } b = r_{n+1}(\alpha + 2p - 1).
\]
It also follows from the lemma that \( \alpha > 1 \) and \( r_{n+1}(\alpha - 1) = b \). Define \( u := \text{ord}_{n+1}(\lambda \cdot (n+1)^{b+1}) + 1 \). Then there is an integer \( \rho \geq 1 \) such that \( \rho \cdot (n+1)^{u-1} = \lambda \cdot (n+1)^{b+1} \) and \( n + 1 \nmid \rho \). Define \( t := b + 1 \), then we see that \( \alpha = \rho \cdot (n+1)^{u-1} - \mu \cdot (n+1)^{t-1} + 1 \) and \( u > t \).

Substituting this in \( QX^2 = \pi^{(n)}(1, \alpha + 2p - 1) = Q \cdot S \cdot \pi^{(n)}(\alpha, \alpha + 2p - 1) \) gives us Equality 6.

We also have \( T_t^{(n)} \cdot S = T_{\text{ord}_{n+1}(\alpha+p-1)+1}^{(n)} = T_u^{(n)} \).

Example 3.8.

- An example of (a) is the following: we have \( \pi^{(2)}(1, 7) = 2 \cdot (2232)^2 \). Take \( n = 2, t = 2, \mu = 1 \), then we see that \( (B_2^{(2)})^1 = 2223 \), and the string \( X = 2232 \) is a conjugate of 2223.

- An example of (b) is the string \( \pi^{(2)}(1, 12) = 2232222 \cdot (32223)^2 \). When we take \( n = 2, t = 2, u = 3, \mu = 1, \rho = 1 \) and \( T_t^{(n)} \cdot S = (3) \) in (b), then we see that \( \pi^{(2)}(1, 1 \cdot 3^2 + 1 \cdot 3^1) = \pi^{(2)}(1, 12) = 2 \cdot 2 \cdot 23 \cdot 2 \cdot 2 \cdot 23 \cdot 2 \cdot 2 \cdot 23 \cdot 2 \cdot 2 \cdot 2 = QX^2 \).

Substrings \( X^3 \), and therefore also \( X^k \) for \( k \geq 3 \), are easier than squares:

**Corollary 3.9.** For \( m, x \geq 1 \), suppose that \( \pi^{(n)}(1, x) \) has a suffix \( X^3 \). There are \( n, t \geq 1 \) with \( n \geq m \) such that \( X \) is a conjugate of a power of \( B_t^{(n)} \).

**Proof.** Let \( n \) be the smallest element of \( X \). Then by Lemma 3.5, there is a number \( y \) such that \( X^3 \) is a suffix of \( \pi^{(n)}(1, y) \). Now \( X^2 \) is also a suffix of \( \pi^{(n)}(1, y) \), so we can apply Lemma 3.7. If (a) is the case, then we are done. So suppose that (b) is the case. Then there are \( Q, t, u, \mu, \rho, S \) as in the Lemma.

We see that
\[
X = S \cdot \pi^{(n)}(\rho \cdot (n+1)^u - \mu \cdot (n+1)^{t-1} + 1, \rho \cdot (n+1)^{u-1} - 1) \cdot n \cdot T_u^{(n)}.
\]
This string must be a suffix of \( Q \). Therefore, \( n \cdot T_u^{(n)} \cdot S \) is a suffix of \( QS \). But
\[
QS = \pi^{(n)}(1, \rho \cdot (n+1)^{u-1} - \mu \cdot (n+1)^{t-1} + 1) \cdot n \cdot T_t^{(n)}.
\]
Since \( T_u^{(n)} \) and \( S \) contain no element \( n \), we see that \( n \cdot T_u^{(n)} \cdot S \) must equal \( n \cdot T_t^{(n)} \). That is a contradiction.
4 The iota functions

So far we have seen two kinds of prefixes of the level-$m$ Gijswijt sequences: the strings $T^{(m-1)}_t$ and the strings $\pi^{(m)}(1,x)$. In this section, we will explore the connection between them. It shall turn out that for each $m \geq 2$, strings $T^{(m-1)}_t$ are a subset of the strings $\pi^{(m)}(1,x)$. We shall link the indices $t$ and $x$ in Definition 4.6 with the function $\iota_{m-1}$. We shall then list some basic properties of this function.

Since $\iota_m$ connects $A^{(m)}$ and $A^{(m+1)}$, this function is vital for proving the estimates from van de Bult et al. [1, pp. 19–23]. In the following lemmas, we shall prove some properties of the strings $(B^{(m)}_x)^{a_x} \cdots (B^{(m)}_1)^{a_1}$. By Lemma 3.4, those are exactly the $\pi$ strings.

Recall from Lemma 3.1 that the strings $T^{(m-1)}_t$ are exactly the (nonempty) prefixes of $A^{(m)}$ that have curling number smaller than $m$. Before investigating these strings further, we shall characterize the prefixes of $A^{(m)}$ with curling number at most $m$. Those turn out to be the $\pi$ strings.

**Lemma 4.1.** For $m \in \mathbb{Z}_{\geq 1}$, the prefixes $S$ of $A^{(m)}$ with $C(S) \leq m$ are exactly the strings of the form $(B^{(m)}_x)^{a_x} \cdots (B^{(m)}_2)^{a_2} : (B^{(m)}_1)^{a_1}$ with $a_1,a_2,\ldots,a_x \in \{0,1,\ldots,m\}$, for some $x \geq 1$.

**Proof.** By the definition of $A^{(m)}$, the prefixes with curling number at most $m$ are the prefixes that are followed by an $m$. By Equation 5, these prefixes are exactly the strings $\pi^{(m)}(1,x)$ for $x \geq 0$. Now this lemma follows from Lemma 3.4. \hfill \Box

**Example 4.2.** The prefix 3333433 of $A^{(3)}$ has curling number 2. This prefix is equal to $B^{(3)}_2(B^{(3)}_1)^2$. The string 222322232223233 is a prefix of $A^{(2)}$ with curling number 2; it equals $B^{(2)}_3$. But the prefix 222 of $A^{(2)}$ has curling number 3, which is larger than 2. The string 222 cannot be written as $(B^{(2)}_x)^{a_x} \cdots (B^{(2)}_1)^{a_1}$ with $a_1,\ldots,a_x \in \{0,1,2\}$.

The following lemma gives a lower bound for the curling numbers of the strings $(B^{(m)}_x)^{a_x} \cdots (B^{(m)}_1)^{a_1}$.

**Lemma 4.3.** For $m \geq 2$ and $a_1,\ldots,a_x \in \{0,1,\ldots,m\}$, the following holds:

(a) $C((B^{(m)}_x)^{a_x} \cdots (B^{(m)}_1)^{a_1}) \geq \max(a_1,a_2,\ldots,a_x)$.

(b) Let $X$ be a conjugate of $B^{(m)}_y$ for some integer $y \geq 1$, and $l \geq 2$ an integer such that $X^l$ is a suffix of $(B^{(m)}_x)^{a_x} \cdots (B^{(m)}_1)^{a_1}$. Then we have $y \leq x$ and $l \leq a_y$.

**Proof.**

(a) For each $y \in \{1,2,\ldots,x\}$, the string $P := (B^{(m)}_{y-1})^{a_{y-1}} \cdots (B^{(m)}_1)^{a_1}$ is a prefix of $B^{(m)}_y$ (which follows from Lemma 3.4), so write $B^{(m)}_y = PQ$. Now $(B^{(m)}_y)^{a_y} \cdot P$ is a suffix of $(B^{(m)}_x)^{a_x} \cdots (B^{(m)}_1)^{a_1}$, and it is equal to $(PQ)^{a_y} \cdot P = P \cdot (QP)^{a_y}$. Therefore, $C((B^{(m)}_x)^{a_x} \cdots (B^{(m)}_1)^{a_1}) \geq a_y$.  

19
(b) Since \( \beta^{(m)}(y) = l(X) < l(X^t) < l((B_x^{(m)})^{a_x+1}) < \beta^{(m)}(x + 1) \), it follows that \( y \leq x \).

Now since \( T^{(m)}_z \) is not a substring of \( X^t \) for all \( z > y \), it follows that \( X^t \) is a proper suffix of \( T^{(m)}_z \cdot (B_y^{(m)})^{a_y} \cdot (B_y^{(m)})^{a_y-1} \cdots (B_1^{(m)})^{a_1} \), for the smallest number \( z > y \) such that \( a_z > 0 \). If such a \( z \) does not exist, then \( X^t \) is a suffix of \( (B_y^{(m)})^{a_y} \cdot (B_y^{(m)})^{a_y-1} \cdots (B_1^{(m)})^{a_1} \). In both cases, the number of \( m \)'s in \( X^t \) is at most the number of \( m \)'s in \( (B_y^{(m)})^{a_y} \cdot (B_y^{(m)})^{a_y-1} \cdots (B_1^{(m)})^{a_1} \), which is smaller than the number of \( m \)'s in \( (B_y^{(m)})^{a_y+1} \). Therefore, \( l < a_y + 1 \), hence \( l \leq a_y \).

\( \square \)

It follows from Lemmas 4.1 and 4.3 (a) that all strings \((B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1}\) such that there is an \( m \) among \( a_1, \ldots, a_x \), have curling number \( m \). Therefore, the remaining strings we have to consider are the strings \((B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1}\) with \( a_i \in \{0, 1, \ldots, m - 1\} \) for all \( i \). These are the only prefixes of \( A^{(m)} \) that may be equal to a \( T^{(m-1)}_t \).

In the following lemma, we establish a subset of these strings, for which the curling number is always smaller than \( m \).

**Lemma 4.4. For** \( m \geq 2 \) and \( a_1, \ldots, a_x \in \{0, 1, \ldots, m - 1\} \) such that \( a_1 \neq 0 \), we have \( C((B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1}) < m \).

**Proof.** We shall prove this by contradiction, so suppose that \( Z := (B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1} \) has a nonempty suffix of the form \( X^m \). Now since \( m \geq 2 \), we can write \( Z = QX^2 \). Since \( a_1 \neq 0 \), the last element of \( Z \) is \( m \), hence \( m \in X \). By Lemma 3.4, we can write \( Z \) as a string \( \pi^{(m)}(1, a) \). Now we can apply Lemma 3.7. Since the last element of \( X \) is \( m \), we can not have case (b), for then \( QX^2 \) would end with the last element of \( T^{(m)}_t \) which is larger than \( m \) since \( t \geq 2 \). Therefore, case (a) holds, so \( X \) equals a conjugate of \( (B_y^{(m)})^\mu \) for some integers \( y \geq 1, \mu \geq 1 \). Now, there is a conjugate \( C \) of \( B_y^{(m)} \) such that \( X = C^\mu \). The string \( X^m = C^{m\mu} \) is a suffix of \( Z \). But then it follows from Lemma 4.3 (b) that \( m \cdot \mu \leq a_y \leq m - 1 \). That is a contradiction.

\( \square \)

Combining the above lemmas, we conclude the following about the strings \( T^{(m-1)}_t \):

**Theorem 4.5. For all** \( m, t \geq 2 \) there are \( a_1, \ldots, a_x \in \{0, 1, \ldots, m - 1\} \) such that \( T^{(m-1)}_t = (B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1} \). Moreover, all combinations \( a_1, \ldots, a_x \in \{0, 1, \ldots, m - 1\} \) with \( a_1 \neq 0 \) occur in this way.

**Proof.** For all \( m, t \geq 2 \), by Lemma 3.1, \( T^{(m-1)}_t \) is a prefix of \( A^{(m)} \) with curling number smaller than \( m \). Combining Lemmas 4.1 and 4.3 (a), we see that there are \( a_1, \ldots, a_x \in \{0, 1, \ldots, m - 1\} \) such that \( T^{(m-1)}_t = (B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1} \). The last statement follows from Lemma 4.4.

\( \square \)
In Theorem 4.5, let $a$ be the number that has expansion $a_x \ldots a_1$ in base $m$. When $\chi^{(m)}(a)$ is defined as the number with $(m+1)$-ary expansion $a_x \ldots a_1$, it follows from Lemma 3.4 that the string $T_t^{(m-1)}$ in the theorem equals $\pi^{(m)}(1, \chi^{(m)}(a))$. Therefore, we can make the following definition:

**Definition 4.6.** For all $m, t \geq 1$, let $\iota_m(t)$ be the unique number in $\mathbb{Z}_{\geq 0}$ such that

$$T_t^{(m)} = \pi^{(m+1)}(1, \chi^{(m+1)}(\iota_m(t))).$$

Here $\chi^{(m)}(a)$ is defined as the number $\sum_i a_i \cdot (m+1)^{i-1}$, where $a = \sum_i a_i \cdot m^{i-1}$ is the $m$-ary expansion of $a$.

The values of $\iota_1$ can be found at Entry A357065 of the OEIS [8].

**Remark 4.7.** It follows from Definition 4.6 and Lemma 3.1 that an integer $p \geq 0$ is in the image of $\iota_m$ if and only if $\pi^{(m+1)}(1, \chi^{(m+1)}(p))$ is equal to some string $T_t^{(m)}$. By Lemma 3.1, this is equivalent to $C(\pi^{(m+1)}(1, \chi^{(m+1)}(p))) \leq m$. We shall often use this.

Since the strings $\pi^{(m+1)}(1, x)$ are defined using the strings $T_t^{(m+1)}$, it follows from Definition 4.6 that $\iota_m$ connects the $T_t^{(m)}$ strings with the $T_t^{(m+1)}$. So in some way, $\iota_m$ is a connection between the level-$m$ and level-$(m + 1)$ Gijswijt sequences.

**Lemma 4.8.** The following holds for all $m \in \mathbb{Z}_{\geq 1}$:

(a) We have $\iota_m(1) = 0, \iota_m(2) = 1$ and $\iota_m$ is a strictly increasing function.

(b) The image of $\iota_m$ contains all integers $a \in \mathbb{Z}_{\geq 1}$ that are not divisible by $m + 1$.

**Proof.**

(a) Since $T_1^{(m)} = \varepsilon = \pi^{(m+1)}(1, \chi^{(m+1)}(0))$, we see that $\iota_m(1) = 0$, and since

$$T_2^{(m)} = m + 1 = \pi^{(m+1)}(1, \chi^{(m+1)}(1))$$

we see that $\iota_m(2) = 1$. Also, we know that for all $t \geq 2$, $T_t^{(m)}$ is a proper prefix of $T_{t+1}^{(m)}$. Therefore, we must have $\chi^{(m+1)}(\iota_m(t)) < \chi^{(m+1)}(\iota_m(t+1))$. Also, $\chi^{(m+1)}$ is strictly increasing. From this it follows that $\iota_m$ is strictly increasing.

(b) Let $a \in \mathbb{Z}_{\geq 1}$ be a number not divisible by $m + 1$. Now in the $(m + 1)$-ary expansion $a_x \ldots a_1$ of $a$, we see that $a_1 \neq 0$. Note that $a_x \ldots a_1$ is also the $(m + 1 + 1)$-ary expansion of $\chi^{(m+1)}(a)$. It follows from Theorem 4.5 that

$$(B_x^{(m+1)})^{a_x} \ldots (B_1^{(m+1)})^{a_1}$$

occurs as a $T_t^{(m)}$ string. By Lemma 3.4, this equals $\pi^{(m+1)}(1, \chi^{(m+1)}(a))$. Hence $a$ is in the image of $\iota_m$. 

\[\square\]

21
Table 7: Values of $\iota_1$

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|
| $\iota_1(x)$ | 0 | 1 | 2 | 3 | 5 | 7 | 8 | 9 | 10 | 11 | 13 | 15 |

| $x$ | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\iota_1(x)$ | 16 | 17 | 18 | 19 | 21 | 23 | 24 | 25 | 26 | 27 | 29 | 31 |

We know from Lemma 4.8 (a) that $\iota_m$ is increasing and $\iota_m(1) = 0$. Therefore we can make the following definition:

**Definition 4.9.** For integers $m \geq 1, p \geq 0$ define $\iota_m^{-1}(p)$ as the largest integer $x \geq 1$ such that $\iota_m(x) \leq p$.

Now $\iota_m^{-1}(p)$ equals the size of the intersection $\text{Im}(\iota_m) \cap [0, p]$, and we have

$$\iota_m^{-1}(\iota_m(p)) = p$$

for all $m \geq 1, p \geq 1$. Also, for all $m \geq 1$ and $p \geq 1$, we know by Lemma 4.8 (b) that either $p$ or $p - 1$ is in the image of $\iota_m$ (or both), hence

$$\iota_m(\iota_m^{-1}(p)) \in \{p - 1, p\} \text{ for all } m \geq 1, p \geq 1.$$ (7)

We shall now calculate some explicit values of $\iota_1$:

**Example 4.10.** We have $\iota_1^{-1}(31) = 24$, and the first values of $\iota_1$ are given by Table 7.

**Proof.** Recall from Remark 4.7 that a number $p \geq 0$ is in the image of $\iota_1$ if and only if $C\left(\pi(2)(1, \chi(2)(p))\right) = 1$. We divide the set $\{0, 1, \ldots, 31\}$ into four subsets, for which we shall prove whether or not their elements are in the image of $\iota_1$:

- It follows from Lemma 4.8 (b) that all sixteen odd integers in $[1, 31]$ are in the image of $\iota_1$.
- For $n = 4, 12, 20, 28$, we know that $n \equiv 4 \mod 8$, hence $\chi(2)(n) \equiv 9 \mod 27$. So by lemma 3.4 the string $B_3^{(2)}$ is a suffix of $\pi(2)(1, \chi(2)(n))$. Since $B_3^{(2)} = 22232223222333$ has a square suffix 33, it follows that $C\left(\pi(2)(1, \chi(2)(n))\right) > 1$, hence $n \notin \text{Im}(\iota_1)$.
- For $n = 6, 14, 22, 30$, we have $n \equiv 6 \mod 8$, hence $n \equiv 12 \mod 27$. So by lemma 3.4 the string $B_3^{(2)}B_2^{(2)}$ is a suffix of $\pi(2)(1, \chi(2)(n))$. Since $B_3^{(2)}B_2^{(2)} = 222322232223332223$ has a square suffix 3222332223, it follows that $C\left(\pi(2)(1, \chi(2)(n))\right) > 1$, hence $n \notin \text{Im}(\iota_1)$.
For \( n = 0, 2, 8, 10, 16, 18, 24, 26 \), the string \( \pi^{(2)}(1, \chi^{(2)}(n)) \) equals respectively

\[
\begin{align*}
\varepsilon & \quad 2223 \\
B_4^{(2)} & = 222322232233222332223222322233334 \\
B_4^{(2)} \cdot 2223 & \\
(B_4^{(2)})^3 \cdot 3 & = B_5^{(2)} \\
(B_4^{(2)})^3 \cdot 32223 & \\
(B_4^{(2)})^3 \cdot 3 \cdot B_4^{(2)} & \\
(B_4^{(2)})^3 \cdot 3 \cdot B_4^{(2)} \cdot 2223 & .
\end{align*}
\]

It follows by immediate calculation that those strings have curling number 1. Therefore, \( n \in \text{Im}(\iota_1) \).

We see that the set \( \text{Im}(\iota_1) \cap [0, 31] \) is equal to

\[
\{0, 1, 2, 3, 5, 7, 8, 9, 10, 11, 13, 15, 16, 17, 18, 19, 21, 23, 24, 25, 26, 27, 29, 31\},
\]

which has 24 elements. Therefore, \( \iota_1^{-1}(31) = 24 \).

**Example 4.11.** It follows by a computer calculation that \( \iota_1^{-1}(4096) = 2834 \): for this, calculate the number of proper prefixes with curling number 1 of the string \( B_{13}^{(2)} \). In Subsection 8.2, we calculate the larger value \( \iota_1^{-1}(279) = 1 + 418090195952691922788353 \).
5 The level-$m$ expansion

In this section, we shall show how the function $\tau_m$ can be obtained from the functions $\tau_{m+1}, \tau_{m+2}, \ldots$. We do this by defining the level-$m$ expansion of an integer $a \geq 1$. Then we will prove Theorem 5.4 on $\tau_m$ with $m \geq 2$. The case $m = 1$ turns out to be more difficult. This is due to Lemma 3.7: square suffixes can occur in two forms, while cube suffixes (Corollary 3.9) can only occur in one form. Finally, we prove the statement on $\tau_1$ in Theorem 5.10. Finally, we shall give an implementation of the theorems in Python.

**Definition 5.1.** Let $m \geq 1$ and $a \geq 1$. Let $n$ be the last element of $\pi^{(m+1)}(1, \chi^{(m+1)}(a))$. Define the numbers $a_{m+1}, a_{m+2}, \ldots, a_n$ recursively in the following way:

- Take $a_{m+1} := a$;
- When $a_l$ is constructed for $l < n$, define $a_{l+1} := \tau_l(\text{ord}_a(a_l) + 1)$.

The sequence $a_{m+1}, a_{m+2}, \ldots, a_n$ is called the level-$m$ expansion of $a$.

Note that $n$ does not have to be the largest element of $\pi^{(m+1)}(1, \chi^{(m+1)}(a))$. For example, $m = 1$ and $a = 3$ gives $\pi^{(2)}(1, \chi^{(2)}(3)) = 22232$, so $n = 2$.

In the following lemma, we list some basis properties of the level-$m$ expansion.

**Lemma 5.2.** Let $m \geq 1$ and $a \geq 1$.

(a) The level-$m$ expansion of $a$ is well-defined.

(b) For all $l \in [m+1, n]$, the string $\pi^{(l)}(1, \chi^{(l)}(a_l))$ is the longest suffix of $\pi^{(m+1)}(1, \chi^{(m+1)}(a))$ that is in $\mathbb{Z}_{\geq 1}^*$.

(c) We could also define $a_{n+1} := \tau_n(\text{ord}_a(a_n) + 1)$; then $a_{n+1} = 0$.

**Proof.** Let $P := \pi^{(m+1)}(1, \chi^{(m+1)}(a))$. We will prove by induction on $l$ that $a_l$ is well-defined and that Part (b) holds for $l$. For $l = m + 1$, both statements are trivial. So suppose that they hold for $l = k$ with $k \in [m + 1, n - 1]$. Since $P$ ends with $n$, it follows that the longest suffix of $P$ that is in $\mathbb{Z}_{\geq 1}^*$ is nonempty. So by the induction hypothesis, $\pi^{(k)}(1, \chi^{(k)}(a_k))$ is nonempty. Therefore, $a_k \geq 1$. From this it follows that $\text{ord}_a(a_k) + 1$ is well-defined and at least 1. Hence $a_{k+1}$ is well-defined.

Now the longest suffix of $P$ that is in $\mathbb{Z}_{\geq k+1}^*$ is $T_{\tau_{k+1}}^{(k)}$. By Definition 4.6, this string equals

$$\pi^{(k+1)}(1, \chi^{(k+1)}(\text{ord}_a(a_k) + 1)).$$

This equals $\pi^{(k+1)}(1, \chi^{(k+1)}(a_{k+1}))$. So the induction hypothesis holds for $l = k + 1$.

To prove Part (c), observe that every step of this proof by induction also holds when we define $a_{n+1} := \tau_n(\text{ord}_a(a_n) + 1)$ and take $k = n$. Hence $a_{n+1}$ is well-defined. Also, since the last element of $P$ is smaller than $n + 1$, it follows that $\pi^{(n+1)}(1, \chi^{(n+1)}(a_{n+1}))$ is empty. Hence $a_{n+1} = 0$. 

The following proposition gives an important example of a level-$m$ expansion:
Proposition 5.3. For all integers \( m, n \) with \( 1 \leq m \leq n - 1 \), we have the following:

(a) We have
\[
\tau_m(t^{(m)}(n)) = (m + 1)^{t^{(m+1)}(n)-1}
\] (8)

(b) The level-\( m \) expansion of \( a := \tau_m(t^{(m)}(n)) \) is:
\[
\tau_m(t^{(m)}(n)), \tau_m(t^{(m+1)}(n)), \ldots, \tau_{n-1}(t^{(n-1)}(n)).
\]

Proof.

(a) We have
\[
\pi^{(m+1)}(1, \chi^{(m+1)}(t_m(t^{(m)}(n)))) = T^{(m)}_{t^{(m)}(n)}
\]
by Definition 4.6
\[
= B^{(m+1)}_{t^{(m)}(n)}
\]
by Lemma 1.2 (b)
\[
= \pi^{(m+1)}(1, \chi^{(m+1)}((m + 1)^{t^{(m+1)}(n)-1}))
\]
by Lemma 3.4

So since \( \chi^{(m+1)} \) is injective, we conclude that \( \tau_m(t^{(m)}(n)) = (m + 1)^{t^{(m+1)}(n)-1} \).

(b) Firstly, the last element of \( \pi^{(m+1)}(1, \chi^{(m+1)}(t_m(t^{(m)}(n)))) = T^{(m)}_{t^{(m)}(n)} \) is \( n \) by definition of \( t^{(m)}(n) \). Now using Part (a), it follows with induction on \( l \) that \( a_l = \tau_{l-1}(t^{(l-1)}(n)) \).

\[ \square \]

For example, for \( m = 1 \) and \( n = 4 \), we have \( \tau_1(t^{(1)}(4)) = 7, \tau_2(t^{(2)}(4)) = 3 \), and \( \tau_3(t^{(3)}(4)) = 2 \). Indeed, the level-1 expansion of 8 is: 8, 3, 1.

By taking \( \tau_{m-1}^{-1} \) of both sides of Proposition 5.3 (a), we see that
\[
t^{(m)}(n) = \tau_{m-1}^{-1}((m + 1)^{t^{(m+1)}(n)-1}).
\] (9)

This allows us to obtain an initial tower of exponents:
\[
t^{(m)}(n) = \tau_{m-1}^{-1}((m + 1)^{-1} + \tau_{m-1}^{-1}(m+2)^{-1} + \cdots + \tau_{m-1}^{-1}(n-1)^{-1}).
\]

In the next section, we shall give precise approximations for the values of \( \tau_l^{-1} \) that occur in above expression. The resulting tower of exponents can be found in Theorem 6.20.

Using the level-\( m \) expansion, we now prove the following theorem on the image of \( \tau_m \) for \( m \geq 2 \):

**Theorem 5.4.** Let \( m \geq 2 \) and \( a \geq 1 \). Let \( a_{m+1}, a_{m+2}, \ldots, a_n \) be the level-\( m \) expansion of \( a \), as defined in Definition 5.1. Define \( \text{maxdigit}_x(y) \) as the largest digit of \( y \) when written in base \( x \). Now we have the following:
\[
C^{(2)}(\pi^{(m+1)}(1, \chi^{(m+1)}(a))) = \max(2, \text{maxdigit}_{m+1}(a_{m+1}), \text{maxdigit}_{m+2}(a_{m+2}), \ldots, \text{maxdigit}_n(a_n)).
\]
Also, \( a \) is in the image of \( \iota_m \) if and only if we have

\[
\max\text{digit}_l(a_l) \leq m
\]  

for all \( l \).

**Proof.** From Corollary 3.9, we obtain that all powers \( X^t \) with \( t \geq 3 \) that are a suffix of \( \pi^{(m+1)}(1, \chi^{(m+1)}(a)) \), must be equal to a conjugate of some string \( (B^{(l)}_u)^{t\mu} \) for some \( l \geq m + 1 \) and \( \mu \geq 1 \). Since \( n \) is the last element of \( \pi^{(m+1)}(1, \chi^{(m+1)}(a)) \), it follows that \( l \leq n \). By Lemma 5.2 (b), this conjugate of \( (B^{(l)}_u)^{t\mu} \) is also a suffix of \( \pi^{(l)}(1, \chi^{(l)}(a_l)) \), which by Lemma 3.4 equals the string \( (B^{(l)}_x)^{c_x} \ldots (B^{(l)}_1)^{c_1} \) when we write \( a_l \) as \( c_x \ldots c_1 \) in base \( l \). Therefore, we see that \( (B^{(l)}_x)^{c_x} \ldots (B^{(l)}_1)^{c_1} \) has a conjugate of \( (B^{(l)}_u)^{t\mu} \) as a suffix. Now by Lemma 4.3(b), it follows that \( x \geq u \) and \( c_u \geq t\mu \). Therefore, we see that \( t \leq \max\text{digit}_l(a_l) \).

Since this holds for all suffixes \( X^t \) with \( t \geq 3 \), we see that

\[
\mathcal{C}^{(2)}(\pi^{(m+1)}(1, \chi^{(m+1)}(a))) = \max(2, \max\text{digit}_{m+1}(a_{m+1}), \max\text{digit}_{m+2}(a_{m+2}), \ldots, \max\text{digit}_n(a_n)).
\]

Combining these, we see that

\[
\mathcal{C}^{(2)}(\pi^{(m+1)}(1, \chi^{(m+1)}(a))) = \max(2, \max\text{digit}_{m+1}(a_{m+1}), \max\text{digit}_{m+2}(a_{m+2}), \ldots, \max\text{digit}_n(a_n)).
\]

By Remark 4.7, the number \( a \) is in the image of \( \iota_m \) if and only if

\[
\mathcal{C}(\pi^{(m+1)}(1, \chi^{(m+1)}(a))) \leq m.
\]

Now since \( m \geq 2 \), this is equivalent to

\[
\mathcal{C}^{(2)}(\pi^{(m+1)}(1, \chi^{(m+1)}(a))) \leq m.
\]

This proves the theorem. \( \square \)

Before we can prove an equivalent of Theorem 5.4 for \( m = 1 \), we need another property of the ruler sequences.

**Lemma 5.5.** Let \( m \geq 2 \) and \( 1 \leq a < b \). We have \( r_m[2a - b + 1, a] = r_m[a + 1, b] \) if and only if there is an integer \( y \) that satisfies the following two conditions:

- In the \( m \)-ary expansion, the \( y \)th digit of \( a \) is at least half the \( y \)th digit of \( b \)
- All other digits are equal in \( a \) and \( b \).

26
Proof. First, suppose that \( r_m[2a - b + 1, a] = r_m[a + 1, b] \). Let \( y - 1 \) be the maximal element of \( r_m[a + 1, b] \). By Lemma 3.3 (a), there is an integer \( \mu \in \{1, 2, \ldots, \lceil \frac{y}{2} \rceil \} \) such that \( b - a = \mu \cdot m^{y-1} \). Hence the first until \((y-1)\)th digits of \( a \) and \( b \) are equal. Since \( y - 1 \) is the maximal element of \( r_m[2a - b + 1, b] \), it follows that there is an integer \( \lambda \geq 0 \) such that \( \lambda \cdot m^y < 2a - b + 1 < b < (\lambda + 1) \cdot m^y \). From this it follows that all digits at higher positions than \( y \) are also equal. Now let \( a = \sum_{i=1}^{\infty} a_i \cdot m^{i-1} \) be the \( m \)-ary expansion of \( a \), and let \( b_y \) be the \( y \)th digit of \( b \). Then \( \sum_{i=y+1}^{\infty} a_i \cdot m^{i-1} = \lambda \cdot m^y < 2a - b + 1 = (\sum_{i=1}^{\infty} a_i \cdot m^{i-1}) - (b_y - a_y) \cdot m^{y-1} + 1 \), hence \( (b_y - 2a_y) \cdot m^{y-1} \leq \sum_{i=1}^{y-1} a_i \cdot m^{i-1} < m^{y-1} \). From this it follows that \( 2a_i \geq b_i \).

Conversely, suppose that the two conditions on the digits hold. Then the existence of \( \lambda \) such that \( \lambda \cdot m^y < 2a - b + 1 < b < (\lambda + 1) \cdot m^y \) follows immediately. Since \( m^{y-1} \mid b - a \), it follows that \( r_m(i) = r_m(i + b - a) \) for all \( i \in [2a - b + 1, a] \).

According to the lemma, we can make the following definition:

**Definition 5.6.** For \( m \geq 2 \), let \( S_m \) be the set of pairs of integers \( (a, b) \) that satisfy the conditions of Lemma 5.5.

The reason to define these sets, is that they are related to a certain pattern in the \( T \) strings. We shall show this in Lemma 5.8. To prove that lemma, we need the following:

**Lemma 5.7.** For integers \( m \geq 2 \) and \( a, b \geq 1 \), we have \( (\chi^{(m)}(a), \chi^{(m)}(b)) \in S_{m+1} \) if and only if \( (a, b) \in S_m \).

**Proof.** This follows from the fact that the \((m+1)\)-ary expansion of \( \chi^{(m)}(x) \) equals the \( m \)-ary expansion of \( x \).

**Lemma 5.8.** For \( m, t, u \geq 1 \) with \( t < u \), there is a suffix \( S \) of \( T^{(m)}_t \) such that 
\[
T^{(m)}_t S = T^{(m)}_u \text{ if and only if } (t_m(t), t_m(u)) \in S_{m+1}.
\]

**Proof.** If \( t = 1 \), then \( T^{(m)}_t \) is empty, so there is no such suffix \( S \). Also, \( t_m(t) = 0 \), so \( (t_m(t), t_m(u)) \notin S_{m+1} \). Now let \( t \geq 2 \). Firstly, observe that \( T^{(m)}_t = \pi^{(m+1)}(1, \chi^{(m+1)}(t_m(t))) \) by Definition 4.6. Using the same Definition, we get \( T^{(m)}_u = T^{(m)}_t \cdot \pi^{(m+1)}(\chi^{(m+1)}(t_m(t)) + 1, \chi^{(m+1)}(t_m(u))) \). Therefore, the special suffix \( S \) exists if and only if
\[
L := \pi^{(m+1)}(1, \chi^{(m+1)}(t_m(t)))
\]
is a suffix of \( \pi^{(m+1)}(1, \chi^{(m+1)}(t_m(t))) \). Since \( L \) starts with an \( m+1 \), this is the case if and only if
\[
\pi^{(m+1)}(\chi^{(m+1)}(t_m(t)) + 1, \chi^{(m+1)}(t_m(u))) = \pi^{(m+1)}(c, \chi^{(m+1)}(t_m(t)))
\]
for some integer \( c \). Now by the definition of the \( \pi \) strings, this is equivalent to 
\[
r_m[\chi^{(m+1)}(t_m(t)) + 1, \chi^{(m+1)}(t_m(u))] = r_m[2c, \chi^{(m+1)}(t_m(t))] \].
Since the lengths of these intervals must be equal, this is in turn equivalent to
\[
r_m[2 \cdot \chi^{(m+1)}(t_m(t)) - \chi^{(m+1)}(t_m(u)) + 1, \chi^{(m+1)}(t_m(t))].
\]
By Definition 5.6, this is equivalent to
\[ (\chi^{(m+1)}(t_m(1)), \chi^{(m+1)}(t_m(2))) \in S_{m+2}. \]

By Lemma 5.7, this is equivalent to \((t_m(1), t_m(2)) \in S_{m+1}\).

\[ \blacksquare \]

The condition on the T strings in this lemma, is exactly what we saw in Lemma 3.7, Case (b). We can now apply this to obtain the following:

**Lemma 5.9.** Let \( m \geq 2 \) and \( x \geq 1 \). The string \( \pi^{(m)}(1, \chi^{(m)}(x)) \) has a suffix \( X^2 \) with \( m \in X \) if and only if one of the following holds:

(a) We have \( \text{maxdigit}_m(x) \geq 2 \).

(b) Write \( x = \sum x_i \cdot (m+1)^{i-1} \) in \( m \)-ary expansion. Let \( t \) be the smallest integer \( i \) such that \( x_i > 0 \). Let \( u \) be the second smallest integer \( i \) such that \( x_i > 0 \). The number \( u \) has to exist, and we must have \((t_m(1), t_m(2)) \in S_{m+1}\).

**Proof.** First, suppose that the string \( \pi^{(m)}(1, \chi^{(m)}(x)) \) has a suffix \( X^2 \) with \( m \in X \). We apply Lemma 3.7. Suppose that we are in Case 3.7 (a). Then \( X \) is a conjugate of some string \( (B_t^{(m)})^\mu \). Using Lemma 3.4 and Lemma 4.3 (b), we see that \( \text{maxdigit}_{m+1}(\chi^{(m)}(x)) \geq 2\mu \), so Case (a) of this lemma holds.

Now suppose that we are in Case 3.7 (b). Then there are integers \( t, u \) with \( 2 \leq t < u \) and integers \( \mu, \rho \geq 1 \) with \( \mu \leq m \) and \( m+1 \nmid \rho \) such that \( \chi^{(m)}(x) = \rho \cdot (m+1)^{u-1} + \mu \cdot (m+1)^{t-1} \). From this we see that \( t \) and \( u \) are the locations of the first and second nonzero digits of \( \chi^{(m)}(x) \) in its \( (m+1) \)-ary expansion. Therefore, they are also the locations of the first and second nonzero digits of \( x \) in its \( m \)-ary expansion. It also follows from Case 3.7 (b) that there is a nonempty suffix \( S \) of \( T_t^{(m)} \) such that \( T_t^{(m)} \cdot S = T_u^{(m)} \). Now Lemma 5.8 shows that \((t_m(1), t_m(2)) \in S_{m+1}\). Hence Case (b) of this lemma holds.

Conversely, suppose that Case (a) of this lemma holds. Then \( \text{maxdigit}_{m+1}(\chi^{(m+1)}(x)) \geq 2 \). Now Lemma 3.4 and Lemma 4.3 (a) tell us that \( C(\pi^{(m)}(1, \chi^{(m)}(x)) \geq 2 \). Examining the proof of Lemma 4.3 (a), the square suffix that is constructed there does always contain an \( m \).

Finally, suppose that Case (b) of this lemma holds. From \((t_m(1), t_m(2)) \in S_m \) we obtain that \( t_m(1) \geq 1 \), hence \( t \geq 2 \). Now by Lemma 5.8, there is a suffix \( S \) of \( T_t^{(m)} \) such that \( T_t^{(m)} \cdot S = T_u^{(m)} \). Also, we can write \( \chi^{(m)}(x) = \rho \cdot (m+1)^{u-1} + \mu \cdot (m+1)^{t-1} \) with \( m+1 \nmid \rho \) and \( 1 \leq \mu \leq m \). Let
\[ X := S \cdot \pi^{(m)}(\rho \cdot (m+1)^{u-1} + \mu \cdot (m+1)^{t-1} + 1, \rho \cdot (m+1)^{u-1} - 1) \cdot m \cdot T_t^{(m)}. \]
Then \( X^2 \) is a suffix of \( \pi^{(m)}(1, \chi^{(m)}(x)) \) and \( m \in X \).

\[ \blacksquare \]

Now we are ready to prove the \( \nu_1 \) version of Theorem 5.4.

28
Theorem 5.10. Let \( m \geq 1 \) and \( a \geq 1 \). Let \( a_{m+1}, a_{m+2}, \ldots, a_n \) be the level-m expansion of \( a \). Let \( t_l \) be the location of the first nonzero digit of \( a_l \) in \( l \)-ary expansion of \( a \), and \( u_l \) the second (if it exists). (in the same way as in Lemma 5.9) Now we have
\[
C(\pi^{(m+1)}(1, \chi^{(m+1)}(a))) = 1
\] (11)
if and only if for all \( l \in [m+1, n] \), we have
\[
\text{maxdigit}_l(a_l) \leq 1 \quad \text{and if } u_l \text{ exists: } (t_l(u_l), t_l(u_l)) \notin S_{l+1}.
\] (12)

For \( m = 1 \), Equation 11 is equivalent to \( a \in \text{Im}(\iota_1) \).

Proof. By Lemma 5.2 (b), we know for all \( l \) that the longest suffix of \( \pi^{(m+1)}(1, \chi^{(m+1)}(a)) \) that is in \( \mathbb{Z}_{p^l}^* \), is \( \pi^{(l)}(1, \chi^{(l)}(a_l)) \). Now for any square suffix \( X^2 \) of \( \pi^{(m+1)}(1, \chi^{(m+1)}(a)) \), we can denote its minimal element by \( l \). Then \( X^2 \) is also a square suffix of \( \pi^{(l)}(1, \chi^{(l)}(a_l)) \). Now apply Lemma 5.9 with \( l \) instead of \( m \), and \( x := a_l \). This proves that such a suffix \( X^2 \) exists if and only if Equation 12 is false. Since this is true for all \( l \), the first part of the theorem follows.

The second part follows from Remark 4.7.

Remark 5.11. Note that \( a_{l+1} = \iota_l(t_l) \) for all \( l \).

5.1 Implementation of the theorems

We have implemented Theorems 5.4 and 5.10 in Algorithm [6]. Together, the theorems determine the curling numbers of all the relevant prefixes of the level-m Gijswijt sequences. Also, they determine how values of \( \iota_m \) can be deduced from values of \( \iota_{m+1}, \iota_{m+2}, \ldots \). Fortunately, the number of values needed is always finite.

To actually calculate the value of \( \iota_m(x) \), the following steps have to be taken.

- Since \( \iota_m \) is increasing, we need to check for \( a = 0, 1, 2, 3, \ldots \) whether \( a \) is in the image of \( \iota_m \). We stop when we have found \( x \) values that are in the image. The last value is \( \iota_m(x) \).

- For each value of \( a \), we can use Theorem 5.4 or 5.10.

- However, this theorem may need several values of \( \iota_l \) for values \( l > m \). Therefore, we repeat the previous steps for these values.

In the actual algorithm, we perform these tasks in the reversed direction: we start by calculating a value of \( \iota_5 \), then \( \iota_4, \ldots, \iota_1 \).

Luckily, the level-m expansion of \( a \) is very rapidly decreasing: we have
\[
a_{l+1} = \iota_l(\text{ord}_l(a_l) + 1) \approx \text{ord}_l(a_l) \leq \log_l(a_l).
\]
Therefore, we have to calculate less values of the image of \( \iota_{l+1} \) than of \( \iota_l \).
After calculating values of $\iota_m$, the algorithm also calculates values of $\tau^{(m)}$. This is done using the following relations: with Definition 4.6 we can relate $\tau^{(m)}(t)$ to the length of a $\pi^{(m+1)}$ string. Then with Definition 3.2, we connect this to values of $\tau^{(m+1)}$. Therefore, we can start by calculating values of $\tau^{(5)}$, then of $\tau^{(4)}, \ldots, \tau^{(1)}$.

Finally, we use Equations 2 and 1 to calculate values of $\sigma^{(1)}$ and $\beta^{(1)}$.

Note that $\tau^{(1)}, \sigma^{(1)}, \beta^{(1)}$ are respectively Entries A092432, A091579, A091411 in the OEIS. In the standard algorithm to calculate these sequences, the curling numbers of strings have to be calculated. This is not the case for our new algorithm. Instead, it uses number-theoretic notions (calculating the $m$-ary expansions of numbers, and calculating the maximum of the digits and the locations of the first and second nonzero digits). This alternative view on the level-$m$ Gijswijt sequences might be useful for further research.
6 The constants $\nu_m$

In this section, we shall prove that for all $m \in \mathbb{Z}_{\geq 1}$, the fraction $\frac{x}{\iota_m(x)}$ converges as $x \to \infty$. We will denote the limit by $\nu_m$:

**Definition 6.1.** For $m \geq 1$, define $\nu_m := \lim_{x \to \infty} \frac{x}{\iota_m(x)}$.

These are the constants $\nu_m$ that will occur in our expression for $\phi^{(m)}(n)$ (Theorem 7.11). We shall prove in Propositions 6.5 (b) and 6.13 (c) that this definition is justified.

We will need to make a distinction between the cases $m \geq 2$ and $m = 1$, where the case $m \geq 2$ is easier. For $m \geq 2$, we deduce from Theorem 5.4 that the image of $\iota_m$ can be characterized by a set $V_m$ (see Definition 6.3). In Proposition 6.5 (b) we obtain the equation

$$\nu_m = \frac{m}{m+1} \sum_{v \in V_m} \frac{1}{(m+1)^v}.$$  

For $m = 1$ we conclude from Theorem 5.10 that the image of $\iota_1$ can be characterized by two sets $Q \subset \mathbb{Z}_{\geq 0}$ and $R \subset (\mathbb{Z}_{\geq 0})^2$ (see Corollary 6.11 and Definition 6.10). In Lemma 6.13 (c), we prove the equation

$$\nu_1 = \sum_{(a,b) \in R} \frac{1}{2^{b+1}}.$$  

We will prove that the irrationality measures of the constants $\nu_m$ are at least $m + 1$ (see Theorems 6.8 and 6.21). We will do this by constructing rational approximations to these constants. The fractions will come from patterns in $V_m, R, Q$ (see Lemmas 6.7 and 6.14). Also, we shall use similar approximations to prove a tower of exponents in Theorem 6.20. This tower of exponents is the first ingredient for Theorem 7.11.

The approximations used for Theorem 6.20, are $\nu_m \approx \frac{m^{-1}((m+1)^{l-1})}{(m+1)^l}$. The approximations used to prove that $\mu(\nu_m) \geq m + 1$ are $\nu_m \approx \frac{m^{-1}((m+1)^{l-2})}{(m+1)^{l-1}}$. (see Remarks 6.9 and 6.23).

We finish the section by calculating values of $\nu_m$ up to 20 decimals.

6.1 The case $m \geq 2$

We start by extracting the following information from Theorem 5.4:

**Corollary 6.2.** For $m \geq 2$, whether a number $p \in \mathbb{Z}_{\geq 1}$ is in the image of $\iota_m$ only depends on $\text{ord}_{m+1}(p)$. Specifically, we have $p \in \text{Im}(\iota_m)$ if and only if $C(T^{(m+1)p}_{\text{ord}_{m+1}(p)+1}) \leq m$.

**Proof.** Let $p_{m+1}, \ldots, p_n$ be the level-$m$ expansion of $p$. Then we always have $\text{maxdigit}_{m+1}(p_{m+1}) \leq m$. Inequality 10 holds for all $l$ if and only if it holds for all $l \geq m+2$. Since $a_{m+2}, \ldots, a_n$ only depend on $\text{ord}_{m+1}(p)$, this proves the first statement.
Now $p \in \text{Im}(\iota_1)$ if and only if Inequality 10 holds for $l = m + 2, m + 3, \ldots, n$. Using the other part of Theorem 5.4, we see that this is equivalent to

$C^{(2)}(\pi^{(m+2)}(1, \chi^{(m+2)}(p_{m+2}))) \leq m$. Using that $m \geq 2$, we see that $p \in \text{Im}(\iota_m)$ if and only if

$C(\pi^{(m+2)}(1, \chi^{(m+2)}(p_{m+2}))) \leq m$. By the definitions of $p_{m+2}$ and $\iota_m$, we see that this is equivalent to

$C(T_{\text{ord}_{m+1}(p)+1}) \leq m$.

Using this corollary, we can make the following definition:

**Definition 6.3.** For $m \in \mathbb{Z}_{\geq 2}$, define $V_m$ as the set of $n \in \mathbb{Z}_{\geq 0}$ such that the numbers $p \in \mathbb{Z}_{\geq 1}$ with $\text{ord}_{m+1}(p) = n$ are in the image of $\iota_m$.

The first elements of the sets $V_m$ for small $m$ are given in Table 9. Observe that $V_m$ seems to start the same as the sequence $\chi^{(m+1)}(1), \chi^{(m+1)}(2), \ldots$. However, eventually they start to differ. For example, $\chi^{(3)}(27) = 64$ but $64 \notin V_2$.

**Remark 6.4.** Since $C(T_1^{(m+1)}) = C(\varepsilon) = 1$, we obtain from Corollary 6.2 that $0 \in V_m$ for all $m \geq 2$. Therefore, all numbers that are not divisible by $m + 1$ are in the image of $\iota_m$. This is exactly Lemma 4.8 (b) for $m \geq 2$.

When we replace $\text{ord}_{m+1}(p)$ by $x$ in Corollary 6.2, we obtain a criterion to test whether a number $x$ is in $V_m$. Since for all $n$, the curling number of $T_{t^{(m+1)(n)}}$ is 1, it follows that $V_m$ is infinite.

**Proposition 6.5.** Let $m \geq 2$.

(a) For all $p \in \mathbb{Z}_{\geq 1}$ we have

$$\iota_m^{-1}(p) = 1 + \sum_{v \in V_m} \left( \left\lfloor \frac{p}{(m+1)^v} \right\rfloor - \left\lfloor \frac{p}{(m+1)^{v+1}} \right\rfloor \right).$$

(b) In Definition 6.1, $\nu_m$ is well-defined, and we have:

$$\nu_m = \frac{m}{m + 1} \cdot \sum_{v \in V_m} \frac{1}{(m + 1)^v}.$$

(c) Let $n \geq m + 1$ and $t := t^{(m+1)(n)} - 1$. Then we have

$$\iota_m^{-1}((m+1)^t) - 1 = [\nu_m \cdot (m + 1)^t].$$

**Proof.**

(a) Observe that

$$\left\lfloor \frac{p}{(m+1)^v} \right\rfloor - \left\lfloor \frac{p}{(m+1)^{v+1}} \right\rfloor$$

is the number of positive integers that are at most $p$ and have exactly $v$ factors $m+1$. Therefore, $\sum_{v \in V_m} \left( \left\lfloor \frac{p}{(m+1)^v} \right\rfloor - \left\lfloor \frac{p}{(m+1)^{v+1}} \right\rfloor \right)$ is the number of positive integers that are in the image of $t_m$ and at most $p$. Now the statement follows since $t_m$ is strictly increasing and $t_m(1) = 0$. 32
(b) Consider the difference
\[
\left| \frac{1}{p} \cdot \left( 1 + \sum_{v \in V_m} \left( \left[ \frac{p}{(m+1)^v} \right] - \left[ \frac{p}{(m+1)^{v+1}} \right] \right) \right) - \frac{m}{m+1} \cdot \sum_{v \in V_m} \frac{1}{(m+1)^v} \right|.
\]
This equals
\[
\left| \frac{1}{p} + \sum_{v \in V_m} \frac{1}{p} \left( \left[ \frac{p}{(m+1)^v} \right] - \left[ \frac{p}{(m+1)^{v+1}} \right] \right) - \frac{m}{m+1} \cdot \frac{1}{(m+1)^v} \right|,
\]
which is \(O\left(\frac{\ln(p)}{p}\right)\) as \(p \to \infty\). Hence it converges to 0. Using Part (a), it follows that
\[
\lim_{p \to \infty} \frac{\iota_m^{-1}(p)}{p} = \frac{m}{m+1} \cdot \sum_{v \in V_m} \frac{1}{(m+1)^v}.
\]
Now the lemma follows since \(\iota_m^{-1}(\iota_m(x)) = x\) for all \(x\).

(c) We know that \((m+1)^t\) is in the image of \(\iota_m\) by Equation 8. Therefore, \(t \in V_m\). Now it follows from Part (a) that
\[
\iota_m^{-1}((m+1)^t) = 1 + \sum_{v \in V_m} \left( \left[ \frac{(m+1)^t}{(m+1)^v} \right] - \left[ \frac{(m+1)^t}{(m+1)^{v+1}} \right] \right)
= 1 + \frac{m}{m+1} \cdot \sum_{v \in V_m, v \leq t} \frac{(m+1)^t}{(m+1)^v} + \frac{1}{m+1}
\geq 1 + \nu_m \cdot (m+1)^t - \frac{m}{m+1} \cdot \sum_{v=t+1}^{\infty} \frac{(m+1)^t}{(m+1)^v} + \frac{1}{m+1}\text{ by Part (b)}
= 1 + \nu_m \cdot (m+1)^t.
\]
Also, the value of Equation 13 is at most \(1 + \nu_m \cdot (m+1)^t + \frac{1}{m+1}\). Now the statement follows since \(\iota_m^{-1}((m+1)^t) - 1\) is an integer.

We shall later use Proposition 6.5 (c) as a building block for Theorem 7.11.

Now, we will prove that the numbers \(\nu_m\) for \(m \geq 2\) are transcendental, and that \(\nu_m\) has irrationality measure at least \(m+1\). For this, we need the following two lemmas.

**Lemma 6.6.** For \(m \geq 2\) and \(y \geq 1\), and for \(\mu\) with \(1 \leq \mu < (m+1) \cdot \iota_m^{-1}((m+2)^y) - 1\), we have \(\iota_m+1(\mu + \iota_m^{-1}((m+2)^y)) = (m+2)^y + \iota_m+1(\mu + 1) < (m+2)^{y+1}\).
Proof. For all $0 \leq i \leq m+1$ and $1 \leq j \leq (m+2)^y$, with $(i,j) \neq ((m+1),(m+2)^y)$, we know that $\text{ord}_{m+2}(i \cdot (m+2)^y + j) = \text{ord}_{m+2}(j)$. Therefore, by Corollary 6.2, $i \cdot (m+2)^y + j$ is in the image of $t_{m+1}$ if and only if $j$ is. Now since $t_{m+1}(1) = 0$, the number of elements of the intersection $\text{Im}(t_{m+1}) \cap [1,(m+2)^y]$ is $\epsilon_{m+1}^{-1}((m+2)^y) - 1$, and therefore the intersections $\text{Im}(t_{m+1}) \cap [x,x + (m+2)^y - 1]$ have the same number of elements for $x = 1,2,\ldots,(m+1) \cdot (m+2)^y$. Now the equality follows since $t_{m+1}$ is strictly increasing. It also follows that $t_{m+1}((m+2) \cdot (t_{m+1}^{-1}((m+2)^y) - 1)) < (m+2)^{y+1}$.

Lemma 6.7. For $m \geq 2$ and $y \geq 1$, and for $\mu$ with $1 \leq \mu < m \cdot (t_{m+1}^{-1}((m+2)^y) - 1)$, the number $\mu + t_{m+1}^{-1}((m+2)^y) - 1$ is in $V_m$ if and only if $\mu$ is. Moreover, $(m+1) \cdot (t_{m+1}^{-1}((m+2)^y) - 1) \notin V_m$.

Proof. Let $p_{m+1}, \ldots, p_l$ and $q_{m+1}, \ldots, q_n$ be the level-$m$ expansions of 

$$(m+1)^{\mu+t_{m+1}^{-1}((m+2)^y)-1}$$

and $(m+1)^\nu$. Now we have $p_{m+2} = t_{m+1}(\mu+t_{m+1}^{-1}((m+2)^y))$ and $q_{m+2} = t_{m+1}(\mu+1)$. Note that we add one to $\mu$ because this also happens in the definition of the level-$m$ expansion: $a_{l+1} := u_i(\text{ord}_i(a_i) + 1)$. By Lemma 6.6, $p_{m+2}$ and $q_{m+2}$ have the same $(m+2)$-adic order. Therefore, we obtain that $p_{m+3} = q_{m+3}$ and therefore the other numbers are also equal, and $l = n$.

Also, the first digits of $p_{m+2}$ and $q_{m+2}$ in base $m+2$ are at most $m$, which follows from repeatedly applying Lemma 6.6. All other digits of $p_{m+2}$ and $q_{m+2}$ are the same as those of $q_{m+2}$. Therefore, $\text{maxdigit}_{m+2}(p_{m+2}) \leq m$ if and only if $\text{maxdigit}_{m+2}(q_{m+2}) \leq m$. Now we see from Theorem 5.4 that $p_{m+1} \in \text{Im}(t_m)$ if and only if $q_{m+1} \in \text{Im}(t_m)$. Hence $\mu + t_{m+1}^{-1}((m+2)^y) - 1 \in V_m$ if and only if $\mu \in V_m$.

When we take instead $\mu = m \cdot (t_{m+1}^{-1}((m+2)^y) - 1)$, then we see in the above reasoning that $p_{m+2} = m \cdot (m+2)^y + t_m(t_{m+1}^{-1}((m+2)^y)) \in \{(m+1) \cdot (m+2)^y, (m+1) \cdot (m+2)^y - 1\}$. In both cases, $\text{maxdigit}_{m+2}(p_{m+2}) = m+1$. Therefore, $(m+1) \cdot (t_{m+1}^{-1}((m+2)^y) - 1) \notin V_m$.

Theorem 6.8. For all integers $m \geq 2$, the number $\nu_m$ is transcendental with irrationality measure at least $m+1$.

Proof. Let $n > m + 2$, and define $t := t^{(m+1)}(n) - 1$. Now we know by Equation 8 that $(m+1)^t$ is in the image of $t_m$. Therefore, $t \in V_m$. Let $y := t^{(m+2)}(n) - 1$. By Equation 9, we have $t = t_m^{-1}((m+2)^y) - 1$. So by Lemma 6.7, we have $(m+1) \cdot t \notin V_m$.

We have $(m+1)^t \geq (m+1)^{2t-1} > 1$. We know from Proposition 6.5 (b) that

$$\xi := \nu_m - \frac{m}{m+1} \cdot \sum_{v \in V_m, v \leq (m+1)t} \frac{1}{(m+1)^v} \in \left[0, \frac{1}{(m+1)^{t+1}}\right].$$

Now by Lemma 6.7, and using that $t \in V_m$ and $0 \in V_m$, the summation on the left-hand side is equal to

$$\sum_{\lambda=0}^{m} \frac{1}{(m+1)^{\lambda t}} \cdot \sum_{\mu \in V_m, 0 \leq \mu \leq t-1} \frac{1}{(m+1)^\mu}.$$
which equals
\[
\left(\frac{(m+1)^t}{(m+1)^t - 1} - \frac{1}{((m+1)^t - 1) \cdot (m+1)^m \cdot t}\right) \cdot \frac{a_n}{(m+1)^t - 1}
\]
when we define
\[
a_n := \sum_{\mu \in V_m, 0 \leq \mu \leq t-1} (m+1)^{t-1-\mu}.
\]
Now \(a_n\) is an integer. Since \(0 \in V_m\), we have \(a_n \geq (m+1)^{t-1}\). Since the summation is multiplied by \(\frac{m}{m+1}\) in the definition of \(\xi\), we obtain that
\[
\nu_m - \frac{m \cdot a_n}{(m+1)^t - 1} = \frac{m \cdot a_n}{((m+1)^t - 1) \cdot (m+1)^{(m+1)t}} + \xi.
\]
Since \(a_n \geq (m+1)^{t-1}\) and \(\xi \leq \frac{1}{(m+1)^{(m+1)t} \cdot (m+1)^{(m+1)t}}\), this expression has to be negative. Furthermore, from the definition of \(a_n\) we see that \(a_n \leq \frac{1}{m} (m+1)^t\). Therefore,
\[
\frac{m \cdot a_n}{((m+1)^t - 1) \cdot (m+1)^{(m+1)t}} < \frac{1}{((m+1)^t - 1)^{m+1}}.
\]
We conclude that
\[
0 < \left|\nu_m - \frac{m \cdot a_n}{(m+1)^t - 1}\right| < \frac{1}{((m+1)^t - 1)^{m+1}}
\]
for all \(n > m + 2\). Since \((m+1)^t \to \infty\) as \(n \to \infty\), it follows for the irrationality measure that \(\mu(\nu_m) \geq m + 1\). Now we obtain from Lemma A.3 that \(\nu_m\) is transcendental.

\[\square\]

Remark 6.9. Let \(t := \ell(m+1)(n) - 1\) for an integer \(n\). In Proposition 6.5 (c), we essentially proved that \(\frac{\nu_m^{\ell(m+1)(n)} - 1}{(m+1)^{t}}\) is a good approximation for \(\nu_m\). In Theorem 6.8, we proved a better approximation for \(\nu_t\). The fraction \(\frac{m}{(m+1)^{t-1}}\) can be rewritten using Equation 13: it is equal to \(\frac{\nu_m^{\ell(m+1)(n)} - 2}{(m+1)^{t-1}}\).

6.2 The case \(m = 1\)

While the case \(m = 1\) is harder than \(m \geq 2\), we are still able to roughly follow the structure from the previous paragraph. At the end of this section, we shall arrive at our main results, which are Theorem 6.20 and Theorem 6.21.

We start by defining two sets:

Definition 6.10. Define \(Q\) as the set of integers \(\alpha_1 \in \mathbb{Z}_{\geq 0}\) such that \(C(T^{(2)}_{\alpha_1}) = 1\). Also, define \(R\) as the set of pairs of integers \((\alpha_1, \alpha_2)\) with the following three properties:

(a) \(\alpha_1 \in Q\).
(b) \( \alpha_1 < \alpha_2 \).

(c) There is no suffix \( S \) of \( T^{(2)}_{\alpha_1+1} \) such that \( T^{(2)}_{\alpha_1+1} \cdot S = T^{(2)}_{\alpha_2+1} \).

The first elements of \( Q \) and \( R \) are given in Table 10.

In Corollary 6.2, we applied Theorem 5.4 to obtain the sets \( V_m \). In the following corollary, we use Theorem 5.10 to prove that the image of \( \iota_1 \) is determined by the sets \( Q \) and \( R \).

**Corollary 6.11.** We have the following:

- The number 0 is in the image of \( \iota_1 \).
- For \( \alpha_1 \in \mathbb{Z}_{\geq 0} \), the number \( 2^{\alpha_1} \) is in the image of \( \iota_1 \) if and only if \( \alpha_1 \in Q \).
- For all integers \( \alpha_1, \alpha_2 \) with \( 0 \leq \alpha_1 < \alpha_2 \) and all odd numbers \( \lambda \geq 1 \), the number \( \lambda \cdot 2^{\alpha_2} + 2^{\alpha_1} \) is in the image of \( \iota_1 \) if and only if \((\alpha_1, \alpha_2) \in R \).

**Proof.** We know from Lemma 4.8 (a) that \( 0 \in \text{Im}(\iota_1) \). Now let \( a \geq 1 \). Let \( a_2, a_3, \ldots, a_n \) be the level-\( m \) expansion of \( a \), and define \( t_i, u_i \) as in Theorem 5.10. Define \( \alpha_1 := \text{ord}_{2}(a) = t_2 - 1 \) and, if it exists, let \( \alpha_2 := u_2 - 1 \). Now by Theorem 5.10, Equation 12 holds for \( l = 3, 4, \ldots, n \) if and only if \( \mathcal{C}(\pi^{(3)}(1, \chi^{(3)}(a_3))) = 1 \). Since \( a_3 = t_2(\text{ord}_{2}(a)+1) = t_2(\alpha_1+1) \), we can use Definition 4.6 to see that \( \pi^{(3)}(1, \chi^{(3)}(a_3)) = T^{(2)}_{\alpha_1+1} \). Therefore, Equation 12 holds for \( l = 3, 4, \ldots, n \) if and only if \( \mathcal{C}(T^{(2)}_{\alpha_1+1}) = 1 \). This is equivalent to \( \alpha_1 \in Q \). If \( \alpha_2 \) does not exist, then Equation 12 automatically holds for \( l = 2 \). Hence \( a \in \text{Im}(\iota_1) \) if and only if \( \alpha_1 \in Q \).

If \( \alpha_2 \) does exist, then Equation 12 holds for \( l = 2 \) if and only if \((t_2(\alpha_1+1), t_2(\alpha_2+1)) \not\in S_3 \). So \( a \in \text{Im}(\iota_1) \) if and only if the latter holds and \( \alpha_1 \in Q \). By Definition 6.10 and Lemma 5.8, this is equivalent to \((\alpha_1, \alpha_2) \in R \).

This finishes the proof. \( \square \)

For all integers \( a \geq 0 \), we have either \( a = 0 \), or \( a = 2^{\alpha_1} \) for some integer \( \alpha_1 \geq 0 \), or \( a = \lambda \cdot 2^{\alpha_2} + 2^{\alpha_1} \) for an odd number \( \lambda \) and integers \( \alpha_1, \alpha_2 \) with \( 0 \leq \alpha_1 < \alpha_2 \). Therefore, \( Q \) and \( R \) determine the image of \( \iota_1 \).

**Remark 6.12.** We know by Corollary 6.2 and Definition 6.3 that for \( m \geq 2 \) and \( a \geq 0 \), we have the following:

\[
a \in V_m \iff \mathcal{C}(T^{(m+1)}_{a+1}) \leq m.
\]

For the set \( Q \) from Definition 6.10, we have:

\[
a \in Q \iff \mathcal{C}(T^{(2)}_{a+1}) = 1.
\]

Therefore, \( Q \) is in some way the equivalent for \( m = 1 \) of the sets \( V_m \).

For \( m \geq 2 \), we know by Corollary 6.2 that whether a number \( p \) is in the image of \( \iota_m \) depends only on \( \text{ord}_{m+1}(p) \). However, this is not true for \( m = 1 \). For example, the numbers \( p = 2, 6 \) have the same number of factors 2, but \( 2 \in \text{Im}(\iota_1) \) and \( 6 \not\in \text{Im}(\iota_1) \) (by Example 4.10).
It follows easily from the definition of $R$ that $(0, \alpha_2) \in R$ for all integers $\alpha_2 > 0$. Using Corollary 6.11, we obtain that all positive odd integers are in the image of $\iota_1$. This is exactly Lemma 4.8 (b) for $m = 1$.

Since $\iota_1$ is a strictly increasing function, we can use Corollary 6.11 to calculate values of $\iota_1^{-1}$, and with this we can prove a formula for $\nu_1$.

**Proposition 6.13.**

(a) For all integers $p \geq 0$, we have

$$\iota_1^{-1}(p) = 1 + \left( \sum_{a \in Q, 2^a \leq p} 1 \right) + \sum_{(a,b) \in R, 2^b < p} \left( \frac{p + 2^b - 2^a}{2^{b+1}} \right).$$

(b) For all integers $p \geq 0$, we have

$$\iota_1^{-1}(2^p) = 1 + \left( \sum_{a \in Q, a \leq p} 1 \right) + \sum_{(a,b) \in R, b < p} 2^{p-b-1}.$$

(c) In Definition 6.1, $\nu_1$ is well-defined, and we have:

$$\nu_1 = \sum_{(a,b) \in R} \frac{1}{2^{b+1}}.$$

**Proof.**

(a) We know that $\iota_1^{-1}(p)$ is the number of integers that are in the image of $\iota_1$ and at most $p$. Therefore, we have to count the number of elements in the three categories of Corollary 6.11, that are at most $p$. Since $p \geq 0$, the first category has exactly 1 element. The number of elements of the second category is exactly $\sum_{a \in Q, 2^a \leq p} 1$.

For each pair $(a, b) \in R$, the number of odd integers $\lambda \geq 1$ such that $\lambda \cdot 2^b + 2^a \leq p$, is exactly $\left\lfloor \frac{p + 2^b - 2^a}{2^b+1} \right\rfloor$. Therefore, the number of elements in the third category is $\sum_{(a,b) \in R} \left\lfloor \frac{p + 2^b - 2^a}{2^b+1} \right\rfloor$. Observe that for all $(a, b) \in R$ with $2^b \geq p$, we have $\left\lfloor \frac{p + 2^b - 2^a}{2^b+1} \right\rfloor = 0$. Therefore, we can add the restriction that $2^b < p$.

Adding the three numbers gives us the desired result.

(b) This follows from replacing $p$ by $2^p$ in Part (a).

(c) For $p \geq 1$, consider the difference

$$\left\lfloor \frac{1}{p} \cdot \left( 1 + \sum_{a \in Q, 2^a \leq p} 1 \right) + \sum_{(a,b) \in R, 2^b < p} \left( \frac{p + 2^b - 2^a}{2^{b+1}} \right) \right\rfloor - \sum_{(a,b) \in R} \frac{1}{2^{b+1}}.$$
This equals
\[
\left| \frac{1}{p} \cdot \left( 1 + \sum_{a \in Q, 2^a \leq p} 1 \right) + \sum_{(a,b) \in R, 2^b < p} \left( \frac{p + 2^b - 2^a}{2^{b+1}} \right) - \frac{1}{2^{b+1}} \right|,
\]
which is \(O\left(\frac{\ln(p)^2}{p}\right)\) as \(p \to \infty\). Therefore, it converges to 0. Using Part (a), it follows that
\[
\lim_{p \to \infty} \frac{t_i^{-1}(p)}{p} = \sum_{(a,b) \in R} \frac{1}{2^{b+1}}.
\]
Now the lemma follows since \(t_i^{-1}(t_i(x)) = x\) for all \(x\).

\[\square\]

We shall use the following patterns in \(Q\) and \(R\):

**Lemma 6.14.** For \(n \geq 3\), define \(t := t^{(2)}(n) - 1\). Now we have the following:

(a) For all integers \(a \in [0, t - 1]\), we have \(a \in Q\) if and only if \(a + t \in Q\).

(b) For all integers \(a, b\) with \(0 \leq a < b < t\), we have \((a, b) \in R\) if and only if \((a + t, b + t) \in R\).

(c) For all integers \(a \in Q\) with \(a < t\) and \(b \in [t, 2t - 1]\), we have \((a, b) \in R\).

(d) We have \((t, 2t) \notin R\).

**Proof.** By Lemma 1.2 (b), we know that \((T^{(2)}_{t^{(2)}(n)})^3 = (B^{(3)}_{t^{(3)}(n)})^3\), which is a prefix of \(A^{(3)}\). For all proper prefixes \(P\) of \(T^{(2)}_{t^{(2)}(n)}\), the strings \(T^{(2)}_{t^{(2)}(n)} \cdot P\) and \((T^{(2)}_{t^{(2)}(n)})^2 \cdot P\) have at most two elements \(n\), so a suffix of the form \(X^3\) cannot contain an \(n\). Therefore, \(C(P) \leq 2\) if and only if \(C(T^{(2)}_{(t^{(2)}(n)) \cdot P}) \leq 2\) if and only if \(C((T^{(2)}_{t^{(2)}(n)})^2 \cdot P) \leq 2\). It follows from Lemma 3.1 that
\[
T^{(2)}_{t^{(2)}(n) + i} = T^{(2)}_{t^{(2)}(n)} \cdot T^{(2)}_{i+1} \text{ for all } i \in \{0, 1, 2, \ldots, 2 \cdot t^{(2)}(n) - 3\}. \tag{15}
\]

(a) For all \(i \in \{0, 1, \ldots, t - 1\}\), the string from Equation 15 contains exactly one element \(n\), which is the last element of \(T^{(2)}_{t^{(2)}(n)}\). Therefore, we see that \(C(T^{(2)}_{t^{(2)}(n) + i})\) has a square suffix if and only if \(T^{(2)}_{i+1}\) has a square suffix. Now it follows from the definition of \(Q\) that \(i + t \in Q\) if and only if \(i \in Q\).

(b) Let \(a, b\) be integers such that \(0 \leq a < b \leq t - 1\). We already know from Part (a) that \(a \in Q\) if and only if \(a + t \in Q\). Hence for proving that \((a, b) \in R\) if and only if \((a + t, b + t) \in R\), it suffices to prove that \(T^{(2)}_{a + 1}\) has a suffix \(S\) such that \(T^{(2)}_{a + 1} \cdot S = T^{(2)}_{b + 1}\), if and only if \(T^{(2)}_{a + t + 1}\) has a suffix \(S'\) such that \(T^{(2)}_{a + t + 1} \cdot S' = T^{(2)}_{b + t + 1}\). By twice applying Equation 15, we see immediately that the existence of \(S\) implies the existence of \(S'\).
For the other direction, observe that $T^{(2)}_{a+t+1}$ and $T^{(2)}_{b+t+1}$ have exactly one element $n$. Therefore, an $S'$ would contain no $n$. Hence $S'$ (if it exists) must be a suffix of $T^{(2)}_{a+1}$, and using Equation 15 we see that $S$ exists too. We conclude that $(a, b) \in R$ if and only if $(a + t, b + t) \in R$.

(c) Let $a \in Q$ with $a < t$ and $b \in [t, 2t - 1]$. To prove that $(a, b) \in R$, we must show that there is no suffix $S$ of $T^{(2)}_{a+1}$ such that $T^{(2)}_{a+1} \cdot S = T^{(2)}_{b+1}$. For this, observe that $T^{(2)}_{a+1}$ contains no element $n$, while $T^{(2)}_{b+1}$ does contain an element $n$. This completes our proof.

(d) We have $T^{(2)}_{2t+1} = T^{(2)}_{t+1}T^{(2)}_{t+1}$, so there is a suffix $S$ of $T^{(2)}_{t+1}$ such that $T^{(2)}_{t+1} \cdot S = T^{(2)}_{2t+1}$.

Equation 15 implies that
\[
S^{(2)}_{t^{(2)}(n)+i} = S^{(2)}_{i+1} \text{ for all } i \in \{0, 1, \ldots, 2 \cdot t^{(2)}(n) - 4\}. \tag{16}
\]

We shall use a generalized version of this in the proof of Lemma 7.6.

Remark 6.15. Define a morphism between two sets of words $V$ and $W$ over some alphabet as a function $\chi : V \to W$ such that for all words $v_1, v_2$ in $V$, we have $\chi(v_1 \cdot v_2) = \chi(v_1) \cdot \chi(v_2)$. Now Equation 16 implies the following. Define $\chi$ as the morphism $\mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 2}$ such that $\chi(2) = B^{(2)}_{t^{(2)}(n)}$ and $\chi(x) = x$ for all $x > 2$. Then for $i = 0, 1, \ldots, 2 \cdot t^{(2)}(n) - 3$, we have $\chi(B^{(2)}_{i+1}) = B^{(2)}_{t^{(2)}(n)+i}$.

We will now make some estimates for $\nu_1$.

Lemma 6.16. For $x \geq 1$, define $q_x := \sum_{a \in Q, a < x} (1)$ and $r_x := \sum_{(a, b) \in R, b < x} \frac{1}{2^{b+1}}$. Now we have
\[
r_x \leq \nu_1 \leq \frac{1}{2x} \cdot (1 + q_x) + r_x.
\]

Proof. By Proposition 6.13 (c), we know that $\nu_1 = \sum_{(a, b) \in R} \frac{1}{2^{b+1}}$. This is equal to
\[
r_x + \sum_{(a, b) \in R, a < b} \frac{1}{2^{b+1}} + \sum_{(a, b) \in R, x \leq a} \frac{1}{2^{b+1}}.
\]
Since the second and third summation are at least 0, the first inequality of the lemma follows immediately. To prove the second inequality, we will bound the second and third summations from above. The second summation is at most
\[
\sum_{a \in Q, a < x} \sum_{b \geq x} \frac{1}{2^{b+1}} = \sum_{a \in Q, a < x} \frac{1}{2^{x}} = \frac{q_x}{2x}.
\]
The third summation is at most
\[
\sum_{x \leq a < b} \frac{1}{2^{b+1}} = \frac{1}{2x}.
\]
This proves the lemma.
Lemma 6.17. For \( n \geq 3 \), let \( t := t^{(2)}(n) - 1 \). Define \( q_x, r_x \) as in Lemma 6.16. Now we have the following:

\[
\left( \frac{1}{2^t} - \frac{1}{2^{2t}} \right) \cdot q_t + (1 + \frac{1}{2^t}) \cdot r_t \leq \nu_1 \leq \left( \frac{1}{2^{2t+1}} + \frac{1}{2^t} \right) \cdot q_t + (1 + \frac{1}{2^t}) \cdot r_t.
\]

Proof. We have

\[
r_{2t} = r_t + \sum_{(a,b) \in R, a < t \leq b < 2t} \frac{1}{2^{b+1}} + \sum_{(a,b) \in R, t \leq a < b \leq 2t} \frac{1}{2^{b+1}}
\]

\[
= \sum_{a \in Q, a < t} 2^{t-1} \frac{1}{2^{b+1}} + (1 + \frac{1}{2^t}) \cdot r_t \quad \quad \text{by Lemma 6.14 (b) and (c)}
\]

\[
= \left( \frac{1}{2^t} - \frac{1}{2^{2t}} \right) \cdot q_t + (1 + \frac{1}{2^t}) \cdot r_t.
\]

By Lemma 6.14 (a), we have \( q_{2t} = 2q_t \). Now apply Lemma 6.16 to \( x := 2t \). We can make the upper bound of the lemma a bit stronger: in Equation 17, we assumed the worst-case scenario where \((a,b) \in R\) for all \( a < 2t \leq b \). However, by Lemma 6.14 (d), we have \((t, 2t) \notin R\). Therefore, the upper bound can be lowered by \( \frac{1}{2^{2t+1}} \). We see that \( \nu_1 \in [r_{2t}, r_{2t} + \frac{1}{2^t}(1 + q_t) - \frac{1}{2^{2t+1}}] \). This lemma follows by substituting our formulas for \( r_{2t} \) and \( q_{2t} \).

Proposition 6.18. For all integers \( n \geq 4 \), we have

\[
i^{-1}(2^t) - 1 = \lceil \nu_1 \cdot 2^t \rceil
\]

where \( t := t^{(2)}(n) - 1 \).

Proof. Firstly, notice that \( t \in Q \) by Equation 8. Let \( q_x, r_x \) be as in Lemma 6.16. Then from Proposition 6.13 (b), it follows that \( i^{-1}(2^t) - 1 = 1 + q_t + 2^t \cdot r_t \).

For \( n = 4 \), it follows from direct calculation that \( r_7 \in [0.69, 0.70] \) (see Algorithm [6]). From Lemma 6.16, it follows that \( \nu_1 \in [0.69, 0.71] \). Now, we have \( t = t^{(2)}(4) - 1 = 3 \), so it follows that \( \lceil \nu_1 \cdot 2^t \rceil = 6 \). Also, we have \( i^{-1}(2^t) - 1 = i^{-1}(8) - 1 = 6 \) by Example 4.10. Hence the proposition holds for \( n = 4 \).

Now suppose that \( n \geq 5 \). We have seen that \( i^{-1}(2^t) - 1 = 1 + q_t + 2^t \cdot r_t \). Now by Lemma 6.17, we see that

\[
\nu_1 \cdot 2^t - i^{-1}(2^t) + 1 \in [-1 + r_t - \frac{1}{2^t} \cdot q_t, \frac{1 + 2q_t}{2^{t+1}} - 1 + r_t].
\]

To examine this interval, we need a few estimates. We have \( t \geq t^{(2)}(5) - 1 = 80 - 1 = 79 \), \( r_{79} \leq r_t \) and \( q_t \leq t \). Furthermore, we have \((1, 2) \notin R\), hence \( r_t \leq -\frac{1}{8} + \sum_{b=1}^{\infty} \frac{b}{2^b} = \frac{7}{8} \). Therefore,

\[
\nu_1 \cdot 2^t - i^{-1}(2^t) + 1 \in [-1 + r_{79} - \frac{t}{2^t} \cdot 2^t + \frac{2t + 1}{2^{t+1}} - 1, \frac{1}{8}].
\]

Since \( r_{79} \geq 0.69 \) and \( t \geq 79 \), this interval is contained in \((-1, 0)\). The proposition follows since \( i^{-1}(2^t) - 1 \) is an integer. \( \Box \)
Remark 6.19. Actually, this proposition also holds for \( n = 2 \) and \( n = 3 \), but we shall not need it.

We shall combine Propositions 6.5 (c) and 6.18 to find an expression for \( t^{(m)}(n) \).

**Theorem 6.20.** For all integers \( m, n \) with \( 1 \leq m \leq n - 2 \) and \( (m,n) \neq (1,3) \), we have

\[
t^{(m)}(n) = 1 + \lfloor \nu_m \cdot (m + 1) \rfloor_{\nu_{m+1} \cdot (m+2)-\nu_{n-2} \cdot (n-1)}.
\]

**Proof.** We first prove the following claim:

**Claim 6.20.1.** For all integers \( m, n \) with \( 1 \leq m \leq n - 2 \) and \( (m,n) \neq (1,3) \), we have

\[
t^{(m)}(n) - 1 = \lfloor \nu_m \cdot (m + 1) \rfloor_{\nu_{m+1} \cdot (m+2)-\nu_{n-2} \cdot (n-1)}
\]  \hspace{1cm} (19)

**Proof.** By Equation 9, the left-hand side of Equation 19 is equal to \( \nu_m^{-1}((m+1)\nu_{m+1}(m+2)-\nu_{n-2}(n-1)) - 1 \). Now the claim follows from Propositions 6.5 (c) and 6.18.

By applying the claim \( n - m - 1 \) times, we obtain that

\[
t^{(m)}(n) - 1 = \lfloor \nu_m \cdot (m + 1) \rfloor_{\nu_{m+1} \cdot (m+2)-\nu_{n-2} \cdot (n-1)}
\]

The theorem follows since \( t^{(n-1)}(n) = 2 \).

**Theorem 6.21.** The constant \( \nu_t \) has irrationality measure at least 2.

**Proof.** Let \( n \geq 5 \) and \( t := t^{(2)}(n) - 1 \). Define \( q_t, r_t \) as in Lemma 6.16. Now Lemma 6.17 states that

\[
\nu_t \in \left[ \left( \frac{1}{2^t} - \frac{1}{2^{2t-1}} \right) \cdot q_t + \left( 1 + \frac{1}{2^t} \right) \cdot r_t, \frac{1}{2^{2t+1}} + \left( \frac{1}{2^t} + \frac{1}{2^{2t}} \right) \cdot q_t + \left( 1 + \frac{1}{2^t} \right) \cdot r_t \right].
\]

It follows that \( \nu_t - \frac{q_t + 2^t \cdot r_t}{2^t - 1} \) is an element of

\[
\left[ \left( \frac{1}{2^t(2^t - 1)} - \frac{1}{2^{2t}} \right) \cdot q_t - \frac{1}{2^t(2^t - 1)} \cdot r_t, \frac{1}{2^{2t+1}} - \frac{1}{2^t(2^t - 1)} \cdot q_t - \frac{1}{2^{2t}} \cdot r_t \right].
\]

As we saw in the proof of Proposition 6.18, we have \( t \geq 79 \) and \( r_t \geq 0.69 \) and \( r_t \leq \frac{7}{8} \).

Therefore, the upper bound of the interval is negative, hence \( \nu_t - \frac{q_t + 2^t \cdot r_t}{2^t - 1} \) is negative.

As \( n \to \infty \), we have \( t \to \infty \). Also, we have \( q_t = O(t) \). Therefore, we see that

\[
0 < \left| \nu_t - \frac{q_t + 2^t \cdot r_t}{2^t - 1} \right| = O\left( \frac{t}{2^{2t}} \right).
\]  \hspace{1cm} (20)

The theorem follows since \( 2^t \cdot r_t \) is an integer.

Remark 6.22. It follows from Lemma A.2 that \( \nu_t \) is irrational.
Table 8: Values of $\nu_m$. See Entry A357066 of the OEIS [8].

| $m$ | $\nu_m$ |
|-----|---------|
| 1   | 0.69167220878112615338... |
| 2   | 0.97499818801525412389... |
| 3   | 0.99706744867944596417... |
| 4   | 0.99974398361495135688... |
| 5   | 0.99998213871077214353... |
| 6   | 0.999999592006611157...   |
| 7   | 0.9999994784593543295...   |
| 8   | 0.999999977056224067...    |
| 9   | 0.999999999999999999999... |
| 10  | 0.99999999999681369182...   |

Table 9: Values of $V_m$.

| $m$ | $V_m$ |
|-----|-------|
| 2   | $\{0,1,2,4,5,6,8,9,10,16,17,18,20,21,22,24,25,26,32,33,34,36,37,38,40,41,42,65,...\}$ |
| 3   | $\{0,1,2,3,5,6,7,8,10,11,12,13,15,16,17,18,25,26,27,28,30,31,32,33,35,...\}$ |
| 4   | $\{0,1,2,3,4,6,7,9,10,12,13,14,15,16,18,19,20,21,22,24,25,26,27,28,...\}$ |
| 5   | $\{0,1,2,3,4,5,7,8,9,10,11,12,14,15,16,17,18,19,21,22,23,24,25,26,28,...\}$ |
| 6   | $\{0,1,2,3,4,5,6,8,9,10,11,12,13,14,16,17,18,19,20,21,22,24,25,26,27,...\}$ |
| 7   | $\{0,1,2,3,4,5,6,7,9,10,11,12,13,14,15,16,18,19,20,21,22,23,24,25,27,...\}$ |
| 8   | $\{0,1,2,3,4,5,6,7,8,10,11,12,13,14,15,16,17,18,20,21,22,23,24,25,26,28,...\}$ |
| 9   | $\{0,1,2,3,4,5,6,7,8,9,11,12,13,14,15,16,17,18,19,20,22,23,24,25,26,...\}$ |
| 10  | $\{0,1,2,3,4,5,6,7,8,9,10,12,13,14,15,16,17,18,19,20,21,22,24,25,26,...\}$ |

Remark 6.23. In Proposition 6.18, we essentially proved that $\frac{\nu_1(2^t)-1}{2^t}$ is a good approximation for $\nu_1$. By Proposition 6.13 (b), this equals $\frac{1+\nu_1+2^t\nu_1}{2^{t+1}}$. In Inequality 20, we proved that $\frac{x+2^t\nu_1}{2^{t+1}}$ is an even better approximation for $\nu_1$.

Algorithm [6] computes the values of $\nu_m$ for $m \in [1, 10]$ up to 20 decimals; see Table 8. For $m \geq 2$, we do this by first calculating the first elements of $V_m$, which are by the second part of Corollary 6.2 exactly the numbers $x \geq 0$ such that $C(T_{x+1}) \leq m$. Then, we use Proposition 6.5 (b).

For $\nu_1$, we calculate the first 20 decimals using Example 4.11, Proposition 6.13 (b) with $p = 79$ and Lemma 6.16.
Van de Bult et al. [1, p. 19] conjectured that there are constants $\epsilon_m$ for $m \in \mathbb{Z}_{\geq 1}$ such that for each $m$, $\beta^{(m)}(n) \approx \epsilon_m \cdot (m+1)^{n-1}$ as $n \to \infty$. We shall prove that the relationship between the two sides is $\sim$ and even a bit stronger (Equation 28).

Intuitively, this can be explained as follows: since $B^{(m)}_{n+1} = (B^{(m)}_n)^{m+1} \cdot S^{(m)}_n$, we have $\beta^{(m)}(n+1) = (m+1) \cdot \beta^{(m)}(n) + \sigma^{(m)}(n)$, so if $\sigma^{(m)}(n)$ is relatively small, then it follows that $\beta^{(m)}(n+1)$ is approximately equal to $(m+1) \cdot \beta^{(m)}(n+1)$. Iterating this we obtain that $\beta^{(m)}(n) \approx (m+1)^{n-1}$, and the contribution from the $\sigma^{(m)}(n)$ gives us the extra factor $\epsilon_m$.

In reality, the proof is a bit harder. However, as a first step we can use the above equality to conclude that $(m+1) \cdot \beta^{(m)}(n) \leq \beta^{(m)}(n+1)$, so $\beta^{(m)}(n) \leq \beta^{(m)}(n+1) / (m+1)$. With induction we obtain the following:

$$\beta^{(m)}(t) \leq \frac{\beta^{(m)}(u)}{(m+1)^{u-t}} \text{ for all } t, u \text{ with } 1 \leq t \leq u.$$  \hfill (21)

Then we see that

$$\frac{\beta^{(m)}(t)}{(m+1)^{t-1}} \leq \frac{\beta^{(m)}(u)}{(m+1)^{u-1}} \text{ for all } t, u \text{ with } 1 \leq t \leq u.$$  \hfill (22)

Hence, if we can prove that $\beta^{(m)}(n) = O((m+1)^{n-1})$ as $n \to \infty$, then it follows that $\beta^{(m)}(n) / (m+1)^{n-1}$ converges for $n \to \infty$, which is what we want to prove.

The idea of the proof (Lemma 7.1) is as follows. We know that the strings $B^{(m)}_t$ are built from $m$’s and strings $S^{(m)}_u$. The $S^{(m)}_u$ strings are in turn substrings of $A^{(m+1)}$. Actually, they turn out to be substrings of some $B^{(m+1)}_t$, where $t'$ is much smaller than $t$. When we repeat this argument, we eventually reach some $B^{(m)}_1$, which contains only 1 element. In this way, we obtain that $\beta^{(m)}(n) = O((m+1)^{n-1})$.

An important tool in the proof are the iota functions, which we have defined in Definition 4.6. From Lemma 4.8 we know that $\iota_m(2) = 1$ and that all numbers that are not divisible by $m+1$ are in the image of $\iota_m$. After some investigation, this gives us the following inequality for $n \geq 2$:

$$\iota_m(n) \leq n - 2 + \left\lceil \frac{n-1}{m} \right\rceil.$$  \hfill (23)

From this we obtain two inequalities:

$$\iota_m(n) \leq 2n - 3 \text{ for all } m \geq 1, n \geq 2.$$  \hfill (24)
\( t_m(n) < \frac{m+1}{m} \cdot n \) for all \( m \geq 1, n \geq 2 \). \hfill (25)

Using this, we can estimate \( \sigma^{(m)} \) in terms of \( \beta^{(m+1)} \): for all \( m, n \in \mathbb{N} \) we have

\[
\sigma^{(m)}(n) \leq \tau^{(m)}(n + 1)
= l \left( \pi^{(m+1)}(1, \chi^{(m+1)}(t_m(n + 1))) \right)
\leq l \left( \pi^{(m+1)}(1, \chi^{(m+1)}((m + 1)^{\log_{m+1}(t_m(n+1)))}) \right)
= \beta^{(m+1)}(\log_{m+1}(t_m(n + 1))) + 1)
\leq \beta^{(m+1)}(\log_{m+1}(2n - 1]) + 1)
\]

by Definition 4.6

by Inequality 24,

hence

\[
\sigma^{(m)}(n) \leq \beta^{(m+1)}(\log_{m+1}(2n - 1]) + 1).
\hfill (26)
\]

We are now ready to prove the main lemma.

**Lemma 7.1.** There are numbers \( M_7, M_8, \ldots \in \mathbb{R}_{>0} \) such that

\[
\frac{\beta^{(m)}(n)}{(m+1)n^{-1}} < M_n \text{ for all } m \geq 1, n \geq 7.
\]

Here the numbers \( M_7, M_8, \ldots \) are as defined in the proof of this lemma.

**Proof.** We shall start with \( n = 7 \). Take \( M_7 := 4 \). From the values of \( \beta^{(m)}(t) \) in Table 4 we see that \( \beta^{(m)}(7) < 4 \cdot (m + 1)^6 \) for \( m = 1, 2, 3, 4, 5 \); and for \( m \geq 6 \), we know (see Lemma 4.1 from van de Bult et al. [1, p. 19]) that \( \beta^{(m)}(7) = (m + 1)^6 + (m + 1)^5 + (m + 1)^4 + (m + 1)^3 + (m + 1)^2 + (m + 1)^1 + (m + 1)^0 < (\sum_{i=0}^{\infty} \frac{1}{(m+1)^i}) \cdot (m + 1)^6 < 4 \cdot (m + 1)^6 \).

Therefore, \( M_7 = 4 \) works.

We shall now recursively define \( M_n \) for \( n = 8, 9, \ldots \).

So suppose that for a fixed \( k \geq 7 \), we have defined \( M_k \) and \( \beta^{(m)}(k) < M_k \cdot (m + 1)^{k-1} \) for all \( m \geq 1 \). Then for all \( m \geq 1 \), we have:

\[
\frac{\beta^{(m)}(k+1)}{(m + 1)^{k}} = \frac{\beta^{(m)}(k)}{(m + 1)^{k-1}} + \frac{\sigma^{(m)}(k)}{(m + 1)^{k}}
\leq M_k + \frac{\beta^{(m+1)}(\log_{m+1}(2k - 1]) + 1)}{(m + 1)^{k}} \text{ by Inequality 26}
\leq M_k + \frac{\beta^{(m+1)}(k)}{(m + 2)^{k-\log_{m+1}(2k-1)]-1}} \cdot (m + 1)^{k} \text{ by Inequality 21 (see below)}
\leq M_k + \frac{M_k \cdot (m + 2)^{\log_{m+1}(2k-1)}}{(m + 1)^{k}}
\leq M_k \left( 1 + \frac{(m + 1)^{\log_{2}(3)-\log_{m+1}(2k-1)]}}{(m + 1)^{k}} \right) \text{ (see below)}
\leq M_k(1 + (m + 1)^{\log_{2}(3)-\log_{2}(2k-1)]-k}) \leq M_k(1 + 2^{\log_{2}(3)-\log_{2}(2k-1)]-k}) \text{ (see below)}
= M_{k+1}.
\]
Here we used that \([\log_{m+1}(2k-1)]+1 \leq k\), \(\log_2(3) \geq \log_{m+1}(m+2)\), and \(\log_2(3) \cdot [\log_2(2k-1)] - k \leq 0\) for all \(k \geq 7\) and \(m \geq 1\). Proving these inequalities is straightforward. \(\square\)

From the lemma and Inequality 22, we obtain that \(\frac{\beta^{(m)}(n)}{(m+1)^{n-1}} < M\) for all \(m, n \in \mathbb{Z}_{>1}\), where we define \(M = 4 \prod_{k=7}^\infty (1 + 2\log_2(3) [\log_2(2k-1)] - k)\). Now it follows that \(M\) is smaller than 20; we omit the proof. Hence \(\frac{\beta^{(m)}(n)}{(m+1)^{n-1}} < 20\) for all \(m, n \in \mathbb{Z}_{>1}\).

**Theorem 7.2.** For all \(m \in \mathbb{Z}_{>1}\) there is a constant \(\epsilon_m \in \mathbb{R}_{>0}\) such that \(\beta^{(m)}(n) \sim \epsilon_m \cdot (m+1)^{n-1}\) for \(n \to \infty\), and we have \(\epsilon_m < 20\).

**Proof.** Fix \(m \in \mathbb{Z}_{>1}\). Now by Inequality 22, the value of \(\frac{\beta^{(m)}(n)}{(m+1)^{n-1}}\) is non-decreasing as \(n \to \infty\). But we also know that this value is smaller than 20 for all \(m, n \in \mathbb{Z}_{>1}\). Hence the sequence is convergent, and the statement follows for \(\epsilon_m := \lim_{n \to \infty} \frac{\beta^{(m)}(n)}{(m+1)^{n-1}}\). \(\square\)

**Corollary 7.3.** For all \(m \in \mathbb{Z}_{>1}\), the series \(\sum_{n=1}^\infty \frac{\sigma^{(m)}(n)}{(m+1)^n}\) is convergent and has value \(\epsilon_m - 1\).

**Proof.** From iterating the equation \(\beta^{(m)}(n+1) = (m+1) \cdot \beta^{(m)}(n) + \sigma^{(m)}(n)\), we obtain that \(\frac{\beta^{(m)}(n)}{(m+1)^{n-1}} = 1 + \frac{\sigma^{(m)}(1)}{(m+1)} + \frac{\sigma^{(m)}(2)}{(m+1)^2} + \cdots + \frac{\sigma^{(m)}(n-1)}{(m+1)^{n-1}}\) for all \(m, n \geq 1\). Now the statement follows from Theorem 7.2. \(\square\)

We see, since \(\frac{\beta^{(m)}(n)}{(m+1)^{n-1}} = 1 + \sum_{i=1}^{n-1} \frac{\sigma^{(m)}(i)}{(m+1)^i} < 1 + \sum_{i=1}^\infty \frac{\sigma^{(m)}(i)}{(m+1)^i} = \epsilon_m\), that

\[
\beta^{(m)}(n) < \epsilon_m \cdot (m+1)^{n-1}
\]

for all \(m \geq 1, n \geq 1\). We shall now prove additional estimates which we shall need for our main theorems.

**Lemma 7.4.** For all \(m \in \mathbb{Z}_{>1}\), the following holds.

(a) For all integers \(n \geq 1\),

\[
\sum_{i=n}^\infty \frac{\sigma^{(m)}(i)}{(m+1)^i} < \epsilon_{m+1} \cdot (m+2) \cdot (2i-1)^{\log_{m+1}(m+2)} \cdot (m+1)^i.
\]

(b) We have

\[
\sum_{i=n}^\infty \frac{\sigma^{(m)}(i)}{(m+1)^i} = O \left( \frac{n^{\log_{m+1}(m+2)}}{(m+1)^{n-1}} \right) \text{ as } n \to \infty.
\]
Proof. (a) We have:

\[
\sum_{i=n}^{\infty} \frac{\sigma^{(m)}(i)}{(m+1)^i} \leq \sum_{i=n}^{\infty} \frac{\beta^{(m+1)}([\log_{m+1}(2i-1)] + 1)}{(m+1)^i}
\]

by Inequality 26

\[
< \sum_{i=n}^{\infty} \frac{\epsilon_{m+1} \cdot (m + 2)^{\log_{m+1}(2i-1)}}{(m+1)^i}
\]

by Inequality 27

\[
< \sum_{i=n}^{\infty} \frac{\epsilon_{m+1} \cdot (m + 2)^{\log_{m+1}(2i-1)+1}}{(m+1)^i}
\]

\[
= \epsilon_{m+1} \sum_{i=n}^{\infty} \frac{(m + 2) \cdot (2i - 1)^{\log_{m+1}(m+2)}}{(m+1)^i}
\]

(b) Using Part (a), we see that

\[
\sum_{i=n}^{\infty} \frac{\sigma^{(m)}(i)}{(m+1)^i} < \epsilon_{m+1} \cdot \sum_{i=n}^{\infty} \frac{(m + 2) \cdot (2i - 1)^{\log_{m+1}(m+2)}}{(m+1)^i}
\]

\[
< \epsilon_{m+1} \cdot (m + 2) \cdot 2^{\log_{m+1}(m+2)} \cdot \frac{1}{(m+1)^{n-1}} \cdot \sum_{i=n}^{\infty} \frac{i^{\log_{m+1}(m+2)}}{(m+1)^{i-n+1}}
\]

\[
< \epsilon_{m+1} \cdot (m + 2) \cdot 2^{\log_{m+1}(m+2)} \cdot \frac{1}{(m+1)^{n-1}} \cdot \sum_{j=1}^{\infty} \frac{(n \cdot j)^{\log_{m+1}(m+2)}}{2^j}
\]

\[
< \epsilon_{m+1} \cdot (m + 2) \cdot 2^{\log_{m+1}(m+2)} \cdot \sum_{j=1}^{\infty} \frac{j^2}{2^j} \cdot \frac{n^{\log_{m+1}(m+2)}}{(m+1)^{n-1}}.
\]

Since the first four factors in the last line are finite constants, this value is \(O\left(\frac{n^{\log_{m+1}(m+2)}}{(m+1)^{n-1}}\right)\) as \(n \to \infty\).

From Lemma 7.4 (b), using that \(\beta^{(m)}(n) = \epsilon_m \cdot (m + 1)^{n-1} - (m + 1)^{n-1} \cdot \sum_{i=n}^{\infty} \frac{\sigma^{(m)}(i)}{(m+1)^i}\), we see that

\[
\beta^{(m)}(n) = \epsilon_m \cdot (m + 1)^{n-1} - O \left(\frac{n^{\log_{m+1}(m+2)}}{(m+1)^{n-1}}\right).
\]  \(\text{(28)}\)

Lemma 7.5. For all \(m \geq 1\) we have \(\epsilon_m < 3.5\).

Proof. It follows by direct calculation that \(1 + \sum_{i=81}^{80} \frac{\sigma^{(1)}(i)}{2^i} < 3.49\). Also, it follows from Lemma 7.4 (a) that \(\sum_{i=81}^{\infty} \frac{\sigma^{(1)}(i)}{2^i} < \epsilon_2 \cdot \sum_{i=81}^{\infty} \frac{3(2i-1)^{\log_{m+1}(3)}}{2^i} < \epsilon_2 \cdot \frac{1}{2000}\). Since \(\epsilon_2 < 20\) by Theorem 7.2, we obtain by Corollary 7.3 that \(\epsilon_1 = 1 + \sum_{i=1}^{\infty} \frac{\sigma^{(1)}(i)}{2^i} < 3.49 + \frac{20}{2000} = 3.5\).

For \(m = 2, 3, 4, \ldots, 20\), in the same way the inequality \(\epsilon_m < 2\) can be obtained, from which it follows that \(\epsilon_m < 3.5\).
Now for $m > 20$, we know that $\sigma^{(m)}(i) = 1$ for $i = 1, 2, \ldots, m$. Therefore, we obtain that
\[
\epsilon_m = 1 + \sum_{i=1}^{m} \frac{1}{(m+1)^i} + \sum_{i=m+1}^{\infty} \frac{\sigma^{(m)}(i)}{(m+1)^i},
\]
by Corollary 7.3
\[
< 2 + \epsilon_{m+1} \cdot \sum_{i=m+1}^{\infty} \frac{(m+2) \cdot (2i - 1)^{\log_{m+1}(m+2)}}{(m+1)^i},
\]
by Lemma 7.4 (a)
\[
< 2 + 20 \cdot \sum_{i=m+1}^{\infty} \frac{(m+2) \cdot (2i - 1)^{\log_{m+1}(m+2)}}{(m+1)^i},
\]
by Theorem 7.2
\[
< 2 + 20 \cdot \sum_{i=21}^{\infty} \frac{22 \cdot (2i - 1)^{\log_{21}(22)}}{21^i}
\]
< 3.5.

Lemma 7.6. For all integers $m \geq 1$ and $n \geq m + 1$, we have
\[
\sigma^{(m)}((m+1) \cdot (t^{(m)}(n) - 1)) - \sigma^{(m)}((m+1) \cdot (n) - 1) \geq \beta^{(m+1)}(t^{(m+1)}(n)).
\]

Proof. Firstly, analogous to the proof of Equation 16, it can be shown that
\[
S^{(m)}_{t^{(m)}(n)-1+i} = S^{(m)}_{i} \text{ for all } i \in \{1, 2, \ldots, m \cdot (t^{(m)}(n) - 1) - 1\}.
\]
(29)
We know that $S^{(m)}_1 \cdot S^{(m)}_2 \cdot \ldots \cdot S^{(m)}_{t^{(m)}(n)-1} = T^{(m)}_{t^{(m)}(n)}$, which equals $B^{(m+1)}_{t^{(m+1)}(n)}$, by Lemma 1.2 (b).

Now by Equation 29 it follows that
\[
S^{(m)}_1 \cdot S^{(m)}_2 \cdot \ldots \cdot S^{(m)}_{t^{(m)}(n)-1} \cdot S^{(m)}_{t^{(m)}(n)-1} = (B^{(m+1)}_{t^{(m+1)}(n)})^{m+1}.
\]
Now consider $S^{(m)}_{(m+1) \cdot (t^{(m)}(n)-1)}$. We have
\[
C(S^{(m)}_1 \cdot \ldots \cdot S^{(m)}_{t^{(m)}(n)-1}) = C(T^{(m)}_{1+(m+1) \cdot (t^{(m)}(n)-1)}) \leq m.
\]
It follows that
\[
C(S^{(m)}_1 \cdot S^{(m)}_2 \cdot \ldots \cdot S^{(m)}_{t^{(m)}(n)-2} \cdot S^{(m)}_{(m+1) \cdot (t^{(m)}(n)-1)}) \leq m.
\]
Therefore, there is an integer $t$ such that $T^{(m)}_{t} = S^{(m)}_1 \cdot \ldots \cdot S^{(m)}_{t^{(m)}(n)-2} \cdot S^{(m)}_{(m+1) \cdot (t^{(m)}(n)-1)}$.

Now $t \geq t^{(m)}(n)$, hence $S^{(m)}_{t^{(m)}(n)-1}$ is a prefix of $S^{(m)}_{(m+1) \cdot (t^{(m)}(n)-1)}$.

Now we see that $(B^{(m+1)}_{t^{(m+1)}(n)})^{m+1}$ is a prefix of $T^{(m)}_{1+(m+1) \cdot (t^{(m)}(n)-1)}$. Since $A^{(m+1)}$ starts with $(B^{(m+1)}_{t^{(m+1)}(n)})^{m+2}$, we see that $T^{(m)}_{1+(m+1) \cdot (t^{(m)}(n)-1)}$ must be longer than $(B^{(m+1)}_{t^{(m+1)}(n)})^{m+2}$ in order to have curling number at most $m$. Therefore, $S^{(m)}_{t^{(m)}(n)-1} \cdot B^{(m+1)}_{t^{(m+1)}(n)}$ is a prefix of $S^{(m)}_{(m+1) \cdot (t^{(m)}(n)-1)}$. From this the lemma follows. □
Lemma 7.7. For all $m, n$ with $n \geq m + 2$, we have

$$0 < \epsilon_m - \frac{\beta^{(m)}(t^{(m)}(n)) - 1}{(m + 1)t^{(m)(n)-1} - 1} < \sum_{i=(m+1):(t^{(m)}(n)-1)}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i}.$$ 

Proof. Define:

$$A := \sum_{i=1}^{t^{(m)(n)-1}} \left( (m + 1)^{t^{(m)(n)-1-i}} \sigma^{(m)}(i) \right)$$
$$B := \sum_{i=1}^{(m+1):(t^{(m)}(n)-1)} \frac{\sigma^{(m)}(i)}{(m + 1)^i}$$
$$C := \frac{\sigma^{(m)}(m + 1) \cdot (t^{(m)}(n) - 1) - \sigma^{(m)}(t^{(m)}(n) - 1)}{(m + 1)^{(m+1):(t^{(m)}(n)-1)}}$$
$$D := \frac{A}{(m + 1)^{(m+1):(t^{(m)}(n)-1)} \cdot ((m + 1)^{t^{(m)(n)-1}} - 1)}.$$

We have

$$B - C = \sum_{j=1}^{m+1} \frac{1}{(m + 1)^{j-t^{(m)(n)-1}}} \cdot \sum_{i=1}^{t^{(m)(n)-1}} \left( (m + 1)^{t^{(m)(n)-1-i}} \sigma^{(m)}(i) \right) \text{ by Equation 29}$$
$$= \frac{A}{(m + 1)^{t^{(m)(n)-1} - 1}} - D. \quad (30)$$

Therefore, we have

$$\epsilon_m = \frac{\beta^{(m)}(t^{(m)}(n)) - 1}{(m + 1)t^{(m)(n)-1} - 1}$$
$$= \epsilon_m - \frac{A + (m + 1)^{t^{(m)(n)-1} - 1}}{(m + 1)^{t^{(m)(n)-1} - 1}}$$
$$= \sum_{i=1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} \cdot \frac{A}{(m + 1)^{t^{(m)(n)-1} - 1}} \text{ by Corollary 7.3}$$
$$= \sum_{i=(m+1):(t^{(m)}(n)-1)+1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} + C - D \text{ by Equation 30.} \quad (31)$$

Therefore, we see that
\[
\epsilon_m - \frac{\beta^{(m)}(t^{(m)}(n)) - 1}{(m + 1)t^{(m)(n)-1} - 1} \\
= \sum_{i=(m+1)-t^{(m)}(n)-1+1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} + C - D \quad \text{by Equation 31} \\
< \sum_{i=(m+1)-t^{(m)}(n)-1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} \quad \text{by definition of } C
\]

This proves the second inequality.

It follows from Corollary 7.3 that \( A < (m + 1)^{t^{(m)(n)-1}} \cdot (\epsilon_m - 1) \). Since \( m + 1 \geq 2 \) and \( t^{(m)}(n) - 1 \geq t^{(m)}(m + 2) - 1 \geq 2 \), it follows that \( A < ((m + 1)^{t^{(m)(n)-1}} - 1) \cdot \frac{4}{3}(\epsilon_m - 1) \).

Now by Lemma 7.5, we see that \( A < ((m + 1)^{t^{(m)(n)-1}} - 1) \cdot 4 \). So we obtain the following:

\[
C \geq \frac{\beta^{(m+1)}(t^{(m+1)}(n))}{(m + 1)(m+1)-t^{(m)(n)-1}} \\
\geq 4 \quad \text{since } n \geq m + 2 \\
\geq A \\
\geq (m + 1)^{(m+1)-t^{(m)(n)-1}} \cdot ((m + 1)^{t^{(m)(n)-1}} - 1) \\
= D.
\]

Now we see that

\[
\epsilon_m - \frac{\beta^{(m)}(t^{(m)}(n)) - 1}{(m + 1)t^{(m)(n)-1} - 1} \\
= \sum_{i=(m+1)-t^{(m)}(n)-1+1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} + C - D \quad \text{by Equation 31} \\
> \sum_{i=(m+1)-t^{(m)}(n)-1+1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} \quad \text{since } C > D \\
> 0.
\]

This proves the first inequality.

Remark 7.8. Let \( t := t^{(m)}(n) - 1 \). The first inequality of Lemma 7.7 is equivalent to

\[
0 < 1 + \sum_{i=1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} - \frac{(m + 1)^t + (m + 1)^t \cdot \sum_{i=1}^{t} \frac{\sigma^{(m)}(i)}{(m + 1)^i}}{(m + 1)^t - 1} \\
= \sum_{i=t+1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} - \frac{\sum_{i=1}^{t} \frac{\sigma^{(m)}(i)}{(m + 1)^i}}{(m + 1)^t - 1}
\]
Multiplying by \((m + 1)^t - 1\), this is equivalent to

\[
0 < (m + 1)^t \sum_{i=t+1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i} - \sum_{i=1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i},
\]

so

\[
\sum_{i=1}^{\infty} \frac{\sigma^{(m)}(i + t)}{(m + 1)^i} > \sum_{i=1}^{\infty} \frac{\sigma^{(m)}(i)}{(m + 1)^i}.
\]

When we replace \(>\) by \(\geq\), it turns out that this inequality holds for all integers \(t\). This follows from the inequality \(\tau^{(m)}(u + v + 1) \geq \tau^{(m)}(u + 1) + \tau^{(m)}(v + 1)\). Since our proof for this is quite long, and since we do not need it for our results, we omit it here.

Now we can prove the last building block for our tower of exponents:

**Proposition 7.9.** For \(m \in \mathbb{Z}_{\geq 1}\) and \(n \geq m + 2\) with \((m, n) \neq (1, 3)\), we have

\[
\beta^{(m)}(t^{(m)}(n)) = [1 - \epsilon_m + \epsilon_m \cdot (m + 1)^{t^{(m)}(n) - 1}].
\]

**Proof.** Multiplying Lemma 7.7 by \((m + 1)^{t^{(m)}(n) - 1} - 1\), we obtain:

\[
0 < 1 - \epsilon_m + \epsilon_m \cdot (m + 1)^{t^{(m)}(n) - 1} - \beta^{(m)}(t^{(m)}(n))
\]

\[
< ((m + 1)^{t^{(m)}(n) - 1} - 1) \cdot \sum_{i=(m+1):(t^{(m)}(n)-1)} \frac{\sigma^{(m)}(i)}{(m + 1)^i}.
\]

By Lemmas 7.4 (a) and 7.5, this is smaller than

\[
3.5 \cdot (m + 1)^{t^{(m)}(n) - 1} \cdot \sum_{i=(m+1):(t^{(m)}(n)-1)} \frac{(m + 2) \cdot (2i - 1)^{\log_{m+1}(m+2)}}{(m + 1)^i}.
\]

When \(m \geq 3\), we have \(t^{(m)}(n) \geq t^{(m)}(m + 2) = m + 2\). Now the value of Expression 33 is smaller than 1. When \(m = 1, 2\) and \(n \geq 5\), we have \(t^{(m)}(n) \geq 80\), and again the value of Expression 33 is smaller than 1. We omit the proof for this; however, see Algorithm [4].

Now it remains to check that the proposition holds for \((m, n) = (1, 4), (2, 4)\). With a proof similar to Lemma 7.5, it can be shown that \(\epsilon_1 < 3.49\) and \(\epsilon_2 < 1.6\). Therefore, \(1 - \epsilon_1 + \epsilon_1 \cdot 64 < 221\) and \(1 - \epsilon_2 + \epsilon_2 \cdot 27 < 43\). So since \(t^{(1)}(4) = 7, t^{(2)}(4) = 4, \beta^{(1)}(7) = 220, \beta^{(2)}(4) = 42\), we see that \(1 - \epsilon_m + \epsilon_m \cdot (m + 1)^{t^{(m)}(n) - 1} - \beta^{(m)}(t^{(m)}(n)) < 1\) in all cases.

Since \(\beta^{(m)}(t^{(m)}(n))\) is an integer, this proves the proposition.

\[\Box\]

**Remark 7.10.** The value that is not covered in Proposition 7.9 is: \(\beta^{(1)}(t^{(1)}(3)) = 9\) while \([1 + \epsilon_1 + \epsilon_1 \cdot 2^{t^{(1)}(3) - 1}] = 11\).

**Theorem 7.11.** For all integers \(m, n\) with \(1 \leq m \leq n - 2\) and \((m, n) \neq (1, 3)\), we have

\[
\phi^{(m)}(n) = [1 - \epsilon_m + \epsilon_m \cdot (m + 1)^{\nu_m \cdot (m+1)^{\nu_{m+1} \cdot (m+2)^{\nu_{m+2} \cdot \ldots_{\nu_{n-2} \cdot (n-1)^{\ldots}}}}}].
\]
\textbf{Proof.} This follows from Theorem 6.20 combined with Proposition 7.9, using that 
\( \phi^m(n) = \beta^m(t^m(n)) \) by Lemma 1.2 (a).

\textbf{Remark 7.12.} The values that are excluded in the theorem are: \( \phi^m(m) = 1 \) and \( \phi^m(m+1) = m+2 \) for all \( m \geq 1 \), and \( \phi^1(3) = 9 \).

\textbf{Theorem 7.13.} For all integers \( m \geq 1 \), the number \( \epsilon_m \) has irrationality measure at least \( m+1 \).

\textbf{Proof.} Let \( m \geq 1 \). For all \( n \geq m+2 \) with \( (m, n) \neq (1, 3) \), we define \( p_n := \beta^m(t^m(n)) - 1 \) and \( q_n := (m+1)^{t^m(n)-1} - 1 \). Now we obtain the following by Lemma 7.7:

\[
0 < \left| \epsilon_m - \frac{p_n}{q_n} \right| < \sum_{i=(m+1)\cdot(t^m(n)-1)}^{\infty} \frac{\sigma^{(m)}(i)}{(m+1)^i}
= O\left( \frac{(m+1)\cdot(t^m(n)-1)\log_{m+1}(m+2)}{(m+1)^{(m+1)\cdot(t^m(n)-1)-1}} \right) \quad \text{by Lemma 7.4 (b)}
= O\left( \frac{1}{((m+1)^{t^m(n)-1} - 1)^{m+1-\epsilon}} \right) \quad \text{for all } \epsilon > 0.
\]

We see that \( 0 < \left| \epsilon_m - \frac{p_n}{q_n} \right| = O\left( \frac{1}{q_n^{m+1-\epsilon}} \right) \) as \( n \to \infty \). Also, we have \( q_n \to \infty \) as \( n \to \infty \). Therefore, \( \mu(\epsilon_m) \geq m+1 \).

\textbf{Remark 7.14.} It follows from Lemma A.3 that \( \epsilon_m \) is transcendental for \( m \geq 2 \), and from Lemma A.2 that \( \epsilon_1 \) is irrational.

\textbf{Remark 7.15.} We used the same estimates for proving Proposition 7.9 and Theorem 7.13: both use Lemma 7.7.

We have computed the values of \( \epsilon_m \) for \( m \in [1, 10] \) up to 20 decimals; see Table 11. We used Corollary 7.3. The code can be found at [4].
Table 11: Values of $\epsilon_m$. See Entry A357067 and A357068 of the OEIS.

8 Further results

In this section, we prove some further results on the Gijswijt sequences. The three subsections are independent from each other.

8.1 Glue and the rho function

Here, we shall prove the conjectures from van de Bult et al. [1, pp. 14–19] on the glue strings $S_t^{(m)}$. The key to proving these conjectures is a new definition of $P$ strings (Definition 8.1). Also, we shall need the definition of the iota functions and the sets $V_m$.

Since we have already proven a lot of interesting estimates, we will not prove the estimates from Conjecture 4.3 of van de Bult et al. [1]; instead we focus on the qualitative results.

Consider the glue strings $S_t^{(1)}$ from Table 1.1. The distinct strings that occur are 2, 223, 2232, 223222332, 22322233222322232223334. We ignore the string 2232 for a moment, since it is similar to 223. We can denote the remaining strings as follows:

$$2, 2B_1^{(2)}S_1^{(2)}, 2B_1^{(2)}S_1^{(2)}B_2^{(2)}S_2^{(2)}, 2B_1^{(2)}S_1^{(2)}B_2^{(2)}S_2^{(2)}B_3^{(2)}S_3^{(2)}.$$  

This suggests the following definition:

**Definition 8.1.** For $m, t \in \mathbb{Z}_{\geq 1}$, define $P_t^{(m)}$ as the string

$$P_t^{(m)} := mB_1^{(m)}S_1^{(m)}B_2^{(m)}S_2^{(m)} \ldots B_{t-1}^{(m)}S_{t-1}^{(m)}.$$  

We will prove in Theorem 8.3 that all glue strings $S_t^{(m)}$ are either of the form $P_u^{(m+1)}$ or of the form $P_u^{(m+1)} \cdot (m + 1)$.

Note that all strings $P_t^{(m)}$ are prefixes of the infinite sequence

$$mB_1^{(m)}S_1^{(m)}B_2^{(m)}S_2^{(m)}B_3^{(m)}S_3^{(m)} \ldots.$$  

In Table 12, some of the values of $P$ are shown.
For \( m = 1 \) we have \( P_t^{(1)} = B_t^{(1)} \), but those strings do not occur as glue strings since there is no \( A^{(0)} \).

The following lemma makes the connection between the strings \( B_t^{(m)} \) and the strings \( P_t^{(m)} \) for \( m \geq 2 \).

**Lemma 8.2.** For all integers \( m \geq 2 \) and \( a \geq 1 \) we have

\[
\prod_{1 \leq n \leq a} \left( P_{\text{ord}_m(n)+1}^{(m)} \right) = (B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1},
\]

where \( a_x \ldots a_1 \) is the \( m \)-ary expansion of \( a \).

**Proof.** Fix \( m \geq 2 \). We shall prove the statement by induction on \( a \). For \( a = 1 \), both strings equal \( m \). Now suppose that the lemma holds for \( a = k \). Let \( a_x \ldots a_1 \) and \( b_x \ldots b_1 \) be the \( m \)-ary expansions of \( k \) and \( k+1 \), respectively. (In the special case that \( k+1 \) is a power of \( m \), we define \( a_x := 0 \).) Let \( d := \text{ord}_m(k+1) \). Now we know that \( b_1 = b_2 = \cdots = b_d = 0 \), and \( a_1 = a_2 = \cdots = a_d = m - 1 \). Furthermore, we have \( b_{d+1} = a_{d+1} + 1 \) and \( b_i = a_i \) for \( i > d + 1 \). Now it follows that

\[
\prod_{1 \leq n \leq k+1} \left( P_{\text{ord}_m(n)+1}^{(m)} \right) = \prod_{1 \leq n \leq k} \left( P_{\text{ord}_m(n)+1}^{(m)} \right) \cdot P_{d+1}^{(m)}
\]

\[
= (B_x^{(m)})^{a_x} \cdots (B_1^{(m)})^{a_1} m B_1^{(m)} S_1^{(m)} \cdots B_d^{(m)} S_d^{(m)} \quad \text{induction hypothesis and Definition 8.1}
\]

\[
= (B_x^{(m)})^{a_x} \cdots (B_{d+2}^{(m)})^{a_{d+2}} (B_{d+1}^{(m)})^{a_{d+1}} (B_d^{(m)})^{m-1} \cdots (B_1^{(m)})^{m-1} B_1^{(m)} B_2^{(m)} S_1^{(m)} \cdots B_d^{(m)} S_d^{(m)}
\]

\[
= (B_x^{(m)})^{b_x} \cdots (B_{d+2}^{(m)})^{b_{d+2}} (B_{d+1}^{(m)})^{b_{d+1}+1} (B_d^{(m)})^{m-1} (B_2^{(m)})^{m-1} (B_2^{(m)})^{m-1} B_2^{(m)} S_2^{(m)} \cdots B_d^{(m)} S_d^{(m)}
\]

\[
= \cdots
\]

\[
= \prod_{1 \leq i \leq a} P_{\text{ord}_m(i)+1}^{(m)}
\]

This completes the induction step. \( \square \)

Combining Lemma 8.2 with Lemma 3.4, we obtain that

\[
\pi^{(m)}(1, \chi^{(m)}(a)) = \prod_{1 \leq i \leq a} P_{\text{ord}_m(i)+1}^{(m)}.
\]
Letting \( a = \tau_m(t) \) and using Definition 4.6, we obtain the following:

\[
T_t^{(m)} = \pi^{(m+1)}(1, \chi^{(m+1)}(\tau_m(t))) = \prod_{1 \leq i \leq \tau_m(t)} P_{\text{ord}_{m+1}(i)+1}^{(m+1)}.
\] (34)

Using this, we can characterize the glue strings \( S_t^{(m-1)} \):

**Theorem 8.3.** For all \( m, t \geq 1 \), there is a \( u \geq 1 \) such that either \( S_t^{(m)} = P_u^{(m+1)} \) or \( S_t^{(m)} = P_u^{(m+1)} \cdot (m + 1) = P_u^{(m+1)} \cdot 1^{(m+1)} \), and in the last case we always have \( u \geq 2 \). Moreover, when we fix \( m \geq 1 \) and write the \( u \)'s corresponding to \( S_t^{(m)} \), \( S_{t+1}^{(m)} \), . . . in a sequence, we see that \( S_t^{(m)} = P_u^{(m+1)} \cdot 1^{(m+1)} \), then we obtain the ruler sequence \( r_{m+1} + 1 \).

**Proof.** For all \( t \geq 1 \) we have \( T_t^{(m)} = T_t^{(m)} = S_t^{(m)} \). Using Equation 34 we see that \( S_t^{(m)} = \prod_{t+1 \leq i \leq \tau_m(t+1)} a_{\text{ord}_{m+1}(i)+1}^{(m+1)} \). Using Lemma 4.8 (a) and (b) we see that either \( \tau_m(t+1) = \tau_m(t) + 1 \), or \( \tau_m(t+1) = \tau_m(t) + 2 \), and in the second case we have \( m + 1 \mid \tau_m(t) + 1 \), hence \( m + 1 \mid \tau_m(t) + 2 \). Now the first part of the theorem follows when we define \( u := \text{ord}_{m+1}(\tau_m(t) + 1) + 1 \).

This theorem proves a number of conjectures from van de Bult et al. [1, pp. 14–19]. There, the lengths \( \sigma(m)(1), \sigma(m)(2), \ldots \) of the glue strings \( S_t^{(m)} \) were studied. As we see from the Theorem, these are all equal to either \( l(P_u^{(m+1)}) \) or \( l(P_u^{(m+1)}) + 1 \) for some \( u \). In van de Bult et al. [1, p. 16], the number \( l(P_u^{(m+1)}) \) is essentially denoted by \( \rho^{(m)}(u-1) \), and van de Bult et al. conjectured [1, Conjecture 4.2] that \( \rho^{(m)}(0) = 1 \) and \( \rho^{(m)}(n+1) = \rho^{(m)}(n) + \beta^{(m+1)}(n+1) + \sigma^{(m+1)}(n+1) \) for all \( n \in \mathbb{Z}_{\geq 0} \). This follows directly from our definition of \( P_1^{(m+1)} \), since \( P_1^{(m+1)} = m + 1 \) and \( P_{n+1}^{(m+1)} = P_n^{(m+1)} B_n^{(m+1)} S_n^{(m+1)} \) for all \( n \in \mathbb{Z}_{\geq 1} \).

Also, van de Bult et al. conjectured [1, Conjecture 4.1] for \( m = 1 \) that replacing (‘smoothing’) the values of \( \sigma(m)(a) \) that equal a \( \rho^{(m)}(n) + 1 \) by the two numbers \( \rho^{(m)}(n) \) and \( 1 \), transforms the sequence \( \sigma^{(m)}(1), \sigma^{(m)}(2), \ldots \) into the ruler sequence \( (\rho^{(m)} \circ r_{m+1})(x) \) (see Entry A091839 in the OEIS [8]). This follows from the second part of our theorem.

Note that the values of \( l(P_u^{(2)}) \) are given by Entry A091588 in the OEIS.

Van de Bult et al. [1, p. 16] observed that the glue strings \( S_t^{(m)} \) for \( m \geq 2 \), are in some way simpler than for \( m = 1 \). Recall from Theorem 8.3 that each glue string \( S_t^{(m)} \) either equals \( P_u^{(m+1)} \) for some \( u \geq 1 \), or \( P_u^{(m+1)} \cdot (m + 1) \) for some \( u \geq 1 \). We actually have the following for \( m \geq 2 \): for each \( u \), only one of the two strings \( P_u^{(m+1)} \) and \( P_u^{(m+1)} \cdot (m + 1) \) can occur as a glue string \( S_t^{(m)} \). This follows from Definition 6.3: if \( u - 1 \in V_m \), then all \( t \) with \( \text{ord}_{m+1}(t + 1) = u \) are in the image of \( \tau_m \), so \( P_u^{(m+1)} \cdot (m + 1) \) can not occur as a glue string. On the other hand, if \( u - 1 \notin V_m \), then all \( t \) with \( \text{ord}_{m+1}(t + 1) = u \) are not in the image of \( \tau_m \), hence \( P_u^{(m+1)} \) can not occur as a glue string.
This is not the case for \( m = 1 \). For example, we have \( S_2^{(1)} = 223 = P_2^{(2)} \) and \( S_5^{(1)} = 2232 = P_2^{(2)} P_1^{(2)} \).

### 8.2 The first 5 and 44 in Gijswijt’s sequence

Here, we use the theory from the previous sections to find the position of the first 5 in Gijswijt’s sequence. From Proposition 6.13 (b), we know that

\[
\iota_1^{-1}(2^p) = 1 + \sum_{a \in Q, a \leq p} + \sum_{(a,b) \in R, b \leq p-1} 2^{p-b-1}.
\]

The first elements of the sets \( Q \) and \( R \) can easily be computed. Since we have to calculate the elements up to \( x \), as opposed to \( 2^x \), this allows us to calculate inverse values of \( \iota_1 \) for very high inputs. We obtain by direct calculation (see Algorithm [6]) that

\[
\iota_1^{-1}(2^{79}) = 1 + 418090195952691922788353.
\]

Since \( t^{(2)}(5) = 80 \) (which we saw in the introduction), it follows from Equation 9 that

\[
t^{(1)}(5) = \iota_1^{-1}(2^{79}) = 1 + 418090195952691922788353.
\]

Now we obtain by Proposition 7.9 that

\[
\phi^{(1)}(5) = \beta^{(1)}(t^{(1)}(5)) = \lfloor 1 - \epsilon_1 + \epsilon_1 \cdot 2^{418090195952691922788353} \rfloor.
\]

This shows that the estimate \( \phi^{(1)}(5) \approx \epsilon_1 \cdot 2^{418090195952691922788353} \) from van de Bult et al. [1, p. 23] is very accurate.

We can also calculate where the first two consecutive 4’s occur in Gijswijt’s sequence. For this, we use the definition of the \( \pi \) strings.

The first \( T^{(3)}_1 \) that contains 44 is \( T^{(3)}_3 \), which equals 44. Now \( \pi^{(3)}(1, 4^{3-1}) = \pi^{(3)}(1, \chi^{(3)}(9)) \). Since \( \iota_2(10) = 9 \), the first \( T^{(2)}_1 \) that contains 44 is \( T^{(2)}_10 \), and it occurs at the end. Now \( \pi^{(2)}(1, 3^{10-1}) = \pi^{(2)}(1, \chi^{(2)}(512)) \) and \( \iota_1(355) = 511 \) while \( \iota_1(356) = 513 \). Therefore, \( T^{(1)}_{355} \) does not contain a substring 44, but \( T^{(1)}_{356} \) contains exactly 1 substring 44. This string ends with 442. From this it follows that the first occurrence of 44 in Gijswijt’s sequence is at places \( [\beta^{(1)}(356) - 2, \beta^{(1)}(356) - 1] \). We can calculate the value of \( \beta^{(1)}(356) \) using Algorithm [5] (by changing the first parameter to 513). As a result, the first occurrence of 44 in Gijswijt’s sequence is at places

\[ [x - 2, x - 1] \]

with \( x = 255 895 648 634 818 208 370 064 452 304 769 558 261 700 170 817 472 823 398 081 655 524 438 021 806 620 809 813 295 008 281 436 789 493 636 146 \).
8.3 The density of $n$ in $A^{(m)}$

Using the results from Section 7, we can prove that each natural number $n \geq m$ has an asymptotic density in $A^{(m)}$. For Gijswijt’s sequence, this was conjectured in a comment at Entry A090822 of the OEIS [8]. After proving this in Theorem 8.6, we shall also show that $A^{(m)}$ has a mean value.

We first need the following lemma.

Lemma 8.4. For all $m$, $\tau^{(m)}(n)$ grows polynomially with $n$.

Proof. The $\sigma^{(m)}(n)$ in Inequality 26 can be replaced by $\tau^{(m)}(n+1)$, as can be seen in the derivation immediately above Inequality 26. Now using Inequality 27, we obtain that

$$
\tau^{(m)}(n+1) \leq \beta^{(m+1)}([\log_{m+1}(2n-1)] + 1) \leq \epsilon_{m+1} \cdot (m+2)^{[\log_{m+1}(2n-1)]}.
$$

This is smaller than

$$
\epsilon_{m+1} \cdot (m+2)^{2\log_{m+2}(2n-1)+1} = O(n^2).
$$

Recall the following definition:

Definition 8.5. For a string $S$ and a number $n$, let $\#(n \in S)$ be the number of occurrences of $n$ in $S$.

Now we are ready to prove the theorem:

Theorem 8.6. For all $m \geq 1$ and $n \geq m$, the set

$$
\{ p \in \mathbb{Z}_{\geq 1} \mid A^{(m)}(p) = n \}
$$

has asymptotic density equal to some real number $\delta^{(m)}(n) > 0$.

Proof. We first prove the following claim:

Claim 8.6.1. The limit

$$
\delta := \lim_{t \to \infty} \frac{\#(n \in B^{(m)}_t)}{\beta^{(m)}(t)}
$$

exists and is positive.

Proof. Observe that for all $t$, $\#(n \in B^{(m)}_{t+1}) \geq (m+1) \cdot \#(n \in B^{(m)}_t)$. Therefore, we obtain that

$$
\frac{\#(n \in B^{(m)}_u)}{(m+1)^{u-1}} \geq \frac{\#(n \in B^{(m)}_t)}{(m+1)^{t-1}}
$$

whenever $u \geq t$. Since $A^{(m)}$ contains all $n \geq m$, it follows that $\frac{\#(n \in B^{(m)}_u)}{(m+1)^{u-1}} > 0$ for some $t$. We see that the limit $\lim_{t \to \infty} \frac{\#(n \in B^{(m)}_t)}{\beta^{(m)}(t)}$ exists and is positive. Using Theorem 7.2, we deduce that

$$
\lim_{t \to \infty} \frac{\#(n \in B^{(m)}_t)}{\beta^{(m)}(t)}
$$

exists as well and is positive. \qed
Since $A^{(m)} = \pi^{(m)}(1, \infty)$, we can write every nonempty prefix $P$ of $A^{(m)}$ as $P = \pi^{(m)}(1, x) \cdot m \cdot Q$, where $Q$ is a prefix of $T_{\text{ord}_{m+1}(x+1)+1}^{(m)}$. By Lemma 3.4, this equals

$$(B_t^{(m)})^{\alpha_t} (B_{t-1}^{(m)})^{\alpha_{t-1}} \cdots (B_1^{(m)})^{\alpha_1} \cdot m \cdot Q$$

for some integers $\alpha_1, \ldots, \alpha_t \in \{0, 1, \ldots, m\}$ with $\alpha_t \neq 0$. When we let $l(P)$ go to infinity, the variable $t$ also goes to infinity. Now we have

$$\#(n \in m \cdot Q) + \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} \alpha_i \cdot \#(n \in B_i^{(m)})$$

$$\leq 1 + \tau^{(m)}(\text{ord}_{m+1}(x+1) + 1) + \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} \alpha_i \cdot \beta^{(m)}(i)$$

$$\leq 1 + \tau^{(m)}(t + 1) + \beta^{(m)}(1 + \left\lfloor \frac{t}{2} \right\rfloor)$$

$$= o(\beta^{(m)}(t))$$

by Lemma 8.4 and Theorem 7.2

$$= o(\#(n \in B_i^{(m)}))$$

by Claim 8.6.1.

Therefore, we see that

$$\frac{\#(n \in P)}{l(P)} \sim \frac{\#(n \in (B_t^{(m)})^{m} \cdots (B_{\lfloor \frac{t}{2} \rfloor + 1})^{\alpha_{\lfloor \frac{t}{2} \rfloor + 1})}{l ((B_t^{(m)})^{\alpha_t} \cdots (B_{\lfloor \frac{t}{2} \rfloor + 1})^{\alpha_{\lfloor \frac{t}{2} \rfloor + 1})}$$

as $l(P) \to \infty$. Now observe that $t \to \infty$ implies that $\lfloor \frac{t}{2} \rfloor + 1 \to \infty$. So we can use Claim 8.6.1 to conclude that $\frac{\#(n \in P)}{l(P)}$ converges to $\delta$ as well. 

We now prove some basic properties of the constants $\delta^{(m)}(n)$:

**Proposition 8.7.** For all $m$, we have

$$\delta^{(m)}(m) = \frac{1}{\epsilon_m}.$$  

Furthermore, let $n > m$. Then we have

$$\delta^{(m)}(n) = \frac{1}{\epsilon_m} \cdot \sum_{i=1}^{\infty} \frac{\#(n \in S_i^{(m)})}{(m+1)^i}.$$  

**Proof.** The first statement follows from $\delta^{(m)}(m) = \lim_{t \to \infty} \frac{(m+1)^{t-1}}{\beta^{(m)}(t)}$ while

$$\epsilon_m = \lim_{t \to \infty} \frac{\beta^{(m)}(t)}{(m+1)^t}.$$  

For $n > m$, we know that $B_t^{(m)}$ consists of $m$’s and copies of $S_1^{(m)}, \ldots, S_{t-1}^{(m)}$. Here $S_i^{(m)}$ occurs exactly $(m+1)^{t-i-1}$ times. Therefore, we have $\#(n \in B_{t}^{(m)}) = \sum_{i=1}^{t-1} (m+1)^{t-i-1}$. $\#(n \in S_i^{(m)})$. It follows that $\frac{\#(n \in B_{t}^{(m)})}{(m+1)^{t-1}} = \sum_{i=1}^{t-1} \frac{\#(n \in S_i^{(m)})}{(m+1)^i}$. Now the statement follows when we multiply by $\frac{(m+1)^{t-1}}{\beta^{(m)}(t)}$ and take the limit $t \to \infty$. 

57
To prove that $A^{(m)}$ has a mean value, we need the following lemma.

**Lemma 8.8.** Fix $m$. There are constants $c_m, c_{m+1}, \ldots > 0$ with $\sum_{n=m}^{\infty} nc_n < \infty$ such that for all $n \geq m$ and $t \geq 1$, we have:

$$\#(n \in A^{(m)}[1, t]) \leq c_n \cdot t.$$  

**Proof.** Define $c_m := 1$. Now take $n > m$. The strings $T^{(m)}_1, \ldots, T^{(m)}_{\pi^{(m)}(n)-1}(n)$ do not contain an $n$. We know that $A^{(m)} = \pi^{(m)}(1, \infty) = \prod_{i=1}^{\infty} m \cdot T^{(m)}_{\pi^{(m)}(i)+1}$. Let $A'$ be the sequence obtained as follows: replace every $T^{(m)}_{\pi^{(m)}(i)+1}(i)$ where $(m+1)^{\pi^{(m)}(n)}$ divides $i$, by the string consisting of $n$'s with the same length. Now the density of $n$ in $A'[1, t]$ (that is, the number of occurrences divided by the length of the string) is at least a large as the density in $A^{(m)}[1, t]$. Also, the largest possible densities occur at the end of the $\pi'(1, u)$ blocks, with $u$ an integer. Since this string is builded from $B'$ blocks (see Lemma 3.4), we can in turn pick the $B'$ block $B_v'$ that is used that has the largest density of $n$. This density is smaller than

$$\frac{1}{\beta^{(m)}(v)} \cdot \sum_{i=\pi^{(m)}(n)}^{v} \frac{\tau^{(m)}(i) \cdot (m+1)^{v-i}}{(m+1)^{v-i-1}}\leq \sum_{i=\pi^{(m)}(n)}^{\infty} \frac{\tau^{(m)}(i)}{(m+1)^{i-1}} := c_n.$$  

Now we have

$$\sum_{n=m}^{\infty} nc_n = c_m + \sum_{i=m+1}^{\infty} \frac{(i-m) \cdot \tau^{(m)}(i)}{(m+1)^{i-1}}.$$  

By Lemma 8.4, the numerators grow polynomially with $i$, while the denominators grow exponentially with $i$. Therefore, this sum is finite. 

**Theorem 8.9.** We have $\sum_{n=m}^{\infty} \delta^{(m)}(n) = 1$. Also, $A^{(m)}$ has a mean value equal to $\sum_{n=m}^{\infty} n \cdot \delta^{(m)}(n)$.  

**Proof.** Let $c_m, c_{m+1}, \ldots$ be as in Lemma 8.8. For all $x, t$, we have:

$$\sum_{n=m}^{x} \frac{1}{t} \cdot \#(n \in A^{(m)}[1, t]) \in [1 - \sum_{n=x+1}^{\infty} c_x, 1].$$  

Taking the limits $t \to \infty$ and then $x \to \infty$ proves the first part of the theorem.
Also, we have for all $x$ and $t$:

$$
\frac{1}{t} \sum_{i=1}^{t} A^{(m)}(i) \in \left[ \frac{1}{t} \sum_{n=m}^{x} n \cdot \#(n \in A^{(m)}[1, t]), \frac{1}{t} \sum_{n=m}^{x} n \cdot \#(n \in A^{(m)}[1, t]) + \sum_{n=x+1}^{\infty} nc_n \right].
$$

Therefore,

$$
\limsup_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} A^{(m)}(i) \leq \sum_{n=m}^{x} n \cdot \delta^{(m)}(n) + \sum_{n=x+1}^{\infty} nc_n
$$

and

$$
\liminf_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} A^{(m)}(i) \geq \sum_{n=m}^{x} n \cdot \delta^{(m)}(n).
$$

Taking the limit $x \to \infty$, we conclude that

$$
\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} A^{(m)}(i) = \sum_{n=m}^{\infty} n \cdot \delta^{(m)}(n).
$$

This proves the second part of the theorem. \qed
9 Open questions

In Definition 6.1 and Theorem 7.2, we have defined the constants $\nu_m$ and $\epsilon_m$ for $m \geq 1$, and we proved in Theorems 6.8 and 7.13 that those numbers are transcendental for $m \geq 2$. The irrationality measures of $\nu_m$ and $\epsilon_m$ are at least $m + 1$. We have also proved in Theorems 6.21 and 7.13 that $\nu_1$ and $\epsilon_1$ are irrational with irrationality measures at least 2. This leaves us with the following questions:

Question 9.1. Are $\nu_1$ and $\epsilon_1$ transcendental?

Question 9.2. For $m \geq 1$, what are the irrationality measures of $\nu_m$ and $\epsilon_m$?

Maybe Theorems 5.4 and 5.10 could be useful for future research: there, the iota functions (which contain all information of the Gijswijt sequences) are derived from each other without calculating a curling number.

We could also ask which number stands on the googolplexth position of Gijswijt’s sequence:

Question 9.3. What is the value of $A^{(1)}(10^{10^{100}})$?

A calculation using the values of $\sigma^{(1)}(i)$ and $\beta^{(1)}(i)$ for

$$i = 1, 2, \ldots, \log_2(10) \cdot 10^{100}$$

would involve a number of calculations on the order of $10^{100}$, which is impossible in practice. We do not know whether our new results can lead to a faster method of calculation, whether new insights are needed, or whether it is not possible at all to calculate $A^{(1)}(10^{10^{100}})$ in any reasonable amount of time.

However, we can prove that $A^{(1)}(10^{10^{100}})$ is at most 5:

Lemma 9.4. We have $A^{(1)}(10^{10^{100}}) \leq 5$.

Proof. We shall prove that $\phi^{(1)}(6) > 10^{10^{100}}$, from which the lemma follows. For this, we use Theorem 7.11:

$$\phi^{(1)}(6) = [1 - \epsilon_1 + \epsilon_1 \cdot 2^{[\nu_1 - 2^{\nu_2 - 3^{\nu_3 - 4^{\nu_4 - 5}}]}]}.$$  

Using the values of $\epsilon_1, \nu_1, \nu_2, \nu_3, \nu_4$ which we calculated in Tables 8 and 11, we see that this is larger than

$$-4 + 2^{0.6 \cdot 2^{0.9 \cdot 3^{0.9 \cdot 4}}},$$

which is in turn larger than

$$2^{2^{3^{100}}},$$

which is larger than $10^{10^{100}}$.

Hence the value of $A^{(1)}(10^{10^{100}})$ is 1, 2, 3, 4 or 5.

Finally, we could try to generalize our results on the irrationality measures of $\epsilon_m$ to the constants $\delta^{(m)}(n)$. By Proposition 8.7, we know that $\delta^{(m)}(m) = \frac{1}{\epsilon_m}$, so $\delta^{(m)}(m)$ and $\epsilon_m$ have the same irrationality measure. For $n > m$, this is less trivial.
Question 9.5. For $n > m$, what is the irrationality measure of $\delta^{(m)}(n)$?

In Lemma 7.7, we approximated $\epsilon_m$ by $\frac{\beta^{(m)}(t^{(m)}(n))}{(m+1)t^{(m)}(n)-1}$. Therefore, we obtain the approximation $\delta^{(m)}(m) \approx \frac{(m+1)t^{(m)}(n)-1}{\beta^{(m)}(t^{(m)}(n))-1}$. Note that $(m+1)t^{(m)}(n) = \#(m \in B^{(m)}_{t^{(m)}(n)})$.

This suggests that we can approximate $\delta^{(m)}(n)$ as follows:

$$\delta^{(m)}(n) \approx \frac{\#(n \in B_{t^{(m)}(n)}) - 1}{\beta^{(m)}(t^{(m)}(n))-1}.$$ 

Following this strategy, it might be possible to prove that $\mu(\delta^{(m)}(n)) \geq m + 1$.

9.1 The Curling Number Conjecture

We have proved a number of conjectures from van de Bult et al. [1]. However, they also posed another conjecture [1, Finiteness Conjecture]. In an article by Chaffin et al. [2], this conjecture is called the Curling Number Conjecture.

Conjecture 9.6 (The Curling Number Conjecture [2, Conjecture 1]). If one starts with any initial sequence of integers $S$, and extends it by repeatedly appending the curling number of the current sequence, the sequence will eventually reach 1.

The techniques in this article seem insufficient to prove this conjecture. Chaffin et al. [2] have obtained partial results on the conjecture, proving it for a subset of initial sequences. We have found new partial results, which we hope to present in another article.
A Approximation by rationals

We use the following result from Roth on approximation of algebraic numbers by rationals.

**Theorem A.1.** [Roth’s theorem [7, p. 2]] Let \( \alpha \) be any algebraic irrational number. If the equation

\[
|\alpha - \frac{h}{q}| < \frac{1}{q^\kappa}
\]

has infinitely many solutions in integers \( h \) and \( q (q > 0) \), then \( \kappa \leq 2 \).

To prove that numbers are irrational, we use the following lemma:

**Lemma A.2.** Let \( \alpha \) be a real number such that there exist integers \( a_n, b_n \) with \( b_n > 0 \) for all \( n \in \mathbb{Z}_{>1} \), for which \( \lim_{n \to \infty} b_n = \infty \), \( \alpha \neq \frac{a_n}{b_n} \) for all \( n \), and \( |\alpha - \frac{a_n}{b_n}| = o\left( \frac{1}{b_n} \right) \) as \( n \to \infty \). Then \( \alpha \) is irrational.

**Proof.** We prove this by contradiction. So suppose that \( \alpha \) is rational. Then there are integers \( a, b \) with \( b > 0 \) such that \( \alpha = \frac{a}{b} \). Now we have \( \left| \frac{a b_n - b a_n}{b b_n} \right| = o\left( \frac{1}{b_n} \right) \) as \( n \to \infty \). Since the left-hand side is unequal to zero, it follows that it is at least \( \frac{1}{b b_n} \). But since \( \frac{1}{b} \) is a constant, that expression is not \( o\left( \frac{1}{b_n} \right) \) as \( n \to \infty \). Therefore, \( \alpha \) is irrational. \( \square \)

Now we can prove that a number is transcendental in the following way:

**Lemma A.3.** Let \( \alpha \) be a real number with irrationality measure \( \mu(\alpha) > 2 \). Then \( \alpha \) is transcendental.

**Proof.** Since \( \mu(\alpha) > 2 \), there are infinitely many solutions \( p, q \) of the equation \( 0 < |\alpha - \frac{p}{q}| < \frac{1}{q^2} \). Therefore, it follows by Lemma A.2 that \( \alpha \) is irrational. Now it follows by contradiction with Theorem A.1 that \( \alpha \) is not algebraic, since otherwise we would have \( \mu(\alpha) \leq 2 \). Hence \( \alpha \) is transcendental. \( \square \)

62
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63