The $a_{3/2}$ heat kernel coefficient for oblique boundary conditions

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August 1, 2018

Abstract

We present a method for the calculation of the $a_{3/2}$ heat kernel coefficient of the heat operator trace for a partial differential operator of Laplace type on a compact Riemannian manifold with oblique boundary conditions. Using special case evaluations, restrictions are put on the general form of the coefficients, which, supplemented by conformal transformation techniques, allows the entire smeared coefficient to be determined.
1 Introduction

The general topic of heat-kernel expansions and eigenvalue asymptotics of an operator $L$ on a $D$-dimensional Riemannian manifold $\mathcal{M}$ has been an important issue for more than 20 years. In mathematics this interest stems, in particular, from the well-known connection that exists between the heat-equation and the Atiyah-Singer index theorem [1]. In physics the expansion stands out in different domains of quantum field theory, as it contains, for example, information on the scaling and divergence behaviour [2, 3, 4, 5, 6].

Of particular interest is the case where $L$ is a Laplacian-like operator. When the manifold $\mathcal{M}$ has a boundary $\partial \mathcal{M}$ one has to impose boundary conditions that guarantee the self-adjointness and the ellipticity of the operator $L$. The traditional, and simplest, conditions are Dirichlet and Neumann, and also a generalization of the latter referred to as Robin conditions. In this case, the heat-kernel coefficients $a_k$ have both a volume and a boundary part [7, 8]. Writing the expansion in its usual form

$$K(t) \sim \sum_{k=0,1/2,1,...} a_k t^{k-D/2}$$ (1)

one has the split

$$a_k = \int_\mathcal{M} dx \ b_k + \int_{\partial \mathcal{M}} dy \ c_k.$$ (2)

Here, the volume part, $b_k$, does not depend on the boundary conditions, whereas $c_k$ exhibits a nontrivial dependence on the chosen boundary condition.

Whereas the calculation of the volume part nowadays is nearly automatic [9, 10, 11], the analysis of $c_k$ is in general much more difficult. Only quite recently has the coefficient $c_2$ for Dirichlet and for Robin boundary conditions been found [12, 13, 14, 15, 16, 17, 18]. $c_{5/2}$ for manifolds with totally geodesic boundaries is given in [19].

One of the approaches is based on the functorial methods most systematically used by Branson and Gilkey [12]. Conformal transformation techniques give relations between the numerical multipliers in the heat-kernel coefficients. However, on its own, this method is unable to determine the coefficients fully. Additional information is needed, coming from other functorial relations or special case calculations [12].
In contrast to this traditional boundary conditions relatively little is known about oblique boundary conditions. These more general conditions take the form
\[ B = \nabla_N + \frac{1}{2}(\Gamma^a \tilde{\nabla}_a + \tilde{\nabla}_a \Gamma^a) - S, \] and involves tangential (covariant) derivatives, \( \tilde{\nabla}_a \), computed from the induced metric on the boundary, \( \Gamma^a \) is a bundle endomorphism valued boundary vector field and \( S \) is a hermitian bundle automorphism. Finally, \( \nabla_N \) is the outward normal derivative at the boundary. In order to ensure symmetry of the operator \( L \) together with the boundary condition
\[ BV|_{\partial M} = 0, \] on a section of some vector bundle, one has to impose \( (\Gamma^a)^\dagger = -\Gamma^a \) and \( S^\dagger = S \). This kind of boundary conditions arises naturally if one requires invariance of the boundary conditions under infinitesimal diffeomorphisms [20, 21, 22] or Becchi-Rouet-Stora-Tyutin transformations [23]. Furthermore (4) is suggested by self-adjointness theory [24, 25] and string theory [26, 27].

Although these boundary conditions have been subject of classical analysis (see e.g. [1, 28, 29, 30]) the explicit determination of the heat-kernel coefficients has hardly begun. Some of the lower coefficients (up to \( a_1 \)) have been evaluated by McAvity and Osborn [24] using their extension of the recursion method developed by De Witt for closed Riemannian manifolds. Another approach has been expounded in [31] based on the functorial methods used in [12]. In [31] the coefficients in the very special case of a flat ambient manifold with a totally geodesic, flat boundary have been computed. However, the combined knowledge of the coefficient with a totally geodesic, flat boundary and the functorial method relations does not allow for the determination of the full (under certain assumptions, see section II) \( a_{3/2} \) coefficient. The reason behind this is that no knowledge of the extrinsic curvature terms at the boundary is obtained via this example. This hinders the determination of all the universal constants.

It is here that the approach of special case evaluation on manifolds with non-vanishing extrinsic curvature becomes important. A specific class of manifolds with this property is the generalized cone, a particular curved manifold whose boundary is not geodesically embedded. For this kind of
manifold techniques have been developed earlier for the evaluation of heat-kernel coefficients and functional determinants [32] (see also [33, 34, 35, 36, 37]) and have been applied to Dirichlet and Robin boundary conditions. Recently the generalisation to smeared heat-kernel coefficients was found [38].

This generalisation turns out to be important because functorial techniques (apart from other things) yield relations between the smeared and non-smeared case. The information one can get on the “smeared side” (these are terms containing normal derivatives of the smearing function $f$) is crucial to find the full “non-smeared” side (even present for $f = 1$). This has been demonstrated by determining the full $a_{5/2}$ coefficient containing the whole group of extrinsic curvature terms [39, 41].

When employing the formalism of [32, 33] to oblique boundary conditions, the tangential derivatives in (3) cause added complication. For the case of the 4 dimensional ball these have been overcome in [39] and the coefficients up to $a_2$ were found for constant $\Gamma^i$ (for new considerations on this specific manifold see [40]). Here we will continue this analysis by generalizing it to the arbitrary dimensional as well as to the smeared case. As a result enough universal constants will emerge from special cases in order to find the full $a_{3/2}$ coefficient for oblique boundary conditions.

The organisation of the paper is as follows. In the next section we state in detail under which assumptions we are going to determine the $a_{3/2}$ coefficient (purely Abelian problem, covariantly constant $\Gamma^i$). For this situation the general form of the coefficients has been stated in [34] and we explain which universal constants and which relations among them can be obtained from the generalized cone. In section 3 the explicit calculation on the cone is performed and the information predicted found. Having this information at hand the functorial techniques [12] are applied to this boundary condition [41]. To obtain as much information as possible even for the case of covariantly constant $\Gamma^i$ the structure and conformal properties of the non-covariantly constant $\Gamma^i$ will be needed and displayed. At the end, we will see that one relation is missing which will be obtained in section 5 by dealing with the manifold $B^2 \times T^{d-1}$ ($B^2$ being the two dimensional ball and $T^{d-1}$ the $(d - 1)$-dimensional torus). The conclusions summarize the main ideas and give further possible applications.
2 Restrictions from the generalized cone

Before actually doing the calculation on the generalized cone we discuss the general structure of the coefficients \([31]\) and see what restrictions can be obtained from the coefficients found on our specific manifold. First one has to state clearly the assumptions under which the structure of the coefficients is formulated. We follow here \([31]\), otherwise the situation is considerably more complicated \([42]\). The assumptions are as follows:

(i) The problem is purely Abelian, i.e. the matrices \(\Gamma^i\) commute: \([\Gamma^i, \Gamma^j] = 0\).

(ii) The matrix \(\Gamma^2 = h_{ij} \Gamma^i \Gamma^j\) which automatically commutes with \(\Gamma^j\) by virtue of (i), commutes also with the matrix \(S\): \([\Gamma^2, S] = 0\).

(iii) The matrices \(\Gamma^i\) are covariantly constant with respect to the (induced) connection on the boundary: \(\hat{\nabla}_i \Gamma^j = 0\).

Under these assumptions we consider the Laplace-like operator

\[
L = -g^{ij} \nabla_i \nabla_j - E
\]

where \(E\) is an endomorphism of the smooth vector bundle \(V\) over \(M\) and \(\nabla\) is a connection, together with the boundary conditions \([3]\). Then the general form of the heat-kernel coefficients is

\[
a_{1/2}(f) = (4\pi)^{-1/2} \frac{1}{D} Tr(\delta f)[\partial M] \tag{6}
\]

\[
a_1(f) = (4\pi)^{-D/2} \frac{1}{6} \left\{ Tr(6fE + fR)[M] + Tr \left\{ f(b_0 K + b_2 S) + b_1 f_{;N} + f \sigma_1 K_{ab} \Gamma^a \Gamma^b \right\} [\partial M] \right\} \tag{7}
\]

\[
a_{2/2}(f) = (4\pi)^{-1/2} \frac{1}{384} Tr \left[ f(c_0 E + c_1 R + c_2 R^a_{;N} + c_3 K^2 + c_4 K_{ab} K^{ab} + c_7 S K + c_8 S^2) + f_{;N}(c_5 K + c_9 S) + c_0 f_{;NN} \right] [\partial M] + Tr \left[ f(\sigma_2 (K_{ab} \Gamma^a \Gamma^b)^2 + \sigma_3 K_{ab} \Gamma^a \Gamma^b K + \sigma_4 K_{ab} K^b \Gamma^a \Gamma^b + \lambda_1 K_{ab} \Gamma^a \Gamma^b S + \mu_1 R_{ab} \Gamma^a \Gamma^b + \mu_2 R^c_{ab} \Gamma^a \Gamma^b + b_1 \Omega_{ab} \Gamma^a + \beta_1 f_{;N} K_{ab} \Gamma^a \Gamma^b \right] [\partial M] \tag{8}
\]

Here and in the following \(f[M] = \int_M dx f(x)\) and \(f[\partial M] = \int_{\partial M} dy f(y)\), with \(dx\) and \(dy\) being the Riemannian volume elements of \(M\) and \(\partial M\).
addition, the semi-colon denotes differentiation with respect to the Levi-Civita connection of $\mathcal{M}$, $\Omega$ is the connection of $\nabla$ and $K_{ab}$ the extrinsic curvature. Finally, our sign convention for the Riemann tensor is

$$R^i_{jkl} = -\Gamma^i_{jk,l} + \Gamma^i_{jl,k} + \Gamma^i_{nk} \Gamma^a_{jl} - \Gamma^i_{nl} \Gamma^a_{jk}$$

(see for example [13]).

Although $a_{1/2}$ and $a_1$ have been determined previously [24, 31, 39], we have included them in the list to explain clearly our procedure. The terms in $a_{3/2}$ are grouped together such that the first two lines, $c_0$ up to $c_9$, contain the type of geometric invariants already present for Robin boundary conditions, whereas all the other terms are due only to the tangential derivatives in the boundary condition.

Our aim in the next section will be to put restrictions on the universal constants of eqs. (1)-(8) by calculating the coefficients of the conformal Laplacian ($E = -(d - 1)R/(4d)$) on the bounded generalized cone. By this we mean the $D = (d + 1)$-dimensional space $\mathcal{M} = I \times N$ with the hyperspherical metric [44]

$$ds^2 = dr^2 + r^2 d\Sigma^2,$$

(9)

where $d\Sigma^2$ is the metric on the manifold $N$ and $r$ runs from 0 to 1. $N$ will be referred to as the base of the cone. If it has no boundary then it is the boundary of $\mathcal{M}$.

It is clear that a special case calculation will be simplified considerably by taking a constant $\Gamma^a$, say $\Gamma^d = ig$, with the real constant $g$. In order that this is covariantly constant one might think of taking a flat base manifold $N$. The most natural such manifold where much is known about all required spectral properties is the torus. We thus choose $N = T^d$, namely the equilateral $d$-dimensional torus with perimeter $L = 2\pi$ and metric $d\Sigma^2 = dx_1^2 + ... + dx_d^2$. Its volume is $\text{vol}(T^d) = (2\pi)^d$ and the basic geometrical tensors read

$$R^{ij}_{\ k\ l} = \frac{1}{r^2}(\delta^i_j \delta^k_l - \delta^i_k \delta^j_l), \quad K^a_b = \delta^a_b.$$

Furthermore, we choose a specific smearing function $f = f(r)$ which will allow the calculation to be effected but which contains nevertheless all the information concerning the universal constants one can obtain. A possible choice is [38] (see section 4 for the applications in the given context)

$$f(r) = f_0 + f_1r^2 + f_2r^d.$$
For this special setting, the coefficients will have the following appearance:

\[
\frac{(4\pi)^{d/2}}{(2\pi)^d} a_{1/2}(f) = \delta f(1)
\]

\[
\frac{(4\pi)^{D/2}}{(2\pi)^d} 6a_1(f) = \frac{1}{2}(d-3)(d-1) \left[ \frac{f_0}{d-1} + \frac{f_1}{d+1} + \frac{f_2}{d+3} \right]
+ b_0 f(1)d + b_1 f;N(1) + b_2 f(1)S - \sigma_1 f(1)g^2
\]

\[
\frac{(4\pi)^{d/2}}{(2\pi)^d} 384a_{3/2}(f) = f(1) \left[ c_0(d-1)^2/4 - c_1 d(d-1) + c_3 d^2 + c_4 d + c_7 S d 
+ c_8 S^2 + \sigma_2 g^4 - \sigma_3 d g^2 - \sigma_4 g^2 
- \lambda_1 S g^2 - \mu_2 g^2 (1-d) \right]
+ f;N(1) \left[ c_5 d + c_9 S - \beta_1 g^2 \right] + c_6 f;N;N(1)
\]

Thus, by doing the calculation on the manifold \( \mathcal{M} = I \times T^d \) with \( f(r) = f_0 + f_1 r^2 + f_2 r^4 \) and by comparing terms containing a specific number of normal derivatives of \( f \) together with a fixed number of powers in \( d \) and \( S \) the following information can be extracted,

\[
\begin{align*}
    a_{1/2} & \quad \delta, \\
    a_1 & \quad b_0, b_1, b_2, \sigma_1, \\
    a_{3/2} & \quad c_3 - c_1 + c_0/4, \Gamma^2 (\sigma_3 - \mu_2) + c_4 + c_1 - c_0/2, c_5, c_6, c_7, c_8, c_9, \beta_1, \lambda_1, \\
    & \quad \Gamma^4 \sigma_2 + \Gamma^2 \sigma_4 + \Gamma^2 \mu_2 + c_0/4.
\end{align*}
\]

The considerable amount of information that one derives from this example is apparent. \( a_{1/2} \) and \( a_1 \) are completely determined without any additional input and from \( a_{3/2} \) one gets 10 of 18 unknowns. There is good hope that the remaining information can be found from functorial techniques.

Having, therefore, a good motivation, we can embark on special case calculations and see afterwards if the functorial relations can complete the information.

3 Non-smeared generalised cone calculation

So let us turn to the spectral analysis of the conformal Laplacian on the described generalized cone together with the boundary condition (3). The
The conformal Laplacian is
\[ \Delta_M - \frac{d - 1}{4d} R = \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} + \frac{(d - 1)^2}{4r^2} + \frac{1}{r^2} \Delta_N \]  
(13)
with eigenfunctions
\[ \frac{J_\nu(\alpha r)}{r^{(d-1)/2}} \exp\{i(x_1 n_1 + \ldots + x_d n_d)\}, \quad \vec{n} \in \mathbb{Z}^d. \]  
(14)

The index \( \nu \) equals
\[ \nu = \left( n_1^2 + \ldots + n_d^2 \right)^{1/2} \]  
(15)
and the eigenvalues \( \alpha \) are determined through (3) by
\[ \alpha J_\nu'(\alpha) + (u + gn_d) J_\nu(\alpha) = 0. \]  
(16)

Here \( u = 1 - D/2 - S \).

For the determination of the heat-kernel coefficients we follow the approach developed in [32, 33]. The basic object is the zeta function of \( M \),
\[ \zeta(s) = \sum \alpha^{-2s} \]  
(17)
and the relation
\[ a_{k/2} = \Gamma((D - k)/2) \text{Res} \zeta_M((D - k)/2) \]  
(18)
between the coefficients and the zeta function is used. In addition the Epstein type zeta function defined by
\[ E_k(s) = \sum_{\vec{n} \in \mathbb{Z}^d / \{\vec{0}\}} \frac{n_d^k}{(n_1^2 + \ldots + n_d^2)^s} \]  
(19)
will turn out to be very useful. Obviously it is connected with the spectrum of the Laplacian on the base manifold \( \mathcal{N} \), the \( n_d^d \)-powers arise from the tangential derivatives in (3).

The starting point of the analysis of \( \zeta_M \) is the contour integral representation,
\[ \zeta(s) = \sum_{\vec{n} \in \mathbb{Z}^d} \int_{\gamma} \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln (k J_\nu'(k) + (u + gn_d) J_\nu(k)), \]  
(20)
where γ must enclose all the solutions of (16) on the positive real axis. It is the appearance of the \( n_d \) in the last term that causes the added complications compared to [32, 33].

In the following analysis the index \( \nu = 0 \) would require a separate treatment. Its contribution has the rightmost pole at \( s = 1/2 \) because it is associated with the zeta function of a second order differential operator in one dimension. Because we are dealing with arbitrary dimensions this pole (and all other poles to the left of it) are irrelevant for our goal due to the relation (18). For convenience therefore, we will continue without including the \( \nu = 0 \) contribution and will still use the same notation, \( \zeta(s) \).

Shifting the contour to the imaginary axis, the zeta function (with the zero mode \( \nu = 0 \) omitted, as explained) reads

\[
\zeta(s) = \sum_{\vec{n} \in \mathbb{Z}^d \setminus \{0\}} \int_0^\infty dz (z^s)^{-2s} \frac{\partial}{\partial z} \log z^{-\nu} \left[ z^\nu I_\nu'(z\nu) + (u + gn_d)I_\nu(z\nu) \right]. \tag{21}
\]

As shown in detail in [32, 33], the heat-kernel coefficients are determined solely by the asymptotic contributions of the Bessel functions as \( \nu \to \infty \). In the given consideration more care is needed since terms like \( n_d/\nu \) have to be counted as of order \( \nu^0 \).

Using the uniform asymptotic expansion of the Bessel function [45] one encounters the expression

\[
\ln \left\{ 1 + \left( 1 + \frac{gn_d}{\nu} \right) t \right\} = \left[ \sum_{k=1}^\infty \frac{v_k(t)}{\nu^k} + \frac{ut}{\nu} + \left( u + gn_d \right) t \sum_{k=1}^\infty \frac{u_k(t)}{\nu^k} \right] = \sum_{j=1}^\infty T_j(u, g, t) \tag{22}
\]

whereby the \( T_j \) are defined and \( t = 1/(1 + z^2) \). For the Olver polynomials, \( u_k \) and \( v_k \), see [45].

Asymptotically one finds

\[
\zeta(s) = A_{-1}(s) + A_0(s) + A_1(s) + \sum_{j=1}^\infty A_j(s), \tag{23}
\]

where \( A_{-1}(s) \) and \( A_0(s) \) are the same as in Robin boundary conditions [32], namely

\[
A_{-1}(s) = \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s+1)} E_0(s-1/2), \tag{24}
\]

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Asymptotically one finds

\[
\zeta(s) = A_{-1}(s) + A_0(s) + A_1(s) + \sum_{j=1}^\infty A_j(s), \tag{23}
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where \( A_{-1}(s) \) and \( A_0(s) \) are the same as in Robin boundary conditions [32], namely

\[
A_{-1}(s) = \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s+1)} E_0(s-1/2), \tag{24}
\]
\[ A_0(s) = \frac{1}{4} E_0(s). \]  \hspace{1cm} (25)

The new quantities are
\[ A_+(s) = \frac{\sin \pi s}{\pi} \sum_{\vec{n} \in \mathbb{Z}^d / \{0\}} \int_0^\infty dz \frac{(z\nu)^{-2s}}{\nu^2} \frac{\partial}{\partial z} \ln \left( 1 + \frac{g_{n_d}t}{\nu} \right), \]  \hspace{1cm} (26)

and
\[ A_j(s) = \frac{\sin \pi s}{\pi} \sum_{\vec{n} \in \mathbb{Z}^d / \{0\}} \int_0^\infty dz \frac{(z\nu)^{-2s}}{\nu^2} \frac{\partial}{\partial z} T_j(u, g, t). \]  \hspace{1cm} (27)

In order to proceed it is convenient to express \( T_j \) as the finite sum
\[ T_j = \sum_{a,b,c} f^{(j)}_{a,b,c} \frac{\delta \epsilon t^a}{(1 + \delta t)^b}, \]  \hspace{1cm} (28)

with \( \delta = g_{n_d}/\nu \). The \( f^{(j)}_{a,b,c} \) are easily determined via an algebraic computer programme.

The next steps are to perform the \( z \)-integrations by the identity,
\[ \int_0^\infty dz \frac{z^{x-2s}}{(1 + \delta t)^y} = \frac{1}{2} \frac{\Gamma(1 - s)}{\Gamma(y)} \sum_{k=0}^\infty (-1)^k \frac{\Gamma(y + k)\Gamma(s - 1 + (x + k)/2)}{k!\Gamma((x + k)/2)} \delta^k, \]  \hspace{1cm} (29)

and then do the \( \vec{n} \)-summation to write everything in terms of the Epstein functions (19). Performing these steps one gets first
\[ A_+(s) = \frac{1}{2\Gamma(s)} \sum_{n=1}^\infty \frac{\Gamma(s + n)}{\Gamma(n + 1)} E_{2n}(s + n)g^{2n}. \]  \hspace{1cm} (30)

For \( c \) even and \( c \) odd in (28) slightly different representations appear such that some more notation is unfortunately necessary. We write
\[ A_j(s) = \sum_{a,b,c} f^{(j)}_{a,b,c} A^{a,b,c}_j(s) \]  \hspace{1cm} (31)

and get
\[ A^{a,b,c}_j(s) = -\frac{1}{\Gamma(s)} \sum_{n=0}^\infty \frac{\Gamma(b + 2n)}{\Gamma(b)\Gamma(2n + 1)} \frac{\Gamma(s + a/2 + n)}{\Gamma(a/2 + n)} E_{2n+c}(s + n + (j + c)/2)g^{2n+c}. \]  \hspace{1cm} (32)
for $c$ even and

$$A_j^{a,b,c}(s) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{\Gamma(b+2n+1) \Gamma(s+(a+1)/2+n)}{\Gamma(b)(2n+1)! \Gamma((a+1)/2+n)} E_{2n+c+1}(s+n+(j+c+1)/2)g^{2n+c+1}. \quad (33)$$

for $c$ odd.

We need the residues of $A_{-1}, A_0, A_+$ and $A_j$, but this is not too difficult, because the Epstein zeta functions are very well studied objects. For us the relevant properties are

$$E_n(s) = 0 \quad \text{for } n \text{ odd} \quad (34)$$

and for $s = l + d/2$ the residue is

$$\text{Res } E_2(l + d/2) = \frac{\pi^{(d-1)/2} \Gamma(l + 1/2)}{\Gamma(d/2 + l)}. \quad (35)$$

A nice feature of the calculation is, that when using the above results (34) and (35) in (31) and (32), Res $A_+((D-k)/2)$ and Res $A_j((D-k)/2)$ reduce to the series representation of the generalized hypergeometric function $\left[\frac{\partial}{\partial z}\right]$

$$pFq(\alpha_1, \alpha_2, ..., \alpha_p; \beta_1, \beta_2, ..., \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k...(\alpha_p)_k z^k}{(\beta_1)_k(\beta_2)_k...(\beta_q)_k k!}. \quad (36)$$

For example, for $A_+$ only one contributions arises which, usefully normalized, reads

$$\Gamma((D-1)/2) \left(\frac{4\pi}{2\pi}\right)^{d/2} \text{Res } A_+((D-1)/2)$$

$$= \frac{1}{2} \left\{ _2F_1(1/2, d/2; d/2; g^2) - 1 \right\}$$

$$= \frac{1}{2} \left\{ (1 - g^2)^{-1/2} - 1 \right\}. \quad (37)$$

In this case the intermediate step in terms of the hypergeometric function is artificial of course but useful in general.

In order to give the contributions of $A_j^{a,b,c}$, we have to distinguish between even and odd $j$. Although a bit lengthy we find it useful to state these results.
Let us stress, that we have already, in principle, determined an arbitrary number of heat-kernel coefficients for the Laplacian on a generalized cone with oblique boundary boundary conditions in the non-smeared case. For \( c \) odd and \( j \) odd we find

\[
\Gamma((D - 1 - j)/2) (4\pi)^{D/2} (2\pi)^d \text{Res } A_j^{a,b,c}((D - 1 - j)/2) = 2 \frac{\Gamma((1+c)/2)}{\Gamma((a+1)/2)\Gamma(b)} \left( \frac{D+c}{2} \right)^{a-j-c} g^c \left( \frac{d}{dg} \right)^{b-1} g^b 3F_2(1, (d+a+1-j)/2, 1+c/2; (a+1)/2, (D+c)/2; g^2),
\]

which contributes to the coefficient \( a(1+j)/2 \). For the specific values of \( b, c, j \) and \( k \) needed the hypergeometric function always reduces to a simple algebraic or a hyperbolic function. The above result neatly summarizes all this information in one equation.

For \( c \) odd and \( j \) even \( A_j^{a,b,c} \) also contributes to the coefficient \( a(j+1)/2 \) and the relevant result is \( 1/(2\sqrt{\pi}) \) times the above (note that in this case the normalization is \( (4\pi)^{d/2} \)).

Furthermore, for \( c \) even and \( j \) odd the analogous result is

\[
\Gamma((D - 1 - j)/2) (4\pi)^{D/2} (2\pi)^d \text{Res } A_j^{a,b,c}((D - 1 - j)/2) = -2 \frac{\Gamma((1+c)/2)}{\Gamma(a/2)\Gamma(b)} \left( \frac{d+c}{2} \right)^{a-j-c} g^c \left( \frac{d}{dg} \right)^{b-1} g^{b-1} 3F_2(1, (d+a-j)/2, (1+c)/2; a/2, (d+c)/2; g^2),
\]

For \( c \) even and \( j \) even the same rules as above hold, furthermore the same comments.

The above results allow for a direct evaluation of the coefficients by an algebraic computer program such as Mathematica. However, before stating the results let us describe the necessary modifications when dealing with the smeared case.
4 Generalization to the smeared heat-kernel coefficients

The inclusion of a smearing function that depends only on the radial variable results in the smeared zeta function,

\[ \zeta(f; s) = \sum_{\vec{n} \in \mathbb{Z}^d/\{0\}} \int \frac{dk}{2\pi i} k^{-2s} \int_0^1 dr f(r) \bar{J}_\nu^2(kr) r \frac{\partial}{\partial k} \ln(kJ'_\nu(k) + (u + gn_d)J_\nu(k)). \]

(40)

For \( f(r) \) a polynomial, we have shown how to analyse (40) for Dirichlet and Robin boundary conditions in [38] and so for the general procedure see this reference. The generalisation to oblique boundary conditions is obtained here.

The bar in (40) signifies normalized. Explicitly

\[ \bar{J}_\nu(kr) = \sqrt{2k} \left( (u + gn_d)^2 + k^2 - \nu^2 \right)^{-1/2} J_\nu(kr). \]

(41)

For \( f(r) = \sum_{n=0}^N f_n r^{2n} \)

(42)

we need normalization integrals of the type

\[ S[1 + 2p] = \int_0^1 dr J_\nu^2(\alpha r)r^{1+2p}. \]

(43)

These can be treated using Schafheitlin’s reduction formula [47], which for the present case gives the recursion

\[ S[1 + 2p] = \frac{2p}{2p+1} \frac{\nu^2 - p^2}{\alpha^2} S[2p - 1] + \frac{1}{2p+1} \left( 1 + \frac{2p(u + p)}{\alpha^2 + (u + gn_d)^2 - \nu^2} \right), \]

(44)

starting with \( S[1] = 1 \). So \( S[1 + 2p] \) has the following form

\[ S[1 + 2p] = \sum_{m=0}^p \left( \frac{\nu}{\alpha} \right)^{2m} \sum_{l=0}^m \gamma_{ml} \nu^{-2l} \]

\[ + \frac{1}{\alpha^2 + (u + gn_d)^2 - \nu^2} \sum_{m=0}^{p-1} \left( \frac{\nu}{\alpha} \right)^{2m} \sum_{l=0}^n \delta_{ml} \nu^{-2l}. \]

(45)
The numerical coefficients $\gamma_{ml}^p$ and $\delta_{ml}^p$ are easily determined recursively. As a result, apart from characteristic differences, the smeared zeta function $\zeta_M(f, s)$ takes a similar form as $\zeta_M(s)$.

It will turn out convenient to divide $\zeta_M(f, s)$ into different pieces characterized below. First, respecting the structure in (45), we define

$$\zeta_\gamma^p(f, s) = \sum_{m=0}^{p} \sum_{l=0}^{m} \gamma_{ml}^p \sum_{\vec{n} \in \mathbb{Z}^d / \{0\}} \nu^{2m-2l} \int \frac{dk}{2\pi i} k^{-2(s+m)} \frac{\partial}{\partial k} \ln(k J'_\nu(k) + (u + g_n d) J_\nu(k))$$

and

$$\tilde{\zeta}_\delta^p(f, s) = \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta_{ml}^p \sum_{\vec{n} \in \mathbb{Z}^d / \{0\}} \nu^{2m-2l} \int \frac{dk}{2\pi i} k^{-2(s+m)} \frac{\partial}{\partial k} \ln(k J'_\nu(k) + (u + g_n d) J_\nu(k)),$$

where the contour $\gamma$ has to be chosen so as to enclose the zeros of only $k J'_\nu(k) + (u + g_n d) J_\nu(k)$. Thus the poles of $S[1 + 2p]$, located at $k = \pm \sqrt{\nu^2 - (u + g_n d)^2}$, must be outside the contour. It is important to locate the contour properly because, when deforming it to the imaginary axis, contributions from the pole at $k = \sqrt{\nu^2 - (u + g_n d)^2}$ arise.

The index $p$ refers to the fact that these are the contributions coming from the power $\nu^{2p}$ in (12). In order to obtain the full zeta function, the $\sum_{p=0}^{N} f_p \zeta^p$ has to be done.

The first piece, $\zeta_\gamma^p$, may be given just by inspection. Comparing (44) with the non-smeared zeta function (20) the contour integral is the same as previously once $s \to s + m$ has been put. Due to the additional factor $\nu^{2m-2l}$ the argument of the base zeta function has to be raised by $l - m$. For explanatory purposes let us give as an explicit example

$$A_{-1}^\gamma(f, s) = \frac{1}{4\sqrt{\pi}} \sum_{p=0}^{N} f_p \sum_{m=0}^{p} \sum_{l=0}^{m} \gamma_{ml}^p \frac{\Gamma(s + m - 1/2)}{\Gamma(s + m + 1)} E_0(s + l - 1/2).$$

In exactly the same way, $A_0^\gamma(f, s)$, $A_1^\gamma(f, s)$ and $A_j^\gamma(f, s)$ are obtained from (25), (30), (32) and (33). This is the stage where the properties (34) and (35) are used and the contributions to the heat-kernel coefficients in terms of hypergeometric function emerge. They will not be displayed, however,
explicitly, because the structure is the one already seen in (38) and the way they are obtained is identical to the procedure described in section 3.

We continue with the analysis of $\tilde{\zeta}_p$, where several additional complications occur. Shifting the contour to the imaginary axis we get the pieces

$$\zeta_p(f, s) = \frac{\sin \pi s}{\pi} \sum_{\vec{n} \in \mathbb{Z}^d/(\vec{0})} \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta^p_{ml} (-1)^m \nu^{-2s-2l}$$

(49)

$$\zeta_{shift}(f, s) = -\frac{1}{2} \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta^p_{ml} \sum_{\vec{n} \in \mathbb{Z}^d/(\vec{0})} \nu^{2m-2l} \nu^{2s-(u+gn_d)^2-s-m-1/2}$$

(50)

the last one arising on moving the contour over the pole at $k = \sqrt{\nu^2 - (u+gn_d)^2}$.

In dealing with $\zeta_p(f, s)$ one can use, as done after eq. (21), the uniform asymptotics of the Bessel functions and define analogously to (23) the asymptotic contributions $A_{i, \delta}(f, s)$. We will illustrate the calculation by dealing with

$$A_{-1, \delta}(f, s) = \frac{\sin \pi s}{\pi} \sum_{\vec{n} \in \mathbb{Z}^d/(\vec{0})} \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta^p_{ml} (-1)^m \nu^{-2s-2l+1}$$

By using the expansion

$$\frac{1}{(u+gn_d)^2 - \nu^2(1+z^2)} = -\sum_{i=0}^{\infty} \frac{(u+gn_d)^{2i}}{\nu^{2i+2}(1+z^2)^{i+1}}$$

the above integrals are recognised as representations of Beta functions [46].

As an intermediate result one gets

$$A_{-1, \delta}(f, s) = \sum_{i=0}^{\infty} \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta^p_{ml} \frac{\Gamma(s + i + m + 1/2)}{\Gamma(s + m + 1) \Gamma(i + 1/2)} \frac{(u+gn_d)^{2i}}{\nu^{2s+2l+2i+1}}.$$  

(51)
Complicated as the expression (51) is, we have to remind ourselves of our initial goal, namely, the determination of the heat-kernel coefficients up to $a_{3/2}$. Thus we need to determine only the residues of (51) at $s = D/2, (D - 1)/2, D/2 - 1$ and $(D - 3)/2$.

In general, the numbers $\delta_{ml}^p$ contain terms independent of $n_d$ and linear in $n_d$,

$$\delta_{ml}^p = \delta_{ml0}^p + \delta_{ml1}^p n_d.$$  

Furthermore it is clear that the higher the power of $n_d$ the more to the right the pole of the associated term will be. Thus in addition consider the expansion

$$(u + gn_d)^{2i} = g^{2i} n_d^{2i} + 2iug^{2i-1} n_d^{2i-1} + O(n_d^{2i-2})$$

in powers of $n_d$. With eq. (53) it is then obvious that the rightmost pole in $A_{-1,\delta}^p(f, s)$ due to the $O(n_d^{2i-2})$ term is situated at $s = (D - 4)/2$ and contributes only to $A_2$. For our immediate purposes it is thus sufficient to take into consideration only the above two terms. As a result

$$A_{-1,\delta}^p(f, s) = F_1^p(f, s) + F_2^p(f, s)$$

with

$$F_1^p(f, s) = \frac{1}{2} \sum_{i=0}^{p-1} \sum_{m=0}^{\infty} \delta_{ml0}^p \frac{\Gamma(s + i + m + 1/2)}{\Gamma(s + m + 1) \Gamma(i + 1/2)}$$

$$g^{2i} E_{2i}(s + l + i + 1/2),$$

$$F_2^p(f, s) = \frac{1}{2} \sum_{i=0}^{p-1} \sum_{m=0}^{\infty} \delta_{ml1}^p \frac{\Gamma(s + i + m + 1/2)}{\Gamma(s + m + 1) \Gamma(i + 1/2)}$$

$$g^{2i} E_{2i}(s + l + i + 1/2).$$

Use of the residues of the base zeta function $E_{2i}$ then easily gives the following normalized contributions (we use the notation $\delta_{ml}^p = 0$ for $l < 0$)

$$\Gamma(D/2 - k)(4\pi)^{D/2} (2\pi)^d \text{Res} \ F_1^p(f, D/2 - k) = \sum_{m=k-1}^{p-1} \delta_{m(k-1)0}^p \frac{\Gamma(\frac{D}{2}m + 1 - k)}{(D/2 - k)_{m+1}}$$

$$\text{Res} \ F_2^p(1, D/2 - k + m + 1/2, d/2; g^2)$$

(54)
\[
\Gamma(D/2 - k) \frac{(4\pi)^{D/2}}{(2\pi)^d} \text{Res } F_2^p(f, D/2 - k) = u \sum_{m=k-1}^{p-1} \delta_{m(k-1)} \frac{(d/2)_{m+1-k}}{(D/2 - k)_{m+1}} \\
g \frac{d}{dg} 2F_1(1, D/2 - k + m + 1/2, d/2; g^2)
\] (55)

For our purposes, only \(k = 1\) is relevant, but we have given these general results to show that in principle one could go further.

The same kind of argument allows one to show that the relevant parts in the other \(A_i^\delta(f, s)\) can all be represented in terms of hypergeometric functions.

Finally we are left with the treatment of \(\zeta_{\text{shift}}^p(f, s)\), eq. (50). Here, instead of the uniform asymptotics of \(I_\nu\) we need it for the function \(J_\nu\). The expansion given in [45] (see also [38]) suggests for \(\nu \to \infty\),

\[
\frac{\partial}{\partial k} \ln(kJ_\nu'(k) + (u + gn_d)J_\nu(k)) \bigg|_{k=\sqrt{\nu^2-(u+gn_d)^2}} \sim \sum_{l=0}^{\infty} e_l \frac{(u + gn_d)^{2l+1}}{\nu^{2l+1}},
\] (56)

with the coefficients \(e_l\) to be determined. Here the problem appears, that every value of \(l\) contributes to the pole of \(\zeta_{\text{shift}}^p(f, s)\) already at \(s = (D - 1)/2\). Thus, the asymptotic expansion of the left hand side to any power in \(n_d/\nu\) is needed, apparently an extremely difficult problem on asymptotics of special functions. However, given that \(\zeta_{\text{shift}}^p(f, s)\) contributes only for \(p > 0\) we will show in the next section how to circumvent a direct evaluation of eq. (56).

5 Results and conformal techniques

After having shown in detail several aspects of the special case calculation on the generalized cone let us collect the information about the universal constants appearing in eqs. (10) – (12).

In order to answer the open question about eq. (50) let us consider first \(a_{1/2}\) and take \(f(r) = f_0\). For this case only \(A_0(s)\) and \(A_+(s)\) contributes and the answer is

\[
\frac{(4\pi)^{d/2}}{(2\pi)^d}a_{1/2}(f) = f_0 \frac{1}{4} \left( \frac{2}{\sqrt{1-g^2}} - 1 \right).
\] (57)

Comparison with (10) gives the correct universal constant

\[
\delta = \frac{1}{4} \left( \frac{2}{\sqrt{1+1^2}} - 1 \right)
\] (58)
and the present special case evaluation determines the $a_{1/2}$ coefficient for a general manifold (which was clear of course). Thus, for the case of the cone, taking $f(r) = f_0 + f_1 r^2$ we know that

$$\frac{(4\pi)^{d/2}}{(2\pi)^d} a_{1/2}(f) = (f_0 + f_1) \frac{1}{4} \left( \frac{2}{\sqrt{1 - g^2}} - 1 \right).$$

Taking into account all terms but the unknown contribution from $\zeta_{\text{shift}}^p(f, s)$ we find

$$(f_0 + f_1) \frac{1}{4} \left( \frac{2}{\sqrt{1 - g^2}} - 1 \right) + \frac{g^2}{3d(1 - g^2)^{3/2}} f_1.$$

Since the first piece is the correct answer, the last piece has to be cancelled by the contribution of $\zeta_{\text{shift}}^p(f, s)$. Use of the expansion (56) and (50) gives

$$\Gamma \left( \frac{D - 1}{2} \right) \frac{(4\pi)^{d/2}}{(2\pi)^d} \text{Res} \zeta_{\text{shift}}^p(f, (D - 1)/2) = -\frac{g^2}{3d} f_1 \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{3}{2} \right)_{i+l} \frac{(d+1)}{2} \frac{i!}{(d+1)_i} e_l g^{2i+2l}$$

which leads to the condition

$$\sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{3}{2} \right)_{i+l} \frac{(d+1)}{2} \frac{i!}{(d+1)_i} e_l g^{2i+2l} = \frac{1}{(1 - g^2)^{3/2}}.$$ 

In order that the left hand side should be nothing other than a complicated series expansion of the right hand side one has to conclude that $e_l = (1/2)_l/l!$ and one finds the surprisingly easy expansion

$$\left. \frac{\partial}{\partial k} \ln(k J_{\nu}(k) + (u + gn_d)) \right|_{k = \sqrt{\nu^2 - (u + gn_d)^2}} = \sum_{l=0}^{\infty} \frac{(1/2)_l (u + gn_d)^{2l+1}}{l! \nu^{2l+1}}.$$ (59)

But having expansion (59) at hand the contribution of $\zeta_{\text{shift}}^p(f, s)$ to any pole can be determined so that now for the cone complete knowledge for $a_1$ and $a_{3/2}$ is available.

We turn now to $a_1$. Dealing first with $f(r) = f_0$, the linear term in $d$ defines $b_0$, the linear term in $S$ defines $b_2$, the term independent of $d$ and
$S$ defines $\sigma_1$. Dealing afterwards with $f(r) = f_1 r^2$ the additional piece is immediately identified with $b_1$. As a result we obtain the correct answer

\[
b_0 = 2 - 6 \left( -\frac{1}{1 + \Gamma^2} + \frac{\text{Arctanh}(\sqrt{1 + \Gamma^2})}{\sqrt{-\Gamma^2}} \right),
\]

\[
b_1 = 3 - 6 \text{Arctanh}(\sqrt{-\Gamma^2}),
\]

\[
b_2 = \frac{12}{1 + \Gamma^2},
\]

\[
\sigma_1 = \frac{6}{\Gamma^2} \left( -\frac{1}{1 + \Gamma^2} + \frac{\text{Arctanh}(\sqrt{-\Gamma^2})}{\sqrt{-\Gamma^2}} \right),
\]

which shows that the ideas involved in our special case calculation are indeed correct. Especially the example once more determines the complete coefficient for a general smooth manifold with boundary.

Proceeding in the same way as described for $a_1$, we obtain the following universal constants for $a_{3/2}$,

\[
c_5 = \frac{1}{\Gamma^4} \left[ 2 \left( -\left( \Gamma^2 \left( 144 - \frac{160}{\sqrt{1 + \Gamma^2}} \right) \right) + 32 \left( -1 + \frac{1}{\sqrt{1 + \Gamma^2}} \right) 
+ \Gamma^4 \left( -15 + \frac{80}{\sqrt{1 + \Gamma^2}} \right) \right) \right],
\]

\[
c_6 = \frac{1}{\Gamma^4} \left[ 8 \left( 32 - \frac{32}{\sqrt{1 + \Gamma^2}} + \Gamma^4 \left( -3 - \frac{8}{\sqrt{1 + \Gamma^2}} \right) - \Gamma^2 \left( -36 + \frac{52}{\sqrt{1 + \Gamma^2}} \right) \right) \right]
+ \frac{32 \left( 5 \Gamma^4 - 8 \left( -1 + \sqrt{1 + \Gamma^2} \right) - 4 \Gamma^2 \left( -4 + 3 \sqrt{1 + \Gamma^2} \right) \right)}{\Gamma^4 \sqrt{1 + \Gamma^2}},
\]

\[
c_7 = \frac{192 \left( 1 - \sqrt{1 + \Gamma^2} - \Gamma^2 \left( -2 + \sqrt{1 + \Gamma^2} \right) \right)}{\Gamma^2 (1 + \Gamma^2)^{3/2}},
\]

\[
c_8 = \frac{192}{(1 + \Gamma^2)^{3/2}}.
\]

\[
c_9 = \frac{-192}{\Gamma^2} \left( 1 - \frac{1}{\sqrt{1 + \Gamma^2}} \right),
\]

\[
\beta_1 = \frac{-32 \left( 5 \Gamma^4 - 8 \left( -1 + \sqrt{1 + \Gamma^2} \right) - 4 \Gamma^2 \left( -4 + 3 \sqrt{1 + \Gamma^2} \right) \right)}{\Gamma^6 \sqrt{1 + \Gamma^2}},
\]

18
respectively the following relations among them,

\[
c_3 - c_1 + c_0/4 = \frac{1}{\Gamma^4(1 + \Gamma^2)^{3/2}} \left[ \Gamma^2 \left( 240 - 224 \sqrt{1 + \Gamma^2} \right) + \Gamma^4 \left( 336 - 207 \sqrt{1 + \Gamma^2} \right) 
\right.
\]

\[
- 32 \left( -1 + \sqrt{1 + \Gamma^2} \right) - 5 \Gamma^6 \left( -16 + 3 \sqrt{1 + \Gamma^2} \right),
\]

(67)

\[
\Gamma^2(\sigma_3 - \mu_2) + c_4 + c_1 - c_0/2 = \frac{6}{\Gamma^4(1 + \Gamma^2)^{3/2}} \left[ 32 \left( -1 + \sqrt{1 + \Gamma^2} \right) 
\right.
\]

\[
+ \Gamma^6 \left( -48 + 7 \sqrt{1 + \Gamma^2} \right) + 16 \Gamma^2 \left( -10 + 9 \sqrt{1 + \Gamma^2} \right)
\]

\[
+ \Gamma^4 \left( -192 + 119 \sqrt{1 + \Gamma^2} \right),
\]

(68)

\[
\sigma_2 + \frac{1}{\Gamma^2}(\sigma_4 + \mu_2) + \frac{c_0}{4\Gamma^4} = -\frac{8}{\Gamma^8(1 + \Gamma^2)^{3/2}} \left[ 32 \left( -1 + \sqrt{1 + \Gamma^2} \right) 
\right.
\]

\[
+ \Gamma^6 \left( -32 + 3 \sqrt{1 + \Gamma^2} \right) + 8 \Gamma^2 \left( -15 + 13 \sqrt{1 + \Gamma^2} \right)
\]

\[
+ 3 \Gamma^4 \left( -42 + 25 \sqrt{1 + \Gamma^2} \right).
\]

(69)

Where applicable, in the limit \( \Gamma \to 0 \) the results for Robin boundary conditions are reproduced as a check.

This serves as a very good input for applying the techniques of [12]. First we use a result on product manifolds [13], which in our case gives

\[
c_0 = 96 \left( -1 + \frac{2}{\sqrt{1 + \Gamma^2}} \right),
\]

(70)

\[
c_1 = 16 \left( -1 + \frac{2}{\sqrt{1 + \Gamma^2}} \right),
\]

(71)

Together with eq. (67) this also determines \( c_3 \),

\[
c_3 = \frac{1}{\Gamma^4(1 + \Gamma^2)^{3/2}} \left[ \Gamma^2 \left( 240 - 224 \sqrt{1 + \Gamma^2} \right) + \Gamma^4 \left( 320 - 199 \sqrt{1 + \Gamma^2} \right) 
\right.
\]

\[
+ \Gamma^6 \left( 64 - 7 \sqrt{1 + \Gamma^2} \right) - 32 \left( -1 + \sqrt{1 + \Gamma^2} \right). 
\]

(72)
The remaining task is to apply the functorial techniques of [12]. For details of the technique itself see this reference and for the modifications when tangential derivatives are involved see [31].

The basic equations are the conformal-variation formulae

\[
\frac{d}{d\epsilon} \left|_{\epsilon=0} a_{n/2}(1, e^{-2\epsilon f} L) = (D - n) a_{n/2}(f, L), \right.
\]

\[
\frac{d}{d\epsilon} \left|_{\epsilon=0} a_{n/2}(e^{-2\epsilon f} H, e^{-2\epsilon f} L) = 0 \text{ for } D = n + 2, \right.
\]

with an arbitrary smooth function \( H \). For a collection of variational formulae again see [12]. The additional relation \( \Gamma^i(\epsilon) = e^{-\epsilon f} \Gamma^i \) is given in [31]. Setting to zero the coefficients of all terms in (73) for \( n = 3 \) one obtains, for example [31],

| Term | Coefficient |
|------|-------------|

\( f_{i,N} \quad 0 = \frac{1}{2} (D - 2)c_0 - 2(D - 1)c_1 - (D - 1)c_2 - (D - 3)c_6 - \Gamma^2 \mu_1 \)

\( Kf_{i,N} \quad 0 = \frac{1}{2} (D - 2)c_0 - 2(D - 1)c_1 - c_2 + 2(D - 1)c_3 + 2c_4 - \frac{1}{2} (D - 2)c_7 - (D - 3)c_5 + \Gamma^2 \sigma_3 - \Gamma^2 \mu_2 \)

The first of these determines \( c_2 \) and \( \mu_1 \), namely

\[
c_2 = \frac{8}{\Gamma^2} \left( 12 \frac{12}{\sqrt{1 + \Gamma^2}} + \Gamma^2 \left( 1 - \frac{8}{\sqrt{1 + \Gamma^2}} \right) \right), \quad (75)
\]

\[
\mu_1 = \frac{96 (2 + \Gamma^2 - 2 \sqrt{1 + \Gamma^2})}{\Gamma^4 \sqrt{1 + \Gamma^2}}. \quad (76)
\]

The second together with (60) gives

\[
c_4 = \frac{2}{\Gamma^4} \left( \Gamma^4 \left( 5 - \frac{32}{\sqrt{1 + \Gamma^2}} \right) + \Gamma^2 \left( 48 - \frac{32}{\sqrt{1 + \Gamma^2}} \right) + 32 \left( -1 + \frac{1}{\sqrt{1 + \Gamma^2}} \right) \right). \quad (77)
\]

In addition we get

\[
b_1 = 0. \quad (78)
\]

Disappointing as it is, under the given assumptions these are the only new universal constants the functorial techniques yield. But due to the restrictions
imposed we have not yet exploited all information available. For example one
has the variational formula
\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} R^i_{\mu\nu} \xi^\mu \xi^\nu = -2f R^i_{\mu\nu} \xi^\mu \xi^\nu - (D - 3) K_{ij} \Gamma^i \Gamma^j f,_{iN}
\]
\[-\Gamma^2 K f,_{iN} - (D - 3) f,_{ij} \Gamma^i \Gamma^j - \Gamma^2 f,_{ll}.
\]
Up to now we have not compared coefficients involving tangential deriva-
tives, because for covariantly constant \( \Gamma^i \) these pieces integrate to zero. But
if we relax the condition of covariantly constant \( \Gamma^i \), eq. (79) shows that two
additional eqs. for the universal constant \( \mu_2 \) would arise. (In contrast the
variations of the terms associated with the missing \( \sigma_2, \sigma_3 \) and \( \sigma_4 \) do not con-
tain any tangential derivatives.) In order to exploit this observation we have
to generalize eq. (8) to when \( \nabla_i \Gamma^i \neq 0 \). This is done as usual by building up
all possible independent geometrical terms with certain homogeneity prop-
ties [12]. For the boundary conditions under consideration eq. (8) has to
be supplemented by the following terms,
\[
\frac{384}{(4\pi)^{1/2}} a^{\text{cov}}_{3/2}(f) = Tr[f (\gamma_1 \Gamma^i \Gamma^j i, j + \gamma_2 \Gamma^i \Gamma^j i, j + \gamma_3 \Gamma^i \Gamma^j i, j
\]
\[+ \gamma_4 \Gamma^i \Gamma^j + \gamma_5 \Gamma^i \Gamma^j + \gamma_6 \Gamma^i \Gamma^j + \gamma_7 \Gamma^i \Gamma^j + \gamma_8 \Gamma^i \Gamma^j + \gamma_9 \Gamma^i \Gamma^j + \gamma_{10} \Gamma^i \Gamma^j + \gamma_{11} \Gamma^i \Gamma^j + \gamma_{12} \Gamma^i \Gamma^j + \gamma_{13} \Gamma^i \Gamma^j + \gamma_{14} \Gamma^i \Gamma^j + \gamma_{15} \Gamma^i \Gamma^j]
\]
\[(\partial M). \] (80)
The term \( \Gamma^i \Gamma^j \) is not added because due to the Gauss-Codacci relation one
has
\[
\Gamma^i \Gamma^j = \Gamma^i \Gamma^j + R^i_{\mu\nu} \Gamma^k \Gamma^j + K K^i \Gamma^k \Gamma^j - K^i K^j \Gamma^k \Gamma^j
\]
so that this term is linearly dependent on the others already displayed. Be-
cause of the simple conformal transformation property of \( \Gamma^i \) it is relatively
easy to find the variational formulas of all invariants in eq. (80). Even
though we know none of the \( \gamma_i \), setting to zero the coefficients of the tangen-
tial derivatives terms in (79), we find the unambiguous answer
\[
\mu_2 = 0.
\]
As a consequence eq. (60) shows
\[
\sigma_3 = \frac{1}{\Gamma^6 (1 + \Gamma^2)^{3/2}} \left[ 32 \left( -5 \Gamma^6 + 8 \left( -1 + \sqrt{1 + \Gamma^2} \right) + 6 \Gamma^4 \left( -5 + 3 \sqrt{1 + \Gamma^2} \right) \right) \right]
\]
This is really all we can get from the lemmas and the specific example of the generalized cone, $I \times T^d$. We can obtain information only about the combination $\Gamma^2 \sigma_2 + \sigma_4$. Thus some additional input is needed to accomplish the goal of finding all universal constants in (8).

6 Oblique boundary conditions on $B^2 \times T^{d-1}$

One possibility of finding the remaining information is to look for an example which is able to separate the contributions of $(K_{ij} \Gamma^i \Gamma^j)^2$ and $K_{ij}K_{ij} \Gamma^i \Gamma^j$. The reason that all types of generalized cones with metric (4) fail to do so, is that $K_{ij} = \delta_{ij}$. As a result, the contraction $K_{ij} \Gamma^i \Gamma^j$ equals $\Gamma^2$ and also $K_{ij}K_{ij} \Gamma^i \Gamma^j = \Gamma^2$. Having a term like $\sigma_2(\Gamma^2) \Gamma^4 + \sigma_4(\Gamma^2) \Gamma^2 = g(\Gamma^2)$, with $g(\Gamma^2)$ a known function of $\Gamma^2$, there is no possibility uniquely determining $\sigma_2$ or $\sigma_4$ because, as indicated, these also depend on $\Gamma^2$. It is clear that this problem is not a result of having chosen only one non-vanishing component $\Gamma^i$. For a generalized cone, these properties are generic. So we are forced to leave this class of examples.

The difference between the invariants associated with $\sigma_2$ and $\sigma_4$ is that the first contains fourth powers of $\Gamma^i$ whereas the second one only squares. If we deal with two instead of one nonvanishing component, say $\Gamma^d = g$ and $\Gamma^i = g_d$, $i \neq d$, $g_d, g$ constants, and if the extrinsic curvature could be a projector on one of them, say $K_{dd} = 1$, $K_{ij} = 0$ for $(i, j) \neq (d, d)$, then $(K_{ij} \Gamma^i \Gamma^j)^2 = g_d^4$ and $K_{ij}K_{ij} \Gamma^i \Gamma^j = g_d^2$. By simply comparing powers of $g_d$ the universal constant $\sigma_2$ could be determined being the only one with $g_d^4$.

Using the metric

$$ds^2 = dr^2 + d\Sigma^2$$

and $K_{ab} = -\Gamma^r_{ab}$, $\Gamma^r_{ab}$ being the Christoffel symbol, it is seen that keeping the manifold topologically as $I \times T^d$, the metric

$$d\Sigma^2 = dx_1^2 + ... + dx_{d-1}^2 + r^2 dx_d^2$$

will have the above property. This is clearly the flat manifold $B^2 \times T^{d-1}$ and the eigenvalue problem is easily solved. With the notation $\vec{n}_i^2 = n_1^2 + ... + n_d^2$,
the eigenfunctions are
\[ J_{|n|d}(r \sqrt{\alpha^2 - \vec{n}_t^2}) e^{i(x_{1n_1} + \ldots + x_{dn_d})}, \vec{n}_t^2 \in \mathbb{Z}^d \]
with eigenvalues \( \alpha^2 \). The boundary condition takes the form
\[ \sqrt{\alpha^2 - \vec{n}_t^2} J'_{|n|d}(\sqrt{\alpha^2 - \vec{n}_t^2}) + (g_d n_d + gn - S) J_{|n|d}(\sqrt{\alpha^2 - \vec{n}_t^2}) = 0, \quad (83) \]
where we have used \( n = n_i \) (the result is the same for any \( i \in \{1, \ldots, d-1\} \)).

Our main interest is to determine \( \sigma_2 \) and \( \sigma_4 \). The calculation involving two tangential derivatives will be seen to be sufficiently difficult so that we will restrict ourselves to what is strictly necessary, namely we will not bother to do the smeared calculation but will be content with the sufficient choice \( f(r) = 1 \). The information derived about \( c_3 + c_4, c_7, c_8, \lambda_1 \) will serve as a further check of the previous calculation, the new quantities, \( \sigma_2 \) and \( \sigma_3 + \sigma_4 \) will complete our analysis of \( a_{3/2}(f) \), for covariantly constant \( \Gamma^i \).

One can proceed very much as before. Starting with a contour representation similar to eq. (20) and shifting the contour to the imaginary axis, one gets
\[ \zeta(s) = \frac{\sin \pi s}{\pi} \sum_{\vec{n} \in \mathbb{Z}^d / \{0\}} \int_{|n|}^\infty dk (k^2 - \vec{n}_t^2)^{-s} \frac{\partial}{\partial k} \ln \left( k I'_{|n|d}(k) + [g_d n_d + gn - S] I_{|n|d}(k) \right). \quad (84) \]
It is seen, that \( \vec{n}_t^2 \) acts effectively as a mass of the field and comparing with the previously treated example \( n_d \) plays the role of \( \nu \).

The general procedure of dealing with \( \zeta(s) \) is the same as in section 3. However several complications arise, and the situation is sufficiently different so as to warrant further description.

One starts from the uniform asymptotic expansion of the Bessel function \[ \text{[46]} \] and eq. (22) is found with the replacements \( gn_d \rightarrow g_d n_d + gn, u \rightarrow -S \) and \( \nu \rightarrow |n_d| \). Asymptotically one again finds eq. (23) with the characteristic differences already described. The several new features arising are now dealt with by looking at
\[ A_+(s) = \frac{\sin \pi s}{\pi} \sum_{\vec{n} \in \mathbb{Z}^d / \{0\}} \sum_{n_d = -\infty}^\infty \int_{|n|/n_d}^\infty dz \left[ z^2 n_d^2 - \vec{n}_t^2 \right]^{-s} \frac{\partial}{\partial z} \ln \left( 1 + \frac{gn + g_d n_d}{|n_d| \sqrt{1 + z^2}} \right). \quad (85) \]
The integral is nothing but a hypergeometric function \[46\] and we get

\[ A_+(s) = \frac{1}{2\Gamma(s)} \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(s + (l + 1)/2)}{\Gamma((l + 3)/2)} \]

\[ \sum_{\vec{n}_t \in \mathbb{Z}^{d-1}/\{0\}} \sum'_{n_d = -\infty} (gn + gd^2n_d)^{l+1}(\vec{n}_t^2)^{-s-(l+1)/2} \]

\[ 2F_1 \left( \frac{l+3}{2}, s + \frac{l+1}{2}, -\frac{2}{\vec{n}_t^2}; -\frac{|n_d|^2}{\vec{n}_t^2} \right), \quad (86) \]

The apparent difficulty is to extract the meromorphic structure of multiple sums of hypergeometric functions. This is very effectively done by using the Mellin-Barnes integral representation of \(2F_1\) \[16\],

\[ 2F_1(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt \frac{\Gamma(\alpha + t)\Gamma(\beta + t)\Gamma(-t)}{\Gamma(\gamma + t)} (-z)^t, \]

where the contour is chosen such that the poles of the function \(\Gamma(\alpha + t)\) and \(\Gamma(\beta + t)\) lie to the left of the path of integration and the poles of the function \(\Gamma(-t)\) lie to the right of it. When using this integral representation it is seen that the sum over \(\vec{n}_t\) leads to \((d-1)\)-dimensional Epstein type zeta functions, whereas the sum over \(n_d\) gives a Riemann zeta function. The relevant zeta function is again of the Epstein type \[19\] which we now write as

\[ E_{l,2l}(s) = \sum_{\vec{n}_t \in \mathbb{Z}^{d-1}/\{0\}} (\vec{n}_t^2)^{-s}n_t^{2l}. \quad (87) \]

As a result of the described steps one arrives at

\[ A_+(s) = \frac{1}{\Gamma(s)} \sum_{l=1}^{\infty} \sum_{k=0}^{l} \frac{1}{l!} \left( \frac{2l}{2k} \right) g^2 g_d^{2l-2k} \]

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt \Gamma(s + l + t)\Gamma(-t)\zeta_R(-2t - 2l + 2k) E_{l,2k}(s + l + t), \quad (88) \]

where the contour (depending on \(l\) and \(k\)) is such that the poles of \(\zeta_R\) lie to the right of the contour, the poles of \(E_{l,2k}\) to the left of it. This Mellin-Barnes representation allows the meromorphic structure of \(A_+(s)\) to be read off by closing the contour to the left. We then encounter poles of \(\Gamma(s + l + t)\) at
\[ t = -s - l - m, \quad m \in \mathbb{N}_0 \text{ with residues } \Gamma(s + l + m) \zeta_R(2s + 2k + 2m) E_{t,2k}(m). \]

The right-most pole lies at \( s = 1/2 \) and it is clear that the poles of the \( \Gamma \)-function are irrelevant for our purposes. However, the pole of the Epstein function is situated at \( t = (d-1)/2 - l - s + k \) and gives relevant contributions.

Keeping only these terms,

\[
A_+(s) \sim \frac{\pi^{(d-1)/2}}{\Gamma(s)} \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{k=0}^{l} \left(\frac{2l}{2k}\right) \frac{\Gamma(k + 1/2)}{\sqrt{\pi}} \Gamma(s + l - k - (d - 1)/2) \\
\zeta_R(2s - d + 1) g^{2k} g_d^{2l-2k}.
\]  

The right most pole at \( s = d/2 \) comes from the Riemann zeta function. Using the relations \[10\]

\[
\frac{\Gamma(x)}{\Gamma(2x)} = \frac{\sqrt{\pi}}{2^{2x-1} \Gamma(x + 1/2)}
\]

and

\[
\frac{(2l)!}{l!2^{2l}} = \frac{(2l - 1)!}{2^l} = \frac{\Gamma(l + 1/2)}{\sqrt{\pi}}
\]

one gets

\[
\text{Res } A_+(d/2) = \frac{\pi^{d/2}}{2 \Gamma(d/2)} \left\{ (1 - g^2 - g_d^2)^{-1/2} - 1 \right\}.
\]

The calculation shows the manner in which \( \Gamma^2 = g^2 + g_d^2 \) is built up. Together with the contribution of \( A_0(s) \) one finds the correct coefficient \( a_{1/2} \).

The next pole in \( (74) \) at \( s = (d - 1)/2 \) comes from the \( \Gamma \)-function for \( k = l \). Then

\[
\text{Res } A_+((d - 1)/2) = -\frac{\pi^{(d-1)/2}}{2 \Gamma((d - 1)/2)} \left\{ (1 - g^2)^{-1/2} - 1 \right\}.
\]

This piece is cancelled by another contribution, but the example shows the way other contributions than \( \Gamma^2 \) appear.

There are no further (interesting) poles due to the zeroes of \( \zeta_R(s) \) at \( s = -2m, \quad m \in \mathbb{N} \).

The basic characteristics, namely that the integrals are hypergeometric functions and that the eigenvalue sums may be dealt with by Mellin-Barnes
integral representations of these, are present for all other \( A_j(s) \). The final results can all be written in terms of \( {}_2F_1 \) or \( {}_3F_2 \) or their derivatives, very much as in section 3. For the sake of space it is impossible to give further details.

In summary we confirm our results for the universal constants \( c_3 + c_4, c_7, c_8 \) and \( \lambda_1 \). Most important, we determine the remaining constants to be

\[
\sigma_2 = \frac{1}{\Gamma^8 (1 + \Gamma^2)^2} \left[ -48 \left( -5 \Gamma^6 + 16 \left( -1 + \sqrt{1 + \Gamma^2} \right) + 8 \Gamma^2 \left( -5 + 4 \sqrt{1 + \Gamma^2} \right) \right) + \Gamma^4 \left( -30 + 16 \sqrt{1 + \Gamma^2} \right) \right] \tag{92}
\]

\[
\sigma_4 = \frac{32 \left( -\Gamma^4 + 16 \left( -1 + \sqrt{1 + \Gamma^2} \right) + 2 \Gamma^2 \left( -7 + 3 \sqrt{1 + \Gamma^2} \right) \right)}{\Gamma^6 \sqrt{1 + \Gamma^2}} \tag{93}
\]

and have thereby achieved the goal of determining \( a_{3/2}(f) \) in eq. (8).

### 7 Conclusions

In this article we have developed a technique for the calculation of smeared heat-kernel coefficients on generalized cones or manifolds of the type \( B^n \times T^{D-n} \) for operators of Laplace type with oblique boundary conditions. These boundary conditions arise in response to questions in quantum gravity, gauge theory and string theory \([21, 23, 24, 26, 27, 31]\). The asymptotic properties of the generalized boundary conditions encoded in the asymptotic heat-kernel expansions are considerably more involved than the corresponding ones for the traditional conditions. This article makes an attempt to provide a practical approach for the calculation of coefficients to any order needed although we suspect that the involved analysis, complicated results and restrictions indicate that further analysis along the lines of this paper should not be lightly undertaken without strong motivation.

The direct calculation of higher coefficients for general curved manifolds with arbitrary smooth boundaries becomes very difficult and impractical. In the approach promoted here (see also \([38]\)) this analysis is avoided. Based on the observation that functorial methods give relations between the numerical multipliers in the heat-kernel coefficients \([12]\) the remaining task is to find as many multipliers as needed by other means. A rich source of information is special case calculation. Done in a systematic fashion in arbitrary dimensions
and with a smearing function as general as necessary, we have seen that simple comparison with the general form of the coefficient yields the encoded information on the universal constants. Several checks on the constants found are provided by the conformal relations as well as by the different examples treated. A systematic feature of the approach is the use of algebraic computer programs.

It is clear that the approach can also be applied to the calculation of $a_2$, however, more than 100 terms are involved and a very real additional effort is necessary as well as a substantial motivation. Generalization to covariantly nonconstant $\Gamma^i$ is desirable and can be attacked by taking different base manifolds (eg. a sphere) or by considering simple dependences of $\Gamma^i$ on one of the tangential variables. Finally, it is hoped that also for the non-Abelian setting new information can be obtained in the spirit of the present work.

Acknowledgments
We wish to thank Michael Bordag for very helpful discussions.
This investigation has been partly supported by the DFG under contract number BO1112/4-2.

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