Slicing convex sets and measures by a hyperplane

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Abstract

Given convex bodies $K_1, \ldots, K_d$ in $\mathbb{R}^d$ and numbers $\alpha_1, \ldots, \alpha_d \in [0, 1]$, we give a sufficient condition for existence and uniqueness of an (oriented) halfspace $H$ with $\text{Vol}(H \cap K_i) = \alpha_i \cdot \text{Vol} K_i$ for every $i$. The result is extended from convex bodies to measures.
1 Transversal spheres

A well known result in elementary geometry states that there is a unique sphere which contains a given set of $d+1$ points in general position in $\mathbb{R}^d$. A similar thing happens with $d$-pointed sets and hyperplanes. What happens if we consider convex bodies instead of points?

These questions are the main motivation for the present paper. The first result in this direction is due to H. Kramer and A.B. Németh \cite{7}. They used the following, very natural definition.

A family $\mathcal{F}$ of connected sets in $\mathbb{R}^d$ is said to be well separated, if for any $k \leq d+1$ distinct elements, $K_1, \ldots, K_k$, of $\mathcal{F}$ and for any choice of points $x_i \in \text{conv}K_i$, the set $\text{aff}\{x_1, \ldots, x_k\}$ is a $(k-1)$-dimensional flat. Here $[k]$ stands for the set $\{1, 2, \ldots, k\}$. It is well known (cf. \cite{4} and \cite{1}), and also easy to check the following.

**Proposition 1** Assume $\mathcal{F} = \{K_1, \ldots, K_n\}$ is a family of connected sets in $\mathbb{R}^d$. The following conditions are equivalent:

1. The family $\mathcal{F}$ is well separated.

2. For every pair of disjoint sets $I, J \subset [n]$ with $|I| + |J| \leq d+1$, there is a hyperplane separating the sets $K_i$, $i \in I$ from the sets $K_j$, $j \in J$.

By an elegant application of Brouwer’s fixed point theorem, Kramer and Németh proved the following:

**Theorem KN.** Let $\mathcal{F}$ be a well separated family of $d+1$ compact convex sets in $\mathbb{R}^d$. Then there exists a unique Euclidean ball which touches each set and whose interior is disjoint from each member of $\mathcal{F}$.

Denote by $B(x, r)$, resp. $S(x, r)$, the Euclidean ball and sphere of radius $r$ and center $x$. We say that the sphere $S(x, r)$ supports a compact set $K$ if
S(x, r) ∩ K ≠ ∅ and either K ⊂ B(x, r) or K ∩ int B(x, r) = ∅. This definition is due to V. Klee, T. Lewis, and B. Von Hohenbalken \[5\]. They proved the following:

**Theorem KLH.** Let \( F = \{ K_1, K_2, \ldots, K_{d+1} \} \) be a well separated family of compact convex sets in \( \mathbb{R}^d \), and let \( I, J \) be a partition of \([d+1]\). Then there is a unique Euclidean sphere \( S(x, r) \) that supports each element of \( F \) in such a way that \( K_i \subset B(x, r) \) for each \( i \in I \) and \( K_j \cap \text{int} B(x, r) = \emptyset \) for each \( j \in J \).

The case \( I = \emptyset \) corresponds to Theorem KN. We are going generalize these results. Let \( Q^d = [0, 1]^d \) denote the unit cube of \( \mathbb{R}^d \). Given a well separated family \( F \) of convex sets in \( \mathbb{R}^d \), a sphere \( S(x, r) \) is said to be transversal to \( F \) if it intersects every element of \( F \). Finally, a convex body in \( \mathbb{R}^d \) is a convex compact set with nonempty interior.

**Theorem 1** Let \( F = \{ K_1, \ldots, K_{d+1} \} \) be a well separated family of convex bodies in \( \mathbb{R}^d \), and let \( \alpha = (\alpha_1, \ldots, \alpha_{d+1}) \in Q^{d+1} \). Then there exists a unique transversal Euclidean sphere \( S(x, r) \) such that \( \text{Vol} (B(x, r) \cap K_i) = \alpha_i \cdot \text{Vol} (K_i) \) for every \( i \in [d+1] \).

**Remark 1.** The transversality of \( S(x, r) \) only matters when \( \alpha_i \) is equal to 0 or 1; otherwise the condition \( \text{Vol} (B(x, r) \cap K_i) = \alpha_i \cdot \text{Vol} (K_i) \) plus convexity guarantees that \( S(x, r) \) intersects \( K_i \).

### 2 Transversal hyperplanes and halfspaces

In a similar direction, S.E. Cappell, J.E. Goodman, J. Pach, R. Pollack, M. Sharir, and R. Wenger \[3\] proved an analogous theorem for the case of supporting hyperplanes, which can be seen as spheres of infinite radius. Given a family \( F \) of sets in \( \mathbb{R}^d \), a hyperplane will be called **transversal** to \( F \) if it intersects each member of \( F \). The following result is a special case of Theorem 3 of Cappell et al. \[3\] (cf \[2\] as well):
Theorem C. Let \( F = \{K_1, \ldots, K_d\} \) be a well separated family of compact convex sets in \( \mathbb{R}^d \) with a partition \( I, J \) of the index set \([d]\). Then there are exactly two hyperplanes, \( H_1 \) and \( H_2 \), transversal to \( F \) such that both \( H_1 \) and \( H_2 \) have all \( K_i \) (\( i \in I \)) on one side and all \( K_j \) (\( j \in J \)) on the other side.

Theorem C was also proved by Klee et al. [6] using Kakutani’s extension of Brouwer’s fixed point theorem. We are going to formulate this theorem in a slightly different way, more suitable for our purposes. So, we need to introduce new notation and terminology.

A halfspace \( H \) in \( \mathbb{R}^d \) can be specified by its outer unit normal vector, \( v \), and by the signed distance, \( t \in \mathbb{R} \), of its bounding hyperplane from the origin. Thus, there is a one-to-one correspondence between halfspaces of \( \mathbb{R}^d \) and pairs \((v, t) \in S^{d-1} \times \mathbb{R}\). We denote the halfspace \( \{x \in \mathbb{R}^d : \langle x, v \rangle \leq t\} \) by \( H(v \leq t) \). Analogously we write \( H(v = t) = \{x \in \mathbb{R}^d : \langle x, v \rangle = t\} \), which is the bounding hyperplane of \( H(v \leq t) \). Furthermore, given a set \( K \subset \mathbb{R}^d \), a unit vector \( v \) and a scalar \( t \), we denote the set \( H(v = t) \cap K \) by \( K(v = t) \), analogously \( K(v \leq t) = H(v \leq t) \cap K \).

Suppose next that \( F = \{K_1, \ldots, K_d\} \) is a well separated family of convex sets in \( \mathbb{R}^d \). Assume \( a_1 \in K_1, \ldots, a_d \in K_d \). The unit normal vectors to the unique transversal hyperplane containing these points are \( v \) and \( -v \). We want to make the choice between \( v \) and \( -v \) unique and depend only on \( F \). We first make it depend on \( a_1, \ldots, a_d \). Define \( v = v(a_1, \ldots, a_d) \) as the (unique) unit normal vector to \( \text{aff}\{a_1, \ldots, a_d\} \) satisfying

\[
\det \begin{vmatrix}
  v & a_1 & a_2 & \cdots & a_d \\
  0 & 1 & 1 & \cdots & 1
\end{vmatrix} > 0,
\]

in other words, the points \( v + a_1, a_1, a_2, \ldots, a_d \), in this order, are the vertices of a positively oriented \( d \)-dimensional simplex. Clearly, with \( -v \) in place of \( v \) the determinant would be negative. This gives rise to the map \( v : K \longrightarrow S^{d-1} \) where \( K = K_1 \times \cdots \times K_d \). This definition seems to depend on the choice of the \( a_i \), but in fact, it does not. Write \( H(v = t) = \text{aff}\{a_1, \ldots, a_d\} \).

**Proposition 2** Under the previous assumption, let \( b_i \in K_i(v = t) \) for each \( i \). Then \( v(a_1, \ldots, a_d) = v(b_1, \ldots, b_d) \)

**Proof.** This is simple. The homotopy \((1 - \lambda)a_i + \lambda b_i \ (\lambda \in [0, 1]) \) moves the \( a_i \).
to the $b_i$ continuously, and keeps $(1 - \lambda)a_i + \lambda b_i$ in $K_i(v = t)$. The affine hull of the moving points remains unchanged, and does not degenerate because $\mathcal{F}$ is well separated. So their outer unit normal remains $v$ throughout the homotopy. □

The previous proposition is also mentioned by Klee et al. [6]. With this definition, a transversal hyperplane to $\mathcal{F}$ determines $v$ and $t$ uniquely. We call $H(v = t)$ a positive transversal hyperplane to $\mathcal{F}$, and similarly, $H(v \leq t)$ is a positive transversal halfspace to $\mathcal{F}$.

**Theorem 2** Let $\mathcal{F} = \{K_1, \ldots, K_d\}$ be a family of well separated convex bodies in $\mathbb{R}^d$, and let $\alpha = (\alpha_1, \ldots, \alpha_d) \in Q^d$. Then there is a unique positive transversal halfspace, $H$, such that $\text{Vol}(K_i \cap H) = \alpha_i \cdot \text{Vol}(K_i)$ for every $i \in [d]$.

Theorem C follows since the partition $I, J$ gives rise to $\alpha, \beta \in Q^d$ via $\alpha_k = 1$ if $k \in I$, otherwise $\alpha_k = 0$, and $\beta_k = 1$ if $k \in J$, otherwise $\beta_k = 0$. By Theorem [2] there are unique positive transversal halfspaces $H(\alpha)$ and $H(\beta)$ with the stated properties. Their bounding hyperplanes satisfy the statement of Theorem C and they are obviously distinct. We mention, however, that Theorem C will be used in the proof of the unicity part of Theorem [2].

**Remark 2.** When all $\alpha_i = 1/2$, the existence of such a halfspace is guaranteed by Borsuk’s theorem, even without the condition of convexity or $\mathcal{F}$ being well separated. (Connectivity of the sets implies that the halving hyperplane is a transversal to $\mathcal{F}$.) The case of general $\alpha_i$, however, needs some extra condition as the following two examples show. If all $K_i$ are equal, then each oriented hyperplane section cuts off the same amount from each $K_i$, so $\alpha_1 = \cdots = \alpha_d$ must hold. The second example consists of $d$ concentric balls with different radii, and if the radius of the first ball is very large compared to those of the others and $\alpha_1$ is too small, then a hyperplane cutting off $\alpha_1$ fraction of the first ball is disjoint from all other balls. Thus no hyperplane transversal exists that cuts off an $\alpha_1$ fraction of the first set.

**Remark 3.** Cappell et al. prove, in fact, a much more general theorem [3]. Namely, assume that $\mathcal{F}$ is well separated and consists of $k$ strictly convex sets, $k \in \{2, \ldots, d\}$ and let $I, J$ be a partition of $[k]$. Then the set of all
supporting hyperplanes separating the $K_i$ ($i \in I$) from the $K_j$ ($j \in J$) is homeomorphic to the $(d - k)$-dimensional sphere.

3 Extension to measures

Borsuk’s theorem holds not only for volumes but more generally for measures. Similarly, our Theorem 2 can and will be extended to nice measures that we are to define soon. We need a small piece of notation.

Let $\mu$ be a finite measure on the Borel subsets of $\mathbb{R}^d$ and let $v \in S^{d-1}$ be a unit vector. Define

$$t_0 = t_0(v) = \inf \{t \in \mathbb{R} : \mu(H(v \leq t)) > 0\},$$
$$t_1 = t_1(v) = \sup \{t \in \mathbb{R} : \mu(H(v \leq t)) < \mu(\mathbb{R}^d)\}.$$

Note that $t_0 = -\infty$ and $t_1 = \infty$ are possible.

Let $H(s_0 \leq v \leq s_1)$ denote the closed slab between the hyperplanes $H(v = s_0)$ and $H(v = s_1)$. Define the set $K$ by

$$K = \bigcap_{v \in S^{d-1}} H(t_0(v) \leq v \leq t_1(v)).$$

$K$ is called the support of $\mu$. Note that $K$ is convex (obviously) and $\mu(\mathbb{R}^d \setminus K) = 0$.

Definition 1 The measure $\mu$ is called nice if the following conditions are satisfied:

(i) $t_0(v)$ and $t_1(v)$ are finite for every $v \in S^{d-1},$

(ii) $\mu(H(v = t)) = 0$ for every $v \in S^{d-1}$ and $t \in \mathbb{R},$

(iii) $\mu(H(s_0 \leq v \leq s_1)) > 0$ for every $v \in S^{d-1}$ and for every $s_0, s_1$ satisfying $t_0(v) \leq s_0 < s_1 \leq t_1(v).$
If \( \mu \) is a nice measure, then its support is full-dimensional since, by (ii), it is not contained in any hyperplane.

The function \( t \mapsto \mu(K(v \leq t)) \) is zero on the interval \((-\infty, t_0]\), is equal to \( \mu(K) \) on \([t_1, \infty)\), strictly increases on \([t_0, t_1]\), and, in view of (iii), is continuous. Assume \( \alpha \in [0, 1] \). Then there is a unique \( t \in [t_0, t_1] \) with

\[
\mu(K(v \leq t)) = \alpha \cdot \mu(K).
\]

Denote this unique \( t \) by \( g(v) \); this way we defined a map \( g : S^{d-1} \to \mathbb{R} \). The following simple lemma is important and probably well known.

**Lemma 1** For fixed \( \alpha \in [0, 1] \) the function \( g \) is continuous.

**Proof.** When \( \alpha = 1 \), \( g(v) \) is the support functional of \( K \), which is not only continuous but convex (when extended to all \( v \in \mathbb{R}^d \)). Similarly, \( g \) is continuous when \( \alpha = 0 \).

Assume now that \( 0 < \alpha < 1 \). Let \( v_0 \in S^{d-1} \) be an arbitrary point. In order to prove the continuity of \( g \) at \( v_0 \) we show first that \( K(v = g(v)) \) and \( K(v_0 = g(v_0)) \) have a point in common whenever \( v \) is close enough to \( v_0 \).

Obviously, \( K(v_0 = g(v_0)) \) is a \((d-1)\)-dimensional convex set lying in the hyperplane \( H(v_0 = g(v_0)) \). Then, for every small enough neighbourhood of \( v_0 \), and for each \( v \) in such a neighbourhood, the supporting hyperplane of \( K \) with unit normal \( v \) (and \(-v\)) is also a supporting hyperplane of \( K(v_0 \geq g(v_0)) \) (and \( K(v_0 \leq g(v_0)) \)).

Assume \( s_\nu \leq S_\nu \) and let \( H(v = s_\nu) \) and \( H(v = S_\nu) \) be the two supporting hyperplanes (with normal \( v \)) to \( K(v_0 = g(v_0)) \) which is a \((d-1)\)-dimensional convex set. Since \( K(v_0 = g(v_0)) \) is a \((d-1)\)-dimensional convex set, condition (iii) implies that \( s_\nu < S_\nu \). It follows that

\[
K(v \leq s_\nu) \subset K(v_0 \leq g(v_0)) \subset K(v \leq S_\nu),
\]

and so

\[
\mu(K(v \leq s_\nu)) \leq \mu(K(v_0 \leq g(v_0)) \leq \mu(K(v \leq S_\nu)).
\]
As \( \mu(K(v_0 \leq g(v))) = \alpha \cdot \mu(K) \), we have \( s_v \leq g(v) \leq S_v \). Consequently, \( K(v = g(v)) \) and \( K(v_0 = g(v_0)) \) have a point, say \( z = z(v) \), in common. This \( z(v) \) is not uniquely determined but that does not matter.

It is easy to finish the proof now. Clearly \( g(v) = \langle v, z(v) \rangle \) and \( g(v_0) = \langle v_0, z(v_0) \rangle \) for all \( v \) in a small neighbourhood of \( v_0 \). Assume the sequence \( v_n \) tends to \( v_0 \). We claim that every subsequence, \( v'_{n'} \), of \( v_n \) contains a subsequence \( v''_{n''} \) such that \( \lim g(v''_{n''}) = g(v_0) \), which evidently implies the continuity of \( g \) at \( v_0 \).

For the proof of this claim observe first that, since \( K(v_0 = g(v_0)) \) is compact, \( z(v_{n'}) \) contains a convergent subsequence \( z(v_{n''}) \) tending to \( z_0 \), say. Taking limits gives \( z_0 \in K(v_0 = g(v_0)) \). Then \( g(v_{n''}) = \langle v_{n''}, z(v_{n''}) \rangle \to \langle v_0, z_0 \rangle = g(v_0) \). □

Theorem 2 is extended to measures in the following way.

**Theorem 3** Suppose \( \mu_i \) is a nice measure on \( \mathbb{R}^d \) with support \( K_i \) for all \( i \in [d] \). Assume the family \( F = \{K_1, \ldots, K_d\} \) is well separated and let \( \alpha = (\alpha_1, \ldots, \alpha_d) \in Q^d \). Then there is a unique positive transversal halfspace, \( H \), such that \( \mu_i(K_i \cap H) = \alpha_i \cdot \mu_i(K_i) \), for every \( i \in [d] \).

**Corollary 1** Assume \( \mu_i \) are finite measures on \( \mathbb{R}^d \) satisfying conditions (i) and (ii) of Definition 7. Let \( K_i \) be the support of \( \mu_i \) for all \( i \in [d] \). Suppose the family \( F = \{K_1, \ldots, K_d\} \) is well separated and let \( \alpha = (\alpha_1, \ldots, \alpha_d) \in Q^d \). Then there is a positive transversal halfspace, \( H \), such that \( \mu_i(K_i \cap H) = \alpha_i \cdot \mu_i(K_i) \), for every \( i \in [d] \).

The corollary easily follows from Theorem 3; we omit the simple details.

Theorem 2 is a special case of Theorem 3 when \( \mu_i \) is the Lebesgue measure (or volume) restricted to the convex body \( K_i \) for all \( i \in [d] \) and the family \( F \) is well separated. Also, Theorem C is a special case of Theorem 3 when \( \mu_i \) and \( K_i \) are the same as above, and, for a given partition \( I, J \) of \( [d] \), we set \( \alpha_i = 1 \) for \( i \in I \), and \( \alpha_j = 0 \) for \( j \in J \). Theorem 11 follows from Theorem 3 via “lifting to the paraboloid”. This is explained in the last section.
4 Proof of Theorem 3.

In the proof we will use Brouwer’s fixed point theorem. We will define a continuous mapping from a topological ball to itself, such that a fixed point of this map yields a halfspace with the desired properties. Set $K = K_1 \times \ldots \times K_d$. Given a point $x = (x_1, \ldots, x_d) \in K$ we consider the hyperplane $\text{aff} \{x_1, \ldots, x_d\}$. Since the family $\mathcal{F}$ is well separated, this hyperplane is well defined for each $x \in K$. Let $H(v \leq t)$ be the (unique) positive transversal halfspace whose bounding hyperplane is $\text{aff} \{x_1, \ldots, x_d\}$.

In Section 2 we defined the map $v : K \rightarrow S^{d-1}$ which is the properly chosen unit normal to $\text{aff} \{x_1, \ldots, x_d\}$. Clearly, this function is continuous.

We prove existence first. We start with the case when $\alpha_i \in (0, 1)$ for every $i \in [d]$. We turn to the remaining case later by constructing a suitable sequence of halfspaces.

Let $g_i : S^{d-1} \rightarrow \mathbb{R}$ be the function such that for each $v \in S^{d-1}$, $g_i(v)$ is the real number for which $\mu_i(K_i(v \leq g_i(v))) = \alpha_i \cdot \mu_i(K_i)$ for each $i \in [d]$. Each $g_i$ is a continuous function by Lemma 1. Let $h : S^{d-1} \rightarrow K$ be the function sending $v \mapsto (s_1, \ldots, s_d)$ where $s_i$ is the Steiner point of the $(d-1)$-dimensional section, $K_i(v = g_i(v))$ for each $i \in [d]$. As is well known, the family of sections $K_i(v = t)$ depend continuously (according to the Hausdorff metric) on the corresponding family of hyperplanes, $\{H(v = t)\}$ whenever every section is $(d-1)$-dimensional, which is obviously the case because $\alpha_i \in (0, 1)$. It is also well known that the function that assigns to a compact convex set its Steiner point is continuous. Hence, $h$ is a continuous function.

It follows that $f := h \circ v : K \rightarrow K$ is a continuous function. As $K$ is a compact convex set in $\mathbb{R}^d \times \ldots \times \mathbb{R}^d$ Brouwer’s fixed point theorem implies the existence of a point $x \in K$ such that $f(x) = x$. Consider a fixed point, $x = (x_1, \ldots, x_d)$, of $f$. Then the halfspace $H(v \leq t)$ whose bounding hyperplane is $\text{aff} \{x_1, \ldots, x_d\}$ is a positive transversal halfspace to $\mathcal{F}$ and it has the required properties.

Next we prove existence for vectors $\alpha = (\alpha_1, \ldots, \alpha_d) \in Q^d$ that may have 0, 1
components as well. Consider the sequence \( \{\alpha^n\} \subset Q^d \), \( \alpha^n = (\alpha^n_1, \ldots, \alpha^n_d) \) (defined for every \( n \geq 2 \)), such that for every entry \( \alpha_i = 0 \) we define \( \alpha_i^n = 0 \), for every entry \( \alpha_i = 1 \) we define \( \alpha_i^n = 1 - \frac{1}{n} \), and for every entry \( \alpha_i \notin \{0, 1\} \) we define \( \alpha_i^n = \alpha_i \). Also, for every \( n \geq 2 \) we consider the unique positive transversal halfspace \( H(v_n \leq t_n) \) with \( \mu_i(K_i(v_n \leq t_n)) = \alpha_i^n \cdot \mu_i(K_i) \), for each \( i \). The compactness of \( K \) implies that the set of all possible \((v, t) \in S^{d-1} \times \mathbb{R}\) such that the hyperplane \( H(v = t) \) is transversal to \( F \) is compact. Thus there exists a convergent subsequence \( \{(v_{n'}, t_{n'})\} \) which converges to a point \((v, t) \in S^{d-1} \times \mathbb{R}\). Clearly, \( H(v \leq t) \) is a positive transversal halfspace to \( F \) which satisfies \( \mu_i(K_i(v \leq t)) = \alpha_i \cdot \mu_i(K_i) \) for every \( i \).

Next comes uniqueness. We start with the 0, 1 case, that is, when \( \alpha = (\alpha_1, \ldots, \alpha_d) \) with all \( \alpha_i \in \{0, 1\} \). Such an \( \alpha \) defines a \( \beta \in Q^d \) via \( \beta_i = 1 - \alpha_i \) for every \( i \). By the previous existence proof there is a unique positive transversal halfspace \( H(v \leq t) \) for \( \alpha \) and another one \( H(u \leq s) \) for \( \beta \). These two halfspaces are distinct, first because \( u = v \) is impossible, and second because of the following fact which implies that \( u \neq -v \).

**Proposition 3** For every pair of points \((a_1, \ldots, a_d)\) and \((b_1, \ldots, b_d)\) in \( K \), \( v(a_1, \ldots, a_d) \) and \( -v(b_1, \ldots, b_d) \) are distinct.

**Proof.** Assume \( v(a_1, \ldots, a_d) = -v(b_1, \ldots, b_d) \). Then the affine hulls of the \( a_i \) and the \( b_i \) are parallel hyperplanes. We use the same homotopy as in the proof of Proposition 2. As \( \lambda \) moves from 0 to 1, the moving points \((1 - \lambda)a_i + \lambda b_i\) stay in \( K_i \), and their affine hull remains parallel with \( \text{aff}\{a_1, \ldots, a_d\} \). So the outer normal remains unchanged throughout the homotopy. A contradiction. \( \square \)

The condition \( \mu_i(K_i(v \leq t)) = \alpha_i \mu_i(K_i) \) implies, in the given case, that all \( K_i \) (\( i \in I \)) are in \( H(v \leq t) \) and all \( K_j \) (\( j \in J \)) are in \( H(v \geq t) \). Thus \( H(v = t) \) is a transversal hyperplane satisfying the conditions of Theorem C with partition \( I, J \) where \( I = \{i \in [d] : \alpha_i = 0\} \) and \( J = \{j \in [d] : \alpha_j = 1\} \). The same way, \( H(u = s) \) is a transversal hyperplane satisfying the conditions of Theorem C with the same partition \( J, I \).

The uniqueness of \( H(v \leq t) \) follows now easily. If we had two distinct positive transversal halfspaces \( H(v_1 \leq t_1) \) and \( H(v_2 \leq t_2) \) for \( \alpha \), then we would have
four distinct transversal hyperplanes with $K_i$ ($i \in I$) on one side and $K_j$ ($j \in J$) on the other side, contradicting Theorem C.

Now we turn to uniqueness for general $\alpha$. Assume that there are two distinct positive transversal halfspaces $H(v_1 \leq t_1)$ and $H(v_2 \leq t_2)$ for $\alpha$. Their bounding hyperplanes cannot be parallel. Define $M = H(v_1 \leq t_1) \cap H(v_2 \geq t_2)$ and $N = H(v_1 \geq t_1) \cap H(v_2 \geq t_2)$. The partition $I, J$ of the index set $[d]$ is defined as follows: $i \in I$ if $M \cap \text{int } K_i \neq \emptyset$ and $j \in J$ if $M \cap \text{int } K_j = \emptyset$. Set $K'_i = M \cap K_i$ for every $i \in I$ and $K'_j = N \cap K_j$ for every $j \in J$. Let $\mathcal{F}'$ be the family consisting of all the convex bodies $K'_i$ ($i \in I$) and $K'_j$ ($j \in J$). It is quite easy to see that no member of $\mathcal{F}'$ is empty. Moreover, $\mathcal{F}'$ is evidently well separated. Given the partition $I, J$, define $\gamma$ by $\gamma_i = 1$ for $i \in I$ and $\gamma_j = 0$ for $j \in J$. Then there are two transversal halfspaces (with respect to $\mathcal{F}'$), namely $H(v_k \leq t_k)$ for $k = 1, 2$ satisfying $\mu_i(K_i(v_k \leq t_k)) = \gamma_i \mu_i(K_i)$ for every $i$. But every $\gamma_i \in \{0, 1\}$ and we just established uniqueness in the 0, 1 case.

□

5 Proof of Theorem 2

We will use the well-known technique of lifting the problem from $\mathbb{R}^d$ to a paraboloid in $\mathbb{R}^{d+1}$, and then apply Theorem 3.

In this section we change notation a little. A point in $\mathbb{R}^d$ is denoted by $x = (x_1, \ldots, x_d)$, a point in $\mathbb{R}^{d+1}$ is denoted by $\mathbf{x} = (x_1, \ldots, x_d, x_{d+1})$. The projection of $\mathbf{x}$ is $\pi(\mathbf{x}) = (x_1, \ldots, x_d)$, and the lifting of $x$ is $\ell(x) = (x_1, \ldots, x_d, |x|^2)$ where $|x|^2 = x_1^2 + \cdots + x_d^2$. Clearly, $\ell(x)$ is contained in the paraboloid

$$P = \{\mathbf{x} \in \mathbb{R}^{d+1} : \mathbf{x} = (x_1, x_2, \ldots, x_d, |x|^2)\}.$$  

A set $K \subset \mathbb{R}^d$ lifts to $\ell(K) = \{\ell(x) \in P : x \in K\}$. Also, $\pi(\ell(K)) = K$.

A hyperplane is called non-vertical if $\pi(H) = \mathbb{R}^d$. The lifting gives a bijective relation between non-vertical hyperplanes in $\mathbb{R}^{d+1}$ (intersecting $P$) and $(d - 1)$-dimensional spheres in $\mathbb{R}^d$ in the following way. Assume $S = S(u, r)$ is the
sphere centered at \( u \), with radius \( r \) in \( \mathbb{R}^d \). Of course, \( \ell(S) \subset P \), but more importantly,

\[
\ell(S) = P \cap H
\]

where \( H \) is the hyperplane with equation \( x_{d+1} = 2\langle u, x \rangle + r^2 - |u|^2 \). Conversely, given a non-vertical hyperplane \( H \) with equation \( x_{d+1} = 2\langle u, x \rangle + s \) where \( s = r^2 - |u|^2 \) with some \( r > 0 \),

\[
\pi(H \cap P) = S(u, r).
\]

As a first application of this lifting, here is a simple proof of a slightly stronger version of Theorem KLH (we can replace the convexity assumption by connectedness). Consider a family of \( d + 1 \) well separated connected compact sets in \( \mathbb{R}^d \) and a partition of the sets into two classes. Lift the family into the paraboloid, and for each lifted set, consider its convex hull. This gives a \( (d + 1) \)-element family of convex bodies in \( \mathbb{R}^{d+1} \). The lifted family is well separated. This can be seen using Proposition \( \mathbb{P} \) the lifting of the separating \( (d - 1) \)-dimensional planes of the original family yield (vertical) separating hyperplanes of the corresponding lifting. Thus Theorem 3 applies to the lifted family (with the obviously induced partition) and gives a hyperplane \( H \) such \( H \cap P \) projects onto a sphere \( S \) in \( \mathbb{R}^d \) satisfying the requirements of Theorem \( \mathbb{P} \) We omit the straightforward detail.

We apply Theorem \( \mathbb{X} \) to the paraboloid lifting to obtain Theorem \( \mathbb{I} \) in the same way. The family \( \mathcal{F} = \{K_1, \ldots, K_{d+1}\} \) lifts to the family \( \ell(\mathcal{F}) = \{\ell(K_1), \ldots, \ell(K_{d+1})\} \), and we define the measures \( \mu_i \) via

\[
\mu_i(C) = \text{Vol} \pi(C \cap \ell(K_i)),
\]

where \( C \) is a Borel subset of \( \mathbb{R}^{d+1} \). Clearly, \( \mu_i \) is finite and \( \ell(\mathcal{F}) \) is well separated. Its support is \( \text{conv} \ell(K_i) \). It is easy to see that \( \mu_i \) is a nice measure by checking that it satisfies all three conditions.

Thus Theorem \( \mathbb{X} \) applies and guarantees the existence of a unique positive transversal (to \( \ell(\mathcal{F}) \)) halfspace \( H \subset \mathbb{R}^{d+1} \) with \( \mu_i(H \cap \ell(K_i)) = \alpha_i \cdot \mu_i(K_i) \) for each \( i \). This translates to the ball \( B = \pi(H \cap P) \) and sphere \( S = \pi(H^0 \cap P) \) (where \( H^0 \) is the bounding hyperplane of \( H \)) as follows: \( S \) is a transversal sphere of the family \( \mathcal{F} \) and \( \text{Vol}(B \cap K_i) = \alpha_i \cdot \text{Vol} K_i \). Unicity of \( S \) follows readily. \( \square \)
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