EXPLICIT CONSTRUCTION AND UNIQUENESS FOR
UNIVERSAL OPERATOR ALGEBRAS OF DIRECTED GRAPHS

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Abstract. Given a directed graph, there exists a universal operator algebra and universal C*-algebra associated to the directed graph. In this paper we give intrinsic constructions of these objects. We provide an explicit construction for the maximal C*-algebra of an operator algebra. We also discuss uniqueness of the universal algebras for finite graphs, showing that for finite graphs the graph is an isomorphism invariant for the universal operator algebra of a directed graph. We show that the underlying undirected graph is a Banach algebra isomorphism invariant for the universal C*-algebra of a directed graph.

There has been significant work in the study of operator algebras associated to combinatorial objects (e.g. groups, semigroups, and graphs). We have continued this study in [4] where the universal operator algebra of a directed graph and the universal C*-algebra of a directed graph were introduced and described. The aim of this paper is twofold: first we refine the construction of the universal operator algebras of directed graphs, then we discuss invariants of the universal algebras of finite directed graphs.

First we use ideas from [2] to define intrinsic norms on OA(Q) the universal operator algebra of a directed graph. This allows a more concrete construction than was given in [4]. We also describe a construction of the maximal C*-envelope of an operator algebra, see [11]. This construction is defined intrinsically using the free product operator algebra construction of Blecher and Paulsen [2]. This suggests that the maximal C*-envelope is not as mysterious as is presumed. In fact having a canonical construction should allow a more detailed study of the maximal operator algebra of a directed graph in particular cases.

Kribs and Power show in [8] that the graph is a complete unitary invariant for the Toeplitz quiver algebra of a directed graph. Recent work on these Toeplitz quiver algebras by Katsoulis and Kribs, [7] and by Solel [13], has demonstrated that the graph is a complete isomorphism invariant for these algebras. In this paper we extend the techniques of [7] to show that for finite graphs the graph is a complete isomorphism invariant for OA(Q). This fact is perhaps not surprising, although the technique requires more subtlety than in [7] and [13]. For the universal C*-algebra of a finite directed graph we are able to show that the underlying directed graph is an isomorphism invariant for the algebra. This is very surprising since the Cuntz-Krieger algebra of a directed graph is not classified by the graph.

Before proceeding, we would like to emphasize a difference between the operator algebras in the present paper and those defined in [10]. When we construct the

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universal operator algebra of a directed graph we consider representations which send vertices to projections. We do not assume that the projections are orthogonal, as was implicit in [10]. This difference provides examples which differ significantly from the Toeplitz quiver algebras.

We remind the reader of some definitions which can be found in [4]. If $Q$ is a directed graph we will let $V$ and $E$ be the vertex and edge sets, respectively. We let $W(Q)$ be the set of finite words in $E \cup V$ and make $W(Q)$ a semigroup by defining multiplication via concatenation.

We would also like to clarify the construction of the universal operator algebra of a directed graph from [4]. It was implicit in the construction that the set $w(Q)$ was the set of all finite words in $V(Q) \cup E(Q)$ subject to the relations $r(e)e = e = es(e)$. Here $r : E \cup V \to V$ and $s : E \cup V \to V$ are the range and source map extended to $E \cup V$ by defining $r(v) = v = s(v)$ for all $v \in V$.

We point out the connection between Section 2 and the results in [11]. There the Gelfand-Naimark seminorm is defined on a Banach $^*$-algebra. It turns out that the seminorm we define in Section 2 is equal to the Gelfand-Naimark seminorm. This yields a more direct approach the material in Section 2. We have used both methods, we concretely define the seminorm to emphasize that the norm is intrinsic, and then we reference Palmer’s work for completeness.

We now emphasize an important fact from the latter parts of this paper. In sections 4, 5 and 6 we restrict our attention to finite graphs.

1. Intrinsic norms for $OA(Q)$

We begin by adapting a technique of [2] to provide intrinsic norms on $OA(Q)$. Recall that for a monoid $M$ an intrinsic norm is described in [2] for the universal operator algebra associated to a monoid, denoted $O(M)$. Traditionally the norm on a universal operator algebra of an object is defined by taking the supremum over all representations of the object as an operator algebra. Sometimes we are able to define this norm without reference to the representations. We will call such a norm an intrinsic norm. We will construct an intrinsic norm for the universal operator algebra of a directed graph.

Let $M$ be a semigroup without identity. It is well known [6] that there is a monoid $M^+$ and a homomorphism $\tau : M \to M^+$ which is one-to-one. It is a consequence of the universal properties of the unitization [9] of an operator algebra that there is a completely isometric inclusion $\tilde{\tau} : O(M) \to O(M^+)$ induced by $\tau$. Further, by the definitions of $O(M^+)$ and $O(M)$ we know that $\tilde{\tau}$ is a completely isometric isomorphism onto its range, in particular $\|x\|_{O(M)} = \|\tau(x)\|_{O(M^+)}$. The following lemma now follows.

Lemma 1. If $M$ is a semigroup without identity having no zero divisors, then $O(M)^+ = O(M^+)$.

Now for $U \in M_n(\mathcal{C}M)$ define

$$\|U\|_n = \inf\{\|A_0\|\|A_1\| \cdots \|A_m\|\}$$

where $A_0 \in M_{n,k}, A_m \in M_{k,n}$, $A_i \in M_k$ and $U = A_0M_1A_2 \cdots M_mA_m$ where $M_i$ is a diagonal $k \times k$ matrix with entries in $M^+$. It is a consequence of [2] that if

$$N = \{x \in \mathcal{C}M : \|x\|_1 = 0\}$$

then $O(M)/N$ is an operator algebra with the matricial norms given by $\|\cdot\|_n$. 
We now turn to the context of universal operator algebras of directed graphs. We will denote by \( W(Q) \) the set of all finite words in the alphabet given by \( E(Q) \cup V(Q) \). We will denote the range and source map by \( r \) and \( s \) respectively and we will extend their definitions to all of \( V(Q) \cup E(Q) \) by defining \( r(v) = s(v) = v \) for all \( v \in V(Q) \).

**Definition 1.** Let \( Q \) be a directed graph and let \( w \in W(Q) \). We say that a word \( w_1w_2 \cdots w_n \in W(Q) \) is reduced if \( r(w_i) \neq w_{i-1}, s(w_i) \neq w_{i+1}, s(w_{i-1}) \neq w_i \), and \( r(w_{i+1}) \neq w_i \). We denote by \( w(Q) \), the set of reduced words in \( W(Q) \).

**Proposition 1.** For a directed graph \( Q \), \( w(Q) \) is a semigroup. Further \( w(Q) \) has an identity if and only if \( V(Q) \) is a singleton.

**Proof.** Certainly \( W(Q) \) is a semigroup. Further the operation of reducing a word is terminating and locally confluent and hence each word \( w \in W(Q) \) has a unique reduced word \( w(Q) \) associated to it. It follows that \( w(Q) \) is a semigroup, with operation given by concatenation followed by reduction.

Now if \( V(Q) \) has a single vertex \( v \), then for an arbitrary edge \( e \) the reduced word for \( ve \) is \( e \), the reduced word for \( ev \) is \( e \) and the reduced word for \( vv \) is \( v \). It follows that \( v \) will serve as an identity element in \( w(Q) \).

If \( w(Q) \) has an identity element \( \iota \) then \( \iota^2 = \iota \) and hence \( \iota \) corresponds to a vertex, since vertices give rise to the only idempotents in \( w(Q) \). On the other hand let \( v \) be a vertex in \( Q \). Then since \( \iota \) is an identity \( \iota v = v \) and hence \( \iota = v \).

We let \( \| \cdot \|_n \) denote the matricial norm on \( O(w(Q)) \). It is a consequence of the proof of Proposition 2.1 in [4] that the subspace \( \{ x : \| x \|_1 = 0 \} \) is the trivial subspace and hence \( \| \cdot \|_n \) yields a norm on \( \mathbb{C}w(Q) \).

**Theorem 1.** Let \( Q \) be a directed graph, then \( O(w(Q)) \) is completely isometrically isomorphic to \( OA(Q) \).

**Proof.** Recalling the construction of \( OA(Q) [4] \) the algebra \( \mathbb{C}w(Q) \) is a dense subalgebra of \( OA(Q) \). Further the norm on \( \mathbb{C}w(Q) \) is the universal norm induced by representations of \( \mathbb{C}w(Q) \). The result now follows.

It follows that matricial norms can be defined on \( OA(Q) \) in an intrinsic manner by defining matricial norms on \( \mathbb{C}w(Q) \) as for the semigroup operator algebra. This provides an intrinsic characterization of the norm on \( OA(Q) \) and perhaps makes the construction of \( OA(Q) \) less mysterious.

**Example.** Let \( T \) be the directed graph with two vertices and a single edge connecting the vertices. Labelling the vertices as \( v_0 \) and \( v_1 \) and the edge as \( t \), with \( r(t) = v_1 \) and \( s(t) = v_0 \), we can see that \( OA(T) \) is the norm closed algebra generated by the span of elements of the set

\[
\{(v_0)^{\delta_0}(v_1v_0)^{l_1}(v_1v_0)^{m_1}(v_1v_0)^{l_2}t^{n_2}(v_1v_0)^{m_2} \cdots (v_1v_0)^{l_k}t^{n_k}(v_1v_0)^{m_k}(v_1)^{\delta_1}\}
\]

where \( \delta_0, \delta_1 \in \{0, 1\} \) and \( l_i, m_i, n_i \geq 0 \). This provides an alternate method from [4] where this algebra is described as the quotient of three free products.

2. **Intrinsic norms for \( C^*_{m}(A) \)**

We now look to build \( C^*_{m}(A) \) in a manner intrinsic to \( A \), without reference to the completely contractive representations of \( A \). This removes the need to define \( C^*_{m}(A) \) by reference to all completely contractive representations of \( A \), as is done in [4]. In particular we will have a concrete construction of the algebra \( C^*_{m}(A) \) which
should lead to a better understanding of the maximal $C^*$-algebra of an operator algebra. We begin by letting $A$ be a unital operator algebra. We recall the intrinsic characterization of the operator algebraic free product of two operator algebras.

**Construction** (Blecher-Paulsen [2] Theorem 4.1). For $A$ and $B$ operator algebras with a common subalgebra $D$ and for $x$ in the algebraic free product of $A$ and $B$ amalgamated over $D$ we define

$$
\|x\|_{OA} = \inf \{ \|x_1\| \|x_2\| \cdots \|x_n\| : x_1 \ast x_2 \cdots \ast x_n = x \}
$$

where the $x_i$ are elements of either $M_{k_i}(A)$ or $M_{j_i}(B)$ with $j_i, k_i \in \mathbb{N}$, and $\ast$ the free product matrix multiplication. Completing the algebraic free product with respect to this norm yields an operator algebra $A \ast_{OA} B$ with the following universal property.

Universal property: If $\tau : A \to X$ and $\sigma : B \to X$ are completely contractive such that $\sigma|_D = \tau|_D$ then there is $\tau \ast \sigma : A \ast_{OA} B \to X$ completely contractive with $\tau \ast \sigma|_A = \tau$ and $\tau \ast \sigma|_B = \sigma$.

Using this construction we will be able to build $C^*_m(A)$ intrinsically. We begin with a definition. Recall that the diagonal of an operator algebra $A \cap A^*$ is independent of representation and hence the diagonal can be found by taking any faithful representation and finding the diagonal in that particular representation.

**Definition 2.** Let $A$ be a unital operator algebra and let $A^\#$ be the canonically associated adjoint algebra. We write $F(A) = A \ast_{\Delta(A)} A^\#$ for the operator algebraic free product amalgamated over the diagonal, $\Delta(A)$, of $A$.

**Remark 1.** We will use $\#$ to denote the formal adjoint, and $\ast$ to represent an adjoint in a $C^*$ algebra. The reason is to minimize confusion between elements of $F$ and elements of $C^*_m(A)$.

**Remark 2.** Where it will not cause confusion we suppress the $A$ in the notation that follows.

We now construct a $C^*$ semi-norm on $F$.

**Definition 3.** For $y \in M_n(F)$ we say that $y \geq 0$ if

$$
y = \sum_{i=1}^m y_i^\# y_i
$$

for some set $\{y_i\}_{i=1}^m \in M_{k_i,n}(F)$, where $k_i$ is a positive integer.

We record some elementary lemmas involving positive elements of $F$ and $M_2(F)$. In what follows we will omit the formal product symbol where it is inferred.
Lemma 2. Let \( x \in F \) and \( t \in \mathbb{R}^+ \), then
\[
\begin{bmatrix}
t & x \\
x^# & t
\end{bmatrix} \geq 0 \text{ if and only if } t^2 - x^#x \geq 0.
\]

Proof. If \( t^2 - x^#x \geq 0 \) then clearly
\[
\begin{bmatrix}
t^2 & 0 \\
0 & t^2 - x^#x
\end{bmatrix} \geq 0
\]
and hence
\[
\begin{bmatrix}
t & x \\
x^# & t
\end{bmatrix} = \frac{1}{\sqrt{t}} \begin{bmatrix} 1 & 0 \\ x^# & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^2 - x^#x \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{t} \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{t}} \geq 0.
\]

Now if \( \begin{bmatrix} t & x \\ x^# & t \end{bmatrix} \geq 0 \) then
\[
0 \leq \begin{bmatrix} 1 & 0 \\ -\frac{x^#}{t} & 1 \end{bmatrix} \begin{bmatrix} t & x \\ x^# & t \end{bmatrix} \begin{bmatrix} 1 & -\frac{x^#}{t} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{x^#}{t} & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t - x^#x \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{t}{x^#x} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & t - x^#x \end{bmatrix}.
\]
It follows that \( t^2 - x^#x \geq 0 \).

The next two lemmas will allow us to show that the semi-norm we define later is actually a C* semi-norm.

Lemma 3. For \( x \in F \) and \( t \in \mathbb{R}^+ \)
\[
\begin{bmatrix}
t & x \\
x^# & t
\end{bmatrix} \geq 0 \text{ if and only if } \begin{bmatrix} t^2 & x^#x \\ x^#x & t^2 \end{bmatrix} \geq 0.
\]

Proof. If \( \begin{bmatrix} t^2 & x^#x \\ x^#x & t^2 \end{bmatrix} \geq 0 \) then it follows that
\[
0 \leq \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 & x^#x \\ x^#x & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2t^2 - 2x^#x & 0 \\ 0 & 0 \end{bmatrix}
\]
which implies that \( t^2 - x^#x \geq 0 \) which by the previous lemma yields one direction of the result.

Now if \( \begin{bmatrix} t & x \\ x^# & t \end{bmatrix} \geq 0 \) then by the previous lemma we have \( t^2 - x^#x \geq 0 \) and clearly \( t^2 + x^#x \geq 0 \). It follows that \( \begin{bmatrix} t^2 + x^#x & 0 \\ 0 & t^2 - x^#x \end{bmatrix} \geq 0 \) and hence
\[
0 \leq \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t^2 + x^#x & 0 \\ 0 & t^2 - x^#x \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2 \begin{bmatrix} t^2 & x^#x \\ x^#x & t^2 \end{bmatrix}.
\]
Lemma 4. If \( x, y \in \mathcal{F}, s, t \in \mathbb{R}^+ \),
\[
\begin{bmatrix}
    s & x \\
    x^# & s
\end{bmatrix} \geq 0, \text{ and } \begin{bmatrix}
    t & y \\
    y^# & t
\end{bmatrix} \geq 0
\]
then
\[
\begin{bmatrix}
    st & xy \\
    y^#x^# & st
\end{bmatrix} \geq 0.
\]

Proof. Notice by the first lemma that \( s^2 - x^#x \geq 0 \) and hence \( s^2y^#y - y^#x^#xy \geq 0 \). But notice also that \( t^2 - y^#y \geq 0 \) hence \( s^2t^2 - s^2y^#y \geq 0 \) and it follows that \( s^2t^2 - y^#x^#xy \geq 0 \) and the first lemma gives the result. ■

The next lemma follows trivially from the definition. The next four are the last steps in providing a \( C^* \)-seminorm on \( \mathcal{F} \).

Lemma 5. Let \( x, y \in \mathcal{F} \) and \( s, t \in \mathbb{R}^+ \) with
\[
\begin{bmatrix}
    s & x \\
    x^# & s
\end{bmatrix} \geq 0 \text{ and } \begin{bmatrix}
    t & y \\
    y^# & t
\end{bmatrix} \geq 0
\]
then
\[
\begin{bmatrix}
    s + t & x + y \\
    x^# + y^# & s + t
\end{bmatrix} \geq 0.
\]

Lemma 6. Let \( x \in \mathcal{F} \) and \( s \in \mathbb{R}^+ \) then
\[
\begin{bmatrix}
    s & x \\
    x^# & s
\end{bmatrix} \geq 0 \text{ if and only if } \begin{bmatrix}
    s & x^# \\
    x & s
\end{bmatrix} \geq 0.
\]

Proof. Notice that
\[
\begin{bmatrix}
    s & x \\
    x^# & s
\end{bmatrix} = \begin{bmatrix}
    0 & 1 \\
    1 & 0
\end{bmatrix}\begin{bmatrix}
    s & x^# \\
    x & s
\end{bmatrix}\begin{bmatrix}
    0 & 1 \\
    1 & 0
\end{bmatrix}
\]
and the result is immediate. ■

Lemma 7. Let \( x \in \mathcal{F} \) and \( \lambda \in \mathbb{C}, s \in \mathbb{R}^+ \) then
\[
\begin{bmatrix}
    |\lambda|s & \lambda x \\
    \lambda x^# & |\lambda|s
\end{bmatrix} \geq 0 \text{ if and only if } \begin{bmatrix}
    s & x^# \\
    x & s
\end{bmatrix} \geq 0.
\]

Proof. Notice first that \( |\lambda|^2s^2 - |\lambda|^2x^#x \geq 0 \) if and only if \( s^2 - x^#x \geq 0 \). Now by the first lemma the result is established. ■

Lemma 8. Let \( x \in \mathcal{F} \) then
\[
\inf \left\{ t : \begin{bmatrix}
    t & x \\
    x^# & t
\end{bmatrix} \geq 0 \right\} \leq \|x\|_{OA}.
\]

Proof. Notice that this is equivalent to showing that \( \|x\|^2 - x^#x \geq 0 \) in \( \mathcal{F} \). This will follow by an induction. We can begin by letting \( x_1x_2 \) be a matrix factorization of \( x \), and we define \( t_i = \|x_i\| \) in the appropriate matrix algebra. Then notice that
\[
t_{1}^{2}t_{2}^{2} - x^#x = t_{2}^{2} - x_{2}^#x_{1}^#x_{1}x_{2} = x_{2}^#(t_{2}^{2} - x_{1}^#x_{1})x_{2} + t_{1}^{2}(t_{2}^{2} - x_{2}^#x_{2})
\]
which is sum of positive elements of and hence
\[
t_{2}^{2}t_{1}^{2} \geq \inf \left\{ t : \begin{bmatrix}
    t & x \\
    x^# & t
\end{bmatrix} \geq 0 \right\}.
\]
Repeating the process for larger matrix factorization tells us that for any factorization \( x = x_1 x_2 \cdots x_n \) we have
\[
\inf \left\{ t : \begin{bmatrix} t & x \\ x^\# & t \end{bmatrix} \right\} \geq \|t_1\| \cdots \|t_n\|.
\]
As the factorization is arbitrary the result follows.

We are now in a position to define a \( C^* \) seminorm on \( \mathcal{F} \).

**Definition 4.** Let \( x \in \mathcal{F} \) then define
\[
\gamma(x) = \inf \left\{ t : \begin{bmatrix} t & x \\ x^\# & t \end{bmatrix} \geq 0 \right\}.
\]

**Proposition 2.** The function \( \gamma \) is a \( C^* \) seminorm on \( \mathcal{F} \) and \( \mathcal{F} / \ker \gamma \) is isomorphic to \( C^*_m(A) \).

**Proof.** The statement that \( \gamma \) is a \( C^* \) seminorm follows from the series of lemmas preceding the definition. We need only show that \( \mathcal{F} / \ker \gamma \cong C^*_m(A) \). Notice that \( q : A \to \mathcal{F} / \ker \gamma \) is completely contractive and hence the induced map \( q : \mathcal{F} \to \mathcal{F} / \ker \gamma \) is a completely contractive quotient homomorphism. Notice that the inclusion \( i : A \to \mathcal{F} \) is completely isometric, and further \( q \circ i : A \to \mathcal{F} / \ker \gamma \) sends \( A \) to a generating subalgebra of \( \mathcal{F} / \ker \gamma \). It follows by the universal property for \( C^*_m(A) \) that there exists an onto * homomorphism \( \tilde{q} : C^*_m(A) \to \mathcal{F} / \ker \gamma \).

We also know that there exists a completely contractive homomorphism \( \pi : \mathcal{F} \to C^*_m(A) \). Now if \( \varepsilon - x^\# x \geq 0 \) for all \( \varepsilon > 0 \), then \( \pi(\varepsilon - x^\# x) \geq 0 \) for all \( \varepsilon > 0 \). In particular \( \varepsilon - \pi(x)^* \pi(x) \geq 0 \) for all \( \varepsilon > 0 \). Now as \( C^*_m(A) \) is a \( C^* \) algebra it follows that \( \pi(x) = 0 \). It follows that \( \ker \gamma \subseteq \ker \pi \). Hence there is a completely contractive homomorphism \( \pi : \mathcal{F} / \ker \gamma \to C^*_m(A) \). Notice that \( \pi \circ \tilde{q}(x) = x \) for all \( x \in A \) it follows that \( C^*_m(A) \cong \mathcal{F} / \ker \gamma \).

This would seem to imply the Blecher-Ruan-Sinclair Theorem (BRS Theorem), see [12, corollary 16.7]. Recall though that the BRS-theorem was implicit in constructing the operator algebraic free product [2] and hence this does not provide an alternate approach to the BRS-theorem.

Notice that in [11] the function defined above is defined for a general Banach *-algebra. There it is shown that the quotient is the maximal \( C^* \) algebra representation of the Banach *-algebra. In particular we can use the general theory of Banach-* algebras to get at the same result [11, proposition 11.4]. We need only show that \( \mathcal{F} \) is indeed a Banach-* algebra.

**Proposition 3.** Let \( A \) be an operator algebra then \( \mathcal{F} = A *_{\Delta(A)} A^\# \) is a Banach *-algebra.

**Proof.** We know that \( \mathcal{F}(A) \) is an operator algebra and hence a Banach algebra. Now let \( x \) be in \( A *_{alg} A^\# \) and let \( \varepsilon > 0 \). By definition there exists \( A_1 \in M_{k_1}(A), A_2 \in M_{k_2}(A), \ldots, A_n \in M_{k_n}(A) \) and \( B_1 \in M_{j_1}(A^\#), B_2 M_{j_2}(A^\#), \ldots, B_n \in M_{j_n}(A^\#) \) such that \( A_1 * B_1 * A_2 * B_2 * \cdots * A_n * B_n = x \) and
\[
\|x\|_{OA} \leq \|A_1\|_{M_{k_1}(A)} \|B_1\|_{M_{j_1}(A^\#)} \cdots \|B_n\|_{M_{j_n}(A^\#)} \leq \|x\|_{OA} + \varepsilon.
\]
Now notice that \( x^\# = B_n^\# \cdots A_1^\# \) and
\[
\|A_1\|_{M_{k_1}(A)} \|B_1\|_{M_{j_1}(A^\#)} \cdots \|B_n\|_{M_{j_n}(A^\#)}
= \|B_n\|_{M_{j_n}(A)} \|A_n\|_{M_{k_n}(A^\#)} \cdots \|A_1\|_{M_{k_1}(A^\#)}.
\]
It follows that \( \|x^\#\|_{OA} \leq \|x\|_{OA} \). A similar argument tells us that \( \|x^\#\|_{OA} \leq \|x\|_{OA} \) and hence \( \|x^\#\|_{OA} = \|x\|_{OA} \).

Now if \( \{x_n\} \) is Cauchy, then \( \{x_n^\#\} \) is cauchy and hence convergent. Now if \( x_n \to x \) then \( \|x_n - x\|_{OA} \to 0 \). By uniqueness of limits it follows that \( \|x_n^\# - x^\#\|_{OA} \to 0 \). Hence \( ||\cdot||_{OA} \) is continuous with respect to \( # \) and hence \( F \) is a Banach * algebra. ■

In \( \mathbb{K} \) \( \gamma \) is called the Gelfand-Naimark seminorm and the ideal \( \ker \gamma \) is called the reducing ideal of \( F \).

In this section we have constructed \( GC_m^*(A) \) intrinsically for an operator algebra \( A \). This construction, in particular, applies to the algebra \( GC_m^*(Q) \) where \( Q \) is a directed graph. Given a directed graph \( Q \), we can use Theorem 3.3 in [4] to recognize \( GC_m^*(Q) \) as a maximal \( C^* \) envelope of \( OA(Q) \). Proposition 2 then gives us an intrinsic seminorm on \( OA(Q) \ast OA(Q)^* \) which yields the algebra \( GC_m^*(Q) \).

3. Idempotents in \( OA(Q) \) and \( GC_m^*(Q) \)

We remind the reader of an example from [4] and a result concerning the \( K \)-groups of \( OA(Q) \) and \( GC_m^*(Q) \).

Example. We will denote by \( V_n \) the graph with \( n \) vertices and no edges. \( OA(V_n) \) is equal to the unamalgamated free product of copies of \( \mathbb{C} \).

Proposition 4. Let \( Q \) be a directed graph, then there is norm continuous homotopy from \( OA(Q) \) onto \( OA(V(Q)) \) and also from \( GC_m^*(Q) \) onto \( OA(V(Q)) \).

Corollary 1. Let \( Q \) be a finite directed graph. Then

\[
K_0(OA(Q)) = K_0(GC_m^*(Q)) = \mathbb{Z}^{|V(Q)|}. 
\]

Proof. The previous proposition tells us that

\[
K_0(OA(Q)) = K_0(V(Q)) = K_0(GC_m^*(Q)).
\]

By applying a result of Cuntz [3] to the algebra \( OA(V(Q)) \) we get

\[
K_0(OA(V(Q))) = K_0(\mathbb{C})^{|V(Q)|} = \mathbb{Z}^{|V(Q)|}. \tag*{■}
\]

It follows that the \( K_0 \)-groups count the number of vertices. Hence the number of vertices is a Banach algebra invariant of the algebra. We will see in the next two sections that more is true. The maximal ideal space will allow us to not only count the vertices but it will also be used to identify the projections \( \{P_v : v \in V(Q)\} \).

4. The maximal ideal space of \( OA(Q) \) and \( GC_m^*(Q) \) for finite graphs

For the remainder of this chapter we will only be concerned with finite graphs. For a Banach algebra \( A \), we denote the maximal ideal space by \( M_A \). By \( \mathcal{P}(X) \) we mean the power set of \( X \) and we let \( \mathcal{P}(X) = \mathcal{P}(X) \setminus \{\emptyset\} \). For \( k \in \mathbb{N} \) we let \( \mathbb{D}^k \) be the cartesian product of \( k \) copies of \( \mathbb{D} \), and we let \( \mathbb{D}^0 = \{0\} \). If \( S \subseteq V(Q) \) and \( S \neq \emptyset \) we let

\[
\mathcal{E}(S) := \{e \in E(Q) : r(e), s(e) \in S\}. 
\]

Lastly, for \( S \) a nonempty subset of \( V(Q) \), we define \( n(S) = |\mathcal{E}(S)| \).

Proposition 5. The set \( M_{OA(Q)} \) is homeomorphic to

\[
\bigsqcup_{S \in \mathcal{P}(V(Q))} \mathbb{D}^{|\mathcal{E}(S)|}.
\]
Proof. We begin by letting \( \varphi \) be a multiplicative linear functional and fixing an enumeration of \( E(Q) \). Now \( \varphi \) is uniquely determined by \( \varphi(P_v) \) and \( \varphi(T_e) \) where \( v \in V(Q) \) and \( e \in E(Q) \). It is clear that \( \varphi(P_v) \in \{0,1\} \) where \( v \in V(Q) \). Further we have that

\[
\| \varphi(T_e) \| \leq \| \varphi \| \| T_e \| \leq 1
\]

for all edges \( e \in E(Q) \) and hence \( \varphi(T_e) \in \mathbb{D} \).

Fix \( \varphi \in M_{OA(Q)} \) and let

\[
S_\varphi := \{ v \in V(Q) : \varphi(P_v) = 1 \}.
\]

Then \( \varphi \) is determined with a fixed ordering on \( E(S_\varphi) \) by the \( n(S_\varphi) \)-tuple

\[
[\varphi(T_{e_{S_1}}), \varphi(T_{e_{S_2}}), \ldots, \varphi(T_{e_{S_{n(S)}}})].
\]

Thus the map \( \varphi \mapsto [\varphi(T_{e_{S_1}}), \varphi(T_{e_{S_2}}), \ldots, \varphi(T_{e_{S_{n(S)}}})] \) gives a map of the maximal ideal space into

\[
\bigcup_{S \in \mathcal{P}(V(Q))} \mathbb{D}^{n(S)}.
\]

We claim that this correspondence is onto. Uniqueness follows by definition.

To prove that the correspondence is onto, let \( S \in \mathcal{P}(V(Q)) \) be nonempty and take \( \lambda \in \mathbb{D}^{n(S)} \). We define \( \varphi_\lambda : Q \to \mathbb{C} \) by

\[
\varphi_\lambda(v) = \begin{cases} 1 & v \in S \\ 0 & \text{else} \end{cases}.
\]

Then define \( \varphi_\lambda(e_i) = \lambda_i \) for \( e_i \in \{ e : r(e) \in S \text{ and } s(e) \in S \} \), and \( \varphi_\lambda(e) = 0 \) otherwise. It is easy to see that \( \varphi_\lambda \) is a contractive representation of \( Q \). Now by the universal property of \( OA(Q) \) there exists a unique completely contractive homomorphism, which we also call \( \varphi_\lambda \), with \( \varphi_\lambda : OA(Q) \to \mathbb{C} \). It follows that the correspondence is onto.

We now turn to continuity. If \( \varphi_\lambda \to \varphi \) then \( \varphi_\lambda(T_e) \to \varphi(T_e) \) and \( \varphi_\lambda(P_v) \to \varphi(P_v) \) for each edge \( e \) and vertex \( v \). It follows that the correspondence will preserve the set \( S \) and the \( n(S) \) tuples will converge pointwise. Thus the correspondence induces a continuous map between \( M_{OA(Q)} \) and \( \bigcup_{S \in \mathcal{P}(V(Q))} \mathbb{D}^{n(S)} \). Now since we have a one to one and onto continuous map from a space which is Hausdorff and compact we have that the inverse map is also continuous and the homeomorphism is established.

In fact we have established that \( M_{OA(Q)} \) is a compact Hausdorff space with a connected component for each nonempty \( S \subseteq V(Q) \).

Example. Let \( Q \) be the graph

![Graph](image)

Since \( Q \) has 3 vertices there are seven connected components in the maximal ideal space. The component corresponding to \( v_1 \) has two copies of \( \mathbb{D} \) since there are two edges with range and source equal to \( v_1 \). The component corresponding to \( v_2 \) and the component corresponding to \( v_3 \) are both singleton sets since neither vertex has an edge which enters and leaves the vertex. The component corresponding to the
pair \( \{v_1, v_2\} \) has three copies of \( \mathbb{D} \) one for each of the edges, \( t_1, t_2, \) and \( t_3 \). The pair \( \{v_2, v_3\} \) also yields a singleton set. The pair \( \{v_1, v_3\} \) has two copies of \( \mathbb{D} \). The final component corresponding to \( \{v_1, v_2, v_3\} \) has 3 copies of \( \mathbb{D} \) since there are three total edges. The maximal ideal space is then homeomorphic to 
\[ \mathbb{D}^2 \sqcup \mathbb{D}^0 \sqcup \mathbb{D}^0 \sqcup \mathbb{D}^3 \sqcup \mathbb{D}^0 \sqcup \mathbb{D}^2 \sqcup \mathbb{D}^3. \]

**Definition 5.** For a finite directed graph \( Q \) we let \( N_Q \) be the number of connected components of \( M_{OA(Q)} \).

We can actually define several invariants of the algebra by using combinatorial arguments and the structure of the maximal ideal space in a fairly simple manner.

**Proposition 6.** For a finite directed graph \( Q \),
\[ |V(Q)| = \log_2(N_Q + 1) \]
and
\[ |E(Q)| = \max\{n(S)\} \]
where \( n(S) = |\mathcal{E}(S)| \).

**Proof.** Each connected component of \( M_{OA(Q)} \) is associated uniquely to a nonempty subset of \( V(Q) \). It follows that \( N_Q + 1 = |\mathcal{P}(V(Q))| = 2^{|V(Q)|} \), and the first formula is established. Secondly as \( V(Q) \in \mathcal{P}(V(Q)) \) there is a connected component of \( M_{OA(Q)} \) associated to the set \( V(Q) \). But \( n(V(Q)) \) is the number of edges emanating from and ending in \( V(Q) \), which is the total number of edges. Since \( n(S) \) is less than or equal to the total number of edges for all \( S \subseteq V(Q) \), we have the second formula and the corollary is established.

**Corollary 2.** Suppose \( OA(Q_1) \) and \( OA(Q_2) \) are algebraically isomorphic. Then 
\[ |V(Q_1)| = |V(Q_2)|, \quad |E(Q_1)| = |E(Q_2)|. \]

**Proof.** If the algebras \( OA(Q_1) \) and \( OA(Q_2) \) are isomorphic, then the spaces \( M_{OA(Q_1)} \) and \( M_{OA(Q_2)} \) are homeomorphic. Hence by the formulas established in the previous Proposition the corollary follows.

Actually, more is true. We say that an edge \( e \in E(Q) \) is a loop edge if \( s(e) = r(e) \). We can use calculations to find the number of loop edges and non loop edges in the graph from combinatorial facts about \( M_{OA(Q)} \).

**Proposition 7.** Let \( Q \) be a finite directed graph. If \( \alpha \) is the number of loop edges in \( Q \) and \( \beta \) is the number of non loop edges in \( Q \) then \( \alpha \) and \( \beta \) can be calculated uniquely from \( M_{OA(Q)} \).

**Proof.** If \( n \) is the number of vertices in \( Q \), then for an edge \( e \) there will be a copy of \( \mathbb{D} \) for every subset of \( S \subseteq V(Q) \) with \( r(e), s(e) \in S \). Thus if \( r(e) = s(e) \), since there are \( 2^{n-1} \) nonempty subsets of \( V(Q) \) containing \( r(e) \), there are \( 2^{n-1} \) copies of \( \mathbb{D} \in M_{OA(Q)} \) for each loop edge. If \( r(e) \neq s(e) \) there are \( 2^{n-2} \) subsets of \( V(Q) \) which contain \( s(e) \) and \( r(e) \). Thus, there are \( 2^{n-2} \) copies of \( \mathbb{D} \in M_{OA(Q)} \) for each edge which is not a loop. If \( \alpha \) is the number of loop edges, and \( \beta \) is the number of non loop edges, then \( \alpha(2^{n-1}) + \beta(2^{n-2}) = \sum_{S \in V(Q)} n(S) \). Now assume that there are \( \alpha' \) and \( \beta' \), a different combination of loop edges and non loop edges respectively, such that \( \alpha(2^{n-1}) + \beta(2^{n-2}) = \alpha'(2^{n-1}) + \beta'(2^{n-2}) \). Since the number of edges is fixed at \( n \) we know that \( (n - \beta)(2^{n-1}) + \beta(2^{n-2}) = (n - \beta')(2^{n-1}) + \beta'(2^{n-2}) \).
Simplifying we get that $\beta(2^{n-2} - 2^{n-1}) = \beta'(2^{n-2} - 2^{n-1})$ and hence $\beta = \beta'$. It follows that in a finite graph the number of loop edges and the number of non loop edges is an isomorphism invariant which can be calculated directly from information about the set $M_{OA(Q)}$. ■

We now look at the algebra $GC^*_m(Q)$. Recall that this is the universal $C^*$ algebra of the directed graph $Q$ which is constructed by looking at $*$ representations of the graph $Q_{V(Q)}^*$. The universal properties will allow us to identify the maximal ideal space of $GC^*_m(Q)$.

**Proposition 8.** The set $M_{GC^*_m(Q)}$ is homeomorphic to $M_{OA(Q)}$. In fact for an operator algebra $A$, $M_A$ is homeomorphic to $M_{C^*_m(A)}$.

**Proof.** Since $GC^*_m(Q) = C^*_m(OA(Q))$ we will prove the more general result. If $\varphi : C^*_m(A) \to C$ is a multiplicative linear functional then $\varphi|_A \to C$ is also a multiplicative linear functional. Further every multiplicative linear functional $\pi : A \to C$ is completely contractive and hence there exists a unique multiplicative linear functional $\tilde{\pi} : C^*_m(A) \to C$ such that $\tilde{\pi}|_A = \pi|_A$. It follows that there is a one to one correspondence between maximal ideals of $A$ and $C^*_m(A)$. That the maps are continuous is trivial. ■

Proposition 7 applies also to $GC^*_m(Q)$ and hence the number of vertices, loop edges and non loop edges are isomorphism invariants for $GC^*_m(Q)$.

5. **Uniqueness of $GC^*_m(Q)$ for finite graphs**

In this section we are interested in uniqueness of $GC^*_m(Q)$. We begin with definitions. Let $A$ be a Banach algebra, and let $\varphi$ be a multiplicative linear functional. We let

$$P(\varphi) := \{ x \in A : x^2 = x \text{ and } \varphi(x) = 1 \}.$$ 

**Definition 6.** For $X$ a connected component of $M_A$ we say that $X$ has degree 1 if for every $\varphi \in X$, $|P(\varphi)| = 1$. We say that $X$ has degree $k$ for $k > 1$ if there are exactly $k$ degree 1 components $X_j$ such that $P(\varphi) \cap \mathfrak{g}(\tau) \neq \emptyset$ for all $\varphi \in X, \tau \in X_j$.

In the context of universal graph operator algebras the preceding definition will be useful in establishing uniqueness. We use it now to identify the number of vertices associated to a particular connected component of $M_{GC^*_m(Q)}$. Recall that to each set $S \in \mathbb{P}(X)$ there is a connected component in $M_{GC^*_m(Q)}$.

It is a consequence of Proposition 7 and Proposition 8 that for $\varphi, \tau \in X$, a connected component of $M_{GC^*_m(Q)}$, $P(\varphi) = P(\tau)$.

**Proposition 9.** Let $X$ be a connected component in $M_{GC^*_m(Q)}$, then $X$ has degree $k \geq 1$ if and only if there is a set of disjoint vertices $V := \{ v_1, v_2, \ldots, v_k \} \subseteq V(Q)$ such that $X$ is the component associated to the set $V \subseteq \mathbb{P}(X)$.

**Proof.** Let $S \in \mathbb{P}(V(Q))$. We will show that if $|S| = k$ then the associated component, $X$, has degree $k$. Let $\{ v_1, v_2, \ldots, v_k \} = S$ and denote by $P_{v_i}$ the projection associated to $v_i$. Now define a contractive representation $\pi : Q \to C$ by sending $v_i$ to 1 for each $i$ and everything else to 0. The induced completely contractive map will be a multiplicative linear functional associated to the component $X$. Now notice that $P(\varphi) = \{ P_{v_1}, P_{v_2}, \ldots, P_{v_k} \}$ and hence $|P(\varphi)| = k$. The result follows. ■
Definition 7. For $X$ a connected component of degree $k$ in $M_{GC_m^*(Q)}$ we let

$$P_X := \bigcup_{Y \text{ degree 1}} \left( \bigcap_{e \in X, \tau \in Y} (P(\varphi) \cap P(\tau)) \right).$$

If $X$ has degree two and $Y$ and $Z$ are components of degree one, then we say that $X$ is the component associated to $Y$ and $Z$ if $P_X = P_Y \cup P_Z$.

We are now in a position to describe the main result of this section. Starting with the graph $Q$ we build an associated undirected graph $\hat{Q}$. Recall that an undirected graph is a 3-tuple $(V, E, n)$, where $V$ is a set of vertices, $E$ is the set of all pairs of vertices, and $n : E \to \mathbb{N}$ is a continuous map. The map $n\{v, w\}$ will specify how many edges connect the pair of vertices $v$ and $w$. For a directed graph $Q$ we let $V(\hat{Q}) = V(Q)$ and $n\{v, w\}$ be the number of edges $e$ with $\{r(e), s(e)\} = \{v, w\}$. The graph $\hat{Q}$ can be thought of as the graph obtained from $Q$ by removing the directions on each edge. We will show that $\hat{Q}$ is an isomorphism invariant for $GC_m^*(Q)$.

We say that an edge $e$ in a graph is a loop edge if $r(e) = s(e)$.

Theorem 2. Let $Q_1$ and $Q_2$ be finite directed graphs. The algebras $GC_m^*(Q_1)$ and $GC_m^*(Q_2)$ are isomorphic as Banach algebras if and only if the graphs $\hat{Q}_1$ and $\hat{Q}_2$ are isomorphic.

Proof. We begin by assuming that two algebras $GC_m^*(Q_1)$ and $GC_m^*(Q_2)$ are isomorphic. It is well known that if two Banach algebras are isomorphic as Banach algebras then there is an induced homomorphism between their maximal ideal spaces. It follows from Corollary 2 that the number of vertices in $Q_1$ is equal to the number of vertices in $Q_2$. The homeomorphism will clearly preserve degree. Let $X$ be a degree two component in $M_{GC_m^*(Q_1)}$ or $M_{GC_m^*(Q_2)}$. We can, by Proposition 9, identify the degree one components, $Y$ and $Z$, which correspond to $X$. For clarity of presentation we will write a degree two component with corresponding degree one components $Y$ and $Z$ as $X_{Y,Z}$.

For an arbitrary connected component $W$ of $M_{GC_m^*(Q)}$ or $M_{GC_m^*(Q')}$ we let $n(W)$ be the number of copies of $\overrightarrow{W}$ in $W$. If $Y$ is a degree one component then $n(Y)$ is the number of loop edges. For a degree two component $n(X_{Y,Z}) - (n(Y) + n(Z))$ is the number of edges $e$ in the graph with $r(e) \neq s(e)$ and $\varphi(r(e)) = 1 = \varphi(s(e))$ for all $\varphi \in X_{Y,Z}$. Now $n(W)$ is also invariant under isomorphism and hence $\hat{Q}_1$ is isomorphic to $\hat{Q}_2$.

For the converse, assume that $\hat{Q}_1$ and $\hat{Q}_2$ are isomorphic. Then there is a 1-1 correspondence between the sets $V(Q_1)$ and $V(Q_2)$. There is also a 1-1 correspondence between the sets $E(Q_1)$ and $E(Q_2)$. We build a new directed graph $\overrightarrow{Q}$ from $\hat{Q}$ by setting $V(Q) = V(\hat{Q})$, $E(Q) = E(\hat{Q})$, $r((v, w)) = v$, and $s((v, w)) = w$. The assignment of range and source will not change the graph $\overrightarrow{Q}$. It is easy to see that $\overrightarrow{Q}_1 *_{V(Q_1)} \overrightarrow{Q}_1$ is isomorphic to $\overrightarrow{Q}_2 *_{V(Q_2)} \overrightarrow{Q}_2$. The result now follows from the construction of $GC_m^*(Q)$. □.
There is no reason to expect a stronger uniqueness result. For example, the two graphs

and

have isomorphic universal $C^*$ algebras, even though the graphs are not isomorphic. On the other hand any uniqueness may be considered surprising since $Q$ is not an invariant for $C^*(Q)$. For an example we point the reader to [5] where it is shown that the distinct graphs

yield isomorphic $C^*$-algebras.

6. **Uniqueness of $OA(Q)$ for finite graphs**

We use the definitions from the previous section in establishing the uniqueness of the algebra $OA(Q)$. Once again we have most of the information about our directed graph embedded in the maximal ideal space. The only complication that remains is identifying the directions on the edges with distinct source and range. We will use ideas similar to those in [7] to build the original graph $Q$ from information about a class of representations of $OA(Q)$.

As in Section 5 we will need to identify the degree one components of $M_{OA(Q)}$. The same arguments work in this context so we do not repeat them here. We need a few preliminary results before addressing uniqueness.

Let $T_2$ be the algebra of $2 \times 2$ upper triangular matrices. If $A$ is an operator algebra we say that a representation $\pi : A \to T_2$ is a two dimensional nest representation if $\pi$ is onto. Let $X$ and $Y$ be degree one connected components in $M_A$. We say that a two dimensional nest representation $\pi$ of $OA(Q)$ has the projection property for $X$ and $Y$ if

$$
\pi(P_X) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\pi(P_Y) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
$$
and if \( x \) is an idempotent not contained in \( P_X \cup P_Y \) then
\[
\pi(x) = 0.
\]

**Definition 8.** For \( X \) and \( Y \), degree one connected components in \( M_{OA(Q)} \), let
\[
K_{X,Y} := \cap \{ \ker(\pi) : \pi \text{ has the projection property for } X \text{ and } Y \}.
\]

If \( \pi : OA(Q) \to T_2 \) has the projection property for \( X \) and \( Y \), then there is an induced map \( \pi_q : OA(Q)/K_{X,Y} \to T_2 \) which is a two dimensional nest representation with the projection property for \( X \) and \( Y \).

**Definition 9.** If \( X \) and \( Y \) are degree one connected components in \( M_{OA(Q)} \) let \( R_{X,Y} \) be the set of all cosets \( OA(Q) + K_{X,Y} \in OA(Q)/K_{X,Y} \) such that
\[
(\pi_q(OA(Q) + K_{X,Y}))^2 = 0
\]
for all \( \pi \) with the projection property for \( X \) and \( Y \).

**Lemma 9.** \( R_{X,Y} \) is a closed two sided ideal in \( OA(Q)/K_{X,Y} \).

**Proof.** The fact that \( (\pi_q(OA(Q) + K_{X,Y}))^2 = 0 \) implies that \( \pi_q(OA(Q) + K_{X,Y}) \) is strictly upper triangular. Now if \( B + K_{X,Y} \) is another coset in \( OA(Q)/K_{X,Y} \) then
\[
\pi_q((A + K_{X,Y})(B + K_{X,Y})) = \pi_q(A + K_{X,Y})\pi_q(B + K_{X,Y})
\]
\[
= \begin{bmatrix} 0 & a & \lambda_1 & \lambda_2 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & a & \lambda_1 & \lambda_2 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Hence \( (\pi_q((A + K_{X,Y})(B + K_{X,Y})))^2 = 0 \). Similar arguments for multiplication on the right by an ideal element shows that \( R_{X,Y} \) is a two sided ideal. Closure is automatic since \( \pi_q \) is continuous.

We now describe \( R_{X,Y} \) for a finite directed graph \( Q \).

**Proposition 10.** Let \( Q \) be a finite graph, and \( v, w \in V(Q) \). We denote the connected components of \( M_{OA(Q)} \) associated to \( \{v\} \) and \( \{w\} \) by \( V \) and \( W \), respectively. There are \( n \) edges with range \( v \) and source \( w \) if and only if \( R_{V,W} \) has a minimal generating set of cardinality \( n \).

**Proof.** Let \( V \) and \( W \) be the sets described. Then it is clear that \( P_e A P_w + K_{V,W} = A + K_{V,W} \) for all \( A \in R_{V,W} \). Further notice that \( T_e + K_{V,W} = P_v T_e P_w + K_{V,W} \in R_{V,W} \) for all edges \( e \) with \( s(e) = w, r(e) = v \). A quick calculation tells us that if \( \pi \) has the projection property for \( V \) and \( W \) then for each edge \( e \) with \( r(e) = v, s(e) = w \) there is an \( a_e \) such that
\[
\pi(T_e) = \begin{bmatrix} 0 & a_e \\ 0 & 0 \end{bmatrix}.
\]

If \( r(e) = s(e) = v \) then
\[
\pi(T_e) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}
\]
for some \( \lambda \in \mathbb{C} \). Similarly if \( r(e) = s(e) = w \) then
\[
\pi(T_e) = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}
\]
for some \( \lambda \in \mathbb{C} \). Lastly if \( r(e) \neq v \) or \( s(e) \neq w \) then \( \pi(T_e) = 0 \). Now, letting
\[
N = \{ e : r(e) = v, s(e) = w \}
\]
we have that \( \{ T_e + K_{V,W} : e \in N \} \) is a linearly independent generating set for \( R_{V,W} \). In particular, a typical element of \( R_{V,W} \) is contained in the closure of the linear span of
\[
R := \{ T^n_T g^m + K_{V,W} : s(f) = r(f) = r(e), e \in N, r(g) = s(g) = s(e), n, m \geq 0 \}.
\]

Now let \( X = \{ x_1, x_2, \ldots, x_m \} \) be a generating set for \( R_{V,W} \). We will show that \( |X| \geq n \). Let \( e \) be an an edge in \( N \). Now define a representation \( \pi_e : Q \to T_2 \) by first letting
\[
\begin{align*}
\pi_e(s(e)) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
\pi_e(r(e)) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
\pi_e(e) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{align*}
\]
and sending all other edges and vertices to zero. There will be a completely contractive extension \( \tilde{\pi}_e : OA(Q) \to T_2 \) and \( \tilde{\pi}_e \) will have the projection property for \( V \) and \( W \). It follows then that \( (\tilde{\pi}_e)_q : OA(Q)/K_{V,W} \to T_2 \) is well defined. Now there exists \( x_i \in X \) such that \( \| (\tilde{\pi}_e)_q(x_i) \| > 0 \) hence \( x_i = \alpha e T_e + k_e \) where \( \alpha e \neq 0 \in \mathbb{C} \) and \( k_e \) is in the kernel of \( \tilde{\pi}_e \).

It follows that for each \( i, x_i = \{ (\sum_{e \in N} \alpha e T_e) + k_i \} \) where \( k_i \in \cap_{e \in N} \ker(\pi_e) \). The set \( \{ T_{x_i} \}_{i \in N} \cup \cap_{e \in N} \ker(\pi_e) \) is a linearly independent subset of \( OA(Q)/\cap_{e \in N} \ker(\pi_e) \) and it follows that \( |X| > n \).

We now prove a classification theorem for universal operator algebras of directed graphs.

**Theorem 3.** Let \( Q_1 \) and \( Q_2 \) be finite directed graphs. The algebras \( OA(Q_1) \) and \( OA(Q_2) \) are isomorphic as Banach algebras if and only if \( Q_1 \) and \( Q_2 \) are isomorphic as directed graphs.

**Proof.** Certainly if \( Q_1 \) and \( Q_2 \) are isomorphic then \( OA(Q_1) \) and \( OA(Q_2) \) are isomorphic by uniqueness of the extension of a directed graph morphism. Let the map \( \pi : OA(Q_1) \to OA(Q_2) \) be a bounded isomorphism then \( \pi \) induces a homeomorphism, which we also call \( \pi \). Further the algebra \( OA(Q_1)/K_{V,W} \) is isomorphic to \( OA(Q_2)/K_{\pi(V),\pi(W)} \), and the result follows.

We note here some differences between the proof here and the proof of uniqueness for quiver algebras given in [H]. For the quiver algebras, since the projections are orthogonal, all of the connected components of the maximal ideal space are degree one. This simplifies the quiver algebra result. Also, Katsoulis and Kribs, use a Fourier expansion for elements of the quiver algebra and hence they do not need to restrict the class of two dimensional nest representations that they use. Although the proofs are significantly different the ideas are similar. Using the maximal ideal space and the two dimensional nest representations we construct the underlying graph from the algebra using only Banach algebra properties.

Extending the uniqueness results to infinite graphs may be more complicated. A better understanding of the maximal ideal space is vital. On the other hand it
is a consequence of the description of the $K$-theory that if $Q$ is an infinite graph, then $OA(Q) \not\sim OA(Q')$ for any finite graph $Q'$ and similarly $GC_m(Q) \not\sim OA(Q')$ for any finite graph $Q'$.

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