Thermodynamics and phase transitions of nonlinear electrodynamics black holes in an extended phase space

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Received December 3, 2018
Revised April 5, 2019
Accepted April 17, 2019
Published April 29, 2019

Abstract. We investigate the thermodynamic behavior of asymptotically AdS nonlinear electrodynamics (NLED) black holes in an extended phase space, where the cosmological constant $\Lambda = -3/l^2$ is interpreted as a thermodynamic pressure. For a generic NLED black hole, we find that the Smarr relation is satisfied in the extended phase space if the dimensionful parameters in the NLED Lagrangian are considered as the thermodynamic variables. In a canonical ensemble with the charge $Q$ fixed, we then investigate the phase structure and transitions of Born-Infeld AdS black holes, endowed with a parameter $a > 0$, which encodes the strength of the nonlinearities. In the $a/l^2$-$Q/l$ phase space, the phase diagrams are obtained, which provides a new viewpoint towards the black holes' phase structure and critical behavior. It shows that the critical line and the region, where a reentrant phase transition occurs, are both finite and terminated at some point. Through the continuation to $a < 0$, we extend the analysis to a new type of NLED black holes, dubbed iBorn-Infeld AdS black holes. For the iBorn-Infeld AdS black holes, the critical line and the reentrant phase transition region are semi-infinite and extend to $Q/l = \infty$. We also examine the thermal and electrical stabilities of the Born-Infeld and iBorn-Infeld AdS black holes.

Keywords: GR black holes, gravity

ArXiv ePrint: 1808.04506v2
1 Introduction

Black holes are among the most intriguing concepts of general relativity, which could have a deep impact upon the understanding of quantum gravity. Understanding the statistical mechanics of black holes has been a subject of intensive study for several decades. In the pioneering work [1–3], Hawking and Bekenstein found that black holes possess the temperature and the entropy. Analogous to the laws of thermodynamics, the four laws of black hole mechanics were established in [4].

Studying the phase transitions of AdS black holes is primarily motivated by the AdS/CFT correspondence [5]. Hawking and Page showed that a first-order phase transition occurs between Schwarzschild AdS black holes and thermal AdS space [6], which was later understood as a confinement/deconfinement phase transition in the context of the AdS/CFT correspondence [7]. For Reissner-Nordstrom (RN) AdS black holes, authors of [8, 9] showed that their critical behavior is similar to that of a Van der Waals liquid gas phase transition.

Later, the asymptotically AdS black holes have been studied in the context of extended phase space thermodynamics, where the cosmological constant is interpreted as a thermodynamic pressure [10, 11]. In this case, the black hole mass should be understood as the enthalpy instead of the internal energy [12]. The P-V criticality study has been explored for various AdS black holes [13–18]. It showed that the P-V critical behavior of AdS black holes is similar to that of a Van der Waals liquid gas system. A reentrant phase transition occurs if, as one monotonically changes a thermodynamic variable, the system undergoes two (or more) phase transitions and returns to a state macroscopically similar to the initial state. In the context of the extended phase space, a reentrant phase transition has been observed for some AdS black holes, e.g., 4D BI-AdS black holes [19], higher dimensional singly spinning Kerr-AdS black holes [20], AdS black holes in Lovelock gravity [16], AdS black holes in dRGT massive gravity [21], hairy AdS black holes [22].

Nonlinear electrodynamics (NLED) is an effective model incorporating quantum corrections to Maxwell electromagnetic theory. Coupling NLED to gravity, various NLED charged black holes were derived and discussed in a number of papers [23–29] (for a brief review...
see [30]). In particular, for general NLED theories, the general static and spherically symmetric black hole solution with an electric field was first given in [31, 32]. Later, this consideration was generalized to the case with both electric and magnetic charges [33]. As pointed out in [33, 34], the existence of a globally regular NLED black hole solution with a nonzero electric charge requires that the NLED Lagrangian is strongly non-Maxwell in the weak field limit. Moreover, the thermodynamics of NLED black holes in the extended phase space has been considered in the literature, e.g., power Maxwell invariant black holes [35–37], non-linear magnetic-charged dS black hole [38].

The effects of NLED have been extensively studied in cosmological and astrophysical contexts, e.g., the avoidance of the singularity in some cosmological models [39], a potential variation of the fine structure constant [40], a potential acceleration of the universe late-time accelerate expansion [41], the modification of the gravitational redshift and the electrodynamics propagation of photons from super strongly magnetized compact objects [42, 43], an available framework for generating the primordial magnetic fields and gravitational baryogenesis [44–46]. On the other hand, due to the efficient discharge in the surrounding plasma, an astrophysical black hole usually does not have any significant electric charge. Hence, it seems that NLED is not likely to play an important role for astrophysical black holes. However, there are still some scenarios where NLED may have some effects on black hole astrophysics. For example, the existence of highly collimated jets in active galactic nuclei is most easily explained by extracting energy from a rotating supermassive black hole via strong magnetic fields associated with the innermost accretion disk [47]. The effects of NLED could appear in this jet formation model, which usually requires numerical simulations since accretion flows are known to be turbulent. In [48], the dynamics of a charged particle around a NLED regular black hole was investigated in the presence of magnetic fields. In addition, huge electromagnetic fields can appear during the late stages of supernovae, and it is possible that a short-lived rotating NLED black hole forms [49, 50]. In [50], it showed that the neutrino dynamics in core-collapse supernovae could be significantly changed in the spacetime of the NLED black hole.

Among various NLED, there is a famous string-inspired one: Born-Infeld electrodynamics, which encodes the low-energy dynamics of D-branes. Born-Infeld electrodynamics incorporates maximal electric fields and smoothes divergences of the electrostatic self-energy of point charges. The Born-Infeld AdS (BI-AdS) black hole solution was first obtained in [51, 52]. The thermodynamic behavior and phase transitions of BI-AdS black holes were studied in a canonical ensemble [53] and in a grand ensemble [54]. The critical behavior and thermodynamics of BI-AdS black holes in various gravities were also investigated in [55–63]. In the extended phase space, the thermodynamic phase structure and critical behavior of 4D and higher dimensional BI-AdS black holes were studied in [19, 64], respectively. In [65], the thermodynamics of a 4D BI-AdS black hole has recently been discussed in an alternative case, in which the charge of the black hole is varied and the pressure is fixed. The reentrant phase transition has been observed in 4D BI-AdS black holes while there was no reentrant phase transition for the system of higher dimensional BI-AdS black holes. Although the properties of BI-AdS black holes are thoroughly investigated in the literature, their electrical stabilities have been rarely reported.

Recently, a new type of NLED black holes, namely iBorn-Infeld AdS (iBI-AdS) black holes, have been considered in [66] as holographic models behaving as prototypes of Mott insulators. The Lagrangian of the iBorn-Infeld field can be obtained from that of the Born-Infeld field by extending the BI parameter \(a\) in eq. (3.1)) to a negative real number. In [67], it
showed that the nonlinearity correction tends to reduce/increase the strength of the repulsive force between two electrons for the Born-Infeld/iBorn-Infeld field. So it is natural to expect that the iBI-AdS black hole is dual to a theory with strong interactions between electrons, which could lead to Mott-like behavior. Moreover, the negative magneto-resistance and the Mott insulator to metal transition induced by a magnetic field can be realized at low temperatures in the iBorn-Infeld holographic models. Compared to the Born-Infeld case, the iBorn-Infeld case leads to a much richer transport behavior in the dual theory. As shown in [66], iBI-AdS black holes satisfy the constraints to ensure consistency in the form of ghosty perturbations and/or gradient instabilities at the decoupling limit. However, the thermodynamic behavior and phase structure of iBI-AdS black holes have yet to be discussed.

In this paper, we first investigate the thermodynamic behavior of generic NLED black holes in the extended phase space. Then, we turn to study the phase structure and critical behavior of BI-AdS and iBI-AdS black holes by studying the phase diagrams in the $Q/l-a/l^2$ plane. The rest of this paper is organized as follows. In section 2, we derive the NLED black hole solution, compute its Euclidean action and discuss thermodynamic properties of the black hole. We find that the Smarr relation is satisfied after including dimensionful couplings in NLED in the extended phase space. In section 3, we study the phase structure and critical behavior of BI-AdS black holes. The phase diagram for BI-AdS black holes in the $Q/l-a/l^2$ plane is given in figure 4, from which one can read the black hole’s phase structure and critical behavior. We further explore thermal and electrical stabilities of BI-AdS black holes. In section 4, the phase structure and critical behavior of iBI-AdS black holes are investigated, which can be read from the phase diagram in the $Q/l-a/l^2$ plane, figure 10. We also study thermal and electrical stabilities of iBI-AdS black holes. We summarize our results in section 5. In appendix, we present an alternative derivation of the Smarr relation for NLED black holes.

2 NLED black hole

In this section, we first derive the asymptotically AdS black hole solution in the Einstein-NLED gravity. After its Gibbs free energy is obtained via calculating the Euclidean action, we then discuss the thermodynamic properties of the black hole, e.g., the Smarr relation, thermal and electrical stabilities.

2.1 Black hole solution

Consider a 4-dimensional model of gravity coupled to a nonlinear electromagnetic field $A_\mu$ with the action given by

$$\begin{align*}
S_{\text{Bulk}} &= \int d^4x \sqrt{-g} \left[ R - 2\Lambda + L(s, a_i) \right],
\end{align*}$$

where the cosmological constant $\Lambda = -\frac{2}{l^2}$, and we take $16\pi G = 1$ for simplicity. In the action (2.1), we assume that the generic NLED Lagrangian $L(s, a_i)$ is a function of $s$ and the parameters $a_i$, where we build an independent nontrivial scalar using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and none of its derivatives:

$$\begin{align*}
s &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}.
\end{align*}$$

The parameters $a_i$ characterize the effects of nonlinearity in the NLED. We also assume that the NLED Lagrangian would reduce to the Maxwell Lagrangian for small fields:

$$\begin{align*}
L(s, a_i) &\approx s \text{ as } s \to 0.
\end{align*}$$

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Varying the action (2.1) with respect to $g_{ab}$ and $A_a$, we find that the equations of motion are

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} - \frac{3}{l^2} g_{\mu \nu} = \frac{T_{\mu \nu}}{2},$$

$$\nabla_\mu G^{\mu \nu} = 0,$$  \hspace{1cm} (2.4)

where $T_{\mu \nu}$ is the energy-momentum tensor:

$$T_{\mu \nu} = g_{\mu \nu} \mathcal{L} (s, a_i) + \frac{\partial \mathcal{L} (s, a_i)}{\partial s} F_\mu^\rho F_{\nu \rho},$$  \hspace{1cm} (2.5)

and we define the auxiliary fields $G^{\mu \nu}$:

$$G^{\mu \nu} = - \frac{\partial \mathcal{L} (s, a_i)}{\partial F_\mu^\nu} = \frac{\partial \mathcal{L} (s, a_i)}{\partial s} F^{\mu \nu}.$$  \hspace{1cm} (2.6)

To construct a black hole solution with asymptotic AdS spacetime, we take the following ansatz for the metric and the NLED field

$$ds^2 = -f (r) dt^2 + \frac{dr^2}{f (r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

$$A = A_t (r) dt.$$  \hspace{1cm} (2.7)

The equations of motion then take the form:

$$-1 + f (r) - \frac{3r^2}{l^2} + rf' (r) = \frac{r^2}{2} \left[ \mathcal{L} (s, a_i) + A_t' (r) G^{rt} \right],$$  \hspace{1cm} (2.8)

$$2f' (r) - \frac{6r}{l^2} + rf'' (r) = r \mathcal{L} (s, a_i),$$

$$[r^2 G^{rt}]' = 0,$$  \hspace{1cm} (2.10)

where

$$s = \frac{A_t^2 (r)}{2}$$

and

$$G^{rt} = - \frac{\partial \mathcal{L} (s, a_i)}{\partial s} A_t' (r).$$  \hspace{1cm} (2.11)

It can show that eq. (2.9) can be derived from eqs. (2.8) and (2.10). Eq. (2.10) leads to

$$G^{tr} = \frac{q}{r^2},$$  \hspace{1cm} (2.12)

where $q$ is a constant. Via eqs. (2.11) and (2.12), $A_t' (r)$ is determined by

$$\mathcal{L}' \left( \frac{A_t^2 (r)}{2}, a_i \right) A_t' (r) = \frac{q}{r^2}.$$  \hspace{1cm} (2.13)

Moreover, integrating eq. (2.9) leads to

$$f (r) = 1 - \frac{m}{r} + \frac{r^2}{l^2} - \frac{1}{2r} \int_r^\infty drr' \left[ \mathcal{L} \left( \frac{A_t^2 (r)}{2}, a_i \right) - A_t' (r) \frac{q}{r^2} \right],$$  \hspace{1cm} (2.14)

where $m$ is a constant. For large values of $r$, one finds that

$$f (r) = 1 - \frac{m}{r} + \frac{r^2}{l^2} + \frac{q^2}{4r^2} + \mathcal{O} (r^{-4}),$$  \hspace{1cm} (2.15)
which reduces to the behavior of a RN-AdS black hole. At the horizon \( r = r_+ \) where \( f(r_+) = 0 \), the Hawking temperature of the black hole is given by

\[
T = \frac{f'(r_+)}{4\pi}.
\]  

(2.16)

Hence at \( r = r_+ \), eq. (2.8) gives

\[
T = \frac{1}{4\pi r_+} \left\{ 1 + \frac{3r_+^2}{l^2} + \frac{r_+^2}{2} \left[ \mathcal{L}\left( \frac{A_t^2(r_+)}{2}, a_i \right) - A_t'(r_+) \frac{q}{r_+^2} \right] \right\}.
\]  

(2.17)

The charge \( Q \) of the black hole can be expressed in terms of the constants \( q \). In fact, if we turn on the external current \( J^\mu \), the action would include an interaction term:

\[
S_I = 4\pi \int d^4x \sqrt{-g} J^\mu A_\mu.
\]  

(2.18)

The equation of motion for \( A_\mu \) then becomes

\[
\nabla_\nu G^{\mu \nu} = 4\pi J^\mu.
\]  

(2.19)

The charge passing through a spacelike hypersurface \( \Sigma \) is given by

\[
Q = -\int_{\Sigma} d^3x \sqrt{\gamma} \sigma_\mu J^\mu,
\]  

(2.20)

where \( \gamma_{ij} \) is the induced metric, and \( \sigma^\mu \) is the unit normal vector of \( \Sigma \). Using Stokes’s theorem and eq. (2.19), we can express the charge as a boundary integral:

\[
Q = -\frac{1}{4\pi} \int_{\partial \Sigma} d^2x \sqrt{\tilde{\gamma}} n_\mu \sigma_\nu G^{\mu \nu},
\]  

(2.21)

where \( \partial \Sigma \) is the boundary of \( \Sigma \), \( \tilde{\gamma}_{ij} \) is the induced metric, and \( n_\mu \) is the unit outward-pointing normal vector. For the metric in eq. (2.7), \( \Sigma \) and \( \partial \Sigma \) can be a constant-\( t \) hypersurface and a two-sphere at \( r = \infty \), respectively. In this case, one has

\[
\sigma_\mu = \left( -f^{1/2}, 0, 0, 0 \right) \quad \text{and} \quad n_\mu = \left( 0, f^{-1/2}, 0, 0 \right).
\]  

(2.22)

Thus, the charge of the black hole given by eq. (2.21) becomes

\[
Q = \frac{1}{4\pi} \int d\theta d\phi r^2 \sin \theta \frac{q}{r^2} = q,
\]  

(2.23)

where we use eq. (2.12). The gauge potential measured with respect to the horizon is

\[
\Phi = 4\pi \int_{r_+}^\infty A_t'(r) = 4\pi A_t(\infty),
\]  

(2.24)

where we fix the gauge field \( A_t(r) \) at the horizon to be zero, i.e., \( A_t(r_+) = 0 \). The electrostatic potential \( \Phi \) plays a role as the conjugated variable to \( Q \) in black hole thermodynamics.

For an asymptotically AdS space, the mass may be extracted by comparison to a reference background, e.g., vacuum AdS. Similar to the charge of the black hole, the mass can also be determined by the Komar integral

\[
M = 4 \int d\theta d\phi r^2 \sin \theta (\sigma_\mu n_\nu \nabla^\mu K^\nu) - M_{\text{AdS}},
\]  

(2.25)
where $K^\mu = (1, 0, 0, 0)$ is the Killing vector associated with $t$, and $M_{\text{AdS}}$ is Komar integral associated with $K^\mu$ for vacuum AdS space

$$M_{\text{AdS}} = 4 \int d\theta d\phi r^2 \sin \theta \left( \frac{r}{l^2} \right).$$

(2.26)

At spatial infinity, one can use eq. (2.15) to calculate

$$\sigma_\mu n_\nu \nabla^\mu K^\nu = \frac{1}{2} f' (r) = \frac{m}{2 r^2} + \frac{r}{l^2} + O (r^{-3}).$$

(2.27)

So the mass of the black hole is

$$M = 8 \pi m.$$  

(2.28)

### 2.2 Euclidean action calculation

In the Euclidean path integral approach to quantum gravity [68, 69], one can identify the Euclidean path integral with the thermal partition function:

$$Z = \int \mathcal{D}g e^{-S_E (g)}.$$  

(2.29)

In the semiclassical approximation, the dominant contribution to the path integral comes from the classical solution, and hence one has

$$Z \simeq e^{-S_E}.$$  

(2.29)

Here, $S_E$ is the on-shell action which is obtained by substituting the classical solution to the action. In asymptotically AdS spaces, $S_E$ needs to be regulated to cancel the divergences coming from the asymptotic region. In the background-subtraction method, one can regularize $S_E$ by subtracting a contribution from a reference background. In [9], the background-subtraction method was used to compute $S_E$ for RN-AdS black holes. On the other hand, there is the counterterm subtraction method [70, 71], in which the action $S_E$ is regularized in a background-independent fashion by adding a series of boundary counterterms to the action. Specifically, the Kounterterms method [72, 73] has been proposed as a regularization scheme for gravity in asymptotically AdS spaces. In [74], the Euclidean action was computed for black hole solutions of AdS gravity coupled to the Born-Infeld electrodynamics using the Kounterterms method.

We now follow the method in [74] to calculate the Euclidean action for the asymptotically AdS NLED black hole solution (2.7) in a canonical ensemble, in which the temperature and charge of the black hole are fixed. The regularized action is then given by

$$S_R = S_{\text{Bulk}} + S_{\text{ct}} + S_{\text{surf}},$$

(2.30)

where the boundary terms are

$$S_{\text{ct}} = \frac{l^2}{4} \int d^3 y B_3 \text{ and } S_{\text{surf}} = - \int d^3 y \sqrt{\gamma} n_\nu G^{\mu \nu} A_\mu,$$

(2.31)

$B_3$ is the 2nd Chern form, and $n^\mu$ is the unit outward-pointing normal vector of the boundary. Since the asymptotically AdS spacetime has constant curvature in the asymptotic region, it showed in [74] that after including the boundary term $S_{\text{ct}}$, the action was stationary around
the classical solution under arbitrary variations of the metric $g_{\mu \nu}$. To keep the charge of the black hole fixed instead of the potential, the boundary term $S_{\text{surf}}$ has to be added. In fact, varying the action with respect to $A_\mu$ gives

$$\delta S_R = \text{EOM} - \int d^3y \sqrt{\gamma} n_\nu \delta G^{\mu \nu} A_\mu = \text{EOM} - \int d^3y \frac{\sqrt{\gamma} n_\nu A_t}{r^2} \delta Q.$$  

For the Euclidean continuation of the action $S^E = iS_R$, the horizon at $r = r_+$ is shrunk to a point, and the manifold spans between $r = r_+$ and $r = \infty$. To avoid a conical singularity at the origin of the radial coordinate, one requires to identify the Euclidean time $\tau (\equiv -it)$ as $\tau \sim \tau + \beta$, where the period $\beta = T^{-1}$ is the inverse of the Hawking temperature $T$. The Euclidean continuation of the counter term $S_{\text{ct}}$ was calculated in [74]:

$$S_{\text{ct}}^E = -4\pi \beta \frac{r^2}{l^2} f' (r) \left[ f (r) - 1 \right] |_{r = \infty}.$$  

(2.32)

The bulk action is

$$S_{\text{Bulk}}^E = -\int_0^\beta d\tau \int d\Omega \int_{r_+}^\infty dr r^2 \left[ R + \frac{6}{l^2} + \mathcal{L} (s, a_i) \right] = 4\pi \beta \left[ r^2 f' (r) \right] |_{r_+} - \beta q \Phi,$$  

(2.33)

where we use eqs. (2.9) and (2.24). The boundary term $S_{\text{surf}}^E$ is

$$S_{\text{surf}}^E = \int_0^\beta d\tau \int d\Omega f^{1/2} (r) r^2 \sin \theta (n_\nu G^{\mu \nu} A_\mu) |_{r = \infty} = \beta q \Phi,$$  

(2.34)

where we use eq. (2.24), $f (r) \to 1$ as $r \to \infty$, and $n_\mu = (0, f^{-1/2}, 0, 0)$. To sum up all terms, the Euclidean action $S^E$ is given by

$$S^E = -16\pi^2 r_+^2 + 4\pi \beta \left\{ r^2 f' (r) - l^2 f' (r) \left[ f (r) - 1 \right] \right\} |_{r = \infty} = \beta (M - TS),$$  

(2.35)

where the entropy of the black hole is

$$S = 16\pi^2 r_+^2,$$  

(2.36)

and eq. (2.15) gives

$$\lim_{r \to \infty} \left\{ r^2 f' (r) - l^2 f' (r) \left[ f (r) - 1 \right] \right\} = 2m.$$  

(2.37)

Since the Euclidean action is calculated at fixed $Q$, $P \left( = 6/l^2 \right)$ and $T$, we can associate it with the Gibbs free energy:

$$F = M - TS.$$  

(2.38)

### 2.3 Thermodynamics

Here, we study the thermodynamics of the NLED AdS black hole solution in the extended phase space. In such perspective on black hole thermodynamics, one needs to include the cosmological constant $\Lambda$ as a pressure term and interpret the mass of the black hole as a gravitational version of chemical enthalpy. Furthermore, as noted in Lovelock gravity [75] and Born-Infeld electrodynamics [19], any dimensionful coupling should be promoted to a thermodynamic variable and hence introduces an associated conjugate, which would add an extra term in the first law and Smarr relation.
In terms of the horizon radius $r_+$, the mass $M$ can be written as

$$M = 8\pi \left\{ r_+ + \frac{r_+^3}{l^2} - \frac{1}{2} \int_{r_+}^{\infty} dr r^2 \left[ \mathcal{L} \left( \frac{A_t^2(r)}{2}, a_i \right) - A_t'(r) \frac{Q}{r^2} \right] \right\}.$$  
(2.39)

So the derivatives of the mass in terms of the entropy and the charge are, respectively,

$$\frac{\partial M}{\partial S} = \frac{1}{4\pi r_+} \frac{\partial m}{\partial r_+} = T, \quad (2.40)$$

and

$$\frac{\partial M}{\partial Q} = 8\pi \left[ -\frac{1}{2} \int_{r_+}^{\infty} dr r^2 \mathcal{L}' \left( \frac{A_t^2(r)}{2}, a_i \right) A_t'(r) \frac{\partial A_t'(r)}{\partial r} + \frac{\Phi}{8\pi} + \frac{Q}{8\pi} \frac{\partial \Phi}{\partial Q} \right] = \Phi, \quad (2.41)$$

where we use eqs. (2.13) and (2.24). Since the pressure $P = 6/l^2$, one has

$$\frac{\partial M}{\partial P} = \frac{4\pi}{3} r_+^3 \equiv V, \quad (2.42)$$

where $V$ is the thermodynamic volume. For a dimensionful coupling $a_i$ in $\mathcal{L}(s, a_i)$, we can introduce an associated conjugate $A_i$:

$$A_i = \frac{\partial M}{\partial a_i}. \quad (2.43)$$

Therefore, the extended first law takes the form

$$dM = TdS + VdP + \Phi dQ + \sum_i A_i da_i. \quad (2.44)$$

Performing the dimensional analysis, we assume that $[a_i] = L^{c_i}$. The Euler scaling argument [12] gives the Smarr relation for the black hole

$$M = 2(TS - VP) + \sum_i c_i a_i A_i + Q\Phi. \quad (2.45)$$

As a check, the Smarr relation is derived directly from the definitions of the thermodynamic quantities of the black hole in the appendix.

Till now, our expressions for the thermodynamic quantities, e.g., the Gibbs free energy $F$, the enthalpy $M$, are functions of the horizon radius $r_+$ (the entropy $S$), the charge $Q$ and the AdS radius $l$ (the pressure $P$). However, in a canonical ensemble with fixed $T$, $Q$ and $P$, we need to express the thermodynamics quantities in terms of $T$, $Q$ and $P$. In doing so, the equation of state (2.17) is solved for $r_+$:

$$r_+ = r_+(T, Q, P, a_i).$$

Interestingly, the equation of state (2.17) can be rewritten as

$$\tilde{T} = \frac{1}{4\pi \tilde{r}_+} \left\{ 1 + 3\tilde{r}_+^2 + \frac{1}{2} \tilde{r}_+^2 \left[ \mathcal{L} \left( \frac{\tilde{A}_t^2(\tilde{r}_+)}{2}, \tilde{a}_i \right) - \tilde{A}_t'(\tilde{r}_+) \frac{\tilde{Q}}{\tilde{r}_+^2} \right] \right\}, \quad (2.46)$$

where we define

$$\tilde{T} = Tl, \quad \tilde{r}_+ = r_+/l, \quad \tilde{Q} = Q/l, \quad \tilde{a}_i = a_i l^{-c_i} \quad \text{and} \quad \tilde{A}_t'(r_+) = l A_t'(r_+). \quad (2.47)$$
and \( \tilde{A}'(r_+) \) is determined by
\[
\mathcal{L}' \left( \frac{\tilde{A}'(r_+)}{2} \tilde{a}_i \right) \tilde{A}'(r_+) = \frac{Q}{r_+^2}.
\] (2.48)

Solving eq. (2.46), we find that \( \tilde{r}_+ \) can be expressed as a function of \( \tilde{T} \), \( \tilde{Q} \) and \( \tilde{a}_i \):
\[ \tilde{r}_+ = \tilde{r}_+ (\tilde{T}, \tilde{Q}, \tilde{a}_i) \]. With \( \tilde{r}_+ = \tilde{r}_+ (\tilde{T}, \tilde{Q}, \tilde{a}_i) \), we can express the thermodynamic quantities in terms of \( \tilde{T} \), \( \tilde{Q} \) and \( \tilde{a}_i \). For example, the Gibbs free energy can be rewritten as
\[ F = F/l = \tilde{F} (\tilde{T} \tilde{Q}, \tilde{a}_i). \]

The rich phase structure of the black hole comes from solving eq. (2.46), i.e., \( \tilde{T} = \tilde{T}(\tilde{r}_+, \tilde{Q}, \tilde{a}_i) \), for \( \tilde{r}_+ \). If \( \tilde{T}(\tilde{r}_+, \tilde{Q}, \tilde{a}_i) \) is a monotonic function with respect to \( \tilde{r}_+ \) for some values of \( \tilde{Q} \) and \( \tilde{a}_i \), there is only one branch for the black hole. More often, with fixed \( \tilde{Q} \) and \( \tilde{a}_i \), there exists a local minimum/maximum for \( \tilde{T}(\tilde{r}_+, \tilde{Q}, \tilde{a}_i) \) at \( \tilde{r}_+ = \tilde{r}_{+\min}/\tilde{r}_{+\max} \). In this case, there is more than one branch for the black hole. In figure 1a, we plot two branches, namely small BH and large BH, around a local minimum of \( \tilde{T} = \tilde{T}_{\min} \). The Gibbs energies of these two branches are displayed in the right panel of figure 1a. Since \[ \partial \tilde{F}(\tilde{T}, \tilde{Q}, \tilde{a}_i)/\partial \tilde{T} = -16\pi^2 \tilde{r}_+^2 \], the upper branch is small BH while the lower one is large BH, which means that the large BH branch is thermodynamically preferred. Similarly, two branches around a local maximum of \( \tilde{T} = \tilde{T}_{\max} \), small BH and large BH, are shown in figure 1b. The upper/lower branch in the right panel of figure 1b is large/small BH since it has more/less negative slope. So the small BH branch is thermodynamically preferred in this case. In general, one needs to figure out how the existence of the local extrema of eq. (2.46) depends on values of \( \tilde{Q} \) and \( \tilde{a}_i \) to study the phase structure of the black hole.

After the black hole’s branches are obtained, it is interesting to consider their thermodynamic stabilities against thermal and electrical fluctuations. In a canonical ensemble, the
Consider the case in which \( \varepsilon \) branch and \( F \) are concave downward/upward in right panels of figure 1. For the large BH branch as \( \tilde{\varepsilon} \), Eq. (2.50) shows that in figure 1a, \( \partial \tilde{\varepsilon} \rightarrow \tilde{T}, \tilde{Q}, \tilde{a}_i \) are concave downward/upward in right panels of figure 1.

The second quantity is

\[
\varepsilon_T = (\frac{\partial Q}{\partial \Phi})_T = \frac{1}{\tilde{T}} \left( \frac{\partial \Phi(\tilde{r}_+, \tilde{Q}, \tilde{a}_i)}{\partial Q} \right) = -1, \tag{2.50}
\]

which describes how the black hole’s electrostatic potential responds to its charge. Here, \( \tilde{r}_+ \) is understood as \( \tilde{r}_+(\tilde{T}, \tilde{Q}, \tilde{a}_i) \). For a positive value of \( \varepsilon_T \), as more charges are placed on the black hole, its potential increases, which makes it harder to move the system from equilibrium. The electrical stability of the branch then follows from \( \varepsilon_T \geq 0 \). For the potential \( \Phi \), it is natural to expect that \( \partial \Phi(\tilde{r}_+, \tilde{Q}, \tilde{a}_i) / \partial \tilde{r}_+ < 0 \). We also assume \( Q > 0 \), and hence \( \tilde{A}_t(r_+) > 0 \), which gives

\[
\frac{\partial \tilde{T}(\tilde{r}_+, \tilde{Q}, \tilde{a}_i)}{\partial Q} = -\frac{1}{8\pi \tilde{r}_+} \tilde{A}_t'(r_+) < 0. \tag{2.51}
\]

Eq. (2.50) shows that in figure 1a, \( \varepsilon_T^{-1} \rightarrow +\infty \) for the small BH branch and \( \varepsilon_T^{-1} \rightarrow -\infty \) for the large BH branch as \( \tilde{T} \rightarrow \tilde{T}_{\max} \). Similarly in figure 1b, \( \varepsilon_T^{-1} \rightarrow -\infty \) for the small BH branch and \( \varepsilon_T^{-1} \rightarrow +\infty \) for the large BH branch as \( \tilde{T} \rightarrow \tilde{T}_{\min} \).

Finally, we turn to the critical point, which is an inflection point and obtained by

\[
\frac{\partial \tilde{T}(\tilde{r}_+, \tilde{Q}, \tilde{a}_i)}{\partial \tilde{r}_+} = 0 \quad \text{and} \quad \frac{\partial^2 \tilde{T}(\tilde{r}_+, \tilde{Q}, \tilde{a}_i)}{\partial \tilde{r}_+^2} = 0. \tag{2.52}
\]

Solving the above equations gives

\[
\tilde{r}_{+,c} = \tilde{r}_{+,c}(\tilde{a}_i), \quad \tilde{Q}_c = \tilde{Q}_c(\tilde{a}_i) \quad \text{and} \quad \tilde{T}_c = \tilde{T}_c(\tilde{a}_i). \tag{2.53}
\]

Defining the specific volume \( v = \frac{r_+}{8\pi} \) [19], one finds that

\[
\rho_c = \frac{P_c v_c}{T_c} = \frac{3 \tilde{r}_{+,c}(\tilde{a}_i)}{4\pi \tilde{T}_c(\tilde{a}_i)}. \tag{2.54}
\]

Consider the case in which \( \mathcal{L}(s, a_i) \) is a power series expansion of \( s \):

\[
\mathcal{L}(s, a_i) = s + \frac{a_1}{2} s^2 + \frac{a_2}{3} s^3 + \cdots. \tag{2.55}
\]
For small values of $a_i$, we find

$$
\tilde{Q}_c = \frac{1}{3} + \frac{7}{18} \tilde{a}_1 + \frac{11}{216} (\tilde{a}_1^2 + 16\tilde{a}_2) + \cdots ,
$$

$$
\tilde{T}_c = \frac{1}{\pi} \frac{\sqrt{2}}{3} - \frac{\tilde{a}_1}{3\sqrt{6} \pi} - \frac{9\tilde{a}_1^2 + 32\tilde{a}_2}{36\sqrt{6} \pi} + \cdots ,
$$

$$
\tilde{r}_{+,c} = \frac{1}{\sqrt{6}} - \frac{7\tilde{a}_1}{6\sqrt{6}} + \frac{27\tilde{a}_1^2 - 352\tilde{a}_2}{72\sqrt{6}} + \cdots ,
$$

(2.56)

$$
\tilde{r}_{+,c} = \frac{1}{\sqrt{6}} - \frac{7\tilde{a}_1}{6\sqrt{6}} + \frac{27\tilde{a}_1^2 - 352\tilde{a}_2}{72\sqrt{6}} + \cdots .
$$

where 

$$
\tilde{Q}_c = Q_c \sqrt{P_c/6}, \quad \tilde{T}_c = T_c \sqrt{6/P_c}, \quad \tilde{r}_{+,c} = r_{+,c} \sqrt{P_c/6} \quad \text{and} \quad \tilde{a}_i = (P_c/6)^i a_i.
$$

The leading value of $\rho_c$ is $3/8$, which reproduces the critical value $\rho_c$ of RN-AdS black holes.

### 3 Born-Infeld AdS black hole

Born-Infeld electrodynamics is described by the Lagrangian density

$$
\mathcal{L}(s) = \frac{1}{a} \left( 1 - \sqrt{1 - 2as} \right),
$$

(3.1)

where the coupling parameter $a$ is related to the string tension $\alpha'$ as $a = (2\pi \alpha')^2 > 0$. When $a = 0$, we can recover the Maxwell Lagrangian. Solving eq. (2.13) for $A'_t(r)$ gives

$$
A'_t(r) = \frac{Q}{\sqrt{r^4 + aQ^2}}.
$$

(3.2)

It follows that the potential of the black hole is

$$
\Phi = \frac{4\pi Q}{r_+} \, _2F_1 \left( \frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{aQ^2}{r_+^4} \right),
$$

(3.3)

where $\, _2F_1(a, b; c; x)$ is the hypergeometric function.

The equation of state (2.46) becomes

$$
\tilde{T}(\bar{r}_+) \equiv \frac{h(\bar{r}_+)}{4\pi \bar{r}_+} = \frac{1}{4\pi \bar{r}_+} \left( 1 + 3\bar{r}_+^2 - \frac{1}{2} \frac{\bar{Q}^2}{\bar{r}_+^2 + \sqrt{\bar{r}_+^4 + \bar{a}\bar{Q}^2}} \right),
$$

(3.4)

where $\bar{a} = a/l^2$, and we define $h(\bar{r}_+)$ for later use. Noting that $h(0) = 1 - \frac{\bar{Q}}{2\sqrt{\bar{a}}}$, $h(\infty) \to +\infty$ and $h(\bar{r}_+)$ is a strictly increasing function, one finds that

$$
\bar{Q}^2 \geq 4a: \quad \tilde{T}(\bar{r}_+) = 0 \quad \text{has only one solution} \quad \bar{r}_+ = \bar{r}_e \geq 0,
$$

$$
\bar{Q}^2 < 4a: \quad \tilde{T}(\bar{r}_+) > 0 \quad \text{for} \quad \bar{r}_+ \geq 0,
$$

where $\bar{r}_e$ corresponds to an extremal black hole.
Table 1. Solutions of $\bar{T}''(\bar{r}_+)=0$ and the local extrema of $\bar{T}'(\bar{r}_+)$ in various cases, where $\bar{r}_+ = \left( x_i^2 - \bar{a} \bar{Q}^2 \right)^{1/4}$.

To study the behavior of the local extrema of $\bar{T}(\bar{r}_+)$, we consider the equation $\bar{T}''(\bar{r}_+) = 0$, which becomes
\[
z(x) \equiv x^3 - \frac{3\bar{Q}^2}{2} x^2 + \bar{a}\bar{Q}^4 = 0,
\]
with $x = \sqrt{\bar{a}\bar{Q}^2 + \bar{r}_+^4}$. Since $\lim_{x \to \pm \infty} z(x) = \pm \infty$, $z'(0) = 0$ and $z'(\bar{Q}^2) = 0$, $z(x)$ has a local maximum of $z(0) = \bar{a}\bar{Q}^4 > 0$ at $x = 0$ and a local minimum of $z(\bar{Q}^2) = (-\bar{Q}^2/2 + \bar{a})\bar{Q}^4$ at $x = \bar{Q}^2$. If the local minimum is not greater than zero ($Q^2 \geq 2a$), there are two positive real roots $x_1 \geq \bar{Q}^2 > x_2 > 0$ to eq. (3.5). Otherwise ($Q^2 < 2a$), eq. (3.5) has no positive real roots. To make $\bar{r}_+ = (x^2 - \bar{a}\bar{Q}^2)^{1/4}$ real, we also require that $x \geq \sqrt{\bar{a}\bar{Q}}$. For $x_1$, one always has that $x_1 \geq \bar{Q}^2 > \sqrt{\bar{a}\bar{Q}}$ since $Q^2 \geq 2a$. To have $x_2 \geq \sqrt{\bar{a}\bar{Q}}$, we need to have $z(\sqrt{\bar{a}\bar{Q}}) \leq 0 \Rightarrow Q^2 \geq 4a$. With the solutions of $\bar{T}''(\bar{r}_+) = 0$, it is easy to analyze the existence of the local extrema of $\bar{T}'(\bar{r}_+)$, results of which are summarized in table 1.

When solving eq. (3.4) for $\bar{r}_+$ in terms of $\bar{T}$, the inverse function $\bar{r}_+(\bar{T})$ is often a multivalued function. The parameters $\bar{a}$ and $\bar{Q}$ determine the number of the branches of $\bar{r}_+(\bar{T})$ and the phase structure of the black hole. In what follows, we find six regions in the $\bar{a}$-$\bar{Q}$ plane, in each of which the black hole has the distinct behavior of the branches and the phase structure:

- Region I: $Q^2 \geq 4a$ and $\bar{T}'(\bar{r}_1) \geq 0$. In this region, $\bar{T}'(\bar{r}_+) \geq \bar{T}'(\bar{r}_1) \geq 0$ and hence $\bar{T}(\bar{r}_+)$ is an increasing function. So there is only one branch for $\bar{r}_+(\bar{T})$, which is thermally stable. Since $\bar{T}(\bar{r}_+) = 0$ has a solution in this region, this branch can extend to zero temperature. For a black hole with $\bar{a} = 0.01$ and $\bar{Q} = 0.4$ in this region, we plot the radius $\bar{r}_+$, the Gibbs energy $\bar{F}$ and the isothermal permittivity $c_p^{-1} l$ as functions of $\bar{T}$ in figure 2a, which shows that this black hole is electrically stable for small enough and large enough $\bar{T}$. However for large enough $\bar{a}$, the black hole is always electrically stable.

- Region II: $Q^2 \geq 4a$ and $\bar{T}'(\bar{r}_1) < 0$. In this region, $\bar{T}'(\bar{r}_+) = 0$ has two solutions $\bar{r}_+ = \bar{r}_{\text{max}}$ and $\bar{r}_{\text{min}}$ with $\bar{r}_{\text{max}} < \bar{r}_1 < \bar{r}_{\text{min}}$. Since $\bar{T}(\pm \infty) = \pm \infty$, $\bar{T}(\bar{r}_+)$ has a local maximum of $T_{\text{max}} = \bar{T}(\bar{r}_{\text{max}})$ at $\bar{r}_+ = \bar{r}_{\text{max}}$ and a local minimum of $T_{\text{min}} = \bar{T}(\bar{r}_{\text{min}})$ at $\bar{r}_+ = \bar{r}_{\text{min}}$. There are three branches for $\bar{r}_+(\bar{T})$: small BH for $0 \leq \bar{T} \leq T_{\text{max}}$, intermediate BH for $T_{\text{min}} \leq \bar{T} \leq T_{\text{max}}$ and large BH for $\bar{T} \geq T_{\text{min}}$, which are displayed in the left panel of figure 2b. The Gibbs free energies of the three branches are plotted in the middle panel, which shows that there is a first order phase transition between small BH and large BH occurring at $\bar{T} = T_{\text{first}}$ with $T_{\text{min}} \leq T_{\text{first}} \leq T_{\text{max}}$. Both the
small BH and large BH branches are thermally stable. As explained in section 2, $c_r^{-1}$ of small BH goes to $-\infty$ as $\tilde{T} \to \tilde{T}_{\text{max}}$ while that of large BH goes to $+\infty$ as $\tilde{T} \to \tilde{T}_{\text{min}}$. The right panel shows that small BH is electrically unstable while large BH is almost electrically stable.

- **Region III:** $4a > Q^2 > 2a$, $\tilde{T}'(\tilde{r}_1) < 0$, $\tilde{T}'(\tilde{r}_2) > 0$ and $\tilde{T}(\tilde{r}_{\text{min}2}) < \tilde{T}(\tilde{r}_{\text{min}1})$. In this region, $\tilde{T}'(\tilde{r}_+) = 0$ has three solutions $\tilde{r}_+ = \tilde{r}_{\text{max}}$, $\tilde{r}_{\text{min}1}$ and $\tilde{r}_{\text{min}2}$ with $\tilde{r}_{\text{min}2} < \tilde{r}_1 < \tilde{r}_{\text{max}}$. So $\tilde{T}(\tilde{r}_+)$ has a local maximum of $\tilde{T}_{\text{max}} = \tilde{T}(\tilde{r}_{\text{max}})$ at $\tilde{r}_+ = \tilde{r}_{\text{max}}$, a local minimum of $\tilde{T}_{\text{min1}} = \tilde{T}(\tilde{r}_{\text{min1}})$ at $\tilde{r}_+ = \tilde{r}_{\text{min1}}$ and a global minimum of $\tilde{T}_{\text{min2}} = \tilde{T}(\tilde{r}_{\text{min2}})$ at $\tilde{r}_+ = \tilde{r}_{\text{min2}}$. There are four branches for $\tilde{r}_+(\tilde{T})$: intermediate BH for $\tilde{T} \geq \tilde{T}_{\text{min2}}$, small BH for $\tilde{T}_{\text{min2}} \leq \tilde{T} < \tilde{T}_{\text{max}}$, intermediate+ BH for $\tilde{T}_{\text{min1}} \leq \tilde{T} < \tilde{T}_{\text{max}}$ and large BH for $\tilde{T} \geq \tilde{T}_{\text{min1}}$, which are displayed in the left panel of figure 3a. Note that there is no black hole solution when $\tilde{T} < \tilde{T}_{\text{min2}}$. The Gibbs free energies of the four branches are plotted in the middle panel, which shows that there is a first order phase transition between small BH and large BH occurring at $\tilde{T} = \tilde{T}_{\text{first}}$ with $\tilde{T}_{\text{min2}} \leq \tilde{T}_{\text{first}} \leq \tilde{T}_{\text{max}}$. Both the small BH and large BH branches are thermally stable, while intermediate $\pm$ BH branches are not. Similarly to Region II, the right panel shows that small BH is electrically unstable while large BH is almost electrically stable.

- **Region IV:** $4a > Q^2 > 2a$, $\tilde{T}'(\tilde{r}_1) < 0$, $\tilde{T}'(\tilde{r}_2) > 0$, $\tilde{T}_{\text{min2}} \geq \tilde{T}_{\text{min1}}$ and $\tilde{F}_S(\tilde{T}_{\text{min2}}) < \tilde{F}_L(\tilde{T}_{\text{min2}})$, where $\tilde{F}_S/L$ is the Gibbs free energy of the small/large BH branch. In this region, $\tilde{T}(\tilde{r}_+)$ has a local maximum of $\tilde{T}_{\text{max}} = \tilde{T}(\tilde{r}_{\text{max}})$ at $\tilde{r}_+ = \tilde{r}_{\text{max}}$, a local minimum of $\tilde{T}_{\text{min2}} = \tilde{T}(\tilde{r}_{\text{min2}})$ at $\tilde{r}_+ = \tilde{r}_{\text{min2}}$ and a global minimum of $\tilde{T}_{\text{min1}} = \tilde{T}(\tilde{r}_{\text{min1}})$ at $\tilde{r}_+ = \tilde{r}_{\text{min1}}$. There are four branches for $\tilde{r}_+(\tilde{T})$: intermediate$-$ BH for $\tilde{T} \geq \tilde{T}_{\text{min2}}$, small BH for $\tilde{T}_{\text{min2}} \leq \tilde{T} < \tilde{T}_{\text{max}}$, intermediate$+$ BH for $\tilde{T}_{\text{min1}} \leq \tilde{T} < \tilde{T}_{\text{max}}$ and large BH for
(a) Region III: \(a/l^2 = 0.01\) and \(Q/l = 0.195\). There is a first order phase transition between small BH and large BH.

(b) Region IV: \(a/l^2 = 0.01\) and \(Q/l = 0.188\). The arrows in the inset indicate increasing \(\tilde{T}\). As \(\tilde{T}\) increases, the black hole jumps from the large BH branch to the small BH one, corresponding to the zeroth order phase transition between small BH and large BH. Further increasing \(\tilde{T}\), there would be a first order phase transition returning to large BH. Here we observe the LBH/SBH/LBH reentrant phase transition.

(c) Region V: \(a/l^2 = 0.01\) and \(Q/l = 0.185\). There is no phase transition.

(d) Region VI: \(a/l^2 = 0.01\) and \(Q/l = 0.15\). There is no phase transition.

**Figure 3.** Plots of \(\tilde{r}_+\), \(\tilde{F}\) and \(\epsilon_T^{-1} l\) against \(\tilde{T}\) for the BI-AdS black holes in Regions III, IV, V and VI. The black holes in these regions are Schwarzschild-like type since they only exist for large enough \(\tilde{T}\). The blue and green branches are always thermally stable. Small BH is electrically unstable while large BH is almost electrically stable.

\(\tilde{T} \geq \tilde{T}_{\text{min1}}\), which are displayed in the left panel of figure 3b. The Gibbs free energies of the four branches are plotted in the middle panel. As \(\tilde{T}\) increases from \(\tilde{T}_{\text{min1}}\), the black hole follows direction of arrows in the inset. It shows that there is a finite jump in the Gibbs free energy leading to a zeroth order phase transition from large BH to small BH, followed by a first order phase transition returning to large BH. This LBH/SBH/LBH phase transition corresponds to a reentrant phase transition.
Figure 4. The six regions and the critical line in the $\tilde{a}-\tilde{Q}$ plane for BI-AdS black holes. The critical line consists of $\tilde{Q}_{12}(\tilde{a})$ (the blue dashed line), $\tilde{Q}_{36}(\tilde{a})$ (the blue solid line) and $\tilde{Q}_{56}(\tilde{a})$ (the red line), where $\tilde{Q}_{ij}(\tilde{a})$ is the boundary Region $i$ and Region $j$. With fixed $\tilde{Q}$ and $\tilde{a}$, the black hole moves along the curve $\tilde{Q}_1(\tilde{a}) = \frac{\tilde{Q}}{\sqrt{\sqrt{\alpha}}}$ by varying $P$.

- Region V: $4a > Q^2 > 2a$, $\tilde{T}'(\tilde{r}_1) < 0$, $\tilde{T}'(\tilde{r}_2) > 0$, $\tilde{T}_{\text{min}2} > \tilde{T}_{\text{min}1}$ and $\tilde{F}_S(\tilde{T}_{\text{min}2}) \geq \tilde{F}_L(\tilde{T}_{\text{min}2})$. As shown in the left panel of figure 3c, the four branches of $\tilde{r}_+(\tilde{T})$ in this region are the same as in Region IV. However, the middle panel shows that the large BH branch is always thermodynamically preferred for $\tilde{T} \geq \tilde{T}_{\text{min}1}$, and hence there is no phase transition in this region.

- Region VI: $4a > Q^2 > 2a$ and $\tilde{T}'(\tilde{r}_1) > 0$ or $\tilde{T}'(\tilde{r}_2) < 0$; or $Q^2 < 2a$. It can show that $\tilde{T}'(\tilde{r}_+)$ has a global minimum at $\tilde{T}_{\text{min}} = \tilde{T}(\tilde{r}_{\text{min}})$ at $\tilde{r}_+ = \tilde{r}_{\text{min}}$. As shown in the left panel of figure 3d, there are two branches for $\tilde{r}_+(\tilde{T})$: large BH and intermediate BH for $\tilde{T} \geq \tilde{T}_{\text{min}}$. The middle panel shows that the large BH branch is always thermodynamically preferred for $\tilde{T} \geq \tilde{T}_{\text{min}}$, and hence there is no phase transition in this region. This region is similar to the Schwarzschild-AdS case. The large BH branch is thermally stable and almost electrically stable.

In figure 4a, we plot these six regions in the $\tilde{a}-\tilde{Q}$ plane. It is interesting to note that the boundary between the region in which $\tilde{T}(\tilde{r}_+)$ has $n$ extrema and that in which $\tilde{T}(\tilde{r}_+)$ has $n + 2$ extrema is the critical line, determined by

$$\frac{\partial \tilde{T}(\tilde{r}_+, \tilde{Q}, \tilde{a})}{\partial \tilde{r}_+} = 0 \quad \text{and} \quad \frac{\partial^2 \tilde{T}(\tilde{r}_+, \tilde{Q}, \tilde{a})}{\partial \tilde{r}_+^2} = 0.$$  

(3.6)
In the case of varying $P$ with fixed $Q$ and $a$, the system moves along $\tilde{Q}_l(\tilde{a})$, which is displayed for various values of $Q/\sqrt{\tilde{a}}$. For $Q/\sqrt{\tilde{a}} < \sqrt{2}$, there is no phase transition in the system. For $Q/\sqrt{\tilde{a}} > \sqrt{2}$, there are one critical point and the corresponding LBH/SBH first order phase transition. In addition, for $\sqrt{2} < Q/\sqrt{\tilde{a}} < 2$, there is a LBH/SBH zeroth order phase transition occurring in Region IV, corresponding to the LBH/SBH/LBH reentrant phase transition. There are 3 such boundaries in figure 4a, i.e., $\tilde{Q}_{12}(\tilde{a})$, $\tilde{Q}_{36}(\tilde{a})$, $\tilde{Q}_{56}(\tilde{a})$, where $\tilde{Q}_{ij}(\tilde{a})$ is the boundary Region $i$ and Region $j$. As shown in figure 4a, the critical line has two branches: $\tilde{Q}_{c1}(\tilde{a}) = \{\tilde{Q}_{12}(\tilde{a}), \tilde{Q}_{36}(\tilde{a})\}$ and $\tilde{Q}_{c2}(\tilde{a}) = \tilde{Q}_{56}(\tilde{a})$. We plot these two branches of the critical line in figure 4b, where $\tilde{Q}_{c1}(\tilde{a})$ is the blue line, and $\tilde{Q}_{c2}(\tilde{a})$ is the red line. Note that $\tilde{Q}_{c1}(\tilde{a})$ and $\tilde{Q}_{c2}(\tilde{a})$ meet and terminate at $\{\tilde{a}_c, \tilde{Q}_c\} \simeq \{0.069, 0.37\}$, which is represented by the black point in figure 4. However, the middle panel of figure 3c shows that the branch $\tilde{Q}_{c2}(\tilde{a})$ is not physical since it does not globally minimize the Gibbs free energy. So the critical line has only one physical branch, $\tilde{Q}_{c1}(\tilde{a})$, which is marked by the blue line. For $\tilde{a} \leq \tilde{a}_1 \simeq 0.030$, $\tilde{Q}_{c1}(\tilde{a})$ is depicted by the blue dashed line in figure 4b. This part of $\tilde{Q}_{c1}(\tilde{a})$ is reminiscent of RN-AdS black holes.

We now discuss the critical behavior and phase structure of black holes in two cases. In the first case, $Q$ and $a$ are fixed parameters, and the AdS radius $l$ (the pressure $P$) varies. With fixed values of $Q$ and $a$, varying $l$ would generate a curve in the $\tilde{a}$-$\tilde{Q}$ plane, which is determined by

$$\tilde{Q}_l(\tilde{a}) = \frac{Q}{\sqrt{\tilde{a}}} \sqrt{\tilde{a}}. \quad (3.7)$$

In figure 4b, we plot $\tilde{Q}_l(\tilde{a})$ for various values of $Q/\sqrt{\tilde{a}}$. It shows that, for $Q/\sqrt{\tilde{a}} < \sqrt{2}$, there is no critical point for black holes. For $Q/\sqrt{\tilde{a}} > \sqrt{2}$, there exists one physical critical point. Moreover, the critical behavior is reminiscent of RN-AdS black holes for $Q/\sqrt{\tilde{a}} > 2$. Note that $\tilde{Q}_l(\tilde{a})$ intersects the unphysical branch $\tilde{Q}_{c2}(\tilde{a})$ for $1.6948 > Q/\sqrt{\tilde{a}} > \sqrt{2}$. The phase structure of $\tilde{Q}_l(\tilde{a})$ can be read from figure 5. It shows that for $Q/\sqrt{\tilde{a}} < 2$, $\tilde{Q}_l(\tilde{a})$ is always in Region VI, and hence there is no first order phase transition. For $Q/\sqrt{\tilde{a}} > 2$, as one starts from $P = 0$, $\tilde{Q}_l(\tilde{a})$ is in Region II, in which there is a first order phase transition between small BH and large BH. Further increasing $P$, $\tilde{Q}_l(\tilde{a})$ goes through the critical line and enters the Region I, in which there is no phase transition. This behavior is reminiscent of that of the RN-AdS black hole. For $\sqrt{2} < Q/\sqrt{\tilde{a}} < 2$, as $P$ increases from $P = 0$, $\tilde{Q}_l(\tilde{a})$
The phase diagram in the $\tilde{Q}$-$\tilde{T}$ plane for the BI-AdS black hole with $a/l^2 = 0.01$. The first order phase transition line separating large BH and small BH is displayed by the brown line, and it terminates at the critical point, marked by the black dot. There is also a zeroth order phase transition line, depicted by the red line. All phases in the diagram are thermally stable. However, the phases in the yellow region are electrically unstable. Large BH above the first order phase transition line is always electrically stable except in the region around the critical point, which are highlighted in the right panel. It shows that the critical point is in the yellow region.

starts from Region VI, crosses the unphysical critical line and enters Region V, during which no phase transition occurs. Further increasing $P$, $\tilde{Q}_l(\tilde{a})$ enters Region IV, in which there is a reentrant phase transition occurring for some range of $P$. As $P$ continuously increases, $\tilde{Q}_l(\tilde{a})$ enters Region III, in which a first order phase transition occurs, crosses the critical line and returns to Region V. The critical behavior and phase structure in this case have been discussed in [19], which are correctly reproduced here.

In the second case, $a$ and $P$ $(l)$ are fixed parameters, and one varies $Q$. Figure 4 shows that for $a/l^2 > \tilde{a}_c$, there is no critical point, and no phase transition occurs. For $a/l^2 < \tilde{a}_c$, there is one critical point. As one increases $Q$ from $Q = 0$, the black hole experiences different regions, in which there occur no phase transition $\rightarrow$ the LBH/SBH/LBH reentrant phase transition $\rightarrow$ the LBH/SBH first order phase transition $\rightarrow$ no phase transition. For $a/l^2 < \tilde{a}_1$ and large enough values of $Q$, the black hole is in Regions I and II, in which the phase transition behavior is reminiscent of the RN-AdS black hole. The critical behavior and phase structure in this case have also been studied in [65].

The phase diagram of the BI-AdS black hole for $a/l^2 = 0.01$ is displayed in the $\tilde{Q}$-$\tilde{T}$ plane in figure 6. There are a LBH/SBH first order transition for some range of $\tilde{Q}$ and a LBH/SBH zeroth order phase transition for some smaller range of $\tilde{Q}$. The zeroth and first order phase transitions are marked by the red and brown lines, respectively. The first order phase transition line terminates at the critical point, represented by the black point. No BH region means that no black hole solutions exist. As discussed before, the black hole solution in the phase diagram is thermally stable. However, the solution in the yellow region is unstable to electrical fluctuations. Small BH below the first order phase transition line is always electrically unstable while large BH above the line is almost electrically stable. The right panel of figure 6 shows that large BH is only electrically unstable in the region around the critical point. Note that the black hole solution at the critical point is electrically unstable.

---

1In [19], their $b$ is our $\frac{1}{4\sqrt{2}}$. 
Figure 7. The phase diagram in the $\tilde{Q}-\tilde{T}$ plane for the BI-AdS black hole with $a/l^2 = 0.1$. There are no phase transitions. The black hole in the yellow region is electrically unstable.

The phase diagram of the BI-AdS black hole for $a/l^2 = 0.1$ is displayed in the $\tilde{Q}-\tilde{T}$ plane in figure 7, which is simpler than that for $a/l^2 = 0.01$. Figure 4 shows that when $a/l^2 > \tilde{a}_c \simeq 0.069$ (the black dot), the black hole is in Regions I or VI, and hence no phase transition occurs. At low temperatures, the black hole solution is electrically unstable for small enough values of $Q/l$.

4 iBorn-Infeld AdS black hole

We now consider an iBorn-Infeld field with the Lagrangian density

$$L(s) = -\frac{1}{a}(1 - \sqrt{1 + 2as}), \quad (4.1)$$

where $a > 0$. For an iBorn-Infeld AdS (iBI-AdS) black hole solution, $f(r)$ in the black hole solution (2.7) is given by

$$f(r) = 1 - \frac{M}{8\pi r} + \frac{r^2}{l^2} - \frac{Q^2}{6\sqrt{r^4 - aQ^2} + 6r^2} + \frac{Q^2}{3r^2} \frac{2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; \frac{aQ^2}{r^4}\right)}{r^4 - aQ^2}, \quad (4.2)$$

where $M$ and $Q$ are the mass and the charge of the black hole, respectively. For $A'_r(r)$, one has

$$A'_r(r) = \frac{Q}{\sqrt{r^4 - aQ^2}}, \quad (4.3)$$

which gives the potential of the black hole

$$\Phi = \frac{4\pi Q}{r^+} 2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; \frac{aQ^2}{r^+}\right). \quad (4.4)$$

This iBI-AdS black hole solution has a singularity at $r = r_s$, where we define

$$r_s \equiv a^{1/4}Q^{1/2}. \quad (4.5)$$

To study the nature of this singularity, we compute the corresponding Ricci scalar:

$$R = \frac{2}{a} - 12 + \frac{1}{a r^2} \frac{r_s^4 - 2r^4}{\sqrt{r^4 - r_s^4}}, \quad (4.6)$$
which becomes divergent at \( r = r_s \). So the singularity at \( r = r_s \) is a physical singularity, and one requires that \( r > r_s \).

The equation of state (2.46) becomes

\[
\tilde{T}(\tilde{r}_+) \equiv \frac{h(\tilde{r}_+)}{4\pi \tilde{r}_+} = \frac{1}{4\pi \tilde{r}_+} \left( 1 + 3\tilde{r}_+^2 - \frac{1}{2} \frac{\tilde{Q}^2}{\tilde{r}_+^2 + \sqrt{\tilde{r}_+^4 - \tilde{a} \tilde{Q}^2}} \right),
\]

where \( \tilde{r}_+ > \tilde{r}_s \equiv \tilde{a}^{1/4} \tilde{Q}^{1/2} \), \( \tilde{a} = a/l^2 \) and \( \tilde{Q} = Q/l \). It can show that \( h(\tilde{r}_+) \) is a strictly increasing function and \( h(\infty) \to +\infty \). At \( \tilde{r}_+ = \tilde{r}_s \), one has

\[
h(\tilde{r}_s) = 1 + 3\tilde{a}^{1/2} \tilde{Q} - \frac{\tilde{Q}}{2\tilde{a}^{1/2}}. \tag{4.7}
\]

For \( h(\tilde{r}_s) \leq 0 \), which reduces to

\[
\tilde{a} < \frac{1}{6} \quad \text{and} \quad \tilde{Q} \geq \frac{2\sqrt{\tilde{a}}}{1 - 6\tilde{a}}, \tag{4.8}
\]

\( \tilde{T}(\tilde{r}_+) = 0 \) has one solution \( \tilde{r}_+ = \tilde{r}_c \), at which the black hole becomes extremal. In this case, the black hole is RN type. For \( h(\tilde{r}_s) > 0 \), which reduces to

\[
\tilde{a} < \frac{1}{6} \quad \text{and} \quad \tilde{Q} < \frac{2\sqrt{\tilde{a}}}{1 - 6\tilde{a}} \quad \text{or} \quad \tilde{a} \geq \frac{1}{6}, \tag{4.9}
\]

the temperature of the black hole has a positive minimum value, and the black hole is Schwarzschild-like type.

The equation \( \tilde{T}'(\tilde{r}_+) = 0 \) becomes

\[
z(x) \equiv x^3 - \frac{3\tilde{Q}^2}{2} x^2 - \tilde{a} \tilde{Q}^4 = 0, \tag{4.10}
\]

where \( x = \sqrt{\tilde{r}_+^4 - \tilde{a} \tilde{Q}^2} > 0 \). It can show that \( z(x) \) has a local maximum of \( z(0) = -\tilde{a} \tilde{Q}^4 < 0 \) at \( x = 0 \) and a local minimum of \( z(\tilde{Q}^2) = (-\tilde{Q}^2/2 - \tilde{a}) \tilde{Q}^4 < 0 \) at \( x = \tilde{Q}^2 \). So \( z(x) = 0 \) always admits one single positive real root \( x = x_1 > 0 \). Since \( \lim_{\tilde{r}_+ \to \tilde{r}_s} \tilde{T}'(\tilde{r}_+) = +\infty \) and \( \lim_{\tilde{r}_+ \to +\infty} \tilde{T}'(\tilde{r}_+) = 3\sqrt{\tilde{a}} \tilde{Q} \), \( \tilde{T}'(\tilde{r}_+) \) always has a global minimum of \( \tilde{T}'_{\min} = \tilde{T}'(\tilde{r}_1) \) at \( \tilde{r}_+ = \tilde{r}_1 \equiv (x_1^2 + \tilde{a} \tilde{Q}^2)^{1/4} \). In what follows, we also find six regions in the \( \tilde{a}-\tilde{Q} \) plane for iBI-AdS black holes, in each of which the black hole has the distinct behavior of the branches and the phase structure:

- **Region I:** \( h(\tilde{r}_s) \leq 0 \) and \( \tilde{T}'_{\min} \geq 0 \). Since \( \tilde{T}'(\tilde{r}_+) \geq \tilde{T}'_{\min} \geq 0 \), \( \tilde{T}(\tilde{r}_+) \) is an increasing function in this region. So there is only one thermally stable branch for \( \tilde{r}_+(T) \). We plot the radius \( \tilde{r}_+ \), the Gibbs energy \( \tilde{F} \) and the isothermal permittivity \( c \tilde{T}^{-1} l \) as functions of \( \tilde{T} \) in figure 8a for a black hole with \( \tilde{a} = 0.01 \) and \( \tilde{Q} = 0.5 \) in this region. Moreover, this black hole is electrically stable for small enough and large enough values of \( \tilde{T} \). However for large enough \( \tilde{a} \), the black hole is always electrically stable.

- **Region II:** \( h(\tilde{r}_s) \leq 0 \) and \( \tilde{T}'_{\min} < 0 \). In this region, \( \tilde{T}'(\tilde{r}_+) = 0 \) has two solutions \( \tilde{r}_+ = \tilde{r}_{\max} \) and \( \tilde{r}_{\min} \) with \( \tilde{r}_{\max} < \tilde{r}_1 < \tilde{r}_{\min} \). Since \( \tilde{T}(+\infty) = +\infty \), \( \tilde{T}(\tilde{r}_+) \) has a local maximum of \( \tilde{T}_{\max} \equiv \tilde{T}(\tilde{r}_{\max}) \) at \( \tilde{r}_+ = \tilde{r}_{\max} \) and a local minimum of \( \tilde{T}_{\min} \equiv \tilde{T}(\tilde{r}_{\min}) \) at
at \( \tilde{r}_+ = \tilde{r}_{\text{min}} \). There are three branches for \( \tilde{r}_+(\tilde{T}) \): small BH for \( 0 \leq \tilde{T} \leq \tilde{T}_{\text{max}} \), intermediate BH for \( \tilde{T}_{\text{min}} \leq \tilde{T} \leq \tilde{T}_{\text{max}} \) and large BH for \( \tilde{T} \geq \tilde{T}_{\text{min}} \), which are displayed in the left panel of figure 8b. The middle panel shows that there is a first order phase transition between small BH and large BH occurring at \( \tilde{T}_{\text{min}} \leq \tilde{T} \leq \tilde{T}_{\text{max}} \). Both the small BH and large BH branches are thermally stable. The right panel shows that large BH is almost electrically stable. However, the electrical stability of small BH depends on the values of \( \tilde{a} \). For small enough \( \tilde{a} \), e.g., \( \tilde{a} = 0.01 \), small BH is electrically stable for \( \tilde{T} < \tilde{T}_1 \) and unstable for \( \tilde{T} > \tilde{T}_1 \) with some \( \tilde{T}_1 > 0 \).

- Region III: \( h(\tilde{r}_s) > 0 \) and \( \tilde{T}_{\text{min}}' < 0 \). As shown in figure 9a, the black hole’s temperature has a minimum of \( \frac{h(\tilde{r}_s)}{4\pi} \) at \( \tilde{r}_+ = \tilde{r}_s \). There is only one branch for \( \tilde{r}_+(\tilde{T}) \) in this region, which is thermally stable. Similarly to Region I, the black hole is electrically unstable for some finite range of \( \tilde{T} \) for small enough \( \tilde{a} \). However for large enough \( \tilde{a} \), the black hole is always electrically unstable.

- Region IV: \( h(\tilde{r}_s) > 0 \), \( \tilde{T}_{\text{min}}' < 0 \) and \( \tilde{T}_{\text{min}}' > \tilde{T}(\tilde{r}_s) \). In this region, \( \tilde{T}'(\tilde{r}_+) = 0 \) has two solutions \( \tilde{r}_+ = \tilde{r}_{\text{max}} \) and \( \tilde{r}_+ = \tilde{r}_{\text{min}} \) with \( \tilde{r}_s < \tilde{r}_{\text{max}} < \tilde{r}_1 < \tilde{r}_{\text{min}} \). So \( \tilde{T}(\tilde{r}_+) \) has a local maximum of \( \tilde{T}_{\text{max}} \) at \( \tilde{r}_+ = \tilde{r}_{\text{max}} \), a local minimum of \( \tilde{T}_{\text{min}} \) at \( \tilde{r}_+ = \tilde{r}_{\text{min}} \) and a global minimum of \( \tilde{T}(\tilde{r}_s) \) at \( \tilde{r}_+ = \tilde{r}_s \). There are three branches for \( \tilde{r}_+(\tilde{T}) \): small BH for \( \tilde{T}(\tilde{r}_s) \leq \tilde{T} \leq \tilde{T}_{\text{max}} \), intermediate BH for \( \tilde{T}_{\text{min}} \leq \tilde{T} \leq \tilde{T}_{\text{max}} \) and large BH for \( \tilde{T} \geq \tilde{T}_{\text{min}} \), which are displayed in the left panel of figure 9b. There is a first order phase transition between small BH and large BH occurring at \( \tilde{T}_{\text{min}} \leq \tilde{T} \leq \tilde{T}_{\text{max}} \). Both the small BH and large BH branches are thermally stable, while the intermediate BH branch is not. The electrical stability of black holes in this region is similar to that in Region II.
(a) Region III: $a/l^2 = 0.05$ and $Q/l = 0.5$. There is no phase transition.

(b) Region IV: $a/l^2 = 0.05$ and $Q/l = 0.23$. There is a first order phase transition between small BH and large BH.

(c) Region V: $a/l^2 = 0.05$ and $Q/l = 0.184$. The arrows in the inset indicate increasing $\tilde{T}$. As $\tilde{T}$ increases, the black hole jumps from the large BH branch to the small BH one, corresponding to the zeroth order phase transition between small BH and large BH. Further increasing $\tilde{T}$, there would be a first order phase transition returning to large BH. Here we observe the LBH/SBH/LBH reentrant phase transition.

(d) Region VI: $a/l^2 = 0.05$ and $Q/l = 0.17$. There is no phase transition.

Figure 9. Plots of $\tilde{r}_+$, $\tilde{F}$ and $\epsilon_T^{-1}$ against $\tilde{T}$ for the iBI-AdS black holes in Regions III, IV, V and VI. The temperature of the black holes in these regions has a minimum value greater than zero. The blue and green branches are always thermally stable. It shows that for large enough $\tilde{a}$, e.g., $\tilde{a} = 0.05$, the small BH branch is always electrically unstable. 

- Region V: $h(\tilde{r}_+) > 0$, $\tilde{T}'_{\min} < 0$, $\tilde{T}_{\min} \leq \tilde{T}(\tilde{r}_+)$ and $\tilde{F}_S(\tilde{r}_+) < \tilde{F}_L(\tilde{r}_+)$, where $\tilde{F}_S/L$ is the Gibbs free energy of the small/large BH branch. In this region, $\tilde{T}(\tilde{r}_+)$ has a local maximum of $\tilde{T}_{\max}$ at $\tilde{r}_+ = \tilde{r}_{\max}$, a global minimum of $\tilde{T}_{\min}$ at $\tilde{r}_+ = \tilde{r}_{\min}$ and a local minimum of $\tilde{T}(\tilde{r}_+)$ at $\tilde{r}_+ = \tilde{r}_s$. Figure 9c shows that there are three branches of $\tilde{r}_+(\tilde{T})$ in this region. The Gibbs free energies of the three branches are plotted in the middle.
Figure 10. The six regions in the $\tilde{a}$-$\tilde{Q}$ plane, each of which possesses the distinct behavior of the branches and the phase structure for iBI-AdS black holes. The LBH/SBH/LBH reentrant phase transition occurs in Region V. The LBH/SBH first order phase transition occurs in Regions II and IV. No phase transitions occur in Regions I, III, and VI.

Panel. As $\tilde{T}$ increases from $\tilde{T}_{\text{min}}$, the black hole follows direction of arrows in the inset. It shows that at $T = \tilde{T}(\tilde{r}_s)$, there is a finite jump in the Gibbs free energy leading to a zeroth order phase transition from large BH to small BH. Further increasing $\tilde{T}$, a first order phase transition returning to large BH occurs at $\tilde{T}_1(\tilde{r}_s) \leq \tilde{T} \leq \tilde{T}_{\max}$. This LBH/SBH/LBH transition corresponds to a reentrant phase transition.

- Region VI: $h(\tilde{r}_s) > 0$, $\tilde{T}_{\text{min}}' < 0$, $\tilde{T}_{\text{min}}(\tilde{r}_s)$ and $\tilde{F}_S(\tilde{r}_s) \geq \tilde{F}_L(\tilde{r}_s)$. As shown in the left panel of figure 9d, there are three branches of $\tilde{r}_+(\tilde{T})$ in this region. The middle panel shows that the large BH branch is always thermodynamically preferred for $\tilde{T} \geq \tilde{T}_{\text{min}}$, and hence there is no phase transition in this region.

These six regions are plotted in the $\tilde{a}$-$\tilde{Q}$ plane in figure 10, from which the critical line can be read. In fact, the critical line is determined by $\tilde{T}'(\tilde{r}_1) = 0$ and hence is composed of $\tilde{Q}_{12}(\tilde{a})$ and $\tilde{Q}_{34}(\tilde{a})$, where $\tilde{Q}_{ij}(\tilde{a})$ is the boundary between Region i and Region j. The critical line is plotted in figure 11a, and $\tilde{Q}_{12}(\tilde{a})/\tilde{Q}_{34}(\tilde{a})$ is depicted by the blue dashed/solid line. The inset in figure 11a demonstrates that for $\tilde{a} \leq \tilde{a}_1 \approx 0.02$, the critical line is $\tilde{Q}_{12}(\tilde{a})$, on which the black hole is RN type. Moreover, the critical line of the iBI-AdS black hole in the $\tilde{a}$-$\tilde{Q}$ plane is semi-infinite while that of the BI-AdS black hole is a finite line. According to figures 8b and 9b, the critical line is physical since it globally minimizes the Gibbs free energy.

In the case of varying $P$ with fixed values of $Q$ and $a$, the system moves along $\tilde{Q}_1(\tilde{a}) = (Q/\sqrt{a})\sqrt{a}$ in the $\tilde{a}$-$\tilde{Q}$ plane, which is plotted for various values of $Q/\sqrt{a}$ in figure 11. Figure 11a shows that there always exists one physical critical point. For $Q/\sqrt{a} \leq \tilde{Q}_1 \approx 2.28$, the inset in figure 11a shows that the critical point occurs for a RN type black hole. The boundary $\tilde{Q}_{45}(\tilde{a})$ is displayed by the red line in figure 11. If $\tilde{Q}_1(\tilde{a})$ intersects $\tilde{Q}_{45}(\tilde{a})$, the black hole on $\tilde{Q}_1(\tilde{a})$ can be in Region V for some range of $P$. The numerical result and figure 11b show that when $Q/\sqrt{a} < 2$, $\tilde{Q}_1(\tilde{a})$ always intersects $\tilde{Q}_{45}(\tilde{a})$, and there is a reentrant phase transition occurring for some range of $P$. Thus for $Q/\sqrt{a} < 2$, as $P$ continuously increases from $P = 0$, the black hole on $\tilde{Q}_1(\tilde{a})$ experiences the following regions: Region VI (no phase
transition) $\rightarrow$ Region V (the LBH/SBH/LBH reentrant phase transition) $\rightarrow$ Regions II or IV (the LBH/SBH first order phase transition) $\rightarrow$ Regions I or III (no phase transition). For $Q/\sqrt{a} > 2$, there is a critical point on $\bar{Q}_1 (\bar{a})$ occurring at $P = P_c$. For $P < P_c$, the black hole on $\bar{Q}_1 (\bar{a})$ with $Q/\sqrt{a} > 2$ is in Regions II or IV, and there is a first order phase transition between small BH and large BH. For $P > P_c$, the black hole is in Regions I or III, and no phase transition occurs.

In the case of varying $Q$ with fixed values of $P$ and $a$, the system moves along a constant-$\bar{a}$ line in the $a-\bar{Q}$ plane. Figure 11a shows that constant-$\bar{a}$ lines always intersect the critical line and the boundary $\bar{Q}_{45} (\bar{a})$. As one increases $Q$ from $Q = 0$, the black hole on a constant-$\bar{a}$ line experiences the following regions: Region VI (no phase transition) $\rightarrow$ Region V (the LBH/SBH/LBH reentrant phase transition) $\rightarrow$ Regions II or IV (the LBH/SBH first order phase transition) $\rightarrow$ Regions I or III (no phase transition). For $\bar{a} \geq 1/6$, the black hole on a constant-$\bar{a}$ line is always Schwarzschild-like type.

The phase diagrams of the iBI-AdS black holes for $a/l^2 = 0.01$ and $a/l^2 = 0.1$ are displayed in the $\bar{Q}-\bar{T}$ plane in figure 12, where we have the LBH/SBH first order phase transition lines (the brown lines), the LBH/SBH zeroth order phase transition lines (the red lines) and the critical points (the black dots). For $a/l^2 = 0.01$, figures 6 and 12a show that the phase diagram of the iBI-AdS black hole is similar to that of the BI-AdS black hole. Moreover, figure 12d shows that the phase diagram of the iBI-AdS black hole with $a/l^2 = 0.1$ is similar to that with $a/l^2 = 0.01$, in the way that they have the LBH/SBH zeroth and first order phase transitions. This is expected from figure 11a, which shows that the $a/l^2 = 0.1$ line intersects the critical line and the boundary $\bar{Q}_{45} (\bar{a})$. However, as shown in figure 7, there is no phase transition in the phase diagram of the BI-AdS black hole with $a/l^2 = 0.1$. All phases in figure 12 are thermally stable. The black hole in the yellow region is electrically unstable. As with the critical point of the BI-AdS black hole, the critical point of the
Figure 12. The phase diagrams in the $\tilde{Q}$-$\tilde{T}$ plane for the iBI-AdS black holes with $a/l^2 = 0.01$ and $a/l^2 = 0.1$. The first order phase transition lines separating large BH and small BH are displayed by the brown lines, and they terminate at the critical points, marked by black dots. There are also zeroth order phase transition lines, depicted by the red lines. All phases in the diagram are thermally stable. However, the phases in the yellow region are electrically unstable.

iBI-AdS black hole is also electrically unstable, which is highlighted in figure 12c. Figures 6 and 7 show that the BI-AdS black hole very close to the boundary of No BH regions is always electrically unstable. However, figure 12 shows that the iBI-AdS black hole very close to the boundary of No BH regions can be electrically stable for large enough values of $Q/l$.

5 Conclusion

We have investigated the thermodynamic behavior of NLED AdS black holes in an extended phase space, which includes the conjugate pressure/volume quantities, any dimensionful couplings $a_i$ in NLED and their associated conjugates $A_i$. For a generic NLED black hole, we first computed its Euclidean action to obtain the Gibbs free energy. To obtain consistency of the
Critical line

The critical line has a physical branch and an unphysical one, which have finite length and both terminate at \( \{ \tilde{a}_c, \tilde{Q}_c \} \simeq \{0.069, 0.37\} \).

Reentrant phase transition region

This region has a finite area and extends to the infinity \( \tilde{Q} = +\infty \).

Varying \( P \) with fixed \( Q \) and \( a \) case (\( \tilde{Q}_l(\tilde{a}) \) line)

There exists one physical critical point for \( \tilde{Q}/\sqrt{\tilde{a}} > \sqrt{2} \). The reentrant phase transition occurs for \( \sqrt{2} < \tilde{Q}/\sqrt{\tilde{a}} < 2 \).

Varying \( Q \) with fixed \( P \) and \( a \) case (constant-\( \tilde{a} \) line)

A physical critical point and reentrant phase transition occur for \( \tilde{a} < \tilde{a}_c \). A physical critical point and reentrant phase transition occur for all values of \( l \) and \( a \).

| BI-AdS BH | iBI-AdS BH |
|-----------|------------|
| Critical line | The critical line is a semi-infinite line and extends to the infinity \( \tilde{Q} = +\infty \). |
| Reentrant phase transition region | This region extends to the infinity \( \tilde{Q} = +\infty \). |
| Varying \( P \) with fixed \( Q \) and \( a \) case (\( \tilde{Q}_l(\tilde{a}) \) line) | There exists one physical critical point for \( \tilde{Q}/\sqrt{\tilde{a}} > \sqrt{2} \). The reentrant phase transition occurs for \( \sqrt{2} < \tilde{Q}/\sqrt{\tilde{a}} < 2 \). |
| Varying \( Q \) with fixed \( P \) and \( a \) case (constant-\( \tilde{a} \) line) | A physical critical point and reentrant phase transition occur for \( \tilde{a} < \tilde{a}_c \). A physical critical point and reentrant phase transition occur for all values of \( l \) and \( a \). |

Table 2. Critical behavior and phase structure for BI-AdS and iBI-AdS black holes.

Smarr relation, we found that it is necessary to include the conjugate pairs \((a_i, A_i)\). It showed that the black hole’s temperature \( T \), charge \( Q \), horizon radius \( r_+ \) (thermodynamic volume \( V \)), the AdS radius \( l \) (pressure \( P \)) and the dimensionful couplings \( a_i \) can be connected by

\[
TL = \tilde{T} \left( r_+/l, Q/l, a_i l^{-c_i} \right),
\]

where \( c_i \) is the dimension of \( a_i \). In the canonical ensemble with fixed \( T \) and \( Q \), we found that the critical behavior and phase structure of the black hole are determined by \( \tilde{Q} \equiv Q/l \) and \( \tilde{a}_l \equiv a_i l^{-c_i} \).

For BI-AdS black holes, we examined their critical behavior and phase structure, whose dependence on \( \tilde{Q} \) and \( \tilde{a} \) was plotted in figure 4. There are six regions in figure 4, and each region has different phase behavior. Specially, the LBH/SBH/LBH reentrant phase transition occurs in Region IV. For iBI-AdS black holes, we displayed the dependence of their critical behavior and phase structure on \( \tilde{Q} \) and \( \tilde{a} \) in figure 10, where there are also six regions. The LBH/SBH/LBH reentrant phase transition occurs in Region V. We summarize the results of the critical behavior and phase structure for BI-AdS and iBI-AdS black holes in table 2.

The thermodynamically preferred phases, along with the zeroth and first phase transitions and critical points, were displayed in figures 6 and 7 for BI-AdS black holes and in figure 12 for iBI-AdS black holes. We examined thermal and electrical stabilities of the black holes and found that all the thermodynamically preferred phases are thermally stable. However, the thermodynamically preferred phases in yellow regions in these figures were found to be electrically unstable. The possible equilibrium phases residing in the yellow regions were discussed in [9], which listed extremal black holes, anti-de Sitter space and black holes surrounded by a gas of particles as candidates. However, this question still remains open. In [66, 67, 76], the electrical transport behavior of the dual theory has been discussed for BI-AdS and iBI-AdS black holes in the context of gauge/gravity duality. It might be inspiring to explore possible equilibrium phases residing in yellow regions from a holographic perspective.

Acknowledgments

We are grateful to Zheng Sun and Zhipeng Zhang for useful discussions and valuable comments. This work is supported in part by NSFC (Grant No. 11005016, 11875196 and 11375121).
A Derivation of Smarr relation

In this appendix, we directly derive the Smarr relation from the definitions of the thermodynamic quantities of the black hole. The Lagrangian \( \mathcal{L}(s, a_i) \) is a function of \( s \) and the parameters \( a_i \). Performing the dimensional analysis, we find

\[
[\mathcal{L}] = L^{-2}, \quad [s] = L^{-2}, \quad [a_i] = L^c. \tag{A.1}
\]

Euler’s theorem says

\[
2\mathcal{L}(s, a_i) = 2s\mathcal{L}'(s, a_i) - c_i a_i \frac{\partial \mathcal{L}(s, a_i)}{\partial a_i}. \tag{A.2}
\]

For a dimensionful coupling \( a_i \), we have

\[
c_i a_i A_i = -4\pi c_i a_i \frac{\partial}{\partial a_i} \int_{r_+}^{\infty} dr r^2 \mathcal{L}(\frac{A_i^2}{2}, a_i) + c_i a_i Q \frac{\partial \Phi}{\partial a_i} \\
= -c_i a_i \left[ Q \frac{\partial \Phi}{\partial a_i} + 4\pi \int_{r_+}^{\infty} dr r^2 \frac{\partial \mathcal{L}}{\partial a_i}(\frac{A_i^2}{2}, a_i) \right] + c_i a_i Q \frac{\partial \Phi}{\partial a_i} \\
= 8\pi \int_{r_+}^{\infty} dr r^2 \left[ \mathcal{L}(\frac{A_i^2}{2}, a_i) - \frac{A_i^2}{2} \frac{\partial \mathcal{L}}{\partial s}(\frac{A_i^2}{2}, a_i) \right] \tag{A.3}
\]

\[
= 8\pi \int_{r_+}^{\infty} dr r^2 \mathcal{L}(\frac{A_i^2}{2}, a_i) - Q\Phi.
\]

Therefore, one has

\[
2(TS - VP) + c_i a_i A_i + Q\Phi \\
= 8\pi r_+ \left\{ 1 + \frac{r_+^2}{l^2} + \frac{r_+^2}{2} \left[ \mathcal{L}(\frac{A_i^2}{2}, a_i) - A_i'(r_+) \frac{Q}{r_+^2} \right) \right\} + 8\pi \int_{r_+}^{\infty} dr r^2 \mathcal{L}(s, a_i) \\
= M - Q\Phi + 8\pi \left\{ \frac{3}{2} \int_{r_+}^{\infty} dr r^2 \mathcal{L}(\frac{A_i^2}{2}, a_i) + \frac{r_+^3}{2} \left[ \mathcal{L}(\frac{A_i^2}{2}, a_i) - A_i'(r_+) \frac{Q\Phi}{r_+^2} \right) \right\} \\
= M,
\]

where in the last equation, we use

\[
\int_{r_+}^{\infty} dr r^2 \mathcal{L}(\frac{A_i^2}{2}, a_i) = -\frac{r_+^3}{3} \mathcal{L}(\frac{A_i^2}{2}, a_i) - \frac{1}{3} \int_{r_+}^{\infty} dr r^2 \mathcal{L}'(\frac{A_i^2}{2}, a_i) A_i'(r) A_i''(r) \\
= -\frac{r_+^3}{3} \mathcal{L}(s) + \frac{q}{3} A_i'(r_+) r_+ + \frac{q\Phi}{12\pi}.
\]
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