Simultaneous Small Noise Limit for Singularly Perturbed Slow-Fast Coupled Diffusions.

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October 17, 2018

Abstract

We consider a simultaneous small noise limit for a singularly perturbed coupled diffusion described by

\begin{align*}
    dX_t^\varepsilon &= b(X_t^\varepsilon, Y_t^\varepsilon)dt + \varepsilon^{\alpha}dB_t, \\
    dY_t^\varepsilon &= -\frac{1}{\varepsilon}\nabla_y U(X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{s(\varepsilon)}{\sqrt{\varepsilon}}dW_t,
\end{align*}

where \(B_t, W_t\) are independent Brownian motions on \(\mathbb{R}^d\) and \(\mathbb{R}^m\) respectively, \(b : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d\), \(U : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}\) and \(s : (0, \infty) \to (0, \infty)\). We impose regularity assumptions on \(b, U\) and let \(0 < \alpha < 1\). When \(s(\varepsilon)\) goes to zero slower than a prescribed rate as \(\varepsilon \to 0\), we characterize all weak limit points of \(X^\varepsilon\) as \(\varepsilon \to 0\), as solutions to a differential equation driven by a measurable vector field. Under an additional assumption on the behaviour of \(U(x, \cdot)\) at its global minima we characterize all limit points as Filippov solutions to the differential equation.

AMS Classification: 60J60, 60G35.

Keywords: Averaging principle, Slow-Fast motion, Carathéodory solution, Filippov solution, Small noise limit, Nonlinear filter, Spectral gap, Reversible diffusion.

1 Introduction

In this article we consider the simultaneous small noise limit for a singularly perturbed coupled slow-fast diffusion given by

\begin{align*}
    dX_t^\varepsilon &= b(X_t^\varepsilon, Y_t^\varepsilon)dt + \varepsilon^{\alpha}dB_t, \\
    dY_t^\varepsilon &= -\frac{1}{\varepsilon}\nabla_y U(X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{s(\varepsilon)}{\sqrt{\varepsilon}}dW_t,
\end{align*}

where \(B_t, W_t\) are independent Brownian motions on \(\mathbb{R}^d\) and \(\mathbb{R}^m\) respectively, \(b : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d\), \(U : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}\) and \(s : (0, \infty) \to (0, \infty)\). We impose regularity assumptions on \(b, U\) and let \(0 < \alpha < 1\). When \(s(\varepsilon)\) goes to zero slower than a prescribed rate as \(\varepsilon \to 0\), we show that in the simultaneous small-noise limit all weak limit points \(X_t^\varepsilon\) satisfy

\[\frac{d}{dt}X_t = \int_{\mathbb{R}^d} b(X_t, y)d\nu_t^{0, X_t}(dy)\]

where \(\nu_t^{0, X_t}(dy)\) is a probability measure supported on finitely many global minima of \(U(X_t, \cdot)\) (see Theorem 1.3). If an additional assumption on the behaviour of \(U(x, \cdot)\) at its global minima is made then we show that \(\nu_t^{0, X_t}(dy)\) is time independent and given by a determinantal formula arising from
Laplace’s principle. Consequently, for this class of $U$, we show that every limit point is a generalized Filippov solution to (3) driven by a vector field (see Theorem 1.6). In Section 1.1 we state the model, assumptions made, and the two results precisely and in Section 1.2 we discuss examples of $U$ that satisfy the required assumptions.

The factor $\frac{1}{\varepsilon}$ in the drift term in (2), intuitively suggests that the $Y^\varepsilon$ process is the “fast moving” process as $\varepsilon \to 0$ and that the “slow moving” process $X^\varepsilon$ will see an averaging of $Y$ in this limit. The study of averaging principle in various dynamical systems dates back to the work of Khasminskii and others, summarized in, e.g., Freidlin and Wentzell [FW12], Kabanov and Pergamenshchikov [KP03]. The dynamical systems considered there involve a “slow process” $X^\varepsilon$ as a solution to an ordinary differential equation (i.e. (1) with no $B_t$ term) coupled with the fast process $Y^\varepsilon$ given by a stochastic differential equation with absence of small noise (i.e. (2) with $s(\varepsilon) = 1$). In this setting, under further assumptions on $b, U$, the averaging principle leading to characterization of limit points, normal deviations, and large deviations from the averaging principle are detailed in [FW12, Chapter 7]. The ground work for this lies in understanding the long-term behavior of solutions to (2) (for fixed $\varepsilon$) and is laid out in [FW12 Chapters 4-6]. We shall rely on this foundation in prescribing assumptions for $U$ in our main results.

Large deviations and generalizations to “full dependence” systems were considered in the works of Veretennikov in [Ver93, Ver94, Ver99]. Motivated by questions from homogenization, Veretennikov considered the fast process (2) with $s(\varepsilon) = 1$ but with presence of small noise for the slow process (i.e. (1) with $\alpha = \frac{1}{2}$) and established a large deviation principle (LDP) for $X^\varepsilon$ as $\varepsilon \to 0$. One can characterize the limit points of $X^\varepsilon$ as $\varepsilon \to 0$ as the set where the rate function is equal to zero (see [Ver99, Remark 3]). In [Lip96], Liptser considered the joint distribution of the slow process and of the empirical process associated with the fast variable in the one-dimensional setting and derived an LDP. This was recently generalized to multidimensional and full dependence systems by Puhalskii in [Puh16]. The diffusions driving the slow and the fast processes in [Puh16] do not have to be uncorrelated.

In related works, Spiliopoulos in [Spi13, Spi14], Morse and Spiliopoulos in [MS17], and Gailus and Spiliopoulos in [GS17] considered a class of coupled diffusions with multiple time scales in the full dependence setting. Contained therein, after suitable relabelling of the parameters and appropriate choice of coefficients, are results that will apply to (1)-(2) for specific $b, \nabla y U$ and with $s(\varepsilon) = \varepsilon^{\alpha - \frac{1}{2}}$. Thus, when $\alpha < \frac{1}{2}$ the fast process then undergoes stochastic homogenization (i.e. (2) with $s(\varepsilon) \to \infty$ as $\varepsilon \to 0$), when $\alpha > \frac{1}{2}$ the fast process has a small noise limit (i.e. (2) with $s(\varepsilon) \to 0$ as $\varepsilon \to 0$) and when $\alpha = \frac{1}{2}$ this corresponds to $s(\varepsilon) = 1$ in (2). In [Spi13], an LDP is shown for the slow process under periodicity assumptions for all the three regimes. Without the periodicity assumption on the coefficients, in [Spi14] fluctuation results for the slow process are shown in the homogenization and $s(\varepsilon) = 1$ regimes, while in [MS17] moderate deviations for the slow process are shown for these two regimes. In [GS17] parameter estimation results are obtained when $s(\varepsilon) = 1$.

Our model falls in the complement of the above. To the best of our knowledge the case where no periodicity assumptions are made and when both slow and fast motions are subjected to small noise limits (i.e. $\alpha > 0$ and $s(\varepsilon) \to 0$) has not been studied in the literature. In this paper, we provide a first step towards understanding this regime. Since we do not impose any periodicity assumptions on the coefficients this does not allow us to restrict dynamics on a torus. Thus we have to handle the nontrivial technicalities that come with a noncompact state space which requires a new approach.

Our motivation to study this problem comes from a general philosophy of a selection principle for ill-posed dynamics, attributed to Kolmogorov in [ER85], that adds noise to the dynamics and looks at the small noise limit for candidate ‘physical’ solution(s). This philosophy has been variously used in nonlinear circuits [Sas83], evolutionary games [FY90], and underlies the notion of ‘viscosity solutions’ [FS06]. The problems of ‘averaging’ two time scale diffusions in the limit of infinite time scale separation on the one hand [KP03] and of small noise asymptotics for diffusions in the vanishing noise limit on the other hand [FW12] have been extensively studied. Our aim here is to analyze the co-occurrence of the two when the time scale separation and the small noise variance are controlled by the same parameter $\varepsilon > 0$. 

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Our first result characterizes any limit point $X$ as a solution to a differential equation given by \( (3) \). Inside this result is contained the interesting observation that the small noise limit in the faster time scale requires the noise variance to scale in an inverse logarithmic fashion, or slower (see Remark 2.2). In hindsight, this is similar to the phenomenon observed in optimization algorithms that track the stationary distribution \( \text{CHS87, GM91, HS90} \) where the spectral gap determines the convergence rate. So intuitively speaking not only does the invariant distribution concentrate as the noise decreases, but also the approach to it slows down because of the scaling of the second eigenvalue of the infinitesimal generator with the noise variance. This observation appears to be new under the additional phenomenon of averaging due to multiple time scales present in the dynamic itself.

For characterizing the limiting measure in \( (3) \) we impose restrictions on the behaviour of $U$ at its global minima. We are then able to identify any limit point as a Filippov solution to a differential equation. In particular we are able to establish an interesting connection between small noise limits with two time scales and the theory of differential equations driven by discontinuous vector fields, as in the spirit of single time scale case in \( \text{BOQ09} \). In the single time scale case, there is already a considerable body of interesting results, see \( \text{BPS2, DF14, CH83, BK10} \), though a conclusive theory is still wanting.

We also make an unconventional use of nonlinear filtering theory in proving our main result. Nonlinear filtering comes naturally into play once we replace the drift of the slow diffusion by its conditional expectation given the history of the fast process. It is then viewed as the ‘observation process’ in nonlinear filtering parlance. We extend the available well-posedness results for nonlinear filters to the case when the drift of the ‘observation’ process also depends on itself in addition to the ‘signal’ process.

We prove this in the appendix of this article in Proposition 1.1, and this result is of independent interest (see Remark 1.7).

We are now ready to state our assumptions and main results in the next subsection.

### 1.1 Main Result

We use the following notation throughout. For $n \geq 1$, $C_b(\mathbb{R}^n)$ is the space of real valued bounded continuous functions on $\mathbb{R}^n$, $C^2(\mathbb{R}^n)$ is the space of real valued functions with continuous partial derivatives up to second order, $C^2_b(\mathbb{R}^n) \subset C^2(\mathbb{R}^n)$ are functions in $C^2(\mathbb{R}^n)$ that are bounded along with their first and second order partial derivatives, and $C^2_b(\mathbb{R}^n) \subset C^2_b(\mathbb{R}^n)$ are functions in $C^2_b(\mathbb{R}^n)$ that in addition vanish at infinity along with their first and second order partial derivatives.

We use $\| \cdot \|_2$ for the $L_2$ norm and $\| \cdot \|_{\infty}$ for the sup norm. For a Polish space $S$, $\mathcal{P}(S)$ is the Polish space of probability measures on $S$ with the Prohorov topology. For $n \geq 1$, $x \in \mathbb{R}^n$, $\| x \|$ is the usual Euclidean norm, $\langle \cdot, \cdot \rangle$ is the usual inner product, and $B_1$ is the closed ball of unit radius centered at the origin in that Euclidean space. We use $\nabla, D^2$ to denote respectively the gradient and the Hessian in variable $z$.

We shall now define the model precisely. Let $0 < \alpha < 1$, $T > 0$, $d \geq 1$, $m \geq 1$, $x_0 \in \mathbb{R}^d, y_0 \in \mathbb{R}^m$ be fixed. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space on which $\{B_t\}_{t \geq 0}$ and $\{W_t\}_{t \geq 0}$ are independent standard Brownian motions on $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively. For $0 \leq t \leq T$ and $\varepsilon > 0$, consider the coupled system of stochastic differential equations given by

\[
X_t^\varepsilon = x_0 + \int_0^t b(X_s^\varepsilon, Y_s^\varepsilon)ds + \varepsilon^\alpha B_t, \quad (4)
\]

\[
Y_t^\varepsilon = y_0 - \frac{1}{\varepsilon} \int_0^t \nabla_y U(X_s^\varepsilon, Y_s^\varepsilon)ds + \frac{s(\varepsilon)}{\sqrt{\varepsilon}} W_t, \quad (5)
\]

where $b : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^d$, $U : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}$, $s : (0, \infty) \rightarrow (0, \infty)$.

We shall make the following assumptions.
(B1) \( b \in C_b(\mathbb{R}^d \times \mathbb{R}^m) \) is locally Lipschitz continuous in \( y \)-variable and is uniformly (w.r.t. \( y \)) Lipschitz continuous in \( x \)-variable, i.e. \( \exists K_1 > 0 \) such that \( \forall x, x' \in \mathbb{R}^d, y \in \mathbb{R}^m \)

\[
\| b(x, y) - b(x', y) \| \leq K_1 \| x - x' \| .
\]  

(U1) \( U \in C^2(\mathbb{R}^d \times \mathbb{R}^m) \). Further, \( \nabla_y U(x, y) \) is uniformly (w.r.t. \( y \)) Lipschitz continuous in \( x \)-variable, i.e. \( \exists K_2 > 0 \) such that \( \forall x, x' \in \mathbb{R}^d, y \in \mathbb{R}^m \),

\[
\| \nabla_y U(x, y) - \nabla_y U(x', y) \| \leq K_2 \| x - x' \| .
\]  

(U2) There exist \( R > 0, M > 0, K_3 > 0 \) such that, for all \( x \in \mathbb{R}^d \),

\[
K_3 \| \xi \|^2 \leq \langle \xi, D_x^2 U(x, y) \xi \rangle \text{ for } \xi \in \mathbb{R}^m \text{ and } y \in \mathbb{R}^m, \| y \| > R, \]

\[
\sup_{\| y \| \leq R} \max \{ \| U(x, y) \|, \| \nabla_y U(x, y) \|, \| D_x^2 U(x, y) \| \} \leq M, \text{ and } \] (9)

\[
\sup_{y \in \mathbb{R}^m} \left[ \frac{1}{4\pi \varepsilon} \left( 4\Delta_y U(x, y) - \frac{4\| \nabla U(x, y) \|}{a} \right) + 2U(x, y) \right] \leq Ms^{\frac{1}{4}}, \]

for \( a \leq 1, s \geq 1, \text{ for some } \eta > 1 \). (10)

**Remark 1.1** The assumptions (B1) and (U1) immediately imply the local existence and uniqueness of a strong solution for the coupled slow-fast small diffusions (4) and (5). These along with (8) and (9) in assumption (U2) imply nonexplosiveness of the system, and thus global existence and uniqueness. Assumption (10) is needed to ensure ultracontractivity (see [BGL14, Page 363]).

Further, using just (3) and (4) in assumption (U2) we can establish that there exists a nonnegative continuous function \( g : (0, \infty) \to (0, \infty) \) such that

\[
\sup_{z,y \in \mathbb{R}^m: \| z - y \| = r} -\frac{1}{r} \langle \nabla_y U(x, z) - \nabla_y U(x, y), z - y \rangle \leq g(r), \text{ for all } r > 0,
\]

with

\[
\Gamma := \int_0^\infty g(s)ds < \infty.
\] (11)

For completeness, we provide a proof of (11) in Lemma A.1, Appendix A Along with (10), this is used to obtain a gradient estimate for the fast process.

(U3) We assume that \( U(x, \cdot) \) has finitely many critical points for each \( x \). For later use, we introduce the following notation for global minima for each \( x \) : with \( L(x) \) denoting the number of global minima of \( U(x, \cdot) \), write

\[
\arg \min U(x, \cdot) := \{ y_1(x), \ldots, y_{L(x)}(x) \}.
\] (12)

Fix \( x \in \mathbb{R}^d \). Consider the action functional associated with the ordinary differential equation,

\[
y(t) = y_0 - \int_0^t \nabla_y U(x, y(s))ds,
\]

defined as follows. For \( \varphi \in C([0, T]; \mathbb{R}^m) \), write

\[
ST(\varphi) = \begin{cases} 
\int_0^T \| \varphi(s) + \nabla_y U(x, \varphi(s)) \|^2 ds & \text{where } \varphi \text{ is absolutely continuous} \\
\infty & \text{otherwise}
\end{cases}
\]

with \( \int_0^T \| \varphi(s) \|^2 ds < \infty \),
Here the dependence of $S_T(\varphi)$ on $x$ is suppressed. Define
\[
V(y_i, y_j(x)) = \inf \left\{ S_T(\varphi) \mid T > 0, \varphi(0) = y_i(x), \varphi(T) = y_j(x), \varphi(s) \in \mathbb{R}^m \setminus \cup_{k \neq i, j} \{y_k(x)\}, 0 \leq s \leq T \right\}.
\]
Write $L$ for $L(x)$ and define $[L] := \{1, 2, \ldots, L\}$. For $W \subset [L]$, a graph with node set $[L]$ and directed edges $m \to n$ with $m \in [L] \setminus W$, $n \in [L]$, $n \neq m$, is said to be a $W$-graph if
- each $m \in [L] \setminus W$ is the initial point of exactly one arrow, and
- there are no cycles in the graph.
Let $G(l), l = 1, 2, \ldots, L$, denote the set of all $W$-graphs with $W$ containing $l$ elements. Set
\[
V^l(x) = \min_{x \in G(l)} \sum_{(m \to n) \in x} \tilde{V}(y_m(x), y_n(x))
\]
The additional assumption we require is the following:
\[
0 \leq \Lambda := \sup_{x \in \mathbb{R}^d} [V^1(x) - V^2(x)] < \infty. \tag{13}
\]

**Remark 1.2** Assumption (U3) has two purposes. The first purpose is as in [Ven72] and [FW12] to enable the averaging principle for the fast process. The second purpose is as in [HS90] to obtain spectral gap estimates for speed of convergence of the fast process to its invariant measure and to control its rate of equilibration in the small noise limit via the decay of $s(\epsilon)$, see (22). This brings us to our next assumption.

(S1) Our next assumption is on the decay rate of $s(\cdot)$ at 0. We assume that
\[
s(\epsilon) \geq \sqrt{\frac{C \ln(1 + \frac{1}{\epsilon})}{1 - \alpha}} \quad \text{with} \quad C > \frac{2(2 + 2\Gamma)}{1 - \alpha} \quad \text{and} \quad \lim_{\epsilon \to 0} s(\epsilon) = 0. \tag{14}
\]

We are now ready to state the first of the two main results. Recall $C$ from (14), $\Gamma$ from (11) and $\Lambda$ from (13).

**Theorem 1.3** Assume (B1), (U1), (U2), (U3) and (S1). Then for any sequence $\epsilon_n \downarrow 0$ there is a further subsequence, $\epsilon_{n_k} \downarrow 0$, along which $\{X_{i, t}^{\epsilon_{n_k}}, 0 \leq t \leq T\}$ converges in law on $C([0, T]; \mathbb{R}^d)$ to $\{X_t, 0 \leq t \leq T\}$ which is almost surely a solution to
\[
X_t = x_0 + \int_0^t \int b(X_s, y)\nu_0^{0, X_r}(dy)ds \tag{15}
\]
where $\nu_0^{0, X_r}(dy)$ is a probability measure supported on $\arg \min U(X_{s, \cdot})$.

**Remark 1.4** In the proof of the result we shall show that the mapping $s \to \int b(X_{s, y})\nu_0^{0, X_r}(dy)$ is almost surely uniformly bounded and integrable. Thus $X$ is also a Carathéodory solution\(^1\) to
\[
\frac{d}{dt} X_t = \int b(X_t, y)\nu_t^{0, X_r}(dy),
\]
with $X_0 = x_0$.

\(^1\)Carathéodory solutions relax the classical requirement that the solution must follow the direction of the vector field at all times: the differential equation need not be satisfied on a set of measure zero on $[0, T]$. See [SB96] for a precise definition.
Consider the differential equation given by
\[ \varepsilon a \text{ further subsequence, } F \]
\[ \text{Recall the set } \{ X_t \}_{t \geq 0} \text{ characterizes all limit points as Filippov solutions of a differential equation.} \]
We refer the reader to [BOQ09] for motivation and various equivalent definitions of Filippov solutions. Towards this we recall a well known concept of a solution to a differential equation driven by a measurable function, namely the Filippov solution.

Definition 1.5 Consider the differential equation given by
\[ \frac{d}{dt} x(t) = h(x(t)), \ t \geq 0, \ x(0) = x_0, \quad (16) \]
where \( h : \mathbb{R}^d \to \mathbb{R}^d \) is a measurable function with at most linear growth. Define the ‘enlargement’ \( h_E(\cdot) \) of \( h(\cdot) \) to be the set-valued map
\[ h_E(x) := \bigcap_{N \subset \mathbb{R}^d, \text{Leb}(N) = 0} \bigcap_{\delta > 0} \overline{\text{co}}(h((x + \delta B_1) \setminus N)), \quad (17) \]
where \( \text{Leb} \) denotes Lebesgue measure and \( \overline{\text{co}}(\cdot) \) denotes the closed convex hull. An absolutely continuous function \( x : [0, \infty) \to \mathbb{R}^d \) is a Filippov solution to \((16)\) if it is a solution to the following differential inclusion
\[ \frac{d}{dt} x(t) \in h_E(x(t)), \ \forall t \geq 0, \]
with \( x(0) = x_0. \)

We refer the reader to [BOQ09] for motivation and various equivalent definitions of Filippov solutions. See [SB96] for a comparison of Carathéodory solutions and Filippov solutions. Our next result characterizes all limit points as Filippov solutions of a differential equation.

Recall the set \( F \) from (U4).

Theorem 1.6 Assume (B1), (U1), (U2), (U3), (U4) and (S1). Then for any sequence \( \varepsilon_n \downarrow 0 \) there is a further subsequence, \( \varepsilon_{n_k} \downarrow 0 \), along which \( \{ X_t^{\varepsilon_{n_k}}, 0 \leq t \leq T \} \) converges in law on \( C([0, T]; \mathbb{R}^d) \) to \( \{ X_t, 0 \leq t \leq T \} \) which belongs almost surely to the set of Filippov solutions to
\[ \frac{d}{dt} X_t = h(X_t), \ \forall t \geq 0, \quad (18) \]
with \( X_0 = x_0 \) and \( h : \mathbb{R}^d \to \mathbb{R}^d \) is defined almost everywhere as follows:
\[ h(x) = \sum_{i=1}^{L(x)} b(x, y_i(x)) \frac{(\text{Det} [D_y^2 U(x, y_i(x))])^{\frac{1}{2}}}{\sum_{j=1}^{L(x)} (\text{Det} [D_y^2 U(x, y_j(x))])^{\frac{1}{2}}}, \quad (19) \]
for all \( x \in F \).
Under (U4), the set $F^c$ has Lebesgue measure 0, and hence the function $h$ is almost everywhere given by the determinantal formula. In general $h$ will not be continuous in $x$, and we will need to consider Filippov solutions of (13). In some circumstances, however, we may be able to get a classical solution.

As we will see later in the proofs, the measure $\nu_t^{0,X_t}$ in Theorem 1.3 may depend on the subsequence and consequently no uniqueness claim is being made about the measure in Theorem 1.3. If $\min U(X_t, \cdot)$ is a singleton then the measure $\nu_t^{0,X_t}$ must be the Dirac measure on the minimizer. In most other cases Theorem 1.3 applies.

**Future Directions:** We conclude this section by mentioning a few possible extensions and open problems. The case when $\alpha > 1$ and there are no periodicity assumptions for the coupled diffusion in (H1)–(H3) remains open. So does the case when there is so called “full dependence”, when the coefficients in front of the respective Brownian motions depend on both the slow and the fast processes. We did not introduce coefficients in front of the driving diffusion process primarily because we wanted to illustrate the possible limits when small noise phenomena are present in both time scales. Our approach of using nonlinear filtering to characterise limit points can be generalized to this setting but the spectral gap estimates for the fast processes which are not reversible will not be available.

There is a possibility of weakening the assumptions on $U$. Assumption (U4) imposes a strict behavior of $U(x, \cdot)$ around its global minima. One can try to handle the case when $D^2(x, y(x))$ is singular by applying a generalization of Laplace’s method (see [AH10]). Further, from the proof of Theorem 1.3 we will be able to infer that, if the rate of convergence of $\|X_{t^n}^c - X_t\|$ as $n \to \infty$ is understood, then we can characterize $\nu_t^{0,X_t}$ without assumption (U4). However, such a rate seems hard to capture given the two timescales and the interdependence of $X_t^c$ on $Y_{t^n}^\alpha$. Towards this an LDP as in [Ver00] or fluctuation results as in [Spi14] when $0 < \alpha < 1$ will have to be understood first. Several constants are assumed to be universal in (U1)and (U2), weakening these should be possible and in some cases even our current proof may hold for a restricted set of $\alpha$.

1.2 Examples

In this section we explore specific examples of $U$ that will help us understand the assumptions used in Theorem 1.3 and Theorem 1.6.

1.2.1 Weak Convergence and a Classical Solution

Under assumption (U4), if $L(x) \equiv L$, if $y(x)$ were Lipschitz in $x$ for $1 \leq i \leq L$, and if $F = \mathbb{R}^d$, then his Lipschitz. The ordinary differential equation (13) is then well-posed and has a unique solution. We can then strengthen Theorem 1.6 to say that the process $X^c$ converges weakly to $X$. We now present an example to illustrate this.

**Example 1** Assume (S1) holds for the function $U_1$ given below. Take $m = d = 1$, and let $b$ be any function that satisfies (B1). Consider $U_1 : \mathbb{R}^2 \to \mathbb{R}$ given by

$$U_1(x, y) = \begin{cases} y^4 - 2y^2 \left(\frac{1/2+x^2}{1+x^2}\right) + 1, & |y| \leq 10 \\ y^4 - 2y^2 + 1, & |y| \geq 20, \end{cases}$$

and for $10 < |y| < 20$ define:

$$U_1(x, y) := (1 - \varphi(|y|)) \left[y^4 - 2y^2 \left(\frac{1/2+x^2}{1+x^2}\right) + 1\right] + \varphi(|y|) \left[y^4 - 2y^2 + 1\right]$$

with $\varphi : \mathbb{R}_+ \to [0, 1]$ such that $\varphi(|y|) = 0$ for $|y| \leq 10$ and $\varphi(|y|) = 1$ for $|y| \geq 20$, and $\varphi(\cdot)$ is a $C^2$ function with both $\varphi'(|y|)$ and $\varphi''(|y|)$ taking the values 0 at $|y| = 10$ and 20.
It is easy to see that (U1) holds. Further, \( \nabla_y U_1(x, y) \) and \( D_y^2 U_1(x, y) \) are continuous for all \( x \) and \( y \). \( \text{(8)} \) holds for \( |y| > R = 20 \), and \( \text{(9)} \) holds for a sufficiently large \( M \) with \( R = 20 \). To see that \( \text{(10)} \) holds, for \( |y| \geq R = 20 \), one verifies that the left-hand side of \( \text{(10)} \) is a sixth degree polynomial in \( y \) with leading coefficient being negative. Optimizing over \( y \) we get the upper bound to be \( M \) for suitably large \( M \). Thus one can choose \( \eta = 2 \) to make \( \text{(10)} \) hold. Hence \( \text{(U2)} \) also holds.

Choose \( g \) suitably so that for each \( x \), the critical points \( y \) satisfying \( \nabla_y U_1(x, y) = 0 \) also satisfy \( |y| \leq 10 \). To find the critical points, we may then equate \( \nabla_y U_1(x, y) = 4y\left(y^2 - \frac{1/2 + x^2}{1 + x^2}\right) = 0 \). There are then exactly three such points for each \( x \). The global minima of \( U_1(x, \cdot) \) are then attained at
\[
y_1(x) = \sqrt{\frac{1/2 + x^2}{1 + x^2}}, \quad y_2(x) = -\sqrt{\frac{1/2 + x^2}{1 + x^2}}
\]
yielding \( L(x) = 2 \) for all \( x \). The point \( y = 0 \) is a local maximum for all \( x \).

The quantity \( V_1(x) \), by symmetry, is the action functional for moving from \(-\sqrt{\frac{1/2 + x^2}{1 + x^2}}\) to \( \sqrt{\frac{1/2 + x^2}{1 + x^2}} \) and \( V_2(x) = 0 \). By considering constant velocity paths, it is easy to verify that action functional is bounded as a function of \( x \) and hence Assumption \( \text{(U3)} \) holds.

Finally, \( L(x) \equiv 2 \) and \( D_y^2 U_1(x, y_1(x)) = D_y^2 U_1(x, y_2(x)) = 8(1/2 + x^2) \geq 4 \) for all \( x \in \mathbb{R} \). Thus \( \text{(U4)} \) also holds with \( F^c = \emptyset \).

Theorem 1.6 then implies \( X_t \) is a Filippov solution to \( \text{(12)} \) which for this example reduces to
\[
\frac{d}{dt} X_t = \frac{1}{2} b \left( X_t - 2y\left(\frac{1/2 + X_t^2}{1 + X_t^2}\right) + \frac{1}{2} b \left( X_t - \sqrt{\frac{1/2 + X_t^2}{1 + X_t^2}}\right) \right), \quad X_0 = x_0.
\tag{20}
\]

Further, from \( \text{(B1)} \), we note that the driving function above is globally Lipschitz. This implies that every limit point \( X \) is given by the unique classical solution to the differential equation \( \text{(20)} \). Consequently we have that \( \{X_t^\infty, t \in [0, T]\} \) converges in law to the unique solution to the differential equation \( \text{(20)} \).

### 1.2.2 Merging and Creation of Global Minima

We now discuss two illustrative examples where the number of global minima \( L(x) \) varies with \( x \). As \( x \) varies, global minima may merge or new global minima may emerge. We begin with an example where global minima merge. In such an event \( D_y^2 U_1(x, y_1(x)) \) could have a vanishing determinant resulting in a nonempty \( F^c \) in assumption \( \text{(U4)} \).

**Example 2** Assume \( \text{(S1)} \) holds for the function \( U_2 \) below. Take \( m = d = 1 \), and let \( b \) be any function that satisfies \( \text{(B1)} \). Similar to Example 1 consider \( U_2 : \mathbb{R}^2 \to \mathbb{R} \) given by
\[
U_2(x, y) = \begin{cases} 
y^4 - 2y^2\frac{x^2}{1 + x^2} + 1, & |y| \leq 10 
y^4 - 2y^2 + 1, & |y| \geq 20,
\end{cases}
\]
and for \( 10 < |y| < 20 \) define:
\[
U_2(x, y) := (1 - \varrho(|y|)) \left[ y^4 - 2y^2\frac{x^2}{1 + x^2} + 1 \right] + \varrho(|y|) \left[ y^4 - 2y^2 + 1 \right]
\]
with \( \varrho : \mathbb{R}_+ \to [0, 1] \) such that \( \varrho(|y|) = 0 \) for \( |y| \leq 10 \) and \( \varrho(|y|) = 1 \) for \( |y| \geq 20 \), and \( \varrho(\cdot) \) is a \( C^2 \) function with both \( \varrho'(\cdot) \) and \( \varrho''(\cdot) \) taking the values \( 0 \) at \( |y| = 10 \) and \( 20 \).

Again, choose \( g \) suitably so that for each \( x \), the critical points \( y \) satisfying \( \nabla_y U_2(x, y) = 0 \) also satisfy \( |y| \leq 10 \), and so we may equate \( \nabla_y U_2(x, y) = 4y(y^2 - x^2)/(1 + x^2) = 0 \). The global minimum is then:
Thus $L(x) = 2$ when $x \neq 0$ and the global minima $y_1(x)$ and $y_2(x)$ merge as $x \to 0$ yielding $L(0) = 1$.

Following the arguments in Example 1, we can conclude that (B1), (U1), (U2), (U3) hold. Furthermore, $D^2_u U(x, y(x))$ is positive definite for all $x \neq 0$ and singular only at $x = 0$, (U4) also holds with $F^c = \{0\}$. From Theorem 1.6 we know that all limits points are characterized by Filippov solutions to (18) with

$$h(x) = \frac{1}{2} b \left( x, \frac{x}{\sqrt{1 + x^2}} \right) + \frac{1}{2} b \left( x, \frac{-x}{\sqrt{1 + x^2}} \right), \text{ for all } x \neq 0. \tag{21}$$

From Theorem 1.5, we know that $X$ solves (19).

With $L(0) = 1$, we must also have $\nu^0_{1, y} \in \delta_0$ whenever $X_t = 0$. So we may define $h(0) = b(0, 0)$ and we have from assumption (B1) that the $h$ in (21) with $h(0) = b(0, 0)$ is a Lipschitz continuous function. As in Example 1 we obtain convergence in law to the unique solution to the differential equation

$$\frac{d}{dt} X_t = \frac{1}{2} b \left( X_t, \frac{X_t}{\sqrt{1 + X_t^2}} \right) + \frac{1}{2} b \left( X_t, \frac{-X_t}{\sqrt{1 + X_t^2}} \right), \quad X_0 = x_0.$$ 

Recall that in Example 1 has $L(x) = 2$ global minima for all $x$ (i.e. no creation or merging) and Example 2 has $L(x) = 2$ global minima for all $x \neq 0$, but they merge as $x \to 0$ to give $L(0) = 1$. In the following example, we consider a different variation where a new global minimum is created.

Example 3 Consider

$$U_3(x, y) = U_1(x, y) + \phi(x)y^4 1\{y \geq 0\},$$

where $U_1$ is as in Example 1 and $\phi(x)$ is any smooth and strictly increasing function that is strictly positive when $x > 0$, equals 0 when $x = 0$, strictly negative when $x < 0$, and $\phi(x) \geq -1/2$ for all $x$. Note that this is a perturbation of $U_1$. When $x > 0$, the perturbation term $\phi(x)y^4 1\{y \geq 0\}$ lifts the graph $U(x, \cdot)$ for $y > 0$ but leaves it unchanged for $y \leq 0$, and therefore the left minimum of $U_1(x, \cdot)$ is the unique global minimum of $U_3(x, \cdot)$. Similarly, when $x < 0$, the perturbation pushes the graph gently down for $y > 0$, leaves it unchanged for $y \leq 0$, and therefore the unique global minimum of $U_3(x, \cdot)$ is strictly positive. When $x = 0$ however, we get $U_3(0, \cdot) = U_1(0, \cdot)$ and we therefore have two global minima.

Thus $L(x) = 1$ for all $x \neq 0, L(0) = 2$, and assumption (U4) holds with $F^c = \{0\}$. It is easy to see that all assumptions for Theorem 1.6 hold and Theorem 1.7 applies. However, if $b(0, 1/\sqrt{2}) \neq b(0, -1/\sqrt{2})$, the resulting $h$ in (19) has $h(0-) \neq h(0+)$. So we will not in general have a classical solution to (13), but we do have a generalized solution, namely, the Filippov solution.

More generally, for any nonconstant $b(\cdot, \cdot)$, one can choose $0 \leq \phi(\cdot) \leq 1/2$ arising from a Lipschitz-continuous distance function with distance taken from a suitable generalized Cantor-type set, so Theorem 1.7 applies. In this case as well the nature of $h$ will be such that we can at best ensure that all limit points are generalized Filippov solutions to (15).

An example of $U(\cdot, \cdot)$ that does not satisfy (U4) is $U_4(x, y) = \phi(x)y^2 + y^4$, where $\phi^{-1}(0)$ has positive Lebesgue measure.

Layout of the paper: The rest of the paper is organized as follows. In Section 2 we present three key results: Proposition 2.1 (establishes a spectral gap bound for the rescaled fast process (22)), Proposition 2.3 (identifies limit points), and Proposition 2.4 (characterizes a given limit point).
Our approach is inspired by the foundations laid in [FW12]. The fast moving $Y^\varepsilon_t$ process approaches its stationary distribution $\nu_{\varepsilon,Y^n}$ and this in turn approaches a limiting measure $\nu_{0,Y^n}$ as $\varepsilon \to 0$. Thus the slow process $X^\varepsilon_t$ as $\varepsilon \to 0$ will now observe an averaging principle in $Y^\varepsilon_t$ as determined by $\nu_{0,Y^n}$. To make the above rigorous we will need to quantify to what extent $Y^\varepsilon_t$ has equilibrated to $\nu_{\varepsilon,Y^n}$ along with the rate of convergence of $\nu_{\varepsilon,Y^n}$ to $\nu_{0,Y^n}$ as $\varepsilon \to 0$. However implementing this program of analysis turns out to be delicate due to the presence of small noise limit dictated by $s(\varepsilon)$. We will see this manifest itself in the spectral gap estimate for the fast process, which we will establish first.

Fix $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^m$, $s \geq t$. Consider the stochastic differential equation

$$Z_{s,t,x}^\varepsilon = y - \int_t^s \nabla_y U(x,Z_{r,t,x}^\varepsilon)dr + s(\varepsilon)(W_s - W_t), s \geq t,$$  

(22)

with $W_t$ being a Brownian motion. One may view the above stochastic differential equation as being obtained from (9) by first freezing $X^\varepsilon \equiv x$, then scaling time by $\varepsilon$ and setting $W_t := \frac{1}{\sqrt{\varepsilon}} W_{\varepsilon t}, t \geq 0$. The small noise limit in (22) (i.e., $s(\varepsilon) \to 0$ as $\varepsilon \to 0$) has been well studied in the literature. Hwang and Sheu [HS90] gave explicit decay rates for the second eigenvalue of the Fokker-Planck operator associated with the generator of (22) and provided connections to simulated annealing (where the exact formulation of $s(\varepsilon)$ can be identified). The small noise phenomenon in (22) can be used to identify the global minima of the function $U$ and has applications in simulated annealing (see [HS90]). These and the other physical phenomena of metastability have been explored by Bovier et al. in [BEGK00, BEGK01, BEGK04] and by Eckhoff in [Eck05]. Recently in [BB09], limits of invariant measures of (22) under the small noise limit were understood via a control theoretic approach.

For $s \geq t$, $f \in C^2_b(\mathbb{R}^m)$, consider the Feller semigroup of the process $Z_{s,t,x}^\varepsilon$ defined by

$$T_{s,t,x}^\varepsilon f(y) = \mathbb{E}_y \left[ f(Z_{s,t,x}^\varepsilon) \right]$$

with the corresponding generator given by

$$L^\varepsilon f(y) = \frac{s(\varepsilon)^2}{2} \Delta f(y) - \langle \nabla_y U(x, y), \nabla f(y) \rangle.$$

(23)

Our first proposition describes the invariant measure of $Z_{s,t,x}^\varepsilon$ and provides a uniform rate of convergence to stationarity using a spectral gap estimate.
Proposition 2.1 (Spectral Gap Estimate) Let $x \in \mathbb{R}^d$, $y \in \mathbb{R}^m$ and $t \in [0, T]$.

(a) The stochastic differential equation (22) has a unique strong solution equipped with a unique invariant probability measure $\nu^{x,t}(dy)$ given by

$$\nu^{x,t}(dy) := C(\varepsilon, x)^{-1} e^{-\frac{2\varepsilon(s-t)}{c_1(\varepsilon, x)}} dy,$$

where $0 < C(\varepsilon, x) < \infty$ is the normalizing factor.

(b) Fix $\delta > 0$. For all sufficiently small $\varepsilon$, there exists a $c_1 > 0$ such that for all $s > t + 1$ and $f \in C^2_b(\mathbb{R}^m)$

$$\| T^{x,\varepsilon,t} f - \nu^{x,t}(f) \|_{\infty} \leq \| f \|_{\infty} e^{-c_1(s-t)} \exp\left(-\frac{(s+\delta)}{c_1(\varepsilon, x)}\right),$$

where $0 \leq \Lambda < \infty$ is as in (13).

Remark 2.2 The spectral gap for reversible diffusion (22) is proved in [HS90, Theorem 3.1] and from this (25) will follow in the $L^2$ sense. We however need the estimate in the infinity norm and the spectral gap to be independent of $x \in \mathbb{R}^d$. These are achieved respectively by ultracontractivity due to (10) of assumption (U2) and (13) of assumption (U3) resulting in an extra factor $\exp\left\{\frac{c_1}{s(\varepsilon, x)^2}\right\}$.

As we see later, we will choose $s-t$ to be $\varepsilon^{-\theta}$ for some $\theta > 0$. Hence $s(\varepsilon)$ as in (13) of Assumption (S1) ensures that the process has mixed and the right-hand side of (25) goes to zero.

Our next step is to establish tightness of $X^\varepsilon$ along with tightness of conditional laws of $Y^\varepsilon$ given $X^\varepsilon$ in a specific topology. For $s > 0$, set $F^\varepsilon_s$ as the completion of $\bigcap_{s' > s} \sigma(X^\varepsilon_u, u \leq s')$. Define $\pi^\varepsilon_s \in \mathcal{P}(\mathbb{R}^m)$ via

$$\pi^\varepsilon_s(f) := \mathbb{E}[f(Y^\varepsilon_s) | F^\varepsilon_s], \quad \forall f \in C_b(\mathbb{R}^m).$$

Using [Won71, Theorem 4.1], one can rewrite (1) in the form

$$X^\varepsilon_t = x_0 + \int_0^t \int b(X^\varepsilon_s, y) \pi^\varepsilon_s(dy) ds + \varepsilon^{\alpha} \eta^\varepsilon_t,$$

where $\eta^\varepsilon_t$ is an $\mathbb{R}^d$-valued Wiener process under $\mathbb{P}$. Let $\mathbb{R}^m_+$ denote the one point compactification of $\mathbb{R}^m$. We equip

$$\widehat{\mathcal{P}} = \{ \zeta : (\zeta : [0, T] \to \mathcal{P}(\mathbb{R}^m)) \text{ and is measurable} \}$$

with the coarsest topology that renders continuous the maps

$$\zeta \in \widehat{\mathcal{P}} \to \int_0^s g(a) \int f(y) \zeta(a)(dy) da$$

for all $0 \leq u < s \leq T$, $g \in L^2([u, s], f \in C(\mathbb{R}^m_+))$. We will view $(X^\varepsilon, \pi^\varepsilon) := (X^\varepsilon_t, \pi^\varepsilon_t)_{t \in [0, T]}$ as elements of $C([0, T]; \mathbb{R}^d) \times \widehat{\mathcal{P}}$.

The above approach towards topologizing the path space of the conditional density of $Y^\varepsilon$ is borrowed from the relaxed control framework in control theory. This is described in Chapter 2 of [ABG12]. More specifically, the topology is compact and metrizable as explained in [ABG12 Section 2.3]. Our next proposition asserts tightness and identifies a limit point with which we will work.

Proposition 2.3 (A limit point) The laws of $\{(X^\varepsilon, \pi^\varepsilon) : 0 < \varepsilon < 1\}$ are tight in the space $\mathcal{P}(C([0, T]; \mathbb{R}^d) \times \widehat{\mathcal{P}})$. Further, there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that

(a) $(X^{\varepsilon_n}, \pi^{\varepsilon_n}) \to (X, \pi)$ weakly as $n \to \infty$, 

(b) there exists a filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), random processes 
\((\tilde{X}^\varepsilon_n, \tilde{\pi}^\varepsilon_n, \tilde{\eta}^\varepsilon_n) \overset{d}{=} (X^\varepsilon_n, \pi^\varepsilon_n, \eta^\varepsilon_n)\) and \((\tilde{X}, \tilde{\pi}, \tilde{\eta}) \overset{d}{=} (X, \pi, \eta)\) such that 

\[(X^\varepsilon_n, \pi^\varepsilon_n, \eta^\varepsilon_n) \to (\tilde{X}, \tilde{\pi}, \tilde{\eta}) \text{ a.s., and} \quad \tilde{E} \left[ \sup_{t \in [0, T]} \| X^\varepsilon_n_t - \tilde{X}_t \|^2 \right] \to 0 \tag{28} \]

as \(n \to \infty\).

From the above result we have a candidate limit point for \(X^\varepsilon\) and a limit point for the conditional density \(\pi^\varepsilon\) in the \(\varepsilon \to 0\) limit. As indicated earlier we will use filtering theory to understand the limit point of \(\pi^\varepsilon\). The chosen topology enables the use of the spectral gap estimate to identify how \(s(\varepsilon)\) should decay to 0 as \(\varepsilon \to 0\) in order to establish that any limit point of \(\pi^\varepsilon\) coincides with a probability measure supported on the \(\arg \min \{U(X_t, \cdot)\}\).

One could directly show tightness of \(Y^\varepsilon\) but characterizing the limit point does not seem to be straightforward (except in the case when \(U(x, \cdot)\) has a unique global minimum). However, the above leads to a much simpler approach to the averaging result because it enables us to avoid reliance on empirical measures of the fast process (which are more difficult to handle). Further, as discussed in the introduction, the probability measure-valued process of conditional laws has its own well defined evolution given by the Fujisaki-Kunita-Kallianpur equation of nonlinear filtering (see Proposition B.1). This facilitates the characterization of its weak limit points in a straightforward manner, which is our next result.

**Proposition 2.4 (Characterization of \(\pi\))** Let \(\gamma = \min\{1 - \alpha, \frac{1}{2}\}\). Let \(\varepsilon_n > 0, \tilde{X}, \tilde{\pi}\) be as constructed in Proposition 2.3. There exists a subsequence \(\varepsilon_{n_k}\) such that for all \(f \in C_0^\infty(\mathbb{R}^m)\):

(a) 

\[\lim_{k \to \infty} \frac{1}{\varepsilon_{n_k}} \int_t^{t + \varepsilon_{n_k}} \tilde{\pi}_s^\varepsilon_n (f) ds = \tilde{\pi}_t (f) \tag{29}\]

for almost every \(t \in [0, T]\), almost surely;

(b) 

\[\lim_{k \to \infty} \left\| \nu^{\varepsilon_{n_k}, \tilde{X}^n_{t+\varepsilon_{n_k}}} (f) - \frac{1}{\varepsilon_{n_k}} \int_t^{t + \varepsilon_{n_k}} \tilde{\pi}_s^\varepsilon_n (f) ds \right\| = 0, \tag{30}\]

for all \(t \in [0, T]\) almost surely; and

(c) Almost surely, for almost every \(t \in [0, T]\), \(\nu^{\varepsilon_{n_k}, \tilde{X}^n_{t+\varepsilon_{n_k}}} \) converges weakly to a probability measure \(\nu^{0, \tilde{X}_t}_{\cdot} \) supported on \(\arg \min U(\tilde{X}_t, \cdot)\) and further \(\tilde{\pi}_t = \nu^{0, \tilde{X}_t}_{\cdot} \).

(d) If (U4) holds and if \(\tilde{X}_t \in F\), then the measure \(\nu^{0, \tilde{X}_t}_{\cdot}\) from (c) is given by

\[\nu^{0, \tilde{X}_t}_{\cdot}(\cdot) = \frac{\left(\text{Det} \left[ D^2_s U(x, y_i(\tilde{X}_t)) \right] \right)^{\frac{1}{2}}}{\sum_{j=1}^{L(\tilde{X}_t)} \left(\text{Det} \left[ D^2_s U(x, y_j(\tilde{X}_t)) \right] \right)^{-\frac{1}{2}}}. \]

The above proposition contains the main architecture of the proof of Theorem 1.3. It works with the sequence \(\{\varepsilon_n\}\) and the associated limit point from Proposition 2.3. Part (a) shows that convergence of the conditional densities holds in the small time-averaged limit along a subsequence. The topology borrowed from [ABG12] is made use of in this step. Part (b) contains the key step that is used to understand the two “limits”, first one in which the fast process approaches stationarity resulting in the averaging phenomenon and the second one in which the stationary measure approaches its limit due to the presence of small noise in \(\nu\). In Proposition 5.2 we show a second moment estimate. It is
here that we critically benefit from the filtering theory approach, understand the role played by the decay rate of $s(\varepsilon)$ to zero as $\varepsilon \to 0$, and observe the need to choose $C$ large enough to achieve the result.

Part (c) characterizes all subsequential weak limits of $\nu_t^\varepsilon \bar{X}^t_\varepsilon$ as measures supported on $\arg \min U(\bar{X}_t, \cdot)$ denoted by $\nu_t^0 \bar{X}_t$. This confirms that the measures $\bar{\pi}_t$, known to be supported on $\mathbb{R}^n$, are actually supported on $\arg \min U(\bar{X}_t, \cdot)$ for almost every $t \in [0, T]$. Finally, in Part (d), assumption (U4) is used to enable the implementation of Laplace’s principle to arrive at a determinantal formula for subsequential limits.

We note that the characterization of $\bar{\pi}$ may change with the choice of the subsequence taken in the previous parts and consequently there is no uniqueness claim being made about the measure $\nu_t^0 \bar{X}_t$, under (U1)-(U3) alone. Of course, if $\arg \min U(\bar{X}_t, \cdot)$ is a singleton, it is perforce unique, being the Dirac measure on the minimizer. In part this motivated assumption (U4) under which a modification of the Laplace’s method holds and the probability assigned by $\nu_t^0 \bar{X}_t$ to each global minima is proportional to $\left(\text{Det} \left[D^2 U(\bar{X}_t, y_i(\bar{X}_t))\right]\right)^{-\frac{1}{2}}$ provided the Hessian (in $y$) of $U$ at all global minima of $U(\bar{X}_t, \cdot)$ are uniformly positive definite in a neighborhood of $\bar{X}_t$.

We are now ready to present the proofs of Theorem 1.3 and Theorem 1.6. We will begin by setting up common notation required for both and will then present the proof of each. From (27) we have that

$$X^t_\varepsilon = x_0 + \int_0^t \int b(X^\varepsilon_s, y)\pi^\varepsilon_s(dy)ds + \varepsilon^n \eta^n_t, \quad t \in [0, T].$$

Let $\varepsilon_n \to 0$ denote the subsequence identified in Proposition 2.4. So there exist a probability space and processes $(\bar{X}^\varepsilon, \bar{\pi}^\varepsilon, \bar{\eta}^\varepsilon, \bar{X}, \bar{\pi}, \bar{\eta})$ such that

- $(X^\varepsilon, \pi^\varepsilon, \eta^\varepsilon)$ and $(\bar{X}^\varepsilon, \bar{\pi}^\varepsilon, \bar{\eta}^\varepsilon)$ have the same law for $n \geq 1$;
- $(X, \pi, \eta)$ and $(\bar{X}, \bar{\pi}, \bar{\eta})$ have the same law;
- $\bar{X}^\varepsilon \to \bar{X}$ and $\bar{\eta}^\varepsilon \to \bar{\eta}$ in $C([0, T]; \mathbb{R}^d)$, and $\bar{\pi}^\varepsilon \to \bar{\pi}$ in $\bar{\mathcal{P}}$, a.s.

Set $\xi^\varepsilon_n := \bar{X}^\varepsilon - \varepsilon^n \bar{\eta}^\varepsilon$. Then,

$$\xi^\varepsilon_n = \bar{X}^\varepsilon - \varepsilon^n \bar{\eta}^\varepsilon = x_0 + \int_0^t \int b(\bar{X}^\varepsilon_s, y)\bar{\pi}^\varepsilon_s(dy)ds,$$

(31)

Since $\bar{X}^\varepsilon \to \bar{X}$ and $\bar{\eta}^\varepsilon \to \bar{\eta}$ in $C([0, T]; \mathbb{R}^d)$, a.s., we have

$$\xi^\varepsilon_n \to \bar{X} \text{ in } C([0, T]; \mathbb{R}^d), \text{ a.s.}$$

(32)

For all $s \in [0, T]$, define

$$\delta_{n,s} := \left\| \int b(\bar{X}^\varepsilon_s, y)\bar{\pi}^\varepsilon_s(dy) - \int b(\bar{X}_s, y)\bar{\pi}_s(dy) \right\|$$

and define

$$\tau_{n,s} := \left\| \int b(\bar{X}_s, y)\bar{\pi}_s(dy) - \int b(\bar{X}, y)\pi(dy) \right\|. $$

(33)

(34)

By Proposition 2.4(c), $\bar{\pi}_s = \nu_0^0 \bar{X}_s$ for almost every $s \in [0, T]$, a.s., with $\nu_0^0 \bar{X}_s$ being a probability measure supported on $\arg \min U(\bar{X}_s, \cdot)$. We can therefore write

$$\tau_{n,s} = \left\| \int b(\bar{X}_s, y)\bar{\pi}_s(dy) - \int b(\bar{X}_s, y)\nu_0^0 \bar{X}_s(dy) \right\| \text{ for almost every } s \in [0, T].$$

(35)
Proof of Theorem 1.3. Observe that by (31), (33), and (35), with some simple algebra we have

\[
\|\tilde{X}_t - x_0\| = \left\| \int_0^t \int b(\tilde{X}_s, y) \nu_s^0(\tilde{X}_s) (dy) ds \right\| \leq \|\tilde{X}_t - \tilde{\xi}^n\| + \int_0^t \delta_{n,s} ds + \int_0^t \tau_{n,s} ds. \tag{36}
\]

By the Lipschitz property of \(b\) in (B1), we have for all \(s \in [0, T]\),

\[
\int_0^t \delta_{n,s} ds \leq K \int_0^t \|\tilde{X}^n_s - \tilde{X}_s\| ds \leq KT \sup_{s \in [0, T]} \|\tilde{X}^n_s - \tilde{X}_s\|
\]

and so

\[
\int_0^t \delta_{n,s} ds \to 0 \text{ for all } t \in [0, T] \text{ a.s.} \tag{37}
\]

By Proposition 2.3(b), as noted earlier, \(\tilde{\pi}^n \to \tilde{\pi}\) in \(\tilde{\mathcal{P}}\). By Proposition 2.4(c), \(\tilde{\pi}_s\) is supported on \(\arg\ min\ \{U(\tilde{X}_s, \cdot)\}\) which is a finite set for each \(s \geq 0\). Consequently, using the topology on \(\tilde{\mathcal{P}}\) it is standard to see that for \(b \in C_b(\mathbb{R}^m)\)

\[
\left\| \int h(y)\tilde{\pi}^n_s(dy) - \int h(y)\tilde{\pi}_s(dy) \right\| \to 0 \text{ almost every } s \in [0, T].
\]

As \(b(\tilde{X}_s, \cdot)\) is a bounded (though random) continuous function, we then have

\[
\left\| \int b(\tilde{X}_s, y)\tilde{\pi}^n_s(dy) - \int b(\tilde{X}_s, y)\tilde{\pi}_s(dy) \right\| \to 0 \text{ almost every } s \in [0, T].
\]

By (34) and (35), this is the same as \(\tau_{n,s} \to 0\) for almost every \(s \in [0, T]\), a.s. An application of the dominated convergence theorem then yields that

\[
\int_0^t \tau_{n,s} ds \to 0 \text{ for all } t \in [0, T] \text{ a.s.} \tag{38}
\]

So using (36) and by (32), (37), and (38) we have

\[
\tilde{X}_t = x_0 + \int_0^t b(\tilde{X}_s, y) \nu_s^0(\tilde{X}_s)(dy) ds
\]

for all \(t \in [0, T]\) a.s. This completes the proof. \(\square\)

The method of proof for Theorem 1.3 is adapted from Theorem 4 in [BOQ09] with some key differences. We present it next.

Proof of Theorem 1.6. From Proposition 2.3(b), we have

\[
\bar{E} \left[ \sup_{t \in [0, T]} \|\tilde{X}^n_t - \tilde{X}_t\|^2 \right] \to 0 \text{ as } n \to \infty. \tag{39}
\]

Since

\[
\|\tilde{\xi}^n_t - \tilde{X}_t\|^2 = \|\tilde{X}^n_t - \epsilon^n_n \bar{\eta}_t - \tilde{X}_t\|^2 \leq 2\|\tilde{X}^n_t - \tilde{X}_t\|^2 + 2\epsilon^n_n \|\bar{\eta}^n_t\|^2,
\]

this together with the facts \(\bar{E}[\sup_{t \in [0, T]} \|\bar{\eta}^n_t\|^2] < \infty\), \(\epsilon^n_n \to 0\), and (39) yields

\[
\bar{E} \left[ \sup_{t \in [0, T]} \|\tilde{\xi}^n_t - \tilde{X}_t\|^2 \right] \to 0 \text{ as } n \to \infty. \tag{40}
\]

Observe now that since

\[
\tilde{\xi}^n_t = \tilde{X}^n_t - \epsilon^n_n \bar{\eta}_t = x_0 + \int_0^t b(\tilde{X}^n_s, y)\tilde{\pi}^n_s(dy) ds,
\]

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we can write
\[
\frac{d}{dt}\tilde{\xi}^n = \int b(\tilde{X}^n_t, y)\pi^n_t(dy),
\]
and in view of the boundedness of \(b\) in Assumption (B1), there is a \(0 < c_1 < \infty\) such that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \tilde{\xi}^n_t \right\|^2 \right] \leq c_1 \quad \text{and} \quad \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \frac{d}{dt}\tilde{\xi}^n_t \right\|^2 \right] \leq c_1, \quad \forall n \geq 1. \tag{41}
\]
In view of (40) and (41), there is a subsequence that converges weakly in the space
\[W^{1,2} := \left\{ Z \in L^2([0,T] \times \tilde{\Omega}; \mathbb{R}^d), Z' \in L^2([0,T] \times \tilde{\Omega}; \mathbb{R}^d) \right\},\]
that is, there is some process \(U\) such that
\[
\tilde{\xi}^n \to \tilde{X} \quad \text{in } L^2, \quad \mathbb{E} \left[ \int_0^T \frac{d}{dt}\tilde{\xi}^n_t \phi(t)dt \right] \to \mathbb{E} \left[ \int_0^T U_t \phi(t)dt \right],
\]
for any (nonrandom) \(\phi \in W^{1,2}\). We next argue that \(U = \frac{d}{dt}\tilde{X}\). Integrating the left-hand side above by parts, we get
\[
\mathbb{E} \left[ \tilde{\xi}^n_T \phi(T) - x_0 \phi(0) \right] - \mathbb{E} \left[ \int_0^T \tilde{\xi}^n_t \frac{d}{dt}\phi(t)dt \right] \to \mathbb{E} \left[ \tilde{X}_T \phi(T) - x_0 \phi(0) \right] - \mathbb{E} \left[ \int_0^T \tilde{X}_t \frac{d}{dt}\phi(t)dt \right],
whence
\[
\mathbb{E} \left[ \int_0^T \frac{d}{dt}\tilde{X}_t \phi(t)dt \right] = \mathbb{E} \left[ \int_0^T U_t \phi(t)dt \right].
\]
Since \(\phi \in W^{1,2}\) was arbitrary, with the only restriction that it is nonrandom, we have established that \(U_t = \frac{d}{dt}\tilde{X}_t\) for almost every \(t \in [0,T]\), a.s. Thus \(\tilde{\xi}^n \to \tilde{X}\) weakly in \(W^{1,2}\).

Recall definition of \(\delta_{n,t}\) and \(\tau_{n,t}\) from (53) and (54), respectively. Let \(\delta_n := \sup_{t \in [0,T]} \|\tilde{X}^n_t - \tilde{X}_t\|\). As discussed earlier, by the Lipschitz property of \(b\) in (B1), we then have for all \(t \in [0,T]\),
\[
\delta_{n,t} \leq TK\delta_n
\]
Using (53), (54), (55) and the triangle inequality we see that the derivative \(\frac{d}{dt}\tilde{\xi}^n\) satisfies, for almost every \(t \in [0,T]\),
\[
\frac{d}{dt}\tilde{\xi}^n = \int b(\tilde{X}^n_t, y)\pi^n_t(dy)
\]
\[\in \int b(\tilde{X}_t, y)\nu^0_0\tilde{X}_t(dy) + (\tau_{n,t} + \delta_{n,t})\tilde{B}_1 \]
\[\leq \int b(\tilde{X}_t, y)\nu^0_0\tilde{X}_t(dy) + (\tau_{n,t} + K\delta_n)\tilde{B}_1 \]
\[= \int b(\tilde{X}_t, y)\nu^0_0\tilde{X}_t(dy) + \gamma_{n,t}\tilde{B}_1 \quad \text{(where } \gamma_{n,t} = \tau_{n,t} + K\delta_n)\). \tag{42}
\]
Let \(h(\cdot)\) be as defined in (10). By assumption (U4) and Proposition (23)(d), whenever \(\tilde{X}_t \in F\), we have that
\[
\int b(\tilde{X}_t, y)\nu^0_0\tilde{X}_t(dy) = \sum_{j=1}^{L(\tilde{X}_t)} b(\tilde{X}_t, y_{j}(\tilde{X}_t)) \left\{ \frac{\det \left[ D^2 b(\tilde{X}_t, y_{j}(\tilde{X}_t)) \right]}{\sum_{j=1}^{L(\tilde{X}_t)} \left[ \det \left[ D^2 b(\tilde{X}_t, y_{j}(\tilde{X}_t)) \right] \right]} \right\} = h(\tilde{X}_t).
\]
Now consider the enlargement \(h_E\) of \(h\) defined in (17) as the smallest upper semi-continuous set-valued map with closed convex values such that \(h(x) \in h_E(x)\) for almost all \(x \in R^d\).
Define \( f, g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) by
\[
f(t, x) = h(x) \quad \text{and} \quad g(t, x) = \begin{cases} f(h(\bar{X}_t, y)) & \text{if } \bar{X}_t = x \text{ and } x \in F \\ h(x) & \text{otherwise} \end{cases}
\]
for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\). We know that \( g = f \) a.e. on \( \mathbb{R}_+ \times \mathbb{R}^m \) and consequently by [BOQ09, Proposition 2(ii)] we have \( g_E = f_E \). As \( f \) does not depend on \( t \) it is easy to see that the enlargement \( f_E(t, x) = h_E(x) \) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \). Therefore, from (12), we have
\[
\frac{d}{dt} \hat{\xi}_n^i \in h_E(\bar{X}_t) + \gamma_{n,t}B_1, \text{ for almost every } t \in [0, T].
\]

From the proof of Theorem 1.3 we have \( \tau_{n,t} \to 0 \) for almost every \( t \in [0, T] \), a.s. By the a.s. convergence of \( \bar{X}^i \) to \( \bar{X} \) in \( C([0, T]; \mathbb{R}^d) \), we also have \( \delta_n \to 0 \). Thus \( \gamma_{n,t} = \tau_{n,t} + K\delta_n \to 0 \) for almost every \( t \in [0, T] \), a.s.

Take \( \tilde{\gamma}_{m,t} = \sup_{m \geq n} \gamma_{m,t} \). We then have
\[
\frac{d}{dt} \hat{\xi}_n^i \in h_E(\bar{X}_t) + \gamma_{n,t}B_1 \quad \text{for almost every } t \in [0, T], \forall n \geq 1,
\]
and \( \tilde{\gamma}_{n,t} \to 0 \) for almost every \( t \in [0, T] \), a.s. Since \( \frac{d}{dt} \hat{\xi}_n^i \to \frac{d}{dt} \bar{X} \) weakly in \( L^2([0, T] \times \bar{\Omega}; \mathbb{R}^d) \), a.s., and on account of [BOQ09], by Mazur’s lemma ([RRO06, Lemma 10.19]), there exists \( \{Z_n\}_{n \geq 1} \) such that
\[
Z_n \to \frac{d}{dt} \bar{X} \text{ in } L^2([0, T] \times \bar{\Omega}; \mathbb{R}^d) \text{ as } n \to \infty,
\]
and
\[
Z_{n,t} \in \text{co} \left( \bigcup_{m \geq n} \left\{ h_E(\bar{X}_t) + \tilde{\gamma}_{m,t}B_1 \right\} \right) \quad \text{for almost every } t \in [0, T].
\]

By passing to a further subsequence, we have \( Z_{n,t} \to \frac{d}{dt} \bar{X}_t \) for almost every \( t \in [0, T] \), a.s. Thus almost surely and for almost every \( t \in [0, T] \), we have:
\[
\frac{d}{dt} \bar{X}_t \in \bigcap_{n \geq 1} \text{co} \left( \bigcup_{m \geq n} \left\{ h_E(\bar{X}_t) + \gamma_{m,t}B_1 \right\} \right) = \bigcap_{n \geq 1} \left\{ h_E(\bar{X}_t) + \tilde{\gamma}_{n,t}B_1 \right\} \quad \text{(because } \tilde{\gamma}_{n,t} \text{ neighborhood contains all others for } m \geq n\text{)}
\]
\[
= \bigcap_{n \geq 1} \left\{ h_E(\bar{X}_t) + \tilde{\gamma}_{n,t}B_1 \right\} = h_E(\bar{X}_t) \quad \text{(because } h_E(\bar{X}_t) \text{ is already convex and so is its } \tilde{\gamma}_{n,t} \text{ neighborhood)}
\]
\[
= h_E(\bar{X}_t) \quad \text{(because } h_E(\bar{X}_t) \text{ is also closed).}
\]

By suitably modifying \( \frac{d}{dt} \bar{X} \) on a Lebesgue null set, we establish that \( \frac{d}{dt} \bar{X}_t \in h_E(\bar{X}_t) \) for all \( t \in [0, T] \). Finally, since \( X \) and \( \bar{X} \) have the same law, we conclude that, almost surely, \( \frac{d}{dt} X_t \in h_E(X_t) \) for all \( t \in [0, T] \).

We now argue that any limit point in law is almost surely a Filippov solution to (13). Let \( \delta_n \to 0 \). Along a subsequence, \( X^{\delta_n} \) converges weakly to a limit point \( X \) as \( \delta_n \to 0 \). There is a further subsequence along which Proposition 2.3 and Proposition 2.4 hold. Imitating the steps of the proof of the first part above along this subsequence, we see that the limit point \( X \) is almost surely a Filippov solution to (13). \( \square \)
3 Proof of Proposition 2.1

A spectral gap estimate is shown in [HS90, Theorem 3.1]. To convert the estimate in our setting and to the required $L_\infty$ norm as stated in Proposition 2.1(b) will require ultracontractivity bounds. For this we will need one additional notation. For $1 \leq p, q \leq \infty$, write $\| \cdot \|_{(p,q)}$ for the $L_p \to L_q$ operator norm, with $L_p$ being the space of functions whose $p$-th power is integrable. Our first lemma establishes ultracontractivity.

Lemma 3.1 Let $x \in \mathbb{R}^d$, $\varepsilon > 0$ and $\eta > 1$ be as in (10). For $0 < t_0 < 1$, there exists $c_1 > 0$ such that

$$\|T_{t_0}^0 \cdot, x\|_{(1, \infty)} < \exp \left( \frac{c_1 t_0^{\frac{n}{\varepsilon}}}{s(\varepsilon)^p} \right).$$  \hbox{(44)}

Proof: Fix $0 < t_0 < 1$. The result follows directly from (10) of Assumption (U3) with $a = s(\varepsilon)^2$, [BGL14, Proposition 7.3.1], and [BGL14, Corollary 7.1.4] with $W(\cdot) = 2U(x, \cdot)/s(\varepsilon)^2$ and the growth function

$$\Phi(r) = \frac{C}{s(\varepsilon)^2}(1 + r^{\frac{n}{\varepsilon-1}}), \text{ with } r \in (0, \infty) \text{ and } C \equiv C(M, m, \eta).$$

In particular, see the discussion in [BGL14, p. 363] explaining the choice of the above growth function $\Phi$ in [BGL14, eqn. (7.3.1)], which satisfies an entropy-energy inequality ([BGL14, Defn. 7.1.1]) by virtue of (10) with $a = s(\varepsilon)^2$ and [BGL14, Proposition 7.3.1]. Then [BGL14, Corollary 7.1.4] yields [14].

Proof of Proposition 2.1 From (8), (9) and (10) with $a = 1$, we may conclude

$$C(x, \varepsilon) := \int_{\mathbb{R}^d} \exp \left\{ -2 \frac{U(x, y)}{s(\varepsilon)^2} \right\} dy < \infty, \text{ (45)}$$

$$\| \nabla_y U(x, y) \|_\infty \to \infty \text{ as } \| y \|_\infty \to \infty, \text{ (46)}$$

$$\| \nabla_y U(x, \cdot) \|^2 - \Delta_y U(x, \cdot) \text{ is bounded below,} \text{ (47)}$$

$$U(x, y) \to \infty \text{ as } \| y \|_\infty \to \infty, \text{ uniformly in } x. \text{ (48)}$$

So, part (a) follows from the results of Appendix A, along with the fact that $L^{c,x}$ in (23) is a self-adjoint operator on $L^2(\nu^{c,x})$ and $\nu^{c,x}(\mathbb{R}^n) = 1$.

(b) Using a standard result on spectral gap (see discussion on [HS90, p. 273]), we have for all $s \geq t$

$$\|T_{s-t}^0 \cdot, x - \nu^{c,x}\|_{(2,2)} \leq e^{-(s-t)\lambda_2^c(x)}, \text{ (49)}$$

where $\lambda_2^c(x)$ is the second largest eigenvalue of $L^{c,x}$. Let $t_0 = \frac{1}{2}, s \geq t + \frac{1}{2}$. Using Lemma 3.1

$$\|T_{s-t}^{t_0} \cdot, x - \nu^{c,x}\|_{(\infty, \infty)} = \|T_{s-t}^{0} \cdot, x - \nu^{c,x}\|_{(\infty, \infty)} \leq \|T_{s-t}^{0} \cdot, x - \nu^{c,x}\|_{(2, \infty)} \leq \|T_{s-t}^{0} \cdot, x - \nu^{c,x}\|_{(2,2)} \leq \|T_{s-t}^{0} \cdot, x - \nu^{c,x}\|_{(1, \infty)} \|T_{s-t}^{0} \cdot, x - \nu^{c,x}\|_{(2,2)} \leq e^{-(s-t)\lambda_2^c(x)} e^{-s(t-\frac{1}{2})\lambda_2^c(x)}.$$

So for all $f \in C^2_b(\mathbb{R}^n)$, there exists $c_2 > 0$ such that

$$\|T_{s-t}^{t_0} \cdot, x f - \nu^{c,x}(f)\|_\infty \leq \|f\|_\infty e^{-(s-t)\lambda_2^c(x)} e^{-s(t-\frac{1}{2})\lambda_2^c(x)}.$$

\hbox{(50)}
As \((\ref{4}), (\ref{5}), (\ref{6}), (\ref{7})\) hold, from \([\text{HS90}]\) Theorem 3.1, we obtain that for any \(\delta > 0\),
\[
\lambda^2(x) \geq \exp \left\{ \frac{\epsilon (1) - \epsilon (2) + \delta_1}{s(\epsilon)^2} \right\}
\]
for all sufficiently small \(\epsilon\). Using \((\ref{13})\), from assumption \((\text{U3})\), and \((\ref{18})\), we have for some \(c_3 > 0\)
\[
\|T^{\epsilon,x}_s f - \nu^{\epsilon,x}(f)\|_\infty \leq \|f\|_\infty e^{\epsilon^{1/2} - (s-t-\frac{1}{2})} \exp(-\frac{\Delta + \varepsilon}{4\epsilon^2}) \leq \|f\|_\infty e^{\epsilon^{1/2} - (s-t) \exp(-\frac{\Delta + \varepsilon}{4\epsilon^2})}. \quad (51)
\]

4 Proof of Proposition 2.3

It is easy to obtain fourth moment bounds for \(X^\varepsilon\) from the assumption \((\text{B1})\), this readily implies tightness, and consequently part (a). Part (b) is a standard application of Skorohod’s Theorem. As indicated earlier the key nuance in the Proposition is the topology on \(\bar{\mathcal{P}}\). One of the facts we shall crucially use is that \(\bar{\mathcal{P}}\) is compact and metrizable in this topology. This and other applications to control theoretic setting are discussed in detail in \([\text{ABG12}]\).

Proof of Proposition 2.3
(a) Let \(0 < \varepsilon < 1\) and \(0 \leq t < T\). As \(X^\varepsilon_t\) solves \((\ref{27})\), we have
\[
\|X^\varepsilon_t - X^\varepsilon_s\| = \left\| \int_s^t \int_{\mathbb{R}^d} b(X^\varepsilon_y, y)\pi^\varepsilon_y(dy)dr + c^\varepsilon(\eta^\varepsilon_t - \eta^\varepsilon_s) \right\| \\
\leq \|b\|_\infty (t-s) + \varepsilon^\alpha \|\eta^\varepsilon_t - \eta^\varepsilon_s\|.
\]
We can then conclude that
\[
\mathbb{E} \left[ \|X^\varepsilon_t - X^\varepsilon_s\|^4 \right] \leq c_1 |t-s|^2
\]
for \(0 \leq s < t < T\). By \([\text{Bil68}]\) (12.51) and Theorem 12.3 we have that the laws of \(\{X^\varepsilon : \varepsilon \in (0,1]\}\) are tight in \(\mathcal{P}(C([0,T];\mathbb{R}^d))\). Further, we note that \(\bar{\mathcal{P}}\) is compact and metrizable \([\text{ABG12}]\) Section 2.3, Theorem 2.3.1. This implies the tightness of the laws of \((X^\varepsilon, \pi^\varepsilon)\) in \(\mathcal{P}(C([0,T];\mathbb{R}^d) \times \bar{\mathcal{P}})\). Hence there exists a sequence \(\varepsilon_n \downarrow 0\) such that \((X^{\varepsilon_n}, \pi^{\varepsilon_n})\) converges weakly to \((X, \pi)\) as \(n \to \infty\).

(b) Let \(\{\varepsilon_n\}_{n \geq 1}\) be the sequence mentioned in part (a). Using Skorohod’s theorem \([\text{Bor95}]\) Theorem 2.2.2, p. 23], there exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and processes \((\tilde{X}^{\varepsilon_n}, \tilde{\pi}^{\varepsilon_n}, \tilde{\eta}^{\varepsilon_n}, \tilde{X}, \tilde{\pi}, \tilde{\eta})\) such that
\[
\text{Law of } (\tilde{X}^{\varepsilon_n}, \tilde{\pi}^{\varepsilon_n}, \tilde{\eta}^{\varepsilon_n}) = \text{Law of } (X^{\varepsilon_n}, \pi^{\varepsilon_n}, \eta^{\varepsilon_n}),
\]
\[
\text{Law of } (\tilde{X}, \tilde{\pi}, \tilde{\eta}) = \text{Law of } (X, \pi, \eta),
\]
and \((\tilde{X}^{\varepsilon_n}, \tilde{\pi}^{\varepsilon_n}, \tilde{\eta}^{\varepsilon_n}) \to (\tilde{X}, \tilde{\pi}, \tilde{\eta})\) almost surely. Further, using Fatou’s lemma followed by Doob’s inequality we have
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} \|\tilde{X}^{\varepsilon_n}_s - \tilde{X}_s\|^2 \right] \leq \liminf_{\delta \to 0} \mathbb{E} \left[ \sup_{0 \leq s \leq T} \|\tilde{X}^{\varepsilon_n}_s - \tilde{X}^\delta_s\|^4 \right] \\
\leq 2^4 \left( \|b\|_\infty 4^4 + (c^\alpha + \delta^\alpha)^4 \mathbb{E} \left[ \sup_{0 \leq s \leq T} \|B_s\|^4 \right] \right) \\
\leq c_1 \left( \|b\|_\infty + \mathbb{E}[\|B_T\|^4] \right) < \infty.
\]
This implies that the family
\[
\left\{ \sup_{0 \leq s \leq T} \|\tilde{X}^{\varepsilon_n}_s - \tilde{X}_s\|^2 : n \geq 1 \right\}
\]
is uniformly integrable. This implies \((\ref{23})\).
5 Proof of Proposition 2.4

The proof of this proposition consists of many steps. Part (a) uses the topology on $\mathcal{F}$ and fundamental theorem of calculus to choose an appropriate subsequence. Part (b) and Part (c) require some technical preparation which we describe in detail first, before proving Proposition 2.4.

For Part (b), we prove a second moment estimate in Proposition 5.2. Using this second moment estimate we will be able to identify the required rate of decay of $s(\varepsilon) \to 0$ as $\varepsilon \to 0$. This will ensure that the second moment goes to zero and consequently a further subsequence goes to zero almost surely. Proof of Proposition 5.2 will require a gradient estimate for the semigroup of $Z^{\lambda, t, x}$ which satisfies (22). We present that first.

**Lemma 5.1** Recall $\Gamma$ from (17). There exists $\varepsilon_0 > 0$, such that for all $f \in C_b^2(\mathbb{R}^m)$, $s \geq t$, $0 < \varepsilon < \varepsilon_0$,

$$\|\nabla T_{s-t}^{\lambda, x}(f)\|_{\infty} \leq \frac{b}{\sqrt{s-t}}.$$

**Proof:** Using (11) and [PW06, Theorem 3.4] we have that for any $f \in C_b^2(\mathbb{R}^m)$,

$$\|\nabla T_{s-t}^{\lambda, x}(f)\|_{\infty} \leq \frac{1 + 2s(\varepsilon)^2}{2s(\varepsilon)^2 \sqrt{s-t}} \exp \left( \frac{\Gamma}{2s(\varepsilon)^2} \right) \|f\|_{\infty}.$$  

We may choose $\varepsilon_0 > 0$ so that $1 + 1/(2s(\varepsilon_0)^2) \leq e^{\Gamma(2s(\varepsilon_0)^2)}$, and so (52) holds. For any $f \in C_b^2(\mathbb{R}^m)$ the result follows by considering positive and negative parts of $f$. \qed

We now present the key second moment estimate.

**Proposition 5.2** Let $0 < \Delta < \infty$ be as in (U3) and let $\delta > 0$ be fixed. There exist $c_1, c_2 > 0$ such that for all $f \in C_b^2(\mathbb{R}^m)$, for all sufficiently small $\varepsilon > 0$, $t \geq 0$, $s > t + 1$, and $\kappa > 0$,

$$\mathbb{E} \left[ \nu_{\varepsilon}^{\Delta, X_t^\varepsilon} \right] \leq c_1 \|f\|_4^2 \left[ e^{\frac{c_2}{\kappa}} - 2(s-t) \exp \left( -\frac{4(1+\kappa) - \varepsilon}{\kappa(1+\kappa)} \right) + (s-t) \kappa \left( \frac{\varepsilon^2}{\kappa^2} + \frac{\varepsilon^{-2\alpha+2}}{\kappa} \right) \right].$$  

**Remark 5.3** The first term inside the bracket in (53) arises from the spectral gap estimate obtained earlier and it specifies the rate at which the fast process approaches its stationary measure. The second term inside the bracket in (53) is from the gradient estimate obtained in Lemma 5.1. So for both these terms to go to zero, we need to impose a rate of decay to 0 on $s(\varepsilon)$ and use Assumption (S1). The third term contains the scaling factor provided by the nonlinear filtering equation and here we require $0 < \alpha < 1$ for this term to go to 0.

**Proof of Proposition 5.2** Let $f \in C_b^2(\mathbb{R}^d)$, $t \geq 0, s > t + 1, \varepsilon > 0, x \in \mathbb{R}^d$ be given. For $0 < \kappa < 1$, define for notational convenience

$$\nu_{t}^{\varepsilon, \kappa} (f) = \frac{1}{\kappa} \int_t^{t+\kappa} \pi_{\varepsilon}^u (f) \, du.$$

By the fundamental theorem of calculus,

$$T_{s-t}^{\varepsilon, x} f = f + \int_t^s \mathcal{L}^{\varepsilon, x} (T_{u-t}^{\varepsilon, x} f) \, du.$$  

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We then readily note that
\[\nu^e_x(f) - \bar{\nu}^e_t(f) = \nu^e_x \left( T^e_{s,t,x} f - \int_t^s \mathcal{L}^e_x (T^e_{u,t,x} f) du \right) - \bar{\nu}^e_t \left( T^e_{s,t,x} f - \int_t^s \mathcal{L}^e_x (T^e_{u,t,x} f) du \right).\]

As \( f \in C^2_b(\mathbb{R}^d) \) and \( \nu^e_x \) is an invariant measure, we have
\[\nu^e_x \left( \int_t^s \mathcal{L}^e_x (T^e_{u,t,x} f) du \right) = \int_t^s \nu^e_x (\mathcal{L}^e_x (T^e_{u,t,x} f)) du = 0. \quad (55)\]

Using (55), we may rewrite
\[\nu^e_x(f) - \bar{\nu}^e_t(f) = \nu^e_x \left( T^e_{s,t,x} f - \bar{\nu}^e_t \left( T^e_{s,t,x} f - \int_t^s \nu^e_x (\mathcal{L}^e_x (T^e_{u,t,x} f)) du \right) \right)
\]\
\[= \nu^e_x \left( T^e_{s,t,x} f - \nu^e_x (f) \right) - \bar{\nu}^e_t \left( T^e_{s,t,x} f - \nu^e_x (f) \right) + \int_t^s \bar{\nu}^e_t (\mathcal{L}^e_x (T^e_{u,t,x} f)) du\]
\[= \nu^e_x \left( T^e_{s,t,x} f - \nu^e_x (f) \right) - \bar{\nu}^e_t \left( T^e_{s,t,x} f - \nu^e_x (f) \right) + \int_t^s \left( \frac{1}{\kappa} \int_t^{t+\kappa} \pi^e_x (\mathcal{L}^e_x (T^e_{u,t,x} f)) du \right)
\[+ \int_t^s \left( \frac{1}{\kappa} \int_t^{t+\kappa} \pi^e_x \left( \mathcal{L}^e_x (T^e_{u,t,x} f) - \mathcal{L}^e_x (X^e_r(t,t+x)) \right) dr \right) du\]
\[+ \int_t^s \left( \frac{1}{\kappa} \int_t^{t+\kappa} \pi^e_x \left( \mathcal{L}^e_x (T^e_{u,t,x} f) \right) dr \right) du. \quad (56)\]

Now, the measure valued process \( \pi^e_x \) is the unique solution to the (Fujisaki-Kallianpur-Kunita) nonlinear filtering equation
\[\pi^e_t = f(y_0) + \frac{1}{\varepsilon} \int_0^t \pi^e_x (\mathcal{L}^e_x (X^e_r(t,t+x)) dr + \varepsilon^{-1} \int_0^t \pi^e_x (f b(X^e_r, \cdot)) - \pi^e_x (f b(X^e_r, \cdot), d\tilde{B}_r), \quad (57)\]

where \( \tilde{B} \) is a standard Brownian motion; see Proposition [3.1] in Appendix 4. Using the definition of \( \mathcal{L}^e_x \) from [23] and the FKK equation [57] in [50], we have,
\[\nu^e_x(f) - \bar{\nu}^e_t(f) = \nu^e_x \left( T^e_{s,t,x} f - \nu^e_x (f) \right) - \bar{\nu}^e_t \left( T^e_{s,t,x} f - \nu^e_x (f) \right) + \int_t^s \left( \frac{1}{\kappa} \int_t^{t+\kappa} \pi^e_x (\mathcal{L}^e_x (T^e_{u,t,x} f)) du \right)
\[+ \int_t^s \left( \frac{1}{\kappa} \int_t^{t+\kappa} m^e_x (T^e_{u,t,x} f) - m^e_x (T^e_{u,t,x} f) \right) dr \right) du\]
\[+ \int_t^s \left( \frac{1}{\kappa} \int_t^{t+\kappa} \pi^e_x (f b(X^e_r, \cdot)) - \pi^e_x (f b(X^e_r, \cdot), d\tilde{B}_r) \right) du. \]

We shall now replace \( x \) in above by \( X^e_r \). To do this one needs to be careful only in the last term. Here we observe that as \( t > 0 \) is fixed, \( t \leq r \leq t + \kappa \), using definition of \( \pi^e_x \) and the stochastic integral, we

\footnote{Proposition [3.1] presented in Appendix, is a more general nonlinear filtering equation and could be of independent interest.}
may replace $x$ by $X^\varepsilon_t$. For the other terms, the substitution is trivial. So we have,

\[
\nu^\varepsilon,X^\varepsilon_t(f) - \hat{\nu}^\varepsilon(X^\varepsilon_t(f)) = \nu^\varepsilon,X^\varepsilon_t \left( T^\varepsilon_{s:t} f - \nu^\varepsilon,X^\varepsilon_t(f) \right) - \hat{\nu}^\varepsilon(X^\varepsilon_t(f)) T^\varepsilon_{s:t} \nu^\varepsilon,X^\varepsilon_t(f) \]

\[
+ \int_t^s \left( \frac{1}{\kappa} \int_t^{t+\kappa} \pi^\varepsilon_t((\nabla_y U(X^\varepsilon_t, - \nabla_y U(X^\varepsilon_t - \nabla_y T^\varepsilon_{s:t} X^\varepsilon_t(f))) dr \right) du
\]

\[
+ \frac{\alpha}{\kappa} \int_t^s \pi^\varepsilon_t(T(\varepsilon_{s:t} X^\varepsilon_t(f)) - \pi^\varepsilon_t(T(\varepsilon_{s:t} X^\varepsilon_t(f)))) du
\]

\[
- \int_t^s \left( \frac{e^{-\alpha + 1}}{\kappa} \int_t^{t+\kappa} \pi^\varepsilon_t(X^\varepsilon_t f b(X^\varepsilon_t, - \pi^\varepsilon_t(T(\varepsilon_{s:t} X^\varepsilon_t(f)) \pi^\varepsilon_t(b(X^\varepsilon_t, - \pi^\varepsilon_t(T(\varepsilon_{s:t} X^\varepsilon_t(f)))) du
\]

\[
= I + II + III - IV. \quad (58)
\]

So,

\[
E \left[ \nu^\varepsilon,X^\varepsilon_t(f) - \hat{\nu}^\varepsilon(X^\varepsilon_t(f)) \right]^2 \leq 16 \left[ I^2 + II^2 + III^2 + IV^2 \right]. \quad (59)
\]

For the first term in (58), i.e., I, by Proposition 2.1 we have that for sufficiently small $\varepsilon > 0$

\[
E[I^2] \leq E \left[ \left| \nu^\varepsilon,X^\varepsilon_t \left( T^\varepsilon_{s:t} X^\varepsilon_t(f) - \nu^\varepsilon,X^\varepsilon_t(f) \right) \right|^2 \right]
\]

\[
\leq 4 \left( E \left| \nu^\varepsilon,X^\varepsilon_t \left( T^\varepsilon_{s:t} X^\varepsilon_t(f) - \nu^\varepsilon,X^\varepsilon_t(f) \right) \right|^2 \right) + E \left[ \left| \hat{\nu}^\varepsilon(X^\varepsilon_t(f)) \right|^2 \right]
\]

\[
\leq 8 \sup_{x \in \mathbb{R}^d} \| T^\varepsilon_{s:t} X^\varepsilon_t(f) - \nu^\varepsilon(X^\varepsilon_t(f)) \|_\infty^2
\]

\[
\leq c_{32} \frac{\varepsilon}{\kappa} \exp \left( \frac{\alpha + \alpha}{\kappa} \right) \| f \|_\infty^2. \quad (60)
\]

For the second term in (58), i.e., II, using (U1) and (62), we have that

\[
E[II^2] \leq E \left[ \left\| \int_t^s \left( \frac{1}{\kappa} \int_t^{t+\kappa} \pi^\varepsilon_t((\nabla_y U(X^\varepsilon_t, - \nabla_y U(X^\varepsilon_t, - \nabla_y T^\varepsilon_{s:t} X^\varepsilon_t(f))) dr \right) du \right\|^2 \right]
\]

\[
\leq c_4 K_2(s - t)^2 \sup_{u \in (s,t)} \| \nabla T^\varepsilon_{s:t} X^\varepsilon_t(f) \|_\infty^2 E \left[ \frac{1}{\kappa} \int_t^{t+\kappa} \| X^\varepsilon_t - X^\varepsilon_t \|^2 dr \right]
\]

\[
\leq c_5 \varepsilon (s - t) \| f \|_\infty^2 \frac{1}{\kappa} \int_t^{t+\kappa} \left( \kappa^2 \| b \|_\infty^2 + \varepsilon^2 \| B_r - B_r \|_\infty^2 \right) dr
\]

\[
\leq c_6 \varepsilon (s - t) \| f \|_\infty^2 \frac{1}{\kappa} \int_t^{t+\kappa} \left( \kappa^2 \| b \|_\infty^2 + \varepsilon^2 \| B_r - B_r \|_\infty^2 \right) dr
\]

\[
\leq c_7 \varepsilon (s - t)^2 \| f \|_\infty^2. \quad (61)
\]

For the third term in (58), i.e., III, as $\pi^\varepsilon$ is a probability measure, we have by triangle inequality and the semigroup property,

\[
E[III^2] \leq E \left[ \frac{\varepsilon^2}{\kappa} \left\| \int_t^s \pi^\varepsilon_t(T^\varepsilon_{s:t} X^\varepsilon_t(f) - \pi^\varepsilon_t(T^\varepsilon_{s:t} X^\varepsilon_t(f))) du \right\|^2 \right]
\]

\[
\leq c_8 \frac{\varepsilon^2}{\kappa^2} (s - t)^2 \| f \|_\infty^2. \quad (62)
\]

For the fourth term in (58), i.e., IV, using Jensen’s inequality and a standard second moment estimate, we have

\[
E[IV^2] \leq E \left[ \left( \int_t^s \frac{\varepsilon^{\alpha + 1}}{\kappa} \int_t^{t+\kappa} \pi^\varepsilon_t(T^\varepsilon_{s:t} X^\varepsilon_t(f) - \pi^\varepsilon_t(T^\varepsilon_{s:t} X^\varepsilon_t(f))) du \right)^2 \right]
\]

\[
\leq c_9 (s - t)^2 \frac{\varepsilon^{2\alpha + 2}}{\kappa} \| f \|^2_\infty \| b \|^2_\infty = c_9 (s - t)^2 \frac{\varepsilon^{2\alpha + 2}}{\kappa} \| f \|^2_\infty. \quad (63)
\]

So from (61), (62), (63), (64), (65) we have the result.

We are now ready to prove the main result of this section.
Proof of Proposition 2.4: Let $f \in C^2_b(\mathbb{R}^m)$. Let $\varepsilon_n > 0, \tilde{X}, \tilde{\pi}$ be as constructed in Proposition 2.3.

Recall that $\gamma = \min\{1 - \alpha, \frac{1}{2}\}$.

(a) By the topology of $\tilde{P}$, we have for any $\eta > 0$

$$\lim_{n \to \infty} \int_t^{t+\eta} \tilde{\pi}^\varepsilon_r(f)dr = \int_t^{t+\eta} \tilde{\pi}_r(f)dr, \text{ a.s.}$$

By [AS08 Proposition 7.5.7], we have

$$\lim_{\eta \to 0} \lim_{n \to \infty} \frac{1}{\eta} \int_t^{t+\eta} \tilde{\pi}^\varepsilon_r(f)dr = \tilde{\pi}_t(f), \text{ almost every } t \in [0,T], \text{ a.s.}$$

Hence for each $k \in \mathbb{N}$, there exists a $\eta_k > 0$ and $\varepsilon(k, \eta_k) > 0$ such that

$$\forall \varepsilon_n \leq \varepsilon(k, \eta_k), \text{ almost every } t \in [0,T], \text{ a.s. and furthermore, } \eta_k \to 0 \text{ as } k \to \infty.$$  \hspace{1cm} (64)

For each $k \geq 1$, choose $n_k$ sufficiently large so that $\varepsilon_{n_k} \leq \varepsilon(k, \eta_k)$ and $\varepsilon_{n_k} \leq \eta_k$. Then by construction we have the following:

$$\forall k \geq 1, \text{ almost every } t \in [0,T], \text{ a.s. The result follows.}$$  \hspace{1cm} (65)

(b) Proposition 2.3(b) implies that

$$\mathbb{E} \left| \nu^{\varepsilon_n, \tilde{X}_t^n}(f) - \frac{1}{\varepsilon_n} \int_t^{t+\varepsilon_n} \tilde{\pi}^\varepsilon_r(f)dr \right|^2 = \mathbb{E} \left| \nu^{\varepsilon_n, \tilde{X}_t^n}(f) - \frac{1}{\varepsilon_n} \int_t^{t+\varepsilon_n} \tilde{\pi}_r^n(f)dr \right|^2.$$  \hspace{1cm} (66)

Using Proposition 5.2 with $\kappa := \varepsilon_n$ with $\gamma = \min\{1 - \alpha, \frac{1}{2}\}$ and $\delta$ to be chosen soon, we have for sufficiently large $n$

$$\mathbb{E} \left| \nu^{\varepsilon_n, \tilde{X}_t^n}(f) - \frac{1}{\varepsilon_n} \int_t^{t+\varepsilon_n} \tilde{\pi}^\varepsilon_r(f)dr \right|^2 \leq c_1 \|f\|_\infty^2 \left[ e^{c_5 \ln(1 + \varepsilon_n) - 2s - s - \frac{s}{2} + (1+\varepsilon_n)\frac{\Delta_b}{1 + \varepsilon_n} + \frac{\Delta_b^2}{2(1 + \varepsilon_n)} + e^{-\theta + \gamma} (\varepsilon_n + \varepsilon_n^2) e^{-\frac{2t}{2 + 2\gamma}} + (s-t)^2 (\varepsilon_n^{2-2\gamma} + \varepsilon_n^{2-2\alpha - \gamma}) \right],$$

for all $t \geq 0$ and $s > t + 1$. Substituting in the above $s = t + \varepsilon_n^{-\theta}$, with $\theta > 0$ to be chosen soon and $\varepsilon_n^2 \geq \frac{C}{\ln(1 + \varepsilon_n)}$, we have

$$\mathbb{E} \left| \nu^{\varepsilon_n, \tilde{X}_t^n}(f) - \frac{1}{\varepsilon_n} \int_t^{t+\varepsilon_n} \tilde{\pi}^\varepsilon_r(f)dr \right|^2 \leq c_2 \|f\|_\infty^2 \left[ e^{c_5 (\ln(\varepsilon_n) - 2s - s - \frac{s}{2} + (1+\varepsilon_n)\frac{\Delta_b}{1 + \varepsilon_n} + \frac{\Delta_b^2}{2(1 + \varepsilon_n)} + e^{-\theta + \gamma} (\varepsilon_n + \varepsilon_n^2) e^{-\frac{2t}{2 + 2\gamma}} + (s-t)^2 (\varepsilon_n^{2-2\gamma} + \varepsilon_n^{2-2\alpha - \gamma}) \right].$$

If we can choose $\delta$ and $\theta$ such that

$$\frac{\Lambda + \delta}{C} < \theta < \min\{2\gamma - \frac{2\Gamma}{C}, \gamma + 2\alpha - \frac{2\Gamma}{C}, 1 - \gamma, 1 - \alpha - \frac{\gamma}{2}\}$$  \hspace{1cm} (67)
then this would imply
\[ \lim_{n \to \infty} \mathbb{E} \left| v^{\varepsilon_n, X^n}(f) - \frac{1}{\varepsilon_n^2} \int_t^{t+\varepsilon_n} \pi^{\varepsilon_n}(f) dr \right|^2 = 0. \]

which would then imply, along with (66) that there exists a further subsequence \( \{\varepsilon_{n_k}\}_{k \geq 0} \) such that (67) holds.

So to complete the proof we need to find small enough \( \theta > 0, \delta > 0 \) so that (67) holds. This will be possible if
\[ \frac{\Lambda}{C} < 2\gamma - 2\Gamma, \quad \frac{\Lambda}{C} < \gamma + 2\alpha - 2\Gamma, \quad \frac{\Lambda}{C} < 1 - \gamma, \text{ and } \frac{\Lambda}{C} < 1 - \alpha - \frac{\gamma}{2}. \]

The above will be true if
\[ C \geq \max \left\{ \frac{\Lambda + 2\Gamma}{2\gamma}, \frac{\Lambda + 2\Gamma}{\gamma + 2\alpha}, \frac{\Lambda}{1 - \gamma}, \text{ and } \frac{\Lambda}{1 - \alpha - \frac{\gamma}{2}} \right\}. \tag{68} \]

As \( \gamma = \min\{1 - \alpha, \frac{1}{2}\} \), we have for \( 0 < \alpha < 1 \) that
\[ 1 - \gamma \geq \frac{1}{2} \quad \text{and} \quad \frac{1 - \alpha}{2} \leq \gamma \leq 1 - \alpha. \]

So, (68) will be true if
\[ C > \max \left\{ \frac{\Lambda + 2\Gamma}{1 - \alpha}, \frac{2\Lambda + 4\Gamma}{1 + 3\alpha}, 2\Lambda, \text{ and } \frac{2\Lambda}{1 - \alpha} \right\}. \tag{69} \]

In (S1) we require \( C > \frac{2(\Lambda + 2\Gamma)}{1 - \alpha} \), so (69) is true.

For part (c) we will need to understand how to characterize weak limit points of \( \nu^{\varepsilon_n, x^n} \) when \( \varepsilon_n \to 0 \) and for deterministic \( x^n \to x \). For part (d), under (U4), we will need to verify that the above sequence of measures obeys Laplace’s principle. We present these results about deterministic sequence of invariant measures in Lemma C.1 of Appendix C. We now use the result in Lemma C.1 to finish the proof.

(c) Let \( \varepsilon_{n_k} \) be a subsequence along which (29) and (30) hold. Using Proposition 2.3(b), there is a null set \( N \) such that \( \tilde{X}^{\varepsilon_{n_k}} \to \tilde{X} \) as \( k \to \infty \), (29) and (30) hold for all \( \omega \in N^c \) for almost every \( t \in [0, T] \).

For a fixed \( \omega \in N^c \), using Lemma C.1 of Appendix C with \( x^{\varepsilon_{n_k}} = \tilde{X}^{\varepsilon_{n_k}}(\omega) \) and \( x = \tilde{X}_t(\omega) \), we conclude that \( \nu^{\varepsilon_{n_k}, \tilde{X}_t^{\varepsilon_{n_k}}(\omega)} \), \( k \geq 1 \) are tight. Let \( \varepsilon_{n_k} \), be a subsequence along which \( \nu^{\varepsilon_{n_k}, \tilde{X}_t^{\varepsilon_{n_k}}(\omega)} \) converges weakly to a measure (again by Lemma C.1) supported on \( \arg\min U(\tilde{X}_t(\omega), \cdot) \), which by (U1) is a finite set. Let us denote this measure by \( \nu_0^{\tilde{X}_t} \). Further, from (29) and (30), \( \nu^{\varepsilon_{n_k}, \tilde{X}_t^{\varepsilon_{n_k}}(\omega)}(f) \) converges to \( \tilde{\pi}_t(f) \) for \( f \in C_b^2(\mathbb{R}^m) \) and for almost every \( t \in [0, T] \). It is now standard to see that \( \tilde{\pi}_t(f) = \nu_0^{\tilde{X}_t}(f) \) for all \( f \in C_b^2(\mathbb{R}^m) \) and for almost every \( t \in [0, T] \), and consequently that \( \tilde{\pi}_t = \nu_0^{\tilde{X}_t} \) for almost every \( t \in [0, T] \).

(d) Follows immediately from (c) and Lemma C.1(b). \( \square \)

**Appendix A  Existence, Uniqueness, and Gradient Estimates**

In this section we show that the coupled system (4) and (5) has a unique strong solution. We begin with a technical lemma.
Lemma A.1 Under (U1), (3) and (4) in assumption (U2) there is $K_4 > 0$ and $R' \geq R$ such that
\[ \langle \nabla_y U(x,y), y \rangle > K_4 \| y \|^2, \quad \| y \| > R'. \tag{70} \]
Also, there exists a nonnegative continuous function $g : (0, \infty) \to (0, \infty)$ such that
\[ \sup_{z,y \in \mathbb{R}^m : \| z - y \| = r} - \frac{1}{r} \langle \nabla_y U(x,z) - \nabla_y U(x,y), z - y \rangle \leq g(r), \quad \text{for all } r > 0, \]
with $\Gamma := \int_0^\infty g(s) ds < \infty. \tag{71}$

Proof: We proceed as follows. Let $B_a$ denote the closed ball of radius $a$ centred at the origin. For any $y$ with $\| y \| > R$, writing $\nabla_y U(x,y) = \nabla_y U(x,0) + \int_0^1 D^2_y U(x,t y) y \, dt$, we have
\[
\langle \nabla_y U(x,y), y \rangle = \langle \nabla_y U(x,0), y \rangle + \int_0^1 \langle y, D^2_y U(x,t y) y \rangle \, dt \geq -M \| y \| - R^/\| y \|^2 dt + \int_0^1 K_3 \| y \|^2 dt = -M \| y \| - MR \| y \| + K_3 \| y \|^2 (1 - R/\| y \|) = K_3 \| y \|^2 - (M + MR + K_3 R) \| y \| \geq K_4 \| y \|^2
\]
for any $K_4 > 0$ and $\| y \| \geq R' \geq R + M(1 + R)/K_3 + K_4/K_3$. In the second inequality above, we have used (3) for the line segment joining 0 to $y$ that lies within $B_R$ and (S) for the remaining line segment. This establishes (70).

Next, for any $y, z \in \mathbb{R}^m$, define $t_0(y,z)$ to be the fractional length of the line segment joining $y$ to $z$ that is within $B_R$. Take $R_1 = R(1 + 2Mm/K_3)$. With $r = \| y - z \|$, we can write
\[
\frac{1}{r} \langle \nabla_y U(x,z) - \nabla_y U(x,y), z - y \rangle = \frac{1}{r} \int_0^1 \langle (z - y), D^2_y U(x,y + t(z - y))(z - y) \rangle \, dt \geq \frac{1}{r} \int_0^1 \langle (z - y), D^2_y U(x,y + t(z - y))(z - y) \rangle 1_{B_R}(y + t(z - y)) \, dt + \frac{1}{r} \int_0^1 \langle (z - y), D^2_y U(x,y + t(z - y))(z - y) \rangle 1_{B_{R_1}}(y + t(z - y)) \, dt \geq -Mmr + \frac{1}{r} \left( 1 - t_0(y,z) \right) K_3 r^2 \tag{72}
\]
\[
= r(\| y - z \| - t_0(y,z)(Mm + K_3)) \geq \begin{cases} -Mmr & \text{if } y, z \in B_{R_1} \\ 0 & \text{otherwise.} \end{cases} \tag{73}
\]
The inequality in (72) follows because:

(a) from (2), on account of $\| D^2_y U(x,y + t(z - y)) \| \leq M$ when $y + t(z - y) \in B_R$, we easily obtain the simple inequality $\langle (z - y), D^2_y U(x,y + t(z - y))(z - y) \rangle \geq -Mmr^2$ using which the first term is obtained; and

(b) from (S), $\langle (z - y), D^2_y U(x,y + t(z - y))(z - y) \rangle \geq K_3 r^2$ when $y + t(z - y)$ is outside $B_R$.

The inequality in (73) follows from the easily verifiable fact
\[
t_0(y,z) \leq 2R/(R + R_1) = K_3/(K_3 + Mm) \tag{74}
\]

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for every \(y, z\) such that one of them is outside \(\mathbb{B}_{R_1}\).

From (73), it is clear that we may take \(g(\cdot)\) to be any continuous function that dominates the function \(M_{\alpha r} \cdot 1\{r \leq 2R_1\}\), and there is at least one such \(g(\cdot)\) that satisfies \(\int_0^\infty g(s)ds < \infty\). \(\Box\)

Given Brownian motion \(B_t\) on \(\mathbb{R}^d\) and an independent Brownian motion \(W_t\) on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a strong solution to the coupled system (4) and (5) is a continuous process \((X_t^\epsilon, Y_t^\epsilon)\) that is adapted to the complete filtration generated by \(B, W\), and satisfies (4) and (5). We say that strong uniqueness holds for the coupled system (4) and (5) if whenever \((X_t^\epsilon, Y_t^\epsilon)\) and \((\tilde{X}_t^\epsilon, \tilde{Y}_t^\epsilon)\) are two strong solutions of the coupled system (4) and (5) with the common initial condition \(x_0, y_0\), then \(\mathbb{P}((X_t^\epsilon, Y_t^\epsilon) = (\tilde{X}_t^\epsilon, \tilde{Y}_t^\epsilon)\text{ for all }t \geq 0) = 1\).

**Lemma A.2** Assume (B1), (U1) and (U2). Let \(\varepsilon > 0, 0 < \alpha < 1\) and \(s(\varepsilon) > 0\) be given. The coupled system given by (7) and (8) has a unique strong solution.

**Proof** By assumptions (B1) and (U1), we know that \(b : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d\) and \(\nabla_y U : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m\) are locally Lipschitz functions. By [IW89] page 178 Theorem 3.1 there exist a unique strong solution, \((X_t^\epsilon, Y_t^\epsilon)_{0 \leq t < \zeta}\) where

\[
\zeta = \inf\{t \geq 0: \|X_t^\epsilon\|^2 + \|Y_t^\epsilon\|^2 = \infty\}.
\]

We will now establish nonexplosiveness of the process. Let \(f : \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty)\) be given by

\[
f(x, y) = \|x\|^2 + \|y\|^2.
\]

Let

\[
\sigma_n = \inf\{t \geq 0: \|X_t^\epsilon\|^2 + \|Y_t^\epsilon\|^2 = n\}.
\]

Clearly \(\sigma_n \leq \zeta\) almost surely for all \(n \geq 1\). Let \(t > 0\) be given. Applying Ito’s formula at time \(\sigma_n \wedge t\), we obtain that

\[
\mathbb{E}[f(X_{\sigma_n \wedge t}^\epsilon, Y_{\sigma_n \wedge t}^\epsilon)] = f(x_0, y_0) + \mathbb{E} \int_0^{\sigma_n \wedge t} 2 \left(\langle X_r^\epsilon, b(X_r^\epsilon, Y_r^\epsilon) \rangle - \frac{1}{\varepsilon} \nabla_y U(X_r^\epsilon, Y_r^\epsilon)\right) dr + (ms(\varepsilon)^2/\varepsilon + d2\alpha)\mathbb{E}(\sigma_n \wedge t).
\]

Using the fact that \(b\) is bounded from assumption (B1) we have, for \(r > 0\),

\[
\langle X_r^\epsilon, b(X_r^\epsilon, Y_r^\epsilon) \rangle \leq c_1 \|X_r^\epsilon\| \|b\|_\infty.
\]

Using (7) from assumption (U2) and (70) derived above we have, for \(r > 0\),

\[
- \langle Y_r^\epsilon, \frac{1}{\varepsilon} \nabla_y U(X_r^\epsilon, Y_r^\epsilon) \rangle < \begin{cases} c_2(M, R) & \text{if } \|Y_r\| \leq R \\ 0 & \text{if } \|Y_r\| > R. \end{cases}
\]

Substituting (76) and (77) in (75) we have

\[
\mathbb{E}(f(X_{\sigma_n \wedge t}^\epsilon, Y_{\sigma_n \wedge t}^\epsilon)) \leq f(x_0, y_0) + \mathbb{E} \int_0^{\sigma_n \wedge t} (2c_1 \|X_r^\epsilon\| \|b\|_\infty + 2c_2(M, R)) dr + (ms(\varepsilon)^2/\varepsilon + d2\alpha)\mathbb{E}(\sigma_n \wedge t)
\]

\[
\leq f(x_0, y_0) + c_3 \int_0^{t} \mathbb{E}[X_r^\epsilon] \mathbb{E}[d_r] + c_t t
\]

\[
\leq f(x_0, y_0) + c_5 \int_0^{t} (1 + r) dr + c_4 t
\]

\[
\leq c_6 + c_7 t + c_8 t^2,
\]

where the penultimate inequality uses \(c_3 \mathbb{E}[\|X_r^\epsilon\|] \leq c_5 (1 + r)\) for a suitable \(c_5\), a fact that follows from (7) and the boundedness assumption on \(b\) in (U1). As \(\sigma_n \to \zeta\) almost surely, the above would imply

\[
\mathbb{E}[f(X_{\zeta \wedge t}^\epsilon, Y_{\zeta \wedge t}^\epsilon)] \leq c_6 + c_7 t + c_8 t^2.
\]

Thus if \(\zeta < t\) then we have a contradiction, as the left-hand side is infinity and the right-hand side is finite. As \(t > 0\) was arbitrary, we have \(\zeta = \infty\) almost surely. This establishes nonexplosiveness of the process and completes the proof of strong uniqueness. \(\Box\)
Appendix B Nonlinear Filtering Equation

Let $x_0 \in \mathbb{R}^d$, $y_0 \in \mathbb{R}^m$, $\sigma_1 > 0$ and $\sigma_2 > 0$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $\{B_t\}_{t \geq 0}$ and $\{W_t\}_{t \geq 0}$ be Brownian motions on $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively. In this section, we consider the coupled diffusion $(X_t, Y_t)_{t \in [0,T]}$ on $\mathbb{R}^d \times \mathbb{R}^m$ described by

\begin{align}
X_t &= x_0 + \int_0^t b_1(X_s, Y_s)ds + \sigma_1 B_t, \\
Y_t &= y_0 + \int_0^t b_2(X_s, Y_s)ds + \sigma_2 W_t,
\end{align}

where $0 \leq t \leq T$, $b_1 : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ and $b_2 : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$. We will make the following assumptions:

- $b_1 \in C_b(\mathbb{R}^d \times \mathbb{R}^m)$ is locally Lipschitz continuous in $y$-variable and is uniformly (w.r.t. $y$) Lipschitz continuous in $x$-variable, i.e. $\exists K_1 > 0$ such that $\forall x, x' \in \mathbb{R}^d, y \in \mathbb{R}^m$
  \[ \| b_1(x, y) - b_1(x', y) \| \leq K_1 \| x - x' \|. \]

- $b_2 \in C^1(\mathbb{R}^d \times \mathbb{R}^m)$. Further, $b_2(x, y)$ is uniformly (w.r.t. $y$) Lipschitz continuous in $x$-variable, i.e. $\exists K_2 > 0$ such that $\forall x, x' \in \mathbb{R}^d, y \in \mathbb{R}^m$
  \[ \| b_2(x, y) - b_2(x', y) \| \leq K_2 \| x - x' \|. \]

- There exists $R > 0, M > 0$ such that for all $x \in \mathbb{R}^d$
  \[ \sup_{\|y\| \leq R} \| b_2(x, y) \| \leq M. \]

Using the above assumptions in the same proof as in Lemma 2.2 it is standard to see that the above coupled system has a unique strong solution. With $f \in C_b^2(\mathbb{R}^m)$, for $y \in \mathbb{R}^m$ let

\[ \mathcal{L}_b^2 f(y) = \frac{\sigma_2^2}{2} \Delta f(y) + \langle b_2(x, \cdot), \nabla f(y) \rangle, \]

where $x$ is treated as a parameter. Let $\mathcal{F}^X_t = \sigma(X_s, s \leq t)$, $\mathcal{F}^X = \bigvee_{t \geq 0} \mathcal{F}^X_t$. Define $\mathcal{F}^{X,Y}_t$ and $\mathcal{F}^{X,Y}$ analogously. Define $\pi_t(dy)$ as the conditional law of $Y_t$ given $\mathcal{F}^X_t$ so that

\[ \pi_t(f) := E[f(Y_t)|\mathcal{F}^X_t] \quad \text{for } f \in C_b^2(\mathbb{R}^m). \]

**Proposition B.1 (Nonlinear Filtering Equation)** The measure valued process $\pi$ is the unique solution to the (Fujisaki-Kallianpur-Kunita) nonlinear filtering equation

\[ \pi_t(f) = f(y_0) + \int_0^t \pi_s(\mathcal{L}^X_b f)ds + \frac{1}{\sigma_1} \int_0^t \left( \pi_s(b_1(X_s, \cdot)) - \pi_s(f) \pi_s(b_1(X_s, \cdot)) \right) d\tilde{B}_s, \quad f \in C_b^2(\mathbb{R}^m), \]

where $\tilde{B}$ is a standard Brownian motion.

**Remark B.2** $\tilde{B}$ is explicitly defined later in the proof. It is called the ‘innovations process’ and, under mild technical conditions, is known to generate the same increasing $\sigma$-fields as $B$ [AM81].
We will closely mimic the arguments in [BC09, Chapter 3] proved for the case when \( b_1(x, y) \) is a function of the first argument alone. In our setting, \( b_1(x, y) \) is a function of both arguments.

Set

\[
\Lambda_s = \exp\left\{ -\frac{1}{\sigma_1^2} \int_0^s \langle b_1(X_u, Y_u), dB_u \rangle - \frac{1}{2\sigma_1^4} \int_0^s b_1(X_u, Y_u)^2 \, du \right\}, \quad s \geq 0, \tag{82}
\]

and

\[
\bar{B}_s = B_s + \frac{1}{\sigma_1} \int_0^s b_1(X_u, Y_u) \, du, \quad s \geq 0.
\]

Define the probability measure \( Q \) by

\[
\frac{dQ}{dP} \big|_{\mathcal{F}^{X,Y}_t} = \Lambda_s, \quad s > 0.
\]

This consistently defines \( Q \) on \( \mathcal{F}^{X,Y} \). As \( b_1 \) is bounded, by the Cameron-Martin-Girsanov theorem, it follows that \( \bar{B} \) is an \( \mathbb{R}^d \)-valued standard Brownian motion under \( Q \). Under \( Q \), the joint process \((X, Y)\) given by (79) - (80) takes the form

\[
X_t = x_0 + \sigma_1 \bar{B}_t, \quad Y_t = y_0 + \int_0^t b_2(X_s, Y_s) \, ds + \sigma_2 W_t. \tag{83}
\]

Before we begin the proof we need some preliminary lemmas.

**Lemma B.3** For \( t > 0 \), let \( Z \) be a \( Q \)-integrable \( \mathcal{F}^{X,Y}_t \)-measurable \( \mathbb{R}^d \)-valued random variable. Then

\[
E^Q[Z|\mathcal{F}^X_t] = E^Q[Z|\mathcal{F}^X].
\]

**Proof:** Set

\[
\tilde{\mathcal{F}}^X_t = \sigma(X_{t+s} - X_t, s \geq 0).
\]

Then \( \mathcal{F}^X = \tilde{\mathcal{F}}^X_t \vee \mathcal{F}^X_t \), and since \( X_s = \sigma_1 \bar{B}_s \), an \( \{\mathcal{F}^X_s\} \)-Wiener process under \( Q \), \( \tilde{\mathcal{F}}^X_t \) is independent of \( \mathcal{F}^X_t \) under \( Q \). Hence

\[
E^Q[Z|\mathcal{F}^X_t] = E^Q[Z|\tilde{\mathcal{F}}^X_t \vee \mathcal{F}^X_t] = E^Q[Z|\mathcal{F}^X].
\]

This completes the proof of the lemma. \( \square \)

**Lemma B.4** Let \( \{\alpha_t, \ t \geq 0\} \) be an \( \{\mathcal{F}^{X,Y}_t\} \)-progressively measurable \( \mathbb{R} \)-valued process such that

\[
E^Q\left[ \int_0^t \alpha_s^2 \, ds \right] < \infty \forall \ t > 0.
\]

Then

\[
E^Q\left[ \int_0^t \alpha_s dX_s \big| \mathcal{F}^X \right] = \int_0^t E^Q[\alpha_s|\mathcal{F}^X] \, dX_s.
\]

**Proof:** Using Lemma B.3 it follows that

\[
E^Q\left[ \int_0^t \alpha_s dX_s \big| \mathcal{F}^X \right], \ E^Q[\alpha_t|\mathcal{F}^X]
\]

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are $\mathcal{F}_t^X$-measurable. Hence using the ‘density result’ of Krylov and Rozovskii, see [BC09, Lemma B.39, p.355], it is enough to show

$$E^Q \left[ \beta_t E^Q \left[ \int_0^t \alpha_s dX_s \mid \mathcal{F}_t \right] \right] = E^Q \left[ \beta_t \int_0^t E^Q[\alpha_s | \mathcal{F}_t] dX_s \right]$$

(84)

for all process $\beta(\cdot)$ of the form

$$\beta_t = 1 + \int_0^t i(\beta_s r_s, dX_s)$$

for a deterministic $r \in L^\infty([0, t]; \mathbb{R}^d)$. Consider

$$E^Q \left[ \beta_t E^Q \left[ \int_0^t \alpha_s dX_s \mid \mathcal{F}_t \right] \right] = E^Q \left[ \beta_t \int_0^t \alpha_s dX_s \right]
=E^Q \left[ \int_0^t \alpha_s dX_s \right] + E^Q \left[ \left( \int_0^t i(\beta_s r_s, dX_s) \right) \left( \int_0^t \alpha_s dX_s \right) \right]
= \sigma^2 t E^Q \left[ \int_0^t i\beta_s r_s \alpha_s ds \right]
= \sigma^2 t E^Q \left[ E^Q \left[ \int_0^t i\beta_s r_s \alpha_s | \mathcal{F}_t \right] dX_s \right]
= E^Q \left[ \left( \int_0^t i\beta_s r_s, dX_s \right) \left( \int_0^t E^Q[\alpha_s | \mathcal{F}_t] dX_s \right) \right]
= E^Q \left[ \beta_t \int_0^t E^Q[\alpha_s | \mathcal{F}_t] dX_s \right]$$

This completes the proof of the lemma. \(\square\)

**Lemma B.5** Let $x \in \mathbb{R}^d$. Let $\{\alpha_t, t \geq 0\}$ be $\{\mathcal{F}_t^X\}$-progressively measurable process such that

$$E^Q \left[ \int_0^t \alpha^2_s d(M^f)_{s} \right] < \infty, f \in C^2_b(\mathbb{R}^m), t \geq 0,$$

where

$$M^f_t = f(Y_t) - f(y_0) - \int_0^t \mathcal{L}_2^f(Y_s) ds,$$

and $\langle M^f \rangle_t$ is its quadratic variation. Then

$$E^Q \left[ \int_0^t \alpha_s dM^f_s \mid \mathcal{F}_t \right] = 0.$$

**Proof:** Via Itô’s formula, we first obtain

$$dM^f_t = \sigma_2(\nabla f(Y_t), dW_t).$$

(85)

Under $P$, this is driven by a Brownian motion independent of $B_t$, which leads to

$$\langle M^f, X \rangle_t = 0, \ P - \text{almost surely}$$

and hence $Q$—almost surely. Using this, the proof follows along the lines of the proof of Lemma B.4 \(\square\)

Set

$$\bar{\Lambda}_t = \Lambda_t^{-1}, t \geq 0,$$

and for $g \in C^2(\mathbb{R}^d \times \mathbb{R}^m)$ with a little abuse of notation denote

$$\pi_t(g) := \pi_t(g(X_t, \cdot)) = E[g(X_t, Y_t) | \mathcal{F}^X_t]$$

We then have the following.
Lemma B.6 (Kallianpur-Striebel formula) For \( g \in C^2(\mathbb{R}^d \times \mathbb{R}^m) \),

\[
\pi_t(g) = \frac{E^Q[\tilde{\Lambda}_t g | \mathcal{F}_t^X]}{E^Q[\tilde{\Lambda}_t | \mathcal{F}_t^X]},
\]

Proof: In view of Lemma [13.3] it is enough to show that

\[
\pi_t(g)E^Q[\tilde{\Lambda}_t | \mathcal{F}_t^X] = E^Q[\tilde{\Lambda}_t g | \mathcal{F}_t^X].
\]

Since both left and right sides are \( \mathcal{F}_t^X \)-measurable, it is enough to show that

\[
E^Q \left[ \beta \pi_t(g)E^Q \left[ \tilde{\Lambda}_t | \mathcal{F}_t^X \right] \right] = E^Q \left[ \beta \tilde{\Lambda}_t g \right] \tag{86}
\]

for all \( \mathcal{F}_t^X \)-measurable \( \beta \). This is now easily verified since, for such \( \beta \), we have

\[
E^Q \left[ \beta \pi_t(g)E^Q \left[ \tilde{\Lambda}_t | \mathcal{F}_t^X \right] \right] = E^Q \left[ \beta \pi_t(g) \tilde{\Lambda}_t \right] = E \left[ \beta \pi_t(g) \right] = E \left[ E \left[ \beta g | \mathcal{F}_t^X \right] \right] = E \left[ E \left[ \beta g \right] \right] = E^Q \left[ \tilde{\Lambda}_t \beta g \right].
\]

This completes the proof of the lemma. \( \square \)

We are now ready to prove Proposition [13.4] We shall derive first the Zakai equation solved by certain unnormalized conditional laws. Then we shall show existence to the Fujisaki-Kallianpur-Kunita) nonlinear filtering equation [33], followed by uniqueness.

Proof of Proposition [13.4] Observe that \( \{ \Lambda_t, t \geq 0 \} \) is given by the solution of the SDE

\[
\Lambda_t = 1 - \int_0^t \Lambda_s \sigma_1^{-1} \langle b_1(X_s, Y_s), dB_s \rangle,
\]

for \( t \geq 0 \). Hence by a routine application of Itô’s formula it follows that

\[
\tilde{\Lambda}_t = 1 + \int_0^t \tilde{\Lambda}_s \sigma_1^{-2} \langle b_1(X_s, Y_s), dX_s \rangle. \tag{87}
\]

From this, since \( X_t \) is driven by \( B_t \) and \( Y_t \) is driven by \( W_t \), for \( f \in C^2_b(\mathbb{R}^m) \), the cross-variation \( \langle \tilde{\Lambda}, f(Y) \rangle_t = 0 \) \( P \)-a.s. and hence \( Q \)-a.s. Using Itô’s formula again, we get

\[
\tilde{\Lambda}_t f(Y_t) = f(y_0) + \int_0^t \tilde{\Lambda}_s [\mathcal{L}^X f(Y_s)] ds + \sigma_2 \langle \nabla f(Y_s), dBW_s \rangle + \int_0^t f(Y_s) d\tilde{\Lambda}_s,
\]

and hence, using [33] and (87), we get

\[
\tilde{\Lambda}_t f(Y_t) = f(y_0) + \int_0^t \tilde{\Lambda}_s [\mathcal{L}^X f(Y_s)] ds + \int_0^t \tilde{\Lambda}_s dM_t^f + \sigma_1^{-2} \int_0^t \tilde{\Lambda}_s f(Y_s) \langle b_1(X_s, Y_s), dX_s \rangle. \tag{88}
\]

Taking conditional expectation \( E^Q[ \cdot | \mathcal{F}^X ] \) in (88) we have using Lemma [13.5] we have

\[
E^Q \left[ \tilde{\Lambda}_t f(Y_t) | \mathcal{F}^X \right] = f(y_0) + E^Q \left[ \int_0^t \tilde{\Lambda}_s [\mathcal{L}^X f(Y_s)] ds | \mathcal{F}^X \right] + \sigma_1^{-2} E^Q \left[ \int_0^t \tilde{\Lambda}_s f((Y_s) \langle b_1(X_s, Y_s), dX_s \rangle | \mathcal{F}^X \right],
\]

and using Lemma [13.4] we have the above is

\[
= f(y_0) + \int_0^t E^Q \left[ \tilde{\Lambda}_s [\mathcal{L}^X f(Y_s)] | \mathcal{F}^X \right] ds + \sigma_1^{-2} \int_0^t (E^Q \left[ \tilde{\Lambda}_s f((Y_s) b_1(X_s, Y_s)) | \mathcal{F}^X \right], dX_s), \tag{89}
\]

For \( g \in C(\mathbb{R}^d \times \mathbb{R}^m) \) denoting

\[
\rho_t(g) = \pi_t(g) E^Q[\tilde{\Lambda}_t | \mathcal{F}^X].
\]
in [58] and using Lemma 3.6 we arrive at the Zakai equation

$$\rho_t(f) = f(y_0) + \int_0^t \rho_s(\mathcal{L}_t^X f) ds + \sigma_1^{-2} \int_0^t \rho_s(fb_1(X_s, \cdot), dX_s).$$  \tag{90}$$

For $1 :=$ the constant function identically equal to 1, we see that $\rho_t(1) = E^Q[\bar{\Lambda}_t | \mathcal{F}_t^X]$, and hence

$$\pi_t(f) = \frac{\rho_t(f)}{\rho_t(1)}.  \tag{91}$$

The nonnegative measure valued process $\{\rho_t\}_{t \geq 0}$ is called the process of unnormalized conditional laws in view of (91).

Now we are ready to prove the existence theorem for the Fujisaki-Kallianpur-Kunita (FKK) equation, [51]. From the Zakai equation (90) we get

$$\rho_t(f) = f(y_0) + \int_0^t \rho_s(1)\pi_s(\mathcal{L}_t^X f) ds + \sigma_1^{-2} \int_0^t \rho_s(1)(\pi_s(fb_1(X_s, \cdot)), dX_s),$$  \tag{92}$$

In particular, one can deduce that

$$\rho_t(1) = 1 + \sigma_1^{-2} \int_0^t \rho_s(1)(\pi_s(b_1(X_s, \cdot)), dX_s).$$  \tag{93}$$

Using Itô’s formula, we get

$$\frac{1}{\rho_t(1)} = 1 - \sigma_1^{-2} \int_0^t \frac{1}{\rho_s(1)}(\pi_s(b_1(X_s, \cdot)), dX_s) + \sigma_1^{-2} \int_0^t \frac{1}{\rho_s(1)}||\pi_s(b_1(X_s, \cdot))||^2 ds,$$  \tag{94}$$

Note that the cross-variation

$$\langle \rho(f), \frac{1}{\rho(1)} \rangle_t = -\int_0^t \sigma_1^{-2} \langle \pi_s(b_1), \pi_s(b_1 f) \rangle ds,$$  \tag{95}$$

Itô’s formula, for the product of $\rho_t(f)$ and $\frac{1}{\rho_t(1)}$ we get

$$\frac{\rho_t(f)}{\rho_t(1)} = f(y_0) + \int_0^t \rho_s(1)\rho_s(f) \frac{1}{\rho_s(1)} + \int_0^t \rho_s(1) d\rho_s(f) + \langle \rho(f), \frac{1}{\rho(1)} \rangle_t$$

Substituting (92), (93), and (94) in the above we have

$$\frac{\rho_t(f)}{\rho_t(1)} = f(y_0) + \int_0^t \rho_s(f) \left[ -\frac{1}{\rho_s(1)} \sigma_1^{-2}(\pi_s(b_1(X_s, \cdot)), dX_s) + \sigma_1^{-2} \frac{1}{\rho_s(1)} ||\pi_s(b_1(X_s, \cdot))||^2 ds, \right]$$

$$+ \int_0^t \frac{1}{\rho_s(1)} \left[ \rho_s(1) \pi_s(\mathcal{L}_t^X f) ds + \sigma_1^{-2} \rho_s(1)(\pi_s(fb_1(X_s, \cdot)), dX_s) \right]$$

$$- \int_0^t \sigma_1^{-2} \langle \pi_s(b_1), \pi_s(b_1 f) \rangle ds.$$

From (91) and simple algebra in the above we have

$$\pi_t(f) = f(y_0) + \int_0^t \pi_s(\mathcal{L}_t^X f) ds + \sigma_1^{-2} \int_0^t \langle \pi_s(fb_1) - \pi_s(f)\pi_s(b_1), dX_s - \pi_s(b_1) ds \rangle$$  \tag{96}$$

Let

$$I_t = X_t - \int_0^t \pi_s(b_1) ds,$$

the so called ‘innovation process’. For $0 \leq s < t$, we have

$$E[I_t - I_s | \mathcal{F}_s^X] = E \left[ \int_s^t E[\pi_u(b_1(X_u, Y_u) - \pi_u(b_1(X_u, \cdot)) | \mathcal{F}_u^X] du \right] \mathcal{F}_s^X$$

$$= \int_s^t E \left[ b_1(X_u, Y_u) - E[b_1(X_u, Y_u) | \mathcal{F}_u^X] \right] \mathcal{F}_s^X du$$

$$= 0.$$
Thus \( \{I_t | t \geq 0\} \) is an \( \{\mathcal{F}_t^X\} \)-martingale with mean 0 and quadratic variation \( \sigma_t^2t \). Thus by Levy’s characterization, \( I \) is a scaled Brownian motion. Define

\[
\tilde{B}_t := \sigma_t^{-1}I_t, \ t \geq 0.
\]

From (93) we have enough to cover our problem. So \( \tilde{B}_t \) is a \( \{\mathcal{F}_t^X\} \)-adapted standard Brownian motion under \( P \). Therefore we have shown that,

\[
\pi_t(f) = f(y_0) + \int_0^t \pi_s(L^X_s f) ds + \sigma_t^{-1} \int_0^t \langle \pi_s(fb_1) - \pi_s(f)\pi_s(b_1), d\tilde{B}_s \rangle,
\]

with \( \tilde{B}_s \) being a standard Brownian motion. Thus solutions \( \pi, \rho \) of FKK, resp. Zakai equations are in one-one correspondence and uniqueness of the FKK equation in the sense of martingale problem follows from Theorem 3.3 of Kurtz and Ocone [KÖSS]. Note that while Kurtz and Ocone [KÖSS] cite nonlinear filtering as an example of this theorem, they consider the classical formulation (see [KÖSS, Theorem 4.1]) which is more restrictive than ours. However the aforementioned theorem ([KÖSS, Theorem 3.3]) is general enough to cover our problem.

From (93) we have

\[
\rho_t(1) = \exp \left\{ \sigma_1^{-2} \int_0^t \langle \pi_s(b_1(X_s, \cdot)), dX_s \rangle - \frac{\sigma_1^{-4}}{2} \int_0^t \| \pi_s(b_1(X_s, \cdot)) \|^2 ds \right\}.
\]

Since \( \pi_t(f) = \frac{\rho_t(f)1}{\rho_t(1)} \),

\[
\rho_t(f) = \pi_t(f)\rho_t(1) = \pi_t(f) \exp \left\{ \sigma_1^{-2} \int_0^t \langle \pi_s(b_1(X_s, \cdot)), dX_s \rangle - \frac{\sigma_1^{-4}}{2} \int_0^t \| \pi_s(b_1(X_s, \cdot)) \|^2 ds \right\}.
\]

Thus solutions \( \pi, \rho \) of FKK, resp. Zakai equations are in one-one correspondence and uniqueness of one implies that of the other.

**Remark B.7** It is interesting to note that some of the earlier uniqueness arguments for the classical framework such as one using multiple Wiener integral expansion due to [Kum83] or via the Clark-Davis ‘pathwise’ filter as in [Hau85], do not work for our case. (The latter would work only if \( b_1(x, \cdot) = \nabla F(x, \cdot) \) for a suitable \( F \).)

### Appendix C Laplace’s principle

We now characterize weak limit points of the sequence of invariant measures for the fast process \( \nu^n\varepsilon, x^n \) when \( \varepsilon \to 0 \) and for deterministic \( x^n \to x \). This is used in the proof of Proposition 2.4(c,d).

**Lemma C.1** Let \( n \geq 1, 0 < \varepsilon_n < 1, x^n \in \mathbb{R}^d \), and \( x \in \mathbb{R}^d \). Suppose \( \varepsilon_n \to 0 \) and \( x^n \to x \) as \( n \to \infty \).

(a) Then the sequence of measures \( \nu^n\varepsilon, x^n \) is tight and any limit point is supported on \( \arg \min \{U(x, \cdot)\} \).

(b) Assume (U4) and let \( x \in D^0_L \) for some \( L \geq 1 \). Then \( \nu^n\varepsilon, x^n \) converges weakly to \( \nu^{0,x} \), where \( \nu^{0,x} \) is given by

\[
\sum_{i=1}^L \delta_{y_i(x)} \left( \frac{\det \left[ \partial^2 U(x, y_i(x)) \right]}{\sum_{j=1}^L \det \left[ \partial^2 U(x, y_j(x)) \right]} \right)^{-\frac{1}{2}}.
\]
As we will now show that any limit point exists. Let \( z \) and \( \nu \) be the measures from (U1) with a suitable relabelling of the minima if necessary, we have

\[
\nu_{x^n} (B(z, r)) = \frac{\int_{B(z, r)} e^{-2 \frac{U(x^n, y) - U(x^n, y_0)}{\epsilon_n} x^n} dy}{\int_{B(y_i(x), r)} e^{-2 \frac{U(x^n, y) - U(x^n, y_0)}{\epsilon_n} x^n} dy} \leq \frac{|B(z, r)| e^{-\frac{\delta}{2\epsilon_n}}}{{|B(y_i(x), r)| e^{-\frac{\delta}{2\epsilon_n}}} = e^{-\frac{\delta}{2\epsilon_n} r^2}.
\]

Therefore,

\[
\lim_{n \to \infty} \nu_{x^n} (B(z, r)) = 0.
\]

Hence any limit point \( \nu \) is supported on the \( \text{arg min} \{U(x, \cdot)\} \).

(b) Let \( x \in D^2 \) for some \( L \geq 1 \). Under (U4) the global minima \( y_i(x) \), \( 1 \leq i \leq L \) are nondegenerate, i.e., the matrix \( D^2_y U(x, y_i(x)) \) is positive definite for \( 1 \leq i \leq L \). Since \( D^2_y \) is open, using \( U \in C^2(\mathbb{R}^m \times \mathbb{R}^d) \) in (U1), with a suitable relabelling of the minima if necessary, we have \( y_i(x^n) \to y_i(x) \) \( \forall 1 \leq i \leq L \) as \( n \to \infty \). Let \( B_i \) be the ball centered at \( y_i(x) \) with radius 1 for each \( i \). Let \( n \) be sufficiently large.
so that \( y_i(x^{\varepsilon_n}) \) are in a ball centered at \( y_i(x) \) with radius \( \frac{1}{2} \). Let \( B_i^n \) be ball centered at \( y_i(x^{\varepsilon_n}) \) with radius \( \frac{1}{2} \) for each \( i \). Note that \( \nabla_y U(x^{\varepsilon_n}, y_i(x^{\varepsilon_n})) = 0 \) and \( U(x^{\varepsilon_n}, y_i(x^{\varepsilon_n})) = \min U(x^{\varepsilon_n}, \cdot) := u_{\min} \) (say) as it does not depend on \( i \). Using Taylor’s expansion up to second order, we have that for each \( y \in B_i \) there is a \( \tilde{y}_i(x^{\varepsilon_n}) \) \( \in B_i \) such that

\[
U(x^{\varepsilon_n}, y) = U(x^{\varepsilon_n}, y_i(x^{\varepsilon_n})) + \frac{1}{2}(y - y_i(x^{\varepsilon_n}))^T D^2_y U(x^{\varepsilon_n}, \tilde{y}_i(x^{\varepsilon_n}))(y - y_i(x^{\varepsilon_n}))
\]

\[
= u_{\min} + \frac{1}{2}(y - y_i(x^{\varepsilon_n}))^T D^2_y U(x^{\varepsilon_n}, \tilde{y}_i(x^{\varepsilon_n}))(y - y_i(x^{\varepsilon_n})).
\]

The above and standard fact about Gaussian random variables implies:

\[
\int_{B_i} e^{-\frac{U(x, y^{\varepsilon_n})}{2\varepsilon_n}} dy \leq e^{-\frac{u_{\min}}{2\varepsilon_n}} \int_{B_i^n} e^{-\frac{(y - y_i(x^{\varepsilon_n}))^T D^2_y U(x^{\varepsilon_n}, \tilde{y}_i(x^{\varepsilon_n}))(y - y_i(x^{\varepsilon_n}))}{2\varepsilon_n}} dy
\]

\[
= e^{-\frac{u_{\min}}{2\varepsilon_n}} \left( \frac{(2\pi)^m s(\varepsilon_n)^2}{2} \text{Det} \left( D^2_y U(x^{\varepsilon_n}, y_i(x^{\varepsilon_n}))^{-1} \right) \right)^{\frac{1}{2}};
\]

and

\[
\int_{B_j} e^{-\frac{U(x, y^{\varepsilon_n})}{2\varepsilon_n}} dy \geq e^{-\frac{u_{\min}}{2\varepsilon_n}} \int_{B_j^n} e^{-\frac{(y - y_i(x^{\varepsilon_n}))^T D^2_y U(x^{\varepsilon_n}, \tilde{y}_i(x^{\varepsilon_n}))(y - y_i(x^{\varepsilon_n}))}{2\varepsilon_n}} dy
\]

\[
= e^{-\frac{u_{\min}}{2\varepsilon_n}} \left( \frac{(2\pi)^m s(\varepsilon_n)^2}{2} \text{Det} \left( D^2_y U(x^{\varepsilon_n}, y_i(x^{\varepsilon_n}))^{-1} \right) \right)^{\frac{1}{2}} P \left( Z^m \in \frac{1}{s(\varepsilon_n)} A_{i,n} \right),
\]

where \( A_{i,n} = \{ z \in \mathbb{R}^m : \| (D^2_y U(x^{\varepsilon_n}, y_i(x^{\varepsilon_n})))^{-1/2} z \| \leq \frac{1}{2\varepsilon_n} \} \) and \( Z^m \) is a standard \( m \)-dimensional Gaussian random variable. Note that \( A_{i,n} \) is a bounded set in \( \mathbb{R}^m \) and by (U1), \( U \in C^2(\mathbb{R}^d \times \mathbb{R}^m) \). So as \( n \to \infty \) we have

\[
P \left( Z^m \in \frac{1}{s(\varepsilon_n)} A_{i,n} \right) \to 1 \quad \text{and} \quad \left( \text{Det} \left( D^2_y U(x^{\varepsilon_n}, y_i(x^{\varepsilon_n}))^{-1} \right) \right)^{\frac{1}{2}} \to \left( \text{Det} \left( D^2_y U(x, y_i(x))^{-1} \right) \right)^{\frac{1}{2}}
\]

for all \( i \). Therefore using a standard sandwich argument we can conclude that, for balls \( B_i \) and \( B_j \), \( 1 \leq i, j \leq L \),

\[
\frac{\nu^{\varepsilon_n, x^{\varepsilon_n}}(B_i)}{\nu^{\varepsilon_n, x^{\varepsilon_n}}(B_j)} = \int_{B_i} e^{-\frac{U(x, y^{\varepsilon_n})}{2\varepsilon_n}} dy \to \left( \frac{\text{Det} \left( D^2_y U(x, y_i(x))^{-1} \right) \right)^{\frac{1}{2}}}{\left( \text{Det} \left( D^2_y U(x, y_j(x))^{-1} \right) \right)^{\frac{1}{2}}} \quad \text{as} \quad n \to \infty.
\]

From (a) we know that the sequence of measures \( \{ \nu^{\varepsilon_n, x^{\varepsilon_n}} \}_{n \geq 1} \) are tight and all limit points are measures supported on the arg \( \min U(x, \cdot) \). Consequently by (106) we have that any limit point \( \nu^{0, x} \) is given by

\[
\nu^{0, x}(\cdot) = \sum_{i=1}^{L} \sum_{j=1}^{L} \frac{(\text{Det} \left[ D^2_y U(x, y_j(x))^{-1} \right])^{-\frac{1}{2}}}{\text{Det} \left[ D^2_y U(x, y_i(x))^{-1} \right])^{\frac{1}{2}}} \delta_{y_i(x)}(\cdot).
\]

Since all subsequential limit points are the same we have the result.

**Acknowledgements:** Research of S.R.A. was supported in part by ISF-UGC grant, research of V.S.B. was supported in part by a J. C. Bose Fellowship, research of K.S.K. was supported in part by the grant MTR/2017/000416 from SERB and research of R.S. was supported in part by RBCCPS-IISc. S.R.A., V.S.B. and R.S. would like to thank the International Centre for Theoretical Sciences (ICTS) for hospitality during the Large deviation theory in statistical physics: Recent advances and future challenges (Code:ICTS/Prog-ldt/2017/8). The authors thank Sanjoy Mitter for pointing out the reference [KOSZ], Laurent Miclo and Patrick Cattiaux for suggestions on the spectral gap estimate in Proposition 2.3(b), and Konstantinos Spiliopoulos for pointing out several references in the literature.
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