On representations of Lie algebras compatible with a grading

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Abstract

The paper extends existing Lie algebra representation theory related to Lie algebra gradings. The notion of a representation compatible with a given grading is defined and applied to finite-dimensional representations of $\mathfrak{sl}(n, \mathbb{C})$ in relation with its $\mathbb{Z}_2$-gradings. For representation theory of $\mathfrak{sl}(n, \mathbb{C})$ the Gel’fand-Tseitlin method turned out very effective.

1 Introduction

Contractions of Lie algebras are usually defined inside the Lie algebra structure. Remaining inside the framework of Lie algebras, a slightly different notion of graded contractions was proposed in [15]. Nevertheless, for physical applications only representations of Lie algebras are relevant. In this matter the only existing mathematical theory [16] is in a very preliminary shape. There has been a short note [17], but nobody has systematically investigated the representations of Lie algebras related to their gradings. Our paper can be considered as a starting point of such an investigation.

1.1 Graded contraction of Lie algebras

A grading of a Lie algebra $L$ is a decomposition $\Gamma$ of the vector space $L$ into vector subspaces $L_j$, $j \in \mathcal{J}$, such that $L$ is a direct sum of these subspaces $L_j$, and, for any pair of indices $j, k \in \mathcal{J}$, there exists $l \in \mathcal{J}$ such that $[L_j, L_k] \subseteq L_l$. We denote the grading by

$$\Gamma : L = \bigoplus_{j \in \mathcal{J}} L_j.$$ 

Clearly, for any Lie algebra $L \neq \{0\}$, there exists the trivial grading $\Gamma : L = L$, i.e. the Lie algebra is not split up at all. The opposite extreme of splitting the Lie algebra into as many subspaces $L_j$ as possible is called fine grading. Let us note that in our definition of grading we do not exclude trivial subspaces $L_i = \{0\}$. It follows directly from the definition that for any grading $\Gamma : \bigoplus_{j \in \mathcal{J}} L_j$ and any automorphism $g \in \text{Aut} L$ the decomposition $\Gamma' : \bigoplus_{j \in \mathcal{J}} g(L_j)$ is a grading as well. The gradings $\Gamma$ and $\Gamma'$ are called equivalent.

Now we describe a specific type of a grading, namely a so-called group grading. The most notorious case of a group grading is the $\mathbb{Z}_2$-grading introduced by E. Inönü and E. Wigner when decomposing a Lie algebra $L$ into two non-zero grading subspaces $L_0$ and $L_1$, where

$$[L_0, L_0] \subseteq L_0, \quad [L_0, L_1] \subseteq L_1, \quad [L_1, L_1] \subseteq L_0.$$ 

(1)
A grading $\Gamma : L = \bigoplus_{j \in J} L_j$ is called a **group grading** if the index set $J$ can be embedded into a semigroup $G$ (whose binary operation is denoted by $+$), and, for any pair of indices $j, k \in J$, it holds that

$$[L_j, L_k] \subseteq L_{j+k}. \quad (2)$$

Since we allow even trivial subspaces in decomposition of $L$, as index set of the group grading may be used directly the semigroup $G$. In this case we will speak about $G$-grading $\Gamma$.

A grading $\Gamma : L = \bigoplus_{i \in J} L_i$ of a Lie algebra $L$ is a starting point for study of **graded contractions** of the Lie algebra. This method for finding contractions of Lie algebras was introduced in [15, 16]. Since we will focus in this paper on group gradings only, we assume in the sequel that the indices of grading subspaces belong to a group $G$, i.e., $\Gamma$ is a $G$-grading of $L$.

In this type of contraction, we define new Lie brackets by prescription

$$[x, y]_{\text{new}} := \varepsilon_{j,k}[x, y], \text{ where } x \in L_j, y \in L_k. \quad (3)$$

The complex or real parameters $\varepsilon_{j,k}$, for $j, k \in G$, must be determined in such a way that the vector space $L$ with the binary operation $[.,. ]_{\text{new}}$ forms again a Lie algebra. Antisymmetry of Lie brackets demands that $\varepsilon_{j,k} = \varepsilon_{k,j}$. If moreover, the coefficients $\varepsilon_{j,k}$ fulfill a system of quadratic equations:

$$\varepsilon_{i,j}\varepsilon_{i+j,k} = \varepsilon_{j,k}\varepsilon_{j+k,i} = \varepsilon_{k,i}\varepsilon_{k+i,j} \quad \text{for all } i, j, k \in G \quad (4)$$

then the vector space $L$ with new brackets $[x, y]_{\text{new}}$ satisfies the Jacobi identities as well. This new Lie algebra will be denoted by $L^\varepsilon$.

**Example 1.** For a $\mathbb{Z}_2$-grading of a Lie algebra $L$, the system of equations (4) has a very simple form

$$(\varepsilon_{00} - \varepsilon_{01})\varepsilon_{01} = 0 = (\varepsilon_{00} - \varepsilon_{01})\varepsilon_{11}$$

There exist infinitely many solutions $\varepsilon = (\varepsilon_{jk})$ of this system. Nevertheless for many solutions, the contracted algebras $L^\varepsilon$ are isomorphic. It can by shown that only four solutions

$$(\varepsilon_{jk}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

give mutually non-isomorphic Lie algebras $L^\varepsilon$. The contracted algebra obtained by the first solution is the semidirect sum of $L_0$ with a commutative algebra $L_1$. The second solution is the direct sum of $L_0$ and the commutative algebra $L_1$. The third solution leads to the Lie algebra of the Heisenberg type, or equivalently, it is a central extension of the commutative algebra $L_1$ by the commutative algebra $L_0$. The fourth solution is an Abelian Lie algebra.

### 1.2 Representations of graded contractions

Let us focus on the question of representation of the contracted Lie algebra $L^\varepsilon$. We will describe a method [17], which enables to find a representation of $L^\varepsilon$ by modifying a given representation of the original algebra $L$.

**Definition 1.1.** Let $r : L \rightarrow \text{End} V$ be a representation of the Lie algebra $L$ and $\Gamma : L = \bigoplus_{i \in G} L_i$ be its $G$-grading. We say that the representation $r$ is **compatible with the $G$-grading**, if there exists a decomposition of the vector space $V$ into a direct sum $V = \bigoplus_{i \in J} V_i$ such that

$$r(X_i)V_j \subset V_{i+j} \quad \text{for each } i, j \in G \text{ and any } X_i \in L_i. \quad (5)$$
Suppose we are given a representation $r$ of a contracted Lie algebra $L^\varepsilon$. Let us define
\[
  r^\varepsilon(X_i)v_j := \psi_{i,j}r(X_i)v_j \quad \text{for each } i,j \in G, \text{ any } X_i \in L_i \text{ and any } v_j \in V_j,
\]
where $\psi_{i,j}$ are unknown parameters. The requirement that $r^\varepsilon$ is a representation of $L^\varepsilon$ formally means
\[
  r^\varepsilon([X_i,X_j])v_k = [r^\varepsilon(X_i),r^\varepsilon(X_j)]v_k = (r^\varepsilon(X_i)r^\varepsilon(X_j) - r^\varepsilon(X_j)r^\varepsilon(X_i))v_k
\]
for any $X_i \in L_i, X_j \in L_j$, and $v_k \in V_k$. Using equations (3) and (6) and the relation (5) we obtain
\[
  \psi_{j,k}\psi_{i,j+k}r(X_i)r(X_j) - \psi_{i,k}\psi_{j,i+k}r(X_i)r(X_j) = \varepsilon_{i,j}\psi_{i+j,k}r([X_i,X_j])
\]
Since $r$ is a representation of $L$, we know that $r(X_i)r(X_j) - r(X_j)r(X_i) = r([X_i,X_j])$. Therefore, the choice of parameters $\psi_{i,j}$ satisfying
\[
  \psi_{j,k}\psi_{i,j+k} = \psi_{i,k}\psi_{j,i+k} = \varepsilon_{i,j}\psi_{i+j,k}
\]
implies that $r^\varepsilon$ defined by (6) is representation of the contracted Lie algebra $L^\varepsilon$. Comparing (7) and (1) we see that the systems of quadratic equations for parameters $\psi_{i,j}$ has at least one solution, namely $\psi_{i,j} = \varepsilon_{i,j}$ for each pair $i,j$. Therefore the mapping $r^\varepsilon : L^\varepsilon \mapsto \text{End} \ V$ defined by (6) is a representation of the graded Lie algebra $L^\varepsilon$. Usually, there exist also other solutions of the system (7), and therefore more representations of the same contracted algebra $L^\varepsilon$.

**Example 2.** Consider a $\mathbb{Z}_2$-grading of a Lie algebra $L$ and its representation $r$ which is compatible with the grading. For the corresponding decomposition of the vector space $V = V_0 \oplus V_1$ we may construct a basis $\mathcal{B}$ of $V$ by putting together the bases of $V_0$ and $V_1$. In such a basis $\mathcal{B}$, the grading relations (1) give explicitly
\[
  r(X_0) = \begin{pmatrix} A(X_0) & 0 \\ 0 & B(X_0) \end{pmatrix} \quad \text{and} \quad r(X_1) = \begin{pmatrix} 0 & C(X_1) \\ D(X_1) & 0 \end{pmatrix}.
\]
In the sequel, we will illustrate all notions on the Lie algebra $L^\varepsilon$, which is obtained by contraction from $\mathbb{Z}_2$-grading of a Lie algebra $L$ by the first solution
\[
  (\varepsilon_{jk}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\]
given in Example 1. For this Lie algebra $L^\varepsilon$ the commutation relations have the form
\[
  [x,y]_{\text{new}} = [x,y], \text{ if } x,y \in L_0 \text{ or if } x \in L_0, y \in L_1 \text{ and } [x,y]_{\text{new}} = 0, \text{ if } x,y \in L_1.
\]
In this case the system of equations (7) is
\[
  \psi_{00}\psi_{00} = \psi_{00}, \quad \psi_{10}\psi_{01} = \psi_{00}\psi_{10} = \psi_{10}, \\
  \psi_{01}\psi_{01} = \psi_{01}, \quad \psi_{11}\psi_{00} = \psi_{01}\psi_{11} = \psi_{11}, \\
  \psi_{10}\psi_{11} = 0.
\]
All solutions (up to equivalence of representations) of this system are
\[
  (\psi_{jk}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
The representations $r^\varepsilon$ of the contracted Lie algebra $L^\varepsilon$ in the chosen basis $B$ of the vector space $V$ have the form

\[
\begin{align*}
  r(X_0) &= \begin{pmatrix}
  \psi_{00}A(X_0) & 0 \\
  0 & \psi_{01}B(X_0)
  \end{pmatrix} \\
  r(X_1) &= \begin{pmatrix}
  0 & \psi_{11}C(X_1) \\
  \psi_{10}D(X_1) & 0
  \end{pmatrix},
\end{align*}
\]

where for parameters $(\psi_{ij})$ one may choose one of the six solutions. Let us mention that only two first solutions are interesting since the elements of subalgebra $L_1$ are in the remaining solutions represented by zero operators.

### 1.3 Some open problems

The four notions introduced above: grading, group grading, graded contraction and representation compatible with $G$-grading, immediately lead to many questions.

1. How many inequivalent gradings has a given Lie algebra $L$?
2. Do notions of grading and group grading coincide? On which Lie algebras?
3. Is the semigroup $G$ assigned to a group grading $\Gamma$ uniquely? If not, does there exist a semigroup $G$ to be preferred?
4. Does there exist a $G$-grading of $L$ for a given Lie algebra $L$ and a given semigroup $G$? How many inequivalent $G$-gradings of $L$ one can find?
5. How many non-isomorphic graded contractions can be found for a given $G$-grading of $L$?
6. Which irreducible representations of a Lie algebra $L$ are compatible with its $G$-grading?

No question of this list has found a satisfactory answer yet. Let us briefly summarize the achievements of numerous papers in this direction.

It seems, from the theoretical point of view, that the most important question to be solved is the second question on our list. In [7], Elduque contradicted the longstanding belief that any grading is a group grading. He found a Lie algebra of dimension 17 and described its grading which cannot be indexed by any semigroup $G$. The algebra which served to Elduque as counterexample is not simple. But even in case of simple finite-dimensional Lie algebras the situation is not clear. Up to now no non-group grading of a simple Lie algebra was found. Non-existence of non-group grading for any simple Lie algebra was claimed in the seminal work [18]. But Elduque’s result showed a gap in the proof of this statement. Therefore, the lists of fine inequivalent gradings of simple Lie algebras which were given in [9], [6], [4] and [5] are complete only as lists of group gradings. To show that these lists are complete in general sense, one needs to prove

**Conjecture:** Any grading of a simple Lie algebra is a group grading.

It is not difficult to answer negatively the first part of the third question. Namely, for any group grading $\Gamma$ there exist more non-isomorphic semigroups suitable for labeling subspaces $L_i$. As shown in [4], any group grading of a simple finite dimensional Lie algebra can be indexed by a finitely generated Abelian group $G_I$. Moreover, surjective homomorphisms of $G_I$ are in one-to-one correspondence with coarsenings of the grading $\Gamma$. The Abelian group $G_I$ with this property is given uniquely up to isomorphisms.

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From the point of view of the result Draper and Martín in [4], the question seems to be less difficult for simple finite-dimensional Lie algebras. We can restrict our considerations to Abelian finitely generated groups, i.e. to additive groups $G = Z_{n_1} \otimes Z_{n_2} \otimes \ldots \otimes Z_{n_r} \otimes Z^k$ for some integers $n_1, n_2, \ldots, n_r, k$. Nevertheless, till now the answer is known only for the simplest groups $G$, namely $Z_{n_1}, Z$ and $Z_2 \otimes Z_2$, [2].

The task to decide whether two Lie algebras given by their structural constants are isomorphic or not is far from being simple. Therefore only several examples of low-dimensional algebras and their graded contractions are well understood, [12].

As we have already said, the first step in construction of representation of graded contractions were made in [16], [15] and [17]. In these works, authors showed for specific type of $G$-gradings of simple Lie algebras that any irreducible representation is compatible with $G$-grading. Moreover, they gave also a recipe how to find suitable decomposition of vector space $V$ satisfying (5).

The article is organized as follows. For representation theory the distinction of gradings and group gradings is important. This open problem is approached in Section 2. It contains a modest improvement of the just mentioned conjecture: we show that the conjecture is true if a grading consists of two subspaces only.

The central part of the paper (Sections 3 and 4) is devoted to representations compatible with a grading. Explicit results are obtained for finite-dimensional representations of $sl(3, \mathbb{C})$ compatible with $\mathbb{Z}_2$-gradings generated either by an inner automorphism of order 2 or by an outer automorphism of order 2.

Our goal is to enlarge the family of gradings of $L$ for which one can decide about compatibility with the given representation of $L$. These results are illustrated on the simple Lie algebra $sl(3, \mathbb{C})$.

## 2 Grading versus group grading

In this section we concentrate on simple Lie algebras of finite dimension, say $k \in \mathbb{N}$. In particular, we will use the fact that $L$ is perfect, i.e.

$$[L, L] := \{[x, y] \mid x, y \in L\} = L. \quad (8)$$

At first we recall several ingredients needed in the proof of the theorem. We will work with the adjoint representation which to any $x \in L$ assigns the linear operator $M_x$ given on the vector space $L$ by the prescription

$$M_x y := [x, y]. \quad (9)$$

Since $L$ is a simple algebra, the adjoint representation is irreducible and according to the Burnside theorem

$$\dim \{M_{x_1} M_{x_2} \ldots M_{x_n} \mid n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in L\}_{\text{lin}} = k^k \quad (10)$$

The following theorem is a special case of the conjecture we mentioned in Section 1.1.

**Theorem 2.1.** Let $L$ be a simple finite dimensional Lie algebra and let

$$\Gamma : L = L_a \oplus L_b$$

be its grading into two nontrivial subspaces. Then $\Gamma$ is a $\mathbb{Z}_2$-grading.
Proof. Since $\Gamma$ is a grading, there exist letters $x, y, z \in \{a, b\}$ such that

$$[L_a, L_a] \subset L_x, \quad [L_a, L_b] \subset L_y, \quad \text{and} \quad [L_b, L_b] \subset L_z. \quad (11)$$

The property $\text{iii}$ guarantees that both letters $a$ and $b$ occur among $x, y, z$, i.e.,

$$\{a, b\} = \{x, y, z\} \quad (12)$$

According to occurrences of inclusions $[L_c, L_c] \subset L_c$, we will discuss separately three cases:

- There exists only one letter $c \in \{a, b\}$, such that $[L_c, L_c] \subset L_c$. Without loss of generality, we may assume $c = a$. Then necessarily for the second index $[L_b, L_b] \subset L_a$. Using (12), the relations (11) have the form

$$[L_a, L_a] \subset L_a, \quad [L_a, L_b] \subset L_b, \quad \text{and} \quad [L_b, L_b] \subset L_a$$

(which is equivalent to $[L_a, L_a] \subset L_a, \quad [L_a, L_b] \subset L_b$, and $[L_b, L_b] \subset L_a$).

If we identify $a$ with 0 and $b$ with 1 in the group $\mathbb{Z}_2$, we see, that our grading fulfills $[L_i, L_j] \subset L_{i+j}$, as desired.

- For both letters $c = a$ and $c = b$ we have $[L_c, L_c] \subset L_c$. Since the role of $a$, and $b$ is symmetric, we may write

$$[L_a, L_a] \subset L_a, \quad [L_a, L_b] \subset L_a, \quad \text{and} \quad [L_b, L_b] \subset L_b \quad (13)$$

Let us denote $\dim L_a = k_a$ and $\dim L_b = k_b$. Clearly $k_a + k_b = k$. Since $L = L_a \oplus L_b$, a basis of $L_a$ and a basis of $L_b$ form together a basis of $L$. Let us denote this basis $L$. Let us consider the form matrices of operators $M_i$ in this base. In fact we will identify $M_x$ with its matrix. The definition (9) and the relations (13) imply

$$M_x = \begin{pmatrix} M_{x1}^{11} & M_{x1}^{12} \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad x \in L_a \quad \text{and} \quad M_y = \begin{pmatrix} M_{y1}^{11} & 0 \\ 0 & M_{y2}^{12} \end{pmatrix} \quad \text{for} \quad y \in L_b$$

It means that $M_x$ and $M_y$ are both block upper triangular matrices. This contradicts (10).

- It remains to discuss the case

$$[L_a, L_a] \subset L_b, \quad [L_a, L_b] \subset L_b, \quad \text{and} \quad [L_b, L_b] \subset L_a \quad (14)$$

Now $M_x$ and $M_y$ have the form

$$M_x = \begin{pmatrix} 0 & M_{x1}^{21} \\ M_{x1}^{21} & 0 \end{pmatrix} \quad \text{for} \quad x \in L_a \quad \text{and} \quad M_y = \begin{pmatrix} 0 & M_{y1}^{21} \\ M_{y1}^{21} & 0 \end{pmatrix} \quad \text{for} \quad y \in L_b \quad (15)$$

We may assume $[L_a, L_a] \neq 0$, otherwise we may write $[L_a, L_a] = 0 \subset L_a$ and this case was already discussed. It means that there exists $x_0 \in L_a$ such that $M_{x0}^{21} \neq 0$. Denote the rank of $M_{x0}^{21}$ by $h \geq 1$. The bases of $L_a$ and $L_b$ were chosen arbitrarily. Now we may assume without loss of generality that we have chosen these bases in such a way that

$$M_{x0}^{21} = \begin{pmatrix} I_h & 0 \\ 0 & 0 \end{pmatrix},$$

where $I_h$ is the unit matrix of size $h \times h$, and the blocks of zeros complete the matrix to the size of $M_{x0}^{21}$, which is $k_b \times k_a$. 

Let us compute for any \( y \in L_b \) the commutator \([M_{x_0}, M_y]\). We obtain
\[
\begin{pmatrix}
0 & 0 \\
M_{x_0}^{21} & M_{x_0}^{22}
\end{pmatrix}
\begin{pmatrix}
0 & M_{y}^{12} \\
M_{y}^{21} & 0
\end{pmatrix}
-\begin{pmatrix}
0 & M_{y}^{12} \\
M_{y}^{21} & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
M_{x_0}^{21} & M_{x_0}^{22}
\end{pmatrix}
= \begin{pmatrix}
-M_{x_0}^{21}M_{x_0}^{22} & -M_y^{12}M_{y}^{22} \\
M_{x_0}^{22}M_{x_0}^{21} & M_{x_0}^{12}M_{y}^{21}
\end{pmatrix}
\]
Since \([x_0, y] \in L_b\), we have
\[
M_{x_0}^{21}M_{y}^{12} = 0 = \begin{pmatrix}
I_h & 0 \\
0 & 0
\end{pmatrix}M_{y}^{12}
\]
It implies that for any \( y \in L_b \) the first \( h \) rows of the matrix \( M_y^{12} \) and therefore also of the matrix \( M_y \) contain only zeros. Taking into account the form of \( M_x \), we see that any element of the Lie algebra is represented by the matrix whose first row is zero and this contradicts (10).

3 Representations compatible with grading
3.1 Group gradings and automorphisms
The simplest way how to find a group grading of a Lie algebra is to decompose the vector space \( L \) into eigensubspaces of a diagonalizable automorphism \( g \in \text{Aut} L \), [9]. For any pair of its eigenvectors \( x_\lambda \) and \( x_\mu \) corresponding to eigenvalues \( \lambda \) and \( \mu \), respectively, we have
\[
g([x_\lambda, x_\mu]) = [g(x_\lambda), g(x_\mu)] = \lambda \mu [x_\lambda, x_\mu]).
\]
Thus the commutator \([x_\lambda, x_\mu]\) is either zero or an eigenvector corresponding to the eigenvalue \( \lambda \mu \). Let us denote by \( \sigma(g) \) the spectrum of the automorphism \( g \) and by \( L_\lambda \) the eigensubspace corresponding to \( \lambda \in \sigma(g) \). The decomposition
\[
\Gamma : L = \bigoplus_{\lambda \in \sigma(g)} L_\lambda
\]
is a group grading, where as a semigroup \( G \) one can use the multiplicative semigroup generated by the spectrum of \( g \). Clearly, if \( h \in \text{Aut} L \) then the decomposition of \( L \) into eigensubspaces of the automorphism \( hgh^{-1} \) is \( L = \bigoplus_{\lambda \in \sigma(g)} h(L_\lambda) \), i.e., the grading given by automorphisms \( g \) and \( hgh^{-1} \) are equivalent. Therefore, the automorphisms \( g \) and \( hgh^{-1} \) are called equivalent as well. Note however that different inequivalent automorphisms may give the same grading.

Similarly, if \( g_1, g_2, \ldots, g_r \) are mutually commuting automorphisms of \( L \), then the decomposition of \( L \) into common eigensubspaces of all these automorphisms is a group grading of \( L \). The suitable semigroup for indexing of this grading is \( G_1 \otimes G_2 \otimes \ldots \otimes G_r \), where \( G_i \) is the semigroups generated by the spectrum of \( g_i \).

Furthermore, for Lie algebras over the complex field \( \mathbb{C} \), any group grading can be obtained by the described procedure. Let us stress, that this is not the case for real Lie algebras.

3.2 Group grading determined by one automorphism
Let \( \Gamma \) be a grading of the form (16), i.e. obtained by decomposition of \( L \) into eigensubspaces of a single automorphism \( g \). We may assume that \( g \) has finite order, say \( g^k = Id \). For its spectrum we have
\[
\sigma(g) \subset \{ e^{i2\pi \ell/d} | \ell = 0, 1, 2, \ldots, k - 1 \} =: G.
\]

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It means that $\Gamma$ is a $G$-grading. Let us consider an irreducible $d$-dimensional representation $r$ of the Lie algebra $L$. Our aim is to discuss the question of compatibility of $r$ with the $G$-grading.

Let $R_g$ be a non-singular matrix in $\mathbb{C}^{d \times d}$ such that

$$r(g(x)) = R_g r(x) R_g^{-1} \quad \text{for all } x \in L. \quad (18)$$

As $g^k = Id$, the previous equality gives

$$r(x) = r(g^k(x)) = R_g^k r(x) R_g^{-k} \quad \text{or} \quad [R_g, r(x)] = 0 \quad \text{for all } x \in L.$$ 

Since the representation $r$ is irreducible, by Schur’s lemma $R_g^k = \alpha Id$, for some $\alpha \in \mathbb{C}$. Of course, any nonzero multiple of $R_g$ satisfies the relation (18) as well. Therefore without loss of generality, we may assume that

$$R_g^k = Id, \quad \text{where } k \text{ is the order of the automorphism } g. \quad (19)$$

This normalization guarantees that the spectrum of the matrix $R_g$ and the spectrum of the automorphism $g$ belong to the same group $G$. In particular, since $R_g^k$ is the identity, the matrix $R_g$ is diagonalizable. Let $V = \oplus_{\lambda \in G} V_\lambda$ denote the decomposition of the column space $\mathbb{C}^d$ into eigen-subspaces of the matrix $R_g$. We will show, that this decomposition is exactly the decomposition required in the definition 1.1.

Let us consider some $\mu \in \sigma(g)$ so that $g(x_\mu) = \mu x_\mu$ for all $x_\mu \in L_\mu$. From (18) for $x = x_\mu$, one can deduce

$$r(g(x_\mu)) R_g = r(\mu x_\mu) R_g = \mu r(x_\mu) R_g = R_g r(x_\mu)$$

and after multiplying by the column vector $v_\lambda \in V_\lambda$

$$\mu \lambda r(x_\mu)v_\lambda = R_g r(x_\mu)v_\lambda.$$ 

The last equality means that the column $r(x_\mu)v_\lambda$ is either zero, or it is an eigenvector of the matrix $R_g$ corresponding to the eigenvalue $\mu \lambda$. Therefore

$$r(x_\mu)V_\lambda \subset V_{\mu \lambda} \quad \text{for any } \lambda, \mu \in G \quad \text{and any } x_\mu \in L_\mu.$$ 

This is the relation (3), just written in the multiplicative form. Of course, our multiplicative group $G$ defined in (17) is isomorphic with the additive group $\mathbb{Z}_k$.

We have seen, that the matrix $R_g$ with the properties (18) and (19) guarantees the compatibility of the grading of $L$ with the representation of the Lie algebra $L$. Such matrix $R_g$ will be called simulation matrix of the automorphism $g$. The matrix $R_g$ depends on the chosen automorphism $g$ and on the chosen representation $r$. The idea how to find the simulation matrix is more straightforward if $g \in \text{Aut } L$ is an inner automorphism. In this case it is natural to search for $R_g$ among matrices in the representation of the corresponding Lie group. This idea was already presented in [15] and [17], where $R_g$ was a representation of power of an element of finite order [13]. Nevertheless, we show that it is possible to find the simulation matrix $R_g$ even for an outer automorphism $g$ as well. In the sequel, we will concentrate on the Lie algebras $sl(n, \mathbb{C})$. The reason is, that these algebras (with the exception of $o(8, \mathbb{C})$) are the only simple classical Lie algebras for which the group of automorphisms contains an outer automorphism as well.
4 Representations of sl\((n, \mathbb{C})\) compatible with \(\mathbb{Z}_2\)-grading

According to Theorem [22] any graded decomposition of \(sl(n, \mathbb{C})\) into two parts is a \(\mathbb{Z}_2\)-grading. Any such grading is uniquely related by an automorphism of order 2. We will identify the Lie algebra \(sl(n, \mathbb{C})\) with \(\{X \in \mathbb{C}^{n \times n} \mid \text{tr}X = 0\}\).

Let us first recall the structure of \(\text{Aut} \, sl(n, \mathbb{C})\) as described in [11]:

- for any inner automorphism \(g\) there exists a matrix \(A \in SL(n, \mathbb{C}) := \{A \in \mathbb{C}^{n \times n} \mid \det A = 1\}\) such that
  \[ g(X) = Ad_AX = AXA^{-1} \quad \text{for any } X \in sl(n, \mathbb{C}); \]
- the mapping given by the prescription
  \[ Out_tX := -X^T \quad \text{for any } X \in sl(n, \mathbb{C}) \]
is an outer automorphism of order 2;
- any outer automorphism \(g\) is a composition of an inner automorphism and the automorphism \(Out_t\).

The second ingredient for construction of simulating matrices of automorphisms is the knowledge of finitedimensional irreducible representations of \(sl(n, \mathbb{C})\). These representations are well described by Gel’fand-Tsetlin formalism [8, 14, 3]. Any irreducible representation \(r\) of \(sl(n, \mathbb{C})\) is in one-to-one correspondence with an \(n\)-tuple \((m_{1,n}, m_{2,n}, \ldots, m_{n,n})\) of non-negative integer parameters \(m_{1,n} \geq m_{2,n} \geq \ldots \geq m_{n,n} = 0\). The dimension of the representation space of \(r = r(m_{1,n}, m_{2,n}, \ldots, m_{n,n})\) is given by the number of triangular patterns

\[
\mathbf{m} = \begin{pmatrix}
m_{1,n} & m_{2,n} & m_{3,n} & \cdots & m_{n,n} \\
m_{1,n-1} & m_{2,n-1} & m_{3,n-1} & \cdots & m_{n-1,n-1} \\
m_{1,n-2} & m_{2,n-2} & \cdots & m_{n-2,n-2} & \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
m_{1,2} & m_{2,2} & \cdots & \cdots & m_{1,1}
\end{pmatrix}
\]

in which the numbers \(m_{i,j} \in \mathbb{Z}\) satisfy \(m_{i,j+1} \geq m_{i,j} \geq m_{i+1,j+1}\) for all \(1 \leq i \leq j \leq n - 1\). To any such pattern \(\mathbf{m}\), we assign the basis vector \(\xi(\mathbf{m})\). The representation \(r\) is fully determined by the action \(r(E_{k\ell})\) on all basis vectors \(\xi(\mathbf{m})\) for any \(k, \ell = 1, 2, \ldots, n\). (We have adopted notation \(E_{k\ell}\) for the matrices \(n \times n\) with elements \((E_{k\ell})_{ij} = \delta_{ik}\delta_{ij}\).) This action can be found e.g. in [8]. For reader convenience the representation is described in Appendix.

4.1 Inner automorphisms of order two

Any inner automorphism \(g\) of order two is associated by equality \(g = Ad_A\) with a group element \(A \in SL(n, \mathbb{C})\) such that \(A\) does not belong to the center \(Z[SL(n, \mathbb{C})]\) and \(A^2\) belongs to the center. If we denote \(\omega = e^{i\pi}\), then the center can be written explicitly \(Z[SL(n, \mathbb{C})] = \{\omega^{2\ell}I_n \mid \ell = 0, 1, \ldots, n - 1\}\). A simple calculation shows that any such element \(A \in SL(n, \mathbb{C})\) is up to equivalence one of the matrices

\[
A_{n,s} := \omega^{\eta(s)} \begin{pmatrix} I_{n-s} & 0 \\ 0 & -I_s \end{pmatrix} \quad \text{where } s = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \eta(s) = \begin{cases} 0 & \text{if } s \text{ is even} \\ 1 & \text{if } s \text{ is odd} \end{cases}
\]
These matrices may be rewritten by using elements of the Lie algebra $sl(n, \mathbb{C})$ as follows:

$$A_{n,s} = \exp(X_{n,s}) \quad \text{with} \quad X_{n,s} = i\pi \left( \frac{n(s)}{n} I_{n-s} 0 \begin{array}{c} 0 \end{array} \frac{n(s)}{n} I_{s} + M_{s} \right),$$

where $M_{s} = \text{diag}(-1, -1, \ldots, (-1)^{s}) \in \mathbb{C}^{s \times s}$. One can use notation of $E_{kk}$ and write

$$X_{n,s} = i\pi \left( \frac{n(s)}{n} \sum_{k=1}^{n} E_{kk} + \sum_{k=n-s+1}^{n} (-1)^{n-s+1-k} E_{kk} \right).$$

If $r$ is any representation of the Lie algebra $sl(n, \mathbb{C})$, then $R_{A_{n,s}} := \exp(r(X_{n,s}))$ satisfies

$$R_{A_{n,s}} r(X)(R_{A_{n,s}})^{-1} = r(A_{n,s}XA_{n,s}^{-1}) = r(Ad_{A_{n,s}}X) \quad \text{and} \quad (R_{A_{n,s}})^2 = Id.$$

Therefore, the matrix $R_{A_{n,s}} := \exp(r(X_{n,s}))$ is the simulation matrix of the inner automorphism $g = Ad_{A_{n,s}}$. We have shown

Theorem 4.1. Any $\mathbb{Z}_{2}$-grading of the Lie algebra $sl(n, \mathbb{C})$ obtained by an inner automorphism and any irreducible representation of $sl(n, \mathbb{C})$ are compatible.

Using (20) and the explicit form of the Gel’fand-Tseitlin representation we obtain for any basis vector $\xi(\mathbf{m})$

$$r(X_{n,s})\xi(\mathbf{m}) = i\pi \left( \frac{n(s)}{n} r_{n}(\mathbf{m}) + 2 \sum_{k=1}^{s-1} (-1)^{k-1} r_{n-s+k}(\mathbf{m}) - r_{n-s}(\mathbf{m}) - (-1)^{n(s)} r_{n}(\mathbf{m}) \right) \xi(\mathbf{m}).$$

So we arrived at the explicit form of the simulation matrix of the automorphism $g = Ad_{A_{n,s}}$

$$R_{A_{n,s}}\xi(\mathbf{m}) = e^{i\pi \left( (\frac{n(s)}{n} - 1) r_{n}(\mathbf{m}) - r_{n-s}(\mathbf{m}) \right) \xi(\mathbf{m})}.$$

4.2 Outer automorphisms of order two

Any outer automorphism of order two on $sl(n, \mathbb{C})$ is up to equivalence the automorphism $Out_{1}(X) = -X^{T}$, and thus we will focus only on it.

It is well known, that for an irreducible representation $r$ characterized in the Gel’fand-Tseitlin formalism by the $n$-tuple $(m_{1,1}, m_{2,1}, \ldots, m_{n,n})$, the mapping $-r^{T}$ is an irreducible representation as well. This representation is equivalent to the so called contragradient representation $r^{c}$ which is characterized by the $n$-tuple $(m'_{1,1}, m'_{2,1}, \ldots, m'_{n,n})$, where

$$m'_{i,n} = m_{i,1} - m_{n-i+1,n} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Let us consider a triangular pattern $\mathbf{m}$ filled by indices $m_{i,j}, 1 \leq i \leq j \leq n$, and associated with the basis vector $\xi(\mathbf{m})$ of the representation $r$. To any such pattern $\mathbf{m}$, we may assign the unique triangular pattern $\mathbf{m}'$ with indices $m'_{i,j} := m_{i,1} - m_{j-i+1,j}$. It is easy to check that $m'_{i,j}$ satisfies the necessary inequalities for $\mathbf{m}'$ to be a correct pattern of the contragradient representation $r^{c}$. Let us define the linear mapping $J$ of the representation space of $r$ onto the representation space of $r^{c}$ by

$$J \xi(\mathbf{m}) := (-1)^{\sum_{i} m_{i,1}} \xi(\mathbf{m}').$$
Now from the formulae in the Appendix one sees that
\[
(r(E_{ij}))^T = r(E_{ji}) = r(E_{ij}^T).
\] (21)

Using this fact one can prove by the direct verification that the mapping \(J\) satisfies
\[
-J \cdot r^T(X) = r^c(X) \cdot J \quad \text{for any } X \in \mathfrak{sl}(n, \mathbb{C}).
\] (22)

Let us return to our original task. We are looking for the simulation matrix of the automorphism \(g = \text{Out}_I\), i.e., we are looking for a matrix \(R_g\) of order two such that
\[
r(\text{Out}_I(X)) = -r(X^T) = R_g r(X) R_g^{-1}.
\]

According to \((21)\), we have \(r(X^T) = (r(X))^T\) and therefore the existence of the simulation matrix \(R_g\) means equivalence of the representations \(r\) and \(-r^T\), i.e. equivalence of \(r\) and its contragradient representation \(r_c\). The Gel’fand-Tseitlin result says that it is possible if and only if \(n\)-tuples \((m_1,n,m_2,n,\ldots,m_n,n)\) and \((m_1',n,m_2',n,\ldots,m_n',n)\) coincide. In this case the simulation matrix \(R_g\) is equal to \(J\). We have deduced

**Theorem 4.2.** A \(\mathbb{Z}_2\)-grading of the Lie algebra \(\mathfrak{sl}(n, \mathbb{C})\) obtained by an outer automorphism is compatible with an irreducible representation \(r\) of \(\mathfrak{sl}(n, \mathbb{C})\) if and only if the representation is self-contragradient.

If we do not insist on the irreducibility of the representation \(r\), the class of representations compatible with \(\mathbb{Z}_2\)-grading obtained by the automorphism \(\text{Out}_I\) is larger. Of course, if for a representation \(r_1\) it is possible to find a simulation matrix \(R^{(1)}\) and for a representation \(r_2\) a simulation matrix \(R^{(2)}\), then the direct sum \(R^{(1)} \oplus R^{(2)}\) is the simulation matrix for the direct sum \(r_1 \oplus r_2\). To avoid the discussion of all such obvious cases, we will describe only those representations \(r\) with simulation matrices \(R\), for which the operator set \(\{R\} \cup \{r(X) \mid X \in \mathfrak{sl}(n, \mathbb{C})\}\) is irreducible, whereas the set \(\{r(X) \mid X \in \mathfrak{sl}(n, \mathbb{C})\}\) is reducible.

If \(r_0\) is a \(d\)-dimensional irreducible representation of \(\mathfrak{sl}(n, \mathbb{C})\) then the \(2d\)-dimensional representation \(r := r_0 \oplus (-r_0^T)\) assigns to \(X\) the matrix
\[
r(X) = \begin{pmatrix} r_0(X) & 0 \\ 0 & -(r_0(X))^T \end{pmatrix}
\]
and therefore
\[
r(\text{Out}_I(X)) = \begin{pmatrix} -(r_0(X))^T & 0 \\ 0 & r_0(X) \end{pmatrix} = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} r(X) \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}.
\]

The matrix \(\begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}\) is the simulation matrix of \(\text{Out}_I\). It is easy to see that the simulation matrix together with all \(r(X)\) form an irreducible set.

**4.3 \(\mathbb{Z}_2\)-grading of \(\mathfrak{sl}(3, \mathbb{C})\)**

Let us illustrate the conclusions of two previous sections on the Lie algebra \(\mathfrak{sl}(3, \mathbb{C})\). On this algebra there exists only two inequivalent automorphisms of order two. In our notation \(g_1 = \text{Ad}_{A_{31}}\) with
\[
A_{31} = \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{where } \omega = e^{i\pi}
\]
and $g_2 = \text{Out}_l$. The corresponding $\mathbb{Z}_2$-gradings are

$$\Gamma_1 : sl(3, \mathbb{C}) = \{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -a-d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \} \oplus \{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & 0 \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \},$$

$$\Gamma_2 : sl(3, \mathbb{C}) = \{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{C} \} \oplus \{ \begin{pmatrix} a & b & c \\ d & e & 0 \\ c & e & -a-d \end{pmatrix} \mid a, b, c, d, e \in \mathbb{C} \}.$$

The first grading $\Gamma_1$ is compatible with any irreducible representation. The simulation matrix $R_{g_1}$ of the automorphism $g_1 = Ad_{A_3}$ acts on the Gel’fand-Tseitlin triangular patterns as follows

$$R_{g_1} \left( \begin{array}{ccc} m_{13} & m_{23} & 0 \\ m_{12} & m_{22} & \end{array} \right) = e^{-\frac{i\pi}{2} (m_{13}+m_{23})} e^{-i\pi (m_{12}+m_{22})} \left( \begin{array}{ccc} m_{13} & m_{23} & 0 \\ m_{12} & m_{22} & \end{array} \right).$$

The irreducible representations compatible with the second grading are only self-contragradient representations, i.e., representations $r = r(2\ell, \ell, 0)$. In such representation, the operator $J$ is defined by

$$J \left( \begin{array}{ccc} 2\ell & \ell & 0 \\ m_{12} & m_{22} & \end{array} \right) = (-1)^{\ell+m_{12}+m_{22}+m_{11}} \left( \begin{array}{ccc} 2\ell & \ell & 0 \\ 2\ell-m_{22} & 2\ell-m_{12} & \end{array} \right).$$

The lowest-dimensional non-trivial self-contragradient representation is $r = r(2, 1, 0)$. Its dimension is 8 and has the following explicit form on the basis vectors:

$$R_{g_2} \left( \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 1 & \end{array} \right) = \left( \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & \end{array} \right), \quad R_{g_2} \left( \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & \end{array} \right) = \left( \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 1 & \end{array} \right), \quad R_{g_2} \left( \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 1 & \end{array} \right) = - \left( \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 1 & \end{array} \right),$$

$$R_{g_2} \left( \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & \end{array} \right) = - \left( \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 0 & \end{array} \right), \quad R_{g_2} \left( \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & \end{array} \right) = - \left( \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 0 & \end{array} \right), \quad R_{g_2} \left( \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 1 & \end{array} \right) = \left( \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 1 & \end{array} \right).$$

If the representation $r = r(m_{13}, m_{23}, 0)$ is not self-contragradient, then the grading $\Gamma_2$ is compatible with the reducible representation

$$r \otimes (X) := \left( \begin{array}{ccc} r(X) & 0 \\ 0 & -(r(X))^{T} \end{array} \right)$$

and the corresponding simulation matrix on the double-dimensional space is $J = \sigma_1 \otimes I$, where $I$ is the identity operator on the representation space of representation $r$. 
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Appendix

Let us give explicit description of irreducible representation of $sl(n, \mathbb{C})$ in Gel’fand Tseitlin formalism. Since any $E_{k\ell}$ can be obtained by commutation relations from $E_{k,k}, E_{k,k-1}$ and $E_{k-1,k}$, only formulas for $r(E_{k,k}), r(E_{k,k-1})$ and $r(E_{k-1,k})$ are needed.

$$r(E_{k,k})\xi(m) = (r_k - r_{k-1})\xi(m),$$

where $r_k = m_{1,k} + \ldots + m_{k,k}$ for $k = 1, 2, \ldots, n$ and $r_0 = 0$,

$$r(E_{k-1,k})\xi(m) = a_{k-1}^1 \xi(m_{k-1}^1) + \ldots + a_{k-1}^{k-1} \xi(m_{k-1}^{k-1}),$$

where $m_{k-1}^1$ denotes the triangular pattern obtained from $m$ replacing $m_{j,k-1}$ by $m_{j,k-1} - 1$,

$$a_{k-1}^j = \left[-\frac{\prod_{i=1}^{k} (m_{i,k} - m_{j,k-1} - i + j + 1) \prod_{i \neq j}^{k-2} (m_{i,k-1} - m_{j,k-1} - i + j + 1)(m_{i,k-1} - m_{j,k-1} - i + j)}{\prod_{i \neq j}^{k-1} (m_{i,k-1} - m_{j,k-1} - i + j + 1)(m_{i,k-1} - m_{j,k-1} - i + j)}\right]^{1/2}$$

and

$$r(E_{k-1,k})\xi(m) = b_{k-1}^1 \xi(m_{k-1}^1) + \ldots + b_{k-1}^k \xi(m_{k-1}^k),$$

where $m_{k-1}^1$ denotes the triangular pattern obtained from $m$ replacing $m_{j,k-1}$ by $m_{j,k-1} + 1$,

$$b_{k-1}^j = \left[-\frac{\prod_{i=1}^{k} (m_{i,k} - m_{j,k-1} - i + j) \prod_{i \neq j}^{k-2} (m_{i,k} - m_{j,k-1} - i + j - 1)(m_{i,k} - m_{j,k-1} - i + j)}{\prod_{i \neq j}^{k-1} (m_{i,k} - m_{j,k-1} - i + j + 1)(m_{i,k} - m_{j,k-1} - i + j)}\right]^{1/2}$$

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