The Point Interpolation Method Using Quadratically Consistent Three-Point Integration Scheme for Free Vibration Analysis

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Abstract. The formulation of quadratically consistent three-point integration scheme based on point interpolant method (PIM-QC3) is proposed for free vibration analysis of two-dimensional solids. Based on the condition with regard to nodal shape function and its derivatives for arbitrary-order approximations, PIM-QC3 is developed using point interpolant method and T6-Scheme for node selection. Thus, complex calculation of inverse matrix at every quadrature points is avoided. And the shape functions including corrected derivatives of nodal shape functions at quadrature points, adopt quadratic basis, meet the so-called differentiation of the approximation consistency as well as the discrete divergence consistency. In addition, the Kronecker delta property is possessed. Some problems for the free vibration of two-dimensional solids are employed to examine the accuracy and efficiency of the PIM-QC3. The numerical results indicate that PIM-QC3 is a very accurate, stable and highly efficient method for the structural analysis of free vibration.

Keywords: Quadratically consistent, Corrected derivatives, Free vibration.

1. Introduction
Node based meshfree method has made great progress over the last twenty years. Various types of meshfree methods have been developed, such as element-free Galerkin (EFG) method [1], the reproducing kernel particle method (RKPM) [2], the point interpolation method (PIM) [3], meshless local Petrov-Galerkin method (MLPG) [4] and so on. Among these meshfree methods, the element-free Galerkin (EFG) method utilizing moving least-squares (MLS) [1] and the reproducing kernel particle method [2] were widely studied.

Meshfree methods have some distinct advantages over FEM. Only nodal information is needed in constructing meshfree approximants, avoiding large meshing work in discret ing domain, and so on. However, there are two adverse effects need to be dealt with. The first is the difficultly in enforcing essential boundary conditions for most meshfree approximants. As absence of Kronecker delta property of MLS approximation, some approaches, such as the penalty method [5], Lagrange multipliers method [6] are proposed to address the issue.

The second and maybe more important issue is relatively low computational efficiency of meshfree methods. High-order Gauss quadrature is necessary for integrating the Galerkin weak form for EFG. To overcome this disadvantage, lots of efficient approaches were proposed. Dyka [7] developed the stress-
point method to overcome the problem of tension instability in SPH methods and increase accuracy. Beissel and Belytschko [8] first presented the thought of nodal integration, which performs integration only at the nodes. However, the value of the introduced parameter in this approach usually needs numerical experiments. Nagashima [9] presented a nodal integration approach by employing a Taylor’s expansion of displacement fields and stabilization terms introduced do not have artificial parameter. Nevertheless, for the EFG with quadratic basis its stabilization effect is poor.

The stabilized conforming nodal integration (SCNI) method developed by Chen et al [10] is widely applied in meshfree fields. The proposed SCNI can pass linear patch while Gauss integration method fails. And greatly improved the accuracy and convergent rates are obtained. However, the SCNI fails to pass quadratic patch test and Puso et al. [11] pointed that at the domain boundaries saw-tooth mode still may present for SCNI. Duan et al. [12] presented the condition with regard to nodal shape function and its derivatives for meshfree method according to the divergence theorem. The EFG with QC3 scheme, which could obtain quadratic consistency, was developed. Great improvements are acquired in accuracy and convergence by the EFG with QC3 and it is more efficient than SCNI. Nevertheless, as the EFG with QC3 employs moving least-squares approximation, corrected nodal derivatives at each quadrature point still require inverse computation of matrix $A$. This will consume some CPU time. And also, EFG with QC3 could not enforce the essential boundary conditions directly.

This paper aims at developing an accurate, efficient and quadratically consistent meshfree method for structural analysis of free vibration. Based on the discrete divergence consistency (DDC), three-point integration scheme using point interpolation method with quadratic approximation is derived. The essential boundary conditions can be imposed easily like FEM. A series of free vibration problems are analyzed to show the validity and efficiency of the proposed method.

2. Discretized Equations for Free Vibration Analysis based on Point Interpolant Method (PIM)

The displacement functions $u(x)$ of any point $x$ in the domain can be expressed as

$$u(x) = \sum_{i=1}^{n} p_i(x) a_i = \hat{p}(x) a$$  \hspace{1cm} (1)$$

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where $p_i(x)$ is polynomial basis function of $x = [x, y]^T$, $a_i$ is the unknown interpolation coefficient. The vector of interpolation coefficients $a$ is written as

$$a = [a_1, a_2, \ldots, a_n]^T$$ \hspace{1cm} (2)$$

The $p(x)$ is a basis function vector, and the complete quadratic polynomial basis is expressed as,

$$p(x) = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}^T$$ \hspace{1cm} (3)$$

To obtain unknown the coefficients, the displacement of node $i$ inside the local support domain is considered to be equal to the displacement function. Then the displacement of node $i$ can be written as

$$u_i = \hat{p}(x_i) a$$ \hspace{1cm} (4)$$
Where $u_i$ is the displacement value of node $i$ at coordinates $x = x_i$. Considering the displacements of $n$ nodes, we derive

$$U = P_a a$$

(5)

Where $U = [u_1 \ u_2 \ \cdots \ u_n]^T$, $P_a = [p(x_1) \ p(x_2) \ \cdots \ p(x_n)]$

Solving Equation (5), the coefficient vector $a$ can be determined

$$a = P_n^{-1} U$$

(6)

Substituting Equation (6) back into Equation (1) yields

$$u(x) = p^T(x) P_n^{-1} U_j = \sum_{i=1}^{n} \varphi_i u_i = \Phi(x) U$$

(7)

Where $\Phi(x)$ is the vector of PIM shape functions,

$$\Phi(x) = [\varphi_1(x) \ \varphi_2(x) \ \cdots \ \varphi_n(x)]$$

(8)

Quadratic basis is selected in this paper. And T6-Scheme [13] is employed for node selection in constructing approximation of an interested point. The procedure of establishing the PIM shape function is straightforward and addresses the singularity issue of moment matrix for special cases. Also, PIM shape functions generated using quadratic basis are quadratic consistent [13]. In addition, it is easy to enforce the prescribed nodal essential boundary conditions since the shape functions possess Kronecker property.

By employing the standard Galerkin procedure, the eigenvalue problem associated with the free vibration analysis is obtained as

$$K\phi - \omega^2 M\phi = 0$$

(9)

$$K_{ij} = \int_{\Omega} B_i^T D B_j d\Omega, \ B_i = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \end{bmatrix}$$

(10)

$$M_{ij} = \int_{\Omega} N_i^T \rho N_j d\Omega, \ N_i = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}$$

(11)

Where $K$, $M$ and $D$ are the stiffness matrix, mass matrix and the matrix of material constants respectively. $\omega$ represents circular or natural frequency and $\phi$ refers to the eigenvector. And the natural frequency $\omega$ can be determined by solving Equation (9).

3. Quadratically Consistent Three-point Integration Scheme Based on Point Interpolant Method (PIM-QC3)

Based on the consistency requirements for the shape functions and their derivatives presented in [12], both DAC and DDC requirements should be fulfilled, which denote the differentiation of the
approximation consistency as well as discrete divergence consistency separately. The former can be written as

$$p_j(x) = \sum_{i} p(x_i) N_{i,j}(x)$$  \hspace{2cm} (12)

As PIM shape functions possess quadratic consistency with regard to the basic functions \( p(x) = [1 \ x \ y \ x^2 \ xy \ y^2]^T \) \([3][13]\), the quadratic DAC requirement could be satisfied like MLS shape functions \([12]\). And virtually, the DDC requirement explains the condition about the meshfree shape function and corresponding derivatives, whose weak form is given by

$$\int_{\Omega_s} N_{i,s}(x) q(x) d\Omega = \int_{\Gamma_s} N_{i}(x) q(x) n_i d\Gamma - \int_{\Omega_b} N_{i}(x) q_{,i}(x) d\Omega$$  \hspace{2cm} (13)

Where \( \Omega_s \) is background triangle cell bounded by \( \Gamma_s \), \( n_i \) is the unit outward normal on the boundary \( \Gamma_s \), \( q(x) = p_{,i}(x) \cup p_{,j}(x) \). For the quadratic basis functions \( p(x) = [1 \ x \ y \ x^2 \ xy \ y^2]^T \), corresponding \( q(x) = p_{,i}(x) \cup p_{,j}(x) = [1 \ x \ y]^T \).

To solve Equation (13) numerically, six nodes quadratically consistent three-point (QC3) integration scheme on PIM was proposed, as illustrated in Figure 1. In the QC3 scheme, there are three integral points per background triangle cell. The six dark nodes \( i_1-i_6 \) for cell \( i \), which are utilized for establishing displacement field function, are support nodes based on T6-Scheme: three nodes which connect the cell \( i \) and the other nodes that are distant vertexes of three adjacent cells. The three blue crosses denote integral points for the domain integrations in the background triangle cell. The red dots on the edges of the cell \( i \) denote the boundary integration points of one-dimension Gauss integration. Thus, Equation (13) could be subdivided into three equations and from which the derivatives at corresponding integral points can be obtained.

The Equation (13) can be discretized as

$$\sum_{H=1}^{3} W_H \tilde{N}_{i,j}(x_H) q(x_H) = \sum_{L=1}^{3} \sum_{G=1}^{3} N_{i,L}(x_G) q(x_G) n_i^L W_G - \sum_{H=1}^{3} W_H N_{i}(x_H)q_{,i}(x_H)$$  \hspace{2cm} (14)
To be distinguished from the classical shape function derivative \( \frac{\partial N_j}{\partial x} \), \( \tilde{N}_j(x) \) is used to denote corrected derivative. \( W_H \) is the domain integration weight of integration point \( x_H \), and \( W_G \) is boundary integration weight of integration point \( x_G \). Taking \( x \)-derivatives for an example, and substituting \( q(x) = \begin{bmatrix} 1 & x & y \end{bmatrix}^T \), \( q_{,x}(x) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \) and \( q_{,y}(x) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \) into Equation (14), we have

\[
W d_x = f_x
\]  

Where

\[
W = \begin{bmatrix} W'_{1,1} & W'_{1,2} & W'_{1,3} \\
W'_{2,1} & W'_{2,2} & W'_{2,3} \\
W'_{3,1} & W'_{3,2} & W'_{3,3} \end{bmatrix} \quad d_x = \begin{bmatrix} \tilde{N}_{J,1}(x_1) \\
\tilde{N}_{J,2}(x_2) \\
\tilde{N}_{J,3}(x_3) \end{bmatrix} 
\]  

\[
f_x = \begin{bmatrix} \sum_{L=1}^{3} \sum_{G=1}^{2} N_j(x_G) n^L_x n^L_w G \\
\sum_{L=1}^{3} \sum_{G=1}^{2} N_j(x_G) x_G n^L_x n^L_w G - \sum_{H=1}^{3} W_H N_j(x_H) \\
\sum_{L=1}^{3} \sum_{G=1}^{2} N_j(x_G) y_G n^L_y n^L_w G \end{bmatrix}
\]  

By solving Equation (15), the corrected \( \tilde{N}_{J,1}(x) \) at three integral points are obtained. The \( y \)-derivatives \( \tilde{N}_{J,2}(x) \) can be obtained in the same way.

Utilizing the calculated corrected derivatives, the stiffness matrix of PIM-QC3 in Equation (10) can be written as

\[
K_{ij} = \int_\Omega B_j^T D B_j d\Omega = \sum_{H=1}^{3} W_H \bar{B}^T \bar{B}\bar{B}^T \bar{B} \bar{B}^T (x_H) 
\]  

Where

\[
\bar{B}_j = \begin{bmatrix} \tilde{N}_{J,1} & 0 & \tilde{N}_{J,2} \\
0 & \tilde{N}_{J,1} & \tilde{N}_{J,2} \end{bmatrix} 
\]  

As \( p(x) = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}^T \) adopted in determining the nodal derivatives are a quadratic basis, the PIM-QC3 integration scheme meets the quadratic DDC. Meanwhile the PIM-QC3 integration scheme can also fulfill the requirement of quadratic DAC [12]. And if only one integration point is employed at the center of each cell, and \( q(x) = \begin{bmatrix} 1 \end{bmatrix}^T \), then the discrete version of Equation (13) will be simplified into one equation as follows

\[
\tilde{N}_{J,1}(x) = \frac{1}{A_\Omega} \sum_{L=1}^{3} \sum_{G=1}^{2} N_j(x_G) n^L_x n^L_w G
\]
Where $A$ and $\chi_c$ are the cell area and cell center respectively. This only meets the linear DDC requirement, like SCNI [10].

4. Numerical Examples

The proposed QC3 integration scheme is denoted as PIM-QC3, one quadrature point per cell for linear DDC is called PIM-LC1, QC3 integration scheme based on MLS approximation [12] is denoted as EFG-QC3. While LFEM refers to linear finite element method, using 1 and 3 integration points in the standard triangle (ST) integration which is adopted in point interpolant method with classical shape function derivatives, are denoted as PIM-ST1 and PIM-ST3 respectively.

In this section, PIM-QC3 is applied for free vibration analysis of two-dimensional structures. The standard international unit system is adopted in this paper except specially mentioned.

4.1. Free vibration analysis of a variable cross-section beam

The beam with a variable cross-section shown in Figure 2 is used for free vibration analysis. The parameters needed in the computation are taken as $E = 3 \times 10^7$, $\nu = 0.3$, thickness $t = 1$ and the mass density $\rho = 1$. Plane stress condition is considered.

The problem domain is modelled by regular and irregular nodal distributions respectively, with 45 nodes ($8 \times 8$ triangle cells), as plotted in Figure 2.

![Figure 2. A cantilever beam with variable cross-sections.](image)

The first ten natural frequencies are listed in Tables 1 and 2, and relative frequency errors are plotted in Figure 3 for regular and irregular nodal distributions respectively. The reference solutions for this problem are taken as the numerical results computed by FEM Quad 4 element with a very dense mesh ($40 \times 80$). From Tables 1 and 2, it can be seen that the frequencies given by the PIM-QC3 agree very well with reference solutions and the PIM-QC3 performs best among all the methods for regular distributions as well as irregular nodal distributions. Moreover, as shown in Figure 3 and Figure 4, PIM-QC3 still shows the high accuracy for irregular distributed nodes, while other methods especially for PIM-ST1 and PIM-LC1 produce poor results.
Table 1. Computed frequencies (Hz) of the variable cross-section cantilever beam for a regular nodal distribution with 45 nodes.

| Mode | PIM-QC3 | PIM-LC1 | PIM-ST3 | PIM-ST1 | LFEM | Reference |
|------|---------|---------|---------|---------|------|-----------|
| 1    | 53.47   | 52.56   | 52.78   | 52.33   | 58.26| 53.6      |
| 2    | 165.69  | 161.75  | 158.72  | 161.34  | 180.5| 168.17    |
| 3    | 178.24  | 177.54  | 176.69  | 176.96  | 185.75| 179.97    |
| 4    | 318.09  | 305.3   | 291.39  | 301.87  | 371.62| 326.87    |
| 5    | 445.27  | 433.12  | 398.73  | 410.95  | 473.53| 461.89    |
| 6    | 471.7   | 446.09  | 433.91  | 432.85  | 562.38| 491.95    |
| 7    | 554.36  | 525.53  | 472.22  | 435.57  | 636.31| 589.03    |
| 8    | 628.44  | 574.43  | 495.32  | 497.97  | 741.74| 668.91    |
| 9    | 636.15  | 595.8   | 516.39  | 514.69  | 782.96| 675.35    |
| 10   | 684.08  | 625.62  | 565.88  | 544.38  | 825.82| 727.81    |

Table 2. Computed frequencies (Hz) of the variable cross-section cantilever beam for an irregular nodal distribution with 45 nodes.

| Mode | PIM-QC3 | PIM-LC1 | PIM-ST3 | PIM-ST1 | LFEM | Reference |
|------|---------|---------|---------|---------|------|-----------|
| 1    | 53.43   | 52.27   | 51.5    | 49.77   | 58.35| 53.6      |
| 2    | 166.2   | 160.3   | 156.97  | 156.67  | 180.69| 168.17    |
| 3    | 178.08  | 176.53  | 166.34  | 164.59  | 187.44| 179.97    |
| 4    | 319     | 299.21  | 294.08  | 295.52  | 372.6 | 326.87    |
| 5    | 443.8   | 404.37  | 400.5   | 366.35  | 473.71| 461.89    |
| 6    | 468.61  | 413.62  | 419.27  | 379.2   | 568.85| 491.95    |
| 7    | 552.71  | 426.38  | 504.38  | 401.62  | 638.56| 589.03    |
| 8    | 636.21  | 492.77  | 551.02  | 433.92  | 744.66| 668.91    |
| 9    | 643.71  | 511.45  | 572.49  | 459.42  | 794.88| 675.35    |
| 10   | 683.1   | 529.05  | 607.39  | 477     | 831.84| 727.81    |

Figure 3. Relative frequency error (a) regular nodal distributions (b) irregular nodal distributions.

4.2. Free vibration analysis of a cantilever beam

In this example, a cantilever beam is considered as shown in Figure 4. The parameters are listed as follows: height $D=10\text{mm}$, length $L=100\text{mm}$, Young’s module $E=2.1\times 10^4\text{kgf/mm}^2$, Poisson ratio $\nu=0.3$, thickness $t=1\text{mm}$, mass density $\rho=8.0\times 10^{-6}\text{kgf s}^2/\text{mm}^4$. Also, the plane stress condition is assumed.
Three types of regular nodal distributions with 63 nodes (4×20 cells), 124 nodes (6×30 cells) and 205 nodes (8×40 cells) as shown in Figure 5 are adopted to compute frequencies respectively, and the corresponding results of first 8 natural frequencies are listed in Table 3-5 separately. From Tables 3-5, it can be seen that the results obtained by PIM-QC3 agree very well with the reference values. Even for the coarse nodal distributions, the presented PIM-QC3 still gives the more accurate frequencies which are much closer to the reference values compared with the other methods. And the relative errors of natural frequencies of the first two modes for three kinds of regular nodal distributions are plotted in Figure 6. It is obviously shown that the computed frequencies given by PIM-QC3 are far more accurate than PIM-LC1, PIM-ST3, PIM-ST1 and LFEM.

![Figure 4](image1.png)

**Figure 4.** Free vibration analysis of a cantilever beam.

![Figure 5](image2.png)

**Figure 5.** Three regular nodal distributions for domain discretization of the cantilever beam: (a) 63 nodes (b) 124 nodes (c) 205 nodes.

| Mode | PIM-QC3 | PIM-LC1 | PIM-ST3 | PIM-ST1 | LFEM | Reference [14] |
|------|---------|---------|---------|---------|------|----------------|
| 1    | 824.1   | 751.4   | 819     | 742.2   | 1119.3 | 823            |
| 2    | 4958.0  | 4544.3  | 4898.5  | 4486.9  | 6617.1 | 4937           |
| 3    | 12798.6 | 12035.8 | 12748   | 11846.1 | 12849.7 | 12824          |
| 4    | 13088.8 | 12793.4 | 12800.1 | 12769.3 | 17162  | 13005          |
| 5    | 23766.5 | 21925   | 22800.6 | 21553.2 | 30745.1 | 23632          |
| 6    | 36140.9 | 33310.7 | 33903.9 | 32598   | 38620.2 | 36040          |
| 7    | 38169.9 | 37982.9 | 37953.2 | 37866.1 | 46401.5 | 38442          |
| 8    | 49579.6 | 45528.6 | 45199.8 | 44323.6 | 63385.9 | 49616          |
Table 4. Frequency results (Hz) of the cantilever beam with a regular 124 nodes distribution.

| Mode | PIM-QC3 | PIM-LC1 | PIM-ST3 | PIM-ST1 | LFEM   | Reference[14] |
|------|---------|---------|---------|---------|--------|---------------|
| 1    | 822.3   | 790.5   | 818.9   | 786.3   | 966.9  | 823           |
| 2    | 4929.9  | 4742.7  | 4882.4  | 4716.9  | 5768.9 | 4937          |
| 3    | 12809.2 | 12469.8 | 12730.7 | 12385.8 | 12838.2| 12824         |
| 4    | 12967.7 | 12810.2 | 12798.4 | 12798.8 | 15112  | 13005         |
| 5    | 23487.3 | 22590.4 | 22844.3 | 22449.6 | 27331.5| 23632         |
| 6    | 35680.2 | 34269.6 | 34321.9 | 34018.9 | 38530.1| 36040         |
| 7    | 38325.0 | 38262.6 | 38212.7 | 38194.7 | 41560.1| 38442         |
| 8    | 48905.3 | 46868.1 | 46462.7 | 46472.4 | 57134.8| 49616         |

Table 5. Frequency results (Hz) of the cantilever beam with a regular 205 nodes distribution.

| Mode | PIM-QC3  | PIM-LC1  | PIM-ST3  | PIM-ST1  | LFEM    | Reference[14] |
|------|----------|----------|----------|----------|---------|---------------|
| 1    | 822.0    | 804.2    | 819.7    | 801.2    | 906.8   | 823           |
| 2    | 4928.7   | 4823.1   | 4900     | 4805     | 5425.6  | 4937          |
| 3    | 12815.2  | 12683.7  | 12799.9  | 12623.7  | 12833.2 | 12824         |
| 4    | 12967.5  | 12816.1  | 12835.4  | 12813.8  | 14254.6 | 13005         |
| 5    | 23511.2  | 22997.6  | 23141.3  | 22911.4  | 25852.8 | 23632         |
| 6    | 35757.0  | 34942.6  | 34977.7  | 34806.4  | 38493.2 | 36040         |
| 7    | 38373.8  | 38337.4  | 38299.9  | 38296.2  | 39398.6 | 38442         |
| 8    | 49069.3  | 47884.2  | 47670    | 47692.9  | 54244.9 | 49616         |

Figure 6. The relative errors of the natural frequencies of the first two modes.

The first six eigenmodes, obtained by PIM-QC3 using the regular nodal distributions with 63 nodes (4×20 cells) are shown in Figure 7. As seen in Figure 7, no spurious non-zero energy modes occurred and these mode shapes are consistent well with those of Liu [15].
Model 1                                               Model 2
Model 3                                               Model 4
Model 5                                               Model 6

Figure 7. First six vibration modes of the cantilever beam obtained by PIM-QC3.

An irregular nodal distribution with 63 nodes (4×20 cells), shown in Figure 8, is also utilized for modal analysis. The computed natural frequencies are separately shown in Table 6. It is seen that results obtained by PIM-QC3 are also the more accurate than that given by PIM-LC1, PIM-ST3, PIM-ST1 and LFEM. Even for the irregular nodal distributions, PIM-QC3 still has better performance.

Figure 8. An irregular nodal distribution.

Table 6. Frequency results (Hz) of the cantilever beam for the irregular nodal distribution.

| Mode | PIM-QC3  | PIM-LC1  | PIM-ST3  | PIM-ST1  | LFEM    | Reference [14] |
|------|----------|----------|----------|----------|---------|----------------|
| 1    | 820.7    | 716.6    | 815.5    | 699.9    | 1113.2  | 823            |
| 2    | 4939.5   | 4327.2   | 4974.6   | 4231.1   | 6544.6  | 4937           |
| 3    | 12603.1  | 11408.4  | 12043.7  | 11540.2  | 12848.2 | 12824          |
| 4    | 13052.5  | 12287.7  | 13171.1  | 11932.3  | 17259.8 | 13005          |
| 5    | 24095    | 20779.6  | 24722.3  | 22206.2  | 30942.3 | 23632          |
| 6    | 37206.3  | 31920.5  | 36263.6  | 32870.1  | 38632.2 | 36040          |
| 7    | 37755    | 36834.3  | 38938.8  | 34888.8  | 47675.5 | 38442          |
| 8    | 52147.4  | 43941.3  | 52546.7  | 46301.8  | 64023.8 | 49616          |

To compare efficiency, the relative errors of frequencies - CPU time curves for the first mode obtained by PIM-QC3, PIM-LC1, PIM-ST3, PIM-ST1, LFEM as well as EFG-QC3 are plotted in Figure 9. Four kinds of regular nodal distributions with 63 nodes (4×20 cells), 124 nodes (6×30 cells), 205 nodes (8×40 cells) and 306 nodes (10×50 cells) are used respectively. And PIM-QC3 is obviously the most efficient method of all the six methods. Compared with EFG-QC3, to achieve the same accuracy, the time cost by PIM-QC3 is far less than that of EFG-QC3. And for the same computing time, the PIM-QC3 gives a more accurate solution than EFG-QC3.

5. Conclusion
In this paper, the quadratically consistent three-point integration scheme based on point interpolant method (PIM-QC3) is applied to investigate the free vibrations of two-dimensional solids. PIM-QC3 formulation is derived by using point interpolant method and T6-Scheme for node selection based on the condition with regard to shape function and the derivatives of nodal shape function. The PIM-QC3 could meet both requirements of the DAC and the DDC. This guarantees high accuracy of the proposed
method and more efficient. In addition, it can apply essential boundary conditions conveniently like FEM since the shape functions of PIM-QC3 possess Kronecker delta property. The PIM-QC3 manifests outstanding performance for free vibration problems. The natural frequencies obtained by PIM-QC3 are both closer to reference solutions for coarse and regular nodal distributions. Even for irregular nodal distributions, the results given by PIM-QC3 are still precise, and their accuracy is evidently better than those given by PIM-LC1, PIM-ST3, PIM-ST1 and LFEM. And in regard to the efficiency of computation, PIM-QC3 is also a most efficient method among EFG-QC3, PIM-LC1, PIM-ST3, PIM-ST1 and LFEM. Numerical examples above show that the PIM-QC3 presented in this paper is simple to implement, and very accurate, stable and efficient for free vibration analysis in two-dimensional solids.

Figure 9. Computational efficiency of the first natural frequency.

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References
[1] T. Belytschko, Y. Y. Lu, L. Gu, Element-free Galerkin methods, Int. J. Numer. Meth. Eng. 37 (1994) 229 - 256.
[2] W. K. Liu, S. Jun, Y. F. Zhang, Reproducing kernel particle methods, Int. J. Numer. Meth. Eng. 20 (1995) 1081 – 1106.
[3] G. R. Liu, Y. T. Gu, A point interpolation method for two-dimensional solids, Int. J. Numer. Meth. Eng. 50 (2001) 937 - 951.
[4] J. Sladek, V. Sladek, M. Repka, P. L. Bishay, Static and dynamic behavior of porous elastic materials based on micro-dilatation theory: A numerical study using the MLPG method. INT. J. Solids. Struct. 96 (2016) 126 - 135.
[5] T. Zhu, S. N. Atluri, A modified collocation method and a penalty formulation for enforcing the essential boundary conditions in the element free Galerkin method. Comput. Mech. 21 (1998) 211 - 222.
[6] G. Ventura, An augmented Lagrangian approach to essential boundary conditions in meshless
methods, Int. J. Numer. Meth. Eng. 53 (2001) 825 - 842.

[7] C. T. Dyka, P. W. Randles, R. P. Ingel, Stress points for tension instability in SPH, Int. J. Numer. Meth. Eng. 40 (1997) 2325 - 2341.

[8] S. Beissel, T. Belytschko, Nodal integration of the element-free Galerkin method, Comput. Method. Appl. M. 139 (1996) 49 - 74.

[9] T. Nagashima, Node-by-node meshless approach and its applications to structural analyses, Int. J. Numer. Meth. Eng. 46 (1999) 341 - 385.

[10] J. S. Chen, C. T. Wu, S. Yoon, Y. You, A stabilized conforming nodal integration for Galerkin mesh-free methods, Int. J. Numer. Meth. Eng. 50 (2001) 435 - 466.

[11] M. A. Puso, J. S. Chen, E. Zywicz, W. Elmer, Meshfree and finite element nodal integration methods, Int. J. Numer. Meth. Eng. 74 (2008) 416 - 446.

[12] Q. L. Duan, X. K. Li, H. W. Zhang, T. Belytschko, Second-order accurate derivatives and integration schemes for meshfree methods, Int. J. Numer. Meth. Eng. 92 (2012) 399 - 424.

[13] G. R. Liu, Meshfree Methods: Moving Beyond the Finite Element Method, 2nd Ed, Taylor & Francis, Boca Raton, FL, 2009.

[14] G. R. Liu, Y. T. Gu, A local radial point interpolation method (LRPIM) for free vibration analyses of 2-D solids, J. Sound. Vib. 246 (2001) 29 - 46.

[15] G. R. Liu, T. Nguyen-Thai, K. Y. Lam, An edge-based smoothed finite element method (ES-FEM) for static, free and forced vibration analyses of solids, J. Sound. Vib. 320 (2009) 1100-1130.