THE COMPLETION OF THE HYPERSPACE OF FINITE SUBSETS,
ENDOWED WITH THE $\ell^1$-METRIC

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ABSTRACT. For a metric space $X$, let $FX$ be the space of all nonempty finite subsets of $X$ endowed with the largest metric $d^F_X$ such that for every $n \in \mathbb{N}$ the map $X^n \to FX$, $(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$, is non-expanding with respect to the $\ell^1$-metric on $X^n$. We study the completion of the metric space $F^1X = (FX, d^F_X)$ and prove that it coincides with the space $Z^1X$ of nonempty compact subsets of $X$ that have zero length (defined with the help of graphs). We prove that each subset of zero length in a metric space has 1-dimensional Hausdorff measure zero. A subset $A$ of the real line has zero length if and only if its closure is compact and has Lebesgue measure zero. On the other hand, for every $n \geq 2$ the Euclidean space $\mathbb{R}^n$ contains a compact subset of 1-dimensional Hausdorff measure zero that fails to have zero length.

1. Introduction

Given a metric space $X$ with metric $d_X$, denote by $KX$ the space of all nonempty compact subsets of $X$, endowed with the Hausdorff metric $d_{KX}$ defined by the formula

$$
\quad d_{KX}(A, B) = \max\{\max_{a \in A} \min_{b \in B} d_X(a, b), \min_{b \in B} \max_{a \in A} d_X(b, a)\}.
$$

The metric space $KX$, called the hyperspace of $X$, plays an important role in General Topology [3 §3.2], [7 §4.5.23] and Theory of Fractals [6 §2.5], [8 §9.1]. It is well-known [7 §4.5.23] that for any complete (and compact) metric space $X$ hyperspace $KX$ is complete (and compact). The hyperspace $KX$ contains an important dense subspace $FX$ consisting of nonempty finite subsets of $X$. The density of $FX$ in $KX$ implies that for a complete metric space $X$, the hyperspace $KX$ is a completion of the hyperspace $FX$.

In [2 §30] it was shown that the Hausdorff metric $d_FX$ on $FX$ coincides with the largest metric on $FX$ such that for every $n \in \mathbb{N}$ the map $X^n \to FX$, $x \mapsto x[n] := \{x(i) : i \in n\}$, is non-expanding, where $X^n$ is endowed with the $\ell^\infty$-metric

$$
\quad d^\infty_X(x, y) = \max_{i \in n} d_X(x(i), y(i)).
$$

Here we identify the natural number $n$ with the set $\{0, \ldots, n - 1\}$ and think of the elements of $X^n$ as functions $x : n \to X$.

Let us recall that a function $f : Y \to Z$ between metric spaces $(Y, d_Y)$ and $(Z, d_Z)$ is non-expanding if $d_Z(f(y), f(y')) \leq d_Y(y, y')$ for any $y, y' \in Y$.

It is well-known that the $\ell^\infty$-metric $d^\infty_X$ on $X^n$ is the limit at $p \to \infty$ of the $\ell^p$-metrics $d^p_X$ on $X^n$, defined by the formula:

$$
\quad d^p_X(x, y) = \left(\sum_{i=1}^n d_X(x(i), y(i))^p\right)^{\frac{1}{p}} \text{ for } x, y \in X^n.
$$

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Given any metric space \((X, d)\) and any number \(p \in [1, \infty]\), let \(d^p_{FX}\) be the largest metric \(d^p_{FX}\) on the set \(FX\) such that for every \(n \in \mathbb{N}\) the map \(X^n \to FX, \ x \mapsto x[n],\) is non-expanding with respect to the \(p\)-metric \(d^p_{X^n}\) on \(X^n\). The metric \(d^p_{FX}\) was introduced in [2], where it was shown that \(d^p_{FX}\) is a well-defined metric on \(FX\) such that
\[
\ell = d^\infty_{FX} \leq d^p_{FX} \leq d^1_{FX},
\]
where \(d^p_{FX}\) stands for the Hausdorff metric on \(FX\).

By \(F^pX\) we will denote the metric space \((FX, d^p_{FX})\). So, \(F^\infty X\) coincides with the hyperspace \(FX\) endowed with the Hausdorff metric.

As we already know, for any complete metric space \(X\), the completion \(\hat{F}\) of the metric space \(F^\infty X\) can be identified with the hyperspace \(KX\) endowed with the Hausdorff metric. In this paper we study the completion \(\hat{F}^1\) of the metric space \(F^1X = (FX, d^1_{FX})\) and show that it can be identified with the space \(Z^1X\) of nonempty compact subsets of zero length in \(X\).

Sets of zero length are defined with the help of graphs.

By a **graph** we understand a pair \(\Gamma = (V, E)\) consisting of a set \(V\) of vertices and a set \(E\) of edges. Each edge \(e \in E\) is a nonempty subset of \(V\) of cardinality \(|e| \leq 2\). A graph \((V, E)\) is finite if its set of vertices \(V\) is finite. In this case the set of edges \(E\) is finite, too.

For a graph \(\Gamma = (V, E)\), a subset \(C \subseteq V\) is **connected** if for any vertices \(x, y \in C\) there exists a sequence of vertices \(c_0, \ldots, c_n \in C\) such that \(c_0 = x, c_n = y\) and \(c_{i-1}, c_i \in E\) for every \(i \in \{1, \ldots, n\}\). The maximal connected subsets of \(V\) are called the **connected components** of the graph \(\Gamma\). It is easy to see that two connected components of \(\Gamma\) either coincide or are disjoint.

For a vertex \(x \in V\) by \(\Gamma(x)\) we shall denote the unique connected component of the graph \(\Gamma\) that contains the point \(x\).

By a **graph in a metric space** \((X, d_X)\) we understand any graph \(\Gamma = (V, E)\) with \(V \subseteq X\). In this case we can define the **total length** \(\ell(\Gamma)\) of \(\Gamma\) by the formula
\[
\ell(\Gamma) = \sum_{\{x, y\} \in E} d_X(x, y).
\]

If \(E\) is infinite, then by \(\sum_{\{x, y\} \in E} d_X(x, y)\) we understand the (finite or infinite) number
\[
\sup_{E' \subseteq E} \sum_{\{x, y\} \in E'} d_X(x, y).
\]

For a subset \(C \subseteq X\) by \(\overline{C}\) we denote the closure of \(C\) in the metric space \((X, d_X)\).

Given a subset \(A\) of a metric space \(X\), denote by \(\Gamma_X(A)\) the family of graphs \(\Gamma = (V, E)\) with finitely many connected components such that \(V \subseteq X\) and \(A \subseteq \overline{V}\). Observe that the family \(\Gamma_X(A)\) contains the complete graph on the set \(A\) and hence \(\Gamma_X(A)\) is not empty.

The set \(A\) is defined to have **zero length in \(X\)** if for any \(\varepsilon > 0\) there exists a graph \(\Gamma \in \Gamma_X(A)\) of total length \(\ell(\Gamma) < \varepsilon\).

In Proposition\[ ]\, we shall prove that each set \(A\) of zero length in a metric space \(X\) is totally bounded and has 1-dimensional Hausdorff measure equal to zero.

For a metric space \(X\), denote by \(ZX\) the family of nonempty compact subsets of zero length in \(X\). It is clear that each finite subset of \(X\) has zero length, so \(FX \subseteq ZX \subseteq KX\).

Now we define the metric \(d^p_{ZX}\) on the set \(ZX\). Given two compact sets \(A, B \in ZX\), let \(\Gamma_X(A, B)\) be the family of graphs \(\Gamma = (V, E)\) in \(X\) such that
\begin{enumerate}
  \item \(A \cup B \subseteq \overline{V}\);
  \item \(\Gamma\) has finitely many connected components;
  \item for every connected component \(C\) of \(\Gamma\) we have \(A \cap \overline{C} \neq \emptyset \neq B \cap \overline{C}\).
\end{enumerate}
The conditions (i),(ii) imply that \( A \cup B \subseteq \overline{V} = \bigcup_{x \in V} \overline{\Gamma(x)} \).

Observe that the family \( \Gamma_X(A, B) \) contains the complete graph on the set \( A \cup B \) and hence is not empty.

For two compact subsets \( A, B \in ZX \), let \[
\ell_1(\Gamma) := \inf_{\Gamma \in \Gamma_X(A, B)} \ell(\Gamma).
\]

By a completion of a metric space \( X \) we understand any complete metric space containing \( X \) as a dense subspace. The following theorem is the main result of this paper.

**Theorem 1.** Let \( X \) be a metric space and \( d_X \) be its metric.

1. The function \( d_1^X \) is a well-defined metric on \( ZX \).
2. \( d_k^X(A, B) \leq d_1^X(A, B) \) for any \( A, B \in ZX \).
3. \( d_1^X(A, B) = d_1^F_X(A, B) \) for any finite sets \( A, B \in FX \).
4. \( FX \) is a dense subset in the metric space \( Z^1X := (ZX, d_1^Z_X) \).
5. If the metric space \( X \) is complete, then so is the metric space \( Z^1X = (ZX, d_1^Z_X) \).
6. If \( Y \) is a dense subspace in \( X \), then \( d_1^Y(A, B) = d_1^Z_X(A, B) \) for any \( A, B \in ZY \).
7. If \( X \) is a completion of the metric space \( X \), then \( Z^1X \) is a completion of the metric space \( F^1X \).

The proof of Theorem 1 is divided into seven lemmas.

**Lemma 1.** \( d_k^X(A, B) \leq d_1^Z_X(A, B) \) for any \( A, B \in ZX \).

**Proof.** To derive a contradiction, assume that \( d_k^X(A, B) > d_1^Z_X(A, B) \) for some compact sets \( A, B \in ZX \). By the definition of \( d_1^Z_X \), there exists a graph \( \Gamma \in \Gamma_X(A, B) \) such that \( \ell(\Gamma) < d_k^X(A, B) \). Choose a positive real number \( \varepsilon \) such that \( \ell(\Gamma) + 2\varepsilon < d_k^X(A, B) \). Since \( \Gamma \) has finitely many connected components and \( A \cup B \subseteq \overline{V} \), for any point \( a \in A \) there exists a connected component \( C \) of the graph \( \Gamma \) such that \( a \in \overline{C} \). By the definition of the family \( \Gamma_X(A, B) \), the intersection \( \overline{C} \cap B \) contains some point \( b' \in B \). Since \( a, b' \in \overline{C} \), there are points \( c, c' \in C \) such that \( d_X(a, c) < \varepsilon \) and \( d_X(b', c') < \varepsilon \). Since the set \( C \) is connected in the graph \( \Gamma = (V, E) \), there exists a sequence \( c_0, \ldots, c_n \in C \) of pairwise distinct points such that \( c_0 = c, c_n = c' \) and \( \{c_i-1, c_i\} \in E \) for all \( i \in \{1, \ldots, n\} \). Since the points \( c_0, \ldots, c_n \) are pairwise distinct, the edges \( \{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\} \) of the graph \( \Gamma \) are pairwise distinct and then
\[
d_X(a, b') \leq d_X(a, c_0) + \sum_{i=1}^{n} d_X(c_{i-1}, c_i) + d_X(c_n, c') < \varepsilon + \ell(\Gamma) + \varepsilon.
\]

Then \( \min_{b \in B} d_X(a, b) \leq d_X(a, b') < 2\varepsilon + \ell(\Gamma) \) and \( \max_{a \in A} \min_{b \in B} d_X(b, a) < 2\varepsilon + \ell(\Gamma) \). By analogy we can prove that \( \max_{b \in B} \min_{a \in A} d_X(b, a) < 2\varepsilon + \ell(\Gamma) \). Then
\[
d_k^X(A, B) = \max_{a \in A, b \in B} \min_{a \in A} d_X(b, a) \leq 2\varepsilon + \ell(\Gamma) < d_k^X(A, B),
\]
which is a desired contradiction completing the proof of the lemma. \( \square \)

**Lemma 2.** \( d_1^Z_X \) is a well-defined metric on \( ZX \).

**Proof.** Given any sets \( A, B, C \in ZX \), we need to verify the three axioms of metric:

1. \( 0 \leq d_1^Z_X(A, B) < \infty \) and \( d_1^Z_X(A, B) = 0 \) iff \( A = B \),
2. \( d_1^Z_X(A, B) = d_1^Z_X(B, A) \),
3. \( d_1^Z_X(A, B) \leq d_1^Z_X(A, C) + d_1^Z_X(C, B) \).
1. First we show that \( d_{\mathcal{Z}X}^1(A, A) = 0 \) for any \( A \in \mathcal{Z}X \). Since the set \( A \) has zero length, for any \( \varepsilon > 0 \) there exists a graph \( \Gamma = (V, E) \) in \( X \) with finitely many connected components such that \( A \subseteq \overline{V} \) and \( \ell(\Gamma) < \varepsilon \). Replacing \( \Gamma \) by a suitable subgraph, we can assume that the closure of each connected component of \( \Gamma \) intersects the set \( A \). Then \( A \in \Gamma_X(A, A) \) and hence \[ d_{\mathcal{Z}X}^1(A, A) \leq \ell(\Gamma) < \varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, \( d_{\mathcal{Z}X}^1(A, A) = 0 \).

If sets \( A, B \in \mathcal{Z}X \) are distinct, then by Lemma \( \square \) \[ d_{\mathcal{Z}X}^1(A, B) \geq d_{\mathcal{K}X}(A, B) > 0 \] (as the Hausdorff metric \( d_{\mathcal{K}X} \) is a metric).

The proof of the first axiom of metric will be complete as soon as we check that \( d_{\mathcal{Z}X}^1(A, B) < \infty \) for any \( A, B \in \mathcal{Z}X \). Since the sets \( A, B \) have zero length, there exist graphs \( \Gamma_A = (V_A, E_A) \) and \( \Gamma_B = (V_B, E_B) \) with finitely many connected components such that \( A \subseteq \overline{V_A} \), \( B \subseteq \overline{V_B} \) and \( \ell(\Gamma_A) + \ell(\Gamma_B) < 1 \). Let \( D \) be a finite subset of \( V_A \cup V_B \) intersecting every connected component of the graphs \( \Gamma_A \) and \( \Gamma_B \). Consider the graph \( \Gamma = (V, E) \) where \( V = V_A \cup V_B \) and \( E = E_A \cup E_B \cup E_D \) where \( E_D := \{ \varepsilon \subseteq D : |\varepsilon| = 2 \} \). It is easy to see that the graph \( \Gamma \) is connected and belongs to the family \( \Gamma_X(A, B) \). Then \[ d_{\mathcal{Z}X}^1(A, B) \leq \ell(\Gamma) \leq \ell(\Gamma_A) + \ell(\Gamma_B) + \sum_{\{x, y\} \in E_D} d_X(x, y) < \infty. \]

2. The definition of the distance \( d_{\mathcal{Z}X}^1 \) implies that \( d_{\mathcal{Z}X}^1(A, B) = d_{\mathcal{Z}X}^1(B, A) \) for any \( A, B \in \mathcal{Z}X \).

3. Finally we check the triangle inequality for \( d_{\mathcal{Z}X}^1 \). Given any \( A, B, C \in \mathcal{Z}X \) and \( \varepsilon > 0 \), it suffices to show that \[ d_{\mathcal{Z}X}^1(A, C) \leq d_{\mathcal{Z}X}^1(A, B) + d_{\mathcal{Z}X}^1(B, C) + 4\varepsilon. \]

By the definition of the distances \( d_{\mathcal{Z}X}^1(A, B) \) and \( d_{\mathcal{Z}X}^1(B, C) \), there exist graphs \( \Gamma \in \Gamma_X(A, B) \) and \( \Gamma' \in \Gamma_X(B, C) \) such that \( \ell(\Gamma) < d_{\mathcal{Z}X}^1(A, B) + \varepsilon \) and \( \ell(\Gamma') < d_{\mathcal{Z}X}^1(B, C) + \varepsilon \). By the definition of the families \( \Gamma_X(A, B) \) and \( \Gamma_X(B, C) \), the graphs \( \Gamma = (V, E) \) and \( \Gamma' = (V', E') \) have finitely many connected components and their closures meet the sets \( A, B \) and \( B, C \), respectively.

Fix a finite set \( D \subseteq V \) intersecting all connected components of the graph \( \Gamma \) and a finite set \( D' \subseteq V' \) intersecting all connected components of the graph \( \Gamma' \). Fix a function \( f : D \to B \) assigning to each point \( x \in D \) a point \( f(x) \in B \cap \overline{\Gamma(x)} \). Since \( B \subseteq \overline{V} = \bigcup_{x \in V} \overline{\Gamma(x)} \), for every \( b \in B \) there exists a point \( g(b) \in V \) such that \( b \in \overline{\Gamma(g(b))} \). Since \( b \in \overline{\Gamma(g(b))} \) we can replace \( g(b) \) by a suitable point in the connected component \( \Gamma(g(b)) \) and additionally assume that \( d(b, g(b)) < \varepsilon/|D| \). Next, do the same for the graph \( \Gamma' \): choose a function \( f' : D' \to B \) such that \( f(x) \in B \cap \overline{\Gamma'(x)} \) for every \( x \in D' \), and a function \( g' : B \to V' \) such that \( b \in \overline{\Gamma'(g'(b))} \) and \( d(b, g'(b)) < \varepsilon/|D'| \) for every \( b \in B \). Consider the graph \( \Gamma'' = (V'', E'') \) where \( V'' = V \cup V' \) and \( E'' = E \cup E' \cup \{ \{ f(x), g'(f(x)) \} : x \in D \} \cup \{ \{ f'(x), g(f'(x)) \} : x \in D' \} \).

It can be shown that \( \Gamma'' \in \Gamma_X(A, C) \) and hence \[ d_{\mathcal{Z}X}^1(A, C) \leq \ell(\Gamma'') \leq \ell(\Gamma) + \ell(\Gamma') + \sum_{x \in D} d(f(x), g'(f(x))) + \sum_{x \in D'} d(f'(x), g(f'(x))) < (d_{\mathcal{Z}X}^1(A, B) + \varepsilon) + (d_{\mathcal{Z}X}^1(B, C) + \varepsilon) + |D| \cdot \frac{\varepsilon}{|D|} + |D'| \cdot \frac{\varepsilon}{|D'|} = d_{\mathcal{Z}X}^1(A, B) + d_{\mathcal{Z}X}^1(B, C) + 4\varepsilon. \]
Lemma 3. \(d^1_{ZX}(A, B) = d^1_{FX}(A, B) = \inf_{\Gamma \in \Gamma^f_X(A, B)} \ell(\Gamma)\) for all \(A, B \in FX\).

Proof. Fix any finite sets \(A, B \in FX\) and put \(I = \inf_{\Gamma \in \Gamma_X(A, B)} \ell(\Gamma)\) and \(I_t = \inf_{\Gamma \in \Gamma^f_X(A, B)} \ell(\Gamma)\). The equality \(d^1_{FX}(A, B) = I_t\) was proved in Theorem 30.4 in [2]. So, it suffices to show that \(I = I_t\). The inequality \(I \leq I_t\) is trivial and follows from the inclusion \(\Gamma^f_X(A, B) \subseteq \Gamma_X(A, B)\). The inequality \(I_t \leq I\) will follow as soon as we show that \(I_t \leq I + 5\varepsilon\) for any \(\varepsilon > 0\). Given any \(\varepsilon > 0\), find a graph \(\Gamma \in \Gamma_X(A, B)\) such that \(\ell(\Gamma) < I + \varepsilon\).

By the definition of the family \(\Gamma_X(A, B)\), for every \(a \in A\) we can find a point \(v(a) \in V\) such that \(a \in \overline{\Gamma(v(a))}\) and \(B \cap \overline{\Gamma(v(a))}\) contains some point \(\beta(a)\). Since \(\beta(a) \in \overline{\Gamma(v(a))}\), there exists a point \(u(a) \in \Gamma(v(a))\) such that \(d_X(u(a), \beta(a)) < \varepsilon/|A|\). Since \(a \in \overline{\Gamma(f(x))}\), we can replace \(v(a)\) by a suitable point in the connected component \(\Gamma(v(a))\) and additionally assume that \(d_X(a, v(a)) < \varepsilon/|A|\). Since the points \(v(a), u(a)\) belong to the same connected component of the graph \(\Gamma\), there exist a number \(n_a \in \mathbb{N}\) and a sequence \(v_0(a), \ldots, v_{n_a}(a) \in V\) such that \(v_0(a) = v(a)\), \(v_{n_a}(a) = u(a)\) and \(\{v_{i-1}(a), v_i(a)\} \in E\) for every \(i \in \{1, \ldots, n_a\}\).

Now do the same with the set \(B\): for every point \(b \in B\) choose points \(\alpha(b) \in A\) and \(v'(b), u'(b) \in V\) such that \(b \in \overline{\Gamma(v'(b))}\), \(\alpha(b) \in A \cap \overline{\Gamma(v'(b))}\), \(d_X(b, v'(b)) < \varepsilon/|B|\), \(u'(b) \in \Gamma(v'(b))\), and \(d_X(\alpha(b), u'(b)) < \varepsilon/|B|\). Since the points \(v'(b), u'(b)\) belong to the same connected component of the graph \(\Gamma\), there exist \(m_b \in \mathbb{N}\) and a sequence \(v'_0(b), \ldots, v'_m(b) \in V\) such that \(v'_0(b) = v'(b)\), \(v'_{m_b}(b) = u'(b)\) and \(\{v'_{i-1}(b), v'_i(b)\} \in E\) for every \(i \in \{1, \ldots, m_b\}\).

Now consider the finite graph \(\Gamma' = (V', E')\) with the set of vertices

\[V' = A \cup B \cup \bigcup_{a \in A} \{v_i(a) : 1 \leq i \leq n_a\} \cup \bigcup_{b \in B} \{v'_i(b) : 1 \leq i \leq m_b\}\]

and the set of edges

\[E' = \left( \bigcup_{a \in A} \{\{a, v(a)\}, \{u(a), \beta(a)\}, \{v_{i-1}(a), v_i(a)\} : 1 \leq i \leq n_a\} \right) \cup \left( \bigcup_{b \in B} \{\{b, v'(b)\}, \{u'(b), \alpha(b)\}, \{v'_{i-1}(b), v'_i(b)\} : 1 \leq i \leq m_b\} \right).\]

It is easy to see that \(\Gamma' \in \Gamma^f_X(A, B)\) and hence

\[I_t \leq \ell(\Gamma') \leq \ell(\Gamma) + \sum_{a \in A} (d_X(a, v(a)) + d_X(u(a), \beta(a))) + \sum_{b \in B} (d_X(b, v'(b)) + d_X(\alpha(b), u'(b))) < I + \varepsilon + 2\varepsilon + 2\varepsilon = I + 5\varepsilon.\]

\[\square\]

Lemma 4. For any dense subset \(Y \subseteq X\), the set \(FY\) is dense in the metric space \(Z^X = (ZX, d^1_{ZX})\).

Proof. Given any \(A \in ZX\) and \(\varepsilon > 0\), it suffices to find a set \(B \in FY\) such that \(d^1_{ZX}(A, B) < 2\varepsilon\). Since \(\ell(A) = 0\), there exists a graph \(\Gamma = (V, E)\) in \(X\) such that \(\Gamma\) has finitely many connected components, \(A \subseteq \overline{V}\) and \(\ell(A) < \varepsilon\). Choose a finite set \(B' \subseteq V\) that meets each connected component of the graph \(\Gamma\) and consider the subset \(B'' = \{b \in B' : \overline{\Gamma(b)} \cap A \neq \emptyset\}\). It is easy to see that \(\Gamma \in \Gamma_X(A, B'')\) and hence \(d^1_{ZX}(A, B'') \leq \ell(\Gamma) < \varepsilon\).

Using the density of the set \(Y\) in \(X\), choose a finite set \(B \subseteq Y\) and a surjective function \(f : B'' \to B\) such that \(d_X(x, f(x)) < \varepsilon/|B''|\) for all \(x \in B''\). Consider the graph \(\Gamma' = (V', E')\)
with the set of vertices \( V' = B'' \cup f(B'') \) and the set of edges \( E' = \{ \{ x, f(x) \} : x \in B'' \} \). Observe that \( \Gamma' \in \Gamma_X(B'', B) \) and hence \( d_{ZX}^1(B, B'') \leq \ell(\Gamma') < \sum_{x \in B''} d_X(x, f(x)) < \varepsilon \). Then
\[
d_{ZX}^1(A, B) \leq d_{ZX}^1(A, B'') + d_{ZX}^1(B'', B) < \varepsilon + \varepsilon = 2\varepsilon.
\]

**Lemma 5.** If the metric space \( X \) is complete, then so is the metric space \( Z^1X \).

**Proof.** We need to prove that each Cauchy sequence in the space \( Z^1X \) is convergent. Since the space \( F^1X \) is dense in \( Z^1X \) (see Lemmas 3, 4), it suffices to prove that each Cauchy sequence in \( F^1X \) converges to some set \( A \in ZX \). So, fix a Cauchy sequence \( \{ A_n \}_{n \in \omega} \subseteq F^1X \). Since \( d_{FX} = d_{FX}^\infty \leq d_{FX}^1 \), the sequence \( \{ A_n \}_{n \in \omega} \) remains Cauchy in the Hausdorff metric \( d_{FX} \). By the completeness of the hyperspace \( KX \), the sequence \( \{ A_n \}_{n \in \omega} \) converges (in the Hausdorff metric \( d_{KX} \)) to some nonempty compact set \( A \in KX \). It remains to show that \( A \in ZX \) and the sequence \( \{ A_n \}_{n \in \omega} \) converges to \( A \) in the metric space \( Z^1X \).

Given any \( \varepsilon > 0 \), use the Cauchy property of the sequence \( \{ A_n \}_{n \in \omega} \) and find an increasing number sequence \( \{ n_k \}_{k \in \omega} \) such that
\[
d_{FX}^1(A_{n_k}, A_i) < \frac{\varepsilon}{2^{k+1}}
\]
for any \( k \in \omega \) and \( i \geq n_k \). By Lemma 3 for every \( k \in \omega \) there exists a graph \( \Gamma_k \in \Gamma^1_X(A_{n_k}, A_{n_{k+1}}) \) such that \( \ell(\Gamma_k) < \frac{\varepsilon}{2^{k+1}} \). Now consider the graph \( \Gamma = (V, E) \) with \( V = \bigcup_{k \in \omega} V_k \) and \( E = \bigcup_{k \in \omega} E_k \) and observe that each connected component of the graph \( \Gamma \) meets the finite set \( A_{n_0} \), which implies that \( \Gamma \) has finitely many connected components. Taking into account that \( A \) is the limit of the sequence \( \{ A_{n_k} \}_{k \in \omega} \) in the Hausdorff metric, we conclude that
\[
A \subseteq \bigcup_{k \in \omega} A_{n_k} \subseteq V
\]
and the closure of each connected component of \( \Gamma \) meets the set \( A \). Then \( \Gamma \in \Gamma_X(A) \) and
\[
\ell(A) \leq \ell(\Gamma) \leq \sum_{k \in \omega} \ell(\Gamma_k) < \sum_{k \in \omega} \frac{\varepsilon}{2^{k+1}} = \varepsilon.
\]
This shows that \( \ell(A) = 0 \) and \( A \in ZX \).

It remains to show that the sequence \( \{ A_n \}_{n \in \omega} \) converges to \( A \) in the metric space \( Z^1X \). Since this sequence is Cauchy, it suffices to show that the subsequence \( \{ A_{n_k} \}_{k \in \omega} \) converges to \( A \). For every \( k \in \omega \), consider the graph \( \Gamma_k = (\tilde{V}_k, \tilde{E}_k) \) with the set of vertices \( \tilde{V}_k = \bigcup_{i=k}^\infty V_k \) and the set of edges \( \tilde{E}_k = \bigcup_{i=k}^\infty E_k \). It can be shown that \( \Gamma_k \in \Gamma_X(A, A_{n_k}) \) and hence
\[
d_{ZX}^1(A, A_{n_k}) \leq \ell(\tilde{\Gamma}_k) \leq \sum_{i=k}^\infty \ell(\Gamma_i) \leq \sum_{i=k}^\infty \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2^k} \xrightarrow{k \to \infty} 0,
\]
which means that the sequence \( \{ A_{n_k} \}_{k \in \omega} \) converges to \( A \) in the metric space \( Z^1X \).

**Lemma 6.** If \( Y \) is a dense subspace of \( X \), then \( d_{ZX}^1(A, B) = d_{ZY}^1(A, B) \) for every \( A, B \in ZY \).

**Proof.** The inequality \( d_{ZX}^1(A, B) \leq d_{ZY}^1(A, B) \) is trivial and follows from the inclusion \( \Gamma_Y(A, B) \subseteq \Gamma_X(A, B) \).

Assuming that \( d_{ZX}^1(A, B) < d_{ZY}^1(A, B) \), find \( \varepsilon > 0 \) such that \( d_{ZX}^1(A, B) + 7\varepsilon < d_{ZY}^1(A, B) \). Using Lemma 4, choose finite sets \( A', B' \in FY \) such that \( d_{ZY}^1(A, A') < \varepsilon \) and \( d_{ZY}^1(B, B') < \varepsilon \). Then also \( d_{ZX}^1(A, A') \leq d_{ZY}^1(A, A') < \varepsilon \) and \( d_{ZX}^1(B, B') \leq d_{ZY}^1(B, B') < \varepsilon \). Applying the
triangle inequality, we obtain
\[ d_{ZX}(A', B') < d_{ZX}'(A', A) + d_{ZX}(A, B) + d_{ZX}(B, B') \leq 2 \varepsilon + d_{ZX}(A, B) < 2 \varepsilon + d_{ZY}(A, B) - 7 \varepsilon \leq d_{ZY}(A', A') + d_{ZY}(A', B') + d_{ZY}(B', B') - 5 \varepsilon < \varepsilon + d_{ZY}(A', B') + \varepsilon - 5 \varepsilon = d_{ZY}(A', B') - 3 \varepsilon. \]
By Lemma 3, there exists a finite graph \( \Gamma = (V, E) \) such that
\[ \ell(\Gamma) < d_{ZX}(A', B') + \varepsilon. \]
Since \( Y \) is dense in \( X \), we can find a function \( f : V \to Y \) such that \( f(x) = x \) if \( x \in Y \) and \( d_X(f(x), x) < \varepsilon/|E| \) if \( x \in V \setminus Y \). Consider the graph \( \Gamma' = (V', E') \) with the set of vertices \( V' = f(V) \) and the set of edges \( E' = \{ \{f(x), f(y)\} : \{x, y\} \in E\} \). Observe that the graph \( \Gamma' \) belongs to the family \( \Gamma_X(A', B') \) and hence
\[ d_{ZY}(A', B') \leq \ell(\Gamma') = \sum_{\{x', y'\} \in E'} d_X(x', y') \leq \sum_{\{x, y\} \in E} d_X(f(x), f(y)) = \sum_{\{x, y\} \in E} (d_X(f(x), x) + d_X(x, y) + d_X(y, f(y)) < \sum_{\{x, y\} \in E} (|E'| + d_X(x, y) + |E'| < 2 \varepsilon + \sum_{\{x, y\} \in E} d_X(x, y) = 2 \varepsilon + \ell(\Gamma) < 2 \varepsilon + d_{ZX}(A', B') + \varepsilon < d_{ZY}(A', B'), \]
which is a desired contradiction showing that \( d_{ZX}(A, B) = d_{ZY}(A, B) \).

**Lemma 7.** If \( \hat{X} \) is a completion of \( X \), then the complete metric space \( Z^1\hat{X} \) is a completion of the metric space \( F^1X \).

**Proof.** By Lemma 3 the metric space \( Z^1\hat{X} \) is complete. By Lemmas 3 and 6 for any \( A, B \in FX \) we have
\[ d_{FX}(A, B) = d_{ZX}'(A, A) + d_{ZX}(A, B) = d_{ZX}(A, B), \]
so the metric space \( F^1X \) is a subspace of the complete metric space \( Z^1\hat{X} \). By Lemma 4 the space \( F^1X \) is dense in \( Z^1\hat{X} \). This means that \( Z^1\hat{X} \) is a completion on \( F^1X \).

Now we discuss the interplay between zero length and 1-dimensional Hausdorff measure. A subset \( A \) of a metric space \( X \) is defined to have 1-dimensional Hausdorff measure zero if for any \( \varepsilon > 0 \) there exists a countable set \( C \subseteq X \) and a function \( \epsilon : C \to (0, 1] \) such that \( \sum_{c \in C} \epsilon(c) < \varepsilon \) and \( A \subseteq \bigcup_{c \in C} B(c, \epsilon(c)) \). Here and further on by \( B(x, \delta) = \{ y \in X : d_X(x, y) < \delta \} \) and \( B[x, \delta] = \{ y \in X : d_X(x, y) \leq \delta \} \) we denote respectively the open and closed balls of radius \( \delta \) around a point \( x \) in the metric space \( (X, d_X) \).

**Proposition 1.** If a subset \( A \) of a metric space \( (X, d_X) \) has zero length, then it is totally bounded, its closure has zero length and also \( \hat{A} \) has 1-dimensional Hausdorff measure zero.

**Proof.** If \( A \) has zero length, then for every \( \varepsilon > 0 \) there exists a graph \( \Gamma = (V, E) \) in \( X \) that has finitely many connected components such that \( \ell(\Gamma) < \varepsilon \) and \( A \subseteq \overline{\Gamma} \). Then also \( \hat{A} \subseteq \overline{\Gamma} \), which means that \( \hat{A} \) has zero length. To see that \( \hat{A} \) has 1-dimensional Hausdorff measure zero, choose a finite set \( D \subseteq \Gamma \) that meets each connected component of \( \Gamma \) in a single point. Then \( \{\Gamma(x)\}_{x \in D} \) is a finite disjoint cover of \( \Gamma \). For every \( x \in D \) let \( \epsilon(x) = \sup_{y \in \Gamma(x)} d_X(x, y) \) and observe that \( V \subseteq \bigcup_{x \in D} B(x, \epsilon(x)) \). The connectedness of \( \Gamma(x) \) implies that \( \epsilon(x) \leq \ell(\Gamma(x)) \).
and \( \sum_{x \in D} \epsilon(x) \leq \ell(\Gamma) < \varepsilon \). Choose any \( \delta > 0 \) such that \( |D| \cdot \delta + \sum_{x \in D} \epsilon(x) < \varepsilon \) and observe that

\[
\mathcal{A} \subseteq \mathcal{V} \subseteq \bigcup_{x \in D} B[x, \epsilon(x)] \subseteq \bigcup_{x \in D} B(x, \epsilon(x) + \delta).
\]

Since \( \sum_{x \in D} (\epsilon(x) + \delta) = |D| \cdot \delta + \sum_{x \in D} \epsilon(x) < \varepsilon \), and \( \varepsilon \) is arbitrary, the set \( \mathcal{A} \) has 1-dimensional Hausdorff measure zero. \( \square \)

For subsets of the real line we have the following characterization.

**Proposition 2.** For a subset \( A \) of the real line the following conditions are equivalent:

1. \( A \) has zero length;
2. the closure \( \overline{A} \) is compact and has zero length;
3. the closure \( \overline{A} \) is compact and has 1-dimensional Hausdorff measure zero;
4. the closure \( \overline{A} \) is compact and has Lebesgue measure zero.

**Proof.** The implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) were proved in Proposition 1. The implication (3) \( \Rightarrow \) (4) follows from the definition of the Lebesgue measure (as the 1-dimensional Hausdorff measure) on the real line.

To prove that (4) \( \Rightarrow \) (1), assume that the closure \( \overline{A} \) is compact and has Lebesgue measure zero. Take any \( \varepsilon > 0 \). Using the compactness of the set \( \overline{A} \) and the regularity of the Lebesgue measure, construct inductively a decreasing sequence \( (U_k)_{k \in \omega} \) of bounded open neighborhoods of \( \overline{A} \) such that for every \( k \in \omega \) the following conditions are satisfied:

- \( \overline{U}_{k+1} \subseteq U_k \);
- the set \( U_k \) has Lebesgue measure \( \lambda(U_k) < \varepsilon/2^k \);
- \( U_k = \bigcup_{i=1}^{n_k} (a_{i,k}, b_{i,k}) \) for some \( n_k \in \mathbb{N} \) and real numbers \( a_{1,k} < b_{1,k} \leq \cdots \leq a_{n_k,k} < b_{n_k,k} \) such that \( A \cap (a_{i,k}, b_{i,k}) \neq \emptyset \) for every \( i \in \{1, \ldots, n_k\} \).

For every \( k \in \omega \) let

\[
a'_{i,k} := \min \{a_{j,k+1} : j \in \{1, \ldots, n_{k+1}\}, a_{i,k} < a_{j,k+1}\}
\]

and observe that \( a'_{i,k} \leq \min (A \cap (a_{i,k}, b_{i,k})) \) and hence \( |a_{i,k} - a'_{i,k}| \leq |a_{i,k} - b_{i,k}| \). For every \( k \in \mathbb{N} \), let

\[
\Omega_k = \{i \in \{1, \ldots, n_k - 1\} : \exists j \in \{1, \ldots, n_{k-1}\} \ (b_{i,k}, a_{i+1,k}) \subseteq (a_{j,k-1}, b_{j,k-1})\}.
\]

Consider the graph \( \Gamma = (V, E) \) with the set of vertices

\[
V = \bigcup_{k \in \omega} \{a_{i,k}, b_{i,k} : 1 \leq i \leq n_k\}
\]

and the set of edges

\[
E = \{\{a_{i,k}, b_{i,k}\}, \{a_{i,k}, a'_{i,k}\} : k \in \omega, \ i \in \{1, \ldots, n_k\}\} \cup \{\{b_{i,k}, a_{i+1,k}\} : k \in \mathbb{N}, \ i \in \Omega_k\}.
\]
It is easy to see that $A \subseteq \overline{A} \subseteq \overline{V}$ and each connected component of the graph $\Gamma$ intersects the set $\{a_{i,0} : 1 \leq i \leq n_0\}$. Therefore, $\Gamma$ has finitely many connected components. Also

$$\ell(\Gamma) \leq \sum_{k=0}^{\infty} \sum_{i=1}^{n_k} (|a_{i,k} - a_{i,k}^0| + |a_{i,k}^0 - a_{i,k}^b|) + \sum_{k=1}^{\infty} \sum_{i \in \Omega_k} |a_{i+1,k} - b_{i,k}| <$$

$$2 \cdot \sum_{k=0}^{\infty} \sum_{i=1}^{n_k} |a_{i,k} - a_{i,k}^0| + \sum_{k=1}^{\infty} \sum_{j=1}^{n_{k-1}} |b_{i,k-1} - a_{i,k-1}| = 3 \cdot \sum_{k=0}^{\infty} \sum_{i=1}^{n_k} |a_{i,k} - a_{i,k}^0| \leq$$

$$3 \cdot \sum_{k=0}^{\infty} \lambda(U_k) < 3 \sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} = 3\varepsilon,$$

which implies that the set $A$ has zero length.

\[ \square \]

**Proposition 3.** For the real line $X = \mathbb{R}$, the identity inclusion $\mathbb{Z}^1 X \rightarrow KX$ is a topological embedding.

**Proof.** Because of Lemma 1 it suffices to prove that for every $A \in \mathbb{Z}X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $B \in \mathbb{Z}X$ the inequality $d_{kX}(A, B) < \delta$ implies $d_{kX}^2(A, B) < \varepsilon$.

By Proposition 2 the set $A$ is compact and has Lebesgue measure zero. By the regularity of the Lebesgue measure on the real line, there exists an open neighborhood $U$ of $A$ in $\mathbb{R}$ such that $U = \bigcup_{i=1}^{n} (a_i, b_i)$ for some sequence $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, such that $\sum_{i=1}^{n} |b_i - a_i| < \frac{1}{\delta} \varepsilon$. By the proof of Proposition 2 there exists a graph $\Gamma_A = (V_A, E_A)$ such that $A \subseteq V_A$, $\ell(\Gamma_A) < 3 \cdot \frac{1}{\delta} \varepsilon = \frac{3}{\delta} \varepsilon$, and each connected component of $\Gamma_A$ intersects the set $\{a_i\}_{i=1}^{n}$. Find $\delta > 0$ such that every set $B \in KX$ with $d_{kX}(A, B) < \delta$ is contained in $U$. Take any set $B \in \mathbb{Z}X$ with $d_{kX}(A, B) < \delta$. Then $B \subseteq V_A$ and by the proof of Proposition 2 there exists a graph $\Gamma_B = (V_B, E_B)$ with finitely many components such that $B \subseteq V_B \subseteq U$ and $\ell(\Gamma_B) < 3 \cdot \frac{1}{\delta} \varepsilon = \frac{3}{\delta} \varepsilon$. Let $D \subseteq V$ be a finite set intersecting each connected component of the graph $\Gamma_B$.

For every $i \in \{1, \ldots, n\}$, write the set $\{a_i\} \cup (D \cap (a_i, b_i))$ as $\{a_{i,0}, \ldots, a_{i,m_i}\}$ for some points $a_{i,0} < \cdots < a_{i,m_i}$. It follows that $a_{i,1} = a_i$ and $a_{i,m_i} \leq b_i$, which implies $\sum_{j=1}^{m_i} |a_{i,j} - a_{i,j-1}| \leq |b_i - a_i|$. Consider the graph $\Gamma = (V, E)$ with the set of vertices $V = V_A \cup V_B$ and the set of edges

$$E = E_A \cup E_B \cup \bigcup_{i=1}^{n} \{a_{i,j-1}, a_{i,j} : j \in \{1, \ldots, m_i\}\}.$$

It can be shown that $\Gamma \in \Gamma X(A, B)$ and hence

$$d_{kX}^2(A, B) \leq \ell(\Gamma) \leq \ell(\Gamma_A) + \ell(\Gamma_B) + \sum_{i=1}^{n} \sum_{j=1}^{m_i} |a_{i,j} - a_{i,j-1}| < \frac{\delta}{3} \varepsilon + \frac{\delta}{3} \varepsilon + \sum_{i=1}^{n} |b_i - a_i| < \frac{\delta}{3} \varepsilon + \frac{\delta}{3} \varepsilon < \varepsilon.$$

\[ \square \]
By [5] §3.2, for every $n \in \mathbb{N}$, every bounded set $X \subseteq \mathbb{R}^n$ with nonempty interior has $\dim_B(X) = n$.

In the following proposition we endow the hyperspace $FX$ with the Hausdorff metric.

**Proposition 4.** Let $X$ be a metric space and $Y \subseteq X$ be a subspace of $X$ such that $\dim_B(Y) > 1$. Then for any $l \in \mathbb{N}$ there exists a nonempty finite subset $A \subseteq Y$ such that $d_{FX}(A, \{x\}) \geq l$ for any singleton $\{x\} \subseteq X$.

**Proof.** To derive a contradiction, assume that there exists $l \in \mathbb{N}$ such that for any finite set $A \subseteq Y$ there exists $x \in X$ such that $d_{FX}(A, \{x\}) < l$.

We are going to show that $N_2(\varepsilon) = (2l + 1)/\varepsilon$ for every $\varepsilon \in (0, 1]$. Given any $\varepsilon \in (0, 1]$, use the Kuratowski-Zorn Lemma and find a maximal subset $M$ in $Y$, which is $2\varepsilon$-separated in the sense that $d_X(y, z) \geq 2\varepsilon$ for any distinct points $y, z \in M$. The maximality of the set $M$ implies that $Y \subseteq \bigcup_{y \in M} B(y, 2\varepsilon)$.

We claim that $|M| \leq (1 + 2l)/\varepsilon$. To derive a contradiction, assume that $|M| > (1 + 2l)/\varepsilon$. In this case we can find a finite subset $A \subseteq M$ such that $|A| > (1 + 2l)/\varepsilon$. The choice of the number $l$ ensures that $d_{X}(A, \{x\}) < l$ for some $x \in X$. By Lemma 3 there exists a finite graph $\Gamma \in \Gamma_X(\{x\}, A)$ such that $\ell(\Gamma) < l$. Since each connected component of the graph $\Gamma$ meets the singleton $\{x\}$, the graph $\Gamma = (V, E)$ is connected. Replacing $\Gamma$ by a minimal connected subgraph, we can assume that $\Gamma$ is a tree.

By Lemma 3 (proved below), there exists a sequence $v_0, \ldots, v_n \in V$ such that

(i) $V = \{v_0, \ldots, v_n\}$;
(ii) $\{\{v_{i-1}, v_i\} : 1 \leq i \leq n\} \subseteq E$;
(iii) for every $e \in E$ the set $\{i \in \{1, \ldots, n\} : \{v_{i-1}, v_i\} = e\}$ contains at most two elements.

Choose a sequence of real numbers $t_0, \ldots, t_n$ such that $t_0 = 0$ and $t_i - t_{i-1} = d_X(v_i, v_{i-1})$ for every $i \in \{1, \ldots, n\}$. The condition (iii) implies that $t_n \leq 2\ell(\Gamma) < 2l$. Then the set $T = \{t_0, \ldots, t_n\}$ has

$$N_{\varepsilon}(T) < 1 + \frac{t_n}{\varepsilon} < 1 + \frac{2l}{\varepsilon} \leq \frac{1 + 2l}{\varepsilon}.$$ 

Taking into account that the map $T \to V$, $t_i \mapsto v_i$, is non-expanding, we conclude that $N_{\varepsilon}(A) \leq N_{\varepsilon}(V) \leq N_{\varepsilon}(T) < (1 + 2l)/\varepsilon$. Since the set $A$ is $2\varepsilon$-separated, it has cardinality $|A| = N_{\varepsilon}(A) < (1 + 2l)/\varepsilon$, which contradicts the choice of $A$.

This contradiction shows that $|M| \leq (1 + 2l)/\varepsilon$ and then $N_{2\varepsilon}(\varepsilon) \leq |M| \leq (1 + 2l)/\varepsilon$ for any $\varepsilon > 0$. Taking the upper limit at $\varepsilon \to +0$, we obtain the upper bound

$$\dim_B(Y) = \lim_{\varepsilon \to +0} \frac{\ln N_{\varepsilon}(Y)}{\ln(1/\varepsilon)} = \lim_{\varepsilon \to +0} \frac{\ln N_{2\varepsilon}(\varepsilon)}{-\ln(1/(2\varepsilon))} \leq \lim_{\varepsilon \to +0} \frac{\ln((1 + 2l)/\varepsilon)}{-\ln(1/(2\varepsilon))} = 1,$$

which contradicts our assumption. \(\square\)

**Lemma 8.** For any finite tree $\Gamma = (V, E)$, there exists a sequence $v_0, \ldots, v_n \in V$ such that

(i) $V = \{v_0, \ldots, v_n\}$,
(ii) $\{\{v_{i-1}, v_i\} : 1 \leq i \leq n\} = E$, and
(iii) for every edge $e \in E$ the set $\{i \in \{1, \ldots, n\} : \{v_{i-1}, v_i\} = e\}$ contains at most two elements.

**Proof.** This lemma will be proved by induction on the cardinality $|V|$ of the tree $V$. If $|V| = 1$, then let $v_0$ be the unique vertex of $X$ and observe that the sequence $v_0$ has the properties (i)–(iii). Assume that for some $k \geq 2$ the lemma has been proved for all trees on $< k$ vertices. Let $\Gamma = (V, E)$ be any tree with $|V| = k$. By [5] 1.5.1, the tree $\Gamma$ has exactly $k - 1$ edges. Consequently, there exists a vertex $v \in V$ having a unique neighbor $u \in V \setminus \{v\}$ in the
tree \((V,E)\). Put \(V' = V \setminus \{v\}, \ E' = E \setminus \{\{u,v\}\}\) and observe that \((V',E')\) is a tree on \(k-1\) vertices. By the inductive assumption, there exists a sequence \(v'_1, \ldots, v'_n \in V'\) such that \(V' = \{v'_1, \ldots, v'_n\}\), \(\{v'_{i-1}, v'_i\} : i \in \{1, \ldots, n\}\) = \(E'\), and for every \(v \in E'\) the set \(\{i \in \{1, \ldots, n\} : v'_{i-1}, v'_i\} = e\) contains at most two elements.

Find an index \(j \in \{1, \ldots, n\}\) such that \(v'_j = u\) and consider the sequence \(v_0, \ldots, v_{n+1}\), where \(v_i = v'_i\) for \(i \leq j\), \(v_{j+1} = v\), and \(v_i = v'_{i-2}\) for \(i \in \{j+1, \ldots, n+2\}\). It is easy to see that the sequence \(v_0, \ldots, v_{n+2}\) has the properties (i)–(iii).

Proposition \(\[\]\) implies the following corollary, in which by \(FX\) we denote the hyperspace of nonempty finite subsets of \(X\), endowed with the Hausdorff metric.

**Corollary 1.** Let \(X\) be a metric space. If for some point \(x \in X\) the identity map \(FX \rightarrow F^1X\) is continuous at \(\{x\}\), then the point \(x\) has a neighborhood \(O_x \subseteq X\) with box-counting dimension \(\dim_B(O_x) \leq 1\).

**Proof.** Assuming that the identity map \(FX \rightarrow Z^1X\) is continuous at \(\{x\}\), we can find \(\delta > 0\) such that for any set \(A \in FX\) with \(d_{FX}(A, \{x\}) < \delta\) we have \(d_{FX}(A, \{x\}) < 1\). Let \(O_x := B(x, \delta)\). Assuming that \(\dim_B(O_x) > 1\), we can apply Proposition \(\[\]\) and find a finite set \(A \subseteq O_x\) such that \(d_{FX}(A, \{x\}) > 1\). On the other hand, the inclusion \(A \subseteq O_x = B(x, \delta)\) implies that \(d_{FX}(A, x) < \delta\) and hence \(d_{FX}(A, \{x\}) < 1\) by the choice of \(\delta\). This contradiction shows that \(\dim_B(O_x) \leq 1\). \(\square\)

Finally, we present an example showing that the equivalence \((2) \iff (3)\) in Proposition \(\[\]\) does not hold for higher-dimensional Euclidean spaces.

**Example 1.** Assume that \(X\) is a complete metric space such that every nonempty open set \(U \subseteq X\) has box-counting dimension \(\dim_B(U) > 1\). Then every nonempty open set \(U\) contains a compact subset \(A \subseteq U\) such that \(A\) has 1-dimensional Hausdorff measure zero but fails to have zero length.

**Proof.** Choose any point \(x_0 \in U\) and a positive number \(\varepsilon_0\) such that \(B[x_0, \varepsilon_0] \subseteq U\). Put \(A_0 = \{x_0\}\). For every \(n \in \mathbb{N}\) we shall inductively choose a finite subset \(A_n \subseteq X\), a positive real number \(\varepsilon_n\), and a map \(r_n : A_n \rightarrow A_{n-1}\), satisfying the following conditions:

\[(i) \ A_{n-1} \subseteq A_n; \]
\[(ii) \ \varepsilon_n \leq \frac{1}{2^n |A_n|}; \]
\[(iii) \ B[x, \varepsilon_n] \cap B[y, \varepsilon_n] = \emptyset \text{ for any distinct points } x, y \in A_n; \]
\[(iv) \ r_n(x) = x \text{ for any } x \in A_{n-1}; \]
\[(v) \ B[x, \varepsilon_n] \subseteq B(r_n(x), \varepsilon_{n-1}) \text{ for any } x \in A_{n-1}; \]
\[(vi) \ d_{FX}(\{x\}, r_n^{-1}(x)) > n \text{ for every } x \in A_{n-1}. \]

Assume that for some \(n \in \mathbb{N}\) we have constructed a set \(A_{n-1}\) and a number \(\varepsilon_{n-1} > 0\) satisfying the condition (iii). By our assumption, for every \(y \in A_{n-1}\) the ball \(B(y, \varepsilon_{n-1})\) has \(\dim_B(B(y, \varepsilon_{n-1}) > 1\). By Proposition \(\[\]\) the ball \(B(y, \varepsilon_{n-1})\) contains a finite subset \(A'_y\) such that \(d_{FX}(A'_y, \{y\}) > n\). The definition of the metric \(d_{FX}\) implies that \(d_{FX}(A'_y \cup \{y\}, \{y\}) = d_{FX}(A'_y, \{y\}) > n\). Let \(A_n = \bigcup_{y \in A_{n-1}} \{y\} \cup A'_y\) and \(r_n : A_n \rightarrow A_{n-1}\) be the map assigning to each point \(x \in A_n\) the unique point \(y \in A_{n-1}\) such that \(x \in A'_y \cup \{y\}\). It is clear that the \(A_n\) satisfies the inductive condition (i) and the function \(r_n\) satisfies the conditions (iv), (vi). Now choose any number \(\varepsilon_n\) satisfying the conditions (ii), (iii) and (v). This completes the inductive step.

After completing the inductive construction, consider the compact set

\[A = \bigcap_{n \in \omega} \bigcup_{x \in A_n} B[x, \varepsilon_n] \subseteq U\]
in $X$. We claim that the set $A$ has 1-dimensional Hausdorff measure zero. Given any $\varepsilon > 0$, find $n \in \omega$ such that $\frac{2^n}{n} < \varepsilon$ and observe that $A \subseteq \bigcup_{x \in A_n} B(x, 2\varepsilon_n)$ and
\[
\sum_{x \in A_n} 2\varepsilon_n < \sum_{x \in A_n} \frac{2}{2^n|A_n|} = \frac{2}{2^n} < \varepsilon,
\]
writing that the 1-dimensional Hausdorff measure of $A$ is zero.

Assuming that $A$ has zero length, we calculate the distance $d_{2X}^1(A, A_0) < \infty$ and find a graph $\Gamma \in \Gamma_X(A, A_0)$ such that $\ell(\Gamma) < \infty$. Since each component of $\Gamma$ intersects the singleton $A_0 = \{x_0\}$, the graph $\Gamma$ is connected. Take any integer number $n > \ell(\Gamma)$ and conclude that for every $x \in A_{n-1}$ we have $\{x\} \cup r_n^{-1}(x) \subseteq A \subseteq \overline{V}$ and hence $\Gamma \in \Gamma_X(\{x\}, r_n^{-1}(x))$. By Lemma 8
\[
d_{2X}^1(\{x\}, r_n(x)) = d_{2X}^1(\{x\}, r_n(x)) \leq \ell(\Gamma) < n,
\]
which contradicts the inductive condition (vi). This contradiction shows that the set $A$ fails to have zero length. □

**Remark 1.** There are interesting algorithmic problems related to efficient calculating the distance $d_{2X}^1(A, B)$ between nonempty finite subsets $A, B$ of a metric space. For a nonempty finite subset $A$ of the Euclidean plane $\mathbb{R}^2$ and a singleton $B = \{x\} \subset \mathbb{R}^2$, the problem of calculating the distance $d_{2X}^1(A, B)$ reduces to the classical Steiner’s problem [4] of finding a tree of the smallest length that contains the set $A \cup B$. This problem is known [9] to be computationally very difficult. On the other hand, for nonempty finite subsets of the real line, there exists an efficient algorithm [1] of complexity $O(n \ln n)$ calculating the distance $d_{FR}^1(A, B)$ between two sets $A, B \in FR$ of cardinality $|A| + |B| \leq n$. Also there exists an algorithm of the same complexity $O(n \ln n)$ calculating the Hausdorff distance $d_{FR}(A, B)$ between the sets $A, B$. Finally, let us remark that the evident brute force algorithm for calculating the Hausdorff distance $d_{2X}(A, B)$ between nonempty finite subsets of an arbitrary metric space $(X, d_X)$ has complexity $O(|A| \cdot |B|)$. Here we assume that calculating the distance between points requires a constant amount of time.

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