Comparative study of Stochastic Taylor Methods and Derivative-Free Methods for Stochastic Differential Equations

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Abstract. Ordinary differential equations (ODEs) have been widely used to model the dynamical behaviour of biological and physical systems. However, modelling these systems using deterministic model such as ODEs is inadequate as the system is subjected to the uncontrolled factors of environmental noise. Stochastic differential equations (SDEs) which are originating from the irregular Brownian motion can be applied to model such systems that subjected to the uncontrolled factors of noisy behaviour. Numerical methods are required to approximate the solution of the model due to the complexity of the equation. Theoretically, methods with a higher order of convergence lead to a better approximation of the solutions. Though, the implementation of the methods might not reflect the theoretical finding as generating a lot of random numbers might contribute to the instability of the methods. This research is aimed to investigate the performance of stochastic Taylor methods of Euler-Maruyama, Milstein scheme, and derivative-free methods of second and fourth-order stochastic Runge-Kutta in approximating the solution of SDEs. Four types of mathematical models which include the Black-Scholes model, logistic model, stochastic Gompertz model, and prey-predator model are simulated using the aforementioned numerical methods. Numerical solutions of the Black-Scholes model are compared with the analytical solution meanwhile, the numerical solutions of the logistic model and stochastic Gompertz models are compared with the experimental data of the fermentation process and cancer cell growth, respectively. The simulated results of the prey-predator model are compared with the experimental data of the interaction between cancer cells (prey) and anticancer Chondroitin Sulfate (predator). The prediction performance of the methods is measured using global error and root mean square errors (RMSE).

1. Introduction

A real-life problem can be modeled by using the mathematical equation of ordinary differential equations (ODEs). However, in many applications, the corresponding simulated trajectory of ODEs do not reflect the underlying systems. Although the observed state seems to follow the predicted trajectory of ODEs, it is subjected to the uncontrolled fluctuations of environmental noise. For instance, an insight throughout the COVID 19 outbreak trend all over the world shows the number of infected people daily is going up and down which demonstrates the presence of uncontrolled factors. Modeling approaches can account for such uncertainty by explicitly incorporating the uncontrolled factors into the deterministic model of ODEs, hence a stochastic differential equation (SDE) is established. In this paper, we consider the general form of one-dimensional autonomous SDEs

\[ dx(t) = f(x(t))dt + g(x(t))dW(t) \]  \hspace{1cm} (1)
for \( t \in [0,T] \) with initial value of \( x(t_0) = x_0 \). The stochastic process \( x = \{x(t), 0 \leq t \leq T\} \) is a unique solution of SDE (1) which consists of a slowly varying component of the drift function, \( f(x(t)) \) and a rapidly fluctuating component of the diffusion function, \( g(x(t)) \). \( W(t) \) corresponds to the random variable called Wiener process that is Gaussian distributed with mean 0 and variance of \( t - s \). Equation (1) can be written explicitly in an integral form of

\[
x(t) = x(t_0) + \int_{t_0}^{t} f(x(s))ds + \int_{t_0}^{t} g(x(s))dW(s)
\]

where the second integral in (2) is an Ito stochastic integral with respect to the Wiener process. There are two ways in developing strong convergence methods for SDEs. The first way is by expanding stochastic Taylor up to a certain order and truncating the stochastic Taylor series expansion. The second way is by expanding the general formulation of the stochastic Runge-Kutta method in the form of the Taylor series and truncate the series based on the measurement of local truncation error. The other method is obtained by comparing the stochastic Taylor series expansion of the derivative-free method and the its corresponding Taylor series of the exact solution. Euler and Milstein's schemes are two methods that were developed based on the truncation of the stochastic Taylor series. The Euler-Maruyama method was developed by Maruyama in 1950 and the method was applied to solve SDEs in multiple fields of studies. It is the simplest method for SDEs and ideally suited for implementation on a digital computer. To increase the order of convergence, Milstein in 1975, derived a Milstein method by truncating the stochastic Taylor series expansion at a second-order term. By adding more terms to the scheme, a higher-order method can be developed. However, it requires the computation of partial derivative for drift and diffusions functions, hence contribute to the complexity in the simulation. Rumelin in 1982 proposed the stochastic Runge-Kutta (SRK) method which is a derivative-free method to solve SDEs. This method is based on the increment of the Wiener process, \( J \). The method is proven to converge to the actual solution, but the order of convergence cannot be increased to more than one. It was Burrage and Burrage [4], who proposed a new generalization of stochastic Runge-Kutta formulation by introducing two Wiener process in the formula. The new formulation of the SRK method overcame the order barrier stated in Rumelin. Based on this formulation, two schemes have been proposed: two-stage stochastic Runge-Kutta (SRK2) and four-stage stochastic Runge-Kutta (SRK4). Theoretically, these two methods will provide a good quality of prediction compare to stochastic Taylor methods of Euler-Maruyama, Milstein scheme, and derivative-free methods of second and fourth-order stochastic Runge-Kutta in approximating the solution of SDEs. In section 2, numerical methods of SDEs are presented. Section 3 concerns several models of SDEs including Black-Scholes, logistic, Gompertz, and prey-predator models. Next in section 4, numerical simulations for four different models by using the aforementioned methods are performed. Global error and RMSE are computed to measure the performance of the methods and the prediction quality of the models, respectively.

2. Numerical Methods of Stochastic Differential Equations

Numerical methods of SDEs are typically based on time discretization points \( 0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T \) in the interval \([0,T]\) with a time step of \( h = T / N \), where \( N \) is an integer. Numerical methods used in this paper are presented in the next subsection.

2.1 Stochastic Taylor Methods

Stochastic Taylor methods can be derived from stochastic Taylor series expansion of the stochastic chain rule of either Ito or Stratonovich formula. In this section, we consider the stochastic Taylor methods of Euler-Maruyama and Milstein schemes. Stochastic Taylor expansion of SDEs can be obtained by
expanding the drift function, \( f(x(t)) \) and the diffusion function, \( g(x(t)) \) in a Taylor series form. Euler-Maruyama method is given by

\[
y(t_{n+1}) = y(t_n) + f(y(t_n))J_0 + g(y(t_n))J_1
\]

where Weiner processes \( J_0 = h \) and \( J_1 = \sqrt{dt}N(0,1) \). The Euler-Maruyama (EM) is the simplest approximation method in SDEs with the order of convergence is 0.5. The method is a modification of the Euler method in ODEs with an additional term of stochastic integral. A method with a strong order of 1.0 was formulated by Milstein [4]. This leads to the following scheme of

\[
y(t_{n+1}) = y(t_n) + f(y(t_n))J_0 + g(y(t_n))J_1 + \frac{1}{2} g'(y(t_n))J_1^2
\]

where \( J_1^2 = \Delta W^2_n \) is a Weiner process. The order of convergence of the numerical methods of SDEs that were developed by truncating the stochastic Taylor expansion up to 1.0 order of convergence. To achieve higher-order method, the scheme becomes more impractical as more high order partial derivatives of the drift and diffusion functions are needed. High order derivatives of the drift and diffusion term lead to the zero term in Taylor series. To overcome the difficulty, it thus requires the development of a derivative-free method.

### 2.2 Derivative-Free Methods

General class of s-stage SRK method was introduced by Burrage [6]. The general class of the SRK method is expanded to a stochastic Taylor expansion and the local truncation error is measured by comparing the stochastic Taylor expansion of exact and s-stage stochastic Runge-Kutta (SRK) method, henceforth obtain two stage of SRK (SRK2) and four stage of SRK (SRK4) with the order of convergence 1.0 and 1.5, respectively. The general formulation of s-stage SRK is formulated as

\[
Y_i = y_n + h \sum a_{ij} f(Y_j) + \sum h^{(i)}_j J_1 + h^{(2)}_j J_{10} / h g(Y_j)
\]

for \( i=1,...,s \) and \( s \) is referred to the number of stages, \( a_{ij}, b^{(i)}_j\) and \( b^{(2)}_j \) are elements of \( s \times s \) matrix \( A, B^{(i)} \) and \( B^{(2)} \) respectively, \( \alpha = (\alpha_1,...,\alpha_s) \) and \( \gamma = (\gamma_1,...,\gamma_s) \) are row vectors and \( Y_i \) are intermediate stages of \( y_n \) for \( n \) iteration and \( J_{10} \) is a Weiner process. To ensure the SRK method has a low order of convergence, the values of \( a_{ij}, b^{(i)}_j, b^{(2)}_j, \alpha \) and \( \gamma \) are developed in such a way local errors are minimized.

After derivations, the numerical method of SRK2 is written as

\[
Y_n = y_n
\]

\[
Y_2 = y_n + hf(Y_1) + \frac{2}{3} J_1 g(Y_1)
\]

\[
y_{n+1} = y_n + h \left( \frac{1}{4} f(Y_1) + \frac{2}{4} f(Y_2) \right) + \frac{1}{4} J_1 g(Y_1) + \frac{3}{4} J_1 g(Y_2)
\]

and SRK-4 is written as
\[ Y_i = y_n + h a_1, f(Y_i) + \left( b_1^{(1)} j_1 + b_1^{(2)} j_1 \right) g(Y_i) \]
\[ Y_2 = y_n + h (a_2, f(Y_i) + a_2, f(Y_i)) + \left( b_2^{(1)} j_1 + b_2^{(2)} j_1 \right) g(Y_i) + \left( b_2^{(1)} j_2 + b_2^{(2)} j_2 \right) g(Y_i) \]
\[ Y_3 = y_n + h (a_3, f(Y_i) + a_3, f(Y_i)) + \left( b_3^{(1)} j_1 + b_3^{(2)} j_1 \right) g(Y_i) + \left( b_3^{(3)} j_1 + b_3^{(4)} j_2 \right) g(Y_i) \]
\[ Y_4 = y_n + h (a_4, f(Y_i) + a_4, f(Y_i)) + \left( b_4^{(1)} j_1 + b_4^{(2)} j_1 \right) g(Y_i) + \left( b_4^{(3)} j_1 + b_4^{(4)} j_1 \right) g(Y_i) + \left( b_4^{(5)} j_1 + b_4^{(6)} j_1 \right) g(Y_i) \]
\[ y_n+1 = y_n + h (\alpha_1, f(Y_i) + \alpha_2, f(Y_i) + \alpha_3, f(Y_i) + \alpha_4, f(Y_i)) + \left( \gamma_1^{(1)} j_1 + \gamma_1^{(2)} j_1 \right) g(Y_i) + \left( \gamma_2^{(1)} j_1 + \gamma_2^{(2)} j_1 \right) g(Y_i) \]

In Tableau form, the SRK4 scheme can be written as

\[ A = \begin{bmatrix}
0 \\
\frac{1}{2} \\
0 \\
0 \\
0
\end{bmatrix}, \quad B^{(1)} = \begin{bmatrix}
0 \\
-0.72429163 \\
0.42373534 \\
-0.19944370 \\
-1.5784755
\end{bmatrix} \]
\[ B^{(2)} = \begin{bmatrix}
2.7002000410 \\
1.75261649 \\
-2.918524118 \\
0 \\
0
\end{bmatrix}, \quad \alpha^s = \begin{bmatrix}
1 \\
1 \\
1 \\
6 \\
6
\end{bmatrix} \]
\[ \gamma^{(1)T} = \begin{bmatrix}
-0.78007 \\
0.073637 \\
1.4865 \\
0.21992
\end{bmatrix} \]
\[ \gamma^{(2)T} = \begin{bmatrix}
1.69395 \\
1.63610 \\
-3.02400 \\
-0.306049
\end{bmatrix} \]

3. Stochastic Model
3.1 Stochastic Logistic Model for Fermentation Process
The stochastic Logistic Model is used to measure the growth of populations with a finite carrying capacity. The model is
\[ dy(t) = \mu_{\text{max}} \left( 1 - \frac{y(t)}{y_{\text{max}}} \right) y(t) dt + \sigma \left( y(t) \right)^2 dW(t) \]
where \( y(t) \) is population growth, \( \mu_{\text{max}} \) is the growth rate, \( y_{\text{max}} \) is the carrying capacity, and \( \sigma \) is volatility. In this study, the values of the parameters used are \( \mu_{\text{max}} = 0.4848, \sigma = 0.0028, y_{\text{max}} = 3.525 \) and \( t = [0,300] \).

3.2 Stochastic Gompertz Model for Cancer Growth
Gompertz model is a model to explain the growth of the population dynamics. The model is expressed as
\[ dy(t) = \left( a - b \log y(t) \right) y(t) dt + \sigma y(t) dW(t) \]
where \( y(t) \) is the size of cancer growth, \( a \) is initial growth rate, \( b \) is retardation constant, and \( \sigma \) is noise. In this study, the values of the parameters used are \( a = 0.08446, b = -0.2, \sigma = 0.0615, y_0 = 23 \), and \( t = [48,50] \).
3.3 Black-Scholes Model in Finance
The Black-Scholes model is used to model stock prices. Black-Scholes Model could be written in the form of SDE as
\[ dy(t) = \lambda y(t) dt + \sigma y(t) dW(t) \]
where \( y(t) \) is approximated stock price, \( \lambda \) is continuously compounded interest rate and \( \sigma \) is the volatility parameter. For the Black-Scholes model, the exact solution that could be derived from the stochastic differential equation is
\[ y(t) = y_0 \exp\left(\left(\lambda - 0.5\sigma^2\right)t + \sigma W(t)\right) \]

3.4 Stochastic Prey-Predator Model for Cancer and Anticancer
Stochastic Lotka-Volterra models are used to model behavior between prey and predator and in this case, are between the cancer cell and anticancer therapeutics which are HeLa and MCF-7. The stochastic Lotka-Volterra with additive noise is
\[
\begin{align*}
    dx(t) &= (ax(t) - bx(t)y(t))dt + \sigma_1 dW_1(t) \\
    dy(t) &= (px(t)y(t) - cy(t))dt + \sigma_2 dW_2(t)
\end{align*}
\]
The stochastic Lotka-Volterra with multiplicative noise is given by
\[
\begin{align*}
    dx(t) &= (ax(t) - bx(t)y(t))dt + \sigma x(t)dW_1(t) \\
    dy(t) &= (px(t)y(t) - cy(t))dt + \sigma_2 y(t)dW_2(t)
\end{align*}
\]
where \( x(t) \) is cancer growths, \( y(t) \) is the concentration of anticancer, \( a, b, c \) and \( p \) are the parameters associated with growth and death rate, \( \sigma_1 \) and \( \sigma_2 \) are noise factor for the stochastic growth, with additive noise model (13) and multiplicative noise model (14) [7].

4. Numerical Simulations
This section presents the numerical algorithm used in approximating 500 simulations of stochastic models (8) – (12). The simulations are done for all four models and the results are displayed in the form of average plots. Two errors which are RMSE and global error are calculated to determine which methods perform well for each model. The simulations are done using MATLAB and the code was adjusted from [8].

4.1 Numerical Algorithm
This subsection presents the numerical algorithm in solving SDE.

1. Start.
2. Define step-size, \( \Delta \), end-step, \( N \), and end-simulation, \( M \), initial parameters and value.
3. Define drift function \( f(y(t)) \), and diffusion function \( g(y(t)) \).
4. Generate \( M \) by \( N \) random number.
5. Perform simulation. For simulation \( \leq \) end-simulation.
   For step \( \leq \) end-step.
   Calculate random increment \( \Delta W_n = \sum_{k=j-1}^{j} dW_{nk} \).
   Calculate \( J_{10} = \sqrt{\Delta} / 2 \left( \Delta W_{10} + \left( \Delta W_{20} / \sqrt{3} \right) \right) \) (for SRK4)
   Perform simulation using the EM (2), Milstein (3), SRK2 (5), or SRK4 (6).
   Calculate the local error, \( e_n = y_n - x_n \).
6. Calculate RMSE, \( RMSE = \left[ \sum_{m=1}^{M} \sum_{n=1}^{N} \left( e_{nm} \right)^2 / NM \right]^{1/2} \) and global error, \( e_f = |y_N - x_N| \)
7. Print Result.
8. Stop.
4.2 Result

Figure 1. Result of actual solution, approximation using EM methods, Milstein methods, SRK2, and SRK4 for stock prices using Black-Scholes Model.

Figure 1 shows the average of the simulated results of the Black-Scholes model (11) using EM, Milstein, SRK2, and SRK4 methods with a step size of $2^{-12}$ and using the average of 500 simulated paths. For all methods, we can see that, at 0.1 to 0.6 years, the simulated results using four different methods show consistency with the exact solution. However, they are slightly inconsistent after 0.7 years. Overall, all methods show an increasing trend that follows the trend in the exact solution.

Figure 2. Observed Data, approximation using EM methods, Milstein methods, SRK2, and SRK4 for stock prices for cervical cell growth.

Figure 2 shows the average of the simulated results of Gompertz model (10) using EM, Milstein, SRK2, and SRK4 methods with a step size of 0.002 and using the average of 500 simulated paths. For all methods, we can see that the prediction of all methods is almost similar and converges toward the actual solution. Overall, all methods show an increasing trend which follows the trend in the observed data.

Figure 3. Observed Data, approximation using EM methods, Milstein methods, SRK2, and SRK4 for stock prices for cell growth of Clostridium Acetobutylicium in fermentation process using Stochastic Logistic Model.
Figure 3 shows the average of the simulated results of the Logistic model (9) using EM, Milstein, SRK2, and SRK4 methods with a step size of 0.001 and using the average of 500 simulated paths. For all methods, we can see that the prediction of all methods is almost similar and converges toward the observed data. Overall, there is a slight difference between the data and the simulated results for all methods in the first 30 hours due to inaccuracy in models, rather than difference in methods’ performance. Data shows a different trend as it plateaus temporarily at $t = 20$ before increases to the population limit. The cell growth in the fermentation process reaches an equilibrium state for $t = 20$.

Figure 4. Observed Data, approximation using EM methods, Milstein methods, SRK2, and SRK4 for stock prices for cancer cell growth as prey in Figure 4 (a) and predator in Figure 4 (b) with HeLa Cell Lines simulated using additive noise as predator in Lotka-Volterra model.

Figure 4 shows the average of the simulated results of the Lotka-Volterra model for additive noise (13) for HeLa Cell Lines using EM, Milstein, SRK2, and SRK4 methods. For all methods, we can see that the prediction of EM, SRK2, and SRK4 methods converges toward the observed data, while Milstein method diverges in Figure 4(b). Overall, all methods show inverse relationship trends expected in the prey-predator model as time increases, which shows the effectiveness of the anticancer (predator) in decreasing the cancer cells (prey) from exponentially increasing as depicted in Figure 4.

Figure 5. Observed Data, approximation using EM methods, Milstein methods, SRK2, and SRK4 for stock prices for cancer cell growth as prey in Figure 5 (a) and predator in Figure 5 (b) with HeLa Cell Lines simulated using multiplicative noise as predator in Lotka-Volterra model.
Figure 5 shows the average of the simulated results of the Lotka-Volterra model for multiplicative noise (14) using HeLa Cell Lines using EM, Milstein, SRK2, and SRK4 methods. For all methods, we can see the prediction of all methods convergences toward the observed data. Overall, all methods show inverse relationship trends expected in the prey-predator model as time increases which shows the effectiveness of the anticancer (predator) in decreasing the cancer cells (prey) to exponentially increase as in Figure 5.

Figure 6. Observed Data, approximation using EM methods, Milstein methods, SRK2, and SRK4 for stock prices for cancer cell growth as prey in Figure 6 (a) and predator in Figure 6 (b) with MCF-7 Cell Lines simulated using additive noise as predator in Lotka-Volterra model.

Figure 6 shows the average of the simulated results of the Lotka-Volterra model for additive noise (13) using MCF-7 Cell Lines using EM, Milstein, SRK2, and SRK4 methods. For all methods, we can see that the prediction of all methods diverges slightly away from the actual solution which indicates a high global error for all methods (see Table 1). Overall, as time increases, the cancer cell (prey) decreases slightly but the anticancer (predator) diverges from data and remains to decrease. This shows the anticancer (predator) is used to maintain or slightly decrease the cancer cell (prey) from increasing as in Figure 6.

Figure 7. Observed Data, approximation using EM methods, Milstein methods, SRK2, and SRK4 for stock prices for cancer cell growth as prey in Figure 7 (a) and predator in Figure 7 (b) with MCF-7 Cell Lines simulated using multiplicative noise as predator in Lotka-Volterra model.

Figure 7 shows the average of the simulated results of the Lotka-Volterra model for multiplicative noise (14) using MCF-7 Cell Lines using EM, Milstein, SRK2, and SRK4 methods. For all methods, we can
see that the prediction of all methods slightly decreases as time increases. This shows the anticancer (predator) is used to maintain or slightly decrease the cancer cell (prey) from increasing as in Figure 7.

Table 1 shows the error measurement (RMSE and global error) for each method. Based on Table 1, we can see that the RMSE of the Black Scholes model for SRK2 and SRK4 were slightly similar. However global error of SRK4 indicates low values compared to the other methods. For the Gompertz model, the RMSE of all four methods are similar but the global error of SRK4 shows a significantly low value from other methods. For Logistic model, all four methods produce similar results in both RMSE and global error, although SRK4 produce the least error in both error calculation even if in a small margin. For Lotka-Volterra model, HeLa cell lines with additive noise, SRK4 produces least error in RMSE for prey, and both error for the predator, while EM and SRK2 both produces least error in global error for prey. For HeLa cell lines with multiplicative noise, the Milstein method produces the least errors for predators and SRK4 produces the least errors for prey. In this case, total error calculation is made for all four methods and the Milstein method produces the least error overall. For MCF-7 cell lines with additive noise, SRK4 produces the least errors in RMSE for prey and both errors for the predator, while EM, Milstein, and SRK2 produce the least amount of errors in global error for prey. For MCF-7 cell lines with multiplicative noise, although SRK4 produces the least errors for the predator, from the total error calculation, Milstein and SRK2 produce the least error overall.

Table 1. Error results

|                       |        |        |        |        |
|-----------------------|--------|--------|--------|--------|
|                       | EM     | Milstein | SRK2   | SRK4   |
| Black Scholes         | RMSE   | 2.2887  | 2.2536  | 2.2233  | 2.2244  |
|                       | Global Error | 3.6705  | 3.2829  | 2.6596  | 2.0165  |
| Gompertz              | RMSE   | 14.4978 | 14.4979 | 14.4982 | 14.5001 |
|                       | Global Error | 13.6168 | 13.1865 | 13.1722 | 12.9215 |
| Logistic              | RMSE   | 0.6296  | 0.6297  | 0.6296  | 0.6296  |
|                       | Global Error | 0.1987  | 0.1986  | 0.1986  | 0.1985  |
| HeLa + Additive noise | Prey   | RMSE    | 0.2154  | 0.0853  | 0.0772  | 0.0757  |
|                       |        | Global Error | 0.0015  | 0.0128  | 0.0015  | 0.0024  |
|                       | Predator | RMSE    | 0.2374  | 0.3484  | 0.1670  | 0.1663  |
|                       |        | Global Error | 0.1734  | 0.3596  | 0.1734  | 0.1658  |
| HeLa + Multiplicative Noise | Prey | RMSE    | 0.0445  | 0.0369  | 0.0368  | 0.0357  |
|                       |        | Global Error | 0.0182  | 0.0223  | 0.0182  | 0.0181  |
|                       | Predator | RMSE    | 0.0524  | 0.0319  | 0.033   | 0.0332  |
|                       |        | Global Error | 0.0259  | 0.0202  | 0.0259  | 0.0259  |
| MCF-7 + Additive noise | Prey   | RMSE    | 0.1939  | 0.0204  | 0.0204  | 0.0203  |
|                       |        | Global Error | 0.0028  | 0.0028  | 0.0028  | 0.0029  |
|                       | Predator | RMSE    | 0.2297  | 0.0457  | 0.0457  | 0.0451  |
|                       |        | Global Error | 0.0703  | 0.0696  | 0.0696  | 0.0685  |
| MCF-7 + Multiplicative noise | Prey | RMSE    | 0.0387  | 0.0110  | 0.0110  | 0.0115  |
|                       |        | Global Error | 0.0140  | 0.0152  | 0.0152  | 0.0169  |
|                       | Predator | RMSE    | 0.0412  | 0.0170  | 0.0170  | 0.0168  |
|                       |        | Global Error | 0.0175  | 0.0151  | 0.0151  | 0.0136  |
Conclusions

This research was conducted to compare the performance of numerical methods in solving stochastic differential equations. The numerical methods studied include the Euler-Maruyama method, Milstein method, second stage, and fourth stage stochastic Runge-Kutta. Then, the performance of these methods in approximating solutions in the Black-Scholes model, Gompertz model, Stochastic Logistic model, and stochastic Lotka-Volterra model was studied. The study is expected to show the implementation issues regarding the performance of higher-order methods caused by multiple Weiner processes. In this case, the low order method might perform better compared to the high order method. Based on the simulated results, it can be observed that SRK4 shows better performance in the Black-Scholes model, Gompertz model, Logistic model, and both Lotka-Volterra for additive noise, as theoretically the order of convergence is 1.5, which is the highest order amongst the four methods. SRK4 although it has the highest order of convergence, but in terms of implementation, this method drops in performance for both the Lotka-Volterra model using multiplicative noise as it is susceptible towards models with higher reliant on Weiner processes. Hence, we can conclude that in implementation, high order method with multiple Weiner processes may suffer in performance when subjected to high error models such as models with multiplicative noise.

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