Abstract. We show that Wahl’s conjecture holds in all characteristics for a minuscule $G/P$.

Let $X$ be a non-singular projective variety over $\mathbb{C}$. For ample line bundles $L$ and $M$ over $X$, consider the natural restriction map (called the Gaussian)

$$H^0(X \times X, I_\Delta \otimes p_1^* L \otimes p_2^* M) \to H^0(X, \Omega^1_X \otimes L \otimes M)$$

where $I_\Delta$ denotes the ideal sheaf of the diagonal $\Delta$ in $X \times X$, $p_1$ and $p_2$ the two projections of $X \times X$ on $X$, and $\Omega^1_X$ the sheaf of differential 1-forms of $X$; note that this map is induced by the natural projection $I_\Delta \longrightarrow I_\Delta / I_\Delta^2$ by identifying the $O_\Delta$-module $I_\Delta / I_\Delta^2$ with $\Omega^1_X$. Wahl conjectured in [11] that this map is surjective when $X = G/P$ for $G$ a complex semisimple algebraic group and $P$ a parabolic subgroup of $G$. Wahl’s conjecture was proved by Kumar in [4] using representation theoretic techniques. In [5], the authors considered Wahl’s conjecture in positive characteristics, and observed that Wahl’s conjecture will follow if there exists a Frobenius splitting of $X \times X$ which compatibly splits the diagonal and which has the maximum possible order of vanishing along the diagonal; this stronger statement was formulated as a conjecture in [5] (see §3 for a statement of this conjecture) which we shall refer to as the LMP-conjecture in the sequel. Subsequently, in [8], Mehta-Parameswaran proved the LMP-conjecture for the Grassmannian. Recently, Lakshmibai-Raghavan-Sankaran (cf.[7]) extended the result of [8] to symplectic and orthogonal Grassmannians. In this paper, we show that the LMP conjecture (and hence Wahl’s conjecture) holds in all characteristics for a minuscule $G/P$ (of course, if $G$ is the special orthogonal group $SO(m)$, then one should not allow characteristic 2). The main philosophy of the proof is the same as in [8,7]: it consists in reducing the LMP conjecture for a $G/P$, $P$ a parabolic subgroup to the problem of finding a section $\varphi \in H^0(G/B, K_{G/B}^{-1})$ ($K_{G/B}$ being the canonical bundle on $G/B$) which has maximum possible order of vanishing along $P/B$. This problem is further reduced to computing the order of vanishing (along $P/B$) of the highest weight vector $f_d$ in $H^0(G/B, L(\omega_d))$, for every fundamental weight $\omega_d$ of $G$. For details, see §4.

It should be remarked that though the spirit of this paper is the same as that of [8,7], the methods used (for computing the order of vanishing of sections) in this paper differ from those of [8,7]. Of course, the methods used in this paper may also be used for proving the results of [8,7]. Thus our methods provide an alternate proof of the results of [8,7]; we have given the details in §9.

As a by-product of our methods, we obtain a nice combinatorial realization of the order of vanishing (along $P/B$) of $f_d$ as being the length of the shortest path through extremal weights in the weight lattice connecting the highest weight

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(namely, $i(\omega_d), i$ being the Weyl involution) in $H^0(G/B, L(\omega_d))$ and the extremal weight $-\tau(\omega_d), \tau$ being the element of largest length in $W_P$, the Weyl group of $P$ (see Remark 5.4, Remark 9.6).

This paper is organized as follows: In §1 we fix notation. In §2 we recall some basic definitions and results about Frobenius splittings, and also the canonical section $\sigma \in H^0(G/B, K^{1-p})$. In §3 we recall the results of §2 about splittings for blow-ups and also the LMP conjecture. In §4 we describe the steps leading to the reduction of the proof of the LMP conjecture to computing $\text{ord}_{P/B} \sigma$ (the order of vanishing of $\sigma$ along $P/B$). In §5 a further reduction is carried out. In §6 the details are carried out for $D_n, E_6, E_7$ respectively. In §7 we give the details for the remaining minuscule $G/P$'s.

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1. Notation

Let $k$ be the base field which we assume to be algebraically closed of positive characteristic; note that if Wahl’s conjecture holds in infinitely many positive characteristics, then it holds in characteristic zero also, for the Gaussian is defined over the integers. Let $G$ be a simple algebraic group over $k$ (if $G$ is the special orthogonal group, then characteristic of $k$ will be assumed to be different from 2). Let $T$ be a maximal torus in $G$, and $R$ the root system of $G$ relative to $T$. We fix a Borel subgroup $B, B \supset T$; let $S$ be the set of simple roots in $R$ relative to $B$, and let $R^+$ be the set of positive roots in $R$. We shall follow [1] for indexing the simple roots. Let $W$ be the Weyl group of $G$; then the $T$-fixed points in $G/B$ (for the action given by left multiplication) are precisely the cosets $e_w := wB, w \in W$. For $w \in W$, we shall denote the associated Schubert variety (the closure of the $B$-orbit through $e_w$) by $X(w)$.

2. Frobenius Splittings

Let $X$ be a scheme over $k$, separated and of finite type. Denote by $F$ the absolute Frobenius map on $X$: this is the identity map on the underlying topological space $X$ and is the $p$-th power map on the structure sheaf $\mathcal{O}_X$. We say that $X$ is Frobenius split, if the $p$-th power map $F^\#: \mathcal{O}_X \to F_!\mathcal{O}_X$ splits as a map of $\mathcal{O}_X$-modules (see [9] §1, Definition 2], [2] Definition 1.1.3]). A splitting $\sigma : F_!\mathcal{O}_X \to \mathcal{O}_X$ compatibly splits a closed subscheme $Y$ of $X$ if $\sigma(F_!\mathcal{I}_Y) \subseteq \mathcal{I}_Y$ where $\mathcal{I}_Y$ is the ideal sheaf of $Y$ (see [9] §1, Definition 3], [2] Definition 1.1.3]).

Now let $X$ be a non-singular projective variety, and $K$ its canonical bundle. Using Serre duality (and the observation that $F^*L \cong L^p$ for an invertible sheaf $L$ on $X$), we get a $(k$-semilinear) isomorphism of $H^0(X, \mathcal{H}om(F_!\mathcal{O}_X, \mathcal{O}_X))$ ($= \mathcal{H}om_{\mathcal{O}_X}(F_!\mathcal{O}_X, \mathcal{O}_X)$) with $H^0(X, K^{1-p})$ (see [9] Page 32], [2] Lemma 1.2.6 and §1.3]). Thus to find splittings of $X$, we are led to look at a $\sigma$ in $H^0(X, K^{1-p})$ such that the associated homomorphism $F_*\mathcal{O}_X \to \mathcal{O}_X$ is a splitting of $F^\#$; in the sequel, following [9], we shall refer to this situation by saying the element $\sigma \in H^0(X, K^{1-p})$ splits $X$. 
Remark 2.1. By local computations, it can be seen easily that if $\sigma \in H^0(X, K_1 ^{-1})$ vanishes to order $> d(p - 1)$ along a subvariety $Y$ of codimension $d$ for some $1 \leq d \leq \dim X - 1$, then $\sigma$ is not a splitting of $X$. Hence we say that a subvariety $Y$ is compatibly split by $\sigma$ with maximum multiplicity if $\sigma$ is a splitting of $X$ which compatibly splits $Y$ and which vanishes to order $d(p - 1)$ generically along $Y$.

We will often use the following Lemma:

Lemma 2.2. Let $f : X \rightarrow Y$ be a morphism of schemes such that $f^* : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is an isomorphism.

1. If $X$ is Frobenius split, then so is $Y$.
2. If $Z$ is compatibly split, then so is the scheme-theoretic image of $Z$ in $Y$.

For a proof, see [2], Lemma 1.1.8 or [9], Proposition 4.

2.3. The section $\sigma \in H^0(X, K_1 ^{-1})$. As above, let $X = G/B$. In [9], a section $s' \in H^0(X, K_1 ^{-1})$ giving a splitting for $X$ is obtained by inducing it from a section $s \in H^0(Z, K_2 ^{-1})$ which gives a splitting for $Z$, the Bott-Samelson variety. It turns out that up to a non-zero scalar multiple, $s'$ equals $\sigma^{p-1}$ where $\sigma \in H^0(X, K_1 ^{-1})$.

In fact, one has an explicit description of $\sigma$: We have, $K_1 ^{-1} = L(2\rho)$ where $\rho$ denotes half the sum of positive roots (here, for an integral weight $\lambda$, $L(\lambda)$ denotes the associated line bundle on $X$). Let $f^+, f^-$ denote respectively a highest, lowest weight vector in $H^0(X, L(\rho))$ (note that $f^+, f^-$ are unique up to scalars). Then $\sigma$ is the image of $f^+ \otimes f^-$ under the map $H^0(X, L(\rho)) \otimes H^0(X, L(\rho)) \rightarrow H^0(X, L(\rho))$ given by multiplication of sections.

See [2] §2.3] for details.

3. Splittings and Blow-ups

Let $Z$ be a non-singular projective variety and $\sigma$ a section of $K_1 ^{-1}$ (where $K$ is the canonical bundle) that splits $Z$. Let $Y$ be a closed non-singular subvariety of $Z$ of codimension $c$. Let $ord_Y \sigma$ denote the order of vanishing of $\sigma$ along $Y$. Let $\pi : \tilde{Z} \rightarrow Z$ denote the blow up of $Z$ along $Y$ and $E$ the exceptional divisor (the fiber over $Y$) in $\tilde{Z}$.

A splitting $\tilde{\tau}$ of $\tilde{Z}$ induces a splitting $\tau$ on $Z$, in view of Lemma 2.2 (since, $\pi_* \mathcal{O}_{\tilde{Z}} \rightarrow \mathcal{O}_Z$ is an isomorphism). We say that $\sigma$ lifts to a splitting of $\tilde{Z}$ if it is induced thus from a splitting $\tilde{\sigma}$ of $\tilde{Z}$ (note that the lift of $\sigma$ to $\tilde{Z}$ is unique if it exists, since $\tilde{Z} \rightarrow Z$ is birational and two global sections of the locally free sheaf $\mathcal{H}om_{\mathcal{O}_{\tilde{Z}}} (F_* \mathcal{O}_Z, \mathcal{O}_{\tilde{Z}})$ that agree on an open set must be equal).

Proposition 3.1. With notation as above, we have

1. $ord_Y \sigma \leq c(p - 1)$.
2. $If ord_Y \sigma = c(p - 1)$ then $Y$ is compatibly split.
3. $ord_Y \sigma \geq (c - 1)(p - 1)$ if and only if $\sigma$ lifts to a splitting $\tilde{\sigma}$ of $\tilde{Z}$; moreover, $ord_Y \sigma = c(p - 1)$ if and only if the splitting $\tilde{\sigma}$ is compatible with $E$.

Proof. Assertion (1) follows in view of Remark 2.1 (since $\sigma$ is a splitting). Assertion (2) follows from the local description as in [9] Proposition 5. For a proof of assertion (3), see [5], Proposition 2.1. \qed
Now let $Z = G/P \times G/P$, and $Y$ the diagonal copy of $G/P$ in $Z$. We have:

**Theorem 3.2** (cf. [5]). Assume that the characteristic $p$ is odd. If $E$ is compatibly split in $Z$, or, equivalently, if there is a splitting of $Z$ compatibly splitting $Y$ with maximal multiplicity, then the Gaussian map is surjective for $X = G/P$.

Let us recall (cf. [5]) the following conjecture:

**LMP-Conjecture** For any $G/P$, there exists a splitting of $Z$ that compatibly splits the diagonal copy of $G/P$ with maximal multiplicity.

4. **Steps leading to a proof of LMP-conjecture for a minuscule $G/P$**

Our proof of the LMP-conjecture for a minuscule $G/P$ is in the same spirit as in [5]. We describe below a sketch of the proof.

I. **The splitting $\lambda$ of $G \times^B G/B$:** For a Schubert variety $X$ in $G/B$, using the $B$-action on $X$, we may form the twisted fiber space

$$G \times^B X := G \times X/(gb, b^{-1}x) \sim (g, x), \quad g \in G, b \in B, x \in X$$

For $X = G/B$, we have a natural isomorphism

$$f : G \times^B G/B \cong G/B \times G/B, \quad (g, xB) \mapsto (gB, gxB)$$

We have (cf. [16]) that there exists a splitting for $G \times^B G/B(\cong G/B \times G/B)$ compatibly splitting the $G$-Schubert varieties $G \times^B X$. In fact, by [2], Theorem 2.3.8, we have that this splitting is induced by $\sigma^{p-1}$ (where $\sigma$ is as in [24], as in that subsection, one identifies $\sigma$ with $f^+ \otimes f^-$). We shall denote this splitting of $G \times^B G/B$ by $\lambda$.

II. **Order of vanishing of $\lambda$ along $G \times^B P/B$:** Let $P$ be a (standard) parabolic subgroup. From the description of $\lambda$, it is clear that the order of vanishing of $\lambda$ along $G \times^B P/B$ equals $(p - 1)\text{ord}_{P/B} \sigma$, where $\text{ord}_{P/B} \sigma$ denotes the order of vanishing of $\sigma$ along $P/B$. For simplicity of notation, let us denote this order by $q$.

III. **Reduction to computing the order of vanishing of $\sigma$ along $P/B$:**

Consider the natural surjection $\pi : G/B \times G/B \rightarrow G/P \times G/P, (g_1B, g_2B) \mapsto (g_1P, g_2P)$. Then under the identification $f : G \times^B G/B \cong G/B \times G/B$, we have that $\pi$ induces a surjection

$$G \times^B P/B \rightarrow \Delta_{G/P}$$

where $\Delta_{G/P}$ denotes the diagonal in $G/P \times G/P$. We now recall the following Lemma from [5].

**Lemma 4.1.** Let $f : X \rightarrow Y$ be a morphism of schemes such that $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is an isomorphism. Let $X_1$ be a smooth subvariety of $X$ such that $f$ is smooth (submersive) along $X_1$. If $X_1$ is compatibly split in $X$ with maximum multiplicity, then the induced splitting of $Y$ has maximum multiplicity along $f(X_1)$.

**Main Reduction:** Hence the LMP-conjecture will hold for a $G/P$ if we could show that $q$ equals $(p - 1)\text{dim} G/P$, equivalently that $\text{ord}_{P/B} \sigma$ equals $\text{dim} G/P$ ($= \text{codim}_{G/B} P/B$).

**Definition 4.2.** A fundamental weight $\omega$ is called minuscule if $\langle \omega, \beta \rangle = \frac{2\omega(\beta)}{\langle \beta, \beta \rangle} \leq 1$ for all $\beta \in R^+$; the maximal parabolic subgroup associated to $\omega$ is called a minuscule parabolic subgroup.
In the following sections, we prove that \( \text{ord}_{P/B} \sigma \) equals \( \dim G/P \) for a minuscule \( G/P \). This is in fact the line of proof for the Grassmannian in [3], and for the symplectic and orthogonal Grassmannians in [7]; as already mentioned, our methods (for computing the order of vanishing of sections) differ from those of [3, 7]. In the following section, we describe the main steps involved in our approach; but first we include the list of all of the minuscule fundamental weights, following the indexing of the simple roots as in [1]:

- Type \( A_n \) : Every fundamental weight is minuscule
- Type \( B_n \) : \( \omega_n \)
- Type \( C_n \) : \( \omega_1 \)
- Type \( D_n \) : \( \omega_1, \omega_{n-1}, \omega_n \)
- Type \( E_6 \) : \( \omega_1, \omega_6 \)
- Type \( E_7 \) : \( \omega_7 \).

There are no minuscule weights in types \( E_8, F_4 \), or \( G_2 \).

5. Steps leading to the determination of \( \text{ord}_{P/B} \sigma \)

We first describe explicit realizations for \( f^+, f^- \), and then describe the main steps involved in computing the order of vanishing of \( \sigma \) along \( P/B \).

**Explicit realizations for \( f^+, f^- \):** We shall denote a maximal parabolic subgroup corresponding to omitting a simple root \( \alpha_i \) by \( P_i \); also, we follow the indexing of simple roots, fundamental weights etc., as in [1]. Let \( \omega_1, \ldots, \omega_l \) be the fundamental weights (\( l \) being the rank of \( G \)). For \( 1 \leq d \leq l \), let \( V(\omega_d) \) be the Weyl module with highest weight \( \omega_d \). One knows (see [3] for instance) that the multiplicity of \( \omega_d \) (in \( V(\omega_d) \)) is 1. Then \( w(\omega_d), w \in W \) give all the extremal weights in \( V(\omega_d) \), and these weights again have multiplicities equal to 1; of course, it suffices to run \( w \) over a set of representatives of the elements of \( W/W_{P_i} \). Given \( w \in W \), let us fix representatives \( w^{(d)}, 1 \leq d \leq l \) for \( w_{P_i}, 1 \leq d \leq l \). Let us fix a highest weight vector in \( V(\omega_d) \) and denote it by \( q_{\omega(d)} \); denote \( w^{(d)} \cdot q_{\omega(d)} \) by \( q_w \). Note in particular that \( q_{\omega_0} \) is a lowest weight vector, \( \omega_0 \) being the element of largest length in \( W \).

Recall the following well-known fact (see [3])

\[
H^0(G/P_d, L(\omega_d)) \cong V(\omega_d)^*
\]

where \( V(\omega_d)^* \) is the linear dual of \( V(\omega_d) \). In particular, \( H^0(G/P_d, L(\omega_d)) \) may be identified with the Weyl module \( V(i(\omega_d)) \), \( i \) being the Weyl involution (equal to \( -w_0 \), as an element of Aut \( R \)), and thus the extremal weights in \( H^0(G/P_d, L(\omega_d)) \) are given by \( -w^{(d)}(\omega_d) \). We may choose extremal weight vectors \( p_w \) in \( H^0(G/P_d, L(\omega_d)) \), of weight \( -w^{(d)}(\omega_d) \), in such a way that under the canonical \( G \)-invariant bilinear form \( (, \) ) on \( H^0(G/P_d, L(\omega_d)) \times V(\omega_d) \), we have,

\[
(p_{\omega(d)}, q_{\omega(d)}) = \delta_{\omega(d), \omega(d)}
\]

As a consequence, we have, for \( \tau \in W \),

\[
(\star) \quad p_{\omega(d)} | X(\tau) \neq 0 \iff \rho^{(d)} \in X(\tau(d))
\]

Now \( \rho \) being \( \omega_1 + \cdots + \omega_l \), we may take \( f^+ \) (resp. \( f^- \) ) to be the image of \( f_1^+ \otimes \cdots \otimes f_l^+ \) (resp. \( f_1^- \otimes \cdots \otimes f_l^- \) ) under the canonical map

\[
H^0(G/B, L(\omega_1)) \otimes \cdots \otimes H^0(G/B, L(\omega_l)) \rightarrow H^0(G/B, L(\rho))
\]
given by multiplication of sections. Hence we may choose
\[ f^+ = \prod_{1 \leq d \leq l} p_{w_0^{(d)}}^\sigma, \quad f^- = \prod_{1 \leq d \leq l} p_{e^{(d)}}. \]
Thus, \( \sigma \) may be taken to be
\[ \sigma = (\prod_{1 \leq d \leq l} p_{w_0^{(d)}})(\prod_{1 \leq d \leq l} p_{e^{(d)}}). \]
Now \( eB \) belongs to every Schubert variety, and hence in view of (*) \( p_{e^{(d)}} | X \neq 0, 1 \leq d \leq l \), for any Schubert variety \( X \). In particular,
\[ p_{e^{(d)}} |_{P/B} \neq 0, 1 \leq d \leq l. \]
Hence we obtain
\[ \text{ord}_{P/B} \sigma = \text{ord}_{P/B}(\prod_{1 \leq d \leq l} p_{w_0^{(d)}}) = \sum_{1 \leq d \leq l} \text{ord}_{P/B} p_{w_0^{(d)}}. \]
Thus we are reduced to computing \( \text{ord}_{P/B} p_{w_0^{(d)}} \).

**Computation of** \( \text{ord}_{P/B} p_{w_0^{(d)}} \): Let us denote the element of largest length in \( W_P \) by \( \tau_P \) or just \( \tau \) (\( P \) having been fixed). Since the \( B \)-orbit through \( \tau \) (we are denoting \( e_\tau \) by just \( \tau \)) is dense open in \( P/B \), we have
\[ \text{ord}_{P/B} p_{w_0^{(d)}} = \text{ord}_{e_\tau} p_{w_0^{(d)}} \]
where the right hand side denotes the order of vanishing of \( p_{w_0^{(d)}} \) at the point \( e_\tau \). Hence
\[ \text{ord}_{P/B} \sigma = \sum_{1 \leq d \leq l} \text{ord}_{e_\tau} p_{w_0^{(d)}}. \]
Thus, our problem is reduced to computing \( \text{ord}_{e_\tau} p_{w_0^{(d)}} \); to compute this, we may as well work in \( G/P_d \). We shall continue to denote the point \( \tau P_d \) (in \( G/P_d \)) by just \( \tau \).

The affine space \( \tau B^- \tau^{-1} \cdot \tau P_d \) (\( B^- \) being the Borel subgroup opposite to \( B \)) is open in \( G/P_d \), and gives a canonical affine neighborhood for the point \( \tau (= \tau P_d) \); further, the point \( \tau P_d \) is identified with the origin. The affine co-ordinates in \( \tau B^- \tau^{-1} \cdot \tau P_d \) may be indexed as \( \{ x_\gamma, \gamma \in \tau(R^- \setminus R_{P_d}^-) \} \) (here, \( R_{P_d}^- \) denotes the set of negative roots of \( P_d \)). We recall the following two well known facts:

**Fact 1:** For any \( f \in H^0(G/P_d, L(\omega_d)) \), the evaluations of \( \left. \frac{\partial f}{\partial x_{\gamma}} \right|_{x_{\gamma}} \) and \( X_\gamma f \) at \( \tau (= \tau P_d) \) coincide, \( X_\gamma \) being the element in the Chevalley basis of \( \text{Lie} G \) (the Lie algebra of \( G \)), associated to \( \gamma \).

**Fact 2:** For \( f \in H^0(G/P_d, L(\omega_d)) \), we have that \( \text{ord}_{e_\tau} f \) is the degree of the leading form (i.e., form of smallest degree) in the local polynomial expression for \( f \) at \( \tau \).

In the sequel, for \( f \in H^0(G/P_d, L(\omega_d)) \), we shall denote the leading form in the polynomial expression for \( f \) at \( \tau \) by \( LF(f) \). Thus we are reduced to determining \( LF(p_{w_0^{(d)}}), 1 \leq d \leq l \).

### 5.1. Determination of \( LF(p_{w_0^{(d)}}) \):

Toward the determination of \( LF(p_{w_0^{(d)}}), 1 \leq d \leq l \), we first look for \( \gamma_i \)'s in \( \tau(R^- \setminus R_{P_d}^-) \) such that \( X_{\gamma_1}^{n_1} \cdots X_{\gamma_r}^{n_r} p_{w_0^{(d)}}, n_i \geq 1 \), is a non-zero multiple of \( p_\tau \) (note that any monomial in the local expression for \( p_{w_0^{(d)}} \) arises from such a collection of \( \gamma_i \)'s and \( n_i \)'s, in view of Fact 1; also note that
Only if \( \gamma_i \) could repeat itself one or more times in \( X_{\gamma_1}^{n_1} \cdots X_{\gamma_r}^{n_r} p_{w_0}^{(a)} \). Consider such an equality:

\[
X_{\gamma_1}^{n_1} \cdots X_{\gamma_r}^{n_r} p_{w_0}^{(a)} = c p_{r}, c \in k^*.
\]

Weight considerations imply

\[
\sum_{1 \leq j \leq r} n_j \gamma_j + i(\omega_d) = -\tau(\omega_d),
\]

\( i \) being the Weyl involution. Writing \( \gamma_j = -\tau(\beta_j) \), for a unique \( \beta_j \in R^+ \setminus R_{P_d}^+ \), we obtain

\[
\tau(\omega_d) + i(\omega_d) = \sum_{1 \leq j \leq r} n_j \tau(\beta_j),
\]

i.e.,

\[
\omega_d + \tau(i(\omega_d)) = \sum_{1 \leq j \leq r} n_j \beta_j
\]

(note that \( \tau = \tau^{-1} \).

Also, using the facts that \( p_r = c \tau p_{r}^{(a)} \) (for some non-zero scalar \( c \)), and \( X_{\gamma_i} = \tau X_{\beta_i} \tau^{-1} \), we obtain that \( X_{\gamma_1}^{n_1} \cdots X_{\gamma_r}^{n_r} p_{w_0}^{(a)} \) is a non-zero scalar multiple of \( p_r \) if and only if \( X_{\beta_1}^{n_1} \cdots X_{\beta_r}^{n_r} p_{w_0}^{(a)} \) is a non-zero scalar multiple of \( p_{\tau}^{(a)} \). Thus,

\[
ord_{p_{\tau}^{(a)}} p_{w_0}^{(a)} = ord_{p_r^{(a)}} p_{w_0}^{(a)}.
\]

Hence, we obtain that \( ord_{p_{w_0}^{(a)}} \) equals \( \min \{ \sum_{1 \leq j \leq r} n_j \} \) such that there exist

\[
\{ \beta_1, \cdots, \beta_r; n_1, \cdots, n_r, \beta_j \in R^+ \setminus R_{P_d}, n_j \geq 1 \}
\]

with \( X_{\beta_1}^{n_1} \cdots X_{\beta_r}^{n_r} p_{\tau}^{(a)} \) being a non-zero scalar multiple of \( p_{\tau}^{(a)} \).

Thus in the following sections, for each minuscule \( G/P \), we carry out Steps 1 & 2 below. Also, in view of the results in [8, 7], we shall first carry out (Steps 1 & 2 below) for the following miniscule \( G/P \)'s:

I. \( G \) of Type \( D \), \( P = P_1 \)
II. \( G \) of type \( E_6 \), \( P = P_1, P_6 \)
III. \( G \) of type \( E_7 \), \( P = P_7 \).

**Remark 5.2.** In view of the fact that for \( G \) of type \( E_6 \), \( G/P_1 \cong G/P_6 \), we will restrict our attention to just \( G/P_1 \) when \( G \) is of type \( E_6 \).

**Remark 5.3.** We need not consider \( Sp(2n)/P_1 \) (which is minuscule), since it is isomorphic to \( \mathbb{P}^{2n-1} \); further, as is easily seen, \( \mathbb{P}^N \times \mathbb{P}^N \) has a splitting which compatibly splits the diagonal with maximum multiplicity (one may also deduce this from [8] by identifying \( \mathbb{P}^N \) with the Grassmannian of 1-dimensional subspaces of \( k^{N+1} \)). It should be remarked (as observed in [8]) that for \( G = Sp(2n) \), \( \sigma \) (as above) does not have maximum multiplicity along \( P_1/B \).

**Step 1:** For each \( 1 \leq d \leq l \), we find the expression \( \sum_{1 \leq j \leq l} c_j \alpha_j, c_j \in \mathbb{Z}^+ \) for \( \omega_d + \tau(i(\omega_d)) \) as a non-negative integral linear combination of simple roots; in fact, as will be seen, we have that \( c_j \neq 0, \forall j \).

**Step 2:** We show that \( \min \{ \sum_{1 \leq j \leq r} n_j \} \) (with notation as above) is given as follows:

\[
D_n, E_6: \min \{ \sum_{1 \leq j \leq r} n_j \} = c_1
\]
Towards proving this, we observe that in $D_n$, $E_6$, coefficient of $\alpha_1$ in any positive root is less than or equal to one, while in $E_7$, the coefficient of $\alpha_7$ in any positive root is less than or equal to one (see [1]). Hence for any collection $\{\beta_1, \cdots, \beta_r; n_1, \cdots, n_r, \ n_j \geq 1\}$ as above, we have

$$\sum_{1 \leq j \leq r} n_j \geq \begin{cases} \ c_1, & \text{if type } D_n \text{ or } E_6 \\ \ c_7, & \text{if type } E_7 \end{cases}$$

We then exhibit a collection $\{\beta_1, \cdots, \beta_r\}$, $\beta_j \in R^+ \setminus R^+_P$, $1 \leq j \leq r$ such that

(a) $\omega_d + \tau(i(\omega_d)) = \sum_{1 \leq j \leq r} \beta_j$

(b) The reflections $s_{\beta_j}$'s (and hence the Chevalley basis elements $X_{-\beta_j}$'s) mutually commute.

(c) For any subset $\{\delta_1, \cdots, \delta_s\}$ of $\{\beta_1, \cdots, \beta_r\}$, $X_{\delta_1} \cdots X_{\delta_s} p_{\tau w^0_d}$ is an extremal weight vector (in $H^0(G/P_d, L(\omega_d))$), and $X_{\delta_1}^{n_1} \cdots X_{\delta_s}^{n_s} p_{\tau w^0_d}$ is a lowest weight vector (i.e., a non-zero scalar multiple of $p_{w^0_d}$).

(d) $\sum_{1 \leq j \leq r} n_j = \begin{cases} \ c_1, & \text{if type } D_n \text{ or } E_6 \\ \ c_7, & \text{if type } E_7 \end{cases}$

(e) We then conclude (by the foregoing discussion) that

$$\text{ord}_\tau p_{w^0_d} = \begin{cases} \ c_1, & \text{if type } D_n \text{ or } E_6 \\ \ c_7, & \text{if type } E_7 \end{cases}$$

Remark 5.4. Thus we obtain a nice realization for $\text{ord}_\tau p_{w^0_d}$ (the order of vanishing along $P/B$ of $p_{w^0_d}$) as being the length of the shortest path through extremal weights in the weight lattice connecting the highest weight (namely, $i(\omega_d)$ in $H^0(G/B, L(\omega_d))$) and the extremal weight $-\tau(\omega_d)$.

Remark 5.5. For the sake of completeness, we have given the details for the remaining $G/P$'s in [11].

For the convenience of notation, we make the following

Definition 5.6. Define $m_d$ to be $\text{ord}_\tau p_{w^0_d} (= \text{ord}_\tau p_{w^0_d})$

We shall treat the cases I,II,III above, respectively in the following three sections. In the following sections, we will be repeatedly using the following:

Fact 3: Suppose $p_\theta$ is an extremal weight vector in $H^0(G/P_d, L(\omega_d))$ of weight $\chi (= -\theta(\omega_d))$, and $\beta \in R$ such that $(\chi, \beta^*) = r$, for some positive integer $r$. Then $X^r_{-\beta} p_\theta$ is a non-zero scalar multiple of the extremal weight vector $p_{w_d \theta}$ (here, $(,)$ is a $W$-invariant scalar product on the weight lattice, and $(\chi, \beta^*) = \frac{2(\chi, \beta)}{(\beta, \beta)}$).

The above fact follows from sl(2)-theory (note that $p_\theta$ is a highest weight vector for the Borel sub group $\theta B^{-1}B^{-1}$, $B^{-1}$ being the Borel subgroup opposite to $B$).
6. THE MINUSCULE $SO(2n)/P_1$

Let the characteristic of $k$ be different from 2. Let $V = k^{2n}$ together with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$. Taking the matrix of the form $(\cdot, \cdot)$ (with respect to the standard basis $\{e_1, \ldots, e_{2n}\}$ of $V$) to be $E$, the anti-diagonal $(1, \ldots, 1)$ of size $2n \times 2n$. We may realize $G = SO(V)$ as the fixed point set $SL(V)^\sigma$, where $\sigma : SL(V) \to SL(V)$ is given by $\sigma(A) = E(A)^{-1}E$. Set $H = SL(V)$.

Denoting by $T_H$ (resp. $B_H$) the maximal torus in $H$ consisting of diagonal matrices (resp. the Borel subgroup in $H$ consisting of upper triangular matrices) we see easily that $T_H, B_H$ are stable under $\sigma$. We set $T_G = T_H^\sigma, B_G = B_H^\sigma$. Then it is well known that $T_G$ is a maximal torus in $G$ and $B_G$ is a Borel subgroup in $G$.

We have a natural identification of the Weyl group $W$ of $G$ as a subgroup of $S_{2n}:

$W = \{ (a_1 \cdots a_{2n}) \in S_{2n} \mid a_i = 2n + 1 - a_{2n+1-i}, 1 \leq i \leq 2n, \text{ and } m_w \text{ is even} \}$

where $m_w = \# \{ i \leq n \mid a_i > n \}$. Thus $w = (a_1 \cdots a_{2n}) \in W_G$ is known once $(a_1 \cdots a_n)$ is known. In the sequel, we shall denote such a $w$ by just $(a_1 \cdots a_n)$; also, for $1 \leq i \leq 2n$, we shall denote $2n + 1 - i$ by $i'$.

For details see [6].

Let $P = P_{\alpha_1}$. We preserve the notation from the previous section; in particular, we denote the element of largest length in $W_P$ by $\tau$. We have

$$\tau = \begin{cases} (12'3' \cdots (n-1)'n), & \text{if } n \text{ is even} \\ (12'3' \cdots (n-1)'n'), & \text{if } n \text{ is odd} \end{cases}$$

Steps 1 & 2 of §5.1. As in [1], we shall denote by $\epsilon_j, 1 \leq j \leq n$, the restriction to $T_G$ of the character of $T_H$, sending a diagonal matrix $diag\{t_1, \cdots, t_n\}$ to $t_j$.

Case 1: Let $d \leq n - 2$. Then $w_d = \epsilon_1 + \cdots + \epsilon_d$ (cf. [1]). We have, $i(w_d) = w_d$, and $\tau(w_d) = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_d$. Hence

$$w_d + \tau(i(w_d)) = 2\epsilon_1 = 2(\epsilon_1 + \cdots + \epsilon_{n-2}) + \alpha_{n-1} + \alpha_n$$

(note that an element in $T_G$ is of the form $diag\{t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}\}$, and hence $\epsilon_j^\prime = -\epsilon_j$ (we follow [1] for denoting the simple roots)). We let $\{\beta_1, \beta_2\} \subset R^+ \setminus R^+_d$ be any (unordered) pair of the form $\{\epsilon_1 - \epsilon_j, \epsilon_1 + \epsilon_j, d+1 \leq j \leq n\}$. Clearly, $s_{\beta_1}, s_{\beta_2}$ commute (since, $(\beta_1, \beta_2^\prime) = 0$), and $w_d + \tau(i(w_d)) = \beta_1 + \beta_2$. Also,

$$-\tau w_0^{(d)}(w_d, \beta_j^\prime) = (\tau(w_d), \beta_j^\prime) = (1,1,1,2);$$

$$-\tau w_0^{(d)}(\omega_d) - \beta_j = \beta_m^\prime = (\tau(w_d) - \beta_j, \beta_m^\prime) = 1, \text{ for } j, m \in \{1, 2\}, \text{ and } j, m \text{ distinct;}$$

$$-\tau w_0^{(d)}(\omega_d) - \beta_1 - \beta_2 = \tau(\omega_d) - \beta_1 - \beta_2 = -\omega_d = -(\epsilon_1 + \cdots + \epsilon_d).$$

From this, (a)-(c) in Step 2 of §5.1 follow for the above choice of $\{\beta_1, \beta_2\}$: (d) in Step 2 is obvious. Hence $m_d = 2, 1 \leq d \leq n - 2$ (recall $m_d$ from Definition §5.6).

Case 2: $d = n - 1$. We have, $i(\omega_{n-1})$ equals $\omega_{n-1}$ or $\omega_n$, according as $n$ is even or odd.

If $n$ is even, then $\tau(i(\omega_{n-1})) = \tau(\omega_{n-1}) = \frac{1}{2} (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{(n-1)'}) - \epsilon_n$.

If $n$ is odd, then $\tau(i(\omega_{n-1})) = \tau(\omega_{n-1}) = \frac{1}{2} (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{(n-1)'}) + \epsilon_{n'}$

$$= \frac{1}{2} (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{(n-1)'}) - \epsilon_n.$$

Thus in either case, $\tau(i(\omega_{n-1})) = \frac{1}{2} (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{(n-1)'}) - \epsilon_n$. Hence

$$\omega_{n-1} + \tau(i(\omega_{n-1})) = \epsilon_1 - \epsilon_n = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1}.$$
which is clearly a root in $R^+ \setminus R^+_{P_n}$. Hence taking $\beta_1$ to be $\epsilon_1 - \epsilon_n$, we find that $\\{\beta_1\}$ (trivially) satisfies (a)-(c) in Step 2 of [5.1] follow; (d) in Step 2 is obvious. Hence $m_{n-1} = 1$.

**Case 3: $d = n$.** Proceeding as in Case 2, we have,

$$\omega_n + \tau(i(\omega_n)) = \epsilon_1 + \epsilon_n = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n$$

which is clearly a root in $R^+ \setminus R^+_{P_n}$. As in case 2, we conclude that $m_n = 1$.

**Theorem 6.1.** The LMP conjecture holds for the minuscule $G/P$, $G, P$ being as above.

**Proof.** From [4] (see “Main reduction” in that section), we just need to show that $\sum_{1 \leq d \leq n} \text{ord}_P w_d(\alpha)$ equals $\text{codim}_{G/B} G/P$. From the above computations, and the discussion in [5] we have $\sum_{1 \leq d \leq n} \text{ord}_P w_d(\alpha) = \sum_{1 \leq d \leq n} m_d = 2n - 2$ which is precisely $\text{codim}_{G/B} G/P$. \hfill \Box

7. Exceptional Group $E_6$

Let $G$ be simple of type $E_6$. Let $P = P_1$. As in the previous sections, let $\tau$ be the unique element of largest length in $W_P$.

**Step 1 & 2 of [5.1]** Note that $\tau$ is the unique element of largest length inside the Weyl group of type $D_5$; we have that $D_5$ sits inside of $E_6$ as

$$\begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \quad \circ \\
\circ
\end{array}$$

Thus, for $2 \leq j \leq 6$, we have $\tau(\alpha_j) = -i(\alpha_j)$, $i$ being the Weyl involution of $D_5$; in this case, we have

$$(*) \quad \tau(\alpha_2) = -\alpha_3, \tau(\alpha_3) = -\alpha_2, \tau(\alpha_j) = -\alpha_j, j = 4, 5, 6.$$ 

Thus, using the Tables in [1], to find $\tau(i(\omega_d)), 1 \leq d \leq 6$, as a linear sum (with rational coefficients) of the simple roots, it remains to find $\tau(\alpha_1)$.

Let $\tau(\alpha_1) = \sum_{1 \leq j \leq 6} a_j \alpha_j, a_j \in \mathbb{Z}$. Since $\alpha_1 \notin R_P$ (the root system of $P$), we have that $\tau(\alpha_1) \notin R_P$. Hence $\alpha_1 \neq 0$; further, $\tau(\alpha_1) \in R^+_{E_6}$ (since, clearly, $l(\tau s_{\alpha_1}) = l(\tau) + 1$). Hence, $a_1 > 0$; in fact, we have, $a_1 = 1$ (since any positive root in the root system of $E_6$ has an $\alpha_1$ coefficient $\leq 1$). Using $(*)$ above, and the following linear system, we determine the remaining $a_j$’s:

$$\begin{array}{c}
2a_2 - a_4 = \langle \tau(\alpha_1), \alpha_2 \rangle = \langle \alpha_1, \tau(\alpha_2) \rangle = \langle \alpha_1, -\alpha_3 \rangle = 1 \\
2a_3 - a_1 - a_4 = \langle \tau(\alpha_1), \alpha_3 \rangle = \langle \alpha_1, \tau(\alpha_3) \rangle = \langle \alpha_1, -\alpha_2 \rangle = 0 \\
2a_4 - a_2 - a_3 - a_5 = \langle \tau(\alpha_1), \alpha_4 \rangle = \langle \alpha_1, \tau(\alpha_4) \rangle = \langle \alpha_1, -\alpha_4 \rangle = 0 \\
2a_5 - a_4 - a_6 = \langle \tau(\alpha_1), \alpha_5 \rangle = \langle \alpha_1, \tau(\alpha_5) \rangle = \langle \alpha_1, -\alpha_5 \rangle = 0 \\
2a_6 - a_5 = \langle \tau(\alpha_1), \alpha_6 \rangle = \langle \alpha_1, \tau(\alpha_6) \rangle = \langle \alpha_1, -\alpha_6 \rangle = 0
\end{array}$$

Either one may just solve the above linear system or use the properties of the root system of type $E_6$ to quickly solve for $a_j$’s. For instance, we have, $a_6 \neq 0$; for, $a_6 = 0$ would imply (working with the last equation and up) that $a_4 = 0$ which in turn would imply (in view of the first equation) that $a_2 = \frac{1}{2}$, not possible. Hence
\( a_6 \neq 0 \), and in fact equals 1 (for the same reasons as in concluding that \( a_1 = 1 \)). Once again working backward in the linear system, \( a_5 = 2, a_4 = 3 \); hence from the first equation, we obtain, \( a_2 = 2 \). Now the second equation implies that \( a_3 = 2 \). Thus we obtain

\[
\tau(a_1) = a_1 + 2a_2 + 3a_3 + 2a_4 + 2a_5 + a_6 = \left( \begin{array}{ccccc}
1 & 2 & 2 & 1 \\
2 & 2 & 1 \\
\end{array} \right).
\]

For \( d \in \{1, \ldots, 6\} \), we shall now describe \( \{\beta_1, \ldots, \beta_r \mid \beta_i \in R^+ \setminus R_{\alpha_i}^+\} \) which satisfies the conditions (a)-(d) in Step 2 of \( \S 5.1 \).

For convenience, we list the fundamental weights here:

\[
\omega_1 = \frac{1}{3} \left( \begin{array}{cccc}
4 & 5 & 6 & 2 \\
3 & 4 & 2 & 1 \\
\end{array} \right),
\omega_2 = \left( \begin{array}{cccc}
1 & 2 & 3 & 1 \\
2 & 2 \\
\end{array} \right),
\omega_3 = \frac{1}{3} \left( \begin{array}{cccc}
5 & 10 & 12 & 8 \\
6 & 10 & 5 \\
\end{array} \right),
\omega_4 = \left( \begin{array}{cccc}
4 & 6 & 4 & 2 \\
3 & 4 & 2 \\
\end{array} \right),
\omega_5 = \frac{1}{3} \left( \begin{array}{cccc}
4 & 8 & 12 & 10 \\
6 & 10 & 5 \\
\end{array} \right),
\omega_6 = \frac{1}{3} \left( \begin{array}{cccc}
2 & 4 & 6 & 5 \\
3 & 4 & 2 \\
\end{array} \right).
\]

**Case 1:** \( d = 1 \).

We have, \( i(\omega_1) = \omega_6 \). Hence using (from above), the expression for \( \omega_6 \) as a (rational) sum of simple roots, and the expressions for \( \tau(\alpha_j), j = 1, \cdots, 6 \), we obtain

\[
\tau(i(\omega_1)) + \omega_1 = \left( \begin{array}{cccc}
2 & 2 & 2 & 1 \\
1 & 0 \\
\end{array} \right).
\]

We let \( \{\beta_1, \beta_2\} \) be the unordered pair of roots:

\[
\left( \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 0 \\
\end{array} \right),
\left( \begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 0 \\
\end{array} \right).
\]

Clearly \( \beta_1, \beta_2 \) are in \( R^+ \setminus R_{\alpha_i}^+ \), and the reflections \( s_{\beta_1}, s_{\beta_2} \) commute (since \( \beta_1 + \beta_2 \) is not a root). Further, \( \tau(i(\omega_1)) + \omega_1 = \beta_1 + \beta_2 \). Also,

\[
\langle -\tau w_0^{(1)}(\omega_1), \beta_j^* \rangle = \langle \tau(i(\omega_1)), \beta_j^* \rangle = \langle \beta_1 + \beta_2 - \omega_1, \beta_j^* \rangle = 1, j = 1, 2;
\]

\[
\langle -\tau w_0^{(1)}(\omega_1) - \beta_i, \beta_j^* \rangle = \langle \beta_j - \omega_1, \beta_j^* \rangle = 1, j, l \in \{1, 2\}, \text{ and } j, l \text{ distinct;}
\]

\[
-\tau w_0^{(1)}(\omega_1) - \beta_1 - \beta_2 = -\omega_1.
\]

From this, (a)-(c) in Step 2 of \( \S 5.1 \) follow for the above choice of \( \{\beta_1, \beta_2\} \); (d) in Step 2 is obvious. Hence \( m_1 = 2 \) (cf. Definition \( 5.6 \)).

**Case 2:** \( d = 2 \).

We have \( i(\omega_2) = \omega_2 \). As in case 1, using the expression for \( \omega_2 \) as a (rational) sum of simple roots, and the expressions for \( \tau(\alpha_j), j = 1, \cdots, 6 \), we obtain

\[
\tau(\omega_2) + \omega_2 = \left( \begin{array}{cccc}
2 & 2 & 3 & 2 \\
2 & 2 & 1 \\
\end{array} \right).
\]

We let \( \{\beta_1, \beta_2\} \) be the unordered pair of roots:

\[
\left( \begin{array}{cccc}
1 & 1 & 2 & 1 \\
1 & 1 & 0 & 0 \\
\end{array} \right).
\]
Clearly $\beta_1, \beta_2$ are in $R^+ \setminus R^+_P$, and the reflections $s_{\beta_1}, s_{\beta_2}$ commute (since $\beta_1 + \beta_2$ is not a root). Further, $\tau(\omega_2) + \omega_2 = \beta_1 + \beta_2$. We have
\[
\langle -\tau w_0^{(2)}(\omega_2), \beta_j^+ \rangle = \langle \tau(\omega_2), \beta_j^+ \rangle = \langle \beta_1 + \beta_2 - \omega_2, \beta_j^+ \rangle = 1, \quad j = 1, 2;
\]
\[
\langle -\tau w_0^{(2)}(\omega_2) - \beta_i, \beta_j^+ \rangle = \langle \beta_j - \omega_2, \beta_j^+ \rangle = 1, \quad j, l \in \{1, 2\}, \quad \text{and } j, l \text{ distinct};
\]
\[
-\tau w_0^{(2)}(\omega_2) - \beta_1 - \beta_2 = -\omega_2.
\]

As in case 1, (a)-(c) in Step 2 of §5.1 follow for the above choice of $\{\beta_1, \beta_2\}$; (d) in Step 2 is also clear. Hence $m_2 = 2$.

The discussion in the remaining cases are similar; in each case we will just give the expression for $\tau(i(\omega_d)) + \omega_d$ as an element in the root lattice, and the choice of $\{\beta_1, \cdots, \beta_r\}$ in $R^+ \setminus R^+_P$, which satisfy the conditions (a)-(d) in Step 2 of §5.1. Then deduce the value of $m_d$.

**Case 3:** $d = 3$.
We have $i(\omega_3) = \omega_5$. Further,
\[
\tau(\omega_5) + \omega_3 = \begin{pmatrix} 3 & 4 & 4 & 2 & 1 \\ 2 \end{pmatrix}.
\]

We let $\{\beta_1, \beta_2, \beta_3\}$ be the unordered triple of roots:
\[
\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 1 & 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 \end{pmatrix}.
\]

Then we have $\tau(\omega_3) + \omega_3 = \beta_1 + \beta_2 + \beta_3$. Reasoning as in case 1, we conclude $m_3 = 3$.

**Case 4:** $d = 4$.
We have $i(\omega_4) = \omega_4$. Further,
\[
\tau(\omega_4) + \omega_4 = \begin{pmatrix} 4 & 5 & 6 & 4 & 2 \\ 3 \end{pmatrix}.
\]

We let $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ be the unordered quadruple of roots:
\[
\begin{pmatrix} 1 & 2 & 2 & 2 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 \end{pmatrix}.
\]

Then we have $\tau(\omega_4) + \omega_4 = \beta_1 + \beta_2 + \beta_3 + \beta_4$. Reasoning as in case 1, we conclude $m_4 = 4$.

**Case 5:** $d = 5$.
We have $i(\omega_5) = \omega_3$. Further
\[
\tau(\omega_3) + \omega_5 = \begin{pmatrix} 3 & 4 & 5 & 4 & 2 \\ 2 \end{pmatrix}.
\]

We let $\{\beta_1, \beta_2, \beta_3\}$ be the unordered triple of roots:
\[
\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 \end{pmatrix}.
\]

We have, $\tau(\omega_3) + \omega_3 = \beta_1 + \beta_2 + \beta_3$. We proceed as in case 1, and conclude $m_5 = 3$. 

Case 6: $d = 6$.

We have, $i(\omega_6) = \omega_1$. Further, $\tau(\omega_1) + \omega_6 = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 \\ 2 & 0 & 1 & 1 & 1 \end{pmatrix}$. We let $\{\beta_1, \beta_2\}$ be the unordered pair of roots:
$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 2 & 3 & 2 & 1
\end{pmatrix}.
$$
We have $\tau(\omega_1) + \omega_6 = \beta_1 + \beta_2$. Proceeding as in case 1, we conclude $m_6 = 2$.

**Theorem 7.1.** The LMP conjecture holds for the minuscule $G/P$, $G, P$ being as above.

**Proof.** Proceeding as in the proof of Theorem 6.1, we just need to show that $\sum_{1 \leq d \leq 6} \text{ord}_\tau p_{w_0}^{(d)}$ equals $\text{codim}_{G/B} P/B$. From the above computations, we have $\sum_{1 \leq d \leq 6} \text{ord}_\tau p_{w_0}^{(d)}$ equals 16 which is precisely $\text{codim}_{G/B} P/B$. $\square$

8. Exceptional Group $E_7$

Let $G$ be simple of type $E_7$. Let $P$ be the maximal parabolic subgroup associated to the fundamental weight $\omega_7$ (the only minuscule weight in $E_7$). We preserve the notation of the previous sections; in particular, $\tau$ will denote the unique element of largest length in $W_P$.

**Step 1 & 2 of §5.1** Note that $\tau$ is the unique element of largest length inside the Weyl group of type $E_6$; $E_6$ sits inside $E_7$ in the natural way:

```
4 3 5 6
1 2
```

Thus, for $\alpha_i$, $1 \leq i \leq 6$, we have, $\tau(\alpha_i) = -i(\alpha_i)$, where $i$ is the Weyl involution on $E_6$. To be very precise, we have,

(\*) $\tau(\alpha_1) = -\alpha_6$, $\tau(\alpha_3) = -\alpha_5$, $\tau(\alpha_4) = -\alpha_i$, $i = 2, 4, 6$.

Thus, using (\!*), to find $\tau(i(\omega_d))$, $1 \leq d \leq 7$, as a linear sum (with rational coefficients) of the simple roots, it remains to find $\tau(\alpha_7)$. Towards computing $\tau(\alpha_7)$, we proceed as in §7. Let $\tau(\alpha_7) = \sum_{i=1}^{7} a_i \alpha_i$, where $a_i \in \mathbb{Z}$. Since $\alpha_7 \notin R_P$ (the root system of $P$), we have that $\tau(\alpha_7) \notin R_P$. Hence $a_7 \neq 0$; further, $\tau(\alpha_7) \in R^+$ (since, clearly, $l(\tau s_{\alpha_7}) = l(\tau) + 1$). Hence, $\alpha_7 > 0$; in fact, we have, $a_7 = 1$ (since any positive root in the root system of $E_7$ has an $\alpha_7$ coefficient $\leq 1$). Using (\*) above, and the following linear system, we determine the remaining $a_j$'s:

$$
\begin{align*}
2a_1 - a_4 &= \langle \tau(\alpha_7), \alpha_1^* \rangle = \langle \alpha_7, \tau(\alpha_1^*) \rangle = \langle \alpha_7, -\alpha_6^* \rangle = 1 \\
2a_2 - a_4 &= \langle \tau(\alpha_7), \alpha_2^* \rangle = \langle \alpha_7, \tau(\alpha_2^*) \rangle = \langle \alpha_7, -\alpha_6^* \rangle = 0 \\
2a_3 - a_4 - a_1 &= \langle \tau(\alpha_7), \alpha_3^* \rangle = \langle \alpha_7, \tau(\alpha_3^*) \rangle = \langle \alpha_7, -\alpha_5^* \rangle = 0 \\
2a_4 - a_2 - a_3 - a_5 &= \langle \tau(\alpha_7), \alpha_4^* \rangle = \langle \alpha_7, \tau(\alpha_4^*) \rangle = \langle \alpha_7, -\alpha_4^* \rangle = 0 \\
2a_5 - a_4 - a_6 &= \langle \tau(\alpha_7), \alpha_5^* \rangle = \langle \alpha_7, \tau(\alpha_5^*) \rangle = \langle \alpha_7, -\alpha_3^* \rangle = 0 \\
2a_6 - a_5 - a_7 &= \langle \tau(\alpha_7), \alpha_6^* \rangle = \langle \alpha_7, \tau(\alpha_6^*) \rangle = \langle \alpha_7, -\alpha_1^* \rangle = 0 
\end{align*}
$$

The fact that $a_7 = 1$ together with the last equation implies $a_5 \neq 0$ (and hence $a_6 \neq 0$, again from the last equation; note that all $a_i \in \mathbb{Z}^+$). Similarly, from the
first equation, we conclude $a_3 \neq 0$ (and hence $a_1 \neq 0$). From the first and third equations, we conclude $a_4 \neq 0$ (and hence $a_2 \neq 0$, in view of the second equation). Thus, all $a_i$’s are non-zero. The fifth equation implies that $a_4, a_6$ are of the same parity, and are in fact both even (in view of the second equation); hence $a_6 = 2$. Now working with the last equation and up, we obtain $a_5 = 3, a_4 = 4, a_2 = 2, a_3 = 3, a_1 = 2$.

Thus

$$\tau(\alpha_7) = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ 2 \end{pmatrix}.$$  

We proceed as in §7. Of course, the Weyl involution for $E_7$ is just the identity map. For each maximal parabolic subgroup $P_d, 1 \leq d \leq 7$, we will give the expression for $\tau(i(\omega_d)) + \omega_d (= \tau(\omega_d) + \omega_d)$ as an element in the root lattice, and the choice of $\{\beta_1, \cdots, \beta_r\}$ in $R^+ \setminus R^{+}_{P_d}$ which satisfy the conditions (a)-(d) in Step 2 of §5.1. Then deduce the value of $m_d$.

For convenience, we list the fundamental weights here:

$$\omega_1 = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ 2 \end{pmatrix},$$

$$\omega_2 = \frac{1}{2} \begin{pmatrix} 4 & 8 & 12 & 9 & 6 & 3 \\ 7 \end{pmatrix}, \omega_3 = \begin{pmatrix} 3 & 6 & 8 & 6 & 4 & 2 \\ 4 \end{pmatrix},$$

$$\omega_4 = \begin{pmatrix} 4 & 8 & 12 & 9 & 6 & 3 \\ 6 \end{pmatrix}, \omega_5 = \frac{1}{2} \begin{pmatrix} 6 & 12 & 18 & 15 & 10 & 5 \\ 9 \end{pmatrix},$$

$$\omega_6 = \begin{pmatrix} 2 & 4 & 6 & 5 & 4 & 3 \\ 3 \end{pmatrix}, \omega_7 = \frac{1}{2} \begin{pmatrix} 2 & 4 & 6 & 5 & 4 & 3 \\ 3 \end{pmatrix}.$$

**Case 1: $d = 1$.**

We have $\tau(\omega_1) + \omega_1 = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 2 \\ 2 \end{pmatrix}$. We let $\{\beta_1, \beta_2\}$ be the unordered pair of roots:

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 & 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 \end{pmatrix}.$$  

Then we have $\tau(\omega_1) + \omega_1 = \beta_1 + \beta_2$, and $m_1 = 2$.

**Case 2: $d = 2$.**

$Q = P_2$. We have $\tau(\omega_2) + \omega_2 = \begin{pmatrix} 2 & 4 & 6 & 5 & 4 & 3 \\ 3 \end{pmatrix}$. We let $\{\beta_1, \beta_2, \beta_3\}$ be the unordered triple of roots:

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 1 \\ 1 \end{pmatrix}.$$  

Then we have $\tau(\omega_2) + \omega_2 = \beta_1 + \beta_2 + \beta_3$, and $m_2 = 3$.

**Case 3: $d = 3$.**

We have $\tau(\omega_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 \end{pmatrix}$; thus $\tau(\omega_3) + \omega_3 = \begin{pmatrix} 3 & 6 & 8 & 6 & 5 & 4 \\ 4 \end{pmatrix}$.

We let $\{\beta_i, i = 1, \cdots, 4\}$ be the unordered quadruple of roots:

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 1 & 1 \end{pmatrix}.$$
Then we have $\tau(\omega_3) + \omega_3 = \sum_{1 \leq i \leq 4} \beta_i$, and $m_3 = 4$.

**Case 4:** $d = 4$.

We have $\tau(\omega_4) + \omega_4 = \begin{pmatrix} 4 & 8 & 12 & 10 \ 6 \end{pmatrix}$. We let $\{\beta_i, i = 1, \ldots, 6\}$ be the unordered 6-tuple of roots:

\[
\begin{pmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 2 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 \end{pmatrix}.
\]

Then we have $\tau(\omega_4) + \omega_4 = \sum_{1 \leq i \leq 6} \beta_i$, and $m_4 = 6$.

**Case 5:** $d = 5$.

We have $\tau(\omega_5) + \omega_5 = \begin{pmatrix} 3 & 6 & 10 & 9 & 7 & 5 \\ 5 \end{pmatrix}$. We let $\{\beta_i, i = 1, \ldots, 5\}$ be the unordered quintuple of roots:

\[
\begin{pmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 \end{pmatrix}.
\]

Then we have $\tau(\omega_5) + \omega_5 = \sum_{1 \leq i \leq 5} \beta_i$, and $m_5 = 5$.

**Case 6:** $d = 6$.

We have $\tau(\omega_6) + \omega_6 = \begin{pmatrix} 2 & 5 & 8 & 7 & 6 & 4 \\ 4 \end{pmatrix}$. We let $\{\beta_i, i = 1, \ldots, 4\}$ be the unordered quadruple of roots:

\[
\begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 \end{pmatrix}.
\]

Then we have $\tau(\omega_6) + \omega_6 = \sum_{1 \leq i \leq 4} \beta_i$, and $m_6 = 4$.

**Case 7:** $d = 7$.

We have $\tau(\omega_7) + \omega_7 = \begin{pmatrix} 2 & 4 & 6 & 5 & 4 & 3 \\ 3 \end{pmatrix}$. We let $\{\beta_1, \beta_2, \beta_3\}$ be the unordered triple of roots:

\[
\begin{pmatrix} 0 & 1 & 2 & 2 & 2 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 1 \\ 1 \end{pmatrix}.
\]

Then we have $\tau(\omega_7) + \omega_7 = \beta_1 + \beta_2 + \beta_3$, and $m_7 = 3$.

We have $\sum_{1 \leq d \leq 7} m_d = 27$ which is equal to codim$_{G/B} P/B$. Hence we obtain

**Theorem 8.1.** The LMP conjecture holds for the minuscule $G/P$, $G, P$ being as above.
9. The remaining minuscule $G/P$’s

In this section, we give the details for the remaining $G/P$’s along the same lines as in \cite{[4], [7], [8]} therefore providing an alternate proof for the results of \cite{[8], [7]}. We fix a maximal parabolic subgroup $P$ of $G$, denote (as in the previous sections), the element of largest length in $W_P$ by $\tau$. Also, as in the previous sections (cf. Definition \ref{def:5.6}), we shall denote $\text{ord}_e p_{\tau w_0}(\omega) = \text{ord}_e p_{w_0}(\omega)$ by $m_d, 1 \leq d \leq n$.

9.1. The simple root $\alpha$. While computing $m_d$, as in \cite{[8], [7], [4]} in each case, we work with a simple root $\alpha$ which occurs with a non-zero coefficient $c_\alpha$ in the expression for $\omega_d + \tau(i(\omega_d))$ (as a non-negative integral linear combination of simple roots) and which has the property that in the expression for any positive root (as a non-negative integral linear combination of simple roots), it occurs with a coefficient $\leq 1$. This $\alpha$ will depend on the type of $G$, and we shall specify it in each case. Then as seen in \cite{[4], [7], [8]} $m_d \geq c_\alpha$. We shall first exhibit a set of roots $\beta_1, \cdots, \beta_r, r = c_\alpha$ in $R^+ \setminus R^+_P$, satisfying (a)-(c) in Step 2 of \cite{[4], [7], [8]} and then conclude that $m_d = c_\alpha$. It will turn out (as shown below) that in all cases, $\sum_{1 \leq d \leq n} m_d$ equals $\text{codim}_{G/B} P/B$, thus proving the LMP conjecture (and hence Wahl’s conjecture).

9.2. Grassmannian. Let $G = SL(n)$. In this case, every maximal parabolic subgroup is minuscule. Let us fix a maximal parabolic subgroup $P := P_d$; we may suppose that $c \leq n - c$ (in view of the natural isomorphism $G/P_e \cong G/P_{n-c}$). Identifying the Weyl group with the symmetric group $S_n$, we have

$$\tau = (c \, c - 1 \, \cdots \, 1 \, n \, n - 1 \, \cdots \, c + 1).$$

Let $\epsilon_j, 1 \leq j \leq n$ be the character of $T$ (the maximal torus consisting of diagonal matrices in $G$), sending a diagonal matrix to its $j$-th diagonal entry. Note that $\sum_{1 \leq j \leq n} \epsilon_j = 0$ (writing the elements of the character group additively, as is customary); this fact will be repeatedly used in the discussion below. Also, for $1 \leq d \leq n - 1$, we have $i(\omega_d) = \omega_{n-d}$. We observe that in the expression for a positive root (as a non-negative integral linear combination of simple roots), any simple root occurs with a coefficient $\leq 1$. For each $P_d$, we shall take $\alpha$ to be $\alpha_d$.

**Case 1:** Let $d < c$. We have (cf.\cite{[1]})

$$\omega_d + \tau(i(\omega_d)) = \omega_d + \tau(\omega_{n-d}) = (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_1 + \cdots + \epsilon_c + \epsilon_{n-d+1} + \cdots + \epsilon_{c+d+1}) = \epsilon_1 + \cdots + \epsilon_d - (\epsilon_{c+1} + \cdots + \epsilon_{c+d}) = (\epsilon_1 - \epsilon_{-d}) + (\epsilon_2 - \epsilon_{d-1}) + \cdots + (\epsilon_d - \epsilon_{c+1}).$$

(note that $d < c \leq n - c$, and hence $n - d > c$). From the last expression, it is clear that $c_\alpha = d$ ($c_\alpha$ being as in \cite{[4], [7], [8]}); note that every one of the roots in the last expression belongs to $R^+ \setminus R^+_P$. We now let $\beta_1, \cdots, \beta_d$ be the unordered $d$-tuple of roots:

$$\epsilon_1 - \epsilon_{c+d}, \epsilon_2 - \epsilon_{c+d-1}, \cdots, \epsilon_d - \epsilon_{c+1}.$$  

Then it is easily checked that the above $\beta_j$’s satisfy (a)-(c) in Step 2 of \cite{[4], [7], [8]} and we have $m_d = d$ (in fact any such grouping will also work).
Case 2: Let \( c \leq d \leq n - c \). We have
\[
\omega_d + \tau(i(\omega_d)) = (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_1 + \cdots + \epsilon_c + \cdots + \epsilon_{c+d+1})
\]
\[
= \epsilon_1 + \cdots + \epsilon_c - (\epsilon_{d+1} + \cdots + \epsilon_{d+c})
\]
\[
= (\epsilon_1 - \epsilon_{d+c}) + (\epsilon_2 - \epsilon_{d+c-1}) + \cdots + (\epsilon_c - \epsilon_{d+1}).
\]
Hence \( c_n = c \). We now let \( \beta_1, \cdots, \beta_n \) be the unordered \( c \)-tuple of roots:
\[
\epsilon_1 - \epsilon_{d+c}, \epsilon_2 - \epsilon_{d+c-1}, \cdots, \epsilon_c - \epsilon_{d+1}.
\]
Then it is easily checked that the above \( \beta_j \)'s satisfy (a)-(c) in Step 2 of \S 5.1 and we have \( m_d = c \).

Case 3: Let \( n - c < d \leq n - 1 \). We have
\[
\omega_d + \tau(i(\omega_d)) = (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_c + \cdots + \epsilon_{c+d+1-n})
\]
\[
= -(\epsilon_{d+1} + \cdots + \epsilon_n) + (\epsilon_c + \cdots + \epsilon_{c+d+1-n})
\]
\[
= (\epsilon_c - \epsilon_{d+1}) + (\epsilon_{c-1} - \epsilon_{d+2}) + \cdots + (\epsilon_{d+1-n} - \epsilon_n)
\]
(note that \( c \leq n - c < d \)). Hence \( c_n = n - d \). We now let \( \beta_1, \cdots, \beta_{n-d} \) be the unordered \((n-d)\)-tuple of roots:
\[
\epsilon_c - \epsilon_{d+1}, \epsilon_{c-1} - \epsilon_{d+2}, \cdots, \epsilon_{c+d+1-n} - \epsilon_n.
\]
Then it is easily checked that the above choice of \( \beta_j \)'s satisfy (a)-(c) in Step 2 of \S 5.1 and we have \( m_d = n - d \).

From the above computations, we have,
\[
\sum_{1 \leq d \leq n-1} m_d = (1+\cdots+c-1)+(n-2c+1)c+(1+\cdots+c-1) = c(n-c) = \text{codim}_{G/B} P/B.
\]

9.3. Lagrangian Grassmannian. Let \( V = K^{2n} \) together with a nondegenerate, skew-symmetric bilinear form \((\cdot, \cdot)\). Let \( H = SL(V) \) and \( G = Sp(V) = \{ A \in SL(V) \mid A \text{ leaves the form } (\cdot, \cdot) \text{ invariant } \} \). Taking the matrix of the form (with respect to the standard basis \( \{ e_1, \ldots, e_{2n} \} \) of \( V \)) to be
\[
E = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}
\]
where \( J \) is the anti-diagonal \((1, \ldots, 1)\) of size \( n \times n \), we may realize \( Sp(V) \) as the fixed point set of a certain involution \( \sigma \) on \( SL(V) \), namely \( G = H^\sigma \), where \( \sigma : H \longrightarrow H \) is given by \( \sigma(A) = E(\frac{1}{2}A)^{-1}E^{-1} \). Denoting by \( T_H \) (resp. \( B_H \)) the maximal torus in \( H \) consisting of diagonal matrices (resp. the Borel subgroup in \( H \) consisting of upper triangular matrices) we see easily that \( T_H, B_H \) are stable under \( \sigma \). We set \( T_G = T_H^\sigma, B_G = B_H^\sigma \). Then it is well known that \( T_G \) is a maximal torus in \( G \) and \( B_G \) is a Borel subgroup in \( G \). We have a natural identification of the Weyl group \( W \) of \( G \) as a subgroup of \( S_{2n} \):
\[
W = \{(a_1 \cdots a_{2n}) \in S_{2n} \mid a_i = 2n + 1 - a_{2n+1-i}, 1 \leq i \leq 2n\}.
\]
Thus \( w = (a_1 \cdots a_{2n}) \in W_G \) is known once \( (a_1 \cdots a_n) \) is known.

For details see \[6\].

In the sequel, we shall denote such a \( w \) by just \((a_1 \cdots a_n)\); also, for \( 1 \leq i \leq 2n \), we shall denote \( 2n + 1 - i \) by \( i' \). The Weyl involution is the identity map. In type \( C_n \), we have that in the expression for a positive root (as a non-negative integral linear combination of simple roots), the simple root \( \alpha_n \) occurs with a coefficient \( \leq 1 \). For all \( P_d, 1 \leq d \leq n \), we shall take \( \alpha \) (cf. \S 6.1) to be \( \alpha_n \).
Let $P = P_n$. Then $\tau = (n \cdots 1)$. Let $1 \leq d \leq n$. We have
\[ \omega_d + \tau(\omega_d) = (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_n + \cdots + \epsilon_{n+1-d}). \]

If $d < n + 1 - d$, then each $\epsilon_j$ in the above sum are distinct. Hence writing
\[ \omega_d + \tau(\omega_d) = (\epsilon_1 + \epsilon_n) + (\epsilon_2 + \epsilon_{n-1}) + \cdots + \epsilon_d + \epsilon_{n+1-d} \]
we have that each root in the last sum belongs to $R^+ \setminus R^+_P$ (since each of the roots clearly involves $\alpha_d$); further, each of the roots involves $\alpha_n$ with coefficient equal to 1. Hence $c_\alpha = d$. We let $\beta_1, \cdots, \beta_d$ be the unordered $d$-tuple of roots:
\[ \epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_{n-1}, \cdots, \epsilon_d + \epsilon_{n+1-d}. \]

If $d \geq n + 1 - d$, then
\begin{align*}
\omega_d + \tau(\omega_d) &= (\epsilon_1 + \cdots + \epsilon_{n-d}) + (\epsilon_n + \cdots + \epsilon_{d+1}) + (2\epsilon_{n-d+1} + \cdots + 2\epsilon_d) \\
&= (\epsilon_1 + \epsilon_n) + (\epsilon_2 + \epsilon_{n-1}) + \cdots + (\epsilon_{n-d} + \epsilon_{d+1}) + 2\epsilon_{n-d+1} + \cdots + 2\epsilon_d.
\end{align*}

Again, we have that each root in the last sum belongs to $R^+ \setminus R^+_P$; and involves $\alpha_n$ with coefficient equal to 1. Hence we obtain that $c_\alpha = d$. We let $\beta_1, \cdots, \beta_d$ be the unordered $d$-tuple of roots:
\[ \epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_{n-1}, \cdots, \epsilon_{n-d} + \epsilon_{d+1}, 2\epsilon_{n-d+1}, \cdots, 2\epsilon_d. \]

Then it is easily checked that in both cases, the $\beta_j$'s satisfy (a)-(c) in Step 2 of §9.1 and we have $m_d = d$.

Hence
\[ \sum_{1 \leq d \leq n} m_d = \sum_{1 \leq d \leq n} d = \left(\frac{n+1}{2}\right) = \text{codim}_{G/B} P/B. \]

9.4. Orthogonal Grassmannian. Since $SO(2n+1)/P_n \cong SO(2n+2)/P_{n+1}$, and $SO(2n)/P_n \cong SO(2n)/P_{n-1}$, we shall give the details for the orthogonal Grassmannian $SO(2n)/P_n$. Thus $G = SO(2n), P = P_n$, and $\tau = (n \cdots n - 1 \cdots 1)$.

In type $D_n$, we have that in the expression for a positive root (as a non-negative integral linear combination of simple roots), the simple root $\alpha_n$ occurs with a coefficient $\leq 1$. For all $P_d, 1 \leq d \leq n$, we shall take $\alpha$ (cf. §5.1) to be $\alpha_n$.

Case 1: Let $1 \leq d \leq n - 2$. We have,
\[ \omega_d + \tau(\omega_d) = (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_n + \cdots + \epsilon_{n+1-d}). \]

If $d < n + 1 - d$, then as in §9.3 we have
\[ \omega_d + \tau(\omega_d) = (\epsilon_1 + \epsilon_n) + (\epsilon_2 + \epsilon_{n-1}) + \cdots + (\epsilon_d + \epsilon_{n+1-d}). \]

We have that each root in the last sum belongs to $R^+ \setminus R^+_P$; and involves $\alpha_n$ with coefficient equal to 1. Hence $c_\alpha = d$. We let $\beta_1, \cdots, \beta_d$ be the unordered $d$-tuple of roots:
\[ \epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_{n-1}, \cdots, \epsilon_d + \epsilon_{n+1-d}. \]

If $d \geq n + 1 - d$, then
\begin{align*}
\omega_d + \tau(\omega_d) &= (\epsilon_1 + \cdots + \epsilon_{n-d}) + (\epsilon_n + \cdots + \epsilon_{d+1}) + (2\epsilon_{n-d+1} + \cdots + 2\epsilon_d) \\
&= (\epsilon_1 + \epsilon_{n+1-d}) + (\epsilon_2 + \epsilon_{n+2-d}) + \cdots + (\epsilon_d + \epsilon_n).
\end{align*}
Again we have that each root in the last sum belongs to $R^+ \setminus R^+_{P_1}$, and involves $\alpha_n$ with coefficient equal to 1. Hence we obtain that $c_\alpha = d$. We let $\beta_1, \ldots, \beta_d$ be the unordered $d$-tuple of roots:

$$\begin{align*}
(\epsilon_1 + \epsilon_{n+1-d}), (\epsilon_2 + \epsilon_{n+2-d}), \ldots, (\epsilon_d + \epsilon_n).
\end{align*}$$

Then it is easily checked that in both cases, the $\beta_j$’s satisfy (a)-(c) in Step 2 of §5.1 and we have $m_d = d$.

**Case 2:** $d = n - 1$. We have,

$$\omega_{n-1} + \tau(i(\omega_{n-1})) = \begin{cases} 
(\epsilon_2 + \cdots + \epsilon_{n-1}), & \text{if } n \text{ is even} \\
(\epsilon_1 + \cdots + \epsilon_{n-1}), & \text{if } n \text{ is odd}.
\end{cases}$$

Hence expressing $\omega_{n-1} + \tau(i(\omega_{n-1}))$ as a non-negative linear integral combination of positive roots, we obtain

$$\omega_{n-1} + \tau(i(\omega_{n-1})) = \begin{cases} 
\sum_{2 \leq i \leq \frac{n+1}{2}} \epsilon_i + \epsilon_{n+1-i}, & \text{if } n \text{ is even} \\
\sum_{1 \leq i \leq \frac{n-1}{2}} \epsilon_i + \epsilon_{n-i}, & \text{if } n \text{ is odd}.
\end{cases}$$

Thus $\omega_{n-1} + \tau(i(\omega_{n-1}))$ is a sum of $\frac{n-1}{2} \alpha$ or $\frac{n-1}{2} \beta$ roots in $R^+ \setminus R^+_{P_{n-1}}$, according as $n$ is even or odd; further, each of them involves $\alpha_n$ with coefficient one. Hence

$$c_\alpha = \begin{cases} 
\frac{n-2}{2}, & \text{if } n \text{ is even} \\
\frac{n-1}{2}, & \text{if } n \text{ is odd}.
\end{cases}$$

We let $\beta_1, \ldots, \beta_r, r = c_\alpha$ be the unordered $r$-tuple of roots:

$$\beta_i = \begin{cases} 
\epsilon_i + \epsilon_{n+1-i}, & \text{if } i \text{ is even} \\
\epsilon_i + \epsilon_{n-i}, & \text{if } i \text{ is odd}.
\end{cases}$$

Clearly, the above $\beta_j$’s satisfy (a)-(c) in Step 2 of §5.1

$$m_{n-1} = \begin{cases} 
\frac{n-2}{2}, & \text{if } n \text{ is even} \\
\frac{n-1}{2}, & \text{if } n \text{ is odd}.
\end{cases}$$

**Case 3:** $d = n$. Proceeding as in Case 2, we have,

$$\omega_n + \tau(i(\omega_n)) = \begin{cases} 
(\epsilon_1 + \cdots + \epsilon_n), & \text{if } n \text{ is even} \\
(\epsilon_2 + \cdots + \epsilon_n), & \text{if } n \text{ is odd}.
\end{cases}$$

Hence we obtain

$$\omega_n + \tau(i(\omega_n)) = \begin{cases} 
\sum_{1 \leq i \leq \frac{n+1}{2}} \epsilon_i + \epsilon_{n+1-i}, & \text{if } n \text{ is even} \\
\sum_{2 \leq i \leq \frac{n+1}{2}} \epsilon_i + \epsilon_{n+2-i}, & \text{if } n \text{ is odd}.
\end{cases}$$

Thus $\omega_{n-1} + \tau(i(\omega_{n-1}))$ is a sum of $\frac{n}{2}$ or $\frac{n-1}{2}$ roots in $R^+ \setminus R^+_{P_n}$, according as $n$ is even or odd; further, each of them involves $\alpha_n$ with coefficient one. Hence

$$c_\alpha = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even} \\
\frac{n-1}{2}, & \text{if } n \text{ is odd}.
\end{cases}$$
We let \( \beta_1, \ldots, \beta_r, r = c_\alpha \) be the unordered \( r \)-tuple of roots:

\[
\beta_i = \begin{cases} 
\epsilon_i + \epsilon_{n+1-i}, & 1 \leq i \leq \frac{n}{2}, \text{ if } n \text{ is even} \\
\epsilon_i + \epsilon_{n+2-i}, & 2 \leq i \leq \frac{n+1}{2}, \text{ if } n \text{ is odd}.
\end{cases}
\]

Clearly, the above choice of \( \beta_j \)'s satisfies (a)-(c) in Step 2 of §5.1.

\[
m_n = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even} \\
\frac{n-1}{2}, & \text{if } n \text{ is odd}.
\end{cases}
\]

Combining cases 2 and 3, we obtain \( m_{n-1} + m_n = n-1 \), in both even and odd cases. Combining this with the value of \( m_d, 1 \leq d \leq n-2 \) as obtained in case 1, we obtain

\[
\sum_{1 \leq d \leq n} m_d = \sum_{1 \leq d \leq n-2} d + n - 1 - \binom{n}{2} = \text{codim}_{G/B} P/B.
\]

Thus combining the results of §6, §7, §8, §9, we obtain

**Theorem 9.5.** The LMP conjecture and Wahl’s conjecture hold for a minuscule \( G/P \).

**Remark 9.6.** Thus in these cases again, we obtain a nice realization for \( \text{ord}_{P} p_{\omega_d^0(0)} \) (the order of vanishing along \( P/B \) of \( p_{\omega_d^0(0)} \)) as being the length of the shortest path through extremal weights in the weight lattice connecting the highest weight (namely, \( i(\omega_d) \) in \( H^0(G/B, L(\omega_d)) \)) and the extremal weight \( -\tau(\omega_d) \).

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