Features of stability loss of structures on an elastic foundation

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Abstract. The features of stability loss for some models of structures on the elastic foundation (e.g. ground) modeled using the Winkler’s foundation were researched. The first model of the structure is given in form of Mises’s truss. The second one is presented as the sloping arc stressed by horizontal load. The pressure related to the deformation was considered. Researches to see how system loses stability were made. In case of Mises’s truss, the solution is obtained in algebraic form. In case of arc the solution is represented in integral-differential form. Methods to solve them are suggested. Computational investigations are made. The method of getting the critical values for load was given. It was discovered that for the lengths big enough the stability loss is accompanied by the “crackling”.

Keywords: postbuckling, post-buckling, stability, stability loss, elasticity, theory of elasticity, elastic foundation, building constructions, deformation, load, critical load, rod, rod system, arc, arc compression.

1 Introduction

Much attention has been paid to the problems of losing stability of the structures and tasks of this kind have not lost their relevance to this day.

The work [1] is devoted to the consideration of problems about compressing rod systems in elastic foundation. The author showed that for the big lengths, unlike the problems of compression without an elastic foundation, after losing stability the deformation value increased while the load increased too.

In the work [3] authors considered problem of elastic transitions from one equilibrium state to another for thin-walled structures in the form of a thin plate previously bent under the action of the compressive force.

Works [2], [4] provide concepts, methodologies of analysis and design and their applications. They also contain problems and formulas for the buckling loads of some structural elements.

Works [5-8] are related to the studying stressed thin plates and shells. They provide information on how compression influences these elements [9-15].

Works [16-17], are devoted to researching fiber composite materials [18-23]. Especially how they act when they are stressed by the compressive forces. They provide the results and methodologies of analytical and experimental researches.

And in this work, we consider the problems of losing stability of structures on an elastic foundation.
2 Methods and Materials

2.1 Post-buckling behavior of the rod system

At first, let us study the problem of compressing symmetrical rod system with an elastic connection (figure 1). We will model the elastic foundation using the Winkler foundation as shown in the figure 1.

![Figure 1. Model elastic foundation using the Winkler foundation.](image)

Also, let us assume that the rods can be inclined before the compression; we will denote the angle of initial inclination as $\theta_0$. In case of symmetry, consider the right part of the system only (figure 2).

![Figure 2. Right part of the system.](image)

Let us say that the reaction of the system $r$ is proportional to the vertical offset of the left end of the rod:

$$r = k y_0 = k s (\sin(\theta_0 + \theta) - \sin \theta_0).$$

We get the support’s reaction $R$, according to the equilibrium condition:

$$R = \int_0^L r ds = \frac{L^2 k (\sin(\theta_0 + \theta) - \sin \theta_0)}{2}$$

where $L$ is the length of the rod and $s$ is the length counted from the support.

Let’s assume that the links are connected using the elastic spring so that moment $M_x$ is proportional to the angle $\theta_0$:

$$M_x = G \theta_0.$$  

The condition that the moment sum in $O$-support equals to zero gives us the relation for load $P$ (Eq. (3.1)).

Then let us consider the relation of value for load $P$ of $\theta_0$ (Eq. (3.1)) using various values for angle $\theta_0$ (figure 3). Before plotting the graph (figure 3), it was assumed that $G = 5000$ N/cm$^2$, $L = 27$ cm, $k = 1$ N/cm$^2$. 

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Figure 3. The relation of load $P$ for various initial deformations.

Now let us try to get the relation for the critical pressure $P_{\text{crit}}$. To do it let us find the first derivative of $P$ (Eq. (1.4)), equate it to zero and find out the critical value for the angle $\theta_{\text{d}}$ by solving the resulting equation.

$$
\frac{2kL^2 \cos \theta_{\text{d}} \cot (\theta_{\text{d}} + \theta_{\text{0}})}{3} + \frac{2G}{L \sin (\theta_{\text{d}} + \theta_{\text{0}})} - \frac{2G \theta_{\text{d}} \cot (\theta_{\text{d}} + \theta_{\text{0}})}{L \sin (\theta_{\text{d}} + \theta_{\text{0}})} - \frac{kL^2 \cos \theta_{\text{d}} \cot (\theta_{\text{d}} + \theta_{\text{0}})}{3\sin^2 (\theta_{\text{d}} + \theta_{\text{0}})} = 0
$$

(1.4)

The equation cannot be solved in given form (Eq. (1.4)). Let us expand the trigonometric functions into the Maclaurin series and find out the roots (Eq. (3.2)). To choose the correct value from Eq. (3.2) we should get the second derivative $P$ of $\theta_{\text{d}}$ and put the values $\theta_{\text{d,crit}}$ from Eq. (3.2) into it. The pressure when the system loses stability is achieved when the second derivative becomes negative.

2.2 The problem of arc compression

Let us consider the arc on an elastic foundation (e.g. ground), compressed by the load $P$ (figure 4).

Figure 4. Arc on an elastic foundation.

We denote the length counted from the left support of arc as $s$, the full length of arc as $L$, the initial tangent angle to the arc axis as $\theta_{\text{0}}$ and the increment to the initial tangent angle $\theta_{\text{0}}$ as $\theta_{\text{d}}$ (figure 4). For the given system (figure 4) we have next relations.
\[ \frac{dz}{ds} = \cos(\alpha(s) + \beta(s)), \quad \frac{dy}{ds} = \sin(\alpha(s) + \beta(s)). \]  

From relations (2.1) we get.

\[ z_i = \int_0^s \cos(\alpha(s) + \beta(s))ds, \quad y_i = \int_0^s \sin(\alpha(s) + \beta(s))ds. \]

We assume that the support reaction is proportional to the vertical offset of the \( ds \) cut.

\[ r(s) = ky_i(s), \]

where

\[ y_c = y - y_0 = \int_0^s (\sin(\alpha + \beta) - \sin\alpha)ds. \]

Express the angles \( \alpha \) and \( \beta \) using the \( \theta_0 \) and \( \theta_0 \) angles.

\[ \alpha(s) = \theta_0 - \frac{2s\theta_0}{L}, \quad \beta(s) = \theta_0 - \frac{2s\theta_0}{L}. \]

Let us consider the left part of the arc cut on distance \( z_i \) from the left support (figure 5).

![Figure 5. Left part of the arc.](image)

To figure out the bending moment \( M_i^x \) we use the following relation.

\[ M_i^x = Rz_i - Q(z_i - z_i) - Py_i. \]

Now let us figure out \( Q_i \).

\[ Q_i = \int_0^s dQ = \int_0^s r(s)ds = k \int_0^s y_i(s)ds. \]

Following the equilibrium condition, we may get the relation for \( R \).

\[ R = \frac{1}{2} \int_0^s rds = \frac{k}{2} \int_0^s y_i(s)ds. \]

Reactive loads moment in area of \( A \)-support is defined as.

\[ M_i^z = \int_0^s z(s)r(s)ds = k \int_0^s z(s)y_i(s)ds. \]

By the other side, let us define it using \( Q_i \).

\[ M_i^z = Qz_i^c. \]

Let us rewrite the relation for \( M_i^z \) now.

\[ M_i^z = Rz_i - Py_i - Qz_i + M_i^j. \]

Bending equation for the arc element (figure 5).

\[ EJ \frac{d\beta}{ds} = M_i^j. \]
Now, after putting the Eqs. (2.7), (2.10) and (2.11) into the Eq. (2.12), we get the relation for load $P$ (Eq. 3.3).

To find out the value $P$ we use the collocations method. Imagine that we divide the arc into the $n$ parts. The value $n$ should be found from the condition where the difference between the average value of load for $n+1$ and $n$ is not greater than 5%. Let us plot the graph of average values of $P_{avg}$ (figure 6). Assume that:

$$
E = 13 \times 10^3 \gamma_{\alpha}, L = 0.13m, k = 6 \times 10^5 \gamma_{\alpha},
$$

$$
J = \frac{\pi d^4}{64}, d = 0.0005m, \theta_0 = 0.031rad, \theta_e = 0.2rad
$$

(2.13)

![Graph of average values of $P_{avg}$](image1.png)

**Figure 6.** Graph of average values of $P_{avg}$.

As we can see on the graph (figure 6), to achieve proper accuracy we can get $n = 5$. The relation $P$ of $\theta_e$ for the various $\theta_0$ figured out using the Eq. (3.3) is in the figure 7.

![The relation of $P_{avg}$ for various initial deformations.](image2.png)

**Figure 7.** The relation of $P_{avg}$ for various initial deformations.

### 3 Results

The relation for load $P$ in case of Mises’s truss.
\[
P = \frac{1}{y_0 + y_0} \left( \frac{2G\theta_0 + RL\cos(\theta_0 + \theta_0)}{3} - \frac{RL\cos(\theta_0 + \theta_0)}{3} \right) = \frac{2G\theta_0}{L\sin(\theta_0 + \theta_0)} + \frac{L^2k\cot(\theta_0 + \theta_0)\sin(\theta_0 + \theta_0) - \sin(\theta_0)}{3}
\]

The relation for the critical deformation angle.

\[
\theta_{crit} = -144G\theta_0 - 24kL^2\theta_0 + 24G\theta_0^3 + 28kl^2\theta_0^3 - 10kL^3 + kL^3\theta_0^3 + (72kL^2\theta_0 - 50kL^2\theta_0^3 + 7kL^2\theta_0^3)N - (72G\theta_0 - 72kL^2\theta_0 + 100kL^2\theta_0^3 - 21kL^2\theta_0^3)N^2 - (48G - 24kL^3 + 100kL^2\theta_0^3 - 35kL^2\theta_0^3 - 35kL^2\theta_0^3)N^3 + (35kL^2\theta_0 - 50kL^2\theta_0)N^4 + (21kL^2\theta_0 - 10kL^3)N^5 + 7kL^2\theta_0\theta_0^4 + kL^3N^7,
\]

\(N \in \{1, 2, 3, 4, 5, 6, 7\}\)

The relation for load \(P\) in case of sloping arc.

\[
P = \frac{(R - Q)z_i + M_j' - EJ\frac{dB}{ds}}{y_i} = k\left(\frac{1}{2} \int_0^y y_i(s) ds - k \int_0^y y_i(s) ds z_i(s) + k \int_0^y z_i(s) y_i(s) ds - EJ\frac{dB}{ds}\right)
\]

4 Discussions
As we can see in the figure, when the system gets the critical pressure it loses stability with a “crackling”. Moreover, as we can see in the figure, our first statement that the system loses stability with a “crackling” confirmed again. Nevertheless, let us note that the statement is not correct for the small lengths.

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