Stability Analysis and Error Estimation of Variable Coefficient Convection-Diffusion Equation: Generalized Numerical Fluxes

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Abstract. In this paper, we study the discontinuous Galerkin (DG) method with generalized numerical fluxes for one-dimensional variable coefficient convection-diffusion equation. For the convection term, we choose the upwind numerical flux, and for the diffusion term, we choose a special type of generalized numerical fluxes, thus we first show that the $L^2$ stability of the DG scheme. Then, by introducing the projection method, we are able to show $k$ order optimal error estimates for the DG scheme. Finally, Numerical experiment is provided to verify the theoretical results.

1. Introduction.
Convection-diffusion equations are a class of mathematics models that can describe many physical phenomena in the fields of chemistry, fluid mechanics, aerodynamics, etc. Therefore, the numerical solutions of convection-diffusion equations has always been one of the important topics in the study of the numerical solution of partial differential equations.

The discontinuous Galerkin (DG) method is mainly used for the spatial discretization of equations. This method was first proposed by Reed and Hill[1] in the United States in 1973 when they studied the steady-state linear neutron transport problem. In 2012, Meng[2]Shu, Zhang, and Wu refered one-dimensional nonlinear conservation laws, proved that when the upwind flux is used, the error can reach $k + 3/2$ order superconvergence. In 2015, Meng[3], Shu and Wu gave the DG scheme of the upwind-biased numerical flux for one-dimensional hyperbolic laws and obtained the $k + 1$ order optimal error estimate. In 2017, Cao[4], Li, Yang, and Zhang studied the superconvergence properties of the DG method with upwind-biased flux for linear hyperbolic laws, and obtained $2k + 1$ order superconvergence. In 2019, Li[5], Zhang, Meng, and Wu gave a DG scheme for selecting the upwind numerical flux for linear hyperbolic laws with degenerate variable coefficients, the stability of the scheme and the $k + 1$ order optimal error estimate are obtained. In the same year, Fu[6], Cheng, Li and Xu analyzed the DG method for several partial differential equations with higher order spatial derivatives. Identify a sub-family of numerical flux by selecting the coefficients in the linear combination, so that the solution of the proposed DG method and some auxiliary variables have the best accuracy under the norm. In 2020, Li[7], Zhang, Meng, Wu and Zhang studied the DG methods...
with generalized numerical fluxes for nonlinear hyperbolic laws, and by constructing and analyzing a special piecewise global projection, gave the optimal order error estimate of the DG scheme.

In this paper, we concentrate on DG method with generalized fluxes for one-dimensional variable coefficient convection-diffusion equation.

\[ u_t + a(x)u_x = b(x)u_x + f(x,t), \quad (x,t) \in I \times (0,T), \]

\[ u(x,0) = u_0(x), \quad \text{at} \ x \in I. \]  

(1)

where \( u_0(x) \) is a smooth function, \( I = [c,d] \), \( a(x), b(x) \in W^{1,\infty}[c,d] \) are assumed to be sufficiently smooth with respect to \( x \) for simplicity, the article assumes \( a(x) < 0, \ b(x) > 0 \), and periodic boundary conditions are considered.

The organization of this paper is as follows. In Section 2, the DG scheme of the variable coefficient convection-diffusion equation under a special type of generalized numerical fluxes and the stability analysis is presented. In Section 3, by introducing a special global projection, the order optimal error estimate is obtained. In Section 4, numerical experiment is shown, which confirms the order of optimal error estimates. Concluding remarks are given in section 5.

2. The DG scheme with generalized numerical fluxes.

2.1. Basic notation.

In this paper, we use the following usual notation of the DG methods. Denote computational interval as \( I = [c,d] \), and it is divided into \( N \) cells \( I_j = [x_{j-1/2}, x_{j+1/2}] \) for \( j = 1, \ldots, N \), where \( c = x_{1/2} < x_1 < \cdots < x_{N-1/2} = d \) and the mean value of the function \( x \) is \( x_j = \frac{1}{2}(x_{j-1/2} - x_{j+1/2}) \). The tessellation of \( I \) is denoted as \( I_h = \{ I_j \}_{j=1}^N \) and the length of \( I_h \) is denoted as \( h_j = x_{j+1/2} - x_{j-1/2} \). In what follows, we define \( \| \omega^+ - \omega^- \| \) and \( \| \omega^+ + \omega^- \| \) as jump and the cell center of \( \omega \) at cell boundaries point.

We choose the following discontinuous finite element space

\[ V_h^k = \{ \omega : \omega \big|_{I_j} \in P_i(I_j), j = 1, \ldots, N \}, \]

where \( P_i(I_j) \) denotes the space of polynomials of degree up to \( k \) defined on cell \( I_j \).

The broken Sobolev space on \( I_h \) is denoted as

\[ W^{1,p}(I_h) = \{ u \in L^2(I) : u \big|_{I_j} \in W^{1,p}(I_j), j = 1, \ldots, N \}, \]

and the norms are denoted as \( \| u \|_{W^{1,p}(I_h)} = \max_{1 \leq j \leq N} \| u \|_{W^{1,p}(I_j)} \) and \( \| u \|_{W^{1,p}(I_h)} = \left( \sum_{j=1}^N \| u \|_{W^{1,p}(I_j)}^p \right)^{1/p} \). The notation \( H^1(I_h) = W^{1,2}(I_h) \), \( L^2(I) = H^0(I_h) \) are adopted. In addition, the boundary norms are denoted as \( \| u \|_{L^2(I_h)}^2 = \sum_{j=1}^N \| u \|_{L^2(I_j)}^2 \) and \( \| u \|_{L^2(I_h)} = \left( u_{j+1/2}^+ + u_{j-1/2}^- \right)^2 \).

Since the complexity of the equations expression in the proof, thus we introduce the DG discrete operator, \( \forall w, v \in V_h^k, c(x) \in W^{1,\infty}(I) \)

\[ H_f(w,v;c) = (c(x)w,v)_{x_{j+1/2}} - c\widehat{w}v^-_{x_{j+1/2}} + c\widehat{w}v^+_{x_{j+1/2}}, \]

which is denoted on \( I_j = \left[ x_{j-1/2}, x_{j+1/2} \right] \). Using \( H_f \) to represent the sum of \( H_f \) with respect to \( I_j \), then
\[ H(w, v; c) = \sum_{j=1}^{N} H_j(w, v; c). \]

The following inverse properties will be used for variable coefficient convection-diffusion equation. For all \( v_h \in V_h^k \), there holds, for more details, see[8].

\[ (i) \| (v_h) \|_{L^2(I)} \leq C h^{-1} \| v_h \|_{L^2(I)}, \]

\[ (ii) \| v_h \|_{L^2(I)} \leq C h^{-\frac{1}{2}} \| v_h \|_{L^2(I)}, \]

\[ (iii) \| v_h \|_{L^2(I)} \leq C h^{-\frac{1}{2}} \| v_h \|_{L^2(I)}. \]

2.2. The DG scheme

Refering to the variable coefficient convection-diffusion equation (1), the DG scheme is as follows: \( \forall t \in (0, T] \), find \( u_h(t) \in V_h^k \) such that for any \( v_h \in V_h^k \) and \( j = 1, \cdots, N \) there holds

\[ \int_I (u_h) v_h dx - \int_I a(x) u_h (v_h)_x dx + a \hat{u}_h v_h^+ |_{j-\frac{1}{2}} - a \hat{u}_h v_h^- |_{j-\frac{1}{2}} + \int_I b(x)(u_h)_x (v_h)_x dx \]

\[ - b(x)(u_h)_x v_h^- |_{j-\frac{1}{2}} + b(x)(u_h)_x v_h^+ |_{j-\frac{1}{2}} = \int_I f v_h dx, \]

where \( \hat{u}_h, b(x)(u_h)_x \) are numerical fluxes. Since the selection of the convection term and the diffusion term is independent of each other, the selection of the numerical fluxes in this paper is as follows,

for the convection term, we choose

\[ \hat{u}_h = u_h^+, \]

(3)

for the diffusion term, we choose

\[ b(x)(u_h)_x = b(x)(u_h)_x + \phi[u_h]. \]

(4)

Let’s verify that the semi-discrete discontinuous finite element scheme has \( L^2 \) stability by selecting the above two numerical fluxes.

Theorem 1. ( \( L^2 \) stability) The DG scheme (2) is \( L^2 \) stable under numerical fluxes (3) and (4), namely

\[ \| u_h(t) \|_{L^2(I)} \leq C \left( \| u_h(0) \|_{L^2(I)} + \int_0^T \| f \|_{L^2(I)} dt \right). \]

(5)

Proof. From the choice of numerical fluxes and the definition of DG discrete operator, the following formula is established

\[ \int_I (u_h) v_h dx - H^+(u_h, v_h; a) + \int_I b(x)(u_h)_x (v_h)_x dx - (b(x)((u_h)_x + \phi[u_h]) v_h^- |_{j-\frac{1}{2}} \]

\[ + (b(x)(u_h)_x + \phi[u_h]) v_h^+ |_{j-\frac{1}{2}} = \int_I f v_h dx. \]

(6)

for any \( v_h \in V_h^k \) and \( j = 1, \cdots, N \). Taking \( v_h = u_h \) in equation (6) and summing up over all \( j \), then equation (6) can be written as

\[ \frac{1}{2} \frac{d}{dt} \| u_h \|_{L^2(I)}^2 - H^+(u_h, u_h; a) + \int_I (b(x)(u_h)_x^2 dx + \sum_{j=1}^{N} (b(x)((u_h)_x + \phi[u_h]) [v_h]) |_{j-\frac{1}{2}} = \int_I f v_h dx. \]

(7)

By the periodic boundary conditions, we have

\[ u_h^+ = \{ u_h \}^+ + \frac{1}{2} \{ u_h \}. \]

(8)
Hence

\[ H^+(u_h, u_{h}; a) = \sum_{j=1}^{N} \left[ \int_{I_j} a(x) \left( \frac{u_h^2}{2} \right)_x dx + au_h \left[ [u_h] ]_{j+\frac{1}{2}} \right] \right] \]

\( \left[ [u_h] ]_{j+\frac{1}{2}} \right] = \left[ \frac{1}{2} \left[ \frac{u_h^2}{2} \right] \right]_{j+\frac{1}{2}} + \frac{1}{2} \left[ \frac{u_h^2}{2} \right]_{j+\frac{1}{2}} \]

In addition, due to \( \left[ [u_h] ]_{j+\frac{1}{2}} \right] = a \left( \frac{u_h^2}{2} \right)_{j+\frac{1}{2}} - a \left( \frac{u_h^2}{2} \right)_{j+\frac{1}{2}} \),

by the Newton Leibniz formula, we have

\[ H^+(u_h, u_{h}; a) = \sum_{j=1}^{N} \left[ \int_{I_j} a(x) \left( \frac{u_h^2}{2} \right)_x dx + a \{ [u_h] ]_{j+\frac{1}{2}} \right] + a \left( \frac{1}{2} [u_h]^2 \right)_{j+\frac{1}{2}} \]

Since \( a(x) \leq 0 \), thus \( a \left( \frac{1}{2} [u_h]^2 \right)_{j+\frac{1}{2}} \leq 0 \).

Then, by the Cauchy Schwartz’s inequality, we get

\[ H^+(u_h, u_{h}; a) \leq \sum_{j=1}^{N} \left[ \int_{I_j} a(x) \left( \frac{u_h^2}{2} \right)_x dx + a \left( \frac{1}{2} [u_h]^2 \right)_{j+\frac{1}{2}} \right] \]

Thus, equation (7) becomes

\[ \frac{1}{2} \frac{d}{dt} \| u_h \|_{L^2(I(t))}^2 = -\int b(x) (u_h)^2 dx - \sum_{j=1}^{N} (b(x)(u_h)_{x} + \phi [u_h] (u_h)]_{j+\frac{1}{2}} + H^+(u_h, u_{h}; a) + \int f u_h dx \]

\[ \leq -\int b(x) (u_h)^2 dx - \sum_{j=1}^{N} (b(x)(u_h)_{x} + \phi [u_h] (u_h)]_{j+\frac{1}{2}} + \sum_{j=1}^{N} (\phi b(x) [u_h]^2)_{j+\frac{1}{2}} + C \| u_h \|_{L^2(I(t))}^2 \]

By Young inequality, after some simple algebraic calculations

\[ \left( (u_h)_{j+\frac{1}{2}} \right) \leq \frac{C}{h} \int_{I_j} (u_h)^2 dx \]

Taking \( \delta = \frac{C}{h} \), in equation(12), we have

\[ \frac{1}{2} \frac{d}{dt} \| u_h \|_{L^2(I(t))}^2 \leq -\frac{C}{h} \sum_{j=1}^{N} (b(x)(u_h)_{x} + \phi [u_h] (u_h)]_{j+\frac{1}{2}} + \| f \|_{L^2(I(t))} \| u_h \|_{L^2(I(t))} + \| u_h \|_{L^2(I(t))}^2 \]

The \( L^2 \) stability result can be obtained by integrating the above inequality with respect to time between 0 and \( T \). This ends the proof.
3. Optimal error estimates.

This section is devoted to proof of optimal error estimates of DG methods with (2) for variable coefficient convection-diffusion equation.

Let’s first introduce the following projection . For \( h \in H^1(\Omega_h) \), \( P_h u \in V_h^k \) is the unique piecewise polynomial satisfying

\[
\int_{I_j} (P_h^+ u) \phi dx = \int_{I_j} u \phi dx, \quad \forall \phi \in P^{k-1}(I_j),
\]

where \( P_h^+ u = u^+ \), at \( x_j \), \( P_h^- u = u^- \), at \( x_j \).

for \( j = 1, \ldots, N \), if \( u \in H^{k+1}(\Omega_h) \), then there holds the following optimal approximation property:

\[
\|u - P_h^+ u\|_{L^2(I_j)} + h^2 \|u - P_h^- u\|_{L^2(I_j)} \leq C h^{k+1} \|\eta\|_{H^{k+1}(\Omega)}.
\]

where \( C \) is independent of \( h \), for more details, see[9][10]. As for the initial discretization, we usually use the following standard \( L^2 \) projection \( \pi_h \), and we have

\[
\|u_0 - \pi_h u\|_{L^2(I_j)} \leq C h^{k+1} \|\eta\|_{H^{k+1}(\Omega)}.
\]

We are now ready to proof the optimal error estimates with the fluxes (3) and (4).

Theorem 2 (error estimate). Let the exact solution \( u \) of (1.1) is sufficiently smooth with bounded derivatives, i.e., \( \|u\|_{L^2(I_j)} \) and \( \|u_t\|_{L^2(I_j)} \) are bounded uniformly for any time \( t \in [0, T] \). Let \( u_h \) be the DG solution with numerical fluxes (3) and (4) for variable coefficient convection-diffusion equation. which corresponds, respectively, to the periodic boundary problem. Assume \( \phi \geq \frac{C}{h} \). For regular triangulations, if the finite element space \( V_h^k \) of degree \( k \geq 0 \) is used, then for \( T > 0 \), there holds the following error estimate:

\[
\|u(T) - u_h(T)\|_{L^2(I_j)} \leq Ch^k.
\]

where \( C \) is independent of \( h \).

Proof. We will finish the proof with the following two steps.

Step 1: Error equation. Since the exact solution \( u \) satisfies the DG scheme (2) and by Galerkin orthogonality, there holds the following error equation at each element,

\[
\int_{I_j} (e) v_h dx - H^+ (e, v_h; a) + \int_{I_j} b(x) e_x v_h dx + (b(x)(e)_x + \phi[e])[v_h]|_{x_j} = 0.
\]

for any \( v_h \in V_h^k \) and \( j = 1, \ldots, N \), denoting the error as \( e = u - u_h \) and \( \eta = u - P_h u, \xi = P_h u - u_h \), taking \( v_h = \xi \in V_h^k \) in equation (20) and summing up over all \( j \), the error equation (20) can be written as

\[
\frac{d}{dt} \|\xi\|_{L^2(I_j)}^2 + \int_{I_j} H^+ (\xi, \xi; a) - H^+ (\eta, \xi; a) + \int_{I_j} b(x)(\xi)_x dx + \int_{I_j} b(x)(\eta) dx + \sum_{j=1}^N (b(x)(\xi)_x + \phi[\xi])[\xi]|_{x_j} = 0.
\]

Step 2: Estimate of \( \eta, \xi \) terms. Let

\[
G_h(\xi, \xi) = -H^+ (\xi, \xi; a) + \int_{I_j} b(x)(\xi)_x dx + \sum_{j=1}^N (b(x)(\xi)_x + \phi[\xi])[\xi]|_{x_j}.
\]
\[
G_h(\eta, \xi) = -H^+ (\eta, \xi; a) + \int b(x) \eta, \xi dx + \sum_{j=1}^{N} (b(x)((\eta)^2 + \phi[\eta])(\xi^2))|_{j+\frac{1}{2}}. 
\]

(23)

Since \(\xi = \{\xi_j\} + \frac{1}{2} [\xi] \), combining with the conclusions in the stability analysis process to obtain

\[
H^+ (\xi, \xi; a) = \sum_{j=1}^{N} \int a(x) \left( \frac{x^2}{2} \right) dx + a \xi^2 |_{j+\frac{1}{2}}
\]

(24)

In addition, due to \(a(x) \leq 0\), thus \(a \frac{1}{2} [\xi^2] |_{j+\frac{1}{2}} \leq 0\), by the Cauchy Schwartz’s inequality, we have

\[
H^+ (\xi, \xi; a) \leq \sum_{j=1}^{N} \left[ \int - \frac{a(x)}{2} \xi dx \right] \leq C \|a'(x)\|_{L^2(\Omega)} \cdot \|\xi\|_{L^2(\Omega)} = C \|\xi\|_{L^2(\Omega)}
\]

(26)

Then, equation (22) becomes

\[
-G_h(\xi, \xi) \leq C \|\xi\|_{L^2(\Omega)} - \int b(x)(\xi^2)dx - \sum_{j=1}^{N} (b(x)(\xi^2)|_{j+\frac{1}{2}}) - \sum_{j=1}^{N} (\phi b(x)(\xi^2)|_{j+\frac{1}{2}}).
\]

(27)

By the inverse inequality(iii), it is easy to show that

\[
\left( \xi^2 \right)|_{j+\frac{1}{2}} \leq \frac{C}{h} \int \xi^2 dx,
\]

(28)

by Young inequality, after some simple algebraic calculations

\[
-G_h(\xi, \xi) \leq - \int b(x)(\xi^2)dx + \sum_{j=1}^{N} (b(x)(\xi^2)|_{j+\frac{1}{2}}) - \sum_{j=1}^{N} (\phi b(x)(\xi^2)|_{j+\frac{1}{2}} + \|\xi\|_{L^2(\Omega)}
\]

(29)

Taking \(\delta = \sqrt{\frac{C}{h}}, \phi \geq \delta^2 = \frac{C}{h}\) in equation(29), we have

\[
-G_h(\xi, \xi) \leq - \frac{C}{h} \sum_{j=1}^{N} (b(x)(\xi^2)|_{j+\frac{1}{2}} + \|\xi\|_{L^2(\Omega)}
\]

(30)

From the properties of projection (17) and Young inequality, we get

\[
-G_h(\eta, \xi) = - \sum_{j=1}^{N} (\phi b(x)(\eta^2)|_{j+\frac{1}{2}}) \leq \sum_{j=1}^{N} \frac{C}{h} (b(x)(\eta^2)|_{j+\frac{1}{2}}) + \sum_{j=1}^{N} \frac{C}{h} b(x)(\xi^2)|_{j+\frac{1}{2}}.
\]

(31)

Adding equation (30) and equation (31) to obtain
\[-G_n(\xi, \xi) - G_n(\eta, \xi) \leq \sum_{j=1}^{N} \frac{C}{h} b(x) |\eta_j^2|_{/2} + C\|\xi\|_{L^2(\Omega)}^2 \leq Ch^{2k} + C\|\xi\|_{L^2(\Omega)}^2. \tag{32}\]

Finally, application of the Cauchy Schwartz’s inequality, properties of projection and Gronwall’s inequality lead to

\[\|\xi\|_{L^2(\Omega)} \leq Ch^k, \tag{33}\]

where \(C\) is independent of \(h\).

Thus, Theorem 3.1 can be obtained by using Triangle inequality and taking into account \(\|\eta(t)\|_{L^2(\Omega)} \leq Ch^{k+1}\).

4. Numerical experiments.

In this section, we provide a numerical experiment to validate the theoretical results. For the example, the third order TVD Runge–Kutta time discretization is used with a suitable time step.

Example 4.1 Consider the following variable coefficient convection-diffusion equation:

\[u_t + 2\cos x \cdot u + (3\sin x + 0.5)u_x - (\cos x + 2u_{xx} = f(x, t), \]

\[f(x, t) = e^{-1}(\cos x + 3\cos 2x - 0.5\sin x), \]

\[u(x, 0) = u_0(x), \tag{34}\]

then it can be seen that the true solution of the variable coefficient convection diffusion equation (34) under periodic boundary conditions is \(u(x, t) = e^{-1} \cos x\), where \(x \in [0, 2\pi], t \in (0, T], \phi = \frac{10}{h}\).

The calculation results are shown in Table 1.

| Orders, \(k\) | Mesh, \(N\) | \(L^2\) error | order |
|-------------|-------------|----------------|-------|
| 1           | 20          | 0.24E-03       | 0.93  |
|             | 40          | 0.22E-01       | 1.01  |
|             | 80          | 0.24E-02       | 0.98  |
| 2           | 20          | 0.36E-02       | 1.94  |
|             | 40          | 0.46E-03       | 2.03  |
|             | 80          | 0.63E-04       | 1.97  |

It can be seen from the table that when \(T = 1, \phi = 10/\ h\) and \(CFL = 1\), the error reaches the \(k\) order convergence accuracy and as the mesh is refined, it is still the \(k\) order convergence accuracy.

5. Concluding remarks

In this paper, an analysis of the \(L^2\) stability and error estimates to DG method using generalized numerical fluxes applied to variable coefficient convection-diffusion equation is carried out. Our analysis is valid for uniform regular meshes and for polynomials of degree \(d\). The main technicality is the analysis of the parameters in the generalized numerical fluxes. By choosing a suitable numerical experiment, we obtain the error reaches the \(k\) order convergence accuracy, thus we confirm the theoretical results. Future work includes study \(L^2\) stability and error estimates of high order nonlinear equations with fluxes (3) and (4).
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References
[1] Reed, W.H., Hill, T.R. (1973) Triangular mesh methods for the neutron transport equation. Los Alamos Report, Los Alamos Scientific Laboratory, Los Alamos, LA-UR-73-479.
[2] Meng, X., Shu, C.W., Zhang, Q., and Wu, B. (2012) Superconvergence of discontinuous Galerkin methods for scalar nonlinear conservation laws in one space dimension. SIAM Journal on Numerical Analysis, 50:2336-2356.
[3] Meng, X., Shu, C.W. and Wu, B. (2015) Optimal error estimates for discontinuous Galerkin methods based on upwind-biased fluxes for linear hyperbolic equations. Mathematics of Computation, 85:1225-1261.
[4] Cao, W., Li, D., Yang, Y. and Zhang, Z. (2017) Superconvergence of discontinuous Galerkin methods based on upwind-biased fluxes for 1D linear hyperbolic equations. ESAIM Mathematics Model on Numerical Analysis, 51:467-486.
[5] Li, J., Zhang, D., Meng, X. and Wu, B. (2019) Analysis of discontinuous Galerkin methods with upwind-biased fluxes for one dimensional linear hyperbolic equations with degenerate variable coefficients. Journal of Scientific Computing, 78:1305-1328.
[6] Fu, P., Cheng, Y., Li, Y. and Xu, Y. (2019) Discontinuous Galerkin methods with optimal accuracy for one dimensional linear PDEs with high order spatial derivatives. Journal of Scientific Computing, 78:816-863.
[7] Li, J., Zhang, D., Meng, X. and Wu, B. and Zhang, Q. (2020) Discontinuous Galerkin methods For Nonlinear Scalar Conservation Laws: Generalized Local Lax-Friedrichs Numerical Fluxes. SIAM Journal on Numerical Analysis, 58:1-20.
[8] Ciarlet, P.G. (1978) The Finite Element Method for Elliptic Problems. 1st. Amsterdam, North Holland, 110-332.
[9] Sun, Y.H. (2018) Error estimates of the discontinuous finite element method for nonlinear convection-diffusion equations. Thesis, Harbin Institute of Technology, Harbin.
[10] Castillo, P., Cockburn, B., Schotzau, D. and Schwab, C. (2002) Optimal a priori error estimates for the hp-version of the local discontinuous Galerkin method for convection-diffusion problems. Mathematics of Computation. 238: 455–478.