On the Second Hankel Determinant of Logarithmic Coefficients for Certain Univalent Functions

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Abstract. In this paper, we investigate the sharp bounds of the second Hankel determinant of logarithmic coefficients for the starlike and convex functions with respect to symmetric points in the open unit disk.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Then $\mathcal{H}$ is a locally convex topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ such that $f(0) = 0$ and $f'(0) = 1$. A function $f$ is said to be univalent in a domain $\Omega \subseteq \mathbb{C}$, if it is one-to-one in $\Omega$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{D}$. If $f \in \mathcal{S}$ then it has the following series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$  (1.1)

A function $f$ from $\mathcal{S}$ belongs to the class $\mathcal{S}^*$, called starlike function, if $f(\mathbb{D})$ is a starlike domain with respect to the origin. Moreover, a function $f \in \mathcal{S}$ is called convex function if $f(\mathbb{D})$ is a starlike domain with respect to each point. The class of such functions is denoted by $\mathcal{C}$.

In [20], Sakaguchi introduced the class of functions that are starlike with respect to symmetric points. A function $f \in \mathcal{A}$ is said to be starlike with respect to symmetric points if for any $r$ close to 1, $r < 1$, and any $z_0$ on the
circle $|z| = r$, the angular velocity of $f(z)$ about the point $f(-z_0)$ is positive at $z_0$ as $z$ traverses the circle $|z| = r$ in the positive direction, i.e.,

$$\text{Re} \left( \frac{z_0 f'(z_0)}{f(z_0) - f(-z_0)} \right) > 0, \quad |z_0| = r.$$ 

Denote by $S^*_S$ the class of all functions in $S$ which are starlike with respect to symmetric points. Functions $f$ in the class $S^*_S$ are characterized by

$$\text{Re} \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in \mathbb{D}.$$ 

It is known that the functions in $S^*_S$ are close-to-convex and hence are univalent (see [20]). Note that the class of functions starlike with respect to symmetric points obviously includes the classes of convex functions and odd starlike functions with respect to the origin. The notion of starlike functions with respect to $N$-symmetric points has been studied in [20]. In 2002, Nezhmetdinov and Ponnusamy [14] proved that $S^*_S \not\subset S^*$ and $S^* \not\subset S^*_S$.

We also consider the class of convex functions with respect to symmetric points studied by Das and Singh [7] in 1971. A function $f \in \mathcal{A}$ is said to be convex with respect to symmetric points if, and only if,

$$\text{Re} \left( \frac{(zf'(z))'}{(f(z) - f(-z))'} \right) > 0, \quad z \in \mathbb{D}.$$ 

The logarithmic coefficients $\gamma_n$ of $f \in S$ are defined by

$$F_f(z) := \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}. \quad (1.2)$$

The logarithmic coefficients $\gamma_n$ play a central role in the theory of univalent functions. A very few exact upper bounds for $\gamma_n$ seem to have been established. The significance of this problem in the context of Bieberbach conjecture was pointed by Milin [13] in his conjecture. Milin [13] conjectured that for $f \in S$ and $n \geq 2$,

$$\sum_{m=1}^{n} \sum_{k=1}^{m} (k|\gamma_k|^2 - \frac{1}{k}) \leq 0,$$

which led De Branges, by proving this conjecture, to the proof of Bieberbach conjecture [5]. For the Koebe function $k(z) = z/(1 - z)^2$, the logarithmic coefficients are $\gamma_n = 1/n$. Since the Koebe function plays the role of extremal function for most of the extremal problems in the class $S$, it is expected that $|\gamma_n| \leq 1/n$ holds for functions in $S$. But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function $f$ in the class $S$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ (see [8, Theorem 8.4]). By
differentiating (1.2) and equating coefficients we obtain
\[
\gamma_1 = \frac{1}{2} a_2, \\
\gamma_2 = \frac{1}{2} \left( a_3 - \frac{1}{2} a_2^2 \right), \\
\gamma_3 = \frac{1}{2} \left( a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right). 
\] (1.3)

If \( f \in S \), it is easy to see that \( |\gamma_1| \leq 1 \), because \( |a_2| \leq 2 \). Using the Fekete-Szegő inequality [8, Theorem 3.8] for functions in \( S \) in (1.3), we obtain the sharp estimate
\[
|\gamma_2| \leq \frac{1}{2} \left( 1 + 2e^{−2} \right) = 0.635 \ldots.
\]

For \( n \geq 3 \), the problem seems much harder, and no significant bound for \( |\gamma_n| \) when \( f \in S \) appear to be known. In 2017, Ali and Allu [1] obtained the initial logarithmic coefficients bounds for close-to-convex functions. In 2020, Ponnusamy et al. [17] computed the sharp estimates for the initial three logarithmic coefficients for a subclass of \( S^* \). The problem of computing the bound of the logarithmic coefficients is also considered in [6,18,19,22] for several subclasses of close-to-convex functions. In 2021, Zaprawa [23] obtained the sharp bounds of the initial logarithmic coefficients’ \( |\gamma_n| \) for functions in the classes \( S^*_S \) and \( K_S \).

For \( q, n \in \mathbb{N} \), the Hankel determinant \( H_{q,n}(f) \) of Taylor’s coefficients of function \( f \in A \) of the form (1.1) is defined by
\[
H_{q,n}(f) = \begin{vmatrix}
 a_n & a_{n+1} & \cdots & a_{n+q-1} \\
 a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)}
\end{vmatrix}.
\]

The Hankel determinant for various order is also studied recently by several authors in different contexts; for instance see [3,15,16,21]. One can easily observe that the Fekete-Szegő functional is the second Hankel determinant \( H_{2,1}(f) \). Fekete-Szegő then further generalized the estimate \( |a_3 - \mu a_2^2| \) with \( \mu \) real for \( f \in S \) [8, Theorem 3.8].

Identifying the widespread applications of logarithmic coefficients, recently, Kowalczyk and Lecko [12] together proposed the study of the Hankel determinant whose entries are logarithmic coefficients of \( f \in S \), which is given by
\[
H_{q,n}(Ff/2) = \begin{vmatrix}
 \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\
 \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\
 \vdots & \vdots & \ddots & \vdots \\
 \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)}
\end{vmatrix}.
\]
Kowalczyk and Lecko [12] obtained the sharp bound of second Hankel determinant of \( Ff/2 \), i.e., \( H_{2,1}(Ff/2) \) for starlike and convex functions. The
problem of computing the sharp bounds of $H_{2,1}(F_f/2)$ has been considered in [4] for various subclasses of $S$.

Suppose that $f \in S$ given by (1.1). Then the second Hankel determinant of $F_f/2$ by using (1.3), is given by

$$H_{2,1}(F_f/2) = \gamma_1 \gamma_3 - \gamma_2^2 = \frac{1}{4} \left( a_2 a_4 - a_3^2 + \frac{1}{12} a_2^4 \right).$$

(1.4)

In this paper, we calculate the sharp bounds for $H_{2,1}(F_f/2)$ for functions in the classes $S^*_S$ and $K_S$. We also provide examples of functions to illustrate these results.

2. Main Results

Let $B_0$ denote the class of analytic functions $w : \mathbb{D} \rightarrow \mathbb{D}$ such that $w(0) = 0$. Functions in $B_0$ are known as Schwarz functions. A function $w \in B_0$ can be written as a power series

$$w(z) = \sum_{n=1}^{\infty} c_n z^n.$$  

For two functions $f$ and $g$ that are analytic in a domain $\mathbb{D}$, we say that the function $f$ is subordinate to $g$ in $\mathbb{D}$ and written as $f(z) \prec g(z)$ if there exists a Schwarz function $w \in B_0$ such that

$$f(z) = g(w(z)), \quad z \in \mathbb{D}.$$  

In particular, if the function $g$ is univalent in $\mathbb{D}$, then $f \prec g$ if, and only if, $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

To prove our results, we need the following lemma for the Schwarz functions.

**Lemma 2.1 [9].** Let $w(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2, \quad \text{and} \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|^2}.$$  

We obtain the following sharp bound for $H_{2,1}(F_f/2)$ for functions in the class $S^*_S$.

**Theorem 2.2.** Let $f \in S^*_S$. Then

$$|H_{2,1}(F_f/2)| \leq \frac{1}{4}.$$  

The inequality is sharp.

**Proof.** Let $f \in S^*_S$ be of the form (1.1). Then by the definition of subordination there exists a Schwarz function $w(z) = \sum_{n=1}^{\infty} c_n z^n$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 + w(z)}{1 - w(z)}.$$  

(2.1)
By comparing the coefficients on both sides of (2.1) we get
\[ a_2 = c_1, \]
\[ a_3 = c_2 + c_1^2, \]
\[ a_4 = \frac{1}{2} \left( c_3 + 3c_1c_2 + 2c_1^3 \right). \]

By substituting the above expression for \( a_2, a_3, \) and \( a_4 \) in (1.4) and then further simplification gives
\[ H_{2,1}(F_f/2) = \gamma_1 \gamma_3 - \gamma_2^2 \]
\[ = \frac{1}{4} \left( a_2a_4 - a_3^2 + \frac{1}{12}a_2^4 \right) \]
\[ = \frac{1}{48} \left( c_1^4 + 6c_1c_3 - 12c_2^2 - 6c_1^2c_2 \right). \]

From equation (2.2) and Lemma 2.1, we obtain
\[ 48|H_{2,1}(F_f/2)| \leq |c_1|^4 + 6|c_1| \left( 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|} \right) + 6|c_1|^2|c_2| + 12|c_2|^2. \]

Now writing \( x = |c_1| \) and \( y = |c_2| \) in (2.3), we obtain
\[ 48|H_{2,1}(F_f/2)| \leq F(x, y), \]
where
\[ F(x, y) = x^4 + 6x \left( 1 - x^2 - \frac{y^2}{1 + x} \right) + 6x^2y + 12y^2. \]

In view of Lemma 2.1, the region of variability of a pair \((x, y)\) coincides with the set
\[ \Omega = \{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2 \}. \]

Therefore, we need to find the maximum value of \( F(x, y) \) over the region \( \Omega \). The critical points of \( F \) satisfies the conditions
\[ \frac{\partial F}{\partial x} = 4x^3 - 18x^2 + 12xy - \frac{6y^2}{(1 + x)^2} + 6 = 0 \]
\[ \frac{\partial F}{\partial y} = x^2 + x^3 + 4y + 2xy = 0, \]
which has no solution in the interior of \( \Omega \). Hence the function \( F(x, y) \) cannot have a maximum in the interior of \( \Omega \). Since \( F \) is continuous on a compact set \( \Omega \), the maximum of \( F \) attains boundary of \( \Omega \). On the boundary of \( \Omega \), we have
\[ F(x, 0) = x^4 - 6x^3 + 6x \leq 2.4378 \text{ for } 0 \leq x \leq 1, \]
\[ F(0, y) = 12y^2 \leq 12 \text{ for } 0 \leq y \leq 1, \]
and
\[ F(x, 1 - x^2) = x^4 - 12x^2 + 12 \leq 12 \text{ for } 0 \leq x \leq 1. \]
Thus combining all the above cases we obtain
\[ \max_{(x,y)\in \Omega} F(x,y) = 12 \]
and hence from (2.4) we have
\[ |H_{2,1}(F_f/2)| \leq \frac{1}{4}. \tag{2.5} \]

To prove the equality in (2.5), we consider the function
\[ f_1(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \cdots, \quad z \in \mathbb{D}. \]
A simple computation shows that \( f_1 \) belongs to the class \( S^*_S \) and
\[ |H_{2,1}(F_{f_1}/2)| = \frac{1}{4} \]
and hence equality holds in (2.5). This completes the proof. \( \square \)

Here we provide an example that associates to Theorem 2.2.

**Example 2.3.** Consider the function
\[ f_2(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots \]
It is easy to see that the function \( f \) belongs to the class \( S^*_S \). It is easy to see that
\[ |H_{2,1}(F_{f_2}/2)| = \frac{1}{12} \leq \frac{1}{4}. \]

In the following result, we estimate the sharp bound for \( H_{2,1}(F_f/2) \) for functions in the class \( K_S \).

**Theorem 2.4.** Let \( f \in K_S \) be of the form (1.1). Then
\[ |H_{2,1}(F_f/2)| \leq \frac{1}{36}. \]
The inequality is sharp.

**Proof.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be a function in \( K_S \), then there exists a Schwarz function \( w(z) = \sum_{n=1}^{\infty} c_n z^n \) such that
\[ \frac{2(zf'(z))'}{(f(z) - f(-z))'} = \frac{1 + w(z)}{1 - w(z)}. \tag{2.6} \]
First note that by equating coefficients in (2.6) we have,
\[ a_2 = \frac{1}{2} c_1, \]
\[ a_3 = \frac{1}{3} (c_2 + c_1^2), \]
\[ a_4 = \frac{1}{8} (c_3 + 3c_1 c_2 + 2c_1^3). \]
A simple computation using (1.4) gives,
\[ H_{2,1}(F_f/2) = \frac{1}{2304} \left( 11c_1^4 + 36c_1 \left( 1 - c_1^2 - \frac{c_2}{1 + c_1} \right) + 20c_1^2 c_2 + 64c_2^2 \right). \tag{2.7} \]
Following the same method as used in the proof of Theorem 2.2, we obtain
\[ |H_{2,1}(F_f/2)| \leq \frac{1}{2304} \left( 11|c_1|^4 + 36|c_1| \left( 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|} \right) + 20|c_1|^2|c_2| + 64|c_2|^2 \right), \] (2.8)

where
\[ 0 \leq |c_1| \leq 1 \text{ and } 0 \leq |c_2| \leq 1 - |c_1|^2. \]

Now by replacing $|c_1|$ by $x$ and $|c_2|$ by $y$ in (2.8) gives
\[ 2304|H_{2,1}(F_f/2)| \leq G(x, y), \] (2.9)

where
\[ G(x, y) = 11x^4 + 36x \left( 1 - x^2 - \frac{y^2}{1 + x} \right) + 20x^2y + 64y^2. \]

In view of Lemma 2.1, the region of variability of a pair $(x, y)$ coincides with the set
\[ \Omega = \{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2 \}. \]

Thus we need to find the maximum value of $G(x, y)$ over the region $\Omega$. The critical points of $G$ satisfies the conditions
\[ \frac{\partial G}{\partial x} = 44x^3 - 108x^2 + 40xy - \frac{36y^2}{(1 + x)^2} + 36 = 0, \]
and
\[ \frac{\partial G}{\partial y} = 5x^2 + 5x^3 + 32y + 14xy = 0, \]

which has no solution in the interior of $\Omega$. By using the elementary calculus, we can show that the maximum of $G(x, y)$ should exists on the boundary of $\Omega$. It is easy to see that on the boundary line $x = 0$ and $0 \leq y \leq 1$, we have $G(0, y) = 64y^2$ and its maximum on this line is equal to 64. Similarly, on the boundary line $y = 0$ and $0 \leq x \leq 1$, we have $G(x, 0) = 11x^4 - 36x^3 + 36x$ and its maximum on this line is 15.512. Finally, on the boundary curve $y = 1 - x^2$ and $0 \leq x \leq 1$, we have $G(x, 1 - x^2) = 19x^4 - 72x^2 + 64$ and its maximum on this curve is 64. Thus, combining all the above cases yields
\[ \max_{(x,y)\in\Omega} G(x, y) = 64 \]

and hence from (2.9) we obtain
\[ |H_{2,1}(F_f/2)| \leq \frac{1}{36}. \] (2.10)

For the sharpness of the inequality (2.10) we consider the function
\[ f_3(z) = \frac{1}{2} \log \frac{1 + z}{1 - z} = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots \]
which belongs to the class $\mathcal{K}_S$. A simple computation shows that $|H_{2,1}(F_{f_3}/2)| = 1/36$ and hence the inequality in (2.10) is sharp. This completes the proof. \hfill $\Box$

In the following example we construct a function that agree with Theorem 2.4.

Example 2.5. Consider the function

$$f_4(z) = -\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots.$$ 

A simple computation shows that

$$\text{Re} \, \left( \frac{(zf_4'(z))^r}{(f_4(z) - f_4(-z))^r} \right) = \frac{1}{2} \text{Re} \, \left( \frac{1 + z}{1 - z} \right) > 0,$$

and hence the function $f_4 \in \mathcal{K}_S$. It is easy to see that

$$|H_{2,1}(F_{f_4}/2)| = \frac{11}{576} \leq \frac{1}{36}.$$

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Declarations

Conflict of Interest The authors declare that there is no conflict of interest regarding the publication of this paper.

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