GENERAL TRANSFORMATIONS BETWEEN THE HEUN AND GAUSS HYPERGEOMETRIC FUNCTIONS

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Abstract. It is known that Heun’s functions (or differential equations) can be reduced to Gauss hypergeometric functions by rational changes of its independent variable only if its parameters, including the fourth singular point location parameter and the accessory parameter, take special values. We present all hypergeometric-to-Heun transformations with two or three free continuous parameters up to fractional-linear transformations.

1. Introduction. The ordinary differential equations for the general Gauss hypergeometric function and the Heun function are second order Fuchsian differential equations on the Riemann sphere \( \mathbb{P}^1 \) with 3 or 4 regular singular points, respectively. Any second order Fuchsian equation with 3 or 4 singularities can be transformed to them by Möbius transformations [4].

1.1. The hypergeometric equation. The differential equation for the general Gauss hypergeometric series

\[
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n
\]

has the form

\[
\frac{d^2 y(z)}{dz^2} + \left( \frac{c}{z} + \frac{a + b - c + 1}{z - 1} \right) \frac{dy(z)}{dz} + \frac{a b}{z (z - 1)} y(z) = 0.
\]
It has regular singularities at $z = 0, 1$ and $\infty$. The local exponents are given by

$$0, 1 - c \text{ at } z = 0; \quad 0, c - a - b \text{ at } z = 1; \quad \text{and } a, b \text{ at } z = \infty.$$

Recall that if $\lambda$ is a local (characteristic) exponent at a regular singularity $z = t$ of the Fuchsian differential equation, then in a neighborhood of $z = t$ there is a local solution $y(z) = (z - t)^\lambda \sum_{k=0}^{\infty} \alpha_k (z - t)^k$. Accordingly, the hypergeometric equation (2) with general $a, b, c$ has the following local bases of solutions:

- at $z = 0$:
  $$2F_1 \left( \frac{a}{c}, \frac{b}{c} \bigg| z \right), \quad z^{1-c} 2F_1 \left( 1 + a - c, 1 + b - c \bigg| \frac{z}{2-c} \right),$$

- at $z = 1$:
  $$2F_1 \left( 1 + a + b - c, 1 - z \bigg| \frac{c}{1 + a + b - c} \right), \quad (1 - z)^{c-a-b} 2F_1 \left( c - a, c - b \bigg| 1 + c - a - b \bigg| \frac{1}{1 - z} \right),$$

- at $z = \infty$:
  $$z^{-a} 2F_1 \left( \frac{a, 1 + a - c}{1 + a - b} \bigg| \frac{z}{1} \right), \quad z^{-b} 2F_1 \left( b, 1 + b - c \bigg| 1 - a + b \bigg| \frac{1}{z} \right).$$

The local exponent differences at singular points are (up to a sign) equal to $1 - c, c - a - b$ and $a - b$, respectively. In this paper we assume that the local exponent differences are non-negative. The information about singularities and local exponents is encoded in the Riemann scheme for (2):

$$P = \begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & a \\
1 - c & c - a - b & b
\end{pmatrix}.$$

Recall that Fuchsian equations with 3 singularities are defined uniquely by their singularities and local exponents. This is not generally true for classes of Fuchsian equations with more singularities; accessory parameters are then needed to specify a Fuchsian equation.

The following Pfaff and Euler fractional-linear transformations [2, Th. 2.2.5] can be applied to 6 local solutions:

$$2F_1 \left( \frac{a, b}{c} \bigg| z \right) = (1 - z)^{-a} 2F_1 \left( \frac{a, c - b}{c} \bigg| \frac{z}{z - 1} \right),$$

$$= (1 - z)^{-b} 2F_1 \left( \frac{c - a, b}{c} \bigg| \frac{z}{z - 1} \right),$$

$$= (1 - z)^{c-a-b} 2F_1 \left( \frac{c - a, c - b}{c} \bigg| z \right). \quad (3)$$

In total we have $6 \times 4 = 24$ different hypergeometric solutions for a general hypergeometric equation; they are referred to as 24 Kummer’s solutions. For integer values of the parameters $a, b, c$ or the local exponents, the structure of 24 solutions may degenerate [14]. The 24 solutions are related by a group of transformations that permutes the singular points or interchanges the local exponents at the singular points. The group is a semidirect product of the symmetric group $S_3$ and $(\mathbb{Z}/2\mathbb{Z})^3$, but the interchange of local exponents at infinity does not change the hypergeometric function. These transformations of the hypergeometric
equation can be presented as transformations of Riemann’s $P$-symbols; for example

$$P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & a \\ 1 - c & c - a - b & b \end{array} \right\} = (1 - z)^{c - a - b} P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & c - b \\ 1 - c & a + b - c & c - a \end{array} \right\}$$

$$= P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & a \\ c - a - b & 1 - c & b \end{array} \right\}.$$

For a basic theory of hypergeometric functions see [2], [4, Vol. I].

1.2. The Heun equation. The Heun equation is defined by

$$\frac{d^2y(x)}{dx^2} + \left( \frac{c}{x} + \frac{d}{x - 1} + \frac{a + b - c - d + 1}{x - t} \right) \frac{dy(x)}{dx} + \frac{abx - q}{x(x - 1)(x - t)} \frac{y(x)}{x} = 0. \quad (4)$$

The equation has 4 regular singular points: $x = 0, x = 1, x = t, x = \infty$. The parameters $a, b, c, d$ determine the local exponents at them, while the parameter $q$ is an accessory parameter. The Riemann scheme of the Heun equation is

$$P \left\{ \begin{array}{ccc} 0 & 1 & t \\ 0 & 0 & 0 \\ 1 - c & 1 - d & c + d - a - b \\ a & x \end{array} \right\}.$$ 

A local solution at $x = 0$ is defined by a power series which coefficients satisfy a three-term recurrence relation. For comparison, the $2F_1$ local solutions of the hypergeometric equation satisfy a similar two-term recurrence relation. In the following we shall denote the solution of (4) with the local exponent 0 and the value 1 at $x = 0$ by

$$H_n \left( t \mid a, b \left\| c, d \mid x \right. \right). \quad (5)$$

This function reduces to the Gauss hypergeometric function $1F_1$ if $d = a + b - c + 1$ and $q = abt$; notice the reduction of equation (14) to (2) then. The function is identical to the constant 1 if $ab = 0$ and $q = 0$. Note that the parameters $a, b$ are symmetric and give the local exponents at $x = \infty$, whereas the parameters $c$ and $d$ are not interchangeable and determine the non-zero local exponents at $x = 0$ and $x = 1$, respectively.

Similarly as with the hypergeometric equation, permutation of the 4 singular points and interchange of local exponents at them act on Heun’s equation by changing the parameters, including the location $t$ of the fourth singular point and the accessory parameter $q$. The acting group is a semi-direct product of $S_4$ and $(\mathbb{Z}/2\mathbb{Z})^4$. The group has $4! \times 2^4 = 384$ elements, but the interchange of the local exponents at $x = \infty$ does not change Heun’s equation. Correspondingly, there are $384/2 = 192$ solutions expressible in terms of Heun’s function (5); they are presented in [8]. Here we present a basic generic structure of the 192 solutions.

Permutation of the singular points $x = 1, x = t, x = \infty$ or interchange of local exponents at these points, are realized by two-term fractional-linear transformations of Heun’s function. First of all, we have transformations that do not move the singular points, but only interchange the local exponents at $x = 1$ and $x = t$. These transformations neither change
the argument $x$ nor the location parameter $t$:

$$
H_n \left( \frac{t}{q} \Bigg| \begin{array}{c} a, b \\ c; d \end{array} \bigg| x \right) = (1 - x)^{1-d} H_n \left( \frac{t}{q-a(d-1)t} \bigg| \begin{array}{c} a-d+1, b-d+1 \\ c; 2-d \end{array} \big| x \right)
$$

$$
= (1 - x)^{1-d} \left( 1 - \frac{x}{t} \right)^{c+d-a-b} H_n \left( \frac{t}{q_1} \bigg| \begin{array}{c} c+d-a, c+d-b \\ c; d \end{array} \big| x \right)
$$

$$
= (1 - x)^{1-d} \left( 1 - \frac{x}{t} \right)^{c+d-a-b} H_n \left( \frac{t}{q_2} \bigg| \begin{array}{c} c-a+1, c-b+1 \\ c; 2-d \end{array} \big| x \right),
$$

(6)

where $q_1 = q - c(a + b - c - d)$, $q_2 = q - c(a + b - c - d + dt - t)$. Permutations of the three singular points $x = 1$, $x = t$, $x = \infty$ are realized by the following fractional-linear transformations:

$$
H_n \left( \frac{t}{q} \bigg| \begin{array}{c} a, b \\ c; d \end{array} \bigg| x \right) = H_n \left( \frac{1}{t} \bigg| \begin{array}{c} a, b \\ q/t \bigg| c; a + b - c - d + 1 \big| x \right)
$$

$$
= (1 - x)^{-a} H_n \left( \frac{t/(t-1)}{(act - q)/(t-1)} \bigg| \begin{array}{c} a, a-d+1 \\ c; a-b+1 \end{array} \big| \frac{x}{x-1} \right)
$$

$$
= (1 - x)^{-a} H_n \left( \frac{1/(1-t)}{(q-ac)/(t-1)} \bigg| \begin{array}{c} a, c+d-b \\ c; a-b+1 \end{array} \big| \frac{x}{x-t} \right)
$$

$$
= (1 - x)^{-a} H_n \left( \frac{1-t}{ac-q} \bigg| \begin{array}{c} a, c+d-b \\ c; d \end{array} \big| \frac{1-t}{x-t} \right)
$$

$$
= (1 - x)^{-a} H_n \left( \frac{1-1/t}{ac-q/t} \bigg| \begin{array}{c} a, a-d+1 \\ c; a+b-c-d+1 \big| \frac{(t-1)x}{t(x-1)} \right).
$$

(7)

Combined with the transformations of 4 functions above, we get $6 \times 4 = 24$ expressions for the local solution $f$ at $x = 0$. Like with the hypergeometric equation, we generally have two local solutions at each of the 4 singular points expressible in terms of Heun’s function. Here are general bases of local solutions at $x = 0$, $x = 1$, $x = t$ and $x = \infty$:

$$
H_n \left( \frac{t}{q} \bigg| \begin{array}{c} a, b \\ c; d \end{array} \bigg| x \right), \quad x^{1-c} H_n \left( \frac{t}{q_1} \bigg| \begin{array}{c} a-c+1, b-c+1 \\ 2-c; d \end{array} \big| x \right),
$$

$$
H_n \left( \frac{1-t}{ab-q} \bigg| \begin{array}{c} a, b \\ d; c \end{array} \bigg| 1-x \right), \quad (1 - x)^{1-d} H_n \left( \frac{1-t}{q_2} \bigg| \begin{array}{c} a-d+1, b-d+1 \\ 2-d; c \end{array} \big| 1-x \right),
$$

$$
H_n \left( \frac{1-1/t}{ab-q/t} \bigg| \begin{array}{c} a, b \\ d; c \end{array} \bigg| 1-x \right), \quad (1 - x)^{c+d-a-b} H_n \left( \frac{1-1/t}{q_3} \bigg| \begin{array}{c} c+d-a, c+d-b \\ c+d-a-b+1; c \end{array} \big| 1-x \right),
$$

$$
H_n \left( \frac{1/t}{q_4} \bigg| \begin{array}{c} a, a-c+1 \\ a-b+1; d \end{array} \big| \frac{1}{x} \right), \quad x^{-b} H_n \left( \frac{1/t}{q_5} \bigg| \begin{array}{c} b, b-c+1 \\ b-a+1; d \end{array} \big| \frac{1}{x} \right),
$$

(8)

where

$$
q_1 = q - (c-1)(a+b-c-d+dt+1),
$$

$$
q_2 = ab - q - (d-1)(a+b-ct-d+1),
$$

$$
q_3 = ab - q/t + (c/t - c-d)(a+b-c-d),
$$

$$
q_4 = q/t + a(a-b/t - c-d + dt + t + 1),
$$

$$
q_5 = q/t + b(b-a/t - c-d + dt + t + 1).
$$
Each of these 8 functions can be expressed in 24 ways by fractional-linear transformations (6)–(7), giving the 192 solutions in [8]. Any 3 of the 8 functions are related by a linear three-term connection formula (since the order of Heun’s equation is 2), though their coefficients are not known in general (unlike Kummer’s solutions of the hypergeometric equation). The group of transformations of Heun’s equation can be represented by two-term transformations of Riemann’s $P$-symbols, or by the action on the 4 signed local exponent differences $(1 - c, 1 - d, c + d - a - b, b - a)$. A transformation can multiply some of the local exponent differences by $-1$, and permute them.

The Heun equation contains a large number of interesting special cases, in particular, the Lamé equation, which is of considerable importance in mathematical physics [4, 10]. The Heun equation appears in problems such as diffusion, wave propagation, magnetohydrodynamics, heat and mass transfer, particle physics and cosmology of the very early universe. Heun functions are much less well understood than hypergeometric functions. In particular, no general elementary integral representation of Heun functions is known.

2. Pull-back transformations of differential equations. For an ordinary linear differential equation a rational pull-back transformation has the form

$$z \mapsto \varphi(x), \quad y(z) \mapsto Y(x) = \theta(x) y(\varphi(x)),$$

where $\varphi(x)$ is a rational function and $\theta(x)$ is a radical function, i.e., a product of powers of rational functions. The term $\theta(x)$ is called a prefactor in the terminology of Maier in [7]. Geometrically, transformation (9) pull-backs a differential equation on the projective line $\mathbb{P}^1_x$ to a differential equation on the projective line $\mathbb{P}^1_z$, with respect to the finite covering $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ determined by the rational function $\varphi(x)$. Here we let $\mathbb{P}^1_x, \mathbb{P}^1_z$ denote the complex projective lines with rational parameters $x, z$ respectively. In [1, 15] these pull-back transformations are called RS-transformations.

In this paper we are interested in general pull-back transformations of the hypergeometric equation to the Heun equation. These transformations typically give two-term transformation formulas

$$H_n \left( \begin{array}{c} t \\ q \end{array} \middle| \begin{array}{c} a, b \\ c, d \end{array} \middle| x \right) = \theta(x) \left. 2_1F_1 \left( \begin{array}{c} A, B \\ C \end{array} \middle| \varphi(x) \right) \right|$$

between the Heun and Gauss hypergeometric functions. These identities give reductions of Heun equations (with special values of the parameters) to the better understood hypergeometric functions. It is generally desirable to have expressions of Heun’s and especially Lamé functions in terms of more elementary functions.

We classify all such transformations with 2 or 3 free continuous parameters, up to fractional-linear transformations (3), (6)–(7) of hypergeometric and Heun’s functions. A similar problem is considered by Maier in [7], where all transformations without the prefactor $\theta(x)$ and coming from pull-back transformations of their differential equations are classified, with any number of free parameters. In this paper, we allow prefactors $\theta(x)$. There is an overlap between our and Maier’s lists. We plan to classify all transformations with 1 or 0 free continuous parameters in future papers. The remainder of this section gives a technical background of our project.

The relevant coverings $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ typically branch only above the 3 singular points
\( z = 0, z = 1, z = \infty \) of the hypergeometric equation. We describe the behavior of singularities under general pull-back transformations in Subsection 2.1 below. The restriction that the transformed differential equation must have only 4 singular points turns out to be very limiting. Pull-back transformations (9) between hypergeometric equations or between Heun equations are classified similarly. In [1], [15] the same methodology is applied to obtain pull-back transformations to isomonodromic \( 2 \times 2 \) Fuchsian systems corresponding to algebraic solutions of the Painlevé VI equation.

2.1. General action of a pull-back transformation. The following definition is introduced in [12]. An irrelevant singularity for an ordinary differential equation is a regular singularity which is not logarithmic, and where the local exponent difference is equal to 1. An irrelevant singularity can be turned into a non-singular point after a suitable pull-back transformation (9) with \( \phi(x) = x \). For comparison, an apparent singularity is a regular singularity which is not logarithmic, and where the local exponents are integers. Recall that a logarithmic point is a singular point \( x = t \) where there is only one local solution of the form \((x-t)^{\lambda}(1 + \sum_{k=1}^{\infty} \alpha_k(x-t)^k)\). For us, a relevant singularity is a singular point which is not an irrelevant singularity.

A pull-back transformation of a Fuchsian equation gives a Fuchsian equation again, usually with more singularities. We are interested in pull-back transformations of the hypergeometric or Heun’s equation to Heun’s (or hypergeometric) equation. Since any second order Fuchsian equation with 3 or 4 singularities can be transformed to the hypergeometric or Heun’s equation by a Möbius transformation, the essential problem is to look for pull-back transformations to a Fuchsian equation with 4 (or 3) singular points. Relevant singular points and local exponent differences for the pull-backed equation are not influenced by the prefactor \( \theta(x) \). The number of relevant singularities is determined by the covering \( \varphi \) alone. Finding suitable coverings is our essential problem.

The behavior of regular singular points of a Fuchsian equation under a pull-back transformation is summarized in the following results from [12].

**Lemma 2.1.** Let \( \varphi : P^1_x \to P^1_z \) be a finite covering. Denote a Fuchsian equation on \( P^1_z \) by \( H_1 \), and the pull-back transformation of \( H_1 \) under (9) by \( H_2 \). Let \( S \) denote a point on \( P^1_x \), and let \( k \) denote the branching index of \( \varphi \) at \( S \).

1. The local exponents for \( H_2 \) at \( S \) are equal to \( k\alpha_1 + \gamma \), \( k\alpha_2 + \gamma \), where:
   - \( \alpha_1, \alpha_2 \) are the local exponents for \( H_1 \) at \( \varphi(S) \in P^1_z \);
   - \( \gamma \) is the local exponent of the radical function \( \theta(x) \) at \( S \).

2. If the point \( \varphi(S) \) is non-singular for \( H_1 \), then the point \( S \) is not a relevant singularity for \( H_2 \) if and only if \( k = 1 \) (i.e., the covering \( \varphi \) does not branch at \( S \)).

3. If the point \( \varphi(S) \) is a singular point for \( H_1 \), then the point \( S \) is not a relevant singularity for \( H_2 \) if and only if the following two conditions hold:
   - The point \( \varphi(S) \) is not logarithmic.
   - The local exponent difference at \( \varphi(S) \) is equal to \( 1/k \).

**Proof.** The first statement is mentioned in the proof of [12, Lemma 2.4]. The other statements are parts 2 and 3 of [12, Lemma 2.4]. \( \blacksquare \)
Lemma 2.2. Let \( \varphi : \mathbb{P}^1 \to \mathbb{P}^1 \) be a finite covering of degree \( d \). Denote a set of 3 points on \( \mathbb{P}^1 \) by \( \Delta \). If all branching points of \( \varphi \) lie above \( \Delta \), then there are exactly \( d + 2 \) distinct points on \( \mathbb{P}^1 \) above \( \Delta \). Otherwise there are more than \( d + 2 \) distinct points above \( \Delta \).

Proof. Part 1 of [12, Lemma 2.5]; follows from the Hurwitz formula [5, Corollary IV.2.4].

Suppose we start with a hypergeometric equation \( H_1 \) on \( \mathbb{P}^1 \); let \( \Delta \) denote the set \( \{ 0, 1, \infty \} \) of its singularities. It follows from the above lemmas that to keep the number of singularities of the pull-backed equation low, we should allow the branching points of \( \varphi \) only above \( \Delta \) (because the outside branching points would be relevant singularities, and we would get more than \( d + 2 \) distinct points above \( \Delta \)), and we should restrict some local exponent differences of \( H_1 \) to the values \( 1/k \), where \( k \) is an integer. Recall that a covering of \( \mathbb{P}^1 \) by a Riemann surface is a Belyi covering [11] if it branches only above a set of three points. It follows that relevant coverings \( \varphi \) will typically be Belyi coverings. Non-Belyi coverings can occur only in the following rather degenerate situations.

Lemma 2.3. Suppose we have a pull-back transformation [9] of a hypergeometric equation \( H_1 \) to Heun’s equation, and there is a branching point of \( \varphi : \mathbb{P}^1 \to \mathbb{P}^1 \) that does not lie above \( \{ 0, 1, \infty \} \in \mathbb{P}^1 \). Then one of the following statements holds:

- \( H_1 \) has actually only 2 relevant singular points;
- Two of the 3 local exponent differences of \( H_1 \) are equal to \( 1/2 \);
- \( H_1 \) has a basis of algebraic solutions.

Proof. Let \( d \) denote the degree of \( \varphi \). By Lemma 2.2 there are at least \( d + 3 \) distinct points above \( \{ 0, 1, \infty \} \in \mathbb{P}^1 \). At most 3 of them can be singularities of the pull-backed equation (because the outside branching points are singularities). Therefore we have at least \( d \) non-singular points above \( \{ 0, 1, \infty \} \). If we restrict a local exponent difference of \( H_1 \) to \( 1/k \) with a positive integer \( k \), we have at most \( d/k \) non-singular points above that point. We can choose \( k = 1 \), but then the point is non-logarithmic only if it is not a relevant singularity by [12, Lemma 5.1]. Otherwise we can restrict two local exponent differences to \( 1/2 \), or restrict all 3 local exponent differences to \( 1/k, 1/\ell, 1/m \) with \( 1/k + 1/\ell + 1/m \geq 1 \) and \( k, \ell, m \) integers. If \( 1/k + 1/\ell + 1/m = 1 \) and we optimally arrange the non-singular points above \( \{ 0, 1, \infty \} \), we have only those \( d \) non-singular points above \( \{ 0, 1, \infty \} \), contradicting Lemma 2.2. The case \( 1/k + 1/\ell + 1/m > 1 \) gives algebraic solutions.

Note that the monodromy group of \( H_1 \) in the three exceptional cases of Lemma 2.3 is, respectively, a cyclic group, a dihedral group, or a finite group.

Apart from these special cases, the rational argument \( \varphi(x) \) of a transformation [10] between the Heun and hypergeometric functions will define a Belyi covering, and the pull-backed equation will have 4 singular points and \( d-2 \) non-singular points above the singular locus \( \{ 0, 1, \infty \} \) of the hypergeometric equation, where \( d \) is the degree of \( \varphi(x) \). For comparison, typical transformations between hypergeometric functions have 3 singular and \( d-1 \) non-singular above the same locus.

We are interested in pull-back transformations to the Heun equations with at least two continuous parameters. Therefore we want to restrict none or just 1 local exponent difference of a starting hypergeometric equation. We may ignore the possibilities with \( k = 1 \), because
non-logarithmic hypergeometric equations with a local exponent difference 1 have only one continuous parameter by \[12, \text{Lemma 5.1}\].

Referring to a Belyi covering \(\varphi\) of degree \(d\), we denote the branching pattern above the three critical points by three partitions of \(d\) separated by the equality sign, just as in [12]. For example, a degree 3 Belyi covering can have the branching pattern \(2 + 1 = 3 = 2 + 1\), meaning one fiber with a single point with the maximal branching index, and two fibers with a simple branching point and a non-branching point. The total number of elements in the three partitions must be equal to \(d + 2\).

To describe transformation of local exponent differences under a pull-back transformation, we extend the notation \((\alpha_1, \beta_1, \gamma_1) \xrightarrow{d} (\alpha_2, \beta_2, \gamma_2)\) in [12] used for transformations between hypergeometric equations. In particular, we write \((\alpha_1, \beta_1, \gamma_1) \xrightarrow{d} (\alpha_2, \beta_2, \gamma_2, \delta_2)\) to denote a pull-back transformation of degree \(d\) of a hypergeometric equation with the local exponent differences \(\alpha_1, \beta_1, \gamma_1\) to Heun’s equation with the local exponent differences \(\alpha_2, \beta_2, \gamma_2, \delta_2\). The arrow follows the direction of the covering \(\varphi: \mathbb{P}^1_x \rightarrow \mathbb{P}^1_z\). The order of local exponent differences on both sides of the arrow is irrelevant for us, as we do not assign the local exponent to particular singularities with the notation.

2.2. Adjusting the prefactor and other technical issues. Once we have a suitable covering \(\varphi: \mathbb{P}^1_x \rightarrow \mathbb{P}^1_z\) and choose local exponents of the starting Fuchsian equation, the prefactor \(\theta(x)\) for a target pull-back transformation (9) is determined up to a constant multiple, and it is straightforward to derive it. For identification of hypergeometric or Heun solutions of both equations, the prefactor is normalized by the condition \(\theta(0) = 1\). A direct pull-back transformation with \(\theta(x) = 1\) usually gives a differential equation with no local exponent at the points above \(z = \infty\) equal to 0, whereas standard hypergeometric or Heun equations have local exponents 0 at all points except \(x = \infty\). Besides, all regular points on \(\mathbb{P}^1_x\) must have the local exponents 0 and 1, hence the local exponents at irrelevant singularities must be shifted to 0, 1 rather than to 0, −1. Irrelevant singularities may occur above non-infinite \(z\)-points if the local exponents at them were restricted to 0, \(-1/k\) with \(k\) a positive integer.

The prefactor must shift the local exponents at the points above \(z = \infty\) and all irrelevant singularities so that we would have the canonical local exponent value 0 everywhere except possibly at \(x = \infty\). The prefactor does not change the local exponent differences or the characteristic sum of all local exponents. (Recall that the sum of local exponents is equal to 1 for the hypergeometric equation and to 2 for the Heun equation by the Fuchs relation). The prefactor has the form \(\theta(x) = \prod (x - \sigma_i)^{-\xi_i}\), where \(\sigma_i\) are all the points where the local exponents need to be shifted, and \(\xi_i\) is the local exponent at \(\sigma_i\) to be shifted to 0. The local exponents at \(x = \infty\) then shift by the sum of all \(\xi_i\)’s. It is convenient to use Riemann’s \(P\)-notation to keep track of changes of local exponents. Intermediate \(P\)-symbols would display irrelevant singularities as well.

The prefactor is not needed, (i.e., \(\theta(x) = 1\)) if there is only one point \(x = \infty\) above \(z = \infty\), and there are no irrelevant singularities above other \(z\)-points. The rational function \(\varphi(x)\) defining the covering is then a polynomial.

Maier [7] classified all transformations between hypergeometric and Heun equations with-
out a prefactor. He obtained a list of five indecomposable transformations

\[(\alpha, \beta, \gamma) \leftarrow (\alpha, \alpha, 2\beta, 2\gamma), \quad \left(\frac{1}{2}, \alpha, \beta\right) \leftarrow \left(\frac{1}{2}, \alpha, 2\alpha, 3\beta\right), \quad \left(\frac{1}{2}, \alpha, \beta\right) \leftarrow \left(\alpha, \alpha, \alpha, 3\beta\right), \quad \left(\frac{1}{2}, \frac{1}{2}, \alpha\right) \leftarrow \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 4\alpha\right), \quad \left(\frac{1}{2}, \frac{1}{2}, \alpha\right) \leftarrow \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 5\alpha\right),\]

and two composite transformations (of degree 4 and 6):

\[(\frac{1}{2}, \alpha, \beta) \leftarrow (\alpha, \alpha, 2\beta) \leftarrow (\alpha, \alpha, 2\alpha, 4\beta), \quad \left(\frac{1}{2}, \frac{1}{2}, \alpha\right) \leftarrow (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2\alpha) \leftarrow (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 6\alpha).\]

Besides, more transformations without a prefactor are given for the degenerate Heun equation with \(ab = q = 0\). The function \(\varphi(x)\) does not have to be a polynomial then, as the points above \(z = \infty\) immediately have a local exponent 0.

To derive a two-term identity between hypergeometric or Heun solutions from a pull-back transformation of their Fuchsian equations, we choose a singular point on \(P^4_\mathbb{C}\) (above the singular locus on \(P^4_\mathbb{C}\)) of the pull-back equation as \(x = 0\), and assign the label \(z = 0\) to its image on \(P^4_\mathbb{C}\). The rational function \(\varphi(x)\) is determined once we label the other singular points on \(P^4_\mathbb{C}\) and \(P^4_\mathbb{C}\). The prefactor \(\Theta(x)\) is determined if additionally the local exponents for singularities on \(P^4_\mathbb{C}\) are fixed. The formula identifies the local solutions at \(x = 0\) and \(z = 0\) with the local exponent 0 and the value 1; it holds in a neighborhood of \(x = 0\) once we set \(z = \varphi(x)\).

A choice of \(x = 0\) and local exponents at \(z = 0\) determines a two-term identity up to fractional-linear transformations [3, 9, 10]. If the local exponent difference at \(z = 0\) is restricted to \(1/k\) with an integer \(k\), the choices \(0, 1/k\) and \(0, -1/k\) of local exponents give different identities. Changing the sign of the local exponent difference at \(z = 0\) basically gives an identity between the other two local solutions (with non-zero local exponents) at \(x = 0\) and \(z = 0\). For transformations between hypergeometric functions, this situation is captured by [12, Lemma 2.3]. Here is a reformulation for identities between Heun and hypergeometric functions.

**Lemma 2.4.** Suppose that we have identity [10] coming from a pull-back transformation between the corresponding hypergeometric and Heun equations. Then \(\varphi(x)^{1-C} \sim Kx^{1-C}\) as \(x \to 0\) for some constant \(K\), and the following identity holds:

\[
\text{He}
\left[
\frac{t}{q_1} \left| \frac{1 + a - c, 1 + b - c}{2 - c; d} \right| x \right] = \Theta(x) \, {}_2F_1\left(\frac{1 + A - C, 1 + B - C}{2 - C} \left| \varphi(x) \right. \right),
\]

where \(q_1 = q - (c - 1)(a + b - c - d + dt + 1)\) and \(\Theta(x) = \theta(x) \varphi(x)^{1-C} / Kx^{1-C}\).

**Proof.** A straightforward identification of the other canonical local solutions of both equations at \(x = 0\) and \(z = 0\).

If the local exponent difference at \(z = 0\) is an unrestricted parameter, changing its sign gives essentially the same two-term identity. More generally, the following choices of \(x = 0\) give the same two-term identities up to the fractional-linear transformations and change of parameters:

- the \(z\)-points with the same branching index and above the same point of \(P^4_\mathbb{C}\);
- the \(z\)-points with the same branching index, if they are in different fibers with the same branching pattern, and either the local exponents at the corresponding \(z\)-points are the same, or the local exponent differences at both \(z\)-points are free parameters.
If the pull-backed equation is Heun’s equation, its accessory parameter can be obtained at the latest stage, by considering power series expansions at $x = 0$ in a supposed two-term identity and comparing the first couple terms in the power series.

**Lemma 2.5.** Suppose that we have identity (10) coming from a pull-back transformation between the corresponding hypergeometric and Heun equations, with

$$\varphi(x) = \lambda x + O(x^2), \quad \theta(x) = 1 + \mu x + O(x^2)$$

as $x \to 0$. Then $q = c t \left( \mu + \frac{A B \lambda}{C} \right)$.

**Proof.** Expanding both sides of (10) in the power series at $x = 0$ gives

$$1 + \frac{q}{c t} x + O(x^2) = 1 + \mu x + \frac{A B \lambda}{C} x + O(x^2).$$

In particular, if the covering $\varphi(x)$ branches at $x = 0$ and the prefactor $\theta(x)$ is absent, then $q = 0$ (because $\lambda = \mu = 0$); see formulas (16) and (26) below.

A pull-back transformation (9) between hypergeometric and (or) Heun equations might fail to produce two-term identities between hypergeometric and (or) Heun solutions only if all singular points of the transformed equation lie above non-singular points of the starting equation. Lemma 2.3 then basically applies. So far we do not know an example of this kind. On the other hand, transformation identities between hypergeometric and Heun functions might formally exist without a pull-back transformation between their equations. For example, the linear function $\binom{2}{\frac{1}{3}}(z)$ can be formally transformed to any (hypergeometric or Heun) polynomial. Part 2 of [12, Lemma 2.1] indicates that this situation can occur only if we start with a hypergeometric function actually satisfying a first order Fuchsian equation.

**2.3. Transformations between hypergeometric functions.** Pull-back transformations between hypergeometric equations typically give algebraic transformations between hypergeometric functions of the form

$$2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \bigg| x \right) = \theta(x) 2F_1 \left( \begin{array}{c} A, B \\ C \end{array} \bigg| \varphi(x) \right).$$

The classical transformations were obtained by Gauss, Goursat, Riemann, Kummer. Here is an example of a cubic transformation, with one free parameter $a$:

$$2F_1 \left( \begin{array}{c} 3a, \frac{1}{3} - a \\ 2a + \frac{5}{6} \end{array} \bigg| x \right) = \left(1 - 4x\right)^{-3a} 2F_1 \left( \begin{array}{c} a, a + \frac{1}{3} \\ 2a + \frac{5}{6} \end{array} \bigg| \frac{27x}{(4x-1)^3} \right).$$

The local exponents are transformed as $(1/2, 1/3, \alpha)^3$ to $(1/2, \alpha, 2\alpha)$, with $\alpha = 1/6 - 2a$.

Quadratic transformations have two free parameters. The transformation of local expo-
nents is \((1/2, \alpha, \beta) \leftrightarrow (\alpha, \alpha, 2\beta)\). Here are hypergeometric formulas:

\[
2F_1\left( \begin{array}{c}
2a, 2b \\
a + b + \frac{1}{2}
\end{array} \middle| x \right) = 2F_1\left( \begin{array}{c}
a, b \\
a + b + \frac{1}{2}
\end{array} \middle| 4x(1-x) \right), \quad (13)
\]

\[
2F_1\left( \begin{array}{c}
2a, a - b + \frac{1}{2} \\
a + b + \frac{1}{2}
\end{array} \middle| x \right) = (1-x)^{-2a}2F_1\left( \begin{array}{c}
a, b \\
a + b + \frac{1}{2}
\end{array} \middle| \frac{4x}{(x-1)^2} \right), \quad (14)
\]

\[
2F_1\left( \begin{array}{c}
2a, b \\
2b
\end{array} \middle| x \right) = \left(1 - \frac{x}{2}\right)^{-2a}2F_1\left( \begin{array}{c}
a, a + \frac{1}{2} \\
b + \frac{1}{2}
\end{array} \middle| \frac{x^2}{(2-x)^2} \right). \quad (15)
\]

The first two formulas are related by fractional-linear transformations \((9)\), whereas \((15)\) is not equivalent up to the fractional-linear transformations (on either \(\mathbb{P}^1\) or \(\mathbb{P}^1_x\)), as noted by Askey \([3]\) and Maier \([9\, \text{Remark 4.1.2}]\). The dividing difference is the choice of the point \(x = 0\): it is a non-branching point in \((13)-(14)\) but a branching point in the last formula.

But formula \((15)\) can be derived from \((13)\) by the following argument. The functions

\[
x^{-2a}2F_1\left( \begin{array}{c}
2a, a - b + \frac{1}{2} \\
2a - 2b + 1
\end{array} \middle| \frac{1}{x} \right), \quad (1-x)^{-2a}2F_1\left( \begin{array}{c}
a, a + \frac{1}{2} \\
1 + a - b
\end{array} \middle| \frac{1}{(1-2x)^2} \right)
\]

are among the 24 Kummer solutions of the differential equations for, respectively, the left-hand side and the right-hand side of \((13)\). Therefore the two functions satisfy the same Fuchsian equation of order 2. We multiply both functions by \(x^{2a}\), make the substitutions \(x \rightarrow 1/x\) and \(b \rightarrow a - b + 1/2\) and obtain the two functions in \((15)\), up to a constant multiple on the right-hand side. Those two functions satisfy the same Fuchsian equation of order 2, have the same value and the same local exponent at a regular singular point (with a non-integer local exponent difference in general), so they must be equal in a neighborhood of \(x = 0\), and \((15)\) follows.

Pull-back transformations of the hypergeometric equations, and subsequently, algebraic transformations of the Gauss hypergeometric functions, are systematically studied and classified in a series of papers by the second author; see \(12\) and references therein. Here is an example of a non-classical transformation \((1/2, 1/3, 1/7) \leftrightarrow (1/3, 1/7, 2/7)\):

\[
2F_1\left( \begin{array}{c}
\frac{5}{42}, \frac{19}{42} \\
\frac{1}{2}
\end{array} \middle| x \right) = \left(1 - \frac{19}{9}x - \frac{343}{243}x^2 + \frac{16807}{6561}x^3\right)^{-1/28} \times
\]

\[
2F_1\left( \begin{array}{c}
\frac{1}{87}, \frac{20}{87} \\
\frac{6}{7}
\end{array} \middle| \frac{x^2 (1-x) (49x - 81)^7}{4(16807x^3 - 9261x^2 - 13851x + 6561)^3} \right).
\]

These identities can be verified by checking the power series at \(x = 0\). But the common region of convergence usually appears to be small. For example, the last identity does not hold at \(x = 1\) or \(x = 81/49\) for the standard analytic branches of \(2F_1\) functions, as can be checked numerically.

2.4. Quadratic transformations between the Heun and hypergeometric equations. By the Hurwitz formula, a quadratic covering \(\varphi: \mathbb{P}^1_x \rightarrow \mathbb{P}^1\) has two ramification points. Application of a quadratic pull-back transformation to the general hypergeometric equation gives a Fuchsian equation with 4 singular points, if the two ramification points lie above \(\{0, 1, \infty\} \subset \mathbb{P}^1_x\). Due to Möbius transformations on \(\mathbb{P}^1_x\), we may assume that the points \(x = 0, x = 1, x = \infty\)
are among those singular points, so that the pullbacked equation is Heun’s equation. It follows that quadratic transformations of general Gauss hypergeometric functions can be expressed in terms of Heun functions. Here are explicit formulas:

\[
\begin{align*}
\text{Hn}\left(\begin{array}{c}
-1 \\
0
\end{array} \bigg| \begin{array}{c}
2a, 2b \\
2c - 1; a + b - c + 1
\end{array} \right| x &= \text{}_2\text{F}_1\left(\begin{array}{c}
a, b \\
c
\end{array} \bigg| x^2 \right), \\
\text{Hn}\left(\begin{array}{c}
2 \\
4ab
\end{array} \bigg| \begin{array}{c}
2a, 2b \\
c; 2a + 2b - 2c + 1
\end{array} \right| x &= \text{}_2\text{F}_1\left(\begin{array}{c}
a, b \\
c
\end{array} \bigg| x(2 - x) \right), \\
\text{Hn}\left(\begin{array}{c}
\frac{1}{2} \\
2ab
\end{array} \bigg| \begin{array}{c}
2a, 2b \\
c; c
\end{array} \right| x &= \text{}_2\text{F}_1\left(\begin{array}{c}
a, b \\
c
\end{array} \bigg| 4x(1 - x) \right).
\end{align*}
\]

They were first indicated by Kuijken in [6]. Other possible polynomials \(\varphi(x)\) for quadratic transformations between hypergeometric and Heun equations are \(1 - x^2\), \((1 - x)^2\), \((2x - 1)^2\).

Fractional-linear transformations of \(P\)-symbols for the 192 Heun functions and the related Kummer’s 24 hypergeometric functions give a set of another 30 rational functions of degree 2 that transform the general hypergeometric equation to Heun’s equations (with a prefactor, in general). The 30 rational functions are given in [6] in the context of the degenerate case \(ab = q = 0\). If we are not concerned about the prefactor, there is no distinction between quadratic transformations to degenerate or non-degenerate Heun equations.

Like for hypergeometric quadratic transformations [13]–[15], we have two different choices for \(x = 0\): a branching point and a non-branching point. Accordingly, identities [17] and [18] are related by fractional-linear transformations [6], [6]–[7], whereas identity [16] cannot be related to them by the fractional-linear transformations. To derive [16] from [17], one can observe that the functions

\[
\text{Hn}\left(\begin{array}{c}
-1 \\
0
\end{array} \bigg| \begin{array}{c}
2a, 2b \\
2a + 2b - 2c + 1; c
\end{array} \right| 1 - x \right), \quad \text{Hn}\left(\begin{array}{c}
1 - x \\
1 - x
\end{array} \bigg| \begin{array}{c}
1 - x \\
1 - x
\end{array} \right| 1 - x \right), \quad \text{Hn}\left(\begin{array}{c}
2b \\
2b
\end{array} \bigg| \begin{array}{c}
2b \\
2b
\end{array} \right| x \right),
\]

satisfy the same Heun equation as both sides of [17], and have the same local exponent and value at \(x = 1\). Therefore they must be generally equal; formula [16] is then obtained after the substitution \(x \mapsto 1 - x, c \mapsto a + b - c + 1\).

### 2.5. Heun-to-Heun transformations

Existence of quadratic and quartic transformations was pointed out by Erdélyi in [1] Vol. 3]. Examples of these transformations are given by Maier in [3] Section 4]. Here are two alternative formulas of the quadratic transformations:

\[
\begin{align*}
\text{Hn}\left(\begin{array}{c}
s^2 \\
q_1
\end{array} \bigg| \begin{array}{c}
2a, 2a - b + 1 \\
b; 2a - b + 1
\end{array} \right| x \right) &= \left(1 + \frac{x}{s}\right)^{-2a} \text{Hn}\left(\begin{array}{c}
\frac{4s(x - 1)}{4s(x + 1)} \\
b; \frac{1}{2}
\end{array} \bigg| \frac{4sx}{(x + s)^2} \right), \\
\text{Hn}\left(\begin{array}{c}
s^2 \\
\frac{2ab + 4bs(x - 1)}{2s - 1}
\end{array} \bigg| \begin{array}{c}
2a, b \\
b; b
\end{array} \right| x \right) &= \left(1 - \frac{x}{s}\right)^{-2a} \text{Hn}\left(\begin{array}{c}
\frac{1}{4s(x - 1)} \\
b; \frac{1}{2}
\end{array} \bigg| \frac{x(x - 1)}{2a - b + 1} \right).
\end{align*}
\]

where \(q_1 = (1 + s)^2q - 2abs\). They are related by a series of fractional-linear transformations (and reparametrizations). The transformation of local exponents is given by the expression \((1/2, 1/2, \alpha, \beta) \leftrightarrow (\alpha, \alpha, \beta, \beta)\); all choices for \(x = 0\) give two-term identities related by fractional-linear transformations [6]–[7].

A quartic transformation can be obtained by composing two versions of the last quadratic transformation. In the composition, we have to restrict particularly \(s \mapsto 1/2s, b \mapsto 2a + 1/2\).
and \(a \rightarrow 2a, b \rightarrow 2a + 1/2\) in the two versions. Remarkably, transformation of the parameters \(t\) and \(q\) simplifies greatly. After setting \(t = s^2/(2s - 1)\) in the composition, we recognize the transformation
\[
\text{Hn}\left(\frac{t}{4q} \left| \begin{array}{c} 4a, 2a + \frac{1}{2} \\ 2a + \frac{1}{2}; 2a + \frac{1}{2} \\ \end{array} \right| x \right) = \left(1 - \frac{x^2}{t}\right)^{-2a} \text{Hn}\left(\frac{t}{q} \left| \begin{array}{c} a, a + \frac{1}{2} \\ 2a + \frac{1}{2}; 2a + \frac{1}{2} \\ \end{array} \right| \frac{4tx(x-1)(x-t)}{(x^2-t)^2} \right)
\]
as in [9, Theorem 4.2]. We note that a similar composition of quadratic transformations for Painlevé VI functions is comparably simpler than the quadratic transformations as well [15, Lemma 6.1]. The local exponents are transformed under the composition as
\[
\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \alpha\right) \mapsto \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \alpha\right) \mapsto \left(\frac{1}{2}, \alpha, \alpha, \alpha\right).
\]
The composite degree 4 covering happens to be a Belyi covering. It is ramified over the 3 points with the local exponent difference 1/2; the branching pattern is 2 + 2 = 2 + 2 = 2 + 2. Up to the Möbius transformation \(z \rightarrow 1/z\), the starting Heun equation for the quartic transformation is a general Lamé equation.

Generally, Heun-to-Heun transformations of degree \(d\) must have at most 4 singular points and at least \(2d - 2\) non-singular points above the 4 singular points of the starting Heun equation, because the Hurwitz formula implies that a covering \(\mathbb{P}^1 \rightarrow \mathbb{P}^1\) has at least \(2d + 2\) points above any 4 points of \(\mathbb{P}^1\). (An indicative situation is adding one fiber in the context of Lemma [22].) It is not hard to see that the distribution of singular and non-singular points is impossible for \(d = 3\) or \(d > 4\), unless all local exponent differences of the starting equation are equal to 1/2 (or we allow it to have less singularities). We are then in a special case of the Lamé equation; possible branching patterns indicate that the transformed equation has the same local exponents. Identification of solutions of the Lamé equation with the local exponent differences \((1/2, 1/2, 1/2, 1/2)\) with the generic indefinite elliptic integral indicates that transformations of these Lamé equations exist in any degree \(d\); they arise from the isogenies between generic elliptic curves. Here is an example of a cubic transformation between respective Lamé functions:
\[
\text{Hn}\left(\frac{s^2(s-2)}{q (1-2s)^2} \left| \begin{array}{c} 0, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \\ \end{array} \right| x \right) = \text{Hn}\left(\frac{s^2(s-2)^3}{q (1-2s)^2} \left| \begin{array}{c} 0, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \\ \end{array} \right| \frac{x(x+s(s-2))^2}{((1-2s)x+s^2)^2} \right),
\]
with \(s \neq 0, 1/2, 1, 2\). The branching pattern is a generic picture of 4 simple branching points in 4 different fibers. Note that there is no prefactor, even if the argument is not a polynomial.

3. Transformations between the Heun and hypergeometric functions. Here we classify pull-back transformations [9] between hypergeometric and Heun equations with 2 or 3 free continuous parameters (in the equations), up to fractional-linear transformations of their \(P\)-symbols. Subsequently, we present all corresponding Heun-to-hypergeometric transformation formulas [10] with 2 or 3 free continuous parameters, up to fractional-linear transformations [8], [6]–[7].

Let \(H_1\) denote the starting hypergeometric equation, and \(d\) denote the degree of an assumed pull-back transformation \(\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1\).

If we assume 3 continuous parameters, we put no restriction on the local exponent differences of \(H_1\). Therefore all points above \(z = 0, z = 1, z = \infty\) will be singular for the
transformed equation. We have at least $d + 2$ singular points by Lemma 2.1 and we wish the transformed equation to be the Heun equation with four singularities, hence, $d + 2 \leq 4$ leading to $d \leq 2$. If the degree of the covering is $d = 1$, we have the fractional-linear transformations which do not change the number of singular points. If $d = 2$, we have a quadratic transformation $(\alpha, \beta, \gamma) \mapsto (2\alpha, \beta, \beta, 2\gamma)$. These transformations are discussed in Subsection 2.4 in full generality.

We obtain new transformations with two continuous parameters if some points above $\{0, 1, \infty\} \subset \mathbb{P}^1_z$ are non-singular. (Otherwise we would have a specialization of a pull-back transformation of the general hypergeometric equation.) Therefore we assume $d > 2$ and restrict a local exponent difference of $H_1$ to the value $1/k$, where $k$ is a positive integer. We also assume $k > 1$ because of the remark following Lemma 2.3 with a reference to [12, Lemma 5.1]. The transformed equation has at most $\lfloor d/k \rfloor$ non-singular points above $\{0, 1, \infty\} \subset \mathbb{P}^1_z$, therefore it has at least $d + 2 - \lfloor d/k \rfloor$ singular points by Lemmas 2.1 and 2.2. We want to have $d + 2 - \lfloor d/k \rfloor \leq 4$. The left-hand side can only decrease when we drop the floor rounding. After ignoring the floor rounding and dividing the inequality by $d$ we get the Diophantine inequality

$$\frac{2}{d} + \frac{1}{k} \geq 1. \quad (22)$$

We have the following solutions with $k > 1$, $d > 2$:

$$(k, d) \in \{(2, 3), (2, 4), (3, 3)\}.$$

For each possibility $(d, k)$ we can write down possible branching patterns for $\varphi$ such that the supposed transformed equation would have at most 4 singular points. The possible local exponent differences, the degree of the covering and branching patterns are presented in the first four columns of Table 1.

Next we have to determine possible rational functions $\varphi(x)$ with the branching patterns in the fourth column of Table 1. In general, a covering with a given branching pattern may not exist, or there might be several different coverings with the same branching pattern. All our particular coverings appear in [12, Table 1] except $2 + 2 = 2 + 2 = 2 + 2$, but the latter branching pattern determines the quartic Heun-to-Heun transformations of Subsection 2.5. Therefore all rational functions $\varphi(x)$ are known in principle. Because the possible degree $d$ is so small, one can recompute all $\varphi(x)$’s using a straightforward ansatz method with unde-
the cubic argument in (25) is the same as in (23)–(24) even if the fiber for Lemma 2.4. The arguments of the first four transformations are polynomials. Notice that however, the branching pattern in both fibers and the branching order for essentially the same covering.

$x$ non-equivalent choices for the local exponent difference at $x$ where

determined, one can choose the local exponents of $H_1$ determined, one can choose the local exponents of $H_1$ to $1/k$, $1/\ell$ with $k, \ell$ positive integers, and would similarly get the diophantine inequalities $d - \lfloor d/k \rfloor - \lfloor d/\ell \rfloor \leq 2$ and $2/d + 1/k + 1/\ell \geq 1$. In particular, the degree bound for transformations of $H_1$ with the local exponent differences $(1/2, 1/3, \alpha)$ is $d \leq 12$. We leave the study of these transformations for the next paper.

3.1. The transformations $(1/2, \alpha, \beta) \rightarrow^3 (1/2, \alpha, 2\alpha, 3\beta)$. Up to fractional-linear transformations, we have the following identities:

$$H_n \left( \frac{1}{2} \left| \begin{array}{c} 3a, 3b \\ a + b + \frac{1}{2} \end{array} \right| x \right) = \phi \left( \frac{a, b}{1} \left| \begin{array}{c} x(4x - 3)^2 \\ \frac{1}{2} \right. \right), \quad (23)$$

$$H_n \left( \frac{1}{2} \left| \begin{array}{c} 3a, 3b \\ a + b + \frac{1}{2} \end{array} \right| x \right) = \left( 1 - \frac{4x}{3} \right) \phi \left( \frac{a + \frac{1}{3}, b + \frac{1}{3}}{\frac{3}{2}} \left| \begin{array}{c} x(4x - 3)^2 \\ \frac{1}{2} \right. \right), \quad (24)$$

$$H_n \left( \frac{3}{2} \left| \begin{array}{c} 3a, 3b \\ a + b + \frac{1}{2} \end{array} \right| x \right) = \phi \left( \frac{a, b}{1} \left| \begin{array}{c} x(4x - 3)^2 \\ \frac{1}{2} \right. \right), \quad (25)$$

$$H_n \left( \frac{3}{2} \left| \begin{array}{c} 3a, 3b \\ a + b + \frac{1}{2} \end{array} \right| x \right) = \phi \left( \frac{a, b}{1} \left| \begin{array}{c} x^2(x + 3) \\ \frac{1}{2} \right. \right), \quad (26)$$

$$H_n \left( \frac{3}{2} \left| \begin{array}{c} 3a, 3b \\ a + b + \frac{1}{2} \end{array} \right| x \right) = \left( 1 + \frac{3x}{4} \right)^{-2a} \phi \left( \frac{a, b}{1} \left| \begin{array}{c} x^3 \\ \frac{1}{2} \right. \right), \quad (27)$$

where $q_1 = (9ab + 3a + 3b - 1)/4$, $q_2 = 6a^2 + 6ab - a$. The five formulas represent the five non-equivalent choices for the local exponent difference at $x = 0$. The choices for the local exponent at $x = 0$ are, respectively: $1/2, -1/2, \alpha, 2\alpha, 3\beta$. The first two identities are related by Lemma 2.5. The arguments of the first four transformations are polynomials. Notice that the cubic argument in (25) is the same as in (23)–(24) even if the fiber for $x = 0$ is different. However, the branching pattern in both fibers and the branching order for $x = 0$ is the same, so the same configuration of the singular points $x = 0$, $x = 1$, $x = \infty$ is possible (even if the local exponents at the respective points are different). The argument in (26) is related to $x(4x - 3)^2$ by the affine transformation $x \mapsto (x + 3)/4$, giving us other point as $x = 0$ on essentially the same covering.
3.2. The transformations \((1/2, \alpha, \beta) \leftarrow (\alpha, 2\alpha, 4\beta)\). Up to fractional-linear transformations, we have the following identities:

\[
\begin{align*}
\text{Hn} \left( \frac{1}{2} \middle| \frac{4a}{8ab} + \frac{4b}{2}; a + b + \frac{1}{2} \middle| x \right) &= 2F_1 \left( \frac{a}{a + b + \frac{1}{2}} \middle| -16x(x-1)(2x-1)^2 \right), \\
\text{Hn} \left( 0 \middle| \frac{4a}{2a + 2b}; a + b + \frac{1}{2} \middle| x \right) &= 2F_1 \left( \frac{a}{a + b + \frac{1}{2}} \middle| 4x^2(1 - x^2) \right), \\
\text{Hn} \left( 0 \middle| \frac{4a}{4b - 1}; 2a - b + 1 \middle| x \right) &= \left( 1 - \frac{x^2}{2} \right)^{-2a} \frac{ab}{b + \frac{1}{2}} 2F_1 \left( \frac{a}{b + \frac{1}{2}} \middle| \frac{x^4}{(x^2 - 2)^2} \right)
\end{align*}
\]

There are indeed three non-equivalent choices for the local exponent difference at \(x = 0\), which are \(\alpha, 2\alpha, 3\beta\), respectively. The pull-back covering is a composition of two quadratic transformations: \((\frac{1}{2}, \alpha, \beta) \leftarrow (\alpha, 2\beta) \leftarrow (\alpha, 2\alpha, 4\beta)\). Accordingly, the three identities are compositions of, respectively, \((13)\) and \((13)\), \((13)\) and \((16)\), or \((15)\) and \((16)\).

3.3. The transformations \((1/2, \alpha, \beta) \leftarrow (\alpha, 3\alpha, 3\beta)\). Up to fractional-linear transformations, we have the following identities:

\[
\begin{align*}
\text{Hn} \left( \frac{9}{9} \middle| \frac{4a}{3a + 3b - 1}; a + b + \frac{1}{2} \middle| x \right) &= \left( 1 - \frac{8x}{9} \right)^{-3a} \frac{ab}{b + \frac{1}{2}} 2F_1 \left( \frac{a}{b + \frac{1}{2}} \middle| \frac{64x^3(1 - x)}{(9 - 8x)^3} \right), \\
\text{Hn} \left( \frac{-1}{8} \middle| \frac{4a}{a + b + \frac{1}{2}; 3a + 3b - 1} \middle| x \right) &= (1 + 8x)^{-3a} \frac{ab}{b + \frac{1}{2}} 2F_1 \left( \frac{a}{b + \frac{1}{2}} \middle| \frac{64x(1 - x)^3}{(1 + 8x)^3} \right)
\end{align*}
\]

where \(q_1 = 9a^2 + 9ab - 3a/2\), \(q_2 = 3a^2 - 5ab + 3a/2\). The choice between \(\alpha\) and \(\beta\) for the local exponent difference at \(x = 0\) gives identities related by fractional-linear transformations, just as the choice between \(3\alpha\) and \(3\beta\). Hence we have only two transformation formulas.

3.4. The possibility \((1/2, \alpha, \beta) \leftarrow (2\alpha, 2\alpha, 3\beta)\). We demonstrate here that there is no covering for the branching pattern \(2 + 2 = 2 + 2 = 3 + 1\). The three fibers are assumed to be \(z = 0, 1, \infty\), of course. We choose the points above \(z = \infty\) to be \(x = 0\) and \(x = \infty\). (Normally, we would put the point \(x = 0\) above \(z = 0\), so that we would have the rational function for transformation formula immediately. But for a negative result we have no reason to do that.) Suppose that the two points above \(z = 0\) are the roots of \(x^2 + ax + b\), with \(a, b\) to be determined. Then the rational function defining the covering should have the form

\[
\varphi(x) = C_1 \frac{(x^2 + ax + b)^2}{x},
\]

for some constant \(C_1\). The branching pattern above \(z = 1\) implies that there are two points above \(z = 1\). To compute the branching points we need to compute \(\varphi'(x) = 0\). We have

\[
\varphi'(x) = C_1 \frac{(x^2 + ax + b)(3x^2 + ax - b)}{x^2}.
\]

The points above \(z = 1\) must be the roots of \(3x^2 + ax - b\). We must have \(1 - \varphi(x)\) equal, up to a constant multiple, to

\[
1 - \varphi(x) = C_2 \frac{(3x^2 + ax - b)^2}{x}.
\]
The derivative of this expression must vanish at the same branching points above \( z = 0 \) and \( z = 1 \) as well. The derivative is equal to
\[
C_2 \frac{(3x^2 + ax - b)(9x^2 + ax + b)}{x^2}.
\]
We have to conclude that the polynomials \( x^2 + ax + b \) and \( 9x^2 + ax + b \) must be proportional, which is possible only in the degenerate case \( a = b = 0 \). Hence a contradiction.

3.5. The transformations \((1/2, \alpha, \beta) \leftarrow ^1 (2\alpha, 2\alpha, 2\beta, 2\beta)\). Up to fractional-linear transformations, we have one identity:
\[
\text{Hn} \left( \begin{array}{c} -1 \\ 0 \end{array} \right| \begin{array}{c} 4a, 2a - 2b + 1 \\ 2a + 2b; 2a - 2b + 1 \end{array} | x \right) = (1 - x^2)^{-2a} \text{F}_1 \left( \begin{array}{c} a, b \\ a + b + \frac{1}{2} \end{array} \right| - \frac{4x^2}{(x^2 - 1)^2} \right),
\]
as the choice of the local exponent differences \( 2\alpha \) or \( 2\beta \) for \( x = 0 \) gives equivalent identities. The identity is a composite of \([14]\) and \([16]\). In fact, the pull-back transformation can be expressed in several ways as a composition of quadratic transformations:
\[
(1/2, \alpha, \beta) \leftarrow ^2 (2\alpha, \alpha, 2\beta),
\]
\[
(1/2, \alpha, \beta) \leftarrow ^2 (2\alpha, \beta, \beta),
\]
\[
(1/2, \alpha, \beta) \leftarrow ^2 (1/2, 1/2, 2\alpha, 2\beta).
\]
In the first two cases the second arrow denotes a quadratic Heun-to-hypergeometric transformation. In the last case, we have a quadratic Heun-to-Heun transformation. The first two compositions imply a hypergeometric formula like
\[
\text{F}_1 \left( \begin{array}{c} a, \beta \\ 2a \end{array} | x(2 - x) \right) = (1 - x)^{-b} \text{F}_1 \left( \begin{array}{c} 2a - b, b \\ a + \frac{1}{2} \end{array} \right| \frac{x^2}{4(x - 1)} \right),
\]
which is a quadratic-quadratic transformation with two free parameters. This kind of transformation is presented in [2] p. 128-130].

3.6. The transformations \((1/3, \alpha, \beta) \leftarrow ^3 (\alpha, 2\alpha, \beta, 2\beta)\). Up to fractional-linear transformations, we have the following identities:
\[
\text{Hn} \left( \begin{array}{c} 3a, 2a + b \\ a + b + \frac{1}{3}; 2a - 2b + 1 \end{array} | x \right) = (1 - x)^{-2a} \text{F}_1 \left( \begin{array}{c} a, b \\ a + b + \frac{1}{3} \end{array} \right| - \frac{x(x - 9)^2}{27(x - 1)^2} \right),
\]
\[
\text{Hn} \left( \begin{array}{c} 3a, 2a + b \\ 2a + 2b - \frac{1}{3}; a + b + \frac{1}{3} \end{array} | x \right) = (1 - \frac{9x}{8})^{-2a} \text{F}_1 \left( \begin{array}{c} a, b \\ a + b + \frac{1}{3} \end{array} \right| \frac{27x^2(x - 1)}{8 - 9x^2} \right),
\]
where \( q_1 = 18a^2 - 9ab + 6a, q_2 = 4a^2 + 4ab - 2a/3 \). The choice between \( \alpha \) and \( \beta \) for the local exponent difference at \( x = 0 \) gives identities related by fractional-linear transformations, just as the choice between \( 2\alpha \) and \( 2\beta \). Hence we have only two transformation formulas.

3.7. The transformations \((1/3, \alpha, \beta) \leftarrow ^3 (\alpha, \alpha, \alpha, 3\beta)\). Up to fractional-linear transformations, we have the following identities:
\[
\text{Hn} \left( \begin{array}{c} -\omega \\ 3(1 - \omega)ab \end{array} \right| \begin{array}{c} 3a, 3b \\ a + b + \frac{1}{3}; a + b + \frac{1}{3} \end{array} | x \right) = \text{F}_1 \left( \begin{array}{c} a, b \\ a + b + \frac{1}{3} \end{array} \right| 3(2\omega + 1)x(x - 1)(x + \omega) \right),
\]
\[
\text{Hn} \left( \begin{array}{c} \omega + 1 \\ 3(\omega + 2)ab \end{array} \right| \begin{array}{c} 3a, a + b + \frac{1}{3} \\ 3b; 2a - b + \frac{2}{3} \end{array} | x \right) = (1 + \frac{\omega - 1}{3}x)^{-3a} \text{F}_1 \left( \begin{array}{c} a, a + \frac{1}{3} \\ b + \frac{2}{3} \end{array} \right| \frac{x^3}{(x - \omega - 2)^3} \right),
\]
where $\omega$ is the root of $\omega^2 + \omega + 1 = 0$. The choices for the local exponent at $x = 0$ are $\alpha$ and $3\beta$. To relate the argument in (37) to [7, formula (3.6a)], note that

$$3(2\omega + 1)x(x - 1)(x + \omega) = 1 - (1 - (\omega + 2)x)^3.$$ 

The other two-parameter coverings of [7] are represented in Subsections 3.1 and 3.2 above.

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