Spin Gauge Theory of Gravity in Clifford Space: A Realization of Kaluza-Klein Theory in 4-Dimensional Spacetime

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Abstract

A theory in which 4-dimensional spacetime is generalized to a larger space, namely a 16-dimensional Clifford space (C-space) is investigated. Curved Clifford space can provide a realization of Kaluza-Klein. A covariant Dirac equation in curved C-space is explored. The generalized Dirac field is assumed to be a polyvector-valued object (a Clifford number) which can be written as a superposition of four independent spinors, each spanning a different left ideal of Clifford algebra. The general transformations of a polyvector can act from the left and/or from the right, and form a large gauge group which may contain the group U(1)×SU(2)×SU(3) of the standard model. The generalized spin connection in C-space has the properties of Yang-Mills gauge fields. It contains the ordinary spin connection related to gravity (with torsion), and extra parts describing additional interactions, including those described by the antisymmetric Kalb-Ramond fields.

1 Introduction

Kaluza-Klein idea for the unification of the gravitational and electromagnetic interaction by extending the dimensionality of spacetime is very fascinating. In eighties it attracted
much attention in its modern formulation in which higher dimensional curved spacetimes $V_n, n > 5$, were considered. This enabled to treat Yang-Mills fields as being incorporated in the metric tensor of $V_n$. When confronting the theory with phenomenology, serious difficulties have arisen. In particular, there is a notorious problem that a charged particle has an effective 4-dimensional mass which is of the order of the Planck mass. A search for a realistic Kaluza-Klein theory has not been successful so far.

In the meantime the focus of attention has switched to string theory (see e.g. [1]) which has turned out to be very promising in unifying gravity with gauge interactions. A lot of fascinating results have been obtained in the last ten years, such as a discovery that string theory contains higher dimensional objects (D-branes) and that there must be a single underlying theory, the so called M-theory, which unifies different known types of string theory (see, e.g., [2], and references therein). The presence of branes in string theory has led to the revival [3] of the old idea [4] that our 4-dimensional universe is a 4-dimensional surface, a world manifold of a 3-brane, living in a higher dimensional spacetime (see also refs. [5, 6, 7]).

There is a number of works which further illuminate strings and branes from the theoretical point of view. For instance, Aurilia et al. [8], following the original proposal by Schild [9] and Eguchi [10], formulated $p$-branes in terms of the tangent $(p + 1)$-multivectors to a $p$-brane’s world manifold. So they obtained the quantum propagator of a bosonic $p$-brane in the quenched minisuperspace approximation [11] which lead to a novel —Clifford algebra based— unified description [12, 13, 5, 16, 17] of $p$-branes for different values of $p$. The background space that emerged in such a framework turned out to be a Clifford space.

Since the seminal Hestenes’s works [18] Clifford algebra is becoming increasingly popular in physics [19]. It has been realized that Clifford algebra is not only a useful tool for description of the known geometry and physics, but it provides a lot of room for new geometry and new physics. As an intermediate step into that new direction several authors have investigated, in a number of illuminating and penetrating works [20–22], the idea that Clifford algebra provides a framework for unification of fundamental interactions. Common to all those works is that they consider the Clifford bundle of spacetime, and so the generators of Clifford algebra and other related quantities depend on position in spacetime. But other researchers [23, 12, 5, 14, 15, 17, 26, 25] have formulated a different theory which is based on the concept of Clifford space ($C$-space). This is the space of oriented $r$-dimensional areas which, in the case of flat $C$-space, can be described by polyvectors $X$. 

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superpositions of $r$-vectors, where $r$ takes the discrete values from 0 to 4. A position in $C$-space is described by sixteen coordinates $x^M = (s, x^\mu, x^{\mu\nu}, x^{\mu\nu\rho}, x^{\mu\nu\rho\sigma})$ which are a generalization of the usual four spacetime coordinates $x^\mu$, $\mu = 0, 1, 2, 3$. The role of Clifford space as the arena for physics has been investigated in refs. [23, 12, 5, 14, 15, 17, 26, 25].

If $C$-space is flat, we have essentially a generalization of special relativity to the new degrees of freedom which reside in $C$-space. The new degrees of freedom, i.e., the polyvector coordinates $x^M$, if associated with a physical object, encode the information not only about the object’s center of mass position, but also about its extensions and orientation. A detailed “shape” or configuration of the extended object is not encoded by $x^M$, only a partial information about the shape is encoded [17]. The extended objects are observed in 4-dimensional spacetime, they exist in spacetime, and their extended nature is “sampled” by the coordinates $x^M$, which denote position in 16-dimensional $C$-space. The latter space is just like a multidimensional configuration space associated with a many particle system which, of course, still resides in spacetime. So we have both at once: 4-dimensional spacetime, and 16-dimensional $C$-space.

In particular, the extended objects can be just the fundamental (closed) branes. It is well known [27, 28] (see also a recent systematic exposition [29] and refs. [30] that the elements of the right or left minimal ideals of Clifford algebra can be used to represent spinors. Therefore, a coordinate polyvector $X$ automatically contains not only bosonic, but also spinor coordinates. In refs. [31, 32] it was proposed to formulate string theory in terms of polyvectors, and thus avoid using a higher dimensional spacetime. Spacetime can be 4-dimensional, whilst the extra degrees of freedom (“extra dimensions”) necessary for consistency of string theory are in Clifford space.

In this paper we investigate the possibility that the arena itself is to become a part of the play. We propose that Clifford space should be considered as a dynamical, in general curved, space, analogous to the spacetime of general relativity. Since a dynamical (curved) Clifford space has 16 dimensions, it provides a realization of Kaluza-Klein idea. This approach has seeds in refs. [33, 25], but explicitly it was formulated in refs. [31, 34, 35].

We will first investigate some basic aspects of the classical general relativity like theory in $C$-space. We point out that the geodetic equation in $C$-space contains the terms that can be interpreted as being related to the extra interactions (besides the ordinary gravitational one). Then we pass to quantum theory and discuss the concept of polyvector valued wave function which can be written in a basis spanning four independent left ideals of
Clifford algebra. We write the Dirac-like equation in curved $C$-space by introducing a generalized \textit{spin connection} which, in general, depends on position $X$ in $C$-space. We formulate the corresponding action principle and derive the Noether currents in which the charges that generate gauge transformations are on the same footing as the spin and angular momentum. We show that the generalized spin connection contains Yang-Mills fields describing fundamental interactions, and also the antisymmetric Kalb-Ramond fields that have an important role in string theory. So we provide a unified framework for description of Yang-Mills fields, including Kalb-Ramond fields.

\section{Clifford space as a generalization of spacetime}

Let $V_n$ be a spacetime manifold\footnote{Following the old tradition, I use the notation $M_n$ for $n$-dimensional (flat) Minkowski space, and $V_n$ for a more general spacetime manifold, which may have curvature and torsion. In modern, especially mathematically oriented literature \cite{37,38} a more precise notation is used.}. For the sake of generality we keep its dimension $n$ arbitrary, but later we eventually assume the physically observed value $n = 4$. Every point $P$ of $V_n$ can be parametrized\footnote{How precisely this is achieved introducing the charts and atlas has been explained in many works, therefore I do not repeat it here.} by arbitrarily chosen coordinates $x^\mu(P)$. The subtleties related to the identification of spacetime point were pointed out by Einstein in his “hole argument” \cite{39}, and subsequently discussed and refined by many authors, of whom let me mention here DeWitt \cite{40} and Rovelli \cite{41}. However, what is clear is that in spacetime physical \textit{events} can happen, and that the events can be unambiguously identified. If in spacetime there is a network of events or, in an idealized limit, an infinitely dense collection of events, e.g., the DeWitt-Rovelli “reference fluid” \cite{40,41}, then the spacetime points can be identified. A quantity of interest is the \textit{distance} between two events. The distance between two infinitely close events in a manifold endowed with metric is given by

\begin{equation}
\text{d}s^2 = g_{\mu\nu}\text{d}x^\mu\text{d}x^\nu
\end{equation}

Actually this is the square of the distance, and $g_{\mu\nu}(x)$ is the metric tensor.

Let us now consider the \textit{square root} of the above quadratic form. Obviously it is $\sqrt{g_{\mu\nu}\text{d}x^\mu\text{d}x^\nu}$. But the latter expression is not linear in $\text{d}x^\mu$. We would like to define an object which is \textit{linear} in $\text{d}x^\mu$ and whose square is eq. (1). Let such object be given by the
expression\textsuperscript{3}

\[
dx = dx^\mu \gamma_\mu
\]

It must satisfy

\[
dx^2 = \gamma_\mu \gamma_\nu \, dx^\mu dx^\nu = \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) \, dx^\mu dx^\nu = g_{\mu\nu} \, dx^\mu dx^\nu
\]  \hspace{1cm} (3)

from which it follows that

\[
\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu}
\]  \hspace{1cm} (4)

This is the defining relation for the generators \(\gamma_\mu\) of Clifford algebra. At every point \(x \in V_n\) the objects \(\gamma_\mu, \mu = 1, 2, ..., n\), form a complete set of \textit{basis vectors}, and they span a vector space\textsuperscript{4}, called the tangent vector space \(T_x(V_n)\).

Basis vectors \(\gamma_\mu\) occurring in eq. (2) are tangent to coordinate curves and they form a \textit{coordinate frame} at \(x \in V_n\). More general frames, that are not tangent to coordinate curves, also exist. Of particular interest are manifolds such that at every point \(x \in V_n\), or at least at every \(x \in \mathcal{R}\) of a region \(\mathcal{R} \in V_n\), one can construct an orthonormal frame.

Eq. (4) says that the \textit{symmetric part} of the Clifford (or geometric) product \(\gamma_\mu \gamma_\nu\) is the \textit{inner product} determining the metric tensor \(g_{\mu\nu}\).

The \textit{antisymmetric part} is the \textit{wedge product} determining a bivector representing an oriented area:

\[
\gamma_\mu \wedge \gamma_\nu \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \equiv \frac{1}{2} [\gamma_\mu, \gamma_\nu]
\]  \hspace{1cm} (5)

This can be continued to the antisymmetric product of 3, 4, ..., \(n\) basis vectors \(\gamma_\mu\). So the basis vectors \(\gamma_\mu\) generate at the point \(x \in V_n\) the Clifford algebra \(\mathcal{C}_\ell_n\) with the basis elements

\[
\gamma_M \equiv \gamma_{\mu_1\mu_2...\mu_r} = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r} \equiv \frac{1}{r!} [\gamma_{\mu_1}, \gamma_{\mu_2}, ..., \gamma_{\mu_r}], \quad r = 0, 1, 2, ..., n
\]  \hspace{1cm} (6)

defined as the antisymmetrized or wedge product.

An important insight that we acquire by involving Clifford algebra into the game is the following:

\textsuperscript{3}Here the \textit{scalar components} \(dx^\mu\) should not be confused with the differential 1-forms.

\textsuperscript{4}The symbol \(V_n\) used here for a spacetime manifold should not be confused with a symbol for a vector space. Since the objectives and the emphasis of research in physics is different from that in mathematics or mathematical physics, certain discrepancies of notation are sometimes unavoidable. The symbol \(M_4\) is reserved for Minkowski space (a usual practice in physics), whilst \(\mathcal{M}\) is reserved for the “membrane space”, i.e., the infinite dimensional space of \(r\)-loops, considered in ref. \[5\]. The latter space is a generalization of \(C\)-space, and provides a possible geometric principle behind the string theory, and is according to my expectations related to the conjectured \(M\)-theory which unifies different string theories.
• The “square root” of the distance quadratic form $ds^2$ is a vector.

• Vectors are Clifford numbers. A complete set of linearly independent vectors generate Clifford algebra.

• Vectors are objects which, like distance, are invariant under general coordinate transformations.

The last point above comes from the fact that under a general coordinate transformation both the components $dx^\mu$ and the basis vectors $\gamma_\mu$ change in such manner that $dx = dx^\mu \gamma_\mu$ remains invariant.

From eqs. (2), (4) we see that in order to obtain basis vectors $\gamma_\mu$ no differentiation is needed in their definition, if we assume that the metric tensor $g_{\mu\nu}(x)$ at the point $x \in V_n$ is already given. By definition $\gamma_\mu$ are Clifford numbers satisfying relations (4). This is a very direct, and intuitively clear definition of tangent vectors and the inner product.

In most modern mathematical works on differential geometry, tangent vectors are identified with differential operator. Such identification certainly has advantages, otherwise it would not have been so widely adopted, but it also has serious drawbacks, as pointed out by Hesteness \[42\]. He wrote: “It is sufficient to note that if tangent vectors [defined as differential operators] are not allowed to generate a geometric algebra at first place, then the algebra must be artificially imposed on the manifold later on, because it is absolutely essential for spinors and quantum mechanics.”

It has been shown in many lucid works that Clifford numbers $\gamma_\mu$, $\mu = 1, 2, ..., n$, are very useful for description of geometry and physics, in particular of special and general relativity. But $\gamma_\mu$ alone are not the whole story. They generate Clifford algebra, spanned by the basis (6), which brings multivectors ($r$-vectors) representing oriented areas, volumes, etc., and their superpositions, called polyvectors, into the game. This provides a framework for a theory which goes beyond the current special or general theory of relativity.

Clifford algebra enables description of, e.g., $r$-dimensional surfaces associated with extended objects. In order to make contact with physics, we have to specify which extended objects we wish to describe. Let us choose to describe “fundamental” extended objects, such as strings and branes. Instead of describing an extended object by infinite number of degrees of freedom, which is one extreme, or only by the center of mass coordinates, which is another extreme, we can describe it by a finite number of “polyvector” coordinates\[^5\]

\[^5\]In particular cases those coordinates can be considered as components of a polyvector. In general this
In order to formalize the theory we generalize the notion of event which, in general, is no longer a point event in an n-dimensional spacetime. Instead it is an extended event, described by a set of “polyvector” coordinates $x^{\mu_1\ldots\mu_r}$, $r = 0, 1, \ldots, n$. Spacetime is thus replaced by a larger space, the space of extended events, called Clifford space or C-space.

In the following we will first review the concept of flat C-space and then generalize it to that of curved C-space.

### 2.1 Flat Clifford space

Let us now consider flat spacetime manifold, i.e., the Minkowski space $M_n$. The object $dx = dx^\mu \gamma_\mu$, defined in eq. (2), represents a vector joining two points $P$ and $P_0$, with coordinates $x^\mu$ and $x^\mu + dx^\mu$, respectively. In flat manifold this relation can be extended to finite separation of points:

$$x(P) - x(P_0) = (x^\mu(P) - x^\mu(P_0)) \gamma_\mu$$

Choosing $x^\mu(P_0) = 0$, we have

$$x(P) = x^\mu(P) \gamma_\mu$$

which is a vector joining the coordinate origin $P_0$ and a point $P$. In other words, a vector $x$ represents an oriented line, with one end at the origin $P_0$ and the other end at a point $P$. If we keep the point $P_0$ fixed and $P$ variable, we can say that $x(P)$ denotes a point $P$ of the manifold $M_n$, or, speaking in physical terms, a point event in $M_n$. We may then simplify the notation, and write $x(P) \equiv x$, $x^\mu(P) \equiv x^\mu$, etc.

In eqs. (7), (8), only the basis vectors $\gamma_\mu$ occur. Since the latter objects generate Clifford algebra, with the basis (6), one can envisage the following generalization of eqs. (7):

$$X(E) - X(E_0) = (x^M(E) - x^M(E_0)) \gamma_M$$

where $E$ generalizes the notion of point events $P$. Choosing $x^M(E_0) = 0$, we obtain the corresponding generalization of eq. (8):

$$X(E) = x^M(E) \gamma_M \equiv x^M \gamma_M = \sigma 1 + x^\mu \gamma_\mu + x^{\mu_1 \mu_2} \gamma_{\mu_1 \mu_2} + \ldots + x^{\mu_1 \ldots \mu_n} \gamma_{\mu_1 \ldots \mu_n}$$

is not so, but for illustrative reasons it is sometimes convenient to keep the name “polyvector” coordinates (see also a description at the beginning of Sec. 2.2).
where $\mu_1 < \mu_2 < \ldots \ldots$. The latter object is a Clifford valued polyvector, a superposition of multivectors $^7$ ($r$-vectors). It denotes the position of a point in an $2^n$-dimensional Clifford space ($C$-space). The series terminates at a finite grade, depending on the dimension $n$. From the point of view of $C$-space the $r$-vector coordinates $x^M \equiv x^{\mu_1 \ldots \mu_r}$, $r = 0, 1, 2, \ldots, n$, (called also ‘polyvector coordinates’, since being components of a polyvector), denote a point. But from the point of view of spacetime they denote an extended event $E$ associated, e.g., with a closed $(r - 1)$-dimensional brane (called also $(r - 1)$-loop) enclosing an $r$-dimensional surface, or a superposition of such objects for different grades $r$. In the case of a closed string (1-loop) embedded in a target spacetime of $n$-dimensions, one represents the projections of the closed string (1-loop) onto the embedding spacetime coordinate planes by the variables $x^{\mu \nu}$. The latter quantities we call bivector coordinates of the 1-loop. Similarly for closed higher dimensional loops $^8$.

The precise shape of an $(r - 1)$-loop is not determined by the $r$-vector coordinates $x^{\mu_1 \ldots \mu_r}$. Only the orientated areas enclosed by the loops are determined. These are thus collective coordinates of a loop. They do not describe all the degrees of freedom of a loop, but only its collective degrees of freedom—area and orientation—common to a family of loops. Namely, in flat $C$-space, one can obtain the $r$-vector coordinates $x^{\mu_1 \ldots \mu_r}$ by integrating the oriented $r$-area elements over an oriented $r$-surface enclosed by the $(r - 1)$-loop. For $r$ fixed, there exist a family of $(r - 1)$-loops, all having the same $x^{\mu_1 \ldots \mu_r}$. Therefore, although the space of loops is infinite dimensional, we obtain a finite dimensional description, if we identify all loops having the same coordinates $x^{\mu_1 \ldots \mu_r}$. This holds for the particular choice of the $C$-space metric, the diagonal generalized Minkowski metric, defined in eq. (12).

The space of all possible extended events is $2^n$-dimensional $C$-space. Since the points of $C$-space are interpreted as extended events “sitting” in spacetime—whose dimension we eventually fix to the physical value $n = 4$—we have that $C$-space is a physical space, accessible to direct observation. An observer, according to this theory, can observe 16-dimensional $C$-space. In this respect the status of $C$-space is analogous to that of the multidimensional configuration space of a many particle system. For instance, a system of

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$^6$If we do not restrict indices to $\mu_1 < \mu_2 < \ldots$, then the factors $1/2!, \ 1/3!, \ldots$ respectively, have to be included in front of every term in eq. (10).

$^7$Although Hestenes and others use the term ‘multivector’ for a generic Clifford number, we prefer to call it ‘polyvector’, and reserve the name ‘multivector’ for objects of definite grade. So our nomenclature is in agreement with the one used in the theory of differential forms, where ‘multivectors’ mean objects of definite grade, and not a superposition of objects with different grades.

$^8$A more detailed discussion can be found in ref [17].
$N$ particles can be described in terms of a $3N$-dimensional configuration space, or a $6N$-dimensional phase space. But basically we still have $3 + 1$ dimensions, since the $N$-particle system is “sitting” in 4-dimensional spacetime. So, although $C$-space is 16-dimensional, we may still say that it describes the physics in 4-dimensional spacetime. This physics, however, is no longer the ordinary, physics, but a generalized physics.

In a series of preceding works [5],[12]–[17], [23]–[26],[31]–[35] it has been proposed to construct the extended relativity theory in $C$-space by a natural generalization of the notion of spacetime interval in Minkowski space to $C$-space:

$$dS^2 \equiv |dX|^2 \equiv dX^\dagger \ast dX = dx_M dx^N G_{MN} \equiv dx^M dx_M$$ \hspace{1cm} (11)

where the metric of $C$-space is given by

$$G_{MN} = \gamma^\dagger_M \ast \gamma_N$$ \hspace{1cm} (12)

In flat $C$-space, one can choose coordinates $x^M$ such that $G_{MN} = \eta_{MN} = \text{diag}(1,1,...,-1,-1,...)$. The operation $\dagger$ reverses the order of vector:

$$(\gamma_{\mu_1} \gamma_{\mu_2} ... \gamma_{\mu_r})^\dagger = \gamma_{\mu_r} ... \gamma_{\mu_2} \gamma_{\mu_1}$$ \hspace{1cm} (13)

Indices are lowered and raised by $G_{MN}$ and its inverse $G^{MN}$. The following relation is satisfied:

$$G^{MJ} G_{JN} = \delta^M_N.$$ \hspace{1cm} (14)

Considering the definition (12) for the $C$-space metric, one could ask why just that definition, which involves reversion, and not a slightly different definition, e.g., without reversion. That reversion is necessary for consistency we can demonstrate by the following example. Let us take a polyvector which has only the 2-vector component different from zero:

$$x^N = (0, 0, x^{\alpha\beta}, 0, 0, ..., 0).$$ \hspace{1cm} (15)

Then the covariant components are

$$x_M = G_{MN} x^N = \frac{1}{2} G_{M[\alpha\beta]} x^{\alpha\beta}$$ \hspace{1cm} (16)

Since the metric $G_{MN}$ is block diagonal, so that $G_{M[\alpha\beta]}$ differs from zero only if $M$ is bivector index, we have

$$x_M = x_{\mu\nu} = \frac{1}{2} G_{[\mu\nu][\alpha\beta]} x^{\alpha\beta}.$$ \hspace{1cm} (17)
From the definition (13) we find

$$G_{[\mu\nu][\alpha\beta]} = (\gamma_\mu \wedge \gamma_\nu)^\dagger * (\gamma_\alpha \wedge \gamma_\beta) = (\gamma_\nu \wedge \gamma_\mu) * (\gamma_\alpha \wedge \gamma_\beta) = g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}$$  \hspace{1cm} (18)

where, in particular, \(g_{\mu\nu}\) can be equal to Minkowski metric \(\eta_{\mu\nu}\). Inserting (18) into (17) we obtain

$$x_{\mu\nu} = \frac{1}{2} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) x^{\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} x^{\alpha\beta}$$  \hspace{1cm} (19)

From the fact that the usual metric \(g_{\mu\nu}\) lowers the indices \(\mu, \nu, \alpha, \beta,...\), so that

$$g_{\mu\alpha} g_{\nu\beta} x^{\alpha\beta} = x_{\mu\nu}$$  \hspace{1cm} (20)

It follows that Eq. (19) is just an identity.

Had we defined the \(C\)-space metric without employing the reversion, then instead of Eq. (18) and (19) we would have \(G_{[\mu\nu][\alpha\beta]} = -(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})\) and \(x_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} x^{\alpha\beta} = -x_{\mu\nu}\), which is a contradiction\(^9\).

Eq. (11) is the expression for the line element in \(C\)-space. If \(C\)-space is generated from the basis vectors \(\gamma_\mu\) of spacetime \(M_n\) with signature \((+ - - - - - - ...)\), then the signature of \(C\)-space is \((+ + + - - - - ...)\), where the number of plus and minus signs is the same, namely, \(2^n / 2\). This has some important consequences that were investigated in ref. [32].

We assume that \(2^n\)-dimensional Clifford space is the arena in which physics takes place. We can take \(n = 4\), so that the spacetime from which we start is just the 4-dimensional Minkowski space \(M_4\). The corresponding Clifford space has then 16 dimensions. In \(C\)-space the usual points, lines, surfaces, volumes and 4-volumes are all described on the same footing and can be transformed into each other by rotations in \(C\)-space (called polydimensional rotations):

$$x^M = L^M {}_N x^N$$  \hspace{1cm} (21)

subjected to the condition \(|dX'|^2 = |dX|^2\).

### 2.2 Curved \(C\)-space

So far we have considered flat \(C\)-space. Let us now assume that in general \(C\)-space can be curved. As we have the special relativity in flat spacetime and general relativity in curved spacetime, so we have now the (extended) special relativity in flat, and the (extended)
general relativity in curved $C$-space. The notion of $C$-space generalises that of spacetime. As a physical spacetime is dynamical in the sense of being a solution to the Einstein equations, so according to the proposed theory also a physical $C$-space is dynamical, being a solution to the generalized Einstein equations, and can be flat or curved.

Points of a curved $C$-space are described by the coordinates $x^M \equiv x^{\mu_1 \ldots \mu_r}$, $r = 0, 1, 2, ..., n$, which we will keep on calling polyvector coordinates, as we did in the case of flat $C$-space. “Polyvector” is here just a name, and it is by no means intended to assert that, from strict mathematical point of view, $x^M$ are components of a polyvector\textsuperscript{10}, joining a fixed point $E_0$ (the “origin”) with coordinates $x^M(E_0) = 0$ and a variable point $E$ with coordinates $x^M(E) \equiv x^M$. The name “polyvector” is used here in a loose, “physical” (not mathematical) sense, just to remind us of the fact that locally, within a sufficiently small region around a chosen point, a curved $C$-space can be approximated with a flat $C$-space, a point of which can be described by a polyvector. Moreover, a curved $C$-space manifold can be considered as a deformation of flat $C$-space manifold\textsuperscript{11}. Since the points of flat $C$-space have the interpretation of being superpositions of oriented $(r - 1)$-loops (the extended events), the same interpretation is in place, if we deform the flat $C$-space into a curved $C$-space.

But in curved $C$-space, unlike in the flat one, coordinates $x^{\mu_1 \ldots \mu_r}$ cannot be “calculated” by performing the integration of $r$-area elements of an $r$-surface enclosed by the $(r - 1)$-loop. With every event $E \in C$ we associate coordinates, denoted $x^M$, $M = 1, 2, ..., 16$, or equivalently, $x^{\mu_1 \ldots \mu_r}$, $r = 0, 1, 2, 3, 4$. For this purpose, like in any differential manifold, we introduce charts that cover different regions $R$ of the manifold $C$, and perform the mapping $x^M: E \in R \rightarrow x^M(E) \in \mathbb{R}^{16}$, where $x^M$, $M = 1, 2, ..., 16$ are real numbers. By assuming that $C$-space is endowed with metric, we can calculated the distances between pairs of points $E, E'$.  

\textsuperscript{10}However, $x^M$ can be considered\textsuperscript{12} as components of a polyvector field $A^M(X)\gamma_M(X)$, such that in a given coordinate system we have $A^M(X) = x^M$. Then at every point $E \in C$, the object $X(E) = x^M(E)\gamma_M(E)$ is a tangent polyvector. So we have one-to-one correspondence between the points $E$ of the $C$-space manifold and the polyvector field $X(E) = x^M(E)\gamma_M(E)$ which we will call the coordinate polyvector field. So although the manifold is curved, every point in it can be described by a tangent polyvector at that point, whose components are equal to the coordinates of that point. We warn the reader not to confuse the tangent polyvector $X(E)$ at a point $E$ with the polyvector joining the points $E_0$ and $E$, a concept which is ill defined in curved manifold $C$.

\textsuperscript{11}Suppose we have a $C$-space manifold endowed with the metric $G(\epsilon)$ and connection $D(\epsilon)$ which depend on a parameter (or a set of parameters) $\epsilon$, so that for $\epsilon \neq 0$ the manifold is curved, whilst for $\epsilon \rightarrow 0$ it approaches to flat manifold. Then we say that $\epsilon$ is a deformation parameter (or a set of parameters), by means of which we deform a flat $C$-space into a curved one.
However, physically, just the reverse procedure is adopted. Namely, a smooth, differential manifold, such as, e.g., $C$-space, is of course a mathematical idealization. What we have physically, is just a set of extended events, that can be unambiguously identified, since they are associated with, e.g., collisions branes with $C$-space photon [26]: the collision region is an extended region in spacetime, and can be given arbitrary coordinates $x^M$ (“house numbers” [13]). By measuring the distances between the nearby extended events within a network of extended events, we can approximately recover the metric. In order to measure the distance in $C$-space we suitably generalize the concept of light clocks [14], which now operate with $C$-space light rays (based on $C$-space photons). We will postpone a detailed description to a future paper, since it will require more practical experience with the applications and consequences of the special relativity in $C$-space. However, after consulting the already existing literature [14] [17] [26], an interested reader can straightforwardly work out the basic principles of $C$-space light clocks and distance measurements. This is beyond the scope of the present paper which aims at pointing out the potential of the concept of curved $C$-space for the unification of the fundamental interactions. Once a sufficient motivation is gained as a result of such studies, we can go back and work further on what we have left unsettled.

The distance between two infinitely close extended events is given by the expression [11] (“the line element”) in which there occurs the metric tensor [12], given in terms of the coordinate basis elements $\gamma_M$. In flat $C$-space the coordinate basis elements $\gamma_M$ can be chosen so that at every point $X(\mathcal{E}) \in C$ they have the same, definite ($r$-vector) grade, and also so that at every point $X(\mathcal{E}) \in C$ the metric tensor is diagonal. This is not the case in curved $C$-space. In curved $C$-space, the orientation of a polyvector can change from point to point. More precisely this means that, if we perform the parallel transport of a polyvector $A = A^M \gamma_M$ from a point $\mathcal{E}$ to $\mathcal{E}'$, the result depends on the path of transport. If we perform the parallel transport of $A(\mathcal{E})$ from a point $\mathcal{E}$ along a closed path back to $\mathcal{E}$, we obtain a polyvector $A'(\mathcal{E})$ which differs from $A(\mathcal{E})$. In particular this means that, if initially we have, e.g., a vector at $\mathcal{E}$, then after a round trip parallel transport we can end up with, e.g., a bivector at $\mathcal{E}$, or more generally, with a superposition of bivectors, vectors, 3-vectors, etc.

At every point $X \in C$ the basis elements, that is, the basis polyvectors, $\gamma_M$ span a tangent space $T_X(C)$ which has the structure of Clifford algebra $\mathcal{C}\ell_n$. In a neighbourhood of a point $X$ the tangent space $T_X(C)$ models the manifold $C$, i.e., it provides an approximate
description of $C$. But in the case of curved $C$ such description becomes less and less accurate, if we increase the neighbourhood of $X$.

Basis polyvectors $\gamma_M$ are tangent to coordinate curves of $C$-space and they form a coordinate frame at a point $X \in C$. In general, a set of $2^n$ linearly independent polyvector fields on a region $\mathcal{R}$ of $C$-space will be called a frame field. Of particular interest are:

(i) **Coordinate frame field** $\{\gamma_M\}$. Basis elements $\gamma_M$, $M = 1, 2, ..., 2^n$ depend on position $X$ in $C$-space. The relation (6) with wedge product can hold only locally at a chosen point $X$, but in general it cannot be preserved globally at all points $X \in \mathcal{R}$ of curved $C$-space. The scalar product of two basis elements determines the metric tensor of the frame field $\{\gamma_M\}$ according to eq. (12).

(ii) **Orthonormal frame field** $\{\gamma_A\}$. Basis elements $\gamma_A$, $A = 1, 2, ..., 2^n$ also depend on $X$. At every point $X$ they are defined as the wedge product and they determine diagonal metric

$$\gamma_A^\dagger \ast \gamma_B = \eta_{AB}$$

The orthonormal frame field thus consists of the elements

$$\{\gamma_A\} = \{1, \gamma_{a_1}, \gamma_{a_1a_2}, ..., \gamma_{a_1a_2...a_n}\}, \quad a_1 < a_2 < ... < a_r, \quad r = 1, 2, ..., n$$

where

$$\gamma_{a_1a_2...a_r} = \gamma_{a_1} \wedge \gamma_{a_2} \wedge ... \wedge \gamma_{a_r} \equiv \frac{1}{r!}[\gamma_{a_1}, \gamma_{a_2}, ..., \gamma_{a_r}]$$

is the antisymmetrized or wedge product.

The relation between the two sets of basis elements is given in term of the $C$-space vielbein:

$$\gamma_M = e_M^A \gamma_A$$

All quantities in eq. (25) depend on position $X$ in $C$-space.

Explicitly, eq. (25) reads

$$\gamma = e_1^{\circ} 1 + e_1^{a_1} \gamma_{a_1} + e_1^{a_1a_2} \gamma_{a_1a_2} + ... + e_1^{a_1...a_n} \gamma_{a_1...a_n}$$

$$\gamma_{\mu_1} = e_{\mu_1} 1 + e_{\mu_1}^{a_1} \gamma_{a_1} + e_{\mu_1}^{a_1a_2} \gamma_{a_1a_2} + ... + e_{\mu_1}^{a_1...a_n} \gamma_{a_1...a_n}$$

$$\gamma_{\mu_1\mu_2} = e_{\mu_1\mu_2} 1 + e_{\mu_1\mu_2}^{a_1} \gamma_{a_1} + e_{\mu_1\mu_2}^{a_1a_2} \gamma_{a_1a_2} + ... + e_{\mu_1\mu_2}^{a_1...a_n} \gamma_{a_1...a_n}$$

$$\vdots$$

$$\gamma_{\mu_1...\mu_n} = e_{\mu_1...\mu_n} 1 + e_{\mu_1...\mu_n}^{a_1} \gamma_{a_1} + e_{\mu_1...\mu_n}^{a_1a_2} \gamma_{a_1a_2} + ... + e_{\mu_1...\mu_n}^{a_1...a_n} \gamma_{a_1...a_n}$$

(26)
Although here we keep the notation $\gamma_M \equiv \gamma_{\mu_1...\mu_r}$, the latter quantities have no definite grade. They had definite grade in flat $C$-space, where according to eq. (6) they are equal to the wedge product of basis vectors. The latter equality is no longer valid in general, it can hold only locally at a given point $X$ of curved $C$-space, but not in its neighbourhood. But for mnemonic reasons we can retain the notation $\gamma_{\mu_1...\mu_r}$ and $x^{\mu_1...\mu_r}$ even in the neighbourhood of that point.

The relation [25] introduced above is a local transformation from an orthonormal basis $\{\gamma_A\}$ to a coordinate basis $\{\gamma_M\}$ of a curved Clifford space. A similar relation was considered by Pezzaglia [24], but his interpretation was different, because he imposed certain constraints in order to satisfy the Clifford algebra relations. Crawford [22] considered an analogous relation to that of Pezzaglia, with the difference that the transformation was from one orthonormal basis to another orthonormal basis, and his quantities depended on spacetime position $x$ only (and not on $X$). Hence our quantity $e^A_M$ is a different object from Pezzaglia’s “geobein” or Crawford’s “drehbein”, although the basic idea is similar, i.e., gauging Clifford algebra.

We have thus a curved Clifford space ($C$-space). A point of $C$-space is described by coordinates $x^M$. A coordinate basis at a point $X$ is $\{\gamma_M|_X\}$, whilst an orthonormal basis is $\{\gamma_A|_X\}$. The tetrad field is given by the scalar product $e^A_M = \gamma^*_M \gamma^A$. In particular, $\gamma_M \equiv \gamma_{\mu_1...\mu_r}$ at $X \in C$ can be multivectors of definite grade, i.e., defined as a wedge product $\gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_r}$. But such property can hold only locally at $X$, and cannot be preserved globally at all points $X$ of our curved Clifford space.

Corresponding to each polyvector field we define a differential operator, a generalized directional derivative, which we call derivative and denote $\partial_{\gamma_M} \equiv \partial_M$, whose action depends on the quantity it acts on:\footnote{This operator is a generalization to curved $C$-space of the derivative $\partial_\mu$ which acts in an $n$-dimensional space $V_n$, and was defined by Hestenes [13] (who used a different symbol, namely $\Box_\mu$).}

(i) $\partial_M$ maps scalars $\phi$ into scalars

$$\partial_M \phi = \frac{\partial \phi}{\partial x^M} \quad (27)$$

Thus $\partial_M$, when acting on scalar fields, is just the ordinary partial derivative.

(ii) $\partial_M$ maps Clifford numbers into Clifford numbers. In particular, it maps a coordinate basis Clifford number $\gamma_N$ into another Clifford number which can, of course, be expressed
as a linear combination of $\gamma_J$:

$$\partial_M \gamma_N = \Gamma^J_{MN} \gamma_J$$  \hspace{1cm} (28)

The above relation defines the coefficients of connection $\Gamma^J_{MN}$ for the coordinate frame field $\{\gamma_M\}$.

An analogous relation we have for the local frame field:

$$\partial_M \gamma_A = -\Omega^B_A \gamma_B$$  \hspace{1cm} (29)

where $\Omega^B_A \gamma_M$ are coefficients of connection for the local frame field $\{\gamma_A\}$.

When the derivative $\partial_M$ acts on a polyvector valued field $A = A^N \gamma_N$ we obtain

$$\partial_M (A^N \gamma_N) = \partial_M A^N \gamma_N + A^N \partial_M \gamma_N = (\partial_M A^N + \Gamma^N_{MK} A^K) \gamma_N \equiv D_M A^N \gamma_N$$  \hspace{1cm} (30)

where $D_M A^N \equiv \partial_M A^N + \Gamma^N_{MK} A^K$ are components of the covariant derivative in the coordinate basis, i.e., the ‘covariant derivative’ of the tensor analysis. Here $A^N$ are scalar components of $A$, and $\partial_M A^N$ is just the partial derivative of a scalar field with respect to $X^M$:

$$\partial_M \equiv \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial x^{\mu_1}}, \frac{\partial}{\partial x^{\mu_1 \mu_2}}, \ldots, \frac{\partial}{\partial x^{\mu_1 \ldots \mu_n}} \right)$$  \hspace{1cm} (31)

The derivative $\partial_M$ behaves as a partial derivative when acting on scalar fields, and it defines a connection when acting on a polyvector field $\gamma_M$ or $\gamma_A$. It has turned out very practical to use the easily writable symbol $\partial_M$ which —when acting on a polyvector field— has to be understood in the sense of eq. (28)–(30). Especially when doing long calculation (which is usually the job of theoretical physicists) it is much easier and quicker to write $\partial_M$ than $\Box_M$, $\nabla_M$, $D_{\gamma_M}$, $\nabla_{\gamma_M}$ which all are symbols used in the literature. Also conceptually, it is perhaps not so wrong to use the same symbols $\partial_{\gamma_M} \equiv \partial_M$, which so far has been reserved for the components of the directional derivative acting on scalar fields, that is, the partial derivative of a scalar field. If one tries to perform the partial derivative of, e.g., a vector field in a curved manifold, one finds that this cannot be done, unless one specifies how to compare vectors at different points of the manifold. A common prescription is to perform a parallel transport of a vector from a point $P'$, along a chosen curve, e.g., along a coordinate line, to a point $P$, where the two vectors can be compared (i.e., substracted). This is the well known procedure, therefore we do not attempt to repeat it in detail here. What we would like to stress is that the same procedure works in the case of an arbitrary geometric object, including a scalar. That is, the same definition of the derivative which
holds for an arbitrary geometric object, also holds for a scalar (in which case the definition of parallel transport is trivial). Therefore we may retain the same, unique, symbol \( \partial \gamma_M \equiv \partial M \) for such operator even when it acts on generic multivector or tensor fields. Here let me remind the reader that in mathematics there are other cases in which the same symbol for an operation was retained after extending the set within which the operation acted. Thus, historically, multiplication was first defined as acting within the set of (positive) integers. But subsequently multiplication was generalized to act within increasingly more general sets of numbers, such as rational numbers, real numbers, complex numbers, Clifford numbers, etc.. And yet the same symbol, namely juxtaposition, has been mostly used in algebraic expressions. We write, e.g., \((a + b)c\) regardless of whether, \(a, b, c,\) are integers, real numbers, complex numbers, Clifford numbers, etc.. Just imagine how complicated mathematics would have become, if at every generalization of numbers, we would have invented and used a different symbol for multiplication. Perhaps a maximal possible mathematical rigour would indeed require such notational distinctions, but development of practical calculus and hence the development of physics and engineering would have been slowed down, if not completely blocked. After performing many practical calculations\(^1\), it is my insight that in the case of Clifford valued fields usage of a unique symbol \( \partial_M \) for derivative is a rational choice which enables a significant simplification and thus renders the geometric calculus based on Clifford algebra potentially more attractive to a wider group of physicists.

There are also other reasons for usage of the unique symbols \( \partial \gamma_M \equiv \partial M \). Suppose that we decide to use a symbol such as, e.g., \( D \gamma_M \equiv D_M \) instead of \( \partial \gamma_M \). The calculation in eq. (30) would then read

\[
D \gamma_M (A^N \gamma_N) = D \gamma_M A^N \gamma_N + A^N D \gamma_M \gamma_N = (\partial_M A^N + \Gamma_{MK}^N A^K) \gamma_N
\]

where we have used the property that \( D \gamma_M \) acts on the terms in a product according to the Leibnitz rule, and identified \( D \gamma_M A^N = \partial_M A^N \). In order to make a contact with the tensor analysis, which is widely used in physics, we would need to invent yet another symbol for the combination \( \partial_M A^N + \Gamma_{MK}^N A^K \). The notation \( D_M A^N \) could not be used for the latter combination, because we already have \( D \gamma_M A^N \equiv D_M A^N = \partial_M A^N \). So we would need three different symbols, \( \partial_M, D_M, \) and, say, \( \nabla_M \). Moreover, since one says (which

\(^{13}\)See, e.g., how quantum theory in curved space can be elegantly formulated by using the definition \( p = -i\hbar \gamma^\mu \partial_\mu \) for the momentum operator, which enables, amongst others, a resolution of the notorious ordering ambiguities [45].
is a common practice in modern differential geometry) that $D_{\gamma M}$ is 'covariant derivative', why then $(D_{\gamma M} A^N)\gamma_N = (\partial_M A^N)\gamma_N$ in eq. (32) does not transform covariantly (in the sense this word is normally understood in physics) under general coordinate transformations, nor does so the term $\Gamma^N_{MK} A^K \gamma_N$? How can $D_{\gamma M} \gamma_N$ be called the 'covariant derivative' of $\gamma_N$, if $D_{\gamma M} \gamma_N$ does not transform covariantly\(^{14}\)? Only the sum of the two terms occurring in eq. (32) transforms covariantly. If so, then it makes sense not to insist that the operator $D_{\gamma M}$ is covariant derivative, but say simply that it is the derivative, as Hestenes \(^{18}\) did. When acting on an object (e.g., a vector field), expanded according to $A^N \gamma_N$, it automatically acts as a covariant derivative\(^{15}\), but not so if acting on the components $A^N$, or on basis elements $\gamma_N$. If so, it is then better not to use in eq. (32) the the symbol $D_{\gamma M}$ (or whatever other symbol for covariant derivative), but use the same symbol as it is used for the partial derivative, that is, $\partial_M$, with understanding that the definition of such operator is now suitably generalized.

The derivative $\partial_M$ is defined with respect to a coordinate frame field \(\{\gamma_M\}\) in C-space.

We can define a more fundamental derivative $\partial$ by

$$\partial = \gamma^M \partial_M$$

(33)

This is the gradient in C-space and it generalizes the ordinary gradient $\gamma^\mu \partial_\mu$, $\mu = 0, 1, 2, ..., n - 1$, discussed by Hestenes \(^{18}\).

Besides the basis elements $\gamma_M$ and $\gamma_A$, we can define the reciprocal elements $\gamma^M$, $\gamma^A$ by the relations

$$(\gamma^M)^{\dagger} \gamma_N = \delta^M_N, \quad (\gamma^A)^{\dagger} \gamma_B = \delta^A_B$$

(34)

**Curvature.** We define the curvature of C-space in the analogous way as in the ordinary spacetime, namely by employing the commutator of the derivatives \(^{18}\) \[15\] \[25\]. Using

\(^{14}\)It is true that the relation $D_M \gamma_N = \Gamma^J_{MN} \gamma_J$ as a whole is covariant under general coordinate transformations of $x^\mu$, i.e., in new coordinates it has the same form as in the old coordinates, but this does not hold for the terms $D_M \gamma_N$ and $\Gamma^J_{MN} \gamma_J$ taken alone. Namely, under a transformation $\gamma_N \rightarrow a_N \gamma_J$, we have $D_M \gamma_N \rightarrow a_M a^J D_M (a_N \gamma_J) = a_M a^J D_M \gamma_J + (a_M \partial_M a^J) \gamma_J$, which certainly is not a covariant transformations property.

\(^{15}\)Remember that in tensor analysis, the “covariant derivative” when acting according to the Leibnitz rule to individual terms in a product of tensor field components retains its covariant character. For instance, for the product of vector field components we have $D_\mu (A^\nu B^\rho) = (D_\mu A^\nu)B^\rho + A^\nu (D_\mu B^\rho)$, where $D_\mu A^\nu$ and $D_\mu B^\rho$ transform covariantly under general coordinates transformations. In the case of a quadratic form, which transforms as a scalar, we have $D_\mu (A^\nu B^\rho g_{\alpha\beta}) = (D_\mu A^\nu)B^\rho g_{\alpha\beta} + A^\nu (D_\mu B^\rho) g_{\alpha\beta}$, where we have used $D_\mu g_{\alpha\beta} = 0$. If we act on the quadratic form with the partial derivative, we have $\partial_\mu (A^\nu B^\rho g_{\alpha\beta}) = (\partial_\mu A^\nu)B^\rho g_{\alpha\beta} + A^\nu (\partial_\mu B^\rho) g_{\alpha\beta} + A^\nu B^\rho \partial_\mu g_{\alpha\beta}$ which, after expressing $\partial_\mu g_{\alpha\beta}$ by making use of $D_\mu g_{\alpha\beta} = 0$, turns out to be the same as $D_\mu (A^\nu B^\rho g_{\alpha\beta})$. 

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eq. (28) we have
\[ [\partial_M, \partial_N]_J = R_{MNJ}^K \gamma_K \] (35)

where
\[ R_{MNJ}^K = \partial_M \Gamma_{NJ}^K - \partial_N \Gamma_{MJ}^K + \Gamma^R_{NJ} \Gamma_{MR}^K - \Gamma^R_{MJ} \Gamma_{NR}^K \] (36)
is the curvature of C-space. Using (35) we can express the curvature according to
\[ (\gamma^K)^* \star ([\partial_M, \partial_N]_J) = R_{MNJ}^K \] (37)

An analogous relation we have if the commutator of the derivatives operates on a local basis elements and use eq. (29):
\[ [\partial_M, \partial_N]_A = R_{MNA}^B \gamma_B \] (38)

where
\[ R_{MNA}^B = -(\partial_M \Omega_A^B - \partial_N \Omega_A^B + \Omega_A^C \Omega_C^B - \Omega_A^C \Omega_C^B) \] (39)

In view of eq. (25) an arbitrary polyvector can be expanded according to
\[ A = A^M \gamma_M = A^M e^A_M \gamma_A \] (40)

In particular we have
\[ \partial_N \gamma_M = \Gamma^I_{MN} \gamma_I = \partial_N(e^A_M \gamma_A) = \partial_N e^A_M \gamma_A + e^A_M \partial_N \gamma_A \] (41)

Using (28), (29), we obtain from (41) the following relation which involves C-space vielbein field and the coefficients of connection for the frame filed \( \{ \gamma_M \} \) and \( \{ \gamma_A \} \), respectively:
\[ \partial_N e^C_M - \Gamma^J_{NM} e^C_J - e^A_M \Omega_A^C \gamma_N = 0 \] (42)

This is a generalization of the well known relation in an ordinary curved spacetime. In eq. (12) \( \Omega_A^C \) extends the notion of the spin connection coefficients \( \omega_a^c \).

From (12) we obtain
\[ \partial_M e^C_N - \partial_N e^C_M + e^A_M \Omega_A^C \gamma_N - e^A_N \Omega_A^C \gamma_M = T_{MNJ} e^C_J \] (43)

where
\[ T_{MNJ} = \Gamma^J_{MN} - \Gamma^J_{NM} \] (44)
is the C-space torsion.
In general, the torsion is different from zero. In particular, when torsion vanishes, we find the following expression for the connection coefficients:

$$\Omega_{BCM} = \frac{1}{2} e_A^M (\Delta_{[AB]C} - \Delta_{[BC]A} + \Delta_{[CA]B}) \quad (45)$$

where

$$\Delta_{[AB]C} \equiv e_A^M e_B^N (\partial_M e_{NC} - \partial_N e_{MC}) \quad (46)$$

generalizes the notion of the Ricci rotation coefficients.

3 On the general relativity in C-space

The basic idea of the novel theory that we are exploring here is that the concept of spacetime should be replaced by that of Clifford space. Although the name “Clifford space” could sound very mathematical and thus not much related to physics, just the contrary is true. Clifford space (C-space) is the very space in which physics takes place. By considering so far only spacetime we have omitted a big portion of a very relevant physics which has been sitting just around the corner. Namely, as we have shown in this and previous works, spacetime is just the start. From its basis we can build a larger space, which is Clifford space. And the latter space, according to the view held in this and other papers [23]– [26], [12]–[17], is just as physical as the spacetime of general relativity. Since C-space has more than four dimensions (namely 16, if built on 4-dimensional spacetime), it can serve as a realization of the Kaluza-Klein theory.

We have thus a 16-dimensional, “physical”, continuum, whose points are described by coordinates $$x^M = (s, x^{\mu_1}, x^{\mu_2\mu_3}, ..., x^{\mu_1...\mu_n})$$. Using a frame field, say, a coordinate frame field $$\{\gamma_M\}$$, the metric is given by $$G_{MN} = \gamma_M^\dagger * \gamma_N$$. The basis elements $$\gamma_M = e_A^M \gamma_A$$, $$M = (0, [\mu_1], [\mu_1\mu_2], ..., [\mu_1...\mu_n])$$, $$\mu_1 < \mu_2 < ... < \mu_r$$, or equivalently, the metric $$G_{MN}$$, are considered as dynamical variables of the theory. From the curvature, defined in eqs. (34)–(37) we can form a kinetic term for $$\gamma_M$$, or equivalently, $$G_{MN}$$.

In addition, we also have sources. The first straightforward possibility is to introduce a single parameter $$\tau$$ and consider a mapping

$$\tau \rightarrow x^M = X^M(\tau) \quad (47)$$

where $$X^M(\tau)$$ are 16 embedding functions that describe a worldline in C-space. From the point of view of C-space, $$X^M(\tau)$$ describe a worldline of a “point particle”: at every value
of \( \tau \) we have a point in \( C \)-space. But from the perspective of the underlying 4-dimensional spacetime, \( X^M(\tau) \) describe an extended object, sampled by the center of mass coordinates \( X^\mu(\tau) \) and the coordinates \( X^{\mu_1\mu_2}(\tau), \ldots, X^{\mu_1\mu_2\mu_3\mu_4}(\tau) \). They are a generalization of the center of mass coordinates in the sense that they provide information about the object’s 2-vector, 3-vector, and 4-vector extension and orientation. For instance, in the case of a closed string we have a 2-dimensional surface enclosed by a 1-dimensional line. Integrating over the oriented area elements, we obtain a finite effective oriented area given in terms of bivector coordinates \( X^{\mu\nu} \). Such bivector coordinates provide an approximate description of a closed string; they do not provide a complete description of the string, but nevertheless, they provide a better approximation, than the mere center of mass coordinates \([17]\).

As already discussed in Sec. 2.2, the integration described above and in ref. [17] makes sense in flat spacetime. Then one can choose as simple a metric tensor as possible, i.e., the diagonal metric tensor, and perform the integration over an \( r \)-surface enclose by an \( (r - 1) \)-loop which gives the polyvector coordinates \( x^{\mu_1 \cdots \mu_r} \). In curved spacetime this cannot be done. One has to consider an extended object, e.g., a brane, as associated with an extended event \( \mathcal{E} \), to which we can assign arbitrary coordinates (“house numbers” according to Wheeler), i.e., perform a mapping \( x^M : \mathcal{E} \rightarrow \mathbb{R}^{16} \). Suppose now that there is a network of extended events, in an idealized limit, an infinitely dense collection of extended events forming altogether a 16-dimensional manifold \( C \). If we measure the distances between the neighbouring extended events, we obtain the metric tensor. This is how we can arrive, starting from physics, at the abstract notion of a \( C \)-space manifold, endowed with metric (see also Sec. 2.2).

Let us assume that the classical action contains a term which describes the “point particle” in \( C \)-space and a kinetic term which describes the dynamics of the \( C \)-space itself:

\[
I[X^M, G_{MN}] = \int d\tau (\dot{X}^M \dot{X}^N G_{MN})^{1/2} + \frac{\kappa}{16\pi} \int [dx] R \tag{48}
\]

Here \( \dot{X}^M \equiv dX^M/d\tau \), \( \kappa \) a constant, \([dx] \equiv d^{16}x \) the measure on \( C \)-space, and \( R = R_{MNJ}^N \) the curvature scalar (see eqs. \([35]\), \([36]\), \([37]\)) of \( C \)-space, analogous to the curvature scalar of the ordinary general relativity. The action is invariant under local (pseudo) rotations in tangent space \( T_X(C) \), and under general coordinate transformations in \( C \)-space.

Variation of the action \([48]\) with respect to \( X^M \) gives the geodetic equation in \( C \)-space:
\[
\frac{1}{\sqrt{\dot{X}^2}} \frac{d}{d\tau} \left( \frac{\dot{X}^M}{\sqrt{\dot{X}^2}} \right) + \Gamma^M_{JK} \frac{\dot{X}^J \dot{X}^K}{\dot{X}^2} = 0 \quad (49)
\]

Varying (48) with respect to \( G_{MN} \) gives the C-space Einstein’s equations

\[
R^{MN} - \frac{1}{2} G^{MN} R = 8\pi\kappa \int d\tau \delta^{(C)}(x - X(\tau)) \dot{X}^M \dot{X}^N \quad (50)
\]

where \( \delta^{(C)} \) is the \( \delta \)-function in C-space.

When looking from the 4-dimensional spacetime, the equation of geodesic (49) contains besides the usual gravitation also other interactions. They are encoded in the metric components \( G_{MN} \) of C-space. Gravity is related to the components \( G_{\mu\nu} \), \( \mu, \nu = 0, 1, 2, 3 \), while the gauge fields due to other interactions are related to the components \( G_{\tilde{M}\tilde{N}} \), where the index \( \tilde{M} \neq \nu \) assumes 12 possible values, excluding the four values \( \nu = 0, 1, 2, 3 \). In addition, there are also interactions due to the components \( G_{\tilde{M}N} \), but they have not the property of the ordinary Yang-Mills fields.

If we now consider the known fundamental interactions of the standard model we see that besides gravity we have 1 photon described by the abelian gauge field \( A_\mu \), 3 weak gauge bosons described by gauge fields \( W^a_\mu \), \( a = 1, 2, 3 \), and 8 gluons described \( A^c_\mu \), \( c = 1, 2, \ldots, 8 \). Altogether there are 12 gauge fields.

Interestingly, the number of mixed components \( G_{\mu\tilde{M}} = (G_{\mu[\alpha]}, G_{\mu[\alpha\beta]}, G_{\mu[\alpha\beta\rho]}, G_{\mu[\alpha\beta\rho\sigma]})(C) \) of the C-space metric tensor \( G_{MN} \) coincides with the number of gauge fields in the standard model\(^{16}\). For fixed \( \mu \), there are 12 mixed components of \( G_{\mu\tilde{M}} \) and 12 gauge fields \( A_\mu, W^a_\mu, A^c_\mu \). This coincidence is fascinating and it may indicate that the known interactions are incorporated in curved Clifford space.

Good features of C-space are the following:

(i) We do not need to introduce extra dimensions of spacetime. We stay with 4-dimensional spacetime manifold \( V_4 \), and yet we can proceed \( \text{à la Kaluza-Klein} \). The extra degrees of freedom are in C-space, which is the space of extended events associated with extended objects residing in spacetime manifold \( V_4 \).

(ii) We do not need to compactify the extra “dimensions”. The extra dimensions of C-space, namely \( s, x^{\mu\nu}, x^{\mu\nu\rho}, x^{\mu\nu\rho\sigma} \) are not just like the ordinary dimensions of spacetime considered in the usual Kaluza-Klein theories. The coordinates \( x^{\mu\nu}, x^{\mu\nu\rho}, x^{\mu\nu\rho\sigma} \)

\(^{16}\)The numbers of the independent indices \( [\alpha], [\alpha\beta], [\alpha\beta\rho], [\alpha\beta\rho\sigma] \) are respectively 1, 6, 4, 1 which sums to 12.
are related to oriented $r$-surfaces, $r = 2, 3, 4$, by which we sample extended objects. Those degrees of freedom are in principle not hidden from our direct observation, therefore we do not need to compactify such “internal” space.

(iii) The number of the mixed metric components $G_{\mu\bar{M}}$ (for fixed $\mu$) is 12, precisely the same as the number of gauge fields in the standard model.

We will not go in further details, since they have already been written down in Kaluza-Klein theories, although with a different interpretation of the extra dimensions. However, one cannot expect to obtain a realistic theory, with correct coupling constants, within the realm of a classical theory.

4 The generalized Dirac equation in curved $C$-space

4.1 Spinors as members of left ideals of Clifford algebra

How precisely the curved $C$-space is related to Yang-Mills gauge fields can be demonstrated by considering a generalization of the Dirac equation to curved $C$-space.

Let $\Phi(X)$ be a polyvector valued field over coordinate polyvector field $X = x^M \gamma_M$:

$$\Phi = \phi^A \gamma_A$$

where the basis elements $\gamma_A$, $A = 1, 2, ..., 16$, form an orthonormal frame field on a region $\mathcal{R}$ of $C$-space$^{17}$ (see eq. (23)) and $\phi^A$ are the projections (components) of $\Phi$ onto $\gamma_A$. We will suppose that in general $\phi^A$ are complex-valued scalar quantities.

We interpret the imaginary unit $i$ in the way that is usual in quantum theory, namely that $i$ lies outside the Clifford algebra of spacetime and hence commutes with all $\gamma_M$. This is different from the point of view hold by many researchers of the geometric calculus based on Clifford algebra (see, e.g., [18, 19]). They insist that $i$ has to be defined geometrically, so it must be one of the elements of the set $\{\gamma_A\}$, such that its square equals $-1$. An alternative interpretation, also often assumed, is that $i$ is the pseudoscalar unit of a higher dimensional space. For instance, if our spacetime is assumed to be 4-dimensional, then $i$ is the pseudoscalar unit of a 5-dimensional space. The problem then arises about a physical interpretation of the extra dimension. This is not the case that we adopt. Instead we adopt the view, first proposed in [5], that $i$ is the bivector of the 2-dimensional phase space $P_2$.

$^{17}$Of particular interest are such $C$-spaces in which there exist global orthonormal frames fields.
spanned by $e_q$, $e_p$, so that $Q \in P_2$ is equal to $Q = e_q e_q + p e_p$, $e_q Q = q + i p$, $i = e_q e_p$. So our $i$ is also defined geometrically, but the space we employ differs from the spaces usually considered in defining $i$. Taking into account that there are four spacetime dimensions, the total phase space is thus the direct product $M_4 \times P_2 = P_8$, so that any element $Q \in P_8$ is equal to $Q = e^q e^q + p e^p$, $e^q Q = (x^\mu + i p^\mu)e_\mu$. This can then be generalized to Clifford space by replacing $x^\mu, p^\mu$ by the corresponding Clifford space variables $x^M, p^M$. In a classical theory, we can just consider $x^\mu$ only (or $x^M$ only), and forget about $p^\mu (p^M)$, since $x^\mu$ and $p_\mu$ are independent. In quantum theory, $x^\mu$ and $p_\mu (x^M$ and $p_M$) are complementary variables, therefore we cannot formulate a theory without at least implicitly involving the presence of momenta $p_\mu (p_M)$. Consequently, wave functions are in general complex valued. Hence the occurrence of $i$ in quantum mechanics is not perplexing, it arises from phase space.

We adopt here the conventional interpretation of quantum mechanics: no hidden variables, Böhmian potential, etc., just the Born statistical interpretation and Bohr-Von Neumann projection postulate. The formalism described here works for the Everett interpretation as well.

Instead of the basis\(^\text{18}\) $\{\gamma_A\}$ one can consider another basis, which is obtained after multiplying $\gamma_A$ by 4 independent primitive idempotents \(^\text{28}\):

\[
P_i = \frac{1}{4}(1 + a_i \gamma_A + b_i \gamma_B + c_i \gamma_C), \quad i = 1, 2, 3, 4
\]

such that

\[
P_i = \frac{1}{4}(1 + a_i \gamma_A)(1 + b_i \gamma_B), \quad \gamma_A \gamma_B = \gamma_C, \quad c_i = a_i b_i
\]

Here $a_i$, $b_i$, $c_i$ are complex numbers chosen so that $P_i^2 = P_i$. For explicit and systematic construction see \(^\text{28, 29}\).

By means of $P_i$ we can form minimal ideals of Clifford algebra. A basis of a left (right) minimal ideal is obtained by taking one of $P_i$ and multiply it from the left (right) with all 16 elements $\gamma_A$ of the algebra:

\[
\gamma_A P_i \in \mathcal{I}_i^L, \quad P_i \gamma_A \in \mathcal{I}_i^R
\]

\(^\text{18}\)We will use ‘basis’ and ‘frame’ as synonyms. In order to simplify notation and wording, we will be sloppy in distinguishing objects from the corresponding fields, e.g., (poly)vectors from (poly)vector fields, frames from frame fields, (generalized) spinors from (generalized) spinor fields etc. From the context it should not be difficult to understand when the talk is about an object taken at a point $X \in C$, and when about an infinite dimensional object, a field. Thus $\gamma_A, A = 1, 2, ..., 16$, can mean either orthonormal polyvectors taken at a point $X \in C$, or polyvector fields, depending on the context. Similarly $\{\gamma_A\}$ can mean either an orthonormal frame (basis), or orthonormal frame field. Thus we avoid the notation $\{\gamma_A|X\}$, which is unnecessarily clumsy within a physically oriented paper. The situation, of course is different in mathematically oriented papers, where maximal possible rigour and very precise notation are requested.
Here $\mathcal{I}_i^L$ and $\mathcal{I}_i^R$, $i = 1, 2, 3, 4$ are four independent minimal left and right ideals, respectively. For a fixed $i$ there are 16 elements $\gamma_A P_i$ or $P_i \gamma_A$, but only four amongst them are different, the remaining elements are just repetition—apart from constant factors—of those four different elements.

Let us denote those different elements $\xi_{\alpha i}$, $\alpha = 1, 2, 3, 4$. They form a basis of the $i$-th left ideal.

As an illustration let us provide an example. Let

\[ P_1 = \frac{1}{4}(1 + \gamma_0 + i\gamma_{12} + i\gamma_{012}) \]  
\[ P_2 = \frac{1}{4}(1 + \gamma_0 - i\gamma_{12} - i\gamma_{012}) \]  
\[ P_3 = \frac{1}{4}(1 - \gamma_0 + i\gamma_{12} - i\gamma_{012}) \]  
\[ P_4 = \frac{1}{4}(1 - \gamma_0 - i\gamma_{12} + i\gamma_{012}) \]  

In short,

\[ P_i = \frac{1}{4}(1 \pm \gamma_0)(1 \pm i\gamma_{12}) \]  

where 4 different choices of sign give 4 different idempotents $P_i$.

The basis of the first left ideal is

\[ \xi_{11} = P_1 = \frac{1}{4}(1 + \gamma_0 + i\gamma_{12} + i\gamma_{012}) \]  
\[ \xi_{21} = -\gamma_{13} P_1 = \frac{1}{4}(-\gamma_{13} - \gamma_{013} + i\gamma_{23} + i\gamma_{023}) \]  
\[ \xi_{31} = -\gamma_3 P_1 = \frac{1}{4}(-\gamma_3 + \gamma_{03} - i\gamma_{123} + i\gamma_{0123}) \]  
\[ \xi_{41} = -\gamma_1 P_1 = \frac{1}{4}(-\gamma_1 + \gamma_{01} + i\gamma_2 - i\gamma_{02}) \]  

All sixteen basis elements $\gamma_A = (1, \gamma_{a_1}, \gamma_{a_1 a_2}, \gamma_{a_1 a_2 a_3}, \gamma_{a_1 a_2 a_3 a_4})$, $a_1 < a_2 < ... < a_r$, $r = 0, 1, 2, 3, 4$, take place in equation (60) defining $\xi_{\alpha 1}$. For $i = 2, 3, 4$ we have expressions analogous to (60), with suitably changed signs and order of indices $\alpha = 1, 2, 3, 4$. Altogether there are 16 basis elements $\xi_{\alpha i}$, $i = 1, 2, 3, 4$. The basis $\{\xi_{\alpha i}\}$ is complete. Every Clifford number can be expanded either in terms of $\gamma_A$ or in terms of $\xi_{\alpha i} = (\xi_{\alpha 1}, \xi_{\alpha 2}, \xi_{\alpha 3}, \xi_{\alpha 4})$:

\[ \Phi = \phi^A \gamma_A = \Psi = \psi^{\alpha i} \xi_{\alpha i} = \psi^{\bar{A}} \xi_{\bar{A}} \]  

In the last step we introduced a single spinor index $\bar{A}$ which runs over all 16 basis elements that span 4 independent left minimal ideals so that $\xi_{\bar{A}} = (\xi_{\alpha 1}, \xi_{\alpha 2}, \xi_{\alpha 3}, \xi_{\alpha 4})$. Explicitly,
eq. (61) reads
\[ \Psi = \psi A \xi A = \psi a1 \xi a1 + \psi a2 \xi a2 + \psi a3 \xi a3 + \psi a4 \xi a4 \] (62)

Eq. (62) or (61) represents a direct sum of four independent 4-component spinors, each in a different left ideal \( I_L \).

The spinor basis elements \( \xi A \) are related to the Clifford algebra basis elements \( \gamma A \) according to
\[ \xi A = H B A \gamma B \] (63)
where \( H B A \) is a matrix that can be read from eqs. (60), and the analogous equation for the remaining three left ideals. An element \( \xi A \) of the spinor basis is a superposition of the elements \( \gamma A \). The coefficients \( H B A \) of the superposition contain real or imaginary numbers.

The metric of the local flat tangent space is given by the scalar product
\[ \gamma A ^* \gamma B = \langle \gamma A \gamma B \rangle_0 = G AB \] (64)

which involves the reversion operation \( \gamma A ^* \) (it reverses the order of vectors entering \( \gamma A \equiv \gamma a1...ar \)).

Analogously we can define the metric in terms of the spinor basis elements \( \xi A \):
\[ \xi A ^* \xi B = \langle \xi A \xi B \rangle_0 = \frac{1}{n} Z AB 1 \] (65)

1 being the unit element of the Clifford algebra. Reversion here acts on all basis elements entering the definition of \( \xi A \) (eq. (63)): not only on \( \gamma A \), but also on the imaginary unit \( i \equiv e_q e_p \), the bivector of phase space, (occurring in \( H B A \), e.g., in eq. (60)), so that \( i^* = -i \). This is just the complex conjugation \( i^* = -i \).

The occurrence of \( n = \delta \mu \nu = 4 \) (i.e., the dimension of the spacetime from which we generate Clifford algebra) comes from a choice of normalization constant in the definition of \( \xi A \) (e.g., in eq. (60)). Let us define an operation \( \langle \rangle S \), distinct from \( \langle \rangle_0 \) with the properties:

a) for the unit element 1 and an arbitrary Clifford number \( A \) we have
\[ \langle 1 \rangle_S = n \] (66)
\[ \langle A \rangle_S = n \langle A \rangle_0 \] (67)

b) for the product of Clifford numbers we have the cyclic behavior:
\[ \langle AB \rangle_S = \langle BA \rangle_S , \quad \langle ABC \rangle_S = \langle CBA \rangle_S , \quad \langle A_1 A_2...A_k \rangle_S = \langle A_k A_{k-1}...A_1 \rangle_S \] (68)
Using the operation \( \langle A \rangle_S \) we find
\[
\langle \xi_A^\dagger \xi_B \rangle_S = Z_{\tilde{A}\tilde{B}} \tag{69}
\]

Introducing the inverse matrix \( Z^{\tilde{B}\tilde{A}} \) according to
\[
Z^{\tilde{A}\tilde{C}} Z_{\tilde{C}\tilde{B}} = \delta_{\tilde{B}}^{\tilde{A}} \tag{70}
\]
we have \( \xi_{\tilde{A}} = Z^{\tilde{A}\tilde{C}} \xi_{\tilde{C}} \), \( \xi_{\tilde{A}}^\dagger = Z^{\tilde{A}\tilde{C}} \xi_{\tilde{C}}^\dagger \) and
\[
\langle \xi_{\tilde{A}}^\dagger \xi_{\tilde{B}} \rangle_S = \delta_{\tilde{B}}^{\tilde{A}} \tag{71}
\]

With respect to the operation \( \langle \ )_S \) the spinor basis elements \( \xi_{\tilde{A}} \) are orthonormal in the above sense.

Relations (69), (71) can be directly verified from eq. (60) and similar relations for \( \xi_{\alpha i} \), \( i = 2, 3, 4 \).

Matrix elements of an arbitrary Clifford number \( A \) in the (generalized) spinor basis can be calculated according to
\[
\langle \xi_{\tilde{A}}^\dagger A \xi_B \rangle_S \equiv A_{\tilde{A}\tilde{B}} \quad \text{and} \quad \langle \xi_{\tilde{A}}^\dagger A \xi_B \rangle_S \equiv A_{\tilde{B}A} \tag{72}
\]

For \( A \) we may take the basis elements \( \gamma_A \), and in particular the generators \( \gamma_a \), \( a = 0, 1, 2, 3 \). So we obtain the Dirac matrices as a particular case:
\[
\langle \xi_{\tilde{A}}^\dagger \gamma_a \xi_B \rangle_S = (\gamma_a)_{\alpha\beta} \quad \text{and} \quad \langle \xi_{\tilde{A}}^\dagger \gamma_a \xi_B \rangle_S = (\gamma_a)^{\alpha\beta} \tag{73}
\]
where \( \alpha, \beta \) are the 4-spinor indices, and \( \xi_\alpha \equiv \xi_{\alpha 1} \), \( \xi_\beta \equiv \xi_{\beta 1} \).

The quadratic form of a polyvector \( \Phi = \phi^A \gamma_A = \psi^A \xi_{\tilde{A}} \) is
\[
\Phi^\dagger \Phi = \langle \Phi^\dagger \Phi \rangle_S = \langle \phi^{*A} \gamma_A^\dagger \gamma_B^\alpha \phi^B \rangle_S = \phi^{*A} G_{AB} \phi^B \tag{74}
\]
or
\[
\langle \Psi^\dagger \Psi \rangle_S = \langle \psi^{*\tilde{A}} \psi_{\tilde{B}} \xi_{\tilde{A}}^\dagger \xi_{\tilde{B}} \rangle_S = \psi^{*\tilde{A}} Z_{\tilde{A}\tilde{B}} \psi_{\tilde{B}} \tag{75}
\]
In eq. (36) we take the scalar part of the expression, whilst in eq. (37) we perform the operation (67). The latter operation is equivalent to taking the trace of the matrix representing the Clifford number.
Explicitly, for the basis (60) of the first left ideal we have

\[
\langle \xi^\alpha \xi_\beta \rangle_S = z_{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\] (76)

and

\[
\langle \xi^{\alpha} \xi_\beta \rangle_S = \delta^{\alpha}_{\beta}
\] (77)

For the basis \(\xi^\alpha_A = \xi_{\alpha i}\), spanning all four ideals, we have

\[
\langle \xi^\dagger_A \xi^\alpha_B \rangle_S = Z^\dagger_{AB} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\] (78)

We can calculate the matrix elements of the basis Clifford numbers \(\gamma_A\) as follows:

\[
\langle \xi^\dagger_A \gamma_A \xi^\alpha_B \rangle_S = (\gamma_A)^\dagger_B = \delta^i_j \otimes (\gamma_A)^\alpha_{\beta}
\] (79)

For instance, the matrix elements of basis vectors \(\gamma_a\) are given by

\[
\langle \xi^\dagger_A \gamma_a \xi^\alpha_B \rangle_S = (\gamma_a)^\dagger_B = \delta^i_j \otimes (\gamma_a)^\alpha_{\beta}
\] (80)

where

\[
(\gamma_a)^\alpha_{\beta} = \langle \xi^{\alpha} \gamma_a \xi^\beta \rangle_S
\] (81)

are the usual Dirac matrices. Using the spinor basis (60) we obtain from (81) just the matrices in the Dirac representation.

Notice that we define the Dirac matrices by taking one contravariant and one covariant spinor index. Such convention is embraced by the relations (76),(77), according to which the unit matrix 1 is replaced by \(\delta^{\alpha}_{\beta}\). The Clifford algebra relations

\[
\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab} \mathbf{1}
\] (82)

can be written in the matrix form according to

\[
\langle \xi^{\alpha} (\gamma_a \gamma_b + \gamma_b \gamma_a) \xi^\beta \rangle_S = 2\eta_{ab} \langle \xi^{\alpha} \mathbf{1} \xi^\beta \rangle_S
\] (83)

Let us now take into account the relation

\[
\xi^\rho \xi^\rho = \mathbf{1}
\] (84)

\[^{19}\text{We omit the index } i, \text{ denoting the ideal, since the same relation holds for each ideal.}\]
which can be directly calculated from eq. (60), and insert it into eq. (83). We obtain

\begin{equation}
\langle \xi^\alpha \xi^\delta \xi^\rho \rangle_S + (a \xrightarrow{\to} b) = 2 \eta_{ab} \delta^\alpha_{\beta}
\end{equation}

which can be written as

\begin{equation}
(\gamma_a)^\alpha_{\beta} (\gamma_b)^\rho_{\beta} + (\gamma_b)^\alpha_{\beta} (\gamma_a)^\rho_{\beta} = 2 \eta_{ab} \delta^\alpha_{\beta}
\end{equation}

or shortly,

\begin{equation}
\gamma_a \gamma_b + \gamma_b \gamma_a = 2 \eta_{ab} 1
\end{equation}

In particular, using (60), we find

\begin{equation}
\langle \xi^\alpha \xi^\beta \rangle_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \end{pmatrix} = (\gamma_0)^\alpha_{\beta} = z_{\alpha\beta}
\end{equation}

The Dirac matrix \( \gamma_0 = (\gamma_0)^\alpha_{\beta} \) has in this representation the same form as the spinor metric \( z_{\alpha\beta} \).

The quadratic form (75) explicitly reads

\begin{equation}
\langle \Psi^\dagger \Psi \rangle_S = \psi^* \tilde{A} Z_{\tilde{A} \tilde{B}} \tilde{\psi} \tilde{B} = \psi^* \alpha Z_{\alpha i \beta j} \psi \beta j
\end{equation}

The sector of the above expression which belongs to one particular left ideal is obtained by fixing the ideal indices \( i = j \). If, for simplicity, we omit the latter indices, we obtain for one particular left ideal the following quadratic form

\begin{equation}
\psi^* \alpha z_{\alpha\beta} \psi \beta = \psi^* \alpha \psi \alpha = \bar{\psi}^\alpha \psi \alpha
\end{equation}

where

\begin{equation}
\bar{\psi}^\alpha = \psi^* \alpha = \psi^* \alpha z_{\alpha\beta}
\end{equation}

Since the components of the matrix \( z_{\alpha\beta} \) are equal to the components of the matrix \( \gamma_0 \), the latter expression corresponds to the definition of the Dirac adjoint:

\begin{equation}
\bar{\psi} = \psi^\dagger \gamma_0
\end{equation}

where \( \dagger \) denotes Hermitian conjugation of a column spinor \( \psi \).

We have thus shown the place of the ordinary Dirac (column) spinors and their adjoints within the framework of such generalized geometric (Clifford algebra based) approach. The Dirac matrix \( \gamma_0 \) entering the definition of the Dirac adjoint spinor corresponds —in this particular representation— to the metric in the spinor space.
Correspondence with the Dirac bra and ket notation

The Dirac bra and ket notation is commonly used in quantum theory. The following relations hold:

\[
\xi^\alpha = |\xi^\alpha\rangle,
\xi_\alpha = \langle\xi^\alpha| \tag{93}
\]

\[
\xi^{\alpha\dagger} = \langle\xi^\alpha|,
\xi^{\alpha\dagger}_\alpha = \langle\xi^\alpha| \tag{94}
\]

\[
\langle\xi^{\alpha\dagger}\xi_\beta\rangle = \langle\xi^{\alpha}\xi_\beta\rangle = \delta^\alpha_\beta \tag{95}
\]

\[
\langle\xi^{\dagger}_\alpha \xi_\beta\rangle = \langle\xi^{\alpha}_\alpha \xi_\beta\rangle = z_{\alpha\beta} \tag{96}
\]

\[
\langle\xi^{\alpha}\rangle = z^{\alpha\beta}\langle\xi_\beta\rangle = \langle\xi^{\alpha}\gamma_0\rangle \tag{97}
\]

\[
\xi_\alpha\xi^{\alpha\dagger} = |\xi^\alpha\rangle\langle\xi^\alpha| = 1 \tag{98}
\]

\[
\langle\xi^{\alpha\dagger}A\xi_\beta\rangle = \langle\xi^{\alpha}\xi_\beta\rangle = (A)_{\alpha\beta} \tag{99}
\]

\[
\langle\xi^{\alpha\dagger}A\xi_\beta\rangle = \langle\xi^{\alpha}\xi_\beta\rangle = (A)^\alpha_\beta \tag{100}
\]

\[
\Psi = \psi^\alpha\xi_\alpha = |\xi^\alpha\rangle\langle\xi^\alpha|\Psi\rangle \tag{101}
\]

where \(\psi^\alpha = \langle\xi^{\alpha\dagger}\Psi\rangle = \langle\xi^\alpha|\Psi\rangle\).

The above relations hold for one ideal. They can be extended to all four ideals by replacing the spinor indices \(\alpha, \beta\) with the generalized spinor indices \(\tilde{A}, \tilde{B}\).

### 4.2 Extending the Dirac equation to curved Clifford space

In refs. [14, 15, 13] it was proposed that the polyvector valued wave function satisfies the Dirac equation in \(C\)-space:

\[
\partial\Psi \equiv \gamma^M\partial_M\Psi = 0 \tag{102}
\]

The latter equation is just the square root of the “massless” Klein-Gordon equation in \(C\)-space, \(\partial\partial\Psi = 0\), considered in ref. [13], the scalar part of which was considered by Castro [13]. Castro’s equation in turn generalizes Pezzaglia’s equation [23] which is based on the Dixon [47] generalization of the Einstein relation for spinning bodies: \(p_\mu p^\mu - S_{\mu\nu}S^{\mu\nu} = m^2\).

The derivative \(\partial_M\) is the same derivative introduced in eqs. (27)–(31). Now it acts on the object \(\Psi\) which, according to eq. (61), is expanded in terms of the 16 basis elements \(\xi_A\) which, in turn, can be written as a superposition of basis elements \(\gamma_A\) of Clifford algebra. The action of \(\partial_M\) on \(\gamma_A\) is given in (25). An analogous expression holds if \(\partial_M\) operates on the spinor basis elements \(\xi_A\):

\[
\partial_M\xi_{\tilde{A}} = \Gamma_M^\tilde{B}_{\tilde{A}}\xi_{\tilde{B}} \tag{103}
\]
where $\Gamma_M \tilde{B}_A$ are components of the generalized spin connection, i.e., the components of the connection of curved $C$-space for the generalized spinor frame field $\{\xi_{\tilde{A}}\}$. Using the expansion (61) and eq. (103) we find
\[
\partial \Psi = \gamma^M \partial_M (\psi^{\tilde{A}} \xi_{\tilde{A}}) = \gamma^M (\partial_M \psi^{\tilde{A}} + \Gamma_M \tilde{A}_B \psi^B) \xi_{\tilde{A}} \equiv \gamma^M \xi_{\tilde{A}} D_M \psi^{\tilde{A}} = 0 \quad (104)
\]
This is just a generalization of the ordinary Dirac equation in curved spacetime. Instead of curved spacetime, spin connection and the Dirac spinor, we have now curved Clifford space, generalized spin connection and the generalized spinor $\psi^{\tilde{A}}$ which incorporates 4 independent Dirac spinors, as indicated in eq. (61).

We may now use the relations (71),(72),(79) and project eq. (104) onto its component form
\[
(\gamma^M)^{\tilde{C}}_{\tilde{A}} (\partial_M \psi^{\tilde{A}} + \Gamma_M \tilde{A}_B \psi^B) = 0 \quad (105)
\]
The spinor indices $\tilde{A}$, $\tilde{B}$ can be omitted and eq. (105) written simply as
\[
\gamma^M (\partial_M + \Gamma_M) \psi = 0 \quad (106)
\]
where
\[
\gamma^M = (\gamma^M)^{\tilde{A}}_{\tilde{B}}, \quad \Gamma_M = \Gamma_M \tilde{A}_B \quad (107)
\]
We see that in the geometric form of the Dirac equation (102) spin connection is automatically present through the operation of the derivative $\partial_M$ on a polyvector field $\Psi$ written as a superposition of basis spinors $\xi_{\tilde{A}}$. The reader has to be careful (i) not to confuse our symbol $\partial_M$, when acting on a polyvector-valued (Clifford-valued) object, with a partial derivative, acting on the scalar-valued matrix components of a polyvector (see Sec 4.3), (ii) not to miss the fact that $\Psi$ in eq. (102) is a Clifford algebra valued object, not just a component spinor, and (iii) not hastily think that eq. (102) lacks covariance. In fact the simple eq. (102) is equivalent to eq. (104).

4.3 The difference between $\partial_M \gamma_A$ and $\partial_M \gamma_A$

We have already stressed that the action of the derivative $\partial_M$ is different when operating on a scalar or on a Clifford algebra valued object. We have considered the objects such as $\gamma_A$ which form the basis of Clifford algebra, or $\xi_{\tilde{A}}$ which form the generalized spinor basis, spanning all four left ideals. The derivatives $\partial_M \gamma_A$ and $\partial_M \xi_{\tilde{A}}$ are given in eqs. (29) and (103), respectively. In those cases we have applied the derivative $\partial_M$ on the Clifford
algebra valued objects $\gamma_A$ and $\xi_{\hat{A}}$. From the latter objects we can obtain the scalar valued matrix elements by using eq. (72). For instance,

$$\langle \xi_{\hat{A}}^\dagger \gamma_A \xi_B \rangle_S = (\gamma_A)_{\hat{A}}^\dagger_B \equiv \gamma_A \quad (108)$$

$$\langle \xi_{\hat{A}}^\dagger \gamma_M \xi_B \rangle_S = (\gamma_M)_{\hat{A}}^\dagger_B \equiv \gamma_M \quad (109)$$

Taking the derivative, we have

$$\partial_M \gamma_A = \partial_M \langle \xi^\dagger_{\hat{A}} \gamma_A \xi_D \rangle_S = \langle \partial_M \xi^\dagger_{\hat{A}} \gamma_A \xi_D + \xi^\dagger_{\hat{A}} \gamma_A \partial_M \xi_D + \xi^\dagger_{\hat{A}} \partial_M \gamma_A \xi_D \rangle_S$$

$$= \langle -\Gamma_M \gamma_A \xi_D + \xi^\dagger_{\hat{A}} \gamma_A \xi_B \Gamma_M \hat{A}_D - \Omega_B M \xi^\dagger \gamma_B \xi_D \rangle$$

$$= -\Gamma_M \gamma_A + \gamma_A \Gamma_M - \Omega_B M \gamma_B \quad (110)$$

$$\partial_M \gamma_N = \partial_M \langle \xi^\dagger_{\hat{N}} \gamma_N \xi_D \rangle_S = \langle \partial_M \xi^\dagger_{\hat{N}} \gamma_N \xi_D + \xi^\dagger_{\hat{N}} \gamma_N \partial_M \xi_D + \xi^\dagger_{\hat{N}} \partial_M \gamma_N \xi_D \rangle_S$$

$$= \langle -\Gamma_M \gamma_N \xi_D + \xi^\dagger_{\hat{N}} \gamma_N \xi_E \Gamma_M \hat{A}_D + \Gamma_M \xi^\dagger \gamma_J \xi_D \rangle$$

$$= -\Gamma_M \gamma_N + \gamma_N \Gamma_M + \Gamma_M \gamma_J \quad (111)$$

In the above calculation we have distinguished between the derivative of $\xi_{\hat{A}}$ and $\xi_{\hat{A}} = Z_{\hat{A}}^\hat{B} \xi_{\hat{B}}$:

$$\partial_M \xi_{\hat{A}} = \Gamma_M \xi_{\hat{C}} \xi_{\hat{C}} \quad \partial_M \xi_{\hat{A}} = -\Gamma_M \xi_{\hat{C}} \xi_{\hat{C}} \quad (112)$$

The latter relations are consistent with

$$\langle \xi_{\hat{A}}^\dagger \xi_B \rangle_S = \delta_{\hat{A}}^B$$

$$\langle \partial_M \xi_{\hat{A}}^\dagger \xi_B + \xi_{\hat{A}}^\dagger \partial_M \xi_B \rangle_S = \langle -\Gamma_M \xi_{\hat{C}} \xi_{\hat{C}} \xi_{\hat{C}} + \xi_{\hat{A}}^\dagger \Gamma_M \xi_{\hat{C}} \xi_{\hat{C}} \rangle_S = -\Gamma_M \xi_{\hat{B}}^\dagger + \Gamma_M \xi_{\hat{B}}^\dagger = 0 \quad (113)$$

If we take into account the relation $\gamma_N = e_N^A \gamma_A$ (see eq. (25)) and insert eq. (110) into eq. (111), then we obtain

$$(\partial_M e_N^A - \Gamma_M^J e_J^A - \Omega_B M e_N^B) \gamma_A = 0 \quad (114)$$

From the latter equation it follows

$$\partial_M e_N^A - \Gamma_M^J e_J^A - \Omega_B M e_N^B = 0 \quad (115)$$
In eqs. (111), (115) we recognize a generalization of the corresponding relations for the Dirac matrices and vielbein in curved spacetime, namely

\[
\partial_\mu \gamma_\nu - \Gamma^\rho_{\mu \nu} \gamma_\rho = [\gamma_\nu, \Gamma_\mu] \tag{116}
\]

\[
\partial_\mu e^a_\nu - \Gamma^a_{\mu \nu} e^a_\rho - \omega^a_{\mu \rho} e^b_\nu = 0 \tag{117}
\]

In fact, the latter relations are just a particular case of the relations (111), (115). If we restrict our general geometric spinors to one ideal only, and assume that the vielbein and the affinity of C-space do not mix all 16 components of the Clifford algebra, but only those four that belong to spacetime, we obtain just the ordinary theory of spinors in curved spacetime.

The procedure that we have developed here demonstrates that the symbol \(\partial_M\) for the derivative indeed cannot be confused with some other derivative operator, because there are no other derivative operators within the framework of our formalism\(^{20}\). We have shown that when the derivative operates on the matrices, e.g., the Dirac matrices, we obtain the expressions (111)–(117) which are different from the expressions (28), (29), (112) which hold when \(\partial_M\) operates on the Clifford algebra valued objects \(\gamma_M, \gamma_A, \xi_{\tilde{A}}\), etc. In short, there is a clear distinction between the abstract Clifford numbers and their representation by matrices. This becomes manifest, e.g., under the action of the derivative \(\partial_M\).

5 Yang-Mills gauge fields as the spin connection in C-space

5.1 Local (pseudo) rotations in C-space

Let us define the generators of local (pseudo) rotations in a tangent C-space \(T_X(C)\) according to

\[
\Sigma_{AB} = -\Sigma_{BA} = \begin{cases} 
\gamma_A \gamma_B, & \text{if } A < B \\
-\gamma_A \gamma_B, & \text{if } A > B \\
0, & \text{if } A = B
\end{cases} \tag{118}
\]

Here we assume that the basis elements \(\gamma_A, \gamma_B\) and the indices \(A, B\) are ordered according to the rule suggested in eq. (23).

\(^{20}\)In ref. [45] we distinguished between two sorts of derivative, but in Sec. 4.3 we have seen that there are in fact only two sorts of objects, namely \(\gamma_M\) and matrices \(\gamma_M\).
We also have \( \Sigma_{AB} = f_{AB}^C \gamma_C \), where \( f_{AB}^C \) are constants. Remember that by \( \mathfrak{o} \) we denote a scalar component of a polyvector, so that for \( C = \mathfrak{o} \) we have \( \gamma_{\mathfrak{o}} \equiv 1 \).

A generic transformation in a tangent \( C \)-space \( T_X(C) \) which maps a polyvector \( \Psi \) into another polyvector \( \Psi' \) is given by
\[
\Psi' = R \Psi S
\]
where
\[
R = e^{i \Sigma_{AB} \alpha^{AB}} = e^{\gamma_A \alpha_A} \quad \text{and} \quad S = e^{i \Sigma_{AB} \beta^{AB}} = e^{\gamma_A \beta_A}
\]
Here \( \alpha^{AB} \) and \( \beta^{AB} \), or equivalently, \( \alpha^A = f_{CD}^A \alpha^{CD} \) and \( \beta^A = f_{CD}^A \beta^{CD} \), are parameters of the transformation.

In general, eq. (119) allows for the transformation which maps a basis element \( \gamma_A \) into a mixture of basis elements:
\[
\gamma_A \rightarrow \gamma'_A = R \gamma_A S = L_A^B \gamma_B
\]
In particular, we have the following three interesting cases:
(i) \( \alpha^{AB} \neq 0 \), \( \beta^{AB} = -\alpha^{AB} \). Then we have
\[
\Psi' = R \Psi R^{-1}
\]
This is the transformation which preserves the Clifford algebra relations
\[
[\gamma_A, \gamma_B] = C_{AB}^C \gamma_C
\]
so that for the transformed elements \( \gamma'_A = R \gamma_A R^{-1} \) we have
\[
[\gamma'_A, \gamma'_B] = C_{AB}^C \gamma'_C
\]
with the same structure constants \( C_{AB}^C \). This means that the transformation (122) maps one basis element \( \gamma_A \) into another basis element \( \gamma'_A \), e.g., a basis vector into a bivector, a 3-vector into 1-vector, etc.
(ii) \( \alpha^{AB} \neq 0 \), \( \beta^{AB} = 0 \). Then we have
\[
\Psi' = R \Psi
\]
This is the transformation which maps a basis spinor \( \xi_{\alpha i} \) into another basis spinor \( \xi'_{\alpha i} \) belonging to the same left ideal:
\[
\xi_{\alpha i} \subset \mathcal{T}_i^L \rightarrow \xi'_{\alpha i} = R \xi_{\alpha i} \subset \mathcal{T}_i^L
\]
(iii) $\alpha^{AB} = 0, \beta^{AB} \neq 0$. Then

$$\Psi' = \Psi S$$  \hspace{1cm} (127)

This is the transformation that maps a right ideal into the same right ideal:

$$\xi_{\alpha i} \subset \mathcal{T}^R_i \rightarrow \xi'_{\alpha i} = \xi_{\alpha i} S \subset \mathcal{T}^R_i$$  \hspace{1cm} (128)

In general, for the transformation (119) we have

$$\Psi' = \psi^{\tilde{A}} R \xi^{A} S = \psi^{\tilde{A}} U^{\tilde{B}} \xi_{\tilde{B}} = \psi^{\tilde{A}} \xi_{\tilde{A}}$$  \hspace{1cm} (129)

where

$$\psi^{\tilde{A}} = U^{\tilde{B}} \psi_{\tilde{B}}$$  \hspace{1cm} (130)

The latter transformations, in general, mixes right and left ideals. Eq. (130) can be considered as a matrix equation in the space spanned by the generalized spinor indices $\tilde{A}, \tilde{B}$:

$$\psi' = U \psi$$  \hspace{1cm} (131)

where $U$ is a $16 \times 16$ matrix, whilst $\psi$ and $\psi'$ are columns with 16 elements.

It is illustrative to calculate the matrix elements of $\Psi'$ with respect to the spinor basis of, say, the first left ideal $\xi^\alpha \equiv \xi^{\alpha 1}$. So we obtain matrices $\psi^{\alpha \beta}$ and $\psi^{\alpha \beta}$ representing the polyvectors $\Psi'$ and $\Psi$ respectively; the index $\beta$ says which column, i.e., which left ideal (it stands now instead of the index $i$ or $j$):

$$\langle \xi^{\gamma 4} \Psi' \xi^{\delta} \rangle_S = \langle \xi^{\gamma 4} R \Psi S \xi^\delta \rangle_S = \langle \xi^{\gamma 4} R \xi^{\alpha 1} \Psi \xi^{\beta} \xi^{\alpha 1} S \xi^{\delta} \rangle_S$$

$$= R^{\gamma \alpha} \psi^{\alpha \beta} S_{\beta} = U_{(\gamma \delta)} \psi^{\alpha \beta} = U^{\tilde{B}} \tilde{C} \psi^{\tilde{C}}$$  \hspace{1cm} (132)

where

$$U^{\tilde{B}} \tilde{C} \equiv U_{(\gamma \delta)} (\alpha \beta) = R^{\gamma \alpha} S_{\beta}$$  \hspace{1cm} (133)

In matrix notation this reads

$$U = R \otimes S^T$$  \hspace{1cm} (134)

where $R$ and $S$ are $4 \times 4$ matrices representing the Clifford numbers $R$ and $S$. That is, $U$ is the direct product of $R$ and the transpose $S^T$ of $S$, and it belongs, in general, to the group $GL(4, C) \times GL(4, C)$. The group is local, because the basis elements $\gamma_A$ entering the definition (65) depend on position $X$ in $C$-space according to the relations (29), and also the group parameters $\alpha^A, \beta^A$ in general depend on $X$.  

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The most general gauge group here is \( \text{GL}(4, \mathbb{C}) \times \text{GL}(4, \mathbb{C}) \). The first piece belongs to the left transformations \( R \) and the second piece to the right transformations \( S \) of eq. (119). The group \( \text{GL}(4, \mathbb{C}) \times \text{GL}(4, \mathbb{C}) \) that we started from is subjected to further restrictions resulting from the requirement that the transformations (119) should leave the quadratic form \( \Psi^\dagger \Psi \) invariant. So we have

\[
\Psi'^\dagger \Psi' = \langle \Psi'^\dagger \Psi' \rangle_S = \langle S^\dagger \Psi'^\dagger R^\dagger RS \rangle_S = \langle \Psi'^\dagger \Psi \rangle_S = \Psi'^\dagger \Psi
\]

provided that

\[
R^\dagger R = 1 \quad \text{and} \quad S^\dagger S = 1.
\]

Using the exponential expression (120) we have

\[
R = e^{\gamma_A \alpha^A}, \quad R^{-1} = e^{-\gamma_A \alpha^A}
\]

\[
R^\dagger = \left( e^{\gamma_A \alpha^A} \right)^\dagger = (1 + \gamma_A \alpha^A + \frac{1}{2!} (\gamma_A \alpha^A)^2 + \ldots) + e^{(\gamma_A \alpha^A)^\dagger} = e^{\gamma_A^\dagger \alpha^A}
\]

where, according to our definition, reversion acts also on imaginary number \( i \).

After taking into account eqs. (137), (138), the condition (136) which reads \( R^\dagger = R^{-1} \) transforms into

\[
\gamma^\dagger_A \alpha^A = -\gamma_A \alpha^A
\]

Since

\[
\gamma^\dagger_A = \gamma_A \quad \text{if} \quad A \in \{A\}_+, \quad \gamma^\dagger_A = -\gamma_A \quad \text{if} \quad A \in \{A\}_-
\]

we have that \( \gamma^\dagger_A \) is equal to plus or minus \( \gamma_A \), depending on the grade, i.e., on the type of the multivector index \( A \), and consequently the parameters \( \alpha^A \) are either real or imaginary\(^{21} \).

If we represent the Clifford numbers \( R, S \) by \( 4 \times 4 \) matrices

\[
R = R^\alpha_{\beta} = \langle \xi^\dagger \xi_{\beta} \rangle_S, \quad S = S^\alpha_{\beta} = \langle \xi^\dagger S \xi_{\beta} \rangle_S
\]

where \( \xi^\alpha \equiv \xi^\alpha_1, \xi^\beta \equiv \xi^\beta_1 \) then the condition (136) reads

\[
\langle \xi^\dagger R^\dagger R \xi_{\beta} \rangle_S = \delta^\alpha_{\beta} \quad \text{and} \quad \langle \xi^\dagger S^\dagger S \xi_{\beta} \rangle_S = \delta^\alpha_{\beta}
\]

\(^{21}\)Later we will show (see eq. (189)) that it is convenient to redefine the Clifford basis so that its elements are invariant under reversion: \( \gamma^\dagger_A = \gamma_A \). Then the parameters \( \alpha^A \) can be kept imaginary for all \( A \), if the form \( R = \exp [\gamma_A \alpha^A] \) is used, or real, if the form \( R = \exp [i \gamma_A \alpha^A] \) is used.
which, after lowering the index \( \alpha \) with the metric \( z_{\alpha \beta} \), becomes

\[
\langle \xi_\alpha^{\dagger} R^{\dagger} R \xi_\beta \rangle_S = z_{\alpha \beta}
\]

(143)

Using \( \xi_\delta \xi_\delta^{\dagger} = \xi_\delta^{\dagger} \xi_\delta = \xi_\delta^{\dagger} \xi_\delta = 1 \) we have

\[
\langle \xi_\alpha^{\dagger} R^{\dagger} \xi_\delta^{\dagger} z_{\delta \gamma} \xi_\gamma^{\dagger} R \xi_\beta \rangle_S = \langle \xi_\alpha^{\dagger} R^{\dagger} \xi_\delta^{\dagger} \rangle_S z_{\delta \gamma} \langle \xi_\gamma^{\dagger} R \xi_\beta \rangle_S = z_{\alpha \beta}
\]

(144)

This can be written as

\[
\langle (\xi_\alpha^{\dagger} R \xi_\alpha)^{\dagger} \rangle_S z_{\delta \gamma} \langle \xi_\gamma^{\dagger} R \xi_\beta \rangle_S = z_{\alpha \beta}
\]

(145)

or

\[
R^{\dagger} z R = z
\]

(146)

where \( R^{\dagger} = \langle (\xi_\alpha^{\dagger} R \xi_\alpha)^{\dagger} \rangle_S \) is the Hermitian conjugate of matrix \( R = \langle \xi_\alpha^{\dagger} R \xi_\alpha \rangle_S \)

The transformations (119), in particular, transform a scalar, or a vector field into a spinor field. In this respect they have the role of supersymmetric transformations. Supersymmetry is automatically present in our approach. A detailed investigation of this fascinating possibility is beyond the scope of this paper. A different approach to supersymmetry and Clifford algebra was proposed in refs. [48].

While reversion \( ^{\dagger} \) refers to a Clifford valued object, say \( A \), hermitian conjugation \( ^{\dagger} \) refers to the matrix representing \( A \). So we have

\[
\langle \xi^{\alpha \dagger} A \xi_\beta \rangle_S = A^{\alpha \beta} = A
\]

(147)

\[
\langle (\xi^{\alpha \dagger} A \xi_\beta)^{\dagger} \rangle_S = \langle \xi^{\dagger \beta} A^{\dagger} \xi^{\alpha} \rangle_S = A^{\star \alpha} = A^{\dagger}
\]

(148)

Complex conjugation \( ^\star \) comes from the reversion which, according to our definition operates also on the (commuting) imaginary unit \( i \) (the bivector of phase space) entering the definition of \( \xi_\alpha \) (see eq. [60]).

5.2 Invariance of the generalized Dirac equation under local (pseudo) rotations in \( C \)-space

We will now explore the invariance properties of the generalized Dirac equation (102). An \( X \)-dependent polyvector valued field can be expanded either in terms of the Clifford algebra basis \( \{ \gamma_A \} \), or in terms of the ideal basis \( \{ \xi_{\tilde A} \} \), or in terms of the \( C \)-space coordinate
basis \{\gamma_M\}, defined by eq. (25). For illustration, let us first consider the latter case. Using (28), (31), we have
\[
\partial \Phi = \gamma^M \partial_M (\phi^N \gamma_N) = \gamma^M (\partial_M \phi^N + \Gamma^N_{MJ} \phi^J) \gamma_N = \gamma^M \gamma^N D_M \phi^N
\] (149)
where \(D_M\) is the covariant derivative.

In another coordinate basis \{\gamma'_M\} we have
\[
\partial' \Phi' = \gamma'_M \partial'_M (\phi'^N \gamma'_N) = \gamma'^M (\partial'_M \phi'^N + \Gamma'^{N'}_{M'J} \phi'^J) \gamma'_N = \gamma'^M \gamma'^N D'_M \phi'^N
\] (150)

The new and old basis elements and components are related by general coordinate transformations in \(C\)-space:
\[
\phi'^N = C^N_J \phi^J, \quad \gamma'_N = C^J_N \gamma_J
\] (151)
where \(C^N_J = \partial x'^N / \partial x^J\). Requiring
\[
\partial \Phi = \partial' \Phi'
\] (152)
and using eqs. (149), (151) we find the following transformation law for the \(C\)-space affine connection:
\[
\Gamma^{J'}_{MN} = C^R_M C^S_N C^J_R \Gamma^{K'}_{RS} + \partial_M C^R_N C^J_R
\] (153)
The latter relation generalizes the well known transformation property of the affine connection.

Similarly, for \(\Phi = \Psi = \psi^A \tilde{\xi}_A\) we have
\[
\partial' \Psi' = \partial \Psi = \gamma^M \partial_M (\psi^A \tilde{\xi}_A') = \gamma^M \partial_M (\psi^A \tilde{\xi}_A)
\] (154)
Using eqs. (103) and (150) and
\[
\partial'_{M'} \tilde{\xi}'_A = \Gamma^{B'}_{M} \tilde{\xi}'_B \text{ and } \tilde{\xi}'_A = U^C \tilde{\xi}'_C
\] (155)
we obtain
\[
\Gamma^{A'}_{ M B'} = C^M_N U^B_D U^C \tilde{\xi}_A \Gamma^{D'}_{ M' C} + \partial_M U^D_{A'} U_D \tilde{B}
\] (156)
which is the transformation law for the generalized spin connection (i.e., the spin connection in \(C\)-space). The relation (156) is analogous to the relation (153).
In a special case when we do not perform a general coordinate transformation \[ (151), \]
but only a local (pseudo) rotations \[ (130), (155), \] we have
\[
\Gamma^\dagger_{\hat{M}\hat{A}} = U_{\hat{D}}^\dagger B U_{\hat{C}}^\dagger \Gamma^\dagger_{\hat{M}\hat{C}} + \partial_M U_{\hat{D}}^\dagger U_{\hat{A}}^\dagger B
\]
(157)

In matrix notation\(^{22}\) this reads
\[
\Gamma_M = U\Gamma^\dagger_M U^{-1} + U\partial_M U^{-1}
\]
(158)

By renaming, respectively, \( \Gamma_M, U \) into \( \Gamma^\dagger_M, U^{-1} \), and vice versa, eq.\( (158) \) assumes a more familiar form
\[
\Gamma^\dagger_M = U^{-1}\Gamma_M U + U^{-1}\partial_M U
\]
(159)

We see that \( \Gamma_M \) transforms as a non abelian gauge field. We have thus demonstrated that the generally covariant Dirac equation in 16-dimensional curved \( C \)-space contains the coupling of spinor fields \( \psi^{\hat{A}} \) with non abelian gauge fields \( \Gamma^\dagger_{\hat{M}\hat{A}} \) which altogether form the spin connection in \( C \)-space. The theory presented here is essentially a generalization of the one considered by Pezzaglia \[ 23 \], where the higher multivector derivatives were not included, and the geometric interpretation of the imaginary unit \( i \) was different from ours. Pezzaglia in turn takes Crawford’s ideas \[ 22 \], but interprets them geometrically.

In eqs.\( (152), (154) \) we considered passive transformations, that is the transformations which change components and the basis elements, so that
\[
\Phi' = \phi'^M \gamma'_{M} = \phi^M \gamma_M = \Phi
\]
(160)
\[
\Psi' = \psi'^{\hat{M}\hat{A}} \xi'_{\hat{A}} = \psi^{\hat{M}\hat{A}} \xi_{\hat{A}} = \Psi
\]
(161)

Similarly for the derivative:
\[
\partial' = \gamma'^M \partial'_{M} = \gamma^M \partial_M
\]
(162)

The invariance of the Dirac equation, written in the geometric form \( \partial \Psi = 0 \), is straightforward, since the polyvector objects \( \partial \) and \( \Psi \), which have values in Clifford algebra, do not change under passive transformations.

Under active transformations either the components or the basis elements change:
\[
\Psi' = \psi^{\hat{M}\hat{A}} \xi'_{\hat{A}} = \psi^{\hat{M}} U^{\hat{C}}_{\hat{A}} \xi'_{\hat{C}} = \psi^{\hat{M}\hat{C}} \xi'_{\hat{C}}
\]
(163)

\(^{22}\)The objects are considered as matrices in the generalized spinor indices \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \).
where
\[ \xi^\Lambda = U^\Lambda \tilde{\xi} \] (164)
and
\[ \psi^\tilde{\Lambda} = U^\tilde{\Lambda} \tilde{\psi} \] (165)

Under the transformation (165) we have
\[ D_M' \psi^\Lambda_A = \frac{\partial x^N}{\partial x'^M} U_A^\Lambda B D_N \psi^B \] (166)

where
\[ D_M' \psi^\Lambda_A = \partial_M' \psi^\Lambda_A + \Gamma_M^\Lambda B \psi^B \] and \[ D_M \psi^\tilde{\Lambda} = \partial_M \psi^\tilde{\Lambda} + \Gamma_M^\tilde{\Lambda} B \psi^B \] (167)

are, respectively, the transformed and the “original” covariant derivative, defined in eq. (104). For the reason of completeness, we have considered in eq. (166) also the general coordinate transformations of the \( C \)-space coordinates \( x^M = (s, x^\mu, x^{\mu\nu}, \ldots) \). The latter transformations are independent from local transformations, such as (164) or (165) which affect the generalized spinor indices \( \tilde{\Lambda}, \tilde{\Lambda}, \ldots \), or from an analogous local transformation which affects the local (flat) polyvector indices \( A, B, \ldots \). If we perform only a local transformation (165) and no general coordinate transformation, i.e., if we take \( \partial x^N / \partial x'^M = \delta^N_M \), then the transformation of the covariant derivative (166) reads simply
\[ D_M' \psi^\Lambda_A = U_A^\Lambda B D_M \psi^B \] (168)

The covariant derivative transforms in the same way as the field \( \psi^\Lambda \). This is the well known property of gauge theories.

**Action**

The \( C \)-space Dirac equation (102) can be derived from the action
\[ I[\Psi, \Psi^\dagger] = \int d^{2n}X \sqrt{|G|} i \Psi^\dagger \partial \Psi = \int d^{2n}X \sqrt{|G|} i \psi^* \tilde{B} B \gamma^M \xi \partial_M \psi^\Lambda \] (169)

where \( d^{2n}X \sqrt{|G|} \) is the invariant volume element of the \( 2^n \)-dimensional \( C \)-space, \( G \equiv \det G_{MN} \) being the determinant of the \( C \)-space metric.

Taking the scalar part of the action (169) we obtain the matrix form of the action:
\[ I[\psi^* B, \psi^\Lambda] = \int d^{2n}X \sqrt{|G|} i \psi^* \tilde{B} B \gamma^M \xi \partial_M \psi^\Lambda \] (170)

where \( (\gamma^M)_{B \Lambda} = \langle \xi^\Lambda \gamma^M \xi \rangle_s \). The quadratic form under the integral can be written in the form
\[ \psi^* \tilde{B} Z_B C (\gamma^M) \tilde{\xi}_A \partial_M \psi^\Lambda = \psi^* (\gamma^M) \tilde{\xi}_A \partial_M \psi^\Lambda = \bar{\psi} \gamma^M \partial_M \psi \] (171)
Here \( \psi \equiv \psi^\hat{A} \) represents a column of contravariant components with the generalized spinor index \( \hat{A} \), while \( \bar{\psi} \equiv \psi^*_\hat{C} \) represents a row of complex conjugate covariant components. Writing the indices \( \hat{A}, \hat{B}, ... \) in the form \( \hat{A} = \alpha_i, \hat{B} = \beta_j \), and fixing the ideal indices \( i, j \) to one chosen ideal, e.g., \( i = j = 1 \) and omitting the index 1, then we find that eq. (171) contains the ordinary spinor quadratic form
\[
\psi^* \gamma^\alpha \beta \partial_M \psi^\beta = \bar{\psi} \gamma^M \partial_M \psi
\]
(172)
If \( M = \mu, \mu = 0, 1, 2, 3 \), we obtain the form \( \bar{\psi} \gamma^\mu \partial_\mu \psi \) entering the ordinary Dirac action.

The action (169) is invariant under passive transformations (161), and passive general coordinate transformations \( x^M \rightarrow x'^M(x^N) \) (which lead to (160)–(162)). Passive transformations do not transform a geometric object, such as \( \Psi \) or \( \partial \Psi \). Since they transform the basis elements and the corresponding components, for instance \( \xi^{\hat{A}} \) and \( \psi^{\hat{A}} \), they only show how the components look in different reference frames.

We will now show that the scalar part of the action (169) is invariant under the active transformations as well. We have
\[
\langle \Psi^\dagger \gamma^C e_C^M \partial_M \Psi \rangle_S \rightarrow \langle \Psi^\dagger \gamma^C e_C^M \partial_M \Psi' \rangle_S = \psi^* \bar{B} \gamma^A \xi^{\hat{A}} S e_C^M D_M \psi^{\hat{A}} \]
(173)
Examining the matrix elements we find
\[
\langle \xi^\dagger_B \gamma^C \xi^{\hat{A}} \rangle_S = \langle \xi^\dagger_C U^* \gamma^C U^\dagger \xi^{\hat{A}} S \rangle_S = U^* \gamma^C U^\dagger \xi^{\hat{A}} \bar{A} = U^* \gamma^C \xi^{\hat{A}} \bar{A} L^C_{\bar{A}}
\]
(174)
Here \( L^C_{\bar{A}} \) is a local “rotation” (i.e., a Lorentz transformation in \( C \)-space) acting on local basis elements \( \gamma^C \). On the other hand
\[
\langle \xi^\dagger_B \gamma^C \xi^{\hat{A}} \rangle_S = \langle S^4 \xi_B^\dagger R^\gamma^C R^\xi A \rangle_S = \langle \xi^\dagger_B \gamma^C \xi^{\hat{A}} \rangle_S = \langle \gamma^C \rangle_{\bar{B} \bar{A}}
\]
(175)
In the last step of the above equation we have taken into account the fact that the operation \( \langle \rangle_S \) (to be distinguished from the operation \( \langle \rangle_0 \)) leaves the cyclically permutated product of geometric objects (Clifford numbers) invariant under a transformation (119), then we used \( SS^t = 1 \) (postulated in Sec. 5.1), and \( R^\gamma^C R = \gamma^C \).

Comparing equations (174) and (175), after raising \( \bar{B} \) and lowering \( \bar{C} \), we have
\[
U^* \gamma^A \bar{C} U^\dagger \bar{A} L^C_{\bar{A}} = \langle \gamma^C \rangle_{\bar{B} \bar{A}}
\]
(176)
i.e.,

\[ U^{-1} \gamma^A U L_A^C = \gamma^C \]  

(177)

The latter relation is a generalization of the well-known transformation properties of gamma matrices under Lorentz transformations. Writing \( \tilde{A} = 0i, \tilde{B} = \beta j \), and fixing the ideal indices \( i, j \) to one chosen ideal, e.g., \( i = j = 1 \), and omitting index 1, we obtain for \( A = a, a = 0, 1, 2, 3, \)

\[ U^* \gamma^\beta (\gamma^a) \gamma^\delta U \delta_a^c = (\gamma^c)^\beta_a \]  

(178)

which is just the ordinary relation.

Using eq. (174) we find that the quadratic form (173) is invariant under active transformations.

Alternatively, we may transform the components, while keeping the basis elements unchanged:

\[ \langle \Psi^\dagger \partial \Psi \rangle_S = \langle \psi^\dagger \tilde{B} \gamma^A D_A^\dagger \delta^\tilde{A} \psi \rangle_S = \psi^\dagger D^\tilde{A} \gamma^C D_A^\dagger \psi^C \]  

(179)

\[ = \psi^\dagger D^\tilde{A} \gamma^C D_A^\dagger L_A^C D_C \psi \]  

Above we have used

\[ \gamma^A D_M \psi \tilde{A} = e^M_A \gamma^A D_M \psi \tilde{A} = \gamma^A D_A \psi \tilde{A} \rightarrow \gamma^A D_A \psi \tilde{A} \]  

(180)

and

\[ D_A^\dagger \psi \tilde{A} = L_A^C D_C \psi \tilde{A} \]  

(181)

We also used eq. (176).

Eqs. (173)–(177) and (179) demonstrate the invariance of the scalar part of the action (169) under active local rotations in \( C \)-space.

\textit{Noether’s current}

Let us perform the variation of the action (169) by retaining the term which results from the variation of the boundary. We obtain:

\[ \delta I = \int d^2 x \left\{ \sqrt{|G|} \left( \delta \Psi^\dagger \gamma^M \partial_M \psi + \Psi^\dagger \gamma^M \partial_M \delta \Psi + \partial_M \left( \sqrt{|G|} \Psi^\dagger \gamma^N \partial_N \psi \delta x^M \right) \right) \right\} \]

\[ = \int d^2 x \left\{ \sqrt{|G|} \left( \delta \Psi^\dagger \gamma^M \partial_M \psi - \partial_M \left( \sqrt{|G|} \Psi^\dagger \gamma^M \right) \delta \Psi \right) \right. \]

\[ + \partial_M \left[ \sqrt{|G|} \left( \Psi^\dagger \gamma^M \delta \Psi + \Psi^\dagger \gamma^N \partial_N \psi \delta x^M \right) \right] \]  

(182)
If we fix the boundary, i.e., if we take $\delta x^M = 0$, then from eq. (182) we read the equations of motion:

$$\gamma^M \partial_M \Psi = \gamma^A e_A^M \partial_M \Psi = 0$$  \hspace{1cm} (183)

and

$$(\partial_M \Psi \dagger) \gamma^M = e_A^M \partial_M \Psi \dagger \gamma^A = 0$$  \hspace{1cm} (184)

where we have used $\partial_M (\sqrt{|G|} \gamma^M) = 0$. Now, performing the operation of reversion on equation (183) we obtain

$$(\partial_M \Psi \dagger) (\gamma^M) \dagger = e_A^M \partial_M \Psi \dagger (\gamma^A) \dagger = 0$$  \hspace{1cm} (185)

which is not the same as eq. (183). The equations of motion for $\Psi$ and its reverse, $\Psi \dagger$, are therefore only consistent if all $\gamma^M$'s entering the equations satisfy either (i) $(\gamma^M) \dagger = \gamma^M$, or (ii) $(\gamma^M) \dagger = -\gamma^M$. Expanding $\gamma^M = e_A^M \gamma^A$ we see that only those Clifford basis numbers $\gamma^A$ which are either even ($(\gamma^A) \dagger = \gamma^A$), or odd ($(\gamma^A) \dagger = -\gamma^A$) under reversion can simultaneously enter the equations of motion. From our action (169) we thus obtain two different classes of equations of motion, one class for those values of the multivector indices $A$ for which $\gamma^A$ is even, and the other class for those $A$ for which $\gamma^A$ is odd under reversion. This is so, because our action is not invariant under reversion.

An alternative is to consider the action which is invariant under reversion:

$$I[\Psi, \Psi \dagger] = \frac{1}{2} \int d^{2n} X \sqrt{|G|} \left[ i \Psi \dagger \partial \Psi + (i \Psi \dagger \partial \Psi) \dagger \right]$$  \hspace{1cm} (186)

Then the corresponding equations of motion are

$$(\gamma^M + (\gamma^M) \dagger) \partial_M \Psi = 0 \text{ and } \partial_M \Psi \dagger (\gamma^M + (\gamma^M) \dagger) = 0$$  \hspace{1cm} (187)

Under reversion the first of equations (187) is transformed into the other, and vice versa. Therefore the equations of motion (182) and their reversed equations are consistent. However, for those $A$-values for which $(\gamma^A) \dagger = -\gamma^A$, the terms in the equations vanish. The terms odd under reversion do not contribute to the equations of motion. In this respect the situation corresponding to the action (186) is the same as that of the action (169). The difference is that in the case of the action (169) there are two classes of equations of motion, while in the case of the action (186) the class with odd $\gamma^A$'s is missing.

This problem is avoided if we redefine the Clifford basis so that instead of

$$\{\gamma_A\} = \{\gamma_{a_1...a_r}\}, \quad r = 1, 2, ..., n$$  \hspace{1cm} (188)
we have
\[ \{ \gamma_A \} = \{ i^{r(r-1)/2} \gamma_{a_1...a_r} \} \] (189)
where \( \gamma_{a_1...a_r}, r = 1, 2, ..., n \), are defined as the antisymmetrized products \( (24) \). Explicitly, for \( n = 4 \), we have
\[ \{ \gamma_A \} = \{ 1, \gamma_{a_1}, i\gamma_{a_1a_2}, -i\gamma_{a_1a_2a_3}, -\gamma_{a_1a_2a_3a_4} \} \] (190)
The newly defined basis elements are invariant under reversion:
\[ \gamma_A^\dagger = \gamma_A \] (191)
Using eq. (25) we also have that the coordinate basis elements are invariant under reversion:
\[ \gamma_M^\dagger = e_M A \gamma_A^\dagger = e_M A \gamma_A = \gamma_M \] (192)
Similarly for \( \gamma^A \) and \( \gamma^M \). Then we have that the square of the generalized Dirac operator is equal to the generalized Klein-Gordon operator:
\[ \partial^A \partial^B \Psi = \gamma^A \gamma^B \partial_a \partial_B \Psi = \gamma^A \gamma^B \partial_a \partial_B \Psi = G^{AB} \partial_a \partial_B \Psi \] (193)
If we define the \( C \)-space Dirac equation by using the new basis (189), then the reversed equation (185) coincides with eq. (184); the actions (186) and (169) become equivalent.

In eq. (182) the quantity \( \delta \Psi = \Psi'(X) - \Psi(X) \) is the variation of the field at fixed point \( X = x^M \gamma_M \). Following the usual procedure we introduce the total variation
\[ \tilde{\delta} \Psi = \Psi'(X') - \Psi(X) = \delta \Psi + \partial_M \Psi \delta x^M \] (194)
which takes into account the variation of the point \( X \) as well. Assuming that the equations of motion are fulfilled, and inserting eq. (194) into eq. (182), we obtain
\[ \delta I = i \int d^{2n} X \partial_M \left[ \sqrt{|G|} (\Psi^\dagger \gamma^M \tilde{\delta} \Psi - \Psi^\dagger \gamma^M \partial_N \Psi \delta x^N) \right] \] (195)
Here
\[ G^M \equiv i (\Psi^\dagger \gamma^M \tilde{\delta} \Psi - \Psi^\dagger \gamma^M \partial_N \Psi \delta x^N) \] (196)
is the generator for the corresponding transformation of our generalized Dirac like action in \( C \)-space.
Let us now consider the following transformation

\[ \Psi'(X') = R \Psi(X) S = e^{\frac{1}{4} \Sigma_{AB} \alpha^{AB}} \Psi(X) e^{\frac{1}{4} \Sigma_{CD} \beta^{CD}} \]  

(197)

\[ x'^{M} = e^{A}_{M} x^{A} = e^{A}_{M} L^{A}_{B} x^{B} \]  

(198)

This are the general “rotations” (Lorentz transformations) in C-space. The corresponding infinitesimal transformations (for infinitesimal parameters \( \alpha^{AB} \) and \( \beta^{AB} \)) are

\[ \bar{\delta} \Psi = \Psi'(X') - \Psi(X) = \frac{1}{4} \Sigma_{AB} \alpha^{AB} \Psi(X) + \Psi(X) \frac{1}{4} \Sigma_{AB} \beta^{AB} \]  

(199)

\[ \bar{\delta} x^{M} = e^{A}_{M} \epsilon^{A}_{B} x^{B} \]  

(200)

Here \( L^{A}_{B} \) is a “Lorentz” transformation in the tangent C-space at a point \( X \), and \( \epsilon^{AB} = -\epsilon^{BA} \) are infinitesimal parameters.

Inserting eqs. (199), (200) into eq. (196) we obtain

\[ G^{M} = i \Psi^{\dagger} \gamma^{M} \left( \frac{1}{4} \Sigma_{AB} \alpha^{AB} \Psi + \Psi \Sigma_{AB} \beta^{AB} \right) - i \Psi^{\dagger} \gamma^{M} (x_{A} \partial_{B} - x_{B} \partial_{A}) \Psi \epsilon^{AB} \]  

(201)

where

\[ x_{A} \partial_{B} = e^{A}_{N} e^{B}_{J} x_{N} \partial_{J} \]  

(202)

Parameters \( \epsilon^{AB} \) on the one hand, and \( \alpha^{AB} \), \( \beta^{AB} \) on the other hand, denote the same local transformation (“rotation”) in C-space. In order to obtain the relation between the two sets of parameters, we proceed as follows.

According to eqs. (51), (61), a polyvector can be written either in terms of the polyvector components \( \phi^{A} \), or the generalized spinor components \( \psi^{\hat{A}} \). The two sets of components are related as

\[ \psi^{\hat{A}} = H^{\hat{A}}_{A} \phi^{A}, \quad \phi^{A} = H^{A}_{\hat{B}} \psi^{\hat{B}} \]  

(203)

where \( H^{\hat{A}}_{A} = \langle \xi^{\hat{A}} \gamma_{A} \rangle_{S} \) and \( H^{A}_{\hat{B}} = \langle \gamma^{A} \xi_{\hat{B}} \rangle_{S} \) satisfy

\[ H^{\hat{A}}_{A} H^{A}_{B} = \delta^{\hat{A}}_{B}, \quad H^{A}_{\hat{A}} H^{\hat{A}}_{B} = \delta^{A}_{B} \]  

(204)

The transformation of \( \psi^{\hat{A}} \) then reads

\[ \psi^{\hat{A}}(X') = H^{\hat{A}}_{A} \phi^{A}(X') = H^{\hat{A}}_{A} L^{A}_{B} \phi^{B}(X) = H^{\hat{A}}_{A} L^{A}_{B} H^{B}_{\hat{B}} \psi^{\hat{B}}(X) = U^{\hat{A}}_{\hat{B}} \psi^{\hat{B}}(X) \]  

(205)

where

\[ U^{\hat{A}}_{\hat{B}} = H^{\hat{A}}_{A} L^{A}_{B} H^{B}_{\hat{B}} \]  

(206)
The infinitesimal transformation is

$$\psi^{i} \delta \tilde{A} (X') = H^{\tilde{A}} \psi^{i} (X)$$

or

$$\bar{\delta} \psi^{i} \tilde{A} = H^{\tilde{A}} \psi^{i} (X)$$

Writing the indices $\tilde{A}, \tilde{B}$ in the form of double indices, $\tilde{A} = \gamma \delta, \tilde{B} = \alpha \beta$ (where the “ideal” indices $i, j$ are now replaced by $\alpha, \beta$, that is, they are written by the same symbol as spinor indices), the latter relation can be written as

$$\bar{\delta} \psi^{\gamma \delta} = H^{\gamma \delta} \psi^{\alpha \beta}$$

On the other hand, the same infinitesimal transformation is given in eq. (199), which in component form reads

$$\bar{\delta} \psi^{\gamma \delta} = \frac{1}{4} \left( \Sigma_{AB} \right)^{\gamma} \alpha \alpha \psi^{\alpha \delta} + \psi^{\gamma \delta} \frac{1}{4} \left( \Sigma_{AB} \right)^{\beta \alpha \delta} = T^{\gamma \delta} \psi^{\alpha \beta}$$

where

$$T^{\gamma \delta} \alpha \beta = \frac{1}{4} \left( \Sigma_{AB} \right)^{\gamma} \alpha \alpha \psi^{\alpha \delta} + \delta^{\gamma} \alpha \frac{1}{4} \left( \Sigma_{AB} \right)^{\beta \alpha \delta}$$

In matrix notation the above relations read

$$\bar{\delta} \psi = T \psi$$

$$T = \frac{1}{4} \Sigma_{AB} \otimes 1 \alpha \alpha + 1 \otimes \frac{1}{4} \Sigma_{AB} \beta \beta$$

Eq. (210) can be directly compared with eq. (209) and so we obtain

$$H^{\gamma \delta} \alpha \beta = \frac{1}{4} \left( \Sigma_{AB} \right)^{\gamma} \alpha \alpha \beta \beta + \delta^{\gamma} \alpha \frac{1}{4} \left( \Sigma_{AB} \right)^{\beta \alpha \beta} = T^{\gamma \delta} \alpha \beta$$

Using eqs. (204) we can isolate the $\epsilon_{A} B$ occurring in eq. (213):

$$\epsilon^{C} \frac{D}{D} = H^{C \gamma \delta} \left[ \frac{1}{4} \left( \Sigma_{AB} \right)^{\gamma} \alpha \alpha \beta \beta + \delta^{\gamma} \alpha \frac{1}{4} \left( \Sigma_{AB} \right)^{\beta \alpha \beta} \right] H^{D \alpha \beta}$$

Alternatively, we can derive the above relation from

$$X' = x^{A} \gamma A = x^{A} R \gamma S = L^{A} B x^{B} \gamma A$$

Multiplying the latter equation by $\gamma^{C} \gamma$ and taking the scalar part, we find

$$x^{A} \left( \gamma^{C} \gamma \right) \gamma A R S = L^{C} B x^{B}$$
Since $x^A$ is arbitrary, we have

$$\langle \gamma^C \delta R \gamma_D S \rangle_S = L^C_D$$  (217)

For an infinitesimal transformation we obtain

$$\langle \gamma^C \delta^{1/4}(\Sigma_{AB} \alpha^{AB} \gamma_D + \gamma_D \Sigma_{AB} \beta^{AB}) \rangle_S = \epsilon^C_D$$  (218)

Using

$$\gamma^C = H^C \xi^C, \quad \gamma_D = H_D \xi_D$$  (219)

eq (218) becomes

$$\epsilon^C_D = H^C \xi^C \left( \langle \xi^C \xi_D \Sigma_{AB} \rangle_S \alpha^{AB} + \langle \xi^C \xi_D \Sigma_{AB} \rangle_S \beta^{AB} \right) H_D \xi_D$$  (220)

Using the cyclic property of the operation $\langle \rangle_S$, namely,

$$\langle \xi^C \xi_D \Sigma_{AB} \rangle_S = \langle \xi_D \xi^C \Sigma_{AB} \rangle_S = \delta^C_D \alpha_{\beta} \gamma^{\delta}_{\gamma} \delta^{\alpha}_{\beta}$$  (221)

we obtain

$$\langle \xi^C \xi_D \Sigma_{AB} \rangle_S = \langle \xi^C \xi_D \Sigma_{AB} \rangle_S = \Sigma_{AB} \gamma^{\delta}_{\gamma} \delta^{\alpha}_{\beta}$$  (222)

$$\langle \xi_D \xi_{AB} \gamma^{\delta}_{\gamma} \delta^{\alpha}_{\beta} \rangle_S = \delta^C_D \alpha_{\beta} \gamma^{\delta}_{\gamma} \delta^{\alpha}_{\beta}$$  (223)

Inserting the latter relation into eq. (220) we see that it is the same as eq. (214).

Let us now use relation (213). Then for the matrix elements of the generator (201) we obtain

$$\langle \xi^C \xi_D \gamma^M \delta \Psi \xi_D \rangle_S = \langle \xi^C \xi_D \gamma^M \delta \Psi \xi_D \rangle_S = \langle \gamma^M \Psi \rangle_{\rho \gamma} \delta \psi^\delta = i \psi^{* \sigma \rho} (\gamma^M)_{\rho \gamma} T^\gamma_{\alpha \beta} \psi_{\alpha \beta}$$  (224)

where $T^\gamma_{\alpha \beta}$ is given in eq. (211).

Let us now return to the generator (201). Sandwiching eq. (196) between the spinors $\xi^C$ and $\xi_D \equiv \xi^{C1}$, and taking the scalar part, we obtain for the first term

$$i \langle \xi^C \xi_D \gamma^M \delta \Psi \xi_D \rangle_S = i \langle \xi^C \xi_D \gamma^M \delta \Psi \xi_D \rangle_S = \langle \gamma^M \Psi \rangle_{\rho \gamma} \delta \psi^\delta = i \psi^{* \sigma \rho} (\gamma^M)_{\rho \gamma} T^\gamma_{\alpha \beta} \psi_{\alpha \beta}$$  (225)

where $T^\gamma_{\alpha \beta}$ is given in eq. (211).

Let us now use relation (213). Then for the matrix elements of the generator (201) we obtain

$$\langle \xi^C \xi_D \gamma^M \delta \Psi \xi_D \rangle_S = \langle \xi^C \xi_D \gamma^M \delta \Psi \xi_D \rangle_S = \langle \gamma^M \Psi \rangle_{\rho \gamma} \delta \psi^\delta = i \psi^{* \sigma \rho} (\gamma^M)_{\rho \gamma} T^\gamma_{\alpha \beta} \psi_{\alpha \beta}$$  (226)

In both parts that contribute to the generator we have now the same parameters $\epsilon^{AB}$ of an infinitesimal rotation in $C$-space. If we require that under a transformation generated by $G^M$ the action remains invariant, i.e., $\delta I = 0$, then $G^M$ is conserved generator:
\[(1/\sqrt{|G|})\partial_M(\sqrt{|G|}G^M) = D_M G^M = 0.\] Considering flat C-space, we may choose a coordinate system in which metric \(G_{MN}\) and \(\det G_{MN}\equiv G \neq 0\) are constant, so that the conservation law is simply \(\partial_M G^M = 0.\) Since the transformation parameters \(\epsilon^{AB}\) are arbitrary we also have

\[\partial_M J^M_{AB} = 0\] (226)

where

\[\langle \xi^\dagger J^M_{AB} \xi^\delta \rangle_S = i\psi^\dagger \sigma^\rho (\gamma^M)_\rho \left[ H^\gamma_A H_B \delta_\beta + (x_A \partial_B - x_B \partial_A) \delta^\gamma_\alpha \delta^\delta_\beta \right] \psi^\alpha \beta \] (227)

is the Noether current belonging to our generalized Dirac action (169). It consists of the generalized spin part and the generalized orbital part, and is conserved in flat C-space.

Alternatively, we can express the \(\epsilon^{AB}\) occurring in the second term of the generator (204) in terms of \(\alpha^{AB}\) and \(\beta^{AB}\) according to eq. (214). The generator (201) then reads

\[G^M = i\psi^\dagger \gamma^M 1/4 (\Sigma_{AB} \alpha^{AB} \Psi + \Psi \Sigma_{AB} \beta^{AB}) + i\psi^\dagger \gamma^M (x_C \partial_D - x_D \partial_C) \Psi H^C \gamma^\gamma_\alpha (\Sigma_{AB})^\gamma_\alpha \delta^\delta_\beta H^\alpha^{BD} \] (228)

From the fact that parameters \(\alpha^{AB}\) and \(\beta^{AB}\) are arbitrary, we read the corresponding Noether current. It consists of the left part due to the left transformations (125)

\[J^M_{AB}(\text{left}) = i\psi^\dagger \gamma^M 1/4 \Sigma_{AB} \Psi + i\psi^\dagger \gamma^M (x_C \partial_D - x_D \partial_C) \Psi H^C \gamma^\gamma_\alpha (\Sigma_{AB})^\gamma_\alpha H^{\alpha^{BD}} \] (229)

and the part due to the right transformations (127)

\[J^M_{AB}(\text{right}) = i\psi^\dagger \gamma^M \Psi 1/4 \Sigma_{AB} + i\psi^\dagger \gamma^M (x_C \partial_D - x_D \partial_C) \Psi H^C \gamma^\gamma_\alpha \delta^\delta_\beta (\Sigma_{AB})^\delta_\beta H^{\alpha^{BD}} \] (230)

In this approach the charges that generate gauge transformations are on the same footing as the spin and orbital angular momentum. This provides a natural unification of spin and charges, a program that has been pursued for many years by Mankoč [16], using different methods and by employing extra spacetime dimensions.

5.3 The gauge field potentials and gauge field strengths

5.3.1 Gauge field potentials as parts of C-space spin connection

Using eq. (118) we can express the spin connection in terms of the generators \(\Sigma_{AB}\):

\[\Gamma_M = 1/4 \Omega^{AB}_N \Sigma_{AB} = A_M^A \gamma_A, \quad A_M^A = 1/4 \Omega^{CD}_N f_{CD}^A \] (231)
The matrices representing the Clifford numbers $\Gamma_M$ can be calculated according to

$$\langle \xi^\dagger \Gamma_M \xi_B \rangle_S = \Gamma_M \bar{\xi}^A \bar{\xi}^B \equiv \Gamma_M$$

(232)

This is the spin connection entering the $C$-space Dirac equation written in the forms (104)–(106).

The $C$-space Dirac equation can be split according to

$$[\gamma^\mu (\partial_\mu + \Gamma_\mu) + \gamma^\bar{M} (\partial_{\bar{M}} + \Gamma_{\bar{M}})] \psi = 0$$

(233)

where $M = (\mu, \bar{M})$, and $\bar{M}$ assumes all the values except $M = \mu = 0, 1, 2, 3$.

From eq. (231) we read that the gauge fields $\Gamma_M$ contain:

(i) The spin connection of the 4-dimensional gravity $\Gamma_\mu^{(4)} = \frac{1}{8} \Omega^{ab}_\mu [\gamma_a, \gamma_b]$

(ii) The Yang-Mills fields $A_\mu \bar{A}_\gamma A$, where we have split the local index according to $A = (a, \bar{A})$. For $\bar{A} = \bar{o}$ (i.e., for the scalar) the latter gauge field is just that of the U(1) group.

(iii) The antisymmetric potentials $A^o_M \equiv A_M = A_\mu, A_{\mu\nu}, A_{\mu\nu\rho}, A_{\mu\nu\rho\sigma}$, if we take indices $A = o$ (scalar) and $M = \mu, \mu\nu, \mu\nu\rho, \mu\nu\rho\sigma$.

(iv) Non abelian generalization of the antisymmetric potentials $A^{a\mu\nu\rho\sigma}_M$ and $A^{\bar{A}\mu\nu\rho\sigma}$.

We see that the $C$-space spin connection contains all physically interesting fields, including the antisymmetric gauge field potentials which occur in string and brane theories.

A caution is in order here. From eq. (231) it appears as if there were only 16 independent generators $\gamma_A$ in terms of which a gauge field is expressed. But inspecting the generators (118) we see that there are more than 16 different rotations in $C$-space. However, some of them, although being physically different transformations, turn out to be mathematically described by the same objects. For instance, the generators $\Sigma_{ab} = \frac{1}{2} [\gamma_a, \gamma_b]$ and $\Sigma_{\bar{a}\bar{b}} = \frac{1}{2} [\gamma_{\bar{a}} \gamma_5, \gamma_{\bar{b}} \gamma_5] = \frac{1}{2} [\gamma_{\bar{a}}, \gamma_{\bar{b}}]$ are equal, although the corresponding transformations, i.e., a rotation in the subspace $M_4$ and the rotation in the dual space $\bar{M}_4$ are in principle independent. Similarly we have $\Sigma_{11} = \frac{1}{2} [\gamma_1, \gamma_5 \gamma_1] = \gamma_5 = \Sigma_{\bar{o}\bar{o}} = 1 \gamma_5$. Such degeneracy of the transformations is removed by the fact that the transformation can act on the generalized spinor polyvector $\Psi$ either from the left or from the right (according to (119)).

Returning to the Dirac equation (233) we see that besides the part having essentially the same form as the ordinary Dirac equation in the presence of minimally coupled 4-dimensional spin connection and Yang-Mills fields, there is also an extra term which can have the role of mass, if $\psi$ is an eigenstate of the operator $\gamma^\bar{M} (\partial_{\bar{M}} + \Gamma_{\bar{M}})$. Since the
metric signature of \( C \)-space is \([31]\) \((8+, 8-)\), and the signature of the “internal” space is \((7+, 5-)\), the mass is not necessarily of the order of the Planck mass; it can be small due to cancellations of the positive and negative contributions.

### 5.3.2 Gauge field strengths as parts of \( C \)-space curvature

Using eq. (103) we can calculate the curvature according to

\[
[\partial_M, \partial_N]A^\xi_\tilde{\xi} = R_{MN}^{\tilde{\xi} \tilde{\xi}} B \xi_\tilde{B}
\]

(234)

where

\[
R_{MN}^{\tilde{\xi} \tilde{\xi}} = \partial_M \Gamma_N^{\tilde{\xi} \tilde{\xi}} - \partial_N \Gamma_M^{\tilde{\xi} \tilde{\xi}} + \Gamma_M^{\tilde{\xi} \tilde{\xi} \tilde{\gamma}} \Gamma_N \tilde{\gamma} - \Gamma_N^{\tilde{\xi} \tilde{\xi} \tilde{\gamma}} \Gamma_M \tilde{\gamma}
\]

(235)

This is the relation for the Yang-Mills field strength. In matrix notation it reads:

\[
R_{MN} = \partial_M \Gamma_N - \partial_N \Gamma_M + [\Gamma_M, \Gamma_N]
\]

(236)

Inserting eqs. (231), (232) into eq. (236) we obtain

\[
F_{MN}^{\xi_\tilde{\xi}} = \partial_M A_N^{\xi_\tilde{\xi}} - \partial_N A_M^{\xi_\tilde{\xi}} + A_M^{\xi_\tilde{\xi}} A_N^{\xi_\tilde{\xi}} C_{BC}^{\xi_\tilde{\xi}}
\]

(237)

where \( C_{BC}^{\xi_\tilde{\xi}} \) are the structure constants of the Clifford algebra:

\[
[\gamma_A, \gamma_B] = C_{AB}^{\xi_\tilde{\xi}} \gamma_C
\]

(238)

From the curvature we can form the invariant expressions, for instance

\[
R_{MN}^{\xi_\tilde{\xi} \tilde{\gamma}} (\gamma_M^{\xi_\tilde{\xi} \gamma} \gamma_N^{\xi_\tilde{\gamma} \gamma} \gamma_B) = R_{MN}^{\xi_\tilde{\xi} \gamma \gamma} (\gamma_M^{\xi_\tilde{\xi} \gamma} \gamma_N^{\xi_\tilde{\gamma} \gamma} \gamma_B) = R_{MN}^{\xi_\tilde{\xi} \gamma \gamma} e^M_A e^N_B = R
\]

(239)

which is linear in curvature, and

\[
R_{MN}^{\xi_\tilde{\xi}} R_{MN}^{\xi_\tilde{\xi}} = R_{MN}^{\xi_\tilde{\xi} \gamma \gamma} R_{MN}^{\xi_\tilde{\xi} \gamma \gamma}
\]

(240)

which is quadratic in curvature. Instead of the form (240) we can use (237) and take its square:

\[
F_{MN}^{\xi_\tilde{\xi}} F_{MN}^{\xi_\tilde{\xi}} = R_{MN}^{\xi_\tilde{\xi} \gamma \gamma} R_{MN}^{\xi_\tilde{\xi} \gamma \gamma}
\]

(241)

The action for our system thus contains the term (169) for the generalized Dirac field \( \psi^{\tilde{A}} \) and the kinetic term for gauge fields:

\[
I[A_M^{\xi_\tilde{\xi}}] = \int dx^2 \sqrt{|G|} (\alpha R + \beta F_{MN}^{\xi_\tilde{\xi}} F_{MN}^{\xi_\tilde{\xi}})
\]

(242)

Here \( \alpha \) and \( \beta \) are the coupling constants. The first term in the action (242) is a generalization of the Einstein gravity, whereas the second term generalizes the higher derivative gravity to \( C \)-space. Other terms of the \( R^2 \) type can also be added to the action (242).
5.3.3 Conserved charges and isometries

In curved space in general there are no conserved quantities, unless there exist isometries which are described by Killing vector fields. Suppose we have a curved Clifford space which admits $K$ Clifford numbers $k^\alpha = k^\alpha_M \gamma^M$, $\alpha = 1, 2, \ldots, K$; $M = 1, 2, \ldots, 16$, where the components satisfy the condition for isometry, namely

$$D_N k^\alpha_M + D_M k^\alpha_N = 0 \quad (243)$$

the covariant derivative being defined in eq. (149). We assume that such $C$-space with isometries is not given ad hoc, but is a solution to the generalized Einstein equations that arise from the action which contains the “matter” term (169) and the field term (242).

Taking a coordinate system in which $k^\alpha_{\mu} = 0$, $k^\alpha_{\bar{M}} \neq 0$, $\mu = 0, 1, 2, 3$, $\bar{M} \neq \mu$, the metric and vielbein can be written as

$$G_{MN} = \left( \begin{array}{cc} G_{\mu\nu} & G_{\mu\bar{M}} \\ G_{\bar{M}\nu} & G_{\bar{M}\bar{N}} \end{array} \right) \quad , \quad e^A_M = \left( \begin{array}{c} e^a_{\mu} \\ e^{\bar{A}}_{\bar{M}} \end{array} \right) \quad (244)$$

Here

$$e^{\bar{A}}_{\bar{M}} = 0 \quad (245)$$

whilst the components $e^A_{\mu}$ can be written in terms of Killing vectors and gauge fields $W^\alpha_{\mu}(x^\mu)$:

$$e^A_{\mu} = e^{\bar{A}}_{\bar{M}} k^\alpha_{\bar{M}} W^\alpha_{\mu}, \quad \partial_{\bar{M}} W^\alpha_{\mu} = 0 \quad (246)$$

If we set the $C$-space torsion to zero and calculate the connection $\Omega_{ABM}$, given in eq. (45), by using eqs. (244)–(246), we obtain an analogous result as given, e.g., in ref. [49]:

$$\Omega_{\bar{M}\bar{N}\mu} = \frac{1}{2} k^\alpha_{[\bar{M},\bar{N}]} W^\alpha_{\mu} \quad (247)$$

where $k^\alpha_{[\bar{M},\bar{N}]} \equiv \partial_{\bar{N}} k^\alpha_{\bar{M}} - \partial_{\bar{M}} k^\alpha_{\bar{N}}$

Let us now rewrite the $C$-space Dirac equation by using eqs. (243)–(247). Omitting the terms due to the $C$-space torsion, we obtain

$$\left[ \gamma^{(4)}_{\mu} \left( \partial_\mu - \Omega_{ab\mu} \frac{1}{8} [\gamma^a, \gamma^b] - q^\alpha W^\alpha_{\mu} + \ldots \right) + \gamma^{\bar{M}} \partial_{\bar{M}} + \ldots \right] \psi = 0 \quad (248)$$

where $\gamma^{(4)}_{\mu} = e^a_{\mu} \gamma^a$ are 4-dimensional basis vectors in coordinate frame, and

$$q^\alpha = k^\alpha_{\bar{M}} \partial_{\bar{M}} + \frac{1}{8} k^\alpha_{[\bar{M},\bar{N}]} e^\bar{M}_{A} e^B_{\bar{N}} \Sigma^{AB} \quad (249)$$

---

23 This is a $C$-space analog of the Kaluza-Klein splitting usually performed in the literature. See, e.g., [49, 50].
are the charges, conserved due to the presence of isometries $k^\alpha \bar{M}$. They are the sum of the coordinate part and the contribution of the spin angular momentum in the “internal” space, spanned by $\gamma^M$. The coordinate part is the projection of the linear momentum onto the Killing vectors, and can in particular be just the orbital angular momentum of the “internal” part of $C$-space. The first term that contributes to the charge $q^\alpha$ comes from the vielbein according to eq. (246), whilst the second term comes from the connection according to eq. (247). Both terms couple to the same 4-dimensional gauge fields $W_\mu^\alpha$, where the index $\alpha$ denotes which gauge field (which Killing vector), and should not be confused with the spinorial index, used in Dirac matrices.

In eq. (248) we explicitly displayed only the most relevant terms which contain the ordinary vierbein $e^a_\mu$ and spin connection $\Omega_{ab\mu}$ (describing gravity and torsion), and also Yang-Mills gauge fields $W_\mu^\alpha$ which, as shown in eqs. (246), (247), occur in the $C$-space vielbein and in the $C$-space “spin” connection. We omitted the terms arising from the $C$-space torsion.

6 Discussion

A motivation for this study is a very promising possibility that the generalized spinors (polyvectors) $\Psi = \psi^A \xi_A$ provide a representation of the gauge group $U(1) \times SU(2) \times SU(3)$ of the standard model, and that this group is incorporated in the group $GL(4,\mathbb{C}) \times GL(4,\mathbb{C})$ formed by the transformations (119). The general spinors $\Psi$ in fact form a representation of the group $GL(4,\mathbb{C}) \times GL(4,\mathbb{C})$ which is further restricted by the requirement that the quadratic form $\langle \Psi^\dagger \Psi \rangle = \psi^A \bar{Z}_{\tilde{A}\tilde{B}} \psi_{\tilde{B}}$ should be invariant under the transformations (119). The generalized spinor metric $Z_{\tilde{A}\tilde{B}} = \langle \xi^\dagger_A \xi_B \rangle_S$ encompasses sixteen basis spinors, four for each left ideal. If the basis spinors are constructed according to the lines as suggested in eqs. (52)–(60), one finds that there are several possibilities, giving different signatures of the spinor metric. Two cases of possible signature that arise in such a construction are of particular interest:

Case (i):

$$\text{Sig}(Z_{\tilde{A}\tilde{B}}) = \begin{pmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{pmatrix}$$  \hspace{1cm} (250)
Case (ii):

\[ \text{Sig}(Z_{\overline{A}\overline{B}}) = \begin{pmatrix} + & - & - & - \\ + & - & - & - \\ - & + & + & + \\ - & + & + & + \end{pmatrix} \]  

(251)

We see that, while in Case (i) the right ideals have signature (++−−) or (−−++), whilst in Case (ii) the right ideals have signature (+−−−) or (−+++).

In Case (i) the group of the left transformations \( R \in \{ R \} \) contains \( U(2,2) \), and the group of the right transformations \( S \in \{ S \} \) contains \( U(2,2) \) as well.

In Case (ii) the group of the left transformations still contains \( U(2,2) \), whereas the group of the right transformations contains \( U(1,3) \).

A subgroup \( SU(3) \) naturally occurs in \( U(1,3) \), just as \( SO(3) \) occurs within the Lorentz group \( SO(1,3) \). We have here a possibility of associating leptons with the first left ideal, and three color states of quarks with the remaining three left ideals (columns). The group \( SU(3) \), which operates from the right, mixes three color states. In addition there occurs a transformation, analogous to a boost, and it mixes leptons and quarks. However, such transitions, according to our model, cannot occur spontaneously; we expect that they require a huge amount of energy, and also higher grade components of momentum, therefore normally they cannot be observed\(^{24}\).

Our total group \( \{ R \} \times \{ S \} \), in Case (ii), contains subgroups \( SL(2,C) \) and \( U(1) \times SU(2) \times SU(3) \). The former group describes the Lorentz transformations in Minkowski space (which is a subspace of flat \( C \)-space\(^{25}\)), whilst the latter group coincides with the gauge group of the standard model which describes electroweak and strong interaction. Whether this indeed provides a description of the standard model remains to be fully investigated. But there is further evidence in favor of the above hypothesis in the fact that a polyvector field \( \Psi = \psi^{\overline{A}} \xi_{\overline{A}} \) has 16 complex components. Altogether it has 32 real components. This number matches, for one generation, the number of independent states for spin, weak isospin and color (together with the corresponding antiparticle states) in the standard model. A complex polyvector field \( \Psi \) has thus enough degrees of freedom to form a representation of the group \( GL(4,C) \times GL(4,C) \) which contains the Lorentz group \( SL(2,C) \) and the group of the standard model \( U(1) \times SU(2) \times SU(3) \). The generators of the

\(^{24}\)In ref. (17) we demonstrated that boosts mixing, for instance, vector and bivector or threevector coordinates indeed require large amounts of energy.

\(^{25}\)If \( C \)-space manifold is curved, then we consider one of its tangents spaces \( T_X(C) \), everyone of which contains Minkowski space as a subspace.
group are given by $\Sigma_{AB}$ defined in eq. [89].

In our model, assigning the generalized spin metric signature [251], we do not need artificially kill pieces of an SU(2n) in order to obtain SU(3). Instead, we have SU(1,3) operating from the right, and SU(3) is a natural subgroup. Since starting from a different model, to obtain SU(3) most authors have to do very unmotivated “turning-off” of degrees of freedom. Chisholm and Farwell [21] do have polyvector wavefunctions, but they do not fully utilize all the possible degrees of freedom. Rather they simulate a column spinor, with the rest of the matrix filled with zero. Instead they consider extra dimensions of spacetime. Hence they ended up with a really big Clifford algebra operating only from the left. A different approach, using octonions, is proposed by Dixon [51], and recently by Dray and Manogue [52].

7 Conclusion

Spacetime manifold $V_4$ can be elegantly described by means of the basis vectors which are generators of Clifford algebra. The latter algebra describes a geometry which goes beyond spacetime: the ingredients are not only points, but also 2-surfaces, 3-volumes, 4-volumes and scalars. All those geometric objects form a 16-dimensional manifold, called Clifford space, or shortly, $C$-space. It is quite possible that the arena for physics is not spacetime, but Clifford space. And the arena itself can become a part of the play, if we assume that Clifford space is curved, and that its curvature is a dynamical quantity entering the action functional. We have thus a higher dimensional curved differential manifold. From now on we can proceed à la Kaluza-Klein. Since the “extra dimensions” are assumed to be related to the physical degrees of freedom, due to the extended nature of physical objects, there is no need to compactify the 12-dimensional “internal” part of $C$-space.

The theory that we pursue here has not only the prospects for providing a clue to the unification of fundamental interactions. It provides a framework for a generalized relativistic dynamics, including generalized gravity, which might find its useful applications in astrophysics and cosmology, which are fast developing fields, where many surprises has already taken place, and more are to be expected on the way.

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References

[1] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory* (Cambridge University Press, Cambridge, 1987); M. Kaku, *Introduction to Superstrings* (Springer-Verlag, New York, 1988); U. Danielsson, *Rep. Progr. Phys.* 64, 51 (2001)

[2] J. Polchinski, “Lectures on D-branes,” arXiv:hep-th/9611050; W. I. Taylor, “Lectures on D-branes, gauge theory and M(atrix),” arXiv:hep-th/9801182; H. Nicolai and R. Helling, “Supermembranes and M(atrix) theory,” arXiv:hep-th/9809103; J.H.Schwarz, Phys. Rep. 315, 107 (1999).

[3] L. Randall and R. Sundrum, *Phys. Rev. Lett.* 83, 4690 (1999) arXiv:hep-th/9906061; M. J. Duff, “Benchmarks on the brane,” arXiv:hep-th/0407175

[4] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B125 (1983) 136; K. Akama, in Proceedings of the Symposium on Gauge Theory and Gravitation, Nara, Japan, eds. Kikkawa, N. Nakanishi and H. Nariai (Springer-Verlag, 1983), hep-th/0001113; M. Visser, *Phys. Lett.* B159, 22 (1985); G. W. Gibbons and D. L. Wiltshire, *Nucl. Phys.* B287, 717 (1987); M. Pavšič *Phys. Lett.* A116, 1 (1986) arXiv:gr-qc/0101075; Nuov. Cim. A95, 297 (1986); M. Pavšič and V. Tapia, “Resource letter on geometrical results for embeddings and branes,” arXiv:gr-qc/0010045

[5] M. Pavšič, *The Landscape of Theoretical Physics: A Global View; From Point Particles to the Brane World and Beyond, in Search of Unifying Principle* (Kluwer Academic, Dordrecht 2001).

[6] M. Pavšič, *Phys. Lett.* A283, 8 (2001) arXiv:hep-th/0006184; *Grav. Cosmol.* 2, 1 (1996) arXiv:gr-qc/9511020; *Found. Phys.* 26, 159 (1996) arXiv:gr-qc/9506057; *Found. Phys.* 24, 1495 (1994).

[7] T. Regge and C. Teitelboim, in: Proceedings of the Marcel Grossman Meeting (Trieste, 1975); M. Pavšič, *Class. Quant. Grav.* 2 869 (1985); *Phys. Lett.* A107, 66 (1985). D.Maia, *Phys. Rev.* D31, 262 (1985); *Class. Quant. Grav.* 6, 173 (1989); V. Tapia, *Class. Quant. Grav.* 6, L49 (1989); T. Hori, *Phys. Lett.* B222, 188 (1989); A. Davidson and D. Karasik, *Mod. Phys. Lett.* A 13, 2187 (1998); A. Davidson, *Class. Quant. Grav.* 16, 653 (1999); A. Davidson, D. Karasik and Y. Lederer, *Class. Quant. Grav.* 16, 1349 (1999).

[8] A. Aurilia, A. Smailagic and E. Spallucci, *Phys. Rev.* D47, 2536 (1993); A. Aurilia and E. Spallucci, *Class. Quant. Grav.* 10, 1217 (1993).

[9] A. Schild, *Phys. Rev.* D16, 1722 (1977).

[10] T. Eguchi, *Phys. Rev. Lett.* 44, 126 (1980).

[11] S. Ansoldi, A. Aurilia, C. Castro and E. Spallucci, Phys. Rev. D64, 026003 (2001) arXiv:hep-th/0105027.

[12] C. Castro, *Chaos, Solitons and Fractals* 10, 295 (1999); 11, 1663 (2000); 12, 1585 (2001); “The Search for the Origins of M Theory: Loop Quantum Mechanics, Loops/Strings and Bulk/Boundary Dualities”, arXiv: hep-th/9809102.
[13] C. Castro, *Found. Phys.* **30**, 1301 (2000).

[14] M. Pavšić, “Clifford Algebra as a Usefull Language for Geometry and Physics”, *Proceedings of the 38. Internationale Universitätswochen für Kern- und Teichenphysik*, Schladming, January 9–16, 1999, p. 395.

[15] M. Pavšić, *Found. Phys.* **31**, 1185 (2001) [arXiv:hep-th/0011216]; M. Pavšić, NATO Sci. Ser. II **95**, 165 (2003) [arXiv:gr-qc/0210060].

[16] A. Aurilia, S. Ansoldi and E. Spallucci, *Class. Quant. Grav.* **19**, 3207 (2002) [arXiv:hep-th/0205028].

[17] M. Pavšić, *Found. Phys.* **33**, 1277 (2003) [arXiv:gr-qc/0211085].

[18] D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966); D. Hestenes and G. Sobcyk, *Clifford Algebra to Geometric Calculus* (D. Reidel, Dordrecht, 1984).

[19] P. Lounesto, *Clifford Algebras and Spinors* (Cambridge University Press, Cambridge, 2001); B. Jancewicz, *Multivectors and Clifford Algebra in Electrodynamics* (World Scientific, Singapore, 1988); R. Porteous, *Clifford Algebras and the Classical Groups* (Cambridge University Press, 1995); W. Baylis, *Electrodynamics, A Modern Geometric Approach* (Boston, Birkhauser, 1999); A. Lasenby and C. Doran, Geometric Algebra for Physicists (Cambridge U. Press, Cambridge 2002); *Clifford Algebras and their applications in Mathematical Physics, Vol 1: Algebras and Physics*, eds by R. Ablamowicz, B. Fauser; Vol 2: Clifford analysis, eds by J. Ryan, W. Sprosig (Birkhauser, Boston, 2000); A.M.Moya, V.V Fernandez and W.A. Rodrigues, *Int. J. Theor. Phys.* **40**, 2347 (2001) [arXiv: math-ph/0302007]; “Multivector Functions of a Multivector Variable” [arXiv: math.GM/0212223]; Multivector Functionals [arXiv: math.GM/0212224]; W.A. Rodrigues, Jr, J. Vaz, Jr, Adv. Appl. Clifford Algebras **7**, 457 (1997); E.C de Oliveira and W.A. Rodrigues, Jr, *Ann. der Physik* **7**, 654 (1998). *Phys. Lett A291*, 367 (2001). W.A. Rodrigues, Jr, J.Y.Lu, *Foundations of Physics* **27**, 435 (1997).

[20] F.D. Smith, Jr, *Intern. J. Theor. Phys.* **24**, 155 (1985); **25**, 355 (1985); G. Trayling and W.E. Baylis, *Int. J. Mod. Phys.* **A16**, Suppl. 1C (2001) 900; *J. Phys. A: Math. Gen.* **34**, 3309 (2001); G. Roepstorff, “A class of anomaly-free gauge theories,” [arXiv:hep-th/0005079] “Towards a unified theory of gauge and Yukawa interactions,” [arXiv:hep-ph/0006065] “Extra dimensions: Will their spinors play a role in the standard model?,” [arXiv:hep-th/0310092] F. D. Smith, “From sets to quarks: Deriving the standard model plus gravitation from simple operations on finite sets,” [arXiv:hep-ph/9708379] .

[21] J.S.R. Chisholm and R.S. Farwell, *J. Phys. A: Math. Gen.* **20**, 6561 (1987); **33**, 2805 (1999); **22**, 1059 (1989); J.S.R. Chisholm, *J. Phys. A: Math. Gen.* **35**, 7359 (2002); *Nuov. Cim. A82*, 145 (1984); 185; 210; “Properties of Clifford Algebras for Fundamental Particles”, in *Clifford (Geometric) Algebras*, W. Baylis ed., Birkhauser (1996), Chapter 27, pp. 365–388.

[22] J.P. Crawford, *J. Math. Phys.* **35**, 2701 (1994); in *Clifford (Geometric) Algebras*, W. Baylis ed., Birkhauser (1996), Chapters 21–26, pp. 297–364; *Class. Quant. Grav.* **20**, 2945 (2003).

[23] W. Pezzaglia, “Physical Applications of a Generalized Geometric Calculus,” in *Dirac Operators in Analysis* (Pitman Research Notes in Mathematics, Number 394), J. Ryan and D. Struppa eds., (Longmann 1997) pp.191–202 [arXiv: gr-qc/9710027]. “Dimensionally Democratic calculus and Principles of Polydimensional Physics,” in *Clifford Algebras and their
Applications in Mathematical Physics, R. Ablamowicz and B. Fauser eds. (Birkhauser 2000), pp.101–123,[arXiv: gr-qc/9912025]; “Classification of Multivector Theories and Modifications of the Postulates of Physics”, in Clifford Algebras and their Applications in Mathematical Physics, Brackx, Delanghe & Serras eds., (Kluwer 1993) pp.317–323, [arXiv: gr-qc/9306006].

[24] W. M. . Pezzaglia and A. W. Differ, “A Clifford Dyadic superfield from bilateral interactions of geometric multispin Dirac theory,” Adv. Appl. Cliff. Alg. Proc. Suppl. 4 (S1) (1994) pp. 437–446 [arXiv:gr-qc/9311015]; W. M. Pezzaglia, “Polydimensional Relativity, a Classical Generalization of the Automorphism Invariance Principle,” in V.Dietrich et al. (eds.) Clifford Algebras and their Applications in Mathematical Physics, 305-317 (Kluwer Academic Publishers, 1998) [arXiv:gr-qc/9608052]; W. M. Pezzaglia and J. J. Adams, “Should metric signature matter in Clifford algebra formulations of physical theories?,” Summary of talk at American Mathematical Society, Oregon State University, Corvallis, Oregon, (1997)(unpublished) [arXiv:gr-qc/9704048]; W. Pezzaglia, Found. Phys. Lett. 5, 57 (1992).

[25] C. Castro and M. Pavšič, Phys. Lett. B539, 133 (2002) [arXiv:hep-th/0110079].

[26] C. Castro and M. Pavšič, Progr. Phys. 1, 31 (2005).

[27] M. Riesz, “Sur certaines notions fondamentales en théorie quantiques relativiste’, in Dixième Congrès Math. des Pays Scandinaves, Copenhagen, 1946 (Jul. Gjellerups Forlag, Copenhagen, 1947), pp. 123-148; M. Riesz, Clifford Numbers and Spinors, E. Bolinder and P. Lounesto (eds.), (Kluwer 1993).

[28] S. Teitler, Supplemento al Nuovo Cimento III, 1 (1965) and references therein; Supplemento al Nuovo Cimento III, 15 (1965); J. Math. Phys. 7, 1730 (1966); 7, 1739 (1966).

[29] N. S. Mankoč Borštnik and H. B. Nielsen, J. Math. Phys. 43, 5782 (2002) [arXiv:hep-th/0111257]; J. Math. Phys. 44, 4817 (2003) [arXiv:hep-th/0303224].

[30] W.A. Rodrigues, Jr., J. Math. Phys. 45, 2908 (2004).

[31] M. Pavšič, “Clifford space as a generalization of spacetime: Prospects for unification in physics,” [arXiv:hep-th/0411053]

[32] M. Pavšič, Found. Phys. 35, 1617 (2005) [arXiv:hep-th/0501222].

[33] C. Castro and M. Pavšič, Int. J. Theor. Phys. 42, 1693 (2003) [arXiv:hep-th/0203191].

[34] C. Castro, Found. Phys. 35, 971 (2005).

[35] M. Pavšič, Phys. Lett. B614, 85 (2005) [arXiv:hep-th/0412255].

[36] M. Kalb and D. Ramond, Phys. Rev. D9, 2273 (1974).

[37] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, Analysis, Manifolds and Physics (North-Holland Publ. Co, Amsterdam 1982)

[38] W.A. Rodrigues, Jr., and Q.A.G. Souza, Int. J. Mod. Phys. D 14 (2005) [arXiv: 0411085]
[39] A. Einstein, Annalen der Physik 49, 769 (1916); J. Stachel, “Einstein,s Search for General Covariance, 1912-1915”, paper read at the Ninth International Conference on General Relativity and Gravitation, Jena 1980; published in Einstein and the History of General Relativity, Einstein Studies, Vol.1, eds. D. Howard and J. Stachel (Birkhauser, Boston, 1985), pp.63-100; J. Norton, How Einstein found his Field Equations: 1912-1915, Historical Studies in the Physical Sciences 14, 252 (1984); Found. Phys. 19, 1215 (1989); Rep. Prog. Phys. 56, 791 (1993); J. Norton, “Einstein, the Hole Argument and the Reality of Space”, in Measurement, Realism and Objectivity, ed. J. Forge (Reidel, Dordrecht, 1987).

[40] B. S. DeWitt, in Gravitation: An Introduction to Current Research (Editor L. Witten, Wiley, New York, 1962)

[41] C. Rovelli, Classical and Quantum Gravity 8, 297 (1991); 8, 317 (1991)

[42] D. Hestenes, Am. J. Phys. 71 (7), 691 (2003).

[43] R. F. Marzke and J. A. Wheeler, in Gravitation and Relativity (Editors Hong-Yee Chiu and William F. Hoffmann, W.A. Benjamin, Inc., New York, Amsterdam, 1964), p. 40.

[44] J. L. Anderson, Principles of Relativity Physics (Academic Press, New York 11967).

[45] M. Pavšič, Class. Quant. Grav. 20, 2697 (2003) arXiv:gr-qc/0111092.

[46] N. S. Mankoč-Borštnik, “Unification of spins and charges in Grassmann space enables unification of all interactions,” arXiv:hep-th/9512050; J. Math. Phys. 36, 1593 (1995); “Unification of spins and charges enables unification of all interactions,” Preparad for Workshop on What Comes Beyond the Standard Model, Bled, Slovenia, 29 Jun - 9 Jul 1998; N. Mankoč-Borštnik and H. B. Nielsen, Phys. Rev. D 62, 044010 (2000) arXiv:hep-th/9911032; Int. J. Theor. Phys. 40, 315 (2001); A. Borštnik-Bračič and N. Mankoč-Borštnik, “The approach unifying spins and charges in SO(1,13) and its predictions,” Prepared for Euresco Conference on What Comes Beyond the Standard Model? Symmetries Beyond the Standard Model, Portorož, Slovenia, 12-17 Jul 2003 N. Mankoč-Borštnik, D. Lukman and H. B. Nielsen, “An example of Kaluza-Klein-like theories leading after compactification to massless spinors coupled to a gauge field: Derivations and proofs,” Prepared for 7th Workshop on What Comes Beyond the Standard Model, Bled, Slovenia, 19-30 Jul 2004

[47] W. G. Dixon, Proc. Roy. Soc. Lond. A314, 499 (1970).

[48] S. J. J. Gates, W. D. . Linch and J. Phillips, “When superspace is not enough,” arXiv:hep-th/0211034; M. Faux and S. J. . Gates, Phys. Rev. D71, 065002 (2005) arXiv:hep-th/0408004; M. R. de Traubenberg, “Clifford algebras in physics,” arXiv:hep-th/0506011; C. Castro, “Polyvector Super-Poincaré Algebra, M, F Theory Algebras and Generalized Supersymmetry in Clifford-Spaces” (Preprint, February, 2005).

[49] J. F Luciani, Nucl. Phys. B135, 111 (1978).

[50] E. Witten, Nucl. Phys. B186, 412 (1981).

[51] G. Dixon, Phys. Rev. D29, 1276 (1984); Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics, (Kluwer, 1994); “Division Algebras, (1,9)-Space-time, Matter-antimatter Mixing”, arXiv: hep-th/930303039; Nuov. Cim. B105, 349 (1990); Division Algebras, 26 Dimensions; 3 Families”, arXiv: hep-th/9902016.
[52] C.A. Manogue and T. Dray, Mod. Phys. Lett. A14, 99 (1999) [arXiv: hep-th/9807044]; T. Dray and C.A. Manogue, “Quaternionic Spin”, arXiv: hep-th/9910010.