On the breaking of collinear factorization in QCD

Jeffrey R. Forshaw, Michael H. Seymour, Andrzej Siódmok

Consortium for Fundamental Physics, School of Physics & Astronomy, University of Manchester, Oxford Road, Manchester M13 9PL. U.K.
E-mail: Jeffrey.Forshaw@manchester.ac.uk, Michael.Seymour@manchester.ac.uk, Andrzej.Siodmok@manchester.ac.uk

ABSTRACT: We investigate the breakdown of collinear factorization for non-inclusive observables in hadron-hadron collisions. For pure QCD processes, factorization is violated at the three-loop level and it has a structure identical to that encountered previously in the case of super-leading logarithms. In particular, it is driven by the non-commutation of Coulomb/Glauber gluon exchanges with other soft exchanges. Beyond QCD, factorization may be violated at the two-loop level provided that the hard subprocess contains matrix element contributions with phase differences between different colour topologies.

KEYWORDS: qcd, jet
1 Introduction

Factorization theorems are a cornerstone of QCD calculations. In particular, it is the inclusive collinear factorization theorem that allows the calculation of cross sections in hadron-hadron collisions as the convolution of universal (i.e. process independent) parton distribution functions and (perturbatively calculable) partonic cross sections.

The factorization theorem has been proven for sufficiently inclusive observables in the final state of the scattering of colourless hadrons \cite{1}. However, it is often assumed that partonic scattering amplitudes also factorize, e.g. in the calculation of amplitudes to high orders and their resummation to all orders. Moreover, this assumed factorization forms the basis upon which parton shower Monte Carlo event generators are constructed. In neither of these cases is the factorization guaranteed. Indeed, in a recent paper, Catani, de Florian and Rodrigo (CdFR) explicitly wrote down amplitude and cross section contributions that violate collinear factorization for non-inclusive processes with two incoming coloured partons \cite{2}.

In this paper we aim to shed light on the breakdown of collinear factorization. To that end, we re-consider the results presented in CdFR. Most of their paper concerns the question of whether there are factorization-violating terms in higher-order corrections to QCD amplitudes. The answer, for initial-state (space-like, SL) collinear branching in a process with two incoming coloured partons, is definitely Yes. CdFR derived an all-orders expression for the singularities of the collinear splitting amplitude and gave its explicit realization at one- and two-loop order. We discuss the structure of these results and isolate the factorization violating terms in a way which makes it clear that their physical origin is the Coulomb interaction between the two incoming partons long before the hard interaction or between two outgoing partons much later than the hard interaction.

CdFR then discuss whether these factorization violations at amplitude level lead to factorization violation at cross section level. The answer at one-loop order is definitely No.
CdFR does however calculate a contribution that violates cross section factorization at two-loop order. We show below that this result is, in fact, only non-zero for certain electroweak processes. For QCD scattering processes, factorization violation effects cancel and the answer at two-loop order remains No. In the following, we argue that this will not continue to be the case at still higher orders, and we do expect factorization violation at three-loop order in the SL collinear limit in parton-parton cross sections. We specify the form we expect for these terms.

This paper is set out as follows. In the remainder of this introduction, we briefly recall our work on the logarithmic structure of high-order corrections to non-inclusive QCD observables such as dijet production with a jet veto (often referred to as ‘gaps between jets’) [3, 4]. As we shall see, this is relevant because the “super-leading logarithms” we identified there have the identical structure and physical origin as the factorization violating terms discussed here. Then, in section 2, we recap some of the formalism used by CdFR in deriving results in the SL collinear limit. In section 3 we review the results of CdFR for the one-loop corrections to factorization of SL collinear emission at amplitude level. We rewrite the final result in a form in which it is manifest that the physical origin of the factorization violating term arises from an effective Coulomb interaction between the two incoming partons. We show that it does not contribute to physical cross sections at this order. In section 4 we consider the two-loop corrections at amplitude level. We confirm the CdFR results and again rewrite them in a form in which the connection with Coulomb gluons is manifest. In section 5 we show that the factorization violating terms do not contribute to QCD cross sections at this order either. We discuss the conditions under which they could give a non-zero contribution and, within the Standard Model, show that these are only satisfied by certain electroweak processes. Finally, in section 6 we make some concluding remarks, including a prediction of the form of the term that will violate the cross section factorization of collinear singularities at three-loop order (for coloured incoming partons).

In Refs. [3, 4] we studied the structure of the ‘gaps-between-jets’ cross section in QCD. As is well known (see for example the series of papers by Sterman and collaborators [5–7]), “leading” logarithmic terms $\sim \alpha_s^n \log^n(Q/Q_0)$, where $Q$ is the hard scale and $Q_0$ the veto scale, can be summed to all orders by a matrix-valued evolution in colour space. The anomalous dimension matrix that drives this receives a real (strictly, Hermitian) contribution from virtual gluons with light-like momenta and in what follows we shall refer to these as ‘eikonal gluons’. It also receives an imaginary (strictly, anti-Hermitian) contribution from space-like virtual gluons that correspond to rescattering between the partons long before or after the hard scattering process. We refer to these as ‘Coulomb gluons’, although they are also referred to as Glauber gluons in the literature.

In Ref. [3], we considered the first correction to this picture. Specifically, we allowed for one additional gluon, either real or virtual, with momentum corresponding to phase space regions lying outside the jet veto region\(^1\). Following the work of Dasgupta and Salam, [8],

---

\(^1\)The virtual corrections in the anomalous dimension matrix are instead integrated over the region of phase space where real emissions are forbidden, i.e. inside the jet veto region.
we expected to find a tower of additional leading logarithms, called non-global logarithms. Physically these arise from the requirement that although radiation outside the veto region is allowed, secondary radiation from it back into the veto region is not. We did find these non-global logarithms. To our surprise we also found that, even when the ‘out-of-gap’ gluon is far forward (i.e. collinear to an incoming parton), the subsequent colour evolution differs, depending on whether the low-angle gluon is real or virtual. This triggers a breakdown of the ‘plus prescription’ and to the conclusion that collinear logarithms can only be factorized into the incoming parton density functions at or below the scale $Q_0$. In subsequent conference presentations [9], we described this as a breakdown of QCD coherence and discussed further the fact that it signals a breakdown of collinear factorization for processes with incoming coloured partons. In the gaps-between-jets calculation, this mis-cancellation means that the integration over the momentum of the out-of-gap gluon is double-logarithmic and leads to the appearance of super-leading logarithms: one out-of-gap gluon contributes a tower $\sim \alpha_s^n \log^{n+1}(Q/Q_0)$, $n \geq 4$, and we speculated that the leading behaviour at high order is actually $\sim \alpha_s^n \log^{2n-3}(Q/Q_0)$. This speculation was subsequently confirmed, at $n = 5$ with two out-of-gap gluons, in a fixed-order calculation [10]. Ref. [4] re-performed the calculation using the colour basis independent notation [11] also used by CdFR and showed that it is a general feature of QCD scattering amplitudes. These effects were also considered for event shape variables in Ref. [12] where they were called “coherence violating logarithms” since, depending on the class of event shapes, they can be super-leading, leading or sub-leading. Importantly, these effects are generic to all non-inclusive final state observables (and they are certainly not unique to non-global observables).

As emphasized in Ref. [4], the origin of the mis-cancellation between the evolution of a system with a real or virtual collinear gluon comes entirely from a mismatch between the colour matrices of the Coulomb gluon contributions to the two amplitudes. For inclusive observables, this would only lead to a phase (a purely anti-Hermitian transformation in colour space) in the amplitude and would not contribute to physical cross sections. However, the non-Abelian nature of QCD means that the colour matrix of the Coulomb gluon contribution does not commute with that of other soft (eikonal) gluon exchanges that generate the Hermitian part of the evolution in the case of non-inclusive observables. This non-commutation can (and does) lead to Coulomb-induced factorization violation in physical cross sections. This occurs first at fourth order relative to the hard process, since we need one collinear emission, two Coulomb exchanges (to give a real Hermitian contribution, $\sim (i\pi)^2$) and one real exchange (for them to not commute with).

In the following, we will show that the physical origin of the non-factorizing terms identified by CdFR is identical to that of the super-leading logarithms and that the same argument about their leading effect on physical cross sections applies: they can contribute to fourth-order corrections to hard QCD processes, i.e. three-loop corrections to the single-collinear splitting functions.
2 Space-like collinear factorization

At tree level, the factorization theorem for QCD amplitudes in the multiparton collinear limit reads

\[ \langle M^{(0)} \rangle \approx S p^{(0)} \langle \overline{M}^{(0)} \rangle. \] (2.1)

\( M^{(0)} \) is the tree-level amplitude\(^2\) for some process involving \( n \) coloured partons. We follow the convention of CdFR in defining all momenta as outgoing, so incoming partons have \( p_0^i < 0 \). To be concrete, when discussing processes with two incoming partons (as we are throughout this paper) we assume that partons 1 and \( m+1 \) are incoming and all others are outgoing. \( \langle M^{(0)} \rangle \) is the representation of \( M^{(0)} \) in colour space \([11]\). Equation (2.1) defines the factorization theorem of the amplitude \( M^{(0)} \) in the multicollinear limit in which \( m \) of the partons become collinear (in the SL case of interest, one of the \( m \) is the incoming parton 1). The accuracy is as specified in CdFR: in the multicollinear case it gives the dominant contribution in which all scalar products among the \( m \) collinear partons are of the same order and all vanish together, and does not include sub-dominant (but still singular) contributions. In the particular case of the two-parton collinear limit there are no sub-dominant corrections and this formula captures the entire singular collinear behaviour. \( \overline{M}^{(0)} \) is the tree-level amplitude for a process in which the \( n - m \) non-collinear partons’ momenta and colours are the same as in \( M^{(0)} \), but the \( m \) collinear partons are replaced by a single (on-shell) parton with momentum

\[ \tilde{p}^\mu = p_{\text{jet}}^\mu - \frac{p_{\text{jet}}^\mu n^\mu}{2p_{\text{jet}} \cdot n}, \quad p_{\text{jet}} = \sum_{i=1}^{m} p_i, \] (2.2)

where \( n \) is a light-like \((n^2 = 0)\) vector introduced to define the collinear limit. In the SL case of interest, \( \tilde{p}^0 < 0 \). Flavour conservation in the collinear limit defines the flavour of parton \( \tilde{p} \) and colour conservation implies that it has colour

\[ T_{\tilde{p}} = \sum_{i=1}^{m} T_i = - \sum_{j=m+1}^{n} T_j. \] (2.3)

Note that in general, colour conservation means that, when acting on physical states, the colour operators obey

\[ \sum_{i=1}^{n} T_i = 0, \] (2.4)

from which the second equality of Eq. (2.3) follows.

\( S p^{(0)} \) is an operator describing the collinear splitting \( \tilde{p}(T_{\tilde{p}}) \rightarrow p_1(T_1) + p_2(T_2) + \ldots + p_m(T_m) \). It can be represented by a non-square matrix in colour space, since it acts on the space of \( n - m + 1 \) partons to produce a state in the space of \( n \) partons. Equation (2.1) is defined as “strict” factorization, because \( S p^{(0)} \) depends only on the colours and momenta

\(^2\)Strictly speaking, we mean the lowest order non-zero amplitude, which for some processes could mean a loop amplitude, for example \( gg \rightarrow \gamma g \).
of the \( m \) collinear partons and \( \mathcal{M}^{(0)} \) depends only on the colours and momenta of the non-collinear partons and \( \tilde{P} \).

CdFR show that the structure of Eq. (2.1) continues to all orders:

\[
\left| \mathcal{M} \right| \approx S_p \left| \mathcal{M} \right|, \tag{2.5}
\]

where the kets \( \left| \mathcal{M} \right| \) and \( \left| \mathcal{M} \right| \) and the operator \( S_p \) have perturbative (loop) expansions starting from 0 loops, i.e.

\[
\left| \mathcal{M}^{(l)} \right| = \sum_{l'=0}^{l} S_{p}^{(l')} \left| \mathcal{M}^{(l-l')} \right| + \left| \mathcal{M}^{(l)}_{\text{fin.}} \right|, \tag{2.6}
\]

Equation (2.5) is a factorization, but it is described as a “generalized” factorization, because although \( \mathcal{M} \) still depends only on the colours and momenta of the non-collinear partons and \( \tilde{P} \), \( S_p \) depends in general on both the collinear and non-collinear partons.

In this paper we study the infrared poles (i.e. \( \epsilon \) poles in dimensional regularization) of the multi-collinear splitting matrix \( S_p \) at various orders\(^3\). In fact, the only ingredients that we need to do this are Eq. (2.5) and the results for the singularities of the on-shell amplitudes \( \left| \mathcal{M}(l) \right| \) and \( \left| \mathcal{M}^{(0)} \right| \). These have been known at 1 loop \( (l = 1) \) for many years, and were first written down in the colour basis independent notation in Ref. [11]. They were written down for the two-loop case in Ref. [13] and have recently been explored at higher loops [14–18]. We define the singular parts of the \( l \)-loop amplitude as

\[
\left| \mathcal{M}^{(l)} \right| = \sum_{l'=1}^{l} I^{(l')} \left| \mathcal{M}^{(l-l')} \right| + \left| \mathcal{M}^{(l)}_{\text{fin.}} \right|, \tag{2.7}
\]

with analogous definitions for operators \( \mathcal{T}^{(l')} \) for the reduced amplitude \( \left| \mathcal{M}^{(l)} \right| \). We give explicit expressions for these operators in the appropriate sections below.

We can use the factorization of amplitudes in the collinear limit to derive the factorization of cross sections:

\[
\left\langle \mathcal{M} \left| \mathcal{M} \right\rangle \approx \left\langle \mathcal{M} \left| P \right. \mathcal{M} \right\rangle, \quad P = S_p^\dagger S_p. \tag{2.8}
\]

If \( S_p \) obeys strict factorization, then \( P \) does too. On the contrary, it is possible that strict factorization is broken at the amplitude level, but nevertheless the factorization-violating terms do not contribute to physical cross sections. In fact this is true for time-like (TL) collinear splitting, less trivially for SL collinear splitting in processes with only one incoming coloured parton (e.g. deep inelastic scattering) and, less trivially still, for general processes at one-loop level and QCD processes at two-loop level, as we will show.

We close this section by noting a property of QCD tree-level matrix elements \( \left| \mathcal{M}^{(0)} \right| \) (and also \( S_p^{(0)} \)) that we will make use of in the following. In the colour basis independent

\(^3\)Of course, taking \( m = 2 \) we recover the two-parton collinear limit. As shown in CdFR, in this case it is possible to calculate the non-factorizing terms exactly in \( d \) dimensions but, while important for detailed calculations, we do not believe that this illustrates any additional physics, so we work in the multicollinear case throughout.
notation, one works in an orthonormal basis for the colour states of a given scattering process. It is natural (and has been done in all calculations to date) to define these basis states as combinations of generators, delta functions and structure constants ($i f^{abc}$ and $d^{abc}$) with real coefficients. It was noticed in Ref. [19] that, in these natural bases, all anomalous dimension matrices calculated in the literature are (complex) symmetric matrices. In Ref. [20] it was proved that this is because the matrix representation of any colour operator $T_i \cdot T_j$ (which is Hermitian in any orthonormal basis) is real in these bases, and hence symmetric. In the following we will exploit the existence of such a basis, even if it is not necessary that it be specified explicitly.

The vector representation of the amplitude $|\mathcal{M}(0)\rangle$ is given by its projections onto the basis vectors of the space. An important point in the following is the fact that tree-level QCD amplitudes do not introduce any additional phases into this representation. That is, all the elements of the vector representation of $|\mathcal{M}(0)\rangle$ have the same phase and so the outer product $|\mathcal{M}(0)\rangle \langle \mathcal{M}(0)|$ (which is called the “hard matrix” in the language of Sterman et al.) can be represented as a matrix with real entries. Since it is also Hermitian it must be symmetric. The same argument applies to $\mathcal{S}p(0)$. It is a tree-level QCD operator whose matrix representation is a transformation between two orthonormal spaces (of different dimensions). If both of those spaces are represented by natural bases, the matrix representation of $\mathcal{S}p(0)$ is real.

3 Space-like collinear factorization at one loop

At one loop [11], the operator $I$ can be written

$$I^{(1)} = \frac{\alpha_s}{2\pi} \frac{1}{2} \left\{ - \sum_{i=1}^{n} \left( \frac{1}{\epsilon^2} C_i + \frac{1}{\epsilon} \gamma_i \right) - \frac{1}{\epsilon} \sum_{i<j}^{n} T_i \cdot T_j \ln \left( \frac{-s_{ij} - i0}{\mu^2} \right) \right\}, \quad (3.1)$$

with an exactly analogous expression for $\overline{I}^{(1)}$. Examining the analytic structure of the logarithm, we see that when $i$ and $j$ are both incoming partons, or both outgoing partons, it contributes an imaginary part. Physically, this imaginary part corresponds to an absorptive contribution coming from on-shell parton-parton scattering long before, or long after, the hard scattering process. One can rewrite this equation to make the imaginary part manifest, for example as

$$I^{(1)} = \frac{\alpha_s}{2\pi} \frac{1}{2} \left\{ - \sum_{i=1}^{n} \left( \frac{1}{\epsilon^2} C_i + \frac{1}{\epsilon} \gamma_i + \frac{i\pi}{\epsilon} C_i \right) - \frac{1}{\epsilon} \sum_{i<j}^{n} T_i \cdot T_j \ln \left( \frac{|s_{ij}|}{\mu^2} \right) + \frac{2i\pi}{\epsilon} T_s^2 \right\}, \quad (3.2)$$

where

$$T_s = T_1 + T_{m+1} = - \sum_{i \neq 1, m+1}^{n} T_i \quad (3.3)$$

\footnote{To avoid a cluttered notation, we do not explicitly indicate that $I$, and all the related operators we consider, are renormalized and only have poles of IR origin. Thus we drop the $R$ superscript used by CdFR, as well as the argument of $\alpha_s$, which is the $\overline{\text{MS}}$ coupling evaluated at $\mu^2$. Finally, we also drop the argument of $I(\epsilon)$, except where it is not $\epsilon$, when we explicitly write it.}
is the total colour in the s channel. In this form, one can see that the contribution comes symmetrically from initial-state and final-state interactions, which, since colour is conserved, are equal. It is more useful for practical calculation to write this entirely in terms of the initial-state contribution. Moreover, purely Abelian imaginary parts $\sim i\pi C_i$ cannot contribute to cross sections. Therefore, in the following, we shall write the physically important imaginary part as $\overline{\Delta}_C^{(1)} = \frac{\alpha_s}{2\pi} T_1 \cdot T_{m+1}$.

Writing the operator as in Eq. (3.2), we are separating different regions of the integration over gluon loop momenta: “eikonal” gluons (i.e. with on-shell momenta) give rise to the logarithm, whilst “Coulomb” gluons (i.e. with space-like momenta) correspond to on-shell external parton-parton scattering and give rise to the $i\pi$ term. It will turn out that the factorization breaking contributions can be understood entirely in terms of the non-trivial colour operators associated with Coulomb and eikonal gluon exchanges.

The singularities of the one-loop splitting matrix, defined by

$$S_p^{(1)} = I_C^{(1)} S_p^{(0)} + S_p^{(1)\text{fin.}},$$

(3.4)
can now be extracted using

$$I_C^{(1)} S_p^{(0)} = I^{(1)} S_p^{(0)} - S_p^{(0)} \overline{T}^{(1)},$$

(3.5)
which we write as

$$I_C^{(1)} = I^{(1)} - \overline{T}^{(1)}.$$  

(3.6)
After some colour algebra, including the colour conservation condition in Eq. (2.3), we obtain

$$I_C^{(1)} = \frac{\alpha_s}{2\pi} \left\{ \left( \frac{1}{\epsilon^2} C_\beta + \frac{1}{\epsilon} \gamma_\beta \right) - \sum_{i=1}^{m} \left( \frac{1}{\epsilon^2} C_i + \frac{1}{\epsilon} \gamma_i - \frac{2}{\epsilon} C_i \ln |z_i| \right) - \frac{i\pi}{\epsilon} \left( C_\beta - C_1 + \sum_{i=2}^{m} C_i \right) - \frac{1}{\epsilon} \sum_{i \neq \ell} T_i \cdot T_\ell \ln \left| \frac{s_{i\ell}}{z_i |z_\ell| \mu^2} \right| \right\} + \overline{\Delta}_C^{(1)},$$

(3.7)
with

$$\overline{\Delta}_C^{(1)} = \frac{\alpha_s}{2\pi} \left\{ 2 \times \frac{i\pi}{\epsilon} T_{m+1} \cdot (T_1 - T_\beta) \right\}.$$  

(3.8)
All notation not explicitly defined here is taken over directly from CdFR, but in brief: $C_i$ is the Casimir of particle type $i$, $\gamma_i$ is a flavour dependent real constant and $z_i = p_i \cdot n/|\vec{n}|$, is positive for the incoming collinear parton ($i = 1$) and negative for the outgoing collinear partons ($i = 2 \ldots m$).

Note that Eq. (3.7) is exactly equal to CdFR Eq. (5.31), but it is rewritten in such a way that our $\overline{\Delta}_C^{(1)}$ is different to their $\Delta_m^{(1)}$ in Eq. (5.32). The terms explicitly written in Eq. (3.7) obey strict factorization: they depend only on the kinematics and colours of the collinear partons. On the other hand, $\overline{\Delta}_C^{(1)}$ violates strict factorization since it depends on

$\overline{\Delta}_C^{(1)}$.
the colour of a non-collinear parton, i.e. the other incoming parton, \( m + 1 \). Written in this form, it is very clear that the violation of strict factorization observed by CdFR is directly related to a mismatch between the colour structures of the Coulomb gluon contributions in the full \( (T_{m+1} \cdot T_1) \) and factorized \( (T_{m+1} \cdot T_{\tilde{P}}) \) matrix elements. All other terms obey strict factorization. This mismatch was illustrated diagrammatically in Figure 1 of Ref. [4] and in the associated discussion.

Turning to the factorization theorem for cross sections, we have

\[
P^{(1)} = Sp^{(0)\dagger} Sp^{(1)} + Sp^{(1)\dagger} Sp^{(0)} = Sp^{(0)\dagger} \left( I_C^{(1)} + I_C^{(1)\dagger} \right) Sp^{(0)} + P^{(1)\text{fin.}} \tag{3.9}
\]

\[
\equiv Sp^{(0)\dagger} I_P^{(1)} Sp^{(0)} + P^{(1)\text{fin.}}, \tag{3.10}
\]

\[
I_P^{(1)} = \frac{\alpha_s}{2\pi} \left\{ \left( \frac{1}{\epsilon^2} C_P + \frac{1}{\epsilon} \gamma_P \right) - \sum_{i=1}^m \left( \frac{1}{\epsilon^2} C_i + \frac{1}{\epsilon} \gamma_i - 2 \frac{1}{\epsilon} C_i \ln |z_i| \right) - \frac{1}{\epsilon} \sum_{i,j \neq k} T_i \cdot T_j \ln \frac{|s_{ij}|}{|z_i||z_j|} \mu^2 \right\}. \tag{3.11}
\]

We see that the divergent terms do now have a factorized structure. This is because \( \Delta_C^{(1)} \), the non-factorizing term in \( I_C^{(1)} \), is anti-Hermitian and it has cancelled in the combination \( I_C^{(1)} + I_C^{(1)\dagger} \). The only possible source of factorization violation is in the finite term,

\[
P^{(1)\text{fin.}} = Sp^{(0)} I_C^{(1)\text{fin.}} + \text{h.c}, \tag{3.12}
\]

since in the multi-parton collinear limit it is not (yet) proven that \( Sp^{(1)\text{fin.}} \) contains only (finite) factorizing terms. In the two-parton limit on the other hand, \( Sp^{(1)\text{fin.}} \) has the same structure as \( Sp^{(0)} \) and hence factorizes.

In summary, in this section, we have presented results that completely agree with those of CdFR, but we believe that our writing of the non-factorizing contribution to \( I_C^{(1)} \), Eq. (3.8), displays its physical origin more clearly.

4 Space-like collinear factorization at two loops: amplitude level

At two loops [13], the operator \( I \) is

\[
I^{(2)} = -\frac{1}{2} \left[ I^{(1)} \right]^2 + \frac{\alpha_s}{2\pi} \left\{ \frac{1}{\epsilon} b_0 \left[ I^{(1)}(2\epsilon) - I^{(1)} \right] + K I^{(1)}(2\epsilon) \right\} + \left( \frac{\alpha_s}{2\pi} \right)^2 \frac{1}{\epsilon} \sum_{i=1}^n H_i^{(2)}, \tag{4.1}
\]

with an exactly analogous expression for \( \overline{T}^{(2)} \). \( K \) and \( H_i^{(2)} \) are more (real) constants that may be found in CdFR. Recall that we suppress the \( \epsilon \) argument of \( I^{(l)} \), except where its argument is not \( \epsilon \).

The singularities of the two-loop splitting operator can then be extracted as

\[
Sp^{(2)} = I_C^{(2)} Sp^{(0)} + I_C^{(1)} Sp^{(1)} + \overline{Sp}^{(2)\text{div.}} + \text{finite terms}, \tag{4.2}
\]

where

\[
\overline{Sp}^{(2)\text{div.}} = \left[ \overline{T}^{(1)}, Sp^{(1)\text{fin.}} \right], \tag{4.3}
\]
which vanishes if there are no factorization violating terms in $S\!P^{(1)\text{fin}}$.

We can extract $I^{(2)}_C$ as
\begin{equation}
I^{(2)}_C = I^{(2)} - \mathbf{T}^{(2)} + \mathbf{T}^{(1)} (I^{(1)} - \mathbf{T}^{(1)}) ,
\end{equation}
i.e.
\begin{equation}
I^{(2)}_C = -\frac{1}{2} \left[ I^{(1)}_C \right]^2 + \frac{\alpha_s}{2\pi} \left\{ + \frac{1}{\epsilon} b_0 \left[ I^{(1)}_C (2\epsilon) - I^{(1)}_C \right] + K I^{(1)}_C (2\epsilon) + \frac{\alpha_s}{2\pi} \frac{1}{\epsilon} \left( \sum_{i\in C} H^{(2)}_i - H^{(2)}_P \right) \right\} + \frac{1}{2} \left[ \mathbf{T}^{(1)}, I^{(1)}_C \right].
\end{equation}

Note that all of the parts that are non-trivial colour operators are determined by one-loop operators. Thus, at two-loop order, the violations of factorization can be understood in terms of one-loop violations. To this end, it is useful to write
\begin{equation}
I^{(1)}_C = I^{(1)\text{fact.}}_C + \Delta^{(1)}_C.
\end{equation}
$I^{(1)\text{fact.}}_C$ depends only on the collinear partons’ colours and momenta and has both Hermitian and anti-Hermitian parts, while $\Delta^{(1)}_C$ depends on the colours of collinear and non-collinear partons and is purely anti-Hermitian. We then have
\begin{equation}
I^{(2)\text{fact.}}_C = -\frac{1}{2} \left[ I^{(1)\text{fact.}}_C \right]^2 + \frac{\alpha_s}{2\pi} \left\{ + \frac{1}{\epsilon} b_0 \left[ I^{(1)\text{fact.}}_C (2\epsilon) - I^{(1)\text{fact.}}_C \right] + K I^{(1)\text{fact.}}_C (2\epsilon) \right\} + \frac{\alpha_s}{2\pi} \frac{1}{\epsilon} \left( \sum_{i\in C} H^{(2)}_i - H^{(2)}_P \right)\right\} ,
\end{equation}
and a factorization violating contribution to $I^{(2)}_C$ of
\begin{equation}
\Delta^{(2)}_C \equiv I^{(2)}_C - I^{(2)\text{fact.}}_C.
\end{equation}
\begin{equation}
= \frac{1}{2} \left\{ I^{(1)\text{fact.}}_C \Delta^{(1)}_C + \Delta^{(1)}_C I^{(1)\text{fact.}}_C + \left( \Delta^{(1)}_C \right)^2 \right\} + \frac{1}{2} \left[ \mathbf{T}^{(1)}, \Delta^{(1)}_C \right]
\end{equation}
\begin{equation}
+ \frac{\alpha_s}{2\pi} \left\{ + \frac{1}{\epsilon} b_0 \left[ \Delta^{(1)}_C (2\epsilon) - \Delta^{(1)}_C \right] + K \Delta^{(1)}_C (2\epsilon) \right\} .
\end{equation}

Note that, unlike $\Delta^{(1)}_C$, which is purely anti-Hermitian, $\Delta^{(2)}_C$ contains both Hermitian and anti-Hermitian parts.

In summary, in this section, we have shown results that completely agree with those of CdFR, but again they are slightly rewritten to isolate more clearly the physical origin of the non-factorizing contributions to $I^{(2)}_C$: they arise entirely from non-factorizing contributions that are already present in the one-loop results.

5 Space-like collinear factorization at two loops: cross section level

In order to compute cross sections, we need
\begin{equation}
P^{(2)} = S\!P^{(0)\dagger} S\!P^{(2)} + S\!P^{(1)\dagger} S\!P^{(1)} + S\!P^{(2)\dagger} S\!P^{(0)}
\equiv P^{(2)}_f + P^{(2)}_{n.f} + \text{finite terms},
\end{equation}
\begin{equation}
\equiv P^{(2)}_f + P^{(2)}_{n.f} + \text{finite terms},
\end{equation}
with a factorizing contribution of
\[
P^{(2)} = S p^{(0)\dagger} \left\{ I^{(2)\text{fact.}} C + \left[I^{(1)\text{fact.}} C\right]^2 + I^{(1)\text{fact.}} C^{(1)\dagger} + I^{(1)\text{fact.}} C^{(1)\dagger} \right\} S p^{(0)}.
\] (5.3)

Any factorization violation at two-loop order arises from the remainder:
\[
P^{(2)\text{n.f.}} = S p^{(0)\dagger} \left\{ \Delta^{(2)} C + \Delta^{(2)\dagger} C + \left(\Delta^{(1)} C^{(1)\dagger}\right)^2 - \Delta^{(1)} I^{(1)\text{fact.}} C^{(1)} + I^{(1)\text{fact.}} \Delta^{(1)} \right\} S p^{(0)}
\]
\[+ S p^{(0)\dagger} \left\{ I^{(1) \text{fin.}} C p + S p^{(2)\text{div.}}\right\} + \text{h.c. + finite terms}.
\] (5.4)

The first line contains all of the IR divergences associated with divergent operators acting on the tree-level splitting operator, while the second line contains possibly non-factorizing divergences due to non-factorizing finite parts of the one-loop splitting operator (recall that \(S p^{(2)\text{div.}}\) depends on \(S p^{(1)\text{fin.}}\)).

Inserting the expression in Eq. (4.9) into the first line of Eq. (5.4), we obtain
\[
P^{(2)\text{n.f.}} = \frac{1}{2} S p^{(0)\dagger} \left\{ \left(\hat{T}^{(1)} + T^{(1)}\right)^{\dagger} + I^{(1)\text{fact.}} C p + I^{(1)\text{fact.}} C^{(1)\dagger}\right\} \hat{\Delta}^{(1)}
\]
\[+ S p^{(0)\dagger} \left\{ I^{(1) \text{fin.}} C p + S p^{(2)\text{div.}}\right\} + \text{h.c. + finite terms}.
\] (5.5)

This important expression has a very simple physical interpretation. It expresses the fact that Coulomb exchange (which is the physics of \(\Delta^{(1)} C^{(1)}\)) does not commute with eikonal exchanges between any two partons in the matrix element (which is the only relevant physics of \(T^{(1)}\) and \(I^{(1)\text{fact.}} C^{(1)}\) since the Coulomb exchanges in these operators cancel after adding the Hermitian conjugate\(^6\)).

We can relate this commutator to that which is explicit in the derivation of super-leading logarithms in Ref. [4] by considering the limit in which the partons of the reduced matrix element divide into two clusters that are separated by a large interval in rapidity, \(Y\). In this case, the only large \(s_{ij}\) are associated with the case where partons \(i\) and \(j\) are on opposite sides of the interval. Taking the leading \(Y\) behaviour is therefore equivalent to neglecting \(I^{(1)\text{fact.}} C^{(1)}\) and simplifying
\[
\hat{T}^{(1)} + T^{(1)} \approx \frac{\alpha_s}{2\pi} \frac{2}{\epsilon} Y T_t^2,
\] (5.6)

where \(T_t\) is the colour exchanged across the rapidity interval. Equation (5.5) then simplifies to
\[
P^{(2)\text{n.f.}} \propto i\pi Y S p^{(0)\dagger} \left[T_t^2, T_{m+1} \cdot (T_1 - T_P)\right] S p^{(0)}.
\] (5.7)

This equation embodies exactly the same physical result as Eq. (4.13) of Ref. [4]: the violation of factorization is driven by the non-vanishing commutator of the eikonal gluon contribution with the mismatch between the Coulomb gluon contributions of the full and factorized matrix elements. To make this more precise, we should explain the connection between the notation used here and that used in Ref. [4]. In that paper, the operator \(t_1^\dagger t_2^\dagger\) is the colour part of \(S p^{(0)}\), \(T_1 \cdot T_2\) is \(T_{m+1} \cdot T_1\) and \(t_1 \cdot t_2\) is \(T_{m+1} \cdot T_{P}\). Finally, since

---

\(^6\)For that reason the superscript ‘fact.’ is actually redundant in Eq. (5.5).
\( T_{m+1} \cdot T_{\bar{p}} \) and \( T_i^2 \) commute with \( S_p^{(0)} \) (in the sense defined more precisely in Eq. (4.12) of Ref. [4]), Eq. (5.7) can be written identically to Eq. (4.13) of Ref. [4].

In general, the commutator in Eq. (5.5) is non-zero. However, we can exploit its properties as a matrix in colour space to show that its expectation value on a QCD tree amplitude \( S_p^{(0)} |\mathcal{M}^{(0)}\rangle \) is zero. As discussed at the end of section 2, there exists a basis in which the matrix representation of any colour operator \( T_i \cdot T_j \) is real. Therefore, in such a basis, the commutator in Eq. (5.5) is purely imaginary. Since it is also Hermitian, it must be antisymmetric. As we have noted, in such a basis QCD tree amplitudes are real and hence \( |\mathcal{M}^{(0)}\rangle \langle \mathcal{M}^{(0)}| \) is symmetric. The final step of the argument is to note that \( \langle \mathcal{M}^{(0)}|A|\mathcal{M}^{(0)}\rangle \) can be written as \( \text{Tr} \left[ (|\mathcal{M}^{(0)}\rangle \langle \mathcal{M}^{(0)}|)A \right] \)

\[
\text{Tr} [SA] = 0 \tag{5.8}
\]

for any symmetric, \( S \), and antisymmetric, \( A \), matrices. Since there exists a basis in which the result is zero it must be zero in any basis.

Therefore, although the commutator in Eq. (5.5) is non-zero, its expectation value on a QCD tree amplitude is zero and

\[
P^{(2)}_{\text{n.f.}} = \text{second line} \quad \text{(pure QCD processes).} \tag{5.9}
\]

We recall that the terms in the “second line” of Eq. (5.4) arise from divergent one-loop corrections to finite one-loop corrections to the splitting operator, which are not yet proven to factorize. We have shown that, apart from this possible source, factorization violating terms in the one- and two-loop splitting amplitudes do not contribute to factorization violation of QCD cross sections at the one- or two-loop level.

CdFR considered the most general case, in which it is true that the expectation value of the non-factorizing term is non-zero, and did not consider the specific case of QCD tree amplitudes. As mentioned in Section 2, it is the fact that the Feynman rules of QCD do not introduce any phases, so all of the elements of the colour vector \( S_p^{(0)} |\mathcal{M}^{(0)}\rangle \) have the same phase, that leads to this result. In order to have a non-zero result, one must have a hard process with more than one colour flow, with non-trivial phase differences between them. Within the Standard Model, the only \( 2 \to 2 \) processes that satisfy these conditions are \( q\bar{q} \to q\bar{q} \) and \( q\bar{q} \to q\bar{q} \) in which an \( s \)-channel colour singlet propagator (\( W \) and \( Z \) respectively) introduces a non-trivial phase, which can then interfere with \( t \)-channel octet (gluon) or singlet (\( \gamma \) or \( Z \)) exchange.

6 Conclusion and discussion

We have explored the results presented in CdFR and rewritten them in a way that makes clear that the origin of the breakdown in collinear factorization is a mismatch in the non-Abelian colour matrix for Coulomb gluon exchange between the two incoming partons in the full matrix element and the sub-process matrix element it is factorized into, at one loop. At two-loop level, no fundamentally new non-factorizing effects enter, and the two-loop violation of strict factorization at the amplitude level is entirely determined by the one-loop one. This mechanism is identical to the one studied in Refs. [3, 4].
At one-loop level the violation of factorization in the amplitude is anti-Hermitian and cannot contribute to physical cross sections. At two-loop level the sum of all contributions corresponding to two Coulomb gluon exchanges (in the amplitude, the conjugate amplitude and one in each) similarly cancels. However, a non-zero commutator between one Coulomb gluon exchange and one eikonal exchange gives a Hermitian term that may contribute to physical cross sections. We find that this contribution does vanish for pure QCD processes. Again, this mirrors the conclusions of Refs. [3, 4]. Ref. [4] in particular has a discussion of the cancellation of these terms in physical cross sections.

In Refs. [3, 4], we concluded that the non-cancellation of Coulomb gluon effects does have a physical effect on non-inclusive QCD observables at the three-loop level. We showed that this comes about because only by exchanging two Coulomb gluons can we obtain a non-zero effect on physical cross sections and only via a non-zero commutator between these gluons and a further eikonal gluon can we avoid their complete cancellation as a pure phase effect. This then induced a mis-cancellation between the real and virtual initial-state collinear emissions (i.e. the breakdown of the ‘plus-prescription’), which resulted in a “super-leading” logarithm.

The all-order development of CdFR can be used at three loops to produce the results for the operator \( I_C^{(3)} \), its non-factorizing part \( \Delta_C^{(3)} \), and its effect on physical cross sections \( P_{\text{n.f.}}^{(3)} \). Exactly the mechanism just described does indeed occur: the commutator between two non-factorizing one-loop Coulomb contributions and one factorizing eikonal contribution produces a non-factorizing term in the cross section that does not cancel. In order to display the results, we define one additional piece of notation: the Coulomb gluon parts of \( I^{(1)} \) and \( \overline{T}^{(1)} \) are defined as \( \tilde{\Delta}^{(1)}_1 \) and \( \tilde{\Delta}^{(1)}_\overline{P} \) respectively (and therefore \( \tilde{\Delta}^{(1)}_C = \tilde{\Delta}^{(1)}_1 - \tilde{\Delta}^{(1)}_\overline{P} \)).

We obtain terms that are in one-to-one correspondence with those obtained for the first non-zero super-leading logarithm in Ref. [4]. In the 2-parton collinear limit, in which the eikonal part of \( I^{(1)}_C \) is a pure number, we obtain

\[
P_{\text{n.f.}}^{(3)} \sim \frac{1}{6} Sp^{(0)\dagger} \left( \left[ \tilde{\Delta}^{(1)}_1, \left[ \tilde{\Delta}^{(1)}_\overline{P}, \overline{T}^{(1)} + T^{(1)} \right] \right] - \left[ \tilde{\Delta}^{(1)}_\overline{P}, \left[ \tilde{\Delta}^{(1)}_1, \overline{T}^{(1)} + T^{(1)} \right] \right] \right) Sp^{(0)}
+ \frac{1}{2} Sp^{(0)\dagger} \left( \left[ \tilde{\Delta}^{(1)}_1, \tilde{T}^{(1)} \right] \right) - \left[ \tilde{\Delta}^{(1)}_\overline{P}, \left[ \tilde{T}^{(1)} \right] \right] \left) \right) Sp^{(0)}.
\]

The structure of the first line is identical to the contribution to super-leading logarithms from configurations in which the out-of-gap gluon has the highest \( k_{\perp} \) (see Eq. (4.18) of Ref. [4]) and is driven by the mismatch between the double-commutator of one eikonal exchange and two Coulomb exchanges in the full and factorized matrix element. The structure and relative normalization of the second line are identical to the contribution to super-leading logarithms from configurations in which the out-of-gap gluon has the second-highest \( k_{\perp} \) (see Eq. (4.20)7 of Ref. [4]) and is driven by the mismatch between the commutator of one eikonal exchange and one Coulomb exchange in the full and factorized matrix element, with a second Coulomb exchange in the factorized matrix element giving an overall real Hermitian contribution. To obtain the result in the multi-parton collinear limit, one simply replaces \( \overline{T}^{(1)} \) by \( T^{(1)} + I^{(1)}_C \).

7This is Eq. (4.19) in the arxiv version of the paper.
We also note that the non-vanishing double-commutators in Eq. (6.1) are identical to those presented in Ref. [22] (Eq. (4.10)), where they triggered the failure of gluon Reggeization at NNLL.

We should note that the results we have presented do not imply any mis-cancellation of the singularities of cross sections in hadron-hadron collisions, because at the longest distance scales, which contribute to the infrared poles of the full hadron-hadron cross section, the incoming particles are the colourless hadrons and the colour factor associated with the factorization violation is zero. This was a crucial step in the proof of the factorization theorem in Ref. [1]: they were able to isolate the effect of the Coulomb/Glauber region of gluon momentum into factorization-violating terms that, when summed over all diagrams, gave zero. Furthermore, we fully expect that the factorization breaking effects we have been discussing will cancel in inclusive observables, in accord with the theorem derived by Collins, Soper and Sterman [1] and more recently by Aybat and Sterman [21]. The cancellation ought to occur upon accounting for real emissions with the same cut-diagram topology as the virtual corrections we consider here. However, these cancellations will not be complete in non-inclusive observables. In the case of super-leading logarithms, the factorization breaking is accompanied by an uncancellation soft-collinear double logarithm in an otherwise single logarithmic observable. The question of at which logarithmic order factorization breaking effects enter has been discussed by Banfi, Salam and Zanderighi [12]. In general, we expect to be able to factorize collinear logarithms only up to a scale \( \mu \) below which real emissions are summed inclusively, and that the violations of factorization discussed here will contribute to the values of physical observables above that scale.

Finally, we may remark that the factorization-breaking physics we have been exploring is not unrelated to the role of the underlying event, or of gap survival in diffractive physics. For an observable that is inclusive below some scale \( \mu \), we may factorize the incoming partons at that scale. The evolution in scales between \( \mu \) and the high scale of a hard process will ‘dress’ these incoming partons with collinear accompaniments and interactions between these systems will violate the collinear factorization theorem, as we have discussed. These accompanying systems will interact with each other in an exactly analogous way to the remnants in hadron-hadron collisions. Thus, the factorization violations discussed in this paper will have effects that may be considered as perturbatively-calculable contributions to the underlying event. Furthermore, diffractive scattering is not power suppressed and can lead to important contributions in studies of the final state. However, it is well known that diffractive scattering in hadron-hadron collisions does not obey a factorization theorem: as a result one is led to the phenomenological idea of gap survival. In the high energy limit of scattering amplitudes, Coulomb gluon exchange is responsible for building the perturbative manifestation of the Pomeron and the eikonal exchange triggers the breakdown of factorization and leads to the filling of a would-be rapidity gap; in this case the factorization violating effects we have discussed can be thought of as the perturbative tail of the gap survival effect.
Acknowledgements

We are grateful to Mrinal Dasgupta and the other members of the Manchester “QCD club” for extensive discussions of CdFR. Thanks also to Stefano Catani for his comments. This work was funded in part by the Lancaster-Manchester-Sheffield Consortium for Fundamental Physics under STFC grant ST/J000418/1 and in part (MHS) by an IPPP Associateship.

References

[1] J. C. Collins, D. E. Soper and G. F. Sterman, “Soft Gluons and Factorization”, Nucl. Phys. B 308 (1988) 833.

[2] S. Catani, D. de Florian and G. Rodrigo, “Space-like (versus time-like) collinear limits in QCD: Is factorization violated?”, JHEP 1207 (2012) 026 [arXiv:1112.4405v1 [hep-ph]].

[3] J. R. Forshaw, A. Kyrieleis and M. H. Seymour, “Super-leading logarithms in non-global observables in QCD”, JHEP 0608 (2006) 059 [hep-ph/0604094].

[4] J. R. Forshaw, A. Kyrieleis and M. H. Seymour, “Super-leading logarithms in non-global observables in QCD: Colour basis independent calculation”, JHEP 0809 (2008) 128 [arXiv:0808.1269 [hep-ph]].

[5] N. Kidonakis, G. Oderda and G. F. Sterman, “Evolution of color exchange in QCD hard scattering”, Nucl. Phys. B 531 (1998) 365 [hep-ph/9803241].

[6] G. Oderda and G. F. Sterman, “Energy and color flow in dijet rapidity gaps”, Phys. Rev. Lett. 81 (1998) 3591 [hep-ph/9806530].

[7] C. F. Berger, T. Kucs and G. F. Sterman, “Energy flow in interjet radiation”, Phys. Rev. D 65 (2002) 094031 [hep-ph/0110004].

[8] M. Dasgupta and G. P. Salam, “Accounting for coherence in interjet E(t) flow: A Case study”, JHEP 0203 (2002) 017 [hep-ph/0203009].

[9] A. Kyrieleis, “Super-leading logarithms in gaps-between-jets”, hep-ph/0606274, 41st Rencontres de Moriond: QCD and Hadronic Interactions, 18–25 Mar. 2006. A. Kyrieleis, J. R. Forshaw and M. H. Seymour, “Breakdown of QCD coherence?”, PoS DIFF 2006 (2006) 031 [hep-ph/0612202], Diffraction 2006: 4th International Workshop on Diffraction in High-Energy Physics, 5–10 Sept. 2006. M. H. Seymour, “Breakdown of Coherence?”, arXiv:0710.2733 [hep-ph], 12th International Conference on Elastic and Diffractive Scattering: Forward Physics and QCD, 21–25 May 2007. J. R. Forshaw and M. H. Seymour, “Soft gluons and superleading logarithms in QCD”, Nucl. Phys. Proc. Suppl. 191 (2009) 257 [arXiv:0901.3037 [hep-ph]], Ringberg Workshop on New Trends in HERA Physics 2008, 5–10 Oct. 2008.

[10] J. Keates and M. H. Seymour, “Super-leading logarithms in non-global observables in QCD: Fixed order calculation”, JHEP 0904 (2009) 040 [arXiv:0902.0477 [hep-ph]].

[11] S. Catani and M. H. Seymour, “A General algorithm for calculating jet cross-sections in NLO QCD”, Nucl. Phys. B 485 (1997) 291 [Erratum-ibid. B 510 (1998) 503] [hep-ph/9605323].

[12] A. Banfi, G. P. Salam and G. Zanderighi, “Phenomenology of event shapes at hadron colliders”, JHEP 1006 (2010) 038 [arXiv:1001.4082 [hep-ph]].
[13] S. Catani, “The Singular behaviour of QCD amplitudes at two loop order”, Phys. Lett. B 427 (1998) 161 [hep-ph/9802439].

[14] L. J. Dixon, L. Magnea and G. F. Sterman, “Universal structure of subleading infrared poles in gauge theory amplitudes”, JHEP 0808 (2008) 022 [arXiv:0805.3515 [hep-ph]].

[15] T. Becher and M. Neubert, “Infrared singularities of scattering amplitudes in perturbative QCD”, Phys. Rev. Lett. 102 (2009) 162001 [arXiv:0901.0722 [hep-ph]].

[16] E. Gardi and L. Magnea, “Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes”, JHEP 0903 (2009) 079 [arXiv:0901.1091 [hep-ph]].

[17] T. Becher and M. Neubert, “On the Structure of Infrared Singularities of Gauge-Theory Amplitudes”, JHEP 0906 (2009) 081 [arXiv:0903.1126 [hep-ph]].

[18] L. J. Dixon, E. Gardi and L. Magnea, “On soft singularities at three loops and beyond”, JHEP 1002 (2010) 081 [arXiv:0910.3653 [hep-ph]].

[19] M. H. Seymour, “Symmetry of anomalous dimension matrices for colour evolution of hard scattering processes”, JHEP 0510 (2005) 029 [hep-ph/0508305].

[20] M. H. Seymour and M. Sjödahl, “Symmetry of anomalous dimension matrices explained”, JHEP 0812 (2008) 066 [arXiv:0810.5756 [hep-ph]].

[21] S. M. Aybat and G. F. Sterman, “Soft-Gluon Cancellation, Phases and Factorization with Initial-State Partons”, Phys. Lett. B 671 (2009) 46 [arXiv:0811.0246 [hep-ph]].

[22] V. Del Duca, C. Duhr, E. Gardi, L. Magnea and C. D. White, “The Infrared structure of gauge theory amplitudes in the high-energy limit”, JHEP 1112 (2011) 021 [arXiv:1109.3581 [hep-ph]].