ON ANALYSIS OF THE EXPONENTIAL MAP OF VOLUME-PRESERVING Diffeomorphism Group on Closed Orientable Surfaces through the Vorticity

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Abstract. We study the exponential map of group of volume-preserving diffeomorphisms on closed orientable surfaces via the vorticity formulation of the incompressible Euler equation. We present an alternative, fluid dynamical proof of the theorem of Ebin–Misiołek–Preston: the exponential is a nonlinear Fredholm mapping of index zero. We extend Shnirelman’s rigidity result for the exponential map from 2-dimensional flat torus to arbitrary orientable closed surfaces. That is, we prove that the exponential map is Fredholm quasiregular.

1. Introduction

The space of volume-preserving diffeomorphisms on a Riemannian manifold \((M, g)\) is an important topic of research in global analysis. On the one hand, it forms an infinite-dimensional Lie group that embeds into the space of \(L^2\)-maps from \((M, g)\) to \(\mathbb{R}^N\), where \(N\) is sufficiently large. On the other hand, it connects naturally to PDEs in mathematical hydrodynamics: as established by Arnold in his seminal paper in 1966, the geodesic equation on this infinite-dimensional Lie group is precisely the Euler equation describing the motion of an incompressible, inviscid fluid on \((M, g)\):

\[
\begin{cases}
\frac{dv}{dt} + v \cdot \nabla v + \nabla p = 0 & \text{in } [0, T] \times M, \\
v|_{t=0} = v_0 & \text{on } M
\end{cases}
\] (1.1)

for timespan \(T > 0\), velocity \(v \in \Gamma(TM)\), and pressure \(p : M \to \mathbb{R}\).

For an \(n\)-dimensional closed (i.e., compact, without boundary) Riemannian manifold \((M, g)\), write \(\text{Diff}(M)\) for the group of diffeomorphisms on \(M\), and define

\[
\text{SDiff}(M, g) = \left\{ \phi \in \text{Diff}(M) : \phi^\# \text{dVol}_g = \text{dVol}_g \right\},
\]

where \(\phi^\#\) is the pullback under \(\phi\) and \(\text{dVol}_g\) is the Riemannian volume form on \((M, g)\). We shall also consider the space \(H^s \text{SDiff}(M, g)\) of volume-preserving diffeomorphisms with Sobolev \(H^s\)-regularity. When \(s > \left\lfloor \frac{n}{2} \right\rfloor + 1\), \(H^s \text{SDiff}(M, g)\) is an infinite-dimensional Lie group and \(H^s \text{SDiff}(M, g) \hookrightarrow H^s (M, \mathbb{R}^N)\) as a Banach submanifold. Here \(m\) is any natural number such that \((M, g)\) isometrically embeds into \(\mathbb{R}^m\), whose existence follows from Nash embedding; see [17]. Its Lie algebra is identified with the space of incompressible vectorfields on \((M, g)\); i.e.,

\[
T_{[0]} H^s \text{SDiff}(M, g) = \{ v \in H^s(M, TM) : \text{div } v = 0 \}.
\] (1.2)

Here and hereafter, for any vector bundle \(E\) over \(M\) and \(X = W^{2,p}, H^s, \ldots\), we shall use \(X(M, E)\) to designate the space of \(E\)-sections in the regularity class \(X\). One also write \(\Gamma(E)\)

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for the space of $E$-sections (with unspecified sufficient regularity; usually assumed to be $C^\infty$ in this paper). The divergence operator $\operatorname{div}: \mathbf{H}^s(M,TM) \to \mathbf{H}^{s-1}(M,\mathbb{R})$ is defined as usual via duality. See Appendix A.

In 1970, Ebin–Marsden [11] proved that the infinite-dimensional manifold $\mathbf{H}^s\operatorname{Diff}(M,g)$ admits a smooth Levi-Civita connection and that, given an initial velocity in $T_p\mathbf{H}^s\operatorname{Diff}(M,g)$, the geodesic equation is locally well-posed. Thus, there is a well defined exponential map

$$\operatorname{Exp}: \mathcal{N} \subset T_{id}\mathbf{H}^s\operatorname{Diff}(M,g) \to \mathbf{H}^s\operatorname{Diff}(M,g)$$

(1.3)

where $\mathcal{N}$ is a neighbourhood of 0. It is classically known that $\operatorname{Exp}$ is smooth and invertible in the vicinity of 0. Recall [12] for a characterisation of $T_{id}\mathbf{H}^s\operatorname{Diff}(M,g)$.

From now on, we denote by $(\Sigma,g)$ a closed orientable surface. For $M = \Sigma$, the domain $\mathcal{N}$ of $\operatorname{Exp}$ in (1.3) can be taken as the whole Lie algebra $T_{id}\mathbf{H}^s\operatorname{Diff}(\Sigma,g)$. The global behaviour of $\operatorname{Exp}$ on $T_{id}\mathbf{H}^s\operatorname{Diff}(\Sigma,g)$, however, has enormous subtleties. Misiolek [13] proved the existence of conjugate points along geodesics on $\mathbf{H}^s\operatorname{Diff}(\Sigma = T^2)$, where $T^2$ is flat torus. In particular, there exists $v \in T_{id}\mathbf{H}^s\operatorname{Diff}(T^2)$ with $\dim \ker(d_v\operatorname{Exp}) \geq 1$ (“mono-conjugate”) for

$$d_v\operatorname{Exp}: T_vT_{id}\mathbf{H}^s\operatorname{Diff}(T^2) \to T_{\operatorname{Exp}(v)}\mathbf{H}^s\operatorname{Diff}(T^2).$$

See also Shnirelman [21] for analogous result on 3-ball.

On the other hand, for the volume-preserving diffeomorphism group on 2D closed surfaces, $\operatorname{Exp}$ is just a little worse than being conjugate point-free: as shown by Ebin–Misiolek–Preston [12] Theorem 1], $\operatorname{Exp}: T_{id}\mathbf{H}^s\operatorname{Diff}(\Sigma,g) \to \mathbf{H}^s\operatorname{Diff}(\Sigma,g)$ is Fredholm of index zero.

**Definition 1.1.** Let $X$ and $Y$ be Banach spaces. A smooth, nonlinear map $f: X \to Y$ is **Fredholm** if the Fréchet derivative $d_pf: X \to Y$ is a Fredholm operator at each $p \in X$. The **Fredholm index** of $f$, denoted as $\operatorname{Ind}(f)$, is the index of $d_pf = df|_p$.

Here, recall that

$$\operatorname{Ind}(f) := \operatorname{Ind}(d_pf) = \dim \ker(d_pf) - \dim \operatorname{coker}(d_pf).$$

It is independent of $p$ when the domain of $f$ is connected; e.g., in our case the domain is a Banach space. The study of Fredholmness of **nonlinear** maps is pioneered by Smale [23].

A refined characterisation of $\operatorname{Exp}$ in the case $\Sigma = T^2$ was obtained in [22] by Shnirelman, who proved that $\operatorname{Exp}$ possesses certain global rigidity in its geometrical structures. More precisely, the nonlinear functional-analytic notions below were introduced by Shnirelman in 1970s:

**Definition 1.2** (Shnirelman [20]). Let $X$ and $Y$ be Banach spaces, and let $F: X \to Y$ be a continuous map that is nonlinear in general. Then

1. $F$ is said to be a **ruled map** if $X$ is foliated into affine subspaces $X^\alpha \subset X$ indexed by $\mathbb{R}^k$, i.e., $X = \bigsqcup_{\alpha \in \mathbb{R}^k} X^\alpha$, such that $F|_{X^\alpha}: X^\alpha \to Y$ is affine for each $\alpha$, and that $F|_{X^\alpha}$ depends continuously on $\alpha$.
2. $F$ is **quasiruled** if it can be approximated locally uniformly by ruled maps $\{F_k\}$.
3. $F$ is **Fredholm quasiruled** if $F$ is quasiruled and the approximating ruled maps $\{F_k\}$ as in (2) above satisfy the following conditions: $Y^\alpha := \operatorname{range}(F_k|_{X^\alpha})$ is closed for each $\alpha$, codim $Y^\alpha = \operatorname{codim} X^\alpha = k$, $F_k|_{X^\alpha}: X^\alpha \to Y^\alpha$ are bijective, and $(F_k|_{X^\alpha})^{-1}$ are locally uniformly continuous for all sufficiently large $k$. (Here $k$ and $X^\alpha$ are as in (1) above.)
4. Assume that $X^-, Y^-$ are dense, compactly embedded subsets of $X$ and $Y$, respectively. A continuous map $f: X^- \to Y^-$ is said to be **quasilinear** if (i), it can be extended to
a continuous map \( F : X \to Y \), and (ii), there exist continuous affine maps \( \{ B_u : X^\to \to Y^\to \} \) which depend continuously on the parameter \( u \), such that \( A(u) = B_u(u) \) for each \( u \in X^\to \).

Fredholm quasi-ruled maps have nice topological–geometrical properties. For instance, they admit a definition of topological degree, thus leading to no-wandering properties (preservation of domain). Moreover, a notion of Fredholm quasi-ruled Banach manifolds can be naturally defined; it forms a category, for which Fredholm quasi-ruled maps are morphisms. As a primary example, \( H^sSDiff(M, g) \) for \( s > \left\lfloor \frac{n}{2} \right\rfloor + 1 \) is a Fredholm quasi-ruled Banach manifold. See [20] for details.

In [22], by using tools from “microglobal” analysis (coined word: microlocal + global analysis), Shnirelman proved that \( \text{Exp} \) is Fredholm quasi-ruled when \( \Sigma = T^2 \), hence the geodesic flow on \( H^sSDiff(T^2) \) for \( s > 2 \) is a Fredholm quasi-ruled flow.

To summarise, the following results have been established concerning the exponential map on volume-preserving diffeomorphism group of surfaces. See Ebin–Misiolek–Preston [12, Theorem 1] & Shnirelman [22, Theorem 3.1]:

**Theorem 1.3.** Let \( (\Sigma, g) \) be a compact surface. Then \( \text{Exp} : T_{Id}H^sSDiff(\Sigma, g) \to H^sSDiff(\Sigma, g) \) is a nonlinear Fredholm map of index zero as long as \( s > 2 \). Moreover, for \( (\Sigma, g) = T^2 \) = the flat torus, \( \text{Exp} \) is Fredholm quasi-ruled.

At the end of the paper, Shnirelman [22, p.S395] remarked —

“Note that we have used the global flat structure on the torus. The proof of the analogous theorem for the fluid motion on a curved surface should require additional devices.”

The goal of this note is to address this problem. We establish the following.

**Theorem 1.4.** Theorem 1.3 is valid for any closed orientable surface \( (\Sigma, g) \). That is, \( \text{Exp} : T_{Id}H^sSDiff(\Sigma, g) \to H^sSDiff(\Sigma, g) \) is a Fredholm quasi-ruled map of index zero whenever \( s > 2 \).

The main difficulty for proving Theorem 1.4, i.e., for extending Theorem 1.3 to general compact orientable surface \( \Sigma \) other than \( T^2 \), lies in the lack of Fourier analytic tools on \( \Sigma \). To our remedy, however, paradiﬀerential calculus has been developed recently on complete Riemannian manifolds with doubling property, lower volume bound for unit balls, and Poincaré inequality. See Bernicot [5] and Bernicot–Sire [6]. We will collect some of these tools in §2 below.

Next, as vorticity is being transported by the Euler flow on \( (\Sigma, g) \), it is handy to express \( \text{Exp}(v) \) for \( v \in T_{Id}H^sSDiff(\Sigma, g) \) as a function of the vorticity \( \omega = dv \in \mathcal{H}^{s-1}(\Sigma, \wedge^2 T^*\Sigma) \) which, by duality, can be identified with a scalar field. The recovery of \( v \) from \( \omega \) is achieved via a pseudodiﬀerential operator of order \(-1\), known as the Biot–Savart operator in the Euclidean case. We shall develop its analogue on \( (\Sigma, g) \) in §3. This will be done via the Green’s operator for the Laplace–Beltrami operator \( \Delta_g : \Omega^1(\Sigma) \to \Omega^2(\Sigma) \) on differential 1-forms. See the classical work [10] by de Rham.

The tools elaborated in §§2 & 3 enable us to prove Theorem 1.4 in §4. We shall essentially follow Shnirelman’s work [22] for the case \( \Sigma = T^2 \); nevertheless, various new arguments are presented to deal with the non-flat geometry.

Apart from the proof for the rigidity (i.e., Fredholm quasiregular) property of \( \text{Exp} \) on closed orientable surfaces, this paper also provides an alternative proof for (the orientable case of)
Ebin–Misiołek–Preston [12, Theorem 1], which states that $\text{Exp}$ on closed 2-dimensional surfaces is Fredholm of index zero. Our new proof is fluid dynamical in nature, which exploits the vorticity formulation of the 2-dimensional incompressible Euler equation. For future investigations, we hope to generalise our approach to non-orientable closed surfaces and to surfaces-with-boundary.

**Notations.** Throughout the paper, $(\Sigma, g)$ is a closed orientable Riemannian manifold of dimension 2. The superscript/subscript hash # always denotes the pullback/pushforward, while the superscript asterisk * is reserved for adjoint operator. For a constant $C$, by writing $C = C(M, g)$ we mean that $C$ depends only on the geometry of $(M, g)$. An inner product is denoted as $\cdot$ if it is taken with respect to a Euclidean metric. We shall write $\nabla$ for Riemannian gradient and $D$ for Euclidean gradient. The index $s$ is always a number strictly greater than $2 = \left\lfloor \frac{\dim \Sigma}{2} \right\rfloor + 1$, hence $H^s \hookrightarrow L^\infty$ over $\Sigma$. The index $\sigma$ is an arbitrary number strictly greater than $s$.

2. **Paradifferential calculus on $(\Sigma, g)$**

In this work, we make essential use of the theory of paradifferential calculus for Riemannian manifolds with sub-Laplacian structures (for which compact surface $(\Sigma, g)$ is a special case) developed by Bernicot–Sire [6].

2.1. **Geometric assumptions.** Let $(M, g)$ be a Riemannian manifold and let $\mathcal{L}$ be a sub-Laplacian on $(M, g)$. That is, $\mathcal{L} = -\sum_{k=1}^K X_k^2$ where $\{X_k\}$ are real-valued vectorfields.

**Assumption 2.1.** Suppose that $(M, g, \mathcal{L})$ satisfy the following (see [6, p.941]):

1. $(M, g)$ is (uniform) doubling, i.e., there exists a uniform constant $C_1 > 0$ such that
   $$\text{Vol}_g(B(x, 2r)) \leq C_1 \text{Vol}_g(B(x, r))$$
   for any $x \in M$ and $r > 0$;
2. $(M, g)$ supports the 1-Poincaré inequality, namely that there exists a uniform constant $C_2 > 0$ such that for each $f \in C^\infty_0(M)$, $r > 0$, and any $Q$ = geodesic ball of radius $r$,
   $$\int_Q |f - f_Q| \text{d}Vol_g \leq C_2 r \int_Q |\nabla f| \text{d}Vol_g.$$
3. There exists a uniform constant $C_3 > 0$ such that $\text{Vol}_g(B(x, 1)) \geq C_3$ for all $x \in M$;
4. All the local Riesz transforms $\mathcal{R}_I$ and $\mathcal{T}_I$ for arbitrary multiindex $I$ are $L^p \to L^p$-bounded for each $p \in [1, \infty[$;
5. The sub-Laplacian $\mathcal{L}$ satisfies [7, Assumption 1.11]: $\mathcal{L}$ is injective and of type $\omega$ for some $\omega \in [0, \pi/2]$ on $L^2$, and there exists $\delta > 1$ such that
   a) For every $z \in S_{\pi/2-\omega}$, the linear operator $e^{-z\mathcal{L}}$ is given by a kernel $p_z$ such that
   $$|p_z(x, y)| \leq \frac{C_4}{\text{Vol}_g(B(x, \sqrt{|z|}))} \left(1 + \frac{d_g(x, y)}{\sqrt{|z|}}\right)^{-\log_2(C_1)-2N-\delta};$$
   b) $\mathcal{L}$ has a bounded $H_\infty$-calculus on $L^2$;
   c) The Riesz transform $\mathcal{R} := \nabla \mathcal{L}^{-1/2}$ is $L^p \to L^p$-bounded for each $p \in ]1, \infty[$.

Here and hereafter, $\nabla$ denotes the Riemannian gradient, $B(x, r)$ denotes the geodesic balls, $d_g$ is the Riemannian distance, $|\cdot|$ is the Riemannian length of vectors, and Vol$_g$ is the Riemannian volume on $(M, g)$. All the $L^p$-spaces and averaged integrals $f$ are taken with respect
to the Riemannian volume form $d\text{Vol}_g$. The number $N \geq 0$ in (5a) is the uniform constant in

$$
\text{Vol}_g(B(y, r)) \leq C_5 \left( 1 + \frac{d_g(x, y)}{r} \right)^N \text{Vol}_g(B(x, r))
$$

for all $x, y \in M$ and $r > 0$, \hspace{1cm} (2.1)

where $C_5$ is another uniform constant. By $\mathcal{L}$ being of type $\omega$ on $L^2$ we mean that $\mathcal{L}$ is a closed operator with spectrum in the sector $S_\omega := \{ z \in \mathbb{C} : |\arg(z)| \leq \omega \} \cup \{ 0 \}$, such that for each $\nu > \omega$ there is a constant $c_\nu$ such that

$$
\| (\mathcal{L} - \lambda \mathbf{1})^{-1} \|_{L^2 \to L^2} \leq \frac{c_\nu}{|\lambda|}
$$

for all $\lambda \notin S_\omega$.

Also, that $\mathcal{L}$ has a bounded $H_\infty$-calculus on $L^2$ means that there exists $c_\nu'$, such that for each $b \in H_\infty(\text{int } S_\nu)$ with $\nu > \omega$, there holds

$$
\| b \|_{L^2 \to L^2} \leq c_\nu' \| b \|_{L^\infty}.
$$

Here $H_\infty$ denotes the space of bounded holomorphic functions. Moreover, for each multiindex $I$, set $X_I := \prod_{i \in I} X_i$; then the local Riesz transforms are defined as

$$
\mathcal{R}_I := X_I (1 + |\mathcal{L}|)^{-\frac{|I|}{2}}, \hspace{1cm} \overline{\mathcal{R}_I} := (1 + |\mathcal{L}|)^{-\frac{|I|}{2}} X_I.
$$

The constants $C_1, C_2, c_3, C_4, C_5$, and $N$ all depend only on the geometry of $(M, g)$. The constants $c_\nu$ and $c_\nu'$ depend in addition on $\nu$.

Note that any closed manifold $(M, g)$ satisfies Assumption 2.1. Indeed, it is known that a Riemannian manifold $(M, g)$ with bounded geometry — that is, $\| \nabla^k \text{Riem}_g \|_{L^\infty} \leq C(k, M, g) < \infty$ for each $k \in \mathbb{N}$, $\text{Ric}_g > -\infty$, and the injectivity radius is positive — satisfies (2) and (5). In fact, the Poincaré inequality holds for any $R$-manifold with bounded geometry

$$
\rho(x) := \int_x^\infty \rho(y) \, dy
$$

as well as

$$
\overline{\psi} := \frac{\psi(x)}{x}, \hspace{1cm} \overline{\phi} := -\int_x^\infty \overline{\psi}(y) \, dy, \hspace{1cm} \psi(\bullet) := \psi(\bullet), \hspace{1cm} \phi(\bullet) := \phi(\bullet).
$$

Under Assumption 2.1, the construction of paraproduct can be generalised to $(M, g, \mathcal{L})$. Paradifferential calculus has been an important tool for the analysis of propagation of singularities in nonlinear PDEs over $\mathbb{R}^n$, pioneered by Bony \cite{Bony} and Alinhac \cite{Alinhac}.

Our presentation of the theorem below, which is essentially a generalisation of Bony’s decomposition, follows Bernicot–Sire \cite{BernicotSire} Definition 4.1, Theorem 4.5, and Corollary 5.2.

**Theorem 2.2.** Let $(M, g, \mathcal{L})$ be a Riemannian manifold with sub-Laplacian satisfying Assumption 2.1. For $f, h : C^\infty_0(M) \to \mathbb{R}$, define the paraproduct of $f$ by $h$ by

$$
\Pi_h(f) := -\int_0^\infty \overline{\psi}(t \mathcal{L}) \left[ t \mathcal{L} \overline{\phi}(t \mathcal{L}) f \overline{\phi}(t \mathcal{L}) h \right] \frac{dt}{t} - \int_0^\infty \overline{\phi}(t \mathcal{L}) \left[ \psi(t \mathcal{L}) f \overline{\phi}(t \mathcal{L}) h \right] \frac{dt}{t}.
$$

(1) For $p, q \in [1, \infty]$ with $0 < p^{-1} + q^{-1} = \frac{1}{r}$, $N > \frac{\log_2(C_1)}{2}$, and $s \in [0, 2N - 4]$, the mapping

$$
(f, h) \mapsto \Pi_h f
$$

is bounded from $W^{s,p} \times L^q$ into $W^{s,r}$.
(2) If, in addition, \( p \in ]1, \infty[ , \varepsilon > 0, s > \frac{\log_2(C_1)}{p} \), and \( s < \frac{N - 2}{2} \), the mapping

\[
\text{Rest}(f, h) := fh - \Pi f h - \Pi_h f
\]

satisfies

\[
\|\text{Rest}(f, h)\|_{W^{2s-\delta, p}} \lesssim \|f\|_{W^{s+p, p}} \|h\|_{W^{s+p, p}}.
\]

**Remark 2.3.** The parameter \( N \in \mathbb{N} \) is taken such that \( N \gg 1 \). This would validate the choice of \( s \in ]0, \min \{2N - 4, \frac{N - 2}{2}\} [ \), for which both statements in Theorem 2.2 hold, as well as the requirement that \( N > \frac{\log_2(C_1)}{2} \). We are uncertain if in [22] this parameter must be the same as in (2.1) and Assumption 2.1 (5a). But, for our purpose, on a compact manifold \((M, g)\) we can take \( N \gg 1 \) by enlarging \( C_1(M, g) \) and \( C_2(M, g) \) in (2.1) and Assumption 2.1 (5a).

The number \( \log_2(C_1) \) is the homogeneous dimension of \((M, g)\); here \( C_1 \) is the (uniform) doubling constant in Assumption 2.1 (1).

On a closed manifold \((M, g)\) we may construct the paraproduct via the arguments in Shnirelman [22, Equation(2.2)], which are taken from [20] by the same author:

\[
fh \approx D^{-k} \nabla^k (fh)
\]

\[
= \Pi_{h,f} + D^{-k} (f \nabla^k h) + D^{-k} (k \nabla h \cdot \nabla^{k-1} f + \ldots + k \nabla f \cdot \nabla^{k-1} h).
\]

Here \( D^{-k} \) can be rigorously defined via (local) Riesz transforms. In [22] \( \Pi_h f \) is denoted as \( T_h f \); here we choose the former notation to avoid confusion with the tangent bundle \( T \Sigma \).

For any \( k \in \mathbb{N} \) let us consider \( X_I = \prod_{i \in I} X_i \), where the multi-index \( I \) has valency \(|I| = k\).

Recall the sub-Laplacian \( \mathcal{L} = -\sum_{i=1}^K X_i^2 \). We obtain by Assumption 2.1 (4), the definition of local Riesz transform in (2.2), and the Leibniz rule, that

\[
fh \approx \mathcal{R}_I (fh)
\]

\[
= (1 + \mathcal{L})^{-|I|/2} X_I (fh)
\]

\[
= (1 + \mathcal{L})^{-|I|/2} (h X_I f) + (1 + \mathcal{L})^{-|I|/2} (f X_I h)
\]

\[
+ (1 + \mathcal{L})^{-|I|/2} \left( \sum_{J+K=I, 1 \leq |J| \leq |I|} X_J f X_K h \right).
\]

(2.4)

On closed manifold \((M, g)\) we may take \( \mathcal{L} = -\Delta_g \). This is the Hodge Laplacian with \( \mathcal{L} = \sum_{i=1}^M \nabla_i \nabla_i \) in local co-ordinates. For \( h \in L^\infty(M; \mathbb{R}) \) and \( f \in H^\sigma(M; \mathbb{R}) \), the first line means that \( fh - \mathcal{R}_I (fh) \in H^\sigma(M; \mathbb{R}) \) for some \( \sigma > s \).

**2.3. Paracomposition.** As another tool for paradifferential calculus, we can define the paracomposition on \((M, g, \mathcal{L})\) based on the paraproduct. For \( \varpi \in \text{Diff}(M) \) and \( f : M \to \mathbb{R} \), consider

\[
f \circ \varpi \approx \mathcal{R}_I (f \circ \varpi)
\]

\[
= (1 + \mathcal{L})^{-|I|/2} X_I (f \circ \varpi).
\]

(2.5)
To be explicit, we first compute for a single vectorfield $X \in \Gamma(TM)$ that
\[ X(f \circ \omega) = d(f \circ \omega)(X) = (df \circ \omega)[d\omega(X)] = \langle d\omega(X), \nabla f \circ \omega \rangle, \]
where $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$. Taking one further derivative with respect to $Y \in \Gamma(TM)$, we have
\[ YX(f \circ \omega) = Y\langle d\omega(X), \nabla f \circ \omega \rangle = \langle (\nabla_Y d\omega)(X) + d\omega(\nabla_Y X), \nabla f \circ \omega \rangle + [\nabla \nabla f \circ \omega](d\omega(X), d\omega(Y)). \]
Here $\nabla \nabla f \in \Gamma(T^*M \otimes T^*M)$ is given by $\nabla \nabla f(Z, W) := \nabla_Z \nabla_W f$. A simple induction shows that for any multiindex $I = (i_1, \ldots, i_k)$ one has
\[ X_I(f \circ \omega) = \langle X_{i_k}X_{i_{k-1}} \cdots X_{i_2}d\omega(X_{i_1}), \nabla f \circ \omega \rangle + \left[ \frac{1}{k!} \left( \nabla \cdots \nabla f \right) \circ \omega \right] (d\omega(X_{i_1}), d\omega(X_{i_2}), \ldots, d\omega(X_{i_k})) + \text{Rem}. \tag{2.6} \]
We denote by $\nabla \cdots \nabla f := \nabla^k f \in \Gamma \left( T^*M \otimes k \right)$ the map $\nabla^k f(Z_1, \ldots, Z_k) := \nabla Z_k \nabla Z_{k-1} \cdots \nabla Z_1 f$ for any $Z_1, \ldots, Z_k \in \Gamma(TM)$. The remainder $\text{Rem}$ consists of those terms containing no more than $(k - 1)$ derivatives on both $\omega$ and $f$; i.e., those involving $\nabla X_i X_k$ for some $X_k, X_i \in X_I$.

**Definition 2.4.** Let $(M, g, \mathcal{L})$ be a Riemannian manifold with sub-Laplacian satisfying Assumption 2.7, $\omega \in \text{Diff}(M)$, and $f : M \to \mathbb{R}$. The paracomposition of $f$ with $\omega$ is defined as
\[ \mathcal{K}_\omega f := (1 + \mathcal{L})^{-\frac{|I|}{2}} \nabla^k f \circ \omega \tag{2.7} \]
\[(d\omega(X_{i_1}), d\omega(X_{i_2}), \ldots, d\omega(X_{i_k})). \]
Note that the paracomposition $\mathcal{K}_\omega f$ carries only the singularities of $f$.

By (2.3) and (2.6) we have
\[ f \circ \omega \approx \mathcal{K}_\omega f + (1 + \mathcal{L})^{-\frac{|I|}{2}} \sum_{j=1}^{m} (D_j f \circ \omega) \left( X_{i_k}X_{i_{k-1}} \cdots X_{i_2}X_{i_1} \omega^j \right) \]
\[ \equiv \sum_{j=1}^{m} (D_j f \circ \omega) \left( X_j \omega^j \right), \tag{2.7} \]

The second term on the right-hand side of (2.7) can be simplified as follows. Observe that the diffeomorphism $\omega \in C^\infty \text{Diff}(M)$ can be regarded as a map from $M$ into $\mathbb{R}^m$, if $(M, g)$ embeds isometrically into $\mathbb{R}^m$ by Nash’s embedding [17]. Then, view $\omega = (\omega^1, \ldots, \omega^m)^\top$ and identify $f$ with an arbitrary Lipschitz extension $f : \mathbb{R}^m \to \mathbb{R}$, without relabelling. In this way
\[ \langle X_{i_k}X_{i_{k-1}} \cdots X_{i_2}d\omega(X_{i_1}), \nabla f \circ \omega \rangle = \sum_{j=1}^{m} (D_j f \circ \omega) (X_{i_k}X_{i_{k-1}} \cdots X_{i_2}X_{i_1} \omega^j) \]
\[ \equiv \sum_{j=1}^{m} (D_j f \circ \omega) (X_j \omega^j), \tag{2.8} \]
where $D$ is the Euclidean gradient on $\mathbb{R}^m$. Taking $(1 + \mathcal{L})^{-\frac{|I|}{2}}$ on both sides and recalling the paraproduct (2.4), we can rephrase (2.7) as
\[ f \circ \omega \approx \mathcal{K}_\omega f + \sum_{j=1}^{m} \Pi_{D_j f \circ \omega} (\omega^j) + (1 + \mathcal{L})^{-\frac{|I|}{2}} \{\text{Rem}\}, \tag{2.8} \]
with $\text{Rem}$ containing $\leq (k - 1)$ derivatives on $\omega$ and $f$. 

7
For our applications in the subsequent developments, we shall apply the construction and corresponding estimates for the paracomposition in the following form.

**Corollary 2.5.** Let \((\Sigma, g)\) be a smooth, oriented, closed surface smoothly isometrically embedded in \(\mathbb{R}^m\). Let \(\varpi \in H^s\text{SDiff}(\Sigma)\) and \(f \in H^\ell(\Sigma; \mathbb{R})\) with \(s > 2\) and \(\ell > 1\). Then

\[
f \circ \varpi - K \varpi f - \sum_{j=1}^{m} \Pi D_j f \varpi (w^j) \in H^\varrho(\Sigma) \quad \text{for some } \varrho > \ell.
\]

**Proof.** It follows directly from the \(H^\varrho \to H^\varrho\) boundedness of local Riesz transform in (2.5), the expression (2.8), Definition 2.4 of paracomposition, and Theorem 2.2 for paraproduct. \(\square\)

**Remark 2.6.** In the case that \((M, g)\) is a closed Riemannian manifold, our constructions above for paraproduct and paracomposition are intrinsic. Thus, if \((M, g)\) is smoothly isometrically embedded into some Euclidean domain \(\mathbb{R}^m\), then they agree with usual paradifferential calculus constructions on \(\mathbb{R}^m\), with suitable restrictions and pullbacks to (the embedded image of) \(\Sigma\).

### 3. Biot–Savart Operator

Let \(v \in \Gamma(T\Sigma)\) be a divergence-free vectorfield. Define its vorticity as

\[
\omega := *\left[ d\left( v^q \right) \right], \tag{3.1}
\]

with the Hodge star \(* : \Omega^2(\Sigma) \to \Omega^0(\Sigma)\). See Appendix A for details. We shall also write (3.1) more compactly as \(\omega = *dv^q\), and shall refer to the mapping \(v \mapsto \omega\) as the curl or rot operator.

#### 3.1. Vorticity is transported.

As with the Euclidean flat case, the vorticity will never aggregate or deplete for any time:

**Lemma 3.1.** Assume that \(v \in \Gamma(T\Sigma)\) satisfies the Euler equation. Its vorticity \(\omega\) is transported along the Lagrangian trajectories. That is,

\[
\partial_t \omega + \nabla_v \omega = 0 \quad \text{on } [0, \infty) \times \Sigma.
\]

**Proof.** We make use of a nice expression for the curl/rot operator on surfaces; see also Samavaki–Tuomela [19, p.11 and Appendix B]. Let \(J \in \Gamma(T^\ast \Sigma \otimes T\Sigma)\) be the almost complex structure arising from the Riemannian metric \(g\). In local co-ordinates, \(J = dx^1 \otimes e_2 - dx^2 \otimes e_1\). Then for each vectorfield \(u \in \Gamma(T\Sigma)\) one has

\[
*\left[ d\left( u^q \right) \right] = J^\alpha_\beta \nabla_\alpha u^\beta,
\]

where \(\nabla\) is the Levi-Civita connection on \(\Sigma\). Note also that \(\nabla J = 0\).

Let us take \(\sharp, d, \text{ and } *\) to the Euler equation

\[
\partial_t v + \nabla_v v + \nabla p = 0
\]

in this given order. As \(\nabla\) is dual to \(d\) and \(d^2 = 0\), the pressure term is eliminated. We obtain from the definition of vorticity in (3.1) that

\[
\partial_t \omega + J^\alpha_\beta \nabla_\alpha \left( \nabla_v v \right)^\beta = 0. \tag{3.2}
\]

To proceed, we first note that

\[
\left( \nabla_v v \right)^\beta = \nabla_v \left( v^\beta \right).
\]
as both sides are equal to \( v^\gamma \partial_\gamma \theta^\beta + v^\gamma \theta^\delta \Gamma^\beta_{\gamma \delta} \), where \( \Gamma^\beta_{\gamma \delta} \) are the Christoffel symbols for \( \nabla \). Then
\[
J^\alpha_\beta \nabla_\alpha (\nabla v)^\beta = J^\alpha_\beta (\nabla_\alpha v^\gamma) \left( \nabla_\gamma v^\beta \right) + J^\alpha_\beta v^\gamma \nabla_\alpha \nabla_\gamma v^\beta \\
= J^\alpha_\beta (\nabla_\alpha v^\gamma) \left( \nabla_\gamma v^\beta \right) + J^\alpha_\beta v^\gamma \nabla_\alpha \nabla_\gamma v^\beta - J^\alpha_\beta v^\gamma v^\delta \text{Riem}^\beta_{\gamma \alpha \delta},
\]
thanks to the definition of Riemann curvature tensor \( \text{Riem}^\beta_{\gamma \alpha \delta} \). On 2D surface the last term vanishes, as \( J^\alpha_\beta v^\gamma \text{Riem}^\beta_{\gamma \alpha \delta} = K \text{J}_{\gamma \alpha} v^\gamma v^\delta = 0 \). Here \( K \) is the Gauss curvature of \((\Sigma, g)\), and the above identity holds by the antisymmetry of \( J \). The second term satisfies
\[
J^\alpha_\beta v^\gamma \nabla_\alpha \nabla_\gamma v^\beta = v^\gamma \nabla_\gamma \left( J^\alpha_\beta \nabla_\alpha v^\beta \right) = v^\gamma \nabla_\gamma \omega = \nabla_\gamma \omega.
\]
Finally, for the first term on the right-hand side, we compute directly to get
\[
J^\alpha_\beta (\nabla_\alpha v^\gamma) \left( \nabla_\gamma v^\beta \right) = \left( \nabla_1 v^1 \right) \left( \nabla_1 v^2 \right) + \left( \nabla_1 v^2 \right) \left( \nabla_2 v^2 \right) - \left( \nabla_2 v^1 \right) \left( \nabla_1 v^1 \right) - \left( \nabla_2 v^2 \right) \left( \nabla_2 v^1 \right) \\
= \left[ \left( \nabla_1 v^1 \right) + \left( \nabla_2 v^2 \right) \right] \left[ \left( \nabla_1 v^2 \right) - \left( \nabla_2 v^1 \right) \right] \\
\equiv (\text{div} \, v) \omega.
\]
In this way, \( (3.2) \) becomes \( \partial_\gamma \omega + (\text{div} \, v) \omega + \nabla_\gamma \omega = 0 \). So the proof is complete by the divergence-free condition for \( v \).

3.2. The Biot–Savart operator. Now let us discuss how to invert the curl/rot operator on closed orientable surface \((\Sigma, g)\). That is, we look for an operator
\[
\mathcal{S} : C^\infty(\Sigma, \mathbb{R}) = \Omega^0(\Gamma) \rightarrow \Gamma(T\Sigma)
\]
defined by
\[
\mathcal{S}(\omega) = v \quad \text{such that } \star \text{d} v^\beta = \omega \text{ and div} \, v = 0. \tag{3.3}
\]
The solution for \( v \) is in general non-unique. To make the solution operator \( \mathcal{S} \) well defined, for initial data \( v_0 \in \Gamma(T\Sigma) \) we further require that
\[
v \text{ is cohomologous to } v_0 \text{ where } \text{div} \, v_0 = 0. \tag{3.4}
\]
We call \( \mathcal{S} \) specified by \( (3.3) \) & \( (3.4) \) the Biot–Savart operator on \((\Sigma, g)\). It is also widely denoted as \( \mathcal{S} = \text{rot}^{-1} \) in the literature; see \cite{22} and the references cited therein.

To determine \( \mathcal{S} \), we take \( \star \text{d} \) to the vorticity in \( (3.1) \) to obtain
\[
\star \text{d} \omega = \star \text{d} \star \text{d} v^\beta = d^* \text{d} v^\beta = \Delta_g v^\beta.
\]
The last equality follows from the definition of Laplace–Beltrami operator
\[
\Delta_g = dd^* + d^* d
\]
and that \( v \) is divergence-free (i.e., \( d^* v^\beta = 0 \)). Thus we obtain the following expression for \( \mathcal{S} \):
\[
\mathcal{S} \omega = [\Delta_g^{-1} (\star \text{d} \omega)]^b, \tag{3.5}
\]
where \( \Delta_g^{-1} \) is the inverse of Laplace–Beltrami operator acting on \( \Omega^1(\Sigma) \) and restricted to the cohomology class of \((v_0)^\beta \).
Example 3.2. For $\Sigma = T^2$, the Biot–Savart operator can be expressed explicitly as follows on the Fourier side; here $v_0 = 0$.

$$\hat{\mathcal{S}}\omega(\xi) = -\sqrt{-1}\frac{\hat{\omega}(\xi)}{|\xi|^2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\xi.$$  

See [22, Equation (3.6)].

For a general closed orientable surface $(\Sigma, g)$, we make use of Green’s operators to find $\mathcal{S}$. See, for instance, the classical work of de Rham [10, p.134, Chapter V, Theorem 23]. There exist linear operators $\mathcal{H}$ and $\mathcal{G}$ on $\Omega^1(\Sigma)$, such that $\mathcal{H}$ is the projection onto $\text{Harm}^1(\Sigma)$, the space of harmonic 1-forms on $\Omega$, and $\mathcal{G}$ satisfies

$$\mathcal{G}\Delta_g = \Delta_g^\mathcal{G} = 1 - \mathcal{H}.$$  

Both $\mathcal{H}$ and $\mathcal{G}$ commute with $d$, $d^*$, and $\star$. The projection $\mathcal{H}$ has a $C^\infty$-kernel, and the kernel $G(x,y) : \Sigma \times \Sigma \to \mathbb{R}$ for $\mathcal{G}$ satisfies smooth off-diagonal and of order $O(|\log r|)$ for $r = d_g(x,y)$. In our setting, since $v^\sharp$ is cohomologous to $v_0^\sharp$ and they are both $d^*$-free, so

$$v^\sharp - v_0^\sharp \in \text{Harm}^1(\Sigma).$$  

Thus we have

$$\mathcal{G}(\star d\omega) = (1 - \mathcal{H})v^\sharp = v^\sharp - \mathcal{H}v_0^\sharp,$$

which leads to

$$v = \mathcal{S}\omega = \left(\mathcal{G}(\star d\omega) + \mathcal{H}v_0^\sharp\right)^\flat.$$  

(3.6)

Let us determine the principal symbol of $\mathcal{S}$. It is well-known that the symbol for $d$ is

$$\sigma(d) = 2\pi\sqrt{-1}\xi \wedge \bullet$$  

on any differentiable manifold. When specialising to compact surface, we denote by $J$ the almost complex structure on $(\Sigma, g)$, where $(\Sigma, g)$ is viewed as a complex manifold. Then the principal symbol for $d\star$ is equal to (the left-multiplication by) $2\pi\sqrt{-1}J\xi$. On the other hand, it is well-known that the principal symbol is multiplicative:

$$\sigma_{\text{ppl}}(AB) = \sigma_{\text{ppl}}(A)\sigma_{\text{ppl}}(B)$$  

(3.8)

for pseudo-differential operators between vector bundles over a given compact manifold. See, e.g., [18, p.84, Proposition 3.1.6]. In our case the bundle is $T^*\Sigma$, and the multiplication on the right-hand side of (3.8) is taken with respect to the metric. Recall that Green’s operator $\mathcal{G}$ is a parametrix for the Laplace–Beltrami $\Delta_g$:

$$\mathcal{G} \circ \Delta_g = \Delta_g \circ \mathcal{G} = 1 + \mathcal{H}.$$  

But the projection $\mathcal{H} : \Omega^1(\Sigma) \to \text{Harm}^1(\Sigma)$ is smooth; i.e., of class $\Psi^{-\infty}$ as a pseudo-differential operator. Hence

$$\sigma_{\text{ppl}}(\mathcal{G})(x,\xi) = \frac{1}{\sigma_{\text{ppl}}(\Delta_g)(x,\xi)} = \frac{-1}{(2\pi)^2|\xi|^2_g}$$  

for all $(x,\xi) \in T^*\Sigma$.

By virtue of (3.6), we obtain by applying (3.8) and $\mathcal{H} \in \Psi^{-\infty}$ once more that

$$\sigma_{\text{ppl}}(\mathcal{S})(x,\xi) = -\sqrt{-1}\frac{J\xi}{2\pi|\xi|^2_g}$$  

(3.9)

for all $(x,\xi) \in T^*\Sigma$.  

Here and hereafter, we view $J$ as a section of the endomorphism bundle $\text{End}(T^*\Sigma)$ instead of a $(1,1)$-tensor through the duality $\text{End}(T^*\Sigma) \cong T^*\Sigma \otimes T\Sigma$.

As we have fixed the cohomology class, the $\Delta_g$ is invertible with inverse $\mathcal{S}$, hence its Fredholm index is zero.

4. Proof of Fredholm quasiruledness

4.1. The quantity $W$. With the toolbox of paradifferential calculus on $(\Sigma,g)$ in [32] at hand, we prove Theorem [11] by adapting the arguments in [22 §3]. When $\Sigma = T^2$, Shnirelman [22] considered the quantity

$$W(\varphi)(x,t) := \Pi_{d(\varphi^{-1}_t)}(\varphi_t - 1)(x),$$

where $\varphi_t$ is the integral flow of the Eulerian velocity $v$. This $W$ is a good quantity with suitably controlled divergence and curl. One recovers $\varphi_t$ from the curl of $W$ via the Biot–Savart operator.

Remark 4.1. Here and hereafter, for matrix-valued function $A = \{A_j\}_{q \leq i \leq t; 1 \leq j \leq m}$ and vector-valued function $V = (V^1, \ldots, V^m)^\top$, the paraproduct $\Pi_AV$ is the vector-valued function given by $$(\Pi_AV)^i := \sum_{j=1}^m \Pi_{A_j}V^j$$ for each $i \in \{1, 2, \ldots, \ell\}$.

We look for an analogue of $W$ in [11] on a generic surface $(\Sigma, g)$. Note, however, that $d\varphi^{-1}_t \in \text{End}(T\Sigma)$ and $\varphi_t \in \text{SDiff}(\Sigma,g)$, so the paraproduct $\Pi_{d\varphi^{-1}_t}\varphi_t$ does not make sense unless $\Sigma$ is flat. We overcome this issue by first embedding $(\Sigma, g)$ isometrically into $\mathbb{R}^m$, and then extending $\varphi_t$ to a map $\mathbb{R}^m \to \mathbb{R}^m$ by Whitney extension.

Definition 4.2. Given a smooth, connected, orientable closed surface $(\Sigma, g)$ and volume-preserving diffeomorphisms $\{\varphi_t\} \subset \text{H}^s\text{SDiff}(\Sigma,g)$ for $s > 2$. Let $\iota : (\Sigma, g) \hookrightarrow \mathbb{R}^m$ be a $C^\infty$-isometric embedding for some $m \geq 3$. Note that $\iota \circ \varphi_t : \Sigma \to \mathbb{R}^m$ is an vector-valued $\text{H}^s$-function.

- Take $\Phi_t : \mathbb{R}^m \to \mathbb{R}^m$ to be any $H^s$-extension of $\iota \circ \varphi_t$.
- Put $d\Phi_t(z)/dt = V(\Phi_t(z))$ for each $z \in \mathbb{R}^m$.
- Take $\Psi_t : \mathbb{R}^m \to \mathbb{R}^m$ to be any $H^s$-extension of $\iota \circ \varphi^{-1}_t$.
- Set $$W_t := \Pi_{D\Psi_t}\Phi_t : \mathbb{R}^m \to \mathbb{R}^m.$$ (4.2)

The existence of $\iota$ follows from Nash’s $C^k$-isometric embedding theorem [17], and the existence of $\Phi_t$ and $\Psi_t$ follows from the Besov (hence Sobolev) version of the Whitney extension theorem; see Jonsson–Wallis [13]. Here $\Phi_t$ and $\Psi_t$ are $\text{H}^s$-vectorfields on $\mathbb{R}^m$ such that

$$\Phi_t \circ \iota = \iota \circ \varphi_t \quad \text{and} \quad \Psi_t \circ \iota = \iota \circ \varphi^{-1}_t \quad \text{almost everywhere on} \ \Sigma. $$

The notion of almost everywhere is understood with respect to $d\text{Vol}_g$. The Riemannian volume measure $d\text{Vol}_g$ is Ahlfors 2-regular and supported in $\Sigma$, embedded via $\iota$ as a closed set in $\mathbb{R}^m$; hence, the assumptions for [13] p.146, Main Theorem] are verified. One may consider $\iota \circ \varphi_t = [(\iota \circ \varphi_t)^1, \ldots, (\iota \circ \varphi_t)^m]^\top$ and apply to each component the Whitney extension to obtain $\Phi_t$, and similarly for $\Psi_t$. In Euclidean co-ordinates

$$\langle W_t \rangle^i_j = \sum_{i=1}^m \Pi_{D_i(\Psi_t)^j}(\Phi_t)^i_j \quad \text{for each} \ j \in \{1, 2, \ldots, m\},$$

thanks to Remark [4.1]. In addition, one has

$$V \circ d\iota = d\iota \circ v \quad \text{on} \ \Gamma(T\Sigma).$$ (4.3)
4.2. **Time derivative of $W$.** The following is an adaptation of the arguments on [22, pp. S393–S394]. As a caveat, we remark that $\Psi_t = \Phi_t^{-1}$ is not required to hold on the whole space $\mathbb{R}^m$, which maybe overdetermined in general. We have only defined $\Phi_t$ to be an extension of $\varphi_t$ and $\Psi_t$ an extension of $\varphi_t^{-1}$. That is, $\Psi_t = \Phi_t^{-1}$ is only imposed on $\iota(\Sigma)$.

**Lemma 4.3.** When restricted to $T[\iota(\Sigma)]$, the time derivative of $W_t$ satisfies

$$
\frac{\partial W_t}{\partial t} = \iota_\# \left\{ \left( \Pi_{d\varphi_t^{-1}} \circ K_{\varphi_t} \circ \mathcal{S} \circ K_{\varphi_t^{-1}} \right) (\omega) \right\} + \text{Remainder},
$$

where Remainder $\in H^{s-1}(T[\iota(\Sigma)])$ for some $s > 0$, and $\Pi, K$ denote respectively the paraproduct and paracomposition as in [22].

**Proof.** We apply the Leibniz rule to $W_t$ defined in (4.2) to get

$$
\frac{\partial}{\partial t} W_t = \Pi_{d\varphi_t} \Phi_t + \Pi_{d\Psi_t} \frac{\partial \Phi_t}{\partial t} = J_1 + J_2.
$$

For the second term, since $\varphi_t = \iota^{-1} \circ \Phi_t \circ \iota$ on $\iota(\Sigma)$ and $\varphi_t$ is the integral flow of $v$:

$$
\frac{\partial}{\partial t} \varphi_t(x) = v(\varphi_t(x)),
$$

we have

$$
J_2 = \Pi_{d\Psi_t} \left\{ d\iota \circ (v \circ \varphi_t) \circ d\iota^{-1} \right\} = \Pi_{d\Psi_t} (V \circ \Phi_t).
$$

(4.4)

Here, we make use of the identification of tangent spaces via $d\iota : T\Sigma \to T\mathbb{R}^m$ and $d\iota^{-1} : T\iota(\Sigma) \to T\Sigma$, as well as (4.3).

For the first term, as $\Psi_t = \iota \circ \varphi_t^{-1} \circ \iota^{-1}$ on $\iota(\Sigma)$, we may compute as follows:

$$
\left( \frac{\partial}{\partial t} D\Psi_t \right) = d\iota \circ \left( \frac{\partial}{\partial t} d\varphi_t^{-1} \right) \circ d\iota^{-1}
$$

$$
= -D\Psi_t \bullet \left\{ d\iota \circ d \left( \frac{\partial}{\partial t} \varphi_t \right) \circ d\iota^{-1} \right\} \bullet D\Psi_t
$$

$$
= -D\Psi_t \bullet \left\{ d\iota \circ d (v \circ \varphi_t) \circ d\iota^{-1} \right\} \bullet D\Psi_t
$$

$$
= -D\Psi_t \bullet \left\{ d\iota \circ ((dv) \circ \varphi_t) \circ d\iota^{-1} \right\} \bullet D\Psi_t
$$

$$
= -D\Psi_t \bullet \left\{ d\iota \circ ((dv) \circ \varphi_t) \circ d\iota^{-1} \right\}.
$$

In the last line we make use of the identities

$$
d\iota \circ d\varphi_t \circ d\iota^{-1} = d(\iota \circ \varphi_t \circ \iota^{-1}) = D\Phi_t = (D\Psi_t)^{-1}.
$$

Thus, on the tangent bundle of $\iota(\Sigma)$, it holds that

$$
J_1 = -\Pi_{d\Psi_t} \bullet \left\{ d\iota \circ ((dv) \circ \varphi_t) \circ d\iota^{-1} \right\} \left\{ \iota \circ \varphi_t \circ \iota^{-1} \right\}
$$

$$
= -\Pi_{d\Psi_t} \bullet \left\{ d\iota \circ ((dv) \circ \varphi_t) \circ d\iota^{-1} \right\} \Phi_t.
$$

Moreover, by (4.3) we have

$$
d\iota \circ ((dv) \circ \varphi_t) \circ d\iota^{-1} = DV \circ \Phi_t.
$$

So we deduce that

$$
J_1 = -\Pi_{d\Psi_t} \bullet [DV \circ \Phi_t] \Phi_t.
$$

(4.5)
Putting together (4.5) and (4.4), we conclude that
\[ \frac{\partial}{\partial t} W_t = \Pi_{D\Phi_t} (V \circ \Phi_t) - \Pi_{D\Phi_t} \Phi_t. \] (4.6)
We point out one possible source of confusion here — take, for instance, the expression \( d_t \circ (v \circ \varphi_t) \circ d_t^{-1} \) in the computation for \( J_2 \) above. In the strict sense, the second composition is different from the other two; one should view it via \( v \circ \varphi_t(x) = v(\varphi_t(x)) := v|_{\varphi_t(x')} \). This agrees with the understanding of vectorfield \( v \in \Gamma(T\Sigma) \) as a map \( v : \Sigma \to T\Sigma \). In the end, all the compositions in (4.6) are of this kind.

Now, by virtue of the convention in Remark 4.1 we make use of the paracomposition formula (2.8) to infer that
\[ \Pi_{D\Phi_t} (V \circ \Phi_t) = \Pi_{D\Phi_t} \{ K_{\Phi_t} V + \Pi_{DV \circ \Phi_t} \Phi_t \}. \]
Moreover, the product of paraproduct satisfies
\[ (\Pi_{D\Phi_t} \circ \Pi_{DV \circ \Phi_t} - \Pi_{D\Phi_t} \Phi_t) \Phi_t \in H^{s+1} \] (4.7)
for \( \Phi_t \in H^s(\mathbb{R}^m, \mathbb{R}^m) \). It is crucial to have the paraproduct structure in (4.7), which carries only the singularity of \( \Phi_t \) and smears out the singularities of \( D\Phi_t \) and \( DV \) in the subscripts.

Thus, we may further continue (4.6) as
\[ \frac{\partial}{\partial t} W_t = \Pi_{D\Psi} (K_{\Phi_t} V) + \text{Remainder}, \]
with Remainder \( \in \mathbf{H}^{\sigma-1} (\mathcal{T}[\iota(\Sigma)]) \) for some \( \sigma > s \). From the definition of \( \Phi_t, \Psi_t, \) and \( V \) as extensions of \( \varphi_t, \varphi_t^{-1}, \) and \( v, \) respectively, as well as the fact that \( \iota : (\Sigma, g) \hookrightarrow \mathbb{R}^m \) is an isometric embedding (see Remark 2.6), we conclude that
\[ \frac{\partial}{\partial t} W_t \big|_{\mathcal{T}[\iota(\Sigma)]} = \iota_{\#} \left\{ (\Pi_{d(\varphi_t^{-1}) K_{\varphi_t}}) v \right\} + \text{Remainder}, \]
where Remainder \( \in \mathbf{H}^{s-1} (\mathcal{T}[\iota(\Sigma)]) \) for some \( \sigma > s \).

Finally, let us expand the principal term on the right-hand side of the above equality via the definition of \( v \). Recall that
\[ \frac{\partial}{\partial t} \varphi_t = v \circ \varphi_t = \mathcal{G} [\omega \circ \varphi_t^{-1}] \circ \varphi_t. \]
So
\[ \Pi_{d(\varphi_t^{-1})} K_{\varphi_t} v = \Pi_{d(\varphi_t^{-1})} K_{\varphi_t} \mathcal{G} [\omega \circ \varphi_t^{-1}]. \]
A further application of the paracomposition (2.7) gives us
\[ \omega \circ \varphi_t^{-1} = K_{\varphi_t^{-1}} \omega + \Pi_{d\omega \varphi_t^{-1}} \varphi_t^{-1} + \text{Remainder (I)} \]
\[ = K_{\varphi_t^{-1}} \omega + \text{Remainder (II)} \]
for Remainder (I), (II) \( \in \mathbf{H}^{\sigma-1} (\Sigma, g) \) for some \( \sigma > s \). Note here that \( \Pi_{d\omega \varphi_t^{-1}} \varphi_t^{-1} \) carries only the singularities of \( \varphi_t^{-1} \), thus is of regularity \( H^s \). The proof is now complete. \( \square \)

4.3. Div and curl of \( \partial W_t / \partial t \). In the previous subsection we investigated the vectorfield
\[ U_t := \iota_{\#} \left\{ \frac{\partial W_t}{\partial t} \right\} \in \mathbf{H}^{s-1}(\Sigma; T\Sigma). \] (4.8)
Recall that we first defined \( W_t : \mathbb{R}^m \rightarrow \mathbb{R}^m \) as a flow on \( \mathbb{R}^m \) which, by construction, restricts to a one-parameter family of diffeomorphisms on \( \iota(\Sigma) \) along the integral curve of \( \varphi_t \). Thus \( U_t \) in (4.8)
is indeed a well-defined vectorfield on $\Sigma$. Moreover, since $\iota$ is a smooth isometric embedding and $W_t$ has the same $H^s$-regularity as $\varphi_t$, we have $U_t \in H^{s-1}(\Sigma; T\Sigma)$.

Now, following [22 §3], let us focus on the divergence and curl of $U_t$; that is,

$$d^* U_t \in H^{s-2}(\Sigma; \mathbb{R}) \quad \text{and} \quad \ast d \left[ (U_t)^2 \right] \in H^{s-2}(\Sigma; \mathbb{R}).$$

We shall conclude the proof for our main Theorem 1.4 by analysing these two quantities. More precisely, we prove that if the curl of $U_t$ is Fredholm and if the divergence of $U_t$ equals zero modulo a remainder of higher regularity, then $\text{Exp}$ is Fredholm quasiruled.

**Proposition 4.4.** Let $(\Sigma, g)$ be a closed, connected, orientable surface, and let $s > 2$. Define

$$\mathcal{B} : H^{s-1}(\Sigma; \mathbb{R}) \times H^{s-1}(\Sigma; \mathbb{R}) \rightarrow H^{s-1}(\Sigma; \mathbb{R}),$$

$$(\zeta, \omega) \mapsto \ast d \left[ \left( \Pi_{\iota\Xi^{-1}} \circ \mathcal{K}_{\Xi} \circ \mathcal{G} \circ K_{\Xi^{-1}} \right)(\omega) \right]^2$$

where

$$\Xi = \Xi_t = \int_0^t \mathcal{G} \left( \omega \circ \Xi^{-1}_\tau \right) \circ \Xi_\tau \, d\tau.$$

Suppose that

$$\mathcal{B}_t(\omega) := \mathcal{B}(\omega, \omega)$$

is a Fredholm mapping of index zero. Suppose also that

$$d^* U_t \in H^{\sigma-2}(\Sigma; \mathbb{R})$$

for some $\sigma > s$. Then $\text{Exp} : T_{\text{Id}}H^s\text{SDiff}(\Sigma, g) \rightarrow H^s\text{SDiff}(\Sigma, g)$ is a Fredholm quasiruled map of index zero.

For notational convenience we shall sometimes suppress the variable $t$, when all the relevant identities/estimates are kinematic, i.e., holding pointwise in $t$.

**Proof.** We first notice by Lemma 4.3 that

$$\ast d \left[ (U_t)^2 \right] \approx \ast d \left[ \left( \Pi_{\iota\varphi_t^{-1}} \circ \mathcal{K}_{\varphi_t} \circ \mathcal{G} \circ K_{\varphi_t^{-1}} \right)(\omega) \right]^2$$

modulo a remainder term of higher regularity (namely, $H^{\sigma-2}$ for some $\sigma > s$). As $\Sigma$ is compact, the Fredholm property is invariant under such higher regularity perturbations by the Rellich lemma. Moreover, recall that $\frac{\partial}{\partial t} \varphi_t = \mathcal{G} \left( \omega \circ \varphi_t^{-1} \right) \circ \varphi_t$, which give us

$$\mathcal{B}_t(\omega) := \mathcal{B}(\omega, \omega) = \ast d (U_t)^2 = \ast d \left[ \iota^\# \left( \frac{\partial W_t}{\partial t} \right) \right]^2 + \text{Remainder}$$

with Remainder $\in H^{\sigma-2}(\Sigma; \mathbb{R})$. This space embeds compactly into $H^{\sigma-2}(\Sigma; \mathbb{R})$, so the remainder has no effect on Fredholm index.

Consider now the Bochner integral

$$\mathcal{I}(t) := \int_0^t \mathcal{B}_t(\omega, \bullet) \, d\tau.$$

This defines a function for each $t$, which equal to the curl of $W_t$ when restricted to $\iota(\Sigma)$. More precisely, denoting by $D$ the Euclidean gradient, one has

$$\mathcal{I}(t) = \star \circ \iota^\# \left\{ D \left[ (W_t)^2 \right] \right\}$$

$$= \star \circ \iota^\# \left\{ D \left[ \left( \Pi_{D\Phi_t} \Phi_t \right)^2 \right] \right\} \in H^{\sigma-2}(\Sigma; \mathbb{R}).$$

(4.11)
Here, $D \left[ (W_t)^2 \right]$ is a differential 2-form on $\mathbb{R}^m$, whose restriction to $\iota(\Sigma)$ is also a 2-form on this 2-dimensional submanifold. Recall also that $\Psi_t, \Phi_t$ are extensions of $\varphi_{t}^{-1}$ and $\varphi_t$, respectively. The Hodge star in (4.11) is a mapping $* : H^{s-2}(\Sigma; \Lambda^2 T^* \Sigma) \rightarrow H^{s-2}(\Sigma; \mathbb{R})$.

We can recover $\varphi_t \in H^s SDiff(\Sigma, g)$ from $I(t)$ as follows. Notice from (4.11) that

$$\iota_\# \varphi_t = \Phi_t \circ \iota = \Pi_{D\Psi} \circ \iota_\# \circ \overline{\mathcal{G}}(I(t)).$$

Here $\overline{\mathcal{G}}$ is the inverse of $* \circ \iota_\# \circ \mathcal{G} \circ \iota_\#$ in (4.11), which is nothing but the curl/rot operator on $\iota(\Sigma)$. So $\overline{\mathcal{G}}$ is the corresponding Biot–Savart operator, and is well-defined thanks to the assumption $d^*U_t \in H^{s-2}(\Sigma; \mathbb{R})$ for $\sigma > s$. More precisely, let us solve for $\varsigma := \overline{\mathcal{G}}(I(t))$ such that

$$\overline{d}\varsigma = I(t) \quad \text{and} \quad \overline{\mathcal{F}}\varsigma = 0,$$

where $\overline{d} = \iota_\# \circ D$ is the exterior differential on $\iota(\Sigma)$, and $\overline{\mathcal{F}}$ is the corresponding codifferential with respect to the metric $\iota_\#g$. Moreover, we fix any harmonic 1-form $s_0 \in H^1(\iota(\Sigma))$ and require $\varsigma$ and $s_0$ to be cohomologous. Then, denoting by $\overline{\mathcal{G}}$ the Green’s matrix for $(\iota(\Sigma), \iota_\#g)$, by $\overline{\mathcal{A}}$ the corresponding Laplace–Beltrami operator, and by $\mathcal{H}_{\varsigma_0}$ the projection onto the cohomology class of $s_0$, one obtains

$$\varsigma = \overline{\mathcal{G}} \left\{ \overline{\mathcal{A}} \int_0^t U_\tau \, d\tau - \overline{d} \overline{d}^* \int_0^t U_\tau \, d\tau \right\} + \mathcal{H}_{\varsigma_0}(s).$$

See §3 for details. The projection $\mathcal{H}_{\varsigma_0}$ is smooth and the Green’s operator $\overline{\mathcal{G}}$ is of order $-2$. In addition, by assumption we have $\overline{d}^* U_t \in H^{s-2}(\Sigma; \mathbb{R})$ for $\sigma > s$. Thus

$$I(t) = \overline{d} \int_0^t U_\tau \, d\tau \in H^{s-2}(\Sigma; \mathbb{R})$$

$$\implies \varsigma = \overline{\mathcal{G}}(I(t)) \in H^{s-1}(\Sigma; T\Sigma) + H^{s-1}(\Sigma; T\Sigma) \quad \text{for some } \sigma > s.$$

The discussions in the previous paragraph and the definition of $I(t)$ in (4.10) yield that

$$\varphi_t = \iota_\# \circ \Pi_{D\Psi} \circ \iota_\# \circ \overline{\mathcal{G}}(I(t))$$

$$= \iota_\# \circ \Pi_{D\Psi} \circ \iota_\# \circ \overline{\mathcal{G}} \left( \int_0^t \overline{B}(\omega(\tau, \bullet)) \, d\tau \right),$$

(4.12)

where $I(t) \in H^{s-2}(\Sigma; \mathbb{R})$. By construction of the Biot–Savart operator $\overline{\mathcal{G}}$ and the paraproduct (see Theorem 2.2), we find that $\iota_\# \circ \Pi_{D\Psi} \circ \iota_\# \circ \overline{\mathcal{G}}$ maps $H^{s-2}(\Sigma; \mathbb{R})$ continuously into $H^s SDiff(\Sigma, g)$, and each of the four mappings in the composition are Fredholm of index zero.

Therefore, if $\overline{B}(\omega)$ is Fredholm of index zero (hence so is its Bochner integral $I(t)$), then for each $t$ the mapping

$$\Gamma_t : H^{s-2}(\Sigma; \mathbb{R}) \rightarrow H^s SDiff(\Sigma, g),$$

$$\omega \mapsto \varphi_t$$

is also Fredholm of index zero. Then

$$\text{Exp}(v) \equiv \Gamma_t \left( \ast d \left( v^2 \right) \right) : [v_0] \subset T_{Id} H^s SDiff(\Sigma, g) \rightarrow H^s SDiff(\Sigma, g)$$

is Fredholm of index zero too. We restrict the domain of $\text{Exp}$ to

$$[v_0] := \left\{ v \in T_{Id} H^s SDiff(\Sigma, g) : v^2 \text{ and } (v_0)^2 \text{ are cohomologous} \right\},$$

in order to ensure the invertibility of the differential $d$. Moreover, as $\overline{B}(\omega)$ is Fredholm quasilinear, $\text{Exp}$ is Fredholm quasiruled. The proof for the proposition is now complete. \hfill \Box
Therefore, to establish the main Theorem 4.4 it remains to verify the two assumptions in Proposition 4.4. This shall be carried out in the remaining two subsections.

4.4. Divergence of $U_t$. We prove the following

**Lemma 4.5.** Let $U_t$ be as in (4.3). Then $d^* U_t$ takes values in a compact subset of $H^{s-2}(\Sigma; \mathbb{R})$.

**Proof.** By Lemma 4.3 we have

$$d^* U_t = d^* \left\{ \left( \Pi_{d\varphi_t^{-1}} \circ K_{\varphi_t} \circ \mathcal{G} \circ K_{\varphi_t^{-1}} \right) (\omega) \right\} + \text{Remainder},$$

with $\text{Remainder} \in H^{s-2}(\Sigma; \mathbb{R})$ for some $\sigma > s$.

Observe also that

$$d^* \left\{ \left( \Pi_{d\varphi_t^{-1}} \circ K_{\varphi_t} \circ \mathcal{G} \circ K_{\varphi_t^{-1}} \right) (\omega) \right\} = \Pi_{d\varphi_t^{-1}} \left\{ d^* \left( K_{\varphi_t} \circ \mathcal{G} \circ K_{\varphi_t^{-1}} (\omega) \right) \right\} + \text{Remainder},$$

with $\text{Remainder} \in H^{s-1}(\Sigma; \mathbb{R})$, since the commutator $[d^*, \pi_{d\varphi_t^{-1}}]$ is a pseudodifferential operator of order 0.

Next, thanks to Corollary 2.25 with $\rho = s - 1$ therein,

$$K_{\varphi_t} \circ \mathcal{G} \circ K_{\varphi_t^{-1}} (\omega) = \left[ \mathcal{G} \circ K_{\varphi_t^{-1}} (\omega) \right] \circ \varphi_t + \text{Remainder},$$

with $\text{Remainder} \in H^{s-1}(\Sigma; \mathbb{R})$ for some $\sigma > s$. Indeed, the paracomposition leads to a term

$$[\text{BAD}] := \Pi_{\partial \left[ \mathcal{G} \circ K_{\varphi_t^{-1}} (\omega) \right] \circ \varphi_t};$$

for simplicity we abbreviate it as $\Pi A \varphi_t$. However, $\omega \in H^{s-1}$ and hence $A \in H^{s-2}$, for which we cannot directly apply the estimate in Theorem 2.2 it needs $A \in L^\infty$ thereof, but $H^{s-2}$ does not embed in $L^\infty$ when merely assuming $s > 2$.

To this end, we show by means of hands-on estimates that for $s = 2 + \varepsilon$, $\rho = 1 + 2\varepsilon$ for any $\varepsilon \in [0, 1]$, we can indeed bound the $H^\rho$-norm of $[\text{BAD}]$. Note that

$$[\text{BAD}] = (1 - \Delta_g)^{-\varepsilon/2} \left( \left\langle A, \Delta^{\varepsilon/2} \varphi_t \right\rangle \right).$$

Then, by Sobolev embedding, we have $A \in H^\varrho \hookrightarrow L^\infty$ and $\Delta^{\varepsilon/2} \varphi_t \in H^{1-\varepsilon} \hookrightarrow L^2$, so Cauchy–Schwarz gives us $\left\langle A, \Delta^{\varepsilon/2} \varphi_t \right\rangle \in L^2$. (Here $\langle \bullet, \bullet \rangle$ is a quadratic function with bounded coefficient.) The $H^\varrho \hookrightarrow H^\varrho$ boundedness of the Riesz transform shows that $[\text{BAD}]$ is bounded in $H^\varrho = H^{1+2\varepsilon}$, which is compactly embedded into $H^{s-1} = H^{1+\varepsilon}$ by the Rellich lemma.

Finally, by construction of the Biot–Savart operator in (4.4) $\mathcal{G}$ takes values in the $d^*$-free part of differential 1-forms. So we have

$$d^* \left\{ \left[ \mathcal{G} \circ K_{\varphi_t^{-1}} (\omega) \right] \circ \varphi_t \right\} = 0.$$

In summary, the above computations yield that $d^* U_t = 0 + \text{Remainder} \in H^{s-2}(\Sigma; \mathbb{R})$ for some $\sigma > s$. The proof is complete by the Rellich lemma.

4.5. $\tilde{B}$ is Fredholm. Finally, we prove that

**Lemma 4.6.** The mapping $\tilde{B} : H^{s-1}(\Sigma; \mathbb{R}) \to H^{s-1}(\Sigma; \mathbb{R})$ defined in Proposition 4.4 is Fredholm of index zero.
Proof. The previous arguments show that

\[ \tilde{\mathcal{B}}(\omega) = \ast d \left[ \left( \Pi_{d\varphi_t^{-1}} \circ \mathcal{K}_{\varphi_t} \circ \mathcal{G} \circ \mathcal{K}_{\varphi_t^{-1}} \right) (\omega) \right]^2 + \text{Remainder}, \]

where \( \text{Remainder} \in H^{s-1}(\Sigma; \mathbb{R}) \) for \( s > 0 \), thus having no effect on the Fredholm index. [MAIN] is a pseudodifferential operator of order zero; this is because \( \Pi, \mathcal{K} \) are of order 0, \( \mathcal{G} \) is of order \(-1\), and \( d \) is of order \(1\). Thus, to prove the thesis, it suffices to check that [MAIN] has a nonvanishing principal symbol, viewed as a function defined on the cotangent bundle \( T^*\Sigma \).

Recall that the principal symbol for the Biot–Savart operator \( \mathcal{G} \) has already been computed in (3.9). Moreover, the conjugation with \( \mathcal{K}_{\varphi_t} \) contributes a factor of \( (d\varphi_t^*)^{-1} \) to the symbol, where the asterisk denotes the operator adjoint. Using once again the symbol for \( d \) (see (3.7)) and its expression on \((\Sigma, g)\) via the almost complex structure \( J \), we have

\[
\sigma_{ppl}([\text{MAIN}]) (x, \xi) = 2\pi \sqrt{-1} \left( \frac{-\sqrt{-1}}{2\pi} \cdot \left\langle \frac{J (d\varphi_t^*)^{-1} \xi}{(d\varphi_t^*)^{-1} \xi} \right\rangle_g \right).
\]

It is clear from (4.13) that the principal symbol for [MAIN] is 0-homogeneous in the fibre variable \( \xi \). Then it suffices to show the ellipticity of [MAIN]; that is, \( \sigma_{ppl}([\text{MAIN}]) (x, \xi) = 0 \) implies that \( \xi = 0 \).

For this purpose, we infer from (4.13) and the identity \((d\varphi_t^{-1})^* = (d\varphi_t^*)^{-1}\) that

\[
\sigma_{ppl}([\text{MAIN}]) (x, \xi) = \frac{\left\langle (d\varphi_t^*)^{-1} J\xi, (d\varphi_t^*)^{-1} J\xi \right\rangle_g}{\left\langle (d\varphi_t^*)^{-1} \xi \right\rangle_g^2}.
\]

Since \((\Sigma, g)\) is a compact orientable Riemannian manifold of dimension two, it is Kähler; hence, there exists a symplectic 2-form \( \omega \in \Omega^2(\Sigma) \) such that \((g, J, \omega)\) is a compatible triple. As a consequence, one has

\[
\sigma_{ppl}([\text{MAIN}]) (x, \xi) = \frac{\omega \left( (d\varphi_t^*)^{-1} \xi, (d\varphi_t^*)^{-1} J\xi \right)}{\left\langle (d\varphi_t^*)^{-1} \xi \right\rangle_g^2}.
\]

That is,

\[
\sigma_{ppl}([\text{MAIN}]) (x, \xi) = \tilde{\omega} (\xi, J\xi) \left\langle (d\varphi_t^*)^{-1} \xi \right\rangle_g^2
\]

where \( \tilde{\omega} \) is the pushforward 2-form:

\[
\tilde{\omega} := (\varphi_t)_\# \omega \in \Omega^2(\Sigma).
\]
This is well-defined as $\varphi$ is a diffeomorphism, and $\tilde{\omega}$ is also a symplectic form (i.e., closed and nondegenerate). Then $\tilde{\omega}(\bullet, J\bullet)$ defines a Riemannian metric on $\Sigma$; call it $\tilde{g}$. Thus

$$
\sigma_{\text{pl}}([\text{MAIN}]) (x, \xi) = \frac{|\xi|^2_{\tilde{g}}}{| (d\varphi^*)^{-1} \xi |^2_g} = \frac{\tilde{g}(\xi, \xi)}{g\left( (d\varphi^*)^{-1} \xi, (d\varphi^*)^{-1} \xi \right)}
$$

This proves that $[\text{MAIN}]$ is elliptic; hence, $\tilde{B}$ is Fredholm of index zero. \hfill \Box

### 4.6. Completion of the proof

Theorem 1.4 now follows from Proposition 4.4, Lemma 4.5, and Lemma 4.6.

### A. Geometric preliminaries

We collect a few notations and preliminaries on differential geometry in the appendix.

For an $n$-dimensional Riemannian manifold $(M, g)$, we denote by $\sharp: TM \to T^* M, v \mapsto v^\sharp$ the canonical isomorphism between tangent and cotangent bundles — that is, if $\{\partial_1, \ldots, \partial_n\} \subset \Gamma(TM)$ is a local co-ordinate frame such that $g = g_{ij} dx^i \otimes dx^j$ with respect to its dual coframe $\{dx^1, \ldots, dx^n\} \subset \Gamma(T^*M)$, then $v = v^i \partial_i$ if and only if $v^\sharp = v_j dx^j$, where $v_j = g_{ij} v^i$. The inverse operator of $\sharp$ is denoted as $\flat$.

Also, write $\Omega^j(M) := \Gamma(\bigwedge^j T^* M)$ for the space of differential $j$-forms on $M$. For the exterior differential $d: \Omega^j(M) \to \Omega^{j+1}(M)$, define as usual the codifferential $d^* : \Omega^{j+1}(M) \to \Omega^j(M)$ as the $L^2$-adjoint of $d$, namely that with respect to the $L^2$-inner product $\langle \bullet, \bullet \rangle$ induced by $g$. In this setting,

$$
\text{div} \, v = d^* (v^\sharp) \quad \text{for} \quad v \in \Gamma(TM).
$$

Moreover, the Laplace–Beltrami operator $\Delta_g$ defined on the space of differential $p$-forms $\Omega^p(M)$ is defined by

$$
\Delta_g = dd^* + d^* d.
$$

We may also write $(\bullet, \bullet) = \langle \bullet, \bullet \rangle_g$ to stress that the inner product is taken with respect to $g$.

For $j \in \{0, 1, \ldots, n\}$, let $\ast : \Omega^j(M) \to \Omega^{n-j}(M)$ be the Hodge star operator. We emphasise that $\ast$ in our work is taken with respect to the Riemannian metric $g$. That is, for any $\omega \in \Omega^j(M)$, define $\ast \omega$ as the $(n-j)$-form such that for any $\zeta \in \Omega^j(M)$,

$$
\zeta \wedge \ast \omega = \langle \zeta, \omega \rangle \ \text{dVol}_g.
$$

Here $\text{dVol}_g \in \Omega^n(M)$ is the Riemannian volume form with respect to $g$.

For a Riemannian manifold $(M, g)$, $\nabla$ always denotes the Levi-Civita connection, $B(x, r)$ denotes the geodesic balls, $d_g$ is the Riemannian distance, $|\bullet|$ is the Riemannian length of vectors, and $\text{Vol}_g$ is the Riemannian volume on $(M, g)$. In addition, we write $\text{Riem}_g$, $\text{Ric}_g$, and $\text{sec}_g$ for the Riemann, Ricci, and sectional curvatures on $(M, g)$, respectively.
The above discussions on ♯, ♫, ♭, ♯♯, ♦♦, *... all carry over to geometric quantities with Sobolev regularity. The superscripts/subscripts indicating the dependence on \( g \) will be suppressed when there is no danger of confusion for the metric.

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