A CAUCHY–DIRAC DELTA FUNCTION

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Abstract. The Dirac δ function has solid roots in 19th century work in Fourier analysis and singular integrals by Cauchy and others, anticipating Dirac's discovery by over a century, and illuminating the nature of Cauchy's infinitesimals and his infinitesimal definition of δ.

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1. DIRAC ON INFINITY

The specialisation of the scientific disciplines since the 19th century has led to a schism between the scientists' pragmatic approaches (e.g., using infinitesimal arguments), on the one hand, and mathematicians' foundational preferences, on the other. The widening schism between mathematics and physics has been the subject of much soul-searching, see, e.g., S.P. Novikov [61].

In an era prior to such a schism, a seminal interaction between physics and mathematics was foreshadowed in a remarkable fashion, exploiting infinitesimals, in Cauchy’s texts from 1827. Cauchy defined the so-called Dirac delta function in terms of infinitesimals, and applied it to the evaluation of singular integrals and in Fourier analysis.

Let us first review P. Dirac’s well-known discussion of $\delta$. Dirac writes in his § 10:

Consider now the single integral

$$\int \langle \xi' x | \xi'' y \rangle d\xi''.$$  \hspace{1cm} (29)

From the orthogonality theorem, the integrand here must vanish over the whole range of integration except the one point $\xi'' = \xi'$. [...] in general $\langle \xi' x | \xi' y \rangle$ must be infinitely great in such a way as to make (29) non-vanishing and finite. The form of infinity required for this will be discussed in § 15 (Dirac 1958, [19, p. 39]).

Note that Dirac explicitly speaks of “infinitely great” values of the integrand in his formula (29), and does not shy away from discussing “the form of infinity required”. Dirac resumes his discussion of $\delta$ in his § 15 in the following terms:

Our work in § 10 led us to consider quantities involving a certain kind of infinity. To get a precise notation for dealing with these infinities, we introduce a quantity $\delta(x)$ depending on a parameter $x$ satisfying the conditions

$$\begin{cases} \int_{-\infty}^{\infty} \delta(x) dx = 1 \\ \delta(x) = 0 \text{ for } x \neq 0 \end{cases}$$

(Dirac 1958, [19, p. 58]).

Dirac views his $\delta$ as a way of “dealing with these infinities”. Dirac is not content with merely providing such an algebraic definition, and proceeds to explain how one can “get a picture of $\delta(x)$”: 
take a function of the real variable $x$ which vanishes everywhere except inside a small domain, of length $\epsilon$ say, surrounding the origin $x = 0$, and which is so large inside this domain that its integral over this domain is unity [\ldots] Then in the limit $\epsilon \to 0$ this function will go over into $\delta(x)$ (ibid.).

Here Dirac is using the expression “in the limit” in a generic sense. Furthermore,

The most important property of $\delta(x)$ is exemplified by the following equation,

$$\int_{-\infty}^{\infty} f(x)\delta(x) = f(0),$$

where $f(x)$ is any continuous function of $x$ (Dirac [19, p. 59]).

The above equation is identical to one appearing in Cauchy a century earlier. An additional intriguing formula appears in the context of Dirac’s discussion of the discontinuity of the principal value of the log function:

$$\frac{d}{dx} \log x = \frac{1}{x} - i\pi\delta(x),$$

(1.1)

accompanied by a comment to the effect that “this particular combination of reciprocal function and $\delta$ function plays an important role in the quantum theory of collision processes” (Dirac [19, p. 61]).

2. Modern Cauchy scholarship

The earliest mention of Cauchy’s $\delta$ appears to be in H. Freudenthal.

2.1. Freudenthal’s biography. Cauchy’s anticipations of $\delta$ were alluded to in 1971 by Freudenthal, who mentioned $\delta$ functions twice in his Cauchy biography for the Dictionary of Scientific Biography [27]. They were the subject of scholarly attention by J. Lützen (1982, [57]) and D. Laugwitz (1989, [52]; 1992 [53]). A spate of recent articles have

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1Dirac does not specify a function, merely mentioning that the procedure works “provided there are no unnecessarily wild variations”. A century earlier Cauchy did specify a function (see formula (3.2) and footnote [11]).

2Dirac’s “limit” can be interpreted mathematically either in terms of distribution theory à la Sobolev and Schwartz, or in the sense of an ultraproduct.

3See formula (3.1).

4The imaginary part of the principal value of $\log(x)$ in (1.1) is $0$ for $x > 0$ and $i\pi$ for $x < 0$. Thus, $\text{Im}(\log(x)) = -i\pi H(x)$ where $H(x)$ is the Heaviside function (which equals $-1$ for negative $x$ and $0$ for positive $x$), whose derivative is $\delta(x)$ in the sense discussed in Section 9.
undertaken a re-evaluation of Cauchy’s foundational contribution to the development of infinitesimal analysis: Bråting (2007, [9]), Barany (2011, [4]), Blaszczyk et al. (2012, [5]), Borovik et al. (2011, [7]), Katz & Katz [41, 39]. The present text will focus on Cauchy’s anticipations of the Dirac delta.

J. Lützen [57] traces the origins of distribution theory (traditionally attributed to S. Sobolev and L. Schwartz), in 19th century work of Fourier, Kirchhoff, and Heaviside. An accessible introduction to this area may be found in J. Dieudonné [18].

2.2. Dieudonné’s review. Dieudonné is one of the most influential mathematicians of the 20th century. A fascinating glimpse into his philosophy is provided by his 1984 review [18] of J. Lützen’s book. At the outset, Dieudonné poses the key question:

One […] may well wonder why it took more than 30 years for distribution theory to be born, after the theory of integration had reached maturity (Dieudonné [18, p. 374]).

This remark inscribes itself in a tradition of a long-established and pervasive dogma, to the effect that 20th century physicists were the first to invent the delta function. Thus, M. Bunge, responding to Robinson’s lecture The metaphysics of the calculus, evoked the physicist’s custom to refer to the theory of distributions for the legalization of the various delta ‘functions’ which his physical intuition led him to introduce (Bunge in Robinson, 1967 [65, p. 44-45]; [66, p. 553-554]).

But did Dirac introduce the delta function? Laugwitz [52, p. 219] notes that probably the first appearance of the (Dirac) delta function is in the 1822 text by Fourier [24].

In his review of J. Lützen’s book for Math Reviews, F. Smithies notes:

Chapter 4, on early uses of generalized functions, covers fundamental solutions of partial differential equations, Hadamard’s “partie finie”, and many early uses of the delta function and its derivatives, including various attempts to create a rigorous theory for them (Smithies, [72]).

At the end of his review, Smithies mentions Cauchy: “In spite of the thoroughness of his coverage, [Lützen] has missed one interesting event—A. L. Cauchy’s anticipation of Hadamard’s ‘partie finie’ ”, but says not a word about Cauchy’s infinitesimally-defined delta functions.
2.3. Gilain on limits. Cauchy’s use of infinitesimals is not always reported accurately in the literature. Thus C. Gilain claims that

On sait que Cauchy définissait le concept d’infiniment petit à l’aide du concept de limite, qui avait le premier rôle (voir Analyse algébrique, p. 19 . . . ) (Gilain, 1989, [28, footnote 67]).

Here Gilain is referring to Cauchy’s Collected Works, Série 2, Tome 3, p. 19, corresponding to (Cauchy 1821, [10, p. 4]). Both of Gilain’s claims are erroneous, as we show below. Already in 1973, Hourya Benis Sinaceur warned:

on dit trop rapidement que c’est Cauchy qui a introduit la ‘méthode des limites’ entendant par là, plus ou moins vaguement, l’emploi systématique de l’épsilonisation (Sinaceur 1973, [70, p. 108]).

Sinaceur pointed out that Cauchy’s definition of limit resembles, not that of Weierstrass, but rather that of Lacroix dating from 1810 (see Sinaceur [70, p. 109]).

Cauchy’s primary notion is that of variable quantities, introduced in the following terms:

On nomme quantité variable celle que l’on considère comme devant recevoir successivement plusieurs valeurs différentes les unes des autres (Cauchy 1821, [10, p. 4]).

Next, Cauchy exploits his primary notion to evoke his kinetic concept of limit in the following terms:

Lorsque les valeurs successivement attribuées à une même variable s’approchent indéfiniment d’une valeur fixe, de manière à finir par en différer par aussi peu qu’on voudra, cette dernière est appelée la limite de toutes les autres (ibid.).

Finally, Cauchy proceeds to define infinitesimals in the following terms:

Lorsque les valeurs numériques successives d’une même variable décroissent indéfiniment, de manière à s’abaisser au-dessous de tout nombre donné, cette variable devient ce qu’on nomme un infinité petit ou une quantité infiniment petite. Une variable de cette espèce a zéro pour limite (ibid.).

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5Translation: “We know that Cauchy defined the concept of infinitely small by means of the concept of limit, which played the primary role” (Gilain).
Thus, Cauchy defined both infinitesimals and limits in terms of variable quantities. Neither is the limit concept primary, nor are infinitesimals defined in terms of limits, contrary to Gilain’s claims.

2.4. Hawking and Gray. In his 2007 anthology, S. Hawking reproduces Cauchy’s *infinitesimal* definition of continuity on page 639:

the function \( f(x) \) remains continuous with respect to \( x \) between the given bounds[6] if, between these bounds, an infinitely small increment in the variable always produces an infinitely small increment in the function itself (Hawking, [33, p. 639]).

But Hawking also claims on the same page 639, in a comic *non-sequitur*, that Cauchy “was particularly concerned to banish infinitesimals”.

In a similar vein, historian J. Gray lists *continuity* among concepts Cauchy allegedly defined using careful, if not altogether unambiguous, *limiting* arguments (Gray 2008, [32, p. 62]) [emphasis added—authors]. But in point of fact *limits* appear in Cauchy’s definition only in the sense of the *extremities* (endpoints, bounds) of the domain of definition.\(^7\) Contrary to Gray’s claim, the arguments Cauchy used to define continuity were not “limiting” but rather infinitesimal.

2.5. Laugwitz replies. Dieudonné’s query mentioned in Subsection 2.2 is answered by Laugwitz,\(^8\) who argues that objects such as delta functions (and their potential applications) disappeared from the literature due to the elimination of infinitesimals, in whose absence they could not be sustained. Laugwitz notes that

Cauchy’s use of delta function methods in Fourier analysis and in the summation of divergent integrals enables us to analyze the change of his attitude toward infinitesimals (Laugwitz 1989, [52, p. 232]).

3. From Cauchy to Dirac

A function of the type generally attributed to Dirac (1902–1984) was specifically described in 1827 by Cauchy in terms of infinitesimals. More specifically, Cauchy uses a unit-impulse, infinitely tall, infinitely

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6The term *bounds* is Hawking’s translation of Cauchy’s *limites*.

7See footnote 6.

8Laugwitz reports having been influenced by Freudenthal’s allusion to Cauchy’s work on delta functions in [27, p. 136].
narrow delta function, as an integral kernel. Thus, in 1827, Cauchy used infinitesimals in his definition of a “Dirac” delta function \[11\] p. 188]. Here Cauchy uses infinitesimals \(\alpha\) and \(\epsilon\), where \(\alpha\) is, in modern terms, the “scale parameter” of the “Cauchy distribution”, whereas \(\epsilon\) gives the size of the interval of integration. Cauchy wrote \[11\] p. 188]:

Moreover one finds, denoting by \(\alpha\), \(\epsilon\) two infinitely small numbers,

\[
\frac{1}{2} \int_{a-\epsilon}^{a+\epsilon} F(\mu) \frac{\alpha d\mu}{\alpha^2 + (\mu - a)^2} = \frac{\pi}{2} F(a) \tag{3.1}
\]

(Cauchy’s 1815-1827 text is analyzed in more detail in Section \[4\]). A formula equivalent to (3.1) was proposed by Dirac a century later. The expression

\[
\frac{\alpha}{\alpha^2 + (\mu - a)^2} \tag{3.2}
\]

(for real \(\alpha\)) is known as the Cauchy distribution in probability theory. Here Cauchy specifies a function which meets the criteria as set forth by Dirac a century later. In modern terminology, the function is called the probability density function, and the parameter \(\alpha\) is referred to as the scale parameter. Cauchy integrates \(F\) against the kernel (3.2) as in (3.1) so as to extract the value of \(F\) at the point \(a\), exploiting the characteristic property of a delta function. Thus, a Cauchy distribution with an infinitesimal scale parameter produces an entity with Dirac-delta function behavior, exploited by Cauchy already in 1827.

Laugwitz notes that Cauchy’s formula (3.1) is satisfied when \(\epsilon \geq \alpha^{1/2}\) (as well as for all positive real values of \(\epsilon > 0\)). Today we would write the integral in the formally more elegant fashion as an integral over an infinite domain:

\[
\frac{1}{2} \int_{-\infty}^{\infty} F(\mu) \frac{\alpha d\mu}{\alpha^2 + (\mu - a)^2} = \frac{\pi}{2} F(a), \tag{3.1}
\]

but Cauchy, ever the practical man rather than a formalist, saw no reason to bother with subjunctive integration over domains where the function is perceptually indistinguishable from zero anyway.

Furthermore, Laugwitz documents Cauchy’s use of

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9See Laugwitz \[52\] p. 230].
10See main text at footnote 8.
11See footnote 11 for a discussion of Dirac’s criterion.
12From the modern viewpoint, formula (3.1) holds up to an infinitesimal error; thus, the right hand side is the standard part of the left hand side, see Section 8.
13Modulo the proviso of footnote 12.
an explicit delta function not contained under an integral sign (Laugwitz 1989, [52, p. 231]) [emphasis added—authors],

contrary to a claim in Dieudonné’s text. An occurrence of a delta function not under an integral sign in Cauchy’s work is discussed in Section 5.

In his 1908 book *Elementary Mathematics from an Advanced Standpoint*, Felix Klein points out that the

naïve [perceptual] methods always rise to unconscious importance whenever in mathematical physics, mechanics, or differential geometry a preliminary theorem is to be set up. You all know that they are very serviceable then (Klein [50, p. 211]).

These remarks apply perfectly well to Cauchy’s δ, as well. On the other hand, Klein is perfectly aware of the situation on the ground:

To be sure, the pure mathematician is not sparing of his scorn on these occasions. When I was a student it was said that the differential, for a physicist, was a piece of brass which he treated as he did the rest of his apparatus (ibid).

Additional remarks by Klein, showing the importance he attached to this vital connection, appear in Section 10.

4. CAUCHY’S NOTE XVIII

Cauchy’s lengthy work *Théorie de la propagation des ondes à la surface d’un fluide pesant d’une profondeur indéfinie* [11] was written in 1815. The manuscript was published in 1827 as a 300-page text, with a number of additional Notes at the end. The running shortened title used throughout is *Mémoire sur la théorie des ondes*.

Note XVIII, entitled *Sur les intégrales définies singulières et les valeurs principales des intégrales indéterminées*, starts on page 288. Cauchy recalls the notion of a singular definite integral, describing it in terms of an integrand that becomes “infinite or indeterminate”. He continues by denoting by ε an “infinitely small number”15 and by a, b

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14See discussion of Dieudonné’s claim in footnote 20
15Cauchy’s use of term “number”, rather than “quantity”, in this context is interesting. In general he refers to infinitesimals as “quantities”.
two positive constants. On page 289, after choosing an additional “infinitely small number” \( \alpha \), Cauchy writes down the integral

\[
\frac{1}{2} \int_{a-\epsilon}^{a+\epsilon} F(\mu) \frac{\alpha d\mu}{\alpha^2 + (\mu - a)^2} = \frac{\pi}{2} F(a)
\]

(already reproduced as formula (3.1) above), which he denotes by (2). Cauchy proceeds to point out that, since the integrand of his equation (2) is sensiblement égale à zéro [essentially equal to zero] for all values of \( \mu \) qui ne sont pas très rapprochées de \( a \) [which are not too close to \( a \)], it follows that the integrals appearing in his earlier Note VI reduce to singular integrals determined by his equation (2).

Note XVIII then proceeds to discuss principal values and to offer alternative derivations of a number of earlier results, and is concluded on page 299.

5. Cauchy’s 1827 Mémoire

An additional occurrence of a delta function occurs in Cauchy’s brief 1827 text Mémoire sur les développements des fonctions en séries périodiques [12]. The text contains an (a priori doomed) attempt to prove the convergence of Fourier series under the sole assumption of continuity. What concerns us here is his, correct, use of infinitesimals at a certain stage in the argument. Cauchy opens his mémoire with a discussion of the importance of what are known today as Fourier series, in “a large number of problems of mathematical physics” [12, p. 12]. On page 13, Cauchy denotes by \( \epsilon \) un nombre infiniment petit [an infinitely small number], lets \( \theta = 1 - \epsilon \), and lets \( x \) be between 0 and \( a = 2\pi \). On page 14, he points out that the expression

\[
1 + \frac{1}{e^{-i(x-\mu)} - \theta} + \frac{1}{e^{i(x-\mu)} - \theta}
\]

(his notation is slightly different) “will be essentially zero, except when \( \mu \) differs very little from \( x \”). Note that the expression \( (5.1) \) appearing on Cauchy’s page 14, does not occur under the integral sign (it was exploited as a kernel in the last formula on the previous page 13).

Cauchy then sets \( \mu = x + iw \) and concludes that the integral will be essentially reduced to

\[
f(x) \cdot \int_{\frac{x-\epsilon}{\epsilon}}^{\frac{2\pi-x}{\epsilon}} \left( \frac{1}{1 + iw} + \frac{1}{1 - iw} \right) dw = 2\pi f(x).
\]

\[16\]Cauchy wrote \( a \) where we wrote \( 2\pi \).
6. Triumvirate history

Studies seeking to document continuity between Cauchy’s infinitesimals and modern set-theoretic implementations of infinitesimals have not always been viewed sympathetically by commentators.

In 1949, C. Boyer hagiographically described G. Cantor, R. Dedekind, and K. Weierstrass as “the great triumvirate” [8, p. 298]. The “triumvirate” historian tends to view the history of mathematics as an ineluctable march toward the radiant future of Weierstrassian epsilontics. Such a stance leaves little room for a sympathetic view of a continuity between infinitesimals as practiced prior to the triumvirate, on the one hand, and 20th century set-theoretic implementations of infinitesimals, on the other. J. Dauben wrote:

In bringing historians of the calculus to task, Robinson was particularly critical of Carl Boyer’s *The Concepts of the Calculus*. The history of mathematics was on shaky ground, Robinson felt, if it chose to pass judgment on earlier theories based upon currently fashionable prejudices. Nonstandard analysis cast a new light on the history of the calculus, and Robinson was interested to see how it might appear if reexamined without assuming that infinitesimals were wrongheaded or at all lacking in rigor (Dauben 1995, [17, p. 349]).

Dauben appears sympathetic to Robinson’s interest in re-examining the history of infinitesimals. It is all the more surprising therefore to read a text Dauben authored a few years earlier. Here Dauben presents a list of authors, including D. Laugwitz, who “have used nonstandard analysis to rehabilitate or ‘vindicate’ earlier infinitesimalists”, and concludes:

Leibniz, Euler, and Cauchy [...] had, in the views of some commentators, “Robinsonian” nonstandard infinitesimals in mind from the beginning. The most detailed and methodically [sic] sophisticated of such treatments to date is that provided by Imre Lakatos; in what follows, it is his analysis of Cauchy that is emphasized (Dauben 1988, [16, p. 179]).

However, Lakatos’s treatment was certainly not “the most detailed and methodically sophisticated” one by the time Dauben’s text appeared in 1988. Thus, in 1987, Laugwitz had published a detailed scholarly study of Cauchy in *Historia Mathematica* (Laugwitz [51]). Laugwitz’s text in *Historia Mathematica* appears to be the published version of his 1985
preprint Cauchy and infinitesimals. Laugwitz’s 1985 preprint does appear in Dauben’s bibliography (Dauben, 1988 [16, p. 199]), indicating that Dauben was familiar with it. It is odd to suggest, as Dauben seems to, that a scholarly study published in Historia Mathematica would countenance a view that Cauchy could have had “Robinsonian” non-standard infinitesimals in mind from the beginning”. Surely Dauben has committed a strawman fallacy here.

Rather, Lakatos, Laugwitz, Cutland et al. [15], Bråting (2007, [9]) and others have argued that infinitesimals as employed by Cauchy have found set-theoretic implementations in the framework of modern theories of infinitesimals, just as Kanovei had done for Euler in 1988 [36]. The existence of such implementations indicates that the historical infinitesimals were less prone to contradiction than has been routinely maintained by triumvirate historians, who invariably cite Berkeley’s flawed empiricist critique. The issue is dealt with in more detail by Katz & Katz [41], [39], [42], [40]; Blaszczyk et al. [5]; Borovik et al. [7]; Katz & Leichtnam [43]; and Katz & Sherry [45, 46].

What criteria can we employ to evaluate the achievements of modern theories of infinitesimals? Remarkably, just such a criterion was explicitly provided by both Felix Klein and Abraham Fraenkel, as discussed in Section [7].

7. Klein–Fraenkel criterion

In 1908, Klein formulated a criterion of what it would take for a theory of infinitesimals to be successful. Namely, one must be able to prove a mean value theorem for arbitrary intervals, including infinitesimal ones:

The question naturally arises whether [...] it would be possible to modify the traditional foundations of infinitesimal calculus, so as to include actually infinitely small quantities in a way that would satisfy modern demands as to rigor; in other words, to construct a non-Archimedean system. The first and chief problem of this analysis would be to prove the mean-value theorem

\[ f(x + h) - f(x) = h \cdot f'(x + \vartheta h) \]

from the assumed axioms. I will not say that progress in this direction is impossible, but it is true that none of the investigators have achieved anything positive (Klein 1908, [50, p. 219]).
In 1928, A. Fraenkel [25] pp. 116-117 formulated a similar requirement in terms of the mean value theorem. Such a Klein–Fraenkel criterion is satisfied by the Hewitt-Łoś-Robinson theory. Indeed, the mean value theorem is true for an arbitrary hyperreal interval by the transfer principle. Fraenkel’s opinion of Robinson’s theory is on record:

my former student Abraham Robinson had succeeded in saving the honour of infinitesimals - although in quite a different way than Cohen[17] and his school had imagined (Fraenkel 1967, [26, p. 107]).

8. Set-theoretic implementation of infinitesimals

In Section 6 we clarified the role of modern theories of infinitesimals in interpreting the work of historical infinitesimalists. Here we present some details of a particular set-theoretic implementation of infinitesimals as developed through the work of Hewitt, Łoś, and Robinson. For an alternative approach to infinitesimals, see P. Giordano [29].

In 1961, Robinson [64] constructed an infinitesimal-enriched continuum, suitable for use in calculus, analysis, and elsewhere, based on earlier work by E. Hewitt [35], J. Łoś [56], and others. In 1962, W. Luxemburg [58] popularized a presentation of Robinson’s theory in terms of the ultrapower construction[18] in the mainstream foundational framework of the Zermelo-Fraenkel set theory with the axiom of choice. Namely, the hyperreal field is the quotient of the collection of arbitrary sequences, where a sequence

\[ \langle u_1, u_2, u_3, \ldots \rangle \] (8.1)

converging to zero generates an infinitesimal. Arithmetic operations are defined at the level of representing sequences; e.g., addition and multiplication are defined term-by-term.

To motivate the construction of the hyperreals, we will start with the ring \( \mathbb{Q}^N \) of sequences of rational numbers. Let \( \mathcal{C}_\mathbb{Q} \subset \mathbb{Q}^N \) denote the subring consisting of Cauchy sequences. The reals are by definition the

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17The reference is to Hermann Cohen (1842–1918), whose fascination with infinitesimals elicited fierce criticism by both Cantor and B. Russell. For an analysis of Russell’s non-sequiturs, see Ehrlich [20] and Katz & Sherry [45].

18Note that both the term “hyper-real”, and an ultrapower construction of a hyperreal field, are due to E. Hewitt in 1948, see [35, p. 74]. Luxemburg [58] clarified its relation to the competing construction of Schmieden and Laugwitz [68], also based on sequences, but using the ideal \( F_{ez} \). Dauben [17, p. 395] mistakenly suggests that it was Luxemburg who initiated the ultrapower approach to the hyperreals using free ultrafilters.
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\[ \mathbb{R} := \mathbb{C}_Q / \mathcal{F}_{null} \]  \hspace{1cm} (8.2)

where \( \mathcal{F}_{null} \) contains all null sequences. An infinitesimal-enriched extension of \( \mathbb{Q} \) may be obtained by modifying (8.2) as follows. We consider a subring \( \mathcal{F}_{ez} \subset \mathcal{F}_{null} \) of sequences that are “eventually zero”, i.e., vanish at all but finitely many places. Then the quotient \( \mathbb{C}_Q / \mathcal{F}_{ez} \) naturally surjects onto \( \mathbb{R} = \mathbb{C}_Q / \mathcal{F}_{null} \). The elements in the kernel of the surjection \( \mathbb{C}_Q / \mathcal{F}_{ez} \to \mathbb{R} \) are prototypes of infinitesimals. Note that the quotient \( \mathbb{C}_Q / \mathcal{F}_{ez} \) is not a field, as \( \mathcal{F}_{ez} \) is not a maximal ideal. It is more convenient to describe the modified construction using the ring \( \mathbb{R}^N \) rather than \( \mathbb{C}_Q \).

We therefore redefine \( \mathcal{F}_{ez} \) to be the ring of sequences in \( \mathbb{R}^N \) that eventually vanish, and choose a \textit{maximal} proper ideal \( \mathcal{M} \) so that we have

\[ \mathcal{F}_{ez} \subset \mathcal{M} \subset \mathbb{R}^N. \]  \hspace{1cm} (8.3)

Then the quotient \( \mathbb{III} := \mathbb{R}^N / \mathcal{M} \) is a hyperreal field. The foundational material needed to ensure the existence of a maximal ideal \( \mathcal{M} \) satisfying (8.3) is weaker than the axiom of choice. This concludes the construction of a hyperreal field \( \mathbb{III} \) in the traditional foundational framework, Zermelo-Fraenkel set theory with the axiom of choice (ZFC).

Let \( I \subset \mathbb{III} \) be the subring consisting of infinitesimal elements (i.e., elements \( e \) such that \( |e| < \frac{1}{n} \) for all \( n \in \mathbb{N} \)). Denote by \( I^{-1} \) the set of inverses of nonzero elements of \( I \). The complement \( \mathbb{III} \setminus I^{-1} \) consists of all the finite (sometimes called \textit{limited}) hyperreals. Constant sequences provide an inclusion \( \mathbb{R} \subset \mathbb{III} \). Every element \( x \in \mathbb{III} \setminus I^{-1} \) is infinitely close to some real number \( x_0 \in \mathbb{R} \). The \textit{standard part function}, denoted “\( \text{st} \)”, associates to every finite hyperreal, the unique real infinitely close to it:

\[ \text{st} : \mathbb{III} \setminus I^{-1} \to \mathbb{R}, \quad x \mapsto x_0. \]

If \( x \) happens to be represented by a Cauchy sequence \( \langle x_n : n \in \mathbb{N} \rangle \), so that \( x = [\langle x_n \rangle] \), then the standard part can be expressed in terms of the ordinary limit:

\[ \text{st}(x) = \lim_{n \to \infty} x_n. \]

More advanced properties of the hyperreals such as saturation were proved later (see Keisler [19] for a historical outline). A helpful “semicolon” notation for presenting an extended decimal expansion of a hyperreal was described by A. H. Lightstone [54]. See also P. Roquette [67] for infinitesimal reminiscences. A discussion of infinitesimal optics is in

\footnote{Such a construction is usually attributed to Cantor, and is actually due to Méray 1869, [59] who published three years earlier than E. Heine.}
K. Stroyan [73], J. Keisler [48] and others. Applications of infinitesimal-enriched continua range from aid in teaching calculus [21, 37, 38] to the Boltzmann equation (see L. Arkeryd [2, 3]) and mathematical physics (see Albeverio et al. [11]). Edward Nelson [60] in 1977 proposed an alternative to ZFC which is a richer (more stratified) axiomatisation for set theory, called Internal Set Theory (IST), more congenial to infinitesimals than ZFC. The traditional construction of the reals out of Cauchy sequences can be factored through the hyperreals (see Giordano et al. [30]). The hyperreals can be constructed out of integers (see Borovik et al. [6]).

9. Heaviside function

Having clarified the connection between historical infinitesimals and their modern set-theoretic implementations in Sections 6 and 8, we return to delta functions and their integrals.

Dieudonné’s review of Lützen’s book is assorted with the habitual, and near-ritual on the part of some mathematicians, expression of disdain for physicists:

However, a function such as the Heaviside function on $\mathbb{R}$, equal to 1 for $x \geq 0$ and to 0 for $x < 0$, has no weak derivative, in spite of its very mild discontinuity; at least this is what the mathematicians would say, but physicists thought otherwise, since for them there was a “derivative” $\delta$, the Dirac “delta function” (Dieudonné [18, p. 377]) [the quotation marks are Dieudonné’s—authors].

Dieudonné then proceeds to make the following remarkable claims:

Of course, there was before 1936 no reasonable mathematical definition of these objects; but it is characteristic that they were never used in bona fide computations except under the integral sign giving formulas such as

$$\int \delta(x - a) f(x) = f(a).$$

Are Dieudonné’s claims accurate? Dieudonné’s claim that, before 1936, delta functions occurred only under the integral sign, is contradicted

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20 We are not certain what bona fide calculations are exactly, but at any rate Dieudonné’s claim that delta functions were never used except under an integral sign, would be inaccurate; see main text at footnote 14.

21 Here we have simplified Dieudonné’s formula in [18, p. 377], by restricting to the special case $n = 0$. 
A CAUCHY–DIRAC DELTA FUNCTION

Figure 9.1. Heaviside function

by Cauchy’s use of a delta function not contained under an integral sign, over a hundred years earlier (see Section 5).

Are the physicists so far off the mark in speaking of the delta function as the derivative of the Heaviside function? Is it really true that there was no reasonable mathematical definition before 1936, as Dieudonné claims? In fact, Cauchy had a reasonable mathematical definition of a delta function, though of course both set theory and set-theoretic implementations of his ideas were still decades away.

Thus, consider the zigzag $Z \subset \mathbb{R}^2$ in the $(x, y)$-plane given by the union

$$Z = (\mathbb{R} \times \{-\frac{\pi}{2}\}) \cup (\{0\} \times [-\frac{\pi}{2}, \frac{\pi}{2}]) \cup (\mathbb{R} \times \{\frac{\pi}{2}\}) ,$$

(9.1)
thought of as the physicist’s Heaviside function, see Figure 9.1.

Now consider the graph of $\arctan(x)$ in the $(x, y)$-plane, and compress it toward the $y$-axis by means of a sequence of functions $\arctan(nx)$, or $\arctan(x/\alpha)$ where $\alpha = \frac{1}{n}$. Their derivatives $F_\alpha(x)$ satisfy

$$\int_{-\infty}^{\infty} F_\alpha = \pi$$

by the fundamental theorem of calculus. In an infinitesimal-enriched continuum (see Section 8), we can assign an infinitesimal value to $\alpha$. Then the graph

$$\Gamma_{\arctan(x/\alpha)}$$
of $\arctan(x/\alpha)$ is “appreciably indistinguishable” from the plane zigzag $Z \subset \mathbb{R}^2$ of formula (9.1). Instead of attempting to differentiate the

$^{22}$In modern notation, this relation would be expressed by the fact that the standard part “st” of the graph $\Gamma_{\arctan(x/\alpha)}$ coincides with the zigzag:

$$\text{st} (\Gamma_{\arctan(x/\alpha)}) = Z \subset \mathbb{R}^2.$$
zigzag itself as physicists are alleged to do, we differentiate its infinitesimal approximation \( \arctan(x/\alpha) \), and note that we obtain precisely Cauchy’s delta function appearing in formula (3.1), against which \( F \) is integrated.

10. **Klein’s remarks on physics**

As in the case of Cauchy’s delta function, infinitesimals provide an intuitive point of entry to key phenomena in both mathematics and physics. In a similar vein, Klein discussed infinitesimal oscillations of the pendulum in his *Elementary Mathematics from an Advanced Standpoint*. Klein presents the derivation of the pendulum law by pointing out that

\[
\frac{d^2 \phi}{dt^2} = \frac{g}{\ell} \sin \phi \quad \text{(Klein [50, p. 187]).}
\]

Here \( g \) is the gravitational constant, while \( \ell \) is the length of the thread by which the pendulum is suspended, and \( \phi \) is the angle of deviation from the normal. Klein continues:

For small amplitudes we may replace \( \sin \phi \) by \( \phi \) without serious error. This gives for the so called infinitely small oscillation of the pendulum

\[
\frac{d^2 \phi}{dt^2} = \frac{g}{\ell} \phi \quad \text{(ibid.).}
\]

Klein proceeds to write down the general solution

\[ \phi = C \cos \sqrt{\frac{g}{\ell}} (t - t_0), \]

and points out that the duration of a complete oscillation, i.e., the period

\[ T = 2\pi \sqrt{\frac{\ell}{g}}, \]

is independent of the amplitude \( C \). Reflecting upon the teaching practices at the time, Klein muses over the incongruity of

the curious phenomenon that one and the same teacher, during one hour, the one devoted to mathematics, makes the very highest demands as to the logical exactness of all conclusions. In his judgment [...] his demands are not satisfied by the infinitesimal calculus. In the next hour, however, that devoted to physics, he accepts the most questionable conclusions and makes the most daring applications of infinitesimals (ibid.).

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Here the function \( \arctan(x/\alpha) \) is the mathematical counterpart of the physicist’s Heaviside function. Of course, Cauchy did not have the notion of a standard part function (see Section 3), to express the idea that an error term is infinitesimal. Instead, he used the expression *sensiblement nulle* (sensibly nothing), see [52, p. 231].

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\[ ^{23} \text{Takeuti [74], Giorello [31], Lightstone and Wong [55], and later Todorov [75] and Péraire [63] have developed this theme further. Yamashita [76] provides a bibliography of articles dealing with hyperreal delta functions.} \]
Klein’s lament concerning the existence of such artificial boundaries is echoed by Novikov [61].

11. Charles Sanders Peirce’s framework

The customary set-theoretic framework that has become the reflexive litmus test of mathematical rigor in most fields of modern mathematics (with the possible exception of the field of mathematical logic) makes it difficult to analyze Cauchy’s use of infinitesimals, and to evaluate its significance. We will therefore use a conceptual framework proposed by C. S. Peirce in 1897, in the context of his analysis of the concept of continuity and continuum, which, as he felt at the time, is composed of infinitesimal parts, see [33, p. 103]. Peirce identified three stages in creating a novel concept:

there are three grades of clearness in our apprehensions of the meanings of words. The first consists in the connexion of the word with familiar experience. . . . The second grade consists in the abstract definition, depending upon an analysis of just what it is that makes the word applicable. . . . The third grade of clearness consists in such a representation of the idea that fruitful reasoning can be made to turn upon it, and that it can be applied to the resolution of difficult practical problems [62] (see Havenel 2008, [33, p. 87]).

The “three grades” can therefore be summarized as

1. familiarity through experience;
2. abstract definition aimed at applications;
3. fruitful reasoning “made to turn” upon it, with applications.

To apply Peirce’s framework to Cauchy’s notion of infinitesimal, we note that grade (1) is captured in Cauchy’s description of continuity of a function in terms of “varying by imperceptible degrees”. Such a turn of phrase occurs both in his letter to Coriolis (see Cauchy 1837, [13], and in his 1853 text [14, p. 35]. At Grade (2), Cauchy describes infinitesimals as generated by null sequences (see [9]), and defines continuity in terms of an infinitesimal \( x \)-increment resulting in an infinitesimal change in \( y \). Finally, at stage (3), Cauchy fruitfully applies the crystallized notion of infinitesimal both in Fourier analysis and in evaluation of singular integrals, by means of a “Dirac” delta function

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24 Note that both Cauchy’s original French “par degrés insensibles”, and its correct English translation “by imperceptible degrees”, are etymologically related to sensory perception.
defined in terms of a (Cauchy) distribution with an infinitesimal scaling parameter.

From the viewpoint of the Peircian framework, and contrary to Dieudonné’s claim, Cauchy did have a reasonable mathematical definition of a, Dirac, delta function, though both set-theory and set-theoretic implementations of infinitesimal-enriched continua in which Cauchy’s definition could be made operative were still decades away. Cauchy’s definition could be compared to the definitions of the fundamental concepts of infinitesimal calculus furnished by Newton and Leibniz. The founders of the calculus, similarly to Cauchy, lacked a set-theoretic formalisation of a continuum, and yet are (rightfully) given credit for the fundamental concepts they introduced. Cauchy exploited infinitesimals both in his definition of continuity in 1821, and in his definition of a notion closely related to uniform convergence in 1853 (see Katz & Katz [39] and Blaszczyk et al. [5]). The centrality of infinitesimals in Cauchy’s approach to analysis is further clarified through his use thereof in defining “Dirac” delta functions.

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