ON A CONJECTURE CONCERNING SOME AUTOMATIC CONTINUITY THEOREMS

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ABSTRACT. Let A and B be commutative locally convex algebras with unit. A is assumed to be a uniform topological algebra. Let Φ be an injective homomorphism from A to B. Under additional assumptions, we characterize the continuity of the homomorphism $\Phi^{-1}/\text{Im}\Phi$ by the fact that the radical (or strong radical) of the closure of $\text{Im}\Phi$ has only zero as a common point with $\text{Im}\Phi$. This gives an answer to a conjecture concerning some automatic continuity theorems on uniform topological algebras.

1. INTRODUCTION. Let A and B be commutative locally convex algebras with unit. A is assumed to be a uniform topological algebra. Let Φ be an injective homomorphism from A to B. Under which conditions is $\Phi^{-1}/\text{Im}\Phi$ continuous?

Under additional assumptions such as:

(1) A is weakly regular and functionally continuous, B an lmc algebra, and $(\text{Im}\Phi)^\sim$ (the closure of $\text{Im}\Phi$) is a semisimple Q-algebra; or

(2) A is weakly $\sigma^*$-compact-regular, and $(\text{Im}\Phi)^\sim$ is a strongly semisimple Q-algebra;

it is shown in [5] that $\Phi^{-1}/\text{Im}\Phi$ is continuous, which improves earlier results by Bedaa, Bhatt and Oudadess ([2]).

The following examples show that the hypothesis $(\text{Im}\Phi)^\sim$ is a Q-algebra in (1) and (2) cannot be omitted.

Example 1. Let $A = C[0,1]$ be the algebra of all complex continuous functions on the closed unit interval $[0,1]$. A is a uniform Banach algebra under the supnorm. Since $M(A)$ is homeomorphic to $[0,1]$, it follows that A is weakly regular. Consider $B = C[0,1]$. For any countable compact subset $K$ of $[0,1]$, and $f \in B$, we put $p_K(f) = \sup \{ |f(x)|, x \in K \}$. B is a complete uT-algebra under the system $(p_K)_{K \text{ compact}}$. Consider $\Phi: A \rightarrow B$, $\Phi(f) = f$. Then $(\text{Im}\Phi)^\sim = B$ is semisimple but not a Q-algebra. Clearly $\Phi^{-1}/\text{Im}\Phi$ is not continuous.

Example 2. Let $A = C_b(R)$ be the algebra of all complex continuous bounded functions on the real line. A is a uniform Banach algebra under the supnorm. A is weakly $\sigma^*$-compact-regular [2, Remark (4)]. Let $B = C(R)$ be the algebra of all complex continuous functions on $R$, with the compact-open topology. Consider $\Phi: A \rightarrow B$, $\Phi(f) = f$. Then $(\text{Im}\Phi)^\sim = C(R)$ is strongly semisimple but not a Q-algebra. Clearly $\Phi^{-1}/\text{Im}\Phi$ is not continuous.

In [2], the authors conjectured that the semisimplicity of $(\text{Im}\Phi)^\sim$ in (1) (and strong semisimplicity of $(\text{Im}\Phi)^\sim$ in (2)) can be omitted. According to the proofs in [2] and [5], the
semisimplicity of \((\text{Im}\Phi)^\sim\) in (1) can be replaced by \(\text{Im}\Phi \cap R((\text{Im}\Phi)^\sim) = \{0\}\), and the strong semisimplicity of \((\text{Im}\Phi)^\sim\) in (2) can be replaced by \(\text{Im}\Phi \cap \text{SR}((\text{Im}\Phi)^\sim) = \{0\}\).

In this paper, we show that if A is weakly regular and functionally continuous, B an lmc algebra, and \((\text{Im}\Phi)^\sim\) is a Q-algebra, then the continuity of \(\Phi^{-1}/\text{Im}\Phi\) is equivalent to \(\text{Im}\Phi \cap R((\text{Im}\Phi)^\sim) = \{0\}\). We also show that if A is weakly \(\sigma^*\)-compact-regular, B has continuous product, and \((\text{Im}\Phi)^\sim\) is a Q-algebra, then the continuity of \(\Phi^{-1}/\text{Im}\Phi\) is equivalent to \(\text{Im}\Phi \cap \text{SR}((\text{Im}\Phi)^\sim) = \{0\}\).

2. PRELIMINARIES. All algebras considered are over the field \(\mathbb{C}\), commutative, and having a unit element. A topological algebra is an algebra which is also a Hausdorff topological vector space such that the multiplication is separately continuous. A locally convex algebra (lc algebra) is a topological algebra whose topology is locally convex. A locally multiplicatively convex algebra (lmc algebra) is a topological algebra whose topology is determined by a family of submultiplicative seminorms. A uniform seminorm on an algebra A is a seminorm p such that \(p(x^2) = p(x)^2\) for all \(x \in A\). Such a seminorm is submultiplicative [4]. A uniform topological algebra (uT-algebra) is a topological algebra whose topology is determined by a family of uniform seminorms. A uniform normed algebra is a normed algebra \((A, \|\|)\) such that \(\|x^2\| = \|x\|^2\) for all \(x \in A\). Let A be an algebra and \(x \in A\), we denote by \(sp_a(x)\) the spectrum of \(x\) and \(r_a(x)\) the spectral radius of \(x\). For an algebra A, \(M^*(A)\) denotes the set of all nonzero multiplicative linear functionals on A. For a topological algebra A, \(M(A)\) denotes the set of all nonzero continuous multiplicative linear functionals on A. A topological algebra A is functionally continuous if \(M^*(A) = M(A)\). A topological algebra is a Q-algebra [7] if the set of invertible elements is open. A topological algebra is weakly regular [2] if given a closed subset F of \(M(A)\), \(F \neq M(A)\), there exists a nonzero \(x \in A\) such that \(f(x) = 0\) for all \(f \in F\). A topological algebra A is weakly \(\sigma^*\)-compact-regular [2] if given a compact subset K of \(M^*(A)\), \(K \neq M^*(A)\), there exists a nonzero \(x \in A\) such that \(f(x) = 0\) for all \(f \in K\). We use R(A) to denote the radical of an algebra A. If \(R(A) = \{0\}\), we say that A is semisimple. Let A be a topological algebra with \(M(A) \neq \emptyset\), the set \(\{x \in A, f(x) = 0\text{ for all }f \in M(A)\}\) is called the strong radical of A and denoted by \(\text{SR}(A)\). If \(\text{SR}(A) = \{0\}\), we say that A is strongly semisimple. Let A be an lmc algebra, if A is complete or a Q-algebra, then \(R(A) = \text{SR}(A)\).

3. RESULTS

Theorem 3.1. Let A be a weakly regular, functionally continuous, uT-algebra. Let B be an lmc algebra, and let \(\Phi\): \(A \rightarrow B\) be a one-to-one homomorphism such that \((\text{Im}\Phi)^\sim\) is a Q-algebra. Then the following are equivalent:

1. \(\Phi^{-1}/\text{Im}\Phi\) is continuous.
2. \(\text{Im}\Phi\) is functionally continuous.
3. \(\Phi^*: M((\text{Im}\Phi)^\sim) \rightarrow M(A), \Phi^*(f) = f \circ \Phi\), is surjective.
4. \(\text{Im}\Phi \cap R((\text{Im}\Phi)^\sim) = \{0\}\).

Proof: (1) \(\Rightarrow\) (2): Let \(F \in M^*(\text{Im}\Phi)\), \(F = F \circ \Phi \circ (\Phi^{-1}/\text{Im}\Phi)\) is continuous since \(F \circ \Phi\) and \(\Phi^{-1}/\text{Im}\Phi\) are continuous.
(2) => (3): Let \( f \in M(A) \) and \( F = f \circ (\Phi^{-1}/\text{Im} \Phi) \). If \( \Phi^* \) is surjective, then \( F \in M(\text{Im} \Phi) \) and \( f = F \circ \Phi \). By (i), there exists \( F \) such that \( F \in M(\text{Im} \Phi) \). Since \( \text{Im} \Phi \) is compact, \( F \) is continuous at 0 [7, Proposition 13.5]. Then \( \Phi^{-1}/\text{Im} \Phi \) is continuous. 

(3) => (1): By [5, Theorem 2.1], the topology of \( A \) is determined by a family \( \{p_s, s \in S\} \) of submultiplicative seminorms such that (i) for all \( x \in A \) and \( s \in S \) with \( p_s(x) = 1 \), there exists \( f \in M(A) \) such that \( |f(x)| = 1 \). Let \( s \in S \) and \( y \in \text{Im} \Phi \) with \( p_s(\Phi^{-1}(y)) \neq 0 \). By (i), there exists \( f \in M(A) \) such that \( |f(\Phi^{-1}(y))| = p_s(\Phi^{-1}(y)) \). Since \( \Phi^{-1} \) is surjective, there exists \( F \in M((\text{Im} \Phi)\bar{\mathbb{Q}}) \) such that \( f = F \circ \Phi \). We have \( p_s(\Phi^{-1}(y)) = |f(\Phi^{-1}(y))| = |F(y)| \leq r_C(y) \), where \( C = (\text{Im} \Phi)\bar{\mathbb{Q}} \). Since \( C \) is a \( \mathbb{Q} \)-algebra, \( r_C \) is continuous at 0 [7, Proposition 7.5]. Then \( \Phi^{-1}/\text{Im} \Phi \) is continuous. 

(3) => (4): Let \( y \in \text{Im} \Phi \cap R((\text{Im} \Phi)\bar{\mathbb{Q}}) \), there exists \( x \in A \) such that \( y = \Phi(x) \) and \( F(\Phi(x)) = 0 \) for all \( F \in M((\text{Im} \Phi)\bar{\mathbb{Q}}) \). Then \( f(x) = 0 \) for all \( f \in M(A) \) since \( \Phi^* \) is surjective. Hence \( x = 0 \) and so \( y = \Phi(x) = 0 \). 

Theorem 3.2. Let \( A \) be a weakly \( \sigma^* \)-compact-regular, \( uT \)-algebra. Let \( B \) be an \( lc \) algebra with continuous product, and \( \Phi: A \to B \) be a one-to-one homomorphism such that \( (\text{Im} \Phi)^{-1} \) is a \( \mathbb{Q} \)-algebra. The following are equivalent: 

(1) \( \Phi^{-1}/\text{Im} \Phi \) is continuous. 

(2) \( \text{Im} \Phi \cap \text{SR}(\text{Im} \Phi^{-1}) = \{0\} \). 

(3) \( \Phi^*: M((\text{Im} \Phi)^{-1}) \to M^*(A), \Phi^*(f) = f \circ \Phi, \) is surjective. 

Proof: (1) => (2): The topology of \( A \) is determined by a family \( \{p_u, u \in U\} \) of uniform seminorms. For each \( u \in U \), let \( N_u = \{x \in A, p_u(x) = 0\} \) and \( A_u \) be the Banach algebra obtained by completing \( A/N_u \) in the norm \( \|x\|_u = p_u(x), x_u = x + N_u \). It is clear that \( A_u \) is a uniform Banach algebra. For each \( u \in U \), let \( M_u(A) = \{f \in M(A), \|f(x)\| \leq p_u(x) \text{ for all } x \in A\} \). Let \( u \in U \) and \( x \in A \), \( p_u(x) = \|x\|_u = r_u(x_u) = \sup \{\|g(x_u)\|, g \in M(A_u)\} = \sup \{|f(x)|, f \in M_u(A)\} \text{ by [7, Proposition 7.5]} \). Let \( f \in M_u(A) \), \( f \circ \Phi \in M((\text{Im} \Phi)^{-1}) \) since \( \Phi^{-1}/\text{Im} \Phi \) is continuous and \( B \) has continuous product. Then \( p_u(\Phi^{-1}(y)) \leq \sup \{|F(y)|, F \in M((\text{Im} \Phi)^{-1})\} \text{ for all } u \in U \) and \( y \in \text{Im} \Phi \). Let \( y \in \text{Im} \Phi \cap \text{SR}(\text{Im} \Phi^{-1}) \), we have \( p_u(\Phi^{-1}(y)) = 0 \) for all \( u \in U \), then \( \Phi^{-1}(y) = 0 \) and so \( y = 0 \). 

(2) => (3): \( \Phi^* \) is continuous. Since \( (\text{Im} \Phi)^{-1} \) is a \( \mathbb{Q} \)-algebra, \( M((\text{Im} \Phi)^{-1}) \) is compact [6, p.187], thus \( \Phi^*(M((\text{Im} \Phi)^{-1})) \) is compact. Suppose that \( \Phi^* \) is not surjective. Since \( A \) is \( \sigma^* \)-compact-regular, there exists a nonzero \( x \in A \) such that \( f(\Phi(x)) = 0 \) for all \( f \in M((\text{Im} \Phi)^{-1}) \). Then \( f(x) = 0 \) for all \( f \in M(A) \) since \( \Phi^* \) is surjective. Hence \( x = 0 \) and so \( y = \Phi(x) = 0 \). 

(3) => (1): Similar to the proof of (3) => (1) in Theorem 3.1.
Example. Let $A = C[0,1]$ be the algebra of all complex continuous functions on the closed unit interval $[0,1]$. $A$ is a uniform Banach algebra under the supnorm $\|\cdot\|$, $A$ is also weakly regular. By [3], there exists a norm $\|\cdot\|$ on $C[0,1]$ such that $C[0,1]$ is an incomplete normed algebra. It is well known that $\|\cdot\| \leq \|\cdot\|$.

Let $B$ be the completion of $C[0,1]$ under the norm $\|\cdot\|$. Consider $\Phi: A \to B$, $\Phi(f) = f$, we have $(\text{Im} \Phi)^\perp = B$. If $B$ is semisimple, then $\Phi$ is continuous, and consequently the norms $\|\cdot\|$ and $\|\cdot\|$ are equivalent, a contradiction. Since $\|\cdot\| \leq \|\cdot\|$, $\Phi^{-1}/ \text{Im} \Phi$ is continuous and so $\text{Im} \Phi \cap \text{R}((\text{Im} \Phi)^\perp) = \{0\}$ by Theorem 3.1.

Remark. The algebra $A$ considered in the above example is also $\sigma\ast$-compact-regular and $\text{Im} \Phi \cap \text{SR}((\text{Im} \Phi)^\perp) = \{0\}$ but $\text{SR}((\text{Im} \Phi)^\perp) \neq \{0\}$.

The following result is an application of Theorem 3.1.

Theorem 3.3. Let $A$ be a functionally continuous normed algebra. Then the following assertions are equivalent:

1. $A$ is a uniform normed algebra.
2. $A$ has a largest closed, idempotent, absolutely convex, bounded subset.

Proof. (1) $\Rightarrow$ (2): Let $\|\cdot\|$ be a uniform norm defining the topology of $A$. Let $B = \{x \in A, \|x\| \leq 1\}$, $B$ is a closed, idempotent, absolutely convex, bounded subset of $A$. Let $C$ be an idempotent bounded subset of $A$. There exists $M > 0$ such that $\|x\| \leq M$ for all $x \in C$. Let $x \in C$, $\|x\| = \|x^n\|^n \leq M^n$ for all $n \geq 1$, then $\|x\| \leq 1$ i.e. $x \in B$.

(2) $\Rightarrow$ (1): Let $B$ be a largest closed, idempotent, absolutely convex, bounded subset of $A$. By [1, Proposition 2.15], we have $A = A(B) = \{tx, t \in \mathbb{C} \text{ and } x \in B\}$. Let $\|\cdot\|_B$ be the Minkowski functional of $B$, $(A, \|\cdot\|_B)$ is a normed algebra. By [1, Proposition 2.15], $\beta = \|\cdot\|_B$ where $\beta$ is the radius of boundedness, then $(A, \|\cdot\|_B)$ is a uniform algebra since $\beta(x^n) = \beta(x)^n$ for all $x \in A$. Let $A_1$ be the completion of $A$ under the original norm. It is clear that $\Phi: (A, \|\cdot\|_B) \to A_1$, $\Phi(x) = x$, is continuous, and consequently $(A, \|\cdot\|_B)$ is functionally continuous. We now remark that we have proved the equivalence of (1), (2) and (3) in Theorem 3.1 without the condition that $A$ is weakly regular. Using this remark, $\Phi^{-1}/ \text{Im} \Phi$ is continuous, then $\Phi$ is a homeomorphism (into), so $A$ is a uniform normed algebra.

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