$P - V$ Criticality of Conformal Gravity Holography in Four Dimensions

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We examine the critical behaviour i.e. $P - V$ criticality of conformal gravity (CG) in an extended phase space in which the cosmological constant should be interpreted as a thermodynamic pressure and the corresponding conjugate quantity as a thermodynamic volume. The main potential point of interest in CG is that there exists a non-trivial Rindler parameter ($a$) in the spacetime geometry. This geometric parameter has an important role to construct a model for gravity at large distances where the parameter “$a$” actually originates. We also investigate the effect of the said parameter on the black hole (BH) thermodynamic equation of state, critical constants, Reverse Isoperimetric Inequality, first law of thermodynamics, Hawking-Page phase transition and Gibbs free energy for this BH. We speculate that due to the presence of the said parameter, there has been a deformation in the shape of the isotherms in the $P - V$ diagram in comparison with the charged-AdS (anti de-Sitter) BH and the chargeless-AdS BH. Interestingly, we find that the critical ratio for this BH is $\rho_c = \frac{P_c}{V_c} = \frac{\sqrt{3}}{2} \left( \frac{3\sqrt{2}}{2} - 2\sqrt{3} \right)$, which is greater than the charged AdS BH and Schwarzschild-AdS BH i.e. $\rho_{CG}^{S-A_{AdS}} : \rho_{RN-A_{AdS}}^{\text{charged}} : \rho_{RN-A_{AdS}}^{\text{charged}} = 0.67 : 0.50 : 0.37$. The symbols are defined in the main work. Moreover, we observe that the critical ratio has a constant value and it is independent of the non-trivial Rindler parameter (a). Finally, we derive the reduced equation of state in terms of the reduced temperature, the reduced volume and the reduced pressure respectively.

Keywords: $P - V$ Criticality, Conformal gravity, Rindler acceleration

1. Introduction

The study of thermodynamic properties of BH in the AdS space has gained much more attention in recent years due to the seminal work of Hawking and Page\textsuperscript{1}. The AdS case is particularly interesting because of gauge-gravity duality via dual conformal field theory (CFT). This duality indicates that the spherically symmetric charged AdS BH admits critical behaviour which is similar to the liquid-gas phase transition. It was explicitly described by Chamblin et al.\textsuperscript{2,3} They proved that there should exist the first order phase transition in the conventional phase space in case of Reissner-Nordström-AdS (RN-AdS) BH. The critical behaviour of this BH has been studied there in details. Furthermore they emphasized that this behaviour is quite analogous to the Van-der-Waals (VdW’s) type liquid-gas phase transition.

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Now if one could treat the cosmological constant as a thermodynamic pressure, the Arnowitt-Deser-Misner (ADM) mass of the AdS BH as an enthalpy of the thermodynamic system and the thermodynamically conjugate quantity as a thermodynamic volume then one could study the critical behaviour in the extended phase space. This is an active area of research over the past few years.

Moreover, one could investigate the critical behaviour in the extended phase space which is quite analogous to VdW’s like liquid-gas phase transition. This thermodynamic behaviour for RN-AdS BH has been explicitly studied by Kubizňák-Mann by using the extended phase-space formalism. They postulated the analogy between VdW’s fluid-gas system and charged AdS BH. They also derived the BH equation of state, and determined the critical constants in comparison with the liquid-gas system. The critical constants and critical exponents were calculated therein. It has been shown that they coincide with those of the liquid-gas systems. The critical behaviour for different types of BH has been studied elaborately in the extended phase space and the same could be found in Ref.

In the present study, we wish to examine the critical behaviour i.e. $P - V$ criticality of CG holography in four dimensions (4D) by treating the cosmological parameter as the thermodynamic pressure and its conjugate parameter as thermodynamic volume. We derive the BH equation of state in terms of BH temperature and specific thermodynamic volume. At the critical point, we calculate the critical constants. It has been shown that the critical ratio has a constant value which is quite different from RN-AdS BH. We would also derive the first law of BH thermodynamics, Reverse Isoperimetric Inequality and Gibbs free energy. Moreover, we would discuss the Hawking-Page phase transition. It has been shown that the Hawking-Page phase transition temperature as a function of Rindler parameter. Finally, we would derive the reduced equation of state.

Furthermore, we would prove that the three critical constants namely the critical pressure, the critical temperature and the critical volume do all depend on the Rindler parameter. The BH thermodynamic equation of state also depends on the said parameter. Interestingly, the critical ratio is independent of the Rindler parameter. Due to the presence of this parameter the shape of the isotherms in the $P-V$ diagram of CG BH is completely different from that of the Schwarzschild-AdS spacetime and the RN-AdS space-time. This is one of the interesting observations of this work.

The CG is a fascinating theory of gravity and has a non-trivial Rindler term. This subleading term indeed originates during the formulation of an effective model for gravity at large distances. This novel parameter produces an acceleration which is the already known Rindler acceleration $(a).$ For example, it produces an anomalous acceleration in geodesics of a test particle. It could be observed in various “anomalous” systems like galaxy-cluster, star-galaxy, Earth-satellite etc. In Ref. the author described an effective 2D field theory for infra-red (IR) gravity to explain the anomalous acceleration of a test particle in the gravitational field of
a central object. He also derived an action\[ for this gravity in the IR region as

\[ I = - \int \sqrt{-g} d^2x \left( \Phi^2 R + 2(\partial \Phi)^2 - 6\Lambda \Phi^2 + 8a\Phi + 2 \right). \tag{1} \]

where $\Phi$ is a scalar field, $\Lambda$ is a cosmological constant and $a$ is Rindler term which generates the Rindler acceleration in a geodesics. This acceleration also depends on the scale of the system that one may use. The value of $a$ in the Ref.\[ 19\] has been given $a \approx 10^{-62} - 10^{-61}$ for anomolous acceleration. For Pioneer acceleration, the value of $a$ is approximately $a \approx 10^{-61}$ whereas for modified-Newtonian-dynamics acceleration the value of $a$ is around $a \approx 10^{-62}$. The main motivation behind this work is to investigate the effect of the Rindler acceleration on the thermodynamic properties of the CG holography in four dimensions.

The other aspect is that CG is a theory of gravity at large distances.\[ 19\] Like other higher curvature theory, it is also a re-normalizable theory having ghost.\[ 20, 21\] On the other hand, the Einstein’s general theory of gravity has no ghost i.e. ghost free gravity but two loop non-renormalizable.\[ 22\] To explain galactic rotating curves without dark matter, Mannheim was first to study this theory phenomenologically.\[ 23\] It emerges as a counter term in AdS/CFT (conformal field theory) correspondence.\[ 24, 25\]

In the quantum gravity context, CG has been studied by 't Hooft.\[ 26\] Malda-\[ 27\] cen showed that by imposing appropriate boundary condition it is possible to eliminate the ghost term. The most important feature of this theory is that it depends only on the (Lorentz) angles but not on the distance. It should be noted that CG is a higher-derivative theory but the entropy obeys the area law.\[ 28\] Another striking feature of CG is that the AdS boundary condition is weaker than the Starobinsky boundary conditions.\[ 15\] This may admit a non-trivial asymptotic Rindler term concurrently with Grumiller’s proposal ‘an effective model for gravity at large distances’ given in 2016\[ 19\]. Thus one could observe a new critical behaviour of CG BH compared to Einstein’s gravity due to the non-trivial geometrical Rindler parameter.

The structure of the paper is as follows. In Sec. 2, we have described the thermodynamic properties of CG holography in four dimensions. Finally, we have given the conclusions in Sec. 3.

2. Thermodynamic Properties of CG Holography in Four Dimensions

Before begining the main investigation, we would like to review some basic features of CG. It is a theory derived from the action of the Weyl conformal curvature. The

\[a\] The equation of motion, the line element and two dimensional Ricci scalar for this IR gravity see the Ref.\[ 19\] (for details).
Weyl action concurrently with an electromagnetic field is given by

\[ I = -\frac{1}{4} \int \sqrt{-g} \, d^4x \left( \beta C_{abcd} C^{abcd} + F^{cd} F_{cd} \right). \] (2)

where \( C_{abcd} \) is denoted as the conformal curvature tensor and \( F_{ab} = A_{a,b} - A_{b,a} \) is denoted as electromagnetic field tensor. Using Eq. (2), one could write the Bach-Maxwell equations by varying of \( g_{ab} \) and \( A_b \) as

\[ \nabla^a F_{ab} = 0 \] (4)

Therefore the most static, spherically symmetric solution of Eq. (5) can be written as

\[ ds^2 = -F(r) dt^2 + \frac{dr^2}{F(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \] (5)

where the metric function has the general form given by

\[ F(r) = \lambda r^2 + \mu r + \delta + \frac{\eta}{r}. \] (6)

and

\[ A \equiv A_a dx^a = \frac{q}{r} dt \] (7)

where \( \lambda, \mu, \delta, \eta \) are constants satisfied by the following relation

\[ 3 \mu \eta - \delta^2 + 1 + \frac{3q^2}{2\beta} = 0. \] (8)

It should be noted that the spherically symmetric solutions of CG were first introduced by Bach in 1921. Later Buchdahl (1953) considered a particular case \( \mu = q = 0, \delta = 1 \) by using Eq. (6).

In 2010, Grumiller proposed an effective model for gravity of a central object at a large distance. For the large radius expansions he found a solution which consists of a cosmological constant and an additional parameter which is called the “Rindler parameter”. This novel parameter could produce acceleration analogous to that observed in different anomalous systems (i.e. Sun-Pioneer spacecraft, star-galaxy etc.).

Subsequently in 2014, the same author gave a spherically symmetric solution of CG in another form by substituting the values of \( \lambda, \mu, \delta, \eta \) in Eq. (5) of Ref. as \( \lambda = -\frac{3}{4}, \mu = 2a, \delta = \sqrt{1 - 12am} \) and \( \eta = -2M \) and without considering the Maxwell field i.e. keeping \( q = 0 \). Interestingly, the values of these parameters satisfied the Eq. (5). Therefore the form of metric function can be written as

\[ F(r) = \sqrt{1 - 12aM} - \frac{2M}{r} + 2ar - \frac{\Lambda}{3} r^2. \] (9)

bThe meaning of this term is that “the difference between the observed trajectory of a test particle in the gravitational field of a central object and the calculated trajectory”.

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Where \( a \) is the Rindler parameter and \( a \geq 0 \). In the limit \( a = 0 \), one finds the Schwarzschild-AdS space-time. In the limit \( aM \ll 1 \), one obtains the Grumiler space-time. When we set the parameters \( a = \Lambda = 0 \), we obtain the Schwarzschild solutions, with \( M \) being the BH mass. The numerical values of \( \Lambda \), \( a \) and \( M \) (in Planck units) are considered as \( \Lambda \approx 10^{-123} \), \( a \approx 10^{-61} \) and \( M \approx 10^{38} M_\star \), where \( M_\star = 1 \) for the sun, and therefore it implies \( aM \approx 10^{-23} M_\star \ll 1 \) for all types of BHs or galaxies in our Universe.

Let us now put \( -\Lambda \ell^2 = \frac{1}{3\ell^2} \) for the AdS case. The BH event horizon, \( r_+ \) can be obtained by imposing the condition \( \mathcal{F}(r_+) = 0 \) i.e.

\[
r_+^3 + 2a\ell^2 r_+^2 + \sqrt{1 - 12aM\ell^2 r_+ - 2M\ell^2} = 0 .
\]

By solving the above equation, one obtains the mass parameter in terms of the event horizon radius as

\[
M = \frac{r_+}{2} \left[ \sqrt{1 - 3a^2 r_+^2 - \frac{6ar_+^4}{\ell^2} - ar_+ + \frac{r_+^2}{\ell^2}} \right] .
\]

In the limit \( a = 0 \), we get the ADM mass for Schwarzschild-AdS BH. It is given by

\[
M = \frac{r_+}{2} \left[ 1 + \frac{r_+^2}{\ell^2} \right] .
\]

It indicates that the mass parameter is a function of event horizon radius and is strictly increasing function. But for CG BH the mass parameter is a function of both the event horizon radius and the Rindler parameter. It seems that due to the Rindler acceleration the mass function first increases when the horizon radius increases then it decreases sharply. It is reverse in nature due to the presence of charge parameter and it could be observed from the Fig. 1. In the presence of charge parameter, the mass function becomes

\[
M = \frac{r_+}{2} \left[ 1 + \frac{Q^2}{r_+^2} + \frac{r_+^2}{\ell^2} \right] .
\]

The BH temperature is given by

\[
T = \frac{1}{4\pi r_+} \left( \sqrt{1 - 3a^2 r_+^2 - \frac{6a r_+^4}{\ell^2} + ar_+ + 3\frac{r_+^2}{\ell^2}} \right) .
\]

It indicates that the BH temperature is dependent on the Rindler parameter. When \( a = 0 \), one obtains the temperature of the famous Schwarzschild-AdS BH. The maximum and minimum value of the above temperature could be found from the following condition

\[
\left( \frac{\partial T}{\partial r_+} \right) |_{a,\ell} = 0 .
\]

which gives

\[
54ar_+^7 + 36a^2\ell^2r_+^6 - 9\ell^2r_+^4 + 6a\ell^4r_+^3 + \ell^6 = 0 .
\]
It is a non-trivial task to determine the roots of the 7\textsuperscript{th} order polynomial equation. Rather, it is easy to see what happens if the limit is \(a = 0\). One obtains

\[
 r_+ = \frac{\ell}{\sqrt{3}} = \frac{1}{\sqrt{8\pi \rho}}.
\] 

where the temperature is \(T = T_{\text{min}} = \frac{\sqrt{3}}{2\pi}\rho = \frac{0.275}{\rho}\) and a single BH is formed with horizon radius \(r_+\). When \(T < T_{\text{min}}\), there are no BHs formed. While for \(T > T_{\text{min}}\), there exists a small and a large BH. Their radii could be calculated from the Eq. (14) (in the limit \(a = 0\)). We have plotted the Eq. (16) in the Fig. 2 to distinguish the stable and the unstable region. By introducing the charge parameter we get the temperature of RN-AdS BH. We do not write the explicit expressions but our aim is to show the variation of this temperature with the event horizon radius and compared it with our model. It could be found from the Fig. 3. The entropy\(^{[13]}\) of the BH using Wald’s formalism is

\[
 S_i = \frac{A_i}{4\ell^2}.
\]

where the area of the BH is given by

\[
 A_i = 4\pi r_i^2.
\]

Interestingly, the entropy relation fulfills the area law despite the fact that CG is a higher derivative gravity.

Since in this work we are interested to study the \(P-V\) criticality in the extended phase space therefore one could define the cosmological constant as thermodynamic pressure and the corresponding conjugate variable as thermodynamic volume i.e.

\[
 P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi \ell^2}.
\]
Fig. 2. Variation of $y = 54a r_+^2 + 36a^2 r_+^6 - 9a^2 r_+^4 + 6a^4 r_+^4 + 6$ with $r_+$ for CG BH. For $a = 0$, the red one curve implies the instability, whereas for $a = 1, 2$: the green one and the yellow one signals a stability.

Fig. 3. Variation of $T$ with $r_+$ for Schwarzschild-AdS BH, CG BH and RN-AdS BH. From the left figure, it follows that the parameter $a$ is modified the temperature curve in comparison with the Schwarzschild-AdS BH. In case of charged AdS BH, it is completely different in shape.

and

$$V = \left( \frac{\partial M}{\partial P} \right)_S.$$  \hspace{1cm} (21)
Now in the extended phase space, the mass parameter becomes
\[ M = \frac{r_+}{2} \left[ \sqrt{1 - 3a^2r_+^2} - 16\pi aPr_+^3 - ar_+ + \frac{8\pi P}{3}r_+^2 \right]. \tag{22} \]
and the thermodynamic volume should be
\[ V = \left( \frac{\partial M}{\partial P} \right)_S = \frac{4}{3} \pi r_+^3 \left[ 1 - \sqrt{1 - \frac{3ar_+}{\sqrt{1 - 3a^2r_+^2} - 16\pi PaP_+^3}} \right]. \tag{23} \]

It is quite strange that due to the Rindler acceleration the thermodynamic volume gets modified and it has been shown in Fig. 4. It also indicates that the thermodynamic volume depends upon the non-trivial Rindler parameter. This is probably the first counter example of any spherically symmetric BH whose thermodynamic volume is
\[ V \neq \frac{4}{3} \pi r_+^3. \tag{24} \]

In general, we know that the thermodynamic volume for spherically symmetric BH (For example, RN-AdS BH) is given by
\[ V = \frac{4}{3} \pi r_+^3. \tag{25} \]

In Eq. (23), when Rindler parameter goes to zero value then one obtains the Eq. (25). It should be noted that Eq. (23) is one of the interesting results of this work. The above Eq. (23) represents the thermodynamic volume modified by the non-trivial
Rindler parameter. This means that this geometric term modified the thermodynamic volume of this BH in CG. This is also quite interesting.

The first law of thermodynamics should read

\[ dM = T dS + V dP + \chi da . \]  

(26)

where \( \chi \) is the physical quantity associated with the parameter \( a \). It should be defined as

\[ \chi = \left( \frac{\partial M}{\partial a} \right)_{S,P} = -\frac{r_+^2}{2} \left[ 1 + \frac{3r_+ \left( a + \frac{8\pi P r_+}{3} \right)}{\sqrt{1 - 3a^2 r_+^2 - 16\pi a P r_+^4}} \right] . \]  

(27)

It implies that the Rindler parameter modified the first law of thermodynamics in the extended phase space.

Another novel feature of the thermodynamic volume is so called Reverse Isoperimetric Inequality\(^9\) which is satisfied for all BHs except super-entropic BHs.\(^10\) It has been conjectured that the thermodynamic volume \( V \) and the horizon area \( A \) always satisfy the isoentropic ratio i.e.

\[ R = \left( \frac{3V}{4\pi} \right)^{\frac{1}{3}} \left( \frac{4\pi}{A} \right)^{\frac{1}{2}} \geq 1 , \]  

(28)

For Schwarzschild-AdS BH, it is maximized. But interestingly, in our case this ratio is calculated to be

\[ R = \left[ 1 - \frac{3ar_+}{\sqrt{1 - 3a^2 r_+^2 - 16\pi P a r_+^4}} \right]^{\frac{1}{2}} \leq 1 , \]  

(29)

and found that it always violates the Reverse Isoperimetric Inequality. Probably this is a second counter example (after Ref.\(^10\)) of violation of the conjecture \( R \geq 1 \).

It could be seen from Fig. 5. From the Eq. (29), we could say that the Rindler parameter modifies the value of Reverse Isoperimetric Inequality. It should be noted that when \( a = 0 \), one obtains the result for ‘maximally entropic’ Schwarzschild-AdS BH\(^10\).

Finally, the Gibbs free energy in the extended phase space should read

\[ G = H - TS = \frac{r_+}{4} \left[ \sqrt{1 - 3a^2 r_+^2 - 16\pi P a r_+^4} - 3ar_+ - \frac{8\pi P}{3} r_+^2 \right] . \]  

(30)

The Gibbs free energy and BH temperature both depend on the parameter \( a \). The global stability properties of small and large BHs could be determined by studying the features of \( G \). From Eq. (30) it follows that when \( G = 0 \) and \( r_+ = 0 \), the BH is in a pure radiation phase. The minimum value of the \( G \) is at the origin which suggests that

\[ r_+^4 - 4\pi T \ell^2 r_+^3 + (4\pi^2 T^2 + 12\pi a T + 12a^2) \ell^4 r_+^2 - \ell^4 = 0 . \]  

(31)

To find the exact numerical value of \( r_+ \) from the above equation it is a very difficult task. Instead, one could observe the variation of this function with \( r_+ \) for different
Fig. 5. Variation of $R$ with $r_+$ for CG BH for different values of Rindler parameter.

Fig. 6. Variation of $y = r_+^4 - 4\pi T\ell^2 r_+^3 + (4\pi^2 T^2 + 12\pi aT + 12a^2) \ell^4 r_+^2 - \ell^4$ with $r_+$ for CG BH. It could be seen from the figure due to the Rindler acceleration the Hawking-Page (HP) phase transition for CG BH gets modified.

values of $a$ and $T$ as presented graphically (See the Fig. 6). While for $a = 0$, one could easily determine the value of $r_+$ and HP phase transition temperature. From Eq. (31), we get

$$r_+^4 - 4\pi T\ell^2 r_+^3 + 4\pi^2 T^2 \ell^4 r_+^2 - \ell^4 = 0 .$$  (32)
which is reduced to more simplified form as
\[ (r_+^2 - 2\pi T \ell r_+ + \ell^2) (r_+^2 - 2\pi T \ell^2 r_+ - \ell^2) = 0 \]  \hspace{1cm} (33)
we discard the second one and from the first one we find 
\[ r_+ + \ell = \sqrt{\frac{3}{8\pi P}} \]
when 
\[ T = T_{HP} = \frac{1}{2\pi} = \frac{\sqrt{\frac{P}{4\pi}}}{2\pi} \]

Where \( T_{HP} \) is called the famous Hawking-Page (HP) critical phase transition temperature. For \( T > T_{HP} \), the large BH is globally stable and for \( T < T_{HP} \), the small BH is thermodynamically unstable, while the larger one is locally stable. But in our case, due to the Rindler parameter we expect that the \( T_{HP} \) gets modified by the factor \( a \). We could not find the exact HP temperature due to the quartic nature of the polynomial equation, but we could say that it must be a function of Rindler parameter. In Fig. 7 we have drawn the Gibbs free energy for different values of temperature for CG BH. In Fig. 8 we have plotted the Gibbs free energy for different values of temperature for RN BH in comparison with CG BH. Now we turn to the main work. Using Eqs. (14) and (20), the Hawking temperature could be rewritten as
\[ T = \frac{1}{4\pi r_+} \left[ ar_+ + 8\pi P r_+^2 + \sqrt{1 - 3a^2r_+^2 - 16\pi aPr_+^3} \right] \]  \hspace{1cm} (34)

Using this equation one could obtain the BH equation of state as
\[ 64\pi^2 r_+^4 P^2 + 32\pi r_+^3 (a - 2\pi T) P + (16\pi^2 r_+^2 T^2 - 8\pi a r_+^2 T + 4a^2 r_+^2 - 1) = 0 \]  \hspace{1cm} (35)
This is a quadratic equation of \( P \). Solving this equation one could find the equation of state for this AdS BH
\[ P = \frac{T}{2r_+} - \frac{a}{4\pi r_+} \pm \frac{\sqrt{1 - 8\pi a Tr_+^2}}{8\pi r_+^2} \]  \hspace{1cm} (36)

The variation of \( P - V \) diagram could be seen from the Fig. 9 for Schwarzschild-AdS BH. For RN-AdS BH & CG BH, it could be seen from Fig. 11 & Fig. 12. One could easily observe from this plot that due to the presence of the Rindler parameter there has been a deformation of shape of the isotherms in the \( P - V \) diagram in comparison with charged-AdS BH and chargeless-AdS BH.

It follows from the Eq. (36) that due to the presence of the Rindler term the BH equation of state gets modified. First, we consider the lower sign (the negative one) for \( P - V \) criticality. The \( P - V \) criticality for upper sign should be considered in the Appendix section. In terms of specific volume \( v = 2r_+ \) one obtains the BH equation of state as
\[ P = \frac{T}{v} - \frac{a}{2\pi v} - \frac{\sqrt{1 - 2\pi av^2 T}}{2\pi v^2} \]  \hspace{1cm} (37)
where \( r_+ \) is the root of the equation \[ 16\pi aPr_+^3 + 12a^2 r_+^5 - (1 + 24aPV)r_+^6 - \left( \frac{9V}{2\pi} \right) a^2 r_+^5 + \left( \frac{1V}{2\pi} \right) \frac{9V}{\pi} a r_+^3 + \left( \frac{27V}{16\pi^2} \right) r_+^3 - \left( \frac{1V}{2\pi} \right)^2 = 0 \]
In the limit \( a = 0 \), \( r_+ = \left( \frac{1V}{2\pi} \right)^{\frac{1}{3}} \).
Fig. 7. Variation of $G$ with $r_+$ for CG BH. It follows from the figure the variation of $G$ with $r_+$ without the parameter, $a$ and with the parameter, $a$ is qualitatively different.

In the limit $a = 0$, one has the equation of state for Schwarzschild-AdS BH\cite{29} as

$$P = \frac{T}{v} - \frac{1}{2\pi v^2}, \quad (38)$$

where $v = 2r_+ = 2 \left( \frac{3V}{4\pi} \right)^{\frac{1}{3}}$.

The variation of $P - r_+$ diagram could be seen from the Fig.\ref{fig:7}a. Each isotherm corresponds to a maximum value of $P$ at $r_+ = \frac{1}{2\pi T}$. As we have mentioned earlier, for a particular temperature $T = T_{min}$, the value of $r_+ = r_{min} = \frac{1}{2\pi T_{min}} = \frac{l}{\sqrt{3}} = \frac{1}{\sqrt{8\pi P}}$. It should be noted that $G$ exhibits an inflection point at $r_{min}$. For greater value of this temperature we would find the radii for one large and one small BH.
From Eq. (35), we can obtain the radii of large and small BH for CG BH. It is a very difficult task to determine the exact roots of the quartic equation numerically. So we can plot this function graphically for various values of $a, P$ and $T$ (See Fig. 10). But in the limit $a = 0$, the Eq. (35) reduces to the following form

$$64\pi^2 P^2 r_+^4 - 64\pi^2 TP r_+^3 + 16\pi^2 T^2 r_+^2 - 1 = 0.$$  

It could be more simplified as

$$(8\pi Pr_+^2 - 4\pi Tr_+ + 1) (8\pi Pr_+^2 - 4\pi Tr_+ - 1) = 0.$$  

The first equation gives the radii of large and small Schwarzschild-AdS BH which is given by

$$r_{\text{large}} = \frac{T}{4P} \left[ 1 + \sqrt{1 - \frac{2P}{\pi T^2}} \right]$$  

$$r_{\text{small}} = \frac{T}{4P} \left[ 1 - \sqrt{1 - \frac{2P}{\pi T^2}} \right]$$  

where $T_{\text{min}} = \sqrt{\frac{2P}{\pi}}$. The second one gives another set of radii of large and small BH

$$r_{\text{large}} = \frac{T}{4P} \left[ 1 + \sqrt{1 + \frac{2P}{\pi T^2}} \right]$$  

$$r_{\text{small}} = \frac{T}{4P} \left[ 1 - \sqrt{1 + \frac{2P}{\pi T^2}} \right]$$  

This is unphysical because in the discriminant part the value of $T$ becomes imaginary. Since we are unable to find the exact root of $r_+$ due to the quartic nature of

![Fig. 8. Variation of $G$ with $r_+$ for RN-AdS BH.](image)
Fig. 9. Variation of $P$ with $r_+$ for Schwarzschild-AdS BH and RN-AdS space-time. The above figures indicate that the $P - r_+$ diagram for $Q = 0$ and $Q = 1, 2, 3$ are qualitatively different in nature. For $Q = 0$, the isotherms have no inflection point. Whereas for $Q = 1, Q = 2$ and $Q = 3$, the isotherms have inflection point. This is the main differences between Schwarzschild-AdS BH and RN-AdS space-time.

Eq. (49) if we set $T = \frac{a}{2\pi}$, and assuming if it is the HP phase transition temperature for CG BH, the Eq. (35) reduces to the following form

$$64\pi^2 P^2 r_+^4 + 4a^2 r_+^2 - 1 = 0$$ (45)
Fig. 10. Variation of $z = (8\pi P)^2 r_+^4 + 32\pi P (a - 2\pi T) r_+^3 + (16\pi^2 T^2 - 8\pi a T + 4a^2) r_+^2 - 1$ and $v = (8\pi P_c)^2 r_+^4 + 32\pi P (a - 2\pi T_c) r_+^3 + (16\pi^2 T_c^2 - 8\pi a T_c + 4a^2) r_+^2 - 1$ with $r_+$ for CG BH.

Using the above equation, one obtains the radii of large and small BH for CG gravity

$$r_{\text{large}} = \sqrt{a^4 + 16\pi^2 P^2 + a^2}$$

$$r_{\text{small}} = \sqrt{a^4 + 16\pi^2 P^2 - a^2}$$

Hence one may conclude that the HP phase transition temperature for CG BH must be the function of Rindler parameter i.e. $T_{HP} = f(a)$. This further implies that the HP phase transition temperature depends on the Rindler parameter. The discussion of fluid analogue of Schwarzschild-AdS BH could be found in Ref.[29]

The critical constants could be determined by applying the following conditions at the inflection point

$$\frac{\partial P}{\partial v} \bigg|_{T=T_c} = 0 .$$

$$\frac{\partial^2 P}{\partial v^2} \bigg|_{T=T_c} = 0 .$$

Solving Eq. (48), one can obtain

$$T_c = \frac{a}{2\pi} + \frac{1 - \pi a v_c^2 T_c}{\pi v_c \sqrt{1 - 2\pi a v_c^2 T_c}} .$$

and solving Eq. (49), one can find

$$T_c = \frac{a}{2\pi} + \frac{3 - 9\pi a v_c^2 T_c + 4\pi^2 a^2 v_c^4 T_c^2}{2\pi v_c \left(1 - 2\pi a v_c^2 T_c\right)^{3/2}} .$$
In the limit $a = 0$, one obtains the critical constants for Schwarzschild-AdS BH

$$P_c = \frac{1}{2\pi v_c^2} \quad \text{(52)}$$

$$T_c = \frac{1}{\pi v_c} \quad \text{(53)}$$

If we choose $v_c = 1$, we find that the critical values are $P_c = \frac{1}{2\pi}$ and $T_c = \frac{1}{\pi}$. The interesting case that we observe an HP phase transition between small and large BHs\cite{Witten} Witten\cite{Witten} explained that the HP phase transition is dual to the QCD confinement/deconfinement phase transition.

Using Eq. (50) and Eq. (51), one could derive the critical Hawking temperature

$$T_c = \frac{1}{3\pi^2 a^2 v_c^2} \quad \text{(54)}$$
Using Eq. (50) and Eq. (54), one obtains the critical volume

\[ v_c = \frac{3\sqrt{2} - 2\sqrt{3}}{3a} . \]  

Finally, using Eq. (37) and Eq. (54), we could find the critical pressure

\[ P_c = \frac{\sqrt{3}}{2\pi v_c^2} . \]  

In terms of \( a \), the critical values are

\[ P_c = \frac{3\sqrt{3}}{4\pi (5 - 2\sqrt{6})} a^2 \]  
\[ v_c = \frac{3\sqrt{2} - 2\sqrt{3}}{3a} \]  
\[ T_c = \frac{2\pi (5 - 2\sqrt{6})}{a} . \]

It may be noted that \( P_c, v_c \) and \( T_c \) are strictly dependent upon the parameter \( a \).

From the critical constants, one could derive the critical ratio for CG BH

\[ \rho_c = \frac{P_cv_c}{T_c} = \frac{\sqrt{3}}{2 \left(3\sqrt{2} - 2\sqrt{3}\right)} . \]  

which is a constant value as it was already expected and which was found earlier for charged-AdS BH as

\[ \rho_c = \frac{P_cv_c}{T_c} = \frac{3}{8} . \]  

For Schwarzschild-AdS BH, the \( \rho_c \) is calculated to be

\[ \rho_c = \frac{P_cv_c}{T_c} = \frac{1}{2} . \]
using Eq. (53).

It should be noted that for Schwarzschild-AdS BH the isotherm in $P - r_+$ diagram is quite different from charged-AdS BH. It may be observed from the Fig. (9-a). The only interesting feature that happens in the Schwarzschild-AdS spacetime is the HP phase transitions between small and large BH that we have described earlier.

Therefore the ratio of $\rho_c$ for these BHs should read

$$\frac{\rho_c^{CG}}{\rho_c^{Sch-AdS}} : \frac{\rho_c^{RN-AdS}}{\rho_c^{Sch-AdS}} = 0.67 : 0.50 : 0.37.$$  (63)

It immediately follows that $\rho_c^{CG} > \rho_c^{Sch-AdS} > \rho_c^{RN-AdS}$. The “law of corresponding states” becomes

$$2\Theta = \left(3\sqrt{2} - 2\sqrt{3}\right) \Phi \left[\sqrt{3}\Xi + \frac{\sqrt{1 - \frac{2}{3}\Theta \Phi^2}}{\Phi^2}\right].$$  (64)

where $\Theta$, $\Phi$ and $\Xi$ could be defined as

$$\Theta = \frac{T}{T_c}$$  (65)

$$\Phi = \frac{v}{v_c}$$  (66)

$$\Xi = \frac{P}{P_c}.$$  (67)

and these quantities like $\Theta$, $\Phi$ and $\Xi$ are called the reduced temperature, the reduced volume and the reduced pressure respectively. Thus the Eq. (64) is called the reduced equation of state.

3. Conclusions

We have analyzed the critical behaviour of CG BH in four dimensions in the context of extended phase space, a thermodynamic model which has garnered increasing interest over the past few years for providing more complete description of the thermodynamic processes of BHs in spacetimes with cosmological constants. The spacetime geometry that we have considered in this work contained a non-trivial Rindler term which produces anomalous acceleration in a geodesics of a test particle. It could be observed in various anomalous systems like star-galaxy, Earth-Satellite etc.

We have examined the effect of this novel parameter on the thermodynamic behaviour. Due to this novel parameter, we observed that in the $P - V$ diagram, the silhouette of the isotherms of CG BH is quite distinguished from the charged-AdS BH and Schwarzschild-AdS BH. We also derived the BH thermodynamic equation of state, critical constant, Reverse Isoperimetric Inequality, first law of thermodynamics, Hawking-Page phase transition and Gibbs free energy. We speculated that all these thermodynamic quantities are strictly dependent upon this novel parameter. More importantly, the Rindler acceleration term modified the first law of BH
thermodynamics. We also observed that the HP phase transition temperature is a function of the said parameter. Furthermore, we computed the critical ratio for CG BH in comparison with RN-AdS BH and it complied with the inequality $\rho_{c}^{CG} > \rho_{c}^{Sch-AdS} > \rho_{c}^{RN-AdS}$. Interestingly, the critical ratio is independent of the Rindler acceleration. Furthermore, we derived the reduced equation of state in terms of the reduced temperature, the reduced volume and the reduced pressure respectively. To summarize, the Rindler acceleration has an effect on the thermodynamic properties of the CG in $d = 4$.

4. Appendix

In the appendix section, we have examined that the $P − V$ criticality for the BH equation state of CG BH which corresponds to

$$P = \frac{T}{2r_+} - \frac{a}{4\pi r_+} + \frac{\sqrt{1 - 8\pi a T r_+^2}}{8\pi r_+^2}.$$  \hspace{1cm} (68)

In terms of specific volume $v = 2r_+$, the equation of state could be written as

$$P = \frac{T}{v} - \frac{a}{2\pi v} + \frac{\sqrt{1 - 2\pi a v^2 T}}{2\pi v^2} .$$  \hspace{1cm} (69)

It should be noted that when $a = 0$, we could not find the BH equation of state for Schwarzschild-AdS BH. Yet, we would see what happens in the critical constants if we use the positive sign instead of negative sign. Doing all the calculations as we have done previously, we have found that the critical constants as

$$P_c = -\frac{3\sqrt{3}}{4\pi (5 + 2\sqrt{6})} a^2 ,$$

$$v_c = \frac{3\sqrt{2} + 2\sqrt{3}}{3a} ,$$

$$T_c = \frac{a}{2\pi (5 + 2\sqrt{6})} .$$

It is unusual that the critical pressure is negative. Being more usual the critical constants are dependent on the Rindler parameter. Then the critical ratio is given by

$$\rho_c = -\frac{\sqrt{3}}{2} \left( 3\sqrt{2} + 2\sqrt{3} \right) .$$  \hspace{1cm} (73)

It is also a peculiar result that the critical constant is negative.

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$\ell = 1$

From right to left:

$a = 1$
$a = 2$
$a = 3$
$a = 5$