DERIVED CATEGORIES FOR GROTHENDIECK CATEGORIES OF ENRICHED FUNCTORS

GRIGORY GARKUSHA AND DARREN JONES

Dedicated to Mike Prest on the occasion of his 65th birthday

ABSTRACT. The derived category \( \mathcal{D}[\mathcal{C}, \mathcal{V}] \) of the Grothendieck category of enriched functors \([\mathcal{C}, \mathcal{V}]\), where \( \mathcal{V} \) is a closed symmetric monoidal Grothendieck category and \( \mathcal{C} \) is a small \( \mathcal{V} \)-category, is studied. We prove that if the derived category \( \mathcal{D}(\mathcal{V}) \) of \( \mathcal{V} \) is a compactly generated triangulated category with certain reasonable assumptions on compact generators or \( K \)-injective resolutions, then the derived category \( \mathcal{D}[\mathcal{C}, \mathcal{V}] \) is also compactly generated triangulated. Moreover, an explicit description of these generators is given.

1. INTRODUCTION

Enriched categories generalize the idea of a category by replacing Hom-sets with objects from a monoidal category. In practice the Hom-sets often have additional structure that should be respected, e.g., that of being a topological space of morphisms, or a chain complex of morphisms. They have plenty of uses and applications. For example, Bondal–Kapranov [BK] construct enrichments of some triangulated categories over chain complexes ("DG-categories") to study exceptional collections of coherent sheaves on projective varieties. Today, DG-categories have become an important tool in many branches of algebraic geometry, non-commutative algebraic geometry, representation theory, and mathematical physics (see a survey by Keller [Kel]). There are also applications in motivic homotopy theory. For example, Dundas–Røndigs–Østvær [DRO1, DRO2] use enriched category theory to give a model for the Morel–Voevodsky category \( \text{SH}(k) \). In [GP1,GP2,GP3] enrichments of smooth algebraic varieties over symmetric spectra have been used in order to develop the theory of "K-motives" and solve a problem for the motivic spectral sequence.

In [AG] the category of enriched functors \([\mathcal{C}, \mathcal{V}]\) was studied, where \( \mathcal{V} \) is a closed symmetric monoidal Grothendieck category and \( \mathcal{C} \) is a small category enriched over \( \mathcal{V} \). It was shown that \([\mathcal{C}, \mathcal{V}]\) is a Grothendieck \( \mathcal{V} \)-category with a set of generators \( \{ \mathcal{V}(c, -) \odot g_i \mid c \in \text{Ob}\mathcal{C}, i \in I \} \), where \( \{g_i\}_I \) is a set of generators of \( \mathcal{V} \). The category \([\mathcal{C}, \mathcal{V}]\) is called in [AG] the Grothendieck category of enriched functors. Basic examples are given by categories of additive functors \((\mathcal{B}, \text{Ab})\) or DG-modules \(\text{Mod}\mathcal{A}\) over a DG-category \(\mathcal{A}\). An advantage of this result is that we can recover some well-known theorems for Grothendieck categories in the case where \( \mathcal{V} = \text{Ab} \).

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Another advantage is that \( \mathcal{V} \) can also contain some rich homological or homotopical information, which is extended to the category of enriched functors \( [\mathcal{C}, \mathcal{V}] \). This homotopical information is of great utility to study the derived category \( \mathbf{D}(\mathcal{C}_R) \) of the category of generalized modules \( \mathcal{C}_R = (\text{mod} R, \text{Ab}) \) over a commutative ring \( R \). It was proven in [AG] that \( \mathbf{D}(\mathcal{C}_R) \) is essentially the same as a unital algebraic stable homotopy category in the sense of Hovey–Palmieri–Strickland [HPS] except that the compact objects do not have to be strongly dualizable, but must have a duality. Moreover, this duality is nothing but the classical Auslander–Gruson–Jensen Duality extended to compact objects of \( \mathbf{D}(\mathcal{C}_R) \) (see [AG] for details).

In this paper we investigate the problem of when the derived category \( \mathbf{D}(\mathcal{C}, \mathcal{V}) \) of the Grothendieck category \( [\mathcal{C}, \mathcal{V}] \) is compactly generated triangulated and give an explicit description of the compact generators. The importance of this problem is that the general localization theory of compactly generated triangulated categories becomes available for \( \mathbf{D}(\mathcal{C}, \mathcal{V}) \) in that case. Namely, we prove the following result (see Theorem 6.2):

**Theorem.** Let \( (\mathcal{V}, \otimes, e) \) be a closed symmetric monoidal Grothendieck category such that the derived category of \( \mathcal{V} \) is a compactly generated triangulated category with compact generators \( \{P_j\}_{j \in J} \). Further, suppose we have a small \( \mathcal{V} \)-category \( \mathcal{C} \) and that any one of the following conditions is satisfied:

1. each \( P_j \) is \( \mathbf{K} \)-projective, in the sense of Spaltenstein [Sp];
2. for every \( \mathbf{K} \)-injective \( Y \in \text{Ch}[\mathcal{C}, \mathcal{V}] \) and every \( c \in \mathcal{C} \), the complex \( Y(c) \in \text{Ch}(\mathcal{V}) \) is \( \mathbf{K} \)-injective;
3. \( \text{Ch}(\mathcal{V}) \) has a model structure, with quasi-isomorphisms being weak equivalences, such that for every injective fibrant complex \( Y \in \text{Ch}[\mathcal{C}, \mathcal{V}] \) the complex \( Y(c) \) is fibrant in \( \text{Ch}(\mathcal{V}) \).

Then \( \mathbf{D}(\mathcal{C}, \mathcal{V}) \) is a compactly generated triangulated category with compact generators \( \{\mathcal{V}_c(e, -) \otimes Q_j \mid c \in \mathcal{C}, j \in J\} \) where, if we assume either (1) or (2), \( Q_j = P_j \) or if we assume (3) then \( Q_j = P'_j \) a cofibrant replacement of \( P_j \).

The formulations of the first two statements of the theorem have nothing to do with model categories and use the terminology of the classical homological algebra only. However, in practice these statements are normally covered by the situation when \( \text{Ch}(\mathcal{V}) \) is equipped with a “projective model structure with certain finiteness conditions” or when every evaluation functor \( E_{c} : \text{Ch}[\mathcal{C}, \mathcal{V}] \to \text{Ch}(\mathcal{V}), c \in \mathcal{C} \), is right Quillen. In this case we should be able to extend homological/homotopical information from \( \text{Ch}(\mathcal{V}) \) to \( \text{Ch}[\mathcal{C}, \mathcal{V}] \). To this end, we need the following result proved in Theorems 5.2 and 5.3.

**Theorem.** Let \( \mathcal{V} \) be a closed symmetric monoidal Grothendieck category and \( \mathcal{C} \) be a small \( \mathcal{V} \)-category. Then the category of chain complexes \( \text{Ch}(\mathcal{V}) \) is closed symmetric monoidal Grothendieck and the category \( \text{Ch}[\mathcal{C}, \mathcal{V}] \) is naturally isomorphic to the category \( [\mathcal{C}, \text{Ch}(\mathcal{V})] \), where \( \mathcal{C} \) is enriched over \( \text{Ch}(\mathcal{V}) \) by the obvious complexes concentrated in degree zero.

As an application of the theorems, we can generate numerous (closed symmetric monoidal) compactly generated triangulated categories which are of independent interest. Moreover, several important results of [AG] are extended from \( \mathbf{D}(\mathcal{C}_R) \) to \( \mathbf{D}(\mathcal{C}, \mathcal{V}) \). Other applications are expected in the study of pure-injectivity of compactly generated triangulated categories, in the
telescope conjecture for compactly generated triangulated categories and in the study of Voevodsky’s triangulated categories of motives. The flexibility of the theorems is that we can vary $\mathcal{V}$ in practice. Furthermore, $\mathcal{V}$ itself can contain rich homological/homotopical structures, in which case we can use the homological algebra and the Bousfield localization theory of $\mathcal{D}[\mathcal{C}, \mathcal{V}]$ together with homological/homotopical structures of $\mathcal{V}$. In order to operate with such structures in practice, we need the above theorems.

### 2. Enriched Category Theory

In this section we collect basic facts about enriched categories we shall need later. We refer the reader to [Bor2, R] for details. Throughout this paper the quadruple $(\mathcal{V}, \otimes, \text{Hom}, e)$ is a closed symmetric monoidal category with monoidal product $\otimes$, internal Hom-object $\text{Hom}$ and monoidal unit $e$. We sometimes write $[a, b]$ to denote $\text{Hom}(a, b)$, where $a, b \in \text{Ob} \mathcal{V}$. We have structure isomorphisms

$$ a_{abc} : (a \otimes b) \otimes c \to a \otimes (b \otimes c), \quad l_a : e \otimes a \to a, \quad r_a : a \otimes e \to a $$

in $\mathcal{V}$ with $a, b, c \in \text{Ob} \mathcal{V}$.

**Definition 2.1.** A $\mathcal{V}$-category $\mathcal{C}$, or a category enriched over $\mathcal{V}$, consists of the following data:

1. a class $\text{Ob} \mathcal{C}$ of objects;
2. for every pair $a, b \in \text{Ob} \mathcal{C}$ of objects, an object $\mathcal{V}_{\mathcal{C}}(a, b)$ of $\mathcal{V}$;
3. for every triple $a, b, c \in \text{Ob} \mathcal{C}$ of objects, a composition morphism in $\mathcal{V}$,

$$ c_{abc} : \mathcal{V}_{\mathcal{C}}(a, b) \otimes \mathcal{V}_{\mathcal{C}}(b, c) \to \mathcal{V}_{\mathcal{C}}(a, c); $$

4. for every object $a \in \mathcal{C}$, a unit morphism $u_a : e \to \mathcal{V}_{\mathcal{C}}(a, a)$ in $\mathcal{V}$.

These data must satisfy the natural associativity and unit axioms.

When $\text{Ob} \mathcal{C}$ is a set, the $\mathcal{V}$-category $\mathcal{C}$ is called a small $\mathcal{V}$-category.

**Definition 2.2.** Given $\mathcal{V}$-categories $\mathcal{A}, \mathcal{B}$, a $\mathcal{V}$-functor or an enriched functor $F : \mathcal{A} \to \mathcal{B}$ consists in giving:

1. for every object $a \in \mathcal{A}$, an object $F(a) \in \mathcal{B}$;
2. for every pair $a, b \in \mathcal{A}$ of objects, a morphism in $\mathcal{V}$,

$$ F_{ab} : \mathcal{V}_{\mathcal{A}}(a, b) \to \mathcal{V}_{\mathcal{B}}(F(a), F(b)) $$

in such a way that the following axioms hold:

- for all objects $a, a', a'' \in \mathcal{A}$, diagram (2.1) below commutes (composition axiom);
- for every object $a \in \mathcal{A}$, diagram (2.2) below commutes (unit axiom).

\[
\begin{array}{ccc}
\mathcal{V}_{\mathcal{A}}(a, a') \otimes \mathcal{V}_{\mathcal{A}}(a', a'') & \xrightarrow{c_{a,a',a''}} & \mathcal{V}_{\mathcal{A}}(a, a'') \\
F_{a,a'} \otimes F_{a'',a''} \downarrow & & \downarrow F_{a,a''} \\
\mathcal{V}_{\mathcal{B}}(Fa, Fa') \otimes \mathcal{V}_{\mathcal{B}}(Fa', Fa'') & \xrightarrow{c_{Fa,Fa',Fa''}} & \mathcal{V}_{\mathcal{B}}(Fa, Fa'')
\end{array}
\]
**Definition 2.3.** Let $\mathcal{A}$, $\mathcal{B}$ be two $\mathcal{V}$-categories and $F, G : \mathcal{A} \to \mathcal{B}$ two $\mathcal{V}$-functors. A $\mathcal{V}$-natural transformation $\alpha : F \Rightarrow G$ consists in giving, for every object $a \in \mathcal{A}$, a morphism $\alpha_a : e \to \mathcal{V}_\mathcal{B}(F(a), G(a))$ in $\mathcal{V}$ such that diagram below commutes, for all objects $a, a' \in \mathcal{A}$.

\[
\begin{array}{ccc}
\mathcal{V}_\mathcal{A}(a, a') & \xrightarrow{\alpha_a \otimes e} & \mathcal{V}_\mathcal{A}(a, a') \\
\mathcal{V}_\mathcal{B}(Fa, Ga) \otimes \mathcal{V}_\mathcal{B}(Ga, Ga') & \xrightarrow{c_{Fa\otimes Ga'}} & \mathcal{V}_\mathcal{B}(Fa, Ga') \\
\mathcal{V}_\mathcal{B}(Fa, Fa) & \xrightarrow{F_{a'}} & \mathcal{V}_\mathcal{B}(Fa, Fa')
\end{array}
\]
such that for all \( A \in \text{Ob} \mathcal{V} \), and \( c \in \text{Ob} \mathcal{C} \), the \( \mathcal{V} \)-functor \( \text{act}(-, A) : \mathcal{C} \to \mathcal{C} \) is left \( \mathcal{V} \)-adjoint to \( \text{coact}(A,-) \) and \( \text{act}(c,-) : \mathcal{V} \to \mathcal{C} \) is left \( \mathcal{V} \)-adjoint to \( \mathcal{V}_C(c,-) \).

If \( \mathcal{C} \) is a small \( \mathcal{V} \)-category, \( \mathcal{V} \)-functors from \( \mathcal{C} \) to \( \mathcal{V} \) and their \( \mathcal{V} \)-natural transformations form the category \( [\mathcal{C}, \mathcal{V}] \) of \( \mathcal{V} \)-functors from \( \mathcal{C} \) to \( \mathcal{V} \). If \( \mathcal{V} \) is complete, then \( [\mathcal{C}, \mathcal{V}] \) is also a \( \mathcal{V} \)-category whose morphism \( \mathcal{V} \)-object \( \mathcal{V}_{[\mathcal{C}, \mathcal{V}]}(X,Y) \) is the end
\[
\int_{\text{Ob} \mathcal{C}} \mathcal{V}(X(c),Y(c)).
\]

**Lemma 2.5.** Let \( \mathcal{V} \) be a complete closed symmetric monoidal category, and \( \mathcal{C} \) be a small \( \mathcal{V} \)-category. Then \( [\mathcal{C}, \mathcal{V}] \) is a closed \( \mathcal{V} \)-module.

**Proof.** See [DRO1, 2.4]. \( \square \)

Given \( c \in \text{Ob} \mathcal{C} \), \( X \mapsto X(c) \) defines the \( \mathcal{V} \)-functor \( \text{Ev}_c : [\mathcal{C}, \mathcal{V}] \to \mathcal{V} \) called evaluation at \( c \). The assignment \( c \mapsto \mathcal{V}_C(c,-) \) from \( \mathcal{C} \) to \( [\mathcal{C}, \mathcal{V}] \) is again a \( \mathcal{V} \)-functor \( [\mathcal{C}, \mathcal{V}]^{\text{op}} \to [\mathcal{C}, \mathcal{V}] \), called the \( \mathcal{V} \)-Yoneda embedding. \( \mathcal{V}_C(c,-) \) is a representable functor, represented by \( c \).

**Lemma 2.6** (The Enriched Yoneda Lemma). Let \( \mathcal{V} \) be a complete closed symmetric monoidal category and \( \mathcal{C} \) a small \( \mathcal{V} \)-category. For every \( \mathcal{V} \)-functor \( X : \mathcal{C} \to \mathcal{V} \) and every \( c \in \text{Ob} \mathcal{C} \), there is a \( \mathcal{V} \)-natural isomorphism \( X(c) \cong \mathcal{V}_C(X(c,-),c) \).

**Lemma 2.7.** If \( \mathcal{V} \) is a bicomplete closed symmetric monoidal category and \( \mathcal{C} \) is a small \( \mathcal{V} \)-category, then \( [\mathcal{C}, \mathcal{V}] \) is bicomplete. (Co)limits are formed pointwise.

**Proof.** See [Bor2, 6.6.17]. \( \square \)

**Corollary 2.8.** Assume \( \mathcal{V} \) is bicomplete, and let \( \mathcal{C} \) be a small \( \mathcal{V} \)-category. Then any \( \mathcal{V} \)-functor \( X : \mathcal{C} \to \mathcal{V} \) is \( \mathcal{V} \)-naturally isomorphic to the coend
\[
X \cong \int_{\text{Ob} \mathcal{C}} \mathcal{V}_C(c,-) \otimes X(c).
\]

3. **The Closed Symmetric Monoidal Structure for Chain Complexes**

In this paper we deal with closed symmetric monoidal Grothendieck categories. Here are some examples.

**Example 3.1.** (1) Given any commutative ring \( R \), the triple \( (\text{Mod} R, \otimes_R, R) \) is a closed symmetric monoidal Grothendieck category.

(2) More generally, let \( X \) be a quasi-compact quasi-separated scheme. Consider the category \( \text{Qcoh}(\mathcal{O}_X) \) of quasi-coherent \( \mathcal{O}_X \)-modules. By [III 3.1] \( \text{Qcoh}(\mathcal{O}_X) \) is a locally finitely presented Grothendieck category, where quasi-coherent \( \mathcal{O}_X \)-modules of finite type form a family of finitely presented generators. The tensor product on \( \mathcal{O}_X \)-modules preserves quasi-coherence, and induces a closed symmetric monoidal structure on \( \text{Qcoh}(\mathcal{O}_X) \).

(3) Let \( R \) be any commutative ring. Let \( C' = \{ C'_n, \partial'_n \} \) and \( C'' = \{ C''_n, \partial''_n \} \) be two chain complexes of \( R \)-modules. Their tensor product \( C' \otimes_R C'' = \{ (C' \otimes_R C'')_n, \partial_n \} \) is the chain complex defined by
\[
(C' \otimes_R C'')_n = \bigoplus_{i+j=n} (C'_i \otimes_R C''_j),
\]
and

\[ \partial_n (t'_i \otimes s''_j) = \partial'_i (t'_i) \otimes s''_j + (-1)^i t'_i \otimes \partial''_j (s''_j), \quad \text{for all } t'_i \in C'_i, \ s''_j \in C''_j, \ (i + j = n), \]

where \( C'_i \otimes_R C''_j \) denotes the tensor product of \( R \)-modules \( C'_i \) and \( C''_j \). Then the triple \( (\text{Ch}(\text{Mod} R), \otimes_R, R) \) is a closed symmetric monoidal category. It is Grothendieck by [AG, 3.4]. Here \( R \) is regarded as a complex concentrated in the zeroth degree.

(4) \( (\text{Mod} kG, \otimes, k) \) is closed symmetric monoidal Grothendieck category, where \( k \) is a field and \( G \) is a finite group.

(5) Given a field \( F \), the category \( N\text{SwT}/F \) of Nisnevich sheaves with transfers [SV, Section 2] is a closed symmetric monoidal Grothendieck category with

\[ \{Z_{tr}(X) \mid X \text{ is an } F\text{-smooth algebraic variety}\} \]

a family of generators.

In this section we prove the following natural fact, as the authors were unable to find a complete account in the literature. We find it necessary to give such a complete account as it will be important to our analysis. The authors do not pretend to originality here.

**Theorem 3.2.** Let \( \mathcal{V} \) be a closed symmetric monoidal Grothendieck category. Then the category of chain complexes over \( \mathcal{V} \), denoted \( \text{Ch}(\mathcal{V}) \), is closed symmetric monoidal Grothendieck.

**Proof.** Firstly, by [AG, 3.4] given \( \mathcal{V} \) Grothendieck, we have that \( \text{Ch}(\mathcal{V}) \) is also Grothendieck. It remains to define the closed symmetric monoidal structure on \( \text{Ch}(\mathcal{V}) \). Denote the tensor product of \( \mathcal{V} \) by \( \otimes \) and its unit object by \( e \). Further denote the associativity isomorphism \( a \), the left unitor isomorphism by \( l \) and the right unitor by \( r \) respectively. We also assign \( s \) to mean the symmetry isomorphism in \( \mathcal{V} \).

Given \( X, Y \in \text{Ch} \mathcal{V} \) we define \( X \otimes Y \) as the chain complex with entries

\[ (X \otimes Y)_n := \bigoplus_{n=p+q} X_p \otimes Y_q. \]

Throughout this proof we tacitly assume that the category \( \text{Gr} \mathcal{V} \) of \( \mathbb{Z} \)-graded objects in \( \mathcal{V} \) is closed symmetric monoidal. This follows from Day’s theorem [Day] and literally repeats [AG Example 4.5]. The differential \( d^X \otimes Y^n_n : (X \otimes Y)_n \to (X \otimes Y)_{n-1} \) determined by its action on each summand as

\[ d^X \otimes Y^n_{(p,q)} : X_p \otimes Y_q \to (X_{p-1} \otimes Y_q) \oplus (X_p \otimes Y_{q-1}) \]

followed by inclusion into \( (X \otimes Y)_{n-1} \) such that

\[ d^X \otimes Y^n_{(p,q)} = d^X_p \otimes id^Y_q + (-1)^p id^X_p \otimes d^Y_q, \]
It does indeed define a chain complex as we see by

\[
\begin{array}{c}
X_p \otimes Y_q \\
\downarrow \quad f_p \otimes g_q \\
X' \otimes Y' \quad q \\
\end{array}
\]

\[
\begin{array}{ccc}
d^p_q \otimes id^p_q + (-1)^p \alpha_p \circ d^p_q & \rightarrow & (X_{p-1} \otimes Y_q) \oplus (X_p \otimes Y_{q-1}) \\
\downarrow (f_{p-1} \otimes g_q) & & \downarrow (f_p \otimes g_q) \\
d^p_{q-1} \otimes id^p_{q-1} + (-1)^p \alpha_{p-1} \circ d^p_{q-1} & \rightarrow & (X'_{p-1} \otimes Y'_{q-1}) \oplus (X'_p \otimes Y'_{q-1})
\end{array}
\]

which commutes on each summand for all choices \( p, q \), hence \( f \circ g \) is consistent with the differential and as \( \circ \) is clearly a functor on graded objects, we can thus conclude that \( \circ \) is a bifunctor \( \text{Ch}(\mathcal{Y}) \times \text{Ch}(\mathcal{Y}) \rightarrow \text{Ch}(\mathcal{Y}) \).

Now we are in a position to define our structure isomorphisms. Given chain complexes \( X, Y, Z \in \text{Ch}(\mathcal{Y}) \) we define an associativity isomorphism

\[
\alpha : (X \circ Y) \circ Z \rightarrow X \circ (Y \circ Z).
\]

For \( n \in \mathbb{Z} \) we define \( \alpha_n = \bigoplus_{n=i+j+k} a^{X, Y, Z_k} \), where \( a^{X, Y, Z_k} : (X_i \otimes Y_j) \otimes Z_k \rightarrow X_i \otimes (Y_j \otimes Z_k) \) is the component of the natural associativity isomorphism in \( \mathcal{Y} \). Since we know that \( \alpha \) is a natural isomorphism, a direct sum of its components is also a natural isomorphism, and further we can say that this \( \alpha \) will satisfy the relevant coherence conditions as it will hold at each degree. However, we need to check that these \( \alpha_n \) give a chain map, i.e. these are consistent with the
differential. We have:
\[ d_{(i,j,k)}^{X \otimes (Y \otimes Z)} = d_{ij}^X \otimes id_Y \otimes id_Z + (-1)^{ij} id_X \otimes d_{j+k}^Y \otimes id_Z = d_{ij}^X \otimes id_Y \otimes (d_{j+k}^Y \otimes id_Z + (-1)^{ij} id_Y \otimes d_{k}^Z) \]
\[ = d_{ij}^X \otimes id_Y \otimes (d_{j+k}^Y \otimes id_Z) + (-1)^{ij} id_X \otimes (id_Y \otimes d_{k}^Z) \]
\[ = d_{ij}^X \otimes (id_Y \otimes id_Z) + (-1)^{ij} id_X \otimes (id_Y \otimes d_{k}^Z) \]

and
\[ d_{(i,j,k)}^{(X \otimes Y) \otimes Z} = d_{i+j}^{X \otimes Y} \otimes id_Z + (-1)^{ij} id_{i+j}^{X \otimes Y} \otimes d_{k}^Z \]
\[ = (d_{i}^{X} \otimes id_{j}^{Y}) + (-1)^{ij} id_{i}^{X} \otimes d_{j}^{Y} \otimes id_{k}^Z + (-1)^{ij} id_{i+j}^{X \otimes Y} \otimes d_{k}^Z \]
\[ = (d_{i}^{X} \otimes id_{j}^{Y}) \otimes id_{k}^Z + (-1)^{ij} (id_{i}^{X} \otimes d_{j}^{Y}) \otimes id_{k}^Z + (-1)^{ij} id_{i+j}^{X \otimes Y} \otimes d_{k}^Z \]
\[ = (d_{i}^{X} \otimes id_{j}^{Y}) \otimes id_{k}^Z + (-1)^{ij} (id_{i}^{X} \otimes d_{j}^{Y}) \otimes id_{k}^Z + (-1)^{ij} (id_{i}^{X} \otimes id_{j}^{Y}) \otimes d_{k}^Z \]

which agree up to a change of brackets (i.e. by applying \(a^{X,Y,Z}\)), hence \(\alpha_n\)-s give a chain map.

We define a unit object for our new tensor product, which we denote by \(e\) as being the chain complex with \(e\) in zeroth degree and 0 in every other degree and note that
\[ (X \otimes e)_n = X_n \otimes e \quad \text{and} \quad d_n^{X \otimes e} = d_n^X \otimes id_e \]

for all \(n \in \mathbb{Z}\) as tensoring with zero is zero and a direct sum is unchanged by adding zeros. Thus we define \(\rho_X^Y = \rho_X^Y\), the fact that this is a chain map follows directly from the naturality of \(r\) and moreover is itself a natural transformation in \(\text{Ch}(\mathcal{V})\). Coherence conditions for the right unitor are satisfied at each degree by properties of \(\mathcal{V}\), hence hold in \(\text{Ch}(\mathcal{V})\).

Similarly, note that
\[ (e \otimes Y)_n = e \otimes Y_n \quad \text{and} \quad d_n^{e \otimes Y} = id_e \otimes d_n^Y \]

and hence define the left unitor \(\lambda^Y\) as \(\lambda_n^Y = \alpha_n\) which satisfies the relevant conditions by a similar argument.

Next, consider chain complexes \(X, Y \in \text{Ch}(\mathcal{V})\). We want to define a map
\[ \sigma^{X,Y} : X \otimes Y \to Y \otimes X. \]
We shall consider this map to consist of \(\sigma_n^{X,Y} : \bigoplus_{n=p+q} X_p \otimes Y_q \to \bigoplus_{n=q+p} Y_q \otimes X_p\) for \(n \in \mathbb{Z}\). They are completely determined by its action on each summand
\[ \sigma_{(p,q)}^{X,Y} : X_p \otimes Y_q \to Y_q \otimes X_p, \]
and these we define to be
\[ \sigma_{(p,q)}^{X,Y} = (-1)^{pq} s_{p,q}^{X,Y}, \]
the components of the symmetry isomorphism in \(\mathcal{V}\) multiplied by \((-1)^{pq}\). Such a map is a natural isomorphism as it is in \(\mathcal{V}\), and satisfies the coherence conditions as it will do so on each
component. However, we need to check if this map is indeed a chain map. This is demonstrated by the following commutative diagram:

\[
\begin{array}{ccc}
X_p \otimes Y_q & \xrightarrow{d^X_p \otimes 1 + (-1)^p id^Y_q \otimes d^Y_q} & (X_{p-1} \otimes Y_q) \oplus (X_p \otimes Y_{q-1}) \\
Y_q \otimes X_p & \xrightarrow{(-1)^q id^X_p \otimes d^Y_q + d^X_q \otimes id^Y_p} & (Y_{q-1} \otimes X_p) \oplus (Y_q \otimes X_{p-1})
\end{array}
\]

Thus \( \text{Ch}(\mathcal{Y}) \) is symmetric monoidal and Grothendieck. We next define an internal Hom-object \( \text{Hom}(X, Y) \) for \( X, Y \in \text{Ch}(\mathcal{Y}) \), as having in each degree \( n \in \mathbb{Z} \)

\[
\text{Hom}(X, Y)_n := \prod_p [X_p, Y_{p+n}],
\]

where \([X_p, Y_{p+n}] := \mathcal{Y}(X_p, Y_{p+n})\). To define its differential \( d_n^{\text{Hom}(X, Y)} : \text{Hom}(X, Y)_n \to \text{Hom}(X, Y)_{n-1} \), it is enough to define this map to each factor by first projecting onto \( p \) and \( p-1 \), and then one gets

\[
d_n^{\text{Hom}(X, Y)} : [X_p, Y_{p+n}] \times [X_{p-1}, Y_{p+n-1}] \to [X_p, Y_{p+n-1}]
\]

to be

\[
c_n^{\text{Hom}(X, Y)} = [id^X_p, d^Y_{p+n}] - (-1)^n [d^X_p, id^Y_{p+n-1}].
\]

Again, we need to verify that this defines a differential. We check this on each factor \( p \) and \( p-1 \), using the following diagram

\[
\begin{array}{ccc}
[X_p, Y_{p+n}] & \xrightarrow{[id^X_p, d^Y_{p+n}]} & [X_p, Y_{p+n-1}] \\
[X_{p-1}, Y_{p-1+n}] & \xrightarrow{(-1)^n [d^X_p, id^Y_{p+n-1}]} & [X_{p-1}, Y_{p-1+n-1}] \\
[X_{p-1}, Y_{p-1+n}] & \xrightarrow{[id^X_{p-1}, d^Y_{p-1+n}]} & [X_{p-1}, Y_{p-1+n-1}] \\
[X_{p-2}, Y_{p-2+n}] & \xrightarrow{(-1)^n [d^X_{p-1}, id^Y_{p-1+n}]} & [X_{p-2}, Y_{p-2+n-1}]
\end{array}
\]

We have

\[
[d^X_p, d^Y_{p+n-1}] \circ ([id^X_p, d^Y_{p+n}] - (-1)^n [d^X_p, id^Y_{p+n-1}])
\]

\[
= 0 - (-1)^n [d^X_p, d^Y_{p+n-1}]
\]

\[
- (-1)^n [d^X_p, id^Y_{p+n-2}] \circ ([id^X_{p-1}, d^Y_{p-1+n}] - (-1)^n [d^X_{p-1}, id^Y_{p-1+n-1}])
\]

\[
= -(-1)^n [d^X_p, d^Y_{p+n-1}] + 0
\]

which sums to zero. Hence \( d \circ d = 0 \) and \( \text{Hom}(X, Y) \) is a chain complex.
To define a closed structure on \( \text{Ch}(\mathcal{Y}) \), it is necessary that \( \text{Hom}(X, Y) \) is functorial. It is apparent that the internal Hom-object of \( \mathcal{Y} \) and the product are functors on graded objects. We need only to check consistency with differentials. Given \( f' : X' \to X \) and \( g : Y \to Y' \), define \( \text{Hom}(f', g)_n := \prod_p [f'_p, g_{p+n}] \) at each degree \( n \in \mathbb{Z} \) and then consider the following commutative diagram

\[
\begin{array}{ccc}
[X_p, Y_{p+n}] \times [X_{p-1}, Y_{p-1+n}] & \xrightarrow{[id^X_p, d^Z_{p+n}] - (-1)^n [d^X_p, id^Z_{p+n-1}]} & [X_p, Y_{p-1+n}] \\
\downarrow [f'_p, g_{p+n}] \times [f'_{p-1}, g_{p-1+n}] & & \downarrow [f'_p, g_{p-1+n}] \\
[X'_p, Y'_{p+n}] \times [X'_{p-1}, Y'_{p-1+n}] & \xrightarrow{[id^X'_p, d^Z'_{p+n}] - (-1)^n [d^X'_p, id^Z'_{p+n-1}]} & [X'_p, Y'_{p-1+n}] 
\end{array}
\]

Lastly, we need to make sure that our definition of the internal Hom-object of chain complexes satisfies the following isomorphism

\[
\varphi : \text{Hom}(X \odot Y, Z) \cong \text{Hom}(X, \text{Hom}(Y, Z)),
\]

natural in \( X, Y, Z \in \text{Ch}(\mathcal{Y}) \).

Given a chain map \( k : X \odot Y \to Z \), we know that this is uniquely determined on each degree by maps on each summand. The collection of maps \( k_{(p,q)} : X_p \otimes Y_q \to Z_{p+q} \) for all \( p, q \in \mathbb{Z} \) determines \( k \) uniquely. Using the closed structure of \( \mathcal{Y} \), we can derive a collection \( \phi(k_{(p,q)}) : X_p \to [Y_q, Z_{p+q}] \), where \( \phi(k_{(p,q)}) \) is the adjunction map in \( \mathcal{Y} \) corresponding to \( k_{(p,q)} \), which is sufficient information to define maps

\[
\varphi(k)_p : X_p \to \prod_q [Y_q, Z_{p+q}]
\]

and construct \( \varphi(k) : X \to \text{Hom}(Y, Z) \) as \( (\varphi(k)_p)_{p \in \mathbb{Z}} \). Thus we have established a one-to-one correspondence between \( k \) and \( \varphi(k) \). As usual, we need to check that this identification is compatible with the differentials. More precisely, let us check that \( \varphi(k) \) is a morphism of complexes. We know that \( k \in \text{Hom}(X \odot Y, Z) \) if and only if for all integers \( p, q \)

\[
\begin{array}{ccc}
X_p \otimes Y_q & \xrightarrow{k_{(p,q)}} & Z_{p+q} \\
\downarrow [d^X_p \otimes id^Y_q + (-1)^p id^X_p \otimes d^Y_q] & & \downarrow [d^Z_{p+q}] \\
(X_{p-1} \otimes Y_q) \oplus (X_p \otimes Y_{q-1}) & \xrightarrow{k_{(p-1,q)} + k_{(p,q-1)}} & Z_{p+q-1}
\end{array}
\]

commutes. Our adjunction in \( \mathcal{Y} \) will lead to equalities

\[
[id^Y_q, d^Z_{p+q}] \circ \phi(k_{(p,q)}) = \phi(d^Z_{p+q} \circ k_{(p,q)})
\]

\[
= \phi(k_{(p-1,q)} \circ (d^X_p \otimes id^Y_q) + (-1)^p k_{(p,q-1)} \circ (id^X_p \otimes d^Y_q))
\]

\[
= \phi(k_{(p-1,q)} \circ d^X_p + (-1)^p \phi(k_{(p,q-1)} \circ (id^X_p \otimes d^Y_q))
\]

\[
= \phi(k_{(p-1,q)} \circ d^X_p + (-1)^p [id^X_p, d^Z_{p+q-1}] \circ \phi(k_{(p,q-1)}).)
\]

So we must have

\[
\phi(k_{(p-1,q)} \circ d^X_p = [id^Y_q, d^Z_{p+q}] \circ \phi(k_{(p,q)}) - (-1)^p [id^Y_q, d^Z_{p+q-1}] \circ \phi(k_{(p,q-1)}).
\]
If we write it as a commutative diagram, we get

\[
\begin{array}{ccc}
X_p & \xrightarrow{(\phi(k_{p,q}), \phi(k_{p,q-1}))} & [Y_q, Z_{p+q}] \times [Y_{q-1}, Z_{p+q-1}] \\
\downarrow d_p^X & & \downarrow [id_{[Y_q, Z_{p+q}]}, d_p^Z_{p+q} - (-1)^p d_p^Z_{p+q-1}]
\end{array}
\]

for all integers \(p, q\). In other words, \(\phi(k)\) is a chain map if and only if so is \(k\), as required.

Next, we have to determine whether our identification is natural. Consider maps \(f: X \to X', g: Y \to Y'\) and \(h: Z \to Z'\), in \(\text{Ch} (\mathcal{V}')\) and \(k: X' \otimes Y' \to Z'\). Given that two chain maps are equal if and only if they are equal on each degree, we fix a degree \(n \in \mathbb{Z}\) and calculate

\[
\phi(h \circ k \circ (f \otimes g))_n = \prod_q \phi(h_{n+q} \circ (k_{n,q}) \circ (f_n \otimes g_q)) \\
= \prod_q [g_q, h_{n+q}] \circ \phi(k_{n,q}) \circ f_n \\
= \text{Hom}(g, h)_n \circ \phi(k)_n \circ f_n \\
= (\text{Hom}(g, h) \circ \phi(k) \circ f)_n
\]

Thus we have the desired naturality. We also have automatically that these isomorphisms are additive, and hence an adjunction in the Grothendieck category \(\text{Ch} (\mathcal{V}')\).

We conclude that if \(\mathcal{V}'\) is a closed symmetric monoidal Grothendieck category, then the category of chain complexes \(\text{Ch} (\mathcal{V}')\) is closed symmetric monoidal Grothendieck with the structure detailed above, as was to be proved. \(\square\)

The preceding theorem leads to the following natural definition.

**Definition 3.3.** A category \(\mathcal{C}\) enriched over \(\text{Ch} (\mathcal{V}')\) is said to be a differential graded \(\mathcal{V}'\)-category or just a DG \(\mathcal{V}'\)-category. \(\mathcal{C}\) is small if its objects form a set. Ordinary DG-categories are recovered as DG \(\mathcal{V}'\)-categories with \(\mathcal{V}' = \text{Ab}\).

The category of differential graded \(\mathcal{V}'\)-modules or just DG \(\mathcal{V}'\)-modules is the category \([\mathcal{C}, \text{Ch} (\mathcal{V}')]\) of enriched functors from a DG \(\mathcal{V}'\)-category \(\mathcal{C}\) to \(\text{Ch} (\mathcal{V}')\). Ordinary DG-modules over a DG-category are recovered as DG \(\mathcal{V}'\)-modules with \(\mathcal{V}' = \text{Ab}\).

Given any complete closed symmetric monoidal category \(\mathcal{V}\) and any small \(\mathcal{V}'\)-category \(\mathcal{C}\), \([\mathcal{C}, \mathcal{V}']\) is a closed \(\mathcal{V}'\)-module by Lemma 2.5. We write \(\otimes\) for the corresponding functor \([\mathcal{C}, \mathcal{V}'] \otimes \mathcal{V}' \to [\mathcal{C}, \mathcal{V}']\).

**Corollary 3.4.** Given a closed symmetric monoidal Grothendieck category \(\mathcal{V}\) with a family of generators \(\{g_i\}_I\) and small differential graded \(\mathcal{V}'\)-category \(\mathcal{C}\), the category of differential graded \(\mathcal{V}'\)-modules \([\mathcal{C}, \text{Ch} (\mathcal{V}')]\) is Grothendieck with the set of generators \(\{\text{Ch} (\mathcal{V}') \otimes (c, -) \otimes D^n g_i \mid c \in \mathcal{C}, i \in I, n \in \mathbb{Z}\}\), where each \(D^n g_i \in \text{Ch} (\mathcal{V}')\) is the complex which is \(g_i\) in degree \(n\) and \(n - 1\) and 0 elsewhere, with interesting differential being the identity map.

**Proof.** By the preceding theorem \(\text{Ch} (\mathcal{V}')\) is a closed symmetric monoidal Grothendieck category. By the proof of [AG] 3.4 its set of generators is given by the family of complexes \(\{D^n g_i \mid i \in I, n \in \mathbb{Z}\}\). Our statement now follows from [AG] 4.2. \(\square\)
4. The enriched structure

Suppose \( \mathcal{Y} \) is a closed symmetric monoidal Grothendieck category and \( \mathcal{C} \) is a small \( \mathcal{Y} \)-category. In order to get some information about \( \textbf{Ch}[\mathcal{C}, \mathcal{Y}] \), we shall identify this category with \( [\mathcal{C}, \textbf{Ch}(\mathcal{Y})] \) (see Theorem 5.4) if we regard \( \mathcal{C} \) as trivially a \( \textbf{Ch}(\mathcal{Y}) \)-category, where for each \( a, b \in \mathcal{C} \) we define the chain \( \textbf{Ch}(\mathcal{Y})a,b \) as having in zeroth degree the \( \mathcal{Y} \)-object \( \mathcal{Y}a,b \) and zero in every other degree. But first we need to collect some facts about \( \textbf{Ch}(\mathcal{Y}) \).

It is known (see [Bor2]) that a closed symmetric monoidal category canonically carries the structure of a category enriched over itself. It will be important for us to describe the unit and composition morphisms in the case of \( \textbf{Ch}(\mathcal{Y}) \) explicitly, using the unit and composition morphisms belonging to \( \mathcal{Y} \).

We begin by describing the unit. Given \( a \in \mathcal{C} \) and any \( F \in [\mathcal{C}, \textbf{Ch}(\mathcal{Y})] \), the unit morphism \( u_{F(a)} : e \to \text{Hom}(F(a), F(a)) \), where \( e \) is the unit object of the tensor product on \( \textbf{Ch}(\mathcal{Y}) \) defined in the proof of Theorem 3.2, reduces to a single morphism in degree zero, \( u_{F(a)} : e \to \prod_p [F(a) \otimes F(a)_p] \) with \( e \) the unit object of the tensor product on \( \mathcal{Y} \). Moreover, \( u_{F(a)} = (u_{F(a)_p})_{p \in \mathbb{Z}} : e \to \prod_p [F(a) \otimes F(a)_p] \) where \( u_{F(A)_p} \) is nothing but the unit morphism in \( \mathcal{Y} \) associated to the \( \mathcal{Y} \)-object \( F(a)_p \) for \( p \in \mathbb{Z} \).

Next, in order to describe the composition morphism, we need to first understand the evaluation morphism in \( \textbf{Ch}(\mathcal{Y}) \) in terms of evaluation in \( \mathcal{Y} \). Thus we consider \( A, B \in \textbf{Ch}(\mathcal{Y}) \), and denote \( \text{ev}_{A,B} : \text{Hom}(A,B) \otimes A \to B \) the evaluation morphism in \( \textbf{Ch}(\mathcal{Y}) \). This evaluation morphism is defined to be adjunct to the identity morphism on \( \text{Hom}(A,B) \). Hence, we are able to calculate this morphism explicitly by maps in \( \mathcal{Y} \) as follows. Consider the projection maps

\[
\text{pr}_{s,t} : \prod_p [A_p \otimes B_{p+t}] \to [A_s \otimes B_{s+t}], \quad s,t \in \mathbb{Z}.
\]

We can calculate \( \text{ev}_{A,B} \) by applying the adjunction \( \phi^{-1} \) in \( \mathcal{Y} \) (see Theorem 3.2). We have then that

\[
(\text{ev}_{A,B})_n = \bigoplus_{s+t=n} \phi^{-1}(\text{pr}_{s,t}) : \bigoplus_{s+t=n} \left( \prod_p [A_p \otimes B_{p+t}] \right) \otimes A_s \to B_{s+t}
\]

Next, consider the following commutative diagram

\[
\begin{array}{ccc}
\prod_p [A_p \otimes B_{p+t}] & \xrightarrow{\text{pr}_{s,t}} & [A_s \otimes B_{s+t}] \\
\downarrow \text{pr}_{s,t} & & \downarrow \text{id} \\
[A_s \otimes B_{s+t}] & \xrightarrow{\text{id}} & [A_s \otimes B_{s+t}]
\end{array}
\]

and apply the adjunction in \( \mathcal{Y} \), and deduce the following commutative diagram

\[
\begin{array}{ccc}
(\prod_p [A_p \otimes B_{p+t}]) \otimes A_s & \xrightarrow{\phi^{-1}(\text{pr}_{s,t})} & B_{s+t} \\
\downarrow \text{pr}_{s,t} \otimes \text{id} & & \downarrow \text{id} \\
[A_s \otimes B_{s+t} \otimes A_s] & \xrightarrow{\text{ev}_{A_s,B_{s+t}}} & B_{s+t},
\end{array}
\]
where $\text{ev}_{A_s, B_{s+t}}$ is the evaluation morphism in $\mathcal{V}$. Thus

$$(\text{ev}_{A,B})_n = \bigoplus_{s+t=n} \phi^{-1}(\text{pr}_{s,s+t}) = \bigoplus_{s+t=n} \text{ev}_{A_s, B_{s+t}} \circ (\text{pr}_{s,s+t} \otimes \text{id}).$$

We are now in a position to describe the composition morphism

$$\text{Hom}(A, B) \circ \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$$

explicitly. Following Borceux [Bor2, Diagram 6.6] this map is defined to be adjoint to the composite

$$\text{ev}_{B,C} \circ \sigma \circ (\text{ev}_{A,B} \circ 1) \circ (\text{id}_{\text{Hom}(A,B)} \circ \sigma) : \text{Hom}(A, B) \circ \text{Hom}(B, C) \circ A \rightarrow C,$$

where $\sigma$ is the swapping chain isomorphism described in the proof of Theorem [3.2]. Furthermore, this composite at degree $n \in \mathbb{Z}$ is determined by a collection of morphisms with $r + p + q = n$,

$$(\text{ev}_{B,C})_{r+p+q} \circ (-1)^{(r+p)qsw} \circ ((\text{ev}_{A,B})_{r+p} \otimes \text{id}) \circ ((-1)^{qr}sw \otimes \text{id}).$$

Using our description of evaluation we may consider the following diagram, where the rightmost path is any morphism from the collection above.
It is clear that this diagram is commutative, and we may take the leftmost path and apply the adjunction in $\mathcal{V}$. Thus we are able to conclude that the composition morphism in $\text{Ch} (\mathcal{V})$ at each degree

\[(c_{\text{Ch} \mathcal{V}})_n : \bigoplus_{p+q=n} \left( \prod_i [A_i, B_{i+p}] \otimes \prod_j [B_j, C_{j+q}] \right) \to \prod_r [A_r, C_{r+p+q}]\]

for $n \in \mathbb{Z}$ is determined by morphisms

\[(-1)^{pq} c_{A_r, B_{r+p}, C_{r+p+q}} \circ (\text{pr}_{[A_i, B_{p+i}]} \otimes \text{pr}[B_{p+i}, C_{p+q+i}])\]  \hfill (4.1)

Hence, composition in $\text{Ch} (\mathcal{V})$ is the same as first taking projections and then composing in $\mathcal{V}$, up to a sign $(-1)^{pq} = (-1)^{qr} (-1)^{(r+p)q}$. 
5. Identifying Chain Complexes with Enriched Functors

We shall work with a closed symmetric monoidal Grothendieck category \( \mathcal{V} \), and consider a small \( \mathcal{V} \)-category \( \mathcal{C} \). It is evident that \( \mathcal{C} \) can be regarded as trivially a \( \text{Ch}(\mathcal{V}) \)-category, where for each \( a, b \in \mathcal{C} \) we define the chain \( \text{Ch}(\mathcal{V})_a(a, b) \) as having in zeroth degree the \( \mathcal{V} \)-object \( \mathcal{V}((a, b)) \) and zero in every other degree.

**Definition 5.1.** Consider the trivial \( \text{Ch}(\mathcal{V}) \)-enrichment on \( \mathcal{C} \) introduced above. We define the enriched functor category \( [\mathcal{C}, \text{Ch}(\mathcal{V})] \) as a category with objects \( \text{Ch}(\mathcal{V}) \)-functors \( F : \mathcal{C} \to \text{Ch}(\mathcal{V}) \) and the morphisms in \( [\mathcal{C}, \text{Ch}(\mathcal{V})] \) are defined as \( \text{Ch}(\mathcal{V}) \)-natural transformations.

Note that for any \( \text{Ch}(\mathcal{V}) \)-functor \( F : \mathcal{C} \to \text{Ch}(\mathcal{V}) \) and \( a, b \in \mathcal{C} \), \( F_a, b : \text{Ch}(\mathcal{V})_a(a, b) \to \text{Hom}(F(a), F(b)) \) is, by definition, a morphism in \( \text{Ch}(\mathcal{V}) \) of the form:

\[
\cdots \rightarrow 0 \rightarrow 0 \rightarrow \mathcal{V}_a(a, b) \rightarrow 0 \rightarrow 0 \rightarrow \cdots
\]

Using the definition of the complex \( \text{Hom}(F(a), F(b)) \) (see the proof of Theorem [5.2]), we see that \( F_a, b \) reduces to the single non-trivial map

\[
\mathcal{V}_a(a, b) \to \prod_p [F(a)_p, F(b)_p]
\]

in degree 0 with the property that

\[
[\text{id}_p^{F(a)}, d_p^{F(b)}] \circ (F_a, b)_p - [d_p^{F(a)}, \text{id}_p^{F(b)}] \circ (F_a, b)_{p-1} = 0
\]

(5.1)

for every \( p \in \mathbb{Z} \).

**Lemma 5.2.** [Bor2, 6.2.8] Given any closed symmetric monoidal category \( \mathcal{V} \) and \( \mathcal{V} \)-functors \( X, Y : \mathcal{C} \to \mathcal{V} \), a \( \mathcal{V} \)-natural transformation \( \alpha : X \to Y \) can be defined as a collection of maps \( \alpha(a) : X(a) \to Y(a) \) in \( \mathcal{V} \) such that

\[
\begin{array}{ccc}
\mathcal{V}(a, b) & \xrightarrow{\alpha(a,b)} & [X(a), X(b)] \\
\downarrow & & \downarrow \text{id} \alpha(b) \\
[Y(a), Y(b)] & \xrightarrow{[\alpha(a), \text{id}]} & [X(a), Y(b)]
\end{array}
\]

commutes for all \( a, b \in \mathcal{C} \).

**Definition 5.3.** The category of chain complexes \( \text{Ch}[\mathcal{C}, \mathcal{V}] \) over the category of enriched functors \( [\mathcal{C}, \mathcal{V}] \) is defined as having objects \( G \), consisting of collections of \( \mathcal{V} \)-functors \( G_n : \mathcal{C} \to \mathcal{V} \) and \( \mathcal{V} \)-natural transformations \( d_n^G : G_n \Rightarrow G_{n-1} \) for \( n \in \mathbb{Z} \) with the property that \( d_2 = 0 \). This category is defined with morphisms \( g : G \to G' \) being collections of \( \mathcal{V} \)-natural transformations \( g_n : G_n \Rightarrow G'_n \) that commute with the differentials.

We are now in a position to prove the main result of the section.

**Theorem 5.4.** Let \( \mathcal{V} \) be a closed symmetric monoidal Grothendieck category and \( \mathcal{C} \) be a small \( \mathcal{V} \)-category. Then the category \( \text{Ch}[\mathcal{C}, \mathcal{V}] \) is naturally isomorphic to the category \( [\mathcal{C}, \text{Ch}(\mathcal{V})] \).
Proof. We split the proof into several steps.

Step 1. Given any $\text{Ch}(\mathcal{V})$-functor $F \in \left[\mathcal{C}, \text{Ch}(\mathcal{V})\right]$ we can associate a chain complex $G \in \text{Ch}[\mathcal{C}, \mathcal{V}]$ to $F$ in the following canonical way.

Firstly, we define the objects that constitute $G$ as a collection of $\mathcal{V}$-functors $G_n : \mathcal{C} \to \mathcal{V}$ such that $G_n(c) := F(c)_n$. Further define the actions on morphisms of these $G_n$ as maps $(G_n)_{a,b} : \mathcal{V}_{\mathcal{C}}(a,b) \to [G_n(a), G_n(b)]$ being equal to the $n$-th factor of the only non-trivial component of the map $F_{a,b}$, precisely the morphisms $(F_{a,b})_n : \mathcal{V}_{\mathcal{C}}(a,b) \to [F(a)_n, F(b)_n]$, for each $a, b \in \mathcal{C}$ and $n \in \mathbb{Z}$ (see p. [15]).

We are able to see that $G_n$ constitute valid $\mathcal{V}$-functors $\mathcal{C} \to \mathcal{V}$ because $F$ is a $\text{Ch}(\mathcal{V})$-functor if and only if

$$
\text{Ch}(\mathcal{V})_{\mathcal{C}}(a,b) \odot \text{Ch}(\mathcal{V})_{\mathcal{C}}(b,c) \xrightarrow{\epsilon_{\text{Ch}(\mathcal{V})}} \text{Ch}(\mathcal{V})_{\mathcal{C}}(a,c)
$$

and

$$
\text{Hom}(F(a), F(b)) \odot \text{Hom}(F(b), F(c)) \xrightarrow{\epsilon_{\text{Ch}(\mathcal{V})}} \text{Hom}(F(a), F(c))
$$

commute in $\text{Ch}(\mathcal{V})$ for all $a, b, c \in \mathcal{C}$. This reduces to the following diagrams

$$
\mathcal{V}_{\mathcal{C}}(a,b) \otimes \mathcal{V}_{\mathcal{C}}(b,c) \xrightarrow{\epsilon_{\mathcal{V}}} \mathcal{V}_{\mathcal{C}}(a,c)
$$

and

$$
\pi_p [F(a)_p, F(b)_p] \otimes \pi_p [F(b)_p, F(c)_p] \xrightarrow{c} \pi_p [F(a)_p, F(c)_p]
$$

in $\mathcal{V}$ for all $a, b, c \in \mathcal{C}$ and every $p \in \mathbb{Z}$. Here $c$ is the map determined by the collection of morphisms

$$
c_{F(a)_p, F(b)_p, F(c)_p} \circ (\text{pr}_{F(a)_p, F(b)_p} \otimes \text{pr}_{F(b)_p, F(c)_p}),
$$

with $p \in \mathbb{Z}$, as detailed in the previous section by $\text{[4.1]}$. Therefore, we have that commutativity of those diagrams is equivalent to commutativity of the following diagrams in $\mathcal{V}$

$$
\mathcal{V}_{\mathcal{C}}(a,b) \otimes \mathcal{V}_{\mathcal{C}}(b,c) \xrightarrow{\epsilon_{\mathcal{V}}} \mathcal{V}_{\mathcal{C}}(a,c)
$$

and

$$
\pi_p [G_p(a), G_p(b)] \otimes [G_p(b), G_p(c)] \xrightarrow{c} [G_p(a), G_p(c)]
$$
and

$$e \xrightarrow{u_{a}} \mathcal{Y}_\mathcal{C}(a,a) \xrightarrow{(G_p)_{a,b}} [G_p(a), G_p(b)]$$

for all $a, b, c \in \mathcal{C}$ and every $p \in \mathbb{Z}$. We see that $G_p$ are $\mathcal{Y}$-functors.

Next, define the differential of $G$ as the $\mathcal{Y}$-natural transformations $d^G_n : G_n \Rightarrow G_{n-1}$ associated with the collection of maps $d^G_n(a) := d^F_n(a)$ using Lemma 5.2. Furthermore $F_{a,b}$ is such that $\text{Ch}(\mathcal{Y}) \circ (a, b) \Rightarrow \text{Hom}(F(a), F(b))$ is a chain map, equivalently that $d^\text{Hom}_{F(a), F(b)} \circ F_{a,b} = 0$, for all $p \in \mathbb{Z}$. By (5.1) we have that

$$[\text{id}^F_p(a), d^F_p(b)] \circ (F_{a,b})_{p} = [d^F_p(a), \text{id}^F_{p-1}] \circ (F_{a,b})_{p-1}.$$ 

This is the same as saying that

$$\mathcal{Y}_\mathcal{C}(a,b) \xrightarrow{(G_{p-1})_{a,b}} [G_{p-1}(a), G_{p-1}(b)] \xrightarrow{[\text{id}^F_p(a), d^F_p(b)]} [G_p(a), G_p(b)]$$

commutes for all $p \in \mathbb{Z}$, hence $d^G_n$ define $\mathcal{Y}$-natural transformations. This defines a valid differential as $d^G_n(a) \circ d^G_{n+1}(a)$ is determined by $d^F_n(a) \circ d^F_{n+1}(a) = 0$ for all $a \in \mathcal{C}$ and $n \in \mathbb{Z}$. Thus we have associated to $F \in [\mathcal{C}, \text{Ch}(\mathcal{Y})]$ a chain complex $G \in \text{Ch}[\mathcal{C}, \mathcal{Y}]$.

Step 2. Now, given any $\text{Ch}(\mathcal{Y})$-natural transformation in $[\mathcal{C}, \text{Ch}(\mathcal{Y})]$ we associate a chain map in $\text{Ch}[\mathcal{C}, \mathcal{Y}]$ in the following canonical way.

Given $f : F \Rightarrow F'$ with $F, F' \in [\mathcal{C}, \text{Ch}(\mathcal{Y})]$, we can associate a chain map $g : G \Rightarrow G'$ where $G, G' \in \text{Ch}[\mathcal{C}, \mathcal{Y}]$ are the chain complexes of Step 1 associated to the respective functors $F$ and $F'$. Using Lemma 5.2, we can determine $f$ by a family of maps $f(a) : F(a) \Rightarrow F'(a) \in \mathcal{C}(\mathcal{Y})$ such that for all $a \in \mathcal{C}$ the following square commutes

$$\text{Ch}(\mathcal{Y}) \circ (a, b) \xrightarrow{F_{a,b}} \text{Hom}(F(a), F(b)) \xrightarrow{\text{Hom}(f(a), \text{id}_b)} \text{Hom}(F(a), F'(b)).$$

As above, it reduces to commutativity of

$$\mathcal{Y}_\mathcal{C}(a,b) \xrightarrow{F_{a,b}} \prod_p [F(a)_p, F(b)_p] \xrightarrow{\prod_p [f(a)_p, \text{id}_p]} \prod_p [F'(a)_p, F'(b)_p].$$

for all $a, b \in \mathcal{C}$ and $p \in \mathbb{Z}$. Thus we define $g_p(a) := f(a)_p$ and see that
are those chain complexes associated to the functors $F$. Further the graded map $g_p$ is determined by the maps $(g_p)_p : G(a) \to G(b)$ in degree zero, the desired structure map is determined by $g_p$. Hence $g_p$ defined in this manner are $\mathscr{V}$-natural transformations by Lemma 5.2. Further the graded map $g := (g_p)_{p \in \mathbb{Z}}$ is in fact a map of chain complexes, because $(g_{p-1})(a) \circ d^G_p(a) = f(a)_{p-1} \circ d^F_p(a) = d^F_{p-1}(a) \circ (g_p)(a)$. Therefore, we have associated to a map $f : F \Rightarrow F'$, with $F, F' \in [\mathscr{C}, \text{Ch}(\mathscr{V})]$, a chain map $g : G \to G'$ where $G, G' \in \text{Ch}[\mathscr{C}, \mathscr{V}]$ are those chain complexes associated to the functors $F$ and $F'$ respectively.

Step 3. Given any chain complex $G \in \text{Ch}[\mathscr{C}, \mathscr{V}]$ we can associate a $\text{Ch}(\mathscr{V})$-functor $F \in [\mathscr{C}, \text{Ch}(\mathscr{V})]$ to $G$ in the following canonical way.

Firstly, we define an action on objects by $F : \mathscr{C} \to \text{Ch}(\mathscr{V})$. Given $c \in \mathscr{C}$ define a chain complex $F(c)$ with components $F(c)_n := G(a)$ equipped with a differential $d^F(c)$ defined by components $d^F_n := d^G_n$. This is a valid chain complex as $d^G_n \circ d^G_{n+1}(c) = 0$ for all $n \in \mathbb{Z}$ and $c \in \mathscr{C}$. Next we define an action on morphisms. Given objects $a, b \in \mathscr{C}$ we define the chain map $F_{a,b} : \text{Ch}(\mathscr{V})(a,b) \to \text{Hom}(F(a), F(b))$ as follows. Since $\text{Ch}(\mathscr{V})(a,b)$ is concentrated in degree zero, the desired structure map is fully determined by $\mathscr{V}(a,b) \to \prod_p \text{Hom}(F(a)_p, F(b)_p)$ being the maps $(G_p)_{a,b} : \mathscr{V}(a,b) \to [F(a)_p, F(b)_p]$. For this to be a valid chain map we must satisfy the following relation

$$[\text{id}^F_p(a), d^F_p(b)] \circ (F_{a,b})_p - [d^F_p(a), \text{id}^F_p(b)] \circ (F_{a,b})_{p-1} = [\text{id}^G_p(a), d^G_p(b)] \circ (G_{a,b})_p - [d^G_p(a), \text{id}^G_{p-1}(b)] \circ (G_{p-1})_{a,b} = 0$$

for every $p \in \mathbb{Z}$. This relation indeed holds by Lemma 5.2 as $d^G_p$ are $\mathscr{V}$-natural transformations. Moreover, we must verify the enriched composition and unit laws for $F$ to be a $\text{Ch}(\mathscr{V})$-functor.

This is, more precisely, establishing the commutativity of the following diagrams

$$\begin{array}{ccc}
\text{Ch}(\mathscr{V})(a,b) & \overset{\epsilon_{\mathscr{C}(\mathscr{V})}}{\longrightarrow} & \text{Ch}(\mathscr{V})(a,c) \\
\text{Hom}(F(a), F(b)) \downarrow F_{a,b} \circ F_{b,c} & & \downarrow F_{a,c} \\
\text{Hom}(F(a), F(b), F(c)) & \overset{\epsilon_{\text{Hom}(F(a), F(c))}}{\longrightarrow} & \text{Hom}(F(a), F(c))
\end{array}$$

and

$$\begin{array}{ccc}
\epsilon & \overset{u_a}{\longrightarrow} & \text{Ch}(\mathscr{V})(a,a) \\
\text{Hom}(F(a), F(a)) \downarrow F_{a,a} & & \downarrow F_{a,a} \\
\text{Hom}(F(a), F(a)) & \overset{\epsilon_{\text{Hom}(F(a), F(a))}}{\longrightarrow} & \text{Hom}(F(a), F(a))
\end{array}$$
for all \(a, b, c \in \mathcal{C}\) (see Step 1). By definition of \(F\) we see that these commute if and only if,

\[
\mathcal{V}_\mathcal{C}(a, b) \otimes \mathcal{V}_\mathcal{C}(b, c) \xrightarrow{e_{\mathcal{V}}} \mathcal{V}_\mathcal{C}(a, c)
\]

\[
\prod_p [F(a)_p, F(b)_p] \otimes \prod_p [F(b)_p, F(c)_p] \xrightarrow{e} \prod_p [F(a)_p, F(c)_p]
\]

and

\[
e \xleftarrow{u_a} \mathcal{V}_\mathcal{V}(a, a)
\]

\[
\prod_p [F(a)_p, F(a)_p]
\]

commute in \(\mathcal{V}\) for all \(a, b, c \in \mathcal{C}\) and every \(p \in \mathbb{Z}\). It follows that these diagrams commute if and only if

\[
\mathcal{V}_\mathcal{C}(a, b) \otimes \mathcal{V}_\mathcal{C}(b, c) \xrightarrow{e_{\mathcal{V}}} \mathcal{V}_\mathcal{C}(a, c)
\]

\[
\prod_p [G_p(a)_p, G_p(b)_p] \otimes \prod_p [G_p(b)_p, G_p(c)_p] \xrightarrow{e_{\mathcal{V}}} \prod_p [G_p(a)_p, G_p(c)_p]
\]

and

\[
e \xleftarrow{u_a} \mathcal{V}_\mathcal{V}(a, a)
\]

\[
\prod_p [G_p(a)_p, G_p(a)_p]
\]

commute in \(\mathcal{V}\) for all \(a, b, c \in \mathcal{C}\) and every \(p \in \mathbb{Z}\), which indeed commute as \(G_p\) are \(\mathcal{V}\)-functors.

Step 4. Now given any chain map in \(\mathbf{Ch}[\mathcal{C}, \mathcal{V}]\), we associate a \(\mathbf{Ch}(\mathcal{V})\)-natural transformation in \(\mathcal{C}, \mathbf{Ch}(\mathcal{V})\) in the following canonical way.

Consider a chain map \(g : G \Rightarrow G'\) with \(G, G' \in \mathbf{Ch}[\mathcal{C}, \mathcal{V}]\). We can associate a \(\mathbf{Ch}(\mathcal{V})\)-natural transformation \(f : F \Rightarrow F'\) with \(F, F' \in \mathcal{C}, \mathbf{Ch}(\mathcal{V})\) being those functors associated to \(G\) and \(G'\) respectively. By Lemma 5.2 we can determine \(g\) at each component \(n \in \mathbb{Z}\) by a family of maps \(g_n(a) : G(a) \to G'(a) \in \mathbf{Ch}(\mathcal{V})\) such that for all \(a, b \in \mathcal{C}\)

\[
\mathcal{V}_\mathcal{C}(a, b) \xrightarrow{(G_n)_a,b} [G_n(a)_p, G_n(b)_p]
\]

\[
[G'_n(a)_p, G'_n(b)_p] \xrightarrow{[g_n(a), id]} [G_n(a)_p, G_n(b)_p]
\]

is commutative. Thus we set \(f(a)_p := g_p(a)\) and deduce that

\[
\mathcal{V}_\mathcal{C}(a, b) \xrightarrow{F_n_{a,b}} \prod_p [F(a)_p, F(b)_p]
\]

\[
\prod_p [F'(a)_p, F'(b)_p] \xrightarrow{\prod_p [f(a)_p, id]} \prod_p [F(a)_p, F'(b)_p]
\]
is commutative. However, in order to say that \( f \) is a map in \([\mathcal{E}, \text{Ch}(\mathcal{V})]\), we must verify that all \( f(a) \) belong to \( \text{Ch}(\mathcal{V}) \) to claim that

\[
\begin{array}{ccc}
\text{Ch}(\mathcal{V})_\mathcal{E} (a, b) & \xrightarrow{F_{a,b}} & \text{Hom}(F(a), F(b)) \\
F_{a,b}' & \downarrow & \text{Hom}(\text{id}_a, f(b)) \\
\text{Hom}(F'(a), F'(b)) & \xrightarrow{\text{Hom}(f(a), \text{id}_b)} & \text{Hom}(F(a), F'(b))
\end{array}
\]

commutes. But this reduces to the fact that \( f(a)_{p-1} \circ d^F_p(a) = (g_{p-1})_p(a) \circ d^G_p(a) = d^{G'}_{p-1}(a) \circ (g_p)(a) = d^{F'(a)}_{p-1} \circ f(a)_p \).

**Conclusion.** We have defined an association between objects and morphisms of the categories \( \text{Ch}[\mathcal{E}, \mathcal{V}] \) and \([\mathcal{E}, \text{Ch}(\mathcal{V})]\), and further claim that it is functorial and an isomorphism of categories. Functoriality can be seen from Lemma 5.2 and the fact that composition of natural transformations is determined by the composition on each component. Clearly, this is an isomorphism of categories by the very construction, as required. \(\square\)

Beke [Be, 3.13] and Hovey [Hov, 2.2] defined a proper cellular model structure on \( \text{Ch}(\mathcal{A}) \) for every Grothendieck category \( \mathcal{A} \), where cofibrations are the monomorphisms, and weak equivalences the quasi-isomorphisms. We also call it the injective model structure. Its fibrant objects are \( \mathbf{K} \)-injective complexes in the sense of Spaltenstein [Sp]. In particular, \( \text{Ch}[\mathcal{E}, \mathcal{V}] \) has the injective model structure \((g: G \to G' \text{ in } \text{Ch}[\mathcal{E}, \mathcal{V}] \text{ is a quasi-isomorphism if and only if } g(a): G(a) \to G'(a) \text{ is a quasi-isomorphism in } \text{Ch}(\mathcal{V}) \text{ for all } a \in \mathcal{E})\).

However, it is hard to deal with the injective model structure for some particular computations. Instead we want to transfer homotopy information from \( \text{Ch}(\mathcal{V}) \) to \( \text{Ch}[\mathcal{E}, \mathcal{V}] \) by using the identification \( \text{Ch}[\mathcal{E}, \mathcal{V}] \cong [\mathcal{E}, \text{Ch}(\mathcal{V})] \) from the preceding theorem.

Suppose \( \text{Ch}(\mathcal{V}) \) possesses a weakly finitely generated monoidal model structure in the sense of [DRO1] in which weak equivalences are the quasi-isomorphisms. Following [DRO1] Section 4] a morphism \( f \text{ in } [\mathcal{E}, \text{Ch}(\mathcal{V})] \) is a pointwise fibration if \( f(c) \) is a fibration in \( \text{Ch}(\mathcal{V}) \) for all \( c \in \mathcal{E} \). It is a cofibration if it has the left lifting property with respect to all pointwise acyclic fibrations.

We have the following application of the preceding theorem.

**Theorem 5.5.** Let \( \mathcal{V} \) be a closed symmetric monoidal Grothendieck category and \( \mathcal{E} \) be a \( \mathcal{V} \)-category. Suppose \( \text{Ch}(\mathcal{V}) \) is a weakly finitely generated monoidal model structure with respect to the tensor product \( \otimes \) of Theorem 4.2 and the monoid axiom holds for \( \text{Ch}(\mathcal{V}) \). Then

1. \( \text{Ch}[\mathcal{E}, \mathcal{V}] \) with the classes of quasi-isomorphisms, cofibrations and pointwise fibrations defined above is a weakly finitely generated \( \text{Ch}(\mathcal{V}) \)-model category.
2. \( \text{Ch}[\mathcal{E}, \mathcal{V}] \) is a monoidal \( \text{Ch}(\mathcal{V}) \)-model category provided that \( \mathcal{E} \) is a symmetric monoidal \( \text{Ch}(\mathcal{V}) \)-category. In this case the tensor product of \( F, G \in \text{Ch}[\mathcal{E}, \mathcal{V}] \) is given by

\[
F \otimes G = \int^{(a,b) \in \mathcal{E} \otimes \mathcal{E}} F(a) \circ G(b) \otimes \text{Ch}(\mathcal{V})_\mathcal{E} (a \otimes b, -).
\]
Here $\text{Ch}(\mathcal{V})\langle a \otimes b, - \rangle$ is regarded as a complex concentrated in zeroth degree. The internal Hom-object is defined as

$$\text{Hom}(F, G)(a) = \int_{b \in \mathcal{C}} \text{Hom}_{\text{Ch}(\mathcal{V})}(F(b), G(a \otimes b)).$$

(3) The pointwise model structure on $\text{Ch}[\mathcal{C}, \mathcal{V}]$ is right proper if $\text{Ch}(\mathcal{V})$ is right proper, and left proper if $\text{Ch}(\mathcal{V})$ is strongly left proper in the sense of $\text{[DRO1, 4.6]}$.

Proof. In all statements we use Theorem 5.4. The first statement follows from $\text{[DRO1, 4.2]}$. The second statement follows from $\text{[DRO1, 4.4]}$ and Day’s Theorem $\text{[Day]}$ for tensor products and internal Hom-objects. Finally, the third statement follows from $\text{[DRO1, 4.8]}$. □

Example 5.6. Suppose $\mathcal{V} = \text{Mod}R$ with $R$ a commutative ring and $\mathcal{C} = \text{mod}R$, the category of finitely presented $R$-modules. Then $\mathcal{C}$ and $\text{Ch}(\text{mod}R)$ together with the projective model structure satisfies the assumptions of Theorem 5.5 and all statements are then true for $\text{Ch}(\text{mod}R, \text{Mod}R)$. Since $\text{[mod}R, \text{Mod}R]$ is isomorphic to the category of generalized modules $\mathcal{C}_R = (\text{mod}R, \text{Ab})$ consisting of the additive functors from mod$R$ to Abelian groups (see $\text{[AG, 6.1]}$), Theorem 5.5 recovers $\text{[AG, 6.3]}$ stating similar model structures for $\text{Ch}(\mathcal{C}_R)$.

6. COMPACT GENERATORS FOR THE DERIVED CATEGORY

We consider the following situation when $\mathcal{V}$ is a closed symmetric monoidal Grothendieck category such that its derived category $\text{D}(\mathcal{V})$ is compactly generated triangulated. We show that $\text{D}(\mathcal{C}, \mathcal{V})$ is also compactly generated in many reasonable cases with $\mathcal{C}$ a small $\mathcal{V}$-category.

Example 6.1. (1) Given a commutative ring $R$, the category of $R$-modules is a closed symmetric monoidal Grothendieck category. Moreover, the derived category of $R$-modules $\text{D}(\text{Mod}R)$ is compactly generated triangulated. The compact generators are those complexes which are quasi-isomorphic to a bounded complex of finitely generated projective modules. Such complexes are called perfect complexes.

(2) Given a finite group $G$ and a field $k$, $(\text{Mod}kG, \otimes_k, k)$ is a closed symmetric monoidal Grothendieck category. The derived category $\text{D}(\text{Mod}kG)$ is compactly generated triangulated. Its compact objects are given by the perfect complexes.

(3) The category of Nisnevich sheaves with transfers $\text{NSwT}/F$ over a field $F$ is a closed symmetric monoidal Grothendieck category. The derived category $\text{D}(\text{NSwT}/F)$ is compactly generated triangulated. Its compact generators are given by complexes $\mathbb{Z}_{tr}(X)[n]$ (the sheaf $\mathbb{Z}_{tr}(X)$ concentrated in the $n$th degree), where $X$ is an $F$-smooth algebraic variety (see, e.g., $\text{[GP1, p. 241]}$).

Theorem 6.2. Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal Grothendieck category such that the derived category of $\mathcal{V}$ is a compactly generated triangulated category with compact generators $\{P_j\}_{j \in J}$. Further, suppose we have a small $\mathcal{V}$-category $\mathcal{C}$ and that any one of the following conditions is satisfied:

1. each $P_j$ is $K$-projective, in the sense of Spaltenstein $\text{[Sp]}$;
2. for every $K$-injective $Y \in \text{Ch}[\mathcal{C}, \mathcal{V}]$ and every $c \in \mathcal{C}$, the complex $Y(c) \in \text{Ch}(\mathcal{V})$ is $K$-injective;
3. \( \text{Ch}(\mathcal{V}) \) has a model structure, with quasi-isomorphisms being weak equivalences, such that for every injective fibrant complex \( Y \in \text{Ch}[\mathcal{C}, \mathcal{V}] \) the complex \( Y(c) \) is fibrant in \( \text{Ch}(\mathcal{V}) \).

Then \( \mathbf{D}[\mathcal{C}, \mathcal{V}] \) is a compactly generated triangulated category with compact generators \( \{ \mathcal{V} \mathcal{C}(c, -) \odot Q_j \mid c \in \mathcal{C}, j \in J \} \) where, if we assume either (1) or (2), \( Q_j = P_j \) or if we assume (3) then \( Q_j = P_j' \) a cofibrant replacement of \( P_j \).

**Proof.** We write \( (c, -) \) to denote \( \mathcal{V} \mathcal{C}(c, -) \). Assuming any of the three conditions, suppose \( c \in \mathcal{C}, X \in \text{Ch}[\mathcal{C}, \mathcal{V}] \) and take any \( Q_j \). Clearly, if we denote the injective fibrant replacement of \( X \) by \( X_f \) (recall that every object is cofibrant in the injective model structure see \([\text{Be}, \text{Hov}]\)), then

\[
\mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, X) \cong \mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, X_f) \cong \mathbf{K}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, X_f).
\]

By the tensor hom adjunction in \( \mathcal{V} \) and the Yoneda lemma, we have

\[
\mathbf{K}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, X_f) \cong \mathbf{K}(\mathcal{V})(Q_j, X_f(c)).
\]

Next by assuming either (1) or (2), and the definition of \( \mathbf{K} \)-projective (\( \mathbf{K} \)-injective respectively) complexes we see

\[
\mathbf{K}(\mathcal{V})(Q_j, X_f(c)) \cong \mathbf{D}(\mathcal{V})(Q_j, X_f(c)).
\]

If, however, we assume (3) then \( Q_j \) is cofibrant, \( X_f(c) \) is fibrant in \( \text{Ch}(\mathcal{V}) \) and this natural isomorphism holds also.

Since the arrow \( X(c) \to X_f(c) \) is a quasi-isomorphism, then

\[
\mathbf{D}(\mathcal{V})(Q_j, X_f(c)) \cong \mathbf{D}(\mathcal{V})(Q_j, X(c)).
\]

Hence we have established a natural isomorphism

\[
\mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, X) \cong \mathbf{D}(\mathcal{V})(Q_j, X(c)).
\]

With this isomorphism in hand the family \( \{ (c, -) \odot Q_j \mid c \in \mathcal{C}, j \in J \} \) is a collection of compact generators can be verified as follows.

First, we verify that \( \{ (c, -) \odot Q_j \mid c \in \mathcal{C}, j \in J \} \) is a family of generators for \( \mathbf{D}[\mathcal{C}, \mathcal{V}] \). Precisely, if \( \mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, A) = 0 \) for all \( j \in J \) and \( c \in \mathcal{C} \), then we must show that \( A \cong 0 \). Assume \( \mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, A) = 0 \) thus \( \mathbf{D}(\mathcal{V})(Q_j, A(c)) = 0 \) which implies \( A(c) \cong 0 \), for all \( c \in \mathcal{C} \). We use the fact that \( \{ Q_j \}_j \) is a family of generators in \( \mathbf{D}(\mathcal{V}) \). Therefore \( A \) is pointwise acyclic and hence is acyclic itself, then \( A \cong 0 \) in \( \mathbf{D}[\mathcal{C}, \mathcal{V}] \) as required.

We now verify compactness, precisely we must demonstrate the following natural isomorphism

\[
\mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, \bigoplus_i B_i) \cong \bigoplus_i \mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, B_i).
\]

We have the following natural isomorphisms

\[
\mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, \bigoplus_i B_i) \cong \mathbf{D}(\mathcal{V})(Q_j, \bigoplus_i B_i(c))
\cong \bigoplus_i \mathbf{D}(\mathcal{V})(Q_j, B_i(c))
\cong \bigoplus_i \mathbf{D}[\mathcal{C}, \mathcal{V}]((c, -) \odot Q_j, B_i).
Here we use the fact that direct sums commute with evaluation and our assumption about the compactness of $Q_j$. Hence, $\{ (c, -) \circ Q_j \mid c \in \mathcal{C}, j \in J \}$ is indeed a family of compact generators for $\mathbf{D}[\mathcal{C}, \mathcal{V}]$.

\begin{remark}

Though conditions (1)-(2) of the preceding theorem have nothing to do with model structures, one should stress that condition (1) normally occurs whenever $\text{Ch}(\mathcal{V})$ has a projective model structure with generating (trivial) cofibrations having finitely presented domains and codomains. Condition (2) is typical for the injective model structure on $\text{Ch}(\mathcal{V})$, which always exists by [Be, Hov], and when $\mathcal{C} = \{ * \}$, a singleton with $R := \mathcal{V}_c(*, +)$ a flat ring object of $\mathcal{V}$ (i.e. the functor $R \otimes -$ is exact on $\mathcal{V}$). Finally, condition (3) is most common in practice. It often assumes intermediate model structures on $\text{Ch}(\mathcal{V})$, i.e. model structures which are between the projective and injective model structures. This situation is often recovered from Theorem \ref{thm:main-theorem}.
\end{remark}

We conclude the paper with the following observation. Given a closed symmetric monoidal Grothendieck category $\mathcal{V}$ and a small symmetric monoidal $\mathcal{V}$-category $\mathcal{C}$, then $[\mathcal{C}, \mathcal{V}]$ is also a closed symmetric monoidal Grothendieck $\mathcal{V}$-category by [AG, 4.2]. If $\mathbf{D}(\mathcal{V})$ has $K$-projective compact generators $\{ P_j \}_j$ then the proof of Theorem \ref{thm:main-theorem} shows that $\mathbf{D}[\mathcal{C}, \mathcal{V}]$ has a family of $K$-projective compact generators given by $\{ (c, -) \circ P_j \mid c \in \mathcal{C}, j \in J \}$. Thus we are able to iterate this process as follows. If we set $\mathcal{V}_1 := [\mathcal{C}, \mathcal{V}]$ and are given a small symmetric monoidal $\mathcal{V}_1$-category $\mathcal{C}_1$, we can conclude that $\mathbf{D}[\mathcal{C}_1, \mathcal{V}_1]$ is also compactly generated having $K$-projective compact generators. We can then set $\mathcal{V}_2 = [\mathcal{C}_1, \mathcal{V}_1]$ and repeat this procedure as many times as necessary to generate as many examples as we desire.

For instance, starting with $\mathcal{V} = \text{Mod} R$, where $R$ is commutative and $\mathcal{C} = \text{mod} R$ (see Example \ref{ex:mod-ring}), set $\mathcal{V}_1 := [\text{mod} R, \text{Mod} R] \cong \mathcal{C}_R$ and $\mathcal{C}_1 := \text{fp}(\mathcal{V}_1)$, where $\text{fp}(\mathcal{V}_1)$ consists of finitely presented objects of $\mathcal{V}_1$. Then $\mathcal{V}_2 = [\text{fp}(\mathcal{V}_1), \mathcal{V}_1]$ is a closed symmetric monoidal locally finitely presented Grothendieck category. Its finitely presented generators are given by $\mathcal{V}_1(a, -) \circ c$, where $a, c \in \mathcal{C}_1$. We use here natural isomorphisms

\[ \text{Hom}_{\mathcal{V}_2}(\mathcal{V}_1(a, -) \circ c, \lim_j X_j) \cong \text{Hom}_{\mathcal{V}_1}(c, \mathcal{V}_1(\mathcal{V}_1(a, -), \lim_j X_j)) \cong \text{Hom}_{\mathcal{V}_1}(c, \lim_j X_j(a)) \cong \lim_j \text{Hom}_{\mathcal{V}_1}(c, X_j(a)) \cong \lim_j \text{Hom}_{\mathcal{V}_2}(\mathcal{V}_1(a, -) \circ c, X_j) \]

and the fact that $\mathcal{C}_1$ is closed under tensor product in $\mathcal{V}_1$. Moreover, $\mathbf{D}(\mathcal{V}_2) = \mathbf{D}[\mathcal{C}_1, \mathcal{V}_1]$ is also compactly generated having $K$-projective compact generators. Iterating this, we can define a closed symmetric monoidal locally finitely presented Grothendieck category $\mathcal{V}_n = [\text{fp}(\mathcal{V}_{n-1}), \mathcal{V}_{n-1}]$ for all $n > 1$. And then $\mathbf{D}(\mathcal{V}_n) = \mathbf{D}[\text{fp}(\mathcal{V}_{n-1}), \mathcal{V}_{n-1}]$ is compactly generated having $K$-projective compact generators.

\section*{References}

[AG] H. Al Hwaeer, G. Garkusha, \textit{Grothendieck categories of enriched functors}, J. Algebra \textbf{450} (2016), 204-241.

[Be] T. Beke, \textit{Sheafifiable homotopy model categories}, Math. Proc. Cambridge Phil. Soc. \textbf{129} (2000), 447-475.

[BK] A. I. Bondal, M. M. Kapranov, \textit{Enhanced triangulated categories}, Mat. Sb. \textbf{181}(5) (1990), 669-683, English transl. in Math. USSR-Sb. \textbf{70}(1), 93-107.

[Bor2] F. Borceux, \textit{Handbook of Categorical Algebra 2}, Cambridge University Press, Cambridge, 1994.

[Day] B. Day, \textit{On closed categories of functors}, In Reports of the Midwest Category Seminar, IV, Springer, Berlin, 1970, pp. 1-38.
[DRO1] B. I. Dundas, O. Røndigs, P. A. Østvær, *Enriched functors and stable homotopy theory*, Doc. Math. 8 (2003), 409-488.

[DRO2] B. Dundas, O. Røndigs, P. A. Østvær, *Motivic functors*, Doc. Math. 8 (2003), 489-525.

[GP1] G. Garkusha, I. Panin, *K-motives of algebraic varieties*, Homology, Homotopy Appl. 14(2) (2012), 211-264.

[GP2] G. Garkusha, I. Panin, *The triangulated category of K-motives DK_{eff}(k)*, J. K-theory 14(1) (2014), 103-137.

[GP3] G. Garkusha, I. Panin, *On the motivic spectral sequence*, J. Inst. Math. Jussieu 17(1) (2018), 137-170.

[Hov] M. Hovey, *Model category structures on chain complexes of sheaves*, Trans. Amer. Math. Soc. 353(6) (2001), 2441-2457.

[HPS] M. Hovey, J. H. Palmieri, N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. 128 (1997), no. 610.

[II] L. Illusie, *Existence de ré solutions globales*, Sem. Geom. algebrique Bois Marie 1966/67, SGA VI, Lecture Notes in Mathematics 225, 1971, pp. 160-221.

[Kel] B. Keller, *On differential graded categories*, International Congress of Mathematicians, Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151-190.

[R] E. Riehl, *Categorical homotopy theory*, New Mathematical Monographs 24, Cambridge University Press, Cambridge, 2014.

[Sp] N. Spaltenstein, *Resolutions of unbounded complexes*, Compos. Math. 65(2) (1988), 121-154.

[SV] A. Suslin, V. Voevodsky, *Bloch–Kato conjecture and motivic cohomology with finite coefficients*, The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 117-189.

DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, FABIAN WAY, SWANSEA SA1 8EN, UNITED KINGDOM
E-mail address: g.garkusha@swansea.ac.uk

DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, FABIAN WAY, SWANSEA SA1 8EN, UNITED KINGDOM
E-mail address: darrenalexanderjones@gmail.com