ABSTRACT. We give explicit and inductive formulas for the construction of a Lorentz covariant renormalization in the EG approach. This automatically provides for a covariant BPHZ subtraction at totally spacelike momentum useful for massless theories.

INTRODUCTION

Our introduction is rather brief since this paper mainly is a continuation of [BPP99]. There we gave a formula for a Lorentz invariant renormalization in one coordinate. We use the same methods to construct a covariant solution for more variables.

We review the EG subtraction and give some useful formulas in section 1. The cohomological analysis that leads to a group covariant solution is reviewed in section 2. In sections 3, 4 we give an inductive construction for these solutions in the case of tensorial and spinorial Lorentz covariance. Since the symmetry of the one variable problem is absent in general some permutation group calculus will enter. Necessary material is in the appendix. In the last section 5 we give a covariant BPHZ subtraction for arbitrary (totally spacelike) momentum by Fourier transformation.

1. THE EG SUBTRACTION

We review the extension procedure in the EG approach [EG73]. A more general introduction can be found in [Pra99], a generalization to manifolds in [BF96, BF99]. Let $D^{0}(\mathbb{R}^{d})$ be the subspace of test functions vanishing to order $\omega$ at 0. As usual, any operation on distributions is defined by the corresponding action on test functions.

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Define
\[ W_{(\omega;w)} : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}^\omega(\mathbb{R}^d), \quad \varphi \to W_{(\omega;w)}\varphi, \]
\[ (W_{(\omega;w)}\varphi)(x) = \varphi(x) - w(x) \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \partial_\alpha (\varphi w^{-1})(0), \quad (1) \]
with \( w \in \mathcal{D}(\mathbb{R}^d), w(0) \neq 0 \). If \( 0^t \) is a distribution on \( \mathcal{D}(\mathbb{R}^d \setminus \{0\}) \) with singular order \( \omega \) then \( 0^t \) can be defined uniquely on \( \mathcal{D}^\omega(\mathbb{R}^d) \), and
\[ \langle t_{(\omega;w)}, \varphi \rangle = \langle 0^t, W_{(\omega;w)}\varphi \rangle \quad (2) \]
defines an extension – called renormalization – of \( 0^t \) to the whole testfunction space, \( t_{(\omega;w)} \in \mathcal{D}'(\mathbb{R}^d) \). With the Leibnitz rule
\[ \partial_\gamma (fg) = \gamma! \sum_{\mu + \nu = \gamma} \frac{1}{\mu!\nu!} \partial_\mu f \partial_\nu g, \quad (3) \]
we find
\[ \partial_\gamma \left( w \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \partial_\alpha (\varphi w^{-1})(0) \right)(0) = \partial_\gamma \varphi(0), \quad (4) \]
for \( |\gamma| \leq \omega \), verifying the projector properties of \( W \). The \( W \)-operation \((1)\) is simplified if we require \( w(0) = 1 \) and \( \partial_\alpha w(0) = 0 \), for \( 0 < |\alpha| \leq \omega \) (this was our assumption in \([BPP99]\)). It can be achieved by the following \( V \)-operation \((\partial_\mu w^{-1} \text{ means } \partial_\mu (w^{-1}))\):
\[ V_\omega : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d), \quad (V_\omega w)(x) = w(x) \sum_{|\mu| \leq \omega} \frac{x^\mu}{\mu!} \partial_\mu w^{-1}(0), \quad (5) \]
where \( w(0) \neq 0 \) is still assumed. We can write \( W \) as
\[ (W_{(\omega;w)}\varphi)(x) = \varphi(x) - \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} V_\omega - |\alpha| w \partial_\alpha \varphi(0). \quad (6) \]

The extension \((2)\) is not unique. We can add any polynomial in derivatives of \( w \) up to order \( \omega \):
\[ \langle t_{(\omega;w)}, \varphi \rangle = \langle \tilde{t}_{(\omega;w)}, \varphi \rangle + \sum_{|\alpha| \leq \omega} \frac{d^\alpha}{\alpha!} \partial_\alpha \varphi(0), \quad (7) \]
or rearranging the coefficients
\[ = \langle \tilde{t}_{(\omega;w)}, \varphi \rangle + \sum_{|\alpha| \leq \omega} \frac{c^\alpha}{\alpha!} \partial_\alpha (\varphi w^{-1})(0) \quad (8) \]
\[ \text{Since } W_{(\omega;w)}(wx^\alpha) = W_{(\omega;w)}(x^\alpha V_{\omega - |\alpha| w}) = 0 \text{ for } |\alpha| \leq \omega, \text{ c resp. } a \text{ are given by} \]
\[ a^\alpha = \langle \tilde{t}_{(\omega;w)}, x^\alpha V_{\omega - |\alpha| w} \rangle, \quad c^\alpha = \langle \tilde{t}_{(\omega;w)}, x^\alpha w \rangle. \quad (9) \]
They are related through:

\[ a^\alpha = c^\alpha \sum_{|\mu| \leq \omega - |\alpha|} \frac{\epsilon^\mu}{\mu!} \partial_\mu w^{-1}(0), \quad c^\alpha = a^\alpha \sum_{|\mu| \leq \omega - |\alpha|} \frac{\epsilon^\mu}{\mu!} \partial_\mu w(0), \quad 1 \leq |\alpha| \leq \omega, \]

\[ a^0 = \sum_{|\mu| \leq \omega} \frac{\epsilon^\mu}{\mu!} \partial_\mu w^{-1}(0), \quad c^0 = \sum_{|\mu| \leq \omega} \frac{\epsilon^\mu}{\mu!} \partial_\mu w(0). \quad (10) \]

The equation for \( a \) follows from the Leibnitz rule in (8), while the equation for \( c \) is derived from (9).

In quantum field theory the coefficients \( a \) are called counter terms. They are not arbitrary. They have to be chosen in such a way that Lorentz covariance of \( 0_t \) is preserved in the extension. This follows in the next sections. The remaining freedom is further restricted by discrete symmetries like permutation symmetry or \( C, P \) and \( T \) symmetries. At the end gauge invariance or renormalization constraints will fix the extension uniquely. But up to now there is no local prescription of the latter.

2. The \( G \)-covariant extension

We will first define the notion of a \( G \)-covariant distribution. So let \( G \) be a linear transformation group on \( \mathbb{R}^d \) i.e. \( x \mapsto gx, \, g \in G \). Then

\[ x^\alpha \mapsto g^\alpha p^\beta = (gx)^\alpha \]

(11)

denotes the corresponding tensor representation. \( G \) acts on functions in the following way:

\[ (g\varphi)(x) = \varphi(g^{-1}x), \]

(12)

so that \( \mathcal{D} \) is made a \( G \)-module. We further have

\[ g(\varphi\psi) = (g\varphi)(g\psi), \]

(13)

\[ x^\alpha \partial_\alpha (g^{-1}\varphi) = (gx)^\alpha g^{-1} \partial_\alpha \varphi, \]

(14)

\[ x^\alpha \partial_\alpha (g^{-1}\varphi)(0) = (gx)^\alpha \partial_\alpha \varphi(0). \]

(15)

Now assume we have a distribution \( 0_t \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\}) \) that transforms covariantly under the Group \( G \) as a density, i.e.

\[ 0_t(gx)|\det g| = D(g)^0_t(x), \]

(16)

where \( D \) is the corresponding representation. That means:

\[ \langle 0_t, g\psi \rangle = \langle D(g)^0_t, \psi \rangle = D(g) \langle 0_t, \psi \rangle. \]

(17)
We will now investigate what happens to the covariance in the extension process. We compute:

\[
D(g) \langle t(\omega; w), g^{-1} \varphi \rangle - \langle t(\omega; w), \varphi \rangle
= D(g) \langle 0 t, W(\omega; w) g^{-1} \varphi \rangle - \langle 0 t, W(\omega; w) \varphi \rangle
\]

\[
\leq D(g) \left( 0 t, g^{-1} \varphi - \sum_{|\alpha| \leq \omega} \frac{\chi^\alpha}{\alpha!} V_{\omega - |\alpha|} \varphi \partial_\alpha (g^{-1} \varphi)(0) \right) - \langle 0 t, W(\omega; w) \varphi \rangle
\]

\[
\leq D(g) \left( 0 t, g^{-1} \varphi - \sum_{|\alpha| \leq \omega} \frac{\chi^\alpha}{\alpha!} (gV_{\omega - |\alpha|} \varphi) \partial_\alpha \varphi(0) \right) - \langle 0 t, W(\omega; w) \varphi \rangle
\]

\[
= \sum_{|\alpha| \leq \omega} b^\alpha(g) \frac{\partial_\alpha \varphi(0)}{\alpha!}
\]

Then (25) defines a map from \( G \) to a finite dimensional complex vectorspace. Now we follow \[SP82], \[Sch95\][chapter 4.5]: Applying two transformations

\[
b^\alpha(g_1 g_2) = \langle 0 t, x^\alpha(\mathbb{I} - g_1 g_2)(V_{\omega - |\alpha|} w) \rangle
\]

\[
= \langle 0 t, x^\alpha(\mathbb{I} - g_1 + g_1 (\mathbb{I} - g_2))(V_{\omega - |\alpha|} w) \rangle
\]

\[
= b^\alpha(g_1) + | \text{det } g_1 | \langle 0 t, g_1 x, (g_1 x)^\alpha (\mathbb{I} - g_2)(V_{\omega - |\alpha|} w) \rangle,
\]

and omitting the indices we see \( b(g_1 g_2) = b(g_1) + D(g_1) g_1 b(g_2) \), which is a 1-cocycle for \( b(g) \). Its trivial solutions are the 1-coboundaries

\[
b(g) = (\mathbb{I} - D(g)) a,
\]

and these are the only ones if the first cohomology group of \( G \) is zero. In that case we can restore \( G \)-covariance by adding the following counter terms:

\[
\langle t^{G - \text{cov}}(\omega; w), \varphi \rangle = \langle t(\omega; w), \varphi \rangle + \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} a^\alpha(w) \partial_\alpha \varphi(0).
\]

The task is to determine \( a \) from (27) and (22):

\[
\langle 0 t, x^\alpha(\mathbb{I} - g)(V_{\omega - |\alpha|} w) \rangle = [(\mathbb{I} - D(g)) a]^\alpha
\]

3. **Tensorial Lorentz Covariance**

The first cohomology group of \( L^1_+ \) vanishes \[Sch95\][chapter 4.5 and references there]. We determine \( a \) from the last equation. The most simple solution appears in the case of Lorentz invariance in one coordinate. This situation was completely analyzed in \[BPP99\] for \( \partial_\alpha w(0) = \delta^0_\alpha \). The following two subsections generalize the results to arbitrary \( w, w(0) \neq 0 \).
3.1. **Lorentz invariance in \( \mathbb{R}^4 \).** If we expand the index \( \alpha \) into Lorentz indices \( \mu_1, \ldots, \mu_n \), (29) is symmetric in \( \mu_1, \ldots, \mu_n \) and therefore \( a \) will be, too. We just state our result from [BPP99] which is modified through the generalization for the choice of \( w \):

\[
a^{(\mu_1 \ldots \mu_n)} = \frac{(n-1)!!}{(n+2)!!} \sum_{s=0}^{n-1} \frac{(n-2s)!!}{(n-2s-1)!!} \eta^{\mu_1 \mu_2 \ldots \mu_{2s-1} \mu_{2s}} \times \langle 0, t, (x^2)^s x^\mu_{2s+1} \ldots x^\mu_{n-1} \left(x^2 \partial^\mu_{2s} - x^\mu_{2s} x^\beta \partial_\beta \right) V_{\omega_{n-w}} \rangle,
\]

if we choose the fully contracted part of \( a \) to be zero in case of \( n \) being even. We used the notation

\[
b^{(\mu_1 \ldots \mu_n)} = \frac{1}{n!} \sum_{\pi \in S_n} b^{\mu_1(\pi(1)) \ldots \mu_n(\pi(n))},
\]

\[
b^{[\mu_1 \ldots \mu_n]} = \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi) b^{(\mu_1(\pi(1)) \ldots \mu_n(\pi(n)))},
\]

for the totally symmetric resp. antisymmetric part of a tensor.

3.2. **Dependence on \( w \).** Performing a functional derivation of the Lorentz invariant extension with respect to \( w \), only Lorentz invariant counter terms appear.

**Definition.** The functional derivation is given by:

\[
\frac{\delta}{\delta \psi} F(g, \psi) \equiv \left. \frac{d}{d \lambda} F(g + \lambda \psi) \right|_{\lambda = 0}.
\]

This definition implies the following functional derivatives:

\[
\langle \frac{\delta}{\delta w} t_{(0,w)}(\varphi), \psi \rangle = - \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \langle t_{(0,w)}(x^\alpha \psi), \partial_\alpha (\varphi w^{-1}) \rangle (0),
\]

\[
\langle \frac{\delta}{\delta w} \langle S, V_{\omega w} \rangle, \psi \rangle = \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \langle S, W_{(0,w)}(x^\alpha \psi) \rangle \partial_\alpha w^{-1} (0),
\]

for any distribution \( S \).

**Proof.** We show how to derive the first relation. Inserting the definition we find:

\[
\left. \frac{d}{d \lambda} t_{(0,w+\lambda \psi)}(\varphi) \right|_{\lambda = 0} = \langle 0, t, - \psi \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \partial_\alpha (\varphi w^{-1}) (0) + w \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \partial_\alpha (\varphi \psi w^{-2}) (0) \rangle,
\]
using Leibnitz rule and rearranging the summation in the second term,

\[
\begin{align*}
&= \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \left\langle 0_t, -\psi x^\alpha \partial_\alpha (\phi w^{-1}) (0) \\
&\quad + w x^\alpha \partial_\alpha (\phi w^{-1}) (0) \sum_{|\nu| \leq \omega - |\alpha|} \frac{x^\nu}{\nu!} \partial_\nu (\psi w^{-1}) (0) \right\rangle \\
&= - \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \langle 0_t, x^\alpha W_{(\omega-|\alpha|):w} \psi \rangle \partial_\alpha (\phi w^{-1}) (0) \\
&= - \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \langle t_{(\omega,w)} , x^\alpha \psi \rangle \partial_\alpha (\phi w^{-1}) (0),
\end{align*}
\]

where we used the relation \( x^\alpha W_{(\omega-|\alpha|):w} \psi = W_{(\omega,w)} (x^\alpha \psi) \) on the last line. The second equation follows from a similar calculation. \( \square \)

We calculate the dependence of \( a \) on \( w \). With (32) we get:

\[
\frac{\delta}{\delta w} a^{(\mu_1 \ldots \mu_n)} (w) = (n-1)!! \sum_{s=0}^{n/2} (n-2s)!! \sum_{|\alpha| \leq \omega-n} \frac{1}{\alpha!} \frac{1}{\beta!} \langle 0_t, (x^2)^s x^{\mu_{2s+1}} \ldots x^{\mu_n} \left( x^2 \partial^{\mu_n} - x^{\mu_n} x^\beta \partial_\beta \right) W_{(\omega-|\alpha|-n):w} (x^\beta \psi) \rangle \partial_\beta w^{-1} (0).
\]

To condense the notation we again use \( \beta \) as a multiindex. Since \( W_{(\omega-|\alpha|-n):w} (x^\beta \psi) \) is sufficient regular, we can put the \( x \)'s and derivatives on the left and the same calculation like in [BPP99] applies. The result is

\[
\langle \frac{\delta}{\delta w} a^{(\mu_1 \ldots \mu_n)} (w), \psi \rangle = \sum_{|\beta| \leq \omega-n} \frac{\partial_\beta w^{-1} (0)}{\beta!} \left[ \langle t_{(\omega,w)} , x^{\mu_1} \ldots x^{\mu_n} x^\beta \psi \rangle + \left\{ \begin{array}{ll}
0, & n \text{ odd,} \\
\frac{2(n-1)!!}{(n+2)!!} \langle t_{(\omega,w)} , (x^2)^{\frac{n}{2}} x^\beta \psi \rangle \eta^{(\mu_1 \mu_2 \ldots \mu_{n-1} \mu_n)}, & n \text{ even.}
\end{array} \right. \right]
\]

Using this result and (31) we find:

\[
\langle \frac{\delta}{\delta w} \langle t_{(\omega,w)}^{\text{linv}}, \phi \rangle , \psi \rangle = - \sum_{n=0}^{\omega} \frac{d_n}{n!} x^{\frac{n}{2}} \phi (0),
\]

\[
d_n \doteq \frac{2(n-1)!!}{(n+2)!!} \sum_{|\beta| \leq \omega-n} \frac{1}{\beta!} \langle t_{(\omega,w)} , (x^2)^{\frac{n}{2}} x^\beta \psi \rangle \partial_\beta w^{-1} (0),
\]

where we set \( d_0 = 1 \).
3.3. **General Lorentz covariance.** If the distribution \(0_t x^\alpha\) depends on more than one variable, \(0_t x^\alpha\) will not be symmetric in all Lorentz indices in general. Since \(x^\alpha\) transforms like a tensor, it is natural to generalize the discussion to the case, where \(0_t\) transforms like a tensor, too. Assume rank \(\text{rank}(0_t) = r\), then \(D(g)g\) is the tensor representation of rank \(p = r + n\), in \((29)\). From now on we will omit the indices. So if \(t \in D(\mathbb{R}^{4m} \setminus \{0\})\), we denote by \(\tilde{x}\) – formerly \(x^\alpha\) – a tensor of rank \(n\) built of \(x_1, \ldots, x_m\).

To solve \((29)\) we proceed like in \([BPP99]\). Since the equation holds for all \(g\) we will solve for \(a\) by using Lorentz transformations in the infinitesimal neighbourhood of \(I\).

If we take \(\theta_{\alpha\beta} = \theta_{[\alpha\beta]}\) as six coordinates these transformations read:

\[
g \approx I + \frac{1}{2} \theta_{\alpha\beta} l^{\alpha\beta},
\]

with the generators

\[
(l^{\alpha\beta})^\mu_v = \eta^{\alpha\nu} \delta^\beta_v - \eta^{\beta\nu} \delta^\alpha_v.
\]

Then, for an infinitesimal transformation one finds from \((29)\):

\[
B_{\alpha\beta} = 2 \left( 0_t, \tilde{x} \sum_{j=1}^m x_j^{\alpha\beta} (V_{\omega^n w}) \right) = (l^{\alpha\beta} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes l^{\alpha\beta}) a,
\]

\(\alpha, \beta\) being Lorentz four-indices. In \([BPP99]\) our ability to solve that equation heavily relied on the given symmetry, which is in general absent here. Nevertheless we can find an inductive construction for \(a\), corresponding to equation \((29)\) in \([BPP99]\).

We build one Casimir operator on the r.h.s. (the other one is always zero, since we are in a \((1/2, 1/2)^{\otimes p}\) representation).

**The case \(p = 1\).** Just to remind that \(p\) is the rank of \(\tilde{x}\), this occurs if either \(t\) is a vector and \(\tilde{x} = 1\), \((n = 0)\), or \(t\) is a scalar and \(\tilde{x} = x_1, \ldots, x_m\). \((35)\) gives:

\[
\frac{1}{2} l_{\alpha\beta} B_{\alpha\beta} = \frac{1}{2} l_{\alpha\beta} l^{\alpha\beta} a = -3I a,
\]

since the Casimir operator is diagonal in the irreducible \((1/2, 1/2)\) representation.

**The case \(p = 2\).** We get

\[
\frac{1}{2} (l_{\alpha\beta} \otimes I + I \otimes l_{\alpha\beta}) B_{\alpha\beta} = (-6I + l_{\alpha\beta} \otimes l^{\alpha\beta}) a.
\]

Since \(a\) is a tensor of rank 2, let us introduce the projector onto the symmetric resp. antisymmetric part and the trace:

\[
P_{S}^{\rho\sigma} = \frac{1}{2} (\delta^\rho_p \delta^\sigma_\sigma + \delta^\rho_\sigma \delta^\sigma_p), \quad P_{A}^{\rho\sigma} = \frac{1}{2} (\delta^\rho_p \delta^\sigma_\sigma - \delta^\rho_\sigma \delta^\sigma_p), \quad P_{\eta}^{\rho\sigma} = \frac{1}{4} \eta^{\rho\nu} \eta_{\nu\sigma},
\]

\[
P^2 = P, \quad P_S + P_A = I, \quad P_S - P_A = \tau,
\]

where \(\tau\) denotes the permutation of the two indices. Using \((34)\), we find

\[
\frac{1}{2} l_{\alpha\beta} \otimes l^{\alpha\beta} = 4P_\eta - \tau.
\]
Now we insert (40) into (37). The trace part will be set to zero again. Acting with $P_A$ and $P_S$ on the resulting equation gives us two equations for the antisymmetric and symmetric part respectively. This yields:

$$a = -\frac{1}{16}(P_S + 2P_A)(l_{\alpha\beta} \otimes I + I \otimes l_{\alpha\beta})B^{\alpha\beta}. \tag{41}$$

**Inductive assumption.** Now we turn back to equation (29). We note that any contraction commutes with the (group) action on the rhs. Hence, if we contract (35), we find on the rhs:

$$\eta_{ij}(l_{\alpha\beta} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes l_{\alpha\beta})a =$$

$$(l_{\alpha\beta} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes l_{\alpha\beta})(\eta_{ij}a),$$

where $i, j$ denote the positions of the corresponding indices, and the hat means omission. Therefore the rank of (35) is reduced by two and we can proceed inductively.

**Induction step.** Multiplying (35) with the generator and contracting the indices yields:

$$\left(3pI + 2 \sum_{\tau \in S_p} \tau\right) a = -\frac{1}{2}(l_{\alpha\beta} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes l_{\alpha\beta})B^{\alpha\beta} + 8 \sum_{i<j \leq p} P_{ni,j}a. \tag{42}$$

The transposition $\tau$ acts on $a$ by permutation of the corresponding indices. For a general $\pi \in S_p$, the action on $a$ is given by: $\pi a^{\mu_1 \cdots \mu_p} = d^{\pi x^{-1}(1) \cdots x^{-1}(p)} a^{\mu_1 \cdots \mu_p}$. In order to solve this equation we consider the representation of the symmetric groups. We give a brief summary of all necessary ingredients in appendix A. So let $k_\tau = \sum_{\tau \in S_p} \tau$ be the sum of all transpositions of $S_p$. Then $k_\tau$ is in the center of the group algebra $A_{S_p}$. It can be decomposed into the idempotents $e_{(m)}$ that generate the irreducible representations of $S_p$ in $A_{S_p}$.

$$k_\tau = h_\tau \sum_{(m) \vdash p} \frac{1}{f_{(m)}} \chi_{(m)}(\tau)e_{(m)}. \tag{43}$$

The sum runs over all partitions $(m) = (m_1, \ldots, m_r), \sum_{i=1}^r m_i = p, m_1 \geq m_2 \geq \cdots \geq m_r$ and $h_\tau = \frac{1}{2}p(p - 1)$ is the number of transpositions in $S_p$. $\chi_{(m)}$ is the character of $\tau$ in the representation generated by $e_{(m)}$ which is of dimension $f_{(m)}$. We use (43), the orthogonality relation $e_{(m)}e_{(m')} = \delta_{(m)(m')}$, and the completeness $\sum_{(m)} e_{(m)} = I$ in (42). The expression in brackets on the l.h.s may be orthogonal to some $e_{(m)}$. The corresponding $e_{(m)}a$ contribution will be any combinations of $\eta$'s and $\epsilon$'s – $\epsilon$ being the totally antisymmetric tensor in four dimensions – transforming correctly and thus can
be set to zero. We arrive at

\[ a = \sum_{(m) \neq 0} \frac{e_{(m)}}{c_{(m)}} \left( -\frac{1}{2} (\mathbb{I}_{\alpha\beta} \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \mathbb{I}_{\alpha\beta}) B_{\alpha\beta} + 8 \sum_{i<j} P_{ij} a_{ij} \right), \]

\[ c_{(m)} = 3p + p(p-1) \frac{\chi_{(m)}(\tau)}{f_{(m)}} = 3p + \sum_{i=1}^{r} \left( b_{i}^{(m)} (b_{i}^{(m)} + 1) - a_{i}^{(m)} (a_{i}^{(m)} + 1) \right), \quad (44) \]

with \( a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_r) \) denoting the characteristics of the frame \((m)\), see appendix A. Let us take \( p = 4 \) as an example:

| idempotent Young frame | dimension | character |
|------------------------|-----------|-----------|
| \( e_{(4)} \)          |           | \( f_{(4)} = 1 \) | \( \chi_{(4)}(\tau) = 1 \) |
| \( e_{(3,1)} \)        |           | \( f_{(3,1)} = 3 \) | \( \chi_{(3,1)}(\tau) = 1 \) |
| \( e_{(2,2)} \)        |           | \( f_{(2,2)} = 2 \) | \( \chi_{(2,2)}(\tau) = 0 \) |
| \( e_{(2,1,1)} \)      |           | \( f_{(2,1,1)} = 3 \) | \( \chi_{(2,1,1)}(\tau) = -1 \) |
| \( e_{(1,1,1,1)} \)    |           | \( f_{(1,1,1,1)} = 1 \) | \( \chi_{(1,1,1,1)}(\tau) = -1 \) |

We find for \( (42) \)

\[ a = \frac{1}{48} (2e_{(4)} + 3e_{(3,1)} + 4e_{(2,2)} + 6e_{(2,1,1)}) \times \text{r.h.s}(42). \quad (45) \]

We see that no \( e_{(1,1,1,1)} \) appears in that equation. It corresponds to the one dimensional \( \text{sgn} \)-representation of \( S_4 \), so \( e_4 a \propto \epsilon \).

## 4. Spinorial Lorentz Covariance

In this section we follow [SU92]. The most general representation of \( \mathcal{L}^+ \) can be built of tensor products of \( SL(2,\mathbb{C}) \) and \( SL(2,\mathbb{C}) \) and direct sums of these. A two component spinor \( \Psi \) transforms according to

\[ \Psi^A = g^A_{\ B} \Psi^B, \quad (46) \]
where $g$ is a $2 \times 2$-matrix in the $SL(2, \mathbb{C})$ representation of $L_+^\dagger$. For the complex conjugated representation we use the dotted indices, i.e.

$$\tilde{\Psi}^X = g^X_\bar{Y} \tilde{\Psi}^Y,$$

with $g^X_\bar{Y} = g^Y_\bar{X}$ in the $SL(2, \mathbb{C})$ representation. The indices are lowered and raised with the $\varepsilon$-tensor.

$$\varepsilon_{AB} = \varepsilon_{\bar{A}\bar{B}} = \varepsilon_{\bar{A}B} = \delta_B^A.$$  \hfill (48)

$$\varepsilon_{AB}\varepsilon_{AC} = \varepsilon_{BA}\varepsilon_{CA} = \delta_B^C.$$  \hfill (49)

We define the Van-der-Waerden symbols with the help of the Pauli matrices $\sigma_\mu$ and $\tilde{\sigma}_\mu = \sigma^\mu$: \hfill (50)

$$\sigma^{AX} = \frac{1}{\sqrt{2}}(\sigma_\mu)^{AX}, \quad \sigma^{A\bar{X}} = \frac{1}{\sqrt{2}}(\sigma^\mu)_{A\bar{X}}.$$  \hfill (50)

They satisfy the following relations:

$$\sigma^{AX} \sigma^{A\bar{X}} = \eta_{\mu\nu}, \quad \sigma^{A\bar{X}} \sigma^{B\bar{Y}} = \varepsilon_{AB}\varepsilon_{\bar{X}\bar{Y}}.$$  \hfill (51)

With the help of these we can build the infinitesimal spinor transformations

$$g \approx I + \frac{1}{2} \theta_{\alpha\beta} S^{\alpha\beta},$$  \hfill (52)

with the generators

$$(S^{\alpha\beta})^A_B \equiv \sigma^{[\alpha A}_{\bar{B}]\bar{X}} S^{\beta_\bar{X}}.$$  \hfill (53)

Note that the $\sigma$’s are hermitian: $\overline{\sigma^{A\bar{X}}} = \sigma^{\bar{X}A}$. Again we define the projectors for the tensor product. But we have only two irreducible parts:

$$P_S^{AB} = \frac{1}{2}(\delta^A_C \delta^B_D + \delta^A_D \delta^B_C), \quad P_\varepsilon^{AB} = \frac{1}{2} \varepsilon^{AB} \varepsilon_{CD},$$

$$P^2 = P_S P_S + P_\varepsilon = I.$$  \hfill (55)

We get the following identities:

$$S^{\alpha\beta} S_{\alpha\beta} = S^{\alpha\beta} S^\beta_\alpha = -3I,$$  \hfill (56)

$$S^{\alpha\beta} \otimes S_{\alpha\beta} = S^{\alpha\beta} \otimes S^\beta_\alpha = 4P_\varepsilon - I,$$  \hfill (57)

$$S^{\alpha\beta} \otimes S_{\alpha\beta} = S^{\alpha\beta} \otimes S^\beta_\alpha = 0.$$  \hfill (58)

In order to have (29) in a pure spinor representation we have to decompose the tensor $\tilde{x}$ into spinor indices according to

$$x^{AX} = x^\mu \sigma^{AX}_\mu.$$  \hfill (59)
Assume $t\tilde{x}$ transforms under the $u$-fold tensor product of $SL(2, \mathbb{C})$ times the $v$-fold tensor product of $\tilde{SL}(2, \mathbb{C})$ then, for infinitesimal transformations, (29) yields:

$$B^{\alpha\beta} = (S^{\alpha\beta} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes \overline{S}^{\alpha\beta})a,$$

with $B^{\alpha\beta}$ from equation (55) in the corresponding spinor representation. The sum consists of $u$ summands with one $S^{\alpha\beta}$ and $v$ summands with one $\overline{S}^{\alpha\beta}$ with $u, v > n$. Multiplying again with the generator and contracting the indices gives twice the Casimir on the r.h.s. Inserting (56-58) yields:

$$
\left( S^{\alpha\beta} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes \overline{S}^{\alpha\beta} \right) B^{\alpha\beta} = \left( -3(u+v)I + 2 \sum_{1 \leq i < j \leq u} (4P_{\epsilon_{ij}} - I) + 2 \sum_{1 \leq i < j \leq v} (4P_{\epsilon_{ij}} - I) \right) a. \tag{61}
$$

The sum over $u$ runs over $\frac{u^2}{2}(u-1)$ possibilities and similar for $v$, so we find the induction:

$$a = \frac{1}{u(u+2) + v(v+2)} \left[ -(S^{\alpha\beta} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes \overline{S}^{\alpha\beta})B^{\alpha\beta} + 8 \left( \sum_{1 \leq i < j \leq u} P_{\epsilon_{ij}} + \sum_{1 \leq i < j \leq v} P_{\epsilon_{ij}} \right) a \right]. \tag{62}
$$

It already contains the induction start for $a^{(AB)}$, $a^{(XY)}$ and $a^{AX}$.

## 5. General covariant BPHZ subtraction

In this section we shrink the distribution space to $S'$ since we are dealing with Fourier transformation. Let $x, q, p \in \mathbb{R}^{4m}$. BPHZ subtraction at momentum $q$ corresponds to using $w = e^{iq}$ in the EG subtraction [Pra99].

$$
\hat{t}(\omega, e^{iq}) \equiv \left< t(\omega, e^{iq}), e^{ip} \right> = \left< 0, e^{ip} - \sum_{|\alpha| \leq \omega} \frac{(p-q)^\alpha}{\alpha!} \partial^\alpha e^{iq} \right>. \tag{64}
$$

It is normalized at the subtraction point $q$, i.e.: $\partial^\alpha \hat{t}(\omega, e^{iq}) (q) = 0, |\alpha| \leq \omega$. This is always possible for $q$ totally spacelike, $(\sum_{j \in I} q_j)^2 < 0, \forall I \subset \{1, \ldots, m\}$ [EG73, Düt]. In massive theories one can put $q = 0$ and has the usual subtraction at zero momentum which preserves covariance. But this leads to infrared divergencies in the massless case. There we can use the results from above to construct a covariant BPHZ subtraction for momentum $q$ by adding $\sum_{|\alpha| \leq \omega} \frac{i^{|\alpha|}}{\alpha!} a^{\alpha} p_\alpha$ to (54), according to equation (28). For $|\beta| \geq \omega + 1$, $0t\alpha^\beta$ is a well defined distribution on $S$ and so is $\partial_\beta \hat{t}$. 
5.1. Lorentz invariance on $\mathbb{R}^4$. We have

$$V_k e^{iq} = e^{iq} \sum_{m=0}^{k} \frac{1}{m!} (-i qx)^m, \quad \partial_\sigma V_k e^{iq} = i q_\sigma e^{iq} \frac{1}{k!} (-i qx)^k. \quad (65)$$

Inserting this into (30) we find:

$$a^{\mu_1...\mu_n} = \frac{i^3}{6} D^3 F \Rightarrow \omega = 2.$$

Example. Take the setting sun in massless scalar field theory:

$$0 \tau = \frac{1}{6} D^3 F \Rightarrow \omega = 2.$$

$$a^{\mu} = -i \frac{1}{3} (q_\sigma q_\rho \partial^\sigma \partial^\mu - q_\rho q_\sigma \partial^\sigma \Box) \tilde{0}_t(q),$$

$$a^{\mu\nu} = \frac{1}{4} (q_\rho \partial^\rho \partial^\mu \partial^\nu - q_\rho \partial^\rho \partial^\nu \Box) \tilde{0}_t(q),$$

and adding $ip_\mu a^{\mu} - \frac{1}{2} p_\mu p_\nu a^{\mu\nu}$ restores Lorentz invariance of the setting sun graph subtracted at $q$.

5.2. General induction. We only have to evaluate $B^{\alpha\beta}$ with $w = e^{iq}$ and plug the result into the induction formulas (62) and (44).

$$B^{\alpha\beta} = 2 p^{\gamma} (-)^{\omega+1} \sum_{j=1}^{m} \sum_{|\gamma| = \omega - n} \frac{q_j^\gamma}{\gamma!} \tilde{0}_t(q). \quad (67)$$

Here, $q_j$ are the $m$ components of $q$ hence $\gamma$ is a $4m$ index and $\alpha, \beta$ are four indices. The tensor (spinor) structure of $\tilde{0}$ is given by $\tilde{x}$ in (55).

6. Summary and Outlook

The subtraction procedure in EG renormalization makes use of an auxiliary (test) function and hence breaks Lorentz covariance, since no Lorentz invariant test function exists. But this symmetry can be restored by an appropriate choice of counterterms. We give an explicit formula for their calculation in lowest order and an inductive one for higher orders. Using the close relationship to BPHZ subtraction this directly translates into a covariant subtraction at totally spacelike momentum.

We expect our solution to be useful for all calculations for which the central solution ($w = 1$, see [Sch95]) does not exist, namely all theories that contain loops of only massless particles.

7. Acknowledgment

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APPENDIX A. REPRESENTATION OF THE SYMMETRIC GROUPS

Everything in this brief appendix should be found in any book about representation theory of finite groups. We refer to [Sim96, Boe70, FH91]. The group algebra $A_{S_p}$ consists of elements
\[ a = \sum_{g \in S_p} \alpha(g) \cdot g, \quad b = \sum_{g \in S_p} \beta(g) \cdot g, \]
where $\alpha, \beta$ are arbitrary complex numbers. The sum of two elements is naturally given by the summation in $\mathbb{C}$ and the product is defined through the following convolution:
\[ ab = \sum_{g_1,g_2} \alpha(g_1)\beta(g_2) \cdot g_1 g_2. \]

The group algebra is the direct sum of simple two-sided ideals:
\[ A_{S_p} = I_1 \oplus \cdots \oplus I_k, \]
and $k$ is the number of partitions of $p$. Every ideal $I_j$ contains $f_j$ equivalent irreducible representations of $S_p$. $I_j$ is generated by an idempotent $e_j \in A_{S_p}$:
\[ I_j = A_{S_p} e_j \quad e_j^2 = e_j. \]

These idempotents satisfy the following orthogonality and completeness relations:
\[ e_j e_i = \delta_{ji} \quad \sum_{j=1}^k e_j = \mathbb{I}. \]

The center of $A_{S_p}$ consists of all elements $\sum_{j=1}^k \alpha_j e_j, \alpha_j \in \mathbb{C}$.

Every permutation of $S_p$ can be uniquely (modulo order) written as a product of disjoint cycles. Since two cycles are conjugated if and only if their length is the same, the number of conjugacy classes is equal to the number of partitions of $p$. Denoting the $j$'th conjugacy class by $c_j$ we build the sum of all elements of one class
\[ k_j = \sum_{\pi \in c_j} \pi \in A_{S_p} \]
which is obviously in the center of $A_{S_p}$, too. So we can expand $k_i$ in the basis $e_j$:
\[ k_i = h_i \sum_{j=1}^k \frac{1}{f_j} \chi_j(c_i) e_j, \]
where $\chi_j(c_i)$ is the character of the class $c_i$ in the representation generated by $e_j$ and $h_i$ is the number of elements of $c_i$. The dimension of that representation is equal to the multiplicity $f_j$.

The construction of the idempotents can be carried out via the
Young tableaux. A sequence of integers \((m) = (m_1, \ldots, m_r), m_1 \geq m_2 \geq \cdots \geq m_r\) with \(\sum_{j=1}^{r} m_j = p\) gives a partition of \(p\). To every such sequence we associate a diagram with

\[
\begin{array}{ccccccc}
\text{\(m_1\) boxes} & & & & & & \\
\text{\(m_2\) boxes} & & & & & & \\
\vdots & & & & & &
\end{array}
\]

called a Young frame \((m)\). Let us take \(p = 5\) as an example:

\[
\begin{array}{ccccccc}
\text{\(5\)} & & & & & & \\
\text{\(4, 1\)} & & & & & & \\
\text{\(3, 2\)} & & & & & & \\
\text{\(3, 1, 1\)} & & & & & & \\
\text{\(2, 2, 1\)} & & & & & & \\
\text{\(2, 1, 1, 1\)} & & & & & & \\
\text{\(1, 1, 1, 1, 1\)} & & & & & &
\end{array}
\]

An assignment of numbers \(1, \ldots, p\) into the boxes of a frame is called a Young tableau. Given a tableau \(T\), we denote \((m)\) by \((m)(T)\). If the numbers in every row and in every column increase the tableau is called standard. The number of standard tableaux for the frame \((m)\) is denoted by \(f(m)\). It is equal to the dimension of the irreducible representation generated by the idempotent \(e(m)\). We will now answer the question

How to construct \(e(m)\). Set

\[
\begin{align*}
\mathcal{R}(T) & \doteq \{\pi \in S_p | \pi \text{ leaves each row of } T \text{ setwise fixed}\}, \\
\mathcal{C}(T) & \doteq \{\pi \in S_p | \pi \text{ leaves each column of } T \text{ setwise fixed}\},
\end{align*}
\]

and build the following objects:

\[
\begin{align*}
P(T) & \doteq \sum_{p \in \mathcal{R}(T)} p, & Q(T) & \doteq \sum_{q \in \mathcal{C}(T)} \text{sgn}(q)q,
\end{align*}
\]

then

\[
e(T) \doteq \frac{f(m)}{p!} P(T) Q(T)
\]

is a minimal projection in \(A_{S_p}\) (generates a minimal left ideal). The central projection (generating the simple twosided ideal) is given by

\[
e(m) \doteq \frac{f(m)}{p!} \sum_{T | (m)(T) = (m)} e(T).
\]

Example \(p = 3\). The frame \[
\begin{array}{ccc}
\text{\(1\)} & \text{\(2\)} & \text{\(3\)}
\end{array}
\]

has only one standard tableau \[
\begin{array}{ccc}
\text{\(1\)} & \text{\(2\)} & \text{\(3\)}
\end{array}
\]

All different tableaux in (78) lead to the same idempotent (77) which is just the sum of all permutations.

\[
e(3) = \frac{1}{6} (1 + (12) + (13) + (23) + (123) + (132)).
\]
For the frame \[
\begin{array}{c}
\end{array}
\] we only need the column permutations in (77). We find
\[
e_{(1,1,1)} = \frac{1}{6} (\mathbb{I} - (12) - (13) - (23) + (123) + (132)).
\]
The frame \[
\begin{array}{c}
\end{array}
\] has two standard tabelaux. For the tableaux \[
\begin{array}{c}
1 & 2 & 3 \\
3 & 2 & 1
\end{array}
\end{array},
\begin{array}{c}
1 & 3 & 2 \\
2 & 3 & 1
\end{array},
\begin{array}{c}
2 & 1 & 3 \\
3 & 1 & 2
\end{array},
\begin{array}{c}
2 & 3 & 1 \\
3 & 2 & 1
\end{array},
\begin{array}{c}
3 & 1 & 2 \\
2 & 1 & 3
\end{array},
\]
we find:
\[
e_{(2,1)} = \frac{2^2}{(3!)^2} \left\{ (\mathbb{I} + (12))(\mathbb{I} - (13)) + (\mathbb{I} + (13))(\mathbb{I} - (12)) + (\mathbb{I} + (12))(\mathbb{I} - (23)) + (\mathbb{I} + (23))(\mathbb{I} - (12)) + (\mathbb{I} + (13))(\mathbb{I} - (23)) + (\mathbb{I} + (23))(\mathbb{I} - (13)) \right\}
\]
\[
= \frac{1}{3} \left\{ 2\mathbb{I} - (123) - (132) \right\}.
\]
Up to order \(p = 4\) the central idempotents are given by the sum of minimal projectors of the standard tabelaux – they are orthogonal.

The characters in the irreducible \((m)\) representation can be computed through
\[
\chi_{(m)}(s) = \frac{f_{(m)}}{p!} \sum_{T \mid (m)} \sum_{p \in \mathcal{R}(T)} \sum_{q \in \mathcal{L}(T) \mid pq = s} \text{sgn}(q).
\]
Many other useful formulas can be derived from the Frobenius character formula. Interchanging rows and columns in a frame \((m)\) leads us to the dual frame \(\tilde{(m)}\). For the characters one finds: \(\chi_{\tilde{(m)}}(s) = \text{sgn}(s)\chi_{(m)}(s)\). There is a nice formula for the characters of the transpositions in \([\text{FH91}]\):

Define the rank \(r\) of a frame to be the length of the diagonal. Let \(a_i\) and \(b_i\) be number of boxes below and to the right of the \(i\)th box, reading from lower right to upper left. Call \[
\begin{array}{cccc}
a_1 & \cdots & a_r \\
b_1 & \cdots & b_r
\end{array}
\]
the characteristics of \((m)\), e.g.
\[
\begin{array}{cccc}
X & X & | \\
X & | & X
\end{array},
\]
\[
r = 3, \text{ characteristics } = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 2 & 5 \end{pmatrix}.
\]
Then
\[
\chi_{(m)}(\tau) = \frac{f_{(m)}}{p(p+1)} \sum_{i=1}^{r} (b_i(b_i+1) - a_i(a_i+1)).
\]
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