Approximation Algorithms for **Round-UFP** and **Round-SAP**

Debajyoti Kar  
Department of Computer Science and Engineering  
Indian Institute of Technology, Kharagpur, India  
debajyoti.kar@iitkgp.ac.in

Arindam Khan  
Department of Computer Science and Automation  
Indian Institute of Science, Bengaluru, India  
arindamkhan@iisc.ac.in

Andreas Wiese  
School of Business and Economics, Operations Analytics  
Vrije Universiteit, Amsterdam, Netherlands  
a.wiese@vu.nl

Abstract

We study **Round-UFP** and **Round-SAP**, two generalizations of the classical **Bin Packing** problem that correspond to the unsplittable flow problem on a path (UFP) and the storage allocation problem (SAP), respectively. We are given a path with capacities on its edges and a set of tasks where for each task we are given a demand and a subpath. In **Round-UFP**, the goal is to find a packing of all tasks into a minimum number of copies (rounds) of the given path such that for each copy, the total demand of tasks on any edge does not exceed the capacity of the respective edge. In **Round-SAP**, the tasks are considered to be rectangles and the goal is to find a non-overlapping packing of these rectangles into a minimum number of rounds such that all rectangles lie completely below the capacity profile of the edges.

We show that in contrast to **Bin Packing**, both the problems do not admit an asymptotic polynomial-time approximation scheme (APTAS), even when all edge capacities are equal. However, for this setting, we obtain asymptotic \((2 + \varepsilon)\)-approximations for both problems. For the general case, we obtain an \(O(\log \log n)\)-approximation algorithm and an \(O(\log \log \frac{1}{\delta})\)-approximation under \((1 + \delta)\)-resource augmentation for both problems. For the intermediate setting of the **no bottleneck assumption** (i.e., the maximum task demand is at most the minimum edge capacity), we obtain absolute 12- and asymptotic \((16 + \varepsilon)\)-approximation algorithms for **Round-UFP** and **Round-SAP**, respectively.

2012 ACM Subject Classification  Theory of computation → Design and analysis of algorithms

Keywords and phrases  Approximation Algorithms, Scheduling, Rectangle Packing.

Digital Object Identifier  10.4230/LIPIcs.CVIT.2016.23

Acknowledgements  We thank Waldo Galvez, Afrouz Jabal Ameli, Siba Smarak Panigrahi and Arka Ray for helpful initial discussions.

1 Introduction

The unsplittable flow on a path problem (UFP) and the storage allocation problem (SAP) are two well-studied problems in combinatorial optimization. In this paper, we study **Round-UFP** and **Round-SAP**, which are two related natural problems that also generalize the classical **Bin Packing** problem.

In both **Round-UFP** and **Round-SAP**, we are given as input a path \(G = (V, E)\) and a set of \(n\) jobs \(J\). We assume that \(\{v_0, v_1, \ldots, v_m\}\) are the vertices in \(V\) from left to right and then for each \(i \in \{1, \ldots, m\}\) there is an edge \(e_i := \{v_{i-1}, v_i\}\). Each job \(j \in J\) has integral
demand $d_j \in \mathbb{N}$, a source $v_{s_j} \in V$, and a sink $v_{t_j} \in V$. We say that each job $j$ spans the path $P_j$ which we define to be the path between $v_{s_j}$ and $v_{t_j}$. For every edge $e \in E$, we are given an integral capacity $c_e$. A useful geometric interpretation of the input path and the edge capacities is the following (see Figure 1): consider the interval $[0,m]$ on the $x$-axis and a function $c: [0,m] \rightarrow \mathbb{N}$. Each edge $e_k$ corresponds to the interval $[k-1,k)$ and each vertex $v_i$ corresponds to the point $i$. For edge $e_k$, we define $c(x) = c_{e_k}$ for each $x \in [k-1,k)$.

In the ROUND-UFP problem, the objective is to partition the jobs $J$ into a minimum number of sets $J_1, ..., J_k$ (that we will denote by rounds) such that the jobs in each set $J_i$ form a valid packing, i.e., they obey the edge capacities, meaning that $\sum_{i \in J_i, e \in P_i} d_i \leq c_e$ for each $e \in E$. In the ROUND-SAP problem, we require to compute additionally for each set $J_i$ a non-overlapping set of rectangles underneath the capacity profile, corresponding to the jobs in $J_i$ (see Figure 1). Formally, we require for each job $j \in J_i$ to determine a height $h_i$ with $h_i + d_i \leq c_e$ for each edge $e \in P_i$, yielding a rectangle $R_i = (s_j, t_j) \times (h_i, h_i + d_i)$, such that for any two jobs $j, j' \in J_i$ we have that $R_i \cap R_{i'} = \emptyset$. Again, the objective is to minimize the number of rounds.

Note that unlike ROUND-SAP, in ROUND-UFP we do not need to pack the jobs as contiguous rectangles. Hence, intuitively in ROUND-UFP we can slice the rectangles vertically and place different slices at different heights. See Figure 1 for the differences between the problems.

ROUND-UFP and ROUND-SAP arise naturally in the setting of resource allocation with connections to many fundamental optimization problems, e.g., wavelength division multiplexing (WDM) and optical fiber minimization (see [5] for more details and practical motivations). The path can represent a network with a chain of communication links in which we need to send some required transmissions in a few rounds so that they obey the given edge capacities. The edges can also correspond to discrete time slots, each slot models a job that we might want to execute, and the edge capacities model the available amount of a resource shared by the jobs like energy or machines. ROUND-UFP models routing in optical networks, where each copy of the resource corresponds to a distinct frequency. As the number of available distinct frequencies is limited, minimizing the number of rounds for a given set of requests is a natural objective. ROUND-SAP is motivated by settings in which jobs need a contiguous portion of an available resource, e.g., a consecutive portion of the computer memory or a frequency bandwidth. Another application is ad-placement, where each job is an advertisement that requires a contiguous portion of the banner [43].
ROUND-UFP and ROUND-SAP are APX-hard as they contain the classical Bin Packing problem as a special case when $G$ has only one single edge. However, while for Bin Packing there exists an asymptotic polynomial time approximation scheme (APTAS)\footnote{For formal definitions of notions like asymptotic approximation ratio, (asymptotic) polynomial time approximation scheme, etc. we refer to Appendix A.}, it is open whether such an algorithm exists for ROUND-UFP or ROUND-SAP. The best known approximation algorithm for ROUND-UFP is a $O(\min\{\log n, \log m, \log \log c_{\text{max}}\})$-approximation \cite{35}. For the special case of ROUND-UFP of uniform edge capacities, Pal \cite{47} gave a 3-approximation. Elbassioni et al. \cite{22} gave a 24-approximation algorithm for the problem under the no-bottleneck assumption (NBA) which states that the maximum task demand is upper-bounded by the minimum edge capacity. A result in (the full version of) \cite{13} states that any solution to an instance of UFP can be partitioned into at most 80 sets of tasks such that each of them is a solution to the corresponding SAP instance. This immediately yields approximation algorithms for ROUND-SAP: a 240-approximation for the case of uniform capacities, a 1920-approximation under the NBA, and a $O(\min\{\log n, \log m, \log \log c_{\text{max}}\})$-approximation for the general case. These are the best known results for ROUND-SAP.

1.1 Our Contributions

First, we show that both ROUND-SAP and ROUND-UFP, unlike the classical Bin Packing problem, do not admit an APTAS, even in the uniform capacity case. We achieve this via a gap preserving reduction from the 3D matching problem. We create a numeric version of the problem and define a set of hard instances for both ROUND-SAP and ROUND-UFP. Together with a result of Chlebik and Chlebikova \cite{16}, we derive an explicit lower bound on the asymptotic approximation ratio for both problems. Our hardness result holds even for the case in which in the optimal packing no round contains more than $O(1)$ jobs, i.e., a case in which we can even enumerate all possible packings in polynomial time.

For the case of uniform edge capacities, we give asymptotic $(2 + \varepsilon)$-approximation algorithms for both ROUND-UFP and ROUND-SAP, and absolute $(2.5 + \varepsilon)$ and 3-approximation algorithms for the two problems, respectively. This improves upon the previous absolute 3- and 240-approximation algorithms mentioned above. Note that for both problems our factor of 2 is a natural threshold: in many algorithms for UFP and SAP \cite{11, 3, 33, 34, 43, 44}, the input tasks are partitioned into tasks that are relatively small and relatively large (compared to the edge capacities). Then, both sets are handled separately with very different sets of techniques. This inherently loses a factor of 2. Our algorithms are based on a connection of our problems to the dynamic storage allocation (DSA) problem and we show how to use some known deep results for DSA \cite{12} in our setting. In DSA the goal is to place some given tasks as non-overlapping rectangles, minimizing the height of the resulting packing. Hence, it is somewhat surprising that also for ROUND-UFP (where we do not have this requirement) it yields the needed techniques for an improved approximation.

For the general cases of ROUND-UFP and ROUND-SAP we give an $O(\log \log \min\{m, n\})$-approximation algorithm. Depending on the concrete values of $n$, $m$, and $c_{\text{max}}$, this constitutes an up to exponential improvement compared to the best known result for ROUND-UFP \cite{35} and for ROUND-SAP (by the reasoning via \cite{43} above). We divide the input tasks into two sets: tasks that use a relatively large portion of the capacity of at least one of their edges (large tasks) and tasks that use a relatively small portion of the capacity of all of their edges (small tasks). For each large task, we fix a corresponding rectangle that is drawn at the
maximum possible height underneath the capacity curve (even for Round-UFP for which we do not need to represent the given tasks as rectangles) and we seek a solution in which in each round these corresponding rectangles are non-overlapping. It follows from known results that this loses at most a factor of $O(1)$ \cite{11,35}. Then, we solve a configuration-LP for our problem which we solve via a separation oracle. We take the integral parts of its solution and show that the fractional parts yield several instances of our original problem in which each point overlaps at most $O(\log m)$ rectangles. Using a recent result by Chalermsook and Walczak \cite{13} on the coloring number of rectangle intersection graphs, we obtain an $O(\log \log m)$-approximation. For the small tasks of a given Round-UFP instance, a result in \cite{22} yields an $O(1)$-approximation, which yields our $O(\log \log m)$-approximation for Round-UFP. Finally, we use the result in \cite{43} mentioned above in order to turn this algorithm even into a $O(\log \log m)$-approximate solution for Round-SAP. Then we study the setting of resource augmentation, i.e., where we can increase the edge capacities by a factor of $1 + \delta$ while the compared optimal solution does not have this privilege. In this case, we show that we can reduce the given problem to the setting in which the edge capacities are in the range $[1, 1/\delta)$. Applying the algorithm from \cite{35} then yields a $O(\log \log \frac{1}{\delta})$-approximation for this case for Round-UFP, and with a similar argumentation as before also for Round-SAP.

Furthermore, for the case of the NBA we improve the absolute approximation ratio from 24 to 12 for Round-UFP, and from 1920 to 24 for Round-SAP, and we obtain even an asymptotic $(16 + \varepsilon)$-approximation for Round-SAP. For Round-SAP we show that we can reduce the general case to the case of uniform edge capacities, losing only a factor of 8. Thus, future improvements for the case of uniform edge capacities will directly yield improvements for the case of the NBA. For Round-UFP we partition the input jobs into several sets. For some of them we reduce the given setting to the case of unit job demands and integral edge capacities and invoke an algorithm from \cite{46} for this case. For the others, we show that a simple greedy routine works sufficiently well.

If in Round-UFP we are given a tree instead of a path, we obtain the Round-Tree problem. The best known result for it under the NBA is a 64-approximation \cite{22}. We are not aware of any better result for uniform edge capacities. We improve the best known approximation ratio under the NBA to 55 and also provide a 5.5-approximation algorithm for the case of uniform edge capacities.

See Table 1 for an overview of our results. Due to space limitations, many proofs had to be moved to the appendix.

1.2 Other Related Work

Adamy and Erlebach \cite{3} studied Round-UFP for uniform edge capacities in the online setting and gave a 195-approximation algorithm, which was subsequently improved to 10 \cite{15,6}. Without the NBA, Epstein et al. \cite{23} showed that no deterministic online algorithm can achieve a competitive ratio better than $\Omega(\log \log n)$ or $\Omega(\log \log (c_{\max}/c_{\min}))$. They also gave a $O(\log c_{\max})$-competitive algorithm, where $c_{\max}$ is the largest edge capacity. Without the NBA, recently Jahanjou et al. \cite{55} gave a $O(\min(\log m, \log \log c_{\max}))$-competitive algorithm.

Round-UFP and Round-SAP are related to many fundamental optimization problems. For example, Round-SAP can be interpreted as an intermediate problem between two-dimensional bin packing (2BP) and the rectangle coloring problem (RC). In 2BP, the goal is to find an axis-parallel nonoverlapping packing of a given set of rectangles (which we can translate in both dimensions) into minimum number of unit square bins. If all edges have the same capacity then Round-SAP can be seen as a variant of 2BP in which the horizontal coordinate of each item is fixed and we can choose only the vertical coordinate. For 2BP,
Table 1 Overview of our results. We distinguish the settings according to uniform edge capacities, the no-bottleneck-assumption (NBA), general edge capacities, and general edge capacities with \((1 + \delta)\)-resource augmentation (r.a.). Also, we distinguish between absolute approximation ratios and asymptotic approximation ratios. All listed previous results are absolute approximation ratios.

| Problem       | Edge capacities | Previous approximation | Improved approximation |
|---------------|-----------------|------------------------|------------------------|
| Round-UFP     | uniform         | 3 \[47\]              | asymp. \(2 + \varepsilon\), abs. \(2.5 + \varepsilon\) |
| Round-SAP     | uniform         | 240 \[17, 14\]        | asymp. \(2 + \varepsilon\), abs. 3 |
| Round-UFP     | NBA             | 24 \[22\]             | abs. 12 |
| Round-SAP     | NBA             | 1920 \[22, 43\]       | asymp. \(16 + \varepsilon\), abs. 24 |
| Round-UFP     | general         | \(O(\log\min\{n, m, \log c_{\text{max}}\})\) \[35\] | abs. \(O(\log\log\min\{n, m\})\) |
| Round-SAP     | general         | \(O(\log\min\{n, m, \log c_{\text{max}}\})\) \[35\] | abs. \(O(\log\log\min\{n, m\})\) |
| Round-UFP     | general with r.a. | \(O(\log\min\{n, m, \log c_{\text{max}}\})\) \[35\] | abs. \(O(\log\log(1/\delta))\) |
| Round-SAP     | general with r.a. | \(O(\log\min\{n, m, \log c_{\text{max}}\})\) \[35\] | abs. \(O(\log\log(1/\delta))\) |
| Round-Tree    | uniform         | 64 \[22\]             | asymp. 5.1, abs. 5.5 |
| Round-Tree    | NBA             | 64 \[22\]             | asymp. 49, abs. 55 |

present best asymptotic approximation guarantee is 1.406 \[9\]. On the other hand, in RC, all rectangles are fixed and the goal is to color the rectangles using a minimum number of colors such that no two rectangles of the same color intersect. For RC, recently Chalermsook and Walczak \[13\] have given a polynomial-time algorithm that uses only \(O(\omega \log \omega)\) colors, where \(\omega\) is the clique number of the corresponding intersection graph (and hence a lower bound on the number of needed colors). Another related problem is Dynamic Storage Allocation (DSA), where the objective is to pack the given tasks (with fixed horizontal location) such that the maximum vertical height, \(\max_i (h_i + d_i)\) (called the makespan) is minimized. The current best known approximation ratio for DSA is \((2 + \varepsilon)\) \[12\].

In a sense Round-UFP and Round-SAP are ‘Bin Packing-type’ problems, and their corresponding ‘Knapsack-type’ problems are UFP and SAP, respectively, where each task has an associated profit and the goal is to select a subset of tasks which can be packed into one single round satisfying the corresponding valid packing constraints. There is a series of work \[33, 8, 10, 4, 34, 32\] in UFP, culminating in a PTAS \[31\]. It is maybe surprising that Round-UFP does not admit an APTAS, even though UFP admits a PTAS. For SAP, the currently best polynomial time approximation ratio is \(2 + \varepsilon\) \[43\], which has been recently improved to \(1.997 + \varepsilon\) \[44\] for the case of uniform capacities, and also a quasi-polynomial time \((1.997 + \varepsilon)\)-approximation is known for quasi-polynomially bounded input data.

There are many other related problems, such as two-dimensional knapsack \[26, 36, 28\], strip packing \[27, 25, 21\], maximum independent set of rectangles \[2, 13, 12, 29\], guillotine separability of rectangles \[39, 38, 40\], weighted bipartite edge coloring \[11\], maximum edge disjoint paths \[19\], etc. We refer the readers to \[37, 18\] for an overview of these problems.

## 2 Preliminaries

Let \(OPT_{UFP}\) and \(OPT_{SAP}\) denote the optimal number of rounds required to pack all jobs of a given instance of Round-UFP and Round-SAP, respectively. By simple preprocessing, we can assume that each vertex in \(V\) corresponds to endpoint(s) of some job(s) in \(J\), and hence \(m \leq 2n - 1\). For each job \(j\) denote by \(b_j\) the minimum capacity of the edges in \(P_j\), i.e., \(\min\{c_e : e \in P_j\}\) which we denote as the bottleneck capacity of \(j\). Job \(j\) is said to pass through...
edge \( e \) if \( e \in P \). The load on edge \( e \) is defined as \( l_e := \sum_{i \in P} d_i \), the total sum of demands of all jobs passing through \( e \). Let \( L := \max_e l_e \) denote the maximum load. The congestion \( r_e \) of edge \( e \) is defined as \( r_e := [l_e/c_e] \), and \( r := \max_e r_e \) denotes the maximum congestion. Clearly, \( r \) is a lower bound on \( OPT_{UFP} \) and \( OPT_{SAP} \).

3 Lower Bounds

A simple reduction from the Partition problem shows that it is NP-hard to obtain a better approximation ratio than \( 3/2 \) for the classical Bin Packing problem. However, in the resulting instances, the optimal solutions use only two or three bins. On the other hand, Bin Packing admits an APTAS \([20]\) and thus, for any \( \varepsilon > 0 \), a \( (1 + \varepsilon) \)-approximation algorithm for instances in which \( OPT \) is sufficiently large. Since Round-SAP and Round-UFP are generalizations of Bin Packing (even if \( G \) has only a single edge), the lower bound of \( 3/2 \) continues to hold. However, maybe surprisingly, we show that unlike Bin Packing, Round-SAP and Round-UFP do not admit APTASes, even in the case of uniform edge capacities. More precisely, we provide a lower bound of \( (1 + 1/1398) \) on the asymptotic approximation ratio for Round-SAP and Round-UFP via a reduction from the 2-Bounded Occurrence Maximum 3-Dimensional Matching (2-B-3-DM) problem.

In 2-B-3-DM, we are given as input three pairwise disjoint sets \( X := \{x_1, x_2, \ldots, x_q\} \), \( Y := \{y_1, y_2, \ldots, y_q\} \), and \( Z := \{z_1, z_2, \ldots, z_q\} \) and a set of triplets \( T \subseteq X \times Y \times Z \) such that each element of \( X \cup Y \cup Z \) occurs in exactly two triplets in \( T \). Note that \( |X| = |Y| = |Z| = q \) and \( |T| = 2q \). A matching is a subset \( M \subseteq T \) such that no two triplets in \( M \) agree in any coordinate. The goal is to find a matching of maximum cardinality (denoted by \( OPT_{3DM} \)). Chlebik and Chlebikova \([16]\) gave the following hardness result.

\[\text{Theorem 1} \quad\text{[16]}.\quad \text{For 2-B-3-DM there exists a family of instances such that for each instance } K \text{ of the family, either } OPT_{3DM}(K) < \alpha(q) := [0.9690082645q] \text{ or } OPT_{3DM}(K) \geq \beta(q) := [0.979338843q], \text{ and it is NP-hard to distinguish these two cases.}\]

Hardness of 2-B-3-DM has been useful in inapproximability results for various (multidimensional) packing, covering, and scheduling problems, e.g., vector packing \([30]\), geometric bin packing \([4]\), geometric bin covering \([17]\), generalized assignment problem \([14]\), etc. Similar to these results, we also use gadgets based on a reduction from 2-B-3-DM to the 4-Partition problem. However, the previous techniques are not directly transferable to our problem due to the inherent differences between these problems. Therefore, we first use the technique from \([30]\) to associate certain integers with the elements of \( X \cup Y \cup Z \) and \( T \) and then adapt the numeric data in a different way to obtain the hard instances.

Let \( \rho = 32q \) and let \( V \) be the set of 5q integers defined as follows: \( x_i' = i \rho + 1 \), for \( 1 \leq i \leq |X| \), \( y_j' = j \rho^2 + 2 \), for \( 1 \leq j \leq |Y| \), \( z_k' = k \rho^3 + 4 \), for \( 1 \leq k \leq |Z| \), \( \tau' = \rho^4 - k \rho^3 - j \rho^2 - i \rho + 8 \), for each triplet \( \tau = (x_i, y_j, z_k) \in T \). Define \( \gamma = \rho^4 + 15 \). The following result is due to Woeginger \([50]\).

\[\text{Lemma 2} \quad\text{[50]}.\quad \text{Four integers in } V \text{ sum up to the value } \gamma \text{ if and only if (i) one of them corresponds to some element } x_i \in X, \text{ one to some element } y_j \in Y, \text{ one to an element } z_k \in Z, \text{ and one to some triplet } \tau \in T, \text{ and if (ii) } \tau = (x_i, y_j, z_k) \text{ holds for these four elements.}\]

Now, we create a hard instance, tailor-made for our problems. We define that our path \( G = (V, E) \) has 40000\( \gamma \) vertices that we identify with the numbers \( 0, 1, \ldots, 40000 \). For each \( x_i \in X \) (respectively \( y_j \in Y \), \( z_k \in Z \) ), we specify two jobs \( a_{X,i} \) and \( a'_{X,i} \) (respectively \( a_{Y,j} \), \( a'_{Y,j} \), and \( a_{Z,k}, a'_{Z,k} \)), which will be called peers of each other. Each job \( j \) is specified by a triplet \((s_j, t_j, d_j)\). We define
\[
a_{X,i} = (0, 20000\gamma - 4x_i', 999\gamma + 4x_i') \text{ and } a_{X,i}' = (20000\gamma - 4x_i', 40000\gamma, 1001\gamma - 4x_i'),
\]
\[
a_{Y,j} = (0, 20000\gamma - 4y_j', 999\gamma + 4y_j') \text{ and } a_{Y,j}' = (20000\gamma - 4y_j', 40000\gamma, 1001\gamma - 4y_j'), \text{ and}
\]
\[
a_{Z,k} = (0, 20000\gamma - 4z_k', 999\gamma + 4z_k') \text{ and } a_{Z,k}' = (20000\gamma - 4z_k', 40000\gamma, 1001\gamma - 4z_k').
\]
For each \( \tau_i \in \mathcal{T} \), we define two jobs \( b_i \) and \( b_i' \) (also peers) by:
\[
\begin{align*}
& b_i = (0, 19001\gamma - 4\gamma_i', 999\gamma + 4\gamma_i') \text{ and } b_i' = (19001\gamma - 4\gamma_i', 40000\gamma, 1001\gamma - 4\gamma_i').
\end{align*}
\]
Finally let \( D \) be a set of \( 5q - 4\beta(q) \) dummy jobs each specified by \( (0, 40000\gamma, 2997\gamma) \). We define that each edge \( e \in E \) has a capacity of \( c_e := c^* := 4000\gamma \). This completes the reduction.

For any job \( j = (s_i, t_i, d_i) \) we define its width \( w_j := t_i - s_i \).

Let \( A_X := \{ a_{X,i} \mid 1 \leq i \leq q \} \) and \( A_X' := \{ a_{X,i}' \mid 1 \leq i \leq q \} \). The sets \( A_Y, A_Y', A_Z, A_Z' \) are defined analogously. Let \( A := A_X \cup A_Y \cup A_Z \) and \( A' := A_X' \cup A_Y' \cup A_Z' \). Finally let \( B := \{ b_i \mid 1 \leq l \leq 2q \} \) and \( B' := \{ b_i' \mid 1 \leq l \leq 2q \} \).

To provide some intuition, we first give an upper bound on the number of jobs that can be packed in a round. All following lemmas, statements, and constructions hold for both \textsc{Round-SAP} and \textsc{Round-UFP}.

\begin{lemma}
In any feasible solution any round can contain at most 8 jobs.
\end{lemma}

We say that a round is \textit{nice} if it contains exactly 8 jobs. It turns out that such a round corresponds exactly to one element \( \tau_i = (x_i, y_i, z_k) \in \mathcal{T} \). We say that the jobs \( a_{X,i}, a_{X,i}', a_{Y,j}, a_{Y,j}', a_{Z,k}, a_{Z,k}', b_i, \) and \( b_i' \) correspond to \( \tau_i = (x_i, y_i, z_k) \).

\begin{lemma}
We have that a round is nice if and only if there is an element \( \tau_i = (x_i, y_i, z_k) \in \mathcal{T} \) such that the round contains exactly the jobs that correspond to \( \tau_i = (x_i, y_i, z_k) \).
\end{lemma}

Given an optimal solution \( \text{OPT}_{3DM} \) to 2-B-3-DM with \( |\text{OPT}_{3DM}| \geq \beta(q) \), we construct a solution as follows:

1. Let \( \mathcal{M} \) be any subset of \( \text{OPT}_{3DM} \) with \( |\mathcal{M}| = \beta(q) \). Create \( \beta(q) \) nice rounds corresponding to the elements in \( \mathcal{M} \), i.e., for each element \( \tau_i = (x_i, y_i, z_k) \in \mathcal{M} \), create a round containing the jobs that correspond to \( \tau_i = (x_i, y_i, z_k) \).
2. For each \( \tau_i \in \mathcal{T} \setminus \mathcal{M} \) create a round containing \( b \) and \( b' \) along with a dummy job.
3. For each \( x_i \in \mathcal{X} \) (respectively \( y_j \in \mathcal{Y} \), \( z_k \in \mathcal{Z} \)) not covered by \( \mathcal{M} \), pack \( a_{X,i} \) and \( a_{X,i}' \) (respectively \( a_{Y,j}, a_{Y,j}' \) and \( a_{Z,k}, a_{Z,k}' \)) together with one dummy job in one round.

\begin{lemma}
If \( |\text{OPT}_{3DM}| \geq \beta(q) \) then the constructed solution is feasible and it uses at most \( 5q - 3\beta(q) \) rounds.
\end{lemma}

\textbf{Proof}. One can easily check that all constructed rounds are feasible. In step (1) we construct exactly \( \beta(q) \) rounds. In step (2), we construct \( |\mathcal{T}| - \beta(q) = 2q - \beta(q) \) rounds, since \( |\mathcal{T}| = 2q \). In step (3), we construct \( 3|\mathcal{T}| - |\mathcal{M}| = 3q - 3|\text{OPT}_{3DM}| \) rounds. Hence, overall we construct at most \( 5q - 3|\text{OPT}_{3DM}| \leq 5q - 3\beta(q) \) rounds.

\textbf{Proof}

Conversely, assume that \( |\text{OPT}_{3DM}| < \alpha(q) \) and that we are given any feasible solution to our constructed instance. We want to show that it uses at least \( 5q - 3\beta(q) + \frac{1}{2}(\beta(q) - \alpha(q)) \) rounds. For this, a key property of our construction is given in the following lemma.

\begin{lemma}
If a round contains a dummy job, then it can have at most three jobs: at most one dummy job, at most one job from \( A \cup B \), and at most one job from \( A' \cup B' \).
\end{lemma}

Let \( n_g \) denote the number of nice rounds in our solution, \( n_d \) the number of rounds with a dummy job, and \( n_b \) the number of remaining rounds. Note that each of the latter rounds can contain at most \( 7 \) jobs each. Since all jobs in \( A \cup A' \cup B \cup B' \) need to be assigned to a round, we have that \( 8n_g + 7n_b + 2n_d \geq 6q + 2|\mathcal{T}| = 10q \). Since the nice rounds correspond to a matching of the given instance of 2-B-3-DM, we have that \( n_g \leq \alpha(q) \). Using this, we lower-bound the number of used rounds in the following lemma.
Lemma 7. If $|OPT_{3DM}| < \alpha(q)$ then the number of rounds in our solution is $n_d + n_g + n_h \geq (5q - 3\beta(q)) + \frac{1}{4}(\beta(q) - \alpha(q))$.

**Proof.** Since $8n_g + 7n_b + 2n_d \geq 6q + 2|T| = 10q$ and $n_d = 5q - 4\beta(q)$, we obtain that $8n_g + 7n_b \geq 8\beta(q)$. Thus $n_g + n_b \geq \frac{5}{4}\beta(q) - \frac{1}{4}n_g$. Since $n_d \leq \alpha(q)$ the number of rounds is at least $n_d + n_g + n_b \geq 5q - 3\beta(q) + \frac{5}{4}\beta(q) - \frac{1}{4}(\beta(q) - \alpha(q))$. ▶

Now Lemmas 5 and 7 yield our main theorem.

Theorem 8. There exists a constant $\delta_0 > 1/1398$, such that it is NP-hard to approximate Round-UFP and Round-SAP in the case of uniform edge capacities with an asymptotic approximation ratio less than $1 + \delta_0$.

### Algorithms for Uniform Capacity Case

In this section, we provide asymptotic $(2 + \varepsilon)$-approximation for Round-SAP and Round-UFP for the case of uniform edge capacities. We distinguish two cases, depending on the value of $d_{\text{max}} := \max_{i \in J} d_i$ compared to $L$.

#### 4.1 Case 1: $d_{\text{max}} \leq \varepsilon^7 L$

First, we invoke an algorithm from [12] for the dynamic storage allocation (DSA) problem. Recall that in DSA the input consists of a set of jobs like in Round-SAP and Round-UFP, but without upper bounds of the edge capacities. Instead, we seek to define a height $h_i$ for each job $j$ such that the resulting rectangles for the jobs are non-overlapping and the makespan $\max_i (h_i + d_i)$ is minimized. The maximum load $L$ is defined as in our setting.

We invoke the following theorem on our input jobs $J$ with $\delta := \varepsilon$.

Theorem 9 ([12]). Assume that we are given a set of jobs $J'$ such that $d_i \leq \delta_0^2 L$ for each job $j \in J'$. Then there exists an algorithm that produces a DSA packing of $J'$ with makespan at most $(1 + \kappa \delta)L$, where $\kappa > 0$ is some global constant independent of $\delta$.

Let $\xi$ denote the makespan of the resulting solution to DSA and let $c^*$ denote the (uniform) edge capacity. For each $h \in \mathbb{R}$, we define the horizontal line $\ell_h := \mathbb{R} \times \{h\}$. A job $j$ is said to be sliced by $\ell_h$ if for the computed packing of the jobs it holds that $h_i < h < h_i + d_i$. Now we will transform this into Round-SAP or Round-UFP packing.

We define a set of rounds $\Gamma_1$. The set $\Gamma_1$ contains a round for each integer $i$ with $0 \leq i \leq \lfloor \xi/c^* \rfloor$ and this round contains all jobs lying between $\ell_{ic^*}$ and $\ell_{(i+1)c^*}$. Thus, $|\Gamma_1| \leq \lfloor (1 + \kappa \varepsilon)L/c^* \rfloor + 1 \leq (1 + \kappa \varepsilon)r + 1$. There are two subcases.

*Subcase A: Assume that $r > 1/(2\kappa \varepsilon)$. In this case $|\Gamma_1| \leq (1 + 3\kappa \varepsilon)r$. We define a set of rounds $\Gamma_2$ as follows. For each integer $i$ with $1 \leq i \leq \lfloor \xi/c^* \rfloor$, $\Gamma_2$ contains all jobs that are sliced by $\ell_{ic^*}$. Thus $|\Gamma_2| \leq \lfloor (1 + \kappa \varepsilon) L/c^* \rfloor \leq (1 + \kappa \varepsilon)r$. Hence, the total number of rounds is bounded by $(2 + 4\kappa \varepsilon)r \leq (2 + O(\varepsilon))OPT_{UFP} \leq (2 + O(\varepsilon))OPT_{SAP}$.*

*Subcase B: Assume that $r \leq 1/(2\kappa \varepsilon)$. Now $\xi \leq (1 + \kappa \varepsilon)L$, and therefore $\xi - L \leq \kappa \varepsilon L \leq c^*/2$. Hence, we have $|\Gamma_1| \leq r + 1$ and the $(r + 1)$th round is filled up to a capacity of at most $c^*/2$ on each edge. Now the total load of the set of jobs that are sliced by $\ell_{ic^*}$ is at most $r \cdot \varepsilon^7 L$. We now invoke the following result on DSA to this set of jobs.*

Theorem 10 ([30]). Let $J'$ be a set of jobs with load $L$. Then a DSA packing of $J'$ of makespan at most $3L$ can be computed in polynomial time.
Thus the makespan of the computed solution is at most $3r \cdot \varepsilon^7 L \leq 3 \cdot \frac{1}{c \varepsilon^2} \cdot 1 \cdot \frac{c^*}{\varepsilon^2} \leq c^*/2$, if $\varepsilon$ is small enough. Hence these jobs can be added to the $(r+1)^{\text{th}}$ round of $\Gamma_1$. Therefore, we get a packing of $J$ using at most $r + 1 \leq OPT_{UFP} + 1 \leq OPT_{SAP} + 1$ rounds.

### 4.2 Case 2: $d_{\text{max}} > \varepsilon^7 L$

For this case, we have $c^* \geq d_{\text{max}} > \varepsilon^7 L$ and therefore $r = \lfloor L/c^* \rfloor \leq 1/\varepsilon^7$. We partition the input jobs into large and small jobs by defining $J_{\text{large}} := \{j \in J | d_j > \varepsilon^{56} L\}$ and $J_{\text{small}} := \{j \in J | d_j \leq \varepsilon^{56} L\}$.

We start with the small jobs $J_{\text{small}}$. First, we apply Theorem 9 to them with $\delta := \varepsilon^{56}$ and obtain a DSA packing $P$ for them. We transform it into a solution to Round-SAP with at most $r + 1$ rounds as follows: we introduce a set $\Gamma_1$ consisting of $r + 1$ rounds exactly as in the previous case (when $r \leq 1/(2\varepsilon^7)$). The $(r + 1)^{\text{th}}$ round would be filled up to a capacity of at most $\kappa \varepsilon^8 L \leq \kappa \varepsilon^7 c^*$.

Again applying Theorem 10 to the remaining jobs, we get a DSA packing of makespan at most $3r \cdot \varepsilon^{56} L \leq 3 \cdot (1/\varepsilon^7) \cdot \varepsilon^{56} \cdot (c^*/\varepsilon^7) \leq c^*/2$, and therefore these jobs can be packed inside the $(r+1)^{\text{th}}$ round. Hence, there exists a packing of $J_{\text{small}}$ using at most $r + 1 \leq OPT_{UFP} + 1 \leq OPT_{SAP} + 1$ rounds.

Now we consider the large jobs $J_{\text{large}}$. Our strategy is to compute an optimal packing for them via dynamic programming (DP). Intuitively, our DP orders the jobs in $J_{\text{large}}$ non-decreasingly by their respective source vertices and assigns them to the rounds in this order. Since the jobs are large, each edge is used by at most $1/\varepsilon^{56}$ large jobs, and using interval coloring one can show easily that at most $1/\varepsilon^{56} = O(1)$ rounds suffice (e.g., we can color the jobs with $1/\varepsilon^{56}$ colors such that no two jobs with intersecting paths have the same color).

In our DP we have a cell for each combination of an edge $e$ and the assignment of all jobs passing through $e$ to the rounds. Given this, the corresponding subproblem is to assign additionally all jobs to the rounds whose paths lie completely on the right of $e$.

For Round-SAP we additionally want to bound the number of possible heights $h_j$. To this end, we restrict ourselves to packings that are normalized which intuitively means that all jobs are pushed up as much as possible. Formally, we say that a packing for a set of jobs $J'$ inside a round is normalized if for every $j \in J'$, either $h_j = d_j = c^*$ or $h_j = d_j = h_{j'}$ for some $j' \in J'$ such that $P_j \cap P_{j'} \neq \emptyset$ (see Figure 4 in Appendix B.5).

\textbf{Lemma 11.} Consider a valid packing of a set of jobs $J' \subseteq J_{\text{large}}$ inside one round. Then there is also a packing for $J'$ that is normalized.

Now the important insight is that in a normalized packing of large jobs, the height $h_j$ of a job $j$ is the difference of (the top height level) $c^*$ and the sum of at most $1/\varepsilon^{56}$ jobs in $J_{\text{large}}$. Thus, the number of possible heights is bounded by $n^{O(1/\varepsilon^{56})}$ and we can compute all these possible heights before starting our DP.

\textbf{Lemma 12.} Given $J_{\text{large}}$ we can compute a set $\mathcal{H}$ of $n^{O(1/\varepsilon^{56})}$ values such that in any normalized packing of a set $J' \subseteq J_{\text{large}}$ inside one round, the height $h_j$ of each job $j \in J'$ is contained in $\mathcal{H}$.

Now we can compute the optimal packing via a dynamic program as described above, which yields the following lemma.

\textbf{Lemma 13.} Consider an instance of Round-UFP or Round-SAP with a set of jobs $J'$ satisfying the following conditions:

(i) The number of jobs using any edge is bounded by $\omega$.

(ii) In the case of Round-SAP there is a given set $\mathcal{H}$ of allowed heights for the jobs.
Then we can compute an optimal solution to the given instance in time \((n|\mathcal{H}'|)^{O(\omega)}\).

We invoke Lemma 13 with \(J' := J_{\text{large}}\), \(\omega = 1/\varepsilon^{56}\), and in the case of Round-SAP we define \(\mathcal{H}'\) to be the set \(\mathcal{H}\) due to Lemma 12. This yields at most \(OPT\) rounds in total for the large jobs \(J_{\text{large}}\). Hence, we obtain a packing of \(J\) using at most \(2 \cdot OPT + 1\) rounds.

Case 1 and 2 together yields our main theorem for the case of uniform edge capacities.

**Theorem 14.** For any \(\varepsilon > 0\), there exist asymptotic \((2 + \varepsilon)\)-approximation algorithms for Round-SAP and Round-UFP, assuming uniform edge capacities.

We now derive some bounds on the absolute approximation ratios. If \(OPT_{SAP} = 1\), our algorithm would return a packing using at most \((2 + \varepsilon) \cdot 1 + 1\) rounds, and hence at most 3 rounds. If \(OPT_{SAP} \geq 2\), then our algorithm uses at most \((2 + \varepsilon) \cdot OPT_{SAP} + OPT_{SAP}/2 = (2.5 + \varepsilon)OPT_{SAP}\) rounds. Hence, we obtain the following result.

**Theorem 15.** There exists a polynomial time 3-approximation algorithm for Round-SAP, assuming uniform edge capacities.

For Round-UFP, it is easy to check whether \(OPT_{UFP} = 1\) by checking whether \(L \leq \varepsilon^{c}\). Otherwise, \(OPT_{UFP} \geq 2\) and similar as above, the number of rounds used would be at most \((2.5 + \varepsilon) \cdot OPT_{UFP}\). This gives an improvement over the result of Pal [17].

**Theorem 16.** For any \(\varepsilon > 0\), there exists a polynomial time \((2.5 + \varepsilon)\)-approximation algorithm for Round-UFP, assuming uniform edge capacities.

## 5 General Case

In this section, present our algorithms for the general cases of Round-UFP and Round-SAP. We begin with our \(O(\log \log \min\{n, m\})\)-approximation algorithms where we consider Round-UFP first and describe later how to extend our algorithm to Round-SAP. We split the input jobs into large and small jobs. We define \(J_{\text{large}} := \{j \in J | d_j > b_j/4\}\) and \(J_{\text{small}} := \{j \in J | d_j \leq b_j/4\}\). For the small jobs, we invoke a result by Elbassioni et al. [22] that yields a 16-approximation.

**Theorem 17** [22]. We are given an instance of Round-UFP with a set of jobs \(J'\) such that \(d_j \leq \frac{3}{4}b_j\) for each job \(j \in J'\). Then there is a polynomial time algorithm that computes a \(16\)-approximate solution to \(J'\).

Now consider the large jobs \(J_{\text{large}}\). For each job \(j \in J_{\text{large}}\), we define a rectangle \(R_j = (s_j, b_j - d_j) \times (t_j, b_j)\). Note that \(R_j\) corresponds to the rectangle for \(j\) in Round-SAP if we assign \(j\) the maximum possible height \(h_j\) (which is \(h_j := b_j - d_j\)). We say that a set of jobs \(J' \subseteq J_{\text{large}}\) is top-drawn (underneath the capacity profile), if their rectangles are pairwise non-overlapping, i.e., \(R_j \cap R_{j'} = \emptyset\) for any \(j, j' \in R_j\). If a set of jobs \(J'\) is top-drawn, then it clearly forms a feasible round in Round-UFP. However, not every feasible round of Round-UFP is top-drawn. Nevertheless, we look for a solution to Round-UFP in which the jobs in each round are top-drawn. The following lemma implies that this costs only a factor of 8 in our approximation ratio.

**Lemma 18** [11]. Let \(J' \subseteq J_{\text{large}}\) be a set of jobs packed in a feasible round for a given instance of Round-UFP. Then \(J'\) can be partitioned into at most 8 sets such that each of them is top-drawn.
Let $R_{\text{large}} := \{ R_i | i \in J_{\text{large}} \}$ denote the rectangles corresponding to the large jobs and let $\omega_{\text{large}}$ be their clique number, i.e., the size of the largest set $R' \subseteq R_{\text{large}}$ such that all rectangles in $R'$ pairwise overlap. As a consequence of Helly’s theorem (see [1] for details), for such a set of axis-parallel rectangles $R'$ there must be a point in which all rectangles in $R'$ overlap. Note that we need at least $\omega_{\text{large}}$ rounds since we seek a solution with only top-drawn jobs in each round.

We first reduce the original instance to the case where there are only $O(m^2)$ many distinct job demands. For each $i \in \{1, \ldots, m\}$, let $p_i$ denote the point $(i, c_i)$. We draw a horizontal and a vertical line segment passing through $p_i$ and lying completely under the capacity profile (see Figure 2(a)). This divides the region underneath the capacity profile into at most $m^2$ regions. Let $H$ denote the set of horizontal lines and $V$ denote the set of vertical lines drawn. Thus, the top edge of any rectangle corresponding to a large job must touch a line in $H$. Now consider any rectangle $R_i$ corresponding to a job $j$. Let $h \in H$ be the horizontal line segment lying just below the bottom edge of $R_i$. We increase the value of $d_i$ so that the the bottom edge of $R_i$ now touches the line segment $h$ (see Figure 2(b)). Since the rectangles were top-drawn, the clique number of this new set of large rectangles (denoted by $R'_{\text{large}}$) does not change. Also any feasible packing of $R'_{\text{large}}$ is a feasible packing of $R_{\text{large}}$.

Note that $R'_{\text{large}}$ contains at most $m^4$ distinct types of jobs: the endpoints $s_i$ and $t_i$ can be chosen in $\binom{m+1}{2} \leq m^2$ ways and the top and bottom edges of $R_i$ must coincide with two lines from $H$, which can be again chosen in $\binom{m}{2} \leq m^2$ ways. Let $U$ denote the number of types of job of the given instance and let $R'_{\text{large}} = \bigcup_{j=1}^{m} J_j$ be the decomposition of $R'_{\text{large}}$ into the $U$ distinct job types.

We now formulate the configuration LP for this instance. Let $C$ denote the set of all possible configurations of a round containing jobs from $R'_{\text{large}}$, drawn as top-drawn sets. For each $C \in \mathcal{C}$, we introduce a variable $x_C$, which stands for the number of rounds having configuration $C \in \mathcal{C}$. We write $J_k \preceq C$ if configuration $C$ contains a job from $J_k$ (note that $C$ can contain at most one job from $J_k$). Then the relaxed configuration LP and its dual (which contains a variable $y_k$ for each set $J_k$) are as follows.

Minimize $\sum_{C \in \mathcal{C}} x_C$ subject to $\sum_{C : J_k \preceq C} x_C \geq |J_k|$, $k = 1, \ldots, U$.

Maximize $\sum_{k=1}^{U} |J_k|y_k$ subject to $\sum_{k : J_k \preceq C} y_k \leq 1$, $C \in \mathcal{C}$.

$x_C \geq 0$, $C \in \mathcal{C}$,

$y_k \geq 0$, $k = 1, \ldots, U$.

The dual LP can be solved via the ellipsoid method with a suitable separation oracle. We interpret $y_k$ as the weight of each job in $J_k$. Given $(y_k)_{k \in \{1, \ldots, U\}}$, the separation problem asks whether there exists a configuration where jobs are drawn as top-drawn sets and the total weight of all the jobs in the configuration exceeds 1. For this, we invoke the following result of Bonsma et al. [11].
Theorem 19 ([11]). Given an instance of UFP with a set of jobs $J'$, the maximum-weight top-down subset of $J'$ can be computed in $O(nm^3)$ time.

Let $(x^*_C)_{C \subseteq C}$ be an optimal basic solution of the primal LP. By the rank lemma, there are at most at most $U$ configurations $C$ for which $x^*_C$ is non-zero. For each non-zero $x^*_C$, we introduce $\lceil x^*_C \rceil$ rounds with configuration $C$, thus creating at most $8 \cdot OPT_{UFP}$ rounds (due to Lemma 18). Now let $R''_{\text{large}} \subseteq R'_\text{large}$ be the large jobs that are yet to be packed and let $\omega''_{\text{large}}$ be their clique number (and note that $\omega''_{\text{large}} \leq 8 \cdot OPT_{UFP}$). In particular, a feasible solution to the configuration LP for the rectangles in $R''_{\text{large}}$ is to select one more round for each configuration $C$ with $x^*_C > 0$. Therefore, we conclude that $\omega''_{\text{large}} \leq U \leq m^4$ since there are at most $U$ configurations $C$ with $x^*_C > 0$ and for each point, each configuration contains at most one rectangle covering this point.

Our strategy is to invoke the following theorem on $R''_{\text{large}}$.

Theorem 20 ([13]). Given a set of rectangles with clique number $\omega$, in polynomial time, we can compute a coloring of the rectangles using $O(\omega \log \omega)$ colors such that no two rectangles of the same color intersect.

Thus, if $\omega''_{\text{large}} = O(\log m)$ then we obtain an $O(\log \log m)$-approximation as desired. However, it might be that $\omega''_{\text{large}}$ is larger. In that case, we partition $R''_{\text{large}}$ into $\omega''_{\text{large}}/\log m$ sets, such that each of them has a clique size of $O(\log m)$.

Lemma 21. There is a randomized polynomial time algorithm that w.h.p. computes a partition $R''_{\text{large}} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_{\omega''_{\text{large}}/\log m}$ such that for each such $\mathcal{R}_i$, the corresponding clique size is at most $O(\log m)$.

Proof. We split the rectangles $R''_{\text{large}}$ uniformly at random into $\omega''_{\text{large}}/\log m$ sets $\mathcal{R}_1, \ldots, \mathcal{R}_{\omega''_{\text{large}}/\log m}$. Thus the expected clique size in each set $\mathcal{R}_i$ at any point $p$ under the profile is at most $\log m$. Using the Chernoff bound, the probability that the clique size at $p$ is more than $8 \log m$ is at most $2^{-8 \log m} = 1/m^8$. As before, we draw the set of horizontal and vertical lines $H$ and $V$, respectively, under the capacity profile, dividing the region underneath the profile into at most $m^2$ regions. Clearly, the clique number must be the same at all points inside any such region. Thus the probability that there exists a point $p$ under the capacity profile where the clique size is more than $8 \log m$ is at most $m^2/m^8 \leq 1/m^6$. Hence using union bound, probability that clique size is more than $8 \log m$ at some point in some set $\mathcal{R}_i$ is at most $1/m^2$ (since $\omega''_{\text{large}} \leq m^4$).

We apply Theorem 20 to each set $\mathcal{R}_i$ separately and thus obtain a coloring with $O(\log m \log \log m)$ colors. Thus, for all sets $\mathcal{R}_i$ together we use at most $O(\omega''_{\text{large}} \log m)$ colors. We pack the jobs from each color class to a separate round for our solution to Round-UFP. This yields an $O(\log \log m)$-approximation, together with Theorem 17. Since $m \leq 2n-1$ after our preprocessing, our algorithms are also $O(\log \log n)$-approximation algorithms.

Theorem 22. There exists a randomized $O(\log \log \min\{n, m\})$-approximation algorithm for Round-UFP for general edge capacities.

In order to obtain an algorithm for Round-SAP, we invoke the following lemma due to [13] to each round of the computed solution to Round-UFP.

Lemma 23 ([23]). Let $J'$ be the set of jobs packed in a feasible round for a given instance of Round-UFP. Then in polynomial time we can partition $J'$ into $O(1)$ sets and compute a height $h_j$ for each job $j \in J'$ such that each set yields a feasible round of Round-SAP.
This yields a solution to ROUND-SAP with only \(O(OPT_{UFP} \log \log m) \leq O(OPT_{SAP} \log \log m)\) many rounds.

**Theorem 24.** There exists a randomized \(O(\log \log \min\{n, m\})\)-approximation algorithm for ROUND-SAP for general edge capacities.

### 5.1 An \(O(\log \log \frac{1}{\delta})\)-approximation algorithm with \((1 + \delta)\)-resource augmentation

We show that if we are allowed a resource augmentation of a factor of \(1 + \delta\) for some \(\delta > 0\), we can get an \(O(\log \log \frac{1}{\delta})\)-approximation for both ROUND-SAP and ROUND-UFP. Consider ROUND-UFP first.

**Lemma 25.** Let \(J^{(i)} := \{j \in J \mid b_j \in [1/\delta^i, 1/\delta^{i+1}]\}\). For packing jobs in \(J^{(i)}\), it can be assumed that the capacity of each edge lies in the range \([1/\delta^i, 2/\delta^{i+1}]\).

Hence using the following theorem, we get a \(O(\log \log \frac{1}{\delta})\)-approximate solutions for each \(J^{(i)}\), which in particular uses at most \(O(OPT_{UFP} \log \log \frac{1}{\delta})\) rounds.

**Theorem 26** ([35]). There is a polynomial time \(O(\log \log \frac{c_{\max}}{c_{\min}})\)-approximation algorithm for ROUND-UFP.

Next we argue that we can combine the rounds computed for the sets \(J^{(i)}\). More precisely, we show that if we take one round from each set \(J^{(0)}, J^{(2)}, J^{(4)}, \ldots\) and form their union, then they form a feasible round for the given instance under \((1 + \delta)\)-resource augmentation. The same holds if we take one round from each set \(J^{(1)}, J^{(3)}, J^{(5)}, \ldots\).

**Lemma 27.** Take one computed round for each set \(J^{(2k)}\) with \(k \in \mathbb{N}\) or one computed round from each set \(J^{(2k+1)}\) with \(k \in \mathbb{N}\), and let \(J'\) be their union. Then \(J'\) is a feasible round for the given instance of ROUND-UFP under \((1 + \delta)\)-resource augmentation.

Thus, due to Lemma 27 we obtain a solution with at most \(O(OPT_{UFP} \log \log \frac{1}{\delta})\) rounds for the overall instance. As earlier, we take the given ROUND-UFP solution and apply Lemma 23 to it, which yields a solution to ROUND-SAP with at most \(O(OPT_{SAP} \log \log \frac{1}{\delta})\) rounds.

**Theorem 28.** There exists an \(O(\log \log \frac{1}{\delta})\)-approximation algorithm for ROUND-SAP and ROUND-UFP for general edge capacities and \((1 + \delta)\)-resource augmentation.

### 6 Algorithms for the no-bottleneck-assumption

In this section, we present a \((16 + \varepsilon)\)-approximation algorithm for ROUND-SAP and a 12-approximation for ROUND-UFP, both under the no-bottleneck-assumption (NBA).

#### 6.1 Algorithm for ROUND-SAP

For our algorithm for ROUND-SAP under NBA, we first scale down all job demands and edge capacities so that \(c_{\min} := \min_{e \in E} c_e = 1\). Since the NBA holds, this implies that \(d_i \leq 1\) for each job \(i \in J\). Then, we scale down all edge capacities to the nearest power of 2, i.e., for each edge \(e \in E\) we define a new rounded capacity \(c'_e := 2^{\lceil \log c_e \rceil}\). We introduce horizontal lines whose \(y\)-coordinates are integral powers of 2 (in contrast to the lines with uniform spacing that we used in Section 4). Formally, we define a set of lines \(L := \{\ell_{2k} \mid k \in \mathbb{N}\}\). Let
OPT′_SAP denote the optimal solution for the rounded down capacities (c'_e)_{e \in E} under the additional constraint that there must be no job whose rectangle intersects a line in \( \mathcal{L} \).

We now show that given a valid Round-SAP packing \( \mathcal{P} \) of a set of jobs \( J' \) for the edge capacities \( (c_e)_{e \in E} \), there exists a valid packing of \( J' \) into 4 rounds \( \mathcal{R}^1, \mathcal{R}^2, \mathcal{R}^3, \mathcal{R}^4 \), under profile \( (c'_e)_{e \in E} \) such that no job is sliced by a line in \( \mathcal{L} \) (see Figure 5 in Appendix B.9). Let \( J^{(i)} \) be the set of jobs with bottleneck capacity in \([2^i, 2^{i+1})\). For \( i \geq 1 \), we pack \( J^{(i)} \) as follows:

1. Place each job \( j \) lying completely below \( \ell_{2i} \) into \( \mathcal{R}^1 \) at height \( h'_i := h_j \). (Note that here \( h_j \) denotes the height of the bottom edge of \( j \) in \( \mathcal{P} \).)
2. Place each job \( j \) lying completely between \( \ell_{2i+1} \) and \( \ell_{2i} \) into \( \mathcal{R}^2 \) with \( h'_i := h_j - 2^{i-1} \).
3. Place each job \( j \) lying completely between \( \ell_{2i-1} \) and \( \ell_{2i+1} \) into \( \mathcal{R}^3 \) with \( h'_i := h_j - 2^i \).
4. Place each job \( j \) sliced by \( \ell_{2i} \) into \( \mathcal{R}^4 \) with \( h'_i := 2^i - d_i \); and place each job \( j \) sliced by \( \ell_{2i-1} \) into \( \mathcal{R}^4 \) with \( h'_i := 3 \cdot 2^{i-2} - d_i \) for \( i \geq 2 \) and \( h'_i := 1 - d_i \) for \( i = 1 \).

Finally, we pack \( J^{(0)} \) as follows:
1. Place each job \( j \) lying completely below \( \ell_1 \) into \( \mathcal{R}^1 \) with \( h'_1 := h_j \).
2. Place each job \( j \) lying completely between \( \ell_1 \) and \( \ell_2 \) into \( \mathcal{R}^2 \) with \( h'_1 := h_j - 1 \).
3. Place each job \( j \) sliced by \( \ell_1 \) into \( \mathcal{R}^3 \) with \( h'_1 := 1 - d_i \).

It can be checked easily that the above algorithm yields a feasible packing of \( J' \) in which no job is sliced by a line in \( \mathcal{L} \). Thus by losing at most a factor of 4 in our approximation ratio, we can convert any valid Round-SAP packing to a packing under the profile \( (c'_e)_{e \in E} \) that satisfies the latter constraint as well.

\textbf{Lemma 29.} We have that \( OPT'_SAP \leq 4 \cdot OPT_{SAP} \)

We shall now obtain a valid packing of \( J \) for the edge capacities \( (c'_e)_{e \in E} \). Let \( c'_\text{max} := \max_{e \in E} c'_e \) and for each \( i \in \{0, 1, \ldots, \log c'_\text{max} \} \) let \( J^{(i)} \subseteq J \) denote the set of jobs with bottleneck capacity \( 2^i \) according to \( (c'_e)_{e \in E} \). For each set \( J^{(i)} \) we create a new (artificial) instance with uniform edge capacities: in the instance for \( J^{(0)} \) all edges have capacity 1, and for each \( i \in \{1, 2, \ldots, \log c'_\text{max} \} \) in the instance for \( J^{(i)} \) all edges have capacity \( 2^{i-1} \). For each \( i \in \{0, 1, 2, \ldots, \log c'_\text{max} \} \) denote by \( OPT^{(i)} \) the number of rounds needed in the optimal solution to the instance for \( J^{(i)} \). Since in the solution \( OPT'_SAP \) no rectangle is intersected by a line in \( \mathcal{L} \), for each set \( J^{(i)} \) we can easily rearrange the jobs in \( J^{(i)} \) in \( OPT'_SAP \) such that we obtain a solution for \( J^{(i)} \) with at most \( 2 \cdot OPT'_SAP \) rounds.

\textbf{Lemma 30.} For each \( i \in \{0, 1, 2, \ldots, \log c'_\text{max} \} \) it holds that \( OPT^{(i)} \leq 2 \cdot OPT'_SAP \).

For each set \( J^{(i)} \) we invoke our asymptotic \((2 + \varepsilon)\)-approximation algorithm for Round-SAP for uniform edge capacities (see Section 4) and obtain a solution \( \Gamma^{(i)} \) which hence uses \( \Gamma^{(i)} \leq (2 + \varepsilon) \cdot OPT^{(i)} + O(1) \leq (2 + \varepsilon) \cdot OPT_{SAP} + O(1) \) rounds. Finally, we combine the solutions \( \Gamma^{(i)} \) for all \( i \in \{0, 1, 2, \ldots, \log c'_\text{max} \} \) to one global solution of \( J \). The key insight for this is that if some edge \( e \) has a (rounded) capacity of \( c'_e = 2^k \), then in one round we can place the solution of one round of each of the solutions \( \Gamma^{(0)}, \Gamma^{(1)}, \ldots, \Gamma^{(k-1)} \).

\textbf{Lemma 31.} Given a solution \( \Gamma^{(i)} \) for the set \( J^{(i)} \) for each \( i \in \{0, 1, 2, \ldots, \log c'_\text{max} \} \), in polynomial time we can construct a solution for \( J \) and the edge capacities \( (c'_e)_{e \in E} \) with at most \( \max^{(i)}(\Gamma^{(i)}) \) many rounds.

Using the asymptotic \((2 + \varepsilon)\)- (resp. absolute 3-) approximation for Round-SAP under uniform edge capacities (from Theorems 14 and 15), this yields the following theorem.

\textbf{Theorem 32.} For any \( \varepsilon > 0 \), there exists an asymptotic \((16 + \varepsilon)\)-approximation and an absolute \(2\varepsilon\)-approximation algorithm for Round-SAP under the NBA.
6.2 Algorithm for Round-UFP

In this section, we present a 12-approximation for Round-UFP under NBA. In Round-UFP, it is not clear how to bootstrap the algorithm for the uniform case as we did for Round-SAP, since in the optimal solution it might not be possible to draw the jobs as non-overlapping rectangles. Instead, our algorithm refines combinatorial properties from [22] to obtain an improved approximation ratio. We will use the following algorithm due to [46] as a subroutine.

\textbf{Theorem 33} (46). \textit{Given an instance of Round-UFP in which all job demands are equal to 1 and all edge capacities are integral. Then $OPT_{UFP} = r$ and in polynomial time we can compute a packing into $OPT_{UFP}$ many rounds.}

Via scaling, we assume that $c_{\min} = 1$ and the demand of each job is at most 1. Let $J_{\text{large}} := \{i \in J \mid d_i > 1/2\}$ and $J_{\text{small}} := J \setminus J_{\text{large}}$. For each job $i \in J_{\text{small}}$ we round up its demand to the next larger power of $1/2$, i.e., we define its rounded demand $d'_i := 2^{\lceil \log d_i \rceil}$. For each $i \in \mathbb{N}$, let $J^{(i)}$ denote the set of jobs whose demands after rounding equal $\frac{1}{2^i}$, i.e., $J^{(i)} = \{i \in J \mid d'_i = \frac{1}{2^i}\}$. For each edge $e$ and each $i \in \mathbb{N}$, we count how many jobs in $J^{(i)}$ use $e$ and we define $n_{e,i} := |\{i \in J^{(i)} \mid e \in P_i\}|$. We partition each set $J^{(i)}$ into the sets $J^{(i)}_0 = \{i \in J^{(i)} \mid 2e \not\in P_i \}$ and $J^{(i)}_1 = J^{(i)} \setminus J^{(i)}_0$. Let $n_{e,i}^0 := |\{i \in J^{(i)}_0 \mid e \in P_i\}|$ and $n_{e,i}^1 := |\{i \in J^{(i)}_1 \mid e \in P_i\}|$. Clearly, $n_{e,i}^0 < 4r$ for each edge $e$ and each $i$.

First, we compute a packing for $J_{\text{small}}$. For the (small) jobs in the sets $J^{(i)}$, we use a packing method that ensures that inside each round we have at most one job from each set $J^{(i)}$. Since $n_{e,i}^0 < 4r$ for each edge $e$ and each $i$, this needs at most $4r$ rounds. Moreover, by a geometric sum argument this yields a valid packing inside each round (the job demands sum up to at most $1 = c_{\min}$). For the jobs in $J^{(i)}_1$, we partition the available capacity inside each round among the sets $J^{(i)}_1$ and then invoke the algorithm due to Theorem 33 for each set $J^{(i)}_1$ separately, which also needs at most $4r$ rounds, and thus at most $8r$ rounds in total.

\textbf{Lemma 34}. \textit{For the jobs in $J_{\text{small}}$ we can find a packing into at most $8r$ rounds.}

\textbf{Proof}. For packing jobs in $\bigcup_i J^{(i)}_1$, we introduce a set $T$ of $4r$ rounds. For each $i$, we consider the jobs in $J^{(i)}_1$ in non-decreasing order of their left endpoints and assign a job to the first round in $T$ where it does not overlap with any job from $J^{(i)}_1$ that have been assigned till now. Since $n_{e,i}^0 \leq 4r$, such an assignment is always possible. Thus over any edge $e$ inside any round of $T$, at most 1 job from each set $J^{(i)}_1$ can be present. Thus the load on edge $e$ is at most $\sum_i \frac{1}{2^i} \leq 1 \leq c_e$, and hence this is a valid packing.

For packing the jobs in $\bigcup_i J^{(i)}_1$, we introduce a set $S$ of $4r$ rounds. Consider a set $J^{(i)}_1$. Inside each of these rounds, to each edge $e$ we assign a capacity of $\frac{1}{2^i} \cdot \lfloor \frac{n_{e,i}}{2^i} \rfloor$ to $J^{(i)}_1$. The resulting congestion of any edge having non-zero capacity is $\frac{\sum_i \lfloor n_{e,i}/2^i \rfloor}{\sum_i 2^i} \leq \frac{n_{e,i}}{2^i} \leq 2r \leq 4r$. Also since each job in $J^{(i)}_1$ has demand equal to $\frac{1}{2^i}$, the assigned capacity of each edge is an integral multiple of the demand. Thus using Theorem 33 jobs in $J^{(i)}_1$ can be packed into at most $4r$ rounds with these capacities. When we do this procedure for each set $J^{(i)}_1$, we obtain that the total load on each edge $e$ inside any round is at most $\sum_i \frac{1}{2^i} \lfloor n_{e,i}/2^i \rfloor \leq \sum_i \frac{1}{2^i} \frac{n_{e,i}}{2^i} \leq \frac{1}{r} \sum_i n_{e,i} \leq c_e$. Thus this is a valid packing.

Therefore, we pack all jobs in $J_{\text{small}}$ into $8r$ rounds.

For the jobs in $J_{\text{large}}$ we round up their demands to 1 and the edge capacities to the respective nearest integer. This increases the congestion $r$ by at most a factor of 4. Then we invoke Theorem 33.
Lemma 35. The jobs in $J_{\text{large}}$ can be packed into $4r$ rounds.

Overall, this yields a 12-approximation algorithm for Round-UFP under the NBA.

Theorem 36. There is a polynomial time 12-approximation algorithm for Round-UFP under the NBA.

7 Algorithms for Round-Tree

Extending the results for Round-UFP, using results on path coloring [24] and multicommodity demand flow [15], we obtain the following results for Round-Tree (see App. C).

Theorem 37. For Round-Tree, there exists a polynomial-time asymptotic (resp. absolute) 5.1- (resp. 5.5-) approximation algorithm for uniform edge capacities and an asymptotic (resp. absolute) 49- (resp. 55-) approximation algorithm for the general case under the NBA.

References

1. Proving fractional helly’s theorem for boxes and rectangles. stackexchange. URL: https://math.stackexchange.com/questions/3451613/proving-fractional-hellys-theorem-for-boxes-and-rectangles.
2. Anna Adamaszek, Sariel Har-Peled, and Andreas Wiese. Approximation schemes for independent set and sparse subsets of polygons. *J. ACM*, 66(4):29:1–29:40, 2019.
3. Udo Adamy and Thomas Erlebach. Online coloring of intervals with bandwidth. In *WAOA*, pages 1–12. Springer, 2003.
4. Aris Anagnostopoulos, Fabrizio Grandoni, Stefano Leonardi, and Andreas Wiese. A mazing 2+epsilon approximation for unsplittable flow on a path. In *SODA*, pages 26–41, 2014.
5. Matthew Andrews and Lisa Zhang. Bounds on fiber minimization in optical networks with fixed fiber capacity. In *Proceedings IEEE 24th Annual Joint Conference of the IEEE Computer and Communications Societies.*, volume 1, pages 409–419. IEEE, 2005.
6. Yossi Azar, Amos Fiat, Meital Levy, and NS Narayanaswamy. An improved algorithm for online coloring of intervals with bandwidth. *Theoretical Computer Science*, 363(1):18–27, 2006.
7. Nikhil Bansal, José R Correa, Claire Kenyon, and Maxim Sviridenko. Bin packing in multiple dimensions: inapproximability results and approximation schemes. *Mathematics of operations research*, 31(1):31–49, 2006.
8. Nikhil Bansal, Zachary Friggstad, Rohit Khandekar, and Mohammad R Salavatipour. A logarithmic approximation for unsplittable flow on line graphs. *ACM Transactions on Algorithms (TALG)*, 10(1):1–15, 2014.
9. Nikhil Bansal and Arindam Khan. Improved approximation algorithm for two-dimensional bin packing. In *SODA*, pages 13–25, 2014.
10. Jatin Batra, Naveen Garg, Amit Kumar, Tobias Mömke, and Andreas Wiese. New approximation schemes for unsplittable flow on a path. In *SODA*, pages 47–58, 2015.
11. Paul S. Bonsma, Jens Schulz, and Andreas Wiese. A constant factor approximation algorithm for unsplittable flow on paths. In *FOCS*, pages 47–56, 2011.
12. Adam L. Buchsbaum, Howard J. Karloff, Claire Kenyon, Nick Reingold, and Mikkel Thorup. OPT versus LOAD in dynamic storage allocation. In *STOC*, pages 556–564, 2003.
13. Parinya Chalermsook and Bartłomiej Walczak. Coloring and maximum weight independent set of rectangles. In *SODA*, pages 860–868, 2021.
14. Chandra Chekuri and Sanjeev Khanna. A polynomial time approximation scheme for the multiple knapsack problem. *SIAM Journal on Computing*, 35(3):713–728, 2005.
15. Chandra Chekuri, Marcelo Mydlarz, and F. Bruce Shepherd. Multicommodity demand flow in a tree and packing integer programs. *ACM Trans. Algorithms*, 3(3):27, 2007.
Miroslav Chlebík and Janka Chlebíková. Complexity of approximating bounded variants of optimization problems. *Theoretical Computer Science*, 354(3):320–338, 2006.

Miroslav Chlebík and Janka Chlebíková. Hardness of approximation for orthogonal rectangle packing and covering problems. *Journal of Discrete Algorithms*, 7(3):291–305, 2009.

Henrik I. Christensen, Arindam Khan, Sebastian Pokutta, and Prasad Tetali. Approximation and online algorithms for multidimensional bin packing: A survey. *Computer Science Review*, 24:63–79, 2017.

Julia Chuzhoy and Shi Li. A polylogarithmic approximation algorithm for edge-disjoint paths with congestion 2. *Journal of the ACM*, 63(5):1–51, 2016.

W Fernandez De La Vega and George S. Lueker. Bin packing can be solved within $1 + \varepsilon$ in linear time. *Combinatorica*, 1(4):349–355, 1981.

Max A Deppert, Klaus Jansen, Arindam Khan, Malin Rau, and Malte Tutas. Peak demand minimization via sliced strip packing. In *APPROX/RANDOM*, volume 207, pages 21:1–21:24, 2021.

Khaled M. Elbassioni, Naveen Garg, Divya Gupta, Amit Kumar, Vishal Narula, and Arindam Pal. Approximation algorithms for the unsplittable flow problem on paths and trees. In *FSTTCS*, pages 267–275, 2012.

Leah Epstein, Thomas Erlebach, and Asaf Levin. Online capacitated interval coloring. *SIAM Journal on Discrete Mathematics*, 23(2):822–841, 2009.

Thomas Erlebach and Klaus Jansen. The complexity of path coloring and call scheduling. *Theoretical Computer Science*, 255(1-2):33–50, 2001.

Waldo Gálvez, Fabrizio Grandoni, Afrouz Jabal Ameli, Klaus Jansen, Arindam Khan, and Malin Rau. A tight $(3/2+\varepsilon)$ approximation for skewed strip packing. In *APPROX/RANDOM*, pages 44:1–44:18, 2020.

Waldo Gálvez, Fabrizio Grandoni, Sandy Heydrich, Salvatore Ingala, Arindam Khan, and Andreas Wiese. Approximating geometric knapsack via l-packings. In *FOCS*, pages 260–271, 2017.

Waldo Gálvez, Fabrizio Grandoni, Salvatore Ingala, and Arindam Khan. Improved pseudo-polynomial-time approximation for strip packing. In *FSTTCS*, pages 9:1–9:14, 2016.

Waldo Gálvez, Fabrizio Grandoni, Arindam Khan, Diego Ramirez-Romero, and Andreas Wiese. Improved approximation algorithms for 2-dimensional knapsack: Packing into multiple l-shapes, spirals, and more. In *SoCG*, volume 189, pages 39:1–39:17, 2021.

Waldo Gálvez, Arindam Khan, Mathieu Mari, Tobias Mömke, Madhusudhan Reddy Pittu, and Andreas Wiese. A $(2+\varepsilon)$-approximation algorithm for maximum independent set of rectangles. *CoRR*, abs/2106.00623, 2021. URL: https://arxiv.org/abs/2106.00623.

Jordan Gergov. Algorithms for compile-time memory optimization. In *SODA*, pages 907–908, 1999.

Fabrizio Grandoni, Tobias Mömke, and Andreas Wiese. A PTAS for unsplittable flow on a path. 2022. personal communication.

Fabrizio Grandoni, Tobias Mömke, and Andreas Wiese. Unsplittable flow on a path: The game! In *SODA*, page To appear, 2022.

Fabrizio Grandoni, Tobias Mömke, Andreas Wiese, and Hang Zhou. To augment or not to augment: Solving unsplittable flow on a path by creating slack. In *SODA*, pages 2411–2422, 2017.

Fabrizio Grandoni, Tobias Mömke, Andreas Wiese, and Hang Zhou. A $(5/3 + \varepsilon)$-approximation for unsplittable flow on a path: placing small tasks into boxes. In *STOC*, pages 607–619, 2018.

Hamidreza Jahanjou, Erez Kantor, and Rajmohan Rajaraman. Improved algorithms for scheduling unsplittable flows on paths. In *ISAAC*, pages 49:1–49:12, 2017.

Klaus Jansen and Guochuan Zhang. On rectangle packing: maximizing benefits. In *SODA*, pages 204–213, 2004.

Arindam Khan. *Approximation algorithms for multidimensional bin packing*. PhD thesis, Georgia Institute of Technology, 2015.
Approximation Algorithms

In this subsection, we define notions related to approximation algorithms.

Definition 38 (Approximation Guarantee). For a minimization problem \( \Pi \), an algorithm \( A \) has approximation guarantee of \( \alpha \) (\( \alpha > 1 \)), if \( A(I) \leq \alpha \cdot \text{OPT}(I) \) for all input instance \( I \) of \( \Pi \). For a maximization problem \( \Pi' \), an algorithm \( A \) has approximation guarantee of \( \alpha \) (\( \alpha > 1 \)), if \( \text{OPT}(I) \leq \alpha \cdot A(I) \) for all input instance \( I \) of \( \Pi' \).

This is also known as absolute approximation guarantee. There is another notion of approximation called asymptotic approximation which we define next.

Definition 39 (Asymptotic Approximation Guarantee). For a minimization problem \( \Pi \), an algorithm \( A \) has asymptotic approximation guarantee of \( \alpha \) (\( \alpha > 1 \)), if \( A(I) \leq \alpha \cdot \text{OPT}(I) + o(\text{OPT}(I)) \) for all input instance \( I \) of \( \Pi \).

Definition 40 (Polynomial Time Approximation Scheme (PTAS)). A minimization problem \( \Pi \) admits PTAS if for every constant \( \varepsilon > 0 \), there exists a \((1 + \varepsilon)-approximation\) algorithm with running time \( O(n^{f(1/\varepsilon)}) \), for any function \( f \) that depends only on \( \varepsilon \).

If the running time of a PTAS is \( O(f(1/\varepsilon) n^c) \) for some function \( f \) and a constant \( c \) that is independent of \( 1/\varepsilon \), we call it Efficient PTAS (EPTAS). If the running time of a PTAS is polynomial in both \( n \) and \( 1/\varepsilon \), we call it Fully PTAS (FPTAS). Asymptotic analogue of PTAS, EPTAS, FPTAS are known as APTAS, AEPTAS, AFPTAS, respectively.
B. Omitted Proofs

B.1 Proof of Lemma 6

We first state some inequalities on the job dimensions.

Lemma 41. The following inequalities hold:

(i) $d_a > 999\gamma$, $\forall a \in A$.
(ii) $d_b > 1001\gamma$, $\forall b \in B$.
(iii) $d_v' > 1000\gamma$, $\forall a' \in A'$.
(iv) $d_v > 997\gamma$, $\forall b' \in B'$.

Proof. (i) Follows from definition of job dimensions.

(ii) Let $b \in B$. Then $d_b \geq 999\gamma + 4 \cdot (\rho^4 - q\rho^3 - q\rho^2 - q\rho + 8)$, as $i, j, k \leq q$. Thus $d_b - 1001\gamma \geq 4 \cdot (32^4q^4 - q \cdot 32^2q^2 - q \cdot 32q + 8) - 2 \cdot (32^2q^4 + 15) \geq 4 \cdot (32^2q^4 - 32^2q^2 - 32q + 8) + 2 > 0$ (using the fact that $\rho = 32q$ and $\gamma = \rho^4 + 15$).

(iii) Let $a' \in A'$. Then $d_{a'} \geq 1001\gamma - 4 \cdot (q^3 + 4)$. Thus $d_{a'} = 1000\gamma \geq 28.32^2q^4 - 1 > 0$.

(iv) Let $b' \in B'$. Then $d_{b'} \geq 1001\gamma + 4 \cdot (\rho^4 - \rho^3 - \rho^2 - \rho + 8)$. Thus $d_{b'} = 997\gamma > (4\gamma - 4 \cdot \rho^4) + 4 \cdot (\rho^3 + \rho^2 + \rho - 8) > 0$.

Thus the demand of any job in $A \cup A' \cup B \cup B'$ is at least $997\gamma$. Also recall that the demand of any dummy job is $2997\gamma$. Suppose a round $R$ contains a dummy job. Consider the leftmost edge $e_1$. If $R$ contains at least 2 jobs from $A \cup B$, the total sum of demands of the jobs inside $R$ over $e_1$ would be at least $2997\gamma + 2 \cdot 997\gamma > 4000\gamma = c^*$, a contradiction. Similarly $R$ can contain at most 1 job from $A' \cup B'$. Thus any round containing a dummy job can contain at most 1 job each from $A \cup B$ and $A' \cup B'$.

B.2 Proof of Lemma 3

From Lemma 6, a round containing a dummy job can contain at most 3 jobs. Suppose a round $R$ does not contain a dummy job. If it has at least 5 jobs from $A \cup B$, the load on the leftmost edge $e_1$ inside round $R$ would be at least $5 \cdot 997\gamma > 4000\gamma = c^*$, a contradiction. Similarly since every job in $A' \cup B'$ passes through the rightmost edge $e_m$, there can be at most 4 such jobs inside $R$. Thus $R$ can contain at most 8 jobs.

B.3 Proof of Lemma 4

The if part follows directly since the 8 jobs corresponding to a triplet $\tau_1 = (x_i, y_j, z_k) \in T$ (4 from $A \cup B$ and their peers) can be packed together inside one round as shown in Figure 3.

![Figure 3](image_url) A nice round corresponding to the triplet $\tau_1 = (x_i, y_j, z_k)$

We begin the proof of the only if part by stating some properties of the job dimensions.
Lemma 42. The following conditions hold:
(i) $w_a + d_a = 20999\gamma$, $\forall a \in A$.
(ii) $w_{a'} + d_{a'} = 21001\gamma$, $\forall a' \in A'$.
(iii) $w_b + d_b = 20000\gamma$, $\forall b \in B$.
(iv) $w_{b'} + d_{b'} = 22000\gamma$, $\forall b' \in B'$.

Proof. Can be verified easily.

Lemma 43. The following statements hold:
1. The demands of four jobs in $A \cup B$ (or $A' \cup B'$) sum to $4000\gamma$ iff they correspond to a triplet.
2. The widths of two jobs in $A \cup A' \cup B \cup B'$ sum to $4000\gamma$ iff they are peers.

Proof. (i) Follows directly from Lemma 2.

(ii) The if part follows from the definition of the job dimensions. For the only if part, note that all the $5q$ integers in $V$ are distinct. Also $x'_i < y'_j < z'_k < \tau'_l$, $\forall i,j,k,l \leq q$ and $l \leq 2q$. It follows that all job widths are distinct. Thus for any $j \in A \cup A' \cup B \cup B'$ having width $w_j$, the unique job in $A \cup A' \cup B \cup B'$ having width $4000\gamma - w_j$ is its peer. Hence the claim follows.

Lemma 44. $s_{b'} < t_a$, $\forall a \in A$, $b' \in B'$.

Proof. Let $a \in A$ and $b' \in B'$. Then $t_a \geq 20000\gamma - 4 \cdot (q\rho^3 + 4) \geq 20000\gamma - 4\gamma = 19996\gamma$, since $\gamma = \rho^3 + 15 \geq q\rho^3 + 4$. Also clearly $s_{b'} < 19001\gamma < 19996\gamma$. Thus we have $s_{b'} < t_a$. 

Now let $R$ be a nice round. Since any ROUND-SAP packing is also a valid ROUND-UFP packing, it suffices to prove the required result for ROUND-UFP packings. From the proof of Lemma 3, it follows that $R$ must contain 4 jobs each from $A \cup B$ and $A' \cup B'$.

We begin by showing that $R$ can contain at most 1 job from $B$. Suppose at least 2 jobs from $B$ are present. From Lemma 44(i) and (ii), the sum of demands of all the 4 jobs from $A \cup B$ is then $> 2 \cdot 1001\gamma + 2 \cdot 999\gamma = 4000\gamma = c^*$, a contradiction. Now we show that at least one job from $B$ must be present. Suppose not, then there are 4 jobs from $A$. Lemma 44 then implies that no job from $B'$ can be present. Thus there must be 4 jobs from $A'$. But from Lemma 44(iii), the sum of demands of these jobs from $A'$ would be $> 4 \cdot 1000\gamma = 4000\gamma = c^*$, a contradiction. Hence, $R$ must have exactly 1 job from $B$ and thus 3 jobs from $A$.

As shown above, $R$ cannot contain 4 jobs from $A'$ and thus must have at least 1 job from $B'$. We now show that exactly 1 job from $B'$ must be present. Lemma 44 implies any job in $B'$ must share at least one edge with any job in $A$. Let $e$ be such a common edge. Therefore, if at least 2 jobs from $B'$ are present, sum of demands of all jobs inside $R$ over $e$ would be $> 3 \cdot 999\gamma + 2 \cdot 997\gamma > 4000\gamma = c^*$ (from Lemma 44), a contradiction. Thus $R$ has exactly 1 job from $B'$ and thus 3 jobs from $A'$.

Now consider the leftmost edge $e_1$. Let the 4 jobs from $A \cup B$ lying over $e_1$, from top to bottom, be $i_1, i_2, i_3, i_4$. Similarly let the 4 jobs from $A' \cup B'$ lying over the rightmost edge $e_m$ be $i_5, i_6, i_7, i_8$ from top to bottom. Then the following inequalities must hold:

$$\sum_{i=1}^{4} d_{i} \leq 4000\gamma, \quad \sum_{i=5}^{8} d_{i} \leq 4000\gamma, \quad \sum_{i=1}^{8} w_{i} \leq 4 \cdot 40000\gamma = 160000\gamma \quad (1)$$

Adding the three inequalities, we get $\sum_{i=1}^{8} (d_{i} + w_{i}) \leq 168000\gamma$. Also Lemma 42 implies that $\sum_{i=1}^{8} (d_{i} + w_{i}) = 3 \cdot 20999\gamma + 3 \cdot 21001\gamma + 1 \cdot 20000\gamma + 1 \cdot 22000\gamma = 168000\gamma$. Thus (1)
must be satisfied with equality. Lemma [13] (i) then implies that the four jobs from $A \cup B$ must correspond to a triplet. Also since equality holds in (1), Lemma [13] (ii) implies that their corresponding peers must be present. Hence the result follows.

### B.4 Proof of Theorem [8]

Let $OPT$ denote either $OPT_{SAP}$ or $OPT_{UFP}$. From Lemma [5] $OPT \leq 5q - 3\beta(q)$ whenever $OPT_{3DM} \geq \beta(q)$, and from Lemma [2] $OPT \geq (5q - 3\beta(q)) + \frac{1}{3}(\beta(q) - \alpha(q))$ whenever $OPT_{3DM} \leq \alpha(q)$. Let $\delta_0 = \frac{1}{2} \frac{\beta(q) - \alpha(q)}{5q - 3\beta(q)}$. Suppose there exists a polynomial time algorithm $A$ for ROUND-SAP or ROUND-UFP and a constant $C$ such that for instances with $OPT > C$, $A$ returns a packing using at most $(1 + \delta_0)OPT$ rounds. Then for any corresponding instance $K$ of 2-B-3-DM, we could distinguish whether $OPT_{3DM} < \alpha(q)$ or $OPT_{3DM} \geq \beta(q)$ in polynomial time, contradicting Theorem [1]. For $\alpha(q) = [0.9690082645q]$ and $\beta(q) = [0.979338843q]$, a simple calculation will show that $\delta_0 > 1/1398$.

### B.5 Proof of Lemma [11]

![Figure 4 A normalized packing](image-url)

We sort the jobs in non-increasing order of their $h_i + d_i$ values and push them up until they either touch the capacity profile or the bottom of some other job.

### B.6 Proof of Lemma [13]

We first consider ROUND-SAP. First we guess the value of $OPT_{SAP}$ as $\kappa$, where $\kappa \in \{1, 2, \ldots, n\}$. Each DP cell consists of the following ($\kappa + 2$) attributes.

- an edge $e \in E$,
- a function $f_e$ that assigns a round to each job passing through $e$,
- $\kappa$ functions $g^1_e, g^2_e, \ldots, g^\kappa_e$, one for each round, where $g^i_e$ assigns the vertical location $h_j$ to each job $j$ assigned to the $i^{th}$ round by $f_e$, $\forall i \in \{1, 2, \ldots, \kappa\}$.

For each edge $e_k$, let $J^e_k := \{j \in J' | s_j < k\}$, i.e., the set of jobs that either pass through or end before $e_k$. We define $DP(e, f_e, g^1_e, g^2_e, \ldots, g^\kappa_e) = 1$ if and only if there exists a valid packing of $J^e$ using $\kappa$ rounds such that the positions of all jobs in $J^e$ are exactly the same as those assigned by the functions $f_e, g^1_e, g^2_e, \ldots, g^\kappa_e$. Thus the recurrence for the DP is given by $DP(e, f_e, g^1_e, g^2_e, \ldots, g^\kappa_e) = 1$ if there exist functions $f_{e_{j-1}}, g^1_{e_{j-1}}, g^2_{e_{j-1}}, \ldots, g^\kappa_{e_{j-1}}$ such that $DP(e_{j-1}, f_{e_{j-1}}, g^1_{e_{j-1}}, g^2_{e_{j-1}}, \ldots, g^\kappa_{e_{j-1}}) = 1$ and $(f_{e_{j}}, g^1_{e_{j}}, g^2_{e_{j}}, \ldots, g^\kappa_{e_{j}})$ and $(f_{e_{j-1}}, g^1_{e_{j-1}}, g^2_{e_{j-1}}, \ldots, g^\kappa_{e_{j-1}})$ are consistent with each other, $\forall j \in \{2, \ldots, m\}$. Here by consistency, we mean that any job $j$ that passes through both $e_j$ and $e_{j-1}$ must be assigned the same round (say the $i^{th}$ round) by $f_{e_j}$ and $f_{e_{j-1}}$ and the same value of $h_j$ by $g^i_{e_j}$ and $g^i_{e_{j-1}}$.

Finally, we bound the running time. Since each job can be assigned to any of the $\kappa$ rounds and have at most $|H'|$ distinct positions inside a round, the number of DP cells
per edge is bounded by \((\kappa|H'|)^\omega\). Also determining each DP entry requires visiting all DP cells corresponding to the edge on the immediate left of the current edge. Thus the time required to determine whether \(J\) can be packed using \(\kappa\) rounds is bounded by 
\[
m \cdot (\kappa|H'|)^\omega \cdot (\kappa|H'|)^\omega \leq (n|H'|)^{O(\omega)}.
\]

For \(\text{ROUND-UFPP}\), the positions of the jobs inside a round does not matter. Thus we simply have a 2-cell DP, consisting of an edge \(e\) and a function \(f_e\) that allocates a round to each job passing through \(e\). The recurrence is given by 
\[
\text{DP}(e_j, f_{e_j}) = 1 \text{ if there exists } f_{e_{j-1}} \text{ such that } \text{DP}(e_{j-1}, f_{e_{j-1}}) = 1 \text{ and } f_{e_j} \text{ and } f_{e_{j-1}} \text{ are consistent with each other.}
\]
Clearly the running time is bounded by \(n^{O(\omega)}\).

### B.7 Proof of Lemma 25

Consider any edge \(e \in E\) inside any round \(R\). Let \(e_L\) and \(e_R\) be the first edges on the left and right of \(e\) respectively that have capacity at most \(1/\delta^{i+1}\). Observe that any job in \(J^{(i)}\) must pass through at least one edge having capacity at most \(1/\delta^{i+1}\). Thus the load on edge \(e\) inside round \(R\) is at most the sum of the loads on edges \(e_L\) and \(e_R\), which is at most \(2/\delta^{i+1}\). Also since the bottleneck capacity of any job in \(J^{(i)}\) is at least \(1/\delta^i\), any edge with capacity less than \(1/\delta^i\) can be contracted (i.e. its capacity can be made 0).

### B.8 Proof of Lemma 27

Let \(R^{(2k)}\) be the computed round for each set \(J^{(2k)}\), for \(k \in \mathbb{N}\). Suppose we are allowed a resource augmentation of \(1 + \gamma\), for some \(\gamma > 0\). Now since the bottleneck capacity of any job in \(J^{(i)}\) is at least \(\frac{1}{\delta^i}\), on resource augmentation, the capacity of any such edge would increase by at least \(\frac{\gamma}{\delta^i}\). Given the packing inside round \(R^1\), we shift up the height \(h_i\) of each job \(j\) by \(\frac{\gamma}{\delta^i}\) and place it in a round \(R\) of the original capacity profile.

Now the jobs from \(J^{(i-2)}\) do not go above \(2\frac{\gamma}{\delta^{i+1}} + \frac{\gamma}{\delta^i}\). We choose \(\gamma\) such that \(2\frac{\gamma}{\delta^{i+1}} + \frac{\gamma}{\delta^i} \geq \frac{\gamma}{\delta^{i-1}}\), from which we get \(\gamma \geq 2\frac{\delta^i}{\delta^{i-1}} = O(\delta)\). Thus one round from each set \(J^{(2k)}\) can be packed together. Similarly one round from each set \(J^{(2k+1)}\) can be packed together.

### B.9 Proof of Lemma 29

\begin{figure}
\centering
\begin{subfigure}{0.49\textwidth}
\centering
\includegraphics[width=\textwidth]{packing_p}
\caption{Packing \(P\)}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\centering
\includegraphics[width=\textwidth]{round_r1}
\caption{Round \(R^1\)}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\centering
\includegraphics[width=\textwidth]{round_r2}
\caption{Round \(R^2\)}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\centering
\includegraphics[width=\textwidth]{round_r3}
\caption{Round \(R^3\)}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\centering
\includegraphics[width=\textwidth]{round_r4}
\caption{Round \(R^4\)}
\end{subfigure}
\caption{Figure for Lemma 29. The yellow, green and pink jobs belong to \(T^{(0)}\), \(T^{(1)}\) and \(T^{(2)}\), respectively.}
\end{figure}
Given a valid Round-SAP packing $\mathcal{P}$ of a set of jobs $J'$ for the given edge capacities $(c_e)_{e \in E}$, we construct a valid packing of $J'$ into 4 rounds $\mathcal{R}^1, \mathcal{R}^2, \mathcal{R}^3, \mathcal{R}^4$, under profile $(c'_e)_{e \in E}$ such that no job is sliced by $\mathcal{L}$. Let $J^{(i)} := \{ j \in T : 2^i \leq b_j < 2^{i+1} \}, \forall i \in \{0, 1, \ldots, \log c_{\max} \}$. Note that jobs in $J^{(i)}$ will have bottleneck capacity equal to $2^i$ in $(c'_e)_{e \in E}$. For $i \in \{1, \ldots, \log c_{\max} \}$, we pack $J^{(i)}$ as follows:

- Place each job $j$ lying completely below $\ell_2$ into $\mathcal{R}^1$ at height $h'_j := h_j$. (Note that here $h_j$ denotes the height of the bottom edge of $j$ in $\mathcal{P}$.)
- Place each job $j$ lying completely between $\ell_2$ and $\ell_{3 \cdot 2^i - 1}$ into $\mathcal{R}^2$ with $h'_j := h_j - 2^{i-1}$.
- Place each job $j$ lying completely between $\ell_{3 \cdot 2^i - 1}$ and $\ell_{2^{i+1}}$ into $\mathcal{R}^3$ with $h'_j := h_j - 2^i$.
- Place each job $j$ sliced by $\ell_2$ into $\mathcal{R}^4$ with $h'_j := 2^i - d_j$.
- Place each job $j$ sliced by $\ell_{3 \cdot 2^i - 1}$ into $\mathcal{R}^4$ with $h'_j := 3 \cdot 2^{i-2} - d_j$ for $i \geq 2$ and $h'_j := 1 - d_j$ for $i = 1$.

Note that the jobs allocated to $\mathcal{R}^1$ are placed at the same vertical height as in $\mathcal{P}$. For $\mathcal{R}^2$, we have maintained the invariant that all jobs placed between $\ell_{2^i-1}$ and $\ell_{2^i}$ lie completely between $\ell_2$ and $\ell_{3 \cdot 2^i-1}$ and had bottleneck capacity in the interval $[2^i, 2^{i+1})$ in $\mathcal{P}$. Similarly for $\mathcal{R}^3$, jobs placed between $\ell_{2^i-1}$ and $\ell_{2^i}$ lie completely between $\ell_{3 \cdot 2^{i-1}}$ and $\ell_{2^{i+1}}$ and had bottleneck capacity in the interval $[2^i, 2^{i+1})$ in $\mathcal{P}$. Observe that the region below $\ell_1$ is currently empty in both $\mathcal{R}^2$ and $\mathcal{R}^3$ (this will be utilized for packing jobs in $J^{(0)}$). Finally, $\mathcal{R}^4$ contains the jobs sliced by $\ell_2$ and $\ell_{3 \cdot 2^i - 1}$.

As we have ensured that jobs from $J^{(i)}$ have been placed only in the region between the slicing lines $\ell_{2^i-1}$ and $\ell_2$, in $\mathcal{R}^2$ and $\mathcal{R}^3$, there is no overlap between jobs from different $J^{(i)}$’s in $\mathcal{R}^2$ and $\mathcal{R}^3$. Also since the demand of each job is at most 1, there is no overlapping of jobs from different $J^{(i)}$’s inside $\mathcal{R}^4$. Now we pack the jobs in $J^{(0)}$ (jobs with bottleneck capacity less than 2), as follows:

- Place each job $j$ lying completely below $\ell_1$ into $\mathcal{R}^1$ with $h'_j := h_j$.
- Place each job $j$ lying completely between $\ell_1$ and $\ell_2$ into $\mathcal{R}^2$ with $h'_j := h_j - 1$.
- Place each job $j$ sliced by $\ell_1$ into $\mathcal{R}^3$ with $h'_j := 1 - d_j$.

We continue to maintain the same invariant for $\mathcal{R}^1$. For $\mathcal{R}^2$ and $\mathcal{R}^3$, we used the empty region below $\ell_1$ to pack jobs from $J^{(0)}$. Note that no job is sliced by $\mathcal{L}$ in the packing obtained. Hence the lemma holds.

### B.10 Proof of Lemma 30

For $i = 0$, $OPT^{(0)} \leq OPT_{SAP}^{(i)}$ from definition and we are done. For $i \geq 1$, given a valid packing under uniform capacity profiles of height $2^i$ with no job sliced by $\mathcal{L}$ (call it type 1), we obtain a valid packing under uniform capacity profiles of height $2^{i-1}$ (type 2) using at most twice the number of rounds, thus proving that $OPT^{(i)} \leq 2 \cdot OPT_{SAP}^{(i)}$. For each round of type 1, introduce 2 rounds of type 2, denoted by $\mathcal{R}^1$ and $\mathcal{R}^2$. Note that since $\ell_{2^{i-1}} \in \mathcal{L}$, no job is sliced by $\ell_{2^{i-1}}$ in any round of type 1. We pack all jobs lying above $\ell_{2^{i-1}}$ into $\mathcal{R}^1$ and all jobs below $\ell_{2^{i-1}}$ into $\mathcal{R}^2$.

### B.11 Proof of Lemma 31

Let $\mathcal{R}^i \in \Gamma_i$ be the round chosen for each $i$. The crucial observation is that for any $j \in J_i$, $\mathcal{P}_j$ only contains edges of capacity at least $2^i$. Thus each $\mathcal{R}^i$, $1 \leq i \leq \log c_{\max}'$ can be placed between the slicing lines $\ell_{2^{i-1}}$ and $\ell_2$ under the profile $I'$. Finally we place $\mathcal{R}^0$ between $\ell_0$ and $\ell_1$. Observe that no job is sliced by $\mathcal{L}$ in this packing generated.
We round up the demand of each job in $J_{\text{large}}$ to 1 and round down the capacity of each edge to the nearest integer. Thus the congestion increases by at most a factor of 4. Hence, by applying Theorem 33, jobs in $J_{\text{large}}$ can be packed into at most $4r$ bins.

### C Round-UFP on Trees

#### C.1 Uniform capacities

First, we consider the Round-Tree problem where all edges of $G_{\text{tree}}$ have the same capacity $c^*$. Let $Z_L := \{ j \in J \mid d_j > c^*/2 \}$ and $Z_S := J \setminus Z_L$.

For the jobs in $Z_L$, any two jobs sharing at least one common edge must be placed in different tree-rounds. Thus the problem reduces to the Path-Coloring problem on trees and we have the following result due to Erlebach and Jansen [24] (for asymptotic approximation ratio) and Raghavan and Upfal [48] (for absolute approximation ratio).

> Lemma 45. There exists a polynomial time asymptotic (resp. absolute) $1.1$- (resp. 1.5-) approximation algorithm for packing jobs in $Z_L$.

Now we consider packing of jobs in $Z_S$. First, we fix any vertex $v_{\text{root}}$ as the root of $G_{\text{tree}}$. For any $j \in J$, let $\theta_j$ denote the least common ancestor of $v_{s_j}$ and $v_{t_j}$. For any two vertices $u, v \in V(G_{\text{tree}})$, let $P_{u-v}$ denote the unique path starting at $u$ and ending at $v$. The level of a vertex $v \in G_{\text{tree}}$ is defined as the number of edges present in $P_{\text{root}-v}$. We sort the jobs in $Z_S$ in non-decreasing order of the levels of $\theta_j$ and apply the First-Fit algorithm, i.e., we assign $j$ to the first tree-round in which it can be placed without violating the edge-capacities. Let $\Gamma$ denote the set of tree-rounds used.

> Lemma 46. $|\Gamma| \leq 4r$.

**Proof.** For any job $j$, let $e^1_j$ and $e^2_j$ denote the first edges on the paths $P_{\theta_j-v_{s_j}}$ and $P_{\theta_j-v_{t_j}}$, respectively. For any tree-round $\mathcal{R} \in \Gamma$ and edge $e \in E(G_{\text{tree}})$, let $l^e_{\mathcal{R}}$ denote the sum of demands of all jobs assigned to $\mathcal{R}$ that pass through $e$.

Let $\mathcal{R}^1, \mathcal{R}^2, \ldots, \mathcal{R}^{(|\Gamma|)}$ be the tree-rounds used to pack $Z_S$. Observe that only $l^e_{\mathcal{R}^i}$ and $l^e_{\mathcal{R}^{i+1}}$, $\forall 1 \leq i \leq |\Gamma|$, need to be considered while determining the tree-round for $j$, as we are considering jobs in $Z_S$ in non-decreasing order of the levels of $\theta_j$. Let $j_1$ be the first job for which $\mathcal{R}^{j_1}$ is opened for the first time. Clearly then at least one of $l^e_{\mathcal{R}^{j_1}}$ or $l^e_{\mathcal{R}^{j_1+1}}$ must exceed $c^*/2$, $\forall i = 1, 2, \ldots, |\Gamma| - 1$. Thus $\frac{c^*}{2} < l^e_{\mathcal{R}^{j_1}} + l^e_{\mathcal{R}^{j_1+1}}$. Summing over all $i$ from 1 to $|\Gamma| - 1$, we get $\frac{c^*}{2}(|\Gamma| - 1) < l_{e^1_{j_1}} + l_{e^2_{j_1}} \leq 2L$. Thus, $|\Gamma| < 4r + 1$. Since $|\Gamma|$ is an integer, we get $|\Gamma| \leq 4r$. ▶

Combining Lemma 45 and Lemma 46, we have the following theorem.

> Theorem 47. There exists a polynomial-time asymptotic (resp. absolute) $5.1$- (resp. 5.5-) approximation algorithm for Round-Tree with uniform edge capacities.

#### C.2 Arbitrary capacities (under NBA)

We now consider the Round-Tree problem. Chekuri et al. [15] proved the following result.

> Theorem 48. [15] If all job demands are equal to 1, then they can be packed into at most $4r$ tree-rounds.
We state the following result which enables us to obtain an improved approximation algorithm.

▶ Lemma 49. Let $\mathcal{H} := \{ j \in J \mid \frac{1}{m_2} c_{\min} < d_j \leq \frac{1}{m_2} c_{\min}\}$, for some $m_1 > m_2 \geq 1$. Then there exists a valid Round-Tree packing of $\mathcal{H}$ using less than $\frac{4m_1(m_2+1)}{m_2} M + 4$ tree-rounds.

Proof. We scale up the demand of each job in $\mathcal{H}$ to $\frac{1}{m_2} c_{\min}$ and scale down the capacity of each edge in $E$ to the nearest integral multiple of $\frac{1}{m_2} c_{\min}$. Let $\mathcal{H}'$ denote the new set of identical demand (of $\frac{1}{m_2} c_{\min}$) jobs and let $I'$ denote the new profile. For $e \in E$, let $I_e^H$ and $r_e^H$ denote the original load and congestion of edge $e$ due to the jobs in $\mathcal{H}$ and let $I_e^{H'}$ and $r_e^{H'}$ denote the new load and congestion, respectively. Then $l_e^{H'} < \frac{m_1}{m_2} l_e^H$ and $c_e' \geq \frac{n_1}{n_2 - 1} c_e$. Thus $r_e^{H'} = \left[\frac{l_e^{H'}}{c_e'}\right] \leq \left[\frac{n_1 l_e^H}{n_2 - 1 c_e'}\right] < \frac{n_1 (m_2 + 1)}{n_2} \frac{l_e^H}{c_e'} + 1 \leq \frac{n_1 (m_2 + 1)}{n_2} r_e^H + 1$.

Finally, we scale all demands and edge capacities by $\frac{1}{m_2} c_{\min}$ so that now all jobs have unit demands and all capacities are integers. Applying Theorem 48 to this set of jobs, we obtain the number of tree-rounds used is at most $4r^{H'} < \frac{4m_1(m_2+1)}{m_2} M + 4$.

Let $J_L := \{ j \in J \mid d_j > b_1/5\}$ and $J_R := J \setminus J_L$.

▶ Lemma 50. The jobs in $J_L$ can be packed into at most $31r + 6$ tree-rounds.

Proof. Let $Q_M := \{ j \in J_L \mid \frac{1}{2} c_{\min} < d_j \leq \frac{1}{2} c_{\min}\}$ and $Q_L := J_L \setminus Q_M$. Apply Lemma 49 to $Q_M$ with $n_1 = 5$ and $n_2 = 2$ and to $Q_L$ with $n_1 = 2$ and $n_2 = 1$. This yields valid packings of $Q_M$ and $Q_L$ using at most $15r + 3$ and $16r + 3$ tree-rounds, respectively (since the number of rounds is an integer and Lemma 49 is strict). Thus the total number of tree-rounds used is at most $31r + 6$.

▶ Lemma 51. The jobs in $J_S$ can be packed into at most $18r$ tree-rounds.

Proof. First, we introduce some terminologies. An edge $e \in E(G_{tree})$ is said to be of class $k$ if $(5/2)^k \leq c_e < (5/2)^{(k+1)}$ and we denote $c(l) := k$. For any $u, v \in V(G_{tree})$, the critical edge of $P_{u \rightarrow v}$, denoted by crit($P_{u \rightarrow v}$) is defined as the first edge having the minimum capacity among all the edges of $P_{u \rightarrow v}$.

We again fix any vertex ($v_{\text{root}}$) as the root of $G_{tree}$. As previously, for any $j \in J$, let $\theta_j$ denote the least common ancestor of $v_{s_j}$ and $v_{t_j}$. We maintain $18r$ tree-rounds, $R^1, R^2, \ldots, R^{18r}$. For any job $j \in J_S$, let $e_j^1 := \text{crit}(P_{v_{s_j} \rightarrow v_{t_j}})$ and $e_j^2 := \text{crit}(P_{v_{t_j} \rightarrow v_{s_j}})$. We consider the jobs in $J_S$ in non-decreasing order of the levels of $\theta_j$ and place $j$ in a tree-round $R$ in which both the following conditions hold:

(i) The sum of demands of all jobs that have already been assigned to $R$ and pass through $e_j^1$ is at most $c_{e_j^1}/9$.

(ii) The sum of demands of all jobs that have already been assigned to $R$ and pass through $e_j^2$ is at most $c_{e_j^2}/9$.

First, we show that such a tree-round must always exist. Suppose to the contrary, there exists a job $j$ that cannot be assigned by the above algorithm. Then in each of the $18r$ tree-rounds, either (i) or (ii) must fail. For any tree-round $R$ and edge $e \in E(G_{tree})$, let $l_e^R$ denote the sum of demands of jobs assigned to $R$ that pass through $e$. Let $\chi$ be the number of tree-rounds in which $l_{e_j^1}^R \geq \frac{1}{9} c_{e_j^1}$. Thus in at least $18r - \chi$ tree-rounds, $l_{e_j^1}^R > \frac{1}{9} c_{e_j^1}$. So $l_{e_j^1}^R > \sum_{i=1}^{18r} l_{e_j^1}^i > \chi \cdot \frac{1}{9} c_{e_j^1}$. Since $r \geq l_{e_j^1}/c_{e_j^1}$, we get $\chi < 9r$. Again $l_{e_j^2}^R > \sum_{i=1}^{18r} l_{e_j^2}^i > (18r - \chi) \cdot \frac{1}{9} c_{e_j^2}$ and since $r \geq l_{e_j^2}/c_{e_j^2}$, we get $\chi > 9r$, a contradiction.
Now we show that the above algorithm produces a valid packing. Consider any tree-round \( R \) and edge \( e = \{u, v\} \), such that \( \text{level}(v) = \text{level}(u) + 1 \). Let \( S^e_R \) denote the set of jobs assigned to \( R \) that pass through \( e \). Let \( j_{\text{up}} \in S^e_R \) be the last job such that at least one of \( e^1_{j_{\text{up}}} \) or \( e^2_{j_{\text{up}}} \) lies on \( P_{\text{root} - v} \). Thus the sum of demands of all such jobs such that at least one of \( e^1_{j_{\text{up}}} \) or \( e^2_{j_{\text{up}}} \) lies on \( P_{\text{root} - v} \), including \( j_{\text{up}} \), is at most

\[
\frac{(5/2)^{\text{cl}(e)+1}}{9} + \frac{c_e}{5} \leq \frac{43}{90} c_e.
\]

All the remaining jobs must satisfy \( \text{level}(\theta_j) > \text{level}(u) \). So the critical edge is of lower level. Thus the sum of demands of all such jobs is at most

\[
\sum_{i=0}^{\text{cl}(e)-1} \left( \frac{(5/2)^{i+1}}{9} + \frac{(5/2)^{i+1}}{5} \right) \leq \frac{14}{45} \sum_{i=0}^{\text{cl}(e)-1} (5/2)^{i+1} \leq \frac{14}{27} c_e.
\]

Since \( 43/90 + 14/27 < 1 \), the total sum of demands of jobs in \( S^e_R \) does not exceed \( c_e \). Hence, the packing is valid. ▶

Note that Lemma 50 yields a packing of \( J_L \) using at most \( 31r + 6 \leq 37r \) tree-rounds. Together with Lemma 51, we have the following theorem.

▶ **Theorem 52.** There exists a polynomial time asymptotic (resp. absolute) 49- (resp. 55-) approximation algorithm for \( \text{Round-Tree} \).

We note that with some careful choice of parameters, the asymptotic approximation ratio could be improved to 48.292, but it is unlikely to be improved further using the present technique.