An Improved Level Set Method for Reachability Problems in Differential Games

Wei Liao, Taotao Liang, Pengwen Xiong, Senior Member, IEEE, Chen Wang, Aiguo Song, Senior Member, IEEE, and Peter X. Liu, Fellow, IEEE

Abstract—This study focuses on reachability problems in differential games. An improved level set (LS) method for computing reachable tubes (RTs) is proposed in this article. The RT is described as a sub-LS of a value function, which is the viscosity solution of a Hamilton–Jacobi (HJ) equation with running cost. We generalize the concept of RTs and propose a new class of RTs, which are referred to as cost-limited one. In particular, a performance index can be specified for the system, and a set of initial states of the system’s evolutions that can reach the target set before the performance index grows to a given allowable cost is referred to as a cost-limited RT (CRT). Such an RT can be obtained by specifying the corresponding running cost function for the HJ equation. Different nonzero sub-LSs of the viscosity solution of the HJ equation at a certain time point can be used to characterize the CRTs with different allowable costs (or the RTs with different time horizons), thus reducing the storage space consumption. The validity and accuracy of the suggested technique are demonstrated via some examples.

Index Terms—Differential game, Hamilton–Jacobi (HJ) equation, optimal control, reachability, viscosity solution.

I. INTRODUCTION

DIFFERENTIAL games, a classical category of problems in control theory, involve the modeling and analysis of conflicts among multiple players in a dynamic system [1]. It is generally accepted that the concept of differential games was first introduced by Isaacs [2]. It has been widely applied in several fields, such as engineering [3], [4], [5] and economics [6].

In recent years, some studies have introduced reachability analyses in differential game problems [7], [8], [9]. In the context of differential games, reachability analysis typically involves two players, where one player seeks to steer the system state away from the target set, while the other attempts to steer the system state toward it. The aim of reachability analysis is to characterize a reachable tube (RT), i.e., a set of states from which the system’s trajectories can touch the target set in a certain time horizon [7], [10]. By applying reachability analysis, a variety of engineering problems can be solved, especially those involving system safety [7], [11], [12], [13].

However, computing RT is challenging. Except for some solvable problems [14], the exact RT computation is generally intractable. Therefore, a common practice is to compute RT’s approximate expressions. Several approaches have been developed over the years, including ellipsoidal methods [15], [16], [17], polyhedral methods [18], [19], [20], [21], and state-space discretization methods. While the first two categories are capable of solving high-dimensional issues, they impose strong constraints on the forms of dynamic systems and are therefore largely utilized to handle linear issues.

The primary advantage of state-space discretization-based techniques is that they impose fewer constraints on the form of a dynamic system and can be utilized to handle nonlinear issues. This category of methods is mostly comprised of the level set (LS) [7], [10], [22] and distance fields over grids (DFOGs) [23], [24], [25] methods. In engineering, the former is more prevalent than the latter.

The LS method solves the Hamilton–Jacobi (HJ) partial differential equation (PDE) numerically without considering a running cost function, and the RT is expressed as the zero sub-LS of the solution [10]. On this basis, several toolboxes have been built [26], [27], [28], which have significantly aided in the development of computation for RT. Additionally, they have been used to address a variety of practical technical challenges, including flight control systems [29], [30], [31], air traffic systems [8], and ground traffic systems [32], [33].

While the LS approach has evolved into a mature, dependable, and widely used tool over the years, it does have a few drawbacks. It poses very high storage space requirements. To
save RT with a given time horizon, one needs to save the solution of the HJ equation at a certain time point; the storage space grows significantly with an increase in the dimension of the problem. To save RTs with different time horizons, one needs to save the solutions of the HJ equation at different time points, which in turn leads to a significant increase in storage space consumption.

In addition, since its inception, differential games have been inextricably linked to optimal control problems. Different forms of performance indexes can be specified in both optimal control and differential game problems, and the time consumption is only a special form. A performance index in its most generic form typically contains a Lagrangian [34], which is the time integral of a running cost. Thus, it is important to build a novel framework that can analyze whether a target set can be reached within an allowable performance index. Moreover, such a framework is of significant engineering importance.

Consider the following scenarios: airplane forced landing and vehicle path planning with limited fuel. In these scenarios, the primary concern is not whether the destination can be reached before the fuel runs out. The fuel consumption rate can be considered as a running cost function.

Theoretically, the running cost function can be considered as the time derivative of a newly added state, thus transforming such a problem into one that can be handled by LS methods. However, this would increase the dimensionality of the state space and thus significantly increase memory and time consumption.

To compensate for the shortcomings of the LS method and determine reachability under different forms of performance indexes, we propose an improved LS method to solve reachability problems and a more generalized definition of RT. We thus aim to make the following contributions.

1) We propose an improved LS method for computing the RT. In the proposed method, RT is represented as a nonzero sub-LS of a value function, which is the viscosity solution of an HJ PDE with constant 1 as a running cost function. The RTs with different time ranges are represented as a cluster of nonzero sub-LSs of this value function, and thus reduce the consumption of storage space.

2) We define a cost-limited RT (CRT). A running cost function can be specified for the system. CRT is a set of states that can be steered toward the target set before the performance index, the time integral of the running cost, increases to the allowable cost. In the context of CRTs, time consumption is a special performance index, and RT is also a special case of CRTs.

3) We generalize the reachability problem and present the definition of CRT, and also present how to compute the CRT by the improved LS method.

4) We give two numerical examples to demonstrate the effectiveness of the proposed algorithm in Section V.

5) We summarize the full study in Section VI.

II. PRELIMINARIES

A. Problem Formulation

Let and denote the collections consisting of measurable functions from to and , respectively. We consider the following control system with state and two players, Player I and Player II, with inputs and and vehicle path planning with limited fuel. In these scenarios, we allow Player II, the opponent [7]. In this study, because the target set represents a significant engineering importance.

Moreover, such a framework is of significant importance. We thus aim to make the following contributions.

1) We define a cost-limited RT (CRT). A running cost function can be considered

2) We describe the improved LS method proposed in this article in Section III.

B. LS Method

In the classical LS method, the viscosity solution of the following HJ PDE of a time-dependent value function

is numerically solved:

subject to the following expression

We assume that the function is Lipschitz continuous and bounded. Then, fixing the state at and given control inputs and , the evolution of system (1) is determined by a continuous trajectory and satisfies

We denote the target set as and assume that Player I seeks to steer the system state toward the target set, and Player II seeks to steer the system state away from it.

In a differential game, different information patterns provide advantages to different players; in particular, players who play nonanticipative strategies have an advantage over their opponents [7]. In this study, because the target set represents a safe portion of the state space, we prefer to underapproximate the RTs (or CRTs) of the target set rather than overapproximate them. To consider the worst case, we allow Player II, the player trying to steer the system state away from the target set, to use nonanticipative strategies. The set containing all the nonanticipative strategies for Player II is denoted as

Then, given a horizon , the definition of the RT is as follows [7].

Definition 1 (RT): RT is the set with the following expression

We give two numerical examples to demonstrate the effectiveness of the proposed algorithm in Section V.

We summarize the full study in Section VI.
where \(d(\cdot)\) is bounded and Lipschitz continuous and satisfies \(S = \{s \in \mathbb{R}^n | d(s) \leq 0\}\). The viscosity solution is approximated using a Cartesian grid of the state space. The RT can then be characterized by a zero sub-LS of \(V(\cdot, 0)\)

\[
R_S(T) = \{s \in \mathbb{R}^n | V(s, 0) \leq 0\}.
\]

(6)

Denote the number of grid points in the \(i\)th dimension of the Cartesian grid as \(N_i\). The storage space consumed to store \(R_S(T)\) is proportional to \(\prod_{i=1}^n N_i\).

It should be noted that, for \(T_1, \ldots, T_M \in [0, \infty)\), the expressions of the RTs with these time horizons are as follows:

\[
R_S(T_1) = \{s \in \mathbb{R}^n | V(s, T_1) \leq 0\}
\]

\[
\ldots
\]

\[
R_S(T_M) = \{s \in \mathbb{R}^n | V(s, T_M) \leq 0\}.
\]

(7)

Since the value functions \(V(\cdot, T-T_1), \ldots, V(\cdot, T-T_M)\) are each different, the storage space consumption required to save these RTs is proportional to \(M \prod_{i=1}^n N_i\).

C. Viscosity Solution

In the LS method, the HJ equation is a powerful tool for studying reachability problems. However, in general, the value function is not smooth enough to satisfy the HJ equation. In fact, the lack of smoothness of the value function is a general case rather than an exception. Therefore, if we use the LS method we need to find the weak solution of the HJ equation [35].

Literature [36] provides a weak solution known as viscosity solution, which remains stable under any reasonable relaxation or approximation of the equation. In addition, this literature proves in detail the existence and uniqueness of the viscosity solution. So far, the viscosity solution has been widely used in the reachability analysis based on the HJ equation [7], [10], [37].

III. IMPROVED LS METHOD

Consider the scenario in which Player I selects \(a(\cdot)\) to facilitate the system state’s entry into the target set as quickly as possible, while Player II chooses a nonanticipative strategy \(\beta[a](\cdot)\) to avoid or delay the entry of the system state into the target set as much as possible. In this case, if a trajectory touches the target set in time \(T\), the initial state lies within the RT. Therefore, a value function \(W(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}\) can be built as

\[
W(s) = \left\{ \begin{array}{ll}
\sup_{\beta[a]} \inf_{f} & I_{f}(s(t))dt \\
\text{s.t.} & \dot{s}(t) = f(s(t), a(t), \beta[a](t)) \forall t \in [0, t_f] \\
& s(0) = s \\
& a(t) \in \mathcal{A} \forall t \in [0, t_f] \\
& \beta[a](t) \in \mathcal{B} \forall t \in [0, t_f] \\
& s(t_f) \in S.
\end{array} \right.
\]

(8)

In the preceding equation, \(t_f\) denotes the free final time.

Remark 1: It is possible that for some initial states \(s\) and Player II’s strategies, the trajectory initialized from \(s\) will never reach the target set no matter what Player I does. In this case, \(W(s)\) is obviously infinity.

Then, the RT can be characterized by the \(T\) sub-LS of \(W(\cdot)\), i.e.,

\[
R_S(T) = \{s \in \mathbb{R}^n | W(s) \leq T \}.
\]

(9)

The value function in (8) involves infinite boundary condition applied along a boundary which cannot always be determined a priori, which leads to the difficulty of approximating this value function globally. Therefore, we do not insist on approximating this value function globally, but rather in the region of interest to us. For this purpose, a modified dynamic system and a modified running cost function are built as follows:

\[
\dot{s} = \tilde{f}(s, a, b) = \begin{cases}
(f(s, a, b), s \notin S) & s \in S \\
0 & 0, s \in S.
\end{cases}
\]

(10)

\[
I(s) = \begin{cases}
1, s \notin S \\
0, s \in S.
\end{cases}
\]

(11)

Specifying the initial time \(t\) and state \(s\), and \(a(\cdot) \in \mathcal{A}\), and \(b(\cdot) \in \mathcal{B}\), the evolution of system (10) is also a continuous trajectory \(\tilde{\phi}_{s,t}^{a,b}(\cdot) : [t, \infty) \rightarrow \mathbb{R}^n\).

Then, given a \(\bar{T} \in [0, \infty)\), we define a value function as follows:

\[
W_{\bar{T}}(s, \tau) = \left\{ \begin{array}{ll}
\sup_{\beta[a]} \inf_{f} & I_{f}(s(t))dt \\
\text{s.t.} & \dot{s}(t) = f(s(t), a(t), \beta[a](t)) \forall t \in [\tau, \bar{T}] \\
& s(\tau) = s \\
& a(t) \in \mathcal{A} \forall t \in [\tau, \bar{T}] \\
& \beta[a](t) \in \mathcal{B} \forall t \in [\tau, \bar{T}].
\end{array} \right.
\]

(12)

Then, the following theorem holds.

Theorem 1: \(\forall \bar{T} < T\) and \(s \in \{s_0 \in \mathbb{R}^n | W_{\bar{T}}(s_0, 0) \leq \bar{T}\}\), \(W_{\bar{T}}(s, 0) = W(s)\).

Proof: This theorem is just a special case of Theorem 3, which is given and proved in the next section.

Theorem 1 indicates that, for \(T_1, \ldots, T_M < \gamma T\), the RTs \(R_S(T_1), \ldots, R_S(T_M)\) can be represented as a family of sub-LSs of \(W_{\bar{T}}(\cdot, 0)\) and all RTs can be saved by saving \(W_{\bar{T}}(\cdot, 0)\)

\[
R_S(T_1) = \{s \in \mathbb{R}^n | W_{\bar{T}}(s, 0) \leq T_1 \}
\]

\[
\ldots
\]

\[
R_S(T_M) = \{s \in \mathbb{R}^n | W_{\bar{T}}(s, 0) \leq T_M \}.
\]

(13)

Theorem 2: The value function \(W_{\bar{T}}(\cdot, \cdot)\) is the viscosity solution of the following HJ PDE:

\[
\frac{\partial W_{\bar{T}}}{\partial s}(s, t) + \min_{a \in \mathcal{A}} \max_{b \in \mathcal{B}} \left\{ I(s) + \frac{\partial W_{\bar{T}}}{\partial s}(s, t)\tilde{f}(s, a, b) \right\} = 0.
\]

(14)

Proof: This theorem is just a special case of Theorem 4, which is given and proved in the next section.

IV. GENERALIZATION OF REACHABILITY PROBLEM

A. Definition of Cost-Limited Reachable Tube

As described earlier, a common performance index typically includes a Lagrangian. We represent the running cost function as \(c(\cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}\), and assume the following.
Assumption 1: \( \forall s \in \mathbb{R}^n, a \in A, b \in B, c(s, a, b) > 0 \) holds, and we denote the minimum value of the running cost function as \( \gamma \), i.e., \( \min_{a \in A, b \in B} c(s, a, b) = \gamma \).

The performance index of the trajectory starting from \( s \) at time \( t_0 \) under control inputs \( a(\cdot) \) and \( b(\cdot) \) in time interval \([t_0, t_1]\) is denoted as
\[
\mathcal{J}_0^W(s, a, b) = \int_{t_0}^{t_1} c(\phi_{s, t_0}^{a, b}(t), a(t), b(t)) dt.
\]

Then, given an allowable cost \( J \) and a target set \( \mathcal{S} \), CRT can be defined.

Definition 2 (CRT): CRT is the set with the following expression
\[
R^c_S(J) = \{ s \in \mathbb{R}^n | \exists \tau \in [0, \infty), \exists a(\cdot) \in \mathcal{A}_0 \forall \beta \in \mathcal{B}_0, \phi_{s, 0}^{a, \beta}(\tau) \in \mathcal{S} \land \mathcal{J}_0^W(s, a, \beta[a]) \leq J \}
\]

where “\( \land \)” is the logical operator “AND.” Intuitively, according to Assumption 1, the performance index \( \mathcal{J}_0^W(s, a, b) \) increases with an increase in \( t \). Furthermore, \( R^c_S(J) \) is a set of states that can be transferred into the target set \( \mathcal{S} \) under any Player II’s nonanticipative strategy \( \beta[\cdot] \) before the performance index increases to the allowable cost \( J \).

Consider the case where Player I attempts to move the system state toward the target set at the lowest possible cost, and Player II plays a nonanticipative strategy to avoid the entry of the system state into the target set or to maximize the cost during state transition. If a trajectory touches the target set before the performance index increases to the given allowable cost \( J \), then its initial state is within \( R^c_S(J) \). Similarly to (8), we construct a value function as follows:
\[
W^c(s) = \left\{ \begin{array}{ll}
\sup_{\beta[\cdot]} \inf_{a[\cdot]} \int_0^T c(s(t), a(t), \beta[a](t)) dt & \text{s.t.} \\
\bar{s}(t) = f(s(t), a(t), \beta[a](t)) \forall t \in [0, T] \\
\bar{s}(0) = s \\
a(t) \in A \forall t \in [0, T] \\
\beta[a](t) \in B \forall t \in [0, T] \\
\bar{s}(T) \in \mathcal{S}.
\end{array} \right.
\]

In this case, the value of \( W^c(s_0) \) may also be infinity, as stated in Remark 1. The CRT can be characterized by the \( J \)-sub-LS of \( W^c(\cdot) \)
\[
R^c_S(J) = \{ s \in \mathbb{R}^n | W^c(s) \leq J \}.
\]

Remark 2: If \( c(s, a, b) \equiv 1 \); Then, \( \int_0^T c(s(t), a(t), b(t)) dt = T \). The performance index in (17) degenerates into the time consumption in (8). Similarly, CRT \( R^c_S(J) \) is degenerated into RT \( R^c_S(J) \).

B. CRT Computation

Similar to (11), the running cost function \( c(\cdot, \cdot, \cdot) \) can be modified as follows:
\[
c(s, a, b) = \begin{cases} 
c(s, a, b), & s \notin \mathcal{S} \\
0, & s \in \mathcal{S}.
\end{cases}
\]

Remark 3: When the trajectory \( \hat{\phi}_{s, t}^{a, b}(\cdot) \) evolves outside the target set \( \mathcal{S} \), it coincides with the trajectory \( \phi_{s, t}^{a, b}(\cdot) \), and \( \hat{c}(s, a, b) \) is equal to \( c(s, a, b) \). When the trajectory \( \hat{\phi}_{s, t}^{a, b}(\cdot) \) touches the target set \( \mathcal{S} \), it is frozen on the boundary under the dynamics (10), and \( \hat{c}(s, a, b) \) is equal to zero.

Then, given a \( \tilde{T} \in [0, \infty) \), we construct the following value function by replacing running cost function \( I(\cdot) \) with \( \hat{c}(\cdot, \cdot, \cdot) \) on the basis of (12):
\[
W_\tilde{T}^c(s, \tau) = \left\{ \begin{array}{ll}
\sup_{\beta[\cdot]} \inf_{a[\cdot]} \int_0^{\tilde{T}} \hat{c}(s(t), a(t), \beta[a](t)) dt & \text{s.t.} \\
\hat{s}(t) = f(s(t), a(t), \beta[a](t)) \forall t \in [\tau, \tilde{T}] \\
\hat{s}(\tau) = s \\
a(t) \in A \forall t \in [\tau, \tilde{T}] \\
\beta[a](t) \in B \forall t \in [\tau, \tilde{T}].
\end{array} \right.
\]

Then, the following theorem holds.

Theorem 3: For any \( \tilde{J} < \gamma \tilde{T} \) and any \( s \in \{ s_0 \in [0, \infty) | W_\tilde{T}^c(s_0, 0) \leq \tilde{J} \} \), \( W_\tilde{T}^c(s_0) = W^c(s) \) holds.

Proof: We denote the strategies selected by the two players in (17) as \( a^*(\cdot) \) and \( \beta^*[\cdot] \) and the strategies selected by the two players in (20) as \( \hat{a}^*(\cdot) \) and \( \hat{\beta}^*[\cdot] \).

According to Remark 3, if the trajectory \( \hat{\phi}_{s, 0}^{a, b}(\cdot) \) cannot reach the target state \( \mathcal{S} \) within time \( \tilde{T} \), then its performance index satisfies
\[
\int_0^{\tilde{T}} c(\phi_{s, 0}^{a, b}(t), a(t), b(t)) dt 
\leq \int_0^{\tilde{T}} \min_{s \in \mathbb{R}^n, a \in A, b \in B} c(s, a, b) dt = \gamma \tilde{T}.
\]

Therefore, \( W^c(s_0) \leq \tilde{J} < \gamma \tilde{T} \) implies that there exists a \( \tau \in [0, \tilde{T}] \), such that \( \hat{\phi}_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(\tau) \in \mathcal{S} \). Furthermore, according to Remark 3, once a trajectory touches the boundary of the target set, it stays there. Therefore
\[
W_\tilde{T}^c(s_0) = \int_0^{\tilde{T}} \hat{c}(\phi_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(t), \hat{a}^*(t), \hat{\beta}^*[\hat{s}^*]) dt 
= \int_0^{\tilde{T}} \hat{c}(\hat{\phi}_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(t), \hat{a}^*(t), \hat{\beta}^*[\hat{s}^*]) + \int_\tau^{\tilde{T}} 0 dt.
\]

Outside the target set, the dynamical systems \( \hat{f}(\cdot, \cdot, \cdot) \) and \( f(\cdot, \cdot, \cdot) \) are identical, and the running cost functions \( \hat{c}(\cdot, \cdot, \cdot) \) and \( c(\cdot, \cdot, \cdot) \) are also identical. Therefore, the trajectory \( \hat{\phi}_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(\cdot) \) can also reach the target set at time \( \tau \). Then
\[
W^c(s_0) = \int_0^{\tilde{T}} c(\phi_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(t), a^*(t), \hat{\beta}^*[\hat{s}^*](t)) dt
\]
and for any \( \tau \in [0, \tilde{T}] \), \( a^*(t) = \hat{a}^*(t) \) and \( \hat{\beta}^*[\hat{s}^*](t) = \hat{\beta}^*[\hat{s}^*](t) \) and \( \phi_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(t) = \phi_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(t) \). Then, we have
\[
W_\tilde{T}^c(s_0) = \int_0^{\tilde{T}} \hat{c}(\phi_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(t), a^*(t), \hat{\beta}^*[\hat{s}^*](t)) dt 
= \int_0^{\tilde{T}} (\phi_{s_0}^{a^*, \hat{\beta}^*[\hat{s}^*]}(t), a^*(t), \hat{\beta}^*[\hat{s}^*](t)) dt = W^c(s).
\]

Remark 4: Set the running cost function to \( c(s, a, b) \equiv 1 \), in which case \( \hat{c}(s, a, b) = I(s) \) and \( \gamma = 1 \). By replacing \( \tilde{T} \) with \( \tilde{T} \), Theorem 3 is transformed into Theorem 1.
Theorem 3 indicates that, for $J_1, \ldots, J_M < \gamma \tilde{T}$, the CRTs $R_{\mathcal{S}}(J_1), \ldots, R_{\mathcal{S}}(J_M)$ can be represented as a family of sub-LSSs of $W_F^c(s, 0)$ and all CRTs can be saved by saving $W_F^c(s, 0)$

$$R_{\mathcal{S}}(J_1) = \left\{ s \in \mathbb{R}^n | W_F^c(s, 0) \leq J_1 \right\}$$

$$\cdots$$

$$R_{\mathcal{S}}(J_M) = \left\{ s \in \mathbb{R}^n | W_F^c(s, 0) \leq J_M \right\}.$$  \hspace{1cm} (25)

Lemma 1: Given $0 \leq t < t + \Delta t \leq \tilde{T}$ and $s \in \mathbb{R}^n$

$$W_F^c(s, t) = \sup_{\beta[\cdot] \in \mathcal{B}} \inf_{a(\cdot) \in \mathcal{A}} \left\{ \int_t^{t + \Delta t} \hat{c}(\phi_{a}, \beta[\cdot](\tau), a(\tau), \beta[\cdot](\tau)) d\tau + W_F^c(\phi_{a}, \beta[\cdot](t + \Delta t), t + \Delta t) \right\}.$$  \hspace{1cm} (26)

Proof: Let

$$\tilde{W}_F(s, t) = \sup_{\beta[\cdot] \in \mathcal{B}} \inf_{a(\cdot) \in \mathcal{A}} \left\{ \int_t^{t + \Delta t} \hat{c}(\phi_{a}, \beta[\cdot](\tau), a(\tau), \beta[\cdot](\tau)) d\tau + W_F^c(\phi_{a}, \beta[\cdot](t + \Delta t), t + \Delta t) \right\}.$$  \hspace{1cm} (27)

and fix $\epsilon > 0$. Then, there exists $\tilde{\beta}[\cdot] \in \mathcal{B}$ such that

$$\tilde{W}_F(s, t) \leq \inf_{a(\cdot) \in \mathcal{A}} \left\{ \int_t^{t + \Delta t} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), a(\tau), \tilde{\beta}[\cdot](\tau)) d\tau + W_F^c(\phi_{a}, \tilde{\beta}[\cdot](t + \Delta t), t + \Delta t) \right\} + \epsilon.$$  \hspace{1cm} (28)

Also, for each $s' \in \mathbb{R}^n$

$$W_F^c(s', t + \Delta t) = \sup_{\beta[\cdot] \in \mathcal{B}, a(\cdot) \in \mathcal{A}} \int_{s', t + \Delta t}^T \hat{c}(\phi_{a}, \beta[\cdot](\tau), a(\tau), \beta[\cdot](\tau)) d\tau.$$  \hspace{1cm} (29)

Thus, there exists $\tilde{\beta}[\cdot] \in \mathcal{B}$ for which

$$\tilde{W}_F(s', t + \Delta t) \leq \inf_{a(\cdot) \in \mathcal{A}} \int_{s', t + \Delta t}^T \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), a(\tau), \tilde{\beta}[\cdot](\tau)) d\tau + \epsilon.$$  \hspace{1cm} (30)

Define $\beta[\cdot] \in \mathcal{B}$ in this way: for each $a(\cdot) \in \mathcal{A}$ set

$$\beta[\cdot](\tau) = \begin{cases} \tilde{\beta}[\cdot](\tau), & \tau \in [t, t + \Delta t] \\ \beta[\cdot](\tau), & \tau \in (t + \Delta t, \tilde{T}] \end{cases}.$$  \hspace{1cm} (31)

Consequently, for any $a(\cdot) \in \mathcal{A}$, (29) and (30) imply

$$\tilde{W}_F(s, t) \leq \inf_{a(\cdot) \in \mathcal{A}} \int_t^{T} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), a(\tau), \tilde{\beta}[\cdot](\tau)) d\tau + \epsilon.$$  \hspace{1cm} (32)

So that

$$\inf_{a(\cdot) \in \mathcal{A}} \int_t^{T} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), a(\tau), \tilde{\beta}[\cdot](\tau)) d\tau \geq \tilde{W}_F(s, t) - 2\epsilon.$$  \hspace{1cm} (33)

Hence

$$W_F^c(s, t) \geq \tilde{W}_F(s, t) - 2\epsilon.$$  \hspace{1cm} (34)

On the other hand, there exists $\tilde{\beta}[\cdot] \in \mathcal{B}$, for which

$$W_F^c(s, t) \leq \inf_{a(\cdot) \in \mathcal{A}} \int_t^{T} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), a(\tau), \tilde{\beta}[\cdot](\tau)) d\tau + \epsilon.$$  \hspace{1cm} (35)

Then

$$W_F^c(s, t) \leq \inf_{a(\cdot) \in \mathcal{A}} \left\{ \int_t^{t + \Delta t} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), a(\tau), \tilde{\beta}[\cdot](\tau)) d\tau + W_F^c(\phi_{a}, \tilde{\beta}[\cdot](t + \Delta t), t + \Delta t) \right\}.$$  \hspace{1cm} (36)

and consequently there exists $\tilde{a}(\cdot) \in \mathcal{A}$ such that

$$W_F^c(s, t) \geq \left\{ \int_t^{t + \Delta t} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), \tilde{a}(\tau), \tilde{\beta}[\cdot](\tau)) d\tau + W_F^c(\phi_{a}, \tilde{\beta}[\cdot](t + \Delta t), t + \Delta t) \right\} - \epsilon.$$  \hspace{1cm} (37)

Now define $\tilde{a}(\cdot) \in \mathcal{A}$ by

$$\tilde{a}(\tau) = \begin{cases} \tilde{a}(\tau), & \tau \in [t, t + \Delta t] \\ a(\tau), & \tau \in (t + \Delta t, \tilde{T}] \end{cases}.$$  \hspace{1cm} (38)

and then define $\tilde{\beta}[\cdot] \in \mathcal{B}$ by

$$\tilde{\beta}[\cdot](\tau) = \tilde{\beta}[\cdot](\tau).$$  \hspace{1cm} (39)

Hence

$$W_F^c(\phi_{a}, \tilde{\beta}[\cdot](t + \Delta t), t + \Delta t) \geq \inf_{a(\cdot) \in \mathcal{A}} \int_t^{t + \Delta t} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), a(\tau), \tilde{\beta}[\cdot](\tau)) d\tau + \epsilon.$$  \hspace{1cm} (40)

and there exists $\tilde{\alpha}[\cdot] \in \mathcal{A}$ for which

$$W_F^c(\phi_{a}, \tilde{\beta}[\cdot](t + \Delta t), t + \Delta t) \geq \int_t^{t + \Delta t} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), \tilde{\alpha}(\tau), \tilde{\beta}[\cdot](\tau)) d\tau - \epsilon.$$  \hspace{1cm} (41)

Set $a(\cdot) \in \mathcal{A}$ as

$$a(\tau) = \begin{cases} \tilde{a}(\tau), & \tau \in [t, t + \Delta t] \\ \tilde{a}(\tau), & \tau \in (t + \Delta t, \tilde{T}] \end{cases}.$$  \hspace{1cm} (42)

Then, (37) and (41) yield

$$W_F^c(s, t) \geq \int_t^{t + \Delta t} \hat{c}(\phi_{a}, \tilde{\beta}[\cdot](\tau), a(\tau), \tilde{\beta}[\cdot](\tau)) d\tau - 2\epsilon.$$  \hspace{1cm} (43)

Therefore, (35) implies

$$W_F^c(s, t) \leq \tilde{W}_F(s, t) + 3\epsilon.$$  \hspace{1cm} (44)

Equations (34) and (44) complete the proof. \hspace{1cm} ■

Theorem 4: The value function $W_F^c(s, \cdot)$ is the viscosity solution of the following HJ PDE:

$$\min_{a(\cdot) \in \mathcal{A}} \max_{b(\cdot) \in \mathcal{B}} \left\{ \hat{c}(s, a, b) + \frac{\partial W_F^c}{\partial s}(s, a) \hat{f}(s, a, b) \right\} = 0.$$  \hspace{1cm} (45)

Proof: A nonanticipative strategy is equivalent to allowing Player II to choose $b(t)$ based on knowledge of $a(t)$ for all
Algorithm 1 Method to Compute CRTs

1. **Inputs:** Dynamic system (1), sets of players’ achievable control inputs $A$ and $B$, running cost function $c(\cdot, \cdot, \cdot)$, allowable costs $J_1, \ldots, J_M$, target set $\mathcal{S}$, computational domain $\Omega$, number of grids $N_x \times N_y \times \cdots$;
2. $\gamma \leftarrow \min_{s \in \mathbb{R}^a, a \in A, b \in B} c(s, a, b)$;
3. $J_{\text{max}} \leftarrow \max(J_1, \ldots, J_M)$, $\bar{T} \leftarrow J_{\text{max}} + \epsilon; \epsilon$ is a small positive number to ensure $\gamma \bar{T} > J_{\text{max}}$;
4. Solve the viscosity solution $W^c_f$ of Eq. (45);
5. $R^c_S(J_1) \leftarrow \left\{ s \in \mathbb{R}^n | W^c_f(s, 0) \leq J_1 \right\}$, ..., $R^c_S(J_M) \leftarrow \left\{ s \in \mathbb{R}^n | W^c_f(s, 0) \leq J_M \right\}$;
6. **Return** $R^c_S(J_1), \ldots, R^c_S(J_M)$;

$\tau \in [t, \bar{T}]$ [7]. Therefore, for a small enough $\Delta t$, (26) can be rewritten as

$$W^c_f(s, t) = \min_{a \in A, b \in B} \left\{ \hat{c}(s, a, b) \Delta t + W^c_f(s + \hat{f}(s, a, b) \Delta t, t + \Delta t) \right\}. \tag{46}$$

The Taylor expansion of $W^c_f(\cdot, \cdot)$ at $(s, t)$ yields

$$W^c_f(s + \hat{f}(s, a, b) \Delta t, t + \Delta t) = W^c_f(s, t) + \frac{\partial W^c_f}{\partial s}(s, t) \hat{f}(s, a, b) \Delta t + \frac{\partial W^c_f}{\partial t}(s, t) \Delta t. \tag{47}$$

Substituting (47) into (46) yields

$$\frac{\partial W^c_f}{\partial t}(s, t) + \min_{a \in A, b \in B} \left\{ \hat{c}(s, a, b) + \frac{\partial W^c_f}{\partial s}(s, t) \hat{f}(s, a, b) \right\} = 0. \tag{48}$$

On the other hand

$$W^c_f(s, \bar{T}) = \int_{t}^{\bar{T}} \hat{c}(s(t), a(t)), \beta[a(t)]dt = 0. \tag{49}$$

Equations (47) and (49) complete the proof.

**Remark 5:** Set the running cost function to $c(s, a, b) \equiv 1$, in which case $\hat{c}(s, a, b) = \|s\|$ and $W^c_f(\cdot, \cdot, \cdot)$ degenerate into $W_f(\cdot, \cdot)$. Therefore, Theorem 2 is a special case of Theorem 4.

Theorems 3 and 4 imply that for any $J \leq \gamma \bar{T}$, the $J$-sub-LS of the solution of the HJ PDE in (45) describes $R^c_S(J)$. The work [38] presents in detail the viscosity solution of the HJ PDE. It is first necessary to specify a rectangular region in the state space as the computational domain, and then discretize it into a Cartesian grid. The accuracy of the viscosity solution depends on the number of grids. Algorithm 1 describes our method.

V. NUMERICAL EXAMPLES

Two examples are provided in this section to demonstrate the suggested method’s validity. The first one is the RT computation of a two-dimensional (2-D) system, that may be solved analytically. It investigate the influence of grid size on computational accuracy. The second example, based on a pursuit–evasion game, shows the practical application of CRT.
The variations in the relative volume errors with the number of grids are shown in Fig. 2. It can be seen that the variation in the computational accuracy of the proposed method with the number of grids is comparable to that of the LS method, and there is no significant effect of the time step size on the computational accuracy.

Although the proposed method is not superior to the LS method in terms of accuracy, it has a significant advantage in terms of storage space consumption. Taking the four time horizons $T_1 = 0.25$, $T_2 = 0.5$, $T_3 = 0.75$, and $T_4 = 1$ as an example, in the proposed method, only $W_S^l(\cdot,0)$ needs to be saved to save all four RTs, $R_S(T_1)$, $R_S(T_2)$, $R_S(T_3)$, and $R_S(T_4)$, see Fig. 3(a). In contrast, in the classical LS method, if the terminal condition of HJ PDE is set to $V(s,T_4) = \ell(s)$, in order to save all four RTs, one needs to save $V(\cdot, T_4 - T_1)$, $V(\cdot, T_4 - T_2)$, $V(\cdot, T_4 - T_3)$, and $V(\cdot, 0)$, see Fig. 3(b). The proposed method consumes only a quarter of the storage space of the LS method for the same grid points’ number.

### B. Pursuit–Evasion Game

Two flight vehicles move on a plane without obstacles. Both vehicles are modeled as a simple mass point with fixed linear velocity $v$ and controllable angular velocity. The pursuer, Player I, tries to capture the evader while the evader, Player II, tries to get away from the pursuer. When the distance between two vehicles is less than $r$, we consider that the pursuer captures the evader. We aim to compute the set of initial states from which the pursuer can capture the evader within the given allowable cost. Translating into reachability terms, $s = [x, y, \theta]^T \in \mathbb{R}^2 \times [0, 2\pi]$ is the system state, the target set $S = \{[x, y, \theta]^T | x^2 + y^2 \leq r^2 \}$. The evolution of the system can be described by the following equation:

$$
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
-v + v \cos \theta + ay \\
v \sin \theta - ax \\
b - a
\end{bmatrix}
$$

(53)

where $a \in A = [-1, 1]$ and $b \in B = [-1, 1]$ are control inputs of Player I and Player II, respectively. The variables in the preceding equation are described in Fig. 4.

The parameters in this example are set as $v = 5$, $r = 5$.

The running cost function is

$$
c(s, a, b) = 1 + \lambda \left( x^2 + y^2 + a^2 + b^2 \right)
$$

(55)

and the given allowable costs are $J_1 = 0.5$, $J_2 = 1$, $J_3 = 1.5$, and $J_4 = 2$.

We investigate two different situations: $\lambda = 0$ and $\lambda = 0.1$. In the former one, the running cost $c(s, a, b) \equiv 1$, and the issue degenerates into RT computation, which can be addressed using the classical LS method. Our method’s solver parameters are provided in Table II.

A comparison of the suggested approach’s outputs to those obtained using the classical LS method is presented in Fig. 5. The computational domain and grid point count in the classical LS approach are identical to those in our method. As can be observed, the suggested technique almost exactly matches the envelope of the RT computed using the classical LS method.
The latter case concerns the CRT computation. CRTs under $J_1$, $J_2$, $J_3$, and $J_4$ are shown in Fig. 6.

VI. CONCLUSION

This article proposes a new method for computing the RTs for nonlinear differential games with two players. In this method, an HJ equation with a running cost function is numerically solved, and the RT is described as a sub-LS of the viscosity solution of the HJ equation.

This article also introduces a novel variant of the RT known as the CRT, which is a set of states that can touch the target set before the performance index grows to a given allowable cost. Such an RT can be obtained by specifying the corresponding running cost function for the HJ equation.

Another advantage of the proposed method is that it reduces the storage space consumption. The CRTs with different allowable costs (or the RTs with different time ranges) are represented as a cluster of sub-LSs of the viscosity solution of the HJ equation at some time point, whereas in the classical LS method, saving multiple RTs requires saving the solution of the HJ equation at multiple time points, which can lead to several times the storage space consumption of the proposed method.

The main drawback of the presented method, and also of the classical LS method, is that the complexity of the algorithm increases exponentially with the state space’s dimensionality [7], [40]. Some measures have been taken to overcome this drawback, such as using projection to decompose a high-dimensional reachable set into several low-dimensional reachable sets [22], [41], [42] and decomposing a high-dimensional system into multiple cascade low-dimensional systems based on the time scale principle and solving them sequentially [43], [44], [45]. These will be considered in our future work.

REFERENCES

[1] B. Luo, Y. Yang, and D. Liu, “Policy iteration Q-learning for data-based two-player zero-sum game of linear discrete-time systems,” IEEE Trans. Cybern., vol. 51, no. 7, pp. 3630–3640, Jul. 2021, doi: 10.1109/TCYB.2020.2970969.
Wei Liao received the B.S. degree in aircraft design from Nanchang Hangkong University, Nanchang, China, in 2012, and the Ph.D. degree in aircraft design from the Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 2021. He is currently a Lecturer with the Advanced Manufacturing School, Nanchang University, Nanchang. His research focuses on optimal control theory, flight control system, and reinforcement learning.

Taotao Liang received the B.S. and Ph.D. degrees in aircraft design from the Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 2016 and 2022, respectively. He is currently a Lecturer with the College of General Aviation and Flight, Nanjing University of Aeronautics and Astronautics. His research focuses on the landing gear design of hypersonic aircrafts, safety analysis of the aerial vehicles, and fault diagnosis of aircraft systems.

Pengwen Xiong (Senior Member, IEEE) received the B.S. degree in electronic information engineering from the North University of China, Taiyuan, China, in 2009, and the Ph.D. degree in instrument science and technology from Southeast University, Nanjing, China, in 2015. From 2013 to 2014, he visited the Laboratory for Computational Sensing and Robotics, Johns Hopkins University, Baltimore, MD, USA. From 2017 to 2019, he was a Postdoctoral Research Fellow with the School of Instrument Science and Engineering, Southeast University. From 2022 to 2023, he was a Visiting Scholar with the Department of Automation, University of Science and Technology, Hefei, China. He is currently a Professor and the Vice Dean of the School of Advanced Manufacturing, Nanchang University, Nanchang, China. His research interests include robotic sensing and controlling.

Chen Wang received the B.S. degree in mechanical engineering and the Ph.D. degree in flight vehicle design from the Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 2014 and 2020, respectively. He is currently an Associate Professor with the School of Astronautics, Nanjing University of Aeronautics and Astronautics. He is major in the design of deep space exploration landing mechanism, spacecraft mechanism design, and dynamics research. Dr. Wang was selected for the Young Elite Scientists Sponsorship Program by the China Association for Science and Technology and won the Second Prize of the Science and Technology of Jiangsu Province.

Aiguo Song (Senior Member, IEEE) received the B.S. degree in automatic control and the M.S. degree in measurement and control from the Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 1990 and 1993, respectively, and the Ph.D. degree in measurement and control from Southeast University, Nanjing, in 1996. From 1996 to 1998, he was an Associate Researcher with the Intelligent Information Processing Laboratory, Southeast University, where he is currently a Professor with the Department of Instrument Science and Engineering. His research interests include haptic display, robot tactile sensor, and telerehabilitation robot. Prof. Song is a member of the Chinese Instrument and Control Association.

Peter X. Liu (Fellow, IEEE) received the B.Sc. degree in mechanical engineering and the M.Sc. degree in instrumentation and control engineering from Northern Jiaotong University, Beijing, China, in 1992 and 1995, respectively, and the Ph.D. degree in electrical and computer engineering from the University of Alberta, Edmonton, AB, Canada, in 2002. He has been with the Department of Systems and Computer Engineering, Carleton University, Ottawa, ON, Canada, since July 2002, where he is currently a Professor. His interests include interactive networked systems and teleoperation, haptics, surgical simulation, control, and intelligent systems. Dr. Liu has served as an Associate Editor for several journals, including the IEEE/ASME TRANSACTIONS ON MECHATRONICS, IEEE TRANSACTIONS ON CYBERNETICS, IEEE TRANSACTIONS ON AUTOMATION SCIENCE AND ENGINEERING, IEEE/CAA JOURNAL OF AUTOMATICA SINICA, and IEEE TRANSACTIONS ON INSTRUMENTATION AND MEASUREMENT. He is a licensed member of the Professional Engineers of Ontario, and a Fellow of the Engineering Institute of Canada and the Canadian Academy of Engineering.