Stationary solutions to a system of size-structured populations with nonlinear growth rate

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(Received 29 September 2010; final version received 7 May 2011)

We study stationary solutions to a system of size-structured population models with nonlinear growth rate. Several characterizations of stationary solutions are provided. It is shown that the steady-state problem can be converted into different problems such as two types of eigenvalue problems and a fixed-point problem. In the two-species case, we give an existence result of nonzero stationary solutions by using the fixed-point problem.

Keywords: system of size-structured populations; stationary solutions; nonlinear growth rate

AMS Subject Classification: 92D25; 35F25

1. Introduction

In this paper, we study stationary solutions to the following system of size-structured population models:

\[
\begin{align*}
    u^i_t + (V^i(x, P(t))u^i)_x &= G^i(u(\cdot, t))(x), & x \in [0, l^i), & 0 \leq t \leq T, \\
    V^i(0, P(t))u^i(0, t) &= F^i(u(\cdot, t)), & 0 \leq t \leq T, \\
    u^i(x, 0) &= u^i_0(x), & x \in [0, l^i), \\
    P(t) &= (P^1(t), \ldots, P^N(t)), & P^i(t) = \int_0^{l^i} w^i(x)u^i(x, t) \, dx.
\end{align*}
\]

(1)

Here, the function \(u^i(x, t)\) \((i = 1, \ldots, N)\) stands for the population density of the \(i\)th species with respect to size \(x \in [0, l^i)\) at time \(t \in [0, T]\), where \(0 < l^i < \infty\) is the maximum size of the \(i\)th species; \(P^i(t) = \int_0^{l^i} w^i(x)u^i(x, t) \, dx\) represents a weighted total population of the \(i\)th species at time \(t\) with a weight function \(w^i \in W^{1,\infty}(0, l^i; \mathbb{R}_+)\); the function \(V^i(x, P(t))\) is the growth rate of the \(i\)th species depending on the individual’s size \(x\) and every total population \(P(t)\), and hence nonlinear. \(F^i\) and \(G^i\) are given mappings corresponding to the birth and ageing functions, respectively.

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ISSN 1751-3758 print/iSSN 1751-3766 online
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http://dx.doi.org/10.1080/17513758.2011.587546
http://www.tandfonline.com
Example 1.1 (species interaction model; cf. [11]) An $N$-species interaction model is formulated by taking the vital rates as follows. The growth rate $V^i(x, P(t))$ is supposed to depend on each weighted total population $P^i(t) = \int_0^l w^i(x)u^i(x, t) \, dx$. The functions $F^i$ and $G^i$ are given by
\[
F^i(u(\cdot, t)) = \int_0^l b^i(x, P(t))u^i(x, t) \, dx,
\]
\[
G^i(u(\cdot, t))(x) = -\left[ \sum_{j=1}^N m^{ij}(x, P^j(t)) \right] u^i(x, t).
\]
Here, $b^i(x, P(t))$ is the fertility rate of the $i$th species depending on each weighted total population, $m^{ij}(x, P^j(t))$ stands for the mortality rate of the $i$th species caused by the interaction with the $j$th species.

Example 1.2 (subpopulation model; cf. [1]) Consider a model, where the population is divided into $N$ subclasses. In this model, $l^i = l$ for $i = 1, \ldots, N$. The growth rate $V^i(x, |P(t)|)$ is supposed to depend on the total weighted population size $|P(t)| = \sum_{i=1}^N \int_0^l w^i(x)u^i(x, t) \, dx$. The functions $F^i$ and $G^i$ are given by
\[
F^i(u(\cdot, t)) = \sum_{j=1}^N \int_0^l b^{ij}(x, |P(t)|)u^j(x, t) \, dx,
\]
\[
G^i(u(\cdot, t))(x) = -m^i(x, |P(t)|)u^i(x, t).
\]
Here, $b^{ij}(x, |P(t)|)$ denotes the reproduction rate of the $i$th subclass from the $j$th subclass of the population and $m^i(x, |P(t)|)$ is the mortality rate depending on $|P(t)|$.

We consider the existence problem of nonzero stationary solutions to the general system (1) containing both multi-species interaction models and subpopulation models as above. The existence problem is shown to be reduced to some equivalent problems such as (1) the existence of zero eigenvalue of a certain nonlinear operator, (2) the existence of eigenvalue 1 of a so-called net reproduction $N \times N$ matrix, and (3) the existence of a nonzero fixed point of a certain nonlinear mappings from $\mathbb{R}^{2N}$ into itself. In case of $N = 2$, we establish the existence of a nonzero stationary solution using the fixed-point problem in the spirit of Prüss [9] and Webb [11].

For single species models, stationary solutions have been discussed in [4]. Stability properties of stationary solutions have been investigated by semigroup theory by Farkas and Hagen [5]. Recently, Borges et al. [3] and Farkas et al. [6] have studied the existence problem of nonzero stationary solutions for certain structured population models by using zero eigenvalue problems. Also, Walker [10] has studied nonzero stationary solutions for age- and spatially structured population models by using a fixed-point problem.

In Section 2, our assumptions and preliminary results are presented. Some characterizations of stationary solutions are provided in Section 3, and then we show the existence of nonzero stationary solutions for two species and/or two subclass models in Section 4.

2. Preliminaries

Let $l = \max\{l^1, \ldots, l^N\}$ and put $L^i_0 := \{\phi = (\phi^1, \ldots, \phi^N) \in L^1(0, l; \mathbb{R}^N) \mid \phi^i(x) = 0 \text{ a.e. } x \in (l^i, l)\}$, $L^{1}_{0,+} := \{\phi \in L^1_0 \mid \phi^i(x) \geq 0 \text{ a.e. } x \in (0, l)\}$. $L^1_0$ is equipped with the norm $\|\phi\|_{L^1_0} = \int_0^l |\phi^i(x)| \, dx$. For $\phi \in L^1_0$, the operator $A^i \phi := \sum_{j=1}^N b^{ij}(x, |P(t)|)u^j(x, t)$ is given by
\[
A^i \phi = \sum_{j=1}^N b^{ij}(x, |P(t)|)u^j(x, t).
\]
\[ \int_0^{|x|_N} \phi(x) \, dx, \text{ where } |x|_N = \sum_{i=1}^N |x_i| \text{ for } x := (x^1, \ldots, x^N) \in \mathbb{R}^N. \]

In what follows, we assume the following hypotheses:

(V) For each \(i = 1, \ldots, N\), \(V^i \in C([0, 1] \times \mathbb{R}^N)\) and satisfies the following,

(i) There exists \(\bar{V} > 0\) such that \(0 < V^i(x, P) \leq \bar{V}\) for \((x, P) \in [0, l^i) \times \mathbb{R}^N; V^i(x, P) = 0\) for \((x, P) \in [l^i, 1] \times \mathbb{R}^N.\)

(ii) There is a nondecreasing function \(c: [l^i, \infty) \rightarrow [l^i, \infty)\) such that

\[ |V^i(x, P) - V^i(\hat{x}, \hat{P})| \leq V_i(|x - \hat{x}| + |P - \hat{P}|_N), \]

\[ |V^i(x, P) - V^i(x, \hat{P})| \leq V_i|P - \hat{P}|_N \]

for \((x, P), (\hat{x}, \hat{P}) \in [0, 1] \times \mathbb{R}^N, |P|_N, |\hat{P}|_N \leq r.\)

(F) For each \(i = 1, \ldots, N\), \(F^{i}\) has the following form:

\[ F(\phi)^i = \sum_{j=1}^N \int_0^{l^i} \beta^{ij}(x, P\phi)\phi^j(x) \, dx \]

for \(\phi \in L^1_0, \phi = (P^{i}\phi, \ldots, P^{N}\phi)\) with \(P^{i}\phi = \int_0^{l^i} w^i(x)\phi^i(x) \, dx\) and \(\beta^{ij} : [0, l^i) \times \mathbb{R}^N \rightarrow [0, \infty)\) satisfies the following conditions.

(i) There is a nondecreasing function \(c^{1}_{\beta} : [0, \infty) \rightarrow [0, \infty)\) such that

\[ |\beta^{ij}(x, P) - \beta^{ij}(\hat{x}, P)| \leq c^{1}_{\beta}(|P|_N)|x - \hat{x}| \]

for \(x, \hat{x} \in [0, l^i)\) and \(P \in \mathbb{R}^N.\)

(ii) There is a nondecreasing function \(c^{2}_{\beta} : [0, \infty) \rightarrow [0, \infty)\) such that

\[ 0 \leq \beta^{ij}(x, P) \leq c^{2}_{\beta}(|P|_N) \]

for all \(x \in [0, l^i)\) and \(P \in \mathbb{R}^N.\)

(iii) There is a nondecreasing function \(c^{3}_{\beta} : [0, \infty) \rightarrow [0, \infty)\) such that

\[ |\beta^{ij}(x, P) - \beta^{ij}(x, \hat{P})| \leq c^{3}_{\beta}(r)|P - \hat{P}|_N \]

for all \(x \in [0, l^i), P, \hat{P} \in \mathbb{R}^N\) with \(|P|_N, |\hat{P}|_N \leq r.\)

(G) For each \(i = 1, \ldots, N\), \(G^{i}\) has the following form:

\[ G(\phi)^i(x) = -\mu^{i}(x, P\phi)\phi^i(x), \quad \text{a.e. } x \in (0, l^i), \]

for \(\phi \in L^1_0, \phi = (P^{i}\phi, \ldots, P^{N}\phi)\) with \(P^{i}\phi = \int_0^{l^i} w^i(x)\phi^i(x) \, dx\) and \(\mu^{i} : [0, l^i) \times \mathbb{R}^N \rightarrow [0, \infty)\) satisfies the following conditions.

(i) There is a nondecreasing function \(c^{1}_{\mu} : [0, \infty) \rightarrow [0, \infty)\) such that

\[ |\mu^{i}(x, P) - \mu^{i}(\hat{x}, P)| \leq c^{1}_{\mu}(|P|_N)|x - \hat{x}| \]

for all \(x, \hat{x} \in [0, l^i), P \in \mathbb{R}^N.\)

(ii) There is a nondecreasing function \(c^{2}_{\mu} : [0, \infty) \rightarrow [0, \infty)\) such that

\[ 0 \leq \mu^{i}(x, P) \leq c^{2}_{\mu}(|P|_N) \]

for all \(x \in [0, l^i), P \in \mathbb{R}^N.\)
for all $x \in [0, l')$, $P, \hat{P} \in \mathbb{R}^N$ with $|P|_N, |\hat{P}|_N \leq r$.

(iv) There exists a positive constant $\underline{\mu} > 0$ such that $\mu^i(x, P) \geq \underline{\mu}$ for all $x \in [0, l')$, $P \in \mathbb{R}^N$.

Remark 2.1 For $(x, P) \in (-\infty, 0) \times \mathbb{R}^N$, put $V^i(x, P) = V^i(0, P)$ and for $(x, P) \in (l', \infty) \times \mathbb{R}^N$, put $V^i(x, P) = 0$. Then, $V^i(x, P)$ is extended to $\mathbb{R} \times \mathbb{R}^N$ and $V$ satisfies the Lipschitz condition $|V(x, P) - V(\hat{x}, P)| \leq V_r|x - \hat{x}|$ for $x, \hat{x} \in \mathbb{R}$, $P \in \mathbb{R}^N$ and $|P|_N \leq r$.

For each $P \in C([0, T]; \mathbb{R}^N)$, we define the characteristic curve $\varphi^i_P(t; t_0, x_0)$ through $(x_0, t_0) \in [0, l') \times [0, T]$ by the unique solution of the following differential equation:

$$x'(t) = V^i(x(t), P(t)), \quad t \in [0, T], \quad x(t_0) = x_0 \in [0, l').$$

For $(x_0, t_0) \in [l', l] \times [0, T]$, let $\varphi^i(t; t_0, x_0)x = x_0$, $t \in [0, T]$. Let $z^i_P(t) := \varphi^i_P(t; 0, 0)$. For $x_0 < z^i_P(t_0)$, the initial time $\tau^i_P = \tau^i_P(t_0, x_0)$ is defined implicitly by the relation

$$\varphi^i_P(\tau^i_P; t_0, x_0) = 0.$$

For $\phi \in L^1_0$, put

$$G^i_v(\phi)(x) := G^i(\phi')(x) - V^i(x, P)\phi^i(x), \quad \text{a.e. } x \in (0, l),$$

where $P\phi = (P^1\phi, \ldots, P^N\phi)$ and $P^i\phi = \int_0^l w^i(x)\phi^i(x) \, dx$. Integrating Equation (1) along the characteristic curve, we come up with the following definition of solutions.

**Definition 2.2** $u \in C([0, T]; L^1_{0,+})$ is said to be a solution of Equation (1) if it satisfies

$$u^i(x, t) = \begin{cases} 
\frac{F^i(u(\cdot, \tau^i_P))}{V^i(0, P(\tau^i_P))} + \int_{\tau^i_P}^t G^i_v(u(\cdot, s))(\varphi^i_P(s; t, x)) \, ds, & \text{a.e. } x \in (0, z^i_P(t)) \\
\varphi^i_P(0; t, x) + \int_0^t G^i_v(u(\cdot, s))(\varphi^i_P(s; t, x)) \, ds, & \text{a.e. } x \in (z^i_P(t), l),
\end{cases}$$

$$P(t) = (P^1(t), \ldots, P^N(t)), \quad P^i(t) = \int_0^l w^i(x)u^i(x, t) \, dx,$$

where $\tau^i_P = \tau^i_P(t, x)$.

Under assumptions (V), (F), and (G), we can show that for any initial value $u_0 \in L^1_{0,+}$, there exists a unique local solution $u \in C([0, T]; L^1_{0,+})$ of Equation (1) for some $T > 0$. Further, assume that there exists $\omega_0 \in \mathbb{R}$ such that for each $j = 1, \ldots, N,$

$$\sum_{i=1}^N \beta^{ij}(x, P) - \mu^j(x, P) \leq \omega_0 \quad \text{a.e. } x \in (0, l'), \ P \in \mathbb{R}^N_+.$$

Then, there exists a unique global solution $u \in C([0, \infty); L^1_{0,+})$ of Equation (1) satisfying

$$\|u(\cdot, t)\|_{L^1_0} \leq e^{\omega t}\|u_0\|_{L^1_0}.$$

The above facts are proved by similar arguments as in [7]. We will discuss the existence results for more general dynamic models than Equation (1) elsewhere [8].
3. Stationary solutions

In view of Definition 2.2, we define stationary solutions as follows.

**Definition 3.1** The function \( \hat{\phi} \in L^1_{0,+} \) is said to be a stationary solution of Equation (1) if \( \hat{\phi} \) satisfies

\[
\hat{\phi}^i(x) = \begin{cases} 
\frac{F^i(\hat{\phi})}{V^i(0, P\hat{\phi})} + \int_{\tau_p}^t G^i_V(\hat{\phi})(\varphi^i_p(s; \tau, t, x)) \, ds & \text{for a.e. } x \in (0, z^i_p(t)) \\
\hat{\phi}^i(\varphi^i_p(0; t, x)) + \int_0^t G^i_V(\hat{\phi})(\varphi^i_p(s; t, x)) \, ds & \text{for a.e. } x \in (z^i_p(t), l^i), 
\end{cases}
\]

(2)

\[
P(t) \equiv P\hat{\phi} = ((P\hat{\phi})^1, \ldots, (P\hat{\phi})^N), \quad (P\hat{\phi})^i = \int_0^{l^i} w^i(x) \hat{\phi}^i(x) \, dx.
\]

Define an operator \( A \) from \( L^1_{0,+} \) to \( L^1_{0,+} \) by

\[
A\phi(x) = (A\phi^1(x), \ldots, A\phi^N(x)), \quad (A\phi^i(x)) = V^i(x, P\phi)(\phi^i)'(x) - G^i_V(\phi)(x) \quad \text{a.e. } x \in (0, l^i)
\]

for \( \phi \in D(A) \), where

\[
D(A) = \{ \phi \in L^1_{0,+} \cap W^{1,1}_{\text{loc}}([0, l]; \mathbb{R}^N) \mid V^i(\cdot, P\phi)(\phi^i)'(\cdot) \in L^1(0, l^i), V^i(0, P\phi)\phi^i(0) = F^i(\phi), \quad i = 1, \ldots, N \}.
\]

Then, we have

**Proposition 3.2** \( \hat{\phi} \in L^1_{0,+} \) and \( \hat{\phi} \) is a stationary solution to Equation (1) if and only if \( \hat{\phi} \in D(A) \) and \( A\hat{\phi} = 0 \).

**Proof** Suppose that \( \hat{\phi} \in D(A) \) and \( A\hat{\phi} = 0 \). Then, we have

\[
V^i(x, P\hat{\phi})(\hat{\phi}^i)'(x) = G^i_V(\hat{\phi}(x), \text{ a.e. } x,
\]

\[
V^i(0, P\hat{\phi})\hat{\phi}^i(0) = F^i(\hat{\phi}).
\]

It follows that

\[
\frac{d}{ds} \hat{\phi}^i(\varphi^i_p(s; t, x)) = (\hat{\phi}^i)'(\varphi^i_p(s; t, x)) \frac{d}{ds} \varphi^i_p(s; t, x)
\]

\[
= (\hat{\phi}^i)'(\varphi^i_p(s; t, x)) V^i(\varphi^i_p(s; t, x), P\hat{\phi}) = G^i_V(\hat{\phi})(\varphi^i_p(s; t, x)).
\]

Then, for a.e. \( x < z^i_p(t) \),

\[
\hat{\phi}^i(x) = \hat{\phi}^i(\varphi^i_p(\tau_p; t, x)) + \int_{\tau_p}^t G^i_V(\hat{\phi})(\varphi^i_p(s; \tau, t, x)) \, ds
\]

\[
= \hat{\phi}^i(0) + \int_{\tau_p}^t G^i_V(\hat{\phi})(\varphi^i_p(s; \tau, 0)) \, ds
\]

and for a.e. \( x > z^i_p(t) \),

\[
\hat{\phi}^i(x) = \hat{\phi}^i(\varphi^i_p(0; t, x)) + \int_0^t G^i_V(\hat{\phi})(\varphi^i_p(s; t, x)) \, ds.
\]

Thus, \( \hat{\phi} \) satisfies Equation (2).
Conversely, suppose \( \hat{\phi} \in L^1_{0,+} \) and \( \hat{\phi} \) is a stationary solution. For \( x, x + h \in [0, l') \), there exist \( s, s' \in [0, \infty) \) such that \( x < z_p(s), x + h = \phi_p(s'; s, x) < z_p(s') \). Then, we have

\[
\frac{1}{h} [\dot{\phi}^i(x + h) - \dot{\phi}^i(x)] = \frac{1}{h} \int_s^{s'} G^i_V(\hat{\phi})(\phi^i_p(\sigma; s, x)) \, d\sigma.
\]

Changing variable \( \xi = \phi^i_p(\sigma; s, x) \), we have \( d\xi / d\sigma = V^i(\xi, \hat{\phi}) \), and hence

\[
\frac{1}{h} [\dot{\phi}^i(x + h) - \dot{\phi}^i(x)] = \frac{1}{h} \int_x^{x + h} G^i_V(\hat{\phi})(\xi) \frac{1}{V^i(\xi, \hat{\phi})} \, d\xi.
\]

Since \( \xi \mapsto G^i_V(\hat{\phi})(\xi) / V^i(\xi, \hat{\phi}) \in L^1_{\text{loc}}[0, l') \), we conclude that \( \dot{\phi}^i \) is differentiable a.e. and \( (\dot{\phi}^i)'(x) = G^i_V(\hat{\phi})(x) / V^i(x, \hat{\phi}) \), a.e. \( x \). Thus, \( \hat{\phi} \in W^1_{\text{loc}}([0, l); \mathbb{R}^N) \) and in addition, \( V^i(\cdot, \hat{\phi})(\dot{\phi}^i)'(\cdot) = G^i_V(\hat{\phi})(\cdot) \in L^1(0, l') \). Since \( x \in (0, z_p(s)) \),

\[
\dot{\phi}^i(x) = \frac{F^i(\hat{\phi})}{V^i(0, \hat{\phi})} + \int_{0}^{x} G^i_V(\phi^i_p(\sigma; s, x)) \, d\sigma
\]

\[
= \frac{1}{V^i(0, \hat{\phi})} \left[ \frac{F^i(\hat{\phi})}{V^i(0, \hat{\phi})} + \int_{0}^{x} G^i_V(\xi) \frac{1}{V^i(\xi, \hat{\phi})} \, d\xi \right].
\]

Hence, \( \dot{\phi}^i(0) = F^i(\hat{\phi}) / V^i(0, \hat{\phi}) \). Consequently, \( \hat{\phi} \in D(A) \) and \( A\hat{\phi} = 0 \).

**Remark 3.3** Putting \( P = P\phi \), the operator \( A \) can be written as a quasi-linear form \( A\phi = A_P\phi \) with \( A_P \) being a linear operator for each fixed \( P \in \mathbb{R}^N \). If we find \( P_\ast \in \mathbb{R}^N \) for which the operator \( A_P \) has zero eigenvalue, then letting \( \phi_\ast \in D(A_P) \) be a corresponding eigenvector, a nonzero stationary solution \( \phi \) is constructed by putting its \( i \)-th component as \( \phi^i = P_\ast^i \phi_\ast^i / (P\phi_\ast)^i \). Such an approach was used recently for similar structured single-species population models by Borges et al. [3] and Farkas et al. [6].

Suppose now that \( \hat{\phi} \in D(A) \) and \( A\hat{\phi} = 0 \). Then, we have

\[
V^i(x, \hat{\phi})(\dot{\phi}^i)'(x) = G^i_V(\hat{\phi})(x) - V^i(x, \hat{\phi})\dot{\phi}^i(x)
\]

\[
= -\mu^i(x, \hat{\phi})\dot{\phi}^i(x) - V^i(x, \hat{\phi})\dot{\phi}^i(x) \quad \text{a.e. } x \in (0, l'), \tag{3}
\]

\[
V^i(0, \hat{\phi})\dot{\phi}^i(0) = F^i(\hat{\phi}) = \sum_{j=1}^{N} \int_{0}^{l} \beta^{ij}(x, \hat{\phi})\dot{\phi}^j(x) \, dx. \tag{4}
\]

Define \( \Pi^i : [0, l') \times \mathbb{R}^N \to [0, \infty), i = 1, \ldots, N, \) by

\[
\Pi^i(x, P) := \exp \left( -\int_{0}^{x} \frac{\mu^i(\xi, P)}{V^i(\xi, P)} \, d\xi \right).
\]

Solving the differential equation (3), we obtain

\[
\dot{\phi}^i(x) = \hat{\phi}^i(0) \exp \left( -\int_{0}^{x} \frac{\mu^i(\xi, \hat{\phi}) + V^i(\xi, \hat{\phi})}{V^i(\xi, \hat{\phi})} \, d\xi \right)
\]

\[
= \hat{\phi}^i(0)\Pi^i(x, \hat{\phi}) \frac{V^i(0, \hat{\phi})}{V^i(x, \hat{\phi})}. \tag{5}
\]
It follows from Equation (4) that
\[
V^i(0, P\hat{\phi})\hat{\phi}^i(0) = \sum_{j=1}^{N} \int_{0}^{l} \beta^{ij}(x, P\hat{\phi})\hat{\phi}^j(0)\Pi^i(x, P\hat{\phi}) \frac{V^j(0, P\hat{\phi})}{V^j(x, P\hat{\phi})} \, dx. \tag{6}
\]

Actually, the converse is also true and we have

**Proposition 3.4** Let \( \hat{\phi} \in L^1_{0,+} \). Then, \( \hat{\phi} \in D(A) \) and \( A\hat{\phi} = 0 \) if and only if each component \( \hat{\phi}^i \) of \( \hat{\phi} \) satisfies Equations \((5)\) and \((6)\).

**Proof** It remains to show the if part. Let \( \hat{\phi} \in L^1_{0,+} \) satisfy Equations (5) and (6). It follows from Equation (5) that
\[
(\hat{\phi}^i)'(x) = -\frac{1}{V^i(x, P\hat{\phi})} \left\{ \mu^i(x, \hat{\phi})\hat{\phi}^i(x) + V^i(x, P\hat{\phi})\hat{\phi}^i'(x) \right\} \quad \text{a.e. } x. \tag{7}
\]
Hence, \( \hat{\phi} \in W^{1,1}_{\text{loc}}([0, l]; \mathbb{R}^N) \), and by (G)(ii) and Equation (7),
\[
\int_{0}^{l} |V^i(x, P\hat{\phi})(\hat{\phi}^i)'(x)| \, dx \leq \{c_2^2(\|\hat{\phi}\|_{L^1}) + L_V\} \|\hat{\phi}\|_{L^1}.
\]
By Equations (5) and (6), we have
\[
V^i(0, P\hat{\phi})\hat{\phi}^i(0) = \sum_{j=1}^{N} \int_{0}^{l} \beta^{ij}(x, P\hat{\phi})\hat{\phi}^j(0)\Pi^i(x, \hat{\phi}) \frac{V^j(0, P\hat{\phi})}{V^j(x, P\hat{\phi})} \, dx = F^i(\hat{\phi}).
\]
Therefore, we find that \( \hat{\phi} \in D(A) \), and from Equation (7), we obtain
\[
[A\hat{\phi}]^i(x) = V^i(x, P\hat{\phi})(\hat{\phi}^i)'(x) - G^i_V(\hat{\phi})(x) = 0.
\]
This completes the proof. \(\blacksquare\)

We now put
\[
\tilde{\beta}^{ij}(x, P) = \beta^{ij}(x, P) \frac{V^j(0, P)}{V^j(0, P)}, \quad \tilde{\Pi}^i(x, P) = \frac{\Pi^i(x, P)}{V^i(x, P)}.
\]
Let \( \hat{\phi} \in L^1_{0,+} \) be a stationary solution. Noting that \( \hat{\phi} \in D(A) \), put \( a^i = \hat{\phi}^i(0) \) and \( P_* = P\hat{\phi} \). Then, it follows from Equation (6) that
\[
a^i = \sum_{j=1}^{N} a^j \int_{0}^{l} \tilde{\beta}^{ij}(x, P_*)\tilde{\Pi}^j(x, P_*) \, dx. \tag{8}
\]
We now define the matrix \( \tilde{R}(P) \) having its \((i,j)\)th entry
\[
\tilde{R}^{ij}(P) := \int_{0}^{l} \tilde{\beta}^{ij}(x, P)\tilde{\Pi}^j(x, P) \, dx, \tag{9}
\]
which corresponds to the net reproduction rate of the \(i\)th species from the \(j\)th species. Let \( a = (a^1, \ldots, a^N)^T \), where \(T\) stands for the transpose. Then, relation (8) can be written as
\[
a = \tilde{R}(P_*)a
\]
Hence, we have the following two propositions.
Proposition 3.5  If $\tilde{R}(P)$ does not have eigenvalue 1 for each $P \in \mathbb{R}^N$, then only the zero stationary solution exists.

Proposition 3.6  Suppose that $\tilde{R}(P_*)$ has eigenvalue 1 for some $P_* \in \mathbb{R}_+^N$ and $a_* \in \mathbb{R}_+^N$ is a corresponding eigenvector. Then, $\hat{\phi} \in L^1_{0,+}$ defined by $\hat{\phi}^i(x) = a_*^i V^i(0, P_*) \tilde{\Pi}^i(x, P_*)$ is a nonzero stationary solution.

Let $Q(P)$ be the diagonal matrix defined by

$$Q(P) = \begin{pmatrix} Q^1(P) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & Q^N(P) \end{pmatrix}, \quad Q^i(P) = V^i(0, P) \int_0^l \tilde{\Pi}^i(x, P) \, dx.$$

Define a mapping $f : \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R}_+^N \times \mathbb{R}_+^N$ by

$$f(P, a) := (Q(P)a, \tilde{R}(P)a). \quad (10)$$

Suppose that $(P_*, a_*)$ is a fixed point of the mapping $f$. Then, we have

$$P_*^i = a_*^i V^i(0, P_*) \int_0^l \tilde{\Pi}^i(x, P_*) \, dx,$$

$$a_*^i = \sum_{j=1}^N a_*^j \int_0^l \tilde{\beta}^{ij}(x, P_*) \tilde{\Pi}^j(x, P_*) \, dx.$$

Put $\hat{\phi}(x) = (\hat{\phi}^1(x), \ldots, \hat{\phi}^N(x))$ with

$$\hat{\phi}^i(x) = a_*^i V^i(0, P_*) \tilde{\Pi}^i(x, P_*).$$

Then, it is shown that $\hat{\phi} \in L^1_{0,+}$ and $\hat{\phi}$ is a stationary solution of Equation (1). Indeed,

$$(P \hat{\phi})^i = \int_0^l \hat{\phi}^i(x) \, dx = V^i(0, P_*) a_*^i \int_0^l \tilde{\Pi}^i(x, P_*) \, dx = P_*^i,$$

$$\hat{\phi}^i(0) = V^i(0, P_*) a_*^i \tilde{\Pi}^i(0, P_*) = a_*^i,$$

and so, Equations (5) and (6) hold. By Propositions 3.2 and 3.4, $\hat{\phi} = (\hat{\phi}^1, \ldots, \hat{\phi}^N)$ is a stationary solution. Conversely, let $\hat{\phi} \in L^1_{0,+}$ is a stationary solution. Then, put $P_*^i = \int_0^l \hat{\phi}^i(x) \, dx$ and $a_*^i = \hat{\phi}^i(0)$. Here, note that $\hat{\phi} \in L^1_{0,+} \cap W^{1,1}_{\text{loc}}([0, l]; \mathbb{R}_+^N)$ by Proposition 3.2 and hence $\hat{\phi}^i(0)$ is meaningful. It is easy to see that $(P_*, a_*) = (P_*^1, \ldots, P_*^N, a_*^1, \ldots, a_*^N)$ satisfies $f(P_*, a_*) = (P_*, a_*)$. Thus, we have

Proposition 3.7  Finding a nonzero stationary solution of Equation (1) is equivalent to finding a nonzero fixed point of the mapping $f$ defined by Equation (10).

4. Existence of stationary solutions in two-species case

In this section, we restrict ourselves to the two-species case, i.e. $N = 2$ and establish the existence of a nonzero stationary solution to Equation (1). In view of Proposition 3.7, we will seek a nonzero fixed point of the mapping $f : \mathbb{R}^4 \to \mathbb{R}^4$ defined by Equation (10).
To do so, we will use the following fixed-point theorem due to Amann [2, Theorem 12.3] (see also [11, Proposition 4.2]).

**Proposition 4.1** Let \( X \) be a Banach space and \( C \) be a closed convex cone in \( X \). For \( \sigma > 0 \), let \( C_{\sigma} := \{ x \in C : \| x \| \leq \sigma \} \), and let \( f : C_{\sigma} \to C \) be such that \( f \) is continuous and \( f(C_{\sigma}) \) has compact closure in \( C \). Suppose that

(a) \( f(x) \neq \lambda x \) for all \( x \in C_{\sigma} \) such that \( \| x \| = \sigma \) and for all \( \lambda > 1 \).

(b) There exists \( \tau \in (0, \sigma) \) and \( x_1 \in C \), \( x_1 \neq 0 \) such that \( x - f(x) \neq \lambda x_1 \) for all \( x \in C_{\sigma} \) such that \( \| x \| = \tau \) and for all \( \lambda > 0 \).

Then, there exists \( x_0 \in \{ x \in C : \tau \leq \| x \| \leq \sigma \} \) such that \( f(x_0) = x_0 \).

In the case of \( N = 2 \), the matrix \( \tilde{R}(P) \) for \( P \in \mathbb{R}^2 \) is written as

\[
\tilde{R}(P) = \begin{pmatrix}
\tilde{R}^{11}(P) & \tilde{R}^{12}(P) \\
\tilde{R}^{21}(P) & \tilde{R}^{22}(P)
\end{pmatrix},
\tilde{R}^{ij}(P) := \int_0^l \tilde{\beta}^{ij}(x, P) \tilde{\Pi}^j(x, P) \, dx.
\tag{11}
\]

Note that \( \tilde{R}(P) \) has two real eigenvalues \( \lambda_1(P) \) and \( \lambda_2(P) \) such that

\[
\lambda_1(P) := \frac{T_p - \sqrt{T_p^2 - 4D_p}}{2} \leq \lambda_2(P) := \frac{T_p + \sqrt{T_p^2 - 4D_p}}{2},
\]

where \( T_p = \tilde{R}^{11}(P) + \tilde{R}^{22}(P) \) and \( D_p = \tilde{R}^{11}(P)\tilde{R}^{22}(P) - \tilde{R}^{12}(P)\tilde{R}^{21}(P) \). In fact, the characteristic equation is

\[
0 = \det(\lambda - \tilde{R}(P)) = \det \begin{pmatrix}
\lambda - \tilde{R}^{11}(P) & -\tilde{R}^{12}(P) \\
-\tilde{R}^{21}(P) & \lambda - \tilde{R}^{22}(P)
\end{pmatrix} = \lambda^2 - T_p \lambda + D_p
\]

and \( T_p^2 - 4D_p = (\tilde{R}^{11}(P) - \tilde{R}^{22}(P))^2 + 4\tilde{R}^{12}(P)\tilde{R}^{21}(P) \geq 0 \). Then our existence result of nonzero stationary solutions to Equation (1) is stated as follows.

**Theorem 4.2** Suppose \( N = 2 \) and that

(i) \( \lambda_2(0) > 1 \), \( \lambda_1(0) \neq 1 \) and \( \tilde{R}^{11}(0) \neq 1 \) (or \( \tilde{R}^{22}(0) \neq 1 \)),

(ii) there exists \( r > 0 \) such that \( \lambda_2(P) < 1 \) for all \( P \in \mathbb{R}^2 \) satisfying \( |P|_2 \geq r \).

Then, there exists a nonzero stationary solution \( \tilde{\varphi} \in L_{0+}^1 \) of Equation (1).

**Proof** Let us consider the mapping \( f : \mathbb{R}^4 \to \mathbb{R}^4 \) defined by Equation (10), i.e. for \( P = (P^1, P^2) \in \mathbb{R}^2 \) and \( a = (a^1, a^2) \),

\[
f(P, a) := (Q(P)a, \tilde{R}(P)a),
\]

where \( Q(P) \) is the matrix defined by

\[
Q(P) = \begin{pmatrix}
Q^{11}(P) & 0 \\
0 & Q^{22}(P)
\end{pmatrix}, \quad Q^i(P) = V^i(0, P) \int_0^l \tilde{\Pi}^i(x, P) \, dx
\]

and \( \tilde{R}(P) \) is the matrix defined by Equation (11).
Since $\tilde{\beta}^{ij}(x, P), \tilde{\Pi}^i(x, P), V^i(0, P)$ are continuous in $P$, $f(P, a)$ is shown to be continuous. Note that the following estimate holds:
\[
\int_0^l \tilde{\Pi}^i(x, P) \, dx \leq \int_0^l \frac{1}{V^i(x, P)} \exp \left( -\mu \int_0^x \frac{1}{V^i(y, P)} \, dy \right) \, dx \leq \mu^{-1},
\]
where $\mu$ appears in assumption (G)(iv). Then, from assumption (F)(ii), we have
\[
|f(P, a)|_4 \leq \frac{\tilde{V} + 2c^2_\beta(|P|_2)}{\mu} |a|_2.
\]
(12)

Let $\sigma := r + 2c^2_\beta(r)/V_*(r)$ with $V_*(r) = \inf\{V^i(0, P) \mid |P|_2 \leq r, \ i = 1, 2\}$, and let $C_\sigma = \{(P, a) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \mid |P|_2 + |a|_2 \leq \sigma\}$. The estimate (12) shows that $f(C_\sigma)$ is bounded and hence has compact closure in $\mathbb{R}_+^2 \times \mathbb{R}_+^2$.

To show that condition (a) in Proposition 4.1 holds, let $(P, a) \in C_\sigma$ such that $|(P, a)|_4 = \sigma$ and assume that there exists $\lambda > 1$ such that $f(P, a) = \lambda (P, a)$. Then,
\[
\lambda P^i = a^i V^i(0, P) \int_0^l \tilde{\Pi}^i(x, P) \, dx,
\]
(13)
\[
\lambda a^i = \sum_{j=1}^2 a^j \tilde{\beta}^{ij}(P).
\]
(14)

Note that $a^i = 0$ if and only if $P^i = 0$ by Equation (13). Thus, the vector $a = (a^1, a^2)$ must be nonzero and relation (14) implies that $a$ is an eigenvector of $\tilde{R}(P)$ associated with $\lambda > 1$. Then, by hypothesis (ii), we have $|P|_2 < r$. From Equations (12)–(14), we have
\[
\sigma = |P|_2 + |a|_2 \leq r + \sum_{i=1}^2 \sum_{j=1}^2 \frac{1}{\lambda} a^i \int_0^l \tilde{\beta}^{ij}(x, P) \tilde{\Pi}^i(x, P) \, dx
\]
\[
\leq r + c^2_\beta(r) \sum_{i=1}^2 \sum_{j=1}^2 \frac{1}{\lambda} a^i \int_0^l \tilde{\Pi}^i(x, P) \, dx
\]
\[
= r + c^2_\beta(r) \sum_{i=1}^2 \sum_{j=1}^2 \frac{P^i}{V^i(0, P)} \leq r + \frac{2c^2_\beta(r)r}{V_*(r)} = \sigma.
\]

This is a contradiction and condition (a) in Proposition 4.1 is satisfied.

Next, we will show that hypothesis (b) of Proposition 4.1 holds. The following two cases have to be considered: cases $\lambda_1(0) < 1 < \lambda_2(0)$ and $1 < \lambda_1(0) \leq \lambda_2(0)$. Note that if $\lambda_1(0) < 1 < \lambda_2(0)$, then $1 - T_0 + D_0 < 0$, and if $1 < \lambda_1(0) \leq \lambda_2(0)$, then $1 - T_0 + D_0 > 0$. By continuity, we can take sufficiently small $\tau \in (0, \sigma)$ such that if $|P|_2 \leq \tau$, then $1 - T_p + D_p < 0$ and $\lambda_2(P) > 1$ hold in case of $\lambda_1(0) < 1 < \lambda_2(0)$, and $1 - T_p + D_p > 0$ and $\lambda_2(P) > 1$ hold in case of $1 < \lambda_1(0) \leq \lambda_2(0)$.

First, consider cases $\lambda_1(0) < 1 < \lambda_2(0)$ and $\tilde{R}^{11}(0) > 1$. In this case, let $x_1 = (0, 0, 1, 0) \in \mathbb{R}^4$. To show that hypothesis (b) of Proposition 4.1 holds, suppose that there exists $\lambda > 0$ and $(P, a) \in C_\sigma$ such that $|(P, a)|_4 = \tau$ and $(P, a) - f(P, a) = \lambda x_1$. Then, we have
\[
(I - \tilde{R}(P))a = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]
where \(I\) is the unit matrix. From this, we have
\[
(1 - T_P + D_P) a^1 = \lambda (1 - \tilde{R}^{22}(P)),
\]
\[
(1 - T_P + D_P) a^2 = \lambda \tilde{R}^{21}(P).
\]

Since \(1 - T_P + D_P < 0\), Equations (15) and (16) imply that \(\tilde{R}^{22}(P) > 1\) and \(a^2 = \tilde{R}^{21}(P) = 0\). But then, \(1 - T_P + D_P = (1 - \tilde{R}^{11}(P))(1 - \tilde{R}^{22}(P)) > 0\) and this contradicts \(1 - T_P + D_P < 0\).

Secondly, consider cases \(\lambda_1(0) < 1 < \lambda_2(0)\) and \(\tilde{R}^{11}(0) < 1\). Suppose that there exists \(\lambda > 0\) and \((P, a) \in C_\sigma\) such that \(|(P, a)|_4 = \tau\) and \((P, a) - f(P, a) = \lambda x_1\). Then, we have
\[
(1 - \tilde{R}(P)) a = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

From these equations, we have
\[
(1 - T_P + D_P) a^1 = \lambda \tilde{R}^{12}(P),
\]
\[
(1 - T_P + D_P) a^2 = \lambda (1 - \tilde{R}^{11}(P)).
\]

Since \(1 - T_P + D_P < 0\), Equation (17) implies that \(a^1 = \tilde{R}^{12}(P) = 0\). Then, \(a^2 > 0\) and by Equation (18), we have \(0 > (1 - T_P + D_P) a^2 = \lambda (1 - \tilde{R}^{11}(P)) \geq 0\), which is a contradiction.

Thirdly, consider the case \(1 < \lambda_1(0) \leq \lambda_2(0)\). Suppose that there exists \(\lambda > 0\) and \((P, a) \in C_\sigma\) such that \(|(P, a)|_4 = \tau\) and \((P, a) - f(P, a) = \lambda x_1\). Then, we have
\[
(\tilde{R}(P) - I) a = -\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Recall that \(\text{det}(\tilde{R}(P) - I) = 1 - T_P + D_P > 0\). If \(\tilde{R}^{11}(P) > 1\), then the matrix \(\tilde{R}(P) - I\) is positive definite and this fact contradicts Equation (19). If \(\tilde{R}^{11}(P) < 1\), then we have \(\tilde{R}^{22}(P) > 1\). In fact, \(\tilde{R}^{11}(P) + \tilde{R}^{22}(P) = T_P \geq 2 + \sqrt{T_P^2 - 4D_P} \geq 2\), and hence, \(\tilde{R}^{22}(P) > 1\). Then, Equation (19) is rewritten as
\[
(\tilde{R}(P) - I) a' = -\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]
where
\[
\tilde{R}(P) = \begin{pmatrix} \tilde{R}^{22}(P) & \tilde{R}^{21}(P) \\ \tilde{R}^{12}(P) & \tilde{R}^{11}(P) \end{pmatrix}, \quad a' = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}.
\]

Since \(\tilde{R}(P) - I\) is positive definite and \(a' \in \mathbb{R}_+^2\), Equation (20) yields the contradiction.

Consequently, condition (b) in Proposition 4.1 is satisfied in any case. Thus, by Proposition 4.1, there exists \((P_*, a_*) \in C_\sigma\) such that \(f(P_*, a_*) = (P_*, a_*)\) and \((P_*, a_*) \neq (0, 0)\). This completes the proof. \(\blacksquare\)

**Remark 4.3** In the case of species interaction as in Example 1.1, we take \(\tilde{R}^{ij}(P) = 0\) if \(i \neq j\). Then, it is easily seen that \(\tilde{R}^{11}(P) = \lambda_1(P)\) (or \(\lambda_2(P)\), respectively) and \(\tilde{R}^{22}(P) = \lambda_2(P)\) (or \(\lambda_1(P)\), respectively). Let \(\tilde{R}^{11}(P) = \lambda_1(P)\) and \(\tilde{R}^{22}(P) = \lambda_2(P)\). Then, assumptions (i) and (ii) in Theorem 4.2 read as \(\tilde{R}^{22}(0) > 1\), \(\tilde{R}^{11}(0) \neq 1\), and \(\tilde{R}^{22}(P) < 1\) for sufficiently large \(|P| > r\). Thus, Theorem 4.2 is compatible with Webb [11, Theorem 4.1], and roughly speaking, Theorem 4.2 says that if one of the two species satisfies that the net reproduction rate is greater than 1 when the total population size is zero, and less than 1 when the total population size is sufficiently large, then there exists a nonzero stationary solution.
Acknowledgements

I would like to thank two anonymous referees for their careful reading of the manuscript and for their useful comments.

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