TETRAHEDRA IN HYPERBOLIC SPACE AND HILBERT SPACES WITH PICK KERNELS

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Abstract. We study of the relation between the geometry of sets in complex hyperbolic space and Hilbert spaces with complete Pick kernels. We focus on the geometry associated with assembling sets into larger sets and of assembling Hilbert spaces into larger spaces. Model questions include describing the possible triangular faces of a tetrahedron in $\mathbb{C}H^n$ and describing the three dimensional subspaces of four dimensional Hilbert spaces with Pick kernels. Our novel technical tool is a complex analog of the cosine of a vertex angle.

1. Introduction and Summary

We begin with an informal overview, precise statements are in the later sections. This work is part of a program introduced in [ARSW] and [Ro] of studying the close relation between the geometry of finite dimensional reproducing kernel Hilbert spaces (RKHS) with the complete Pick property (CPP) and the geometry of finite sets in complex hyperbolic space, $\mathbb{C}H^n$, and real hyperbolic space, $\mathbb{R}H^n$. We focus on assembly questions, questions about constructing a set or space from designated smaller parts. The following two questions, which are essentially equivalent, are specific examples.

Question 1: Given four triangles in complex hyperbolic space, $\mathbb{C}H^n$, is there a tetrahedron in hyperbolic space with faces congruent to those triangles?

In Euclidean space of any dimension the necessary and sufficient condition for three positive numbers to be the side lengths of a triangle is that they satisfy the triangle inequality and if that holds then those lengths determine the triangle up to congruence. On the other hand, given four Euclidean triangles with matching side lengths there might not be a tetrahedron with faces congruent to those triangles, or, what is the same thing, edges the same lengths as the sides of those triangles. For instance the numbers $\{4, 4, 4, 4, 7\}$ are not the edge lengths of a tetrahedron. A necessary and sufficient condition for there to be a tetrahedron is the nonnegativity of the determinant of the associated Cayley-Menger matrix, a matrix with entries constructed using the side lengths. This is an elegant condition but it is an inequality for a sixth degree polynomial in the lengths, [WD].

The complexity of that answer results from the choice of side lengths as data. If instead we select a vertex and use the lengths of sides meeting at that vertex and the angular structure at the vertex, either vertex angles or dihedral angles, as data then the necessary and sufficient conditions are simpler and also hold for tetrahedra in real hyperbolic space. They are presented in Section 6.

In real or complex hyperbolic space the necessary and sufficient condition for three lengths to be the side lengths of a triangle is that they satisfy the strong triangle inequality, STI, (2.2) [Ro]. In $\mathbb{C}H^1$ and in $\mathbb{R}H^n$, $n \geq 1$, those lengths determine the congruence class of the the triangle, but that is not true in $\mathbb{C}H^n$, $n > 1$. However it was shown by Brehm [Br] that in $\mathbb{C}H^n$ the three side lengths together with a fourth number, for instance the angular invariant, suffice to specify the congruence class of a triangle, and he gives the explicit conditions those four numbers must satisfy, we recall the details in Theorem 3.1. In Theorem 3.2 we see that the congruence class of a tetrahedron in $\mathbb{C}H^n$ is determined by nine numbers; Question 1 asks for the constraints those numbers must satisfy.

References
Questions about point sets in $\mathbb{CH}^n$ are equivalent to questions about Hilbert spaces with reproducing kernels which have the complete Pick property. In Section 3.2 we show that Question 1 is equivalent to the following:

**Question 2:** Given four three dimensional RKHS with the CPP, is there a four dimensional space with the CPP whose regular three dimensional subspaces are rescalings of the given spaces?

This work began as an effort to better understand an example due to Quiggin which shows that natural necessary conditions in Question 2 are not sufficient. That example is analyzed in Section 6.3.

We will describe sets in $\mathbb{CH}^n$ using two functionals which are invariant under hyperbolic isometries, the pseudohyperbolic distance between pairs of points, $\delta(a,b)$, and $\text{kos}_a(b,c)$, a functional of triples of points which is a complex analog of the cosine of the vertex angle at $a$. The "same" functionals are also defined for tuples of kernel functions in any RKHS and are invariant under rescaling of the space. In a RKHS $\delta(a,b)$ is related to the angle between kernel functions and $\text{kos}_a(b,c)$ describes the geometry of the projection of one kernel function onto the linear span two others. Accepting the guidance of Klein’s Erlangen Program, these invariant quantities should be regarded as geometric descriptors of sets in $\mathbb{CH}^n$ and of RKHS.

In Theorem 5.1 we use those functionals to give geometric descriptions of finite sets in $\mathbb{CH}^n$ and of a class of Hilbert spaces. Using those descriptions we build associated moduli spaces and to study assembly questions.

Here is an overview of the contents. In the next section we give background information about hyperbolic geometry and about Hilbert spaces with reproducing kernels; in the section after that we recall results connecting those topics. In Section 4 we discuss various geometric roles of the functional $\text{kos}$. In Section 5 we describe finite sets $X$ in $\mathbb{CH}^n$, and also an associated class of Hilbert spaces, using a version of spherical coordinates with values of $\text{kos}$ substituting for cosines of angles. That description involves the positive semidefiniteness of a matrix $A$, $A \succeq 0$, which has the form $A = (\text{kos}_i(i,j))$. Submatrices of $A$ encode the geometry of subsets of $A$ and Sylvester’s criterion lets us recast the fact that $A \succeq 0$ as statements about those submatrices. We use those facts to relate the geometry of $X$ to the geometry of its subsets. In Section 6 we specialize the results from Section 5 to four point sets and four dimensional spaces and answer Questions 1 and 2. In both cases a crucial part of the answer is a condition of the form $\det A \geq 0$. If our set $X$ is inside a copy of $\mathbb{RH}^k$ inside $\mathbb{CH}^n$ then our results specialize as results about sets in real hyperbolic space. There is then a fundamental simplification, the value of $\text{kos}$ at vertex simplifies to the cosine of the vertex angle. We develop that theme in detail for four point sets in Section 6.5 where we give conditions on the vertex angles and on the dihedral angles at a vertex of a real hyperbolic tetrahedron. The brief final section contains a few remarks.

2. **Background**

2.1. **Hyperbolic Geometry.** Our background reference for complex hyperbolic geometry is [Go]. We will use the ball model of complex hyperbolic space $\mathbb{CH}^n$. In that model the manifold for $\mathbb{CH}^n$ is the unit ball, $\mathbb{B}_n \subset \mathbb{C}^n$, and the geometry is determined by the transitive group of biholomorphic automorphisms of the ball, $\text{Aut} \mathbb{B}_n$. For each $\alpha \in \mathbb{B}_n$ there is a unique involution $\phi_\alpha \in \text{Aut} \mathbb{B}_n$ which satisfies $\phi_\alpha(\alpha) = 0$. The group $\text{Aut} \mathbb{B}_n$ is generated by those involutions together with the
unitary maps. We will say two sets $Z, W \subset \mathbb{C}^n$ are congruent, $Z \sim W$, if there is a $\phi \in \text{Aut} \mathbb{B}_n$ with $\phi(Z) = W$. If the sets are ordered then, absent other comment, we suppose that $\phi$ respects the ordering. Congruence is an equivalence relation and we are particularly interested in congruence equivalence classes.

The pseudohyperbolic metric $\delta$ on $\mathbb{C}^n$ is defined by $\forall \alpha, \beta \in \mathbb{C}^n$

$$\delta(\alpha, \beta) = |\phi_\alpha(\beta)| = |\phi_\beta(\alpha)|.$$  

For $n = 1$ the formula is $\delta(\alpha, \beta) = \left| \frac{\alpha - \beta}{1 - \bar{\beta}\alpha} \right|$. It satisfies the strong triangle inequality, STI; for $x, y, z \in \mathbb{B}_n$

$$\frac{|\delta(x, z) - \delta(z, y)|}{1 - \delta(x, z)\delta(z, y)} \leq \delta(x, y) \leq \frac{\delta(x, z) + \delta(z, y)}{1 + \delta(x, z)\delta(z, y)}.$$  

Equivalently, $\delta$ is the distance on $\mathbb{B}_n$ which satisfies $\delta(0, z) = |z|$ for $z \in \mathbb{B}_n$ and is $\text{Aut} \mathbb{B}_n$ invariant [DW]. It is not a length metric; the length metric it generates is the Bergman metric for the ball normalized to agree with the Euclidean metric to second order at the origin.

The space $\mathbb{C}^n$ contains various totally geodesic submanifolds of interest here. These include the classical geodesics which are totally geodesic copies of $\mathbb{R}^2$, and also includes totally geodesic copies of $\mathbb{C}^1$, sometimes called complex geodesics. Every pair of points is contained in a unique classical geodesic which is in turn contained in a unique complex geodesic. There are also totally geodesic copies of the real hyperbolic plane $\mathbb{H}^2$ inside $\mathbb{C}^n$. In particular set $\mathbb{R}^2 = \{ (x, y, 0, \ldots, 0) : x, y \in \mathbb{R} \} \subset \mathbb{C}^n$ and $BK_2 = \mathbb{R}^2 \cap CH^n$. That intersection is a totally geodesically embedded submanifold whose induced geometry is that of the Beltrami-Klein model of $\mathbb{H}^2$ with constant curvature $-1/4$. The geometry of $BK_2$ is not conformal with the Euclidean geometry of the containing $\mathbb{R}^2$. However the two geometries are conformally equivalent at the origin, in particular angles with vertex at the origin have the same size in both geometries. There is a useful picture of $BK_2$ in [Go, pg. 83] and there is a discussion of the geometry of that model (although the version with curvature $-1$) as well as the more familiar Poincaré disk model in Appendices C and B of [J]. The automorphic images of $BK_2$ are also totally geodesically embedded submanifolds of real dimension two, and they, together with the complex geodesics, are the only such.

Similar statements hold for totally geodesic submanifolds of higher dimension. In particular there are higher dimensional analogs of $BK_2$ and at the origin the induced geometry on $BK_r$ is conformal with the Euclidean geometry of the containing $\mathbb{R}^r$. More information on these topics is in [Go] and [Bl].

If $G$ is a complex geodesic and $x \in CH^n$ we define the metric projection of $x$ onto $G$, $P_G x$, to be that point in $G$ which is closest to $x$ in the pseudohyperbolic metric.

For $z, w \in \mathbb{B}_r$ we define the kernel function $k$ by

$$k_z(w) = k(w, z) = \frac{1}{1 - \langle w, z \rangle}.$$  

(We write $\langle \cdot, \cdot \rangle$ for the inner product on $\mathbb{C}^n$ to distinguish from the inner products on the general Hilbert spaces we consider.)

There is a fundamental identity which describes the interaction of the involutive automorphisms with the kernel functions [Ru]: for any $y, z, w \in \mathbb{B}_r$

$$\langle \cdot, \cdot \rangle,$$
(2.4) \[ k(\phi_y(z), \phi_y(w)) = \frac{k(y, y)^{1/2}}{k(z, y)} \frac{k(y, y)^{1/2}}{k(y, w)} k(z, w). \]

By a triangle in \( \mathbb{C}^n \) we mean an ordered set of three distinct points called vertices, or those vertices together with the sides, the geodesic segments connecting the vertices. The length of a side is the \( \delta \) distance between the corresponding vertices. Similarly for tetrahedra. We do not require any nondegeneracy on the sets, for instance three points could be in a single geodesic.

### 2.2. Hilbert Spaces with Reproducing Kernels

Our background references for Hilbert spaces are [AM], [Ro], [ARSW2]. Except for the spaces \( DA_r \) introduced later in this section, all the Hilbert spaces in this paper are finite dimensional.

An \( n \)-dimensional reproducing kernel Hilbert space, RKHS, is an \( n \)-dimensional Hilbert space \( H \) together with a distinguished basis of vectors called reproducing kernels, \( RK(H) = \{h_1\}_{i=1}^n \). For any \( v \in H \) we write \( \hat{v} \) for its normalized version; \( \hat{v} = v/\|v\| \). For \( h_i, h_j \in RK(H) \) we write \( h_{ij} \) for their inner product and \( \hat{h}_{ij} \) for the inner product of their normalizations.

\[
\begin{align*}
&h_{ij} = \langle h_i, h_j \rangle, \quad \hat{h}_{ij} = \left\langle \hat{h}_i, \hat{h}_j \right\rangle = \left\langle \frac{h_i}{\|h_i\|}, \frac{h_j}{\|h_j\|} \right\rangle.
\end{align*}
\]

The Gram matrix of \( H \) is the matrix \( Gr(H) = (h_{ij})_{i,j=1}^n \).

A regular subspace of \( H \) is a subspace \( J \) spanned by a subset \( S \) of \( RK(H) \) and regarded as a RKHS by setting \( RK(J) = S \).

We now recall the Drury-Arveson spaces, \( DA_r \) (sometimes denoted \( H^2 \) and sometimes the Hardy spaces), some basic references are [Ar], [AM], and [Sh]. With \( k_z(\cdot) \) the functions from (2.3), \( DA_r \) is the infinite dimensional RKHS with kernel functions \( \{k_z : z \in \mathbb{B}_r = \mathbb{C}^r \} \) and inner product given by \( \langle k_s, k_t \rangle = k_s(t) \).

We are particularly interested in finite dimensional regular subspaces of \( DA_r \). For \( Z = \{z_j\}_{j=1}^n \subset \mathbb{C}^r \) let \( DA_r(Z) \) be the regular subspace of \( DA_r \) spanned by the kernel functions \( \{k_{z_j}\}_{j=1}^n \). We abbreviate them by \( \{k_i\} \) and set \( k_{ij} = \langle k_i, k_j \rangle \).

If \( r' > r \) there are natural inclusions of \( \mathbb{C}^r, \mathbb{B}_r, \) and \( \mathbb{C}^r \) into the corresponding objects with \( r' \). These inclusions interact in harmless ways with the constructions we have discussed and will discuss. For instance, given \( Z \subset \mathbb{B}_{r'} \), the natural inclusion of \( \mathbb{B}_r \) into \( \mathbb{B}_{r'} \) takes \( Z \) to a set \( Z' \subset \mathbb{B}_{r'} \). There is then an obvious natural map between \( DA_r(Z) \) and \( DA_{r'}(Z') \) which preserves all the structure of interest here. Going forward we will identify such pairs of sets and of spaces and drop the subscripts on \( DA_r \) and \( DA_{r'}(Z) \). In particular when we consider a finite set \( X \) in some \( \mathbb{C}^n \) we will generally suppose \( n \) is sufficiently large and not be more specific.

#### 2.2.1. Rescaling

Given finite dimensional RKHS, \( G, H \) with \( RK(H) = \{h_\alpha\}_{\alpha \in A} \), \( RK(G) = \{g_\beta\}_{\beta \in B} \) we say \( G \) is a rescaling of \( H \) if there is a one to one map \( \theta \) of \( A \) onto \( B \) and a nonvanishing complex valued function \( \gamma \) defined on \( A \) such that for all \( \alpha_1, \alpha_2 \) in \( A \),

\[
\langle h_{\alpha_1}, h_{\alpha_2} \rangle = \langle \gamma(\alpha_1)g_{\theta(\alpha_1)}, \gamma(\alpha_2)g_{\theta(\alpha_2)} \rangle = \gamma(\alpha_1)\overline{\gamma(\alpha_2)} \langle g_{\theta(\alpha_1)}, g_{\theta(\alpha_2)} \rangle.
\]

If \( A \) and \( B \) are ordered then, unless we specify otherwise, we suppose that \( \theta \) is monotone increasing. If \( G_H \) is a regular subspace of \( G \) and \( H \sim G_H \) we will write \( H \hookrightarrow G \) or \( H \sim G_H \subset G \) and will say that we have a rescaling of \( H \) into \( G \) with image \( G_H \).
Rescaling is an equivalence relation between spaces $H$ and $G$; we denote it by $H \sim G$. We are especially interested in rescaling equivalence classes.

2.2.2. Assuming Irreducibility. In [AM] pg. 79 a RKHS $H$ is called irreducible if no two elements of $RK(H)$ are parallel and no two are orthogonal. Irreducibility is preserved by rescaling and is inherited by regular subspaces.

With our definition of finite dimensional RKHS the first condition is automatic. The second is equivalent to requiring that neither $H$ not any of its regular subspaces can be written as a nontrivial orthogonal direct sum of two RKHS. It is also equivalent to the requirement that no entry of $Gr(H)$ is zero.

All the spaces we consider will be assumed irreducible and we abbreviate that class by RKHSI. For those spaces the definitions of invariants in this section proceed without exceptions.

If $H$ is a finite dimensional space in RKHSI we will write $H \in RK$.

2.2.3. The CPP. The CPP is a property which some $H \in RK$ have and some do not. General information about it is in [AM], [Sh], and [ARSW2]. We will see in Theorem 2.2.2 that the CPP characterizes the class of Hilbert spaces for which Question 2 can be studied by focusing on Question 1, and for our purposes here we could have used the statements of that theorem as our definition of the CPP. However we briefly recall the classical definition.

A multiplier operator on $H \in RK$ is a linear operator $M$ whose adjoint $M^*$ is diagonalized by the kernel functions of $H$. Passing from a regular subspace $J$ of $H$ to the larger $H$ corresponds to an enlargement of the set of kernel functions. Hence the adjoint $M^*_J$ of any multiplier $M_J$ on $J$ can easily be extended to an operator $M^*_\text{ext}$ on $H$ which is diagonalized by the kernel functions of $H$. By virtue of that structure $M^*_\text{ext}$ will be the adjoint of a multiplier operator $M^*_\text{ext}$ on $H$, and it is then immediate that $\|M^*_\text{ext}\| \geq \|M^*_J\|$. If $H \in RK$ and for any regular subspace $J$ of $H$ and any multiplier $M_J$ on $J$ we can select $M^*_\text{ext}$ so that $\|M^*_\text{ext}\| = \|M^*_J\|$ then $H$ is said to have the Pick property. If additionally it is true that matrices of multiplier operators acting on spaces of tuples of elements of $J$ always have such an extension with equal norm then $H$ is said to have the complete Pick property, CPP.

If $H \in RK$ has the CPP then we write $H \in CPP$. If $H \in CPP$ then it is straightforward that any regular subspace of $H$ also has the CPP. Also it is not hard to see that if $H \sim J$ then $J \in CPP$. We will use both facts without mention.

It is fundamental that the spaces $DA$ all have the CPP and hence so do the finite dimensional regular subspaces $DA(Z)$. In the other direction we see in Theorem 2.2.2 that if $H \in CPP$ then there is a $Z \subset CH^n$ with $H \sim DA(Z)$. Furthermore, for finite $X, W$ we have $X \sim W$ if and only if $DA(X) \sim DA(W)$; one direction follows from earlier remarks and both directions are contained in Theorem 2.2.2.

2.3. Invariants. In this section we suppose $H \in RK$ and introduce several functions of tuples of elements from $RK(H)$. The values of the functionals are determined by entries of the Gram matrix of $H$ and hence are unchanged if $H$ is a regular subspace of some $H' \in RK$ and we regarded the functionals as acting on elements of $RK(H')$. Also, it will be clear from their formulas that these particular functionals are invariant under rescaling of $H$.

Such functionals can also be regarded as functionals of tuples of points in $CH^n$ using the following scheme. If, for instance, $F$ is defined on pairs of kernel functions
then for \( x, y \in \mathbb{CH}^n \) we define \( F \) acting on \( x, y \) by selecting a finite \( X \) containing both points, regarding \( F \) as acting on \( RK(DA(X)) \) and setting \( F(x, y) = F(k_x, k_y) \). By the previous remarks this definition does not depend on the choice of \( X \) and we will not mention \( X \) again. (We introduced \( X \) only to avoid using the infinite dimensional space \( DA \).) We noted that automorphisms of \( \mathbb{CH}^n \) induce rescalings of the \( DA(X) \) and that the functionals we will consider are rescaling invariant. Hence the functionals, regarded as acting on tuples of points in \( \mathbb{CH}^n \), are invariant under automorphisms; they depend only on the congruence class of the tuple of arguments. Going forward we will use the same names and notation for the functionals acting on a \( RK(H) \) and for the induced functionals acting on points in \( \mathbb{CH}^n \).

\[ \delta : \text{For } H \in \mathcal{RK} \text{ and } h_1, h_2 \in RK(H) \text{ we define } \delta_H \text{ by} \]

\begin{equation}
\delta_H(i, j) = \delta_H(h_i, h_j) = \sqrt{1 - \frac{|h_{ij}|^2}{h_{ii}h_{jj}}} = \sqrt{1 - |h_{ij}|^2}.
\end{equation}

This function is a metric on \( RK(H) \) \cite[Lemma 9.9]{AM}, \cite{ARSW}. Clearly \( \delta \) is invariant under rescaling.

The metric \( \delta_{DA} \) on \( \mathbb{B}_n \) equals the pseudohyperbolic metric \( \delta \). To see this use the definition of \( k \) to rewrite (2.3) as, for \( w, x, y, z \in \mathbb{CH}^n \),

\begin{equation}
1 - \frac{k(w, y)k(y, z)}{k(y, y)k(w, z)} = \langle \phi_y(z), \phi_y(w) \rangle.
\end{equation}

Taking \( z = w \) we see

\begin{equation}
\delta^2(y, w) = |\phi_y(w)|^2 = 1 - \frac{|k(y, w)|^2}{k(y, y)k(w, w)} = \delta_{DA}^2(y, w).
\end{equation}

The first equality in (2.3) is the definition of \( \delta \), the second is the special case of (2.7), the third is the definition of \( \delta_{DA}^2 \). Alternatively, for any \( z \in \mathbb{B}_n \) note that \( \delta(0, z) = |z| = \delta_{DA}(0, z) \) and both \( \delta \) and \( \delta_{DA} \) are invariant under automorphisms; hence the two metrics are equal.

Going forward we will write \( \delta \) for the various \( \delta_{DA}(z) \) and for the pseudohyperbolic metric. Using (2.8) we can express \( \delta \) in terms of coordinates; for \( y, w \in \mathbb{B}_n \)

\begin{equation}
\delta^2(y, w) = 1 - \frac{(1 - \langle y, y \rangle)(1 - \langle w, w \rangle)}{|1 - \langle y, w \rangle|^2}.
\end{equation}

\[ \alpha : \text{For } H \in \mathcal{RK} \text{ we define the angular invariant } \alpha \text{ by, for } k_1, k_2, k_3 \in RK(H) \]

\begin{equation}
\alpha(1, 2, 3) = \alpha(k_1, k_2, k_3) = -\arg \langle k_1, k_2 \rangle \langle k_2, k_3 \rangle \langle k_3, k_1 \rangle = -\arg k_{12}k_{23}k_{31}.
\end{equation}

(In general situations care is needed in selecting a branch of \( \arg \). However for \( k \in RK(DA(Z)) \), \( \Re k > 0 \) and that lets us avoid problems.) Notice that \( \alpha \) satisfies a cocycle identity; if \( k_4 \) is a fourth kernel function then

\begin{equation}
\alpha(1, 2, 3) = \alpha(2, 3, 4) + \alpha(3, 4, 1) - \alpha(4, 1, 2) = 0.
\end{equation}

We discuss the geometry associated with \( \alpha \) in Section 4.5. More about this invariant is in \cite{Go}, \cite{CO}, \cite{Qi}, \cite{M}.

\[ \text{kos} : \text{For } H \in \mathcal{RK} \text{ we define } \text{kos} \text{, a functional of triples of kernel functions. For } k_1, k_2, k_3 \in RK(H), k_1 \neq k_2, k_3, \text{ set} \]

\begin{equation}
\text{kos}_1(k_2, k_3) = \frac{1}{\delta_{12}\delta_{13}} \left( 1 - \frac{k_{21}k_{13}}{k_{11}k_{23}} \right),
\end{equation}

and note the symmetry \( \text{kos}_1(2, 3) = \overline{\text{kos}_1(3, 2)} \).
If $H = DA(X)$ for some $X \subset \mathbb{C}^n$ then kos is related to the geometry of $X$. Recall that for $x \in \mathbb{C}^n$ $\phi_x$ is the ball involution which interchanges $x$ and 0. We can use (2.7) and (2.8) to obtain

\begin{equation}
\text{kos}_x(y, z) = \frac{1}{\delta(x, y)\delta(x, z)} \langle \langle \varphi_x(y), \varphi_x(z) \rangle \rangle = \langle \langle \hat{\varphi_x}(y), \hat{\varphi_x}(z) \rangle \rangle.
\end{equation}

In particular, if $x_1$ is at the origin then $\delta(x_1, w) = |w|$ and $\phi_{x_1}$ is the identity. In that case

\begin{equation}
\text{kos}_{x_1}(y, z) = \left\langle \left\langle \frac{y}{|y|}, \frac{z}{|z|} \right\rangle \right\rangle = \langle \langle \hat{y}, \hat{z} \rangle \rangle.
\end{equation}

(Although kos is invariant under automorphisms of $\mathbb{B}_n$ this formula is an inhomogeneous representation and is not invariant.)

If the vectors $y$ and $z$ were in $\mathbb{R}^n$ this would be the inner product of unit vectors in $\mathbb{R}^n$ and hence would equal the cosine of the angle between the segments 0$y$ and 0$z$. That is the source of the name kos.

If dim $H = n$ then for $1 \leq s \leq n$ we define the $(n - 1) \times (n - 1)$ matrices

\begin{equation}
\text{KOS}(H, s) = \text{KOS}(\text{Gr}(H), s) = (\text{kos}_s(i, j))_{i, j \neq s}^{n}.
\end{equation}

\begin{equation}
\text{MQ}(H, s) = (\delta_{si}\delta_{sj} \text{kos}_s(i, j))_{i, j \neq s}^{n} = \left(1 - \frac{k_{1i}k_{1j}}{k_{ij}k_{ss}}\right)_{i, j \neq s}^{n}.
\end{equation}

We also write KOS($X$, $s$) for KOS($DA(X)$, $s$).

$X$: In [KR] Korányi and Reimann introduced a functional of 4-tuples of points in $\partial \mathbb{B}_n$, the ideal boundary of $\mathbb{C}^n$. Their definition extends in a natural way to an automorphism invariant functional on 4-tuples of points in $\mathbb{C}^n$, and more generally to a rescaling invariant functional of 4-tuples of kernel functions in a RKHSI.

**Definition 2.1.** Suppose $H \in \mathcal{RK}$. Given $\{k_i\}_{i=1}^{4} \subset \mathcal{RK}(H)$ the Korányi-Reimann cross ratio, $X(k_1, k_2, k_3, k_4)$, is defined as

\[ X(k_1, k_2, k_3, k_4) = \frac{k_{31}k_{42}}{k_{32}k_{41}}. \]

Although $X$ is often used when describing and studying the geometry of sets in $\mathbb{C}^n$, for instance [KR], [Ga], [CG], and the invariants we discussed earlier in this section can be defined using $X$, we will not use $X$ below.

Gr($H$): Having discussed several invariants we should emphasize that for $H \in \mathcal{RK}$ the Gram matrix, $\text{Gr}(H)$, is not invariant under rescaling of $H$. However $\text{Gr}(H)$ is stable under passage from $H$ to a regular subspace $J$; $\text{Gr}(J)$ is the submatrix of $\text{Gr}(H)$ obtained by retaining all and only the rows and columns of $\text{Gr}(H)$ which are built using reproducing kernels of $H$ which are also in $J$.

2.4. Matrix Notation. For an $n \times n$ matrix $A$ we write $A \succ 0$ if $A$ is positive definite and $A \succ 0$ if it is positive semidefinite. We say $B$ is a principal submatrix of $A$ if it is obtained from $A$ by removing certain rows and also the corresponding columns. Note that if $H \in \mathcal{RK}$ then $J$ is a regular subspace of $H$ if and only $\text{Gr}(J)$ is a principal submatrix of $\text{Gr}(H)$. We denote the set of all principal submatrices of $A$ by $\mathcal{PS}(A)$. If $B \in \mathcal{PS}(A)$ and the rows and columns retained in $B$ are, for some specified $k \leq n$, those with indices $j$, $1 \leq j \leq k$ then $B$ is said to be a leading
principal submatrix. The determinants of those matrices are called principal minors and leading principal minors respectively.

3. The CPP and Point Sets in $\mathbb{CH}^n$

We will often use a model triangle $\Gamma$ or a model tetrahedron $\Delta$ which have convenient coordinates. Our model triangle is $\Gamma \subset \mathbb{CH}^2$:

$$\Gamma = \{ x_1, x_2, x_3 \} = \{ (0, 0), (a, 0), (x, b) \},$$

$$a > 0, b \geq 0, \ x \in \mathbb{C},$$

$$0 < a, \ |x|^2 + b^2 < 1.$$ 

Our model tetrahedron is $\Delta \subset \mathbb{CH}^3$:

$$\Delta = \{ x_1, x_2, x_3, x_4 \} = \{ (0, 0, 0), (a, 0, 0), (x, b, 0), (y, z, c) \},$$

$$a > 0, b, c \geq 0, \ x, y, z \in \mathbb{C},$$

$$|a|^2, \ |x|^2 + |b|^2, \ |y|^2 + |z|^2 + |c|^2 < 1.$$ 

To a three dimensional $H \in \mathcal{R}K$ with $\mathcal{R}K(H) = \{ h_1 \}_{i=1}^3$ we associate the following data sets:

$$S = \{ |\widehat{h}_{12}|, |\widehat{h}_{23}|, |\widehat{h}_{13}|, \alpha_{123} \},$$

$$S' = \{ \delta_{12}, \delta_{13}, \delta_{23}, \alpha_{123} \},$$

$$S'' = \{ \delta_{12}, \delta_{13}, \cos(2, 3) \}.$$ 

And for convenience we set

$$\Gamma_{abc} = \left| \frac{\widehat{h}_{ab}\widehat{h}_{bc}}{\widehat{h}_{ca}} \right| = \sqrt{\frac{(1 - \delta_H^2(a, b))(1 - \delta_H^2(b, c))}{(1 - \delta_H^2(c, a))}}.$$ 

All these quantities are unchanged by rescaling $H$. Also note that the $\Gamma$’s can be computed using entries from $\text{Gr}(H)$, or directly from the data in $S$ or in $S'$.

The following describes three point sets $X$ in $\mathbb{CH}^2$ and the associated $\text{DA}(X)$ spaces.

**Theorem 3.1** ([Br] [AM] [Ro]). Given a three dimensional $H \in \mathcal{R}K$ the following are equivalent:

1. $H \in \text{CPP}$.
2. There is a three point set $X$ in $\mathbb{CH}^2$ with $H \sim \text{DA}(X)$.
3. There is a $\Gamma$ as in (3.1) with $H \sim \text{DA}(\Gamma)$.
4. Let $J$ be the regular subspace of $H$ spanned by $\{ h_1, h_2 \}$. Let $M$ be the multiplier on $J$ of norm one specified by the action of its adjoint,

$$M^* h_1 = 0,$$

$$M^* h_2 = \delta_H(h_1, h_2) h_2;$$

then $M$ extends to a multiplier of norm one on $H$.
5. $\text{KOS}(H, 1) \geq 0$.

(3.4) 

$$|\cos(2, 3)| \leq 1.$$
(7) $S$ and the $\Gamma$’s defined from $S$ using (3.5) satisfy
\begin{equation}
\Gamma_{123} + \Gamma_{231} + \Gamma_{312} \leq 2 \cos \alpha_{123}.
\end{equation}

(8) $S'$ and the $\Gamma$’s defined from $S'$ using (3.4) satisfy (3.5).
Furthermore $X$ sits inside a complex geodesic if and only if $\det \text{KOS}(H,1) = 0$, or equivalently $|\cos_1(2,3)| = 1$, or the $b$ coordinate of $\Gamma$ in (3) equals 0.

Conversely, given data $S$ and $\Gamma$’s defined from $S$ using (3.3) such that (3.3) holds, or data $S'$ and $\Gamma$’s defined from $S'$ using (3.4) satisfy (3.5), or data $S''$ for which (3.4) holds, there is triangle $X$ in $\mathbb{CH}^n$, unique up to congruence, which has those parameters.

Proof. All of this is in the references mentioned except for the statement about $X$ being in a complex geodesic. That fact is implicit in the proof of Theorem 16 in [Ro].

Thus each of $S$, $S'$, or $S''$ could be used to describe a triangle. The equivalence of using $S$ or $S'$ is clear. Passing between $S'$ and $S''$ is described in Section 4.2.

Some aspects of the previous theorem extend to larger sets and spaces. The next result is an amalgam of the fact that up to rescaling the $H \in \mathbb{CPP}$ are exactly the spaces $\text{DA}(X)$ for $X$ a finite set in $\mathbb{CH}^n$ [AM], the fact mentioned earlier that automorphisms of the ball induce rescalings of spaces $\text{DA}(X)$, and the description of congruence classes of finite sets in $\mathbb{CH}^n$ given in [BE], [G], [HS], and [Ro].

**Theorem 3.2.** An $n$ dimensional $H \in \mathbb{RK}$ satisfies $H \in \mathbb{CPP}$ if and only if there is an $X = \{x_i\}_{i=1}^n \subset \mathbb{CH}^{n-1}$ with $H \sim \text{DA}(X)$.

Given an $n$ dimensional $H' \in \mathbb{CPP}$ with $H' \sim \text{DA}(X')$ for $X' = \{x_i'\}_{i=1}^n$ the following are equivalent:

1. $H \sim H'$.
2. $X \sim X'$.
3. $\text{DA}(X) \sim \text{DA}(X')$.
4. All the triangles of $X$ are congruent to the triangles of $X'$; i.e. for $1 \leq i, j, k \leq n$ there is a ball automorphism taking $\{x_i, x_j, x_k\}$ to $\{x_i', x_j', x_k'\}$.
5. The triangles of $X$ which have one vertex at $x_1$ are congruent to the corresponding triangles of $X'$; for $1 < i < j \leq n$ the triangles $\{x_1, x_i, x_j\}$ and $\{x_1', x_i', x_j'\}$ are congruent.
6. The regular three dimensional subspaces of $H$ are rescalings of the corresponding regular three dimensional subspaces of $H'$.
7. The regular three dimensional subspaces of $H$ which contain $k_{x_1}$, are rescalings of the corresponding three dimensional subspaces of $H'$.

A consequence is

**Corollary 3.3.** There are bijections between the class of $n$ dimensional $H \in \mathbb{CPP}$ modulo rescaling, the class of $n$ dimensional $\mathbb{RK}$ of the form $\text{DA}(X)$ modulo rescaling, and the class of $n$ point sets $X$ in $\mathbb{CH}^k$ modulo congruence. The correspondences respect inclusions between spaces and between sets.

If we use $S$, $S'$, or $S''$ to characterize the triangles in (4) then we obtain $O(n^3)$ real numbers which describe $X$ up to congruence. However that list has repetitions
and redundancies. Restricting to the triangles listed in (5) and adjusting for the fact that some side lengths are listed twice produces a list of \((n - 1)^2\) independent real parameters which determine \(X\) up to congruence. That number is optimal; triangles are determined by 4 parameters, tetrahedra by 9. There are also constraints on the parameters, the inequality (3.5) for triangles is the simplest example. We give analogous constraints for tetrahedra in Theorem 6.6 below. Also, noting the previous corollary, the same parameters (or, perhaps, similarly named parameters) describe spaces \(DA(X)\) and spaces in \(CPP\).

3.1. Assembly and Coherence. It may be that we have several spaces \(\{J_i\}\) and a rescaling of each with a \(H_i\) contained in \(H; J_i \sim H_i \subset H\). If that holds then there are coherence conditions connecting the \(\{J_i\}\) with each other and the \(\{J_i\}\) are said to be a coherent set of spaces. Informally the conditions are that subspaces of various \(J_i\) which are rescalings of the same subspace of \(H\) must be rescalings of each other. Rather than give a detailed definition we will describe the situation in detail in the context of Question 2.

For any four dimensional \(H \in CPP\) with \(RK(H) = \{h_i\}_{i=1}^4\) denote the four regular three dimensional subspaces \(\{H_i\}_{i=1}^4\) by

\[
RK(H_i) = \{h_r : 1 \leq r \leq 4, \ r \neq i\}.
\]

Suppose we have four three dimensional spaces \(\{J_i\}_{i=1}^4 \subset RK\). Question 2 asks for necessary and sufficient conditions on the \(\{J_i\}_{i=1}^4\) to insure that there is an \(H \in CPP\) with for \(1 \leq i \leq 4\), \(H_i \sim J_i\). If there are such rescalings then we suppose the indices on the \(j_{ir}\) have been chosen so that for each \(i, r\) the rescaling of \(H_i\) and \(J_i\) matches the kernel function \(h_r\) of \(H_i\) with the kernel function \(j_{ir}\) of \(J_i\).

Now notice that, given the existence of the rescalings there must be relationships between some subspaces of the various \(\{J_i\}\). The relationships are all of the same form, we will describe one particular case. The space \(J_1\), with kernel functions \(\{j_{12}, j_{13}, j_{14}\}\), is a rescaling of \(H_1\) which has kernel functions \(\{h_2, h_3, h_4\}\) with \(j_{1s}\) pairing with \(h_s\); similarly for \(J_3\) and \(H_3\) with \(\{j_{31}, j_{32}, j_{34}\}\) and \(\{h_1, h_3, h_4\}\). The subspace \(J_{13}\) of \(J_1\) with kernel functions \(\{j_{12}, j_{14}\}\) and the subspace \(J_{31}\) of \(J_3\) with kernel functions \(\{j_{32}, j_{34}\}\) are both rescalings of the subspace \(H_{34}\) with kernel functions \(\{h_2, h_4\}\), and hence \(J_{13} \sim J_{31}\). Define \(J_{pq}\) similarly for \(1 \leq p \neq q \leq 4\). The coherence conditions on the \(\{J_i\}\) are the collection of all relations \(J_{pq} \sim J_{qp}\). They are necessary for the rescalings of \(\{J_i\}\) into \(H\). If the \(\{J_i\}\) satisfy these conditions, even if we do not know that there is an \(H\), we write \(\{J_i\} \Rightarrow H\). If furthermore we know there is an \(H\) we write \(\{J_i\} \Rightarrow H\). We use the same notation if there are more \(\{J_i\}\) of they have larger dimension.

In general the dimension of the overlap sets may be larger than two. (However for spaces with the CPP Condition (7) of Theorem 3.2 insures that the conditions on overlap of two and three dimensional sets imply the coherence conditions for the larger sets.) Thus the statement \(\{J_i\} \Rightarrow H\) is the statement that there is a description (perhaps only implicit) of a type of target space \(H\) and an assembly scheme for constructing such an \(H\) from overlapping copies of the \(\{J_i\}\) and, furthermore, the \(\{J_i\}\) satisfy the coherence conditions which are necessary for such an assembly to be possible.

If \(\{J_i\} \subset CPP\) then by Corollary 3.3 these statements about spaces, subspaces, rescalings, values of invariants and coherence are equivalent to statements about sets in \(CH^m\), subsets, congruences, values of invariants and an appropriate notion of coherence, and we will use the same language and notation in that context. For
instance given sets \( \{Y_i\} \) in \( \mathbb{C}H^n \) we will write \( \{Y_i\} \Rightarrow X \) if there are congruences of each \( Y_i \) into some \( X \) which satisfy a preassigned scheme saying which points from the \( \{Y_i\} \) are to be mapped to which points of \( X \). We write \( \{Y_i\} \Rightarrow \ast \) if we do not know there is an \( X \) but do have the congruences among various subsets \( \{Y_i\} \) which would be mapped to the same subsets of \( X \). Sometimes we will pass between the formulations with spaces and with sets.

If we have \( \{J_i\} \Rightarrow H \) we would like obtain information about \( H \) from the spaces \( \{J_i\} \) together with the coherence data \( \{J_i\} \Rightarrow \ast \). We know from Theorem 3.2 that the values of \( \delta_{ij} \) and \( \text{kos}(j,k) \) for \( H \) describe \( H \) up to rescaling and would like to compute them from the \( \{J_i\} \). Given \( \{J_i\} \Rightarrow \ast \) we can construct the imputed value of \( \delta_H(a,b) \) by selecting any \( J_s \) whose image under the rescaling \( J_s \sim H \) contains the kernel functions \( h_a \) and \( h_b \) of \( RK(H) \). We write \( j_{ra} \) and \( j_{rb} \) for the elements of \( RK(J_s) \) which correspond to \( h_a \) and \( h_b \) under that rescaling map and use \( \delta_{J_s(j_{ra},j_{rb})} \) as our constructed value. Note that this is defined even if there is no \( H \), however if there is an \( H \) then the rescaling \( J_s \sim H \) insures that this value is \( \delta_H(a,b) \). Also, if there is another possible choice for \( J_s \) then the coherence conditions insure it will produce the same value. If there is no such \( J_s \) then we have no candidate for the value \( \delta_H(a,b) \). The procedure for constructing our candidate for \( \text{kos}_H(b,c) \) is the same except that we need to select a \( J_s \) whose image would contain the three elements of \( RK(H) \) with indices \( a, b, \) and \( c \).

Using the candidate values of \( \text{kos}_H(b,c) \) we construct a matrix, perhaps only partially defined, which we denote \( \text{KOS}(\{J_i\},a) \). If \( J_i = DA(X_i) \) we may write \( \text{KOS}(\{X_i\},a) \) for \( \text{KOS}(\{J_i\},a) \). If there is an \( H \) with \( \{J_i\} \Rightarrow H \) then the defined values in this matrix will equal those in \( \text{KOS}(H,a) \). However \( \text{KOS}(\{J_i\},a) \) is constructed from \( \{J_i\} \) and the coherence data, without needing \( H \). Hence comparing properties of \( \text{KOS}(\{J_i\},a) \) to properties any \( \text{KOS}(H,a) \) must have is a test of the possibility there is such an \( H \).

### 3.2. Equivalence of the Two Questions.

Using the previous theorems we can see that the two questions in the introduction are equivalent. Those theorems give us the following facts:

1. Given a set of triangles \( \{Y_i\}_{i=1}^4 \) in \( \mathbb{C}H^n \) there are three dimensional \( \{J_i\}_{i=1}^4 \subset CPP \) such that

\[
J_i \sim DA(Y_i), \quad i = 1, \ldots, 4,
\]

In the other direction, given 3 dimensional \( \{J_i\}_{i=1}^4 \subset CPP \) there are \( \{Y_i\}_{i=1}^4 \) such that \( (3.6) \) holds. In either case \( (3.6) \) continues to hold if the \( \{J_i\} \) are replaced by rescalings \( \{J'_i\} \) or if the \( \{Y_i\} \) are replaced by congruent sets \( \{Y'_i\} \).

2. Given a tetrahedron \( X \) in \( \mathbb{C}H^n \) there is an \( H \in CPP \) such that

\[
H \sim DA(X).
\]

In the other direction, given \( H \in CPP \) there is an \( X \) such that \( (3.7) \) holds. In either case \( (3.7) \) continues to hold if \( H \) is replaced by a rescaling \( H' \) or \( X \) is replaced by a congruent \( X' \).

**Proposition 3.4.** Given triangles \( \{Y_i\}_{i=1}^4 \) in \( \mathbb{C}H^n \) and Hilbert spaces \( \{J_i\}_{i=1}^4 \subset CPP \) which are related as in \( (3.7) \), there is an \( X \) with \( \{Y_i\} \Rightarrow X \) if and only if there is an \( H \in CPP \) with \( \{J_i\} \Rightarrow H \). In that case \( H \) and \( X \) satisfy \( (3.7) \).
Proof. This follows from statements (1) and (2) above together with the observation that for a finite $X \subset \mathbb{CH}^n$ the regular subspaces of $DA(X)$ are exactly the spaces $DA(Y)$ for $Y$ a subset of $X$. □

In the proposition the assumption that (3.6) holds insures the $J_i \in CPP$. Even with that condition, having $H \in RK$ and $\{J_i\} \Rightarrow H$ is not enough to insure that $H \in CPP$. This is shown by, for instance, Quiggin’s example which is discussed in Section 6.3.

4. Geometry and Kos

In this section we describe various relations between values of kos and the geometry of sets in $\mathbb{CH}^n$.

4.1. Evaluating Kos. Because the functional kos is an automorphism invariant we can study it for a general triple by first using an automorphism to place our triple in the configuration $\Gamma$ described in (3.1):

$\{x_1, x_2, x_3\} = \{(0,0), (a,0), (x,b)\}$.

In that case computing using (2.14) gives

$$ (4.1) \quad \cos_1(2,3) = \left\langle \frac{(a,0)}{\|a,0\|}, \frac{(x,b)}{\|x,b\|} \right\rangle = \frac{ax}{\|x_2\| \|x_3\|} = \bar{x}. $$

An invariant formulation of that statement will let us evaluate $\cos_1(2,3)$ for a general triple. Suppose $\{x_1, x_2, x_3\}$ is a given triple and $G(1,2)$ is the complex geodesic which contains $x_1$ and $x_2$. Recall that $P_{G(1,2)}$ is the metric projection onto $G(1,2)$. Here is the invariant statement.

Proposition 4.1. Set $P_{G(1,2)}x_3 = y$. Writing $\cos(x_1, x_2, x_3) = re^{i\theta}, r > 0, -\pi \leq \theta < \pi$ and angle for the hyperbolic angle we have

$$ (4.2) \quad r = \frac{\delta(x_1, y)}{\delta(x_1, x_3)}, $$

$$ (4.3) \quad \theta = \text{angle}(x_1x_2, x_1y). $$

Proof. We need to show the formula is correct for $\Gamma$ and that it is invariant. For $\Gamma$ the complex geodesic $G(1,2)$ is the unit disk in the first coordinate line and in that case it is elementary to show that the metric projection of $x_2$ onto $G(1,2)$ is $(x,0)$. Also, for angles with vertex at the origin the Euclidean angle and hyperbolic angle are the same. That is enough to establish that both (4.2) and (4.3) are correct for $\Gamma$.

To see that (4.2) is invariant note that the statement $P_{G(1,2)}x_3 = y$ is invariant as is $\delta$. The equality (4.3) is more subtle because there is no natural definition of the angle of intersection for two geodesics in $\mathbb{CH}^n$. However the geodesic segments $x_1P_{G(1,2)}x_3$ and $x_1x_2$ are both in $G(1,2)$, and any complex geodesic is conformally equivalent to the classical Poincaré disk which does carry an invariant notion of angle between intersecting curves. Using that notion of angle we see that (4.3) is also invariant. □

In some cases there is a simple relation between the values of kos and the geometry of triangles.

Given a triangle $T = \{x_1, x_2, x_3\} \subset \mathbb{CH}^2$ we use an automorphism to suppose that $T$ is in $\mathbb{CH}^2$ and $x_1$ is at the origin. Regard $\mathbb{CH}^2$ as $\mathbb{B}_2$ inside $\mathbb{C}^2$, denote by $V$ the real linear span of the points of $T$, and set $W = V \cap \mathbb{B}_2$. It may be that $W$ is a
totally geodesic submanifold of $\mathbb{C}H^2$. In that case, with the origin in $W$, there are three possibilities. First, $W$ may be a classical geodesic in which case it will be a line segment through the origin. Second, $W$ may be a complex geodesic which contains the origin. We can suppose it is the unit disk in the plane of the first coordinate and hence it is the classical Poincaré disk and it has constant negative curvature $-1$. The final possibility is that $W$ is a totally real totally geodesic disk and hence, after an automorphism, we can suppose it is $BK_2 = \{(r, s) \in \mathbb{R}_2 : r, s \in \mathbb{R}\}$, the Beltrami-Klein disk of constant curvature $-1/4$.

In the first case, noting (4.2) and (4.3), we then see that $\cos_1(2, 3) = \pm 1$. The value $-1$ occurs when $x_1$ separates the other two points, the value $+1$ when it does not. In the second case $x_3$ is in $G(1, 2)$ and so $P_{G(1, 2)} x_3 = x_3$. From (4.2), we see that $|\cos_1(x_2, x_3)| = 1$ and then from (4.3) that $\cos_1(2, 3) = e^{i\gamma}$ where $\gamma$ is the Euclidean angle between the segments $x_1 x_2$ and $x_1 x_3$. Thus $\gamma$ is the Euclidean and the hyperbolic angle at vertex $x_1$. Similarly the values of $\cos_3$ at the other two vertices give the other angles. The congruence class of a triangle in a plane of constant negative curvature is determined by its angles and hence in this case also by the three values of $\cos$. Finally, in the third case the triangle is in a totally real vector space and we are in the situation discussed in Proposition 6.11 below.

From (2.14) we see that $\cos_2(x_2, x_3) = \cos \theta$ where $\theta$ is the hyperbolic angle of the triangle at $x_1$. Similarly for the other two vertices. Hence, as in the previous case, we know the three angles of a triangle in a plane of constant negative curvature and that determines the congruence class of the triangle. Finally, looking backward we see that the first case is the second and third cases holding simultaneously.

In each of these cases the argument can be reversed. If $\cos_1(2, 3) = \pm 1$ then $W$ is a line through the origin, if $|\cos_1(x_2, x_3)| = 1$ then $P_{G(1, 2)} x_3 = x_3$ and hence the triangle lies in a complex geodesic, and finally, noting Proposition 6.10 below, if $\cos_3(x_2, x_3)$ is real then after automorphism the triangle is in the position described.

4.2. Kos and other Invariants. It is a result of Brehm [Br] that equality of the data sets $S'$ is a congruence criterion for triangles in $\mathbb{C}H^n$. That criterion and variations are often used in describing the geometry of finite sets in $\mathbb{C}H^n$, [BE], [HS], [G], [CG], [Rq]; but here we focus on the congruence criterion given by the data set $S''$. It is straightforward to pass between $S'$ and $S''$. The parameters are invariant so we can assume we are in the model case and the triangle is $\Gamma = \{x_1, x_2, x_3\} = \{(0, 0), (a, 0), (x, b)\}$.

First suppose we have the data $S''$, that is $a = \delta(x_1, x_2)$, $\omega = \delta(x_1, x_3)$, and $\cos_1(2, 3)$ are known. Using those values and (2.14) we also know $a x$. Hence using (2.20) we can find the third side length using

$$\delta^2(x_2, x_3) = 1 - \frac{(1 - \|x_2\|^2)(1 - \|x_3\|^2)}{|1 - \langle x_2, x_3 \rangle|^2} = 1 - \frac{(1 - a^2)(1 - \omega^2)}{|1 - ax|^2}.$$

To finish determining $S'$ we need to find $\beta$, the angular invariant. For this triangle $\beta = \arg(1 - ax)$ and, as we just mentioned, $ax$ is known; hence $\beta$ is known.

In the other direction, to go from $S'$ to $S''$ we need to find $\cos_1(2, 3)$. In this case we know the three side lengths and hence, noting (4.3), we know $|1 - ax|$. We also know the angular invariant $\beta = \arg(1 - ax)$. Combining these two we know $ax$. With that information, and using (4.3), we can find $\cos_1(2, 3)$.
Given points \( \{x_1, x_2, x_3\} \) we are describing the geometry of the intersection of the geodesics \( x_1x_2 \) and \( x_1x_3 \) at \( x_1 \) using the complex number \( \cos_1(2, 3) \). There are other complex parameters and pairs of real parameters that are used for the same purpose. For instance in his geometric analyses of the ball model of \( \mathbb{CH}^n \) Goldman uses angles, \( \phi, \theta \) related to \( \cos \) by \( \cos \phi = \cos \theta = \Re \cos_1(2, 3) \) \cite{Go}.

### 4.3. Kos and Vertices

For a bivalent vertex in \( \mathbb{R}^n \) the cosine of the vertex angle carries all the intrinsic geometric data about the vertex. The geometry of a trivalent vertex is determined by the geometry of the three component bivalent vertices. The values of \( \cos_\theta \) provide similar information for vertices in \( \mathbb{CH}^n \).

First we consider sets in \( \mathbb{R}^n \). By a vertex \( V \) in \( \mathbb{R}^n \) we mean a collection of two or more line segments, rays, with a common starting point the vertex point, \( V \), also simply called the vertex. We call \( V \) bivalent if it has two rays, trivalent if there are three, multivalent in general. We suppose the rays of a bivalent vertex are ordered. The two rays of a bivalent vertex span an affine plane and form an angle in that plane, the vertex angle. We say two vertices are (Euclidean) congruent if there is a Euclidean isometry placing the second vertex point at the same position as the first and so that the initial segments of the rays of the second coincide with the initial segments of the rays of the first.

**Proposition 4.2.** Two bivalent vertices in \( \mathbb{R}^n \) are congruent if and only if the cosines of their vertex angles are equal.

Suppose now we are given \( \mathcal{S} \), a set of three triangles \( \{T_i\}_{i=1}^3 \) each contained in some \( \mathbb{R}^n \). Suppose for \( i = 1, 2, 3 \) that \( T_i \) has vertices \( \{x_{ia}, x_{ib}, x_{ic}\} \), that the Euclidean length of the side \( x_{ia}x_{ib} \) is \( l_{ia} \) and similarly for \( l_{ib} \), and that \( W_i \) is the bivalent vertex in \( T_i \) with vertex point \( x_{ia} \) and rays ordered similarly to the sides of the triangle: \( x_{ia}x_{ib} \) first, \( x_{ia}x_{ib} \) second. It may or may not be possible to join these three triangles as faces of a tetrahedron in some \( \mathbb{R}^n \). That is there may be a tetrahedron \( \{y_1, y_2, y_3, y_4\} \) in some \( \mathbb{R}^n \) with the triangular face \( \{y_1, y_2, y_3\} \) congruent to \( T_1 \), \( \{y_1, y_3, y_4\} \) congruent to \( T_2 \), and \( \{y_1, y_2, y_4\} \) congruent to the triangle \( T_3 = \{x_{ia}, x_{ib}, x_{ic}\} \). (That last triangle has the same vertices as \( T_3 \) but their order is different. That distinction will be significant when we consider the complex case.) Certainly a necessary condition for building a tetrahedron is that certain side lengths match.

**Definition 4.3.** The triangles \( \mathcal{S} \) are said to be a matched set if there are numbers \( \{L_i\}_{i=1}^3 \) such that

\[
(4.5) \quad l_{1a} = l_{2a} = L_1, \quad l_{2b} = l_{3a} = L_2, \quad l_{3b} = l_{1a} = L_3.
\]

**Proposition 4.4.** Given a matched set of three Euclidean triangles \( \mathcal{S} \) the following are equivalent.

1. The triangles of \( \mathcal{S} \) are congruent to the faces of a tetrahedron.
2. If \( \mathcal{S}' \) is another matched set of three triangles (with associated data denoted by primes) and for \( i = 1, 2, 3 \) the bivalent vertex \( W_i \) is congruent to the bivalent vertex \( W_i \), then the triangles of \( \mathcal{S}' \) are congruent to the faces of a tetrahedron. That is, the previous conclusion holds for any choice of the lengths \( \{L_i\} \).
There is a trivalent vertex $V$ whose three component bivalent vertices are congruent to the three bivalent vertices $\{W_i\}_{i=1}^3$.

In each case the tetrahedron or the trivalent vertex is unique up to congruence.

In sum, if the side lengths match then the possibility of putting the triangles together is determined by the geometric combinatorics of the vertex angles, the specific lengths $\{L_i\}$ play no role.

Analogous results hold in $\mathbb{CH}^n$. By a vertex $V$ in $\mathbb{CH}^n$ we mean a collection of two or more geodesic segments, rays, $R_i$, with a common starting point the vertex point $V$. If there are two rays we assume they are ordered. Again a vertex may be bivalent, trivalent, or multivalent. We say two vertices are congruent if there is an automorphism placing the second vertex point on the first and so that the initial segments of the two sets of rays overlap.

The geometry of a bivalent vertex in $\mathbb{CH}^n$ can be described by two real numbers or one complex number. We will use the complex quantity $\text{kos}(V)$ which we define to be $\text{kos}_V(x_1,x_2)$ where $V$ is the vertex point and $x_i$ is chosen on the ray $R_i$, $i = 1, 2$. The formula (2.11) shows that this value does not depend on those choices. Thus if $\{x_1,x_2,x_3\}$ is a triangle and $V$ the bivalent vertex with vertex point $x_1$ then $\text{kos}(V) = \text{kos}_1(2,3)$.

If we regard $\text{kos}(V)$ as a substitute for the cosine of the angle of a bivalent vertex in Euclidean space then statement (8) in Theorem 3.1 is the analog of the classical side–angle–side congruence criterion for Euclidean triangles. We also have analogs of the two previous results.

**Proposition 4.5.** Two bivalent vertices, $V, W$, in $\mathbb{CH}^n$ are congruent if and only if $\text{kos}(V) = \text{kos}(W)$.

**Proof.** If the vertices are congruent then the conclusion is clear. In the other direction, pick $\varepsilon$ small and pick $x$ and $y$ on the two rays of $V$ at distance $\varepsilon$ from the vertex point. Pick $u$ and $v$ similarly on the rays of $W$. From (8) in Theorem 3.1 we see that the triangle formed by the vertex point of $V$ together with $x$ and $y$ is congruent to the triangle formed by the vertex point of $W$ together with the points $u, v$. That congruence of triangles also gives the required congruence of the vertices.

There is an interesting contrast between the real and complex cases. For a bivalent vertex $V$ let $V_{\text{rev}}$ be the vertex obtained from $V$ by reversing the order of the two rays. For vertices in $\mathbb{RH}^n$ $V_{\text{rev}}$ is congruent to $V$. However if $V \subset \mathbb{CH}^n$ then $V_{\text{rev}}$ is congruent to $V^*$, the vertex obtained by conjugating the coordinates of $V$, but generally is not congruent to $V$. To see this, compute that $\text{kos}(V_{\text{rev}}) = \text{kos}(V^*) = \text{kos}(V) = \text{kos}(V^*)$.

Suppose now we are given $\mathcal{S}$, a set of three triangles each in $\mathbb{CH}^n$. We continue the earlier notation and terminology, but now we use the pseudohyperbolic side lengths $\delta(\cdot,\cdot)$. With that change the analog of Proposition 4.4 holds.

**Proposition 4.6.** Given a matched set of triangles $\mathcal{S}$ in $\mathbb{CH}^n$ the statements (1), (2), and (3) of Proposition 4.4 are equivalent. If the conditions hold then the congruence class of the tetrahedra and of the trivalent vertex are uniquely determined.

**Proof.** It is immediate that (2) $\implies$ (1) $\implies$ (3). To see that (1) $\implies$ (2) suppose from (1) that we have $\mathcal{S}$ and the associated tetrahedron $\Lambda$ and suppose that we are
given a new set of lengths \( \{L'_i\} \) from (2). Use an automorphism to replace \( \Lambda \) with the tetrahedron \( \Delta \) with coordinates given by (3.2). For \( i = 2, 3, 4 \) select \( \gamma_i > 0 \) so that the tetrahedron \( \Delta' = \{x_1, \gamma_2 x_2, \gamma_3 x_3, \gamma_4 x_4\} \) has \( \|\gamma_i x_i\| = L'_i \). The bivalent vertices of \( \Delta' \) at the origin are obtained from those of \( \Delta \) by changing the lengths of the rays which does not change the congruence class of the vertices. Hence those vertices have the desired congruence classes. Also the lengths of the edges of \( \Delta' \) which meet at the origin match the \( \{L'_i\} \). Thus by the results on \( S'' \) in Theorem 3.4 the triangles of \( \mathcal{S}' \) are congruent to the faces of \( \Delta' \), establishing (2). To show that \( (3) \implies (1) \) first use an automorphism to place the trivalent vertex \( V \) at the origin with its rays in the directions of the rays of \( \Delta \) of (3.2). Next select a point on each ray whose distance from the origin is the appropriate \( L_i \). The triangles with vertex at the origin are, again by Theorem 5.1, congruent to the triangles of \( \mathcal{S} \) and hence the origin together with those three new points are the vertices of the tetrahedron required to show that (1) holds.

For uniqueness, first consider two trivalent vertices \( W \) and \( W' \) which satisfy Condition (3). Pick a small \( \varepsilon \) and a pick point on each ray at distance \( \varepsilon \) from the vertex. Let \( \Sigma \) be the tetrahedron determined by those four points and let \( \Sigma' \) be the similar tetrahedron constructed using \( W' \). We will be done if we show \( \Sigma \) and \( \Sigma' \) are congruent. The argument which shows they are congruent also gives the uniqueness in statements (1) and (2). First note that the results on \( S'' \) in Theorem 3.4 insures that the triangular faces of \( \Sigma \) meeting at the vertex point \( W \) are congruent to those in \( \Sigma' \) meeting at the vertex point of \( W' \). By Condition (5) of Theorem 5.2 this is enough to show the tetrahedra are congruent.

\[ \square \]

Thus, as in \( \mathbb{R}^n \), the possibility of putting the triangles together is determined by the geometric combinatorics of the vertex angles, now described by kos. Again the lengths \( \{L_i\} \) play no role.

In the Corollary 5.4 we will see a similar result for congruence of multivalent vertices in \( \mathbb{C} \mathbb{H}^n \).

4.4. Kos and Congruence of Triangles. Angles are commonly used in discussing congruence of Euclidean triangles but the angle cosines are an equivalent parameter. Motivated by the analogy between cos and kos we briefly look at the role of kos in congruence criteria for triangles in complex hyperbolic space.

Recall the standard naming conventions for Euclidean triangles. The abbreviation SAS refers to the length of two sides and the size of the angle between them, similarly for the other combinations of S and A. The Euclidean results are that SSS, SAS, ASA, and AAS each give enough information to determine the congruence class of a triangle, the data SSA can give two different congruence classes, and AAA only determines the triangle up to similarity. These and related topics, for Euclidean, real hyperbolic, and spherical geometries are discussed in detail in Section 7 of [J].

From [J] we see that the results for triangles in \( \mathbb{R} \mathbb{H}^2 \) are the same as for \( \mathbb{R}^2 \) with one exception, in \( \mathbb{R} \mathbb{H}^2 \) the data set AAA determines the congruence class of a triangle. The version of \( \mathbb{R} \mathbb{H}^2 \) with curvature \(-1\) is isometric to \( \mathbb{C} \mathbb{H}^1 \) and knowing the value of kos at a vertex of a triangle in \( \mathbb{C} \mathbb{H}^1 \) is equivalent to knowing the angle of the vertex of that triangle in \( \mathbb{R} \mathbb{H}^2 \). Hence all the results mentioned for \( \mathbb{R} \mathbb{H}^2 \) also hold for triangles in \( \mathbb{C} \mathbb{H}^1 \).
The results for triangles in \( \mathbb{R}H^2 \) also hold for triangles in \( \mathbb{R}H^n \). To see this note that any triangle in \( \mathbb{R}H^n \) is inside a totally geodesically embedded copy of \( \mathbb{R}H^2 \) and the analysis can be done in that copy. However the analogous statement in complex hyperbolic space is not true; not every triangle in \( \mathbb{C}H^n \) is contained in a complex geodesic and hence there is no natural notion of the cosine of the angle at a vertex. We will try to proceed using the value of \( \cos \) at a vertex as a substitute. A parameter count shows this situation will be different. We see from Theorem 3.1 that the congruence class of a triangle in a complex geodesic is determined by 3 real parameters, but to specify the class of a general triangle requires 4 parameters. Hence it is not surprising that SSS, which only has three parameters, does not give a congruence criterion for triangles in \( \mathbb{C}H^n \).

To see the failure for triangles in \( \mathbb{C}H^2 \) consider the triangles

\[
T_t = \left\{ (0,0), \left( \frac{1}{2}, 0 \right), \left( \frac{t e^{i \theta}}{2}, \frac{(2 - t^2)^{1/2}}{2} \right) \right\}, \quad 1 \leq t \leq \sqrt{2}, \quad \cos \theta = \frac{t^2 + 7}{8t}.
\]

Computation shows they all have the same side lengths, however triangles in the form (3.1), as these are, are only congruent if they are identical.

On the other hand the SAS data set for triangles in \( \mathbb{C}H^n \) does determine their equivalence class. This is a consequence of Theorem 3.1 because the data set SAS is the data set \( S'' \) of that theorem.

In summary, the congruence class of a triangle in \( \mathbb{C}H^1 \) is determined by each of the data sets mentioned except SSA. In \( \mathbb{C}H^n \), \( n > 1 \) SAS is sufficient to determine the congruence class, neither SSA nor SSS is sufficient, and the situation with the other data sets, ASA, AAS, and AAA, is not known.

4.5. *Kos and Area.* Let \( T = \{x, y, z\} \) be the vertices of a triangle in \( \mathbb{C}H^1 \) with geodesic segments as sides. The invariant area of \( T \), \( \text{Area}(T) \), can be evaluated using the angular invariant, \( 2\alpha(x, y, z) = \text{Area}(T) \) [Go 1.3.6], and by invariance a similar result holds if \( T \) is contained in a complex geodesic. The next proposition and corollary hold for triangles in a complex geodesic but for convenience we present it for sets in \( \mathbb{C}H^1 \).

**Proposition 4.7.** If \( T = \{x_1, x_2, x_3\} \subset \mathbb{C}H^1 \) then

\[
\pi - \arg (\cos_1(2, 3) \cos_2(3, 1) \cos_3(1, 2)) = 2\alpha(x_1, x_2, x_3) = \text{Area}(T)
\]

**Proof.** We identify \( \mathbb{C}H^1 \) with the unit disk in the complex plane. From (2.13) we see that \( \cos_1(2, 3) \) is a positive multiple of

\[
\kappa_{123} = \langle \langle \phi_{x_1}(x_2), \phi_{x_1}(x_3) \rangle \rangle.
\]

Hence in computing the left hand side, \( LHS \), of (4.6) we can replace \( \cos(2, 3) \) with \( \kappa_{123} \) and similarly for other indices. The conformal involutions of the disk are given by Blaschke factors. Hence, noting that for \( a, b \) in the disk \( \langle \langle a, b \rangle \rangle = ab \) we find that

\[
\kappa_{123} = \frac{x_1 - x_2}{1 - x_1 x_2}\frac{x_1 - x_3}{1 - x_1 x_3}
\]

Thus

\[
LHS = \pi - \arg \frac{-\prod |x_i - x_j|^2}{\prod (1 - x_i x_j)^2}
\]
with both products over the index pairs \((1, 2), (2, 3), (3, 1)\). The positive factor \(\Pi |x_i - x_j|^2\) does not affect the value of \(\arg\) and hence we continue with

\[
LHS = \pi - (\pi + 2 \arg \Pi k(x_i, x_j)) = 2\alpha
\]

the last equality by (2.10).

To finish we need to show that one of the first two terms in (4.6) equals Area\((T)\). In fact there are separate reasons why each equals Area\((T)\). We mentioned that the relation between \(\alpha\) and Area\((T)\) is in [Go]. For the other case let \(\gamma_i\) be the angle at vertex \(x_i\). We see from Proposition 4.1 that because \(T \subset \mathbb{C}H^1\) we have \(\cos(2\gamma_1) = e^{i\gamma_1}\) where \(\gamma_1\) is the angle at \(x_1\) of the triangle \(T\). Similarly for the other indices. Using that we see that \(LHS\) in (4.6) equals \(\pi - (\gamma_1 + \gamma_2 + \gamma_3)\) which equals Area\((T)\) by the classical result based on the Gauss-Bonnet theorem. \(\square\)

This result extends in the usual way to convex hyperbolic polygons. The interior of the polygon can be triangulated, the area of each triangular piece computed using the expressions in (4.6) and the total area computed as a sum. There will be cancellations generated by the fact that for kernel functions \(k_{xy} = k_{yx}\). The final result will involve the boundary vertices considered cyclically. Let Area\((x_1, ..., x_n)\) denote the invariant area of the interior of the polygon with vertices \(\{x_i\}_{i=1}^n \subset \mathbb{C}H^1\) and with the indices continued cyclically, \(x_{n+1} = x_1\), etc. We have

**Corollary 4.8.**

\[
(n - 2)\pi - \arg \prod_{i=1}^n \cos(i + 1, i + 2) = -2 \arg \prod_{i=1}^n k_{x_i, x_{i+1}} = \text{Area}(x_1, ..., x_n).
\]

If \(T\) is not in a complex geodesic then the first two terms in (4.6) need not be equal. In that more general case \(2\alpha\) equals the symplectic area of \(T\), the integral of the symplectic form of \(\mathbb{C}H^n\) over a real two manifold bounded by the sides of \(T\) [HM]. That quantity is also equal to the area of the triangle in the complex geodesic \(G\) containing \(x_1\) and \(x_2\) and with vertices \(x_1, x_2, \) and \(P_Gx_3\) [Go]. There is a nice discussion of the geometry associated to the angular invariant in [CO] and that paper presents the result we used for triangles in \(CH^1\) as the simplest special case of the rich relationship between the angular invariant and Hermitian geometry. More about that relationship is in [BI, BIW, C, CO, HM, Go, Ro], and the references there.

In the previous proposition (4.9) can be rewritten using \(\arg(abc) = \Im \log a + \Im \log b + \Im \log c\). If that is done then the following proposition appears to somehow be a pair with the previous one. As before the result can be formulated invariantly but we just present the cleaner case. The set \(BK_2 = \{(x, y, 0, ..., 0) \in \mathbb{B}_n : x, y \in \mathbb{R}\}\) inside \(\mathbb{C}H^2 = \mathbb{B}_n\) is a totally geodesically embedded copy of the Beltrami-Klein model of \(\mathbb{R}H^2\) which has constant curvature \(-1/4\). As such it carries a natural area measure.

**Proposition 4.9.** Suppose \(T = \{x_1, x_2, x_3\} \subset BK_2\). Then

\[
4(\pi - (\cos^{-1}\cos(2, 3) + \cos^{-1}\cos(1, 2) + \cos^{-1}\cos(3, 1))) = \text{Area}(T).
\]

**Proof.** This is a consequence of two facts. First, taking note of (2.14), \(\cos^{-1}\cos(2, 3)\) is the angle between the geodesics \(x_1x_2\) and \(x_1x_3\), and similarly for the other indices. Secondly, by the Gauss-Bonnet theorem, the area of a triangle with angles \(\alpha, \beta, \gamma\) in a plane of constant curvature \(-1/4\) is \(4(\pi - (\alpha + \beta + \gamma))\). \(\square\)
It would be interesting to have a general result which unifies the two propositions.

4.6. Kos and the Multiplier Algebra. We discuss briefly a role of kos the study of multiplier algebras of RKHS; more information is in [Ro] and [Ha, Sec. 3].

Suppose \( H \in RK \) and let \( M = M(H) \) be the algebra of multiplier operators on \( H \) normed by the operator norm. Introduce coordinates on \( M \) by associating the multiplier \( M \) with the \( n \)-tuple \( \lambda = (\lambda_1, \ldots, \lambda_n) \) where \( M^*k_j = \lambda_jk_j \), \( j = 1, \ldots, n \). Let \( (\mathcal{M})_1 \) be the unit ball of \( M \) and let Slice\(_1\) \((\mathcal{M})_1\) be the slice \((\mathcal{M})_1 \cap \{\lambda : \lambda_1 = 0\}\). The discussion in [Ro] and in [Ha, Sec. 3] gives the following.

**Proposition 4.10** ([Ha, Sec. 3]). If \( H \in CPP \) then the point on Slice\(_1\) \((\mathcal{M})_1\) with maximum value of Re \( \lambda_j \), \( 2 \leq j \leq n \), has coordinates

\[
(0, \delta_{12} \cos(1, 2), \ldots, \delta_{1n} \cos(1, n)).
\]

Thus if \( H = DA(X) \) then the geometry of \((\mathcal{M}(H))_1\) is enough to reconstruct the Gram matrix Gr\((H)\) and hence also enough to describe \( H \) and \( X \).

5. Finite Sets in \( CH^n \)

5.1. Describing Sets by Their Triangles. It is possible to give parametric descriptions of congruence classes of sets \( X = \{x_i\}_{i=1}^{n+1} \subset CH^n \) using the distances \( \delta(i, j) \) and the angular invariants \( \alpha(i, j, k) \). The description of triangles using the data set \( S' \) is the simplest example. Using those results and their proofs it is also possible to specify a unique model set in each congruence class, the triangles \( \Gamma \) in (3.1) and the tetrahedra \( \Delta \) in (3.2) are examples. Those invariants and related ones can also be used to describe more general geometric structures associated to \( CH^n \), see for instance [C] and [CG].

When those descriptions are specialized to sets in real hyperbolic space, \( X \subset RH^n \subset CH^n \), the angular invariant vanishes and we obtain descriptions of polyhedra in \( RH^n \) in terms of edge lengths. Here we use kos rather than the angular invariant and obtain descriptions of sets in \( RH^n \) and \( CH^n \) which emphasize a different type of geometric data; the basic example is the use of \( S'' \) to describe triangles. Our description generalizes the idea of describing polyhedra in \( RH^n \) using a mix of edge lengths and cosines of vertex angles. The approach is well suited to describing the relationship between the geometry of a set and its subsets. In particular it lets give explicit answers to the questions in the introduction.

Suppose we are given \( X = \{x_i\}_{i=1}^{n+1} \subset CH^n \) and we connect \( x_1 \) to each of the other \( x_i \) by a geodesic \( \gamma_{1i} \). The point \( x_1 \) will be the vertex point of an \( n \)-valent vertex \( V \) which is composed of the bivalent vertices \( V_{ij}, 2 \leq i, j \leq n + 1 \) having \( \gamma_{1i} \) as a first ray and \( \gamma_{1j} \) as a second. We will describe \( X \) using the distances between \( x_1 \) and the other points and the numbers \( K_{ij} = \cos(V_{ij}) \). Specifically, recalling the definition (2.15), we define the \( n \)-vector \( \varrho(X) \) and the \( n \times n \) matrix \( M(X) \) by

\[
\rho(X) = (\delta(x_1, x_2), \ldots, \delta(x_1, x_{n+1})) \text{, and}
\]

\[
M(X) = (K_{ij})_{i,j=2}^{n+1} = (\cos(V_{ij}))_{i,j=2}^{n+1} = KOS(DA(X), 1).
\]

**Theorem 5.1.** Given \( X = \{x_i\}_{i=1}^{n+1} \subset CH^n \):

1. Each entry of \( \varrho(X) \) is between 0 and 1, \( M(X) \) has 1’s on the diagonal and \( M(X) \succ 0 \).
(2) Conversely, given an $n$-tuple $\sigma$ of numbers between 0 and 1, and a matrix $N$ with 1’s on the diagonal and $N \succ 0$, there is an $X \subset \mathbb{CH}^n$ with $M(X)$ 
$= N$ and $g(X) = \sigma$.

(3) Given $Y \subset \mathbb{CH}^n$, $X \sim Y$ if and only if $M(X) = M(Y)$ and $g(X) = g(Y)$.

Proof. In (1) the claim for $\rho(X)$ is clear. The matrix $M(X)$ is invariant under automorphisms of $\mathbb{CH}^n$ and hence we can suppose $x_1$ is at the origin. Let $\hat{X}$ be the set of radial projections of the remaining points onto the unit sphere, $\hat{X} = \{\hat{x}_2, ..., \hat{x}_{n+1}\} \subset \partial \mathbb{B}_n$. We then see from (2.14) that $K_{ij} = \langle (\hat{x}_i, \hat{x}_j) \rangle$ for $2 \leq i, j \leq n + 1$. Thus $M(X)$ is the Gram matrix of the set of vectors $\hat{X}$ and hence is positive semidefinite. For (2), a matrix with the properties of $N$ must be the Gram matrix of a set $W = \{w_i\}_{i=2}^{n+1} \subset \mathbb{C}^n$, unique up to unitary equivalence. The 1’s on the diagonal of $N$ insure that $W \subset \partial \mathbb{B}_n$. We now form $X$ by designating the origin as $x_1$ and for $2 \leq i \leq n + 1$ picking $x_i$ on the line segment $[0, w_i]$ with $|x_i| = \sigma_i$. It is straightforward that $X$ has the required properties.

For (3), first suppose $Y$ is congruent to $X$; that is $Y$ is the image of $X$ under an automorphism of the ball. The entries of $g$ and $M$ are automorphism invariants and this gives the desired equalities. In the other direction suppose the data associated with $X$ equals the data associated with $Y$. Without loss of generality we can suppose $x_1$ is at the origin in which case $M(X)$ is the Gram matrix of the set $\hat{X} \subset \partial \mathbb{B}_n$. Similarly for $Y$ and $\hat{Y}$. Thus $\hat{X}$ and $\hat{Y}$ have the same Gram matrix and hence there is a unitary map $\mathbb{C}^n$ which takes $\hat{X}$ to $\hat{Y}$. That unitary is also an automorphism $\mathbb{CH}^n$ and so takes each geodesic from the origin to a point of $\hat{X}$ to a geodesic connecting the origin to a point of $\hat{Y}$. Given the further assumption that $\rho(X) = \rho(Y)$ it must take $X$ to $Y$, as required. \hfill \Box

It is possible to start with this result and describe a unique $X$ in each congruence class, for instance by moving $X$ so that $x_1$ is at the origin and then constructing an orthonormal basis for $\mathbb{C}^n$ in which the matrix of coordinates of $X$ is lower triangular. Theorem 7 of [Ro] is an instance of this approach carried out in detail.

Corollary 5.2. If $H \in \mathcal{RK}$ then $H \in \mathcal{CP}$ is and only if $\text{KOS}(H, 1) \succ 0$.

Proof. This is immediate from the previous theorem and Theorem 3.2. \hfill \Box

We say $X$ is in general position if it is not contained in a totally geodesic copy of $\mathbb{CH}^{n-1}$ inside of $\mathbb{CH}^N$. From the previous theorem and proof we have

Corollary 5.3. $X$ is in general position if and only if $M(X) \succ 0$.

We can use the previous theorem to describes $n$--valent vertices in $\mathbb{CH}^n$. Suppose $V$ is an $n$--valent vertex in $\mathbb{CH}^n$ with vertex point $x_1$ and rays $\{\gamma_i\}_{i=2}^{n+1}$. We are only interested in congruence classes and hence we suppose $x_1$ is at the origin. For $i = 2, ..., n + 1$ select a point $x_i$ on $\gamma_i$ and set $X = \{x_i\}_{i=2}^{n+1}$. From the previous theorem we know the congruence class of $X$ is determined by $M(X)$ and $g(X)$. Looking at that proof we see that knowing $M(X)$ is equivalent to knowing the congruence class of the projected set $\hat{X}$. From the definitions we see that knowing $\hat{X}$ is equivalent to knowing $V$. Hence, defining $M(V)$ to be $M(X)$ we have the following corollary of the previous theorem:

Corollary 5.4. Given an $n$--valent vertex $V$ in $\mathbb{CH}^n$:

(1) $M(V)$ has 1’s on the diagonal and $M(V) \succ 0$. 

(2) Given \( N \geq 0 \) with \( 1 \)'s on the diagonal there is a \( V \) in \( \mathbb{C}^n \) with \( \mathcal{M}(V) = N \).

(3) Given \( Z = \{ z_i \}_{i=2}^{n+1} \subset \partial \mathbb{B}_n \) there is a \( V \) in \( \mathbb{C}^n \) with \( \mathcal{M}(V) \) equal to the Gram matrix of \( Z \).

(4) Given other such vertex \( W \), \( V \) is congruent to \( W \) if and only if \( \mathcal{M}(V) = \mathcal{M}(W) \).

Neither the definitions of \( \rho(X) \) and \( \mathcal{M}(X) \) nor the previous theorem required that \( x_1 \) is at the origin. However any \( X \) is congruent to a set with \( x_1 \) at the origin and in that case \( \mathcal{M}(X) \) is the Gram matrix of the set \( \tilde{X} \), and \( \delta(x_1, x_i) = |x_i| \). In that case \( \mathcal{M}(X) \) has the information about the direction of travel from the origin to the other points of \( X \) and \( \rho(X) \) has the information about the distances to be traveled. Using those descriptions it is clear that the sets of data \( \rho \) and \( \mathcal{M} \) are independent of each other, each can be specified freely without regard to the other. 

In sum, the set
\[
\mathfrak{S}_{n+1} = (0, 1)^n \times \left\{ \text{positive definite } n \times n \text{ matrices with ones on the diagonal} \right\}
\]
is a moduli space for the congruence classes of ordered \( (n + 1) \)-tuples in complex hyperbolic space and the second factor is a moduli space for \( n \)-valent vertices. The first factor, \( \rho(X) \), consists of \( n \) real numbers, and the second, \( \mathcal{M}(X) \), is determined by \( n(n - 1)/2 \) complex numbers. Together they give the expected total of \( n^2 \) real parameters.

For \( n + 1 = 3 \) the description is quite simple. From Theorem 3.1 we know that the congruence class of the triangle \( X = \{ x_1, x_2, x_3 \} \) is described by the set \( S'' = (\delta_{12}, \delta_{13}, \cos(2, 3)) \). In the notation of the previous theorem
\[
\rho(X) = (\delta_{12}, \delta_{13}) , \quad \mathcal{M}(X) = \begin{pmatrix} 1 & \cos(2, 3) \\ \cos(3, 2) & 1 \end{pmatrix}.
\]

It is also straightforward to describe the parameters for sets contained in \( \mathbb{C} \mathbb{H}^1 = \mathbb{B}_1 \), the Poincare disk, or in \( \mathbb{R}^2 = B K_2 \), the Beltrami-Klein model of \( \mathbb{R}^2 \). For \( X \subset \mathbb{B}_1 \) we use the ambient complex coordinate and write \( X = \{ r_s e^{i \theta_s} \}_{s=1}^{n+1} \) with \( r_1 = 0 \). Using the computations in Section 4.1 we see that
\[
\rho(X) = (r_2, ..., r_{n+1}) , \quad \mathcal{M}(X) = (\exp i (\theta_s - \theta_t))_{s,t=2}^{n+1}.
\]

The automorphisms of \( \mathbb{C} \mathbb{H}^1 \) which fix the base point are rotations. They do not change the entries in \( \mathcal{M}(X) \) or the congruence class of \( X \). On the other hand, complex conjugation, which is not in Aut \( \mathbb{B}_1 \), can change the matrix entries and the congruence class.

For \( Y \subset B K_2 \) we use the polar coordinates of the containing \( \mathbb{R}^2 \). We have \( Y = \{ (r_s, \theta_s) \}_{s=1}^{n+1} \) with \( r_1 = 0 \). Now using Section 4.1 gives
\[
\rho(Y) = (r_2, ..., r_{n+1}) , \quad \mathcal{M}(Y) = (\cos(\theta_s - \theta_t))_{s,t=2}^{n+1}.
\]

In this case the map \( (r, \theta) \to (r, -\theta) \), which looks like complex conjugation, is the restriction of an element of Aut \( \mathbb{B}_2 \) to \( B K_2 \), namely the map \( (z, w) \to (z, -w) \). That map changes the sign of the \( \theta \)'s but that does not change \( \mathcal{M}(Y) \) or the congruence class of \( Y \).
5.2. **Comparison with the McCullough-Quiggin Theorem.** The McCullough-Quiggin theorem as presented in [AM] is for possibly infinite dimensional RKHS $H$. If $H$ is finite dimensional the theorem takes the following simpler form:

**Theorem 5.5.** If $H \in \mathcal{R}K$ then $H \in \mathcal{CPP}$ if and only if

$$\text{KOS}(H, s) \succeq 0, 1 \leq s \leq \dim H.$$ 

**Proof.** This follows from the result in [AM] after two observations. First, that result uses matrices MQ as in (2.16) while we use matrices KOS from (2.15). The matrices are related to each other through multiplication by diagonal matrices with positive entries. Hence they have the same positivity properties and the choice between them in this context is one of convenience. Second, the theorem in [AM] requires $\text{KOS}(J, r) \succeq 0$ for each $J$ which is a finite dimensional regular subspace of $H$. However if $H$ is finite dimensional we can apply the hypotheses to the maximal matrices $\text{KOS}(H, s)$. Every $J$ we need to consider is a regular subspace of $H$ and hence the matrix $\text{KOS}(J, r)$ is a principal submatrix of some $\text{KOS}(H, s)$. If $\text{KOS}(H, s) \succeq 0$ that implies $\text{KOS}(J, r) \succeq 0$ and hence we do not need a separate hypothesis for the smaller matrix. 

In finite dimensions Corollary 5.2 is an improvement on the McCullough-Quiggin theorem, it requires a condition on $\text{KOS}(H, 1)$ not on the full set of $\text{KOS}(H, s)$. The proof of Theorem 5.1 allows the construction of a set $X$ with $\mathcal{D}A(X) \sim H$ using the data in $\text{KOS}(H, 1)$. The fact that $\text{KOS}(H, s) \succeq 0$ for the other $s$ is then obtained by working with $\mathcal{D}A(X_R)$ for $X_R$ which is a renumbering of $X$.

On the other hand Theorem 5.1 is more superficial than Theorem 5.5. In the proof of Theorem 5.1 the relationship between $H \in \mathcal{CPP}$ and $H = \mathcal{D}A(X)$ is taken as known, it was imported into the discussion at Theorem 3.2. In contrast, the proof of Theorem 5.5 in [AM] is part the substantial work of establishing that relationship.

5.3. **Some Assembly Required.** In Theorem 3.1 we identified descriptors of the congruence classes of triangles in $\mathbb{C}^{n}$. In Theorem 5.2 we saw that the geometry of a finite set $X \subset \mathbb{C}^{n}$ is determined by the geometry of its included triangles. In Theorem 5.1 we saw how to organize that information into a vector $\rho(X)$ and a matrix $\mathcal{M}(X)$ which together describe $X$. If $Y$ is a subset of $X$ then $\rho(Y)$ is a subvector of $\rho(X)$ and $\mathcal{M}(Y)$ is a submatrix of $\mathcal{M}(X)$. We now combine those facts with our results about $X$ to study the relations between the geometry of subsets of $X$ and properties of submatrices of $\mathcal{M}(X)$. The vector $\rho(X)$ has a surprisingly little role in the analysis.

The following classical result will let us relate the fact that $\mathcal{M}(X) \succeq 0$ to the principal submatrices $\mathcal{PS}(\mathcal{M}(X))$ of $\mathcal{M}(X)$.

**Lemma 5.6 (Sylvester’s Criterion).** An $n \times n$ matrix $A$ satisfies $A \succeq 0$ if and only if $\det A \geq 0$ for all $A \in \mathcal{PS}(A)$, that is, if and only if the principal minors of $A$ are nonnegative. $A \succ 0$ if and only if the leading principal minors are positive.

To use this with $A = \text{KOS}(\mathcal{D}A(X), 1)$ we want information about its principal submatrices, $\mathcal{PS}(\text{KOS}(\mathcal{D}A(X), 1))$.

**Lemma 5.7.** If $H \in \mathcal{R}K$ then the matrices in $\mathcal{PS}(\text{KOS}(H, 1))$ are the matrices $\text{KOS}(J, 1)$ for $J$ which are regular subspaces of $H$ which contain the kernel function.
In particular if \( H = DA(X) \) then they are the matrices \( \text{KOS}(DA(Y), 1) \) for \( Y \) a subset of \( X \) which contains the distinguished point.

We want to know if sets \( \{ Y_i \} \) are congruent in specified ways to subsets of a larger set \( X \); that is, can the \( \{ Y_i \} \) be assembled into \( X \)? In the earlier notation the question is: given \( \{ Y_i \} \Rightarrow \ast \) is it true that \( \{ Y_i \} \Rightarrow X \)? Question 1 of the introduction is an example.

The previous results give general tools for analyzing such questions; we will look at three specific variations. Question 1 deals with 3 point subsets of 4 point sets. We first generalize that to 3 point subsets of \( n + 1 \) point sets and then to \( n \) point inside a set of size \( n + 1 \). In the final variation we consider assembling two 4 point sets along 3 point subsets to form a set with 5 points. That is an ad hoc example chosen to show ways these ideas play out in more complicated situations.

The results are in the language of the previous theorem. In each case there are no nontrivial conditions on \( \rho(X) \), the operational conditions concern determinants of matrices from \( \mathcal{PSM}(X) \). In the first case the conditions involve all of those matrices, in the second case only one, and in the third, more intricate, case the conclusion involves conditions on 4 of the 15 determinants.

5.3.1. Variation 1. We know from Theorem 3.2 that the congruence class of a set \( X \) is determined by the congruence classes of the triangles in \( X \) which contain a specified base point. We now ask if a given set of triangles can be congruent to those faces of some tetrahedron \( X \). We want to know if we can map triangles \( \{ T_i \} \) in \( \mathbb{C}H^n \),

\[
T_j = \{ y_{j1}, y_{j,j+1}, y_{j,j+2} \}, \quad j < n
\]

\[
T_n = \{ y_{n1}, y_{n,n+1}, y_{n2} \},
\]

into a set \( X = \{ x_1, ..., x_{n+1} \} \) with each \( y_{jk} \) is mapped to \( x_k \). That is, each image has its first vertex at \( x_1 \) and the images fill \( X \) with each segment \( x_1x_t \) in \( X \) covered twice. The coherence conditions are that two triangle sides that cover the same \( x_1x_t \) must be the same length.

Recall that \( \text{KOS}(\{ Y_i \}, 1) \) is the same as \( \text{KOS}(\{ DA(Y_i) \}, 1) \).

**Theorem 5.8.** Given the coherence data \( \{ T_i \} \Rightarrow \ast \) just described, the following are equivalent:

1. \( \exists X, \{ T_i \} \Rightarrow X \),
2. \( \text{KOS}(\{ T_i \}, 1) \triangleright 0 \),
3. \( \forall A \in \mathcal{PS}(\text{KOS}(\{ T_i \}, 1)), \det A \geq 0 \),
4. \( \forall S \subset \{ 1, ..., n \}, 1 \in S, \det \text{KOS}(\{ T_i \}_{y \in S}, 1) \geq 0 \).

**Proof.** This is a direct consequence of Theorem 5.1 and the two previous lemmas. \( \square \)

Taking note of Corollary 3.3 we also have the same result for spaces \( J_i \in \mathcal{CPP} \) and the question of moving from \( \{ J_i \} \Rightarrow \ast \) to \( \{ J_i \} \Rightarrow H \) with \( H \in \mathcal{CPP} \).

For both the \( \{ T_i \} \) and the \( \{ J_i \} \) the condition \( \det A \geq 0 \) is automatic for those \( A \) that are \( 1 \times 1 \) and is insured by Theorem 3.1 if \( A \) is \( 2 \times 2 \). If the \( \{ J_i \} \) are assumed to be in \( \mathcal{RK} \) but not necessarily in \( \mathcal{CPP} \) then the situation is more complicated, see the comment after Theorem 6.8.
5.3.2. Variation 2. Suppose we are given \( \{Y_i\}_{i=1}^n \), sets of size \( n \) in \( \mathbb{C}^n \). Write \( Y_i = \{y_{ij} : 1 \leq j \leq n + 1, j \neq i + 1\} \). We impose the coherence conditions \( \{Y_i\} \implies ?? \) that would hold if there were a set

\[
X = \{x_1, \ldots, x_{n+1}\}
\]

and congruences \( Y_i \rightsquigarrow X \) which mapped the points \( y_{is} \) to \( x_s \), for all \( s \neq i \). This coherence condition \( \{Y_i\} \implies ?? \) requires strong interrelations between the \( Y_i \). Given \( Y_r \) and \( Y_s \) there are \( Y_{rs} \subset Y_r \) and \( Y_{sr} \subset Y_s \) both of size \( n - 1 \) with \( Y_{rs} \sim Y_{sr} \). Also, note that every \( A \in \mathcal{PS}(\text{KOS}(\{Y_i\}, 1)) \) which is not maximal, \( A \neq \text{KOS}(\{Y_i\}, 1) \), satisfies \( A \in \mathcal{PS}(\text{KOS}(Y_r, 1)) \) for some individual \( Y_r \). We know \( \text{KOS}(Y_r, 1) \geq 0 \) and hence, by Sylvester’s criterion \( \det A \geq 0 \). In sum, the only \( A \in \mathcal{PS}(\text{KOS}(\{Y_i\}, 1)) \) for which we do not know \( \det A \geq 0 \) is the matrix \( \text{KOS}(\{Y_i\}, 1) \) itself. This discussion, together with Theorem 5.3.1 and the previous two lemmas, complete the proof of the following:

**Theorem 5.9.** Given \( \{Y_i\} \implies ?? \), there is an \( X \) so that \( \{Y_i\} \implies X \) if and only if

\[
\det \text{KOS}(\{Y_i\}, 1) \geq 0.
\]

5.3.3. Variation 3. In the previous two variations the coherence requirements on the \( \{Y_i\} \) were minimal and maximal. We now look at an intermediate case which is rich enough to display some structure and simple enough for explicit computations.

Suppose we have two four point sets in \( \mathbb{C}^n \), \( Y_A = \{a_1, a_2, a_3, a_4\} \) and \( Y_B = \{b_1, b_3, b_4, b_5\} \). The coherence requirements, \( \{Y_A, Y_B\} \implies ?? \), are the congruences that would hold if we had maps \( Y_A, Y_B \rightsquigarrow X = \{x_1, \ldots, x_5\} \) which respect the subscripts of the points. If that holds then the triangles \( \{a_1, a_3, a_4\} \) and \( \{b_1, b_3, b_4\} \) are congruent, and that congruence is the only coherence requirement.

We should not expect to fill the matrix \( \text{KOS}(\{Y_A, Y_B\}, 1) \). The set \( X \) has 5 points and so is determined by \( (5 - 1)^2 = 16 \) real parameters. On the other hand, each of \( Y \)'s provides 9 parameters but 4 of those are pinned by the fact that two triangles are congruent, leaving 14. This suggests our description is two real or one complex parameter short of being able to fully describe \( X \). In fact we cannot construct the entry \( \text{kos}_1(2, 5) \) in the matrix \( \text{KOS}(\{Y_A, Y_B\}, 1) \). Doing so would require a \( Y \) which contains points with subscripts \( \{1, 2, 5\} \), and neither of the \( Y \)'s satisfy that condition. To move forward we introduce a new parameter \( z \) and fill the matrix \( \text{KOS}(\{Y_A, Y_B\}, 1) \) to a matrix \( Y = \text{KOS}(\{Y_A, Y_B, z\}, 1) \) obtained from \( \text{KOS}(\{Y_A, Y_B\}, 1) \) by putting \( z \) in the place where the \( \text{kos}_1(2, 5) \) entry would be, and \( \bar{z} \) where \( \text{kos}_1(5, 2) \) would be. The values of \( z \) for which \( \text{KOS}(\{Y_A, Y_B, z\}, 1) \geq 0 \), if any, will parameterize inequivalent possible constructions of the desired \( X \).

We need to study the determinants of the matrices in \( \mathcal{PS}(Y) \). The matrix \( Y \) is a \( 4 \times 4 \) matrix with rows and columns indexed by the set \( \{2, 3, 4, 5\} \). The matrices in \( \mathcal{PS}(Y) \) are determined by the 15 nonempty subsets of that index set. We denote those matrices by \( Y \) with subscripts denoting the rows, and hence also columns, of \( Y \) that are retained. There are 4 single element subsets to consider, for each of them the resulting matrix has the single entry 1 and hence a positive determinant. There are 6 possibilities with two subscripts. The matrix \( Y_{34} \) will be a submatrix of both \( \text{KOS}(Y_A, 1) \) and \( \text{KOS}(Y_B, 1) \) and hence, by Sylvester’s criterion, will have a positive determinant. The matrix \( Y_{32} \) is not a submatrix of \( \text{KOS}(Y_B, 1) \), but it is a submatrix of \( \text{KOS}(Y_A, 1) \) and that is enough to insure it has a positive determinant. The same holds for \( Y_{24} \) and a similar argument applies \( Y_{35} \) and \( Y_{45} \) but with the roles of \( A \) and \( B \) reversed. The remaining matrix of that size is \( Y_{25} \); it cannot
be studied using either $Y_A$ or $Y_B$. It is a $2 \times 2$ matrix with 1’s on the diagonal and $z$ and $\bar{z}$ as off diagonal elements, and it must be dealt with separately. If we do not know about $\det Y_{25}$ then we also cannot know the positive semidefinite nature of the matrices with it as a submatrix; and hence those matrices must also be studied separately. They are $Y_{253}$, $Y_{254}$, and $Y_{2534} = Y$. The two remaining submatrices are $Y_{234} = \text{KOS}(Y_A, 1)$ and $Y_{345} = \text{KOS}(Y_B, 1)$ which we know are positive semidefinite.

**Theorem 5.10.** There is an $X$ so that $\{Y_A, Y_B\} \Rightarrow X$ if and only if there is a $z$ so that $Y = \text{KOS}(\{Y_A, Y_B, z\}, 1) \succcurlyeq 0$, equivalently if and only if $Y$ and the submatrices $Y_{25}$, $Y_{253}$, and $Y_{254}$ have positive determinants.

6. Tetrahedra

In the previous section we considered sets of $n + 1$ points. For $n = 2$ those results give the equivalence of Conditions (1), (5), and (6) of Theorem 3.1. Now we consider $n = 3$. We will make the previous conditions more explicit and give answers to Questions 1 and 2 of the introduction. We also use the results to analyze a family of four dimensional RKHS introduced by Quiggin.

6.1. **Question 1.** We want to know if a set of four triangles in $\mathbb{C}H^n$, $\{T_i\}_{i=1}^4$, might be congruent to the four faces of a tetrahedron $X = \{x_i\}_{i=1}^4 \subset \mathbb{C}H^3$. Actually it suffices to consider only three triangles and we begin by presenting the argument for that.

**Lemma 6.1.** If $\{T_i\}_{i=1}^4 \Rightarrow X$ then the congruence class of $X$ determines and is determined by the congruence classes of the $\{T_i\}_{i=1}^4$.

**Proposition 6.2.** Proof. That the congruence class of the faces determines the congruence class of the tetrahedron is Condition (5) of Theorem 3.2. In the other direction if we have the coordinates of $X$ then we can read off the coordinate description of the triangular faces. If we only know the description of $X$ as given in Theorem 5.2 – the designation of a distinguished vertex $x_1$, the vector $\rho(X)$, and the matrix $M(X)$ – then we can read off the data sets $S'$ for the three triangular faces of $X$ which meet at $x_1$. The needed side lengths are entries of $\rho(X)$ and the values of kos at the bivalent vertices at $x_1$ are entries of $M(X)$. By Theorem 3.3 those data sets determine the congruence class of the three triangles. To find the congruence class of the fourth triangle we first note coherence properties the set of four triangular faces must satisfy; various pairs of sides must have the same lengths, and the angular invariants must satisfy the cocycle condition (2.11). The proof is completed by the next lemma which shows that that data also suffices to determines the congruence class of the fourth face.

**Proposition 6.3.** Suppose $\{T_i\}_{i=1}^4$ is a set of four triangles in $\mathbb{C}H^n$ which satisfy the matching side length conditions and the cocycle condition (2.11) then

**Lemma 6.4.** (1) The congruence classes of $\{T_i\}_{i=1}^3$ determines the congruence class of $T_4$.

(2) The congruence classes of three triangular faces $\{F_i\}_{i=1}^3$ of a tetrahedron determines the congruence class of the fourth face $F_4$.

(3) There is a tetrahedron $R_4$ with $\{T_i\}_{i=1}^4 \Rightarrow R_4$ if and only if there is a tetrahedron $R_3$ with $\{T_i\}_{i=1}^3 \Rightarrow R_3$. 

□
Proof. For the first statement, if we know the side lengths of \( \{ T_i \}_{i=1}^3 \) then using the matching side length condition we also know the side lengths of \( T_4 \). By the cocycle condition \( 213 \) the values of the angular invariants for the \( \{ T_i \}_{i=1}^3 \) determines the angular invariant for \( T_3 \). Hence by the results in Theorem \( 3.1 \) for the data set \( S' \) the congruence class of \( T_4 \) is determined.

The second statement follows because the collection of faces of a tetrahedron automatically satisfy the matching side length conditions and the cocycle condition.

One half of the third statement is automatic. In the other direction, suppose we are given \( \{ T_i \}_{i=1}^4 \) and know \( \{ T_i \}_{i=1}^3 \Rightarrow R_3 \) for a tetrahedron \( R_3 \). Denote the faces of \( R_3 \) by \( \{ F_i \}_{i=1}^4 \) with the first three congruent to the \( \{ T_i \}_{i=1}^3 \). By the second statement the congruence class of \( F_4 \) is determined by the \( \{ F_i \}_{i=1}^3 \). By the first statement the congruence class of \( T_4 \) is determined by the \( \{ T_i \}_{i=1}^3 \). Furthermore, those two determinations use the same argument, in one case with data from the \( \{ T_i \} \) in the other case data from the \( \{ F_i \} \). But the \( \{ T_i \} \) and the \( \{ F_i \} \) are congruent so those two data sates are the same. Hence \( T_4 \sim F_4 \) and hence \( R_3 \) is the required \( R_4 \).

We now make two reductions which simplify Question 1. First, taking note of the previous lemma, we can replace \( \{ T_i \}_{i=1}^4 \) which satisfy a matching side length condition and a cocycle condition with a set of three triangles \( \{ T_2, T_3, T_4 \} \) which satisfy a matching side length condition. Second, by Proposition \( 4.6 \) knowing if those three triangles can be assembled as faces of a tetrahedron is independent of knowing the side lengths and is determined by the values of kos at the distinguished vertices.

Thus we are reduced to a situation of being told \( \{ T_i \}_{i=1}^3 \Rightarrow ?? \) and asking if there is a tetrahedron \( X \) with \( \{ T_i \}_{i=1}^3 \Rightarrow X \). That notation includes the assumption that the congruence relations between the vertices of the triangles, or between the subspaces of the associated \( DA \) spaces, have been specified. We will describe those relations in detail for this particular question.

We start with three triangles \( \{ T_i \}_{i=2}^4 \) in \( \mathbb{C}H^k \) and we denote their coordinates by \( T_i = \{ t_{i1}, t_{i3}, t_{ib} \} \). We want to know if we can assemble the \( \{ T_i \}_{i=2}^4 \) into a tetrahedron \( X = \{ x_{i1} \}_{i=1}^4 \) using congruences which take \( \{ t_{21}, t_{2a}, t_{2b} \} \) to the triangular face \( \{ x_{1}, x_{2}, x_{3} \} \), \( \{ t_{31}, t_{3a}, t_{3b} \} \) to the triangular face \( \{ x_{1}, x_{3}, x_{4} \} \), and \( \{ t_{41}, t_{4a}, t_{4b} \} \) to the triangular face \( \{ x_{1}, x_{4}, x_{2} \} \). For that to happen certain points must be identified, for instance \( t_{21}, t_{31}, t_{41} \) would be identified at the point \( x_{1} \). Also certain side lengths will match, for instance because two triangle sides will coincide as the edge \( x_{1}x_{2} \) of \( X \) we must have \( \delta(t_{21}, t_{2a}) = \delta(t_{41}, t_{4b}) \).

If there is no \( X \) then the \( \{ x_{i} \} \) are only bookkeeping symbols but the same considerations lead to the same coherence rules.

We now use Theorem \( 5.1 \) to see that \( \{ T_i \}_{2}^4 \Rightarrow X \) if and only if \( M(\{ T_i \}) \not\Rightarrow 0 \). First, suppose \( M(\{ T_i \}) \not\Rightarrow 0 \). The side length data from the \( \{ T_i \} \) is enough to construct \( \rho^* \) the candidate for \( \rho(X) \). By the second statement of that theorem, given that \( M(\{ T_i \}) \not\Rightarrow 0 \) then there is an \( X \) with \( M(\{ T_i \}) = M(X) \) and \( \rho^* = \rho(X) \). From \( M(\{ T_i \}) = M(X) \) and \( \rho^* = \rho(X) \) and the condition in Theorem \( 3.2 \) on the data set \( S'' \) we see that the faces of \( X \) are congruent to the \( \{ T_i \} \) and thus \( \{ T_i \}_{2}^4 \Rightarrow X \).

In the other direction, if we knew \( \{ T_i \}_{2}^4 \Rightarrow X \) we would know \( M(\{ T_i \}) = M(X) \) and hence \( M(\{ T_i \}) \not\Rightarrow 0 \).
The entries of $\mathcal{M}(X)$ would be values of $K_{ij} = \cos(V_{ij})$ for the bivalent vertices of $X$ at $x_i$. If $\{T_i\}_{i=1}^4 \rightarrow X$ then those vertices are congruent to vertices of the $\{T_i\}_{i=4}$. Specifically, writing $\cos(T_i)$ for the value of $\cos$ at the vertex $t_{i1}$ in the triangle $T_i$ we would have $K_{23} = \cos(T_2), K_{34} = \cos(T_3)$, and $K_{24} = \cos(T_4)$. (The final complex conjugation because $T_4$ was placed “backwards” in terms of our numbering of the edges.) Thus $\mathcal{M}(\{T_i\})$ is given by

$$
\mathcal{M}(\{T_i\}) = \begin{pmatrix}
1 & \cos(T_2) & \cos(T_3) \\
\cos(T_2) & 1 & 1 \\
\cos(T_3) & 1 & 1
\end{pmatrix}.
$$

We will be interested in positivity properties of matrices of that form.

**Lemma 6.5.** Suppose we have \(\{a_i\}_{i=1}^3 \subset \mathbb{C}\) and

\[
\mathcal{N} = \begin{pmatrix}
1 & a_1 & a_2 \\
\bar{a}_1 & 1 & a_3 \\
\bar{a}_2 & \bar{a}_3 & 1
\end{pmatrix}.
\]

Suppose that for some $i$, $|a_i| < 1$ or that for all $i$, $|a_i| \leq 1$. The following are equivalent:

1. $0 \leq \mathcal{N}$,
2. $0 \leq \det \mathcal{N}$,
3. $0 \leq 1 + 2 \Re a_1 \bar{a}_2 a_3 - |a_1|^2 - |a_2|^2 - |a_3|^2$,
4. $|a_1 \bar{a}_2 - a_3|^2 \leq (1 - |a_1|^2)(1 - |a_2|^2)$.

**Proof.** By Sylvester’s criterion the first condition implies the second. The second, third conditions are equivalently defined. That the third and fourth are equivalent can be seen by expanding both sides of the fourth statement giving

$$|a_1 \bar{a}_2|^2 - 2 \Re a_1 \bar{a}_2 a_3 + |a_3|^2 \leq 1 - |a_1|^2 - |a_2|^2 + |a_1 a_2|^2.$$

Cancellation and rearrangement shows that is equivalent to the third statement.

To go back to the first statement we want to use Sylvester’s criterion. That states that the first statement is a consequence of the nonnegativity of the seven principal minors. Three are the determinants of the $1 \times 1$ matrices given by the diagonal entries and they are positive. One is the positivity of $\det \mathcal{N}$ which is insured by the second statement. The other three requirements are equivalent to the conditions that each $|a_i| \leq 1$ and hence if we had assumed the $|a_i| \leq 1$ then we would be done. Otherwise note that the right hand side of the third statement is nonnegative. That lets us see that if, for instance, $|a_1| < 1$ then $|a_2| \leq 1$ and a similar argument applies to $a_3$. Thus we have reduced to the case of all $|a_i| \leq 1$. \hfill $\square$

We have collected all of the pieces to answer Question 1.

**Theorem 6.6.** With the numbering and naming scheme just described

1. If $\{T_i\} \Rightarrow X$ then $\mathcal{M}(\{T_i\}) = \mathcal{M}(X) \geq 0$.
2. If $\mathcal{M}(\{T_i\}) \geq 0$ then there is an $X$ with $\{T_i\} \Rightarrow X$ in which case $\mathcal{M}(\{T_i\}) = \mathcal{M}(X)$.
3. The previous lemma applies to the statements $\mathcal{M}(\{T_i\}) \geq 0$ and $\mathcal{M}(X) \geq 0$. In particular $\mathcal{M}(X) \geq 0$ if and only if

$$|K_{34} - \overline{K_{23}K_{24}}|^2 \leq (1 - |K_{23}|^2)(1 - |K_{24}|^2).$$
Proof. The first two statements follow from the discussion before the theorem together with Theorem 5.1. The third statement follows from the previous lemma together with statement (6) of Theorem 3.1 which insures that each \(|K_{ij}| \leq 1\). □

The condition (6.2) was obtained by specializing general results. It is interesting to also see how it follows from working directly with coordinates. We can assume \(X = \Delta\) as described in (6.2) in which case the \(K_{ij}\) can be computed using the points \(\{\hat{x}_i\}_{i=2}^4\) on the unit sphere with coordinates

\[
\hat{x}_2 = (1, 0, 0), \quad \hat{x}_3 = (\xi, \beta, 0), \quad \hat{x}_4 = (\eta, \zeta, \gamma);
\]

\[
|\xi|^2 + |\beta|^2 = |\eta|^2 + |\zeta|^2 + \gamma^2 = 1.
\]

(6.3)

For \(2 \leq i, j \leq 4\) we have \(K_{ii} = 1, K_{ij} = \overline{K_{ji}}\). The rest of the story is given by

\[
K_{23} = \hat{\xi}, \quad K_{24} = \hat{\eta}, \quad K_{34} = \hat{\xi}\hat{\beta} + \hat{\zeta}.
\]

We also have

\[
|\beta|^2 = 1 - |K_{23}|^2; \quad |\hat{\gamma}|^2 = 1 - |K_{24}|^2 - \gamma^2.
\]

Hence, noting that for all \(i, j, |K_{ij}| \leq 1\), we must have

\[
|K_{34} - \overline{K_{23}K_{24}}|^2 \leq (1 - |K_{23}|^2)(1 - |K_{24}|^2).
\]

(6.4)

which is (6.2).

In the other direction it is not hard to start from \(\{K_{ij}\}\) which satisfy these conditions and find coordinates of points on the sphere which generate this data.

The first two statements in Theorem 6.6 are special cases of both Theorem 5.8 and Theorem 5.9. The next result can be seen as a simpler variant of Theorem 5.10.

If \(T_2\) and \(T_3\) of \(\{T_i\}_{i=2}^4\) are specified then we can use the previous result to give conditions on \(T_4\) which insure that \(\{T_i\}_{i=2}^4 \Rightarrow X\). Specifically if we are given a \(T_2 \) and \(T_3\) and the coherence conditions then the side lengths of \(T_2\) and \(T_3\) are enough to fully specify \(\rho(X)\). The value of \(\text{kos}\) at the distinguished vertex of \(T_2\) will give the value of \(K_{23}\) for the matrix \(\mathcal{M}(\{T_2, T_3\})\). Similarly for \(T_3\) and \(K_{24}\). For us to assemble the triangles into the tetrahedron the lengths of the sides of \(T_4\) must match appropriately lengths of sides of \(T_2\) and \(T_3\). However the data from \(T_2\) and \(T_3\) is not enough to compute \(K_{42}\) which would be the value of \(\text{kos}\) at its distinguished vertex of \(T_4\). Hence the congruence class of \(T_4\) is indeterminate. To go forward we set \(K_{42} = z\) and using \(z\) we complete the matrix \(\mathcal{M}(\{T_2, T_3\})\) to a matrix \(\mathcal{M}(\{T_2, T_3, z\})\) which has no missing values. We can then apply the previous theorem to that matrix.

Corollary 6.7. If \(\{T_2, T_3\} \Rightarrow ??\) then there is a third triangle \(T_4\) and a tetrahedron \(X\) with \(\{T_2, T_3, T_4\} \Rightarrow X\) if and only if the Euclidean ball in \(\mathbb{C}^1\)

\[
B = B \left(\overline{K_{23}K_{24}}, (1 - |K_{23}|^2)^{1/2}(1 - |K_{24}|^2)^{1/2}\right)
\]

is nonempty. In that case the pairing of \(z\) with the value \(K_{42}\) establishes a one to one correspondence between \(z \in B\) and the congruence class of the possible third triangle \(T_4\). If \(B\) is empty then there are no such \(T_4\).

Proof. Putting \(z\) into (6.2) gives \(|z - \overline{K_{23}K_{24}}|^2 \leq (1 - |K_{23}|^2)(1 - |K_{24}|^2)|. □
6.2. Question 2. We saw in Section 6.2 that Question 2 is equivalent to Question 1. Having answered Question 1 we now reconfigure that answer to apply to Question 2. We will be informal.

We start with three dimensional \( \{ J_i \}_{i=1}^4 \subset \text{CPP} \). We assume the matching side length conditions and the cocycle condition, now defined using the invariants \( \delta \) and \( \alpha \) defined directly from reproducing kernels of the \( \{ J_i \}_{i=1}^4 \). We want to know if there is a four dimensional \( H \in \text{CPP} \) whose four three dimensional regular subspaces are rescalings of the \( \{ J_i \} \). As in the previous section it suffices to consider only the three spaces \( \{ J_i \}_{i=1}^4 \). Given those spaces we can construct the matrix \( \mathcal{M}(\{ J_i \}) \), the matrix that will equal \( \text{KOS}(H, 1) \) if there is an \( H \). The entries of \( \mathcal{M}(\{ J_i \}) \) are the values \( \{ L_{ij} \} \) of kos for the \( \{ J_i \} \), we obtain

\[
(6.5) \quad \mathcal{M}(\{ J_i \}) = \begin{pmatrix}
1 & L_{23} & L_{24} \\
L_{32} & 1 & L_{34} \\
L_{42} & L_{43} & 1
\end{pmatrix}.
\]

Theorem 6.8. Given \( \{ J_i \}_{i=2}^4 \subset \text{CPP} \) and \( \{ J_i \}_{i=1}^4 \Rightarrow ??? \) there is a four dimensional \( H \in \text{CPP} \) with \( \{ J_i \}_{i=2}^4 \Rightarrow H \) if and only if \( \mathcal{M}(\{ J_i \}) \geq 0 \).

Because \( \{ J_i \}_{i=1}^4 \subset \text{CPP} \) we know from Theorem 6.1 that each \( |L_{ij}| \leq 1 \). Hence Lemma 6.5 can be applied and that gives several conditions equivalent to \( \mathcal{M}(\{ J_i \}) \geq 0 \) including \( \det \mathcal{M}(\{ J_i \}) \geq 0 \). However if we only know that \( \{ J_i \}_{i=2}^4 \subset \text{RK} \) then deriving \( \mathcal{M}(\{ J_i \}) \geq 0 \) from \( \det \mathcal{M}(\{ J_i \}) \geq 0 \) requires the additional assumption that the \( |L_{ij}| \leq 1 \). (However this is not actually a different formulation. By Theorem 6.1 adding the assumptions that \( |L_{ij}| \leq 1 \) is equivalent to passing from the assumption that \( \{ J_i \}_{i=2}^4 \subset \text{RK} \) to the assumption that \( \{ J_i \}_{i=2}^4 \subset \text{CPP} \).

6.3. Quiggin’s Example. If a three dimensional RKHSI \( H \) has the Pick property then it has the complete Pick property. This can be seen from the implication (4) \( \Rightarrow \) (2) in Theorem 6.1 and is not hard to prove directly. It is then natural to ask if a four dimensional \( H \) with the Pick proper must have the \( \text{CPP} \). It was shown by Quiggin in his thesis that this is not true, \( Q \), \( AM \) Page 94]. He produced a family of four dimensional RKHSI \( H_x, 0 < x < 1 \), with the Pick property (and hence whose three dimensional regular subspaces each have the Pick property, and hence also the \( \text{CPP} \)) and showed that the space \( H_{1/4} \) failed the \( \text{CPP} \).

Here we consider the spaces \( \{ H_x \} \) using the results from the previous sections. We do not verify that the \( \{ H_x \} \) have the Pick property (that is done in \( Q \) pg, 84]), but we will show that for each \( H_x \) the regular three dimensional subspaces have the \( \text{CPP} \). We also show that none of the \( H_x \) have the \( \text{CPP} \).

Following Quiggin we introduce a family \( \{ H_x : 0 < x < 1 \} \subset \text{RK} \) of four dimensional spaces by specifying their Gram matrices, \( \text{Gr}(H_x) \). For \( 0 < x < 1 \) and \( s = (1 - x) \sqrt{x} \) set

\[
\text{Gr}(H_x) = \begin{pmatrix}
1 & x & x & x + is \\
x & 1 & x - is & x \\
x & x + is & 1 & x \\
x - is & x & x & 1
\end{pmatrix}.
\]

To show this is the Gram matrix of a RKHSI we need to show that \( \text{Gr}(H_x) \geq 0 \). By Lemma 5.6 we can do that by checking the signs of the leading principal minors.
They are
\[(1 + x)^2 (1 - x)^4, (1 + x)(1 - x)^2, (1 + x)(1 - x), 1\]
and, by inspection, are all positive for 0 < x < 1. (Those computations and the determinant computations below were done using computer algebra.)

Earlier we used the matrices $K_{OS}(\text{Gr}(H_x), 1)$ from (6.5). Here for ease in computing we use the matrices $MQ(H_x, 1) = (\delta_{ij} \delta_{1j} \cos(\theta(i, j)))_{i,j=2}^4$ mentioned in (2.10). The two have determinants of the same sign as do their square submatrices.

From the definitions we have
\[
MQ(H_x, 1) = \begin{pmatrix}
1 - x^2 & 1 - \frac{x^2}{x - is} & 1 - x - is \\
1 - \frac{x^2}{x + is} & 1 - x^2 & 1 - x - is \\
1 - x + is & 1 - x + is & (1 - x)(1 + x^2)
\end{pmatrix}
\]

Fix $x$. We want to know that $\{J_{xi}\}_{i=1}^4$, the regular three dimensional subspaces of $H_x$, have the CPP. By (5) of Theorem (3.1) we know that $J_{x1} \in CPP$ if the matrix $J_2$ obtained by deleting the first row and first column of $MQ(H_x, 1)$ satisfies $J_2 \not\geq 0$.

That will follow if we show $\det J_2 \geq 0$. Similarly for $J_{x2}$ and $J_{x4}$. For $J_{x1}$ we follow the same path but starting with $MQ(H_x, 2)$ rather than $MQ(H_x, 1)$. To show that $H_x \not\in CPP$ we will show that $\det MQ(H_x, 1) < 0$ and hence $MQ(H_x, 1) \not\geq 0$ fails. All these things can be seen in the explicit formulas for the determinants. Note that for 0 < x < 1, we have $x^2 - x + 1 > 0$. We have
\[
\det J_1 = \det J_2 = \det J_3 = x^2 (x + 1) (x - 1)^2,
\]
\[
\det J_4 = \frac{x^3(x + 1)(x - 1)^2}{x^2 - x + 1},
\]
\[
\det MQ(H_x, 1) = -\frac{x^3(x + 1)^2(x - 1)^4}{x^2 - x + 1}.
\]

This shows that the matching distances property together with the cocycle property are not sufficient to insure that a set of four three dimensional spaces with the CPP can be assembled into a four dimensional space with the CPP.

**Proposition 6.9.** Fix $x$, 0 < x < 1, let $\{J_{xi}\}_{i=1}^4$ be the three dimensional regular subspaces of $H_x$. Then

1. The $\{J_i\}_{i=1}^4 \subseteq CPP$ and $\{J_i\}_{i=1}^4 \not\subseteq$?.
2. There is an $H \in RK$ with $\{J_i\}_{i=1}^4 \Rightarrow H$.
3. There is no $H \in CPP$ with $\{J_i\}_{i=1}^4 \Rightarrow H$.

**Proof.** We verified above that the $\{J_i\}$ all have the CPP. The coherence is automatic because the $\{J_i\}$ are the three dimensional regular subspaces of a four dimensional RKHSI. The second statement is tautological, $H = H_x$ will suffice. It is included to emphasize that there is no obstruction to assembling the $\{J_i\}$ into a containing RKHSI $H$, just not one with the CPP.

Because $\{J_{xi}\} \subseteq CPP$ there are triangles $\{T_{xi}\}$ in $\mathbb{C}^n$ with $J_{xi} \sim DA(T_{xi})$. The coherence of the $\{J_{xi}\}$ insures that the $\{T_{xi}\}$ satisfy the coherence conditions for assembly as a tetrahedron. Hence the previous result can be recast as a result about the possible assembly of those triangles.
6.4. Tetrahedra in $\mathbb{R}^k$. The study of polyhedra in $\mathbb{R}^k$ is a very rich topic which is considered in many places; for instance the books [Co], [Fi], [An], surveys [J], [MP], and research papers [HR]. [Di] [W]. The relation between HSRK and sets in $\mathbb{R}^k$ is studied in [HM], [M] and [Ro] Sec. 7. Here we look at what happens when the results related to Theorem 5.1 are specialized to tetrahedra in $\mathbb{R}^k$.

We will call sets in $\mathbb{C}^n$ that are totally geodesic submanifolds isometric to $\mathbb{R}^k$ copies of $\mathbb{R}^k$. A four point set in a copy of $\mathbb{R}^k$ is a real hyperbolic tetrahedron. For those sets the value of $\cos$ at a bivalent vertex specializes as the the cosine of the vertex angle. This leads to simplifications of some of the previous results and to new geometric questions.

There are two technical points to discuss before going forward. First, there is an ambiguity in saying that two such sets in a copy $\mathbb{R}^k$ are congruent. We could mean that there is an automorphism of $\mathbb{C}^n$ which takes one to the other. Alternatively we might mean that there is a hyperbolic automorphism of $\mathbb{R}^k$ which takes one set to the other, with no mention of an automorphism of $\mathbb{C}^n$. Fortunately these two notions are equivalent [Go]. A similar comment holds for sets inside two different copies of $\mathbb{R}^k$ inside different copies $\mathbb{C}^n$.

Second, we have been studying $X$ using invariants derived from $DA(X)$ but have only defined $DA(X)$ for $X$ inside a copy of $\mathbb{R}^k$ inside some $\mathbb{C}^n$. To have definitions that apply to an $X$ in any version of $\mathbb{R}^k$ we select $\Lambda$, an isometric map of that $\mathbb{R}^k$ onto a copy of $\mathbb{R}^k$ inside a $\mathbb{C}^n$, and then define $DA(X)$ to be $DA(\Lambda(X))$. The construction depends on the choice of $\Lambda$ but changing $\Lambda$ produces a rescaling of $DA(X)$ and that does not change the values of the invariants we work with.

6.5. Sets Inside Copies of $\mathbb{R}^k$. If $S$ is a copy of $\mathbb{R}^k$ with $k = 1$ then $S$ is an ordinary geodesic, if $k = 2$ then, recalling $BK_2$ from Section 2.1, $S = \phi(BK_2)$ for some $\phi \in \text{Aut} \mathbb{B}_n$. The cases $k \geq 2$ follow a similar pattern, those sets are automorphic images of the set $BK_k$ obtained by intersecting $\mathbb{B}_n$ with a copy of $\mathbb{R}^k$ which is inside $\mathbb{C}^n$. These ideas are discussed in more detail in [Go] and [Bl].

The fact that $X \subset \mathbb{C}^n$ is inside such an $S$ is invariant under automorphisms of the ambient $\mathbb{C}^n$. There are various intrinsic characterizations of sets $X$ with this property; the following proposition is consequence of [Bl] Lemma 2.1.

Proposition 6.10. $\{x_i\}_{i=1}^s \subset \mathbb{C}^n$ is inside a copy of $\mathbb{R}^k$ if and only if all the numbers $\cos_i(p, q)$ are real.

Recall that there is a notion of the hyperbolic angle of intersection for curves in $\mathbb{R}^2$.

Corollary 6.11. The triangle $T = \{x_1, x_2, x_3\} \subset \mathbb{C}^n$ is in a copy of $\mathbb{R}^2$ if and only if $\cos_1(2, 3)$ is real. In that case $\cos_1(2, 3) = \cos(va_{23})$ where $va_{23}$ is hyperbolic at the vertex formed by the hyperbolic geodesics $x_1x_2$ and $x_1x_3$.

Proof. Using the model triangle $\Gamma$ in (3.1) it is easy to check that if $\cos_1(2, 3)$ is real then the coordinates of the points of $\Gamma$ are real and hence also so are the other values of $\cos$. It then follows from the previous proposition that $T$ is in a copy of $\mathbb{R}^k$. Because $\Gamma$ only has three points we can take $k = 2$. Using (2.13) we see that $\cos_1(2, 3) = \langle \hat{x}_2, \hat{x}_3 \rangle$. That inner product equals the cosine of the Euclidean angle at the origin of $\mathbb{R}^2$ between the segments $0\hat{x}_2$ and $0\hat{x}_3$. At the origin the Euclidean...
metric on $\mathbb{R}^2$ is conformal with the hyperbolic metric and hence the Euclidean angle whose cosine we found is also the hyperbolic angle. \hfill \Box

6.5.1. **Vertex Angles.** Angles such as $va_{23}$ in the previous corollary are called vertex angles. (On a polyhedron they are also sometimes called face angles or dihedral angles.)

The previous corollary shows that if $X = \{x_1, x_2, x_3, x_4\}$ is in a copy of $\mathbb{R}^n$ then $\cos(va_{ij})$ is the cosine of the vertex angle at the vertex $x_j x_i$. Denoting that angle by $va_{ij}$ the entries of $M(X) = \text{KOS}(DA(X), 1)$ are $K_{ij} = \cos(va_{jk})$.

Theorem 6.6 applies to the matrix and the formulas simplify slightly because the $K_{ij}$ are real. To emphasize this change we introduce a modified notation for the matrix $\text{KOS}(DA(X), 1)$. We set

$$M_{CV}(X) = \text{KOS}(DA(X), 1) = \begin{pmatrix} 1 & \cos va_{23} & \cos va_{24} \\ \cos va_{32} & 1 & \cos va_{34} \\ \cos va_{42} & \cos va_{43} & 1 \end{pmatrix}.$$ 

and note that $\cos va_{ij} = \cos va_{ji}$. Thus $M_{CV}(X) = M(X)$, the added subscript is a reminder that the matrix entries are cosines of vertex angles.

If we have triangles $\{T_i\}_{i=2}^4$ in a copy of $\mathbb{R}^n$ which satisfy the coherence conditions for assembly into a tetrahedron, $\{T_i\}_{i=2}^4 \supseteq ??$ we set $M_{CV}(\{T_i\}_{i=2}^4) = M(\{T_i\}_{i=2}^4)$. Thus if $\{T_i\}_{i=2}^4 \supseteq X$ then $M_{CV}(\{T_i\}_{i=2}^4) = M_{CV}(X)$. Those entries are real and hence, by the previous proposition, if there is an $X$ then it is in a real hyperbolic space.

Theorem 6.6 applies in this context and gives

**Corollary 6.12.** Given triangles $\{T_i\}_{i=1}^4$ in $\mathbb{R}^n$ with $\{T_i\}_{i=2}^4 \supseteq ??$ there is a tetrahedron $X$ in $\mathbb{R}^n$ such that $\{T_i\}_{i=2}^4 \supseteq X$ if and only if $M_{CV}(\{T_i\}_{i=2}^4) \supseteq 0$. That condition is equivalent to each of the following three conditions:

\begin{align}
&\text{(6.7) } \det M_{CV}(\{T_i\}_{i=2}^4) \geq 0 \\
&\text{(6.8) } (\cos va_{34} - \cos va_{23} \cos va_{42})^2 \leq (1 - \cos^2 va_{23})(1 - \cos^2 va_{42}) \\
&\text{(6.9) } \left(\frac{\cos va_{34} - \cos va_{23} \cos va_{42}}{\sin va_{23} \sin va_{42}}\right)^2 \leq 1,
\end{align}

\textbf{Proof.} The conditions $M_{CV}(\{T_i\}_{i=2}^4) \supseteq 0$, (6.7), and (6.8) are repeats of parts of Theorem 6.6. The final statement is a rewriting of (6.8) which will be useful later. \hfill \Box

Each of these conditions is necessary and sufficient conditions for the vertex angles of a tetrahedron in $\mathbb{R}^2$. We will see a much simpler equivalent condition in Proposition 6.18.

6.5.2. **Dihedral Angles.** Going forward we will suppose that the tetrahedra we consider in $\mathbb{R}^k$ are nondegenerate, they are not congruent to tetrahedra in $\mathbb{R}^2$. It will be convenient now realize $\mathbb{R}^3$ as the Poincaré ball model. In that model the manifold is the real three ball, $\mathbb{B}_3$, the group action is the group of conformal automorphisms of the ball, and the "sphere at infinity" of $\mathbb{R}^3$ is the Euclidean two sphere $S^2 = \partial \mathbb{B}_3$. These choices let us use Euclidean coordinate geometry.

If $X = \{x_i\}_{i=1}^4$ is a tetrahedron in $\mathbb{R}^3$ then we know from Proposition 6.6 that the vertex angles $\{va_{ij}\}$ describe the congruence class of the trivalent vertex at $x_1$. That vertex can also be described using the angles between faces, the **dihedral**
angles. In fact the dihedral angles are used more commonly than vertex angles in describing real hyperbolic polyhedra; see for instance [W], [FG], [HR], [Roe], and the references there. We now look briefly at describing the dihedral angles of a tetrahedron in our framework and at their relation to the vertex angles.

The work in this section is influenced by the work of Roeder in [Roe] and there are overlaps. In [Roe] tetrahedra are described using the matrix of negatives of cosines of dihedral angles. The matrix \( \mathcal{M}_{CDA}(X) \) we introduce below is a submatrix of that matrix and some of the results about submatrices in Theorem 1 in [Roe] match some of the results in Theorem 6.17 below. There are also differences. In particular the work in [Roe] restricts attention to tetrahedra with non-obtuse dihedral angles.

In the next few paragraphs we will use the indices \( r, s, t \) to denote three different indices from the set \( \{2, 3, 4\} \). Given the tetrahedron \( X \) in \( \mathbb{H}^3 \) with \( x_1 \) at the origin denote the triangular face with vertices \( \{x_1, x_i, x_j\} \) by \( F_{ij} \). The dihedral angle along edge \( s, d_1, s = 2, 3, 4 \), is the angle between the faces \( F_r \) and \( F_{st} \). If this were a Euclidean tetrahedron we could find \( d_1 \) by finding the inward pointing unit normals, \( n_r, s = 2, 3, 4 \), for the face \( F_r \), and using the fact that \(- \cos d_1 = \langle n_r, s, t \rangle \). The requirement that \( n_r \) be inward pointing is the requirement that \( \langle n_r, x_i \rangle \geq 0 \). However the formula for \( d_1 \) is unchanged if the normals are replaced by their negatives and hence it is enough to construct the normals so that the inner products \( \langle n_r, x_i \rangle \) all have the same sign. Taking note of the fact that \( \langle a, b \times c \rangle = \langle b, c \times a \rangle \) for vectors in \( \mathbb{R}^3 \) we see that the choices

\[
\langle d_1 \rangle = \frac{x_r \times x_s}{\|x_r \times x_s\|} = \frac{\hat{x}_r \times \hat{x}_s}{\|\hat{x}_r \times \hat{x}_s\|}
\]

insures that the sign of \( \langle n_r, x_i \rangle \) is unchanged by cyclic permutation of the indices. Hence we can compute

\[
\cos d_1 = -\langle \langle n_r, s, t \rangle \rangle
= -\left\langle \frac{\hat{x}_r \times \hat{x}_s}{\|\hat{x}_r \times \hat{x}_s\|} \frac{\hat{x}_s \times \hat{x}_t}{\|\hat{x}_s \times \hat{x}_t\|} \right\rangle
= -\frac{\langle \langle \hat{x}_r, \hat{x}_s \rangle \rangle \langle \langle \hat{x}_s, \hat{x}_t \rangle \rangle - \langle \langle \hat{x}_r, \hat{x}_t \rangle \rangle \langle \langle \hat{x}_s, \hat{x}_s \rangle \rangle}{\|\hat{x}_r \times \hat{x}_s\| \|\hat{x}_s \times \hat{x}_t\|}
= \frac{\cos \angle_{\hat{x}_r, \hat{x}_s} \cos \angle_{\hat{x}_s, \hat{x}_t}}{\sin \angle_{\hat{x}_r, \hat{x}_s} \sin \angle_{\hat{x}_s, \hat{x}_t}}.
\]

This was a Euclidean computation but the hyperbolic geometry at the origin is conformal to the Euclidean geometry. Hence the appropriate hyperbolic computations, using tangent lines to geodesics and tangent planes to faces, all with their tangencies at the origin, leads to that Euclidean computation.

We will study the dihedral angles of the \( X \) at \( x_1 \) using the matrix

\[
\mathcal{M}_{CDA}(X) = \begin{pmatrix}
1 & -\cos d_{a_{23}} & -\cos d_{a_{24}} \\
-\cos d_{a_{32}} & 1 & -\cos d_{a_{34}} \\
-\cos d_{a_{42}} & -\cos d_{a_{43}} & 1
\end{pmatrix}.
\]

Here the subscript is a reminder that the matrix entries are the negatives of cosines of dihedral angles.

Our computation of dihedral angles gives one statement of the hyperbolic law of cosines.
**Definition 6.13.** Suppose that for \(2 \leq i,j \leq 4\) we are given angles \(\{va_{ij}\}\) and \(\{da_{ij}\}\) with, for all \(i,j\), \(va_{ij} = va_{ji}\), \(da_{ij} = da_{ji}\), \(va_{ii} = 0\), \(da_{ii} = \pi\); set \(VA_{ii} = 0\), \(DA_{ii} = \pi\). Define
\[
\begin{align*}
DA_{rs} &= \frac{\cos va_{rs} - \cos va_{rs} va_{ts}}{\sin va_{rs} \sin va_{ts}}, \\
VA_{rs} &= \frac{\cos da_{rs} + \cos da_{ts} \cos da_{ts}}{\sin da_{rs} \sin da_{ts}}.
\end{align*}
\]

The vertex angles and dihedral angles of a hyperbolic tetrahedron in \(\mathbb{H}^n\) are related by those two formulas.

**Lemma 6.14** (Hyperbolic Law of Cosines). If \(X = \{x_1\}_{i=1}^4 \subset \mathbb{H}^n\) has vertex angles at \(x_1\) \(\{va_{ij}\}_{i,j=2}^4\) and dihedral angles \(\{da_{ij}\}_{i,j=2}^4\), then, with the notation \(\text{(6.12)}\) and \(\text{(6.13)}\),
\[
\cos da_{ij} = DA_{ij}, \quad \cos va_{ij} = VA_{ij}, \quad 2 \leq i,j \leq 4.
\]

These formulas are classical, for instance [Co], [J]. They can be obtained by doing Euclidean vector computations. Alternatively they are consequences of the spherical law of cosines applied to the triangle \(X\) along with the relation between the geometry of \(X\) and of \(\hat{X}\). Details are in [Roe] and Chapter 5 of [An] and the references there.

We separated the definition from the lemma because we can use the definitions even if the \(va_{ij}\) are not known to be data from a tetrahedron. If we have a set of triangles \(\{T_i\}_{i=2}^4\) in \(\mathbb{H}^k\) which satisfy the coherence conditions for assembly into a tetrahedron, \(\{T_i\}_{i=2}^4 \Rightarrow ??\), then we can construct the matrix \(M_{\text{CVA}}(\{T_i\}_{i=2}^4)\). Using that data in \(\text{(6.12)}\) we can compute imputed values of the \(DA_{ij}\). If \(\{T_i\}_{i=2}^4 \Rightarrow X\) for a tetrahedron \(X\) then those values will be cosines of angles and satisfy \(|D_{ij}| \leq 1\). In fact the condition \(|D_{ij}| \leq 1\) is also sufficient for there to be an \(X\); comparing the formula \(\text{(6.12)}\) with \(\text{(6.9)}\) we obtain

**Corollary 6.15.** If \(\{T_i\}_{i=2}^4 \Rightarrow ??\) and \(DA_{ij}\) is computed using the \(va_{ij}\) values from \(M_{\text{CVA}}(\{T_i\}_{i=2}^4)\) and \(\text{(6.12)}\) then there is a tetrahedron \(X\) with \(\{T_i\}_{i=2}^4 \Rightarrow X\) if and only if for some \(i,j\) \(|DA_{ij}| \leq 1\).

That corollary is a special case of a more general observation. If we have \(\{T_i\}_{i=2}^4 \Rightarrow ??\) then we can construct \(M_{\text{CVA}}(\{T_i\}_{i=2}^4)\). Using the data from that matrix and the formula \(\text{(6.12)}\) we can construct \(M_{\text{CDA}}(\{T_i\}_{i=2}^4)\), a matrix that would be \(M_{\text{CDA}}(X)\) if we knew there were an \(X\). Both \(M_{\text{CVA}}(\{T_i\}_{i=2}^4)\) and \(M_{\text{CDA}}(\{T_i\}_{i=2}^4)\) are matrices to which Lemma \(6.5\) applies. The formulas \(\text{(6.12)}\) and \(\text{(6.13)}\) give equivalences between the conditions from Lemma \(6.5\) which insure \(M_{\text{CVA}}(\{T_i\}_{i=2}^4) \geq 0\) and the conditions from that lemma which would insure, \(M_{\text{CDA}}(\{T_i\}_{i=2}^4) \geq 0\).

Those equivalences are an instance of a systematic duality between results about vertex angles and results about dihedral angles. Here is another instance. Given a set of three angles \(\Gamma = \{\gamma_2, \gamma_3, \gamma_4\}\) it may or may not be true that those angles can arise as the vertex angles of a nondegenerate trivalent vertex \(\mathbb{H}^k\). If they can we say that \(\Gamma\) are \textit{good vertex angles} and write \(\Gamma \in \mathcal{GVA}\). It also may or may not be true that those angles can arise as a set of dihedral angles of a nondegenerate trivalent vertex in \(\mathbb{H}^k\). If so we call them \textit{good dihedral angles} and write \(\Gamma \in \mathcal{GDA}\). We have the following duality.
Proposition 6.16. Given the set of angles $\Gamma = \{\gamma_2, \gamma_3, \gamma_4\}$ and the set $\Pi - \Gamma = \{\pi - \gamma_2, \pi - \gamma_3, \pi - \gamma_4\}$, then $\Gamma \in \mathcal{GVA}$ if and only if $\Pi - \Gamma \in \mathcal{GDA}$.

Note that $\Pi - (\Pi - \Gamma) = \Gamma$ and hence the proposition is symmetric in the two sets of angles.

Proof. First suppose $\Gamma \in \mathcal{GDA}$. In that case there is a tetrahedron $X$ with dihedral angles $\Gamma$ and the negatives of the cosines of elements of $\Gamma$ as entries in $\mathcal{M}_{CDA}(X)$. We want to find a tetrahedron $Y$ which shows $\Pi - \Gamma \in \mathcal{GVA}$. To do that consider the matrix $\mathcal{M}_{CDA}(X)$. Because it was constructed as a Gram matrix (of three normal vectors) we know it is positive semidefinite. Hence from Theorem 6.6 here is a tetrahedron $Y$ with $\mathcal{M}_{CVA}(Y) = \mathcal{M}_{CDA}(X)$. Noting the identity $-\cos \gamma = \cos(\pi - \gamma)$ we see that the entries of $\mathcal{M}_{CVA}(Y)$ are the cosines of the angles in $\Pi - \Gamma$. Hence this is the required $Y$.

Suppose now that $\Gamma \in \mathcal{GVA}$, specifically $\Gamma$ are the vertex angles of a tetrahedron $X$. As before the matrix $\mathcal{M}_{CDA}(X)$ equals $\mathcal{M}_{CVA}(Y)$ for some tetrahedron $Y = \{y_i\}_{i=1}^4$. Let $\Gamma_Y$ be the set of dihedral angles of $Y$ and hence $\Gamma_Y \in \mathcal{GDA}$. We will be finished if we show $\Pi - \Gamma_Y = \Gamma$ or, equivalently, that the cosines of elements of $\Gamma$ are the negatives of the cosines of elements of $\Gamma_Y$. To see this we analyze the construction of the cosines of the dihedral angles of $Y$. The normal vectors to a face of $Y$ will be parallel to a vector $y_i^* \times y_j^*$. By construction $y_i^*$ is parallel to $x_i^* \times x_j^*$ where neither $r$ nor $q$ is equal to $i$. Thus the normal vector will be parallel to $(x_i^* \times x_j^*) \times (x_r^* \times x_q^*)$ where, also, neither $p$ nor $q$ is equal to $j$. Because we only have three indices the normal must be parallel to $(x_i^* \times x_j^*) \times (x_k^* \times x_s^*)$ where $k$ is the index that is not $i$ or $j$. The normal is perpendicular to the first factor and hence in the plane spanned by $x_i^*$ and $x_j^*$. Similarly, looking at the second factor we see the normal is in the span of $x_i^*$ and $x_k^*$. Hence it is in the intersection of those spans which is the span of $x_k^*$. We want the inward facing normal and hence it is $x_k^*$. The inner products of these normal vectors are, by definition, the negatives of the cosines of the dihedral angles of $Y$; they are also, by inspection, the cosines of the vertex angles of $X$. Thus the two are equal, as we wanted. \qed

(The previous results were stated and proved in the language of vertex angles and dihedral angles of a hyperbolic tetrahedron. However the results can also be formulated and proved in the language of spherical geometry. The previous result is the relationship between the angles in the spherical triangle $\hat{X} = \{\hat{x}_2, \hat{x}_3, \hat{x}_4\} \subset \mathbb{S}_2$ and the angles in its polar dual. Alternatively the results are about the Euclidean geometry of $\mathbb{R}^3$ associated with the cross product.)

Using the details of the previous proof lets us give a version of Theorem 6.6 for dihedral angles. Let $\mathcal{L} = (L_{ij})_{i,j=2}^4$ be the symmetric matrix with real entries given by

$$
\mathcal{L} = \begin{pmatrix}
1 & L_{23} & L_{24} \\
L_{32} & 1 & L_{24} \\
L_{42} & L_{42} & 1
\end{pmatrix}
$$

Theorem 6.17. The matrix $\mathcal{L}$ is the matrix $\mathcal{M}_{CDA}(X)$ of a tetrahedron $X$ in $\mathbb{R}^k$ if and only $\mathcal{L} \succ 0$. 
Proof. We saw in the previous proof that for any $X$ we have $\mathcal{M}_{-CDA}(X) \succ 0$. Suppose now we have $\mathcal{L} \succ 0$ then by Theorem 6.6 $\mathcal{L} = \mathcal{M}_{CVA}(Y)$ for a tetrahedron $Y$ in $\mathbb{R}H^k$. We now follow the pattern of the previous proof. From $\mathcal{M}_{CVA}(Y)$ we obtain $\mathcal{M}_{-CDA}(Y)$, and then, by Theorem 6.6 $\mathcal{M}_{-CDA}(Y) = \mathcal{M}_{CVA}(Z)$ for some $Z$. Now consider the matrix $\mathcal{M}_{-CDA}(Z)$. The argument in the previous proof shows that this type of repeated passage from vertex angles to dihedral angles reproduces the original data; in particular $\mathcal{L} = \mathcal{M}_{-CDA}(Z)$ which shows that $\mathcal{L}$ is of the required form. □

If we suppose that the $|L_{ij}| \leq 1$, for instance if the $\{L_{ij}\}$ are the negatives of cosines of angles, then we can apply Lemma 6.5 to obtain various statements equivalent to $\mathcal{L} \succ 0$. If $\mathcal{L} = (L_{ij})$ with $L_{ij} = -\cos \theta_{ij}$ for some angles $\theta_{ij} = \theta_{ji}$ and $\theta_{ii} = \pi$ then those angles are the dihedral angles at the vertex of a tetrahedron in $\mathbb{R}H^k$ if and only if $\mathcal{L} \succ 0$ and that happens if and only if

\[
(6.14) \quad (\cos \theta_{34} + \cos \theta_{23} \cos \theta_{42})^2 \leq (1 - \cos^2 \theta_{23})(1 - \cos^2 \theta_{42})
\]

(or the similar result with a different distinguished index). This condition and the earlier (6.8) are analogs of (6.2) for the values of cos at the vertex of a tetrahedron in $\mathbb{C}H^n$. However in this case the fact that we are working with real numbers allows very substantial simplifications.

Proposition 6.18. (1) If $\Gamma = \{\alpha, \beta, \gamma\} \subset (0, \pi)$ then $\Gamma \in \mathcal{GVA}$ if and only if

\[
(6.15) \quad \alpha \leq \beta + \gamma.
\]

That inequality is sometimes called the "triangle inequality for angles".

(2) If $\Gamma = \{A, B, C\} \subset (0, \pi)$ then $\Gamma \in \mathcal{GDA}$ if and only if

\[
(6.16) \quad \pi \leq A + B + C
\]

Proof. We start with (6.8) and then pass to equivalent formulations:

\[
(\cos \alpha - \cos \beta \cos \gamma)^2 \leq (1 - \cos^2 \beta)(1 - \cos^2 \gamma)
\]

\[
|\cos \alpha - \cos \beta \cos \gamma| \leq \sin \beta \sin \gamma
\]

\[
-\sin \beta \sin \gamma + \cos \beta \cos \gamma \leq \cos \alpha \leq \sin \beta \sin \gamma + \cos \beta \cos \gamma
\]

\[
\cos (\beta + \gamma) \leq \cos |\beta - \gamma|
\]

If $\gamma + \beta < \pi$ then the three angles in the previous line are in the range $(0, \pi)$ where the cosine is monotone decreasing. In that case the first inequality gives $\alpha \leq \beta + \gamma$. In the other case we have $\alpha \leq \pi \leq \beta + \gamma$. In both cases we have (6.15).

For the dihedral angles we first note that $-\cos A = \cos (\pi - A)$. We then follow the previous pattern and obtain

\[
\cos (B + C) \leq \cos (\pi - A) \leq \cos |B - C|.
\]

Following the same analysis as before we obtain $\pi - A < B + C$ which is equivalent to (6.16).

These are elementary results in real hyperbolic geometry and certainly can be given much more direct proofs. On the other hand this approach gives interesting insight into how results such as Theorem 6.6 can be seen as Extensions to complex hyperbolic geometry of basic facts from real hyperbolic geometry.
6.6. Factorization. The expressions $\det M_{CA}(X)$ and $\det M_{CVA}(X)(X)$ arise in hyperbolic trigonometry where they are called amplitudes. Among their many properties are trigonometric factorizations. The formulas can be obtained by trigonometric analysis \cite[pg. 107]{F}, \cite[(2.88) - (2.94)]{J} or, as Roeder points out \cite{Roe}, by direct computation with complex exponentials.

**Proposition 6.19.** Set

\[
s = (va_{23} + va_{24} + va_{34})/2
\]

\[S = (da_{23} + da_{24} + da_{34})/2\]

then

\[
1 + 2 \cos va_{23} \cos va_{24} \cos va_{34} - \cos^2 va_{23} - \cos^2 va_{24} - \cos^2 va_{34}
\]

\[= 4 \sin(s) \sin(s - va_{23}) \sin(s - va_{24}) \sin(s - va_{34})
\]

and

\[
1 - 2 \cos da_{23} \cos da_{24} \cos da_{34} - \cos^2 da_{23} - \cos^2 da_{24} - \cos^2 da_{34}
\]

\[= -4 \cos(S) \cos(S - da_{23}) \cos(S - da_{24}) \cos(S - da_{34})
\]

Given a tetrahedron $X \subset \mathbb{H}^n$ the left hand side of (6.17) is $\det M_{CVA}(X)$ and hence by (6.7) must be positive. We can use that as a starting point and study the signs of the individual factors and derive (6.15). Similarly in \cite{Roe} Roeder analyzes the sign of the factors in (6.18) and obtains (6.16) for acute angles $A, B, C$.

7. Final Comments

**Cayley Equations:** If we replace the trigonometric variables in the statements $\det M_{CA}(X) = 0$ and $\det M_{CVA}(X)(X) = 0$ with algebraic variables we obtain two algebraic equations:

\[
1 - 2xyz - x^2 - y^2 - z^2 = 0,
\]

\[p(x, y, z) = 1 + 2xyz - x^2 - y^2 - z^2 = 0.
\]

These equations were studied by Cayley in his classic study of cubic equations and sometimes carry his name \cite{Cay}. For us the region in $\mathbb{R}^3$ where the variables have absolute value at most one and $p(x, y, z) > 0$ parameterizes nondegenerate tetrahedra in $\mathbb{H}^3$. The boundary surface $\Omega$, where $p(x, y, z) = 0$, correspond to degenerate tetrahedra. The smooth points of $\Omega$ correspond to simple degenerations, degenerate tetrahedra that become nondegenerate when a single vertex is moved a small amount. The singular points of $\Omega$ correspond to more complicated, nongeneric, degeneracies. For instance, let $T$ be a triangle $\{w, y, z\}$ in the ball model of $\mathbb{H}^3$, $\mathbb{B}^3$, which is in the plane $\mathbb{H}^2$ specified by the vanishing of the third coordinate and which has the origin of that plane in its interior. Form tetrahedra $X_\varepsilon$ by adjoining a fourth vertex, which will be the distinguished vertex $x_1$, with Euclidean coordinates $(0, 0, \varepsilon)$ for a small positive $\varepsilon$. The $X_\varepsilon$ are proper tetrahedra but the limiting $X_0$ is a degenerate tetrahedron. The vertices of $X_0$ are all in the plane $\mathbb{H}^2$, and include the origin of that plane as the distinguished vertex. Thus for $X_0$ the values of $\cos_1$ are the cosines of the angles formed by connecting the origin to the other vertices. We are in a plane so those angles sum to $2\pi$. Thus the corresponding $(x, y, z)$ values are $(\cos \alpha, \cos \beta, \cos(2\pi - \alpha - \beta))$ for some angles $\alpha$ and $\beta$. These
values satisfy (7.1), as can be checked by noting that equality holds in (6.9). That point is a smooth point on the surface \( \Omega \) and the degeneracy of \( X_0 \) can be removed by moving the vertex at the origin slightly to obtain an \( X_\varepsilon \).

The singular points of \( \Omega \) are the points \((\pm 1, \pm 1, \pm 1)\) with an even number of minus signs. The corresponding tetrahedra have four points on a single real geodesic. Those tetrahedra are have nongeneric degeneracy, they remain degenerate if any of the points is moved slightly.

Some related discussion is in [H].

**Larger Polyhedra:** A polyhedron in \( RH^n \) is determined by a set of vertex points together with combinatoric data telling which vertices are connected by edges, which edges bound faces, etc. For tetrahedra the combinatorics are trivial, every pair of vertices bounds an edge, every triple of edges bound a face. Hence the study of tetrahedra is essentially equivalent to the study of the geometry of four point sets. Also, for four point sets we obtained relatively clean results because the dimensions were small.

For larger polyhedra we obtain information that is more complicated. This is seen for instance in Theorem 5.10. Also the information from Theorem 5.1 and related results says nothing about the polyhedral combinatorics the set might have. It would be interesting if there were a natural way to encode the polyhedral structure as a layer of structure in the Hilbert spaces \( DA(X) \).

The appropriate definition of a polyhedron in \( CH^n \) is not clear. An indication that the situation is very different from \( RH^n \) can already be seen with triangles. Any three points in \( RH^n \) are contained in a totally geodesically embedded \( RH^2 \) and that allows a natural definition of the face bounded by the geodesics connecting the vertices. However there is no similar construction for triples of points in \( CH^n \), in particular there is no natural notion of the face of a triangle. This shows up, for instance, in the fact that symplectic area of a triangle, a substitute for classically defined area, is defined in a way that is independent of the choice of real two manifold joining the geodesic edges.

(Some thought has been given to defining and describing polyhedra in \( C^n \) but that work does not seem directly relevant here.)

**Vertices at Infinity:** The study of tetrahedra in real hyperbolic space, and to some extent in complex hyperbolic space, is not restricted to classical bounded tetrahedra but also includes consideration of ideal tetrahedra, tetrahedra with one or more vertices in the ideal boundary (i.e. the “sphere at infinity”, \( \partial B_n \)). Although some of the previous discussion extends to those contexts it is not clear if there are spaces similar to the \( DA(X) \) which are useful in studying those geometries. It may be that the adjoining ideal boundary points to hyperbolic space is analogous to adjoining vectors of infinite norm to the \( DA(X) \).

There is analysis of congruence of finite sets in the closure \( CH^n \) in several places including [Go], [HS], [G], and [CG].

**The Physics Literature:** The question of characterizing triples of triangles in \( RH^3 \) which can be assembled as part of a tetrahedron is also studied in the physics literature, sometimes with the name ”closure questions”, for instance [CL], [BDGL], [HHR]. In contrast to the work here, those papers make substantial use of the descriptive and analytical properties of the automorphism group.

**Other Hilbert Spaces, Other Geometries:** The relation between RKHSI and geometry extends to general RKHSI and includes relations to other classical
geometries. We have focused on the relation between the DA spaces and hyperbolic geometry. There are similar relations between the Segal–Bargmann–Fock spaces and the Hermitian geometry of \(\mathbb{C}^n\), and between the Hilbert spaces of spin coherent states \([BZ]\), and the geometry of complex spheres and projective spaces.

More general relations between geometry and spaces such as \(DA(X)\) for \(X\) in \(\mathbb{R}H^n\) and \(\mathbb{C}H^n\) are suggested by the work in \([M]\).

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