On the rank of the 2-class group of some imaginary triquadratic number fields

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Received: 10 September 2020 / Accepted: 28 December 2020 / Published online: 16 January 2021
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Abstract
Let \(d\) be an odd square-free integer and \(\zeta_8\) a primitive 8-th root of unity. The purpose of this paper is to investigate the rank of the 2-class group of the fields \(L_d = \mathbb{Q}(\zeta_8, \sqrt{d})\).

Keywords 2-rank · 2-class group · Imaginary triquadratic number fields · Real quadratic field

Mathematics Subject Classification 11R11 · 11R16 · 11R18 · 11R27 · 11R29

1 Introduction

Let \(k\) be a number field and \(p\) a prime integer. Let \(\text{Cl}_p(k)\) denote the \(p\)-class group of \(k\), that is the \(p\)-Sylow subgroup of its ideal class group \(\text{Cl}(k)\) in the wide sense. Class groups of number fields have been studied for a long time, and there are many very interesting (and very difficult) problems concerning their behavior. An interesting invariant is the \(p\)-rank of \(\text{Cl}_p(k)\), i.e. the dimension of \(\text{Cl}(k)/\text{Cl}(k)^p\) as a vector space over the field with \(p\) elements \(\mathbb{F}_p\).

In this work, we investigate the 2-rank of the class groups of some imaginary triquadratic number fields. To the best of our knowledge, there is no study in this setting and it would be an interesting task to develop such investigations. Note that the odd part of the class group of a number field is much better understood, since it is known to be isomorphic to the direct product of the odd parts of the class groups of its quadratic subfields (see [22]).

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In the literature, there are many studies that investigated the 2-rank of the class group of a number field \( k \), let us quote some. For a quadratic field \( k \), using Gauss’s genus theory, one can easily deduce the rank of \( \text{Cl}_2(k) \). For biquadratic fields, the authors of [1, 3] determined all positive integers \( d \) such that \( \text{Cl}_2(k) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), where \( k = \mathbb{Q}(\sqrt{d}, -\sqrt{\varepsilon}) \) and \( \varepsilon = 1, 2 \). The papers [4, 5] investigated the rank of \( \text{Cl}_2(k) \) for the fields \( k = \mathbb{Q}(\sqrt{d}, \sqrt{m}) \), where \( m \) is a prime and \( d \) a positive square-free integer. In the same direction, [7] classified all fields \( k = \mathbb{Q}(\sqrt{d}, i) \) such that \( \text{Cl}_2(k) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \) or \( (\mathbb{Z}/2\mathbb{Z})^3 \). In [9, 11], E. Brown and C. Parry determined some imaginary quartic cyclic number fields \( k \) such that the rank of \( \text{Cl}_2(k) \) is at most 3. Finally, [24] determined all imaginary biquadratic fields whose 2-class group is cyclic.

Let \( d \) be an odd square-free integer and \( \zeta_8 \) a primitive 8-th root of unity. In the present work, we are interested in the rank of the 2-class group of the imaginary triquadratic number fields \( L_d := \mathbb{Q}(i, \sqrt{2}, \sqrt{d}) = \mathbb{Q}(\zeta_8, \sqrt{d}) \). Methods and techniques used are based on genus theory, ambiguous class number formula, the properties of the norm residue symbol, units of some number fields and some other results of algebraic number theory.

The structure of this paper is the following. In Sect. 2, we recall the ambiguous class number formula and the relations allowing to calculate the rank of the 2-class group of \( L_d \). Next, we compute the number of prime ideals of \( K = \mathbb{Q}(\zeta_8) \) ramified in \( L_d = K(\sqrt{d}) \). Thereafter, in Sect. 3, we recall the definition and some properties of the quadratic norm residue symbol. In Sect. 4, we investigate the rank of \( \text{Cl}_2(L_d) \) according to classes (mod 8) in which the prime divisors of \( d \) lie. As applications, in Sect. 5 we determine the integers \( d \) such that \( \text{Cl}_2(L_d) \) is trivial, cyclic or of rank 2, and we thus deduce the integers \( d \) satisfying \( \text{Cl}_2(L_d) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \), we shall say \( \text{Cl}_2(L_d) \) is of type (2, 2). We end this section by giving the 2-part of the class number of \( L_d \) in terms of those of its biquadratic subfields.

Notations

Let \( k \) be a number field. Throughout this paper, we use the following notations.

- \( d \): An odd square-free integer,
- \( \zeta_n \): An \( n \)-th primitive root of unity,
- \( K := \mathbb{Q}(\zeta_8) \),
- \( L_d := K(\sqrt{d}) \),
- \( \delta_d \): The discriminant of \( k \),
- \( \mathcal{O}_k \): The ring of integers of \( k \),
- \( \text{Cl}(k) \): The class group of \( k \),
- \( \text{Cl}_2(k) \): The 2-class group of \( k \),
- \( h(k) \): The class number of \( k \),
- \( h_2(k) \): The 2-class number of \( k \),
- \( h_2(m) \): The 2-class number of a quadratic field \( \mathbb{Q}(\sqrt{m}) \),
- \( N \): The norm map for the extension \( L_d/K \),
- \( E_k \): The unit group of \( \mathcal{O}_k \),
- \( e_d \): The integer defined by \( (E_K : E_K \cap N(L_d)) = 2e_d \),
- \( \left( \frac{a}{p} \right) \): The quadratic norm residue symbol over \( K \),
- \( \left( \frac{\cdot}{4} \right) \): The biquadratic residue symbol,
- \( \bar{\alpha} \): The coset of \( \alpha \) in \( E_K/(E_K \cap N(L_d)) \),
- \( r_2(d) \): The rank of the 2-class group of \( L_d \),
- \( \varepsilon_m \): The fundamental unit of \( \mathbb{Q}(\sqrt{m}) \), where \( m \) is a positive square-free integer,
\[ W_k \]: The set of roots of unity contained in \( k \),  
\[ \omega_k \]: The cardinality of \( W_k \),  
\[ k^+ \]: The maximal real subfield of \( k \),  
\[ \mathbb{Q}_k \]: The Hasse’s index, that is \( (E_k : W_kE_k^*) \), if \( k/k^+ \) is CM,  
\[ Q(k'/k) \]: The unit index of a biquadratic extension \( k'/k \),  
\[ q(k) := (E_k : \prod E_{k_i}^*) \], where \( k_i \) are the quadratic subfields of a multiquadratic field \( k \).

## 2 Preliminaries

Let \( K := \mathbb{Q}(\zeta_k) \) and \( L_d := K(\sqrt{d}) \), where \( d \) is an odd square-free integer. Note that \( \mathcal{O}_K \) is a principal ideal domain. Denote by \( Am(L_d/K) \) the group of ambiguous ideal classes of \( L_d/K \), that are classes of \( \text{Cl}(L_d) \) fixed under any element of \( \text{Gal}(L_d/K) \), and by \( Am_2(L_d/K) \) its 2-Sylow subgroup. In fact, \( Am_2(L_d/K) = \{ \epsilon \in \text{Cl}(L_d) : \epsilon^2 = 1 \} \) is an elementary 2-group of order \( 2^{\epsilon_d(d)} \).

It is well known, according to the ambiguous class number formula for a cyclic extension of prime degree (cf. [17]), that

\[
|Am(L_d/K)| = h(K) \frac{2^{r_d - 1}}{(E_k : E_k \cap N(L_d))}. 
\]

It follows that

\[
|Am_2(L_d/K)| = \frac{2^{r_d - 1}}{(E_k : E_k \cap N(L_d))} = 2^{r_d - 1 - \epsilon_d}, \tag{1}
\]

where \( t_d \) is the number of finite and infinite primes of \( K \) which ramify in \( L_d/K \) and \( \epsilon_d \) is defined by \( (E_k : E_k \cap N(L_d)) = 2^{\epsilon_d} \). So the 2-rank of \( \text{Cl}(L_d) \) verifies the relation:

\[
r_2(d) = t_d - 1 - \epsilon_d. \tag{2}
\]

Let us now determine the ideals of \( K \) that ramify in \( L_d \). Let \( \delta_{L_d/K} \) denote the generator of the relative discriminant of the extension \( L_d/K \). We have:

**Proposition 2.1** Let \( d \) be an odd square-free integer. Then the relative discriminant of \( L_d/K \) is generated by \( \delta_{L_d/K} = d \).

**Proof** Assume that \( d \equiv 1 \) (mod 4). Thus \( \delta_{\mathbb{Q}(\sqrt{d})} = d \), \( \delta_K = 2^8 \) and \( \mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \mathbb{Z}[\frac{1+\sqrt{d}}{2}] \).

Since \( \delta_{\mathbb{Q}(\sqrt{d})} \) and \( \delta_K \) are coprime, then \( \mathcal{O}_{L_d} = \mathcal{O}_K[\frac{1+\sqrt{d}}{2}] \). Hence \( \delta_{L_d/K} = \text{disc}_{L_d/K}(1, \frac{1+\sqrt{d}}{2}) = d \). If \( d \equiv 3 \) (mod 4), then \( -d \equiv 1 \) (mod 4). As \( L_d = L_{-d} \), the previous case completes the proof.

The following result is easily deduced.

**Corollary 2.2** Let \( d \) be an odd square-free integer.
1. The discriminant of $L_d$ is $\delta_L = 2^{16} \cdot d^4$, and thus the primes that ramify in $L_d$ are exactly 2 and the prime divisors of $d$.

2. The ring of integers of $L_d$ is given by

$$\mathcal{O}_{L_d} = \begin{cases} 
\mathbb{Z}[\zeta_8, \frac{1 + \sqrt{d}}{2}] & \text{if } d \equiv 1 \pmod{4}, \\
\mathbb{Z}[\zeta_8, \frac{1 + \sqrt{-d}}{2}] & \text{if } d \equiv 3 \pmod{4}.
\end{cases}$$

Next, we compute the number of prime ideals of $K$ that ramify in $L_d$.

**Theorem 2.3** Let $d$ be an odd square-free integer. Then the number of prime ideals of $K$ that ramify in $L_d$ is $2(q + r)$, where $r$ is the number of prime integers dividing $d$ and $q$ is the number of those which are congruent to 1 (mod 8).

**Proof** Using the theorem of the cyclotomic reciprocity law ([27, Theorem 2.13]), one can verify what follows:

- If $p$ is a prime integer such that $p \equiv 1 \pmod{8}$, then there are exactly 4 prime ideals of $K$ lying over $p$.
- If $p \not\equiv 1 \pmod{8}$, then there are exactly 2 primes of $K$ above $p$.

By Proposition 2.1, the prime ideals of $K$ that ramify in $L_d$ are exactly the divisors of $d$ in $\mathcal{O}_K$. So the number of those primes is $4q + 2(r - q) = 2(q + r)$ as desired. □

Since $K$ is a biquadratic field, $e_d \in \{0, 1, 2\}$. The next corollary follows directly from the previous theorem and the formula (2).

**Corollary 2.4** Let $d$ be an odd square-free integer. Then $h(L_d)$ is even in the following two cases:

1. $|d|$ is composite.
2. $|d|$ is a prime congruent to 1 (mod 8).

The next lemma provides the unit group of $K$.

**Lemma 2.5** The unit group of $K$ is given by $E_K = \langle \zeta_8, \varepsilon_2 \rangle$.

**Proof** The field $K = \mathbb{Q}(\zeta_8) = \mathbb{Q}(i, \sqrt{2})$ is a Galois extension of $\mathbb{Q}$ whose Galois group is an elementary 2-group of order 4. We have, $\mathbb{Q}(\sqrt{2})$ is the real quadratic subfield of $K$ with fundamental unit $\varepsilon_2 = 1 + \sqrt{2}$, which is also a fundamental unit of $K$. Thus by [16, Theorem 42], the Hasse’s unit index is equal to 1, i.e., $(E_K : E_{\mathbb{Q}(\sqrt{2})} W_K) = 1$, where $W_K$ is the set of roots of unity contained in $K$. □
3 Quadratic norm residue symbol

To compute the index $e_d$ appearing in the formula (2), we will use the quadratic norm residue symbol, so we have to recall its definition and some of its properties (cf. [18, Chapter II, Theorem 3.1.3]). Let $k$ be a number field and $\beta \in k^*$ a square-free element in $k$. Let $f$ be the conductor of $k(\sqrt{\beta})/k$. For any prime $\mathfrak{p}$ of $k$ (finite or infinite), we denote by $f_\mathfrak{p}$ the largest power of $\mathfrak{p}$ dividing $f$. Let $\alpha \in k^*$, according to the approximation theorem there exists $\alpha_0 \in k$ such that

$$\alpha_0 \equiv \alpha \pmod{f_\mathfrak{p}} \quad \text{and} \quad \alpha_0 \equiv 1 \pmod{f/f_\mathfrak{p}}.$$ 

If $(\alpha_0) = \mathfrak{p}^n\mathfrak{B}$ with $n \in \mathbb{Z}$ and $(\mathfrak{B}, \mathfrak{p}) = 1$ ($n = 0$ if $\mathfrak{p}$ is infinite), set

$$\left( \frac{\alpha, k(\sqrt{\beta})}{\mathfrak{p}} \right) = \left( \frac{\alpha k(\sqrt{\beta})}{\mathfrak{B}} \right).$$

where $\left( \frac{k(\sqrt{\beta})}{\mathfrak{B}} \right)$ is the Artin map applied to $\mathfrak{B}$. For $\alpha \in k^*$ and a prime (finite or infinite) $\mathfrak{p}$ of $k$, the quadratic norm residue symbol is defined by

$$\left( \frac{\alpha, \beta}{\mathfrak{p}} \right) = \frac{\left( \frac{\alpha, k(\sqrt{\beta})}{\mathfrak{p}} \right)(\sqrt{\beta})}{\sqrt{\beta}} \in \{\pm 1\}. $$

If the prime $\mathfrak{p}$ is unramified in $k(\sqrt{\beta})/k$, we set

$$\left( \frac{\beta}{\mathfrak{p}} \right) = \frac{\left( \frac{k(\sqrt{\beta})}{\mathfrak{p}} \right)(\sqrt{\beta})}{\sqrt{\beta}} \in \{\pm 1\}. $$

Note that the norm residue symbol may be defined more generally for an extension $k(\sqrt[m]{\beta})/k$, where $m \in \mathbb{N}^*$, $k$ is a number field containing the $m$-th root of unity and $\beta \in k^*$. The quadratic norm residue symbol verifies the following properties that we shall use later.

1. $\left( \frac{a_1 a_2, \beta}{\mathfrak{p}} \right) = \left( \frac{a_1, \beta}{\mathfrak{p}} \right) \left( \frac{a_2, \beta}{\mathfrak{p}} \right)$. 
2. $\left( \frac{a, \beta}{\mathfrak{p}} \right) = \left( \frac{\beta, a}{\mathfrak{p}} \right)$. 
3. If $\mathfrak{p}$ is unramified in $k(\sqrt{\beta})/k$ and appears with exponent $e$ in the decomposition of $(\alpha)$, then $\left( \frac{\alpha, \beta}{\mathfrak{p}} \right) = \left( \frac{\beta}{\mathfrak{p}} \right)^e$. 
4. If $\mathfrak{p}$ is unramified in $k(\sqrt{\beta})/k$ and does not appear in the decomposition of $(\alpha)$, then $\left( \frac{\alpha, \beta}{\mathfrak{p}} \right) = 1$. 
5. $\prod_{\mathfrak{p} \in PL} \left( \frac{a, \beta}{\mathfrak{p}} \right) = 1$, where $PL$ is the set of all finite and infinite primes of $k$. 
6. Let $k_1$ be a finite extension of $k$, $\alpha \in k_1^*$ and $\beta \in k^*$. Denote by $\mathfrak{p}$ a prime ideal of $k$ and by $\mathfrak{B}$ a prime ideal of $k_1$ above $\mathfrak{p}$. Thus...
\[
\prod_{\mathfrak{p}|d} \left( \frac{\alpha, \beta}{\mathfrak{p}} \right) = \left( \frac{N_{K/k}(\alpha), \beta}{\mathfrak{p}} \right).
\]

4 The rank of the 2-class group of \( L_d \)

In the present section we compute, \( r_2(d) \), the rank of the 2-class group of \( L_d \). Since \( L_d = L_{-d} = L_{2d} \), then without loss of generality, we suppose that \( d \) is an odd positive square-free integer. We shall compute \( r_2(d) \) distinguishing two cases according to the classes (mod 8) in which the prime divisors of \( d \) lie. Let us start with the following results.

**Lemma 4.1** [4] Let \( k \) be a number field, \( d \) a positive square-free integer and \( \alpha \in k^* \) such that the ideal \( \alpha \mathcal{O}_k \) is the norm of a fractional ideal of \( k(\sqrt{d}) \). Then, \( \alpha \) is norm in \( k(\sqrt{d})/k \) if and only if \( \left( \frac{\alpha, d}{\mathfrak{p}} \right) = 1 \) for all primes \( \mathfrak{p} \) of \( k \) ramified in \( k(\sqrt{d}) \).

Thus, to compute \( e_d \), it suffices to compute the quadratic norm residue symbols \( \left( \frac{\varepsilon_{8,d}}{\mathfrak{p}} \right) \) and \( \left( \frac{\varepsilon_{2,d}}{\mathfrak{p}} \right) \) for the prime ideals \( \mathfrak{p} \) of \( K \) dividing \( d \) (see Proposition 2.1 and Lemma 2.5).

The next lemma follows directly from properties 1, 4 and 5 of the norm residue symbol.

**Lemma 4.2** Let \( d \) be an odd positive square-free integer and \( \mathfrak{p} \) a prime of \( K \) dividing \( d \). Denote by \( \alpha \) a unit of \( K \). Then

1. \( \left( \frac{\alpha, d}{\mathfrak{p}} \right) = \left( \frac{\alpha, p}{\mathfrak{p}} \right) \), where \( p \) is the prime contained in \( \mathfrak{p} \).
2. \( \prod_{\mathfrak{p}|d} \left( \frac{\alpha, d}{\mathfrak{p}} \right) = 1. \)

4.1 Case 1: The prime divisors of \( d \) are in the same coset of \( \mathbb{Z}/8\mathbb{Z} \)

Let \( d \) be an odd positive square-free integer such that the primes \( p \mid d \) are in the same coset (mod 8).

**Lemma 4.3** Let \( p \) be a prime such that \( p \not\equiv 1 \) (mod 8) and denote by \( \mathfrak{p}_K \) any prime ideal of \( K \) above \( p \).

1. If \( p \equiv 3 \) (mod 8), then \( \left( \frac{\varepsilon_2, p}{\mathfrak{p}_K} \right) = -1 \) and \( \left( \frac{\varepsilon_8, p}{\mathfrak{p}_K} \right) = -1. \)
2. If \( p \equiv 5 \) (mod 8), then \( \left( \frac{\varepsilon_2, p}{\mathfrak{p}_K} \right) = -1 \) and \( \left( \frac{\varepsilon_8, p}{\mathfrak{p}_K} \right) = 1. \)
3. If \( p \equiv 7 \pmod{8} \), then \( \left( \frac{\varepsilon_2 \cdot p}{\mathfrak{p}_K} \right) = 1 \) and \( \left( \frac{\varepsilon_8 \cdot p}{\mathfrak{p}_K} \right) = 1 \).

**Proof** Suppose that \( p \) is congruent to 3 (mod 8). Let \( \mathfrak{p}_{Q(i)} \) and \( \mathfrak{p}_{Q(\sqrt{2})} \) be the prime ideals of \( Q(i) \) and \( Q(\sqrt{2}) \) respectively above \( p \). Note that these two prime ideals are totally decomposed in \( K \). As \( p \) is inert in both \( Q(i) \) and \( Q(\sqrt{2}) \), then

\[
\left( \frac{\varepsilon_8 \cdot p}{\mathfrak{p}_K} \right) = \left( \frac{1+i \cdot p}{\mathfrak{p}_K} \right) \left( \frac{\sqrt{2} \cdot p}{\mathfrak{p}_K} \right) = \left( \frac{1+i}{\mathfrak{p}_K} \right) \left( \frac{\sqrt{2}}{\mathfrak{p}_K} \right) \left( \frac{\sqrt{2}}{\mathfrak{p}_{Q(\sqrt{2})}} \right) = -\left( \frac{2}{p} \right) = -1,
\]

and

\[
\left( \frac{1+\sqrt{2} \cdot p}{\mathfrak{p}_K} \right) = \left( \frac{1+\sqrt{2}}{\mathfrak{p}_K} \right) = \left( \frac{1+\sqrt{2}}{\mathfrak{p}_{Q(\sqrt{2})}} \right) = \left( \frac{1}{p} \right) = -1.
\]

We similarly prove the other cases. \( \square \)

**Theorem 4.4** Let \( p > 2 \) be a prime such that \( p \not\equiv 1 \pmod{8} \).

1. If \( p \equiv 3 \) or 5 (mod 8), then the 2-class group of \( L_p \) is trivial, i.e., \( r_2(p) = 0 \).
2. If \( p \equiv 7 \pmod{8} \), then the 2-class group of \( L_p \) is cyclic nontrivial, i.e., \( r_2(p) = 1 \).

**Proof** By Theorem 2.3, we have \( r_2(p) = 2 - 1 - e_p = 1 - e_p \), thus \( e_p \in \{0, 1\} \). According to the previous Lemma 4.3, we have \( e_p = 1 \) if \( p \equiv 3 \) or 5 (mod 8), and \( e_p = 0 \) otherwise. Hence, the results. \( \square \)

**Theorem 4.5** Let \( d > 2 \) be a composite odd square-free integer. Denote by \( r \) the number of distinct primes dividing \( d \).

1. If all the primes dividing \( d \) are congruent to 3 (mod 8) (resp. 5 (mod 8)), then \( r_2(d) = 2r - 2 \).
2. If the primes dividing \( d \) are congruent to 7 (mod 8), then \( r_2(d) = 2r - 1 \).

**Proof** Assume that all the primes \( p | d \) are congruent to 3 (mod 8). Let \( \mathfrak{p}_K \) be a prime ideal of \( K \) above \( p \). By Lemmas 4.2 and 4.3, we have

\[
\left( \frac{\varepsilon_8 \cdot d}{\mathfrak{p}_K} \right) = \left( \frac{\varepsilon_8 \cdot p}{\mathfrak{p}_K} \right) = -1 \text{ and } \left( \frac{\varepsilon_2 \cdot d}{\mathfrak{p}_K} \right) = \left( \frac{\varepsilon_2 \cdot p}{\mathfrak{p}_K} \right) = -1,
\]

thus \( \left( \frac{\varepsilon_2 \cdot d}{\mathfrak{p}_K} \right) = 1 \). It follows that \( \overline{\varepsilon_8} = \overline{\varepsilon_2} \) in \( E_K/(E_K \cap N(L_p)) \). We infer that \( E_K/(E_K \cap N(L_p)) = \{1, \overline{\varepsilon_8}\} \) and \( e_d = 1 \). By Theorem 2.3, the number of prime ideals of \( K \) ramified in \( L_d \) is \( 2r \). Hence, \( r_2(d) = 2r - 1 - 1 = 2r - 2 \).

The other cases are similarly proved. \( \square \)
In what follows, we will compute $r_2(d)$ when all the primes $pkl$ are congruent to 1 (mod 8). For this, we need the following lemmas.

**Lemma 4.6** Let $p$ be a prime such that $p \equiv 1 \pmod{8}$. Then $e_p = 0$ or 1. More precisely,

1. $e_p = 0$ if and only if $p \equiv 1 \pmod{16}$ and $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4$.
2. $e_p = 1$ if and only if $p \equiv 9 \pmod{16}$ or $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$.

**Proof** Let $p \in \mathfrak{p}_{\mathbb{Q}(i)} \subseteq \mathfrak{p}_K$ be two prime ideals of $\mathbb{Q}(i)$ and $K$ respectively. We shall calculate the two symbols $\left(\frac{\zeta_8 \sqrt{2}}{\mathfrak{p}_K}\right)$ and $\left(\frac{\varepsilon_2 \sqrt{2}}{\mathfrak{p}_K}\right)$. We have

\[
\left(\frac{\zeta_8 \sqrt{2}}{\mathfrak{p}_K}\right) = \left(\frac{1+i}{\mathfrak{p}_K}\right) \left(\frac{\sqrt{2}}{\mathfrak{p}_K}\right) = \left(\frac{1+i}{\mathfrak{p}_K}\right) \left(\frac{\sqrt{2}}{\mathfrak{p}_K}\right) = \left(\frac{2}{a+b}\right) \left(\frac{\sqrt{2}}{\mathfrak{p}_K}\right),
\]

where $a$ and $b$ are two integers such that $p = a^2 + b^2$ (see [23, page 154]). Since

\[
\left(\frac{\zeta_8 \sqrt{2}}{\mathfrak{p}_K}\right) = \left(\frac{\zeta_8 \sqrt{2}}{\mathfrak{p}_{\mathbb{Q}(\sqrt{2})}}\right) = \left(\frac{\zeta_8 \sqrt{2}}{\mathfrak{p}_{\mathbb{Q}(\sqrt{2})}}\right) = (-1)^{\frac{p-1}{8}},
\]

where $\mathfrak{p}_{\mathbb{Q}(\sqrt{2})}$ is an ideal of $\mathbb{Q}(\sqrt{2})$ lying over $p$ (see [11, page 21]), then

\[
\left(\frac{\sqrt{2}}{\mathfrak{p}_K}\right) = (-1)^{\frac{p-1}{8}} \left(\frac{\zeta_8 \sqrt{2}}{\mathfrak{p}_K}\right) = (-1)^{\frac{p-1}{8}} \left(\frac{\zeta_8}{\mathfrak{p}_K}\right) = (-1)^{\frac{p-1}{8}} \left(\frac{\varepsilon_2}{\mathfrak{p}_{\mathbb{Q}(\sqrt{2})}}\right).
\]

On the other hand, by [19, page 323] and [23, page 160], we have

\[
\left(\frac{2}{a+b}\right) = \left(\frac{2}{p}\right)_4 \left(\frac{p}{2}\right)_4 = \left(\frac{\varepsilon_2}{\mathfrak{p}_{\mathbb{Q}(\sqrt{2})}}\right).
\]

Hence

\[
\left(\frac{\zeta_8 \sqrt{2}}{\mathfrak{p}_K}\right) = (-1)^{\frac{p-1}{8}}.
\]

If $\zeta_8$ and $\varepsilon_2$ are not norms in $L_p/K$, we claim that $\overline{\zeta_8} = \overline{\varepsilon_2}$ in $E_K/(E_K \cap N(L_p))$. Indeed $(\zeta_8 \sqrt{2})$ and $(\varepsilon_2 \sqrt{2})$ do not depend on $\mathfrak{p}_K$ so $(\zeta_8 \sqrt{2}) = 1$ for all $\mathfrak{p}_K$. It follows that $\overline{\zeta_8} \overline{\varepsilon_2} = \overline{1}$, then $\overline{\zeta_8} = \overline{\varepsilon_2}$ as claimed. \hfill $\square$

From the previous proof, we have the following remarks.

**Remark 4.7** Let $p$ be a prime such that $p \equiv 1 \pmod{8}$. Then, for any prime ideal $\mathfrak{p}_K$ of $K$ above $p$, we have $\left(\frac{\zeta_8}{\mathfrak{p}_K}\right) = (-1)^{\frac{p-1}{8}}$ and $\left(\frac{\varepsilon_2}{\mathfrak{p}_K}\right) = \left(\frac{2}{p}\right)_4 \left(\frac{p}{2}\right)_4$.

**Lemma 4.8** Let $d = p_1 \cdots p_r > 2$ be an odd composite square-free positive integer such that $p_i \equiv 1 \pmod{8}$ for all $i$. Consider the following assertions:

(a) $\zeta_8$ and $\varepsilon_2$ are not norms in $L_{p_i}/K$ and $L_{p_j}/K$ respectively for some $i \neq j$. 

\[\text{ Springer} \]
(b) \( \zeta_8 \) is a norm in \( L_{p_i}/K \) or \( \epsilon_2 \) is a norm in \( L_{p_i}/K \).
(c) \( \zeta_8 \) and \( \epsilon_2 \) are not norms in \( L_d/K \).
(d) \( \zeta_8 \neq \overline{\epsilon_2} \) in \( E_K/(E_K \cap N(L_d)) \).

Then (a) and (b) hold if and only if (c) and (d) hold.

**Proof** Assume that (a) and (b) hold, then \( \left( \frac{\zeta_8 \cdot d}{p_i} \right) = -1 = \left( \frac{\epsilon_2 \cdot d}{p_j} \right) \), where \( p_i \) (resp. \( p_j \)) is a prime ideal of \( K \) lying above \( p_i \) (resp. \( p_j \)). So by Lemma 4.2 we get

\[
\left( \frac{\zeta_8}{p_i} \right) = \left( \frac{\epsilon_2}{p_j} \right) = -1
\]

and (c) follows. If \( \left( \frac{\zeta_8 \cdot d}{p_i} \right) = 1 \), then \( \left( \frac{\zeta_8 \cdot \epsilon_2 \cdot d}{p_i} \right) = \left( \frac{\zeta_8 \cdot p_i}{p_i} \right) \left( \frac{\epsilon_2 \cdot d}{p_j} \right) = -1 \). Thus \( \zeta_8 \epsilon_2 \neq 1 \). Hence \( \zeta_8 \neq \overline{\epsilon_2} \). (We similarly treat the case \( \left( \frac{\zeta_8 \cdot d}{p_i} \right) = 1 \). So the assertion (d).

Conversely, suppose (c) and (d) hold. Since \( \zeta_8 \) and \( \epsilon_2 \) are not norms in \( L_d/K \), then there exist \( i \) and \( j \) such that \( \left( \frac{\zeta_8}{p_i} \right) = -1 \) and \( \left( \frac{\epsilon_2}{p_j} \right) = -1 \). Suppose this is true only for \( i = j \), then for all \( k \), we get \( \left( \frac{\zeta_8 \cdot d}{p_i} \right) \left( \frac{\epsilon_2 \cdot d}{p_j} \right) = \left( \frac{\zeta_8 \cdot \epsilon_2 \cdot d}{p_i} \right) = 1 \). So \( \zeta_8 \epsilon_2 \) is a norm in \( L_d/K \), which contradicts (d). Thus (a) holds. Suppose that (b) is not verified, then for all \( i \neq j \) satisfying (a) and not verifying (b), we have

\[
\left( \frac{\zeta_8 \cdot p_i}{p_i} \right) = \left( \frac{\epsilon_2 \cdot p_j}{p_j} \right) = -1
\]

Thus for all \( k \), \( \left( \frac{\zeta_8 \cdot \epsilon_2 \cdot d}{p_i} \right) = 1 \), i.e., \( \left( \frac{\zeta_8 \cdot d}{p_i} \right) = 1 \) for all prime \( p \) of \( K \) ramified in \( L_d \). It follows that, \( \zeta_8 \epsilon_2 \) is a norm in \( L_d/K \), which contradicts (d). Hence (b) holds too, which ends the proof of the lemma.

Now we give an analogous of Lemma 4.6 for a composite square-free integer \( d > 2 \).

**Lemma 4.9** Let \( d = p_1 \cdots p_r \) be a composite odd square-free integer such that \( p_i \equiv 1 \pmod{8} \) for all \( i \in I = \{1, \ldots, r\} \). Then

1. \( e_d = 0 \iff \forall i \in I, p_i \equiv 1 \pmod{16} \) and \( \left( \frac{2}{p_i} \right)_4 = \left( \frac{p_i}{2} \right)_4 \).
2. \( e_d = 1 \) if and only if one of the following assertions holds
   i. \( \forall i \in I, p_i \equiv 1 \pmod{16} \) and \( \exists j \in I, \left( \frac{2}{p_j} \right)_4 \neq \left( \frac{p_i}{2} \right)_4 \).
   ii. \( \exists i \in I, p_i \equiv 9 \pmod{16} \) and \( \forall j \in I, \left( \frac{2}{p_j} \right)_4 = \left( \frac{p_i}{2} \right)_4 \).
   iii. \( \exists (i, j) \in I^2 \), \( p_j \equiv 9 \pmod{16} \) and \( \left( \frac{2}{p_j} \right)_4 \neq \left( \frac{p_i}{2} \right)_4 \) and all the couples \((i, j)\) satisfying the last condition satisfy also \( \left( \frac{2}{p_i} \right)_4 \neq \left( \frac{p_j}{2} \right)_4 \).
3. \( e_d = 2 \) if and only if there exist \( i \neq j \in I \) such that \( p_i \equiv 9 \pmod{16} \), \( \left( \frac{2}{p_i} \right)_4 \neq \left( \frac{p_j}{2} \right)_4 \) and \( \left( \frac{2}{p_i} \right)_4 = \left( \frac{p_j}{2} \right)_4 \) or \( p_j \equiv 1 \pmod{16} \).

**Proof** We have:
• The first assertion is deduced from Lemma 4.2, Remark 4.7 and the fact that $e_d = 0$ if and only if both $\zeta_8$ and $\epsilon_2$ are norms in $L_d/K$.
• The second assertion is deduced too from Lemmas 4.2, 4.8, Remark 4.7 and the fact that $e_d = 1$ if and only if $\{(\zeta_8 \text{ is a norm and } \epsilon_2 \text{ is not}) \text{ or } (\zeta_8 \text{ is not a norm and } \epsilon_2 \text{ is}) \text{ or } (\text{both } \zeta_8 \text{ and } \epsilon_2 \text{ are not norms and } \overline{\epsilon_2} \neq \overline{\zeta_8})\}$.
• The last assertion is a result of Lemmas 4.2, 4.8 and the fact that $e_d = 2$ if and only if $\zeta_8$ and $\epsilon_2$ are not norms and $\overline{\epsilon_2} \neq \overline{\zeta_8}$.

Now, we can easily deduce the following theorem.

**Theorem 4.10** Let $d > 2$ be an odd square-free integer such that all its prime divisors are congruent to 1 (mod 8). If $r$ is the number of these distinct primes divisors, then

$$r_2(d) = 4r - 1 - e_d,$$

where $e_d$ is given by Lemmas 4.6 and 4.9.

We close this subsection with some numerical examples.

**Example 4.11**

1. For $d = 73 \cdot 89 \cdot 97$, we have $73 \equiv 89 \equiv 9 \pmod{16}$, $97 \equiv 1 \pmod{16}$, $(\frac{73}{2})_4 = (\frac{89}{2})_4 = -\left(\frac{97}{2}\right)_4 = -1$ and $(\frac{2}{73})_4 = (\frac{2}{89})_4 = -(\frac{2}{97})_4 = 1$. Thus, by Theorem 4.10, the rank of the 2-class group of $L_d := \mathbb{Q}(\sqrt{2}, i, \sqrt{73 \cdot 89 \cdot 97})$ equals $4 \cdot 3 - 3 = 9$ (see the third item of Lemma 4.9), and the class group of $L_d$ by PARI/GP is of type $(1224, 8, 4, 4, 2, 2, 2, 2, 2, 2, 2)$.

2. For $d = 73 \cdot 89 \cdot 113$, we have $73 \equiv 89 \equiv 9 \pmod{16}$, $113 \equiv 1 \pmod{16}$, $(\frac{73}{2})_4 = (\frac{89}{2})_4 = -\left(\frac{113}{2}\right)_4 = -1$ and $(\frac{2}{73})_4 = (\frac{2}{89})_4 = (\frac{2}{113})_4 = 1$. So by Theorem 4.10, the rank of the 2-class group of $L_d := \mathbb{Q}(\sqrt{2}, i, \sqrt{73 \cdot 89 \cdot 113})$ (resp. $\mathbb{Q}(\sqrt{2}, i, \sqrt{73 \cdot 97 \cdot 113})$) is $4 \cdot 3 - 2 = 10$ (resp. $4 \cdot 2 - 2 = 6$) (see the second item of Lemma 4.9), and the class group of $L_d$ by PARI/GP is of type $(384, 32, 2, 2, 2, 2, 2, 2, 2, 2, 2)$ (resp. $(912, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$).

3. For $d = 353 \cdot 257 \cdot 113$, we have $353 \equiv 257 \equiv 113 \equiv 1 \pmod{16}$ and $(\frac{353}{2})_4 = (\frac{257}{2})_4 = (\frac{113}{2})_4 = (\frac{2}{353})_4 = (\frac{2}{257})_4 = (\frac{2}{113})_4 = 1$. So by Theorem 4.10, the rank of the 2-class group of $L_d := \mathbb{Q}(\sqrt{2}, i, \sqrt{353 \cdot 257 \cdot 113})$ (resp. $\mathbb{Q}(\sqrt{2}, i, \sqrt{257 \cdot 113})$) is $4 \cdot 3 - 1 = 11$ (resp. $4 \cdot 2 - 1 = 7$) (see the first item of Lemma 4.9), and the class group of $L_d$ by PARI/GP is of type $(408, 204, 2, 2, 2, 2, 2, 2, 2, 2, 2)$ (resp. $(4368, 8, 2, 2, 2, 2, 2, 2, 2, 2, 2)$).

### 4.2 Case 2: The prime divisors of $d$ are not in the same coset of $\mathbb{Z}/8\mathbb{Z}$

In this subsection, we will make use of Lemmas 4.2, 4.3 and Remark 4.7 to determine $r_2(d)$ for any odd composite square-free integer $d > 2$ for which the prime divisors are not in the same coset of $\mathbb{Z}/8\mathbb{Z}$. Since the number $t_d$ of prime ideals of $K$ ramified in $L_d$ is determined by Theorem 2.3, we shall give the rank of the 2-class group of $L_d$ in terms of $t_d$. 

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Theorem 4.12 Let $d > 2$ be an odd composite square-free integer such that the primes dividing $d$ are not in the same coset (mod 8).

1. If there exist two prime divisors $p_1$ and $p_2$ of $d$ such that $p_1 \equiv -p_2 \equiv 5$ (mod 8), then $r_2(d) = t_d - 3$.

2. If $d$ is divisible by a prime congruent to 3 (mod 8) and none of the other prime divisors of $d$ is congruent to 5 (mod 8), then $r_2(d) = t_d - 2$ or $t_d - 3$. More precisely, $r_2(d) = t_d - 3$ if and only if there exists a prime $p \equiv 1$ (mod 8) dividing $d$ such that $\left(\frac{2}{p}\right)_4 = -1$.

3. If $d$ is divisible by a prime congruent to 5 (mod 8) and none of the other prime divisors of $d$ is congruent to 3 (mod 8), then $r_2(d) = t_d - 2$ or $t_d - 3$. More precisely, $r_2(d) = t_d - 3$ if and only if there exists a prime $p \equiv 1$ (mod 8) dividing $d$ such that $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$.

4. If all the primes dividing $d$ are congruent to $\pm 1$ (mod 8), then $r_2(d) = t_d - 1 - e_{d_1}$, where $d_1$ is the product of all the primes $p|d$ such that $p \equiv 1$ (mod 8). Note that $e_{d_1}$ is given by Lemmas 4.6 and 4.9.

Proof By formula (2), the rank of the 2-class group of $L_d$ is $r_2(d) = t_d - 1 - e_d$. As $K$ is a biquadratic number field, then $e_d \in \{0, 1, 2\}$. On the other hand, the symbols $\left(\frac{\zeta}{p}\right)$ and $\left(\frac{e_{d_1}}{p}\right)$ are trivial for any prime ideal $p$ of $K$ lying over a prime $p \equiv 7$ (mod 8) (see Lemmas 4.2, 4.3) so we will ignore them.

1. Let $p_1$ be a prime of $K$ above $p_1$. By Lemmas 4.2 and 4.3, the units $\zeta_8$ and $\epsilon_2$ are not norms in $L_d/K$ and $\left(\frac{\zeta_8}{P_1}\right) = \left(\frac{\epsilon_2}{P_1}\right) = -1$, so $\zeta_8 \neq \epsilon_2$ in $E_K/(E_K \cap N(L_d))$. Thus $e_d = 2$. Hence the first item.

2. Let $p_2$ be a prime dividing $d$ such that $p_1 \equiv 3$ (mod 8). Assume that $d$ is divisible by prime $p_2 \equiv 1$ (mod 8). Denote by $p_1$ and $p_2$ two prime ideals of $K$ lying over $p_1$ and $p_2$ respectively. By Lemmas 4.2, 4.3 and Remark 4.7, we have $\left(\frac{\zeta}{p_1}\right) = \left(\frac{\zeta}{p_2}\right) = -1$, 
$$\left(\frac{e_{d_1}}{P_1}\right) = -1, \left(\frac{e_{d_1}}{P_2}\right) = \left(\frac{e_{d_1}}{P_2}\right),$$

Hence $\zeta_8$ and $\epsilon_2$ are not norms in $L_d/K$, and so $e_d \neq 0$, which implies that $e_d \in \{1, 2\}$. We have $e_d = 2$ if and only if $\zeta_8 \neq \epsilon_2$ in $E_K/(E_K \cap N(L_d))$ and this, by Lemma 4.9, can only happen for primes $p \equiv 1$ (mod 8). Since $\left(\frac{\zeta_8}{p_2}\right) = (−1)^{\frac{p_2 - 1}{4}} \left(\frac{2}{p_2}\right)_4 \left(\frac{p_2}{2}\right)_4 = \left(\frac{2}{p_2}\right)_4$, then $e_d = 2$ if and only if $\left(\frac{2}{p}\right)_4 = -1$ for some prime $p|d$ such that $p \equiv 1$ (mod 8). The second item follows. We similarly prove the third item. The fourth item is immediate.

We close this subsection with the following numerical examples.

Example 4.13

1. For $d = 7 \cdot 3 \cdot 113$ (resp. $d = 7 \cdot 3 \cdot 17$), we have $\left(\frac{113}{2}\right)_4 = \left(\frac{3}{113}\right)_4 = -\left(\frac{2}{17}\right)_4 = 1$. So by the second item of the previous theorem the rank of the 2-class group of $L_d = \mathbb{Q}(i, \sqrt{2}, \sqrt{7} \cdot 3 \cdot 113)$ (resp. $L_d = \mathbb{Q}(i, \sqrt{2}, \sqrt{7} \cdot 3 \cdot 17)$) is $8 - 2 = 6$ (resp.
8 − 3 = 5), and the class group of \(L_d\) by PARI/GP is of type (42, 2, 2, 2, 2) (resp. (12, 2, 2, 2, 2)).

2. For \(d = 7 \cdot 5 \cdot 17\) (resp. \(d = 7 \cdot 5 \cdot 113\)), we have \(\left(\frac{17}{2}\right)_4 = -\left(\frac{2}{17}\right)_4 = \left(\frac{113}{2}\right)_4 = \left(\frac{2}{113}\right)_4 = 1\). So, by the third item of the previous theorem the rank of the 2-class group of \(L_d = \mathbb{Q}(i, \sqrt{2}, \sqrt{7 \cdot 5 \cdot 17})\) is 8 − 3 = 5 (resp. \(L_d = \mathbb{Q}(i, \sqrt{2}, \sqrt{7 \cdot 5 \cdot 113})\) is 8 − 2 = 6), and the class group of \(L_d\) by PARI/GP is of type (20, 2, 2, 2) (resp. (42, 6, 2, 2, 2, 2)).

3. For \(d = 7 \cdot 17\) (resp. \(d = 7 \cdot 113\)), we have \(\left(\frac{17}{2}\right)_4 = -\left(\frac{2}{17}\right)_4 = \left(\frac{113}{2}\right)_4 = \left(\frac{2}{113}\right)_4 = 1\).

Then, by the last item of the previous theorem the rank of the 2-class group of \(L_d = \mathbb{Q}(i, \sqrt{2}, \sqrt{7 \cdot 17})\) is 6 − 1 − 1 = 4 (resp. \(L_d = \mathbb{Q}(i, \sqrt{2}, \sqrt{7 \cdot 113})\) is 6 − 1 − 0 = 5), and the class group of \(L_d\) by PARI/GP is of type (20, 2, 2, 2) (resp. (64, 2, 2, 2, 2)).

5 Applications

In this section, we will determine the integers \(d\) such that the 2-class group of \(L_d\) is trivial, cyclic or isomorphic to \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\). For this, we have to recall the following results.

**Lemma 5.1** ([21]) Let \(k'/k\) be a biquadratic extension of CM-type, then

\[
h(k') = \frac{Q_{k'}}{Q_k} \cdot \frac{\omega_{k'}}{\omega_k} \cdot \frac{h(k_1)h(k_2)h(k'^+)}{h(k)^2}.
\]

Where \(k_1, k_2\) and \(k'^+\) are the three sub-extensions of \(k'/k\).

The result below, gives the class number of a multiquadratic number field in terms of those of its quadratic subfields.

**Proposition 5.2** ([26]) Let \(k\) be a multiquadratic number field of degree \(2^n\), \(n \in \mathbb{N}\), and \(k_i\) the \(s = 2^n − 1\) quadratic subfields of \(k\). Then

\[
h(k) = \frac{1}{2^n}(E_k : \prod_{i=1}^{s} E_k) \prod_{i=1}^{s} h(k_i),
\]

with

\[
v = \begin{cases} 
n(2^n - 1); & \text{if } k \text{ is real}, 
(n - 1)(2^{n-2} - 1) + 2^{n-1} - 1 & \text{if } k \text{ is imaginary}. \end{cases}
\]

**Lemma 5.3** ([24]) Let \(S\) be the set of odd primes that are ramified in \(\mathbb{Q}(\sqrt{d})\) and \(S_0\) its subset consisting of those primes that are congruent to 1 (mod 4). Let \(s\) and \(s_0\) be the cardinality of \(S\) and \(S_0\) respectively. Then the rank of the 2-class group of \(k = \mathbb{Q}(\sqrt{d}, i)\) equals

- \(s + s_0\) if \(d\) is even and \(p \equiv 1\) (mod 8) for all \(p \in S_0\) (the case \(S_0 = \emptyset\) is included here).
- \(s + s_0 - 1\) if \(d\) is even and there exists \(p \in S_0\) satisfying \(p \equiv 5\) (mod 8), or \(d\) is odd and \(p \equiv 1\) (mod 8), for all \(p \in S_0\) (the case \(S_0 = \emptyset\) is included here).
- \(s + s_0 - 2\) if \(d\) is odd and there exists \(p \in S_0\) satisfying \(p \equiv 5\) (mod 8).
5.1 Fields $L_d$ with trivial or cyclic 2-class group

Using Theorem 2.3 and the theorems in the previous sections, one can easily deduce the following results.

**Theorem 5.4** The 2-class group of $L_d$ is trivial if and only if $d$ is a prime congruent to either 3 or 5 (mod 8).

**Theorem 5.5** The 2-class group of $L_d$ is cyclic nontrivial if and only if $d$ takes one of the following forms

1. $d = q \equiv 7 \pmod{8}$ is a prime,
2. $d = qp$, where $q \equiv 3 \pmod{8}$ and $p \equiv 5 \pmod{8}$ are primes.

5.2 Fields $L_d$ with 2-class group of rank 2

By Theorem 2.3 and the theorems in the previous sections, we easily deduce the following results.

**Theorem 5.6** The rank of the 2-class group of $L_d$ equals 2 if and only if $d$ takes one of the following forms

1. $d = q_1 q_2$, with $q_1 \equiv q_2 \equiv 3 \pmod{8}$,
2. $d = p_1 p_2$, with $p_1 \equiv p_2 \equiv 5 \pmod{8}$,
3. $d = q_1 q_2$, with $q_1 \equiv 3 \pmod{8}$ and $q_2 \equiv 7 \pmod{8}$,
4. $d = pq$, with $p \equiv 5 \pmod{8}$ and $q \equiv 7 \pmod{8}$,
5. $d = p \equiv 1 \pmod{8}$ is a prime satisfying $p \equiv 9 \pmod{16}$ or $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$,

where $p_1, q_1, p$ and $q$ are prime integers.

5.3 Fields $L_d$ with 2-class group of type (2, 2)

Now we shall determine the integers $d$ for which the 2-class group of $L_d$ is of type (2, 2). The main theorem of this subsection is the following. For the proof see Propositions 5.8, 5.9, 5.10, 5.13 and 5.14 below.

**Theorem 5.7** Let $d > 2$ be an odd square-free integer. The 2-class group of $L_d = \mathbb{Q}(\sqrt{d}, \sqrt{2}, i)$ is of type (2, 2) if and only if $d$ takes one of the following forms

1. $d = p$, with $p \equiv 1 \pmod{16}$ and $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$.
2. $d = pq$, with $p \equiv 5 \pmod{8}$, $q \equiv 7 \pmod{8}$ and $\left(\frac{p}{q}\right)_4 = -1$.
3. $d = q_1 q_2$, with $q_1 \equiv 3 \pmod{8}$, $q_2 \equiv 7 \pmod{8}$ and $\left(\frac{q_1}{q_2}\right)_4 = -1$.  

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where $q_1$, $q$ and $p$ are primes.

To prove this theorem, we shall check all the items of Theorem 5.6.

**Proposition 5.8** Let $p$ be a prime such that $p \equiv 1 \pmod{8}$. Then

$$\text{Cl}_2(L_p) = (2, 2) \text{ if and only if } p \equiv 1 \pmod{16} \text{ and } \left( \frac{2}{p} \right)_4 \neq \left( \frac{p}{2} \right)_4.$$ 

Moreover, $h_2(L_p) = h_2(-2p)$ if and only if $\left( \frac{2}{p} \right)_4 \neq \left( \frac{p}{2} \right)_4$.

**Proof** Let $p \equiv 1 \pmod{8}$ be a prime. Set $L_p^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p})$, $K = \mathbb{Q}(\sqrt{2}, i)$ and $K' = \mathbb{Q}(\sqrt{2}, \sqrt{-p})$. By applying Lemma 5.1 to the extension $L_p^+/\mathbb{Q}(\sqrt{2})$, we have

$$h(L_p) = \frac{Q_{L_p} \omega_{L_p}}{Q_K \omega_K} \cdot \frac{h(L_p^+) h(K) h(K')}{h(\mathbb{Q}(\sqrt{2}))^2}.$$ 

We have $h(\mathbb{Q}(\sqrt{2})) = h(K) = 1$. By [6, Théorème 3], $Q_{L_p} = 1$ and by Lemma 2.5 $Q_K = 1$. Since $\omega_{L_p} = \omega_K = 8$ and $\omega_{K'} = 2$, then by passing to the 2-part in the above equality we get

$$h_2(L_p) = \frac{1}{2Q_K} h_2(L_p^+) h_2(K').$$

As $\varepsilon_2$ has a negative norm, so by the item (2) of Section 3 of [2] we obtain that $E_{K'} = (-1, \varepsilon_2)$. This in turn implies that $q(K') = Q_{K'} = 1$. From which we infer, by Proposition 5.2, that $h_2(K') = \frac{1}{2} \cdot 1 \cdot h_2(2) h_2(-p) h_2(2)(-2p) = \frac{1}{2} h_2(-p) h_2(-2p)$. It follows, by the equality (3), that

$$h_2(L_p) = \frac{1}{4} h_2(L_p^+) h_2(-p) h_2(-2p).$$

(4)

Keep the notations of [10, Theorem 2], by this theorem we have $h_2(-p) = 4$ if and only if $\left( \frac{p}{2} \right)_4 = -1$. From the proof of [10, Theorem 1], one deduces easily that $\left( \frac{\varepsilon_2}{p} \right)_4 = \left( \frac{2}{p} \right)_4 \left( \frac{p}{2} \right)_4.$

Therefore, $h_2(-p) = 4$ if and only if $\left( \frac{2}{p} \right)_4 \neq \left( \frac{p}{2} \right)_4$.

Thus by Theorem 5.6, we have only two cases to check.

- If $\left( \frac{2}{p} \right)_4 \neq \left( \frac{p}{2} \right)_4 = (-1)^{\frac{p-1}{2}}$, then by [20, Theorem 2], $h(L_p^+)$ is odd. So

$$h_2(L_p) = \frac{1}{4} h_2(-p) h_2(-2p).$$

As $h_2(-p) = 4$, then $h_2(L_p) = h_2(-2p)$. From [25] we deduce that $h_2(-2p) = 4$ if and only if $\left( \frac{p}{2} \right)_4 = 1$.

- If $p \equiv 9 \pmod{16}$ and $\left( \frac{2}{p} \right)_4 = \left( \frac{p}{2} \right)_4 = -1$, then by [6, Théorème 10], the rank of the 4-class group is equal to 1 and from Theorem 4.10, we infer that $h_2(L_p)$ is divisible by 8. Thus $\text{Cl}_2(L_p) \neq (2, 2)$. Which achieves the proof.

**Proposition 5.9** Let $d = p_1 p_2$ where $p_1$ and $p_2$ are two primes such that $p_1 \equiv p_2 \equiv 5 \pmod{8}$. Then $h_2(L_d) \equiv 0 \pmod{8}$ and $\text{Cl}_2(L_d) \neq (2, 2)$.  

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Proof Consider the following diagram (Fig. 1)

By Kuroda’s class number formula (see [22]), we have

\[ h_2(L_d) = \frac{1}{8} Q(L_d/Q(i))h_2(K_1)h_2(K_2)/h_2(Q(i))^2 \]

Using Lemma 5.3, we get 8 and 4 divide \( h_2(K_1) \) and \( h_2(K_2) \) respectively. Hence, 8 divides \( h_2(L_d) \) and so \( Cl_2(L_d) \) is not elementary by Theorem 5.6. 

\[ \square \]

Proposition 5.10 Let \( d = q_1q_2 \) where \( q_1 \) and \( q_2 \) are two primes such that \( q_1 \equiv q_2 \equiv 3 \pmod{8} \). Then \( h_2(L_d) \equiv 0 \pmod{8} \) and \( Cl_2(L_d) \neq (2, 2) \).

Proof By Kuroda’s class number formula (see [22]), we have

\[ h_2(L_d) = \frac{1}{4} Q(L_d/Q(i))h_2(k_1)h_2(k_2), \]

where \( k_1 = \mathbb{Q}(\sqrt{q_1q_2}, i) \) and \( k_2 = \mathbb{Q}(\sqrt{2q_1q_2}, i) \). Note that \( h_2(k_1) \) is divisible by 2 (see Lemma 5.3). On the other hand by Lemma 5.3 the rank of the 2-class group of \( k_2 \) is 2 and by [1, 7] its 2-class group is not of type (2, 2) or (2, 4), so \( h_2(k_2) \) is divisible by 16.

Hence \( h_2(L_d) \) is divisible by 8. So the result. 

\[ \square \]

To continue, we need the following two lemmas.

Lemma 5.11 Let \( d = pq \) with \( p \equiv 5 \pmod{8} \) and \( q \equiv 7 \pmod{8} \) are primes. Then \( \{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\} \) is a fundamental system of units of both \( L_d \) and \( L_p^+ = \mathbb{Q}(\sqrt{pq}, \sqrt{2}) \). Moreover,

1. \( E_{L_d} = \langle \varepsilon_8, \varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}} \rangle \).
2. \( Q_{L_d} = 1 \) and \( q(L_d) = 4 \).

Proof Let \( \varepsilon_{2pq} = x + y\sqrt{2pq} \) and \( \varepsilon_{pq} = a + b\sqrt{pq} \) be the fundamental units of \( \mathbb{Q}(\sqrt{2pq}) \) and \( \mathbb{Q}(\sqrt{pq}) \) respectively. It is well known that \( N(\varepsilon_{2pq}) = N(\varepsilon_{pq}) = 1 \). Then \( a^2 - 1 = b^2pq \) and \( x^2 - 1 = 2y^2pq \). Hence the unique prime factorization in \( \mathbb{Z} \) implies that one of the numbers: \( x \pm 1 \) (resp. \( a \pm 1 \)), \( p(x \pm 1) \) (resp. \( p(a \pm 1) \)) and \( 2p(x \pm 1) \) (resp. \( 2p(a \pm 1) \)) is a prime factor of \( \varepsilon_{2pq} \) or \( \varepsilon_{pq} \) or \( \sqrt{\varepsilon_{pq}\varepsilon_{2pq}} \) or \( 2 \sqrt{\varepsilon_{pq}\varepsilon_{2pq}} \) or \( 4 \sqrt{\varepsilon_{pq}\varepsilon_{2pq}} \) or \( 8 \sqrt{\varepsilon_{pq}\varepsilon_{2pq}} \) or \( 16 \sqrt{\varepsilon_{pq}\varepsilon_{2pq}} \) or \( 32 \sqrt{\varepsilon_{pq}\varepsilon_{2pq}} \) or \( 64 \sqrt{\varepsilon_{pq}\varepsilon_{2pq}} \).

\[ L_d = \mathbb{Q}(\sqrt{2}, i, \sqrt{d}) \]

\[ K = \mathbb{Q}(\sqrt{2}, i) \]

\[ K_2 = \mathbb{Q}(\sqrt{d}, i) \]

\[ K_1 = \mathbb{Q}(\sqrt{2d}, i) \]

\[ \mathbb{Q}(i) \]

Fig. 1 \( L_{p_1p_2}/\mathbb{Q}(i) \)
square in \( \mathbb{N} \). We claim that \( x + 1, a + 1, x - 1 \) and \( a - 1 \) are not squares in \( \mathbb{N} \), otherwise we get for \( b = b_1b_2 \) and \( y = y_1y_2 \)

\[
\begin{cases}
  a + 1 = b_1^2 \\
  a - 1 = pqb_2^2
\end{cases}
\quad \text{and} \quad \begin{cases}
  x + 1 = y_1^2 \\
  x - 1 = 2pqy_2^2.
\end{cases}
\]

Hence

\[
1 = \left( \frac{b_1^2}{p} \right) = \left( \frac{a+1}{p} \right) = \left( \frac{a+1 \pm 2}{p} \right) = \left( \frac{\pm 2}{p} \right) = \left( \frac{2}{p} \right) = -1
\]

and

\[
1 = \left( \frac{y_1^2}{q} \right) = \left( \frac{x+1}{q} \right) = \left( \frac{x+1 \pm 2}{q} \right) = \left( \frac{\pm 2}{q} \right) = \left( \frac{2}{q} \right) = -1,
\]

which is absurd. Thus by [8, Proposition 3.3] \( \{ \epsilon_2, \epsilon_{pq}, \sqrt{\epsilon_{pq}^2} \} \) is the fundamental system of units of both \( L_d \) and \( L_d^+ \). So \( E_{L_d} = \langle \zeta_8, \epsilon_2, \epsilon_{pq}, \sqrt{\epsilon_{pq}^2} \rangle \) and \( Q_{L_d} = 1 \). As \( \prod_{i=1}^7 E_{k_i} = \langle i, \epsilon_2, \epsilon_{pq}, \epsilon_{2pq} \rangle \), then \( q(L_d) = 4 \).

\[\square\]

**Lemma 5.12** Let \( d = q_1q_2 \) with \( q_1 \equiv 3 \pmod{8} \) and \( q_2 \equiv 7 \pmod{8} \) are primes. Then \( \{ \epsilon_2, \epsilon_{q_1q_2}, \sqrt{\epsilon_{q_1q_2}^2} \} \) is a fundamental system of units of both \( L_d \) and \( L_d^+ = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{2}) \). Moreover,

1. \( E_{L_d} = \langle \zeta_8, \epsilon_2, \epsilon_{q_1q_2}, \sqrt{\epsilon_{q_1q_2}^2} \rangle \)
2. \( Q_{L_d} = 1 \) and \( q(L_d) = 4 \).

**Proof** We proceed as in the proof of Lemma 5.11.

\[\square\]

**Proposition 5.13** Let \( d = pq \) where \( p \) and \( q \) are two primes such that \( p \equiv 5 \pmod{8} \) and \( q \equiv 7 \pmod{8} \). Then

\[ Cl_2(L_d) = (2, 2) \] if and only if \( \left( \frac{p}{q} \right) = -1 \), otherwise, \( h_2(L_d) \equiv 0 \pmod{16} \).

**Proof** By Lemma 5.11 we have \( q(L_d) = 4 \). It follows by Proposition 5.2 that we have

\[
h_2(L_d) = \frac{1}{2^5} q(L_d)h_2(pq)h_2(-pq)h_2(2pq)h_2(-2pq)h_2(2)h_2(-2)h_2(-1) = \frac{1}{2^5} h_2(pq)h_2(-pq)h_2(2pq)h_2(-2pq).
\]

- Assume that \( \left( \frac{q}{p} \right) = -1 \), then by [15, Corollaries 19.6 and 19.7] \( h_2(pq) = h_2(-pq) = h_2(2pq) = 2 \) and by [19, p. 353] \( h_2(-2pq) = 4 \). Therefore the above equation gives

\[
h_2(L_d) = \frac{1}{2^5} q(L_d)h_2(pq)h_2(-pq)h_2(2pq)h_2(-2pq) = \frac{1}{2^5} \cdot 2 \cdot 2 \cdot 2 \cdot 4 = 4.
\]

- Assume that \( \left( \frac{p}{q} \right) = 1 \). By [15, Corollary 19.7] we have \( h_2(pq) = h_2(2pq) = 2 \). Then as above we have

\[
h_2(L_d) = \frac{1}{2} h_2(-pq)h_2(-2pq).
\]
By Proposition $B'_10$ of [19, p. 353], $h_2(-2pq)$ is divisible by 8. By [15, Corollaries 19.6 and 18.4] $h_2(-pq)$ is divisible by 4. Therefore in this case $h_2(L_d)$ is divisible by 16. Hence Theorem 5.6 completes the proof. □

**Proposition 5.14** Let $d = q_1 q_2$ where $q_1$, $q_2$ are two primes such that $q_1 \equiv 3 \pmod{8}$ and $q_2 \equiv 7 \pmod{8}$. Then

$$\text{Cl}_2(L_d) = (2, 2) \text{ if and only if } \left(\frac{q_1}{q_2}\right) = -1,$$

otherwise, $h_2(L_d)$ is divisible by 16.

**Proof** The proof is similar to the one of Proposition 5.13. □

**Example 5.15** The examples are given by using PARI/GP software and they confirm our results.

1. We have $\left(\frac{2}{17}\right)_4 = -\left(\frac{17}{2}\right)_4 = -1$ and the 2-class group of $\mathbb{Q}(\sqrt{17}, \sqrt{2}, i)$ is of type $(2, 2)$.
2. The 2-class groups of the fields $\mathbb{Q}(\sqrt{21}, \sqrt{2}, i)$ and $\mathbb{Q}(\sqrt{35}, \sqrt{2}, i)$ are of type $(2, 2)$.
3. The 2-class group of the field $\mathbb{Q}(\sqrt{19 \cdot 31}, \sqrt{2}, i)$ is not, since the 2-part of its class number is divisible by 16. In fact $\left(\frac{19}{31}\right) = -\left(\frac{19}{23}\right) = 1$.

### 5.4 The 2-part of the class number of some biquadratic fields

In this subsection we use the previous results to give the 2-class number of $L_d$ in terms of that of $\mathbb{Q}(\sqrt{2}, \sqrt{-d})$.

**Theorem 5.16** Let $d = pq$ where $p, q$ are two primes such that $p \equiv 5 \pmod{8}$ and $q \equiv 7 \pmod{8}$. Then

$$h_2(L_d) = h_2(k),$$

with $k = \mathbb{Q}(\sqrt{2}, \sqrt{-d})$. Moreover,

$$\text{Cl}_2(k) = (2, 2) \text{ if and only if } \left(\frac{p}{q}\right) = -1.$$

**Proof** Applying Lemma 5.1 to the extension $L_d / \mathbb{Q}(\sqrt{2})$, we get

$$h_2(L_d) = \frac{Q_{L_d}}{Q_k Q_{\mathbb{Q}}} \frac{1}{2} h_2(L_d^+) h_2(k).$$

By Proposition 5.2, Lemma 5.11 and the settings on values of class numbers of quadratic fields given in the proof of Proposition 5.13 we obtain

$$h_2(L_d^+) = \frac{1}{4} q(L_d^+) h_2(pq) h_2(2pq) h_2(2) = \frac{1}{4} \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 2.$$
By Lemmas 2.5 and 5.11 (resp. [2, p. 19]), we have $Q_{f_d} = Q_K = 1$ (resp. $Q_k = 1$). Thus $h_2(L_d) = h_2(k)$. Since by [24, Proposition 2], the rank of the 2-class group of $k$ is 2, we have the equivalence by Proposition 5.13.

Theorem 5.17 Let $d = q_1 q_2$ where $q_1$, $q_2$ are two primes such that $q_1 \equiv 3 \pmod{8}$ and $q_2 \equiv 7 \pmod{8}$. Then

$$h_2(L_d) = \frac{1}{2} h_2(k),$$

with $k = \mathbb{Q}(\sqrt{2}, \sqrt{-d})$. Moreover,

Cl$_2(k) = (2, 2, 2)$ if and only if $\left(\frac{q_1}{q_2}\right) = -1$.

Proof We proceed as in the proof of Theorem 5.16.

Remark 5.18 As a continuation of this work we interested in the cases where Cl$_2(L_d)$ is of type $(2^n, 2^m)$ or $(2, 2, 2)$, where $n \geq 1$ and $m > 1$, and based on this work, we studied the problem of the Hilbert 2-class field tower of these fields. Note also that, we generalized some of our results on $\mathbb{Q}(\sqrt{-p}, \sqrt{d})$ to the fields $\mathbb{Q}(\sqrt{2p}, \sqrt{d})$, for $m \geq 4$ (cf. [12–14]).

Acknowledgements The authors are very grateful to the reviewer for his/her careful and meticulous reading of the paper.

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