Fast Near-Optimal Heterogeneous Task Allocation via Flow Decomposition

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Abstract—Multi-robot systems are uniquely well-suited to performing complex tasks such as patrolling and tracking, information gathering, and pick-up and delivery problems, offering significantly higher performance than single-robot systems. A fundamental building block in most multi-robot systems is task allocation: assigning robots to tasks (e.g., patrolling an area, or servicing a transportation request) as they appear based on the robots’ states to maximize reward. In many practical situations, the allocation must account for heterogeneous capabilities (e.g., availability of appropriate sensors or actuators) to ensure the feasibility of execution, and to promote a higher reward, over a long time horizon. To this end, we present the FlowDec algorithm for efficient heterogeneous task-allocation achieving an approximation factor of at least 1/2 of the optimal reward. Our approach decomposes the heterogeneous problem into several homogeneous subproblems that can be solved efficiently using min-cost flow. Through simulation experiments, we show that our algorithm is faster by several orders of magnitude than a MILP approach.

I. INTRODUCTION

A central problem in many multi-robot applications, including patrolling, information gathering, and pick-up and delivery problems, is task allocation: that is, to assign robots to outstanding, spatially-distributed tasks (e.g. patrolling an area or servicing a transportation request) based on the robots’ states (e.g., position and power-level) and potentially heterogeneous capabilities (e.g., availability of appropriate sensors or actuators), and accounting not only for current tasks but also for the likelihood that future tasks will appear.

In this paper we consider a task-allocation setting where mobile heterogeneous robots need be assigned to time-varying reward sets. In our formulation, robots are divided into homogeneous fleets based on their ability to collect private reward sets (where each fleet is associated with a unique private set). There is also a shared reward set that can be collected by robots in any fleet. A robot is rewarded for a specific reward set if (i) it resides in the spatial vicinity of the reward, (ii) the reward is either shared or private to the robot’s specific fleet, and (iii) no other robot is assigned to this reward.

A number of problems of interest fall in this setting, including object tracking, intruder following and imaging a scientific phenomenon. We are particularly motivated by data-gathering and agile science applications for planetary science where multiple spacecraft detect events of interest and then perform follow-up scientific observations. A number of concepts have been proposed in this setting, including multi-spacecraft constellations to study the Martian atmosphere and the dust cycle [1], networks of balloons to detect seismic and volcanic events on Venus [2–4], and swarms of small spacecraft to study small bodies [5]. In all of these applications, the heterogeneous task-allocation problem is central: a set of robots with heterogeneous sensing capabilities is tasked with observing a variety of scientific events of interest (e.g. dust storms, volcanism, or changes in a body’s surface); a dynamic model that approximately predicts the spatio-temporal distribution and evolution of the phenomenon of interest is available; and, while certain tasks (e.g., radio science or medium-resolution imaging) can be performed by all agents, other specialized tasks (e.g., hyperspectral imaging, deployment of sondes, or sampling) can only be performed by a subset of the robots.

A. Related work

Task allocation is an enabling subroutine for applications like closely-coupled coordination (e.g., deciding which robot takes which place in a formation) and for loosely-coupled coordination (e.g., which robot traverses to a new spot to observe an interesting target). Unfortunately, most variants of the problem are known to be computationally prohibitive to solve [6, 7]. As a result, many approaches for task allocation tend to scale poorly with the problem size (e.g., the number of robots) or provide no guarantees on the solution quality or runtime. Furthermore, heterogeneous robot capabilities (e.g., ground vehicles and aerial drones jointly working to achieve a mutual goal), impose additional challenges for designing practical high-quality solution approaches [8–12].

Several of algorithm types have been proposed specifically for task allocation by the robotics community (see survey in [13]). In auction-based algorithms [6, 14], robots bid on tasks based on their state and capabilities. Auction-based algorithms can be readily implemented in a distributed fashion and naturally accommodate heterogeneous robots. Spatial partitioning algorithms [15] rely on partitioning the workspace into regions and assigning each region to one or multiple robots. Tasks within a region are assigned to the robot (or robots) responsible for that region. Spatial partitioning algorithms capture the likelihood of occurrence of future events. Team-forming and temporal partitioning algorithms [16] group heterogeneous robots in teams so that each team is capable of performing all the tasks that might arise. Mixed-integer linear programming (MILP) approaches [17] explicitly represent the task-allocation problem as a mixed-integer program, and can readily capture a variety of constraints including heterogeneity. Markov chain-based
algorithms [18] model the robots’ motion via a stochastic policy prescribed by a Markov chain optimized according to a given cost function. While all those approaches cover a wide range of problems and techniques, they generally either do not scale well with the problem size or provide weak theoretical guarantees on the solution quality or runtime.

The predictive nature of our problem strongly relates to the notion of online algorithms [19–21], which are designed for problems in which the input is revealed gradually, while optimizing a goal function (e.g., minimizing cost). The typical benchmark for online algorithms is the worst-case ratio between the solution of the online algorithm and the optimal solution for the offline case in which the entire input is given a priori. This is termed as the algorithm’s competitive ratio. The k-server problem [22, 23], which has been studied in this online context, bears some resemblance to our setting. In particular, it can be viewed as the centralized case consisting of a single fleet of $k > 1$ homogeneous agents, where the goal is to collect all the rewards while minimizing travel cost in an online fashion. A recent paper introduced an online randomized algorithm for the problem which achieves a competitive ratio of $O(\log^6 k)$ [24]. The weighted variant of the problem, which is closer to our heterogeneous setting, requires an even larger competitive ratio of $\Omega(2^{\sqrt{k} - 1})$ [25].

Applications of homogeneous task allocation have been extensively explored recently within the setting of transportation and logistics. For instance, the operation of an autonomous mobility-on-demand system requires to assign ground vehicles to routes in order to fulfill passenger demand, while potentially accounting for road congestion [26–28]. A recent work develops efficient package delivery framework consisting of multiple drones in which drones are assigned to packages and delivery routes [29]. This work proposes utilizing public-transit vehicles on which the drones can hitchhike in order to conserve their limited energy, and thus noticeably increase their service range. From a broader perspective, task allocation can be viewed through the lens of the vehicle routing problem (VRP) [30] or the orienteering problem (OP) [31]. However, save a few special cases, VRP and OP are typically approached with MILP formulations that scale poorly, or by heuristics that do not provide optimality guarantees.

The problem that we address in this paper can be represented as a multi-agent pathfinding (MAPF) problem [32]. The goal in MAPF is to compute a collection of paths for the agents to minimize minimize execution time, while accounting for inter-agent conflict constraints. Unfortunately, no polynomial-time approximation algorithms that can solve general MAPF instances exist [33], to the best of our knowledge. A recent approach for MAPF termed conflict-based search [34–37] earned popularity due to its efficiency in moderately-sized instances. However, it does not provide run-time guarantees and does not scale well in settings that require considerable amount of coordination among agents.

B. Contribution

We present an efficient approximation algorithm, termed FLOWDEC, for heterogeneous task-allocation in the context of maximizing the collection of time-varying rewards. From the theoretical perspective we prove that our algorithm achieves an approximation factor of at least 1/2 of the optimal reward in polynomial time. Our algorithm also exhibits good performance in practice. Specifically, in simulation experiments, we demonstrate that our algorithm achieves a speedup of several orders of magnitude over a MILP approach, and its runtime scales modestly with the problem size. Moreover, we demonstrate that the runtime is insensitive to the number of agents present in each fleet.

From an algorithmic standpoint, the FLOWDEC algorithm decomposes the heterogeneous problem into several homogeneous subproblems, that can be solved efficiently using min-cost flow [38], without significant degradation in solution quality. Our approach shares some similarity with a recent work that also considers homogeneous decomposition [39], albeit the problem setting (computing multiple travelling-salesman routes in an undirected and time-invariant graph), as well as the algorithmic techniques and analysis that are developed there are quite different from ours.

C. Organization

The organization of this paper is as follows. In Section II we provide basic definitions and the problem formulation. In Section III we first study the homogeneous subproblem. Then, in Section IV we describe the FLOWDEC algorithm and provide its theoretical analysis. In Section V we provide experimental results demonstrating the good performance and scalability of our approach. We conclude with a discussion of future research directions in Section VI.

II. Preliminaries and Problem Formulation

We first describe the problem ingredients and then proceed to a formal definition of the problem. The robots’ workspace is represented by a directed graph $G = (V, E)$, with vertices $i \in V$ denoting physical locations for the robots, and edges $(i, j) \in E$ denoting transitions between locations (See example in Figure 1.) We use $E^+_i$ to denote the set of outgoing neighbors of a vertex $i \in V$, namely $\{j \in V | (i, j) \in E\}$. We similarly define $E^-_i$ to represent the set of incoming neighbors, i.e., $\{j \in V | (j, i) \in E\}$.

We consider a discrete-time, finite-horizon framework, where the horizon is specified by a positive integer $T$. We
use \( \tau \in [0..T] \) to denote a given time step, where \([r..r']\)
denotes an integer interval \( \{k \in \mathbb{N} : r \leq k \leq r'\} \) between
two integers such that \( r < r' \).

Throughout this paper, we refer to the robots as “robots” or
“agents” interchangeably. The set of all agents is denoted by
\( A \), which is subdivided into \( F \) disjoint fleets \( A^1, \ldots, A^F \),
where agents within the same fleet are assumed to have
equatable capabilities. We denote by \( a_f = |A^f| \) the number of agents in a given fleet \( f \in F \), where \( F := [1..F] \)
is the set of fleet indices. The agents are mobile and transition
from one vertex \( i \in V \) to another \( j \in V \) every time step,
assuming that \((i, j) \in E \). For a given fleet \( f \in F \), the
positions of its agents at time \( \tau \in [0..T] \) are specified by
the vector \( p^f[i, \tau] \in [0..a_f] \) \( i \in V \), where for a given vertex \( i \in V \),
the value \( p^f[i, \tau] \) specifies the number of agents of \( A^f \) located
at vertex \( i \). See example in Figure 1.

### A. Shared and private rewards

The problem consists of allocating agents to rewards along
the time-expanded vertices of \( G \). For now, we assume that the
distributions of rewards are known in advance. We discuss a
predictive extension where the reward sets are not known in advance
in Section II-C.

There are \( F + 1 \) types of reward sets \( R^0, R^1, \ldots, R^F \),
determine the values gained by the agents for visiting
any given vertex \( i \in V \) at time \( \tau \in [0..T] \), and there are
constraints that specify which fleets \( f \in F \) can collect
a specific reward \( R^f \) of type \( t \in [0..F] \). Specifically, rewards
of type \( t = 0 \) are considered to be shared, i.e., can be
collected by any robot of any fleet \( f \in F \). In particular, for
a given vertex \( i \in V \) and time step \( \tau \in [0..T] \), the system
1 gains the reward \( R^0[i, \tau] \) if \( \sum_{f \in F} p^f[i, \tau] > 1 \), i.e., at least one
agent (from any fleet) visits the vertex \( i \) at time \( \tau \). In contrast,
tasks of type \( t \neq 0 \) are considered to be private, and can only
be gained via agents from the particular fleet \( A^f \) such that
\( f = t \). That is, when \( t \neq 0 \) then for any \( i \in V, \tau \in [0..T] \),
the system gains the reward \( R^f[i, \tau] \) if \( p^f[i, \tau] > 1 \), i.e., at least
one agent from \( A^f \) visits the vertex \( i \) at time \( \tau \).

### B. Problem formulation

We provide a formal definition of our problem in the form of an integer program (Table I). The input to the problem consists of
the workspace graph \( G \), time horizon \( T \in \mathbb{N}_{>0} \), agent fleets \( A^1, \ldots, A^F \) with \( F \in \mathbb{N}_{>0} \) with
known initial positions \( p^0[i] \) for all \( i \in V \), and rewards
sets \( R^0, \ldots, R^F \). The goal of this work is to obtain a
task-allocation scheme, which maximizes the total collected
reward. The task-allocation scheme consists of (i) specifying the
locations of all agents for every time step \( \tau \in [0..T] \) and
(ii) assigning agents to rewards.

The solution is described through two types of decision
variables. The integer variable \( x^f[i, j, \tau] \in [0..a_f] \) denotes a transition
of agents in fleet \( f \in F \) from vertex \( i \in V \) to
\( j \in V \), at time \( \tau \in [0..T-1] \), assuming that \((i, j) \in E \). The
decision variables \( y^f[i, \tau] \in [0..1] \) and \( z^f[i, \tau] \in [0..1] \) indicate
whether an agent from fleet \( f \in F \) is assigned to collect the
shared reward \( R^0[i, \tau] \), or private reward \( R^f[i, \tau] \), respectively.

The objective function is given in Equation (1a). Equation
(1b) ensures that agents will start at their initial positions
as specified by \( p_0 \). Equation (1c) ensures the flow conservation
of agents. Equations (1d), (1e), ensure that an agent from

\[
\begin{align}
\text{maximize} & \quad \sum_{j \in V} \sum_{\tau \in [0..T]} \sum_{f \in F} \left( R^0[j, \tau] \cdot y^f[j, \tau] + R^f[j, \tau] \cdot z^f[j, \tau] \right) \\
\text{subject to} & \quad \sum_{i \in \mathcal{E}^f} x^f[i, j, \tau] = p^0[j, \tau], \quad \forall j \in V, f \in F, \ (1b) \\
& \quad \sum_{i \in \mathcal{E}^f} x^f[i, j, \tau] - \sum_{i \in \mathcal{E}^f} x^f[i, j, \tau] = 0, \quad \forall j \in V, f \in F, \ (1c) \\
& \quad \sum_{i \in \mathcal{E}^f} x^f[i, j, \tau] = y^f[j, \tau], \quad \forall j \in V, f \in F, \ (1d) \\
& \quad \sum_{i \in \mathcal{E}^f} x^f[i, j, \tau] = z^f[j, \tau], \quad \forall j \in V, f \in F, \ (1e) \\
& \quad y^f[j, \tau] \leq 1, \quad \forall j \in V, f \in F, \ (1f) \\
& \quad z^f[j, \tau] \leq 1, \quad \forall j \in V, f \in F, \ (1g)
\end{align}
\]

| Table I | Definition of the heterogeneous task-allocation problem. |
|------------------|-----------------------------------------------|
| fleet \( f \) is assigned to a shared reward \( R^0[j, \tau] \) only if of the |
| agents of this fleet is at vertex \( j \) in time \( \tau \) (equations (1d), (1e) |
| similarly enforce this condition with respect to private |
| reward sets). Equations (1f), (1g) limit the number of agents |
| assigned to every reward type in a given vertex to 1. |

### C. Predictive setting

The above formulation can be extended to the predictive
setting, where we are given the values of the reward sets
for the first time step \( R^0, \ldots, R^0 \), and a stochastic model
of the evolution of rewards with respect to time. To exploit
the aforementioned deterministic formulation, it is straightforward
to show that by plugging into the problem defined in
Table I the expected values of the stochastic rewards, the
solution would maximize the expected gained reward.

This gives rise to a receding-horizon implementation:
given the current state of the system (e.g., locations of agents
and values of current rewards) and the current time step \( \tau \) we
predict the reward values \( R^0, \ldots, R^F \) for \( T \) time steps into
the future, and compute a corresponding solution \( x^\tau, y^\tau, z^\tau \).

Then we execute this solution for the first time step, obtain
current values of the reward, and repeat this process in
the next time step \( \tau + 1 \). For the simplicity of presentation, we
shall focus on the static setting of the problem from now on.
We do note that our experiments are for the predictive case.

### D. Discussion

The problem formulation in Table I and our FlowDec
algorithm can be straightforwardly extended to a setting
where the goal is to maximize the reward minus the travel cost, where the latter is represented by costs assigned to the edges of $G$.

Our current problem formulation and algorithm do not account for collisions between agents that can arise when two or more agents reside in the same vertex of the graph, as is required in a MAPF formulation wherein agents are typically operating in close proximity to each other. We do mention that a typical approach in autonomy frameworks (see, e.g., [40, 41]) is to handle high-level decisions (e.g., task allocation) at a different level than low-level collision avoidance. In this respect, our method can be combined with a lower-level collision-avoidance module, e.g., conflict-based search for MAPF [34], to safely execute our allocation scheme. Such a methodology was recently used in warehouse logistics and drone delivery [29, 35–37].

### III. Algorithm for the Homogeneous Case

In preparation to the FlowDec algorithm for the heterogeneous problem, we describe a key ingredient, which is an efficient solution to the homogeneous problem. The homogeneous problem consist of maximizing the collected reward for a single fleet $f \in \mathcal{F}$ of homogeneous agents and a private reward set $R_f$. Our main insight is that an optimal solution to the homogeneous problem can be found efficiently by solving a min-cost flow (MCF) problem [38].

Given the graph $G$, time horizon $T$, agent fleet $A_f$, initial positions $p_0$, and private reward set $R_f$, maximize $R_f(x, z) := \sum_{j \in V} \sum_{\tau \in [0..T]} R_f[j] \cdot z[j]$, subject to (1b), (1c), (1g), (1b), (1i), with respect to $A_f$.

**TABLE II:** Definition of the heterogeneous task-allocation optimization.

Denote by $\mathcal{H}(R^f, p_0)$ the homogeneous optimization problem described in Table II, for a private reward set $R_f$ and initial positions $p_0$ of a fleet $f \in \mathcal{F}$. Note that the problem of assigning the set of shared tasks to all the agents $A_f$ while ignoring the assignment of private tasks, can be viewed as the homogeneous problem $\mathcal{H}(R^0, p)$. The following lemma states that an optimal solution for the homogeneous problem can be obtained in (low-degree) polynomial time.

**Lemma 1** (Efficient solution of $\mathcal{H}$). For a given fleet $f \in \mathcal{F}$, private reward set $R_f$, and initial positions $p_0$, the optimal solution for the homogeneous problem $\mathcal{H}(R^f, p_0)$ can be computed in $O((T^2mn\log(Tn) + T^4n^2\log(Tn))$ time, where $m = |\mathcal{E}|$, $n = |V|$. This bound also holds for the homogeneous problem $\mathcal{H}(R^0, p_0)$ with respect to the shared reward $R_f$.

**Proof.** We show that the homogeneous problem can be transformed into a min-cost flow (MCF) problem [38], for which an optimal solution can be efficiently found. In the remainder of this proof we define the ingredients for MCF, formally define the problem, describe its relation to the homogeneous problem, and finally discuss its complexity.

MCF is defined over a directed graph $G = (V, E)$, which has to designated vertices $s, g \in V$, that represent the source and sink, respectively. In addition, for every edge $e \in E$ we have the attributes $u_e > 0$ and $c_e \in \mathbb{R}$ which represent the edge’s capacity and traversal cost, respectively. We also have a variable $h^*$ which denotes the total flow that emerges from the source $s$ and needs to arrive to the sink $g$.

The goal in MCF is to assign flow values $h_e \in [0,\infty)$, which represent the number of agents traversing $e$, to all edges $e \in E$ in order to minimize the expression $\sum_{e \in E} h_e c_e$ under the constraints

\[
\begin{align}
    h_e & \in \{0, u_e\}, \forall e \in E, \quad (3a) \\
    \sum_{e \in E^+_v} h_{sv} = \sum_{e \in E^-_v} h_{vg} = h^*, \quad (3b) \\
    \sum_{e \in E^+_v} h_{ve} - \sum_{e \in E^-_v} h_{ve} = 0, \forall v \in V \setminus \{s, g\}. \quad (3c)
\end{align}
\]

Constraint (3a) ensures that edge capacities are maintained; Constraint (3b) ensures that the flow leaving $s$ and entering $g$ would be equal to $h^*$; Constraint (3c) ensures flow conservation.

For a given instance of $\mathcal{H}(R^f, p_f)$ we construct an equivalent instance of the MCF problem where flows correspond to agent locations and their reward assignments. In particular, we define the graph $G = (V, E)$, such that

\[V = \{s, g\} \cup \{(v^1_i, w^1_i) | i \in V, \tau \in [0..T]\}.\]

That is, the vertex set consists of the source $s$ and sink $g$ vertices, as well as two copies $v^1_i, w^1_i$ of every vertex $i \in V$ from the workspace graph $G$ for each time step $\tau \in [0..T]$.

The edge set of $G$ is a union of several edge sets, i.e.,

\[E = E_s \cup E_g \cup E_0 \cup E_R \cup E_1,\]

where

\[E_s = \{(s, v^1_i) | \exists q \in A, p^0_0[i] \neq 0\},\]

\[E_g = \{(w^1_i, g) | i \in V\},\]

\[E_0 = \{(v^1_i, w^1_i) | i \in V, \tau \in [0..T]\},\]

\[E_R = \{(v^1_i, w^1_i) | i \in V, \tau \in [0..T], R_f[i] \neq 0\},\]

\[E_E = \{(w^1_i, v^1_{i+1}) | (i, j) \in E, \tau \in [0..T-1]\}.\]

Namely, edges in $E_s$ connect the source vertex $s$ to all the start vertices of the agents. Edges in $E_g$ connect all the vertices in the final time step $T$ to the sink node $g$. Edges in $E_0$ connect every two copies $v^1_i, w^1_i$ of a vertex $i \in V$ for a given time step $\tau$. Agents traversing the latter set of edges will receive no reward. On the other hand, every edge $(v^1_i, w^1_i) \in E_R$ will be used to represent the collection of the reward $R_f[i]$ by the agent traversing this edge. Edges in $E_E$ simulate the traversal of agents along the edges of $E$ between one time step to the next.

Next, we assign costs for the edges in $E$. In particular, for any $e \in E \setminus E_R$ we set $c_e = 0$. For a given edge $e' \in (v^1_i, w^1_i) \in E_R$ we set $c_{e'} := -R_f[i]$. The final ingredient to the MCF problem is represented by the edge capacities $u$. For any $e \in E_s \cup E_0 \cup E_E$ we set the infinite capacity $u_e := \infty$. To ensure that the correct number of agents will arrive at the starting positions, we specify for a given $e' = (s, v^1_i) \in E_s$ the capacity $u_{e'} := \sum_{e'' \in A} p^0_0[i]$. Finally, to ensure that every reward will be collected by at most one agent, we specify $u_{e''} := 1$ for any $e'' \in E_R$.

By definition the above MCF formulation is equivalent to the homogeneous problem, with one slight change. Namely, MCF permits fractional assignments to the flow variables $h$, whereas the values of $x, z$ are integral. However, if all edge capacities are integral (which is the case in our setting),
the linear relaxation of MCF enjoys a totally-unimodular constraint matrix form [42]. Hence, the fractional solution will necessarily have an integer optimal solution.

The MCF problem above can be solved in \(O((M' + N)\log N(M + N \log N))\) time using Orlin's MCF algorithm [38, 43], where \(M = |E| = O(Tn)\), \(N = |V| = O(Tn)\), and \(M'\) is the number of \(G\) edges whose capacity is finite. In our case \(M' = O(Tn)\), which implies that the total time complexity for solving the above MCF formulation is \(O(T^2 \min n \log n + T^2n^2 \log^2 n)\).

IV. ALGORITHM FOR THE HETEROGENEOUS CASE

We present an efficient algorithm, which we call \textsc{FlowDec}, for the heterogeneous task-allocation problem described in Table I. The \textsc{FlowDec} algorithm decomposes the problem into several homogeneous subproblems that are solved using min-cost flow, as described in Section III. In the remainder of this section we describe \textsc{FlowDec}, and determine its approximation and runtime guarantees.

A. The \textsc{FlowDec} algorithm

The \textsc{FlowDec} algorithm (Algorithm 1) accepts as input the agent fleets \(A^1, \ldots, A^F\), initial positions \(p_0\), and reward sets \(R := \{R^0, R^1, \ldots, R^F\}\). Recall that we wish to find an assignment \(x, y, z\) such that the expression \(R(x, y, z)\) is maximized. Also recall that for a given fleet \(f \in F\), timestep \(\tau \in [0..T]\), and vertices \(i, j \in V\), \(x^f[i, j]\) represents the transitions of the agents in \(A^f\) from \(i\) to \(j\), and \(y^f[i], z^f[i]\) indicate whether those agents are assigned to collect the rewards \(R^f_1[i]\) and \(R^f_2[j]\), respectively, at vertex \(i\). For a given fleet \(f \in F\), denote by \(x^f\) the corresponding values of the \(x\) assignment for agents belonging to fleet \(f\), i.e., \(x^f = \{x^f_\tau\}_{\tau \in [0..T-1]}\). The sets \(y^f, z^f\) are similarly defined.

\textsc{FlowDec} computes two candidate solutions using the subroutines \textsc{PrivateFirst} and \textsc{SharedFirst}, respectively, and returns the one that yields the larger reward of the two. Next we elaborate on those two subroutines.

Algorithm 1: \textsc{FlowDec}(\(R, p_0\))

1. \((x, y, z) \leftarrow \textsc{PrivateFirst}(R, p_0)\);
2. \((x', y', z') \leftarrow \textsc{SharedFirst}(R, p_0)\);
3. if \(R(x, y, z) > R(x', y', z')\) then
4. \hspace{1em} return \((x, y, z)\);
5. \hspace{1em} return \((x', y', z')\);

The subroutine \textsc{PrivateFirst} (Algorithm 2) prioritizes the assignment of private rewards over shared rewards. This is achieved by assigning to each fleet \(f \in F\) a new reward set \(R^f\) that combines the private reward set \(R^f_1\) and the shared reward set \(R^f_2\), where the latter is rescaled by \(F^{-1}\), i.e., \(R^f_2[j] = R^f_2[j] + R^f_1[j] \cdot F^{-1}\). An assignment over \(R^f\) for every \(f \in F\) is then obtained by solving the homogeneous problem \(H(R^f, p_0)\). Note that \(z^f\) implicitly encodes both an assignment to a private reward and shared reward. That is, an agent assigned to perform a reward \(R^f_1[j]\) can be interpreted as being assigned to both \(R^f_1[j]\) and \(R^f_2[j]\). In lines 5-9 the solutions of the individual fleets are combined to eliminate cases where several agents (from different fleets) are assigned to the same shared reward. To do so, for a given time step \(\tau\) and vertex \(j\), we iterate over all fleets \(f \in F\) and assign \(y^f[j] = 1\) for the first agent we encounter that is assigned to a shared reward in the corresponding vertex.

Algorithm 2: \textsc{PrivateFirst}(\(R, p_0\))

1. \(R^f = R^f_1 + R^f_2 \cdot F^{-1}, \forall f \in F\);
2. \((x^f, z^f) \leftarrow H(R^f, p_0), \forall f \in F\);
3. \(x \leftarrow \{x^f_f \}_{f \in F}, z \leftarrow \{z^f_f\}_{f \in F};\)
4. \(y \leftarrow \{0\}_{f \in F}[\tau \in [0..T]};\)
5. for \(\tau \in [0..T], j \in V\) do
6. \hspace{1em} for \(f \in F\) do
7. \hspace{2em} if \(z^f[j] == 1\) then
8. \hspace{3em} \(y^f_\tau[j] \leftarrow 1;\)
9. \hspace{2em} break;
10. return \((x, y, z)\);

The \textsc{SharedFirst} subroutine (Algorithm 3) prioritizes the assignment of shared rewards, by first computing an assignment for all the agents \(A\) to the shared reward set \(R^S\), to maximize the total reward. This is achieved by solving the homogeneous problem \(H(R^S, p_0)\). It then generates an updated private reward set \(R^p\) for every fleet \(f \in F\), where the value of a reward \(R^p_1[j]\) is equal to \(R^p_2[j]\) in case that \(R^p_2[j]\) was not assigned to \(f\), and otherwise equal to \(R^p_1[j] + R^p_2[j]\), for every time step \(\tau\) and vertex \(j\). Next, for every fleet \(f\) the private assignment \((x^f, z^f)\) over \(R^f\) is computed by solving \(H(R^f, p_0)\). Note that \(z\) here represents simultaneously assignments for shared and private rewards.

Algorithm 3: \textsc{SharedFirst}(\(R, p_0\))

1. \((\bar{x}, \bar{y}) \leftarrow H(R^p, p_0);\)
2. for \(f \in F, \tau \in [0..T], j \in V\) do
3. \hspace{1em} \(R^f_{\tau}[j] \leftarrow R^f_{\tau}[j] + R^p_{\tau}[j] \cdot \bar{y}^f_{\tau}[j];\)
4. \hspace{1em} \((x^f, z^f) \leftarrow H(R^p, p_0), \forall f \in F;\)
5. \hspace{1em} \(x \leftarrow \{x^f_f \}_{f \in F}, z \leftarrow \{z^f_f\}_{f \in F};\)
6. return \((x, z, z);\)

B. Analysis of \textsc{FlowDec}

We prove that the \textsc{FlowDec} algorithm is guaranteed to achieve a solution within a constant factor of the optimum. Let \((x, y, z)\) be a solution of \textsc{FlowDec}. We use \(R(x, y, z)\) to represent the portion of the total reward \(R(x, y, z)\) that is attributed to fleet \(f \in F\) (where assignment values of agents in other fleets are set to 0). Similarly, denote by \(\hat{R}(x, y, 0)\) and \(\hat{R}(x, 0, z)\) the shared and private portion of the total reward, respectively, where 0 represents the zero vector (whose dimension will be clear from context).

Let \((X, Y, Z)\) be a solution to the heterogeneous problem (Table I) that maximizes the expression \(R(X, Y, Z)\) and define \(\text{OPT} := R(X, Y, Z) = S^* + P^*\), where \(S^* := R(X, Y, 0)\) and \(P^* := R(X, 0, Z) = \sum_{f \in F} R(X^f, 0, Z^f)\).

We are ready to state our main theoretical contribution:

**Theorem 1** (Approximation factor of \textsc{FlowDec}). Let \((x, y, z)\) be the solution returned by \textsc{FlowDec}. Then \(R(x, y, z) \geq \text{OPT} \cdot \frac{1}{3}\).
Before proceeding to the proof we establish two intermediate results, concerning the guarantees of PrivateFirst and SharedFirst, when considered separately. In particular, we show that each of the subroutines provide complementary approximations with respect to $S^*$ and $P^*$, and FlowDec enjoys the best of both worlds. See illustration in Figure 2.

Claim 1 (PrivateFirst’s solution quality). Let $(x, y, z)$ be the solution returned by PrivateFirst. Then $\mathcal{R}(x, y, z) \geq P^* + F^{-1} \cdot S^*$.

Proof. Fix a fleet $f \in \mathcal{F}$ and note that
\[
\mathcal{R}(x^f, y^f, z^f) \geq \mathcal{R}(x^f, 0, Z^f) + F^{-1} \cdot \mathcal{R}(X^f, Y^f, 0),
\]
since PrivateFirst is free to choose $(x^f, z^f) := (x^f, Y^f + Z^f)$ (line 2 in Algorithm 2). Hence,
\[
\mathcal{R}(x, y, z) = \sum_{f \in \mathcal{F}} \mathcal{R}(x^f, y^f, z^f) \\
\geq \sum_{f \in \mathcal{F}} \mathcal{R}(x^f, 0, Z^f) + F^{-1} \sum_{f \in \mathcal{F}} \mathcal{R}(X^f, Y^f, 0) \\
= P^* + F^{-1} \cdot S^*.
\]

Claim 2 (SharedFirst’s solution quality). Let $(x, y, z)$ be a solution returned by SharedFirst. Then $\mathcal{R}(x, y, z) \geq S^*$.

Proof. The proof follows from the inequality $\mathcal{R}(\bar{x}, \bar{y}, 0) \geq \mathcal{R}(X, Y, 0)$, where $\bar{x}, \bar{y}$ are defined Algorithm 3, line 1.

We are ready for the main proof:

Proof of Theorem 1. Claims 1 and 2 imply that FlowDec obtains a solution $(x, y, z)$ with reward at least $\max \{P^* + F^{-1} \cdot S^*, S^*\}$. If either of $S^*$ or $P^*$ is zero, then FlowDec would return the optimal solution. Thus, assume instead that $S^*$ and $P^*$ are positive. Thus, there exists $\Delta > 0$ such that $P^* = \Delta S^*$. The approximation factor of FlowDec can be expressed as follows:
\[
\frac{\mathcal{R}(x, y, z)}{\mathcal{R}(X, Y, Z)} = \frac{\max \{\Delta \cdot S^* + F^{-1} \cdot S^*, S^*\}}{\Delta S^* + S^*} \\
= \max \left\{\frac{\Delta + F^{-1} \cdot \frac{1}{\Delta + 1}}{\Delta + 1}, \frac{1}{\Delta + 1}\right\} \\
\geq \max \left\{\frac{\frac{F-1}{F} + F^{-1} \cdot \frac{1}{F} + 1}{2F - 1}, \frac{1}{2F - 1}\right\} = \frac{F}{2F - 1},
\]
where the last inequality follows from $\arg\min_{\Delta > 0} \max \left\{\frac{\Delta + F^{-1} \cdot \frac{1}{\Delta + 1}}{\Delta + 1}, \frac{1}{\Delta + 1}\right\} = (F - 1)/F$.

We conclude with a runtime analysis of FlowDec.

Corollary 1 (FlowDec runtime). FlowDec can be implemented in $O(FT^2n^2 \log n + FT^2n^2 \log^2 n)$ time.

Proof. The bottleneck of FlowDec is solving multiple homogeneous subproblems. Since PrivateFirst and SharedFirst solve in total $2F + 1$ homogeneous problems, respectively, and each such computation requires $O(T^2mn \log n + T^2n^2 \log^2 n)$ time (Lemma 1), the total runtime follows.

V. EXPERIMENTAL RESULTS

We validate our theoretical results from the previous section through simulation experiments. We show that our FlowDec approach is faster than a MILP approach by several orders of magnitude, and we observe that the approximation factors that FlowDec achieves in practice are higher than the worst-case lower bound of $\frac{1}{2F - 1}$. Additionally, we observe experimentally that the running time of FlowDec is insensitive to the number of agents in each fleet.

A. Implementation and scenario details

The results were obtained using a commodity laptop with 2.80GHz × 4 core i7-7600U CPU, and 16GB of RAM. We implemented the FlowDec algorithm in C++, using the network-simplex algorithm for min-cost flow in the LEMON Graph Library [44, 45]. For comparison, we used the MILP implementation in CPLEX [46].

We tested both implementations on a predictive problem formulation, as described in Section II-C. We consider the problem of heterogeneous tracking of multiple moving objects, where the value of rewards are chosen to incentivise agents to visit graph vertices where objects are located. In particular, for a given graph $\mathcal{G}$, time horizon $T$, number of fleets $F$, and initial object count $I \in \mathbb{N}$, we chose uniformly at random for each reward type $t \in [0..F]$, $I$ vertices of $\mathcal{G}$ (with repetitions) which represent initial locations of objects to be tracked as part of the reward set $R^t$. In particular, we set the value of $R^t_0[i]$ to be the number of objects at a given vertex $i \in V$. For the subsequent time steps $\tau \in [1..T]$ we set $R^t_\tau$ to be the expected reward assuming that the objects move to neighboring vertices via a random walk. Initial agent locations are chosen in a uniform random fashion.

B. Results

We compare the performance of the MILP approach and our FlowDec algorithm, in terms of solution quality and runtime. We then study the scalability of the FlowDec algorithm on larger test cases for which the MILP approach has timed out.

1) Comparison between FlowDec and MILP: In this setup, we fix the graph $\mathcal{G}$ to be a $10 \times 10$ grid, set the initial number of tracked objects for every task to be $I = 3$, and set the number of agents within each fleet $f \in \mathcal{F}$ to be $a_f = 5$. In Table III we report the running time of the MILP solution and the FlowDec algorithm, as well as the approximation factor that was achieved by FlowDec, for scenarios of varying sizes. We set the time horizon $T$ and the fleet number
$F$ to be in the range $[2 \ldots 128]$. The reported running times are averaged over 20 randomly-generated scenarios for each parameter combination. The reported approximation factor is the minimum result over the 20 runs. We terminate the run of each algorithm if it exceeds 10 minutes.

In terms of running time, we observe that the MILP approach behaves similarly to FLOWDEC only for the smallest test cases, e.g., when $T = 2$. However, as the problem size increases the running time of the MILP approach grows significantly faster than that of FLOWDEC. For instance, already for $T = 4$, when $F = 2$ FLOWDEC is nearly 3 times faster than the MILP approach, and when $F = 128$ it is more than 20 times faster. As $T$ is increased we encounter more scenarios in which the MILP implementation is forced to time out, whereas FLOWDEC finishes fairly quickly. For example, when $T = 16$, FLOWDEC finishes within a few seconds, for all fleet sizes, whereas the MILP approach times out for $F = 128$, which yields a speedup of at least 100 for FLOWDEC. For $T = 128$, the MILP approach is able to solve only the smallest scenarios, whereas FLOWDEC solves all of them. In terms of solution quality, we observe that FLOWDEC typically achieves approximation factors that are larger than the theoretical lower bound, which suggests that this bound is loose. We note that the smallest approximation factor that we observed with FLOWDEC is 0.64 (for $F = 32$).

2) Scalability of FLOWDEC: We rerun the previous experiment for a larger $50 \times 50$ graph to test how the runtime of the FLOWDEC algorithm is affected by its different parameters. We report in Table IV the runtime of the algorithm, where $I = 3$, $a_f = 5$ were set as in the previous experiment. In accordance with the theoretical complexity bound in Corollary 1, we observe that the runtime increases linearly with the number of fleets $F$. The runtime increases linearly

| time hor. | type  | 2    | 4    | 8    | 16   | 32   | 64   | 128  |
|----------|------|-----|-----|-----|-----|-----|-----|-----|
| MILP     | 0.00 | 0.01| 0.01| 0.02| 0.03| 0.06| 0.12| 0.26|
| FLOW     | 0.01 | 0.01| 0.02| 0.03| 0.06| 0.12| 0.26|      |
| APX      | 1.00 | 1.00| 0.93| 0.86| 0.84| 0.75| 0.75|      |
| MILP     | 0.02 | 0.04| 0.07| 0.16| 0.39| 0.90| 1.77|      |
| FLOW     | 0.01 | 0.03| 0.05| 0.09| 0.17| 0.35| 0.67|      |
| APX      | 1.00 | 0.97| 0.87| 0.86| 0.77| 0.74| 0.85|      |
| MILP     | 0.11 | 0.26| 0.36| 1.39| 3.62| 12  | 21  |      |
| FLOW     | 0.09 | 0.14| 0.25| 0.46| 0.96| 1.35| 2.05|      |
| APX      | 0.96 | 0.92| 0.82| 0.80| 0.84| 0.88| 0.94|      |
| MILP     | 0.45 | 1.40| 3.78| 20  | 92  | 455 | 592 |      |
| FLOW     | 0.10 | 0.19| 0.36| 0.67| 1.27| 2.54| 5.92|      |
| APX      | 0.96 | 0.93| 0.88| 0.92| 0.89| 0.95|      |      |
| MILP     | 1.5  | 5.65| 20  | 284 | 567 | -   | -   |      |
| FLOW     | 0.29 | 0.52| 1.00| 1.87| 3.76| 7.11| 16  |      |
| APX      | 0.94 | 0.89| 0.82| 0.88| 0.94|      |      |      |
| MILP     | 4.78 | 21  | 205 | -   | -   | -   | -   |      |
| FLOW     | 0.84 | 1.50| 2.78| 5.22| 12  | 20  | 43  |      |
| APX      | 0.97 | 0.91| 0.92| 0.90| 0.94|      |      |      |
| MILP     | 14.20| -   | -   | -   | -   | -   | -   |      |
| FLOW     | 2.48 | 4.85| 8.25| 17  | 31  | 59  | 115 |      |
| APX      | 0.96 | -   | -   | -   | -   | -   |      |      |

TABLE III: Comparison between FLOWDEC and a MILP approach in terms of runtime and solution quality for $10 \times 10$ grid graphs. For every combination of number of fleets $F$ and time horizon $T$ we report in the “MILP” and “FLOW” rows the corresponding running times (in seconds). The label “-” indicates that MILP did not finish within the 10-minute time limit. In the “APX” row we report the approximation factor of FLOWDEC, i.e., the quotient between the reward values obtained by FLOWDEC and MILP, respectively.

TABLE IV: Running time (seconds) of FLOWDEC for a $50 \times 50$ grid as a function of the number of fleets $F$ and time horizon $T$. The with the time horizon $T$ as well, which suggests that the theoretical quadratic increase is overly conservative.

Finally, we note that the runtime of FLOWDEC is only mildly affected by the size of each individual fleet, e.g., as we increase the value of $a_f$ from 5 to 500, we observe a modest increase of $\%10$ to the runtime. This is in contrast to the MILP approach which is highly sensitive to this value.

VI. Conclusion

We presented a near-optimal algorithm, termed FLOWDEC, for heterogeneous task allocation and demonstrated its good performance through extensive simulation tests. Our work suggests a few interesting directions for future research, which we highlight below. From an algorithmic perspective we plan to explore whether the approximation factor of FLOWDEC can be improved by introducing a third subroutine which would better estimate the optimal reward for scenarios in which the subroutines SHAREDFIRST and PRIVATEFIRST under approximate (see Figure 2). We also plan to extend our approach to account for collision-avoidance constraints by exploiting the fact that those constraints can be captured via a MCF-based solution for the homogeneous problem. Finally, we plan to extend our algorithm to a distributed setting, by potentially relying on a dual decomposition that would be applied to the homogeneous subproblems [47, Chapter 7].

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