Dynamic Programming Principle for Backward Doubly Stochastic Recursive Optimal Control Problem and Sobolev Weak Solution of The Stochastic Hamilton-Bellman Equation

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Abstract

In this paper, we study backward doubly stochastic recursive optimal control problem where the cost function is described by the solution of a backward doubly stochastic differential equation. We give the dynamical programming principle for this kind of optimal control problem and show that the value function is the unique Sobolev weak solution for the corresponding stochastic Hamilton-Jacobi-Bellman equation.

Keywords: Backward double stochastic differential equation; Dynamic programming principle; Recursive optimal control; Hamilton-Jacobi-Bellman equation; Sobolev weak solution

1 Introduction

Backward stochastic differential equation (BSDE in short) has been introduced by Pardoux and Peng [3]. Independently, Duffie and Epstein [2] introduced BSDE from economic background. In [2] they presented a stochastic differential recursive utility which is an extension of the standard additive utility with the instantaneous utility depending not only on the instantaneous consumption rate but also on the future utility. The recursive optimal control problem is presented as a kind of optimal control problem whose cost functional is described by the solution of BSDE. In [4] they gave the formulation of recursive utilities and their properties from the BSDE point of view. In 1992, Peng [6] got the Bellman’s dynamic programming principle for this kind of problem and proved that the value function is a viscosity solution of one kind of quasi-linear second-order partial differential equation (PDE in short) which is the well-known as Hamilton-Jacobi-Bellman equation. Later in 1997 he virtually generalized these results to

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a much more general situation, under Markovian and even Non-Markovian framework. In this
cinese version, Peng used the backward semigroup property introduced by a BSDE under
Markovian and Non-Markovian framework. He also proved that the value function is a viscos-
ity solution of a generalized Hamilton-Jacobi-Bellman equation. In 2007, Wu and Yu [7] gave
the dynamic programming principle for one kind of stochastic recursive optimal control problem
with the obstacle constraint for the cost functional described by the solution of a reflected BSDE
and showed that the value function is the unique viscosity solution of the obstacle problem for
the corresponding Hamilton-Jacobi-Bellman equation.

In 1994, Pardoux and Peng first studied the backward doubly stochastic differential equa-
tions (BDSDE in short). There are two different directions of stochastic integral in the equations
involving with two independent standard Brownian motions: a standard (forward) \(dW_t\) and a
backward \(dB_t\). They had proved existence and uniqueness result of this equation and estab-
lished the connection between BDSDE and a classical solution for stochastic partial differential
equation (SPDE in short) under smoothness assumption on the coefficients. And then, Bally
and Matoussi [1] gave the probabilistic representation of the solution in Sobolev space of semi-
linear stochastic PDEs in terms of BDSDE. Shi and Gu [16] gave the comparison theorem of
BDSDE. Then Auguste and Modeste [10] got the uniqueness and existence of reflected BDSDE’s
olutions.

In our paper, we study a stochastic recursive optimal control problem where the control
system is described by the classical stochastic differential equation, however, the cost function
is described by the solution of a backward doubly stochastic differential equation. This kind of
recursive optimal control problem has some practical meaning. For example, in an arbitrage-
free incomplete financial market, there may exist so called informal trading such as “insider
trading”. An individual has access to insider information would have an unfair edge over other
vestors, who do not have the same access, and could potentially make larger ‘unfair’ profits than
their fellow investors. This phenomenon could be described by a BDSDE in a financial market
models. More specifically, there are two kinds of investors with different levels of information
about the future price evolution in a market influenced by an additional source of randomness.
The ordinary trader only has the “public information”—market prices of the underlying assets
ained in the filtration \(\mathcal{F}^W_t\). However, an insider who has access to a larger filtration \(\mathcal{F}^W_t \lor \mathcal{F}^R_{t\wedge T}\), which includes insider information. For instance, an insider knows the functional law of
the price process or he knows in advance that a significant change has occurred in the business
olicy or scope of a security issue or he could estimate if his portfolio is better than others.
We would like to emphasize that BDSDE techniques provide powerful instruments to analyze
the problem of portfolio optimization of an insider trader. For an insider trader, his investment
strategy still satisfies the property that locally optimal is equal to globally optimal.

The problem we are most interested in is whether the dynamic programming principle still
holds for this recursive optimal control problem. The good news is that it can be accomplished
by the properties of the BDSDE. Compared with the Hamilton-Jacobi-Bellman (HJB in short)
equation in paper[6][7], the corresponding HJB we get is a SPDE in a Markovian framework.
In the stochastic case where the diffusion is possibly degenerate, the HJB equation may in
general have no classical solution. To overcome this difficulty, Crandall and Lions introduced
the so-called viscosity solutions in the early 1980s. Obviously, the research on the viscosity
solution on HJB equations have yielded fruitful results. However, the viscosity solution of the
HJB equation cannot give an reasonable probabilistic interpretation on a pair of solution \((Y,Z)\)
of BSDE considering that relationship do not established between the Z part of the solution
and the HJB equation. Here, we study a different kind of weak solution for HJB equations in
a Sobolev space, in which part \( Z \) is spontaneously contained in the weak definition. Wei and Wu [11] have proved that the value function is the unique Sobolev weak solution of the related HJB equation by virtue of the nonlinear Doob-Meyer decomposition theorem introduced in the study of BSDEs.

In this paper, we consider the issue on Sobolev weak solution of HJB equation connected with BDSDE. Since that we cannot find a Doob-Meyer decomposition theorem in BDSDE, it is a point that how to establish the equation like Lemma 4.1. and 4.2. in [11]. Inspired by the [10] we bring a increasing process into the equation in order to push the cost functional upward in a minimum force.

The paper is organized as follows. Preliminaries and assumption are introduced in Section 2. In Section 3 we formulate a stochastic recursive optimal control problem where the cost function is described by the solution of a BDSDE. We show that the celebrated dynamic programming principle still holds for this kind of optimization problem. In Section 4 we prove that the value function of this problem is the unique weak solution in a Sobolev space for the corresponding stochastic Hamilton-Jacobi-Bellman equation.

2 Preliminaries and assumption

In this section, we give some preliminary results of the BDSDE which are useful for the dynamic programming principle for the recursive optimal control problem.

Let \( (\Omega, \mathcal{F}, \mathcal{P}) \) be a probability space, and \( T > 0 \) be an arbitrarily fixed constant throughout this paper. Let \( \{W_t; 0 \leq t \leq T\} \) and \( \{B_t; 0 \leq t \leq T\} \) be two mutually independent standard Brownian Motion processes with values respectively in \( \mathbb{R}^d \) and \( \mathbb{R}^l \), defined on \( (\Omega, \mathcal{F}, \mathcal{P}) \). Let \( \mathcal{N} \) denote the class of \( \mathcal{P} \)-null sets of \( \mathcal{F} \). For each \( t \in [0, T] \), we define

\[
\mathcal{F}_t := \mathcal{F}_t^W \lor \mathcal{F}_{t,T}^B,
\]

where for any process \( \{\eta_t\}, \mathcal{F}_{s,t}^\eta = \sigma \{\eta_r - \eta_s; s \leq r \leq t\} \lor \mathcal{N}, \mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta \). Let \( \mathcal{F}_t^\eta \) be the complete filtration generated by the Brownian motion \( W_t - W_t, t \leq r \leq t' \) \lor \mathcal{N}.

Note that the collection \( \{\mathcal{F}_t; t \in [0, T]\} \) is neither increasing nor decreasing, so it does not constitute a filtration.

Let us introduce some notations:

\[
\mathcal{L}^p = \{\xi \text{ is an } \mathcal{F}_T - \text{measurable random variable s.t. } E(|\xi|^p) < +\infty, \quad p \geq 2\},
\]

\[
\mathcal{H}^p = \{\psi_t, \quad 0 \leq t \leq T \} \text{ is a predictable process s.t. } E(\int_0^T |\varphi|^2 dt)^{\frac{p}{2}} < +\infty, \quad p \geq 2\},
\]

\[
\mathcal{S}^p = \{\varphi_t, \quad 0 \leq t \leq T \} \text{ is a predictable process s.t. } E(\sup_{0 \leq t \leq T} |\varphi|^p) < +\infty, \quad p \geq 2\}.
\]

and the following BDSDE:

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{1}
\]

Here

\[
f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k,
\]

\[
g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^{k \times l}.
\]
and $f,g$ satisfying

\((H1)\) for all \((y,z) \in \mathbb{R} \times \mathbb{R}^d,\)

\[
f(\cdot,y,z) \in M^P(0,T;\mathbb{R}^k); \quad g(\cdot,y,z) \in M^P(0,T;\mathbb{R}^{k\times l}),
\]

\((H2)\) for some \(L > 0\) and \(0 < \alpha < 1\) all \(y,y' \in \mathbb{R}, z,z' \in \mathbb{R}^d, a.s.\)

\[|f(t,y,z) - f(t,y',z')|^2 \leq L(|y-y'|^2 + |z-z'|^2), \quad |g(t,y,z) - g(t,y',z')| \leq L|y-y'|^2 + \alpha |z-z'|^2.\]

There exists \(C\) such that for all \((t,x,y,z) \in [0,T]\times \mathbb{R}^d \times \mathbb{R}^{k\times d},\)

\[gg^*(t,x,y,z) \leq zz^* + C(||g(t,x,o,o)||^2 + |y|^2)I.\]

\((H3)\) \(\xi \in L^p.\)

We notice that there are two independent Brownian motions \(W\) and \(B\) in (1), where the \(dW\) integral is a formed Itô’s integral and \(dB\) integral is a backward Itô’s integral. The extra noise \(B\) in the equation can be thought of as some extra information that can not be detected in the market in general, but is available to the particular investor. The problem then is to show how this investor can take advantage of such extra information to optimize the utility, but by taking actions that are completely “legal”, in the sense that the investor has to choose the optimal strategy in the usual class of the admissible portfolios.

Then form Theorem 1.1 in [5], then there exists a unique solution \(\{(Y_t,Z_t), 0 \leq t \leq T\} \in \mathcal{S}^p(0,T;\mathbb{R}^k) \times \mathcal{H}^p(0,T;\mathbb{R}^{k\times d}).\)

Now we give two more accurate estimates of the solutions. They are very important and necessary for the dynamic programming principle of our optimal control problem and play an important role for the continuation properties of value function \(u(t,x)\) about \(t\) and \(x\). The proof is complicated and technical, some technique derive from [1].

**Proposition 2.1** Let\(\{(Y_t,Z_t), 0 \leq t \leq T\} \in \mathcal{S}^p(0,T;\mathbb{R}^k) \times \mathcal{H}^p(0,T;\mathbb{R}^{k\times d}),\)

\(0 \leq t \leq T\) be the solution of the above BDSDE, then for some \(p > 2, \xi \in L^p(\Omega,\mathcal{F}_T,T;\mathbb{R}^k)\) and

\[
E \int_0^T (|f(t,0,0)|^p + |g(t,0,0)|^p)dt < \infty,
\]

we have

\[
E \left\{ \sup_{t \leq s \leq T} |Y_s|^p + \left( \int_0^T |Z_s|^2 dt \right)^{\frac{p}{2}} \right\} < \infty. \tag{2}
\]

**Proposition 2.2** Let \((\xi,f,g)\) and \((\xi',f',g')\) be two triplets satisfying the above assumption. Suppose \((Y,Z)\) is the solution of the BDSDE \((\xi,f,g)\) and \((Y',Z')\) is the solution of the BDSDE \((\xi',f',g')\). Define

\[
\triangle \xi = \xi - \xi', \quad \triangle f = f - f', \quad \triangle g = g - g',
\]

\[
\triangle Y = Y - Y', \quad \triangle Z = Z - Z'.
\]

Then there exists a constant \(C\) such that

\[
E \left\{ \sup_{t \leq s \leq T} |\triangle Y_s|^p + \left( \int_0^T |\triangle Z_s|^2 dt \right)^{\frac{p}{2}} \right\} \leq CE \{|\triangle \xi|^p\}. \tag{3}
\]
3 Formulation of the problem and the Dynamic Programming Principle

In this section, we first formulate a backward doubly stochastic recursive optimal control problem, and then we prove that the dynamic programming principle still holds for this kind of optimization problem.

We introduce the admissible control set $U$ defined by

$$U := \{ v(\cdot) \in H^p | v(\cdot) \text{ take value in } U \subset \mathbb{R}^k \}.$$ 

An element of $U$ is called an admissible control. Here $U$ is a compact subset of $\mathbb{R}^k$, however this restriction is often satisfied in practical applications.

For a given admissible control, we consider the following control system

$$\begin{aligned}
\int dX_t^{t, \xi, \nu} &= b(s, X_s^{t, \xi, \nu}, v_s)ds + \sigma(s, X_s^{t, \xi, \nu}, v_s)dW_s, \quad s \in [t, T], \\
X_t^{t, \xi} &= \zeta,
\end{aligned}$$

(4)

Where $t \geq 0$ is regarded as the initial time and $\zeta \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ as the initial state.

The mappings $b : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times d}$.

satisfy the following conditions:

(H3.1) $b$ and $\sigma$ are continuous in $t$.

(H3.2) for some $L > 0$, and all $x, x' \in \mathbb{R}^n$, $v, v' \in U$, a.s.

$$|b(t, x, v) - b(t', x', v')| + |\sigma(t, x, v) - \sigma(t, x', v')| \leq L(|x - x'| + |v - v'|).$$

Obviously, under the above assumption, for any $v(\cdot) \in U$, control system (4) has a unique strong solution $\{X_s^{t, \xi, \nu} \in H^p(0, T; \mathbb{R}^k), 0 \leq t \leq s \leq T\}$, and we also have the following estimates.

**Proposition 3.1** For all $t \in [0, T]$, $\zeta, \zeta' \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, $v(\cdot), v'(\cdot) \in U$,

$$\mathbb{E}^{\mathcal{F}_t} \left\{ \sup_{t \leq s \leq T} |X_s^{t, \xi, \nu}|^p \right\} \leq C_p (1 + |\zeta|^p),$$

(5)

$$\mathbb{E}^{\mathcal{F}_t} \left\{ \sup_{t \leq s \leq T} |X_s^{t, \xi, \nu} - X_s^{t, \xi', \nu'}|^p \right\} \leq C_p |\zeta - \zeta'|^p + C \mathbb{E}^{\mathcal{F}_t} \left\{ \int_t^T |v_s - v'_s|^p ds \right\}. \quad (6)$$

Where the constant $C_p$ also depends on $L$.

**Proposition 3.2** For all $t \in [0, T]$, $x \in \mathbb{R}^n$, $v(\cdot) \in U$, $\delta \in [0, T - t]$,

$$\mathbb{E} \left\{ \sup_{t \leq s \leq t + \delta} |X_s^{t, \xi, \nu} - x|^p \right\} \leq C_p \delta^p,$$

(7)

where the constant $C$ also depends on $x$ and $L$.

Now for any given admissible control $v(\cdot)$, we consider the following BDSDE:
over, we get the following estimates for the solution from Proposition 2.1 and 2.2.

\[
Y_s^{\xi,\zeta^u} = \Phi(X_s^{\xi,\zeta^u}) + \int_s^T f(r, X_r^{\xi,\zeta^u}, Y_r^{\xi,\zeta^u}, Z_r^{\xi,\zeta^u}, v_r)dr \\
+ \int_s^T g(r, X_r^{\xi,\zeta^u}, Y_r^{\xi,\zeta^u}, Z_r^{\xi,\zeta^u})dB_r - \int_s^T z_r^{\xi,\zeta^u}dW_r, \quad t \leq s \leq T, \tag{8}
\]

where
\[
\Phi : \mathbb{R}^n \to \mathbb{R}^n, \\
f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathcal{U} \to \mathbb{R}^n, \\
g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^{n \times l},
\]

and they satisfy the following conditions:

(H3.3) \(f\) and \(h\) are continuous in \(t\).

(H3.4) for some \(L > 0\) and \(0 < \alpha < 1\) all \(x, x' \in \mathbb{R}^n; y, y' \in \mathbb{R}; z, z' \in \mathbb{R}^d; v, v' \in \mathcal{U}\)
\[
|f(t, x, y, z, v) - f(t, x', y', z', v')| + |\Phi(x) - \Phi(x')| \\
\leq L(|x - x'| + |y - y'| + |z - z'| + |v - v'|).
\]

(H3.5) The function \(g \in L^2(R^d, \rho dx)\).

(H3.6) \(\forall (y, z) \in R \times R^d, f(\cdot, y, z) \in H^2; g(\cdot, y, z) \in H^2\).

(H3.7) \(f\) is measurable in \((t, x, y, z, v)\) and for any \(r \in [t, T]\),
\[
E \int_0^T |f(r, 0, 0, 0, v_r)|^2dr \leq M,
\]
functions \(f\) and \(g\) are continuous and controlled by \(C(1 + |x| + |y| + |z| + |v|)\).

Then there exists a unique solution \((Y^{\xi,\zeta^u}, Z^{\xi,\zeta^u}) \in S^p(0, T; \mathbb{R}^k) \times H^p(0, T; \mathbb{R}^{k \times d})\). Moreover, we get the following estimates for the solution from Proposition 2.1 and 2.2.

**Proposition 3.3**

\[
E \left\{ \sup_{t \leq s \leq T} |Y_s^{\xi,\zeta^u}|^p + \left( \int_t^T |Z_r^{\xi,\zeta^u}|^2dr \right)^{\frac{p}{2}} \right\} \leq EC_p(1 + |\xi|^q). \tag{9}
\]

**Proposition 3.4**

\[
E \left\{ \sup_{0 \leq s \leq T} |Y_s^{\xi,\zeta^u} - Y_s^{\xi',\zeta'^u}|^p + \left( \int_0^T |Z_r^{\xi,\zeta^u} - Z_r^{\xi',\zeta'^u}|^2dr \right)^{\frac{p}{2}} \right\} \leq E \left\{ C_p(1 + |\xi|^q + |\xi'|^q)(t - t')^{\frac{p}{2}} + |\xi - \xi'|^p + \int_0^T |v_r - v_r'|^pdr \right\}. \tag{10}
\]

The proof is complicated and technical, we put in the Appendix.

Given a control process \(v(\cdot) \in \mathcal{U}\), we introduce the associated cost functional:
\[ J(t, x; v(\cdot)) := Y_s^{t,x,v}|_{s=t}, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad (11) \]

and we define the value function of the stochastic optimal control problem

\[ u(t, x) := \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)), \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad (12) \]

Now we continue our study of the control problem (12) and prove that the celebrated dynamic programming principle still holds for this optimization problem. Some proof ideas come from the proof of the dynamic programming principle for recursive problem given by Peng in Chinese version [6], and Wu and Yu in [7].

Now we introduce the following subspace of \( \mathcal{U} \):

\[
\mathcal{U}^t := \left\{ v(\cdot) \in \mathcal{U} \mid v(s) \text{ is } \mathcal{F}_{s,t}^W \text{ progressively measurable, } \forall t \leq s \leq T \right\},
\]

\[
\mathcal{U}^t := \left\{ v_s = \sum_{j=1}^N v^j s I_{A_j} | v^j_s \in \mathcal{U}^t, \{A_j\}_{j=1}^N \text{ is a partition of } (\Omega, \mathcal{F}^W_t) \right\}.
\]

Firstly we will prove that:

**Proposition 3.5** Under the assumptions (H3.1)-(H3.4), the value function \( u(t, x) \) defined in (12) is \( \mathcal{F}^B_{t,T} \) measurable.

**Proof.** First we can prove:

\[ \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)) = \text{ess sup}_{v(\cdot) \in \mathcal{U}^t} J(t, x; v(\cdot)), \]

\( \mathcal{U}^t \) is the subset of \( \mathcal{U} \), then

\[ \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)) \geq \text{ess sup}_{v(\cdot) \in \mathcal{U}^t} J(t, x; v(\cdot)). \]

We need to consider the inverse inequality. For any \( v(\cdot), \bar{v}(\cdot) \in \mathcal{U} \), for the Proposition 3.4, we know

\[ E \left\{ |Y_t^{t,x,v} - Y_t^{t,x,\bar{v}}|^p \right\} \leq CE \int_t^T |v_r - \bar{v}_r|^p \, ds. \]

Note that \( \mathcal{U}^t \) is dense in \( \mathcal{U} \), then for each \( v(\cdot) \in \mathcal{U} \), there exists a sequence \( \{v_n(\cdot)\}_{n=1}^\infty \in \mathcal{U}^t \) such that

\[ \lim_{n \to \infty} E \left\{ |Y_t^{t,x,v_n} - Y_t^{t,x,v}|^p \right\} = 0. \]

So there exists a subsequence, we denote without loss of generality \( \{v_n(\cdot)\}_{n=1}^\infty \) such that

\[ \lim_{n \to \infty} Y_t^{t,x,v_n} = Y_t^{t,x,v}, \quad \text{a.s.}, \]

so that

\[ \lim_{n \to \infty} J(t, x, v_n(\cdot)) = J(t, x, v(\cdot)), \quad \text{a.s.}. \]
By the arbitrariness of $v(\cdot)$ and the definition of essential supremum, we get
\[
\text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)) \geq \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)),
\]
then we obtain (3.11).
Second, we want to prove
\[
\text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)) = \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)).
\]
Obviously,
\[
\text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)) \geq \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)).
\]

In order to get the inverse inequality, we need the following Lemma:

**Lemma 3.6**

\[
X^{t,x;\sum_{j=1}^N v^j I_{A_j}} = \sum_{j=1}^N I_{A_j} X^{t,x,v^j}, \quad Y^{t,x;\sum_{j=1}^N v^j I_{A_j}} = \sum_{j=1}^N I_{A_j} Y^{t,x,v^j}
\]

\[
\forall v(\cdot) \in \mathcal{U}, \text{ we have}
\]
\[
J(t, x; v(\cdot)) = J(t, x; \sum_{j=1}^N v^j I_{A_j}) = \sum_{j=1}^N I_{A_j} J(t, x; v^j(\cdot)),
\]

because
\[
\text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)) = \text{ess sup}_{v(\cdot) \in \mathcal{U}} \sum_{j=1}^N I_{A_j} J(t, x; v^j(\cdot))
\]
\[
\leq \sum_{j=1}^N \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v^j(\cdot)) = \sum_{j=1}^N \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)) = \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)),
\]
then we can get
\[
\text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)) \leq \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot)).
\]
However, when $v(\cdot) \in \mathcal{U}$, the cost functional $J(t, x; v(\cdot))$ is $\mathcal{F}_{t,T}$ measurable.
So
\[
u(t, x) = \text{ess sup}_{v(\cdot) \in \mathcal{U}} J(t, x; v(\cdot))
\]
is $\mathcal{F}_{t,T}$ measurable.

Next we will discuss the continuity of value function $u(t, x)$ with respect to $x$ and $t$. We have the following estimates:
Lemma 3.7  For each $t \in [0, T]$, $x$ and $x' \in \mathbb{R}^n$, we have

\begin{align}
(i) \quad & \mathbb{E}|u(t, x) - u(t', x')|^p \leq C_p(1 + |x|^q + |x'|^q)(|x - x'|^p + |t - t'|^{\frac{p}{2}});
(ii) \quad & \mathbb{E}|u(t, x)|^p \leq C_p(1 + |x|^q). \tag{13}
\end{align}

Proof. Using the estimates: $\mathbb{E}(\sup_{t \leq s \leq T}|Y^t_{s,x}v|^p) \leq C_p(1 + |X|^q)$, for each admissible control $v(\cdot) \in \mathcal{U}$, we have

\[
\mathbb{E}|J(t, x; v(\cdot))|^p \leq C_p(1 + |x|^q)
\]

and

\[
\mathbb{E}|J(t, x; v(\cdot)) - J(t', x; v'(\cdot))|^p \leq C_p(1 + |x|^q + |x'|^q)(|t - t'|^{\frac{p}{2}} + |x - x'|^{p}).
\]

On the other hand, for each $\varepsilon > 0$, $\exists v(\cdot), v'(\cdot) \in \mathcal{U}$ such that:

\[
J(t, x; v'(\cdot)) \leq u(t, x) \leq J(t, x, v(\cdot)) + \varepsilon,
\]

\[
J(t', x'; v(\cdot)) \leq u(t', x') \leq J(t', x', v'(\cdot)) + \varepsilon.
\]

Form the estimate (10) we can get:

\[-C_p(1 + |x|^q) - \varepsilon \leq \mathbb{E}|J(t, x; v'(\cdot))|^p \leq \mathbb{E}|u(t, x)|^p \leq \mathbb{E}|J(t, x; v(\cdot))|^p + \varepsilon \leq C_p(1 + |x|^q) + \varepsilon.\]

Form the arbitrariness of $\varepsilon$, we can obtain (ii).

Similarly,

\[
J(t, x; v'(\cdot)) - J(t', x'; v'(\cdot)) - \varepsilon \leq u(t, x) - u(t', x') \leq J(t, x; v(\cdot)) - J(t', x'; v(\cdot)) + \varepsilon.
\]

\[
|u(t, x) - u(t', x)| \leq \max\{|J(t, x; v(\cdot)) - J(t', x'; v'(\cdot))|, |J(t, x; v(\cdot)) - J(t', x'; v(\cdot))|\} + \varepsilon.
\]

\[
\mathbb{E}|u(t, x) - u(t, x)|^p \\
\leq C \max\{\mathbb{E}|J(t, x; v(\cdot)) - J(t', x'; v'(\cdot))|^p, \mathbb{E}|J(t, x; v(\cdot)) - J(t', x'; v(\cdot))|^p\} + C\varepsilon^p \\
\leq C_p(1 + |x|^q + |x'|^q)(|x - x'|^p + |t - t'|^{\frac{p}{2}}) + C\varepsilon^p.
\]

Then we can obtain (i). \hfill \Box

For the value function of our recursive optimal control problem, we have:

Lemma 3.8  \forall t \in [0, T], \forall v(\cdot) \in \mathcal{U}, for all $\zeta \in L^p(\Omega, \mathcal{F}_t, P;)$, we have

\[J(t, \zeta; v(\cdot)) = Y^t,\zeta; v(\cdot).\]
Proof. We first study a simple case: $\zeta$ is the following form: 
$$
\zeta = \sum_{i=1}^{N} I_{A_i} x_i,
$$
where $\{A_i\}_{i=1}^{N}$ is a finite partition of $(\Omega, \mathcal{F}_t^W)$, and $x_i \in \mathbb{R}^n$ for $1 \leq i \leq N$, so

$$
Y^t,\zeta;v = Y^t,\sum_{i=1}^{N} I_{A_i} x_i;v = \sum_{i=1}^{N} I_{A_i} Y^t, x_i;v.
$$

From the definition of cost functional. We deduce that

$$
Y^t,\zeta;v = \sum_{i=1}^{N} I_{A_i} Y^t, x_i;v = \sum_{i=1}^{N} I_{A_i} J(t, x_i; v(\cdot)) = J(t, \zeta; v(\cdot)).
$$

Therefore, for simple functions, we get the desired result.

Given a general $\zeta \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we can choose a sequence of simple function $\{\zeta_i\}$ which converges to $\zeta$ in $L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. Consequently, we have:

$$
E\{|Y^t,\zeta;v - Y^t,\zeta_i;v|^p\} \leq E\{C_p(1 + |\zeta|^q + |\zeta_i|^q)(|\zeta - \zeta_i|^p)\} \to 0, \text{ as } i \to \infty.
$$

So

$$
E\{|J(t, \zeta; v(\cdot)) - J(t, \zeta_i; v(\cdot))|^p\} \leq E\{C_p(1 + |\zeta|^q + |\zeta_i|^q)(|\zeta - \zeta_i|^p)\} \to 0, \text{ as } i \to \infty.
$$

With the help of $Y^t,\zeta;v = J(t, \zeta; v(\cdot))$, the proof is completed.

For the value function of our recursive optimal control problem, we have

Lemma 3.9: Fixed $t \in [0, T)$ and $\zeta \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, for each $v(\cdot) \in \mathcal{U}$, we have:

$$
u(t, \zeta) \geq Y^t,\zeta;v(\cdot).
$$

On the other hand, for each $\varepsilon > 0$, there exists an admissible control $v(\cdot) \in \mathcal{U}$ such that:

$$
u(t, \zeta) \leq Y^t,\zeta;v(\cdot) + \varepsilon, \text{ a.s.}
$$

Now we start to discuss the (generalized) dynamic programming principle for our recursive optimal control problem.

Firstly we introduce a family of (backward) semigroups which is original from Peng’s idea in [6].

Given the initial condition $(t, x)$, an admissible control $v(\cdot) \in \mathcal{U}$, a positive number $\delta \leq T - t$ and a real-value random variable $\eta \in L^p(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R})$, we denote

$$
G^{t,x,v}_{t,t+\delta}[\eta] := Y_t,\eta,
$$

where $(Y_s, Z_s)$ is the solution of the following double BSDE with the horizon $t + \delta$:
\[ Y_s = \xi + \int_s^{t+\delta} f(r, Y_r, Z_r)dr + \int_s^{t+\delta} g(r, Y_r, Z_r)dB_r - \int_s^{t+\delta} Z_r dW_r, \quad t \leq s \leq t + \delta. \]

Obviously,
\[
G_{t,t}^{d,\bar{v},\bar{v}}[\Phi(X_T^{t,\bar{v}})] = G_{t,t+\delta}^{d,\bar{v},\bar{v}}[Y_t^{t+\delta,\bar{v}}].
\]

Then our (generalized) dynamic programming principle holds.

**Theorem 3.11**  
Under the assumption (H3.1)-(H3.4), the value function \( u(t, x) \) obeys the following dynamic programming principle: for each \( 0 < \delta \leq T - t, \)
\[
u(t, x) = \text{ess sup}_{v(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[u(t + \delta, X_{t+\delta}^{t,\bar{v}})].
\]

**Proof.** We have
\[
u(t, x) = \text{ess sup}_{v(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[\Phi(X_T^{t,\bar{v}})] = \text{ess sup}_{v(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[Y_{t+\delta}^{t+\delta,\bar{v}}] = \text{ess sup}_{v(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[Y_{t+\delta}^{t+\delta,\bar{v}}].
\]

Form Lemma 3.10 and the comparison theorem of double BDSDE
\[
u(t, x) \leq \text{ess sup}_{v(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[u(t + \delta, X_{t+\delta}^{t,\bar{v}})].
\]

On the other hand, from Lemma 3.10, for every \( \varepsilon > 0, \) we can find an admissible control \( \bar{v}(\cdot) \in \mathcal{U} \) such that
\[
u(t + \delta, X_{t+\delta}^{t,\bar{v}}) \leq Y_{t+\delta}^{t+\delta,\bar{v}} + \varepsilon.
\]

For each \( v(\cdot) \in \mathcal{U}, \) we denote \( \bar{v}(s) = I_{\{s \leq t+\delta\}}v(s) + I_{\{s > t+\delta\}}\bar{v}(s). \) From the above inequality and the comparison theorem, we get
\[
Y_{t+\delta}^{t+\delta,\bar{v}} \geq u(t + \delta, X_{t+\delta}^{t,\bar{v}}) - \varepsilon, \quad u(t, x) \geq \text{ess sup}_{\bar{v}(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[u(t + \delta, X_{t+\delta}^{t,\bar{v}}) - \varepsilon].
\]

By Proposition 2.2, there exists a positive constant \( C_0 \) such that
\[
u(t, x) \geq \text{ess sup}_{\bar{v}(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[u(t + \delta, X_{t+\delta}^{t,\bar{v}})] - C_0\varepsilon.
\]

Therefore, letting \( \varepsilon \downarrow 0, \) we obtain
\[
u(t, x) \geq \text{ess sup}_{\bar{v}(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[u(t + \delta, X_{t+\delta}^{t,\bar{v}})].
\]

Because \( \bar{v}(\cdot) \) acts only on \([t, t + \delta]\) for \( G_{t,t+\delta}^{d,\bar{v}} \) from the definition of \( \bar{v}(\cdot) \) and the arbitrariness of \( v(\cdot), \) we know that the above inequality can be written as
\[
u(t, x) \geq \text{ess sup}_{v(\cdot) \in \mathcal{U}} G_{t,t+\delta}^{d,\bar{v}}[u(t + \delta, X_{t+\delta}^{t,\bar{v}})],
\]
which is our desired conclusion. \( \Box \)
4 Sobolev weak solutions for the HJB equations corresponding to the stochastic recursive control problem

In this section we consider the Sobolev weak solution for the SHJB equation related to the stochastic recursive optimal control problem.

We give some preliminary results of the BDSDE which are useful for the sobolev weak solutions for the recursive optimal control problem. In order to facilitate understanding and narration, we divided it into several parts.

Part I
Consider the control system defined by (4)

\[
\begin{aligned}
dx_s^{t,x,v} &= b(s, x_s^{t,x,v}, v_s) ds + \sigma(s, x_s^{t,x,v}, v_s) dW_s, \quad s \in [t, T], \\
x_t^{t,x,v} &= x.
\end{aligned}
\]

satisfying the following conditions:
(H4.1) The coefficient \(b\) is 2 times continuously differentiable in \(x\) and all their partial derivatives are uniformly bounded, \(\sigma\) is 3 times continuously differentiable in \(x\) and all their partial derivatives are uniformly bounded, and 

\[|b(t, x, v)| + |\sigma(t, x, v)| \leq K(1 + |x|),\]

where \(K\) is a constant.

And the cost function defined by the following BSDE:

\[
Y_s^{t,\zeta,v} = h(x_T^{t,\zeta,v}) + \int_T^s f(r, x_r^{t,\zeta,v}, Y_r^{t,\zeta,v}, Z_r^{t,\zeta,v}, u_r) dr + \int_T^s g(r, x_r^{t,\zeta,v}, Y_r^{t,\zeta,v}, Z_r^{t,\zeta,v}, u_r) dB_r - \int_s^T Z_r^{t,\zeta,v} dW_r,
\]

where

\[
\begin{aligned}
h &: \mathbb{R}^n \to \mathbb{R}, \\
f &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}, \\
g &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R},
\end{aligned}
\]

satisfying the conditions as same as that denoted in Chapter 3.

Obviously, under the above assumptions(H3.4)(H3.5)(H3.7)and(H4.1), for a given control \(v(\cdot) \in U\), there exists a unique solution \((Y^{t,\zeta,v}, Z^{t,\zeta,v}) \in S^2(0, T; \mathbb{R}) \times H^2(0, T; \mathbb{R}^d)\). We introduce the associated cost functional:

\[
J(t, x; v) := Y_s^{t,x,v} |_{s=t}, \quad (t, x) \in [0, T] \times \mathbb{R}^n,
\]

and define the value function of the stochastic optimal control problem

\[
u(t, x) := \text{ess sup}_{v \in U} J(t, x; v), \quad (t, x) \in [0, T] \times \mathbb{R}^n,
\]

According to the conclusion in previous chapter, we know that the celebrated dynamic programming principle still holds for this recursive stochastic optimal control problem. We therefore deduce the following HJB equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t, x) + \sup_{v \in U} \{ \mathcal{L}(t, x; v) u(t, x) + f(t, x, u(t, x), \sigma \nabla u(t, x), v) + g(t, x, u(t, x), \sigma \nabla u(t, x)) dB_t \} &= 0, \\
u(T, x) &= h(x),
\end{aligned}
\]

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where $\mathcal{L}$ is a family of second order linear partial differential operators,

$$
\mathcal{L}(t, x, \varphi) = \frac{1}{2} tr[\sigma(t, x, v_t)\sigma(t, x, v_t)^T D^2 \varphi] + \langle b(t, x, v_t), D\varphi \rangle.
$$

**Part II**

We define the weight function $\rho$ is continuous positive on $R^d$ satisfying $\int_{R^d} \rho(x) dx = 1$ and $\int_{R^d} |x|^2 \rho(x) dx < \infty$.

Denote by $L^2(R^d, \rho(x) dx)$ the weighted $L^2$-space with weight function endowed with the norm

$$
\|u\|_{L^2(R^d, \rho(x) dx)} = \left( \int_{R^d} |u(x)|^2 \rho(x) dx \right)^{\frac{1}{2}}.
$$

We set $D := \{u : R^d \rightarrow R \text{ such that } u \in L^2(R^d, \rho(x) dx) \text{ and } \frac{\partial}{\partial x_i} u \in L^2(R^d, \rho(x) dx) \}$, where $\frac{\partial}{\partial x_i}$ is derivative with respect to $x$ in the weak sense. Note that $D$ equipped with the norm

$$
\|u\|_D = \left[ \int_{R^d} |u(x)|^2 \rho(x) dx + \sum_{1 \leq i \leq d} \int_{R^d} \left| \frac{\partial u}{\partial x_i} \right|^2 \rho(x) dx \right]^{\frac{1}{2}}
$$

is a Hilbert space, which is a classical Dirichlet space. Moreover, $D$ is a subset of the Sobolev space $H_1(R^d)$.

We set $H := \{u : u \in L^2(R^d, \rho(x) dx) \text{ and } (\sigma^* \nabla u) \in L^2(R^d, \rho(x) dx) \}$ equipped with the norm

$$
\|u\|_H = \left[ \int_{R^d} |u(x)|^2 \rho(x) dx + \int_{R^d} |(\sigma^* \nabla u(x))|^2 \rho(x) dx \right]^{\frac{1}{2}}.
$$

We say $u \in L^2([0, T], H)$ if $\int_0^T \|u(t)\|_H^2 dt < \infty$.

Let $T$ be a strictly positive real number and $U$ a nonempty compact set of $R^k$.

**Part III**

Then, we introduce some equivalence norm.

The solution of SDE generates a stochastic flow, and the inverse flow is denoted by $x_{s,t}^{l,x,v}$. It is known from [9] that $x \rightarrow x_{s,t}^{l,x,v}$ is differentiable and we denote by $J(x_{s,t}^{l,x,v})$ the determinant of the Jacobian matrix of $x_{s,t}^{l,x,v}$, which is positive and $J(x_{s,t}^{l,x,v}) = 1$. For $\varphi \in C_C^\infty(R^d)$ we define a process $\varphi_t : \Omega \times [0, T] \times R^d \rightarrow R$ by $\varphi_t(s, x) = \varphi(x_{s,t}^{l,x,v}) J(x_{s,t}^{l,x,v})$. Following Kunita [13], we can define the composition of $u \in L^2(R^d)$ with the stochastic flow by $(u \circ x_{s,t}^{l,x,v}, \varphi) = (u, \varphi_t(s, \cdot))$. Indeed, by a change of variable, we have

$$
(u \circ x_{s,t}^{l,x,v}, \varphi) = \int_{R^d} u(y) \varphi(x_{s,t}^{l,x,v}) J(x_{s,t}^{l,x,v}) dy = \int_{R^d} u(x_{s,t}^{l,x,v}) \varphi(x) dx.
$$

In [11], V. Bally and A. Matoussi proved that $\varphi_t(s, x)$ is a semimartingale and admits the following lemma 4.1 and lemma 4.2.

**Lemma 4.1.** For $\varphi \in C_C^2(R^d)$, we have

$$
\varphi_t(s, x) = \varphi(x) - \sum_{j=1}^d \int_t^s \sum_{i=1}^d \frac{\partial}{\partial x_i} (\sigma_{i,j}(r, x) \varphi_t(r, x)) dW_t^j + \int_t^s L_r \varphi_t(r, x) dr,
$$

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where $L^*_t$ is the adjoint operator of $L_t$.

The next lemma, known as the norm equivalence result and proved in [1], plays an important role in the proof of the main result.

**Lemma 4.2.** Assume that (H1) holds. Then for any $v \in \mathcal{U}$ there exist two constants $c > 0$ and $C > 0$ such that for every $t \leq s \leq T$ and $\varphi \in L^1(R^d, \rho(x)dx)$

$$c \int_{R^d} |\varphi(x)|\rho(x)dx \leq \int_{R^d} E(|\varphi(x)|)\rho(x)dx \leq C \int_{R^d} |\varphi(x)|\rho(x)dx.$$ 

Moreover, for every $\psi \in L^1([0,T] \times R^d; dt \otimes \rho(x)dx)$,

$$c \int_{R^d} \int_t^T |\psi(s, x)|ds\rho(x)dx \leq \int_{R^d} \int_t^T E(|\psi(s, x)|)ds\rho(x)dx \leq C \int_{R^d} \int_t^T |\psi(s, x)|ds\rho(x)dx.$$ 

The constants $c$ and $C$ depend on $T$, on $\rho$ and on the bounds of derivatives of the $b$ and $\sigma$. The proof is similar to the proof of Proposition 5.1 in [1], hence we omit it.

Now we define the notion of a solution to the SHJB equation (17).

**Definition 4.1** We say that $V$ is a weak solution of the equation (17), if

(i) $V \in L^2([0, T]; H)$, i.e.,

$$\int_0^T \|V(t)\|^2_{H^1} dt = \int_0^T \left( \int_{R^d} |V(t, x)|^2 \rho(x)dx + \int_{R^d} |(\sigma^* \nabla V)(t, x)|^2 \rho(x)dx \right) dt < \infty.$$ 

(ii) For any nonnegative $\varphi \in C_0^1([0, T] \times R^d)$ and for any $v \in \mathcal{U}$,

$$\int_{R^d} \int_s^T (V(r, x), \partial_r \varphi(r, x))drdx + \int_{R^d} (V(s, x), \varphi(s, x))dx \geq \int_{R^d} (h(x), \varphi(T, x))dx + \int_{R^d} \int_s^T (f(r, x, V, \sigma^* \nabla V, v_r), \varphi(r, x))drdx + \int_{R^d} \int_s^T (g(r, x, \sigma^* \nabla V, \varphi(r, x))dBrdx + \int_{R^d} \int_s^T (L^*_r V(r, x), \varphi(r, x))drdx,$$

where $(L_r V(r, x), \varphi(r, x)) = \int_{R^d} (\frac{1}{2} \nabla \sigma \sigma^* \nabla \varphi + V \text{div}(b - A) \varphi)dx$ with $A_i = \frac{1}{2} \sum_{k=1}^d \frac{\partial a_{ki}}{\partial x_k}$.

(iii) For any nonnegative $\varphi \in C_0^1([0, T] \times R^d)$ and for any small $\varepsilon > 0$, there exists a control $v' \in \mathcal{U}$, such that

$$\int_{R^d} \int_s^T (V(r, x), \partial_r \varphi(r, x))drdx + \int_{R^d} (V(s, x), \varphi(s, x))dx - \varepsilon \leq \int_{R^d} (h(x), \varphi(T, x))dx + \int_{R^d} \int_s^T (f(r, x, V, \sigma^* \nabla V, v_r'), \varphi(r, x))drdx + \int_{R^d} \int_s^T (g(r, x, \sigma^* \nabla V, \varphi(r, x))dBrdx + \int_{R^d} \int_s^T (L^*_r V(r, x), \varphi(r, x))drdx.$$ 

**Lemma 4.3.** Let $(\xi, f, g)$ and $(\xi', f', g')$ be two parameters of BDSDEs, each one satisfies all the assumptions (H1), (H2) and (H3) with the exception that the Lipschitz condition could be satisfied by either $f$ or $f'$ only and suppose in addition the following

$$\xi \leq \xi', \text{ a.s.}, \quad f(t, y, z) \leq f'(t, y, z), \text{ a.s. a.e.} \quad \forall (y, z) \in R \times R^d. \quad (20)$$ 

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Let \((Y, Z)\) be a solution of the BDSDE with parameter \((\xi, f, g)\) and \((Y', Z')\) a solution of the BDSDE with parameter \((\xi', f', g')\). Then

\[
Y_t \leq Y'_t, \quad \text{a.e.} \quad \forall \ 0 \leq t \leq T. \tag{21}
\]

The proof is similar to the proof in [16].

**Lemma 4.4.** Let \((H3.4)(H3.5)(H3.7)\) and \((H4,1)\) hold, then for any \(v \in \mathcal{U}\), the value function satisfies

\[
V(s, x_s^{t,x,v}) \geq E\left\{ \int_s^{s'} f(r, x_r^{t,x,v}, y_r^{t,x,v}, z_r^{t,x,v}, v_r)dr + g(r, x_r^{t,x,v}, y_r^{t,x,v}, z_r^{t,x,v})dB_r \right\}
\]

\[
\forall t \leq s \leq s' \leq T. \tag{22}
\]

and for any small \(\varepsilon > 0\), there exists a \(v' \in \mathcal{U}\), such that

\[
V(s, x_s^{t,x,v}) - \varepsilon \leq E\left\{ \int_s^{s'} f(r, x_r^{t,x,v'}, y_r^{t,x,v'}, z_r^{t,x,v'}, v_r)dr + g(r, x_r^{t,x,v'}, y_r^{t,x,v'}, z_r^{t,x,v'}) dBr + V(s', x_{s'}^{t,x,v'}) | F_t \right\}
\]

\[
\forall t \leq s \leq s' \leq T. \tag{23}
\]

**Proof.** According to the theory of dynamic programming principle we have got above,

\[
V(s, x_s^{t,x,v}) = \operatorname{ess sup}_{v \in \mathcal{U}} G_{s,s'}^{t,x,v} \left[ V(s', x_{s'}^{t,x,v'}) \right], \quad \forall t \leq s \leq s' \leq T. \tag{24}
\]

Then we set

\[
G_{s,s'}^{t,x,v} \left[ V(s', x_{s'}^{t,x,v'}) \right] := \tilde{y}_s^{t,x,v}
\]

is the solution of following BDSDE:

\[
\tilde{y}_s^{t,x,v} = V(s', x_{s'}^{t,x,v}) + \int_s^{s'} f(r, x_r^{t,x,v}, \tilde{y}_r^{t,x,v}, \tilde{z}_r^{t,x,v}, v_r)dr + \int_s^{s'} g(r, x_r^{t,x,v}, \tilde{y}_r^{t,x,v}, \tilde{z}_r^{t,x,v})dBr - \int_s^{s'} \tilde{z}_r^{t,x,v} dW_r, \quad \text{i.e.,}
\]

\[
\tilde{y}_s^{t,x,v} = E\left\{ \int_s^{s'} f(r, x_r^{t,x,v}, \tilde{y}_r^{t,x,v}, \tilde{z}_r^{t,x,v}, v_r)dr + g(r, x_r^{t,x,v}, \tilde{y}_r^{t,x,v}, \tilde{z}_r^{t,x,v})dBr + V(s', x_{s'}^{t,x,v'}) | F_t \right\}. \tag{26}
\]

Then it is no hard to finish the proof. \(\square\)

**Lemma 4.5.** For each \(t \in [0, T]\), \(x\) and \(x' \in \mathbb{R}^n\), we have

(i) \((Y_t^n - V_t^-) \to 0 \) in \( S^2\);
(ii) \((Y_t^p - V_t^-) \to 0 \) in \( S^2\).
Proof. The proof is similar to the proof in [10]. Since $Y^n_t \geq Y^0_t$, we can replace $V_t$ by $V_t \lor Y^0_t$, so assume that $E\left(\sup_{t \leq T} V^2_t\right) < \infty$; we first want to compare a.s. $Y_t$ and $\bar{S}_t$ for all $t \in [0, T]$, while we do not know yet that $Y$ is a.s. continuous. From the comparison theorem for BDSDE’s, we have that a.s. $Y^n_t \geq \bar{Y}^n_t$, $0 \leq t \leq T$ $n \in N$, where $\{\bar{Y}^n_t, \bar{Z}^n_t; 0 \leq t \leq T\}$ is the unique solution of the BDSDE:

$$\bar{Y}^n_t = \xi + \int_t^T f(s, X_s, Y^n_s, Z^n_s, V_s) ds + n \int_t^T (V_t - \bar{Y}^n_s) ds + \int_t^T g(s, X_s, Y^n_s, Z^n_s) dB_s - \int_t^T \bar{Z}^n_s dW_s.$$  

(28)

Let $\nu$ be a stopping time such that $0 \leq \nu \leq T$. Then

$$\bar{Y}^n_t = E^{\mathcal{F}_\nu}[e^{-n(T-\nu)} \xi + \int_\nu^T e^{-n(s-\nu)} f(s, X_s, Y^n_s, Z^n_s, V_s) ds + n \int_\nu^T e^{-n(s-\nu)} V_s ds] + \int_\nu^T e^{-n(s-\nu)} g(s, X_s Y^n_s, Z^n_s) dB_s.$$  

(29)

It is easily seen that

$$e^{-n(T-\nu)} \xi + n \int_\nu^T e^{-n(s-\nu)} V_s ds \to \xi 1_{\nu=T} + V_\nu 1_{\nu<T},$$

a.s. and in $L^2$, and the conditional expectation converges also in $L^2$. Moreover,

$$\left| \int_\nu^T e^{-n(s-\nu)} f(s, Y^n_s, Z^n_s) ds \right| \leq \frac{1}{\sqrt{2n}} \left( \int_0^T f^2(s, Y^n_s, Z^n_s) ds \right)^{\frac{1}{2}},$$

hence $E^{\mathcal{F}_\nu} \int_\nu^T e^{-n(s-\nu)} f(s, Y^n_s, Z^n_s) ds \to 0$ in $L^2$, as $n \to \infty$ and

$$E \left( \int_\nu^T g(s, Y^n_s, Z^n_s) dB_s \right)^2 \leq \frac{c}{4n} E \int_0^T e^{-2n(s-\nu)} g^2(s, Y^n_s, Z^n_s) ds \leq \frac{c}{4n} E \int_0^T g^4(s, Y^n_s, Z^n_s) ds \to 0.$$

Consequently, $\bar{Y}^n_t \to \xi 1_{\nu=T} + S_\nu 1_{\nu<T}$ in mean square, and $Y_\nu \geq V_\nu$ a.s. From this and the section theorem in Dellacherie and Meyer [15], it follows that a.s.

$$Y^n_t \geq V_t, \quad 0 \leq t \leq T.$$

Hence $(Y^n_t - V_t)^- \leq 0, 0 \leq t \leq T$, and from Dini’s theorem the convergence is uniform in $t$. Since $(Y^n_t - V_t)^- \leq (V_t - Y^0_t)^+ \leq |V_t| + |Y^0_t|$, we have

$$\lim_{n \to +\infty} E \left( \sup_{0 \leq t \leq T} |Y^n_t - V_t|^2 \right) = 0.$$

by the dominated convergence theorem. 

\[ \square \]
Before lemma 4.6, we now introduce the BDSDE with increasing process:

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + K_T - K_t + \int_t^T g(s, Y_s, Z_s) \, dB_s - \int_t^T Z_s \, dW_s, 0 \leq t \leq T. \]  

(30)

The solution of the equation is triple \((Y, Z, K)\) of \(\mathcal{F}_t\) measurable and take value in \((R, R^d, R_+)\) and satisfying

\((H4.2)\) \(Z \in \mathcal{H}^2\).

\((H4.3)\) \(Y \in S^2\), and \(K_T \in \mathcal{L}^2\).

\((H4.4)\) \(K_t\) is a continuous and increasing process, \(K_0 = 0\) and \(\int_0^T (Y_t - V_t) \, dK_t = 0\).

**Lemma 4.6.** We assume \((H3.4)/(H3.5)/(H3.7)/(H4.1)-(H4.4)\), then \(V(s, x^{l,x,v}_t)\) is a \(g\)-supersolution and \(E|V(s, x^{l,x,v}_t)|^2 < \infty\). Moreover there exists a unique increasing process \((K^{l,x,v}_T)\) with \(K^{l,x,v}_0 = 0\) and \(E[(K^{l,x,v}_T)^2] < \infty\) such that \(V(s, x^{l,x,v}_t)\) coincides with the unique solution \(y^{l,x,v}_t\) of the BSDE:

\[ y^{l,x,v}_t = V(T, x^{l,x,v}_T) + \int_t^T f(r, x^{l,x,v}_r, y^{l,x,v}_r, Z^{l,x,v}_r, v_r) \, dr + K^{l,x,v}_T - K^{l,x,v}_t \]

\[ + \int_t^T g(r, x^{l,x,v}_r, y^{l,x,v}_r, Z^{l,x,v}_r) \, dB_r - \int_t^T Z^{l,x,v}_r \, dW_r. \]  

(31)

where \(Z^{l,x,v}_r = \sigma^* \nabla V(r, x^{l,x,v}_r)\) in the sense of Definition 4.1.

**Proof.** Since the solution of the BDSDE is no longer a super-martingale, the method of proof in Lemma 4.1[11] will fail in our situation. The ideas of proof comes from the the properties of BDSDE and limitation theory. According to the penalization method and the comparasion theorem

\[ f_n(s, x, y, z, v) = f(s, x, y, z, v_s) + n(y - V_s)^+. \]  

(32)

For each \(n \in \mathcal{N}\), we denote \((Y^n, Z^n)\) the unique pair of \(\mathcal{F}_t\) measureable process with valued in \(R \times R^d\) is the solution of

\[ Y^n_t = V(T, X_T) + \int_t^T f(s, X, Y^n_s, Z^n_s, V_s) \, ds + n \int_t^T (Y^n_s - V_s)^- \, ds \]

\[ + \int_t^T g(s, X, Y^n_s, Z^n_s) \, dB_s - \int_t^T Z^n_s \, dW_s. \]  

(33)

We denote

\[ K^n_t = \int_0^t (Y^n_s - V_s)^- \, ds. \]

First we prove \((Y, Z)\) is the limit of \((Y^n, Z^n)\). We know \(f_n(t, y, z) \leq f_{n+1}(t, y, z)\), from comparision theorem, \(Y^n_t \leq Y^{n+1}_t, 0 \leq t \leq T\). Therefore

\[ Y^n_t \uparrow Y_t, \quad 0 \leq t \leq T, \quad \text{a.e.} \]  

(34)

Moreover \(y^{l,x,v}_t\) is bounded by \(V(t, x^{l,x,v}_t)\) and according to the result from [10]

\[ E \left( \sup_{0 \leq t \leq T} |Y^n_t|^2 \right) + E \int_t^T |Z^n_s|^2 \, ds + E \left[ (K^n_T)^2 \right] \leq c, \quad n \in \mathcal{N}. \]  

(35)
It follows from the Fatou lemma that $E \left( \sup_{0 \leq t \leq T} |Y_t|^2 \right) \leq c$, then by the dominated convergence,

$$E \int_0^T (Y_t - Y_n^t)^2 \, dt \to 0, \quad \text{as} \quad n \to \infty. \tag{36}$$

Next, we desire to prove $E \int_0^T (Z_t - Z_n^t)^2 \, dt \to 0, \quad \text{as} \quad n \to \infty$. Applying Itô’s formula to the process $|Y^n_t - Y^p^n_t|^2$.

$$|Y^n_t - Y^p_t|^2 + \int_t^T |Z^n_s - Z^p_s|^2 \, ds$$

$$= 2 \int_t^T \left[ f \left( s, X_s, Y^n_s, Z^n_s, V_s \right) - f \left( s, X_s, Y^p_s, Z^p_s, V_s \right) \right] (Y^n_s - Y^p_s) \, ds$$

$$+ \int_t^T \left| g \left( s, X_s, Y^n_s, Z^n_s \right) - g \left( s, X_s, Y^p_s, Z^p_s \right) \right|^2 \, ds$$

$$+ 2 \int_t^T \left| g \left( s, X_s, Y^n_s, Z^n_s \right) - g \left( s, X_s, Y^p_s, Z^p_s \right) \right| (Y^n_s - Y^p_s) \, dB_s - 2 \int_t^T (Y^n_s - Y^p_s) (Z^n_s - Z^p_s) \, dW_s$$

$$+ \int_t^T (Y^n_s - Y^p_s) \, d(K^n_s - K^p_s).$$

$$E \left( |Y^n_t - Y^p_t|^2 \right) + E \int_t^T |Z^n_s - Z^p_s|^2 \, ds$$

$$\leq 2KE \int_t^T \left( |Y^n_s - Y^p_s|^2 + |Y^n_s - Y^p_s| \cdot |Z^n_s - Z^p_s| \right) \, ds + KE \int_t^T |Y^n_s - Y^p_s|^2 \, ds$$

$$+ \alpha E \int_t^T |Z^n_s - Z^p_s|^2 \, ds + 2E \int_t^T (Y^n_s - V_s)^- \, dK^n_s + 2E \int_t^T (Y^p_s - V_s)^- \, dK^n_s$$

$$\leq \left( 3K + K^2 \frac{2}{1 - \alpha} \right) E \int_t^T |Y^n_s - Y^p_s|^2 \, ds + \frac{1 + \alpha}{2} E \int_t^T |Z^n_s - Z^p_s|^2 \, ds$$

$$+ 2E \int_t^T (Y^n_s - V_s)^- \, dK^n_s + 2E \int_t^T (Y^p_s - V_s)^- \, dK^n_s.$$
Now we begin to prove $Y$ is continuous.

\[
|Y^*_t - Y^*_n|^2 + \int_t^T |Z^*_s - Z^n_s|^2 \, ds
\]

\[
= 2 \int_t^T [f(s, X_s, Y^n_s, Z^n_s, V_s) - f(s, X_s, Y^*_s, Z^*_s, V_s)] (Y^*_n - Y^*_s) \, ds
\]

\[
+ \int_t^T |g(s, X_s, Y^n_s, Z^n_s) - g(s, X_s, Y^*_s, Z^*_s)|^2 \, ds
\]

\[
+ 2 \int_t^T [g(s, X_s, Y^n_s, Z^n_s) - g(s, X_s, Y^*_s, Z^*_s)] (Y^*_n - Y^*_s) \, dB_s - 2 \int_t^T (Y^*_n - Y^*_s) (Z^n_s - Z^*_s) \, dW_s
\]

\[
+ 2 \int_t^T (Y^*_n - Y^*_s) \, d(K^*_s - K^*_n).
\]

From Burkholder-Davis-Gundy inequality,

\[
E \sup_{0 \leq t \leq T} |Y^*_t^n - Y^*_t|^2 \leq \frac{1}{2} E \sup_{0 \leq t \leq T} |Y^*_t^n - Y^*_t|^2 + cE \int_0^T \left(||Y^*_t^n - Y^*_t|| + ||Z^n_s - Z^*_s||\right) ds
\]

\[
+ \left(E \sup_{0 \leq t \leq T} \left|(Y^*_t^n - V_t)^{-1}\right|^2 \cdot E |K^*_t|^2\right)^{\frac{1}{2}} + \left(E \sup_{0 \leq t \leq T} \left|(Y^*_t^n - V_t)^{-2}\right| \cdot E |K^*_t|^2\right)^{\frac{1}{2}}.
\]

We get $E \left(\sup_{0 \leq t \leq T} |Y^*_t^n - Y^*_t|^2\right) \to 0$ as $n, p \to \infty$. $Y^n$ convergence uniformly in $t$ to $Y$, a.s. hence $Y$ is continuous.

In addition, we have denoted that $K^*_n$ is a increasing process with $E \left((K^*_n)^2\right) \leq C$, it is obvious that $K_T < \infty$, a.s.

\[
E \left(\sup_{0 \leq t \leq T} |K^*_t^n - K^*_t|^2\right) \leq c \left(E \sup_{0 \leq t \leq T} |Y^*_t^n - Y^*_t|^2 + E|Y_0^n - Y_0^p|^2\right)
\]

\[
+ E \int_0^T (f(s, X_s, Y^n_s, Z^n_s, V_s) - f(s, X_s, Y^*_s, Z^*_s, V_s))^2 ds
\]

\[
+ E \left(\sup_{0 \leq t \leq T} |\int_0^t g(s, Y^n_s, Z^n_s) - g(s, Y^*_s, Z^*_s) dB_s|\right)
\]

\[
+ E \left(\sup_{0 \leq t \leq T} |\int_0^t (Z^n_s - Z^*_s) dW_s|\right).
\]
From the Lipschitz conditions and the Burkholder-Davis-Gundy inequality, we have

$$E \left( \sup_{0 \leq t \leq T} (K_t^n - K_t^p)^2 \right) \to 0, \quad \text{as} \quad n, p \to \infty. \quad (38)$$

It remains to check that \( \int_0^T (Y_t - V_t) \, dK_t = 0 \).

According to (36) and (38), we have

$$\int_0^T (Y_s^n - V_s) \, dK_s^n \to \int_0^T (Y_s - V_s) \, dK_s$$

as \( n \to \infty \). Moreover \( Y_t \leq V_t \), a.s.

we obtain

$$\int_0^T (Y_s^n - V_s) \, dK_s^n = -n \int_0^T \left| (Y_s^n - V_s) \right|^2 \, ds \leq 0, \quad \text{a.s.}$$

Finally, we take the limit of both sides of the equation of (33), then we have equation(31). The proof of the uniqueness are derived from the proof of the Proposition 1.6 in the [12].

If there exist another solution \( K_t^{kr,x,v} \) and \( Z_t^{kr,x,v} \) satisfying equation(33), then we apply Itô formula to \((y_t - y_t)^2 \equiv 0\) on the \([0, T]\) and take expectation

$$E \int_t^T \left| Z_s^{t,x,v} - Z_s^{t,x,v} \right|^2 \, ds + E \left[ (K_T^{t,x,v} - K_T^{t,x,v}) - (K_T^{t,x,v} - K_T^{t,x,v}) \right]^2 = 0.$$

therefore \( Z_t^{t,x,v} \equiv Z_t^{r,x,v}, K_t^{t,x,v} \equiv K_t^{r,x,v} \) for any \( t \in [0, T] \).

We device that \( E \int_0^T (Z_t^n - Z_t^p)^2 \, dt \to 0 \), by the lemma 4.1 in [13], we know that

$$Z_t^{t,x,v} = \sigma^* \nabla \bar{y}_t^{t,x,v} = \sigma^* \nabla V \left( r, x_t^{t,x,v} \right).$$

Then it remain to prove that \( \bar{y}_t^{t,x,v} = V(r, x_t^{t,x,v}) \). From the BDSDE(13), we have

$$K_t^{t,x,v,n} - K_t^{t,x,v,n} = y_t^{t,x,v,n} - V(T, x_T^{t,x,v}) - \int_t^T f(r, x_r^{t,x,v}, y_r^{t,x,v,n}, z_r^{t,x,v,n}, v_r) \, dr$$

$$- \int_t^T g(r, x_r^{t,x,v}, y_r^{t,x,v,n}, z_r^{t,x,v,n}) \, dBr + \int_t^T Z_r^{t,x,v,n} \, dW_r$$

$$\leq |y_t^{t,x,v,n}| + |V(T, x_T^{t,x,v})| + \int_t^T f(r, 0, 0, 0, v_r) \, dr$$

$$+ \int_t^T (K|x_r^{t,x,v}| + K|y_r^{t,x,v,n}| + K|Z_r^{t,x,v,n}|) \, dBr + \int_t^T Z_r^{t,x,v,n} \, dW_r$$

$$\leq |V(t, x)| + |V(T, x_T^{t,x,v})| + \int_t^T f(r, 0, 0, 0, v_r) \, dr$$

$$+ \int_t^T (K|x_r^{t,x,v}| + K|y_r^{t,x,v,n}| + K|Z_r^{t,x,v,n}|) \, dBr$$

\[ 20 \]
on the other hand, we use Itô’s formula to $|y_r^{t,x,v}|^2$.

$$
|y_r^{t,x,v}|^2 + E \int_t^T |Z_r^{t,x,v}|^2 \, dr
=E|V(T,x_T^{t,x,v})|^2 + 2E \int_t^T y_r^{t,x,v} f(r,x_r^{t,x,v},y_r^{t,x,v},Z_r^{t,x,v},v_r) \, dr
+E \int_t^T |g(r,x_r^{t,x,v},y_r^{t,x,v},Z_r^{t,x,v})|^2 \, dr + 2E \int_t^T y_r^{t,x,v} dK_r^{t,x,v}
\leq E|V(T,x_T^{t,x,v})|^2
+ 2E \int_t^T |y_r^{t,x,v}|(|f(r,0,0,0,v_r)| + K|x_r^{t,x,v}| + K|y_r^{t,x,v}| + K|Z_r^{t,x,v}|) \, dr
+E \int_t^T (|g(r,0,0,0)| + K|x_r^{t,x,v}| + K|y_r^{t,x,v}| + K|Z_r^{t,x,v}|)^2 \, dr
+ 2E \int_t^T y_r^{t,x,v} dK_r^{t,x,v}
\leq E|V(T,x_T^{t,x,v})|^2 + \int_t^T |f(r,0,0,0,v_r)|^2 \, dr + E \int_t^T |y_r^{t,x,v}|^2 \, dr
+E \int_t^T K^2|y_r^{t,x,v}|^2 + |x_r^{t,x,v}|^2 \, dr + E \int_t^T (2K^2 + 2K)|y_r^{t,x,v}|^2 + \frac{1}{2} |Z_r^{t,x,v}|^2 \, dr
+E \int_t^T |g(r,0,0,0)|^2 + 4K^2|x_r^{t,x,v}|^2 + 4K^2|y_r^{t,x,v}|^2 + 4K^2|Z_r^{t,x,v}|^2 \, dr
+ 2E[K_r^{t,x,v} \sup_{t \leq s \leq T} |y_s^{t,x,v}|]
$$

because for any $v_r$, $E \int_t^T |f(r,0,0,0,v_r)|^2 \, dr \leq M$. We observe that $y_r^{t,x,v}$ is dominated by $|y_r^{t,x,v}| + |V(r,x_r^{t,x,v})|$. From equation (33) we have

$$
E|K_r^{t,x,v}|^2 \leq 13|V(t,x)|^2 + 13E|g(x_T^{t,x,v})|^2 + 13E \int_t^T |f(r,0,0,0,v_r)|^2 \, dr
+ 13E \int_t^T (K^2|x_r^{t,x,v}|^2 + K^2|y_r^{t,x,v}|^2 + K^2|V(r,x_r^{t,x,v})|^2 + K^2|Z_r^{t,x,v}|^2) \, dr
+ 13 \int_t^T |g(r,0,0,0)|^2 \, dr + 13E \int_t^T |Z_r^{t,x,v}|^2 \, dr.
$$

Thus we can define a $C_3(t,T,x,v)$, independent of $n$, such that

$$
E|K_r^{t,x,v}|^2 \leq C_3(t,T,x,v) + 8(K^2 + 1)E \int_t^T |Z_r^{t,x,v}|^2 \, dr.
$$

On the other hand, we use Itô’s formula to $|y_r^{t,x,v}|^2$.
\[ \begin{align*}
\leq E|V(T, x_T^{t,x,v})|^2 + E \int_t^T |f(r, 0, 0, 0, v_r)|^2 dr + (4K^2 + 1)E \int_t^T |x_r^{t,x,v}|^2 dr \\
+ (7K^2 + 2K + 1)E \int_t^T [|y_r^{t,x,v}]^2 + |V(r, x_r^{t,x,v})|^2 | dr + (4K^2 + \frac{1}{2})E \int_t^T |Z_r^{t,x,v}|^2 dr \\
+ 4E \int_t^T |g(r, 0, 0, 0)|^2 dr + \frac{1}{32(K^2 + 1)}E|K_r^{t,x,v}|^2 + 64(K^2 + 1)E \sup_{t \leq s \leq T} [|y_s^{t,x,v}]^2 + |V(s, x_s^{t,x,v})|^2 |.
\end{align*} \]

Then we can define a \( C_4(t, T, x, v) \) satifying

\[ E \int_t^T |Z_r^{t,x,v}|^2 dr \leq C_4(t, T, x, v) + \frac{1}{16(K^2 + 1)}E|K_r^{t,x,v}|^2. \]

Then we have

\[ E \left| K_r^{t,x,v} \right|^2 \leq 2C_3(t, T, x, v) + 16(K^2 + 1)C_4(t, T, x, v), \]

it follows that

\[ n^2 \int_t^T \left( V(s, x_s^{t,x,v}) - y_s^{t,x,v} \right)^2 ds \leq 2C_3(t, T, x, v) + 16(K^2 + 1)C_4(t, T, x, v). \]

Let \( n \to \infty \), we get \( y_r^{t,x,v} = V(r, x_r^{t,x,v}) \).

On the other hand, for any small \( \varepsilon > 0 \), there exists a control \( \nu' \in \mathcal{U}, V(r, x_r^{t,x,v'}) \) satifying

\[ V(s, x_s^{t,x,v'}) \leq Y_s^{t,s,x,v'} + \varepsilon. \]

\[ \Box \]

**Lemma 4.7.** We assume (H3.4)-(H3.5)(H3.7)(H4.1)-(H4.4), then \( V(s, x_s^{t,x,v'}) \) is a q- supersolution. Same as the proof of lemma 3.6, there exists a unique increasing process \( (A_r^{t,x,v'}) \) with \( A_r^{t,x,v'} = 0 \) and \( E|A_r^{t,x,v'}|^2 < \infty \) such that \( V(s, x_s^{t,x,v'}) \) coincides with the unique solution \( y_r^{t,x,v} \) of the BSDE:

\[ y_r^{t,x,v'} = V(T, x_T^{t,x,v'}) + \varepsilon + \int_t^T f(r, x_r^{t,x,v'}, y_r^{t,x,v'}, Z_r^{t,x,v'}, \nu_r') dr \\
- (K_r^{t,x,v'} - K_r^{t,x,v'}) + \int_t^T g(r, x_r^{t,x,v'}, y_r^{t,x,v'}, Z_r^{t,x,v'}) dB_r - \int_t^T Z_r^{t,x,v'} dW_r. \]

where \( Z_r^{t,x,v'} = \sigma^* \nabla V(r, x_r^{t,x,v'}) \) in the sense of Definition 4.1.

The proof of Lemma 4.7. is similar to that of Lemma 4.6.

**Theorem 4.1.** Under the assumption (H3.4)-(H3.5)(H3.7)(H4.1)-(H4.4), the value function \( V(t, x) \) defined in \([16]\) is the unique Sobolev solution of the PDE \([17]\).

**Proof.** Existence: In the stochastic recursive optimal control problem, the value function \( V(t, x) \) defined by \([16]\) satisfies the Bellman’s dynamic programming principle. By Lemma 4.6 and
Lemma 4.7 we know that, for any \( v \in U \), there have a unique increasing process \( A_t^{t,x,v} \), \( V(s,x_t^{t,x,v}) \) satisfy the following BDSDE:

\[
V(s,x_s^{t,x,v}) = V(T,x_T^{t,x,v}) + \int_s^T f(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}),v_r)dr + (K_T^{t,x,v} - K_s^{t,x,v})
+ \int_s^T g(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}))dBr - \int_s^T \sigma^*\nabla V(r,x_r^{t,x,v})dW_r. \tag{41}
\]

So it follows easily that

\[
V(s,x_s^{t,x,v}) \geq V(T,x_T^{t,x,v}) + \int_s^T f(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}),v_r)dr
+ \int_s^T g(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}))dBr - \int_s^T \sigma^*\nabla V(r,x_r^{t,x,v})dW_r. \tag{42}
\]

On the other hands, for any small \( \varepsilon > 0 \) there exists a control \( v' \in U \), such that \( V(s,x_s^{t,x,v'}) \) satisfies the following BDSDE:

\[
V(s,x_s^{t,x,v'}) - \varepsilon = V(T,x_T^{t,x,v'}) + \int_s^T f(r,x_r^{t,x,v'},V(r,x_r^{t,x,v'}),\sigma^*\nabla V(r,x_r^{t,x,v'}),v_r')dr
- (K_T^{t,x,v'} - K_s^{t,x,v'}) + \int_s^T g(r,x_r^{t,x,v'},V(r,x_r^{t,x,v'}),\sigma^*\nabla V(r,x_r^{t,x,v'}))dBr
- \int_s^T \sigma^*\nabla V(r,x_r^{t,x,v'})dW_r. \tag{43}
\]

Then we have

\[
V(s,x_s^{t,x,v'}) - \varepsilon \leq V(T,x_T^{t,x,v'}) + \int_s^T f(r,x_r^{t,x,v'},V(r,x_r^{t,x,v'}),\sigma^*\nabla V(r,x_r^{t,x,v'}),v_r')dr
+ \int_s^T g(r,x_r^{t,x,v'},V(r,x_r^{t,x,v'}),\sigma^*\nabla V(r,x_r^{t,x,v'}))dBr - \int_s^T \sigma^*\nabla V(r,x_r^{t,x,v'})dW_r. \tag{44}
\]

We can deduce by the equivalence of norm result (Lemma 4.2) that \( V \in L^2([t,T],H) \). Indeed, in the stochastic recursive optimal control problem, the cost function can be regarded as a solution of BSDE:

\[
Y_t^{t,x,v} = h(x_T^{t,x,v}) + \int_s^T f(r,x_r^{t,x,v},Y_r^{t,x,v},Z_r^{t,x,v},v_r)dr + \int_s^T g(r,x_r^{t,x,v},y_r^{t,x,v},z_r^{t,x,v})dBr
- \int_s^T Z_r^{t,x,v}dW_r.
\]

By usual estimates of BSDEs, (H3.4)(H3.5)(H3.7)(H4.1)-(H4.4) we know that

\[
\int_{\mathbb{R}^d} E(|Y_t^{t,x,v}|^2) + \int_t^T |Z_r^{t,x,v}|^2 dr \rho(x)dx
\leq K \int_{\mathbb{R}^d} E|h(x_T^{t,x,v})|^2 \rho(x)dx + K \int_{\mathbb{R}^d} \int_t^T E|f(r,x_r^{t,x,v},0,0,v_r)|^2dr \rho(x)dx
\leq KC \int_{\mathbb{R}^d} |h(x)|^2 \rho(x)dx + KC \int_{\mathbb{R}^d} \int_t^T |f(r,x,0,0,v_r)|^2 dr \rho(x)dx
\]

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By virtue of the same techniques, because (44) holds, so for any nonnegative \( \varphi \in C_c^\infty (\mathbb{R}^d) \), we have

\[
\int_{\mathbb{R}^d} (V(s, x_s^{t, x, v}), \varphi(x))dx \\
\geq \int_{\mathbb{R}^d} (h(x), \varphi(T, x))dx + \int_{\mathbb{R}^d} \int_s^T \left( f(r, x_s^{t, x, v}, V(r, x_s^{t, x, v}, \sigma \nabla V(r, x_s^{t, x, v}, v_r), \varphi(x))drdx \\
+ \int_{\mathbb{R}^d} \int_s^T \left( g(r, x, V, \sigma \nabla V(r, x_s^{t, x, v})), \varphi(x)drdx - \int_{\mathbb{R}^d} \int_s^T \left( \sigma \nabla V(r, x_s^{t, x, v}), \varphi(x) )dW_r, dx. \right) (46)
\]

It turns out that

\[
\int_{\mathbb{R}^d} (V(s, x), \varphi(s, x))dx \\
\geq \int_{\mathbb{R}^d} (h(x), \varphi(T, x))dx + \int_{\mathbb{R}^d} \int_s^T \left( f(r, x, V(r, x), \sigma \nabla V(r, x), v_r), \varphi(x))drdx \\
+ \int_{\mathbb{R}^d} \int_s^T \left( g(r, x, V, \sigma \nabla V(r, x)), \varphi(r, x)drdx - \int_{\mathbb{R}^d} \int_s^T \left( \sigma \nabla V(r, x), \varphi(r, x) )dW_r, dx. \right) (47)
\]

Furthermore, using Lemma 1.1 we have that

\[
-\int_{\mathbb{R}^d} \int_s^T \sigma \nabla V(r, x) \varphi(r, x) dW_r, dx \\
= -\int_{\mathbb{R}^d} \sum_{j=1}^d \int_s^T \sum_{i=1}^d \sigma_{i,j}(r, x) \frac{\partial V}{\partial x_i} (r, x), \varphi(x)) dW_r^j \\
= \int_{\mathbb{R}^d} \int_s^T (\mathcal{L}^c V(r, x), \varphi(r, x))dr - \int_{\mathbb{R}^d} \int_s^T (V(r, x), \partial_r \varphi(r, x))drdx. \quad (48)
\]

Taking (48) into (47), we have that

\[
\int_{\mathbb{R}^d} \int_s^T (V(r, x), \partial_r \varphi(r, x))drdx + \int_{\mathbb{R}^d} (V(s, x), \varphi(s, x))dx \\
\geq \int_{\mathbb{R}^d} (h(x), \varphi(T, x))dx + \int_{\mathbb{R}^d} \int_s^T \left( f(r, x, V(r, x), \sigma \nabla V(r, x), v_r), \varphi(x))drdx \\
+ \int_{\mathbb{R}^d} \int_s^T \left( g(r, x, V, \sigma \nabla V(r, x)), \varphi(r, x)drdx - \int_{\mathbb{R}^d} \int_s^T \left( \mathcal{L}^c V(r, x), \varphi(r, x) )dW_r, dx. \right) (49)
\]

By virtue of the same techniques, because (44) holds, so for any nonnegative \( \varphi \in C_c^\infty (\mathbb{R}^d) \), we take \( \varepsilon = \frac{\varepsilon'}{\int_{\mathbb{R}^d} \varphi(x)dx} \), then

\[
\int_{\mathbb{R}^d} (V(s, x_s^{t, x, v'}), \varphi(x))dx - \varepsilon'
\]
Uniqueness: Let \( V \) be another solution of the PDE \( (17) \). By Definition 4.1, one gets that for any \( v \in \mathcal{U} \),

\[
\int_{R^d} \int_s^T (\nabla_r (r, x, v), \partial_r \varphi (r, x)) dr dx + \int_{R^d} (\nabla (s, x), \varphi (s, x)) dx
\]

\[
\geq \int_{R^d} (h(x), \varphi(T, x)) dx + \int_{R^d} \int_s^T (f(r, x, V(r, x), \varphi(r, x))) dr dx
\]

\[
+ \int_{R^d} \int_s^T (g(r, x, \nabla V(r, x), \varphi(r, x))) dr dx + \int_{R^d} \int_s^T (\mathcal{L}_y^\sigma V(r, x), \varphi(r, x)) dr dx.
\]

Taking (48) into (51), we obtain

\[
\int_{R^d} \int_s^T (\nabla_r (r, x, v), \partial_r \varphi (r, x)) dr dx + \int_{R^d} (\nabla (s, x), \varphi (s, x)) dx - \varepsilon'
\]

\[
\leq \int_{R^d} (h(x), \varphi(T, x)) dx + \int_{R^d} \int_s^T (f(r, x, V(r, x), \varphi(r, x))) dr dx
\]

\[
+ \int_{R^d} \int_s^T (g(r, x, \nabla V(r, x), \varphi(r, x))) dr dx + \int_{R^d} \int_s^T (\mathcal{L}_y^\sigma V(r, x), \varphi(r, x)) dr dx.
\]

(52)

By Lemma 4.5 in \( [13] \), we have

\[
\int_{R^d} \int_s^T (\nabla_r (r, x, v), \partial_r \varphi (r, x)) dr dx
\]

\[
= \int_{R^d} \sum_{j=1}^d \int_s^T (\nabla_r (r, x, \varphi(r, x), \partial_{x_j} \varphi(r, x))) dr dx + \int_{R^d} \int_s^T (\mathcal{L}_y^\sigma V(r, x), \varphi(r, x)) dr dx
\]

\[
= \int_{R^d} \int_s^T (\nabla_r (r, x, \varphi(r, x))) dr dx + \int_{R^d} \int_s^T (\mathcal{L}_y^\sigma V(r, x), \varphi(r, x)) dr dx.
\]

(54)
Taking (54) into (53), we get
\[
\int_{R^d} (\nabla (s, x), \varphi (s, x)) dx + \int_{R^d} \int_{s}^{T} (\sigma^* \nabla V)(r, x) \varphi (r, x) dW_r dx \\
+ \int_{R^d} \int_{s}^{T} (L_\nu^r \nabla (r, x), \varphi (r, x)) dr dx \\
\geq \int_{R^d} (h(x), \varphi (T, x)) dx + \int_{R^d} \int_{s}^{T} (f (r, x, \nabla (r, x), \sigma^* \nabla V(r, x), v_r), \varphi (r, x)) dr dx \\
+ \int_{R^d} \int_{s}^{T} (g (r, x, \nabla (r, x), \sigma^* \nabla V(r, x)), \varphi (r, x)) dB r dx \\
- \int_{R^d} \int_{s}^{T} (\sigma^* \nabla V)(r, x) \varphi (r, x) dW_r dx.
\] 
So
\[
\int_{R^d} (\nabla (s, x), \varphi (s, x)) dx \\
\geq \int_{R^d} (h(x), \varphi (T, x)) dx + \int_{R^d} \int_{s}^{T} (f (r, x, \nabla (r, x), \sigma^* \nabla V(r, x), v_r), \varphi (r, x)) dr dx \\
+ \int_{R^d} \int_{s}^{T} (g (r, x, \nabla (r, x), \sigma^* \nabla V(r, x)), \varphi (r, x)) dB r dx \\
- \int_{R^d} \int_{s}^{T} (\sigma^* \nabla V)(r, x) \varphi (r, x) dW_r dx.
\] 
By Definition 4.1. we also have that, for any small \( \varepsilon > 0 \), there exists a control \( v' \in \mathcal{U} \), we have
\[
\int_{R^d} \int_{s}^{T} (\nabla (r, x), \partial_v \varphi (r, x)) dr dx + \int_{R^d} (\nabla (s, x), \varphi (s, x)) dx - \varepsilon \\
\leq \int_{R^d} (h(x), \varphi (T, x)) dx + \int_{R^d} \int_{s}^{T} (f (r, x, \nabla (r, x), \sigma^* \nabla V(r, x), v'_r), \varphi (r, x)) dr dx \\
+ \int_{R^d} \int_{s}^{T} (g (r, x, \nabla (r, x), \sigma^* \nabla V(r, x)), \varphi (r, x)) dB r dx \\
+ \int_{R^d} \int_{s}^{T} (L_\nu^r \nabla (r, x), \varphi (r, x)) dr dx.
\] 
Taking (54) into (56), we have
\[
\int_{R^d} (\nabla (s, x), \varphi (s, x)) dx - \varepsilon \\
\leq \int_{R^d} (h(x), \varphi (T, x)) dx + \int_{R^d} \int_{s}^{T} (f (r, x, \nabla (r, x), \sigma^* \nabla V(r, x), v'_r), \varphi (r, x)) dr dx \\
+ \int_{R^d} \int_{s}^{T} (g (r, x, \nabla (r, x), \sigma^* \nabla V(r, x)), \varphi (r, x)) dB r dx \\
- \int_{R^d} \int_{s}^{T} (\sigma^* \nabla V)(r, x) \varphi (r, x) dW_r dx.
\] 
Let us make the change of variable \( y = \frac{t_r x_v}{x_v} \) in each term of (55), then
\[
\int_{R^d} (\nabla (s, x), \varphi (s, x)) dx = \int_{R^d} (\nabla (s, \frac{t_r x_v}{x_v}), \varphi (y)) dy,
\]
\[
\int_{\mathbb{R}^d} (h(x), \varphi(T, x)) \, dx = \int_{\mathbb{R}^d} (h(x_t^{y, v}), \varphi(y)) \, dy, \quad (59)
\]

\[
\int_{\mathbb{R}^d} \int_s^T \left( f(r, x, \nabla(r, x), \sigma^* \nabla \nabla(r, x), v_r) \right) \, dr \, dx = \\
\int_{\mathbb{R}^d} \int_s^T \left( f(r, x_t^{y, v}, \overline{\nabla}(r, x_t^{y, v}), (\sigma^* \nabla \nabla)(r, x_t^{y, v}), v_r) \right) \, dr \, dy
\]

\[
\int_{\mathbb{R}^d} \int_s^T \left( (\sigma^* \nabla \nabla)(r, x_t^{y, v}), \varphi(x) \right) \, dr \, dy + \\
\int_{\mathbb{R}^d} \int_s^T g(r, x_t^{y, v}, \overline{\nabla}(r, x_t^{y, v}), \sigma^* \nabla \nabla(r, x_t^{y, v})), \varphi(y)) \, dy \, dr
\]

So \((55)\) becomes

\[
\int_{\mathbb{R}^d} \overline{\nabla}(s, x_s^{y, v}) \varphi(y) \, dy
\]

\[
\geq \int_{\mathbb{R}^d} \int_s^T h(x_t^{y, v}) \varphi(y) \, dy \, dr + \\
\int_{\mathbb{R}^d} \int_s^T \left( f(r, x_t^{y, v}, \overline{\nabla}(r, x_t^{y, v}), (\sigma^* \nabla \nabla)(r, x_t^{y, v}), v_r) \right) \, dr \, dy
\]

Let

\[
\overline{y}_s^{y, v} = h(x_t^{y, v}) + \int_s^T f(r, x_t^{y, v}, \overline{\nabla}(r, x_t^{y, v}), (\sigma^* \nabla \nabla)(r, x_t^{y, v}), v_r) \, dr
\]

Then

\[
\overline{\nabla}(s, x_s^{y, v}) = \overline{y}_s^{y, v} + \int_s^T g(r, x_t^{y, v}, \overline{\nabla}(r, x_t^{y, v}), \sigma^* \nabla \nabla(r, x_t^{y, v})), \varphi(y)) \, dy \, dr
\]

\[
\int_{\mathbb{R}^d} \left( (\sigma^* \nabla \nabla)(r, x_t^{y, v}), \varphi(x) \right) \, dr \, dy + \\
\int_{\mathbb{R}^d} \int_s^T \left( (\sigma^* \nabla \nabla)(r, x_t^{y, v}), \varphi(y)) \, dy \, dr
\]

\[
\int_{\mathbb{R}^d} \int_s^T \left( (\sigma^* \nabla \nabla)(r, x_t^{y, v}), \varphi(y)) \, dy \right) \, dr
\]
\[ - \int_s^T (\sigma^* \nabla \bar{V})(r, x_r^{t,y,v}) dW_r. \] (66)

Here \( \bar{V}(s, x_s^{t,y,v}) - y_s^{t,y,v} \geq 0 \), so by the comparison theorem of the BDSDE, we know that the \( \bar{V}(r, x_r^{t,y,v}) \) is the g-supersolution of the BSDE (15). So we have

\[ \bar{V}(s, x_s^{t,y,v}) \geq Y_s^{s,x_s^{t,y,v},v}. \] (67)

Let us make the same change of variable \( y = \tilde{x}_r^{t,x,v'} \) in each term of [57], so [57] becomes

\[
\begin{align*}
\int_{\mathbb{R}^d} \bar{V}(s, x_s^{t,y,v'}) \varphi(y) dy - \varepsilon & \leq \int_{\mathbb{R}^d} h(x_T^{t,y,v'}) \varphi(y) dy + \int_s^T \int_{\mathbb{R}^d} f(r, x_r^{t,y,v'}, \nabla \bar{V}(r, x_r^{t,y,v'}), (\sigma^* \nabla \bar{V})(r, x_r^{t,y,v'}), v_r', \varphi(y)) dy dr \\
& + \int_s^T \int_{\mathbb{R}^d} g(r, x_r^{t,y,v'}, \nabla \bar{V}(r, x_r^{t,y,v'}), (\sigma^* \nabla \bar{V})(r, x_r^{t,y,v'})), \varphi(y)) dy d Br \\
& - \int_s^T \int_{\mathbb{R}^d} (\sigma^* \nabla \bar{V})(r, x_r^{t,y,v'}) \varphi(y) dy dW_r. \tag{68}
\end{align*}
\]

Since \( \varphi \) is arbitrary, we have proven that for almost every \( y \)

\[
\begin{align*}
\bar{V}(s, x_s^{t,y,v'}) & - \frac{\varepsilon}{\int_{\mathbb{R}^d} \varphi(y) dy} \\
& \leq h(x_T^{t,y,v'}) + \int_s^T f(r, x_r^{t,y,v'}, \nabla \bar{V}(r, x_r^{t,y,v'}), (\sigma^* \nabla \bar{V})(r, x_r^{t,y,v'}), v_r') dr \\
& + \int_s^T g(r, x_r^{t,y,v'}, \nabla \bar{V}(r, x_r^{t,y,v'}), (\sigma^* \nabla \bar{V})(r, x_r^{t,y,v'})) d Br - \int_s^T (\sigma^* \nabla \bar{V})(r, x_r^{t,y,v'}) dW_r. \tag{69}
\end{align*}
\]

Let \( \tilde{V}(s, x_s^{t,y,v'}) = \bar{V}(s, x_s^{t,y,v'}) - \frac{\varepsilon}{\int_{\mathbb{R}^d} \varphi(y) dy} \).

Then

\[
\begin{align*}
\tilde{V}(s, x_s^{t,y,v'}) + \frac{\varepsilon}{\int_{\mathbb{R}^d} \varphi(y) dy} & \leq h(x_T^{t,y,v'}) + \int_s^T f(r, x_r^{t,y,v'}, \tilde{V}(r, x_r^{t,y,v'}), \frac{\varepsilon}{\int_{\mathbb{R}^d} \varphi(y) dy}, (\sigma^* \nabla \tilde{V})(r, x_r^{t,y,v'}), v_r') dr \\
& + \int_s^T g(r, x_r^{t,y,v'}, \tilde{V}(r, x_r^{t,y,v'}), \frac{\varepsilon}{\int_{\mathbb{R}^d} \varphi(y) dy}, (\sigma^* \nabla \tilde{V})(r, x_r^{t,y,v'})) d Br \\
& - \int_s^T (\sigma^* \nabla \tilde{V})(r, x_r^{t,y,v'}) dW_r + \frac{\varepsilon}{\int_{\mathbb{R}^d} \varphi(y) dy}. \tag{70}
\end{align*}
\]

Define

\[
K^{t,y,v'} = h(x_T^{t,y,v'}) + \int_s^T f(r, x_r^{t,y,v'}, \tilde{V}(r, x_r^{t,y,v'}), \frac{\varepsilon}{\int_{\mathbb{R}^d} \varphi(y) dy}, (\sigma^* \nabla \tilde{V})(r, x_r^{t,y,v'}), v_r') dr \\
+ \int_s^T g(r, x_r^{t,y,v'}, \tilde{V}(r, x_r^{t,y,v'}), \frac{\varepsilon}{\int_{\mathbb{R}^d} \varphi(y) dy}, (\sigma^* \nabla \tilde{V})(r, x_r^{t,y,v'})) d Br. \tag{71}
\]

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\[
- \int_s^T (\sigma^* \nabla \tilde{V})(r, x_t^{t,y,v'})dW_r + \frac{\varepsilon}{\int_{R^d} \varphi(y)dy}.
\]

By (70), we can know that
\[
\tilde{V}(s, x_s^{t,y,v'}) + \frac{\varepsilon}{\int_{R^d} \varphi(y)dy} = h(x_T^{t,y,v'}) - (K^{t,y,v'} - \tilde{V}(s, x_s^{t,y,v'}) - \frac{\varepsilon}{\int_{R^d} \varphi(y)dy})
\]
\[
+ \int_s^T f(r, x_r^{t,y,v'}, \tilde{V}(r, x_r^{t,y,v'})) + \frac{\varepsilon}{\int_{R^d} \varphi(y)dy}, (\sigma^* \nabla \tilde{V})(r, x_r^{t,y,v'}, v_r')dr
\]
\[
+ \int_s^T g(r, x_r^{t,y,v'}, \tilde{V}(r, x_r^{t,y,v'})) + \frac{\varepsilon}{\int_{R^d} \varphi(y)dy}, (\sigma^* \nabla \tilde{V})(r, x_r^{t,y,v'})dBr
\]
\[
- \int_s^T (\sigma^* \nabla \tilde{V})(r, x_r^{t,y,v'})dW_r + \frac{\varepsilon}{\int_{R^d} \varphi(y)dy}.
\] (72)

Because \(K^{t,y,v'} - \tilde{V}(s, x_s^{t,y,v'}) - \frac{\varepsilon}{\int_{R^d} \varphi(y)dy} \geq 0\), so by the comparison theorem of BDSDEs, we knows that
\[
\tilde{V}(s, x_s^{t,y,v'}) + \frac{\varepsilon}{\int_{R^d} \varphi(y)dy} \leq Y_s^{t,x_s^{t,y,v'}, v'} + \frac{\varepsilon}{\int_{R^d} \varphi(y)dy}.
\]

So
\[
\tilde{V}(s, x_s^{t,y,v'}) \leq Y_s^{t,x_s^{t,y,v'}, v'} + \frac{\varepsilon}{\int_{R^d} \varphi(y)dy}.
\] (73)

Finally combining (67) and (73), we know that
\[
\tilde{V}(t, y) = \sup_{v \in U} Y_t^{t,y,v}.
\]

Thus \(\tilde{V}(t, y)\) is also the value of \(\sup_{v \in U} J(t, y, v)\), from uniqueness of the solution of cost functional and the uniqueness of supremum, we get uniqueness of weak solution for PDEs (17), i.e. \(\tilde{V}(t, x) = V(t, x)\).

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