The Thirring Model 40 Years Later⋆

N. Ilieva∗♯ and W. Thirring

Institut für Theoretische Physik
Universität Wien
and
Erwin Schrödinger International Institute
for Mathematical Physics

Abstract

Solutions to the Thirring model are constructed in the framework of algebraic quantum field theory. It is shown that for all positive temperatures there are fermionic solutions only if the coupling constant is \( \lambda = \sqrt{2(2n + 1)\pi} \), \( n \in \mathbb{N} \), otherwise solutions are anyons. Different anyons (which are uncountably many) live in orthogonal spaces, so the whole Hilbert space becomes non-separable and in each of its sectors a different Urgleichung holds. This feature certainly cannot be seen by any power expansion in \( \lambda \). Moreover, if the statistic parameter is tied to the coupling constant it is clear that such an expansion is doomed to failure and will never reveal the true structure of the theory.

On the basis of the model in question, it is not possible to decide whether fermions or bosons are more fundamental since dressed fermions can be constructed either from bare fermions or directly from the current algebra.

Invited talk at the
XI International Conference
PROBLEMS OF QUANTUM FIELD THEORY
In memory of
D.I. Blokhintsev

July 1998, Dubna, Russia

⋆ Work supported in part by “Fonds zur Förderung der wissenschaftlichen Forschung in Österreich” under grant P11287-PHY;

∗ Permanent address: Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul.Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria

♯ E–mail address: ilieva@pap.univie.ac.at
1 Introduction

After T.D. Lee had constructed a model of a soluble QFT \[1\] many people tried to find other examples; but to solve a nontrivial relativistic QFT seemed out of the question. The idea that Bethe’s ansatz \[2\] could be successfully used to solve also Heisenberg’s “ Urgleichung” \[3\] reduced to one space one time dimension then led to a soluble relativistic field theory – the Thirring model \[4\]. During the years, this model has not only been extensively studied but has also been actively used for analysis, testing and illustration of various phenomena in two–dimensional field theories.

It is not our purpose to review the enormous literature on the subject but we rather focus on the very starting point – Heisenberg’s Urgleichung. With no bosons present in it at all, it represents the ultimate version of the opinion that fermions should enter the basic formalism of the fundamental theory of elementary particles that is usually taken for granted.

The opposite point of view, namely that a theory including only observable fields, necessarily uncharged bosons, is capable of describing evolution and symmetries of a physical system, being the kernel of algebraic approach to QFT \[5\], also enjoys an enthusiastic support. As we will see, there is no possibility to judge this matter on the basis of the model in question, since both formulations can be equally well used to construct the physically relevant objects – the dressed fermions.

In any case, before claiming that an “Urgleichung” of the type

\[
\mathcal{A}\psi(x) = \lambda \psi(x)\bar{\psi}(x)\psi(x)
\]

(1.1)
determines the whole Universe one should see whether it determines anything mathematically and it is our aim in the present note to discuss the elements needed to make its solution well defined. In fact we shall first consider only one chiral component and we shall restrict ourselves to the two–dimensional spacetime, so that this component depends only on one light cone coordinate. Also the bose–fermi duality takes place there and we want to make use of it. This phenomenon amounts to the fact that in certain models formal functions of fermi fields can be written that have vacuum expectation values and statistics of bosons and vice versa, the equivalence being understood within perturbation theory.

The bose–fermi duality is actually well established when the construction of bosons out of fermions is considered. The problem of rigorous definitions of operator–valued distributions and eventually operators having the basic properties of fermions by taking functions of bosonic fields is rather more delicate. On the level of operator valued distributions solutions have been given by Dell’Antonio et al. \[6\] and Mandelstam \[7\] and on the level of operators in a Hilbert space — by Carey and collaborators \[8, 9\] and in a Krein space by Acerbi, Morchio and Strocchi \[10\].
Thus our goal is to give a precise meaning to the following three ingredients

(a) \[ [\psi^*(x), \psi(x')]_+ = \delta(x-x'), \quad [\psi(x), \psi(x')]_+ = 0 \] CAR
(b) \[ \frac{1}{i} \frac{d}{dx} \psi(x) = \lambda_j(x) \psi(x) \] Urgleichung (1.2)
(c) \[ j(x) = \psi^*(x) \psi(x) \] Current

Eq. (1.2b) involves (derivatives of) objects which are according to (1.2a) rather discontinuous. Therefore it is expedient to pass right away to the level of operators in Hilbert space since the variety of topologies there provides a better control over the limiting procedures. In general norm convergence can hardly be hoped for but we have to strive at least for strong convergence such that the limit of the product is the product of the limits. With \( \psi_f \) for \( f \in L^2(\mathbb{R}) \) and \( \langle . | . \rangle \) the scalar product in \( L^2(\mathbb{R}) \). This shows that \( \psi_f \)'s are bounded and form the \( C^*- \) algebra CAR. There the translations \( x \to x+t \) give an automorphism \( \tau_t \) and we shall use the corresponding KMS–states \( \omega_\beta \) and the associated representation \( \pi_\beta \) to extend CAR. Though there \( j = \infty \), one can give a meaning to \( j \) as a strong limit in \( \mathcal{H}_\beta \) by smearing \( \psi(x) \) over a region \( \varepsilon \) to \( \psi_\varepsilon(x) \) and define

\[ j_f = \int dx f(x) \lim_{\varepsilon \to 0} \lim_{R \to \infty} \langle \psi_\varepsilon^*(x) \psi_\varepsilon(x) - \omega_\beta (\psi_\varepsilon^*(x) \psi_\varepsilon(x)) \rangle, \quad f : \mathbb{R} \to \mathbb{R} \]

These limits exist in the strong resolvent sense and define self–adjoint operators which determine with

\[ e^{ijf} e^{ijg} = e^{\frac{4}{\pi i} \int dx (f(x) g'(x) - f'(x) g(x))} e^{ijf + ig} \] (1.4)

the current algebra \( \mathcal{A}_c \). Its Weyl structure is the same for all \( \beta > 0 \) and \( \omega_\beta \) extends to \( \mathcal{A}_c \).

To construct the interacting fermions which on the level of distributions look like

\[ \Psi(x) = Z e^{i\lambda \int_{-\infty}^x dx' j(x')} \lim_{\varepsilon \to 0} \lim_{R \to \infty} \Psi_{\varepsilon,R}(x) \]

(with some renormalization constant \( Z \)) poses both infrared \( (R \to \infty) \) and ultraviolet \( (\varepsilon \to 0) \) problems. So an extension of \( \pi(\mathcal{A}_c)^* \) is needed to accommodate such a kind of objects.

There are two equivalent ways of handling the infrared problem. Since the automorphism generated by the unitaries \( \Psi_{\varepsilon,R}(x) \) converges to a limit \( \gamma \) for \( R \to \infty \), one can form with it the crossed product \( \mathcal{A}_c = \mathcal{A}_c \hat{\otimes} \mathbb{Z} \), so that in \( \mathcal{A}_c \) there are unitaries with the properties which the limit should have \([1, 12]\). On the other hand, the symplectic form in (1.4) and the state \( \omega_\beta \) can be defined for the limiting element \( \Psi_{\varepsilon}(x) \) and this we shall do in what follows.

In any case \( \mathcal{H}_\beta \) assumes a sectorial structure, the subspaces \( \mathcal{A}_c \sum_{i=1}^n \Psi_{\varepsilon}(x_i)|\Omega\rangle \) for different \( n \) are orthogonal and thus may be called \( n \)–fold charged sectors. The \( \Psi_{\varepsilon}(x) \)'s
have the property that for $|x_i - x_j| > 2\varepsilon$ they obey anyon statistics with parameter $\lambda^2$ and an Urgleichung (1.2b) where $j(x)$ is averaged over a region of length $\varepsilon$ below $x$.

Then, by removing the ultraviolet cut–off the sectors abound and the subspaces $\mathcal{A}_c \Psi(x) | \Omega \rangle$ become orthogonal for different $x$, so $\tilde{H}_\beta$ becomes non–separable. To get canonical fields of the type (1.3) one has to combine $\varepsilon \downarrow 0$ with a field renormalization $\Psi_\varepsilon \rightarrow \varepsilon^{-1/2} \Psi_\varepsilon$ such that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1/2} \int dx f(x) \Psi_\varepsilon(x) = \Psi_f$$

converge strongly in $\tilde{H}_\beta$ and satisfy (1.2b) in sense of distributions.

However, the objects so constructed are in general anyons and only for particular values of the coupling constant, $\lambda = \sqrt{2(2n + 1)}\pi$, $n \in \mathbb{N}$, they are fermions, so that the coupling constant is tied to the statistic parameter. Thus we find that there is indeed some magic about the Urgleichung inasmuch as on the quantum level it allows fermionic solutions by this construction only for isolated values of the coupling constant $\lambda$ whereas classically $\Psi(x) = Z e^{i\lambda \int_{-\infty}^x dx' j(x')}$ solves (1.2b) for any $\lambda$. This feature can certainly not be seen by any power expansion in $\lambda$.

By a symmetry $\alpha$ of a physical system an automorphism of the algebra $\mathcal{A}$ which describes it is understood. The algebraic chain of inclusions we construct gives an example of a symmetry destruction, that is, for a given extension $\mathcal{B}$ of the algebra $\mathcal{A}$, $\mathcal{B} \supset \mathcal{A}$, $\beta|_\mathcal{A} = \alpha$ for some $\alpha \in \text{Aut} \mathcal{A}$. This phenomenon is related to the spontaneous collapse of a symmetry [13] and in contrast to the spontaneous symmetry breaking [14], it cannot occur in a finite–dimensional Hilbert space.

2 Bosons out of fermions: the CAR-algebra, its KMS-states and associated v. Neumann algebras

Let us consider the C*-algebra $\mathcal{A}^l$ formed by the bounded operators

$$\psi_f = \int_{-\infty}^\infty dx \psi(x) f(x) = \int_{-\infty}^\infty \frac{dp}{2\pi} \tilde{\psi}(p) \tilde{f}(p), \quad \tilde{f}(p) = \int_{-\infty}^\infty dx e^{ipx} f(x)$$

(2.1)

with $\psi(x)$, $x \in \mathbb{R}$, being operator-valued distributions which satisfy

$$[\psi^*(x), \psi(x')]_+ = \delta(x-x'), \quad (2.2)$$

so, describing the left movers (we have assigned a superscribed to the relevant quantities, $x$ stands for $x-t$) and $f \in L^2(\mathbb{R})$. This algebra is characterized by

$$[\psi^*_f, \psi_g]_+ = \langle f | g \rangle = \int dx f^*(x) g(x). \quad (2.3)$$

Translation $\tau_t$ define an automorphism of $\mathcal{A}^l$

$$\tau_t \psi_f = \psi_{f_t}, \quad f_t(x) = f(x-t). \quad (2.4)$$
\( \mathcal{A}' \) inherits the norm from \( L^2(\mathbb{R}) \) such that \( \tau_t \) is (pointwise) normcontinuous in \( t \) and even normdifferentiable for the dense set of \( f \)'s for which
\[
\lim_{\delta \downarrow 0} \frac{f(x + \delta) - f(x)}{\delta} = f'(x)
\]
exists in \( L^2(\mathbb{R}) \)
\[
\frac{d}{dt} \tau_t \psi_f \bigg|_{t=0} = -\psi_{f'}.
\]
(2.5)
The \( \tau \)-KMS-states over \( \mathcal{A}' \) are given by
\[
\omega_\beta(\psi_f^* \psi_g) = \int_{-\infty}^{\infty} dp \frac{\tilde{f}^*(p) \tilde{g}(p)}{2\pi(1 + e^{\beta p})} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2\pi} \int \frac{dx dx' f^*(x) g(x')}{i(x - x') - n\beta + \varepsilon}, \quad \varepsilon \downarrow 0,
\]
(2.6)
\[
\omega_\beta(\psi_g^* \psi_f) = \omega_\beta(\psi_f^* \tau_\beta \psi_g).
\]
With each \( \omega_\beta \) are associated a representation \( \pi_\beta \) with cyclic vector \( |\Omega\rangle \), \( \omega_\beta(a) = \langle \Omega|a|\Omega\rangle \) in \( \mathcal{H}_\beta = \mathcal{A}'|\Omega\rangle \) and a v. Neumann algebra \( \pi_\beta(\mathcal{A}')' \). It contains the current algebra \( \mathcal{A}_c' \) which gives the formal expression \( j(x) = \psi^*(x)\psi(x) \) a precise meaning.

To show this, let us recall two lemmas (for the proofs see [12]) which make the whole construction transparent:

**Lemma (2.7)**
If the kernel \( K(k, k') : \mathbb{R}^2 \rightarrow C \) is as operator \( \geq 0 \) and trace class \( (K(k, k) \in L^1(\mathbb{R})) \), then \( \forall \beta \in \mathbb{R}^+ \)
\[
\lim_{M \rightarrow \pm \infty} B_M := \lim_{M \rightarrow \pm \infty} \frac{1}{(2\pi)^2} \int dk dk' K(k, k') \tilde{\psi}^*(k + M) \tilde{\psi}(k' + M) =
\]
\[
= \frac{1}{(2\pi)^2} \int dk dk' \lim_{M \rightarrow \pm \infty} K(k, k') \omega_\beta(\tilde{\psi}^*(k + M) \tilde{\psi}(k' + M)) =
\]
\[
= \begin{cases} 
\frac{1}{2\pi} \int dk \ K(k, k) & \text{for } M \rightarrow +\infty \\
0 & \text{for } M \rightarrow -\infty 
\end{cases}
\]
in the strong sense in \( \mathcal{H}_\beta \).

However, if \( \int |K|^2 \) keeps increasing with \( M \), then \( B_M - \langle B_M \rangle \) may nevertheless tend to an (unbounded) operator.

**Lemma (2.8)**
If
\[
B_M = \frac{1}{(2\pi)^2} \int dk dk' \tilde{f}(k - k') \Theta(M - |k|) \Theta(M - |k'|) \tilde{\psi}^*(k) \psi(k')
\]
with \( \tilde{f} \) decreasing faster than an exponential and being the Fourier transform of a positive function, then the difference \( B_M - \omega_\beta(B_M) \) is a strong Cauchy sequence \( \rightarrow 0 \) for \( M \rightarrow \infty \) on a dense domain on \( \mathcal{H}_\beta \).
Remarks (2.9)

1. (2.7) substantiates the feeling that for $k > 0$ most levels are empty and for $k < 0$ most are full.

2. $B_M$ is a positive operator and by diagonalizing $K$ one sees
   $$\|B_M\| = \|K\|_1 = \frac{1}{2\pi} \int dk\ K(k, k).$$

3. As just mentioned, $\|B_M\| < 2M\tilde{f}(0)$ and $f(x) \geq 0$ is not a serious restriction since any function is a linear combination of positive functions.

4. Since the limit $j_f$ is unbounded the convergence is not on all of $H_\beta$, however since for the limit $j_f$ holds $\tau_i\beta\tau_j f = j_{e^{\beta p}f}$, the dense domain is invariant under $j_f$. Thus we have strong resolvent convergence which means that bounded functions of $B_M$ converge strongly. Also the commutator of the limit is the limit of the commutators.

Thus we conclude that the limit exists and is selfadjoint on a suitable domain. We shall write it formally

$$j_f = \int_{-\infty}^{\infty} \frac{dk dk'}{(2\pi)^2} \tilde{f}(k - k') : \tilde{\psi}(k)^*\tilde{\psi}(k') :$$  \hspace{1cm} (2.10)

Next we show that the currents so defined satisfy the CCR with a suitable symplectic form $\sigma$ \cite{15} \cite{16}.

Theorem (2.11)

$$[j_f, j_g] = i\sigma(f, g) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2} p\tilde{f}(p)\tilde{g}(-p) = \frac{i}{4\pi} \int_{-\infty}^{\infty} dx (f'(x)g(x) - f(x)g'(x)).$$

Proof: For the distributions $\tilde{\psi}(k)$ we get algebraically

$$[\tilde{\psi}(k)^*\tilde{\psi}(k'), \tilde{\psi}(q)^*\tilde{\psi}(q')] = 2\pi \left[ \tilde{\psi}(k)^*\tilde{\psi}(q')\delta(q-k') - \tilde{\psi}(q)^*\tilde{\psi}(k')\delta(k-q') \right]$$

and for the operators after some change of variables

$$\frac{1}{(2\pi)^3} \int dk dp dp' \tilde{f}(p)\tilde{g}(p')\tilde{\psi}(k)^*(p + p' + k)\tilde{\psi}(k)\Theta(M - |k|)\Theta(M - |p + p' + k|).$$

For fixed $p$ and $p'$ and $M \to \infty$ we see that the allowed region for $k$ is contained in $(M - |p| - |p'|, M)$ and $(-M, -M + |p| + |p'|)$. Upon $k \to k \pm M$ we are in the situation of (2.7), thus we see that the commutator of the currents (2.10) is bounded uniformly
in $M$ if $\tilde{f}$ and $\tilde{g}$ decay faster than exponentials and converges to the expectation value. This gives finally
\[
\int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2} \tilde{f}(p)\tilde{g}(-p) \int dk \Theta(M - |k|) \left[ \Theta(M - |k-p|) - \Theta(M - |k+p|) \right] \frac{1}{1 + e^{\beta k}} \rightarrow \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2} p\tilde{f}(p)\tilde{g}(-p).
\]

Remarks (2.12)

1. Since the $j_f$'s satisfy the CCR they cannot be bounded and it is better to write (2.11) in the Weyl form for the associated unitaries
\[
e^{ij_f} e^{ij_g} = e^{i2\sigma(g,f)} e^{ij_f} e^{ij_g} = e^{i\sigma(g,f)} e^{ij_f} e^{ij_g}.
\]

2. The currents $j_f$ are selfadjoint, so the unitaries $e^{i\alpha j_f}$ generate 1–parameter groups — the local gauge transformations
\[
e^{-i\alpha j_f} \psi \bigstar e^{i\alpha j_f} = \psi e^{i\alpha f}.
\]

3. The state $\omega_\beta$ can be extended to $\bar{\omega}_\beta$ over $\pi_\beta(\mathcal{A})''$ and $\bar{\tau}_t, \tilde{\tau} \in \text{Aut } \pi_\beta(\mathcal{A})''$ with $\bar{\tau}_t j_f = j_{f_t}$. Furthermore $\bar{\omega}_\beta$ is $\bar{\tau}$–KMS and is calculated to be ([12], see also [17])
\[
\bar{\omega}_\beta(e^{ij_f}) = \exp \left[ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2} \frac{p}{1 - e^{-\beta p}} |\tilde{f}(p)|^2 \right].
\]

4. A physically important symmetry of the algebra $\mathcal{A}$, the parity $P$,
\[
P \in \text{Aut } \mathcal{A}, \quad P \psi = \psi P, \quad P f(x) = f(-x)
\]
is destroyed in $\pi_\beta$, since
\[
[j(x), j(x')] = -\frac{i}{2\pi} \delta'(x-x')
\]
is not invariant under $j(x) \rightarrow j(-x)$. Thus $P \notin \text{Aut } \pi_\beta(\mathcal{A})''$ and $\bar{\omega}_\beta$ is not $P$–invariant.

5. The extended shift automorphism $\bar{\tau}_t$ is not only strongly continuous but for suitable $f$'s also differentiable in $t$ (strongly on a dense set in $\mathcal{H}_\beta$)
\[
\frac{1}{i} \frac{d}{dt} \bar{\tau}_t e^{ij_f} = \left[ j_{f_t} + \frac{1}{2} \sigma(f_t, f'_t) \right] e^{ij_{f_t}} = e^{ij_{f_t}} \left[ j_{f_t} - \frac{1}{2} \sigma(f_t, f'_t) \right] = \frac{1}{2} \left[ j_{f_t} e^{ij_{f_t}} + e^{ij_{f_t}} j_{f_t} \right].
\]

6. The symplectic structure is formally independent on $\beta$ [18], however for $\beta < 0$ it changes its sign, $\sigma \rightarrow -\sigma$, and for $\beta = 0$ (the tracial state) it becomes zero.
Thus starting from a CAR-algebra $\mathcal{A}_l$, we identified in $\pi_\beta(\mathcal{A}_l)'\prime\prime$ bosonic fields – the currents, which satisfy CCR’s. The crucial ingredient needed was the appropriately chosen state. Here we have used the KMS–state (which is unique for the CAR algebra). Another possibility would be to introduce the Dirac vacuum (filling all negative energy levels in the Dirac sea). This is what has been done in the thirties [15, 19], in order to achieve stability for a fermion system, and recovered later by Mattis and Lieb [20] in the context of the Luttinger model. Thus as an additional effect the appearance of an anomalous term in the current commutator (later called Schwinger term) had been discovered that actually enables bosonization of these two–dimensional models.

3 Extensions of $\mathcal{A}_c$: fermions out of bosons

So far $\mathcal{A}_c^l$ was defined for $j_\xi$’s with $f \in C^\infty_0$, for instance. The algebraic structure is determined by the symplectic form $\sigma(f, g)$ (2.11) which is actually well defined also for the Sobolev space, $\sigma(f, g) \rightarrow \sigma(\bar{f}, \bar{g}), \bar{f}, \bar{g} \in H_1, H_1 = \{ f : f, f' \in L^2 \}$. Also $\bar{\omega}_\beta$ can be extended to $H_1$, since $\bar{\omega}_\beta(e^{ij}) > 0$ for $\bar{f} \in H_1$. The anticommuting operators we are looking for are of the form $e^{ij}$, with $f(x) = 2\pi \Theta(x_0 - x) \not\in H_1$. Still one can give $\sigma(f, g)$ a meaning for such an $f$. However, the corresponding state $\omega_\beta$ exhibits singular behaviour for both $p \rightarrow 0$ and $p \rightarrow \infty$, so that

$$\omega_\beta(e^{ij}) = \exp \left[ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dp}{p(1 - e^{-\beta p})} \right] = 0$$

and thus such an operator would act in $\mathcal{H}_\beta$ as zero. Therefore an approximation of $\Theta$ by functions from $H_1$ would result in unitaries that converge weakly to zero.

This situation can be visualized by the following

**Example** (3.1)
Consider the $H_1$–function $\Phi_{\delta,\varepsilon}(x)$,

$$\Phi_{\delta,\varepsilon}(x) := \varphi_{\varepsilon}(x) - \varphi_{\varepsilon}(x + \delta) \in H_1,$$

with

$$\varphi_{\varepsilon}(x) := \begin{cases} 1 & \text{for } x \leq -\varepsilon \\ -x/\varepsilon & \text{for } -\varepsilon \leq x \leq 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

as an approximation to the step function,

$$\lim_{\delta \rightarrow \infty, \varepsilon \rightarrow 0} \Phi_{\delta,\varepsilon}(x) = \Theta(x).$$

Then

$$\tilde{\Phi}_{\delta,\varepsilon}(p) = \frac{1 - e^{ip\varepsilon}}{\varepsilon p^2(1 - e^{ip\delta})}$$
There is a natural extension $\bar{\omega}$ idealized element $e$ for $\beta/\delta$. But $e$ and $8$ since $H$ is total in $\mathcal{H}_\beta$, the divergence of $U$ generated by $\bar{H}$ it as a sign that one should enlarge $A$ scalar field in $(1+1)$ dimensions and various remedies have been proposed [21]. We take the quasifree automorphisms on $A$ converges since $\tau$ to zero.

$\bar{\omega}$ $U_t$ is the countable orthogonal sum of sectors with $n$ particles created by $\bar{H}$, and work in the Hilbert space $\mathcal{H}$ generated by $\bar{A}_c^t$ on the natural extension of the state. Thus we add to $A_c^t$, the idealized element $e^{i2\pi j\varphi_e} = U_\pi$ and keep $\sigma$ and $\omega_\beta$ as before. Equivalently we take the automorphism $\gamma$ generated by $U_\pi$ and consider the crossed product $\bar{A}_c^t = A_c^t \bowtie \mathbf{Z}$. There is a natural extension $\bar{\omega}$ to $\bar{A}_c^t$ and a natural isomorphism of $\mathcal{H}$ and $\bar{A}_c^t(\bar{\omega})$. Here $\mathcal{H}$ is the countable orthogonal sum of sectors with $n$ particles created by $U_\pi$. Thus,

$$\langle \omega | e^{ij} U_\pi | \Omega \rangle = 0$$

(3.2)

means that $U_\pi$ leads to the one-particle sector, in general

$$\langle \omega | U_\pi^{sn} e^{ij} U_\pi^{in} | \Omega \rangle = \delta_{nm} \omega_\beta(\gamma^n e^{ij}).$$

The quasifree automorphisms on $A_c^t$ (e.g. $\tau_t$) can be naturally extended to $\bar{A}_c^t$, $\tau_t U_\pi = e^{i2\pi \varphi_{e,t}}$, $\varphi_{e,t}(x) = \varphi_e(x+t)$ and since $\varphi_e - \varphi_{e,t} \in H_1 \forall t$, this does not lead out of $\bar{A}_c^t$.

$U_\pi$ has some features of a fermionic field since

$$\sigma(\varphi_e, \tau_t \varphi_e) = -\sigma(\varphi_e, \tau_\pi \varphi_e) = \frac{1}{4\pi} \left\{ \begin{array}{ll} \frac{1}{2} \frac{1}{x^2 - t^2} & \text{for } t > \varepsilon \\ 1 - \frac{t^2}{\varepsilon^2} & \text{for } 0 \leq t \leq \varepsilon \end{array} \right. .$$

(3.3)
More generally we could define $U_\alpha = e^{i\sqrt{2\pi j} \varphi \varepsilon}$ and get from (3.3) with

$$\text{sgn}(t) = \Theta(x) - \Theta(-x) = \begin{cases} 
1 & \text{for } t > 0 \\
0 & \text{for } t = 0 \\
-1 & \text{for } t < 0.
\end{cases}$$

**Proposition (3.4)**

\begin{align*}
U_\alpha \tau_t U_\alpha &= \tau_t(U_\alpha)U_\alpha e^{i\alpha \text{sgn}(t)/2}, \\
U_\alpha^* \tau_t U_\alpha &= \tau_t(U_\alpha)^*U_\alpha^* e^{i\alpha \text{sgn}(t)/2} \forall |t| > \varepsilon.
\end{align*}

**Remark (3.5)**

We note a striking difference between the general case of anyon statistics and the two particular cases — Bose ($\alpha = 2 \cdot 2n \pi$) or Fermi ($\alpha = 2(2n + 1) \pi$) statistics. Only in the latter two cases parity $P$ (2.12:4) is an automorphism of the extended algebra generated through $U_\alpha$. Thus $P$ which was destroyed in $\mathcal{A}_c^\varepsilon$ is now recovered for two subalgebras.

The particle sectors are orthogonal in any case

$$\langle \Omega | U_\alpha^{*n} e^{ij} U_\alpha^{m} | \Omega \rangle = 0 \quad \forall n \neq m, f \in H_1.$$  

Furthermore, sectors with different statistics are orthogonal $\langle \Omega | U_\alpha^{*} U_{\alpha'} | \Omega \rangle = 0, \alpha \neq \alpha'$, thus if we adjoin $U_\alpha, \forall \alpha \in \mathbb{R}_+, \mathcal{H}_\beta$ becomes nonseparable.

Next we want to get rid of the ultraviolet cut–off and let $\varepsilon$ go to zero. Proceeding the same way we can extend $\sigma$ and $\tau_t$ but keeping $\omega$ the sectors abound. The reason is that $\varphi_{\varepsilon} \xrightarrow{\varepsilon \to 0} \Theta(x)$ and

$$\|\Theta - \Theta_1\|^2 = \int_{-\infty}^{\infty} \frac{dp}{1 - e^{-\beta p}} \frac{|1 - e^{it p}|^2}{p^2}$$

is finite near $p = 0$ but diverges logarithmically for $p \to \infty$. This means that $e^{ij} e^{ij \omega} |\Omega\rangle$, $f \in H_1$ gives a sector where one of these particles (fermions, bosons or anyons) is at the point $x = 0$ and is orthogonal to $e^{ij} e^{ij \omega} |\Omega\rangle \forall t \neq 0$. Thus the total Hilbert space is not separable and the shift $\tau_t$ is not even weakly continuous, so there is no chance to make sense of $\frac{d}{dt} \tau_t e^{ij \omega}$.

So far, only one chiral component has been considered. When both chiralities are present, no significant changes arise in the construction described. The only point demanding for some care is the anticommutativity between left- and right- moving fermions, which asks for an even larger extension of the current algebra by extending its test–functions space.

So, the Weyl algebra $\mathcal{A}_c = \mathcal{A}_c^r \otimes \mathcal{A}_c^l$ is now generated by the unitaries

$$W(f_r, f_l) = e^{i \int (f_r(x)j_r(x) + f_l(x)j_l(x))dx},$$

with $\sigma(f_l, g_l)$ given by (2.11) and $\sigma(f_r, g_r) = -\sigma(f_l, g_l)$. The minimal extension of $\mathcal{A}_c$ is then obtained by adding two idealized elements,

$$U^*_\pi := W(\alpha, 2\pi(1 - \varphi_{\varepsilon})) \quad \text{and} \quad U^r_\pi := W(2\pi \varphi_{\varepsilon}, c_r)$$
\[ c_l - c_r = (2k + 1)\pi, \quad k \in \mathbb{Z} \quad (\text{e.g. } c_r = \pi/2 = -c_l) \]

They generate for \( \epsilon \to 0 \) (not inner) automorphisms of \( \mathcal{A}_c \)

\[
U^\tau_\pi \colon \gamma_\tau(\mathcal{U}) W(f_r, f_l) = e^{if_r(\epsilon(0))} e^{\frac{\pi}{4} \int_{-\infty}^{\infty} f_r(y)dy} W(f_r, f_l)
\]

in which for the two subalgebras \{\( W(\bar{f}, \bar{f}) \)\} and \{\( W(\bar{f}, -\bar{f}) \)\}, \( \bar{f} \in \mathcal{D}_o \subset \mathcal{C}_o^{\infty} \), the vector and axial gauge transformations can easily be traced back.

In addition to the obvious replacement of (3.4), also the following relation holds

\[
U^\tau_\pi U^\dagger_\pi = -U^\dagger_\pi U^\tau_\pi.
\]

Thus we can identify the chiral components of the fermion field with the so constructed unitaries

\[
\psi_r(x) = \exp \left\{ i2\pi \int_{-\infty}^{\infty} \varphi_\epsilon(y-x) j_r(x')dx' \pm \frac{i\pi}{2} \int_{-\infty}^{\infty} j_l(x')dx' \right\}
\]

\[
\psi_l(x) = \exp \left\{ \mp \frac{i\pi}{2} \int_{-\infty}^{\infty} j_r(x')dx' + i2\pi \int_{-\infty}^{\infty} \varphi_\epsilon(x-y) j_l(x')dx' \right\}
\]

In general, we could define an extension of the algebra \( \mathcal{A}_c \) through the abstract elements \( U^\tau_\alpha \)

\[
U^\tau_\alpha := W(\sqrt{2\pi\alpha} \varphi_\epsilon, \frac{1}{2}\sqrt{\frac{2\pi\alpha}{2}})
\]

\[
U^\dagger_\alpha := W(-\frac{1}{2}\sqrt{\frac{2\pi\alpha}{2}}, \sqrt{2\pi\alpha}(1 - \varphi_\epsilon))
\]

Propositon (3.4) extends also for the non–chiral model generalization. As expected, admitting arbitrary values for \( \alpha \), we get a very rich field structure where definite statistic behaviour is preserved only within a given field class (fixed value of \( \alpha \)), so that even “different” fermions (with different “2 x odd” values of \( \alpha \)) do not anticommute but instead follow the general fractional statistics law.

### 4 Anyon fields in \( \pi_o(\bar{\mathcal{A}}_c)'' \)

Next we shall use another ultraviolet limit to construct local fields which obey some anyon statistics. Of course quantities like

\[
[\Psi^*(x), \Psi(x')]_\alpha := \Psi^*(x)\Psi(x') e^{\frac{\sqrt{2\pi\alpha}}{2} \text{sgn}(x'-x)} + \Psi(x')\Psi^*(x) e^{-i\frac{2\pi\alpha}{2} \text{sgn}(x'-x)} = \delta(x-x')
\]

will only be operator valued distributions and have to be smeared to give operators. Furthermore in this limit the unitaries we used so far have to be renormalized so that \( \delta(x-x') \) gets a factor 1 in front. A candidate for \( \Psi(x) \) will be \( (\alpha \in (0, 4\pi)) \)

\[
\Psi(x) := \lim_{\epsilon \to 0} n(\epsilon) \exp \left[ i\sqrt{2\pi\alpha} \int_{-\infty}^{\infty} dy \varphi_\epsilon(x-y) j(y) \right]
\]
with \( \varphi_\varepsilon \) from (3.1) and \( n(\varepsilon) \) a suitably chosen normalization. With the shorthand \( \varphi_{\varepsilon,x}(y) = \varphi_\varepsilon(x - y) \) we can write
\[
\Psi_\varepsilon(x) \Psi_\varepsilon(x') = \exp \{ i \, 2\pi \alpha \sigma(\varphi_{\varepsilon,x}, \varphi_{\varepsilon,x'}) \} \exp \{ i \sqrt{2\pi \alpha} j \varphi_{\varepsilon,x'} - \varphi_{\varepsilon,x} \},
\]
\[
\Psi_\varepsilon(x') \Psi_\varepsilon^*(x) = \exp \{ -i \, 2\pi \alpha \sigma(\varphi_{\varepsilon,x}, \varphi_{\varepsilon,x'}) \} \exp \{ i \sqrt{2\pi \alpha} j \varphi_{\varepsilon,x'} - \varphi_{\varepsilon,x} \}.
\]
We had in (3.3)
\[
4\pi \sigma(\varphi_{\varepsilon,x}, \varphi_{\varepsilon,x'}) = \operatorname{sgn}(x - x') \left\{ \Theta(|x - x'| - \varepsilon) + \Theta(\varepsilon - |x - x'|) \right\} \frac{(x - x')^2}{\varepsilon^2}
\]
and thus
\[
\left[ \Psi_\varepsilon(x), \Psi_\varepsilon(x') \right]_\alpha = 2n(\varepsilon)^2 \cos \left[ \operatorname{sgn}(x - x') \left( \frac{\pi}{2} - \frac{\alpha}{4} (1 - D_\varepsilon(x - x')) \right) \right] \exp \left[ i \alpha j \varphi_{\varepsilon,x'} - \varphi_{\varepsilon,x} \right].
\]
Note that for \(|x - x'| \geq \varepsilon\) the argument of the \( \cos \) becomes \( \pm \pi/2 \), so the \( \alpha \)-commutator vanishes, in agreement with (3.4). To manufacture a \( \delta \)-function for \(|x - x'| \leq \varepsilon\) we note that \( \cos (...) > 0 \) and \( \omega_\beta(\exp(i\alpha)) > 0 \), so we have to choose \( n(\varepsilon) \) such that
\[
2n^2(\varepsilon) \varepsilon \int_{-1}^{1} d\delta \cos \left( \frac{\pi}{2} - \frac{\alpha}{4} (1 - \delta^2) \right) \cdot \omega_\beta \left( \exp \left[ i \alpha j \varphi_{\varepsilon,x_\delta - \varphi_{\varepsilon,x}} \right] \right) = 1
\]
and to verify that for \( \varepsilon \downarrow 0 \) \( \left[ \right]_\alpha \) converges strongly to a \( c \)-number. For the latter to be finite we have to smear \( \Psi(x) \) with \( L^2 \)-functions \( g \) and \( h \):
\[
\int dx' dx' g(x) h(x') \left[ \Psi_\varepsilon^*(x), \Psi_\varepsilon(x') \right]_\alpha = \int dx dx' g(x) h(x') 2n(\varepsilon)^2 \cos(\varepsilon) \exp \left[ i \alpha j \varphi_{\varepsilon,x'} - \varphi_{\varepsilon,x} \right].
\]
This converges strongly to \( \langle g|h \rangle \) if for \( \varepsilon \downarrow 0 \)
\[
\langle \exp \left[ -i \alpha j \varphi_{\varepsilon,x'} - \varphi_{\varepsilon,x} \right] \exp \left[ i \alpha j \varphi_{\varepsilon,y'} - \varphi_{\varepsilon,y} \right] \rangle - \langle \exp \left[ -i \alpha j \varphi_{\varepsilon,x'} - \varphi_{\varepsilon,x} \right] \rangle \langle \exp \left[ i \alpha j \varphi_{\varepsilon,y'} - \varphi_{\varepsilon,y} \right] \rangle \to 0
\]
for almost all \( x, x', y, y' \). Now
\[
\langle e^{-ij_\alpha} \rangle = \langle e^{-ij_\alpha} \rangle \exp \left[ \int_{-\infty}^{\infty} \frac{dp}{1 - e^{-\beta p}} \tilde{a}(-p) \tilde{b}(p) \right].
\]
In our case this last factor is
\[
\int_{-\infty}^{\infty} \frac{dp}{1 - e^{-\beta p}} \frac{1 - e^{i|p|\varepsilon}}{\varepsilon^2 p^4} (e^{ipx} - e^{-ipx}) (e^{-ipy} - e^{-ipy'}) =
\]
\[
= \int_{-\infty}^{\infty} \frac{dp}{p^3(1 - e^{-\beta p/\varepsilon})} (e^{ipx/\varepsilon} - e^{ipx'/\varepsilon}) (e^{-ipy/\varepsilon} - e^{-ipy'/\varepsilon}).
\]
For fixed \( \beta \neq 0 \) and almost all \( x, x', y, y' \) this converges to zero for \( \varepsilon \to 0 \) by Riemann-Lebesgue. In the same way one sees that \( \exp \left[ i \alpha j \varphi_{\varepsilon,x} + \varphi_{\varepsilon,x'} \right] \) converges strongly to zero and that the \( \Psi_{\varepsilon,g} \) are a strong Cauchy sequence for \( \varepsilon \to 0 \). To summarize we state
Theorem (4.1)

$\Psi_{\varepsilon,g}$ converges strongly for $\varepsilon \to 0$ to an operator $\Psi_g$ which for $\alpha = 2\pi$ satisfies

$$[\Psi_g^*, \Psi_h]_+ = \langle g | h \rangle, \quad [\Psi_g, \Psi_h]_+ = 0.$$  

If $\text{supp } g < \text{supp } h$,

$$\Psi_g^* \Psi_h e^{i\frac{2\pi \alpha}{4}} + \Psi_h^* \Psi_g e^{-i\frac{2\pi \alpha}{4}} = 0 \quad \forall \alpha.$$  

Furthermore we have to verify the claim (1.5) that also for $\Psi_g$ the current $j_f$ induces the local gauge transformation $g(x) \to e^{2i\alpha f(x)}g(x)$. For finite $\varepsilon$ we have

$$e^{ij_f \Psi_{\varepsilon,g} e^{-ij_f}} = \Psi_{\varepsilon, e^{i2\pi \alpha (f, \varphi_\varepsilon)}}$$

and for $\varepsilon \downarrow 0$ we get $\sigma(f, \varphi_\varepsilon) \to \frac{1}{2\pi} f(0)$, so that $\sigma(f, \tau_\varepsilon \varphi_\varepsilon) = \frac{1}{2\pi} f(x)$.

To conclude we investigate the status of the “Urgleichung” in our construction. It is clear that the product of operator valued distributions on the r.h.s. can assume a meaning only by a definite limiting prescription. Formally it would be

$$\Psi(x) \Psi^*(x) \Psi(x) = [\Psi(x), \Psi^*(x)]_+ + \Psi(x) - \Psi^*(x) \Psi(x)^2 = \delta(0) \Psi(x) - 0.$$  

From (2.11,5) we know

$$\frac{1}{i} \frac{\partial}{\partial x} \Psi_\varepsilon(x) = \sqrt{\frac{2\pi \alpha}{2}} [j(x), \Psi_\varepsilon(x)]_+, \quad j(x) = \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x} dy \ j(y).$$  

Using $j_{\varphi'} e^{ij_\varphi} = \frac{1}{i} \frac{\partial}{\partial \alpha} e^{i\frac{2\pi}{2} \sigma(\varphi', \varphi)} e^{ij_\varphi + \alpha \varphi'} |_{\alpha = 0}$ one can verify that the limit $\varepsilon \downarrow 0$ exists for the expectation value with a total set of vectors and thus gives densely defined (not closable) quadratic forms. They do not lead to operators but we know from (2.7) that they define operator valued distributions for test functions from $H_1$. Thus one could say that in the sense of operator valued distributions the Urgleichung holds

$$\frac{1}{i} \frac{\partial}{\partial x} \Psi(x) = \sqrt{\frac{2\pi \alpha}{2}} [j(x), \Psi(x)]_+. \quad (4.2)$$

The remarkable point is that the coupling constant $\lambda$ in (1.1) is related to the statistics parameter $\alpha$. For fermions one has a solution only for $\lambda = \sqrt{2\pi}$. Of course one could for any $\lambda$ enforce fermi statistics by renormalizing the bare fermion field $\psi \to \sqrt{Z} \psi$, $j \to Zj$ with a suitable $Z(\lambda)$ but this just means pushing factors around. Alternatively one could extend $\mathcal{A}_c$ by adding $e^{i\sqrt{2\pi \alpha} j_{\varphi_\varepsilon}}$, for all $\alpha \in \mathbb{R}_+$. Then one gets in $\mathcal{H}_\omega$ uncountably many orthogonal sectors, one for each $\alpha$, and in each sector a different Urgleichung holds. Thus different anyons live in orthogonal Hilbert spaces and $e^{i\sqrt{2\pi \alpha} j_{\varphi_\varepsilon}}$ is not even weakly continuous in $\alpha$. If $\alpha$ is tied to $\lambda$ it is clear that an expansion in $\lambda$ is doomed to failure and will never reveal the true structure of the theory.
5 Internal symmetries

We shall briefly discuss what happens in the case of a fermion multiplet. Then,

\[ \{ \psi_i^*(x), \psi_k(y) \} = \delta_{ik} \delta(x - y), \quad i = 1, 2, ..., N \]

The CAR–algebra so defined possesses an obvious \( U(N) \) symmetry

\[ \psi \longrightarrow U \psi, \quad U \in U(N) \]

In analogy with the previous case we construct quadratic forms \( j_k(x) = \psi_k^*(x)\psi_k(x) \) which satisfy anomalous commutation relations:

\[ [j_k(x), j_m(y)] = -\frac{i}{2\pi} \delta_{km} \delta'(x - y) \quad (5.1) \]

and give rise to operators \( j_f = \int j_k(x)f_k(x)dx, \quad f \in H_1 \).

Denote the corresponding Weyl operators with \( W(f_1, ..., f_N) \)

\[ W(f_1, ..., f_N) := \exp\{ i \sum_{n=1}^N j_{f_n} \} \]

The current algebra (5.1) has a (global) \( O(N) \) symmetry,

\[ j \rightarrow Mj = j', \quad M \in O(N) \quad (5.2) \]

Genuine anticommuting fields can now be identified in an extension of the current algebra quite similar to the one for the non–chiral one–flavour case. More precisely, we have to allow for the existence in the new algebra of the following elements

\[ U_{\pi k} = e^{i2\pi \int_{-\infty}^{\infty} \varphi(x-x')j_k(x')dx' + i \sum_{n=1, n \neq k}^N c_k \int_{-\infty}^{\infty} j_n(x')dx'} =: \psi_k(x), \]

\[ c_k \in \mathbb{R}, \quad c_k - c_n = (2l + 1)\pi, \quad l \in \mathbb{Z}, \quad \forall k, n \quad (5.3) \]

For the elements so defined the following relation holds

\[ \psi_k^*(y)\psi_i^*(y) = \psi_i^*(y)\psi_k^*(x)e^{-i\pi \delta_{ki} \text{sgn}(y-x)}e^{-i(1-\delta_{kl})(c_k-c_l)} \]

which would then lead (after an appropriate renormalisation) to the desired CAR’s.

How should one consider the element, obtained through the same ansatz, but after a transformation (5.2) of the currents, i.e.

\[ \psi_k'(x) = e^{i2\pi \int_{-\infty}^{\infty} M_{kl}j_l(x')dx' + (\ldots)} \quad (5.4) \]

Has it something to do with the \( U(N) \)-transformed fermion \( \psi_k' \), i.e.

\[ e^{i2\pi \int_{-\infty}^{\infty} M_{kl}j_l(x')dx'} \longleftrightarrow U_{kl}\psi_l(x) \]
What one notices is that the $O(N)$–transformation, being (of course) an isomorphism of the algebra $\mathcal{A}_c$, is no longer an automorphism of the latter.

We are faced with the similar situation also for the $U$–invariance we mentioned at the beginning. Here, e.g. for $U(1)$, so

$$\psi_k(x) \rightarrow e^{i\alpha} \psi_k(x)$$

we get

$$\psi'_k(x) = e^{i\alpha} e^{i2\pi \int_{-\infty}^{x} j_k(x')dx'} = e^{i2\pi \int_{-\infty}^{x} j'_k(x')dx'}$$

with

$$j'_k(x') = j_k(x') + \frac{1}{2\pi} \bar{\alpha}(x'), \quad \bar{\alpha}(k) : \int_{-\infty}^{x} \bar{\alpha}(x') = \alpha(x)$$

In the local case this still remains an automorphism of the extended algebra, which is in agreement with the very idea of the construction presented, while for a global transformation this is no longer the case, so for this fermion construction there is no global $U(1)$ symmetry present.

However, on the passage from the CAR–algebra $\mathcal{A}$ to the current algebra $\mathcal{A}_c$ contained in the v.Neumann algebra $\pi_\beta(\mathcal{A})''$ the parity has been broken, but also as an isomorphism of $\mathcal{A}_c$.

Thus we are in a situation in which a particular (physically motivated) extension $\bar{\mathcal{A}}_c$, of the algebra of observables (the current algebra $\mathcal{A}_c$) is constructed, $\bar{\mathcal{A}}_c \supset \mathcal{A}_c$, such that $\beta \in \text{Aut} \bar{\mathcal{A}}_c : \beta|_{\mathcal{A}} = \alpha$ for some $\alpha \in \text{Aut} \mathcal{A}_c$. This phenomenon we call symmetry destruction. It is related to the spontaneous collapse of a symmetry, discussed by Buchholz and Ojima in the context of supersymmetry \[13\] and is seen to be a field effect since in contrast to the spontaneous symmetry breaking \[14\], it cannot occur in a finite–dimensional Hilbert space.

6 Concluding remarks

To summarize we gave a precise meaning to eq.(1.2a,b,c) by starting with bare fermions, $\mathcal{A} = \text{CAR}(\mathbb{R})$. The shift $\tau_t$ is an automorphism of $\mathcal{A}$ which has KMS–states $\omega_\beta$ and associated representations $\pi_\beta$. In $\pi_\beta(\mathcal{A})''$ one finds bosonic modes $\mathcal{A}_c$ with an algebraic structure independent on $\beta$. Taking the crossed product with an outer automorphism of $\mathcal{A}_c$ or equivalently augmenting $\mathcal{A}_c$ by an unitary operator to $\bar{\mathcal{A}}_c$ we discover in $\bar{\pi}_\beta(\mathcal{A}_c)''$ anyonic modes which satisfy the Urgleichung in a distributional sense. For special values of $\lambda$ they are dressed fermions distinct from the bare ones. From the algebraic inclusions $\text{CAR}(\text{bare}) \subset \pi_\beta(\mathcal{A})'' \supset \mathcal{A}_c \subset \bar{\mathcal{A}}_c \subset \bar{\pi}_\beta(\bar{\mathcal{A}}_c)'' \supset \text{CAR}(\text{dressed})$ one concludes that in our model it cannot be decided whether fermions or bosons are more fundamental. One can construct the dressed fermions either from bare fermions or directly from the current algebra and our original question remains open like the one whether the egg or the hen was first.
Acknowledgements

The authors are grateful to H. Narnhofer and R. Haag for helpful discussions on the subject of this paper.

N.I. acknowledges the financial support from the “Fonds zur Förderung der wissenschaftlichen Forschung in Österreich” under grant P11287–PHY and the hospitality at the Institute for Theoretical Physics of University of Vienna.

References

[1] T.D. Lee, Phys. Rev. 95, 1329 (1954).
[2] H. Bethe, Z. Physik 71, 205 (1931).
[3] W. Heisenberg, Z. Naturforsch. 9a, 292 (1954).
[4] W. Thirring, Ann. Phys. 3, 91 (1958).
[5] R. Haag, Local Quantum Physics (Springer, Berlin Heidelberg, 1993).
[6] G. Dell’Antonio, Y. Frischman, D. Zwanziger, Phys. Rev. D6, 988 (1972).
[7] S. Mandelstam, Phys. Rev. D11, 3026 (1975).
[8] A.L. Carey, S.N.M. Ruijsenaars, Acta Appl. Math. 10, 1 (1987).
[9] A.L. Carey, C.A. Hurst, D.M. O’Brien, J. Math. Phys. 24, 2212 (1983).
[10] F. Acerbi, G. Morchio, F. Strocchi, Lett. Math. Phys. 26 (1992) 13; 27, 1 (1993).
[11] N. Ilieva, H. Narnhofer, ÖAW Sitzungsber. II 205, 13 (1996).
[12] N. Ilieva, W. Thirring, Eur. Phys. J. C (to appear) [hep-th/9808103].
[13] D. Buchholz and I. Ojima, Nucl. Phys. B498, 228 (1997).
[14] H. Narnhofer, W. Thirring, Spontaneously Broken Symmetries, Vienna Preprint UWThPh–1996–56, Ann. Inst. Henri Poincaré (to appear).
[15] P. Jordan, Z. Phys. 93, 464 (1935); 98, 759 (1936); 99, 109 (1936).
[16] J. Schwinger Phys. Rev. 128, 2425 (1962).
[17] H.J. Borchers, J. Yngvason, Modular Groups of Quantum Fields in Thermal States, ESI Preprint, ESI–551–1998.
[18] H. Grosse, W. Maderner, C. Reitberger, in *Generalized Symmetries in Physics*, Eds. H.–D. Döbner, V. Dobrev, and A. Ushveridze (World Scientific, Singapore, 1994).

[19] M. Born, N. Nagendra–Nath, *Proc. Ind. Acad. Sci.* 3, 318 (1936).

[20] D.C. Mattis and E. Lieb, *J. Math. Phys.* 6, 304 (1965).

[21] F. Strocchi, *General Properties of QFT, Lecture Notes in Physics – vol.51* (World Scientific, Singapore, 1993).

[22] H. Narnhofer, W. Thirring, *Lett. Math. Phys.* 27, 133 (1993).