ON GROUND STATES OF ROZIKOV MODEL ON THE CAYLEY TREE

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Abstract. In this paper we consider a model on a Cayley tree which has a finite radius of interactions, the model was first considered by Rozikov. We describe a set of periodic ground states of the model.

The Cayley tree.

The Cayley tree $\mathbb{I}_k$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that each vertex of which lies on $k + 1$ edges. Let $\mathbb{I}_k = (V, L, i)$, where $V$ is the set of vertexes of $\mathbb{I}_k$, $L$ is the set of edges of $\mathbb{I}_k$, and $i$ is the incidence function associating to each edge $l \in L$ its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then $x$ and $y$ are called nearest neighboring vertexes, and we write $\langle x, y \rangle$. A collection of the pairs $\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_d, y \rangle$ is called a path from $x$ to $y$. The distance $d(x, y), x, y \in V$ is the length of the shortest path from $x$ to $y$ in $V$.

For the fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\},$ $V_n = \{x \in V \mid d(x, x^0) \leq n\},$ $L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$

It is known (see e.g., [2]) that there exists a one-to-one correspondence between the set $V$ of vertices of the Cayley tree of order $k \geq 1$ and the group $G_k$, of the free products of $k + 1$ cyclic groups $\{e, a_i\}, i = 1, \ldots, k + 1$ of the second order (i.e. $a_i^2 = e, a_i^{-1} = a_i$) with generators $a_1, a_2, \ldots, a_{k+1}$.

Configuration Space and the model

We consider models where the spin takes values in the set $\Phi = \{1, 2, \ldots, q\}, q \geq 2$. For $A \subseteq V$ a spin configuration $\sigma_A$ on $A$ is defined as a function $x \in A \rightarrow \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$. We denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$. Also we define a periodic configuration as a configuration $\sigma \in \Omega$ which is invariant under a subgroup of shifts $F_k \subset G_k$ of finite index.

More precisely, a configuration $\sigma \in V$ is called $F_k$– periodic if $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in F_k$. 
For a given periodic configuration the index of the subgroup is called the period of the configuration. A configuration that is invariant with respect to all shifts is called translational-invariant.

For $A \subset V$ let us define a generalized Kronecker symbol (see [6]) as the function $U(\sigma_A) : \Omega_A \rightarrow \{|A| - 1, |A| - 2, \ldots, |A| - \min\{|A|, |\Phi|\}\}$, by

$$U(\sigma_A) = |A| - |\sigma_A \cap \Phi|,$$

where as before $\Phi = \{1, 2, \ldots, q\}$ and $|\sigma_A \cap \Phi|$ is the number of different values of $\sigma_A(x), x \in A$.

For instance if $\sigma_A$ is a constant configuration then $|\sigma_A \cap \Phi| = 1$.

Note that if $|A| = 2$, say, $A = \{x, y\}$, then $U(\{\sigma(x), \sigma(y)\}) = \delta_{\sigma(x)\sigma(y)}$,

$$\delta_{\sigma(x)\sigma(y)} = \begin{cases} 1, & \sigma(x) = \sigma(y), \\ 0, & \sigma(x) \neq \sigma(y). \end{cases}$$

Fix $r \in N$ and put $r' = \lceil \frac{r+1}{2} \rceil$, where $\lceil a \rceil$ is the integer part of $a$. Denote by $M_r$ the set of all balls $b_r(x) = \{y \in V : d(x, y) \leq r'\}$ with radius $r'$, i.e. $M_r = \{b_r(x) : x \in V\}$.

We consider the energy of the configuration $\sigma \in \Omega$ is given by the formal Hamiltonian

$$H(\sigma) = -J \sum_{b \in M_r} U(\sigma_b),$$

where $J \in R$. This Hamiltonian was first considered by Rozikov [6].

**Ground states**

The ground states for the model defined on $\mathbb{Z}^d$ can, for example, be found in [3], [7].

**Definition 1.** A configuration $\varphi$ is called the ground states of relative Hamiltonian $H$, if

$$U(\varphi_b) = U^{\min} = \min\{U(\sigma_b) : \sigma_b \in \Omega_b\} \text{ for any } b \in M_r.$$

In [1], [5] the ground states of Ising and Potts models with competing interactions of radius $r = 2$ on the Cayley tree were described.

Let $GS(H)$ be the set of all ground states, and let $GS_p(H)$ be the set of all periodic ground states.

**Theorem 1.** a) If $J > 0$, then for all $r \geq 1$ and $k \geq 2$ the set $GS(H)$ consists only configurations $\{\sigma^{(i)} : i = 1, 2, \ldots, s\}$, where $\sigma^{(i)} \equiv i, \forall x \in V$;
b) Let \( r = 2, J < 0, q \geq 2^m \) and \( k \in \{2^{m-1} - 1, \ldots, q - 2\} \), \( m = 3, 4, \ldots \) then there exists a normal subgroup \( F \) of index \( 2^m \), such, that any \( F \)– periodic configuration \( \sigma \) is a ground state for Hamiltonian \( H \) i.e. \( \sigma \in GS_p(H) \).

**Proof** a) Easily follows from (1), (2) and Definition 1.

b) Since \( J < 0 \) to construct a ground state it is necessary to consider configurations \( \sigma \) with a condition, that \( U(\sigma_b) = 0 \) for all \( b \in M \), i.e. on any ball \( b \in M \) the configuration \( \sigma \) is such that \( \sigma(x) \neq \sigma(y) \) if \( x \neq y \). Therefore we will construct a normal subgroup \( F \) of index \( 2^m \) such, that any element of the set \( S_1(e) = \{e, a_1, \ldots, a_{k+1}\} \) is not equivalent (with respect to \( F \)) to each other element of the set. Since \( k + 2 \leq q \) we get \( k \leq q - 2 \). Consider a normal subgroup \( F \) of index \( 2^m \), such that \( F = F_{A_1} \cap \cdots \cap F_{A_m} \) where \( F_{A_i} = \{x \in G_k : \sum_{j \in A_i} \omega_j(x) - \text{even}\} \), and \( \omega_x(a_i) \) is the number of letter \( a_i \), in nondeductible word \( x \), \( A_i \subset \{1, \ldots, k+1\}, i = 1, \ldots, m \).

Now we shall construct \( A_i, i = 1, \ldots, m \), so that all elements of any ball \( b \in M \) were from different classes of equivalency.

Let’s consider all possible configurations \( \alpha : \{1, 2, \ldots, m\} \rightarrow \{e, o\} \) (where "e" designates "even" and "o" designates "odd"). Let’s notice, that number of such configurations is equal to \( 2^m \). From them choose half, i.e. \( 2^{m-1} \) configurations with following properties: or the number of letters "e" in a configuration is more than number of letters "o", or the number of letters "e" in a configuration is equal to number of letters "o" and among the last there are no configurations coinciding at replacement "o" on letters "o". Let’s denote these \( 2^{m-1} \) configurations by

\[
\alpha_0 = \{e, e, e, \ldots, e\} = (\alpha_0_1, \alpha_0_2, \ldots, \alpha_0_m)
\]

\[
\alpha_1 = \{o, e, e, \ldots, e\} = (\alpha_1_1, \alpha_1_2, \ldots, \alpha_1_m)
\]

\[
\alpha_2 = \{e, o, e, \ldots, e\} = (\alpha_2_1, \alpha_2_2, \ldots, \alpha_2_m)
\]

\[
\alpha_3 = \{e, e, o, \ldots, e\} = (\alpha_3_1, \alpha_3_2, \ldots, \alpha_3_m)
\]

\[
\ldots \ldots
\]

\[
\alpha_{2^{m-1}} = \{o, e, e, \ldots, o\} = (\alpha_{2^{m-1}}_1, \alpha_{2^{m-1}}_2, \ldots, \alpha_{2^{m-1}}_m).
\]

We can define sets \( A_i, i = 1, 2, \ldots, m \), as follows

\[
A_i = \{j \in \{1, 2, \ldots, k\} : \alpha_{ji} \text{– odd} \} \cup \{k + 1\}, \quad i = 1, 2, \ldots, m.
\]
Let’s notice, that $A_i, i = 1, 2, \ldots m$, make sense if $k + 1 \geq 2^{m-1}$ i.e. $k \geq 2^{m-1} - 1$. Check, that $F = F_{A_1} \cap \cdots \cap F_{A_m}$, constructed by sets (3), satisfies conditions of the theorem. At first we shall prove, that $S_1(e)$ with respect to $F$ divides into different non-equivalent elements: Denote $S_1(x) = \{ y \in V : d(x,y) = 1 \} = \{ x, xa_1, \ldots, xa_{k+1} \}, \gamma_i(x) = |S_1(x) \cap F_i|$. It is enough to prove, that $\gamma_i(x) = 0$ or 1 for any $x \in V$ and $i = 1, \ldots, m$. By our construction one has $\gamma_i(e) \in \{0,1\}$ for any $i = 1,\ldots, m$. Hence, elements of the set $S_1(e)$ are not equivalent to each others, also they are not equivalent to $e$. Then by Theorem 3 of [4] elements of the set $S_1(x)$ are not equivalent to each others. By Theorem 1 of [4] we get $x \sim xa_i$ (i.e. $x$ and $xa_i$ belong to one class) if and only if $e \sim a_i$. By our construction $e \sim a_i, \forall i = 1, \ldots, k + 1$ hence $x \sim xa_i$; therefore, $\gamma_i(x) = 0$ or 1.

The theorem is proved.

**Theorem 2.** Let $r = 2$. a) if $J > 0$, then $|GS_p(H)| = q$; b) If $J < 0$, then $|GS_p(H)| = C^{k+2}(k + 2)!$

**Proof.** Case a) is trivial. In case b) for a given configuration $\varphi_b$, for which the energy $U(\varphi_b)$ is minimal, we can use Theorem 1 to construct the periodic configurations $\sigma$ with period $2^m$. In each case, the exact number of such ground states coincides with the number of different configurations $\sigma_b$, such that the energy $U(\sigma_b)$ is minimal for any $b \in M$. The theorem is proved.

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