Exact Universal Amplitude Ratios for Two-Dimensional Ising Models and a Quantum Spin Chain

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Let \( f_N \) and \( \xi^{-1}_N \) represent, respectively, the free energy per spin and the inverse spin-spin correlation length of the critical Ising model on a \( N \times \infty \) lattice, with \( f_N \to f_\infty \) as \( N \to \infty \). We obtain analytic expressions for \( a_k \) and \( b_k \) in the expansions:

\[
N(f_N - f_\infty) = \sum_{k=1}^{\infty} a_k/N^{2k-1} \quad \text{and} \quad \xi^{-1}_N = \sum_{k=1}^{\infty} b_k/N^{2k-1}
\]

for square, honeycomb, and plane-triangular lattices, and find that \( b_k/a_k = (2^{2k-1})/(2^{2k-1} - 1) \) for all of these lattices, i.e. the amplitude ratio \( b_k/a_k \) is universal. We also obtain similar result for a critical quantum spin chain and find that such result could be understood from a perturbated conformal field theory.

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Experimental data, analytic and simulational studies of phase transition models, and renormalization group (RG) theory suggest that critical systems can be grouped into universality classes so that the systems in the same class have the same set of critical exponents. RG theory has also been used to propose that critical systems of the same universality class could have universal finite-size scaling functions (UFSSF’s) and universal amplitude ratios and some analytic and numerical calculations of critical systems have supported the idea of universal amplitude ratios. Using Monte Carlo methods and choosing appropriate aspect ratios for lattices of critical systems, Hu, et al. have found UFSSF’s for percolation and Ising models. Despite the success of RG theory and Monte Carlo simulations, it is valuable to have more analytic results which could widen or deepen our understanding of the universality of critical systems. In this Letter we present exact calculations for a set of universal amplitude ratios for the Ising model on square (sq), honeycomb (hc), and plane-triangular (pt) lattices and for a quantum spin chain, which is in the universality class of two-dimensional (2D) Ising models. As far as we know, no previous RG arguments, analytic calculations, or numerical studies predict the existence of this whole set of universal amplitude ratios.

Let \( f_N \) and \( \xi^{-1}_N \) represent, respectively, the free energy per spin and the inverse spin-spin correlation length of the Ising model on a \( N \times \infty \) lattice with periodic boundary conditions, with \( f_N \to f_\infty \) as \( N \to \infty \). In this letter, we obtain analytic equations for \( a_k \) and \( b_k \) in the expansions:

\[
N(f_N - f_\infty) = \sum_{k=1}^{\infty} \frac{a_k}{N^{2k-1}},
\]

\[
\xi^{-1}_N = \sum_{k=1}^{\infty} \frac{b_k}{N^{2k-1}},
\]

for sq, hc, and pt lattices, and find that

\[
b_k/a_k = (2^{2k} - 1)/(2^{2k-1} - 1),
\]

for all of these lattices, i.e. the amplitude ratio \( a_k/b_k \) is universal. We also obtain similar expansions for the critical ground state energy \( E_0 \) and the critical first energy gap \( (E_1 - E_0) \) of a quantum spin chain, which are, respectively, the quantum analogies of the free energy and inverse spin-spin correlation length for the Ising model, and find that the amplitude ratios have the same values. We could physically understand such result from a perturbated conformal field theory.

Consider an Ising ferromagnet on an \( N \times M \) lattice with periodic boundary conditions (i.e. a torus). The Hamiltonian of the system is

\[
\beta H = -J \sum_{<ij>} s_i s_j,
\]

where \( \beta = (k_B T)^{-1} \), the Ising spins \( s_i = \pm 1 \) are located at the sites of the lattice and the summation goes over all nearest-neighbor pairs of the lattice. We consider a transfer matrix acting along the \( M \) direction. If \( \Lambda_0 \) and \( \Lambda_1 \) are the largest and the second-largest
eigenvalues of the transfer matrix, in the limit $M \to \infty$ the free energy per spin, $f_N$, and the inverse longitudinal spin-spin correlation length, $\xi_N^{-1}$, are

$$f_N = \frac{1}{\zeta N} \ln \Lambda_0 \quad \text{and} \quad \xi_N^{-1} = \frac{1}{\zeta} \ln (\Lambda_0/\Lambda_1). \quad (5)$$

Here $\zeta$ is a geometric factor which is $1$, $2/\sqrt{3}$ and $1/\sqrt{3}$ for sq, hc, and pt lattices, respectively \cite{2}. Exact expressions for eigenvalues $\Lambda_0$ and $\Lambda_1$ are available for all lattices under consideration: sq \cite{7,11,12,13}, hc \cite{2,9}, and pt \cite{8}.

We start from the Ising model on the sq lattice. Onsager \cite{7} has obtained expressions for all eigenvalues of the transfer matrix. The two leading eigenvalues are $\Lambda_0 = (2 \sinh 2J)^{N/2} \exp \left( \frac{1}{2} \sum_{r=0}^{N-1} \gamma_{2r+1} \right)$, $\Lambda_1 = (2 \sinh 2J)^{N/2} \exp \left( \frac{1}{2} \sum_{r=1}^{N} \gamma_{2r} \right)$, where $\gamma_k$ is implicitly given by $\cosh \gamma_k = \cosh 2J \coth 2J - \cos (k\pi/N)$. At the critical point $J_c$ of the sq lattice Ising model, where $J_c = \frac{1}{2} \ln (1 + \sqrt{2})$, one then obtains $\gamma_k = 2\psi_{sq} \left( \frac{k\pi}{2N} \right)$. Here

$$\psi_{sq}(x) = \ln \left( \sin x + \sqrt{1 + \sin^2 x} \right). \quad (6)$$

Then the critical free energy $f_N$ and critical spin-spin correlation length $\xi_N$ of Eq. (5) can be written as

$$f_N = \frac{1}{2} \ln 2 + \frac{1}{2N} \sum_{r=0}^{N-1} \gamma_{2r+1}, \quad (7)$$

$$\xi_N^{-1} = \frac{1}{2} \sum_{r=0}^{N-1} (\gamma_{2r+1} - \gamma_{2r}). \quad (8)$$

It is readily seen from Eqs. (6), (7) and (8) that $\xi_N^{-1}$ and $Nf_N$ have odd parity as a function of $N^{-1}$. Therefore, in the following expansions of $\xi_N^{-1}$ and $Nf_N$ as a function of $N^{-1}$, we keep only odd terms.

To write $f_N$ and $\xi_N^{-1}$ in the form of Eqs. (11) and (2), we must evaluate Eqs. (7) and (8) asymptotically. These sums can be handled by using the Euler-Maclaurin summation formula \cite{14}. After a straightforward calculation, we have

$$N(f_N - f_\infty) = \sum_{k=1}^{\infty} \frac{2B_{2k}}{(2k)!} (2^{2k-1} - 1) \left( \frac{\pi}{2N} \right)^{2k-1} \psi_{sq}^{(2k-1)}$$

$$= \frac{\pi}{12N} + \frac{7}{180} \left( \frac{\pi}{2N} \right)^3 + \frac{31}{756} \left( \frac{\pi}{2N} \right)^5 + \frac{10033}{75600} \left( \frac{\pi}{2N} \right)^7 + \ldots, \quad (9)$$

$$\xi_N^{-1} = \sum_{k=1}^{\infty} \frac{2B_{2k}}{(2k)!} (2^{2k-1} - 1) \left( \frac{\pi}{2N} \right)^{2k-1} \psi_{sq}^{(2k-1)}$$

$$= \frac{\pi}{4N} + \frac{1}{12} \left( \frac{\pi}{2N} \right)^3 + \frac{1}{12} \left( \frac{\pi}{2N} \right)^5 + \frac{1343}{5040} \left( \frac{\pi}{2N} \right)^7 + \ldots. \quad (10)$$

Here $B_{2k}$ are the Bernoulli numbers and $\psi_{sq}^{(2k-1)} = \left( d^{2k-1} \psi_{sq}(x) / dx^{2k-1} \right)_{x=0}; b_2 = \pi^3/96$ has been computed previously by Derrida and Seze \cite{13}.

For the Ising model on the honeycomb (hc) lattice, Husimi and Syozi \cite{9} found that

$$\Lambda_0 = (2 \sinh 2J)^N \exp \left( \gamma_1 + \gamma_3 + \ldots + \gamma_{N-1} \right) \quad \text{and}$$

$$\Lambda_1 = (2 \sinh 2J)^N \exp \left( \gamma_2 + \gamma_4 + \ldots + \gamma_{N-2} \right) \quad \text{and}$$

\cite{9}.
\[ \Lambda_1 = (2 \sinh 2J)^N \exp \left( \frac{1}{2} \gamma_0 + \gamma_2 + \ldots + \gamma_{N-2} + \frac{1}{2} \gamma_N \right), \]

where the \( \gamma_r \) are given by \( \cosh \gamma_r = \cosh 2J \cosh 2J^* - \sin^2 \frac{a}{N} - \sin \frac{\pi}{N} (\sin^2 2J \sinh^2 2J^* - \sin^2 \frac{a}{N})^{1/2} \). Here \( J^* \) is defined by \( \cosh (2J - 1)(\cosh 2J^* - 1) = 1 \), so that one has \( J^* = J \) at the critical point \( (J_c = \frac{1}{2} \ln (2 + \sqrt{3})) \) and one then obtains \( \gamma_r = 2\psi_{hc} \left( \frac{2\pi}{2N} \right) \), where

\[ \psi_{hc} (x) = \ln \left\{ A(x) + \sqrt{A^2(x) - 1} \right\}, \]

with \( A(x) = \left( \sqrt{8 + \cos^2 2x} - \cos 2x \right) / 2 \). Using the Euler-Maclaurin summation formula, we can write the free energy \( f_N \) and the inverse spin-spin correlation length \( \xi_N^{-1} \) for the hc lattice as

\[
N(f_N - f_\infty) = \sum_{k=1}^{\infty} \sqrt{5B_{2k}(2^{2k-1} - 1)} \left( \frac{\pi}{2N} \right)^{2k-1} \psi_{hc}^{(2k-1)}
\]

\[ = \frac{\pi}{12N} - \frac{31}{210} \left( \frac{\pi}{3N} \right)^5 + \frac{511}{110} \left( \frac{\pi}{3N} \right)^9 + \ldots, \]

(12)

\[
\xi_N^{-1} = \sum_{k=1}^{\infty} \frac{\sqrt{5B_{2k}(2^{2k-1} - 1)} \left( \frac{\pi}{2N} \right)^{2k-1} \psi_{hc}^{(2k-1)}}{(2k)!}
\]

\[ = \frac{\pi}{4N} - \frac{3}{10} \left( \frac{\pi}{3N} \right)^5 + \frac{93}{10} \left( \frac{\pi}{3N} \right)^9 + \ldots, \]

(13)

where \( \psi_{hc}^{(2k-1)} = \left( d^{2k-1} \psi_{hc}(x) / dx^{2k-1} \right) \bigg|_{x=0} \).

For the case of the pt lattice, we note that one can use the star-triangle transformation to transform the hc to the pt lattice [11]. The amplitudes of the \( N^{-3} \) and \( N^{-7} \) correction terms are identically zero for the hc and the pt lattices.

Above results for the sq, hc, and pt lattices can be summarized as

\[
N(f_N - f_\infty) = \sum_{k=1}^{\infty} \frac{2B_{2k}(2^{2k-1} - 1)}{\zeta(2k)!} \left( \frac{\pi}{2aN} \right)^{2k-1} \psi^{(2k-1)},
\]

(14)

\[
\xi_N^{-1} = \sum_{k=1}^{\infty} \frac{2B_{2k}(2^{2k-1} - 1)}{\zeta(2k)!} \left( \frac{\pi}{2aN} \right)^{2k-1} \psi^{(2k-1)},
\]

(15)

where \( a = 1, \quad \zeta = 1, \quad \psi(x) = \psi_{sq}(x) \) for the sq lattice, \( a = 1, \quad \zeta = 2/\sqrt{3}, \quad \psi(x) = \psi_{hc}(x) \) for the hc lattice, and \( a = 2, \quad \zeta = 1/\sqrt{3}, \quad \psi(x) = \psi_{tr}(x) = \psi_{hc}(x) \) for the pt lattice. The ratios of the nonvanishing amplitudes of the \( N^{-(2k-1)} \) correction terms in the spin-spin correlation length and the free energy expansion, i.e. \( b_k/a_k \), are the same for all three lattices under consideration [13]. Thus we have established Eq. (3).

To check whether Eq. (3) is still valid for other models in the Ising universality class, we proceed to study a quantum spin model on a one-dimensional lattice of \( N \) sites with periodic boundary conditions, whose Hamiltonian is [10]

\[
H = -\lambda \sum_{n=1}^{N} \sigma_n^z - \frac{1}{4\gamma} \sum_{n=1}^{N} \left[ (1 + \gamma)\sigma_{n+1}^x \sigma_n^x + (1 - \gamma)\sigma_{n+1}^y \sigma_n^y \right],
\]

(16)
\[ -E_0 - N\alpha_0 = \sum_{k=1}^{\infty} \frac{2B_{2k}(2^{2k-1} - 1)}{(2k)!} \left( \frac{\pi}{2N} \right)^{2k-1} \psi_q^{(2k-1)} \]

\[ = \frac{\pi}{12N} - \frac{7}{15} \left( \frac{1}{\gamma^2} - \frac{4}{3} \right) \left( \frac{\pi}{4N} \right)^3 - \frac{62}{63} \left( \frac{1}{\gamma^4} - \frac{16}{15} \right) \left( \frac{\pi}{4N} \right)^5 + \ldots, \]

\[ E_1 - E_0 = \sum_{k=1}^{\infty} \frac{2B_{2k}(2^{2k-1} - 1)}{(2k)!} \left( \frac{\pi}{2N} \right)^{2k-1} \psi_q^{(2k-1)} \]

\[ = \frac{\pi}{4N} - \left( \frac{1}{\gamma^2} - \frac{4}{3} \right) \left( \frac{\pi}{4N} \right)^3 - 2 \left( \frac{1}{\gamma^4} - \frac{16}{15} \right) \left( \frac{\pi}{4N} \right)^5 + \ldots, \]

\[ E_2 - E_0 = \sum_{k=1}^{\infty} \frac{8k}{(2k)!} \left( \frac{\pi}{2N} \right)^{2k-1} \psi_q^{(2k-1)} \]

\[ = \frac{2\pi}{N} + 16 \left( \frac{1}{\gamma^2} - \frac{4}{3} \right) \left( \frac{\pi}{4N} \right)^3 - 16 \left( \frac{1}{\gamma^4} - \frac{16}{15} \right) \left( \frac{\pi}{4N} \right)^5 + \ldots, \]

where, \( \psi_q^{(2k-1)} = \left( d^{2k-1} \psi_q(x) / dx^{2k-1} \right)_{x=0}, \psi_q(x) = \sqrt{\sin^2 x - (1 - 1/\gamma^2) \sin^4 x}, \) and \( \alpha_0 \) is a non-universal number \( \alpha_0 = \frac{2}{\pi} \int_0^\pi \psi_q(x) dx = 2 \left[ 1 + \arccos \gamma / (\gamma \sqrt{1 - \gamma^2}) \right] / \pi. \) Thus, the ratios of amplitudes for \( (E_1 - E_0) \) and \( (-E_0) \) also satisfy Eq. (B). Equations (17) and (19) implies also that the ratios \( \bar{r}_k \) of amplitudes for \( (E_2 - E_0) \) and \( (-E_0) \) are \( \gamma \)-independent and given by

\[ \bar{r}_k = \frac{4k}{(2k-1 - 1)B_{2k}}. \]

It is of interest to compare this finding with other results. The exact and numerical estimates [18] of the subdominant correction amplitudes for the sq, hc and pt lattices are presented in Table I, which shows that the numerical values obtained by Queiroz [18] are very close to our exact results. On the basis of conformal invariance, the asymptotic finite-size scaling behavior of the critical free energy and the inverse correlation length is found to be [19]

\[ \lim_{N \to \infty} N^2(f_N - f_\infty) = \frac{c\pi}{6}, \]

\[ \lim_{N \to \infty} N\xi_i^{-1} = \lim_{N \to \infty} N(E_i - E_0) = 2\pi x_i, \]

where \( c \) is the conformal anomaly number and \( x_i \) is the scaling dimension of the \( i \)th scaling field. For the 2D Ising model, we have \( c = 1/2, x_1 = \eta/2 = 1/8 \) and \( x_2 = 1 \) and the leading
the quantum spin chain with the general results of Eqs. (26) and (27) one can find that
\[ E_n = E_{n,c} + g_l \int_{-N/2}^{+N/2} |\phi_l(v)|n > dv + \ldots, \] (25)
where \( E_{n,c} \) are the critical eigenvalues of \( H \). The matrix element \( <n|\phi_l(v)|n> \) can be computed in terms of the universal structure constants (\( C_{nlm} \)) of the operator product expansion [20]:
\[ <n|\phi_l(v)|n> = (2\pi/N)^{x_l} C_{nlm}, \]
where \( x_l \) is the scaling dimension of the conformal field \( \phi_l(v) \). The correlation lengths (\( \xi_n^{-1} = E_n - E_0 \)) and the ground-state energy (\( E_0 \)) can be written as
\[ \xi_n^{-1} = \frac{2\pi}{N} r_n + 2\pi g_l (C_{nlm} - C_{000}) \left( \frac{2\pi}{N} \right)^{x_l-1} + \ldots, \] (26)
\[ E_0 = E_{0,c} + 2\pi g_l C_{000} \left( \frac{2\pi}{N} \right)^{x_l-1} + \ldots. \] (27)
Equations (26) and (27) show that while the amplitude of correction to scaling terms are not universal, ratios of them are. For the 2D Ising model, one finds [24] that the leading finite-size corrections (1/\( N^3 \)) can be described by the Hamiltonian given by Eq. (24) with a single perturbative conformal field \( \phi_l(v) = L_{-2}(v) + L_{-2}(v) \) with scaling dimension \( x_l = 4 \). The universal structure constants \( C_{2l2}, C_{111} \) and \( C_{000} \) can be obtained from the matrix element \( <n|L_{-2}(v) + L_{-2}(v)|n> \), which have already been computed by Reinicke [23] (\( C_{2l2} = 1729/5760, C_{111} = -7/720 \) and \( C_{000} = 49/5760 \)). Equations (26) and (27) implies that the ratios of first-order corrections amplitudes for \( (E_n - E_0) \) and \( (-E_0) \) is universal and equal to \( (C_{000} - C_{nlm})/C_{000} \), which is consistent with Eq. (3) and Eq. (20) for the cases \( n = 1, k = 2 \) \([C_{000} - C_{111}]/C_{000} = 15/7\) and \( n = 2, k = 2 \) \([C_{000} - C_{2l2}]/C_{000} = -240/7\) respectively. Compare the amplitudes of the \( N^{-3} \) correction terms for the Ising model and the quantum spin chain with the general results of Eqs. (29) and (27) one can find that \( g_l = (3/\gamma^2 - 4)/56\pi \) for the quantum spin chain and \( g_l = -1/28\pi \) for the Ising model on the sq lattice. For the Ising model on the hc and pt lattices we find that \( g_l = 0 \), which
indicate that at least two perturbative conformal fields are necessary to generate all finite-size correction terms. Further work has to be done to possibly evaluate exactly all finite-size correction terms from perturbative conformal field theory.

The results of this Letter inspire several problems for further studies: (i) On the basis of perturbated conformal field theory, can one find other universal amplitude ratios? (ii) How do such amplitudes behave in other models, for example in the three-state Potts model? (iii) For the critical Ising model on a large $N \times M$ sq lattice ($M/N$ is a finite number), we have obtained expansions in $N^{-1}$ for the free energy, the internal energy, and the specific heat [24]. It is of interest to extend such expansions to inverse spin-spin correlation length and to hc and pt lattices, and to study whether the amplitude ratios are also universal.

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[15] To check the applicability of these surprising results, we study the anisotropic sq lattice Ising model with coupling constant J and kJ along the horizontal and vertical directions, respectively, with 0 < k < ∞. At the critical point J_c, where J_c is defined by sinh 2J_c sinh 2kJ_c = 1, we obtain \[ \psi_{sq}(x) = \ln \left[ \sin x / \sinh 2J_c + \sqrt{1 + (\sin x / \sinh 2J_c)^2} \right]. \]

We can easily show that Eq. (3) holds for all anisotropy k. Such result is different from the case considered in [4], where the amplitude ratio depends on k.

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TABLES

TABLE I. Comparison of exact (Eqs. (9), (10), (12) – (15)) and numerical [18] values for subdominant finite-size corrections terms in free energy and inverse spin-spin correlation length expansion.
Table I. Izmailian and Hu

|     | square       | honeycomb   | triangular |
|-----|--------------|-------------|------------|
|     | exact        | numerical   | exact      | numerical | exact | numerical |
| \(a_2\) | 0.15072...   | 0.150730(2) | 0          | < 10\(^{-6}\) | 0     | < 10\(^{-6}\) |
| \(b_2\) | 0.322982...  | 0.322987(6) | 0          | < 10\(^{-8}\) | 0     | < 10\(^{-8}\) |
| \(a_3\) | 0.39213...   | 0.385(1)    | -0.18590... | -0.1865(10) | -0.01161... | -0.01165(5) |
| \(b_3\) | 0.79692...   | 0.790(1)    | -0.37780... | -0.3777(2)  | -0.02361... | -0.02360(1) |
| \(a_4\) | 3.13146...   |             | 0          |            | 0     |            |
| \(b_4\) | 6.28759...   |             | 0          |            | 0     |            |
| \(a_5\) | 48.9925...   | 7.03535...  |            | 0.02748... |       |            |
| \(b_5\) | 98.0809...   | 14.0844...  |            |           | 0.05501... |            |