On constructing weight structures and extending them onto idempotent extensions

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Abstract

In this note we describe some new methods for constructing a weight structure $w$ on a triangulated category $C$.

For a given $C$ and $w$ they allow us to give a fairly comprehensive (and new) description of triangulated categories containing $C$ as a dense subcategory (i.e., of subcategories of the idempotent completion of $C$ that contain $C$; we call them idempotent extensions of $C$) such that $w$ extends to them. In particular, any bounded above or below $w$ extends to any idempotent extension of $C$. Our results can also be applied to certain triangulated categories of ("relative") motives.

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Introduction

Weight structures for triangulated categories were independently introduced in [Bon10] and in [Pau08] (where D. Pauksztello has called them co-t-structures); they have quickly found numerous applications to motives, representation theory, and algebraic topology. It is no wonder that the methods of constructing weight structures have also received much attention. The most "popular" result of this kind (cited in dozens of papers) is the following one: Theorem 4.3.2(II) of [Bon10] states that for any full additive subcategory $H$ of a triangulated category $C$ that is negative, i.e., there are only zero $C$-morphisms between $H$ and $H[i]$ for $i > 0$, there exists a unique weight structure $w_D$ on the extension-closure $D$ of $\bigcup_{i \in \mathbb{Z}} H[i]$ in $C$ such that the heart $H_w$ of $w$ contains $H$ (and equals the retraction-closure of $H$ in $D$). So, all bounded weight structures for triangulated categories come from their negative subcategories that strongly generate them.\(^1\)

Yet instead of describing a negative subcategory that strongly generates a given $C$ it is "often" easier to specify $H$ such that $C$ is equivalent to the idempotent completion $\text{Kar}(D)$ of $D$ (certainly, this is only possible if $C$ is idempotent complete; yet this condition is "usually" fulfilled for the triangulated categories of interest). In this case $C$ is endowed with a unique weight structure $w_C$ whose heart is (naturally) equivalent to the idempotent completion of $H$; see Proposition 5.2.2 of ibid.

However, these statements are actual only in the case where there exists "enough of explicit elements" of $\text{Obj} C_w = C_w \leq 0 = C_w \geq 0$. Still in some cases we only have (more or less) explicit descriptions of the classes $C_w \leq 0$ and $C_w \geq 0$, and it is quite difficult to check whether their intersection is "large enough". A collection of examples of this situation may be found in [BoI15], [BoL16], and [Bon16] (cf. §4 below). Now, in the first two of these papers the fact that the couple $(C_w \leq 0, C_w \geq 0)$ does give the corresponding Chow weight structure (the problem here is to verify the existence of weight decompositions for all objects of $C$) was given by a certain gluing argument (that originates from §8 of [Bon10]). However, gluing does not yield the existence of Chow-weight decompositions for the categories of $\delta$-effective motives considered in [Bon16] §3]. This paper (along with [Bon15b]) has motivated the authors to study the question when the existence of weight decompositions for all objects of $C$ follows from the corresponding statement for a certain "generating" subclass of $\text{Obj} C$.

Two statements of this sort are contained in Theorem 2.1.1 below.

This theorem yields (in Theorem 2.2.2 see also Remark 2.2.3(4)) a fairly comprehensive answer to the following question: when a weight structure $w$ for $C$ may be extended to $\text{Kar}(C)$ or, more generally, to some full triangulated subcategory $C'$ of $\text{Kar}(C)$ containing $C$ (so, $C$ is dense in $C'$; we call say that $C'$ is an idempotent extension of $C$). We prove that the corresponding extension necessarily exists (for any idempotent extension of $C'$) whenever $w$ is bounded below or above; this is a very significant generalization of the aforementioned [Bon10 Proposition 5.2.2], whereas our current arguments are substantially

\(^1\)Note that the definition of a weight structure is rather similar to that of a t-structure; yet certainly no analogue of this statement for t-structures is valid.
easier than the ones used for the proof of loc. cit. In the general case we describe the (essentially) maximal idempotent extension $\text{Kar}_{\text{max}}(C)$ of $C$ such that $w$ extends to it, and demonstrate by an example that $\text{Kar}_{\text{max}}(C)$ may be strictly smaller than $\text{Kar}(C)$\footnote{This example is described in \S3. It is also an example of a "rather natural" triangulated category that is not idempotent complete.} Other (motivic) applications of Theorem 2.1.1 can be found in \S3 of [Bon16]; we also describe them briefly in \S4 below. These examples are not directly related to the idempotent completion issues.

Let us now describe the contents of the paper.

In \S1 we introduce some basic (mostly, categorical) notation and recall some of the theory of weight structures. None of the statements of this section are really new.

In \S2 we prove the aforementioned existence of weight structures results.

In \S3 we demonstrate that our results on the extensions of weight structures to idempotent extensions cannot be improved (for weight structures that are not bounded from any side).

In \S4 we describe the "relative motivic" applications of the results of the current paper.

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1 Preliminaries

In \S1.1 we introduce some notation and conventions, and recall some results on triangulated categories.

In \S1.2 we recall some basics on weight structures.

1.1 Some notation and a few results on triangulated categories

For categories $C$ and $D$ we write $D \subset C$ if $D$ is a full subcategory of $C$.

For a category $C$ and $X, Y \in \text{Obj } C$ the notation $C(X, Y)$ is used to denote the set of $C$-morphisms from $X$ to $Y$. We will say that $X$ is a retract of $Y$ if $\text{id}_X$ can be factored through $Y$. Note that if $C$ is triangulated then $X$ is a retract of $Y$ if and only if $X$ is its direct summand.

For a category $C$ the symbol $C^{\text{op}}$ will denote its opposite category.

For a subcategory $D \subset C$ we will say that $D$ is retraction-closed in $C$ if it contains all retracts of its objects in $C$. We will call the smallest retraction-closed subcategory $\text{Kar}_C(D)$ of $C$ containing $D$ (here "Kar" is for Karoubi) the retraction-closure of $D$ in $C$. In this case the class $\text{Obj } \text{Kar}_C(D)$ will also be (abusively) called the retraction-closure of $D$; we will say that this class is retraction-closed in $C$.

The idempotent completion $\text{Kar}(B)$ (no lower index) of an additive category $B$ is the category of "formal images" of idempotents in $B$ (so, $B$ is embedded

2This example is described in \S3. It is also an example of a "rather natural" triangulated category that is not idempotent complete.
into a category that is idempotent complete, i.e., any idempotent endomorphism splits in it).

The symbols $\mathcal{C}$ and $\mathcal{C}'$ will always denote some triangulated categories. We will use the term exact functor for a functor of triangulated categories (i.e., for a functor that preserves the structures of triangulated categories).

A class $D \subset \text{Obj} \mathcal{C}$ will be called extension-closed if $0 \in D$ and for any distinguished triangle $A \to B \to C$ in $\mathcal{C}$ we have the following implication: $A, C \in D \implies B \in D$. In particular, any extension-closed $D$ is strict in $\mathcal{C}$ (i.e., contains all objects of $\mathcal{C}$ isomorphic to its elements).

The full subcategory of $\mathcal{C}$ whose object class is the smallest extension-closed $D \subset \text{Obj} \mathcal{C}$ containing a given $D' \subset \text{Obj} \mathcal{C}$ will be called the extension-closure of $D'$. Sometimes will also abusively use this term for $D$ itself.

Below we will need the following simple fact.

**Lemma 1.1.1.** Let $M, N \in \text{Obj} \mathcal{C}$, $n \geq 0$, and assume that $N$ is a retract of $M$. Then $N$ belongs to the extension-closure of $\{N[2n]\} \cup \{M[i], \ 0 \leq i < 2n\}$

**Proof.** Assume that $M \cong N \bigoplus P$. Then the assertion is given by the (split) distinguished triangles $M[2j] \to N[2j] \to P[2j+1]$ and $M[2j+1] \to P[2j+1] \to N[2j+2]$ for $0 \leq j < n$.

The smallest extension-closed $D \subset \mathcal{C}$ that is also closed with respect to retracts and contains a given $D' \subset \text{Obj} \mathcal{C}$ will be called the envelope of $D'$.

We will say that a class $D \subset \text{Obj} \mathcal{C}$ strongly generates a subcategory $\mathcal{D} \subset \mathcal{C}$ and write $\mathcal{D} = C(D)\mathcal{C}$ if $D$ is the smallest full strict triangulated subcategory of $\mathcal{C}$ such that $D \subset \text{Obj} \mathcal{D}$. Certainly, this condition is equivalent to $\mathcal{D}$ being the extension-closure of $\cup_{j \in \mathbb{Z}} D[j]$.

We will say that $D \subset \text{Obj} \mathcal{C}$ densely generates a subcategory $\mathcal{D} \subset \mathcal{C}$ whenever $\mathcal{D}$ is smallest retraction-closed triangulated subcategory of $\mathcal{C}$ such that $D \subset \text{Obj} \mathcal{D}$. Certainly, this condition is equivalent to $\text{Obj} \mathcal{D}$ being the envelope of $\cup_{j \in \mathbb{Z}} D[j]$.

We will say (following §1.4 of [Tho97]) that a full strict triangulated subcategory $\mathcal{C}$ of a triangulated $\mathcal{C}'$ is dense in it if $\text{Kar}(\mathcal{C}) = \mathcal{C}$. Recall that (according to Theorem 1.5 of [BaS01]) the category $\text{Kar}(\mathcal{C})$ can be naturally endowed with the structure of a triangulated category so that the natural embedding functor $\mathcal{C} \to \text{Kar}(\mathcal{C})$ is exact. Hence if $\mathcal{C}$ is a dense subcategory of $\mathcal{C}'$ then there exists a fully faithful functor $\mathcal{C} \to \text{Kar}(\mathcal{C})$. Moreover, the subcategory $\mathcal{C}_1$ of $\mathcal{C}$ that is strongly generated by some class $D \subset \text{Obj} \mathcal{C}$ is dense in the subcategory $\mathcal{C}_2$ of $\mathcal{C}$ densely generated by this class.

For $X, Y \in \text{Obj} \mathcal{C}$ we will write $X \perp Y$ if $\mathcal{C}(X, Y) = \{0\}$. For $D, E \subset \text{Obj} \mathcal{C}$ we write $D \perp E$ if $X \perp Y$ for all $X \in D, Y \in E$. For $D \subset \text{Obj} \mathcal{C}$ the symbol $D^\perp$ will be used to denote the class

$$\{Y \in \text{Obj} \mathcal{C} : X \perp Y \ \forall X \in D\}.$$  

Dually, $D^\perp$ is the class $\{Y \in \text{Obj} \mathcal{C} : Y \perp X \ \forall X \in D\}$.

In this paper all complexes will be cohomological, i.e., the degree of all differentials is +1. We let $K(\mathcal{D})$ denote the homotopy category of complexes
over an additive category $B$. Its full subcategory of bounded complexes will be denoted by $K^b(B)$.

Since triangulated categories of complexes give examples of weight structures important for the current paper, we recall the following simple statements.

**Proposition 1.1.2.** 1. The full subcategories of $K(B)$ corresponding to classes of $B$-complexes concentrated in degrees $\geq 0$ and $\leq 0$ are idempotent complete.

2. The classes of bounded $B$-complexes that are homotopy equivalent to complexes concentrated in degrees $\geq 0$ and $\leq 0$ are retraction-closed in $K^b(B)$.

**Proof.** 1. This is a part of Theorem 3.1 of [Sch11] (cf. also Proposition 4.2.4 of [Sos15]; one should take $F(-) = \bigcup_{i\geq 0} [-2i]$ in it).

2. See Remark 6.2.2(1) of [Bon10].

\[ \square \]

### 1.2 Weight structures: basics

**Definition 1.2.1.** I. A pair of subclasses $C_{w<0}, C_{w\geq 0} \subset \text{Obj} C$ will be said to define a weight structure $w$ for a triangulated category $C$ if they satisfy the following conditions.

(i) $C_{w<0}$ and $C_{w\geq 0}$ are retraction-closed in $C$ (i.e., contain all $C$-retracts of their objects).

(ii) **Semi-invariance with respect to translations.**

$C_{w<0} \subset C_{w\leq 0}[1]$ and $C_{w\geq 0}[1] \subset C_{w\geq 0}$.

(iii) **Orthogonality.**

$C_{w<0} \perp C_{w\geq 0}[1]$.

(iv) **Weight decompositions.**

For any $M \in \text{Obj} C$, there exists a distinguished triangle $X \to M \to Y \to X[1]$ such that $X \in C_{w<0}$ and $Y \in C_{w\geq 0}[1]$.

II. The category $Hw = C$ whose objects are $C_{w=0} = C_{w\geq 0} \cap C_{w\leq 0}$ and morphisms are $Hw(Z,T) = C(Z,T)$ for $Z,T \in C_{w=0}$, is called the *heart* of $w$.

III. $C_{w>1}$ (resp. $C_{w\leq 1}$, resp. $C_{w=1}$) will denote $C_{w\geq 0}[i]$ (resp. $C_{w<0}[i]$, resp. $C_{w=0}[i]$).

IV. $C_{w>1}$ denotes $C_{w>1} \cap C_{w\leq 1}$; so, this class equals $\{0\}$ if $i > j$.

$C_{w=1}$ will be the category whose object class is $\cup_{i,j \in \mathbb{Z}} C_{i,j}$.

V. We will say that $(C,w)$ is bounded if $C_{w=1} = \text{Obj} C$ (i.e., if $\cup_{i \in \mathbb{Z}} C_{w\leq i} = \text{Obj} C = \cup_{i \in \mathbb{Z}} C_{w\geq 1}$).

Respectively, we will call $\cup_{i \in \mathbb{Z}} C_{w\geq i}$ (resp. $\cup_{i \in \mathbb{Z}} C_{w>1}$) the class of $w$-bounded above (resp. $w$-bounded below) objects; we will say that $w$ is bounded above (resp. bounded below) if all the objects of $C$ satisfy this property.

VI. Let $C$ and $C'$ be triangulated categories endowed with weight structures $w$ and $w'$, respectively; let $F:C \to C'$ be an exact functor.

$F$ is said to be *weight-exact* (with respect to $w, w'$) if it maps $C_{w<0}$ into $C'_{w'<0}$, and sends $C_{w\geq 0}$ into $C'_{w\geq 0}$.

VII. Let $B$ be a full additive subcategory of a triangulated category $C$.

We will say that $B$ is negative (in $C$) if $\text{Obj} B \perp (\cup_{i \geq 0} \text{Obj}(B[i]))$. 

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Remark 1.2.2. 1. A simple (though rather important) example of a weight structure comes from the stupid filtration on \( K(B) \) (or on \( K^0(B) \), on \( K^-(B) \), or on \( K^+(B) \)) for an arbitrary additive category \( B \). In either of these categories we take \( C_{w<0} \) (resp. \( C_{w>0} \)) being the class of objects in \( C \) that are homotopy equivalent to those complexes in \( C \subset K(B) \) that are concentrated in degrees \( \geq 0 \) (resp. \( \leq 0 \)). The only non-trivial axiom check here is that the classes \( C_{w<0} \) and \( C_{w>0} \) are retraction-closed in \( C \); this fact is immediate from Proposition 1.1.4.

The heart of this stupid weight structure is the retraction closure of \( B \) in \( C \).

2. A weight decomposition (of any \( M \in \text{Obj} C \)) is (almost) never canonical. Yet for \( m \in \mathbb{Z} \) some choice of a weight decomposition of \( M \) shifted by \([m]\) is often needed. So we choose a distinguished triangle

\[
\begin{align*}
& w_{\leq m} M \to M \to w_{\geq m+1} M
\end{align*}
\]

with some \( w_{m+1} M \in C_{w>0} \), \( w_{\leq m} M \in C_{w<0} \); we will call it an \( m \)-weight decomposition of \( M \).

We will use this notation below (though \( w_{m+1} M \) and \( w_{\leq m} M \) are not canonically determined by \( M \)). Besides, when we will write arrows of the type \( w_{\leq m} M \to M \) or \( M \to w_{\geq m+1} M \) we will always assume that they come from some \( m \)-weight decomposition of \( M \).

3. In the current paper we use the “homological convention” for weight structures; it was previously used in Wil09, Heb11, Bon14, Bon15a, Bon15b, Bon16, whereas in Bon10 the “cohomological convention” was used. In the latter convention the roles of \( C_{w<0} \) and \( C_{w>0} \) are interchanged, i.e., one considers \( C_{w<0} = C_{w>0} \) and \( C_{w>0} = C_{w<0} \). So, a complex \( X \in \text{Obj} \ A \) whose only non-zero term is the fifth one (i.e., \( X^5 \neq 0 \)) has weight \( -5 \) in the homological convention, and has weight \( 5 \) in the cohomological convention. Thus the conventions differ by “signs of weights”; respectively, \( K(A)_{[i,j]} \) is the retraction closure in \( K(A) \) of the class of complexes concentrated in degrees \( [-j, -i] \).

4. Actually, in Bon10 the first author has also demanded both "halves" of \( w \) to be additive. Yet the proof of Proposition 1.2.3(2) below does not require this additional assumption, whereas this statement certainly implies the additivity of \( C_{w<0} \) and \( C_{w>0} \).

5. The orthogonality axiom in Definition 1.2.1(I) immediately yields that \( Hw \) is negative in \( C \). We will mention certain results in the converse direction in Remark 2.1.2 below.

Let us recall some basic properties of weight structures. Starting from this moment we will assume that \( C \) is (a triangulated category) endowed with a (fixed) weight structure \( w \).

Proposition 1.2.3. Let \( M, M' \in \text{Obj} C \), \( g \in \text{C}(M, M') \).

1. The axiomatics of weight structures is self-dual, i.e., for \( D = C^{op} \) (so \( \text{Obj} D = \text{Obj} C \)) there exists the (opposite) weight structure \( w^{op} \) for which \( D_{w^{op}<0} = C_{w>0} \) and \( D_{w^{op}>0} = C_{w<0} \).
2. $C_{w \geq 0} = (C_{w \leq -1})^\perp$ and $C_{w \leq 0} = C_{w \geq 1}^\perp$.

3. $C_{w \leq 0}$, $C_{w \geq 0}$, and $C_{w = 0}$ are (additive and) extension-closed.

4. The full subcategory $C^+$ (resp. $C^-$) of $C$ whose objects are the bounded below (resp. bounded above) objects of $C$ is a retraction-closed triangulated subcategory of $C$.

5. $C^b$ is the extension-closure of $\cup_{i \in \mathbb{Z}} C_{w = i}$ in $C$.

6. Let $v$ be another weight structure for $C$; assume $C_{w \leq 0} \subset C_{v \leq 0}$ and $C_{w \geq 0} \subset C_{v \geq 0}$. Then $w = v$ (i.e., the inclusions are equalities).

Proof. All of this assertions were proved in [Bon10] (pay attention to Remark 1.2.2(3) above!).

Remark 1.2.4. For $C$ endowed with a weight structure $w$ and a triangulated subcategory $D \subset C$ we will say that $w$ restricts onto $D$ whenever the classes $(C_{w \leq 0} \cap \text{Obj } D, C_{w \geq 0} \cap \text{Obj } D)$ yield a weight structure $w_D$ for $D$. Part 2 of our proposition easily implies that $w$ restricts onto $D$ if and only if the embedding $D \to C$ is weight-exact with respect to a certain weight structure for $D$; if this weight structure exists then it equals $w_D$ as described by the previous sentence.

2 Main results

This is the central section of the paper.

In §2.1 we prove our (new) general results on the existence of weight structures.

In §2.2 we apply these statements to extending weight structures onto idempotent extensions of $C$.

2.1 Extending weight decompositions from strong generators

Theorem 2.1.1. Let $C'$ be a triangulated category. Assume that certain $C'_{w \leq 0}, C'_{w \geq 0} \subset \text{Obj } C'$ satisfy the axioms (ii) and (iii) of Definition 1.2.1 (for $C_{w \leq 0}$ and $C_{w \geq 0}$, respectively). Let us call a $C'$-distinguished triangle $X \to M \to Y[1]$ a pre-weight decomposition of $M$ if $X$ belongs to the envelope $C'_{w \leq 0}$ of $C'_{w \leq 0}$ and $Y$ belongs to the envelope $C'_{w \geq 0}$ of $C'_{w \geq 0}$.

1. Then the following statements are valid.

1. The class of objects possessing pre-weight decompositions is extension-closed (in $C'$). Moreover, if $M, N \in \text{Obj } C'$ possess pre-weight decompositions then any $C'$-extension of $M$ by $N$ possesses a pre-weight decomposition whose components are some extensions of the corresponding components of pre-weight decompositions of $M$ and $N$, respectively.

2. $C'_{w \leq 0} \subset C'_{w \leq 0}[1]$ and $C'_{w \geq 0}[1] \subset C'_{w \geq 0}$.
3. \( C'_{w' \leq 0} \cap C'_{w' \geq 0}[1] \).

4. Let \( C' \) be a subclass of \( \text{Obj} \) \( C' \) such that \( C' \) is the extension-closure of \( C' \) and any object of \( C' \) possesses a pre-weight decomposition. Then \( (C'_{w' \leq 0}, C'_{w' \geq 0}) \) yield a weight structure \( w' \) for \( C' \).

5. Assume (in addition to the assumptions of the previous assertion) that \( C' = \bigcup_{i \in \mathbb{Z}} C[i] \) for some \( C \subset \text{Obj} C' \) and that for any \( c \in C \) there exists \( i_c \in \mathbb{Z} \) such that \( c[i_c] \in C'_{w' \geq 0} \) (resp. \( c[i_c] \in C'_{w' \leq 0} \)). Then this \( w' \) is bounded below (resp. bounded above).

II. Suppose that a class \( C'' \subset \text{Obj} C' \) satisfies the following conditions: \( C' \) is is densely generated by \( C'' \) (see §1.1), pre-weight decompositions exist for \( c[i] \) whenever \( c \in C'' \) and \( i \in \mathbb{Z} \), and for any \( c \in C'' \) there exists \( i_c \in \mathbb{Z} \) such that \( c[i_c] \in C'_{w' \geq 0} \) (cf. assertion I.5).

Then the couple \( (C'_{w' \leq 0}, C'_{w' \geq 0}) \) yields a weight structure \( w' \) for \( C' \) in this case also; this weight structure is bounded below.

Moreover, \( w' \) is also bounded above if we assume in addition that for any \( c \in C'' \) there exists \( i' \in \mathbb{Z} \) such that \( c[i'] \in C'_{w' \leq 0} \).

III. Assume that \( N \in C'_{w' \leq 0} \) is a retract of some \( M \in \text{Obj} C' \) and let \( X \to M \to Y[1] \to X[1] \) be a pre-weight decomposition (of \( M \)). Then the following statements are valid.

1. \( N \) is a retract of \( X \).

2. Suppose that \( N' \in C'_{w' \geq 0} \) is a retract of some \( M' \in \text{Obj} C' \) and let \( A' \to M'[1] \to B'[1] \to A'[1] \) be a pre-weight decomposition of \( M'[1] \). Then \( N' \) is a retract of \( B' \).

3. Let \( A \to X[1] \to B[1] \to A[1] \) be a pre-weight decomposition of \( X[1] \). Then \( B \in C'_{w' \leq 0} \cap C'_{w' \geq 0} \). Moreover, if \( N \) (also) belongs to \( C'_{w' \geq 0} \) then \( N \) is a retract of \( B \).

Proof. I.1. See Remark 1.5.5(1) of [Bon10].

2.3. Obvious from the corresponding properties of \((C', C'_w)\).

4. We use an easy and more or less standard argument; it was first applied to weight structures in (the proof of) Theorem 4.3.2(II.1) of [Bon10].

Certainly \( C'_{w' \leq 0} \) and \( C'_{w' \geq 0} \) are retraction-closed in \( C' \). Axioms (ii) and (iii) of weight structures are fulfilled for \( (C'_{w' \leq 0}, C'_{w' \geq 0}) \) according to the previous assertions.

We only have to verify the existence of weight decompositions (for all objects of \( C' \)). This statement is an immediate consequence of assertion I.1.

5. Immediate from Proposition 1.2.3(1).

II. We take \( C' = \bigcup_{i \in \mathbb{Z}} C''[i] \cup C'_{w' \geq 0}[1] \). Certainly, all elements of \( C' \) possess pre-weight decompositions. According to assertion I.4, the couple \( (C'_{w' \leq 0}, C'_{w' \geq 0}) \) yields a weight structure for \( C' \) if \( C'' \) equals the extension-closure of \( C' \); so we verify the latter fact.

Denote by \( C'' \) the triangulated subcategory of \( C' \) strongly generated by \( C'' \). According to assertion I.1, any \( c \in \text{Obj} C'' \) possesses a pre-weight decomposition, and there (also) exists \( i_c \in \mathbb{Z} \) such that \( c[i_c] \in C'_{w' \geq 0} \). Now, any object of \( C' \) is a retract of an object of \( C'' \); hence it also satisfies the latter property. Applying
Lemma 1.1.1 we easily deduce that $C'$ equals the envelope of $\text{Obj}(C'' \cup C'_{w \geq 0}[1])$; hence it also equals the $C'$-envelope of $C'$. 

Lastly, the boundedness below of $w'$ along with the "moreover" part of the assertion follows immediately from Proposition 1.2.4.

III.1. Recall that $N$ being a retract of $M$ means that $\text{id}_N$ can be factored through $M$. Next, we have $N \perp Y[1]$; hence the corresponding morphism from $N$ into $M$ can be factored through $X$.

2. This assertion can be easily seen to be the categorical dual of the previous one (cf. Proposition 1.2.3(1)).

3. Recall that $C'_{w \leq 0}$ is extension-closed in $C'$. Hence the distinguished triangle $X \to B \to A$ gives $B \in C'_{w \leq 0}$. Next, $B \in C'_{w \geq 0}$ by the definition of a pre-weight decomposition.

Lastly, $N$ is a retract of $X$ according to assertion III.1; hence the "moreover" part of this assertion follows immediately from the previous assertion.

\[ \square \]

Remark 2.1.2. 1. Now we describe an easy application of our theorem.

Let $B$ be an (additive) negative subcategory (see Definition 1.2.1(VII)) of (a triangulated category) $C'$ such that $C'$ is densely generated by $\text{Obj} B$. Then $C'_- = \cup_{i \leq 0} \text{Obj}(B[i])$, $C'_+ = \cup_{i \geq 0} \text{Obj}(B[i])$, and $C'' = \text{Obj} B$ certainly satisfy the conditions of Theorem 2.1.1(II). Thus the corresponding classes $(C'_{w \leq 0}, C'_{w \geq 0})$ yield a bounded weight structure $w'$ for $C'$.

2. Now denote by $C$ the subcategory of $C'$ strongly generated by $B$. Then part I.4 of our theorem immediately implies that $w'$ restricts onto $C$ (in the sense of Remark 1.2.4). Denote the corresponding weight structure for $C$ by $w$. Its existence is precisely Theorem 4.3.2(II.1) of [Bon10].

Now, applying (the "moreover" statement in) part I.1 of our theorem to $(C, w)$ we obtain that for any $M \in \text{Obj} C$ there exists a choice of $X = w_{\leq 0} M$ belonging to the extension-closure of $\cup_{i \leq 0} \text{Obj}(B[i])$. Applying the same part of the theorem to $X$ we obtain the existence of a choice of $w_{\geq 0} X$ belonging to $\text{Obj} B$.

Next, any object $N$ of $C'$ is a retract of some object $M$ of $C$. Taking an arbitrary $N \in C'_{w_{\leq 0} = 0}$ and considering the corresponding $w_{\geq 0} X$ as described above we obtain that $N$ is a retract of $w_{\geq 0} X \in \text{Obj} B$ according to part III.3 of the theorem. Hence $H w'$ equals $\text{Kar}_{C'}(B)$.

Combining this fact with Proposition 1.2.3(5) we also obtain that $\text{Obj} \text{Kar}_{C'}(B)$ strongly generates $C'$.

Furthermore, combining Theorem 2.2.2(1.1) below with Proposition 1.2.3(6) one easily obtains that $w'$ is the only weight structure for $C'$ whose heart contains $B$.

3. For $C'$ as above being idempotent complete our observations yield Proposition 5.2.2 of [Bon10]; so we obtain a new proof of loc. cit. that only relies on §1 of ibid. (and so, it is somewhat easier).

The general case of the statements contained in this remark is completely new.
2.2 On extending weight structures onto idempotent extensions

**Definition 2.2.1.** 1. We will call a triangulated category $C'$ an idempotent extension of $C$ if it contains $C$ and there exists a fully faithful exact functor $C' \to \text{Kar}(C)$.

2. We will say that a weight structure $w$ extends onto an idempotent extension $C'$ of $C$ whenever there exists a weight structure $w'$ for $C'$ such that the embedding $C \to C'$ is weight-exact. In this case we will call $w'$ an extension of $w$.

3. We will say that a triangulated category $C'$ endowed with a weight structure $w'$ is weight-Karoubian if $H^w$ is idempotent complete.

4. We will call a weight-Karoubian category $(C', w')$ a weight-Karoubian extension of $(C, w)$ if $C'$ is an idempotent extension of $C$ and $w'$ is an extension of $w$ onto $C'$.

5. The (triangulated) category $\langle \text{Obj } C \cup \text{Obj Kar}(H^w) \rangle_{\text{Kar}(C)}$ will be denoted by $\text{Kar}^w_{\text{min}}(C)$, and the category $\langle \text{Obj Kar}(C^-) \cup \text{Obj Kar}(C^+) \rangle_{\text{Kar}(C)}$ (see Proposition 1.2.3(4)) will be denoted by $\text{Kar}^w_{\text{max}}(C)$.

Now we study those idempotent extensions of $C$ such that $w$ extends onto them.

**Theorem 2.2.2.** Let $C'$ be an idempotent extension of $C$.

1. Assume that $w'$ is an extension of $w$ onto $C'$. Then $C'_{w \leq 0}$ (resp. $C'_{w' \geq 0}$, resp. $C'_{w' = 0}$) is the retraction-closure of $C_{w \leq 0}$ (resp. $C_{w' \geq 0}$, resp. $C_{w' = 0}$) in $C'$.

2. An extension of $w$ onto $C'$ exists if and only if $C'$ is strongly generated by $\text{Obj } C \cup C_1 \cup C_2$, where $C_1$ consists of certain retracts of objects of $C$ and $C_2$ consists of $C'$-retracts of objects of $C$.

3. An extension $w'$ of $w$ is bounded below (resp. above) if and only if $w$ is.

4. Assume that $w$ is either bounded below or bounded above. Then $w$ extends onto any idempotent extension of $C'$.

5. The categories $\text{Kar}^w_{\text{min}}(C) \subset \text{Kar}^w_{\text{max}}(C)$ are weight-Karoubian extensions of $C$.

6. If $C'$ is a weight-Karoubian extension of $C$ then $H^w$ is equivalent to the idempotent completion of $H^w$.

7. If $w$ extends onto $C'$ then there exists a fully faithful exact functor from $C'$ into $\text{Kar}^w_{\text{max}}(C)$: this functor is weight-exact with respect to the corresponding ("extended") weight structures.

8. If $C'$ is weight-Karoubian then there exists a fully faithful weight-exact functor $\text{Kar}_{w}(C) \to C'$.

**Proof.** 1.1. Since $C'_{w' \leq 0}$, $C'_{w' \geq 0}$, and $C'_{w' = 0}$ are retraction-closed (in $C'$), these classes do contain the retraction closures in question.

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3The latter assumption is certainly equivalent to the following conditions: any object of $C'$ is a retract of some object of $C$; $C$ is dense (see 1.1) in $C'$.
The proof of the converse implication is rather similar to the argument used in Remark 2.1.2(2). Let \( N \) belong to \( \mathcal{C}_{w' \geq 0} \) (resp. to \( \mathcal{C}_{w' \leq 0} \), resp. to \( \mathcal{C}_{w' = 0} \)) be a retract of \( M \in \text{Obj} \mathcal{C} \). Note now that any \( w \)-decomposition of an object of \( \mathcal{C} \) is also its \( w' \)-decomposition. Applying Theorem 2.1.1(III) we obtain that \( N \) is a retract of any choice of \( X = w \leq 0 M \) (resp. of \( w \geq 0 M \), resp. of \( w = 0 X \)), whereas these three objects belong to \( \mathcal{C}_{w \leq 0} \), \( \mathcal{C}_{w \geq 0} \), and \( \mathcal{C}_{w = 0} \), respectively.

2. Assume that an extension of \( w \) onto \( \mathcal{C}' \) exists. Then any object \( M \) of \( \mathcal{C}' \) possesses a weight decomposition with respect to \( w' \). Applying assertion I.1, we obtain that this yields a presentation of \( M \) as an extension of an object \( M_1 \) of \( \text{Kar}_{\mathcal{C}'}(\mathcal{C}^+) \) by an object \( M_2 \) of \( \text{Kar}_{\mathcal{C}'}(\mathcal{C}^-) \). Thus one can take \( C_1 \) being the class of all \( M_1 \) obtained this way, and \( C_2 \) being the class of all \( M_2 \).

To verify the converse implication, for \( \mathcal{C}' \) being strongly generated by \( \text{Obj} \mathcal{C} \cup C_1 \cup C_2 \) we should check that the \( \mathcal{C}' \)-retraction closures \( \mathcal{C}'_{w \geq 0} \) and \( \mathcal{C}'_{w \leq 0} \) of the classes \( \mathcal{C}_{w \geq 0} \) and \( \mathcal{C}_{w \leq 0} \), respectively, yield a weight structure for \( \mathcal{C}' \). For this purpose we apply Theorem 2.1.1(1.4) for \( \mathcal{C}' = \mathcal{C}'_{w \leq 0} \) and \( \mathcal{C}'^+ = \mathcal{C}'_{w \leq 0} \). According to this statement, it suffices to verify that any element of \( \text{Obj} \mathcal{C} \cup (\cup_{i \in \mathbb{Z}} C_1[i]) \cup (\cup_{i \in \mathbb{Z}} C_2[i]) \) possesses a pre-weight decomposition. Certainly, any object of \( \mathcal{C} \) possesses a pre-weight decomposition inside \( \mathcal{C} \). Hence it suffices to verify the existence of pre-weight decompositions for elements of \( \cup_{i \in \mathbb{Z}} C_1[i] \) (since dualization would yield the same assertion for \( \cup_{i \in \mathbb{Z}} C_2[i] \); cf. Proposition 1.3.3(1)).

Thus it suffices to verify the following: for any \( j \in \mathbb{Z} \) and all pairs \((M, N)\), where \( M \in \mathcal{C}_{w \geq j} \) and \( N \) is a \( \mathcal{C}' \)-retract of \( M \), there exists a pre-weight decomposition of \( N \). This fact is certainly true if \( j > 0 \). In the general case we choose \( n \geq 0 \) such that \( j + 2n > 0 \) and recall that \( N \) belongs to the extension-closure of \( \{N[2n]\} \cup \{M[i], 0 \leq i < 2n\} \) (see Lemma 1.1.1). It remains to apply Theorem 2.1.1(1.1).

II.1. Certainly, if all objects of \( \mathcal{C}' \) are \( w' \)-bounded below (resp. above) then all objects of \( \mathcal{C} \) are \( w' \)-bounded below (resp. above); hence they are \( w \)-bounded below (resp. above) also.

The converse implication is immediate from assertion I.1.

2. Immediate from assertion I.2.

III.1. \( w \) extends onto \( \text{Kar}_{\text{min}}^{w}(\mathcal{C}) \) and to \( \text{Kar}_{\text{max}}^{w}(\mathcal{C}) \) according to assertion I.2; these categories are weight-Karoubian according to assertion I.1. Lastly, the existence of weight decompositions in \( \mathcal{C} \) certainly implies that \( \text{Kar}_{\text{min}}^{w}(\mathcal{C}) \subset \text{Kar}_{\text{max}}^{w}(\mathcal{C}) \).

2. Immediate from assertion I.1.

3. The existence of a fully faithful exact functor \( F : \mathcal{C}' \to \text{Kar}_{\text{max}}^{w}(\mathcal{C}) \) is immediate from assertion I.2. \( F \) is weight-exact according to assertion I.1.

4. Certainly, if a weight-Karoubian extension \( \mathcal{C}' \) of \( \mathcal{C} \) is a strict subcategory of \( \text{Kar}(\mathcal{C}) \) then it contains \( \text{Kar}_{\text{Kar}(\mathcal{C})} H \mathcal{C}' \); this implies the existence of a full embedding \( \text{Kar}_{\text{min}}^{w}(\mathcal{C}) \to \mathcal{C}' \). This functor is weight-exact according to assertion I.1.\[\square\]
Remark 2.2.3. 1. In particular, there exists at most one extension of \( w \) onto \( C' \) (so, it may be called "the" extension of \( w \) onto \( C' \)); its heart can be embedded into the idempotent completion of \( Hw \).

2. So, any \((C, w)\) possesses a weight-Karoubian extension. This fact is important for \([Bon15b]\).

3. Certainly, any idempotent complete triangulated category with a weight structure is weight-Karoubian, but the converse fails (for unbounded weight structures). In particular, the categories \( \text{Kar}_\text{min}(C) \) and \( \text{Kar}_\text{max}(C) \) can be distinct from \( \text{Kar}(C) \) (see the example in §3.1).

4. Obviously, part I.2 of the theorem can be reformulated as follows: \( w \) extends to \( C' \) if and only if \( C' \) is strongly generated by \( \text{Obj Kar}^{-}C \cup \text{Obj Kar}^{+}C \).

Assume now that the category \( C \) is essentially small; then its idempotent completion \( D = \text{Kar}(C) \) is also essentially small. Next, the categories \( C' \), \( (\text{Obj } C \cup \text{Obj Kar}^{-}C)_{C'} \), and \( (\text{Obj } C \cup \text{Obj Kar}^{+}C)_{C'} \) may be assumed to be dense in \( D \) for any idempotent extension \( C' \) of \( C \).

Now recall that the Grothendieck group \( K_0(D) \) is defined as follows: it is the Abelian group whose generators are isomorphism classes of objects of \( D \), and such that for any \( D \)-distinguished triangle \( X \to Y \to Z \) the relation \([Y] = [X] + [Z]\) on the classes is fulfilled. Furthermore, sending a subgroup \( H \) of \( K_0(D) \) into the full subcategory of \( D \) whose objects are characterized by the condition \([M] \in H \) one obtains a one-to-one correspondence between the set of subgroups of \( K_0(D) \) and the set of (all) dense subcategories of \( D \); see Theorem 2.1 of \([Tho97]\).

Thus the subcategories \( (\text{Obj } C \cup \text{Obj Kar}^{-}C)_{D} \) and \( (\text{Obj } C \cup \text{Obj Kar}^{+}C)_{D} \) of \( D \) correspond to certain subgroups \( K^{-} \) and \( K^{+} \) of \( K_0(D) \), and one can easily check that \( w \) extends to \( C' \) if and only if for the group \( G = \text{Im}(K_0(C') \to K_0(D)) \) we have \((G \cap K^{-}) + (G \cap K^{+}) = G\).

Yet the authors suspect that this criterion may be rather difficult to apply.

3 Some (counter)examples

By Theorem 2.2.2(II.2), any bounded above (or bounded below) weight structure \( w \) on \( C \) extends onto any idempotent extension of \( C \). In this section we demonstrate that this statement fails for a general \( w \) "for two distinct reasons".

3.1 The category \( \text{Kar}^{w}(C) \) may be "strictly smaller" than \( \text{Kar}(C) \)

Certainly, if \( C^{+} \) and \( C^{-} \) are idempotent complete then \( C \cong \text{Kar}^{w}(C) \). Now we construct an example of this situation with \( C \) not being idempotent complete; it certainly follows that \( \text{Kar}^{w}(C) \) is not equivalent to \( \text{Kar}(C) \) in this situation.

Consider \( C = K(A) \), where \( A \) is an additive category with \( K_{-1}(A) \neq 0 \); here we endow \( A \) with the "trivial" structure of an exact category and define the groups \( K_{i}(A) \) using Definition 8 of \([Sch06]\). Note that for this purpose one can take \( A \) being the category of finitely generated projective modules over a
(commutative) ring $R$ such that $K_{-1}(R) \neq 0$ (see Theorem 5 of ibid.), whereas rings satisfying this condition are well known to exist. Next, Corollary 6 of ibid. implies that in this case $K(A)$ is not idempotent complete.\footnote{Theorem 5, Corollary 6, and Definition 8 in the published version of this paper correspond to Theorem 7.1, Corollary 8.2, and Definition 5.4 in the K-theory archives preprint version, respectively.}

Now take $w$ being the stupid weight structure for $C$ (see Remark \ref{rem:stupid}). Then the categories $C^+$ and $C^-$ are idempotent complete according to Proposition \ref{prop:idempotent}. Thus Kar$_{w \max}(C)$ is equivalent to $C$, whereas Kar$(C)$ is not, and we obtain the desired example.

Lastly, applying Theorem \ref{thm:weight} III.3 we conclude that a weight structure on a triangulated category does not necessarily extend onto its idempotent completion.

This example also demonstrates that there exist rather "natural" triangulated categories that are not idempotent complete.

### 3.2 An idempotent extension inside Kar$_{w \max}(C)$ such that $w$ does not extend onto it

Now we construct an example of $(C, w)$ and an idempotent extension $C'$ of $C$ such that Kar$_{w \min}(C) \subset C' \subset$ Kar$_{w \max}(C)$, but $w$ does not extend onto $C'$. Certainly, $w$ will not be bounded either above or below (cf. Theorem \ref{thm:weight} II.2).

Let $L$ be an arbitrary (fixed) field; denote by $L$-vect the category of finite dimensional $L$-vector spaces. We take for $C$ the subcategory of $K(L$-vect) consisting of those complexes almost all of whose (co)homology are even-dimensional (i.e., only a finite number of homology spaces are of odd dimension). Obviously, the stupid weight structure for $K(L$-vect) restricts onto $C$ (see Remark \ref{rem:stupid}); the heart of this $w$ is equivalent to $L$-vect. Since for any $N \in \text{Obj} K(L$-vect) we have $N \oplus N \in \text{Obj} C$, $K(L$-vect) is an idempotent extension of $C$ such that $w$ extends onto it. Since $K(L$-vect) is also idempotent complete, we have $K(L$-vect) $\cong$ Kar$_{w \max}(C) = \text{Kar}(C)$. Moreover, Kar$_{w \min}(C)$ is equivalent to $C$ since $L$-vect is idempotent complete.

Now, consider $C' \supset C$ consisting of complexes almost all of whose cohomology have the same parity. Certainly, $C' \subset K(L$-vect) $\cong$ Kar$_{w \max}(C)$. Yet any object of $C'$ with odd-dimensional cohomology (for example, $M = \ldots L \to L \to L \to L \to \ldots$) cannot possess a weight decomposition inside $C'$ (since any its weight decomposition is essentially a "stupid" one in $K(L$-vect) according to Remark \ref{rem:stupid}).

### 4 On motivic applications of Theorem \ref{thm:main}

Now we describe the application of Theorem \ref{thm:main} II to various "relative motivic" categories, i.e., we consider certain triangulated categories of motives over
schemes that are separated and of finite type over a certain (fixed) base scheme $B$. We always assume $B$ is Noetherian separated excellent of finite Krull dimension; we will call schemes that are separated and of finite type over $B$ just $B$-schemes, and all $B$-morphisms mentioned below will be assumed to be separated (and of finite type).\footnote{So, for a $B$-scheme $Y$ a $B$-morphism into $Y$ is just a separated morphism of finite type.}

Our main examples will be certain subcategories of motivic categories of the following type:

(i) For any $B$ satisfying these conditions one can consider the categories of Beilinson motives over $B$-schemes; recall that Beilinson motives are certain Voevodsky motives with rational coefficients that were one of the main subjects of \cite{CI12} (that heavily relied on \cite{AY07}).

(ii) If we assume in addition that $B$ is a scheme of characteristic $p$ then for any $\mathbb{Z}[[\frac{1}{p}]]$-algebra $R$ (we set $\mathbb{Z}[[\frac{1}{p}]] = \mathbb{Z}$ if $p = 0$) one can consider $R$-linear $cdh$-motives over $B$-schemes (this is another version of Voevodsky motives that was studied in detail in \cite{CI15}).

(iii) For any $B$, $\mathcal{S}$ being a set of primes containing all primes non-invertible on $B$, $\Lambda = \mathbb{Z}[\mathcal{S}^{-1}]$, and $Y$ being a $B$-scheme one can consider the $\Lambda$-linear version of the homotopy category of modules over the symmetric motivic ring spectrum $K_{G\text{L}} Y$, i.e., following §13.3 of \cite{CI12} (that relied on \cite{RSO10}) one should consider a certain Quillen model for the motivic stable homotopy category $SH(Y)$, then pass to the category of strict left modules over $K_{G\text{L}} Y$ (that is a certain highly structured ring spectrum weakly homotopy equivalent to the Voevodsky’s $K$-theory spectrum $K_{G\text{L}} Y$), and "invert the primes in $\mathcal{S}$" using the corresponding well-known method (see \cite{Ker12} §A.2 or \cite{BoL16, Proposition 1.1.1}). We will denote this category by $DK(Y)$ and call its objects $K$-motives.

(iv) For any $B$ as in (ii), a $B$-scheme $Y$, any set of primes $\mathcal{S}$ containing $p$, and $\Lambda = \mathbb{Z}[\mathcal{S}^{-1}]$ one can similarly take the $\Lambda$-linear version of the category $\text{DMGl}(Y)$ of strict left modules over the Voevodsky’s spectrum $\text{MGl}_Y$.

Actually, any (other) couple $(B, D)$ that fulfills a certain (rather long) list of "axioms" is fine for our purposes; cf. \cite{Bon16, §3}.

All these categories can be endowed with the corresponding Chow weight structures. This fact was established in \cite{Bon14} for the case (i) (whereas in \cite{Heb11} just a little less general result was independently obtained; see below) and in \cite{BoL15} for the case (ii); moreover, the methods of ibid. can actually be used in all the four cases described above; see §2 of \cite{BoL16} and Remark 3.4.3 of \cite{Bon16} for more detail. For simplicity in this paper we will concentrate on the "weights of" compact objects of these motivic categories (though the corresponding weight structures extend to the corresponding "big" categories; see \cite{BoL16, §2.3} and \cite{Bon16, Proposition 1.2.4}). So all the (Chow) weight structures mentioned below will be bounded.

Now we introduce some notation for these categories and try to explain what the Chow weight structure is. For $Y$ being a $B$-scheme the full tensor triangulated subcategory of compact objects in $\mathcal{D}(Y)$ (for $\mathcal{D}(\_)$ being any of the four aforementioned motivic categories) will be denoted by $\mathcal{D}^c(Y)$ and its
tensor unit will be denoted by $1_Y$. We recall that the behaviour of the categories $D^c(-)$ and of the "weights" of their objects is quite similar to that of mixed $\mathbb{Q}$-complexes of étale sheaves and their weights as studied in [BBDS2]. So, for any (separated) $B$-morphism of $B$-schemes $f : X \to Y$ there are two pairs of adjoint functors $f_! : D^c(X) \rightleftarrows D^c(Y) : f^!$ and $f^* : D^c(Y) \rightleftarrows D^c(X) : f_*$. Next (cf. Theorem 5.3.8 of [BBDS2]), if $X$ is regular then the object $1_X$ is of weight $0$; if $f$ is proper then the functor $f_* = f_!$ is weight-exact. Hence $f_*(1_X) \in D^c(Y)_{w_{Chow}(Y) = 0}$ under these assumptions. Moreover, for any $Y$ we have a natural splitting $g_*(1_{\mathbb{P}^1(Y)}) \cong 1_Y \oplus 1_Y(-1)$ for a certain $\otimes$-invertible object $1_Y(-1)$ (and the natural $g : \mathbb{P}^1(Y) \to Y$). We denote the tensor product by the $-n$th power of $1_Y(-1)$ in $D^c(Y)$ by $-(n)$ for any $n \in \mathbb{Z}$; this is a certain version of Tate twist that is denoted by $-(n)[2n]$ in the Voevodsky’s convention introduced in [Voe00a]. Now, the functor $-(n)$ is weight-exact for any $Y$ and any $n \in \mathbb{Z}$. So, $D^c$-retracts of objects of the form $f_*(1_P)(n)$ for $f : P \to Y$ running through all proper morphisms with regular domains and $n \in \mathbb{Z}$ will be called $D$-Chow motives over $Y$; they form an additive subcategory of $\mathbb{H}^{w_{Chow}(Y)}$. In particular, this category is negative in $D^c(Y)$ (see Definition 1.2.1(VII)). Note here that in the case where $Y$ is the spectrum of a perfect field $k$ and $f$ is any smooth morphism, $D^c(Y) = DM_{gm}$ is the category of geometric Voevodsky motives (with coefficients in any ring) then the object $f_! f^!(1_Y)$ is isomorphic to the Voevodsky motif of the variety $P$ (over $k$), and $f_! f^!(1_Y) \cong f_!(1_X)(d)$ whenever all connected components of $X$ are of dimension $d$. Hence our definition of Chow motives generalizes the description of the subcategory of Chow motives inside $DM_{gm}(k)$ (see [Voe00a]).

So, an important question for the construction and study of the Chow weight structure for $D^c(Y)$ is whether these Chow motives strongly generate $D^c(Y)$. To answer it one requires some statement on the "abundance" of proper $Y$-schemes that are regular; thus it is a certain resolution of singularities problem. In the case where $Y = \text{Spec} \ k$, $k$ a characteristic 0 field, it was proved in [Voe00a] (using Hironaka’s resolution of singularities) that the subcategory of $D^c(Y)$ strongly generated by the motives of smooth projective $k$-varieties also contains the motives of all smooth varieties. Thus the category $\langle \text{Obj Chow}(k) \rangle_{DM_{gm}(k)}$ is dense in $DM_{gm}(k)$. Next, a formal argument (essentially using weight decompositions) was applied in [Bon09] to prove that $\langle \text{Obj Chow}(k) \rangle_{DM_{gm}(k)}$ actually equals $DM_{gm}(k)$. This method of the proof can be applied to all of our four examples of $D(Y)$ whenever $Y$ is of characteristic 0 (i.e., if it is a Spec $\mathbb{Q}$-scheme); see Theorem 2.4.3 of [Bon13]. For other $Y$ one needs certain alterations (de Jong’s ones for rational coefficients and Gabber’s ones in general) and somewhat more complicated "formal" arguments. So, if the coefficient ring is not a $\mathbb{Q}$-algebra then our current level of knowledge enables us to prove that $D^c(Y)$ is

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This observation was treated in §3 of [Bon13]. Note however that "weights" for mixed complexes of étale sheaves do not correspond to any weight structures; see Remark 2.5.2 of [Bon13]. On the other hand, we have the (self-dual) perverse $t$-structure $p_{1/2}$ for mixed complexes of sheaves that "respects weights", whereas the existence of its motivic analogue (essentially suggested by Beilinson; cf. [Bon13]) is an extremely hard conjecture (that may be true only for motives with coefficients in a $\mathbb{Q}$-algebra).
strongly generated by \( D \)-Chow motives over \( Y \) only under the assumption that \( Y \) is essentially of finite type over a field; see [Bon11] (cf. also [Kel12] Proposition 5.5.3]) for the case \( Y = \text{Spec} k \) and [BoI15] §2.3 for the general case. For rational coefficients substantially weaker assumptions on \( Y \) are sufficient (cf. [BoI15] §2.4); the corresponding method of constructing \( w_{\text{Chow}} \) was applied in [ILO14]. It appears (though this argument was never written down in the general case; yet cf. [BoI15] §2.4 for a somewhat related reasoning) that these assumptions on \( Y \) (or \( B \)) are also sufficient to ensure (more or less) "easily" that the corresponding Chow weight structure for \( D^c(Y) \) restricts (see Remark 1.2.4) onto the levels of the dimension filtration for \( D^c(Y) \) (that we will describe below).

Now we explain how the weight structure \( w_{\text{Chow}}(Y) \) can be defined if the aforementioned assumptions on \( Y \) are not fulfilled. One of possible definitions of \( w_{\text{Chow}}(Y) \) in the case (i) (for a "general" \( Y \)) was given in [Bon14] §2.3]; it used stratifications of \( Y \). In [BoI15] some equivalent descriptions of \( w_{\text{Chow}}(Y) \) were given (though formally in ibid. only \( D^c(-) \) of type (ii) was studied, the corresponding arguments can be applied to all of our examples of \( D^c(-) \) without any problems). We recall one of these descriptions here.

Note that for any \( B \)-morphism \( f \) (that is not necessarily proper) the functors \( f_* \) and \( f_! \) should possess certain weight-exactness properties with respect to the corresponding Chow weight structures; cf. the 'stabilities' 5.1.14 of [BBDS2]. Thus for any \( f: P \to Y \), where \( P \) is a regular scheme, and \( n \in \mathbb{Z} \) we should have \( f_*(1_P)\langle n \rangle \in D^c(Y)_{w_{\text{Chow}}(Y)\geq 0} \) and \( f_!(1_P)\langle n \rangle \in D^c(Y)_{w_{\text{Chow}}(Y)\leq 0}. \) So we define \( D^c(Y)_{w_{\text{Chow}}(Y)\geq 0} \) (resp. \( D^c(Y)_{w_{\text{Chow}}(Y)\leq 0} \)) as the envelope of \( f_* (1_P)\langle n \rangle \langle i \rangle \) (resp. of \( f_!(1_P)\langle n \rangle \langle -i \rangle \)) for \( f: P \to Y \) running through all \( B \)-morphisms with regular domain, \( n \in \mathbb{Z} \) and \( i \geq 0 \).

However, checking the existence of weight decompositions of objects of \( D^c(Y) \) with respect to \( w_{\text{Chow}}(Y) \) (using any of the descriptions of this weight structure) is not that easy. The authors know two methods for proving this statement. The first one is the gluing of weight structures argument described in [Bon14] §2.3] and [BoI15] §2.1]. Yet this formal argument does not yield much information on "weights" and weight decompositions of ("concrete") objects of \( D^c(Y) \). In particular, if one considers (following [Voe00a]) certain dimension filtration for \( D^c(Y) \) then the gluing argument does not imply that an object belonging to some level of this filtrations possesses a weight decomposition inside this level.

To overcome this difficulty and construct "more or less explicit" \( w_{\text{Chow}} \)-decompositions certain (complicated) "geometric" arguments were described in [BoI15] §3]; they used [ILO14] Theorems IX.1.1, II.4.3.2] and were closely related to Gabber's arguments applied in [ILO14] §XIII] to the study of the con-

\[ \text{Recall that for } Z \text{ being any closed subscheme of } Y \text{ and } U = Y \setminus Z \text{ the categories } D(Y), D(Z), \text{ and } D(U) \text{ along with the natural functors connecting them yield a gluing datum in the sense of §1.4.3 of } \text{[BBDS2]}; \text{ cf. Proposition 1.1.2(10) of } \text{[Bon14]. Furthermore, weight structures can be "glued" in this setting according to Theorem 8.2.3 of } \text{[Bon10]. One also needs certain "continuity" arguments to "glue } w_{\text{Chow}}(Y) \text{ from the Chow weight structures over points of } Y\text{.} \]

\[ \text{The problem is that some of the functors in the aforementioned gluing datum do not respect this filtration.} \]
structibility for complexes of étale sheaves (that were also applied to the study of constructibility of motives in [CiD12 §4.2]). Unfortunately, the corresponding Theorem 3.4.2 of [BoL16] is too complicated to formulate it here. So (following [Bon16 §3.4]) we formulate some of its consequences instead.

Now it is the time to define dimension filtration for \( D^c(Y) \). One of the problems here is that to obtain a "satisfactory" filtration we need some sort of dimension function \( \delta \) on \( B \)-schemes; the reason is that we want some notion of dimension that would satisfy the following property: if \( U \) is open dense in \( X \) then its "dimension" \( \delta(U) \) should be equal to \( \delta(X) \). In the case where \( B \) is a Jacobson scheme all of whose components are equidimensional one may take \( \delta \) being equal to the Krull dimension function. More generally, one may take \( \delta \) being a "true" dimension function as described in [IL014 §XIV.2] (cf. §1.1 of [BoD15]) or being a certain its "variation" as introduced in [BoL16 §3.1] or being a "true" dimension function as described in [ILO14 §XIV.2] (cf. §1.1). In particular, this statement may be applied in the case where \( B \) is of finite type over \( \text{Spec} \mathbb{Z} \) or over \( \text{Spec} k \); yet the latter case is not really interesting to us due to the reasons described above.

The dimension filtration on \( D^c(Y) \) is defined as follows: for any \( j \in \mathbb{Z} \) we take \( D^c_{\leq j}(Y) \) being the subcategory of \( D^c(Y) \) that is densely generated (see [4.11] by \( \{ f_!(1_{P})|\delta(P)\} \) for \( f: P \to Y \) running through all (separated finite type) \( B \)-morphisms with \( \delta(P) \leq j \).

Now we recall the main ingredient of the proof of Theorem 3.4.2(1) of [Bon16].

**Proposition 4.1.** For any \( m \in \mathbb{Z} \) and a \( B \)-morphism \( f: P \to Y \) the object \( M = f_!(1_{P})|\delta(P)\) possesses an \( m \)-weight decomposition such that the corresponding \( w_{\leq m}M \) and \( w_{\geq m+1}M \) belong to \( D^c_{\leq \delta(P)}(Y) \).

**Proof.** This is an easy consequence of [BoL16 Theorem 3.4.3]; see also Proposition 3.4.1(2) of [Bon16] for some more detail.

**Remark 4.2.** 1. Since all objects of \( D^c(Y) \) are \( w_{\text{Chow}}(Y) \)-bounded, combining this proposition with Theorem 2.2.1(II) we obtain that \( w_{\text{Chow}}(Y) \) restricts onto \( D^c_{\leq j}(Y) \) for any \( j \). Thus \( w_{\text{Chow}}(Y) \) also restricts onto the union \( \cup_{j \in \mathbb{Z}} D^c_{\leq j}(Y) \) that we will denote by \( D^c_{\text{eff}}(Y) \). The latter fact is certainly (formally) weaker than the existence of all the restrictions onto \( D^c_{\leq j}(Y) \); yet the authors do not know any its proof that does not rely on [BoL16].

2. We will call \( D^c_{\text{eff}}(Y) \) the subcategory of \( \delta \)-effective objects of \( D^c(Y) \). Note here that \( D^c_{\text{eff}}(Y) \) is the subcategory of \( D^c(Y) \) that is densely generated by \( \{ f_!(1_{P})|\delta(P)\} \) for \( f: P \to Y \) running through all (separated finite type) \( B \)-morphisms.

This definition originates from Definition 2.2.1 of [BoD15]; it is also closely related to an earlier definition from [Peh13 §2].

Recall also that the definition of Voevodsky motives (in [Voe00a]) actually "started from" certain effective motivic categories. It seems that this method does not work so nicely for general ("relative") motivic categories; still our
definition of \( \mathcal{D}_{\text{eff}}(Y) \subset \mathcal{D}(Y) \) essentially generalizes the one of ibid. according to [BoD15, Example 2.3.12].

3. Certainly, having the category \( \mathcal{D}_{\text{eff}}(Y) \) one can also consider the slice filtration of \( \mathcal{D}(Y) \) by the subcategories \( \mathcal{D}_{\text{eff}}(Y)(i) = \mathcal{D}_{\text{eff}}(Y)(i) \) for \( i \) running through integers. Note also that the union \( \cup_{i \in \mathbb{Z}} \mathcal{D}_{\text{eff}}(Y)(i) \) equals \( \mathcal{D}(Y) \) itself. Moreover, Theorem 3.4.2(II) (along with Remark 3.4.3(1)) of [Bon16] easily implies that the intersection of \( \mathcal{D}_{\text{eff}}(Y)(i) \) for all \( i \in \mathbb{Z} \) is zero for \( \mathcal{D} \) being of our "type" (i), (ii), or (iv). On the other hand, \( K \)-motives are periodic, i.e., \( \mathbf{1}_Y(i) \cong \mathbf{1}_Y \) for any \( i \in \mathbb{Z} \) (and so, all \( \mathcal{DK}_{\text{eff}}(Y)(i) \) equals \( \mathcal{DK}(Y) \)).

Furthermore, one can easily note that Proposition 4.1 (combined with Theorem 2.1.1(II)) can be used to prove the existence of \( w_{\text{Chow}}(Y) \)-decompositions for all objects of \( \mathcal{D}_{\text{eff}}(Y) \) and \( \mathcal{D}(Y) \) (i.e., one can easily avoid circularity in this argument).

4. These two types of filtrations for the "whole" \( \mathcal{D}(Y) \) were the main subject of [Bon16] §3.

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