Matrix formulation for non-Abelian families

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We generalize the $K$ matrix formulation to non-trivial non-Abelian families of 2+1D topological orders. Given a topological order $C$, any topological order in the same non-Abelian family as $C$ can be efficiently described by $a = (a_i)$ where $a_i$ are Abelian anyons in $C$, together with a symmetric invertible matrix $K$, $K_{IJ} = k_{IJ} - t_{a_I,a_J}$ where $k_{IJ}$ are integers, $k_{IJ}$ are even and $t_{a_I,a_J}$ are the mutual statistics between $a_I, a_J$. In particular, when $C$ is a root whose rank is the smallest in the family, $K$ becomes an integer matrix. Our results make it possible to generate the data of large numbers of topological orders instantly.

\section*{Introduction:} Topological phases of matter have drawn more and more research interest during recent years. A most remarkable feature of topological phases is that there can be several quantum states which are “topologically” degenerate. Such degeneracy is robust against any local perturbation, thus these states can be employed as qubits that are automatically immune to local noises. Given the possible application in quantum memory and quantum computation, it is then natural to ask how to produce the desired topological degeneracy.

One source of topological degeneracy is to put topological ordered system on a manifold with nontrivial topology\textsuperscript{1–4}. This approach is not ideal: for one reason, it is not easy to shape a physical system into nontrivial manifold such as a torus; for another, to manipulate the degenerate ground states one has to perform non-local operations.

Another source of topological degeneracy is to trap several anyonic quasiparticles. By braiding and fusion of these anyons, it is possible to realize universal topological quantum computation\textsuperscript{5}. For an anyon $i$, we use the quantity $d_i$, called the quantum dimension, to measure the effective topological degeneracy carried by $i$. When there is a large number $N$ of anyon $i$ trapped, the topological degeneracy is of the order $d_i^N$.

Thus for anyons to produce desired topological degeneracy, it is necessary that $d_i > 1$. An anyon with $d_i = 1$ is called Abelian while with $d_i > 1$ is called non-Abelian. If all the anyons in a topological order are Abelian, it is called an Abelian topological order. Clearly Abelian topological phases are useless in the braiding-fusion based topological quantum computation.

In Ref. 6 we proposed the generalized hierarchy construction that can add or remove Abelian anyons to or from any topological order. Two topological orders which can be connected by such construction are of the same “non-Abelian family”, which is the equivalence class up to Abelian topological orders. The non-Abelian family captures the invariants of non-Abelian anyons, and we expect that topological orders in the same non-Abelian family behave similarly in topological quantum computation.

However, the construction in Ref. 6 is performed in a step-by-step manner. Given a topological order $C$, it is not easy to calculate the property of another topological order in the same non-Abelian family that requires several steps of hierarchy constructions from $C$. This letter aims at resolving such difficulty. We showed that given a topological order $C$, any topological order in the same non-Abelian family can be efficiently represented by a sequence of Abelian anyons in $C$ together with a $K$ matrix. When $C$ is the trivial topological order, our result reduces to the original $K$ matrix formulation for Abelian topological orders\textsuperscript{7}.

\section*{One-step generalized hierarchy construction:} We first review and refine the construction proposed in Ref. 6. The main idea is to let Abelian anyons form an effective Laughlin-like state\textsuperscript{8}. This idea dates back to Haldane and Halperin, known as “hierarchy” construction\textsuperscript{9,10}. But below we discuss it at a more general level.

We start with a topological phase $C$. The anyons in $C$ are labeled by $i,j,k,...$. Let $a_c$ be an Abelian anyon in $C$ with topological spin $s_{a_c}$. $s_{a_c}$ determines the self statistics of $a_c$: exchanging two $a_c$ anyons leads to the phase factor $e^{2\pi i s_{a_c}}$. We try to make $a_c$ form the Laughlin state,

$$\langle \{z_a\}|\Psi\rangle = \prod_{a<b}(z_a - z_b)^{M_{c}} \times e^{-\frac{i}{2} \sum |z_a|^2}.$$  

(1)

The resulting topological phase is determined by $C, a_c$ and $M_{c}$, which will be denoted by $C_{a_c,M_{c}}$. Here $z_a, z_b$ are the positions of $a_c$ anyons. $M_{c}$ must be consistent with anyon statistics. Consider exchanging two $a_c$ anyons, we obtain: a phase factor $e^{2\pi i s_{a_c}}$ from the wave function and a phase factor $e^{2\pi i s_{a_c}}$ from anyonic statistics. To be consistent, total phase factor must be 1:

$$\frac{M_{c}}{2} + s_{a_c} \in \mathbb{Z}.$$  

(2)

So we need to take $M_{c} = m_{c} - 2s_{a_c}$, where $m_{c}$ is an even integer.

Anyon $i$ in the phase $C$ may be dressed with a flux $M_{i}$ in the new phase $C_{a_c,M_{c}}$.

$$\psi(i,M_i) = \prod_{b}(\xi_i - z_b)^{M_i} \prod_{a<b}(z_a - z_b)^{M_{c}} \times e^{-\frac{i}{2} \sum |z_a|^2}.$$  

(3)
Here $\xi_i$ is the position of anyon $i$. Thus an anyon in the new phase is represented by a pair $(i, M_i)$. Again, $M_i$ can not be arbitrary. If $a_c$ has trivial mutual statistics with $i$, $M_i$ can be any integer. Otherwise, consider moving $a_c$ around $(i, M_i)$ and we obtain: a phase factor $e^{2\pi i M_i}$ from the flux $M_i$ and a phase factor $e^{2\pi i t_i}$ from the mutual statistics between $a_c$ and $i$. The mutual statistics can be extracted from the $S$ matrix, $e^{2\pi i s_{ij}} = DS_{ac,Mc}/d_t$, $t_{ac} = 2s_{ac}$. To be consistent, total phase factor must be 1:

$$M_i + t_i \in \mathbb{Z}. \quad (4)$$

Since the anyon $a_c$ dressed with a flux $M_c$ is a “trivial excitation” in the new phase:

$$\Psi(a_c, M_c) \sim \prod_{b}^{n}(\xi_{ac} - z_b)^{M_b} \prod_{a<b}^{n+1}(z_a - z_b)^{M_c}$$

$$= \prod_{a<b}^{n+1}(z_a - z_b)^{M_c},$$

$$(a_c, M_c) \sim (1, 0), \quad (5)$$

we have the equivalence relation:

$$(i, M_i) \sim (i \otimes a_c, M_i + M_c). \quad (6)$$

Next we list the data of the resulting topological order $C_{a_c, M_c}$:

- The spin of $(i, M_i)$ is given by the spin of $i$ plus the “spin” of the flux $M_i$:

$$s_{(i, M_i)} = s_i + \frac{M_i^2}{2M_c}. \quad (7)$$

- To fuse anyons $(i, M_i), (j, M_j)$ in the new phase, just fuse $i, j$ as in the old phase, and add up the flux:

$$(i, M_i) \otimes (j, M_j) = \bigoplus_k N_k^{ij}(k, M_i + M_j). \quad (8)$$

And then apply the equivalence relation $(6)$.

- The rank (number of anyon types) of $C_{a_c, M_c}$ is

$$N^{C_{a_c, M_c}} = |M_c|N^{C}. \quad (9)$$

- The quantum dimensions remain the same

$$d_{(i, m)} = d_i. \quad (10)$$

- The $S$ matrix is

$$S^{C_{a_c, M_c}}_{(i, M_i), (j, M_j)} = \frac{1}{\sqrt{|M_c|}} S^C_{ij} e^{-2\pi i \frac{M_i M_j}{|M_c|^2}}. \quad (11)$$

- The chiral central charge is

$$c^{C_{a_c, M_c}} = c^C + \text{sgn} M_c. \quad (12)$$

The one-step hierarchy construction is reversible. In $C_{a_c, M_c}$, choosing $a_c' = (1, 1)$, $s_{a_c'} = \frac{\pi}{2M}$, $m_{a_c}' = -1/M_c$, and repeating the construction, we will go back to $C$. Therefore, hierarchy construction defines an equivalence relation between topological phases. We call the corresponding equivalence classes the “non-Abelian families”. Each non-Abelian family have “root” phases with the smallest rank. Let $C_{Ab}$ denote the full subcategory of all Abelian anyons in $C$, $C$ is a root if$^{1}$ and only if$^{1}$ $C_{Ab}$ is a symmetric fusion category, namely all the Abelian anyons are bosons or fermions with trivial mutual statistics with each other.

Multiple steps of construction and the matrix formulation: Now we consider multiple steps of hierarchy constructions and try to write down the final result at once. Note that in the flux label $M_i$ we need to use the mutual statistics in the previous step, and things get involved when there are multiple steps. To separate out the mutual statistics and thus make things clearer, we use the “integer convention” $(i, m)$, instead of the “flux convention” $(i, M_i)$, where $m = t_i = M_i$. Now consider starting from a topological order $C$ and performing one-step construction $\kappa$ times. The first step we take $a_1 \in C_{Ab}$ and even integer $k_1$. The second step we take an Abelian anyon $(a_2 \in C_{Ab}, k_1)$ and even integer $k_2$, where $k_2$ is an integer. The third step we take an Abelian anyon $(a_3 \in C_{Ab}, k_1, k_2)$ and even integer $k_3$, where $k_3$ is integer. Keep moving on and we see that the steps can be summarized by $a_1$ and $k_1$. Define a corresponding integer symmetric $\kappa$ by $\kappa$ matrix by setting $k_{IJ} = k_{IJ}$. Denote by $t_i,a$ the mutual statistics between anyon $i$ and Abelian anyon $a$ in $C$ (note $e^{2\pi i k_{IJ}}$ is the phase factor of braiding $a$ around $i$), by $k_{IJ}$ the spin of anyon $i$ in $C$, and set $t_{a, a} = 2s_a$. Let the $K$ matrix be

$$K_{IJ} = k_{IJ} - t_{a_1}, \quad (13)$$

Physically, we let the Abelian anyons $a_1$, $I = 1, 2, \ldots, \kappa$ form a multilayer Laughlin-like state

$$\prod (\xi_a - z_b^I)^{K_{IJ}},$$

where $I$ labels the layer and $z_b^I$ is the position of $a_I$ anyon. By a similar argument as in the one-step case, we know that $K_{IJ} + t_{a_1, a_j}$ must be an integer and $K_{IJ} + t_{a_1, a_j}$ must be an even integer.

Though we are using the integer convention, note that similar to the one-step case, it is the combination $K_{IJ} = k_{IJ} - t_{a_1, a_j}$ or $M_c = m_c - t_a$ that determines the final topological order, not the integer $k_{IJ}$ or $m_c$ alone. The meaning of $k_{IJ}$ or $m_c$ depends on the choice of mutual statistics $t_{i,a}$.

The fusion rule and $T, S$ matrices of the resulting topological order after $\kappa$ steps can be calculated efficiently via the $K$ matrix as stated in Theorem 1. This result generalizes the $K$ matrix formulation for Abelian topological orders$^7$.

**Theorem 1.** The topological order constructed from root $C$ via $\kappa$ steps can be summarized by $a_I$ and $K_{IJ}$,
where \( I, J = 1, \ldots, \kappa, a_I \in C_{Ab}, \) det \( K \neq 0, K_{IJ} = K_{JI}, K_{IJ} + t_{a_I,a_J} \) are integers and \( K_{IJ} + t_{a_I,a_J} \) are even. Let \( a \) formally denote the vector \((a_I)\) and \( C_{a,K} \) denote the resulting topological order. \( C_{a,K} \) is as follows:

- Fix a choice of mutual statistics \( t_{i,a_I} \) in \( C \). Let \( t_i \) be the \( \kappa \)-dimensional vector \((t_{i,a_I})\). Anyons are labeled by \((i \in C, I)\) where \( I \) is a \( \kappa \)-dimensional integer vector, subject to the following equivalence relations

\[
(i, I) \sim (i \otimes a_I, I + K - t_i + t_{i \otimes a_I}).
\]

(14)

where \( K_I \) is the \( I \)th column vector of \( K \). For a different choice of mutual statistics, or representative \( i' \sim i, t_{i',a_I} \) differs from \( t_{i,a_I} \) by an integer, and \((i', I + t_i - t_i) \sim (i, I)\). \( C_{a,K} \) does not depend on the choice of mutual statistics or representative in \( C \).

- Fusion is given by

\[
(i, I) \otimes (j, K) = \oplus N^{ij}_a(s, I + k - t_i - t_j + t_s).
\]

(15)

- The spin of \((i, I)\) is

\[
s_{(i, I)} = s_i + \frac{1}{2}(I - t_i)^T K^{-1} (I - t_i).
\]

(16)

- The \( S \) matrix is

\[
S_{(i, I)(j, K)} = \frac{1}{\sqrt{\text{det} K}} S_{ij} e^{-2\pi i(l - t_i)^T K^{-1}(k - t_j)}.
\]

(17)

- The rank is \( N^{C_{a,K}} = \text{det} K \). The chiral central charge is \( e^{\pi \alpha_{C_{a,K}}^+} = e^\alpha + \text{sgn} K \). Here \( \text{sgn} K \) denotes the index of the matrix \( K \), namely the number of positive eigenvalues minus the number of negative eigenvalues.

Proof. We postpone the lengthy proof to Appendix A.

When \( C \) is a root whose Abelian anyons \( C_{Ab} \) is a symmetric fusion category, \( a_I,a_J \) are mutually trivial, and \( t_{a_I,a_J} \) are integers. In particular, we can choose \( t_{a_I,a_J} = 1 \) when \( a_I \) is fermionic, and other \( t_{a_I,a_J} = 0 \). In this case the \( K \) matrix is an integer matrix and \( K_{IJ} \) is even when \( a_I \) is a boson and odd when \( a_I \) is a fermion.

**Equivalence relation of \( C_{a,K} \):** Starting form the same topological order \( C \), different paths of construction may result in the same topological order. It is natural to ask what is the equivalence relation for \((a, K)\). For now, we know three ways to generate equivalent \( C_{a,K} \):

1. The equivalence between the starting point \( F : C \simeq D \) naturally give rise to equivalence \( C_{a,K} \simeq D_{F(a),K} \).

2. “Integer linear recombination” of \( a_I \), \( W \in \text{GL}(\kappa, \mathbb{Z}) \) (namely \( W \) is an integer matrix with \( \text{det} W = \pm 1 \)). \( C_{a,K} \simeq C_{W a,K W^T} \). We call such transformation as the \( \text{GL}(\mathbb{Z}) \) transformation.

3. The reversibility of one-step construction means that the topological order constructed from \( C \) with \((a_I = a_c, \ K = (M_c \ 1) \) is equivalent to \( C \). Also \((a_I, K_{IJ}) \) is equivalent to \((a_I, t_{IJ} - t_a 0 1 0)\), where \( a \) can be any Abelian anyon in \( C_{Ab} \). Note that under \( \text{GL}(\mathbb{Z}) \) transformation, \( (K_{IJ} 0 0 0, -2s_a a_I) \rightarrow (K \oplus (-t_{a,a} 1 1 0)) \).

We have \((a, K) \sim (a \oplus (a I), K \oplus (-t_{a,a} 1 1 0)) \). We refer to \((a 1, -t_{a,a} 1 0)\) as the “trivial bilayer”.

**Conjecture 1.** \( C_{a,K} \) and \( C_{a',K'} \) (with exactly the same chiral central charge, not modulo 8) are equivalent if and only if, up to automorphisms of \( C \) and \( \text{GL}(\mathbb{Z}) \) transformations, \((a \oplus b, K \oplus X) \sim (a' \oplus b', K' \oplus X') \) where \((b, X) \) and \((b', X') \) are direct sums of trivial bilayers \((a 1, -t_{a,a} 1 0)\).

**The formal categorical formulation:** We give the formal basis independent formulation of the above constructions. Let \( C \) be a braided fusion category, \( \alpha_{A,B,C} \) denote the associator and braiding in \( C, C_{Ab} \) denote the Abelian group corresponding to the pointed subcategory \( C_{Ab} \), and \( t : \text{Irr}(C) \times C_{Ab} \rightarrow Q \) denote the mutual statistics between simple objects and pointed ones, namely \( e^{2\pi i h(a, a)} = \frac{1}{\text{Tr} C_{a,b} c_{a,b}} \) in particular, the diagonal entries are related to exchange statistics \( e^{2\pi i h(a, a)} = \text{Tr} C_{a,a} \).

Let \( \mathbb{K}^c \) be a free Abelian group with \( \kappa \) generators. It can be naturally extended to a \( \kappa \) dimensional vector space over \( \mathbb{Q} \). Let \( \mathbb{Z}^c := \text{Hom}(\mathbb{K}^c, \mathbb{Q}) \) denote the “dual space”, the space of \( \mathbb{Q} \)-linear functions. Conventionally, we use \( x, y, \ldots \) to denote elements in \( \mathbb{Z}^c \) and \( f(-), g(-), \ldots \) or simply \( f, g \) when not confusing, to denote functions in \( \mathbb{Z}^c \).

Let \( K : \mathbb{Z}^c \times \mathbb{Z}^c \rightarrow \mathbb{Q} \) be a non-degenerate symmetric bilinear form. It defines an isomorphism from \( \mathbb{Z}^c \) to \( \mathbb{Z}^c \), by \( x \mapsto K(x, -) = K(-, x) \). Denote the inverse map by \( K \), thus

\[
K(K(x, -)) = x, \quad K(K(f), x) = f(x).
\]

(18)
There is then a natural non-degenerate symmetric bilinear form \( \overline{K} \) on \( \mathbb{Z}^e \) induced from \( K \), via
\[
\overline{K}(f,g) = K(\overline{K}(f),\overline{K}(g)) = f(\overline{K}(g)) = g(\overline{K}(f)).
\] (19)

If one chooses a basis of \( \mathbb{Z}^e \) and the corresponding dual basis of \( \overline{\mathbb{Z}}^e \), the matrix of \( K \) and \( \overline{K} \) are inverse to each other.

We also need to choose \( \kappa \) Abelian anyons for each step. This is concluded in a group homomorphism \( a : \mathbb{Z}^e \to \mathbb{C}^\times \). The bilinear form \( K \) needs to satisfy the even integral condition, namely \( \forall x,y, K(x,y) + t(a(x),a(y)) \in \mathbb{Z} \) and \( K(x,x) + t(a(x),a(x)) \in 2\mathbb{Z} \).

For a \( \kappa \) step construction, we first define a semisimple category \( \mathbb{C}^\top_{a,K} \). \( \mathbb{C}^\top_{a,K} \) is graded by \( \mathbb{Z}^e/K(2\ker a,-) \) (not faithful). Take a representative \( f \in \mathbb{Z}^e \), the component \( (\mathbb{C}^\top_{a,K})_f \) is a full subcategory of \( \mathbb{C} \) with simple objects \( i \) satisfying \( f(+) + t(i,a(-)) \in \mathbb{Z} \) [note that \( K(x,-) \) is an integer for \( x \in \ker a \), so this is well defined for \( f + K(2\ker a,-) \)]. Denote the simple objects in \( \mathbb{C}^\top_{a,K} \) by \( i_f \). We then define the tensor product and braiding in \( \mathbb{C}^\top_{a,K} \),
\[
i_f \otimes j_g = (i \otimes j)_{f+g} = \oplus_k \kappa^{ij}_k k_{f+g},
\]
\[
a_{i_f,j_g} k_h = a_{i,j,k},
\]
\[
c_{i_f,j_g} = c_{i,j} e^{i\pi K(f,g)}.
\] (20) (21) (22)

(22) is independent of the choice of representative: \( \forall x \in \ker a \),
\[
c_{i_{f + K(2x,-)},j_g} = c_{i_f,j_g} e^{i\pi K(2x,g)} = c_{i_f,j_g} e^{2\pi i g(x)}.
\] (23)

Since \( t(j,a(x)) = t(j,0) \in \mathbb{Z} \), clearly \( g(x) \in \mathbb{Z} \) as desired. Thus \( \mathbb{C}^\top_{a,K} \) is a braided fusion category graded by \( \mathbb{Z}^e/K(2\ker a,-) \). It is obvious that \( d_{i_f} = d_i \).

Observe that for any \( x \in \mathbb{Z}^e \), \( a(x)_{K(x,-)} \) is a self boson and mutually trivial to any object \( i_f \). \( a(x)_{K(x,-)} \) is a self boson since
\[
\text{Tr} c_{a(x)_{K(x,-)},a(x)_{K(x,-)}} = \text{Tr} c_{a(x),a(x)} e^{i\pi K(x,-,K(x,-))} = e^{i\pi [t(a(x),a(x)) + K(x,x)]} = 1.
\] (24)

\( a(x)_{K(x,-)} \) is in the M"uger center\(^\top\) (mutually trivial to any object \( i_f \)) since
\[
\frac{1}{d_i} \text{Tr} c_{i_f,a(x)_{K(x,-)},a(x)_{K(x,-)},i_f} = e^{2\pi i [t(i,a(x)) + \rho (f,K(x,-))]} = e^{2\pi i [t(i,a(x)) + f(x)]} = 1.
\] (25)

Therefore, \( \{a(x)_{K(x,-)}, x \in \mathbb{Z}^e \} \) generates a symmetric fusion subcategory in the M"uger center of \( \mathbb{C}^\top_{a,K} \) which is equivalent to \( \text{Rep}(a(x)_{K(x,-)}) \simeq \mathbb{Z}^e/2\ker a \). Condense it\(^\top\) (take the category of local modules over \( \text{Fun}(\mathbb{Z}^e/2\ker a) \)) and we obtain the final result \( \mathbb{C}^\top_{a,K} = \mathbb{C}^\top_{a,K}/\text{Fun}(\mathbb{Z}^e/2\ker a) \). In general the associator \( (F \text{ matrix}) \) will change and get complicated after such anyon condensation process. However, since the condensed anyons are in the M"uger center, the braiding and fusion rules are preserved\(^\top\). Thus if we are only interested in the simple data such as fusion rules and \( T,S \) matrices, it is fine to work in the larger category \( \mathbb{C}^\top_{a,K} \).

**Conclusion and outlook:** In this letter we introduced the matrix formulation for non-Abelian families, which makes it possible to generate any topological order in the same non-Abelian family as a given one almost instantly. We have provided a powerful tool, which, on one hand, can help group known topological orders\(^\top\) (or modular tensor categories\(^\top\)) into non-Abelian families, and for simplicity, only the data of one root is necessary to be listed explicitly; on the other hand, one can efficiently generate the data of infinitely many possible unknown topological orders.

The results in Ref. 6 already reduces the classification problem of all 2+1D topological orders to the classification of all root topological orders, namely in which the Abelian anyons have trivial self and mutual statistics. The results in this letter further makes this reduction an efficient and simple algorithm. In the end, we only need to maintain a list of root topological orders. It will be interesting to find the canonical (the simplest) form of \( (a,K) \), and then we will have a simple name for each topological order: the root \( \mathcal{C} \) plus the canonical form of \( (a,K) \). Moreover, after fixing a root \( \mathcal{C} \), we should be able to extract all possible non-Abelian invariants\(^\top\) of this family by studying \( \mathcal{C} \) and the pair \( (a,K) \). These non-Abelian invariants will surely deepen our understanding on topological phases of matter, as well as on the application of topological materials in quantum computation.

Our construction can also be viewed as a generalization of anyon condensation\(^\top\), where anyons are forced to form an effective trivial state, and the condensed anyons are necessarily bosons. We make anyons form effective Laughlin states, and our results imply that the multilayer Laughlin states are the most general type of states Abelian anyons can form. From this point of view, it is natural to ask what kind of nontrivial effective states non-Abelian anyons can form. Further research along this line may reveal more exotic relations between topological phases, by nontrivial condensations of non-Abelian anyons, and further simplify our understanding of topological orders.

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Appendix A: Proof of Theorem 1

We prove the theorem by induction. It is obviously true for $\kappa = 1$. Now assume that it is true for $\kappa - 1$ where $\kappa > 1$. Let $K_0$ be the corresponding $\kappa - 1$ by $\kappa - 1$ matrix. From $\kappa - 1$ to $\kappa$ we choose $a_c = (a_c, I_c)$ and even integer $m_c$. The new $K$ matrix is

\[ K_1 = \begin{pmatrix} K_0 & l_c - t_{a_c} \\ (l_c - t_{a_c})^T & m_c - 2s_{a_c} \end{pmatrix}. \]

The spin of $a_c$ is

\[ s_{a_c} = s_{a_c} + \frac{1}{2} (l_c - t_{a_c})^T K_0^{-1} (l_c - t_{a_c}), \]

and the mutual statistics between ($i, l_0$) and $a_c$ is

\[ t_{(i,l_0)} = t_{i,a_c} + (l_c - t_{a_c})^T K_0^{-1} (l_c - t_{a_c}). \]

First, as long as $m_c - 2s_{a_c} \neq 0$, $K_1$ is invertible with

\[ K_1^{-1} = \begin{pmatrix} K_0^{-1} + \frac{K_0 (l_c - t_{a_c}) (l_c - t_{a_c})^T K_0^{-1}}{m_c - 2s_{a_c}} & -\frac{K_0^{-1} (l_c - t_{a_c})}{m_c - 2s_{a_c}} \\ -\frac{K_0^{-1} (l_c - t_{a_c})}{m_c - 2s_{a_c}} & \frac{1}{m_c - 2s_{a_c}} \end{pmatrix}. \]

Also

\[ \det(K_1) = \det \left( \frac{K_0}{(l_c - t_{a_c})^T m_c - 2s_{a_c}} \right) = \det(K_0)(m_c - 2s_{a_c} - (l_c - t_{a_c})^T K_0^{-1} (l_c - t_{a_c})) = (m_c - 2s_{a_c}) \det(K_0). \]

Thus $\det K$ accounts for the increment of rank, total quantum dimension, as well as the normalization of $S$ matrix. Also $\text{sgn}K = \text{sgn}K_0 + \text{sgn}(m_c - 2s_{a_c})$ accounts for the increment of chiral central charge.

The new anyons are labeled by $(i, l_0, m)$ where $m$ is an integer. Combine $l$ and $m$ into a $\kappa$-dimensional vector $l^T = (l_0^T, m)$. We only need to verify the spin, equivalence relations and fusion rule of $(i, l)$; $S$ matrix follows directly.

The spin of $(i, l_0, m) = (i, l)$ is

\[ s_{(i,l)} = s_{(i,l_0)} + \frac{(m - t_{i,a_c})^2}{2(m_c - 2s_{a_c})} = s_i + \frac{1}{2} (l_0 - t_i)^T K_0^{-1} (l_0 - t_i) + \frac{(m - t_{i,a_c})^2}{2(m_c - 2s_{a_c})}. \]

While (using the same notation for $\kappa - 1$ and $\kappa$ dimensional $t_i$)

\[ \frac{1}{2} (l - t_i)^T K_1^{-1} (l - t_i) = \frac{1}{2} (l_0 - t_i)^T K_1^{-1} \left( \frac{l_0 - t_i}{m - t_{i,a_c}} \right) = \frac{1}{2} \left( (l_0 - t_i)^T K_0^{-1} (l_0 - t_i) + (m - t_{i,a_c})^2 + (t_{i,l_0} - t_{i,a_c})^2 - 2(m - t_{i,a_c}) (t_{i,l_0} - t_{i,a_c}) \right) \left/ m_c - 2s_{a_c} \right. . \]

Indeed we have

\[ s_{(i,l)} = s_i + \frac{1}{2} (l - t_i)^T K_1^{-1} (l - t_i). \]

For $\kappa - 1$ we have equivalence relations

\[ (i, l_0) \sim (i \otimes a_{i}, l_0 + (K_0)_{i} - t_i + t_{i \otimes a_i}). \]

For $(i, l_0, m)$, one equivalence relation comes from condensing $a_c = (a_c, I_c)$ with even integer $m_c$,
Thus we do have

\[(i, l_0, m) \sim (i \otimes a_\kappa, l_0 + l_c - t_i - t_{a_\kappa} + t_{i \otimes a_\kappa}, m + m_c - t_{(i, l_0)} - t_{(i, a_\kappa, t_i)} + t_{(i \otimes a_\kappa, l_0 + t_c - t_i - t_{a_\kappa} + t_{i \otimes a_\kappa})}), \]

where

\[-t_{(i, l_0)} - t_{(i, a_\kappa, l_0) + l_c - t_i - t_{a_\kappa} + t_{i \otimes a_\kappa}}
\]

\[= -t_{i, a_\kappa} - t_{a_\kappa, a_\kappa} + t_{i \otimes a_\kappa, a_\kappa} + (l_c - t_{a_\kappa})^T K_0^{-1} (t_i - l_0 + t_{a_\kappa} - l_c - t_{i \otimes a_\kappa} + l_0 + l_c - t_i - t_{a_\kappa} + t_{i \otimes a_\kappa})
\]

\[= -t_{i, a_\kappa} - t_{a_\kappa, a_\kappa} + t_{i \otimes a_\kappa, a_\kappa}.
\]

Thus

\[(i, l) \sim (i \otimes a_\kappa, l + (K_1)_\kappa - t_i + t_{i \otimes a_\kappa}),\]

where \((K_1)_\kappa^T = (K_0)_\kappa^T, (l_c - t_{a_\kappa}), I = 1, \ldots, \kappa - 1.

Thus

\[(i, l) \sim (i \otimes a_\kappa, l + (K_1)_{i+j} - t_i + t_{i \otimes a_\kappa}),\]

where \((K_1)_{ij}^T = ((K_0)_j^T, (l_c - t_{a_\kappa}), I = 1, \ldots, \kappa - 1.

The fusion of \((i, l_0, m)\) and \((j, k_0, n)\) is

\[(i, l_0, m) \otimes (j, k_0, n) = \oplus N^{ij}_s (s, l_0 + k_0 - t_i - t_j + t_s, m + n - t_{(i, l_0)} - t_{(j, k_0)} + t_{(s, l_0 + k_0 - t_i - t_j + t_s)}),\]

where

\[-t_{(i, l_0)} - t_{(j, k_0)} + t_{(s, l_0 + k_0 - t_i - t_j + t_s)}
\]

\[-t_{i, a_\kappa} - t_{j, a_\kappa} + t_{s, a_\kappa} + (l_c - t_{a_\kappa})^T K_0^{-1} (t_i - l_0 + t_j - k_0 - t_s + l_0 + k_0 - t_i - t_j + t_s)
\]

\[-t_{i, a_\kappa} - t_{j, a_\kappa} + t_{s, a_\kappa}.
\]

The fact that \(t_{a_I, a_I'} = \sum_{PQ} W_{IJ} t_{a_P, a_Q} W_{JQ}\) implies the transformation for the \(K\) matrix. As \(a_I\) are in an Abelian group, it is convenient to write in the additive convention \(a_I' = \sum_J W_{IJ} a_I\), or simply \(a' = W a\). Thus \(C_{a, \kappa} \simeq C_{W a, W K W^T}\) for integer matrix \(W\) with \(\det W = \pm 1\). More precisely, the equivalence is given by \((i, l) \mapsto (i, Wl). Note that \(t_{a_I, a_I'} = (t_{a_I, a_I'}) = W t_{a_I}. It is straightforward to check that this map is compatible with the equivalence relation (14), and preserves fusion (15), spin (16), and S matrix (17).