Temporal Dynamics in Perturbation Theory

V. I. Yukalov and E. P. Yukalova

Department of Mathematics and Statistics
Queen’s University, Kingston, Ontario K7L 3N6, Canada
and
Bogolubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna 141980, Russia

Perturbation theory can be reformulated as dynamical theory. Then a sequence of perturbative approximations is bijective to a trajectory of dynamical system with discrete time, called the approximation cascade. Here we concentrate our attention on the stability conditions permitting to control the convergence of approximation sequences. We show that several types of mapping multipliers and Lyapunov exponents can be introduced and, respectively, several types of conditions controlling local stability can be formulated. The ideas are illustrated by calculating the energy levels of an anharmonic oscillator.

I. INTRODUCTION

Perturbation theory as applied to realistic physical problems practically always yields divergent sequences. There is a number of methods of finding effective limits of such sequences. The most known are the methods of Padé and of Borel summation. Constantly, new techniques appear. We are not aiming to give here a review of those. But we think it would be worth mentioning some of the recently developed interesting methods. Among them are some variants of the Padé method [1,2] and the converse Cesaro method [3]. An analogy of the Kolmogorov perturbation theory in classical mechanics has been constructed for self-adjoint operators [4], which reorganizes the asymptotic series in the usual Rayleigh–Schrödinger perturbation theory to a generalized asymptotic series. The accuracy of perturbation theory is known to improve when a perturbative expansion is accompanied by a variational procedure [5-7]. The latter often gives good results already at the first step of calculations, as in the method of potential envelopes [8,9] or when the zero approximation is very close to the considered problem [10]. The variational principle improves the accuracy of the quasiclassical and Langer methods [11] and of different sequence transformations [12]. The critical behaviour of several spin and vertex models has been studied by combining a variational series expansion and the coherent anomaly method [13-17].

The common drawback of practically all perturbation techniques used for physical problems is the absence of a general method for checking the convergence of perturbation sequences. This is because an explicit expression for the perturbative approximation of arbitrary order is, except a few trivial cases, never available. For some simple problems one can obtain exact numerical solutions. Then the accuracy of perturbation theory can be checked by a direct comparison of its results against the known exact answers. If, when increasing the approximation order, one gets better accuracy, then it is common to say that the procedure is convergent. Strictly speaking, this terminology is not correct, since the classical notion of convergence presupposes that the sequence of approximations has an exact solution as its limit when the approximation order tends to infinity. However, in actual calculations one always has a finite number of approximations, though the approximation order may be quite high. There is a number of examples when the accuracy of a calculational scheme improves till a fixed approximation order but after it worsens resulting in a divergent sequence. Such behaviour is, indeed, a common feature of asymptotic series. In this situation it is more correct to say that the sequence of perturbation theory is semiconvergent.

Note that here and in what follows we use the term "perturbation theory" in its most general meaning: perturbation theory is a regular procedure prescribing a general algorithm for defining a sequence of approximations of arbitrary order. The term "general algorithm" means that there is a general rule for calculating any approximation, although in practice it may happen that because of technical difficulties, one is able to find just a few initial approximations. The algorithm of perturbation theory may include, in addition to a basic expansion procedure, any of various resummation or renormalization tricks. In this sense, the so-called nonperturbative approaches are nothing but particular variants of perturbation theory, supplied with additional conditions.

The main question for any variant of perturbation theory in the case of a complicated problem is how to control the convergence if neither an explicit form of high-order approximations nor the exact solution are available? To overcome this difficulty, an idea has been advanced [18] that the perturbation algorithm can be supplemented by a set
of functions controlling the convergence of the approximation sequence. These functions because of their role, can be called the control, or governing, functions. Perturbation theory employing control functions has been first published in ref.[19] and used for describing anharmonic crystals [20-25]. In the cited papers, control functions were defined by a minimal–difference condition. A variational approach, called the minimal–sensitivity condition [5], has also been applied to several anharmonic models [26-34]. The choice of conditions for control functions has been heuristic.

To justify the option of conditions defining control functions, it was shown [35-39] that perturbation theory can be formulated as renormalization–group theory. Then control functions are to be defined from a fixed–point condition, whose particular variants yield either the minimal–difference or minimal–sensitivity conditions. As far as a renormalization group can be considered as a kind of dynamical system, it was natural to reformulate perturbation theory to the language of dynamical theory [40-42]. This reformulation not only makes the theory more logical but also permits to define stability conditions related to the problem of convergence. In our previous papers we did not pay enough attention to these stability problems. The purpose of the present publication is to compensate this deficiency. We proffer a detailed analysis of the stability conditions showing that, really, there are several types of them, each with a different meaning. These conditions allow to understand intimate features of perturbation sequences and, therefore, to control their properties. Perturbation theory, whose convergence is controlled by control functions and which is supplemented by the stability conditions allowing for a thorough control of the properties of perturbation sequences, can be called the controlled perturbation theory.

II. survey of approach

Before turning to stability conditions, we need to give a brief survey of the method whose stability is to be analysed.

Assume that we are trying to solve a problem whose solution is a function \( f(g) \), such that \( f : \mathbb{D} \to \mathbb{R} \), of a variable \( g \in \mathbb{D} \subset \mathbb{R} \). The first thing we have to do is to introduce control functions. For simplicity, we shall speak here about one set of such functions. The generalization to the case of several sets is straightforward [42]. Incorporate into the perturbation algorithm a parameter \( u \in \mathbb{R} \) whose value is yet undefined. Then we get a sequence \( \{F_k(g, u)\}_{k=0}^{\infty} \) of functions \( F_k : \mathbb{D} \times \mathbb{R} \to \mathbb{R} \). In each term \( F_k(g, u) \), we substitute for the parameter \( u \) a function \( u_k(g) \), such that \( u_k : \mathbb{D} \to \mathbb{R} \) and \( k \in \mathbb{Z}_+ \equiv \{0,1,2,\ldots\} \). This results in a function \( f_k : \mathbb{D} \to \mathbb{R} \) which is

\[
f_k(g) \equiv F_k(g, u_k(g)). \tag{1}
\]

The role of the control functions \( u_k(g) \) is to govern the convergence of the sequence \( \{f_k(g)\}_{k=0}^{\infty} \) to the limit

\[
\lim_{k \to \infty} f_k(g) = f(g). \tag{2}
\]

As is clear, there can be infinite number of different types of functions satisfying (2). This means that (2) defines a class of equivalence

\[
\mathcal{U} \equiv \{u_k(g) : k \in \mathbb{Z}_+\}.
\]

The latter can be explicitly found only if the general form of (1), as \( k \to \infty \), is known. Of course, this can be realised only for a few simple zero–and one–dimensional models [43-45]. Usually, the explicit form of (1) for arbitrary \( k \to \infty \) is not known.

The second step is to narrow the class of equivalence \( \mathcal{U} \) by those control functions that satisfy an evolution equation defining a dynamical system in discrete time, called the approximation cascade. This task is a kind of an inverse problem in the optimal control theory [46,47]. The direct problem in the latter is when an evolution equation is given and one has to find control functions minimizing a performance index. In our case, we need to find an evolution equation itself. To this end, introduce the coupling function \( g_k(f) \), such that \( g_k : \mathbb{R} \to \mathbb{D} \), given by the equation

\[
F_0(g_k(f), u_k(g_k(f))) = f. \tag{3}
\]

As is clear, the function \( g_k^{-1} : \mathbb{D} \to \mathbb{R} \) defined as

\[
g_k^{-1}(g) \equiv F_0(g, u_k(g)) \tag{4}
\]

is inverse with respect to \( g_k(f) \), since

\[
g_k^{-1}(g_k(f)) = f, \quad g_k(g_k^{-1}(g)) = g. \tag{5}
\]
Introduce the function
\[ y_k(f) \equiv f_k(g_k(f)) \] (6)
realizing an endomorphism \( y_k : \mathbb{R} \to \mathbb{R} \) of the measurable space \( \mathbb{R} \). The function (1) can be recovered from (6) by the transformation
\[ f_k(g) = y_k(g_k^{-1}(g)). \] (7)
The sequence \( \{y_k(f)\}_{k=0}^{\infty} \), by construction, is bijective to \( \{f_k(g)\}_{k=0}^{\infty} \).

Let us require that the endomorphism \( y_k \) possess the semigroup properties
\[ y_k \cdot y_p = y_{k+p}, \quad y_0 = 1. \] (8)
This is equivalent to the relation
\[ y_{k+p}(f) = y_k(y_p(f)), \] (9)
having the meaning of an evolution equation with the initial condition \( y_0(f) = f \). Equations analogous to (9) can be met in various physical problems where they are often called the self-similar relations [48]. The requirement (8) narrows the class of equivalence \( U \) to those control functions that provide the validity of the evolution equation (9).

The semigroup
\[ \mathcal{Y} \equiv \{y_k\} : \mathbb{Z}_+ \times \mathbb{R} \to \mathbb{R} \] (10)
of the endomorphisms \( y_k \) defines a dynamical system with the discrete time \( k \in \mathbb{Z}_+ \). In the dynamical theory this is called a semicascade. In our case, the latter is related to the sequence of approximations (7) because of which we call (10) the approximation cascade. The cascade trajectory \( \{y_k(f)\}_{k=0}^{\infty} \) is bijective to the approximation sequence \( \{f_k(g)\}_{k=0}^{\infty} \). The existence of the limit (2), which, according to (7), can be written as
\[ \lim_{k \to \infty} f_k(g) = \lim_{k \to \infty} y_k(g_k^{-1}(g)) = f(g), \] (11)
is equivalent to the existence of an attracting fixed point of the approximation cascade,
\[ \lim_{k \to \infty} y_k(f) = \lim_{k \to \infty} f_k(g_k(f)) = y^*(f). \] (12)

At the third step, we embed the approximation cascade (10) into an approximation flow. This is done as follows. Instead of the discrete variable \( k \in \mathbb{Z}_+ \) consider a continuous variable \( t \in \mathbb{R}_+ = [0, \infty) \). Introduce an endomorphism \( y(t, \cdot) : \mathbb{R} \to \mathbb{R} \) of the measurable space \( \mathbb{R} \), satisfying the semigroup properties (8) for each \( t \in \mathbb{R}_+ \). Then in the place of (9) we have
\[ y(t + t', f) = y(t, y(t', f)). \] (13)
Require that the endomorphism \( y(t, \cdot) \) would satisfy the conditions
\[ y(k, f) = y_k(f) \quad (k \in \mathbb{Z}_+), \]
\[ \lim_{t \to \infty} y(t, f) = y^*(f). \] (14)
The semigroup
\[ \tilde{\mathcal{Y}} \equiv \{y(t, \cdot)\} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \] (15)
of the endomorphisms \( y(t, \cdot) \) is a dynamical system called semiflow. We name (15) the approximation flow. By construction, the semigroup (10) is a subgroup of the semigroup (15). In other words, the approximation cascade (10) is embedded into the approximation flow (15). The flow trajectory contains all the cascade trajectory,
\[ \{y_k(f) \mid k \in \mathbb{Z}_+\} \subset \{y(t, f) \mid t \in \mathbb{R}_+\}. \]
Assume that the embedding of the cascade into the flow is smooth, in the sense that the derivative \( \frac{dy}{dt} \) exists and is piecewise continuous on \( \mathbb{R}_+ \). Then the equation (13) can be rewritten in the differential form

\[
\frac{d}{dt}y(t, f) = v(y(t, f)),
\]

in which the velocity

\[
v(f) \equiv \lim_{t \to 0} \frac{\partial}{\partial t} y(t, f)
\]

is a transformation \( v : \mathbb{R} \to T(\mathbb{R}) \) from \( \mathbb{R} \) to a tangent space \( T(\mathbb{R}) \). Integrating the evolution equation (16), we can cast it into the evolution integral

\[
\int_{y(t_1, f)}^{y(t_2, f)} \frac{dy}{v(y)} = t_2 - t_1.
\]

The fixed point of the approximation flow is defined as a zero of the velocity

\[
v(y^*(f)) = 0.
\]

The fourth step is the definition of quasifixed points, that is, of approximate fixed points. We cannot find fixed points exactly because the velocity of the approximation flow is not known, so we are not able to use (19). From another side, we cannot also define fixed points as the limit (12), since an expression for \( y_k(f) \) at arbitrary \( k \to \infty \) is not available.

To find approximate fixed points, or quasifixed points, we need to have an explicit form of the cascade velocity

\[
v_k(f) \equiv v(y(t, f)) \quad (t \in [k, k+1]).
\]

We may define it as an Euler discretization of the derivative \( \frac{dy}{dt} \). In doing this, we take into account that \( F_k(g, u_k) \) depends on \( k \) directly as well as through \( u_k \). In this way, we may write the cascade velocity (20) as

\[
v_k(f) = V_k(g_k(f), u_k(g_k(f)))
\]

with the finite difference

\[
V_k(g, u_k) = F_{k+1}(g, u_k) - F_k(g, u_k) + (u_{k+1} - u_k) \frac{\partial}{\partial u_k} F_k(g, u_k)
\]

and the coupling function \( g_k \) given by the constraint (3). As the cascade trajectory approaches the fixed point, when \( k \to \infty \), then \( v_k(f) \to 0 \). But to check whether the cascade velocity tends to zero, as \( k \to \infty \), we need to know the form of \( F_k(g, u_k) \) for arbitrary \( k \to \infty \). If the latter would be known, then we could check directly the convergence condition

\[
F_{k+p}(g, u_{k+p}) - F_k(g, u_k) \to 0 \quad (k \to \infty),
\]

in which \( p \geq 1 \). This condition would define those control functions which provide the convergence for the sequence \( \{f_k(g)\}_{k=0}^{\infty} \) of terms (1). Such a condition of defining control functions may be called the asymptotic fitting condition. It has been used for some anharmonic models [43-45]. However, the possibility of finding the general expression for \( F_k(g, u_k) \) at the arbitrary \( k \to \infty \) is rather an extreme exception. Usually, just a few first terms are available only. Therefore, we need to have some working conditions for defining control functions in the case of finite numbers \( k \) of \( F_k \).

One such condition for defining control functions is the minimal–difference condition [18-25],

\[
F_{k+1}(g, u_k) - F_k(g, u_k) = 0.
\]

This condition makes zero only a part of the cascade velocity (21). Therefore (23) may be named the quasifixed–point condition.

Another quasifixed–point condition is the minimal–sensitivity condition [26-34],

\[
\frac{\partial}{\partial u_k} F_k(g, u_k) = 0,
\]
which also makes zero only a part of (21).

A slight generalization of (24) following from (21) is the quasifixed–point condition [42]

\[
(u_{k+1} - u_k) \frac{\partial}{\partial u_k} F_k(g, u_k) = 0.
\]  

The meaning of (25) is as follows: if (24) has a solution for \( u_k \), then this solution gives the control function \( u_k(g) \); when (24) has no solution for \( u_k \), then we put \( u_k = u_{k+1} \).

Defining the control functions from one of the quasifixed–point conditions, we obtain a sequence of quasifixed points \( y_k(f) \) and, respectively, a sequence of their images \( f_k(g) \) that are the sought approximations for \( f(g) \). Several other types of quasifixed–point conditions have also been studied [49,50].

In the fifth step of the considered perturbation theory we find \textit{corrected approximations}. As far as the quasifixed–point conditions do not make the cascade velocity exactly zero, the trajectory does not stop at \( y_k(f) \), though the motion slows down. If we accept the quasifixed–point condition (25), then the cascade velocity (21) in the vicinity of a quasifixed point \( y_k(f) \) becomes

\[
v_k^*(f) = V_k^*(g_k(f), u_k(g_k(f))),
\]  

where

\[
V_k^*(g, u_k) = F_{k+1}(g, u_k) - F_k(g, u_k).
\]

The motion in the interval of \( t \in [k, k+1] \) is described by the approximation flow (15). Substituting into the evolution integral (18) the time limits \( t_1 = k \) and \( t_2 = k + 1 \), we have

\[
\int_{y_k(f)}^{y_{k+1}(f)} \frac{dy}{v(y)} = 1.
\]  

(27)

Using in (27), instead of \( v(y) \), the velocity (26), we get the evolution integral

\[
\int_{y_k(f)}^{y_k^*(f)} \frac{dy}{v_k^*(y)} = 1
\]  

(28)

defining the corrected quasifixed point \( y_k^*(f) \). Making in (28) the substitution \( f \rightarrow g_k^{-1}(g) \), we come, according to (7), to the integral

\[
\int_{f_k(g)}^{f_k^*(g)} \frac{df}{v_k^*(f)} = 1,
\]  

(29)

in which

\[
f_k^*(g) = y_k^*(g_k^{-1}(g))
\]  

(30)

is the corrected \( k \)-order approximation.

III. STABILITY OF CASCADE

A very important feature of the controlled perturbation theory is the possibility to control whether we are approaching the fixed point, that is, the correct answer, even if this exact answer is not known. This possibility is based on the semigroup property (9) of the approximation cascade, according to which each point \( y_k(f) \) is considered as a result of mapping from a previous point \( y_p(f) \) with \( p \leq k - 1 \). We are approaching an attracting fixed point if the mapping is contracting. This, of course, depends on the perturbation algorithm and on the initial approximation. A given perturbation algorithm has a basin of attraction. The mapping can be contracting only if an initial approximation is in the basin of attraction. In iteration theory, the analog of the basin of attraction is the set of normality or Fatou set, while the complement of the Fatou set is called the Julia set. Attracting fixed points are in the Fatou set, while repelling fixed points are in the Julia set. When the iteration is done by means of an entire transcendental function, then every point in the Julia set is a limit point of repelling periodic points, that is, the Julia set is the closure of the set of repelling periodic points [51].
Let us analyse when the mapping corresponding to the approximation cascade is contracting. Consider the change

\[ y_p(f) \rightarrow y_p(f) + \delta y_p(f) \]  

with \( \delta y_p(f) \rightarrow 0 \) and the variation

\[ \delta y_{kp}(f) \equiv y_k(y_p(f) + \delta y_p(f)) - y_k(y_p(f)). \]  

A particular case of (32) is

\[ \delta y_k(f) \equiv y_k(f + \delta f) - y_k(f) = \delta y_{k0}(f), \]  

when the initial condition is changed.

To proceed further, we need to introduce the notation for mapping multipliers. Define the \textit{quasilocal multipliers}

\[ \mu_{kp}(f) \equiv \frac{\delta y_k(y_p(f))}{\delta y_p(f)} \]  

and

\[ \mu_k(f) \equiv \frac{\delta y_k(f)}{\delta f} = \mu_{k0}(f). \]  

They satisfy the relation

\[ \mu_{kp}(f) = \mu_k(y_p(f)) \]  

and have the property

\[ \mu_{0k}(f) = \mu_0(f) = 1. \]  

Other useful relations can be derived basing on the semigroup property

\[ y_{k+p}(f) = y_k(y_p(f)) = y_p(y_k(f)) \]  

and the variational derivative

\[ \frac{\delta y_k(f)}{\delta y_p(f)} = \frac{dy_k(f)/df}{dy_p(f)/df} = \frac{\mu_k(f)}{\mu_p(f)}. \]  

In this way we obtain

\[ \mu_{kp}(f) = \frac{\mu_{k+p}(f)}{\mu_p(f)} \]  

and

\[ \mu_{kp}(f)\mu_p(f) = \mu_{pk}(f)\mu_k(f). \]  

Introduce the \textit{local multiplier}

\[ \mu_k^*(f) \equiv \frac{\delta y_k(f)}{\delta y_{k-1}(f)} = \mu_{k-1}(f). \]  

Using (38), this can be also written as

\[ \mu_k^*(f) = \frac{\mu_k(f)}{\mu_{k-1}(f)} \quad (k \geq 1). \]  

In the case of \( k = 1 \),

\[ \mu_1^*(f) = \mu_1(f). \]
The quasilocal multipliers (34) and (35) can be presented as products

$$\mu_{kp}(f) = \prod_{j=p+1}^{k+p} \mu_j^*(f),$$

$$\mu_k(f) = \prod_{p=1}^{k} \mu_p^*(f)$$

of the local multipliers (40). In another form (42) reads

$$\mu_{kp}(f) = \mu_{k+p}^*(f)\mu_{k-1}^*p(f),$$

$$\mu_k(f) = \mu_k^*(f)\mu_{k-1}(f).$$

With these multipliers, the variation (32) becomes

$$\delta y_{kp}(f) = \mu_{kp}(f)\delta y_p(f).$$

Eq.(44) describes the deviation of the cascade trajectory at the \((k+p)\)–step resulting from the variation \(\delta y_p(f)\) at a \(p\)–step. The mapping, corresponding to (44), is contracting if the condition of quasilocal contraction

$$|\mu_{kp}(f)| < 1$$

holds. This shows that the mapping is effectively contracting after \(k\) steps starting from a \(p\)–step. Equivalently, one may say that the mapping is effectively contracting on the interval \([p, k+p]\).

In particular, for the interval \([0, k]\), we need to deal with the variation (33) which yields

$$\delta y_k(f) = \mu_k(f)\delta f.$$ 

The condition of quasilocal contraction on the interval \([0, k]\) is

$$|\mu_k(f)| < 1,$$

where \(\mu_k(f)\) is the quasilocal multiplier (35) with \(k \geq 1\).

If we are interested in the contraction property at just one step, from \(k-1\) to \(k\), then we have to consider the variation

$$\delta y_k(f) = \mu_k^*(f)\delta y_{k-1}(f),$$

in which \(\mu_k^*(f)\) is the local multiplier (40). The mapping is locally contracting at a \(k\)–step if the condition of local contraction

$$|\mu_k^*(f)| < 1$$

is valid.

The condition (49) is stronger than (47) in the following sense: if \(|\mu_p^*| < 1\) holds for all \(p \in [0, k]\), then (47) follows from this because of the relation (42), though the inverse is not true. It may happen that (47) holds, but for some \(p\) from the interval \([0, k]\) the condition (49) is not valid. In other words, there can exist the effective contraction on an interval \([0, k]\) although not for all steps there can be the local contraction. Simbolically, the relation between the two notions is:

local contraction $\rightarrow$ quasilocal contraction.

The contraction for a mapping is the same as the stability for a cascade. The stability is characterized by Lyapunov exponents. Again, different kinds of such exponents can be defined. The quasilocal Lyapunov exponent

$$\lambda_{kp}(f) = \frac{1}{k} \ln |\mu_{kp}(f)|$$

is related to the quasilocal multiplier (34), and the quasilocal exponent
\[
\lambda_k(f) = \frac{1}{k} \ln |\mu_k(f)| = \lambda_{k0}(f),
\]

(51)
to the multiplier (35). Taking into account (38), we have
\[
\lambda_{kp}(f) = \frac{k+p}{k} \lambda_{k+p}(f) - \frac{p}{k} \lambda_p(f).
\]

(52)
The local Lyapunov exponent
\[
\lambda^*_k(f) \equiv \ln |\mu^*_k(f)| = \lambda_1 k^{-1}(f)
\]

is defined through the local multiplier (40). Because of (42), the quasilocal exponent (51) is an arithmetic average
\[
\lambda_k(f) = \frac{1}{k} \sum_{p=1}^{k} \lambda^*_p(f)
\]

(54)
of the local exponents from (53).

The effective stability on the interval \([p, p+k]\) with \(p \geq 0\) and \(k \geq 1\), means that the condition of quasilocal stability
\[
\lambda_{kp}(f) < 0
\]

(55)
holds. In the case of \(p = 0\), this reduces to
\[
\lambda_k(f) < 0.
\]

(56)
As is evident, (55) and (56) follow from the contraction conditions (45) and (47), respectively.

The condition of local stability at a \(k\)-step, from \(k-1\) to \(k\), reads
\[
\lambda^*_k(f) < 0,
\]

(57)
resulting from (49). Anew, the condition (57) is stronger than (56) in the sense that the stability can exist on an interval but not necessarily at all points of the latter, while if (57) holds for all points of an interval, then (56) follows for this interval.

The maximal Lyapunov exponent, usually employed in dynamical theory, is
\[
\lambda(f) \equiv \lim_{k \to \infty} \lambda_k(f).
\]

(58)
The condition of asymptotic stability implies that
\[
\lambda(f) < 0.
\]

(59)
If the condition (56) is valid for all \(k \geq 1\), then it is stronger than (59). Thus, the relation between the different types of stability is as follows:

local stability \rightarrow quasilocal stability \rightarrow asymptotic stability.

Recollect that the approximation–cascade trajectory \(\{y_k(f)\}_{k=0}^{\infty}\) is, by construction, bijective to the approximation sequence \(\{f_k(g)\}_{k=0}^{\infty}\). Each point \(y_k(f)\) has its image \(f_k(g)\) given by the relations (6) and (7). For the mapping multipliers and Lyapunov exponents introduced above, we may also define their images as functions of \(g\).

The image of the quasilocal multiplier (35) is
\[
M_k(g) \equiv \mu_k(g^{-1}_k(g)).
\]

(60)
This can also be written as
\[
M_k(g) = \mu_k(F_0(g, u_k(g)) = \frac{\delta F_k(g, u_k(g))}{\delta F_0(g, u_k(g))}.
\]

The image of the local multiplier (40) is
Using the properties of multipliers, we may write

\[ M_k^*(g) = \mu_k(F_0(g, u_{k-1}(g))) / \mu_{k-1}(F_0(g, u_{k-1}(g))). \]  

(61)

The contraction conditions (47) and (49) can be reformulated for the multipliers (60) and (61) giving

\[ |M_k(g)| < 1, \quad |M_k^*(g)| < 1, \]  

(62)

respectively.

For the image of the quasilocal Lyapunov exponent (51), we get

\[ \Lambda_k(g) \equiv \lambda_k(g^{-1}(g)) = 1/k \ln |M_k(g)|, \]  

(63)

and for the image of the local Lyapunov exponent (53),

\[ \Lambda_k^*(g) \equiv \lambda_k^*(g^{-1}(g)) = \ln |M_k^*(g)|. \]  

(64)

These are connected with each other, equivalently to (54), through the arithmetic averaging

\[ \Lambda_k(g) = \frac{1}{k} \sum_{p=1}^{k} \Lambda_k^*(g). \]

The stability conditions (56) and (57) can be written for the Lyapunov exponents (63) and (64), so that

\[ \Lambda_k(g) < 0, \quad \Lambda_k^*(g) < 0, \]  

(65)

respectively.

As far as the stability conditions for the Lyapunov exponents are a reformulation of the contraction conditions for the mapping multipliers, in what follows we shall often refer to any of them as to the stability conditions.

IV. STABILITY OF FLOW

The stability of motion analyzed above concerns the stability of an approximation cascade. If the latter is embedded into an approximation flow, we need to check the stability of the flow as well. This can be done by making in the evolution equation (16)

the substitution

\[ y(t, f) \rightarrow y(t, f) + \delta y(t, f) \]  

(66)

implying that \( \delta y(t, f) \rightarrow 0 \). Then we find

\[ \delta y(t, f) = \delta y(t_0, f) \exp \left\{ \int_{t_0}^{t} \lambda(t', f) dt' \right\}, \]  

(67)

where \( t_0 \leq t \), and

\[ \lambda(t, f) \equiv \frac{\delta v(y(t, f))}{\delta y(t, f)} \]  

(68)

is the local Lyapunov exponent for the flow. According to (20), the flow velocity in the time interval \( k \leq t \leq k + 1 \) is given by the corresponding cascade velocity. For this reason, we put \( t_0 = k \) in (67) and get

\[ \delta y(t, f) = \delta y_k(f) \exp \{ \lambda_k (f)(t - k) \}, \]  

(69)

where we took into account that, in compliance with (14), \( y(k, f) = y_k(f) \), and that (68) gives
\[
\tilde{\lambda}_k (f) \equiv \frac{\delta v_k(f)}{\delta y_k(f)}.
\]

The approximation flow near a quasifixed point \(y_k(f)\) is stable if
\[
\tilde{\lambda}_k (f) < 0.
\]

The image of (70) is
\[
\tilde{\Lambda}_k (g) \equiv \frac{\delta v_k(f_k(g))}{\delta f_k(g)} = \frac{\delta y_k(f_k(g))}{\delta f_k(g)} \delta y_k(f).
\]

The condition (71) of the local stability of the flow acquires the form
\[
\tilde{\Lambda}_k (g) < 0.
\]

Note that the local Lyapunov exponents for a cascade and for a flow, into which the cascade is embedded, are, generally, different. This means that (53) does not coincide with (70). Therefore, it may happen that the cascade at a point \(y_k(f)\) is locally stable but the flow at the same point is not, or vice versa. To understand better the distinction between (70) and (53), we may invoke the definition of the cascade velocity as of the finite difference
\[
v_k(f) = y_{k+1}(f) - y_k(f).
\]

Then (70) becomes
\[
\tilde{\lambda}_k (f) = \mu_{k+1}k(f) - \mu k(f).
\]

V. STABILITY AND CONVERGENCE

Since the cascade trajectory is bijective to the approximation sequence, the stability conditions for the cascade should characterize the corresponding convergence properties for the sequence.

The deviation of the trajectory point \(y_k(f)\) from the fixed point \(y^*(f)\) is
\[
\Delta y_k(f) \equiv y_k(f) - y^*(f).
\]

Consider the deviation \(\Delta y_{k+p}(f)\) assuming that \(y_p(f)\) is close to \(y^*(f)\) in the sense that
\[
|y_p(f) - y^*(f)| \ll |y^*(f)|.
\]

Employing the definition of the fixed point,
\[
y_k(y^*(f)) = y^*(f),
\]
The accuracy of an approximation \( f \) error (81). When the condition of quasilocal stability (56) holds, that is when \( \Lambda f = 0 \), the accuracy of \( f \) improves at each step. This also means the convergence of \( \{g_k\} \rightarrow \infty \). The necessary and sufficient condition of convergence is

\[
k\Lambda_k(g) \rightarrow -\infty \quad (k \rightarrow \infty).
\]

For the corresponding quasilocal exponent (51), this reads

\[
k\lambda_k(f) \rightarrow -\infty \quad (k \rightarrow \infty),
\]

which is equivalent to the asymptotic condition

\[
|\mu_k(f)| \rightarrow 0 \quad (k \rightarrow \infty).
\]

The convergence conditions (83)–(85) are weaker than the conditions of quasilocal stability (56), local stability (57) and asymptotic stability (59). The maximal Lyapunov exponent (58) can be zero; nevertheless, the convergence of the approximation sequence will persist provided (84) is valid.

By analogy with the usage of the terms quasilocal or local contraction, as applied to a mapping, and quasilocal or local stability, as applied to a cascade, we may use the terms quasilocal or local convergence, as applied to a sequence. We shall say that a sequence of approximations is quasilocally convergent on the interval \([0, k]\) if the condition of quasilocal stability (56), i.e., \( \Lambda_k(g) < 0 \), holds on this interval. The sequence will be named locally convergent at a \( k \)-step if the condition of local stability (57), that is, \( \Lambda_k^*(g) < 0 \), is valid at this given \( k \). When the conditions of quasilocal and local convergence hold everywhere, that is, are true for all \( k \geq 1 \), then they are stronger than the convergence criterion (63). There is the following relation between different notions of convergence:

\[
\text{local convergence} \rightarrow \text{quasilocal convergence} \rightarrow \text{convergence}.
\]

This notions should not be confused with the point convergence, convergence on an interval and uniform convergence of the sequence \( \{f_k(g)\}_{k=0}^\infty \) with respect to the variable \( g \).
Because of the exponential renormalization of the deviation in (82), one may say, when the corresponding convergence conditions are valid, that there occurs an exponential convergence.

In order to find out when the corrected approximation \( f_k^*(g) \) is better than \( f_k(g) \), consider the deviation

\[
\Delta y_k(f) \equiv y_k^*(f) - y^*(f). \tag{86}
\]

From the evolution equation (16) we obtain

\[
\Delta y_k^*(f) \simeq \Delta y_k(f) \exp\{\tilde{\lambda}_k(f)\}, \tag{87}
\]

where \( \tilde{\lambda}_k(f) \) is defined in (70). The image of (86) is

\[
\Delta f_k^*(g) \equiv f_k^*(g) - f(g). \tag{88}
\]

Whence, (87) gives

\[
|\Delta f_k^*(g)| \simeq |\Delta f_k(g)| \exp\{\tilde{\lambda}_k(g)\}. \tag{89}
\]

Eq. (89) shows that the corrected approximation \( f_k^*(g) \) is more accurate than \( f_k(g) \) when the approximation flow is locally stable, so that condition (73) holds.

Let us also observe when the corrected approximation \( f_k^*(g) \) is better than \( f_{k+1}(g) \). This happens when

\[
|\Delta y_k^*(f)| < |\Delta y_{k+1}(f)|. 
\]

The latter inequality leads to

\[
\tilde{\lambda}_k(f) < \lambda_{k+1}^*(f),
\]

\[
\tilde{\Lambda}_k(g) < \Lambda_{k+1}^*(g). \tag{90}
\]

When the approximation cascade is locally stable at the \((k+1)\)–step, then the improvement of the accuracy for \( f_k^*(g) \), as compared to \( f_k(g) \), can be achieved only if the enveloping approximation flow is stable at its \( k \)–point with the local exponent satisfying (90). If the approximation cascade is locally unstable, so that \( \lambda_{k+1}^*(f) > 0 \), then the corrected approximation \( f_k^*(g) \) can be of much better accuracy than \( f_k(g) \) even in the case of an unstable approximation flow, provided (90) is valid. In the latter case, to easier satisfy (90), the motion should be damped, so that to make \( \tilde{\lambda}_k(f) \) smaller. This can be done by incorporating into the definition of the cascade velocity (21) a damping parameter \( \delta_k \) lowering \( v_k(f) \),

\[
v_k(f) \rightarrow \delta_k v_k(f).
\]

The value of the damping parameter \( \delta_k \) can be found from additional conditions. For example, one may require the coincidence of some asymptotic values for the corrected approximation \( f_k^*(g) \) and for the exact \( f(g) \), of course, if such asymptotic values of \( f(g) \) are available [39,49]. Another option is to put \( \delta_k = \frac{1}{2} \), which corresponds to diminishing twice the step of the calculational procedure.

The possibility of improving the accuracy even for unstable approximation cascades and flows is the main advantage of the corrected approximations.

VI. ANHARMONIC OSCILLATOR

To illustrate the ideas of the approach we choose an anharmonic–oscillator model. The anharmonicity of oscillations plays a very important role in many physical problems, for instance, in anharmonic crystals [52,53].

Suppose we need to find the energy levels of an anharmonic oscillator with the Hamiltonian

\[
H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + g x^4, \tag{91}
\]

in which \( x \in (-\infty, \infty) \) and the coupling, or anharmonicity, parameter \( g \geq 0 \). Note that several problems of quantum mechanics can be reduced to oscillator–type models by a special change of variables [54,55].
Emphasize that our aim here is not simply the calculation of the energy levels but the demonstration how the controlled perturbation theory, formulated as dynamical theory, works for such a touchstone model as (91). In our previous papers [38-40] we have considered solely the first step of the theory as applied to an anharmonic oscillator. One step, of course, does not permit yet to illustrate and exploit in full the underlying ideas. This is why we have again to turn our attention to the model (91) extending the consideration to the higher-order approximations.

It is natural to start from the harmonic oscillator whose Hamiltonian

\[ H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} u^2 x^2 \]  

contains an unknown parameter \( u \). For convenience, we introduce the notation

\[ E_k(g, u) \equiv \left( n + \frac{1}{2} \right) F_k(g, u) \]  

for the \( k \)-order approximation of the spectrum. The quantum index \( n = 0, 1, 2, \ldots \) in \( E_k \) and \( F_k \) is not written explicitly for the sake of brevity.

The sequence \( \{ F_k(g, u) \}_{k=0}^{\infty} \) is to be obtained by the Rayleigh–Schrödinger perturbation theory starting from

\[ F_0(g, u) = u. \]  

In what follows we shall need the notation

\[ \alpha = \alpha(u) \equiv 1 - \frac{1}{u^2}, \]

\[ \beta = \beta(u) \equiv \frac{6\gamma_g}{u^3}, \]

\[ \gamma \equiv \frac{n^2 + n + 1/2}{n + 1/2}. \]  

The first four approximations, under a fixed \( u \), are

\[ F_1(g, u) = u - \frac{u}{4} (2\alpha - \beta), \]

\[ F_2(g, u) = F_1(g, u) - \frac{u}{8} (\alpha^2 - 2\alpha\beta + 2\alpha^2), \]

\[ F_3(g, u) = F_2(g, u) - \frac{u}{16} (\alpha^3 - 4\alpha^2\beta + 10\alpha\beta^2 - 3\beta^3), \]

\[ F_4(g, u) = F_3(g, u) - \frac{u}{32} \left( \frac{5}{4} \alpha^4 - 8\alpha^3\beta + 35\alpha^2\beta^2 - 24\alpha\beta^3 + 4\beta^4 \right), \]  

in which

\[ a = a(\gamma) \equiv \frac{17n^2 + 17n + 21}{(6\gamma)^2}, \]

\[ b = b(\gamma) \equiv \frac{125n^4 + 250n^3 + 472n^2 + 347n + 111}{(n + 1/2)(6\gamma)^3}, \]

\[ c = c(\gamma) \equiv \frac{10689n^4 + 21378n^3 + 60616n^2 + 49927n + 30885}{8(6\gamma)^4}. \]

From the quasifixed-point condition (25) of the form

\[ \frac{\partial}{\partial u_1} F_1(g, u_1) = 0 \]
we get the equation

\[ u_1^3 - u_1 - 6\gamma g = 0 \]  

(97)

for the control function \( u_1(g) \). Eqs. (95) and (97) tell us that

\[ \alpha(u_1) = \beta(u_1). \]

Introducing the notation

\[ \alpha_k \equiv \alpha(u_k) = 1 - \frac{1}{u_k^2}, \quad \beta_k \equiv \beta(u_k), \]  

(98)

we have

\[ F_{k+1}(g, u_1) = F_k(g, u_1) + u_1 A_{1k} \alpha_k^{k+1} \quad (k = 0, 1), \]  

(99)

where

\[ A_{10} = \frac{-1}{4}, \quad A_{11} = \frac{1}{8} (1 - 2a). \]

The equation \( \partial F_2(g, u_2)/\partial u_2 = 0 \) has no real solutions for \( u_2 \), therefore, according to (25), we put

\[ u_2(g) = u_3(g), \]  

(100)

and \( u_3 \) being defined by the equation

\[ \frac{\partial}{\partial u_3} F_3(g, u_3) = 0. \]

The latter yields

\[ u_3^3 - u_3 - 6\gamma_3 g = 0 \]  

(101)

with

\[ \gamma_3 \equiv \kappa \gamma, \quad \alpha_3 = \kappa \beta_3 \]  

(102)

and with \( \kappa \) given by the equation

\[ 5\kappa^3 - 24\kappa^2 + 70a\kappa - 24b = 0. \]  

(103)

Eqs. (101)–(103) make it possible to find

\[ F_{k+1}(g, u_3) = F_k(g, u_3) + u_3 A_{3k} \alpha_k^{k+1} \quad (k \leq 3), \]  

(104)

where

\[ A_{30} = -\frac{1}{4} \left( 2 - \frac{1}{\kappa} \right), \]

\[ A_{31} = -\frac{1}{8} \left( 1 - \frac{2}{\kappa} + 2a_3 \right), \]

\[ A_{32} = -\frac{1}{16} \left( 1 - \frac{4}{\kappa} + 10a_3 - 3b_3 \right), \]

\[ A_{33} = -\frac{1}{32} \left( \frac{5}{4} - \frac{8}{\kappa} + 35a_3 - 24b_3 + 4c_3 \right), \]
and
\[ a_3 \equiv a(\gamma_3), \quad b_3 \equiv b(\gamma_3), \quad c_3 \equiv c(\gamma_3). \]

Definition (3) of the coupling function \( g_k(f) \) because of (94), gives the equation
\[ u_k(g_k(f)) = f. \]  
(105)

From here we find
\[ g_k(f) = f(f^2 - 1) \frac{1}{6\gamma_k}, \]  
(106)
where we took into account that
\[ \alpha(u_k(g_k(f))) = 1 - \frac{1}{f^2} \]  
(107)
and used the notation
\[ \gamma_1 \equiv \gamma, \quad \gamma_2 \equiv \gamma_3 = \kappa \gamma. \]

For the cascade velocity (26) we get
\[ v_k^*(f) = A_k f \left(1 - \frac{1}{f^2}\right)^{k+1} \]  
(108)
with
\[ A_k \equiv A_{kk}, \quad A_{2k} \equiv A_{3k}. \]

The evolution integral (29) becomes
\[ \int_{f_k(g)}^{f_k^*(g)} \frac{f^{2k+1}df}{(f^2 - 1)^{k+1}} = A_k. \]  
(109)
With the notation
\[ f_k^*(g) \equiv \sqrt{1 + z_k^*(g)}, \quad f_k(g) \equiv \sqrt{1 + z_k(g)}, \]  
(110)
integral (109) transforms to
\[ \int_{z_k(g)}^{z_k^*(g)} \frac{(1 + z)^k}{z^{k+1}}dz = 2A_k. \]  
(111)
Employing the binomial formula
\[ (1 + z)^k = \sum_{p=0}^{k} C_k^p z^p; \quad C_k^p \equiv \frac{k!}{(k - p)!p!}, \]
we integrate (111) obtaining the equation
\[ z_k^* = z_k \exp \left\{ \sum_{p=0}^{k-1} \frac{C_k^p}{k - p} \left[ \frac{1}{(z_k^{*})^{k-p}} - \frac{1}{z_k^{*}} \right] + 2A_k \right\}, \]  
(112)
in which
\[ z_k^* = z_k^*(g), \quad z_k = z_k(g). \]
Introducing the polynomial

\[ P_k(x) \equiv \sum_{p=0}^{k-1} \frac{C_p}{k-p} x^{k-p} \]

with

\[ P_1(x) = x, \]
\[ P_2(x) = 2x + \frac{1}{2} x^2, \]
\[ P_3(x) = 3x + \frac{3}{2} x^2 + \frac{1}{3} x^3, \]

we can cast (112) into a more compact form

\[ z_k^* = z_k \exp \left\{ P_k \left( \frac{1}{z_k} \right) - P_k \left( \frac{1}{z_k} \right) + 2A_k \right\}. \] (113)

In this way, the \( k \)-order approximation for the spectrum of the Hamiltonian (91), i.e.

\[ e_k(g) \equiv E_k(g, u_k(g)), \]

owing to notation (93) can be written as

\[ e_k(g) = \left( n + \frac{1}{2} \right) f_k(g) = e_k(n, g) \] (114)

with \( f_k(g) \equiv F_k(g, u_k(g)) \). The corrected \( k \)-order approximation is

\[ e_k^*(g) = \left( n + \frac{1}{2} \right) f_k^*(g) = e_k^*(n, g). \] (115)

The accuracy of these approximations is characterized by their percentage errors

\[ \varepsilon_k(g) \equiv \frac{e_k(g) - e(g)}{e(g)} \cdot 100\%, \]
\[ \varepsilon_k^*(g) \equiv \frac{e_k^*(g) - e(g)}{e(g)} \cdot 100\%, \]

with respect to exact numerical values \( e(g) \).

Table I illustrates the accuracy of the approximations

\[ e_1(g) = E_1(g, u_1(g)); \quad e_2(g) = E_2(g, u_3(g)); \]
\[ e_3(g) = E_3(g, u_3(g)); \quad e_4(g) = E_4(g, u_3(g)), \]

and table II describes the accuracy of the corresponding approximations

\[ e_1^*(g) = E_1^*(g, u_1(g)); \quad e_2^*(g) = E_2^*(g, u_3(g)); \quad e_3^*(g) = E_3^*(g, u_3(g)), \]

where \( E_k^* \) means the right-hand side of (115). The errors \( \varepsilon_k(g) \equiv \varepsilon_k(n, g) \) and \( \varepsilon_k^*(g) \equiv \varepsilon_k^*(n, g) \) depend on the value of the coupling parameter \( g \in \mathbb{R}_+ \) and on the level number \( n \in \mathbb{Z}_+ \). The uniform accuracy of an approximation may be characterized by the maximal error

\[ \varepsilon_k \equiv \sup_{g \in \mathbb{R}_+} \sup_{n \in \mathbb{Z}_+} |\varepsilon_k(n, g)|. \]
or, respectively by

\[ \varepsilon^*_k \equiv \sup_{g \in \mathbb{R}_+} \sup_{n \in \mathbb{Z}_+} |\varepsilon_k^*(n, g)|. \]

From the table I we have

\[ \varepsilon_1 = 2.0\%, \quad \varepsilon_2 = 0.45\%, \quad \varepsilon_3 = 0.84\%, \quad \varepsilon_4 = 0.50\%, \]

and from the table II,

\[ \varepsilon^*_1 = 0.40\%, \quad \varepsilon^*_2 = 0.37\%, \quad \varepsilon^*_3 = 0.65\%. \]

As is seen, \( \varepsilon^*_k < \varepsilon_k \), which means that the corrected approximation (115) improves the accuracy of (114). However, the error do not monotonically decrease as the approximation order \( k \) increases. This should be related to the occurrence of local instabilities in the calculational procedure, which can be detected by the stability analysis.

To analyse the stability, we, first, need to define the trajectory of the approximation cascade, whose points are given by (6). In the considered case we find

\[ y_k(f) = f + f \sum_{p=1}^{k} B_{kp} \left( 1 - \frac{1}{f} \right)^p \]  

(116)

with the coefficients

\[ B_{11} = A_{10}, \]
\[ B_{21} = A_{10}, \quad B_{22} = A_{11}, \]
\[ B_{31} = A_{30}, \quad B_{32} = A_{31}, \quad B_{33} = A_{32}, \]
\[ B_{41} = A_{30}, \quad B_{42} = A_{31}, \quad B_{43} = A_{32}, \quad B_{44} = A_{33}. \]

For the quasilocal multipliers (35) we obtain

\[ \mu_k(f) = 1 + \sum_{p=1}^{k} B_{kp} \left( 1 + \frac{p-1}{f} \right) \left( 1 - \frac{1}{f} \right)^{p-1}. \]  

(117)

The local multipliers (40) can be found from (117) by means of (41).

The numerical analysis shows that the condition of quasilocal contraction (47) holds true,

\[ |\mu_k(f)| < 1 \quad (k = 1, 2, 3, 4), \]  

(118)

and, respectively, the condition of quasilocal stability (56) is valid. This means that all approximations (114), with \( k \geq 1 \) are closer to the exact values than the zero approximation. But the condition of local contraction (49) does not necessary holds for all \( f \in (1, \infty) \) and \( n \in \mathbb{Z}_+ \). Because of this the accuracy of approximations may not improve at each step.

The local Lyapunov exponent (70) for the approximation flow is

\[ \lambda_k(f) = A_k \left( 1 + \frac{2k+1}{f^2} \right) \left( 1 - \frac{1}{f^2} \right)^k. \]  

(119)

Since \( f \in (1, \infty) \), the sign of (119) is defined by that of \( A_k \),

\[ \text{sgn} \lambda_k(f) = \text{sgn} A_k. \]

For \( A_k \) we have the inequalities

\[ -\frac{1}{48} \leq A_1 \leq \frac{1}{144}. \]
\[-0.002496 \leq A_2 \leq 0.003001,\]

\[-0.000525 \leq A_3 \leq -0.000439\]

depending on the quantum number \( n = 0, 1, 2, \ldots \). Therefore (119) yields

\[
\tilde{\lambda}_1 (f) < 0.007, \quad \tilde{\lambda}_2 (f) < 0.003, \quad \tilde{\lambda}_3 (f) < 0.
\]

Thus, at the first two steps the local flow exponents (119) become positive for some energy levels. The positiveness of the local Lyapunov exponents signifies the occurrence of local chaos. However, at the third step the motion stabilizes, since \( \tilde{\lambda}_3 (f) < 0 \) for all \( f \in (1, \infty) \) and \( n \in \mathbb{Z}_+ \). The same inequalities (120) are valid for the images of \( \tilde{\lambda}_k (f) \) given by (72), that is, for \( \Lambda_k (g) \) as functions of \( g \in \mathbb{R}_+ \) and \( n \in \mathbb{Z}_+ \).

VII. CONCLUSION

Perturbation theory can be made convergent by introducing control functions. The reformulation of perturbation theory to the language of dynamical theory makes it possible to control convergence of the approximation sequence by checking stability conditions. The sequence of perturbative approximations is bijective to the trajectory of the approximation cascade. The control functions are defined from quasifixed-point conditions. The terms of the perturbation sequence are images of quasifixed points. Each quasifixed point can be corrected by embedding the approximation cascade into an approximation flow and considering the motion near the given quasifixed point.

The stability of calculational procedure is controlled by mapping multipliers and Lyapunov exponents. Several such characteristics can be introduced each of them being responsible for controlling particular stability properties. The quasilocal multipliers control the quasilocal contraction of a mapping on intervals. The local multipliers control the local contraction of a mapping at each step. The quasilocal Lyapunov exponents control the quasilocal stability of the approximation cascade on intervals. The local Lyapunov exponents control the local stability of the approximation cascade at each point of its trajectory. The classical Lyapunov exponents describe the asymptotic stability of the approximation cascade. The local Lyapunov exponents for the approximation flow, enveloping the approximation cascade, define the local stability of motion near the corresponding quasifixed points. The stability conditions are directly related to the character of convergence of the approximation sequence. Because of the possibility to control the convergence by means of control functions, mapping multipliers and Lyapunov exponents, the developed approach is called the controlled perturbation theory.

As an illustration of the method, the calculation of energy levels for a one-dimensional anharmonic oscillator is accomplished. Higher-order approximations are considered, as compared to our previous papers. The stability analysis showed that, in this case, the approximation cascade is quasilocally stable but not locally stable. In other words, the approximation sequence is quasilocally convergent but not locally convergent. This results in local chaos at the beginning of the cascade trajectory and, respectively, in a nonmonotonic fluctuation of errors for several first approximations. The local chaos can, in principle, be suppressed by introducing a damping parameter diminishing the cascade velocity. However, here we did not study the mechanism of such a suppression of chaos. We hope to do this in future papers.

[1] B.Sarkar and K.Bhattacharyya, J.Math.Phys. 33 (1992) 349.
[2] B.Sarkar and K.Bhattacharyya, Phys. Rev. B 48 (1993) 6913.
[3] B.Sarkar and K.Bhattacharyya, Chem. Phys. Lett. 204 (1993) 211.
[4] W.Scherer, Phys. Rev. Lett. 74 (1995) 1495.
[5] P.M.Stevenson, Phys. Rev. D 23 (1981) 2916.
[6] A.Okopińska, Ann. Phys. 228 (1993) 19.
[7] S.K.Gandhi and A.J.McKane, Nucl. Phys. B 419 (1994) 424.
[8] R.L.Hall, Phys. Rev. A 39 (1989) 5500.
[9] R.L.Hall, J. Math. Phys. 33 (1992) 3472.
[10] F.M.Fernández, Phys. Lett. A 160 (1991) 511.
11] I.C. Goyal, R.L. Gallawa and A.K. Ghatak, J. Math. Phys. 34 (1993) 1169.
[12] E. Weniger, J. Čiček and F. Vinette, J. Math. Phys. 34 (1993) 571.
[13] M. Kolesik and L. Šumaj, Phys. Lett. A 177 (1993) 87.
[14] M. Kolesik and L. Šumaj, J. Stat. Phys. 72 (1993) 1203.
[15] M. Kolesik and L. Šumaj, J. Phys. France 3 (1993) 93.
[16] M. Kolesik, Physica A 202 (1994) 529.
[17] M. Kolesik, Mod. Phys. Lett. B 8 (1994) 113.
[18] V. I. Yukalov, Ph.D. Thesis (Moscow State Univ., Moscow 1973).
[19] V. I. Yukalov, Mosc. Univ. Phys. Bull. 31 (1976) 10.
[20] V. I. Yukalov, Theor. Math. Phys. 28 (1976) 652.
[21] V. I. Yukalov, Physica A 89 (1977) 363.
[22] V. I. Yukalov, Ann. Physik 36 (1979) 31.
[23] V. I. Yukalov, Ann. Physik 37 (1980) 171.
[24] V. I. Yukalov, Ann. Physik 38 (1981) 419.
[25] V. I. Yukalov, Phys. Rev. B 32 (1985) 436.
[26] W. E. Caswell, Ann. Phys. 123 (1979) 153.
[27] P. M. Stevenson, Phys. Rev. D 24 (1981) 1622.
[28] J. K. Killingbeck, J. Phys. A 14 (1981) 1005.
[29] A. Okopińska, Phys. Rev. D 35 (1987) 1835.
[30] K. Vlachos, Phys. Rev. A 47 (1993) 838.
[31] R. Pathak, A. Chandna and K. Bhattacharyya, Phys. Rev. A 48 (1993) 4997.
[32] H. Klemmert, Phys. Lett. A 173 (1993) 332.
[33] H. Klemmert, Phys. Lett. B 300 (1993) 261.
[34] K. Vlachos and A. Okopińska, Phys. Lett. A 186 (1994) 375.
[35] V. I. Yukalov, Int. J. Mod. Phys. B 3 (1989) 1691.
[36] V. I. Yukalov, Physica A 167 (1990) 833.
[37] V. I. Yukalov, Phys. Rev. A 42 (1990) 3324.
[38] V. I. Yukalov, J. Math. Phys. 32 (1991) 1235.
[39] V. I. Yukalov, J. Math. Phys. 33 (1992) 3994.
[40] V. I. Yukalov and E. P. Yukalova, Int. J. Mod. Phys. B 7 (1993) 2367.
[41] V. I. Yukalov and E. P. Yukalova, Nuovo Cimento B 108 (1993) 1017.
[42] V. I. Yukalov and E. P. Yukalova, Physica A 206 (1994) 553.
[43] A. Duncan and H. F. Jones, Phys. Rev. D 47 (1993) 2560.
[44] C. M. Bender, A. Duncan and H. F. Jones, Phys. Rev. D 49 (1994) 4219.
[45] R. Guida, K. Konishi and H. Suzuki, Genova Univ. Preprint GEF-Th 7–1994 (Genova, 1994).
[46] F. L. Lewis, Optimal Control (Wiley, New York, 1986).
[47] L. M. Hocking, Optimal Control (Clarendon, Oxford, 1991).
[48] D. V. Shirikov, A. S. Shumovsky and V. I. Yukalov, JINR Commun. E2–86–460 (Dubna, 1986).
[49] V. I. Yukalov and E. P. Yukalova, Can. J. Phys. 71 (1993) 537.
[50] V. I. Yukalov and E. P. Yukalova, Physica A 198 (1993) 573.
[51] W. Bergweiler, Complex Var. 18 (1991) 57.
[52] V. I. Zubov, M. F. Pascual, J. N. T. Rabelo and A. C. de Faria, Phys. Status Solidi B 182 (1994) 315.
[53] V. I. Zubov, M. P. Lobo and J. N. T. Rabelo, Int. J. Mod. Phys. B 9 (1995) 585.
[54] M. Znojil, Phys. Lett. A 188 (1994) 113.
[55] M. Znojil, Phys. Lett. A 189 (1994) 1.
## TABLE I

The accuracy of the approximations \( e_k(n, g) \) of the controlled perturbation theory for the energy levels of the one-dimensional anharmonic oscillator, as compared to the exact numerical values \( e(n, g) \).

| \( g \) | \( n \) | \( e(n, g) \) | \( \varepsilon_1(n, g) \) | \( \varepsilon_2(n, g) \) | \( \varepsilon_3(n, g) \) | \( \varepsilon_4(n, g) \) |
|---|---|---|---|---|---|---|
| 0 | 0.50726 | 0.006 | \( 10^{-4} \) | \( 10^{-4} \) | \( 10^{-4} \) | \( -10^{-4} \) |
| 1 | 1.5357 | 0.009 | 0.001 | \( -10^{-4} \) | \( 10^{-4} \) | \( -10^{-4} \) |
| 2 | 2.5909 | 0.001 | 0.008 | \( -10^{-4} \) | \( 10^{-4} \) | \( -10^{-4} \) |
| 0.01 | 3 | 3.6711 | \( -0.008 \) | 0.017 | \( -10^{-4} \) | \( 10^{-4} \) | \( -10^{-4} \) |
| 4 | 4.7749 | \( -0.019 \) | 0.027 | \( -10^{-4} \) | \( 10^{-4} \) | \( -10^{-4} \) |
| 5 | 5.9010 | \( -0.030 \) | \( -0.002 \) | \( 10^{-4} \) | \( -10^{-4} \) | \( 10^{-4} \) |
| 6 | 7.0483 | \( -0.040 \) | \( -0.001 \) | \( 0.001 \) | \( 10^{-4} \) | \( -10^{-4} \) |
| 7 | 8.2158 | \( -0.051 \) | \( -0.001 \) | \( 0.001 \) | \( 10^{-4} \) | \( -10^{-4} \) |
| 8 | 9.4030 | \( -0.066 \) | \( -0.004 \) | \( -0.003 \) | \( -0.003 \) | \( -0.003 \) |
| 0.3 | 0 | 0.63799 | 0.57 | 0.10 | 0.01 | \( -0.003 \) |
| 1 | 2.0946 | 0.49 | 0.15 | \( -0.01 \) | \( -0.004 \) | \( -0.004 \) |
| 2 | 3.8448 | \( -0.06 \) | 0.21 | \( -0.24 \) | \( -0.13 \) | \( -0.13 \) |
| 3 | 5.7970 | \( -0.30 \) | 0.20 | \( -0.36 \) | \( -0.20 \) | \( -0.20 \) |
| 4 | 7.9100 | \( -0.40 \) | 0.23 | \( -0.40 \) | \( -0.21 \) | \( -0.21 \) |
| 5 | 10.167 | \( -0.51 \) | \( -0.09 \) | 0.01 | \( -0.01 \) | \( -0.01 \) |
| 6 | 12.540 | \( -0.53 \) | \( -0.02 \) | 0.05 | 0.03 | 0.03 |
| 7 | 15.030 | \( -0.59 \) | \( -0.01 \) | 0.03 | 0.01 | 0.01 |
| 8 | 17.620 | \( -0.63 \) | \( -0.01 \) | 0.03 | 0.01 | 0.01 |
| 1 | 0 | 0.83037 | 1.09 | 0.21 | 0.02 | \( -0.01 \) |
| 2 | 2.7379 | 0.81 | 0.25 | \( -0.03 \) | \( -0.01 \) | \( -0.01 \) |
| 3 | 5.1780 | \( -0.11 \) | 0.25 | \( -0.40 \) | \( -0.22 \) | \( -0.22 \) |
| 4 | 7.9400 | \( -0.40 \) | 0.25 | \( -0.51 \) | \( -0.28 \) | \( -0.28 \) |
| 5 | 10.160 | \( -0.55 \) | 0.24 | \( -0.58 \) | \( -0.32 \) | \( -0.32 \) |
| 6 | 12.540 | \( -0.66 \) | \( -0.12 \) | 0.03 | \( -0.003 \) | \( -0.003 \) |
| 7 | 15.030 | \( -0.69 \) | \( -0.05 \) | 0.05 | 0.01 | 0.01 |
| 8 | 17.620 | \( -0.77 \) | \( -0.06 \) | 0.01 | \( -0.03 \) | \( -0.03 \) |
| 200 | 0 | 3.9309 | 1.97 | 0.44 | 0.04 | \( -0.03 \) |
| 1 | 14.059 | 1.24 | 0.39 | \( -0.05 \) | \( -0.02 \) | \( -0.02 \) |
| 2 | 27.550 | \( -0.24 \) | 0.22 | \( -0.67 \) | \( -0.41 \) | \( -0.41 \) |
| 3 | 4.4009 | \( -0.61 \) | 0.18 | \( -0.79 \) | \( -0.48 \) | \( -0.48 \) |
| 4 | 6.0030 | \( -0.75 \) | 0.19 | \( -0.82 \) | \( -0.48 \) | \( -0.48 \) |
| 5 | 7.8400 | \( -0.85 \) | \( -0.18 \) | 0.03 | \( -0.02 \) | \( -0.02 \) |
| 6 | 9.7900 | \( -0.88 \) | \( -0.10 \) | 0.03 | \( -0.02 \) | \( -0.02 \) |
| 7 | 11.840 | \( -0.88 \) | \( -0.03 \) | 0.05 | 0.01 | 0.01 |
| 8 | 13.990 | \( -0.92 \) | \( -0.03 \) | 0.03 | \( -0.01 \) | \( -0.01 \) |
| 20000 | 0 | 18.137 | 2.01 | 0.45 | 0.04 | \( -0.03 \) |
| 1 | 64.987 | 1.26 | 0.40 | \( -0.05 \) | \( -0.02 \) | \( -0.02 \) |
| 2 | 127.51 | \( -0.25 \) | 0.21 | \( -0.69 \) | \( -0.43 \) | \( -0.43 \) |
| 3 | 199.20 | \( -0.64 \) | 0.16 | \( -0.82 \) | \( -0.50 \) | \( -0.50 \) |
| 4 | 277.00 | \( -0.76 \) | 0.18 | \( -0.84 \) | \( -0.49 \) | \( -0.49 \) |
| 5 | 368.20 | \( -0.83 \) | 0.16 | 0.05 | \( -0.01 \) | \( -0.01 \) |
| 6 | 454.00 | \( -0.95 \) | \( -0.16 \) | \( -0.04 \) | \( -0.09 \) | \( -0.09 \) |
| 7 | 548.90 | \( -0.91 \) | \( -0.05 \) | 0.03 | \( -0.01 \) | \( -0.01 \) |
| 8 | 648.50 | \( -0.93 \) | \( -0.03 \) | 0.03 | \( -0.01 \) | \( -0.01 \) |
The accuracy of the corrected approximations $e_k(n, g)$ for the energy levels of the one-dimensional anharmonic oscillator, compared to the exact values $e(n, g)$.

| $g$ | $n$ | $e(n, g)$ | $e_1^*(n, g)$ | $e_2^*(n, g)$ | $e_3^*(n, g)$ |
|-----|-----|-----------|---------------|---------------|---------------|
| 0.01 | 0  | 0.50726   | 0.005         | $10^{-4}$     | $10^{-4}$     |
|      | 1  | 1.5357    | 0.007         | 0.001         | $-10^{-4}$    |
|      | 2  | 2.5909    | 0.001         | 0.008         | $-0.001$      |
|      | 3  | 3.6711    | $-0.006$      | 0.017         | $-0.003$      |
|      | 4  | 4.7749    | $-0.013$      | 0.027         | $-0.006$      |
|      | 5  | 5.9010    | $-0.021$      | $-0.002$      | $10^{-4}$     |
|      | 6  | 7.0483    | $-0.028$      | $-0.001$      | 0.001         |
|      | 7  | 8.2158    | $-0.035$      | $-10^{-4}$    | 0.001         |
|      | 8  | 9.4030    | $-0.046$      | $-0.004$      | $-0.003$      |

| 0.3  | 0  | 0.63799   | 0.25          | 0.08          | 0.01          |
|      | 1  | 2.0946    | 0.19          | 0.11          | $-0.01$       |
|      | 2  | 2.8448    | $-0.11$       | 0.11          | $-0.22$       |
|      | 3  | 3.7970    | $-0.20$       | 0.07          | $-0.33$       |
|      | 4  | 4.7910    | $-0.21$       | 0.06          | $-0.36$       |
|      | 5  | 5.1677    | $-0.26$       | $-0.04$       | 0.004         |
|      | 6  | 6.1540    | $-0.23$       | 0.02          | 0.04          |
|      | 7  | 7.1530    | $-0.26$       | 0.01          | 0.02          |
|      | 8  | 8.1762    | $-0.27$       | 0.01          | 0.02          |

| 1    | 0  | 0.80797   | 0.28          | 0.16          | 0.01          |
|      | 1  | 2.7379    | 0.19          | 0.16          | $-0.02$       |
|      | 2  | 5.1780    | $-0.19$       | 0.03          | $-0.35$       |
|      | 3  | 7.9400    | $-0.24$       | $-0.04$       | $-0.44$       |
|      | 4  | 10.960    | $-0.26$       | $-0.08$       | $-0.49$       |
|      | 5  | 14.203    | $-0.30$       | $-0.02$       | 0.01          |
|      | 6  | 17.630    | $-0.28$       | 0.02          | 0.03          |
|      | 7  | 21.240    | $-0.32$       | $-0.02$       | $-0.02$       |
|      | 8  | 25.000    | $-0.32$       | $-0.02$       | $-0.02$       |

| 200  | 0  | 3.9399    | $-0.06$       | 0.20          | 0.001         |
|      | 1  | 14.059    | 0.01          | 0.13          | $-0.03$       |
|      | 2  | 27.550    | $-0.37$       | $-0.27$       | $-0.53$       |
|      | 3  | 43.010    | $-0.37$       | $-0.34$       | $-0.62$       |
|      | 4  | 60.030    | $-0.33$       | $-0.35$       | $-0.64$       |
|      | 5  | 78.400    | $-0.33$       | $-0.02$       | $-0.01$       |
|      | 6  | 97.900    | $-0.33$       | $10^{-4}$     | $-0.01$       |
|      | 7  | 118.40    | $-0.30$       | 0.04          | 0.02          |
|      | 8  | 139.90    | $-0.32$       | 0.01          | $-0.01$       |

| 20000| 0  | 18.137    | $-0.09$       | 0.20          | $-10^{-4}$    |
|      | 1  | 64.987    | $0.001$       | 0.13          | $-0.03$       |
|      | 2  | 127.51    | $-0.39$       | $-0.28$       | $-0.54$       |
|      | 3  | 199.20    | $-0.39$       | $-0.37$       | $-0.65$       |
|      | 4  | 278.10    | $-0.34$       | $-0.37$       | $-0.65$       |
|      | 5  | 363.20    | $-0.33$       | $-0.01$       | 0.01          |
|      | 6  | 454.00    | $-0.40$       | $-0.06$       | $-0.08$       |
|      | 7  | 548.90    | $-0.32$       | 0.02          | $-0.001$      |
|      | 8  | 648.50    | $-0.31$       | 0.01          | $-0.003$      |