SPECTRAL MULTIPLIER THEOREMS OF HÖRMANDER TYPE ON HARDY AND LEBESGUE SPACES

PEER CHRISTIAN KUNSTMANN AND MATTHIAS UHL

Abstract. Let $X$ be a space of homogeneous type and let $L$ be an injective, non-negative, self-adjoint operator on $L^2(X)$ such that the semigroup generated by $-L$ fulfills Davies-Gaffney estimates of arbitrary order. We prove that the operator $F(L)$, initially defined on $H^1_L(X) \cap L^2(X)$, acts as a bounded linear operator on the Hardy space $H^1_L(X)$ associated with $L$ whenever $F$ is a bounded, sufficiently smooth function. Based on this result, together with interpolation, we establish Hörmander type spectral multiplier theorems on Lebesgue spaces for non-negative, self-adjoint operators satisfying generalized Gaussian estimates in which the required differentiability order is relaxed compared to all known spectral multiplier results.

Contents

1. Introduction 1
2. Preliminaries 6
2.1. Spaces of homogeneous type 7
2.2. Off-diagonal estimates 8
3. Hardy spaces associated with operators 9
4. Spectral multipliers on the Hardy space $H^1_L(X)$ 12
5. Boundedness of spectral multipliers on $H^p_L(X)$ and $L^p(X)$ 26
6. Proofs of some auxiliary results 29
References 33

1. Introduction

Let $L$ be a non-negative, self-adjoint operator on the Hilbert space $L^2(X)$, where $X$ is a $\sigma$-finite measure space. If $E_L$ denotes the resolution of the identity associated with $L$, the spectral theorem asserts that the operator

$$F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda)$$

(1.1)

is well defined and acts as a bounded linear operator on $L^2(X)$ whenever $F: [0, \infty) \to \mathbb{C}$ is a bounded Borel function. Spectral multiplier theorems provide regularity assumptions on $F$ which ensure that the operator $F(L)$ extends from $L^p(X) \cap L^2(X)$ to a bounded linear operator on $L^p(X)$ for all $p$ ranging in some symmetric interval containing 2.

In 1960, L. Hörmander addressed this question for the Laplacian $L = -\Delta$ on $X = \mathbb{R}^D$ during his studies on the boundedness of Fourier multipliers on $\mathbb{R}^D$. His famous Fourier multiplier theorem ([38, Theorem 2.5]) states that the operator $F(-\Delta)$ is of weak type $(1, 1)$ whenever $F: [0, \infty) \to \mathbb{C}$

2010 Mathematics Subject Classification. 42B15, 42B20, 42B30, 47A60.

Key words and phrases. Spectral multiplier theorems, Hardy spaces, non-negative self-adjoint operators, Davies-Gaffney estimates, spaces of homogeneous type.
is a bounded Borel function such that
\[ \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^s_2} < \infty \] (1.2)
for some \( s > D/2 \). Here and in the following \( \omega \in C_0^\infty(0, \infty) \) is a non-negative function such that
\[ \supp \omega \subset (1/4,1) \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \omega(2^{-n} \lambda) = 1 \quad \text{for all } \lambda > 0. \]

As a consequence, \( F(-\Delta) \) is bounded on \( L^p(\mathbb{R}^D) \) for every \( p \in (1, \infty) \). Note that the so-called Hörmander condition (1.2) does not depend on the special choice of \( \omega \). By considering imaginary powers \((-\Delta)^i\tau\) for \( \tau \in \mathbb{R} \), M. Christ (\cite{christ1990} p. 73) observed that the regularity order in Hörmander’s statement cannot be improved beyond \( D/2 \). This means that for any \( s < D/2 \) there exists a bounded Borel function \( F: [0, \infty) \to \mathbb{C} \) such that the Hörmander condition (1.2) holds, but \( F(-\Delta) \) does not act as a bounded operator on \( L^p(\mathbb{R}^D) \) for the whole range \( p \in (1, \infty) \).

Hörmander’s multiplier theorem was generalized, on the one hand, to other spaces than \( \mathbb{R}^D \) and, on the other hand, to more general operators than the Laplacian. The development began in the early 1990’s. G. Mauceri and S. Meda (\cite{mauceri1992}) and M. Christ (\cite{christ1990}) extended the result to homogeneous Laplacians on stratified nilpotent Lie groups. Further generalizations were obtained by G. Alexopoulos (\cite{alexopoulos1993}) who showed in the setting of connected Lie groups of polynomial volume growth a corresponding statement for the left invariant sub-Laplacian which was in turn extended by W. Hebisch (\cite{hebisch1995}) to integral operators with kernels decaying polynomially away from the diagonal. More historical remarks about spectral multiplier theorems can be found e.g. in \cite{duong2016} and the references therein.

The results in \cite{duong2016} due to X.T. Duong, E.M. Ouhabaz, and A. Sikora marked an important step toward the study of more general operators. In the abstract framework of (subsets of) spaces of homogeneous type \((X,d,\mu)\) with dimension \( D > 0 \) they investigated non-negative, self-adjoint operators \( L \) on \( L^2(X) \) which satisfy pointwise Gaussian estimates, i.e. the semigroup \((e^{-tL})_{t>0}\) generated by \(-L\) can be represented as integral operators
\[ e^{-tL}f(x) = \int_X p_t(x,y)f(y)\,d\mu(y) \quad (f \in L^2(X), t > 0, \mu\text{-a.e. } x \in X) \]
and the kernels \( p_t: X \times X \to \mathbb{C} \) enjoy the following pointwise upper bound
\[ |p_t(x,y)| \leq C \mu(B(x, t^{1/m}))^{-1} \exp\left( -b \left( \frac{d(x,y)}{t^{1/m}} \right)^{m-1} \right) \] (1.3)
for all \( t > 0 \) and all \( x,y \in X \), where \( b, C > 0 \) and \( m \geq 2 \) are constants independent of \( t, x, y \) and \( B(x,r) := \{ y \in X : d(x,y) < r \} \) denotes the open ball in \( X \) with center \( x \in X \) and radius \( r > 0 \). Under these hypotheses X.T. Duong, E.M. Ouhabaz, and A. Sikora proved that the operator \( F(L) \) is of weak type \((1,1)\) whenever \( F: [0, \infty) \to \mathbb{C} \) is a bounded Borel function such that \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^s_2} < \infty \) for some \( s > (D+1)/2 \). Consequently, \( F(L) \) is then bounded on \( L^p(X) \) for all \( p \in (1, \infty) \).

However, the price for the generality lies in the requirement of an additional \( 1/2 \) in the regularity order of the Hörmander condition. Unfortunately, sharp results as for the Laplacian are unknown at present time. In the general situation it is only known that the regularity assumption \( s > D/2+1/6 \) cannot be weakened as an example in \cite{thangavelu1991} by S. Thangavelu shows.

In order to get better multiplier results in the general situation as well, X.T. Duong, E.M. Ouhabaz, and A. Sikora introduced the so-called Plancherel condition (\cite{duong2016} (3.1)) which means the following: There exist \( C > 0 \) and \( \eta \in [2, \infty) \) such that for all \( R > 0, y \in X \), and all bounded Borel functions \( F: [0, \infty) \to \mathbb{C} \) with \( \supp F \subset [0, R] \)
\[ \int_X |K_F(\eta\sqrt{\Theta_\eta}(x,y))|^2 \,d\mu(x) \leq C \mu(B(y,1/R))^{-1} \| F(R\cdot) \|_{L^2}^2, \] (1.4)
whereas
where \( K_{F(\sqrt{L})} : X \times X \to \mathbb{C} \) denotes the kernel of the integral operator \( F(\sqrt{L}) \). The result of X.T. Duong, E.M. Ouhabaz, and A. Sikora reads as follows ([27], Theorem 3.1):

\[ \text{Definition 1.1.} \]

Let \((X,d,\mu)\) be a space of homogeneous type with dimension \( D \) and let \( L \) be a non-negative, self-adjoint operator on \( L^2(X) \) which satisfies pointwise Gaussian estimates. Suppose that the Plancherel condition holds for some \( q \in [2,\infty] \) and that \( F : [0,\infty) \to \mathbb{C} \) is a bounded Borel function with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^s_2} < \infty \) for some \( s > D/2 \). Then the operator \( F(L) \) is of weak type \((1,1)\) and thus bounded on \( L^p(X) \) for all \( p \in (1,\infty) \).

Here, we have set \( H^s_\infty := C^s \), the space of Hölder continuous functions. Sometimes it is not clear whether, or even not true that, a non-negative, self-adjoint operator on \( L^2(X) \) admits pointwise Gaussian estimates and therefore the above results cannot be applied. This occurs, for example, for Schrödinger operators with bad potentials ([14]) or elliptic operators of higher order with bounded measurable coefficients ([23]). Nevertheless, it is often possible to show a weakened version of ([3]), so-called generalized Gaussian estimates.

\[ \text{Definition 1.1.} \]

Let \( 1 \leq p \leq 2 \leq q \leq \infty \) and \( m \geq 2 \). A non-negative, self-adjoint operator \( L \) on \( L^2(X) \) is said to satisfy generalized Gaussian \((p,q)\)-estimates of order \( m \) if there are constants \( b,C > 0 \) such that

\[ \left\| 1_{B(x,t^{1/m})} e^{-tL} 1_{B(y,t^{1/m})} \right\|_{L^p \to L^q} \leq C \mu(B(x,t^{1/m}))^{-(\frac{1}{p} - \frac{1}{q})} \exp \left( -b \left( \frac{d(x,y)}{t^{1/m}} \right)^{\frac{m}{m-1}} \right) \quad (1.5) \]

for all \( t > 0 \) and all \( x,y \in X \). In this case, we will use the shorthand notation \( \text{GGE}_m(p,q) \). If \( L \) satisfies \( \text{GGE}_m(2,2) \), then we also say that \( L \) enjoys Davies-Gaffney estimates of order \( m \) and just write \( \text{DG}_m \). Here, \( 1_{E_1} \) denotes the characteristic function of the set \( E_1 \) and \( \| 1_{E_1} e^{-tL} 1_{E_2} \|_{L^p \to L^q} \) is defined via \( \sup_{f} \| f \|_{L^p} \leq \| 1_{E_1} e^{-tL} (1_{E_2} f) \|_{L^q} \) for Borel sets \( E_1, E_2 \subset X \).

In the case \((p,q) = (1,\infty)\), this definition covers pointwise Gaussian estimates (cf. [11], Proposition 2.9).

In 2003, S. Blunck ([9], Theorem 1.1]) showed a spectral multiplier theorem for non-negative, self-adjoint operators \( L \) on \( L^2(X) \) satisfying \( \text{GGE}_m(p_0,p'_0) \) for some \( p_0 \in [1,2] \), where \( 1/p_0 + 1/p'_0 = 1 \). It guarantees that the operator \( F(L) \) is of weak type \((p_0,p'_0)\) if \( F : [0,\infty) \to \mathbb{C} \) is a bounded Borel function such that \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^s_2} < \infty \) holds for some \( s > (D+1)/2 \). In particular, \( F(L) \) is then bounded on \( L^p(X) \) for all \( p \in (p_0,p'_0) \).

Here, the required regularity order in the Hörmander condition for getting a weak type \((p_0,p'_0)\)-bound is the same as needed for the weak type \((1,1)\)-bound in the corresponding statement for operators enjoying pointwise Gaussian estimates. The proof of S. Blunck relies on the weak type \((p_0,p'_0)\) criterion due to S. Blunck and the first named author ([12], Theorem 1.1]) and it seems to be impossible to weaken the regularity assumptions with this approach directly. However, since for boundedness of \( F(L) \) on \( L^2(X) \) no regularity of \( F \) is needed, one expects, motivated by interpolation, \( s > (D+1)(1/p_0 - 1/2) \) instead of \( s > (D+1)/2 \) as a sufficient regularity assumption in the Hörmander condition when one is interested in boundedness of \( F(L) \) in \( L^p(X) \) for all \( p \in (p_0,p'_0) \).

In order to obtain such a multiplier result, we make use of Hardy spaces which serve as a substitute of Lebesgue spaces. For our purposes we shall consider specific Hardy spaces being associated with the operator \( L \), similarly to the way that the classical Hardy spaces are adapted to the Laplacian. They were originally introduced by P. Auscher, X.T. Duong and A. McIntosh in [3] and revised during the past ten years. We refer to the beginning of Section 3 for a short survey on recent developments.
Definition 1.2. Let $L$ be an injective, non-negative, self-adjoint operator on $L^2(X)$ which satisfies Davies-Gaffney estimates of order $m \geq 2$. Consider the conical square function

$$Sf(x) := \left( \int_0^\infty \int_{B(x,t)} |L^m e^{-tmL} f(y)|^2 \frac{d\mu(y)}{|B(x,t)|} \frac{dt}{t} \right)^{1/2} \quad (f \in L^2(X), x \in X).$$

For $p \in [1, 2]$, the Hardy space $H^p_L(X)$ associated with the operator $L$ is said to be the completion of the set $\{f \in L^2(X) : Sf \in L^p(X)\}$ with respect to the norm

$$\|f\|_{H^p_L(X)} := \|Sf\|_{L^p}.$$ 

By the spectral theorem, it is plain to see that $H^2_L(X) = L^2(X)$ with equivalent norms. Hardy spaces associated with $L$ are known to possess nice properties, for example, they form a complex interpolation scale (cf. Fact 3.3), coincide under the assumption of GGE$_m(p_0, 2)$ with $L^p(X)$ for all $p \in (p_0, 2]$ (cf. Theorem 3.7) and allow spectral multiplier theorems even for all $p \in [1, p_0]$ (cf. Sections 4, 5).

There is an equivalent characterization of the space $H^1_L(X)$ in terms of a molecular decomposition (cf. Theorem 3.5). In order to verify boundedness of an operator on the Hardy space $H^1_L(X)$, one has just to understand the action of the operator on an individual molecule. Such an idea is classical in the more comfortable situation of an atomic decomposition and was used by various authors for obtaining boundedness of spectral multipliers on the Hardy space $H^1_L(X)$. For example, J. Dziubański ([29]) showed a spectral multiplier theorem for Schrödinger operators and, later, J. Dziubański and M. Preisner ([30]) established a generalization to arbitrary operators satisfying pointwise Gaussian estimates of order 2. Recently, X.T. Duong and L.X. Yan ([28]) obtained boundedness of spectral multipliers on the Hardy space $H^1_L(X)$ for operators $L$ satisfying Davies-Gaffney estimates of order 2.

All these authors confined their studies to operators satisfying Davies-Gaffney estimates of order 2 and used essentially that, in this case, the validity of Davies-Gaffney estimates is equivalent to the finite speed propagation property for the corresponding wave equation (cf. e.g. [13] Theorem 3.4). Hence one obtains information on the support of the integral kernel of $\cos(t\sqrt{L})$ and this in turn entails information on the support of the integral kernel of $F(\sqrt{L})$. However, for general $m > 2$, such a relation to finite speed propagation properties fails. We develop the following spectral multiplier theorem on the Hardy space $H^1_L(X)$ for operators $L$ satisfying Davies-Gaffney estimates of arbitrary order $m \geq 2$ (cf. Theorem 4.1).

Theorem 1.3. Let $(X, d, \mu)$ be a space of homogeneous type with dimension $D$ and $L$ an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates of order $m \geq 2$. If a bounded Borel function $F: [0, \infty) \to \mathbb{C}$ satisfies $\sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^2_L} < \infty$ for some $s > (D + 1)/2$, then $F(L)$ can be extended from $H^1_L(X) \cap L^2(X)$ to a bounded linear operator on $H^1_L(X)$.

Based on ideas in [28] by X.T. Duong and L.X. Yan, we give a sufficient criterion for the boundedness of spectral multipliers on $H^1_L(X)$ (cf. Theorem 4.6), which will be achieved by reducing the proof of the boundedness of $F(L)$ in $H^1_L(X)$ to the uniform boundedness of $F(L)a$ in $H^1_L(X)$ for every molecule $a$. In order to derive the above Hörmander type multiplier theorem on $H^1_L(X)$, we use suitable weighted norm estimates that generalize the tools prepared in [27] and compensate for the lack of information on the support caused by the missing finite speed propagation property. We also present an improved spectral multiplier result with an adequate $L^2$-version of the Plancherel condition ([14]) which also works for operators $L$ satisfying Davies-Gaffney estimates. In order to motivate our replacement, we rewrite ([14]) as a norm estimate for the operator $F(\sqrt{L})$ itself

$$\|F(\sqrt{L}) \mathbb{1}_{B(y,1/R)}\|_{L^2} \leq C \frac{|B(y,1/R)|^{1/2}}{R} \|F(R \cdot)\|_{L^q}$$
for all \( R > 0, \ y \in \mathbb{X}, \) and all bounded Borel functions \( F : [0, \infty) \to \mathbb{C} \) with \( \text{supp} \ F \subseteq [0, R] \), where the constants \( C > 0 \) and \( q \in [2, \infty) \) are independent of \( R, y, F \). Inspired by this observation, we introduce our substitute of the Plancherel condition for operators \( L \) which fulfill Davies-Gaffney estimates of order \( m \geq 2 \) as follows:

\[
\| F(\sqrt{L}) \, 1_{B(y, 1/R)} \|_{L^2 \to L^2} \leq C \| F(R \cdot) \|_{L^q}
\]

(1.6)

for all \( R > 0, \ y \in \mathbb{X}, \) and all bounded Borel functions \( F : [0, \infty) \to \mathbb{C} \) with \( \text{supp} \ F \subseteq [0, R] \), where the constants \( C > 0 \) and \( q \in [2, \infty) \) are independent of \( R, y, F \). Having this replacement of (1.4) at hand, we are able to show the following result (cf. Theorem 1.2).

**Theorem 1.4.** Let \( (X, d, \mu) \) be a space of homogeneous type with dimension \( D \) and \( L \) an injective, non-negative, self-adjoint operator on \( L^2(X) \) for which Davies-Gaffney estimates of order \( m \geq 2 \) hold. Suppose that \( L \) fulfills the Plancherel condition (1.6). If \( F : [0, \infty) \to \mathbb{C} \) is a bounded Borel function with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^s_q} < \infty \) for some \( s > \max\{D/2, 1/q\} \), then there exists a constant \( C > 0 \) such that for all \( f \in H^1(X) \)

\[
\| F(L) f \|_{H^1(X)} \leq C \left( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^s_q} + \| F(0) \| \right) \| f \|_{H^1(X)}.
\]

In the same way as for the original Plancherel condition (1.4) the validity of its variant (1.6) for some \( q \in [2, \infty) \) entails that the point spectrum of the considered operator \( L \) is empty. We also present a version of the Plancherel condition that applies for operators with non-empty point spectrum as well (cf. Theorem 1.3). The approach is similar to the one of [27, Theorem 3.2]. Since the Plancherel condition (1.6) always holds for \( q = \infty \) (cf. Lemma 1.5), Theorem 1.3 yields the following multiplier result (cf. Theorem 1.4 b)), in which the same order of differentiability is required as in [28, Theorem 1.1] (which covers the case \( m = 2 \)).

**Theorem 1.5.** Let \( (X, d, \mu) \) be a space of homogeneous type with dimension \( D \) and \( L \) be as in Theorem 1.3. If \( F : [0, \infty) \to \mathbb{C} \) is a bounded Borel function with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{C^\omega} < \infty \) for some \( s > D/2 \), then \( F(L) \) extends to a bounded linear operator on the Hardy space \( H^1(X) \).

Having spectral multiplier theorems on \( H^1(X) \) at hand, we can prove spectral multiplier results on Lebesgue spaces for operators satisfying generalized Gaussian estimates \( GGE_m(p_0, p_0') \) for some \( p_0 \in [1, 2] \) and \( m \geq 2 \). In a first step we combine our multiplier results on the Hardy space \( H^1(X) \) with the interpolation procedure [10, Corollary 4.84] that allows to interpolate the regularity order in the Hörmander condition as well. This yields multiplier results on \( H^p(X) \) for all \( p \in [1, 2] \) (cf. Theorem 5.3). As the spaces \( H^p(X) \) and \( L^p(X) \) coincide for each \( p \in (p_0, 2] \), we obtain spectral multiplier theorems on Lebesgue spaces which read as follows (cf. Theorem 5.4).

**Theorem 1.6.** Let \( (X, d, \mu) \) be a space of homogeneous type with dimension \( D \) and \( L \) be a non-negative, self-adjoint operator on \( L^2(X) \) such that generalized Gaussian estimates \( GGE_m(p_0, p_0') \) hold for some \( p_0 \in [1, 2] \) and \( m \geq 2 \).

a) For fixed \( p \in (p_0, p_0') \) suppose that \( s > (D + 1)/|p - 1/2| \) and \( 1/q < |1/p - 1/2| \). Then, for every bounded Borel function \( F : [0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^s_q} < \infty \), the operator \( F(L) \) is bounded on \( L^p(X) \).

b) Let \( p \in (p_0, p_0') \) and \( s > D/|p - 1/2| \). Then, for any bounded Borel function \( F : [0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{C^\omega} < \infty \), the operator \( F(L) \) is bounded on \( L^p(X) \).

c) In addition, assume that \( L \) fulfills the Plancherel condition (1.6) for some \( q_0 \in [2, \infty) \). Fix \( p \in (p_0, p_0') \). Let \( s > \max\{D, 2/q_0\}/|p - 1/2| \) and \( 1/q < 2/q_0/|p - 1/2| \). Then, for every bounded Borel function \( F : [0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^s_q} < \infty \), the operator \( F(L) \) is bounded on \( L^p(X) \).

The statement a) improves the results [9, Theorem 1.1] of S. Blunck and [39, Theorem 5.6] (see also [40, Theorem 4.95]) of C. Kriegler in which the regularity orders \( s > (D + 1)/2, \ q = 2 \) and
s > D[1/p − 1/2] + 1/2, q = 2 were required, respectively. However, [9] Theorem 1.1 also includes a weak type \((p_0, p_0)\) assertion for \(F(L)\).

We emphasize that in the presence of pointwise Gaussian estimates the aforementioned multiplier theorem due to X.T. Duong, E.M. Ouhabaz and A. Sikora in combination with interpolation would need the same order of regularity for \(F\) as our main result for ensuring the boundedness of \(F(L)\) on \(L^p(X)\) for any \(p \in (p_0, p_0')\). Additionally, in the case \(p_0 = 1\) the statement b) matches [27] Theorem 3.1 which is sharp in the sense that it includes the same regularity assumptions as needed for the Laplacian in Hörmander’s multiplier theorem.

Recently, P. Chen, E.M. Ouhabaz, A. Sikora, and L. Yan obtained a similar spectral multiplier result for operators \(L\) satisfying \(DG_2\) in which the required order of differentiability is the same as ours in c) provided that \(L\) satisfies the so-called Stein-Tomas restriction type condition \([15] \text{(ST}_{p_0,2})\) for some \(p_0 \in [1, 2)\) (cf. [15] Theorem 4.1). This corresponds to the \(L^{p_0} - L^2\)-version of the Plancherel condition \([16]\) and is thus more restrictive than our assumption. The approach in [15] makes no use of Hardy spaces, but uses the result of [9]. On the other hand, the approach relies heavily on the finite speed propagation property and thus the method of proof is restricted to the case \(m = 2\).

Examples of operators to which our results apply but those in [28, 15, 29, 30] are not applicable include higher order elliptic operators in divergence form with bounded complex-valued coefficients on \(\mathbb{R}^D\) (cf. [22, 23]). These operators are given by forms \(a: H^k_2(\mathbb{R}^D) \times H^k_2(\mathbb{R}^D) \to \mathbb{C}\) of the type

\[
a(u, v) = \int_{\mathbb{R}^D} \sum_{|\alpha|=|\beta|=k} a_{\alpha\beta} \partial^\alpha u \overline{\partial^\beta v} \, dx,
\]

where \(a_{\alpha\beta}: \mathbb{R}^D \to \mathbb{C}\) are bounded and measurable functions. We assume \(a_{\alpha\beta} = \overline{a_{\beta\alpha}}\) for all \(\alpha, \beta\) and Garding’s inequality

\[
a(u, u) \geq \delta \|\nabla^k u\|_{L^2}^2 \quad \text{for all } u \in H^k_2(\mathbb{R}^D)
\]

for some \(\delta > 0\), where \(\|\nabla^k u\|_{L^2}^2 := \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^2}^2\). Then \(a\) is a closed symmetric form. The associated operator \(L\) is defined by \(u \in \mathcal{D}(L)\) and \(Lu = f\) if and only if \(u \in H^k_2(\mathbb{R}^D)\) and \(\int_{\mathbb{R}^D} f \overline{v} \, dx = a(u, v)\) for all \(v \in H^k_2(\mathbb{R}^D)\). In the case \(D > 2k\), \(L\) satisfies generalized Gaussian estimates \(\text{GGE}_m(p_0, p_0')\) with \(m := 2k\) and \(p_0 := 2D/(m + D)\) (cf. [22]). It is well-known that \(p_0\) is sharp in the sense that for any \(r \notin [p_0, p_0']\) there exists an operator \(L\) in the given class for which \(e^{-tL}\) cannot be extended from \(L^r(\mathbb{R}^D) \cap L^2(\mathbb{R}^D)\) to a bounded linear operator on \(L^r(\mathbb{R}^D)\) for any \(t > 0\) (cf. e.g. [23] Theorem 10).

In another paper ([41]) we discuss how spectral multiplier theorems of the type presented here apply to the second order Maxwell operator with measurable coefficient matrices and the Stokes operator with Hodge boundary conditions on bounded Lipschitz domains in \(\mathbb{R}^3\) as well as the time-dependent Lamé system equipped with homogeneous Dirichlet boundary conditions.

2. Preliminaries

Throughout the whole article we assume that \((X, d, \mu)\) is a space of homogeneous type with dimension \(D\) as introduced in Section 2.1 below. To avoid repetition, we skip this assumption in all the subsequent statements.

We make use of the notation \(B(x, r) := \{y \in X : d(y, x) < r\}\) for the open ball in \(X\) with center \(x \in X\) and radius \(r \geq 0\). We shall write \(\lambda B(x, r)\) for the \(\lambda\)-dilated ball \(B(x, \lambda r)\) and \(A(x, r, k)\) for the annular region \(B(x, (k + 1)r) \setminus B(x, kr)\), where \(k \in \mathbb{N}_0\), \(\lambda > 0\), \(r > 0\), and \(x \in X\). The volume of a Borel set \(\Omega \subset X\) will be denoted by \(|\Omega| := \mu(\Omega)\).

The symbol \(1_E\) stands for the characteristic function of a Borel set \(E \subset X\), whereas the norm \(\|1_{E_1}T1_{E_2}\|_{L^p \to L^q}\) is defined via \(\sup_{\|f\|_{L^p} \leq 1} \|1_{E_1} \cdot T(1_{E_2} f)\|_{L^q}\) for a bounded linear operator \(T\) on \(L^2(X)\), Borel sets \(E_1, E_2 \subset X\), and \(1 \leq p \leq q \leq \infty\).
For $p \in [1, \infty]$ the conjugate exponent $p'$ is defined by $1/p + 1/p' = 1$ with the usual convention $1/\infty := 0$.
For $q \in (1, \infty)$ and $s \geq 0$, let $H^s_q$ denote the Bessel potential space on $\mathbb{R}$, whereas $H^\infty_s$ stands for the Hölder space $C^s$.

In the proofs, the letters $b, C$ denote generic positive constants that are independent of the relevant parameters involved in the estimates and may take different values at different occurrences. We will often use the notation $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$ for two non-negative expressions $a, b$; $a \cong b$ stands for the validity of $a \lesssim b$ and $b \lesssim a$.

2.1. Spaces of homogeneous type. We use the general framework of spaces of homogeneous type in the sense of Coifman and Weiss [17], i.e. $(X, d)$ is a non-empty metric space endowed with a $\sigma$-finite regular Borel measure $\mu$ with $\mu(X) > 0$ which satisfies the so-called doubling condition, that is, there exists a constant $C > 0$ such that for all $x \in X$ and all $r > 0$
\[ \mu(B(x, 2r)) \leq C \mu(B(x, r)). \] (2.1)

It is easy to see that the doubling condition (2.1) entails the strong homogeneity property, i.e. the existence of constants $C, D > 0$ such that for all $x \in X$, all $r > 0$, and all $\lambda \geq 1$
\[ \mu(B(x, \lambda r)) \leq C \lambda^D \mu(B(x, r)). \] (2.2)

In the sequel the value $D$ always refers to the constant in (2.2) which will be also called dimension of $(X, d, \mu)$. Of course, $D$ is not uniquely determined and for any $D' \geq D$ the inequality (2.2) is still valid. However, the smaller $D$ is, the stronger will be the multiplier theorems we are able to obtain. Therefore, we are interested in taking $D$ as small as possible.

There is a multitude of examples of spaces of homogeneous type. The simplest one is the Euclidean space $\mathbb{R}^D$, $D \in \mathbb{N}$, equipped with the Euclidean metric and the Lebesgue measure. Bounded open subsets of $\mathbb{R}^D$ with Lipschitz boundary endowed with the Euclidean metric and the Lebesgue measure are also spaces of homogeneous type.

We give a short review about well-known results concerning spaces of homogeneous type and start with a simple but useful observation which is a direct consequence of the doubling condition (2.1).

**Fact 2.1.** There exists a constant $C > 0$ such that for all $r > 0$, $x \in X$, and $y \in B(x, r)$
\[ C^{-1} |B(y, r)| \leq |B(x, r)| \leq C |B(y, r)|. \]
Consequently, it holds for any $r > 0$ and any $x \in X$
\[ C^{-1} \leq \int_{B(x, r)} \frac{1}{|B(y, r)|} d\mu(y) \leq C. \] (2.3)

An essential feature of spaces of homogeneous type is the validity of covering results which mean that, as in the Euclidean setting, one can cover a ball of radius $r$ by balls of radius $s$ and their number is bounded from above by a term only involving the ratio $r/s$ and the constants in (2.2) whenever $r \geq s > 0$.

**Lemma 2.2.** For each $r \geq s > 0$ and $y \in X$, there exist finitely many points $y_1, \ldots, y_K$ in $B(y, r)$ such that
\begin{enumerate}
  \item[(i)] $d(y_j, y_k) > s/2$ for all $j, k \in \{1, \ldots, K\}$ with $j \neq k$;
  \item[(ii)] $B(y, r) \subseteq \bigcup_{k=1}^K B(y_k, s)$;
  \item[(iii)] $K \lesssim (r/s)^D$;
  \item[(iv)] each $x \in B(y, r)$ is contained in at most $M$ balls $B(y_k, s)$, where $M$ depends only on the constants in (2.2) and is independent of $r, s, x, y$.
\end{enumerate}

The existence of $y_1, \ldots, y_K \in B(y, r)$ with the properties (i) and (ii) is well-known (cf. e.g. [6] Lemmas 6.1, 6.2 or [17] pp. 68 ff.). It can be easily shown that (iii) and (iv) are valid for such a family of points.
2.2. Off-diagonal estimates. We collect some properties of two-ball estimates in the next statement which are proved in [13, Proposition 2.1].

Fact 2.3. Let \(1 \leq p \leq q \leq \infty\), \(r > 0\), \(\omega > 1\), and \(g(\lambda) := Ce^{-b\lambda}\omega\) for some constants \(b, C > 0\). Suppose that \(T\) is a bounded linear operator on \(L^2(X)\). Then the following assertions are equivalent:

a) For all \(x, y \in X\), it holds
\[
\|1_{B(x,r)}T1_{B(y,r)}\|_{L^p_{x} \rightarrow L^q_y} \leq |B(x,r)|^{-\left(\frac{1}{p} - \frac{1}{q}\right)} g\left(\frac{d(x,y)}{r}\right).
\]

b) For all \(x, y \in X\) and all \(u, v \in [p, q]\) with \(u \leq v\), it holds
\[
\|1_{B(x,r)}T1_{B(y,r)}\|_{L^u_{x} \rightarrow L^v_y} \leq |B(x,r)|^{-\left(\frac{1}{p} - \frac{1}{q}\right)} g\left(\frac{d(x,y)}{r}\right).
\]

c) For all \(x \in X\) and all \(k \in \mathbb{N}\), it holds
\[
\|1_{B(x,r)}T1_{A(x,r,k)}\|_{L^p_{x} \rightarrow L^q_y} \leq |B(x,r)|^{-\left(\frac{1}{p} - \frac{1}{q}\right)} g(k).
\]

d) For all balls \(B_1, B_2 \subset X\) and all \(\alpha, \beta \geq 0\) with \(\alpha + \beta = \frac{1}{p} - \frac{1}{q}\), it holds
\[
\|1_{B_1}v_{\alpha}^\ast T v_{\beta}^\ast 1_{B_2}\|_{L^p_{x} \rightarrow L^q_y} \leq g\left(\frac{\text{dist}(B_1, B_2)}{r}\right),
\]
where \(\text{dist}(B_1, B_2) := \inf\{d(x,y) : x \in B_1, y \in B_2\}\) and \(v_r(x) := |B(x,r)|\) for \(x \in X\).

This statement is written modulo identification of \(g\) and \(\bar{g}\), where \(\bar{g}(\lambda) = ag(c\lambda)\) for some constants \(a, c > 0\) independent of \(r\), \(\omega\), \(T\).

Since the estimate stated in c) involves an annular set \(A(x,r,k)\), we call bounds of this kind estimates of annular type. A very useful feature of generalized Gaussian estimates is that they can be extended from real times \(t > 0\) to complex times \(z \in \mathbb{C}\) with \(\text{Re} z > 0\). The following result is taken from [10, Theorem 2.1] whose proof relies on the Phragmén-Lindelöf theorem.

Fact 2.4. Let \(m \geq 2\), \(1 \leq p \leq 2 \leq q \leq \infty\), and \(L\) be a non-negative, self-adjoint operator on \(L^2(X)\). Assume that there are constants \(b, C > 0\) such that for any \(t > 0\) and \(x, y \in X\)
\[
\|1_{B(x,t^l/m)}e^{-tL}1_{B(y,t^l/m)}\|_{L^p_{x} \rightarrow L^q_y} \leq C |B(x,t^{1/m})|^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \exp\left(-b\left(\frac{d(x,y)}{t^{1/m}}\right)^{m-1}\right).
\]

Then there exist constants \(b', C' > 0\) such that for all \(x, y \in X\) and all \(z \in \mathbb{C}\) with \(\text{Re} z > 0\)
\[
\|1_{B(x,r_z)}e^{-zL}1_{B(y,r_z)}\|_{L^p_{x} \rightarrow L^q_y} \leq C' |B(x,r_z)|^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \exp\left(-b'\left(\frac{d(x,y)}{r_z}\right)^{m-1}\right),
\]
where \(r_z := (\text{Re} z)^{1/m-1}|z|\) for each \(z \in \mathbb{C}\) with \(\text{Re} z > 0\).

Here the radius of the balls in the above two-ball estimate for \(e^{-zL}\) depends on the value of \(z\). The next lemma provides two-ball estimates with balls of arbitrary radius \(r > 0\) by the cost of an additional factor involving the ratio of \(r\) and \(r_z\) as well as the dimension of the underlying space of homogeneous type. Also a corresponding version for estimates of annular type is given. We postpone the proof to Section 6.

Lemma 2.5. Suppose that the assumptions of Fact 2.4 are fulfilled and, as before, define \(r_z := (\text{Re} z)^{1/m-1}|z|\) for each \(z \in \mathbb{C}\) with \(\text{Re} z > 0\).
a) There exist constants $b', C' > 0$ such that for all $r > 0$, $x, y \in X$, and $z \in \mathbb{C}$ with $\text{Re} z > 0$
\[
\| \mathbbm{1}_{B(x,r)} e^{-zL} \mathbbm{1}_{B(y,r)} \|_{L^p \to L^q} \leq C' |B(x,r)|^{-\frac{1}{p} - \frac{1}{q}} \left( 1 + \frac{r}{r_z} \right)^{\frac{d(\frac{1}{p} - \frac{1}{q})}{\frac{d}{2} - \frac{1}{q}}} \left( \frac{|z|}{\text{Re} z} \right)^{\frac{d(\frac{1}{p} - \frac{1}{q})}{\frac{d}{2} - \frac{1}{q}}} \exp \left( -b' \left( \frac{d(x,y)}{r_z} \right)^{\frac{m}{m-\frac{d}{2}}} \right).
\]

b) There exist constants $b''$, $C'' > 0$ such that for all $k \in \mathbb{N}$, $r > 0$, $x \in X$, and $z \in \mathbb{C}$ with $\text{Re} z > 0$
\[
\| \mathbbm{1}_{B(x,r)} e^{-zL} \mathbbm{1}_{A(x,r,k)} \|_{L^p \to L^q} \leq C'' |B(x,r)|^{-\frac{1}{p} - \frac{1}{q}} \left( 1 + \frac{r}{r_z} \right)^{\frac{d(\frac{1}{p} - \frac{1}{q})}{\frac{d}{2} - \frac{1}{q}}} \left( \frac{|z|}{\text{Re} z} \right)^{\frac{d(\frac{1}{p} - \frac{1}{q})}{\frac{d}{2} - \frac{1}{q}}} k^D \exp \left( -b'' \left( \frac{r}{r_z} k \right)^{\frac{m}{m-D}} \right).
\]

In Section 3 we consider specific Hardy spaces associated with an operator $L$. For defining and working with these spaces it is enough to require a special form of two-ball estimates on $L^2(X)$ for the semigroup $(e^{-tL})_{t>0}$ generated by $-L$, so-called Davies-Gaffney estimates.

**Definition 2.6.** Let $m \geq 2$. We say that a family $\{S_t : t > 0\}$ of bounded linear operators acting on $L^2(X)$ satisfies Davies-Gaffney estimates of order $m$ if there exist constants $b,C > 0$ such that for all $t > 0$ and all $x,y \in X$,
\[
\| \mathbbm{1}_{B(x,t^{1/m})} S_t \mathbbm{1}_{B(y,t^{1/m})} \|_{L^2 \to L^2} \leq C \exp \left( -b \left( \frac{d(x,y)}{t^{1/m}} \right)^{\frac{m}{m-\frac{d}{2}}} \right).
\]

In order to indicate the validity of Davies-Gaffney estimates of order $m$, we later use the abbreviation $DG_m$. If $\{S_t : t > 0\} = (e^{-tL})_{t>0}$ is a semigroup on $L^2(X)$ generated by $-L$, we shall also say that $L$ satisfies Davies-Gaffney estimates when the semigroup $(e^{-tL})_{t>0}$ enjoys this property.

Estimates of the type (2.4) were first introduced by E.B. Davies [21] inspired by ideas of M.P. Gaffney [32]. They hold naturally for many operators, including large classes of self-adjoint, elliptic differential operators or Schrödinger operators with real-valued potentials (cf. e.g. [18]).

Davies-Gaffney estimates were extensively studied in the recent series of papers [4], [5], [6], [7] by P. Auscher and J.M. Martell (see also [18], [25], [34]). We mention that in the literature one usually finds a slightly different definition of Davies-Gaffney estimates in which the validity of (2.4) is required for all open subsets of $X$. It is known that the definitions coincide for $m = 2$ (cf. [18], Lemma 3.1).

Finally, we quote a statement originally given in [34] Proposition 3.1 for operators satisfying $DG_2$. However, with some minor modifications the proof can be adapted to include Davies-Gaffney estimates of arbitrary order $m \geq 2$ as well. For a detailed proof we refer to Section 6

**Lemma 2.7.** Let $m \geq 2$ and $L$ be a non-negative, self-adjoint operator on $L^2(X)$. If $L$ fulfills Davies-Gaffney estimates $DG_m$, then for each $K \in \mathbb{N}$ the family of operators
\[
\{(tL)^K e^{-tL} : t > 0\}
\]

satisfies also Davies-Gaffney estimates $DG_m$ with constants depending only on $K$ and the constants in the doubling condition (2.2) and the Davies-Gaffney condition (2.4) for the semigroup $(e^{-tL})_{t>0}$.

3. HARDY SPACES ASSOCIATED WITH OPERATORS

Quite recently, a theory of Hardy spaces associated with certain operators was introduced, similar to the way that classical Hardy spaces are adapted to the Laplacian. We refer to [25] for a survey on the recent development and only mention that their origin lies in the paper [3] by P. Auscher, X.T. Duong, and A. McIntosh, who defined the Hardy space $H^1_1(\mathbb{R}^D)$ associated with an operator $L$ which has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^D)$ and for which the semigroup
operators have a pointwise Poisson upper bound. Afterwards, the assumptions on the associated operator were relaxed. S. Hofmann and S. Mayboroda ([36]) defined Hardy spaces associated with second order divergence form elliptic operators on \( \mathbb{R}^D \) with complex coefficients. S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea, and L.X. Yan ([34]) developed a theory of Hardy spaces adapted to non-negative, self-adjoint operators \( L \) on \( L^2(X) \) which satisfy Davies-Gaffney estimates in the setting of spaces of homogeneous type. X.T. Duong and J. Li ([25]) considered even non-self-adjoint operators and introduced Hardy spaces associated with operators which have a bounded holomorphic functional calculus on \( L^2(X) \) and generate an analytic semigroup on \( L^2(X) \) satisfying Davies-Gaffney estimates of order 2.

Throughout this section, let \( L \) be an injective, non-negative, self-adjoint operator on \( L^2(X) \) which satisfies Davies-Gaffney estimates \( \text{DG}_m \) for some \( m \geq 2 \). We summarize the most important facts about Hardy spaces associated with \( L \). For more details and proofs of the statements, we refer to [36, 37, 25, 31, 8, 14, and 24]. The proofs given there carry over with only minor changes to our more general setting.

**Definition 3.1.** Let \( p \in [1, 2] \). Put \( \psi_0(z) := ze^{-z} \), \( z \in \mathbb{C} \), and consider the conical square function

\[
S_f(x) := \left( \int_0^\infty \int_{B(x,t)} |\psi_0(t^m L) f(y)|^2 \frac{d\mu(y)}{|B(x,t)|} \frac{dt}{t} \right)^{1/2} \quad (f \in L^2(X), x \in X).
\]

The Hardy space \( H^p_L(X) \) associated with \( L \) is defined to be the completion of

\[
\{ f \in L^2(X) : Sf \in L^p(X) \}
\]

with respect to the norm

\[
\| f \|_{H^p_L} := \| Sf \|_{L^p}.
\]

The definition of Hardy spaces associated with operators is also possible for \( p \in (0, 1) \) or \( p \in (2, \infty) \). In addition, other functions than \( \psi_0 \) can be considered. More information on this can be found in the aforementioned literature.

By using Fubini’s theorem and the spectral theorem it can be verified that \( H^2_L(X) = L^2(X) \) with equivalent norms. Additionally, the set \( H^1_L(X) \cap L^2(X) \) is dense in \( H^1_L(X) \). Note that in the special case of \( X = \mathbb{R}^D \) and \( L = -\Delta \) this definition yields the Hardy space \( H^p(\mathbb{R}^D) \) as introduced by E.M. Stein and G. Weiss ([10]). Similar to classical Hardy spaces, Hardy spaces associated with operators form a complex interpolation scale. This can be verified by viewing these spaces via the framework of tent spaces and by using the interpolation properties of tent spaces (cf. [37, Lemma 4.20]).

**Fact 3.2.** Suppose that \( 1 \leq p_0 < p < p_1 \leq 2 \) with \( 1/p = (1 - \theta)/p_0 + \theta/p_1 \) for some \( \theta \in (0, 1) \). Then it holds

\[
[H^p_{L_0}(X), H^p_{L_1}(X)]_\theta = H^p_L(X).
\]

It is well-known that the classical Hardy space \( H^1(\mathbb{R}^D) \) possesses an atomic decomposition. This property carries over to Hardy spaces associated with injective, non-negative, self-adjoint operators \( L \) satisfying Davies-Gaffney estimates of order 2 (cf. [34, Theorem 4.1]). Besides the atomic decomposition of tent spaces, the proof in [34] relies heavily on the equivalence between the Davies-Gaffney estimates \( \text{DG}_2 \) for \( L \) and the finite speed propagation property for the corresponding wave equation \( Lu + u_{tt} = 0 \) (cf. e.g. [13, Theorem 3.4]). Unfortunately, it is not possible to deduce a result similar to the finite speed propagation property for operators \( L \) that fulfill \( \text{DG}_m \) for some \( m > 2 \) and thus it seems not to be clear whether an atomic decomposition of \( H^1_L(X) \) for these operators \( L \) is possible. Nevertheless, in the general situation one can decompose the Hardy space \( H^1_L(X) \) by considering molecules instead of atoms.

**Definition 3.3.** Let \( M \in \mathbb{N} \) and \( \varepsilon > 0 \). A function \( a \in L^2(X) \) is said to be an \((M, \varepsilon, L)\)-molecule if there exist a function \( b \in \mathcal{D}(L^M) \) and a ball \( B \subset X \) with radius \( r \) such that
(i) \( a = L^M b \);
(ii) for every \( k \in \{0, 1, \ldots, M\} \) and \( j \in \mathbb{N}_0 \), it holds

\[ \left\| (r^mL)^k b \right\|_{L^2(U_j(B))} \leq r^{mM}2^{-j\varepsilon} \mu(2^j B)^{-1/2}, \]

(3.1)

where the dyadic annuli \( U_j(B) \) are defined by

\[ U_0(B) := B \quad \text{and} \quad U_j(B) := 2^j B \setminus 2^{j-1} B \quad \text{for all} \quad j \in \mathbb{N}. \]

(3.2)

In this situation we sometimes refer to \( a \) as being an \((M, \varepsilon, L)\)-molecule associated with \( B \).

In the literature (cf. e.g. [34], [25], [37]) authors mostly study the case when \( m = 2 \) and typically use the terminology “\((1, 2, M, \varepsilon)\)-molecule associated with \( L^2 \)” instead of \((M, \varepsilon, L)\)-molecule. Next, we give the definition of the molecular Hardy spaces associated with \( L \) (cf. e.g. [31], [14]).

**Definition 3.4.** Fix \( M \in \mathbb{N} \) and \( \varepsilon > 0 \). Let \( f \in L^1(X) \). We call \( f = \sum_{j=0}^{\infty} \lambda_j m_j \) a **molecular \((M, \varepsilon, L)\)-representation** of \( f \) if \((\lambda_j)_{j \in \mathbb{N}_0} \in \ell^1 \) is a numerical sequence, \( m_j \) is an \((M, \varepsilon, L)\)-molecule for any \( j \in \mathbb{N}_0 \), and the sum \( \sum_{j=0}^{\infty} \lambda_j m_j \) converges in \( L^2(X) \). Define

\[
H^1_{L,\text{mol},M,\varepsilon}(X) := \{ f \in L^1(X) : f \text{ has a molecular } (M, \varepsilon, L)\text{-representation} \}
\]

with the norm given by

\[
\| f \|_{H^1_{L,\text{mol},M,\varepsilon}} := \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j| : \sum_{j=0}^{\infty} \lambda_j m_j \text{ is a molecular } (M, \varepsilon, L)\text{-representation of } f \right\}.
\]

The **molecular Hardy space** \( H^1_{L,\text{mol},M,\varepsilon}(X) \) **associated with** \( L \) is said to be the completion of \( H^1_{L,\text{mol},M,\varepsilon}(X) \) with respect to the norm \( \| \cdot \|_{H^1_{L,\text{mol},M,\varepsilon}} \).

As a direct consequence of the definition, we note that \( H^1_{L,\text{mol},M_2,\varepsilon}(X) \subset H^1_{L,\text{mol},M_1,\varepsilon}(X) \) for \( \varepsilon > 0 \) and \( M_1, M_2 \in \mathbb{N} \) with \( M_1 \leq M_2 \). In addition, the Hardy space \( H^1_{L,\text{mol},M,\varepsilon}(X) \) is contained in \( L^1(X) \) because the \( L^1(X) \)-norm of \((M, \varepsilon, L)\)-molecules is uniformly bounded by a constant depending only on \( \varepsilon \) and the constants in the doubling condition.

One can show the following characterization. For a proof, we refer to [25 Theorem 3.12] (see also [14] for \( X = \mathbb{R}^D \)).

**Theorem 3.5.** Assume that \( M \in \mathbb{N} \) with \( M > \frac{D}{2m} \) and \( \varepsilon \in (0, mM - D/2] \). Then

\[
H^1_{L,\text{mol},M,\varepsilon}(X) = H^1_L(X)
\]

with equivalent norms

\[
\| f \|_{H^1_{L,\text{mol},M,\varepsilon}} \approx \| f \|_{H^1_L}.
\]

where implicit constants depend only on \( \varepsilon, M \) or the constants in the Davies-Gaffney and the doubling condition.

In particular, every function \( f \in H^1_L(X) \cap L^2(X) \) admits a molecular \((M, \varepsilon, L)\)-representation.

A detailed examination of the proof due to X.T. Duong and J. Li shows the following

**Corollary 3.6.** Let \( \varepsilon > 0 \) and \( M \in \mathbb{N} \) with \( M > \frac{D}{2m} \). Then every \((M, \varepsilon, L)\)-molecule \( a \) belongs to \( H^1_L(X) \) and there is a constant \( C > 0 \) depending only on \( \varepsilon, M \) and the constants in the Davies-Gaffney (2.4) and the doubling condition (2.2) such that for all \((M, \varepsilon, L)\)-molecules \( a \):

\[
\| a \|_{H^1_L} \leq C.
\]
Thanks to $H^1_L(X) \subset L^1(X)$ and $H^2_L(X) = L^2(X)$, Fact 3.2 yields that $H^p_L(X) \subset L^p(X)$ for each $p \in (1,2)$. The question under which assumptions on $L$ the reverse inclusion holds is settled for the classical Hardy spaces $H^p(\mathbb{R}^D)$. It is well-known that they can be identified with the Lebesgue spaces $L^p(\mathbb{R}^D)$ for any $p \in (1,\infty)$ (see e.g. [15, p. 220]). However, if $L$ is an injective, non-negative, self-adjoint operator on $L^2(\mathbb{R}^D)$ which satisfies Davies-Gaffney estimates $D_{m,n}$ for some $m \geq 2$ and $p \in (1,2)$, then $H^p_L(\mathbb{R}^D)$ may or may not coincide with $L^p(\mathbb{R}^D)$ (see e.g. [37, Proposition 9.1 (v), (vi)], where Riesz transforms were studied).

P. Auscher, X.T. Duong, and A. McIntosh showed in [3, Theorem 6] that pointwise Gaussian estimates (1.3) imply $H^p_L(\mathbb{R}^D) = L^p(\mathbb{R}^D)$ for all $p \in (1,2]$. By reasoning similar to P. Auscher in [2, Proposition 6.8], who sketched a proof in the case of second order divergence form operators on $\mathbb{R}^D$, one can show a corresponding result for operators satisfying only generalized Gaussian estimates. In the case $m = 2$ this is already stated in [37, Proposition 9.1 (v)], with a reference to [2] for the proof.

**Theorem 3.7.** Assume that $L$ is an injective, non-negative, self-adjoint operator on $L^2(X)$ which fulfills generalized Gaussian estimates $GGE_{m,p_0,p'_0}$ for some $p_0 \in [1,2)$ and $m \geq 2$. Then, for each $p \in (p_0,2]$, the Hardy space $H^p_L(X)$ and the Lebesgue space $L^p(X)$ coincide and their norms are equivalent.

By density it is enough to establish the estimate $\|Sf\|_{L^p} \equiv \|f\|_{L^p}$ for every $f \in L^p(X) \cap L^2(X)$. This is divided into three steps. In a first step, which is the main work, one verifies that $\|Sf\|_{L^p} \lesssim \|f\|_{L^p}$ for all $f \in L^p(X) \cap L^2(X)$ and $p \in (p_0,2]$. In a second step it is shown that this estimate is actually valid for any $p \in (2,p_0')$. In the final step three one can deduce the reverse inequality $\|f\|_{L^p} \lesssim \|S\psi f\|_{L^p}$ for all $f \in L^p(X) \cap L^2(X)$ and $p \in (p_0,2]$ by a dualization argument based on the bound obtained in the second step.

The idea of the proof of the first step consists in establishing a weak type $(p_0,p_0)$-estimate for the square function $S$. As technical difficulties arise, which are caused by the definition of $S$ via an area integral, one studies the properties of what may be called Littlewood-Paley-Stein $g^*_\lambda$-function adapted to $L$

$$g^*_\lambda(f)(x) := \left( \int_0^\infty \int_X \left( \frac{s^{1/m}}{d(x,y) + s^{1/m}} \right)^D \left| \Lambda_L e^{-sL} f(y) \right|^2 \frac{d\mu(y)}{|B(x,s^{1/m})|} \right)^{1/2}$$

for $\lambda > 0$, $x \in X$, and $f \in L^2(X)$. It turns out that $g^*_\lambda$ is better suited than $S$ as far as Fubini arguments are concerned because it contains an integral over the full space. Since $g^*_\lambda$ controls $S$ for any $\lambda > 1$, it suffices to verify a weak type $(p_0,p_0)$-estimate for $g_\lambda$. A detailed proof can be found in [49, Section 4.4].

### 4. Spectral multipliers on the Hardy space $H^1_L(X)$

In this section, we formulate and prove Hörmander type spectral multiplier results on $H^1_L(X)$, where $L$ is an injective, non-negative, self-adjoint operator on $L^2(X)$ which satisfies Davies-Gaffney estimates of arbitrary order $m \geq 2$. We will state three versions, namely a more classical one, presented in Theorem 4.1, and two including a Plancherel condition which leads to weakened regularity assumptions on the involved function, given in Theorem 4.2 and Theorem 4.3.

**Theorem 4.1.** Let $L$ be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates of order $m \geq 2$.

1. If $s > (D+1)/2$ and $F: [0,\infty) \to \mathbb{C}$ is a bounded Borel function with

$$\sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^2_\psi} < \infty,$$

(4.1)
then $F(L)$ can be extended from $H^1_L(X) \cap L^2(X)$ to a bounded linear operator on $H^1_L(X)$. More precisely, there exists a constant $C > 0$ such that

$$\|F(L)\|_{H^1_L(X) \to H^1_L(X)} \leq C \left( \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^2} + |F(0)| \right).$$

b) If $s > D/2$ and $F : [0, \infty) \to \mathbb{C}$ is a bounded Borel function with

$$\sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{C^s} < \infty,$$

then $F(L)$ extends to a bounded linear operator on the Hardy space $H^1_L(X)$. To be more precise, there is a constant $C > 0$ such that

$$\|F(L)\|_{H^1_L(X) \to H^1_L(X)} \leq C \left( \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{C^s} + |F(0)| \right).$$

In the special case $m = 2$ the statement b) corresponds to [28, Theorem 1.1].

Theorem 4.2. Let $L$ be an injective, non-negative, self-adjoint operator on $L^2(X)$ for which Davies-Gaffney estimates of order $m \geq 2$ hold. Suppose that there exist $C > 0$ and $q \in [2, \infty]$ such that for any $R > 0$, $y \in X$, and any bounded Borel function $F : [0, \infty) \to \mathbb{C}$ with supp $F \subseteq [0, R]$

$$\|F(\sqrt{L}) \mathbb{1}_{B(y,1/R)}\|_{L^2 \to L^2} \leq C \|F(R\cdot)\|_{L^q}.$$ \hspace{1cm} (4.3)

If $s > \max\{D/2, 1/q\}$ and $F : [0, \infty) \to \mathbb{C}$ is a bounded Borel function with

$$\sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q} < \infty,$$

then there exists a constant $C > 0$ such that for all $f \in H^1_L(X)$

$$\|F(L)f\|_{H^1_L} \leq C \left( \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q} + |F(0)| \right) \|f\|_{H^1_L}.$$

As already mentioned in the introduction, in general the assertion of Theorem 4.2 is false without the Plancherel condition (4.3). But (4.3) always holds for $q = \infty$, as Lemma 4.5 shows (cp. [27, Lemma 2.2] for a similar result). Consequently, Theorem 4.1 (b) follows from Theorem 4.2.

On the one hand, (4.3) ensures that the class of functions for which the multiplier result applies is extended. However, on the other hand, the validity of (4.3) for some $q \in [2, \infty)$ entails the emptiness of the point spectrum of $L$. Indeed, according to the Plancherel condition (4.3), one has for all $0 \leq a \leq R$ and $y \in X$

$$\|\mathbb{1}_{\{a\}}(\sqrt{L}) \mathbb{1}_{B(y,1/R)}\|_{L^p \to L^2} \leq |B(y,1/R)|^{-\left(\frac{1}{q_0} - \frac{1}{q} \right)} \|\mathbb{1}_{\{a\}}(R\cdot)\|_{L^q} = 0$$

and therefore $\mathbb{1}_{\{a\}}(\sqrt{L}) = 0$. Due to $\sigma(L) \subseteq [0, \infty)$, it follows that the point spectrum of $L$ is empty. In order to treat operators with non-empty point spectrum as well, one may introduce some variation of the Plancherel condition (4.3). This approach originates in [20] and was also used in [27] or [15]. For $N \in \mathbb{N}$, $q \in [1, \infty)$, and a bounded Borel function $F : \mathbb{R} \to \mathbb{C}$ with supp $F \subseteq [-1, 2]$ define the norm $\|F\|_{N,q}$ via the formula

$$\|F\|_{N,q} := \left( \frac{1}{N} \sum_{k=-N}^{2N} \sup_{\lambda \in [\frac{k}{N}, \frac{k+1}{N})} |F(\lambda)|^q \right)^{1/q}.$$

Theorem 4.3. Let $L$ be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying DG$_m$ for some $m \geq 2$. Fix $\kappa \in \mathbb{N}$ and $q \in [2, \infty]$. Suppose that there is $C > 0$ such that for any $N \in \mathbb{N}$, $y \in X$, and any bounded Borel function $F : \mathbb{R} \to \mathbb{C}$ with supp $F \subseteq [-1, N+1]$

$$\|F(\sqrt{L}) \mathbb{1}_{B(y,1/N)}\|_{L^2 \to L^2} \leq C \|F(N\cdot)\|_{N,\kappa,q}.$$
In addition, assume that for every \(\varepsilon > 0\) there is \(C > 0\) such that for all \(N \in \mathbb{N}\) and all bounded Borel functions \(F: \mathbb{R} \to \mathbb{C}\) with \(\text{supp} F \subseteq [-1, N + 1]\)

\[
\|F(\sqrt[n]{\xi})\|_{L^2(\xi \in \{0\})}^2 \leq C N^k d(\xi, \mu) \leq CN^{k+\varepsilon} \|F(N\cdot)\|_{N^k-q}^2. \tag{4.4}
\]

Let \(s > \max\{D/2, 1/q\}\). Then, for any bounded Borel function \(F: \mathbb{R} \to \mathbb{C}\) with

\[
\sup_{n \in \mathbb{N}} \|\omega F(2^n \cdot)\|_{H^s} < \infty,
\]

there exists a constant \(C > 0\) such that for all \(f \in H^s_0(X)\)

\[
\|F(L)f\|_{H^s} \leq C \left( \sup_{n \in \mathbb{N}} \|\omega F(2^n \cdot)\|_{H^s} + \|F\|_{L^\infty} \right) \|f\|_{H^s}.
\tag{4.5}
\]

The proof of Theorem 4.3 is essentially based on the approach in [27, Theorem 3.2]. We omit the details here, and only mention that one can use the following \(L^2\)-version of [27, Lemma 4.3 b)] which can be proven in a similar way as Lemma 4.10 below.

\[\text{Lemma 4.4. Let } L, \kappa, q \text{ be as in Theorem 4.3. For } \xi \in C^\infty_c([-1, 1]) \text{ and } N \in \mathbb{N} \text{ define the function } \xi_N \text{ via the formula } \xi_N(\lambda) := N(\xi(N\lambda)). \text{ Then for any } s \geq 2/q, \varepsilon > 0, \text{ and any } \xi \in C^\infty([-1, 1]) \text{ there exists a constant } C > 0 \text{ such that}
\]

\[
\left\| (F \ast \xi_{N^{-1}})(\sqrt[n]{\xi}) \mathbb{1}_{B(1, \sqrt[n]{X})} \right\|_{L^2(\xi \in \{0\})} \leq C \|F(N\cdot)\|_{H^s}^{1/2 + \varepsilon} \tag{4.6}
\]

for all \(N \in \mathbb{N}\) with \(N > 8\), all \(y \in X\), and all bounded Borel functions \(F: \mathbb{R} \to \mathbb{C}\) with \(\text{supp} F \subseteq [N/4, N]\) and \(F(N\cdot) \in H^s_q\).

The rest of this section is devoted to the proofs of Theorems 4.1 a) and 4.2. We start with the aforementioned statement concerning the validity of the Plancherel condition 4.3 for \(q = \infty\).

\[\text{Lemma 4.5. Let } L \text{ be a non-negative, self-adjoint operator on } L^2(X) \text{ which satisfies } DG_m \text{ for some } m \geq 2. \text{ Then there exists a constant } C > 0 \text{ such that for all } R > 0, y \in X, \text{ and bounded Borel functions } F: [0, \infty) \to \mathbb{C} \text{ with } \text{supp} F \subseteq [0, R]^2
\]

\[
\|F(\sqrt[n]{\xi}) \mathbb{1}_{B(y, 1/R)}\|_{L^2 \to L^2} \leq C \|F\|_{L^\infty}.
\]

\[\text{Proof. Let } R > 0, y \in X \text{ and } F: [0, \infty) \to \mathbb{C} \text{ be a bounded Borel function whose support is contained in } [0, R]. \text{ For any } \lambda \geq 0 \text{ define } G_1(\lambda) := F(\sqrt[n]{\xi}) e^{\lambda/R^m} \text{ and } G_2(\lambda) := e^{-\lambda/R^m}, \text{ so that we can write } F(\sqrt[n]{\xi}) = G_1(L)G_2(L). \text{ Observe that } \text{supp} G_1 \subseteq [0, R^m] \text{ and thus } \|G_1(L)\|_{L^2 \to L^2} \leq \|G_1\|_{L^\infty} \leq e \|F\|_{L^\infty}. \text{ As } L \text{ fulfills } DG_m, \text{ we deduce with the help of Fact 2.3 that}
\]

\[
\|G_2(L) \mathbb{1}_{B(y, 1/R)}\|_{L^2 \to L^2} \leq \sum_{k=0}^{\infty} \|\mathbb{1}_{A(y, 1, R/K)} e^{-\lambda/R^m} L \mathbb{1}_{B(y, 1/R)}\|_{L^2 \to L^2} \lesssim \sum_{k=0}^{\infty} e^{-b k^m R^m} \lesssim 1.
\]

Combining these estimates gives the desired bound

\[
\|F(\sqrt[n]{\xi}) \mathbb{1}_{B(y, 1/R)}\|_{L^2 \to L^2} \leq \|G_1(L)\|_{L^2 \to L^2} \|G_2(L) \mathbb{1}_{B(y, 1/R)}\|_{L^2 \to L^2} \lesssim \|F\|_{L^\infty}.
\]

\[\square\]

Now, we provide a criterion for the boundedness of spectral multipliers on the Hardy space \(H^1_0(X)\). Our result, presented in Theorem 4.6 below, generalizes the statement [28, Theorem 3.1] due to X.T. Duong and L.X. Yan which merely works under Davies-Gaffney estimates of order \(m = 2\). Afterwards we check that the assumption (4.7) holds whenever the involved function \(F\) satisfies the assumptions of one of the above theorems.
Theorem 4.6. Let $L$ be an injective, non-negative, self-adjoint operator on $L^2(X)$ which satisfies Davies-Gaffney estimates $\text{DG}_m$ for some $m \geq 2$. Further, let $F : [0,\infty) \to \mathbb{C}$ be a bounded Borel function. Assume that there exist an integer $M > D/m$ and constants $C_F > 0$, $\delta > D/2$ such that

$$\left\| 1_{U_j(B)} F(L)(I - e^{-r^m L})^M 1_B \right\|_{L^2 \to L^2} \leq C_F 2^{-j\delta}$$

(4.7)

for every $j \in \mathbb{N} \setminus \{1\}$ and every ball $B \subset X$ with radius $r$. As usual, $U_j(B)$ stands for the dyadic annular set as defined in (3.2). Then the operator $F(L)$ extends from $H^1_L(X) \cap L^2(X)$ to a bounded linear operator on $H^1_L(X)$. More precisely, there exists a constant $C > 0$ such that

$$\left\| F(L) \right\|_{H^1_L(X) \to H^1_L(X)} \leq CC_F.$$  

The strategy of proof consists in reducing the statement to uniform boundedness of the $H^1_L(X)$-norm of $F(L)a$ for every $(2M, \tilde{\varepsilon}, L)$-molecule $a$. Recall that $a$ can be rewritten as $a = L^{2Mb}$ for some $b \in \mathcal{D}(L^{2M})$. By the lack of information on the support of $L^{k}b$ for $k \in \{0, 1, \ldots, 2M\}$, we cannot apply (4.7) directly. Instead we shall choose $\tilde{\varepsilon}$ large enough and use an estimate of annular type furnished by the next lemma whose proof is postponed to Section 6.

Lemma 4.7. Suppose that the operator $L$ and the function $F$ have the same properties as in Theorem 4.6. Then there exists a constant $C > 0$ such that

$$\left\| 1_{U_j(B)} F(L)(I - e^{-r^m L})^M 1_{U_i(B)} \right\|_{L^2 \to L^2} \leq CC_F 2\delta 2^{-j-i\delta}$$

(4.8)

for every $i, j \in \mathbb{N} \setminus \{1\}$ and every ball $B \subset X$ with radius $r$.

Next, we provide the technical result that an integrated version of the regularization operator $(I - e^{-r^m L})^M$ satisfies $L^2(X)$-norm estimates of annular type if $L$ fulfills $\text{DG}_m$. This will be achieved with a similar reasoning as in the proof of the preceding statement (cf. Section 6).

Lemma 4.8. Let $K \in \mathbb{N}$ and $L$ be an injective, non-negative, self-adjoint operator on $L^2(X)$ which fulfills Davies-Gaffney estimates $\text{DG}_m$, for some $m \geq 2$. For $M \in \mathbb{N}$ and $r > 0$ define the operator

$$P_{m,M,r}(L) := r^{-m} \int_r^{\sqrt{2}r} s^{m-1}(I - e^{-s^m L})^M ds.$$  

(4.9)

Then there exist $b, C > 0$ such that for any $i, j \in \mathbb{N}_0$ and arbitrary balls $B \subset X$ of radius $r$

$$\left\| 1_{U_j(B)} P_{m,M,r}(L)^K 1_{U_i(B)} \right\|_{L^2 \to L^2} \leq C \exp\left(-b 2^{j-i}\right).$$

(4.10)

Here, the constants $b, C$ depend exclusively on $m, K, M$ and the constants appearing in the Davies-Gaffney and doubling condition.

With the preceding lemmas at hand, we are prepared for the proof of Theorem 4.6. Here, we rely to a large extent on the proof of [28 Theorem 3.1].

Proof of Theorem 4.6. Let $F : [0,\infty) \to \mathbb{C}$ be a bounded Borel function such that (4.7) holds for some constants $C_F > 0$, $\delta > D/2$, and $M \in \mathbb{N}$ with $M > D/m$.

First of all, we note that the operator $F(L)$ can be defined via (1.1) on the set $H^1_L(X) \cap L^2(X)$ which is dense in $H^1_L(X)$ (cf. Definition 3.1).

Let $\bar{\delta} \in (D/2, \min\{\delta, mM - D/2\})$ be fixed. Define $\varepsilon := \bar{\delta} - D/2 > 0$ and $\tilde{\varepsilon} := D + \bar{\delta}$. We claim that, for every $(2M, \tilde{\varepsilon}, L)$-molecule $a$, $F(L)a$ is, up to multiplication by a constant independent of $a$, an $(M, \varepsilon, L)$-molecule. The conclusion of Theorem 4.6 is then an immediate consequence of Corollary 3.6. Indeed, by Theorem 3.5 every $f \in H^1_L(X) \cap L^2(X)$ admits a molecular $(2M, \tilde{\varepsilon}, L)$-representation, i.e. there exist a scalar sequence $(\lambda_j)_{j \in \mathbb{N}_0} \in \ell^1$ and a sequence $(m_j)_{j \in \mathbb{N}_0}$ of $(2M, \tilde{\varepsilon}, L)$-molecules such that

$$f = \sum_{j=0}^{\infty} \lambda_j m_j.$$
in $L^2(X)$ and

$$
\|f\|_{H^1_L} \cong \sum_{j=0}^{\infty} |\lambda_j|,
$$

with implicit constants independent of $f$. Therefore, we have

$$
\|F(L)f\|_{H^1_L} \leq \sum_{j=0}^{\infty} |\lambda_j| \|F(L)m_j\|_{H^1_L}.
$$

But by the claim above, $F(L)m_j$ is a constant multiple of an $(M, \varepsilon, L)$-molecule. Hence, we conclude from Corollary 3.6 that the $H^1_L(X)$-norm of $F(L)m_j$ is bounded by a constant $C > 0$ being independent of $j$. Thus, once the above claim is proved, the boundedness of $F(L)$ on $H^1_L(X)$ is shown since

$$
\|F(L)f\|_{H^1_L} \leq \sum_{j=0}^{\infty} |\lambda_j| \|F(L)m_j\|_{H^1_L} \leq C \sum_{j=0}^{\infty} |\lambda_j| \cong \|f\|_{H^1_L}
$$

for any $f \in H^1_L(X) \cap L^2(X)$ and $H^1_L(X) \cap L^2(X)$ is dense in the Hardy space $H^1_L(X)$.

Now we proceed with the proof of the claim stated above. Let $a$ be an $(2M, \varepsilon, L)$-molecule. According to Definition 3.3 we find a function $b \in D(L^{2M})$ and a ball $B \subset X$ such that $a = L^{2M}b$ and (3.1) hold. By the spectral theorem for $L$, we may write

$$
F(L)a = L^M \left( F(L) L^M b \right).
$$

In particular, $F(L) L^M b$ belongs to $D(L^M)$. For the proof that $F(L)a$ is a constant multiple of an $(M, \varepsilon, L)$-molecule it remains to check (ii) from Definition 3.3, i.e. the existence of a constant $C > 0$ such that for all $j \in \mathbb{N}_0$ and all $k \in \{0, 1, \ldots, M\}$

$$
\left\| (r^m L^k \left( F(L) L^M b \right) \right\|_{L^2(U_j(B))} \leq CC_F r^m 2^{-j \varepsilon} \mu(2^j B)^{-1/2},
$$

where $r$ denotes the radius of the ball $B$.

For $j \in \{0, 1, 2\}$, we employ the boundedness of $F(L)$ on $L^2(X)$ as well as the properties of the $(2M, \varepsilon, L)$-molecule $a$. For any $k \in \{0, 1, \ldots, M\}$, this leads to

$$
\left\| (r^m L^k \left( F(L) L^M b \right) \right\|_{L^2(U_j(B))} \leq r^{mk} \|F(L)\|_{L^2 \to L^2} r^{-m(M+k)} \| (r^m L) L^{M+k} b \|_{L^2}
\leq \|F\|_{L^\infty} r^{-mM} \sum_{i=0}^{\infty} \| (r^m L)^{i} b \|_{L^2(U_j(B))}
\leq \|F\|_{L^\infty} r^{-mM} \sum_{i=0}^{\infty} r^{-2mM} 2^{-i \varepsilon} \mu(2^i B)^{-1/2}
\lesssim \|F\|_{L^\infty} r^{-mM} \mu(B)^{-1/2}
\lesssim \left( \|F\|_{L^\infty} 2^{D+2\varepsilon} \right) r^{mM} 2^{-j \varepsilon} \mu(2^j B)^{-1/2}.
$$

Now assume that $j \geq 3$. We start by representing the identity on $L^2(X)$ with the help of the operators $e^{-r_{||r-mL}}$ and $P_{m,M,r}(L)$, where the latter has been defined in (4.9). Applying this to $(r^m L)^k \left( F(L) L^M b \right)$, the procedure produces a regularizing effect for the operator $F(L)$ and finally permits us to insert the assumption (4.17) in the version of Lemma 4.7 and the Davies-Gaffney estimates in the form of Lemma 4.8. Inspired by [36] (8.7), (8.8), we use the elementary equations

$$
1 = mr^{-m} \int_{r}^{r/2r} s^{m-1} ds
$$
and
\[ 1 = (1 - e^{-s^m \lambda})^M - \sum_{\nu=1}^{M} \binom{M}{\nu} (-1)^\nu e^{-\nu s^m \lambda} \quad (\lambda \geq 0, s > 0) \]
to deduce, via the spectral theorem for \( L \),
\[ I = m r^{-m} \int_{r}^{\sqrt{2} r} s^{m-1} (I - e^{-s^m L})^M ds + \sum_{\nu=1}^{M} \nu C_{\nu, M} m r^{-m} \int_{r}^{\sqrt{2} r} s^{m-1} e^{-\nu s^m L} ds, \quad (4.13) \]
where \( C_{\nu, M} := \frac{(-1)^{\nu+1}}{\nu} \binom{M}{\nu}. \) Further, we have \( \partial_s e^{-\nu s^m L} = -\nu m s^{m-1} L e^{-\nu s^m L} \) and therefore
\[ \nu m L \int_{r}^{\sqrt{2} r} s^{m-1} e^{-\nu s^m L} ds = e^{-\nu r^m L} - e^{-2\nu r^m L} = e^{-\nu r^m L}(I - e^{-\nu r^m L}) \]
\[ = e^{-\nu r^m L}(I - e^{-r^m L}) \sum_{\eta=0}^{\nu-1} e^{-\eta r^m L}. \quad (4.14) \]
By recalling the definition of \( P_{m, M, r}(L) \) and by inserting the equation (4.14) into (4.13), we end up with the following formula for the identity on \( L^2(X) \)
\[ I = m P_{m, M, r}(L) + \sum_{\nu=1}^{M} C_{\nu, M} r^{-m} L^{-1}(I - e^{-r^m L}) \sum_{\eta=0}^{\nu-1} e^{-\eta r^m L}. \]
Expanding the identity \( I^M \) by means of the binomial formula leads to
\[ I = \left( m P_{m, M, r}(L) \right)^M \]
\[ + \sum_{l=1}^{M} \frac{M!}{l!} \left( \sum_{\nu=1}^{M} C_{\nu, M} r^{-m} L^{-1}(I - e^{-r^m L}) \sum_{\eta=0}^{\nu-1} e^{-\eta r^m L} \right)^l \left( m P_{m, M, r}(L) \right)^{M-l} \]
\[ = m^M P_{m, M, r}(L)^M + \sum_{l=1}^{M} r^{-ml} L^{-l}(I - e^{-r^m L})^l P_{m, M, r}(L)^{M-l} \sum_{\nu=1}^{(2M-1)l} C_{l, \nu, m, M} e^{-\nu r^m L} \]
for appropriate constants \( C_{l, \nu, m, M} \) depending on the subscripted parameters.

Now fix \( k \in \{0, 1, \ldots, M\} \). The above identity allows us to represent \( (r^m L)^k (F(L)L^M b) \) in the following way
\[ (r^m L)^k (F(L)L^M b) = m^M r^{mk} P_{m, M, r}(L)^M F(L)(L^{M+k} b) \]
\[ + \sum_{l=1}^{M} r^{mk-ml} L^{-l}(I - e^{-r^m L})^l P_{m, M, r}(L)^{M-l} \sum_{\nu=1}^{(2M-1)l} C_{l, \nu, m, M} e^{-\nu r^m L} F(L)(L^{M+k} b) \]
\[ =: \sum_{l=0}^{M} C_{l, M, r}^{(k)}. \]
We shall establish an adequate bound on \( \| G_{l, M, r}^{(k)} \|_{L^2(U_j(B))} \) by distinguishing the three cases \( l = 0, l \in \{1, \ldots, M-1\}, \) and \( l = M \).

Case 1: \( l = 0 \).
First, we write for \( \mu \text{-a.e. } x \in X \)
\[
\left| G_{0,M,r}^{(k)}(x) \right| = m^{M} r^{mk} |P_{m,M,r}(L)(P_{m,M,r}(L)^{M-1} F(L)(L^{M+k}b))(x)| \leq m^{M} r^{mk} r^{-m} \int_{r}^{\sqrt{2}r} s^{m-1} \left| P_{m,M,r}(L)^{M-1}(F(L)(I - e^{-s^m L})^{M}(L^{M+k}b))(x) \right| \, ds
\]
\[
\leq \sum_{i=0}^{\infty} m^{M} r^{mk} r^{-m} \times \int_{r}^{\sqrt{2}r} s^{m-1} \left| P_{m,M,r}(L)^{M-1}(1_{U_i(B)}(F(L)(I - e^{-s^m L})^{M}(L^{M+k}b)))(x) \right| \, ds.
\]

As seen in Lemma 4.8, the operator \( P_{m,M,r}(L)^{M-1} \) enjoys the off-diagonal estimate (4.10). This yields
\[
\left\| G_{0,M,r}^{(k)} \right\|_{L^2(U_j(B))} \leq m^{M} r^{mk} \sum_{i=0}^{\infty} r^{-m} \times \int_{r}^{\sqrt{2}r} s^{m-1} \left\| P_{m,M,r}(L)^{M-1}(1_{U_i(B)}(F(L)(I - e^{-s^m L})^{M}(L^{M+k}b))\right\|_{L^2(U_j(B))} \, ds
\]
\[
\leq r^{mk} \sum_{i=0}^{\infty} \exp(-b2^{j-i}) r^{-m} \int_{r}^{\sqrt{2}r} s^{m-1} \left\| F(L)(I - e^{-s^m L})^{M}(L^{M+k}b)\right\|_{L^2(U_i(B))} \, ds.
\]

In order to apply Lemma 4.7, we first observe that for every \( s \in [r, \sqrt{2}r) \) the ball \( U_0(B) \) is contained in \( U_0(B(x_B, s)) \) and the annulus \( U_i(B) \) in \( U_{i-1}(B(x_B, s)) \cup U_i(B(x_B, s)) \) for each \( i \in \mathbb{N} \) where \( x_B \) denotes the center of \( B \). These inclusions give for every \( s \in [r, \sqrt{2}r] \)
\[
\left\| F(L)(I - e^{-s^m L})^{M}(L^{M+k}b)\right\|_{L^2(U_i(B))} \leq \sum_{\nu=1}^{i} \left( \left\| F(L)(I - e^{-s^m L})^{M}(1_{B(x_B, s)}L^{M+k}b)\right\|_{L^2(U_{\nu}(B(x_B, s)))} \right.
\]
\[
\leq \sum_{\nu=1}^{i} \left( \left\| F(L)(I - e^{-s^m L})^{M}(1_{B(x_B, s)}L^{M+k}b)\right\|_{L^2(U_{\nu}(B(x_B, s)))} \right) \left( \sum_{\eta=1}^{\infty} \left\| F(L)(I - e^{-s^m L})^{M}(1_{U_\eta(B(x_B, s))}L^{M+k}b)\right\|_{L^2(U_{\nu}(B(x_B, s)))} \right).
\]

Due to (4.7), the first summand in the bracket is bounded by
\[
C_F 2^{-i\delta} \left\| L^{M+k}b\right\|_{L^2(B(B(x_B, s)))} \leq C_F 2^{-i\delta} \left( \left\| L^{M+k}b\right\|_{L^2(B)} + \left\| L^{M+k}b\right\|_{L^2(U_i(B))} \right).
\]

By recalling the properties of the \((2M, \bar{r}, L)\)-molecule \( a \), we obtain
\[
\left\| L^{M+k}b\right\|_{L^2(B)} = r^{-m(M+k)} \left\| (r^m L)^{M+k}b\right\|_{L^2(B)} \leq r^{-m(M+k)} r^{2m} \mu(B)^{-1/2} = r^{mM-mk} \mu(B)^{-1/2}
\]
as well as
\[
\left\| L^{M+k}b\right\|_{L^2(U_i(B))} = r^{-m(M+k)} \left\| (r^m L)^{M+k}b\right\|_{L^2(U_i(B))} \leq r^{-m(M+k)} r^{2m} 2^{-\bar{r}} \mu(2B)^{-1/2} \leq r^{mM-mk} \mu(B)^{-1/2}.
\]
Hence, we have the bound
\[ \| F(L)(I - e^{-s^mL})^M(1_B(x_B,s))L^{M+k}b \|_{L^2(U_\eta(B(x_B,s)))} \lesssim C_{F}r^{mM-mk}2^{-\nu\delta}\mu(B)^{-1/2}. \tag{4.16} \]

The series in the bracket of (4.15) can be estimated with the help of Lemma 4.7,
\[ \sum_{\eta=1}^{\infty} \| F(L)(I - e^{-s^mL})^M(1_B(x_B,s))L^{M+k}b \|_{L^2(U_\eta(B(x_B,s)))} \lesssim \sum_{\eta=1}^{\infty} C_{F}r^{2nD}2^{-[\nu-\eta]\delta} \| L^{M+k}b \|_{L^2(U_\eta(B(x_B,s)))}. \]

Since \( a \) is an \((2M,\bar{\nu},L)\)-molecule, we obtain
\[ \| L^{M+k}b \|_{L^2(U_\eta(B(x_B,s)))} \leq r^{-m(M+k)} \| (r^mL)^{M+k}b \|_{L^2(U_\eta(B(x_B,r)))} + \| (r^mL)^{M+k}b \|_{L^2(U_{\eta+1}(B(x_B,r)))} \]
\[ \leq r^{-m(M+k)} \left( r^{2mM}2^{-\bar{\nu}\delta}\mu(B(x_B,r)^{-1/2} + r^{2mM}2^{-(\eta+1)\bar{\nu}}\mu(2^{\eta+1}B(x_B,r)^{-1/2}) \right) \]
\[ \lesssim r^{mM-mk}2^{-\bar{\nu}\delta}\mu(B(x_B,r)^{-1/2} \]

and thus
\[ \sum_{\eta=1}^{\infty} \| F(L)(I - e^{-s^mL})^M(1_B(x_B,s))L^{M+k}b \|_{L^2(U_\eta(B(x_B,s)))} \lesssim C_{F}r^{mM-mk}\mu(B(x_B,r)^{-1/2} \cdot 2^{-[\nu-\eta]\delta} \sum_{\eta=1}^{\infty} 2^{-\bar{\nu}(\bar{\nu}-D)}2^{-[\nu-\eta]\delta} \]
\[ \lesssim C_{F}r^{mM-mk}2^{-\nu\delta}\mu(B(x_B,r)^{-1/2}. \tag{4.17} \]

In the last step we used the fact that
\[ \sum_{\eta=1}^{\infty} 2^{-\eta(\bar{\nu}-D)}2^{-[\nu-\eta]\delta} = 2^{-\nu(\bar{\nu}-D)} \left( \sum_{n=0}^{0} 2^n(\bar{\nu}-D)2^{-n\delta} + \sum_{n=1}^{\nu-1} 2^n(\bar{\nu}-D)2^{-n\delta} \right) \]
\[ \leq 2^{-\nu\delta} \left( \sum_{n=0}^{0} 2^{-n\delta} + \sum_{n=1}^{\nu-1} 2^{-n(D-\bar{\nu})} \right) \lesssim 2^{-\nu\delta}. \]

In view of the inequalities (4.16) and (4.17), we have the following estimate of (4.15)
\[ \| F(L)(I - e^{-s^mL})^M(L^{M+k}b) \|_{L^2(U_j(B))} \lesssim C_{F}r^{mM-mk}2^{-\bar{\nu}\delta}\mu(B)^{-1/2}. \]

With the help of this bound and the doubling property, we continue
\[ \| G^{(k)}_{0,M,r} \|_{L^2(U_j(B))} \lesssim r^{mk} \sum_{i=0}^{\infty} \exp(\bar{b}2^{(j-i)}) r^{-m} \int_{r}^{\sqrt{\bar{b}r}} s^{m-1} \| F(L)(I - e^{-s^mL})^M(L^{M+k}b) \|_{L^2(U_j(B))} ds \]
\[ \lesssim r^{mk} \sum_{i=0}^{\infty} \exp(\bar{b}2^{(j-i)}) r^{-m} \int_{r}^{\sqrt{\bar{b}r}} s^{m-1} ds C_{F}r^{mM-mk}2^{-\bar{\nu}\delta}\mu(B)^{-1/2} \]
\[ \lesssim C_{F}r^{mM}2^{-\bar{\nu}\delta}\mu(B)^{-1/2} \lesssim C_{F}r^{mM}2^{-j[\delta-D/2]}\mu(2^jB)^{-1/2}. \tag{4.18} \]
In the second to the last step we used, among other things, the following fact which is easily verified by an index shift
\[
\sum_{i=0}^{\infty} \exp(-b 2^{j-i}) 2^{-i\delta} = \sum_{n=-\infty}^{0} \exp(-b 2^{n}) 2^{-(j-n)\delta} + \sum_{n=1}^{j} \exp(-b 2^{n}) 2^{-(j-n)\delta} \\
\leq 2^{-j\delta} \sum_{n=-\infty}^{\infty} \exp(-b 2^{n}) 2^{n\delta} \lesssim 2^{-j\delta}.
\]

\((4.19)\)

**Case 2:** \(l \in \{1, 2, \ldots, M - 1\}\).

We have for \(\mu\text{-a.e. } x \in X\)
\[
|G_{l_M,r}^{(k)}(x)| \leq r^{m(k-l)} \sum_{\nu=1}^{(2M-1)!} |C_{l_M,r}(x)| \int_{r}^{\sqrt{2}r} \left( \frac{r}{s} \right)^{m} |L^{M-l} e^{-\nu r m L} (I - e^{-r m L})| \circ P_{mol} \circ P_{m,M,r}(L)^{M-l-1} (F(L)(I - e^{-s m L})(L^k b))(x) \frac{ds}{s} \\
\lesssim r^{m(k-M)} \sum_{\nu=1}^{(2M-1)!} \int_{r}^{\sqrt{2}r} |(r m L)^{M-l} e^{-\nu r m L} (I - e^{-r m L})| \circ P_{mol} \circ P_{m,M,r}(L)^{M-l-1} \left( \mathbb{1}_{U_i(B)}(F(L)(I - e^{-s m L})(L^k b)) \right)(x) \frac{ds}{s}.
\]

By Lemma 2.1 the operator family \(\{(tL)^{M-l} e^{-\nu t L} : t > 0\}\) satisfies DG\(_m\). After writing \((I - e^{-t L})\) with the help of the binomial formula, it is straightforward to prove that DG\(_m\) also holds for \(\{(tL)^{M-l} e^{-\nu t L} (I - e^{-t L})^{l} : t > 0\}\). Hence, one can show \(L^2(X)\)-norm estimates of annular type similar to those in (6.3) below for operators of the form \((r m L)^{M-l} e^{-\nu r m L} (I - e^{-r m L})^{l}\) whenever \(r\) denotes the radius of the considered ball. Thanks to Lemma 4.8, \(P_{m,M,r}(L)^{M-l-1}\) fulfills (4.10).

If one adapts the arguments given at the end of the proof of Lemma 4.8 (cf. Section 6), one can verify that the composition of these operators enjoys the following version of (4.10)
\[
\|\mathbb{1}_{U_i(B)}(r m L)^{M-l} e^{-\nu r m L} (I - e^{-r m L})^{l} P_{m,M,r}(L)^{M-l-1} \mathbb{1}_{U_i(B)}\|_{2 \rightarrow 2} \leq C \exp(-b 2^{j-i})
\]
for some constants \(b, C > 0\) depending only on \(m, K, M\) and the constants in the Davies-Gaffney and the doubling condition.

This estimate leads to
\[
\|G_{l_M,r}^{(k)}\|_{L^2(U_i(B))} \lesssim r^{m(k-M)} \sum_{\nu=1}^{(2M-1)!} \int_{r}^{\sqrt{2}r} |F(L)(I - e^{-s m L})(L^k b)| \|F(L)(I - e^{-s m L})(L^k b)\|_{L^2(U_i(B))} \frac{ds}{s}.
\]

By employing similar arguments as in Case 1 (just replace \(L^{M+k_b}\) by \(L^k b\)), we conclude that for any \(i \in \mathbb{N}_0\) and \(s \in [r, \sqrt{2}r]\)
\[
\|F(L)(I - e^{-s m L})(L^k b)\|_{L^2(U_i(B))} \lesssim C_F r^{m(2M-M-k_2)2^{-i\delta}} \mu(B)^{-1/2}.
\]

Inserting this estimate into (4.20) yields readily
\[
\|G_{l_M,r}^{(k)}\|_{L^2(U_i(B))} \lesssim C_F r^{m2^{-i(\delta-D/2)}} \mu(2^j B)^{-1/2}.
\]

\((4.22)\)

**Case 3:** \(l = M\).
In this case we have

\[ G^{(k)}_{M,M,r} \equiv r^{m(k-M)} \sum_{\nu=1}^{(2M-1)M} C_{M,\nu,m} e^{-\nu r L} (F(L)(I - e^{-r L})^M(L^k b)) = r^{m(k-M)} \sum_{\nu=1}^{(2M-1)M} C_{M,\nu,m} \sum_{i=0}^{\infty} e^{-\nu r L} \left( \mathbb{1}_{U_i(B)}(F(L)(I - e^{-r L})^M(L^k b)) \right). \]

With the help of (6.3, 6.2) below, and (4.21), (4.19), we obtain

\[
\left\| G^{(k)}_{M,M,r} \right\|_{L^2(U_j(B))} \lesssim r^{m(k-M)} \sum_{i=0}^{\infty} \exp(-b 2^{j-i}) \left\| F(L)(I - e^{-r L})^M(L^k b) \right\|_{L^2(U_i(B))} \\
\lesssim C_{F r^{mM}} \sum_{i=0}^{\infty} \exp(-b 2^{j-i}) 2^{-i\delta} \mu(B)^{-1/2} \\
\lesssim C_{F r^{mM}} 2^{-jD} \mu(B)^{-1/2} \\
\lesssim C_{F r^{mM}} 2^{-jD} \mu(B)^{-1/2}.
\]

This, in combination with (4.12, 4.18, and 4.22), gives the desired estimate (4.11). □

We prepare the proof of Theorem 4.1 a) with the next two lemmas. The first one corresponds to [27] Lemma 4.1 and gives an extension of generalized Gaussian estimates from real times to complex times in some weighted space. This is crucial for our proof of Lemma 4.9 where the operator \( F(\sqrt{L}) \) will be represented in terms of the extended semigroup \((e^{-z L})_{\text{Re } z > 0}\) by a Fourier transform argument taken from [27].

**Lemma 4.9.** Let \( s \geq 0 \). In the situation of Theorem 4.1 a), there exists a constant \( C > 0 \) such that for all \( R > 0 \), \( \tau \in \mathbb{R} \), and \( y \in X \)

\[
\left\| e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)} \right\|_{L^2(X) \rightarrow L^2(X,(1+Rd(-y))^s)\mu) \leq C(1 + \tau^2)^{s/4}.}
\]

**Proof.** According to Fact 2.4 there are constants \( b, C > 0 \) such that for all \( x, y \in X \) and all \( z \in \mathbb{C} \) with \( \text{Re } z > 0 \)

\[
\left\| \mathbb{1}_{B(x,r_z)} e^{-z} \mathbb{1}_{B(y,r_z)} \right\|_{L^2 \rightarrow L^2} \leq C \exp\left( -b \left( \frac{d(x,y)}{r_z} \right) \right),
\]

where \( r_z := (\text{Re } z)^{1/m-1} |z| \). By Fact 2.3, this two-ball estimate is equivalent to the assertion that there exist \( b, C > 0 \) such that for every \( k \in \mathbb{N}_0 \), \( y \in X \), and \( z \in \mathbb{C} \) with \( \text{Re } z > 0 \)

\[
\left\| \mathbb{1}_{A(y,r_z,k)} e^{-z L} \mathbb{1}_{B(y,r_z)} \right\|_{L^2 \rightarrow L^2} \leq C \exp\left( -b k^{\frac{m}{m-1}} \right).}
\]
Now let $R > 0$, $s \geq 0$, $\tau \in \mathbb{R}$, and $y \in X$ be fixed. For $z := (1 + i\tau)R^{-m}$ we calculate $r_z = (1 + \tau^2)^{1/2}/R \geq 1/R$ and obtain

$$
\|e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\|_{L^2(X) \to L^2(X,(1+Rd(-y))^{s/4})} \leq \sum_{k=0}^{\infty} \|1_A(y,r_z,k)e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\|_{L^2(X) \to L^2(X,(1+Rd(-y))^{s/4})} \\
\leq \sum_{k=0}^{\infty} (1 + R(k+1)r_z)^{s/2} \|1_A(y,r_z,k)e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\|_{L^2 \to L^2} \\
\leq \sum_{k=0}^{\infty} (1 + (k+1)(1 + \tau^2)^{1/2})^{s/2} \|1_A(y,r_z,k)e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\|_{L^2 \to L^2} \\
\leq C(1 + \tau)^{s/4 \sum_{k=0}^{\infty} (k+2)^{s/2} \exp(-bk^{-m-1})} \\
\lesssim (1 + \tau^2)^{s/4}.
$$

The second preparatory statement is based on [27] Lemma 4.3 a]) and is used to transfer regularity of a function $F$ to an off-diagonal $L^2$-estimate for $F(\sqrt{L})$. The only difference between (4.23) and (4.24) is in the norm of $F(R \cdot)$.

**Lemma 4.10.** Let $L$ be a non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates $DG_m$ for some $m \geq 2$.

a) Then for any $s \geq 0$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$
\|F(\sqrt{L}) \mathbb{1}_{B(y,1/R)}\|_{L^2(X) \to L^2(X,(1+Rd(-y))^{s})} \leq C \|F(R \cdot)\|_{H^{(s+1)/2+\varepsilon}}
$$

(4.23)

for every $R > 0$, $y \in X$, and every bounded Borel function $F: [0,\infty) \to \mathbb{C}$ with supp $F \subset [R/4,R]$ and $F(R \cdot) \in H^{(s+1)/2+\varepsilon}.$

b) Suppose additionally that $L$ fulfills the Plancherel condition (4.3) for some $q \in [2,\infty]$. Then for any $s \geq 2/q$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$
\|F(\sqrt{L}) \mathbb{1}_{B(y,1/R)}\|_{L^2(X) \to L^2(X,(1+Rd(-y))^{s})} \leq C \|F(R \cdot)\|_{H^{s/2+\varepsilon}}
$$

(4.24)

for every $R > 0$, $y \in X$, and every bounded Borel function $F: [0,\infty) \to \mathbb{C}$ with supp $F \subset [R/4,R]$ and $F(R \cdot) \in H^{s/2+\varepsilon}.$

Proof. Let $R > 0$ and $F: [0,\infty) \to \mathbb{C}$ be a bounded Borel function with supp $F \subset [R/4,R]$. For all $\lambda \geq 0$ define $G(\lambda) := F(R \sqrt{\lambda}) e^{\lambda}$. If $\hat{G}$ denotes the Fourier transform of $G$, then it holds

$$
F(\sqrt{L}) = G(R^{-m}L) e^{-R^{-m}L} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{G}(\tau) e^{-(1-i\tau)R^{-m}L} d\tau
$$
in the strong convergence sense in $L^2(X)$. Thus, Lemma 4.9 and the Cauchy-Schwarz inequality yield for any $y \in X$, $s \geq 0$, and $\varepsilon > 0$ whenever $F(R \cdot) \in H_2^{(s+1)/2+\varepsilon}$

$$
\left\| F(\sqrt{L}) \mathbb{1}_{B(y,1/R)} \right\|_{L^2(X) \to L^2(X,(1+Rd(\cdot,y))^s \, d\mu)} \
\lesssim \int_{-\infty}^{\infty} |\hat{G}(\tau)| \left\| \tau^{-(1-i\tau) R^{-m} L \mathbb{1}_{B(y,1/R)}} \right\|_{L^2(X) \to L^2(X,(1+Rd(\cdot,y))^s \, d\mu)} \, d\tau
$$

$$
\lesssim \int_{-\infty}^{\infty} |\hat{G}(\tau)| (1+\varepsilon)^{s/4} \, d\tau
$$

$$
\lesssim \left( \int_{-\infty}^{\infty} |\hat{G}(\tau)|^2 (1+\varepsilon)^{s/4} \, d\tau \right)^{1/2} \left( \int_{-\infty}^{\infty} (1+\varepsilon)^{s/4} \, d\tau \right)^{1/2}
$$

$$
\lesssim \left\| \hat{G} \right\|_{H_2^{s+1} \to H_2^{s+1}}.
$$

(4.25)

Due to $\text{supp} \, F(R \cdot) \subset [1/4, 1]$, it follows

$$
\left\| \hat{G} \right\|_{H_2^{s+1} \to H_2^{s+1}} \lesssim \left\| F(R \cdot) \right\|_{H_2^{s+1}} \lesssim \left\| F(R \cdot) \right\|_{H_q^{s+1}}
$$

(4.26)

for each $q \in [2, \infty]$. From (4.25) and (4.26) we obtain part a) of the lemma.

Inserting (4.26) in (4.23) leads to a statement in which the required order of differentiability of the function $F(R \cdot)$ is 1/2 larger than that of part b). In order to get rid of this additional 1/2, we make use of the interpolation procedure as described in [27] p. 455 (see also [13]) based on the Plancherel condition (4.3). By a simple scaling argument, we first observe that the claimed bound (4.24) is equivalent to the following estimate

$$
\left\| H(R^{-1/2} \sqrt{L}) \mathbb{1}_{B(y,1/R)} \right\|_{L^2(X) \to L^2(X,(1+Rd(\cdot,y))^s \, d\mu)} \lesssim \left\| H \right\|_{H_q^{s+1} \to H_q^{s+1}}
$$

(4.27)

for any $\varepsilon > 0$, $s \geq 2/q$, $R > 0$, $y \in X$, and any bounded Borel function $H : [0, \infty) \to \mathbb{C}$ with $\text{supp} \, H \subset [1/4, 1] \cup H \subset H_q^{s+1} \to H_q^{s+1}$.

For fixed $R > 0$, $y \in X$, and $\varphi \in L^2(X)$ with $\text{supp} \, \varphi \subset B(y,1/R)$ and $\left\| \varphi \right\|_{L^2} = 1$ define

$$
K_{y,R,q} \cdot E_q \to L^2(X), \quad H \mapsto H(R^{-1/2} \sqrt{L})\varphi,
$$

where $E_q := L^\infty([1/4, 1])$ if $q < \infty$ and $E_q := C^0([1/4, 1])$ if $q = \infty$. According to the Plancherel condition (4.3), we see after rescaling that

$$
\left\| K_{y,R,q}(H) \right\|_{L^2} \lesssim \left\| H \right\|_{L^q([1/4, 1])}
$$

for every $H \in E_q$. Next, for $\alpha > 0$ we denote by $H_q^\alpha([1/4, 1])$ the set of all $H \in H_q^\alpha$ with $\text{supp} \, H \subset [1/4, 1]$. The inequalities (4.23) and (4.26) lead to

$$
\left\| K_{y,R,q}(H) \right\|_{L^2(X,(1+Rd(\cdot,y))^s \, d\mu)} \lesssim \left\| H \right\|_{H_q^{s+1} \to H_q^{s+1}}
$$

(4.28)

for any $s \geq 0$, $\varepsilon > 0$, and $H \in H_q^{s+1} \to H_q^{s+1}$([1/4, 1]). Now complex interpolation yields for every $\theta \in (0, 1)$

$$
\left\| K_{y,R,q}(H) \right\|_{L^2(X,(1+Rd(\cdot,y))^s \, d\mu)} \lesssim \left\| H \right\|_{H_q^{s+1+\varepsilon} \to H_q^{s+1+\varepsilon}}
$$

for any $s \geq 0$, $\varepsilon > 0$, $H \in H_q^{s+1+\varepsilon} \to H_q^{s+1+\varepsilon}([1/4, 1])$, and $\delta > 0$.

Let $s' \geq 2/q$ and $\varepsilon' > 0$ be arbitrary. Take $\theta \in (0, 1)$ and $\delta > 0$ with $(1+\varepsilon')\theta/2 + \delta = \varepsilon'$. Next, choose $s \geq 0$ with $s\theta = s'$. Then inequality (4.28) reads

$$
\left\| K_{y,R}(H) \right\|_{L^2(X,(1+Rd(\cdot,y))^s \, d\mu)} \lesssim \left\| H \right\|_{H_q^{s'+\varepsilon'} \to H_q^{s'+\varepsilon'}}([1/4, 1])}
$$
for any $H \in H^{s'/2+\epsilon'}_q([1/4,1])$. Taking the supremum over all \( \varphi \in L^2(X) \) such that \( \text{supp} \varphi \subset B(y,1/R) \) and \( \| \varphi \|_{L^2} = 1 \) yields
\[
\| H(R^{-1} \sqrt{L}) \mathbb{1}_{B(y,1/R)} \|_{L^2(X) \rightarrow L^2(X,(1+Rd(y)))^{s'}} \lesssim \| H \|_{H^{s'/2+\epsilon'}_q([1/4,1])}
\]
for any $H \in H^{s'/2+\epsilon'}_q([1/4,1])$. This proves \((4.27)\) and thus \((4.24)\). \(\Box\)

**Proof of Theorem 4.1 a.** Let $F : [0, \infty) \rightarrow \mathbb{C}$ be a bounded Borel function. Observe that $F$ satisfies \((4.1)\) if and only if the function $\lambda \mapsto F(\sqrt[\lambda]{L})$ satisfies \((4.1)\). Hence, we can consider $F(\sqrt[\lambda]{L})$ in lieu of $F(L)$ during the proof. First, we write
\[
F(\sqrt[\lambda]{L}) = (F - F(0))(\sqrt[\lambda]{L}) + F(0)I
\]
and notice, after replacing $F$ by $F - F(0)$, that we may assume $F(0) = 0$ in the sequel. Due to the properties of $\omega$, for every $\lambda \geq 0$ we then have the decomposition
\[
F(\lambda) = \sum_{l=-\infty}^{\infty} \omega(2^{-l}\lambda)F(\lambda) = \sum_{l=-\infty}^{\infty} F_l(\lambda),
\]
where $F_l(\lambda) := \omega(2^{-l}\lambda)F(\lambda)$.

Fix $s > D/2$ and $M \in \mathbb{N}$ with $M > 2s/m$. Further, assume that $F$ fulfills the Hörmander condition \((4.1)\) of order $s + 1/2$. For verifying the uniform boundedness of $\sum_{l=-N}^{N} F_l(\sqrt[\lambda]{L})$ in $H^1_q(X)$, we apply Theorem 4.6. To this end, we only need to check that condition \((4.7)\) holds for the operator $\sum_{l=-N}^{N} F_l(\sqrt[\lambda]{L})$ with a constant $C_F$ independent of $N \in \mathbb{N}$.

For each $l \in \mathbb{Z}$ and $r > 0$, we introduce the abbreviations
\[
F_{r,M}(\lambda) := F(\lambda)(1 - e^{-(r\lambda)^m})M,
\]
\[
F_{r,M}^l(\lambda) := F_{r,M}(\lambda)(1 - e^{-(r\lambda)^m})M = \omega(2^{-l}\lambda)F(\lambda)(1 - e^{-(r\lambda)^m})M,
\]
where $\lambda \geq 0$. In this notation, we may write
\[
F(\sqrt[\lambda]{L})(I - e^{-r^mL})^M = F_{r,M}(\sqrt[\lambda]{L}) = \lim_{N \rightarrow \infty} \sum_{l=-N}^{N} F_{r,M}^l(\sqrt[\lambda]{L}). \tag{4.29}
\]

We choose $s' \in (D/2,s)$ and claim that for all $j \in \mathbb{N} \setminus \{1\}$, $l \in \mathbb{Z}$, and balls $B \subset X$ of radius $r$
\[
\| \mathbb{1}_{U_j(B)} F_{r,M}^l(\sqrt[\lambda]{L}) \mathbb{1}_B \|_{L^2 \rightarrow L^2} \lesssim C_{\omega,s} 2^{-j s'} (2^j r)^{-s'} \min \{1,(2^j r)^{mM}\} \max \{1,(2^j r)^{D/2}\}, \tag{4.30}
\]
where $C_{\omega,s} := \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H^{s'+1/2}}$ and the implicit constant depends only on $m, M, s$ and the constants in the Davies-Gaffney and the doubling condition.

This, together with \((4.29)\), shows that for any $j \in \mathbb{N} \setminus \{1\}$ and any ball $B \subset X$ of radius $r$
\[
\| \mathbb{1}_{U_j(B)} F(\sqrt[\lambda]{L})(I - e^{-r^mL})^M \mathbb{1}_B \|_{L^2 \rightarrow L^2} \lesssim C_{\omega,s} 2^{-j s'} \lim_{N \rightarrow \infty} \sum_{l=-N}^{N} (2^j r)^{-s'} \min \{1,(2^j r)^{mM}\} \max \{1,(2^j r)^{D/2}\}
\]
\[
\leq C_{\omega,s} 2^{-j s'} \left( \sum_{l \in \mathbb{Z}} (2^j r)^{D/2-s'} + \sum_{l \in \mathbb{Z}} (2^j r)^{M-M-s'} \right).
\]

As both sums converge and have an upper bound independent of $r$, the estimate \((4.7)\) holds for the function $F(\sqrt[\lambda]{\cdot})$, as desired.
It remains to prove our claim (4.30). Consider a ball $B \subset X$ with center $y \in X$ and radius $r > 0$. First, we observe that $\text{supp } F_{r,M}^l \subset (2^{-l}, 2^l)$. Lemma 4.10 a) then says that for any $l \in \mathbb{Z}$ and any $\varepsilon > 0$

$$\| F_{r,M}^l (\sqrt{L}) \mathbbm{1}_{B(y, 2^{-l})} \|_{L^2(X)} \lesssim \| F_{r,M}^l (2^l \cdot) \|_{H^{2+1/2+\varepsilon}}. \quad (4.31)$$

Let $j \in \mathbb{N} \setminus \{1\}$. For each $x \in U_j(B)$ we obtain, due to $d(x, y) \geq 2^{j-1}r$, the estimate $(1 + 2^d(x, y))^{s'} \geq 2^{(j-1)s'} (2^l r)^{s'}$. Hence, we get for $\varepsilon := s - s' > 0$

$$2^{-s'} 2^{j s'} (2^l r)^{s'} \| \mathbbm{1}_{U_j(B)} F_{r,M}^l (\sqrt{L}) \mathbbm{1}_{B(y, 2^{-l})} \|_{L^2 \to L^2} \leq \| \mathbbm{1}_{U_j(B)} F_{r,M}^l (\sqrt{L}) \mathbbm{1}_{B(y, 2^{-l})} \|_{L^2(X) \to L^2(X, (1+2^d(x, y))^{2s'} d\mu)} \lesssim \| F_{r,M}^l (2^l \cdot) \|_{H^{s\varepsilon}}$$

or equivalently

$$\| \mathbbm{1}_{U_j(B)} F_{r,M}^l (\sqrt{L}) \mathbbm{1}_{B(y, 2^{-l})} \|_{L^2 \to L^2} \lesssim 2^{-j s'} (2^l r)^{-s'} \| F_{r,M}^l (2^l \cdot) \|_{H^{s\varepsilon}}. \quad (4.32)$$

For $l \in \mathbb{Z}$ with $r \leq 2^{-l}$ the left-hand side is an upper bound for $\| \mathbbm{1}_{U_j(B)} F_{r,M}^l (\sqrt{L}) \mathbbm{1}_B \|_{2 \to 2}$.

In the case $l \in \mathbb{Z}$ with $r > 2^{-l}$, we cover $B = B(y, r)$ by balls of radius $2^{-l}$. This procedure eventually leads to an additional factor depending on the ratio of $r$ and $2^{-l}$ and the dimension of the underlying space $X$. By Lemma 2.2 one can construct a family of points $y_1, \ldots, y_K \in B(y, r)$ such that $B(y, r) \subset \bigcup_{\nu=1}^{K} B(y_\nu, 2^{-l})$, $K \lesssim (2^l r)^D$, and every $x \in B(y, r)$ is contained in at most $M$ balls $B(y_\nu, 2^{-l})$, where $M$ depends only on the constants in the doubling condition. Observe that

$$U_j(B(y, r)) \subset \bigcup_{\eta=j-1}^{j+1} U_\eta(B(y_\nu, r))$$

for all $j \in \mathbb{N} \setminus \{1\}$ and $\nu \in \{1, 2, \ldots, K\}$. Therefore, by (4.32), one obtains

$$\| \mathbbm{1}_{U_j(B(y, r))} F_{r,M}^l (\sqrt{L}) \mathbbm{1}_{B(y, 2^{-l})} \|_{L^2 \to L^2} \leq \sum_{\eta=j-1}^{j+1} \| \mathbbm{1}_{U_\eta(B(y_\nu, r))} F_{r,M}^l (\sqrt{L}) \mathbbm{1}_{B(y_\nu, 2^{-l})} \|_{L^2 \to L^2} \lesssim \sum_{\eta=j-1}^{j+1} 2^{-\eta s'} (2^l r)^{-s'} \| F_{r,M}^l (2^l \cdot) \|_{H^{s\varepsilon}} \lesssim 2^{-j s'} (2^l r)^{-s'} \| F_{r,M}^l (2^l \cdot) \|_{H^{s\varepsilon}}.$$
Consider \( g, h \in L^2(X) \) with \( \|g\|_{L^2} = 1 \) and \( \|h\|_{L^2} = 1 \). Then we obtain for every \( j \in \mathbb{N} \setminus \{1\} \) and every \( l \in \mathbb{Z} \) with \( r > 2^{-l} \)
\[
\left| \langle h, 1_{U_j(B(y,r))}F_{r,M}^l(\sqrt{L}) 1_{B(y,r)}g \rangle \right|^2 = \left| \langle 1_{B(y,r)}F_{r,M}^l(\sqrt{L})^* 1_{U_j(B(y,r))}h, g \rangle \right|^2 \\
\leq \left\| 1_{B(y,r)}F_{r,M}^l(\sqrt{L})^* 1_{U_j(B(y,r))}h \right\|_{L^2}^2 \|g\|_{L^2}^2 \\
= \int_{B(y,r)} \left| F_{r,M}^l(\sqrt{L})^* (1_{U_j(B(y,r))}h)(x) \right|^2 d\mu(x) \\
\leq \sum_{\nu=1}^K \int_{B(y,2^{-l})} \left| F_{r,M}^l(\sqrt{L})^* (1_{U_j(B(y,r))}h)(x) \right|^2 d\mu(x) \\
\leq \sum_{\nu=1}^K \left\| 1_{U_j(B(y,r))}F_{r,M}^l(\sqrt{L}) 1_{B(y,2^{-l})} \right\|_{L^2 \rightarrow L^2}^2 \\
\lesssim \sum_{\nu=1}^K (2^{-js'}(2^r)^{-s'}\|F_{r,M}^l(\sqrt{L})\|_{H^{s+1/2}_2})^2.
\]
Thus, by taking the supremum over all such \( g, h \) and by recalling \( \sqrt{K} \lesssim (2^r)^{D/2} \), we deduce
\[
\left\| 1_{U_j(B(y,r))}F_{r,M}^l(\sqrt{L}) 1_{B(y,r)} \right\|_{L^2 \rightarrow L^2} \lesssim (2^r)^{D/2} 2^{-js'}(2^r)^{-s'}\|F_{r,M}^l(\sqrt{L})\|_{H^{s+1/2}_2}.
\]
In summary, we have shown that
\[
\left\| 1_{U_j(B)}F_{r,M}^l(\sqrt{L}) 1_B \right\|_{L^2 \rightarrow L^2} \lesssim \max\{1, (2^r)^{D/2}\} 2^{-js'}(2^r)^{-s'}\|F_{r,M}^l(\sqrt{L})\|_{H^{s+1/2}_2} \tag{4.33}
\]
for any \( j \in \mathbb{N} \setminus \{1\}, l \in \mathbb{Z} \), and any ball \( B \subset X \) of radius \( r \).

If \( \gamma \) is an integer larger than \( s + 1/2 \), then it holds
\[
\|F_{r,M}^l(\sqrt{L})\|_{H^{s+1/2}_2} = \|\lambda \mapsto \omega(\lambda)F(\sqrt{L})^\gamma(1 - e^{-2^r\lambda})^m\|_{H^{s+1/2}_2} \\
\lesssim \|\omega(\sqrt{L})\|_{H^{s+1/2}_2} \|\lambda \mapsto (1 - e^{-2^r\lambda})^M\|_{C^\gamma([\frac{1}{r}, 1])} \\
\lesssim \sup_{n \in \mathbb{Z}} \|\omega(\sqrt{L})^n\|_{H^{s+1/2}_2} \min\{1, (2^r)^{mM}\}. \tag{4.34}
\]
The first inequality is due to [31] Corollary (ii), p. 143], whereas the second inequality follows from [9] Lemma 3.5].

In view of (4.33) and (4.34), the claim (4.30) is confirmed. This completes the proof.

The proof of Theorem 4.2 follows the same lines as that of Theorem 4.1 a) with one small modification. Instead of Lemma 4.10 a) one has to employ part b) of the same lemma to obtain the desired regularity order in the Hörmander condition.

5. BOUNDEDNESS OF SPECTRAL MULTIPLIERS ON \( H^p_L(X) \) AND \( L^p(X) \)

In the preceding section we established spectral multiplier theorems on the Hardy space \( H^p_L(X) \) which ensure the boundedness of the operator \( F(L) \) on \( H^p_L(X) \), where \( F \) is a bounded Borel function satisfying (4.1) or (4.2) and \( L \) is an injective, non-negative, self-adjoint operator on \( L^2(X) \) for which Davies-Gaffney estimates hold. Since self-adjoint operators on \( L^2(X) \) have the functional calculus for arbitrary bounded Borel functions \( \mathbb{R} \to \mathbb{C} \) without any regularity hypothesis, one expects that the regularity assumptions on \( F \) can be weakened when one asks about boundedness of \( F(L) \) on \( H^p_L(X) \) for some \( p \in (1, 2) \). This is actually true, as the interpolation procedure described in [40], Section 4.6.1] shows.
Definition 5.1. Let \( p \in [1, 2] \), \( q \in [2, \infty] \), \( s > 1/q \), and \( L \) be an injective, non-negative, self-adjoint operator on \( L^2(X) \) which fulfills Davies-Gaffney estimates. We say that \( L \) has an \( \mathcal{H}_q^s \) Hörmander calculus on \( H^p_L(X) \) if there exists a constant \( C > 0 \) such that
\[
\|F(L)\|_{H^p_L(X) \to H^p_L(X)} \leq C \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q_s}
\]
for all \( F \in \mathcal{H}_q^s := \{ F : (0, \infty) \to \mathbb{C} \text{ bounded Borel function such that } \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q_s} < \infty \}. \)

Since the Hörmander condition \( \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q_s} < \infty \) contains no information on the value of \( F(0) \), the value \( F(0) \) is not regarded in the so-called Hörmander class \( \mathcal{H}_q^s \). But this causes no problems as long as one studies injective operators.

The interpolation statement concerning the Hörmander calculus, adapted to our present situation, reads as follows (cf. [40, Corollary 4.84]).

Fact 5.2. Let \( L \) be an injective, non-negative, self-adjoint operator on \( L^2(X) \) such that Davies-Gaffney estimates DG\(_m\) hold for some \( m \geq 2 \). Assume that \( L \) has an \( \mathcal{H}_q^s \) Hörmander calculus on the Hardy space \( H^p_1(X) \) for some \( q \geq 2 \) and \( s > 1/q \). Then, for any \( \theta \in (0, 1) \), the operator \( L \) has an \( \mathcal{H}_q^s \) Hörmander calculus on \( [L^2(X), H^p_1(X)]_{\theta} \) whenever \( s_0 > \theta s \) and \( q_0 > q/\theta \).

With the help of this interpolation result, we obtain spectral multiplier theorems on the Hardy space \( H^p_L(X) \) for each \( p \in [1, 2] \). We also state a version including the Plancherel condition which yields a lower regularity order in the Hörmander condition.

Theorem 5.3. Let \( L \) be an injective, non-negative, self-adjoint operator on \( L^2(X) \) satisfying Davies-Gaffney estimates DG\(_m\) for some \( m \geq 2 \).

a) Fix \( p \in [1, 2] \). Let \( s > (D + 1)(1/p - 1/2) \) and \( 1/q < 1/p - 1/2 \). Then \( L \) has an \( \mathcal{H}_q^s \) Hörmander calculus on \( H^p_L(X) \), i.e. for every bounded Borel function \( F : (0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q_s} < \infty \), there exists a constant \( C > 0 \) such that
\[
\|F(L)\|_{H^p_L(X) \to H^p_L(X)} \leq C \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q_s}.
\]

b) Let \( p \in [1, 2] \) and \( s > D(1/p - 1/2) \). Then \( L \) has an \( \mathcal{H}_q^\infty \) Hörmander calculus on \( H^p_L(X) \), i.e. for every bounded Borel function \( F : (0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q_s} < \infty \), there exists a constant \( C > 0 \) such that
\[
\|F(L)\|_{H^p_L(X) \to H^p_L(X)} \leq C \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^\infty_s}.
\]

c) Assume further that \( L \) fulfills the Plancherel condition \([4.5]\) for some \( q_0 \in [2, \infty) \). Fix \( p \in [1, 2] \). Let \( s > \max\{D, 2/q_0\}(1/p - 1/2) \) and \( 1/q < 2/q_0(1/p - 1/2) \). Then \( L \) has an \( \mathcal{H}_q^s \) Hörmander calculus on \( H^p_L(X) \), i.e. for every bounded Borel function \( F : (0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^q_s} < \infty \), there exists a constant \( C > 0 \) such that
\[
\|F(L)\|_{H^p_L(X) \to H^p_L(X)} \leq C \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s}.
\]

Proof. Let \( p \in [1, 2] \). The assertion of part a) follows directly by combining Theorem 4.1 a) and Fact 5.2 with \( \theta := 2(1/p - 1/2) \). A similar reasoning using Theorem 4.1 b) and the embedding \( C^{s+\varepsilon} \hookrightarrow [C^{s_0}, C^{s_1}]_{\theta} \) (for \( \varepsilon > 0 \), \( s_0 < s_1 \), \( \theta \in (0, 1) \) and \( s = (1 - \theta)s_0 + \theta s_1 \)) gives part b).

Suppose that \( L \) additionally satisfies the Plancherel condition \([4.3]\) for some \( q_0 \in [2, \infty) \). Due to Theorem 1.2 \( L \) has an \( \mathcal{H}_q^s \) Hörmander calculus on \( H^p_1(X) \) for each \( s > \max\{D/2, 1/q_0\} \). Now Fact 5.2 with \( \theta := 2(1/p - 1/2) \) yields the assertion of c). 

If the operator \( L \) actually enjoys generalized Gaussian estimates \( \text{GGE}_m(p_0, p') \) for some \( p_0 \in [1, 2] \) and \( m \geq 2 \), then Theorem 3.7 ensures \( H^p_L(X) = L^p(X) \) for every \( p \in (p_0, 2] \). Therefore, we deduce from Theorem 5.3 spectral multiplier results on \( L^p(X) \) as well. The regularity assumptions
in our statement a) are weaker than those of [9] Theorem 1.1 and [39] Theorem 5.6 (or [40] Theorem 4.95), where \( s > (D + 1)/2, q = 2 \) and \( s > D(1/p - 1/2) + 1/2, q = 2 \) were required, respectively.

**Theorem 5.4.** Let \( L \) be a non-negative, self-adjoint operator on \( L^2(X) \) such that generalized Gaussian estimates \( \text{GGE}_{m}(p_0, p_0') \) hold for some \( p_0 \in [1, 2) \) and \( m \geq 2 \).

a) For fixed \( p \in (p_0, p_0') \) suppose that \( s > (D + 1)/p - 1/2 \) and \( 1/q < (1/p - 1/2) \). Then, for every bounded Borel function \( F : [0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_q^s} < \infty \), the operator \( F(L) \) is bounded on \( L^p(X) \). More precisely, there exists a constant \( C > 0 \) such that

\[
\| F(L) \|_{L^p \to L^p} \leq C \left( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_q^s} + |F(0)| \right).
\]

b) Let \( p \in (p_0, p_0') \) and \( s > D(1/p - 1/2) \). Then, for any bounded Borel function \( F : [0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{C^s} < \infty \), the operator \( F(L) \) is bounded on \( L^p(X) \). More precisely, there exists a constant \( C > 0 \) such that

\[
\| F(L) \|_{L^p \to L^p} \leq C \left( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{C^s} + |F(0)| \right).
\]

c) In addition, assume that \( L \) fulfills the Plancherel condition \( 4.3 \) for some \( q_0 \in [2, \infty) \). Fix \( p \in (p_0, p_0') \). Let \( s > \max\{D, 2/q_0\} (1/p - 1/2) \) and \( 1/q < 2/q_0 (1/p - 1/2) \). Then, for every bounded Borel function \( F : [0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_q^s} < \infty \), the operator \( F(L) \) is bounded on \( L^p(X) \). More precisely, there exists a constant \( C > 0 \) such that

\[
\| F(L) \|_{L^p \to L^p} \leq C \left( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_q^s} + |F(0)| \right).
\]

**Proof.** Let \( p \in (p_0, 2) \). We shall prove the three assertions simultaneously. Suppose that \( s > (D + 1)(1/p - 1/2) \) and \( 1/q < 1/p - 1/2 \) for the proof of a) and \( s > D(1/p - 1/2), q = \infty \) for b).

For the proof of part c) suppose that \( L \) fulfills the Plancherel condition \( 4.3 \) for some \( q_0 \in [2, \infty) \) as well as \( s > \max\{D, 2/q_0\} (1/p - 1/2) \) and \( 1/q < 2/q_0 (1/p - 1/2) \).

Since injectivity of \( L \) is not assumed, Theorem 5.3 cannot be applied directly. In order to overcome this difficulty, we use the concept of [42] Proposition 15.2 (see also [19] Theorem 3.8) that provides a decomposition of the space \( L^2(X) \) as the orthogonal sum of the closure of the range \( \mathcal{R}(L) \) of \( L \) and the null space \( N(L) \) of \( L \). The operator \( L \) then takes the form

\[
L = \begin{pmatrix} L_0 & 0 \\ 0 & 0 \end{pmatrix}
\]

with respect to the decomposition \( L^2(X) = \mathcal{R}(L) \oplus N(L) \), where \( L_0 \) is the part of \( L \) in \( \mathcal{R}(L) \), i.e. the restriction of \( L \) to \( \mathcal{D}(L_0) := \{ x \in \mathcal{R}(L) \cap \mathcal{D}(L) : Lx \in \mathcal{R}(L) \} \). But \( L_0 \) is injective on its domain, so that Theorem 5.3 applies to \( L_0 \). This approach was already made in [40] Section 4.6.1 and, as remarked in [40] Illustration 4.87, the decomposition and the interpolation result cited in Fact 5.2 can be combined. Hence, \( L_0 \) has an \( \mathcal{H}_q^s \) Hörmander calculus on \( H_{L_0}^p(X) \). Consider a bounded Borel function \( F : [0, \infty) \to \mathbb{C} \) with \( \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_q^s} < \infty \). Then it holds

\[
F(L) = \begin{pmatrix} (F|_{(0, \infty)})(L_0) & 0 \\ 0 & F(0) \end{pmatrix} I_{N(L)}
\]

on \( H_{L_0}^p(X) \cap L^2(X) \). Because of \( F|_{(0, \infty)} \in \mathcal{H}_q^s \), one has moreover

\[
\| (F|_{(0, \infty)})(L_0) \|_{H_{L_0}^p(X) \to H_{L_0}^p(X)} \leq \sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_q^s}
\]

as well as

\[
\| F(0) I_{N(L)} \|_{H_{L_0}^p(X) \to H_{L_0}^p(X)} \leq |F(0)|.
\]
Since, by Theorem \[\text{Lemma} 3.7\], the spaces \(H^p_{\theta_0}(X)\) and \(L^p(X)\) coincide, the statements a), b) and c) are proven for any \(p \in (p_0, 2)\).

Let \(p \in (2, p_0)\). Due to the self-adjointness of \(L\) on \(L^2(X)\), boundedness of spectral multipliers on \(L^p(X)\) follows by the case proved above and dualization. The claim for \(p = 2\) is trivial. \(\square\)

**Remark 5.5.** (1) The assertions of Theorem 5.4 remain even valid for open subsets \(\Omega\) of \(X\) provided that the ball appearing on the right-hand side of (1.2) is the one in \(X\). The reasoning is standard and relies on an observation quoted in [12, pp. 934-935] by adapting the arguments given in [26, p. 245] (see also [9, p. 452]). For this purpose, one has only to extend an operator \(T: L^p(\Omega) \to L^q(\Omega)\) by zero to the operator \(\tilde{T}: L^p(X) \to L^q(X)\) defined via

\[
\tilde{T}u(x) := \begin{cases} 
T(\mathbb{1}_{\Omega}u)(x) & \text{for } x \in \Omega \\
0 & \text{for } x \in X \setminus \Omega 
\end{cases} \quad (u \in L^p(X), \ \mu\text{-a.e. } x \in X)
\]

and observe that \(\|\tilde{T}\|_{L^p(\Omega) \to L^q(\Omega)} = \|T\|_{L^p(\Omega) \to L^q(\Omega)}\). The modified result allows to cover elliptic operators on irregular domains \(\Omega \subset \mathbb{R}^D\) as well (cf. e.g. [9, Section 2.1]).

(2) Of course, it is possible to apply the same method (complex interpolation with the functional calculus in \(L^2(X)\) and coincidence of \(H^p_L(X)\) and \(L^p(X)\)) also with Theorem 4.3 as a starting point. We do not go into details here.

6. PROOFS OF SOME AUXILIARY RESULTS

In this section, we proof the Lemmata 2.5, 2.7, 4.7 and 4.8.

**Proof of Lemma 2.5** a) In view of Fact 2.4, there are constants \(b, C > 0\) such that

\[
\| 1_{B(x,r)} e^{-L} 1_{B(y,r)} \|_{L^p \to L^q} \leq C |B(x,r)|^{-\left(\frac{1}{p} - \frac{1}{q}\right) \left(\frac{1}{2} - \frac{1}{p}\right)} \exp \left(-b \left(\frac{d(x,y)}{r_{y}}\right)^{\frac{m}{m-1}}\right)
\]

for all \(x, y \in X\) and \(z \in \mathbb{C}\) with \(\text{Re } z > 0\). By Fact 2.3 (with \(T := (|z|/\text{Re } z)^{-D(1/p-1/q)} e^{-L}\)), one finds \(b', C' > 0\) such that

\[
\| 1_{B(z,\rho_{x,z})}^{-\frac{1}{2}} T 1_{B(z,\rho_{x,z})} \|_{L^p \to L^q} \leq C' \exp \left(-b' \left(\frac{\text{dist}(B_1, B_2)}{\rho_{z}}\right)^{\frac{m}{m-1}}\right) \quad (6.1)
\]

for all balls \(B_1, B_2 \subset X\) and all \(z \in \mathbb{C}\) with \(\text{Re } z > 0\), where \(\rho_{x,z} := |B(\cdot, r_z)|\). Let \(r > 0\) be fixed. The doubling property leads to

\[
v_r(x) \lesssim \left(1 + \frac{r}{\rho_{z}}\right)^{D} v_{r_z}(x)
\]

for every \(x \in X\) and \(z \in \mathbb{C}\) with \(\text{Re } z > 0\). Now choose arbitrary \(x, y \in X\) with \(d(x,y) \geq 4r\) and consider the balls \(B_1 := B(x,r)\) and \(B_2 := B(y,r)\). Then it holds

\[
\text{dist}(B_1, B_2) = d(x,y) - 2r \geq \frac{1}{2} d(x,y).
\]

By inserting \(B_1, B_2\) into (6.1) and collecting the estimates above together, one arrives at

\[
\| 1_{B(x,r)} e^{-L} 1_{B(y,r)} \|_{L^p \to L^q} \lesssim \left(1 + \frac{r}{\rho_{z}}\right)^{D} \exp \left(-b' \left(\frac{d(x,y)}{2\rho_{z}}\right)^{\frac{m}{m-1}}\right).
\]

Since \(v_{r}(x) \equiv v_{r}(z)\) for all \(z \in B(x,r)\) (cf. Fact 2.1), one obtains the desired estimate

\[
\| 1_{B(x,r)} e^{-L} 1_{B(y,r)} \|_{L^p \to L^q} \leq C' |B(x,r)|^{-\left(\frac{1}{p} - \frac{1}{q}\right) \left(\frac{1}{2} - \frac{1}{p}\right)} \exp \left(-b' \left(\frac{d(x,y)}{\rho_{z}}\right)^{\frac{m}{m-1}}\right).
\]
for all \( r > 0, z \in \mathbb{C} \) with \( \text{Re} \, z > 0 \), and \( x, y \in X \) with \( d(x, y) \geq 4r \). By the cost of changing the constants \( b', C' \), one is able to remove this restriction on \( d(x, y) \).

b) Our approach mimics that of [13] (i)⇒(3), p. 359]. Observe that it suffices to prove the statement only for every \( k \in \mathbb{N} \setminus \{1\} \). With the help of [13] Lemma 3.4, we can write for each \( k \in \mathbb{N} \setminus \{1\}, r > 0, x \in X \), and \( z \in \mathbb{C} \) with \( \text{Re} \, z > 0 \)

\[
\left\| \mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{A(x,r,k)} \right\|_{L^p \to L^q} \leq \int_X \left\| \mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)} \right\|_{L^p \to L^q} \left\| \mathbb{1}_{A(x,r,k)} \right\|_{L^q \to L^q} v_r(y)^{-1} \, d\mu(y)
\]

By exploiting the bound from part a), we continue our estimation

\[
\lesssim |B(x,r)|^{-\left(\frac{k-1}{p} - \frac{1}{q}\right)} \left(1 + \frac{r}{r_z}\right) \frac{D(\frac{k-1}{p} - \frac{1}{q})}{\text{Re} \, z} \int_{B(x,(k+2)r) \setminus B(x,(k-1)r)} \exp \left(-b' \left(\frac{d(x,y)}{r_z}\right) \left(\frac{m}{m-1}\right)^{\frac{m}{m-1}} \right) v_r(y)^{-1} \, d\mu(y).
\]

Using \( d(x,y) \geq (k-1)r \geq kr/2 \) as well as \( v_r(y)^{-1} \lesssim (k+2)^D v_{(k+2)r}(y)^{-1} \) leads to

\[
\lesssim |B(x,r)|^{-\left(\frac{k-1}{p} - \frac{1}{q}\right)} \left(1 + \frac{r}{r_z}\right) \frac{D(\frac{k-1}{p} - \frac{1}{q})}{\text{Re} \, z} \int_{B(x,(k+2)r) \setminus B(x,(k-1)r)} \exp \left(-2 - \frac{m}{m-1} b' \left(\frac{kr}{r_z}\right)^{\frac{m}{m-1}} \right) (k+2)^D v_{(k+2)r}(y)^{-1} \, d\mu(y)
\]

where the last inequality is thanks to [2.3]. This proves the statement. \( \square \)

**Proof of Lemma 2.7.** Let \( K \in \mathbb{N} \) and \( t > 0 \) be arbitrary. The Cauchy formula gives the representation

\[
(tL)^K e^{-tL} = t^K \frac{(-1)^K K!}{2\pi i} \int_{|z-t|=\eta t} e^{-zL} \frac{dz}{(z-t)^{K+1}},
\]

where \( \eta := 1/2 \sin(\theta/2) \) for some \( \theta \in (0, \pi/2) \). Note that the choice of \( \eta \) ensures that the ball \( \{z \in \mathbb{C} : |z-t| \leq \eta t\} \) is contained in the sector \( \Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\text{arg} \, z| < \theta\} \). According to Lemma 2.5, it holds for every \( x, y \in X \):

\[
\left\| \mathbb{1}_{B(x,t^{1/m})} (tL)^K e^{-tL} \mathbb{1}_{B(y,t^{1/m})} \right\|_{L^2 \to L^2} \leq t^K \frac{K!}{2\pi} \int_{|z-t|=\eta t} \left\| \mathbb{1}_{B(x,t^{1/m})} e^{-zL} \mathbb{1}_{B(y,t^{1/m})} \right\|_{L^2 \to L^2} \frac{|dz|}{|z-t|^{K+1}}.
\]
where \( r_z := (\text{Re} \ z)^{1/m}|z|/\text{Re} \ z \). Due to \( \text{Re} \ z \in [(1 - \eta)t, (1 + \eta)t] \) and \( 1 \leq |z|/\text{Re} \ z \leq 1/\cos \theta \) for all \( z \) belonging to the integration path, we have \( r_z \approx t^{1/m} \) with implicit constants depending only on \( \theta \) or \( m \). Thus, we can finish our estimation as follows

\[
\leq t^{K} \frac{K!}{2\pi} \exp \left( -b' \left( \frac{d(x, y)}{t^{1/m}} \right)^{m-1} \right) \frac{1}{(\eta t)^{K+1}} = K! \frac{1}{\eta^K} \exp \left( -b' \left( \frac{d(x, y)}{t^{1/m}} \right)^{m-1} \right).
\]

\( \square \)

**Proof of Lemma 4.7.** It suffices to check (4.8) only for each \( i, j \in \mathbb{N} \setminus \{1\} \) with \( |j - i| > 3 \) since otherwise (4.8) is valid by the spectral theorem after choosing appropriate constants. Due to the self-adjointness of \( L \), one can swap \( i \) and \( j \) in the term on the left-hand side of (4.8). Hence, it will be enough to show the assertion for every \( i, j \in \mathbb{N} \setminus \{1\} \) with \( j - i > 3 \). By applying [13, Lemma 3.4], (4.7), and the doubling property, we get for each \( r > 0 \) and each \( x \in X \):

\[
\| \mathbb{1}_{U_j(B(x, r))} F(L)(I - e^{-r^m L})^M \mathbb{1}_{U_i(B(x, r))} \|_{L^2 \to L^2} \leq \int_X \left\| \mathbb{1}_{U_j(B(x, r))} F(L)(I - e^{-r^m L})^M \mathbb{1}_{B(z, r)} \|_{L^2 \to L^2} \right\| \frac{d\mu(z)}{|B(z, r)|}.
\]

In the second step we covered \( U_j(B(x, r)) \) by dyadic annuli around the point \( z \). Here, we used, among other things, the elementary inequalities

\[
|2^\alpha - 2^\beta| \geq 2^{(\alpha-\beta)-1} \quad \text{and} \quad 2^\alpha + 2^\beta \leq 2^{\alpha+\beta+1} \quad (6.2)
\]

which are valid for each \( \alpha, \beta \in \mathbb{N}_0 \) with \( \alpha \neq \beta \). With the help of

\[
\sum_{\nu=j-i-3}^{j+i+1} 2^{-\nu \delta} = 2^{3\delta} 2^{-(j-i)\delta} \sum_{\nu=0}^{2i+4} 2^{-\eta \delta} \leq 2^{-(j-i)\delta}
\]

and Fact 2.1 we finish our estimation as follows

\[
\| \mathbb{1}_{U_j(B(x, r))} F(L)(I - e^{-r^m L})^M \mathbb{1}_{U_i(B(x, r))} \|_{L^2 \to L^2} \leq C_F 2^{-(j-i)\delta} \int_{B(x, 2^{i+1}r)} 2^{(i+1)D} \frac{d\mu(z)}{|B(z, 2^{i+1}r)|} \leq C_F 2^{iD} 2^{-(j-i)\delta}.
\]

\( \square \)

**Proof of Lemma 4.8.** Let \( K, M \in \mathbb{N} \), \( r > 0 \), and \( x \in X \). At the beginning, we note that the operator \( P_{m, M, r}(L) \) is bounded on \( L^2(X) \):

\[
\| P_{m, M, r}(L) \|_{L^2 \to L^2} \leq r^{-m} \int_r^{\sqrt{2}r} s^{m-1} \| I - e^{-s^m L} \|_{L^2 \to L^2}^M \ ds \leq r^{-m} \int_r^{\sqrt{2}r} s^{m-1} \| I - e^{-s^m L} \|_{L^2 \to L^2}^M \ ds = \frac{2^M}{m}.
\]
With analogous arguments as in the proof of Lemma 4.7 it is enough to verify (4.10) for each \( i, j \in \mathbb{N}_0 \) with \( j - i > 6 \). To this purpose, fix \( k \in \{1, \ldots, M\} \) and \( s \in [r, \sqrt{2r}] \) for a moment. We shall establish the estimate

\[
\left\| \mathbbm{1}_{U_j(B(x,r))} e^{-ksL} \mathbbm{1}_{U_i(B(x,r))} \right\|_{L^2 \to L^2} \leq C \exp \left( -b(2^{j-1} - 2^{i+2}) \right)
\]

(6.3)

for some constants \( b, C > 0 \) depending only on \( m, M \) and the constants in the Davies-Gaffney or doubling condition, but not on the other parameters.

From the Davies-Gaffney estimates \( DG_m \) we obtain for each \( y \in X \):

\[
\left\| \mathbbm{1}_{B(x,r)} e^{-ksL} \mathbbm{1}_{B(y,r)} \right\|_{L^2 \to L^2} \leq \left\| \mathbbm{1}_{B(x,k^{1/m}s)} e^{-k^{1/m}sL} \mathbbm{1}_{B(y,k^{1/m}s)} \right\|_{L^2 \to L^2}
\]

\[
\lesssim \exp \left( -b \left( \frac{d(x,y)}{k^{1/m}s} \right)^\frac{m}{m-1} \right) \leq \exp \left( -b \left( \frac{2M}{r} \right)^{\frac{m}{m-1}} \right).
\]

Therefore, Fact 2.3 yields for any \( \nu \in \mathbb{N} \):

\[
\left\| \mathbbm{1}_{A(x,r,\nu)} e^{-ksL} \mathbbm{1}_{B(x,r)} \right\|_{L^2 \to L^2} \lesssim \exp \left( -b \nu^{\frac{m}{m-1}} \right) \leq e^{-b \nu}.
\]

By applying [13, Lemma 3.4] and the doubling property, we deduce

\[
\left\| \mathbbm{1}_{U_j(B(x,r))} e^{-ksL} \mathbbm{1}_{U_i(B(x,r))} \right\|_{L^2 \to L^2}
\]

\[
\lesssim \int_X \left\| \mathbbm{1}_{U_j(B(x,r))} e^{-ksL} \mathbbm{1}_{U_i(B(x,r))} \right\|_{L^2 \to L^2} \left\| \mathbbm{1}_{B(z,r)} \mathbbm{1}_{U_i(B(x,r))} \right\|_{L^2 \to L^2} \frac{d\mu(z)}{B(z,r)}
\]

\[
\lesssim \int_{B(x,2^{i+1}r)} \sum_{\nu=2^{i-1} - 2^{i+1}}^{2^{i+1} + 2^{i+1}} \left\| \mathbbm{1}_{A(z,r,\nu)} e^{-ksL} \mathbbm{1}_{B(z,r)} \right\|_{L^2 \to L^2} \frac{d\mu(z)}{B(z,r)}
\]

\[
\lesssim \int_{B(x,2^{i+1}r)} \sum_{\nu=2^{i-1} - 2^{i+1}}^{2^{i+1} + 2^{i+1}} e^{-b\nu} 2^{(i+1)D} \frac{d\mu(z)}{B(z,2^{i+1}r)}
\]

With the help of

\[
\sum_{\nu=2^{i-1} - 2^{i+1}}^{2^{i+1} + 2^{i+1}} e^{-b\nu} \leq \exp \left( -b(2^{j-1} - 2^{i+1}) \right) \sum_{n=0}^{\infty} e^{-bn} = \frac{1}{1 - e^{-b}} \exp \left( -b(2^{j-1} - 2^{i+1}) \right)
\]

and Fact 2.1 we finally arrive at the claimed estimate (6.3)

\[
\left\| \mathbbm{1}_{U_j(B(x,r))} e^{-ksL} \mathbbm{1}_{U_i(B(x,r))} \right\|_{L^2 \to L^2} \lesssim 2^{jD} \exp \left( -b(2^{j-1} - 2^{i+1}) \right) \lesssim \exp \left( -b(2^{j-1} - 2^{i+2}) \right)
\]

In view of the formula

\[
(I - e^{-sL})^M = \sum_{k=0}^{\infty} \binom{M}{k} (-1)^k e^{-ksL}
\]

and the disjointness of \( U_i(B(x,r)) \) and \( U_j(B(x,r)) \), we get from (6.3)

\[
\left\| \mathbbm{1}_{U_j(B(x,r))} P_{m,M,r}(L) \mathbbm{1}_{U_i(B(x,r))} \right\|_{L^2 \to L^2}
\]

\[
\leq \sum_{k=0}^{\infty} \binom{M}{k} r^{-m} \int r^{s^{-1}} s^{m-1} \left\| \mathbbm{1}_{U_j(B(x,r))} e^{-ksL} \mathbbm{1}_{U_i(B(x,r))} \right\|_{L^2 \to L^2} ds
\]

\[
\lesssim \sum_{k=1}^{M} \binom{M}{k} r^{-m} \int r^{s^{-1}} s^{m-1} ds \exp \left( -b(2^{j-1} - 2^{i+2}) \right)
\]

\[
\lesssim \exp \left( -b(2^{j-1} - 2^{i+2}) \right).
\]

(6.4)

Due to the inequality (6.2), the assertion (4.10) for \( K = 1 \) is verified.
The general statement follows by induction, once (4.10) is checked for $K = 2$. This will be achieved by adapting the proof of Lemma 2.3 to the present situation. For the rest of the proof we abbreviate $P := P_{m,M,r}(L)$. Let $f \in L^2(X)$ with $\text{supp} f \subset U_j(B)$ and $\|f\|_{L^2} = 1$ be fixed. We consider the set

$$G := \{ y \in X : \text{dist}(y, U_j(B)) < \frac{1}{2} \text{dist}(U_i(B), U_j(B)) \}$$

$$= \{ y \in X : (2^{j-2} + 2^{-i})r < d(x,y) < (5 \cdot 2^{j-2} - 2^{i-1})r \}$$

and analyze

$$\|1_{U_j(B)} P^2 f\|_{L^2} \leq \|P(1_G \cdot Pf)\|_{L^2(U_j(B))} + \|P(1_{X \setminus G} \cdot Pf)\|_{L^2(U_j(B))}$$

In order to estimate the first term on the right-hand side, we initially exploit the boundedness of $L$ that the off-diagonal estimate (6.4) is applicable, and then one utilizes the boundedness of $P$ on $L^2(X)$ and then cover the set $G$ by dyadic annuli in such a way as to enable us to apply (6.4):

$$\|P(1_G \cdot Pf)\|_{L^2(U_j(B))} \lesssim \|1_G \cdot Pf\|_{L^2} \leq \sum_{k = [\log_2(2^{j-2} + 2^{i-1})]}^{[\log_2(5 \cdot 2^{j-2} - 2^{i-1})] + 1} \|1_{U_k(B)} \cdot Pf\|_{L^2}$$

$$\lesssim \sum_{k = [\log_2(2^{j-2} + 2^{i-1})]}^{[\log_2(5 \cdot 2^{j-2} - 2^{i-1})] + 1} e^{-b(2k-1 - 2i+2)} \|f\|_{L^2}$$

$$\leq \left( (\log_2(5 \cdot 2^{j-2} - 2^{i-1}) + 3 - \log_2(2^{j-2} + 2^{i-1})) \right) e^{-b(2^{j-2} + 2^{i-1})/4 - 2^{i+2}}$$

$$\lesssim e^{-b(2^{j-4} - 2^{i+2})}.$$

Thanks to (6.2), the latter is bounded by a constant times $\exp(-b2^{j-i})$, as desired.

The second summand $\|P(1_{X \setminus G} \cdot Pf)\|_{L^2(U_j(B))}$ can be treated in an analogous manner. One has only to interchange the sequence of the arguments. At first, one covers $X \setminus G$ by dyadic annuli, so that the off-diagonal estimate (6.4) is applicable, and then one utilizes the boundedness of $P$ on $L^2(X)$ as well as (6.2). This gives a similar estimate as before and finishes the proof.

References

[1] G. Alexopoulos: Spectral multipliers on Lie groups of polynomial growth. Proc. Am. Math. Soc. 120, No. 3, 973-979, 1994.

[2] P. Auscher: On necessary and sufficient conditions for $L^p$-estimates of Riesz transforms associated with elliptic operators on $\mathbb{R}^n$ and related estimates. Mem. Am. Math. Soc. 871, 2007.

[3] P. Auscher, X.T. Duong, and A. McIntosh: Boundedness of Banach space valued singular integral operators and Hardy spaces. Unpublished preprint, 2005.

[4] P. Auscher and J.M. Martell: Weighted norm inequalities, off-diagonal estimates and elliptic operators. III: Harmonic analysis of elliptic operators. J. Funct. Anal. 241, No. 2, 703-746, 2006.

[5] P. Auscher and J.M. Martell: Weighted norm inequalities, off-diagonal estimates and elliptic operators. I: General operator theory and weights. Adv. Math. 212, No. 1, 225-276, 2007.

[6] P. Auscher and J.M. Martell: Weighted norm inequalities, off-diagonal estimates and elliptic operators. II: Off-diagonal estimates on spaces of homogeneous type. J. Evol. Equ. 7, No. 2, 265-316, 2007.

[7] P. Auscher and J.M. Martell: Weighted norm inequalities, off-diagonal estimates and elliptic operators. IV: Riesz transforms on manifolds and weights. Math. Z. 260, No. 3, 527-539, 2008.

[8] P. Auscher, A. McIntosh, and E. Russ: Hardy spaces of differential forms on Riemannian manifolds. J. Geom. Anal. 18, No. 1, 192-248, 2008.

[9] S. Blunck: A Hörmander-type spectral multiplier theorem for operators without heat kernel. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2, No. 3, 449-459, 2003.

[10] S. Blunck: Generalized Gaussian estimates and Riesz means of Schrödinger groups. J. Aust. Math. Soc. 82, No. 2, 149-162, 2007.

[11] S. Blunck and P.C. Kunstmann: Weighted norm estimates and maximal regularity. Adv. Differ. Equ. 7, No. 12, 1513-1532, 2002.
[12] S. Blunck and P.C. Kunstmann: Calderón-Zygmund theory for non-integral operators and the $H^\infty$ functional calculus. Rev. Mat. Iberoam. 19, No. 3, 919-942, 2003.

[13] S. Blunck and P.C. Kunstmann: Generalized Gaussian estimates and the Legendre transform. J. Oper. Theory 53 (2), 351-365, 2005.

[14] J. Cao and D. Yang: Hardy Spaces $H^p_\alpha(\mathbb{R}^n)$ Associated to Operators Satisfying k-Davies-Gaffney Estimates Preprint, 2011. URL: http://arxiv.org/abs/1107.5365.

[15] P. Chen, E.M. Ouhabaz, A. Sikora, and L. Yan: Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner-Riesz means. Preprint, 2012. URL: http://arxiv.org/abs/1202.4052.

[16] M. Christ: $L^p$ bounds for spectral multipliers on nilpotent groups. Trans. Am. Math. Soc. 328, No. 1, 73-81, 1991.

[17] R.R. Coifman and G. Weiss: Analyse harmonique non-commutative sur certains espaces homogènes. Lecture Notes in Mathematics 242, Springer, 1971.

[18] T. Coulhon and A. Sikora: Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. Proc. Lond. Math. Soc. (3) 96, No. 2, 507-544, 2008.

[19] M. Cowling, I. Doust, A. McIntosh, and A. Yagi: Banach space operators with a bounded $H^\infty$ functional calculus. J. Aust. Math. Soc., Ser. A 60, No. 1, 51-89, 1996.

[20] M. Cowling and A. Sikora: A spectral multiplier theorem for a sublaplacian on SU(2). Math. Z. 238, No. 1, 1-36, 2001.

[21] E.B. Davies: Heat kernel bounds, conservation of probability and the Feller property. J. Anal. Math. 58, 99-119, 1992.

[22] E.B. Davies: Uniformly elliptic operators with measurable coefficients. J. Funct. Anal. 132, 141-169, 1995.

[23] E.B. Davies: Limits on $L^p$ regularity of self-adjoint elliptic operators. J. Differ. Equations 135, No. 1, 83-102, 1997.

[24] Q. Deng, Y. Ding, and X. Yao: Characterizations of Hardy spaces associated to higher order elliptic operators. J. Funct. Anal. 263, No. 3, 604-674, 2012.

[25] X.T. Duong and J. Li: Hardy spaces associated with operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus. Preprint, 2010.

[26] X.T. Duong and A. McIntosh: Singular integral operators with non-smooth kernels on irregular domains. Rev. Mat. Iberoam. 15, No. 2, 233-265, 1999.

[27] X.T. Duong, E.M. Ouhabaz, and A. Sikora: Plancherel-type estimates and sharp spectral multipliers. J. Funct. Anal. 196, 443-485, 2002.

[28] X.T. Duong and L.X. Yan: Spectral multipliers for Hardy spaces associated with non-negative self-adjoint operators satisfying Davies-Gaffney estimates. J. Math. Soc. Japan 63, No. 1, 295-319, 2011.

[29] J. Dziubański: Spectral multiplier theorem for $H^1$ spaces associated with some Schrödinger operators. Proc. Am. Math. Soc. 127, No. 12, 3605-3613, 1999.

[30] J. Dziubański and M. Preisner: Remarks on spectral multiplier theorems on Hardy spaces associated with semigroups of operators. Rev. Unión Mat. Argent. 50, No. 2, 201-215, 2009.

[31] D. Frey: Paraproducts via $J$. Dziubański and M. Preisner: Remarks on spectral multiplier theorems on Hardy spaces associated with semigroups of operators. Rev. Unión Mat. Argent. 50, No. 2, 201-215, 2009.

[32] D. Frey: Paraproducts via $H^\infty$-functional calculus. To appear in Rev. Mat. Iberoam. URL: http://arxiv.org/abs/1107.4348.

[33] M.P. Gaffney: The conservation property of the heat equation on Riemannian manifolds. Commun. Pure Appl. Math. 12, 1-11, 1959.

[34] W. Hebisch: Functional calculus for slowly decaying kernels. Preprint, 1995. URL: http://www.math.uni.wroc.pl/~hebisch/.

[35] S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea, and L.X. Yan: Hardy spaces associated with non-negative self-adjoint operators satisfying Davies-Gaffney estimates. Mem. Amer. Math. Soc, Vol. 214, No. 1007, 2011.

[36] S. Hofmann and J.M. Martell: $L^p$ bounds for Riesz transforms and square roots associated with second order elliptic operators. Publ. Mat. 47, No. 2, 497-515, 2003.

[37] S. Hofmann and S. Mayboroda: Hardy and BMO spaces associated with divergence form elliptic operators. Math. Ann. 344, No. 1, 37-116, 2009.

[38] S. Hofmann, S. Mayboroda, and A. McIntosh: Second order elliptic operators with complex bounded measurable coefficients in $L^p$, Sobolev and Hardy spaces. Ann. Sci. Éc. Norm. Supér. (4) 44, No. 5, 723-800, 2011.

[39] L. Hörmander: Estimates for translation invariant operators in $L^p$ spaces. Acta Math. 104, 93-140, 1960.

[40] C. Kriegler: Hörmander Type Functional Calculus and Square Function Estimates. Preprint, 2012. URL: http://arxiv.org/abs/1201.4830.

[41] C. Kriegler: Spectral multipliers, $R$-bounded homomorphisms, and analytic diffusion semigroups. Dissertation, Universität Karlsruhe (TH), 2009. URL: http://digbib.ubka.uni-karlsruhe.de/volltexte/1000015866.

[42] P.C. Kunstmann and M. Uhl: $L^p$-spectral multipliers for some elliptic systems. Submitted, 2012.
[42] P.C. Kunstmann and L. Weis: Maximal $L_p$-regularity for parabolic equations, Fourier multiplier theorems and $H^\infty$-functional calculus. Functional analytic methods for evolution equations, Lecture Notes in Mathematics 1855, Springer, 65-311, 2004.

[43] G. Mauceri and S. Meda: Vector-valued multipliers on stratified groups. Rev. Mat. Iberoam. 6, No. 3-4, 141-154, 1990.

[44] G. Schreieck and J. Voigt: Stability of the $L_p$-spectrum of Schrödinger operators with form-small negative part of the potential. “Functional analysis”. Proceedings of the Essen conference, 1991. Marcel-Dekker, New York. Lect. Notes Pure Appl. Math. 150, 95-105, 1994.

[45] E.M. Stein: Singular integrals and differentiability of functions. Princeton Univ. Press, 1970.

[46] E.M. Stein and G. Weiss: On the theory of harmonic functions of several variables, I. The theory of $H^p$-spaces. Acta Math. 103, 25-62, 1960.

[47] S. Thangavelu: Summability of Hermite expansions. I, II. Trans. Am. Math. Soc. 314, No. 1, 119-142, 143-170, 1989.

[48] H. Triebel: Theory of function spaces. Monographs in Mathematics, Vol. 78, Birkhäuser, 1983.

[49] M. Uhl: Spectral multiplier theorems of Hörmander type via generalized Gaussian estimates. Dissertation, Karlsruher Institut für Technologie (KIT), 2011. URL: http://digbib.ubka.uni-karlsruhe.de/volltexte/1000025107

Department of Mathematics, Karlsruhe Institute of Technology (KIT), Kaiserstr. 89, 76128 Karlsruhe, Germany

E-mail address: peer.kunstmann@kit.edu, matthias.uhl@kit.edu