MERIDIAN SURFACE OF WEINGARTEN TYPE IN
4-DIMENSIONAL EUCLIDEAN SPACE \( \mathbb{E}^4 \)

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Abstract. In this paper, we study meridian surfaces of Weingarten type in Euclidean 4-space \( \mathbb{E}^4 \). We give the necessary and sufficient conditions for a meridian surface in \( \mathbb{E}^4 \) to become Weingarten type.

1. INTRODUCTION

A surface \( M \) in \( \mathbb{E}^n \) is called Weingarten surface if there exist a non-trivial function
\[
\Psi(K, H) = 0
\]
(1.1)
between the Gauss curvature \( K \) and mean curvature \( H \) of the surface \( M \). The existence of a non-trivial functional relation \( \Psi(K, H) = 0 \) on a surface \( M \) parametrized by a patch \( X(u, v) \) is equivalent to the vanishing of the corresponding Jacobian determinant, namely
\[
\left| \frac{\partial(K, H)}{\partial(u, v)} \right| = 0.
\]
(1.2)
The condition (1.2) that must be satisfied for the Weingarten surface \( M \) leads to
\[
K_u H_v - K_v H_u = 0
\]
(1.3)
with subscripts denoting partial derivatives.

For the study of these surfaces, W. Kühnel [8] investigated ruled Weingarten surface in a Euclidean 3-space \( \mathbb{E}^3 \). Further, D. W. Yoon [12] classified ruled linear Weingarten surface in \( \mathbb{E}^3 \). Meanwhile, F. Dillen and W. Kühnel [4] and Y. H. Kim and D. W. Yoon [7] gave a classification of ruled Weingarten surfaces in a Minkowski 3-space \( \mathbb{E}^3_1 \). Recently, M. I. Munteanu and I. Nistor [11] and R. Lopez ([9, 10]) studied polynomial translation Weingarten surfaces in a Euclidean 3-space.

The study of meridian surfaces in \( \mathbb{E}^4 \) was first introduced by G. Ganchev and V. Milousheva (See, [5] and [6]). They construct a surface \( M^2 \) in \( \mathbb{E}^4 \) in the following way:
\[
M^2 : X(u, v) = f(u) r(v) + g(u) e_4, \quad u \in I, v \in J
\]
where \( f = f(u), g = g(u) \) are non-zero smooth functions, defined in an interval \( I \subset \mathbb{R} \), such that \((f'(u))^2 + (g'(u))^2 = 1, u \in I \) and \( r = r(v) \) \((v \in J \subset \mathbb{R}) \) is a curve on \( S^2(1) \) parameterized by the arc-length and \( e_4 \) is the fourth vector of the standard orthonormal frame in \( \mathbb{E}^4 \). See also [1] for the classification of meridian surfaces in 4-dimensional Euclidean space \( \mathbb{E}^4 \) which have pointwise 1-type Gauss map.

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In this paper, we study meridian surfaces of Weingarten type in 4-dimensional Euclidean space $\mathbb{E}^4$. We proved the following theorem:

**Main Theorem.** Let $M^2$ be a meridian surface given with the parametrization (3.2). Then $M^2$ is a Weingarten surface if and only if $M^2$ is one of the following surfaces:

i) a planar surface lying in the constant 3-dimensional space spanned by $\{x, y, n_2\}$,

ii) a developable ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$,

iii) a developable ruled surface in a 4-dimensional Euclidean space $\mathbb{E}^4$,

iv) a surface given with the surface patch

$$X(u, v) = \left(\frac{\cos(au + ac_1)}{a} + c_2\right) r(v) +$$

$$+ \left(\frac{2(\sin(au + ac_1) - 1) \sqrt{1 + \sin(au + ac_1)}}{\cos(au + ac_1)}\right) e_4,$$

iv) a surface given with the surface patch

$$X(u, v) = \pm \frac{1}{2} \left(\frac{-a}{(e_1^2)^2} \left((e_1^2)^2 (e_1^2)^2 + 1\right) \right) r(v)$$

$$\pm \frac{1}{2} \left(\frac{4b^2(e_1^2)^2 + a(e_1^2)^4 - 2a(e_1^2)^2(e_1^2)^2 + a}{b^2(e_1^2)^2(e_1^2)^2}(e_1^2)^2 + 1\right) e_4$$

where $a, b, c, c_1, c_2$ are real constants.

## 2. Basic Concepts

Let $M$ be a smooth surface in $\mathbb{E}^n$ given with the patch $X(u, v) : (u, v) \in D \subset \mathbb{E}^2$. The tangent space to $M$ at an arbitrary point $p = X(u, v)$ of $M$ span $\{X_u, X_v\}$. In the chart $(u, v)$ the coefficients of the first fundamental form of $M$ are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle,$$

where $\langle, \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u, v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^\perp M$ where $T_p^\perp M$ is the orthogonal component of $T_pM$ in $\mathbb{E}^n$.

Let $\chi(M)$ and $\chi^1(M)$ be the space of the smooth vector fields tangent to $M$ and the space of the smooth vector fields normal to $M$, respectively. Given any local vector fields $X_1, X_2$ tangent to $M$, consider the second fundamental map $h : \chi(M) \times \chi(M) \rightarrow \chi^1(M)$,

$$h(X_i, X_j) = \nabla^\chi X_i \cdot X_j - \nabla^\chi X_j \cdot X_i \quad 1 \leq i, j \leq 2,$$

where $\nabla$ and $\nabla^\chi$ are the induced connection of $M$ and the Riemannian connection of $\mathbb{E}^n$, respectively. This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal frame field $\{N_1, N_2, \ldots, N_n\}$ of $M$, recall the shape operator $A : \chi^1(M) \times \chi(M) \rightarrow \chi(M)$,

$$A_{N_i}X_j = -\langle \nabla^\chi X_j, N_i \rangle T, \quad X_j \in \chi(M).$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_i}X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = a_{ij}^k, \quad 1 \leq i, j \leq 2; \quad 1 \leq k \leq n - 2.$$
where \( c_{ij}^k \) are the coefficients of the second fundamental form.

The equation (2.2) is called Gaussian formula, and

\[
(2.5) \quad h(X_i, X_j) = \sum_{k=1}^{n-2} c_{ij}^k N_k, \quad 1 \leq i, j \leq 2.
\]

Then the Gauss curvature \( K \) of a regular patch \( X(u, v) \) is given by

\[
(2.6) \quad K = \frac{1}{W^2} \sum_{k=1}^{n-2} (c_{11}^k c_{22}^k - (c_{12}^k)^2).
\]

Further, the mean curvature vector of a regular patch \( X(u, v) \) is given by

\[
(2.7) \quad \vec{H} = \frac{1}{2W^2} \sum_{k=1}^{n-2} (c_{11}^k G + c_{22}^k E - 2c_{12}^k F) N_k,
\]

where \( E, F, G \) are the coefficients of the first fundamental form and \( c_{ij}^k \) are the coefficients of the second fundamental form.

The norm of the mean curvature vector \( H = \left\| \vec{H} \right\| \) is called the mean curvature of \( M \). The mean curvature \( H \) and the Gauss curvature \( K \) play the most important roles in differential geometry for surfaces \cite{3}. Recall that a surface \( M \) is said to be flat (resp. minimal) if its Gauss curvature (resp. mean curvature vector) vanishes identically \cite{2}.

3. Meridian Surfaces in \( \mathbb{E}^4 \)

Let \( \{e_1, e_2, e_3, e_4\} \) be the standard orthonormal frame in \( \mathbb{E}^4 \), and \( S^2(1) \) be a 2-dimensional sphere in \( \mathbb{E}^3 = \text{span}\{e_1, e_2, e_3\} \), centered at the origin \( O \). We consider a smooth curve \( c : r = r(v), v \in J, J \subset \mathbb{R} \) on \( S^2(1) \), parameterized by the arc-length \( (r'(v))^2 = 1 \). We denote \( t(v) = r'(v) \) and consider the moving frame field \( \{t(v), n(v), r(v)\} \) of the curve \( c \) on \( S^2(1) \). With respect to this orthonormal frame field the following Frenet formulas hold good:

\[
(3.1) \quad \begin{align*}
  r'(v) &= t(v); \\
  t'(v) &= \kappa(v) n(v) - r(v); \\
  n'(v) &= -\kappa(v) t(v),
\end{align*}
\]

where \( \kappa \) is the spherical curvature of \( c \).

Let \( f = f(u), g = g(u) \) be non-zero smooth functions, defined in an interval \( I \subset \mathbb{R} \), such that \( (f'(u))^2 + (g'(u))^2 = 1, u \in I \). Now we construct a surface \( M^2 \) in \( \mathbb{E}^4 \) in the following way:

\[
(3.2) \quad M^2 : X(u, v) = f(u) r(v) + g(u) e_4, \quad u \in I, v \in J
\]

The surface \( M^2 \) lies on the rotational hypersurface \( M^3 \) in \( \mathbb{E}^4 \) obtained by the rotation of the meridian curve \( \alpha : u \to (f(u), g(u)) \) around the \( Oe_4 \)-axis in \( \mathbb{E}^4 \). Since \( M^2 \) consists of meridians of \( M^3 \), we call \( M^2 \) a meridian surface (see, \cite{5}).

The tangent space of \( M^2 \) is spanned by the vector fields:

\[
(3.3) \quad \begin{align*}
  X_u(u, v) &= f'(u) r(v) + g'(u) e_4; \\
  X_v(u, v) &= f(u) t(v),
\end{align*}
\]
and hence the coefficients of the first fundamental form of \( M^2 \) are \( E = 1; F = 0; G = f^2(u) \). Without loss of generality we can take \( g'(u) \neq 0 \). Taking into account (3.1), we calculate the second partial derivatives of \( X(u,v) \):

\[
\begin{align*}
X_{uu}(u,v) &= f''(u)r(v) + g''(u)e_4; \\
X_{uv}(u,v) &= f'(u)t(v); \\
X_{vv}(u,v) &= f(u)\kappa(v)n(v) - f(u)r(v).
\end{align*}
\]

Let us denote \( X = X_u, Y = \frac{\nabla X}{f} = t \) and consider the following orthonormal normal frame field of \( M^2 \):

\[
(3.5) \quad N_1 = n(v); \quad N_2 = -g'(u)r(v) + f'(u)e_4.
\]

Thus we obtain a positive orthonormal frame field \( \{X, Y, N_1, N_2\} \) of \( M^2 \). If we denote by \( \kappa_\alpha(u) \) the curvature of the meridian curve \( \alpha(u) \), i.e.

\[
(3.6) \quad \kappa_\alpha(u) = f'(u)g''(u) - g'(u)f''(u) = \frac{-f''(u)}{\sqrt{1 - f'^2(u)}}.
\]

Using \( (3.4) \) and \( (3.5) \) we can calculate the coefficients of the second fundamental form of \( X(u,v) \) as follows:

\[
\begin{align*}
c_{11}^1 &= 0, c_{22}^1 = f(u)\kappa(v), \\
c_{12}^1 &= c_{12}^2 = 0, \\
c_{21}^2 &= \kappa_\alpha(u), \\
c_{22}^2 &= f(u)g'(u).
\end{align*}
\]

**Lemma 1.** Let \( M^2 \) be a meridian surface given with the surface patch \( (3.2) \) then

\[
(3.8) \quad A_{N_1} = \begin{bmatrix} 0 & 0 \\ 0 & \kappa(v) \end{bmatrix}, \quad A_{N_2} = \begin{bmatrix} \kappa_\alpha(u) & 0 \\ 0 & g'(u) \end{bmatrix}.
\]

Further by the use of \( (2.6) \) and \( (2.7) \) with \( (3.7) \), the Gauss curvature is given by

\[
(3.9) \quad K = \frac{\kappa_\alpha(u)g'(u)}{f(u)}.
\]

and the mean curvature vector field of \( M^2 \) becomes

\[
(3.10) \quad \vec{H} = \frac{\kappa(v)}{2f(u)}N_1 + \frac{\kappa_\alpha(u)f(u) + g'(u)}{2f(u)}N_2.
\]

From the equation \( (3.10) \), we get the mean curvature of \( M^2 \)

\[
(3.11) \quad H = \frac{1}{2f(u)}\sqrt{\kappa(v)^2 + (\kappa_\alpha(u)f(u) + g'(u))^2}.
\]
4. Proof of the Main Theorem

Let $M^2$ be meridian surface given with the surface patch (3.2). Then differentiating $K$ and $H$ with respect to $u$ and $v$ one can get

\[ K_v = 0, \quad K_u = -\frac{\left(f(u)f'''(u) - f'(u)f''(u)\right)}{f(u)^2}, \]

\[ H_v = \frac{\kappa(v)\kappa'(v)}{2f(u)\sqrt{\kappa(v)^2 + (\kappa_\alpha(u)f(u) + g'(u))^2}}. \]

Suppose that $M^2$ is a Weingarten surface then by the use of equation (1.3), we get,

\[ -\kappa(v)\kappa'(v)\left(f(u)f'''(u) - f'(u)f''(u)\right) = 0. \]

Thus we distinguish the following cases:

Case I: $\kappa(v) = 0$;

Case II: $\kappa'(v) = 0$;

Case III: $f(u)f'''(u) - f'(u)f''(u) = 0$.

Let us consider these in turn;

Case I: Suppose $\kappa(v) = 0$, i.e. the curve $c$ is a great circle on $S^2(1)$. In this case $N_1 = \text{const}$, and $M^2$ is a planar surface lying in the constant 3-dimensional space spanned by $\{X, Y, N_2\}$. Particularly, if in addition $\kappa_\alpha(u) = 0$, i.e. the meridian curve lies on a straight line, then $M^2$ is a developable surface in the 3-dimensional space span $\{X, Y, N_2\}$.

Case II: Suppose $\kappa'(v) = 0$. This implies that $\kappa(v)$ is nonzero constant. Then we have the following subcases:

Case II(a): $\kappa_\alpha(u) = 0$. In this case $c$ is a circle on $S^2(1)$, then $M^2$ is a developable ruled surface in a 3-dimensional Euclidean space $\mathbb{R}^3$.

Case II(b): $\kappa_\alpha(u)$ is nonzero constant. In this case we obtain the following ordinary differential equation.

\[ \frac{-f''(u)}{\sqrt{1 - f'^2(u)}} = a. \]

Thus, the following expression is obtained from the solution of the differential equation (1.2)

\[ f(u) = \frac{\cos(au + ac_1)}{a} + c_2. \]

Further, using the condition $(f'(u))^2 + (g'(u))^2 = 1$ we get

\[ g(u) = \frac{2(\sin(au + ac_1) - 1)\sqrt{1 + \sin(au + ac_1)}}{\cos(au + ac_1)}. \]

Case III: Suppose $f(u)f'''(u) - f'(u)f''(u) = 0$. Then we have the following subcases:

Case III(a): $f''(u) = 0$. This implies that $\kappa_\alpha(u) = K = 0$, i.e. the meridian curve is part of a straight line and $M^2$ is a developable ruled surface. If in addition
\( \kappa(v) \neq \text{const}, \) i.e. \( c \) is not a circle on \( S^2(1) \), then \( M^2 \) is a developable ruled surface in \( \mathbb{R}^4 \) ([5]).

Case III(b): \( f''(u) \neq 0 \). In this case we obtain the following ordinary differential equation.

\[
(4.3) \quad f(u)f'''(u) - f'(u)f''(u) = 0
\]

Thus, the following expression is obtained from the solution of the differential equation:

\[
f(u) = \pm \frac{1}{2} \sqrt{-\frac{a}{(e^u b)(e^u c)^2}} \left( (e^u)^2 (e^u c)^2 + 1 \right).
\]

Further, using the condition \( (f'(u))^2 + (g'(u))^2 = 1 \) one can get

\[
g(u) = \pm \frac{1}{2} \sqrt{\frac{4b^2 (e^u b)^2 (e^u c)^2 + a (e^u b)^4 (e^u c)^4 - 2a (e^u b)^2 (e^u c)^2 + a}{b^2 (e^u b)^2 (e^u c)^2}}
\]

where \( a, b, c, c_1, c_2 \) are real constants. This completes the proof of the theorem.

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