RELATIVE RESOLUTION AND ITS APPLICATIONS

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Abstract. We present an introduction to the derived and relative resolutions of the moduli of stable maps. We discuss one application and mention a few problems.

1. Introduction

We aim to provide a relative resolution of a singular moduli, preferably by geometric method, not by any algorithm. Our primary example is the Kontsevich moduli space $\overline{M}_g(\mathbb{P}^n, d)$ of degree $d$ stable maps from genus $g$ curves to the projective space $\mathbb{P}^n$. In this note, we will give a hopefully leisurely introduction to some recent works on the two kinds of “resolutions” of $\overline{M}_g(\mathbb{P}^n, d)$ ([VZ08], [HL10], [HL11], [HL12]).

The first is the derived resolution. Its primary purpose is to define the reduced GW invariants which properly count the contributions of the main components of the moduli of stable maps. The derived resolution always exists and has a minimal one, unique up to isomorphism.

The second is the relative resolution. For a smooth stack $M$, we say that a blowup $\tilde{M} \to M$ is smooth if $\tilde{M}$ is smooth. Thus, smooth blowups include blowups along smooth closed centers. If a moduli space has several components, it usually comes equipped with a distinguished one, called the main component.

Definition 1.1. Let $X$ be a Deligne-Mumford stack with a main component $X'$. Let $M$ be a smooth Artin stack and $X \to M$ be a morphism. Suppose that we have a sequence of smooth blowups $\tilde{M} \to M$ such that if let $\tilde{X} = X \times_M \tilde{M}$, then the main component $\tilde{X}' = X' \times_M \tilde{M}$ is smooth and the entire stack $\tilde{X}$ has at worst normal crossing singularities. Then we say $\tilde{X} \to X$ is a resolution of $X$ relative to $\tilde{M} \to M$. If in addition, every irreducible component of $\tilde{X}$ is smooth, then we say it is a strong relative resolution.

Here, a DM stack (not necessarily irreducible) is said to have at worst normal crossing singularities if at any singular point, it is locally equivalent to a union of coordinate subspaces of an affine space. A relative
resolution is a natural extension of the ordinary resolution in that we allow \( \tilde{X} \) to be reducible but require normal crossing singularities for all irreducible components as well as for their intersections, which seems to be the best one can hope for.

For the moduli space \( \overline{M}_g(P^n,d) \), one should interpret a relative resolution as follows. The stack \( \overline{M}_g(P^n,d) \) fibers over a smooth stack \( \mathcal{M} \) such that all its obstructions lie in the fiber direction. A relative resolution is a process to remove all the obstructions in the fiber direction and at the mean time preserve smoothness of the base (whence the requirement that the blowup \( \tilde{\mathcal{M}} \to \mathcal{M} \) is smooth).

For many topological applications, lesser resolution may be sufficient. Let \( E^\bullet \) be a two-term perfect derived object over the DM stack \( X \). Suppose that \( \tilde{\mathcal{M}} \to \mathcal{M} \) is a blowup (not necessarily smooth) such that if let \( \tilde{X} = X \times_{\mathcal{M}} \tilde{\mathcal{M}} \), then the pullback of \( E^\bullet \) to \( \tilde{X} \) becomes locally diagonalizable. Then we say \( \tilde{X} \to X \) is a derived resolution for the object \( E^\bullet \). We let \( \pi : C \to \overline{M}_g(P^n,d) \) be the universal family and \( f : C \to P^n \) be the universal map. The canonical derived objects \( R\pi_* f^* \mathcal{O}_{P^n}(k) \) are of central importance in the Gromov-Witten theory.

Our main results may be summarized as follows. We obtained a strong relative resolution for \( \overline{M}_1(P^n,d) \) ([VZ08], [HL10]). We achieved relative resolutions for \( \overline{M}_2(P^n,d) \) ([HL12]). We constructed the derived resolutions for the canonical objects \( R\pi_* f^* \mathcal{O}_{P^n}(k) \) over \( \overline{M}_g(P^n,d) \) for all \( g \) and \( k > 0 \) ([HL11]). More details are as follows.

When \( g = 1 \), the relative resolution is achieved over a canonical smooth blowup of the smooth Artin stack of weighted curves ([HL10]); for \( g = 2 \), it is relative to a canonical smooth blowup of the smooth Artin stack of pairs of curve and line bundle ([HL12]). The resolution of the main component of \( \overline{M}_1(P^n,d) \) is originally due to Vakil and Zinger ([VZ08]) and was applied to calculate the genus one reduce GW numbers by Zinger ([Zin09]). Chang and Li used the relative resolution to give an algebro-geometric proof of the Li-Zinger formula for genus one GW invariants of quintics ([CL12]). It is expected that the relative resolution of \( \overline{M}_2(P^n,d) \) can be applied to prove the genus two version of the LZ formula. Higher genus cases are considerably harder.

We hope that this note will benefit researchers in the Gromov-Witten theory as well as other mathematicians wandering around. All the works on the derived resolution, reduced invariants, and relative resolutions of stable map moduli are jointly obtained with Jun Li ([HL10], [HL11], [HL12]). Section 4 on the genus one LZ formula is due to a joint work of Chang and Li ([CL12]). I thank Jun Li for collaboration.
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2. Derived and Relative Resolutions: an overview

2.1. A little background. The moduli space $\overline{M}_g(\mathbb{P}^n, d)$ in algebraic geometry was introduced by Kontsevich. Its original primary application is to laid a foundation for the Gromov-Witten theory.

The moduli space $\overline{M}_g(\mathbb{P}^n, d)$ is a moduli compactification of $M_g(\mathbb{P}^n, d)$, the open subset of degree $d$ maps from smooth projective curves of genus $g$ to the projective space $\mathbb{P}^n$. It is not a compactification in
the usual sense since $\overline{M}_g(\mathbb{P}^n, d)$ may contain many components of various dimensions. The closure of $M_g(\mathbb{P}^n, d)$, is the main component, denoted $\overline{M}_g(\mathbb{P}^n, d)'$. It is of a central interest in the GW theory to understand the contribution to the GW invariants of each individual component. For this, it is helpful to understand the structures of the components and how they interact, to certain extent. But, when $n \geq 2$, the spaces $M_g(\mathbb{P}^n, d)$ (hence also $\overline{M}_g(\mathbb{P}^n, d)$) can contain arbitrary singularity types when $g$ varies and $0 < d < 2g - 2$ according to a theorem of Vakil (based upon Mnëv's universality theorem). In addition, the main component $\overline{M}_g(\mathbb{P}^n, d)'$ needs not to be irreducible if $d \leq 2g - 2$. So, it looks rather hopeless to fully grasp the geometric structures of the moduli space.

However, thinking positively, the moduli spaces $\overline{M}_g(\mathbb{P}^n, d)$ provide us a single geometric setting to resolve all possible singularities in algebraic geometry, possibly by a uniform geometric method. Even if one is not interested in resolution of general singularities but is only interested in the GW invariant of hypersurfaces or more generally the GW invariants of complete intersections in $\mathbb{P}^n$, it is still desirable to have a resolution of the singularities of $\overline{M}_g(\mathbb{P}^n, d)$, to certain extent\(^1\). We will explain and hopefully convince the reader that a partial relative resolution of $\overline{M}_g(\mathbb{P}^n, d)$, in a suitable sense, can be applied to study the GW numbers of a quintic threefold and more generally to study the GW invariants of a hypersurface of $\mathbb{P}^n$.

2.2. Where are the singularities? So, to get a “resolution” of singularities of the moduli space $\overline{M}_g(\mathbb{P}^n, d)$, first we need to understand where its singularities are.

A point of the moduli space $\overline{M}_g(\mathbb{P}^n, d)$ is a degree $d$ stable morphism $u : C \to \mathbb{P}^n$, where $C$ is a projective curve of genus $g$ with at worst nodal singularities. Its stability means that its automorphism group is finite. We will often abbreviate a stable map $u : C \to \mathbb{P}^n$ as $[u, C]$. Take any point $[u_0 : C_0]$ in $\overline{M}_g(\mathbb{P}^n, d)$. Fix homogeneous coordinates $[x_0, \ldots, x_n]$ of $\mathbb{P}^n$ with $x_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. Then we can write $u_0 = [s_0, \ldots, s_n]$ where $s_i = u^*x_i \in H^0(u_0^*\mathcal{O}_{\mathbb{P}^n}(1))$. Thus, we see that a deformation of $[u_0, C_0]$ is determined by a combined deformation of the curve $C_0$ and the sections $\{s_0, \ldots, s_n\}$. As the deformation of the (nodal) curve $C_0$ is unobstructed, all possible obstructions to deform the map $[u_0, C_0]$ come from the obstructions of deforming the sections $\{s_0, \ldots, s_n\}$. If in a neighborhood of $[u_0, C_0]$, the rank of $H^0(u_0^*\mathcal{O}_{\mathbb{P}^n}(1))$ remains constant, then one has a vector bundle over the neighborhood with fibers

\(^1\)E.g., in the sense of Definition 1.1, or some weaker ones to be explained in what follows.
After shrinking the neighborhood, we may assume that this vector bundle is trivial; after a trivialization, one sees that there is no obstruction to choose a set of sections \( \{ S_0, \ldots, S_n \} \) of this trivialized bundle, extending the vectors \( \{ s_0, \ldots, s_n \} \) in \( H^0(u^*\mathcal{O}_{\mathbb{P}^n}(1)) \) to its nearby fibers \( H^0(u^*\mathcal{O}_{\mathbb{P}^n}(1)) \). Or else, if the rank of \( H^0(u^*\mathcal{O}_{\mathbb{P}^n}(1)) \) is higher than its nearby general fibers \( H^0(u^*\mathcal{O}_{\mathbb{P}^n}(1)) \), then some sections of \( \{ s_0, \ldots, s_n \} \) may experience obstructions when we try to extend them to nearby fibers. If this happens, we encounter a singularity of \( M_g(\mathbb{P}^n, d) \) at \([u_0, C_0]\).

Informally, we say that a stable map has two directions: the curve direction and the section direction; its curve direction is unobstructed but its section direction may experience obstructions.

Now, to formulate the above informal discussions in formal terms, we need the universal curve \( \pi : C \to M_g(\mathbb{P}^n, d) \) over \( M_g(\mathbb{P}^n, d) \) and the universal map \( f : C \to \mathbb{P}^n \). For a point \([u, C] \in M_g(\mathbb{P}^n, d)\), the fiber \( \pi^{-1}([u, C]) \subset C \) is just the curve \( C \); the map \( f \) restricted to the fiber \( \pi^{-1}([u, C]) = C \) is just the map \( u : C \to \mathbb{P}^n \). To capture the variation of vector space \( H^0(u^*\mathcal{O}_{\mathbb{P}^n}(1)) \), we have the direct image sheaf \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \). In diagrams, this is

\[
\begin{array}{ccc}
\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) & & \mathcal{O}_{\mathbb{P}^n}(1) \\
\pi & \downarrow & \\
M_g(\mathbb{P}^n, d) & \to & \mathbb{P}^n \\
\end{array}
\]

By the above discussions, we have the following important observation. The moduli space \( \overline{M}_g(\mathbb{P}^n, d) \) is smooth at a point \([u, C]\) if and only if the sheaf \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \) is locally free at \([u, C]\). Or, equivalently, the moduli space \( \overline{M}_g(\mathbb{P}^n, d) \) is singular at a point \([u, C]\) if and only if the sheaf \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \) is not locally free at \([u, C]\). (By this characterization, all smooth stable maps must lie in the main component of \( \overline{M}_g(\mathbb{P}^n, d) \). The remainder components contain no smooth points.) Thus, intuitively, one would expect that making the sheaf \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \) locally free will have consequence on the resolution of the moduli space \( \overline{M}_g(\mathbb{P}^n, d) \).

We will elaborate this later sections.

Now, what about the other direct image sheaves \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k) \), \( k > 1 \)?

2.3. The canonical derived objects \( R\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k) \). Well, if we consider a hypersurface \( X \) of degree \( k \) in \( \mathbb{P}^n \) (e.g., a quintic in \( \mathbb{P}^4 \)), we may assume that it is given by the zeros of a section \( s \in H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \).
Then one checks that at least set-theoretically, we have that the moduli space \( \overline{M}_g(X, d) \subset \overline{M}_g(\mathbb{P}^n, d) \) is given by the zeros of the section \( \sigma = \pi_*f^*s \) of the sheaf \( \pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k) \). Thus, in an ideal situation such as if \( \pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k) \) were locally free and \( \overline{M}_g(\mathbb{P}^n, d) \) were of pure and expected dimension, then \( \pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k) \) would possess the Euler class and the fundamental class of \( \overline{M}_g(X, d) \) would be the intersection of the Euler class with the fundamental class of \( \overline{M}_g(\mathbb{P}^n, d) \). This motivates that we should also make \( \pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k) \) locally free so that we can define Euler class and obtain invariants when restricted to the main component of \( \overline{M}_g(\mathbb{P}^n, d) \).

Thus, for either purpose of desingularizing the moduli space \( \overline{M}_g(\mathbb{P}^n, d) \) and making applications to the GW theory of hypersurfaces in \( \mathbb{P}^n \), one would like to resolve the sheaves \( \pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k), k \geq 1 \). However, we should point out here that as it turns out, it is not very helpful to just consider the direct image sheaf \( \pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k) \), it is more natural and mathematically correct to also include the higher direct image sheaf \( R^1\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k) \) and consider the derived object \( R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k) \).

To stress the points, one should bear in mind that when \( k = 1 \) the structures of the object \( R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(1) \) implies structural properties of the underlying moduli space \( \overline{M}_g(\mathbb{P}^n, d) \); when \( k > 1 \), the structures of the object \( R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k) \) will have applications to the GW invariants of the hypersurfaces of degree \( k \) in \( \mathbb{P}^n \). Our treatment for \( R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k), k \geq 1 \), is uniform.

### 2.4. The quintic Calabi-Yau threefolds

We now spend a subsection to discuss the important case of quintic CY threefolds which is related to the object \( R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(5) \). Let \( Q \subset \mathbb{P}^n \) be a smooth quintic threefold. The (virtual) numbers of curves in \( Q \) are central topics in the GW theory for the last two decades. To this date, only the cases of rational and elliptic curves are mathematically known. The case of higher genus is still out of reach.

Consider the moduli space \( \overline{M}_g(Q, d) \) of degree \( d \) stable maps from genus \( g \) curves into the quintic \( Q \). Its expected dimension is zero and possesses a virtual fundamental cycle \( \overline{M}_g(Q, d)^{\text{vir}} \) which is a zero cycle with rational coefficients. Its degree is a rational number, denoted \( N_{g,d} \). This provides a virtual count of genus \( g \), degree \( d \) curves in \( Q \). Let us know examine the meanings of \( N_{g,d} \) via the structures of the moduli space \( \overline{M}_g(Q, d) \). The space \( \overline{M}_g(Q, d) \) consists of stable maps classified as in Figure 1. The domain of every stable map is a genus \( g \) curve. But in its image, one may only see a genus \( h \) curve if a genus \( (g - h) \) subcurve is contracted (the dotted curve in the picture is contracted); such a stable map is a general point in its corresponding
component $\overline{M}_g(Q, d)_h$ of $\overline{M}_g(Q, d)$. Intuitively, this indicates that the virtual number $N_{g, d}$ should contain the contribution of the component of $\overline{M}_g(Q, d)_h$; this contribution should come with two parts: one for $\overline{M}_h(Q, d)$, i.e., the genus $h$ curves that we see in its images, which we denote by $N'_{h, d}$; the other comes from the contracted curves of genus $g - h$, and should be an invariant on the moduli space $\overline{M}_{g-h}$ of stable curves of genus $(g - h)$, which we denote by $c_{g-h}$.

![Figure 1](image)

There are other components, but conjecturally they do not contribute. This explains the idea behind Li-Zinger’s conjecture:

$$N_{g, d} = N'_{g, d} + c_1 N'_{g-1, d} + \cdots + c_{g-1} N'_{1, d} + c_g N'_{0, d}.$$  

What is $N'_{h, d}$? This should be the contribution of the main component $\overline{M}_h(Q, d)'$ of $\overline{M}_h(Q, d)$ whose general points are stable maps with smooth domain curves. How do we rigorously define $N'_{g, d}$? We will explain this in the next couple of sections.

2.5. The derived resolution. To rigorously define $N'_{g, d}$, we need to make the sheaf $\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)$ locally free so that it can provide an Euler class that we can use. As pointed out earlier, it is more correct to consider the object $R\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)$ rather than just the sheaf $\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)$ alone. We can do this in more general setting. So now we examine the properties of the object $R\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(k)$. First, it is a particularly nice kind of object: it is a two term perfect object, meaning locally, it can be presented as a two term complex $[\mathcal{E} \xrightarrow{\phi} \mathcal{F}]$ of locally free sheaves $\mathcal{E}$ and $\mathcal{F}$. (Even though, for $R\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(k)$, a global presentation can be found, we still prefer its local presentations, as in practice, local presentations come more naturally, and more importantly, possess geometric meanings.) Recall that one of our naive aim is to make
the sheaf \(\pi_\ast f^\ast \mathcal{O}_{\mathbb{P}^n}(k)\) locally free. As it turns out, the more functorial way to achieve this is to \textit{locally diagonalize} the object \(\mathbf{R}\pi_\ast f^\ast \mathcal{O}_{\mathbb{P}^n}(k)\) from which the local freeness of \(\pi_\ast f^\ast \mathcal{O}_{\mathbb{P}^n}(k)\) comes as an immediate consequence. So, we explain now the very useful notion of \textit{locally diagonalizable} derived object.

For any scheme \(X\), a homomorphism \(\varphi: \mathcal{O}_X^{\oplus p} \to \mathcal{O}_X^{\oplus q}\) is said to be diagonalizable if we have direct sum decompositions by trivial sheaves

\[\mathcal{O}_X^{\oplus p} = G_0 \oplus \bigoplus_{i=1}^l G_i\] and \[\mathcal{O}_X^{\oplus q} = H_0 \oplus \bigoplus_{i=1}^l H_i\]

with \(\varphi(G_i) \subset H_i\) for all \(i\) such that

1. \(\varphi|_{G_0} = 0\);
2. for every \(1 \leq i \leq l\), \(\varphi|_{G_i}\) equals to \(p_i I_i\) for some \(0 \neq p_i \in \Gamma(\mathcal{O}_X)\) where \(I_i : G_i \to H_i\) is an isomorphism;
3. \(\langle p_i \rangle \nsubseteq \langle p_{i+1} \rangle\).

A homomorphism \(\varphi: E \to F\) between locally free sheaves of a scheme \(X\) is said to be locally diagonalizable if there are trivializations of \(E\) and \(F\) over some open covering of \(X\) such that \(\varphi: E \to F\) is diagonalizable over every open subset.

Now if \(E^\bullet\) is a two-term perfect derived object over a DM stack \(X\), using any local presentation of \([E \xrightarrow{\varphi} F]\) of \(E^\bullet\), we can introduce the notion of locally diagonalizable derived object. One can prove that the notion does not depend on the local presentation ([HL11]).

Not every homomorphism (or two-term perfect derived object) can be locally diagonalized. But if it can be, then there are some good implications. For example, suppose that a homomorphism \(\varphi: E \to F\) (or an object \(E^\bullet\)) is locally diagonalizable, then for every irreducible component \(X'\) of \(X\) with the reduced scheme structure, \(\ker(\varphi|_{X'}) (\mathcal{H}^0(E^\bullet))\) is locally free. Also, if \(f: Y \to X\) is a morphism, then \(f^\ast \varphi (f^\ast E^\bullet)\) is also locally diagonalizable, that is, “locally diagonalizable” has base change property. Note here that the rank of \(\ker \varphi\) depends on the properties of the functions \(p_i\), hence may not be constant over \(X\). Further, when \(X'\) is not reduced, \(\ker(\varphi|_{X'})\) needs not to be locally free. These technical issues lead us to use the “integral” assumption whenever we want to produce a locally free sheaf.

For any \(0 \leq i \leq r := \max(\text{rank } E, \text{rank } F)\), we let \(\mathcal{I}_{\varphi,i}\) be the \(i\)-th determinantal ideal sheaf of \(\varphi: E \to F\). That is, \(\mathcal{I}_{\varphi,i}\) is the ideal sheaf of the zero scheme of \(\wedge^{i+1} E \to \wedge^{i+1} F\). Again, using local presentations of the object \(E^\bullet\), one can introduce the \(i\)-th determinantal ideal sheaf \(\mathcal{I}_{E^\bullet,i}\) of \(E^\bullet\) and show that it is independent of the choice of the presentations. We proved the following.
Theorem 2.1. ([HL11]) Suppose $E^\bullet$ is a two-term perfect derived object over a DM stack $X$. If we let $b : \tilde{X} \to X$ be the blowup of $X$ along the ideal sheaves $\mathcal{I}_{E^\bullet,0}, \ldots, \mathcal{I}_{E^\bullet,r}$ and so on, then the pullback $b^* E^\bullet$ becomes locally diagonalizable over $\tilde{X}$. In particular, $\mathcal{H}^0(b^* E^\bullet)$ is locally free provided $X$ is integral. Further, $\tilde{X}$ is minimal in the sense that if there is another dominating morphism $g : Z \to X$ such that $g^* E^\bullet$ is locally diagonalizable, then $g$ factors uniquely as $Z \to \tilde{X} \to X$.

In fact, we can consider any perfect derived object $E^\bullet$ in the bounded derived category $D^b(M)$ of an integral DM stack $M$ with cohomologies concentrated in the non-negative places. Using the above theorem, we can show

Theorem 2.2. ([HL11]) Let $E^\bullet$ be any perfect derived object over an integral DM stack $X$. Assume that $E^\bullet$ can locally be represented by a complex of locally free sheaves of finite length supported only in non-negative degrees. Then there is another integral DM stack $\tilde{X}$ and a surjective birational morphism $b : \tilde{X} \to X$ such that $H^0(b^* E^\bullet)$ is locally free.

The above allows us to define the Euler class of such a derived object. For this, we suppose further that the cohomologies in positive places $\mathcal{H}^{i>0}(E^\bullet)$ are all torsion sheaves over $X$. Then, we define the Euler class $e(E^\bullet)$ in the Chow group $A_\star X$ of cycles on $X$ by,

$$e(E^\bullet) := b_*(c_r(\mathcal{H}^0(b^* E^\bullet))) \cdot [\tilde{X}],$$

where $r = \text{rank} \mathcal{H}^0(b^* E^\bullet)$. One can show the Euler class $e(E^\bullet)$ is independent of the choice of the resolutions $b : \tilde{X} \to X$.

2.6. The reduced GW numbers of quintics and LZ conjecture.

The above, when applied to the derived object $R\pi'_f^* \mathcal{O}_{\mathbb{P}^4}(5)$ restricted to the primary component $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)'$ of the moduli stack $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$, enables us to construct the modular Euler class when $d > 2g - 2$. Here, the general points of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)'$ are maps with smooth domains, and $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)'$ is irreducible and of the expected dimension when $d > 2g - 2$. In this case, letting $\pi'$ and $f'$ be the restrictions of the projection of the universal family and universal map to $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)'$, respectively, then $R^1\pi'_f f^* \mathcal{O}_{\mathbb{P}^4}(5)$ is a torsion sheaf over $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)'$. Hence, for $d > 2g - 2$, we can define the modular Euler class of $R\pi'_f f^* \mathcal{O}_{\mathbb{P}^4}(5)$ over $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)'$ to be

$$e(R\pi'_f f^* \mathcal{O}_{\mathbb{P}^4}(5)) \in A_\star (\overline{\mathcal{M}}_g(\mathbb{P}^4, d)');$$
for any smooth Calabi-Yau manifold $Q$ in $\mathbb{P}^4$, we define

\begin{equation}
N'_{g,d}(Q) = \deg \left( \mathcal{R} \pi'_{*} f^* \mathcal{O}_{\mathbb{P}^4}(5) \right).
\end{equation}

Using these numbers $N'_{g,d}(Q)$, in [HL11], we rigorously formulate the conjecture of Li and Zinger:

**Conjecture 2.3.** Let $N_{g,d}(Q)$ be the genus $g$, degree $d$ GW-invariants of the smooth quintic $Q \subset \mathbb{P}^4$. Then there are universal constants $c_h$ such that for all $d > 2g - 2$,

\[
N_{g,d}(Q) = \sum_{0 \leq h \leq g} c_h N'_{h,d}(Q).
\]

The $g = 0$ case is trivial: $N_{0,d} = N'_{0,d}$. The $g = 1$ version, $N_{1,d} = N'_{1,d} + \frac{1}{12} N'_{0,d}$, is proved by Li and Zinger (and later by Chang-Li, using algebro-geometric method). The $g \geq 2$ cases are open.

**2.7. Relative resolutions: a general program.** The derived resolutions, i.e., the blowup $b : \tilde{X} \to X$ in Theorem 2.1 (or Theorem 2.2), are unfortunately very singular in general. This is so even when we start with a smooth scheme $X$. For various purpose, a resolution in the usual sense, which, at the mean time, can also serve as a derived resolution, is desirable. This is, of course, much hard to achieve in general.

For the case of $\mathcal{M}_g(\mathbb{P}^n, d)$, we hope to achieve a relative resolution. Recall that a stable map has two directions to deform, one is curve, the other is collection of sections. Intuitively, one could image that the derived resolution removes the obstructions of the section directions but at the cost of destroying the (original) smoothness of the curve direction. The difficulty is to resolve the obstructions of section direction and at the mean time preserve the smoothness of the curve direction. Below, we will make it clear this vague point.

To find out the base of relative resolution of $\mathcal{M}_g(\mathbb{P}^n, d)$, we should begin with analyzing a neighborhood $U$ of a singular point $[u, C] \in \mathcal{M}_g(\mathbb{P}^n, d)$. We use $\pi_U : C_U \to U$ to denote the restriction $C|_U$ of the universal family and $f_U$ the restriction $f|_{C_U}$ of the universal map. So, in diagram, we have

\[
\begin{array}{ccc}
C_U & \xrightarrow{f_U} & \mathbb{P}^n \\
\pi_U \downarrow & & \\
U.
\end{array}
\]

Suppose we plan to resolve $\mathcal{R} \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$ (just remember that the case $k = 1$ ties closely to the local structures of the underlying moduli).
First, we represent \( \mathcal{O}_{n}(k) \) as \( \mathcal{O}_{n}(H) \) by picking a general hypersurface of degree \( k \) meeting the image curve \( f(C) \) (which is of degree \( d \)) at \( m := dk \) many isolated points. If we perturb the map \([u, C]\) slightly, the image curve still meets \( H \) at isolated \( m \) points. Putting all these neighboring maps together, this implies that by replacing \( U \) by a smaller neighborhood if necessary, we may assume that we have \( f^{*}\mathcal{O}_{n}(H) = \mathcal{O}_{C}(D) \) with \( D = D_{1} + \cdots + D_{m} \) where each \( D_{i} \) is a section of the family \( C_{U} \to U \) and they are disjoint. If \( C' \) is a component of \( C \) such that the degree of \( u|_{C'} \) is \( d' \), then there should be \( d'k \) many sections passing through the curve \( C' \subset C_{U} \); if \( C' \) gets contracted by the map \( f \), then \( C' \) meets none of the sections.

Observe that the choice of the general hypersurface \( H \) determines a morphism by assigning the map \([u, C]\) to the pair \((C, \delta_{1} + \cdots + \delta_{m})\) where \( \delta_{i} = C \cap D_{i} \). For this purpose, we introduce the Artin stack \( \mathcal{M}_{g}^{\text{div}} \) of pairs \((C, \delta_{1} + \cdots + \delta_{m})\) where \( C \) is a prestable curve of genus \( g \) and \( \delta_{1}, \ldots, \delta_{m} \) are disjoint smooth points on \( C \). This stack is smooth. By the above, we have a morphism \( U \to V \subset \mathcal{M}_{g}^{\text{div}} \) where \( V \) is a chart of \( \mathcal{M}_{g}^{\text{div}} \). (The morphism is not canonical because it depends on the choice of \( H \)). The geometry of the pairs is easier to get hold of than that of the stable map. This leads to the idea that we should work over the chart \( V \) and then pull back the results to the neighborhood \( U \).

For this, the Artin stack admits a universal curve \( C_{V} \) and a universal divisor \( D = D_{1} + \cdots + D_{m} \) so that we have a fiber square

\[
\begin{array}{ccc}
C_{U} & \xrightarrow{\alpha} & C_{V} \\
\downarrow^{\pi_{U}} & & \downarrow^{\rho_{V}} \\
U & \longrightarrow & V
\end{array}
\]

and \( D_{i} = \alpha^{*}D_{i} \) and \( D = \alpha^{*}D \). This allows us to study the object \( R(\pi_{U})_{*}\mathcal{O}_{C_{U}}(D) \) via the object \( R(\rho_{V})_{*}\mathcal{O}_{C_{V}}(D) \). That is, a derived resolution of \( R(\rho_{V})_{*}\mathcal{O}_{C_{V}}(D) \) will induce a derived resolution of \( R\pi_{*}\mathcal{O}_{C_{U}}(D) \) because of the base change property. The task now is: smoothly blow up \( V \) so that the object \( R(\rho_{V})_{*}\mathcal{O}_{C_{V}}(D) \) becomes locally diagonalizable.

This is what we mean earlier when we mention that to get a true resolution we should resolve the obstructions of the section directions (i.e., to locally diagonalize \( R(\rho_{V})_{*}\mathcal{O}_{C_{V}}(D) \)) and preserve the smoothness of curve directions (i.e., smoothly blowup the chart \( V \)).

Unfortunately, we have no obvious global morphisms from \( \mathcal{M}_{g}(\mathbb{P}^{n}, d) \) to \( \mathcal{M}_{g}^{\text{div}} \). To overcome to this difficulty, one could in principle work over local charts \( V \) of \( \mathcal{M}_{g}^{\text{div}} \) and then (cumbersomely) argue how to patch the data over the charts. Instead, we choose to introduce another Artin
stack so that it admits a global morphism from $\overline{M}_g(\mathbb{P}^n, d)$ and not too much technical information is lost.

This substituting stack is the Artin stack $\mathfrak{P}_g$ of pairs $(C, L)$ of a prestable curve of genus $g$ and a line bundle over $C$. This again is a smooth stack. It comes with a universal curve
\[ \rho_{\mathfrak{P}} : C_{\mathfrak{P}} \longrightarrow \mathfrak{P}_g \]
and a universal line bundle $\mathcal{L}$. The relevant derived object here is $R(\rho_{\mathfrak{P}})_* \mathcal{L}$. We have a (global) morphism (depending on $k$)
\[ \varsigma_k : \overline{M}_g(\mathbb{P}^n, d) \longrightarrow \mathfrak{P}_g, \quad [u, C] \rightarrow (C, u^* \mathcal{O}_{\mathbb{P}^n}(k)). \]

Our earlier local morphism clearly factors through
\[ \begin{array}{ccc}
U & \longrightarrow & \overline{M}_g(\mathbb{P}^n, d) \\
\downarrow & & \downarrow \\
\mathcal{V} & \longrightarrow & \mathfrak{M}_g^{\text{div}} \\
\downarrow \varsigma_k & & \downarrow \\
& \mathfrak{P}_g &
\end{array} \]

The idea now is that we can work over the chart $\mathcal{V}$ (which we found easy to cope with since it conveniently parameterizes a pair of curve and a divisor). Then, according the local smooth blowup of $\mathcal{V}$ we find, we may translate it to the stack $\mathfrak{P}_g$ to obtain the corresponding global smooth blowup, as desired. The goal is that we need to smoothly blow up $\mathcal{V}$ and hence also $\mathfrak{P}_g$ so that $R(\rho_{\mathfrak{P}})_* \mathcal{L}$, hence also $R\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$, becomes locally diagonalizable. The author excepts the following.

**Conjecture 2.4.** Fix $g > 0$ and $k > 0$. There is a canonical sequence of smooth blowups $\overline{\mathfrak{P}}_g^k \rightarrow \mathfrak{P}_g$ of $\mathfrak{P}_g$ such that if we form the cartesian diagram
\[ \overline{M}_g^k(\mathbb{P}^n, d) = \mathfrak{P}_g^k \times_{\mathfrak{P}_g} \overline{M}_g(\mathbb{P}^n, d) \longrightarrow \mathfrak{P}_g^k \]
\[ \overline{M}_g(\mathbb{P}^n, d) \longrightarrow \mathfrak{P}_g, \]
then the derived object $R\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$, when pulling back to $\overline{M}_g^k(\mathbb{P}^n, d)$, becomes locally diagonalizable.

Such a relative derived resolution should be sufficient to apply to the GW theory\(^2\) of hypersurfaces in $\mathbb{P}^n$. Because $\overline{\mathfrak{P}}_g^k$ is smooth, we have

\(^2\)Indeed, an easier partial relative derived resolution is sufficient. This will be explained in a forthcoming paper.
a perfect relative obstruction theory $\widetilde{M}_g^k(\mathbb{P}^n, d)/\mathbb{P}_g^k$ by pulling back a perfect relative obstruction theory $\overline{M}_g(\mathbb{P}^n, d)/\mathbb{P}_g$ (which is known to exist). We may apply this to $R\pi_*\psi^*\mathcal{O}_{\mathbb{P}^4}(5)$ to study the GW invariants of quintic threefolds.

To get a relative resolution of $\overline{M}_g(\mathbb{P}^n, d)$, we specialize the above to the case of $k = 1$.

**Conjecture 2.5.** Fix $g > 0$. There is a canonical sequence of smooth blowups $\mathfrak{P}_g \rightarrow \mathfrak{P}_g$ of $\mathfrak{P}_g$ such that if we form the cartesian diagram

\[
\begin{array}{ccc}
\widetilde{M}_g(\mathbb{P}^n, d) = \mathfrak{P}_g \times_{\mathfrak{P}_g} \overline{M}_g(\mathbb{P}^n, d) & \longrightarrow & \mathfrak{P}_g \\
\downarrow & & \downarrow \\
\overline{M}_g(\mathbb{P}^n, d) & \longrightarrow & \mathfrak{P}_d
\end{array}
\]

(we omit $\varsigma_1$ from the fiber product), then the derived object $R\pi_*\psi^*\mathcal{O}_{\mathbb{P}^n}(1)$, when pulling back to $\widetilde{M}_g^k(\mathbb{P}^n, d)$, becomes locally diagonalizable. Further, assume that $d > 2g - 2$, then we have

1. the main component of $\widetilde{M}_g(\mathbb{P}^n, d)$ is smooth;
2. the entire stack $\widetilde{M}_g(\mathbb{P}^n, d)$ has at worst normal crossing singularities.

The above is a resolution of $\overline{M}_g(\mathbb{P}^n, d)$ relative to $\mathfrak{P}_g$. This sort of relative resolution problem may be characterized as “desingularization by removing relative obstructions, while preserving the smoothness of the base.” The author expects this to be true possibly after substituting $\mathfrak{P}_g$ by another moduli stack, or re-interpret the smoothness.

We can carry out this program for the cases of genus one and two. That is, Conjectures 2.4 and 2.5 hold in this two cases. In fact, our statements are slightly stronger than stated in the conjecture.

### 2.8. Relative resolution: the genus one case.
Ideally, for technical convenience as we discussed earlier, we would prefer to describe a relative resolution of $\overline{M}_g(\mathbb{P}^n, d)$ with $\mathfrak{M}_g^{\text{div}}$ as the base. But, we do not have a global morphism to achieve so. Thus we are forced to compromise to work over the stack $\mathfrak{P}_g$ downstairs. However, the stack $\mathfrak{P}_g$ is not that convenient to work with. Therefore, whenever possible, we should replace $\mathfrak{P}_g$ by another stack that is easier to work with. For genus one, this substituting stack is the smooth Artin stack $\mathfrak{M}_g^{\text{wt}}$ of weighted curves. In general, the stack $\mathfrak{M}_g^{\text{wt}}$ consists of pairs of pre-stable curve of genus $g$ and a nonnegative integer (called weight). Forgetting the weight, we obtain a morphism from $\mathfrak{M}_g^{\text{wt}}$ to the smooth Artin stack $\mathfrak{M}_g$ of pre-stable curves. This morphism is étale, showing the smoothness
of $\mathcal{M}_g^{w_t}$. There is a canonical morphism $\mathcal{P}_g \to \mathcal{M}_g^{w_t}$ by assigning a pair $(C, L) \in \mathcal{P}_g$ to $(C, c_1(L)) \in \mathcal{M}_g^{w_t}$. Thus, for every $k > 0$, we have

$$\overline{M}_g(\mathbb{P}^n, d) \to \mathcal{P}_g \to \mathcal{M}_g^{w_t}, \quad [u, C] \to (C, u^*\mathcal{O}_{\mathbb{P}^n}(k)) \mapsto (C, c_1(u^*\mathcal{O}_{\mathbb{P}^n}(k))).$$

For genus one, the resolution can be achieved relative to the base $\mathcal{M}_1^{w_t}$.

In [VZ08], Vakil and Zinger provided a natural resolution of the main component of $\overline{M}_1(\mathbb{P}^n, d)$. We follow exactly their blowing up procedure, but instead blow up $\mathcal{M}_1^{w_t}$, and then treat the entire moduli space by taking fiber product. In this case, we will only use the morphism

$$\varsigma_1 : \overline{M}_1(\mathbb{P}^n, d) \to \mathcal{M}_1^{w_t}, \quad [u, C] \to (C, c_1(u^*\mathcal{O}_{\mathbb{P}^n}(1))).$$

For any $i > 0$, let $\Theta_i \subset \mathcal{M}_1^{w_t}$ be the close subset such its general points are of the form $(C, w)$ such that $C$ can be obtained from a smooth elliptic curve $E$ by attaching (connected) trees of rational curves at $i$ many disjoint points of $E$ and the weight $w$ restricted to $E$ is zero. Every $\Theta_i$ is a smooth closed substack of codimension $i$.

**Theorem 2.6.** ([VZ08], [HL10]) We let $\tilde{\mathcal{M}}_1^{w_t} \to \mathcal{M}_1^{w_t}$ be the successive blowup of $\mathcal{M}_1^{w_t}$ along the smooth closed substacks $\Theta_2, \Theta_3, \ldots$, and so on. If we let

$$\tilde{M}_1(\mathbb{P}^n, d) = \tilde{\mathcal{M}}_1^{w_t} \times_{\mathcal{M}_1^{w_t}} \overline{M}_1(\mathbb{P}^n, d),$$

then we have

1. every (including the main) irreducible component of $\tilde{M}_1(\mathbb{P}^n, d)$ is smooth;
2. the entire DM stack $\tilde{M}_1(\mathbb{P}^n, d)$ has at worse normal crossing singularities; further
3. the derived object $R\pi_*\mathcal{O}_{\mathbb{P}^n}(k)$, upon pulling back to $\tilde{M}_1(\mathbb{P}^n, d)$, becomes locally diagonalizable for all $k > 0$; in particular
4. for any irreducible component $N$ of $\tilde{M}_1(\mathbb{P}^n, d)$, let $(\pi_N, f_N)$ be the pullback of the universal family $(\pi, f)$ to $N$, then the direct image sheaf $(\pi_N)_*f_N^*\mathcal{O}_{\mathbb{P}^n}(k)$ is locally free for all $k > 0$.

**2.9. Relative resolutions: the genus two case.** The genus two case is already substantially more complicated than the case of genus one. For one thing, unlike $g = 1$, for $g = 2$, there are more than one rounds of canonical sequence of blowups, and the case $k = 1$ requires its own sequence of blowups; for another, the blowup centers are complicated to describe. Because it requires substantial preparation to describe them, we will have to be vague on the blowing up centers here. But, we will give a good hint near the end of the next section.
Theorem 2.7. ([HL12]) There are three rounds of sequences of successive smooth blowups $\overline{M}_2^{\text{wt}} \to \overline{M}_2^{\text{wt}}$ such that if we let

$$\overline{M}_2^k(\mathbb{P}^n, d) = \overline{M}_2^{\text{wt}} \times_{\overline{M}_2^{\text{wt}}} \overline{M}_2(\mathbb{P}^n, d),$$

then we have

(1) the derived object $R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k)$, upon pulling back to $\overline{M}_2^k(\mathbb{P}^n, d)$, becomes locally diagonalizable for all $k \geq 2$; in particular

(2) for any irreducible component $N$ of $\overline{M}_2^k(\mathbb{P}^n, d)$, let $(\pi_N, f_N)$ be the pullback of the universal family $(\pi, f)$ to $N$, then the direct image sheaf $(\pi_N)_*f_N^*\mathcal{O}_{\mathbb{P}^n}(k)$ is locally free for all $k \geq 2$.

Unlike $g = 1$, some of the smooth blowups for $g = 2$ are not blowups along smooth closed centers.

The object $R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}$ (1) underpins the local structure of the underlying moduli space and requires one more sequence of blowups to resolve.

Theorem 2.8. ([HL12]) There is a canonical sequence of smooth blowups of $\overline{M}_2 \to \overline{M}_2^{\text{wt}}$ such that if we let

$$\overline{M}_2(\mathbb{P}^n, d) = \overline{M}_2 \times_{\overline{M}_2^{\text{wt}}} \overline{M}_1(\mathbb{P}^n, d),$$

and suppose $d > 2$, then we have

(1) the main component of $\overline{M}_2(\mathbb{P}^n, d)$ is smooth;

(2) the entire DM stack $\overline{M}_2(\mathbb{P}^n, d)$ has at worse normal crossing singularities; further

(3) the derived object $R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k)$, upon pulling back to $\overline{M}_2(\mathbb{P}^n, d)$, becomes locally diagonalizable for all $k \geq 1$; in particular

(4) for any irreducible component $N$ of $\overline{M}_2(\mathbb{P}^n, d)$, let $(\pi_N, f_N)$ be the pullback of the universal family $(\pi, f)$ to $N$, then the direct image sheaf $(\pi_N)_*f_N^*\mathcal{O}_{\mathbb{P}^n}(k)$ is locally free for all $k \geq 1$.

3. Why does Relative Resolution Work?

In this section, we provide some details to explain how we could potentially derive the modular resolutions for all $g$ and why the cases $g = 1$ and 2 work.

We remind the reader that our goal is to smoothly blow up $\mathcal{M}_g$ so that the derived object $R(\rho_{\mathcal{M}})_*\mathcal{L}$ becomes locally diagonalizable.

---

3We mention here that the first two rounds of sequences of smooth blowups are topological and the third round is geometric. We will be more specific about them in the next section. We believe that the sequences of topological blowups extend to all genera and are sufficient to study the GW numbers of quintics. This will appear in a forthcoming paper.
As in Section 2.7, locally, we can factorize the canonical morphism $M_g(\mathbb{P}^n, d) \to \mathcal{M}_g$ as

$$U \to V \to \mathcal{M}_g$$

where $U$ is an open neighborhood of a point $[u, C] \in M_g(\mathbb{P}^n, d)$ and $V$ is a chart over $\mathcal{M}_g^{\text{div}}$. Any blowup of $\mathcal{M}_g$ induces a blowup of $V$ by taking the fiber product. By now, we get the idea that to carry out our relative resolution program, it is important to know the local structures of the derived object $R(\rho_V)_* \mathcal{O}_{C_V}(D)$ (which is related to $R\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$ by Cohomology and Base Change).

So, we begin to analyze the derived object $R(\rho_V)_* \mathcal{O}_{C_V}(D)$ and pin down a local presentation. This is known to admit a local presentation of the form $[E \xrightarrow{\psi} F]$ where $E$ and $F$ are locally free sheaves. But now, we are not content with such an inexplicit form (for the derived resolution, it suffices since we blow up along determinantal ideals whose explicit forms are not necessary), we need an explicit form so that we can extract the geometry behind the form. For this purpose, shrinking $V$ if necessary, we can choose $g$ many disjoint sections $A_1, \cdots, A_g$ of $C_V \to V$ such that they are disjoint from the divisor $D$ and for any $v \in V$ and any irreducible subcurve $C'$ of $C_v = \rho_{T_v}^{-1}(v)$ of genus $h \leq g$, the set $\{A_1 \cap C', \cdots, A_g \cap C'\}$ consists of $h$ many distinct points in general position (that is, they do not form a special divisor on the curve $C'$). Set $A = \sum_i A_i$. Then from the short exact sequence,

$$0 \to \mathcal{O}_{C_V}(D) \to \mathcal{O}_{C_V}(D + A) \to \mathcal{O}_A(A) \to 0$$

we have a long one

$$(\rho_V)_* \mathcal{O}_{C_V}(D) \to (\rho_V)_* \mathcal{O}_{C_V}(D + A) \xrightarrow{\psi} (\rho_V)_* \mathcal{O}_A(A) \to R^1(\rho_V)_* \mathcal{O}_{C_V}(D)$$

because one shows that $R^1(\rho_V)_* \mathcal{O}_{C_V}(D + A) = 0$ by the assumptions on $A$. Hence, $(\rho_V)_* \mathcal{O}_{C_V}(D + A)$ is locally free of rank $d + 1$ by Riemann-Roch. Also, it is clear that $(\rho_V)_* \mathcal{O}_A(A)$ is locally free of rank $g$ since $A$ are disjoint sections. Thus, we have

$$R(\rho_V)_* \mathcal{O}_{C_V}(D) = [(\rho_V)_* \mathcal{O}_{C_V}(D + A) \xrightarrow{\psi} (\rho_V)_* \mathcal{O}_A(A)].$$

We will call $\psi$ a structural homomorphism; understanding this homomorphism is a key to our program. We can always choose frames of the free sheaves

$$E = (\rho_V)_* \mathcal{O}_{C_V}(D + A) \quad \text{and} \quad F = (\rho_V)_* \mathcal{O}_A(A)$$

so that the homomorphism $\psi$ can be represented by a matrix. Then our task becomes to derive an explicit form for every entry in the matrix. Before we do it, we need some useful lemmas to decompose the homomorphism $\psi$. 

First we choose another general section \( B \); this will help us to isolate a canonical trivial sheaf in the kernel of \( \psi \).

**Lemma 3.1.** We have a natural splitting of locally free sheaves

\[
(\rho_V)_* \mathcal{O}_{C_V}(D + A) = \mathcal{O}_V \oplus (\rho_V)_* \mathcal{O}_{C_V}(D + A - B)
\]

such that \( \mathcal{O}_V \subset \ker \psi \) and \( \psi \) can be naturally decomposed as

\[
\mathcal{O}_V \oplus (\rho_V)_* \mathcal{O}_{C_V}(D + A - B) \xrightarrow{\varphi} (\rho_V)_* \mathcal{O}_A(A)
\]

where \( \varphi : (\rho_V)_* \mathcal{O}_{C_V}(D + A - B) \to (\rho_V)_* \mathcal{O}_A(A) \) is the restriction homomorphism.

Since \( \psi = 0 \oplus \varphi \), we also call \( \varphi \) a structural homomorphism. Observe now that the rank of the locally free sheaf \( (\rho_V)_* \mathcal{O}_{C_V}(D + A - B) \) coincides with the degree of \( D = \sum_{i=1}^m D_i \). Indeed, if we let \( \varphi_j : (\rho_V)_* \mathcal{O}_{C_V}(D_j + A - B) \to (\rho_V)_* \mathcal{O}_A(A) \) be the restriction homomorphism, then we have

**Lemma 3.2.** We have the following natural decomposition of locally free sheaves

\[
(\rho_V)_* \mathcal{O}_{C_V}(D + A - B) = \bigoplus_{j=1}^m (\rho_V)_* \mathcal{O}_{C_V}(D_j + A - B).
\]

Further, the homomorphism \( \varphi \) also decomposes as

\[
\varphi = \bigoplus_{j=1}^m \varphi_j.
\]

This reduces the study of the homomorphism \( \psi \) to each \( \varphi_i \) which can further be reduced to

\[
\varphi_{ij} : (\rho_V)_* \mathcal{O}_{C_V}(D_j + A - B) \to (\rho_V)_* \mathcal{O}_{A_i}(A_i) \cong \mathcal{V}
\]

for all \( 1 \leq i \leq g \). Shrinking \( V \) if necessary, we can assume all sheaves \( (\rho_V)_* \mathcal{O}_{C_V}(D_j + A - B) \) are isomorphic to the trivial sheaf \( \mathcal{O}_V \). Thus, upon fixing frames, we may regard \( \varphi_{ij} \) as a function in \( \Gamma(\mathcal{O}_V) \). We found that this function vanishes at \( v \in V \) if and only if there are separable nodes between \( A_i \cap C_v \) and \( D_j \cap C_v \). Here a node \( q \) of a connected curve \( C \) is separable if \( C \setminus q \) is disconnected; the separable node \( q \) is said to be between two points \( a, \delta \in C \) if \( a \) and \( \delta \) lie in different components of \( C \setminus q \). For any two points \( a, \delta \in C \), we let \( N_{[a, \delta]} \) denote the set of all separable nodes between \( a \) and \( \delta \). Every node \( q \) of \( C \) is associated with a function \( \zeta_q \) in \( \Gamma(\mathcal{O}_V) \), called a node-smoothing parameter. Geometrically, at any point \( v \in (\zeta_q = 0) \), the fiber curve \( C_v \) contains a node corresponding to \( q \); at any point \( v \) away from the locus.
(ζ_q = 0), the curve C_v does not have a node corresponding to q (the
node is smoothed out). This leads us to the following key proposition.

**Proposition 3.3.** Upon fixing frames, we have \( \varphi_{ij} = c_{ij} \prod_{q \in N_{[a_i, \delta_j]}} \zeta_q \) where \( c_{ij} \in \Gamma(\mathcal{O}_\mathcal{V}^*) \).

Putting all together, we obtain a matrix representation of the struc-
tural homomorphism \( \varphi \) by

\[
\Phi = (c_{ij} \prod_{q \in N_{[a_i, \delta_j]}} \zeta_q)_{1 \leq i \leq g, 1 \leq j \leq m}.
\]

This matrix is important because it provides a local equation of
\( \mathcal{M}_g(\mathbb{P}^n, d) \) at the point \([u, C]\) when \( m = d \) (i.e, for \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \)). The idea is roughly that the kernel of the homomorphism \( \varphi \) gives rise to
sections of \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \) but sections of \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \) determine stable maps.

Note that \( \Phi \) is a matrix of size \( g \times d \) (for \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \)). We let \( w_i \) be the column vector \((w_{i1}, \ldots, w_{id}), i = 1, \ldots, n \) with \( w_{ij} \in \mathbb{A}^1 \).

**Theorem 3.4.** ([HL10]) The local equation of a (small) neighborhood
\( U \) of the point \([u, C] \in \mathcal{M}_g(\mathbb{P}^n, d) \) is given by the system of equations

\[
(3.1) \quad \Phi \cdot w^i = 0, \quad i = 1, \ldots, n,
\]

realizing it as a closed subset of the smooth space \( \mathcal{V} \times \mathbb{A}^{dn} \).

The simplest kind of matrices are diagonal ones. In such a case, equations (3.1) become elementary. However, one can not hope in
general to find a diagonal form of \( \Phi \) simply by choosing frames. To
make \( \Phi \) diagonal in general, birational base change is necessary. A
good thing is that the local equation (3.1) pulls back to provide a local
equation under any base change.

**Theorem 3.5.** Let \( f : \mathcal{V}' \to \mathcal{V} \) be any morphism, \( U' = U \times_\mathcal{V} \mathcal{V}' \) and
\( \Phi' \) a matrix representation of the pullback homomorphism \( f^* \varphi \). Then
\( U' \) is defined by the system of equations

\[
(3.2) \quad \Phi' \cdot w^i = 0, \quad i = 1, \ldots, n,
\]

realizing it as a closed subset of \( \mathcal{V}' \times \mathbb{A}^{dn} \).

Now, suppose \( f : \tilde{\mathcal{V}} \to \mathcal{V} \) is a smooth blowup of \( \mathcal{V} \) such that the
derived object \( R(\rho_{\mathcal{V}})_* \mathcal{O}_{\mathcal{C}_\mathcal{V}}(D) \) (related to \( R\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1) \)) becomes diag-
onalizable upon pulling back to \( \tilde{\mathcal{V}} \). Then we can find a diagonal form
for its corresponding matrix \( \Phi \). Write \( \tilde{\Phi} \) as \( \text{diag}(z_1, \ldots, z_g, 0, \ldots, 0) \)
(here, we assume \( d > g \)). Then
Corollary 3.6. The open subset $\tilde{U} = U \times_Y \tilde{V}$ is given by equations

$$z_j \cdot w_j^i = 0, \ 1 \leq j \leq g, \ i = 1, \ldots, n,$$

realizing it as a closed subset of the smooth space $\tilde{V} \times \mathbb{A}^{dn}$.

The main component of $\tilde{U}$ is found by requiring the non-vanishing of $z_j$ over its generic points, hence it is given by

$$w_j^i = 0, \ 1 \leq j \leq g, \ i = 1, \ldots, n,$$

which define a smooth closed subset of the smooth space $\tilde{V} \times \mathbb{A}^{dn}$. This shows the smoothness of the main component. Since $f : V \to \tilde{V}$ is a smooth blowup of $V$, we conclude that $(z_j = 0)$ defines a normal crossing divisor over $V$. Any other irreducible component is defined by a mixed vanishing of prime factors of $z_j$ and $w_j^i$. We conclude that every irreducible component admits at worst normal crossing singularities and they meet in normal crossing way. A priori, we cannot exclude the case when some prime divisor of irreducible component has self-intersection. If in addition, we know that every prime divisor of irreducible component is smooth, then we can conclude that every irreducible component is smooth. This is the case when $g = 1$, but not the case when $g = 2$, hence it should not be the case for all $g \geq 2$.

When $g = 1$, the matrix $\Phi = (c_{ij} \prod_{q \in N_{\{a_i, b_j\}}} \zeta_q)$ in the local equation in (3.1) has only one row. We can absorb all the invertible coefficients $c_{ij}$ into the frames. Then, a simple analysis implies that all the singularities are caused by the presence of nodes. We call these singularities of topological type. These are straightforward to resolve as in Theorem 2.6.

When $g = 2$, the matrix $\Phi = (c_{ij} \prod_{q \in N_{\{a_i, b_j\}}} \zeta_q)$ in the local equation in (3.1) has two rows. We classify the singularities into two types. Topological type: the ones caused solely by the presence of nodes. Geometric type: the ones caused by vanishing of minor determinants of the matrix $(c_{ij})$. This classification works for all genera. In genus two, we prove that the singularities of geometric type means: Weierstrass and conjugate rational tails, double cover of rational curves (hyperelliptic). Thus, one may also call the geometric type the Brill-Noether type. These are not too hard to handle when $g = 2$. In Theorems 2.7 and 2.8, the first two rounds of sequences of smooth blowups are along centers of topological type. Roughly, the first round deals with $(1 \times 1)$ minors; the second round deals with $(2 \times 2)$ minors. Both rounds ignore singularities of geometric type. Then the third rounds resolve singularities of Weierstrass and conjugate rational tails. The fourth round resolves singularities caused by hyperelliptic maps.
For general $g$, it is not hard for the author to “see” how to resolve singularities of topological type. The remainder singularities of geometric type are fortunately of high codimension. I believe that a canonical partial relative resolution that resolves singularities of topological type will be sufficient to prove Conjecture 2.3 (the details will be in a forthcoming publication).

4. Applications to the GW theory

In this section, we simply summarize the approach of [CL12].

The GSW invariants originate from Guffin and Sharpe’s work [GuSh08]. Let $\overline{M}_1(\mathbb{P}^4, d)^p$ be the DM stack of pairs $([u, C], p)$ with $[u, C] \in \overline{M}_1(\mathbb{P}^4, d)$ and $p \in H^0(C, u^*\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_C)$. This is the moduli of stale maps with $p$-fields. Chang and Li [CL11] showed that it admits with a perfect obstruction theory. We shall use the following shorthands

$X = \overline{M}_1(\mathbb{P}^4, d), \quad Y = \overline{M}_1(\mathbb{P}^4, d)^p$.

The polynomial $x_1^5 + \ldots + x_5^5$ induces a cosection of its obstruction sheaf $\sigma: \mathcal{O}_Y \rightarrow \mathcal{O}_Y$, whose degeneracy locus (where $\sigma$ fails to be surjective) is

$\overline{M}_1(Q, d) \subset Y$, \quad $Q = (x_1^5 + \ldots + x_5^5 = 0) \subset \mathbb{P}^4$,

which is proper. The cosection localized virtual class of Kiem-Li ([KL10]) defines a localized virtual cycle $[Y]_{\sigma}^{vir} \in A_0\overline{M}_1(Q, d)$. The GSW-invariant of the quintic $Q$ is

\begin{equation}
N_{1,d}^p = \deg[Y]_{\sigma}^{vir}.
\end{equation}

**Theorem 4.1.** ([CL11]) The GSW-invariant coincides with the GW-invariant of the quintic $Q$ up to a sign:

$N_{1,d}^p = (-1)^{5d} \cdot N_{1,d}$

where $N_{1,d}$ is the (usual) genus 1, degree $d$ GW invariant of the smooth quintic $Q \subset \mathbb{P}^4$.

This theorem translates the problem from over $X$ to over $Y$. To prove Conjecture 2.3, it requires to study the separation of the virtual cycle $[Y]_{\sigma}^{vir}$. For this, we need some structural results on its intrinsic normal cone. So, we let $(f_X, \pi_X) : C_X \rightarrow \mathbb{P}^4 \times X$ be the universal family of $X$; let

$(f_Y, \pi_Y) : C_Y \rightarrow \mathbb{P}^4 \times Y$, \quad $\psi_Y \in \Gamma(Y, \mathcal{P}_Y)$, \quad $\mathcal{P}_Y = f_Y^*\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_{C_Y/Y}$

be the universal family of $Y$. A perfect obstruction theory of $X$ relative to the Artin stack $\mathfrak{M}$ of genus $g$ nodal curves is ([BF97])

$\phi_{X/\mathfrak{M}} : (E_{X/\mathfrak{M}})^{\vee} \rightarrow L_{X/\mathfrak{M}}^*, \quad E_{X/\mathfrak{M}} := R^*\pi_X f_X^*T_{\mathbb{P}^4}$. 
Using the Euler sequence, a perfect obstruction theory of \( X \) relative to \( P_1 \) is (cf. [CL11])

\[
\phi_{X/P_1} : (E_{X/P_1})^\vee \to L_{X/P_1}^\ast, \quad E_{X/P_1} := R^\ast \pi_{X*}(L_X^{\oplus 5}), \quad L_X = f_X^\ast O(1).
\]

Chang and Li also worked out a perfect obstruction theory of \( Y \) relative to \( P_1 \):

\[
\phi_{Y/P_1} : (E_{Y/P_1})^\vee \to L_{Y/P_1}^\ast, \quad E_{Y/P_1} := R^\ast \pi_{Y*}(L_Y^{\oplus 5} \oplus P_Y), \quad L_Y = f_Y^\ast O(1).
\]

The cohomology sheaf

\[
\mathcal{O}_{b_{Y/P_1}} := H^1(E_{Y/P_1}) = R^1 \pi_{Y*}(L_Y^{\oplus 5} \oplus P_Y)
\]

is the relative obstruction sheaf of \( \phi_{Y/P_1} \).

The following theorem will be useful.

**Theorem 4.2.** We let \( \widetilde{M}_1^{\text{wt}} \to \mathcal{M}_1^{\text{wt}} \) be the successive blowup of \( \mathcal{M}_1^{\text{wt}} \) along the smooth closed substacks \( \Theta_2, \Theta_3, \ldots \), and so on. If we let

\[
\widetilde{M}_1(\mathbb{P}^4, d)^p = \widetilde{M}_1^{\text{wt}} \times_{\mathcal{M}_1^{\text{wt}}} \mathcal{M}_1(\mathbb{P}^4, d)^p,
\]

then we have

1. every (including the main) irreducible component of \( \widetilde{M}_1(\mathbb{P}^4, d)^p \) is smooth;
2. the entire DM stack \( \widetilde{M}_1(\mathbb{P}^4, d)^p \) has at worst normal crossing singularities; further
3. the object \( R^\ast \pi_{Y*}(L_Y^{\oplus 5} \oplus P_Y) \), upon pulling back to \( \widetilde{M}_1(\mathbb{P}^4, d)^p \), becomes locally diagonalizable.

This is analogous to Theorem 2.6 and can be proved parallely.

We let \( \widetilde{\mathcal{Y}} \) be \( \widetilde{M}_1(\mathbb{P}^n, d)^p \) and \( \widetilde{P}_1 = \mathcal{M}_1^{\text{wt}} \times_{\mathcal{M}_1^{\text{wt}}} P_1 \). Then we have a canonical morphism \( \widetilde{\mathcal{Y}} \to \widetilde{P}_1 \). By working out the relative perfect obstruction theory of \( \widetilde{\mathcal{Y}} \to \widetilde{P}_1 \), we obtain its obstruction complex \( E_{\widetilde{\mathcal{Y}}/\widetilde{P}_1} \). The intrinsic normal cone \( C_{\widetilde{\mathcal{Y}}/\widetilde{P}_1} \) of \( \widetilde{\mathcal{Y}} \to \widetilde{P}_1 \) is embedded in \( h^1/h^0(E_{\widetilde{\mathcal{Y}}/\widetilde{P}_1}) \).

The blowup \( \widetilde{X} \) is a union of smooth DM stacks: one, denoted \( \widetilde{X}' \), is the proper transform of the main component \( X' \) of \( X \), the rest are indexed by partitions \( \mu \) of \( d \). In formula,

\[
\widetilde{X} = \widetilde{X}' \cup (\cup_{\mu \vdash d} \widetilde{X}_\mu).
\]

Geometrically, general points of \( \widetilde{X}' \) are stable morphisms with smooth domain curves; general points of \( \widetilde{X}_\mu \) lie over stable morphisms whose domain curves are the union of a smooth genus one curve and connected trees of rational curves such that the genus one curve is contracted and
the degrees of the stable morphism on the connected trees form the partition \( \mu \). The blowup \( \tilde{Y} \) has the similar decomposition
\[
\tilde{Y} = Y' \cup (\cup_{\mu \vdash d} \tilde{Y}_\mu).
\]

Theorem 4.2 implies that

**Lemma 4.3.** Let \( C_{\tilde{Y}/\tilde{P}_1} \subset h^1/h^0(E_{\tilde{Y}/\tilde{P}_1}) \) be the intrinsic normal cone embedded in \( h^1/h^0(E_{\tilde{Y}/\tilde{P}_1}) \) via the obstruction theory \( \phi_{\tilde{Y}/\tilde{P}_1} \). Then

1. away from \( \cup_{\mu \vdash d} \tilde{Y}_\mu \), \( C_{\tilde{Y}/\tilde{P}_1} \) is the zero section of \( h^1/h^0(E_{\tilde{Y}/\tilde{P}_1}) \);
2. away from \( Y' \), it is a rank two subbundle stack of \( h^1/h^0(E_{\tilde{Y}/\tilde{P}_1}) \).

Lemma 4.3 allows us to conclude that the cone \( C_{\tilde{Y}/\tilde{D}} \) can be separated as
\[
[C_{\tilde{Y}/\tilde{D}}] = [C'] + \sum_{\mu \vdash d} [C_\mu],
\]
where \( C' \) is an irreducible cycle lying over \( \tilde{Y}_{\text{pri}} \); each \( C_\mu \) lies over \( \tilde{Y}_\mu \). (The cycles \( C_\mu \) need not to be irreducible.) Thus, applying Kiem-Li's cosection localized virtual class, we obtain

\[
\deg 0^1_{\tilde{\sigma},\text{loc}}[C_{\tilde{Y}/\tilde{D}}] = \deg 0^1_{\tilde{\sigma},\text{loc}}[C'] + \sum_{\mu \vdash d} 0^1_{\tilde{\sigma},\text{loc}}[C_\mu],
\]

where the cosection \( \tilde{\sigma} : \mathcal{O}b_{\tilde{Y}} \to \mathcal{O}_{\tilde{Y}} \) is induced from \( \sigma : \mathcal{O}b_{Y} \to \mathcal{O}_Y \).

Now, a general standard argument implies that
\[
\deg 0^1_{\tilde{\sigma},\text{loc}}[C'] = (-1)^{5d} N'_{1,d}.
\]

By some accurate dimensional arguments, one finds the vanishing
\[
\deg 0^1_{\tilde{\sigma},\text{loc}}[C_\mu] = 0
\]
for all \( \mu \neq (d) \) where \( (d) \) be the partition of \( d \) into a single part (i.e., the non-partition of \( d \)). Finally, Chang and Li calculate to prove
\[
\deg 0^1_{\tilde{\sigma},\text{loc}}[C_{(d)}] = \frac{(-1)^{5d}}{12} N_{0,d}.
\]

Putting these together, one gets \( N_{1,d} = N'_{1,d} + \frac{1}{12} N_{0,d} \).

5. Modular Resolution Program

Suppose we have a moduli space \( \mathfrak{M} \) and we let \( \tilde{\mathfrak{M}} \to \mathfrak{M} \) be a blowup. In general, it is not clear how to provide \( \tilde{\mathfrak{M}} \) a modular meaning. There are a few interesting cases where we know that the blowups have geometric meanings. Here we mention one class of such examples.

We let \( \text{Gr}(k,V) \) be the Grassmannian of \( k \)-dimensional subspaces in a vector space \( V \cong k^n \) where \( k \) is the base field \( (0 < k < n) \). Consider the variety \( Q_d \) of degree \( d \) maps from \( \mathbb{P}^1 \) to \( \text{Gr}(k,V) \). It comes with a naive
compactification, the Grothendieck Quot scheme $\overline{Q}_d (= \text{Quot}_{V_{p^1/p^1,k}}^{d,n-k})$.
The compactification is smooth but the boundary $\overline{Q}_d \setminus Q_d$ has rather intricate singularities. It comes equipped with a natural filtration by closed subsets

$$Z_{d,0} \subset Z_{d,1} \subset \cdots \subset Z_{d,d-1} = \overline{Q}_d \setminus Q_d$$

where $Z_{d,r}$ consists of non-locally free sheaves whose torsion parts have degree at least $d - r$.

**Theorem 5.1.** ([HLS11], [Shao11]) The singularities of $Z_{d,r}$ can be resolved by repeatedly blowing up $Z_{d,0}$, $Z_{d,1}$, \ldots, $Z_{d,d-1}$. Consequently, by successively blowing-up the Quot scheme $\overline{Q}_d$ along $Z_{d,0}$, $Z_{d,1}$, \ldots, $Z_{d,d-1}$, we obtain a smooth compactification $\widetilde{Q}_d$ such that the boundary $\widetilde{Q}_d \setminus Q_d$ is a simple normal crossing divisor.

Next, it is natural to ask whether $\widetilde{Q}_d$ admits any modular interpretation. We provide an affirmative answer to this question. For any coherent sheaf $E$ over $P^1$, we let $E_t$ denote the torsion subsheaf of $E$ and $E_f = E/E_t$ denote the locally free part of $E$.

**Definition 5.2.** A complete quotient of $V_{p^1}$ consists of $(E_0, \ldots, E_\ell)$ where $E_0$ is a coherent quotient sheaf of $V_{p^1}$, and for every $1 \leq i \leq \ell$, $E_i$ is a nonsplit extension of $E_i^{-1}$ by $E_i^{t-1}$ such that the last sheaf $E_\ell$ is the unique one that is locally free. (We allow $\ell = 0$.)

**Theorem 5.3.** ([HS13]) The variety $\widetilde{Q}_d$ in Theorem 5.1 parameterizes complete quotients of $V_{p^1}$ of degree $d$ and rank $n-k$.

There are further interesting questions. For $g = 1$, the analogous of $\overline{Q}_d$ is MOP’s moduli $\overline{SQ}_1(\text{Gr}(r,V),d)$ of stable quotients. A smooth blowup $\overline{SQ}_1(\text{Gr}(r,V),d) \to \overline{Q}_1(\text{Gr}(r,V),d)$, resolving all the boundary singularities, is currently being worked out by Thomas D. Maienschein ([Mai14]). The above should generalize to all high genera.

The author believes the following.

**Conjecture 5.4.** For every $g > 0$, there is a smooth moduli stack $\mathcal{M}$ admitting a dominating morphism $\overline{M}_g(\mathbb{P}_n,d) \to \mathcal{M}$; further, there is also another smooth moduli stack $\widetilde{M}$ admitting a birational dominating morphism to $\mathcal{M}$ such that if we let

$$\widetilde{M}_g(\mathbb{P}_n,d) = \widetilde{M} \times_\mathcal{M} \overline{M}_g(\mathbb{P}_n,d),$$

then $\widetilde{M}_g(\mathbb{P}_n,d)$ inherits moduli meaning. Further, suppose $d > 2g - 2$, then we have
(1) the main component of $\tilde{M}_g(\mathbb{P}^n, d)$ is smooth;
(2) the entire stack $\tilde{M}_g(\mathbb{P}^n, d)$ has at worst normal crossing singularities.

This is a kind of reformulation of Conjecture 2.5. Here, the smoothness can subject to re-definition. For $g = 1$, it is believed that $\tilde{M}$ can be taken to be the moduli of weighted genus one curves with certain admissible log structures (this is due to Qile Chen).

Beyond reach, the author ends this note with an unlimitedly wild speculation.

**Conjecture 5.5.** Let $\mathcal{X}$ be a singular moduli stack with the main component $\mathcal{X}'$. Then under some reasonable hypotheses on $\mathcal{X}$, there is another moduli stack $\tilde{\mathcal{X}}$ with a smooth main component $\tilde{\mathcal{X}}'$, at worst normal crossing singularities, and admitting a dominant morphism to $\mathcal{X}$ such that it restricts to a birational morphism $\tilde{\mathcal{X}}' \to \mathcal{X}'$.

Derived from Anna Karenina,

*Regularities are all alike; every singularity is singular in its own way.*

The conjecture would say that every singularity can be resolved because there are geometric reasons to let us achieve it (not because we can implant an algorithm). de Jong’s alteration has this flavor. We hope to furnish more interesting examples of this in the future publications.

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