Fast Track Communication

Black hole horizons and quantum charged particles

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Received 8 February 2015
Accepted for publication 23 April 2015
Published 28 May 2015

Abstract

We point out a structural similarity between the characterization of black hole apparent horizons as stable, marginally outer trapped surfaces (MOTS) and the quantum description of a non-relativistic charged particle moving in given magnetic and electric fields on a closed surface. Specifically, the spectral problem of the MOTS-stability operator corresponds to a stationary quantum particle with a formal fine-structure constant $\alpha$ of negative sign. We discuss how such analogy enriches both problems, illustrating this with the insights into the MOTS-spectral problem gained from the analysis of the spectrum of the quantum charged particle Hamiltonian.

Keywords: black holes, quantum charged particle, spectral geometry

1. Introduction: a formal analogy

Analogies between physical systems, either of mathematical or physical nature, often play a fundamental catalyst role in conceptual and/or technical developments of the respective theories [1]. We discuss here a mathematical analogy between the descriptions of black hole horizons and quantum charged particles (QCPs), that opens a domain of cross-fertilization between quantum mechanics and gravitation theory. More specifically, apparent horizons—namely marginally outer trapped surfaces (MOTS)—possess a stability notion that guarantees their physical consistency as models of black hole horizons. Such MOTS-stability notion [2] admits a spectral characterization in terms of the so-called principal eigenvalue of the operator

$$L_S = -\Delta + 2\Omega^a D_a - \left( |\Omega|^2 - D_a \Omega^a - \frac{1}{2} R_S + G_{ab} k^a \ell^b \right),$$

defined on the apparent horizon $S$. The terms modifying the Laplacian $\Delta$ on $S$ are determined by the intrinsic and extrinsic geometry of the apparent horizon and the gravitational field equations via the Einstein tensor $G_{ab}$ (see next section for details). The relevant remark in the
present context is that under the complexification of the vector $\Omega^a$ and the identifications

$$\Omega^a \leftrightarrow \frac{ie}{\hbar} A^a, \quad R_S \leftrightarrow \frac{4me}{\hbar^2} \phi, \quad G_{ab} k^a \phi_k \leftrightarrow -\frac{2m}{\hbar^2} V,$$

the MOTS-stability operator becomes $\frac{\hbar^2}{2m} L_S \leftrightarrow \hat{H}$, where

$$\hat{H} \equiv -\frac{\hbar^2}{2m} \Delta + \frac{i\hbar}{mc} A^a D_a + \frac{i\hbar}{2mc} D_a A^a + \frac{e^2}{2mc^2} A^a A^a + e\phi + V$$

$$= \frac{1}{2m} \left( -\hbar \phi - \frac{e}{c} A \right)^2 + e\phi + V,$$

is the Hamiltonian of a non-relativistic particle with mass $m$ and charge $e$ moving on $S$ under magnetic and electric fields with vector and scalar potentials given by $A^a$ and $\phi$, and an external potential $V$. This formal mathematical analogy relies on a simple but crucial remark: the derivative and $\Omega^a$ terms in $L_S$ can be collected in a perfect square as follows

$$L_S \psi = \left[ -(D - \Omega)^2 + \frac{1}{2} R_S - G_{ab} k^a \phi_k \right] \psi.$$ 

Beyond its aesthetic appeal, and in spite of the key difference in the self-adjoint nature of the operators, this analogy has the potential to open bridges between the well-studied quantum particle problem and the rich but largely uncharted MOTS subject, with applications ranging from the MOTS-spectral problem to the spinorial formulation of MOTS stability. We focus here on the study of the full $L_S$ spectrum, a challenging problem formulated in [3] in the setting of a black hole/fluid analogy, but with a definite geometric interest on its own.

2. Geometry and stability of MOTS

2.1. Some MOTS geometry

Let us consider a codimension-2 surface $S$, spacelike, closed (compact and without boundary) and orientable, embedded in a $n$-dimensional spacetime $(M, g_{ab})$. The spacetime Levi-Civita connection is denoted by $\nabla_a$ with Einstein curvature tensor $G_{ab}$. Let us denote the induced metric on $\Sigma$ by $q_{ab}$, with Levi-Civita connection $\nabla_a$, Ricci scalar $R_S$, volume form $\epsilon_{a_1...a_{n-2}}$ and associated integration measure $\Sigma$. Let us consider future-oriented null vectors $\ell^a$ and $k^a$ spanning the normal bundle $T^\perp \Sigma$ and normalized as $\ell^a k_a = -1$. This normalization leaves a null-vector-rescaling freedom by a positive function $f > 0$ respecting time orientation, corresponding to a boost transformation

$$\ell'^a = f \ell^a, \quad k'^a = f^{-1} k^a.$$ 

We define the expansion $\theta^{(\ell)}$ and the Hájíček or rotation form $\Omega_a$

$$\theta^{(\ell)} \equiv q^{ab} \nabla_a \ell_b, \quad \Omega_a \equiv -k^c q_{bd} \nabla_d \ell^c,$$

associated with $\ell^a$. Considering $v^a = \alpha \ell^a + \beta k^a$, we can write $q_{ab} \nabla_c v_b = D^a_v \nu_b + \Theta^{(v)}_{ab}$, with $\Theta^{(v)}_{ab} \equiv q_{a} q^{d}_{b} \nabla_{d} v_{c}$ and

$$D^a_v \nu_b = (D_a \nu + \Omega_a \alpha) \ell_b + (D_a \beta - \Omega_a \beta) \nu_b.$$ 

The Hájíček form therefore provides a connection on the normal bundle $T^\perp \Sigma$ for the tangent derivative of normal vectors. From a physical perspective it represents a sort of angular momentum density. Given an axial Killing vector $\phi^a$ on $S$,
\[
J[\phi] = \frac{1}{8\pi} \int_S \Omega_a \phi^a \eta_S, \tag{8}
\]
is the (Komar) angular momentum associated with \(S\). Regarding the expansion, the surface \(S\) is a MOTS if it satisfies the condition: \(\theta^{(\ell)} = 0\). We refer then to \(\ell^a\) as outgoing and to \(k^a\) as ingoing null vectors.

### 2.2. MOTS stability and MOTS-stability operator

A MOTS \(S\) is said to be stable (more properly, stably outermost [2, 4]) in the ingoing \(k^a\) direction if it can be infinitesimally deformed along \(k^a\) into a properly (outer) trapped surface \(S'\), i.e. with \(\theta^{(\ell)}|_{S'} < 0\). Using the deformation operator \(\delta_\ell\) along a normal vector \(v^a\) 1, this amounts to the existence of a function \(\psi > 0\) defined on \(S\) such that \(\delta_\ell \psi^{(\ell)} < 0\). Such a condition admits a spectral characterization in terms of the elliptic operator \(L_S\) defined as \(L_S \psi \equiv \delta_\ell \psi^{(-k)}\), with explicit expression (1). The operator \(L_S\), namely the MOTS-stability operator, is generically non-selfadjoint [in \(L^2(S, \eta_S)\)] due to the \(2\Omega^a D_k\) term. Therefore, in the eigenvalue problem
\[
L_S \psi_n = \lambda_n \psi_n, \tag{9}
\]
the \(\lambda_n\)'s are generically complex. Their real part is bounded below, leading to the definition of the principal eigenvalue \(\lambda_o\) of \(L_S\) as that with smallest real part. Lemmas 1 and 2 in [2] state that: (i) \(\lambda_o\) is real, and (ii) \(S\) is stably outermost iff \(\lambda_o > 0\).

### 2.3. MOTS-gauge symmetry

The MOTS geometry described above does not depend on the choice of null normals (subject to \(\ell^a k_a = -1\)). In this sense, the null vector rescaling freedom (5) is a gauge transformation of the MOTS geometry. We consider now the transformation of the main objects on \(S\) under the rescaling (5).

**Lemma 1.** ([MOTS-gauge transformations]) Under the null normal rescaling \(\ell^a = f \ell^a\), \(k^a = f^{-1} k^a\), with \(f > 0\):

(i) The expansion and Hájíček form transform as
\[
\theta^{(\ell)} = f \theta^{(\ell)}; \quad \Omega'_a = \Omega_a + D_a (\ln f). \tag{10}
\]

(ii) The MOTS-stability operator transforms covariantly
\[
(L_S) \psi = f L_S (f^{-1} \psi), \tag{11}
\]
where \((L_S) \psi \equiv \delta_\ell \psi^{(-k)}\).

(iii) The MOTS-eigenvalue problem is invariant under the additional eigenfunction transformation, \(\psi' = f \psi\)
\[
L_S \psi = \lambda \psi \rightarrow (L_S) \psi' = \lambda \psi'. \tag{12}
\]

1 The variation \(\delta_\ell \theta^{(\ell)}\) of \(\theta^{(\ell)}\) at \(S\), along the vector \(v^a\) at \(S\), is \(\delta_\ell \theta^{(\ell)} \equiv \partial_\ell \theta^{(\ell)} |_{v=0}\), for a one-parameter family of surfaces \(S_\tau\), such that \(S_{\tau=0} = S\), and where \(v^a_\tau = \frac{\partial}{\partial \tau}|_{\tau=0} \) [2, 4].
Proof. Point (i) follows directly by plugging (5) into (6). Regarding point (ii), although it can be obtained by straightforward substitution of (10) into (1), it is simpler to use the definition of $L_{S}$. Considering its action on a function $\psi$

\[
(L_{S})^\ell \psi = \delta_{\psi(-k)} \theta^{\ell} = \delta_{\psi(-k)} \left( f\theta^{\ell} \right),
\]

\[
= \delta_{\psi(-k)} \left( f^{\ell} \right) + f\delta_{\psi(-k)} \theta^{\ell} = f\delta_{\psi(-k)} \theta^{\ell},
\]

\[
= f\delta_{\psi(-k)} \left( f^{\ell} \right) = f\delta_{f^{\ell}(-k)} \theta^{\ell} = L_{S} \left( f^{\ell} \right).
\]

(13)

where in the first line we have used the $\theta^{\ell}$ transformation in (10), the second line uses the Leibniz rule holding for $\delta_{\psi}$ and the MOTS condition, and in the third line we used again the definition of $L_{S}$. Finally, point (iii) follows directly

\[
(L_{S})^\ell \psi' = fL_{S} \left( f^{-1} \psi' \right) = fL_{S} \left( \psi \right) = f\lambda \psi = \lambda \psi'.
\]

(14)

Point (i) just states the invariance of the MOTS notion under (5) and the transformation of $\Omega_{a}$ as a connection under the (multiplicative) Abelian gauge group $\mathbb{R}^{+}$ of positive null rescalings. The latter is consistent with the nature of $\Omega_{a}$ in (7) as a connection in the normal bundle. Point (ii), stating the good (covariant) transformation properties of $L_{S}$ under $\mathbb{R}^{+}$, is the analogue in the present setting of proposition 4 in [5] concerning the free choice of section of stationary black hole horizons. Point (iii) guarantees that the MOTS-eigenvalue problem is well-defined and provides the gauge transformation rule for the associated eigenfunctions. Of course, all these points evoke familiar features of the QCP.

3. MOTS and QCPs

To take a step further from the formal correspondence (2) into a more precise statement, let us review the stationary QCP problem. The Schrödinger equation for a non-relativistic (spin-0) charged particle moving in electromagnetic fields with magnetic vector potential $A_{a}$ and electric potential $\phi$, namely $i\hbar \partial t \Psi = \hat{H} \Psi$ [with $\hat{H}$ in (3)], follows from that of a non-charged particle in an external mechanical potential $V$ via a minimal-coupling prescription

\[
i\hbar \partial_i \rightarrow i\hbar \partial_i - e\phi, \quad -i\hbar D_{a} \rightarrow -i\hbar \sigma = \frac{e}{c} A_{a}.
\]

(15)

The stationary equation for the energy eigenvalues $E$ is then

\[
\left[ \frac{1}{2m} \left( -i\hbar \sigma - \frac{e}{c} A_{a} \right)^{2} + e\phi + V \right] \Psi = E \Psi,
\]

(16)

where $\Psi = e^{-iE \sigma \partial t} \Psi$ (with $\partial t \Psi = 0$). This equation should not depend on the gauge choice of the electromagnetic potentials. The gauge transformation of $A_{a}$ by a total gradient

\[
A_{a} \rightarrow A_{a} - D_{a} \sigma,
\]

(17)

leaves equation (16) invariant if we simultaneously transform $\Psi$ as

\[
\Psi \rightarrow e^{i\sigma (c \hbar)} \Psi,
\]

(18)

2 The electric potential $\phi$ stays invariant $\phi \rightarrow \phi - \frac{1}{c} \partial \sigma$ under gauge transformations compatible with stationarity, $\partial \sigma = 0$. 
i.e. by a (local) phase. Transformations (17) and (18) define the electromagnetic Abelian $U(1)$-gauge symmetry. From these remarks we can state the following similarities between the eigenvalue problems (9) and (16), placing the MOTS-QCP analogy in (2) on a sounder structural basis$^3$:

(i) Abelian gauge symmetry. The QCP eigenvalue problem (16) and the MOTS-spectral problem (9) are respectively invariant under transformations (cf. equations (17), (18) and lemma 1)

$$AODQCP: \mathbf{e} \rightarrow e^{i\sigma\epsilon(\chi)}\mathbf{e},$$

$$\OmegaODMOTS: \mathbf{e} \rightarrow e^{-\sigma}\mathbf{e}.$$ (19)

They both define Abelian symmetries of gauge nature: in the QCP case it is the electromagnetic $U(1)$-gauge transformation ($g^e = g \cdot g'$, with $g = e^{i\sigma\epsilon(\chi)}$) relying on the phase invariance of the wave function, whereas for MOTSs it defines a non-compact $\mathbf{R}^r$-gauge counterpart ($g^e = g \cdot g'$, with $g = f = e^{-\sigma}$) reflecting the in-built null rescaling (boost) freedom of the MOTS geometric description. In brief, the MOTS-spectral problem presents symmetry transformation properties in full analogy with those of the stationary QCP Schrödinger equation.

(ii) Gauge field potential. In this symmetry setting, the 1-form $\Omega$ emerges as the natural gauge field of the $\mathbf{R}^r$-gauge group. This endorses, at a structural level, its purely formal correspondence in (2) with the $A_\mu$ magnetic $U(1)$-gauge field. We note that the normal connection in (7) admits an interpretation as a gauge connection: $\ell D^k_\mu (\psi k_\mu) = -(D_\mu - \Omega_\mu) \psi$.

(iii) Minimal coupling. It is at a ‘dynamical’ level where the analogy proves remarkable: the $\Omega_\mu$ field enters in $L$ via a standard gauge ‘minimal coupling’ mechanism, namely a shift in the Levi-Civita connection with the gauge connection

$$D_\mu \rightarrow D_\mu - \Omega_\mu.$$ (20)

This becomes apparent in the perfect-square version (4) of $L$. Therefore, in full analogy with the minimal coupling mechanism for incorporating the magnetic field in the QCP problem via the shift (15) in the non-charged equations, rotation in a MOTS is switched-on via the minimal coupling (20).

3.1. MOTSs and a negative ‘fine-structure constant’ $\alpha$

Setting $\hbar = m = c = 1$ and introducing a formal complex ‘fine-structure constant’ $\alpha \equiv e^2$, we define the operator family

$$L = \frac{1}{2} \left[ D - i \sqrt{\alpha} \Omega \right]^2 - \frac{\alpha}{4} R_S - \frac{1}{2} G_{ab} k^a q^b,$$

$$= \frac{1}{2} \left[ D + i \sqrt{\alpha} \Omega \right]^2 + \frac{\alpha}{4} R_S - \frac{1}{2} G(k, \ell).$$ (21)

The QCP Hamiltonian corresponds to the (normalized) standard real positive $\alpha = 1$, whereas (half) the MOTS-stability operator corresponds to a negative $\alpha = -1$. Specifically, QCP and MOTS operators are recovered with branch choices: $\hat{H} = L[\sqrt{\alpha} = 1]$ and $L_S / 2 = L[\sqrt{\alpha} = -1]$. In this sense, stable MOTSs can be seen as QCPs with negative ‘fine-structure constant’ $\alpha$. This suggests to import QCP terms to MOTSs.

$^3$ A further point in this list of similarities could be the understanding of MOTS-stability, $\lambda_\alpha > 0$, as a MOTS-counterpart of a positivity condition on the quantum ground state $E_0$, refining quantum stability. This is however delicate, since the operator correspondence (2) does not necessarily preserve eigenvalue signs (see section 4).
Terminology for $L_S$ terms. Regarding terms containing the rotation field $\Omega_a$, we refer to $|\Omega|^2$ as the diamagnetic term, whereas $\Omega \cdot D$ is the paramagnetic term [6]. The divergence $D \cdot \Omega$ is a gauge-fixing term and can be chosen by an appropriate transformation (19). For completeness sake, $\Delta$ is the kinematical term and $G(k, \ell)$ is the external mechanical potential. Finally, $R_S/4$ can be referred to as the electric potential term. To justify its explicit distinction from the external mechanical potential, we consider in the 2-dimensional case the complex scalar $\mathcal{K}$ on $S$ introduced by Penrose and Rindler [7] as

$$\mathcal{K} = \frac{1}{4} R_S + \frac{1}{4} e^{ab} F^\Omega_{ab},$$

(22)

where $F^\Omega_{ab} = D_a \Omega_b - D_b \Omega_a$, namely the curvature of $\Omega_a$. The real and imaginary parts of $\mathcal{K}$ correspond, respectively, to electric and magnetic terms. This gravity/electromagnetic analogy has been used to discuss isolated/dynamical horizon source multipoles [8] and to introduce the notions of ‘vortexes’ and ‘tendexes’ in the analysis of dynamical black holes [9]. The present discussion promotes such analogy to a sounder structural level by identifying the symmetry and minimal coupling similarities in the relevant operators.

4. MOTS-spectral problem

4.1. Analyticity in the ‘fine-structure constant’

The fact that the MOTS-stability operator can be obtained from the QCP Hamiltonian as an analytic continuation of $L[\sqrt{\alpha}]$ ($\sqrt{\alpha} = 1 \rightarrow \sqrt{\alpha} = -i$) raises the following question: can we recover the MOTS-spectrum ($\alpha = -1$) as an analytic extension of the QCP spectrum ($\alpha = 1$) self-adjoint problem?

This question dwells naturally in the perturbation theory of linear operators (where $L[\sqrt{\alpha}]$ defines a self-adjoint holomorphic family of type (A) [10]), but giving a fully general answer defines a difficult problem. A given eigenvalue $\lambda_{\sqrt{\alpha}, n}$ can be analytically continued along its $\sqrt{\alpha}$-path in the complex plane, as long as its evolution does not encounter (for the same $\sqrt{\alpha}$) another eigenvalue $\lambda_{\sqrt{\alpha}, m}$. But checking this is a hard task (even in the explicit example below in 4.2). On the other hand, our particular setting is free of two potential threats for the analyticity discussion, namely boundaries and function pathologies: (i) $S$ has no boundaries (is closed), and (ii) the functions in $L[\sqrt{\alpha}]$, being induced from the ambient smooth geometry, can be taken as regular as needed. As a third point (iii), potential topological issues associated to the underlying $U(1)$ or $\mathbb{R}^+$-fibre bundle are absent since such bundle is trivial (we are excluding here the possibility of a non-trivial NUT charge). Motivated by these points we propose the following:

Analyticity conjecture. Given an orientable closed surface $S$ and the one-parameter family of operators $L[\sqrt{\alpha}]$ defined in (21), the MOTS-spectrum ($\sqrt{\alpha} = -i$) can be recovered as an ‘analytic continuation’ of the QCP spectrum ($\sqrt{\alpha} = 1$).

We present this conjecture as an open problem. In case the conjecture proves to be valid\(^4\), the MOTS-stability spectrum problem would be ‘essentially’ reduced to that of the self-adjoint problem of the stationary non-relativistic QCP.

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\(^4\) Even without a continuation argument, the prescription $\sqrt{\alpha} \rightarrow -i \sqrt{\alpha}$ to recover MOTS spectra from QCPs could still hold.
4.2. An explicit example: ‘MOTS-Landau’ levels

To gain support and insight into the analyticity conjecture, and following the spirit of Landau levels of a QCP in a constant magnetic field (that provides an explicit example illustrating basic features of such quantum systems), we discuss a simple eigenvalue problem that can be explicitly solved in the MOTS case and, independently, in the analogous QCP case.

Let us take a 2-sphere $S = S^2$ with ‘round’ metric $g_{ab} = r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$. Decomposing the Hájíček form as $\Omega_a = e^{i\rho} D_a \omega + D_b \zeta$ with the simplest non-trivial choice $\omega = a \cos \theta, \zeta = 0 \ (a \in \mathbb{R})$, we have: $\Omega = a \sin^2 \theta d\phi$. In vacuum (i.e. $G_{ab} = 8\pi T_{ab} = 0$), the solution to the MOTS-eigenvalue problem for the resulting $L_S$ is given explicitly by

$$\lambda = \frac{1}{r^2} \left[ (\lambda_{tm}(a) + 1 - a^2) + i \ 2am \right], \quad \psi = S_{tm}(a, \cos \theta)e^{i\omega}\psi (23)$$

where $S_{tm}(a, \cos \theta)$ are the prolate spheroidal functions with eigenvalues $\lambda_{tm}(a)$ (21.6.2 in [11]). Standard spherical harmonics are recovered in the limit $a \to 0$: $\lambda_{tm}(a) \to \ell (\ell + 1)$ and $S_{tm}(a, \cos \theta) \to \ell \rho_{tm}(\cos \theta)$. We can now consider the ‘QCP counterpart’ by performing the complex rotation $a \to ia$ in the operator $L_S$. The resolution of the new eigenvalue problem leads to eigenvalues $\bar{\lambda}$ and eigenfunctions $\bar{\psi}$

$$\bar{\lambda} = \frac{1}{r^2} \left( \lambda_{tm}(a) + 1 + a^2 - 2am \right), \quad \bar{\psi} = \bar{S}_{tm}(a, \cos \theta)e^{i\omega}\bar{\psi} (24)$$

where $\bar{S}_{tm}(a, \cos \theta) = S_{tm}(ia, \cos \theta)$ are now the oblate spheroidal functions with eigenvalues $\lambda_{tm}(a)$ (see 21.6.4 and 21.7.5 in [11]; note that $\lambda_{tm}(ia) \notin \mathbb{R}$). Therefore at least in this simple example, that contains the basic qualitative features we expect to be relevant (namely non-trivial rotation and potential obstructions associated with topology), the analyticity conjecture is verified: eigenvalues $\lambda$ (eigenfunctions $\psi$) of the operator $L_S = 2L_{[-ia]} \psi$ can be actually recovered by solving for the ‘rotated’ $a \to ia$ self-adjoint operator $L_{[a]}$, and then inverting the rotation $a \to -i a$ in the resulting eigenvalues $\bar{\lambda}$ (eigenfunctions $\bar{\psi}$).

4.3. Ground state of the charged particle

As an ‘inverse’ application of the conjecture, we consider the use of a MOTS result to calculate the ground state energy $E_o$ of QCPs. In [4] a variational Rayleigh–Ritz-like expression for $\lambda_{\rho_S}$ is presented. This remarkable result does not follow from the Rayleigh–Ritz characterization, since $L_S$ is generically not self-adjoint. The expression for $\lambda_{\rho_S}$ is rather obtained by starting from a min–max characterization by Donsker and Varadhan [12], valid for real but not necessarily self-adjoint operators. If the conjecture above proves true, the ‘inverse rotation’ $\sqrt{\alpha} \to i\sqrt{\alpha}$ in the $\lambda_{\rho_S}$ of [4] results in a QCP $E_o$

$$E_o = \inf_{\psi > 0} \int_S \left[ |D \psi|^2 + \left( e\phi + V + e^2 |D \omega| + \omega \right) \psi^2 \right] \eta_S, (25)$$

where $A_S = z + D_b \zeta$ (with $D_b z = 0$), $\int_S \psi^2 \eta_S = 1$ and $\omega$ satisfies, for a given $\psi > 0$, the constraint equation

$^5 L_{[-ia]}$ assumes implicitly a Hájíček form $\Omega = \sin^2 \theta d\phi$. 

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−Δωψ − 2
ψDψDψ = 2
ψeDψψ.

(26)

This expression for $E_o$, ‘blindly’ transported from the MOTS result, has two virtues as compared with the straightforward evaluation of the Rayleigh–Ritz $E_o = \inf_{\|\psi\|=1} \int_S \psi^* \hat{H} \psi \eta_S$: (i) it is explicitly gauge-invariant, since the term $D_a \hat{A}' (= \Delta \zeta)$ is absent; and (ii) the paramagnetic term is recast as a diamagnetic one, something of potential interest in numerical strategies. Then, the crucial point is that the ‘blind’ expression (25) for QCPs can actually be proved: starting from the (now valid) Rayleigh–Ritz expression and adapting the steps in [4] to the MOTS/QCPs analogy, expression (25) follows. This is remarkable since, although it is known [12] that Rayleigh–Ritz and Donsker–Varadhan expressions coincide when they both apply (namely, self-adjoint real operators), they cannot be generically reduced to one another (in particular in our setting $A_o \neq 0$). Therefore, the fact that (25) holds in the fully general case offers a (second) non-trivial test of the conjecture.

4.4. Semi-classical approach and $L_S$-spectral zeta function

The problem of studying the full spectrum of $L_S$ was formulated in [3] as a probe into the MOTS geometry, with possible applications in horizon (in)stability issues such as super-radiance. In particular, ‘excited eigenvalues’ $\lambda_n$’s control the MOTS dynamical evolution through $L_S C = \mathcal{F}_{ext}$, that relates the reaction of the MOTS (encoded in the function $C$) to an external incoming flux of energy $\mathcal{F}_{ext}$ (see equation (13) in [3]).

In this full MOTS-spectrum context, approximate expressions for eigenvalues and eigenfunctions could be obtained by importing semi-classical tools for QCPs. Starting from the self-adjoint ‘rotated’ $L_S$ and reverting the ‘quantization rule’ $p_i \rightarrow -i \alpha_i$, we introduce the classical Hamiltonian function

$$H_{cl}(\sqrt{\alpha|x,p|} = \left(p - \sqrt{\alpha}\Omega\right)^2 + \frac{1}{2} R_S - G(k, \ell),$$

(27)

defined on the cotangent $T^* S$. Assuming the analyticity conjecture, approximate expressions for the MOTS-spectrum can be obtained by applying semi-classical tools on $H_{cl}(\sqrt{\alpha})$ and then evaluating the explicit results on $\sqrt{\alpha} \rightarrow -i$.

In parallel to these considerations, given an operator $L$ its spectral zeta function $\zeta_L(s)$ provides a powerful and systematic manner of encoding its spectrum. Assuming again the analyticity conjecture, the self-adjoint eigenvalue problem of $L[\sqrt{\alpha}]$ in (21), i.e. $L[\sqrt{\alpha}] \psi = \lambda \sqrt{\alpha}^n \psi$ (with $\alpha > 0$), can be used to formally define a spectral zeta function for $L_S$, i.e. $\xi_{\lambda_S}(s)$, by analytic continuation as follows

$$\xi_{\lambda_S}(s) = \sum_n \left(\frac{1}{2\lambda_S^\alpha n}\right)^s, \xi_{\lambda_S}(s) \equiv \lim_{\sigma \rightarrow -i} \xi_{2\lambda_S[\sqrt{\sigma}]\psi}(s).$$

(28)

As an application, the zeta-regularized determinant of $L_S$ introduced as $\det(L_S) = e^{-\xi_{L_S}(0)}$, could be used to explore near-extremal MOTS properties, due to its vanishing in this limit (since $\lambda_o = 0$ for an extremal MOTS). Other applications, analogous to the Aharonov–Bohm ground state estimations in [15] (with potential implications in the important problem of approximating $\lambda_o$ in Kerr) or the analysis of genericity aspects of MOTS-spectrum statistics

$^6$ WKB techniques could be appropriate in separable problems, but generic cases would need to resort to the rich semi-classical tools developed in quantum chaos studies (e.g. [13, 14]).
result from combining spectral zeta function techniques with the semi-classical tools sketched above.

5. Conclusions and perspectives

We have introduced an analogy between stable MOTS and quantum particles, namely discussing MOTS as formally ‘negative fine-structure constant’ QCPs. The ultimate motivation behind this analogy is to explore the transfer of concepts and tools between both problems. This can prove fruitful for the MOTS-spectral problem by profiting from the accumulated knowledge about QCP stationary states. A conjecture has been proposed that might motivate mathematical studies on the spectrum analytic continuation of the MOTS-stability operator (magnetic Hamiltonian). In parallel, the development of spinor treatments of MOTS-stability can largely benefit from the presented analogy. In particular, using the Lichnerowicz–Weitzenböck formula to mimic Pauli’s approach to spin provides insight into the structure of the MOTS second-order operator $L_S$ (e.g. $L_S$ can be expressed in terms of a codimension-2 Sen connection), whereas Dirac’s approach to spin opens an avenue to a first-order formulation of MOTS-stability (with interest in boundary value problems of spinor equations). The GHP formalism [17] can offer additional insights in this setting. Spinor approaches and further applications, such as Kerr MOTS-stability and superradiance, will be studied in future.

Acknowledgments

I thank A Afriat, C Aldana, V Aldaya, L Andersson, M Ansorg, A Ashtekar, C Barceló, J Bičák, S Dain, M E Gabach-Clement, Q Hummel, B Krishnan, A Marquina, M Mars, G Mena-Marugán, J P Nicolas, I Rácz, M Reiris, L Rezzolla, M Sánchez, J M M Senovilla, W Simon, S Nonnenmacher, J D Urbina and J A Valiente-Kroon. I fondly thank E Alcalá, whose inspiring lively images made this research possible.

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