DECENTRALIZED CRITICAL OBSERVERS OF NETWORKS OF FINITE STATE MACHINES AND MODEL REDUCTION

DAVIDE PEZZUTI, GIORDANO POLA, ELENA DE SANTIS AND MARIA D. DI BENEDETTO

Abstract. This paper deals with the analysis of critical observability and design of observers for networks of Finite State Machines (FSMs). Critical observability is a property of FSMs that corresponds to the possibility of detecting immediately if the current state of an FSM reaches a set of critical states modeling unsafe operations. This property is relevant in safety-critical applications where the timely recovery of human errors and device failures is of primary importance in ensuring safety. A critical observer is an observer that detects on-line the occurrence of critical states. When a large-scale network of FSMs is considered, the construction of such an observer is prohibitive because of the large computational effort needed. In this paper, we propose a decentralized architecture for critical observers of networks of FSMs, where on-line detection of critical states is performed by local critical observers, each associated with an FSM of the network. For the design of local observers, efficient algorithms were provided which are based on on-the-fly techniques. Further, we present results on model reduction of networks of FSMs, based on bisimulation equivalence preserving critical observability. The advantages of the proposed approach in terms of computational complexity are discussed and examples offered.

1. Introduction

In recent years, Cyber-Physical Systems have been intensively investigated by both academic and industrial communities because they offer a solid paradigm for modeling, analysis and control of many next generation large-scale, complex, distributed and networked engineered systems. Among them, safety-critical applications, as for example Air Traffic Management systems, play a prominent role. Ensuring safety in large-scale and networked safety-critical applications is a tough but challenging problem. In particular, complexity is one of the most difficult issues that must be overcome to make theoretical methodologies applicable to real industrial applications.

In this paper, we address the analysis of critical observability and design of observers for networks of Finite State Machines (FSMs). A network of FSMs is a collection of FSMs whose interaction is captured by the notion of parallel composition. Critical observability is a property that corresponds to the possibility of detecting immediately if the current state of an FSM reaches a set of critical states, modeling unsafe operations. This notion has been introduced in [5] for linear switching systems and is relevant in safety-critical applications where the timely recovery of human errors and device failures is of primary importance in ensuring safety.

Current approaches available in the literature to check critical observability are based on regular language theory as in [6] or on the design of the so-called critical observers [2, 5]. The computational complexity of the first approach is polynomial in the number of states of the FSM, while the one of the second is exponential. Although disadvantageous from the computational complexity point of view, the construction of critical observers cannot be avoided at the implementation layer since it is necessary for the automatic on-line detection of critical situations. Motivated by this issue, we elaborated on some results which can reduce, in some cases drastically, the computational effort in designing critical observers for large-scale networks of FSMs. We first propose a decentralized architecture for critical observers of the network, which is composed of a collection of local critical observers, each associated with an FSM of the network. Efficient algorithms for the synthesis of critical observers are proposed, based on on-the-fly techniques for verification and control of FSMs (see e.g. [4, 15]). We then propose some results on model reduction based on bisimulation equivalence [8, 10].

The research leading to these results has been partially supported by the Center of Excellence DEWS.
which plays a fundamental role for mitigating complexity in formal verification. We reduce the original network of FSMs to a smaller one, obtained as the quotient of the original network induced by the bisimulation equivalence. In the reduction process, FSMs composing the network are never composed, a key factor in complexity reduction. We first show that critical observability of the original network implies and is implied by the critical observability of the quotient network. We then show that a decentralized critical observer for the original network can be easily derived from the one designed for the quotient network. The advantages of the proposed approach in terms of computational complexity are discussed and illustrated through examples. Critical observability is closely related to fault diagnosability (see e.g. [18] and the references therein). While critical observability requires the immediate detection of a critical state, diagnosability allows for a finite delay before fault detection; moreover, while critical states are needed to be detected whenever they are reached, faults are detected only the first time they are reached. Decentralized diagnosability for networks of FSMs that are never composed, thereby allowing model reduction at a higher level of abstraction, and therefore being more effective.

This paper is organized as follows. Section 2 introduces notation, networks of FSMs and the critical observability property. Section 3 presents the main results of the paper concerning decentralized critical observers and model reduction. Illustrative examples are reported in Section 4. Section 5 offers concluding remarks and outlook. We include in the Appendix the proofs of some technical results.

2. Networks of Finite State Machines and critical observability

In this section, we start by introducing our notation in Subsection 2.1. We then recall the notions of networks of finite states machines in Subsection 2.2 and critical observability in Subsection 2.3.

2.1. Notation and preliminary definitions. The symbols $\land$ and $\lor$ denote the $And$ and $Or$ logical operators, respectively. The symbol $\mathbb{N}$ denotes the set of nonnegative integers. Given $n, m \in \mathbb{N}$ with $n < m$ let be $[n; m] = [n, m] \cap \mathbb{N}$. The symbol $|X|$ denotes the cardinality of a finite set $X$. The symbol $2^X$ denotes the power set of a set $X$. Given a function $f : X \to Y$ we denote by $f(Z)$ the image of a set $Z \subseteq X$ through $f$, i.e. $f(Z) = \{y \in Y \mid \exists z \in Z \text{ s.t. } y = f(z)\}$; if $X' \subseteq X$ and $Y' \subseteq Y$ then $f|_{X' \to Y'}$ is the restriction of $f$ to domain $X'$ and co-domain $Y'$, i.e. $f|_{X' \to Y'}(x) = f(x)$ for any $x \in X'$ with $f(x) \in Y'$. We now recall from [2] some basic notions of language theory. Given a set $\Sigma$, a sequence $w = \sigma_1\sigma_2\sigma_3...$, with symbols $\sigma_i \in \Sigma$ is called a word in $\Sigma$; the empty word is denoted by $\varepsilon$. The Kleene closure $w^*$ of a finite word $w$ is the collection of finite words $\varepsilon, w, ww, www, ...$. The symbol $\Sigma^*$ denotes the set of all finite words in $\Sigma$, including the empty word $\varepsilon$. The concatenation of two words $u, v \in \Sigma^*$ is denoted by $uv \in \Sigma^*$. Any subset of $\Sigma^*$ is called a language. The projection of a language $L \subseteq \Sigma^*$ onto a subset $\hat{\Sigma}$ of $\Sigma$ is the language $P_{\hat{\Sigma}}(L) = \{t \in \hat{\Sigma}^* \mid \exists w \in L \text{ s.t. } P_{\hat{\Sigma}}(w) = t\}$ where $P_{\Sigma}(w)$ is inductively defined for any $w \in L$ and $\sigma \in \Sigma$ by $P_{\Sigma}(\varepsilon) = \varepsilon$ and $P_{\Sigma}(w\sigma) = P_{\Sigma}(w)\sigma$ if $\sigma \in \hat{\Sigma}$ and $P_{\Sigma}(w\sigma) = P_{\Sigma}(w)$, otherwise.

2.2. Networks of Finite State Machines. We start by recalling the notion of finite state machines.

Definition 2.1. A Finite State Machine (FSM) $M$ is a tuple $(X, X^0, \Sigma, \delta)$ where $X$ is the set of states, $X^0 \subseteq X$ is the set of initial states, $\Sigma$ is the set of input labels and $\delta : X \times \Sigma \to 2^X$ is the transition map.

A state run $r$ of an FSM $M$ is a sequence $x^0 \xrightarrow{\sigma^1} x^1 \xrightarrow{\sigma^2} x^2 \xrightarrow{\sigma^3} x^3 ...$ such that $x^0 \in X^0$, $x^i \in X$, $\sigma^i \in \Sigma$ and $x^{i+1} \in \delta(x^i, \sigma^i)$ for any $x^i$ and $\sigma^i$ in the sequence; the sequence $\sigma^1\sigma^2\sigma^3...$ is called the trace associated with $r$. For $X' \subseteq X$ and $\sigma \in \Sigma$, we abuse notation by writing $\delta(X', \sigma)$ instead of $\bigcup_{x \in X'} \delta(x, \sigma)$. The extended transition map $\hat{\delta}$ associated with $\delta$ is inductively defined for any $w \in \Sigma^*$, $\sigma \in \Sigma$ and $x \in X$ by
δ(x, ε) = {x} and δ(x, wσ) = ∪y∈δ(x,w) δ(y, σ). The language generated by M, denoted L(M), is composed by all traces generated by M, or equivalently, L(M) = {w ∈ Σ∗ | ∃x0 ∈ X0 s.t. δ(x0, w) ≠ ∅}. The class of FSMs in Definition 2.1 is non–deterministic. An FSM M is deterministic if |X0| = 1 and |δ(x, σ)| ≤ 1, for any x ∈ X and σ ∈ Σ. In this paper we are interested in studying whether it is possible to detect if the current state of an FSM M is or is not in a set of critical states C ⊂ X representing unsafe operations. We refer to an FSM (X, X0, Σ, δ) equipped with a set of critical states C by the tuple (X, X0, Σ, δ, C). We also refer to an FSM with outputs by a tuple (X, X0, Σ, δ, Y, H), where Y is the set of output labels and H : X → Y is the output function. For simplicity we call an FSM equipped with critical states or with outputs as an FSM. The operator Ac(·) extracts the accessible part from an FSM M = (X, X0, Σ, δ, C) (resp. M = (X, X0, Σ, δ, Y, H)), i.e. Ac(M) = (X′, X0, Σ, δ′, C′) (resp. Ac(M) = (X′, X0, Σ, δ′, Y, H′)) where X′ = {x ∈ X | ∃x0 ∈ X0 ∧ w ∈ Σ∗ s.t. x ∈ δ(x0, w)}, δ′ = δ′ |X′×Σ→X′, C′ = C ∩ X′ and H′ = H |X′→Y.

Interaction among FSMs is captured by the following notion of composition.

**Definition 2.2.** The parallel composition M1[||]M2 = (X1, X0, Σ, δ1, C1) and M2 = (X2, X0, Σ, δ2, C2) is the FSM Ac(X1, X2, Σ, δ1, C1, C2) where X1 = X1 ∪ X2, X0 = X0 ∪ X0, Σ1 = Σ1 ∪ Σ2, C1 = C1 ∪ C2 and δ1 = δ1 : X1 × Σ → 2X1 is defined for any x1 ∈ X1, x2 ∈ X2 and σ ∈ Σ1 by

\[
\begin{align*}
\delta_1(x_1, σ) & \times δ_2(x_2, σ), \text{if } \delta_1(x_1, σ) \neq \emptyset \& \delta_2(x_2, σ) \neq \emptyset \\
\emptyset \& σ \in Σ_1 \cap Σ_2, & \\
\delta_1(x_1, σ) \times \{x_2\}, \text{if } \delta_1(x_1, σ) \neq \emptyset \& σ \in Σ_1 \setminus Σ_2, & \\
\{x_1\} \times δ_2(x_2, σ), \text{if } δ_2(x_2, σ) \neq \emptyset \& σ \in Σ_2 \setminus Σ_1, & \\
\emptyset, \text{otherwise.}
\end{align*}
\]

By definition, a state (x1, x2) ∈ C1,2, i.e. (x1, x2) is considered critical for M1||M2, if and only if x1 ∈ C1 or x2 ∈ C2. Vice versa, (x1, x2) ∉ C1,2 if and only if x1 ∉ C1 and x2 ∉ C2.

**Proposition 2.3.** [2] The parallel composition operation is commutative up to isomorphisms and associative.

The proof of the above result is reported in the Appendix. By the above property of parallel composition, we may write in the sequel M1||M2||M3, X1,2,3 and C1,2,3 instead of M1||M2||M3, X1,2,3 and C1,2,3 or, instead of (M1||M2)||M3, X1,2,3 and C1,2,3. In this paper we consider a network

\[ N = \{M_1, M_2, ..., M_N\} \]

of N FSMs M1 whose interaction is captured by the notion of parallel composition; the corresponding FSM is given by M(N) = M1||M2||...||MN. The FSM M(N) is well defined because the composition operator || is associative. The definition of parallel composition among an arbitrary number of FSMs is reported in e.g. [2] and coincides with the recursive application of the binary operator ||, as in Definition 2.2. For the computational complexity analysis, we will use in the sequel the number \( n_{max} = max_{i \in [1,N]} |X_i| \) as indicator of the sizes of the FSMs composing the network N. An upper bound to space and time computational complexity in constructing M(N) is \( O(2^N \log(n_{max})) \).

2.3. Critical observability and observers. Critical observability corresponds to the possibility of detecting immediately whether the current state x of a run of an FSM is or is not critical on the basis of the information given by the corresponding trace at state x:

**Definition 2.4.** An FSM M = (X, X0, Σ, δ, C) is critically observable if \([δ(x^0, w) \subseteq C] \vee [δ(x^0, w) \subseteq X \setminus C]\), for any initial state \( x^0 \in X^0 \) and any trace \( w \in L(M) \).

Any FSM M having an initial state that is critical and another initial state that is not critical, is never critically observable. For this reason in the sequel we assume that \( [X^0 \subseteq C] \vee [X^0 \subseteq X \setminus C] \) for any FSM M. An illustrative example follows.
Consider FSMs $M_i = (X_i, X_i^0, \Sigma_i, \delta_i, C_i)$, $i = 1, 2$, depicted in Fig. 1, where $X_1 = \{1, 2, 3, 4\}$, $X_2^0 = \{1\}$, $\Sigma_1 = \{a, b, c, d\}$, $C_1 = \{4\}$, $X_2 = \{5, 6, 7, 8\}$, $X_2^0 = \{5\}$, $\Sigma_2 = \{a, b, e\}$, $C_2 = \{7, 8\}$ and transition maps $\delta_1$ and $\delta_2$ are represented by labeled arrows in Fig. 1 labels on the arrows represent the input label associated with the corresponding transition. FSM $M_1$ is not critically observable because it is possible to reach both noncritical state 3 and critical state 4 starting from the initial state 1, by applying the same input label $a$. FSM $M_2$ is critically observable because by applying traces $b(ab)^*$ and $(b(ab))^a$ to the initial state 5, the state reached is always in $X_2 \setminus C_2$ while by applying any trace other than the previous ones, states reached are always critical.

On-line detection of critical states of critically observable FSMs can be obtained by means of critical observers, as defined hereafter.

**Definition 2.6.** A deterministic FSM $\text{Obs} = (X_{\text{Obs}}, X_{\text{Obs}}^0, \Sigma_{\text{Obs}}, \delta_{\text{Obs}}, Y_{\text{Obs}}, H_{\text{Obs}})$ with output set $Y_{\text{Obs}} = \{0, 1\}$ is a critical observer for an FSM $M = (X, X^0, \Sigma, \delta, C)$ if $\Sigma_{\text{Obs}} = \Sigma$ and for any state run $r : x^0 \xrightarrow{\sigma^1} x^1 \xrightarrow{\sigma^2} x^2 \xrightarrow{\sigma^3} x^3 \ldots$ of $M$, the corresponding (unique) state run $r_{\text{Obs}} : z^0 \xrightarrow{\sigma^1} z^1 \xrightarrow{\sigma^2} z^2 \xrightarrow{\sigma^3} z^3 \ldots$ of $\text{Obs}$ is such that $H_{\text{Obs}}(z^i) = 1$ if $x^i \in C$ and $H_{\text{Obs}}(z^i) = 0$ otherwise, for any state $z^i$ in $r_{\text{Obs}}$.

For later use, we report from e.g. [2] the following construction of observers.

**Definition 2.7.** Given an FSM $M = (X, X^0, \Sigma, \delta, C)$, define the deterministic FSM $\text{Obs}(M)$ as the accessible part $\text{Ac}(\text{Obs}(M))$ of $\Sigma_{\text{Obs}}^* : (X_{\text{Obs}}, X_{\text{Obs}}^0, \Sigma_{\text{Obs}}, \delta_{\text{Obs}}, Y_{\text{Obs}}, H_{\text{Obs}})$, where $X_{\text{Obs}} = 2^X$, $X_{\text{Obs}}^0 = \{X^0\}$, $\Sigma_{\text{Obs}} = \Sigma$, $\delta_{\text{Obs}} : X_{\text{Obs}} \times \Sigma_{\text{Obs}} \rightarrow 2^{X_{\text{Obs}}}$ is defined by $\delta_{\text{Obs}}(z, \sigma) = \bigcup_{x \in z} \delta(x, \sigma)$, $Y_{\text{Obs}} = \{0, 1\}$ and $H_{\text{Obs}}(z) = 1$ if $z \cap C \neq \emptyset$ and $H_{\text{Obs}}(z) = 0$, otherwise.

An upper bound to the space and time computational complexity in constructing $\text{Obs}(M)$ for an FSM $M$ with $n = |X|$ states is $O(2^n)$ from which, an upper bound to the space and time computational complexity in constructing $\text{Obs}(M(N))$ is $O(2^{2N \log(n_{\text{max}}/n_{\text{min}})})$. A direct consequence of Definitions 2.4, 2.6 and 2.7 is the following:

**Proposition 2.8.** The following statements are equivalent:
(i) $M$ is critically observable;
(ii) $\text{Obs}(M) = (X_{\text{Obs}}, X_{\text{Obs}}^0, \Sigma_{\text{Obs}}, \delta_{\text{Obs}}, Y_{\text{Obs}}, H_{\text{Obs}})$ is a critical observer for $M$;
(iii) For any $z \in X_{\text{Obs}}$, if $H_{\text{Obs}}(z) = 1$ then $z \subseteq C$.

An example of the above result follows.

**Example 2.9.** Consider FSMs $M_1$ and $M_2$ in Example 2.5 and depicted in Fig. 1. Corresponding FSMs $\text{Obs}(M_1)$ and $\text{Obs}(M_2)$ are depicted in Fig. 2. FSM $\text{Obs}(M_1)$ does not satisfy conditions in the third statement of Proposition 2.8 while FSM $\text{Obs}(M_2)$ does. Hence, $M_2$ is critically observable while $M_1$ is not; note that this is consistent with Example 2.5.
In this section, we first show that critical observability of all the FSMs composing a network ensures the critical observability of their parallel composition, i.e. of the network itself. However, a network of FSMs can be critically observable even though not all the FSMs composing the network are critically observable. This means that for checking the critical observability of a network we need to compose the FSMs, and this may be problematic especially when dealing with large-scale networks where a large number of FSMs has to be composed. We therefore propose two complementary approaches to tackle this problem. First, in Subsection 3.1 we propose a decentralized architecture for critical observers detecting on-line occurrence of critical states. Then, in Subsection 3.2 we propose the use of bisimulation equivalence for reducing the complexity of the composition. Finally, in Subsection 3.3 we combine these two approaches.

Proposition 3.1. If FSMs $M_1$ and $M_2$ are critically observable then FSM $M_1 || M_2$ is critically observable.

The proof of the above result is reported in the Appendix. The converse implication is not true in general, as shown in the following example.

Example 3.2. Consider FSMs $M_1$ and $M_2$ in Example 2.5 and depicted in Fig. 1. As discussed in Example 2.5, FSM $M_2$ is critically observable while FSM $M_1$ is not. It is readily seen that FSM $M_1 || M_2$, depicted in Fig. 2, is critically observable because by applying traces $b((cd)^*ab)^*$ and $b(cd)^*a(b(cd)^*a)^*$ to the initial state $(1,5)$, the state reached is always in $X_{1,2} \setminus C_{1,2}$ and by applying any trace other than the previous ones, states reached are always in $C_{1,2}$.

As a consequence, critical observability of each FSM composing $\mathcal{N}$ is not necessary for $M(\mathcal{N})$ to be critically observable.

3.1. Design of decentralized critical observers. In this section, we show how decentralized observers can be used for detecting critical states of the network of FSMs $M(\mathcal{N})$. The notion of isomorphism will be used:

Definition 3.3. Two FSMs $M_i = (X_i, X_0^i, \Sigma_i, \delta_i, Y_i, H_i), i = 1, 2$, are isomorphic, denoted $M_1 \simeq M_2$, if $\Sigma_1 = \Sigma_2$, $Y_1 = Y_2$ and there exists a bijective function $\phi : X_1 \to X_2$, such that:

(i) $\phi(X_0^1) = X_0^2$;
(ii) $\phi(X_1) = X_2$;
(iii) $\phi(\delta_1(x_1, \sigma)) = \delta_2(\phi(x_1), \sigma)$, for any accessible state $x_1 \in X_1$ of $M_1$ and $\sigma \in \Sigma_1$;
(iv) $H_1(x_1) = H_2(\phi(x_1))$, for any $x_1 \in X_1$.

Proposition 3.4. Given FSMs $M_i (i \in [1; 4])$, if $M_1 \simeq M_2$ and $M_3 \simeq M_4$ then $M_1 || M_2 \simeq M_3 || M_4$. 
Given \( \mathcal{N} \), consider the collection of deterministic FSMs

\[
(3.1) \quad \text{Obs}(M_i) = (X_{\text{Obs},i}, X^0_{\text{Obs},i}, \Sigma_i, \delta_{\text{Obs},i}, Y_{\text{Obs},i}, H_{\text{Obs},i})
\]

each associated to the FSM \( M_i \) and define the decentralized observer \( \text{Obs}^d(\mathcal{N}) \) as the FSM

\[
\text{Obs}(M_1) \| \text{Obs}(M_2) \| \ldots \| \text{Obs}(M_N)
\]

with output set \( Y_{\text{Obs},d} = \{0, 1\} \) and output function \( H_{\text{Obs}}(z_1, z_2, \ldots, z_N) = \bigvee_{i \in [1:N]} H_{\text{Obs},i}(z_i) \). The following result shows that the decentralized observer \( \text{Obs}^d(\mathcal{N}) \) can be used for detecting on-line critical states of the monolithic FSM \( M(\mathcal{N}) \).

**Theorem 3.5.** \( \text{Obs}^d(\mathcal{N}) =_{\text{iso}} \text{Obs}(M(\mathcal{N})) \).

**Proof.** We start by showing the result for the network \( \mathcal{N}' = \{M_1, M_2\} \) where \( M_i = (X_i, X^0_i, \Sigma_i, \delta_i, C_i) \), i.e.

\[
(3.2) \quad \text{Obs}(M_1) \| \text{Obs}(M_2) =_{\text{iso}} \text{Obs}(M_1 \| M_2).
\]

Let be \( \text{Obs}^d(\mathcal{N}') = (X_{\text{Obs},d}', X^0_{\text{Obs},d}', \Sigma_{\text{Obs},d}', \delta_{\text{Obs},d}', Y_{\text{Obs},d}', H_{\text{Obs},d}) \) and \( \text{Obs}(M(\mathcal{N}')) = (X_{\text{Obs}}, X^0_{\text{Obs}}, \Sigma_{\text{Obs}}, \delta_{\text{Obs}}, Y_{\text{Obs}}, H_{\text{Obs}}) \). Define \( \phi : X_{\text{Obs},d} 
arrow X_{\text{Obs}} \) such that \( \phi((z_1, z_2)) = z_1 \times z_2 \), for any \( (z_1, z_2) \in X_{\text{Obs},d} \) with \( z_i \in X_{\text{Obs},i} \). First of all note that \( \Sigma_{\text{Obs},d} = \Sigma_{\text{Obs}} = \Sigma_1 \cup \Sigma_2 \) and \( Y_{\text{Obs},d} = Y_{\text{Obs}} = \{0, 1\} \). Moreover, with reference to Definition 3.3 we get:

**Condition (i).** By Definitions 2.2 and 2.7 we get \( \phi(X^0_{\text{Obs},d}) = \phi(X^0_{\text{Obs},1} \times X^0_{\text{Obs},2}) = \phi(\{X^0_1\} \times \{X^0_2\}) = \phi(\{(X^0_1, X^0_2)\}) = \{\phi(X^0_1), \phi(X^0_2)\} = X^0_{\text{Obs}} \).

**Conditions (ii) and (iii).** We proceed by induction and show that if \( \phi((z_1, z_2)) = z_1 \times z_2 \) for a pair of accessible states \( (z_1, z_2) \in X_{\text{Obs},d} \) and \( z_1 \times z_2 \in X_{\text{Obs}} \) then \( \phi(\delta_{\text{Obs},d}((z_1, z_2), \sigma)) = \delta_{\text{Obs}}(\phi((z_1, z_2), \sigma)) \) for any \( \sigma \in \Sigma_1 \cup \Sigma_2 \). With reference to Definition 2.2 we have three cases: (case 1) \( \sigma \in \Sigma_1 \cap \Sigma_2 \), (case 2) \( \sigma \in \Sigma_1 \setminus \Sigma_2 \), and (case 3) \( \sigma \in \Sigma_2 \setminus \Sigma_1 \). We start with case 1. By Definitions 2.2 and 2.7 we get \( \phi(\delta_{\text{Obs},d}((z_1, z_2), \sigma)) = \delta_{\text{Obs},d}(z_1 \times z_2, \sigma) = \delta_{\text{Obs}}(\phi((z_1, z_2), \sigma)) \).

**Condition (iv).** Suppose \( H_{\text{Obs},d}(z_1, z_2) = 1 \). By definition of \( H_{\text{Obs},d}(\cdot) \) we get that \( H_{\text{Obs},1}(z_1) = 1 \) or \( H_{\text{Obs},2}(z_2) = 1 \), or by Definition 2.7 equivalently that \( z_1 \cap C_1 \neq \emptyset \) or \( z_2 \cap C_2 \neq \emptyset \). The last conditions imply that \( z_1 \times z_2 \cap C_1 \times C_2 \neq \emptyset \). Since \( C_1 \times C_2 \subseteq C_{1,2} \) we get that \( H_{\text{Obs},d}(z_1, z_2) = 1 \). Suppose now \( H_{\text{Obs},d}(z_1, z_2) = 0 \). By definition of \( H_{\text{Obs},d}(\cdot) \) we get that \( H_{\text{Obs},1}(z_1) = 0 \) or \( H_{\text{Obs},2}(z_2) = 0 \), or by Definition 2.7 equivalently that \( z_1 \cap C_1 = \emptyset \) or \( z_2 \cap C_2 = \emptyset \). The last conditions imply that \( z_1 \times z_2 \cap C_{1,2} = \emptyset \) which, by Definition 2.7 implies \( H_{\text{Obs},d}(z_1, z_2) = 0 \). Hence, the isomorphic equivalence in (3.2) is proven. We now generalize (3.2) to the case of a generic network \( \mathcal{N}' = \{M_1, M_2, \ldots, M_N\} \). By applying recursively the equivalence in (3.2) and by Proposition 3.4 we get \( \text{Obs}(M(\mathcal{N}')) = \text{Obs}(M_1) \| \text{Obs}(M_2) \| \ldots \| \text{Obs}(M_N) =_{\text{iso}} \text{Obs}(M(\mathcal{N})) \)

In the sequel, we will refer to the FSM \( \text{Obs}^d(\mathcal{N}) \) satisfying condition (iii) of Proposition 2.8 as a decentralized critical observer for \( \mathcal{N} \). Fig. 3 shows a possible implementation architecture for \( \text{Obs}^d(\mathcal{N}) \). Observer \( \text{Obs}^d(\mathcal{N}) \) can be obtained as a bank on \( N \) local observers \( \text{Obs}(M_i) \) that act asynchronously. Each local observer \( \text{Obs}(M_i) \) takes as input the trace \( w_i \in L(M_i) \) generated by \( M_i \) in response to the input word \( w \), and sends the output boolean values \( y_i \) to the OR (static) block. The OR block acts as a coordinator and whenever it receives one or more boolean values \( y_i \), as inputs, it outputs boolean value \( y \) as the logical operation \( or \) among \( y_i \). Note that this architecture does not require the explicit construction of the parallel composition of local observers \( \text{Obs}(M_i) \).

Since space and time computational complexity in constructing \( \text{Obs}^d(\mathcal{N}) \) is \( O(2^{n_{\text{max}}}N) \), a direct consequence of Theorem 3.5 is:
Corollary 3.6. An upper bound to space and time computational complexity in constructing \( \text{Obs}(\mathcal{M}(N)) \) is \( O(2^n \max N) \).

An example of application of the above result follows.

Example 3.7. Consider a network of FSMs \( \mathcal{N} = \{M_1, M_2\} \) where FSMs \( M_1 \) and \( M_2 \) are depicted in Fig. 1. Parallel composition \( M_1 || M_2 \) is reported in Fig. 7 and the corresponding centralized observer \( \text{Obs}(M_1 || M_2) \) is reported in Fig. 4. Local observers \( \text{Obs}(M_1) \) and \( \text{Obs}(M_2) \) associated with \( M_1 \) and \( M_2 \) are depicted in Fig. 5 and decentralized observer \( \text{Obs}^d(\mathcal{N}) = \text{Obs}(M_1) || \text{Obs}(M_2) \) in Fig. 6. It is readily seen that observers \( \text{Obs}^d(\mathcal{N}) \) and \( \text{Obs}(M_1 || M_2) \) are isomorphic, as formally proven in Theorem 3.5.

Note that the above upper bound is lower than \( O(2^N \log(n \max)) \).

From Proposition 2.8 and Theorem 3.3, one way to check critical observability and to design decentralized observers of \( \mathcal{M}(N) \) is illustrated in Algorithm 1.

\begin{algorithm}
1: Construct \( N \) local observers \( \text{Obs}(M_i) \);
2: Compose the \( N \) local observers \( \text{Obs}(M_i) \) to get \( \text{Obs}^d(\mathcal{N}) \);
3: Apply Proposition 2.8 to \( \text{Obs}^d(\mathcal{N}) \).
\end{algorithm}

Algorithm 1 has space and time computational complexity \( O(2^n \max N) \). It can be improved from the computational point of view, because:

(D1) It constructs the whole local observer \( \text{Obs}(M_i) \) for each \( M_i \). A more efficient algorithm would construct, for each \( M_i \), only the sub–FSM of \( \text{Obs}(M_i) \) that is interconnected with the other local observers in \( \text{Obs}^d(\mathcal{N}) \).

(D2) It constructs the whole observer \( \text{Obs}^d(\mathcal{N}) \), which is not required at the implementation layer (see Fig. 3). A more efficient algorithm would check critical observability on the basis of local observers.

(D3) It first constructs \( \text{Obs}^d(\mathcal{N}) \) before checking if states \( z \) of \( \text{Obs}^d(\mathcal{N}) \) satisfy condition (iii) of Proposition 2.8. A more efficient algorithm would conclude that \( \mathcal{N} \) is not critically observable when the first state \( z \) not satisfying condition (iii) of Proposition 2.8 shows up.

In order to cope with the aforementioned drawbacks, we now present a procedure that integrates each step of Algorithm 1 in one algorithm. This procedure is based on on–the–fly algorithms for verification and control of FSMs (see e.g. [4, 16]) and is reported in Algorithm 2.

Algorithm 2 makes use of the notion of projected local observers. The projection \( \pi_{\text{Obs}(M_i)}(\text{Obs}^d(\mathcal{M}(\mathcal{N}))) \) of \( \text{Obs}^d(\mathcal{M}(\mathcal{N})) \) onto \( \text{Obs}(M_i) \), as in (3.1), is defined as the FSM \( \text{Ac}(X'_{\text{Obs},i}, X^0_{\text{Obs},i}, \Sigma_i, \delta_{\text{Obs},i}, Y_{\text{Obs},i}, H_{\text{Obs},i}) \) where \( X'_{\text{Obs},i} \) contains states \( z_i \in X_{\text{Obs},i} \) for which there exist states \( z_j \in X_{\text{Obs},j}, j \in [1; N], j \neq i \) such that \((z_1, z_2, ..., z_N)\) is a state of \( \text{Obs}^d(\mathcal{M}(\mathcal{N})) \)
Algorithm 2 Integrated design of decentralized observers

1: Input: FSMs $M_i = (X_i, X_0^i, \Sigma_i, \delta_i, C_i)$, with $i \in [1; N]$;
2: Init: $X_{\text{Obs}, i} := \{X_0^i\}$, $X_0^\text{temp}, i := \{X_0^i\}$, $Y_{\text{Obs}, i} := \{0.1\}$ for any $i \in [1; N]$; $X_{\text{Obs}} := \{X_0^1, X_0^2, \ldots, X_0^N\}$; $X_{\text{temp}} := X_{\text{Obs}}$;
3: while $X_{\text{temp}} \neq \emptyset$ do
4: $\exists_{\text{temp}} := \emptyset$
5: for all $(z_1, z_2, \ldots, z_N) \in X_{\text{temp}}$ do
6: for all $\sigma \in \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_N$ do
7: for all $i \in [1; N]$ do
8: $z_i^+ := \delta_i(z_i, \sigma)$;
9: end for
10: if $\delta_{\text{Obs}, i}((z_1, z_2, \ldots, z_N), \sigma) = \{(z_1^+, z_2^+, \ldots, z_N^+)\}$ then
11: if $(z_1^+, z_2^+, \ldots, z_N^+) \notin X_{\text{Obs}, i}$ then
12: if $[(z_1^+, z_2^+, \ldots, z_N^+) \notin C_1, 2, \ldots, N] \wedge [(z_1^+, z_2^+, \ldots, z_N^+) \notin X_{1, 2, \ldots, N} \setminus C_1, 2, \ldots, N]$ then
13: BREAK: $M(N)$ is not critically observable;
14: end if
15: $\exists_{\text{temp}} := \exists_{\text{temp}} \cup \{(z_1^+, z_2^+, \ldots, z_N^+)\}$;
16: for all $i \in [1; N]$ do
17: if $[z_i^+ \notin \emptyset] \wedge [z_i^+ \notin X_{\text{Obs}, i}]$ then
18: $X_{\text{Obs}, i} := X_{\text{Obs}, i} \cup \{z_i^+\}$;
19: $\delta_{\text{Obs}, i}(z_i, \sigma) := \{(z_i^+, z_i^+)^+\}$;
20: if $z_i^+ \notin C_i$ then
21: $H_{\text{Obs}, i}(z_i^+) := 1$;
22: else
23: $H_{\text{Obs}, i}(z_i^+) := 0$;
24: end if
25: end if
26: end for
27: end if
28: end if
29: end for
30: end for
31: $X_{\text{Obs}} := X_{\text{Obs}} \cup \exists_{\text{temp}}$; $X_{\text{temp}} := \exists_{\text{temp}}$;
32: end while
33: output: $M(N)$ is critically observable;

Projected local observers $\pi_{\text{Obs}(M_i)}(\text{Obs}^d(N)) = (\text{X}_{\text{Obs}, i}, X_0^0, \Sigma_i, \delta_{\text{Obs}, i}, Y_{\text{Obs}, i}, H_{\text{Obs}, i})$.

The input of Algorithm 2 is the collection of FSMs $M_i$ of $N$. The output is the collection of projected local observers $\pi_{\text{Obs}(M_i)}(\text{Obs}^d(N))$ if $M(N)$ is critically observable. In line 2, the initial state and the output set of the projected local observers are defined and their sets of states $X_{\text{Obs}, i}$ are initialized to contain only the initial states. At each iteration, the algorithm processes candidate new states of $\text{Obs}^d(N)$ and adds them to $X_{\text{Obs}, i}$ whenever they are compatible with the parallel composition of $\pi_{\text{Obs}(M_j)}(\text{Obs}^d(N))$ with $j \neq i$. For each aggregate $(z_1, z_2, \ldots, z_N)$ in the temporary set $X_{\text{temp}}$ and for each $\sigma \in \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_N$ (lines 5 and 6), first, successors $z_i^+$ of states $z_i$ in $M_i$ are computed (lines 7–9). If in line 10, according to the definition of the transition map of $\text{Obs}^d(N)$, there is a transition in $\text{Obs}^d(N)$ from aggregate $(z_1, z_2, \ldots, z_N)$ to aggregate $(z_1^+, z_2^+, \ldots, z_N^+)$ with input label $\sigma$, then $(z_1^+, z_2^+, \ldots, z_N^+)$ is further processed. Note that we are not storing information computed in line 10 on the transition map of $\text{Obs}^d(N)$ but only states of $\text{Obs}^d(N)$ which cannot be avoided for computing the projections $\pi_{\text{Obs}(M_j)}(\text{Obs}^d(N))$. By this fact, Algorithm 2 overcomes drawback (D2) of Algorithm 1. Algorithm 2 first checks in line 11 if the aggregate $(z_1^+, z_2^+, \ldots, z_N^+)$ was not processed before. If so, line 12 is processed. By definition of output function $H_{\text{Obs}, i}$, condition $(z_1^+, z_2^+, \ldots, z_N^+) \notin X_{1, 2, \ldots, N} \setminus C_1, 2, \ldots, N$ implies $H_{\text{Obs}, i}(z_1^+, z_2^+, \ldots, z_N^+) = 1$ which, combined with condition $(z_1^+, z_2^+, \ldots, z_N^+) \notin C_1, 2, \ldots, N$, implies that condition (iii) of Proposition 2.8 is not satisfied. Hence, by applying Proposition 2.8, if $(z_1^+, z_2^+, \ldots, z_N^+)$ satisfies such a condition, the algorithm immediately terminates in line 13, concluding that $N$ is not critically observable. By this fact, Algorithm 2 overcomes drawback (D3) of Algorithm 1. If $(z_1^+, z_2^+, \ldots, z_N^+)$ does not satisfy condition in line 12, it is added to $X_{\text{temp}}$ in line 15; the set of states $X_{\text{Obs}, i}$ and the transition map $\delta_{\text{Obs}, i}$ of $\pi_{\text{Obs}(M_j)}(\text{Obs}^d(N))$ are updated in lines 18 and 19. Note that since $\delta_{\text{Obs}, i}$ is updated only if condition in line 10 holds, then Algorithm 2 constructs step-by-step $\pi_{\text{Obs}(M_j)}(\text{Obs}^d(N))$ and not Obs($M_i$). By this fact, Algorithm 2 overcomes drawback (D1) of Algorithm 1. The outputs of states $z_i^+$ are set in lines 20–23. In line
3.2. Model reduction via bisimulation. In this section we propose the use of bisimulation equivalence \[8, 10\] to reduce the computational complexity in checking critical observability and designing observers. We start by recalling the notion of bisimulation equivalence.

**Definition 3.8.** Two FSMs \(M_1 = (X_1, X_0^1, \Sigma_1, \delta_1, C_1)\) and \(M_2 = (X_2, X_0^2, \Sigma_2, \delta_2, C_2)\) are bisimilar, denoted by \(M_1 \cong M_2\), if there exists a relation \(R \subseteq X_1 \times X_2\), called bisimulation relation, such that for any \((x_1, x_2) \in R\) the following conditions are satisfied:

(i) \(x_1 \in X_1^0\) if and only if \(x_2 \in X_2^0\);

(ii) for any \(\sigma \in \Sigma_1\) such that \(\delta_1(x_1, \sigma) \neq \emptyset\) and for any \(x_1^+ \in \delta_1(x_1, \sigma)\) there exists \(x_2^+ \in \delta_2(x_2, \sigma)\) such that \((x_1^+, x_2^+) \in R\).

31, set \(X_{\text{Obs}}\) is updated to \(X_{\text{Obs}} \cup Z_{\text{temp}}^\text{temp}\) and set \(X_{\text{temp}}^\text{temp}\) to \(Z_{\text{temp}}^\text{temp}\); the next iteration then starts. Algorithm 2 terminates when there are no more states in \(X_{\text{temp}}^\text{temp}\) to be processed (see line 5) or condition in line 12 is satisfied. From the above explanation, it is clear that Algorithm 2 terminates in a finite number of states. Moreover it is also clear that in the worst case, the computational complexity of Algorithm 2 is the same as the one in Algorithm 1 i.e. \(O(2^{n_{\text{max}}N})\). This is typical for on-the-fly based algorithms. However, there are practical cases in which Algorithm 2 performs better than Algorithm 1; an example is included in Section 4.
(iii) for any $\sigma \in \Sigma_2$ such that $\delta_2(x_2, \sigma) \neq \emptyset$ and for any $x_2^+ \in \delta_2(x_2, \sigma)$ there exists $x_1^+ \in \delta_1(x_1, \sigma)$ such that $(x_1^+, x_2^+) \in R$;
(iv) $x_1 \in C_1$ if and only if $x_2 \in C_2$.

The above notion of bisimulation equivalence differs from the classical one \cite{8, 10} because of the additional condition (iv). By adding this condition, we get the following result:

**Proposition 3.9.** If FSMs $M_1$ and $M_2$ are bisimilar then:
(i) $M_1$ is critically observable if and only if $M_2$ is critically observable;
(ii) An FSM $\text{Obs}$ is a critical observer for $M_1$ if and only if it is a critical observer for $M_2$.

The proof of the above result is reported in the Appendix. Space and time computational complexities in checking bisimulation equivalence between $M_1$ and $M_2$ with $|X_1| = n_1$ and $|X_2| = n_2$ states are $O(n_1^2 + n_2^2)$ and $O((n_1^2 + n_2^2) \log(n_1 + n_2))$, respectively, see e.g. \cite{9, 7}. Bisimulation equivalence is an equivalence relation on the class of FSMs. We now define the network of FSMs $\mathcal{N}_{\text{min}}$ as the quotient of the original network $\mathcal{N}$ induced by the bisimulation equivalence. More precisely given $\mathcal{N}$, define the following equivalence classes induced by the bisimulation equivalence:

$$E_1 = \{M_{i(1,1)}, M_{i(1,2)}, \ldots, M_{i(1,n_1^+)}\},$$
$$E_2 = \{M_{i(2,1)}, M_{i(2,2)}, \ldots, M_{i(2,n_2^+)}\},$$
$$\ldots,$$
$$E_{N_{\text{min}}} = \{M_{i(N_{\text{min}},1)}, M_{i(N_{\text{min}},2)}, \ldots, M_{i(N_{\text{min}},n_{\text{min}})}\},$$

such that the collection of FSMs $M_{i(k,j)}$ coincides with $\mathcal{N}$ and $M_{i(k,j_1)}, M_{i(k,j_2)} \in E_k$ if and only if $M_{i(k,j_1)} \equiv M_{i(k,j_2)}$. Denote by $M_{i_{\text{min}}^k} \in E_k$ a representative of the equivalence class $E_k$ and define the network of FSMs

$$\mathcal{N}_{\text{min}} = \{M_{i_{\text{min}}^1}, M_{i_{\text{min}}^2}, \ldots, M_{i_{\text{min}}^{N_{\text{min}}}}\}$$

with $M(\mathcal{N}_{\text{min}}) = M_{i_{\text{min}}^1}|M_{i_{\text{min}}^2}|\ldots|M_{i_{\text{min}}^{N_{\text{min}}}}$. The forthcoming results rely upon the following technical lemmas whose proofs are included in the Appendix.

**Lemma 3.10.** Consider FSMs $M_i = (X_i, X_i^0, \Sigma_i, \delta_i, C_i)$ with $i \in [1; 3]$ and suppose that $M_2$ and $M_3$ are bisimilar. Then an FSM $\text{Obs}$ is a critical observer for $M_1||M_2$ if and only if it is a critical observer for $M_1||M_2||M_3$. 
Lemma 3.11. Consider FSMs $M_i = (X_i, X_i^0, \Sigma_i, \delta_i, C_i)$ with $i \in [1; 3]$ and suppose that $M_2$ and $M_3$ are bisimilar. Then $M_1||M_2$ is critically observable if and only if $M_1||M_2||M_3$ is critically observable.

We now have all the ingredients to present the main results of this section.

Theorem 3.12. $M(N)$ is critically observable if and only if $M(N^{\text{min}})$ is critically observable.

Proof. By applying Lemma 3.11 for any $M_i(N^{\text{min}}, j) \in N \setminus N^{\text{min}}$, FSM $M(N^{\text{min}}) = (M_{i_1}^{\text{min}}||M_{i_2}^{\text{min}}||...||M_{i_{n^{\text{min}}-1}}^{\text{min}})$ $M_{i_{n^{\text{min}}}}^{\text{min}}$ is critically observable if and only if FSM $(M_{i_1}^{\text{min}}||M_{i_2}^{\text{min}}||...||M_{i_{n^{\text{min}}-1}}^{\text{min}})||M_{i_{n^{\text{min}}}}^{\text{min}}$ is critically observable (recall that $M_i(N^{\text{min}}, j) \cong M_{i_{n^{\text{min}}}}^{\text{min}}$ for any $j \in [1; n^{\text{min}}]$). Hence, by applying recursively Lemma 3.11 to all other FSMs $M_i(k, j) \in N \setminus N^{\text{min}}$ and by making use of Proposition 2.3 to properly rearrange terms in the composed FSM, the result follows.

□

Theorem 3.13. $\text{Obs}(M(N^{\text{min}}))$ is a critical observer for $M(N^{\text{min}})$ if and only if it is a critical observer for $M(N)$.

Proof. By applying Lemma 3.10 $\text{Obs}(M(N^{\text{min}}))$ is a critical observer for $M(N^{\text{min}}) = (M_{i_1}^{\text{min}}||M_{i_2}^{\text{min}}||...||M_{i_{n^{\text{min}}-1}}^{\text{min}}||M_{i_{n^{\text{min}}}}^{\text{min}})$ if and only if it is a critical observer for $(M_{i_1}^{\text{min}}||M_{i_2}^{\text{min}}||...||M_{i_{n^{\text{min}}-1}}^{\text{min}})||M_{i_{n^{\text{min}}}}^{\text{min}}$ for any FSMs $M_i(N^{\text{min}}, j) \in N \setminus N^{\text{min}}$ (recall that $M_i(N^{\text{min}}, j) \cong M_{i_{n^{\text{min}}}}^{\text{min}}$ for any $j \in [1; n^{\text{min}}]$). Hence, by applying recursively Lemma 3.10 to all other FSMs $M \in N \setminus N^{\text{min}}$ and by making use of Proposition 2.3 to properly rearrange terms in the composed FSM, the result follows.

□

The above results reduce the computational complexity effort since they show that it is possible to consider the reduced network $N^{\text{min}}$ to check critical observability and to design critical observers for the original network $N$. We stress that the model reduction via bisimulation equivalence that we propose here is performed on the reduced network $N^{\text{min}}$ and not on the monolithic FSM $M(N)$, as done for example in [17]; this may allow a drastic computational complexity reduction when several bisimilar FSMs are present in the network.

3.3. Combining design of decentralized critical observers with model reduction. Results in Subsections 3.1 and 3.2 can be combined together as illustrated in Algorithm 3.

Algorithm 3 Integrated design of decentralized observers with model reduction

1: Compute the network $N^{\text{min}}$;
2: Apply Algorithm 2 to $N^{\text{min}}$;
3: if $M(N^{\text{min}})$ is not critically observable then
4: BREAK: $M(N)$ is not critically observable;
5: else
6: $M(N)$ is critically observable;
7: end if
8: Define projected local observers $\pi|_{\text{Obs}(M_i(k, j))}(\text{Obs}^d(N)) = \pi|_{\text{Obs}(M_{i_{n^{\text{min}}}}^{\text{min}})(\text{Obs}^d(N^{\text{min}}))}$ for any $j \in [1; n_k], k \in [1; N^{\text{min}}]$.

As a consequence of Proposition 3.9 (ii), the composition of local observers computed in line 8 of Algorithm 3 is a decentralized critical observer for $N$. We conclude this section with a computational complexity analysis. We focus on computational complexity with respect to parameters $n_{\text{max}}, N$ and $N^{\text{min}}$. A traditional approach to check critical observability of the network $N = \{M_1, M_2, ..., M_N\}$ consists in computing $\text{Obs}(M(N))$, whose space and time computational complexity by Corollary 3.6 is $O(2^{n_{\text{max}}N})$, as reported in the second column of Table 1. Computational complexity analysis of Algorithm 3 follows. In line 1, one needs to check bisimulation equivalence between any pair of FSMs $M_i, M_j$ in $N$ whose space computational complexity is $O(n_{\text{max}}^2N)$ and time computational complexity is $O(n_{\text{max}}^2N^2\log(1/n_{\text{max}}))$. Space and time computational complexities associated with line 2 are $O(2^{n_{\text{max}}N^{\text{min}}})$ and those with line 8 are zero. Resulting computational complexity bounds are reported in the
third column of Table 1. For example, for $N = 10$, $N^{\text{min}} = 7$ and $n_{\text{max}} = 10$, space and time computational complexities in constructing $\text{Obs}(M(\mathcal{N}))$ are $2^{100}$ and the ones in constructing $\text{Obs}^d(\mathcal{N}^{\text{min}})$ are $2^{70}$.

| Complexity | $\text{Obs}(M(\mathcal{N}))$ | $\text{Obs}^d(\mathcal{N}^{\text{min}})$ |
|------------|-----------------------------|----------------------------------|
| Space      | $O(2^{n_{\text{max}}N})$   | $O(n_{\text{max}}^2 N + 2^{n_{\text{max}}N^{\text{min}}})$ |
| Time       | $O(2^{n_{\text{max}}N})$   | $O(n_{\text{max}}^2 N^2 \log(n_{\text{max}}) + 2^{n_{\text{max}}N^{\text{min}}})$ |

**Table 1.** Computational complexity analysis.

4. **ILLUSTRATIVE EXAMPLES**

In this section we illustrate the results of the previous sections through two academic examples. In both examples, the goal is to check if a network $\mathcal{N}$ is critically observable and if so, to design a decentralized critical observer for $\mathcal{N}$. For this purpose we apply Algorithm 3. In the sequel, space complexity of an FSM is computed as $n_1 + n_2$ where $n_1$ is the sum of the data needed to be stored for each transition and $n_2$ is the number of output data associated with states. Date stored for a transition from a state $(z_1, z_2, ..., z_m)$ to a state $(z_1^+, z_2^+, ..., z_m^+)$ with a given input label are counted as $\sum_{i \in [1,m]} |z_i| + \sum_{i \in [1,m^*]} |z_i^+| + 1$. Time complexity is computed as the number of transitions generated in composed FSMs and observers, which represent macro iterations in the algorithms.

**Example 4.1.** Consider $\mathcal{N} = \{M_1, M_2, M_3, M_4\}$ where FSMs $M_1$ and $M_2$ are depicted in Fig. 7 and FSMs $M_3$ and $M_4$ in Fig. 8 and apply Algorithm 3. It is easy to see that $\text{FSMs } M_1$ and $M_2$ are bisimilar with bisimulation relation $R_{13} = \{(1, 9), (2, 10), (3, 11), (3, 12), (4, 13)\}$ and that $\text{FSMs } M_2$ and $M_4$ are bisimilar with bisimulation relation $R_{24} = \{(5, 14), (6, 15), (7, 16), (7, 17), (7, 18), (8, 16), (8, 17), (8, 18)\}$. Equivalence classes induced by the bisimulation equivalence on $\mathcal{N}$ are $\mathcal{E}_1 = \{M_1, M_3\}$ and $\mathcal{E}_2 = \{M_2, M_4\}$. The resulting network $\mathcal{N}^{\text{min}}$ can be chosen as $\{M_1, M_2\}$. By applying Algorithm 2 we get that $\mathcal{N}^{\text{min}}$ is critically observable. The resulting projected local observers $\pi_{\text{Obs}(M_1)}(\text{Obs}^d(\mathcal{N}^{\text{min}}))$ and $\pi_{\text{Obs}(M_2)}(\text{Obs}^d(\mathcal{N}^{\text{min}}))$ are depicted in Fig. 9. (Line 8) Define $\pi_{\text{Obs}(M_i)}(\text{Obs}^d(\mathcal{N}))$ as $\pi_{\text{Obs}(M_i)}(\text{Obs}^d(\mathcal{N}^{\text{min}}))$ for $i = 1, 3$ and as $\pi_{\text{Obs}(M_2)}(\text{Obs}^d(\mathcal{N}^{\text{min}}))$ for $i = 2, 4$. Space and time complexities in constructing projected local observers are 54 and 8. A traditional approach would first construct explicitly $M(\mathcal{N})$ and then construct $\text{Obs}(M(\mathcal{N}))$. Obtained $\text{FSM } M(\mathcal{N}) = M_1||M_2||M_3||M_4$ is depicted in Figure 9. The number of states of $M(\mathcal{N})$ is 21 and the one of $\text{Obs}(M(\mathcal{N}))$ is 6. Resulting space and time complexities are 633 and 39.

**Example 4.2.** Consider $\mathcal{N} = \{M_1, M_2, M_3, M_4, M_5\}$ where FSMs $M_i$, $i \in [1; 4]$ are as in Example 4.1 and $M_5$ is depicted in Fig. 10 with $C_5 = \{20\}$, and apply Algorithm 3. It is easy to see that for any $i \in [1; 4]$, FSMs $M_5$ and $M_i$ are not bisimilar from which, $\mathcal{N}^{\text{min}}$ can be chosen as $\{M_1, M_2, M_3, M_5\}$. In the first iteration of Algorithm 2 starting from the initial state $z = (\{1\}, \{5\}, \{19\})$ with label $b$, state $z^+ = (\{2\}, \{6\}, \{20, 21\})$ is reached. Since state $z^+$ does not satisfy the condition in line 12 of Algorithm 2, Algorithm 2 terminates from which, Algorithm 3 terminates as well, giving as output that $\mathcal{N}$ is not critically observable. Data stored in line 1 of Algorithm 2 are 72 while those in lines 2–9 are 19 from which,
Figure 8. Projected local observers $\pi_{\text{Obs}(M_1)}(\text{Obs}^d(N_{\text{min}}))$ in the left and $\pi_{\text{Obs}(M_2)}(\text{Obs}^d(N_{\text{min}}))$ in the right.

Figure 9. Composed FSM $M_1|M_2|M_3|M_4$.

Figure 10. FSM $M_5$.

space complexity at line 12 is 91. Since Algorithm 2 terminates at the first iteration and no transitions are generated, time complexity is evaluated as 0. A traditional approach would first construct explicitly $M(N)$ to then construct $\text{Obs}(M(N))$. The number of states of $M(N)$ is 24 and the one of $\text{Obs}(M(N))$ is 6. Resulting space and time complexities are 895 and 39.
5. Conclusions

In this paper, we proposed decentralized critical observers for networks of FSMS. On line detection of critical states is performed by local critical observers, each one associated with an FSM of the network. For the design of local observers, efficient algorithms were provided which are based on on-the-fly techniques. Model reduction of networks of FSMS via bisimulation equivalence was shown to facilitate the design of distributed observers for the original network. In some specific applications, as e.g. in Air Traffic Management systems, critical states may be associated to aggregates of FSMS rather than single FSMS. Useful insights in this regard are reported in [12] [11]. We plan to investigate this issue in our future work.

Acknowledgements: We wish to thank Sina Lessanibahri for participating in fruitful discussions at the beginning of this project. We also thank Alessandro D’Innocenzo for fruitful discussions on critical observability and regular languages.

6. Appendix

In this section we report the proof of some technical results.

Proof of Proposition 2.3. Given FSMS $M_i = (X_i, X_i^0, \Sigma_i, \delta_i, C_i)$, with $i = 1, 2, 3$ define $M_i' = (X_i, X_i^0, \Sigma_i, \delta_i)$. FSMS $M_1'||M_2'||M_3'$ are isomorphic with bijective function $\phi : X_{1,2} \rightarrow X_{2,1}$ defined by $\phi(x_1, x_2) = (x_2, x_1)$ for any $(x_1, x_2) \in X_{1,2}$ and by e.g. [2], $(M_i'||M_2'||M_3') = M_1'||M_2'||M_3'$. Moreover, $C_{1,2} = \phi(C_{2,1})$ and for the associativity property, we get:

$$C_{1,(2,3)} = (C_{1,2} \times X_3) \cup (X_{1,2} \times C_3)$$

$$(1)$$

In this section we report the proof of some technical results.

Proof of Proposition 3.4. Set $M_i = (X_i, X_i^0, \Sigma_i, \delta_i, C_i)$, $i = 1, 2$. By contradiction, assume that $M_i||M_2$ is not critically observable. Thus, there exists a pair of state runs $r_1$ and $r_2$ with initial states $(x_0, x_0^0_i)$, $(x_0^0, x_0^0_2)$ such that $\delta_1(x_1, w) \cup \delta_2(x_2, w)$, states $x_1 \in X_{1,2}$, and states $x_2 \in X_{2,1}$, and states $x_1 \in X_{1,2}$ and states $x_2 \in X_{2,1}$. By definition of the projection operator $P_{C_i}(\cdot)$ we then get $[x_1 \in \delta_1(x_1^0, P_{C_i}(w)) \cup [x_1 \in \delta_1(x_1^0, P_{C_i}(w)) \cup [x_2 \in \delta_2(x_2^0, P_{C_i}(w)) \cup [x_2 \in \delta_2(x_2^0, P_{C_i}(w))]$. Moreover, by definition of $C_{1,2}$, we then get $[(x_1^0 \in X_{1,2}) \cup [x_2 \in X_{2,1} \cup [x_2 \in X_{2,1}]$, and a contradiction holds.

Proof of Proposition 3.5, Proof of (i). Set $M_i = (X_i, X_i^0, \Sigma_i, \delta_i, C_i)$, $i = 1, 2$. By contradiction assume that $M_i$ is critically observable and $M_2$ is not critically observable. Hence, there exist a pair of state runs of $M_2$ with initial states $x_0^0, x_0^0_2 \in X_{0,2}$, common trace $w = L(M_2)$, and states $x_1 \in \delta_2(x_0^0, w)$, $x_2 \in \delta_2(x_0^0, w)$ such that (6.1)

$$x_2 \in X_{2,1} \cup x_2 \in X_{2,1}.$$ Let $R$ be a bisimulation relation between $M_1$ and $M_2$. Since $M_1 \cong M_2$, there exist a pair of state runs of $M_1$ with initial states $x_0^0, x_0^0_2 \in X_{0,2}$, common trace $w = L(M_1)$ and states $x_1 \in \delta_1(x_1^0, w)$, $x_1 \in \delta_1(x_1^0, w)$ such that $(x_1, x_2), (x_1, x_2) \in R$ which, by (6.1) and condition (iv) in Definition 3.8 implies $x_1 \in X_1$ and $x_1 \in X_1 \cup C_1$. Thus, $M_1$ is not critically observable and a contradiction holds. The proof of (ii) can be given by using a similar reasoning.

Proof of Lemma 3.10. Let be $\text{Obs} = (X_{\text{Obs}}, X_{\text{Obs}}, \Sigma_{\text{Obs}}, \delta_{\text{Obs}}, Y_{\text{Obs}}, H_{\text{Obs}})$ and $R_{23}$ be a bisimulation relation between $M_2$ and $M_3$. (Sufficiency.) By contradiction assume that $\text{Obs}$ is not a critical observer for $M_1||M_2||M_3$. By Definition 2.6, there exist a state run $r_1 = (x_0^0, x_0^0, x_0^0) \rightarrow (x_0^1, x_0^1, x_0^1) \rightarrow \ldots \rightarrow (x_0^n, x_0^n, x_0^n)$ of $M_1||M_2||M_3$ and the corresponding state run $r_{\text{Obs}} = (z_0 \rightarrow z_1 \rightarrow \ldots \rightarrow z^n)$ of $\text{Obs}$ such that (case
1) \( [x^0_1, x^2_2, x^3_3] \notin C_{1, 2, 3} \land H_{Obs}(z^n) = 1 \) or (case 2) \( [x^0_1, x^2_2, x^3_3] \in C_{1, 2, 3} \land H_{Obs}(z^n) = 0 \). Construct the sequence \( r_{1, 2} : (x^0_1, x^2_2) \xrightarrow{σ^1} (x^1_1, x^2_2) \xrightarrow{σ^2} \cdots \xrightarrow{σ^n} (x^n_1, x^n_2) \), where \( r_{1, 2} \) has been obtained by removing the third component in the states of run \( r \). It is readily seen that \( r_{1, 2} \) is a state run of \( M_1 || M_2 \). Construct the sequence \( \tilde{r}_{1, 2} : (x^0_1, x^2_2) \xrightarrow{σ^1} (x^1_1, x^2_2) \xrightarrow{σ^2} \cdots \xrightarrow{σ^n} (x^n_1, x^n_2) \) such that \( (x_i^n, x_i^n) \in R_{23} \) for any \( i \in [0; n] \). By construction, \( \tilde{r}_{1, 2} \) is a state run of \( M_1 || M_2 \). We start by considering case 1. Since \( (x^0_1, x^2_2, x^3_3) \notin C_{1, 2, 3} \) then \( x^1_1 \notin C_1 \) and \( x^2_2 \notin C_2 \) from which the last state \( (x^n_1, x^n_2) \) of run \( r_{1, 2} \) is such that \( (x^0_1, x^2_2, x^3_3) \notin C_{1, 2} \). Since \( H_{Obs}(z^n) = 1 \), FSM Obs is not a critical observer for \( M_1 || M_2 \) and a contradiction holds. We now consider case 2. Since \( (x^0_1, x^2_2, x^3_3) \in C_{1, 2, 3} \) then (case 2.1) \( (x^0_1, x^2_2) \in C_1 \lor x^3_3 \in C_2 \) or (case 2.2) \( x^3_3 \in C_3 \). We start by considering case 2.1. Since \( (x^0_1, x^2_2) \in C_1 \lor x^3_3 \in C_2 \) or equivalently, the last state \( (x^n_1, x^n_2) \) of run \( r_{1, 2} \) is such that \( (x^0_1, x^2_2) \) is in \( C_{1, 2} \) and by assumption \( H_{Obs}(z^n) = 0 \), a contradiction holds. We conclude with case 2.2. Since \( x^3_3 \in C_3 \), by definition of run \( \tilde{r}_{1, 2} \), state \( x^2_2 \in C_2 \) which implies that the last state \( (x^n_1, x^n_2) \) of run \( \tilde{r}_{1, 2} \) is such that \( (x^n_1, x^n_2) \in C_{1, 2} \). This last condition combined with the assumed condition \( H_{Obs}(z^n) = 0 \), leads to a contradiction, as well. (Necessity.) By contradiction assume that Obs is not a critical observer for \( M_1 || M_2 \). By Definition 2.6, there exist a state run \( r' : (x^0_1, x^2_2, x^3_3) \xrightarrow{σ^1} (x^1_1, x^2_2, x^3_3) \xrightarrow{σ^2} \cdots \xrightarrow{σ^n} (x^n_1, x^n_2, x^n_3) \) of \( M_1 || M_2 \) and the corresponding state run \( r_{obs} : z^0 \xrightarrow{σ^1} z^1 \cdots \xrightarrow{σ^n} z^n \) of Obs such that (case 1) \( [(x^0_1, x^2_2, x^3_3) \notin C_{1, 2} \land H_{Obs}(z^n) = 1] \) or (case 2) \( [(x^0_1, x^2_2, x^3_3) \in C_{1, 2} \land H_{Obs}(z^n) = 0] \). Construct the sequence \( r' : (x^0_1, x^2_2, x^3_3) \xrightarrow{σ^1} (x^1_1, x^2_2, x^3_3) \xrightarrow{σ^2} \cdots \xrightarrow{σ^n} (x^n_1, x^n_2, x^n_3) \), where \( (x^3_3, x^3_3) \in R_{23} \) for any \( i \in [0; n] \). By construction \( r' \) is a state run of \( M_1 || M_2 || M_3 \). We start by considering case 1. By condition (iv) of Definition 3.8, we get \( [(x^0_1, x^2_2) \notin C_{1, 2} \land H_{Obs}(z^n) = 1] \) iff \( x^0_1 \notin C_1 \land x^2_2 \notin C_2 \) iff \( x^0_1 \notin C_1 \land x^2_2 \notin C_2 \land x^2_2 \notin C_3 \) iff \( x^0_1 \notin C_1 \land x^2_2 \notin C_2 \land x^3_3 \notin C_3 \). Since \( H_{Obs}(z^n) = 1 \), FSM Obs is not a critical observer for \( M_1 || M_2 || M_3 \) and a contradiction holds. We now consider case 2. By condition (iv) of Definition 3.8, we get \( [(x^0_1, x^2_2) \in C_{1, 2} \land H_{Obs}(z^n) = 1] \) iff \( x^0_1 \in C_1 \land x^2_2 \in C_2 \) iff \( x^0_1 \in C_1 \land x^2_2 \in C_2 \land x^3_3 \in C_3 \) iff \( x^0_1 \in C_1 \land x^2_2 \in C_2 \land x^3_3 \in C_3 \). Since \( H_{Obs}(z^n) = 0 \), FSM Obs is not a critical observer for \( M_1 || M_2 || M_3 \) and a contradiction holds.

Proof of Lemma 3.10 The result follows by combining Proposition 2.8 and Lemma 3.10.
[13] B. Brandin R. Debouk, R. Malik. A modular architecture for diagnosis of discrete event systems. In Proceedings of the 41st Conference on Decision and Control, Las Vegas, Nevada, USA, pages 417–422, December 2002.

[14] K. W. Schmidt. Verification of modular diagnosability with local specifications for discrete-event systems. IEEE Transactions on Systems, Man and Cybernetics, 43(5):1130–1140, 2013.

[15] R. Su and W.M. Wonham. Global and local consistencies in distributed fault diagnosis for discrete-event systems. IEEE Transactions on Automatic Control, 50(12):1923–1935, 2005.

[16] S. Tripakis and K. Altisen. On-the-fly controller synthesis for discrete and dense-time systems. In World Congress on Formal Methods in the Development of Computing Systems, volume 1708 of Lecture Notes in Computer Science, pages 233 – 252. Springer Verlag, Berlin, September 1999.

[17] S.H. Zad, R.H. Kwong, and W.M. Wonham. Fault diagnosis in discrete-event systems: Framework and model reduction. IEEE Transactions on Automatic Control, 48(7):51–65, July 2003.

[18] J. Zaytoon and S. Lafortune. Overview of fault diagnosis methods for discrete event systems. Annual Reviews in Control, 37(2):308–320, 2013.

1Department of Information Engineering, Computer Science and Mathematics, Center of Excellence DEWS, University of L’Aquila, 67100 L’Aquila, Italy

E-mail address: davide.pezzuti@graduate.univaq.it, {giordano.pola,elena.desantis,mariadomenica.dibenedetto}@univaq.it