Conformal Perturbations of Twisted Dirac Operators and Noncommutative residue

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Abstract

In this paper, we obtain two kinds of Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and conformal perturbations of signature operators by a vector bundle with a non-unitary connection on six-dimensional manifolds with (respectively without)boundary.

Keywords: Conformal perturbations of twisted Dirac operators; conformal perturbations of twisted signature operators; noncommutative residue; non-unitary connection.

1. Introduction

The noncommutative residue found in \cite{1,2} plays a prominent role in noncommutative geometry. For one-dimensional manifolds, the noncommutative residue was discovered by Adler \cite{3} in connection with geometric aspects of nonlinear partial differential equations. For arbitrary closed compact $n$-dimensional manifolds, the noncommutative residue was introduced by Wodzicki in \cite{2} using the theory of zeta functions of elliptic pseudodifferential operators. In \cite{4}, Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action in \cite{5}. In \cite{6}, Kastler gave a brute-force proof of this theorem. In \cite{7}, Kalau and Walze proved this theorem in the normal coordinates system simultaneously. And then, Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator $\text{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of $D^2$ in \cite{8}.

In \cite{9}, Ponge defined lower dimensional volumes of Riemannian manifolds by the Wodzicki residue. Fedosov et al. defined a noncommutative residue on Boutet de Monvel’s algebra and proved that it was a unique continuous trace in \cite{10}. In \cite{11}, Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. In \cite{12}, Wang generalized the Kastler-Kalau-Walze type theorem to the cases of 3, 4-dimensional spin manifolds with boundary and proved a Kastler-Kalau-Walze type theorem. In \cite{12,13,14,15,16}, Y. Wang and his coauthors computed the lower dimensional volumes for 5, 6, 7-dimensional spin manifolds with boundary and also got some Kastler-Kalau-Walze type theorems. In \cite{17}, authors computed $\text{Wres}[(\pi^+D^{-2}) \circ (\pi^+D^{-n+2})]$ for any-dimensional manifolds with boundary, and proved a general Kastler-Kalau-Walze type theorem.

In \cite{18}, J. Wang and Y. Wang proved two kinds of Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and conformal perturbations of signature operators by a vector bundle with a non-unitary connection on four-dimensional manifolds with (respectively without)boundary.

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The motivation of this paper is to establish two Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and conformal perturbations of signature operators with non-unitary connections on six-dimensional manifolds with boundary. We know that the leading symbol of conformal perturbations of twisted Dirac operators is not $\sqrt{-\text{Im}}(\xi)$. This is the reason that we study the residue of conformal perturbations of twisted Dirac operators.

This paper is organized as follows: In Section 2, we recall some basic facts and formulas about Boutet de Monvel’s calculus. In Section 3, we give a Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators on six-dimensional manifolds with boundary. In Section 4 and Section 5, we recall the definition of conformal perturbations of signature operators and compute their symbols, and we give a Kastler-Kalau-Walze type theorems for conformal perturbations of signature operators on six-dimensional manifolds with boundary.

2. Boutet de Monvel’s calculus and noncommutative residue

In this section, we shall recall some basic facts and formulas about Boutet de Monvel’s calculus. Let

$$F : L^2(\mathbb{R}_t) \to L^2(\mathbb{R}_v); \ F(u)(v) = \int e^{-ivt}u(t)dt$$

denote the Fourier transformation and $\varphi(\mathbb{R}^+)$ (similarly define $\varphi(\mathbb{R}^-)$), where $\varphi(\mathbb{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^+); \ f \to f|_{\mathbb{R}^+}; \ \mathbb{R}^+ = \{x \geq 0; x \in \mathbb{R}\}. \quad (2.1)$$

We define $H^+ = F(\varphi(\mathbb{R}^+)); \ H^- = F(\varphi(\mathbb{R}^-))$ which are orthogonal to each other. We have the following property: $h \in H^+ (H^-)$ if and only if $h \in C^\infty(\mathbb{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\text{Im}\xi < 0\} (\{\text{Im}\xi > 0\})$ such that for all nonnegative integer $l$,

$$\frac{d^lh}{dx^l}(\xi) \sim \sum_{k=1}^\infty \frac{d^l}{dx^l}(c_k \xi^k) \quad (2.2)$$

as $|\xi| \to +\infty, \text{Im}\xi \leq 0 (\text{Im}\xi \geq 0)$.

Let $H'$ be the space of all polynomials and $H'' = H_0' \bigoplus H'; \ H = H^+ \bigoplus H^-$. Denote by $\pi^+ (\pi^-)$ respectively the projection on $H^+ (H^-)$. For calculations, we take $H = \tilde{H} = \{\text{rational functions having no poles on the real axis}\}$ ($\tilde{H}$ is a dense set in the topology of $H$). Then on $\tilde{H}$,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \to 0} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \quad (2.3)$$

where $\Gamma^+$ is a Jordan close curve included $\text{Im}\xi > 0$ surrounding all the singularities of $h$ in the upper half-plane and $\xi_0 \in \mathbb{R}$. Similarly, define $\pi^-$ on $\tilde{H}$,

$$\pi^- h = \frac{1}{2\pi} \int_{\Gamma^-} h(\xi) d\xi. \quad (2.4)$$

So, $\pi^-(H^-) = 0$. For $h \in H \bigcap L^1(R)$, $\pi^+ h = \frac{1}{2\pi} \int_{R} h(v) dv$ and for $h \in H^+ \bigcap L^1(R)$, $\pi^- h = 0$. Denote by $B$ Boutet de Monvel’s algebra (for more details, see Section 2 of [14]).

An operator of order $m \in \mathbb{Z}$ and type $d$ is a matrix

$$A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(X,E_1) \bigoplus C^\infty(\partial X,F_1) \\ \longrightarrow \bigoplus C^\infty(X,E_2) \bigoplus C^\infty(\partial X,F_2) \end{array},$$

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where $X$ is a manifold with boundary $\partial X$ and $E_1, E_2 (F_1, F_2)$ are vector bundles over $X (\partial X)$. Here, $P : C^\infty_0 (\Omega, E_1) \to C^\infty (\Omega, E_2)$ is a classical pseudodifferential operator of order $m$ on $\Omega$, where $\Omega$ is an open neighborhood of $X$ and $E_i|X = E_i (i = 1, 2)$. $P$ has an extension: $\mathcal{E}' (\Omega, E_1) \to \mathcal{D}' (\Omega, E_2)$, where $\mathcal{E}' (\Omega, E_1) (\mathcal{D}' (\Omega, E_2))$ is the dual space of $C^\infty (\Omega, E_1) (C^\infty (\Omega, E_2))$. Let $e^+: C^\infty (X, E_1) \to \mathcal{E}' (\Omega, E_1)$ denote extension by zero from $X$ to $\Omega$ and $r^+: \mathcal{D}' (\Omega, E_2) \to \mathcal{D}' (\Omega, E_2)$ denote the restriction from $\Omega$ to $X$, then define

$$\pi^+ P = r^+ P e^+ : C^\infty (X, E_1) \to \mathcal{D}' (\Omega, E_2).$$

In addition, $P$ is supposed to have the transmission property; this means that, for all $j, k, \alpha$, the homogeneous component $p_j$ of order $j$ in the asymptotic expansion of the symbol $p$ of $P$ in local coordinates near the boundary satisfies:

$$\partial_{x_n}^j \partial^\alpha_x p_j (x', 0, 0, +1) = (-1)^{\nu_j} \partial^\alpha_x \partial_{x_n}^j p_j (x', 0, 0, -1),$$

then $\pi^+ P : C^\infty (X, E_1) \to C^\infty (X, E_2)$ by Section 2.1 of [14].

In the following, write $\pi^+ D^{-1} = \left( \begin{array}{cc} \pi^+ D^{-1} & 0 \\ 0 & 0 \end{array} \right)$. Let $M$ be a compact manifold with boundary $\partial M$. We assume that the metric $g^M$ on $M$ has the following form near the boundary

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \quad (2.5)$$

where $g^{\partial M}$ is the metric on $\partial M$. Let $U \subset M$ be a collar neighborhood of $\partial M$ which is diffeomorphic $\partial M \times [0, 1)$. By the definition of $h(x_n) \in C^\infty (\Omega, 0, 1)$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty (\Omega, 1)$ such that $\tilde{h}|_{(0, 1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric $\tilde{g}$ on $\tilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\tilde{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + dx_n^2, \quad (2.6)$$

such that $\tilde{g}|_M = g$. We fix a metric $\tilde{g}$ on $\tilde{M}$ such that $\tilde{g}|_M = g$. Note $\tilde{D}_F$ is the twisted Dirac operator on the spinor bundle $S(TM) \otimes F$ corresponding to the connection $\tilde{\nabla}$.

Now we recall the main theorem in [10].

**Theorem 2.1. (Fedosov-Golse-Leichtnam-Schröhe)** Let $X$ and $\partial X$ be connected, $\dim X = n \geq 3$, $A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$, and denote by $p, b$ and $s$ the local symbols of $P, G$ and $S$ respectively. Define:

$$\overline{\mathrm{Wres}}(A) = \int_X \int_S \mathrm{tr}_F [p_{-n} (x, \xi)] \sigma (\xi) dx + 2\pi \int_{\partial X} \int_S \{ \mathrm{tr} E [\mathrm{tr} b_{-n} (x', \xi')] + \mathrm{tr} F [s_{1-n} (x', \xi')] \} \sigma (\xi') dx', \quad (2.7)$$

Then

a) $\overline{\mathrm{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$;

b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

**3. Conformal perturbations of twisted Dirac operator and Noncommutative residue**

In this section we consider a $n$-dimensional oriented Riemannian manifold $(M, g^M)$ equipped with a fixed spin structure. Let $S(TM)$ be the spinors bundle and $F$ be an additional smooth vector bundle equipped with a non-unitary connection $\nabla^F$. Let $S_1, S_2 \in \Gamma (F)$, $g^F$ be a metric on $F$. We define the dual connection $\nabla^{F,*}$ by

$$g^F (\nabla^{F,*}_{S_1} S_2) + g^F (S_1, \nabla^{F,*}_{S_2}) = X (g^F (S_1, S_2))$$

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for \( X \in \Gamma(TM) \) and define
\[
\nabla^F = \frac{\nabla F + \nabla F^*}{2}, \quad A = \frac{\nabla F - \nabla F^*}{2},
\]
then \( \nabla^F \) is a metric connection and \( \Phi \) is an endomorphism of \( F \) with a 1-form coefficient. We consider the tensor product vector bundle \( S(TM) \otimes F \), which becomes a Clifford module via the definition:
\[
c(a) = c(a) \otimes \text{id}_F, \quad a \in TM,
\]
and which we equip with the compound connection:
\[
\nabla^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla^F.
\]
Let
\[
\nabla^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla^F,
\]
then the spinor connection \( \tilde{\nabla} \) induced by \( \nabla^{S(TM) \otimes F} \) is locally given by
\[
\tilde{\nabla}^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla^F + \text{id}_{S(TM)} \otimes \Lambda.
\]
Let \( \{ e_i \}_{1 \leq i, j \leq n} \) (\( \{ \partial_i \} \) be the orthonormal frames (natural frames respectively ) on \( TM \),
\[
D_F = \sum_{i,j=1}^n g^{ij} c(\partial_i) \nabla^{S(TM) \otimes F}_{\partial_j} = \sum_{j=1}^n c(e_j) \nabla^{S(TM) \otimes F}_{e_j},
\]
where \( \nabla^{S(TM) \otimes F}_{\partial_j} = \partial_j + \sigma_j^* + \sigma_j^F \) and \( \sigma_j^* = \frac{4}{3} \sum_{j,k=1}^n (\nabla^{S(TM)}_{\partial_k} e_j) c(e_j)c(e_k) \), \( \sigma_j^F \) is the connection matrix of \( \nabla^F \), then the twisted Dirac operators \( \tilde{D}_F, \tilde{D}_F^* \) associated to the connection \( \tilde{\nabla} \) as follows.
For \( \psi \otimes \chi \in S(TM) \otimes F \), we have
\[
\tilde{D}_F(\psi \otimes \chi) = D_F(\psi \otimes \chi) + c(A)(\psi \otimes \chi),
\]
\[
\tilde{D}_F^*(\psi \otimes \chi) = D_F(\psi \otimes \chi) - c(A^*)(\psi \otimes \chi),
\]
where \( c(A) = \sum_{i=1}^n c(e_i) \otimes A(e_i) \) and \( c(A^*) = \sum_{i=1}^n c(e_i) \otimes A^*(e_i) \), \( A^*(e_i) \) denotes the adjoint of \( A(e_i) \).
Then, we have obtain
\[
\tilde{D}_F = \sum_{j=1}^n c(e_j) \nabla^{S(TM) \otimes F}_{e_j} + c(A),
\]
\[
\tilde{D}_F^* = \sum_{j=1}^n c(e_j) \nabla^{S(TM) \otimes F}_{e_j} - c(A^*).
\]
Let \( \nabla^TM \) denote the Levi-civita connection about \( g^M \). In the local coordinates \( \{ x_i : 1 \leq i \leq n \} \) and the fixed orthonormal frame \( \{ \tilde{e}_1, \cdots, \tilde{e}_n \} \), the connection matrix \( (\omega_{a,t}) \) is defined by
\[
\nabla^TM(\tilde{e}_1, \cdots, \tilde{e}_n) = (\tilde{e}_1, \cdots, \tilde{e}_n)(\omega_{a,t}).
\]
Let \( c(\tilde{e}_i) \) denote the Clifford action, \( g^{ij} = g(dx_i,dx_j) \nabla^{TM}_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k \), \( \Gamma^k = g^{ij} \Gamma^k_{ij} \) and the cotangent vector \( \xi = \sum \xi_j dx_j \) and \( \xi^i = g^{ij} \xi_i \), by Lemma 1 in [12] and Lemma 2.1 in [12], for any fixed point \( x_0 \in \partial M \), we can choose the normal coordinates \( U \) of \( x_0 \) in \( \partial M \) (not in \( M \)), by the composition formula and (2.2.11) in [12], we obtain in [19],
Lemma 3.1. Let $\tilde{D}_F, \tilde{D}_F$ be the twisted Dirac operators on $\Gamma(S(TM) \otimes F)$, then

$$\sigma^{-1}(\tilde{D}_F)^{-1} = \sigma^{-1}(\tilde{D}_F)^{-1} = \frac{\sqrt{-\gamma}}{|\xi|^2};$$

$$\sigma^{-2}(\tilde{D}_F)^{-1} = \frac{c(\xi)\sigma_0(\tilde{D}_F^*\xi)}{|\xi|^2} + \frac{c(\xi)}{|\xi|^2} \sum_j c(dx_j) \left[ \partial_j [c(\xi)] |\xi|^2 - c(\xi) \partial_j |\xi|^2 \right];$$

$$\sigma^{-2}(\tilde{D}_F)^{-1} = \frac{c(\xi)\sigma_0(\tilde{D}_F^*\xi)}{|\xi|^2} + \frac{c(\xi)}{|\xi|^2} \sum_j c(dx_j) \left[ \partial_j [c(\xi)] |\xi|^2 - c(\xi) \partial_j |\xi|^2 \right];$$

where

$$\sigma_0(\tilde{D}_F) = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_l) c(e_i) c(e_s) c(e_i) + \sum_{j=1}^n c(e_j)(\sigma_j^F - A^*(e_j));$$

$$\sigma_0(\tilde{D}_F) = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_l) c(e_i) c(e_s) c(e_i) + \sum_{j=1}^n c(e_j)(\sigma_j^F + A(e_j)).$$

For convenience, let $\lambda = \sum_{j=1}^n c(e_j)(\sigma_j^F - A^*(e_j))$, $\mu = \sum_{j=1}^n c(e_j)(\sigma_j^F + A(e_j))$. In the following, we will compute the more general case $\text{Wres}[\pi^+(f \tilde{D}_F^{-1}) \circ \pi^+ (f^{-1}(\tilde{D}_F^{-1}) \cdot f \tilde{D}_F^{-1} \cdot f^{-1}(\tilde{D}_F^{-1})^{-1})]$ for nonzero smooth functions $f$, $f^{-1}$. Denote by $\sigma_l(P)$ the $l$-order symbol of an operator $P$. An application of (3.5) and (3.6) in [14] shows that

$$\text{Wres}[\pi^+(f \tilde{D}_F^{-1}) \circ \pi^+ (f^{-1}(\tilde{D}_F^{-1}) \cdot f \tilde{D}_F^{-1} \cdot f^{-1}(\tilde{D}_F^{-1})^{-1})] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM) \otimes F} [\sigma_{-n}(f \tilde{D}_F \cdot \tilde{D}_F f^{-1} \cdot f^{-1}(\tilde{D}_F^{-1})^{-1})] \sigma(\xi)dx + \int_{\partial M} \Phi,$$

where

$$\Phi = \int_{|\xi|=1} \int_{-\infty}^{\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{a+j+k+\ell}}{a! j! k!(j+k+1)!} \text{trace}_{S(TM) \otimes F} \left[ \partial_{x_n}^j \partial_{\xi_n}^k \sigma_1^F(f \tilde{D}_F^{-1})(x',0, \xi', \xi_n) \right] \times \partial_{\xi_n}^\alpha \partial_{\xi_n}^{\beta+1} \partial_{x_n}^\lambda \sigma_l \left( f^{-1}(\tilde{D}_F^{-1}) \cdot f \tilde{D}_F^{-1} \cdot f^{-1}(\tilde{D}_F^{-1}) \right)(x',0, \xi', \xi_n) dx' \xi_n \sigma(\xi')dx',$$

and the sum is taken over $r - k + |a| + \ell - j - 1 = -n = -6$, $r \leq -1$, $\ell \leq -3$.

Note that

$$f^{-1}(\tilde{D}_F^{-1}) \cdot f \tilde{D}_F^{-1} \cdot f^{-1}(\tilde{D}_F^{-1})^{-1}$$

$$= (\tilde{D}_F \cdot f \tilde{D}_F f^{-1} \cdot f^{-1}(\tilde{D}_F^{-1})^{-1})$$

$$= \left( \tilde{D}_F \cdot f \tilde{D}_F f^{-1} \cdot f - \tilde{D}_F \cdot f \tilde{D}_F f^{-1} \cdot f \right)^{-1}$$

$$= \left( f \tilde{D}_F \tilde{D}_F f^{-1} \cdot f \right)^{-1}$$

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In order to get the symbol of operators \( \tilde{D}_F f \cdot \tilde{D}_F f^{-1} \cdot \tilde{D}_F f \). We first give the specification of \( \tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* \), \( \tilde{D}_F^* \tilde{D}_F \) and \( \tilde{D}_F \tilde{D}_F^* \). By (3.9) and (3.10), we have

\[
\tilde{D}_F^* \tilde{D}_F = D_F^2 - c(A)D_F - c(A)c(A^*)
\]

\[
= -g^{ij} \partial_i \partial_j - 2\sigma_j^{(T \otimes F)} \partial_j + \Gamma^k \partial_k + \sum_{j=1}^n \left[ c(A)c(e_j) - c(A^*)c(e_j) \right] e_j - \sum_{j=1}^n c(e_j) \sigma_j^{(T \otimes F)} c(A^*)
\]

\[
- g^{ij} \left[ (\partial_i \sigma_j^{(T \otimes F)}) + \sigma_j^{(T \otimes F)} \sigma_j^{(T \otimes F)} - \Gamma_{ij}^{k} \sigma_k^{(T \otimes F)} \right] + \frac{1}{4}s + \frac{1}{2} \sum_{i \neq j} R^F (e_i, e_j) c(e_i)c(e_j)
\]

\[+
\sum_{j=1}^n \left[ c(A)c(e_j) \right] \sigma_j^{(T \otimes F)} - \sum_{j=1}^n c(e_j) c_j (c(A^*)) - c(A)c(A^*)
\]

(3.20)

and

\[
\tilde{D}_F \tilde{D}_F^* = D_F^2 - c(A^*)D_F + D_Fc(A) - c(A)c(A^*)
\]

\[
= -g^{ij} \partial_i \partial_j - 2\sigma_j^{(T \otimes F)} \partial_j + \Gamma^k \partial_k + \sum_{j=1}^n \left[ c(e_j)c(A) - c(A^*)c(e_j) \right] e_j + \sum_{j=1}^n c(e_j) \sigma_j^{(T \otimes F)} c(A)
\]

\[
- g^{ij} \left[ (\partial_i \sigma_j^{(T \otimes F)}) + \sigma_j^{(T \otimes F)} \sigma_j^{(T \otimes F)} - \Gamma_{ij}^{k} \sigma_k^{(T \otimes F)} \right] + \frac{1}{4}s + \frac{1}{2} \sum_{i \neq j} R^F (e_i, e_j) c(e_i)c(e_j)
\]

\[-\sum_{j=1}^n \left[ c(A^*)c(e_j) \right] \sigma_j^{(T \otimes F)} + \sum_{j=1}^n c(e_j) c_j (c(A)) - c(A^*)c(A).
\]

(3.21)
Combining (3.10) and (3.20), we obtain

\[
\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* = \begin{align*}
\sum_{i,j,l=1}^n c(e_r)(e_r, dx_l) \left( -g^{ij} \partial_i \partial_j + \sum_{r,l=1}^n c(e_r)(e_r, dx_l) \right) & - \sum_{i,j,k=1}^n (\partial g^{ij}) \partial_i \partial_j - \sum_{i,j,k=1}^n g^{ij} \\
\times (4 \sigma_i^{S(TM)\otimes F} \partial_j - 2 \Gamma^{k}_{ij} \partial_k) & + \sum_{r,l=1}^n c(e_r)(e_r, dx_l) - 2 \sum_{i,j=1}^n (\partial g^{ij}) \sigma_i^{S(TM)\otimes F} \partial_j \\
+ \sum_{i,j,k=1}^n g^{ij} (\partial_i \Gamma^{k}_{ij}) \partial_k - 2 \sum_{i,j,k=1}^n g^{ij} (\partial_i \sigma_i^{S(TM)\otimes F} \partial_j) + \sum_{i,j,k=1}^n (\partial g^{ij}) \Gamma^{k}_{ij} \partial_k & + \sum_{r,l=1}^n c(e_r)(e_r, dx_l) \\
\times \partial_i \left\{ - \sum_{i,j,k=1}^n g^{ij} (\partial_i \sigma_i^{S(TM)\otimes F} - \sigma_i^{S(TM)\otimes F} \Gamma^{k}_{ij}) \partial_k - \sum_{r,l=1}^n c(e_r)(e_r, dx_l) \right\} + \sum_{i,j=1}^n c(e_r)(e_r, dx_l) \\
\times c(e_l)(c(e_j)) + \sigma_0 (\tilde{D}_F^*) \sum_{j,l=1}^n (-g^{ij} \partial_i \partial_j) & + \sum_{r,l=1}^n c(e_r)(e_r, dx_l) \left\{ 2 \sum_{j,k=1}^n c(A)(c(e_j)) - c(e_j) c(A^*) \right\} \\
\sum_{j,l=1}^n c(e_l)(c(e_j)) \partial_l \partial_k + \sigma_0 (\tilde{D}_F^*) & - 2 \sigma_i^{S(TM)\otimes F} \partial_j + \Gamma^{k}_{ij} \partial_k + \sum_{j,l=1}^n c(A)(c(e_j)) - c(e_j) c(A^*) \right\} e_j - \sum_{j=1}^n c(e_j) \\
\sum_{j=1}^n \left[ c(A)(c(e_j)) \right] \sigma_j^{S(TM)\otimes F} - \sum_{j=1}^n c(e_j) \sigma_j^{S(TM)\otimes F} c(A^*) & + \frac{1}{2} \sum_{i \neq j} R^{F}(e_i, e_j) c(e_i) c(e_j) \right}\end{align*}
\]

(3.22)

Thus, using (3.19)-(3.22), we get the specification of \( \tilde{D}_F^* \cdot \tilde{D}_F f^{-1} \cdot \tilde{D}_F f \).
\[
\begin{aligned}
\dot{D}_F f \cdot \dot{D}_F f^{-1} & \cdot \dot{D}_F f \\
&= f \cdot \dot{D}_F \dot{D}_F \dot{D}_F + c(df) \dot{D}_F \dot{D}_F - \dot{D}_F \dot{D}_F f \cdot c(df^{-1}) \cdot f + \dot{D}_F \cdot c(df) c(df^{-1}) f \\
&= f \cdot \left\{ \begin{array}{l}
\sum_{i,j,l=1}^{n} c(e_r)(e_r, dx_l)(-g^{ij} \partial_l \partial_j) + \sum_{r,l=1}^{n} c(e_r)(e_r, dx_l) \left\{ -2 \sum_{i,j=1}^{n} (\partial_l g^{ij}) \partial_l \partial_j - \sum_{i,j,k=1}^{n} g^{ij} \right.\\
\times \left( 4 \sigma^2_i D \right) \partial_j - 2 \Gamma^k_{ij} \partial_k \right\} \\
+ \sum_{r,l=1}^{n} c(e_r)(e_r, dx_l) \left\{ -2 \sum_{i,j=1}^{n} (\partial_l g^{ij}) \sigma_i D \partial_j + \sum_{i,j,k=1}^{n} g^{ij} \right. \\
\times \left( \partial_l \Gamma^k_{ij} \right) \partial_k - 2 \sum_{i,j=1}^{n} g^{ij} (\partial_l \sigma_i D) \partial_j + \sum_{i,j,k=1}^{n} \sigma_i D \left( \partial_l \Gamma^k_{ij} \partial_k + \sum_{k,l=1}^{n} \left[ \partial_l \left( c(A) c(e_j) - c(e_j) \right) \right] \partial_k \right) \\
+ \sum_{i,j=1}^{n} c(A) \left[ \partial_l \partial_j + \sum_{k=1}^{n} \left( c(A) c(e_j) - c(e_j) c(A^*) \right) \right] \partial_k \right) + \sum_{i,j=1}^{n} \left[ \partial_l \left( c(A) c(e_j) - c(e_j) c(A^*) \right) \right] \\
\times \sum_{j=1}^{n} \left\{ \begin{array}{l}
\sum_{j=1}^{n} \left[ c(A) c(e_j) \sigma_j \sigma_j D c(A^*) - \sum_{i,j=1}^{n} c(e_j) \sigma_j \sigma_j D c(A^*) \right] + \frac{1}{2} \sum_{i,j=1}^{n} R^F (e_i, e_j) \\
+ \sum_{j=1}^{n} \left[ c(A) c(e_j) \sigma_j \sigma_j D c(A^*) - \sum_{i,j=1}^{n} c(e_j) \sigma_j \sigma_j D c(A^*) \right] + \frac{1}{2} \sum_{i,j=1}^{n} R^F (e_i, e_j) \\
\end{array} \right\} \\
\left\{ - g^{ij} \partial_l \partial_j - 2 \sigma_j \sigma_j D \partial_j + \Gamma^k \partial_k \right\} + \sum_{j=1}^{n} \left[ c(A) c(e_j) - c(e_j) c(A^*) \right] e_j - \sum_{j=1}^{n} c(e_j) \right) \\
\times c(A^*) - g^{ij} \left[ (\partial_l \sigma_i \sigma_j D) + \sigma_i \sigma_j D \sigma_j D \sigma_i D - \Gamma^k \sigma_i \sigma_j D \sigma_i D \right] + \frac{1}{4} s + \frac{1}{2} \sum_{i,j=1}^{n} R^F (e_i, e_j) \\
\times c(e_i) c(e_j) + \sum_{j=1}^{n} \left[ c(A) c(e_j) \sigma_j \sigma_j D c(A^*) - \sum_{i,j=1}^{n} c(e_j) \sigma_j \sigma_j D c(A^*) \right] \right) \right\} \\
\left\{ - g^{ij} \partial_l \partial_j - 2 \sigma_j \sigma_j D \partial_j + \Gamma^k \partial_k \right\} + \sum_{j=1}^{n} \left[ c(e_j) c(A) - c(A^*) c(e_j) \right] e_j + \sum_{j=1}^{n} c(e_j) \sigma_j \sigma_j D c(A) \\
- g^{ij} \left[ (\partial_l \sigma_i \sigma_j D) + \sigma_i \sigma_j D \sigma_j D \sigma_i D - \Gamma^k \sigma_i \sigma_j D \sigma_i D \right] + \frac{1}{4} s + \frac{1}{2} \sum_{i,j=1}^{n} R^F (e_i, e_j) \\
\times c(e_i) - \sum_{j=1}^{n} \left[ c(A^*) c(e_j) \right] \sigma_j \sigma_j D c(A^*) + \sum_{j=1}^{n} c(e_j) c(A) - c(A^*) c(A) \right) \right\} \right) f \cdot c(df^{-1}) \cdot f \\
+ \left\{ \sum_{i,j=1}^{n} g^{ij} (\partial_l \sigma_i \sigma_j D) \partial_j + \sum_{i,j=1}^{n} c(e_j) c(A) - c(A^*) c(A) \right) \right\} \cdot c(df) c(df^{-1}) f.
\end{array} \right\}
\end{aligned}
\]
Let $\partial^i = g^{ij} \partial_j$, $\sigma^j = g^{ij} \sigma_j$, by the above formulas, then we obtain:

**Lemma 3.2.** Let $\tilde{D}_f^*, \tilde{D}_F$ be the twisted Dirac operators on $\Gamma(S(TM) \otimes F)$,

\[
\begin{align*}
\sigma_3(\tilde{D}_f^* \cdot \tilde{D}_f f^{-1} \cdot \tilde{D}_f f) &= f \sigma_3(\tilde{D}_f^* \tilde{D}_f \tilde{D}_f) = \frac{\sqrt{-1}c(\xi)}{\xi^2}; \\
\sigma_2(\tilde{D}_f^* \cdot \tilde{D}_f f^{-1} \cdot \tilde{D}_f f) &= f \sigma_2(\tilde{D}_f^* \tilde{D}_f \tilde{D}_f) + 2c(df)[\xi^2],
\end{align*}
\]  

(3.24) (3.25)

where $\sigma_2(\tilde{D}_f^* \tilde{D}_f \tilde{D}_f) = c(\xi) (4\sigma^k - 2\Gamma^k) \xi_k - \frac{1}{4} \xi^2 h'(0) c(dx_n) + \lambda \xi^2 - 2c(\xi)c(A)c(\xi) - 2[\xi^2 c(A^*)].$

For convenience, we write that $\sigma_2(\tilde{D}_f \tilde{D}_F \tilde{D}_f) = G + \lambda \xi^2 - 2c(\xi)c(A)c(\xi) - 2[\xi^2 c(A^*)]$. In order to get the symbol of operators $\tilde{D}_f f^{-1} \cdot \tilde{D}_f f$. We first give the following formulas:

\[
\tilde{D}_f^0 = (\sqrt{-1})^|\alpha| \partial^\alpha; \quad \sigma(\tilde{D}_f f^{-1} \cdot \tilde{D}_f f) = p_3 + p_2 + p_1 + p_0;
\]

\[
\sigma((\tilde{D}_f f^{-1} \cdot \tilde{D}_f f)^{-1}) = \sum_{j=3}^{\infty} q_j.
\]  

(3.26)

By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma(\tilde{D}_f f^{-1} \cdot \tilde{D}_f f) \circ (\tilde{D}_f f^{-1} \cdot \tilde{D}_f f)^{-1}
\]

\[
= (p_3 + p_2 + p_1 + p_0)(q_3 + q_4 + q_5 + \cdots)
\]

\[
+ \sum_j \partial_{\xi_j} p_3 + p_2 + \partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(D_{x_j} q_3 + D_{x_j} q_4 + D_{x_j} q_5 + \cdots)
\]

\[
= p_3 q_3 + (p_3 q_4 + p_2 q_4 + \sum_j \partial_{\xi_j} p_3 D_{x_j} q_3) + \cdots.
\]  

(3.27)

Then

\[
q_3 = p_3^{-1}; \quad q_4 = -p_3^{-1}[p_2 p_3^{-1} + \sum_j \partial_{\xi_j} p_3 D_{x_j} (p_3^{-1})].
\]  

(3.28)

By Lemma 2.1 in [12] and (3.24), (3.25), we obtain:

**Lemma 3.3.** Let $\tilde{D}_f^*, \tilde{D}_F$ be the twisted Dirac operators on $\Gamma(S(TM) \otimes F)$, then

\[
\begin{align*}
\sigma_{-3}(\tilde{D}_f^* \cdot \tilde{D}_f f^{-1} \cdot \tilde{D}_f f)^{-1} &= f^{-1} \sigma_{-3}(\tilde{D}_f^* \tilde{D}_f \tilde{D}_f)^{-1} = \frac{\sqrt{-1}c(\xi)}{\xi^4};
\end{align*}
\]  

(3.29)

\[
\begin{align*}
\sigma_{-4}(\tilde{D}_f^* \cdot \tilde{D}_f f^{-1} \cdot \tilde{D}_f f)^{-1} &= f^{-1} \sigma_{-4}(\tilde{D}_f^* \tilde{D}_f \tilde{D}_f)^{-1} + \frac{2c(\xi) c(df) c(\xi)}{\xi^6} \
&+ \frac{ic(\xi) \sum_j [c(dx_j) \xi^2 + 2\xi_j c(\xi)] D_{x_j} (f^{-1}) c(\xi)}{\xi^8}.
\end{align*}
\]  

(3.30)

where

\[
\begin{align*}
\sigma_{-4}(\tilde{D}_f^* \tilde{D}_f \tilde{D}_f)^{-1} &= \frac{c(\xi) \sigma_2(\tilde{D}_f^* \tilde{D}_f \tilde{D}_f) c(\xi)}{\xi^8} \sum_j [c(dx_j) \xi^2 + 2\xi_j c(\xi)] \left[\partial_{x_j} [c(\xi)] \xi^2 - 2c(\xi) \partial_{x_j} (|\xi|^2)\right] \\
&= \frac{c(\xi) G c(\xi)}{\xi^8} + \frac{c(\xi) c(\xi)}{\xi^4} \sum_j [c(dx_j) \xi^2 + 2\xi_j c(\xi)] \left[\partial_{x_j} [c(\xi)] \xi^2 - 2c(\xi) \partial_{x_j} (|\xi|^2)\right] \\
&- 2c(\xi) \partial_{x_j} (|\xi|^2)
\end{align*}
\]  

(3.31)
Locally we can use Theorem 2.5 in [19] to compute the interior term of (3.17), then

\[
\int_M \int_{|\xi|=1} \text{traces}_{(TM)\otimes F}[\sigma_{-n}\left((\tilde{D}_F^* f \cdot \tilde{D}_F f^{-1})^{-2}\right)]\sigma(\xi) \, d\xi \\
= 8\pi^3 \int_M \left\{ \text{traces}\left[ -(\frac{s}{12} + c(A^*) c(A) - \frac{1}{4} \sum_i [c(A^*) c(e_i) - c(e_i) c(A)]^2 \\
- \frac{1}{2} \sum_j \nabla_j^F (c(A^*) c(e_j) - \frac{1}{2} \sum_j c(e_j) \nabla_j^F (c(A))] - 2f^{-1} \Delta(f) \\
+ 4f^{-1} \text{traces}\left[ A(\text{grad}_M f) - f^2 \left| \text{grad}_M(f)^2 + 2\Delta(f) \right| \right] \right\} \, d\text{vol}_M. \tag{3.32}
\]

So we only need to compute \( \int_{\partial M} \Phi \).

From the formula (3.18) for the definition of \( \Phi \), now we can compute \( \Phi \). Since the sum is taken over \( r + \ell - k - j - |\alpha| - 1 = -6 \), \( r \leq -1 \), \( \ell \leq -3 \), then we have the \( \int_{\partial M} \Phi \) is the sum of the following five cases:

**case (a) (I)** \( r = -1, \ell = -3, j = k = 0, |\alpha| = 1 \).

By (3.18), we get

\[
\text{case (a) (I)} = - \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{traces}\left[ \partial_{\xi^+} \sigma_{-1}(f \tilde{D}_F^{-1}) \times \partial_{\xi^+} \sigma_{-1}(f^{-1}(\tilde{D}_F^{-1}) \cdot f \tilde{D}_F^{-1}) \\
\cdot f^{-1}(\tilde{D}_F^{-1}) \right](x_0) d\xi_n \sigma(\xi') \, dx' \\
= - \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{traces}\left[ \partial_{\xi^+} \sigma_{-1}(f \tilde{D}_F^{-1}) \times \partial_{\xi^+} \sigma_{-1}(f^{-1}(\tilde{D}_F^{-1}) \cdot f \tilde{D}_F^{-1}) \\
\times d\xi_n \sigma(\xi') \right] dx' \\
= - \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{traces}\left[ \partial_{\xi^+} \sigma_{-1}(f \tilde{D}_F^{-1}) \times \partial_{\xi^+} \sigma_{-1}(f^{-1}(\tilde{D}_F^{-1}) \cdot f \tilde{D}_F^{-1}) \\
\times d\xi_n \sigma(\xi') \right] dx' \tag{3.33}
\]

By Lemma 2.2 in [12] and (3.29), for \( i < n \), we have

\[
\partial_{\xi_i} \sigma_{-3}\left((\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^*)^{-1}\right)(x_0) = \partial_{\xi_i} \left[ \frac{\sqrt{1 - c(\xi)}}{|\xi|^4} \right](x_0) = \sqrt{1 - c(\xi)} |\xi|^{-4}(x_0) - 2\sqrt{1 - c(\xi)} \partial_{\xi_i} \left[ |\xi|^2 \right]|\xi|^{-6}(x_0) = 0. \tag{3.34}
\]

Thus we have

\[
- \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{traces}\left[ \partial_{\xi^+} \sigma_{-1}(f \tilde{D}_F^{-1}) \times \partial_{\xi^+} \sigma_{-1}(f^{-1}(\tilde{D}_F^{-1}) \cdot f \tilde{D}_F^{-1}) \\
\times d\xi_n \sigma(\xi') \right] dx' = 0. \tag{3.35}
\]

By (3.12) and direct calculations, for \( i < n \), we obtain

\[
\frac{\partial_{\xi_i} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1})(x_0)|_{|\xi'|=1} = \partial_{\xi_i} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1})(x_0)|_{|\xi'|=1}}{2(\xi_n - \sqrt{-1})} = \frac{c(dx_i)}{2(\xi_n - \sqrt{-1})} - \frac{\xi_i c(\xi_n - 2\sqrt{-1}c(\xi') + \xi_i c(dx_n))}{2(\xi_n - \sqrt{-1})^2} \tag{3.36}
\]
and we get
\[
\partial_{\xi_n} \sigma_{-3}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* )^{-1} = \frac{\sqrt{-1}c(dx_n)}{[\xi]^4} - \frac{4\sqrt{-1}[\xi_n c(\xi') + \xi_n^2 c(dx_n)]}{[\xi]^6}.
\] (3.37)

Then for \( i < n \), we have
\[
\text{trace} \left[ \partial^2_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* )^{-1} \right](x_0)
\]
\[
= -\xi_i \text{trace} \left[ \frac{c(dx_n)^2}{2(\xi_n - \sqrt{-1})^2} \right] - 4\sqrt{-1}\xi_i \text{trace} \left[ \frac{c(dx_n)^2}{2(\xi_n - \sqrt{-1})[\xi]^6} \right] + 4\sqrt{-1}\xi_i \xi_i (\xi_n - 2\sqrt{-1})
\times \text{trace} \left[ \frac{c(\xi')^2}{2(\xi_n - \sqrt{-1})^2[\xi]^6} \right] + 4\sqrt{-1}\xi_i \xi_i \text{trace} \left[ \frac{c(dx_n)^2}{2(\xi_n - \sqrt{-1})^2[\xi]^6} \right].
\] (3.38)

We note that \( i < n \), \( \int_{|\xi'|=1} \xi_i \sigma(\xi') = 0 \), so
\[
-f \sum_{j<n} \partial_j (f^{-1}) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^2_{\xi_i} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* )^{-1} \right](x_0) d\xi_n \sigma(\xi') dx'
\]
\[
= 0.
\] (3.39)

Then we have case (a) (I) = 0.

case (a) (II) \( r = -1, l = -3, |\alpha| = k = 0, j = 1 \).

By (3.18), we have
\[
\text{case (a) (II)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(f \tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}(f^{-1}(\tilde{D}_F^*)^{-1} \cdot f \tilde{D}_F^{-1}).
\]
\[
f^{-1}(\tilde{D}_F^{-1})^{-1} \right](x_0) d\xi_n \sigma(\xi') dx'
\]
\[
= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* )^{-1} \right](x_0) d\xi_n \sigma(\xi') dx'
\]
\[
-\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* )^{-1} \right](x_0) dx'
\]
\[
\times d\xi_n \sigma(\xi') dx'.
\] (3.40)

By (2.2.23) in \[12\] and (3.12), we have
\[
\pi^+_{\xi_n} \partial_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1})(x_0)|_{|\xi'|=1} = \frac{\partial_{\xi_n} c(\xi')(x_0)}{2(\xi_n - \sqrt{-1})} + \sqrt{-1}h'(0) \left[ \frac{\sqrt{-1}c(\xi')}{4(\xi_n - \sqrt{-1})} + \frac{c(\xi') + \sqrt{-1}c(dx_n)}{4(\xi_n - \sqrt{-1})^2} \right].
\] (3.41)

By (3.29) and direct calculations, we have
\[
\partial_{\xi_n} \sigma_{-3}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* )^{-1} = -\frac{4\sqrt{-1}\xi_n c(\xi') + \sqrt{-1}(1 - 3\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3}
\] (3.42)

and
\[
\partial^2_{\xi_n} \sigma_{-3}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^* )^{-1} = \sqrt{-1} \left[ \frac{(20\xi_n^2 - 4)c(\xi') + 12(\xi_n^2 - \xi_n)c(dx_n)}{(1 + \xi_n^2)^4} \right].
\] (3.43)

Since \( n = 6 \), \( \text{trace}_{S(TM) \otimes F} [\text{id}] = -8\text{dim} F \). By the relation of the Clifford action and \( \text{trace} PQ = \text{trace}QP \), then
\[
\text{trace}[c(\xi')(c(dx_n))] = 0; \text{trace}[c(dx_n)^2] = -8\text{dim} F; \text{trace}[c(\xi')^2](x_0)|_{|\xi'|=1} = -8\text{dim} F; \text{trace}[\partial_{\xi_n} c(\xi')(c(dx_n))] = 0; \text{trace}[\partial_{\xi_n} c(\xi')(c(dx_n))(x_0)|_{|\xi'|=1} = -4h'(0)\text{dim} F.
\] (3.44)
By (3.41)-(3.44), we get
\[
\text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}((\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^*)^{-1}) \right](x_0) = h'(0) \dim F \frac{-8 - 24 \xi_n \sqrt{-1} + 40 \xi_n^2 + 24 \sqrt{-1} \xi_n^3}{(\xi_n - \sqrt{-1})^6 (\xi_n + \sqrt{-1})^4}.
\]
(3.45)

Then we obtain
\[
\frac{1}{2} \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}((\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^*)^{-1}) \right](x_0) d\xi_n \sigma(\xi') d\xi' = -\frac{15}{16} \pi h'(0) \Omega_4 \dim F d\xi'.
\]
(3.46)

On the other hand, by calculations, we have
\[
\pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1})(x_0) |_{|\xi'|=1} = \frac{c(\xi') + \sqrt{-1} e(d\xi_n)}{2 (\xi_n - \sqrt{-1})^6}.
\]
(3.47)

By (3.42), (3.44) and (3.47), we get
\[
\text{trace} \left[ \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}((\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^*)^{-1}) \right](x_0) = -16 \dim F \frac{5 \xi_n^2 \sqrt{-1} - \sqrt{-1} - 3 \xi_n^3 + 3 \xi_n}{(\xi_n - \sqrt{-1})^4 (\xi_n + \sqrt{-1})^4}.
\]
(3.48)

Then we obtain
\[
\frac{-1}{2} f^{-1} \partial_{x_n}(f) \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}((\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^*)^{-1}) \right](x_0) d\xi_n \sigma(\xi') d\xi' = \frac{5 \sqrt{-1} + 44}{4} \pi f^{-1} \partial_{x_n}(f) \cdot \dim F \Omega_4 d\xi'.
\]
(3.49)

where \( \Omega_4 \) is the canonical volume of \( S_4 \).

Combining (3.40), (3.46) and (3.49), we obtain
\[
\text{case (a) II} = -\frac{15}{16} \pi h'(0) \Omega_4 \dim F d\xi' + \frac{5 \sqrt{-1} + 44}{4} \pi f^{-1} \partial_{x_n}(f) \cdot \Omega_4 \dim F d\xi'.
\]
(3.50)

\text{case (a) (III)} \quad r = -1, l = -3, |\alpha| = j = 0, k = 1.

By (3.18), we have
\[
\text{case (a) (III)} = \frac{-1}{2} \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(f \tilde{D}_F^{-1}) \times \partial_{x_n} \partial_{\xi_n} \sigma_{-3}(f^{-1}(\tilde{D}_F)^{-1}) \cdot f \tilde{D}_F^{-1} \right](x_0) d\xi_n \sigma(\xi') d\xi'.
\]
\[
= -\frac{1}{2} \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} (\sigma_{-1}(\tilde{D}_F^{-1})) \times \partial_{x_n} \partial_{\xi_n} \sigma_{-3}(\tilde{D}_F \tilde{D}_F \tilde{D}_F^*)^{-1} \right](x_0) d\xi_n \sigma(\xi') d\xi' -\frac{1}{2} f \partial_{x_n}(f^{-1}) \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{x_n} \sigma_{-3}(\tilde{D}_F \tilde{D}_F \tilde{D}_F^*)^{-1} \right](x_0) d\xi_n \sigma(\xi') d\xi'.
\]
(3.51)

By (2.2.29) in [12], we have
\[
\partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1})(x_0) |_{|\xi'|=1} = \frac{c(\xi') + \sqrt{-1} e(d\xi_n)}{2 (\xi_n - \sqrt{-1})^2}.
\]
(3.52)
By (3.29) and direct calculations, we have
\[
\partial_{\xi_0} \partial_{x_n} \sigma^{-3}((\bar{D}_F^* \bar{D}_F \bar{D}_F^*)^{-1}) = \frac{-4 \sqrt{-1} \xi_0 \partial_{\xi_0} c(\xi')(x_0)}{(1 + \xi_0^2)^3} + \frac{12 \sqrt{-1} h'(0) \xi_0 c(\xi')}{(1 + \xi_0^2)^4} - \frac{\sqrt{-1}(2 - 10 \xi_0^2) h'(0) c(dx_n)}{(1 + \xi_0^2)^4}.
\]
(3.53)

Combining (3.44), (3.52) and (3.53), we have
\[
\text{trace} \left[ \partial_{\xi_0} \pi_{\xi_0} \sigma^{-1}(\bar{D}_F^{-1}) \times \partial_{\xi_0} \partial_{x_n} \sigma^{-3}(\bar{D}_F^* \bar{D}_F \bar{D}_F^*)^{-1}) \right] (x_0)|_{\xi'|=1} = h'(0) \text{dim} F \frac{8 \sqrt{-1} - 32 \xi_n - 8 \sqrt{-1} \xi_n^2}{(\xi_n - \sqrt{-1})^6(\xi_n + \sqrt{-1})^4},
\]
(3.54)
and
\[
\text{trace} \left[ \partial_{\xi_0} \pi_{\xi_0} \sigma^{-1}(\bar{D}_F^{-1}) \times \partial_{\xi_0} \sigma^{-3}(\bar{D}_F^* \bar{D}_F \bar{D}_F^*)^{-1}) \right] (x_0)|_{\xi'|=1} = -4 \text{dim} F \frac{4 \sqrt{-1} \xi_n + 1 - 3 \xi_n^2}{(\xi_n - \sqrt{-1})^6(\xi_n + \sqrt{-1})^3}.
\]
(3.55)

Then
\[
- \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_0} \pi_{\xi_0} \sigma^{-1}(\bar{D}_F^{-1}) \times \partial_{\xi_0} \partial_{x_n} \sigma^{-3}(\bar{D}_F^* \bar{D}_F \bar{D}_F^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' = \frac{25}{16} \pi h'(0) \Omega_4 \text{dim} F dx',
\]
(3.56)
and
\[
- \frac{1}{2} f \partial_{x_n} (f^{-1}) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_0} \pi_{\xi_0} \sigma^{-1}(\bar{D}_F^{-1}) \times \partial_{\xi_0} \sigma^{-3}(\bar{D}_F^* \bar{D}_F \bar{D}_F^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' = \frac{\pi \sqrt{-1}}{16} \cdot f \cdot \partial_{x_n} (f^{-1}) \Omega_4 \text{dim} F dx',
\]
(3.57)
where \( \Omega_4 \) is the canonical volume of \( S_4 \).

Then
\[
\text{case (a) III} = \left[ \frac{25}{16} \pi h'(0) + \frac{\pi \sqrt{-1}}{16} \cdot f \cdot \partial_{x_n} (f^{-1}) \right] \Omega_4 \text{dim} F dx'.
\]
(3.58)

\text{case (b) } r = -1, l = -4, |\alpha| = j = k = 0.
By (3.18), we have
\[
\text{case (b)} = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1} (f \tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-4} (f^{-1} \tilde{D}_F^{-1} \cdot f \tilde{D}_F^{-1} \cdot f^{-1} \tilde{D}_F^{-1}) \right] (x_0) \\
\times d\xi_n \sigma(\xi') dx'
\]
\[
= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1} (f \tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( f^{-1} \sigma_{-4} (\tilde{D}_F^{-1} \tilde{D}_F \tilde{D}_F^{-1})^{-1} + \frac{2 c(\xi) c(df) c(\xi)}{f^2 |\xi|^6} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'
\]
\[
-2i f^{-1} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1} (\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) c(df) c(\xi)}{|\xi|^6} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'
\]
\[
-6i f^{-1} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1} (\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) \sum_j [c(dx_j)] |\xi|^2 + 2 \xi_j c(\xi)] D_{x_j} (f^{-1}) c(\xi)}{|\xi|^8} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'
\]
\[
\times (x_0) d\xi_n \sigma(\xi') dx',
\]
(3.59)

In the normal coordinate, \( g^j(x_0) = \delta^j_i \) and \( \partial_{x_j} (g^a \omega)(x_0) = 0 \), if \( j < n; \partial_{x_j} (g^a \omega)(x_0) = h'(0) \delta_{ij} \), if \( j = n \). So by Lemma A.2 in [12], we have \( \Gamma^a(x_0) = \frac{1}{2} h'(0) \) and \( \Gamma^b(x_0) = 0 \) for \( k < n \). By the definition of \( \delta^k \) and Lemma 2.3 in [12], we have \( \delta^a(x_0) = 0 \) and \( \delta^b = \frac{1}{2} h'(0) c(\tilde{c}_a) c(\tilde{c}_n) \) for \( k < n \). By (3.30), we obtain
\[
\sigma_{-4} (\tilde{D}_F^{-1} \tilde{D}_F \tilde{D}_F^{-1})^{-1} = \frac{-17 - 9 \xi_n^3}{4(1 + \xi_n^4)^3} h'(0) c(\xi') c(dx_n) c(\xi') + \frac{33 \xi_n + 17 \xi_n^3}{2(1 + \xi_n^4)^4} h'(0) c(\xi') + \frac{49 \xi_n + 25 \xi_n^3}{2(1 + \xi_n^4)^4} h'(0) c(dx_n)
\]
\[
+ \frac{1}{(1 + \xi_n^4)^3} c(\xi') c(dx_n) \partial_{x_n} [c(\xi')](x_0) - \frac{3 \xi_n}{(1 + \xi_n^4)^3} \partial_{x_n} [c(\xi')](x_0) - \frac{2 \xi_n}{(1 + \xi_n^4)^3} h'(0) c(dx_n)
\]
\[
+ \frac{1 - 2 \xi_n^2}{(1 + \xi_n^4)^3} h'(0) c(dx_n)(x_0) + \frac{c(\xi) \lambda c(\xi)}{|\xi|^6} - \frac{2 c(\xi) c(A^*) c(\xi)}{|\xi|^4} - \frac{2 c(A)}{|\xi|^4}.
\]
(3.60)

Then
\[
\partial_{\xi_n} \left( \sigma_{-4} (\tilde{D}_F^{-1} \tilde{D}_F \tilde{D}_F^{-1})^{-1} \right) (x_0)
\]
\[
= \frac{59 \xi_n + 27 \xi_n^3}{2(1 + \xi_n^4)^3} h'(0) c(\xi') c(dx_n) c(\xi') + \frac{33 - 180 \xi_n^2 - 85 \xi_n^4}{2(1 + \xi_n^4)^5} h'(0) c(\xi') + \frac{49 \xi_n - 97 \xi_n^3 - 50 \xi_n^5}{2(1 + \xi_n^4)^5} h'(0) c(dx_n)
\]
\[
- \frac{6 \xi_n}{(1 + \xi_n^4)^4} c(\xi') c(dx_n) \partial_{x_n} [c(\xi')](x_0) - \frac{3 - 15 \xi_n^2}{(1 + \xi_n^4)^4} \partial_{x_n} [c(\xi')](x_0) + \frac{4 \xi_n^3 - 8 \xi_n^5}{(1 + \xi_n^4)^4} h'(0) c(dx_n)
\]
\[
+ \frac{2 - 10 \xi_n^2}{(1 + \xi_n^4)^4} h'(0) c(\xi') + \frac{c(dx_n) \lambda c(\xi') + c(\xi') \lambda c(dx_n) + 2 \xi_n c(dx_n) \lambda c(dx_n)}{(1 + \xi_n^4)^3} - \frac{6 \xi_n c(\xi) c(A^*) c(\xi)}{(1 + \xi_n^4)^4} - \frac{2 \xi_n c(A)}{(1 + \xi_n^4)^3}.
\]
(3.61)
By (3.47) and (3.61), we obtain
\[
\text{trace}\left[\pi^+_n \sigma_{-1}(\tilde{D}^{-1}_F) \times \partial_{\xi_n} \sigma_{-4}(\tilde{D}^{-1}_F \tilde{D}_F \tilde{D}_F^{-1})^{-1}\right](x_0)|_{|\xi'|=1} = h'(0) \text{dim} F \frac{4i(-17 - 42\xi_n + 50\xi_n^2 - 16\xi_n^3 + 29\xi_n^4)}{\xi_n^3(\xi + i)^5}
\]
\[
+ \frac{4\xi_n i}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(\xi') \lambda] + \frac{4\xi_n i}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(dx_n) \lambda]
\]
\[
+ \frac{2(\xi_n + i)(1 + \xi_n^2)^3}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(\xi') c(A^*)] + \frac{2(\xi_n + i)(1 + \xi_n^2)^3}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(dx_n) c(A^*)]
\]
\[
+ \frac{2(\xi_n - i)(1 + \xi_n^2)^3}{2(\xi_n - i)(1 + \xi_n^2)^3} \text{trace}[c(\xi') c(A)] + \frac{2(\xi_n - i)(1 + \xi_n^2)^3}{2(\xi_n - i)(1 + \xi_n^2)^3} \text{trace}[c(dx_n) c(A)].
\]

By the relation of the Clifford action and \(\text{trace}QP = \text{trace}QP\), then we have the following equalities
\[
\text{trace}[c(dx_n) \lambda] = \text{trace}[c(dx_n) \sum_{j=1}^n c(e_j)(\sigma_j^F - A^*(e_j))] = \text{trace}[- \text{id} \otimes (\sigma^F_n - A^*(e_n))];
\]
\[
\text{trace}[c(\xi') \lambda] = \text{trace}[c(\xi') \sum_{j=1}^n c(e_j)(\sigma_j^F - A^*(e_j))] = \text{trace}[- \sum_{j=1}^{n-1} \xi_j (\sigma_j^F - A^*(e_j))];
\]
\[
\text{trace}[c(dx_n) c(A^*)] = \text{trace}[c(dx_n) \sum_{j=1}^n c(e_j) \otimes A^*(e_j)] = \text{trace}[- \text{id} \otimes A^*(e_n)];
\]
\[
\text{trace}[c(dx_n) c(A)] = \text{trace}[c(dx_n) \sum_{j=1}^n c(e_j) \otimes A(e_j)] = \text{trace}[- \text{id} \otimes A(e_n)];
\]
\[
\text{trace}[c(\xi') c(A^*)] = \text{trace}[c(\xi') \sum_{j=1}^n c(e_j) \otimes A^*(e_j)] = \text{trace}[- \sum_{j=1}^{n-1} \xi_j A^*(e_j)];
\]
\[
\text{trace}[c(\xi') c(A)] = \text{trace}[c(\xi') \sum_{j=1}^n c(e_j) \otimes A(e_j)] = \text{trace}[- \sum_{j=1}^{n-1} \xi_j A(e_j)].
\]

We note that \(i < n\), \(\int_{|\xi'|=1} \xi_i \sigma(\xi') = 0\), so \(\text{trace}[c(\xi') c(A^*)]\) has no contribution for computing case (b).

By (3.24), then
\[
-i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left[\pi^+_n \sigma_{-1}(\tilde{D}^{-1}_F) \times \partial_{\xi_n} \sigma_{-4}(\tilde{D}^{-1}_F \tilde{D}_F \tilde{D}_F^{-1})^{-1}\right](x_0) d\xi_n \sigma(\xi') dx'
\]
\[
- \frac{129}{16} h'(0) + \frac{3}{2} \text{trace}[\sigma^F_n - A^*(e_n)] - 3\text{trace}[A^*(e_n)] - \text{trace}[A(e_n)] \pi \text{dim} \Omega_4 dx'.
\]

Since
\[
\partial_{\xi_n} \left(\frac{c(\xi)c(df)c(\xi)}{|\xi|^6}\right) = \frac{c(dx_n)c(df)c(\xi') + c(\xi')c(df)c(dx_n) + 2\xi_n c(dx_n)c(df)c(dx_n)}{(1 + \xi_n^2)^3}
\]
\[
- 6\xi_n c(\xi)c(df)c(\xi) \frac{1}{(1 + \xi_n^2)^3}
\]

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and
\[
\frac{i c(\xi) \sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1})c(\xi)}{\xi^8}
\]
\[
= i \left\{ c(dx_n) \sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1})c(\xi') + c(\xi') \sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1})c(dx_n) \\
+ 2\xi_n c(dx_n) \sum_j \left[c(dx_j)|\xi|^2 + 2\xi_j c(\xi)\right] D_{x_j}(f^{-1})c(dx_n)\right\}(1 + \xi_n^2)^{-4} - i \left\{ 8\xi_n c(\xi) \sum_j [c(dx_j)|\xi|^2 \\
+ 2\xi_j c(\xi)] D_{x_j}(f^{-1})c(\xi)\right\}(1 + \xi_n^2)^{-5},
\]
then
\[\text{trace}\left[\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left(\frac{c(\xi)c(df)c(\xi)}{|\xi|^8}\right)\right](x_0) \]
\[= \frac{(4\xi_n i + 2)i}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(\xi')c(df)] + \frac{4\xi_n i + 2}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(dx_n)c(df)].\]
and
\[\text{trace}\left[\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left(\frac{i c(\xi) \sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1})c(\xi)}{|\xi|^8}\right)\right](x_0) \]
\[= \frac{(3\xi_n - i)i}{(\xi_n + i)(1 + \xi_n^2)^4} \text{trace}\left[c(\xi') \sum_j \left[c(dx_j)|\xi|^2 + 2\xi_j c(\xi)\right] D_{x_j}(f^{-1})\right] \\
+ \frac{3\xi_n - i}{(\xi_n + i)(1 + \xi_n^2)^4} \text{trace}\left[c(dx_n) \sum_j \left[c(dx_j)|\xi|^2 + 2\xi_j c(\xi)\right] D_{x_j}(f^{-1})\right].\]

By the relation of the Clifford action and trace\(QP = \text{trace}PQ\), then we have the following equalities
\[\text{trace}\left[c(dx_n)c(df)\right] = -g(dx_n, df);\]
and
\[\text{trace}\left[c(dx_n) \sum_j \left[c(dx_j)|\xi|^2 + 2\xi_j c(\xi)\right] D_{x_j}(f^{-1})\right] \]
\[= \text{trace}(-id)|\xi|^2 \left(i \partial_{x_n}(f)f^{-1}\right) + 2 \sum_j \xi_j \xi_n \text{trace}(-id) \left(i \partial_{x_j}(f)f^{-1}\right) \\
= -8\dim F|\xi|^2 \left(i \partial_{x_n}(f)f^{-1}\right) + 2 \sum_j \xi_j \xi_n \text{trace}(-id) \left(i \partial_{x_j}(f)f^{-1}\right).\]

We note that \(i < n, \int_{|\xi|=1} \xi_i \sigma(\xi') = 0\), so \(\text{trace}\left[c(\xi')c(df)\right], \text{trace}\left[c(\xi') \sum_j \left[c(dx_j)|\xi|^2 + 2\xi_j c(\xi)\right] D_{x_j}(f^{-1})\right]\)
and \(2i \sum_{j} \xi_{j}\partial_{x_{j}}(f)^{-1} \text{tr}[-\text{id}]\) have no contribution for computing case (b). Then we obtain

\[
-2if^{-1} \int_{[\xi'=1]}^{+\infty} \int_{-\infty}^{\infty} \text{tr}\left[ \pi_{\xi_{n}}^{+}\sigma_{-1}(\tilde{D}_{F}^{-1}) \times \partial_{\xi_{n}} \left( \frac{c(\xi)c(df)c(\xi)}{|\xi|^6} \right) \right] (x_{0})d\xi_{n}\sigma(\xi')dx',
\]

\[
= \frac{3}{8f} \pi g(dx_{n}, df)\Omega_{4}dx'.
\]

and

\[
-fi \int_{[\xi'=1]}^{+\infty} \int_{-\infty}^{\infty} \text{tr}\left[ \pi_{\xi_{n}}^{+}\sigma_{-1}(\tilde{D}_{F}^{-1}) \times \partial_{\xi_{n}} \left( \frac{ic(\xi)\sum_{j} c(dx_{j})|\xi|^2 + 2\xi_{j}c(\xi)}{|\xi|^8} \right) \right]
\times (x_{0})d\xi_{n}\sigma(\xi')dx',
\]

\[
= -\frac{15i}{2} \partial_{x_{n}}(f)\pi \dim F\Omega_{4}dx'.
\]

Thus we have

\[
\text{case (b)} = \left\{ -\frac{129}{16} h'(0) + \frac{3}{2} \text{trace}\left[ \sigma_{n}^{F} - A^{*}(e_{n}) \right] - 3\text{trace}\left[ A^{*}(e_{n}) \right] - \text{trace}\left[ A(e_{n}) \right] \right\} \pi \dim F\Omega_{4}dx'
+ \frac{3}{8f} \pi g(dx_{n}, df)\Omega_{4}dx' - \frac{15i}{2} \partial_{x_{n}}(f)\pi \dim F\Omega_{4}dx'.
\]

\[
\text{case (c)} r = -2, l = -3, |\alpha| = j = k = 0.
\]

By (3.18), we have

\[
\text{case (c)} = -i \int_{[\xi'=1]}^{+\infty} \int_{-\infty}^{\infty} \text{tr}\left[ \pi_{\xi_{n}}^{+}\sigma_{-2}(f\tilde{D}_{F}^{-1}) \times \partial_{\xi_{n}} \left( f^{-1}(\tilde{D}_{F}^{-1} \cdot f\tilde{D}_{F}^{-1} \cdot f^{-1}(\tilde{D}_{F}^{-1}) \right) \right] (x_{0})
\times d\xi_{n}\sigma(\xi')dx',
\]

\[
= -i \int_{[\xi'=1]}^{+\infty} \int_{-\infty}^{\infty} \text{tr}\left[ \pi_{\xi_{n}}^{+}\sigma_{-2}(\tilde{D}_{F}^{-1}) \times \partial_{\xi_{n}} \left( (\tilde{D}_{F}^{-1} \cdot f\tilde{D}_{F}^{-1} \cdot f^{-1}(\tilde{D}_{F}^{-1}) \right) \right] (x_{0})d\xi_{n}\sigma(\xi')dx'.
\]

By (3.14), we have

\[
\pi_{\xi_{n}}^{+}\sigma_{-2}(\tilde{D}_{F}^{-1}) = \pi_{\xi_{n}}^{+}\left( \frac{c(\xi)\sigma_{0}(\tilde{D}_{F})c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^8} \sum_{j} c(dx_{j}) \left[ \partial_{x_{j}}[c(\xi)]|\xi|^2 - c(\xi)\partial_{x_{j}}(|\xi|^2) \right] \right),
\]

where

\[
T_{1} = -\frac{1}{4(\xi_{n} - i)^2} \left[ (2 + i\xi_{n})c(\xi')\sigma_{0}(\tilde{D}_{F})c(\xi') + i\xi_{n}c(dx_{n})\sigma_{0}(\tilde{D}_{F})c(dx_{n}) + (2 + i\xi_{n})c(\xi')c(dx_{n}) \times \partial_{x_{n}}[c(\xi')] + ic(dx_{n})\sigma_{0}(\tilde{D}_{F})c(\xi') + ic(\xi')\sigma_{0}(\tilde{D}_{F})c(dx_{n}) - i\partial_{x_{n}}[c(\xi')] \right],
\]

\[
T_{2} = \frac{h'(0)}{2} \left[ \frac{c(dx_{n})}{4i(\xi_{n} - i)} + \frac{c(dx_{n})}{8(\xi_{n} - i)^2} + \frac{3\xi_{n} - 7i}{8(\xi_{n} - i)^2} \left( ic(\xi') - c(dx_{n}) \right) \right].
\]
On the other hand,
\[\pi_{\xi}^+(\frac{e(\xi)\mu(\xi)}{|\xi|^2})(x_0)|_{|\xi'|=1}\]
\[= \frac{-i\xi_n - 2e(\xi')\mu(\xi') - i\int e(dx_n)\mu(\xi') + c(\xi')\mu(\xi dx_n) - i\xi_n e(dx_n)\mu(\xi dx_n)}{4(\xi_n - i)^2}. \tag{3.79}\]

By (3.42) (3.44) and (3.76), then we have
\[\text{tr}\left[ T_1 \times \partial_{\xi_n}\sigma_{\lambda-3}(\tilde{D}_F^*\tilde{D}_F\tilde{D}_F)^{-1}\right]|_{|\xi'|=1}\]
\[= \text{tr}\left\{ \frac{1}{4(\xi_n - i)^2}\left[ \frac{5}{2} h'(0)c(dx_n) - \frac{5i}{2} h'(0)c(\xi') - (2 + i\xi_n)c(\xi')e(dx_n)\sigma_{\xi, c(\xi')} + i\partial_{\xi_n}c(\xi') \right] \right\}
\[\times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2) c(dx_n)}{(1 + \xi_n^2)^3} \right\}
\[= h'(0)\dim F\frac{3 + 12i\xi_n + 3\xi_n^2}{(\xi_n - i)^4(\xi_n + i)^4}. \tag{3.80}\]

Similarly, we have
\[\text{trace}\left[ T_2 \times \partial_{\xi_n}\sigma_{\lambda-3}(\tilde{D}_F^*\tilde{D}_F\tilde{D}_F)^{-1}\right]|_{|\xi'|=1}\]
\[= \text{trace}\left\{ \frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} \left( ic(\xi') - c(dx_n) \right) \right] \right\}
\[\times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2) c(dx_n)}{(1 + \xi_n^2)^3} \right\}
\[= h'(0)\dim F\frac{4i - 11\xi_n - 6i\xi_n^2 + 3\xi_n^3}{(\xi_n - i)^4(\xi_n + i)^4}. \tag{3.81}\]

By (3.79) and (3.80), we obtain
\[-i \int_{|\xi'|=1}^{\pm\infty} \int_{-\infty}^{\infty} \text{trace}\left[ \left( T_1 - T_2 \right) \times \partial_{\xi_n}\sigma_{\lambda-3}(\tilde{D}_F^*\tilde{D}_F\tilde{D}_F)^{-1}\right] |x_0| d\xi_n \sigma(\xi') dx'
\[= -i\dim F h'(0) \int_{|\xi'|=1}^{\pm\infty} \int_{-\infty}^{\infty} \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n - i)^4(\xi_n + i)^3} d\xi_n \sigma(\xi') dx'
\[= -i\dim F h'(0) \frac{2\pi i}{4!} \left[ \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n + i)^3} \right] |_{\xi_n=\Omega_4} dx'
\[= \frac{55}{16} \dim F \pi h'(0) \Omega_4 dx'. \tag{3.82}\]

By (3.55) and (3.56), we have
\[\text{trace}\left[ \pi_{\xi}^+(\frac{e(\xi)\mu(\xi)}{|\xi|^2}) \times \partial_{\xi_n}\sigma_{\lambda-3}(\tilde{D}_F^*\tilde{D}_F\tilde{D}_F)^{-1}\right](x_0)
\[= \frac{(3\xi_n - i)i}{2(\xi_n - i)(1 + \xi_n^2)^3} \text{trace}[e(dx_n)\mu] + \frac{3\xi_n - i}{2(\xi_n - i)(1 + \xi_n^2)^3} \text{trace}[c(\xi')\mu]. \tag{3.83}\]

By the relation of the Clifford action and trace $PQ = \text{trace}QP$, then we have the equalities
\[\text{trace}[e(dx_n)\mu] = \text{trace}\left[ e(dx_n) \sum_{j=1}^{n} e(e_j)(\sigma_j^F + A(e_j)) \right] = \text{trace}\left[ -\text{id} \otimes (\sigma_n^F + A(e_n)) \right]; \tag{3.84}\]
\[\text{trace}[c(\xi')\mu] = \text{trace}\left[ c(\xi') \sum_{j=1}^{n} e(e_j)(\sigma_j^F + A(e_j)) \right] = \text{trace}\left[ -\sum_{j=1}^{n-1} \xi_j(\sigma_j^F + A(e_j)) \right]. \tag{3.85}\]
We note that $i < n$, \( \int_{|\xi'|=1} \xi_i \sigma(\xi') = 0 \), so trace\(c(\xi')\mu\) has no contribution for computing case (c).

Then, we obtain

\[
- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \left( \frac{c(\xi')e(\xi)}{|\xi'|^4} \right) \right] \times \partial_{\xi_n} \sigma_{-3} \left( \left( \tilde{D}_F \tilde{D}_{F'} \tilde{D}_{F''}^{-1} \right) \right) (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{(3 \xi_n - i)^3}{2(\xi_n - i)(1 + \xi_n^2)^3} c(\xi') \mu \mid \partial_{\xi_n} \sigma(\xi') dx'
\]

\[
= - 2 \pi \dim F \text{trace} [\sigma_n^F + A(e_n)] \Omega_4 dx'.
\]

(3.86)

Then

\[
\text{case (c)} = \frac{55}{16} \dim F \pi h'(0) \Omega_4 dx' = 2 \pi \dim F h'(0) \text{trace} [\sigma_n^F + A(e_n)] \Omega_4 dx'.
\]

(3.87)

Now \( \Phi \) is the sum of the case (a), case (b) and case (c), then

\[
\Phi = \left[ - 4 h'(0) - \text{trace} \left( A(e_n) \right) - 3 \text{trace} \left( A^*(e_n) \right) \right] + \frac{3}{2} \text{trace} \left( \sigma_n^F - A^*(e_n) \right)
\]

\[
- 2 \text{trace} \left( \sigma_n^F + A(e_n) \right) + \left( \frac{19 i}{16} + 11 \right) 
\]

\[
- \frac{15 i}{2} \partial_{\xi_n} (f) \pi \dim F \Omega_4 dx'.
\]

(3.88)

By (4.2) in [12], we have

\[
K = \sum_{1 \leq i, j \leq n-1} K_{i, j} \gamma_{\tilde{M}}^{i, j}; K_{i, j} = - \Gamma_{i, j}^n,
\]

and \( K_{i, j} \) is the second fundamental form, or extrinsic curvature. For \( n = 6 \), then

\[
K(x_0) = \sum_{1 \leq i, j \leq n-1} K_{i, j} (x_0) \gamma_{\tilde{M}}^{i, j} (x_0) = \sum_{i=1}^{5} K_{i, i} (x_0) = - \frac{5}{2} h'(0).
\]

(3.89)

Hence we conclude that

**Theorem 3.4.** Let \( M \) be a 6-dimensional compact spin manifolds with the boundary \( \partial M \). Then

\[
\text{Wres} \left[ \pi^+(f \tilde{D}_F^{-1}) \circ \pi^+ \left( f^{-1}(\tilde{D}_F^{-1}) \circ \tilde{D}_{F'}^{-1} \cdot f^{-1}(\tilde{D}_{F''}^{-1}) \right) \right]
\]

\[
= 8 \pi^3 \int_M \left[ \text{trace} \left[ - \frac{s}{12} + c(A^*) c(A) - \frac{1}{4} \sum_i [c(A^*) c(e_i) - c(e_i) c(A)] \right] \right.
\]

\[
- \frac{1}{2} \sum_j \left[ c(e_j) \nabla^2_{\xi_j} (c(A)) \right] - \frac{4 \Delta(f)}{f} + \frac{2 \text{trace} \left( A(\text{grad}_M f) \right)}{f} - f^2 \left[ (\text{grad}_M f)^2 + 2 \Delta(f) \right]
\]

\[
+ \int_{\partial M} \left( \frac{3}{2} \text{trace} \left( \sigma_n^F - A^*(e_n) \right) - 4 h'(0) - \text{trace} \left( A(e_n) \right) - 3 \text{trace} \left( A^*(e_n) \right) - 2 \text{trace} \left( \sigma_n^F \right) 
\]

\[
+ A(e_n) \right) + \left( \frac{19 i}{16} + 11 \right) \partial_{\xi_n} (f) \right] \pi \dim F \Omega_4 + \frac{3 \pi g(dx_n, df)}{8 f} \Omega_4 \right] 
\]

\[
\left. \text{dvol}_M \right. \). \quad (3.90)
\]

where \( s \) is the scalar curvature.
4. Twisted signature operator and its symbol

Let us recall the definition of twisted signature operators. We consider a $n$-dimensional oriented Riemannian manifold $(M, g^M)$. Let $F$ be a real vector bundle over $M$, let $g^F$ be an Euclidean metric on $F$. Let

$$\bigwedge^*(T^*M) = \bigoplus_{i=0}^{n} \bigwedge^i(T^*M)$$  \hspace{1cm} (4.1)

be the real exterior algebra bundle of $T^*M$. Let

$$\Omega^*(M, F) = \bigoplus_{i=0}^{n} \Omega^i(M, F) = \bigoplus_{i=0}^{n} C^\infty(M, \bigwedge^i(T^*M) \otimes F)$$  \hspace{1cm} (4.2)

be the set of smooth sections of $\bigwedge^*(T^*M) \otimes F$. Let $*$ be the Hodge star operator of $g^{TM}$. It extends on $\bigwedge^*(T^*M) \otimes F$ by acting on $F$ as identity. Then $\Omega^*(M, F)$ inherits the following standardly induced inner product

$$\langle \zeta, \eta \rangle_F = \int_M \langle \zeta \wedge *\eta \rangle_F, \quad \zeta, \eta \in \Omega^*(M, F).$$  \hspace{1cm} (4.3)

Let $\hat{\nabla}^F$ be the non-Euclidean connection on $F$. Let $d^F$ be the obvious extension of $\nabla^F$ on $\Omega^*(M, F)$. Let $\delta^F = d^F*$ be the formal adjoint operator of $d^F$ with respect to the inner product. Let $\hat{D}^F$ be the differential operator acting on $\Omega^*(M, F)$ defined by

$$\hat{D}^F = d^F + \delta^F.$$  \hspace{1cm} (4.4)

Let $\omega(F, g^F) = \hat{\nabla}^{F,*} - \hat{\nabla}^F$, $\nabla^{F,*} = \nabla^F + \frac{1}{2} \omega(F, g^F)$.  \hspace{1cm} (4.5)

Then $\nabla^{F,*}$ is an Euclidean connection on $(F, g^F)$.

Let $\nabla^{\bigwedge^*(T^*M)}$ be the Euclidean connection on $\bigwedge^*(T^*M)$ induced canonically by the Levi-Civita connection $\nabla^{TM}$ of $g^{TM}$. Let $\nabla^e$ be the Euclidean connection on $\bigwedge^*(T^*M) \otimes F$ obtained from the tensor product of $\nabla^{\bigwedge^*(T^*M)}$ and $\nabla^{F,*}$. Let $\{e_1, \cdots, e_n\}$ be an oriented (local) orthonormal basis of $TM$. The following result was proved by Proposition in [20].

The following identity holds

$$d^F + \delta^F = \sum_{i=1}^{n} c(e_i) \nabla^e_{e_i} - \frac{1}{2} \sum_{i=1}^{n} \tilde{c}(e_i) \omega(F, g^F)(e_i).$$  \hspace{1cm} (4.6)

Let $\hat{D}^F = \sum_{i=1}^{n} c(e_i) \nabla^e_{e_i}$ and $\omega(F, g^F)$ be any element in $\Omega(M, EndF)$, then we define the generalized twisted signature operators $\hat{D}_F, \hat{D}^*_F$ as follows.

For sections $\psi \otimes \chi \in \bigwedge^*(T^*M) \otimes F$,

$$\hat{D}_F(\psi \otimes \chi) = D_F(\psi \otimes \chi) - \frac{1}{2} \sum_{i=1}^{n} \tilde{c}(e_i) \omega(F, g^F)(e_i)(\psi \otimes \chi),$$  \hspace{1cm} (4.7)

$$\hat{D}^*_F(\psi \otimes \chi) = D^*_F(\psi \otimes \chi) - \frac{1}{2} \sum_{i=1}^{n} \tilde{c}(e_i) \omega^*(F, g^F)(e_i)(\psi \otimes \chi).$$  \hspace{1cm} (4.8)

Here $\omega^*(F, g^F)(e_i)$ denotes the adjoint of $\omega(F, g^F)(e_i)$.

In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$, the connection matrix $c_{s,t} \ (\omega_{s,t})$ is defined by

$$\tilde{\nabla}(\tilde{e}_1, \cdots, \tilde{e}_n) = (\tilde{e}_1, \cdots, \tilde{e}_n)(\omega_{s,t}).$$  \hspace{1cm} (4.9)

Let $M$ be a 6-dimensional compact oriented Riemannian manifold with boundary $\partial M$. We define that $\hat{D}_F: C^\infty(M, \bigwedge^*(T^*M) \otimes F) \to C^\infty(M, \bigwedge^*(T^*M) \otimes F)$ is the generalized twisted signature operator. Take
the coordinates and the orthonormal frame as in Section 3. Let $\epsilon(\hat{e}_j^*)$, $\iota(\hat{e}_j^*)$ be the exterior and interior multiplications respectively. Write
\begin{equation}
\epsilon(\hat{e}_j) = \epsilon(\hat{e}_j^*) - \iota(\hat{e}_j^*), \quad \iota(\hat{e}_j) = \epsilon(\hat{e}_j^*) + \iota(\hat{e}_j^*).
\end{equation}

(4.10)

We’ll compute $\text{tr} \lambda^*(T^*M) \otimes F$ in the frame $\{\hat{e}_{i_1}^* \wedge \cdots \wedge \hat{e}_{i_k}^* \mid 1 \leq i_1 < \cdots < i_k \leq 6\}$. By (3.2) and (4.8) in [12], we have
\begin{align*}
\hat{D}_F &= \sum_{i=1}^n c(e_i) \nabla_e e_i - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i) \\
&= \sum_{i=1}^n c(e_i) \left( \nabla^\gamma_\epsilon(T^*M) \otimes \text{id}_F + \text{id}_{\lambda^*(T^*M)} \otimes \nabla^F \right) - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i) \\
&= \sum_{i=1}^n c(\hat{e}_i) \left[ \hat{c}_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\hat{e}_i) \hat{c}(\hat{e}_s) \hat{c}(\hat{e}_t) - c(\hat{e}_s) c(\hat{e}_t) \right] \otimes \text{id}_F + \text{id}_{\lambda^*(T^*M)} \otimes \sigma^F \omega \\
&- \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i),
\end{align*}

(4.11)

Similarly, we have
\begin{align*}
\hat{D}_F^* &= \sum_{i=1}^n c(\hat{e}_i) \left[ \hat{c}_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\hat{e}_i) \hat{c}(\hat{e}_s) \hat{c}(\hat{e}_t) - c(\hat{e}_s) c(\hat{e}_t) \right] \otimes \text{id}_F + \text{id}_{\lambda^*(T^*M)} \otimes \sigma^F \omega \\
&- \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i).
\end{align*}

(4.12)

For convenience, let $\hat{c}(\omega) = \sum_i \hat{c}(e_i) \omega(F, g^F)(e_i)$ and $\hat{c}(\omega^*) = \sum_i \hat{c}(e_i) \omega^*(F, g^F)(e_i)$, by the composition formula and (2.2.11) in [12], we obtain in [19],

**Lemma 4.1.** Let $\hat{D}_F^*, \hat{D}_F$ be the twisted signature operators on $\Gamma(\wedge^*(T^*M) \otimes F)$, then
\begin{align}
\sigma_1(\hat{D}_F) &= \sigma_1(\hat{D}_F^*) = \sqrt{-1} c(\xi); \\
\sigma_0(\hat{D}_F) &= \sum_{i=1}^n c(\hat{e}_i) \left[ \frac{1}{4} \sum_{s,t} \omega_{s,t}(\hat{e}_i) \hat{c}(\hat{e}_s) \hat{c}(\hat{e}_t) - c(\hat{e}_s) c(\hat{e}_t) \right] \otimes \text{id}_F + \text{id}_{\lambda^*(T^*M)} \otimes \sigma^F \omega \\
&- \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i),
\end{align}

(4.13)
(4.14)

By the composition formula of pseudodifferential operators in Section 2.2.1 of [12], we have

**Lemma 4.2.** The symbol of the twisted signature operators $\hat{D}_F^*, \hat{D}_F$ as follows:
\begin{align}
\sigma_{-1}(\hat{D}_F^{-1}) &= \sigma_{-1}((\hat{D}_F^*)^{-1}) = \sqrt{-1} c(\xi) \frac{1}{|\xi|^2}; \\
\sigma_{-2}(\hat{D}_F^{-1}) &= \frac{c(\xi) \sigma_0(\hat{D}_F) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j} |\xi|^2 \right]; \\
\sigma_{-2}(\hat{D}_F^{-1}) &= \frac{c(\xi) \sigma_0(\hat{D}_F^*) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j} |\xi|^2 \right].
\end{align}

(4.15)
Since $\Psi$ is a global form on $\partial M$, so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$) and compute $\Psi(x_0)$ in the coordinates $\hat{U} = U \times [0,1)$ and the metric $\frac{1}{h(x_0)} g^{\partial M} + dx_n^2$. The dual metric of $g^{\partial M}$ on $\hat{U}$ is $\frac{1}{h(x_0)} g^{\partial M} + dx_n^2$. Write $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $g_{ij}^\partial M = g^\partial M(dx_i, dx_j)$, then

$$[g_{ij}^M] = \begin{bmatrix} \frac{1}{h(x_0)} g_{ij}^\partial M & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{h(x_0)} g_{ij}^\partial M & 0 \\ 0 & 1 \end{bmatrix},$$  \hspace{0.5cm} (4.19)

and

$$\partial_{x_j} g_{ij}^\partial M(x_0) = 0, \hspace{0.5cm} 1 \leq i, j \leq n-1; \hspace{0.5cm} g_{ij}^\partial M(x_0) = \delta_{ij}.$$

(4.20)

Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal frame field in $U$ about $g^{\partial M}$ which is parallel along geodesics and $e_i = \frac{\partial}{\partial x_i}(x_0)$, then $\{\tilde{e}_1 = \sqrt{h(x_0)} e_1, \ldots, \tilde{e}_{n-1} = \sqrt{h(x_0)} e_{n-1}, \tilde{e}_n = dx_n\}$ is the orthonormal frame field in $\hat{U}$ about $g^{\partial M}$. Locally $\Lambda^*(T^*M)\mid \hat{U} \cong \Lambda^\partial C(\frac{\partial}{\partial x_i})$. Let $\{f_1, \ldots, f_n\}$ be the orthonormal basis of $\Lambda^\partial C(\partial M)$. Take a spin frame field $\sigma : \hat{U} \to Spin(M)$ such that $\pi \sigma = \{\tilde{e}_1, \ldots, \tilde{e}_n\}$ where $\pi : Spin(M) \to O(M)$ is a double covering, then $\{[\sigma, f_i], 1 \leq i \leq 4\}$ is an orthonormal frame of $\Lambda^*(T^*M)\mid \hat{U}$. In the following, since the global form $\Psi$ is independent of the choice of the local frame, so we can compute trace $\Lambda^\partial C(\partial M)$ in the frame $\{[\sigma, f_i], 1 \leq i \leq 4\}$. Let $\{E_1, \ldots, E_n\}$ be the canonical basis of $R^n$ and $c(E_i) \in \Lambda C(n) \cong \text{Hom}(\Lambda^\partial C(\partial M), \Lambda^\partial C(\partial M))$ be the Clifford action. By [12], then

$$c'(\tilde{e}_i) = \left(\sigma, c(E_i)\right); \hspace{0.5cm} c'(\tilde{e}_i)[[\sigma, f_i]] = [\sigma, c(E_i)f_i]; \hspace{0.5cm} \partial_{x_i} = [(\sigma, \frac{\partial}{\partial x_i})],$$  \hspace{0.5cm} (4.21)

then we have $\frac{\partial}{\partial x_i} c'(\tilde{e}_i) = 0$ in the above frame. By Lemma 2.2 in [12], we have

**Lemma 4.3.**

$$\partial_{x_j} (\xi^2 \mid g^{\partial M})(x_0) = \begin{cases} 0, & \text{if } j < n; \\ h'(0) \xi^2 \mid g^{\partial M}, & \text{if } j = n, \end{cases}$$

(4.22)

$$\partial_{x_j} (c(\xi))(x_0) = \begin{cases} 0, & \text{if } j < n; \\ \partial_{x_n}(c(\xi))(x_0), & \text{if } j = n, \end{cases}$$

(4.23)

where $\xi = \xi' + \xi_n dx_n$.

Then an application of Lemma 2.3 in [12] shows

**Lemma 4.4.** The symbol of the twisted signature operators $\hat{D}_F, \hat{D}_f$ as follows:

$$\sigma_0(\hat{D}_F) = \theta + \theta^*; \hspace{0.5cm} \sigma_0(\hat{D}_f) = \theta + \bar{\theta},$$

(4.24)

(4.25)

where

$$\theta = -\frac{5}{4} h'(0)c(dx_n) + \frac{1}{4} h'(0) \frac{n-1}{n} \sum_{i=1}^{n-1} c(\tilde{e}_i)\tilde{c}(\tilde{e}_n)\tilde{c}(\tilde{e}_i)(x_0) \otimes \text{id}_F;$$

$$\bar{\theta}^* = \sum_{i=1}^{n} c(\tilde{e}_i)\sigma^{F,e}_i - \frac{1}{2} \sum_{i=1}^{n} c(\tilde{e}_i)\omega^*(F; g^F)(e_i);$$

$$\bar{\theta} = \sum_{i=1}^{n} c(\tilde{e}_i)\sigma^{F,e}_i - \frac{1}{2} \sum_{i=1}^{n} c(\tilde{e}_i)\omega(F; g^F)(e_i).$$

(4.26)

In order to get the symbol of operators $\hat{D}_F f \cdot \hat{D}_f f^{-1} \cdot \hat{D}_F f$. Similar to (3.19)-(3.23), we give the specification of $\hat{D}_F f \cdot \hat{D}_f f^{-1} \cdot \hat{D}_F f$.  

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Combining (4.11) and (4.12), we have
\[
\hat{D}_F f \cdot \hat{D}_F f^{-1} \cdot \hat{D}_F f
\]
\[
= f \cdot \left\{ \sum_{i,j,l=1}^{n} \sum_{r=1}^{n} c(e_r) \langle e_r, dx_i \rangle (-g^{ij} \partial_h \partial_i \partial_j) + \sum_{r,s=1}^{n} c(e_r) \langle e_r, dx_i \rangle \right\} - \sum_{i,j=1}^{n} (\partial_h g^{ij}) \partial_i \partial_j - \sum_{i,k,j=1}^{n} g^{ij}
\]
\[
\times (4 \sigma^\wedge_\Lambda^{(T^*M) \otimes F} \partial_j - 2 \Gamma^k_{ij} \partial_k) \bigg\} + \sigma_0(\hat{D}_F^*) \left\{ - 2 \sigma^\wedge_\Lambda^{(T^*M) \otimes F} \partial_j + \Gamma^k \partial_k - \frac{1}{2} \sum_{j=1}^{n} \left[ \hat{c}(\omega) c(e_j) + c(e_j) \hat{c}(\omega^*) \right] \right\}
\]
\[
\times e_j - g^{ij} \left[ (\partial_i \sigma^\wedge_\Lambda^{(T^*M) \otimes F}) + \sigma^\wedge_\Lambda^{(T^*M) \otimes F} \sigma^\wedge_\Lambda^{(T^*M) \otimes F} \right] + \frac{1}{4} \hat{c}(\omega) c(e^*)
\]
\[
- \frac{1}{2} \sum_{j=1}^{n} \hat{c}(\omega) c(e_j) \sigma^\wedge_\Lambda^{(T^*M) \otimes F} c(e_j) - \frac{1}{2} \sum_{j=1}^{n} c(e_j) e_j \left[ \hat{c}(\omega^*) \right] - \frac{1}{2} \sum_{j=1}^{n} \sigma^\wedge_\Lambda^{(T^*M) \otimes F} c(e_j) \langle e_j, dx_i \rangle \bigg\} \right\} + \frac{1}{2} \sum_{j=1}^{n} \left[ \hat{c}(\omega) c(e_j) + c(e_j) \hat{c}(\omega^*) \right] \langle e_j, dx^k \rangle \partial_k
\]
\[
\times \partial_j - 2 \sum_{i,j=1}^{n} (\partial_h g^{ij}) \sigma^\wedge_\Lambda^{(T^*M) \otimes F} \partial_j - \frac{1}{2} \sum_{j,k=1}^{n} \left[ \partial_j \left( \hat{c}(\omega) c(e_j) + c(e_j) \hat{c}(\omega^*) \right) \right] \langle e_j, dx^k \rangle \partial_k
\]
\[
- \frac{1}{2} \sum_{i,j=1}^{n} \left[ \partial_j \left( \hat{c}(\omega) c(e_j) + c(e_j) \hat{c}(\omega^*) \right) \right] \langle e_j, dx^k \rangle \partial_k
\]
Lemma 4.6. By the above composition formulas, then we obtain:

\[
\sigma_j^{k,e} \Gamma^{k} \partial_j \partial_k - \frac{1}{2} \sum_{j,k=1}^n \left( \hat{c}(w)c(e_j) + c(e_j)\hat{c}(\omega^*) \right) [\partial_k (e_j, dz^k)] \partial_j \right) + c(df) \left\{ -g^{ij} \partial_i \partial_j \right.
\]

\[
-2 \sigma_j^{k,e} \sigma_j^{k,e} \Gamma^{k} \partial_j - \frac{1}{2} \sum_j \hat{c}(\omega)c(e_j) + c(e_j)\hat{c}(\omega^*) \right\} e_j - g^{ij} \left( \partial_i \sigma_j^{k,e} \right) + \frac{1}{2} \sum_j \left( c(e_j) C(e_j) + \frac{1}{4}s \hat{c}(\omega) \right)
\]

\[
+ \frac{1}{4} \hat{c}(\omega)\hat{c}(\omega^*) - \frac{1}{2} \sum_j c(e_j) \sigma_j^{k,e} \hat{c}(\omega^*) + \frac{1}{2} \sum_{i \neq j} R_{ij}^{e} c(e_i, e_j) c(e_j) \right\} \left\{ f \cdot c(df)^{-1} \cdot f + \left\{ \frac{1}{4} g^{ij} \hat{c}(\partial_i) \left[ \partial_j \left( \frac{1}{4} \sum_{s \neq t} \hat{c}(e_s) \hat{c}(e_t) - c(e_s) c(e_t) \right) \right] \right. \right\}
\]

\[-\frac{\hat{c}(\omega)}{2} \right\} c(df) c(df)^{-1} f.
\]

By the above composition formulas, then we obtain:

Lemma 4.5. Let \( \hat{D}_F, \hat{D}_F \) be the twisted signature operators on \( \Gamma(\Lambda^*(T^* M) \otimes F) \), then

\[
s_i (\hat{D}_F f \cdot \hat{D}_F f^{-1} \cdot \hat{D}_F f) = f \sigma_i (\hat{D}_F \hat{D}_F \hat{D}_F) = f \sqrt{-1} c(\xi) |\xi|^2;
\]

(4.28)

\[
s_i (\hat{D}_F f \cdot \hat{D}_F f^{-1} \cdot \hat{D}_F f) = f \sigma_i (\hat{D}_F \hat{D}_F \hat{D}_F) + 2c(df) \left| \xi \right|^2
\]

(4.29)

where \( \sigma_2 (\hat{D}_F \hat{D}_F \hat{D}_F) = c(\xi)(4\sigma^k - 2\Gamma^k) \partial_k - \frac{1}{4}|\xi|^2 h'(0)c(dx_n) + |\xi|^2 \left( \frac{1}{4} h'(0) \sum_{i=1}^5 (c_0 c_0^* c_0^0\xi) (x_0) + \theta^* - \hat{c}(\omega^*) \right) + c(\xi) c(\omega) c(\xi).

For convenience, we write that \( \sigma_2 (\hat{D}_F \hat{D}_F \hat{D}_F) = G + |\xi|^2 \left( p + \theta^* - \hat{c}(\omega^*) \right) + c(\xi) c(\omega) c(\xi). \) By (4.28), (4.29), Lemma 2.1 in [12] and the composition formula of pseudodifferential operators, similar to (3.26)-(3.28), we obtain

Lemma 4.6. Let \( \hat{D}_F, \hat{D}_F \) be the generalized twisted signature operators on \( \Gamma(\Lambda^*(T^* M) \otimes F) \), then

\[
s_{-3} (\hat{D}_F f \cdot \hat{D}_F f^{-1} \cdot \hat{D}_F f) = \frac{\sqrt{-1} c(\xi)}{f |\xi|^4};
\]

(4.30)

\[
s_{-4} (\hat{D}_F f \cdot \hat{D}_F f^{-1} \cdot \hat{D}_F f) = f^{-1} s_{-4} ((\hat{D}_F \hat{D}_F \hat{D}_F) f^{-1}) + \frac{2c(\xi) c(df) c(\xi)}{f^2 |\xi|^6} + \frac{ic(\xi) \sum_j c(df_j) |\xi|^2 + 2\xi c(\xi) D_{dx_j} (f^{-1}) c(\xi)}{|\xi|^8},
\]

(4.31)
where
\[
\sigma_4((\hat{D}_F^* \hat{D}_F \hat{D}_F^* )^{-1}) = \frac{c(\xi)\sigma_2(\hat{D}_F^* \hat{D}_F \hat{D}_F^*)c(\xi)}{|\xi|^8} + \frac{c(\xi)}{|\xi|^{10}} \sum_j \left[ c(dx_j)|\xi|^2 - 2c(\xi)_j c(\xi) \right] \left[ 2c(\xi)_j c(\xi) - 2c(\xi)_j |\xi|^2 \right]
\]
\[
= \frac{c(\xi)Gc(\xi)}{|\xi|^8} + \frac{c(\xi)(p + \partial^* - \hat{c}(\omega^*))c(\xi)}{|\xi|^6} + \frac{\hat{c}(w)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^{10}} \sum_j \left[ c(dx_j)|\xi|^2 - 2c(\xi)_j c(\xi) \right] \left[ -2c(\xi)_j |\xi|^2 \right].
\]

Hence we cite that

**Theorem 4.7.** [19] For even n-dimensional oriented compact Riemannian manifolds without boundary, the following equality holds:

\[
\text{Wres(}\hat{D}_F f \cdot \hat{D}_F f^{-1}(\hat{\omega})) = \frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 2)!} \int_M \left\{ \text{trace} \left[ -\frac{8}{12} + \frac{n}{16} \left[ \hat{c}(\omega^*) - \hat{c}(\omega) \right]^2 - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) + \frac{1}{4} \sum_j \nabla^F_{e_j} (\hat{c}(\omega^*)) e_j \\
+ \frac{1}{4} \sum_j \hat{c}(e_j) \nabla^F_{e_j} (\hat{c}(\omega))^2 + 4f^{-1}\Delta(f) + 8(\text{grad}_M f, \text{grad}_M (f^{-1})) - 5f^{-2} |\text{grad}_M f|^2 \\
+ 2\Delta(f) \right\} \text{dvol}_M.
\]

(4.32)

(4.33)

5. Conformal perturbations of twisted Signature Operators and Noncommutative residue

In the following, we will compute the more general case \(\text{Wres}[\pi^+(f \hat{D}_F^{-1}) \circ \pi^+(f^{-1}(\hat{D}_F^*)) \cdot \hat{D}_F^{-1} \cdot f^{-1}(\hat{D}_F^*)^{-1}]\) for nonzero smooth functions \(f, f^{-1}\). An application of (2.1.4) in [14] shows that

\[
\text{Wres}[\pi^+(f \hat{D}_F^{-1}) \circ \pi^+(f^{-1}(\hat{D}_F^*)) \cdot \hat{D}_F^{-1} \cdot f^{-1}(\hat{D}_F^*)^{-1}] = \int_M \int_{|\xi| = 1} \text{trace}_{\Lambda^*(T^* M) \otimes F} (\hat{D}_F f \cdot \hat{D}_F f^{-1}) \sigma(\xi) dx + \int_{\partial M} \Psi,
\]

(5.1)

where

\[
\Psi = \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{+\infty} (-1)^{|\alpha|+j+k+\ell} \alpha! (j+k+1)! \text{trace}_{\Lambda^*(T^* M) \otimes F} \left[ \partial_{\xi_n} \partial_{\xi_n} \partial_{\xi_n} \sigma(\hat{D}_F^{-1})(x', 0, \xi', \xi_n) \right. \\
\left. \times \partial_{\xi_n} \partial_{\xi_n} \sigma(\hat{D}_F^{-1})(x', 0, \xi', \xi_n) \right] d\xi_n \sigma(\xi') dx',
\]

(5.2)

and the sum is taken over \(r - k + |\alpha| + \ell - j - 1 = -n, r \leq -1, \ell \leq -1\).
Locally we can use Theorem 4.7 to compute the interior term of (5.1), then

\[
\int_M \int_{|\xi|=1} \text{trace}_{\Lambda^* (T^* M) \otimes F} [\sigma_{-4}((\hat{D}_F^* f \cdot \hat{D}_F f^{-1})^{-2})] \sigma(\xi) \, dx
\]

\[
= 8\pi^4 \int_M \left\{ \text{trace} \left[ \frac{-8}{12} + \frac{3}{8} (\hat{c}(\omega^*) - \hat{c}(\omega))^2 - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) - \frac{1}{4} \sum_j \nabla_{e_j}^F (\hat{c}(\omega^*)) c(e_j) \right.ight.
\]

\[
+ \frac{1}{4} \sum_{j} c(e_j) \nabla_{e_j}^F (\hat{c}(\omega)) \left.] + 4f^{-1} \Delta(f) + 8 \langle \text{grad}_M(f), \text{grad}_M(f^{-1}) \rangle - 5f^{-2} \| \text{grad}_M(f) \|^2
\]

\[
+ 2 \Delta(f) \right\} \, dvol_M.
\]

(5.3)

So we only need to compute \( \int_{\partial M} \Psi \). From the remark above, now we can compute \( \Psi \) (see formula (5.2) for the definition of \( \Psi \)). Since the sum is taken over \( r + \ell - k - j - |\alpha| - 1 = -6, \ r \leq -1, \ell \leq -3 \), then we have the \( \int_{\partial M} \Psi \) is the sum of the following five cases:

**case (a) \((1)\)** \( r = -1, \ell = -3, j = k = 0, |\alpha| = 1 \).

By (5.2), we get

\[
\text{case (a) (1) } = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \delta^\alpha_{\xi'} \sigma_{-1}(f \hat{D}_F^{-1}) \times \partial^\alpha_{\xi'} \partial_{\xi} \sigma_{-3}(f^{-1}(\hat{D}_F^*)^{-1} \cdot f \hat{D}_F^{-1}) \cdot f^{-1}(\hat{D}_F^*)^{-1} \right](x_0) d\xi_n \sigma(\xi') \, dx'\]

\[
= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \delta^\alpha_{\xi'} \sigma_{-1}(\hat{D}_F^{-1}) \times \partial^\alpha_{\xi'} \partial_{\xi} \sigma_{-3}(\hat{D}_F^* \hat{D}_F \hat{D}_F^{-1})^{-1} \right](x_0) d\xi_n \sigma(\xi') \, dx' - f \sum_{j<n} \partial_j (f^{-1}) \int\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \delta^\alpha_{\xi'} \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi} \sigma_{-3}(\hat{D}_F^* \hat{D}_F \hat{D}_F^{-1})^{-1} \right](x_0) \, dx' \times d\xi_n \sigma(\xi') \, dx'.
\]

(5.4)

By (3.24) and (4.29), we have \( \sigma_{-3}(\hat{D}_F^* \hat{D}_F \hat{D}_F^{-1})^{-1} = \sigma_{-3}(\hat{D}_F^* \hat{D}_F \hat{D}_F^{-1})^{-1} \).

By (3.34) and Lemma 2.2 in [12], for \( i < n \) we have

\[
\partial_{\xi} \sigma_{-3}(\hat{D}_F^* \hat{D}_F \hat{D}_F^{-1})^{-1}(x_0) = 0.
\]

(5.5)

Thus we have

\[
- \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \delta^\alpha_{\xi'} \sigma_{-1}(\hat{D}_F^{-1}) \times \partial^\alpha_{\xi'} \partial_{\xi} \sigma_{-3}(\hat{D}_F^* \hat{D}_F \hat{D}_F^{-1})^{-1} \right](x_0) d\xi_n \sigma(\xi') \, dx' = 0.
\]

(5.6)

By (3.12) and (4.16), we have \( \sigma_{-1}(\hat{D}_F)^{-1} = \sigma_{-1}(\hat{D}_F)^{-1} \). Similar to (3.36)-(3.38), for \( i < n \), we have

\[
\text{trace} \left[ \delta^\alpha_{\xi'} \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi} \sigma_{-3}(\hat{D}_F^* \hat{D}_F \hat{D}_F^{-1})^{-1} \right](x_0)
\]

\[
= -\xi \text{trace} \left[ \frac{c(\text{d}x_n)^2}{2(\xi_n - \sqrt{-1})^2} \right] - 4\sqrt{-1} \xi_n \xi \text{trace} \left[ \frac{c(\text{d}x_n)^2}{2(\xi_n - \sqrt{-1})^2|\xi|^6} \right] + 4\sqrt{-1} \xi_n \xi_n \text{trace} \left[ \frac{c(\text{d}x_n)^2}{2(\xi_n - \sqrt{-1})^2|\xi|^6} \right] - 2\sqrt{-1} \text{trace} \left[ \frac{c(\xi')^2}{2(\xi_n - \sqrt{-1})^2|\xi|^6} \right] + 4\sqrt{-1} \xi_n \xi_n \text{trace} \left[ \frac{c(\text{d}x_n)^2}{2(\xi_n - \sqrt{-1})^2|\xi|^6} \right]
\]

(5.7)

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We note that \( i < n \), \( \int_{|\xi'|=1} \xi_i \sigma(\xi') = 0 \), so
\[
-f \sum_{j<n} \partial_j(f^{-1}) \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^2_i \pi_{\xi_n}^+ \sigma_{-1}(D_F^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}(D_F^* D_F D_F^* D_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' = 0. \tag{5.8}
\]
Then we have case (a) (I) = 0.

case (a) (II) \( r = -1, l = -3, |\alpha| = k = 0, j = 1 \).

By (5.2), we have
\[
\text{case (a) (II)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(f D_F^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}(f^{-1}(D_F^*)^{-1} \cdot f D_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' = 0. \tag{5.9}
\]
Since \( n = 6 \), \( \text{trace}_{\pi \ast (T \cdot M)}[-\text{id}] = -64\text{dim}F \). By the relation of the Clifford action and \( \text{trace} PQ = \text{trace} QP \), then
\[
\begin{align*}
\text{trace}[c(\xi')c(dx_n)] &= 0; \text{trace}[c(dx_n)^2] = -64\text{dim}F; \text{trace}[c(\xi')^2](x_0)|_{|\xi'|=1} = -64\text{dim}F; \\
\text{trace}[\partial_{x_n} c(\xi')c(dx_n)] &= 0; \text{trace}[\partial_{x_n} c(\xi')(\xi')](x_0)|_{|\xi'|=1} = -32h'(0)\text{dim}F. \tag{5.10}
\end{align*}
\]
Similar to (3.41)-(3.45), then we obtain
\[
\begin{align*}
&\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D_F^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}(D_F^* D_F D_F^* D_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&= -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} 8h'(0)\text{dim}F \frac{8 - 24\xi_n i + 40\xi_n^2 + 24i\xi_n^3}{(\xi_n - i)^6 (\xi_n + i)^4} d\xi_n \sigma(\xi') dx' \\
&= 8h'(0)\text{dim}F \Omega_4 \frac{\pi i}{3!} \left[ \frac{8 + 24\xi_n i - 40\xi_n^2 - 24i\xi_n^3}{(\xi_n + i)^4} \right] (5) |_{\xi_n=i} dx' \\
&= -\frac{15}{2} \pi h'(0)\Omega_4 \text{dim}F dx'. \tag{5.11}
\end{align*}
\]
Similar to (3.47) and (3.48), then we obtain
\[
\begin{align*}
-\frac{1}{2} f^{-1} \partial_{x_n} (f) \int_{|\xi'|=1}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(D_F^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}(D_F^* D_F D_F^* D_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&= (10\pi i + 88\pi)\Omega_4 \text{dim}F \cdot f^{-1} \partial_{x_n} (f) dx', \tag{5.12}
\end{align*}
\]
where \( \Omega_4 \) is the canonical volume of \( S_4 \). Then
\[
\text{case (a) (II)} = -\frac{15}{2} \pi h'(0)\Omega_4 \text{dim}F dx' + (10\pi i + 88\pi)\Omega_4 \text{dim}F \cdot f^{-1} \partial_{x_n} (f) dx', \tag{5.13}
\]

(5.13)
where $\Omega_4$ is the canonical volume of $S_4$.

**case (a) (III)** $r = -1, l = -3, |\alpha| = j = 0, k = 1.$

By (5.2) and an integration by parts, we have

$$
case (a) (III) = -\frac{1}{2} \int_{|\xi'| = 1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(f\hat{D}_F^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3}(f^{-1}(\hat{D}_F)^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'
$$

$$= -\frac{1}{2} \int_{|\xi'| = 1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ (\sigma_{-1}(D_F^{-1})) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3}(D_F^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'

- \frac{1}{2} f\partial_{x_n}(f^{-1}) \int_{|\xi'| = 1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\hat{D}_F^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'.
$$

(5.14)

Similar to (3.52), (3.53) and combining (5.10), we have

$$
\text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D_F^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3}(D_F^{-1}) \right](x_0)|_{|\xi'| = 1} = 8h'(0)\text{dim}F \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^3(\xi + i)^2}.
$$

(5.15)

Then

$$
-\frac{1}{2} \int_{|\xi'| = 1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ (\sigma_{-1}(D_F^{-1})) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3}(D_F^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'

= -\frac{1}{2} \int_{|\xi'| = 1}^{\infty} \int_{-\infty}^{\infty} 8h'(0)\text{dim}F \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^3(\xi + i)^2} d\xi_n \sigma(\xi') dx'

= -8h'(0)\text{dim}F \Omega_4 \frac{\pi i}{4} \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi + i)^2}|_{|\xi'| = 1} dx'

= \frac{25}{2} \pi h'(0)\Omega_4 \text{dim}F dx',
$$

(5.16)

and

$$
-\frac{1}{2} f\partial_{x_n}(f^{-1}) \int_{|\xi'| = 1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\hat{D}_F^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'

= \frac{\pi i}{2} f \cdot \partial_{x_n}(f^{-1}) \Omega_4 \text{dim}F dx',
$$

(5.17)

where $\Omega_4$ is the canonical volume of $S_4$. Then

**case (a) (III)** $= \frac{25}{2} \pi h'(0)\Omega_4 \text{dim}F dx' + \frac{\pi i}{2} f \cdot \partial_{x_n}(f^{-1}) \Omega_4 \text{dim}F dx'$.

(5.18)

**case (b)** $r = -2, l = -3, |\alpha| = j = k = 0.$

By (5.2) and an integration by parts, we have
Hence, then an application of Lemma 4.3 shows

\[ \text{trace}\left[ \pi_{\xi_n}^+ \sigma_{-2}(fD_F^{-1}) \times \partial_{\xi_n} \sigma_{-3}\left(f^{-1}(D_F^{-1}) \cdot fD_F^{-1} \cdot f^{-1}(D_F^{-1})^{-1}\right) \right](x_0) \times d\xi_n \sigma(\xi') dx'. \]

Then an application of Lemma 4.3 shows

\[ \sigma_{-2}(D_F^{-1})(x_0) = \frac{c(\xi)\sigma_0(\tilde{D}_F)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^4} \sum_j c(dx_j) \left[ \partial_x (c(\xi))|\xi|^2 - c(\xi)\partial_x (|\xi|^2) \right](x_0) \]

\[ = \frac{c(\xi)\sigma_0(\tilde{D}_F)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^4} (dx_n) \left[ \partial_x (c(\xi))(x_0) - c(\xi)h'(0)|\xi'|^2 \right]. \]

Hence,

\[ \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_F^{-1})(x_0) := B_1 + B_2 + B_3 + B_4, \]

where

\[ B_1 = -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i \xi_n)c(\xi') \left( -\frac{5}{4} h'(0)c(dx_n)c(\xi') + i\xi_n c(dx_n) \left( -\frac{5}{4} h'(0)c(dx_n) \right)c(dx_n) \right) \right. \]

\[ + \left. \left( (2 + i \xi_n)c(\xi')c(dx_n) \partial_x c(\xi') + ic(dx_n) \left( -\frac{5}{4} h'(0)c(dx_n) \right)c(\xi') + ic(\xi') \left( -\frac{5}{4} h'(0)c(dx_n) \right) \right) \times c(dx_n) - i\partial_x c(\xi') \right] \]

\[ = -\frac{1}{4(\xi_n - i)^2} \left[ \frac{5}{2} h'(0)c(dx_n) - \frac{5i}{2} h'(0)c(\xi') - (2 + i \xi_n)c(\xi')c(dx_n) \partial_x c(\xi') + i\partial_x c(\xi') \right]; \]

\[ B_2 = -\frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{i\partial_x (c(\xi'))}{8(\xi_n - i)^2} + \frac{3x_n - 7i}{8(\xi_n - i)^3} \right][ic(\xi') - c(dx_n)] \]

\[ B_3 = -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i \xi_n)c(\xi')pc(\xi') + i\xi_n c(dx_n) pc(dx_n) + (2 + i \xi_n)c(\xi')c(dx_n) \partial_x c(\xi') \right. \]

\[ + \left. ic(dx_n)pc(\xi') + ic(\xi')pc(dx_n) - i\partial_x c(\xi') \right]; \]

\[ B_4 = -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i \xi_n)c(\xi')vdxn + i\xi_n c(dx_n) vdxn + ic(dx_n) uc(\xi') + ic(\xi') vdxn \right]. \]

On the other hand,

\[ \partial_{\xi_n} \sigma_{-3}(\tilde{D}_F^{-1} \cdot \tilde{D}_F \cdot \tilde{D}_F^{-1}) = -\frac{4i\xi_n c(\xi')}{(1 + \xi_n^2)^3} + \frac{i(1 - 3\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3}. \]

From (5.22) and (5.26), we have

\[ \text{trace}[B_1 \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_F^{-1} \cdot \tilde{D}_F \cdot \tilde{D}_F^{-1})(x_0)]|_{|\xi'|=1} \]

\[ = \text{tr} \left\{ \frac{1}{4(\xi_n - i)^2} \left[ \frac{5}{2} h'(0)c(dx_n) - \frac{5i}{2} h'(0)c(\xi') - (2 + i \xi_n)c(\xi')c(dx_n) \partial_x c(\xi') + i\partial_x c(\xi') \right] \right. \]

\[ \times \left. \left[ \frac{4i\xi_n c(\xi') + (i - 3\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3} \right] \right\} \]

\[ = 8h'(0) \frac{3 + 12i\xi_n + 3\xi_n^2}{(\xi_n - i)^4(\xi_n + i)^3}. \]
Similarly, we obtain
\[
\text{trace}\{B_2 \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F)^{-1})(x)\}\big|_{|\xi'|=1} = \text{tr}\left\{ -\frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}[ic(\xi') - c(dx_n)] \right] \right. \\
\times \left. \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2) c(dx_n)}{(1 + \xi_n^2)^3} \right\} = -8h'(0)\frac{4i - 11\xi_n - 6i\xi_n^2 + 3\xi_n^3}{(\xi_n - i)^2(\xi_n + i)^3}. \quad (5.28)
\]

For the signature operator case,
\[
\text{trace}[c(\xi') p c(\xi') c(dx_n)](x_0) = \text{trace}[pc(\xi') c(dx_n) c(\xi')](x_0) = |\xi'|^2 \text{trace}[p(x_0) c(dx_n)], \quad (5.29)
\]
and
\[
c(dx_n)p(x_0) = -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\xi_i) c(\xi_i^*) c(\xi_n) = -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} [c(\xi_i) c(\xi_i^*) - i c(\xi_i) c(\xi_i^*)] [c(\xi_n) c(\xi_n^*) - i c(\xi_n^*) c(\xi_n^*)]. \quad (5.30)
\]

By Section 3 in [12], then
\[
\text{trace}_{\lambda=(T^* \cdot M)} \{ [c(\xi_i) c(\xi_i^*) - i c(\xi_i) c(\xi_i^*)] [c(\xi_n) c(\xi_n^*) - i c(\xi_n^*) c(\xi_n^*)] \} = a_{n,m}(e_i^*, e_n^*)^2 + b_{n,m} |e_n^*|^2 = b_{n,m}, \quad (5.31)
\]
where
\[
b_{n,m} = \left( \frac{4}{m - 2} \right) + \left( \frac{4}{m} \right) - 2 \left( \frac{4}{m - 1} \right).
\]

Then
\[
\text{tr}_{\lambda=(T^* \cdot M)} \{ [c(\xi_i) c(\xi_i^*) - i c(\xi_i) c(\xi_i^*)] [c(\xi_n) c(\xi_n^*) - i c(\xi_n^*) c(\xi_n^*)] \} = \sum_{m=0}^{6} b_{n,m} = 0. \quad (5.32)
\]

Hence in this case,
\[
\text{trace}_{\lambda=(T^* \cdot M)}[c(dx_n)p(x_0)] = 0. \quad (5.33)
\]

We note that \(\int_{|\xi'|=1} \xi_1 \cdots \xi_{2q+1} \sigma(\xi') = 0\), then \(\text{trace}_{\lambda=(T^* \cdot M)}[c(\xi') p(x_0)]\) has no contribution for computing case (b).

So, we obtain
\[
\text{trace}\left[ B_1 \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F)^{-1})(x)\right|_{|\xi'|=1} = \text{tr}\left\{ -\frac{1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi') p c(\xi') + i\xi_n c(dx_n) p c(dx_n) + (2 + i\xi_n)c(\xi') c(dx_n) \partial_{\xi_n} c(\xi') \right. \\
+ i c(dx_n) p c(\xi') + ic(dx_n) c(\xi') p(dx_n) - i \partial_{\xi_n} c(\xi') \right] \times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2) c(dx_n)}{(1 + \xi_n^2)^3} \right\} = 8h'(0) \text{dim}F\frac{3\xi_n^2 - 3i\xi_n - 2}{(\xi_n - i)^4(\xi_n + i)^3}. \quad (5.34)
\]

Then, we have
\[
\text{trace}\left[ (B_1 + B_2 + B_3) \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F)^{-1}) \right](x_0) = \text{dim}F \frac{3\xi_n^3 + 9i\xi_n^2 + 21\xi_n - 5i}{(\xi_n - i)^3(\xi_n + i)^3}. \quad (5.35)
\]
By the relation of the Clifford action and trace $PQ = \text{trace}QP$, then we have the equalities
\[
\text{trace}[c(\tilde{c})c(dx_n)] = 0, i < n; \quad \text{trace}[c(\tilde{c})c(dx_n)] = -64\dim F, i = n; 
\]
(5.36)
\[
\text{trace}[c(\tilde{c})c(\xi')] = \text{trace}[\tilde{c}(c(dx_n))] = 0. 
\]
(5.37)
Then $\text{trace}[\vartheta c(\xi')]$ has no contribution for computing case (b).

Then, we have
\[
\text{trace}[B_4 \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1})]|_{\xi' = 1} = \text{trace}\left\{ \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi') \vartheta c(\xi') + i\xi_n c(dx_n) \vartheta c(dx_n) + ic(dx_n) \vartheta c(\xi') \right] \right\} \\
\times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3} \\
= \frac{i(3\xi_n - i)}{2(\xi_n - i)^4(\xi_n + i)^3} \text{trace}[c(dx_n)\vartheta] \\
= -32\dim F \frac{1 + 3\xi_n i}{(\xi_n - i)^4} \text{trace}[\sigma_n^{F,e}]. 
\]
(5.38)

From (5.35), we obtain
\[
-i \int_{|\xi'| = 1}^{+\infty} \text{trace} \left[ (B_4 + B_2 + B_3) \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= -8\dim F h'(0) \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \frac{3\xi_n^4 + 9\xi_n^2 i + 21\xi_n - 5i}{(\xi_n - i)^4(\xi_n + i)^3} d\xi_n \sigma(\xi') dx' \\
= -8\dim F h'(0) \frac{2\pi i}{4!} \left[ \frac{3\xi_n^4 + 9\xi_n^2 i + 21\xi_n - 5i}{(\xi_n + i)^3} \right]^{(4)} |_{\xi_n = i} \Omega_4 dx' \\
= \frac{45}{2} \dim F \pi h'(0) \Omega_4 dx'. 
\]
(5.39)

From (5.38), we obtain
\[
-i \int_{|\xi'| = 1}^{+\infty} \text{trace} \left[ B_4 \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_F^* \hat{D}_F \hat{D}_F^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= 32\dim F \text{trace}[\sigma_n^{F,e}] \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \frac{1 + 3\xi_n}{(\xi_n - i)^4(\xi_n + i)^3} d\xi_n \sigma(\xi') dx' \\
= 32\dim F \text{trace}[\sigma_n^{F,e}] \left[ \frac{2\pi i}{3!} \left[ \frac{1 + 3\xi_n}{(\xi_n + i)^3} \right]^{(4)} \right] |_{\xi_n = i} \Omega_4 dx' \\
= -16\dim F \text{trace}[\sigma_n^{F,e}] \Omega_4 dx'. 
\]
(5.40)

Combining (5.19), (5.39) and (5.40), we have
\[
\text{case (b)} = \left[ \frac{45}{2} h'(0) - 16\text{trace}(\sigma_n^{F,e}) \right] \pi \dim F \Omega_4 dx'. 
\]
(5.41)

\[
\text{case (c)} r = -1, l = -4, |\alpha| = j = k = 0. 
\]

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By (5.2) and an integration by parts, we have

\[
\text{case (c)} = -i \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(f\tilde{D}_F^{-1}) \times \partial_{\xi_n} \sigma_{-4} \left( f^{-1}(\tilde{D}_F^*)^{-1} \cdot f \tilde{D}_F^{-1} \cdot f^{-1}(\tilde{D}_F^*)^{-1} \right) \right](x_0) \\
\times d\xi_n \sigma(\xi') dx'
\]

\[
= -i \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \sigma_{-4}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^*)^{-1} \right) \right](x_0) d\xi_n \sigma(\xi') dx'
\]

\[
+ 2i f^{-1} \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi)c(df)c(\xi)}{|\xi|^6} \right) \right](x_0) d\xi_n \sigma(\xi') dx',
\]  

(5.42)

By direct calculations, we have

\[
\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1}) = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}.
\]  

(5.43)

In the normal coordinate, \(g^{ij}(x_0) = \delta^i_j\) and \(\partial_{x_j}(g^{ij})(x_0) = h'(0)\delta^j_0\), if \(j < n; \partial_{x_j}(g^{kn})(x_0) = h'(0)\delta^j_0\), if \(j = n\). So by Lemma 2.2 in [12], we have \(\Gamma^k(x_0) = \frac{1}{2}h'(0)\) and \(\Gamma^k(x_0) = 0 \) for \(k < n\). By the definition of \(\hat{\delta}^k\) and Lemma 2.3 in [12], we have \(\sigma^k(x_0) = 0\) and \(\hat{\delta}^k = \frac{1}{2}h'(0)c(\hat{\epsilon}_k)c(\sigma_n)\) for \(k < n\). By (3.15) in [19], we obtain

\[
\sigma_{-4}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^*)^{-1}(x_0)
\]

\[
= \frac{-17 - 9\xi_n^2}{4(1 + \xi_n^2)} h'(0)c(\xi')c(dx_n) c(\xi') + \frac{33\xi_n^2 + 17\xi_n^3}{2(1 + \xi_n^2)^3} h'(0)c(\xi') + \frac{49\xi_n^2 + 25\xi_n^4}{2(1 + \xi_n^2)^4} h'(0)c(dx_n)
\]

\[
+ \frac{1}{1 + \xi_n^2} c(\xi') c(dx_n) \partial_{x_n}[c(\xi')(x_0)] - \frac{3\xi_n^2}{(1 + \xi_n^2)^3} \partial_{x_n}[c(\xi')(x_0)] - \frac{2\xi_n^2}{(1 + \xi_n^2)^3} h'(0)\xi_n c(\xi')
\]

\[
+ \frac{1 - \xi_n^2}{(1 + \xi_n^2)^3} h'(0) c(dx_n) + \frac{\xi_n d(\xi')}{\xi_n^6} + \frac{\xi_n d(\xi')}{\xi_n^6} + \frac{\xi_n d(\xi')}{\xi_n^6}.
\]  

(5.44)

Then

\[
-i \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \sigma_{-4}(\tilde{D}_F^* \tilde{D}_F \tilde{D}_F^*)^{-1} \right) \right](x_0) d\xi_n \sigma(\xi') dx'
\]

\[
- \frac{i}{2\pi} h'(0) \text{dim} F \Omega_4 dx' + 12\pi \text{trace} \left[ \sigma_{-4} c \right] \text{dim} F \Omega_4 dx' + 4\pi \text{trace} \left[ w(F, g^F)(e_n) \right] \text{dim} F \Omega_4 dx'
\]

\[
- 12\pi \text{trace} \left[ w(F, g^F)(e_n) \right] \text{dim} F \Omega_4 dx'.
\]  

(5.45)

By \(\sigma_{-1}(\tilde{D}_F^{-1}) = \sigma_{-1}(\tilde{D}_F^{-1})\), similar to case (b) in Section 3, and we get

\[
\text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi)c(df)c(\xi)}{|\xi|^6} \right) \right](x_0)
\]

\[
= \frac{(4\xi_n i + 2)i}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(\xi')c(df)] + \frac{4\xi_n i + 2}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{trace}[c(dx_n)c(df)].
\]

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and

\[
\text{trace}\left[ \pi^+_{\xi_n} \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{ic(\xi) \sum_j [c(dx_j) |\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1})c(\xi)}{|\xi|^6} \right) \right] (x_0)
\]

\[
= \frac{(3\xi_n - i)i}{(\xi_n + i)(1 + \xi_n)^4} \text{trace} \left[ c(\xi') \sum_j [c(dx_j) |\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) \right] + \frac{3\xi_n - i}{(\xi_n + i)(1 + \xi_n)^4} \text{trace} \left[ c(dx_n) \sum_j [c(dx_j) |\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) \right].
\]

(5.46)

By the relation of the Clifford action and \(\text{trace}QP = \text{trace}PQ\), then we have the following equalities

\[
\text{trace} \left[ c(dx_n)c(df) \right] = -g(dx_n, df);
\]

and

\[
\text{trace} \left[ c(dx_n) \sum_j [c(dx_j) |\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) \right] = \text{trace}( \text{id} )|\xi|^2 \left( -i\partial_{x_n}(f)f^{-1} \right) + 2 \sum_j \xi_j \xi_n \text{trace}( \text{id} ) \left( -i\partial_{x_j}(f)f^{-1} \right) = -64\text{dim}F |\xi|^2 \left( -i\partial_{x_n}(f)f^{-1} \right) + 2 \sum_j \xi_j \xi_n \text{trace}( \text{id} ) \left( -i\partial_{x_j}(f)f^{-1} \right).
\]

We note that \(i < n\), \(\int_{|\xi'|=1} \xi_n \sigma(\xi') = 0\), so \(\text{trace} [c(\xi')c(df)]\), \(\text{trace} \left[ c(\xi') \sum_j [c(dx_j) |\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) \right]\) and \(2i \sum_j \xi_j \xi_n \partial_{x_j}(f)f^{-1} \text{trace}[\text{id}]\) have no contribution for computing case (b). Then we obtain

\[
-2if^{-1} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{ic(\xi) \sum_j [c(dx_j) |\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1})c(\xi)}{|\xi|^6} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= \frac{3}{8f} \pi g(dx_n, df) \Omega_4 dx'.
\]

(5.47)

and

\[
-f \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_{-1}(\hat{D}_F^{-1}) \times \partial_{\xi_n} \left( \frac{ic(\xi) \sum_j [c(dx_j) |\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1})c(\xi)}{|\xi|^6} \right) \right]
\]

\[
\times (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= -60i\partial_{x_n}(f) \pi \text{dim}F \Omega_4 dx'.
\]

(5.48)

Then we have

\[
\text{case (c)} = \left\{ 12\text{trace} \left[ \sigma^F_n c \right] - \frac{129}{2} h''(0) + 4\text{trace} \left[ w(F, g^F)(e_n) \right] - 12\text{trace} \left[ w^*(F, g^F)(e_n) \right] - 60i\partial_{x_n}(f) \right\} \pi \text{dim}F \Omega_4 dx' + \frac{3}{8f} g(dx_n, df) \pi \Omega_4 dx'.
\]

(5.49)
Now $\Psi$ is the sum of the case (a), case (b) and case (c), then

$$\Psi = \left\{ 4\text{trace}\left[ w(F, g^F)(e_n) \right] - 37h'(0) - 4\text{trace}\left[ \sigma^{F, e}_n \right] - 12\text{trace}\left[ w^*(F, g^F)(e_n) \right] \right. $$

$$+ \left( \frac{19i}{22} + \frac{88}{f} - 60i \right) \partial_{x_n}(f) \right\} \pi \Omega_4 \dim F \, dx' + \frac{3}{8f} g(dx_n, df) \pi \Omega_4 \, dx'. \tag{5.50}$$

By (4.2) in [12], we have

$$K = \sum_{1 \leq i,j \leq n-1} K_{i,j}g_{\partial M}; K_{i,j} = -\Gamma^i_{i,j},$$

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. For $n = 6$, then

$$K(x_0) = \sum_{1 \leq i,j \leq n-1} K_{i,j}(x_0)g^{i,j}_{\partial M}(x_0) = \sum_{i=1}^5 K_{i,i}(x_0) = -\frac{5}{2} h'(0). \tag{5.51}$$

Hence we conclude that

**Theorem 5.1.** Let $M$ be a 6-dimensional compact manifolds with the boundary $\partial M$. Then

$$\text{Wres} \left[ \frac{\pi^+ (f \hat{D}_F^{-1}) \circ \pi^+ (f^{-1}(\hat{D}_F)^{-1}) \cdot f \hat{D}_F^{-1} \cdot f^{-1}(\hat{D}_F)^{-1})}{} \right]$$

$$= 8\pi^3 \int_M \left\{ \text{trace} \left[ -\frac{s}{12} + \frac{3}{8} \hat{c}(\omega^*) - \hat{c}(\omega) \right] - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) - \frac{1}{4} \sum_j \nabla_{\epsilon_j} \left( \hat{c}(\omega^*) \right) \hat{c}(\epsilon_j) \right.$$

$$+ \left. \frac{1}{4} \sum_j \hat{c}(\epsilon_j) \nabla_{\epsilon_j} \left( \hat{c}(\omega) \right) \right\} + 4f^{-1}\Delta(f) + 8 \langle \text{grad}_M(f), \text{grad}_M(f^{-1}) \rangle + 5f^{-2} \left[ |\text{grad}_M(f)|^2 \right. $$

$$+ 2\Delta(f) \right\} \text{dvol}_M + \int_{\partial M} \left\{ 4\text{trace}\left[ w(F, g^F)(e_n) \right] - 4\text{trace}\left[ \sigma^{F, e}_n \right] - 12\text{trace}\left[ w^*(F, g^F)(e_n) \right] $$

$$- 37h'(0) + \left( \frac{19i}{22} + \frac{88}{f} - 60i \right) \partial_{x_n}(f) \right\} \text{dim F} + \frac{3}{8f} g(dx_n, df) \right\} \pi \Omega_4 \text{dvol}_M. \tag{5.52}$$

where $s$ is the scalar curvature.

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