SUBALGEBRAS OF THE CUNTZ C*-ALGEBRA

ALAN HOPENWASSER AND JUSTIN R. PETERS

Abstract. In this paper we exploit the fact that a Cuntz C*-algebra is a groupoid C*-algebra to facilitate the study of non-self-adjoint subalgebras of $O_n$. The Cuntz groupoid is not principal and the spectral theorem for bimodules does not apply in full generality. We characterize the bimodules (over a natural masa) which are determined by their spectra in the Cuntz groupoid; these are exactly the ones which are invariant under the gauge automorphisms and exactly the ones which are generated by the Cuntz partial isometries which they contain.

We investigate analytic subalgebras of $O_n$, $n$ finite, by studying cocycles on the Cuntz groupoid. In contrast to AF groupoids, there are no cocycles which are integer valued or bounded and vanish precisely on the natural diagonal. $O_n$ contains a canonical UHF subalgebra; each strongly maximal triangular subalgebra of the UHF subalgebra has an extension to a strongly maximal triangular subalgebra of $O_n$ and each trivially analytic subalgebra of the UHF subalgebra has a proper analytic extension.

We also study the Volterra subalgebra of $O_n$. We identify the spectrum of the Volterra subalgebra and use this to prove a theorem of Power that the radical is equal to the closed commutator ideal. We also show that the Volterra subalgebra is maximal triangular but not strongly maximal triangular.

1. Introduction

Non-self-adjoint subalgebras of AF C*-algebras now have an extensive theory (see [10] for an introduction); subalgebras of other classes of C*-algebras are less well known. In 1985 Power [9] investigated the Volterra subalgebra of a Cuntz C*-algebra, a topic we shall revisit in this paper using groupoid techniques. More recently, Popescu made use of both the norm closed and the weak operator topology closed

Date: March 26, 2003.

Key words and phrases. Cuntz C*-algebra, Cuntz groupoid, Volterra subalgebra, analytic subalgebra, cocycle.

2000 Mathematics Subject Classification. Primary, 47L40; Secondary, 47L35.

The authors would like to thank Steve Power for helpful comments on the subject matter of this paper and for making a major contribution to Theorem 1.
algebras generated by \( n \) isometries with orthogonal range in his multi-variable version of the von Neumann inequality for a contraction \( \mathfrak{S} \).

And in \[1\], Davidson and Pitts introduced the systematic study of free semigroup algebras (the WOT closed algebra generated by \( n \) isometries with pairwise orthogonal ranges). See \[3\] for one continuation of this theme. When the range projections of the isometries add to the identity, the free semigroup algebra is a weakly closed subalgebra of the weak closure of some representation of \( O_n \); these algebras play an important role in the classification of certain representations of the Cuntz algebra.

By contrast, the subalgebras which we study are only norm closed. More importantly, these subalgebras are bimodules over a natural canonical masa in \( O_n \). (The non-self-adjoint norm closed algebra generated by the generators of \( O_n \) is not such a bimodule.) It is the bimodule property which makes our study amenable to groupoid techniques, so both our methods and our results have little in common with \[3, 4\]. Still, this paper and \[3, 4\] are part of a growing body of work on non-self-adjoint algebras related to the Cuntz and Cuntz-Krieger algebras.

The Cuntz \( C^* \)-algebra \( O_n \) is a groupoid \( C^* \)-algebra \[11, 7\]; the Cuntz groupoid has some similarities to (and one major difference from) the AF groupoids which determine AF \( C^* \)-algebras. In the spirit of much of the work on subalgebras of AF \( C^* \)-algebras, this paper uses groupoid techniques to investigate subalgebras of \( O_n \). The two principal topics studied are the Volterra subalgebra of \( O_n \) (we recover many of Power’s results with simpler proofs as well as a few new facts) and analytic subalgebras of \( O_n \). Analytic subalgebras of groupoid \( C^* \)-algebras are most conveniently defined in terms of cocycles on the groupoid (at the cost of obscuring connections with analyticity); accordingly, our study of analytic subalgebras of \( O_n \) focuses primarily on cocycles on the Cuntz groupoid. In this paper, \( n \) will always be finite.

Further on in the introduction we will give a brief review of those aspects of the Cuntz \( C^* \)-algebra which we need as well as a sketch of the relevant terminology concerning groupoids. In section \[2\] we will give a more detailed review of the Cuntz groupoid and its connection with \( O_n \). In section \[3\] we describe how to obtain a number of interesting subalgebras of \( O_n \) through the use of the Cuntz groupoid. It is here that the most striking difference from AF algebras appears. AF groupoids are \( r \)-discrete principal groupoids and so the spectral theorem for bimodules \[6\] is valid; this establishes a one-to-one correspondence between bimodules over a canonical masa and open subsets of the groupoid. As we show by example, this theorem is not valid in full generality for the
CUNTZ SUBALGEBRAS

In Theorem 1 we characterize which bimodules over a natural masa in $O_n$ do arise from open subsets of the Cuntz groupoid.

In section 4 we discuss cocycles on the Cuntz groupoid and their associated analytic algebras. The main tool is Theorem 2 which establishes a one-to-one correspondence between cocycles and continuous real-valued functions defined on the space of units of the groupoid. Theorem 4 illustrates further differences from the AF context. The most important cocycles used to define analytic subalgebras of AF $C^*$-algebras are either integer valued or bounded. We show, in contrast, that the Cuntz groupoid supports neither integer valued nor bounded cocycles which vanish precisely on the space of units.

The Cuntz $C^*$-algebra contains an $n^\infty$-UHF algebra in a natural way. We show in section 4 how any strongly maximal triangular subalgebra of the $n^\infty$-UHF subalgebra (with natural diagonal) is contained in a strongly maximal triangular subalgebra of $O_n$ (with the same diagonal). Furthermore, the refinement TAF subalgebra of the canonical UHF subalgebra (or any trivially analytic subalgebra, for that matter) has an extension to an analytic subalgebra of $O_n$.

Section 5 is devoted to the Volterra subalgebra of $O_n$. We define this algebra through its spectrum in the Cuntz groupoid and then show that it is equal to the intersection of a natural representation of $O_n$ acting on $L^2[0,1]$ and the usual Volterra nest subalgebra of $L^2[0,1]$. We identify the spectrum of the radical of the Volterra nest subalgebra of $O_n$ and use this to prove Power’s theorem that the radical is equal to the closed commutator ideal. The Volterra subalgebra provides an example of a maximal triangular subalgebra of $O_n$ which is not strongly maximal.

We conclude the introduction with some salient facts about $O_n$ and some terminology concerning groupoids. Cuntz proved in [1] that, up to isomorphism, there is only one $C^*$-algebra generated by $n$ isometries whose range projections are pairwise orthogonal and add to the identity operator. If $S_1, S_2, \ldots, S_n$ are generators of $O_n$, then $S_i^*S_j$ is either 0 (if $i \neq j$) or $I$ (if $i = j$). Using this, it is easy to see that any word in the $S_i$ and the $S_i^*$ can be written in the form $S_{\alpha_1}S_{\alpha_2} \cdots S_{\alpha_k}S_{\beta_j}^* \cdots S_{\beta_1}^*$ for some finite strings $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_j)$ whose coordinates are taken from $\{1,2,\ldots,n\}$.

If $\alpha$ and $\beta$ are finite strings, let $S_{\alpha}S_{\beta}^*$ denote the partial isometry $S_{\alpha_1}S_{\alpha_2} \cdots S_{\alpha_k}S_{\beta_j}^* \cdots S_{\beta_1}^*$. Either string, $\alpha$ or $\beta$ could be empty. If both are empty, $S_{\alpha}S_{\beta}^* = I$. The set of all $S_{\alpha}S_{\beta}^*$ forms an inverse semigroup (the Cuntz inverse semigroup generated by $S_1, \ldots, S_n$). Each element
of the Cuntz inverse semigroup will be referred to as a **Cuntz partial isometry**.

A particularly useful representation of the Cuntz C*-algebra acts on $L^2[0,1]$. If we partition $[0,1]$ into $n$ disjoint subintervals (ignoring endpoints), then the order preserving affine map from $[0,1]$ onto the $i$th subinterval induces an isometry $S_i$ on $L^2[0,1]$. The $S_i$ generate a copy of the Cuntz algebra.

Each partial isometry $S_\alpha S_\beta^*$ is induced by an order preserving affine map from some subinterval of $[0,1]$ onto some other subinterval; we often let these affine maps “stand in” for the partial isometries when discussing them. So, for example, $S_i$ is the partial isometry from $[0,1]$ onto the $i$th subinterval of $[0,1]$ – using the natural order of the indices to match the natural order of the subintervals. In this fashion, $S_{\alpha_1}\cdots S_{\alpha_j}$ is the partial isometry from $[0,1]$ onto the $j$th subinterval of the $i$th subinterval of $[0,1]$. (To be specific about endpoints, the $i$th subinterval of $[0,1]$ is $[\frac{i-1}{n}, \frac{i}{n}]$ and the $j$th subinterval of this is $[\frac{i-1}{n} + \frac{j-1}{n^2}, \frac{i-1}{n} + \frac{j}{n^2}]$.)

This latter “range” interval has length $1/n^2$. The adjoint, $S_{\alpha_1}\cdots S_{\alpha_j}^*$ is, of course, the partial isometry from the $j$th subinterval of the $i$th subinterval of $[0,1]$ onto $[0,1]$. Thus, the range interval for $S_\alpha$ is determined by looking at interval $\alpha_1$ in the partition of $[0,1]$ into $n$ parts, then at interval $\alpha_2$ in the partition of interval $\alpha_1$ into $n$ parts, then at interval $\alpha_3$ in the partition of the preceding interval into $n$ parts, etc. The domain of $S_\alpha^*$ is, of course, the same as the range of $S_\alpha$. Based on this model, (and using the notation $\ell(\alpha)$ to denote the length of $\alpha$, the number of coordinates in the string) if $\ell(\alpha) \geq \ell(\beta)$ we refer to $S_\alpha S_\beta^*$ as being **contractive** (the range interval is shorter than the domain interval); if $\ell(\alpha) \leq \ell(\beta)$ we say that $S_\alpha S_\beta^*$ is **expansive**.

The Volterra subalgebra of $L^2[0,1]$ is defined as the algebra of all operators which leave invariant each projection onto $L^2[0,x]$ viewed as a subspace of $L^2[0,1]$. Let $p_x$ denote this projection and $\mathcal{V}$ the nest of all such projections. Most of the $p_x$’s do not lie in $O_n$; in fact, $p_x \in O_n$ if, and only if, $x$ is an $n$-adic rational in $[0,1]$. However, $\{p_x \mid x \text{ is } n\text{-adic}\}$ is strongly dense in $\mathcal{V}$, so either nest can be used to determine the Volterra nest algebra.

Finally, we briefly review some groupoid terminology. A groupoid is a set $G$ with a partially defined multiplication and an inversion. If $a$ and $b$ can be multiplied, then $(a,b)$ is called a **composable pair**; $G^2$ denotes the set of all composable pairs. The multiplication satisfies an associative law and inversion satisfies $(a^{-1})^{-1} = a$. Elements of the form $a^{-1}a$ and $aa^{-1}$ are called **units**. Units act as left and right identities when multiplied by elements with which they are composable.
Groupoids have range and domain maps defined by $r(a) = aa^{-1}$ and $d(a) = a^{-1}a$. A subset $E \subseteq G$ is said to be a $G$-set if $r$ and $d$ are both one-to-one on $E$.

The Cuntz groupoid $G_n$ (described in detail in section 2) is a locally compact space whose topology is generated by a collection of compact open $G$-sets. The unit space is open, so $G_n$ is $r$-discrete (by definition); but $G_n$ is not a principal groupoid (i.e., it is not an equivalence relation on the unit space).

The groupoid $C^*$-algebra of $G_n$ is built from $C_c(G_n)$, the set of continuous functions on $G_n$ with compact support. A convolution type multiplication and an inversion turn $C_c(G_n)$ into a $^*$-algebra; the definitions of these operations specific to this context are given in section 2.

An auxiliary norm $\| \cdot \|_I$ is put on $C_c(G_n)$ and a $C^*$-norm is defined on $C_c(G_n)$ by $\| f \| = \sup_{\pi} \| \pi(f) \|$, where $\pi$ runs over all $^*$-representations of $C_c(G_n)$ which are decreasing with respect to $\| \cdot \|_I$. The groupoid $C^*$-algebra $C^*(G_n)$ is the completion of $C_c(G_n)$ with respect to the $C^*$-norm.

Extensive treatments of groupoids and their associated $C^*$-algebras can be found in [11] and [7]. We refer the reader to either of those monographs for details. Renault proves [11, Proposition 4.1] that for $r$-discrete groupoids, the $\| \cdot \|_\infty$-norm on $C_c(G)$ is dominated by the $C^*$-norm. As a consequence, any element of $C^*(G)$ can be identified with a continuous function on $G$ which vanishes at infinity. Caveat: not every $C_0$ function on $G$ is associated with an element of the groupoid $C^*$-algebra. We shall make extensive use of this identification throughout this paper; in particular, elements of $C^*(G_n)$ will routinely be viewed as continuous functions on $G_n$ (vanishing at infinity). Both [11] and [7] identify $C^*(G_n)$ as $O_n$; all the same, for the convenience of the reader we indicate the connection between $C^*(G_n)$ and $O_n$ in section 2.

2. Review of the Cuntz groupoid

The following description of the groupoid $G_n$ for the Cuntz algebra, $O_n$ is taken from Paterson’s book [7]. (An earlier description of this groupoid is in Renault’s book [11].) Much of what we outline in this section appears in more general form in [5]. The material here is sufficient for subsequent sections of this paper and should prove convenient for readers unfamiliar with the Cuntz groupoid.

Let $X = \prod_{i=1}^{\infty} \{1, 2, \ldots, n\}$; equip $X$ with the product topology.
For a finite string $\alpha$ of digits in $\{1, 2, \ldots, n\}$, let $\ell(\alpha)$ denote the length of the string. If $\alpha$ is a finite sequence and $\gamma$ is an element of $X$, then $\alpha\gamma$ denotes the element of $X$ obtained by ‘prepending’ $\alpha$ to $\gamma$. As a set, the Cuntz groupoid is

$$G_n = \{(\alpha\gamma, \ell(\alpha) - \ell(\beta), \beta\gamma) \mid \alpha \text{ and } \beta \text{ are finite strings and } \gamma \in X\}.$$ 

Two elements $(x, k, y)$ and $(w, j, z)$ of $G_n$ are composable if, and only if, $y = w$. Multiplication and inversion in the groupoid are given by the formulas:

$$(x, k, y)(j, k, z) = (x, k + j, z) \quad \text{and} \quad (x, k, y)^{-1} = (y, -k, x).$$

Multiplication in $C^*(G_n)$ is given by a convolution formula:

$$f \ast g(x, k, y) = \sum f(x, k_1, z)g(z, k_2, y)$$

The sum is taken over all $k_1, k_2 \in \mathbb{Z}$ and $z \in X$ such that $(x, k_1, z)$ and $(z, k_2, y)$ are in $G_n$ and $k_1 + k_2 = k$. Initially, this formula is used in $C_c(G_n)$; in this context all but finitely many terms in the sum are zero. But the same formula is also valid for all those functions in $C_0(G_n)$ which correspond to elements of $C^*(G_n)$. The adjoint is given by the formula

$$f^*(x, k, y) = \overline{f(y, -k, x)}$$

The Cuntz groupoid is a locally compact groupoid. The sets

$$U_{\alpha, \beta} = \{(\alpha\gamma, \ell(\alpha) - \ell(\beta), \beta\gamma) \mid \gamma \in X\}.$$ 

form a basis for the topology.

A neighborhood basis at a point $(x, k, y) \in G_n$ can be obtained in the following way. Write $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$. There is a positive integer $p$ such that $y_j = x_{j+k}$ for all $j \geq p$. We may as well assume that $p$ is the smallest integer which satisfies this property (although this is not essential). When $k < 0$, we also assume that $-k < p$, so that $j + k \geq 1$ when $j \geq p$. For each $j \geq p$, let

$$\alpha^j = (x_1, \ldots, x_{j+k}),$$

$$\beta^j = (y_1, \ldots, y_j).$$

Then $\ell(\alpha^j) - \ell(\beta^j) = j + k - j = k$. If

$$\gamma^j = (x_{j+k+1}, x_{j+k+2}, \ldots) = (y_{j+1}, y_{j+2}, \ldots),$$

then $(x, k, y) = (\alpha^j \gamma^j, \ell(\alpha^j) - \ell(\beta^j), \beta^j \gamma^j)$. So, $(x, k, y) \in U_{\alpha^j, \beta^j}$, for all $j \geq p$.

On the other hand, let $(\overline{x}, k, \overline{y}) \in G_n$ and suppose that $\overline{x} \neq x$. Then, for some $i$, $\overline{x}_i \neq x_i$. Consequently, for all $j > i$, $(\overline{x}, k, \overline{y}) \notin U_{\alpha^j, \beta^j}$. 

Similarly, if \( \overline{y} \neq y \) or, even more easily, if \( \overline{k} \neq k \), then \( (x, \overline{k}, \overline{y}) \notin U_{\alpha^j, \beta^j} \), for large \( j \). The conclusion is that

\[
\{(x, k, y)\} = \cap_{j=p}^\infty U_{\alpha^j, \beta^j},
\]

so \( \{U_{\alpha^j, \beta^j} \mid j \geq p\} \) forms a neighborhood basis at \( (x, k, y) \).

If we restrict the topology to a basic open set \( U_{\alpha, \beta} \), the association

\[
\gamma \longleftrightarrow (\alpha \gamma, \ell(\alpha) - \ell(\beta), \beta \gamma)
\]

is a homeomorphism between \( X \) and \( U_{\alpha, \beta} \). In particular, each \( U_{\alpha, \beta} \) is compact in the topology on \( G_n \).

The groupoid C*-algebra of the Cuntz groupoid \( G_n \) is the Cuntz C*-algebra \( O_n \) \[11, 7\]. The rest of this section is intended to illuminate the connection between the groupoid and the C*-algebra.

Fix

\[
\alpha = (\alpha_1, \ldots, \alpha_a),
\beta = (\beta_1, \ldots, \beta_b),
\gamma = (\gamma_1, \ldots, \gamma_c),
\delta = (\delta_1, \ldots, \delta_d).
\]

Suppose \( b < c \) and let \( \gamma = (\gamma_{b+1}, \ldots, \gamma_c) \). Since

\[
S_i^*S_j = \begin{cases} 0, & \text{if } i \neq j, \\ I, & \text{if } i = j, \end{cases}
\]

we have

\[
S_\alpha S_\gamma S_\beta S_\delta = \begin{cases} S_\alpha S_\gamma S_\delta^*, & \text{if } (\beta_1, \ldots, \beta_b) = (\gamma_1, \ldots, \gamma_b), \\ 0, & \text{if } (\beta_1, \ldots, \beta_b) \neq (\gamma_1, \ldots, \gamma_b). \end{cases}
\]

In the following, \( \chi_{\alpha, \beta} \) will denote the characteristic function of \( U_{\alpha, \beta} \). This is a continuous function on \( G_n \) with compact support. The convolution, \( \chi_{\alpha, \beta} \ast \chi_{\gamma, \delta} \) is given by the formula

\[
\chi_{\alpha, \beta} \ast \chi_{\gamma, \delta}(x, k, y) = \sum \chi_{\alpha, \beta}(x, i, u) \chi_{\gamma, \delta}(u, j, y),
\]

where \( i + j = k \) and \( u, i, \) and \( j \) run through the countably many choices yielding composable elements of \( G_n \).

Suppose \( u \) is such that \( \chi_{\alpha, \beta}(x, i, u) \chi_{\gamma, \delta}(u, j, y) \neq 0 \). Then

\[
(u_1, \ldots, u_b) = (\beta_1, \ldots, \beta_b) \quad \text{and} \quad (u_1, \ldots, u_c) = (\gamma_1, \ldots, \gamma_c).
\]

With the assumption \( b < c \) in force, this yields, \( (\beta_1, \ldots, \beta_b) = (\gamma_1, \ldots, \gamma_b) \).
Thus, if \((\beta_1, \ldots, \beta_b) \neq (\gamma_1, \ldots, \gamma_b)\), then every term in the convolution sum is zero and we have the correspondence

\[
0 = \chi_{\alpha, \beta} * \chi_{\gamma, \delta} \longleftrightarrow S_{\alpha} S_{\beta}^* S_{\gamma} S_{\delta}^* = 0.
\]

So assume that \((\beta_1, \ldots, \beta_b) = (\gamma_1, \ldots, \gamma_b)\). Recall \(\gamma = (\gamma_{b+1}, \ldots, \gamma_c)\) and \(\ell(\gamma) = c - b\). Suppose \(i, j\) and \(u\) are such that \(\chi_{\alpha, \beta}(x, i, u) \chi_{\gamma, \delta}(u, j, y) \neq 0\). This forces

\[
i = \ell(\alpha) - \ell(\beta) = a - b,
\]

and hence

\[
i + j = a + c - b - d = \ell(\alpha) + \ell(\gamma) - \ell(\delta) = \ell(\alpha \gamma) - \ell(\delta).
\]

Also, as before,

\[
(u_1, \ldots, u_b) = (\beta_1, \ldots, \beta_b), \quad \text{and} \quad (u_1, \ldots, u_c) = (\gamma_1, \ldots, \gamma_c).
\]

In addition,

\[
(x_{a+1}, x_{a+2}, \ldots) = (u_{b+1}, u_{b+2}, \ldots), \quad \text{and} \quad (u_{c+1}, u_{c+2}, \ldots) = (y_{d+1}, y_{d+2}, \ldots).
\]

Hence,

\[
(x_{a+1}, \ldots, x_{a+c-b}) = (u_{b+1}, \ldots, u_c) = (\gamma_{b+1}, \ldots, \gamma_c) = \gamma.
\]

Letting \(\eta = (x_{a+c-b+1}, \ldots)\), we see that \(x = \alpha \gamma \eta\). Now

\[
y = (y_1, \ldots, y_d) = (\delta_1, \ldots, \delta_d) = \delta
\]

and

\[
\eta = (x_{a+c-b+1}, \ldots) = (u_{b+c-b+1}, \ldots) = (u_{c+1}, \ldots) = (y_{d+1}, \ldots),
\]

so \(y = \delta \eta\).

Since \(u\) is completely determined by these conditions, there is only one term in

\[
\sum \chi_{\alpha, \beta}(x, i, u) \chi_{\gamma, \delta}(u, j, y)
\]

which is non-zero if the sum is non-zero. This happens only if \(x = \alpha \gamma \eta\) and \(y = \delta \eta\).

Conversely, if \(x = \alpha \gamma \eta\) and \(y = \delta \eta\), for some \(\eta\) in \(X\), then choose \(u\) to be \(\beta \gamma \eta\); for this choice of \(u\)

\[
\chi_{\alpha, \beta}(x, a - b, u) \chi_{\gamma, \delta}(u, c - d, y) = 1.
\]
(No other choice of \( u \) yields a non-zero value for this product.) Thus, 
\[
\chi_{\alpha, \beta} \ast \chi_{\gamma, \delta}(x, k, y) = 1 \iff \begin{cases} 
    k = \ell(\alpha \gamma) - \ell(\delta), \\
    x = \alpha \gamma \eta, \\
    y = \delta \eta,
\end{cases}
\]
for some \( \eta \in X \).

Thus, 
\[
\chi_{\alpha, \beta} \ast \chi_{\gamma, \delta}(x, k, y) = 1 \iff \chi_{\alpha \gamma, \delta}(x, k, y) = 1
\]
and 
\[
\chi_{\alpha, \beta} \ast \chi_{\gamma, \delta}(x, k, y) = 0 \iff \chi_{\alpha \gamma, \delta}(x, k, y) = 0.
\]
The only possible values for \( \chi_{\alpha, \beta} \ast \chi_{\gamma, \delta} \) are 0 and 1, so \( \chi_{\alpha, \beta} \ast \chi_{\gamma, \delta} = \chi_{\alpha \gamma, \delta} \).
Thus we have the correspondence 
\[
\chi_{\alpha, \beta} \ast \chi_{\gamma, \delta} \leftrightarrow S_{\alpha} S_{\gamma} S_{\delta}^* = S_{\alpha} S_{\beta}^* S_{\gamma} S_{\delta}^*.
\]

In the case in which \( b = c \), omit the \( \gamma \). If \( b > c \), a similar argument yields the correspondence between \( \chi_{\alpha, \beta} \ast \chi_{\gamma, \delta} \) and \( S_{\alpha} S_{\beta}^* S_{\gamma} S_{\delta}^* \). It is now clear that \( \chi_{1,0}, \ldots, \chi_{n,0} \) generate all \( \chi_{\alpha, \beta} \) in \( C^*(G_n) \) and that \( \chi_{1,0}, \ldots, \chi_{n,0} \) are isometries whose range projections sum to \( I \). This illuminates the identification of \( C^*(G_n) \) as \( O_n \).

### 3. The spectral theorem for bimodules

In this section we show how to associate subalgebras of \( O_n \) to certain subsets of the Cuntz groupoid. The spectral theorem for bimodules, proven in [6] for principal groupoids, is extremely useful in the study of non-self adjoint subalgebras of groupoid \( C^* \)-algebras; however, this theorem is not valid in full generality for the Cuntz groupoid. In Theorem 1 we characterize those bimodules over a natural masa for which the spectral theorem is valid. In what follows we assume all bimodules are norm closed.

Keep in mind in the sequel that we view elements of \( C^*(G_n) = O_n \) as continuous functions on \( G_n \) which vanish at infinity. Also, let \( D_0 \) denote the space of units in \( G_n \); i.e., \( D_0 = \{(x, 0, x) \mid x \in X\} \). This is an open subset of \( G_n \); in fact, it is \( U_{\alpha, \beta} \), where both \( \alpha \) and \( \beta \) are empty strings.

If \( P \) is an open subset of \( G_n \), let 
\[
A(P) = \{ f \in C^*(G_n) \mid \text{supp } f \subseteq P \}.
\]
If \( f \in A(P) \), then \( f(x, k, y) = 0 \) for all \( (x, k, y) \in G_n \setminus P \). For any open set \( P \) in \( G_n \), \( A(P) \) is a bimodule over \( A(D_0) \). This is easy to see: when \( f \in A(D_0) \), \( f \) is supported on \( D_0 \), so 
\[
f \ast g(x, k, y) = f(x, 0, x) g(x, k, y)
\]
and
\[ g \ast f(x, k, y) = g(x, k, y) f(y, 0, y). \]
If \( g \) is supported in \( P \) and \( f \in A(D_0) \), then \( f \ast g \) and \( g \ast f \) are also supported in \( P \).

If, in addition, \( P \) satisfies the property
\[ (x, k, y) \in P \text{ and } (y, j, z) \in P \implies (x, k + j, z) \in P, \]
then \( A(P) \) is a subalgebra of \( O_n \). Also, if \( D_0 \subseteq P \), then \( A(D_0) \subseteq A(P) \).

One choice for \( P \) is \( D_0 \). We shall reprove in Proposition 2 the known fact that \( A(D_0) \) is a masa in \( O_n \). Another choice for \( P \) is
\[ P_{UHF} = \{(x, k, y) \in G_n \mid k = 0\}. \]
Then \( A(P_{UHF}) \) is a subalgebra of \( O_n \) which contains \( A(D_0) \); the first part of Theorem 1 shows that \( A(P_{UHF}) \) is generated by the Cuntz partial isometries which it contains. But these are just the \( S_\alpha S_\beta^* \) with \( \ell(\alpha) = \ell(\beta) \); it is well known that the subalgebra of \( O_n \) generated by these Cuntz partial isometries is an \( n^\infty \)-UHF algebra. We shall refer to this subalgebra as the canonical UHF subalgebra of \( O_n \). Note also that there is an obvious isomorphism between \( P_{UHF} \) and the usual groupoid for the \( n^\infty \)-UHF algebra.

If \( B \subseteq O_n \) is an \( A(D_0) \) bimodule, define
\[ \sigma(B) = \{(x, k, y) \mid \text{there is } f \in B \text{ with } f(x, k, y) \neq 0\}. \]
In other words, \( G_n \setminus \sigma(B) = \{(x, k, y) \mid f(x, k, y) = 0 \text{ for all } f \in B\} \).
Clearly, \( \sigma(B) \) is an open subset of \( G_n \).

**Proposition 1.** If \( P \) is an open subset of \( G_n \), then \( \sigma(A(P)) = P \).

**Proof.** Suppose that \( P \) is an open subset of \( G_n \). If \( (x, k, y) \notin P \), then \( f(x, k, y) = 0 \) for all \( f \in A(P) \), so \( (x, k, y) \in G_n \setminus \sigma(A(P)) \). Thus \( \sigma(A(P)) \subseteq P \). On the other hand, suppose that \( (x, k, y) \in P \). Since \( P \) is open, there are \( \alpha \) and \( \beta \) such that \( \ell(\alpha) - \ell(\beta) = k \) and \( (x, k, y) \in U_{\alpha,\beta} \subseteq P \). Now \( \chi_{\alpha,\beta} \in C^*(G_n) \) (and corresponds to the partial isometry \( S_\alpha S_\beta^* \)). Since \( \text{supp } \chi_{\alpha,\beta} \subseteq P \), \( \chi_{\alpha,\beta} \in A(P) \). But \( \chi_{\alpha,\beta}(x, k, y) = 1 \), so \( (x, k, y) \in \sigma(A(P)) \). This shows that \( P \subseteq \sigma(A(P)) \). \( \square \)

**Definition 1.** A bimodule over \( A(D_0) \) is said to be reflexive if \( B = A(\sigma(B)) \).

Note that the reflexive bimodules over \( A(D_0) \) are exactly the ones of the form \( A(P) \) for some open set \( P \subseteq G_n \). If the spectral theorem for bimodules over \( A(D_0) \) were valid in full generality, then every bimodule over \( A(D_0) \) would be reflexive. The following example gives a bimodule
which is not reflexive. The authors thank Steve Power for drawing this example to their attention.

**Example 1.** Let \( P = D_0 \cup U_{(1),0} \). Note that \( D_0 \) is the support set for \( I \) and \( U_{(1),0} \) is the support set for \( S_1 \). Let \( \mathbf{T} \) denote the element \((1,1,1,\ldots)\) in \( X \). Both \((\mathbf{T},0,\mathbf{T})\) and \((\mathbf{T},1,\mathbf{T})\) are elements of \( P \). Let

\[
B = \{ f \in O_n \mid \text{supp } f \subseteq P \text{ and } f(\mathbf{T},0,\mathbf{T}) = f(\mathbf{T},1,\mathbf{T}) \}.
\]

If \( f \in B \) and \( g \in A(D_0) \), then

\[
g \ast f(\mathbf{T},0,\mathbf{T}) = g(\mathbf{T},0,\mathbf{T})f(\mathbf{T},0,\mathbf{T}) = g(\mathbf{T},0,\mathbf{T})f(\mathbf{T},1,\mathbf{T}) = g \ast f(\mathbf{T},1,\mathbf{T}).
\]

Similarly \( f \ast g(\mathbf{T},0,\mathbf{T}) = f \ast g(\mathbf{T},1,\mathbf{T}) \); thus \( B \) is a bimodule. From the definition of \( B \) it is clear that \( \sigma(B) \subseteq P \). But the characteristic function of \( P \) is in \( B \), so \( P \subseteq \sigma(B) \). Thus \( \sigma(B) = P \) and it is trivial that \( B \neq A(P) \).

Note that \( B \) is the bimodule over \( A(D_0) \) generated by \( I + S_1 \).

The next theorem characterizes the reflexive bimodules over \( A(D_0) \). One of the conditions equivalent to reflexivity is invariance under the gauge automorphisms. For each complex number \( \lambda \) of absolute value one, a gauge automorphism of \( O_n \) is determined by its action on the generators: \( \eta_{\lambda}(S_i) = \lambda S_i \). The authors thank Steve Power for pointing out condition (3) and providing a proof that (3) implies (2).

**Theorem 1** (Spectral Theorem for Bimodules). Let \( B \) be a bimodule over \( A(D_0) \). Then the following are equivalent:

1. \( B \) is reflexive.
2. \( B \) is generated by the Cuntz partial isometries which it contains.
3. \( B \) is invariant under all the gauge automorphisms.

**Proof.** We first show (1) implies (2); i.e., any bimodule of the form \( A(P) \) with \( P \) an open subset of \( G_n \) is generated by the Cuntz partial isometries which it contains. Let \( B \) be the bimodule generated by the Cuntz partial isometries in \( A(P) \). Suppose that \( \chi_{\alpha,\beta} \) is a Cuntz partial isometry in \( A(P) \), so that \( U_{\alpha,\beta} \subseteq P \). Let \( f \) be any continuous function supported on \( U_{\alpha,\beta} \). Define a function \( g \) on \( U_{\alpha,\alpha} \) by \( g(\alpha\gamma,0,\alpha\gamma) = f(\alpha\gamma,\ell(\alpha) - \ell(\beta),\beta\gamma) \). Observe that \( f = g \ast \chi_{\alpha,\beta} \). Since \( g \in A(D_0) \), we have \( f \in B \). Thus, \( B \) contains any continuous function supported on a compact open set of the form \( U_{\alpha,\beta} \subseteq P \). Since any compact open subset of \( P \) can be written as a finite union of sets of the form \( U_{\alpha,\beta} \), \( B \) contains any continuous function supported on a compact open subset.
of $P$. Any compact set is contained in a compact open set, so all continuous functions with compact support in $P$ are in $B$. But these are dense (in the $C^*$-norm) in $A(P)$, so $A(P) = B$.

To prove that (2) implies (1), suppose that $B$ is an $A(D_0)$ bimodule and that $B$ is generated by the Cuntz partial isometries which it contains. Let $P = \bigcup U_{\alpha,\beta}$, where the union is taken over all $\alpha, \beta$ such that $\chi_{\alpha,\beta}$ is in $B$. It is obvious that $P \subseteq \sigma(B)$. To see the reverse containment, suppose that $(x, k, y) \notin P$. Then $\chi_{\alpha,\beta}(x, k, y) = 0$ for all Cuntz partial isometries $\chi_{\alpha,\beta}$ in $B$. If $f$ and $g$ are any elements of $A(D_0)$, then $f * \chi_{\alpha,\beta} * g$ also vanishes at $(x, k, y)$, since $f * \chi_{\alpha,\beta} * g(x, k, y) = f(x, 0, x)\chi_{\alpha,\beta}(x, k, y)g(y, 0, y)$. It follows that any element of the bimodule generated by these $\chi_{\alpha,\beta}$, i.e., any element of $B$, also vanishes at $(x, k, y)$. Thus $(x, k, y) \notin \sigma(B)$.

We now know that $P = \sigma(B)$, so $B \subseteq A(P)$. It remains to show that $A(P) = B$. If $\alpha$ and $\overline{\sigma}$ are finite strings with $\ell(\alpha) < \ell(\overline{\alpha})$ and $\alpha_j = \overline{\alpha}_j$ for $j = 1, \ldots, \ell(\alpha)$, we shall call $\overline{\alpha}$ an extension of $\alpha$. If $\overline{\alpha}$ is an extension of $\alpha$ and $\overline{\beta}$ is an extension of $\beta$ and if $\ell(\overline{\alpha}) - \ell(\overline{\beta}) = \ell(\alpha) - \ell(\beta)$, then $U_{\overline{\alpha},\overline{\beta}} \subseteq U_{\alpha,\beta}$. Since $\chi_{\overline{\alpha},\overline{\beta}} = \chi_{\alpha,\overline{\alpha}} \chi_{\alpha,\beta} \chi_{\beta,\overline{\beta}}$, if $\chi_{\alpha,\beta}$ is in $B$ then so is $\chi_{\overline{\alpha},\overline{\beta}}$.

To prove that $A(P) \subseteq B$, it suffices to show that if $\chi_{\gamma,\delta}$ is a Cuntz partial isometry in $A(P)$, then $\chi_{\gamma,\delta}$ is in $B$. (We have already seen that $A(P)$ is generated by the Cuntz partial isometries which it contains.) Let $(x, k, y) \in U_{\gamma,\delta}$. Since $P = \sigma(B)$, there is a Cuntz partial isometry $\chi_{\alpha,\beta}$ in $B$ such that $(x, k, y) \in U_{\alpha,\beta}$. For some extensions $\overline{\alpha}$ of $\alpha$ and $\overline{\beta}$ of $\beta$ with $\ell(\overline{\alpha}) - \ell(\overline{\beta}) = k$, we have $(x, k, y) \in U_{\overline{\alpha},\overline{\beta}} \subseteq U_{\alpha,\beta} \cap U_{\gamma,\delta}$ and, as noted above, $\chi_{\overline{\alpha},\overline{\beta}} \in B$.

Since $U_{\gamma,\delta}$ is compact, we can write it as a finite disjoint union of sets of the form $U_{\overline{\alpha},\overline{\beta}}$ with $\chi_{\overline{\alpha},\overline{\beta}} \in B$. But $\chi_{\gamma,\delta}$ is the sum of the corresponding $\chi_{\overline{\alpha},\overline{\beta}}$; thus $\chi_{\gamma,\delta} \in B$ and we have shown that $A(P) = B$.

(2) implies (3) is trivial: a gauge automorphism maps a Cuntz partial isometry to a scalar multiple of itself. It remains only to prove that (3) implies (2). Cuntz showed in [1] that each element of $O_n$ has a Fourier series with respect to a chosen generator, say $S_1$:

$$a \sim \sum_{-\infty}^{-1} (S_1^*)^{|k|} a_k + \sum_{0}^{\infty} a_k S_1^k.$$

The coefficients $a_k$ all lie in the canonical UHF subalgebra. If $\eta_{\lambda}$ is a gauge automorphism, then

$$\eta_{\lambda}(a) \sim \sum_{-\infty}^{-1} (S_1^*)^{|k|} \lambda^k a_k + \sum_{0}^{\infty} \lambda^k a_k S_1^k.$$
By Remark 1 of section 1.10 of [1], there is a Cesaro convergence of generalized polynomials and, for \( k \geq 0 \),
\[
a_k S^k_1 = \int_{\mathbb{T}} \lambda^k \eta_\lambda (a) d\lambda,
\]
where \( d\lambda \) is normalized Lebesgue measure on \( \mathbb{T} \). A similar formula holds when \( k \) is negative. Thus, if \( a \in B \), then all \( a_k S^k_1 \) and all \( (S^*_1)^k a_k \) lie in \( B \). Note that we may, without loss of generality, assume that each coefficient \( a_k \) satisfies \( a_k = a_k S^k_1 (S^*_1)^k \) when \( k \geq 0 \), with a similar assertion when \( k < 0 \).

Let
\[
B_k = \{ b \in A(\mathcal{P}_{UHF}) \mid b S^k_1 \in B, b = b S^k_1 (S^*_1)^k \}
\]
when \( k \geq 0 \); a similar definition is used when \( k < 0 \). So \( B \) is the closed linear span of all the subspaces \( B_k S^k_1 \) and \( (S^*_1)^k B_{-k} \). It is trivial that \( B_k \) is a left bimodule over \( A(D_0) \) and easy to see that \( B_k \) is a right bimodule. (It suffices to prove this for right multiplication by projections in \( A(D_0) \) and we can limit ourselves to subprojections of the range projection for \( S^k_1 \). But if \( p \) is a subprojection of \( S^k_1 (S^*_1)^k \), then there is a subprojection \( q \) of the domain projection for \( S^k_1 \) such that \( p S^k_1 = S^k_1 q \). For any \( b \in B_k \), \( bp S^k_1 = b S^k_1 q \in B \) and so \( bp \in B_k \).)

Now \( B_k \) is a bimodule over \( A(D_0) \) contained in the canonical UHF algebra \( A(\mathcal{P}_{UHF}) \) and the spectral theorem for bimodules is valid in full generality in this context. So \( B_k \) is generated by the Cuntz partial isometries which it contains. It follows that the same is true for \( B \). □

It is shown in [2, Remark 2.1.8] that \( A(D_0) \) is a masa in \( O_n \); for the convenience of the reader, we reprove this using groupoid techniques.

**Proposition 2.** \( A(D_0) \) is a masa in \( C^*(G_n) = O_n \).

**Proof.** Suppose that \( g \in O_n \) and \( f * g = g * f \), for all \( f \in A(D_0) \). We must show that \( g \in A(D_0) \). When \( f \in A(D_0) \), \( f \) is supported on \( D_0 \), so
\[
f * g(x, k, y) = f(x, 0, x) g(x, k, y)
\]
and
\[
g * f(x, k, y) = g(x, k, y) f(y, 0, y).
\]
If \( x \neq y \), we can find \( f \in A(D_0) \) such that \( f(x, 0, x) \neq f(y, 0, y) \), hence \( g(x, k, y) = 0 \) when \( x \neq y \). If \( k \neq 0 \) then \( \{(x, k, y) \in G_n \mid x = y \} \) has empty interior and so cannot support a non-zero continuous function. Thus, if \( g \) is continuous on \( G_n \) and \( g(x, k, y) = 0 \) whenever \( x \neq y \) and \( k \neq 0 \), then \( g(x, k, x) = 0 \) for all \( x \).
We have shown that if \( g \) commutes with \( f \) then \( g(x, k, y) = 0 \) whenever \( x \neq y \) or \( k \neq 0 \). This means that \( g \) is supported on \( D_0 \) and \( A(D_0) \) is a masa in \( C^*(G_n) \).

\[ \square \]

4. Analytic subalgebras of \( O_n \)

**Definition 2.** A **cocycle** is a continuous function \( d \) from \( G_n \) to \( \mathbb{R} \) which satisfies the identity

\[ d(x, k, y) + d(y, l, z) = d(x, k + l, z) \]

for \((x, k, y)\) and \((y, l, z)\) in \( G_n \).

If \( P = \{(x, k, y) \mid d(x, k, y) \geq 0\} \) is open, then we call \( A(P) \) the **analytic subalgebra** associated with \( d \).

To avoid trivialities, we always assume that \( d \) is not identically equal to 0.

**Remark 1.** There is a standard procedure for associating a one-parameter family of automorphisms to a cocycle \( d \). For each \( t \in \mathbb{R} \), an automorphism \( \eta_t \) is given by the formula:

\[ (\eta_t f)(x, k, y) = e^{itd(x, k, y)} f(x, k, y). \]

Each \( \eta_t \) is a \(*\)-automorphism of \( C_c(G_n) \) onto itself; it is not hard to show that this automorphism preserves the \( C^* \)-norm and so extends to an automorphism of \( O_n \) with the formula above. For each point \((x, k, y) \in G_n \), the map \( f \mapsto f(x, k, y) \) is decreasing with respect to the \( \| \|_\infty \), and hence also norm decreasing with respect to the \( C^* \)-norm. So these maps are continuous linear functionals on \( O_n \). If \( f \in O_n \), we consider all functions of the form \( t \mapsto \rho(\eta_t(f)) \), where \( \rho \) is a linear functional of the type above. Given \( f \in O_n \), it is easy to check that \( t \mapsto \rho(\eta_t(f)) \) is an \( H^\infty \)-function on \( \mathbb{R} \) for all linear functional of this form if, and only if, \( f \) is supported on \( \{(x, k, y) \mid d(x, k, y) \geq 0\} \).

This indicates why the term **analytic subalgebra** is appropriate in the definition above.

**Example 2.** Consider the **dilation cocycle** \( d(x, k, y) = k \). Since each Cuntz partial isometry \( S_\alpha S_\beta^* \) corresponds to the function \( \chi_{\alpha, \beta} \) on the Cuntz groupoid, \( \eta_t(S_\alpha S_\beta^*) = e^{it(\ell(\alpha) - \ell(\beta))} S_\alpha S_\beta^* \). In particular, for each generating isometry \( S_k \), \( \eta_t(S_k) = e^{it} S_k \). This determines the automorphism \( \eta_t \). The dilation cocycle gives rise to the gauge automorphisms which appeared in Theorem \[ \square \]

The backward shift map on \( X = \prod_{1}^{\infty} \{1, \ldots, n\} \) is a useful tool in the study of cocycles on \( G_n \). Define \( S : X \to X \) by \( S(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots) \).
Now let \((x, 0, y) \in P_{UHF}\) and \(k \geq 0\) be given. Then \((x, k, S^k y) \in G_n\). Conversely, if \((x, k, z) \in G_n\) with \(k \geq 0\), there is a \(y \in X\) such that \(z = S^k y\) and \((x, 0, y) \in P_{UHF}\). Indeed, we can take \(y = \alpha z\) where \(\alpha\) is an arbitrary \(k\)-tuple. If \(k > 0\) and \((x, 0, y) \in P_{UHF}\), then \((S^k x, -k, y) \in G_n\). If \((z, -k, y) \in G_n\) with \(k > 0\), then there is \(x \in X\) such that \(z = S^k x\) and \((x, 0, y) \in G_n\).

**Theorem 2.** If \(d\) is a cocycle on \(G_n\), define a continuous real valued function \(f_d\) on \(X\) by

\[
f_d(x) = d(x, 1, Sx), \quad \text{for all } x \in X.
\]

If \(f : X \to \mathbb{R}\) is continuous, define a cocycle \(d_f\) on \(G_n\) by

\[
(1) \quad d_f(x, k, S^k y) = \sum_{j=0}^{k-1} f(S^j x) + \sum_{j=k}^{\infty} [f(S^j x) - f(S^j y)],
\]

\[
(2) \quad d_f(S^k x, -k, y) = -\sum_{j=0}^{k-1} f(S^j y) + \sum_{j=k}^{\infty} [f(S^j x) - f(S^j y)],
\]

where \(k \geq 0\) and \((x, 0, y) \in P_{UHF}\).

These two correspondences are inverse to one another. Thus, there is a one-to-one correspondence between continuous cocycles on \(G_n\) and continuous real valued functions on \(X\).

**Proof.** The continuity of \(f_d\) follows from the continuity of \(d\) and of the map \(x \mapsto (x, 1, Sx)\).

Now suppose that \(f\) is a continuous real valued map on \(X\). By the remarks above, every element of \(G_n\) has one of the two forms \((x, k, S^k y)\) or \((S^k x, -k, y)\) where \(k \geq 0\) and \((x, 0, y) \in P_{UHF}\). In equation \((1)\), the first \(k\) coordinates of \(y\) play no role in either side of the equation; similarly, in equation \((2)\) the first \(k\) coordinates of \(x\) play no role. When \(k = 0\), the two equations agree, since the first sum in the right hand side is absent in either version. Since \((x, 0, y) \in P_{UHF}\), \(S^j x = S^j y\) for all large \(j\); consequently the sums in equations \((1)\) and \((2)\) are finite.

Observe that, with \(k \geq 0\) and \((x, 0, y) \in P_{UHF}\),

\[
d_f(S^k y, -k, x) = -\sum_{j=0}^{k-1} f(S^j x) + \sum_{j=k}^{\infty} [f(S^j x) - f(S^j y)]
\]

\[
= -\left[\sum_{j=0}^{k-1} f(S^j x) + \sum_{j=k}^{\infty} [f(S^j x) - f(S^j y)]\right]
\]

\[
= -d_f(x, k, S^k y).
\]
Now suppose that \( k \geq 0, l \geq 0, (x, 0, y) \in P_{UHF} \) and \((S^k y, 0, S^k z) \in P_{UHF}\). It follows that \((y, 0, z) \in P_{UHF}\). If we prove that
\[
d_f(x, k, S^k y) + d_f(S^k y, l, S^{k+l} z) = d_f(x, k + l, S^{k+l} z).
\]
then the cocycle identity for \( d_f \) on all of \( G_n \) follows easily.

Now,
\[
d_f(x, k, S^k y) + d_f(S^k y, l, S^{k+l} z)
\]
\[
= \sum_{j=0}^{k-1} f(S^j x) + \sum_{j=k}^{\infty} [f(S^j x) - f(S^j y)] + \sum_{i=0}^{l-1} f(S^{i+k} y) + \sum_{i=l}^{\infty} [f(S^{i+k} y) - f(S^{i+k} z)]
\]
\[
= \sum_{j=0}^{k-1} f(S^j x) + \sum_{j=k}^{k+l-1} f(S^j y) - \sum_{j=k}^{k+l-1} f(S^j y) + \sum_{j=k+l}^{\infty} [f(S^j y) - f(S^j z)]
\]
\[
= \sum_{j=0}^{k+l-1} f(S^j x) + \sum_{j=l+1}^{\infty} [f(S^j x) - f(S^j z)]
\]
\[
= d_f(x, k + l, S^{k+l}).
\]

It remains to show that these two maps are inverse to one another. Let \( f : X \to \mathbb{R} \) be continuous. Then
\[
f_{d_f}(x) = d_f(x, 1, Sx) = f(x) + \sum_{j=1}^{\infty} [f(S^j x) - f(S^j x)] = f(x).
\]

It is a bit more complicated to show that the composition in the other order is also the identity.

Let \( d \) be a cocycle on \( G_n \) and define \( f (= f_d) \) by \( f(x) = d(x, 1, Sx) \), for all \( x \in X \). If \((x, 0, y) \in P_{UHF}\), then
\[
f(x) + d(Sx, 0, Sy) = d(x, 1, Sx) + d(Sx, 0, Sy)
\]
\[
= d(x, 1, Sy)
\]
\[
= d(x, 0, y) + d(y, 1, Sy)
\]
\[
= d(x, 0, y) + f(y)
\]

Therefore
\[
f(x) - f(y) = d(x, 0, y) - d(Sx, 0, Sy).
\]

Apply this to \( S^j x \) and \( S^j y \) to obtain
\[
f(S^j x) - f(S^j y) = d(S^j x, 0, S^j y) - d(S^{j+1} x, 0, S^{j+1} y), \quad \text{for any} \ j \geq 0.
\]
Since \((x, 0, y) \in P_{UHF}\), there is \(N \in \mathbb{N}\) such that \(S^n x = S^n y\) (and hence \(d(S^n x, 0, S^n y) = 0\)) for all \(n \geq N\). Addition yields

\[
(4) \quad d(x, 0, y) = \sum_{j=0}^{\infty} [f(S^j x) - f(S^j y)].
\]

The sum is, of course, really a finite sum.

Next, we express \(d(x, k, S^k y)\) in terms of \(f\) (when \(k > 0\)). Just observe that

\[
d(x, k, S^k y) = d(x, 1, S x) + d(S x, 1, S^2 x) + \cdots + d(S^{k-1} x, 1, S^k x) + d(S^k x, 0, S^k y)
\]

\[
= f(x) + f(S x) + \cdots + f(S^{k-1} x) + d(S^k x, 0, S^k y)
\]

\[
= \sum_{j=0}^{k-1} f(S^j x) + \sum_{j=k}^{\infty} [f(S^j x) - f(S^j y)],
\]

where equation \((4)\) is used in the last equality.

We have now shown that \(d\) and \(d_f\) agree on \((x, k, y)\) whenever \(k \geq 0\). But \(d\) and \(d_f\) are both cocycles, so they also agree whenever \(k < 0\), i.e., on all of \(G_n\).

The correspondence between continuous functions on \(X\) and continuous cocycles on \(G_n\) is clearly a linear one. It also preserves uniform convergence on compacta.

**Proposition 3.** Let \(f\) and \(f_n\) \((n \in \mathbb{N})\) be continuous functions on \(X\). Then \(f_n \to f\) uniformly on \(X\) if, and only if, \(d_{f_n} \to d_f\) uniformly on compacta in \(G_n\).

**Proof.** Assume that \(f_n \to f\) uniformly on \(X\). Let \(k \geq 0\) and let \(\alpha\) and \(\beta\) be such that \(\ell(\alpha) - \ell(\beta) = k\). Let \((x, k, S^k y) \in U_{\alpha, \beta}\). Then

\[
d_{f_n}(x, k, S^k y) - d_f(x, k, S^k y) =
\]

\[
\sum_{j=0}^{k-1} [f_n(S^j x) - f(S^j x)] + \sum_{j=k}^{\ell(\alpha)} \left( [f_n(S^j x) - f(S^j x)] - [f_n(S^j y) - f(S^j y)] \right)
\]

and this converges uniformly to 0 on \(U_{\alpha, \beta}\). A similar argument applies when \(\ell(\alpha) - \ell(\beta) < 0\).

For the converse, observe that

\[
f(x) - f_n(x) = d_f(s, 1, S x) - d_{f_n}(x, 1, S x)
\]

and that

\[
\{(x, 1, S x) \mid x \in X\} = \bigcup_{i=1}^{n} U_{(i), \beta}.
\]

Since \(d_{f_n} \to d_f\) uniformly on this set, \(f_n \to f\) uniformly on \(X\). \(\square\)
The following remark will be useful in Theorems 3 and 4.

**Remark 2.** If $A$ is a compact and open subset of $X$, then $A$ is a finite disjoint union of cylinder sets (sets of the form $E_\alpha = \{ x \in X \mid x_1 = \alpha_1, ..., x_k = \alpha_k \}$). Indeed, for each $a \in A$ there is a cylinder set $C_a$ such that $a \in C_a \subseteq A$. Since $A$ is compact, the open cover $\{ C_a \mid a \in A \}$ has a finite subcover. Thus $A$ is a finite union of cylinder sets. Since any cylinder set $E_\alpha$ can be written as a disjoint union of cylinder sets of the form $E_\beta$ where the $\beta$'s all have a specified length greater than the length of $\alpha$, $A$ can be written as a union of disjoint cylinder sets.

Now suppose that $f$ is a continuous function on $X$ which assumes only finitely many values. For each $t$ in the range of $f$, $f^{-1}(t)$ is a compact open subset of $X$, and hence can be written as a finite disjoint union of cylinder sets. It follows that there is a positive integer $N$ such that, for all $x$, $f(x)$ depends only on the first $N$ coordinates of $x$.

**Theorem 3.** If $f$ is a continuous function on $X$ with finite range, then the cocycle $d$ associated with $f$ is locally constant.

**Proof.** By Remark 2 there is a positive integer $N$ such that the value of $f$ at any point of $X$ depends only on the first $N$ coordinates of the point.

Let $k \geq 0$ and $(x, 0, y) \in P_{UHF}$. We shall show that $d$ is constant on a neighborhood of $(x, k, S^k y)$. Let $P$ be an integer greater than $k$ such that $x_i = y_i$ for all $i \geq P$. Let

$$\alpha = (x_1, ..., x_{P+N})$$
$$\beta = (y_{k+1}, ..., y_{P+N}).$$

Then $\ell(\alpha) - \ell(\beta) = k$ and $(x, k, S^k y) \in U_{\alpha,\beta}$. For any $\gamma \in X$,

$$d(\alpha \gamma, k, \beta \gamma) = \sum_{j=0}^{k-1} f(S^j(\alpha \gamma)) + \sum_{j=k}^{P} [f(S^j(\alpha \gamma)) - f(S^{j-k}(\beta \gamma))].$$

With $j \leq P$, the first $N$ coordinates of $S^j(\alpha \gamma)$ are all coordinates of $\alpha$ and the first $N$ coordinates of $S^{j-k}(\beta \gamma)$ are all coordinates of $\beta$; thus the value of $d$ is independent of $\gamma$. This shows that $d$ is constant on $U_{\alpha,\beta}$. For points of the form $(S^k x, -k, y)$, choose a basic open neighborhood $U_{\alpha,\beta}$ for $(y, k, S^k x)$ on which $d$ is constant: then $U_{\beta,\alpha}$ is a neighborhood of $(S^k x, -k, y)$ on which $d$ is constant. Thus, $d$ is locally constant. \qed

**Corollary 1.** If $f$ is a continuous function on $X$ with finite range and $d$ is the corresponding cocycle, then $d^{-1}(0)$ is open. Consequently, $d^{-1}[0, \infty)$ is open and the support set of an analytic algebra.
Proof. Let \((x, k, z) \in G_n\) be such that \(d(x, k, z) = 0\). Let \(U_{\alpha,\beta}\) be a neighborhood of \((x, k, z)\) on which \(d\) is constant. Then \((x, k, z) \in U_{\alpha,\beta} \subseteq d^{-1}(0)\) and \(d^{-1}(0)\) is open. 

The dilation cocycle \(d(x, k, y) = k\) is non-negative on the set

\[
P_+ = \{ (\alpha \gamma, \ell(\alpha) - \ell(\beta), \beta \gamma \mid \gamma \in X, \ell(\alpha) \geq \ell(\beta) \}
\]

\[
= \bigcup \{ U_{\alpha,\beta} \mid \ell(\alpha) \geq \ell(\beta) \}.
\]

This is an open (and closed) subset of \(G_n\) which is clearly closed under the groupoid multiplication. The analytic algebra associated with the dilation cocycle is \(A(P_+)\). Since \(P_+ \cup P_+^{-1} = G_n\) and \(P_+ \cap P_+^{-1} = P_{UHF}\), it follows that \(A(P_+) + A(P_+)^*\) is dense in \(O_n\) and \(A(P_+) \cap A(P_+)^* = A(P_{UHF})\), the canonical UHF subalgebra.

The diagonal algebra in \(A(P_+)\) is not abelian, but a modification gives examples of strongly maximal triangular subalgebras of \(O_n\). Let \(Q\) be an open subset of \(P_{UHF}\) which satisfies the following

1. \(Q \circ Q \subseteq Q\)
2. \(Q \cap Q^{-1} = D_0\)
3. \(Q \cup Q^{-1} = P_{UHF}\)

Then \(A(Q)\) is a strongly maximal triangular subalgebra of the canonical UHF C*-algebra, \(A(P_{UHF})\). Two specific examples are:

\[
Q^{ref} = \{ (\alpha \gamma, 0, \beta \gamma) \in P_{UHF} \mid \alpha \preceq \beta \}
\]

\[
Q^{st} = \{ (\alpha \gamma, 0, \beta \gamma) \in P_{UHF} \mid \alpha \preceq_{r} \beta \}
\]

Here \(\preceq\) is the lexicographic order and \(\preceq_{r}\) is the reverse lexicographic order. \(Q^{ref}\) is the spectrum of the refinement subalgebra of the canonical UHF algebra. and \(Q^{st}\) is the spectrum of the standard subalgebra.

Given \(Q\) as above, let

\[
Q_+ = \{ (x, k, y) \mid \text{either } k = 0 \text{ and } (x, 0, y) \in Q \text{ or } k > 0 \}
\]

It is easy to see that \(Q_+ \circ Q_+ \subseteq Q_+, Q_+ \cup Q_+^{-1} = G_n, \) and \(Q_+ \cap Q_+ = D_0\). \(A(Q_+)\) is a strongly maximal triangular subalgebra of \(O_n\). We shall call \(A(Q_+)\) the contractive extension of \(A(Q)\).

**Proposition 4.** The “contractive” algebras, \(A(P_+)\) and \(A(Q_+)\), are semisimple.

**Proof.** By Theorem 1(b) both \(A(P_+)\) and \(A(Q_+)\) are invariant under the gauge automorphisms. Therefore the radical is invariant under gauge automorphisms and so is generated by the Cuntz partial isometries which it contains. But if \(S_\alpha S_\beta^*\) is in the radical, then so is \(S_\alpha = S_\alpha S_\beta^* S_\beta\)
(since $S_β$ is in $A(P_+)$ or $A(Q_+)$, as appropriate). Thus, if the radical contains a non-zero operator, it contains one which is not quasi-nilpotent. This is impossible, so each algebra is semisimple. \qed

**Example 3.** Let $f : X \to \mathbb{R}$ be given by $f(x) = 1$ for all $x$. Then it is easy to see that $d_f(x, k, z) = k$ for all $(x, k, z) \in G_n$. Thus, the constant function 1 corresponds to the dilation cocycle; the associated analytic subalgebra is the contractive algebra $A(P_+)$. If $f(x) = c$ for all $x$ (with $c \neq 0$), then $d_f(x, k, z) = ck$. Again, the analytic subalgebra is $A(P_+)$.  

**Example 4.** Let $f : X \to \mathbb{R}$ by $f(x) = x_1$. The range of $f$ is $\{1, \ldots , n\}$. When $k \geq 0$ and $(x, 0, y) \in P_{\text{UHF}}$,  

$$d_f(x, k, S^k y) = x_1 + \cdots + x_k + \sum_{i=k+1}^{\infty} (x_i - y_i)$$

and

$$d_f(S^k x, -k, y) = -y_1 - \cdots - y_k + \sum_{i=k+1}^{\infty} (x_i - y_i)$$

For every $k$, there exist $x, z \in X$ such that $d_f(x, k, z) = 0$; i.e. $d^{-1}(0)$ intersects every $k$-level set. In particular, $d^{-1}(0)$ properly contains $D_0$.

In UHF algebras, the $\mathbb{Z}$-analytic subalgebras are those analytic algebras which arise from $\mathbb{Z}$-valued cocycles which vanish precisely on the canonical diagonal. Other important analytic algebras, for example the refinement algebras, arise from bounded cocycles which vanish precisely on the canonical diagonal. Theorem 4 shows that the Cuntz algebras lack precise analogs of these analytic subalgebras of UHF algebras.

**Theorem 4.** There is no cocycle $d$ on the Cuntz groupoid $G_n$ which is bounded and which vanishes precisely on $D_0$. There is no cocycle $d$ which is integer valued and which vanishes precisely on $D_0$.

**Proof.** First suppose that $d$ is a bounded cocycle. Let $f$ be the associated function on $X$ and let $\overline{\text{T}} = (1, 1, \ldots)$. Then $S \overline{\text{T}} = \overline{\text{T}}$. If $f(\overline{\text{T}}) = 0$, then $d^{-1}(0)$ properly contains $D_0$. If $f(\overline{\text{T}}) \neq 0$, then, for all $k > 0$,

$$d(\overline{\text{T}}, k, S^k \overline{\text{T}}) = \sum_{j=0}^{k-1} f(S^j \overline{\text{T}}) = \sum_{j=0}^{k-1} f(\overline{\text{T}}) = kf(\overline{\text{T}})$$

and $d$ is unbounded.

Now suppose that $d$ is integer valued. Then $f$ is also integer valued. Since $f$ is continuous and $X$ is compact, the range of $f$ is finite. By
remark 2, the value of $f$ at $x$ depends only on the first $N$ coordinates of $x$.

We will exhibit $x \neq y$ such that $(x, 0, y) \in G_n$ and $d(x, 0, y) = 0$. Let $\alpha$ be the string consisting of $2 \cdots 2$ repeated $N - 1$ times and let $\gamma$ be the infinite string consisting entirely of 1’s. Set

$$x = \alpha 21 \alpha \gamma \quad \text{and} \quad y = \alpha 12 \alpha \gamma.$$ 

It is routine to show that a string of length $N$ appears as the first $N$ coordinates of one of the points $x, Sx, \ldots, S^{2N-1}x$ if, and only if, it appears as the first $N$ coordinates of one of $y, Sy, \ldots, S^{2N-1}y$; furthermore, the frequency of appearance is the same in both cases. Thus

$$\sum_{j=0}^{2N-1} f(S^j x) = \sum_{j=0}^{2N-1} f(S^j y).$$

Since $S^j x = S^j y$ for $j \geq 2N$, we have

$$d(x, 0, y) = \sum_{j=0}^{\infty} [f(S^j x) - f(S^j y)] = 0.$$ 

Thus, $d^{-1}(0)$ properly contains $D_0$. \hfill \Box

As we pointed out earlier, the standard TUHF algebra is the strongly maximal triangular subalgebra of $A(P_{UHF})$ whose spectrum is $Q_{st} = \{(x, 0, y) \in P_{UHF} \mid x \preceq_r y\}$. (The order is the reverse lexicographic order.) This is the prototypical $\mathbb{Z}$-analytic algebra. The associated cocycle – the standard cocycle – is given by the formula

$$d(x, 0, y) = \sum_{i=1}^{\infty} n^{i-1} (x_i - y_i).$$

**Proposition 5.** The standard cocycle $d$ on $P_{UHF}$ has no extension to a cocycle on $G_n$.

**Proof.** Suppose that $d$ has an extension to $G_n$, which we also denote by $d$, and that $f$ is the associated function on $X$. Since

$$d(x, 0, y) = \sum_{j=0}^{\infty} [f(S^j x) - f(S^j y)],$$

any function which differs from $f$ by an additive constant will yield a cocycle which agrees with the standard cocycle on $P_{UHF}$. Therefore, we may, without loss of generality, assume that $f(\mathbf{T}) = 0$, where $\mathbf{T} =$
Now let $x = (2, 2, \ldots, 2, \overline{1})$, where there are $N$ coordinates in $x$ which are 2. By equation 3 in the proof of Theorem 2,

$$f(x) = d(x, 0, \overline{1}) - d(Sx, 0, \overline{1}) = \sum_{j=1}^{N} n^{j-1} - \sum_{j=1}^{N-1} n^{j-1} = n^{N-1}$$

This shows that $f$ is unbounded, an impossibility for a continuous function on a compact set. Thus the standard cocycle on $P_{UHF}$ admits no extension to $G_n$. □

A cocycle $d$ on $P_{UHF}$ is trivial if there is a continuous function $b$ on $X$ such that $d(x, 0, y) = b(y) - b(x)$. The refinement cocycle, given by

$$d(x, 0, y) = \sum_{i=1}^{\infty} \frac{1}{n^i} (x_i - y_i),$$

is one of the simplest and most important examples of a trivial cocycle. The function $b$ is given by

$$b(x) = \sum_{i=1}^{\infty} \frac{x_i - 1}{n^i};$$

the range of $b$ is $[0, 1]$. The analytic subalgebra of $A(P_{UHF})$ associated with the refinement cocycle is the refinement TUHF algebra; as we noted earlier, the spectrum of this algebra is $Q_{\text{ref}} = \{(x, 0, y) \in P_{UHF} \mid x \preceq y\}$ (lexicographic order). In general, an analytic subalgebra of $A(P_{UHF})$ whose cocycle is trivial is said to be trivially analytic.

**Theorem 5.** Let $A(Q)$ be a trivially analytic subalgebra of $A(P_{UHF})$ with cocycle $d$. Then $d$ has extensions to cocycles on $G_n$. Furthermore, there is an extension so that the corresponding analytic subalgebra of $O_n$ is $A(Q_+)$. 

**Remark 3.** As noted above, the refinement subalgebra of $A(P_{UHF})$ is a trivially analytic subalgebra. Thus $A(Q_{\text{ref}})$ is an analytic subalgebra of $O_n$ which is an “extension” of the refinement TUHF algebra.

**Proof.** Let $b$ be a continuous function on $X$, let $d$ be the cocycle on $P_{UHF}$ given by $d(x, 0, y) = b(y) - b(x)$ and let $A(Q)$ be the analytic subalgebra of $A(P_{UHF})$ associated with the cocycle.

Let $X_0$ be the equivalence class of the point $\overline{1} = (1, 1, 1, \ldots)$. This is a dense set in $X$. We wish to define a function $f$ on $X$ which will yield a cocycle on $G_n$ which extends $d$.

Begin by setting $f(\overline{1}) = 0$. For $x \in X_0$, define

$$f(x) = d(x, 0, \overline{1}) - d(Sx, 0, \overline{1}).$$
Observe that, for any for any \( x, y \in X_0 \),
\[
  f(x) - f(y) = d(x, 0, \overline{1}) - d(Sx, 0, \overline{1}) - d(y, 0, \overline{1}) + d(Sy, 0, \overline{1})
  = d(x, 0, \overline{1}) + d(\overline{1}, 0, y) - [d(Sx, 0, \overline{1}) + d(\overline{1}, 0, Sy)]
  = d(x, 0, y) - d(Sx, 0, Sy)
\]

We claim that \( f \) is uniformly continuous, and therefore has a continuous extension to \( X \). Let \( \epsilon > 0 \). Let \( \rho \) be the metric on \( X \) defined by
\[
  \rho(x, y) = \sum_{i=1}^{\infty} \frac{|y_i - x_i|}{n^i}.
\]
Since \( b \) is uniformly continuous, there is \( \delta > 0 \) such that \( |b(y) - b(x)| < \epsilon/2 \) whenever \( \rho(x, y) < \delta \).

Sets of the form \( U_{\alpha} = \{ \alpha \gamma \mid \gamma \in X \} \) form a neighborhood basis for the topology on \( X \). If the length of \( \alpha \) is \( L \), then \( \rho(x, y) \leq 1/n^L \) whenever \( x \) and \( y \) are in \( U_{\alpha} \). Choose \( L \) so that \( 1/n^L < \delta \). Let \( U_{\alpha} \) be any basic open neighborhood with \( \ell(\alpha) \geq L+1 \). Then, when \( x, y \in U_{\alpha} \cap X_0 \), we have \( Sx, Sy \in U_{S_{\alpha}} \) and \( \ell(S\alpha) \geq L \). Therefore,
\[
  |f(x) - f(y)| = |d(x, 0, y) - d(Sx, 0, Sy)| = |b(y) - b(x) - (b(Sy) - b(Sx))| \\
  \leq |b(y) - b(x)| + |b(Sy) - b(Sx)| < \epsilon.
\]
Denote the extension to \( X \) by \( f \) also. We next show that the relation
\[
  f(x) - f(y) = d(x, 0, y) - d(Sx, 0, Sy)
\]
holds for any \( (x, 0, y) \in P_{UHF} \). Let \( k \) be such that \( x_i = y_i \) for all \( i \geq k \). Let \( \alpha = (x_1, \ldots, x_k) \) and \( \beta = (y_1, \ldots, y_k) \). Then \( U_{\alpha, \beta} \) is a neighborhood of \( (x, 0, y) \) and a \( G \)-set in \( G_n \). In particular, the projection maps onto the first and third coordinates are each homeomorphisms when restricted to \( U_{\alpha, \beta} \). Let \( x_\nu \) and \( y_\nu \) be sequences in \( X_0 \) such that \( (x_\nu, 0, y_\nu) \in U_{\alpha, \beta} \), \( x_\nu \to x \) and \( y_\nu \to y \). It follows that
\[
  d(x_\nu, 0, y_\nu) \to d(x, 0, y), \\
  d(Sx_\nu, 0, Sy_\nu) \to d(Sx, 0, Sy), \\
  f(x_\nu) \to f(x), \text{ and} \\
  f(y_\nu) \to f(y).
\]
Since
\[
  f(x_\nu) - f(y_\nu) = d(x_\nu, 0, y_\nu) - d(Sx_\nu, 0, Sy_\nu)
\]
for every \( \nu \), we have
\[
  f(x) - f(y) = d(x, 0, y) - d(Sx, 0, Sy)
\]
as desired.
By an argument used in Theorem 2

\[ d(x, 0, y) = \sum_{j=0}^{\infty} [f(S^j x) - f(S^j y)]. \]

It now follows immediately that the cocycle \( d_f \) on \( G_n \) induced by the function \( f \) extends the cocycle \( d \) on \( P_{UHF} \).

Fix a value for \( k \) and let \( G_{n,k} = \{(x, j, y) \in G_n \mid j = k \} \). We claim that \( d_f \) is bounded on each \( G_{n,k} \). Indeed, when \( k \geq 0 \), any point in \( G_{n,k} \) can be written in the form \((x, k, S^k y)\) for some \((x, 0, y) \in P_{UHF} \) and

\[
d_f(x, k, S^k y) = \sum_{j=0}^{k-1} f(S^j x) + \sum_{j=k}^{\infty} [f(S^j x) - f(S^j y)]
= \sum_{j=0}^{k-1} f(S^j x) + d(S^k x, 0, S^k y).
\]

Since \( f \) is bounded, \( k \) is fixed, and \( d \) is bounded on \( P_{UHF} \), \( d_f \) is bounded on \( G_{n,k} \). The boundedness of \( d_f \) on \( G_{n,k} \) when \( k < 0 \) follows from the cocycle property.

Finally, we show that there is an extension of \( d \) such that the corresponding analytic subalgebra is \( A(Q_+) \). Indeed, if \( c \) is any constant and \( g = f + c \), then, since the correspondence between functions on \( X \) and cocycles is linear, \( d_g(x, k, y) = d_f(x, k, y) + kc \). In particular, \( d_g(x, 1, y) = d_f(x, 1, y) + c \). Since \( d_f \) is bounded on \( G_{n,1} \), we may pick \( c \) so that \( d_g > 0 \) on \( G_{n,1} \). For \( k > 0 \), any element of \( G_{n,k} \) is a product of \( k \) elements of \( G_{n,1} \); the cocycle property implies that \( d_g > 0 \) on \( G_{n,k} \). The cocycle property also guarantees that \( d_g \) is negative on \( G_{n,k} \) when \( k < 0 \). Thus, \( d_g \geq 0 \) precisely on \( Q_+ \).

□

5. The Volterra Subalgebra of \( O_n \)

The intersection of the Cuntz algebra with the Volterra nest algebra on \( L^2[0,1] \) has been studied by S. C. Power in [2]. We refer to this intersection as the Volterra subalgebra of \( O_n \). In this section, we show that groupoid techniques allow us to obtain many of Power’s results easily, as well as some new information. We first identify an open set \( P_V \) for which \( A(P_V) \) will turn out to be the Volterra subalgebra of \( O_n \).

An element \((\alpha \gamma, \ell(\alpha) - \ell(\beta), \beta \gamma)\) of \( G_n \) is in \( P_V \) if any of the following
five conditions on \( \alpha \) and \( \beta \) hold:

\[
\begin{align*}
(5) & \quad \ell(\alpha) = \ell(\beta) \text{ and } \alpha \preceq \beta, \text{ or} \\
(6) & \quad \ell(\alpha) < \ell(\beta) \text{ and } \alpha \prec (\beta_1, \ldots, \beta_{\ell(\alpha)}), \text{ or} \\
(7) & \quad \ell(\alpha) < \ell(\beta) \text{ and } \alpha = (\beta_1, \ldots, \beta_{\ell(\alpha)}) \text{ and } \beta_{\ell(\alpha)+1} = \cdots = \beta_{\ell(\beta)} = n, \text{ or} \\
(8) & \quad \ell(\alpha) > \ell(\beta) \text{ and } (\alpha_1, \ldots, \alpha_{\ell(\beta)}) \prec \beta, \text{ or} \\
(9) & \quad \ell(\alpha) > \ell(\beta) \text{ and } (\alpha_1, \ldots, \alpha_{\ell(\beta)}) = \beta \text{ and } \alpha_{\ell(\beta)+1} = \cdots = \alpha_{\ell(\alpha)} = 1.
\end{align*}
\]

The order on finite strings used in these conditions is the lexicographic order.

There is an alternate description of \( P_V \). Although the lexicographic order is most commonly used for finite sequences or for sequences in \( X \) which are equivalent in the sense that ‘tails are equal’, the lexicographic order is also useful as a total order on all of \( X \). With this in mind, it is not hard to see that \( P_V \) is the union of the following four sets:

\[
\begin{align*}
R &= \{(x, k, y) \mid x < y\} \\
D_0 &= \{(x, 0, x) \mid x \in X\} \quad \text{(diagonal)} \\
S_e &= \{(x, k, x) \mid k < 0 \text{ and } x_j = n \text{ for all large } j\} \quad \text{(expansive)} \\
S_c &= \{(x, k, x) \mid k > 0 \text{ and } x_j = 1 \text{ for all large } j\} \quad \text{(contractive)}
\end{align*}
\]

Indeed, if \( x < y \) and \( (x, k, y) \in G_n \), let \( j \) be the first index for which \( x_j \neq y_j \). Choose initial segments \( \alpha \) and \( \beta \) of \( x \) and \( y \) such that \( j < \min(\ell(\alpha), \ell(\beta)) \) and \( k = \ell(\alpha) - \ell(\beta) \). Then \( (x, k, y) \) has the form \((\alpha \gamma, \ell(\alpha) - \ell(\beta), \beta \gamma)\) for suitable \( \gamma \) and also satisfies one of conditions (5), (6), or (8). Elements of the form \((x, 0, x)\) clearly satisfy condition (5). If \( k < 0 \) and \( x_j = n \) for all large \( j \), then \( (x, k, x) \) satisfies condition (7); if \( k > 0 \) and \( x_j = 1 \) for all large \( j \), then \( (x, k, x) \) satisfies condition (9).

In the other direction, if \( (x, k, y) \) satisfies condition (7) or condition (8), then \( x < y \). Suppose that \( (x, k, y) = (\alpha \gamma, \ell(\alpha) - \ell(\beta), \beta \gamma) \) satisfies (7). If any of \( \gamma_1, \ldots, \gamma_{\ell(\beta) - \ell(\alpha)} \) are less than \( n \), then \( x < y \). If all of \( \gamma_1, \ldots, \gamma_{\ell(\beta) - \ell(\alpha)} \) equal \( n \) and any \( \gamma_j < n \) with \( j \) between \( \ell(\alpha) - \ell(\beta) + 1 \) and \( 2(\ell(\beta) - \ell(\alpha)) \) then again \( x < y \). Continuing in this fashion, we see that either \( x \prec y \) or \( x = y \) and \( x_j = n \) for all large \( j \). Similarly, if \( (x, k, y) \) satisfies condition (8), then either \( x \prec y \) or \( x_j = 1 \) for all large \( j \). Finally, if \( (x, k, y) \) satisfies (5), then \( k = 0 \) and \( (x, k, y) \) is in \( D_0 \cup R \).

Next we show that \( P_V \) is an open subset of \( G_n \). Since \( D_0 \) and \( \{(x, k, y) \in G_n \mid x < y\} \) are open sets, we need only show that all
points \((x, k, x)\) with \(k < 0\) and a tail consisting of \(n\)'s or with \(k > 0\) and a tail consisting of \(1\)'s lie in open sets contained in \(P_V\). Suppose that \(k > 0\) and \(x = \beta \delta\), where \(\beta = (\beta_1, \ldots, \beta_s)\) and \(\delta = (1, 1, 1, \ldots)\). Let \(\alpha = (\beta_1, \ldots, \beta_s, 1, \ldots, 1)\), where there are \(k\) 1's following \(\beta_s\). Then \(U_{\alpha, \beta} = \{ (\alpha \gamma, k, \beta \gamma) \mid \gamma \in X\}\) is an open subset of \(G_n\). If \(\gamma = (1, 1, \ldots)\), then \((\alpha \gamma, k, \beta \gamma) = (x, k, x)\), so \((x, k, x) \in U_{\alpha, \beta}\). If \(\gamma \neq (1, 1, \ldots)\), then it is easy to see that \(\alpha \gamma \prec \beta \gamma\) and hence \((\alpha \gamma, k, \beta \gamma) \in P_V\). Thus \((x, k, x) \in U_{\alpha, \beta} \subseteq P_V\). A similar argument takes care of points with \(k < 0\) and a tail consisting of \(n\)'s; thus \(P_V\) is open.

While \(P_V\) is an open subset of \(G_n\), it is not closed. To see this, let \(\alpha\) be any finite string and let \(\delta\) be a string of length \(k\) in which some \(\delta_i \neq 1\). Since \(\delta\) is not first in the lexicographic order on \(\{1, \ldots, n\}^k\), there is a string \(\eta\) of length \(k\) such that \(\eta \prec \delta\). For each \(p \in \mathbb{N}\), let

\[
y^p = \alpha \delta \ldots \delta \eta \ldots \quad (p \text{ copies of } \delta),
\]
\[
z^p = \alpha \delta \ldots \delta \eta \ldots \quad (p + 1 \text{ copies of } \delta),
\]
\[
x = \alpha \delta \ldots \quad \text{ (infinitely many copies of } \delta).\]

Since \(y^p \prec z^p\), \((y^p, k, z^p) \in P_V\). Also \((y^p, k, z^p) \rightarrow (x, k, x)\) in the topology on \(G_n\). But \((x, k, x) \notin P_V\), so \(P_V\) is not closed.

When is a partial isometry \(S_\alpha S_\beta^*\) in \(A(P_V)\)? For the case when \(\ell(\alpha) = \ell(\beta)\), i.e., when \(S_\alpha S_\beta^*\) is in the canonical UHF subalgebra, \(S_\alpha S_\beta^*\) is in \(A(P_V)\) if, and only if, \(S_\alpha S_\beta^*\) is also in the Volterra nest subalgebra on \(L^2[0, 1]\). These partial isometries correspond to condition \(\mathfrak{5}\). Expansive partial isometries in \(A(P_V)\) correspond to conditions \(\mathfrak{6}\) and \(\mathfrak{7}\). Condition \(\mathfrak{4}\) gives the expansive partial isometries for which the range interval lies to the left of the domain interval. Condition \(\mathfrak{8}\) gives those expansive partial isometries for which the domain interval is a subinterval of the range interval located at the right end of the range interval. Similarly, conditions \(\mathfrak{5}\) and \(\mathfrak{9}\) correspond to the contractive partial isometries in \(A(P_V)\). Condition \(\mathfrak{8}\) gives the contractive partial isometries for which the range interval lies to the left of the domain interval and condition \(\mathfrak{9}\) describes the case when the range interval is a subinterval of the domain interval which is located at the left end of the domain interval.

From these considerations, we see that the partial isometries \(S_\alpha S_\beta^*\) which are in \(A(P_V)\) are precisely the Cuntz partial isometries which lie in the Volterra nest algebra. Since \(A(P_V)\) is generated by its Cuntz partial isometries, it follows that \(A(P_V)\) is contained in \(O_n \cap \text{Alg } \mathcal{V}\), where \(\mathcal{V}\) is the Volterra nest.

While most of the projections in \(\mathcal{V}\) are not in \(O_n\), those that are are strongly dense in \(\mathcal{V}\). These projections can be described in several ways. The most useful is as follows: let \(T = \{x \in X \mid x_j = n \text{ for all large } j\}\).
For $x \in T$, let $p_x$ be the characteristic function of $\{(y,0,y) \mid y \preceq x\}$. This subset of $P_V$ is compact and, since $x \in T$, also open; thus $p_x$ is a continuous function with compact support and can therefore be viewed as an element of $A(P_V)$.

As an operator on $L^2[0,1]$ (using the representation of the Cuntz algebra described above), $p_x$ is the projection onto the subspace $L^2[0,x]$, where

$$x = \sum_{j=1}^{\infty} \frac{x_j - 1}{n^j}.$$ 

And in terms of the generators of the Cuntz algebra, each $p_x$ has the form $\sum_\alpha S_\alpha S_\alpha^*$, where all $\alpha$ in the summation have the same length, say $k$, and run through an initial segment in the set of $k$-tuples with the lexicographic order.

**Theorem 6.** $A(P_V) = O_n \cap \text{Alg} \mathcal{V}$.

**Proof.** If $f \in O_n$ and $x \in T = \{x \in X \mid x_j = n \text{ for all large } j\}$, then for any $(s,k,t) \in G_n$,

$$(f * p_x)(s,k,t) = f(s,k,t)p_x(t,0,t)$$

$$= \begin{cases} f(s,k,t), & \text{if } t \preceq x, \\ 0, & \text{if } x \prec t \end{cases}$$

and

$$(p_x * f * p_x)(s,k,t) = p_x(s,0,s)f(s,k,t)p_x(t,0,t)$$

$$= \begin{cases} f(s,k,t), & \text{if both } s \preceq x \text{ and } t \preceq x \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose that $f \in O_n \cap \text{Alg} \mathcal{V}$. To prove that $f \in A(P_V)$ we must show that $f$ vanishes on the complement of $P_V$ in $G_n$. First, suppose that $t \prec s$ and $(s,k,t) \in G_n$. If $t$ is an immediate predecessor of $s$, choose $x = t$; otherwise choose $x \in T$ such that $t \prec x \prec s$. In either case, $x \prec s$. Since $f \in \text{Alg} \mathcal{V}$ and $p_x \in \mathcal{V}$, we have $f * p_x = p_x * f * p_x$. It follows from the equations above that $f(s,k,t) = 0$. This leaves the case in which $s = t$. There are two possibilities: $k > 0$ and infinitely many $t_i > 1$, for which the details are below, and $k < 0$ and infinitely many $t_i < n$ (which can be done in a similar fashion).

Suppose that $(t,k,t) \in G_n$ with $k > 0$ and infinitely many $t_i > 1$. Then there are finite strings $\tau$ and $\gamma$ such that one of the components of $\gamma$ is greater than 1, $\ell(\gamma) = k$, and $t = \tau\gamma$. (Here, $\gamma$ denotes the infinite concatenation $\gamma\gamma\gamma\ldots$) If $f(t,k,t) \neq 0$, then there is a neighborhood $U$ of $(t,k,t)$ such that $f$ is non-zero on $U$. For $n$ a positive integer,
Let \( \gamma^n \) denote \( \gamma \ldots \gamma \), the \( n \)-fold concatenation of \( \gamma \) with itself. Then there exists \( n \) such that if \( \alpha = \tau \gamma^{n+1} \) and \( \beta = \tau \gamma^n \), then \( U_{\alpha, \beta} \subseteq U \).

Let \( \delta \) be any sequence of length \( k \) which strictly precedes \( \gamma \) in the lexicographic order. (The existence of such a sequence is guaranteed by the fact that one of the components of \( \gamma \) is greater than 1.) Let \( x = \alpha \delta \) and \( y = \beta \delta \). Then \( y \prec x \) and \( (x, k, y) \in U_{\alpha, \beta} \subseteq U \). By the preceding paragraph, \( f(x, k, y) = 0 \); but this contradicts the fact that \( f \) does not vanish on \( U \). Thus, if \( k > 0 \) and infinitely many \( t_i > 1 \), then \( f(t, k, t) = 0 \). Similarly, if \( k < 0 \) and infinitely many \( t_i < n \), then \( f(t, k, t) = 0 \). Thus, \( f \) vanishes on the complement of \( P_V \) and we have shown that \( O_n \cap \mathrm{Alg} \mathcal{V} \subseteq A(P_V) \). We have already verified the reverse containment, so equality is proven. \( \square \)

**Proposition 6.** The spectrum of the radical in \( A(P_V) \) is \( \{(x, k, y) \in G_n \mid x < y \} \).

**Proof.** Let \( R = \{(x, k, y) \in G_n \mid x < y \} \) and let \( R_0 \) denote the spectrum of \( \mathrm{rad} A(P_V) \), the Jacobson radical of \( A(P_V) \). We first show that \( R \subseteq R_0 \) while the three sets

\[
D_0 = \{(x, 0, x) \mid x \in X \},
\]

\[
S_c = \{(x, k, x) \mid k < 0, x \in X \text{ and } x_j = n \text{, for all large } j \},
\]

\[
S_c = \{(x, k, x) \mid k > 0, x \in X \text{ and } x_j = 1 \text{, for all large } j \}
\]

are all disjoint from \( R_0 \); in other words, \( R = R_0 \).

Let \( (x, k, y) \in R \). Then there exist finite strings \( \alpha \) and \( \beta \) and \( \gamma \in X \) such that \( \ell(\alpha) - \ell(\beta) = k \), \( \alpha \prec \beta \), \( x = \alpha \gamma \) and \( y = \beta \gamma \). The set \( U_{\alpha, \beta} \), the support set for \( S_{\alpha}^* S_{\beta}^* \), is a subset of \( R \). Since the “range interval” for the partial isometry \( S_{\alpha}^* S_{\beta}^* \) lies to the left of the “domain interval”, there is a projection \( p \in \mathcal{V} \) such that \( S_{\alpha}^* S_{\beta}^* = p S_{\alpha}^* S_{\beta}^* p \).

It follows easily that, for any \( T \in A(P_V) \), both \( S_{\alpha}^* S_{\beta}^* T \) and \( T S_{\alpha}^* S_{\beta}^* \) are nilpotent (of order 2). Alternatively, since \( S_{\alpha}^* S_{\beta}^* \) lies in the radical of the Volterra nest algebra, the ideal generated by \( S_{\alpha}^* S_{\beta}^* \) in the Volterra nest algebra consists entirely of quasi-nilpotents; a fortiori, the ideal generated by \( S_{\alpha}^* S_{\beta}^* \) in \( A(P_V) \) consists of quasi-nilpotents. Either way, \( S_{\alpha}^* S_{\beta}^* \) lies in the radical of \( A(P_V) \). Thus \( U_{\alpha, \beta} \subseteq R_0 \); in particular, \( (x, k, y) \in R_0 \). Thus \( R \subseteq R_0 \).

We next show that \( D_0 \cap R_0 = \emptyset \). Suppose that \( (x, 0, x) \in D_0 \cap R_0 \). Then there is a finite string \( \alpha \) such that \( U_{\alpha, \alpha} \subseteq R_0 \). But this implies that \( S_{\alpha}^* S_{\alpha}^* \), a non-zero projection, lies in \( \mathrm{rad} A(P_V) \). This is a contradiction, so \( D_0 \cap R_0 = \emptyset \).

The final two pieces, \( S_c \cap R_0 = \emptyset \) and \( S_c \cap R_0 = \emptyset \), are proven in essentially the same way, so we provide only one of the proofs. Suppose
\((x, k, x) \in S_\epsilon \cap R_0\). Since \(S_\epsilon\) and \(R_0\) are open, there is a neighborhood of \((x, k, x)\) which is a subset of \(R_0\). It follows that there are finite strings \(\alpha\) and \(\beta\) corresponding to the intervals \([a, c]\) and \([b, c]\) such that \(a < b\) and \(S_\alpha S_\beta^* \in \text{rad} A(P_V)\). But if we compress \(S_\alpha S_\beta^*\) to \(L^2[a, c]\), we get a co-isometry, all powers of which have norm 1. So all powers of \(S_\alpha S_\beta^*\) have norm 1, contradicting the assertion that \(S_\alpha S_\beta^* \in \text{rad} A(P_V)\). This shows that \(S_\epsilon \cap R_0 = \emptyset\).

In the proof that \(R \subseteq R_0\), we showed that if \((x, k, y) \in R\), then there is a neighborhood \(U_{\alpha, \beta}\) of \((x, k, y)\) such that \(U_{\alpha, \beta} \subseteq R\) and \(S_\alpha S_\beta^* \in \text{rad} A(P_V)\). We next show that this implies that \(A(R) \subseteq \text{rad} A(P_V)\). Since \(A(R)\) is generated by the Cuntz partial isometries which it contains, it is sufficient to show that each Cuntz partial isometry in \(A(R)\) lies in the radical.

If \(S_\gamma S_\delta^*\) is a Cuntz partial isometry in \(A(R)\), then for each \((x, k, y) \in U_{\gamma, \delta}\) there is a neighborhood \(U_{\alpha, \beta}\) such that \((x, k, y) \in U_{\alpha, \beta} \subseteq U_{\gamma, \delta}\) and \(S_\alpha S_\beta^* \in \text{rad} A(P_V)\). This gives an open cover of \(U_{\gamma, \delta}\), from which we can select a finite subcover. It is routine to arrange that the sets in the subcover are pairwise disjoint (with corresponding Cuntz partial isometries still in the radical). So \(U_{\gamma, \delta}\) can be written as a finite disjoint union of sets of the form \(U_{\alpha, \beta}\) for which \(S_\alpha S_\beta^* \in \text{rad} A(P_V)\). But then \(S_\gamma S_\delta^* \in \text{rad} A(P_V)\), and \(S_\gamma S_\delta^* \in \text{rad} A(P_V)\). Thus \(A(R) \subseteq \text{rad} A(P_V)\).

We now know that \(A(R) \subseteq \text{rad} A(P_V) \subseteq A(R_0)\) and that \(R = R_0\). But then \(A(R) = A(R_0)\); in particular, \(\text{rad} A(P_V) = A(R)\). \(\Box\)

**Remark 4.** If we let \(B\) denote the norm closure of \(A(D_0) + \text{rad} A(P_V)\) in \(A(P_V)\), then \(B\) is a proper subset of \(A(P_V)\). This follows from the fact that the spectrum of \(B\) is \(D_0 \cup R\). While far from being semi-simple, the algebra \(A(P_V)\) falls short of having a radical plus diagonal decomposition.

As mentioned earlier, the Volterra subalgebra \(A(P_V)\) has been studied extensively by Power in \([9]\). Making use of the spectrum, we can obtain several of Power’s results with new proofs which provide a different intuitive insight. For example, Power points out that \(A(P_V)\) is *non-Dirichlet* in the sense that \(A(P_V) + A(P_V)^*\) is not dense in \(O_n\); in other language, \(A(P_V)\) is triangular but not strongly maximal triangular. This is evident from the fact that \(P_V \cup P_V^{-1} \neq G_n\).

We prove the next theorem, which appears in \([9]\), using spectral techniques. In this theorem, \(\text{com} A(P_V)\) denotes the closed ideal generated by all commutators in \(A(P_V)\).
Theorem 7. The radical of $A(P_V)$ is equal to the commutator ideal of $A(P_V)$.

Proof. We begin by showing that $\text{com} \ A(P_V) \subseteq \text{rad} \ A(P_V)$. It suffices to show that $[f, g] \in \text{rad} \ A(P_V)$ for all $f$ and $g$ in $A(P_V)$. To do this, it is enough to prove that $[f, g]$ vanishes on $D_0 \cup S_e \cup S_c$. For any $(x, k, x) \in D_0 \cup S_e \cup S_c$, the product $f * g$ is given by

$$f * g(x, k, x) = \sum_{i \in \mathbb{N}_k} f(x, i, u)g(u, k - i, x)$$

The sum is taken over all $i \in \mathbb{Z}$ and $u$ for which $(x, i, u)$ and $(u, k - i, x)$ lie in $P_V$. But this requires $x \preceq u$ and $u \preceq x$; thus $u = x$. Furthermore, $i$ and $k-i$ cannot have opposite signs; otherwise $x$ would have to possess a tail consisting only of $n$’s and a tail consisting only of $1$’s. If $N_k$ is the set $\{0, \ldots, k\}$ when $k \geq 0$ and the set $\{k, \ldots, 0\}$ when $k \leq 0$, then the formula reduces to

$$f * g(x, k, x) = \sum_{i \in N_k} f(x, i, x)g(x, k - i, x).$$

The change of index $j = k - i$ now yields:

$$f * g(x, k, x) = \sum_{i \in N_k} f(x, i, x)g(x, k - i, x)$$

$$= \sum_{j \in N_k} g(x, j, x)f(x, k - j, x)$$

$$= g * f(x, k, x).$$

This shows that $f * g - g * f$ vanishes on $D_0 \cup S_e \cup S_c$. Thus $[f, g] \in A(R) = \text{rad} \ A(P_V)$.

It remains to show that $\text{rad} \ A(P_V) = A(R) \subseteq \text{com} \ A(P_V)$. By means of the same compactness argument used in the proof of Proposition 6, it suffices to show that for each point $(x, k, y) \in R$, there is a neighborhood $U_{\alpha, \beta}$ of $(x, k, y)$ contained in $R$ such that $S_\alpha S_\beta^* \in \text{com} A(P_V)$.

Let $(x, k, y) \in R$. Then $x \prec y$. As in Proposition 6, there are finite strings $\alpha$ and $\beta$ and an infinite string $\gamma$ such that $\alpha \prec \beta$, $x = \alpha \gamma$, $y = \beta \gamma$, and $k = \ell(\alpha) - \ell(\beta)$. Furthermore, the partial isometry $S_\alpha S_\beta^* \in \text{rad} A(P_V)$. We know that there is a projection $p$ in the Volterra nest such that $S_\alpha S_\beta^* = p S_\alpha S_\beta^* p^\perp$. Since the range interval for $S_\alpha S_\beta^*$ lies to the left of the domain interval, we can even arrange that $p$ corresponds to an initial segment in $[0, 1]$ with $n$-adic right endpoint; in particular, we may assume that $p \in A(P_V)$. We then have $[p, S_\alpha S_\beta^*] = p S_\alpha S_\beta^* - S_\alpha S_\beta^* p = S_\alpha S_\beta^*$. Thus $S_\alpha S_\beta^* \in \text{com} A(P_V)$. \qed
Remark 5. The proof of the preceding theorem in [9] makes use of an intermediate result: the closed commutator ideal is equal to the closed linear span of \( \{ a \in O_n \mid a = p_s a (1 - p_t) \text{ for some } s < t \} \), which we denote by \( J \). Since the generators of \( J \) are obviously in \( \text{rad } A(P_V) = \text{com } A(P_V) \), we can establish this result in our framework by showing that \( \text{rad } A(P_V) \subseteq J \). But if \((x, k, y) \in R\), then (as above) there are finite strings \( \alpha \prec \beta \) such that \((x, k, y) \in U_{\alpha, \beta} \subseteq R \) and the range interval associated with \( S_\alpha S_\beta^* \) lies to the left of the domain interval associated with \( S_\alpha S_\beta^* \). Hence we can find \( s < t \) such that \( S_\alpha S_\beta^* = p_s S_\alpha S_\beta^*(1 - p_t) \). Any Cuntz partial isometry in \( \text{rad } A(P_V) = A(R) \) can be written as a sum of Cuntz partial isometries of this type, so \( A(R) \subseteq J \).

Remark 6. Power also proves that the (commutative) algebra \( A(P_V)/\text{rad } A(P_V) \) is isomorphic to a function algebra. The function algebra is defined on a subset \( Y \) of \( X \times \mathbb{D} \); this subset consists of all points \((x, 0)\) for which \( x \) is not \( n \)-adic together with all points \((x, z)\) where \( x \) is \( n \)-adic and \( z \in \mathbb{D} \), the closed unit disk in \( \mathbb{C} \). We will not reprove this result, but it is worthwhile to state explicitly the homomorphism from \( A(P_V) \) (viewed as functions supported on \( P_V \)) to this function algebra which induces Power’s isomorphism on \( A(P_V)/\text{rad } A(P_V) \).

Let \( S = D_0 \cup S_\alpha \cup S_\beta = P_V \setminus R \) and let \( \Phi \) be the homomorphism on \( A(P_V) \) which induces Power’s isomorphism. For \( f \in A(P_V) \), \( \Phi(f) \) depends only on \( f|_S \). If \( x \) is not \( n \)-adic, then \( \Phi(f)(x, 0) = f(x, 0, x) \). If \( x \) is \( n \)-adic with a tail consisting of 1’s, then

\[
\Phi(f)(x, z) = \sum_{k=0}^{\infty} f(x, k, x)z^k.
\]

And, if \( x \) is \( n \)-adic with a tail consisting of \( n \)'s, then

\[
\Phi(f)(x, z) = \sum_{k=-\infty}^{0} f(x, k, x)z^{-k}
\]

When \( f \in \text{rad } A(P_V) \), \( \Phi(f) = 0 \); therefore \( \Phi \) induces a map on the quotient of \( A(P_V) \) by its radical. This map is an isometric isomorphism of operator algebras. In particular, as in [9], if \( \alpha \) and \( \beta \) are strings corresponding to intervals which share a left endpoint or share a right endpoint, and if \( S_\alpha S_\beta^* \in A(P_V) \), then \( \Phi(S_\alpha S_\beta^*) \) is the monomial \( z^k \), where \( k \) is the absolute value of the index of dilation for \( S_\alpha S_\beta^* \).

We have seen how spectral techniques give new proofs for several of Power’s results on the Volterra subalgebra of \( O_n \). The next proposition contains new information about the Volterra subalgebra.
Proposition 7. $A(P_V)$ is a maximal triangular subalgebra of $O_n$.

Proof. Suppose that $T$ is a triangular subalgebra of $O_n$ which contains $A(P_V)$. To prove that $A(P_V)$ is maximal triangular, we need to show that $T \subseteq A(P_V)$. To do this, it is sufficient to show that if $f \in T$ and if $x$ is an $n$-adic element of $[0, 1]$, then $p_x^f p_x = 0$. It then follows that $f$ leaves invariant every projection in the Volterra nest and so lies in $O_n \cap \text{Alg } V = A(P_V)$.

Observe that $p_x f p_x^+$ leaves invariant every projection in the Volterra nest and that, when $x$ is $n$-adic, $p_x \in O_n$. So $p_x f p_x^+ \in A(P_V) \subseteq T$. Therefore, $g = p_x^f p_x + p_x f p_x^+$ is a self-adjoint element of $T$. Consequently, $g$ lies in the diagonal of $T$. But the diagonal is $A(D_0)$, so $g$ commutes with $p_x$. Therefore $0 = p_x^+ g p_x = p_x^f p_x$ and the maximal triangularity follows.

□

Remark 7. There is no analytic subalgebra $A(P)$ defined by a cocycle $d$ with the property that $P_V \setminus D_0 \subset \{(x, k, y) \in G_n \mid d(x, k, y) > 0\}$. Call this latter set $P^+$ and similarly define $P^-$ and $P^0$.

Now $A(P^+ \cup D_0)$ is a triangular algebra containing $A(P_V)$; by Proposition 7 the two are equal. Write

$$G_n = [(P^+ \cup D_0) \cup (P^- \cup D_0)] \cup [P^0 \setminus D_0]$$

$$= [P_V \cup P_V^{-1}] \cup [P^0 \setminus D_0].$$

As both $P^0$ and $D_0$ are clopen, so is their difference. Since the two terms in square brackets are disjoint, we conclude that $P_V \cup P_V^{-1}$ is closed. But the same argument that was used to show that $P_V$ is not closed also shows that $P_V \cup P_V^{-1}$ is not closed. Thus we have a contradiction. This shows that $A(P_V)$ is not contained in an analytic subalgebra of $O_n$ with diagonal $A(D_0)$.

References

[1] Joachim Cuntz, Simple $C^*$-algebras generated by isometries, Comm. Math. Phys. 57 (1977), no. 2, 173–185. MR 57 #7189
[2] Joachim Cuntz and Wolfgang Krieger, A class of $C^*$-algebras and topological Markov chains, Invent. Math. 56 (1980), no. 3, 251–268. MR 82f:46073a
[3] Kenneth R. Davidson, Elias Katsoulis, and David R. Pitts, The structure of free semigroup algebras, J. Reine Angew. Math. 533 (2001), 99–125. MR 2002a:47107
[4] Kenneth R. Davidson and David R. Pitts, Invariant subspaces and hyper-reflexivity for free semigroup algebras, Proc. London Math. Soc. (3) 78 (1999), no. 2, 401–430. MR 2000k:47005
[5] Alex Kumjian, David Pask, Iain Raeburn, and Jean Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), no. 2, 505–541. MR **98g**:46083

[6] Paul S. Muhly and Baruch Solel, *Subalgebras of groupoid C*-algebras*, J. Reine Angew. Math. **402** (1989), 41–75. MR **90m**:46098

[7] Alan L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser Boston Inc., Boston, MA, 1999. MR **2001a**:22003

[8] Gelu Popescu, *von Neumann inequality for \((B(H)^*)_1\)*, Math. Scand. **68** (1991), no. 2, 292–304. MR **92k**:47073

[9] S. C. Power, *On ideals of nest subalgebras of C*-algebras*, Proc. London Math. Soc. (3) **50** (1985), no. 2, 314–332. MR **86d**:47057

[10] Stephen C. Power, *Limit algebras: an introduction to subalgebras of C*-algebras*, Pitman Research Notes in Mathematics Series, vol. 278, Longman Scientific & Technical, Harlow, 1992. MR **94g**:46001

[11] Jean Renault, *A groupoid approach to C*-algebras*, Springer, Berlin, 1980. MR **82h**:46075

**Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487**

*E-mail address*: ahopenwa@euler.math.ua.edu

**Department of Mathematics, Iowa State University, Ames, IA 50011**

*E-mail address*: peters@iastate.edu