Dirac cohomology of the Dunkl-Opdam subalgebra via inherited Drinfeld properties

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ABSTRACT
In this paper, we define a new presentation for the Dunkl-Opdam subalgebra of the rational Cherednik algebra. This presentation uncovers the Dunkl-Opdam subalgebra as a Drinfeld algebra. We use this fact to define Dirac cohomology for the DO subalgebra. We also formalize generalized graded Hecke algebras and extend a Langlands classification to generalized graded Hecke algebras.

ARTICLE HISTORY
Received 29 May 2018
Revised 15 July 2019
Communicated by Jason Bell

KEYWORDS
Dunkl Opdam subalgebra; Dirac Cohomology; Drinfeld algebra; Langlands classification; Rational Cherednik algebra

2010 MATHEMATICS SUBJECT CLASSIFICATION
20C08; 17B22; 16S80

1. Introduction
We study the Dunkl-Opdam subalgebra, \( \mathbb{H}_{DO} \), of the rational Cherednik algebra associated to \( G(m, 1, n) = S_n \times (\mathbb{Z}_m)^n \), introduced by Dunkl and Opdam [9]. This subalgebra of the rational Cherednik algebra \( \mathcal{H}_t(G(m, 1, n)) \) (Definition 4.1) is independent of the parameter \( t \). In this chapter, we take a closer look at \( \mathbb{H}_{DO} \) and notice that it is similar to both a graded Hecke algebra and a Drinfeld algebra. We extend several results for Hecke algebras and faithful Drinfeld algebras to include the Dunkl-Opdam subalgebra. We construct a new presentation of \( \mathbb{H}_{DO} \):

Theorem. There exists a presentation of \( \mathbb{H}_{DO} \) given by elements \( \{\bar{z}_i : i = 1, \ldots, n\} \) and elements in \( G \) such that:

\[
\begin{align*}
    s_i \varpi_j s_i^{-1} &= s_i(\bar{z}_j), \\
    g_i \bar{z}_j = \bar{z}_j g_i &\quad \forall i, j = 1, \ldots, n, \\
    [\bar{z}_i, \bar{z}_j] &\in C G.
\end{align*}
\]

The explicit formulation of this expression is given in Definition 4.14.

This presentation exposes \( \mathbb{H}_{DO} \) as a Drinfeld algebra. Drinfeld [8] initially defined these algebras (Definition 2.1) with the potential to have non faithful representations. In the literature this has been largely forgotten, perhaps because there appeared to be no natural examples of a non-faithful Drinfeld algebra. The Dunkl-Opdam subalgebra is a naturally occurring non-faithful Drinfeld algebra. Ciubotaru [4] defined Dirac cohomology for faithful Drinfeld algebras and we extend this to non-faithful Drinfeld algebras.
Dezélee introduced the idea of generalized graded Hecke algebra to look at the Dunkl-Opdam subalgebra. We concretely define a class of generalized graded Hecke algebra, which contains Dezélee’s examples. We extend Evans’ [10] Langlands classification to generalized graded Hecke algebras.

**Theorem.** Let $\mathcal{G}_\mathcal{H}$ denote a generalized graded Hecke algebra. A parabolic subalgebra $B$ is denoted by $GH_B$, with $GH_P$, denoting the semisimple part of $GH_P$ (Definition 3.3).

(i) Every irreducible $\mathcal{G}_\mathcal{H}$ module $V$ can be realized as a quotient of $\mathcal{G}_\mathcal{H}(W \times T) \otimes_{\mathcal{G}_\mathcal{H}P} U$, where $U = \tilde{U} \otimes C_\nu$ is such that $\tilde{U}$ is an irreducible tempered $\mathcal{G}_\mathcal{H}P$, module and $C_\nu$ is a character of $\mathcal{S}(\alpha)$ defined by $\nu \in \alpha^+$. 

(ii) If $U$ is as in (i) then $\mathcal{H}(W \times T) \otimes_{\mathcal{G}_\mathcal{H}P} U$ has a unique irreducible quotient to be denoted $\mathcal{J}(P, U)$.

(iii) If $\mathcal{J}(P, \tilde{U} \otimes C_\nu) \cong \mathcal{J}(P', \tilde{U}' \otimes C_{\nu'})$ then $P = P'$, $\tilde{U} \cong \tilde{U}'$ as $\mathcal{G}_\mathcal{H}P$, modules and $\nu = \nu'$.

The notation for the above theorem is introduced in Section 3.

The Dunkl-Opdam subalgebra has a commutative subalgebra $CT \cong (\mathbb{Z}_m)^n$. We decompose representations into weight spaces (Definition 3.5), which then defines weights of a representation. The weights of a $\mathcal{H}_{DO}$ representation come in orbits (Lemma 3.11). We use these weights to highlight that irreducible representations of $\mathcal{H}_{DO}$ are pullbacks of $\mathcal{H}(S_{\alpha_i})$ representations via specific quotients (Lemma 5.5). Define the set $A = \{ a \in \mathbb{N}^m : \sum a_i = n \}$, there is an equivalence of irreducible modules

$$\mathcal{H}_{DO}(G(m, 1, n) - \text{ssmod}) \xrightarrow{F} \bigoplus_{a \in A} \mathcal{H}(S_{a_0}) \otimes \cdots \otimes \mathcal{H}(S_{a_{m-1}}) - \text{ssmod} \quad \text{(Theorem 5.6)}.$$ 

We use this equivalence to describe the Dirac cohomology of a $\mathcal{H}_{DO}$ module $X$ in terms of Dirac cohomology of the associated $\mathcal{H}(S_{\alpha_i})$ modules. Let $F$ and $F^{-1}$ be functors displaying this equivalence of irreducible modules, these are functors between the semisimplification of the respective module categories. We denote the semisimplification of the module categories by the suffix $-\text{ssmod}$.

**Theorem.** Given an irreducible representation $V$ with $\mathbb{C}[T]$ weight space $F(V) = V_{i_0}$, where $a = (a_0, \ldots, a_{m-1})$, let $C$ be a set of coset representatives of $S_n \mod S_P$, the space $F(V)$ is as $a \mathcal{H}_{S_{a_0}} \otimes \cdots \otimes \mathcal{H}_{S_{a_{m-1}}}$ module $F(V) \cong X_{a_0} \otimes \cdots \otimes X_{a_{m-1}}$. The Dirac cohomology of $V$ is

$$\bigoplus_{c \in C} c \left( H_D(X_{a_0}) \otimes \cdots \otimes H_D(X_{a_{m-1}}) \right),$$

where $H_D(X)$ is the type $A$ Dirac cohomology of the $\mathcal{H}_{S_{\alpha_i}}$-module $X$. Let $H_D(\bullet)$ denote the functor taking the relevant module to its Dirac cohomology. We have the following commutative diagram:

$$\begin{align*}
\mathcal{H}(G(m, 1, n) - \text{ssmod}) & \xrightarrow{F} \bigoplus_{a \in A} \mathcal{H}_{S_{a_0}} \otimes \cdots \otimes \mathcal{H}_{S_{a_{m-1}}} - \text{ssmod} \\
\downarrow H_D(\bullet) & \downarrow H_D(\bullet) \\
\mathcal{G}(m, 1, n) - \text{mod} & \xleftarrow{F^{-1}} \mathcal{C}S_{a_0} \otimes \cdots \otimes \mathcal{C}S_{a_{m-1}} - \text{mod}.
\end{align*}$$

The functor $F$ is defined in Lemma 5.4 and the functor $F^{-1}$ is defined in Corollary 6.8.

In Section 2 we study Drinfeld algebras, we focus on the fact that these algebras can be defined with a non-faithful representation. We extend Dirac cohomology defined in [4] to the non-faithful case. In Section 3, we introduce the class of generalized graded Hecke algebras and we extend Evans’ [10] Langlands classification to this class. In Section 4 we introduce the Dunkl-Opdam subalgebra. We highlight that it is a generalized graded Hecke algebra and by introducing a new presentation show that it is also a non-faithful Drinfeld algebra. Section 5 defines a Morita equivalence between the semisimplification of the module categories of DO subalgebra and direct
sums of graded Hecke algebras associated to parabolic symmetric groups. In Section 6 we combine results on Dirac cohomology (Section 2) and the Morita equivalence (Section 5) to describe Dirac cohomology of a $\mathbb{H}_{DO}$ module with Dirac cohomology of its associated module under the Morita equivalence. This highlights that the Morita equivalence behaves well with respect to Dirac cohomology.

2. Drinfeld algebras

In this section, we will define Drinfeld algebras as introduced by Drinfeld. Ciubotaru [4] defined Dirac cohomology for faithful Drinfeld algebras and we will extend Dirac cohomology to non-faithful Drinfeld algebras. The results in this section follow almost verbatim from the proofs in [4] so we will not write them here.

Given a finite group $G$, antisymmetric bilinear forms $b_g$ for $g \in G$ and a representation $(\rho, V)$ of $G$, then we construct an algebra

$$\mathbb{H} = C[G] \rtimes T(V)/R.$$ 

Here $R$ is the two sided ideal of $C[G] \rtimes T(V)$ generated by the relations,

$$g^{-1}vg = \rho(g)(v) \quad \text{for all } g \in G \text{ and } v \in V,$$

and

$$[u, v] = \sum_{g \in G} b_g(u, v)g \quad \text{for all } v, u \in V.$$

We define a filtration on the algebra $C[G] \rtimes T(V)/R$, a vector $v$ has degree 1 and a group element $g \in G$ has degree 0.

**Definition 2.1.** [8] An algebra of the form $\mathbb{H} = C[G] \rtimes T(V)/R$ is a Drinfeld algebra if it satisfies a PBW criterion. That is the associated graded algebra is naturally isomorphic to

$$C[G] \rtimes S(V).$$

Here $\rtimes$ denotes the semi direct product with the natural action of $G$ on $V$.

We state the conditions on the bilinear forms $b_g$ such that $\mathbb{H}$ is a Drinfeld algebra. This was originally stated in [8], and explained for the faithful case in [16]. Define $G(b) = \{g \in G : b_g \neq 0\}$.

**Theorem 2.2.** [8, 16, Theorem 1.9] The algebra $\mathbb{H}$ is a Drinfeld algebra if and only if:

1. For every $g \in G, b_{g^{-1}hg}(u, v) = b_h(\rho(g)(u), \rho(g)(v))$ for every $u, v \in V$,
2. For every $g \in G(b) \setminus \text{Ker}\rho$, then $\text{Ker}b_g = V^{\rho(g)}$ and $\dim(V^{\rho(g)}) = \dim V - 2$,
3. For every $g \in G(b) \setminus \text{Ker}\rho$ and $h \in Z_G(g)$, $\text{Det}(h|_{V^{\rho(g)^{\perp}}}) = 1$, where $V^{\rho(g)^{\perp}} = \{v - \rho(g)(v) : v \in V\}$.

The above statements follow immediately from the proofs given in [16] for the faithful case. The only variation is that (1) in [16] is replaced by the set $\text{Ker}\rho$.

2.1. Non-faithful Drinfeld algebras

In the recent literature, Drinfeld algebras have predominately been considered with $G$ a subgroup of $GL(V)$, however, Drinfeld originally expressed them with a potentially non-faithful representation. To address this disparity and to avoid confusion we will say that a Drinfeld algebra is a
faithful Drinfeld algebra if the representation involved is faithful and we will say that a Drinfeld algebra is non-faithful if the representation is non-faithful. The class of Drinfeld algebras includes both faithful and non-faithful Drinfeld algebras.

### 2.2. The Dirac operator for (non-faithful) Drinfeld algebras

If $V$ has a $G$-invariant symmetric bilinear form then one can define a Dirac operator $D$. In [4] Dirac cohomology is defined for any faithful Drinfeld algebra. Furthermore, an equation involving the square of the Dirac operator is proved [5, Theorem 2.7]. The extension of these theorems to the case of non-faithful representations is clear from the proofs. We will, however, give the equivalent formulation of the theorems in the non-faithful case. In this section, we will denote a Drinfeld algebra by $H$.

#### 2.2.1. The Clifford algebra

Let $\langle , \rangle$ be a $G$-invariant non-degenerate bilinear form on $V$. The Clifford algebra $C(V)$ associated to $V$ and $\langle , \rangle$ is the quotient of the tensor algebra $T(V)$ by the relations

$$v \cdot v' + v' \cdot v = -2\langle v, v' \rangle.$$ 

The Clifford algebra has a filtration by degrees and a $\mathbb{Z}/2\mathbb{Z}$-grading by parity of degrees. In this grading $C(V) = C(V)_0 \oplus C(V)_1$. We define an automorphism $\epsilon : C(V) \to C(V)$ which is the identity on $C(V)_0$ and minus the identity on $C(V)_1$. Let us extend $\epsilon$ to be an automorphism of $H \otimes C(V)$ by defining $\epsilon$ be the identity on $H$. We define an anti-automorphism, the transpose of $C(V)$, such that, $v' = -v$ for all $v \in V$. The Pin group is:

$$\text{Pin}(V) = \{a \in C(V)^\times : \epsilon(a)V a^{-1} \subset V, a^t = a^{-1}\}.$$ 

Let $(\rho, V)$ be a representation of $G$ with $G$-invariant form. This establishes $\rho(G)$ as a subgroup of $O(V)$. The Pin group is a double cover of $O(V)$ with surjection $p : \text{Pin}(V) \to O(V)$. We define the pin double cover of $\rho(G) \subset O(V)$ as

$$\rho(G) : = p^{-1}(\rho(G)) \subset \text{Pin}(V).$$ 

We construct a cover of $G$. Consider the algebra $C[G] \otimes C(V)$, we define $G$ to be the subgroup generated by $g \otimes p^{-1}(\rho(g))$ for all $g \in G$. That is

$$\tilde{G} = \{g \otimes p^{-1}(\rho(g)) | g \in G\}.$$ 

If $\text{Ker}\rho$ is abelian we can express $G$ as a semidirect product. If $\text{Ker}\rho$ is abelian then $G$ is the semi-direct product $\text{Ker}\rho \rtimes _{\rho} (G)$ with cross multiplication:

$$(h, g) \cdot (h', g') = (hp(g'^{-1})h'p(g), gg'),$$ 

for all $g, g' \in \rho(G)$ and $h, h' \in \text{Ker}\rho$.

By construction $G$ embeds in $H \otimes C(V)$ via

$$\Delta : G \to H \otimes C(V).$$ 

For more information on the Clifford algebra see [12] and [15].

#### 2.2.2. The Dirac element

Given any basis $\{v_i\}$ of $V$ and dual basis $\{v^i\}$ with respect to $\langle , \rangle$ we define the Dirac element

$$D = \sum_i v_i \otimes v^i \in H \otimes C(V).$$
We give a formula for $D^2$. This is equivalent to [4, Theorem 2.7]. The only variation being that $\text{ker} \rho$ replaces 1.

For every $g \in G(b)$ set,

$$k_g = \sum_{i,j} b_g(v_i, v_j)v'_i v_j \in C(V),$$

and

$$h = \sum_i v_i v'_i \in \mathbb{H}.$$  

The commutation relation defined for a Drinfeld algebra shows:

$$D^2 = -h \otimes 1 + \frac{1}{2} \sum_{g \in G(b)} g \otimes k_g.$$

This result is [4, Lemma 2.5]. Recall $G(b) = \{g \in G : b_g \neq 0\}$, we write $\widetilde{G}(b)$ for the cover of this subset.

**Lemma 2.3.** Similar to [4, Lemma 2.6] every element $g$ in $G(b)/\text{Ker} \rho$ can be expressed as a product of two reflections. Every element in $G(b) \setminus \text{Ker} \rho$ can be written as a coset representative of $G(b)/\text{Ker} \rho$ conjugated by an element in Kerp. Therefore given $g \in G(b) \setminus \text{Ker} \rho$, there exists an $h \in \text{Ker} p$ and $\alpha, \beta \in V$ such that $g = h^{-1}s_{\alpha}s_{\beta}h$ and the roots $\alpha, \beta$ span the space $(V^p(g))^{\perp}$. We scale $\alpha$ and $\beta$ such that $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1$.

**Proof.** See proof of [4, Lemma 2.6] with $G$ replaced by $G/\text{Ker} (\rho)$.

For every coset representative $g \in G(b)/\text{Ker} (\rho)$ define

$$\widetilde{g} = \alpha \beta \in C(V), c_\widetilde{g} = \frac{b_g(\alpha, \beta)}{1 - \langle \alpha, \beta \rangle^2} \in \mathbb{C}, e_\widetilde{g} = \frac{b_g(\alpha, \beta)\langle \alpha, \beta \rangle}{1 - \langle \alpha, \beta \rangle^2} \in \mathbb{C}.$$  

Every $x \in \widetilde{G}(b)$ can be written as $h^{-1}gh$ where $g$ is a coset representative of $\widetilde{G}(b)/\text{Ker} \rho$ and $h \in \text{Ker} \rho$. **Lemma 2.3** gives $g = s_{\alpha}g$ and $\tilde{g} = \alpha \beta \in C(V)$. We define, for $x = h^{-1}gh \in \widetilde{G}$,

$$\tilde{x} = \tilde{g} = \alpha \beta \in C(V), c_{\tilde{x}} = c_{\tilde{g}}, e_{\tilde{x}} = he_{\tilde{g}}, h^{-1}.$$  

Let us define the Casimir elements, $\Omega_\mathbb{H}$ in $\mathbb{H}$ and $\Omega_\widetilde{G}$ in $\widetilde{G}$.

$$\Omega_\mathbb{H} = h - \sum_{g \in G(b)/\text{Ker} \rho} e_g \in \mathbb{H}^G,$$

$$\Omega_\widetilde{G} = \sum_{h \in \text{Ker} \rho, g \in G(b)/\text{Ker} \rho} h^{-1} \tilde{g} h e_\tilde{g} \in \mathbb{C}[\widetilde{G}]\widetilde{G}.$$  

**Theorem 2.4.** [4, c.f. Theorem 2.7] The square of the Dirac element can be expressed as a sum of the two Casimir elements plus terms from the kernel:

$$D^2 = -\Omega_\mathbb{H} \otimes 1 + \Delta(\Omega_\widetilde{G}) + \frac{1}{2} \sum_{g \in \text{Ker} \rho} k_g.$$  

2.2.3. **Vogan’s morphism**

Let $\Omega_\mathbb{H} = \Omega_\mathbb{H} - \frac{1}{2} \otimes \sum_{h \in \text{Ker} \rho} k_h$, define

$$A = Z_{\mathbb{H} \otimes C(V)}(\Omega_\mathbb{H}) \subset (\mathbb{H} \otimes C(V))\widetilde{G}.$$  

If \( \text{Ker} \rho \cap G(b) = \emptyset \) then \( A = \mathbb{H} \otimes C(V) \). Define a derivation
\[
d : \mathbb{H} \otimes C(V) \to \mathbb{H} \otimes C(V),
\]
\[
d(a) = Da - \epsilon(a)D.
\]
The Dirac operator \( D \) interchanges the trivial and \( \text{Det} \tilde{G} \) - isotypic spaces of \( A \). We define \( d_{\text{triv}} \) and \( d_{\text{Det}} \) to be the restriction of \( d \) to the trivial and \( \text{Det} \tilde{G} \) - isotypic spaces. We state the theorems in [4] but note that the proofs apply verbatim to this case.

**Theorem 2.5.** [4, c.f. Theorem 3.5] The kernel of \( d_{\text{triv}} \) equals:
\[
\text{Ker} d_{\text{triv}} = \text{im} d_{\text{Det}} \oplus \Delta(\mathbb{C}[\tilde{G}])^G.
\]

Since \( d \) is a derivation then \( \text{Ker} d_{\text{triv}} \) is an algebra. The following theorem is the statement of Vogan’s Dirac homomorphism in the non-faithful Drinfeld case

**Theorem 2.6.** [4, c.f. Theorem 3.8] The projection \( \zeta : \text{Ker} d_{\text{triv}} \to \mathbb{C}[\tilde{G}]^G \) defined in Theorem 2.5 is an algebra homomorphism.

Note that since the image of \( \zeta \) is an abelian algebra the morphism must factor through the abelianisation of \( \text{Ker} d_{\text{triv}} \). Recall that when \( b_g = 0 \) for all \( g \in \text{Ker} \rho \) then \( \mathbb{Z}(\mathbb{H}) \otimes 1 \) is contained in \( \text{Ker} d_{\text{triv}} \). With this extra condition we can consider the dual of \( \zeta \) which relates the representations of \( \tilde{G} \) with characters of \( \mathbb{Z} \). \( \mathbb{H} \).

\[
\zeta^* : \text{Irr}(\tilde{G}) = \text{Spec} \mathbb{C}[\tilde{G}]^G \to \text{Spec} \mathbb{Z}(\mathbb{H}).
\]

Here Spec denotes the algebra of characters on a given algebra. This could potentially be used to prove similar results to [3] for complex reflection groups.

### 3. Generalized graded Hecke algebras

Ram and Shepler [16] show that there does not exist a faithful Drinfeld algebra associated to the complex reflection group \( G(m, 1, n) = S_n \rtimes \mathbb{Z}_m^n \). However, they define a candidate for an algebra that is similar to a graded Hecke algebra. Dezéée [7] introduced the term generalized graded Hecke algebra. In Section 4 we show that these algebras are non-faithful Drinfeld algebras. We define a larger group of algebras denoted generalized graded Hecke algebras or GGH for short.

Set \( A = B \) to be the free product of unital associative complex algebras.

**Definition 3.1.** Let \( W \) be a Weyl group generated by simple reflections \( s, z \in \Pi \). \( W \) acts on a commutative group \( T, t \) is a faithful complex \( W \)-representation. \( \langle , \rangle \) is a \( W \)-invariant pairing between the vector spaces \( T^* \) and \( t \). We define a parameter function
\[
\bar{\epsilon} : \Pi \to \mathbb{C}[T].
\]

The generalized graded Hecke algebra \( \mathbb{G} \mathbb{H}(W \rtimes T) \) is the quotient of the algebra
\[
\mathbb{C}[W \rtimes T] * S(t)
\]
by the relations
\[
s_z t = s_z(t) s_z + \langle z, t \rangle \bar{\epsilon}(z), \quad \forall t \in T, z \in \Pi
\]
\[
[h, t] = 0 \quad \forall t \in T, h \in T.
\]

In the case that \( T \) is the trivial group this includes all graded Hecke algebra. In this form, the relations look very similar to the graded Hecke algebras except that the parameter function takes values in \( \mathbb{C}[T] \) instead of \( \mathbb{C} \). In the following section, we will prove a Langlands classification for generalized graded Hecke algebras. This follows Evens’ [10] proof of the Langlands classification for graded Hecke algebras.
3.1. Preliminaries for the Langlands classification

Let \( \{X, R, Y, R, \Pi\} \) be root datum, where \( X \) and \( Y \) are free finitely generated abelian groups and there exists a perfect bilinear pairing between them. The roots \( R \subset X \) and coroots \( \hat{R} \subset Y \) are finite subsets with a bijection between them. Let \( \Pi \) denote the simple roots \( \{\varepsilon_1, \ldots, \varepsilon_n\} \). Positive roots \( R_+ \) (resp. \( \hat{R}^+ \)) are the \( \mathbb{N} \) span of \( \Pi \) (respectively \( \varepsilon \) for \( \varepsilon \in \Pi \)). Let \( t = X \otimes \mathbb{C} \) and \( t^* = Y \otimes \mathbb{C} \) be dual vector spaces, similarly let \( t_{2R}, t_{2R}^* \) be the real spans of \( X \) and \( Y \). Let \( T \) be a finite abelian group such that \( W \) acts on \( T \). Let \( \tilde{c} \) be a function from \( \Pi \) to \( \mathbb{C} \) which is constant on conjugacy classes.

**Definition 3.2.** We define the generalized graded Hecke algebra, associated to the root system of \( W \), \( T \) and \( \tilde{c} \), to be the free product of algebras

\[
\mathcal{GH}(W \ltimes T) \cong \mathbb{C}[W \ltimes T] \ast S(t)
\]

modulo the relations:

\[
s_x x - s_x(x)s_x = \langle x, x \rangle \tilde{c}(x), \forall x \in t, \varepsilon \in \Pi
\]

\[
g x = x g \forall x \in t, g \in T,
\]

and the requirement that \( \mathbb{C}[W \ltimes T] \) and \( S(t) \) are subalgebra.

We will denote a generalized graded Hecke algebra by \( \mathcal{GH} \).

In the case of the generalized graded Hecke algebra associated to \( G(m, n) \), we set \( W = S_m \), \( T = (\mathbb{Z}_m)^n \). The function \( \tilde{c} \) from \( \Pi \) to \( \mathbb{C} \) is defined by \( \tilde{c}(\varepsilon_i - \varepsilon_j) = \sum_{l=0}^{m-1} g_i^{-l} g_j^{-l} \). The element \( g_i \) is the \( i^{th} \) generator of \( (\mathbb{Z}_m)^n \).

**Definition 3.3.** Given a subset \( \Pi_p \) of \( \Pi \) we can define a parabolic subgroup \( W_p \) of \( W \) generated by \( s_x \) for \( x \in \Pi_p \). The corresponding parabolic subalgebra of the generalized graded Hecke algebra \( \mathcal{GH}_p \) is generated by \( s_x \) for \( x \in \Pi \) and \( \mathbb{C}[T] \otimes S(t) \). This is the generalized graded Hecke algebra associated to \( W_p \ltimes T \).

**Definition 3.4.** Define \( a \) to be the vector space \( \{x \in t : \tilde{c}(x) = 0, \varepsilon \in \Pi\} \). Given any parabolic subalgebra \( \mathcal{GH}_p \), set \( a_p = \{x \in t : \tilde{c}(x) = 0, \varepsilon \in \Pi_p\} \). Let \( a_p \) be the perpendicular subspace to \( a_p \) under the pairing of \( t \) and \( t^* \).

Then \( \mathcal{GH}_p \cong \mathcal{GH}_p(W_p \ltimes T) \otimes S(a_p) \), where \( \mathcal{GH}_p(W_p \ltimes T) \) is constructed (Definition 3.2) as the quotient of the algebra:

\[
\mathcal{GH}_p(W_p \ltimes T) \cong \mathbb{C}[W \ltimes T] \ast S(a_p).
\]

The commutative subalgebra \( A = \mathbb{C}T \otimes S(t) \) features in all parabolic subalgebras. For every \( A \) module \( V \) we can consider a weight space decomposition.

**Definition 3.5.** Let \( A = \mathbb{C}T \otimes S(t) \) and \( A^* \) denote characters on this algebra. Given an \( A \) module \( V \) and character \( \mu \otimes \lambda \in A^* \otimes S(t)^* \) define the subspace:

\[
V_{\mu \otimes \lambda} = \{v \in V : y \otimes x(v) = \mu(y) \otimes \lambda(x)(v) \text{ for all } y \otimes x \in A\}.
\]

We can decompose \( V \) into weight spaces:

\[
V = \bigoplus_{\lambda \otimes \mu \in A^*} V_{\mu \otimes \lambda}.
\]

The weights of \( V \) are the \( \mu \otimes \lambda \in A^* \) such that \( V_{\mu \otimes \lambda} \) is non zero.

**Definition 3.6.** Given simple roots \( \varepsilon_1, \ldots, \varepsilon_n \). The fundamental coweights \( x_i \in t \) are such that

\[
\delta_i(x_i) = \delta_{ij} \text{ and } \nu(x_i) = 0 \text{ for all } \nu \in a^+.
\]
Example 3.7. Let \( W = S_n \), \( t = \text{span}\{\epsilon_1, ..., \epsilon_n\}, t^* = \text{span}\{a = \epsilon_1 + \epsilon_2 + ... + \epsilon_n\} \). Let the simple roots be \( x_i = \epsilon_i - \epsilon_{i+1} \) for \( i = 1, ..., n - 1 \). Then the fundamental coweights are

\[
x_i = \sum_{j \leq i} \epsilon_j - \frac{i}{n} (\epsilon_1 + \epsilon_2 + ... + \epsilon_n).
\]

The modification by \( \epsilon_1 + ... + \epsilon_n \) is required so that \( a^* \) is perpendicular to \( x_i \). If we had defined \( t \) to be \( \text{span}\{\epsilon_1, ..., \epsilon_n\}/\text{span}\{\epsilon_1 + ... + \epsilon_n\} \) this would not be required as \( a^* = 0 \).

Definition 3.8. An irreducible \( \mathbb{GH} \) module \( V \) is essentially tempered if for all weights \( \mu \otimes \lambda : \mathbb{CT} \otimes S(t) \to \mathbb{C} \) of \( V \), \( \text{Re}(\lambda(x_i)) \leq 0 \), for all fundamental coweights \( x_i \). The module \( V \) is tempered if \( V \) is essentially tempered and \( \text{Re}(\lambda|_{\mathbb{R}}) = 0 \). Here \( \mathbb{R} \) is the real span of \( x \in X \) perpendicular to the coroots.

Let

\[
a_p^* = \{ \nu \in a_p^* : \text{Re}(\nu(x)) > 0, x \in \Pi - \Pi_p \}.
\]

3.2. The Langlands classification for generalized graded Hecke algebras

Theorem 3.9.

(i) Every irreducible \( \mathbb{GH} \) module \( V \) can be realized as a quotient of \( \mathbb{GH}(W \rtimes T) \otimes_{\mathbb{GH}_p} U \), where \( U = \hat{U} \otimes \mathbb{C}_\nu \) is such that \( \hat{U} \) is an irreducible tempered \( \mathbb{GH}_p \) module and \( \mathbb{C}_\nu \) is a character of \( S(\alpha_p) \) defined by \( \nu \in a_p^* \).

(ii) If \( U \) is as in (i) then \( \mathbb{GH}(W \rtimes T) \otimes_{\mathbb{GH}_p} U \) has a unique irreducible quotient to be denoted \( f(P, U) \).

(iii) If \( f(P, \hat{U} \otimes \mathbb{C}_\nu) \cong f(P', \hat{U}' \otimes \mathbb{C}_\nu) \) then \( P = P' \), \( \hat{U} \cong \hat{U}' \) as \( \mathbb{GH}_p \) modules and \( \nu = \nu' \).

First, we state a couple of lemmas of Langlands and a technical lemma about orbits of weights.

Let \( Z \) be a real inner product space of dimension \( n \). Let \( \{\bar{z}_1, ..., \bar{z}_n\} \) be a basis such that \( (\bar{z}_i, \bar{z}_j) \leq 0 \) whenever \( i \neq j \). Let \( \{\beta_1, ..., \beta_n\} \) be a dual basis. For a subset \( F \) of \( \Pi \), let

\[
S_F = \left\{ \sum_{j \in F} c_j \beta_j - \sum_{i \in F} d_i \bar{z}_i : c_j > 0, d_i \leq 0 \right\}.
\]

Lemma 3.10. [2, IV, 6.11] Let \( x \in Z \). Then \( x \in S_F \) for a unique subset \( F = F(x) \).

If \( x \in Z \) then let \( x_0 = \sum_{j \in F} c_j \beta_j \), where \( x \in S_F \) and \( x = \sum_{j \in F} c_j \beta_j - \sum_{i \in F} d_i \bar{z}_i \). It is clear that if \( x_0 = y_0 \) then \( F(x) = F(y) \). Define a partial order on \( Z \) by setting \( x \geq y \) if \( x - y = \sum_{i \geq 0} t_i \bar{z}_i \).

Lemma 3.11. [2, IV, 6.13] If \( x, y \in Z \) and \( x \geq y \) then \( x_0 \geq y_0 \).

Lemma 3.12. Given an irreducible \( \mathbb{GH}_p \) module \( V \), the set of weights \( \{\lambda \otimes \mu\} \) are all in the same \( W_p \) orbit.

Proof. The group \( W_p \) is the only part of \( \mathbb{GH}_p \) which does not act by eigenvalues on \( V_{\mu \otimes \lambda} \). For any \( \mathbb{GH}_p \) module \( U \) and a weight \( \mu \otimes \lambda \) the subspace

\[
\bigoplus_{w \in W_p} U_{w(\lambda) \otimes w(\mu)}
\]

is a \( \mathbb{GH}_p \) submodule of \( U \).

Proof of (i). The simple coroots \( \bar{z}_1, ..., \bar{z}_n \) will have a dual basis \( \beta_1, ..., \beta_n \) in \( t_{\mathbb{R}} \), relative to the Killing form. Let \( V \) be an irreducible \( \mathbb{GH} \) representation. Let \( \mu \otimes \lambda \) be an \( A \) weight of \( V \) which is
maximal among $Re(\lambda)$. Let $\Pi_p = F = F(Re(\lambda))$. Let $a_i$ (respectively $a_i^*$) be the elements of $t$ (respectively $t'$) perpendicular to $a_p^*$ (respectively $a_p$). The space $t^*$ splits

$$t^* = a_p^* \oplus a_p^*.$$ 

We can restrict characters of $t$ to $a_p$ (respectively $a_i$) by considering the projection of the character in $t^*$ to $a_p^*$ (respectively $a_i^*$).

Let $\nu = \mu \otimes \lambda|_{a_p}$. Since $\lambda$ was considered maximal then by construction $\nu \in a_p^*$. Let $U$ be an irreducible representation of $\mathbb{G}H_p$ appearing in $V$ such that $S(a_p)$ acts by $\nu$. Let $\mu \otimes \phi$ be a $CT \otimes S(a_i)$ weight of $U$. Since $CT \otimes S(t) \cong CT \otimes S(a_i) \otimes S(a_p)$ then $\mu \otimes \phi \otimes \nu = \mu \otimes (\phi + \nu)$ is a $CT \otimes S(a)$ weight of $V$.

$$Re(\phi + \nu) = \sum_{j \in F} c_j b_j - \sum_{j \in F} z_i a_i, c_j > 0,$$

while

$$Re(\lambda) = \sum_{j \in F} c_j b_j - \sum_{j \in F} d_j a_i, c_j > 0, d_i > 0.$$ 

To prove $U$ is a tempered representation of $\mathbb{G}H_p$, it is sufficient to prove that $z_i \geq 0$.

Let $F_2 = \{i \in F : z_i < 0\}$ and $F_1 = F - F_2$. Then $Re(\phi + \nu) \geq \sum_{j \in F} c_j b_j - \sum_{i \in F_1} z_i a_i$. Thus by Lemma 3.11 $Re(\phi + \nu)_0 \geq \sum_{j \in F} c_j b_j = Re(\lambda)_0$. But $Re(\lambda) \geq Re(\phi + \nu)$, hence $Re(\lambda)_0 = Re(\phi + \nu)_0$ and therefore $F(Re(\lambda)_0) = F(Re(\phi + \lambda)_0)$. Thus $\phi + \nu$ is in $S_F$ and $z_i \geq 0$ for all $i$.

The inclusion of $\mathbb{G}H_p$ modules $U \subset V$ induces a nonzero map $\pi : \mathbb{G}H \otimes \mathbb{G}H_p U \rightarrow V$ given by $\pi(h \otimes w) = h.w$. Since $V$ is irreducible, $V$ is a quotient of $\mathbb{G}H \otimes \mathbb{G}H_p U$.

This argument is very similar to the argument given by Evens for the case of graded Hecke algebras. Note that this argument implies that every weight $\mu' \otimes \lambda'$ of $U$ has $F(Re(\lambda')) = F$.

Proof of (ii).

The space $U$ is naturally embedded in $\mathbb{G}H \otimes \mathbb{G}H_p U$. $U$ is a $\mathbb{G}H_p$ module therefore it is invariant under $W_p$. Lemma 3.12 implies that the weights of $\mathbb{G}H \otimes \mathbb{G}H_p U$ are $w(\mu) \otimes w(\lambda)$ where $w \in W$ and $\mu \otimes \lambda$ is a weight of $U$. Considering the weights of $(\mathbb{G}H \otimes \mathbb{G}H_p U)/U$, these are $w(\mu) \otimes w(\lambda)$ where $w \neq 1$ and is a coset representative of $W/W_p$, alternatively $w \in W^p = \{w \in W : w(R_p) \subset R^+\}$. Note that for all $w \in W$ one can write $w$ as a product of $w^p \in W^p$ and $w_p \in W_p$.

Let $\mu \otimes \lambda$ be a weight of $U$, and write

$$Re(\lambda) = \sum_{j \in F} c_j b_j - \sum_{j \in F} d_j a_i, c_j > 0, d_i > 0.$$ 

Then if $w \in W^p$, $Re(w\lambda) = \sum_{j \in F} c_j w b_j - \sum_{j \in F} d_j w a_i$. Define $\rho : t \rightarrow \mathbb{C}$ by $\rho(\tilde{\lambda}) = 1, \tilde{\lambda} \in \Pi$. Since $w : \Pi_p \rightarrow R^+$ then $\rho(w(\lambda)) \geq \rho(\lambda_i)$, for $i \in F$. Since $\beta_i$ is a fundamental weight, $w(\beta_i) \leq \beta_i$, with equality if and only if each expression of $w$ as a product of simple reflections is such that each simple reflection fixes $\beta_i$. If we make this requirement for all $j \notin F$ then this implies $w \in W_p$ hence $w \in W_p \cap W^p = \{1\}$, therefore $w = 1$. Thus we can assume that if $w \in W^p \setminus 1$ then $\rho(Re(w(\lambda))) < \rho(Re(\lambda))$.

Fix a weight $\mu \otimes \lambda$ such that $\rho(Re(\lambda))$ is maximal, then $\mu \otimes \lambda$ can not occur as a weight of $(\mathbb{G}H \otimes \mathbb{G}H_p U)/U$. This implies that if a submodule $Z$ of $\mathbb{G}H \otimes \mathbb{G}H_p U$ contains $\mu \otimes \lambda$ then $Z$ contains $U$ and hence is $\mathbb{G}H \otimes \mathbb{G}H_p U$. Define $I_{max}$ to be the sum of all submodules of $\mathbb{G}H \otimes \mathbb{G}H_p U$ which do not contain $\mu \otimes \lambda$ then $I_{max}$ is maximal and $(\mathbb{G}H \otimes \mathbb{G}H_p U)/I_{max}$ is the unique irreducible quotient.

Proof of (iii). Suppose $\pi : J(P, U) \cong J(P', U')$. Let $\mu \otimes \lambda$ (respectively $\mu' \otimes \lambda'$) be a weight of $U$ (respectively $U'$) which is maximal with respect to $\rho$. Suppose $F(Re(\lambda)) \neq F(Re(\lambda'))$. Then it
follows that $\mu \otimes \lambda$ is not a weight of $U'$ and $\mu' \otimes \lambda'$ is not a weight of $U$. Therefore $\mu \otimes \lambda$ is a weight of $(\mathbb{G}H \otimes \mathbb{G}H_p, U')/U'$ which suggests that $\rho(\text{Re}(\lambda)) < \rho(\text{Re}(\lambda'))$. However, exchanging $\lambda$ with $\lambda'$ suggests $\rho(\text{Re}(\lambda')) < \rho(\text{Re}(\lambda))$, which cannot be the case. Hence $F(\text{Re}(\lambda)) = F(\text{Re}(\lambda'))$ and $P = P'$.

Since $J(P, U) \cong J(P', U')$ is irreducible Lemma 3.12 implies there exists a $w \in W$ such that $w(\mu) \otimes w(\lambda) = \mu' \otimes \lambda'$. If we suppose that $w \not\in W_P$ then $w$ has part of its decomposition in $w^P$, by the proof of (ii) this suggests that

$$
\rho(\text{Re}(w(\lambda))) = \rho(\text{Re}(\lambda')) < \rho(\text{Re}(\lambda)).
$$

However, if this is the case then $\lambda$ is not maximal with respect to $\rho$. Therefore $w(\mu) = \mu'$ where $w \in W_P$, $\pi(U) = U'$ since $U$ (respectively $U'$) is the unique $\mathbb{G}H_p$ submodule which has a weight $\mu_1 \otimes \lambda_1$ such that $\rho(\text{Re}(\lambda)) = \rho(\text{Re}(\lambda_1))$ and $\mu$ is in the same $W_P$ orbit as $\mu_1$. Similarly $\rho(\text{Re}(\lambda')) = \rho(\text{Re}(\lambda_1))$ and $\mu'$ is in the same $W_P$ orbit as $\mu_1$. Hence $U \cong U'$ as $\mathbb{G}H_p$ submodules.

4. Dunkl-Opdam subalgebra

In this section we study the Dunkl-Opdam subalgebra, $\mathbb{H}_{DO}$ defined in [9]. The algebra $\mathbb{H}_{DO}$ is a subalgebra of the rational Cherednik algebra associated to the complex reflection group $G(m, p, n)$. This subalgebra exists for any parameter $\ell$ and its existence is independent of the parameters $c_1, \ldots, c_{n-1}$. We show that $\mathbb{H}_{DO}$ is a naturally occurring example of a non-faithful Drinfeld algebra. Section 2.2 endows $\mathbb{H}_{DO}$ with a Dirac operator. From the defining presentation given by [9], this subalgebra is a generalized Hecke algebra. Therefore, we have a Langlands classification for $\mathbb{H}_{DO}$. This sets up Section 5 in which we describe the representation theory of $\mathbb{H}_{DO}$ as blocks corresponding to multi-partitions and representations of the graded Hecke algebra of type A.

4.1. The rational Cherednik algebra

Dunkl and Opdam [9] introduced the rational Cherednik algebra. Let $G \subset GL(V)$ be a complex reflection group with reflections $S$. Let $\langle \cdot, \cdot \rangle$ be the natural pairing of $V$ and $V^*$. Let $x_i \in V$ be a $\lambda$ eigenvector for $s \in S$ and let $v_i \in V^*$ be a $\lambda^{-1}$ eigenvectors for $s \in S$ such that $\lambda \neq 1$, and $\langle x_i, v_i \rangle = 1$. For every reflection $s \in S$ introduce the parameters $\ell, c_i \in \mathbb{C}$ such that $c_s = c_{s'}$ if $s'$ and $s$ are in the same conjugacy class. The rational Cherednik algebra is defined as the quotient of the associative $\mathbb{C}$ algebra

$$
T(V \oplus V^*) \otimes \mathbb{C}[G]
$$

by the relations

$$
[x,x'] = [y,y'] = 0, \quad \text{for all } x, x', y, y' \in V, y' \in V^*,
$$

$$
[x,y] = \ell \langle x, y \rangle - \sum_{s \in S} c_s \frac{\langle x, y \rangle \langle x, v_s \rangle}{\langle x_s, v_s \rangle} s, \quad \forall x \in V, y \in V^*,
$$

$$
g^{-1} v g = g(v), \quad \forall v \in V \oplus V^*. 
$$

If one restricts to rational Cherednik algebras associated to classical complex groups $G(m, p, n)$ then [9] show there is a set of commuting operators inside the rational Cherednik algebra. The main part of these operators is quadratic on a special basis of $V$ and $V^*$. We give a particular presentation of the rational Cherednik algebra associated to $G(m, 1, n)$.

Define a generating set for $G(m, 1, n)$ consisting of the reflections $\{s_{i, i+1} : i = 1, \ldots, n-1\}$ in $S_n$ and the reflections $\{g_i : i = 1, \ldots, n\}$ which have order $m$, we may write $s_i$ for $s_{i, i+1}$. Let $\eta$ be a
primitive $m^{th}$ root of unity. Given $G(m,1,n)$ acting on $V$ let $x_i \in V$ be the vectors such that $w(x_i) = x_{w(i)}$ for $w \in S_n$ and
\[
g_i(x_j) = \begin{cases} 
\eta^{-1}x_j & \text{if } i = j, \\
x_j & \text{otherwise.}
\end{cases}
\]

Let $\{y_1, \ldots, y_n\} \in V^*$ be the dual basis to $\{x_1, \ldots, x_n\}$. For $G = G(m,1,n)$ there are $m + 1$ conjugacy classes of reflections, for reflections in the conjugacy class of $s_{1,2}$ then let $k \in \mathbb{C}$ denote their parameter. Similarly for reflection conjugate to $g_l^i$ denote the parameter by $c_l \in \mathbb{C}$.

**Definition 4.1.** The rational Cherednik algebra for $G(m,1,n)$ and parameters $k, c_i, t, \mathcal{H}(G(m,1,n))$ is the quotient of the $\mathbb{C}$ algebra $T(V \oplus V^*) \ltimes \mathbb{C}[G(m,1,n)]$ by the relations
\[
[x_i, x_j] = [y_i, y_j] = 0, \\
[x_i, y_i] = t - k \sum_{l=1}^{m-1} \sum_{i \neq j} s_{ij} g_i^{l-1} g_j^l - \sum_{l=1}^{m-1} \gamma_l g_i^l, \\
[y_i, x_j] = \sum_{l=1}^{m-1} s_{ij} g_i^{l-1} g_j^l, \\
g^{-1} vg = g(v), \text{ for all } v \in V \text{ or } V^*.
\]

**4.2. Dunkl-Opdam quadratic operators**

**Definition 4.2.** For $i \leq n$ define elements $\mathcal{H}(G(m,1,n))$
\[
z_i = y_i x_i + k \sum_{l=1}^{m-1} \sum_{i \neq j} s_{ij} g_i^{l-1} g_j^l + \frac{1}{2} \sum_{l=1}^{m-1} c_l g_i^l + \frac{1}{2} t,
\]
\[
x_i y_i = - k \sum_{l=1}^{m-1} \sum_{i \neq j} s_{ij} g_i^{l-1} g_j^l - \frac{1}{2} \sum_{l=1}^{m-1} c_l g_i^l - \frac{1}{2} t.
\]

These operators were defined in [9] and also appeared in [11] for the specialization at $t = 1$. Martino [14] used them to study the blocks of the rational Cherednik algebra.

**Definition 4.3.** The Dunkl-Opdam subalgebra $\mathbb{H}_{DO}(G(m,1,n))$ of the rational Cherednik algebra is the subalgebra generated by $G(m,1,n)$ and $z_i$ for $i = 1, \ldots, n$.

**Remark 4.4.** The following relations hold in $\mathbb{H}_{DO}(G(m,1,n))$
\[
[z_i, z_j] = 0 \quad \text{for } i, j = 1, \ldots, n, \\
[z_i, g_k] = 0, \quad \forall i, k = 1, \ldots, n, \\
[z_i, s_i, i + 1] = 0 \quad \text{for } j \neq i, i + 1, \\
z_i s_i, i + 1 = s_i, i + 1 z_i + 1 - k \epsilon_{ij}.
\]

Here $\epsilon_{ij} = \sum_{l=1}^{m-1} g_i^l g_j^{l-1}$.

In fact $\mathbb{H}_{DO}(G(m,1,n))$ is isomorphic to the $\mathbb{C}[k]$ associative algebra generated by $z_i$ and $G(m,1,n)$ subject to the relations stated in Remark 4.4.

**4.3. Dunkl-Opdam subalgebra admits a non-faithful Drinfeld presentation**

In this section, we derive a new presentation of $\mathbb{H}_{DO}$ which demonstrates that $\mathbb{H}_{DO}$ is a non-faithful Drinfeld algebra. Thus, simultaneously showing that one can associate a Drinfeld algebra to $G(m,1,n)$, also giving a natural example of a non-faithful Drinfeld algebra.
We introduce Jucys-Murphy elements for $G(m, 1, n)$. These are well known, however, the tool that we use here is that we consider two different sets of Jucys-Murphy elements.

**Definition 4.5.** We define Jucys-Murphy elements for $G(m, 1, n)$.

\[
M_i = \sum_{k < i} \sum_{s=0}^{m-1} s_k g_k^{-s} g_i^s,
\]

\[
M_j = \sum_{k > j} \sum_{s=0}^{m-1} s_k g_k^{-s} g_j^s.
\]

The commutator $[M_i, M_j] = 0 = [M_i, M_j]$ by a standard argument using the fact that $\sum_{i \leq j} M_i$ is in the centralizer of the subgroup generated by $\{s_{k-1}, g_k : k \leq i\}$. It should be noted that $[M_i, M_j] \neq 0$ for $i > j$. Furthermore

\[
s_i M_i = M_{i+1} s_i - \sum_{j=0}^{m-1} \zeta_i^{-j} g_i^j,
\]

and

\[
s_i M_j = M_{j+1} s_i + \sum_{j=0}^{m-1} \zeta_j^{-j} g_j^j.
\]

If we adjust $z_i$ by $-kM_i$ or $kM_i$, $\tilde{z}_i = z_i - kM_i$ ($\tilde{z}_i = z_i + kM_i$ respectively) the elements $\tilde{z}_i$ satisfy the relations $s_i \tilde{z}_i \tilde{s}_i = \tilde{z}_{i+1}$ and $g_i \tilde{z}_i g_i^{-1} = \tilde{z}_i$. Hence, we obtain an action of $G(m, 1, n)$ on the set $\tilde{z}_i$. However, the set $\{\tilde{z}_i\}$ no longer commutes. This presentation was given in [9, Corollary 3.6] where the symbol $T_i x_i$ denotes $\tilde{z}_i$. We provide an exposition of the presentation with $\{\tilde{z}_i\}$ using a simple automorphism of $H_{DO}$.

**Corollary 4.6.** [9, Corollary 3.6] Let $\tilde{z}_i = z_i - kM_i$ then $\tilde{z}_i$ and $G$ generate $H_{DO}$ and the following relations define $H_{DO}(G(m, 1, n))$:

\[
[s_j \tilde{z}_i, s_j] = 0,
\]

\[
[s_i \tilde{z}_j, s_i] = \zeta_i^{s_i s_j},
\]

\[
[\tilde{z}_i, \tilde{z}_j] = k(\tilde{z}_i - \tilde{z}_j) \sum_{j=0}^{m-1} \zeta_i^{-j} g_i^{-j} g_j^j.
\]

**Lemma 4.7.** Let $\Phi : H_{DO} \to H_{DO}$ such that

\[
\Phi(z_i) = -z_{n+1-i},
\]

\[
\Phi(s_i) = s_{n-i},
\]

\[
\Phi(g_i) = g_{n+1-i}.
\]

The map $\Phi$ is an automorphism of $H_{DO}$. Furthermore $\Phi(M_i) = M_{n+1-i}$.

**Proof.**

\[
\Phi(s_i z_i - z_{i+1} s_i - k \epsilon_{i+1}) = -s_{n-i} z_{n+1-i} + z_{n-i} s_{n-i} - k \epsilon_{n-i, n+1-i}.
\]

We formally define $\Phi$ as a map from $CG \times S(V) \to H_{DO}$ then $\Phi$ takes the set of defining relations in $H_{DO}$ to itself. $\Phi$ is surjective since it takes generators to generators. Hence we can define $\Phi$ as a automorphism on $H_{DO}$.

Using $\Phi$ we define the presentation of $H_{DO}$ with generators $\{\tilde{z}_i\}$. □
Lemma 4.8. Let \( \hat{z}_j = z_j + kM_j \), then the set \( \hat{z}_j \) and \( G \) generate \( \mathbb{H}_{DO} \). Further the following relations hold:

\[
[\hat{z}_i, \hat{z}_j] = -k(\hat{y}_j - \hat{z}_j) \sum_{s=0}^{m-1} s_i, g_i^{-s} g_j^s,
\]

\[
s_i \hat{z}_j = \hat{z}_{i+1} s_i,
\]

\[
[g_i, \hat{z}_j] = 0, \forall j \neq i, i + 1,
\]

\[
[\hat{g}_i, \hat{z}_j] = 0.
\]

Proof. \( \Phi(\hat{z}_i) = \Phi(z_i - kM_i) = -z_{i+1} - kM_{i+1} = -\hat{z}_{i+1} \). Hence \( \hat{z}_j \) and \( G \) generate \( \mathbb{H}_{DO} \) since they are images of a generating set under the automorphism \( \Phi \). From the definition of \( \Phi \),

\[
[\hat{z}_i, \hat{z}_j] = \Phi\left(-\hat{z}_{i+1}, -\hat{z}_{j+1}\right) = \Phi\left(\hat{z}_{i+1}, \hat{z}_{j+1}\right)
\]

\[
= \Phi(\hat{z}_{i+1} - \hat{z}_{j+1}) \sum_{s=0}^{m-1} s_i, g_i^{-s} g_j^s = -\hat{z}_{i+1} \sum_{s=0}^{m-1} s_i, g_i^{-s} g_j^s.
\]

Similarly

\[
s_i \hat{z}_j = -\Phi(s_{i+1} - s_{j+1}) = -\hat{z}_{i+1} \hat{z}_i.
\]

The relations we gave are images of relations in the second presentation under \( \Phi \). Furthermore, the generators and relations are exactly the images of the generators and relations of a presentation hence 4.8 gives another presentation of \( \mathbb{H}_{DO} \).

We will now work towards a fourth presentation of \( \mathbb{H}_{DO} \). We are aiming for a Drinfeld presentation of \( \mathbb{H}_{DO} \). We observe that the commutators of the set \{\( \hat{z}_j \)\}, and similarly \{\( \hat{z}_j \)\} can be expressed as commutators of the Jucys-Murphy elements and \( z_j \).

Lemma 4.9. The commutator of the operators \( \hat{z}_i \) and \( \hat{z}_j \) are such that:

\[
[\hat{z}_i, \hat{z}_j] = k([z_j, M_j] - [z_i, M_j]).
\]

Similarly for \( \hat{z}_i \) and \( M_j \):

\[
[\hat{z}_i, \hat{z}_j] = k([z_i, M_j] - [z_j, M_j]).
\]

Proof.

\[
[\hat{z}_i, \hat{z}_j] = [z_i - kM_j, z_j - kM_i] = [z_j, z_j] - k[z_j, M_j] + k[z_i, M_j] + k^2 [M_i, M_j],
\]

since \([M_i, M_j] = [M_j, M_i] = [z_i, z_j] = 0\).

Lemma 4.10. For operators \( \hat{z}_i \) and \( \hat{z}_j \),

\[
[\hat{z}_i, \hat{z}_j] + [\hat{z}_j, \hat{z}_i] \in \mathbb{C}G.
\]

Proof. Using Corollary 4.6 and Lemma 4.8 we can expand the commutators:

\[
[\hat{z}_i, \hat{z}_j] + [\hat{z}_j, \hat{z}_i] = (\hat{z}_i - \hat{z}_j) \sum_{s=0}^{m-1} s_i, g_i^{-s} g_j^s - (\hat{z}_i - \hat{z}_j) \sum_{s=0}^{m-1} s_i, g_i^{-s} g_j^s.
\]
Writing out $\hat{z}_i$ and $\hat{z}_j$ in terms of the commuting operators $z_i$ one obtains
\[
(z_i - kM_i - z_j + kM_j) \sum_{s=0}^{m-1} s_i g_t^{-s} g_j^s - (z_i + kM_i - z_j - kM_j) \sum_{s=0}^{m-1} s_i g_t^{-s} g_j^s.
\]
Canceling out the operators $z_i$ we arrive at the element of the group algebra
\[
[\hat{z}_i, \hat{z}_j] + [\hat{z}_j, \hat{z}_i] = k(M_j + M_i - M_i) \sum_{s=0}^{m-1} s_i g_t^{-s} g_j^s \in \mathbb{C}G.
\]

\[\square\]

**Lemma 4.11.** The commutators of $z_i - \frac{k}{2} M_i + \frac{k}{2} M_i$ are in $\mathbb{C}G$.

\[
\left[ z_i - \frac{k}{2} M_i + \frac{k}{2} M_i, z_j - \frac{k}{2} M_j + \frac{k}{2} M_j \right] \in \mathbb{C}G.
\]

**Proof.** Expanding out the commutator linearly:
\[
\left[ z_i - \frac{k}{2} M_i + \frac{k}{2} M_i, z_j - \frac{k}{2} M_j + \frac{k}{2} M_j \right] = [z_i, z_j] + \frac{k}{2} \left( [z_j, M_i] - [z_i, M_j] + [z_i, M_j] - [z_j, M_j] \right)
+ \left( \frac{k}{2} \right)^2 \left( [M_i, M_j] - [M_i, M_j] - [M_j, M_i] + [M_j, M_i] \right)
\]
Using Lemma 4.9:
\[
= \frac{1}{2} \left( [\hat{z}_i, \hat{z}_j] + [\hat{z}_j, \hat{z}_i] \right) - \left( \frac{k}{2} \right)^2 \left( [M_i, M_j] + [M_i, M_j] \right) \in \mathbb{C}G,
\]
\[
= \frac{k}{2} (M_j + M_j - M_i - M_i) \sum_{s=0}^{m-1} s_i g_t^{-s} g_j^s - \left( \frac{k}{2} \right)^2 \left( [M_i, M_j] + [M_i, M_j] \right) \in \mathbb{C}G.
\]

\[\square\]

**Lemma 4.12.** $s_j \left( z_i - \frac{k}{2} M_i + \frac{k}{2} M_j \right) s_j^{-1} = (z_{s_j(i)} - \frac{k}{2} M_{s_j(i)} + \frac{k}{2} M_{s_j(i)})$.

**Proof.** The result follows from compiling these three relations:
\[
s_i z_i = z_{i+1} s_i + k \epsilon_{i,i+1},
\]
\[
s_i M_i = M_{i+1} s_i + \epsilon_{i,i+1},
\]
\[
s_i M_i = M_{i+1} s_i - \epsilon_{i,i+1}.
\]
\[\square\]

**Theorem 4.13.** There exists a presentation of $\mathbb{H}_{DO}$ given by elements $\{\tilde{z}_i : i = 1, ..., n\}$ and elements in $G$ such that:
\[
s_i \tilde{z}_j s_i^{-1} = s_i(\tilde{z}_j),
\]
\[
g_i \tilde{z}_j = \tilde{z}_j g_i \forall i, j = 1, ..., n,
\]
\[
[\tilde{z}_i, \tilde{z}_j] \in \mathbb{C}G.
\]
Proof. Let \( \tilde{z}_i = \frac{1}{2}(\tilde{z}_i + \tilde{z}_j) = z_i - \frac{k}{2}(M_i - M_j) \) then the first two relations follow from Lemma 4.12 and by Lemma 4.11 their commutant is in \( \mathbb{C}G \). One may be worried that we have defined an algebra that surjects onto \( \mathbb{H}_{DO} \) but does not procure an injection. However performing the above arguments in reverse setting \( z_i = \tilde{z}_i + \frac{k}{2}(M_i - M_j) \) shows that the original relations follow from these relations.

\[ \square \]

Definition 4.14. Give \( V \) a basis \( \{v_i\} \) and recall that \( S_n \) act on this basis by permutations. Let \( \theta \) be the homomorphism of \( G(m, 1, n) \) onto \( S_n \). \( (V, \phi) \) is the standard representation of \( S_n \), now define \( (V, \rho) \) to be the representation of \( G \) via the projection onto \( S_n \), that is, \( \rho(g) = \phi(\theta(g)) \). We define skew-symmetric forms on \( V \) for elements in \( G(m, 1, n) \):

\[
b_{i,j,k}(v_p, v_q) = \frac{k^2}{2}((\epsilon_i - \epsilon_j, v_p)(\epsilon_j - \epsilon_k, v_q) - (\epsilon_i - \epsilon_j, v_q)(\epsilon_j - \epsilon_k, v_p)),
\]

for \( 0 < i < j < k \leq n \),

\[
b_{i,j,k}^{l,l'}(v_p, v_q) = b_{i,j,k} \quad \text{for all } l, l' = 0, ..., m - 1
\]

\[
b_g = 0 \quad \text{otherwise.}
\]

Theorem 4.15. The algebra \( \mathbb{H}_{DO} \) is a Drinfeld algebra. More concretely \( \mathbb{H}_{DO} \) is isomorphic to \( \mathbb{C}G\mathcal{T}(V) \) with the relations:

\[
[u, v] = \sum_{g \in G} b_g(u, v)g \quad \forall u, v \in V,
\]

where \( b_g \) are skew-symmetric forms on \( V \) defined in Definition 4.14.

Proof. Conjugating \( b_g \) by \( g_i \) must fix \( b_g \) since \( g_i \) acts trivially on \( V \). Quotienting \( \mathbb{H}_{DO}(G(m, 1, n)) \) by \( g_i - 1 \) gives a quotient isomorphic to the graded Hecke algebra of type A, \( \mathbb{H}(S_n) \). Hence the forms \( b_g \) must agree with, under the quotient the forms that construct the Drinfeld presentation of the graded Hecke algebra for \( S_n \). The forms \( b_{s_{ij}^2g} \) descend to the forms, labeled the same element, defining the graded Hecke algebra as a Drinfeld algebra. Conjugating by various \( g_i^l \) gives the forms \( b_{s_{ij}^2g}^{l,l'} \) above. Since \( b_1 = 0 \) in \( \mathbb{H}(S_n) \) then \( b_k = 0 \) for all \( k \in \ker \rho \). There are no other elements of \( G(m, 1, n) \) such that \( \dim V^g = \dim V - 2 \), therefore the rest of the \( b_g = 0 \) for all \( g \) not mentioned above.

\[ \square \]

4.4. Dunkl-Opdam subalgebra is a generalized graded Hecke algebra

Recall Definition 3.1 of the generalized graded Hecke algebra associated to the root system of \( W \), \( T \) and parameter function \( \tilde{c} \). Let \( \epsilon_{ij} = \sum_{l=1}^{n-1} g_i^l g_j^{-l} \in \mathbb{C}G(m, 1, n) \). The algebra \( \mathbb{H}_{DO} \) is isomorphic to, as a vector space, \( \mathbb{C}[G(m, 1, n)] \otimes S(V) \), with multiplication such that \( \mathbb{C}[G(m, 1, n)] \) and \( S(V) \) are subalgebras and the following cross relations hold:

\[
[z_i, g_k] = 0, \quad \forall i, k = 1, ..., n,
\]

\[
[z_j, s_{i,i+1}] = 0 \quad \text{for } j \neq i, i + 1,
\]

\[
0 = (i, i + 1) z_{i+1} - k \epsilon_{ij}.
\]

If we substitute \( G(m, 1, n) \cong S_n \times (\mathbb{Z}/m\mathbb{Z})^n \) for \( W \times T \) then \( \mathbb{H}_{DO} \) is a generalized Hecke algebra with parameter function

\[
\tilde{c}(s_y) = k \epsilon_{ij} \in \mathbb{C}[(\mathbb{Z}_m)^n].
\]
Since $H_{DO}$ is a generalized graded Hecke algebra we can apply the Langlands classification from Section 3.2. Therefore, we can construct every representation of $H_{DO}$ as a quotient of the module inducted from a tempered module of a parabolic subalgebra.

**Corollary 4.16.** Let $U = U \otimes \mathbb{C}_\nu$ be such that $U$ is a tempered $H_p$ module and $\nu$ is a character of $\lambda^*$. Every irreducible representation of $H_{DO}$ can be constructed as a quotient of a tempered module of a parabolic subalgebra $H_p$. That is it is a quotient of $H_{DO} \otimes H_k U$.

### 5. Constructing the representations of $H(G(m, 1, n))$ from $H(S_n)$

In this section we prove the irreducible representations of $H(G(m, 1, n))$ can be built up from blocks of irreducible representations of the graded Hecke algebras associated to the symmetric group. This is very similar to how one can build the representations of $W(B_n)$ from the pullback of two representations of symmetric groups, $S_a$ and $S_b$, where $a + b = n$.

We denote the usual graded Hecke algebra of type $A_{k-1}$ by $H(S_k)$, $\eta$ denotes a fixed primitive $m^{th}$ root of unity.

We define $\mathbb{N}$ to include zero. Let $A \subset \mathbb{N}^m$ be the set of vectors such that the coordinates sum to $n$. Then let $a = (a_0, \ldots, a_{m-1})$ be a vector in $A$, explicitly $\sum_{i=0}^{m-1} a_i = n$. We define the character $\mu_a \in \mathbb{C}[T]^*$ by

$$\mu_a(g_i) = \eta^i$$

where $\sum_{k=0}^{i-1} a_k < j \leq \sum_{k=0}^i a_k$.

This character takes the first $a_0$ reflections to 1 it then takes the following $a_1$ reflections to $\eta$ then the following $a_2$ to $\eta^2$ and continues in this way. Finally it takes the last $a_{m-1}$ reflections to $\eta^{m-1}$. The set $A$ will become a parametrizing set.

**Example 5.1.** Let $n = 5$ and $m = 3$, define $\omega$ to be a primitive 3rd root of unity. The character of $\mathbb{C}[(\mathbb{Z}/3)^3]$ associated to the vector $(1, 1, 3)$ is such that;

$$\mu_{(1,1,3)}(g_1) = 1, \quad \mu_{(1,1,3)}(g_2) = \omega, \quad \mu_{(1,1,3)}(g_3) = \mu_{(1,1,2)}(g_1) = \mu_{(1,1,3)}(g_3) = \omega^2.$$

If we take the $S_n$ orbits of $\mathbb{C}[T]^*$ then a representative set of these orbits is

$$\{\mu_a | a \in A\}.$$

Let $\text{Irr}(H(G(m, 1, n)))$ be the set of isomorphism classes of irreducible modules for $H(G(m, 1, n))$. We define $\text{Irr}(H(G(m, 1, n))|\mu_a)$ to be the subset of $\text{Irr}(H(G(m, 1, n)))$ consisting of representations that have a weight $\mu_a \otimes \lambda$ for any $\lambda \in S(V)^\perp$. Similarly, we will denote the set of irreducible representations of a complex algebra $B$ by $\text{Irr}(B)$.

**Lemma 5.2.** The irreducible representations of $H(G(m, 1, n))$ split into disjoint sets labeled by $A$,

$$\text{Irr}(H(G(m, 1, n))) = \bigcup_{\mu_a \in A} \text{Irr}(H(G(m, 1, n))|\mu_a).$$

**Proof.** Since every irreducible representation of $H(G(m, 1, n))$ has at least one $\mathbb{C}[T]$ weight then by Lemma 3.12 it must contain one and only one $S_n$ orbit, hence it must contain exactly one $\mu_a$. Therefore, every irreducible representation $V$ is in exactly one of the sets $\text{Irr}(H(G(m, 1, n))|\mu_a)$. 


Let $S_{a_0} \times S_{a_1} \times \cdots \times S_{a_{m-1}}$ be the parabolic subgroup of $S_n$ generated by $s_x$ for

$$
\Pi_a = \left\{ \epsilon_i - \epsilon_{i+1} \sum_{k=0}^{j-1} a_k \leq i < \sum_{k=0}^{j} a_k \text{ for some } j \right\} \subset \Pi.
$$

Fix $c \in A$. The stabilizer, $\text{stab}(\mu_c) \subset \mathbb{H}(G(m, 1, n))$, of the character $\mu_c$ is generated by $\mathbb{C}[T]$, $\mathcal{S}(V)$ and $s_i \in S_{a_0} \times S_{a_1} \times \cdots \times S_{a_{m-1}} \subset S_n$. $\Pi_a$ is equivalent to the set of simple roots $\epsilon_i - \epsilon_{i+1}$ such that $\mu_a(g_i) = \mu_a(g_{i+1})$. This is the parabolic subalgebra associated to the subset $\Pi_a \subset \Pi$ defined in Definition 3.3.

**Lemma 5.3.** The subalgebra $\text{stab}(\mu_a)$ which stabilizes the character $\mu_a$ is isomorphic to $\mathbb{H}(G(a_0, 1, m)) \otimes \mathbb{H}(G(a_1, 1, m)) \otimes \cdots \otimes \mathbb{H}(G(a_{m-1}, 1, m))$.

**Proof.** The subalgebra generated by $S_{a_0} \times S_{a_1} \times \cdots \times S_{a_{m-1}}, \mathbb{C}[T]$ and $S(V)$ certainly contains $\mathbb{H}(G(a_i, 1, m))$ for every $i = 0, \ldots, m - 1$. The algebra $\mathbb{H}(G(a_i, 1, m))$ consists, as a vector space of $S(V_i) \otimes \mathbb{C}[T_i] \otimes S_{a_i}$ where $V_i$ is the span of $e_i$, and $\mathbb{C}[T_i]$ is generated by $g_i$ such that $\sum_{k=0}^{i-1} a_k < j \leq \sum_{k=0}^{i} a_k$. We have $V_0 \oplus \cdots \oplus V_{m-1} = V$ hence $S(V_0) \otimes \cdots \otimes S(V_{m-1}) = S(V)$. Similarly $\mathbb{C}[T_0] \otimes \cdots \otimes \mathbb{C}[T_{m-1}] = \mathbb{C}[T]$, and $\mathbb{C}[S_{a_0}] \otimes \cdots \otimes \mathbb{C}[S_{a_{m-1}}] = \mathbb{C}[S_{a_0} \times \cdots \times S_{a_{m-1}}]$. Hence as a vector space:

$$
\text{stab}(\mu_a) = \bigotimes_{i=0}^{m-1} S(V_i) \otimes \mathbb{C}[T_i] \otimes \mathbb{C}[S_{a_i}].
$$

Each $\mathbb{H}(G(a_i, 1, m))$ is a subalgebra and as vector spaces we have equality, one just needs to check that each subalgebra commutes with the other subalgebras. We already know $S_{a_i}$ and $S_{a_j}$ commute, for $i \neq j$, and $S_{a_i}$ commutes with $\mathbb{C}[T_j]$ because $S_{a_i}$ fixes $T_j$. Similarly $S_{a_i}$ fixes $V_j$ so $s_{a_i} \in S_{a_i}$ commutes with $e_j \in S(V_j)$. 

**Lemma 5.4.** The set of irreducible representations $\text{Irr}(\mathbb{H}(G(m, 1, n))|\mu_a)$ is in natural one-one correspondence with $\text{Irr}(\text{stab}(\mu_a)|\mu_a)$. The bijection $F$ is defined by

$$
F^{-1} : \text{Irr}(\text{stab}(\mu_a)|\mu_a) \rightarrow \text{Irr}(\mathbb{H}(G(m, 1, n))|\mu_a),
$$

$$
F^{-1}(W) = \text{Ind}^{\mathbb{H}(G(m, 1, n))}_{\text{stab}(\mu_a)} W
$$

and

$$
F : \text{Irr}(\mathbb{H}(G(m, 1, n))|\mu_a) \rightarrow \text{Irr}(\text{stab}(\mu_a)|\mu_a)
$$

$$
F(U) = \text{Unique irreducible submodule of } \text{Res}^{\mathbb{H}(G(m, 1, n))}_{\text{stab}(\mu_a)} U \text{ with weight } \mu_a.
$$

This equivalence of irreducible modules defines a functor on the semisimplification of the two module categories. We also denote this functor by $F$ and $F^{-1}$.

For an irreducible module $U$ in $\text{Irr}(\mathbb{H}(G(m, 1, n))|\mu_a)$, $F^{-1}(U)$ is the $\mu_a$-weight space of $U$.

**Proof.** Let $P$ be the corresponding partition defined by the subset $\Pi_a \subset \Pi$ and set $C$ to be the set of coset representatives of the parabolic group $S_P$ in $S_n$. For a $\text{stab}(\mu_a)$ module $W$ the module $W^c$, for $c \in C$, is isomorphic to $W$ as a vector space with the action

$$
b \cdot W^c = c^{-1} W b W^c \text{ for all } b \in \text{stab}(\mu_a).
$$

We must check that $F(W)$ for $W \in \text{Irr}(\text{stab}(\mu_a))$ is an irreducible $\mathbb{H}(G(m, 1, n))$ module, the rest follows. As a vector space $\mathbb{H}(G(m, 1, n)) \cong \bigoplus_{c \in \text{C}} \text{stab}(\mu_a)c$. Here $C$ is a set of coset representative of the parabolic group $S_P$ in $S_n$. The $\text{stab}(\mu_a)$-composition series of $\text{Ind}^{\mathbb{H}(G(m, 1, n))}_{\text{stab}(\mu_a)} W$
consists of the stab($a$) modules $W^c$ where $c \in C$. We must show that $\text{Ind}^{\mathbb{H}(G(m,1,n))}_{\text{stab}(a)} W$ is irreducible. The module $W$ is an irreducible stab($a$) module. If $W$ is an irreducible $G(m,1,a_0) \times \ldots \times G(m,1,a_{m-1})$ module then utilizing Mackey’s criterion for finite groups we need to show that $W^c$ are non isomorphic. However by construction $W$ has only weights containing $\mu_a$, and $W^c$ will only have weights containing $c(\mu_a)$, and since $S_P$ is the stabilizer of $\mu_a$ in $S_n$ then for all $c \neq 1$ we have $c(\mu_a) \neq \mu_a$. Therefore each $W^c$ has a different set of weights and hence are not isomorphic. Hence if $W$ is an irreducible $G(m,1,n)$ module then using Mackey’s irreducibility criterion $\text{Ind}^{\mathbb{H}(G(m,1,n))}_{\text{stab}(a)} W$ is an irreducible $G(m,1,n)$ module and hence $F(W)$ is irreducible as a $\mathbb{H}(G(m,1,n))$ module.

If $W$ is reducible as a $G(m,1,a_0) \times \ldots \times G(m,1,a_{m-1})$ module then $W = \bigoplus V_i$ as irreducible $G(m,1,a_0) \times \ldots \times G(m,1,a_{m-1})$ modules. By the same argument as above the induction of each of these is an irreducible $G(m,1,n)$ module. We have $\text{Ind}^{\mathbb{H}(G(m,1,n))}_{\text{stab}(a)} W = \bigoplus \text{Ind}^{G(m,1,n)}_{G} V_i$ as a $G(m,1,n)$ module. Suppose that $\text{Ind}^{\mathbb{H}(G(m,1,n))}_{\text{stab}(a)} W$ is not irreducible as a $\mathbb{H}(G(m,1,n))$ module then some direct sum of $V_i$’s is a submodule. Suppose $\bigoplus_{i \in I} \text{Ind} V_i$ is a submodule. Therefore $\bigoplus_{i \in I} V_i$ is a stab($a$) submodule of $W$ hence since $W$ is irreducible as a stab($a$) module then $I = \{0,1,\ldots,m-1\}$ and the only irreducible non-trivial submodule is the whole module. Therefore $\text{Ind}^{\mathbb{H}(G(m,1,n))}_{\text{stab}(a)} W$ is irreducible.

It is easy to verify that $F^{-1} \cdot F(V) = V$ using the universal property of induced modules and similarly $F \cdot F^{-1}(W) = W$.

Given a representation of $(V, \pi) \in \text{Irr}(\text{stab}(a)|\mu_a)$ we can explicitly describe how $g_i \in G(m,1,n)$ acts. Since this algebra stabilizes $\mu_a$, this is the only $C[T]$ weight occurring in $V$. Therefore $g_i = \mu_a(g_i)Id$.

Let $x_i = e_i - e_{i+1}$, if we study the relation $s_x x_i = s_x (x_i) s_x + \sum_{l=0}^{m-1} g_s g_{i+1}^{-l} g_s g_{i+1}^{-l}$ in $\mathbb{H}(G(m,1,n))$, on $(V, \pi)$, the element $\sum_{l=0}^{m-1} g_s g_{i+1}^{-l}$ is equal to $\pi(\sum_{l=0}^{m-1} g_s g_{i+1}^{-l}) = \mu_a(g_i) \mu_a(g_{i+1})^{-1}Id$ which then equals $m$ if $\mu_a$ is constant on $g_i$ and $g_{i+1}$ and $\sum_{l=0}^{m-1} g_s g_{i+1}^{-l}$ is zero if $\mu_a(g_i) \neq \mu_a(g_{i+1})$. One can summarize, on any representation in $\text{Irr}(\text{stab}(a)|\mu_a)$

$$\sum_{l=0}^{m-1} g_s g_{i+1}^{-l} = \begin{cases} m & \text{if } s_{c_i-c_{i+1}} \in \text{stab}(\mu_a), \\ 0 & \text{otherwise}. \end{cases}$$

Recall from Lemma 5.3 that stab($a$) is isomorphic to $\mathbb{H}(G(a_0,1,m)) \otimes \ldots \otimes \mathbb{H}(G(a_{m-1},1,m))$. We have shown that if $(V, \pi) \in \text{Irr}(\text{stab}(a)|\mu_a)$ then the relations $s_x x_i = s_x (x_i) s_x + k \sum_{l=0}^{m-1} g_s g_{i+1}^{-l}$ become $s_x x_i = s_x (x_i) s_x + km$ on $V$ for all $x_i \in \Pi$ and the relations $s_x x_i = s_x (x_i) s_x + k \sum_{l=0}^{m-1} g_s g_{i+1}^{-l}$ become $s_x x_i = s_x (x_i) s_x + 0$ on $V$ for all $x_i \notin \Pi$. Hence we can conclude that if $(V, \pi) \in \text{Irr}(\text{stab}(a)|\mu_a)$ then this representation factors through the algebra

$$\mathbb{H}(S_{a_0}) \otimes \ldots \otimes \mathbb{H}(S_{a_{m-1}}),$$

via the quotient by the ideal $I_a = \langle g_i - \mu_a(g_i)Id \rangle$. Explicitly

$$\pi : \mathbb{H}(G(a_0,1,m)) \otimes \ldots \otimes \mathbb{H}(G(a_{m-1},1,m)) \rightarrow \mathbb{H}(S_{a_0}) \otimes \ldots \otimes \mathbb{H}(S_{a_{m-1}}) \rightarrow GL(V).$$

Where $\mathbb{H}(S_n)$ is the usual graded Hecke algebra associated to $S_n$, with parameter $c(z) = mk$. 

Lemma 5.5. The set \( \text{Irr}(\mathbb{H}(G(a_0, 1, m)) \otimes \ldots \otimes \mathbb{H}(G(a_{m-1}, 1, m))|\mu_a) \) is in one-one correspondence with \( \text{Irr}(\mathbb{H}(S_{a_0}) \otimes \ldots \otimes \text{Irr}(\mathbb{H}(S_{a_{m-1}}))) \).

Proof. The irreducible representation in \( \text{Irr}(\mathbb{H}(G(a_0, 1, m)) \otimes \ldots \otimes \mathbb{H}(G(a_{m-1}, 1, m))|\mu_a) \) all occur as pullbacks of the irreducible representations of \( \mathbb{H}(S_{a_0}) \otimes \ldots \otimes \mathbb{H}(S_{a_{m-1}}) \) via the specific quotient of \( \mathbb{H}(G(a_0, 1, m)) \otimes \ldots \otimes \mathbb{H}(G(a_{m-1}, 1, m))|\mu_a \) onto \( \mathbb{H}(S_{a_0}) \otimes \ldots \otimes \mathbb{H}(S_{a_{m-1}}) \), with the ideal
\[
I_a = \langle g_i - \mu_a(g_i)Id | i = 1, \ldots, n > .
\]

Furthermore given a representation \( U \) of \( \mathbb{H}(S_{a_0}) \otimes \ldots \otimes \mathbb{H}(S_{a_{m-1}}) \) one can create a representation in \( \text{Irr}(\mathbb{H}(G(a_0, 1, m)) \otimes \ldots \otimes \mathbb{H}(G(a_{m-1}, 1, m))|\mu_a) \) by pulling back the representation \( U \) from the quotient of \( I_a \).

\[\square\]

Theorem 5.6. The irreducible representations of \( \mathbb{H}(G(m, 1, n)) \) split into blocks which are induced from products of \( \mathbb{H}(S_a) \) representations:
\[
\text{Irr}(\mathbb{H}(G(m, 1, n))) \cong \bigsqcup_{\mu \in \Lambda} \text{Irr}(\mathbb{H}(S_{a_0}) \otimes \ldots \otimes \text{Irr}(\mathbb{H}(S_{a_{m-1}}))).
\]

If one considers a tempered \( \mathbb{H}(G(m, 1, n)) \) module, that is \( V \) such that the \( \mathbb{C}[T] \otimes S(t) \) weights, \( \mu \otimes \lambda \) are such that \( \text{Re}(\lambda(x_i)) \leq 0 \) for all fundamental coweights and that \( \text{Re}(\lambda|_{a_0}) = 0. \) This condition is only dependent on the \( S(t) \) weight \( \lambda \) therefore a tempered \( \mathbb{H}(G(m, 1, n)) \) correspond to a \( \mathbb{H}(S_n) \) tempered module with weight \( \lambda. \) Hence every \( \mathbb{H}(S_n) \) tempered module \( V \) there will correspond to \( m \) different \( \mathbb{H}(G(m, 1, n)) \) tempered modules. Each tempered \( \mathbb{H}(G(m, 1, n)) \) module will be a pullback of the module \( V. \) However the difference between the \( m \) different modules are that the short reflections \( g_i \) will act by \( \eta^j \) for fixed \( j = 1, \ldots, m.\)

This gives a method to parametrize the Langlands data for an irreducible \( \mathbb{H}(G(m, 1, n)) \) module via tempered modules of \( \mathbb{H}(S_a). \) Recall that every irreducible \( \mathbb{H}(G(m, 1, n)) \) module can be realized as a quotients of
\[
\mathbb{H}(G(m, 1, n)) \otimes_{\mathbb{H}(G(m, 1, n), \rho) \otimes U \otimes \mathbb{C}_
u.}
\]

If we fix an irreducible module \( X \) then using the above realization, we associate to it Langlands data \( (P, U). \)

Fix \( P = (p_0, \ldots, p_{m-1}) \) a partition of \( n \) with at most \( m \) parts. The tempered \( \mathbb{H}(G(m, 1, n)) \) modules are the pullbacks of tempered \( \mathbb{H}(S_{a_0}) \otimes \ldots \otimes \mathbb{H}(S_{a_{m-1}}) \) modules. Recall that the size of a partition \( \lambda = \{x_1, \ldots, x_j\} \) is \( \sum_{j=1}^n x_j. \) The tempered modules of graded Hecke algebra with real central character correspond to partitions \( \{e, \phi\} \) where \( \phi \) is nilpotent [6, 3.3], [13]. In the case of \( W = S_n \) \( e \) is always 1 and \( \phi \) is characterized by it’s Jordan form and hence corresponds to a partition. Hence the tempered modules of \( \mathbb{H}(S_{a_0}) \otimes \ldots \otimes \mathbb{H}(S_{a_{m-1}}) \) with real central character will correspond to a set of \( m \) partitions \( \{\lambda_0, \ldots, \lambda_{m-1}\} \) such that the sum of the sizes of the partitions \( \lambda_i \) equals \( a_i. \)

Theorem 5.7. Let \( P = (a_0, \ldots a_{m-1}) \) be a set associated to \( a = (a_0, \ldots, a_m) \) with at most \( m \) parts. We associate to \( P, \) a parabolic subalgebra \( \mathbb{H}_P(G(m, 1, n)) \subset \mathbb{H}(G(m, 1, n)). \) The tempered modules of the parabolic algebra \( \mathbb{H}_P(G(m, 1, n)) \) are built up from tempered modules of each parabolic part \( \mathbb{H}(G(m, 1, a_i)). \) By above a tempered module of \( \mathbb{H}(G(m, 1, a_i)) \) corresponds to a tempered module of \( \mathbb{H}(S_{a_0}). \) The tempered modules of \( \mathbb{H}(S_{a_0}) \) with real central character are labeled by partitions of \( a_i. \) Hence tempered module of \( \mathbb{H}_P(G(m, 1, n)) \) with real central character are labeled by multipartitions \( \{\lambda_0, \ldots, \lambda_{m-1}\} \) with \( m \) partitions such that the size of \( \lambda_i \) equals \( a_i. \) Furthermore, one can construct these tempered modules via the pullback of

\[
\mathbb{H}(G(m, 1, n)) \rightarrow_{\phi} \mathbb{H}(S_{a_0}) \otimes \ldots \otimes (\mathbb{H}(S_{a_{m-1}}) \rightarrow GL(V_{\lambda_0} \otimes \ldots \otimes V_{\lambda_{m-1}}).
6. Dirac cohomology of the Dunkl-Opdam subalgebra

In this section, we will use the description of irreducible representations from Section 5 to describe how the Dirac operator for the Dunkl-Opdam subalgebra acts on irreducible modules. We will show that the Dirac operator $D_{DO}$ for $H_{DO}$ descends to a relevant Dirac operator for a tensor of type A graded Hecke algebras and then describe the Dirac operator in terms of Dirac operators for type A parabolic algebras.

Let $A$ be a Drinfeld algebra $T(V)\times\mathbb{C}[G]/R$. We have an associated Clifford algebra $C(V)$, with respect to the $G$-invariant symmetric product $<,>$. Given a $G$-invariant basis $B$ we defined the Dirac operator to be

$$\sum_{b \in B} b \otimes b^* \in A \otimes C(V).$$

We have two presentations of the Dunkl-Opdam subalgebra, one producing the Lusztig presentation Definition 4.4 with commuting basis elements and the Drinfeld presentation used in Theorem 1 which shows that $H(G(m,1,n))$ is a Drinfeld algebra. We used the Lusztig presentation to show the equivalence of irreducibles of the Dunkl-Opdam subalgebra to a sum of tensors of type A graded Hecke algebras, this uses parabolic subalgebras. However, the Dirac theory developed for the Dunkl-Opdam subalgebra uses the Drinfeld presentation. This Drinfeld presentation does not admit parabolic subalgebras.

Let us recall that to transform between from the Lusztig presentation to the Drinfeld presentation one takes the standard basis $\{z_1, \ldots, z_n\}$ of the reflection representation of $S_n$ which along with $G(m,1,n)$ gives the Lusztig presentation. Then to obtain the Drinfeld presentation we use the generators:

$$\tilde{z}_i = z_i + \frac{k}{2} \sum_{i<j} s_{i,j} \sum_{l=1}^{m-1} g_i^{-l} g_j^l - \frac{k}{2} \sum_{j<i} s_{i,j} \sum_{l=1}^{m-1} g_i^{-l} g_j^l = z_i + \frac{k}{2}(M_i - M_i).$$

Recall $M_i$ and $\tilde{M}_i$ are Jucys-Murphy elements of $G(m,1,n)$ with reverse orderings. The Dirac element in terms of the Drinfeld presentations is:

$$D_{DO} = \sum_{i=1}^{n} \tilde{z}_i \otimes \tilde{z}_i^*.$$ 

In terms of the Lusztig presentation $\{z_i\}$ the Dirac element is

$$D_{DO} = \sum_{i=1}^{n} \left( z_i + \frac{k}{2} \sum_{i<j} s_{i,j} \sum_{l=1}^{m-1} g_i^{-l} g_j^l - \frac{k}{2} \sum_{j<i} s_{i,j} \sum_{l=1}^{m-1} g_i^{-l} g_j^l \right) \otimes z_i^*.$$ 

**Definition 6.1.** Given a $\mathbb{H}$ module $X$ and a spinor $S$ of $C(V)$, then $D_{DO} : X \otimes S \rightarrow X \otimes S$. The Dirac cohomology of $X$ with respect to $S$ is defined to be

$$\text{Ker}(D_{DO})/\text{im}(D_{DO}) \cap \text{Ker}(D_{DO}).$$

Since $D_{DO}$ sgn-commutes with the group $\tilde{G}$ then the Dirac cohomology is naturally a $\tilde{G}$ module.

From Section 5 if $V$ contains a weight $\mu_\lambda$ corresponding to $a = \{a_0, \ldots, a_{m-1}\}$ then we will write $V_{\mu_\lambda}$ for the $\mathbb{C}[T]$-weight space corresponding to the weight $a$. We can decompose $V$ into $\mathbb{C}[T]$ weight spaces,
\[ V = \bigoplus_{c \in C} V^c. \]

**Lemma 5.4** shows that \( V_{\mu_2} \) is the image of the functor \( F \). It is a \( \text{stab}(\mu_2) \) module and is the pull-back of a tensor of \( \mathbb{H}_{S_0} \) modules. A problem that occurs is that the Dirac operator \( \mathcal{D}_{DO} \) does not sit in the subalgebra \( \text{stab}(\mu_2) \). We will look at the Dirac operators already given for the standard type A graded Hecke algebra.

**Definition 6.2.** [1] For the graded Hecke algebra \( \mathbb{H}(S_k) \) the Dirac operator is

\[
D_{S_k} = \sum_{i=1, \ldots, k} \left( z_i + \frac{mk}{2} \sum_{i<j} s_{i,j} - \frac{mk}{2} \sum_{j<i} s_{i,j} \right) \otimes z^*_i.
\]

**Remark 6.3.** We abuse notation here as \( z_i \) in this context denotes the same basis as we have used in the definition of \( \mathbb{H}_{DO} \) but of course it is not in the same algebra. We justify this since all surjections of \( \mathbb{H}_{DO} \) onto \( \mathbb{H}_{S_n} \) preserve this notation. We have used the parameter \( mk \) as opposed to \( k \) for \( \mathbb{H}_{S_n} \) because naturally our map sends \( \mathbb{H}_{DO} \) to \( \mathbb{H}_{S_n} \) with parameter \( mk \).

Recall that the weight space \( V_{\mu_2} \) is naturally a \( \otimes_{i=1}^{m-1} \mathbb{H}_{S_i} \) module, via the functor \( F \) defined in the **Lemma 5.4**. We extend Definition 6.2 to define a Dirac operator for \( \otimes_{i=1}^{m-1} \mathbb{H}_{S_i} \):

\[
D_{S_{a_0} \times \cdots \times S_{m-1}} = D_{S_{a_0}} \otimes \cdots \otimes D_{S_{m-1}}.
\]

Written out explicitly this is

\[
D_{S_{a_0} \times \cdots \times S_{m-1}} = \sum_{i=0}^{m-1} \sum_{j=a_{i-1}}^{a_i} \left( z_j + \frac{mk}{2} \sum_{j<k \leq a_i} s_{j,k} - \frac{mk}{2} \sum_{a_{i-1} < j < k} s_{j,k} \right) \otimes z^*_j.
\]

Here we have associated \( \otimes_{i=1}^{m-1} S(V_i) \) with \( S(\bigoplus V_i) \). Similarly, we have substituted \( C(\bigoplus V_i) = \otimes C(V_i) \). Initially this looks like the Dirac operator for \( \mathbb{H}_{S_n} \), however one should notice that not all of the reflections are involved in this Dirac operator. We highlight this with an example.

**Example 6.4.** Let \( n = 3 \). The Dirac operator for \( \mathbb{H}_{S_3} \) is

\[
\left( z_1 - \frac{mk}{2} (1, 2) - \frac{mk}{2} (1, 3) \right) \otimes z^*_1 + \left( z_2 + \frac{mk}{2} (1, 2) - \frac{mk}{2} (2, 3) \right) \otimes z^*_2 + \left( z_3 + \frac{mk}{2} (1, 3) + \frac{mk}{2} (2, 3) \right) \otimes z^*_3.
\]

However, the Dirac operator for \( \mathbb{H}_{S_1} \otimes \mathbb{H}_{S_2} \subset \mathbb{H}_{S_3} \) is

\[
z_1 \otimes z^*_1 + \left( z_2 - \frac{mk}{2} (2, 3) \right) \otimes z^*_2 + \left( z_3 + \frac{mk}{2} (2, 3) \right) \otimes z^*_3.
\]

One can see that there are four reflections in the \( D_{\mathbb{H}(S_3)} \) not involved in \( D_{\mathbb{H}(S_1)} \times D_{\mathbb{H}(S_2)} \).

Viewing \( V_{\mu_2} \) as a \( \text{stab}(\mu_2) \) module, \( V \) is a sum of twists of \( F(V) \). Let us look at the \( \mathbb{C}[T] \)-invariant element of \( \text{stab}(\mu_2) \) which maps to \( D_{S_{a_0} \times \cdots \times S_{m-1}} \), this is:

\[
\sum_{i=0}^{m-1} \sum_{j=a_{i-1}}^{a_i} \left( z_j + \frac{k}{2} \sum_{l=1}^{m-1} g_j^{-1} g_l \sum_{j<k \leq a_i} s_{j,k} - \frac{k}{2} \sum_{l=1}^{m-1} g_j^{-1} g_l \sum_{a_{i-1} < j < k} s_{j,k} \right) \otimes z_j.
\]

Written this way one notices that this looks very similar to the \( \mathbb{H}_{DO} \) Dirac operator however it excludes the reflections that are not in the parabolic subgroup that stabilizes \( \mu_2 \). The following lemma shows that the difference vanishes on \( V_{\mu_2} \).
Lemma 6.5. Given an irreducible module \( V_{\mu} \) with \( \mathbb{C}[T] \) weight \( \mu \) then on the subspace \( F(V_{\mu}) \) the Dirac operator for \( \mathbb{H}_{DO} \) acts by the Dirac operator \( \mathcal{D}_{S_{m_0} \times \cdots \times S_{m_{m-1}}} \).

Proof. Recall that since \( V_{\mu} \) only has one \( \mathbb{C}[T] \) weight, namely \( \mu \), we can explicitly describe how \( \sum_{i=0}^{m-1} g_i^{-1} g_j^i \) acts on this subspace.

This parametrisation of pairs \( \{i, j\} \) can be described in another way. If the transposition \( s_{i,j} \) stabilizes the character \( \mu_a \) then \( \sum_{i=0}^{m-1} g_i^{-1} g_j^i = m \). However, if \( s_{i,j} \) is not in \( \text{stab}(\mu_a) \) then \( \sum_{i=0}^{m-1} g_i^{-1} g_j^i = 0 \) on the \( \mu_a \)-weight space. Ultimately this means that the Dirac operator \( \mathcal{D}_{DO} \in \mathbb{H}_{DO} \otimes L(V) \) preserves the subspace \( F^{-1}(V) \otimes \mathcal{S} \) since the transpositions included in \( \mathcal{D}_{DO} \) which do not preserve \( V_{\mu} \) are preceded by the element \( \sum_{i=0}^{m-1} g_i^{-1} g_j^i \) which acts by zero in this case. Finally since \( \mathcal{D}_{DO} \) preserves \( V_{\mu_a} \) it equals an element inside \( \text{stab}(\mu_a) \otimes \mathcal{S} \). This is the pull back of \( \mathcal{D}_{S_{m_0} \times \cdots \times S_{m_{m-1}}} \) and hence \( \mathcal{D}_{DO} \) agrees with \( \mathcal{D}_{S_{m_0} \times \cdots \times S_{m_{m-1}}} \) on the \( \mu_a \)-weight space of \( V \).

We have described how the Dirac operator acts on the \( \mu_a \) weight space of \( V \). Since \( \mathcal{D}_{DO} \) is \( G(m,1,n) \) invariant we can describe how it acts on the rest of the weight spaces. As discussed in Lemma 5.4 the other weight spaces are twists of this space by the coset representatives, \( c \in C \) of the parabolic subgroup \( S_P \) in \( S_n \). The group \( S_P \) fixes the \( \mu_a \) weight space. Therefore if \( \mathcal{D}_{DO} \) acts by \( \mathcal{D}_{S_{m_0} \times \cdots \times S_{m_{m-1}}} \) on \( V_{\mu} \) then \( \mathcal{D} \) acts by \( c\mathcal{D}_{S_{m_0} \times \cdots \times S_{m_{m-1}}} c^{-1} \) on \( \mathcal{V}_{\mu} \). Hence \( \text{Ker}(\mathcal{D}_{DO}) \subset V \otimes \mathcal{S} \) is

\[ \bigoplus_{c \in C} c \text{Ker}(\mathcal{D}_{S_{m_0} \times \cdots \times S_{m_{m-1}}}). \]

Similarly since \( \mathcal{D}_{DO} \) acts by \( \mathcal{D}_{S_{m_0} \times \cdots \times S_{m_{m-1}}} \) on \( \text{stab}(\mu_a) \otimes \mathcal{S} \) then

\[ \text{im}(\mathcal{D}_{DO}) = \bigoplus_{c \in C} \text{im}(\mathcal{D}_{S_{m_0} \times \cdots \times S_{m_{m-1}}}). \]

We can describe the Dirac cohomology of an irreducible module \( X \) in terms of the Dirac cohomology of its corresponding \( \mathbb{H}_{S_{m_0}} \otimes \cdots \otimes \mathbb{H}_{S_{m_{m-1}}} \) module.

Theorem 6.6. Given an irreducible representation \( V \) with \( \mathbb{C}[T] \) weight space \( V_{\mu_a} \) then by transforming \( V_{\mu_a} \) to a \( \mathbb{H}_{S_{m_0}} \otimes \cdots \otimes \mathbb{H}_{S_{m_{m-1}}} \) module \( X_{\mu_a} \otimes \cdots \otimes X_{\mu_{m-1}} \) the Dirac cohomology of \( V \) is

\[ \bigoplus_{c \in \mathbb{C} / \mathbb{H}_n} c \left( \text{H}_{D}(X_{\mu_a}) \otimes \cdots \otimes \text{H}_{D}(X_{\mu_{m-1}}) \right), \]

where \( \text{H}_{D}(X) \) is the Dirac cohomology of the \( \mathbb{H}_{S_n} \)-module \( X \).

Definition 6.7. The group \( \widetilde{G} \) for \( \mathbb{H}(G(m,1,n)) \) is isomorphic to \( \mathbb{H}_{S_n} \otimes \mathbb{C}[Z_n]^m \). Since this is a semidirect product irreducible representations of \( S_n \otimes \mathbb{C}[Z_n]^m \) decompose into blocks corresponding to characters \( \mu_a \in (\mathbb{C}[Z_n]^m)^* \) and under this correspondence

\[ \text{Irr}(\mathbb{H}_{S_n} \otimes \mathbb{C}[Z_n]^m|\mu_a) \cong \text{Irr}(\mathbb{H}_{S_{m_0}} \times \cdots \times \mathbb{H}_{S_{m_{m-1}}}). \]

There is a corresponding equivalence between the module category of \( \widetilde{G} \) and \( \bigoplus_{\mu_a \in A} \otimes \mathbb{C}[S_{m_0} \times \cdots \times S_{m_{m-1}}] \) Define \( F \) and \( F^{-1} \) to be the functors exhibiting this equivalence between \( G(m,1,n) \) and \( \bigoplus_{\mu_a \in A} \otimes \mathbb{C}[S_{m_0} \times \cdots \times S_{m_{m-1}}] \), similarly to Lemma 5.4.

Corollary 6.8. Let \( H_D(\bullet) \) denote the functor taking the relevant module to its Dirac cohomology.
We have the following commutative diagram:
Here \( \text{mod} \) denotes the module category and \( \text{ssmod} \) denotes the semisimplification of the module category.

**Funding**

EPRSC grant number [EP/M508111/1].

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