To understand the Abelian dominance and magnetic monopole dominance in low-energy QCD, we rewrite the non-Abelian Wilson loop into the form which is written in terms of its Abelian components or the 't Hooft-Polyakov tensor describing the magnetic monopole. This is performed by making use of a version of non-Abelian Stokes theorem. We propose a modified version of the maximal Abelian (MA) gauge. By adopting the modified MA gauge in QCD, we show that the off-diagonal gluons and Faddeev-Popov ghosts acquire their masses through the ghost–anti-ghost condensation due to four ghost interaction coming from the gauge-fixing term of the modified MA gauge. The asymptotic freedom of the original non-Abelian gauge theory is preserved in this derivation.

1 Introduction

In this talk we discuss quark confinement in QCD based on the Wilson criterion, i.e., the area (decay) law of the Wilson loop. The main subject of this talk is to understand the Abelian dominance in low-energy QCD which was found by Suzuki and Yotsuyanagi based on Monte Carlo simulations of lattice gauge theory under the maximal Abelian (MA) gauge proposed by Kronfeld et al. The Abelian dominance was predicted by Ezawa and Iwazaki immediately after the proposal of the Abelian projection by 't Hooft. The Abelian dominance and the subsequent magnetic monopole dominance is quite important to understand quark confinement from the viewpoint of the dual superconductor picture of the QCD vacuum, since the condensation of magnetic monopole can lead to the dual superconductivity based on the electro-magnetic duality argument.

In this talk we give a pedagogical introduction to a version of non-Abelian Stokes theorem (NAST). This version of the NAST clearly shows (in the operator level) the relationship between the Wilson loop and its Abelian components or the magnetic monopole. This fact is very suggestive of the possible Abelian and monopole dominance after taking the expectation value of the Wilson loop.

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For the Abelian dominance to hold in low-energy region of QCD, it is sufficient to show that the off-diagonal gluons become massive and they can be in a sense neglected in the low-energy region (although the latter statement is not necessarily true as shown in this talk). In order to really understand the Abelian dominance, we need to know the mechanism of the mass generation for the off-diagonal gluons. We propose a modified version of the MA gauge. Then we show that mass generation can be understood as a consequence of taking the (modified) MA gauge. We will give two kinds of explanations. One is given by a rather formal argument based on a novel reformulation of the gauge theory (as a perturbative deformation of a topological quantum field theory (TQFT)) proposed by the author. Another is to examine the explicit form of the gauge fixing term in the MA gauge. We give a prediction on the off-diagonal gluon mass.

2 A version of non-Abelian Stokes theorem (NAST)

We make use of a version of non-Abelian Stokes theorem (NAST) which is obtained by making use of the idea of Dyakonov and Petrov.

**Theorem.** For a closed loop \( C \), we define the non-Abelian Wilson loop operator by

\[
W_C[A] = \text{tr} \left\{ P \exp \left[ ig \oint_C dx^\mu A_\mu(x) \right] \right\} / \text{tr}(1),
\]

where \( P \) is the path-ordered product. Then it is rewritten as

\[
W_C[A] = \int d\mu_C(\xi) \exp \left[ ig \int_C dx^\alpha a^\xi_\alpha(x) \right]
\]

\[
= \int d\mu_C(\xi) \exp \left[ ig \int_S d\sigma^{\mu\nu} f^\xi_{\mu\nu}(x) \right],
\]

where

\[
a^\xi_\mu(x) := \langle \Lambda | A^\xi_\mu(x) | \Lambda \rangle, \quad A^\xi_\mu(x) = \xi^\dagger(x) A_\mu(x) \xi(x) + \frac{i}{g} \xi^\dagger(x) \partial_\mu \xi(x),
\]

and

\[
f^\xi_{\mu\nu} := \partial_\mu a^\xi_\nu - \partial_\nu a^\xi_\mu.
\]

Here \( |\Lambda\rangle \) is the highest-weight state of the representation defining the Wilson loop and the measure \( d\mu_C(\xi) \) is the product measure \( d\mu_C(\xi) = \prod_{x \in C} d\mu(\xi(x)) \) on \( G/\tilde{H} \) with the maximal stability group \( \tilde{H} \). The maximal stability group is
the subgroup leaving the highest-weight state invariant (up to a phase factor) and depends on the $G$ and the representation in question.

For $G = SU(2)$, the $\tilde{H}$ is given by the maximal torus subgroup $H = U(1)$ irrespective of the representation. Hence $G/\tilde{H} = CP^1 = F_1$. For $G = SU(N)(N \geq 3)$, however, $\tilde{H}$ does not necessarily agree with $H = U(1)^{N-1}$ depending on the representation. For $G = SU(3)$, all the representations can be classified by the Dynkin index $[m, n]$. If $m = 0$ or $n = 0$, $\tilde{H} = U(2)$ and $G/\tilde{H} = CP^2$. On the other hand, when $m \neq 0$ and $n \neq 0$, $\tilde{H} = U(1)^2$ and $G/\tilde{H} = F_2$. Here $CP^n$ is the complex projective space and $F_n$ the flag space.

This NAST is obtained by making use of the generalized coherent state. For details, see Perelomov or Feng, Gilmore and Zhang.

For the fundamental representation, the expression (4) is greatly simplified as

$\langle \Lambda | \cdots | \Lambda \rangle = 2 \text{tr}[\mathcal{H} | \cdots ]$, \hspace{1cm} $\mathcal{H} = \frac{1}{2} \text{diag} \left( \frac{N-1}{N}, \frac{-1}{N}, \ldots, \frac{-1}{N} \right)$. \hspace{1cm} (6)

Therefore, $a_\mu = A_\mu^3$ for $G = SU(2)$, and $a_\mu = A_\mu^3 + \frac{1}{\sqrt{3}} A_\mu^8$ for $G = SU(3)$. This implies that the non-Abelian Wilson loop can be expressed by the diagonal (Abelian) components. This is suggestive of the Abelian dominance in the expectation value of the Wilson loop.

The monopole dominance in the Wilson loop is also expected to hold as shown follows. We can rewrite $f_{\mu\nu}^A$ in the NAST as

$f_{\mu\nu}^A = \partial_\mu [n^A A_\nu^A] - \partial_\nu [n^A A_\mu^A] - \frac{1}{g} f^{ABC} n^A \partial_\mu n^B \partial_\nu n^C$, \hspace{1cm} (7)

where

$n^A(x) T^A = \xi(x) H \xi^\dagger(x)$. \hspace{1cm} (8)

The $f_{\mu\nu}^A$ is invariant under the full $G$ gauge transformation as well as the residual $H$ gauge transformation. (Indeed, we can write a manifestly gauge invariant form, see ref. 16.) This is nothing but the 't Hooft-Polyakov tensor of the non-Abelian magnetic monopole, if we identify $n^A(x)$ with the unit vector of the elementary Higgs scalar field in the gauge-Higgs theory:

$n^A(x) \leftrightarrow \hat{\phi}^A(x) := \phi^A(x)/|\phi(x)|$. \hspace{1cm} (9)

This implies that $n^A(x)$ is identified with the composite scalar field and plays the same role as the scalar field in the gauge-Higgs model, even though QCD has no elementary scalar field. This fact could explain why the QCD vacuum can be dual superconductor due to magnetic monopole condensation. By
introducing the magnetic monopole current \( k \) by \( k = \delta^* f \), we have another expression,

\[
W_C[A] = \int d\mu_C(\xi) \exp \left[ ig(N, k^\xi) \right], \quad N := \Delta^{-1} \star dS,
\]

where \( \Delta \) is the Laplacian and \( S \) is the area two-form on the surface spanned by the Wilson loop \( C \). Hence, the Wilson loop can also be expressed by the magnetic monopole current \( k_\mu \).

In the case of SU(2), the Wilson loop in an arbitrary representation (characterized by \( J = 1/2, 1, 3/2, \cdots \)) is written in the form,

\[
W_C[A] = \int d\mu_C(\xi) \exp \left[ ig J \oint_C dx \mu a_\xi(\xi(x)) \right],
\]

where \( a_\xi(\xi) := \text{tr} \{ \sigma_3 [\xi^\dagger(x) A_\mu(x) \xi(x)] + ig^{-1} \xi^\dagger(x) \partial_\mu \xi(x) \} \).

3 The modified MA gauge

When we calculate the expectation value \( \langle W_C[A] \rangle_{YM} \) of the Wilson loop, we must specify the procedure of the gauge fixing. In order to incorporate the magnetic monopole in the non-Abelian gauge theory without the elementary scalar (Higgs) field, we adopt the modified MA gauge to define the gauge-fixed QCD. The gauge fixing (GF) and the Faddeev-Popov (FP) term of the modified MA gauge is given by

\[
S_{GF+FP} = \int d^4x \ i \delta_B \bar{\delta}_B \left[ \frac{1}{2} A_\mu^a(x) A^{\mu a}(x) - \frac{\alpha}{2} i C^a(x) \bar{C}^a(x) \right],
\]

where \( \delta_B (\bar{\delta}_B) \) is the BRST (anti-BRST) transformation. The special case \( \alpha = -2 \) was discussed by several papers. The modified MA gauge fixing term which is the BRST and anti-BRST exact and FP conjugation invariant has a hidden \( OSp(4|2) \) supersymmetry. Due to this supersymmetry, the dimensional reduction of Parisi-Sourlas type takes place.

For simplicity, we discuss only the SU(2) case. For SU(3) case, see ref.\textsuperscript{13}.

\[
S_{GF+FP}' = \int d^4x \left\{ B^a D_\mu [a]^{ab} A^{\mu b} + \frac{\alpha}{2} B^a B^a 
+ i \bar{C}^a D_{\mu [a]}^{\cdots} C^{\mu}] C^b - ig^2 \epsilon^{cd} \epsilon^{eb} \bar{C}^a C^b A^{\mu c} A_\mu^d 
+ i \bar{C}^a \epsilon^{eb} (D_{\mu [a]}^{\cdots} A_\mu^c) C^3 
- \alpha g \epsilon^{ab} i B^a \bar{C}^b C^3 + \frac{\alpha}{4} g^2 \epsilon^{cd} \epsilon^{eb} \bar{C}^a C^b C^c C^d \right\}.
\]
Integrating out the $B^a$ field leads to

$$S'_{GF+FP} = \int d^4x \left\{-\frac{1}{2\alpha}(D_\mu[a]^{ab}A^{\mu b})^2 + i\bar{C}^a D_\mu[a]^{ab}D^\mu b - ig^2\epsilon^{a\mu b\nu}C^a A_\mu C^b - \alpha_4 g^2\epsilon^{a\mu b\nu}C^a C^b A_\mu A_\nu \right\},$$

(15)

A crucial difference of the modified MA gauge from the conventional Lorentz gauge is the necessity of four ghost interactions for renormalizability. Even for $\alpha = 0$, the four ghost interaction term is induced through radiative corrections due to the existence of the $c\bar{c}AA$ vertex.

4 Deformation of a TQFT and the dimensional reduction

The author has proposed a novel reformulation of the gauge theory, i.e., a perturbative deformation of a topological quantum field theory (TQFT). Here a part extracted from the GF+FP term (13) is identified with a TQFT. By making use of the NAST within this reformulation, we have calculated the expectation value of the Wilson loop and obtained the area law.

In this calculation, we have utilized the dimensional reduction from TQFT to NLSM. Actual calculations are performed by 1) the instanton calculus (in the dilute gas approximation) and by 2) the large N expansion (in the leading order). The static interquark potential is obtained as

$$V(R) = \sigma R - \frac{N^2 - 1}{2N} \frac{\alpha_s}{R} f(R) + \text{const.},$$

(16)

where $f(R) \to 1$ as $R \to 0$. The second term of the potential comes from the perturbative deformation part where the coefficient $(N^2 - 1)/(2N)$ for SU(N) was obtained by Prosperi. The string tension $\sigma$ is obtained for the fundamental representation as

$$\sigma = m^2 \exp \left[ -|\alpha| \frac{2\pi^2}{g^2} \right],$$

(17)

In the instanton calculus, the mass dimension $m$ is required by the dimensional reasons for defining the measure of the instanton size. On the other hand, $m$ is equal to the mass of NLSM and calculable in the large N expansion. The coupling constant $g$ should run through the renormalization of the potential $V(R)$. However, this calculation is not so easy. We look for other route.
5 Ghost self-interaction and dynamical mass generation

Integrating out off-diagonal field components \((A^a_\mu, B^a, C^a, \bar{C}^a)\) in Yang-Mills theory in the MA gauge, we can obtain an effective Abelian gauge theory written in terms of only the diagonal components \((a^i_\mu, B^i, C^i, \bar{C}^i)\). This theory called the Abelian projected effective gauge theory (APEGT) is regarded as a low-energy effective theory (LEET) of QCD. The coupling constant of APEGT has the \(\mu\) (renormalization-scale) dependence governed by the \(\beta\) function which is the same as the original Yang-Mills theory. This reflects the asymptotic freedom of the original non-Abelian gauge theory. The other RG functions and the anomalous dimensions have been calculated recently.

In the MA gauge, the renormalizability requires the existence of four ghost interactions. The modified MA gauge determines the strength of four ghost interaction where the modified MA gauge is obtained from the viewpoint of pursuing the maximal symmetry, namely, BRST, anti-BRST, FP conjugation and \(OSp(4|2)\) supersymmetry. The attractive four ghost interaction leads to two types of ghost–anti-ghost condensations

\[
\epsilon^{ab}(i\bar{C}^a(x)C^b(x)) \neq 0, \quad \epsilon^{ab}(i\bar{C}^a(x)C^b(x)) = \frac{\nu}{16\pi} \neq 0. \tag{18}
\]

In the condensed vacuum, the ghost-gluon 4-body interaction,

\[
-ig^2\epsilon^{ad}\epsilon^{cb}\bar{C}^aC^bA^c_\mu A^d_\mu, \tag{19}
\]

leads to a mass term of the off-diagonal gluons,

\[
-ig^2\epsilon^{ad}\epsilon^{cb}\langle i\bar{C}^aC^b \rangle A^c_\mu A^d_\mu = \frac{1}{2}g^2\langle i\bar{C}^cC^c \rangle A^c_\mu A^c_\mu, \tag{20}
\]

Thus this condensation leads to the mass for the off-diagonal gluons

\[
m^2_A = g^2\langle i\bar{C}^a(x)C^a(x) \rangle > 0. \tag{21}
\]

On the other hand, the off-diagonal ghost (and anti-ghost) acquires the mass

\[
m^2_C = \alpha g^2\langle i\bar{C}^a(x)C^a(x) \rangle, \tag{22}
\]

through the four ghost interaction,

\[
\frac{\alpha}{4}g^2\epsilon^{ab}\epsilon^{cd}\bar{C}^a\bar{C}^bC^cC^d = \frac{\alpha}{2}g^2\langle i\epsilon^{ab}\bar{C}^bC^b \rangle^2 = \frac{\alpha}{2}g^2\langle i\bar{C}^aC^a \rangle^2
\]

\[
\to \alpha g^2\langle i\bar{C}^aC^a \rangle i\bar{C}^bC^b. \tag{23}
\]

Note that the introduction of the explicit mass term spoils the renormalizability. It can be shown that the diagonal gluons remain massless. The
mass obtained in this way provides the scale which is comparable to the QCD scale $\Lambda_{QCD}$. This result is consistent with the lattice simulations performed by Amemiya and Suganuma. The dynamical mass generation for the off-diagonal components strongly supports the Abelian dominance in low-energy (or long-distance) QCD.

At least for $G = SU(2)$, the Lagrangian in the modified MA gauge has a novel (continuous) global symmetry, $SL(2, R)$ as found by Schaden. Then the mass generation can be considered as a spontaneous breaking of this symmetry from $SL(2, R)$ to the non-compact Abelian subgroup corresponding to the ghost number charge $Q_c$. This mechanism of mass generation can be called the dynamical Higgs mechanism, since QCD has no elementary scalar field. The associated massless Nambu-Goldstone particles can not be observed, since they have zero norms due to the extended quartet mechanism. In these analyses, we have assumed that the vacuum satisfies the physical condition,

$$Q_B |0\rangle = 0, \quad Q_c |0\rangle = 0, \quad \bar{Q}_B |0\rangle = 0.$$  

(24)

For $G = SU(3)$, the ghost condensation scenario for mass generation of the off-diagonal components can be applied and leads to two different masses for off-diagonal gluons; two of them are heavier than the remaining four off-diagonal gluons, e.g.,

$$m_{A_1} = m_{A_2} = \sqrt{2}m_{A_4} = \sqrt{2}m_{A_5} = \sqrt{2}m_{A_6} = \sqrt{2}m_{A_7}.$$  

(25)

6 Discussion

Finally, we raise the problems to be solved in the future investigations.

1. All the results obtained above are invariant under the residual $U(1)^{N-1}$ Abelian gauge symmetry. However, they may depend on the gauge parameter $\alpha$ of the MA gauge. In the recent work, $\alpha$ was determined by requiring the $\mu$ independence of the effective potential of the order parameter of the ghost condensation as

$$\alpha = b_0/N = 11/3.$$  

(26)

2. The proof of renormalizability of QCD in the (modified) MA gauge has not yet been given when the ghost condensation takes place. In the absence of ghost condensation, the proof of renormalizability was given 15 years ago citeMLP85.

3. For $SU(3)$, no one has shown the existence of a global symmetry whose spontaneous breaking leads to the mass generation of off-diagonal fields (through the ghost condensation). Hence the relationship between the mass
generation and the spontaneous symmetry breaking is not yet understood in a satisfactory level.

4. It is important to show how the dynamical mass of the off-diagonal gluons is related to the mass of the dual gauge field in the dual Abelian gauge theory (e.g., dual Ginzburg-Landau theory\textsuperscript{26}) which is expected to be another LEET of QCD.

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