Two Photon Anti-Coalescence Interference: the Signature of Two Photon Entanglement

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We study a general theory on the interference of two-photon wavepacket in a beam splitter. We find that the symmetry of two-photon spectrum plays an important role in the manners of interference. We distinguish the coalescence and anti-coalescence interferences, and prove that the anti-coalescence interference is the signature of photon entanglement.

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The two-photon interference is one of the interesting topics in quantum optics. One of the reasons is that it concerns the nature of how two photons interfere. Another reason could be its connection with optical quantum information. For example, can we know photon entanglement in two-photon interference? The first experimental observations of two-photon interference in a beam splitter (BS) were reported in 1980s. [1,2] The key of experiment is to prepare two separated and degenerate photons with a same frequency and polarization. This can be done by the spontaneously parametric down-conversion (SPDC) of type I in a crystal, in which a pair of photons, signal and idle, are produced. In the degenerate case, two photons are incident into two ports of a 50/50 BS, no coincidence count is found at the output ports. This effect is called the photon coalescence interference (CI). The early theoretical explanation was based on indistinguishability of two monochromatic single photons and it requires that two photons must meet in BS. In the late experiments, it showed that the CI also occurs even if two photons do not meet. [3,4] In the single photon regime, one might conclude a superluminal propagation. A succeed theoretical explanation is to use two-photon entanglement with the help of conceptual Feynman diagrams in which the pair of photons interfered have to be seen as a whole, the two-photon. [4] Just recently, Santori et al [5] has demonstrated in their experiment, that two independent single-photon pulses emitted by a semiconductor quantum dot show a coalescence interference in a BS. The theoretical description in this case is not related to photon entanglement. [6] A natural question could be does the two-photon interference behave both the two-photon and the two photons ways?

In this paper, we contribute an uniform theoretical description for two-photon interference in BS. The theory is valid for any form of two-photon wavepacket, entangled or un-entangled, and the coincidence probability (CP) can be readily evaluated. We find that the symmetry of two-photon spectrum plays a key role in the manners of interference. We distinguish the coalescence and anti-coalescence interference (ACI), and find out the necessary and sufficient condition for the perfect CI and ACI. We prove that the photon entanglement is irrelevant to CI and necessary for ACI. In the case of the perfect ACI, two photon state is transparent passing the BS.

We consider a two-photon wavepacket in two orthogonal polarization modes, designated by \(H\) and \(V\). The general form is described as

\[
|\Phi_{\text{two}}\rangle = |\Phi_{HH}\rangle + |\Phi_{VV}\rangle + |\Phi_{HV}\rangle + |\Phi_{VH}\rangle, \tag{1a}
\]

\[
|\Phi_{pp}\rangle = \sum_{\omega_1, \omega_2} C_{pp}(\omega_1, \omega_2)a_1^p(\omega_1)a_2^p(\omega_2)|0\rangle, \quad p = H, V \tag{1b}
\]

\[
|\Phi_{HV}\rangle = \sum_{\omega_1, \omega_2} C_{HV}(\omega_1, \omega_2)a_1^H(\omega_1)a_2^V(\omega_2)|0\rangle, \quad H \leftrightarrow V \tag{1c}
\]

in which two photons travel in two given directions, designated by subscripts 1 and 2. In the case of only one polarization available, one has \(|\Phi_{\text{two}}\rangle = |\Phi_{pp}\rangle\).

A beam splitter (BS) performs a linear transform for two input beams. For a lossless BS, the general transformation between the input and output field operators obeys [7]

\[
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix} = S(\theta, \phi_r, \phi_p) \begin{pmatrix}
a_1 \\
a_2
\end{pmatrix}, \quad S(\theta, \phi_r, \phi_p) = \begin{pmatrix}
e^{i\phi_r} \cos \theta & e^{i\phi_p} \sin \theta \\
-e^{-i\phi_r} \sin \theta & e^{-i\phi_p} \cos \theta
\end{pmatrix}, \tag{2}
\]

where \(a_i\) and \(b_i\) are the field annihilation operators for the input and output ports, respectively. The subscript \(i\) \((i = 1, 2)\) symbolizes the port in a same propagation direction. \(\theta\) characterizes both the reflection and the transmission

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Ref. [7]. In this paper, we use, alternatively, a simpler method to evaluate the output wavevector. It is a duplicate of the corresponding transform of wavevectors in the S-picture written as

\[ \text{In result, Eq. (5) is written as} \]

\[ |\Psi\rangle_{\text{out}} = U|\Psi\rangle_{\text{in}} = U f(a_1, a_2, a_2U^{-1})|0\rangle \]

where \( U \equiv U_{a_1} U_{a_2} \) and \( U_{a_1} U_{a_2} = U_a \) is known due to Eq. (2), one may obtain its inverse transform. For a 50/50 BS, one obtains

\[ \tilde{b}_1(\omega) = [e^{i\phi}a_1^\dagger(\omega) - e^{-i\phi}a_2^\dagger(\omega)]/\sqrt{2}, \]

\[ \tilde{b}_2(\omega) = [e^{i\phi}a_1^\dagger(\omega) + e^{-i\phi}a_2^\dagger(\omega)]/\sqrt{2}. \]

Equation (4) is valid for a given polarization.

According to Eq. (3), one obtains the output state for the input state (1) by replacing \( a_j^\dagger \) with \( \tilde{b}_j^\dagger \). Using Eq. (4), for \( |\Phi_{pp}\rangle \) one has

\[ |\Psi_{pp}\rangle_{\text{out}} = U|\Phi_{pp}\rangle_{\text{out}} = (1/2) \sum_{\omega_1, \omega_2} C_{pp}(\omega_1, \omega_2) \left\{ \left[ a_{1p}^\dagger(\omega_1) a_{2p}^\dagger(\omega_2) e^{i\phi} - a_{2p}^\dagger(\omega_1) a_{1p}^\dagger(\omega_2) e^{-i\phi} \right] + \left[ a_{1p}^\dagger(\omega_1) a_{2p}^\dagger(\omega_2) - a_{2p}^\dagger(\omega_1) a_{1p}^\dagger(\omega_2) \right] \right\} |0\rangle, \]

where \( \phi = \phi_1 + \phi_2 \). In the summation, when the frequency variables, \( \omega_1 \) and \( \omega_2 \), are taken in the whole frequency space, \( (\omega_1, \omega_2) \) and \( (\omega_2, \omega_1) \) contribute to two indistinguishable states which should be added together. For doing it, we may take \( \sum_{\omega_1 < \omega_2} + \sum_{\omega_1 = \omega_2} + \sum_{\omega_1 > \omega_2} \), and then exchange the variables \( \omega_1 \) and \( \omega_2 \) in the last summation. In result, Eq. (5) is written as

\[ |\Psi_{pp}\rangle_{\text{out}} = (1/2) \sum_{\omega_1, \omega_2} \left\{ \left[ C_{pp}(\omega_1, \omega_2) + C_{pp}(\omega_2, \omega_1) \right] \left[ a_{1p}^\dagger(\omega_1) a_{2p}^\dagger(\omega_2) e^{i\phi} - a_{2p}^\dagger(\omega_1) a_{1p}^\dagger(\omega_2) e^{-i\phi} \right] \right\} |0\rangle. \]

In the above equation, the first and last terms describe the states in which two photons exit from a same port, and the second term, two photons exit from different ports, causing a “click-click” in the coincidence measurement. Because the output states corresponding to \( |\Phi_{HV}\rangle \) and \( |\Phi_{VH}\rangle \) exist indistinguishable states and should be considered in a coherent superposition, one has

\[ |\Psi_{HV+VH}\rangle_{\text{out}} = U(|\Phi_{HV}\rangle + |\Phi_{VH}\rangle) \]

\[ = (1/2) \sum_{\omega_1, \omega_2} \left\{ \left[ C_{HV}(\omega_1, \omega_2) + C_{VH}(\omega_2, \omega_1) \right] \left[ a_{1H}^\dagger(\omega_1) a_{2V}^\dagger(\omega_2) e^{i\phi} - a_{2H}^\dagger(\omega_1) a_{1V}^\dagger(\omega_2) e^{-i\phi} \right] \right\} |0\rangle. \]

Again, the first term and the second term describe the states in which two photons exit from a same port and different ports, respectively. Note that there is no interference among the states \( |\Phi_{HH}\rangle, |\Phi_{VV}\rangle \) and \( (|\Phi_{HV}\rangle + |\Phi_{VH}\rangle) \), since the two-photon polarization configurations are distinguishable.

Equations (6) and (7) have implied that the symmetry of the spectra plays important role to the manners of interference. The symmetric spectrum reads as

\[ C_{pp}(\omega_1, \omega_2) \equiv C_{pp}(\omega_2, \omega_1), \quad p = H, V, \]

\[ C_{HV}(\omega_1, \omega_2) \equiv C_{VH}(\omega_2, \omega_1). \]

In this case, the second terms in Eqs. (6) and (7) vanish. It causes so called perfectly "coalescence interference" (CI), that is, two photons coalesce together and cause a null coincidence probability. This effect has been paid more attention in the literatures. [1] [2] On the contrary, the anti-symmetric spectrum satisfies condition...
under which two output photons never go together and the CP is unit. We call it the perfect anti-coalescence interference (ACI). Precisely, Eqs. (6) and (7) are respectively reduced to

\[
\begin{align*}
|\Psi_{pp}\rangle_{\text{out}} &= \sum_{\omega_1 < \omega_2} C_{pp}(\omega_1, \omega_2)[a_{1p}^+ (\omega_1) a_{2p}^+ (\omega_2) - a_{1p}^+ (\omega_2) a_{2p}^+ (\omega_1)]|0\rangle \\
&= \sum_{\omega_1, \omega_2} C_{pp}(\omega_1, \omega_2) a_{1p}^+ (\omega_1) a_{2p}^+ (\omega_2)|0\rangle = |\Phi_{pp}\rangle,
\end{align*}
\]

and

\[
\begin{align*}
|\Psi_{HV}\rangle_{\text{out}} &= \sum_{\omega_1, \omega_2} C_{HV}(\omega_1, \omega_2)[a_{1H}^+ (\omega_1) a_{2V}^+ (\omega_2) - a_{1H}^+ (\omega_2) a_{2V}^+ (\omega_1)]|0\rangle \\
&= \sum_{\omega_1, \omega_2} (C_{HV}(\omega_1, \omega_2) a_{1H}^+ (\omega_1) a_{2V}^+ (\omega_2) + C_{VH}(\omega_2, \omega_1) a_{1H}^+ (\omega_1) a_{2V}^+ (\omega_2))|0\rangle = |\Phi_{HV}\rangle.
\end{align*}
\]

That is, the output state are identical to the input. Physically, this two-photon wavepacket is perfectly transparent passing BS. We would like to emphasize that conditions (9a) and (9b) are sufficient and necessary for Eqs. (10) and (11), respectively.

In general, we may measure the coincidence probability (CP) to evaluate the CI or ACI effects. According to Eqs. (6) and (7), the CP is obtained as

\[
P_c = \frac{1}{2} \left\{ 1 - \frac{1}{2} \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 [2C_{HV}(\omega_1, \omega_2) C_{VH}^*(\omega_2, \omega_1) + \sum_{p=H,V} C_{pp}(\omega_1, \omega_2) C_{pp}^*(\omega_2, \omega_1) + c.c.] \right\},
\]

where the three integral terms which may increase or decrease the coincidence indicate the two-photon interference. When all the interference terms vanish, there is no two-photon interference and we have \( P_c = 1/2 \). \( P_c < 1/2 \) and \( P_c > 1/2 \) stand for CI and ACI, respectively. When the anti-symmetric condition (9) is satisfied, \( P_c = 1 \), which witnesses the two-photon transparent state.

The CI can occur for both the entangled and the un-entangled two photon wavepackets, whereas ACI never occurs for the un-entangled two photon state. Assume two independent single-photon wavepackets being in two orthogonal polarizations, each of them is written as \( |\Phi_{c,\omega}\rangle = \sum_{\omega} |C_j H(\omega) a_{1H}^+ (\omega) + C_j V(\omega) a_{1V}^+ (\omega)\rangle \). The coincidence probability is

\[
P_c = \frac{1}{2} \left[ 1 - \left| \int_{-\infty}^{\infty} d\omega [C_{1H}(\omega) C_{2H}^*(\omega) + C_{1V}(\omega) C_{2V}^*(\omega)] \right|^2 \right] \leq \frac{1}{2}.
\]

Therefore, ACI effect never occurs for un-entangled two photon wavepacket.

In the following examples, we show how to demonstrate two photon entanglement in experimental observations. First, we consider the case of one polarization. A two-photon wavepacket is assumed in a symmetric form

\[
Q(\omega_1, \omega_2) = g(\omega_1 + \omega_2 - 2\Omega)f(\omega_1 - \Omega)f(\omega_2 - \Omega),
\]

where \( \Omega \) is the central frequency for both beams, and \( f \) describes the spectral profile for each single-photon. In the SPDC of type I, a pair of converted photons can be in this form [9] However, Eq. (14) can describe two independent single-photon wavepackets either, provided \( g(x) = 1 \). Because of the symmetry of the two-photon spectrum \( Q(\omega_1, \omega_2) \), the perfect CI must occur. In the experiment, beam \( j \) travels a path \( z_j \), and enters the input port of a BS. So that the two photon spectrum at the input ports is \( C(\omega_1, \omega_2) = Q(\omega_1, \omega_2) \exp[i(\omega_1 z_1 + \omega_2 z_2)/\epsilon] \). Assume the single-photon spectrum is a Gaussian \( f(x) = \exp[-x^2/(2\sigma^2)] \) with a bandwidth \( \sigma \), we may obtain the coincidence probability \( P_c = (1/2) \{ 1 - \exp\left[-(\sigma |\Delta z|/\epsilon)^2/2 \right] \} \) where \( \Delta z = z_2 - z_1 \). It shows a famous interference dip. [1] [2] [5] This result is independent of the form of function \( g(x) \), whether in entanglement or not. Therefore, our theory presents an uniform explanation to the "dip".

Nevertheless, the dip can not tell us the entanglement between two photons. In order to show the entanglement, one may introduce an additional phase in two-photon spectrum for changing its symmetry. The method was reported in Refs. [3] [4], in which the authors wanted to show that two photon interference occurs even if they "do not meet".
Let beam 1 splits two parts, one travels a short path $L_s$ and the other, a long path $L_l$. Then these two sub-beams incorporate a beam again which interferes with beam 2 traveling a path $z_2$. The new two-photon spectrum at the input ports of BS is obtained as

$$C(\omega_1, \omega_2) = Q(\omega_1, \omega_2)[e^{i\omega_1 L_1/c} + e^{i\omega_1 L_s/c}]e^{i\omega_2 z_2} = 2Q(\omega_1, \omega_2)e^{i(\omega_1 z_1 + \omega_2 z_2)/c \cos(\omega_1 \Delta L/c)}, \quad (15)$$

where $z_1 = (L_l + L_s)/2$ and $\Delta L = (L_l - L_s)/2$. Substituting the symmetric spectrum (14) into Eq. (15), one obtains

$$C(\nu_1, \nu_2) = 2e^{i\Omega(z_1 + z_2)/c /g(v_1 + v_2)}f(\nu_1)f(\nu_2)e^{i(\nu_1 z_1 + \nu_2 z_2)/c \cos(\nu_1 \Delta L/c + \alpha)}, \quad (16)$$

where $\nu_i = \omega_i - \Omega$, $(i = 1, 2)$ and $\alpha \equiv \Omega \Delta L/c$. In the case $g(x) \to \delta(x)$, which denotes a perfect phase matching in SPDC, at the balanced position $z_1 = z_2$, the above spectrum is symmetric (anti-symmetric) when $\alpha = n\pi$ ($\alpha = [n + (1/2)]\pi$). Let $g(x) = \exp[-x^2/(2\sigma_p^2)]$, we can calculate exactly the CP

$$P_c = \frac{1}{\pi^{1/2}} \left[ 1 - \left( \frac{1}{2B} \right) \left[ \text{cos}(2\alpha) \exp\left( -\frac{x^2}{2\sigma_p^2} \right) + \frac{1}{2B} \left( 1 + \cos(2\alpha) \right) \exp\left( -\frac{x^2}{2\sigma_p^2} \right) \right] \right]. \quad (17)$$

where $B = \frac{1}{2}[1 + \cos(2\alpha)] \exp[-\frac{1}{2\pi^2 \sigma_p^2} \Delta L^2]$ and $\beta \equiv \sigma_p / \sigma$. Figure 1 shows the CP versus $\Delta z(\sigma/c)$ for two-photon state (16). In Fig. 1a, when $\alpha = n\pi$ and $\alpha = [n + (1/2)]\pi$, the CP shows the CI (dashed lines) and ACI (solid lines) effects, respectively. For $\beta = 0$, $g(x) \to \delta(x)$, both CI and ACI are perfect. However, for $\beta = 1 \to \infty$, all the CP curves are almost superposed by showing no interference at the balanced position. For two independent single-photon wavepackets ($\beta = \infty$), when $\Delta L(\sigma/c)$ is smaller, the CI may appear at the balanced position but the ACI never occurs, as shown in Fig. 1b.

In the next example, we consider two photons being in two orthogonal polarizations. Assume a single-photon state produced by a single-photon source $|\Phi_{one}\rangle = \sum_{\omega} f(\omega - \Omega)|a_H^\dagger(\omega) + a_V^\dagger(\omega)|0\rangle$, one sets up an experiment for two independent single-photon wavepackets interfered in a BS. Similarly, in the experiment, beam $j$ travels a path $z_j$, before entering BS. In order to show a different manner of interference, one puts a wave-plate in path 1 which introduces an additional relative phase $\alpha$ to a particular polarization. [10] The two-photon state for these two independent wavepackets is written as

$$\sum_{\omega_1, \omega_2} f(\omega_1 - \Omega)f(\omega_2 - \Omega)e^{i(\omega_1 z_1 + \omega_2 z_2)/c}[a_H^\dagger(\omega_1) + e^{i\alpha}a_V^\dagger(\omega_1)]|a_H^\dagger(\omega_2) + e^{i\alpha}a_V^\dagger(\omega_2)|0\rangle. \quad (18)$$

If $f(x) = \exp[-x^2/(2\sigma_p^2)]$, the CP can be calculated as

$$P_c = (1/2)[1 - (1/2)(1 + \cos \alpha)\exp(-\Delta z(\sigma/c))]^2. \quad (19)$$

On the other hand, for SPDC of type II, the two photon state with polarization entanglement can be generated in the overlap between the o-ray cone and the e-ray cone. [10] The two-photon wavepacket with the polarization entanglement can be described as

$$\sum_{\omega_1, \omega_2} Q(\omega_1, \omega_2)e^{i(\omega_1 z_1 + \omega_2 z_2)/c}[a_H^\dagger(\omega_1)a_V^\dagger(\omega_2) + e^{i\alpha}a_V^\dagger(\omega_1)a_H^\dagger(\omega_2)]|0\rangle, \quad (20)$$

where $Q(\omega_1, \omega_2)$ has been defined in Eq. (14). The phase $\alpha$ depends on the crystal birefringence, but, as mentioned above, it can be set as desired. [10] For $\alpha = 0$ and $\alpha = \pi$, at the balanced case $z_1 = z_2$, Eq. (20) satisfies the symmetric and anti-symmetric conditions, (8b) and (9b), respectively. Let $f(x)$ is a Gaussian, the coincidence probability is obtained as

$$P_c = (1/2)(1 - \cos \alpha \cdot e^{-\Delta z^2/c^2}). \quad (21)$$

Not that the result is independent of the function form $g(x)$, so long as it is symmetric to $\omega_1$ and $\omega_2$. Figures 2a and 2b show the CP versus the normalized path difference for un-entangled and entangled wavepackets, (18) and (20), respectively. Obviously, the interference pattern depends on the phase $\alpha$. For $\alpha = 0$, the both cases show the perfect CI effect. On the contrary, for $\alpha = \pi$, the entangled two-photon wavepacket shows the perfect ACI, whereas the two independent single-photon wavepackets show "no interference". Precisely, in the late case, the CP introduced by three interference terms in Eq. (12) is cancelled exactly. However, for $\alpha = \pi/2$, there is no interference for the entangled wavepacket because of the out of phase of the probability amplitudes.

In conclusion, we have shown that, in the quantum state language, two-photon interference originates from indistinguishability of two-photon states, instead of two photons. This interference mechanism is true for both entangled
and un-entangled two-photon wavepackets. The photon entanglement may affect the manners of interference. We propose that ACI effect is the signature of two-photon entanglement, which might be a simpler way to witness the entanglement, besides the violation of Bell’s inequality.

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Fig.1 In the case of one polarization mode, coincidence probability versus the normalized path difference $\Delta z(\sigma/c)$. (a) for $\Delta L(\sigma/c) = 5$, $\alpha = n\pi$ (dashed) and $(n+1/2)\pi$ (solid), curves 1,2,3,4 and 5 are for $\beta = 0, 0.2, 0.5, 1$ and $\infty$, respectively; (b) for $\beta = \infty$ and $\Delta L(\sigma/c) = 1$, dashed, dotted and solid curves are for $\alpha = n\pi$, $(n+1/4)\pi$ and $(n+1/2)\pi$, respectively.

Fig. 2 In the case of two orthogonal polarization modes, coincidence probability versus the normalized path difference $\Delta z(\sigma/c)$ for $\alpha = 0, \pi/2$, and $\pi$; (a) two independent single-photon wavepackets, and (b) two photon wavepacket with polarization entanglement.

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