Towards a Better Understanding of the Semigroup Tree

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Abstract

In this paper we elaborate on the structure of the semigroup tree and the regularities on the number of descendants of each node observed in [2]. These regularities admit two different types of behavior and in this work we investigate which of the two types takes place in particular for well-known classes of semigroups. Also we study the question of what kind of chains appear in the tree and characterize the properties (like being (in)finite) thereof. We conclude with some thoughts that show how this study of the semigroup tree may help in solving the conjecture of Fibonacci-like behavior of the number of semigroups with given genus.

Keywords: Numerical semigroup, Fibonacci numbers.

1 Introduction

A numerical semigroup is a subset of the non-negative integers \( \mathbb{N}_0 \) which is closed under addition, contains 0 and its complement in \( \mathbb{N}_0 \) is finite. The elements in this complement are called gaps and the number of gaps of a numerical semigroup is its genus. The smallest integer in a numerical semigroup from which all larger integers belong to the numerical semigroup is

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Figure 1: Recursive construction of the numerical semigroups of genus $g$ from the numerical semigroups of genus $g-1$. Generators larger than the conductor are written in bold face.

called the conductor of the numerical semigroup. Notice that the conductor of a numerical semigroup is exactly the largest gap (known as its Frobenius number) plus one.

It can be shown that each numerical semigroup has a unique minimal set of generators. The numerical semigroups of genus $g$ can be obtained from the numerical semigroups of genus $g-1$ by taking out one by one the generators that are larger than or equal to the conductor of each semigroup. This leads to an infinite tree containing all numerical semigroups, with root corresponding to the trivial semigroup and where each level of nodes represents numerical semigroups of genus given by the level. The parent of a numerical semigroup is obtained by adding to the semigroup its Frobenius number. This tree is illustrated in Figure 1 where we used $\langle a_1, \ldots, a_k \rangle$ to denote the numerical semigroup generated by $a_1, \ldots, a_k$. This construction was already considered in [8, 12, 11].

The number $n_g$ of all numerical semigroups of genus $g$ has been studied in [3, 2]. In [3] it is conjectured that $n_g$ asymptotically behaves like the Fibonacci numbers. That is, $n_g \geq n_{g-1} + n_{g-2}$, $\lim_{g \to \infty} (n_{g-1} + n_{g-2})/n_g = 1$, and $n_g/n_{g-1}$ approaches the golden ratio. In [2] the tree of numerical
semigroups is used to derive, for \( g \geq 3 \), the bounds \( 2F_g \leq n_g \leq 1 + 3 \cdot 2^{g-3} \), where \( F_g \) denotes the \( g \)-th Fibonacci number. The goal of this paper is providing results for better understanding the semigroup tree and giving possible directions for attacking the previous conjecture. The bounds given in \cite{2} are a consequence of the fact that only two kinds of generators exist in a numerical semigroup larger than or equal to its conductor. In Section \cite{2} we call these two kinds of generators \textit{weak} and \textit{strong} and we study their existence in three well-known classes of numerical semigroups: symmetric, pseudo-symmetric, and Arf semigroups.

In Section \textsection 3 we analyze which nodes have an infinite number of descendants. For the nodes having a finite number of descendants we give a way to determine the descendant at largest distance; for the nodes having an infinite number of descendants we determine the number of infinite chains in which the semigroup lies. It turns out here that primality and coprimality of integers appear in the scene as discriminating factors. Some results related to weak and strong generators of semigroups lying in infinite chains are also given.

In the last section we give what we think should be future directions for attacking the conjecture on the Fibonacci-like behavior of \( n_g \) and how the results presented in the first sections could help.

\section{Behavior of known classes of numerical semigroups}

The enumeration \( \lambda \) of a numerical semigroup \( \Lambda \) is the unique increasing bijective map \( \mathbb{N}_0 \rightarrow \Lambda \). Usually \( \lambda(i) \) is denoted \( \lambda_i \). It is easy to check that if \( c \) and \( g \) are the conductor and the genus of \( \Lambda \) then \( \lambda_{c-g} = c \) and for \( \lambda_i \geq c \), \( \lambda_i = i + g \). A semigroup for which \( \lambda_1 = c \), i.e. a semigroup of the form \( \{0\} \cup [c, \infty) \), is called \textit{ordinary}.

It was shown in \cite{2} that the next Lemma holds.

\textbf{Lemma 1.} If \( \lambda_{i_1} < \lambda_{i_2} < \ldots < \lambda_{i_n} \) are the generators of a non-ordinary numerical semigroup \( \Lambda \) that are larger than or equal to its conductor then the generators of \( \Lambda \setminus \{\lambda_{ij}\} \) that are larger than or equal to its conductor are either \( \lambda_{ij+1}, \ldots, \lambda_{i_n} \) or \( \lambda_{ij+1}, \ldots, \lambda_{i_n}, \lambda_{ij} + \lambda_1 \).

Motivated by this lemma, we call the generators of a non-ordinary numerical semigroup that are larger than or equal to its conductor, the \textit{effective
generators and we say that an effective generator $\lambda_{ij}$ is strong if the set of effective generators of $\Lambda \setminus \{\lambda_{ij}\}$ is $\lambda_{ij+1}, \ldots, \lambda_{in}, \lambda_{ij} + 1$. An effective generator that is not strong is called a weak generator.

Finally we say that a leave is a node with no descendants, a stick is a node with exactly one descendant and a bush is a node with two or more descendants.

### 2.1 Symmetric semigroups

Symmetric semigroups are those semigroups for which the conductor is twice the genus. Symmetric semigroups and their applications to coding theory have been studied, among others, in [1, 4, 6, 7]. An important property of symmetric semigroups is that if $c$ and $g$ are the genus and the conductor of a symmetric semigroup $\Lambda$ then any integer $i$ is a gap of $\Lambda$ if and only if $c - 1 - i$ is a non-gap.

The semigroups of the form $\langle 2, 2n + 1 \rangle, n \geq 1$ are symmetric. They are called hyperelliptic semigroups.

**Lemma 2.** Hyperelliptic numerical semigroups are sticks and the unique effective generator, which is the conductor plus one, is strong.

Given a numerical semigroup $\Lambda$ with enumeration $\lambda$, the associated $\nu$-sequence is defined by $\nu_i = \# \{j \in \mathbb{N}_0 : \lambda_i - \lambda_j \in \Lambda\}$. It is proven in [7, Theorem 3.8] that

$$\nu_i = i - g(i) + \#D(i) + 1,$$

where $g(i)$ is the number of gaps smaller than $\lambda_i$, and $D(i) = \{l \notin \Lambda | \lambda_i - l \notin \Lambda\}$. Notice that an element $\lambda_i \in \Lambda$ is a generator of $\Lambda$ if and only if $\nu_i = 2$.

**Lemma 3.** For a numerical semigroup with enumeration $\lambda$ and conductor $c$, an element $\lambda_i \geq c$ is a generator if and only if

$$\#D(i) = g - i + 1.$$

**Proof.** It follows from equality (1) and from the fact that a non-gap $\lambda_i$ is a generator if and only if $\nu_i = 2$. \[\square\]

**Lemma 4.** Non-hyperelliptic symmetric semigroups are leaves.
Proof. For a symmetric semigroup with conductor \( c \) and genus \( g \), \( \lambda_i \geq c \) if and only if \( i \geq g \). Hence, by Lemma 3, \( \lambda_i \geq c \) can only be a generator if \( \lambda_i = c \) and \( |D(i)| = 1 \) or if \( \lambda_i = c + 1 \) and \( |D(i)| = 0 \). The first situation is only possible when \( 1 = c - 1 \) because \( 1, c - 1 \notin \Lambda, 1 + (c - 1) = c \) and otherwise, \( |D(i)| > 1 \). But \( 1 = c - 1 \) would mean that \( c = 2 \) and thus the numerical semigroup would be hyperelliptic. The second situation is only possible for hyperelliptic semigroups since for other semigroups, \( 2 \) and \( c - 1 \) are gaps and \( 2 + (c - 1) = c + 1 \). This implies \( |D(i)| > 0 \).

As an example of non-hyperelliptic symmetric semigroup consider \( \Lambda = \{0, 4, 5, 8, 9, 10\} \cup [12, \infty) \). In this case the generators are 4 and 5 and none of them is effective.

2.2 Pseudo-symmetric semigroups

Pseudo-symmetric semigroups are those semigroups for which the conductor is twice the genus minus one. An important property of pseudo-symmetric semigroups analogous to the one for symmetric semigroups is that if \( c \) and \( g \) are the genus and the conductor of a pseudo-symmetric semigroup \( \Lambda \) then any integer \( i \) different from \((c - 1)/2\) is a gap of \( \Lambda \) if and only if \( c - 1 - i \) is a non-gap.

Lemma 5. For a non-ordinary numerical semigroup \( \Lambda \) with enumeration \( \lambda \) and conductor \( c \), a non-gap \( \lambda_k \neq 2\lambda_1 \) is a strong generator if and only if \( \lambda_k \geq c \) and \( \nu_{k+\lambda_1} = 4 \).

Proof. If \( \lambda_k \) is strong, then by definition \( \lambda_k \geq c \). Now, \( \nu_{k+\lambda_1} \geq 4 \) because \( \lambda_{k+\lambda_1} - \lambda_1 = \lambda_k, \lambda_{k+\lambda_1} - \lambda_k = \lambda_1 \), and \( 0, \lambda_1, \lambda_k, \lambda_{k+\lambda_1} \) are different. If \( \nu_{k+\lambda_1} > 4 \) this means that there exists at least one \( \lambda_l \) with \( l \) different from \( k \) such that \( \lambda_{k+\lambda_1} - \lambda_l \in \Lambda \) and \( \lambda_{k+\lambda_1} - \lambda_l \neq \lambda_k \). Then \( \lambda_{k+\lambda_1} \) is not a generator of \( \Lambda \setminus \{\lambda_k\} \).

On the other hand, if \( \lambda_k \geq c \) and \( \nu_{k+\lambda_1} = 4 \) this means that \( \lambda_k \) is a generator. Indeed, if \( \lambda_k = \lambda_l + \lambda_m \) with \( 0 < l < m < k \) then \( \lambda_{k+\lambda_1} - \lambda_l = \lambda_m + \lambda_1 \in \Lambda \), so, \( \nu_{k+\lambda_1} > 4 \). Furthermore, since \( \nu_{k+\lambda_1} = 4 \) this means that \( \lambda_k + \lambda_1 \) can only be subtracted by \( 0, \lambda_1, \lambda_k, \lambda_{k+\lambda_1} \) within the numerical semigroup. Consequently, \( \lambda_k + \lambda_1 \) is a generator of \( \Lambda \setminus \{\lambda_k\} \).

Lemma 6. 1. The unique pseudo-symmetric semigroup of genus \( g \) with only one interval of non-gaps between 0 and the conductor is \( \Lambda_{psg} = \{0, g, g + 1, \ldots, 2g - 3\} \cup [2g - 1, \infty) \).
2. The numerical semigroup $\Lambda_{ps} = \{0, 3\} \cup [5, \infty)$, has 5 and 7 as the only effective generators. The generator 5 is strong and the generator 7 is weak.

3. The numerical semigroup $\Lambda_{ps} = \{0, 4, 5\} \cup [7, \infty)$, has 7 as the only effective generator and it is strong.

4. The numerical semigroup $\Lambda_{ps_g}$, for $g \geq 5$ is a stick, its unique effective generator is $c$, and it is weak.

Proof. The proof of statement 1 follows directly from the main property of pseudosymmetric semigroups. Statements 2 and 3 can be proved by an exhaustive search of generators and by checking which are weak and which are strong.

Since the conductor of $\Lambda_{ps_g}$ is $2g - 1$, every integer larger than or equal to $4g - 2$ will not be a generator. The integer $4g - 3$ is not a generator since $4g - 3 = g + (2g - 3)$. The integer $4g - 4$ is not a generator since $4g - 4 = (2g - 3) + (2g - 1)$. The integers from $2g$ to $4g - 6$ are generated by the interval $g, \ldots, 2g - 3$. So the only effective generator of $\Lambda_{ps_g}$ can be $c = 2g - 1$ and $4g - 5$. It is easy to check that $c$ is a generator. If the integer $4g - 5$ is larger than or equal to $g + (2g - 1)$ then it is not a generator. This is equivalent to $g \geq 4$.

On the other hand, $2g - 1$ is weak if and only if $g + (2g - 1)$ is a sum of two non-gaps strictly smaller than $2g - 1$ and this is equivalent to having $g + g \leq g + (2g - 1) \leq (2g - 3) + (2g - 3)$, which in turn is equivalent to $g \geq 5$. Thus $c$ is a weak generator if $g \geq 5$.

Lemma 7. 1. A numerical semigroup is pseudo-symmetric and has $\lambda_1 = 3$ if and only if it is equal to $\Lambda = \{0, 3, 6, \ldots, 3k, 3(k + 1) - 1, 3(k + 1), 3(k + 2) - 1, 3(k + 2), \ldots, 3(2k - 1) - 1, 3(2k - 1)\} \cup [3(2k - 1) + 2, \infty)$ or $\Lambda = \{0, 3, 6, \ldots, 3k, 3(k + 1), 3(k + 1) + 1, 3(k + 2), 3(k + 2) + 1, \ldots, 3(2k), 3(2k) + 1\} \cup [3(2k) + 3, \infty)$ for some $k$.

2. Each pseudo-symmetric semigroup with $\lambda_1 = 3$ has a unique effective generator, it is $c + 2$ and it is weak.

3. The descendants of a pseudo-symmetric semigroups with $\lambda_1 = 3$ are non-hyperelliptic symmetric semigroups, and thus, leaves.
Proof. 1. From the property of pseudo-symmetric semigroups that any non-negative integer $i$ different from $(c - 1)/2$ is a gap if and only if $c - 1 - i$ is a non-gap we deduce that each pseudo-symmetric semigroup with $\lambda_1 = 3$ must be one of the semigroups above. To see that these semigroups are always pseudo-symmetric, let us compute the genus and the conductor. In the first case we have that up to $3k$ the semigroup $\Lambda$ has exactly $2k$ gaps: 2 gaps per interval $[3i, 3i + 2], 0 \leq i \leq k - 1$. Then from $3k + 1$ to $3(2k - 1)$ there are $k - 1$ gaps: one per interval $[3i + 1, 3(i + 1)], k \leq i \leq 2k - 2$. Together with the gap $3(2k - 1) + 1$ that makes $g = 2k + k - 1 + 1 = 3k$ gaps. Obviously $c = 3(2k - 1) + 2 = 2g - 1$. So $\Lambda$ is pseudo-symmetric. The other case is done analogously and we have $g = 2(k + 1) + k = 3k + 2, c = 3(2k) + 3 = 2g - 1$.

2. An element larger than or equal to the conductor must be $c + i = \lambda_{c-g+i}$ for some $i \geq 0$. Since these semigroups are pseudo-symmetric, $c - g + i = g + i - 1$. Now, by Lemma 3 $\lambda_{g+i-1}$ is a generator if and only if $D(g + i - 1) = 2 - i$. Since $D(g + i - 1) \geq 0$, this means that $i \leq 2$. So, the only elements larger than or equal to the conductor that can be generators are $c, c+1, c+2$. The elements $c$ and $c+1$ cannot be generators, because $c = 3(2k - 1) + 2 = 3(2k - 1) - 1 + 3, c+1 = 3(2k - 1) + 3$ for the first case and similarly is done for the second. Let us show that $c+2$ is a generator. Consider the first case, the second one is done analogously. We have $c+2 = 3(2k - 1) + 4 = 6k + 1$, so it has residue 1 modulo 3. Note that all the non-gaps less than $c+2$ have residues 0 or 2 modulo 3. So, if $c+2$ is not a generator, it is a sum of two non-gaps with residue 2. So we have $c+2 = 3(k+i)-1+3(k+j)-1 = 6k+3i+3j-2$ for some $i, j \geq 1$. But then we have that $i+j = 1$, a contradiction.

To see that $c+2$ is a weak generator, suppose that $\lambda_k = c+2$. Since $\lambda_k > c$, then $\lambda_{k+\lambda_1} = c+5$, so $k+\lambda_1 = c+5-g = g+4$. Assume that $c+2$ is a strong generator, then by (1) we have $\nu_{k+\lambda_1} = k+\lambda_1-g+\#D(k+\lambda_1)+1$ and by Lemma 5 we have $4 = g + 4 - g + \#D(k + \lambda_1) + 1$, thus $1 \leq \#D(k + \lambda_1) + 1 = 0$, a contradiction.

3. The only descendant of $\Lambda$ is obtained by removing $c+2$. The semigroup $\Lambda \setminus \{c + 2\}$ is symmetric since its genus is $g + 1$ and its conductor is $c + 3$, and we have $c + 3 = 2(g + 1)$, since $c = 2g - 1$. It is easy to see that $\Lambda \setminus \{c + 2\}$ is non-hyperelliptic semigroup, and thus a leave, cf. Lemma 4.

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Lemma 8. Each pseudo-symmetric semigroup with $\lambda_1 \neq 3$ and with more than one interval of non-gaps between 0 and the conductor is a leave.

Proof. By Lemma 3, the only cases in which $\lambda_k \geq c$ can be a generator correspond to the next three situations. We used that $\lambda_k \geq c$ if and only if $k \geq c - g$.

- $\lambda_k = c$ if $\#D(k) = 2$. Since there exists more than one interval of non-gaps between 0 and $c$, there exists $i \in \Lambda$, $i \neq 0, c - 2$ such that $i + 1 \not\in \Lambda$. So $c - 1 - i \not\in \Lambda$ (pseudo-symmetric property) and $i + 1, c - i - 1$ are different from 1, $c - 1$ and they also add up to $c$. Hence, $\#D(k) > 2$, a contradiction.

- $\lambda_k = c + 1$ if $\#D(k) = 1$. This case is impossible since $2 \neq c - 1$, both 2 and $c - 1$ are gaps, and they add up to $c + 1$.

- $\lambda_k = c + 2$ if $\#D(k) = 0$. This is impossible if $\lambda_1 \neq 3$ because 3 and $c - 1$ are then gaps and so $D(k) \neq \emptyset$.

As an example of pseudo-symmetric semigroup with $\lambda_1 \neq 3$ and with more than one interval of non-gaps between 0 and the conductor we can take $\Lambda = \{0, 4, 7, 8, 9\} \cup [11, \infty)$. In this case the generators are 4, 7, 9 and none of them is effective.

A numerical semigroup is said to be irreducible if it cannot be expressed as an intersection of two numerical semigroups properly containing it. It was proven in [9] that irreducible semigroups are exactly symmetric and pseudo-symmetric semigroups. Thus we have shown that the only non-leaves corresponding to irreducible numerical semigroups are those treated in Lemmas 2, 6, 7. Moreover the number of effective generators is small and the number of strong generators is even smaller. Therefore, the parts of the semigroup tree in a vicinity of an irreducible semigroup are not ”bushy” and are easily described.

2.3 Arf semigroups

A numerical semigroup $\Lambda$ with enumeration $\lambda$ is said to be Arf if $\lambda_i + \lambda_j - \lambda_k \in \Lambda$ for every $i, j, k \in \mathbb{N}_0$ with $i \geq j \geq k$. Hyperelliptic semigroups are an
example of Arf semigroups. In fact, it was shown in [5] that hyperelliptic semigroups are the only Arf symmetric semigroups. A lot of work has been done related to Arf semigroups. One can see, for instance, [1 10 5].

For the next lemma we use the fact that for an Arf numerical semigroup \( \Lambda \), an element \( \lambda_i \neq 0, \lambda_1 \) is a generator if and only if \( \lambda_i - \lambda_1 \not\in \Lambda \).

**Lemma 9.**
1. Non-hyperelliptic Arf numerical semigroups are bushes.
2. Arf semigroups appear as descendants of semigroups with strong generators when removing one such generator.

**Proof.**
1. For an Arf semigroup we know that if \( i, i+1 \in \Lambda \), then \( i \geq c \).

\[
  j = i + ((i + 1) - i) + ((i + 1) - i) + \ldots + ((i + 1) - i) 
\]

\[
  = i + ((i + 1) - i) + ((i + 1) - i) + \ldots + ((i + 1) - i) 
\]

Thus we know that \( c - 1 \) and either \( c - 2 \) or \( c - 3 \) are gaps. Since \( \Lambda \) is not hyperelliptic, \( \lambda_1 \geq 3 \). Thus, \( c - 1 + \lambda_1 \) and either \( c - 2 + \lambda_1 \) or \( c - 3 + \lambda_1 \) are generators.

2. It follows from the remark previous to the Lemma.

\[ \square \]

It was shown in [10] that at most two of the descendants of Arf semigroups are Arf. For illustrating this, notice that \( \{0,5,7\} \cup [9, \infty) \) has no Arf descendants; \( \{0,5\} \cup [7, \infty) \) has two Arf descendants: \( \{0,5\} \cup [8, \infty) \) and \( \{0,5,7\} \cup [9, \infty) \); \( \{0,5\} \cup [10, \infty) \) has one Arf descendant: \( \{0,5,10\} \cup [12, \infty) \).

## 3 Infinite chains

We say that an infinite sequence of numerical semigroups \( \Lambda_0 = \mathbb{N}_0, \Lambda_1, \Lambda_2, \ldots \) is an *infinite chain* if for each \( i \geq 1 \), \( \Lambda_{i-1} \) can be obtained by adding to \( \Lambda_i \) its Frobenius number. Clearly, a numerical semigroup has infinitely many descendants in the semigroup tree if and only if it lies in an infinite chain.
For the proof of the next lemma we will use that a set of integers $l_1, \ldots, l_m$ generate a numerical semigroup if and only if they are coprime.

**Lemma 10.** Given an infinite chain $(\Lambda_i)_{i \geq 0}$, 

$$\bigcap_{i \geq 0} \Lambda_i = d \cdot \Lambda$$

for some integer $d > 1$ and some numerical semigroup $\Lambda$.

**Proof.** The intersection $\cap_{i \geq 0} \Lambda_i$ satisfies $0 \in \cap_{i \geq 0} \Lambda_i$ and $x + y \in \cap_{i \geq 0} \Lambda_i$ for all $x, y \in \cap_{i \geq 0} \Lambda_i$. Furthermore, all elements in $\cap_{i \geq 0} \Lambda_i$ must be divisible by an integer $d > 1$. Indeed, otherwise we could find a finite set of coprime elements which would generate a numerical semigroup, and this numerical semigroup should be a subset of $\cap_{i \geq 0} \Lambda_i$. Then the infinite chain would not contain any semigroup with genus larger than that of this semigroup, giving a contradiction. Let $d$ be the greatest of the common divisors of $\cap_{i \geq 0} \Lambda_i$.

Then $\frac{1}{d} (\cap_{i \geq 0} \Lambda_i)$ must be a numerical semigroup. 

**Lemma 11.** Given an integer $d > 1$ and a numerical semigroup $\Lambda$ the infinite chain obtained by deleting repetitions in the sequence $\Lambda_j = d \cdot \Lambda \cup \{l \in \mathbb{N} : l \geq j\}$ has intersection $d \cdot \Lambda$.

Consequently, if we denote by $S$ the set of all numerical semigroups, there is a bijection 

$$S \times \mathbb{N}_{\geq 2} \leftrightarrow \{\text{infinite chains}\}$$

In the next theorem we show that the greatest common divisor of the first elements of a numerical semigroup determine whether the numerical semigroup has infinite number of descendants. Notice that since $\lambda_{c-g} = c$, the set $\lambda_0, \ldots, \lambda_{c-g-1}$ is the set of non-gaps smaller than the conductor.

**Theorem 12.** Let $\Lambda$ be a numerical semigroup with enumeration $\lambda$, genus $g$, and conductor $c$, and let $d$ be the greatest common divisor of $\lambda_0, \ldots, \lambda_{c-g-1}$. Then,

1. $\Lambda$ lies in an infinite chain if and only if $d \neq 1$.

2. If $d = 1$ then the descendant of $\Lambda$ with largest genus is the numerical semigroup generated by $\lambda_0, \ldots, \lambda_{c-g-1}$. 

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3. If $d \neq 1$ then $\Lambda$ lies in infinitely many infinite chains if and only if $d$ is not prime.

4. If $d$ is a prime then the number of infinite chains in which $\Lambda$ lies is the number of descendants of $\{\frac{\lambda_0}{d}, \frac{\lambda_1}{d}, \ldots, \frac{\lambda_{c-g-1}}{d}\} \cup \{l \in \mathbb{N}_0 : l \geq \lceil \frac{c}{d} \rceil\}$.

Proof. 1. If $d = 1$ then $\lambda_0, \ldots, \lambda_{c-g-1}$ generate a numerical semigroup $\Lambda'$ and each descendant of $\Lambda$ must contain $\Lambda'$. Thus, the maximum of the genus of the descendants is the genus of $\Lambda'$ which is finite. On the other hand, if $d \neq 1$ then $\lambda_0 = d\tilde{\lambda}_0, \ldots, \lambda_{c-g-1} = d\tilde{\lambda}_{c-g-1}$ with $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{c-g-1}$ coprime. Let $\tilde{\Lambda}$ be the numerical semigroup generated by $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{c-g-1}$. Consider the sequence of semigroups

$$\Lambda_i = d \cdot \tilde{\Lambda} \cup \{l \in \mathbb{N}_0 : l \geq i\}.$$  

By deleting repetitions we obtain an infinite chain that contains $\Lambda$.

2. It follows from the proof of the previous statement.

3. If $d$ is not prime then $d = d_1d_2$ for some $d_1, d_2 > 1$ and, as before,

$$\lambda_0 = d_1d_2\tilde{\lambda}_0, \ldots, \lambda_{c-g-1} = d_1d_2\tilde{\lambda}_{c-g-1}$$

with $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{c-g-1}$ coprime. Let $\tilde{\Lambda}$ be the numerical semigroup generated by $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{c-g-1}$. For each $i \geq 0$ and each $j \geq 0$ define

$$\Lambda_{i,j} = d_1d_2\tilde{\Lambda} \cup \{d_1l \in \mathbb{N}_0 : l \geq i\} \cup \{l \in \mathbb{N}_0 : l \geq j\}.$$  

For each fixed $i \geq \lceil \frac{c}{d_1} \rceil$, by deleting repetitions in the sequence $(\Lambda_{i,j})_{j \geq 0}$ we obtain an infinite chain. Moreover every such chain contains $\Lambda$, as $\Lambda = \Lambda_{i,c}$ if $i \geq \lceil \frac{c}{d_1} \rceil$. For each $i \geq \lceil \frac{c}{d_1} \rceil$ this chain is different. Thus we get infinitely many infinite chains. The complete result in this statement follows from statement 4.

4. Suppose that an infinite chain $(\Lambda_i)_{i \geq 0}$ contains $\Lambda$. It must satisfy $\cap_{i \geq 0} \Lambda_i = d \cdot \tilde{\Lambda}$ for a unique numerical semigroup $\tilde{\Lambda}$ such that

- $d\tilde{\lambda}_0 = \lambda_0, \ldots, d\tilde{\lambda}_{c-g-1} = \lambda_{c-g-1}$,
• \( d\tilde{\lambda}_{c-g} \geq c \), since \( d\tilde{\Lambda} \subseteq \Lambda \).

Thus, \( \tilde{\Lambda} \) is a descendant of \( \{\frac{\lambda_0}{d}, \frac{\lambda_1}{d}, \ldots, \frac{\lambda_{c-g-1}}{d}\} \cup \{l \in \mathbb{N}_0 : l \geq \lceil \frac{c}{d} \rceil\} \).

Lemma 13. Let \( \Lambda \) be a numerical semigroup with enumeration \( \lambda \), genus \( g \), conductor \( c \), and \( \gcd(\lambda_0, \ldots, \lambda_{c-g-1}) = d > 1 \) lying in an infinite chain. Then

1. All non-gaps between \( c \) and \( c + \lambda_1 - 1 \) that are not multiples of \( d \) are generators. Thus \( \Lambda \) has at least \( \lambda_1 - \frac{\lambda_1}{d} \) effective generators.

2. If there are at least two non-gaps between 0 and \( c \), then all non-gaps between \( c \) and \( c + d - 1 \) that are not multiples of \( d \) are strong generators. Thus \( \Lambda \) has at least \( d - 1 \) strong generators.

3. If there is just one non-gap between 0 and \( c \), then there is at least one strong generator.

Proof. 1. If \( c \leq \lambda_k \leq c + \lambda_1 - 1 \), \( \lambda_k \) is not a multiple of \( d \), and there exist \( 0 < i < j \) such that \( \lambda_i + \lambda_j = \lambda_k \) then it must be \( \lambda_j < c \); otherwise \( \lambda_k = \lambda_i + \lambda_j \geq \lambda_1 + c \). But if \( \lambda_i, \lambda_j < c \) then \( \lambda_k = \lambda_i + \lambda_j \) is a multiple of \( d \), since \( \lambda_i \) and \( \lambda_j \) are, a contradiction.

2. If \( c \leq \lambda_k \leq c + d - 1 \), \( \lambda_k \) is not a multiple of \( d \), and there exist \( 1 < i < j \) such that \( \lambda_i + \lambda_j = \lambda_1 + \lambda_k \) then it must be \( \lambda_i < c \). Otherwise \( \lambda_2 + c > \lambda_1 + c + d - 1 \geq \lambda_1 + \lambda_k = \lambda_i + \lambda_j \geq 2c \), a contradiction since \( \lambda_2 \leq c \). But then \( \lambda_i \) is a multiple of \( d \) and \( \lambda_i + \lambda_j = \lambda_1 + \lambda_k \) means that \( \lambda_j \equiv \lambda_k \mod d \). By hypothesis \( \lambda_k \) is not a multiple of \( d \) and so \( \lambda_j \) is not a multiple of \( d \) either and consequently \( \lambda_j \geq c \). But then \( \lambda_k - \lambda_j \leq c + d - 1 - c = d - 1 \), so \( \lambda_j = \lambda_k \) and \( \lambda_i = \lambda_1 + \lambda_k - \lambda_j = \lambda_1 \), a contradiction.

3. For the last statement notice that at least \( c \) or \( c + 1 \) is strong.

Notice that in the second statement of the previous lemma the requirement that there are at least two non-gaps between 0 and \( c \) is necessary. As a counterexample consider the semigroup \( \{0, 8\} \cup [10, \infty) \). In this case, \( d = \lambda_1 = 8 \) and all non-gaps between 10 and \( 10 + \lambda_1 - 1 = 17 \) are generators except for 16 which is a multiple of \( d \). This is a consequence of the first
statement. The second statement fails since 12 is between \( c \) and \( c + d - 1 \) and it is not a multiple of \( d \), but \( 12 + 8 = 10 + 10 \).

4 Future directions for solving the conjecture about the Fibonacci-like behavior of \( n_g \)

In this section we outline some further thoughts on strong/weak generators and how they might help to solve the Fibonacci conjecture. First of all, computational evidence suggests that as \( g \) grows, the portion of strong generators among all effective generators becomes smaller. Namely, the following is conjectured.

**Conjecture 14.** Let \( S_g \) be the number of all strong generators in all numerical semigroups of genus \( g \) and let \( W_g \) be the number of all weak generators in all numerical semigroups of genus \( g \). We conjecture that

\[
\lim_{g \to \infty} \frac{S_g}{W_g} = 0.
\]

Notice that by Lemma 1 if the number of effective generators (and so the number of descendants) of a semigroup is \( k \) and all \( k \) effective generators are weak then the number of effective generators (and so the number of descendants) of its descendants is respectively 0, 1, \ldots, \( k - 1 \). In [2], the tree \( A \) represented in Figure 2 was recursively defined as follows: Its root is labeled as 1 and it has a single descendant which is labeled as 2. This descendant in turn has two descendants labeled as 1 and 3. At each level \( g \), the number of descendants of a node is equal to its label. From level \( g = 2 \) on, if the label of a node is \( k \) then the labels of its descendants are 0, \ldots, \( k - 1 \) except for the node with label \( k = g + 1 \), whose descendants have labels 0, \ldots, \( k - 3, k - 1, k + 1 \).

Because of Lemma 1 and because of the particular structure of ordinary semigroups, the semigroup tree in Figure 2 contains \( A \) as a subtree.

Define \( A_0 = \{1\} \), \( A_1 = \{2\} \) and for \( g \geq 2 \) define \( A_g \) as

\[
A_g = \{g + 1\} \cup \left( \bigcup_{m \in A_{g-1}} \{0, 1, \ldots, m - 1\} \right) \setminus \{g - 2\}.
\]
Figure 2: Tree A. It is a subtree of the tree of numerical semigroups.

The tree A has $A_g$ as the nodes at distance $g$ from its root. Thus, $|A_g| \leq n_g$. It was shown in [2] that $|A_g| = 2F_g$, where $F_i$ denotes the $i$-th Fibonacci number. From this the lower bound $n_g \geq 2F_g$ was deduced.

The next Proposition observes that no matter how a tree behaves at the beginning, if at some point its generation rule coincides with the one of $A$, the Fibonacci behavior is observed from some point on.

**Proposition 15.** Let $l \geq 2$ be an integer and let $L_l$ be a multiset composed of some (maybe with repetitions) numbers $\leq l - 2$, and numbers $l - 1$ and $l + 1$. For $k > l$ define recursively

$$L_k = \{k + 1\} \bigcup \left( \bigcup_{m \in L_{k-1}} \{0, 1, \ldots, m - 1\} \right) \setminus \{k - 2\}.$$ 

Then, for all $k \geq 2l$:

$$|L_k| = |L_{k-1}| + |L_{k-2}|.$$ 

Even more: $|L_k| = 2F_k$.

**Proof.** In [2] it is proven that for $l = 2$ and $L_2 = \{1, 3\}$, the recursively defined sets $L_k$ satisfy $|L_k| = 2F_k$ for all $k \geq 2$. This proves the lemma in the particular case in which $l = 2$ and $L_2 = \{1, 3\}$. 

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Next we will prove that if \( l, l' \) are integers and the multisets \( L_l, L'_l \) satisfy the hypothesis, then \( L_k = L'_k \) for all \( k \geq \max(2l, 2l') \). This, together with the result in [2] will end the proof.

Suppose \( m \in L_s, m \neq s + 1 \). Then \( m \) gives rise to a subset \( \{0, \ldots, m-1\} \subseteq L_{s+1} \) and to a subset in \( L_{s+2} \) whose maximum element is \( m - 2 \) and to a subset in \( L_{s+3} \) whose maximum element is \( m - 3 \) and so on. However, the fact that \( m \in L_s \) does not affect \( L_{s'} \) for \( s' > k + m \). Similarly, the only element in \( L_l \) that affects \( L_k \) for any \( k \geq 2l \) is \( l + 1 \). Consequently, \( L_k = L'_k \) for any \( k \geq 2l \).

A rough idea of future approaches to the Fibonacci problem would be: observe that the number of strong generators becomes negligible compared to all effective generators as \( g \to \infty \), then the semigroup tree behaves more and more like the tree \( A \) from [2]. So roughly speaking we are in the situation of Proposition 15. Pushing this idea further could help to solve the Fibonacci conjecture.

Finally, we would like to mention some computational evidence that suggests that strong generators appear quite regularly. Let \( n^i_g \) be the number of numerical semigroups of genus \( g \) with \( i \) strong generators. Then we conjecture that

\[
n^i_g = 0 \text{ for } i > \left\lfloor \frac{g - 1}{2} \right\rfloor.
\]

It is observed that as \( g \) increases, \( n^{|\frac{g - 1}{2}\!-j|}_g \) approaches a constant for \( g \) even and another constant for \( g \) odd. So, we can define two sequences

\[
e_j = \lim_{k \to \infty} n^{k-1-j}_g, \\
o_j = \lim_{k \to \infty} n^{k-j}_{2k+1}.
\]

The first terms of the sequence \( e \) have been observed to be

\[2, 2, 5, 12, 21, 45.\]

And the first terms of the sequence \( o \) have been observed to be

\[1, 2, 3, 8, 14, 34 - 35.\]

It seems that \( e_j \geq \sum_{i=0}^{j-1} e_i \) and the same for \( o \), so we conjecture in particular that the \( e \)- and \( o \)-sequences are superincreasing.
5 Conclusions

In this paper we went a step further on the study of the structure of the semigroup tree. Namely we described the nodes that correspond to some well-studied classes of numerical semigroups, like symmetric, pseudosymmetric and Arf. Apart from this we also considered what kind of chains appear in the semigroup tree. Namely, when a node (semigroup) belongs to an infinite chain, and when the number of such chains is finite/infinite. We concluded the paper with some conjectures and observations regarding the number of strong generators. These conjectures hopefully can help in tackling the Fibonacci problem.

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