Projectively Quantization Map

Sofiane BOUARROUDJ

CNRS, Centre de Physique Théorique, Luminy, Case 907, F13288 Marseille, Cedex 9, France.
e-mail: sofbo@cpt.univ-mrs.fr

Abstract

Let $M$ be a manifold endowed with a symmetric affine connection $\Gamma$. The aim of this paper is to describe a quantization map between the space of second-order polynomials on the cotangent bundle $T^*M$ and the space of second-order linear differential operators, both viewed as modules over the group of diffeomorphisms and the Lie algebra of vector fields on $M$. This map is an isomorphism, for almost all values of certain constants, and it depend only on the projective class of the affine connection $\Gamma$.

1 Introduction

Quantization procedure proposed in this paper is deals with the space of linear differential operators and the corresponding space of symbols viewed as modules over the group of diffeomorphisms $\text{Diff}(M)$ and the Lie algebra of vectors fields $\text{Vect}(M)$. This method of quantization have been introduced in the recent papers ([11], [5], [7]).

Let $D_{\lambda,\mu}(M)$ be the space of linear differential operators from the space of $\lambda-$densities with the space of $\mu-$densities. The corresponding space of symbols, $\text{Pol}_{\delta}(T^*M)$, is the space of polynomials on $T^*M$ with values in the space of $\delta-$densities, where $\delta = \mu - \lambda$.

We call quantization map, a linear map

\[ Q_{\lambda,\mu} : \text{Pol}_{\delta}(M) \to D_{\lambda,\mu}(M), \]

that is bijective and preserves the principal symbol (see [11], [7], [5]).

There is no quantization map (1.1) equivariant with respect to the action of the group $\text{Diff}(M)$. It is natural to consider a subgroup $G \subset \text{Diff}(M)$ (of finite dimension) and to restrict the action of $\text{Diff}(M)$ on the subgroup $G$. There are two interesting cases:

If $M = \mathbb{R}^n$ is endowed with a flat projective structure, the quantization map is given in [11]. This map is equivariant with respect to the action of the group of projective transformations $\text{SL}_{n+1} \subset \text{Diff}(\mathbb{R}^n)$. If $M = \mathbb{R}^n$ is endowed with a flat conformal structure, the quantization map is given in [7], it is equivariant with respect to the action of the group of conformal diffeomorphisms $SO(p+1,q+1) \subset \text{Diff}(\mathbb{R}^n)$, where $(p+q = n)$. (See also [8], [2], [4], for the one dimensional case.)

A natural and well-known way to define a quantization map is to fix an affine connection on $M$ (see, e.g. [1]). However, there is no canonical quantization map associated to a given connection.

The purpose of this paper is to study the quantization map (1.1) between the space of second-order symbols and the space of second-order linear differential operators satisfying the following properties:
1. It is projectively invariant, i.e. it depend only on the projective class of the affine connection $\Gamma$.

2. If $M = \mathbb{R}^n$ with a flat projective structure, this isomorphism is equivariant with respect to the action of the group of projective transformations $\text{SL}_{n+1}$ (resp. infinitesimal projective transformations $\text{sl}_{n+1}$).

The method used in this paper follows that of the recent preprint [6].

2 Space of linear differential operators

Let $M$ be a manifold of dimension $n$ endowed with an affine connection $\Gamma$. We are interested in defining a two parameter family of $\text{Diff}(M)$–module (resp. $\text{Vect}(M)$–module) on the space of linear differential operators. This space was recently studied in recent papers ([2], [3], [8], [11], [5], [6], [7], [10]).

2.1 Space of tensor densities

For simplicity, we assume $M$ oriented throughout this paper.

The space of tensor densities on $M$, $\mathcal{F}_\lambda(M)$, or $\mathcal{F}$ for simplify, is the space of sections of the line bundle $(\Lambda^n T^*M)^{\otimes \lambda}$, where $\lambda \in \mathbb{C}$. As a vector space, tensor densities are isomorphic to the space of complexified functions, but the structure of $\text{Diff}(M)$-module is different. Let us explicit this action:

Let $f \in \text{Diff}(M)$ and $\phi \in \mathcal{F}_\lambda$. In a local coordinates $(x^i)$, the action is given by

$$f^* \phi = \phi \circ f^{-1} \cdot (J_f^{-1})^\lambda, \quad (2.1)$$

where $J_f = \left| \frac{Df}{Dx} \right|$ is the Jacobian of $f$.

In the case $\lambda = 0, 1$, the action (2.1) is precisely the standard action of $\text{Diff}(M)$ on the space of functions and differential forms of degree $n$ respectively.

Differentiating the action of the flow of a vector field, one gets the corresponding representation of $\text{Vect}(M)$.

$$L_X^\lambda(\phi) = X^i \partial_i(\phi) + \lambda \partial_i(X^i)\phi, \quad (2.2)$$

where $X = X^i \partial_i \in \text{Vect}(M)$.

The formulæ (2.1), (2.2) do not depend on the choice of coordinates.

Let us now recall the definition of covariant derivative on tensor densities (cf. [7]).

Let $\nabla$ be the covariant derivative associated to the affine connection $\Gamma$. If $\phi \in \mathcal{F}_\lambda$, then $\nabla \phi \in \Omega^1(M) \otimes \mathcal{F}_\lambda$, given, in a local coordinates, by the formula:

$$\nabla_i \phi = \partial_i \phi - \lambda \Gamma_i \phi, \quad (2.3)$$

with $\Gamma_i = \Gamma^l_{il}$ (summation is understood on repeated indices).
2.2 Space of linear differential operators

Consider the space of linear differential operators acting on tensor densities

\[ A : \mathcal{F}_\lambda \to \mathcal{F}_\mu. \]  

(2.4)

The action of \( \text{Diff}(M) \) on \( \mathcal{D}(M) \) depends on two parameters \( \lambda \) and \( \mu \). This action is given by the equation:

\[ f_{\lambda,\mu}(A) = f^* \circ A \circ f^* - 1, \]  

(2.5)

where \( f^* \) is the action (2.1) of \( \text{Diff}(M) \) on \( \mathcal{F}_\lambda \).

Differentiating the action of the flow of a vector field, one gets the corresponding representation of \( \text{Vect}(M) \).

\[ L^\lambda_{X,\mu}(A) = L^\mu_X \circ A - A \circ L^\lambda_X, \]  

(2.6)

where \( X \in \text{Vect}(M) \).

These formulæ do not depend on the choice of a system of coordinates.

Notation. Denote \( \mathcal{D}_k(M) \) the space of \( k \)-order linear differential operators. In a local coordinates \( (x^i) \), one can write \( A \in \mathcal{D}_k(M) \)

\[ A = a^{i_1,\ldots,i_k}_{i_1,\ldots,i_k} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}} + \cdots + a^i_j \frac{\partial}{\partial x^i} + a_0, \]  

(2.7)

with the coefficients \( a^{i_1,\ldots,i_k}_{i_1,\ldots,i_k} = a^{i_1,\ldots,i_k}_k(x^1, \ldots, x^n) \in C^\infty(M) \). We have then a filtration

\[ \mathcal{D}^0 \subset \mathcal{D}^1 \subset \cdots \subset \mathcal{D}^k \subset \cdots \]

Denote by \( \mathcal{D}_{\lambda,\mu} \) the module of linear differential operators on \( M \) endowed with the action of \( \text{Diff}(M) \) (resp. \( \text{Vect}(M) \)) given by (2.5) (resp. (2.6)). The space of \( k \)-order linear differential operators, denoted by \( \mathcal{D}_{\lambda,\mu}^k \), is a \( \text{Diff}(M) \)-submodule (resp. \( \text{Vect}(M) \)-submodule) of \( \mathcal{D}_{\lambda,\mu} \).

Remark 2.1 The space of linear differential operators viewed as a module over the group of diffeomorphisms is a classical object (see e.g. \cite{14}). For example, in the case \( M = S^1 \), the space of sturm Liouville operators \( \frac{d^2}{dx^2} + u(x) \) is viewed as a submodule of \( \mathcal{D}^{\frac{1}{2},\frac{1}{2}} \). Also, the modules \( \mathcal{D}_{\frac{1}{2},\frac{1}{2}}^{\frac{1}{2},\frac{1}{2}} \) was considered in \cite{14}.

3 Space of symbols

The space of symbols, \( \text{Pol}(T^*M) \), is the space of functions on the cotangent bundle \( T^*M \) polynomial on the fibers. In a local coordinate system \( (x_i, \xi_i) \), one can write

\[ T = \sum_{l=0}^{k} T^{i_1,\ldots,i_l} \xi_i_1 \cdots \xi_i_l, \]

with \( T^{i_1,\ldots,i_l}(x^1, \ldots, x^n) \in C^\infty(M) \).

One defines a one parameter family of \( \text{Diff}(M) \)-module (resp. \( \text{Vect}(M) \)-module) on the space of symbols by \( \text{Pol}_\delta(T^*M) := \text{Pol}(T^*M) \otimes \mathcal{F}_\delta \).
Let us explicit this action.

Take \( f \in \text{Diff}(M) \) and \( X \in \text{Vect}(M) \). Then, in a local coordinates \((x_i, \xi_i)\), one has:

\[
\begin{align*}
 f_\delta(T) &= f_* T \cdot (J_f)^\delta, \\
 L^\delta_X(T) &= L_X(T) + \delta D(X) T,
\end{align*}
\]

where

\[
 L_X = X^i \partial_i - \xi_j \partial_i (X^j) \partial_{\xi_i}, \quad D(X) = \partial_i X^i.
\]

The space of symbols admits a graduation

\[
 \text{Pol}_\delta(T^* M) = \bigoplus_{k=0}^{\infty} \text{Pol}_{\delta,k}(T^* M),
\]

where \( \text{Pol}(T^* M)_{\delta,k} \) are the homogeneous polynomials of degree \( k \) on \( T^*(M) \). This graduation is \( \text{Diff}(M) \)-invariant.

Throughout this paper, we will identify the space of symbols with the space of symmetric contravariant tensor fields on \( M \).

4 Flat projective structure and projectively equivalent connection

Let \( M \) be a manifold of dimension \( n \). Recall two notions on projective geometry, the notion of flat projective structure and the notion of projectively equivalent connections (see [9]).

4.1 Flat projective structure

A manifold \( M \) admits a flat projective structure if there exists an atlas \( \{ \phi_i \} \) such that the local transformations \( \phi_i \circ \phi_j^{-1} \) are projective transformations.

The most interesting case is when \( M = \mathbb{R}^n \). In this case, the group \( \text{SL}_{n+1} \) acts locally on \( \mathbb{R}^n \) by projective transformations. Choosing a local coordinate system, the Lie algebra \( \text{sl}_{n+1} \) can be identified with the subalgebra of \( \text{Vect}(\mathbb{R}^n) \) generated by the vector fields:

\[
 \partial_i, \quad x^i \partial_j, \quad x^i x^j \partial_j.
\]

The projective Lie algebra \( \text{sl}_{n+1} \) is a maximal subalgebra of the Lie algebra of polynomial vector fields on \( \mathbb{R}^n \) (cf. [11]).

4.2 Projectively equivalent connections

The notion of projectively equivalent connection is an old notion related to projective geometry of the “paths” studied by H. Weyl in [13] and T.Y. Thomas in [12]. Weyl gives the following definition:

Two affine connection without torsion, with Christoffel symbols \( \tilde{\Gamma}^i_{jk} \) and \( \Gamma^i_{jk} \) given on the same system of coordinate \( x^1, \ldots, x^n \), are projectively equivalent, if there exists a differential 1-form with components \( \omega_i \), such that

\[
 \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \omega_k + \delta^i_k \omega_j.
\]
Geometrically, two affine connections without torsion projectively equivalent give the same unparameterized geodesics (cf. [9], [13]).

An affine connection $\Gamma$ is said to be \textit{projectively flat}, if they can be written:

$$\Gamma^i_{jk} = \frac{1}{n+1} \left( \delta^i_j \Gamma_k + \delta^i_k \Gamma_j \right).$$

(4.2)

A manifold $M$ endowed with an affine connection $\Gamma$, admits a flat projective structure if and only if the connection $\Gamma$ is projectively flat (cf. [9]).

5 Main theorems

In this section, we will give the quantization map between the space of second-order symbols and the space of second-order linear differential operators. First, decompose the space of symbols into a direct sum

$$\text{Pol}_{\delta,2}(T^*M) \oplus \text{Pol}_{\delta,1}(T^*M),$$

where $\text{Pol}_{\delta,2}(T^*M)$ is the space of symbols of degree 2, and $\text{Pol}_{\delta,1}(T^*M)$ the space of symbols of degree less or equal 1. We will construct a quantization map on each of these spaces.

In the case of first order symbols, there exists a quantization map that commutes with the action of $\text{Diff}(M)$ and $\text{Vect}(M)$ (see [11], [6]).

\textbf{Theorem 5.1} For any $\delta \neq 1$, the map $Q^{1}_{\lambda,\mu} : \text{Pol}_{\delta,1}(M) \to D^{1}_{\lambda,\mu}(M)$ given by

$$Q^{1}_{\lambda,\mu}(T) = T^i \nabla_i + \alpha \nabla_i(T^i) + T^0$$

(5.1)

where $T = T^i \xi_i + T^0$, and

$$\alpha = \frac{\lambda}{1-\delta}$$

(5.2)

is a projectively invariant isomorphism (i.e. it depends only on the projective class of the affine connection $\Gamma$.)

\textbf{Proof of Theorem 5.1.} Let $\tilde{\Gamma}$ be a symmetric affine connection projectively equivalent to $\Gamma$. Denote by $\tilde{Q}^{1}_{\lambda,\mu}$ the quantization map written with the connection $\tilde{\Gamma}$. We must show

$$\tilde{Q}^{1}_{\lambda,\mu} = Q^{1}_{\lambda,\mu}.$$

We need some formulæ (see [7]):

Recall the covariant derivative on the space of 1-order contravariant tensor fields: Let $T^i$ be a tensor, then one has:

$$\nabla_j(T^i) = \partial_j T^i + \Gamma^i_{ji} T^j - \delta \Gamma_j T^i.$$  

(5.3)

Using the formulæ (4.1), (5.3) one obtains

$$\nabla_i(\phi) = \nabla_i(\phi) - \lambda(n+1) \omega_i \phi, \quad \nabla_i(T^i) = \nabla_i(T^i) + (1-\delta)(1+n) \omega_i T^i.$$
Then after a straightforward calculation:
\[ \tilde{Q}^1_{\lambda,\mu}(T) = Q^1_{\lambda,\mu}(T) + (1 + n)(\alpha(1 - \delta) - \lambda) \omega_i T^i. \]
Hence \( \tilde{Q}^1_{\lambda,\mu} = Q^1_{\lambda,\mu} \) if and only if \( \alpha \) is given as in (5.2).

**Remark 5.2**
1. If \( M \) is endowed with a flat projective structure, the isomorphism (5.1) is the unique provided it preserves the principal symbol (cf. [11]).
2. In the case \( \delta = 1 \), the modules are still isomorphic if \((\lambda, \mu) = (0, 1)\). This isomorphism is given by (5.1) with an arbitrary \( \alpha = 0 \).

Let us give the quantization map on the space of homogeneous symbols of degree 2.

**Theorem 5.3** If \( n \geq 2 \), for any \( \delta \neq \frac{n+3}{n+1}, \frac{n+2}{n+1} \), there exists a projectively invariant isomorphism \( Q^2_{\lambda,\mu} : Pol_{\delta,2}(T^* M) \to D^2_{\lambda,\mu}(M) \) given by
\[ Q^2_{\lambda,\mu}(T) = T^{ij}\nabla_i \nabla_j + \beta_1 \nabla_j T^{ij}\nabla_i + \beta_2 \nabla_i \nabla_j (T^{ij}) + \beta_3 R_{ij}T^{ij}, \]
where \( T(\xi) = T^{ij}\xi_i\xi_j \), the coefficients \( \beta_1, \beta_2, \beta_3 \) are as follows
\[
\beta_1 = \frac{2 + 2\lambda(n + 1)}{2 + (1 + n)(1 - \delta)}
\]
\[
\beta_2 = \frac{\lambda(n + 1)(1 + \lambda(n + 1))}{((1 - \delta)(1 + n) + 1)((1 - \delta)(1 + n) + 2)}
\]
\[
\beta_3 = \frac{\lambda(\mu - 1)(n + 1)^2}{(1 - n)((1 - \delta)(1 + n) + 1)}
\]
and \( R_{ij} \) denote the components of Ricci tensor of the connection \( \Gamma \).

**Corollary 5.4** If \( M \) is endowed with a flat projective structure then:
1. The isomorphism (5.4) has the following form:
\[ Q_{\lambda,\mu}(T) = T^{ij}\partial_i\partial_j + \beta_1 \partial_j T^{ij}\partial_i + \beta_2 \partial_i\partial_j T^{ij}, \]
where the constants \( \beta_1, \beta_2 \) are as in (5.5).
2. It is the unique map equivariant with respect to the action of \( SL_{n+1} \) (resp. \( sl_{n+1} \)) that preserves the principal symbols (cf. [11]).

**Proof of the Theorem 5.3** Let \( \tilde{\Gamma} \) be a connection projectively equivalent to \( \Gamma \). Denote by \( Q^2_{\lambda,\mu} \) the quantization map written with \( \tilde{\Gamma} \).

We need some formulæ (see [7]):

The covariant derivative on the space of 2-order contravariant tensor fields reads:
\[ \nabla_k(T^{ij}) = \partial_k T^{ij} + \Gamma^j_{ik} T^{ij} + \Gamma^i_{kj} T^{il} - \delta \Gamma_k T^{ij}. \]

The second-order term in \( Q^2_{\lambda,\mu} \) reads:
\[ T^{ij}\partial_i\partial_j = T^{ij}\nabla_i \nabla_j + (T^{jk}\Gamma^i_{jk} + 2\lambda T^{ij}\Gamma_j) \nabla_i \]
\[ + T^{ij}(\lambda^2 \Gamma_i \Gamma_j + \lambda \partial_i \Gamma_j), \]
the first-order term in $Q_{\lambda,\mu}^2$ reads:
\[
\partial_j T^{ij} \partial_i = \nabla_j T^{ij} \nabla_i - (T^{jk} \Gamma^i_{jk} + (1 - \delta) T^{ij} \Gamma_j) \nabla_i 
\]
\[
+ \lambda (\nabla_i T^{ij} \Gamma_j) - \lambda T^{ij} (\Gamma^k_{ij} \Gamma_k + (1 - \delta) \Gamma_i \Gamma_j),
\]
(5.9)
and the zero-order part of $Q_{\lambda,\mu}^2$ reads:
\[
\partial_j \partial_j T^{ij} = \nabla_i \nabla_j T^{ij} - 2(1 - \delta) (\nabla_i T^{ij}) \Gamma_j - (\nabla_i T^{jk}) \Gamma^i_{jk} - (1 - \delta) \Gamma^i_{ij} \Gamma_i - (1 - \delta)^2 \Gamma_i \Gamma_j.
\]
(5.10)
Now after calculation one has:
\[
\tilde{Q}_{\lambda,\mu}(T) = Q_{\lambda,\mu}(T) + [2 \beta_2 + (1 + n)(-\lambda \beta_1 + 2 \eta_\delta \beta_2)] \nabla_i T^{ij} \omega_j 
\]
\[
+ [2 \beta_1 - 2 + (1 + n)(-2 \lambda + \eta_\delta \beta_1)] T^{ij} \omega_j \nabla_i 
\]
\[
+ [(1 + n)(-\lambda + \eta_\delta \beta_2) + 2 \beta_2 + (1 - n) \beta_3] T^{ij} \omega_j \partial_i 
\]
\[
+ [(1 + n)(\lambda - \eta_\delta \beta_2) - 2 \beta_2 + \beta_3 (n - 1)] T^{ij} \omega_j \Gamma_i 
\]
\[
+ [(1 + n)^2 (\lambda^2 + \eta_\delta (\delta \beta_2 - \lambda \beta_1)) + 2(1 + n)(\lambda(1 - \beta_1) + \delta \beta_2) + (n - 1) \beta_3] T^{ij} \omega_i \omega_j 
\]
where $\eta_\delta = 1 - \delta$.
Hence, $\tilde{Q}_{\lambda,\mu}^2 = Q_{\lambda,\mu}^2$, if and only if the constants $\beta_1, \beta_2, \beta_3$ are given as in (5.5).

**Proof of the Corollary 5.4.** In this case (see section 4.2), the connection $\Gamma$ can be written, in the coordinates of the flat projective structure, in the form
\[
\Gamma^k_{ij} = \frac{1}{n+1} (\delta^k_{ij} \Gamma_j + \delta^k_{ij} \Gamma_i).
\]
Substituting this formula to the equations (5.8), (5.9), (5.10), and, finally to the map (5.4) one gets the expression (5.6).

The proof of the part 2) is given in [11].

Let us study the particular values of $\delta$ called “resonant”:

**Proposition 5.5** In the resonant case $\delta = \frac{n+2}{n+1}, \frac{n+3}{n+1}$, the modules are still isomorphic with the particular values of $\lambda, \mu, \beta_1, \beta_2, \beta_3$, given in the table I bellow.

**Proof of the proposition 5.5.** Replace the particular values of $\delta$ in the formula (5.11). hence, $Q_{\lambda,\mu}^2 = \tilde{Q}_{\lambda,\mu}^2$, if and only if the constants $\lambda, \mu, \beta_1, \beta_2, \beta_3$ is given as in the table I.

**Remark 5.6** In contrast with the non-resonant case, if $M$ is flat and $\delta = \frac{n+3}{n+1}$, the isomorphism is not unique. There is a family of isomorphisms with arbitrary constant $\beta_2$. 


It would be interesting to obtain an analogue of the formula (5.4) in the case of higher-order differential operators.

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Sofiane BOUARROUDJ
CNRS, Centre de Physique Théorique, Luminy, Case 907, F13288 Marseille, Cedex 9, France.
e-mail: sofbou@cpt.univ-mrs.fr

Abstract

Let $M$ be a manifold endowed with a symmetric affine connection $\Gamma$. The aim of this Letter is to describe a quantization map between the space of second-order polynomials on the cotangent bundle $T^*M$ and the space of second-order linear differential operators, both viewed as modules over the group of diffeomorphisms and the Lie algebra of vector fields on $M$. This map is an isomorphism, for almost all values of certain constants, and it depends only on the projective class of the affine connection $\Gamma$.

1 Introduction

The quantization procedure proposed in this Letter deals with the space of linear differential operators and the corresponding space of symbols viewed as modules over the group of diffeomorphisms $\text{Diff}(M)$ and the Lie algebra of vectors fields $\text{Vect}(M)$. This method of quantization have been introduced in the recent papers ([11], [5], [7]).

Let $\mathcal{D}_{\lambda,\mu}(M)$ be the space of linear differential operators from the space of $\lambda-$densities to the space of $\mu-$densities. The corresponding space of symbols, $\text{Pol}_\delta(T^*M)$, is the space of fiberwise polynomials on $T^*M$ with values in the space of $\delta-$densities over $M$, where $\delta = \mu - \lambda$.

We call quantization map, a linear map

$$Q_{\lambda,\mu} : \text{Pol}_\delta(T^*M) \to \mathcal{D}_{\lambda,\mu}(M),$$

that is bijective and preserves the principal symbol (see [11], [7], [5]).

There is no quantization map (1.1) equivariant with respect to the action of the group $\text{Diff}(M)$. It is natural to consider a subgroup $G \subset \text{Diff}(M)$ (of finite dimension) and to restrict the action of $\text{Diff}(M)$ on the subgroup $G$. There are two interesting cases:

If $M = \mathbb{R}^n$ is endowed with a flat projective structure, the quantization map is given in [11]. This map is equivariant with respect to the action of the group of projective transformations $\text{SL}_{n+1} \subset \text{Diff}(\mathbb{R}^n)$. If $M = \mathbb{R}^n$ is endowed with a flat conformal structure, the quantization map is given in [7], it is equivariant with respect to the action of the group of conformal diffeomorphisms $\text{SO}(p+1,q+1) \subset \text{Diff}(\mathbb{R}^n)$, where $(p+q = n)$. (See also [8], [2], [4], for the one dimensional case.)

A natural and well-known way to define a quantization map is to fix an affine connection on $M$ (see, e.g., [1]). However, there is no canonical quantization map associated to the given connection.
The purpose of this Letter is to study the quantization map (1.1) between the space of symbols of degree less than two and the space of second-order linear differential operators satisfying the following properties:

1. It is projectively invariant, i.e. it depends only on the projective class of the affine connection $\Gamma$.
2. If $M = \mathbb{R}^n$ with a flat projective structure, this isomorphism is equivariant with respect to the action of the group of projective transformations $SL_{n+1}$ (resp. infinitesimal projective transformations $sl_{n+1}$).

The method used in this paper follows that of the recent preprint [6].

2 Space of Linear Differential Operators

Let $M$ be a manifold of dimension $n$ endowed with an affine connection $\Gamma$. We are interested in defining a two parameter family of $Diff(M)$-module (resp. $Vect(M)$-module) on the space of linear differential operators. This space was recently studied in recent papers ([2], [3], [8], [11], [5], [6], [7], [10]).

2.1 Space of Tensor Densities

For simplicity, we assume that $M$ is oriented throughout this Letter.

The space of tensor densities on $M$, $\mathcal{F}_\lambda(M)$, or $\mathcal{F}_\lambda$ for simplify, is the space of sections of the line bundle $(\Lambda^nT^*M)^{\otimes \lambda}$, where $\lambda \in \mathbb{C}$. As a vector space, tensor densities are isomorphic to the space of complexified functions, but the structure of $Diff(M)$-module is different. Let us explicate this action:

Let $f \in Diff(M)$ and $\phi \in \mathcal{F}_\lambda$. In a local coordinates $(x^i)$, the action is given by

$$f^*\phi = \phi \circ f^{-1} \cdot (J_{f^{-1}})_{\lambda}, \quad (2.1)$$

where $J_f = \left| \frac{Df}{Dx} \right|$ is the Jacobian of $f$.

In the case $\lambda = 0, 1$, the action (2.1) is precisely the standard action of $Diff(M)$ on the space of functions and differential forms of degree $n$ respectively.

Differentiating the action of the flow of a vector field, one gets the corresponding representation of $Vect(M)$.

$$L^\lambda_X(\phi) = X^i \partial_i(\phi) + \lambda \partial_l(X^i)\phi, \quad (2.2)$$

where $X = X^i \partial_i \in Vect(M)$.

Formulae (2.1), (2.2) do not depend on the choice of coordinates.

Let us now recall the definition of covariant derivative on tensor densities (cf. [7]). Let $\nabla$ be the covariant derivative associated to the affine connection $\Gamma$. If $\phi \in \mathcal{F}_\lambda$, then $\nabla \phi \in \Omega^1(M) \otimes \mathcal{F}_\lambda$, given, in a local coordinates, by the formula

$$\nabla_i \phi = \partial_i \phi - \lambda \Gamma_i \phi, \quad (2.3)$$

with $\Gamma_i = \Gamma^l_{il}$ (summation is understood on repeated indices).
2.2 Space of Linear Differential Operators

Consider $\mathcal{D}_{\lambda,\mu}(M)$, the space of linear differential operators acting on tensor densities

$$A : \mathcal{F}_\lambda \to \mathcal{F}_\mu.$$ (2.4)

The action of Diff$(M)$ on $\mathcal{D}_{\lambda,\mu}(M)$ depends on two parameters $\lambda$ and $\mu$. This action is given by the equation:

$$f_{\lambda,\mu}(A) = f^* \circ A \circ f^{-1},$$ (2.5)

where $f^*$ is the action (2.1) of Diff$(M)$ on $\mathcal{F}_\lambda$.

Differentiating the action of the flow of a vector field, one gets the corresponding representation of Vect$(M)$.

$$L^\lambda_X(A) = L^\mu_X \circ A - A \circ L^\lambda_X,$$ (2.6)

where $X \in$ Vect$(M)$.

These formulæ do not depend on the choice of a system of coordinates.

Denote $\mathcal{D}^k_{\lambda,\mu}(M)$ the space of $k$-order linear differential operators acting on tensors densities. In a local coordinates $(x^i)$, one can write $A \in \mathcal{D}^k_{\lambda,\mu}(M)$

$$A = a_{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \ldots \frac{\partial}{\partial x^{i_k}} + \ldots + a_{l} \frac{\partial}{\partial x^{i}} + a_0,$$ (2.7)

with the coefficients $a_{i_1 \ldots i_l} = a_{i_1 \ldots i_l}(x^1, \ldots, x^n) \in C^\infty(M)$, with $l = 0, \ldots, k$. We have then a filtration

$$\mathcal{D}^0_{\lambda,\mu}(M) \subset \mathcal{D}^1_{\lambda,\mu}(M) \subset \cdots \subset \mathcal{D}^k_{\lambda,\mu}(M) \subset \cdots$$

The space of $k$-order linear differential operators $\mathcal{D}^k_{\lambda,\mu}(M)$ is a Diff$(M)$-submodule (resp. Vect$(M)$-submodule) of $\mathcal{D}_{\lambda,\mu}(M)$.

Remark 2.1 The space of linear differential operators viewed as a module over the group of diffeomorphisms is a classical object (see e.g., [14]). For example, in the case $M = S^1$, the space of Sturm-Liouville operators $\frac{d^2}{dx^2} + u(x)$, where $u(x) \in \mathcal{F}_2$, is viewed as a submodule of $\mathcal{D}^2_{\frac{1}{2}, \frac{1}{2}}(S^1)$. Also, the modules $\mathcal{D}^k_{\frac{1}{2} - \frac{1}{2}, \frac{1}{2} + \frac{1}{2}}(S^1)$ was considered in [14].

3 Space of Symbols

The space of symbols, Pol$(T^*M)$, is the space of functions on the cotangent bundle $T^*M$ polynomial on the fibers. In a local coordinate system $(x_i, \xi_i)$, one can write

$$T = \sum_{l=0}^{k} T^{i_1 \ldots i_l} \xi_{i_1} \cdots \xi_{i_l},$$

with $T^{i_1 \ldots i_l}(x^1, \ldots, x^n) \in C^\infty(M)$.

One defines a one parameter family of Diff$(M)$–module (resp. Vect$(M)$–module) on the space of symbols by

$$\operatorname{Pol}_\delta(T^*M) := \operatorname{Pol}(T^*M) \otimes \mathcal{F}_\delta.$$
Let us explicit this action.

Take $f \in \text{Diff}(M)$, and $X \in \text{Vect}(M)$. Then, in a local coordinates $(x_i, \xi_i)$, one has:

$$
\begin{align*}
  f_\delta(T) &= f^* T \cdot (J_f^{-1})^\delta, \\
  L_X^\delta(T) &= L_X(T) + \delta D(X) T, \quad T \in \text{Pol}_\delta(T^* M)
\end{align*}
$$

(3.1)

(3.2)

where

$$
L_X = X^i \partial_i - \xi_j \partial_i (X^j) \partial_{\xi_i}, \quad D(X) = \partial_i X^i.
$$

The space of symbols admits a graduation

$$
\text{Pol}_\delta(T^* M) = \bigoplus_{k=0}^{\infty} \text{Pol}_{\delta,k}(T^* M),
$$

where $\text{Pol}_{\delta,k}(T^* M)$ are the homogeneous polynomials of degree $k$ on $T^*(M)$. This graduation is $\text{Diff}(M)$–invariant.

Throughout this Letter, we will identify the space of symbols with the space of symmetric contravariant tensor fields on $M$.

4 Flat Projective Structure and Projectively Equivalent Connection

Let $M$ be a manifold of dimension $n$. Recall two notions on projective geometry, the notion of flat projective structure and the notion of projectively equivalent connections (see [9]).

4.1 Flat projective structure

A manifold $M$ admits a flat projective structure if there exists an atlas $\{\phi_i\}$ such that the local transformations $\phi_i \circ \phi_j^{-1}$ are projective transformations.

The most interesting case is when $M = \mathbb{R}^n$. In this case, the group $\text{SL}_{n+1}$ acts locally on $\mathbb{R}^n$ by projective transformations. Choosing a local coordinate system, the Lie algebra $\mathfrak{sl}_{n+1}$ can be identified with the subalgebra of $\text{Vect}(\mathbb{R}^n)$ generated by the vector fields:

$$
\partial_i, \quad x^i \partial_j, \quad x^i x^j \partial_j.
$$

The projective Lie algebra $\mathfrak{sl}_{n+1}$ is a maximal subalgebra of the Lie algebra of polynomial vector fields on $\mathbb{R}^n$ (cf. [11]).

4.2 Projectively Equivalent Connections

The notion of projectively equivalent connection is an old notion related to projective geometry of the “paths” studied by H. Weyl in [13] and T.Y. Thomas in [12]. Weyl gives the following definition:

Two affine connection without torsion, with Christoffel symbols $\tilde{\Gamma}^i_{jk}$ and $\Gamma^i_{jk}$ given on the same system of coordinate $x^1, \ldots, x^n$, are projectively equivalent, if there exists a differential 1-form with components $\omega_i$, such that

$$
\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \omega_k + \delta^i_k \omega_j.
$$

(4.1)
Geometrically, two affine connections without torsion projectively equivalent give the same unparameterized geodesics (cf. [9], [13]).

An affine connection $\Gamma$ is said to be projectively flat, if they can be written:

$$\Gamma_{jk}^i = \frac{1}{n+1} \left( \delta_j^i \Gamma_k + \delta_k^i \Gamma_j \right).$$

A manifold $M$ endowed with an affine connection $\Gamma$, admits a flat projective structure if and only if the connection $\Gamma$ is projectively flat (cf. [9]).

5 Main Theorems

In this section, we will give a quantization map between the space of symbols of degree less than two and the space of second-order linear differential operators. First, decompose the space of symbols into a direct sum

$$\text{Pol}_{\delta,2}(T^*M) \oplus \text{Pol}_{\delta,1}(T^*M) \oplus \text{Pol}_{\delta,0}(T^*M),$$

where $\text{Pol}_{\delta,k}(T^*M)$, is the space of homogeneous symbols of degree $k$, ($k = 0, 1, 2$). We will construct a quantization map on each of these spaces.

In the case of symbols of degree less than 1, there exists a quantization map that commutes with the action of Diff($M$) and Vect($M$) (see [11], [6]).

**Theorem 5.1** For any $\delta \neq 1$, the map $Q_{\lambda,\mu}^1 : \text{Pol}_{\delta,1}(M) \oplus \text{Pol}_{\delta,0}(M) \rightarrow D_{\lambda,\mu}^1(M)$ given by

$$Q_{\lambda,\mu}^1(T) = T^i \nabla_i + \alpha \nabla_i(T^i) + T^0$$

(5.1)

where $T = T^i \xi_i + T^0$, and

$$\alpha = \frac{\lambda}{1 - \delta}$$

(5.2)

is a projectively invariant isomorphism (i.e. it depends only on the projective class of the affine connection $\Gamma$).

**Proof of Theorem 5.1.** Let $\tilde{\Gamma}$ be a symmetric affine connection projectively equivalent to $\Gamma$. Denote by $\tilde{Q}_{\lambda,\mu}^1$ the quantization map written with the connection $\tilde{\Gamma}$. We must show

$$\tilde{Q}_{\lambda,\mu}^1 = Q_{\lambda,\mu}^1.$$

We need some formulæ (see [7]):

Recall the covariant derivative on the space of 1-order contravariant tensor fields: Let $T^i$ be a tensor, then one has:

$$\nabla_j(T^i) = \partial_j T^i + \Gamma^i_{ji} T^j - \delta_j T^i.$$  \hspace{1cm} (5.3)

Using formulæ (4.1), (5.3) one obtains

$$\tilde{\nabla}_i(T^i) = \nabla_i(T^i) + (1 - \delta)(1 + n) \omega_i T^i.$$

Then after a straightforward calculation:

$$\tilde{Q}_{\lambda,\mu}^1(T) = Q_{\lambda,\mu}^1(T) + (1 + n)(\alpha(1 - \delta) - \lambda) \omega_i T^i.$$  \hspace{1cm} (5.4)

Hence $\tilde{Q}_{\lambda,\mu}^1 = Q_{\lambda,\mu}^1$ if and only if $\alpha$ is given as in (5.2).
**Remark 5.2** 1. If $M$ is endowed with a flat projective structure, the isomorphism (5.1) is the unique provided it preserves the principal symbol (cf. [11]).
2. In the case $\delta = 1$, the modules are still isomorphic if $(\lambda, \mu) = (0, 1)$. This isomorphism is given by (5.1) with an arbitrary $\alpha$.

Let us give the quantization map on the space of homogeneous symbols of degree 2.

**Theorem 5.3** If $n \geq 2$, for any $\delta \neq \frac{n+3}{n+1}, \frac{n+2}{n+1}$, there exists a projectively invariant isomorphism $Q^2_{\lambda, \mu} : \text{Pol}_2(T^*M) \to \mathcal{D}^2_{\lambda, \mu}(M)$ given by

$$Q^2_{\lambda, \mu}(T) = T^{ij} \nabla_i \nabla_j + \beta_1 \nabla_j T^{ij} \nabla_i + \beta_2 \nabla_i \nabla_j (T^{ij}) + \beta_3 R_{ij} T^{ij},$$

(5.4)

where $T(\xi) = T^{ij} \xi_i \xi_j$, the coefficients $\beta_1, \beta_2, \beta_3$ are as follows

$$\beta_1 = \frac{2 + 2\lambda(n + 1)}{2 + (1 + n)(1 - \delta)}$$

$$\beta_2 = \frac{\lambda(n + 1)(1 + \lambda(n + 1))}{((1 - \delta)(1 + n) + 1)((1 - \delta)(1 + n) + 2)}$$

$$\beta_3 = \frac{\lambda (\mu - 1)(n + 1)^2}{(1 - n)((1 - \delta)(1 + n) + 1)}$$

(5.5)

and $R_{ij}$ denote the components of Ricci tensor of the connection $\Gamma$.

**Corollary 5.4** If $M$ is endowed with a flat projective structure then:

1. The isomorphism (5.4) has the form

$$Q_{\lambda, \mu}(T) = T^{ij} \partial_i \partial_j + \beta_1 \partial_j T^{ij} \partial_i + \beta_2 \partial_i \partial_j T^{ij},$$

(5.6)

where the constants $\beta_1, \beta_2$ are as in (5.5).

2. It is the unique map equivariant with respect to the action of $\text{SL}_{n+1}$ (resp. $\text{sl}_{n+1}$) that preserves the principal symbols (cf. [11]).

**Proof of the Theorem 5.3** Let $\tilde{\Gamma}$ be a connection projectively equivalent to $\Gamma$. Denote by $Q^2_{\lambda, \mu}$ the quantization map written with $\tilde{\Gamma}$.

We need some formulæ (see [7]):

The covariant derivative on the space of 2-order contravariant tensor fields reads:

$$\nabla_k(T^{ij}) = \partial_k T^{ij} + \Gamma^i_{lk} T^{lj} + \Gamma^j_{lk} T^{il} - \delta \Gamma^i_k T^{ij}. \quad (5.7)$$

The second-order term in $Q^2_{\lambda, \mu}$ reads:

$$T^{ij} \partial_i \partial_j = T^{ij} \nabla_i \nabla_j + (T^{jk} \Gamma^i_{jk} + 2\lambda T^{ij} \Gamma^i_j) \nabla_i$$

$$+ T^{ij}(\lambda^2 \Gamma_i \Gamma_j + \lambda \partial_i \Gamma_j), \quad (5.8)$$

the first-order term in $Q^2_{\lambda, \mu}$ reads:

$$\partial_j T^{ij} \partial_i = \nabla_j T^{ij} \partial_i - (T^{jk} \Gamma^i_{jk} + (1 - \delta) T^{ij} \Gamma^i_j) \nabla_i$$

$$+ \lambda (\nabla_i T^{ij}) \Gamma_j - \lambda T^{ij} (\Gamma^k_{ij} \Gamma_k + (1 - \delta) \Gamma_i \Gamma_j), \quad (5.9)$$
and the zero-order part of $Q^2_{\lambda,\mu}$ reads:

$$
\partial_j \partial_j T^{ij} = \nabla_i \nabla_j T^{ij} - 2(1-\delta)(\nabla_i T^{ij}) \Gamma_j - (\nabla_i T^{jk}) \Gamma^i_{jk} - T^{ij} \partial_i \Gamma_{ij} - (1-\delta) \partial_i \Gamma_{ij} - 2(1-\delta) \Gamma^k_{ij} \Gamma_k - (1-\delta)^2 \Gamma_i \Gamma_j.
$$

(5.10)

Now after calculation one has:

$$
\tilde{Q}_{\lambda,\mu}(T) = Q_{\lambda,\mu}(T) + [2 \beta_2 + (1+n)(-\lambda \beta_1 + 2 \eta \beta_2)] \nabla_i T^{ij} \omega_j + [2 \beta_1 - 2 + (1+n)(-2 \lambda + \eta \beta_1)] T^{ij} \omega_j \nabla_i
$$

$$
+ [(1+n)(-\lambda + \eta \beta_2) + 2 \beta_2 + (1-n) \beta_3] T^{ij} \partial_i \omega_j
$$

$$
+ [(1+n)(\lambda - \eta \beta_2) - 2 \beta_2 + \beta_3(n-1)] T^{jk} \Gamma^i_{jk} \omega_i
$$

$$
+ [(1+n)^2(\lambda^2 + \eta \delta \beta_2 - \lambda \beta_1)) + 2(1+n)(\lambda(1-\beta_1) + \delta \beta_2) + (n-1) \beta_3] T^{ij} \omega_i \omega_j
$$

(5.11)

where $\eta \delta = 1 - \delta$.

Hence, $\tilde{Q}^2_{\lambda,\mu} = Q^2_{\lambda,\mu}$ if and only if the constants $\beta_1, \beta_2, \beta_3$ are given as in (5.5).

**Proof of the Corollary 5.4.** In this case (see section 4.2), the connection $\Gamma$ can be written, in the coordinates of the flat projective structure, in the form

$$
\Gamma^k_{ij} = \frac{1}{n+1}(\delta^k_i \Gamma_j + \delta^k_j \Gamma_i).
$$

Substituting this formula in Equations (5.8), (5.9), (5.10), and, finally to the map (5.4) one gets the expression (5.6).

The proof of the part 2) is given in [11].

Let us study the particular values of $\delta$ called “resonant”:

**Proposition 5.5** In the resonant case $\delta = \frac{\eta + 2}{n+1}$, the modules are still isomorphic with the particular values of $\lambda, \mu, \beta_1, \beta_2, \beta_3$, given in table I.

**Proof of the proposition 5.5.** Replace the particular values of $\delta$ in the formula (5.11).

hence, $Q^2_{\lambda,\mu} = \tilde{Q}^2_{\lambda,\mu}$, if and only if the constants $\lambda, \mu, \beta_1, \beta_2, \beta_3$ is given as in the table I.

**Remark 5.6** In contrast with the nonresonant case, if $M$ is flat and $\delta = \frac{\eta + 3}{n+1}$, the isomorphism is not unique. There is a family of isomorphisms with arbitrary constant $\beta_2$.

| $\delta$ | $\lambda$ | $\mu$ | $\beta_1$ | $\beta_2$ | $\beta_3$
|----------|-----------|-------|-----------|-----------|-----------|
| $\frac{n+3}{n+1}$ | $-\frac{1}{n+1}$ | $\frac{n+2}{n+1}$ | $2 \beta_2$ | . | $\frac{1}{1-n}$ |
| $\frac{n+2}{n+1}$ | $0$ | $\frac{n+2}{n+1}$ | $2$ | $0$ | $0$ |
| $\frac{n+2}{n+1}$ | $-\frac{1}{n+1}$ | $1$ | $0$ | $0$ | $\frac{1}{1-n}$ |

Table I.
It would be interesting to obtain an analogue of the formula (5.4) in the case of higher-order differential operators.

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