DEHN SURGERIES ON THE FIGURE EIGHT KNOT: AN UPPER BOUND FOR THE COMPLEXITY

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ABSTRACT. We establish an upper bound \( \omega(p/q) \) on the complexity of manifolds obtained by \( p/q \)-surgeries on the figure eight knot. It turns out that if \( \omega(p/q) \leq 12 \), the bound is sharp.

INTRODUCTION

The notion of the complexity \( c(M) \) of a compact 3-manifold \( M \) was introduced in [1]. The complexity is defined as the minimal possible number of true vertices of an almost simple spine of \( M \). If \( M \) is closed and irreducible and \( c(M) > 0 \), then \( c(M) \) is the minimal number of tetrahedra needed to obtain \( M \) by gluing together their faces. The problem of calculating the complexity \( c(M) \) is very difficult. The exact values of the complexity are presently known only for certain infinite series of irreducible boundary irreducible 3-manifolds [2, 3, 4]. In addition, this problem is solved for all closed orientable irreducible manifolds up to complexity 12 (see [5]). Note that the table given in [5] contains 36833 manifolds and is only available in electronic form [6].

The task of finding an upper bound for the complexity of a manifold \( M \) does not present any particular difficulties. To do that it suffices to construct an almost simple spine \( P \) of \( M \). The number of true vertices of \( P \) will serve as an upper bound for the complexity. It is known [7, 2.1.2] that an almost simple spine can be easily constructed from practically any representation of a manifold. The rather large number of manifolds in [6] gives rise to a new task of finding potentially sharp upper bounds for the complexity, i.e. upper bounds that would yield the exact value of the complexity for all manifolds from the table [6]. An important result in this direction was obtained by Martelli and Petronio [8]. They found a potentially sharp upper bound for the complexity of all closed orientable Seifert manifolds. Similar results for infinite families of graph manifolds can be found in [9, 10].

An upper bound \( h(r/s, t/u, v/w) \) for the complexity of hyperbolic manifolds obtained by surgeries on the link \( 6_1^3 \) (in Rolfsen’s notation [11]) with rational parameters \( (r/s, t/u, v/w) \) is given by Martelli and Petronio in [8]. It turns out that the bound is not sharp for a large number of manifolds, as the following two examples show. First, the value of \( h \) is equal to 10 only for 13 of 24 manifolds of complexity 10 obtained by surgeries on \( 6_1^3 \) (see [5]). Second, on analyzing the table [6] we noticed that the bound is not sharp for 44 of 46 manifolds of the type \( 6_1^3(1, 2, v/w) \) with

2000 Mathematics Subject Classification. Primary: 57M99, Secondary: 57M25.

Key words and phrases. Dehn surgery, figure eight knot, upper bound of the complexity.

The research was supported by the Russian Foundation for Basic Research (grant 10-01-91056) and by the Joint Program of the Institute for Mathematics and Mechanics UrO RAN and of the Institute for Mathematics SO RAN.
complexity less or equal to 12. Denote by $4_1(p/q)$ the closed orientable 3-manifold obtained from the figure eight knot $4_1$ by $p/q$-surgery. Since the manifolds $4_1(p/q)$ and $6_3^1(1, 2, p/q + 1)$ are homeomorphic, a potentially sharp upper bound for the complexity of such manifolds become important.

The following theorem is the main result of the paper. To give an exact formulation, we need to introduce a certain $\mathbb{N}$-valued function $\omega(p/q)$ on the set of non-negative rational numbers. Let $p \geq 0, q \geq 1$ be relatively prime integers, let $\lfloor p/q \rfloor$ be the integer part of $p/q$, and let $\text{rem}(p, q)$ be the remainder of the division of $p$ by $q$. As in [7], we denote by $S(p, q)$ the sum of all partial quotients in the expansion of $p/q$ as a regular continued fraction. Now we define:

$$\omega(p/q) = a(p/q) + \max\{\lfloor p/q \rfloor - 3, 0\} + S(\text{rem}(p, q), q),$$

where

$$a(p/q) = \begin{cases} 6, & \text{if } p/q = 4, \\ 7, & \text{if } p/q \in \mathbb{Z} \text{ and } p/q \neq 4, \\ 8, & \text{if } p/q \notin \mathbb{Z}. \end{cases}$$

**Theorem.** For any two relatively prime integers $p \geq 0$ and $q \geq 1$ we have the inequality $c(4_1(p/q)) \leq \omega(p/q)$. Moreover, if $\omega(p/q) \leq 12$, then $c(4_1(p/q)) = \omega(p/q)$.

Note that the restrictions $p \geq 0$ and $q \geq 1$ in the above theorem are inessential, since the knot $4_1$ is equivalent to its mirror image, which implies $4_1(-p/q)$ is homeomorphic to $4_1(p/q)$.

1. Preliminaries

In this section we recall some known definitions and facts that will be used in the paper.

1.1. Theta-curves on a torus. By a theta-curve $\theta \subset T$ on a torus $T$ we mean a graph that is homeomorphic to a circle with a diameter and such that $T \setminus \theta$ is an open disc. It is well known [8] [12] that any two theta-curves on $T$ can be transformed into each other by isotopies and by a sequence of flips (see Fig. 1). Let us endow the set $\Theta(T)$ of theta-curves on $T$ with the distance function $d$ defining for given $\theta, \theta' \in \Theta(T)$ the distance $d(\theta, \theta')$ between them as the minimal number of flips required to transform $\theta$ into $\theta'$.

For calculating the distance between two theta-curves on a torus we use the classical ideal triangulation $F$ (Farey tessellation) of the hyperbolic plane $\mathbb{H}^2$. If we view the hyperbolic plane $\mathbb{H}^2$ as the upper half plane of $\mathbb{C}$ bounded by the circle $\partial \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$, then the triangulation $F$ has vertices at the points of $\mathbb{Q} \cup \{1/0\} \subset \partial \mathbb{H}^2$, where $1/0 = \infty$, and its edges are all the geodesics in $\mathbb{H}^2$ with endpoints the pairs $a/b, c/d$ such that $ad - bc = \pm 1$. For convenience, the images of the hyperbolic

![Figure 1. A flip-transformation](image-url)
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Figure 2. The ideal Farey triangulation of the hyperbolic plane

plane $\mathbb{H}^2$ and of the triangulation $\mathbb{F}$ under the mapping $z \to (z - i)/(z + i)$ are shown in Fig. 2.

Fix some coordinate system $(\mu, \lambda)$ on a torus $T$. We now construct a map $\Psi_{\mu, \lambda}$ from $\Theta(T)$ to the set of triangles of $\mathbb{F}$. To do that we consider the map $\psi_{\mu, \lambda}$ that assigns to each nontrivial simple closed curve $\mu^\alpha \lambda^\beta$ on $T$ the point $\alpha/\beta \in \partial \mathbb{H}^2$. Note that each theta-curve $\theta$ on $T$ contains three nontrivial simple closed curves $\ell_1, \ell_2, \ell_3$, that are formed by the pairs of edges of $\theta$. Since the intersection index of every two curves $\ell_i, \ell_j, i \neq j,$ is equal to $\pm 1$, the points $\psi_{\mu, \lambda}(\ell_1), \psi_{\mu, \lambda}(\ell_2), \psi_{\mu, \lambda}(\ell_3)$ are the vertices of a triangle $\triangle$ of the Farey triangulation, and we define $\Psi_{\mu, \lambda}(\theta)$ to be $\triangle$.

Denote by $\Sigma$ the graph dual to the triangulation $\mathbb{F}$. This graph is a tree because the triangulation is ideal. We now define the distance between any two triangles of $\mathbb{F}$ to be the number of edges of the only simple path in $\Sigma$ that joins the corresponding vertices of the dual graph. The key observation used for the practical calculations is that for any coordinate system $(\mu, \lambda)$ on $T$ the distance between any two theta-curves $\theta, \theta'$ is equal to the distance between the triangles $\Psi_{\mu, \lambda}(\theta), \Psi_{\mu, \lambda}(\theta')$ of the Farey triangulation. The reason is that if $\theta'$ is obtained from $\theta$ via a flip, the corresponding triangles have a common edge.

1.2. Simple and special spines. A compact polyhedron $P$, following Matveev [7], is called simple if the link of each point $x \in P$ is homeomorphic to one of the following 1-dimensional polyhedra:

(a) a circle (the point $x$ is then called nonsingular);
(b) a circle with a diameter (then $x$ is a triple point);
(c) a circle with three radii (then $x$ is a true vertex).

The components of the set of nonsingular points are said to be the 2-components of $P$, while the components of the set of triple points are said to be the triple lines of $P$. A simple polyhedron is special if each of its triple lines is an open 1-cell and each of its 2-components is an open 2-cell.

A subpolyhedron $P$ of a 3-manifold $M$ is a spine of $M$ if $\partial M \neq \emptyset$ and the manifold $M \setminus P$ is homeomorphic to $\partial M \times [0, 1]$, or $\partial M = \emptyset$ and $M \setminus P$ is an open ball. A spine of a 3-manifold is called simple or special if it is a simple or special polyhedron, respectively.
1.3. Relative spines. A manifold with boundary pattern, following Johannson [13], is a 3-manifold $M$ with a fixed graph $\Gamma \subset \partial M$ that does not have any isolated vertices. A manifold $M$ with boundary pattern $\Gamma$ can be conveniently viewed as a pair $(M, \Gamma)$. The case $\Gamma = \emptyset$ is also allowed.

**Definition.** Let $(M, \Gamma)$ be a 3-manifold with boundary pattern. Then a subpolyhedron $P \subset M$ is called a relative spine of $(M, \Gamma)$ if the following holds:

1. $M \setminus P$ is an open ball;
2. $\partial M \subset P$;
3. $\partial M \cap \text{Cl}(P \setminus \partial M) = \Gamma$.

A relative spine is simple if it is a simple polyhedron. Obviously, if $M$ is closed, then any relative spine of $(M, \emptyset)$ is a spine of $M$.

**Example 1.** Let $V$ be a solid torus with a meridian $m$. Choose a simple closed curve $\ell$ on $\partial V$ that intersects $m$ twice in the same direction. Note that $\ell$ decomposes $m$ into two arcs. Consider a theta-curve $\theta_V \subset \partial V$ consisting of $\ell$ and an arc (denote it by $\gamma$) of $m$. Then the manifold $(V, \theta_V)$ has a simple relative spine without interior true vertices. This spine is the union of $\partial V$, a Möbius strip inside $V$, and a part of meridional disc bounded by $\gamma$ (Figure 3a).

Note that among the three nontrivial simple closed curves contained in $\theta_V$, none is isotopic to the meridian $m$ of $V$. On the other hand, applying the flip to $\theta_V$ along $\gamma$, we get a theta-curve $\theta_m \subset \partial V$ containing $m$.

**Example 2.** Let $\theta, \theta'$ be two theta-curves on a torus $T$ such that $\theta'$ is obtained from $\theta$ by exactly one flip. Then the manifold

$$(T \times [0, 1], (\theta \times \{0\}) \cup (\theta' \times \{1\}))$$

has a simple relative spine $R$ with one interior true vertex (in Figure 3b the torus $T$ is represented as a square with the sides identified). Note that $R$ satisfies the following conditions:

1. for each $t \in [0, 1/2)$ a theta-curve $\theta_t$, where $R \cap (T \times \{t\}) = \theta_t \times \{t\}$, is isotopic to $\theta$;
2. for each $t \in (1/2, 1]$ the theta-curve $\theta_t$ is isotopic to $\theta'$;
3. $R \cap (T \times \{1/2\})$ is a wedge of two circles.

1.4. Assembling of manifolds with boundary patterns. Denote by $\mathcal{T}$ the class of all manifolds $(M, \Gamma)$ such that any component $T$ of $\partial M$ is a torus and $T \cap \Gamma$ is a theta-curve. Let $(M, \Gamma)$ and $(M', \Gamma')$ be two manifolds in $\mathcal{T}$ with nonempty
boundaries. Choose two tori $T \subseteq \partial M, T' \subseteq \partial M'$ and a homeomorphism $\varphi : T \to T'$ taking the theta-curve $\theta = T \cap \Gamma$ to the theta-curve $\theta' = T' \cap \Gamma'$. Then we can construct a new manifold $(W, \Xi) \in \mathcal{T}$, where $W = M \cup_{\varphi} M'$, and $\Xi = (\Gamma \setminus \theta) \cup (\Gamma' \setminus \theta')$. In this case we say that the manifold $(W, \Xi)$ is obtained assembling $(M, \Gamma)$ and $(M', \Gamma')$.

Note that if manifolds $(M, \Gamma)$ and $(M', \Gamma')$ have simple relative spines denoted $P$ and $P'$ respectively, with $v$ and $v'$ interior true vertices, then the manifold $(W, \Xi)$ has a simple relative spine $R$ with $v + v'$ interior true vertices. Indeed, $R$ can be obtained by gluing $P$ and $P'$ along $\varphi$ and removing the open disc in $P \cup_{\varphi} P'$ that is obtained by identifying $T \setminus \theta$ with $T' \setminus \theta'$.

To prove the main theorem of the paper we generalize the notion of the assembling by removing the restriction $\varphi(\theta) = \theta'$.

**Lemma.** Let $(M, \Gamma)$ and $(M', \Gamma')$ be two manifolds in $\mathcal{T}$ with nonempty boundaries that admit simple relative spines with $v$ and $v'$ interior true vertices respectively. Then for any homeomorphism $\varphi : T \to T'$ of a torus $T \subseteq \partial M$ onto a torus $T' \subseteq \partial M'$ there exists a simple relative spine of a manifold $(W, \Upsilon)$, where $W = M \cup_{\varphi} M'$ and $\Upsilon = (\Gamma \setminus \theta) \cup (\Gamma' \setminus \theta')$, with $v + v' + d(\varphi(\theta), \theta')$ interior true vertices.

**Proof.** First, by induction on the number $n = d(\varphi(\theta), \theta')$ we prove that there exists a simple relative spine of the manifold

$$(M'', \Gamma'') = (T' \times [0, 1], (\varphi(\theta) \times \{0\}) \cup (\theta' \times \{1\}))$$

with $n$ interior true vertices. If $n = 0$, i.e. the theta-curve $\varphi(\theta)$ is isotopic to the theta-curve $\theta'$, the desired spine is isotopic to the polyhedron $(\varphi(\theta) \times [0, 1]) \cup \partial M''$. Suppose that $n > 0$. As has already been alluded to in the beginning of the section 1.1, there exists a sequence $\{\theta_i\}_{i=0}^n$ of pairwise distinct theta-curves on the torus $T'$ such that $\theta_0 = \varphi(\theta)$, $\theta_n = \theta'$, and $\theta_i$ is obtained from $\theta_{i-1}$ by a flip, for $i = 1 \ldots n$. The induction assumption implies that the manifold

$$(T' \times [0, 1/2], (\theta_0 \times \{0\}) \cup (\theta_{n-1} \times \{1/2\}))$$

has a simple relative spine with $n - 1$ interior true vertices. Furthermore, the simple relative spine of the manifold

$$(T' \times [1/2, 1], (\theta_{n-1} \times \{1/2\}) \cup (\theta_n \times \{1\}))$$

with one interior true vertex is described in the Example 2. Then the desired spine of the manifold $(M'', \Gamma'')$ is obtained by assembling the manifolds (1) and (2) along the identity map on $T' \times \{1/2\}$.

Now, note that the consecutive assemblings of the manifolds $(M, \Gamma)$, $(M'', \Gamma'')$ and $(M', \Gamma')$ along natural homeomorphisms that take each point $x \in T$ to the point $(\varphi(x), 0) \in T' \times \{0\}$, and each point $(y, 1) \in T' \times \{1\}$ to the point $y \in T'$, yield the manifold $(W, \Upsilon)$ and its simple relative spine with $v + v' + d(\varphi(\theta), \theta')$ interior true vertices.

\[\square\]

2. Relative spines of the figure eight knot complement

In this section we construct some simple relative spines of the figure eight knot complement $E(4_1)$. Let us fix a canonical coordinate system on the boundary torus $\partial E(4_1)$ consisting of oriented closed curves $\mu, \lambda$ such that the meridian $\mu$ generates
Figure 4. A minimal spine of the complement of the figure eight knot

$H_1(E(4_1); \mathbb{Z})$ and the longitude $\lambda$ bounds a surface in $E(4_1)$. This system determines the map $\Psi_{\mu,\lambda}$ from $\Theta(T)$ to the set of triangles of the Farey triangulation. Denote by $\triangle^{(i)}$ the triangle of $\mathbb{F}$ with the vertices at $i$, $i+1$, and $\infty$.

**Proposition.** For any $i \in \{0, 1, 2, 3\}$ there exists a theta-curve $\theta^{(i)}$ on the torus $\partial E(4_1)$ such that the manifold $(E(4_1), \theta^{(i)})$ has a simple relative spine with 10 interior true vertices and $\Psi_{\mu,\lambda}(\theta^{(i)}) = \triangle^{(i)}$.

**Proof.** Step 1. Let $P$ be a special spine of an arbitrary compact orientable 3-manifold $M$ whose boundary is a torus, and let $\theta$ be a theta-curve on $\partial M$. We begin the proof by describing a method for constructing a simple relative spine $R(P, \theta)$ of the manifold $M$.

By Theorem 1.1.7 [7], $M$ can be identified with the mapping cylinder of a local embedding $f : \partial M \to P$. Denote by $f_{\theta} : \theta \to P$ the restriction to $\theta$ of the map $f$. Then the union $R(P, \theta)$ of the mapping cylinder of $f_{\theta}$ and of $\partial M$ is a relative spine of $M$, since $\partial M \subset R(P, \theta)$, $\partial M \cap Cl(R(P, \theta) \setminus \partial M) = \emptyset$, and $M \setminus R(P, \theta)$ is homeomorphic to the direct product of the open disc $\partial M \setminus \theta$ with an interval. In general, $R(P, \theta)$ just constructed is not necessarily a simple polyhedron. This can be dealt with by introducing the notion of general position. We say that a theta-curve $\theta \subset \partial M$ is in general position with respect to the map $f$, if the image $f(\theta)$ satisfies the following conditions.

1. $f(\theta)$ contains no true vertices of $P$.
2. For any intersection point $x$ of $f(\theta)$ with the triple lines of $P$ there exists a neighborhood $U(x) \subset P$ such that the intersection $U(x) \cap f(\theta)$ is an arc meeting the set of the triple lines of $P$ transversally exactly at $x$.
3. For any intersection point $x$ of the set $f(\theta)$ with the 2-components of $P$ its inverse image $f_{\theta}^{-1}(x)$ consists of at most two points of $\theta$. Moreover, if $f_{\theta}^{-1}(x)$ consists of exactly two points, then there exists a neighborhood $U(x) \subset P$ such that the inverse image $f_{\theta}^{-1}(U(x) \cap f(\theta))$ of the intersection $U(x) \cap f(\theta)$ is the disjoint union of two arcs $\gamma_1$, $\gamma_2$ of $\theta$, and the images $f(\gamma_1)$, $f(\gamma_2)$ intersect each other transversally at exactly one point $x$. Such a point $x$ is called the self-intersection point of the image $f(\theta)$ of $\theta$.

Obviously, if a theta-curve $\theta$ is in general position with respect to the map $f$, then the relative spine $R(P, \theta)$ of the manifold $M$ is simple.

Step 2. We consider now the minimal special spine $P$ of the manifold $M = E(4_1)$ shown in Figure 4 (see [7] 2.4.2). To construct the theta-curves $\theta^{(i)}$, $i \in \{0, 1, 2, 3\}$, we need to describe certain cell decompositions of the torus $T = \partial M$ and of its universal covering $\tilde{T}$. The local embedding $f : T \to P$ determines a cell decomposition of $T$ as follows.
1. The inverse image $f^{-1}(C)$ of every open $k$-dimensional cell $C$ of $P$ consists of two open 2-cells if $k = 2$, three open arcs if $k = 1$, and four points if $k = 0$.

2. The restriction of $f$ to each of these cells is a homeomorphism onto the corresponding cell of $P$.

Construct the universal covering $\tilde{T}$ of $T$. It can be presented as a plane decomposed into hexagons, see Fig. 5a. The group of covering translations is isomorphic to the group $\pi_1(T) = H_1(T; \mathbb{Z})$. We choose a basis $\tilde{\mu}, \tilde{\lambda}$ as shown in Fig. 5a. It is easy to see that the corresponding elements of $\pi_1(T)$ (which can be also viewed as oriented loops) form the canonical coordinate system $(\mu, \lambda)$ on $T$. If we factor this covering by the translations $\tilde{\mu}, \tilde{\lambda}$, we recover $T$. If we additionally identify the hexagons marked by the letter $A$ with respect to the composition of the symmetry in the dotted diagonal of the hexagon and the translation by $-\tilde{\mu} + \tilde{\lambda}/2$, and do the same for the hexagons marked by the letter $B$, we obtain $P$. The torus $T$ is shown in Fig. 5b as a polygon $D$ composed of four hexagons. Each side of $D$ is identified with some other one via the translation along one of the three vectors $\tilde{\mu}, -2\tilde{\mu} + \tilde{\lambda}$, and $-\tilde{\mu} + \tilde{\lambda}$. The spine $P$ can be presented as the union of two hexagons, see Fig. 6 (right). The edges of the hexagons are decorated with four different patterns. To recover $P$, one should identify the edges having the same pattern.

Step 3. Now for each $i \in \{0, 1, 2, 3\}$ we exhibit a theta-curve $\theta^{(i)} \subset \partial M$ such that the simple relative spine $R(P, \theta^{(i)})$ of $M$ has 10 interior true vertices and $\Psi_{\mu, \lambda}(\theta^{(i)}) = \Delta^{(i)}$.

Consider the wedge of the three arcs on $\tilde{T}$, see Fig. 6 (left). The projections of the arcs onto $T$ yield a theta-curve that we denoted by $\theta^{(0)}$. It can be checked directly that $\theta^{(0)}$ is in general position with respect to the map $f$, and $\Psi_{\mu, \lambda}(\theta^{(0)}) = \Delta^{(0)}$. It remains to note that the set of the interior true vertices of $R(P, \theta^{(0)})$ consists of (a) the two true vertices of the special polyhedron $P$, (b) the images under $f$ of the

![Figure 5. Cell decompositions of $\tilde{T}$ (left) and $T$ (right)](image-url)
The theta-curves $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ satisfying the conclusion of the Proposition are shown in Fig. 7, 8, 9. We point out that among the 10 interior true vertices of $R(P, \theta^{(3)})$ there are 6 intersection points of the set $f(\theta^{(3)})$ with the triple lines of $P$, see Fig. 9 (left), while there are no self-intersection points of the image $f(\theta^{(3)})$ of $\theta^{(3)}$, see Fig. 9 (right). □
3. Proof of the main theorem

Let $p \geq 0$ and $q \geq 1$ be two relatively prime integers. To prove the inequality $c(4_1(p/q)) \leq \omega(p/q)$ it suffices to construct a simple spine of the manifold $4_1(p/q)$ with $\omega(p/q)$ true vertices.

Thurston [16] proved that the manifold $4_1(p/q)$ is hyperbolic except for $p/q \in \{0, 1, 2, 3, 4, \infty\}$. The case $p/q = \infty$ does not satisfy the assumptions of the Theorem. In each of the five remaining cases the non-hyperbolic manifold $4_1(p/q)$ has complexity 7 and $\omega(p/q) = 7$.

Let us construct a simple spine of the hyperbolic manifold $4_1(p/q)$. Recall that the meridian $m$ and the theta-curve $\theta_m$ on the boundary of $(V, \theta_V)$ were fixed in Example 1. Let $(\mu, \lambda)$ be the canonical coordinate system on the boundary torus $\partial E(4_1)$ of the figure eight knot complement $E(4_1)$. Among all homeomorphisms
Recall that for each $i$ the Farey triangulation with the vertices at $\theta_i$ has a common edge with $\triangle_{\theta_i}$ has a common edge with $\theta_i$ and $\theta_i^{(0)}$ be as small as possible. For convenience denote by $z$ the number $\min\{[p/q], 3\}$. By the Proposition, the manifold $(E(\Delta_{\phi}), \partial) \) has a simple relative spine with 10 interior true vertices. Since $E(\Delta_{\phi}) = V \cup_{\phi} E(\Delta_{\phi})$, it follows from Lemma 2 that the manifold $(4_1(p/q), \emptyset)$ has a simple relative spine $Q_{p/q}$ with 10 + $d(\phi_{\theta_i}, \theta_i^{(0)})$ interior true vertices. Moreover, $Q_{p/q}$ is a spine of $4_1(p/q)$, since $\partial 4_1(p/q) = \emptyset$.

Now let us prove that $d(\phi_{\theta_i}, \theta_i^{(0)}) = -2 + \max\{[p/q] - 3, 0\} + S(\phi_{\theta_i}, p/q, q)$. Recall that for each $i \in \{0, 1, 2, 3\}$ the map $\Psi_{\mu, \lambda}$ takes $\phi_{\theta_i}^{(0)}$ to the triangle $\Delta_{\theta_i}$ of the Farey triangulation with the vertices at $i, i + 1,$ and $\infty$. Denote by $\Delta_{\theta_i}$ and $\triangle_{\theta_i}$ the triangles $\Psi_{\mu, \lambda}(\phi_{\theta_i})$ and $\Psi_{\mu, \lambda}(\theta_i)$, respectively. Since the distance between theta-curves on $\partial E(\Delta_{\phi})$ is equal to the distance between the corresponding triangles of $F$, it is sufficient to find $d(\Delta_{\theta_i}, \theta_i^{(0)})$.

The choice of $\phi$ guarantees us that $\Delta_{\theta_i}$ is the closest triangle to $\theta_i^{(0)}$ among all the triangles with a vertex at $p/q$. This implies (see 4 Proposition 4.4) that $d(\Delta_{\theta_i}, \theta_i^{(0)}) = S(p/q) - 1$. Since the theta-curve $\theta_i$ is obtained from $\theta_i$ by exactly one flip and each $\theta_i$ does not contain the meridian $m$, the triangle $\Delta_{\theta_i}$ has a common edge with $\theta_i$ and $p/q$ is not a vertex of $\Delta_{\theta_i}$. Hence, $d(\Delta_{\theta_i}, \theta_i^{(0)}) = S(p/q) - 2$. Analyzing the Farey triangulation, we can notice that $d(\Delta_{\theta_i}, \theta_i^{(0)}) = d(\Delta_{\theta_i}, \theta_i^{(0)}) - d(\theta_i^{(0)}, \theta_i^{(0)})$. Taking into account that $d(\theta_i^{(0)}, \theta_i^{(0)}) = z$, $S(p/q)$ is obtained from $\Delta_{\theta_i}$ by a sequence of moves along boundary curves of length 4 (similar arguments can be found in 11 page 81). The spine $Q_{p/q}$ has the same number of true vertices but possesses a boundary curve of length 4, hence it can be simplified. The result is a new spine of $4_1(p/q)$ with $\omega(\phi_{\theta_i})$ true vertices.

To conclude the proof of the theorem, it remains to note that the table 4 contains 46 hyperbolic manifolds of the type $4_1(p/q)$ satisfying $\omega(\phi_{\theta_i}) \leq 12$. For each of them our upper bound is sharp.
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