Ricci flow and curvature on the variety of flags on the two dimensional projective space over the complexes, quaternions and the octonions.

Man-Wai Cheung Nolan R. Wallach

May 1, 2014

Abstract

For homogeneous metrics on the spaces of the title it is shown that the Ricci flow can move a metric of strictly positive sectional curvature to one with some negative sectional curvature and one of positive definite Ricci tensor to one with indefinite signature.

1 Introduction

In this note, we will show that a metric of the homogeneous Riemannian manifold $SU(3)/T^2$ with strictly positive curvature is deformed to a metric with some negative sectional curvature by the Ricci flow. This result has been announced by Böhm and Wilking [BW] in which they assert that this can be proved using a method similar to the one they used to show that a metric of positive sectional curvature on $Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$ can be flowed to one with some negative Ricci curvature. We use a different and simpler approach to prove this result. We will show that if we initiate the Ricci flow at a metric on the boundary of the metrics with positive sectional then the derivative of the flow of sectional curvature at a plane of zero curvature is negative for all of the examples in [W]. We also show that for all the examples in [W] (dimension 6,12,24) the Ricci flow can cause the Ricci tensor to go from positive definite to signature $(d,2d)$ ($d = 2, 4, 8$). In the concluding remarks to the paper we give a simple variant of Valiev’s necessary and sufficient condition for a homogeneous metric on one of the spaces to have strictly positive sectional curvature. We would like to thank Lei Ni for suggesting that we look at the curvature transition of the 6 dimensional space in [W] under the Ricci flow and for his patience.
as we were trying to understand the subtleties of the argument in [BW] for the twelve dimensional example.

2 Setup

In this section, we will set up the notation for the main calculations and establish the Ricci flow equations in terms of the metric parameters. Set $G = SU(3), Sp(3)$ or compact $F_4$ and let $K$ be respectively a maximal torus, $T^2$, of $SU(3), Sp(1) \times Sp(1) \times Sp(1)$ in $Sp(3)$ or $Spin(8)$ in compact $F_4$. Let $g$ be the Lie algebra of $G$, $\mathfrak{k}$ be the Lie algebra of a $K$. Let $\mathfrak{p}$ be the $Ad(K)$–invariant complement to $\mathfrak{k}$ in $g$. Then $\mathfrak{p}$ can be decomposed into a direct sum of three irreducible inequivalent $K$-invariant subspaces $\mathfrak{p} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{v}_3$.

Consider the $Ad(G)$-invariant inner product $\langle X, Y \rangle_0 = -1/2 \text{Re} \text{tr}(X,Y)$ on $g$ for the first two examples and in the case of $G = F_4$ the unique $Ad(G)$-invariant inner product that agrees with our choice for the imbedded $Sp(3)$ that is compatible with the decompositions. The dimensions of the real vector spaces $\mathfrak{v}_i$ are the same in each case and are respectively $d = 2, 4, 8$.

In each case we may identify the spaces $\mathfrak{v}_i$ with the fields over $\mathbb{R}$: $\mathbb{C}$, $\mathbb{H}$ (the quaternions), $\mathbb{O}$ (the octonions) such that the inner product is $\text{Re}(z\bar{w})$. If $z \in \mathfrak{v}_1, w \in \mathfrak{v}_2$ then $[z, w] \in \mathfrak{v}_3$ and under our identification corresponds to $\overline{z}w$ in $\mathfrak{v}_3$. Similarly with sign changes as in the cross-product $[\mathfrak{v}_i, \mathfrak{v}_j] \in \mathfrak{v}_k$ if $i, j, k$ are distinct. Schur’s Lemma implies that any $K$-invariant inner product on $\mathfrak{p}$ is given by

$$x_1 \langle \ldots, \ldots \rangle_0 |_{\mathfrak{v}_1} + x_2 \langle \ldots, \ldots \rangle_0 |_{\mathfrak{v}_2} + x_3 \langle \ldots, \ldots \rangle_0 |_{\mathfrak{v}_3}$$  \hspace{1cm} (1)

where $x_1, x_2, x_3$ are positive constants. Let $g$ be the Riemannian structure on $M$ corresponding to $(x_1, x_2, x_3)$. We will write $g \leftrightarrow (x_1, x_2, x_3)$. In [AW] it was proved that if $x_1 = x_2 = 1$ then for all examples above the sectional curvature is strictly positive if $0 < x_3 < 1$ or $1 < x_3 < 4/3$. We note

**Lemma 2.1.** If $x_1 = x_2$ then the sectional curvature is is strictly positive if $0 < x_3 < 1$ or $1 < x_3 < 4/3$ and there is some strictly negative curvature if $x_3 > 4/3$.

**Proof.** We need only prove the assertion about negative curvature. We may assume that $x_1 = x_2 = 1$. We consider the embedding of $SU(3)$ into $G$ so that $T^2$ imbeds in $K$ and the the imbedding of the complement to $\text{Lie}(T^2)$ in $\text{Lie}(SU(3), \mathfrak{q}$, imbeds in $\mathfrak{p}$ as $\mathbb{C}$ imbeds in $\mathbb{H}$ or $\mathbb{O}$. We note that if $u, v \in \mathfrak{q}$ then the formula in Lemma 7.3 of [W] reduces the calculation to the case

2
We compute a specific curvature

\[
    u = \begin{bmatrix}
        0 & u_1 & u_2 \\
        -u_1 & 0 & u_3 \\
        -u_2 & -u_3 & 0
    \end{bmatrix},
    v = \begin{bmatrix}
        0 & v_1 & v_2 \\
        -v_1 & 0 & v_3 \\
        -v_2 & -v_3 & 0
    \end{bmatrix}
\]

with \(u_1 = 1, v_1 = -1, u_2 = v_2 = 1/\sqrt{1+x^2}, u_3 = v_3 = x/\sqrt{1+x^2}\) with \(x \in \mathbb{R}\). Then with \(x_1 = x_2 = 1, x_3 = 1 + t\)

\[
    g(R(u,v)v, u) = \frac{2}{1+x^2}(1-3t+(1+t)^2x^2).
\]

So if \(t = \frac{1}{3} + s\) with \(s > 0\) and

\[
    0 < x < \sqrt{\frac{3s}{(1 + \left(\frac{1}{3} + 3s\right)^2)}}
\]

then the curvature corresponding to the two plane \(\text{span}\_\mathbb{R}(u, v)\) is negative. This shows that the there is negative Gaussian curvature for any \(t > \frac{1}{3}\) so my condition is necessary and sufficient.

We also note that Schur’s lemma implies that the Ricci curvature of \(g\), denoted \(\text{Ric}(g)\), is given by

\[
    \text{Ric}(g) = x_1r_1 (\ldots, \ldots) \_0 |v_1 + x_2r_2 (\ldots, \ldots) \_0 |v_2 + x_3r_3 (\ldots, \ldots) \_0 |v_3. \quad (2)
\]

Using the (first) Lemma 7.1 in [W] it is easily seen that \(r_i\) is given by

\[
    r_i = \frac{dx_i^2 - dx_j^2 - dx_k^2 + (10d - 8)x_jx_k}{2x_1x_2x_3} \quad (3)
\]

where \(\{i, j, k\} = \{1, 2, 3\}\).

We note that the Ricci flow preserves left invariant metrics on the spaces \(G/K\) and hence can be considered to be the ordinary differential equation

\[
    \frac{dx_i}{dt} = -2r_ix_i \quad (4)
\]

In particular we see that the set of metrics with \(x_i = x_j\) for some \(i, j\) is preserved by the Ricci flow. Also, permutation of the indices of the \(x_i\) preserves the solutions.
3 The sectional curvature

In this section we will prove that the Ricci flow deforms some metric $g$ with strictly positive curvature into metric with some negative sectional curvature. To start with, we investigate the metric $g_0 \leftrightarrow (1, 1, \frac{4}{3})$ which, in light of Lemma 2.1 is of nonnegative sectional curvature and $g \leftrightarrow (1, 1, u)$, $u > \frac{4}{3}$ has some strictly negative curvature. Using the symmetric invariance of the system (2.4) we note that if we start with $g_0 \leftrightarrow (1, 1, \frac{4}{3})$ under the Ricci flow the metric $g_t \leftrightarrow (x_1(t), x_2(t), x_3(t))$ satisfies $x_1(t) = x_2(t)$. Our strategy to prove that some curvature turns negative is to show that

$$\frac{d}{dt} t = 0 x_3(t) > 0.$$  \hfill (5)

This will say that there exists $\varepsilon > 0$ such that $\frac{1}{x_3(-\varepsilon)} g_{-\varepsilon} \leftrightarrow (1, 1, u)$ with $1 < u < \frac{4}{3}$ and $\frac{1}{x_3(\varepsilon)} g_{\varepsilon} \leftrightarrow (1, 1, v)$ with $v > \frac{4}{3}$, So Lemma 2.1 implies our assertion. We now carry out the calculation.

$$\frac{d}{dt} x_3(t) = \frac{x_3(t)x_1(t) - x_3(t)x_1'(t)}{x_1(t)^2}$$

so (2.4) implies that

$$\frac{d}{dt} x_3(t) = -2 \frac{x_3(t)}{x_1(t)} (r_3 - r_1).$$  \hfill (6)

In the three cases ($d = 2, 4, 8$) we have $-2(r_3 - r_1) = -2 + \frac{4d}{3} > 0$.

We have proved

**Theorem 3.1.** On the three examples of [W] the Ricci flow deforms certain positively curved metrics into metrics with mixed sectional curvatures.

We note that this result for the 12 dimensional example follows from [BW].

4 Change in Ricci curvature.

We first indicate why the method of the last section doesn’t work for Ricci curvature. We consider the case when $x_1 = x_2$ and calculate

$$2(r_1 - r_3) = \frac{-2(1 - \frac{x_3}{x_1})(4d - 4) - d\frac{x_3}{x_1}}{x_3}.$$  \hfill (7)
We therefore see in (light of (3.2)) that if \(0 < \frac{x_3(t)}{x_1(t)} < 1\) then \(\frac{d x_3(t)}{dt} x_1(t) < 0\).

So if we started the Ricci flow with a (positive curvature) initial condition

\[ x_1 = x_2, \frac{x_4}{x_1} < 1 \text{ then } \frac{x_4}{x_1} \text{ is decreasing}. \]

If initially \(1 < \frac{x_4}{x_1} < \frac{4(d-1)}{d}\) then under the flow we would have \(\frac{d x_3(t)}{dt} x_1(t) > 0\). If \(\frac{4(d-1)}{d} < \frac{x_4}{x_1}\) then \(\frac{d x_3(t)}{dt} x_1(t) < 0\).

Thus \(\frac{x_4}{x_1} = 1\) is a repelling (i.e unstable fixed point) and \(\frac{x_4}{x_1} = \frac{4(d-1)}{d}\) is an attractor. The upshot is that if the initial condition is \(x_1 = x_2\) and the sectional curvature is positive then \(\frac{x_4}{x_1} < \frac{4(d-1)}{d}\) for the entire Ricci flow. On the other hand the Ricci tensor for \(x_1 = x_2\) is given by

\[
\frac{10d - 8 - \frac{d x_4}{x_1}}{2} (\langle \ldots, \ldots \rangle_0 | v_1 + \langle \ldots, \ldots \rangle_0 | v_2) + \frac{(8d - 8) - d \left(\frac{x_4}{x_1}\right)^2}{2} \langle \ldots, \ldots \rangle_0 | v_3.
\]

Thus if we begin the Ricci flow with a metric of positive curvature and \(x_1 = x_2\) then \(\frac{x_4}{x_1} < \frac{4(d-1)}{d}\) which implies that

\[
\frac{10d - 8 - \frac{d x_4}{x_1}}{2} > \frac{3d - 2}{d} > 0
\]

and

\[
\frac{(8d - 8) - d \left(\frac{x_4}{x_1}\right)^2}{2} > \frac{4(d-1)(3d-2)}{d} > 0.
\]

This indicates how delicate the methods of [BW] must be.

We observe that if the initial condition satisfies \(x_2 > x_1 > x_3\) then the flow will stay among the homogeneous metrics satisfying this condition.

Since Ricci curvature is invariant under constant scalar multiples of the metric we may assume that our initial metric corresponds to \(x_1 = 1, x_2 = 1 + r, x_3 = s\) and \(s < 1\) (notice that Lemma 2.1 implies that if \(s < 1\) is fixed and \(r\) is sufficiently small then the metric has positive sectional curvature).

We also note that (2.3) implies that if \(x_1 = 1, x_2 = 1 + r, x_3 = s\) then the coefficients of the Ricci curvature are given by

\[
r_1 x_1 = \frac{-2rd - dr^2 + (10d - 8)s + (10d - 8)rs - ds^2}{2(1 + r)s},
\]

\[
r_2 x_2 = \frac{dr + dr^2 + (10d - 8)s - ds^2}{2s}
\]

and

\[
r_3 x_3 = \frac{(8d - 8) + (8d - 8)r - dr^2 + ds^2}{2(1 + r)}.
\]
Thus if $s < 1$ and $0 < r < 1$ then $r_2x_2$ and $r_3x_3$ are strictly positive. If we solve the quadratic equation for $r_1x_1 = 0$ then we have for the cases $d = 2, 4, 8$ respectively

$$r = \sqrt{1 + 8s^2} - (1 - 3s),$$

$$r = \sqrt{1 + 15s^2} - (1 - 4s)$$

and

$$r = \sqrt{1 + \frac{77}{4} s^2} - (1 - \frac{9}{2} s).$$

We note that if we substitute these values of $r$ into the above coefficients of the Ricci tensor then we find that if $s < 1$, $r_2x_2 > 0$ and $r_3x_3 > 0$. So if we show that if we take our initial condition at such a value $\frac{dx}{dt} < 0$ we will have shown that the Ricci flow transitions from positive definite to signature $(d, 2d)$ ($d$ negatives). We therefore study

$$-2 \sum r_i x_i \frac{\partial r_1}{\partial x_i}$$

at these values we find that if $d = 2$ then this expression is negative for

$$0 < s < 1 - \frac{\sqrt{5}}{8} (0.20943058...$$

for $d = 4$ the expression is negative for

$$0 < s < \frac{30 + 5\sqrt{21} - 3\sqrt{5(21 + 4\sqrt{21})}}{30} (0.361437...$$

and for $d = 8$ the expression is negative for

$$0 < s < \frac{693 + 11\sqrt{2737} - 7\sqrt{22(511 + 9\sqrt{2737})}}{616} (0.389089...)$$. This proves

**Theorem 4.1.** *For all the examples in [W] (i.e. the manifold of flags in the two dimensional projective space over $\mathbb{C}, \mathbb{H}$ or $\mathbb{O}$) the Ricci flow of a metric with positive definite Ricci tensor can flow to one with signature $(d, 2d)$.*

### 5 Concluding remarks

In this section we will give a necessary and sufficient condition that the metric corresponding to $(x_1, x_2, x_3)$ have positive curvature for the three types of examples that we have been studying. We compare this condition to what is necessary for positive Ricci curvature and one, thereby, gets a better understanding of the result in [BW].
We first observe that the permutation action of the symmetric group permutes the \((x_1, x_2, x_3)\) that correspond to strictly positive curvature among themselves. We have also completely described the \((x_1, x_2, x_3)\) with some pair \(x_i = x_j\) with \(i \neq j\). Thus we are left with the cases where

\[
\prod_{i<j} (x_i - x_j) \neq 0.
\]

Using the action of the symmetric group just described we may assume that \(x_2 > x_1 > x_3 > 0\) (we chose this order to be consistent with the results of [BW]). Since a multiplication by a positive scalar doesn’t change the sign of curvature we may assume that \(x_1 = 1, x_2 = 1 + r\) and \(x_3 = s\) with \(r > 0\) and \(s < 1\). The following result follows directly from Theorem 3 a) in [V].

**Proposition 5.1.** With the notation above a necessary and sufficient condition that the sectional curvature be positive is

\[
r < \frac{s^2 - 2 + 2\sqrt{1 - s + s^2}}{3}.
\]

**Remark.** We note that if \(0 < s < 1\) then

\[
\frac{s^2}{4} < \frac{s - 2 + 2\sqrt{1 - s + s^2}}{3} < \frac{s^2}{3}
\]

and the expression estimated is monotone increasing. This can be seen in the following graph:

The axes are horizontal, \(s\), and vertical, \(r\), the set points under each curve represent the \(r\) values for each \(s\) value such that \((1, 1 + r, s)\) with \(r > 0\) and \(0 < s < 1\) respectively satisfies the necessary condition above (lowest curve, blue), the necessary and sufficient condition for positive Ricci curvature for the 6 (second curve, red), 12 (third curve, yellow) and 24 (top curve, green).
dimensional examples. We note that to get the full set of metrics with strictly positive curvature satisfying the inequalities \( x_3 \leq x_1 \leq x_2 \) one must allow the points \((s,0), 0 < s < 1\) and \((1,r)\) with \(0 < r < \frac{1}{3}\) (that is add the the original set given in [AW]).

In the argument in [BW] they start their Ricci flow at a metric corresponding to \((x_1,x_2,x_3)\) such that \((x_1,x_2,x_3)/x_1 = (1,1+r,s)\) (in our notation) and \(r > 0, 0 < s < 1\) (the reason for our strange condition). In light of the above \((r,s)\) must be below the blue curve in the graph above. In the Ricci flow (normalized or not) the set \(x_2 > x_1 > x_3\) is preserved. If \((x_1(t),x_2(t),x_3(t))\) is a point in the flow and \((x_1(t),x_2(t),x_3(t))/x_1(t) = (1,1+r(t),s(t))\) then their the curve \((s(t),r(t))\) starts at \(t = 0\) under the lowest (blue) curve (so as to have positive curvature) and it must eventually cross the yellow (second highest) curve in order to have some negative Ricci curvature.

References

[AW] Simon Allof, Nolan R. Wallach, An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures, Bull. Amer. Math. Soc., 81 (1975), 93-97.

[BW] C. Böhm, B. Wilking, Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature, Geom. funct. anal. 17 (2007), 665-681.

[V] F.M. Valiev, Precise estimates for the sectional curvatures of homogeneous Riemannian metrics on Wallach spaces, Siberian Math. Journal 20 (1979), 176-187.

[W] Nolan R. Wallach, Compact homogeneous Riemannian manifolds with strictly positive curvature, Anna. of Math. (2) 96 (1972), 277-295.