FACTORIZATION OF SPANNING TREES
ON FEYNMAN GRAPHS

(revised version)

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Abstract

In order to use the Gaussian representation for propagators in Feynman amplitudes, a representation which is useful to relate string theory and field theory, one has to prove first that each $\alpha$-parameter (where $\alpha$ is the parameter associated to each propagator in the $\alpha$-representation of the Feynman amplitudes) can be replaced by a constant instead of being integrated over and second, prove that this constant can be taken equal for all propagators of a given graph. The first proposition has been proven in one recent letter when the number of propagators is infinite. Here we prove the second one. In order to achieve this, we demonstrate that the sum over the weighted spanning trees of a Feynman graph $G$ can be factorized for disjoint parts of $G$. The same can also be done for cuts on $G$, resulting in a rigorous derivation of the Gaussian representation for super-renormalizable scalar field theories. As a by-product spanning trees on Feynman graphs can be used to define a discretized functional space.

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1. Introduction

In the study of the relationship between field theories and string theories, the $\alpha$-representation for Feynman graphs is a very useful tool [1-4]. In this representation one $\alpha$-parameter is assigned to every propagator and the only integrations to be made are over these parameters, integration over the momenta circulating in the graph having already been made. This is therefore a very economical representation and it has quite a while ago been used to study the renormalization of field theories in the most accurate way [5]. However, it has another nice property; writing a propagator of a scalar field theory as

$$[(P_i - P_j)^2]^{-1} = \int_0^\infty d\alpha \exp[-\alpha(P_i - P_j)^2]$$

(1)

we see that the $\alpha$-parameter is a sliding scale for Gaussians. If we fix $\alpha$ to some constant $\bar{\alpha}$ we have what we call a Gaussian representation (usually called a “Gaussian approximation”) for the propagator. Now, discretized surface theories can be constructed using precisely the Gaussian representation in planar graphs amplitudes and taking $\bar{\alpha}$ to be the same quantity for all propagators of a graph. $\bar{\alpha}$ is then interpreted as proportional to the inverse of the slope of the Regge trajectories of the equivalent string theory [6]. In a recent letter [7] we proved that, indeed, once an integration over an overall scale was made, all the $\alpha_i$’s ($\alpha_i$ being the $\alpha$-parameter of the propagator i) could be replaced by their mean-values $\bar{\alpha}_i$ which, in turn, were demonstrated to be $O(1/I)$, $I$ being the total number of propagators of any one-particle, one vertex irreducible Feynman graph $G$, planar or non-planar, $I \rightarrow \infty$. This was done for any number of Euclidean dimensions where the theory was super-renormalizable. (When the theory is renormalizable we have to make the weak assumption that a logarithm coming from the renormalization of some sub-divergence is provoking only a shift in the coupling constant when the integration is made over the momenta of the legs of the sub-diverging part). However, in any case, there is a second step in the derivation which consists in proving that all the $\bar{\alpha}_i$’s can, in turn, be replaced by a single value $\bar{\alpha}$ for a given graph $G$. This could be demonstrated [7] provided
that the sum over spanning trees of $G$ can be considered as a functional integral, i.e. that the sum could be factorized on disjoint domains of $G$. It is our purpose here to present a rigorous derivation of that statement and thereby ending the proof about the validity of the Gaussian representation (using an unique $\bar{\alpha}$).

In section 2 we present the basics of the $\bar{\alpha}$-representation for Euclidean scalar field theories. We give the general expression for the Feynman graph amplitude $F_G$ of a graph $G$ with $I$ internal lines, $L$ loops in terms of an integral over $I$ $\alpha$-parameters. This integral can be evaluated using the mean-value theorem which states that if a function $f$ is continuous in its arguments $\{\alpha\}$, then

$$\int_\omega f = V \cdot f(\{\bar{\alpha}\}) ,$$

where $V$ is the volume of the connected domain $\omega$ over which the integration extends and $\{\bar{\alpha}\}$ a set of values of the $\alpha$-parameters defining some point in $\omega$. We expect the mean-values $\bar{\alpha}_i$ of the $I$ $\alpha$-parameters to be a priori different. Then, the central result of this article is to demonstrate that all $\bar{\alpha}_i$’s can be replaced by one single value $\bar{\alpha}$ without changing the value of Feynman amplitude (expressed as a function of the $\bar{\alpha}_i$’s via the mean-value theorem). This will amount to showing that the ratio of polynomials $Q_G(P_v, \{\bar{\alpha}_i\})$ defined in section 2 and appearing in the expression of $F_G$ is indeed insensitive to that replacement. Isolating one particular $\bar{\alpha}_i$, $Q_G$ can be set in the form

$$Q_G(P, \{\bar{\alpha}_i\}) = \left(\frac{\bar{\alpha}_i}{\bar{\beta}_i}\right)\left(\frac{\bar{\delta}_i/\bar{\epsilon}_i + \bar{\alpha}_i}{\bar{\alpha}_i/\bar{\beta}_i + \bar{\alpha}_i}\right)$$

where $\bar{\alpha}_i/\bar{\beta}_i$ and $\bar{\delta}_i/\bar{\epsilon}_i$ are ratios of homogeneous polynomials in the mean-values of all $\alpha$-parameters except $\alpha_i$. The proof of the independence on the shift $\bar{\alpha}_i \rightarrow \bar{\alpha}$ will then translate into a proof of the equality of the ratios $\bar{\alpha}_i/\bar{\beta}_i$ and $\bar{\delta}_i/\bar{\epsilon}_i$. Then, an important property of $\bar{\alpha}_i$, $\bar{\beta}_i$, $\bar{\delta}_i$ and $\bar{\epsilon}_i$ is that they can be expressed as sums over products of $\bar{\alpha}_i$’s, $l$ indexing propagators belonging to spanning trees of $G$ (a spanning tree of $G$ is a tree incident with all vertices of $G$). The ratio $\bar{\alpha}_i/\bar{\beta}_i$ involves a sum over trees containing $i$, i.e.
$a_i$, and a sum over trees not containing $i$, i.e. $b_i$. The ratio $\bar{a}_i/\bar{e}_i$ is the ratio of a sum over trees containing $i$ and cut at some other propagator over a sum over trees not containing $i$ and cut at some other propagator (cutting means that the propagator is deleted from the tree, yielding a cut). Then, proving that $\bar{a}_i/b_i = \bar{d}_i/\bar{e}_i$ amounts to proving that the effect of cutting of a propagator can be factorized in the sum over trees. This will be true if the structure of trees is such that their structure far from $i$ is independent from their structure close to $i$. The next sections will be devoted to the proof that the sums over spanning trees can indeed be factorized over domains far apart on $G$.

In section 3 we restrict ourselves to the case of self-avoiding paths on $G$ instead of trees. This is because, aside from simplicity in a first approach, there is always a self-avoiding path linking two vertices of $G$ on any spanning tree of $G$. Thus, spanning trees can be built out of self-avoiding paths. We first give the general strategy for the proof of the equality $\bar{a}_i/b_i = \bar{d}_i/\bar{e}_i$. Then, the following ratio

$$R_i(s_j) = \frac{\sum P(i, s_j)}{\sum P(\bar{i}, s_j)}$$

is proven to be independent of $s_j$ if $s_j$ is a vertex infinitely far from $v_i$ on $G$, $v_i$ being a vertex incident with the propagator $i$. (The notion of distance on $G$ will be discussed later on. By infinitely far we mean that an infinite number of propagators separate $v_i$ from $s_j$). $P(i, s_j)$ is a self-avoiding path linking $v_i$ to $s_j$ going through $i$. $P(\bar{i}, s_j)$ is also a self-avoiding path linking $v_i$ to $s_j$ but not going through $i$. The proof uses the evaluation of $R_i(s_j)$ as a mean-value in a volume $V_j$. In fact $R_i(s_{j+1})$ will turn out to be the average of $R_i(s_j)$. Letting $j \to \infty$, the averaging process, repeated an infinite number of times, removes the dependence on $s_j$ of $R_i(s_j)$. This proof is essential as the same proof will be used to treat $m$-paths, i.e. paths with $m$ connected parts, of which at least one of them is a path $P(i, s_j)$ or a path $P(\bar{i}, s_j)$. We also discuss some possible difficulties associated with the convergence of the averaging process.

In section 4, the main difficulty which could impede convergence is identified with the
fact that the ratio $R_i(s_{j+1})/R_i(s_j)_M$ can be infinitesimally close to one, $R_i(s_j)_M$ being an extremum value of $R_i(s_j)$ when the weight for $R_i(s_j)_M$ is infinite with respect to the sum of the weights for all other $R_i(s_j)$.

We solve this difficulty in the case of spanning trees in $V_j$ directly. Then, $s_j$ is replaced by $\{s_j\}$, a partition of the vertices of the border of $V_j$ with $V_{j+1} - V_j$, where $V_j$ is a volume which increases with $j$. In the averaging procedure $W_m^r(\{s_j\}, \{s_{j+1}\})$ is the weight of $R_m^r(\{s_j\})$ in the evaluation of $R_m^r(\{s_{j+1}\})$. The weight-ratio

$$\sum_{\{s_j\}_M} W_m^r(\{s_j\}_M, \{s_{j+1}\}) / \sum_{\{s_j\}} W_m^r(\{s_j\}, \{s_{j+1}\})$$

is studied, where $\{s_j\}_M$ is a partition corresponding to $R_m^r(\{s_j\}_M)$, i.e. an extremum value of $R_m^r(\{s_j\})$ and $\{s_j\}$ any other partition. When the above weight-ratio becomes infinite we have a convergence problem in the case of spanning trees equivalent to the one for paths mentioned above. We then prove that if a constraint on the construction of the $V_j$’s is imposed which, indeed, is easy to implement, the above weight-ratio takes the same value for all partitions $\{s_{j+1}\}$ of the vertices on the border of $V_{j+1}$ with $G - V_{j+1}$.

This result allows to conclude that if the above weight-ratio is infinite, it is infinite whatever $\{s_{j+1}\}$ and then $R_m^r(\{s_{j+1}\})$ is equal to $R_m^r(\{s_j\}_M)$, implying that convergence has been obtained. When, this weight-ratio is finite, then the convergence of the averaging process is not impeded and a unique value for $R_m^r(\{s_{j+1}\})$ is obtained as $j \to \infty$. In the last sub-section we show how the averaging process works in the case of spanning trees on $G$ (instead of paths or multiple paths).

In section 5, the use of the above proof allow the proof of the factorization theorem. Section 6 will be the conclusion.
2. The $\alpha$-representation

Here, we deal with scalar field theories in $d$ Euclidean dimensions. We study one-line, one-vertex irreducible Feynman graphs with $I$ internal lines (propagators), $L$ loops, external momenta $P_v$ and we take the coupling constant equal to $-1$ in order to simplify. Then, $F_G$, the Feynman amplitude for a graph $G$ and for a field of mass $m$ reads [8]

\[ F_G = h_0 (4\pi)^{-dL/2} \int_0^{h_0} \prod_{i=1}^I d\alpha_i \delta(h_0 - \sum_i \alpha_i) \left[ P_G(\alpha) \right]^{-d/2}. \]

\[ \int_0^\infty d\lambda / \lambda \lambda^{|I-dL/2|} \exp\{-\lambda [Q_G(P_v, \alpha) + m^2 h_0]\} \] (2)

where $P_G(\alpha)$ is a homogeneous polynomial of degree $L$ in the $\alpha_i$’s defined as

\[ P_G(\alpha) = \sum_{T} \prod_{l \notin T} \alpha_l \] (3)

where the sum runs over all the spanning tree $T$ of $G$. (A spanning tree of $G$ is a tree incident with every vertex of $G$). $Q_G(P, \alpha)$ is quadratic in the $P_v$’s and is given by the ratio of a homogeneous polynomial of degree $L+1$ over $P_G(\alpha)$

\[ Q_G(P_v, \alpha) = [P_G(\alpha)]^{-1} \sum_C s_C \prod_{l \in C} \alpha_l \] (4)

where the sum runs over all cuts $C$ of $L+1$ lines that divide $G$ in two connected parts $G_1(C)$ and $G_2(C)$, with

\[ s_C = (\sum_{v \in G_2(C)} P_v)^2 = (\sum_{v \in G_1(C)} P_v)^2 \] (5)

(A cut $C$ is obtained from a tree $T$ by cutting off one line of $T$. Then, the cut $C$ will consist of all lines on $G$ not on $T$ plus the line of $T$ which has been cut). We note that $\lambda$ can be interpreted as an overall scale for the $\alpha$-parameters ((2) indicates that $\sum_i \alpha_i = h_0$ where $h_0$ is arbitrary but taken equal to one in most circumstances). The integration over $\lambda$ gives the overall divergence of $F_G$ for a renormalizable theory. Here, we will limit ourselves to super-renormalizable theories, i.e. $I - dL/2$ will always be positive, giving a convergent integral
\[ I_{\lambda}(Q_G) = \int_0^{\infty} d\lambda/\lambda \lambda^{I-\frac{dL}{2}} \exp\{-\lambda[Q_G(P_v, \alpha) + m^2 h_0]\} \]

\[ = \Gamma(I - \frac{dL}{2}) [Q_G(P_v, \alpha) + m^2 h_0]^{-\frac{(I - \frac{dL}{2})}{2}} \] (6)

Now, the spirit of the demonstration concerning the replacement of the \( \alpha_i \)'s by their mean-values \( \bar{\alpha}_i \) consists in isolating the dependence of the integrand of \( F_G \) on one particular \( \alpha_i \) and in using the mean-value theorem to perform the integration [7]. A discussion of the consistency of the result of this integration then shows that one should have, in any case [7],

\[ \bar{\alpha}_i = O(\frac{h_0}{I}) \] (7)

This property can easily be understood by considering the phase space for the \( I \) variables \( \alpha_i \), which can be found to be equal to \( h_0^{I-1}/(I-1)! \sim (e \frac{h_0}{I})^I \), leaving a phase space for each \( \alpha_i \) of the order of \( e \frac{h_0}{I} \).

Then, one has to show that indeed all \( \bar{\alpha}_i \)'s can be taken equal to some common value \( \bar{\alpha} \). Consequently, we shall define \( \bar{\alpha} \) by

\[ \mathcal{N}_{\mathcal{T}} \bar{\alpha}^\mathcal{T} = \sum_{\mathcal{T}} \prod_{l \notin \mathcal{T}} \bar{\alpha}_l = P_G(\bar{\alpha}) \] (8)

where \( \mathcal{N}_{\mathcal{T}} \) is the number of spanning trees on \( G \). We see from (2) that the expression obtained for \( F_G \) by using the mean value theorem is

\[ F_G = h_0 (4\pi)^{-\frac{dL}{2}} [P_G(\bar{\alpha})]^{-\frac{d}{2}} I_{\lambda}[Q_G(P_v, \{\bar{\alpha}_i\})] h_0^{I-1}/(I-1)! \] (9)

where the factor \( h_0^{I-1}/(I-1)! \) is the volume of the phase space available for the \( \alpha_i \)'s. So, in fact, from (9) it is clear that our goal amounts to showing that \( Q_G(P_v, \{\bar{\alpha}_i\}) \) is not affected by the replacement \( \bar{\alpha}_i \rightarrow \bar{\alpha} \). Let us write

\[ P_G(\bar{\alpha}) = \bar{a}_i + \bar{b}_i \bar{\alpha}_i \] (10)
\[ \sum_{c} s_{c} \prod_{l \in c} \bar{\alpha}_{l} = \bar{d}_{i} + \bar{e}_{i} \bar{\alpha}_{i} \]  

(11)

where \( \bar{a}_{i}, \bar{b}_{i}, \bar{d}_{i} \) and \( \bar{e}_{i} \) do not contain any \( \bar{\alpha}_{i} \) factor. As

\[ Q_{G}(P_{v}, \{\bar{\alpha}_{i}\}) = (\bar{e}_{i}/\bar{b}_{i})(\bar{d}_{i}/\bar{e}_{i} + \bar{\alpha}_{i})/((\bar{a}_{i}/\bar{b}_{i} + \bar{\alpha}_{i}) \]  

(12)

it is easily seen that if

\[ \bar{d}_{i}/\bar{e}_{i} = \bar{a}_{i}/\bar{b}_{i} \]  

(13)

the shift \( \bar{\alpha}_{i} \rightarrow \bar{\alpha} \) will not affect \( Q_{G}(P_{v}, \{\bar{\alpha}_{i}\}) \). Repeating the reasoning for all \( \bar{\alpha}_{i} \)'s shows that \( Q_{G}(P_{v}, \{\bar{\alpha}_{i}\}) \) is invariant under the replacement of all the \( \bar{\alpha}_{i} \)'s by \( \bar{\alpha} \) if (13) is true. It will be the purpose of the next sections to demonstrate that, up to vanishing corrections as \( I \rightarrow \infty \), (13) is indeed true. Then, the shift of \( \log \left[ Q_{G}(P_{v}, \{\bar{\alpha}_{i}\}) \right] \)

\[ \delta Q_{G}(P_{v}, \{\bar{\alpha}_{i}\})/Q_{G}(P_{v}, \{\bar{\alpha}_{i}\}) = Q_{G}^{-1} \sum_{i=1}^{I} \frac{\partial Q_{G}}{\partial \bar{\alpha}_{i}} \delta \bar{\alpha}_{i} \]  

(14)

will be vanishing, because \( \delta \bar{\alpha}_{i} \sim 1/I \) for any \( i \) (\( \delta \bar{\alpha}_{i} = \bar{\alpha} - \bar{\alpha}_{i} \)). Then, defining \( \Delta(\bar{\alpha}) \) as

\[ \Delta(\bar{\alpha}) = \prod_{l \in G} \bar{\alpha}_{l} \]  

(15)

i.e. defined as the product of all \( \bar{\alpha}_{l} \)'s over \( G \), we can write (see (3) and (10))

\[ \bar{a}_{i} = \Delta(\bar{\alpha}) \sum_{T \supseteq i} \prod_{l \in T} \bar{\alpha}_{l}^{-1} \]  

(16)

\[ \bar{b}_{i} = \bar{\alpha}_{i}^{-1} \Delta(\bar{\alpha}) \sum_{T \supsetneq i} \prod_{l \in T} \bar{\alpha}_{l}^{-1} \]  

(17)

In an analogous way \( \bar{d}_{i} \) and \( \bar{e}_{i} \) can be written

\[ \bar{d}_{i} = \Delta(\bar{\alpha}) \sum_{T \supseteq i} \prod_{l \in T} \bar{\alpha}_{l}^{-1} \sum_{k \in T} \bar{\alpha}_{k} s_{C_{k}} \nu^{-1}(C_{k}) \]  

(18)
\[ \bar{e}_i = \bar{\alpha}_i^{-1} \Delta(\bar{\alpha}) \sum_{T \ni i} \prod_{l \in T} \bar{\alpha}_l^{-1} \sum_{k \in T} \bar{\alpha}_k s_{\mathcal{C}_k} \nu^{-1}(\mathcal{C}_k) \]  

(19)

where \( \nu(\mathcal{C}_k) \) counts the number of times the same cut \( \mathcal{C}_k \) is obtained in cutting trees \( T \) at \( k \), \( k \) being among the \( \nu(\mathcal{C}_k) \) propagators binding the connected parts of \( G \), \( G_1(\mathcal{C}_k) \) and \( G_2(\mathcal{C}_k) \) separated by \( \mathcal{C}_k \).

To understand how (18) and (19) can be obtained from (11), let us recall that we are summing in (11) over all possible cuts \( \mathcal{C} \) belonging to \( G \). It is then, useful to note that the same cut \( \mathcal{C}_k \), containing the propagator \( k \), can be obtained by cutting different trees provided these trees have exactly the same structure in \( G_1(\mathcal{C}_k) \) and \( G_2(\mathcal{C}_k) \) and only in this case. That is, they will only differ by the propagator on them linking \( G_1(\mathcal{C}_k) \) and \( G_2(\mathcal{C}_k) \). Let us note by \( \nu(\mathcal{C}_k) \) the number of propagators on \( G \) linking \( G_1(\mathcal{C}_k) \) and \( G_2(\mathcal{C}_k) \). Then, \( \nu(\mathcal{C}_k) \) will count how many trees \( T \) can be cut to yield the same cut \( \mathcal{C}_k \). Dividing by \( \nu(\mathcal{C}_k) \) in (18) and (19) ensures that each cut is only counted once when cutting all possible trees yielding it. \( s_{\mathcal{C}_k} \) is defined by (5) where \( \mathcal{C} = \mathcal{C}_k \). Of course, all possible cuts are generated because \( k \) is taken to be any propagator of \( G \) on \( T \).

Comparing (18) and (19) to (16) and (17) respectively, we see that the sum

\[ \sum_{k \in T} \bar{\alpha}_k s_{\mathcal{C}_k} \nu^{-1}(\mathcal{C}_k) \]

is the factor which distinguishes \((\bar{a}_i, \bar{b}_i)\) from \((\bar{d}_i, \bar{e}_i)\). If this factor can be factorized out of the sum over trees, the relation (13) will be obvious. However, this can only be done if the structure of the trees far from \( i \) is independent of their structure near \( i \) and when, in addition, \( k \) and \( i \) are far apart on \( G \), i.e. if they are separated by an infinite number of propagators on \( G \). When \( I \to \infty \), most of the propagators of \( G \) will be far from \( i \), so that we will be able to neglect in the sum over \( k \), those \( k \) which are in a finite range of \( i \). So our main goal will be, in fact, to show that a factorization occurs in the sum of weighted trees for domains far apart on \( G \).
3. Construction of the spanning trees on $G$: paths on $G$

It will prove to be convenient for the following to rewrite (18) and (19) by inverting the summation order, $T_k^i(C_k)$ being a spanning tree which contains $i$ and $k$ and which cut at $k$ gives a cut $C_k$, containing $k$,

$$\bar{d}_i = \Delta(\bar{\alpha}) \sum_k \bar{\alpha}_k \sum_{T_k^i(C_k)} s_{C_k} \nu^{-1}(C_k) \prod_{l \in T_k^i} \bar{\alpha}_l^{-1}$$

(20)

$$\bar{e}_i = \bar{\alpha}_i^{-1} \Delta(\bar{\alpha}) \sum_k \bar{\alpha}_k \sum_{T_k^i(C_k)} s_{C_k^i} \nu^{-1}(C_k^i) \prod_{l \in T_k} \bar{\alpha}_l^{-1}$$

(21)

where $T_k(C_k^i)$ contains $k$ but not $i$ and gives a cut $C_k^i$ when cut at $k$, $C_k^i$ containing $i$ and $k$. What we want to demonstrate now is that for any $k$ far apart from $i$ on $G$, defining

$$\bar{d}_{i,k} = \sum_{T_k^i(C_k)} s_{C_k} \nu^{-1}(C_k) \prod_{l \in T_k^i} \bar{\alpha}_l^{-1}$$

(22)

$$\bar{e}_{i,k} = \bar{\alpha}_i^{-1} \sum_{T_k^i(C_k)} s_{C_k^i} \nu^{-1}(C_k^i) \prod_{l \in T_k} \bar{\alpha}_l^{-1}$$

(23)

we have

$$\bar{d}_{i,k}/\bar{e}_{i,k} = \bar{a}_i/b_i$$

(24)

up to terms which tend to zero as $I \to \infty$. The proof of (24) naturally entails the validity of (13) because $\bar{d}_i = \Delta(\bar{\alpha}) \sum_k \bar{\alpha}_k \bar{d}_{i,k}$ and $\bar{e}_i = \Delta(\bar{\alpha}) \sum_k \bar{\alpha}_k \bar{e}_{i,k}$ and also because those $k$ within a “volume” $V_j$ containing $i$ and a number of propagators infinitesimal compared to the total number in $G$ contribute to a negligible fraction of the sum, as will be shown at the end of section 5.

A - General strategy for the proof of (24)

We now give the general lines of the proof of the relation (24).
In the first place we consider $G$ as embedded in a $\mathbb{R}^3$-space embedded with a metric. Then, we consider a volume $V_j$ in that space which contains the propagator $i$. When $j$ is finite the number of propagators contained in $V_j$ will be finite. As $j \to \infty$ the number of propagators contained in $V_j$ will tend to infinity. We will however consider the number of propagators on $G$ outside $V_j$ infinite with respect to the number of those inside $V_j$, even as $j \to \infty$. The propagator $k$ is taken to be outside $V_j$. Of course, in the sum over all $k$’s some of them are inside $V_j$, but their number will be infinitesimal with respect to the total number of $k$’s, i.e. the number of propagators in $G$. So the contribution of $k$’s inside $V_j$ will be negligible in the sum over them.

The reason we want to isolate $V_j$ is that we shall see that inside $V_j$ we can sum over sub-trees in it (when a spanning tree on $G$ is cut by the border of $V_j$, the portion of that spanning tree inside $V_j$ has no reason to be connected and is in general composed of several connected pieces that we call sub-trees) independently of the rest of the trees outside it provided the vertices on the border of $V_j$ are partitioned in a definite way, each partition corresponding to a partition of sub-trees in $V_j$. So we get a factorization in the structure of trees on $G$, the sum over all trees in $G$ being factorized into the sum over sub-trees in $V_j$ times the sum over all sub-trees in $G - V_j$ for a given partition of the border of $V_j$ in subsets of vertices, each sub-set being attached to the one sub-tree in $V_j$. Of course, sub-trees in $G - V_j$ (i. e. $G$ minus all propagators and vertices in $V_j$) have to be compatible with those in $V_j$ (not forming loops for instance) in that one should obtain trees in $G$ altogether. However, again this implies only a restriction of the partition of vertices on the border of $V_j$, with respect to sub-trees in $G - V_j$ this time. So given the structure on the border of $V_j$, factorization of trees inside $V_j$ and outside it holds.

However, we still have a dependence on the structure of the partition of the border of $V_j$. That is where comes a first and essential part of the proof.

Let us call $T(i, \{s_j\})$ those sets of sub-trees in $V_j$ which contain the propagator $i$ and
$T(\vec{i}, \{s_j\})$ those sets of sub-trees in $V_j$ which do not contain the propagator $i$, both being attached to a partition $\{s_j\}$ of the border of $V_j$. Let us define the ratio

$$R_i(\{s_j\}) = \frac{\sum_T T(i, \{s_j\})}{\sum_T T(\vec{i}, \{s_j\})}$$ (25)

which is the ratio of the sum of the weights of sub-trees sets $T(i, \{s_j\})$ over the sum of the weights of sub-trees sets $T(\vec{i}, \{s_j\})$. (Each propagator $\ell$ on a sub-tree brings a factor $\bar{\alpha}^{-1}_\ell$ in the weight of any sub-tree). Then, it will be proved in the next sub-section (for self-avoiding paths) and in the next section (for trees themselves) that as $j \to \infty$, i.e. when the radius of $V_j$ grows to infinity, but with $V_j$ still being infinitesimal with respect to $G$, $R_i(\{s_j\})$ tends to some value $R_i^\infty$ independent of the partition $\{s_j\}$. Using the factorization property discussed above, this amounts to say that

$$\bar{a}_i/\bar{b}_i = R_i^\infty \bar{\alpha}_i$$ (26.a)

(see (16) and (17)). Now, the same argument can be used to derive also

$$\bar{d}_{i,k}/\bar{e}_{i,k} = R_i^\infty \bar{\alpha}_i$$ (26.b)

and thereby prove (24) if the additional structure of cutting through $k$ does not interfere with the inside of $V_j$, i.e. if $\nu(C_k)$ and $s_{\mathcal{C}_k}$ are unaffected by the inside of $V_j$.

Let us denote by $S_k$ the surface defined by the cut $C_k$ going through $k$ and dividing $G$ into two separate pieces $G_1(C_k)$ and $G_2(C_k)$. ($S_k$ cuts through all the propagators on $G$ linking $G_1(C_k)$ and $G_2(C_k)$). If $S_k$ does not go through $V_j$, the factorization property then trivially shows that the sums inside $V_j$ are unaffected by $S_k$, there is no interference. If $S_k$ goes through $V_j$, $\nu(C_k)$ counting the propagators in $S_k$ will only be infinitesimally affected by the structure of sub-trees inside $V_j$, as well as $s_{\mathcal{C}_k}$, because the number of propagators in $V_j$ is infinitesimal with respect with their total number in $G$. So there is
only an infinitesimal interference in this case and therefore the relations above prove the
validity of (24).

In the next sub-sections, we simplify in a first approach, replacing trees on $G$ by
self-avoiding paths and prove in this case that the ratios $R_i(\{s_{j+1}\})$ are indeed averages
of $R_i(\{s_j\})$. We also show how the factorization described above works. The averaging
property means that

$$R_i(\{s_j\})_{\text{min}} \leq R_i(\{s_{j+1}\}) \leq R_i(\{s_j\})_{\text{max}}$$  \hspace{1cm} (27)

where $R_i(\{s_j\})_{\text{min}}$ and $R_i(\{s_j\})_{\text{max}}$ are respectively the maximum and the minimum
value of $R_i(\{s_j\})$. If, in the averaging, $R_i(\{s_j\})_{\text{min}}$ or $R_i(\{s_j\})_{\text{max}}$ only have a finite
weight relative to the sum of the weights of the other values of $R_i(\{s_j\})$, then it is clear
that $R_i(\{s_{j+1}\})$ will be different from $R_i(\{s_j\})_{\text{min}}$ or $R_i(\{s_j\})_{\text{max}}$, even having at least a
finite (non-infinitesimal) difference with them. Then, we will have

$$R_i(\{s_{j+1}\})/R_i(\{s_j\})_{\text{max}} = 1 - \eta_1$$  \hspace{1cm} (28.a)

$$R_i(\{s_{j+1}\})/R_i(\{s_j\})_{\text{min}} = 1 + \eta_2$$  \hspace{1cm} (28.b)

$\eta_1$ and $\eta_2$ being positive and non-infinitesimal. As $j \rightarrow \infty$, the interval of variation of
$R_i(\{s_j\})$ we tend to zero, and a value $R_i^\infty$ independent of $\{s_j\}$ will be obtained for $R_i\{s_j\}$.

However, there may be a snag if the weight of $R_i(\{s_j\})_{\text{max}}$ or $R_i(\{s_j\})_{\text{min}}$ is infinite
with respect to the sum of the weights of the other values of $R_i(\{s_j\})$ for some $\{s_{j+1}\}$,
because (28) may not hold. In the next section, we study the weight-ratio

$$W_{m_2}^m(\{s_j\}_2, \{s_{j+1}\})/W_{m_1}^m(\{s_j\}_1, \{s_{j+1}\})$$

of weights corresponding to $R_i^{m_2}(\{s_j\}_2)$ and $R_i^{m_1}(\{s_j\}_1)$ in the evaluation of $R_i^m(\{s_{j+1}\})$
and prove that when a certain constraint (easy to implement) on the construction of the
V_j’s is imposed, it is independent of \{s_{j+1}\}. This allows us to conclude that either \eta_j is finitely different from zero or that when it is infinitesimal, it is so for any \{s_{j+1}\}, ensuring convergence.

B - Self-avoiding paths

Definition

A path \( P(v_1, v_n) \) is defined as the succession of propagators \((v_1v_2), (v_2v_3), \cdots, (v_{n-1}v_n)\) linking \( v_1 \) to \( v_n \), \( v_1, v_2, \cdots, v_n \) being \( n \) vertices on \( G \). In a self-avoiding path \( v_1, v_2, \cdots, v_n \) are all different vertices.

A closed path is constructed when \( v_1 \) and \( v_n \) are the same vertex. A loop is a self-avoiding closed path. \( \square \)

The main tool we will be using now is the fact that, taking a vertex \( v_k \) at one end of the propagator \( k \) and a vertex \( v_i \) at one end of the propagator \( i \), for each spanning tree \( T \) on \( G \), there is one path, and only one, on \( T \), binding \( v_k \) and \( v_i \). Furthermore, this path is self-avoiding, it goes through each vertex it is incident with only once. So the idea is to construct all spanning trees on \( G \) by beginning to construct all self-avoiding paths \( P(v_i, v_k) \) on \( G \) binding \( v_i \) and \( v_k \). Two spanning trees of \( G \) having a different path \( P(v_i, v_k) \) are necessarily different. (If that were not the case, we would have two different paths \( P(v_i, v_k) \) on the same tree, giving a loop on this tree which is forbidden). Of course, for every such path there exist many spanning trees obtained by sprouting branches out of the path. In fact, counting the number of trees associated with one path is the same as counting the number of ways branches can be sprouted out of this path. However, in a first step, we will concentrate our attention on the paths \( P(v_i, v_k) \) themselves, taking into account the effect of branches later on, i.e. in the next section.

Now, let us consider the sum over all paths, each one being weighted by the product of all \( \bar{\alpha}_i^{-1} \) belonging to the propagators along it. We denote by \( P(i, s) \) a path \( P(v_i, s) \) which
goes through the propagator \( i \) and by \( P(i, s) \) a path which does not go through \( i \), both paths relating \( v_i \) and \( s \). Then, the sums over all \( P(i, v_k) \) and \( P(\bar{i}, v_k) \) can be written (the sum over paths \( P, \sum_P \), is a multiple sum, a summation being made for each path \( P(s_p, s_{p+1}) \)),

\[
\sum_P P(i, v_k) = \sum_P \sum_{s_1, \ldots, s_{l-1}} P(i, s_1)P(s_1, s_2)\ldots P(s_{l-1}, v_k) \quad (29.a)
\]

\[
\sum_P P(\bar{i}, v_k) = \sum_P \sum_{s_1, \ldots, s_{l-1}} P(\bar{i}, s_1)P(s_1, s_2)\ldots P(s_{l-1}, v_k) \quad (29.b)
\]

\( s_1, s_2, \ldots, s_{l-1} \) belonging to the ensemble of the border-vertices of closed volumes \( V_1, V_2, \ldots, V_{l-1} \) such that

\[
V_1 \subset V_2 \subset \ldots \subset V_{l-1}
\]

It is important to note that a path \( P(s_j, s_{j+1}) \) will go out of \( V_j \) at \( s_j \) for the first time but will otherwise be entirely contained in \( V_{j+1} \). Moreover, \( P(s_j, s_{j+1}) \) can re-enter \( V_j \) and go out again at some vertex \( s'_j \). We also will take \( V_1 \) to have a border at a finite distance of \( v_i \), and the border of \( V_{j+1} \) to be at a finite distance of \( V_j \) in terms of the minimum number of propagators separating them.

Let us now define this notion of distance on a graph. We are going to embed \( G \) in an Euclidean space where each propagator has a definite length (this is the purpose of this embedding). Therefore, any path will have a definite length in this Euclidean space. The length of the path with the least number of propagators will be the distance between two vertices on \( G \).

Let us also remark that, according to our above description, the distance between two vertices measured in Euclidean space and measured in the least number of propagators joining them have a priori non monotonicity relation between them, because the lengths of propagators can vary from one part to another part of \( G \). We now define in a constructive way the volumes \( V_j \).
C - Construction of the $V_i$'s

We suppose that $G$ is embedded in a 3-dimensional $R_3$ space. Then, we define the sphere $S_1$, i.e. the ensemble of points within a certain radius, centered at the vertex $v_i$, provided a metric has been defined in $R_3$. The radius of $S_1$ will be taken such that $S_1$ contains a finite number of propagators. In general, the two-dimensional border-surface of $S_1$ cut through propagators of $G$. We then deform continuously $S_1$ inwards in such a way that its border slides along the cut propagators until vertices attached to these propagators are met.

**Definition**

The border of $V_1$ is the deformed border of $S_1$. □

As we suppose that $v_i$ is inside $S_1$, it will also be inside $V_1$. Note that the border of $V_1$ may also contain entire propagators linking two vertices on its boundary. All the vertices and propagators belonging to the deformed sphere $S_1$ will belong to $V_1$.

Constructing $V_2$, we start with a sphere $S_2$ of radius larger than that of $S_1$. Then, we deform $S_2$ along the propagators cut by its border-surface until we meet the vertices attached to these propagators. Now, we are tempted to define $V_2$ in the same way as we did for $V_1$, by taking its border to coincide with the deformed sphere $S_2$. However, we will do so only in the case where the part of $G$ between the borders of the deformed $S_2$ and $S_1$ is connected. In general, we expect that, in fact, there will be several connected pieces of $G$ between the borders of $S_2$ and $S_1$. For each of these pieces we will have a corresponding connected piece of $V_2 - V_1$ (meaning $V_2$ from which $V_1$ has been subtracted) containing all the vertices and propagators of this connected piece of $G$. Let us consider in more details a given connected piece in $V_2 - V_1$.

Such a piece, let us call it $(V_2 - V_1)_C$ has a boundary formed by three pieces

i) a piece on the border of the deformed sphere $S_2$, which itself has a closed curve
$C_2$ as boundary,

ii) a piece on the border of the deformed sphere $S_1$, which itself has a closed curve $C_1$ as boundary,

iii) a cylindrical piece having $C_1$ and $C_2$ as boundaries. It should not be crossed by $G$.

In short, each connected $(V_2 - V_1)_C$ will enclose a corresponding connected part of $G$ in $V_2 - V_1$, let us call it $G_C$. Then, $C_2$ encloses all the vertices and propagators of $G_C$ on the surface of the deformed $S_2$. In the same manner $C_1$ encloses all the vertices and propagators of $G_C$ on the border of $V_1$ (which according to the last definition is the surface of the deformed $S_1$). Note that there may be, in some cases, no vertex of $G_C$ either in the surface enclosed by $C_1$ or in the surface enclosed by $C_2$, due to the topology of $G$ ($G$ may not go through the surface of the deformed $S_2$ at the border of $(V_2 - V_1)_C$, or through the piece of border of $V_1$ in common with that of $(V_2 - V_1)_C$).

Another condition we have on the boundary of $(V_2 - V_1)_C$ is that it should not cross the boundary of another connected piece in $V_2 - V_1$. In that way, two different connected volumes of $V_2 - V_1$ do not overlap.

This condition and the properties i), ii) and iii) above define the boundary of one connected part $(V_2 - V_1)_C$.

For the following volumes $V_3, \ldots, V_j$, the constructive process we just described for $V_2$ repeats itself, each connected piece of $V_j - V_{j-1}$ being defined through its boundaries. It may happen that two successive $V_{j-1}, V_j$ have some coinciding part of their borders (on a common part of the deformed $S_{j-1}$ and $S_j$) because the density of propagators may be much larger in other parts of $V_j - V_{j-1}$. This ends our construction of the volumes $V_j$.

D - An averaging theorem

We then have an unequivocal definition for the paths $P(v_i, v_k)$. In general the ratio
$P(i, s_1)/P(\tilde{i}, s_1)$ depends on $s_1$. However, considering sums over paths, we want to prove that

$$R_i(s_j) = \sum_P P(i, s_j)/\sum_P P(\tilde{i}, s_j)$$

(31)

tends to a value independent of $s_j$ as $j \to \infty$. This is the clue for deriving (24) and thereby the factorization property. (It will also be proven in section 5 that when $k$ is outside $V_j$, $j \to \infty$, for a given $k$, the factors $s_{C_k}^{-1}(C_k)$ and $s_{C_k'}^{-1}(C_k')$ are equal). Now we want to prove the following lemma

**Lemma 1**

If $P(s_j, s_{j+1})$ never returns in $V_j$ then,

$$R_i(s_j)_{min} \leq R_i(s_{j+1}) \leq R_i(s_j)_{max}$$

(32)

where $R_i(s_j)_{min}$ and $R_i(s_j)_{max}$ are respectively the minimum and the maximum values of $R_i(s_j)$ for those $s_j$ coupled to $s_{j+1}$ by at least one path $P(s_j, s_{j+1})$.

**Proof**

As a first remark, we can factorize the expression for $\sum_P P(i, s_{j+1})$,

$$\sum_{s_j} \sum_P P(i, s_j)P(s_j, s_{j+1}) = \sum_{s_j} [\sum_P P(i, s_j)] [\sum_P P(s_j, s_{j+1})]$$

because the paths $P(s_j, s_{j+1})$ being entirely in $V_{j+1} - V_j$ never interact with the paths $P(i, s_j)$ contained in $V_j$. Therefore, using (31) we can write

$$R_i(s_{j+1}) = \{\sum_{s_j} [\sum_P P(i, s_j)] [\sum_P P(s_j, s_{j+1})]\} /$$

$$\{\sum_{s_j} [\sum_P P(\tilde{i}, s_j)] [\sum_P P(s_j, s_{j+1})]\}$$

$$= \{\sum_{s_j} R_i(s_j) [\sum_P P(\tilde{i}, s_j)][\sum_P P(s_j, s_{j+1})]\} /$$

$$\{\sum_{s_j} [\sum_P P(\tilde{i}, s_j)][\sum_P P(s_j, s_{j+1})]\}$$

(33)
We see that $R_i(s_{j+1})$ is an average of $R_i(s_j)$ for those $s_j$ coupled to $s_{j+1}$ and therefore (32) is true. □

Our goal is, of course, a repeated use of (32) and as $j$ grows we expect $R_i(s_j)$ to become independent of $s_j$. However, we have restricted the paths $P(s_j, s_{j+1})$ to be in $V_{j+1} - V_j$ in order to avoid an interaction with the paths $P(v_i, s_j)$. If we allow such an interaction to occur, i.e. if $P(s_j, s_{j+1})$ returns in $V_j$, we have to distinguish between the different topologies of $P(s_j, s_{j+1})$. That is, we have to cut $P(s_j, s_{j+1})$ in parts which stay in $V_j$ and parts which stay in $V_{j+1} - V_j$ in order to be able to factorize the sum over paths. Let us call $s^0_j$ the first vertex of $P(s_j, s_{j+1})$ and $s^1_j, s^3_j, ..., s^{2m-3}_j$ the vertices where $P(s_j, s_{j+1})$ re-enters $V_j$. (At $s^2_j, s^4_j, ..., s^{2m-2}_j$ $P(s_j, s_{j+1})$ goes out of $V_j$). We denote by $\{s^l_j\}$ the ensemble $s^0_j, s^1_j, s^2_j, ..., s^{2m-2}_j$ and $M_j$ the maximum number of connected parts of $P(i, s_{j+1})$ in $V_j$. Then, $\sum P(i, s_{j+1})$ can be written

\[
\sum_P P(i, s_{j+1}) = \sum_{s^0_j} \left[ \sum_P P(i, s^0_j) \right] \left[ \sum_P P(s^0_j, s_{j+1}) \right] \\
+ \sum_{m=2}^{M_j} \sum_{\{s^l_j\}} \sum_{\{s^{2l-1}_j\}} \left[ \sum_P P(i, s^0_j) \prod_{l=1}^{m-1} P(s^{2l-1}_j, s^{2l}_j) \right] \\
\left[ \sum_P P(s^{2m-2}_j, s_{j+1}) \prod_{l=1}^{m-1} P(s^{2l-2}_j, s^{2l-1}_j) \right] 
\]

(34)

where the paths $P(i, s^0_j)$, $P(s^{2l-1}_j, s^{2l}_j)$ are contained in $V_j$ and the paths $P(s^0_j, s_{j+1})$, $P(s^{2l-2}_j, s^{2l-1}_j)$ and $P(s^{2m-2}_j, s_{j+1})$ have all their propagators contained in $V_{j+1} - V_j$. $\{s^{2l}_j, s^{2l-1}_j\}$ is the ensemble of couples $(s^{2l}_j, s^{2l-1}_j)$ on $P(s_j, s_{j+1})$. The first term in (34) stands for the case where $P(s_j, s_{j+1})$ never returns in $V_j$ and therefore corresponds to the case of Lemma 1 ($m = 1$). The following terms describe the case where $P(s_j, s_{j+1})$ returns in $V_j$ ($m \geq 2$). Then, for a given $m$, $\{s^l_j\}$, $\{s^{2l}_j, s^{2l-1}_j\}$ the summation over paths has
been factorized into paths contained in $V_j$ and paths contained in $V_{j+1} - V_j$. Notice that the couples $(s_j^{2l}, s_j^{2l-1})$, in fact, select an order among the $s_j^l$'s, their order along the path $P(s_j^0, s_{j+1})$. However, the sum over all orders can be factorized as a sum over all orders of paths in $V_j$ times a sum over all orders of paths in $V_{j+1} - V_j$. So, we could dispense ourselves of tracking down the order of the $s_j^l$'s through the couples $(s_j^{2l}, s_j^{2l-1})$. We do so only in order to make clear how the summation over paths is done. Later on, in the case of spanning trees this order will be meaningless and it will not appear in the summation over all trees. Let us now define by (for $m \geq 2$)

$$R_i^m(s_j) = \left[ \sum_P P(i, s_j^0) \prod_{l=1}^{m-1} P(s_j^{2l-1}, s_j^{2l}) \right] / \left[ \sum_{\bar{P}} P(\bar{i}, s_j^0) \prod_{l=1}^{m-1} \bar{P}(s_j^{2l-1}, s_j^{2l}) \right]$$

the ratio of the part of $\sum_P P(i, s_{j+1})$ in $V_j$ over the part of $\sum_P P(\bar{i}, s_{j+1})$ which also is in $V_j$; $m$, $\{s_j^l\}$ and the couples $\{s_j^{2l}, s_j^{2l-1}\}$ are given.

$\bar{P}$ helps to distinguish the paths which are associated with $P(\bar{i}, s_j^0)$. Of course, $R_i(s_j) \equiv R_i^1(s_j)$.

We remark that the sums over $P$ in (35) have not been factorized. This is, of course, because the paths in $V_j$ have to avoid each other, so that a configuration of one of them affect the summation over the others. Then, we have the following theorem

**Theorem 1**

$$R_i^m(s_j)_{\text{min}} \leq R_i^{m'}(s_{j+1}) \leq R_i^m(s_j)_{\text{max}}$$

where $R_i^m(s_j)_{\text{min}}$ and $R_i^m(s_j)_{\text{max}}$ are respectively the minimum and the maximum values of $R_i^m(s_j)$ viewed as a function of $m$, $\{s_j^l\}$ and $\{s_j^{2l}, s_j^{2l-1}\}$, and for those $s_j^l$ on at least one path $P(i, s_{j+1})$. 
Proof

For the sake of simplicity of notation we will make implicit the sum $\sum_P$ each time the symbol $P$ appears. Then define

$$P^m(i, s_j) = P(i, s^0_j) \prod_{l=1}^{m-1} P(s^2_{j-l}, s^2_{j-l})$$  \hspace{1cm} (37.a)

$$P^m(\bar{i}, s_j) = P(\bar{i}, s^0_j) \prod_{l=1}^{m-1} \bar{P}(s^2_{j-l}, s^2_{j-l})$$ \hspace{1cm} (37.b)

where the symbol $\bar{P}$ is for a path which is associated with $P(\bar{i}, s^0_j)$ and could differ from a path $P$ because they are returning in $V_1$ where the paths $P(i, s^0_j)$ and $\bar{P}(\bar{i}, s^0_j)$ are different.

Writing $P^{m'}(i, s_{j+1})$ we obtain (we remark that, except for $P(i, s^0_j)$ and $\prod_{l=1}^{m-1} P(s^2_{j-l}, s^2_{j-l})$ which are confined in $V_j$, all other $P$’s are in $V_{j+1} - V_j$) (see fig. 1)

$$P^{m'}(i, s_{j+1}) = \sum_{s^0_j} P(i, s^0_j) P(s^0_j, s^0_{j+1}) \prod_{l=1}^{m'-1} P(s^2_{j+1-l}, s^2_{j+1})$$

$$+ \sum_{m=2}^{M_j} \sum_{\{s^l_j\}} P(i, s^0_j) \prod_{l=1}^{m-1} P(s^2_{j-l}, s^2_{j-l}) .$$

$$\sum_{\{t_l\}} P(s^2_{2l-t}, s^2_{2l-t-1}) .$$

$$\sum_{\{u_l\}} P(s^l_{u_l}, s^l_{u_l+1}) .$$

$$\sum_{\{v_l\}} P(s^2_{2l-v}, s^2_{2l-v-1})$$  \hspace{1cm} (38)

with

$$m - t_j + v_{j+1} = m'$$ \hspace{1cm} (39.a)

$$\{s^l_j\} = \{s^2_{2l-1}\} \cup \{s^2_{2l-1}\} \cup \{s^l_u\}$$ \hspace{1cm} (39.b)
The first term in (38) stands for the case where the connected component starting from $v_i$ of $P^m(i, s_{j+1})$, $P(i, s^0_j) P(s^0_j, s_{j+1}^0)$, goes out of $V_j$ at $s^0_j$ and stays in $V_{j+1} - V_j$, with all other connected components $P(s_{j+1}^{2l-1}, s_{j+1}^{2l})$ also staying in $V_{j+1} - V_j$. In this case, there is only one connected part in $V_j$, $P(i, s^0_j)$, of $P^m(i, s_{j+1})$ and therefore this corresponds to $m = 1$. The following terms in (38) describe the other cases with $m$ components $(m \geq 2)$ in $V_j : P(i, s^0_j)$ with $m - 1$ other components $P(s_{j+1}^{2l-1}, s_{j+1}^{2l})$. The connected paths $P(s_{j+1}^{2l-2}, s_{j+1}^{2l-1})$ are never incident with the border of $V_{j+1}$ and end up at the border of $V_j$. The connected paths $P(s_{j+1}^{lu}, s_{j+1}^{lu})$ start on the border of $V_j$ and end up on the border of $V_{j+1}$ while the connected paths $P(s_{j+1}^{2l-2}, s_{j+1}^{2l-1})$ start and end up on the border of $V_{j+1}$ with no incidence with the border of $V_j$.

We remark that for $m' = 1$ we recover expression (34) because the products over the paths $P(s_{j+1}^{2l-1}, s_{j+1}^{2l})$ and $P(s_{j+1}^{2l-2}, s_{j+1}^{2l-1})$ do not exist in that case and the product over $P(s_{j+1}^{lu}, s_{j+1}^{lu})$ is replaced by one unique term $P(s_{j+1}^{2m-2}, s_{j+1})$ in (34).

The relation (39.a) is obtained by counting the number of vertices of $P^m(i, s_j)$ on the border of $V_j$ which is equal to $2m - 1$ and the number of vertices of $P^{m'}(i, s_{j+1})$ on the border of $V_{j+1}$ which is equal to $2m' - 1$ and calculating the difference $2(m' - m)$. This difference comes from the connected paths $P(s_{j+1}^{2l-2}, s_{j+1}^{2l-1})$ which contributes $2v_{j+1}$ and $P(s_{j+1}^{2l-2}, s_{j+1}^{2l-1})$ which contributes $-2t_j$ to it. The same expression as (38) is obtained for $P^{m'}(i, s_{j+1})$ by replacing $P(i, s^0_j)$ by $P(\bar{i}, s^0_j)$ and $P(s_{j+1}^{2l-1}, s_{j+1}^{2l})$ by $\bar{P}(s_{j+1}^{2l-1}, s_{j+1}^{2l})$. Then, writing (sums over $P$, again, are implicit)

$$R^{m'}_i(s_{j+1}) = P^m(i, s_{j+1})/P^{m'}(i, s_{j+1})$$

and using (35) in it we obtain $R^{m'}_i(s_{j+1})$ as an average of $R^m_i(s_j)$ function of the $s_j$'s and $m$, the average being the result of a sum over $m$, \{s_j^l\} and the sets of couples \{s_{j+1}^{2l-1}, s_{j+1}^{2l}\}, \{s_{j+1}^{2l-2}, s_{j+1}^{2l-1}\}, \{s_{j+1}^{lu}, s_{j+1}^{lu}\}$ and \{s_{j+1}^{2l-2}, s_{j+1}^{2l-1}\} aside from the sum over paths $P$. However, once the set \{s_j^l\} is given the partition of the paths in $V_{j+1} - V_j$ does not depend on that of the paths in $V_j$ because there is no interaction between them. Therefore the sum over
all couples can be factorized out as it was already the case for the sum over the couples \( \{s_j^{2l-1}, s_j^{2l}\} \). Consequently, the only variables which are necessary to retain are \( m \) and the set \( \{s_j^l\} \) in the functional dependence of \( R_i^m(s_j) \). As said earlier we retain the dependence on the couples \( \{s_j^{2l}, s_j^{2l-1}\} \) in order to remind ourselves that we had to sum over all orders, this sum being, again, factorizable. Therefore the relation (36) follows. \( \square \)

Again, a repeated use of (36) will allow us to make all the ratios \( R_i^m(s_j) \) converge towards one value independent of \( s_j \) as \( j \to \infty \).

However, we have to be careful about two problems potentially hampering the efficiency of this uniformization of \( R_i^m(s_j) \).

a) As long as \( R_i^m(s_j) \) has not converged, both inequalities (the same as (28.a) and (28.b))

\[
R_i^{m'}(s_j+1)/R_i^m(s_j)_{\text{max}} < 1 - \eta_1 \quad (41.a)
\]

\[
R_i^{m'}(s_j+1)/R_i^m(s_j)_{\text{min}} > 1 + \eta_2 \quad (41.b)
\]

should be satisfied, \( \eta_1 \) and \( \eta_2 \) being two positive non-infinitesimal constants.

b) \( G \) could have the topology of a tree-like structure or "polymer", i.e. many branches could stem out of \( V_1 \) and \( V_{j+1} - V_j \) would be multiply connected. Remember that in Lemma 1 and Theorem 1, we have the restriction that \( s_j \)'s should be on at least one path \( P(i, s_j+1) \). Then, a path going in some connected part of \( V_{j+1} - V_j \) could never go in another connected part of \( V_{j+1} - V_j \) because in order to do so, it would have to return in \( V_1 \) which may be impossible because it would have to go through vertices with which it is already incident. Then, due to this finite volume effect, most paths going along one branch of \( V_{j+1} - V_j \) would never go in another branch. This leaves the possibility for \( R_i^m(s_j) \) of evolving towards a different value along each branch of this tree-like structure, instead of a unique value as we wanted to show.

However, for spanning trees on \( G \) this difficulty is easily removed. The reason for
this is clear: a spanning tree of $G$ is incident with every vertex of $G$. Then, we can think that a spanning tree on $G$ is composed of one path $P(i, s_{j+1})$ and a myriad of paths stemming out of it, the ensemble of all paths going through all vertices of $V_{j+1}$, and then through all connected domains of $V_{j+1} - V_j$. For trees, a separate evolution of the quantity corresponding to $R^m_i(s_{j+1})$ along branches of $V_{j+1} - V_j$ is therefore prohibited. That is the essential difference between paths and spanning trees in our approach.

In the next section we also show how the problem a) is solved by insuring the validity of (41.a) and (41.b).

4. Trees on $G$

A - Convergence of the iteration of mean-value operation

Let us consider $R_i(s_{j+1})$ as given by the mean-value expression (33). For each $s_j$, $R_i(s_j)$ is multiplied by a weight $W(s_j, s_{j+1}) = \sum_p P(\bar{i}, s_j) P(s_j, s_{j+1})/\sum_p P(\bar{i}, s_{j+1})$. Either (41.a) or (41.b) may be violated if and only if in the sum $\sum_{s_j} R_i(s_j)W(s_j, s_{j+1})$ the weight $W(s_j, s_{j+1})$ associated with $R_i(s_j)_{\text{max}}$ or $R_i(s_j)_{\text{min}}$ is infinite with respect to the sum of weights of any other value for $R_i(s_j)$. (By infinite, we mean that the ratios should be infinite).

Indeed, let us verify this and use (33), isolating the contribution of $R_i(s_j)_{\text{max}}$, $s_{jm}$ being the vertex for which $R_i(s_{jm}) = R_i(s_j)_{\text{max}}$

\[
R_i(s_{j+1}) = [R_i(s_j)_{\text{max}} W(s_{jm}, s_{j+1}) + \sum_{s_j \neq s_{jm}} R_i(s_j) W(s_j, s_{j+1})]/
\]

\[
[W(s_{jm}, s_{j+1}) + \sum_{s_j \neq s_{jm}} W(s_j, s_{j+1})] .
\]

We will take $R_i(s_j)/R_i(s_j)_{\text{max}}$ for $s_j \neq s_{jm}$ to have a finite (non-infinitesimal) difference with one (otherwise convergence to a unique value is already achieved) and simplify
the notation, defining

\[ W_m = W(s_{jm}, s_{j+1}) \]  \hspace{1cm} (43.a)

\[ W = \sum_{s_j \neq s_{jm}} W(s_j, s_{j+1}) \]  \hspace{1cm} (43.b)

Then,

\[ \frac{R_i(s_{j+1})}{R_i(s_j)_{\text{max}}} = \frac{[W_m \cdot R_i(s_j)_{\text{max}} + W \bar{R}_i(s_j)]}{[(W_m + W)R_i(s_j)_{\text{max}}]} \]  \hspace{1cm} (44)

with

\[ W \bar{R}_i(s_j) = \sum_{s_j \neq s_{jm}} R_i(s_j) \cdot W(s_j, s_{j+1}) \]  \hspace{1cm} (45)

\( \bar{R}_i(s_j) \) being the mean-value of \( R_i(s_j) \) for \( s_j \neq s_{jm} \). We have

\[ \bar{R}_i(s_j) = (1 - \eta) \cdot R_i(s_j)_{\text{max}} \]  \hspace{1cm} (46)

\( \eta > 0 \), and non-infinitesimal, following our assumption that \( R_i(s_j)/R_i(s_j)_{\text{max}} \) is finitely different from one for \( s_j \neq s_{jm} \). Hence, (44) gives

\[ \frac{R_i(s_{j+1})}{R_i(s_j)_{\text{max}}} = 1 - \eta/(1 + W_m/W) \]  \hspace{1cm} (47)

which shows that (41.a) is satisfied provided the ratio \( W_m/W \) is not infinite. Of course, the same sort of reasoning is also valid for showing that (41.b) is violated only if the weight of \( R_i(s_j)_{\text{min}} \) is infinite with respect to the sum of the other weights.

Of course, it could also be that while having a finite weight, either \( R_i(s_j)_{\text{max}} \) or \( R_i(s_j)_{\text{min}} \) dominates the sum because the number of \( s_j \)'s with \( R_i(s_j) \neq (\text{either } R_i(s_j)_{\text{max}} \text{ or } R_i(s_j)_{\text{min}}) \) is infinitesimal with respect to the number of \( s_j \)'s with \( R_i(s_j) = (\text{either } R_i(s_j)_{\text{max}} \text{ or } R_i(s_j)_{\text{min}}) \). However, in this particular case, the range of variation of
$R_i(s_{j+1})$ would be infinitesimal and its convergence to value independent of $s_{j+1}$ insured, which is what we want to demonstrate to be valid in any case.

Then, to obviate the difficulty mentioned above in the case of infinite weight-ratio $W_m/W$ for either $R_i(s_j)_{\text{max}}$ or $R_i(s_j)_{\text{min}}$ we may show that taking $s_{1j}$ and $s_{2j}$ separated by a finite number of propagators in $V_{j+1} - V_j$ as well as in $V_j$ the weight associated with $R_i(s_{2j})$ can be obtained from the weight associated with $R_i(s_{1j})$ by multiplying it by a finite factor. If $R_i(s_{1j})$ is $R_i(s_j)_{\text{max}}$ or $R_i(s_j)_{\text{min}}$ and $R_i(s_{2j})/R_i(s_{1j})$ is finitely different from one, the ratio $W_m/W$ will then stay finite and the relations (41) will be valid. This can be easily understood because the corresponding paths $P(\bar{i}, s_{j+1})$ can be obtained from each other by a “local” deformation, i.e. by substituting only a finite number of propagators.

In the following, we will consider a proof of the validity of (41) inspired from this idea, but specific to trees. Therefore, we will need to translate the language adopted for paths and multi-paths into the one adopted for spanning trees and multiple-spanning trees. When we consider a spanning tree on $G$, the part of $G$ contained in $V_j$ will in general be a spanning $m$-tree in $V_j$, i.e. a spanning tree in $V_j$ from which $m$ propagators have been removed. The vertices of the border of $V_j$ will be separated into $m$ sub-sets $\{s_j^{m_c}\}$ (because each sub-tree has to be incident with the border of $V_j$ on the border of the deformed $S_j$ in order to be connected to the rest of the spanning tree on $G$), each sub-set $\{s_j^{m_c}\}$ belonging to one of the $m$ sub-trees belonging to the spanning $m$-tree in $V_j$. There will also be sub-trees in $V_{j+1} - V_j$. Some (or all) of them will connect to a sub-tree in $V_j$ to form, as a whole, a spanning $m'$-tree in $V_{j+1}$, i.e. a $m'$-tree which is incident with all the vertices contained in $V_{j+1}$. In the same way the vertices on the border of $V_{j+1}$ will be divided into $m'$ sub-sets, each belonging to a sub-tree of a spanning $m'$-tree in $V_{j+1}$. Therefore, in analogy with the multi-paths situation, we will write

$$R_i^m(\{s_j\}) = \frac{\sum T^m(i, \{s_j\})}{\sum T^m(\bar{i}, \{s_j\})}$$

(48)

where $T^m(i, \{s_j\})$ is the weight for a spanning $m$-tree in $V_j$ going through the propagator
\( i \) and \( T^m(i, \{s_j\}) \) is the weight of a spanning \( m \)-tree in \( V_j \) not going through \( i \).

As for multi-paths the only information conveyed from the structure of \( m \)-trees in \( V_j \) to the sub-trees in \( V_{j+1} - V_j \) is the partition \( \{s_j\} \) of the vertices on the part of the border of \( V_j \) which is on the border of the deformed sphere \( S_j \). This partition is common to the \( m \)-trees of weight \( T^m(i, \{s_j\}) \) and weight \( T^m(\bar{i}, \{s_j\}) \) and is the ensemble of all sub-sets \( \{s_j^{m-}\} \). We write in analogy with (33)

\[
R^{m'}_i(\{s_{j+1}\}) = \sum_{\{s_j\}} R^m_i(\{s_j\}) W^{m'}_m(\{s_j\}, \{s_{j+1}\}) \tag{49.a}
\]

\[
W^{m'}_m(\{s_j\}, \{s_{j+1}\}) = \sum_{T^m} T^m(\bar{i}, \{s_j\}) \sum_{T^n} T^n(\{s_{j'}\}, \{s_{j+1}\}) / N_{m'} \tag{49.b}
\]

\[
N_{m'} = \sum_{T^{m'}} T^{m'}(\bar{i}, \{s_{j+1}\}) \tag{49.c}
\]

where \( T^n(\{s_{j'}\}, \{s_{j+1}\}) \) is the weight of a spanning \( n \)-tree in \( V_{j+1} - V_j \) which together with a \( m \)-tree in \( V_j \) forms a spanning \( m' \)-tree in \( V_{j+1} \). \( \{s_{j'}\} \) may not be identical to \( \{s_j\} \) because not all vertices in \( \{s_j\} \) may be incident with a \( n \)-tree in \( V_{j+1} - V_j \). \( N_{m'} \) is the sum over the weights of all spanning \( m' \)-trees in \( V_{j+1}, T^{m'}(\bar{i}, \{s_{j+1}\}) \), which do not go through \( i \) and correspond to \( \{s_{j+1}\} \).

Some partitions \( \{s_j\}_M \) correspond to an extremum value \( R^m_i(\{s_j\}_M) \) and to ensure the validity of (41) we have to ensure that the weight \( W^{m'}_{m_M}(\{s_j\}_M, \{s_{j+1}\}) \) corresponding to these partitions does not become infinite relative to the sum of weights of the other partitions. Or, alternatively if that case arises, showing that the ratio

\[
\sum_{\{s_j\}_M} W^{m'}_{m_M}(\{s_j\}_M, \{s_{j+1}\}) / \sum_{\{s_j\}} W^{m'}_m(\{s_j\}, \{s_{j+1}\}) \tag{50}
\]

stays infinite whatever \( \{s_{j+1}\} \) is also solves our problem. Then, we would already have convergence to a unique value \( R^{m'}_i \) equal to \( R^m_i(\{s_j\}_M) \). Our strategy will consist in
building a constructive procedure which allows us to calculate the ratio of two weights

\[ \frac{W^{m'_{m_2}}(\{s_j\}_2, \{s_{j+1}\})}{W^{m'_{m_1}}(\{s_j\}_1, \{s_{j+1}\})} \]  

(51)

for two different partitions \(\{s_j\}_1\) and \(\{s_j\}_2\), the partition \(\{s_{j+1}\}\) remaining the same. \(\{s_{j+1}\}\) is the partition of vertices on the part of the border of \(V_{j+1}\) on the deformed sphere \(S_{j+1}\). It consists in sub-sets \(\{s_{j+1}\}^{m'_c}\), \((m'_c = 1, \ldots, m')\) of vertices on a sub-tree of a spanning \(m'\)-tree in \(V_{j+1}\). Now, \(\{s_j\}_2\) can always be obtained from \(\{s_j\}_1\) by a series of minimal modifications, each of which consists in cutting in \(V_{j+1} - V_j\) a self-avoiding path connecting two vertices \(s_1j\) and \(s_2j\) belonging to those forming \(\{s_j\}\) instead of cutting a path staying \(V_j\) relating the same vertices \(s_1j\) and \(s_2j\). Thus, in a minimal modification a propagator in \(V_{j+1} - V_j\) on a \(m'\)-tree in \(V_{j+1}\) is replaced by a propagator in \(V_j\), obtaining another \(m'\)-tree in \(V_{j+1}\) but with the same \(\{s_{j+1}\}\) because \(s_1j\) and \(s_2j\) (and the vertices \(s_{j+1}\) connected to them) stay connected. The reverse operation is also considered as a minimal modification. We will discuss further below this minimal modification.

**B - Proof of the independence of the weight-ratio (51) on \(\{s_{j+1}\}\)**

We know that for spanning trees in \(V_{j+1}\), given two vertices \(s_1j\) and \(s_2j\) of \(\{s_j\}\), there are two possibilities:

i) they are not connected in \(V_j\),

ii) they are connected in \(V_j\) by a self-avoiding path on the spanning tree.

Let us call such a path in \(V_j\) \(P_1(s_{1j}, s_{2j})\). From such a path, spanning trees in \(V_{j+1}\) can be constructed by rooting branches on it, and then second branches rooted on the first branches, and then again branches rooted on these second branches, and so on until every vertex in \(V_{j+1}\) is incident with a branch. Let us suppose for a while that a self-avoiding path \(P_2(s_{2j}, s_{1j})\) is entirely in \(V_{j+1} - V_j\). From this path again spanning trees in \(V_{j+1}\) can be constructed in the same way as for \(P_1(s_{1j}, s_{2j})\). However, on these last spanning trees \(s_{1j}\) and \(s_{2j}\) are disconnected in \(V_j\). We remark that the succession of paths \(P_1(s_{1j}, s_{2j})\)
$P_2(s_{2j}, s_{1j})$ forms a loop $L$ which crosses the border of $V_j$ with $V_{j+1} - V_j$ at $s_{1j}$ and $s_{2j}$. Then,

i) if we cut the loop $L$ by removing one propagator $(v_1v_2)$ on $P_1(s_{1j}, s_{2j})$, $v_1$ and $v_2$ being the end-vertices of the propagator, we get a self-avoiding path on $L$, $P(v_2, v_1)$ from which spanning trees in $V_{j+1}$ can be constructed in which $s_{1j}$ and $s_{2j}$ are not connected in $V_j$. $P(v_2, v_1)$ is considered as $P_2(s_{2j}, s_{1j})$ with two branches on $P_1(s_{1j}, s_{2j})$ rooted at $s_{1j}$ and $s_{2j}$.

ii) If we cut the loop $L$ by removing one propagator $(v'_1v'_2)$ on $P_2(s_{2j}, s_{1j})$, $v'_1$ and $v'_2$ being the end-vertices of this propagator, we get a self-avoiding path $P(v'_1, v'_2)$ from which spanning trees in $V_{j+1}$ can be constructed in which $s_{1j}$ and $s_{2j}$ are connected in $V_j$. $P(v'_1, v'_2)$ is considered as $P_1(s_{1j}, s_{2j})$ with two branches on $P_2(s_{2j}, s_{1j})$ rooted at $s_{1j}$ and $s_{2j}$.

Cutting $(v_1v_2)$ instead of $(v'_1v'_2)$ on $L$ then defines a minimal modification which modifies $\{s_j\}$. Calculating the weight of the sum of spanning $m'$-trees in $V_{j+1}$ obtained by cutting $m' - 1$ propagators of the spanning trees constructed in i) we get a weight $W_{m_1}^{m'}(\{s_j\}_1, \{s_{j+1}\})$ if the cutting is made such as to preserve $\{s_j\}_1$ and $\{s_{j+1}\}$. Calculating the weight of the sum of spanning $m'$-trees in $V_{j+1}$ obtained by cutting $m' - 1$ propagators of the spanning trees constructed in ii) we get a weight $W_{m_2}^{m'}(\{s_j\}_2, \{s_{j+1}\})$ if again the cutting is made such as to preserve $\{s_j\}_2$ and $\{s_{j+1}\}$. We will see that for a given $L$ the ratio the respective contributions from i) and from ii) to the weights is easily obtained. (Here we have assumed that $\{s_j\}_1$ and $\{s_j\}_2$ are related by only one minimal modification). However, to obtain the total contribution to the weights we have, of course, to sum over all allowed $L$.

The justification for the use of the loop $L$ is that it allows us to make a systematic correspondence between paths $P_2(s_{1j}, s_{2j})$ which have some part in $V_{j+1} - V_j$ and paths $P_1(s_{1j}, s_{2j})$ which stay in $V_j$. Let us call $P_2(s_{kj}, s_{k+1j})$ a sub-path of $P_2(s_{1j}, s_{2j})$ in $V_{j+1}$ —
$V_j, s_{kj}$ and $s_{k+1}j$ being vertices of $\{s_j\}$. To $P_2(s_{kj}, s_{k+1}j)$ we associate a path $P_1(s_{kj}, s_{k+1}j)$ in $V_j$ so that $P_1(s_{kj}, s_{k+1}j)$ $P_2(s_{k+1}j, s_{kj})$ will form a loop $L_k$. Let us give an ensemble $\{L_k\}$ of these loops, each being not incident with any another one. Relating the loops $L_k$, we have paths $P(s_{kj}, s_{k'j})$ in $V_j$, with $k' \neq k+1$, which are not incident with $\{L_k\}$ except at $s_{kj}$ and $s_{k'j}$ and which are common to $P_1(s_{1j}, s_{2j})$ and $P_2(s_{1j}, s_{2j})$. It is clear then, that the sets $\{L_k\}$ allows us to make a systematic correspondence between all self-avoiding paths $P_1(s_{1j}, s_{2j})$ and all self-avoiding paths $P_2(s_{1j}, s_{2j})$:  

$$P_1(s_{1j}, s_{2j}) \xrightarrow{\{L_k\}} P_2(s_{1j}, s_{2j})$$

(52)

where $\{L_k\}$ and the paths $P_1(s_{1j}, s_{2j}), P_2(s_{1j}, s_{2j})$ are incident with $\{s_j\}$ at the same vertices.

Now, on each $L_k$ we can make a cut either in $V_j$ or in $V_{j+1} - V_j$ in order to go from a partition where $s_{kj}$ and $s_{k+1}j$ are disconnected in $V_j$ to partition where they are connected in $V_j$. However, we see that for each $k$ the cutting of a loop $L_k$ corresponds to a different minimal modification of the partition $\{s_j\}$. Moreover, each loop $L_k$ has to be cut in order to have a tree. Therefore, to $p$ loops $L_k$ in $\{L_k\}$ we have a set of $p$ cuttings. Here, requiring only one minimal modification to take place, we want to cut $P_1(s_{1j}, s_{2j})$ at most once. Suppose this cut takes place on a given loop $L_k$. Then, we will associate to $P_1(s_{1j}, s_{2j})$ the path $P_2(s_{1j}, s_{2j})$ which differs from $P_1(s_{1j}, s_{2j})$ only on $L_k$, i.e. we will have only one $L_k$ in $\{L_k\}$. Then, $P_1(s_{1j}, s_{2j})$ will consist of the succession of paths $P(s_{1j}, s_{kj})$ $P_1(s_{kj}, s_{k+1}j) P(s_{k+1}j, s_{2j})$, all in $V_j$, see fig. 2. The simplest topology appears when $s_{1j}$ is $s_{kj}$ and $s_{2j}$ is $s_{k+1}j$, $L_k$ becoming the loop $L$ described before. In the following, we will treat this simple case first because the reasoning is almost unchanged passing from $L$ to $L_k$. We now turn to the construction of spanning $m'$-trees in $V_{j+1}$, and first of spanning trees in $V_{j+1}$.

We observe that the branches in i) and ii) are exactly the same. The corresponding spanning trees in $V_{j+1}$ only differ in the way $L$ is cut.
From the spanning trees, \( m' \)-trees in \( V_{j+1} \) are obtained by cutting off \( m' - 1 \) propagators either on \( L \) or on branches. The rule to be observed is that any sub-tree generated by the cutting should be incident with at least one vertex of the border of \( V_{j+1} \) with \( G - V_{j+1} \), i.e. a vertex of \( \{ s_{j+1} \} \). Otherwise, a sub-tree would be isolated from all the others on \( G \) and the ensemble of sub-trees could not form a spanning tree on \( G \) as they should.

Let us cut branches first. For the reason given above a branch can only be cut when it is incident or connected to a vertex of \( \{ s_{j+1} \} \). It is clear that branches will be cut in exactly the same way for spanning trees in i) and ii).

Then, we come to the eventual cutting of \( L \), i.e. \( P(v_2, v_1) \) or \( P(v'_1, v'_2) \). However, remember that we want that in \( \{ s_j \} \) \( P_1(s_{1j}, s_{2j}) \) to be cut only once and in \( \{ s_j \} \) not to be cut at all. Therefore, we don’t allow \( P(v_2, v_1) \) and \( P(v'_1, v'_2) \) to be cut in \( V_j \). However, they can be cut in \( V_{j+1} - V_j \). After the branch-cutting has been completed we focus our attention on those branches which are still incident or connected with a vertex of \( \{ s_{j+1} \} \).

Let us call \( r_k \) the roots on \( L \) of these particular branches and let us order them along \( L \). It is clear that we can cut \( L \) only once between \( r_k \) and \( r_{k+1} \) because otherwise the part of \( L \) between two cut propagators, having only branches rooted on it not incident or connected to \( \{ s_{j+1} \} \), would be disconnected from all other sub-trees on \( G \), which is forbidden.

Moreover, we want the partition \( \{ s_{j+1} \} \) to be the same for \( m' \)-trees constructed from \( P(v_2, v_1) \) or \( P(v'_1, v'_2) \). Cutting \( L \) between two roots \( r_k \) and \( r_{k+1} \) will in general modify \( \{ s_{j+1} \} \) (except when \( L \) is cut only once) and therefore any path \( P(r_k, r_{k+1}) \) on \( L \) should be cut or not cut at the same time for the \( m' \)-trees generated from \( P(v_2, v_1) \) and \( P(v'_1, v'_2) \).

The exception is when only one cut is performed on \( L \), i.e. leaving \( P(v_2, v_1) \) and \( P(v'_1, v'_2) \) uncut, because all roots on \( L \) are still connected and in particular the roots \( r_k \). Then, \( (v_1, v_2) \) can be anywhere on \( P_1(s_{1j}, s_{2j}) \) and \( (v'_1, v'_2) \) anywhere on \( P_2(s_{2j}, s_{1j}) \) in \( V_{j+1} - V_j \).

Let us return to the general case when \( L \) is cut more than once. Cutting \( L \) at \( (v_1v_2) \)
or \((v'_1 v'_2)\) should not modify \(\{s_{j+1}\}\). Therefore, \((v_1 v_2)\) and \((v'_1 v'_2)\) should be on the same path \(P(r_{k_1}, r_{k_2})\) on \(L\), \(r_{k_1}\) and \(r_{k_2}\) being two consecutive roots of type \(r_k\) on \(L\). Of course, \(P(r_{k_1}, r_{k_2})\) should cross the border of \(V_j\) with \(V_{j+1} - V_j\) in order to have \((v_1 v_2)\) in \(V_j\) and \((v'_1 v'_2)\) in \(V_{j+1} - V_j\).

Let us now consider roots on \(L\) of branches in \(V_j\) which are incident with or connected to branches in \(V_j\) incident with at least one vertex of \(\{s_j\}\). Let us call such roots \(r_b\). Again, it is clear that \(L\) can only be cut once between two successive roots \(r_b\) and \(r_{b+1}\) along \(L\) for the same reason as for the roots \(r_k\). Furthermore, any cut on a path \(P(r_b, r_{b+1})\) on \(L\) gives rise to \(m'\)-tree in \(V_{j+1}\) with \(m\)-trees in \(V_j\) corresponding to the same partition \(\{s_j\}\). Of course, the propagator \((v_1 v_2)\) on \(L\) is on such a path (which itself is on \(P(r_{k_1}, r_{k_2})\)) which we will call \(P(r_{b_1}, r_{b_2})\), i.e. \(b_2 = b_1 + 1\), as well as \(k_2 = k_1 + 1\). And the propagator \((v'_1 v'_2)\) is on the intersection of \(P_2(s_{2j}, s_{1j})\) with \(P(r_{k_1}, r_{k_2})\), this intersection being \(P_2(s_{2j}, s_{1j})\) if \(P(r_{k_1}, r_{k_2})\) contains \(P_2(s_{2j}, s_{1j})\) (in which case \(r_{k_1}\) and \(r_{k_2}\) are in \(V_j\)), or a path \(P(r_{k_1}, s_{1j})\) on \(P_2(s_{2j}, s_{1j})\) if \(r_{k_1}\) is in \(V_{j+1} - V_j\), or a path \(P(s_{2j}, r_{k_2})\) on \(P_2(s_{2j}, s_{1j})\) if \(r_{k_2}\) is in \(V_{j+1} - V_j\). In any case let us call this intersection path \(P_{int}\) which of course is always in \(V_{j+1} - V_j\).

Now, \textit{given} \(L\), we can now write easily the ratio of the weights of \(m'\)-trees where \(P_{int}\) is cut to those where \(P(r_{b_1}, r_{b_2})\) is cut, this is (\(L\) cut more than once, \(m' > 1\))

\[
\sum_{\ell_2 \subset P_{int}} \tilde{\alpha}_\ell_2 / \sum_{\ell_1 \subset P(r_{b_1}, r_{b_2})} \tilde{\alpha}_\ell_1
\]

(53)

which is a remarkably simple expression. In this minimal modification \(\{s_j\}_1\) goes to \(\{s_j\}_2\) and \(\{s_{j+1}\}\) is unchanged. In the case when \(m' = 1\), \(L\) is cut only once, the roots \(r_{k_1}\) and \(r_{k_2}\) are irrelevant to determine where \(P_2(s_{2j}, s_{1j})\) should be cut in order to keep \(\{s_{j+1}\}\) unchanged because all roots on \(L\) are in any way connected, and \(P_{int}\) has to be replaced by \(P_2(s_{2j}, s_{1j})\) in the above expression.

We now want to make an important observation, i.e. that in the ratio (53) the only object which may be sensitive to \(\{s_{j+1}\}\), \(\{s_j\}\) being fixed, is \(P_{int}\) through the position of
In what follows, we are going to show that in spite of that, we can constrain the building of the successive volumes $V_j, V_{j+1}, \ldots$ in such a way that the ratio of weights (51) does not depend on a change of $\{s_{j+1}\}$. Then, if it is finite or infinite for one given $\{s_{j+1}\}$ it stays so for any $\{s_{j+1}\}$. As noted before in the discussion of the weight-ratio (50) this solves immediately our convergence problem for $R^m_i(\{s_j\})$.

**Constraint on the construction of the $V_j$’s**

Any propagator going out of $V_j$ is relating a vertex of the border of the deformed sphere $S_j$ to a vertex of the border of the deformed sphere $S_{j+1}$ if the vertices at the ends of this propagator are one in $V_j$ and the other one in $V_{j+1} - V_j$. □

The later provision takes into account the possibility for the borders of the deformed $S_j$ and $S_{j+1}$ to coincide on some domain in which case a propagator going out of $S_j$ would also go out of $S_{j+1}$. This constraint is easy to satisfy because we only need to make the radius of $S_{j+1}$ sufficiently close to that of $S_j$ in order to obey it. It has the following consequence: if a loop $\mathcal{L}$ enters $V_{j+1} - V_j$ at $s_{2j}$ and reenters $V_j$ at $s_{1j}$, then the part of $\mathcal{L}$ in $V_{j+1} - V_j$ is a path

$$(s_{2j}, s_{2j+1}) P(s_{2j+1}, s_{1j+1})(s_{2j+1}, s_{1j})$$

(54)

where $(s_{2j}, s_{2j+1})$ and $(s_{1j+1}, s_{1j})$ are two propagators relating vertices on the border of $V_j$ and $V_{j+1}$. Furthermore any vertex of $P(s_{2j+1}, s_{1j+1})$ not on the border of $V_{j+1}$ is related to vertices of $V_j$ through paths on $P(s_{2j+1}, s_{1j+1})$ going to vertices on the border of $V_{j+1}$, see fig. 3.

An immediate consequence of the structure of $\mathcal{L}$ in $V_{j+1} - V_j$ as shown in (54) is that $s_{1j+1}$ is $r_{k_1}$ or $s_{2j+1}$ is $r_{k_2}$ because being in $\{s_{j+1}\}$ and on $\mathcal{L}$ they are roots of branches $r_{k_1}$ or $r_{k_2}$ when $r_{k_1}$ or $r_{k_2}$ are in $V_{j+1} - V_j$. $P(r_{b_1}, r_{b_2})$ being in $V_j$ is insensitive to $\{s_{j+1}\}$, because once $\{s_j\}$ is fixed, $r_{b_1}$ and $r_{b_2}$ cannot depend on the structure of the $m'$-trees outside $V_j$. Or, said otherwise, the structure of $m$-trees in $V_j$ only depend on $\{s_j\}$. 

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The later provision takes into account the possibility for the borders of the deformed $S_j$ and $S_{j+1}$ to coincide on some domain in which case a propagator going out of $S_j$ would also go out of $S_{j+1}$. This constraint is easy to satisfy because we only need to make the radius of $S_{j+1}$ sufficiently close to that of $S_j$ in order to obey it. It has the following consequence: if a loop $\mathcal{L}$ enters $V_{j+1} - V_j$ at $s_{2j}$ and reenters $V_j$ at $s_{1j}$, then the part of $\mathcal{L}$ in $V_{j+1} - V_j$ is a path

$$(s_{2j}, s_{2j+1}) P(s_{2j+1}, s_{1j+1})(s_{2j+1}, s_{1j})$$

(54)

where $(s_{2j}, s_{2j+1})$ and $(s_{1j+1}, s_{1j})$ are two propagators relating vertices on the border of $V_j$ and $V_{j+1}$. Furthermore any vertex of $P(s_{2j+1}, s_{1j+1})$ not on the border of $V_{j+1}$ is related to vertices of $V_j$ through paths on $P(s_{2j+1}, s_{1j+1})$ going to vertices on the border of $V_{j+1}$, see fig. 3.
incident with \( \{s_{j+1}\} \), these branches being restricted to one vertex. Then, \( P_{\text{int}} \) is simply the propagator \((s_{1j+1} s_{1j})\) or the propagator \((s_{2j} s_{2j+1})\) depending on which part of \( \mathcal{L} \) we choose \( P(r_{k_1}, r_{k_2}) \) to be. However, in any case, \( P_{\text{int}} \) depends only on \( \mathcal{L} \) and no more on \( \{s_{j+1}\} \), because the vertices \( s_{1j+1} \) and \( s_{2j+1} \) will not move on \( \mathcal{L} \) as we change \( \{s_{j+1}\} \). Then, the ratio (53) will depend on \( \mathcal{L} \) and not \( \{s_{j+1}\} \) and the ratio of the weights for two partitions \( \{s\}_1 \) and \( \{s\}_2 \) related by a minimal modification will be, summing over all \( \mathcal{L} \) going out of \( V_j \) at \( s_{2j} \) and reentering \( V_j \) at \( s_{1j} \). (\( \mathcal{L} \) cut more than once, \( m' > 1 \)),

\[
W_{m_1-1}^{m'}(\{s_j\}_2, \{s_{j+1}\})/W_{m_1}^{m'}(\{s_j\}_1, \{s_{j+1}\}) = \bar{\alpha}_{P_{\text{int}}} \sum_{T^{m_1-1}} T^{m_1-1}(i, \{s_j\}_2)/\sum_{T^{m_1-1}} \left( \sum_{\ell_i \subset P(r_{b_1}, r_{b_2})} \bar{\alpha}_{t_i} \right) T^{m_1-1}(i, \{s_j\}_2) \tag{55}
\]

where \( T^{m_1-1}(i, \{s_j\}_2) \) is for the weight of spanning \((m_1-1)\)-trees in \( V_j \) with partition \( \{s_j\}_2 \) and \( \bar{\alpha}_{P_{\text{int}}} \) is for the \( \bar{\alpha}_t \) of the propagator \((s_{1j+1} s_{1j})\) or \((s_{2j} s_{2j+1})\). The sum over the spanning \((m_1-1)\)-trees is provided by cutting in all possible ways compatible with \( \{s_j\}_2 \) \( m_1-2 \) propagators from all spanning trees in \( V_j \), not going through the propagator \( i \), and obtained from all possible paths \( P_2(s_{2j}, s_{1j}) \). Looking at the weight structure in (49.b) we see that the contribution to weights coming from sub-trees in \( V_j - V_j \), \( \sum_{T^n} T^n(\{s'_j\}_1, \{s_{j+1}\}) \), being the same ones for \( W_{m_1-1}^{m'}(\{s_j\}_2, \{s_{j+1}\}) \) and \( W_{m_1}^{m'}(\{s_j\}_1, \{s_{j+1}\}) \), and being factorized, cancels out in (55). The main feature of the expression on the right-hand side of (55) is that it does not depend on \( \{s_{j+1}\} \), which is what we were looking for.

We now have to establish the same property for the contribution coming from spanning \( m' \)-trees in \( V_{j+1} \) where \( \mathcal{L} \) is only cut once. For this case we need to separate two classes of \( m' \)-trees

a) those \( m' \)-trees where for \( \{s_j\}_2 \), \( \mathcal{L} \) is cut on \( P_{\text{int}} \), i.e. on the propagator \((s_{2j} s_{2j+1})\) or \((s_{1j+1} s_{1j})\).

b) those \( m' \)-trees where for \( \{s_j\}_2 \), \( \mathcal{L} \) is cut on the complement of \( P_{\text{int}} \) on \( P_2(s_{2j}, s_{1j}) \), which we will call \( P_{\text{int}}^{\text{comp}} \), but uncut on \( P_{\text{int}} \).
For the $m'$-trees in a) the reasoning is the same as for the $m'$-trees where $L$ is cut more than once and the result (55) is valid for them too. For the $m'$-trees in b), changing \{s_{j+1}\}, *we cannot obtain* $m'$-trees in which $L$ is cut more than once, because $P(r_k, r_{k_2})$ would stay uncut for \{s_j\}_2 - corresponding $m'$-trees, which is forbidden (i.e., for $L$ cut more than once, $P(r_{k_1}, r_{k_2})$ has to be cut for \{s_j\}_1 in $V_j$ and for \{s_j\}_2 in $V_{j+1} - V_j$, i.e. on $P_{\text{int}}$, in order not to change \{s_{j+1}\} passing from \{s_j\}_1 to \{s_j\}_2). We remind the reader that when $L$ is cut only once all roots $r_k$ on $L$ are connected. Then, for the $m'$-trees in b) changing \{s_{j+1}\} always keeps all roots $r_k$ on $L$ connected because $L$ stays cut only once. Let us then take *all the loops* $L$ with the same $P_{\text{int}}^{\text{comp}}$, for these the contribution of the weights will be such that ($L$ cut once, $m' \geq 1$)

$$W_{m_1-1}^{m'}(\{s_j\}_2, \{s_{j+1}\})/W_{m_1}^{m'}(\{s_j\}_1, \{s_{j+1}\}) =\left(\sum_{\ell_2 \subset P_{\text{int}}^{\text{comp}}} \bar{\alpha}_{\ell_2}\right) \sum_{T}^{m_1-1} T^{-1}(\bar{i}, \{s_j\}_2) / \sum_{T}^{m_1-1} \left(\sum_{\ell_1 \subset P(r_{b_1}, r_{b_2})} \bar{\alpha}_{\ell_1}\right) T^{-1}(\bar{i}, \{s_j\}_2) \quad (56)$$

which, again, is independent of \{s_{j+1}\}.

Now, because each contribution to the weight $W_{m_1}^{m'}(\{s_j\}_1, \{s_{j+1}\})$ is multiplied by a factor independent of \{s_{j+1}\} when the corresponding contribution to $W_{m_1-1}^{m'}(\{s_j\}_2, \{s_{j+1}\})$ is taken, we have the following theorem:

**Theorem 2a**

Let us consider a minimal modification of \{s_j\}, \{s_j\}_1 \to \{s_j\}_2 where \{s_{j+1}\} remains unchanged. In \{s_j\}_2, s_{1j} and s_{2j}, vertices of \{s_j\} are connected by a self-avoiding path $P_1(s_{1j}, s_{2j})$ in $V_j$. In \{s_j\}_1, $P_1(s_{1j}, s_{2j})$ is cut once. The $m'$-trees in $V_{j+1}$ corresponding to \{s_j\}_2 are obtained from $P_1(s_{1j}, s_{2j})$. The $m'$-trees in $V_{j+1}$ corresponding to \{s_j\}_1 are obtained from a self-avoiding path $P_2(s_{2j}, s_{1j})$ with all propagators in $V_{j+1} - V_j$, its only vertices on the border of $V_{j+1} - V_j$ with $V_j$ being $s_{1j}$ and $s_{2j}$.

Then, the ratio of the sum of weights of $m'$-trees corresponding to \{s_j\}_2 to the sum of the weights of $m'$-trees corresponding to \{s_j\}_1 is independent of \{s_{j+1}\}. □
We have examined so far the simplest case where the loop $L_k$ was taken as the loop $L$ going out of $V_j$ at $s_{2j}$ and in $V_j$ at $s_{1j}$. In general, as discussed earlier, a path $P_2(s_{1j}, s_{2j})$ may have some parts $P_2(s_{kj}, s_{k+1j})$ in $V_{j+1} - V_j$. Confining ourselves to the minimal modification of $\{s_j\}$, only one such $P_2(s_{kj}, s_{k+1j})$ or one $L_k$ is relevant.

We can repeat the reasoning followed with the loop $L$ for the loop $L_k$, the only change being that two self-avoiding paths in $V_j$, $P(s_{1j}, s_{kj})$ and $P(s_{k+1j}, s_{2j})$ will be rooted at $s_{kj}$ and $s_{k+1j}$ respectively on $L_k$. Then, we have to divide the paths $P_1(s_{1j}, s_{2j})$ into two classes in order to avoid a double-counting:

a) those which are incident with a vertex of $\{s_j\}$ other than $s_{1j}$ and $s_{2j}$ only once

b) those which are incident with at least two vertices $s_{kj}$ and $s_{k'j}$ of $\{s_j\}$.

The paths a) will be associated to the loop $L$. The paths b) will be associated to a loop $L_k$ with $s_{k'j}$ being then noted $s_{k+1j}$. Of course, in this case $P_1(s_{1j}, s_{2j})$ can be incident with other vertices of $\{s_j\}$ as well. We only need to exhaust all pairs of vertices $s_{kj}$, $s_{k'j}$ in order to form all possible loops $L_k$. In this way, all different paths $P_1(s_{1j}, s_{2j})$ are taken into account only once. Theorem 2a applied to each pair $s_{kj}$, $s_{k'j}$ replacing $s_{1j}$ and $s_{2j}$ then provides the independence of the weight ratio (51) on $\{s_{j+1}\}$.

The only thing which is left to prove is that, indeed, any partition $\{s_j\}_2$ can be obtained from another one $\{s_j\}_1$ by a series of minimal modifications. So let us consider the respective situation of two vertices $s_{1j}$ and $s_{2j}$ in $\{s_j\}_1$ and in $\{s_j\}_2$. Suppose that they are unconnected in $\{s_j\}_1$ and connected in $\{s_j\}_2$. Then, it is easy to see that a minimal modification will allow to disconnect them, passing from $\{s_j\}_2$ to $\{s_j\}_3$, leaving weight-ratios corresponding to those in (51) independent of $\{s_{j+1}\}$. Then, we will consider the respective situation of any two other $s_j$’s and repeat the operation until we obtain $\{s_j\}_1$, having always weight-ratios insensitive to $\{s_{j+1}\}$.

Let us make also a remark about the topology of $G$ in $V_{j+1}$. $G$ can be disconnected
into several pieces in $V_{j+1}$, although $G$ itself is taken to be connected and even 1-line and 1-vertex irreducible. When $G$ is multiply connected in $V_{j+1}$ we consider each connected piece separately for the construction of spanning trees on them and the eventual cutting of propagators. The resulting weight for $m'$-trees will simply be the product of the weights of all connected pieces of $G$ in $V_{j+1}$. Of course, in this case $m' > 1$ but the above reasoning is essentially unchanged. We then are able to write the following theorem:

**Theorem 2b**

For any two different partition $\{s_j\}_1$ and $\{s_j\}_2$ of vertices $s_j$ on the border of $V_j$ with $V_{j+1} - V_j$ the ratio of weights (51) is independent of $\{s_{j+1}\}$ if we impose the constraint on the construction of the $V_j$'s described earlier and which can always be satisfied. It follows that the ratio (50)

$$\sum_{\{s_j\}_M} W_{m,M}^{m'} (\{s_j\}_M, \{s_{j+1}\}) / \sum_{\{s_j\}} W_{m}^{m'} (\{s_j\}, \{s_{j+1}\})$$

is also independent of $\{s_{j+1}\}$. Therefore, when this ratio is infinite it is so for all $\{s_{j+1}\}$ and $R_i^{m'}(\{s_{j+1}\})$ takes a unique value $R_i^{m}(\{s_{j+1}\}_M)$. When this ratio is finite, it also stays finite for all $\{s_{j+1}\}$ and therefore the convergence condition (41) is satisfied. \(\square\)

In order to have $R_i^{m}(\{s_j\})$ to converge as $j \to \infty$, we also need to have this ratio finite for some finite value of $j$. This condition is naturally satisfied if we impose the constraint that $V_1$ should only contain a finite number of propagators. This condition, of course, can always be satisfied by choosing a sufficiently small radius for the sphere $S_1$.

**C - Extension of the convergence proof to trees**

We now want to extend the inequality (36) of theorem 1 to the spanning trees in $V_j$ and $V_{j+1}$. For that purpose let us define a (spanning) multiple tree (or $m'$-tree) $T_i^{m'}(i, \{s_{j+1}\})$ contained in $V_{j+1}$, one component of which is going through $i$. $\{s_{j+1}\}$ stands for a partition of all border-vertices of $V_{j+1}$ with $G - V_{j+1}$ with which $T_i^{m'}(i, \{s_{j+1}\})$ is incident. As for
paths, $T^{m'}(i, \{s_{j+1}\})$ has the same properties as $T^{m'}(i, \{s_{j+1}\})$, except that it does not go through $i$ but is still incident with $v_i$ through one to its component-tree. In the following, in order to have a simpler notation we will imply that a summation over all $T$'s of the kind is made whenever the symbol $T$ appears. Then, we can write, $\{s_j\}$ being some partition of vertices of the border of $V_j$ with $V_{j+1} - V_j$ and $\{s_{j+1}\}$ a corresponding partition for $V_{j+1}$, $(T^m(i, \{s_j\}))$ is a $m$-tree contained in $V_j$, $T(\{s_j^i\})$ is a tree in $V_{j+1} - V_j$ with no vertex on the border of $V_{j+1}$ and incident with the border of $V_j$ at $\{s_j^i\}$, $T(\{s_j^u, \{s_{j+1}\}\})$ is a tree in $V_{j+1} - V_j$ incident with the borders of $V_j$ and $V_{j+1}$ at $\{s_j^u\}$ and $\{s_{j+1}\}$ respectively, and $T(\{s_j^{v+1}\})$ is a tree in $V_{j+1} - V_j$ not incident with the border of $V_j$ but incident with the border of $V_{j+1}$ at $\{s_j^{v+1}\})$.

$$T^{m'}(i, \{s_{j+1}\}) = \sum_{m=1}^{M_j} \sum_{\{s_j\}} T^m(i, \{s_j\}).$$

$$\sum_{\{\{s_j^i\}\}} \prod_{t=1}^{x} T(\{s_j^i\}) \sum_{\{\{s_j^u, \{s_{j+1}\}\}\}} \prod_{u=1}^{y} T(\{s_j^u\}, \{s_{j+1}\}) \sum_{\{\{s_j^{v+1}\}\}} \prod_{v=1}^{z} T(\{s_j^{v+1}\})$$

$\{\{s_j^u\}, \{s_j^{v+1}\}\}$ is the ensemble of border-vertices of the $y$ trees $T(\{s_j^u\}, \{s_{j+1}\})$, $\{\{s^i\}\}$ is the ensemble of the border-vertices of the $x$ trees $T(\{s_j^i\})$ and $\{\{s_j^{v+1}\}\}$ is the ensemble of border-vertices of the $z$ trees $T(\{s_j^{v+1}\})$ for $x, y, z \geq 1$ and with

$$y + z \geq m' \geq z$$

When $x = 0$, $y = 0$ or $z = 0$ the corresponding product in (57) is equal to one (no tree). The relation (51) can be explained by saying that $y + z$ is the maximum number of components of $T^{m'}(i, \{s_{j+1}\})$ and $z$ the minimum number of its components. A tree $T(\{s_j^i\})$ in $V_{j+1} - V_j$ should be connected to at least one tree in $V_j$, giving

$$\forall t, \quad \{s_j\} \cap \{s_j^t\} \neq \phi$$

If $\{s_j^{m_c}\}$, $(m_c = 1, \cdots, m)$, denotes the set of border-vertices of $V_j$ of one component-tree of $T^m(i, \{s_j\})$, i.e. if $\{s_j\} = \{\{s_j^{m_c}\}\}$, then, avoiding loops,

$$\{s_j^i\} \cap \{s_j^{m_c}\} = \text{at most one } s_j$$  \hspace{1cm} (60.a)
\{s^u_j \} \cap \{s^{mc}_j \} = \text{at most one } s_j \quad (60.b)

Any component-tree of $T^m(i, \{s_j\})$ should be connected to at least one tree in $V_{j+1} - V_j$ giving

$$\forall m, \quad \{\{s^l_j\} \cup \{s^u_j\}\} \cap \{s^{mc}_j\} \neq \emptyset \quad (60.c)$$

The relations (60) insure that a component-tree of $T^m(i, \{s_j\})$ in $V_j$ should be incident with at least one component tree in $V_{j+1} - V_j$ through at most one vertex. Indeed, (59) and (60) fix the topology of contact between the component-trees in $V_j$ and $V_{j+1} - V_j$ in order to leave no component-tree isolated and in a way that avoids the formation of any loop.

Let us denote $\{\{s^l_j\} \cup \{s^u_j\}\}$ by $\{s^l_j\}$ which will be the ensemble of the vertices on the border of $V_j$ of the trees in $V_{j+1} - V_j$. Then, it is clear that once $\{s_j\}$ and $\{s_j'\}$ have been fixed, the summation over trees in $V_j$ and $V_{j+1} - V_j$ is factorizable because no interaction occurs between the trees in $V_j$ and those in $V_{j+1} - V_j$ (apart from their contact at the common border of $V_j$ and $V_{j+1} - V_j$). Note even that once $\{s_j\}$ is fixed the summation over all $T^m(i, \{s_j\})$ (or over all $T^m(\bar{i}, \{s_j\})$) does not depend on $\{s_j'\}$. Consequently, the sum over all $n$-trees in $V_{j+1} - V_j$, including the sum over $\{s_j'\}$, can be factorized out.

Again, in a way analogous to that of paths let us define

$$R^m_i(\{s_j\}) = \sum_{T^m} T^m(i, \{s_j\}) / \sum_{T^m} T^m(\bar{i}, \{s_j\}) \quad (61)$$

for which we demonstrate as for paths the following

**Theorem 3**

$$R^m_i(\{s_j\})_{min} < R^m_i(\{s_{j+1}\}) < R^m_i(\{s_j\})_{max} \quad (62)$$

where $R^m_i(\{s_j\})_{min}$ and $R^m_i(\{s_j\})_{max}$ are respectively the minimum and the maximum
value of $R^m_i(\{s_j\})$ for all $m$’s or $R^{m'}_i(\{s_{j+1}\})$ has converged to either $R^m_i(\{s_j\})_{\min}$ or $R^m_i(\{s_j\})_{\max}$.

**Proof**

It is clear, using (61) in (57) that $R^{m'}_i(\{s_{j+1}\})$ is a mean-value of $R^m_i(\{s_j\})$ considered as a function of $m$, $\{s_j\}$ and $\{\{s_j\}, \{|s'_j\}|}$. However, due to the factorization property mentioned above (once $\{s_j\}$ is fixed, $\{|s'_j\}|$ does not have an influence over the $m$-trees in $V_j$), the functional dependence of $R^m_i(\{s_j\})$ is indeed restricted to $m$ and $\{s_j\}$. Moreover, theorem 2b either excludes the limiting cases where the inequalities become equalities or makes $R^{m'}_i(\{s_{j+1}\})$ equal to $R^m_i(\{s_j\})_{\min}$ or $R^m_i(\{s_j\})_{\max}$ for any $\{s_{j+1}\}$. It therefore follows that either (62) is true or $R^{m'}_i(\{s_{j+1}\})$ has converged. □

**Remark**

We note that $R^m_i(\{s_j\})$ (and $T^m(i, \{s_j\})$) is a function of a partition $\{s_j\}$ which covers any part of the border of $V_j$ on the surface of the deformed $S_j$ independently of the partition $\{s_{j+1}\}$ of $T^{m'}(i, \{s_{j+1}\})$. Therefore, the sum $\sum_{\{s_j\}}$ in (57) is a sum over the whole part of border of $V_j$ on the deformed surface $S_j$. This is in contrast with the corresponding sums $\sum_{s_j^0}$ or $\sum_{s_j'}$ in (34) for paths which may cover only part of the border of $V_j$ on the deformed surface $S_j$, depending on $s_{j+1}$ in $P(i, s_{j+1})$. Therefore, in the case of trees, the topology of $V_{j+1} - V_j$ does not intervene to possibly limit the range of $\{s_j\}$. Hence, we have a unique value for $R^m_i(\{s_j\})$ and this solves problem b) of the preceding section. The repeated use of (62) will make all $R^m_i(\{s_j\})$ converge towards the same value $R^\infty_i$.

**5. The factorization of trees on G**

We now proceed to the proof of the relation (24), the crucial factorization property of spanning trees on $G$.

Let us consider a volume $V_j$ with $j \to \infty$, and a partition of its border-vertices $\{s_j\}$ and
let us denote by \( R_i^\infty \) the common value to which all \( R_i^m(\{s_j\}) \) tend as \( j \to \infty \). Recalling (16) and (17) we have, \( T^n \) being an \( n \)-tree on \( G-V_j \), (a sum over \( m \) is implied in the sum over all \( \{s_j\} \) and a sum over \( n \) is implied in the sum over all \( \{s'_j\} \) compatible with \( \{s_j\} \))

\[
\bar{a}_i = \Delta(\bar{\alpha}) \sum_{T \supseteq i} \prod_{l \in T} \bar{\alpha}_l^{-1}
= \Delta(\bar{\alpha}) \sum_{\{s_j\},T^m} T^m(i,\{s_j\}) \sum_{\{s'_j\},T^n} T^n(\{s'_j\})
\]  

(63.a)

\[
\bar{b}_i = \bar{\alpha}_i^{-1}\Delta(\bar{\alpha}) \sum_{T \supseteq i} \prod_{l \in T} \bar{\alpha}_l^{-1}
= \bar{\alpha}_i^{-1}\Delta(\bar{\alpha}) \sum_{\{s_j\},T^m} T^m(\bar{i}\{s_j\}) \sum_{\{s'_j\},T^n} T^n(\{s'_j\})
\]  

(63.b)

\( \{s'_j\} \) is a subset of the vertices of \( V_j \) belonging to \( T^n(\{s'_j\}) \), and \( T^m \cup T^n \) form a spanning tree of \( G \). Notice that the sets \( \{s'_j\} \) and \( \{s_j\} \) are in general different, a branch of \( T^m \) can end at one \( s_j \) without being incident at that \( s_j \) with one branch of \( T^n \). For any given \( m \), \( \{s_j\}, T^m(i,\{s_j\}) \) can be replaced by \( R_i^\infty T^m(\bar{i},\{s_j\}) \) and we therefore get

\[
\bar{a}_i/\bar{b}_i = R_i^\infty \bar{\alpha}_i 
\]

(64)

Remember that by a given \( \{s_j\} \) we mean a given \( \{\{s_j^{m_c}\}\} \) where \( \{s_j^{m_c}\} \) is a set of \( s_j \)'s belonging to the same component-tree of \( T^m \). In the same way \( \{s'_j\} \) means \( \{\{s'_j^{n_c}\}\} \) where \( \{s'_j^{n_c}\} \) belongs to one component-tree of \( T_n \).

Looking now at (22) and (23) we can write, using the same notations,

\[
\bar{d}_{i,k} = \sum_{\{s_j\},T^m} s_{C_k} \nu^{-1}(C_k) T^m(i,\{s_j\}) \sum_{\{s'_j\},T^n_k} T^n_k(\{s'_j\})
\]

(65.a)

\[
\bar{e}_{i,k} = \bar{\alpha}_i^{-1} \sum_{\{s_j\},T^m} s_{C_k} \nu^{-1}(C_k^i) T^m(\bar{i},\{s_j\}) \sum_{\{s'_j\},T^n_k} T^n_k(\{s'_j\})
\]

(65.b)
where $T^n_k$ is a $n$-tree belonging to $G - V_j$ and going through the propagator $k$ which is assumed to be outside $V_j$.

We see that, compared to (63), the expressions (65) have exactly the same structure except for the factors $s_{C_k} \nu^{-1}(C_k)$ and $s_{C_k} \nu^{-1}(C^i_k)$. We now want to demonstrate that, indeed,

$$[s_{C_k} \nu^{-1}(C_k)]/[s_{C^i_k} \nu^{-1}(C^i_k)] = 1 + \varepsilon$$

(66)

where $\varepsilon$ is infinitesimal. Let us recall that $\nu(C_k)$ counts the number of propagators on the surface $S_k(C_k)$ cutting $G$ into two disjoint pieces $G_1(C_k)$ and $G_2(C_k)$, the same being true for $\nu(C^i_k)$ replacing $C_k$ by $C^i_k$.

Let us take $j = 1$ for the expressions (65) in order to have a finite number of propagator in $V_1$. First, it is obvious that if $S_k$ does not cut through $V_1$ it will cut only the $T^n$'s which are the same $n$-trees in (65.a) and (65.b) for a given $\{s_j\}$ and we will have $\varepsilon = 0$ in (66).

Now, if $j$ grows, $S_k$ will remain identical in (65.a) and (65.b) because $C_k$ and $C^i_k$ do not depend on $j$, i.e. on the decomposition of a spanning tree in $G$ into a $m$-tree in $V_j$ and a $n$-tree in $G - V_j$. So, $\varepsilon$ will remain equal to zero as $j \to \infty$.

The non-trivial case occurs when $S_k$ cuts through $V_1$. Then, it could be that when $j \to \infty$ the number of propagators cut by $S_k$ in $V_j$ remains finite. This could arise when there are domains in the deformed sphere $S_j$ which are empty of propagators and of a size large enough so that when $S_k$ cuts through them it will contain only a finite number of propagators in $V_j$. An example of this situation is provided when $G$ has a topology such that it consists of infinite ladders (which may join and separate themselves creating a sort of effective field theory of Reggeons).

To study the situation where $S_k$ cuts through $V_1$ let us consider, for a given spanning tree $\mathcal{T}$ on $G$, the sum

$$\Sigma_{\mathcal{T}}(G) = \sum_{k \in \mathcal{T}} \bar{\alpha}_k s_{C_k} \nu^{-1}(C_k)$$

(67)
and a similar sum for the part of $\mathcal{T}$ in $V_1$, $\mathcal{T}_1$

\[
\Sigma_{\mathcal{T}}(V_1) = \sum_{k \in \mathcal{T}_1} \bar{\alpha}_k \ s_{C_k} \nu^{-1}(C_k)
= \langle \bar{\alpha}_k \ s_{C_k} \rangle_{\mathcal{T}_1} \sum_{k \in \mathcal{T}_1} \nu^{-1}(C_k) \quad (68)
\]

where $\langle \bar{\alpha}_k \ s_{C_k} \rangle_{\mathcal{T}_1}$ is the mean-value of $\bar{\alpha}_k \ s_{C_k}$ for all propagators $k$ in $V_1$ belonging to $\mathcal{T}$. The maximum value of $\Sigma_{\mathcal{T}}(V_1)$ is obtained when $\nu(C_k)$ for all $k$’s belonging to $\mathcal{T}_1$ is constant, meaning that the corresponding $S_k$’s each contain a finite number of propagators. Then, $\Sigma_{\mathcal{T}}(V_1)$ is equal to $\langle \bar{\alpha}_k \ s_{C_k} \rangle_{\mathcal{T}_1}$ multiplied by some constant.

We can express $\Sigma_{\mathcal{T}}(G)$ in the same way, writing

\[
\Sigma_{\mathcal{T}}(G) = \langle \bar{\alpha}_k \ s_{C_k} \rangle_{\mathcal{T}} \sum_{k \in \mathcal{T}} \nu^{-1}(C_k) \quad (69)
\]

where $\langle \bar{\alpha}_k \ s_{C_k} \rangle_{\mathcal{T}}$ is the mean-value of $\bar{\alpha}_k \ s_{C_k}$ over $\mathcal{T}$. Now, we can calculate a lower bound for $\sum_{k \in \mathcal{T}} \nu^{-1}(C_k)$.

This comes from the fact that every propagator in $S_k$ has to be incident with a vertex of the sub-trees in $G_1(C_k)$ and $G_2(C_k)$, the parts of $G$ separated by $S_k$. If a branch consisting of $N$ vertices (and $N - 1$ propagators) is separated in a $\phi^n$ field theory, the number of propagators cut is $(n-2)N + 1$ which represents the maximum number of propagators cut for a tree with $N$ vertices (and of course $N - 1$ propagators). The number of propagators in $G_1(C_k)$, for example can be taken to vary from zero to $I - L$. Thus, the following inequality follows

\[
\sum_{k \in \mathcal{T}} \nu^{-1}(C_k) > \frac{1}{n - 2} \int_1^{I - L + 1} dN/(N + 1) = (1/(n - 2)) \log(I - L + 1) \quad (70)
\]

with $(1/(n - 2)) \log(I - L + 1)$ tending to infinity as $I \to \infty$. Therefore, if the ratio

\[
\langle \bar{\alpha}_k \ s_{C_k} \rangle_{\mathcal{T}_1} / \langle \bar{\alpha}_k \ s_{C_k} \rangle_{\mathcal{T}} \quad (71)
\]
stays finite, the ratio

\[ \frac{\sum_{\mathcal{T}}(G)}{\sum_{\mathcal{T}}(V_1)} \quad (72) \]

is infinite as \( I \to \infty \) and the contribution to (65.c) and (65.b) coming from \( S_k \)'s cutting through \( V_1 \) can be neglected. It follows, then, that (66) will be true.

Let us now remark that (71) may be infinite in the case where some internal lines incident with vertices in \( V_1 \) carry a momentum infinite with respect to the momenta of external lines incident with \( G - V_1 \). Then, \( S_k \) may cut \( V_1 \) in such a way as to separate such lines, giving an infinite \( s_{C_k} \), while a cancellation occurs between infinite momenta when \( S_k \) does not cut through \( V_1 \). Note however that if, although infinite, (71) is equal to

\[ \varepsilon_1 \log(I - L) \quad (73) \]

with \( \varepsilon_1 \to 0 \) as \( I \to \infty \), the conclusion reached above, i.e. that \( \sum_{\mathcal{T}}(V_1) \) can be neglected in front of \( \sum_{\mathcal{T}}(G) \), is still valid.

Finally, considering the ratio

\[ \frac{\sum_{\mathcal{T}}(G)}{\sum_{\mathcal{T}}(V_j)} \quad (74) \]

where \( \sum_{\mathcal{T}}(V_j) \) is the sum of \( \bar{\alpha}_k \ s_{C_k} \nu^{-1}(C_k) \) for \( k \)'s belonging to the part of \( \mathcal{T} \) in \( V_j \), we see that we have the inequality

\[ \frac{\sum_{\mathcal{T}}(G)}{\sum_{\mathcal{T}}(V_j)} > \left[ (n - 2)^{-1} \log(I - L)/(I - L)_V \right] < \bar{\alpha}_k \ s_{C_k} >_{\mathcal{T}}/ < \bar{\alpha}_k \ s_{C_k} >_{\mathcal{T}_j} \quad (75) \]

if \( (I - L)_V \) is the number of lines a spanning tree in \( V_j \) (a \( m \)-tree in \( V_j \) has even less lines than \( (I - L)_V \)) and \( < \bar{\alpha}_k \ s_{C_k} >_{\mathcal{T}} \) the mean-value of \( \bar{\alpha}_k \ s_{C_k} \) over \( \mathcal{T}_j \), the \( m \)-tree part of \( \mathcal{T} \) on \( V_j \).

Provided the ratio on the right of (75) is infinite we can neglect in (20) and (21) the sum over \( k, k \) belonging to \( V_j \), even as \( (I - L)_V \) goes to infinity, as was claimed in section 3.
Then, $T^m(i, \{s_j\})$ can be replaced by $R_i^\infty T^m(\bar{i}, \{s_j\})$ in (65.a) and (65.b) as in (63.a) and (63.b) with the result

$$\bar{d}_{i,k}/\bar{e}_{i,k} = R_i^\infty \bar{\alpha}_0 \quad (76)$$

which was sought after. This entails that (24) and thereby (23) are verified. As said in section 2 this, in turn, makes $Q_G(P_v, \{\bar{\alpha}_i\})$ insensitive to the replacement $\bar{\alpha}_i \to \infty$ and a unique $\bar{\alpha}$ can be used to evaluate $F_G$ in a super-renormalizable scalar field theory.

### 6. Conclusion

Our initial aim was to put the Gaussian representation for propagators on a firm footing. Using the well-known $\alpha$-representation we are able to prove that the parameter $\alpha$ which measures the inverse of the variance of that Gaussian can be taken everywhere equal to some unique $\bar{\alpha} = O(1/I)$ where $I$ is the number of internal lines of a Feynman graph $G$. We did this for a super-renormalizable scalar field theory, although we expect the same result to hold for renormalizable theories as well. But, what is more interesting even, is that we were obliged during the derivation to prove a factorization property of spanning trees on $G$, i.e., we can sum over all graphs in a volume $V_j$, and if $j \to \infty$, the structure of the trees outside $V_j$ is independent of the structure of the same trees in $V_1$, a subpart of $V_j$. We can however imagine that $V_j$ itself is an infinitesimal volume relative to the whole volume of $G$. Then, we can interpret our factorization of trees as the factorization of local sums defined on trees. In other words, *trees on a Feynman graph can be used to define a functional integral.* In fact, we assumed such a functional property in our first attempt [7] to derive the relation (13).
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Figure Captions

**Fig. 1** We illustrate the case (see eq. (38)) where \( m' = 3, m = 3 \), i.e. there are 3 paths in \( V_{j+1} \) and 3 paths in \( V_j \) with \( t_j = 1, u_j = 3 \) and \( v_{j+1} = 1 \). \( B(V_{j+1}) \) and \( B(V_j) \) are respectively the borders of \( V_{j+1} \) and \( V_j \).

The path \( P(i, s^0_j) \) relates one end of the propagator \( i \) to \( s^0_j \) without going through \( i \). Paths \( P(s^{2\ell-1}_j, s^{2\ell}_j) \) corresponding to \( \ell = 1 \), relating \( s^1_j \) to \( s^2_j \), and to \( \ell = 2 \), relating \( s^3_j \) to \( s^4_j \) are shown, together with three paths \( P(s^{2u}_{j+1} s^{2u}_{j+1}) \) with \( u = 1, 2, 3 \) and one path \( P(s^{2v-2}_{j+1} s^{2v-1}_{j+1}) \) with \( v = 1 \).

**Fig. 2** The case where one loop \( \mathcal{L}_k \) is present is depicted. The path \( P_1(s_{1j}, s_{2j}) \) in \( V_j \) consists of the successive paths \( P(s_{1j}, s_{kj}) P_1(s_{kj}, s_{k+1j}) P(s_{k+1j}, s_{2j}) \). The propagator \( (v_1 v_2) \) is shown on \( P_1(s_{kj}, s_{k+1j}) \). The path \( P_2(s_{1j}, s_{2j}) \) differs from \( P_1(s_{1j}, s_{2j}) \) by the path \( P_2(s_{kj}, s_{k+1j}) \) in \( V_{j+1} - V_j \) on which the propagator \( (v_1' v_2') \) is shown. The loop \( \mathcal{L}_k \) is formed by the succession of paths \( P_1(s_{kj}, s_{k+1j}) P_2(s_{k+1j}, s_{kj}) \) where the latter is the reverse path of \( P_2(s_{kj}, s_{k+1j}) \).

**Fig. 3** The constraint that any propagator stemming out of \( V_j \) should relate the part of the borders of \( V_j \) and \( V_{j+1} \) (on the deformed spheres \( S_{j+1} \) and \( S_j \) respectively) has been imposed, if this propagator is in \( V_{j+1} \). The path in \( V_j \) \( P_1(s_{1j}, s_{2j}) \) is shown (here \( \mathcal{L}_k \) is simply \( \mathcal{L} \)). In \( V_{j+1} - V_j \), the path \( P_2(s_{1j}, s_{2j}) \) is shown by a thick line. Depicted are the propagators \( (s_{1j} s_{1j+1}) \) and \( (s_{2j} s_{2j+1}) \).
Fig. 2
Fig. 3
Fig. 1