A NEW PROOF OF A FORMULA FOR THE TYPE $A_2$ FUSION RULES

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Abstract. We give a new proof of a formula for the fusion rules for type $A_2$ due to Bégin, Mathieu, and Walton. Our approach is to symbolically evaluate the Kac-Walton algorithm.

1. Introduction

For an affine Lie algebra $\hat{\mathfrak{g}}$, the irreducible integrable $\hat{\mathfrak{g}}$-modules are classified by a highest weight and an integer $\ell$ called the level. The tensor product on $\hat{\mathfrak{g}}$-modules is additive with respect to the level. There exists a second product called the fusion product, which is level-preserving.

The fusion rules of an affine Lie algebra are the full set of structure constants $N^{(\ell)\nu}_{\lambda,\mu}$ that describe how the fusion product of two irreducible integrable level $\ell$ $\hat{\mathfrak{g}}$-modules decomposes into irreducibles. Kac and Walton independently found an algorithm for computing the fusion rules. The Kac-Walton algorithm only uses the combinatorics of the underlying root system, and hence, this algorithm can be used to define a product on $\mathfrak{g}$-modules as well as $\hat{\mathfrak{g}}$-modules. In this case, the algorithm is highly similar to the Racah-Speiser algorithm for tensor product decompositions, which is an algorithmic version of a formula that is variously attributed to Brauer, Klimyk, Steinberg, and Racah; see Section 3 for more discussion.

For Type $A_1$, the fusion rules for any level are easily computed. For Type $A_2$, Bégin, Mathieu, and Walton give a closed formula for the fusion rules for any level in [BMW92]. For other root systems, the fusion rules are known in some special cases. For instance, when the root system rank and level are small, the fusion rules can be computed using a computer; if the level is small, level-rank duality may be used; and if the weights have special properties, additional formulas are known [MS12, SS01, Tud02]. But at the time of this writing, we do not know of any other root systems besides $A_1$ and $A_2$ where the fusion rules are fully known for all weights and levels.

Bégin, Mathieu, and Walton derive their formula for the fusion rules of type $A_2$ using another formula called the depth rule. At the time their paper was published, the depth rule was only a conjecture, but it has since been proven in [FF08]. Unfortunately, extending the approach used in [BMW92] to other root systems has proven difficult.

In this paper, we give a new proof of Bégin, Mathieu, and Walton’s formula for the fusion rules of type $A_2$. Our approach is to symbolically evaluate the Kac-Walton algorithm using the computer algebra system Macaulay2. We hope that our approach can be applied to obtain fusion rules for some other root systems of small rank.

We briefly mention three applications of fusion rules.

Date: August 20, 2014.
One application of the fusion rules is to compute the ranks of vector bundles of conformal blocks [Bea96]. Write \( \ast \) for the involution on the weight lattice given by \(-w_0\), where \( w_0 \) is the longest word in the Weyl group. The ranks of conformal blocks on \( \mathcal{M}_{0,3} \) are related to fusion coefficients by

\[
\text{rank } \mathcal{V}(g, \ell, (\lambda, \mu, \nu)) = N^{(\ell) \nu \ast}_{\lambda, \mu}.
\]

Then, for any \( g \) and \( n \) with \( 3g - 3 + n \geq 0 \), factorization of vector bundles of conformal blocks allows the rank of any conformal block \( \mathcal{V}(g, \ell, \lambda, \nu) \) on \( \mathcal{M}_{g,n} \) to be computed recursively with the fusion rules as the seeds of this recursion.

As a second application, the fusion rules are related to the quantum cohomology of Grassmannians, at least in type A. Specifically, the ring \( \mathcal{F}(\widehat{sl}(n))_k \) with generators indexed by the irreducible integrable level \( k \) \( \widehat{sl}(n) \)-modules and structure constants given by the fusion rules is a quotient of the small quantum cohomology ring \( qH^*(\text{Gr}_{k,n+k}) \) ([KS10]).

Finally, since the fusion coefficients \( N^{(\ell) \nu \ast}_{\lambda, \mu} \) are always dominated by the tensor coefficients \( N^{\nu \ast}_{\lambda, \mu} \), we may view the fusion product as a truncated tensor product. It seems worth investigating whether fusion products could be used to approximate tensors in scientific or engineering applications.

1.1. Outline of the paper. In Section 2 we present a formula for the fusion rules due to Bégin, Mathieu, and Walton. In Section 3 we review the Racah-Speiser and Kac-Walton algorithms. In Section 4 we give our proof of the Bégin-Mathieu-Walton formula.

1.2. Acknowledgements. The first and third authors were supported by scholarships and summer research funding from the Clare Boothe Luce Foundation. The fourth author was supported by summer research funding from the dean of Fordham College at Lincoln Center. The second author would like to thank John Cannon and the Magma group for hosting a visit to the University of Sydney during which the fusion rules were first implemented in Macaulay2. The second author would also like to thank Allen Knutsen and Dan Roozemond for many helpful conversations, Mark Walton for telling him about the reference [FF08], and Dan Grayson and Mike Stillman for their advice in implementing the fusion rules in Macaulay2. Several additional Macaulay2 packages were used in our research, and we would like to thank their authors: Greg Smith, author of the FourierMotzkin package [Smi08]; René Birkner, author of the Polyhedra package [Bir10]; and Josephine Yu, Nathan Ilten, and Qingchun Ren, who shared preliminary versions of their PolyhedralObjects and PolymakeInterface packages.

2. The Bégin-Mathieu-Walton formula

2.1. Notation. Let \( g = sl_3 \). Let \( h \subset g \) be the Cartan subalgebra of diagonal matrices. Let \( \varepsilon_i : h \to \mathbb{C} \) be the function \( \varepsilon_i(H) = h_{i,i} \). Let \( \alpha_1 = \varepsilon_i - \varepsilon_2 \) and \( \alpha_2 = \varepsilon_2 - \varepsilon_3 \). Then \( \Delta = \{\alpha_1, \alpha_2\} \) is a base of the root system of \( g \), and \( \theta = \alpha_1 + \alpha_2 \) is the highest root with respect to \( \Delta \). The Cartan matrix is

\[
\begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}.
\]
Let \( \omega_1 \) and \( \omega_2 \) be the fundamental dominant weights. Then we have
\[
\alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_2 = -\omega_1 + 2\omega_2,
\]
and we may invert this system of equations to obtain
\[
\omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \quad \text{and} \quad \omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2.
\]
The Killing form on the fundamental weights is
\[
(\omega_1, \omega_1) = \frac{2}{3}, \quad (\omega_1, \omega_1) = \frac{1}{3}, \quad (\omega_2, \omega_2) = \frac{2}{3}.
\]

We have \((\theta, \theta) = 2\), and \((a\omega_1 + b\omega_2, \theta) = a + b\).

Let \( \lambda = a\omega_1 + b\omega_2 = (a, b) \), \( \mu = c\omega_1 + d\omega_2 = (c, d) \), and \( \nu = e\omega_1 + f\omega_2 = (e, f) \). Let \( C^+ \) be the fundamental Weyl chamber; then \( C^+ = \{ c_1\omega_1 + c_2\omega_2 : c_1, c_2 \geq 0 \} \). The fundamental Weyl alcove of level \( \ell \) is \( P_\ell = \{ \beta \in C^+ : (\beta, \theta) \leq \ell \} \). Thus, \( \lambda, \mu, \nu \in P_\ell \) if and only if \( a, b, c, d, e, f \geq 0 \) and \( a + b, c + d, e + f \leq \ell \).

2.2. The Bégin-Mathieu-Walton formula. In the exposition below, we combine some of the formulas from [BMW92] to obtain a more self-contained presentation.

**Theorem 2.1.** [BMW92] The fusion rules of type \( A_2 \) are given as follows:
\[
N_{\lambda,\mu}^{(\ell)\nu} = \begin{cases} 
\min\{k_0^{\max}, \ell\} - k_0^{\min} + 1 & \text{if } \ell \geq k_0^{\min} \text{ and } N_{\lambda,\mu}^{\nu} > 0, \\
0 & \text{if } \ell < k_0^{\min} \text{ or } N_{\lambda,\mu}^{\nu} = 0,
\end{cases}
\]

where
\[
A = \frac{1}{3}(2(a + c + f) + (b + d + e)), \\
B = \frac{1}{3}((a + c + f) + 2(b + d + e)), \\
k_0^{\min} = \max\{a + b, c + d, e + f, A - \min(a, c, f), B - \min(b, d, e)\}, \\
k_0^{\max} = \min\{A, B\}, \\
\delta = \begin{cases} 
1 & \text{if } k_0^{\max} \geq k_0^{\min} \text{ and } A, B \in \mathbb{Z}_{\geq 0}, \\
0 & \text{otherwise},
\end{cases} \\
N_{\lambda,\mu}^{\nu} = (k_0^{\max} - k_0^{\min} + 1)\delta.
\]

We note three nice features of this formula. First, note that \( \ell \) enters only in the very last step of the calculation; it does not appear in the definition of \( A, B, k_0^{\min}, k_0^{\max}, \delta \), or \( N_{\lambda,\mu}^{\nu} \).

Second, it is clear that \( N_{\lambda,\mu}^{(\ell)\nu} \) stabilizes for all sufficiently large \( \ell \), specifically once \( \ell \geq k_0^{\max} \) and \( \ell \geq k_0^{\min} \). Finally, this formula allows us to interpret \( N_{\lambda,\mu}^{(\ell)\nu} \) as the number of lattice points in a polytope; specifically, it is the number of integers \( x \) satisfying \( k_0^{\min} \leq x \leq \min\{k_0^{\max}, \ell\} \).
2.3. An equivalent version of the Bégin-Mathieu-Walton formula. We modify the formula from Bégin, Mathieu, and Walton’s paper slightly. We use fewer instances of max and min, and the cases are rewritten slightly to match the output we obtain from the Kac-Walton algorithm.

Define

\[ G(\lambda, \mu, \nu, \ell) := \begin{cases} 
\ell_{0}^{\max} - k_{0}^{\min} + 1 & \text{if } \ell_{0}^{\max} - k_{0}^{\min} \geq -1, \\
0 & \text{otherwise.}
\end{cases} \]

where

\[
A = \frac{1}{3}(2(a + c + f) + (b + d + e)), \\
B = \frac{1}{3}((a + c + f) + 2(b + d + e)), \\
k_{0}^{\min} := \max\{a + b, c + d, e + f, A - a, A - c, A - f, B - b, B - d, B - e\}, \\
\ell_{0}^{\max} = \min\{A, B, \ell\}.
\]

**Proposition 2.2** (Bégin-Mathieu-Walton). If \( A \) and \( B \) are integers, then \( G(\lambda, \mu, \nu, \ell) = N_{\lambda, \mu}^{(\ell)\nu} \).

We view the formula \( G(\lambda, \mu, \nu, \ell) \) above as a continuous piecewise linear function supported on 27 polyhedral cones. As an example of one such cone, to get the expression \( N_{\lambda, \mu}^{(\ell)\nu} = A - a - b + 1 \) above requires that

\[ a + b = \max\{a + b, c + d, e + f, A - a, A - c, A - f, B - b, B - d, B - e\}, \]
\[ \ell = \min\{A, B, \ell\}. \]

This leads to the the inequalities

\[
\begin{align*}
a + b &\geq c + d \\
a + b &\geq e + f \\
a + b &\geq A - a \\
a + b &\geq A - b \\
a + b &\geq A - f \\
a + b &\geq B - b \\
a + b &\geq B - d \\
a + b &\geq B - e \\
A &\leq B \\
A &\leq \ell.
\end{align*}
\]

We also have \( a, b, c, d, e, f, \ell \geq 0 \) and \( a + b, c + d, e + f \leq \ell \). These 20 inequalities determine a polyhedral cone in \( \mathbb{R}^{7} \).

In a similar fashion, we may associate a finitely-generated polyhedral cone to each of the remaining 26 nonzero expressions that may arise from \( G(\lambda, \mu, \nu, \ell) \).
3. The Racah-Speiser and Kac-Walton algorithms

Two references for the Kac-Walton algorithm are [Kac90, Exercise 13.35] and [Wal90]. Its history is described in [Wal90]. The Kac-Walton algorithm is closely related to the Racah-Speiser algorithm for tensor coefficients, and so we recall the Racah-Speiser algorithm first.

3.1. The Racah-Speiser algorithm. Let \( g \) be a Lie algebra. Let \( h \subseteq g \) be a Cartan subalgebra, let \( \Phi \) be the root system determined by \( h \), and let \( \Delta = \{ \alpha_1, \ldots, \alpha_n \} \) be a base of \( \Phi \). Let \( \lambda \) be a dominant integral weight, and let \( V(\lambda) \) be an irreducible finite-dimensional \( g \)-module with highest weight \( \lambda \).

Let \( m_\lambda(\mu) \) denote the dimension of the weight space \( V_\mu \) in the irreducible representation \( V(\lambda) \). We shall refer to the set of pairs \( (\mu, m_\lambda(\mu)) \) as the weight diagram of \( \lambda \); clearly, the character of \( V(\lambda) \) can be computed from the weight diagram, and vice versa. Let \( \Phi^+ \) be the set of positive roots, and let \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \). Let \( C^+ \) be the fundamental Weyl chamber.

Let \( W \) be the Weyl group of \( g \), and for \( w \in W \), let \( w \cdot \varphi \) be the shifted reflection defined by \( w \cdot \varphi = w(\varphi + \rho) - \rho \).

Define the tensor product coefficients \( N^\nu_{\lambda \mu} \) by

\[
V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in C^+} V(\nu)^{\otimes N^\nu_{\lambda \mu}}.
\]

The Racah-Speiser algorithm is described below:

**Algorithm 1** Racah-Speiser algorithm

**Input:** dominant integral weights \( \lambda \) and \( \mu \)

**Output:** the set of tensor product coefficients \( \{ N^\nu_{\lambda \mu} \} \)

Begin with \( N^\nu_{\lambda \mu} = 0 \).

Compute \( WD(\lambda) \), the weight diagram of \( \lambda \).

Translate each weight in \( WD(\lambda) \) by \( \mu \).

For each weight \( \varphi \) in \( WD(\lambda) + \mu \), if \( \varphi \) is not fixed by any shifted reflection \( w \cdot \varphi \) for \( w \in W \), compute an element \( w \) such that \( w \cdot \varphi \in C^+ \) and add \( m_\lambda(\varphi - \mu) \text{sgn}(w) \) to \( N^{w \cdot \varphi}_{\lambda \mu} \).

**return** \( \{ N^\nu_{\lambda \mu} \} \)

For a proof of the correctness of this algorithm, we refer to [GW09, Corollary 7.1.7]; see also [Hum78, Exercise 24.9], where this formula is attributed to Brauer-Klimyk, and [FH91, Exercise 25.31], where this formula is attributed to Racah. After converting Goodman and Wallach’s notation to ours, the main formula of Corollary 7.1.7 is

\[
N^\nu_{\lambda \mu} = \sum_{w \in W} \text{sgn}(w)m_\lambda(\nu + \rho - w(\mu + \rho)).
\]

Since the weight diagram is symmetric under the Weyl group, we have

\[
m_\lambda(\nu + \rho - w(\mu + \rho)) = m_\lambda(w(\nu + \rho) - (\mu + \rho)) = m_\lambda(w \cdot \nu - \mu).
\]
Substituting this into the previous formula gives
\begin{equation}
N^\nu_{\lambda\mu} = \sum_{w \in W} \text{sgn}(w)m_\lambda(w \cdot \nu - \mu),
\end{equation}
and this formula agrees with the calculations described in the Racah-Speiser algorithm.

**Example.** As an example, let $\mathfrak{g} = \mathfrak{sl}_3$, and let $\lambda = (4, 2) = 4\omega_1 + 2\omega_2$ and $\mu = (3, 1) = 3\omega_1 + 1\omega_2$. We use the Racah-Speiser algorithm to compute the decomposition of the tensor product $V(\lambda) \otimes V(\mu)$ into irreducible $\mathfrak{sl}_3$ modules.

**Figure 1.** The Racah-Speiser algorithm. (a) The weight diagram of $\lambda$. (b) The weight diagram of $\lambda$ translated by $\mu$. (c) Reflect into the fundamental chamber. (d) The tensor coefficients.

From this we see that
\begin{align*}
N^\nu_{\lambda\mu} &= 1 \text{ if } \nu \in \{(0,5), (1,3), (1,6), (2,1), (3,5), (4,0), (5,4), (7,0), (7,3), (8,1)\}, \\
N^\nu_{\lambda\mu} &= 2 \text{ if } \nu \in \{(2,4), (3,2), (4,3), (5,1), (6,2)\}, \text{ and otherwise } N^\nu_{\lambda\mu} = 0.
\end{align*}

3.2. **The Kac-Walton algorithm.** Fix an integer $\ell \geq 0$. The Kac-Walton algorithm differs from the Racah-Speiser algorithm by replacing the Weyl group $W$ with the affine Weyl group $\hat{W}$ and the fundamental Weyl chamber $C^+$ by the fundamental Weyl alcove $P_\ell$.
defined below. In contrast with the Weyl group, the affine Weyl group is infinite. However, it can be obtained by adding just one extra generator to the Weyl group. For \( i = 1, \ldots, n \), let \( s_i \) be the reflection across the hyperplanes perpendicular to the simple root \( \alpha_i \). Let \( s_0 \) be the affine linear transformation

\[
s_0(\beta) = \beta + (\ell - (\beta, \theta) + 1)\theta,
\]

where \( \theta \) is the highest root, and \((-,-)\) is the Killing form normalized so that \((\theta, \theta) = 2\). Then \( W = \langle s_1, \ldots, s_n \rangle \), and \( \hat{W} = \langle s_0, \ldots, s_n \rangle \).

The fundamental Weyl alcove of level \( \ell \) is \( P_{\ell} = \{ \beta \in C^+ : (\beta, \theta) \leq \ell \} \).

The Kac-Walton algorithm is described below:

**Algorithm 2 Kac-Walton algorithm**

**Input:** dominant integral weights \( \lambda, \mu \in P_{\ell} \)

**Output:** the set of fusion coefficients \( \{ N^{(\ell)\nu}_{\lambda\mu} \} \)

Begin with \( N^{(\ell)\nu}_{\lambda\mu} = 0 \).

Compute \( WD(\lambda) \), the weight diagram of \( \lambda \).

Translate each weight in \( WD(\lambda) + \mu \).

For each weight \( \varphi \) in \( WD(\lambda) + \mu \), if \( \varphi \) is not fixed by any shifted reflection \( w \cdot \varphi \) for \( w \in \hat{W} \), compute an element \( w \) such that \( w \cdot \varphi \in P_{\ell} \) and add \( m_{\lambda}(\varphi - \mu) \text{sgn}(w) \) to \( N^{(\ell)w\cdot\varphi}_{\lambda\mu} \).

return \( \{ N^{(\ell)\nu}_{\lambda\mu} \} \)

For the purposes of this paper, we shall use the Kac-Walton algorithm to define the fusion coefficients \( N^{(\ell)\nu}_{\lambda\mu} \). For a proof that the Kac-Walton algorithm computes the multiplicities of irreducible level \( \ell \) integrable \( \hat{g} \)-modules in the fusion product, see [Kac90, Exercise 13.35] and [Wal90].

**Example.** As an example, let \( g = sl_3 \), and let \( \lambda = (4, 2) = 4\omega_1 + 2\omega_2 \) and \( \mu = (3, 1) = 3\omega_1 + 1\omega_2 \), and let \( \ell = 7 \). We use the Kac-Walton algorithm to compute the decomposition of the fusion product \( V(\lambda) \otimes_7 V(\mu) \) into irreducible \( sl_3 \) modules. The first two steps are the same as those of the Racah-Speiser algorithm; see Figure 1 (a) and (b).
Figure 2. The Kac-Walton algorithm. Steps (a) and (b) are the same as in Figure 1. (e) Reflect into the fundamental alcove. (f) Fusion coefficients.

From this we see that $N^{(\ell)}_{\mu \nu} = 1$ if $\nu \in \{(0, 5), (1, 3), (1, 6), (2, 1), (4, 0), (4, 3), (5, 1)\}$, $N^{(\ell)}_{\mu \nu} = 2$ if $\nu \in \{(2, 4), (3, 2)\}$, and otherwise $N^{(\ell)}_{\mu \nu} = 0$.

4. OUR PROOF

4.1. A multiplicity formula. Weight diagrams for Type $A_2$ have a very pretty description; in [Hum78, §21.3], Humphreys attributes this description to Antoine and Speiser. The boundary of the weight diagram is a (nonregular) hexagon with all multiplicities equal to one. As one passes from one hexagonal “shell” of the weight diagram to the next “shell” inside it, the multiplicity increases by one, until the shells become triangles, at which point the multiplicity is constant. See Figure 1(a) for an example.

Writing formulas for the pattern described above yields the following:

Lemma 4.1 (Antoine-Speiser). Let $\lambda = (a, b)$ and $\varphi = (x, y)$ be two weights in the fundamental chamber $C^+$, and suppose $a + 2b - x - 2y$ is divisible by 3 (so that $\lambda - \varphi$ is in the root lattice). Then the multiplicity of $\varphi$ in $V(\lambda)$ is

$$m_{\lambda}(\varphi) = \max \left\{ 0, \min \left\{ \frac{1}{3}(a + b - x - 2y) + 1, \frac{1}{3}(2a + b - 2x - y) + 1, a + 1, 2 + b + 1 \right\} \right\}.$$ 

Proof. The most popular way to derive this formula is use the fact that the multiplicity of $\mu$ in $V(\lambda)$ is the number of semistandard Young tableaux of shape $\lambda$ and weight $\mu$ (see e.g. [GW09, Cor. 8.1.7]). This leads to the inequalities printed above.

However, following Exercise 25.15 in [FH91], we wrote our own proof using double induction and Freudenthal’s formula (see e.g. [FH91, Lecture 25]). The first induction is on the
distance to the boundary along a positive root, and the second induction is on the distance
from an arbitrary weight \( \mu \) with distance \( k \) to the boundary to the point \( \lambda - k\theta \). For the full
proof, see our website:

http://faculty.fordham.edu/dswinarski/symbolickacwalton/

We view the formula in Lemma 4.1 as a continuous piecewise linear function supported
on seven cones. As an example of one such cone, to get the multiplicity expression \( \frac{1}{3}(a + b - x - 2y) + 1 \) above requires the inequalities

\[
\frac{1}{3}(a + b - x - 2y) + 1 \geq 0 \\
\frac{1}{3}(a + b - x - 2y) + 1 \leq \frac{1}{3}(2a + b - 2x - y) + 1 \\
\frac{1}{3}(a + b - x - 2y) + 1 \leq a + 1 \\
\frac{1}{3}(a + b - x - 2y) + 1 \leq b + 1.
\]

These inequalities, together with the inequalities \( 0 \leq a, b, x, y \), determine a finitely-generated
polyhedral cone in \( \mathbb{R}^4 \). In a similar fashion, we associate three more cones to the other three
nonzero expressions in Lemma 4.1.

We define three different cones where the multiplicity expression is 0. Observe first that
since \( \lambda \in P_\ell \), we have \( a \geq 0 \) and \( b \geq 0 \), so the expressions \( a + 1 \) and \( b + 1 \) in Lemma 4.1 never
cause the multiplicity to vanish. Thus, we define one cone where \( \frac{1}{3}(a + b - x - 2y) + 1 \geq 0 \)
and \( \frac{1}{3}(2a + b - 2x - y) + 1 \leq 0 \); in the second cone, we have \( \frac{1}{3}(a + b - x - 2y) + 1 \leq 0 \) and
\( \frac{1}{3}(2a + b - 2x - y) + 1 \geq 0 \); and in the third cone, we have \( \frac{1}{3}(a + b - x - 2y) + 1 \leq 0 \) and
\( \frac{1}{3}(2a + b - 2x - y) + 1 \leq 0 \). Thus we obtain seven cones total covering the fundamental Weyl chamber.

Since a weight diagram is symmetric under the Weyl group, we may use the Weyl group
to obtain expressions for the multiplicity in the remaining chambers. This yields a formula
with 42 cones. However, the resulting 42 expressions are not distinct; some of these cones
may be combined, yielding the following formula, which has 14 cones.

**Proposition 4.2.** If \( \lambda - (x, y) \) is in the root lattice, then the multiplicity of \( (x, y) \) in \( V(\lambda) \)
is given by the continuous piecewise polynomial formula printed in Figure 4.2.
Figure 3. Multiplicity expressions on 14 cones. An expression of the form $F(x, y, a, b)$ in the left column represents the inequality $F(x, y, a, b) \geq 0$.

| Cone inequalities | Multiplicity |
|-------------------|--------------|
| $x - y - a + b$, $2x + y - a + b$, $-x + y - 2a - b - 3$ | 0 |
| $-x + y - a - b$, $x + 2y - a + b$, $x + 2y - a - 2b - 3$ | 0 |
| $-2x - y - a + b$, $x + 2y - a - b$, $-x + y - 2a - b - 3$ | 0 |
| $-x + y + a - b$, $-x - 2y + a - b$, $-2x - y - a - 2b - 3$ | 0 |
| $x - y + a - b$, $x - 2y - a + b$, $x - y - a - 2b - 3$ | 0 |
| $2x + y + a - b$, $x + 2y - a + b$, $-2x - y + 2a + b + 3$ | $-(2/3)x - (1/3)y + (2/3)a + (1/3)b + 1$ |
| $-x + y + a - b$, $2x + y + a - b$, $x + 2y - a + b$, $-x - 2y + a + 2b + 3$ | $-(1/3)x - (2/3)y + (1/3)a + (2/3)b + 1$ |
| $-2x - y - a + b$, $x + 2y - a + b$, $-x - y + 2a + b + 3$, $-x + y + a - b$ | $(1/3)x - (1/3)y + (2/3)a + (1/3)b + 1$ |
| $-x + y + a - b$, $-2x - y - a + b$, $-x - 2y + a - b$, $2x + y + a + 2b + 3$ | $(2/3)x + (1/3)y + (1/3)a + (2/3)b + 1$ |
| $x - y - a + b$, $-2x - y - a + b$, $x + 2y + 2a + b + 3$, $-x - 2y + a - b$ | $(1/3)x + (2/3)y + (2/3)a + (1/3)b + 1$ |
| $x - y - a + b$, $2x + y + a - b$, $-x - 2y + a - b$, $-x + y + a + 2b + 3$ | $-1(3)x + (1/3)y + (1/3)a + (2/3)b + 1$ |
| $-x + y + a - b$, $2x + y + a - b$, $-x - 2y + a - b$, $a - b$ | $b + 1$ |
| $-a + b + x - y$, $-a + b + x + 2y$, $-a + b - 2x - y$, $-a + b$ | $a + 1$ |

4.2. Contributing alcoves. Recall that for $w \in \hat{W}$, $w \cdot \beta = w(\beta + \rho) - \rho$.

Lemma 4.3. The alcove $w \cdot P_\ell$ contributes zero to the Kac-Walton algorithm unless $w$ is equivalent in the Weyl group to one of the following 13 elements:

$$\{s_0s_2s_0, s_0s_1s_0, s_1s_2s_1, s_0s_2, s_0s_1, s_2s_0, s_1s_0, s_2s_1, s_1s_2, s_0, s_2, s_1, Id\}.$$

Proof. Let $WD(\lambda)$ denote the weight diagram of $\lambda$, and let $WP_\ell$ denote the W-orbit of $P_\ell$. We hope that this clash of notation will not cause too much confusion.

Since $\lambda \in P_\ell$ and the weight diagram is symmetric under the Weyl group, we have $WD(\lambda) \subset WP_\ell$. Since $\mu \in P_\ell$, we have $WD(\lambda) + \mu$ is contained in the Minkowski sum $WP_\ell + P_\ell$, and we check in turn that the Minkowski sum is contained in the union of the 13 alcoves listed. In Figure 4 below, the regions $P_\ell$, $WP_\ell$, and $WP_\ell + P_\ell$ are shown in increasingly lighter shades of green, respectively, and the 13 alcoves are labeled. \qed
4.3. Our Macaulay2 types and functions. We implemented two new types in Macaulay2 called ConeSupportedExpression and ConeSupportedExpressionSet. These two types are highly specialized for the calculations required here. An object of type ConeSupportedExpression is a hash table recording an expression and a cone on which it is supported. An object of type ConeSupportedExpressionSet is an unordered set of ConeSupportedExpressions. We assume that

1. The dimension of each cone in each ConeSupportedExpression is equal to the dimension of the ambient vector space;
2. no two cones in a ConeSupportedExpressionSet have a full-dimensional intersection;
3. the union of the cones in a ConeSupportedExpressionSet is equal to the ambient vector space.

The multiplicity formula in Figure 4.2 has these three properties, and hence can be implemented as an object of type ConeSupportedExpressionSet.

We implemented methods for adding two ConeSupportedExpressionSets and for multiplying a ConeSupportedExpressionSet by a scalar.

We also wrote a function isUnionConvex to decide whether the union of several cones is convex. One use of this function is to simplify a ConeSupportedExpressionSet; if one
A nonzero expression is supported on two or more cones, and the union of these cones is convex, then we replace these cones by their union, yielding a `ConeSupportedExpressionSet` containing fewer `ConeSupportedExpressions`.

4.4. **The main program.** We use the notation for roots and weights described in Section 2.

We begin with $\lambda = (a, b)$, $\mu = (c, d)$, and $\nu = (e, f)$.

For each word $w$ in the list of contributing alcoves in Lemma 4.3, we compute $w \cdot \nu - \mu = (x, y)$ and use the formulas in Figure 4.2 to compute $m_\lambda(w \cdot \nu - \mu)$ as a `ConeSupportedExpressionSet`. We then compute

$$N_{\lambda \mu}^{\ell \nu} = \sum_w \text{sgn}(w) m_\lambda(w \cdot \nu - \mu),$$

simplifying the intermediate `ConeSupportedExpressionSet` after each addition or subtraction.

The program takes approximately ten minutes to compute its answer. It finds 27 nonzero expressions supported on cones, and computes an additional 82 cones supporting the expression 0.

We checked that the 27 nonzero expressions we obtained and the cones on which they are supported match the nonzero expressions and cones of Bégin, Mathieu, and Walton’s formula. Since our program computes its answer without using Bégin, Mathieu, and Walton’s formula along the way, we obtain a new, independent proof of Proposition 2.2, first proved by Bégin, Mathieu, and Walton in [BMW92]. Notably, our proof does not use the depth rule, which was used in [BMW92].

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