GENERALIZED NASH EQUILIBRIUM PROBLEM BASED ON MALFATTI'S PROBLEM

ENKHBAT RENTSEN
Institute of Mathematics and Digital Technology
Academy of Sciences of Mongolia, Ulaanbaatar, Mongolia

BATTUR GOMPIL
Center of Mathematics for Applications and Department of Applied Mathematics
National University of Mongolia, Ulaanbaatar, Mongolia

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Abstract. In this paper we consider non-cooperative game problem based on the Malfatti's problem. This problem is a special case of generalized Nash equilibrium problems with nonconvex shared constraints. Some numerical results are provided.

1. Introduction. In 1803 Italian mathematician Malfatti [15] posed the following problem: how to pack three non-overlapping disks of maximum total area in a given triangle? Malfatti originally assumed that the solution to this problem are three disks inscribed in a triangle such that each disk tangent to other two and touches two sides of the triangle. Now it is well known that Malfatti’s solution is not optimal. There are many works devoted to solving Malfatti’s problem, for instance [1], [9], [10], [13], [14] and [18]. The most common methods used for finding the best solutions to Malfatti’s problem were algebraic and geometric approaches. In 1994 Zalgaller and Los [19] showed that the greedy arrangement is the best one. Based on trigonometric equations and inequalities, using so called rigid systems they found the best solution to Malfatti’s problem. In [4], Malfatti’s problem was considered as a global optimization problem. Also, high dimensional Malfatti’s problem was examined in [5]. On the other hand, in this paper, we consider a problem based on the Malfatti’s problem from game-theoretic point of view. In other words, we first formulate a generalized Nash equilibrium problem (GNEP for short) which is not equivalent to Malfatti’s problem in general. The GNEP is the extension of the classical Nash equilibrium problem (NEP) in which each player’s strategy set depends on the rival player’s strategies. Since the mid-1990s many efforts have been devoted to the investigation of GNEP (see [6], [8], [12], [11]), because it has many interesting applications in the fields of economics, operational research and engineering. For instance, Wei and Smeers [17] formulated oligopolistic electricity models as GNEPs. The paper is organized as follows. In Section 2, we reformulate Malfatti’s problem as the convex maximization problem. In Section 3, we formulate

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* Corresponding author: Battur Gompil.
three player’s generalized Nash equilibrium problems. Computational experiments were done in Section 4.

2. Malfatti’s Problem and Convex Maximization. In order to formulate Malfatti’s problem as an optimization problem, we need to do following steps.

First, we equivalently formulate the problem in terms of convex sets such as a disk and triangle set. Secondly, we characterize inscribed conditions of disks into a triangle set. For this purpose, we introduce the following sets. Denote by \( B \) a disk and triangle set. Secondly, we characterize inscribed conditions of disks into a triangle set. For this purpose, we introduce the following sets. Denote by \( B(x, r) \) a disk with a center \( x \in \mathbb{R}^2 \) and a radius \( r \in \mathbb{R} \):

\[
B(x, r) = \{ y \in \mathbb{R}^2 \mid \| y - x \| \leq r \},
\]

A triangle set \( D \subset \mathbb{R}^2 \) is given by

\[
D = \{ x \in \mathbb{R}^2 \mid \langle a^i, x \rangle \leq b_i, a^i \in \mathbb{R}^2, b_i \in \mathbb{R}, i = 1, 2, 3 \},
\]

where \( \langle , \rangle \) denotes the scalar product of two vectors in \( \mathbb{R}^2 \), and \( \| \cdot \| \) is Euclidean norm, \( a^i \parallel a^j \), \( i \neq j \); \( i, j = 1, 2, 3 \). In addition, we assume that \( D \) is a bounded set, and \( \text{int} D \neq \emptyset \).

**Theorem 2.1.** \([4]\) \( B(x, r) \subset D \) if and only if

\[
\langle a^i, x \rangle + r\| a^i \| \leq b_i, i = 1, 2, 3
\]

Then Malfatti’s problem is formulated as a modification proposed by Enkhbat [4] as below:

\[
\max G = \pi (x_3^2 + x_6^2 + x_9^2),
\]

\[
\langle a^i, u \rangle + x_3 \| a^i \| \leq b_i, u = (x_1, x_2), i = 1, 2, 3,
\]

\[
\langle a^i, v \rangle + x_6 \| a^i \| \leq b_i, v = (x_4, x_5), i = 1, 2, 3,
\]

\[
\langle a^i, p \rangle + x_9 \| a^i \| \leq b_i, p = (x_6, x_7), i = 1, 2, 3,
\]

\[
(x_4 - x_1)^2 + (x_5 - x_2)^2 \geq (x_3 + x_6)^2,
\]

\[
(x_7 - x_1)^2 + (x_8 - x_2)^2 \geq (x_3 + x_9)^2,
\]

\[
(x_7 - x_4)^2 + (x_8 - x_5)^2 \geq (x_6 + x_9)^2,
\]

\[
x_3 \geq 0, x_6 \geq 0, x_9 \geq 0.
\]

The function \( G \) in (4) denotes a total area of the three disks. Conditions (5) – (7) characterize inscribed conditions of three disks into a triangle set while conditions (8) – (11) describe nonoverlapping conditions of disks. The problem was solved numerically by Algorithm Max \([3]\) in [4].

3. Generalized Nash equilibrium problems. GNEP is a \( N \) players noncooperative game where each player’s strategy set depends on rival players’ strategies. The GNEP was introduced in 1952 by Debreu [2]. We refer to the survey papers [6] and [7]. Consider the generalized Nash equilibrium problem where player \( k \) \((k = 1, \ldots, N)\) controls \( x_k \in \mathbb{R}^{n_k} \) and tries to solve the following optimization problem

\[
P_k(x_{-k}) \max_{x_k} f_k(x_k, x_{-k})
\]

s.t. \( g^k(x_k, x_{-k}) \leq 0 \)
with given \( f_k : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( g^k : \mathbb{R}^n \rightarrow \mathbb{R}^{m_k} \). Here, \( n := n_1 + \cdots + n_k \) denotes the total number of variables, \( m := m_1 + \cdots + m_N \) will be the total number of constraints. Each player \( k \) controls his strategy vector 
\[
x_k := (x_1^k, \ldots, x_{n_k}^k)^T \in \mathbb{R}^{n_k}
\]
of \( n_k \) decision variables. The vector 
\[
x := (x_1, \ldots, x_N)^T \in \mathbb{R}^n
\]
contains the \( n = \sum_{k=1}^N n_k \) decision variables of all players. To emphasize the \( k \)-th player’s variables within \( x \), one can write \((x_k, x_{-k})\) instead of \( x \), where 
\[
x_{-k} := (x_{k'})_{k'=1, k' \neq k} \in \mathbb{R}^{n-k}.
\]
A vector \( x := (x_1, \ldots, x_N)^T \) is called feasible for the GNEP if it satisfies the constraints \( g^k(x) \leq 0 \) for all players \( k = 1, \ldots, N \). A feasible point \( \bar{x} \) is a solution of the GNEP if, for all players \( k = 1, \ldots, N \), we have 
\[
f_k(\bar{x}_k, \bar{x}_{-k}) \geq f_k(x_k, \bar{x}_{-k}), \forall x_k : g^k(x_k, x_{-k}) \leq 0,
\]
i.e., if for all players \( k \), \( \bar{x}_k \) is the solution of the \( k \)-th player’s problem when the other players set their variables to \( \bar{x}_{-k} \). In other words, the problem \( P_k(x_{-k}) \) are \( N \) parallel and parameteric optimization problems. These problems are extremely difficult for solving due to nonconvexities. In practice, convex class is more popular than the class nonconvex of the GNEP. That is why, compared with convex class of the GNEPs, the study of theory and computational methods of nonconvex GNEPs is still in its infancy. Now we extend Malfatti’s problem from a view point of game theory. Assume that three players who correspond to each disk have to maximize their area simultaneously in a given triangle. First, we introduce the following variables:
\[
x_1 = (x_1^1, x_2^1, x_3^1), \quad u = (x_1^1, x_2^1),
x_2 = (x_1^2, x_2^2, x_3^2), \quad v = (x_1^2, x_2^2),
x_3 = (x_1^3, x_2^3, x_3^3), \quad p = (x_1^3, x_2^3).
\]
Then Malfatti’s problem can be rewritten as:
\[
\begin{align*}
\text{max } G &= \pi((x_1^3)^2 + (x_2^3)^2 + (x_2^3)^2), \\
\langle a^i, u \rangle + x_1^i \| a^i \| &\leq b_i, \quad i = 1, 2, 3, \\
\langle a^i, v \rangle + x_2^i \| a^i \| &\leq b_i, \quad i = 1, 2, 3, \\
\langle a^i, p \rangle + x_3^i \| a^i \| &\leq b_i, \quad i = 1, 2, 3, \\
(x_1^1 + x_2^2)^2 - (x_2^1 - x_1^2)^2 - (x_2^1 - x_2^2)^2 &\leq 0, \\
(x_1^1 + x_2^2)^2 - (x_1^3 - x_1^1)^2 - (x_2^1 - x_2^1)^2 &\leq 0, \\
(x_3^1 + x_2^2)^2 - (x_3^1 - x_1^1)^2 - (x_2^2 - x_2^2)^2 &\leq 0, \\
x_1^1 \geq 0, \quad x_2^2 \geq 0, \quad x_3^3 \geq 0.
\end{align*}
\]

Then three players generalized Nash equilibrium problem based on Malfatti’s problem is formulated as follows:
\[
(P_1) \max_{x_1} f_1(x) = \pi(x_1^1)^2, \quad u = (x_1^1, x_2^1)
\]
We denote by the first-order optimality condition for each player's problem.

Here, a stationary point means a point that satisfies the constraints of all players' problems. Therefore, it may be reasonable to try to find a stationary Nash equilibrium, which consists of stationary points of all players' problems. However, from the viewpoint of computation, it is difficult to find an equilibrium that guarantees global maxima of all players' problems. How-ever, numerically, it is possible only to define a Nash equilibrium of this game in the ordinary sense, which is based on a global maximum of each player's problem. How-ever, from the viewpoint of computation, it is difficult to find an equilibrium that guarantees global maxima of all players' problems. Therefore, we can note some relations between constraints in the following:

\[
X_1(x_{-1}) = \{ x_1 \in \mathbb{R}^3 \mid g^1_i(x) = \langle a^i, u \rangle + x^3_i \| a^i \| - b_i \leq 0, \ i = 1, 2, 3, \\
g^1_i(x) = (x^3_i + x^3_2)^2 - (x^1_i - x^1_1)^2 - (x^2_i - x^2_1)^2 \leq 0, \\
g^3_i(x) = (x^3_i + x^3_3)^2 - (x^1_i - x^1_1)^2 - (x^2_i - x^2_2)^2 \leq 0, \\
g^5_i(x) = -x^3_i \leq 0 \}
\]

\[
(P_2) \max_{x_2} f_2(x) = \pi(x^3_2)^2, \ v = (x^2_1, x^2_2)
\]

\[
X_2(x_{-2}) = \{ x_2 \in \mathbb{R}^3 \mid g^2_i(x) = \langle a^i, v \rangle + x^3_i \| a^i \| - b_i \leq 0, \ i = 1, 2, 3, \\
g^2_i(x) = (x^3_i + x^3_3)^2 - (x^1_i - x^1_1)^2 - (x^2_i - x^2_2)^2 \leq 0, \\
g^3_i(x) = (x^3_i + x^3_3)^2 - (x^1_i - x^1_1)^2 - (x^2_i - x^2_3)^2 \leq 0, \\
g^5_i(x) = -x^3_i \leq 0 \}
\]

\[
(P_3) \max_{x_3} f_3(x) = \pi(x^3_3)^2, \ p = (x^3_1, x^3_3)
\]

\[
X_3(x_{-3}) = \{ x_3 \in \mathbb{R}^3 \mid g^3_i(x) = \langle a^i, p \rangle + x^3_i \| a^i \| - b_i \leq 0, \ i = 1, 2, 3, \\
g^3_i(x) = (x^3_i + x^3_3)^2 - (x^1_i - x^1_1)^2 - (x^2_i - x^2_2)^2 \leq 0, \\
g^3_i(x) = (x^3_i + x^3_3)^2 - (x^1_i - x^1_2)^2 - (x^2_i - x^2_3)^2 \leq 0, \\
g^5_i(x) = -x^3_i \leq 0 \}
\]

Now the constraints (16) – (18) will be shared constraints for the problem (P_1) – (P_3). Therefore, we can note some relations between constraints in the following:

\[
g^1_i(x) = g^2_i(x), \\
g^3_i(x) = g^1_i(x), \\
g^5_i(x) = g^3_i(x).
\]

The problem (P_1) – (P_3) is a generalized Nash equilibrium problem with non-convex shared constraints. Therefore this problem is extremely difficult for solving numerically. It is possible only to define a Nash equilibrium of this game in the ordinary sense, which is based on a global maximum of each player's problem. However, from the viewpoint of computation, it is difficult to find an equilibrium that guarantees global maxima of all players' problems. Therefore, it may be reasonable to try to find a stationary Nash equilibrium, which consists of stationary points of all players' problems. Here, a stationary point means a point that satisfies the first-order optimality condition for each player's problem.

We denote by

\[
L^k(x, \lambda^k) := -f_k(x) + \sum_{i=1}^{5} \lambda^k_i g^k_i(x) + \mu_k g^k_0(x), \ k = 1, 2, 3
\]
the Lagrangian of player $k$. Hence we can write down concatenating KKT conditions of problem $(P_1) - (P_3)$ as follows:

$$-\nabla x_i f_i(x) + \sum_{i=1}^{5} \lambda_i^1 \nabla x_i g_i^1(x) + \mu_i \nabla x_i g_i^0(x) = 0. \tag{20}$$

$$\lambda_i^1 g_i^1(x) = 0, \mu_1 g_i^0(x) = 0, \lambda_i \geq 0, g_i^1(x) \leq 0, \mu_i \geq 0, g_i^0(x) \leq 0, \forall i = 1, ..., 5. \tag{21}$$

$$-\nabla x_2 f_2(x) + \sum_{i=1}^{5} \lambda_i^2 \nabla x_2 g_i^1(x) + \mu_2 \nabla x_2 g_i^0(x) = 0, \tag{22}$$

$$\lambda_i^2 g_i^1(x) = 0, \mu_2 g_i^0(x) = 0, \lambda_i \geq 0, g_i^1(x) \leq 0, \mu_2 \geq 0, g_i^0(x) \leq 0, \forall i = 1, ..., 5. \tag{23}$$

$$-\nabla x_3 f_3(x) + \sum_{i=1}^{5} \lambda_i^3 \nabla x_3 g_i^1(x) + \mu_3 \nabla x_3 g_i^0(x) = 0, \tag{24}$$

$$\lambda_i^3 g_i^1(x) = 0, \mu_3 g_i^0(x) = 0, \lambda_i \geq 0, g_i^1(x) \leq 0, \mu_3 \geq 0, g_i^0(x) \leq 0, \forall i = 1, ..., 5. \tag{25}$$

There is the following relationship between the generalized Nash equilibrium problem $(P_1) - (P_3)$ and Malfatti’s problem (12) – (19).

**Proposition 1.** A generalized Nash equilibrium of problem $(P_1) - (P_3)$ satisfies the optimality conditions for Malfatti’s problem.

**Proof.** In order to write down the optimality conditions for problem (12) – (19), we introduce the Lagrangian:

$$L(x, \lambda, \nu, \tau) = -\pi \sum_{i=1}^{3} (x_3^i)^2 + \sum_{i=1}^{3} \lambda_i^1 g_i^1(x) + \sum_{i=1}^{3} \lambda_i^2 g_i^2(x) + \sum_{i=1}^{3} \lambda_i^3 g_i^3(x) +$$

$$+ \nu_1 g_1(x) + \nu_2 g_2(x) + \nu_3 g_3(x) - \tau_1 g_1^1(x) - \tau_2 g_1^2(x) - \tau_3 g_1^3(x).$$

Then the KKT conditions are:

$$\frac{\partial L}{\partial x_i^i} = -2\pi x_i^i + \sum_{i=1}^{3} \lambda_i^1 \frac{\partial g_i^1(x)}{\partial x_i^i} + \sum_{i=1}^{3} \lambda_i^2 \frac{\partial g_i^2(x)}{\partial x_i^i} + \sum_{i=1}^{3} \lambda_i^3 \frac{\partial g_i^3(x)}{\partial x_i^i} +$$

$$+ \nu_1 \frac{\partial g_1(x)}{\partial x_i^i} + \nu_2 \frac{\partial g_2(x)}{\partial x_i^i} + \nu_3 \frac{\partial g_3(x)}{\partial x_i^i} - \tau_j = 0, \text{ for } i = 3, j = 1, 2, 3, \tag{26}$$

$$\frac{\partial L}{\partial x_i^j} = \sum_{i=1}^{3} \lambda_i^1 \frac{\partial g_i^1(x)}{\partial x_i^j} + \sum_{i=1}^{3} \lambda_i^2 \frac{\partial g_i^2(x)}{\partial x_i^j} + \sum_{i=1}^{3} \lambda_i^3 \frac{\partial g_i^3(x)}{\partial x_i^j} +$$

$$+ \nu_1 \frac{\partial g_1(x)}{\partial x_i^i} + \nu_2 \frac{\partial g_2(x)}{\partial x_i^i} + \nu_3 \frac{\partial g_3(x)}{\partial x_i^i} = 0, \text{ for } i \neq 3, i, j = 1, 2, 3, \tag{27}$$

$$g_i^1(x) \leq 0, \lambda_i^1 g_i^1(x) = 0, \nu_i g_i(x) = 0, \nu_i \geq 0, \nu_i \geq 0, \tau_j \geq 0, i, j = 1, 2, 3. \tag{28}$$

$$g_i(x) \leq 0, i, j = 1, 2, 3. \tag{29}$$

$$\tau_j (x_j^i) = 0, x_j^i \geq 0, i, j = 1, 2, 3, \tag{30}$$

$$\lambda_i^1 \geq 0, \nu_i \geq 0, \tau_j \geq 0, \text{ for } i, j = 1, 2, 3. \tag{31}$$

It can be checked that if we sum up (20), (22) and (24) in the concatenating KKT conditions of problem $(P_1) - (P_3)$, then we obtain

$$- \sum_{i=1}^{3} \nabla x_i f_i(x) + \sum_{i=1}^{5} \lambda_i^1 \nabla x_i g_i^1(x) + \sum_{i=1}^{5} \lambda_i^2 \nabla x_i g_i^2(x) + \sum_{i=1}^{5} \lambda_i^3 \nabla x_i g_i^3(x) +$$

$$+ \nu_1 g_1(x) + \nu_2 g_2(x) + \nu_3 g_3(x) - \tau_1 g_1^1(x) - \tau_2 g_1^2(x) - \tau_3 g_1^3(x). \tag{32}$$
linear inequalities: given vertices of a triangle. The triangle set is given by the following system of
Example 1 vertices. solving problem (P) Matlab R2017b. We run the algorithm: 'trust-region-reflective' and 'exitflag=1'. For
are made on an Intel(R) Core(TM) i7, CPU@ 3.20GHz processor workstation under
concatenating KKT conditions which have 27 unknowns of each players. The runs
problems. In computation, we use Matlab’s lsqnonlin() function for solving the
λ must be solved, in some cases with bounds λ ≥ 0 on the Lagrange multipliers
in various variables (some Lagrange multipliers, some the original unknowns) which
4.
Computational Results. The KKT conditions reduced to nonlinear equations
in various variables (some Lagrange multipliers, some the original unknowns) which
must be solved, in some cases with bounds λ ≥ 0 on the Lagrange multipliers
corresponding to inequality constraints. The command lsqnonlin() can solve such
problems. In computation, we use Matlab’s lsqnonlin() function for solving the
concatenating KKT conditions which have 27 unknowns of each players. The runs
are made on an Intel(R) Core(TM) i7, CPU@ 3.20GHz processor workstation under
Matlab R2017b. We run the algorithm: ‘trust-region-reflective’ and ‘exitflag=1’.
For solving problem (P1) – (P3) numerically, we consider the following triangles with
vertices.
Example 1 We consider the following example. A(2, 2), B(4, 19), C(24,8) are
given vertices of a triangle. The triangle set is given by the following system of
linear inequalities:

\[-17x_1 + 2x_2 ≤ -30,\]
\[11x_1 + 20x_2 ≤ 424,\]
\[6x_1 - 22x_2 ≤ -32.\]

Then the generalized Nash problem (P1) – (P3) is formulated as:

\[ (P1) \max_{x_1} f_1(x) = \pi(x_1^3), \]

\[ X_1(x_{-1}) = \{ x_1 ∈ \mathbb{R}^3 | g_1^1(x) = -17x_1^2 + 2x_2 + 17.11x_3^3 + 30 ≤ 0, \]
\[ g_1^2(x) = 11x_1^2 + 20x_2^2 + 22.82x_3^2 - 424 ≤ 0 \]
\[ g_1^3(x) = 6x_1^2 - 22x_2 + 22.80x_3^2 + 32 ≤ 0 \]
\[ g_1^4(x) = (x_3^3 + x_3^2)^2 - (x_1^2 - x_1^3)^2 - (x_2^2 - x_2^3)^2 ≤ 0, \]
\[ g_1^5(x) = (x_3^3 + x_3^2)^2 - (x_1^3 - x_1^2)^2 - (x_2^3 - x_2^2)^2 ≤ 0, \]
\[ g_1^6(x) = -x_3 ≤ 0 \} \]

\[ (P2) \max_{x_2} f_2(x) = \pi(x_2^3), \]

\[ X_2(x_{-2}) = \{ x_2 ∈ \mathbb{R}^3 | g_2^1(x) = -17x_1^2 + 2x_2^2 + 17.11x_3^2 + 30 ≤ 0, \]
\[ g_2^2(x) = 11x_1^2 + 20x_2^2 + 22.82x_3^2 - 424 ≤ 0 \]
\[ g_2^3(x) = 6x_1^2 - 22x_2 + 22.80x_3^2 + 32 ≤ 0 \]
\[ g_2^4(x) = (x_3^3 + x_3^2)^2 - (x_1^2 - x_1^3)^2 - (x_2^2 - x_2^3)^2 ≤ 0, \]
\[ g_2^5(x) = (x_3^3 + x_3^2)^2 - (x_1^3 - x_1^2)^2 - (x_2^3 - x_2^2)^2 ≤ 0, \]
\[ g_2^6(x) = -x_3 ≤ 0 \} \]

\[ (P3) \max_{x_3} f_3(x) = \pi(x_3^3), \]
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\[ X_3(x_{-3}) = \{ x_3 \in \mathbb{R}^3 \mid g_1^3(x) = -17x_1^3 + 2x_2^3 + 17.11x_3^3 + 30 \leq 0, \]
\[ g_2^3(x) = 11x_1^3 + 20x_2^3 + 22.82x_3^3 - 424 \leq 0 \]
\[ g_3^3(x) = 6x_1^3 - 22x_2^3 + 22.80x_3^3 + 32 \leq 0 \]
\[ g_4^3(x) = (x_3^1 + x_3^3)^2 - (x_1^3 - x_1^1)^2 - (x_2^3 - x_2^1)^2 \leq 0, \]
\[ g_5^3(x) = (x_3^1 + x_3^3)^2 - (x_1^3 - x_1^1)^2 - (x_2^3 - x_2^1)^2 \leq 0, \]
\[ g_6^3(x) = -x_3^1 \leq 0 \} \]

For numerical computation, we choose initial point \( x_0(j) = \text{randi}(8,1), \ j = 1, 27 \).
We have found a stationary Nash equilibrium point as the following:

Player 1 controls
\[ x_1^{*1} = 13.1078, \ x_2^{*1} = 9.2945, \ x_3^{*1} = 4.1149, \]

Player 2 controls
\[ x_1^{*2} = 6.8268, \ x_2^{*2} = 13.5090, \ x_3^{*2} = 3.4490, \]

Player 3 controls
\[ x_1^{*3} = 6.0159, \ x_2^{*3} = 6.6644, \ x_3^{*3} = 3.4434. \]

\[ X_1(x_{-1}) = \{ x_1 \in \mathbb{R}^3 \mid g_1^1(x) = -4x_1^1 + 3x_2^1 + 5x_3^1 \leq 0, \]
\[ g_2^1(x) = 6x_1^1 - 8x_2^1 + 10x_3^1 \leq 0, \]
\[ g_3^1(x) = -2x_1^1 + 5x_2^1 + \sqrt{29}x_3^1 \leq 14 \]

**Example 2** For a test purpose, the triangle with vertices \( A(0,0), B(3,4) \) and \( C(8,6) \) has been considered. The triangle set is given by the following system of linear inequalities:
\[ -4x_1 + 3x_2 + 5x_3 \leq 0, \]
\[ 6x_1 - 8x_2 + 10x_3 \leq 0, \]
\[ -2x_1 + 5x_2 + \sqrt{29}x_3 \leq 14. \]

The generalized Nash problem \((P_1) - (P_3)\) is formulated as:
\[ (P_1) \max_{x_1} f_1(x) = \pi(x_1^1)^2, \]

\[ X_1(x_{-1}) = \{ x_1 \in \mathbb{R}^3 \mid g_1^1(x) = -4x_1^1 + 3x_2^1 + 5x_3^1 \leq 0, \]
\[ g_2^1(x) = 6x_1^1 - 8x_2^1 + 10x_3^1 \leq 0, \]
\[ g_3^1(x) = -2x_1^1 + 5x_2^1 + \sqrt{29}x_3^1 \leq 14. \]


\[ g^1_3(x) = (x_3^1 + x_3^2)^2 - (x_1^2 - x_1^1)^2 - (x_2^2 - x_2^1)^2 \leq 0, \]
\[ g^2_3(x) = (x_3^1 + x_3^2)^2 - (x_1^2 - x_1^1)^2 - (x_2^2 - x_2^1)^2 \leq 0, \]
\[ g^3_0(x) = -x_3^1 \leq 0 \]

\[(P_2) \max_{x_2} f_2(x) = \pi(x_3^2)^2,\]
\[X_2(x-2) = \{x_2 \in \mathbb{R}^3 \mid g_1^0(x) = -4x_1^2 + 3x_2^2 + 5x_3^2 \leq 0, \]
\[g_2^0(x) = 6x_1^2 - 8x_2^2 + 10x_3^2 \leq 0, \]
\[g_3^0(x) = -2x_1^2 + 5x_2^2 + 5x_3^2 \leq 14 \]
\[g_4^0(x) = (x_3^1 + x_3^2)^2 - (x_1^2 - x_1^1)^2 - (x_2^2 - x_2^1)^2 \leq 0, \]
\[g_5^0(x) = (x_3^1 + x_3^2)^2 - (x_1^2 - x_1^1)^2 - (x_2^2 - x_2^1)^2 \leq 0, \]
\[g_6^0(x) = -x_3^2 \leq 0 \]

\[(P_3) \max_{x_3} f_3(x) = \pi(x_3^3)^2,\]
\[X_3(x-3) = \{x_3 \in \mathbb{R}^3 \mid g_1^0(x) = -4x_1^2 + 3x_2^2 + 5x_3^2 \leq 0, \]
\[g_2^0(x) = 6x_1^2 - 8x_2^2 + 10x_3^2 \leq 0, \]
\[g_3^0(x) = -2x_1^2 + 5x_2^2 + 5x_3^2 \leq 14 \]
\[g_4^0(x) = (x_3^1 + x_3^2)^2 - (x_1^2 - x_1^1)^2 - (x_2^2 - x_2^1)^2 \leq 0, \]
\[g_5^0(x) = (x_3^1 + x_3^2)^2 - (x_1^2 - x_1^1)^2 - (x_2^2 - x_2^1)^2 \leq 0, \]
\[g_6^0(x) = -x_3^3 \leq 0 \].

From numerical computation, we present here two cases respect to three player’s stationary equilibrium points:

**Case I**
Player 1 controls
\[ x_1^{s1} = 3.7999, \quad x_2^{s1} = 3.0429, \quad x_3^{s1} = 0.1544, \]
Player 2 controls
\[ x_1^{s2} = 4.0198, \quad x_2^{s2} = 3.7631, \quad x_3^{s2} = 0.5987, \]
Player 3 controls
\[ x_1^{s3} = 3.0469, \quad x_2^{s3} = 3.0469, \quad x_3^{s3} = 0.6094. \]

**Case II.**
Player 1 controls
\[ x_1^{s1} = 5.3057, \quad x_2^{s1} = 4.4858, \quad x_3^{s1} = 0.4052, \]
Player 2 controls
\[ x_1^{s2} = 4.4925, \quad x_2^{s2} = 4.0288, \quad x_3^{s2} = 0.5276, \]
Player 3 controls
\[ x_1^{s3} = 3.4339, \quad x_2^{s3} = 3.4339, \quad x_3^{s3} = 0.6868. \]
Example 3 Now we consider the equilateral triangle with vertices $A(0,0)$, $B(3,3\sqrt{3})$ and $C(6,0)$ has been considered. Then triangle set is given by the following system of linear inequalities:

\[-\sqrt{3}x_1 + x_2 + 2x_3 \leq 0,\]
\[\sqrt{3}x_1 + x_2 + 2x_3 \leq 6\sqrt{3},\]
\[-x_2 + x_3 \leq 0.\]

The generalized Nash problem $(P_1) - (P_3)$ is formulated as:

\[(P_1) \max_{x_1} f_1(x) = \pi(x_3^1)^2,\]
\[X_1(x_{-1}) = \{ x_1 \in \mathbb{R}^3 | g_1^1(x) = -\sqrt{3}x_1^1 + x_2^1 + 2x_3^1 \leq 0,\]
\[g_2^1(x) = \sqrt{3}x_1^1 + x_2^1 + 2x_3^1 \leq 6\sqrt{3},\]
\[g_3^1(x) = -x_2^1 + x_3^1 \leq 0\]
\[g_4^1(x) = (x_3^1 + x_3^2)^2 - (x_2^2 - x_1^1)^2 - (x_2^2 - x_1^1)^2 \leq 0,\]
\[g_5^1(x) = (x_3^1 + x_3^3)^2 - (x_2^3 - x_1^1)^2 - (x_2^3 - x_1^1)^2 \leq 0,\]
\[g_6^1(x) = -x_3^1 \leq 0\]
\[(P_2) \quad \max_{x_2} f_2(x) = \pi(x_3^2)^2,\]
\[X_2(x_2) = \{ x_2 \in \mathbb{R}^3 \mid g_1^2(x) = -\sqrt{3}x_1^2 + x_2^2 + 2x_3^2 \leq 0, \]
\[g_2^2(x) = \sqrt{3x_1^2 + x_2^2 + 2x_3^2} \leq 6\sqrt{3}, \]
\[g_3^2(x) = -x_2^2 + x_3^2 \leq 0 \]
\[g_4^2(x) = (x_3^2 + x_3^3)^2 - (x_1^2 - x_1^2)^2 - (x_2^2 - x_2^2)^2 \leq 0, \]
\[g_5^2(x) = (x_3^2 + x_3^3)^2 - (x_1^2 - x_1^2)^2 - (x_2^2 - x_2^2)^2 \leq 0, \]
\[g_6^2(x) = -x_3^2 \leq 0 \} \]

\[(P_3) \quad \max_{x_3} f_3(x) = \pi(x_3^3)^2,\]
\[X_3(x_3) = \{ x_3 \in \mathbb{R}^3 \mid g_1^3(x) = -\sqrt{3}x_1^3 + x_2^3 + 2x_3^3 \leq 0, \]
\[g_2^3(x) = \sqrt{3x_1^3 + x_2^3 + 2x_3^3} \leq 6\sqrt{3}, \]
\[g_3^3(x) = -x_2^3 + x_3^3 \leq 0 \]
\[g_4^3(x) = (x_3^3 + x_3^3)^2 - (x_1^3 - x_1^3)^2 - (x_2^3 - x_2^3)^2 \leq 0, \]
\[g_5^3(x) = (x_3^3 + x_3^3)^2 - (x_1^3 - x_1^3)^2 - (x_2^3 - x_2^3)^2 \leq 0, \]
\[g_6^3(x) = -x_3^3 \leq 0 \} \]

We present here also two cases respect to three player’s stationary equilibrium points:

**Case 1**

Player 1 controls
\[x_1^{*1} = 3.0000, \quad x_2^{*1} = 3.0000, \quad x_3^{*1} = 1.0981, \]

Player 2 controls
\[x_1^{*2} = 1.9019, \quad x_2^{*2} = 1.0981, \quad x_3^{*2} = 1.0981, \]

Player 3 controls
\[x_1^{*3} = 4.0981, \quad x_2^{*3} = 1.0981, \quad x_3^{*3} = 1.0981. \]

**Case 2**

![Figure 4. Case 1](image-url)
Player 1 controls
\[ x_1^* = 1.9019, \quad x_2^* = 1.0981, \quad x_3^* = 1.0981, \]

Player 2 controls
\[ x_1^* = 3.0000, \quad x_2^* = 3.0000, \quad x_3^* = 1.0981, \]

Player 3 controls
\[ x_1^* = 4.0981, \quad x_2^* = 1.0981, \quad x_3^* = 1.0981. \]

**Conclusion.** In this paper, we for the first time examine Malfatti’s problem from a view point of generalized Nash equilibrium approach. We investigate relations between Malfatti’s problem and the generalized Nash equilibrium problem. Numerical results are given.

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E-mail address: renkhbat46@yahoo.com
E-mail address: battur@seas.num.edu.mn