Electron trimer states in conventional superconductors

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We expand the Cooper problem by including a third electron in an otherwise empty band. We demonstrate the formation of a trimer state of two electrons above the Fermi sea and the third electron, for sufficiently strong inter-band attractive interaction. We show that the critical interaction strength is the lowest for small Fermi velocities, large masses of the additional electron and large Debye energy. This trimer state competes with the formation of the two-electron Cooper pair, and can be created transiently via optical pumping.

In a seminal paper, Ref. [1], L.N. Cooper showed that two electrons immersed in a Fermi sea form a bound state for arbitrarily weak attractive interactions. The Cooper problem assumes the dominance of the effective interaction induced by the electron-phonon interaction over the screened Coulomb potential [2–5]. It is modeled as constant in momentum space within a narrow energy range of the order of the Debye energy for the relative kinetic energy of the electrons. This simplified model distills the key features and energy scales of the full interaction induced by electron-phonon coupling that are relevant for the formation of the bound state and its properties. The existence of this bound state indicates that the noninteracting Fermi sea is unstable against pair formation, which suggests the emergence of a superconducting state. A more extensive theory of this state was provided by BCS theory [2,11], which elaborated on the essential ingredients that are necessary for the formation of conventional superconductors, pointed out by the Cooper problem and its solution.

In this Letter, we expand the Cooper problem by including a third electron, as depicted in Fig. 1. We assume that this additional electron, labeled ‘1’, is in an otherwise empty band with quadratic dispersion relation, ε₁. Its spin state is arbitrarily depicted as spin-up. The two electrons ‘2’ and ‘3’ are restricted to be outside of an inert Fermi sea of momentum k_F and of energy E_F. For simplicity, the dispersion ε₂ of the lower band is assumed to be quadratic. We propose to realize this scenario by optically pumping electrons from a lower band into an unoccupied band, using current technology of pump-probe experiments [12–15]. This results in a low, metastable electron density in the upper band. We assume that the effective interaction between the electrons is attractive, following the reasoning of the Cooper problem. We consider the interaction between two electrons to be a negative constant g_ij, with i, j = 1, 2, 3 and i ≠ j, for the incoming and outgoing momentum of particle i smaller than a cutoff Λ_i, and zero otherwise. We choose the values of Λ_1 and Λ_2 = Λ_3 such that

$$E_D = \frac{\hbar^2}{2m_i} \Lambda_i^2 = \frac{\hbar^2}{2m_2} (\Lambda_2^2 - k_F^2),$$

(1)

where m_i is the effective mass of particle i and E_D is the Debye energy, see Fig. 1. For clarity, we allow for three different masses m_1, m_2, and m_3, but are primarily interested in the case m_2 = m_3. A similar restriction will be placed on g_12 and g_13, which we choose to be equal throughout this Letter. For a typical conventional superconductor we have E_D ≪ E_F, implying that Λ_1 ≪ k_F and Λ_2 = k_F, which we will use as small parameters.

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ε that this additional electron, labeled ‘1’, is in an other-

Figure 1. The expanded Cooper problem consists of two electrons in a partially filled band with dispersion ε_2 and a third electron in an empty band with dispersion ε_1. The inert Fermi sea in the lower band has a Fermi energy E_F and Fermi momentum k_F. The three electrons interact attractively via the two-body interactions g_{12}, g_{13}, and g_{23}, with g_{12} = g_{13}. These interactions are cut off in momentum space by the cutoffs Λ_1 and Λ_2, whose magnitudes are chosen to fulfill ε_1(Λ_1) = ε_2(Λ_2) = E_F = E_D, where E_D is the Debye energy.

Further down. The Fermi velocity is $v_F = \hbar k_F/m_2$. We define the length scale

$$r_D = (\Lambda_2 - k_F)^{-1} \approx \frac{\hbar v_F}{E_D}. \quad (2)$$

We note that in the following we consider three-body states with vanishing total momentum.

The main result of our analysis is shown in Fig. 2. We depict whether the lowest energy state is a Cooper pair state or a trimer state, as a function of the interaction parameters $\xi_{23}$ and $\xi_{12} = \xi_{13}$, where $\xi_{23} = 2\mu/(4\pi\hbar^2)g_{23}$ and $\xi_{12} = 2\mu/(4\pi\hbar^2)g_{12}$. The reduced masses are $1/\mu = 1/m_1 + 1/m_3 = 2/m_2$ and $1/\mu = 1/m_1 + 1/m_2$.

For $g_{12} = 0$ we recover the result of the Cooper problem. For any value of $g_{23} < 0$ the electrons in the lower band form a pair. As $g_{12}$ is set to a negative nonzero value, we show that the three electrons form a trimer state beyond a critical value of $g_{12}$, which increases in absolute magnitude as $|g_{23}|$ increases. This trimer formation also occurs for vanishing $g_{23}$. From the perspective of the electrons in the lower band, this can also be considered as a bound state formation that is induced by a third electron in an higher band, i.e., a particle-induced bound state. As we show below, the magnitude of the critical value of $g_{12}$ for trimer formation is controlled by the ratio of the Fermi velocity, and the mass of the electron in the upper band and the Debye energy. If the mass of the electrons in the upper band is heavier, the critical value is reduced. Similarly, a smaller Fermi velocity and a larger Debye energy reduces the critical value.

We note that for the typical parameter regime of conventional superconductors we find one trimer state only [10]. However, we also give an example for a parameter regime in which more than one trimer state exists below.

For a pump-probe experiment, this result implies that a system that is initially either in a superconducting or a metallic state can be transformed into a Fermi liquid of electron trimers when electrons are pumped into a higher band, and the attractive inter-band interaction is sufficiently strong.

To determine the bound states of the three-electron system we consider the Schrödinger equation in momentum space:

$$\left( \begin{array}{c} \hbar^2 k_1^2 & \hbar^2 k_2^2 & \hbar^2 k_3^2 \\ 2m_1 & 2m_2 & 2m_3 \\ \hat{U}_{12} + \hat{U}_{13} + \hat{U}_{23} - E \end{array} \right) \psi = 0,$$

where $k_1$, $k_2$, $k_3$ are the electron momenta, $E$ is the energy [17], and $\psi = \psi(k_1, k_2, k_3)$ is the wave function. The interaction $\hat{U}_{ij}$ between two electrons ‘$i$’ and ‘$j$’ is

$$\hat{U}_{ij} = g_{ij}\theta_{\Lambda}(k_i)\theta_{\Lambda}(k_j) \int \frac{d^3q}{(2\pi)^3}\delta(k_i - q)\theta_{\Lambda}(k_j + q),$$

where $g_{ij} < 0$ and $q$ denotes the momentum transfer [18]. The resulting operators $\hat{U}_{ij}\psi$ are represented in App. A. The cutoff function $\theta_{a,b}(k)$ is defined as

$$\theta_{a,b}(k) = \begin{cases} 1 & \text{for } a \leq |k| \leq b, \\ 0 & \text{otherwise}, \end{cases}$$

for two real numbers $0 \leq a < b$, and $\theta_{0,b}(k) = \theta_{0,0}(k)$. The inert Fermi sea demands the constraints $k_2 > k_F$ and $k_3 > k_F$ on the momenta of electrons ‘2’ and ‘3’, respectively. We consider a singlet state for the electrons ‘2’ and ‘3’ in the following. This system is separable.
Figure 4. The resulting three-body eigenenergies $E$ versus the interaction parameter $|\xi_{12}|$, in units of $r_D$, with $g_{23} = 0$ and $E_D/E_F = 0.02$, for (a) $m_2/m_1 = 1$, which corresponds to the vertical dashed red line in Fig. 2, (b) $m_2/m_1 = 10$, and (c) $m_2/m_1 = 1/10$. In each panel the single blue curve is the numerical solution of the lowest energy trimer state. As $|\xi_{12}|$ increases, the first pair (dimer 12) appears as the lowest energy state of a two-body bound state continuum depicted by dense blue curves. The vertical arrow locates the critical value of the inter-band interaction parameter given by Eq. (9), which is in good agreement with the onset of the numerical result. The dashed red curve shows the analytical approximation for the trimer state, Eq. (D.1), derived in App. D. It represents a good approximation for the asymptotic of the single blue curve, which is depicted by thicker red. The solid red curve shows the analytical approximation for the lowest energy pair (dimer 12), Eq. (D.2), derived in App. D, which is in good agreement with numerics. In panel (c) the green dashed curve is the second analytical approximation (10). The thicker green curve represents a good approximation for the asymptotic of the single blue curve as long as $m_1 \gg m_2$.

as shown in App. A, which results in a system of two coupled integral equations:

\[
\left[\frac{1}{g_{12}} + \int \frac{d^3p_3}{(2\pi)^3} K_1(k_2, p_3; E)\right] F_2(k_2) = -\theta_{k_2, A_2}(k_2) \left[\int \frac{d^3p_3}{(2\pi)^3} K_1(k_2, p_3; E) F_2(p_3) + \int \frac{d^3p_1}{(2\pi)^3} K_2(k_2, p_1; E) F_1(p_1)\right],
\]

\[
\left[\frac{1}{g_{23}} + \int \frac{d^3p_3}{(2\pi)^3} K_3(k_1, p_3; E)\right] F_1(k_1) = -2\theta_{A_1}(k_1) \int \frac{d^3p_3}{(2\pi)^3} K_3(k_1, p_3; E) F_2(p_3),
\]

where $p_i = k_i - q_i$ for $i = 1, 2$, and $p_3 = k_3 + q_3$. The three integral kernels $K_1, K_2, K_3$ and the three functions $F_1, F_2, F_3$ are derived in App. A. Due to the singlet symmetry for electrons ‘2’ and ‘3’ we consider $F_2 = F_3$. We assume $F_i(k) = F_i(p)$, implying $s$-wave symmetry of the state.

To determine whether the lowest energy state is a two-body or a three-body bound state, we use the small parameter $E_D/E_F$ to approximate the full integral equation with an equation that relates $\xi_{23}, \xi_{12},$ and $E$, as described on App. B. We solve this equation numerically for energies near the threshold energy $E_{thr} = 2E_F = \hbar^2k_F^2/m_2$ which results in Fig. 2 depicting a region where electrons ‘2’ and ‘3’ form a Cooper pair, and a second region where the three electrons form a trimer state. The intra-band electrons ‘2’ and ‘3’ can form a Cooper pair for an attractive interaction $\xi_{23}$. The trimer state is only formed when the inter-band interaction $\xi_{12}$ is sufficiently strong. Trimer states of zero total momentum appear as discrete energy levels, whereas dimer states appear as continuum of states.

Next, we solve the full Eqs. (6) and (7) numerically. For that, we reduce the three dimensional integrals over momentum to one dimensional integrals over the absolute value of each momentum. We approximate the integrals by a sum over discrete values according to the Gauss-Legendre quadrature rule [19–21]. The continuous functions $F_1$ and $F_2$ are evaluated at these discrete momentum values. We therefore approximate Eqs. (6) and (7) with a discrete eigenvalue problem, see App. C.

In Fig. 3 we show the resulting eigenenergies $E$ below the threshold energy $E_{thr}$, for $E_D/E_F = 0.02$ and $\xi_{12} = -3r_D$. For small values of $|\xi_{23}|$ the lowest energy state is a trimer state which appears as a single line of solutions. For larger values the lowest energy state is a Cooper pair which appears as the lowest energy state of a two-body bound state continuum.

To estimate the critical value of $g_{12}$ for vanishing $g_{23}$ analytically, we recall that $F_1 \propto g_{23}$ and $F_2 \propto g_{12}$, cf. App. A. With this, Eqs. (6) and (7) reduce to...
Eqs. (D.1) and (D.2), respectively. These solutions are two-body bound state, is derived in App. D, based on electrons in the upper band, for a smaller Fermi velocity, the critical interaction strength is achieved for heavier finite.

The dense curves show the two-body bound state continuum. When $m_2 \gg m_1$ we can see the formation of more than one trimer states, however, the number of the states will remain finite.

\[ \frac{4\pi\hbar^2}{2\mu g_{12}} \frac{\tau}{m_1 \pi k_F} \int_{k_F}^{\Lambda_2} dp_3 p_3 \times \ln \left( \frac{(1 - \frac{\mu}{m_1})p_3^2 + (1 - \frac{\mu}{m_1})k_F^2 + \frac{\mu}{m_1} \Lambda_2^2 - \frac{2\mu}{\tau}E}{p_3^2 - \frac{2\mu}{m_1}k_Fp_3 + k_F^2 - \frac{2\mu}{\tau}E} \right) \approx 0, \]

where $\tau = 1$ describes the system of three electrons and $\tau = 1/2$ corresponds to a system of two electrons ‘1’ and ‘2’ (or ‘3’), see App. D. We evaluate the integral, which is done in App. B and D, and expand to lowest order in $E_D/E_F$. We solve for the interaction parameter $g_{12}$, then choose the threshold condition $E = E_{\text{thr}}$, which finally gives the critical value

\[ |g_{12}^{(c)}| \approx \frac{2\pi^2\hbar^2}{m_1} r_D \approx \frac{2\pi^2\hbar^3 v_F}{m_1 E_D}, \]

where $v_F = \hbar k_F/m_2$. This shows that a lower value of the critical interaction strength is achieved for heavier electrons in the upper band, for a smaller Fermi velocity, and a higher Debye energy.

An approximate analytical solution of Eq. (8), which describes both the trimer state and the lowest energy two-body bound state, is derived in App. D, based on Eqs. (D.1) and (D.2), respectively. These solutions are depicted as the dashed and the continuous line in Figs. (a)-(c). For $m_1 \gg m_2$ and $E_D \ll E_F$ we have $|g_{12}^{(c)}| = 2\mu/(4\pi\hbar^2)|g_{12}^{(c)}| \rightarrow 0$. In this case, a second analytical approximate solution of Eq. (8) can be represented by:

\[ E \approx E_{\text{thr}} + \frac{E_D}{1 - \exp \left( \frac{\pi \frac{\mu}{m_1} r_F}{k_F} \right)}, \]

see App. D. This approximation is shown as a green dashed line in Fig. (c). For larger $m_1/m_2$ Eq. (10) becomes a better approximation. We compare these analytical results with the numerical results for different parameter sets in Fig. In (a) we use $m_1 = m_2$ and $E_D/E_F = 0.02$. In (b) we use $m_2/m_1 = 10$ and $E_D/E_F = 0.02$. In (c) we use $m_2/m_1 = 1/10$ and $E_D/E_F = 0.02$. We observe the formation of a trimer state, that is well approximated by the analytical approximation. In contrast to the Cooper problem where a two-body bound state originates at a vanishing coupling constant [1, 5, 7], here, a trimer state emerges at the critical value $\xi_{12}^{(c)}$.

Finally, for $m_2 \gg m_1$ and large values of the Debye energy that are comparable with Fermi energy, $E_D \sim E_F$, we observe the formation of more than one trimer state [19]. Figure (c) shows the formation of two trimer states for $m_2/m_1 = 10$. Physical systems in this regime are superconductors like fullerides [22] or magnesium diboride [23].

In conclusion, we have demonstrated the formation of a trimer state of electrons in a conventional superconductor, in which an additional electron occupies a higher band. We show this by expanding the Cooper problem of two attractively interacting electrons by adding an additional electron that also interacts attractively with the other electrons. The trimer formation sets in beyond a critical inter-band interaction strength, for which we give an analytical estimate. This demonstrates an instability of the BCS state. We propose to realize this scenario by optically pumping electrons to otherwise single empty band. Out of the initial superconducting or metallic state, a transient state of a Fermi liquid of electron trimers can be formed. We also emphasize that the analogue of the Cooper problem can also be formulated for orders such as antiferromagnetism or charge-density wave orders. Here, we have a two-body problem of an electron and a hole. We emphasize that the analysis of this paper can be extended to any order that is described by a two-Fermi order parameter, and predicts three-fermion bound states for all these orders for the corresponding parameter regimes.

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In this Letter, momentum is measured in units of the Fermi momentum and energy is measured in units of the threshold energy, \( E_{\text{th}} = 2E_F = \frac{\pi^2}{m^*} k_F^2 \), where \( E_F \) is the Fermi energy.

By ‘momentum transfer’ we simply mean the difference of the in-state and out-state momenta of a particle; see, J.R. Taylor, *Scattering Theory: The Quantum Theory of Nonrelativistic Collisions* (Dover, 2006).

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\[ \psi(k_2, k_3) = -\frac{\theta_{\Lambda_1}(-k_2 - k_3)\theta_{\Lambda_2}(k_2)F_3(k_3) + \theta_{\Lambda_1}(-k_2 - k_3)\theta_{\Lambda_3}(k_3)F_2(k_2) + \theta_{\Lambda_2}(k_2)\theta_{\Lambda_3}(k_3)F_1(-k_2 - k_3)}{\frac{\hbar^2(k_2 + k_3)^2}{2m_1} + \frac{\hbar^2k_2^2}{2m_2} + \frac{\hbar^2k_3^2}{2m_3} - E}. \] (A.6)

If the electrons ‘2’ and ‘3’ are in a spin singlet state and \( g_{12} = g_{13} \), then \( F_3 = F_2 \). We also assume \( m_3 = m_2 \) and take into account the Fermi sea condition by \( k_2, k_3 > k_F \). Introducing the variables \( p_1 \equiv -k_2 - p_3 \) and \( p_2 \equiv -k_1 - p_3 \), with \( p_3 \equiv p_3 \), the functions \( F_1 \) and \( F_2 \) now read:

\[
F_1(k_1) = g_{23}\int \frac{d^3p_3}{(2\pi)^3} \theta_{k_F, \Lambda_2}(-k_1 - p_3)\theta_{k_F, \Lambda_2}(p_3)\times \\
\times \psi(-k_1 - p_3, p_3), \tag{A.7}
\]

\[
F_2(k_2) = g_{12}\int \frac{d^3p_3}{(2\pi)^3} \theta_{\Lambda_1}(-k_2 - p_3)\theta_{k_F, \Lambda_2}(p_3)\times \\
\times \psi(k_2, p_3). \tag{A.8}
\]

We insert the ansatz \( (A.6) \) into Eqs. \((A.7)\) and \((A.8)\), and arrive at a system of two coupled integral equations \([6] \) and \([7] \), where the three kernels \( K_1, K_2, \) and \( K_3 \) are given by:

\[
K_1(k_2, p_3; E) = \frac{\theta_{\Lambda_1}(-k_2 - p_3)\theta_{\Lambda_2}(p_3)}{\frac{\hbar^2(k_2 + p_3)^2}{2m_1} + \frac{\hbar^2k_2^2}{2m_2} + \frac{\hbar^2k_3^2}{2m_3} - E}, \tag{A.9}
\]

\[
K_2(k_2, p_1; E) = \frac{\theta_{\Lambda_1}(p_1)\theta_{k_F, \Lambda_2}(-p_1 - k_2)}{\frac{\hbar^2p_1^2}{2m_1} + \frac{\hbar^2k_2^2}{2m_2} + \frac{\hbar^2(p_1 + k_2)^2}{2m_3} - E}, \tag{A.10}
\]

\[
K_3(k_1, p_3; E) = \frac{\theta_{k_F, \Lambda_2}(-k_1 - p_3)\theta_{\Lambda_2}(p_3)}{\frac{\hbar^2k_1^2}{2m_1} + \frac{\hbar^2(k_1 + p_3)^2}{2m_2} + \frac{\hbar^2k_3^2}{2m_3} - E}. \tag{A.11}
\]

### B. Overall behavior of the three-electron system

As mentioned, we choose the values of \( \Lambda_1 \) and \( \Lambda_2 \) according to relation \([1] \). For a typical conventional superconductor we have \( E_D \ll E_F \), implying that \( \Lambda_1 \ll k_F \) and \( \Lambda_2 - k_F \ll k_F \). We recall that \( 0 < k_1 < \Lambda_1 \) and \( k_F < k_2 < \Lambda_2 \). We thus make a first approximation such that \( k_1 \sim 0 \) and \( k_2 \sim k_F \). In addition, because the integral variable \( p_3 \) is varying within the interval \((k_F, \Lambda_2)\) and \( \Lambda_2 - k_F \ll k_F \), we make a second approximation in this interval and assume that the two functions \( F_2(k_3) \) and \( F_2(p_3) \) both remain constant, \( F_2(k_F) \). Thus, we rewrite the system of Eqs. \([6] \) and \([7] \) as follows:

\[
\begin{cases} 
\Omega_1(E, g_{12}; \tau)F_2(k_F) + \Omega_2(E)F_1(0) \approx 0, \\
\Omega_3(E)F_2(k_F) + \Omega_4(E, g_{23})F_1(0) \approx 0. 
\end{cases} \tag{B.1}
\]

We calculate \( \Omega_1(E, g_{12}; \tau) \) to be:
\[
\Omega_4(E, g_{23}) = \frac{4\pi\hbar^2}{2\mu g_{23}} + \frac{1}{2} \Omega_3(E). \quad (B.5)
\]

In order that Eq. (B.1) possesses nontrivial solutions, it is required that

\[
\Omega_1(E, g_{12}; \tau = 1)\Omega_4(E, g_{23}) - \Omega_2(E)\Omega_3(E) = 0, \quad (B.6)
\]

which gives rise to a relation between \(g_{12}\) and \(g_{23}\) through \(E\). Figure 2 shows the result for \(E \approx E_{\text{thr}}\).

C. Numerical solution of the system of two coupled integral equations (6) and (7)

As mentioned, we assume \(F_i(k) = F_i(k)\), which implies that we only consider the isotropic solutions of Eqs. (6) and (7). To solve Eqs. (6) and (7) numerically we therefore replace the three-dimensional integrals over momenta by one-dimensional integrals over absolute values of each momentum. We discretize each integral range such that the grid points \(\{x_j\}, j = 1, 2, \ldots, N\), are the set of zeros of the Legendre polynomials \(P_N(x)\). We approximate the integrals by a truncated sum weighted by \(w_j\):

\[
w_j = \frac{2}{(1 - x_j^2)|P'_N(x_j)|^2}, \quad (C.1)
\]

where \(P'_N(x) = dP_N(x)/dx\) [20][21]. This choice, which is the so-called Gauss-Legendre quadrature rule, scales the range of integration from a given interval \((a, b)\) to \((-1, 1)\), and has order of accuracy exactly \(2N-1\), which is the highest accuracy among the other quadrature choices [19]. We apply the Gauss-Legendre quadrature rule on each integral and construct a matrix equation analog to each integral equation. For given values of \(E\) below \(E_{\text{thr}}\) we calculate the eigenvalues, which then provide the corresponding values of the interaction parameter. The functions \(F_1\) and \(F_2\) are also obtained as the eigenvectors of the matrix equations. Notice that, due to the truncation on each sum, the two-body continuum revealed in Figs. 3, 4, and 5 has a finite range.

D. Derivation and solution of Eq. (8), calculation of \(g_{12}^{(c)}\), and derivation of Eq. (10)

To derive Eq. (8), recall that for vanishing \(g_{23}\) we have \(F_1 = 0\), and therefore Eq. (7) will have no effect anymore. As discussed in App. B, for a conventional superconductor we make an approximation such that \(k_1 \sim 0\) and \(k_2 \sim k_F\). The system of three electrons is then described by Eq. (3) for \(\tau = 1\). Also, the system of two electrons ‘1’ and ‘2’ (or ‘3’) will be described by the same equation when \(\tau = 1/2\). Equation (8) for \(\tau = 1\) is then solved by

\[
\Omega_1(E, g_{12}; \tau = 1) = 0, \quad (D.1)
\]

where the function \(\Omega_1(E, g_{12}; \tau)\) was calculated in App. B, see Eq. (B.2). Equation (D.1) provides now a relation between the interaction parameter \(g_{12}\) and the energy \(E\), see dashed red curves in Fig. 4. For a system of two electrons ‘1’ and ‘2’ (or ‘3’) we set \(\tau = 1/2\) and calculate the integral by the same argument. The lowest energy two-body bound state is obtained by solving

\[
\Omega_1(E, g_{12}; \tau = 1/2) = 0, \quad (D.2)
\]

see solid red curves in Fig. 4.

To calculate the onset of the trimer state analytically, we expand Eq. (D.1) for \(\Lambda_2 - k_F \ll k_F\) and \(\Lambda_1 \ll k_F\) at the threshold energy \(E_{\text{thr}} = 2E_F\), and solve the leading order for the interaction parameter \(g_{12}^{(c)} \equiv g_{12}(E = E_{\text{thr}})\), which results in Eq. (9).

To derive Eq. (10), we notice that the onset of the trimer state for \(m_1 \gg m_2\) leads to the origin, \(|g_{12}^{(c)}| \to 0\), cf. Eq. (9). In this case, to find the asymptotic of the trimer we solve the integral appearing in Eq. (B.2) by changing variable \(x \equiv p_3/k_F\). The integral bounds will be 1 and \(\Lambda_2/k_F\). For a conventional superconductor the upper bound \(\Lambda_2/k_F\) is very close to the lower bound. Therefore, we calculate the integral by making the leading order of the integrand when \(x \to 1\). Solving the result for \(E\) we arrive at Eq. (10), see dashed green curve in Fig. 4 (c).