CONVERGENCE OF SOLUTIONS TO INVERSE PROBLEMS FOR A CLASS OF VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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ABSTRACT. The paper investigates an inverse problem for a stationary variational-hemivariational inequality. The solution of the variational-hemivariational inequality is approximated by its penalized version. We prove existence of solutions to inverse problems for both the initial inequality problem and the penalized problem. We show that optimal solutions to the inverse problem for the penalized problem converge, up to a subsequence, when the penalty parameter tends to zero, to an optimal solution of the inverse problem for the initial variational-hemivariational inequality. The results are illustrated by a mathematical model of a nonsmooth contact problem from elasticity.

1. Introduction. In this paper we consider a class of inverse problems for nonlinear stationary variational-hemivariational inequalities with pseudomonotone operators. This class of inverse problems arises in many practical problems of mechanics and physics when the goal is to determine an unknown parameter in the direct problem from various measurements (observations) of the data.

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The main distinction of these inverse problems is that the corresponding direct problem, being a variational-hemivariational inequality, is highly nonlinear and contains a nonlinear pseudomonotone operator, a convex set of unilateral constraints and two nondifferentiable and nonsmooth functionals, between which at least one is convex. Typical examples of the unilateral constraint condition are represented by the Signorini contact condition and its variants which appear in contact mechanics, see e.g. [22, 24, 25, 29]. In the inverse problems under investigation, the cost functionals are of a general form and there are many possibilities to choose a class of admissible parameters and cost functionals, see, for instance, [9, 15, 16, 17, 18, 19].

The notion of hemivariational inequality has been introduced by Panagiotopoulos in 1981 as the weak formulation of important classes of unilateral and inequality problems in mechanics (see [27]). This notion is based on a concept of the generalized gradient of Clarke (see [3, 4, 22]) and serves as a generalization of variational inequality for a case where the function involved is nonconvex and nonsmooth. Variational-hemivariational inequalities appear naturally in various mechanical problems, for example, the unilateral contact problems in nonlinear elasticity, the problems describing the adhesive and friction effects, the nonconvex semipermeability problems, the masonry structures, and the delamination problems in multilayered composites (see e.g. [22, 26]). They cover boundary value problems for partial differential equations with nonmonotone, possibly multivalued and nonconvex nonlinearities. In the last few years many types of variational and hemivariational inequalities have been investigated (see [8, 28, 30]) and the study of variational-hemivariational inequalities has emerged today as a new and interesting branch of applied mathematics. The penalty method has been used in the study of variational inequalities in [29, 31] and in hemivariational inequalities in [23].

Many problems in applied sciences can conveniently be modeled as variational-hemivariational inequalities involving certain parameters. These parameters are known and they often characterize physical properties of the underlying model. In this context, the direct problem consists in solving variational-hemivariational inequality. In contrast, an inverse problem asks for the identification of parameters when a certain observation or measurement of a solution of the variational-hemivariational inequality is available. In recent years, the field of inverse and identification problems emerged as one of the most vibrant and developing branches of applied and industrial mathematics because of their wide applications, see [1, 2, 6, 10, 11] and the references therein. Inverse coefficient problems have been studied in [9, 10, 14] for variational inequalities, and in [15, 17, 18, 19, 20, 21] for hemivariational inequalities. The stability of inverse problems with respect to perturbations of the direct problem and of cost functional has been studied in [16] and a related optimal control problem has been treated in [13]. Inverse problems to identify parameters in variational-hemivariational inequalities have been studied only recently in [24]. An inverse problem of identifying variable parameters in certain variational and quasi-variational inequalities has been recently studied in [7]. There, interesting results on existence and convergence for optimization problems, penalization and optimality conditions are provided.

The present paper represents a continuation of [24]. First, we comment on existence of optimal solutions to an inverse problem for variational-hemivariational inequality. Then, we study an inverse problem for the penalized variational-hemivariational inequality in which the set of unilateral constraints is removed and a penalty operator is added to the model. For the penalized direct problem, we
also establish the existence of optimal solutions. Finally, we study the convergence of optimal solutions of the penalized problem to an optimal solution of the initial problem, as the positive penalty parameter $\lambda$ tends to zero. Such convergence is important in many inverse problems. It allows to approximate optimal solutions to inverse problem for variational-hemivariational inequality by an optimal solution to a simpler direct problem in which the unilateral constraint is replaced by a problem without constraints.

Since the direct problem is a variational-hemivariational inequality, we are able to incorporate in this setting various complicated physical phenomena modeled by nonmonotone and nondifferentiable potentials which are met in industrial processes. For instance, we refer to [24] for a class of inverse problems for a frictional contact problem from theory of elasticity to which our results can be applied. Moreover, in the last section, we study inverse problems for a unilateral contact problem for elastic materials in which the frictional contact conditions are in a subdifferential form. This contact problem is expressed as a variational-hemivariational inequality for the displacement field, for which our abstract results are applied. The technique and results discussed in this paper could be used in many other parameter identification problems and optimal control problems in mechanics.

The outline of the paper is depicted on Figure 1 and is as follows.

(1) First we study the variational-hemivariational inequality, called the direct problem $(DP)$, see Problem 1, for every fixed parameter $p$. For this problem, we state a result on its unique solvability in Theorem 3.1 and on the continuous dependence of solution with respect to a parameter in Theorems 3.2 and 3.3.

(2) Next, we investigate a penalized variational-hemivariational inequality, called the penalized direct problem $(DP)_\lambda$, see Problem 3, for every fixed penalty parameter $\lambda > 0$ and a fixed parameter $p$. For every fixed $\lambda$ and $p$, in Theorem 4.2, we prove existence and uniqueness of solution $u_\lambda = u_\lambda(p) \in X$ and show that $u_\lambda \to u$, as $\lambda \to 0$, where $u = u(p)$ is the unique solution to the problem $(DP)$.

(3) Then, we study an inverse problem $(IP)_\lambda$ associated with the problem $(DP)_\lambda$, see Problem 5. We prove in Theorem 5.1 a result on existence of optimal solutions $\{(p^*_\lambda, u_\lambda(p^*_\lambda))\}$ to inverse problem $(IP)_\lambda$ for every fixed $\lambda$.

(4) Finally, we prove, in Theorem 5.1, our main result on the convergence of optimal solutions: we can find a subsequence of $\{(p^*_\lambda, u_\lambda(p^*_\lambda))\}$ that converges to an element $(p^*, u(p^*))$ which is an optimal solution to the inverse problem $(IP)$ for $(DP)$. 

![Figure 1. Outline of the paper](image-url)
2. Preliminaries. In this section we recall basic notation and definitions, see [3, 4, 22] for more details.

Let \((X, \| \cdot \|_X)\) be a Banach space, by \(X^*\) we denote its dual and by \(\langle \cdot, \cdot \rangle_X\) the duality pairing between \(X^*\) and \(X\). The space \(X\) endowed with the weak topology is denoted by \(X_w\). We denote by \(\rightharpoonup\) the strong convergence and by \(\rightharpoonup^*\) the weak convergence. The space of linear and bounded operators from a Banach space \(E\) to a Banach space \(F\) will be denoted by \(\mathcal{L}(E, F)\).

Definition 2.1. Given a locally Lipschitz function \(g: X \to \mathbb{R}\) on a Banach space \(X\), we denote by \(g^0(u; v)\) the generalized (Clarke) directional derivative of \(g\) at the point \(u \in X\) in the direction \(v \in X\) defined by

\[
g^0(u; v) = \limsup_{\lambda \to 0^+, w \to u} \frac{g(w + \lambda v) - g(w)}{\lambda},
\]

The generalized gradient of \(g: X \to \mathbb{R}\) at \(u \in X\) is defined by

\[
\partial g(u) = \{ u^* \in E^* | g^0(u; v) \geq \langle u^*, v \rangle \text{ for all } v \in X \}.
\]

A locally Lipschitz function \(g\) is said to be regular in the sense of Clarke at \(u \in X\) if for all \(v \in X\) the one-sided directional derivative \(g'(u; v)\) exists and \(g^0(u; v) = g'(u; v)\).

Recall the following definitions for a multivalued operator defined on a reflexive Banach space \(X\). An operator \(F: X \to 2^{X^*}\) is monotone, if \(\langle v^* - u^*, v - u \rangle_X \geq 0\) for all \(u, v \in X\), \(u^* \in F(u)\), \(v^* \in F(v)\). It is called strongly monotone, if there exist \(c > 0\) and \(r > 1\) such that for all \(u, v \in X\),

\[
\langle v^* - u^*, v - u \rangle_X \geq c \|u - v\|_X^2\text{ for all } u^* \in F(u)\text{ and } v^* \in F(v).
\]

An operator \(F: X \to 2^{X^*}\) is pseudomonotone, if

(a) for all \(u \in X\), the set \(F(u)\) is nonempty, bounded, closed and convex,
(b) the mapping \(F\) is upper semicontinuous from each finite dimensional subspace of \(X\) to \(X^*\) endowed with the weak topology,
(c) if \(\{u_n\} \subset X\), \(u_n \to u\) in \(X\), and \(u^*_n \in F(u_n)\) is such that \(\limsup \langle u^*_n, u_n - u \rangle_X \leq 0\), then for every \(v \in X\), there exists \(u^*(v) \in F(u)\) such that

\[
\langle u^*(v), u - v \rangle_X \leq \liminf \langle u^*_n, u_n - v \rangle_X.
\]

An operator \(F: X \to 2^{X^*}\) is called generalized pseudomonotone, if for any sequence \(\{u_n\} \subset X\) with \(u_n \to u\) in \(X\), and \(u^*_n \in F(u_n)\) with \(u^*_n \to u^*\) such that

\[
\limsup \langle u^*_n, u_n - u \rangle_X \leq 0,
\]
we have \(u^* \in F(u)\) and \(\langle u^*_n, u_n \rangle \to \langle u^*, u \rangle\).

Lemma 2.2. ([22, Proposition 1.3.58]) Let \(X\) be a reflexive Banach space and \(F: X \to 2^{X^*}\) be an operator,
(i) If \(F\) is a pseudomonotone operator, then \(F\) is generalized pseudomonotone.
(ii) If \(F\) is a generalized pseudomonotone operator which is bounded and \(F(u)\) is a nonempty, closed and convex subset of \(X^*\) for all \(u \in X\), then \(F\) is pseudomonotone.

Finally, recall a definition for a single-valued operator. An operator \(F: X \to X^*\) is said to be pseudomonotone, if it is bounded (it send bounded sets into bounded sets) and satisfies the inequality

\[
\langle Fu, u - v \rangle_X \leq \liminf \langle Fu_n, u_n - v \rangle_X\text{ for all } v \in X,
\]

where \(u_n \to u\) in \(X\) with \(\limsup \langle Fu_n, u_n - u \rangle_X \leq 0\).
From [22, Proposition 1.3.66], it is known that an operator \( F: X \to X^* \) defined on a reflexive Banach space is pseudomonotone if and only if \( F \) is bounded and satisfies the following condition: if \( u_n \rightharpoonup u \) in \( X \) and \( \lim \sup \{ Fu_n, u_n - u \} \leq 0 \), then \( Fu_n \rightharpoonup Fu \) in \( X^* \) and \( \lim \{ Fu_n, u_n - u \} = 0 \).

3. Inverse problems for variational-hemivariational inequality. In this section we recall a result from [24] on existence and uniqueness of solution to the Inverse problems for variational-hemivariational inequality. Moreover, we provide two continuous dependence results which will play a crucial role in the study of parameter identification problems.

In what follows, we assume that \((P, \| \cdot \|)\) is a normed space of parameters.

**Problem 1. (Problem (DP))** Given \( p \in \mathcal{P} \), find \( u = u(p) \in K \) such that

\[
\langle A(p, u) - f(p), v - u \rangle_X + \varphi(p, u, v) - \varphi(p, u, u) + \int_0^1 (p; v - u) \geq 0 \quad \text{for all } v \in K.
\]

We need the following hypotheses on the data of Problem 1.

\( K \) is a nonempty, closed, convex subset of \( X \).

\[
A: \mathcal{P} \times X \to X^* \text{ is such that }
\]

(a) \( A(p, \cdot) \) is pseudomonotone for all \( p \in \mathcal{P} \).

(b) there exists \( \alpha_A > 0 \) such that

\[
\langle A(p, u_1) - A(p, u_2), u_1 - u_2 \rangle_X \geq \alpha_A \| u_1 - u_2 \|_X^2
\]

for all \( p \in \mathcal{P}, u_1, u_2 \in X \).

\( \varphi: \mathcal{P} \times K \times X \to \mathbb{R} \) is such that

(a) \( \varphi(p, u, \cdot): X \to \mathbb{R} \) is convex and l.s.c. on \( X \), for all \( p \in \mathcal{P}, u \in K \).

(b) there exist \( \alpha_\varphi, \beta_\varphi \geq 0 \) such that

\[
\varphi(p_1, u_1, v_2) - \varphi(p_1, u_1, v_1) + \varphi(p_2, u_2, v_1) - \varphi(p_2, u_2, v_2) \\
\leq \alpha_\varphi \| u_1 - u_2 \|_X \| v_1 - v_2 \|_X + \beta_\varphi \| p_1 - p_2 \|_\mathcal{P} \| v_1 - v_2 \|_X
\]

for all \( p_1, p_2 \in \mathcal{P}, u_1, u_2 \in K, v_1, v_2 \in X \).

\( j: \mathcal{P} \times X \to \mathbb{R} \) is such that

(a) \( j(p, \cdot) \) is locally Lipschitz for all \( p \in \mathcal{P} \).

(b) there exist \( c_0, c_1, c_2 \geq 0 \) such that

\[
\| \partial j(p, u) \|_{X^*} \leq c_0 + c_1 \| u \|_X + c_2 \| p \|_\mathcal{P}
\]

for all \( p \in \mathcal{P}, u \in X \).

(c) there exist \( \alpha_j, \beta_j \geq 0 \) such that

\[
j^0(p_1, u_1; u_2 - u_1) + j^0(p_2, u_2; u_1 - u_2) \\
\leq \alpha_j \| u_1 - u_2 \|_X^2 + \beta_j \| p_1 - p_2 \|_\mathcal{P} \| u_1 - u_2 \|_X
\]

for all \( p_1, p_2 \in \mathcal{P}, u_1, u_2 \in X \).
\[ f(p) \in X^* \text{ for all } p \in \mathcal{P}. \] (6)

We have the following existence and uniqueness result.

**Theorem 3.1.** Assume that (2)-(6) hold and the following smallness condition is satisfied
\[ \alpha_{\varphi} + \alpha_j < \alpha_A. \] (7)
Then, for all \( p \in \mathcal{P} \), Problem 1 has a unique solution \( u = u(p) \in K \).

Note that a result on existence and uniqueness of solution to the variational-hemivariational inequality in Problem 1 has been recently provided in [23, Theorem 18] under an additional hypothesis on the nonlinear operator \( A \). The proof of Theorem 3.1 can be found in [24] and is based on another surjectivity result (see [12]) than the one used in [23]. Moreover, if the operator \( A(p, \cdot) \) satisfies hypothesis (3)(b) and is bounded and hemicontinuous, then it satisfies also (3)(a), see e.g. [22, Theorem 3.69(i)].

Next, we give continuous dependence results for Problem 1. Consider the following variational-hemivariational inequality corresponding to a sequence of parameters \( \{p_n\} \subset \mathcal{P}, \ n \in \mathbb{N} \).

**Problem 2.** Given \( p_n \in \mathcal{P} \), find \( u_n = u(p_n) \in K \) such that
\[
\langle A(p_n, u_n) - f(p_n), v - u_n \rangle_X + \varphi(p_n, u_n, v) - j^0(p_n, u_n; v - u_n) \geq 0 \text{ for all } v \in K. \] (8)

We need the following hypotheses on the data.

\[
\begin{aligned}
&\text{For any } \{p_n\} \subset \mathcal{P}, \ \{u_n\} \subset X \text{ with } p_n \to p \text{ in } \mathcal{P} \text{ and } u_n \to u \text{ in } X, \\
&\text{and all } v \in X, \text{ we have } \limsup \langle A(p, u_n) - A(p_n, u_n), u_n - v \rangle_X \leq 0. \quad (9)
\end{aligned}
\]

\[
\begin{aligned}
&\text{For any } \{p_n\} \subset \mathcal{P}, \ \{u_n\} \subset X \text{ with } p_n \to p \text{ in } \mathcal{P} \text{ and } u_n \to u \text{ in } X, \\
&\text{and all } v \in X, \text{ we have } \limsup j^0(p_n, u_n; v - u_n) \leq j^0(p, u; v - u). \quad (10)
\end{aligned}
\]

\[
\begin{aligned}
&\text{For any } \{p_n\} \subset \mathcal{P}, \ \{u_n\} \subset K \text{ with } p_n \to p \text{ in } \mathcal{P} \text{ and } u_n \to u \text{ in } X, \\
&\text{and all } v \in X, \text{ we have } \\
&\limsup (\varphi(p_n, u_n, v) - \varphi(p_n, u_n, u_n)) \leq \varphi(p, u, v) - \varphi(p, u, u). \quad (11)
\end{aligned}
\]

For any \( \{p_n\} \subset \mathcal{P} \) with \( p_n \to p \) in \( \mathcal{P} \), we have \( f(p_n) \to f(p) \) in \( X^* \).

\[
\begin{aligned}
&\text{There exist positive constants } l_1 \text{ and } l_2 \text{ such that} \\
&\|A(p, 0)\|_{X^*} \leq l_1 \|p\|_{\mathcal{P}}, \quad |\varphi(p, 0, v)| \leq l_2 \|p\|_{\mathcal{P}} \|v\|_X \quad (13)
\end{aligned}
\]

\[
\begin{aligned}
&\text{for all } p \in \mathcal{P} \text{ and } v \in X. \\
&0 \in K. \quad (14)
\end{aligned}
\]

The first continuous dependence result reads as follows.

**Theorem 3.2.** Assume that hypotheses of Theorem 3.1 hold. Suppose also that (9)-(14) hold and \( \{p_n\} \subset \mathcal{P} \) with \( p_n \to p \) in \( \mathcal{P} \) for some \( p \in \mathcal{P} \). Then the sequence \( \{u_n\} = \{u(p_n)\} \subset K \) of unique solutions to Problem 2 converges weakly in \( X \) to the solution \( u(p) \in K \) of Problem 1.
Next, we provide hypotheses on the operator $A$ and the element $f$ to obtain the second continuous dependence result.

$$\begin{align*}
A: \mathcal{P} \times X &\to X^* \text{ is such that there exists } L_A > 0 \text{ with } \\
\|A(p_1, u) - A(p_2, u)\|_{X^*} &\leq L_A \|p_1 - p_2\|_\mathcal{P} \\
\text{for all } p_1, p_2 \in \mathcal{P} \text{ and } u \in X. 
\end{align*}$$ \hspace{1cm} (15)

$$\begin{align*}
f: \mathcal{P} &\to X^* \text{ is such that there exists } L_f > 0 \text{ with } \\
\|f(p_1) - f(p_2)\|_{X^*} &\leq L_f \|p_1 - p_2\|_\mathcal{P} \\
\text{for all } p_1, p_2 \in \mathcal{P}. 
\end{align*}$$ \hspace{1cm} (16)

**Theorem 3.3.** Assume that hypotheses of Theorem 3.1 hold, and (15) and (16) are satisfied. Then

$$\|u(p_1) - u(p_2)\|_X \leq \frac{L_A + \beta_p + \beta_j + L_f}{\alpha_A - \alpha_p - \alpha_j} \|p_1 - p_2\|_\mathcal{P}$$ \hspace{1cm} (17)

for all $p_1, p_2 \in \mathcal{P}$, where $u(p) \in K$ denotes the unique solution to Problem 1 corresponding to $p \in \mathcal{P}$.

Now we pass to the formulation of inverse problem for Problem 1. In the context of inverse problems, the variational-hemivariational inequality (1) is referred to as the direct problem (DP).

Consider the following inverse problem (IP). Given an admissible subset of parameters $\mathcal{P}_{ad} \subset \mathcal{P}$ and a cost functional $F: \mathcal{P} \times K \to \mathbb{R}$, find a solution $p^* \in \mathcal{P}_{ad}$ to the following problem

$$F(p^*, u(p^*)) = \min \{ F(p, u(p)) \mid p \in \mathcal{P}_{ad} \},$$ \hspace{1cm} (18)

where $u = u(p) \in K$ denotes the unique solution of Problem 1 corresponding to a parameter $p$. A pair $(p^*, u(p^*)) \in \mathcal{P}_{ad} \times K$ which solves (18) is called an optimal solution to the inverse problem.

We are now in a position to state the main result on the existence of solutions to the inverse problem (18). We admit the following hypotheses

$$\mathcal{P}_{ad} \text{ is a compact subset of } \mathcal{P},$$ \hspace{1cm} (19)

$$F: \mathcal{P} \times K \to \mathbb{R} \text{ is l.s.c. on } \mathcal{P}_{ad} \times X_w,$$ \hspace{1cm} (20)

$$F: \mathcal{P} \times K \to \mathbb{R} \text{ is l.s.c. on } \mathcal{P}_{ad} \times X,$$ \hspace{1cm} (21)

where, recall, $X_w$ denotes the space $X$ endowed with the weak topology.

**Theorem 3.4.** Assume that hypotheses of Theorem 3.2 hold, and (19) and (20) are satisfied. Then problem (18) has at least one solution.

Similarly, we have the following result.

**Theorem 3.5.** Assume that hypotheses of Theorem 3.3 hold, and (19) and (21) are satisfied. Then problem (18) has at least one solution.
4. Penalty method for variational-hemivariational inequality. In this section we will study a penalty method for variational-hemivariational inequality formulated in Problem 1. We prove that the penalized variational-hemivariational inequality is uniquely solvable and that its solution converges to the problem with constraints when the penalty parameter approaches zero. Then we show a result on the continuous dependence of the solution on parameter for the penalized variational-hemivariational inequality.

We adopt the following notion of the penalty operator, see [5].

**Definition 4.1.** A single-valued operator \( P: X \to X^* \) is said to be a penalty operator of a set \( K \subset X \) if \( P \) is bounded, demicontinuous, monotone and \( K = \{ x \in X \mid Px = 0_{X^*} \} \).

In the following we provide the well-known example of a penalty operator.

**Example 1.** Let \( X \) be a reflexive Banach space, \( J: X \to X^* \) be the duality mapping, \( I \) denotes the identity map on \( X \), and \( P_K: X \to X \) be the projection operator on \( K \). Then the mapping \( P = J(I - P_K) \) is a penalty operator of \( K \). Recall that if \( X \) is a reflexive Banach space, then it can be equivalently renormed to become a strictly convex space and, therefore, the duality map \( J: X \to 2^{X^*} \) defined by

\[
Jx = \{ x^* \in X^* \mid \langle x^*, x \rangle = \|x^*\|^2_X \leq \|x\|^2_X \} \quad \text{for all } x \in X
\]

is a single-valued operator. For details, see [5, Proposition 1.3.27] and [32, Proposition 32.22].

In what follows, we assume that \( P: \mathcal{P} \times X \to X^* \) is such that \( P(p, \cdot) \) is a penalty operator of \( K \) for each \( p \in \mathcal{P} \). For every fixed \( \lambda > 0 \) and \( p \in \mathcal{P} \), we consider the following penalized problem associated with the variational-hemivariational inequality (1).

**Problem 3.** (Problem \((DP)_\lambda\)) Find an element \( u_\lambda = u_\lambda(p) \in X \) such that

\[
(A(p, u_\lambda) - f(p), v - u_\lambda)_X + \frac{1}{\lambda} \langle P(p, u_\lambda), v - u_\lambda \rangle_X + \varphi(p, u_\lambda, v) - \varphi(p, u_\lambda, u_\lambda) + j^0(p, u_\lambda; v - u_\lambda) \geq 0 \quad \text{for all } v \in X.
\]

We need the following hypothesis on the function \( \varphi \).

\[
\varphi: \mathcal{P} \times X \times X \to \mathbb{R} \text{ is such that}
\]

(a) \( \varphi(p, u, \cdot): X \to \mathbb{R} \) is convex and l.s.c. on \( X \),

for all \( p \in \mathcal{P}, u \in X \).

(b) there exist \( \alpha_\varphi, \beta_\varphi \geq 0 \) such that

\[
\varphi(p_1, u_1, v_2) - \varphi(p_1, u_1, v_1) + \varphi(p_2, u_2, v_1) - \varphi(p_2, u_2, v_2) \leq \alpha_\varphi \|u_1 - u_2\|_X \|v_1 - v_2\|_X + \beta_\varphi \|p_1 - p_2\|_P \|v_1 - v_2\|_X
\]

for all \( p_1, p_2 \in \mathcal{P}, u_1, u_2, v_1, v_2 \in X \).

(c) for any \( \{p_n\} \subset \mathcal{P}, \{u_n\} \subset X \) with \( p_n \to p \) in \( \mathcal{P} \) and \( u_n \to u \) in \( X \), and all \( v \in X \), we have

\[
\limsup \left( \varphi(p_n, u_n, v) - \varphi(p_n, u_n, u_n) \right) \leq \varphi(p, u, v) - \varphi(p, u, u).
\]

Our main result of this section is the following.
Theorem 4.2. Assume that hypotheses (2), (3), (5), (6), (7), (10), (13), (14), (23) are satisfied. Then

(i) for each \( \lambda > 0 \) and \( p \in P \), there exists a unique solution \( u_\lambda = u_\lambda(p) \in X \) to Problem 3.

(ii) for each \( p \in P \), we have \( u_\lambda(p) \to u(p) \) in \( X \), as \( \lambda \to 0 \), where \( u(p) \in K \) is the unique solution to Problem 1.

Note that this theorem was proved in [23, Theorem 20] in a particular case when the function \( \varphi \) is independent of the second variable, i.e., when \( \varphi(p, u, v) = \varphi(p, v) \).

Proof. Let \( \lambda > 0 \) be fixed. For any \( p \in P \), using the properties of the penalty operator \( P(p, \cdot) \) and the fact that the domain of \( P(p, \cdot) \) equals to \( X \), it follows (see Exercise I.9 in Section 1.9 of [5]) that \( P(p, \cdot) \) is bounded, hemicontinuous and monotone. From [32, Proposition 27.6], we deduce that \( P(p, \cdot) \) is a pseudomonotone operator. The proof of (i) follows from (3), (4), (5), and the properties of \( P(p, \cdot) \), ensuring that for any \( p \in P \) the map \( A(p, \cdot) + \frac{1}{\lambda} P(p, \cdot) \) is pseudomonotone.

Next, we pass to the proof of (ii). Let \( p \in P \). We claim that there are an element \( \tilde{u} \in X \) and a subsequence of \( \{ u_\lambda \} \), denoted in the same way, such that \( u_\lambda \to \tilde{u} \) in \( X \), as \( \lambda \to 0 \). To this end, we will establish the boundedness of \( \{ u_\lambda \} \) in \( X \). Using hypotheses (3) and (14), from (22), we have

\[
\alpha_A \| u_\lambda \|^2_X \leq \langle A(p, u_\lambda) - A(p, 0), u_\lambda - 0 \rangle_X
\]

\[
\leq \langle -A(p, 0), u_\lambda \rangle_X + \frac{1}{\lambda} \langle P(p, u_\lambda), -u_\lambda \rangle_X + \varphi(p, u_\lambda, 0) - \varphi(p, u_\lambda, u_\lambda)
\]

\[
+ j^0(p, u_\lambda; -u_\lambda) + \langle f(p), u_\lambda \rangle_X
\]

\[
= \langle -A(p, 0), u_\lambda \rangle_X - \frac{1}{\lambda} \langle P(p, 0) - P(p, u_\lambda), 0 - u_\lambda \rangle_X
\]

\[
+ (\varphi(p, u_\lambda, 0) - \varphi(p, u_\lambda, u_\lambda) + \varphi(p, 0, u_\lambda) - \varphi(p, 0, 0))
\]

\[
+ (\varphi(p, 0, 0) - \varphi(p, 0, u_\lambda)) + (j^0(p, u_\lambda; -u_\lambda) + j^0(p, 0; u_\lambda))
\]

\[
- j^0(p, 0; u_\lambda) + \langle f(p), u_\lambda \rangle_X.
\]

Exploiting hypotheses (4), (5), (13), the monotonicity of \( P(p, \cdot) \), and the estimate

\[
| j^0(p, 0; u_\lambda)| = \| \{ \zeta, u_\lambda \} \| \leq (c_0 + c_2 \| p \|) \| u_\lambda \|_X,
\]

we obtain

\[
\alpha_A \| u_\lambda \|^2_X \leq \alpha_\varphi \| u_\lambda \|^2_X + \alpha_j \| u_\lambda \|^2_X + (c_0 + (c_2 + l_1 + l_2) \| p \| \| p \|_X) \| u_\lambda \|_X + \| f(p) \|_X \cdot \| u_\lambda \|_X.
\]

Therefore, we have

\[
(\alpha_A - \alpha_\varphi - \alpha_j) \| u_\lambda \|^2_X \leq (c_0 + (c_2 + l_1 + l_2) \| p \| \| p \|_X + \| f(p) \|_X \cdot \| u_\lambda \|_X.
\]

which, due to the smallness condition (7), implies that there is a constant \( C > 0 \) independent of \( \lambda \) such that \( \| u_\lambda \|_X \leq C \). Thus, from the reflexivity of \( X \), we deduce, by passing to a subsequence, if necessary, that

\[
u_\lambda \to \tilde{u} \text{ in } X, \text{ as } \lambda \to 0
\]

with some \( \tilde{u} \in X \). This proves the claim.
Next, we show that \( \bar{u} \in K \) is a solution to Problem 1. Since for any \( p \in P, u \in X \), \( \varphi(p, u, \cdot) \) is convex and lower semicontinuous on \( X \), it admits an affine minorant, cf. e.g. [4, Proposition 5.2.25], that is, there are \( l_{p,u} \in X^* \) and \( b_{p,u} \in \mathbb{R} \) such that

\[
\varphi(p, u, v) \geq \langle l_{p,u}, v \rangle + b_{p,u} \quad \text{for all } v \in X.
\]

Then from (3), for every \( v \in X \), we have

\[
\frac{1}{\lambda}\langle P(p, u_\lambda), u_\lambda - v \rangle_X
\leq \langle A(p, u_\lambda) - f(p), v - u_\lambda \rangle_X + \varphi(p, u_\lambda, v) - \varphi(p, u_\lambda, u_\lambda) + j^0(p, u_\lambda; v - u_\lambda)
\]

\[
= -\langle A(p, u_\lambda) - A(p, v), u_\lambda - v \rangle_X + \langle A(p, v) - f(p), v - u_\lambda \rangle_X
\]

\[
+ (\varphi(p, u_\lambda, v) - \varphi(p, u_\lambda, u_\lambda) + \varphi(p, v, u_\lambda) - \varphi(p, v, v))
\]

\[
+ (\varphi(p, v, v) - \varphi(p, v, u_\lambda)) + (j^0(p, u_\lambda; v - u_\lambda) + j^0(p, v; u_\lambda - v))
\]

\[
- j^0(p, v; u_\lambda - v) + (f(p) - A(p, v), u_\lambda - v)\rangle_X
\]

\[
\leq (\alpha_\varphi + \alpha_j)\|u_\lambda - v\|_X^2 + \|\varphi(p, v, v)\| + \|l_{p,v}\|_X^* \|u_\lambda\|_X + |b_{p,v}|
\]

\[
+ (c_0 + c_1\|p\|_p + c_2\|u_\lambda\|_X + \|A(p, v) - f(p)\|_X^*)\|u_\lambda - v\|_X.
\]

Since \( \|u_\lambda\|_X \leq C \), we infer that

\[
\frac{1}{\lambda}\langle P(p, u_\lambda), u_\lambda - v \rangle_X \leq C_v \quad \text{for all } v \in X,
\]

where \( C_v \) depends on \( v \) and is independent of \( \lambda \). Taking \( v = \bar{u} \) in above inequality, we have

\[
\limsup_{\lambda \to 0} \langle P(p, u_\lambda), u_\lambda - \bar{u} \rangle_X \leq 0.
\]

Exploiting the pseudomonotonicity of \( P(p, \cdot) \), for every \( p \in P \), we have

\[
\langle P(p, \bar{u}), \bar{u} - v \rangle_X \leq \liminf_{\lambda \to 0} \langle P(p, u_\lambda), u_\lambda - v \rangle_X \leq 0 \quad \text{for all } v \in X.
\]

Now, let \( t > 0 \) and \( w \in X \) be arbitrary. We choose \( v = \bar{u} + tw \) in the last inequality to get \( \langle P(p, \bar{u}), w \rangle_X \geq 0 \). Since \( w \in X \) is arbitrary, we have \( P(p, \bar{u}) = 0 \) which implies that \( \bar{u} \in K \).

Subsequently, testing (22) with \( v \in K \) and using the monotonicity of \( P(p, \cdot) \) for every \( p \in P \), we have

\[
\langle A(p, u_\lambda), u_\lambda - v \rangle_X \leq -\frac{1}{\lambda}\langle P(p, v) - P(p, u_\lambda), v - u_\lambda \rangle_X
\]

\[
+ \varphi(p, u_\lambda, v) - \varphi(p, u_\lambda, u_\lambda) + j^0(p, u_\lambda; v - u_\lambda) + (f(p), u_\lambda - v)\rangle_X,
\]

which implies

\[
\langle A(p, u_\lambda), u_\lambda - v \rangle_X \leq \varphi(p, u_\lambda, v) - \varphi(p, u_\lambda, u_\lambda) + j^0(p, u_\lambda; v - u_\lambda) + (f(p), u_\lambda - v)\rangle_X
\]

for all \( v \in K \). From hypotheses (10) and (11), we obtain

\[
\limsup_{\lambda \to 0} \langle A(p, u_\lambda), u_\lambda - \bar{u} \rangle_X \leq 0.
\]

This inequality together with (24) and the pseudomonotonicity of \( A(p, \cdot) \) implies

\[
\langle A(p, \bar{u}), \bar{u} - v \rangle \leq \liminf_{\lambda \to 0} \langle A(p, u_\lambda), u_\lambda - v \rangle \quad \text{for all } v \in X.
\]
We are now in a position to pass to the upper limit in (25). From hypotheses (10) and (11), we get
\[
\limsup_{\lambda \to 0} (A(p, u_\lambda), u_\lambda - v)_X \leq \varphi(p, \tilde{u}, v) - \varphi(p, \tilde{u}, \tilde{u}) + j^0(p, \tilde{u}; v - \tilde{u}) + \langle f(p), \tilde{u} - v \rangle_X
\]
for all \(v \in K\). Combining (26) and (27), we have
\[
\langle A(p, \tilde{u}), \tilde{u} - v \rangle_X \leq \varphi(p, \tilde{u}, v) - \varphi(p, \tilde{u}, \tilde{u}) + j^0(p, \tilde{u}; v - \tilde{u}) + \langle f(p), \tilde{u} - v \rangle_X
\]
for all \(v \in K\). Hence, it follows that \(\tilde{u} \in K\) is a solution to Problem 1.

Since Problem 1 has a unique solution \(u \in K\), we deduce that \(\tilde{u} = u\). This implies that every subsequence of \(\{u_\lambda\}\) which converges weakly has the same limit and, therefore, it follows that the whole sequence \(\{u_\lambda\}\) converges weakly to \(\tilde{u}\).

In the final step of the proof, we show that \(u_\lambda \to \tilde{u}\) in \(X\), as \(\lambda \to 0\). We take \(v = \tilde{u} \in K\) in both (26) and (27) to obtain
\[
0 \leq \liminf_{\lambda \to 0} (A(p, u_\lambda), u_\lambda - \tilde{u})_X \quad \text{and} \quad \limsup_{\lambda \to 0} (A(p, u_\lambda), u_\lambda - \tilde{u})_X \leq 0,
\]
respectively, which gives \((A(p, u_\lambda), u_\lambda - \tilde{u})_X \to 0\), as \(\lambda \to 0\). Therefore, using the strong monotonicity of \(A(p, \cdot)\) for all \(p \in P\) and the convergence \(u_\lambda \to u\) weakly in \(X\), we have
\[
\alpha_A \|u_\lambda - \tilde{u}\|^2_X \leq (A(p, u_\lambda) - A(p, \tilde{u}), u_\lambda - \tilde{u})_X = (A(p, u_\lambda), u_\lambda - \tilde{u})_X - (A(p, \tilde{u}), u_\lambda - u)_X \to 0,
\]
as \(\lambda \to 0\). It follows from here that \(u_\lambda \to \tilde{u}\) in \(X\), as \(\lambda \to 0\), which completes the proof of the theorem.

The last result of this section concerns the continuous dependence on parameter of the solution of the penalized variational-hemivariational inequality \((DP)_\lambda\).

Let \(\lambda > 0\) be fixed. Consider the following penalized variational-hemivariational inequality corresponding to a sequence of parameters \(\{p_n\} \subset P, n \in \mathbb{N}\).

**Problem 4.** Given \(p_n \in P\), find \(u_n = u(p_n) \in X\) such that
\[
\langle A(p_n, u_n) - f(p_n), v - u_n \rangle_X + \frac{1}{\lambda} (P(p_n, u_n), v - u_n)_X \geq 0 \quad \text{for all} \quad v \in X.
\]

We introduce the following hypothesis for the dependence of the penalty operator on the parameter.

\[
\left\{
\begin{array}{l}
\text{For any} \ \{p_n\} \subset P, \ \{u_n\} \subset X \ \text{with} \ \ p_n \to p \ \text{in} \ P \ \text{and} \ u_n \to u \ \text{in} \ X, \\
\text{and all} \ v \in X, \ \text{we have}
\end{array}
\right.
\]
\[
\limsup_{n \to \infty} (P(p_n, v), v - u_n)_X \leq (P(p, v), v - u)_X.
\]

The continuous dependence result reads as follows.

**Theorem 4.3.** Assume that hypotheses of Theorem 4.2 hold. Suppose also that (9), (12) and (29) hold, and \(\{p_n\} \subset P\) with \(p_n \to \overline{p}\) in \(P\), as \(n \to \infty\) for some \(\overline{p} \in P\). Then, the sequence \(\{u_n\} = \{u(p_n)\} \subset X\) of unique solutions to Problem 4 converges weakly in \(X\) to the unique solution \(u(\overline{p}) \in X\) of Problem 3.
Proof. Let \( \{p_n\} \subset \mathcal{P} \) with \( p_n \to \overline{p} \) in \( \mathcal{P} \) for some \( \overline{p} \in \mathcal{P} \). Let \( u_n = u(p_n) \in X \) be the unique solution to Problem 4 guaranteed by Theorem 4.2(i). Similarly as in the proof of Theorem 4.2, from the boundedness of \( \{p_n\} \), we obtain that the sequence \( \{u_n\} \) remains bounded in \( X \) as well. Therefore, by the reflexivity of \( X \), we may suppose, passing to a subsequence, if necessary, that

\[
 u_n \to \overline{u} \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty
\]

with \( \overline{u} \in X \). Choosing \( v = \overline{u} \) in (28), we have

\[
\langle A(p_n, u_n), u_n - \overline{u} \rangle_X \leq \frac{1}{\lambda} \langle P(p_n, u_n), \overline{u} - u_n \rangle_X + \langle f(p_n), u_n - \overline{u} \rangle_X \quad (30)
\]

\[
+ \varphi(p_n, u_n, \overline{u}) - \varphi(p_n, u_n, u_n) + j^0(p_n, u_n; \overline{u} - u_n).
\]

Using hypotheses (9)–(12) in (30), we have

\[
\limsup \langle A(\overline{p}, u_n), u_n - \overline{u} \rangle_X \leq \limsup \langle A(\overline{p}, u_n) - A(p_n, u_n), u_n - \overline{u} \rangle_X + \limsup \langle A(p_n, u_n), u_n - \overline{u} \rangle_X
\]

\[
- \frac{1}{\lambda} \liminf \langle P(p_n, \overline{u}) - P(p_n, u_n), \overline{u} - u_n \rangle_X
\]

\[
+ \frac{1}{\lambda} \limsup \langle P(p_n, \overline{u}), \overline{u} - u_n \rangle_X + \limsup \langle f(p_n), u_n - \overline{u} \rangle
\]

\[
+ \limsup (\varphi(p_n, u_n, \overline{u}) - \varphi(p_n, u_n, u_n)) + \limsup j^0(p_n, u_n; \overline{u} - u_n)
\]

\[
\leq 0.
\]

Exploiting the facts that the operator \( A(\overline{p}, \cdot) \) is pseudomonotone, \( u_n \rightharpoonup \overline{u} \) in \( X \) and \( \limsup \langle A(\overline{p}, u_n), u_n - \overline{u} \rangle_X \leq 0 \), from [22, Proposition 1.3.66] (see Preliminaries), we infer

\[
A(\overline{p}, u_n) \rightharpoonup A(\overline{p}, \overline{u}) \quad \text{in} \quad X^*, \quad (31)
\]

\[
\langle A(\overline{p}, u_n), u_n - \overline{u} \rangle_X \to 0, \quad \text{as} \quad n \to \infty. \quad (32)
\]

Next, conditions (31) and (32) imply

\[
\langle A(\overline{p}, u_n), u_n \rangle_X \to \langle A(\overline{p}, \overline{u}), \overline{u} \rangle_X, \quad (33)
\]

\[
\langle A(\overline{p}, u_n), u_n - v \rangle_X = \langle A(\overline{p}, u_n), u_n \rangle_X - \langle A(\overline{p}, u_n), v \rangle_X
\]

\[
\to \langle A(\overline{p}, \overline{u}), \overline{u} \rangle_X - \langle A(\overline{p}, \overline{u}), v \rangle_X = \langle A(\overline{p}, \overline{u}), \overline{u} - v \rangle_X \quad \text{for all} \quad v \in X.
\]

Subsequently, we are in a position to pass to the limit in (28). Let \( v \in X \). Inserting

\[
\langle A(p_n, u_n), u_n - v \rangle_X \leq \frac{1}{\lambda} \langle P(p_n, u_n), v - u_n \rangle_X + \langle f(p_n), u_n - v \rangle_X
\]

\[
+ \varphi(p_n, u_n, v) - \varphi(p_n, u_n, u_n) + j^0(p_n, u_n; v - u_n)
\]

into the inequality

\[
\langle A(\overline{p}, u_n), u_n - v \rangle_X = \langle A(p_n, u_n) - A(\overline{p}, u_n), v - u_n \rangle_X + \langle A(p_n, u_n), u_n - v \rangle_X,
\]

we obtain that the sequence \( \{u_n\} \) is pseudomonotone.
and using the convergence (34), we obtain
\[
\langle A(\bar{p}, \bar{u}), \bar{u} - v \rangle_X = \lim \langle A(\bar{p}, u_n), u_n - v \rangle_X
\]
\[
\leq \lim \sup (A(\bar{p}, u_n) - A(p_n, u_n), u_n - v)\rangle_X
\]
\[
+ \frac{1}{\lambda} \lim \inf (P(p_n, v) - P(p_n, u_n), v - u_n)
\]
\[
+ \frac{1}{\lambda} \lim \sup (P(p_n, v), v - u_n) + \lim \sup (f(p_n), u_n - v)\rangle_X
\]
\[
+ \lim \sup (\varphi(p_n, u_n, v) - \varphi(p_n, u_n, u_n)) + \lim \sup j^0(p_n, u_n; v - u_n)
\]
\[
\leq \frac{1}{\lambda} P(\bar{p}, \bar{u}), v - \bar{u}) + \varphi(\bar{p}, \bar{u}, v) - \varphi(\bar{p}, \bar{u}, \bar{u}) + j^0(\bar{p}, \bar{u}; v - \bar{u})
\]
\[
+ \langle f(\bar{p}), \bar{u} - v \rangle_X.
\]
Since \(v \in X\) is arbitrary, we deduce that \(\bar{u} \in X\) is a solution to Problem 3. Since this problem is uniquely solvable, it follows that \(\bar{u} = u(\bar{p})\). This completes the proof of the theorem.

5. Convergence of optimal solutions. The goal of this section is to establish the existence of solution to the inverse problem for the penalized variational-hemivariational inequality. We also prove a result on the convergence of a sequence of optimal solutions to the inverse problem for the penalized inequality to an optimal solution to the original inverse problem.

Consider an analogue of the inverse problem (18) in which the underlying variational-hemivariational inequality has been replaced by a penalized problem. Let \(\lambda > 0\) be fixed.

**Problem 5.** (Problem (IP)\(\lambda\)) Find a solution \(p^*_\lambda \in \mathcal{P}_{ad}\) to the following problem
\[
F(p^*_\lambda, u(p^*_\lambda)) = \min \{ F(p\lambda, u\lambda(p\lambda)) \mid p\lambda \in \mathcal{P}_{ad} \},
\]
where \(u\lambda(p) \in X\) denotes the unique solution of Problem 3 corresponding to a parameter \(p\).

We give the following existence and convergence result.

**Theorem 5.1.** Assume that hypotheses of Theorem 4.3, and (19), (20) hold. Then,
(i) for every \(\lambda > 0\), Problem 5 has a solution \(p^*_\lambda \in \mathcal{P}_{ad}\).
(ii) If for any \(p \in \mathcal{P}_{ad}\), \(F(p, \cdot)\) satisfies
\[
\lim \inf_{n \to \infty} F(p, u_n) \leq F(p, u) \quad \text{for} \quad u_n \to u, \quad \text{as} \quad n \to \infty,
\]
then there is a subsequence of \(\{(p^*_\lambda, u^*_\lambda)\}\), not relabeled, where \(u^*_\lambda = u\lambda(p^*_\lambda)\) is the unique solution of Problem 3, such that for \(\lambda \to 0\), we have \(p^*_\lambda \to p^*\) in \(\mathcal{P}\) and \(u^*_\lambda \to u^*\) in \(X\), where \(p^*\) is a solution of (18) and \(u^* = u(p^*)\) is the unique solution of Problem 1.
(iii) If, in addition, the inverse problem (18) has a unique optimal solution, then the whole sequence \(\{(p^*_\lambda, u^*_\lambda)\}\) converges to \((p^*, u^*)\) in \(\mathcal{P} \times X\), as \(\lambda \to 0\).

**Proof.** (i) For a fixed \(\lambda > 0\) the existence of a solution \(p^*_\lambda \in \mathcal{P}_{ad}\) follows by arguments similar to those used in the proof of Theorem 3.4.
(ii) Since $P_{ad}$ is compact, for the sequence $\{p^*_\lambda\} \subset P_{ad}$, there exists a subsequence, denoted in the same way, that converges to some $p^* \in P_{ad}$. Let $\{u^*_\lambda\}$ be the corresponding sequence of the penalized solutions, that is, a sequence of solutions of Problem 3 corresponding to $\{p^*_\lambda\}$. Similar to the proof of Theorem 4.2, from the boundedness of $\{p^*_\lambda\}$ we can obtain that the sequence $\{u^*_\lambda\}$ remains bounded in $X$. Consequently, there exists a subsequence, denoted by $\{u^*_\lambda\}$ again, which converges weakly to some $u^* \in X$. From the proof of Theorem 4.2, it follows that $u^* \in K$ and $u^* = u(p^*)$.

Let $p_0 \in P$ be an arbitrary solution of (18) and let $u_0$ be the corresponding solution of Problem 3. For $p_0$, let $u_\lambda(p_0)$ be the unique solution of Problem 3, that is,

$$
(A(p_0, u_\lambda) - f(p_0), v - u_\lambda) + \frac{1}{\lambda}(P(p_0, u_\lambda), v - u_\lambda) + \varphi(p_0, u_\lambda, v) - \varphi(p_0, u_\lambda, u_\lambda)
$$

$$
+ j^0(p_0, u_\lambda; v - u_\lambda) \geq 0 \text{ for all } v \in X.
$$

Then, from Theorem 4.2(ii), we have $u_\lambda(p_0) \rightharpoonup u_0$, as $\lambda \to 0$ and hence $(p_0, u_\lambda(p_0))$ is feasible for Problem 5. Therefore,

$$
F(p^*, u^*) \leq \liminf_{\lambda \to 0} F(p^*, u^*_\lambda) \leq \liminf_{\lambda \to 0} F(p_0, u_\lambda(p_0)) \leq F(p_0, u_0),
$$

which shows that $p^*$ is a solution of (18).

By now we know that $u^*_\lambda \rightharpoonup u^*$, as $\lambda \to 0$. We conclude this proof by showing that, in fact, $u^*_\lambda \to u^*$, as $\lambda \to 0$. We have

$$
(A(p^*_\lambda, u^*_\lambda) - f(p^*_\lambda), u^* - u^*_\lambda) + \frac{1}{\lambda}(P(p^*_\lambda, u^*_\lambda), u^* - u^*_\lambda) + \varphi(p^*_\lambda, u^*_\lambda, u^*)
$$

$$
- \varphi(p^*_\lambda, u^*_\lambda, u^*_\lambda) + j^0(p^*_\lambda, u^*_\lambda; u^* - u^*_\lambda) \geq 0,
$$

and since this equality can be written as follows

$$
(A(p^*_\lambda, u^*_\lambda) - A(p^*_\lambda, u^*), u^* - u^*_\lambda) + \frac{1}{\lambda}(P(p^*_\lambda, u^*_\lambda) - P(p^*_\lambda, u^*), u^* - u^*_\lambda)
$$

$$
\leq \varphi(p^*_\lambda, u^*_\lambda, u^*_\lambda) - \varphi(p^*_\lambda, u^*_\lambda, u^*) - j^0(p^*_\lambda, u^*_\lambda; u^* - u^*_\lambda) + (f(p^*_\lambda), u^* - u^*_\lambda),
$$

we obtain by the monotonicity of $P(p^*_\lambda, \cdot)$ that

$$
\alpha_A\|u^*_\lambda - u^*\|^2_X
$$

$$
\leq (A(p^*_\lambda, u^*_\lambda) - f(p^*_\lambda, u^*_\lambda), u^* - u^*_\lambda) - \frac{1}{\lambda}(P(p^*_\lambda, u^*_\lambda) - P(p^*_\lambda, u^*), u^* - u^*_\lambda)
$$

$$
\leq (A(p^*_\lambda, u^*) - f(p^*_\lambda, u^*_\lambda), u^* - u^*_\lambda) + \varphi(p^*_\lambda, u^*_\lambda, u^*)
$$

$$
- \varphi(p^*_\lambda, u^*_\lambda, u^*_\lambda) + j^0(p^*_\lambda, u^*_\lambda; u^* - u^*_\lambda).
$$

By passing the above inequality to limit as $\lambda \to 0$ and the fact that $u^*_\lambda \rightharpoonup u^*$, we deduce that $\|u^*_\lambda - u^*\|_X \to 0$, as $\lambda \to 0$.

(iii) For the sequence $\{p^*_\lambda\} \subset P_{ad}$, any subsequence, not relabeled, converges to the same limit $p^* \in P_{ad}$. Thus, the whole sequence $\{p^*_\lambda\}$ converges to $p^* \in P_{ad}$. From (ii) the result follows. This completes the proof.

6. Application. In this section we provide a simple example which provides an illustration of the results obtained in Sections 3–5. This example is a simplified model of a nonsmooth contact problem in elasticity which has been investigated.
in [23]. We refer to Section 7 of [23] for a physical description of the model, its interpretation, and a detailed discussion.

The classical formulation of the problem is the following.

**Problem 6.** Find a displacement field \( u: \Omega \to \mathbb{R}^d \), a stress field \( \sigma: \Omega \to \mathbb{S}^d \) and an interface force \( \xi: \Gamma_3 \to \mathbb{R} \) such that

\[
\sigma = A(p, \varepsilon(u)) \quad \text{in } \Omega, \\
\text{Div} \sigma + f_0(p) = 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \Gamma_1, \\
\sigma \nu = f_2(p) \quad \text{on } \Gamma_2, \\
u_\nu \leq g, \quad \sigma_\nu + \xi \leq 0, \quad (u_\nu - g)(\sigma_\nu + \xi) = 0, \quad \xi \in \partial j_\nu(p, u_\nu) \quad \text{on } \Gamma_3, \\
\|\sigma_\tau\| \leq h(p), \quad -\sigma_\tau = h(p) \frac{u_\tau}{\|u_\tau\|} \quad \text{if } u_\tau \neq 0 \quad \text{on } \Gamma_3.
\]

The problem is parametrized by a parameter \( p \in \mathcal{P} \), where, as before, \((\mathcal{P}, \| \cdot \|_\mathcal{P})\) represents a normed space of parameters.

We assume that \( \Omega \) is occupied by the elastic body and it is an open, bounded, connected set in \( \mathbb{R}^d \) (\( d = 2, 3 \)) with a Lipschitz boundary \( \partial \Omega = \Gamma \). The set \( \Gamma \) is partitioned into three disjoint and measurable parts \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) such that \( \text{meas}(\Gamma_1) \) is positive. We use the notation \( \mathbf{x} = (x_i) \) for a generic point in \( \Omega \cap \Gamma \) and \( \nu \) for the outward unit normal at \( \Gamma \). The indices \( i, j, k, l \) run between 1 and \( d \) and, unless stated otherwise, the summation convention over repeated indices is used.

Notation \( \mathbb{S}^d \) stands for the space of second order symmetric matrices on \( \mathbb{R}^d \). For a vector field, notation \( v_\nu \) and \( v_\tau \) represent the normal and tangential components of \( v \) on \( \Gamma \) given by \( v_\nu = v \cdot \nu \) and \( v_\tau = v - v_\nu \nu \). Also, \( \sigma_\nu \) and \( \sigma_\tau \) represent the normal and tangential components of the stress field \( \sigma \) on the boundary, i.e., \( \sigma_\nu = (\sigma v_\nu) \cdot \nu \) and \( \sigma_\tau = \sigma v_\tau - \sigma_\nu \nu \). Here \( \varepsilon(v) = (\varepsilon_{ij}) \) with \( \varepsilon_{ij}(u) = 1/2(u_{ij} + u_{ji}) \) denotes the linearized strain tensor and \( \sigma \) is the stress tensor, \( f_0 \) represents the density of the body forces and surface tractions of density \( f_2 \) act on \( \Gamma_2 \). Moreover, \( g > 0, \partial j_\nu \) denotes the Clarke subdifferential of the given function \( j_\nu \) with respect to its second variable, and \( h \) denotes a positive function, the friction bound. More details and mechanical interpretation on static contact models with elastic materials could be found in the monographs [22, 30].

Consider the spaces \( V \) and \( \mathcal{H} \) defined by

\[
V = \{ v \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_1 \}, \quad \mathcal{H} = L^2(\Omega; \mathbb{S}^d).
\]

On the space \( V \) we consider the inner product \( (u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \) for \( u, v \in V \), and the associated norm \( \| \cdot \|_V \). Recall that, since \( \text{meas}(\Gamma_1) > 0 \), it follows that \( V \) is a real Hilbert space. Moreover, by the Sobolev trace theorem, we have

\[
\|v\|_{L^2(\Gamma; \mathbb{R}^d)} \leq c_k \|\gamma\| \|v\|_V \quad \text{for all } v \in V,
\]

\( c_k > 0 \) being a constant in the Korn inequality and \( \|\gamma\| \) being the norm of the trace operator \( \gamma: V \to L^2(\Gamma; \mathbb{R}^d) \).
In the study of Problem 6 we assume that the elasticity operator $A$ satisfies the following condition.

\[
\begin{align*}
A: \mathcal{P} \times \Omega \times \mathbb{S}^d & \rightarrow \mathbb{S}^d \text{ is such that} \\
(a) \quad & A(p, x, \varepsilon) = (a_{ijkl}(x, p) \varepsilon_{kl}) \text{ for all } \varepsilon = (\varepsilon_{ij}), \ p \in \mathcal{P}, \ \text{a.e.} \ x \in \Omega. \\
(b) \quad & a_{ijkl}(\cdot, p) = a_{ijkl}(\cdot, p) = a_{klij}(\cdot, p) \in L^\infty(\Omega) \text{ for all } p \in \mathcal{P}, \\
& a_{ijkl}(x, p) \varepsilon_{ij} \varepsilon_{kl} \geq m_a \|\varepsilon\|_{\mathbb{S}^d}^2 \text{ for all } \varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^d \text{ with } m_a > 0.
\end{align*}
\]

In addition, the potential function $j_\nu$ and the friction bound $h$ are such that

\[
\begin{align*}
j_\nu: \Gamma_3 \times \mathcal{P} \times \mathbb{R} & \rightarrow \mathbb{R} \text{ is such that} \\
(a) \quad & j_\nu(\cdot, p, r) \text{ is measurable on } \Gamma_3 \text{ for all } p \in \mathcal{P}, r \in \mathbb{R} \text{ and there exists } \tau \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, p, \tau(\cdot)) \in L^1(\Gamma_3) \text{ for all } p \in \mathcal{P}. \\
(b) \quad & j_\nu(x, p, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for all } p \in \mathcal{P}, \ \text{a.e. } x \in \Gamma_3. \\
(c) \quad & |\partial j_\nu(x, p, r)| \leq \tau_0 + \tau_1 |p| |p + \tau_2| r| \\
& \text{ for all } p \in \mathcal{P}, r \in \mathbb{R}, \ \text{a.e. } x \in \Gamma_3, \ \text{with } \tau_0, \tau_1, \tau_2 \geq 0. \\
(d) \quad & j_\nu^0(\mathbf{x}, p_1, r_1; r_2 - r_1) + j_\nu^0(\mathbf{x}, p_2, r_2; r_1 - r_2) \\
& \leq \alpha_{j_\nu} |r_1 - r_2|^2 + \beta_{j_\nu} |p_1 - p_2| |r_1 - r_2| \\
& \text{ for all } p \in \mathcal{P}, r_1, r_2 \in \mathbb{R}, \ \text{a.e. } x \in \Gamma_3 \text{ with } \alpha_{j_\nu}, \beta_{j_\nu} \geq 0.
\end{align*}
\]

\[
\begin{align*}
h: \Gamma_3 \times \mathcal{P} & \rightarrow \mathbb{R}_+ \text{ is such that} \\
(a) \quad & h(\cdot, p) \text{ is measurable on } \Gamma_3 \text{ for all } p \in \mathcal{P}. \\
(b) \quad & \text{there exists } L_h > 0 \text{ such that} \\
& |h(x, p_1) - h(x, p_2)| \leq L_h |p_1 - p_2| p \\
& \text{for all } p_1, p_2 \in \mathcal{P}, \ \text{a.e. } x \in \Gamma_3. \\
(c) \quad & h(x, 0) = 0 \text{ for a.e. } x \in \Gamma_3.
\end{align*}
\]

Finally, we assume that the densities of body forces and surface tractions have the regularity

\[
f_0(p) \in L^2(\Omega; \mathbb{R}^d), \quad f_2(p) \in L^2(\Gamma_3; \mathbb{R}^d) \quad \text{for all } p \in \mathcal{P}.
\]

Next, we introduce the set of admissible displacement fields $U$, and the element $f(p) \in V^*$ defined by

\[
U = \{ \mathbf{v} \in V \mid v_\nu \leq g \ \text{a.e. on } \Gamma_3 \},
\]

\[
(f(p), \mathbf{v}) = (f_0(p), \mathbf{v})_{L^2(\Omega; \mathbb{R}^d)} + (f_2(p), \mathbf{v})_{L^2(\Gamma_3; \mathbb{R}^d)} \quad \text{for all } \mathbf{v} \in V, \ p \in \mathcal{P}.
\]
Problem 7. Find a displacement field \( \mathbf{u} \in U \) such that

\[
(A(p, \varepsilon(u)), \varepsilon(v) - \varepsilon(u)) + \int_{\Gamma_3} h(p)(\|v_r\| - \|u_r\|) \, d\Gamma
+ \int_{\Gamma_3} j^0(v, u; v_r - u_r) \, d\Gamma \geq \langle f(p), v - u \rangle \quad \text{for all } v \in U.
\]

We proceed with the following existence and uniqueness result.

**Theorem 6.1.** Assume hypotheses (42)-(45) and, in addition, suppose the smallness condition

\[
c_k \alpha_j \|\gamma\|^2 < m_a.
\]

Then, for all \( p \in \mathcal{P} \), Problem 7 has a unique solution \( \mathbf{u} = u(p) \in U \).

**Proof.** We shall apply Theorem 3.1 with \( X = V, K = U, A: \mathcal{P} \times V \to V^* \),

\[
\langle A(p, u), v \rangle = (A(p, \varepsilon(u)), \varepsilon(v)) \quad \text{for } u, v \in V,
\]

\[
\varphi: \mathcal{P} \times K \to \mathbb{R}, \quad \varphi(p, v) = \int_{\Gamma_3} h(x, p)\|v_r(x)\| \, d\Gamma \quad \text{for } u, v \in V,
\]

\[
j: \mathcal{P} \times V \to \mathbb{R}, \quad j(p, v) = \int_{\Gamma_3} j^0(x, p, v_r(x)) \, d\Gamma \quad \text{for } v \in V
\]

and with \( f(p) = f(p) \in V^* \) defined by (47) for all \( p \in \mathcal{P} \). To this end, we need to verify hypotheses (2)-(6).

First, from (46), it is clear that \( U \) is nonempty, closed and convex subset of \( V, 0 \in U \), that is, condition (2) holds.

Second, from (42), it follows that the operator \( A \) defined by (49) satisfies condition (3). Indeed, for all \( p \in \mathcal{P} \), we know that \( A(p, \cdot) \in L(V, V^*) \) and it is strongly monotone with constant \( \alpha_A = m_a \) for all \( p \in \mathcal{P} \). Since \( A(p, \cdot) \) is continuous, it is also hemicontinuous which together with boundedness and nonnegativity, by [22, Theorem 3.69], implies that it is also pseudomonotone. Hence (3) is satisfied.

Third, we verify that the function \( \varphi \) given by (50) satisfies condition (4). In fact, observe that \( \varphi \) is independent of the variable \( u \), and it is obvious that \( \varphi(p, \cdot) \) is convex and l.s.c. for all \( p \in \mathcal{P} \). From (44), we prove that \( \varphi \) satisfies condition (4)(b) with constants \( \alpha_\varphi = 0 \) and \( \beta_\varphi = c_k L_h \|\gamma\| \sqrt{\text{meas}(\Gamma_3)} \). Indeed, we have

\[
\varphi(p_1, v_2) - \varphi(p_1, v_1) + \varphi(p_2, v_1) - \varphi(p_2, v_2)
= \int_{\Gamma_3} (h(x, p_1) - h(x, p_2)) (|v_{1r}(x)| - |v_{2r}(x)|) \, d\Gamma
\leq L_h \|p_1 - p_2\| \int_{\Gamma_3} |v_{1r}(x) - v_{2r}(x)| \, d\Gamma
\leq L_h \|p_1 - p_2\| \|v_r\| \sqrt{\text{meas}(\Gamma_3)} \|v_{1r} - v_{2r}\|_{L^2(\Gamma_3; \mathbb{R})}
\leq c_k L_h \|\gamma\| \sqrt{\text{meas}(\Gamma_3)} \|p_1 - p_2\| \|v_1 - v_2\|_V
\]

for all \( p_1, p_2 \in \mathcal{P}, v_1, v_2 \in V \). Therefore, \( \varphi \) satisfies condition (4)(b) with the aforementioned constants. Hence (4) holds.

Fourth, using hypotheses (43), we prove that the function \( j \) defined by (51) satisfies condition (5). Condition (5)(a) is obvious while (5)(b) follows from the following
estimate
\[ \| \partial j(p, v) \| \leq \int_{\Gamma_3} \| \partial j_\nu(x, p, v_\nu(x)) \| \, d\Gamma \leq \int_{\Gamma_3} (\tau_0 + \tau_1 \| p \|_p + \tau_2 \| v_\nu(x) \|) \, d\Gamma \]
\[ \leq (\tau_0 + \tau_1 \| p \|_p) \sqrt{\text{meas}(\Gamma_3)} + \tau_2 \int_{\Gamma_3} \| v_\nu(x) \| \, d\Gamma \]
\[ \leq (\tau_0 + \tau_1 \| p \|_p) \sqrt{\text{meas}(\Gamma_3)} + \tau_2 \sqrt{\text{meas}(\Gamma_3)} \| v_\nu \|_{L^2(\Gamma_3, \mathbb{R}^d)} \]
\[ \leq (\tau_0 + \tau_1 \| p \|_p) \sqrt{\text{meas}(\Gamma_3)} + c_k \tau_2 \| \gamma \| \sqrt{\text{meas}(\Gamma_3)} \| v \|_V \]
for all \( p \in \mathcal{P}, v \in V \). Furthermore, by (43)(d), for all \( p_1, p_2 \in \mathcal{P}, u_1, u_2 \in V \), we easily obtain
\[
j_0(p_1, u_1; u_2 - u_1) + j_0(p_2, u_2; u_1 - u_2)
= \int_{\Gamma_3} (j_0'(x, u_{1\nu}(x); u_{2\nu}(x) - u_{1\nu}(x)) + j_0'(x, u_{2\nu}(x); u_{1\nu}(x) - u_{2\nu}(x))) \, d\Gamma
\leq \int_{\Gamma_3} (\alpha_j \| u_{1\nu}(x) - u_{2\nu}(x) \|^2 + \beta_j \| p_1 - p_2 \|_p \| u_{1\nu}(x) - u_{2\nu}(x) \|) \, d\Gamma
\leq c_k \alpha_j \| \gamma \|^2 \| u_1 - u_2 \|^2_V + c_k \beta_j \| \gamma \| \sqrt{\text{meas}(\Gamma_3)} \| p_1 - p_2 \|_p \| u_1 - u_2 \|_V.\]
Therefore, the function \( j \) defined by (51) satisfies condition (5) with constants
\[ c_0 = \tau_0 \sqrt{\text{meas}(\Gamma_3)}, \quad c_1 = \tau_1 \sqrt{\text{meas}(\Gamma_3)}, \quad c_2 = c_k \tau_2 \| \gamma \| \sqrt{\text{meas}(\Gamma_3)}, \]
\[ \alpha_j = c_k \alpha_j \| \gamma \|^2, \quad \beta_j = c_k \beta_j \| \gamma \| \sqrt{\text{meas}(\Gamma_3)}.\]

Fifth, from hypothesis (45), it is obvious that function \( f \) defined by (47) satisfies (6). Finally, considering the constants involved in the conditions, we see that the smallness condition (7) is a consequence of condition (48). The conclusion of the theorem is now a direct consequence of Theorem 3.1.

Next, we illustrate the continuous dependence results for Problem 7. Consider the following problem corresponding to a given sequence of parameters \( \{p_n\} \subset \mathcal{P}, n \in \mathbb{N}. \)

**Problem 8.** Find a displacement field \( u_n \in U \) such that
\[
(\mathcal{A}(p_n, \varepsilon(u_n)), \varepsilon(v) - \varepsilon(u_n))_H + \int_{\Gamma_3} h(p_n)(\| v \| - \| u_n \|) \, d\Gamma
+ \int_{\Gamma_3} j_0(p_n, u_n; v - u_n) \, d\Gamma \geq \langle f(p_n), v - u_n \rangle \quad \text{for all } v \in U.
\]

We introduce the following hypotheses.
\[
\begin{align*}
\text{(a) For any } \{p_n\} \subset \mathcal{P}, \{u_n\} \subset V \text{ with } p_n \to p \text{ in } \mathcal{P} \text{ and } \\
\{p \mapsto a_{ijkl}(x, p) \text{ is Lipschitz continuous with constant } L_a > 0 \text{ for a.e. } x \in \Omega, \text{ for all } i, j, k, l = 1, \ldots, d, \}
\end{align*}
\]
\[
\begin{align*}
\text{(b) Either } j_\nu(x, p, \cdot) \text{ or } j_\nu(x, p, \cdot) \text{ is regular in the sense of Clarke for all } p \in \mathcal{P}, \text{ a.e. } x \in \Omega.
\end{align*}
\]
Theorem 6.2. Assume that hypotheses of Theorem 6.1 hold. Suppose also that (52)–(54) hold and \( \{p_n\} \subset P \) with \( p_n \to \overline{p} \) in \( P \) for some \( \overline{p} \in P \). Then the sequence \( \{u_n\} = \{u(p_n)\} \subset U \) of unique solutions to Problem 8 converges weakly in \( V \) to the solution \( u(\overline{p}) \in U \) to Problem 7.

Proof. It follows from Theorem 3.2. We will verify that hypotheses (52)–(54) imply conditions (9)–(13). Let \( \{p_n\} \subset P \), \( \{u_n\} \subset V \) with \( p_n \to p \) in \( P \) and \( u_n \to u \) in \( V \), and \( v \in V \). We have

\[
\limsup_n \langle A(p, u_n) - A(p, u_n), u_n - v \rangle_V = \limsup_n \langle A(p, \varepsilon(u_n)) - A(p, \varepsilon(u_n)), \varepsilon(u_n) - \varepsilon(v) \rangle_H \leq L_n \limsup_n \|p_n - p\|_P \|\varepsilon(u_n)\|_H \|\varepsilon(u_n) - \varepsilon(v)\|_H = 0
\]

which implies condition (9). Moreover, since the trace operator \( \gamma: V \to L^2(\Gamma_3; \mathbb{R}^d) \) is compact, we get the convergence \( \gamma u_n \to \gamma u \) in \( L^2(\Gamma_3; \mathbb{R}^d) \). By passing to a subsequence, if necessary, we have \( \gamma u_n(x) \to \gamma u(x) \) for a.e. \( x \in \Gamma_3 \). On the other hand, we recall that for function \( j \), by [22, Theorem 3.47], we have the following inequality

\[
j^0(\overline{p}, v; w) \leq \int_{\Gamma_3} j^0_\beta(\overline{p}, v(x); w(x)) \, d\Gamma \quad \text{for all} \quad \overline{p} \in P, \ v, w \in V
\]

and if, in addition, (53)(b) is assumed, then (55) holds with equality. From (53) and (55), by Fatou’s lemma, we deduce

\[
\limsup_n j^0(p_n, u_n; v - u_n) \leq \limsup_n \int_{\Gamma_3} j^0_\beta(x, p_n, u_{n\nu}(x); v_{\nu}(x) - u_{n\nu}(x)) \, d\Gamma \\
\leq \int_{\Gamma_3} \limsup_n j^0_\beta(x, p_n, u_{n\nu}(x); v_{\nu}(x) - u_{n\nu}(x)) \, d\Gamma \\
\leq \int_{\Gamma_3} j^0_\beta(x, p, u_{\nu}(x); v_{\nu}(x) - u_{\nu}(x)) \, d\Gamma = j^0(p, u; v - u).
\]

Hence, condition (10) is verified. Furthermore, using hypothesis (44) and the continuity of the norm, we obtain

\[
\limsup_n (\varphi(p_n, v) - \varphi(p_n, u_n)) = \limsup_n \int_{\Gamma_3} h(x, p_n)(\|v_{\nu}(x)\| - \|u_{n\nu}(x)\|) \, d\Gamma \\
\leq \int_{\Gamma_3} \limsup_n h(x, p_n)(\|v_{\nu}(x)\| - \|u_{n\nu}(x)\|) \, d\Gamma \\
\leq \int_{\Gamma_3} h(x, p)(\|v_{\nu}(x)\| - \|u_{\nu}(x)\|) \, d\Gamma = \varphi(p, v) - \varphi(p, u).
\]

Thus, condition (11) is satisfied. Condition (13) also holds with constants \( l_1 = 0 \) and \( l_2 = c_n L_h \gamma \|\sqrt{\text{meas}(\Gamma_3)}\| \), as a consequence of (42)(b) and (44)(c). Finally, we apply Theorem 3.2 to conclude the proof.

Moreover, we have the following estimate which provides the second continuous dependence result.

\[
\left\{ \begin{array}{l}
p \to f_0(p) \text{ is Lipschitz continuous with constant } L_{f_0} > 0, \\
p \to f_2(p) \text{ is Lipschitz continuous with constant } L_{f_2} > 0.
\end{array} \right.
\]

(54)
Theorem 6.3. Assume that hypotheses of Theorem 6.1 hold and (52), (54) are satisfied. Then

\[ \|u(p_1) - u(p_2)\|_X \leq \frac{L_u \min\{\alpha_{p_1}, \alpha_{p_2}\} + d_1 + d_2 + L_{f_0} + L_{f_2}}{m_u - c_k \alpha_{j_x}} \|\gamma\|^2 \|p_1 - p_2\|_p \]  

(56)

for all \( p_1, p_2 \in \mathcal{P} \), where \( u(p) \in U \) denotes the unique solution to Problem 7 corresponding to \( p \in \mathcal{P} \), \( d_1 = c_k L_h \|\gamma\| \sqrt{\text{meas}(\Gamma_3)} \), \( d_2 = c_k \beta_{j_x} \|\gamma\| \sqrt{\text{meas}(\Gamma_3)} \), and

\[ \alpha_{p_i} = \frac{c_0 \sqrt{\text{meas}(\Gamma_3)} + (c_k \tilde{c}_2 \|\gamma\| \sqrt{\text{meas}(\Gamma_3)} + d_1)\|p\|_p + f_{p_i}}{m_u - c_k \alpha_{j_x} \|\gamma\|^2} \]

\( f_{p_i} = \|f_0(p_i)\|_{L^2(\Omega; \mathbb{R}^d)} + c_k \|\gamma\|\|f_2(p_i)\|_{L^2(\Gamma_2; \mathbb{R}^d)} \), \( i = 1, 2 \).

Proof. The result is a direct consequence of Theorem 3.3. It is enough to observe that under hypotheses (52) and (54), conditions (15) and (16) hold. Let \( p_1, p_2 \in \mathcal{P} \) and \( u, v \in V \). Then

\[ \langle A(p_1, u) - A(p_2, u), v \rangle_V \leq L_u \|p_1 - p_2\|_p \|u\|_V \|v\|_V. \]

On the other hand, analogously as in the proof of Theorem 4.2 (ii), we can show that

\[ \|u(p)\|_X \leq \alpha_p = \frac{c_0 \sqrt{\text{meas}(\Gamma_3)} + (c_k \tilde{c}_2 \|\gamma\| \sqrt{\text{meas}(\Gamma_3)} + d_1)\|p\|_p + f_p}{m_u - c_k \alpha_{j_x} \|\gamma\|^2}, \]

where

\[ f_p = \|f_0(p)\|_{L^2(\Omega; \mathbb{R}^d)} + c_k \|\gamma\|\|f_2(p)\|_{L^2(\Gamma_2; \mathbb{R}^d)}. \]

Hence

\[ \|A(p_1, u) - A(p_2, u)\|_V \leq C \|p_1 - p_2\|_p \text{ with } C > 0, \]

that is, condition (15) holds. Moreover, by the estimate

\[ \|f_0(p_1) - f_0(p_2), v\|_V \leq \|f_0(p_1) - f_0(p_2), v\|_{L^2(\Omega; \mathbb{R}^d)} + \|f_2(p_1) - f_2(p_2), v\|_{L^2(\Gamma_2; \mathbb{R}^d)} \]

\[ \leq \|f_0(p_1) - f_0(p_2)\|_{L^2(\Omega; \mathbb{R}^d)} \|v\|_{L^2(\Omega; \mathbb{R}^d)} + \|f_2(p_1) - f_2(p_2)\|_{L^2(\Gamma_2; \mathbb{R}^d)} \|v\|_{L^2(\Gamma_2; \mathbb{R}^d)} \]

\[ \leq (L_{f_0} + c_k \|\gamma\| \text{meas}(\Gamma_3)) \|p_1 - p_2\|_p \|v\|_V, \]

it follows that (16) is satisfied. This concludes the proof of the theorem. \( \square \)

From Theorems 6.2 and 6.3, we deduce that following results on the inverse problem (18).

Theorem 6.4. (i) Assume that hypotheses of Theorem 6.2 and (19) are satisfied, and

\[ F: \mathcal{P} \times U \rightarrow \mathbb{R} \text{ is l.s.c. on } \mathcal{P}_{ad} \times V_w \]

holds. Then problem (18) has at least one solution.

(ii) Assume that hypotheses of Theorem 6.3 and (19) are satisfied, and

\[ F: \mathcal{P} \times U \rightarrow \mathbb{R} \text{ is l.s.c. on } \mathcal{P}_{ad} \times V, \]

holds. Then problem (18) has at least one solution.

In what follows we can study, analogously as in Sections 4 and 5, the convergence of the penalty method and the convergence of optimal solutions of the inverse problems associated with Problem 7. We are not repeat here all formulations of statements of the results, this can be easily done using the hypotheses introduced.
above. Note only that given constraint set $U$ in Problem 7, we can define the operator $P: V \to V^*$ by
\[
\langle Pu, v \rangle = \int_{\Gamma_3} (u_\nu - g)_+ v_\nu \, d\Gamma \quad \text{for all } u, v \in V.
\] (57)

It follows from e.g. [22, Theorem 2.21] that $P$ is an example of the penalty operator of the set $U$. This operator is independent of the parameter $p$. Since the trace operator from $V$ into $L^2(\Gamma; \mathbb{R}^d)$ is compact, it can be shown that $P$ satisfies condition (29).

Finally, we conclude that the following results hold for Problem 7.

(a) The unique solvability and the convergence of solutions to penalized problem corresponding to Problem 7 when the penalty operator tends to zero. These results can be stated and proved analogously as in Theorem 4.2.

(b) The continuous dependence on parameter of the penalized problem corresponding to Problem 7. This is a result analogous to that of Theorem 4.3.

(c) Results on the existence of solution to the inverse problem for the penalized inequality and the convergence of optimal solutions to corresponding inverse problems. They are analogous to the one in Theorem 5.1.

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