THE CONTINUUM SELF-SIMILAR TREE

MARIO BONK AND HUY TRAN

Abstract. We introduce the continuum self-similar tree (CSST) and characterize it topologically. We apply this to answer a question of Curien [Cur14] about the topology of the continuum random tree (CRT). We also give a topological characterization of other trees with branch points of finite or infinite valences.

1. Introduction

In this paper, we use the following definition of a (metric) tree.

Definition 1.1. A (metric) tree is a locally connected compact metric space \((T, d)\) containing at least two points such that for all \(a, b \in T\) with \(a \neq b\) the following statements are true:

(i) There exists a continuous injective map \(\alpha : [0, 1] \to T\) such that \(\alpha(0) = a\) and \(\alpha(1) = b\).

(ii) If \(\tilde{\alpha} : [0, 1] \to T\) is another continuous injective map with \(\tilde{\alpha}(0) = a\) and \(\tilde{\alpha}(1) = b\), then \(\tilde{\alpha}([0, 1]) = \alpha([0, 1])\).

In other words, in a tree \(T\) any two distinct points \(a, b \in T\) can be joined by a unique arc with endpoints \(a\) and \(b\). If a tree \((T, d)\) is a geodesic metric space in addition, then it is a compact real tree as defined in [LG06].

If \(T\) is a tree, then for \(x \in T\) we denote by \(\nu_T(x) \in \mathbb{N} \cup \{\infty\}\) the number of (connected) components of \(T \setminus \{x\}\). This number \(\nu_T(x)\) is called the valence of \(x\). If \(\nu_T(x) = 1\), then \(x\) is called a leaf of \(T\). If \(\nu_T(x) \geq 3\), then \(x\) is a branch point of \(T\). If \(\nu_T(x) = 3\), then we also call \(x\) a triple point.

The continuum random tree (CRT) is a random (metric) tree introduced by Aldous [Ald91] when he studied the scaling limits of simplicial trees arising from the critical Galton-Watson process. One can describe the CRT as follows. We consider a sample of Brownian excursion \((e_t)_{0 \leq t \leq 1}\) on the interval \([0, 1]\). For \(s, t \in [0, 1]\), we set

\[
d_{e}(s, t) = e(s) + e(t) - 2 \inf \{e(r) : \min(s, t) \leq r \leq \max(s, t)\}.
\]
Then $d_e$ is a pseudo-metric on $[0, 1]$. We define an equivalence relation on $[0, 1]$ by setting $s \sim t$ if $d_e(s, t) = 0$. Then $d_e$ descends to a metric on the quotient space $T_e = [0, 1]/\sim$. The metric space $(T_e, d_e)$ is almost surely a tree (see [LG06 Sections 2 and 3]). Curien [Cur14] asked the following question.

**Question.** Is the topology of the CRT almost surely constant, that is, are two independent samples of the CRT almost surely homeomorphic?

In this paper, we give a positive answer to this question. For this we introduce a deterministic topological model for the CRT. The starting point are some well-known properties of the CRT. Namely, almost surely for every point $x$ in a sample $T$ of the CRT, the valence $\nu_T(x)$ is either 1, 2 or 3, and the set $\{x : \nu_T(x) = 3\}$ is countable and dense in $T$ [DLG05 Theorem 4.6] or [LG06 Proposition 5.2 (i)]. Moreover, the CRT is self-similar in distribution. Informally, this means that a typical sample $T$ of the CRT can be cut at an appropriate triple point into three different trees such that after being rescaled, each of these trees has the same distribution as the CRT (see [Ald94 Theorem 2] for a precise statement). We construct a (deterministic) tree with an analogous self-similarity. We call this tree the *continuum self-similar tree* (CSST) and denote it by $T$.

One way to construct $T$ is as follows. We start with a line segment $J_0$ of length 2. Its midpoint $c$ subdivides $J_0$ into two line segments of length 1. If we glue to $c$ one of the endpoints of another line segment of the same length, then we obtain a space $J_1$ consisting of three line segments of length 1. The space $J_1$ is equipped with the natural path metric. We now repeat this procedure inductively. At the $n$th step we obtain a tree $J_n$ consisting of $3^n$ line segments of length $2^{-1-n}$. Each of these line segments $s$ is subdivided by its midpoint $c_s$ into two line segment of length $2^{-n}$ and we glue to $c_s$ one endpoint of another line segment of length $2^{-n}$. In this way, we obtain an ascending sequence of geodesic trees $J_0 \subseteq J_1 \subseteq \ldots$. The union $J = \bigcup_{n \in \mathbb{N}_0} J_n$ carries a natural path metric $\varrho$ that agrees with the metric on $J_n$ for each $n \in \mathbb{N}_0$. Then one can define $T$ as completion of the metric space $(J, \varrho)$.

In Section 4 we will actually give another definition of $T$ as the attractor of an iterated function system in the complex plane $\mathbb{C}$. Then we can consider $T$ as a subset of $\mathbb{C}$. See Figure 2 for a graphical representation. The Euclidean metric on $T$ is bi-Lipschitz equivalent to the geodesic metric on $T$ that we obtain if we define $T$ as the completion of $(J, \varrho)$ as outlined above (see the discussion at the end of Section 4 for more details).

The following theorem is one of the main results of this paper.
Theorem 1.2. A tree \((T,d)\) is homeomorphic to the continuum self-similar tree \(T\) if and only if the following conditions are true:

(i) For every point \(x \in T\) we have \(\nu_T(x) \in \{1,2,3\}\).

(ii) The set of triple points \(\{x \in T : \nu_T(x) = 3\}\) is a dense subset of \(T\).

Theorem 1.2 seems to be a folklore statement. For example, it was asked on MathOverflow in 2011 [Mat], but to the best of our knowledge, there is no recorded proof in the literature.

By the preceding discussion, Theorem 1.2 affirmatively answer Curien’s question.

Corollary 1.3. A sample \(T\) of the CRT is almost surely homeomorphic to the CSST.

Informally, this says that the topology of the CRT is (almost surely) constant. Curien’s question was answered implicitly in [CH08]. There the authors used the distributional self-similarity property of the CRT and showed that the CRT is isometric to a metric space with a random metric. It is constructed similarly to the CSST.

An important source of trees is given by Julia sets of postcritically-finite polynomials without periodic critical points in \(\mathbb{C}\). It follows from [DH84] (or see [CG93, Theorem V.4.2]) that the Julia sets of such polynomials are indeed trees. One can show that the Julia set \(J(P)\) of the polynomial \(P(z) = z^2 + i\) (see Figure 1a) satisfies the conditions in Theorem 1.2. Accordingly, \(J(P)\) is homeomorphic to the CSST. In the forthcoming paper [BT18] we prove a stronger statement: If the Julia set \(J(P)\) of a postcritically-finite polynomial \(P\) with no periodic critical points in \(\mathbb{C}\) is homeomorphic to the CSST, then \(J(P)\) is quasi-symmetrically equivalent to the CSST.

We will derive Theorem 1.2 from a more general statement. For its formulation let \(m \in \mathbb{N} \cup \{\infty\}\) with \(m \geq 3\). We consider the family \(\mathcal{T}_m\) consisting of all metric trees \(T\) such that

(i) for every point \(x \in T\) we have \(\nu_T(x) \in \{1,2,m\}\) and

(ii) the set of branch points \(\{x \in T : \nu_T(x) = m\}\) is a dense subset of \(T\).

Then the following statement is true.

Theorem 1.4. Let \(m \in \mathbb{N} \cup \{\infty\}\) with \(m \geq 3\). Then all trees in \(\mathcal{T}_m\) are homeomorphic to each other.

The proof of Theorem 1.4 can be outlined as follows. Fix \(m\) as in the statement and consider a tree \(T \in \mathcal{T}_m\). Then we cut \(T\) into subtrees at
a carefully chosen branch point. This process is repeated inductively. One labels the subtrees obtained in this way by finite words consisting of letters in an alphabet \( \mathcal{A} \). Here \( \mathcal{A} = \{1, 2, \ldots, m\} \) if \( m \in \mathbb{N} \) and \( \mathcal{A} = \mathbb{N} \) if \( m = \infty \). The labels are chosen so that if \( S \) is another tree in \( \mathcal{T}_m \), then one has the same combinatorics (i.e., intersection and inclusion pattern) for the subtrees in \( T \) and \( S \). The desired homeomorphism of \( T \) and \( S \) can then be obtained from general statements that produces a homeomorphism between two spaces, if they admit matching decompositions into pieces satisfying suitable conditions (we will apply Proposition 2.1 for \( m \in \mathbb{N} \) and Proposition 6.2 for \( m = \infty \)).

The so-called stable trees with index \( \alpha \in (1, 2] \) appear in probability as generalizations of the CRT (see [LG06, Section 4] for the definition). For \( \alpha \in (1, 2) \), a sample \( T \) of such a stable tree belongs to \( \mathcal{T}_\infty \) almost surely [LG06, Proposition 5.2 (ii)]. Our Theorem 1.4 covers this situation. In particular, two independent samples of stable trees for fixed \( \alpha \in (1, 2) \) are almost surely homeomorphic. Note that the Julia set of a polynomial never belongs to \( \mathcal{T}_\infty \). This follows from results due to Kiwi (see [Kiw02, Theorem 1.1]).

In the definition of \( \mathcal{T}_m \) one can replace \( m \) with any finite list \( m_1 < m_2 < \cdots < m_s \) of elements in \( \{3, 4, \ldots\} \cup \{\infty\} \). Then \( \mathcal{T}_{m_1, \ldots, m_s} \) consists of all trees whose branch points form a dense subset of \( T \) and have only valences in the set \( \{m_1, \ldots, m_s\} \). In general, one cannot expect that all trees in \( \mathcal{T}_{m_1, \ldots, m_s} \) are homeomorphic to each other. For instance, there exist two non-homeomorphic trees in \( \mathcal{T}_{3,4} \). Figures 1b and 1c provide an example, because one can show that unlike the tree in Figure 1b, the tree in Figure 1c has subarcs that do not contain branch points of valence 3.

In order to state a positive result, we consider the family \( \mathcal{T}^*_{m_1, \ldots, m_s} \) consisting of all trees \( T \) such that

(i) for every point \( x \in T \) we have \( \nu_T(x) \in \{1, 2, m_1, \ldots, m_s\} \) and

(ii) for each (non-degenerate) arc \( \alpha \subseteq T \) and each \( i = 1, \ldots, s \) there exists a point \( x \in \alpha \) with \( \nu_T(x) = m_i \).

Then the following statement is true.

**Theorem 1.5.** Let \( s \in \mathbb{N} \) and \( m_1 < m_2 < \cdots < m_s \) be numbers in \( \{3, 4, \ldots\} \cup \{\infty\} \). Then all trees in \( \mathcal{T}^*_{m_1, \ldots, m_s} \) are homeomorphic to each other.

It is easy to verify that \( \mathcal{T}^*_{m} = \mathcal{T}_m \) for \( m \in \{3, 4, \ldots\} \cup \{\infty\} \) (see the beginning of the proof of Theorem 1.5 in Section 7). So the previous theorem actually implies Theorem 1.4. We will prove Theorem 1.4 first and then point out the necessary changes for the proof of Theorem 1.5.
(a) The Julia set of $z^2 + i$.

(b) The Julia set of $z^2 - 1.222863... + i0.316882...$. Here the critical point $\omega_0 = 0$ has the orbit: $\omega_0 \mapsto \omega_1 \mapsto \omega_2 \mapsto \omega_3 \mapsto \omega_4 \mapsto \omega_5 = \omega_0$.

(c) A tree in $\mathcal{T}_{3,4}$.

Figure 1. Various trees. The first two were simulated by Wolf Jung’s Mandel program.
This paper is organized as follows. In Section 2 we state and prove a general criterion for two metric spaces to be homeomorphic based on the existence of combinatorially equivalent decompositions of the spaces with suitable properties. In Section 3 we collect some general facts about tree that we use later. The CSST is defined precisely and studied in Section 4. In Section 5 we explain how to decompose trees in $T_m$ with $m \in \mathbb{N} \cup \{\infty\}$, $m \geq 3$. Based on this, we then provide a proof Theorem 1.2 for $m \neq \infty$. Section 6 is devoted to the proof of Theorem 1.4 for trees in $T_\infty$. Finally, in Section 7 we explain how a refinement of our methods leads to a justification of Theorem 1.5.

2. Constructing homeomorphisms between spaces

Throughout this paper, we use fairly standard metric space notation. If $(X,d)$ is a metric space, then we denote by $B(a, r) = \{x \in X : d(a, x) < r\}$ the open ball of radius $r > 0$ centered at $a \in X$. If $A, B \subseteq X$, then $\text{diam}(A)$ is the diameter of $A$ and $\text{dist}(A, B)$ the (minimal) distance of $A$ and $B$. Similarly, if $a \in X$, then $\text{dist}(a, B)$ denotes the distance of the point $a$ to the set $B$. Finally, if $\gamma$ is a path in $X$, then $\text{length}(\gamma)$ stands for its length.

Before we discuss trees in more detail and define the CSST, we will establish the following proposition that is key to showing that two trees are homeomorphic. The statement will also give us some guidance for the desired properties of tree decompositions that we will discuss in the following sections. The proposition is inspired by [BM17, Proposition 17.7], which provided geometric conditions for the decomposition of a space that can be used to construct quasisymmetric homeomorphisms.

**Proposition 2.1.** Let $(X,d_X)$ and $(Y,d_Y)$ be compact metric spaces. Suppose that for each $n \in \mathbb{N}$, the space $X$ admits a decomposition $X = \bigcup_{i=1}^{M_n} X_{n,i}$ as a finite union of non-empty compact subsets $X_{n,i}$, $i = 1, \ldots, M_n \in \mathbb{N}$, with the following properties for all $n$, $i$, and $j$:

(i) Each set $X_{n+1,j}$ is the subset of some set $X_{n,i}$.

(ii) Each set $X_{n,i}$ is equal to the union of some of the sets $X_{n+1,j}$.

(iii) $\max_{1 \leq i \leq M_n} \text{diam}(X_{n,i}) \to 0$ as $n \to \infty$.

Suppose that for $n \in \mathbb{N}$ the space $Y$ admits a decomposition $Y = \bigcup_{i=1}^{M_n} Y_{n,i}$ as a union of non-empty compact subsets $Y_{n,i}$, $i = 1, \ldots, M_n$, with properties analogous to (i)–(iii) such that

(2.1) $X_{n+1,j} \subseteq X_{n,i}$ if and only if $Y_{n+1,j} \subseteq Y_{n,i}$

and

(2.2) $X_{n,i} \cap X_{n,j} \neq \emptyset$ if and only if $Y_{n,i} \cap Y_{n,j} \neq \emptyset$
for all $n, i, j$.

Then there exists a unique homeomorphism $f : X \to Y$ such that $f(X_{n,i}) = Y_{n,i}$ for all $n$ and $i$.

In particular, under these assumptions the spaces $X$ and $Y$ are homeomorphic.

**Proof.** We define a map $f : X \to Y$ as follows. For each point $x \in X$, by (ii) there exists a nested sequence of sets $X_{n,i_n}$, $n \in \mathbb{N}$, such that $\{x\} = \bigcap_n X_{n,i_n}$. Then the corresponding sets $Y_{n,i_n}$, $n \in \mathbb{N}$, are also nested by (2.1). Since these sets are non-empty and compact, condition (iii) for the space $Y$ this implies that there exists a unique point $y \in \bigcap_n Y_{n,i_n}$. We define $f(x) = y$.

Then $f$ is well-defined. To see this, suppose we have another nested sequence $X_{n,i'_n}$, $n \in \mathbb{N}$, such that $\{x\} = \bigcap_n X_{n,i'_n}$. Then there exists a unique point $y' \in \bigcap_n Y_{n,i'_n}$. Now $x \in X_{n,i_n} \cap X_{n,i'_n}$ and so $Y_{n,i_n} \cap Y_{n,i'_n} \neq \emptyset$ by (2.2) for all $n \in \mathbb{N}$. By condition (iii) for $Y$, this is only possible if $y = y'$. So $f : X \to Y$ is indeed well-defined.

One can define a map $g : Y \to X$ by a similar procedure. Namely, for each $y \in Y$ we can find a nested sequence $Y_{n,i_n}$, $n \in \mathbb{N}$, such that $\{y\} = \bigcap_n Y_{n,i_n}$. Then there exists a unique point $x \in \bigcap_n X_{n,i_n}$ and if we set $g(y) = x$, we obtain a well-defined map $g : Y \to X$.

It is obvious from the definitions that the maps $f$ and $g$ are inverse to each other. Hence they define bijections between $X$ and $Y$.

Conditions (i) and (ii) imply that if $X_{k,i}$ is a set in one of the decompositions of $X$ and $x \in X_{k,i}$, then there exists a nested sequence $X_{n,i_n}$, $n \in \mathbb{N}$, with $X_{k,i_n} = X_{k,i}$ and $\{x\} = \bigcap_n X_{n,i_n}$. This implies that $f(x) \in Y_{k,i}$ and so $f(X_{k,i}) \subseteq Y_{k,i}$. Similarly, $g(Y_{k,i}) \subseteq X_{k,i}$. Since $g = f^{-1}$, this implies $f(X_{k,i}) = Y_{k,i}$ as desired. It is clear that this last condition together with our assumptions determines $f$ uniquely.

It remains to show that $f$ is a homeomorphism. For this it suffices to prove that $f$ and $f^{-1} = g$ are continuous. Since the roles of $f$ and $g$ are completely symmetric, it is enough to establish that $f$ is continuous.

For this, let $\epsilon > 0$ be arbitrary. By (iii) we can choose $N \in \mathbb{N}$ such that

$$\max\{\text{diam}(Y_{N,i}) : 1 \leq i \leq M_N\} < \epsilon/2.$$ 

Since the sets $X_{N,i}$ are compact, there exists $\delta > 0$ such that

$$\text{dist}(X_{N,i}, X_{N,j}) > \delta,$$

whenever $i, j \in \{1, \ldots, M_N\}$ and $X_{N,i} \cap X_{N,j} = \emptyset$.

Now suppose that $a, b \in X$ are arbitrary points with $d_X(a, b) < \delta$. We claim that then $d_Y(f(a), f(b)) < \epsilon$. Indeed, we can find $i, j \in \{1, \ldots, M_N\}$ such that $a \in X_{N,i}$ and $b \in X_{N,j}$. Since $d_X(a, b) < \delta$, we
then necessarily have $X_{N,i} \cap X_{N,j} \neq \emptyset$ by definition of $\delta$. So $Y_{N,i} \cap Y_{N,j} \neq \emptyset$ by (2.2). Moreover, $f(a) \in f(X_{N,i}) = Y_{N,i}$ and $f(b) \in f(X_{N,j}) = Y_{N,j}$. Hence

$$d_Y(f(a), f(b)) \leq \text{diam}(Y_{N,i}) + \text{diam}(Y_{N,j}) < \epsilon.$$  

The continuity of $f$ follows.  

3. Topology of trees

In this section we fix some terminology and collect some general facts about trees.

An arc $\alpha$ in a metric space is a homeomorphic image of the unit interval $[0, 1] \subseteq \mathbb{R}$. The points corresponding to 0 and 1 are called the endpoints of $\alpha$.

Let $T$ be a tree. Then the last part of Definition 1.1 is equivalent to the requirement that for all points $a, b \in T$ with $a \neq b$, there exists a unique arc in $T$ joining $a$ and $b$, i.e., it has the endpoints $a$ and $b$. We use the notation $[a, b]$ for this unique arc. It is convenient to allow $a = b$ here. Then $[a, b]$ denotes the degenerate arc consisting only of the point $a = b$. Sometimes we want to remove one or both endpoints from the arc $[a, b]$. Accordingly, we define $(a, b) = [a, b] \setminus \{a, b\}$, $[a, b) = [a, b] \setminus \{b\}$ and $[a, b] = [a, b] \setminus \{a\}$. In Section 4 we will not use this notation for arcs in a tree. There $[a, b]$ will always denote the Euclidean line segment joining two points $a, b \in \mathbb{C}$.

A metric space $X$ is called path-connected if any two points $a, b \in X$ can be joined by a path in $X$, i.e., there exists a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$. The space $X$ is arc-connected if any two distinct points in $X$ can be joined by an arc in $X$. The image of a path joining two distinct points in a metric space always contains an arc joining these points (this follows from the fact that every Peano space is arc-connected; see [HY61, Theorem 3.15, p. 116]). In particular, every path-connected metric space is arc-connected.

**Lemma 3.1.** Let $(T, d)$ be a tree. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $a, b \in T$ with $d(a, b) < \delta$ we have $\text{diam}([a, b]) < \epsilon$.

**Proof.** Fix $\epsilon > 0$. Since $T$ is a locally connected, connected, and compact metric space, it is a Peano space. So by the Hahn-Mazurkiewicz theorem there exists a continuous surjective map $\varphi : [0, 1] \to T$ of the unit interval onto $T$ [HY61, Theorem 3.30, p. 129]. By uniform continuity of $\varphi$ we can represent $[0, 1]$ as a union $[0, 1] = I_1 \cup \cdots \cup I_n$ of finitely many closed intervals $I_1, \ldots, I_n \subseteq [0, 1]$ with $\text{diam}(X_k) < \epsilon/2$. 

where $X_k = \varphi(I_k)$ for $k = 1, \ldots, n$. The sets $X_k = \varphi(I_k)$ are compact. This implies that there exists $\delta > 0$ such that $\text{dist}(X_i, X_j) > \delta$, whenever $i, j \in \{1, \ldots, n\}$ and $X_i \cap X_j = \emptyset$.

Now let $a, b \in T$ with $d(a, b) < \delta$ be arbitrary. We may assume $a \neq b$. Then there exist $i, j \in \{1, \ldots, n\}$ with $a \in X := X_i$ and $b = Y := X_j$. By choice of $\delta$ we must have $X \cap Y \neq \emptyset$. As continuous images of intervals, the sets $X$ and $Y$ are path-connected. Since $X \cap Y \neq \emptyset$, the union $X \cup Y$ that contains the points $a$ and $b$ is also path-connected.

This implies that $X \cup Y$ is arc-connected, and so there exists an arc $\alpha \subseteq X \cup Y$ with endpoints $a$ and $b$. The unique such arc in the tree $T$ is $[a, b]$, and so $[a, b] = \alpha \subseteq X \cup Y$. This implies $\text{diam}([a, b]) \leq \text{diam}(X) + \text{diam}(Y) < \epsilon$ as desired. □

Lemma 3.2. Let $(T, d)$ be a tree and $p \in T$. Then the following statements are true:

(i) Each component $U$ of $T \setminus \{p\}$ is an open and arc-connected subset of $T$.

(ii) If $U$ is a component of $T \setminus \{p\}$, then $\overline{U} = U \cup \{p\}$ and $\partial U = \{p\}$.

(iii) Two points $a, b \in T \setminus \{p\}$ lie in the same component of $T \setminus \{p\}$ if and only if $p \notin [a, b]$.

Proof. (i) The set $T \setminus \{p\}$ is open. Since $T$ is locally connected, each component $U$ of $T \setminus \{p\}$ is also open.

For $a, b \in U$ we write $a \sim b$ if $a$ and $b$ can be joined by a path in $U$. Obviously, this defines an equivalence relation on $U$. The equivalence classes are open subsets of $T$. To see this, suppose $a$ and $b$ can be joined by a path $\beta$ in $U$. Then for all points $x$ in a sufficiently small neighborhood $V \subseteq U$ of $b$ we have $[b, x] \subseteq U$ as follows from Lemma 3.1.

So by concatenating $\beta$ with the arc $[b, x]$, we obtain a path $\beta'$ in $U$ that joins $a$ and $x \in V$. This shows that every point $b$ in the equivalence class of $a$ has a neighborhood $V$ that also belongs to this equivalence class.

We see that the equivalence classes of $\sim$ partition $U$ into open sets. Since $U$ is connected, there can only be one such set. It follows that $U$ is path-connected and hence also arc-connected.

(ii) Let $U$ be a component of $T \setminus \{p\}$. We can pick a point $a \in U$. The set $[a, p]$ is connected set, contained in $T \setminus \{p\}$, and meets $U$ in $a$. Hence $[a, p] \subseteq U$. This implies that $p \in \overline{U}$. On the other hand, the set $U \cup \{p\}$ is closed, because its complement is a union of components of
\[ T \setminus \{ p \} \] and hence open by (i). Thus \( U = U \cup \{ p \} \). By (i) no point in \( U \) is a boundary point of \( U \), and so \( \partial U = \{ p \} \).

(iii) If \( a, b \in T \setminus \{ p \} \) and \( p \not\in [a, b] \), then \([a, b] \) is a connected subset of \( T \setminus \{ p \} \). Hence \([a, b] \) lies in a component \( U \) of \( T \setminus \{ p \} \). In particular, \( a, b \in [a, b] \) lie in the same component \( U \) of \( T \setminus \{ p \} \).

Conversely, suppose that \( a, b \in T \setminus \{ p \} \) lie in the same component \( U \) of \( T \setminus \{ p \} \). We know by (i) that \( U \) is arc-connected. Hence there exists a (possible degenerate) arc \( \alpha \subseteq U \) with endpoints \( a \) and \( b \). But the unique such arc in \( T \) is \([a, b] \). Hence \([a, b] = \alpha \subseteq U \subseteq T \setminus \{ p \} \), and so \( p \not\in [a, b] \).

A subset \( S \) of a tree \((T, d)\) is called a subtree of \( T \) if \( S \) equipped with the restriction of the metric \( d \) is also a tree as in Definition 1.1. Every subtree \( S \) of \( T \) contains two points and hence a non-degenerate arc. In particular, every subtree \( S \) of \( T \) is an infinite, actually uncountable set.

The following statement characterizes subtrees.

**Lemma 3.3.** Let \((T, d)\) be a tree. Then a set \( S \subseteq T \) is a subtree of \( T \) if and only if \( S \) contains at least two points and is closed and connected.

**Proof.** If \( S \) is a subtree of \( T \), then \( S \) contains at least two points, and is connected and compact. Hence it is a closed subset of \( T \). Conversely, suppose that \( S \) contains at least two points and is closed and connected. Then \( S \) is compact, because \( T \) is compact.

Suppose that \( a, b \in S, \ a \neq b \), are two distinct points in \( S \). We consider the arc \([a, b] \subseteq T \). Suppose there exists a point \( p \in [a, b] \) with \( p \not\in S \). Then \( p \neq a, b \), and so by Lemma 3.2 (iii), the points \( a \) and \( b \) lie in different components of \( T \setminus \{ p \} \). This is impossible, because the connected set \( S \subseteq T \setminus \{ p \} \) must be contained in exactly one component of \( T \setminus \{ p \} \). This shows that \([a, b] \subseteq S \) and so the points \( a \) and \( b \) can be joined by an arc in \( S \). This arc in \( S \) is unique, because it is unique in \( T \).

It remains to show that \( S \) is locally connected, i.e., every point in \( S \) has arbitrarily small connected relative neighborhoods. To see this, let \( a \in S \) and \( \epsilon > 0 \) be arbitrary. Then by Lemma 3.1 we can find \( \delta > 0 \) such that \([a, x] \subseteq B(a, \epsilon)\) whenever \( x \in B(a, \delta) \). Now let \( N \) be the union of all arcs \([a, x] \) with \( x \in S \cap B(a, \delta) \). These arcs lie in \( S \) and so \( N \) is a connected set contained in \( S \cap B(a, \epsilon) \). Moreover, \( S \cap B(a, \delta) \subseteq N \) and so \( N \) is a connected relative (not necessarily open) neighborhood of \( a \) in \( S \). This shows that \( S \) is locally connected, and we conclude that \( S \) is indeed a subtree of \( T \). □

**Lemma 3.4.** Let \((T, d)\) be a tree, \( p \in T \), and \( U \) a component of \( T \setminus \{ p \} \). Then \( B = U \cup \{ p \} \) is a subtree of \( T \) and \( p \) is a leaf of \( B \).
Let \( \text{Lemma 3.5.} \) is a leaf of \( U \). Since \( U \neq \emptyset \) and \( p \notin U \), the set \( B \) contains at least two points. Hence \( B \) is a subtree of \( T \) by \( \text{Lemma 3.3} \). Since \( B\{p\} = U \) is connected, \( p \) is a leaf of \( B \).

If the subtree \( B = U \cup \{p\} \) is as in the previous lemma, then we call \( B \) a branch of \( p \) in \( T \) (or just a branch of \( p \) if \( T \) is understood).

**Lemma 3.5.** Let \( (T,d) \) be a tree, \( S \subseteq T \) be a subtree of \( T \), and \( p \in S \). Then every branch \( B' \) of \( p \) in \( S \) is contained in a unique branch \( B \) of \( p \) in \( T \). The assignment \( B' \mapsto B \) is an injective map between the sets of branches of \( p \) in \( S \) and in \( T \). If \( p \) is an interior point of \( S \), then this map is a bijection.

In particular, if under the given assumptions \( \nu_T(p) \) is the valence of \( p \) in \( T \) and \( \nu_S(p) \) the valence of \( p \) in \( S \), then \( \nu_S(p) \leq \nu_T(p) \). Here we have equality if \( p \) is an interior point of \( S \).

If \( p \in S \) is a leaf of \( T \), then \( T \) has only one branch \( B \) at \( p \), namely \( B = T \). Hence \( 1 \leq \nu_S(p) \leq \nu_T(p) \leq 1 \), and so \( \nu_S(p) = 1 \). This means that \( p \) is also a leaf of \( S \). More informally, we can say that the property of a point being a leaf in \( T \) is passed to subtrees that contain the point.

**Proof.** If \( B' \) is a branch of \( p \) in \( S \), then \( B' = U' \cup \{p\} \), where \( U' \) is a component of \( S\{p\} \). Then \( U' \) is a connected subset of \( T\{p\} \) and so contained in a unique component \( U \) of \( T\{p\} \). Then \( B = U \cup \{p\} \) is a branch of \( p \) in \( T \) with \( B' \subseteq B \) and it is clear that \( B \) is the unique such branch.

To show injectivity of the map \( B' \mapsto B \), let \( B'_1 \) and \( B'_2 \) be two distinct branches of \( p \) in \( S \). Pick points \( a \in B'_1 \{p\} \) and \( b \in B'_2 \{p\} \). Then \( a \) and \( b \) lie in different components of \( S\{p\} \) and so \( p \in [a,b] \) by \( \text{Lemma 3.2} (\text{iii}) \) applied to the tree \( S \). Hence \( a \) and \( b \) lie in different components of \( T\{p\} \), and so in different branches of \( p \) in \( T \). This implies that \( B'_1 \) and \( B'_2 \) must be contained in different branches of \( p \) in \( T \). This shows that the map \( B' \mapsto B \) is indeed injective.

Now assume in addition that \( p \) is an interior point of \( S \). To show surjectivity of the map \( B' \mapsto B \), we consider a branch \( B \) of \( p \) in \( T \). Pick a point \( a \in B \{p\} \). Then \( [a,p] \subseteq B\{p\} \), because \( B \) is a subtree of \( T \). Since \( p \) is an interior point of \( S \), there exists a point \( x \in [a,p] \) close enough to \( p \) such that \( x \in S\{p\} \). If \( B' \) is the unique branch of \( p \) in \( S \) that contains \( x \), then we have \( x \in B' \cap B \). This implies \( B' \subseteq B \). Hence the map \( B' \mapsto B \) is also surjective, and so a bijection.

**Lemma 3.6.** Let \( T \) be a tree, \( p,a_1,a_2,a_3 \in T \) with \( p \neq a_1,a_2,a_3 \) and suppose that the sets \( [a_1,p) \), \( [a_2,p) \), \( [a_3,p) \) are pairwise disjoint. Then
the points \(a_1, a_2, a_3\) lie in different components of \(T \setminus \{p\}\) and \(p\) is a branch point of \(T\).

**Proof.** The arcs \([a_1, p]\) and \([a_2, p] = [p, a_2]\) have only the point \(p\) in common. So their union \([a_1, p] \cup [p, a_2]\) is an arc and this arc must be equal to \([a_1, a_2]\). Hence \(p \in [a_1, a_2]\) which by Lemma 3.2 (iii) implies that \(a_1\) and \(a_2\) lie in different components of \(T \setminus \{p\}\). A similar argument shows that \(a_3\) must be contained in a component of \(T \setminus \{p\}\) different from the components containing \(a_1\) and \(a_2\). In particular, \(T \setminus \{p\}\) has at least three components and so \(p\) is a branch point of \(T\). The statement follows. 

**Lemma 3.7.** Let \(T\) be a tree such that the branch points of \(T\) are dense in \(T\). If \(a, b \in T\) with \(a \neq b\), then there exists a branch point \(c \in (a, b)\).

**Proof.** We pick a point \(x_0 \in (a, b) \neq \emptyset\). Then \(x_0\) has positive distance to both \(a\) and \(b\). This and Lemma 3.1 imply that we can find \(\delta > 0\) such that for all \(x \in B(x_0, \delta)\) the arc \([x, x_0]\) has uniformly small diameter and so does not contain \(a\) or \(b\).

Since branch points are dense in \(T\), we can find a branch point \(p \in B(x_0, \delta)\). Then \(a, b \notin [p, x_0]\). If \(p \in (a, b)\), we are done.

In the other case, we have \(p \notin (a, b)\). If we travel from \(p\) to \(x_0 \in (a, b)\) along \([p, x_0]\), we meet \([a, b]\) in a first point \(c \in (a, b)\). Then \(a, b, p \neq c\). Moreover, the sets \([a, c), [b, c), [p, c)\) are pairwise disjoint. Hence \(c \in (a, b)\) is a branch point of \(T\) as follows from Lemma 3.6.

**Lemma 3.8.** Let \((X, d)\) be a compact, connected, and locally connected metric space, \(J\) an index set, \(p_i \in T\), and \(U_i\) a component of \(X \setminus \{p_i\}\) for each \(i \in J\). Suppose that

\[U_i \cap U_j = \emptyset\]

for all \(i, j \in J, i \neq j\). Then \(J\) is a countable set. If there exists \(\delta > 0\) such that \(\text{diam}(U_i) > \delta\) for each \(i \in J\), then \(J\) is finite.

Informally, the space \(X\) cannot contain a "comb" with too many long teeth.

**Proof.** We prove the last statement first. We argue by contradiction and assume that \(\text{diam}(U_i) > \delta > 0\) for each \(i \in J\), where \(J\) is an infinite index set. Then we can choose a point \(x_i \in U_i\) such that \(d(x_i, p_i) \geq \delta/2\). The set \(A = \{x_i : i \in J\}\) is infinite and so it must have a limit point \(q \in X\), because \(X\) is compact. Since \(X\) is locally connected, there exists a connected neighborhood \(N\) of \(q\) such that \(N \subseteq B(q, \delta/8)\). Since \(q\) is a limit point of \(A\), the set \(N\) contains infinitely many points in \(A\). In
particular, we can find \(i, j \in J\) with \(x_i, x_j \in N\) and \(i \neq j\). Then
\[
\text{dist}(p_i, N) \geq d(p_i, x_i) - \text{diam}(N) \geq \delta/2 - \delta/4 > 0,
\]
and so \(N \subseteq X \setminus \{p_i\}\). Since the connected set \(N\) meets \(U_i\) in the point \(x_i\), this implies that \(N \subseteq U_i\). Similarly, \(N \subseteq U_j\). This is impossible, because we have \(i \neq j\) and so \(U_i \cap U_j \neq \emptyset\), while \(\emptyset \neq N \subseteq U_i \cap U_j\).

To prove the first statement, note that \(\text{diam}(U_i) > 0\) for each \(i \in J\). Indeed, otherwise \(\text{diam}(U_i) = 0\) for some \(i \in J\). Then \(U_i\) consists of only one point \(a\). Since \(X\) is locally connected, the component \(U_i\) of \(X \setminus \{p_i\}\) is an open set. So \(a\) is an isolated point of \(X\). This is impossible, because the metric space \(X\) is connected and so it does not have isolated points.

Now we write \(J = \bigcup_{n \in \mathbb{N}} J_n\), where \(J_n\) consists of all \(i \in J\) such that \(\text{diam}(U_i) > 1/n\). Then each set \(J_n\) is finite by the first part of the proof. This implies that \(J\) is countable. \(\square\)

We can apply the previous lemma to a tree \(T\) and choose for each \(p_i\) a fixed branch point \(p\) of \(T\). Then it follows that \(p\) can have at most countably many distinct complementary components \(U_i\) and hence there are only countable many distinct branches \(B_i = U_i \cup \{p\}\) of \(p\). Moreover, since \(\text{diam}(B_i) = \text{diam}(\overline{U}_i) = \text{diam}(U_i)\), there can only be finitely many of these branches whose diameter exceeds a given positive number \(\delta > 0\). In particular, we can label the branches of \(p\) by numbers \(n = 1, 2, 3, \ldots\) so that
\[
\text{diam}(B_1) \geq \text{diam}(B_2) \geq \text{diam}(B_3) \geq \ldots.
\]
We now set \(M(p) = \text{diam}(B_3)\) and call \(M(p)\) the weight of the branch point \(p\) in \(T\). So the weight of a branch point \(p\) is the diameter of the third largest branch of \(p\).

**Lemma 3.9.** Let \((T, d)\) be a tree and \(\delta > 0\). Then there are at most finitely many branch points \(p \in T\) with weight \(M(p) > \delta\).

**Proof.** We argue by contradiction and assume that this is not true. Then the set \(E\) of branch points \(p\) in \(T\) with \(M(p) > \delta\) has infinitely many elements. Since \(T\) is compact, the set \(E\) has a limit point \(q \in T\).

Claim. There exists a branch \(Q\) of \(q\) such that the set \(E \cap Q\) is infinite and has \(q\) as a limit point.

Otherwise, \(q\) has infinitely many distinct branches \(Q_n, n \in \mathbb{N}\), that contain a point \(a_n \in E \cap (Q_n \setminus \{q\})\). Then \(a_n\) is a branch point with \(M(a_n) > \delta\) and so at least one of the three branches of \(a_n\) whose diameters exceed \(\delta\) does not contain \(q\). If we denote such a branch of \(a_n\) by \(V_n\), then \(V_n\) is a connected set in \(T \setminus \{q\}\). It meets \(Q_n \setminus \{q\}\), because
$a_n \in (Q_n \setminus \{q\}) \cap V_n$. It follows that $V_n \subseteq Q_n$ and so $\text{diam}(Q_n) \geq \text{diam}(V_n) > \delta$. Since the branches $Q_n$ are distinct, this contradicts Lemma 3.8 (see the discussion after the proof of this lemma). The Claim follows.

We fix a branch $Q$ of $q$ as in the Claim. For each $n \in \mathbb{N}$ we will now inductively construct branch points $p_n \in E \cap (Q \setminus \{q\})$ together with a branch $B_n$ of $p_n$ and an auxiliary compact set $K_n \subseteq T$. They will satisfy the following conditions for each $n \in \mathbb{N}$:

(i) $\text{diam}(B_n) > \delta$,

(ii) the sets $B_1, \ldots, B_n$ are disjoint,

(iii) the set $K_n$ is compact and connected, and

\[ B_1 \cup \cdots \cup B_n \subseteq K_n \subseteq Q \setminus \{q\}. \]

We pick an arbitrary branch point $p_1 \in E \cap (Q \setminus \{q\})$ to start. Then we can choose a branch $B_1$ of $p_1$ that does not contain $q$ and satisfies $\text{diam}(B_1) > \delta$. We set $K_1 = B_1$. Then $K_1$ is a compact and connected set that does not contain $q$ and meets $Q$, because $p_1 \in K_1 \cap Q$. Hence $K_1 \subseteq Q \setminus \{q\}$.

Suppose a branch point $p_k \in E \cap Q$, a branch $B_k$ of $p_k$, and a set $K_k$ with the properties (i)–(iii) have been chosen for all $1 \leq k \leq n$.

Since $q \notin K_n$, we have $\text{dist}(q, K_n) > 0$, and so we can find a branch point $p_{n+1} \in E \cap (Q \setminus \{p\})$ sufficiently close to $q$ such that $p_{n+1} \notin K_n$. This is possible, because $q$ is a limit point of $E \cap (Q \setminus \{q\})$. Since the set $K_n \subseteq T \setminus \{p_{n+1}\}$ is connected, it must be contained in a branch of $p_{n+1}$. Since there are three branches of $p_{n+1} \neq q$ whose diameters exceed $\delta$, we can pick one of them that contains neither $q$ nor $K_n$. Let $B_{n+1}$ be such a branch of $p_{n+1}$. Then $\text{diam}(B_{n+1}) > \delta$ and so (i) is true for $n+1$. We have $B_{n+1} \cap K_n = \emptyset$; so (iii) shows that $B_{n+1}$ is disjoint from the previously chosen disjoint sets $B_1, \ldots, B_n$. This gives (ii).

Since $p_n, p_{n+1} \in Q \setminus \{q\}$, the arc $[p_n, p_{n+1}]$ does not contain $q$ (see Lemma 3.2 (iii)). We also have $p_n \in B_n \subseteq K_n$ and $p_{n+1} \in B_{n+1}$, which implies that the set $K_{n+1} := K_n \cup [p_n, p_{n+1}] \cup B_{n+1} \subseteq Q \setminus \{q\}$ is compact and connected. We have

\[ B_1 \cup \cdots \cup B_n \cup B_{n+1} \subseteq K_n \cup B_{n+1} \subseteq K_{n+1} \subseteq Q \setminus \{q\}, \]

and so $K_{n+1}$ has property (iii).

Continuing with this process, we obtain disjoint branches $B_n$ for all $n \in \mathbb{N}$ that satisfy (i). The last part of Lemma 3.8 implies that this is impossible and we get a contradiction. \[ \square \]
4. Basic properties of the continuum self-similar tree

Here we define the continuum self-similar tree (CSST) and study some of its properties. Unless otherwise specified, all metric notions in this section refer to the Euclidean metric on the complex plane \( \mathbb{C} \).

In this section, \( i \) always denotes the imaginary unit and we do not use this letter for indexing as in the other sections. If \( a, b \in \mathbb{C} \) we denote by \([a, b]\) the Euclidean line segment in \( \mathbb{C} \) joining \( a \) and \( b \). We also use the usual notation for open or half-open line segments. So \([a, b) = [a, b] \setminus \{b\} \), etc.

We consider the following contracting homeomorphisms on \( \mathbb{C} \):

\[
\begin{align*}
  f_1(z) &= \frac{1}{2} z - \frac{i}{2}, \\
  f_2(z) &= -\frac{1}{2} \bar{z} + \frac{i}{2}, \\
  f_3(z) &= \frac{i}{2} \bar{z} + \frac{1}{2},
\end{align*}
\]

Then the following statement is true.

**Proposition 4.1.** There exists a unique non-empty compact set \( T \subseteq \mathbb{C} \) satisfying

\[
T = f_1(T) \cup f_2(T) \cup f_3(T).
\]

In other words, \( T \) is the attractor of the iterated function system \( \{f_1, f_2, f_3\} \) in the plane. Proposition 4.1 follows from well-known results in the literature (see [Kig01, Theorem 1.1.4], for example), but we will provide a proof for completeness and will give an essentially self-contained presentation.

**Definition 4.2.** The CSST is the set \( T \subseteq \mathbb{C} \) as given by Proposition 4.1.

We will see later in this section that \( T \) is a tree that belongs to \( \mathcal{T}_3 \) and so satisfies the conditions in Theorem 1.2. In [CH08] the authors considered an iterated function system very similar to (4.1) whose attractor is a set homeomorphic to \( T \). Related is also Hata’s tree-like set considered in [Kig01, Example 1.2.9].

For the proof of Proposition 4.1 we consider a coding procedure of certain points in the complex plane by words in an alphabet. We first fix some terminology related to this.

We consider a non-empty set \( \mathcal{A} \). Then we call \( \mathcal{A} \) an *alphabet* and refer to the elements in \( \mathcal{A} \) as the *letters* in this alphabet. In this paper we will only use alphabets of the form \( \mathcal{A} = \mathbb{N} \) or \( \mathcal{A} = \{1, 2, \ldots, m\} \) with \( m \in \mathbb{N}, m \geq 3 \). We consider the set \( W(\mathcal{A}) := \mathcal{A}^{\mathbb{N}} \) of infinite sequences in \( \mathcal{A} \) as the set of *infinite words* in the alphabet \( \mathcal{A} \) and write the elements \( w \in W(\mathcal{A}) \) in the form \( w = w_1w_2 \ldots \), where it is understood that \( w_k \in \mathcal{A} \) for \( k \in \mathbb{N} \). Similarly, we set \( W_n(\mathcal{A}) := \mathcal{A}^n \) and consider \( W_n(\mathcal{A}) \) as the set of all words in the alphabet \( \mathcal{A} \) of length
We write the elements $w \in W_n(\mathcal{A})$ in the form $w = w_1 \ldots w_n$ with $w_k \in \mathcal{A}$ for $k = 1, \ldots, n$. We use the convention that $W_0(\mathcal{A}) = \{\emptyset\}$ and consider the only element $\emptyset$ in $W_0(\mathcal{A})$ as the empty word of length 0. Finally, $W^*(\mathcal{A}) := \bigcup_{n \in \mathbb{N}_0} W_n(\mathcal{A})$ is the set of all words of finite length. If $u = u_1 \ldots u_n$ is a finite word and $v = v_1 v_2 \ldots$ is a finite or infinite word in the alphabet $\mathcal{A}$, then we denote by $uv = u_1 \ldots u_n v_1 v_2 \ldots$ the word obtained by concatenating $u$ and $v$. We call $u$ an initial segment and $v$ a tail of the word $w = uv$. If the alphabet $\mathcal{A}$ is understood, then we simply drop $\mathcal{A}$ from the notation. So $W$ will denote the set of infinite words in $\mathcal{A}$, etc.

For the rest of this section, we use the alphabet $\mathcal{A} = \{1, 2, 3\}$. So when we write $W, W_n, W^*$ it is understood that $\mathcal{A} = \{1, 2, 3\}$ is the underlying alphabet. There exists a unique metric $d$ on $W = \{1, 2, 3\}^\mathbb{N}$ with the following property. If we have two words $u = u_1 u_2 \ldots$ and $v = v_1 v_2 \ldots$ in $W$ and $u \neq v$, then for some $n \in \mathbb{N}_0$ we have $u_1 = v_1, \ldots, u_n = v_n$, and $u_{n+1} \neq v_{n+1}$. Then $d(u, v) = 1/2^n$. More informally, two elements $u, v \in W$ are close in this metric precisely if they share a large number of initial letters. The metric space $(W, d)$ is compact and homeomorphic to a Cantor set.

**Figure 2.** The left figure: the continuum self-similar tree and some of its subtrees. The right figure: an illustration of some associated sets.
If \( n \in \mathbb{N}_0 \) and \( w = w_1 w_2 \ldots w_n \in W_n \), we define
\[
 f_w := f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_n},
\]
where we use the maps in (4.1) in the composition. By convention, \( f_\emptyset = \text{id}_C \) is the identity map on \( C \). Note that \( f_w \) is a Euclidean similarity on \( C \) that scales Euclidean distances by the factor \( 2^{-n} \). If \( a, b \in C \), then \( f_w([a, b]) = [f_w(a), f_w(b)] \). We will use this repeatedly in the following.

Let \( H \subseteq C \) be the (closed) convex hull of the four points \( 1, +i, -\frac{1}{2} + \frac{i}{2}, \) and \( -i \) (see Figure 2). We set \( H_k = f_k(H) \) for \( k = 1, 2, 3 \). Then
\[
 H_1 \cup H_2 \cup H_3 = f_1(H) \cup f_2(H) \cup f_3(H) \subseteq H.
\]
This implies that
\[
 (4.3) \quad f_w(H) \subseteq H
\]
for all \( w \in W_n \).

**Lemma 4.3.** There exists a well-defined continuous map \( \pi : W \to C \) given by
\[
 \pi(w) = \lim_{n \to \infty} f_{w_1 w_2 \ldots w_n}(z_0)
\]
for \( w = w_1 w_2 \ldots \in W \) and \( z_0 \in C \). Here the limit exists and is independent of the choice of \( z_0 \in C \).

In the following, \( \pi : W \to C \) will always denote the map provided by this lemma.

**Proof.** Fix \( z_0 \in C \). Then there exists a constant \( C \geq 0 \) such that
\[
 |z_0 - f_k(z_0)| \leq C
\]
for \( k = 1, 2, 3 \). If \( n \in \mathbb{N}_0 \) and \( u \in W_n \), then
\[
 |f_u(a) - f_u(b)| = \frac{1}{2^n} |a - b|
\]
for all \( a, b \in C \). This implies that if \( w = w_1 w_2 \ldots \in W, \ n \in \mathbb{N}, \) and \( u := w_1 w_2 \ldots w_n \in W_n \), then
\[
 |f_{w_1 w_2 \ldots w_n}(z_0) - f_{w_1 w_2 \ldots w_{n+1}}(z_0)| = |f_u(z_0) - f_u(f_{w_{n+1}}(z_0))|
 = \frac{1}{2^n} |z_0 - f_{w_{n+1}}(z_0)| \leq \frac{C}{2^n}.
\]
It follows that \( \{f_{w_1 w_2 \ldots w_n}(z_0)\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C \). Hence this sequence converges and
\[
 \pi(w) = \lim_{n \to \infty} f_{w_1 w_2 \ldots w_n}(z_0)
\]
is well-defined for each \( w = w_1 w_2 \ldots \in W \).
The limit does not depend on the choice of $z_0$. Indeed, if $z_0' \in \mathbb{C}$ is another point, then

$$|f_{w_1w_2...w_n}(z_0) - f_{w_1w_2...w_n}(z_0')| = \frac{1}{2^n}|z_0 - z_0'|,$$

which implies that

$$\lim_{n \to \infty} f_{w_1w_2...w_n}(z_0) = \lim_{n \to \infty} f_{w_1w_2...w_n}(z_0').$$

The definition of $\pi$ shows that if $w = w_1w_2... \in W$ and $n \in \mathbb{N}_0$, then

$$\pi(w) = \pi(w_1w_2...) = f_{w_1...w_n}(\pi(w_{n+1}w_{n+2}...)).$$

If we pick $z_0 \in H$, then (4.3) and the definition of $\pi$ imply that $\pi(W) \subseteq H$. If we combine this with (4.4), then we see that if two words $u, v \in W$ start with the same letters $w_1, ..., w_n$, then

$$|\pi(u) - \pi(v)| \leq f_{w_1...w_n}(\text{diam}(H)) = \frac{1}{2^n} \text{diam}(H).$$

The continuity of the map $\pi$ follows from this and the definition of the metric $d$ on $W$. \hfill $\square$

**Proof of Proposition 4.1.** Let $\pi : W \to \mathbb{C}$ be the map provided by Lemma 4.3 and define $\mathbb{T} = \pi(W) \subseteq \mathbb{C}$. Since $W$ is compact and $\pi$ is continuous, the set $\mathbb{T}$ is non-empty and compact. The relation (4.2) immediately follows from (4.4) for $n = 1$. Note that (4.2) implies that

$$f_w(\mathbb{T}) = f_{w_1}(\mathbb{T}) \cup f_{w_2}(\mathbb{T}) \cup f_{w_3}(\mathbb{T})$$

for each $w \in W_n$, $n \in \mathbb{N}_0$. From this in turn we deduce that

$$\bigcup_{w \in W_n} f_w(\mathbb{T}) = \mathbb{T}$$

for each $n \in \mathbb{N}_0$.

It remains to show the uniqueness of $\mathbb{T}$. Suppose $\mathbb{T} \subseteq \mathbb{C}$ is another non-empty compact set satisfying the analog of (4.2). Then the analogs of (4.5) and (4.6) are also valid for $\mathbb{T}$. This and the definition of $\pi$ using a point $z_0 \in \mathbb{T}$ imply that $\mathbb{T} = \pi(W) \subseteq \mathbb{T}$.

For the converse inclusion, let $a \in \mathbb{T}$ be arbitrary. Using the relation (4.5) for the set $\mathbb{T}$, we can inductively construct an infinite word $w_1w_2... \in W$ such that $a \in f_{w_1w_2...w_n}(\mathbb{T})$ for all $n \in \mathbb{N}$. Since

$$\text{diam}(f_{w_1w_2...w_n}(\mathbb{T})) = \frac{1}{2^n} \text{diam}(\mathbb{T}) \to 0 \text{ as } n \to \infty,$$
the definition of $\pi$ (using a point $z_0 \in \tilde{T}$) implies that $a = \pi(w)$. In particular, $a \in \pi(W) = T$, and so $\tilde{T} \subseteq T$. The uniqueness of $T$ follows. □

In the proof of the previous proposition we have seen that $T = \pi(W)$.

If $p \in T$ and $p = \pi(w)$ for some $w \in W$, then we say that the word $w$ represents $p$.

The following lemma provides some geometric descriptions of $T$.

**Proposition 4.4.** Let $I = [-i, +i]$ be the line segment joining the points $-i, +i \in \mathbb{C}$. For $n \in \mathbb{N}_0$ define

$$J_n = \bigcup_{w \in W_n} f_w(I) \quad \text{and} \quad K_n = \bigcup_{w \in W_n} f_w(H).$$

Then the sets $J_n$ and $K_n$ are compact and satisfy

$$J_n \subseteq J_{n+1} \subseteq T \subseteq K_{n+1} \subseteq K_n$$

for $n \in \mathbb{N}_0$. Moreover, we have

$$\bigcup_{n \in \mathbb{N}_0} J_n = T = \bigcap_{n \in \mathbb{N}_0} K_n.$$ 

**Proof.** It is clear that the sets $J_n$ and $K_n$ are compact for each $n \in \mathbb{N}_0$. Set $I_k = f_k(I)$ for $k = 1, 2, 3$. Then an elementary geometric consideration shows that (see Figure 2)

$$I \subseteq I_1 \cup I_2 \cup I_3 \subseteq H_1 \cup H_2 \cup H_3 \subseteq H.$$

This in turn implies that

$$f_w(I) \subseteq f_{w1}(I) \cup f_{w2}(I) \cup f_{w3}(I)$$

$$\subseteq f_{w1}(H) \cup f_{w2}(H) \cup f_{w3}(H) \subseteq f_w(H)$$

for each $w \in W_n$, $n \in \mathbb{N}_0$. Taking the union over all $w \in W_n$, we obtain

$$J_n \subseteq J_{n+1} \subseteq K_{n+1} \subseteq K_n$$

for all $n \in \mathbb{N}_0$. The set $\tilde{T} = \bigcup_{n \in \mathbb{N}_0} J_n$ is non-empty, compact, and satisfies

$$\bigcup_{k=1,2,3} f_k(\tilde{T}) = \bigcup_{k=1,2,3} f_k \left( \bigcup_{n \in \mathbb{N}_0} J_n \right) = \bigcup_{k=1,2,3} f_k \left( \bigcup_{n \in \mathbb{N}_0} J_n \right)$$

$$= \bigcup_{k=1,2,3} f_k \left( \bigcup_{n \in \mathbb{N}_0} J_n \right) = \bigcup_{n \in \mathbb{N}_0} \bigcup_{k=1,2,3} f_k(J_n)$$

$$= \bigcup_{n \in \mathbb{N}_0} J_{n+1} = \bigcup_{n \in \mathbb{N}_0} J_n = \tilde{T}.$$
Hence $\tilde{T} = T$ by the uniqueness statement in Proposition 4.1. So we have the first equation in (4.8).

Since $0 \in H$, we have $f_w(0) \in f_w(H) \subseteq K_n$ for each $w \in W_n$. Since the sets $K_n$ are compact and nested, this implies that for each $w = w_1 w_2 \ldots \in W$ we have

$$\pi(w) = \lim_{n \to \infty} f_{w_1 \ldots w_n}(0) \in \bigcap_{n \in \mathbb{N}_0} K_n.$$ 

It follows that $T = \pi(W) \subseteq \bigcap_{n \in \mathbb{N}_0} K_n$.

To show the reverse inclusion, let $a \in \bigcap_{n \in \mathbb{N}_0} K_n$ be arbitrary. Then $a \in K_n$ for each $n \in \mathbb{N}_0$, and so there is a word $u_n \in W_n$ such that $a \in f_{u_n}(H)$. Define $z_n = f_{u_n}(0) \in J_n \subseteq T$. Since $0 \in H$, we have $z_n \in f_{u_n}(H)$, and so

$$|z_n - a| \leq \text{diam}(f_{u_n}(H)) = \frac{1}{2n} \text{diam}(H).$$

Hence $z_n \to a$ as $n \to \infty$. Since $z_n \in T$ and $T$ is compact, it follows that $a \in T$. We see that $\bigcap_{n \in \mathbb{N}_0} K_n \subseteq T$. So the second equation in (4.8) is also valid.

The inclusions (4.7) follow from (4.8) and (4.9).

For $u \in W_*$ we define

\begin{equation}
T_u := f_u(T) \subseteq T.
\end{equation}

Note that $T_\emptyset = T$. Since $T = \pi(W)$ and $f_u(\pi(v)) = \pi(uv)$ whenever $u \in W_*$ and $v \in W$ (see (4.4)), the set $T_u$ consists precisely of the points $p \in T$ that can be represented in the form $p = \pi(w)$ with a word $w \in W$ that has $u$ has an initial segment. This implies that if $v \in W_*$ is a finite word with the initial segment $u \in W_*$, then $T_v \subseteq T_u$. It follows from (4.5) that

$$T_u = T_{u1} \cup T_{u2} \cup T_{u3}$$

for each $u \in W_*$ and from (4.6) that

\begin{equation}
T = \bigcup_{u \in W_n} T_u
\end{equation}

for each $n \in \mathbb{N}_0$.

Since $I = [-i, +i] \subseteq T \subseteq H$ and $\text{diam}(I) = \text{diam}(H) = 2$, we have $\text{diam}(T) = 2$. If $n \in \mathbb{N}_0$ and $u \in W_n$, then $f_u$ is a similarity map that
scales distances by the factor $1/2^n$. Hence
\begin{equation}
\text{diam}(T_u) = 2^{1-n}.
\end{equation}

We have $0 = f_1(+i) = f_2(-i) = f_3(-i)$. This implies
\begin{equation}
0 \in T_k = f_k(T) \subseteq f_k(H) = H_k
\end{equation}
for $k = 1, 2, 3$. If $k, l \in \{1, 2, 3\}$ and $k \neq l$, then (see Figure 2)
\begin{equation}
H_k \cap H_l = \{0\},
\end{equation}
and so
\begin{equation}
T_k \cap T_l = \{0\}.
\end{equation}

The next lemma provides a criterion when two infinite words in $W$ represent the same point in $T$ under the map $\pi$. Here we use the notation $\hat{k}$ for the infinite word $kkk\ldots$ for $k \in \{1, 2, 3\}$.

**Lemma 4.5.**
(i) We have $\pi^{-1}(0) = \{\hat{1}, \hat{2}, \hat{3}\}$.

(ii) Let $v, w \in W$ with $v \neq w$. Then $\pi(v) = \pi(w)$ if and only if there exists a finite word $u \in W_\ast$ such that $\pi(v) = \pi(w) = f_u(0)$ and $v, w \in \{u\hat{1}, u\hat{2}, u\hat{3}\}$.

Note that if $v \in W$ and $v \in \{u\hat{1}, u\hat{2}, u\hat{3}\}$ for some $u \in W_\ast$, then $u$ is uniquely determined. This and the lemma imply that each point in $T = \pi(W)$ has at most three preimages under the map $\pi$.

**Proof.** (i) Note that $\hat{1} \in \pi^{-1}(0)$ as follows from
\begin{align*}
f_2(+i) &= +i \quad \text{and} \quad f_1(+i) = 0.
\end{align*}
Similarly, $\hat{2}, \hat{3} \in \pi^{-1}(0)$, because
\begin{align*}
f_1(-i) &= -i, \quad f_2(-i) = 0, \quad \text{and} \quad f_1(-i) = -i, \quad f_3(-i) = 0.
\end{align*}
Hence $\{\hat{1}, \hat{2}, \hat{3}\} \subseteq \pi^{-1}(0)$.

To prove the converse inclusion, suppose that $\pi(w) = 0$ for some $w = w_1w_2\ldots \in W$. We first consider the case $w_1 = 1$. Then $0 = f_1(a)$, where $a := \pi(w_2w_3\ldots)$, and so $a = +i$. Since $+i \in T_2 \setminus (T_1 \cup T_3)$ as follows from (4.13), we must have $w_2 = 2$. Then $+i = f_2(b)$, where $b := w_3w_4\ldots$, and so $b = +i \in T_2 \setminus (T_1 \cup T_3)$. This implies $w_3 = 2$. Repeating the argument, we see that $2 = w_2 = w_3 = \ldots$, and so $w = \hat{1}$.

A very similar argument shows that if $w_1 = 2$, then $w = \hat{2}$, and if $w_1 = 3$, then $w = \hat{3}$.

(ii) Suppose that $\pi(v) = \pi(w)$. Let $u \in W_\ast$ be the longest initial word that $v$ and $w$ have in common. So $v = uv_m+1v_{m+2}\ldots$ and $w =
$uw_{m+1}w_{m+2}\ldots$, where $m \in \mathbb{N}_0$ and $v_{m+1} \neq w_{m+1}$. Since $f_u$ is bijective and

$$
\pi(v) = f_u(\pi(v_{m+1}v_{m+2}\ldots)) = \pi(w) = f_u(\pi(w_{m+1}w_{m+2}\ldots)),
$$

we have

$$
\pi(v_{m+1}v_{m+2}\ldots) = \pi(w_{m+1}w_{m+2}\ldots).
$$

Note that $\pi(v_{m+1}v_{m+2}\ldots) \in T_{v_{m+1}}$ and $\pi(w_{m+1}w_{m+2}\ldots) \in T_{w_{m+1}}$. By (4.15) this implies $\pi(v_{m+1}v_{m+2}\ldots) = \pi(w_{m+1}w_{m+2}\ldots) = 0$. Hence $v_{m+1}v_{m+2}\ldots, w_{m+1}w_{m+2}\ldots \in \{12, 21, 31\}$ by (i), and the statement follows. The converse implication follows from (i). □

Lemma 4.6. 
(i) For each $p \in T$ there exists a (possibly degenerate) arc $\alpha$ in $T$ with endpoints $-i$ and $p$.
(ii) The sets $T$, $T \setminus \{+i\}$, and $T \setminus \{-i\}$ are arc-connected.

Proof. (i) Let $p \in T$. Then $p = \pi(w)$ for some $w = w_1w_2\ldots \in W$.

Let $v_n = w_1\ldots w_n$ and define $a_n = f_{v_n}(-i) \in T$ for $n \in \mathbb{N}_0$. Then $a_0 = f_{\emptyset}(-i) = -i$. For each $n \in \mathbb{N}_0$ we have

$$
[a_n, a_{n+1}] = [f_{v_n}(-i), f_{v_1v_{n+1}}(-i)] = f_{v_n}([-i, f_{v_1v_{n+1}}(-i))].
$$

If $w_{n+1} = 1$, then $f_{v_1v_{n+1}}(-i) = f_1(-i) = -i$; so $a_n = a_{n+1}$ and

$$
[a_n, a_{n+1}] = \emptyset.
$$

If $w_{n+1} \in \{2, 3\}$, then $f_{v_1v_{n+1}}(-i) = 0$; so

$$
[-i, f_{v_1v_{n+1}}(-i)) = [-i, 0) \subseteq T_{1\setminus\{0\}},
$$

and

$$
[a_n, a_{n+1}] = f_{v_n}([-i, 0)) \subseteq f_{v_n}(T_{1\setminus\{0\}}) \subseteq T_{v_n} \subseteq T.
$$

Moreover,

$$
(4.16)
$$

length$([a_n, a_{n+1}]) = \frac{1}{2^n}$ length$([-i, f_{v_1v_{n+1}}(-i)))] = \frac{1}{2^n} \chi(2,3)(w_{n+1})$

Here $\chi_M$ denotes the characteristic function of a set $M$. Let

$$
A_n := \{p\} \cup \bigcup_{k \geq n+1} [a_k, a_{k+1}]
$$

for $n \in \mathbb{N}_0$. By what we have seen above,

$$
[a_k, a_{k+1}] \subseteq T_{v_k} \subseteq T_{v_{n+1}}
$$

for $k \geq n+1$. Since $p = \lim_{k \to \infty} a_k$ and $T_{v_{n+1}}$ is closed, we also have $p \in T_{v_{n+1}}$, and so

$$
A_n \subseteq T_{v_{n+1}}.
$$
This implies that
\[ [a_n, a_{n+1}) \cap A_n = \emptyset \]
for each \( n \in \mathbb{N}_0 \). Indeed, if \( w_{n+1} = 1 \) this is clear, because then \([a_n, a_{n+1}) = \emptyset\).

If \( w_{n+1} = 2 \), then
\[ A_n \subseteq T_{v_{n+1}} = f_{v_n}(f_2(T)) = f_{v_n}(T_2), \]
which implies that
\[ [a_n, a_{n+1}) \cap A_n \subseteq f_{v_n}(T_1 \setminus \{0\}) \cap f_{v_n}(T_2) = f_{v_n}((T_1 \setminus \{0\}) \cap T_2) = \emptyset. \]
If \( w_{n+1} = 3 \), then \([a_n, a_{n+1}) \cap A_n = \emptyset \) by the same reasoning. This shows that the sets
\[ [a_0, a_1), [a_1, a_2), [a_2, a_3), \ldots, \{p\} \]
are pairwise disjoint. As \( n \to \infty \), we have \( a_n \to p \) and also \( \text{diam}(A_n) \to 0 \) by (4.16). Therefore, the union
\[ (4.17) \quad \alpha = [a_0, a_1) \cup [a_1, a_2) \cup [a_2, a_3) \cup \cdots \cup \{p\} \]
is an arc in \( T \) joining \( a_0 = -i \) and \( p \) (if \( p = -i \), this arc is degenerate).

We claim that if \( p \neq +i \), then this arc \( \alpha \) does not contain \(+i\). Otherwise, we must have \(+i \in [a_n, a_{n+1}) \subseteq T_{v_n} \) for some \( n \in \mathbb{N}_0 \). This shows that \(+i\) can be written in the form \(+i = \pi(u)\), where \( u \in W \) is an infinite word starting with the finite word \( v := v_n \) (note that this and the statements below are trivially true for \( n = 0 \)). On the other hand, we have \( f_2(+i) = +i \) which implies that \(+i = \pi(2)\). By Lemma 4.5 (ii), this is only possible if all the letters in \( v \) are 2's. Then \( f_v(+i) = +i \) and it follows that
\[ +i = f_v(+i) \in [a_n, a_{n+1}) = f_v([-i, f_{w_n+1}(-i))]. \]
Since \( f_v \) is a bijection, this implies that \(+i \in [-i, f_{w_n+1}(-i))\). Now \( f_{w_n+1}(-i) \in \{-i, 0\} \), and we obtain a contradiction. So indeed, \(+i \not\in \alpha\).

(ii) Let \( p, q \in T \) with \( p \neq q \) be arbitrary. In order to show that \( T \) is arc-connected, we have to find an arc \( \gamma \) in \( T \) joining \( p \) and \( q \). Now by the construction in (i) we can find arcs \( \alpha \) and \( \beta \) in \( T \) joining \( p \) and \( q \) to \(-i\), respectively. Then the desired arc \( \gamma \) can be found in the union \( \alpha \cup \beta \) as follows. Starting from \( p \), we travel long \( \alpha \) until we first hit \( \beta \), say in a point \( x \). Such a point \( x \) exists, because \(-i \in \alpha \cap \beta \neq \emptyset \). Let \( \alpha' \) be the (possibly degenerate) subarc of \( \alpha \) with endpoints \( p \) and \( x \), and \( \beta' \) be the subarc of \( \beta \) with endpoints \( x \) and \( q \). Then \( \gamma = \alpha' \cup \beta' \) is an arc in \( T \) joining \( p \) and \( q \).

The arc-connectedness of \( T \setminus \{+i\} \) is proved by the same argument. Indeed, if \( p, q \in T \setminus \{+i\} \), then by the remark in the last part of the
proof of (i), the arcs $\alpha$ and $\beta$ constructed as in (i) do not contain $+i$. Then the arc $\gamma \subseteq \alpha \cup \beta$ does not contain $+i$ either.

Finally, to show that $T \setminus \{-i\}$ is arc-connected, we assume that $p, q \in T \setminus \{-i\}$. If $x$ is, as above, the first point on $\beta$ as we travel along $\alpha$ starting from $p$, then it suffices to show that $x \neq -i$, because then $-i \not\in \gamma$. This in turn will follow if we can show that $\alpha$ and $\beta$ have another point in common besides $-i$.

To find such a point, we revisit the above construction. Pick $w = w_1 w_2 \ldots \in W$ and $u = u_1 u_2 \ldots \in W$ such that $p = \pi(w)$ and $q = \pi(u)$. Let $\alpha$ and $\beta$ be the arcs for $p$ and $q$, respectively, as constructed in (i). Then $\alpha$ is as in (4.17) and we can write the other arc $\beta$ as

$$
\beta = \{b_0, b_1\} \cup \{b_1, b_2\} \cup \{b_2, b_3\} \cup \cdots \cup \{q\},
$$

where $b_n = f_{u_1 \ldots u_n}(-i)$ for $n \in \mathbb{N}_0$. Since $p \neq q$, we have $w \neq u$, and so there exists a largest $n \in \mathbb{N}_0$ such that $v := w_1 \ldots w_n = u_1 \ldots u_n$ and $w_{n+1} \neq u_{n+1}$. Then $a_n = b_n = f_v(-i) \in \alpha \cap \beta$. If $a_n = b_n \neq -i$, we are done. So we may assume that $a_n = b_n = f_v(-i) = -i$. Then $a_0 = \cdots = a_n = -i$, and so $w_k = u_k = 1$ for $k = 1, \ldots, n$. This shows that all letters in $v$ are equal to 1.

Since the letters $w_{n+1}$ and $u_{n+1}$ are distinct, one of them is different from 1. We may assume $w_{n+1} \neq 1$. Then $f_{u_{n+1}}(-i) = 0$, and so $(b_n, b_{n+1}) = f_v((\dot{0}, 0)) \subseteq \beta \setminus \{-i\}$. Here we used that $f_v$ is a homeomorphism with $f_v(-i) = -i$.

Since $p = \pi(w) \neq -i = \pi(\dot{1})$, we have $w \neq \dot{1}$ and so there exists a smallest $l \in \mathbb{N}$ such that $w_{n+l} \neq 1$. Then $f_{w_{n+l}}(-i) = 0$ and so a simple computation using $w_{n+1} = \cdots = w_{n+l-1} = 1$ shows that

$$
c := f_{w_{n+1} \ldots w_{n+l}}(-i) = f_{w_{n+1} \ldots w_{n+l-1}}(0) = (2^{1-l} - 1)i \in (-i, 0).
$$

Hence

$$
a_{n+l} = f_v(c) \in f_v((\dot{0}, 0)) \subseteq \beta \setminus \{-i\}.
$$

It follows that $a_{n+l} \in \alpha \cap \beta$ and $a_{n+l} \neq -i$ as desired.\hfill \square

The next lemma will help us to identify the branch points of $T$ once we know that $T$ is a tree.

**Lemma 4.7.**

(i) The components of $T \setminus \{0\}$ are given by the non-empty sets $T_1 \setminus \{0\}$, $T_2 \setminus \{0\}$, $T_3 \setminus \{0\}$.

(ii) If $u \in W_1$, then $T \setminus \{f_u(0)\}$ has exactly three components. The sets $T_{u_1} \setminus \{f_u(0)\}$, $T_{u_2} \setminus \{f_u(0)\}$, $T_{u_3} \setminus \{f_u(0)\}$ are each contained in a different component of $T \setminus \{f_u(0)\}$.

In the proof we will use the following general facts about components of a subset $M$ of a metric space $X$. Recall that a set $A \subseteq M$ is relatively
closed in $M$ if $A = \overline{A} \cap M$, or equivalently, if each limit point of $A$ that belongs to $M$ also belongs to $A$. Each component $A$ of $M$ is relatively closed in $M$, because its relative closure $\overline{A} \cap M$ is a connected subset of $M$ with $A \subseteq \overline{A} \cap M$. Hence $A = \overline{A} \cap M$, because $A$ is a component of $M$ and hence a maximal connected subset of $M$.

If $A_1, \ldots , A_n \subseteq M$ for some $n \in \mathbb{N}$ are non-empty, pairwise disjoint, relatively closed, and connected sets with $M = A_1 \cup \cdots \cup A_n$, then these sets are the components of $M$.

**Proof.** (i) Each of the sets $\mathbb{T}\{+i\}$ and $\mathbb{T}\{-i\}$ is non-empty, and connected by Lemma 4.6 (ii). Therefore, the sets

$$\mathbb{T}_1\{0\} = f_1(\mathbb{T}\{+i\}), \quad \mathbb{T}_2\{0\} = f_2(\mathbb{T}\{-i\}), \quad \mathbb{T}_3\{0\} = f_3(\mathbb{T}\{-i\})$$

are non-empty and connected. They are also relatively closed in $\mathbb{T}\{0\}$ and pairwise disjoint by (4.15). Since $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3$ we have

$$\mathbb{T}\{0\} = (\mathbb{T}_1\{0\}) \cup (\mathbb{T}_2\{0\}) \cup (\mathbb{T}_3\{0\}).$$

This implies that the sets $\mathbb{T}_k\{0\}$, $k = 1, 2, 3$, are the components of $\mathbb{T}\{0\}$. The statement follows.

(ii) We prove this by induction on the length $n \in \mathbb{N}_0$ of the word $u \in W_\ast$. If $n = 0$ and so $u = \emptyset$, this follows from statement (i).

Suppose the statement is true for all words of length $n - 1$, where $n \in \mathbb{N}$. Let $u = u_1 \ldots u_n \in W_n$ be an arbitrary word of length $n$. We set $l := u_1$ and $u' := u_2 \ldots u_n$. Then $u = lu'$. To be specific and ease notation, we will assume that $l = 1$. The other cases $l = 2$ and $l = 3$ are completely analogous.

Note that $f_{u}(0) \neq 0$. Indeed, if

$$0 = f_{u}(0) = f_{u}(\pi(1\hat{2})) = \pi(u1\hat{2}),$$

then $u1\hat{2} \in \{1\hat{2}, 2\hat{1}, 3\hat{1}\}$ by Lemma 4.5 (i). This is only possible $u1 = 1$. This is a contradiction, because $u$ has length $n \geq 1$. Hence $f_{u}(0) \neq 0$.

Since $u_1 = l = 1$, we have $f_{u}(0) \in \mathbb{T}_1\{0\}$.

By induction hypothesis, $\mathbb{T}\{f_{u'}(0)\}$ has exactly three connected components $V_1$, $V_2$, $V_3$, and we may assume that $\mathbb{T}_{u'k}\{f_{u'}(0)\} \subseteq V_k$ for $k = 1, 2, 3$. It follows that

$$f_{l}(\mathbb{T}\{f_{u'}(0)\}) = f_{l}(\mathbb{T}\{f_{u'}(0)\}) = \mathbb{T}_1\{f_{u}(0)\}$$

has exactly three connected components $U_k = f_{l}(V_k) \subseteq \mathbb{T}_1$ with

$$\mathbb{T}_{uk}\{f_{u}(0)\} = \mathbb{T}_{1uk}\{f_{1u'}(0)\} = f_{l}(\mathbb{T}_{u'k}\{f_{u'}(0)\}) \subseteq f_{l}(V_k) = U_k$$

for $k = 1, 2, 3$. 
Let \( k \in \{1, 2, 3\} \). Then we have \( V_k = \overline{V}\cap \mathbb{T}\backslash \{f_u(0)\} \), because \( V_k \) is a component of \( \mathbb{T}\backslash \{f_u(0)\} \) and hence relatively closed in \( \mathbb{T}\backslash \{f_u(0)\} \). This implies that
\[
U_k = f_1(V_k) = f_1(\overline{V}\cap \mathbb{T}\backslash \{f_u(0)\}) = f_1(\overline{V}\cap \mathbb{T}_1\backslash \{f_u(0)\})
\]
\[
= f_1(\overline{V}\cap \mathbb{T}_1\backslash \{f_u(0)\}) = \overline{U}\cap \mathbb{T}_1\backslash \{f_u(0)\}.
\]
Since \( \mathbb{T}_1 \subseteq \mathbb{T} \) is compact, \( U_k \subseteq \mathbb{T}_1 \), and so \( \overline{U} \subseteq \mathbb{T}_1 \), this shows that every limit point of \( U_k \) belongs to \( U_k \). Hence \( U_k \) is relatively closed in \( \mathbb{T}\backslash \{f_u(0)\} \).

Exactly one of the components of \( \mathbb{T}\backslash \{f_u(0)\} \), say \( U_1 \), contains the point \( 0 \in \mathbb{T}\backslash \{f_u(0)\} \). Then \( U_1' := U_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3 \) is a relatively closed subset of \( \mathbb{T}\backslash \{f_u(0)\} \). This set is also connected, because the sets \( U_1 \), \( \mathbb{T}_2 = f_2(\mathbb{T}) \), \( \mathbb{T}_3 = f_3(\mathbb{T}) \) are connected and have the point \( 0 \) in common. Hence the connected sets \( U_1', U_2, U_3 \) are pairwise disjoint, relatively closed in \( \mathbb{T}\backslash \{f_u(0)\} \), and
\[
\mathbb{T}\backslash \{f_u(0)\} = (\mathbb{T}_1 \backslash \{f_u(0)\}) \cup \mathbb{T}_2 \cup \mathbb{T}_3 = U_1' \cup U_2 \cup U_3.
\]
This implies that \( \mathbb{T}\backslash \{f_u(0)\} \) has exactly the three connected components \( U_1', U_2, U_3 \). Moreover, \( \mathbb{T}_{u1}\backslash \{f_u(0)\}, \mathbb{T}_{u2}\backslash \{f_u(0)\}, \mathbb{T}_{u3}\backslash \{f_u(0)\} \) lie in the different components \( U_1', U_2, U_3 \) of \( \mathbb{T}\backslash \{f_u(0)\} \), respectively. This provides the induction step, and the statement follows. \( \square \)

**Proposition 4.8.** The set \( \mathbb{T} \) is a metric tree.

**Proof.** We know that \( \mathbb{T} \) is compact, contains at least two points, and is arc-connected by Lemma \[4.6\].

Let \( p \in \mathbb{T} \) and \( n \in \mathbb{N} \) be arbitrary, and define
\[
N = \bigcup\{\mathbb{T}_u : u \in W_n \text{ and } p \in \mathbb{T}_u\}.
\]
Since each of the sets \( \mathbb{T}_u = f_u(\mathbb{T}) \), \( u \in W_n \), is a compact and connected subset of \( \mathbb{T} \), the set \( N \) is connected. Moreover, since each of the finitely many sets \( \mathbb{T}_u, u \in W_n \), is closed, we can find \( \delta > 0 \) such
\[
\text{dist}(p, \mathbb{T}_u) \geq \delta
\]
whenever \( u \in W_n \) and \( p \not\in \mathbb{T}_u \). Then \( B(p, \delta) \cap \mathbb{T} \subseteq N \) by \[4.11\], and so \( N \) is a connected relative neighborhood of \( p \) in \( \mathbb{T} \). It follows from \[4.12\] that \( \text{diam}(N) \leq 2^{2^{-n}} \). This shows that each point in \( \mathbb{T} \) has arbitrarily small connected neighborhoods in \( \mathbb{T} \). Hence \( \mathbb{T} \) is locally connected.

To complete the proof, it remains to show that the arc joining two given distinct points in \( \mathbb{T} \) is unique. For this we argue by contradiction and assume that there are two distinct arcs in \( \mathbb{T} \) with the same endpoints. By considering suitable subarcs of these arcs, we can reduce to
the following situation: there are arcs \( \alpha, \beta \subseteq \mathbb{T} \) that have the distinct endpoints \( a, b \in \mathbb{T} \) in common, but no other points.

To see that this leads to a contradiction, we represent the points \( a \) and \( b \) by words in \( W \); so \( a = \pi(v) \) and \( b = \pi(w) \), where \( v = v_1 v_2 \ldots \) and \( w = w_1 w_2 \ldots \) are in \( W \). Since \( a \neq b \) and every point in \( \mathbb{T} \) has at most three such representations by Lemma 4.5 (ii), we can find a pair \( v \) and \( w \) representing \( a \) and \( b \) with the largest common initial word, say \( v_1 = w_1, \ldots, v_n = w_n \), and \( v_{n+1} \neq w_{n+1} \) for some maximal \( n \in \mathbb{N}_0 \).

Let \( u = v_1 \ldots v_n = w_1 \ldots w_n \) and

\[
    t = f_u(0) = \pi(u1\overline{2}) = \pi(u2\overline{1}) = \pi(u3\overline{1}).
\]

Then \( t \neq a, b \). To see this, assume that \( t = a \), say. We have \( w_{n+1} \in \{1,2,3\} \), and so, say \( w_{n+1} = 1 \). But then \( a = t = \pi(u1\overline{2}) \) and \( b = \pi(u1w_{n+2} \ldots) \). So \( a \) and \( b \) are represented by words with the common initial segment \( u1 \) that is longer than \( u \). This contradicts the choice of \( v \) and \( w \).

So indeed \( t = f_u(0) \neq a, b \). Moreover \( a = \pi(uw_{n+1} \ldots) \in \mathbb{T}_{uw_{n+1}} \setminus \{t\} \) and similarly \( b \in \mathbb{T}_{uw_{n+1}} \setminus \{t\} \). Since \( v_{n+1} \neq w_{n+1} \), the points \( a \) and \( b \) lie in different components of \( \mathbb{T} \setminus \{t\} \) by Lemma 4.7 (ii). So any arc joining \( a \) and \( b \) must pass through \( t \). Hence \( t \in \alpha \cap \beta \), but \( t \neq a, b \). This contradicts our assumption that the arcs \( \alpha \) and \( \beta \) have no other points than their endpoints \( a \) and \( b \) in common.

If \( M \subseteq \mathbb{T} \), then we denote by \( \partial M \subseteq \mathbb{T} \) the relative boundary of \( M \) in \( \mathbb{T} \).

**Lemma 4.9.** Let \( n \in \mathbb{N} \) and \( u \in W_n \). Then

\[
    \partial \mathbb{T}_u \subseteq \{f_u(-i), f_u(+i)\}. \tag{4.18}
\]

Moreover, if \( p \in \partial \mathbb{T}_u \), then \( p = f_w(0) \) for some word \( w \in W_n \) of length \( \leq n - 1 \).

In particular, the set \( \partial \mathbb{T}_u \) contains at most two points.

**Proof.** We prove this by induction on \( n \). First consider \( n = 1 \). So let \( u = k \in W_1 = \{1,2,3\} \). Then \( \mathbb{T}_k \setminus \{0\} \) is a component \( \mathbb{T} \setminus \{0\} \) by Lemma 4.7 (i). Hence Proposition 4.8 and Lemma 3.2 (i) imply that \( \mathbb{T}_k \setminus \{0\} \) is a relatively open set in \( \mathbb{T} \). So each of its points lies in the relative interior of \( \mathbb{T}_k \) and cannot lie in \( \partial \mathbb{T}_k \). Therefore, \( \partial \mathbb{T}_k \subseteq \{0\} \).

Since

\[
    0 = f_0(0) = f_1(+i) = f_2(-i) = f_3(-i), \tag{4.19}
\]

the statement is true for \( n = 1 \).
Suppose the statement is true for all words in $W_n$, where $n \in \mathbb{N}$. Let $u \in W_{n+1}$ be arbitrary. Then $u = vk$, where $v \in W_n$ and $k \in \{1, 2, 3\}$. By what we have just seen, the set $T_k \setminus \{0\}$ is open in $T$. Hence

$$f_v(T_k \setminus \{0\}) = f_u(T) \setminus \{f_v(0)\} = T_u \setminus \{f_v(0)\}$$

is a relatively open subset of $T$. This implies that

From this we conclude that each point $p$ in $T_u$ is a relatively open subset of $T$. Hence $p$ is a point of $T_v$ for some finite word $w$. Indeed, if $k = 1$, then $+i \notin T_1 \subseteq H_1$, and so $f_v(+i) \notin f_v(T_1) = T_u$. It follows that $\partial T_u \subseteq \{f_v(-i), f_v(0)\}$. Note that the boundary of $T_v$ is not an interior point of $T_v$ in $T$ and hence belongs to the boundary of $T_v$. This and the induction hypothesis imply that

$$\partial T_u \subseteq \{f_v(0)\} \cup \partial T_v \subseteq \{f_v(0), f_v(-i), f_v(+i)\}.$$

From this we conclude that each point $p$ in $\partial T_u \subseteq \{f_v(0)\} \cup \partial T_v$ can be written in the form $f_w(0)$ for an appropriate word $w$ of length $\leq n$. This is clear if $p = f_v(0)$ and follows for $p \in \partial T_v$ from the induction hypothesis.

Now $T_u = f_u(T)$ is compact and so closed in $T$. Hence $\partial T_u \subseteq T_u$. On the other hand, $T_u$ contains only two of the points $f_v(0), f_v(-i), f_v(+i)$. Indeed, if $k = 1$, then $+i \notin T_1 \subseteq H_1$, and so $f_v(+i) \notin f_v(T_1) = T_u$. It follows that $\partial T_u \subseteq \{f_v(-i), f_v(0)\}$. Note that $f_1(-i) = -i$ and $f_1(+i) = 0$, and so

$$f_v(-i) = f_v(f_1(-i)) = f_u(-i) \quad \text{and} \quad f_v(0) = f_v(f_1(+i)) = f_u(+i).$$

Hence

$$\partial T_u \subseteq \{f_u(-i), f_u(+i)\}.$$

Very similar considerations show that if $k = 2$, then

$$\partial T_u \subseteq \{f_v(0), f_v(+i)\} = \{f_u(-i), f_u(+i)\},$$

and if $k = 3$, then

$$\partial T_u \subseteq \{f_v(0)\} = \{f_u(-i)\}.$$

The statement follows. \qed

The next lemma shows that all branch points of $T$ are of the form $f_u(0)$ with $u \in W_*$.

**Lemma 4.10.** The branch points of $T$ are exactly the points of the form $t = f_u(0)$ with some finite word $u \in W_*$. They are triple points of $T$.

**Proof.** By Lemma 4.7 (ii) we know that each point $t = f_u(0)$ with $u \in W_*$ is a triple point of the tree $T$. We have to show that there are no other branch points of $T$.

So suppose that $t$ is a branch point of $T$, but $t \neq f_u(0)$ for each $u \in W_*$. Then we can find (at least) three distinct components $U_1$, \ldots
$U_2, U_3$ of $\mathbb{T}\setminus\{t\}$. Pick a point $x_k \in U_k$ and choose $m \in \mathbb{N}$ such that $|x_k - t| > 2^{-m}$ for $k = 1, 2, 3$. By (4.11) we can find $u \in W_m$ such that $t \in T_u$. Then $t$ is distinct from the points in the relative boundary $\partial T_u$, because they have the form $f_w(0)$ for some $w \in W_*$ (see Lemma 4.9). Hence $t$ is contained in the relative interior of $T_u$ in $\mathbb{T}$. Moreover, $\text{diam}(T_u) = 2^{-m}$, and so $x_k \not\in T_u$. For $k = 1, 2, 3$ let $\alpha_k$ be the arc in $T$ joining $x_k$ and $t$. As we travel from $x_k$ to $t$ along $\alpha_k$, there exists a first point $y_k \in T_u$. Then $y_k \in \partial T_u$ and so $y_k \neq t$. Let $\beta_k$ be the subarc of $\alpha_k$ with endpoints $x_k$ and $y_k$. Then $\beta_k$ is a connected set in $\mathbb{T}\setminus\{t\}$. Since $x_k \in \beta_k$, it follows that $\beta_k \subseteq U_k$, and so $y_k \in U_k$.

This shows that the points $y_1, y_2, y_3$ are distinct and contained in the relative boundary $\partial T_u$. This is impossible, because by Lemma 4.9 the set $\partial T_u$ consists of at most two points. □

**Proposition 4.11.** Each branch point of the tree $\mathbb{T}$ is a triple point, and these triple points are dense in $\mathbb{T}$.

In other words, $\mathbb{T}$ satisfies the conditions in Theorem 1.2 and $\mathbb{T}$ belongs to the class of trees $\mathcal{T}_3$.

**Proof.** By Lemma 4.10 each branch point of $\mathbb{T}$ is a triple point and each set $T_u$ for $u \in W_n$ and $n \in \mathbb{N}$ contains the triple point $t = f_u(0)$. The sets $T_u, u \in W_n$, cover $\mathbb{T}$ and have small diameter for $n$ large. It follows that the triple points are dense in $\mathbb{T}$. □

We will show one more property of $\mathbb{T}$, namely that it is a *quasi-convex* subset of $\mathbb{C}$, i.e., any two points in $\mathbb{T}$ can be joined by a path whose length is comparable to the distance of the points. We first require a lemma.

**Lemma 4.12.** There exists a constant $K > 0$ such that if $p \in \mathbb{T}$ and $\alpha$ is the arc in $\mathbb{T}$ joining 0 and $p$, then

$$\text{length}(\alpha) \leq K|p|.$$  

(4.20)

In particular, the arc $\alpha$ is a rectifiable curve.

**Proof.** Let $p \in \mathbb{T}$ be arbitrary. We may assume that $p \neq 0$. Then $p = \pi(w)$ for some $w = w_1w_2\ldots \in W$. For simplicity we assume $w_1 = 3$. The other cases, $w_1 = 1$ and $w_1 = 2$, are very similar and we will only present the details for $w_1 = 3$.

Since $p \neq 0 = \pi(31)$, we have $w_2w_3\ldots \neq 1$. Hence there exists a smallest number $n \in \mathbb{N}$ such that such that $w_{n+1} \neq 1$. Let $v = w_1\ldots w_n$ be the initial word of $w$ and $v' = w_{n+1}w_{n+2}\ldots$ be the tail of $w$. The word $v$ has form $v = 31\ldots 1$, where the sequence of 1’s could possibly
be empty. Note that $q := \pi(w') \in T_{w_{n+1}} \subseteq T_2 \cup T_3 \subseteq H_2 \cup H_3$. Since

$$c_0 := \text{dist}(-i, H_2 \cup H_3) > 0$$

(see Figure 2), for the distance of $q$ and $-i$ we have $|q + i| \geq c_0$. We also have $f_v(q) = p$, and $f_v(-i) = 0$, because $f_1(-i) = -i$ and $f_3(-i) = 0$. It follows that

$$|p| = |f_v(q) - f_v(-i)| = \frac{1}{2^n}|q + i| \geq \frac{c_0}{2^n}. \quad (4.21)$$

Now define $a_0 = 0 = f_v(-i)$ and $a_k = f_{vw_{n+1}...w_{n+k-1}}(0)$ for $k \in \mathbb{N}$ (here $w_{n+1}...w_{n+k-1} = \emptyset$ for $k = 1$). Note that then

$$a_1 = f_v(0) = f_{w_1...w_n}(0) = f_{31...1}(0) = f_3((2^{1-n} - 1)i) = 1/2^n,$$

and so

$$[a_0, a_1] = [f_v(-i), f_v(0)] = [0, 1/2^n] \subseteq [0, 1] \subseteq T.$$ This also shows that $\text{length}([a_0, a_1]) = 1/2^n$.

For $k \in \mathbb{N}$ we have $f_{w_{n+k}}(0) \in \{-i/2, +i/2, 1/2\}$, and $[0, f_{w_{n+k}}(0)] \subseteq T$. This implies that

$$[a_k, a_{k+1}] = f_{vw_{n+1}...w_{n+k-1}}([0, f_{w_{n+k}}(0)]) \subseteq T$$

and $\text{length}([a_k, a_{k+1}]) = 1/2^{n+k}$ for $k \in \mathbb{N}$. Since $\lim_{k \to \infty} a_k = \pi(w) = p$, we can concatenate the intervals $[a_k, a_{k+1}] \subseteq T$ for $k \in \mathbb{N}_0$, add the endpoint $p$, and obtain a path $\gamma$ in $T$ that joins 0 and $p$ with

$$\text{length}(\gamma) = \sum_{k=0}^{\infty} \frac{1}{2^{n+k}} = \frac{1}{2^{n-1}}.$$ The (image of the) path $\gamma$ will contain the unique arc $\alpha$ in $T$ joining 0 and $p$ and so $\text{length}(\alpha) \leq 1/2^{n-1}$. If we combine this with (4.21), then inequality (4.12) follows with $K = 2/c_0$. \qed

**Proposition 4.13.** There exists a constant $L > 0$ with the following property: if $p, q \in T$ and $\alpha$ is the unique arc in $T$ joining $p$ and $q$, then

$$\text{length}(\alpha) \leq L|p - q|.$$ This shows that $T$ is indeed a quasi-convex subset of $C$.

**Proof.** Let $p, q \in T$ be arbitrary. We may assume that $p \neq q$. Then there are words $u = u_1u_2... \in W$ and $v = v_1v_2... \in W$ such that $p = \pi(u)$ and $q = \pi(v)$. Since $p \neq q$, we have $u \neq v$ and so there exists a smallest number $n \in \mathbb{N}_0$ such that $u_1 = v_1, ..., u_n = v_n$ and $u_{n+1} \neq v_{n+1}$. Let $w = u_1...u_n = v_1...v_n$, $u' = u_{n+1}u_{n+2}... \in W$ and $v' = v_{n+1}v_{n+2}... \in W$. We define $p' = \pi(u')$ and $q' = \pi(v')$. Set $k = u_{n+1}$ and $l = v_{n+1}$. Then $k \neq l$, $p' \in T_k \subseteq H_k$, and $q' \in T_l \subseteq H_l$.
We now use the following elementary geometric estimate: there exists a constant $c_1 > 0$ such that

$$\|x - y\| \geq c_1(\|x\| + \|y\|),$$

whenever $x \in H_k$, $y \in H_l$, $k, l \in \{1, 2, 3\}$, $k \neq l$. Essentially, this follows from the fact that the sets $H_1$, $H_2$, $H_3$ are contained in closed sectors in $\mathbb{C}$ that are pairwise disjoint except for the common point 0.

In our situation, this means that

$$|p' - q'| \geq c_1(|p'| + |q'|).$$

Let $\sigma$ and $\tau$ be the arcs in $\mathbb{T}$ joining 0 to $p'$ and $q'$, respectively. Then by $\sigma \cup \tau$ contains the arc $\alpha'$ in $\mathbb{T}$ joining $p'$ and $q'$. Then it follows from Lemma 4.12 that

(4.22) $\text{length}(\alpha') \leq \text{length}(\sigma) + \text{length}(\tau) \leq K(|p'| + |q'|) \leq L|p' - q'|$

with $L := K/c_1$.

For the similarity $f_w$ we have $f_w(p') = p$ and $f_w(q') = q$. Since $f_w(\mathbb{T}) \subseteq \mathbb{T}$, it follows that $\alpha := f_w(\alpha')$ is the unique arc in $\mathbb{T}$ joining $p$ and $q$. Since $f_w$ scales distances by a fixed factor (namely $1/2^n$), (4.22) implies the desired inequality $\text{length}(\alpha) \leq L|p - q|$. □

By the previous proposition we can define a new metric $\rho$ on $\mathbb{T}$ by setting

$$\rho(p, q) = \text{length}(\alpha)$$

for $p, q \in \mathbb{T}$, where $\alpha$ is the unique arc in $\mathbb{T}$ joining $p$ and $q$. Then the metric space $(\mathbb{T}, \rho)$ is geodesic. Moreover, the metric spaces $\mathbb{T}$ (as equipped with the Euclidean metric) and $(\mathbb{T}, \rho)$ are bi-Lipschitz equivalent by the identity map.

This allows us to reconcile the construction of the CSST $\mathbb{T}$ as discussed in this section with the one outlined in the introduction. Namely, consider the sets $J_n$ for $n \in \mathbb{N}_0$ as in Proposition 4.4. Here $J_0 = I = [-i, +i]$ is a line segment of length 2. Hence each set $f_w(I)$ for $w \in W_n$ is a line segment of length $2^{1-n}$. In the passage from $J_n$ to $J_{n+1}$ we can think of each line segment $f_w(I) = f_w([-i, +i])$ as being replaced with

$$f_{w1}(I) \cup f_{w2}(I) \cup f_{w3}(I) = f_w([-i, 0]) \cup f_w([0, +i]) \cup f_w([0, 1]).$$

So $f_w([-i, +i])$ is split into two intervals $f_w([-i, 0])$ and $f_w([0, +i])$, and at its midpoint $f_w(0)$ a new interval $f_w([0, 1])$ is “glued” to $f_w(0)$. This is exactly the procedure described in the introduction. It is clear that $J_n$ is compact, and one can show by induction based on the replacement procedure just discussed that $J_n$ is connected. Hence each $J_n$ is a subtree of $\mathbb{T}$ by Lemma 3.3. The metric $\rho$ on $J_n$, $n \in \mathbb{N}_0$, and on
$J := \bigcup_{n \in \mathbb{N}_0} J_n$ is just the natural Euclidean path metric on these sets. In particular, $\varrho$ is a geodesic metric on $J$.

By Proposition 4.4 the tree $T$ is the equal to closure $\overline{J}$ in $\mathbb{C}$. Since on $J$ the Euclidean metric and the metric $\varrho$ are comparable, the set $T = \overline{J}$ is homeomorphic to the space obtained from the completion of the geodesic metric metric space $(J, \varrho)$. This is how we described the CSST in the introduction.

We mention without proof that the geodesic metric $\varrho$ on $T$ can be described by using the representation of points in $T$ by words in $W$. Namely, let $u = u_1 u_2 \ldots \in W$. If $u_1 = 2$ or $u_1 = 3$, then

$$\varrho(\pi(u), 0) = \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} \chi_{\{2,3\}}(u_n),$$

and if $u_1 = 1$,

$$\varrho(\pi(u), 0) = \sum_{n=2}^{k} \frac{1}{2^{n-1}} \chi_{\{1\}}(u_n) + \sum_{n=k+1}^{\infty} \frac{1}{2^{n-1}} \chi_{\{2,3\}}(u_n),$$

where $k \in \mathbb{N}$ is the smallest integer such that $u_{k+1} = 3$ if such $k$ exists and $k = \infty$ otherwise. In the latter case, we interpret the second sum as the empty sum.

Finally, if $v = v_1 v_2 \ldots \in W$, and $u \neq v$, let $m \in \mathbb{N}_0$ be the largest integer such that $u_1 = v_1, \ldots, u_m = v_m$. If we define $\bar{u} = u_{m+1} u_{m+2} \ldots$ and $\bar{v} = v_{m+1} v_{m+2} \ldots$, then

$$\varrho(\pi(u), \pi(v)) = \frac{1}{2^m} \left( \varrho(\pi(\bar{u}), 0) + \varrho(\pi(\bar{v}), 0) \right).$$

5. Decomposing trees in $T_m$

In the previous section we have seen that for each $n \in \mathbb{N}$ the CSST admits a decomposition

$$T = \bigcup_{u \in W_n} T_u$$

into subtrees. We will now consider an arbitrary tree in $T_m$, $m \in \mathbb{N} \cup \{\infty\}, m \geq 3$, and find similar decompositions into subtrees. Our goal is to have decompositions for each level $n \in \mathbb{N}$ so that the conditions (i)–(iii) in Proposition 2.1 are satisfied. For the case $m = \infty$ we will need a more refined statement (see Proposition 6.2), because then our decomposition on a given level $n$ will have infinitely many pieces.

Note that each tree class $T_m$ is non-empty. From a purely logical point of view, we will not need the fact that $T_m \neq \emptyset$ for the proof of Theorem 1.4 and so we will only give a sketch for the construction of such trees. If $m \in \mathbb{N}$, $m \geq 3$, then a tree in $T_m$ can be obtained
by essentially the same method as outlined in the introduction for the CSST. The only difference is that instead of gluing one line segment of length $2^{-n}$ to the midpoint $c_s$ of a line segment $s$ of length $2^{1-n}$ obtained in the $n$th step, we glue an endpoint of $m-2$ such segments to $c_s$.

The construction of a tree in $\mathcal{T}_\infty$ is slightly more complicated. Again we start with a line segment $J_0$ of length 2. The midpoint $c$ of $J_0$ divides $J_0$ into two line segments of length $2^{-0}, 2^{-1}, \ldots$. We equip the resulting space $J_1$ with the obvious path metric. Then $J_0 \subseteq J_1$. We repeat this construction for each of the line segments obtained in this way, that is, a line segment $s$ of length $l$ is divided into two at its midpoint $c_s$ and segments of lengths $2^{-1}l, 2^{-2}l, \ldots$ are glued in at $c_s$.

Proceeding in this way, we obtain an ascending sequence of compact geodesic metric spaces $J_0 \subseteq J_1 \subseteq \ldots$. Then $J = \bigcup_{n \in \mathbb{N}_0} J_n$ carries a geodesic metric that agrees with the metric on $J_n$ for each $n \in \mathbb{N}_0$. The completion $\mathcal{T}_\infty$ of $J$ with respect to this metric is a tree in $\mathcal{T}_\infty$.

We now fix $m \in \mathbb{N} \cup \{\infty\}$, $m \geq 3$, for the rest of this section. We consider the alphabet $A = \{1, 2, \ldots, m\}$ if $m \in \mathbb{N}$ and $A = \mathbb{N}$ if $m = \infty$. In the following, words will contain only letters in this fixed alphabet and we use the simplified notation for the sets of words $W, W_n, W_*$ as discussed in Section 4.

Let $T \in \mathcal{T}_m$ be arbitrary. We will now define subtrees $T_u$ of $T$ for all levels $n \in \mathbb{N}$ and all $u \in W_n$. The boundary $\partial T_u$ of $T_u$ in $T$ will consist of one or two points that are leaves of $T_u$ and branch points of $T$. We consider each point in $\partial T_u$ as a marked point in $T_u$ and will assign to it an appropriate sign $-$ or $+$ so that if there are two marked points in $T_u$, then they carry different signs. Accordingly, we refer to the points in $\partial T_u$ as the signed marked points of $T_u$. The same point may carry different signs in different subtrees. We write $p^-$ if a marked point $p$ in $T_u$ carries the sign $-$ and $p^+$ if it carries the sign $+$. To refer to this sign, we also write $\text{sgn}(p, T_u) = -$ in the first and $\text{sgn}(p, T_u) = +$ in the second case. If $T_u$ has exactly one marked point, we call $T_u$ an end-piece and if there are two marked points an arc-piece.

For the construction we will use an inductive procedure on $n$. As in Section 3 (see the discussion before Lemma 3.9, for each branch point $b \in T$, we let $M(b)$ be the diameter of the third largest branch of $b$ in $T$. If $\delta > 0$, then by Lemma 3.9 there are only finitely many branch points $b$ of $T$ with weight $M(b) > \delta$, and in particular there is one for which this quantity is maximal.

For the first step $n = 1$, we choose a branch point $c$ of $T$ with maximal weight $M(c)$. Since $T \in \mathcal{T}_m$, this branch point $c$ has $m = \nu_T(c)$ branches
in \( T \). So we can enumerate the distinct branches by the letters in our alphabet as \( T_k, k \in \mathcal{A} \).

We choose \( c \) as the signed marked point in each \( T_k \), where we set \( \text{sgn}(c, T_1) = + \) and \( \text{sgn}(c, T_k) = - \) for \( k \neq 1 \). So the set of signed marked points is \( \{c^+\} \) in \( T_1 \) and \( \{c^-\} \) in \( T_k, k \neq 1 \). Note that \( \partial T_k = \{c\} \) by Lemma \[3.2\] (ii) and that \( c \) is a leaf in \( T_k \) by Lemma \[3.4\] for each \( k \in \mathcal{A} \).

Suppose that for some \( n \in \mathbb{N} \) and all \( u \in W_n \) we have constructed subtrees \( T_u \) of \( T \) such that \( \partial T_u \) consists of one or two signed marked points that are leaves in \( T_u \) and branch points of \( T \). We will now construct the subtrees of the \((n+1)\)-th level as follows by subdivision of the trees \( T_u \).

Fix \( u \in W_n \). To decompose \( T_u \) into subtrees, we will use a suitable branch point \( c \) of \( T_u \) in \( T_u \setminus \partial T_u \). The choice of \( c \) depends on whether \( \partial T_u \) contains one or two elements, that is, whether \( T_u \) is an end-piece or an arc-piece. We will explain this precisely below, but first record some facts that are true in both cases.

Since \( c \in T_u \setminus \partial T_u \) is an interior point of \( T_u \), there is a bijective correspondence between the branches of \( c \) in \( T \) and in \( T_u \) (see Lemma \[3.5\]). So \( \nu_{T_u}(c) = \nu_T(c) = m \), and we can label the distinct branches of \( c \) in \( T_u \) by \( T_{uk}, k \in \mathcal{A} \). Then we have

\[
T_u = \bigcup_{k \in \mathcal{A}} T_{uk}.
\]

Each set \( T_{uk} \) is a subtree of \( T_u \) and hence also of \( T \). We call these subtrees the \textit{children} of \( T_u \) and \( T_u \) the \textit{parent} of its children. Note that two distinct children of \( T_u \) have only the point \( c \) in common and no other points.

Since \( T_{uk} \) is a subtree of \( T \), it contains all of its boundary points and so \( \partial T_{uk} \subseteq T_{uk} \). We have \( c \in \partial T_{uk} \), because \( c \in T_{uk} \) and every neighborhood of \( c \) contains points in the complement of \( T_{uk} \) as follows from Lemma \[3.2\] (ii) (here it is important that there are at least two branches of \( c \)). If \( p \in T_{uk} \subseteq T_u \) and \( p \notin \{c\} \cup \partial T_u \), then a sufficiently small neighborhood \( N \) of \( p \) belongs to \( T_u \). Since \( T_{uk} \setminus \{c\} \) is relatively open in \( T_u \) (this follows from Lemma \[3.2\] (i)), we can shrink this neighborhood so that \( N \subseteq T_{uk} \). So no point \( p \) in \( T_{uk} \) can be a boundary point of \( \partial T_{uk} \) unless it belongs to \( \{c\} \cup \partial T_u \). We know that \( c \in \partial T_{uk} \). If \( p \in \partial T_u \cap T_{uk} \), then \( p \) is a boundary point of \( T_{uk} \), because every neighborhood of \( p \) contains elements in the complement of \( T_u \) and hence in the complement of \( T_{uk} \subseteq T_u \). These considerations show that

\[
\partial T_{uk} = \{c\} \cup (\partial T_u \cap T_{uk})
\]
for each \( k \in A \). This identity implies that each point in \( \partial T_{u_k} \) is a
branch point of \( T \), because \( c \) is and the points in \( \partial T_u \) are also
branch points of \( T \) by construction on the previous level \( n \). Moreover, each
point \( p \in \partial T_{u_k} \subseteq T_{u_k} \) is a leaf of \( T_{u_k} \), because if \( p = c \), then \( p \) is a leaf
in \( T_{u_k} \) by Lemma 3.4. In the other case, \( p \in \partial T_u \). Then \( p \) is a leaf
of \( T_u \) by construction and hence a leaf of \( T_{u_k} \) by the discussion after
Lemma 3.5.

For the precise choice of the branch point \( c \in T_u \setminus \partial T_u \) we now con-
sider two cases for the set \( \partial T_u \). See also Figure 3.

**Case 1**: \( \partial T_u \) contains precisely one element, say \( \partial T_u = \{a\} \). Note
that \( T_u \) is subtree of \( T \) and so an infinite set. So \( T_u \setminus \partial T_u \neq \emptyset \). All
points in \( T_u \setminus \partial T_u \) are interior points of \( T_u \). Since branch points in \( T \)
are dense (here we use that \( T \in \mathcal{T}_m \)), there exist branch points of \( T \) in
\( T_u \setminus \partial T_u \). We choose a branch point \( c \in T_u \setminus \partial T_u \) with maximal weight
\( M(c) \) among all such branch points. This is possible by Lemma 3.9.

Since \( a \in \partial T_u \subseteq T_u \setminus \{c\} \), precisely one of the children of \( T_u \) contains
\( a \). We now consider two subcases depending on the sign of the marked
point \( a \).

If \( \operatorname{sgn}(a, T_u) = - \), then we choose a labeling of the children so that
\( a \in T_{u_1} \). It then follows from (5.2) that \( \partial T_{u_1} = \{a, c\} \) and \( \partial T_{u_k} = \{c\} \)
for \( k \neq 1 \). We choose signs so that the set of signed marked points is
\( \{a^-, c^+\} \) in \( T_{u_1} \) and \( \{c^-\} \) in \( T_{u_k} \), \( k \neq 1 \).

If \( \operatorname{sgn}(a, T_u) = + \), then we choose a labeling such that \( a \in T_{u_2} \). Then
again by (5.2) we have \( \partial T_{u_2} = \{a, c\} \) and \( \partial T_{u_k} = \{c\} \) for \( k \neq 2 \). We
choose signs so that the set of signed marked points is \( \{c^+\} \) in \( T_{u_1} \),
\( \{c^-, a^+\} \) in \( T_{u_2} \), and \( \{c^-\} \) in \( T_{u_k} \), \( k \neq 1, 2 \).

**Case 2**: \( \partial T_u \) contains precisely two elements, say \( \partial T_u = \{a^-, b^+\} \).
Then we choose a branch point \( c \in (a, b) \) of \( T \) such that it has the
maximal weight \( M(c) \) among all branch points that lie on \( (a, b) \). The
existence of \( c \) is guaranteed by Lemma 3.7 and Lemma 3.9. Note that
\( (a, b) \subseteq T_u \), because \( T_u \) is a subtree of \( T \).

The points \( a \) and \( b \) lie in different branches of \( c \) in \( T_u \) as follows
from Lemma 3.2 (iii). We may assume that labels have been chosen
so that \( a \in T_{u_1} \) and \( b \in T_{u_2} \). Then by (5.2) we have \( \partial T_{u_1} = \{a, c\} \),
\( \partial T_{u_2} = \{c, b\} \), and \( \partial T_{u_k} = \{c\} \), \( k \neq 1, 2 \). We choose signs so that the
set of marked points is \( \{a^-, c^+\} \) in \( T_{u_1} \), \( \{c^-, b^+\} \) in \( T_{u_2} \), and \( \{c^-\} \), in
\( T_{u_k} \) for \( k \neq 1, 2 \).

Note that in all cases \( T_{u_k} \) is a subtree of \( T \) such that \( \partial T_u \) contains one
or two points that are branch points of \( T \) and leaves of \( T_{u_k} \). Moreover,
the signs of the points in each set \( \partial T_u \) are chosen so that these signs
differ if \( \partial T_u \) contains two points.
If $T_u$ has marked point $p^-$, then $p$ is passed to the child $T_{u1}$ with the same sign. Similarly, a marked point $p^+$ of $T_u$ is passed to $T_{u2}$ with the same sign. So marked points are passed to a unique child and they retain their signs.

If we continue this construction, then we obtain subtrees $T_u$ of $T$ for all $u \in W_\star$. Here it is convenient to set $T_\emptyset = T$ with an empty set of marked points.

Lemma 5.1. The following statements are true:

(i) $T = \bigcup_{u \in W_n} T_u$ for each $n \in \mathbb{N}$.

(ii) If $n \in \mathbb{N}$, $u, v \in W_n$, $u \neq v$, and $T_u \cap T_v \neq \emptyset$, then $T_u \cap T_v$ consists of precisely one point $p \in T$ that is a marked point in both $T_u$ and $T_v$.

(iii) For $n \in \mathbb{N}$, $u \in W_n$, and $v \in W_{n+1}$, we have $T_v \subseteq T_u$ if and only if $u = vk$ for some $k \in A$. 

Figure 3. An illustration for the decomposition of subtrees with one marked point (top) or two marked points (bottom).
(iv) For \( u \in W_\ast \) let \( c_u \) be the branch point chosen in the decomposition of \( T_u \) into children. Then \( c_u \neq c_v \) for all \( u, v \in W_\ast \) with \( u \neq v \).

**Proof.** (i) This immediately follows from (5.1) and induction on \( n \).

(ii) We prove this by induction on \( n \). By choice of the subtrees \( T_k \) for \( k \in A = W_1 \) and their marked points this is clear for \( n = 1 \).

Suppose the statement is true for all distinct words of length \( n - 1 \), where \( n \geq 2 \). Now consider two words \( u, v \in W_n \) of length \( n \) with \( u \neq v \) and \( T_u \cap T_v \neq \emptyset \). Then \( u = u'k \) and \( v = v'l \), where \( u', v' \in W_{n-1} \) and \( k, l \in A \).

If \( u' = v' \), then \( T_u \) and \( T_v \) are two of the branches obtained from \( T_{u'} \) and a suitable branch point \( c \in T_{u'} \). In this case, \( \{c\} = T_u \cap T_v \) and \( c \) is a marked point in both \( T_u \) and \( T_v \).

In the other case, \( u' \neq v' \). Then \( T_u \cap T_v \neq \emptyset \), because \( T_u \cap T_v \neq \emptyset \), \( T_u \subseteq T_{u'} \), and \( T_v \subseteq T_{v'} \). By induction hypothesis, \( T_{u'} \cap T_{v'} \) consists of precisely one point \( p \) that is a marked point in both \( T_{u'} \) and \( T_{v'} \). Then necessarily \( T_u \cap T_v = \{p\} \). Moreover, \( p \) is a marked point in both \( T_u \) and \( T_v \), because marked points are passed to children. The statement follows.

(iii) Let \( n \in \mathbb{N} \) and \( u \in W_n \). Then we have \( T_{uk} \subseteq T_u \) for each \( k \in A \) by our construction. Conversely, suppose \( T_v \subseteq T_u \), where \( v = v'k \in W_{n+1} \) with \( v' \in W_n \) and \( k \in A \). Then \( T_{u'} \cap T_u \supseteq T_v \) contains more than one point. By (iii) this implies that \( v' = u \). The statement follows.

(iv) If \( u \in W_n \), \( n \in \mathbb{N}_0 \), then by construction \( c_u \in T_u \) does not lie in the set \( \partial T_u \) of marked points of \( T_u \). By (ii) this implies that \( c_u \notin T_w \) for each \( w \in W_n, w \neq u \). It follows that the points \( c_u, u \in W_n \), are all distinct, and none of them is contained in the union of sets \( \partial T_u \), \( u \in W_n \). By our construction this union is equal to the set of all point \( c_v \), where \( v \in W_\ast \) is a word of length \( \leq n - 1 \). This shows that the branch points \( c_u, u \in W_n \), used to define the subtrees of level \( n + 1 \) are all distinct and distinct from any of the previously chosen branch points for levels \( \leq n \). The statement follows from this. \( \square \)

**Lemma 5.2.** We have

\[
\lim_{n \to \infty} \sup \{ \text{diam}(T_u) : u \in W_n \} = 0.
\]

**Proof.** Let \( \delta_n := \sup \{ \text{diam}(T_u) : u \in W_n \} \) for \( n \in \mathbb{N} \). It is clear that the sequence \( \{\delta_n\} \) is non-increasing. To show that \( \delta_n \to 0 \) as \( n \to \infty \), we argue by contradiction. Then there exists \( \delta > 0 \) such that \( \delta_n \geq \delta \) for all \( n \in \mathbb{N} \). This means that for each \( n \in \mathbb{N} \) there exists \( u \in W_n \)
with
\[(5.3)\quad \text{diam}(T_u) \geq \delta.\]

Note that for each fixed \(n \in \mathbb{N}\) the number of words \(u \in W_n\) with \(\text{diam}(T_u) \geq \delta\) is finite. This is clear if \(\mathcal{A}\) is a finite alphabet, because then \(W_n\) is a finite set. If \(\mathcal{A}\) is infinite and so \(\mathcal{A} = \mathbb{N}\), this follows by induction on \(n\) from Lemma 3.8 and our construction of the trees \(T_u\).

We now use (5.3) to find an infinite word \(w = w_1w_2\ldots \in W\) such that
\[(5.4)\quad \text{diam}(T_{w_1\ldots w_n}) \geq \delta\]
for all \(n \in \mathbb{N}\). The word \(w\) is constructed inductively as follows. There are only finitely many letters \(k \in \mathcal{A}\) with \(\text{diam}(T_k) \geq \delta\) and so one of these letters \(k \in \mathcal{A}\) must have the property that there are arbitrarily long words \(u\) starting with \(k\) such that (5.3) is true.

We define \(w_1 = k\). Note that then \(\text{diam}(T_{w_1}) \geq \delta\). Then there are only finitely many letters \(l \in \mathcal{A}\) such that \(\text{diam}(T_{w_1l}) \geq \delta\). By choice of \(w_1\), one of these letters \(l\) must have the property that there are arbitrarily long words \(u\) starting with \(w_1l\) such that (5.3) is true. We define \(w_2 = l\). Then \(\text{diam}(T_{w_1w_2}) \geq \delta\). Continuing in this manner, we can find \(w = w_1w_2\ldots \in W\) satisfying (5.4).

Obviously,
\[T_{w_1} \supseteq T_{w_1w_2} \supseteq T_{w_1w_2w_3} \supseteq \ldots.\]
So the subtrees \(K_n = T_{w_1\ldots w_n}, \ n \in \mathbb{N}\), of \(T\) form a descending family of compact sets with \(\text{diam}(K_n) \geq \delta\). This implies that
\[K = \bigcap_{n \in \mathbb{N}} K_n\]
is a non-empty compact subset of \(T\) with \(\text{diam}(K) \geq \delta\).

In particular, we can choose \(p, q \in K\) with \(p \neq q\). Then \(p, q \in K_n\) for each \(n \in \mathbb{N}\). Since \(K_n\) is a subtree of \(T\), we then have \([p, q] \subseteq K_n\). Moreover, by Lemma 3.7 there exists a branch point \(x\) of \(T\) contained in \((p, q) \subseteq K_n\). By Lemma 3.2 (iii) the points \(p\) and \(q\) lie in different components of \(K_n \setminus \{x\}\). In particular, \(x\) is not a leaf of \(K_n\) and hence distinct from the marked points of \(K_n\).

By Lemma 3.9 there are only finitely many branch points \(y_1, \ldots, y_s\) of \(T\) distinct from \(x\) with \(M(y_j) \geq M(x) > 0\) for \(j = 1, \ldots, s\). This implies that at most \(s\) of the trees \(K_n\) are end-pieces, i.e., have only one marked point. Indeed, if \(K_n\) an end-piece, then it is decomposed into branches by use of a branch point \(c \in K_n \setminus \partial K_n\) with the largest weight \(M(c)\). The point \(c\) is then a marked point in each of the children of \(K_n\) and in particular in \(K_{n+1}\). Since the branch point \(x \in K_n\) is distinct
from the marked points of $K_n$ and $K_{n+1}$, we have $x \in K_n \setminus \partial K_n$ and $x \neq c$. So $x$ was not chosen to decompose $K_n$, and we must have $M(c) \geq M(x)$. Since the branch points $c$ that appear from end-pieces at different levels $n$ are all distinct as follows from Lemma 3.6, because $(n)$ at different levels can have at most $s$ end-pieces in the sequence $K_n$, $n \in \mathbb{N}$. This implies that there exists $N \in \mathbb{N}$ such that $K_n$ for $n \geq N$ is an arc-piece, and so has precisely two marked points.

Let $a, b \in K_N$ with $a \neq b$ be the marked points of $K_N$. As we travel from $x$ along $[x, a] \subseteq K_N$ towards $a$, there is a first point $x'$ on $[a, b]$. Then $x' \neq a$. Otherwise, $x' = a$. Then $[x, a]$ and $[a, b]$ have only the point $a$ in common, which implies that $[x, a] \cup [a, b]$ is an arc equal to $[x, b]$. Then $a \in (x, b)$, which by Lemma 3.2 (iii) implies that $x, b \in K_N$ lie in different components of $K_N \{a\}$. This contradicts the fact that $a$ is a leaf of $K_N$ and so $K_N \{a\}$ has only one component. Similarly, one can show that $x' \neq b$.

The point $x'$ is a branch point of $T$. This is clear if $x' = x$. If $x' \neq x$, this follows from Lemma 3.6 because $a, b, x \neq x'$ and the arcs $[a, x')$, $[b, x')$, $[x, x')$ are pairwise disjoint.

The tree $K_{N+1}$ is a branch of $K_N$ obtained from a branch point $c \in (a, b)$ of $T$ with largest weight $M(c)$ among all branch points on $(a, b)$. We have $x' \neq c$. Otherwise, $x' = c$. Then $x \neq x'$, because $x' = c$ is a marked point of $K_{N+1}$ and $x$ is distinct from all the marked points in any of the sets $K_n$. This implies that the points $a, b, x$ lie in different components of $K_N \{x'\}$ and hence in different branches of $x'$ in $K_N$. Since $a$ and $b$ are the marked points of $K_N$, the branches containing $a$ and $b$ are arc-pieces and all other branches of $x' = c$ in $K_N$ are end-pieces. The unique branch of $x'$ in $K_N$ containing $x$, which is equal to $K_{N+1}$, must be an end-piece by our choice of the decomposition of $T$. This is impossible by our choice of $N$ and so indeed $x' \neq c$. Note that this implies $M(c) \geq M(x')$.

Since $x' \neq c$, $c \in (a, b)$, and $[x, x') \cap [a, b] = \emptyset$, we have $[x, x') \subseteq K_N \{c\}$. So $x'$ lies in the same branch of $c$ in $K_N$ as $x$, which in $K_{N+1}$. Moreover, depending on whether $c \in (a, b)$ lies on the left or right of $x' \in (a, b)$, we have $x' \in (a, c)$ or $x' \in (c, b)$. In the first case, $[a, c] \subseteq K_{N+1}$ and $a$ and $c$ are the marked points of $K_{N+1}$. In the second case, $[c, b] \subseteq K_{N+1}$ and $c$ and $b$ are the marked points of $K_{N+1}$. So in both cases, if $a'$ and $b'$ are the marked points of $K_{N+1}$, then $x' \in (a', b')$, $[x, x') \subseteq K_{N+1}$, and $[x, x') \cap [a', b'] = \emptyset$.

These facts allow us to repeat the argument for $K_{N+1}$ instead of $K_N$. Again $K_{N+1}$ is decomposed into branches by choice of a branch point $c' \in (a', b')$. We must have $c' \neq x'$, because otherwise we again obtain a contradiction to the fact that $K_{N+2}$ is not an end-piece. This
implies that $M(c') \geq M(x')$. Continuing in this manner, we obtain an infinite sequence of branch points $c, c', \ldots$. By construction these branch points are all distinct and have a weight $\geq M(x')$. This is impossible by Lemma 3.9. We obtain a contradiction that establishes the statement.

The previous argument shows that each branch point $x$ of $T$ will eventually be chosen as a branch point in the decomposition of $T$ into the subtrees $T_u, u \in W_*$. Indeed, otherwise $x$ is distinct from all the marked points of any of the subtrees $T_u, u \in W_*$. This in turn implies that there exists a unique infinite word $w = w_1w_2\ldots \in W$ such that $x \in K_n := T_{w_1\ldots w_n}$ for $n \in \mathbb{N}$. From this one obtains a contradiction as in the last part of the proof of Lemma 5.2.

**Lemma 5.3.** Let $m \in \mathbb{N} \cup \{\infty\}, m \geq 3$, and $T, S \in T_m$. Suppose subtrees $T_u$ of $T$ and $S_u$ of $S$ with signed marked points have been defined for $u \in W_*$ by the procedure described above. Then the following statements are true:

(i) Let $n \in \mathbb{N}$, $u \in W_n$, and $v \in W_{n+1}$. Then $T_v \subseteq T_u$ if and only if $S_v \subseteq S_u$.

(ii) For $n \in \mathbb{N}$ and $u, v \in W_n$ with $u \neq v$ we have $T_u \cap T_v \neq \emptyset$ if and only if $S_u \cap S_v \neq \emptyset$. Moreover, if these intersections are non-empty, then they are singleton sets, say $\{p\} = T_u \cap T_v$ and $\{\tilde{p}\} = S_u \cap S_v$. The point $p$ is a signed marked point in $T_u$ and $T_v$, the point $\tilde{p}$ is a signed marked point in $S_u$ and $S_v$, $\text{sgn}(p, T_u) = \text{sgn}(\tilde{p}, S_u)$, and $\text{sgn}(p, T_v) = \text{sgn}(\tilde{p}, S_v)$.

In (ii) we are actually only interested in the statement that $T_u \cap T_v \neq \emptyset$ if and only if $S_u \cap S_v \neq \emptyset$. The additional claim in (ii) will help us to prove this statement by an inductive argument.

**Proof.** (i) This follows from Lemma 5.1 (iii) applied to the decompositions of $T$ and $S$. Indeed, we have $T_v \subseteq T_u$ if and only if $v = uk$ for some $k \in A$ if and only if $S_v \subseteq S_u$.

(ii) We prove this by induction on $n \in \mathbb{N}$. The case $n = 1$ is clear by how the decompositions were chosen.

Suppose the claim is true for words of length $n - 1$, where $n \geq 2$. Now consider two words $u, v \in W_n$ of length $n$ with $u \neq v$. Then $u = u'k$ and $v = v'l \in W_n$, where $u', v' \in W_{n-1}$ and $k, l \in A$. Since the claim is symmetric in $T$ and $S$, we may assume that $T_u \cap T_v \neq \emptyset$.

If $u' = v'$, then $T_u$ and $T_v$ are two of the branches obtained from $T_{u'}$ and a branch point $c \in T_{v'}$. In this case, $T_u \cap T_v = \{c\}$ and $c$ is a marked point in both $T_u$ and $T_v$. Similarly, $S_u$ and $S_v$ are two of
the branches obtained from \( S_u \) and a branch point \( \tilde{c} \in S_{u'} \). We have \( S_u \cap S_v = \{ \tilde{c} \} \) and \( \tilde{c} \) is a marked point in both \( S_u \) and \( S_v \). Moreover, \( c \) has the same sign in \( T_u \) as \( \tilde{c} \) in \( S_u \). Indeed, by the choice of labeling in the decomposition, this sign is + if \( k = 1 \) and − otherwise. Similarly, \( c \) has the same sign in \( T_v \) as \( \tilde{c} \) in \( S_v \).

In the other case, \( u' \neq v' \). Then \( T_{u'} \cap T_{v'} \neq \emptyset \), because \( T_u \cap T_v \neq \emptyset \), \( T_u \subseteq T_{u'} \), and \( T_v \subseteq T_{v'} \). Then by induction hypothesis, \( T_{u'} \cap T_{v'} \) consists of precisely one point \( p \) that is both a marked point in \( T_{u'} \) and \( T_{v'} \). The set \( S_{u'} \cap S_{v'} \) consists of one point \( \tilde{p} \) that is a marked point in \( S_{u'} \) and \( S_{v'} \). Moreover, we have \( \text{sgn}(p, T_{u'}) = \text{sgn}(\tilde{p}, S_{u'}) \) and \( \text{sgn}(p, T_{v'}) = \text{sgn}(\tilde{p}, S_{v'}) \). Since \( \emptyset \neq T_u \cap T_v \subseteq T_{u'} \cap T_{v'} = \{ p \} \), we then have \( T_u \cap T_v = \{ p \} \).

If \( \text{sgn}(p, T_{u'}) = \text{sgn}(\tilde{p}, S_{u'}) = - \), then \( u = u'1 \), because \( p \in T_u \). Hence \( \tilde{p} \in S_{u'1} = S_u \), because the marked point \( \tilde{p} \) of \( S_{u'} \) with \( \text{sgn}(\tilde{p}, S_{u'}) = - \) is passed to the child \( S_{u'1} \). If \( \text{sgn}(p, T_{u'}) = \text{sgn}(\tilde{p}, S_{u'}) = + \), then \( u = u'2 \) and \( \tilde{p} \in S_{u'2} = S_u \).

Similarly, if \( \text{sgn}(p, T_{v'}) = \text{sgn}(\tilde{p}, S_{v'}) = - \), then \( v = v'1 \) and if \( \text{sgn}(p, T_{v'}) = \text{sgn}(p, S_{v'}) = + \), then \( v = v'2 \), because \( p \in T_v \). In both cases, \( \tilde{p} \in S_u \).

In each of these cases, \( p \) is a marked point in \( T_u \) and \( T_v \), and \( \tilde{p} \) is a marked point in \( S_u \) and \( S_v \). In particular, \( \{ \tilde{p} \} \subseteq S_u \cap S_v \subseteq S_{u'} \cap S_{v'} = \{ \tilde{p} \} \) and so \( S_u \cap S_v = \{ \tilde{p} \} \). So both \( T_u \cap T_v = \{ p \} \) and \( S_u \cap S_v = \{ \tilde{p} \} \) are singleton sets consisting of marked points as claimed. Since signed marked points are passed to children with the same sign, we have

\[
\text{sgn}(p, T_u) = \text{sgn}(p, T_{u'}) = \text{sgn}(\tilde{p}, S_{u'}) = \text{sgn}(\tilde{p}, S_u).
\]

Similarly, we conclude that \( \text{sgn}(p, T_v) = \text{sgn}(\tilde{p}, S_v) \). The statement follows. \( \square \)

We are now ready to prove Theorem 1.4 in the finite case \( m \in \mathbb{N} \), \( m \geq 3 \). We formulate this as a separate statement.

**Proposition 5.4.** Let \( m \in \mathbb{N} \) with \( m \geq 3 \). Then all trees in \( T_m \) are homeomorphic to each other.

**Proof.** Let \( m \) be as in the statement, and \( S, T \in T_m \) be arbitrary. For each \( n \in \mathbb{N} \) we consider the decompositions \( T = \bigcup_{u \in \mathcal{W}_n} T_u \) and \( S = \bigcup_{u \in \mathcal{W}_n} S_u \) as defined earlier in this section. Here of course, \( \mathcal{W}_n = \mathcal{W}_n(\mathcal{A}) \), where \( \mathcal{A} = \{ 1, 2, \ldots, m \} \).

We want to show that decompositions of \( T \) and \( S \) for different levels \( n \in \mathbb{N} \) have the properties in Proposition 2.1. In this proposition the index \( i \) for fixed level \( n \) corresponds to the words \( u \in \mathcal{W}_n \).

The spaces \( T \) and \( S \) are trees and hence compact. The sets \( T_u \) and \( S_u \) appearing in their decompositions are subtrees and hence non-empty.
and compact. Conditions (i), (ii), and (iii) in Proposition 2.1 follow from Lemma 5.1 (iii), (5.1), and Lemma 5.2, respectively. Finally, (2.1) and (2.2) follow from Lemma 5.3 (i) and (ii).

Proposition 2.1 implies $T$ and $S$ are homeomorphic as desired. $\square$

Theorem 1.2 is an immediate consequence.

Proof of Theorem 1.2. As we have seen in Section 4, the CSST $T$ is a metric tree with the properties (i) and (ii) as in the statement (see Proposition 4.8 and Proposition 4.11). In particular, $T \in \mathcal{Z}_3$. Since these properties (i) and (ii) are invariant under homeomorphisms, every metric tree $T$ homeomorphic to $T$ has these properties.

Conversely, suppose that $T$ is a metric tree with the properties (i) and (ii). In other words, $T \in \mathcal{T}_3$. Then $T$ and $T$ are homeomorphic by Proposition 5.4 applied to $m = 3$. $\square$

6. Trees with branch points of infinite valence

In this section, we will complete the proof of Theorem 1.4 by dealing with the remaining case $m = \infty$. Again we formulate this as a separate statement.

Proposition 6.1. All trees in $\mathcal{T}_\infty$ are homeomorphic to each other.

For the proof of this proposition we need a version of Proposition 2.1 adapted to decompositions of the underlying spaces with infinitely many pieces.

Proposition 6.2. Let $(X,d_X)$ and $(Y,d_Y)$ be compact, connected, and locally connected metric spaces. Suppose that for each $n \in \mathbb{N}$ the space $X$ admits a decomposition $X = \bigcup_{i \in M_n} X_{n,i}$ as a union of non-empty, compact, and connected subsets $X_{n,i}$, $i \in M_n \subseteq \mathbb{N}$, satisfying the following conditions for all $n$, $i$, and $j$:

(i) Each set $X_{n+1,j}$ is a subset of some set $X_{n,i}$.
(ii) Each set $X_{n,i}$ is equal to the union of some of the sets $X_{n+1,j}$.
(iii) $\sup_{i \in M_n} \text{diam}(X_{n,i}) \to 0$ as $n \to \infty$.
(iv) If $\alpha \subseteq X$ is an arc and $n \in \mathbb{N}$, then there exists a finite set $J \subseteq M_n$ such that $\alpha \subseteq \{X_{n,i} : i \in J\}$.
(v) If $a \in X$ and $U$ is a component of $X \setminus \{a\}$, then $U$ is arc-connected and for each $n \in \mathbb{N}$ there exists $\delta > 0$ such that

$U \cap B_X(a, \delta) \subseteq \bigcup \{X_{n,i} : i \in M_n \text{ and } a \in X_{n,i}\}$. 
Suppose that for each \( n \in \mathbb{N} \) the space \( Y \) admits a decomposition \( Y = \bigcup_{i \in M_n} Y_{n,i} \) as a union of non-empty, compact, and connected subsets \( Y_{n,i}, i \in M_n \), with properties analogous to (i)--(vi) such that (2.1) and (2.2) are true.

Then there exists a unique homeomorphism \( f: X \rightarrow Y \) such that \( f(X_{n,i}) = Y_{n,i} \) for all \( n \) and \( i \).

In the proof of Proposition 6.1 we will apply this for the index sets \( M_n = \mathbb{N} \), but for the proof of Theorem 1.5 we need this more general version that allows both finite and countably infinite index sets \( M_n \).

Proof. The first part of the proof is essentially the same as the proof of Proposition 2.1. We can again define a map \( f: X \rightarrow Y \) such that if \( x \in X \) and we have a nested sequence of sets \( X_{n,i} \) with \( \{x\} = \bigcap_n X_{n,i} \), then \( \{f(x)\} = \bigcap_n Y_{n,i} \). A inverse map \( g: Y \rightarrow X \) of \( Y \) is defined in a similar way. Then \( f \) is a bijection that satisfies \( f(X_{n,i}) = Y_{n,i} \) for all \( n, i \in \mathbb{N} \). It is uniquely determined by the last requirement.

In order to prove that \( f \) is a homeomorphism, we have to show that \( f \) and \( g = f^{-1} \) are continuous. By symmetry, it suffices again to show that \( f \) is continuous.

Consider an arbitrary point \( a \in X \), and let \( b = f(a) \). We first establish two claims.

Claim 1. If \( U \) is a component of \( X \setminus \{a\} \), then there exists a unique component \( V \) of \( Y \setminus \{b\} \) such that \( f(U) \subseteq V \).

To see this, pick \( x_0 \in U \). Then \( y_0 := f(x_0) \neq b \), because \( f: X \rightarrow Y \) is a bijection. Hence there exists a unique component \( V \) of \( Y \setminus \{b\} \) with \( y_0 \in V \). We want to show that \( f(U) \subseteq V \). For this let \( x \in U \) be arbitrary. Since \( U \) is arc-connected by (v), there exists a possibly degenerate arc \( \alpha \) in \( U \) joining \( x \) and \( x_0 \). Then \( \text{dist}(a, \alpha) > 0 \). Hence by (iii) we can choose \( n \in \mathbb{N} \) such that

\[
\text{diam}(X_{n,i}) < \text{dist}(a, \alpha)
\]

for all \( i \in \mathbb{N} \). By (iv), for this \( n \) we can find finitely many sets \( X_{n,i} \) that cover \( \alpha \). By a standard argument, we can extract from this finite set a chain joining \( x \) and \( x_0 \). More precisely, we can find \( k \in \mathbb{N} \) and \( i_0, \ldots, i_k \in \mathbb{N} \) such that each \( X_{n,i} \) meets \( \alpha \), \( x \in X_{n,i_0} \), \( x_0 \in X_{n,i_k} \), and \( X_{n,i_{l-1}} \cap X_{n,i_l} \neq \emptyset \) for \( l = 1, \ldots, k \). Then each set \( X_{n,i} \) is contained in \( X \setminus \{a\} \) by (6.1). Hence \( Y_{n,i_l} \subseteq Y \setminus \{b\} \) for \( l = 0, \ldots, k \). We also have \( Y_{n,i_{l-1}} \cap Y_{n,i_l} \neq \emptyset \) for \( l = 1, \ldots, k \). Since each of the sets \( Y_{n,i} \) is connected, it follows that

\[
M := Y_{n,i_0} \cup \cdots \cup Y_{n,i_k}
\]
is connected. The set $M$ is contained in $Y \setminus \{b\}$ and contains

$$y_0 = f(x_0) \in f(X_{n,i_0}) = Y_{n,i_0} \subseteq M.$$ 

Hence $M \subseteq V$ and so

$$f(x) \in f(X_{n,i_k}) = Y_{n,i_k} \subseteq M \subseteq V.$$ 

This shows that $f(U) \subseteq V$ as desired.

**Claim 2.** If $U$ and $U'$ are distinct components of $X \setminus \{a\}$, and $V$ and $V'$ are the components of $Y \setminus \{b\}$ such that $f(U) \subseteq V$ and $f(U') \subseteq V'$, then $V \neq V'$.

Note that $V$ and $V'$ exist by Claim 1. Moreover, Claim 1 is also true for $g = f^{-1}$. Hence $f^{-1}(V)$ is contained in a unique component of $X \setminus \{a\}$. This component meets $U \subseteq f^{-1}(V)$ and so is identical to $U$. Hence $f^{-1}(V) \subseteq U$. Similarly, $f^{-1}(V') \subseteq U'$. Since $U \neq U'$, this is only possible if $V \neq V'$.

Note that Claims 1 and 2 applied to $f$ and $g = f^{-1}$ imply that $f$ induces a bijection between the components of $X \setminus \{a\}$ and $Y \setminus \{b\}$.

We are now ready to prove that $f$ is continuous at $a$. Let $\epsilon > 0$ be arbitrary. Then by Lemma 3.8, there are only finitely many components $V$ of $Y \setminus \{b\}$ such that $\text{diam}(V) \geq \epsilon$. Let $V_1, \ldots, V_k$ be these components, where $k \in \mathbb{N}_0$ (so this may be an empty list). By what we have seen, there are finitely many components $U_1, \ldots, U_k$ of $X \setminus \{a\}$ such that $f(U_l) \subseteq V_l$ for $l = 1, \ldots, k$. If $U$ is any component of $X \setminus \{a\}$ distinct from $U_1, \ldots, U_k$, then $U$ is mapped by $f$ into a component $V$ of $Y \setminus \{b\}$ with $\text{diam}(V) < \epsilon$.

By condition (iii) (for the space $Y$) we can choose $N \in \mathbb{N}$ such that $\text{diam}(Y_{N,i}) < \epsilon$ for all $i \in \mathbb{N}$. Then (v) implies that we can find $\delta > 0$ such that

$$U_1 \cup \cdots \cup U_k \cap B(a, \delta) \subseteq \bigcup \{X_{N,i} : i \in \mathbb{N} \text{ and } a \in X_{N,i}\}.$$ 

Now let $x \in X$ with $d_X(a, x) < \delta$ be arbitrary. We may assume that $x \neq a$. Then $x$ lies in one of the components $U$ of $X \setminus \{a\}$.

If $U \neq U_1, \ldots, U_k$, then $f(U)$ lies in a component $V$ of $Y \setminus \{b\}$ with $\text{diam}(V) < \epsilon$. Since $Y$ is locally connected, $V$ is an open subset of $Y$. Since $V$ is a component of $Y \setminus \{b\}$, it is also relatively closed in $Y \setminus \{b\}$ (see the discussion after Lemma 4.7). This implies $b \in \overline{V}$, because otherwise $V = \overline{V} \cap (Y \setminus \{b\}) = \overline{V}$ and so $V$ would be a non-empty, open, and closed subset of $Y$ with $V \neq Y$. This is impossible, because $Y$ is connected. So $b \in \overline{V}$ and we see that

$$d_Y(f(a), f(x)) = d_Y(b, f(x)) \leq \text{diam}(\overline{V}) = \text{diam}(V) < \epsilon.$$
Suppose \( U \) is equal to one of the components \( U_1, \ldots, U_k \). We may assume that \( U = U_1 \) without loss of generality. Then by (6.2) there exists \( i \in \mathbb{N} \) such that \( a, x \in X_{N,i} \). Then \( b, f(x) \in Y_{N,i} \) and so
\[
d_Y(f(a), f(x)) \leq \text{diam}(Y_{N,i}) < \epsilon.
\]
Hence if \( x \in X \) is arbitrary, then the inequality \( d_X(a, x) < \delta \) always implies \( d_X(f(a), f(x)) < \epsilon \). It follows that \( f \) is continuous at \( a \) as desired. \( \square \)

Now suppose \( T \) is a tree in \( T_\infty \). We consider the decompositions
\[
T = \bigcup_{u \in W_n} T_u
\]
for \( n \in \mathbb{N} \) as defined in Section 5. Here \( W_n = W_n(A) \) with \( A = \mathbb{N} \).

Lemma 6.3. The decompositions in (6.3) satisfy the conditions (i)–(v) in Proposition 6.2.

Here for given \( n \in \mathbb{N} \), the index set \( M_n \subseteq \mathbb{N} \) in Proposition 6.2 corresponds to the countable set \( W_n \).

Proof. The given tree \( T \in T_\infty \) is compact, connected, and locally connected. The sets \( T_u, u \in W_n \), are subtrees of \( T \) and hence non-empty, compact, and connected. Conditions (i), (ii), and (iii) in Proposition 6.2 follow from Lemma 5.1 (iii), (5.1), and Lemma 5.2, respectively.

Condition (iv) is proved by induction on \( n \). Let \( \alpha \subseteq T \) be an arc. It is convenient to start the induction at \( n = 0 \), where \( T_\emptyset = T \) with an empty set of marked points. Then the statement is trivially true for \( n = 0 \), because \( \alpha \subseteq T = T_\emptyset \).

Suppose that for some \( n \in \mathbb{N}_0 \), there exist finitely many words \( u_1, \ldots, u_k \in W_n \) such that \( \alpha \subseteq T_{u_1} \cup \cdots \cup T_{u_k} \). Let \( F \) be the finite set consisting of all marked points of the trees \( T_u \) and all the branch points that were used in decomposition of \( T_{u_i} \) into children. Then \( \alpha \setminus F \) consisting of finitely many, non-degenerate, and relatively open sub-arcs of \( \alpha \). Let \( \beta \subseteq \alpha \) be one of these subarcs. Then \( \beta \) meets one of sets \( T_{u_i} \). To be specific, let us assume that \( \beta \cap T_{u_1} \neq \emptyset \). Since \( \beta \) does not meet \( F \), it is disjoint from the set \( \partial T_{u_1} \) of marked points of \( T_{u_1} \). This implies that \( \beta \subseteq T_{u_1} \). Let \( c \in T_{u_1} \) be the branch point used in the decomposition of \( T_{u_1} \) into its children \( T_{u_1i}, i \in \mathbb{N} \). Then \( c \notin \beta \) by choice of \( F \). Since the sets \( T_{u_1i \setminus \{c\}}, i \in \mathbb{N} \), form the connected components of \( T_{u_1 \setminus \{c\}} \), one of these components \( T_{u_1i \setminus \{c\}} \) must meet the non-degenerate arc \( \beta \). Since \( c \notin \beta \), Lemma 3.2 (iii) implies that \( \beta \subseteq T_{u_1i \setminus \{c\}} \) and so \( \beta \subseteq T_{u_1i} \). Since the closures of these finitely many
arcs $\beta$ cover $\alpha$, we see that $\alpha$ is contained in the union of finitely many sets $T_v$ with $v \in W_{n+1}$. Condition (iv) follows.

To see that condition (v) is true, let $a \in T$ and $U$ be a component of $T\setminus\{a\}$. By Lemma 3.2 (i) we know that $U$ is arc-connected.

To prove the second part of the condition, we claim a stronger statement. Namely, that for each $n \in \mathbb{N}_0$ there exist $\delta > 0$ and $u \in W_n$ such that $a \in T_u$ and $U \cap B_X(a, \delta) \subseteq T_u$.

To see this, we use again induction on $n \in \mathbb{N}_0$. The induction beginning is trivial, because $U \subseteq T = T_0$. Suppose the statement is true for some $n \in \mathbb{N}_0$. Then there exist $u \in W_n$ and $\delta' > 0$ such that $a \in T_u$ and $U \cap B_X(a, \delta') \subseteq T_u$. Let $c \in T_u$ be the branch point of $T$ used in the decomposition of $T_u$ into children. We now consider two cases.

If $a \neq c$, then $a \in T_u$ is contained in a unique child $T_v$ of $T_u$, where $v = uk$ with $k \in \mathbb{N}$. Then by Lemma 3.1 we can choose $\delta > 0$ so small that for all $x \in B_X(a, \delta)$ the arc $[x, a]$ stays in $B_X(a, \delta')$ and does not meet $c$.

We claim that $U \cap B_X(a, \delta) \subseteq T_v$. To see this, let $x \in U \cap B_X(a, \delta)$ be arbitrary. By choice of $\delta$, we then have $[x, a] \subseteq B_X(a, \delta')$. The set $[x, a]$ is a connected set in $T\setminus\{a\}$. It meets $U$ in the point $x$ if $x \neq a$. In any case, $[x, a] \subseteq U$, and so $[x, a] \subseteq U \cap B_X(a, \delta')$. This implies that $[x, a] \subseteq T_u$ by choice of $\delta'$. Moreover, $c \notin [x, a]$ by choice of $\delta$, and so $[x, a]$ is a connected set contained in $T_u \setminus\{c\}$ that meets $T_v$. It follows that $[x, a] \subseteq T_v$. In particular, $x \in T_v$ as desired.

The other case is when $a = c$. Since $c$ is an interior point of $T_u$ (in $T$), we may assume that $\delta' > 0$ is so small that $B_X(a, \delta') \subseteq T_u$. Then $a = c \in \overline{U}$ by Lemma 3.2 (ii) and so we can find a point $y \neq c$ close to $c$ such that $y \in U \cap T_u$. Then $y$ is contained in a unique child $T_v$ of $T_u$, where $v = uk$ with $k \in \mathbb{N}$. Again by Lemma 3.1 we can choose $\delta > 0$ such for any pair of points in $B_X(a, \delta)$ the unique arc joining these two points stays in $B_X(a, \delta') \subseteq T_u$. We claim that $U \cap B_X(a, \delta) \subseteq T_v$. To see this, let $x \in U \cap B_X(a, \delta)$ be arbitrary. Since $x$ and $y$ lie in the same component $U$ of $T\setminus\{a\}$, we have $[x, y] \subseteq U$ by Lemma 3.2 (iii). Moreover, by choice of $\delta$ we have $[x, y] \subseteq U \cap B_X(a, \delta') \subseteq T_u$. Since $[x, y]$ does not meet $a = c$, this set must be contained in unique child of $T_u$. Since $y \in T_v$, we have $[x, y] \subseteq T_v$, and so $x \in T_v$ as desired. Condition (v) follows and the proof is complete.

We can now prove the result stated in the beginning of this section.

Proof of Proposition 6.1. Let $T$ and $S$ be two trees in $\mathcal{T}_\infty$. We use the procedure described in Section 5 to decompose $T$ as in (6.3), and use
the same procedure for $S$ to find decompositions
\[ S = \bigcup_{u \in W_n} S_u \]
for $n \in \mathbb{N}$. By Lemma 6.3 applied to $T$ and $S$ the conditions (i)–(v) in Propositions 6.2 are true for these families of decompositions of $T$ and $S$. By Lemma 5.3 (i) and (ii) conditions (2.1) and (2.2) are also true. So Proposition 6.2 implies that $T$ and $S$ are homeomorphic as desired. □

Proof of Theorem 1.4. The statement immediately follows from Propositions 5.4 and 6.1. □

7. Trees with branch points of varying valence

In this section we will prove Theorem 1.5. We will first briefly explain how to construct trees as in this theorem. Let $s \in \mathbb{N}$ and $m_1 < \cdots < m_s$ in $\{3, 4, \ldots\} \cup \{\infty\}$ be fixed. We define $D = s + 1$ and start with the unit line segment $J_0 = [0, 1]$. We partition $J_0 = [0, 1]$ by using $s = D - 1$ equally spaced points $0 < p_1 < \cdots < p_s < 1$. In this way $[0, 1]$ is subdivided into $D$ line segments of equal length $1/D$. To each point $p_i$ with $i < s$ we glue one of the endpoints of $s_i - 2$ additional line segments of length $1/D$. If $m_s < \infty$, we do the same for the point $p_s$. If $m_s = \infty$, we glue infinitely line segments to $p_s$ of length $D^{-1}, D^{-2}, \ldots$. In this way, we obtain a space $J_1$ that carries a natural decomposition as a union of line segments $I$. For each $I$ we repeat the same process as for $[0, 1]$, where we scale all line segments by the length of $I$. This process produces an ascending sequence $J_0 \subseteq J_1 \subseteq \ldots$ of spaces $J_n$. Their union $J = \bigcup_{n \in \mathbb{N}_0} J_n$ carries a natural path metric $\varrho$. Then the completion $T$ of the metric space $(J, \varrho)$ is a tree in $T_{m_1, \ldots, m_s}$.

Proof of Theorem 1.5. Let $s \in \mathbb{N}$ and $m_1 < \cdots < m_s$ in $\{3, 4, \ldots\} \cup \{\infty\}$ be fixed. If $T \in T_{m_1, \ldots, m_s}$ then for each $i \in \{1, \ldots, s\}$ the branch points $x \in T$ with $\nu_T(x) = m_i$ are dense in $T$. To see this, consider an arbitrary non-empty open set $U \subseteq T$. Let $a \in U$. Then $a$ is not an isolated point of $T$, and so by Lemma 3.1 there exists a point $b \in U$ with $b \neq a$ such that the (non-degenerate) arc $\alpha := [a, b]$ is contained in $U$. Since $T \in T_{m_1, \ldots, m_s}$ there exists a branch point $x \in \alpha \subseteq U$ with $\nu_T(x) = m_i$. This shows that these branch points are indeed dense in $T$. Note that an immediate consequence of these considerations and Lemma 3.7 is that $T_m = T_m^*$ for each $m \in \mathbb{N} \cup \{\infty\}$, $m \geq 3$.

In order to show that all trees in $T_{m_1, \ldots, m_s}^*$ are homeomorphic, let $S, T \in T_{m_1, \ldots, m_s}^*$ be arbitrary. We will decompose the trees $S$ and $T$ into subtrees by the procedure described in Section 5. These subtrees
are labeled by words of finite length in the infinite alphabet $A = \mathbb{N}$, but not all finite words will appear as labels of subtrees. We will call the finite words $u \in W_s = W_s(A)$ that appear as such labels admissible words. They will be defined by an inductive process based on their length $n$. We will denote the set of admissible words of length $n$ by $\tilde{W}_n \subseteq W_n = W_n(A)$.

To decompose $T$ into subtrees of level $n = 1$, we choose a branch point $c \in T$ of largest weight $M(c)$. Let $\nu = \nu_T(c) \in \{3, 4, \ldots\} \cup \{\infty\}$. As in Section 5, we obtain subtrees $T_k$ for $k = 1, \ldots, \nu$ if $\nu < \infty$ and for $k \in \mathbb{N}$ if $\nu = \infty$ corresponding to the countably many branches of $c$. Each of these subtrees $T_k$ carries the signed marked point $c$. We declare the admissible words of length 1 to be precisely 1 for $k \in \mathbb{N}$ if $\nu = \infty$ corresponding to the labels of the trees $T_k$ in this decomposition.

We continue this process inductively as in Section 5. Let $n \in \mathbb{N}$ and $u \in \tilde{W}_n$ be an admissible word with a corresponding subtree $T_u$ of $T$ with signed marked points. Then we choose a branch point $c \in T_u \setminus \partial T_u$ of $T$ of highest weight $M(c)$ and the additional requirement that $c \in (a, b)$ if $T_u$ has two distinct marked points $a$ and $b$ (and so $T_u$ is an arc-piece). If $\nu_T(c) = \nu$, then we obtain branches $T_{u1}, \ldots, T_{u\nu}$ of $c$ in $T_u$ if $\nu < \infty$ and branches $T_{uk}$ for all $k \in \mathbb{N}$ if $\nu = \infty$. Accordingly, we declare a word $uk$ with the initial segment $u \in \tilde{W}_u$ and $k \in \mathbb{N}$ as admissible if it correspond to a label of a child of $T_u$. Continuing in this way, we obtain sets $\tilde{W}_n$ of admissible words $u$ of all lengths $n \in \mathbb{N}$ and corresponding subtrees $T_u$ of $T$ with signed marked points.

Now we create subtrees $S_u$ for each each admissible word $u \in \tilde{W}_n$ by an inductive procedure on $n$ such that $S_u$ and $T_u$ are of the same type, i.e., they are both arc-pieces or both end-pieces. For $n = 1$ we choose a branch point $c'$ of $S$ of the same valence as the branch point $c$ used to decompose $T$ into subtrees $T_k$ with $k \in \tilde{W}_1$. Such a branch point $c'$ in $S$ exists, because branch points in $S$ with $\nu_S(c') = \nu_T(c) \in \{m_1, \ldots, m_s\}$ are dense in $S$ by what we have seen above. Then we obtain subtrees $S_u$ with signed marked points for each $u \in \tilde{W}_1$ by the procedure described in Section 5. Note that $T_k$ and $S_k$ for $k \in \tilde{W}_1$ are of the same type (actually, they are all end-pieces). While $c$ is the branch point with the highest weight $M(c)$ in $T$, a corresponding statement is not necessarily true for the branch point $c'$ in $S$.

Suppose that for some $n \in \mathbb{N}$ and some $u \in \tilde{W}_n$ a subtree $S_u$ with signed marked point has been defined that is of the same type as $T_u$. Let $c$ be the branch point of $T_u$ used to decompose $T_u$ into children. Then our assumptions imply that we can choose a branch point $c' \in S_u \setminus \partial S_u$
such that $\nu_S(c') = \nu_T(c)$. Moreover, we can require in addition that $c'$ lies on the open arc $(a,b)$ joining the two distinct marked points $a, b \in \partial S_u$ if $S_u$ and $T_u$ are arc-pieces (corresponding to the similar requirement for $c$ in $T_u$). Then we assign labels and signed marked points to the children of $S_u$ as described in Section 5. Then it is obvious that the children of $T_u$ and $S_u$ with the same label $uk \in \tilde{W}_{n+1}$ are of the same type. Continuing in this manner, we obtain subtrees $S_u$ of $S$ of the same type as $T_u$ for all $u \in \tilde{W}_*$. 

It is clear that we have analogs of Lemma 5.1 for the decompositions of $T$ and $S$ obtained at each level $n$. Moreover, the proof of Lemma 5.2 goes through for the tree $T$ with inessential changes and so

$$\lim_{n \to \infty} \sup \{ \text{diam}(T_u) : u \in \tilde{W}_n \} = 0.$$ 

One cannot expect a similar statement for the subtrees $S_u$ of $S$, because in the decomposition of $S$ into subtrees we matched valences of the branch points of $S$ with the corresponding ones in $T$ and had no control over the weight of these branch points.

To address this issue, we modify the decomposition process and alternate between the trees $T$ and $S$ in the following way. Namely, by (7.1) there exists a level $N_1$ such that the subtrees of $T$ of level $N_1 \in \mathbb{N}$ are small, say

$$\text{diam}(T_u) < 1/2$$

for all $u \in \tilde{W}_{N_1}$. For levels $n \leq N_1$ we do not change anything, but for levels for levels $n > N_1$ we redefine the notion of an admissible word $u \in \tilde{W}_n$ and the corresponding subtrees $T_u$ and $S_u$. For these levels we choose branch points $c$ in $S$ of highest weights subject to the conditions specified in Section 5 and choose corresponding branch points $c'$ in $T$ with matching valences. Then

$$\lim_{n \to \infty} \sup \{ \text{diam}(S_u) : u \in \tilde{W}_n \} = 0,$$

because again the argument in the proof of Lemma 5.2 applies with minor changes. Here it is important that the levels $n \leq N_1$ essentially do not matter (indeed, we may assume that $N > N_1$ for the integer $N$ in the proof of Lemma 5.2). So there exists a level $N'_1 > N_1$ such

$$\text{diam}(S_u) < 1/2$$

for all $u \in W_{N'_1}$.

We now switch back to $T$. Namely, for levels $n \leq N_2$ we do not change anything, but we forget about the admissible words $u \in \tilde{W}_n$ and the corresponding subtrees $T_u$ and $S_u$ for levels $n > N_2$. For levels $n > N_2$ we redefine the notion of admissible words $u \in \tilde{W}_n$ and
the corresponding subtrees $T_u$ and $S_u$. For these levels we now again choose branch points $c$ in $T$ of highest weights subject to the conditions specified in Section 5 and choose corresponding branch points $c'$ in $S$ with matching valences. In this way we can find a level $N_2 > N'_1$ such that

$$\text{diam}(T_u) < 1/4$$

for all $u \in W_{N_2}$. Then we switch back to $S$, etc.

Continuing this process, we obtain a sequence $N_1 < N'_1 < N_2 < \ldots$. For each $n \in \mathbb{N}$ the notion of an admissible word $u$ of length $n$ and the corresponding subtrees $T_u$ and $S_u$ will eventually become fixed and for all $i \in \mathbb{N}$ we have $\text{diam}(T_u) < 1/2^i$ for $u \in W_{N_i}$ and $\text{diam}(S_u) < 1/2^i$ for $u \in W_{N'_i}$. This implies that

$$(7.3) \quad \lim_{n \to \infty} \sup \{ \text{diam}(T_u) : u \in \tilde{W}_n \} = \lim_{n \to \infty} \sup \{ \text{diam}(S_u) : u \in \tilde{W}_n \} = 0.$$ We have analogs of Lemma 5.3 and Lemma 6.3 for the decompositions $T = \bigcup_{u \in \tilde{W}_n} T_u$ and $S = \bigcup_{u \in \tilde{W}_n} S_u$, $n \in \mathbb{N}$.

The proofs apply with no essential changes. Of course, condition (iv) in Proposition 6.2 here follows from (7.3). Proposition 6.2 implies that $T$ and $S$ are homeomorphic as desired.

\begin{flushright}
\[\square\]
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Department of Mathematics, University of California, Los Angeles, CA 90095, USA

E-mail address: mbonk@math.ucla.edu

Institut für Mathematik, Technische Universität Berlin, Sekr. MA 7-1, Strasse des 17. Juni 136, 10623 Berlin, Germany

E-mail address: tran@math.tu-berlin.de