SHORT COMMUNICATIONS

Automorphism Groups of Moishezon Threefolds

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1. INTRODUCTION

The Jordan property plays an important role in the study of automorphism groups of algebraic varieties and complex manifolds. Following [1, Definition 2.1], we say that a group $\Gamma$ is \textit{Jordan} (or has the \textit{Jordan property}) if there is a constant $J = J(\Gamma)$ such that every finite subgroup $G \subset \Gamma$ contains a normal Abelian subgroup $A \subset G$ of index at most $J$.

The groups that enjoy the Jordan property include: the general linear groups $\text{GL}_n(k)$, where $k$ is a field of characteristic zero (see, e.g., [2, Theorem 36.13]); the groups of birational self-maps of rationally connected algebraic varieties (see [3, Theorem 1.8] and [4, Theorem 1.1]); the groups of birational self-maps of nonuniruled algebraic varieties (see [5, Theorem 1.8]); and many diffeomorphism groups of smooth compact real manifolds (see, e.g., [6], [7]). One of the most beautiful results concerning the Jordan property for groups of geometric origin is the following theorem due to Sh. Meng and D.-Q. Zhang.

\textbf{Theorem 1 ([8])}. Let $X$ be a projective variety over a field of characteristic zero. Then the group $\text{Aut}(X)$ is Jordan.

The paper [9] contains a generalization of Theorem 1 to the case of compact Kähler manifolds.

Note that, according to [10], there exist projective surfaces with non-Jordan groups of birational self-maps; furthermore, there exist smooth compact four-dimensional real manifolds with non-Jordan diffeomorphism groups (see [11]). There is a complete classification of two- and three-dimensional projective varieties with non-Jordan groups of birational self-maps over algebraically closed fields of characteristic zero; see [1, Theorem 2.32] and [12, Theorem 1.8]. Moreover, it is known that the birational automorphism group of any nonprojective compact complex surface is Jordan (see [13]), but in higher dimensions, there are no significant results in this direction yet. With this in mind, it is interesting to study automorphism groups of various classes of compact complex manifolds from the point of view of the Jordan property.

Recall that a compact complex space $X$ is said to be \textit{Moishezon} if the transcendence degree of its field of meromorphic functions on $X$ is maximal, that is, equals the dimension of $X$. Every proper algebraic variety is a Moishezon compact complex space. Every Moishezon compact complex space is birational to a projective variety (see [14, Theorem 1] or [15, Theorem 3N]). While every compact complex curve is projective, there exist two-dimensional Moishezon compact complex spaces that are not projective (see, e.g., [16, Example 7.6.26]). A smooth Moishezon compact complex space is called a \textit{Moishezon manifold}. For every Moishezon compact complex space, one can construct a resolution of singularities that is a Moishezon manifold (and even a smooth projective variety); see [15, Theorem 3M]. Every two-dimensional Moishezon manifold is projective (see [17, Corollary IV.6.5]).

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There are well-known examples of three-dimensional nonprojective Moishezon manifolds (see, e.g., [18, Sec. 3] and [16, Examples 7.6.20, 7.6.21]).

The purpose of this paper is to prove the following result, which is, to some extent, analogous to Theorem 1.

**Theorem 2.** Let $X$ be a three-dimensional Moishezon compact complex space. Then the group $\text{Aut}(X)$ is Jordan.

It would be interesting to find out whether there is a generalization of Theorem 2 to the case of Moishezon compact complex spaces of arbitrary dimension.

## 2. SOME PROJECTIVITY CRITERIA

In this section, we collect several assertions on the projectivity of certain Moishezon varieties.

**Definition.** A divisor $A$ is strongly numerically effective if $A \cdot C > 0$ for every curve $C$.

**Lemma 1.** Let $X$ be a Moishezon threefold, and let $A$ be a big strongly numerically effective divisor on $X$. Suppose that, for some $n > 0$, the linear system $|nA|$ has no fixed components. Then $A$ is ample. In particular, the manifold $X$ is projective.

**Proof.** This lemma follows from the Nakai–Moishezon ampleness criterion for Moishezon compact complex spaces (see [14, Theorem 6]). Indeed, by assumption, we have $A \cdot C > 0$ for every curve $C$. Furthermore, since the divisor $A$ is numerically effective and big, one has $A^3 > 0$.

Let $S \subset X$ be an irreducible surface (that is, a two-dimensional compact complex subspace). Then $S$ is a Moishezon compact complex space by [14, Theorem 3]. In particular, $S$ contains curves. The restriction $A_S = A|_S$ is an effective divisor on $S$, and one has $A_S \neq 0$, because $A_S$ has positive intersections with curves on $S$. Therefore, we see that $A^2 \cdot S > 0$. Hence $A$ is ample by the Nakai–Moishezon criterion.

**Lemma 2.** Let $g: \hat{F} \to F$ be a surjective morphism of smooth compact complex surfaces, and let $\hat{A}$ be an ample divisor on $\hat{F}$. Then the divisor $A = g_*\hat{A}$ is ample.

**Proof.** By the adjunction formula, we have $A \cdot C > 0$ for every curve $C$ on $F$, and also $A^2 > 0$. Hence $A$ is ample by the Nakai–Moishezon criterion.

**Remark 1.** Lemma 2 fails in the case where the surface $F$ is singular. However, it still holds if $F$ is a two-dimensional normal compact complex space with $\mathbb{Q}$-factorial singularities.

For a contraction $h: Y \to Z$, by $\text{Aut}(Y; h)$ we denote the subgroup of $\text{Aut}(Y)$ that consists of all automorphisms such that $h$ is equivariant with respect to their action, and by $\text{Aut}(Y; h)'$ we denote the maximal subgroup of $\text{Aut}(Y; h)$ that acts trivially on $H^*(Y, \mathbb{Z})$. Note that $\text{Aut}(Y; h)'$ is a normal subgroup of $\text{Aut}(Y; h)$ and the quotient $\text{Aut}(Y; h)/\text{Aut}(Y; h)'$ has bounded finite subgroups by a classical theorem of Minkowski. Therefore, the group $\text{Aut}(Y; h)$ has bounded finite subgroups if and only if so does $\text{Aut}(Y; h)'$.

**Lemma 3.** Let $Y$ be a Moishezon threefold, and let $h: Y \to Z$ be its contraction to a nonrational curve. Assume that the image of the homomorphism $\text{Aut}(Y; h)' \to \text{Aut}(Z)$ is infinite. Then the threefold $Y$ is projective.
Proof. Note that the fibers of \( h \) are smooth Moishezon surfaces (see [14, Theorem 3]) and are thus projective. Let \( g: \hat{Y} \to Y \) be a birational morphism such that \( \hat{Y} \) is projective; such a morphism always exists (see [15, Theorem 3.2]). Let \( \hat{A} \) be an ample divisor on \( \hat{Y} \). We set \( A = g_\ast \hat{A} \). Then the divisor \( A \) is big. Moreover, \( A \) is ample on the fibers of \( h \) by Lemma 2. By the projection formula, there are at most a \( \text{finite} \) number of curves \( C_i \subset Y \) such that \( A \cdot C_i \leq 0 \) (these curves must be contained in the image of the \( g \)-exceptional divisor).

Note that, for any \( \delta \in \text{Aut}(Y; h)' \), we have
\[
A \cdot \delta(C_i) = A \cdot C_i \leq 0.
\]
Hence the set \( \{C_i\} \) of all such curves is invariant under \( \text{Aut}(Y; h)' \). By our assumptions, \( Z \) is an elliptic curve and the image of \( \text{Aut}(Y; h)' \to \text{Aut}(Z) \) consists of translations. This implies that no curve \( C_i \) can be contained in a fiber of \( h \), i.e., all curves \( C_i \) dominate \( Z \). Indeed, otherwise, the group \( \text{Aut}(Y; h)' \) would preserve the nonempty finite subset of \( Z \) consisting of the images of those curves \( C_i \) which are contained in fibers of \( h \), and thus \( \text{Aut}(Y; h)' \) would be finite.

Therefore, given a sufficiently ample divisor \( D \) on \( Z \), one has \((A + h^\ast D) \cdot C_i > 0 \) for all \( C_i \). Thus, \( A' = A + h^\ast D \) is a big strongly numerically effective divisor. Hence it is ample by Lemma 1, and the threefold \( Y \) is projective.

Lemma 4. Let \( Y \) be a Moishezon threefold, and let \( h: Y \to Z \) be a contraction to a nonruled surface. Assume that the image \( \Upsilon \) of the homomorphism \( \text{Aut}(Y; h)' \to \text{Aut}(Z) \) has bounded finite subgroups. Then the threefold \( Y \) is projective.

Proof. As in the proof of Lemma 3, consider a birational morphism \( g: \hat{Y} \to Y \) such that \( \hat{Y} \) is projective. Let \( \hat{A} \) be an ample divisor on \( \hat{Y} \), and let \( A = g_\ast \hat{A} \). There are at most a finite number of curves \( C_i \subset Y \) such that \( A \cdot C_i \leq 0 \). As in the proof of Lemma 3, the set \( \{C_i\} \) of all such curves is invariant under \( \text{Aut}(Y; h)' \). Suppose that some curve \( C_i \) is contained in a fiber of \( h \). Let \( z = h(C_i) \), let \( \text{Aut}(Z, z) \subset \text{Aut}(Z) \) be the stabilizer of \( z \), and let \( \Upsilon_z = \Upsilon \cap \text{Aut}(Z, z) \). The index \( [\Upsilon : \Upsilon_z] \) is finite. Therefore, \( \Upsilon_z \) has bounded finite subgroups. By [8, Lemma 2.5], the finite subgroups of \( \Upsilon_z/(\Upsilon_z \cap \text{Aut}_0(Z)) \) are bounded, where \( \text{Aut}_0(Z) \) is the connected component of the identity element in \( \text{Aut}(Z) \). On the other hand, since \( Z \) is not ruled, \( \text{Aut}_0(Z) \) is trivial or is an Abelian variety, and hence the stabilizer \( \text{Aut}_0(Z, z) \subset \text{Aut}_0(Z) \) must be trivial. This contradiction shows that no curve \( C_i \) can be contained in a fiber of \( h \). Thus, for a sufficiently ample divisor \( D \) on \( Z \), the divisor \( A' = A + h^\ast D \) is big and strongly numerically effective. Hence \( A' \) is ample by Lemma 1, and the threefold \( Y \) is projective.

Note that the construction of [16, Example 7.6.20] yields an example of a nonprojective Moishezon threefold such that the base of its maximal rationally connected fibration has arbitrary dimension (from 0 to 3).

3. PROOF OF THEOREM 2

In this section, we prove Theorem 2.

Suppose that the group \( \text{Aut}(X) \) is not Jordan. Since \( \text{Aut}(X) \) is a subgroup of the group \( \text{Bir}(X) \) of birational self–maps of \( X \), we conclude that \( \text{Bir}(X) \) is not Jordan either. Note that \( \text{Bir}(X) \) is isomorphic to the group of birational self–maps of some projective variety \( \hat{X} \) birational to the compact complex space \( X \). According to [5, Theorem 1.8(ii)], the variety \( \hat{X} \) is uniruled, and by [3, Theorem 1.8] and [4, Theorem 1.1], it is not rationally connected. Since \( X \) is birational to \( \hat{X} \), we see that \( X \) is also uniruled but not rationally connected. There exists a maximal rationally connected fibration \( f: X \dashrightarrow V \), and one has \( 0 < \dim V < \dim X \); see [19, Theorem 5.5.4]. The compact complex space \( V \) is Moishezon; see [14, Theorem 2]. Note that the maximal rationally connected fibration is defined only as a rational map. Thus, resolving the singularities of \( V \), we may assume that \( V \) is smooth.

One of our main tools is the following result, which is implied by the existence of a canonical resolution of singularities (see [20, Sec. 13]).
\textbf{Theorem 3.} Let $M$ be an irreducible compact complex manifold, and let $W \subset M$ be its compact complex subspace. Then there exists a sequence of blow ups $\pi: \tilde{M} \to M$ with smooth centers such that the union of the proper transform $\pi^{-1}W$ with the exceptional locus $E$ of $\pi$ is a simple normal crossing divisor. Moreover, the morphism $\pi$ is canonical in the sense that every automorphism $M \to M$ preserving $W$ can be extended to an automorphism $\tilde{M} \to \tilde{M}$ commuting with $\pi$.

Theorem 3 allows us to prove the following result.

\textbf{Lemma 5.} Let $X$ be a Moishezon compact complex space, and let $f: X \dashrightarrow V$ be a maximal rationally connected fibration, where $V$ is assumed to be smooth. Suppose that $\dim V \leq 2$. Let $Z$ be the minimal model of $V$. Then there is an $\text{Aut}(X)$-equivariant commutative diagram

$$
\begin{array}{c}
X & \leftarrow & Y \\
\downarrow & & \downarrow h \\
Z & \rightarrow & Y
\end{array}
$$

\begin{itemize}
\item Here $Y$ is a Moishezon manifold, $Z$ is smooth and projective, $Y \to X$ is a birational morphism, and $h: Y \to Z$ is the maximal rationally connected fibration for $Y$.
\end{itemize}

\textbf{Proof.} Since the manifold $V$ is Moishezon and has dimension at most 2, it is projective. Hence its minimal model $Z$ is projective (and smooth) as well.

Recall that the group $\text{Aut}(X)$ acts on $V$ by birational maps (possibly not faithfully). Since $Z$ is a minimal model of $V$, the group $\text{Aut}(V)$ acts on $Z$ biregularly. The composition $\sigma: X \dashrightarrow V$ of $f$ with the contraction $V \to Z$ is an $\text{Aut}(X)$-equivariant map by construction. Consider the closure $\overline{\Gamma}_\sigma$ of the graph of this map in $X \times Z$. Since the action of $\text{Aut}(X)$ on $X \times Z$ is biregular, the action of $\text{Aut}(X)$ on $\overline{\Gamma}_\sigma$ is biregular as well. Finally, let $Y$ be the canonical resolution of singularities of $\overline{\Gamma}_\sigma$ provided by Theorem 3. The action of $\text{Aut}(X)$ on $Y$ is again biregular, which gives us the commutative $\text{Aut}(X)$-equivariant diagram $(*)$. \hfill \square

Let us apply Lemma 5 to our three-dimensional Moishezon compact complex space $X$. We have an embedding $\text{Aut}(X) \subset \text{Aut}(Y)$.

Since the map $h: Y \to Z$ is a maximal rationally connected fibration for $Y$, we see that the group $\text{Aut}(Y)$ acts (possibly not faithfully) on $Z$ and the map $h$ is $\text{Aut}(Y)$-equivariant. Let $\text{Aut}(Y)_h$ be the subgroup of $\text{Aut}(Y)$ consisting of all automorphisms whose action is fiberwise with respect to $h$, and let $\Upsilon \cong \text{Aut}(Y)/\text{Aut}(Y)_h$ be the image of the group $\text{Aut}(Y)$ in $\text{Aut}(Z)$. All finite subgroups of $\text{Aut}(Y)_h$ act faithfully on any nonmultiple fiber of $h$, and thus $\text{Aut}(Y)_h$ is Jordan by [5, Theorem 1.5]. This implies that if $\Upsilon$ has bounded finite subgroups, then the group $\text{Aut}(Y)$ is Jordan as well. Therefore, we shall assume that the group $\Upsilon$ has unbounded finite subgroups.

Suppose that the dimension of $Z$ equals 1. Then $Z$ is a nonrational curve. In this case, $Y$ is projective by Lemma 3.

Suppose that the dimension of $Z$ equals 2. Then $Z$ is a nonruled surface. In this case, $Y$ is projective by Lemma 4.

Therefore, we see that the group $\text{Aut}(X)$ is contained in the automorphism group of the projective variety $Y$. Now Theorem 2 follows from Theorem 1.

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