QED on Curved Background and on Manifolds with Boundaries: Unitarity versus Covariance

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Abstract
Some recent results show that the covariant path integral and the integral over physical degrees of freedom give contradicting results on curved background and on manifolds with boundaries. This looks like a conflict between unitarity and covariance. We argue that this effect is due to the use of non-covariant measure on the space of physical degrees of freedom. Starting with the reduced phase space path integral and using covariant measure throughout computations we recover standard path integral in the Lorentz gauge and the Moss and Poletti BRST-invariant boundary conditions. We also demonstrate by direct calculations that in the approach based on Gaussian path integral on the space of physical degrees of freedom some basic symmetries are broken.

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1. Introduction

A self-consistent approach to quantization of gauge theories was suggested more than a quarter of century ago in a pioneering paper by Faddeev and Popov [1]. In this and subsequent works [2] it was demonstrated that covariant gauge quantization rules can be derived from manifestly unitary reduced phase space approach. A suitable quantization procedure for the case of curved space-time was also elaborated (see [3,4,5]). However contradictions between covariant approach and Hamiltonian approach were reported for gravity and electrodynamics on de Sitter space [6-11] and on manifolds with boundaries [5,12-18]. This is a very serious problem since there is no use of a theory which is non-covariant or not unitary. Proper field theory should admit some kind of probabilistic interpretation and have all fundamental symmetries conserved, i.e. it should posses both unitarity and covariance properties.

Some progress was achieved in overcoming the above mentioned difficulty for unbounded manifolds. It was demonstrated [19] that the path integral over physical degrees of freedom of quantum gravity is equivalent to the covariant result by Taylor and Veneziano [7] provided covariant functional measure is used throughout computations. A correct path integral measure for Hamiltonian QED on Friedmann-Robertson-Walker background was constructed [20]. It was suggested [21] that the discrepancy between the two approaches is due to the fact that on certain manifolds the 3+1 decomposition is ill-defined. It was demonstrated [22] that on a flat space region between two concentric spheres the value of the scaling behavior $\zeta(0)$ is gauge independent. However, to obtain this result it was necessary to include quantization over non-physical degrees of freedom and ghosts. Some recent works show [23,24] that a part of difficulties of quantum field theory on manifolds with boundaries, but not all of them, are due to mistakes in calculations.

Before going into details, let us give some motivations. Consider two path integrals, one over covariant configuration space and the other over physical degrees of freedom (PDF). Suppose, that one of these two path integrals satisfy both covariance and unitarity requirements. Let us try to guess which one most probably gives correct results. It was not demonstrated anywhere that the covariant path integral violates unitarity. Let us check up covariance of the PDF path integral. To this end let us recall the heat kernel expansion
for electrodynamics on unit four-sphere \([11]\) and unit radius four-disk \([13]\).

\[
W_{\text{PDF}} = - \log \int_{\text{PDF}} D A \exp(-S(A)) = - \frac{1}{2} \int_{0}^{\infty} \frac{dt}{t} K(t)
\]

\[
K(t)[S^4] = \frac{1}{3t^2} - \frac{1}{t} + \frac{1}{2\sqrt{\pi}} t - \frac{16}{45} + O(\sqrt{t}) \tag{1.1}
\]

\[
K(t)[\text{Disk}] = \frac{1}{16t^2} - \frac{\sqrt{\pi}}{8t^\frac{3}{2}} + \frac{1}{\sqrt{t}} \left( \frac{53\sqrt{\pi}}{256} + \frac{1}{2\sqrt{\pi}} \right) - \frac{77}{180} + O(\sqrt{t}) \tag{1.2}
\]

An experienced reader would see what is wrong with equation (1.1). The heat kernel expansion of a covariant theory on a closed manifold cannot contain term with half integer powers of proper time. Such a term leads to divergency which cannot be cancelled by local covariant counter-term. This indicates that the covariance property is lost in the PDF formalism on \(S^4\).

The situation with the expression (1.2) is more subtle. On manifolds with boundaries terms with half integer powers of \(t\) are allowed. However, looking at the general form of the heat kernel expansion \([25]\), we see that on the unit disk the \(\sqrt{\pi}\) naturally appears in numerator, but not in denominator as in one of the terms of (1.2). Such terms with \(\pi^{-\frac{3}{2}}\) are not strictly forbidden, but their appearance is quite unsatisfactory. These terms hardly can be reproduced in a covariant theory. We see that in the PDF formulation covariance is most probably violated. Thus the covariant approach is better candidate for a proper theory than the PDF one.

The present paper is devoted to the study of QED on curved space and on manifolds with boundaries paying special attention to covariance and unitarity issues. First we shall derive covariant path integral from the reduced phase space quantization (sec. 2) using covariant path integral measure. This derivation shows that the source of differences between two approaches was in using non-covariant measure on reduced phase space. The proper measure on the space of physical degrees of freedom appeared to be complicated. This fact should not be a surprise (see \([26]\)). We extend the phase space in order to be able to use covariant measure for four-vector fields. We also consider an important particular case of static space-time. In this section we assume that the problems with boundary conditions are somehow regulated. In the next section 3 we give several examples. We demonstrate the loss of invariance properties in PDF formalism for electrodynamics on torus and sphere. We also re-calculate some results connected with criterion of normalizability of
the wave function of the Universe using covariant technique. The section 4 is devoted to manifolds with boundaries. We demonstrate that the procedure of section 2 leads to the Moss and Poletti boundary conditions [14,27] for electrodynamics. We also re-derive these boundary conditions using manifestly covariant procedure. Appendix A contains some additional illustrative material to sec. 3. Useful relations of differential geometry of manifolds with boundaries are collected in the Appendix B.

Now some definitions are in order. A path integral measure $Dv$ on the space $V$ with scalar product $\langle , \rangle$ will be called Gaussian if

$$\int_V Dv \exp(-\langle v, v \rangle) = 1 \quad (1.3)$$

The measure $Dv$ will be called covariant if both the integration space $V$ and scalar product $\langle , \rangle$ posses all necessary invariance properties. In our case these are the Lorentz and diffeomorphism invariance.

The definition (1.3) is written for Euclidean path integral. In the Minkowski signature space the minus sign in exponent should be replaced by $i$. These definitions follow the approach by DeWitt [3] and Polaykov [28].
2. Quantization and the measure

Consider the action for electromagnetic field $A_\mu$

$$S = -\int d^4x \sqrt{-g} \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$  \hspace{1cm} (2.1)

On a Friedmann-Robertson-Walker background ($g_{00} = 0, \ g_{00} = -N^2 = \text{const}$) one can rewrite (2.1) in the form

$$S = \int dt \int d^3x \left[ \sqrt{(-g)} P^i (\partial_0 A_i) - (\mathcal{H} + \lambda \Phi) \right]$$  \hspace{1cm} (2.2)

where $\sqrt{-g} P^i = \sqrt{-g} F_{i0}$ is the momentum conjugate to $A_i$, and

$$H = \int d^3x \mathcal{H} = \int d^3x \sqrt{-g} \frac{1}{4} [2P_i P^i N^2 + F_{ik} F^{ik}]$$  \hspace{1cm} (2.3)

plays the role of Hamiltonian. Last term in (2.2) generates the first class constraint

$$\Phi = - (3)^{\nabla} \nabla_i P^i = 0,$$  \hspace{1cm} (2.4)

where $(3)^{\nabla}$ is the covariant derivative with the respect to three-metric $g_{ik}$. To obtain (2.3) an integration by parts over $x_j$ is needed. We assume that either spatial sections are closed manifolds or all the fields decay rapidly at spatial infinity. The problem of the boundary conditions in the time direction is more subtle. It will be considered in the section 4. The canonical Poisson bracket is

$$\{ A_i(x,t), \sqrt{-g} P^k(y,t) \} = \delta^3(x-y) \delta_i^k.$$  \hspace{1cm} (2.5)

The constraint (2.4) generates gradient transformation of $A_i$. The corresponding gauge freedom can be fixed by imposing the condition

$$\chi = (3)^{\nabla} A^i = 0$$  \hspace{1cm} (2.6)

Note that in general the condition (2.6) does not fix the gauge freedom of the four-dimensional action (2.1) completely. The transformations

$$\delta A_\mu = \partial_\mu \omega(t) \quad \text{with} \quad \partial_i \omega(t) = 0$$  \hspace{1cm} (2.7)

do not change canonical coordinates and thus are allowed by (2.6). If a spatial section is homeomorphic to $R^3$, the transformations (2.7) are excluded by
conditions at spatial infinity. However, these transformations are allowed for compact spatial sections. We shall return to this problem a bit later.

As usual, the reduced phase space can be obtained by solving equations (2.4) and (2.6). This space consists of transverse spatial components of vector potential, $A^T_i$, and transverse components of conjugate momentum, $P^{Tj}$. The canonical bracket (2.5) takes the form

$$\{A^T_i(x, t), \sqrt{-g}P^{Tk}\} = \left(\delta^k_i - \frac{(3)\nabla^k_i}{(3)\Delta}\right)\delta(x - y)$$

We see that the right hand side of (2.8) is now non-trivial. This means that the measure $DA^T_i DP^{Tk}$ should include the determinant of the functional metric induced on the surface defined by the conditions (2.4) and (2.6). This determinant depends upon geometry of space-time manifold, boundary conditions, etc. The evaluation of this determinant is highly complicated task. It is much easier to extend the phase space in order to be able to work with standard inner product of four-vectors

$$<u, v> = \int d^4x \sqrt{-g}v_\mu u^\mu$$

According to the general prescription [1,2] the path integral can be written in the form

$$Z = \int DA_i D\sqrt{-g}P^k D\lambda \delta(\chi) \det\{\chi, \Phi\} \exp(-iS)$$

with the action (2.2). The Lagrange multiplier $\lambda$ is just the weighted zeroth component of the vector potential, $\lambda = \sqrt{-g}A_0$. The fields $A_0$ which do not depend on spatial coordinates do not generate any constraints. Thus one should exclude spatial constants from the integration measure $D\lambda$. This can be done by means of the condition

$$\tilde{\chi} = \partial_0 \int d^3x \sqrt{-g}A_0 = \partial_0 <\sqrt{-g}A_0> = 0$$

Other choices are also possible. From the point of view of the covariant theory, eq. (2.11) is nothing else than fixation of the gauge freedom (2.7). This leads to the appearance of the Jacobian factor $\tilde{J}$ in the integration measure

$$D\lambda = \tilde{J}\delta(\tilde{\chi})DA_0$$
The condition (2.11) appears naturally in a framework of the BRST quantization. Previously it was demonstrated that \( \tilde{J} = \prod_x \det(-\nabla_0^2) \).

Now we are to extend the phase space in order to obtain an integral over four-vector corresponding to \( P^k \). This can be done by inserting the identity

\[
1 = \int \mathcal{D}a \mathcal{D}\sqrt{-g} P^0 \exp(-i \int d^4x \sqrt{-g} [a P^0 - \frac{1}{2} P^{02} N^4])
\]

where \( a \) is a scalar field. After some algebra the path integral (2.10) is represented in the form

\[
Z = \int \mathcal{D}A_\mu \mathcal{D}\sqrt{-g}P^\mu \mathcal{D}a \tilde{J} \delta\{\chi, \Phi\} \delta(\chi) \delta(\tilde{\chi}) \exp(-iS(A_\mu)) + \frac{i}{2} \int d^4x \sqrt{-g} \left( (P_\mu - P_\mu(A))(P^\mu - P^\mu(A))N^2 + a^2 N^{-4} \right)
\]

where \( S(A) \) is the action (2.1), \( P^i(A) \) is the canonical expression for \( P^i \) in terms of \( A_i \), \( P^0(A) = a \).

The main advantage of the expression (2.14) is that now the momentum integration is performed over unconstrained four-vector \( P^\mu \). We can use standard definitions of the covariant Gaussian measures for vector density and scalar field

\[
\int \mathcal{D}\sqrt{-g} v^\mu \exp(i < v, v>) = 1 \tag{2.15}
\]

\[
\int \mathcal{D}a \exp(i \int d^4x \sqrt{-g} a^2) = 1 \tag{2.16}
\]

The integration over \( a \) and \( P^\mu \) produces ill-defined factor \( \prod_x N^{-2} \). This factor can be neglected in dimensional or zeta-regularization because it does not contain derivatives and does not depend on coordinates. It does not contribute to \( \zeta(0) \). Moreover, for Friedmann-Robertson- Walker space-time \( N \) can be absorbed in definition of time variable. Important thing that \( N \) enters path integral in even power and hence does not lead to imaginary factor after continuation to Euclidean space. Using the definitions (2.15) and (2.16) we obtain the following expression for generating functional \( Z \) in terms of the functional integral over \( A_\mu \)

\[
Z = \int \mathcal{D}A_\mu \delta(\chi) \delta(\tilde{\chi}) \det\{\chi, \Phi\} \tilde{J} \exp(-iS(A_\mu))
\]

This is the familiar form of the path integral of electrodynamics with delta-functions of gauge conditions. One can remove delta-functions by introducing
gauge-fixing terms. However, in this case, apart from ill-defined term $\tilde{\chi}^2$, we shall also get non-covariant term $\chi^2$. This would lead to non-covariant 3+1 splitting of $A_\mu$ and working with non-covariant measures.

The most elegant way to evaluate the integral (2.17) using only covariant measures is to introduce the so-called adapted coordinates [30]. Let us make the Hodge-de Rham decomposition of the four-vector $A_\mu$

$$A_\mu = A^\perp_\mu + A^\parallel_\mu, \quad A^\parallel_\mu = \partial_\mu \omega, \quad \nabla^\nu A^\perp_\nu = 0. \quad (2.18)$$

This decomposition is orthogonal with respect to the scalar product (2.9). The Jacobian factor due to the change of variables $A_\mu \to (A^\perp, \omega)$ is just the determinant of the scalar four-Laplacian, $\det(-\Delta)^{\frac{1}{2}}_S$. The measure induced on the subspace $A^\perp$ is Gaussian. Indeed,

$$\int \mathcal{D}A^\perp_\mu \exp(i < A^\perp, A^\perp >) = \int \mathcal{D}A^\perp_\mu \exp(i < A^\perp, A^\perp >) \times$$

$$\times \int \mathcal{D}\omega \exp(i < \omega, (-\Delta)\omega >) \det(-\Delta)^{\frac{1}{2}}_S =$$

$$= \int \mathcal{D}A^\perp_\mu \mathcal{D}\omega \det(-\Delta)^{\frac{1}{2}}_S \exp(i < A^\perp + \partial\omega, A^\perp + \partial\omega >) =$$

$$= \int \mathcal{D}A_\mu \exp(i < A, A >) = 1$$

where we used the definition (2.16) of the covariant measure for scalar field and the property $< A^\perp, \partial\omega >= 0$.

The gauge transformations $\omega$ can be parametrized by the gauge fixing functions $\chi, \tilde{\chi}$. The Jacobian factor appearing due to the change of variables $\omega \to (\chi, \tilde{\chi})$ is $[J \det\{\chi, \Phi\}]^{-1}$. In the adapted coordinates $(A^\perp, \chi, \tilde{\chi})$ the path integral (2.17) takes the form

$$Z = \int \mathcal{D}A^\perp \mathcal{D}\chi \mathcal{D}\tilde{\chi} \det(-\Delta)^{\frac{1}{2}}_S \exp[-iS(A(A^\perp, \chi, \tilde{\chi}))] \delta(\chi) \delta(\tilde{\chi}) \quad (2.20)$$

The action for electrodynamics can be rewritten as

$$S(A) = \frac{1}{2} \int d^4x \sqrt{-g}[A_\mu (g^{\mu\nu} \Delta - \nabla^\mu \nabla^\nu + R^{\mu\nu}) A_\nu] \quad (2.21)$$

It is interesting to note that the operator in the action (2.21) maps four-vectors to transversal four-vectors:

$$\nabla^\mu (g^{\mu\nu} \Delta - \nabla^\mu \nabla^\nu + R^{\mu\nu}) A_\nu = 0 \quad (2.22)$$
The path integral (2.20) is easily evaluated giving

\[ Z = \det(-g^{\mu \nu} \Delta - R^{\mu \nu})^{\frac{-1}{2}} \det(-\Delta)^{\frac{1}{2}}. \]  

(2.23)

The subscript \( \perp \) means that the determinant is evaluated on the space of transversal four-vectors. This is the well-known covariant expression for the path integral of QED. We derived it starting from the canonical Hamiltonian expression. Hence it is both unitary and covariant.

Consider the expression for the path integral in terms of physical degrees of freedom \[ Z_{PDF} = \det(-g^{\mu \nu} \Delta - R^{\mu \nu})^{\frac{1}{T^2}} \]  

(2.24)

where the determinant is restricted to transversal three-vectors. This expression can be obtained from (2.10) by formal integration over \( \lambda, A_i \) and \( P^k \). However, one should suppose that all the measures are Gaussian and integration do not give rise to any new factors. The expression (2.24) cannot be obtained if one uses only the covariant integration measures for four-vectors and scalars. The non-covariance of the path integral (2.24) is manifested through half-integer powers of the proper time in the heat kernel expansion (1.1) on de Sitter space [11]. Other effects of non-covariance of (2.24) will be described in the next section. Note here that (2.24) do not follow from canonical quantization of electrodynamics if one does not make any additional assumptions.

Let us extend the integration in (2.23) to all vector fields. This is done with the help of equation

\[ \det(-g^{\mu \nu} \Delta - R^{\mu \nu})_{\|} = \det(-\Delta)_S \]  

(2.25)

Using orthogonality of the decomposition (2.18) and the fact that the operator under determinant do not mix longitudinal and transversal four-vectors we can represent the path integral (2.23) in the following form

\[ Z = \det(-g^{\mu \nu} \Delta - R^{\mu \nu})^{\frac{1}{2}} \det(-\Delta)_S \]  

(2.26)

The subscript \( V \) means all four-vector fields.

Consider now an important particular case of static space-time, \( M^4 = M^3 \times T^1 \). For the sake of convenience we shall work with the Euclidean
signature space-time. On a static background the configuration space of all vector fields can be decomposed in a direct sum

\[ V = V_T \oplus V_L \oplus V_0 \]  

(2.27)

where \( V_T \) and \( V_L \) are spaces of transversal and longitudinal vector fields respectively, \( V_0 \) is the space of vectors having only zeroth component. Let the eigenvalues of the scalar three-Laplacian be \(-\lambda_l\) while \( l = 0 \) corresponds to constant field, \( \lambda_0 = 0 \) (if constant mode is allowed by boundary conditions on \( M^3 \)). The eigenvalues of the scalar four-Laplacian \( \Delta_S \) are \(-k^2 r^{-2} - \lambda_l\), \( k = 0, \pm 1, \pm 2, \ldots\), \( r \) is the radius of \( T^1 \). After some algebra one can obtain that the eigenvalues of the operator \( g^{\mu\nu} \Delta + R^{\mu\nu} \) on \( V_L \) and \( V_0 \) are also \(-k^2 r^{-2} - \lambda_l\), but on \( V_L \) the \( l = 0 \) harmonics should be excluded since they do not generate longitudinal fields. Hence,

\[
\begin{align*}
\det(-g^{\mu\nu} \Delta - R^{\mu\nu})_L &= \prod_k \prod_{l \neq 0} (r^{-2} k^2 + \lambda_l) \\
\det(-g^{\mu\nu} \Delta - R^{\mu\nu})_0 &= \prod_{(k,l) \neq 0} (r^{-2} k^2 + \lambda_l) \\
\det(-\Delta)_S &= \prod_{(k,l) \neq 0} (r^{-2} k^2 + \lambda_l)
\end{align*}
\]

(2.28)

In the last two equations \( l \) and \( k \) can not be zero simultaneously. Substituting (2.28) in (2.26) we obtain that contribution of scalar ghost is cancelled by contribution of \( V_L \) and \( V_0 \) fields up to the factor

\[
\prod_{k \neq 0} (r^{-2} k^2)^{\frac{1}{2}}
\]

(2.29)

It is easy to see that the expression (2.29) is just the Jacobian factor \( \tilde{J} \) appearing due to the condition (2.11).

\[
Z = \tilde{J}^\frac{1}{2} \det(-g^{\mu\nu} \Delta - R^{\mu\nu})_{T}^{-\frac{1}{2}} = \tilde{J}^\frac{1}{2} Z_{PDF}
\]

(2.30)

We see that on static background the non-covariant expression (2.24) should be modified only by the factor \( \tilde{J}^\frac{1}{2} \). This factor corresponds to one-dimensional field theory and describes less than one degree of freedom in four dimensions. However, this factor contributes to the energy-momentum tensor. Note that
even on non-static background the inclusion of $\bar{J}$ leads to cancellation of manifestly non-covariant terms in the heat kernel expansions (1.1) and (1.2) [20,23]. If $M^3 = R^3$ the scalar modes with $l = 0$ are excluded by conditions at spatial infinity and the expression is correct without any modifications.
3. Examples

3.1. QED on four-torus
Consider electrodynamics on $T^4$. The Euclidean effective action corresponding to the path integral (2.24) over physical degrees of freedom is

$$-\log Z_{PDF} = \frac{1}{2} \left[ 2 \sum_{n_0, n_1, n_2, n_3} ' \log \left( \frac{n_1^2}{r_1^2} + \frac{n_2^2}{r_2^2} + \frac{n_3^2}{r_3^2} + \frac{n_0^2}{r_0^2} \right) + 3 \sum_{n_0} ' \log \frac{n_0^2}{r_0^2} \right]$$  (3.1)

where prime means that all summation indices can not be equal to zero simultaneously. The $r_0, ..., r_3$ are radii of $T^4$. The coefficient 2 before the first sum reflects presence of two helicity components for any non-zero vector $(n_1, n_2, n_3)$. The coefficient 3 before the second sum is due to three independent components of a transversal three-vector for $n_1 = n_2 = n_3 = 0$. The expression (3.1) obviously depends on choice of the $x_0$ direction among coordinates on $T^4$.

The Jacobian factor $\tilde{J}$ (2.29) has the form

$$\log \tilde{J} = \frac{1}{2} \sum_{n_0} ' \log \frac{n_0^2}{r_0^2}$$  (3.2)

Now we can write the covariant effective action corresponding to the path integral (2.30)

$$-\log Z = \sum_{n_0, n_1, n_2, n_3} ' \log \left( \frac{n_1^2}{r_1^2} + \frac{n_2^2}{r_2^2} + \frac{n_3^2}{r_3^2} + \frac{n_0^2}{r_0^2} \right)$$  (3.3)

The same expression can be obtained directly from (2.26). The effective action (3.3) is symmetric under re-labeling coordinates on $T^4$ as it should be.

3.2. Vector field on four-sphere
Consider a massless field theory on sphere $S^{d+1}$ with an axially symmetric metric. If we apply a symmetry-preserving analytical regularization, e.g. the zeta-regularization, the regularized vacuum energy momentum tensor $T_{\mu \nu}^{reg}$ is proportional to $g_{\mu \nu}$ [4] because the $g_{\mu \nu}$ is the only symmetric invariant tensor on $S^{d+1}$.

$$T_{\mu \nu}^{reg} = \frac{1}{d+1} g_{\mu \nu} (T^{reg})^\rho_\rho$$  (3.4)

The right hand side of (3.4) is proportional to the conformal anomaly, which is finite in zeta-regularization. Thus the left hand side should be finite too.
For the simplest case of scalar theory on $S^2$ the relation (3.4) is established in the Appendix A.

In this subsection we shall demonstrate that the energy-momentum tensor for $Z_{PDF}$ is divergent on $S^4$. This means that the rotational symmetry is broken.

The metric on $S^4$ has the form

$$ds^2 = dx_0^2 + \alpha \sin^2 x_0 d\Omega^2$$

(3.5)

where $d\Omega^2$ is the metric on unit $S^3$. We introduced a real squashing parameter $\alpha$. The $\alpha = 1$ corresponds to unit round $S^4$.

By definition

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta W}{\delta g^{\mu\nu}}, \quad W = - \log Z$$

(3.5)

Hence

$$\frac{\delta W}{\delta \alpha}|_{\alpha = 1} = - \frac{1}{2} \int d^4x T_{ij}g^{ij}\sqrt{g} = - \frac{1}{2} < T_i^i >$$

(3.6)

The $T_i^i$ in the right hand side of (3.6) refers to the round unit $S^4$. According to (3.4) it should be finite if the $O(4)$ symmetry is unbroken.

Let us decompose the "physical" fields $A^T$ in series of transversal vector harmonics $Y(l)$ on $S^3$, $l = 1, 2, ...$ (see e.g. [31])

$$A_i^T = \sum_{(l)} f_{(l)}(x_0) Y_{(l)i}(x_k)$$

(3.7)

Substitution of (3.7) in the eigenvalue equation $(g^{ik} \Delta + R^{ik})A_k^T = \lambda A^T_i$ gives an ordinary differential equation for $f_{(l)}$

$$[\partial_0^2 + \cotg(x_0) \partial_0 - \frac{(l + 1)^2}{\alpha \sin^2(x_0)}]f_{(l)} = \lambda f_{(l)}$$

(3.8)

To solve this equation we use the method of the paper [29]. After the change of variables

$$f = h \sin^b(x_0), \quad b = (l + 1)\alpha^{-\frac{1}{2}}$$

$$z = \frac{1}{2} (\cos(x_0) + 1)$$

(3.9)

the equation (3.8) takes the form

$$z(z - 1)h'' + (1 + c)(z - \frac{1}{2})h' + eh = 0,$$
Prime denotes differentiation with the respect to $z$. According to the general prescription let us express $h$ as power series [32]

$$h(z) = \sum_{k=0} a_k z^k$$  \hspace{1cm} (3.11)

By substituting (3.11) in (3.10) we get the recurrent condition for the coefficients $a_k$

$$a_{k+1} = \frac{k(k-1) + (1+c)k + c}{(k+1)(k + \frac{1}{2}(c+1))}$$ \hspace{1cm} (3.12)

The denominator of (3.12) is positive for all $k$. The polynomial eigenfunctions $h_k$ can be found by imposing the condition on the numerator to be equal to zero for some $k$. We obtain the eigenvalues

$$\lambda_{(l)k} = -k^2 - k(2\frac{(l+1)}{\sqrt{\alpha}} + 1) - \frac{(l+1)}{\sqrt{\alpha}} \frac{(l+1)}{\sqrt{\alpha}} + 1)$$ \hspace{1cm} (3.13)

The degeneracies of $\lambda_{(l)k}$ are the same as of corresponding three-dimensional transversal vector spherical harmonics

$$D_l^T = 2l(l+2)$$ \hspace{1cm} (3.14)

In the limit $\alpha = 1$ we obtain correct eigenvalues of the vector Laplacian on $S^4$, $\lambda = -(k + l + 1)(k + l + 2) = -p(p+1)$, $p = k + l + 1 = 2,3...$ The functions $h_k$ become associated Legendre polynomials.

The effective action corresponding to the path integral $Z_{PDF}$ (2.24) is

$$W_{PDF} = \frac{1}{2} \sum_{l,k} D_l^T \log \lambda_{(l)k}$$ \hspace{1cm} (3.15)

In a framework of the zeta-regularization

$$\frac{1}{2} < T_i^i >= \frac{\delta W}{\delta \alpha}_{\alpha=1} = \frac{1}{2} \lim_{s \to 0} \sum_{l,k} D_l^T \left( \frac{d\lambda_{(l)k}}{d\alpha} \lambda_{(l)k}^{-(1+s)} \right)_{\alpha=1} =$$

$$= \frac{1}{2} \lim_{s \to 0} \sum_{p=2}^{\infty} \sum_{l=1}^{p-1} \frac{(p + \frac{1}{2})(l + 1)D_l}{[p(p+1)]^{1+s}}$$

14
\[
= \frac{1}{4} \lim_{s \to 0} \sum_{p=2}^{\infty} \frac{(p^2(p + 1)^2 - 2p(p + 1) - 24)(p + \frac{1}{2})}{[p(p + 1)]^{1+s}}
\]  

(3.16)

In its more rigorous version the zeta-regularization should be applied directly to the effective action. The result for \( < T_i^i > \) is the same up to the multiplier \( \Gamma(s)s \) which is unity in the limit \( s \to 0 \).

The last line of the equation (3.16) can be interpreted as a value at the point \( y = 1 \) of the generalized zeta-function \( \zeta(y) \) of the operator with eigenvalues \( p(p + 1) \) and the degeneracies given by the numerator. We can use the fact that the residue of \( \Gamma(y)\zeta(y) \) at \( y = 1 \) is equal to the coefficient before \( t^{-1} \) in the heat kernel-like expansion of the following sum for small \( t \)

\[
\sum_{p=2}^{\infty} \frac{1}{4}(p^2(p + 1)^2 - 2p(p + 1) - 24)(p + \frac{1}{2}) \exp(-tp(p + 1))
\]  

(3.17)

The asymptotic behavior of (3.17) can be easily find using the equation [10]

\[
\sum (p + \frac{1}{2}) \exp(-tp(p + 1)) = \frac{1}{2t} + O(t^0)
\]  

(3.18)

This gives

\[
\Gamma(1)\text{Res}_{y=1}\zeta(y) = -3 \neq 0
\]  

(3.19)

Hence the energy-momentum tensor

\[
-\frac{1}{2} < T_i^i > = \lim_{s \to 0} \zeta(1 + s)
\]  

(3.20)

is divergent in zeta-regularization. This means that the \( O(4) \) symmetry is broken in the theory described by the path integral \( Z_{PDF} \). One can also verify that the Jacobian factor \( \tilde{J} \) do not contribute to \( < T_i^i > \). Thus the difficulties of the approach based on the integration over physical degrees of freedom can not be resolved by simply multiplying by \( \tilde{J} \) as it was in the static case.

3.3 Normalizability of the wave function of the Universe

Recently [33] a criterion of the normalizability of the wave function of the Universe was suggested. It reads

\[
C > -1
\]  

(3.21)
where \( C \) is the sum of anomalous scalings of all the fields on the Euclidean four-sphere \( S^4 \). In the paper [33] the contribution of vector fields was computed in a framework of an approach based on integration over physical degrees of freedom, which corresponds to the path integral \( Z_{PDF} \) (see Ref.[11])

\[
C_{PDF}^1 = -\frac{16}{45}
\]  

(3.32)

From the above discussion it is clear that the anomalous scaling (3.22) should be replaced by that coming from the covariant path integral (2.26) [11]

\[
C_1 = \frac{59}{45}
\]  

(3.23)

Note that another value of \( C_1 \) in covariant approach exists in the literature [9,10]. The difference is due to miscounting of ghost zero modes [11].

On the same grounds we prefer to use covariant path integral for gravitational field. The corresponding scaling behavior is [7]

\[
C_2 = \frac{329}{45}
\]  

(3.24)

The computations in terms of physical degrees of freedom [8] give \( C_{PDF}^2 = -\frac{661}{45} \). For a review of different approaches to quantum gravity on \( S^4 \) see [19]. For non-gauge scalar and Dirac spinor fields the values of scaling behavior are unambiguous and well known (see e.g. [4,10])

\[
C_0 = \frac{29}{90} - 4\xi + 12\xi^2, \quad C_{\frac{1}{2}} = \frac{11}{90}
\]  

(3.35)

where \( \xi \) is the standard parameter of scalar coupling to gravity.

For the system consisting of graviton, three generations of fermions, gauge fields of the standard model and one complex Higgs doublet we obtain

\[
C = \frac{325}{12} + 48\xi^2 - 16\xi
\]  

(3.26)

instead of the non-covariant result [34]

\[
C_{PDF} = -\frac{179}{12} + 48\xi^2 - 16\xi
\]  

(3.27)

We see that the criterion (3.21) is fulfilled by (3.26) for both minimal (\( \xi = 0 \)) and conformal (\( \xi = \frac{1}{6} \)) coupling, while the \( C_{PDF} \) do not satisfy this criterion. The covariant path integral gives more optimistic predictions.
4. QED in a bounded region

One of the most interesting applications of quantum field theory on manifolds with boundaries is related to quantum cosmology and the Hartle-Hawking wave function of the Universe. In the semiclassical approximation the contribution of the electromagnetic field to the wave function of the Universe is given by the Euclidean path integral \[ Z = \int D\mu(A) \exp(-S(A)) \] (4.1)

where the fields \( A \) satisfy some conditions on the boundary \( \partial M \) of gravitational instanton \( M \). The measure \( D\mu(A) \) includes all Faddeev-Popov determinants.

As in the case of unbounded manifolds the computations in terms of physical degrees of freedom [13] and in terms of four-vector fields [14,15] give different results [5,18]. A part of this contradictions was due to a mistake in analytical formula for heat kernel expansion in the case of mixed boundary conditions (see [23,24]). However, even if correct formula is used the two above mentioned approaches still disagree [24]. From our point of view this difference originates from the fact that the Gaussian measure on the space of physical variables is really non-covariant and should be replaced by another measure obtained by reduction of standard measure on the space of four-vector fields to the subspace of physical variables. An interesting explanation why these measures are different was suggested recently [21,22]. We postpone for a while discussion of this suggestion. In this section we shall derive the covariant Moss and Poletti path integral [14,15,27] starting from the canonical formalism. This will prove unitarity of the path integral [14]. Next we shall derive the same path integral from the manifestly covariant expression by means of the so-called geometric approach to quantization of gauge theories [36].

4.1 Canonical path integral

Let the manifold \( M \) admit a coordinate system such that

\[ ds^2 = dx_0^2 + g_{ik}dx^i dx^k \] (4.2)

and the boundary \( \partial M \) corresponds to a constant value of the Euclidean time \( x_0 \). All canonical conjugate pairs will be defined with respect to \( x_0 \) variable.
To make physical consideration more transparent we consider a simplified case

$$g_{ik}(x_0, x_j) = c^2(x_0)\tilde{g}_{ik}(x_j) \quad (4.3)$$

We suppose also that the metrics $g_{\mu\nu}$ and $\tilde{g}_{ik}$ satisfy the Einstein equations. This ensures that the Ricci tensors are covariantly constant. All these restrictions, however, do not exclude any physically interesting example, such as disk or a spherical segment. We shall use notation $\Gamma(x_0) = c^{-1}\partial_0 c$.

Let us start with the path integral $[35,13]$ over physical components

$$Z = \int (DA^T_i)_D \exp(-S(A^T)) \quad (4.4)$$

where integration is performed over transversal three-vectors satisfying the Dirichlet boundary condition

$$A^T_i|_{\partial M} = 0 \quad (4.5)$$

Unlike Ref. [13] we will not suppose that the measure $(DA^T)_D$ is Gaussian. Instead we enlarge the integration domain selfconsistently in order to obtain integrals over four-vectors. We shall omitt the steps which are identical to that of sec. 2. Special attention will be paid to boundary conditions.

According to the general method [1,2] the path integral (4.4) should be considered as a result of integration over momenta in the reduced phase space path integral. Let us recall the first order form of the Euclidean action for QED

$$S = \int d^4x \sqrt{g}[P^i\partial_0 A_i + \frac{1}{2}(\frac{1}{2}F_{ik}F^{ik} - P_i P^i) + A_0 (3) \nabla_i P^i] \quad (4.6)$$

This action is written in terms of independent variables $P^i = F^{0i}$ and $A_i$, $A_0$ is the Lagrange multiplier. The integration by parts over $x_i$ is always allowed because either spatial slices are closed manifolds or all the fields decay rapidly at spatial infinity.

Boundary conditions for the reduced phase space momenta $P^{Tj}$ are not arbitrary. They are specified by the boundary conditions (4.5) for $A^T_i$ through the equation

$$P^i = g^{ik}\partial_0 A_k \quad (4.7)$$

Note that the eq. (4.7) does not mean that $A_i$ and $P^k$ become dependent variables. This equation rather states a map between two functional spaces
which enables us to interpret $A_i$ and $P^k$ as conjugate variables. As it is demonstrated in the Appendix B, the relation (4.7) leads to the following boundary condition on $P_T^i$

$$(\partial_0 + 3\Gamma)P_T^i|_{\partial M} = 0$$  \hspace{1cm} (4.8)$$

In order to be able to use covariant definition of the path integral measure we are to extend the phase space in such a manner to obtain integration over unconstrained four-vector fields. Boundary conditions for the non-physical components and ghosts should be chosen self-consistently.

The most natural choice for spatial components $A_i$ and $P^i$s is to extend the Dirichlet and Neumann boundary conditions (4.5) and (4.8) to all three-vector fields.

$$A_i|_{\partial M} = 0, \quad (\partial_0 + 3\Gamma)P^i|_{\partial M} = 0$$  \hspace{1cm} (4.9)$$

To generate the constraint $\Phi = -(3)^2\nabla_i P^i = 0$ the Lagrange multiplier should belong to the same functional space and hence satisfy the same boundary condition as $\Phi$, namely

$$(\partial_0 + 3\Gamma)A_0|_{\partial M} = 0$$  \hspace{1cm} (4.10)$$

The boundary conditions for the ghosts are the same as for the parameter $\omega$ of gauge transformation which map the space defined by (4.9) and (4.10) in itself. One can demonstrate that these conditions are (see Appendix B)

$$\omega|_{\partial M} = 0$$  \hspace{1cm} (4.11)$$

Note that the equations (4.9)-(4.11) describe a particular case of more general BRST-invariant boundary conditions proposed by Moss and Poletti [14]

$$A_i|_{\partial M} = 0, \quad (\nabla_0 + k_i^j)A_0|_{\partial M} = 0, \quad \omega|_{\partial M} = 0$$  \hspace{1cm} (4.12)$$

where $k_{ij}$ is the second fundamental form of the boundary.

The last ingredient we need for the case of compact spatial sections is the additional gauge fixing condition (2.11). It is obvious, that to obtain covariant configuration space one should extend boundary conditions (4.12) to spatially constant fields $A_0$ and $\omega$. The operator $1 \otimes \{\Phi, \chi\} + (-\nabla_0^2) \otimes 1$, where the first multiplier acts on $x_i$-independent fields and the second one only on the fields with non-trivial $x_i$-dependence, is non-degenerate ghost operator.
Boundary conditions for the fields $P^0$ and $a$ are not significant because these fields are immediately integrated out. The covariance of the obtained path integral will be stated independently in the next sub-section.

Now we are able to write the Euclidean path integral in the form (see (2.17))

$$Z = \int D\mathcal{A}_\mu \delta(\chi)\delta(\tilde{\chi}) J \exp(-S(A_\mu))$$

(4.13)

where the fields $A_\mu$ and ghosts in the Jacobian factors satisfy the above defined boundary conditions. To obtain Lorentz gauge path integral we shall use the Faddeev-Popov trick. Let us insert the unity in the path integral (4.13)

$$1 = \int D\omega \det(-\Delta)\delta(\nabla^\mu A_\mu(\omega)), \quad A_\mu(\omega) = A_\mu + \partial_\mu \omega$$

(4.14)

with $\omega$ satisfying Dirichlet boundary condition. After change of variables $A_\mu \rightarrow A_\mu(-\omega)$ and integration over $\omega$ with the help of another representation of unity

$$1 = \int D\omega \det\{\Phi, \chi\} \tilde{J}\delta(\chi(A(-\omega)))\delta(\tilde{\chi}(A(-\omega)))$$

(4.14a)

we arrive at

$$Z = \int D\mathcal{A}_\mu \det(-\Delta)\delta(\nabla^\mu A_\mu) \exp(-S(A_\mu))$$

$$= \int D\mathcal{A}^\perp_\mu \det(-\Delta)\tilde{J} \exp(-S(A^\perp_\mu))$$

(4.15)

for derivation of (4.14) and (4.14a) it is essential that Lorentz gauge condition fixes the gauge freedom completely, as well as the conditions $\chi$ and $\tilde{\chi}$. To obtain the last line of equation (4.15) we integrated over longitudinal fields using the fact that due to conditions (4.12) transversal fields are orthogonal to longitudinal ones with the respect to ordinary scalar product without surface terms. Of course, all the fields in (4.15) again satisfy conditions (4.12). The expression (4.15) is the Lorentz gauge path integral proposed by Moss and Poletti. Here we derived it directly from the canonical reduced phase space quantization. This supports conclusion made previously on the basis of BRST-invariance [14] that the path integral (4.15) describes unitary theory.

4.2. Covariance of the path integral
The boundary conditions (4.12) have an advantage [23,25] that the Hodge-de Rham decomposition (2.18) is orthogonal with the respect to ordinary scalar product in the space of vector fields without surface terms and the Laplace operator is self-adjoint. However, these conditions are not manifestly covariant. Different components of \( A_\mu \) obey different types of boundary conditions. In this subsection we re-obtain path integral (4.15), (4.12) by means of manifestly covariant procedure in a framework of the so-called geometric approach to quantization of gauge theories [36,30,37,38].

The starting point is the path integral without gauge fixing terms

\[
Z = \frac{1}{\text{vol } G} \int \mathcal{D}A_\mu \exp(-S(A_\mu))
\]

where \( \text{vol } G \) is infinite volume of the gauge group. We assume that all the components of \( A_\mu \) satisfy Dirichlet boundary condition

\[
A_\mu|_{\partial M} = 0
\]  

and the gauge group is spanned by gradient transformations with the parameter \( \omega \) satisfying Dirichlet boundary too

\[
\omega|_{\partial M} = 0
\]

The boundary conditions (4.17) and (4.18) are manifestly covariant.

The action \( S(A_\mu) \) is independent of pure gauge degrees of freedom. We can integrate over longitudinal fields in (4.16) thus cancelling the volume of the gauge group. To this end let us use the Hodge-de Rham decomposition

\[
A_\mu = A_\mu^\perp + \partial_\mu \omega
\]

\[
\nabla^\mu A_\mu^\perp = 0
\]

Due to the condition (4.18) this decomposition is orthogonal with the respect to ordinary inner product without surface terms.

From (4.17), (4.18) and (4.19) we see that spatial components of \( A^\perp \) satisfy the Dirichlet condition

\[
A^\perp_\mu|_{\partial M} = 0
\]

As it was explained in the previous subsection (see also Appendix B), the condition (4.21) together with (4.18) and (4.19) lead to

\[
(\partial_0 + k_0^i)A^\perp_0|_{\partial M} = 0
\]
To verify consistency of the procedure, let us substitute (4.21) and (4.22) in
the gauge condition on the boundary

$$\nabla^\mu A^\perp_\mu|_{\partial M} = (\partial_0 + k_i^i)A^\perp_0|_{\partial M} + (3)\nabla^i A^\perp_i|_{\partial M} = 0 \quad (4.23)$$

Due to the orthogonality property the path integral measure can be repre-
represented in the form

$$\mathcal{D}A_\mu = \mathcal{D}A^\perp_\mu \mathcal{D}\omega \det(-\Delta)^{\frac{1}{2}} \quad (4.24)$$

Substituting (4.24) in (4.16) and integrating over $\omega$ with the help of identity
\(\text{vol}G = \int \mathcal{D}\omega\) we arrive at

$$Z = \int \mathcal{D}A^\perp_\mu \det(-\Delta)^{\frac{1}{2}} \exp(-S(A^\perp)) \quad (4.25)$$

with the boundary conditions (4.18), (4.21) and (4.22). This is exactly the
Moss and Poletti path integral (4.15). We have demonstrated that their
result is covariant.
5. Conclusions

Let us summarize in short the main results of this paper. We analyzed the reduced phase space quantization procedure on curved background. It was demonstrated that the Hamiltonian quantization plus manifestly invariant measure lead to path integral identical to that obtained in the gauge fixed approach. This procedure was also applied to manifolds with boundaries. It was shown that the Faddeev-Popov trick lead to the Moss and Poletti [14] boundary conditions. The same boundary conditions were also obtained by using manifestly covariant method based on geometric approach to quantization of gauge theories. With the example of QED on four-sphere we demonstrated that the use of Gaussian measure on the space of physical degrees of freedom leads to breakdown of the $O(4)$ rotational symmetry. Some important particular cases were also analyzed. A criterion of normalizability of the wave function of the Universe was applied to the covariant path integral with field content identical to that of the Standard Model.

The main problem we addressed to in this paper was as follows. There are two formulations of path integral for gauge theories on curved background and on manifolds with curved boundaries. The one formulation is manifestly covariant, the other is manifestly unitary. These two formulations disagree. The roots of this disagreement are in use of different path integral measures. If one uses covariant measure in both cases, one gets identical results. The covariant path integral appeared to be also unitary because it was obtained from the Faddeev reduced phase space integral.

The other question is why the two measures are different. The results of this paper support the point of view [21,22] that the reason is that the 3+1 decomposition is ill-defined on curved background and on some manifolds with boundaries. However it is yet unclear whether this effect is pure topological and is entirely due to the presence of singular point-like spatial slices. Anyhow, this problem deserves further investigation. One should also check up independently unitarity of the covariant path integral and find out which fundamental symmetries are broken in the approach based on physical degrees of freedom on manifolds with boundaries.

The results of this paper can be also interesting from the point of view of equivalence of different quantization techniques for gauge theories (see e.g. [30,37,39]). It would be important to derive covariant path integral measure from the action principle and canonical quantization.
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Appendix A

As a simple example of application of the technique of sec. 3.2 to covariant case consider minimaly coupled scalar field on the two-sphere $S^2$. Introduce a deformed metric

$$ds^2 = dx_0^2 + \alpha \sin^2(x_0) dx_1^2$$  \hspace{1cm} (A.1)

The scalar Laplace operator reads

$$\Delta = (\partial_0 + \cot(x_0))\partial_0 + \frac{1}{\alpha} \partial_1^2$$ \hspace{1cm} (A > 2)

The spectrum of $\Delta$ can be found by the same manipulations as in the sec. 3.2.

$$\lambda_{k,l} = -k^2 - k\left(\frac{|l|}{\sqrt{\alpha}} + 1\right) - \frac{|l|}{\sqrt{\alpha}} \left(\frac{|l|}{\sqrt{\alpha}} + 1\right)$$ \hspace{1cm} (A.3)

The $|l|$ is the absolute value of $l$. We have for the energy-momentum tensor

$$\frac{1}{2} < T^1_1 > = \frac{\delta W}{\delta \alpha} |_{\alpha=1} = -\frac{1}{2} \lim_{s \rightarrow 0} \sum_{k,l} \frac{k|l| + l^2 + \frac{1}{2}|l|}{[(k + |l|)(k + |l| + 1)]^{1+s}} =$$

$$= \frac{1}{2} \lim_{s \rightarrow 0} \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{(2n+1)l}{[n(n+1)]^{1+s}} = -\frac{1}{4} \lim_{s \rightarrow 0} \sum_{n=1}^{\infty} \frac{(2n+1)(n+1)n}{[n(n+1)]^{1+s}} =$$

$$=-\frac{1}{4} \lim_{s \rightarrow 0} \sum_{n=1}^{\infty} \frac{2n+1}{[n(n+1)]^s} = -\frac{1}{4} \zeta(0)$$ (2.4)

where $n = |l| + k$. The $\zeta(0)$ is just the zeta-function of the operator (A.2) for $\alpha = 1$. The crucial difference between (A.4) and (3.16) is that now the multiplier $n(n+1)$ in the numerator cancels the one in denominator. Thus the corresponding term in the heat kernel expansion before zeroth power of proper time is equivalent to the residue of $\Gamma(y)\zeta(y)$ at the point $y = 0$ (and not at $y = 1$ as in (3.16)). At $y = 0$ the $\Gamma$-function itself has a pole. Hence the $\zeta(y)$ remains finite at this point. The value of residue is well known (see e.g. [10]); it gives $\zeta(0) = \frac{1}{3}$. It is instructive to compare (A.4) with expression for the conformal anomaly or one-loop scaling behaviour.

$$< T_{\mu}^\mu > = \lim_{s \rightarrow 0} \sum_{k,l} \frac{-\lambda_{k,l}}{[-\lambda_{k,l}]^{1+s}} = \lim_{s \rightarrow 0} \sum_{n=1}^{\infty} \sum_{l=-n}^{n} \frac{1}{[n(n+1)]^{1+s}} =$$
\[
= \lim_{s \to 0} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)^s} = \zeta(0) \quad (A.5)
\]

We have demonstrated that in zeta-regularization

\[
< T^\mu_\mu > = 2 < T^I_I > = \zeta(0) \quad (A.6)
\]

and the energy-momentum tensor is finite as it should be for a covariant theory on \(S^2\) with unbroken \(O(2)\)-symmetry.

**Appendix B**

This Appendix contains necessary information on eigenmodes of the Laplace operator on manifolds with boundaries admitting metric described by eqs. (4.2) and (4.3). The Laplace operator acting on scalar and vector fields has the form

\[
\Delta \phi = (\partial^2_0 + 3\Gamma \partial_0 + (3)\Delta) \phi,
\]

\[
(\Delta A)_0 = [\partial^2_0 + 3\Gamma \partial_0 + (3)\Delta - 3\Gamma^2]A_0 - 2\Gamma (3)^i A_i,
\]

\[
(\Delta A)_i = [\partial^2_0 + \Gamma \partial_0 + (3)\Delta - 3\Gamma^2 + \Gamma(\partial_0 \Gamma)]A_i + 2\Gamma (3)^i A_i A_0
\]

Let us remind that

\[
g_{ik} = c^2(x_0)\tilde{g}_{ik}(x_j); \quad \Gamma = \frac{\partial_0 c}{c}. \quad (B.2)
\]

Since we assumed that the \(g\) and \(\tilde{g}\) are Einsteinian, the terms with Ricci tensors in vector operators simply lead to constant shift of eigenvalues. We can ignore Ricci tensor in what following.

The metric (4.2), (4.3), (B.2) admits separation of variables in the operators (B.1). Let \(\Delta\) be the 3-dimensional Laplace operator built up with metric \(g_{ik}\), \(\Delta = c^{-2} 2\Delta\). Let us denote scalar and transverse vector eigenfunctions of \(\Delta\) as \(Y^S_{(l)}\) and \(Y^T_{(l)}\). The four-dimensional vector and scalar fields can be decomposed in the following series of orthogonal harmonics

\[
\phi(x_0, x_j) = \sum_{(l)} \phi_{(l)}(x_0)Y^S_{(l)}(x_j), \quad A_0(x_0, x_j) = \sum_{(l)} a_{(l)}(x_0)Y^S_{(l)}(x_j),
\]

\[
A_i(x_0, x_j) = \sum_{(l)} v^T_{(l)}(x_0)Y^T_{(l)i}(x_j) + \sum_{(k)} v^T_{(k)}(x_0)\partial_i Y^S_{(l)}(x_j) \quad (B.3)
\]
Substituting (B.3) in the eigenvalue equations $\Delta \phi = \lambda \phi$ and $\Delta A_\mu = \lambda A_\mu$ we obtain ordinary differential equations for $\phi_{(l)}$ and $v^T_{(l)}$ and pairs of coupled ordinary differential equations for $a_{(k)}$ and $v_{(k)}$. It means that scalar and transversal 3-vector can be decomposed in sums of harmonics which are eigenfunctions both $\Delta$ and $\tilde{\Delta}$. Transversal 3-vectors decouple from the other fields.

As far as functional integral is concerned we can establish boundary conditions for separate eigenmodes of the Laplace operator [27].

First let us prove the equation (4.8)

$$\left( \partial_0 + 3\Gamma \right) P^T = \left( \partial_0 + 3\Gamma \right) g^{ik} \partial_k A^T_k = g^{ik} \left( \partial_0 + \Gamma \right) \partial_k A^T_k =$$

$$= g^{ik} \left( \Delta A^T_k - \frac{1}{c^2} \tilde{\Delta} - 3\Gamma^2 + \Gamma (\partial_0 \Gamma) \right) A^T_k$$

(B.4)

where we used decoupling of $A^T_i$ from other vector harmonics and the equation (B.1). Let $A^T_i$ be an eigenmode of $\Delta$ and $\tilde{\Delta}$. When the expression in the last line of (B.4) vanishes term by term on the boundary if $A^T_i$ satisfies Dirichlet boundary condition (4.5).

Next we demonstrate that if the gauge parameter $\omega$ satisfy the Dirichlet condition (4.11) when the pure gauge vector potential $A_\mu = \partial_\mu \omega$ satisfy the conditions (4.9) and (4.10). The Dirichlet condition (4.10) for $A_i$ is obvious. Indeed, differentiation with respect to $x_i$ do not affect the coefficient function $f_{(l)}(x_0)$ which vanishes on the boundary due to (4.11). For the zeroth component we have

$$(\partial_0 + 3\Gamma) \partial_0 \omega = \Delta \omega - \frac{1}{c^2} \tilde{\Delta} \omega$$

(B.5)

Since the harmonics of scalar field $\omega$ can be chosen to be eigenfunctions on both $\Delta$ and $\tilde{\Delta}$, the right hand side of (B.5) vanishes on the boundary term by term due to the Dirichlet condition (4.11).

After obvious rearrangements in the above reasoning on can also obtain the equation (4.22). Let us remind, that on the boundary $k_{ij} = \Gamma g_{ij}$ if the outward pointing normal vector is directed along positive $x_0$ axis.
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