Research article

An analytical method for shallow spherical shell free vibration on two-parameter foundation

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ABSTRACT

The free vibration control differential equation of shallow spherical shell on two-parameter foundation is a four order differential equation. Using the intermediate variable, the four order differential equation is reduced to two lower order differential equations. The first lower order differential equation is a Helmholtz equation. A new method of two-dimensional Helmholtz operator is proposed as shown in the paper in which the Bessel function included in Helmholtz equation needs to be treated appropriately to eliminate singularity. The first lower order differential equation is transformed into the integral equation using the proposed method in the paper. The second lower order differential equation which is a Laplace equation is transformed into the integral equation by existing methods. Then the two integral equations are discretized according to the middle rectangle formula, and the corresponding solutions can be obtained by MATLAB programming. In this paper, the R-function theory is used to select the appropriate boundary equation to eliminate the singularity. Based on the properties of R-function, the combined method of Helmholtz equation and Laplace equation can solve the free vibration problem of irregular shallow spherical shell on two-parameter foundation. Five examples are given to verify the feasibility of the method.

1. Introduction

In this paper, Helmholtz equation is mainly used in the first low order differential equation. The first low order equation combine Helmholtz equation and Green's formula to get the first integral equation. Helmholtz equation is widely used in many fields such as engineering technology, electromagnetic field theory, scattering theory, mechanics and so on. The study of its numerical solution has not only extensive practical significance but also important theoretical value. Traditional numerical methods, such as finite difference method, finite element method and boundary element method, need to establish network related to interpolation node topology. Shifted-Laplacian Preconditioners for Heterogeneous Helmholtz Problems was researched by Oosterlee et al. [1]. A Fourth-Order Kernel-Free Boundary Integral Method for the Modified Helmholtz Equation was discussed by Xie et al. [2]. Helmholtz, Duffing and Helmholtz-Duffing Oscillators: Exact Steady-State Solutions was studied by Kovacic et al. [3]. Analysis of the Shifted Helmholtz Expansion Preconditioner for the Helmholtz Equation was considered by Cocquet et al. [4]. Effect of Helmholtz Oscillation on Auto-shroud for atmospheric-pressure plasma spray Tungsten Carbide Coating was researched by Jin et al. [5].

We used the proposed method of Helmholtz equation to solve the problem of free vibration of simply supported irregular shallow spherical shell on two-parameter foundation. Two-parameter foundation model uses two independent parameters to represent the compression and shear characteristics of soil. It overcomes the defect that Winkler foundation model can't reflect the pressure diffusion, and the mathematical treatment is simpler than that of elastic half space foundation model. If the parameters are selected properly, the mechanical properties of the foundation can be well described. The mechanical problems of spherical shell on two-parameter foundation exist in civil engineering, mechanical engineering, chemical engineering, energy and aerospace industry. Many scholars have done a lot of research on free vibration of shallow and deep ellipsoidal shells having variable...
thickness with and without a top opening was researched by Ko et al. [8]. Static and vibration analysis of cross-ply laminated composite doubly curved shallow shell panels with stiffeners resting on Winkler–Pasternak elastic foundations was discussed by Tran et al. [9]. Free Vibrations of a Ribbed Cylindrical Shell Interacting with an axisymmetric Winkler or Pasternak Foundation was studied by Yu. V. Skosarenko [10]. Influence of reinforcement and the Winkler and Pasternak Foundation on the vibrations of shallow shells with rectangular planform was discussed by Lugovoi et al. [11]. Free vibration analysis of functionally graded doubly curved shell panels resting on elastic foundation in thermal environment was researched by Tran et al. [12]. Free vibration of stiffened functionally graded cylindrical shell resting on Winkler–Pasternak foundation with different boundary conditions under thermal environment was studied by Tran et al. [13]. Vibrations of ribbed shallow rectangular shells on the Winkler and Pasternak Foundation was studied by Lugovoi et al. [14]. Benchmark solution for free vibration of thick open cylindrical shells on Pasternak foundation with general boundary conditions was discussed by Wang et al. [15]. Free vibration analysis of fiber reinforced composite conical shells resting on Pasternak-type elastic foundation was considered by Zarouni et al. [16] using Ritz and Galerkin methods. Vibrations of functionally graded cylindrical shells based on the Winkler and Pasternak Foundation was considered by Shah et al. [17]. Free vibrations of a cylindrical shell partially resting on a two-parameter Pasternak elastic foundation was discussed by S. A. Bochkarev [18]. Analytical solution for free vibration of stiffened functionally graded cylindrical shell structure resting on Winkler–Pasternak foundation was researched by Nguyen et al. [19]. Free vibration analysis of cylindrical shells partially resting on a Winkler foundation, and free vibration of thick open cylindrical shells on Pasternak foundation with general boundary conditions was discussed by Wang et al. [15]. Free vibration analysis of fiber reinforced composite conical shells resting on Pasternak-type elastic foundation was considered by Zarouni et al. [16] using Ritz and Galerkin methods. Vibrations of functionally graded cylindrical shells based on the Winkler and Pasternak Foundation was considered by Shah et al. [17]. Free vibrations of a cylindrical shell partially resting on a two-parameter Pasternak elastic foundation was discussed by S. A. Bochkarev [18]. Analytical solution for free vibration of stiffened functionally graded cylindrical shell structure resting on Winkler–Pasternak foundation was researched by Nguyen et al. [19]. Free vibration analysis of cylindrical shells partially resting on a Winkler foundation, and free vibration of thick open cylindrical shells on Pasternak foundation with general boundary conditions was discussed by Wang et al. [15]. Free vibration analysis of fiber reinforced composite conical shells resting on Pasternak-type elastic foundation was considered by Zarouni et al. [16] using Ritz and Galerkin methods. Vibrations of functionally graded cylindrical shells based on the Winkler and Pasternak Foundation was considered by Shah et al. [17]. Free vibrations of a cylindrical shell partially resting on a two-parameter Pasternak elastic foundation was discussed by S. A. Bochkarev [18]. Analytical solution for free vibration of stiffened functionally graded cylindrical shell structure resting on Winkler–Pasternak foundation was researched by Nguyen et al. [19]. Free vibration analysis of cylindrical shells partially resting on a Winkler foundation, and free vibration of thick open cylindrical shells on Pasternak foundation with general boundary conditions was discussed by Wang et al. [15]. Free vibration analysis of fiber reinforced composite conical shells resting on Pasternak-type elastic foundation was considered by Zarouni et al. [16] using Ritz and Galerkin methods. Vibrations of functionally graded cylindrical shells based on the Winkler and Pasternak Foundation was considered by Shah et al. [17].

R-function theory was first proposed by V I Rvachev, a Ukrainian scholar, in 1963. Its name means Rvachev function. The R-function is a series of real functions whose result is positive or negative only depending on whether the variables are positive or negative, regardless of the specific values of the variables. An R-function always corresponds to a logical expression. In this way, if you know the implicit representation of the parts, by R-functions any shape that is represented by set will also satisfy the first-order normalized equation, many scholars have done some research on the R-function. Such as, Kurpa and Shmatko et al. have done a lot of research on R-function theory. We studied the clamped cutout of the complex form by the Ritz method and the R-functions theory was researched by Shmatko et al. [31]. We studied the R-function theory and applied it to a logical expression. In this way, if you know the implicit representation of the parts, by R-functions any shape that is represented by set will also satisfy the first-order normalized equation, many scholars have done some research on the R-function. Such as, Kurpa and Shmatko et al. have done a lot of research on R-function theory. We studied the clamped cutout of the complex form by the Ritz method and the R-functions theory was researched by Shmatko et al. [31]. We studied the R-function theory and applied it to find the shape function of irregular polygonal shell.

The R-function, the Bessel function, the basic solution and boundary condition are used to transform the two-dimensional Helmholtz operator into integral equation. The free vibration of an elastic arbitrary polygonal shallow spherical shell on a two-parameter foundation model with simply supported periphery is analyzed, and the corresponding mode shapes are calculated by using the frequencies obtained. Examples are given to verify the feasibility of the method. The proposed solution is suitable for shallow spherical shell problem with simply-supported boundary conditions, and the boundary is straight line.

2. Basic equations

The mechanical model of simply supported shallow spherical shell on two-parameter foundation is shown in Figure 1. According to Kirchhoff hypothesis, the governing equations of free vibration of shallow spherical shell on two-parameter foundation are

$$\nabla^4 \varphi(x, t) - \frac{E_B}{R_0} \nabla^2 \varphi(x, t) = 0, \quad x \in \Omega$$

(1)

$$D \nabla^4 w(x, t) + \frac{1}{R_0^4} \nabla^2 \varphi(x, t) - G_p \nabla^2 w(x, t) + k w(x, t) + m \frac{\partial^2 w(x, t)}{\partial t^2} = 0,$$

$$x \in \Omega$$

(2)

in which, $\nabla^4 = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)^2$ is the biharmonic operator, the bending stiffness is $D = \frac{E Bh^3}{12(1-\nu^2)}$, $x = (x_1, x_2), \Omega$ is the domain contained in the middle plane $(x_1 = 0) plane) of the shallow spherical shell in Cartesian coordinate system. Other variables and parameters are shown in Table 1.

The simply supported boundary conditions can be written as

$$w|_f = \nabla^2 w|_f = \varphi|_f = \varphi^2 = 0$$

(3)

Making use of Eq. (1) and Eq. (3), we get

$$\nabla^4 \left( \nabla^2 \varphi - \frac{E_B}{R_0} \varphi \right) = 0$$

(4a)

So we can easily obtain

$$\nabla^4 \varphi = wE_B/R_0$$

(4b)

Substituting Eq. (4b) into Eq. (2) yields.

$$D \nabla^4 w(x, t) + w(x, t)E_B \left[ R_0^4 - G_p \nabla^2 w(x, t) + k w(x, t) + m \frac{\partial^2 w(x, t)}{\partial t^2} \right] = 0,$$

$$x \in \Omega$$

(5a)

We take $w(x, t) = (A \cos(\Omega t) + B \sin(\Omega t)) W(x)$

(5b)

Substitute Eq. (5b) into Eq. (5a), and then we can get

$$D \nabla^4 W(x) + W(x)E_B \left[ R_0^4 - G_p \nabla^2 W(x) + k W(x) = \pi \Omega^2 W(x) \right], \quad x \in \Omega$$

(6)

Figure 1. Mechanical model of simply supported shallow spherical shell on two-parameter foundation.
in which, \( \overline{m} \) is the natural frequency, \( W(x) \) represents the mode function. Other variables and parameters are shown in Table 1.

To decompose Eq. (6), let us introduce the intermediate variables

\[
M = \frac{1}{1 + v} (M_1 + M_2)
\]

(7)

In the formula, the expression of \( M_1 \) and \( M_2 \) is

\[
M_1 = -D \left( \frac{\partial^2 W}{\partial x_1^2} + v \frac{\partial^2 W}{\partial x_2^2} \right) - D \left( \frac{\partial^2 W}{\partial x_2^2} + v \frac{\partial^2 W}{\partial x_1^2} \right)
\]

(8a)

From Eq. (8a) we known,

\[
M_1 + M_2 = -D(1 + v) \left( \frac{\partial^2 W}{\partial x_1^2} + \frac{\partial^2 W}{\partial x_2^2} \right) = -D(1 + v) \nabla^2 W
\]

(8b)

From Eq.(7) and Eq.(8b), we can get

\[
M = -D \nabla^2 W
\]

(8c)

According to (8c)

\[
\nabla^2 M = -D \nabla^4 W
\]

(8d)

By substituting Eq.(8c)and Eq.(8d) into differential Eq. (6), the differential equation of free vibration of shallow spherical shell on two-parameter foundation can be degraded into Helmholtz equation and Laplace equation.

\[
\nabla^2 \frac{G_p}{M} = \frac{\nabla^2}{\overline{m}^2} \left( k - \frac{Eh}{R_b} \right) W
\]

(9)

\[
\nabla^2 W = \frac{M}{D}
\]

(10)

In which, \( \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \) is a Laplace operator, engineering frequency \( f = \frac{\pi}{2\sqrt{D}} \).

Two-parameter foundation model assumes that there is shear interaction between the spring elements. The shear interaction is achieved by connecting the spring element with a vertical element which can only produce transverse shear deformation but not compressible. The mechanical model is divided into three layers. The first layer is the shallow spherical shell, the second layer is the shear layer, and the third layer is the spring layer.

3. Boundary conditions

The deflection on the shallow spherical shell boundary should be zero and the bending moment should be equal to zero under simply supported constraint boundary conditions.

\[
w = 0, \quad W = 0, \quad x \in \Gamma,
\]

(11)

\[
M_1 = 0, \quad M_2 = 0, \quad M = 0, \quad x \in \Gamma,
\]

(12)

4. The integral equations

Using the Solution of Two-Dimensional Helmholtz Operator (as shown in the Appendix), the boundary value problems Eq.(9) and Eq.(11) transformed into integral equation as follow.

\[
M(x) - \frac{1}{2\pi} \int_{\Omega} G_\omega(x, \xi) W(\xi) d\xi d\Omega + \int_{\Omega} M(\xi) K_\omega(x, \xi) d\xi d\Omega
\]

(13)

\[
K_\omega(x, \xi) = -\frac{1}{2\pi} \left( \nabla^2 - \lambda \right) \delta(x, \xi)
\]

(14)

Using the method of Laplace operator in literatures [22, 23, 24, 25], the boundary value problems Eq.(10) and Eq.(12) into transformed integral equation as follow.

\[
W(x) = -\frac{1}{D} \int_{\Omega} G_W(x, \xi) M(\xi) d\xi d\Omega + \int_{\Omega} W(\xi) K_W(x, \xi) d\xi d\Omega
\]

(15)

\[
K_W(x, \xi) = \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial x_2^2} \delta(x, \xi)
\]

(16)

According to the corresponding formulas in the Appendix and literatures [22, 23, 24, 25], we can get

\[
K_\omega(x, \xi) = -\frac{1}{2\pi} \sqrt{j_k \sqrt{4R}} \left\{ \frac{2}{R^2} \left[ 1 + \omega(x) \nabla^2 \omega \right] + \frac{2}{R^2} \left[ \nabla \omega \cdot \nabla \omega \right] - \frac{1}{2} \lambda \nabla^2 \omega \right\}
\]

(17)

(18)

In the formulas, \( k_1 \) is the second kind of first-order deformed Bessel function, \( \omega = \omega(\xi), \nabla = \nabla_\xi \). The singularity of integral kernels is treated

![Figure 2. Shallow spherical shell which front projection is quadrilateral.](image)
The integral Eqs. (13) and (15) are discretized by the method of literature [32], which guarantees the continuity of integral kernels $K_M(x, \xi)$ and $K_W(x, \xi)$.

5. Discretization of integral equations

The integral Eqs. (13) and (15) are discretized by the method of literatures [22, 23, 24, 25]. The integral domain $\Omega$ is divided into several sub-domains $\Omega_i (i = 1, 2, \ldots, N)$. The rectangular formulas are applied in each sub-domain to perform numerical quadrature. The integral Eq.(13) and Eq.(15) can be transformed into two equations as follows.

$$M(x) = \frac{1}{2\pi} \left[ \begin{array}{c} A_{N+N} \ b_{N+N} \\ C_{N+N} \ D_{N+N} \end{array} \right] \left[ \begin{array}{c} M(x_1) \\ M(x_2) \\ \vdots \\ M(x_N) \\ W(x_1) \\ W(x_2) \\ \vdots \\ W(x_N) \end{array} \right] = 0$$

$$A_{N+N} = (a_{ij})_{N+N}, \quad B_{N+N} = (b_{ij})_{N+N}, \quad C_{N+N} = (c_{ij})_{N+N}, \quad D_{N+N} = (d_{ij})_{N+N}$$

When $i = j, a_{ij} = K_M(x_i, \xi_i)A_i - 1$

When $i \neq j, a_{ij} = K_M(x_i, \xi_j)A_j - 1$

$$b_{ij} = \frac{Eh}{2\pi} \left( \begin{array}{c} G(x_i, \xi_i)A_i + \sum_{j=1}^{N} M(\xi_j) K(x_i, \xi_j) A_j \\ \vdots \\ G(x_i, \xi_N)A_N + \sum_{j=1}^{N} M(\xi_j) K(x_i, \xi_N) A_N \end{array} \right)$$

$$c_{ij} = -\sum_{j=1}^{N} M(\xi_j) K(x_i, \xi_j) A_j$$

$$d_{ij} = \sum_{j=1}^{N} W(\xi_j) K(x_i, \xi_j) A_j - 1$$

(Note, according to literature [32], we can get

$$a = \frac{1}{m} \left\{ D \left[ \frac{m \pi}{a_0} \right]^2 + G \left[ \frac{m \pi}{b_0} \right]^2 \right\}^2 + \frac{Eh}{K_0} + k \right\},$$

where $m$ and $n$ are integers, $a_0 = 2a$, $b_0 = 2b$, Engineering frequency’s analytic solutions $f = \frac{m}{2\pi}$.)
And $A_j$ denotes the area of the $j$ subdomain. The condition for a system of homogeneous Eq. (20) to have nontrivial solutions is as follows.

$\begin{bmatrix} A_{N,N} & B_{N,N} \\ C_{N,N} & D_{N,N} \end{bmatrix} = 0$  \hspace{1cm} (22)

The natural frequency $\bar{\omega}$ can be obtained by solving Eq. (22). Then the.

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6. Numerical examples

Example 1 A simply supported shallow spherical shell on two-parameter foundation model is shown in Figure 2. We set $a = b = 1m,$
Poisson’s ratio $\nu = 0.3$, thickness $t = 0.1$ m, Elastic modulus $E = 3 \times 10^9$ Pa, the mass per unit area $m = 780$ kg, the coefficient of elastic foundation $k = 2 \times 10^7$ N/m, radius of curvature $R_b = 5$ m, Shear modulus of foundation $G_F = 2 \times 10^5$ Pa. According to R-function theory [29, 30, 31], just make $
abla_0 = \nabla_1 + \nabla_2 \sqrt{\nabla_1^2 + \nabla_2^2}$ in which $\nabla_1 = \frac{1}{2} \left(a^2 - x^2\right)$ and $\nabla_2 = \frac{1}{2} \left(b^2 - y^2\right)$. Then $\nabla_0 = 0$ is the first-order boundary normalized equation of the shallow spherical shell, $\nabla_1$ and $\nabla_2$ are the sides of shallow spherical shell.

| Method                        | Modal order | 1 frequency (Hz) | 2 frequency (Hz) | 3 frequency (Hz) |
|-------------------------------|-------------|-----------------|-----------------|-----------------|
| The method in this paper      | 25          | 34.864          | 45.932          | 62.488          |
|                               | 49          | 34.751          | 45.084          | 60.541          |
|                               | 81          | 34.709          | 44.707          | 59.488          |
|                               | 121         | 34.689          | 44.559          | 58.922          |

25(5 × 5), 49(7 × 7), 81(9 × 9) and 121 (11 × 11) grid layout schemes are adopted for shallow spherical shell which front projection is quadrilateral. The natural frequency's results are shown in Table 2.

The modes corresponding to each frequency are shown in Figures 3, 4, and 5.

Example 2 A simply supported rectangular shallow spherical shell on two-parameter foundation model is shown in Figure 2. We set $a = 1.25$m, $b = 1$m, other parameters applied are shown in example 1. According to R-function theory [29, 30, 31], just make

$$\nabla_0 = \nabla_1 + \nabla_2 \sqrt{\nabla_1^2 + \nabla_2^2}$$

in which $\nabla_1 = \frac{1}{2} \left(a^2 - x^2\right)$ and $\nabla_2 = \frac{1}{2} \left(b^2 - y^2\right)$. Then $\nabla_0 = 0$ is the first-order boundary normalized equation of the shallow spherical shell, $\nabla_1$ and $\nabla_2$ are the sides of shallow spherical shell.

25(5 × 5), 49(7 × 7), 81(9 × 9) and 121 (11 × 11) grid layout schemes are adopted for shallow spherical shell which front projection is rectangular. The natural frequency's results are shown in Table 3.
The modes corresponding to each frequency are shown in Figures 6, 7, and 8.

**Example 3** A simply supported trapezoidal shallow spherical shell on two-parameter foundation model is shown in Figure 9. We set $a = 1.25\text{ m}$, $b = 1\text{ m}$, $c = 2.25\text{ m}$. Other parameters applied are shown in example 1. According to R-function theory [29, 30, 31], just make

$$
\omega_0 = \omega_1 + \omega_2 + \omega_3 - \sqrt{\omega_1^2 + \omega_2^2} - \sqrt{\omega_1^2 + \omega_3^2} - \sqrt{\omega_2^2 + \omega_3^2} + \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2},
$$

where

$$
\omega_2 = \frac{1}{\sqrt{1+\left(\frac{a}{b}\right)^2}} \left( \frac{a}{b} x + \frac{b}{a} y \right),
$$

$$
\omega_3 = \frac{1}{\sqrt{1+\left(\frac{c}{d}\right)^2}} \left( \frac{c}{d} x - \frac{d}{c} y \right) \quad \text{and} \quad \omega_1 = \frac{\left(\frac{c}{d}\right) y}{a},
$$

Then $\omega_0 = 0$ is the first-order boundary normalized equation of trapezoidal shallow spherical shell which front projection is trapezoidal. $\omega_1 = 0$, $\omega_2 = 0$ and $\omega_3 = 0$ are each side of the shallow spherical shell which front projection is trapezoidal.

$25(5 \times 5)$, $49(7 \times 7)$, $81(9 \times 9)$ and $121(11 \times 11)$ grid layout schemes are adopted for shallow spherical shell which front projection is trapezoidal. The natural frequency’s results are shown in Table 4.

The modes corresponding to each frequency are shown in Figures 10, 11, and 12.

**Example 4** A simply supported L-shaped shallow spherical shell on two-parameter foundation model is shown in Figure 13. We set $a = 1.5\text{ m}$, $b = 2\text{ m}$, $c = 3.5\text{ m}$, $d = 1.2\text{ m}$. Other parameters applied are shown in example 1. According to R-function theory [29, 30, 31], just make

$$
\omega_0 = \omega_1 + \omega_2 + \omega_3 - \sqrt{\omega_1^2 + \omega_2^2} - \sqrt{\omega_1^2 + \omega_3^2} - \sqrt{\omega_2^2 + \omega_3^2} + \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2},
$$

where

$$
\omega_2 = \frac{1}{\sqrt{1+\left(\frac{a}{b}\right)^2}} \left( \frac{a}{b} x + \frac{b}{a} y \right),
$$

$$
\omega_3 = \frac{1}{\sqrt{1+\left(\frac{c}{d}\right)^2}} \left( \frac{c}{d} x - \frac{d}{c} y \right) \quad \text{and} \quad \omega_1 = \frac{\left(\frac{c}{d}\right) y}{a},
$$

Then $\omega_0 = 0$ is the first-order boundary normalized equation of L-shaped shallow spherical shell which front projection is L-shaped.

$25(5 \times 5)$, $49(7 \times 7)$, $81(9 \times 9)$ and $121(11 \times 11)$ grid layout schemes are adopted for shallow spherical shell which front projection is L-shaped. The natural frequency’s results are shown in Table 5.

The modes corresponding to each frequency are shown in Figures 14, 15, and 16.

**Table 5.** Engineering frequencies $f$ of shallow spherical shell which front projection is L-shaped.

| Method                         | Modal order | 1 Frequency (Hz) | 2 Frequency (Hz) | 3 Frequency (Hz) |
|-------------------------------|-------------|------------------|------------------|------------------|
| The method in this paper      | 1           | 27               | 32.958           | 34.791           | 36.968           |
|                               | 2           | 75               | 32.946           | 34.790           | 36.847           |
|                               | 3           | 147              | 32.925           | 34.719           | 36.666           |

![Figure 14](image1.png)  
**Figure 14.** First-order mode shape of shallow spherical shell with L-shaped front projection.

![Figure 15](image2.png)  
**Figure 15.** Second-order mode shape of shallow spherical shell with L-shaped front projection.

![Figure 16](image3.png)  
**Figure 16.** Third-order mode shape of shallow spherical shell with L-shaped front projection.

![Figure 17](image4.png)  
**Figure 17.** Shallow spherical shell which front projection is L-shaped.
The modes corresponding to each frequency are shown in Figures 14, 15, and 16.

**Example 5** A simply supported I-shaped shallow spherical shell on two-parameter foundation model is shown in Figure 17. We set $a = 1.2m$, $b = 1.5m$, $c = 1m$, $d = 1m$. Other parameters applied are shown in example 1. According to R-function theory [29, 30, 31], just make

$$\omega_0 = \omega_1 + \omega_6 - \sqrt{\omega_1^2 + \omega_6^2},$$

in which $\omega_0 = \omega_1 + \omega_2 - \sqrt{\omega_1^2 + \omega_2^2}$ and $\omega_0 = \omega_3 + \omega_4 + \sqrt{\omega_3^2 + \omega_4^2}$, and $\omega_1 = \frac{\omega_1^2 + \omega_2^2}{2} \geq 0$, $\omega_2 = \frac{\omega_3^2 + \omega_4^2}{2} \geq 0$, $\omega_3 = (c - x) \geq 0$, $\omega_4 = (d - y) \geq 0$. Then $\omega_0 = 0$ is the first-order boundary normalized equation of L-shaped shallow spherical shell which front projection is L-shaped. $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 0$ and $\omega_4 = 0$ are the sides of the shallow spherical shell which front projection is L-shaped.

Shallow spherical shell which front projection is L-shaped is arranged in $27(9 \times 3)$, $75(25 \times 3)$ and $147(49 \times 3)$ grids respectively. The natural frequency’s results are shown in Table 5.

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**Table 5.** Engineering frequencies $f$ of shallow spherical shell which front projection is L-shaped.

| Method                        | Modal order | Frequency (Hz) |
|-------------------------------|-------------|----------------|
| The method in this paper      | 1           | 27             |
|                               | 3           | 34.405         |
|                               | 3           | 38.673         |
|                               | 3           | 47.505         |
|                               | 2           | 75             |
|                               | 2           | 34.371         |
|                               | 2           | 38.867         |
|                               | 2           | 48.175         |
|                               | 3           | 147            |
|                               | 3           | 34.318         |
|                               | 3           | 38.768         |
|                               | 3           | 47.427         |

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**Figure 18.** First-order mode shape of shallow spherical shell with L-shaped front projection.

**Figure 19.** Second-order mode shape of shallow spherical shell with L-shaped front projection.

**Figure 20.** Third-order mode shape of shallow spherical shell with L-shaped front projection.

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**7. Conclusion**

Using Eq. (5b), the governing differential Eq. (5a) is transformed into a time-independent fourth-order differential Eq. (6). By referring to the relation between bending moment $M$ and mode $W$ (7-8d), the fourth-order differential Eq. (6) is transformed into two second-order differential Eqs. (9) and (10). According to the method in the appendix, the two-dimensional Helmholtz operator, Laplace operator and quasi-Green function are used to transform two second-order differential equations into two integral Eqs. (13) and (15). While R-function theory can easily find the shell shape function $\omega_0$, $\omega_0$ can be substituted into Eq. (A.23) in the appendix to get $\omega_1$, and $\omega$ is an important component of the integral Eqs. (13) and (15). Then, the integral equation is discretized, the integral
Therefore, using variable substitution \( \lambda \) where

\[
\frac{\partial^2 \nu}{\partial x_1^2} + \frac{\partial^2 \nu}{\partial x_2^2} - \lambda \nu = f_0
\]

Making its boundary condition become

\[
\nu|_\Gamma = \phi_0
\]

Where \( \lambda \) is constant, \( \Gamma = \partial \Omega \) is boundary of \( \Omega \)'s borders, extend the function \( \phi_0 \) to \( \Omega \), establish a function \( \nu \) that is smooth enough in \( \Omega \), make \( \phi|_\Gamma = \phi_0 \). Therefore, using variable substitution \( u = \nu - \phi \), and substituting expressions Eqs. (A.1) and (A.2), the original problem can be transformed into a boundary value problem with homogeneous boundary conditions as follow.

\[
\nabla^2 u - \lambda u = f
\]

\[
u|_\Gamma = 0
\]

Where \( f = f_0 - (\nabla^2 \phi - \lambda \phi) \).

**A2 Quasi-Green’s functions of two-dimensional Helmholtz operators**

Let \( \omega = 0 \) be the first-order normalized equation of boundary \( \Gamma \). Satisfy the following expression [29, 30, 31].

\[
\omega(x) = 0, \ |\nabla \omega| = 1, \ x \in \Gamma
\]

\[
\omega(x) > 0, \ x \in \Omega
\]

Using the basic solution of the problem, the quasi-Green’s function is constructed as follow.

\[
G(x, \xi) = k_0 \left( \sqrt{iR} \right) - e(x, \xi)
\]

\[
e(x, \xi) = k_0 \left( \sqrt{iR} \right)
\]

Where \( k_0 \) is the second kind of zero-order deformed Bessel function. And

\[
r = |\xi - x| = \sqrt{(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2}
\]

\[
\vec{r} = (\xi_1 - x_1) \hat{i} + (\xi_2 - x_2) \hat{j}
\]

\[
R = \sqrt{r^2 + 4\omega(\xi)\omega(x)}
\]
In the formula, \( \hat{t}, \hat{j} \) represent the unit vectors in the direction \( x_1, x_2 \), respectively. Where \( x = (x_1, x_2), \xi = (\xi_1, \xi_2) \). Obviously quasi-Green’s function \( G(x, \xi) \) satisfies the condition

\[ G(x, \xi)|_{|x|} = 0 \quad (A.12) \]

### A3 Integral equation

The boundary value problem Eq. (A.3) and Eq. (A.4) are transformed into integral equation, and Green’s formula of class \( C^2(\Omega) \) function is applied. For all \( U, V \in (\Omega \cup \Gamma) \), there are the following expression.

\[
\int_\Omega \left[ V(\nabla^2 U - \lambda U) - U(\nabla^2 V - \lambda V) \right] d\Omega = \int_\Omega \left( V \frac{\partial U}{\partial n} - U \frac{\partial V}{\partial n} \right) d\Gamma \quad (A.13)
\]

\( u \) and quasi-Green’s function \( G \) in formula (A.3) are used to replace \( U \) and \( V \) in formula (A.13), respectively. Noting that \(- \frac{1}{2} j k_0(\sqrt{\lambda}) \) is the fundamental solution of Helmholtz operator \( \nabla^2 u - \lambda u \), using formulas (A.4) and (A.12), obtain the following expression

\[
u(x) = u_0(x) + \int_\Omega u(\xi) K(x, \xi) d\Omega \quad (A.14)
\]

Where

\[
u_0(x) = -\frac{1}{2\pi} \int_\Omega G(x, \xi) \delta(\xi) d\xi \quad (A.15)
\]

\[
K(x, \xi) = -\frac{1}{2\pi} (\nabla^2 \xi - \lambda) \delta(x, \xi) \quad (A.16)
\]

The expression (A.8) is substituted to the expression (A.16). By using the properties of Bessel function, we can get

\[
K(x, \xi) = -\frac{1}{2\pi} \sqrt{j k_0} \left( \sqrt{j R} \right) \left( -\nabla^2 R + \frac{1}{R} (\nabla R)^2 \right) - \frac{1}{2\pi} j k_0 \left( \sqrt{j R} \right) (|\nabla R|^2 - 1) \quad (A.17)
\]

According to the relationship between \( R \) and \( \omega \) of formula (A.11), formula (A.17) can eventually be transformed to the following form.

\[
K(x, \xi) = -\frac{1}{2\pi} \sqrt{j k_0} \left( \sqrt{j R} \right) \left\{ -\frac{2}{R} \left[ 1 + \omega(x) \nabla^2 \omega \right] + \frac{2}{R^2} \nabla^2 \omega + 2\omega(x) \nabla \omega \right\} - \frac{1}{2\pi} j k_0 \left( \sqrt{j R} \right) \left[ \frac{1}{R^2} + 2\omega(x) \nabla \omega \right]^2 - 1 \} \quad (A.18)
\]

In which, \( \omega = \omega(\xi), \nabla = \nabla_\xi \).

### A4 Elimination of kernel singularity

When \( R = 0 \), namely \( x = \xi \) and \( \omega = 0 \), discontinuity may occur in \( K(x, \xi) \) in expression (A.18). In fact, when \( x = \xi \), Eq. (A.18) can be transformed to the following expression.

\[
K(x, \xi)|_{|x|} = \frac{1}{2\pi} \sqrt{j k_0 (2\omega \sqrt{\omega})} \left\{ \frac{1}{\omega} \left[ |\nabla \omega|^2 - 1 - \omega \nabla^2 \omega \right] - \frac{1}{2\pi} \sqrt{j k_0 (2\omega \sqrt{\omega})} \left[ |\nabla \omega|^2 - 1 \right] \} \quad (A.19a)
\]

In order to make integral kernels \( K(x, \xi) \in C(\Omega \cup \Gamma) \), hypothesis

\[
\omega = \omega_0 + \omega_0^2 \varphi \quad (A.19b)
\]

The formula \( \omega_0 = 0 \) is the first order normalized equation of \( \Gamma = \partial \Omega \), which satisfies Eq. (A.5) and Eq. (A.6). Obviously, \( \omega_0 \) and \( \omega \) are the first-order normalized equations of boundary \( \Gamma \). According to formulas (A.5) and (A.6), equation \( \omega_0 = 0 \), we can get

\[
(\nabla \omega_0)^2 = 1 + \omega_0 \varphi_0, \quad \varphi_0 \in C^0(\Omega \cup \partial \Omega)
\]

It can be obtained from the upper formula.

\[
\varphi_0 = \left[ (\nabla \omega_0)^2 - 1 \right] \omega_0^{-1} \quad (A.20)
\]

By expanding the deformed Bessel function, we can get
\( k_1 \left( 2\alpha \sqrt{\lambda} \right) \approx \frac{1}{2\alpha \sqrt{\lambda}} (\omega \rightarrow 0) \)

\( k_0 \left( 2\alpha \sqrt{\lambda} \right) \approx \ln \frac{1}{\omega \sqrt{\lambda}} \)  

(A.21)

Assuming the function \( \omega \) of form (A.19b), we can obtain the power series of \( \nabla \omega \), \( (\nabla \omega)^2 \) and \( \nabla^2 \omega \) expressed by \( \omega_0 \). By simplifying Eq. (A.20), we can get

\[ \nabla \omega = \nabla \omega_0 + 2\alpha_0 \nabla \omega_0 \phi + \alpha_0^2 \nabla \phi \]

\( (\nabla \omega)^2 = 1 + \alpha_0 \phi_0 + 4\alpha_0 \phi - \phi^2 + O (\alpha_0^3) \)

\[ \nabla^2 \omega = \nabla^2 \omega_0 + 2\phi + O (\omega_0) \]  

(A.22)

Substituting formulas (A.20) and (A.21) into formula (A.22). In order to ensure the continuity of kernel function \( K(x, \xi) \) in the integral domain, only \( \omega = \frac{1}{2} (\nabla^2 \omega_0 - \phi_0) \) is needed. Substitute the expression of \( \phi \) into formula (A.19b), we can get

\[ \omega = \omega_0 + \frac{1}{2} \alpha_0^2 (\nabla^2 \omega - \phi_0) = \omega_0 + \frac{1}{2} \alpha_0^2 \left[ \nabla^2 \omega_0 + 1 - \frac{1}{\omega_0} \right] \]  

(A.23)

Declarations

Author contribution statement

Jiarong Gan: Conceived and designed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Hong Yuan: Conceived and designed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Shanqing Li: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Qifeng Peng: Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

Huanliang Zhang: Analyzed and interpreted the data; Wrote the paper.

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Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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