**Observation of the weak-to-strong transition of quantum measurement in trapped ions**

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**Abstract**

Quantum measurement remains puzzling through the mist of their stormy history from the early birth of quantum mechanics to state-of-the-art quantum technologies. Here, we report the observation of measurement transition from weak to strong in a single trapped \(^{40}\text{Ca}^+\) ion system. By realizing a cat state of the ion's vibrational motion by pre- and post-selecting its internal electronic state, we observed the transition from weak value asymptotics to expectation value asymptotics. More precisely, the weak-to-strong measurement transition is characterized by a universal transition factor \(e^{-\Gamma^2/2}\), where \(\Gamma\) is a dimensionless coupling parameter related to the system-apparatus interaction. This transition, which continuously connects Aharonov’s weak measurement and von Neumann’s strong projective measurement, is capable of opening new experimental possibilities for testing quantum foundations, as well as novel applications for computation in trapped ions and other physical systems.
To date, quantum mechanics has been describing with unprecedented precision a variety of physical, chemical and even biological phenomena. In contrast, fundamental challenges remain, such as the measurement problem which is still considered to be an unsolved puzzle, persisting from the very birth of quantum mechanics. The mathematical formalism of quantum measurement was set forth by von Neumann (1932) [1], by treating both the measured system and measuring apparatus as being quantum, and coupling them through a simple interaction Hamiltonian. This process, also called “premeasurement” is unitary and followed by a non-unitary macroscopic amplification which selects only a single outcome $a_n$ being an eigenvalue of the measured operator $\hat{A}$ with eigenstate $|a_n\rangle$. By repeating this measurement procedure over a large ensemble of similarly prepared systems with state $|i\rangle$ the expectation value of $\hat{A}$ can be represented as $\langle A \rangle_i \equiv \langle i|\hat{A}|i\rangle/\langle i|i \rangle = \sum_{f} |\langle i|f \rangle|^2 \langle A \rangle_f$, when choosing an arbitrary complete orthonormal basis $\sum_{f} |f\rangle\langle f| = I$, $\langle f|f \rangle = 1$. Obviously, $\langle A \rangle_i$ depends only on the initial state and $\langle A \rangle_f \equiv \langle f|\hat{A}|f\rangle/\langle f|f \rangle$ only on the final state.

Instead, weak measurement enables to record the information regarding a pre- and post-selected ensemble into the “weak value”. The weak value was first proposed by Aharonov, Albert and Vaidman (1988) [2], and required both pre-selection (i.e., $|i\rangle$) and post-selection (i.e., $|f\rangle$) of the measured system as shown in Fig.1a. The weak value is defined as $\langle A \rangle_w = \langle f|\hat{A}|i\rangle/\langle f|i\rangle$, where $\hat{A}$ is again the measured observable. Note that the weak value is, in general, a complex number and can sometimes be “superweak” [3] or “anomalous” [4-8] lying well outside the spectrum of the measured operator. This has led to successful demonstrations of weak value amplification (WVA) techniques [9-11]. However, anomalous weak values demand the price of a small $p = |\langle f|i \rangle|^2$ due to the approximate orthogonality of the pre- and post-selected states. Alongside with practical applications, weak values and weak measurements have been linked to many fundamental topics and quantum paradoxes, in theory and experiment [12-18]. Note that the weak value $\langle A \rangle_w$ is quantitatively and conceptually different from the expectation value $\langle A \rangle_s$ [19]. Consequently, the observation of these two values can provide us a direct test whether the performed measurement was weak or strong, especially in the context of experimental realization [19-23]. Additionally, we can easily notice that the weak value would be equal to the expectation value when the pre- and post-selected states coincide. We call this special post-selection an “eigenstate projection”.

Here is the leading question: “How does the measurement outcome range from weak value ($\langle A \rangle_w$) to expectation value ($\langle A \rangle_s$) when one tunes the system-apparatus interaction from weak
to strong?” In this work, we would like to experimentally verify the weak-to-strong measurement transition process in a trapped ion setup in order to unify the weak-value and expectation-value predictions. The trapped ion system has been widely developed as one of the most promising candidates for achieving large-scale quantum simulation and computation [24-26]. In our context it offers us a well-designed measurement setup to observe the outcomes of the tunable interaction process between system and apparatus in the quantum regime. Following the von Neumann’s measurement scheme [1,2], we introduce a generic system-apparatus coupling Hamiltonian $H_I = -\gamma Ap$ in our setup of a single trapped ion. By tuning the coupling strength ($\gamma$) from weak to strong regimes we generate a cat state of the ion’s external vibrational motion as the measuring pointer. The measuring pointer of the ion would read out the observable transition from weak value to expectation value.

Here is the main idea of our measurement setup (which is also explained in Fig.1a): Suppose we would like to measure a Pauli operator (e.g., $A = \sigma_x$) in a two-level (qubit), trapped ions system, pre-selected state $|i\rangle$ and post-selected state $|f\rangle$. Meanwhile, we initialize the pointer state $|\phi\rangle$ as the ion’s axial vibrational ground-state wavepacket. With the pointer coupled to the pre- and post-selected qubit system during the time interval $(0, t)$, we can create a final cat state of the measuring pointer (i.e., a two-wavepacket superposition state). By controlling the coupling strength between the qubit system state and the pointer state of the trapped ion, as well as the duration of the interaction, we are able to readout the coupling-dependent observable $\langle \sigma_x \rangle$ of the system through the spatial displacement of the pointer (see Fig.1d).

**Setup.** We have built our experimental system employing trapped ions to observe the weak-to-strong measurement transition in accordance with [26, 27]. In our demonstration, a single trapped $^{40}\text{Ca}^+$ ion resonantly interacts with one bichromatic laser light having two frequency components ($729 \text{ nm} \pm \omega_z$, the red and blue sidebands of the ion) modulated by an acousto-optic modulator (AOM, $\omega_z = 2\pi \times 1.41 \text{ MHz}$) as depicted in Fig. 1b, c. We start by taking into account only two levels of the internal electronic state as “the measured qubit system” and the axial vibrational mode as “the measuring Gaussian pointer”, which results in the von Neumann coupling form in the Lamb-Dicke approximation [27]

$$H_I = \gamma_0 \sigma_x p,$$  

Here, the effective coupling parameter is $\gamma_0 = \eta \Delta_\omega$ in the Lamb-Dicke regime with Lamb-Dicke parameter $\eta = 0.08$ and the Rabi frequency is set at $\Omega = 2\pi \times 19 \text{ kHz}$. The laser field
strength and the ion ground-state wavepacket size of the axial vibrational mode are characterized by \( \Delta_z = \frac{\hbar}{\sqrt{2m\omega_z}} = 9.47 \text{ nm} \). The Pauli operator \( \sigma_x \) corresponds to the two-level qubit state jumping between the Zeeman sublevel \( S_{1/2} (m_J=1/2), |\downarrow\rangle \), and \( D_{5/2} (m_J=-1/2), |\uparrow\rangle \) with the lifetime of 1.1 s, in a magnetic field of 5.3 G, and correspondingly the momentum operator \( p \) for the ion axial motion oriented along the \( z \) direction [27]. A third rapidly decaying level \( P_{1/2} \) with extremely short-lived lifetime (about 7.1 ns) is used for laser cooling and qubit state readout with a laser field at 397 nm as shown in Fig. 1b. Also, the state reconstruction and spatial displacement of external axial motion after post-selection are measured indirectly by means of photon fluorescence detection, see the Methods or Ref. [27].

The resulting von Neumann Hamiltonian (1) offers the flexibility to split the initial pointer ground-state of axial vibrational motion into two superposed wavepackets. The measurement setup of capturing the cat state of pointer is schematically depicted in Fig. 1a. After Doppler cooling, sideband cooling and optical pumping [26, 27], we prepared the initial qubit state at \( |i\rangle = |\downarrow\rangle \) and its axial motion of the trapped ion in the ground state \( |\phi(z)\rangle = (2\pi\Delta_z^2)^{-1/4} \exp (-z^2/4\Delta_z^2) \) with size \( \Delta_z = 9.47 \text{ nm} \). After applying the system-apparatus interaction (1) for a controllable time duration \( t \), we then post-select the final qubit system at \( |f\rangle = \cos \theta |\uparrow\rangle - \sin \theta |\downarrow\rangle \) by the projective measurement, in which the parameter \( \theta \) is the vector angle between the post- and pre-selections on the Bloch sphere of qubit system. Considering the pre- and post-selection as performed on the qubit system, we can anticipate the outcome of the measured Pauli operator in both of coupling configurations. In the context of the weak measurement scheme, the weak value of the Pauli-x observable is

\[
\langle \sigma_x \rangle_w = \frac{|f|^2 |\sigma_x| |i\rangle \langle i|}{|f\rangle \langle f|} = - \cot \theta \quad (2)
\]

On the other hand, in the standard quantum measurement procedure the expectation value, solely involving the final qubit state, is given by

\[
\langle \sigma_x \rangle_s = \frac{|f|^2 |\sigma_x| |f\rangle \langle f|}{|f\rangle \langle f|} = - \sin 2\theta \quad (3)
\]

Usually, the weak value (2) is not equal to the expectation value (3) except for the eigenstate projection (e.g., at \( \theta = \pi/4 \)). If we project the qubit state as a vector on Bloch sphere, the post-selected final state is orthogonal at angle \( \theta = 0 \) and parallel at \( \theta = \pi/2 \) to the initial state.
Again, we claim based on our experimental demonstration that the essential difference between the weak and strong measurement schemes is completely captured by the observable’s physical value, being a weak value in (2) and an expectation value in (3).

Now, we are ready to utilize the cat state in our setup of a single trapped ion. The measurement process as depicted in Fig. 1a-c can be further described in an exact mathematical form

\[
\langle f | \exp \left( -\frac{i}{\hbar} \int_0^t H_i(t')dt' \right) | i \rangle \otimes | \phi(z) \rangle
\]

\[
= -\left( \frac{\sin \theta + \cos \theta}{2} \right) | \phi(z + \gamma_0 t) \rangle + \left( \frac{\cos \theta - \sin \theta}{2} \right) | \phi(z - \gamma_0 t) \rangle
\]

After normalization with the initial Gaussian pointer $| \phi(z) \rangle$, the final vibrational motion state of the ion (i.e., the cat state) is thus obtained as

\[
| \text{cat}_\theta \rangle = \frac{-\sin \left( \frac{\pi}{4} \right) | \phi(z + \gamma_0 t) \rangle + \cos \left( \frac{\pi}{4} \right) | \phi(z - \gamma_0 t) \rangle}{\sqrt{1 - \cos(2\theta)e^{-\Gamma^2/2}}}
\]

We define the dimensionless factor $\Gamma = \gamma_0 t / \Delta z$ that is the ratio between the corresponding coupling length and the pointer’s size. The Schrödinger cat state consists of two superposed Gaussian wavepackets with central position shifts $(\pm \gamma_0 t)$. In general, we can measure the cat’s spatial displacement, which is

\[
\langle \delta z \rangle_\theta = \frac{\langle \text{cat}_\theta | \hat{\delta} | \text{cat}_\theta \rangle}{\langle \text{cat}_\theta | \text{cat}_\theta \rangle} = -\frac{\gamma_0 t \sin(2\theta)}{1 - \cos(2\theta)e^{-\Gamma^2/2}}
\]

In the weak-coupling regime $\Gamma \ll 1$, we found $\langle \delta z \rangle_\theta |_{\Gamma \rightarrow 0} = -\gamma_0 t \cot(\theta) = \gamma_0 t \langle \sigma_x \rangle_w$, corresponding to the weak-valued Pauli operator (Eq. 2). On the other hand, in the strong-coupling regime $\Gamma \gg 1$, we obtained $\langle \delta z \rangle_\theta |_{\Gamma \rightarrow \infty} = -\gamma_0 t \sin(2\theta) = \gamma_0 t \langle \sigma_x \rangle_s$, in accordance with the expectation value (3). The correspondence in both limits is not a coincidence, it reveals that the essence of quantum measurement stems from creating entangled information between the system and pointer states via von Neumann coupling (Eq.1). The strength of entanglement ($\Gamma$) affects the outcomes of measurement, and as a result, determines the weak value and expectation value in a unified way (Eq.5). At the selection angle $\theta = \pi/4$, we notice that $\langle \delta z \rangle_{\pi/4} = -\gamma_0 t$ and $\langle \sigma_x \rangle_w = \langle \sigma_x \rangle_s$ are special since the final pointer state reduces to a single shifted wavepacket $| \phi(z + \gamma_0 t) \rangle$, and correspondingly the final qubit system state (i.e., $-\rangle = (|1\rangle - |\downarrow\rangle)/\sqrt{2}$) becomes the eigenstate projection of the Pauli-x operator. When
rotating the post-selection angle $\theta$ from 0 to $\pi/4$, one can expect a WVA, but in the range from $\pi/4$ to $\pi/2$, the weak-value is instead suppressed unfortunately in comparison to the corresponding expectation value.

**Weak value versus expectation value.** Fig. 1d shows the relative spatial displacement of the ion’s motional cat state $\langle \delta z \rangle_{\theta} / \gamma_0 t$ as a function of the post-selection angle $\theta$ at different ratios $\Gamma$, where the parameter $\gamma_0 t$ is the central shift of the ion’s wavepacket related to the interaction. As dedicated from the correspondence between $\langle \delta z \rangle_{\theta} / \gamma_0 t$ of the pointer and $\langle \sigma_x \rangle$ of the system, all the possible curves of measured observables lied in between the weak-value limit (asymptotics) (red dashed line) and the expectation-value limit (asymptotics) (black dot-dashed line) are presented in the shadow region of Fig.1d. When decreasing the ratio $\Gamma \to 0$, the cat’s spatial displacement shows “blow-up” characteristics which indicate the anomalous weak value amplification (WVA) at the nearly-orthogonal post-selection regime $\theta \to 0$. This anomalous phenomenon exhibited by weak measurement has been applied to detection of extremely weak signals such as the spin Hall Effect of light [9], and could provide practical advantages in the presence of detector saturation and technical noise [28, 29].

Fig. 1e explains the cat state interaction configuration from the measuring pointer in both limits of weak measurement (WM) and strong measurement (SM). Note that the overlap of two superposed wavepackets is essential for the survival of the weak value observation. Under the condition $\gamma_0 t > \Delta z$, the overlap effect diminishes and the two sidebands are well separated, eventually resulting in the expectation value. The quantum interference in the cat state is relevant in two aspects. First, the interference between the initial ground-state and the final pointer state is registered and forces the weak value with dependence on the system-apparatus initialization, while the expectation value is rather independent of the initial wavefunction. Second, both the coupling strength and the wavepacket size of the measuring pointer are relevant to determine the weak-to-strong measurement transition. The ratio $\Gamma$ between those two parameters can characterize the measurement sensitivity of the system and apparatus coupling.

Note that the spatial displacement of the cat state by itself is also a weak value of the measuring apparatus $\langle z \rangle_w = \langle \text{cat}_\theta | z \phi(z) \rangle / \langle \text{cat}_\theta | \phi(z) \rangle = -\gamma_0 t \cot(\theta)$, which corresponds to the weak value (2) of the Pauli-x operator. This correspondence between system and apparatus is intriguing because it indicates that the weak value $\langle \sigma_x \rangle_w$ always attaches to the weak value
transition is to discussed corresponds to the squeezing technique of the ion measurement implies measuring apparatus. The decrease pointer contribution selection ion effective between in the observed at different post selection, we would like to strong coupling regime (~0.02) to strong coupling regimes (~2.0). Note that the displacement $\langle \delta z \rangle_\theta$ observed at different post-selection angles (the colored points) is mapped to the same position in the theoretical transition curve ($e^{-\Gamma^2/2}$). Consequently, the nearly-perfect agreement between theory and experiment indicates that the weak-to-strong transition is almost universal for Gaussian apparatus, regardless of the specific pre- and post-selections of the measured qubit system [19]. Besides, other alternative transition factors are anticipated theoretically [30-33], for instance, Lorentzian or exponential.

Back to our trapped ion setup (Fig.1c), the measurement transition can be implemented independently in three ways, each corresponding to one of the three parameters $\gamma_0, t, \Delta_z$. The effective coupling parameter $\gamma_0$ determines the coupling strength between the qubit and the ion’s axial vibrational motion (1), and the duration $t$ determines the coupling between pre-selection and post-selection, where both of them are proportional to the interference contribution. More interestingly, the interference factor $\Gamma$ is inversely proportional to the pointer wavepacket size (or point uncertainty) $\Delta z$ that determines the sensitivity of the measuring apparatus. The decrease in the wavepacket size (i.e., squeezing the ground-state) implies high sensitivity of the apparatus and the loss of wavepacket overlap results in strong measurement (Fig. 1e). In a single trapped ion system, the change of pointer sensitivity corresponds to the squeezing technique of the ion’s ground-state of the axial motion as discussed in [24, 34]. Nevertheless, the simplest approach to implement the weak-to-strong transition is to tune the interaction duration ($t$) as we did in our trap ion setup, see the Methods.
State reconstruction at the two coupling limits. Next, we reconstruct the cat state of the ion’s axial motion to demonstrate the landscape of the measured Pauli-x operator (Fig. 3a) during the weak-to-strong transition. The typical cat state reconstructions in both the weak-value and expectation-value asymptotics are shown in Fig. 3b-f. First, Fig. 3b-d presents the probability density distribution of pointer in the weak coupling regime with \( \Gamma = 0.1 \), or 0.04 and angles \( \theta = 0.02, \pi/4, 1.5 \) between pre- and post-selection states, which correspond to nearly-orthogonal, eigenstate projection and nearly-parallel post-selection, respectively. Figs. 3b-d were reconstructed at the weak coupling regime, and the corresponding spatial density distributions for a given qubit post-selection at nearly-orthogonal, eigenstate projection and nearly-parallel respectively are plotted for comparison with different selection angles. Note that the large pointer shift (WVA) can be found in the nearly-orthogonal post-selection, in which all the data points accord well with our predictions (see Fig.3). In our experimental demonstration, we have amplified a tiny spatial displacement of \( \sim 4 \) Å of the trapped ion to 10 nm by post-selecting the appropriate internal electronic qubit state in nearly orthogonal regime \( \theta = 0.02, \gamma_0 = 0.04, t = 4 \) μs, that is, signal amplification by factor of 25 is reported (see Ref. [27]). Note that the trade-off for achieving such large amplification is the small success probability \( p = \sin^2 \theta \approx 4 \times 10^{-4} \).

Secondly, Figs. 3e-g are the measuring pointer cat states reconstructed at strong coupling \( \Gamma = 1.0 \). In the nearly-orthogonal regime, the two sidebands can be distinguished, eventually yielding the expectation-value observation (Fig. 1d,e). At \( \theta = \pi/4 \) there is only a single wavepacket shift to the left with displacement \( (-\gamma_0 t) \) in the entire coupling regime regardless the interaction strength. Finally, at the nearly parallel post-selection \( \theta = 1.5 \), both the weak value and expectation value tend to zero, thus the measuring pointer remains at the initial position of its ground state with no shift (Fig. 3d,g).

To address the point, the measurement landscape (Fig.3) shows that the observables depend on the system-apparatus coupling configuration, such as the pre- and post-selection of the system states and the sensitivity of the measuring pointer of the von Neumann coupling scheme. The outcome of the measurement process in the weak coupling limit could be a complex number due to the interference between the pre- and post-selection states [27]. However, the expectation value is always real, and thus in the weak-to-strong transition the outcome becomes a real number eventually. For our concern, it is notable that the measurement landscape enables us to characterize all the observables comprehensively based on the system-apparatus coupling
procedure either by Aharonov’s weak measurement or von Neumann’s projective measurement scheme.

**Conclusion.** In summary, the weak-to-strong measurement transition predicts the weak value asymptotics and expectation value asymptotics of the measurement can be continuously related. Full control of this transition was confirmed by our experimental demonstration in a single trapped $^{40}\text{Ca}^+$ ion system. The quantum measurement process is investigated in detail by preparing the ion’s ground-state axial motion as the measuring apparatus. The transition factor $\exp\left(-\Gamma^2/2\right)$ is universal for Gaussian-type apparatuses that also have been found in the classical-to-quantum transition of light-matter interactions [35]. Prospectively, our measurement transition from weak values to expectation values offers a promising avenue for reexamining fundamental topics and core issues related to quantum measurements (perhaps even the measurement problem itself), and may also emphasize the versatility of trapped ion systems for future applications.

**Methods.** Some technical details regarding the experimental setup can be found in Ref [27], here we give a brief summary. The axial motional mode of a single trapped ion with secular frequency of $\omega_z = 2\pi \times 1.41$ MHz is cooled to its motional ground state with Doppler cooling and resolved sideband cooling methods. A narrow linewidth laser at 729 nm is used to coherently couple the qubit which we choose the energy levels $S_{1/2}(m_J = -1/2)$ and $D_{5/2}(m_J = -1/2)$ as $\downarrow$ and $\uparrow$ respectively. The bichromatic laser pulse is generated by injecting mixed RF signal with two frequency components resonant with red and blue sidebands of the axial mode to an acousto-optical modulator (AOM). The qubit transition is isolated by a gap of 8.8 MHz from its nearest internal state transitions with a magnetic field of 5.3 G. The beam goes through the two end-caps with almost 0 degree with respect to $z$ axis of the trap resulting in a Lamb-Dicke parameter of $\eta \approx 0.08$. The internal state of the ion is read by using the electron shelving technique with a detection time of 300 µs. The heating rate of the axial motional mode is about 70 quanta per second, and the coherence time of the Fock state superposition $\left(|1\rangle+|0\rangle\right)/\sqrt{2}$ has been measured to be 5.0 ms. Due to the large magnetic fluctuations induced by the AC-power line, we trigger the experimental cycles at 50 Hz and the coherence time of the qubit has been measured using Rasmey fringe to be around 1.1 ms.
To create the weak coupling between the internal state and axis motional state, we employ a bichromatic field (with AOM) to create the Schrödinger cat state of the two superposition states, which can be written as $|\psi\rangle = \frac{1}{\sqrt{2}} (|+\rangle, |\alpha\rangle - |-, -\alpha\rangle)$, where $|+\rangle$ and $|\rangle$ states are the eigenstates of $\sigma_x$, and $\alpha = \eta \omega / 2$ is the displacement proportional to the Lamb-Dicke parameter $\eta$, Rabi strength $\Omega$, and pulse time $t$. The displacement is calibrated experimentally by using the relation $\bar{n} = |\alpha|^2$, and the Rabi strength is set to be $\Omega = 2\pi \times 19$ kHz, as shown in Fig. M1a.

In the trapped ion system, the ion wavepacket cannot be measured directly, therefore a state dependent displacement operator $U_z = \exp(-i k z \sigma_z / 2)$ is used to obtain the probability distribution, and this method could be regard as a general method for reconstructing motional wavepacket of trapped ions [20, 23]. By applying this operator to a quantum qubit state and following the measurement of $\sigma_z$ returns the observable $O(k) = U_z^* \sigma_z U_z = \cos(k z) \sigma_z + \sin(k z) \sigma_y$. Setting the internal qubit state to be eigenstates of $\sigma_z$ and $\sigma_y$, $\langle \sin(k z) \rangle$ and $\langle \cos(k z) \rangle$ can be found respectively. Theoretically, the probability distribution density of wavepacket $|\phi(z)|^2$ could be extracted by using the Fourier transform of $\langle \cos(k z) \rangle + i \langle \sin(k z) \rangle$, however, a small number of experimental data points cannot provide enough information for this method to work. Instead, a constrained least-square optimization method based on convex optimization [36] is employed, the details could be found in Ref [37]. In some cases of our experiments (see Fig.2), we only concern the center shift of the ion wavepacket. The above equation entails that $\frac{d}{dk} \langle O(k) \rangle |_{k=0} = \langle z \sigma_y \rangle$, therefore the center of wavepacket could be found by preparing the internal state to be the eigenstate of $\sigma_y$ and take the slope of fitting pointers in Fig. M1a.

However, we should note that the only direct measurement in trapped ions is the Pauli operator $\sigma_z$. Experimentally the measured qubit state is always rotated to $|\uparrow\rangle$ after post-selection, so we first apply a $\pi / 2$ carrier transition laser pulse to coherently prepare the internal state to be the eigenstate of $\sigma_y$, afterwards the operation $U_z$ is applied with various time pulse (max 10 $\mu$s in the experiment) and $\langle \sin(k z) \rangle$ is then obtained, finally $\langle \delta z \rangle$ can thus be extracted by
fitting the data with a linear model as shown in Fig. M1b. Note that the same method can be used for the measurement of $\langle p \rangle$ for detecting the imaginary part of weak value [27], the only difference for the imaginary weak value is that we apply the operation $U_p = \exp(-ikp\sigma_x / 2)$ after post-selection detection. The success rates of postselection are different for each data point of the measurement $\langle \sin(kz) \rangle$, therefore we should consider different error budget for each data point when we perform the data analysis. Here we use the weighted fitting method instead of regular fitting for extracting the slope, and standard deviation derived from the projection noise of each point is used as the weighting parameter. Fig. M1b shows an example for measuring $\langle \delta Z \rangle$ with parameter $\Gamma = \gamma_\theta / \Delta = \eta \Omega t = 0.4$ and $\theta = 0.2$.

Finally, for comparison, we represent the cat state in phase-space representation with Wigner function distribution defined as $W(z, p) = \int \psi(z + x / 2)\psi^\ast(z + x / 2) e^{-ipx/h} dx$, by substituting the cat state (4), one can obtain

$$
W(z, p) = \left(1 - \cos(20)e^{-i\pi/2}\right)^{-1}\left\{\sin^2\left(\theta + \pi/4\right)W^{(0)}(z + \gamma_0 t, p) + \cos^2\left(\theta + \pi/4\right)W^{(0)}(z - \gamma_0 t, p)\right.
- \cos(20)\cos(2\gamma_0 pt)W^{(0)}(z, p)\right\},
$$

(M1)

where the initial Wigner function $W^{(0)}(z, p) = (2\pi)^{-1}\exp\left(-z^2/4\Delta_z^2 - p^2/4\right)$ describes the initial ground state of the ion axial motion in $(z, p)$ phase space. Correspondingly, the spatial projection of Wigner function leads to the spatial density distribution of the cat state which is $|\psi(z)|^2 = \int W(z, p) dp = \langle \text{cat}_o | \text{cat}_o \rangle$. 

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Competing financial interests
The authors declare no competing financial interests.

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Figures:

(a) The pre- and post-selected states, and the initialization of the apparatus in the two system-apparatus coupling regimes. (b-c) The initialization and measurement of a single trapped ion system in the quantum weak measurement scheme. (d) The phase-diagram of measurement as a function of post-selection angle \( \theta \) ranging from the weak value limit to expectation value limit according to the governing factor \( \Gamma \). The weak value amplification (WVA) emerges in the nearly-orthogonal regime \( \theta \to 0, \Gamma \to 0 \). (e) The Schrödinger cat superposed wavepackets are overlapping, with an interference contribution, in the weak measurement (WM) regime, yet separable, with appearance of two sidebands, in the strong measurement (SM) regime.
Figure 2] The measurement transition from weak value to expectation value asymptotics. The purple curve is the theoretical prediction $\exp\left(-\frac{\Gamma^2}{2}\right)$, where $\Gamma = \frac{\gamma_U t}{\Delta_z}$. The points $\left(\langle\delta z\rangle_0 + \gamma_U t \sin(2\theta)\right) / \left(\langle\delta z\rangle_0 \cos 2\theta\right)$ are drawn from our experimental data (inset: the ion’s axial displacement), taking into account the variation of post-selection angles $\theta = 0.05, 0.1, 0.2$ respectively. The match between prediction and experiment indicates that the weak-to-strong measurement transition has no dependence on postselection of the measured system.
Figure 3] The measurement landscape of the relative measuring pointer’s displacement $\langle \delta z \rangle_\theta / \gamma_0 t$ (Eq. 5 in the text) (a) as a function of transition factor $\Gamma$ and selection angle $\theta$. It corresponds to the observable of Pauli operator $\sigma_x$. State reconstructions of the pointer state are shown in the weak value regime $\Gamma = 0.1$, or $0.04$ (b-d) and expectation value regime $\Gamma = 1.0$ (e-g), respectively. Three typical cases of pre- and post-selection are reconstructed experimentally: the nearly-orthogonal $\theta = 0.02$ (b, e), the eigenstate projection $\theta = \pi/4$ (c, f) and the nearly-parallel $\theta = 1.5$ (d, g), respectively. Especially, (b) shows the weak value amplification of the pointer state that corresponds to the peak of (a) in weak measurement region, and (e) shows the pointing state splitting in strong measurement region. For eigenstate projection, the observable of system assigned a single shifted wavepacket (c, f), and in the nearly-parallel region, both the outcomes are trivial since $\sin 2\theta = \sin 3 \approx 0$ and $\langle \delta z \rangle_\theta = \langle \sigma_x \rangle = 0$. All theoretical red curves of the pointer cat state distribution $|\phi(z)|^2$ (Eq. 4) match nicely to the experimental observations in the yellow histograms (see data analysis in Fig.M1 of Methods). The measurement landscape can characterize the quantum weak-to-strong transition from weak-value limit to expectation value limit.
Figure M1 | (a). Calibration of the displacement $\alpha$ as a function of the bichromatic pulse time with the low Rabi strength. The black dots indicate the value of $\alpha$ obtained from the average phonon number $\pi$. The red line denotes the fitting of data points and the corresponding Rabi strength of the bichromatic field. (b). Data analysis for obtaining the average displacement. The black curve denotes the full simulation of $\langle \sin(kz) \rangle$ with parameters $\Gamma = 0.2$ and $\theta = 0.2$. The red cycles are experimental data points and the red line shows the weighted fitting of data points for extracting the displacement $\langle \delta z \rangle$. 