INTERPOLATING SEQUENCES FOR THE NEVANLINNA AND SMIRNOV CLASSES

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ABSTRACT. We give analytic Carleson-type characterisations of the interpolating sequences for the Nevanlinna and Smirnov classes. From this we deduce necessary and sufficient geometric conditions, both expressed in terms of a certain non-tangential maximal function associated to the sequence. Some examples show that the gap between the necessary and the sufficient conditions cannot be covered. We also discuss the relationship between our results and the previous work of Naftalić for the Nevanlinna class, and Yanagihara for the Smirnov class. Finally, we observe that the arguments used in the previous proofs show that interpolating sequences for “big” Hardy-Orlicz spaces are in general different from those for the scale included in the classical Hardy spaces.

1. INTRODUCTION AND BASIC DEFINITIONS

Let $\Lambda$ be a discrete sequence of points in the unit disk $\mathbb{D}$. For a space of holomorphic functions $X$, the interpolation problem consists of describing the trace of $X$ on $\Lambda$, i.e. the set of restrictions $X|\Lambda$, which is regarded as a sequence space. One way of defining interpolating sequences is to fix a priori a natural trace space $\ell$ and look for conditions ensuring that $X|\Lambda = \ell$. The second possibility is to require the trace space $X|\Lambda$ to be ideal, i.e. $\ell^\infty X|\Lambda \subset X|\Lambda$ (see definitions below). This approach is motivated by the property of unconditional bases to be absolutely convergent (see [Nik97, Section C.3.1 (Volume 2)] for more about this, in particular, Theorem C.3.1.4) and it is natural at least for those spaces that are stable under multiplication by $H^\infty$, the space of bounded holomorphic functions on $\mathbb{D}$.

For many spaces (for instance Hardy and Bergman spaces), both definitions turn out to be equivalent, provided that the a priori fixed trace space is chosen in a natural way. The situation changes for the non-Banach classes we have in mind. In order to illustrate this, we briefly discuss

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the known results for the spaces we will deal with, the Nevanlinna class
\[ N = \left\{ f \in \text{Hol}(\mathbb{D}) : \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{it})| \, dt < \infty \right\} \]
and the related Smirnov class
\[ N^+ = \left\{ f \in N : \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{it})| \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(e^{it})| \, dt \right\} \]

In 1956, in a very interesting article, Naftalević [Na56] described the sequences \( \Lambda \) for which the trace \( N|\Lambda \) coincides with the sequence space \( l_{Na} = \{(a_\lambda)_\lambda : \sup_\lambda (1 - |\lambda|) \log^+ |a_\lambda| < \infty \}. \) The choice of \( l_{Na} \) is motivated by the fact that \( \sup_z (1 - |z|) \log^+ |f(z)| < \infty \) for \( f \in N, \) and this growth is attained. Unfortunately, the growth condition imposed in \( l_{Na} \) forces the sequences to be confined in a finite union of Stolz angles. Consequently a big class of Carleson sequences (i.e. sequences such that \( H^\infty|\Lambda = l^\infty \)), namely those tending tangentially to the boundary, cannot be interpolating in the sense of Naftalević. This does not seem natural, for \( H^\infty \) is in the multiplier space of \( N \) (since \( N \) is an algebra, its multiplier space obviously coincides with itself). Further comments on Naftalević’s result can be found in [HaMa01].

A similar problem occurs in the Smirnov class. In [Ya74], Yanagihara proved that in order that \( N^+|\Lambda \) contains the space \( l_{Ya} = \{(a_\lambda)_\lambda : \sum_\lambda (1 - |\lambda|) \log^+ |a_\lambda| < \infty \}, \) it is sufficient that \( \Lambda \) is a Carleson sequence (he also gave a necessary condition that we will discuss below). However there are Carleson sequences such that \( N^+|\Lambda \) does not embed into \( l_{Ya} [Ya74, \text{Theorem 3}] \).

In conclusion, it seems quite difficult to find a “natural” trace for these spaces. Therefore we consider the following definitions.

**Definition.** A sequence space \( l \) is called ideal if \( \ell^\infty l \subset l \), i.e. whenever \((a_n)_n \in l \) and \((\omega_n)_n \in \ell^\infty \), then also \((\omega_n a_n)_n \in l \).

**Definition.** Let \( X \) be a space of holomorphic functions in \( \mathbb{D} \). A sequence \( \Lambda \subset \mathbb{D} \) is called free interpolating for \( X \) if \( X|\Lambda \) is ideal. We denote \( \Lambda \in \text{Int} X \).

Since the Nevanlinna and Smirnov classes contain the constants, free interpolation for these classes entails the existence of a nonzero function \( f \in N \) vanishing on \( \Lambda \), hence the Blaschke condition \( \sum_\lambda (1 - |\lambda|) < \infty \) is necessary and will be assumed throughout this paper.

**Remark 1.1.** For any function algebra \( X \) containing the constants, \( X|\Lambda \) is ideal if and only if \( \ell^\infty \subset X|\Lambda \).

The inclusion is obviously necessary. In order to see that it is sufficient notice that, by assumption, for any \((\omega_\lambda)_\lambda \in \ell^\infty \) there exists \( g \in X \) such that \( g(\lambda) = \omega_\lambda \). Thus, if \((f(\lambda))_\lambda \in X|\Lambda \), the sequence of values \((\omega_\lambda f(\lambda))_\lambda \) can be interpolated by \( fg \in X \).

It is then clear that \( \text{Int } N^+ \subset \text{Int } N \).

We shall use Remark 1.1 not only for the classes \( N \) and \( N^+ \), but also for “big” Hardy-Orlicz classes (see Section 6).
In order to state our results we need to recall some standard facts about the structure of the Nevanlinna and Smirnov classes (we refer to [Gar81], [Nik02] or [RosRov] for general references). Let $d\sigma$ denote the normalised Lebesgue measure in the unit circle $\mathbb{T}$.

A function $f$ is called outer if it can be written in the form

$$f(z) = C \exp \left\{ \log v(\zeta) d\sigma(\zeta) \right\},$$

where $|C| = 1$, $v > 0$ a.e. on $\mathbb{T}$ and $\log v \in L^1(\mathbb{T})$. Such a function is the quotient $f = f_1/f_2$ of two bounded outer functions $f_1, f_2 \in H^\infty$ with $\|f_i\|_\infty \leq 1$, $i = 1, 2$. In particular, the weight $v$ is given by the boundary values of $|f_1/f_2|$. Setting $w = \log v$, we have

$$\log |f(z)| = P[w](z) := \int_\mathbb{T} P(z, \zeta) w(\zeta) d\sigma(\zeta),$$

where

$$P(z, \zeta) = \text{Re} \left( \frac{\zeta + z}{\zeta - z} \right) = \frac{1 - |z|^2}{|\zeta - z|^2}$$

is the Poisson kernel in $\mathbb{D}$. Formula (1) allows to freely switch between assertions on outer function $f$ and assertions on the associated measures $wd\sigma$.

Another important family of functions in this context are inner functions: $I \in H^\infty$ such that $|I| = 1$ almost everywhere on $\mathbb{T}$. Any inner function $I$ can be factorised into a Blaschke product $B_\Lambda = \prod \lambda_n b_{\lambda_n}$ carrying the zeros $\Lambda = \{\lambda_n\}_n$ of $I$, where $b_{\lambda}(z) = \frac{|\lambda|}{1 - \overline{\lambda} z}$ denotes the usual M"obius transformation, and a singular inner function $S$ defined by

$$S(z) = \exp \left\{ - \int_\mathbb{T} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right\},$$

for some positive Borel measure $\mu$ singular with respect to Lebesgue measure.

According to the Riesz-Smirnov factorisation, any function $f \in N^+$ is represented as

$$f = \alpha \frac{BSf_1}{f_2},$$

where $f_1, f_2$ are outer with $\|f_1\|_\infty, \|f_2\|_\infty \leq 1$, $S$ is singular inner, $B$ is a Blaschke product and $|\alpha| = 1$. Similarly, functions $f \in N$ are represented as

$$f = \alpha \frac{BS_1f_1}{S_2f_2},$$

with $f_i$ outer, $\|f_i\|_\infty \leq 1$, $S_i$ singular inner, $B$ is a Blaschke product and $|\alpha| = 1$.

Given the Blaschke product $B$ with zero-sequence $\Lambda$, denote $B_\lambda = B/b_\lambda$.

Our main results are the following.

**Theorem 1.2.** Let $\Lambda$ be a sequence in $\mathbb{D}$. The following statements are equivalent:

(a) $\Lambda \in \text{Int } N^+$. 

There exists an outer function $g$ such that
\[ |B_\lambda(\lambda)| \geq |g(\lambda)| \text{ for all } \lambda \in \Lambda. \]

There exists a positive function $w \in L^1(\mathbb{T})$ such that
\[ \log \frac{1}{|B_\lambda(\lambda)|} \leq P[w](\lambda) \text{ for all } \lambda \in \Lambda. \]

The trace space is given by:
\[ N^+|\Lambda = l_{N^+} := \{(a_\lambda) : \text{ there exists a function } u \in L^1(\mathbb{T}) \text{ such that } P[u](\lambda) \geq \log^+ |a_\lambda|, \lambda \in \Lambda \}. \]

Note that in (b) there is no harm in assuming that $g \in H^\infty$ with $\|g\|_\infty \leq 1$, i.e. $w < 0$. This is easily achieved by neglecting the denominator in the quotient $g = g_1/g_2$, $\|g_1\|_\infty \leq 1$. Then it is clear from (1) above that the assertions (b) and (c) are essentially the same.

In case $|g|$ is uniformly bounded from below by a positive constant on the sequence $\Lambda$, condition (2) is nothing but the classical Carleson condition for interpolation in $H^\infty$.

According to the Riesz-Smirnov factorisation described above, the essential difference between Nevanlinna and Smirnov functions is the extra singular factor appearing in the denominator in the Nevanlinna case. This is reflected in the corresponding result for free interpolation in $N$.

Given a measure $\mu$ on $\mathbb{T}$, let $P[\mu](z)$ denote the integral obtained from (1) by replacing $w \, d\sigma$ by $d\mu$.

**Theorem 1.3.** Let $\Lambda$ be a sequence in $\mathbb{D}$. The following statements are equivalent:

(a) $\Lambda \in \text{Int } N$.

(b) There exist an outer function $g$ and a singular inner function $S$ such that
\[ |B_\lambda(\lambda)| \geq |g(\lambda)S(\lambda)| \text{ for all } \lambda \in \Lambda. \]

(c) There exists a positive finite measure $\mu$ on $\mathbb{T}$ such that
\[ \log \frac{1}{|B_\lambda(\lambda)|} \leq P[\mu](\lambda) \text{ for all } \lambda \in \Lambda. \]

(d) The trace space is given by:
\[ N|\Lambda = l_N := \{(a_\lambda) : \text{ there exists a positive finite measure } \mu \text{ on } \mathbb{T} \text{ such that } P[\mu](\lambda) \geq \log^+ |a_\lambda|, \lambda \in \Lambda \}. \]

As before, in (b) we can assume $g = g_1 \in H^\infty$, $\|g_1\|_\infty \leq 1$. Let $\mu_S$ denote the singular measure associated with $S$, then the equivalence of (b) and (c) can be checked using the measure $d\mu = \log(1/|g_1|) \, dm + d\mu_S$.

By Smirnov’s theorem, any positive harmonic function on $\mathbb{D}$ is the Poisson extension of some positive finite measure on $\mathbb{T}$. Thus our results are closely related to the following problem.
Problem. Given a Blaschke sequence $\Lambda$, describe the sequences of positive values $\{m_\lambda\}_\lambda$ for which there exists a positive harmonic function $u$ in the unit disk such that $u(\lambda) \geq m_\lambda$ for all $\lambda \in \Lambda$.

A geometric description of such sequences must depend on both the behaviour of $\{m_\lambda\}_\lambda$ and the geometry of $\Lambda$. We only have an answer in two extremal cases: when $\Lambda$ approaches the unit circle non-tangentially, and when it does it very tangentially (in the sense that the arcs $\{\zeta \in \mathbb{T} : |\arg \zeta - \arg \lambda| < (1 - |\lambda|)^{1/2}\}$ are pairwise disjoint).

In what follows we would like to see some geometric implications of the analytic conditions above. To begin with, we would like to state the maybe surprising result that separated sequences are interpolating for the Smirnov class (and hence the Nevanlinna class). Recall that a sequence $\Lambda$ is called separated if

$$\delta(\Lambda) := \inf_{\lambda \neq \lambda'} |b_\lambda(\lambda')| > 0.$$  

For such sequences there always exists an outer function satisfying (2) (see Proposition 3.1), thus the following corollary is immediate from Theorem 1.2.

**Corollary 1.4.** Let $\Lambda$ be a separated Blaschke sequence. Then $\Lambda \in \text{Int } \mathbb{N}^+$ (and hence $\Lambda \in \mathbb{N}$).

More precise geometric conditions can be given in terms of a non-tangential maximal function associated with $\Lambda$. For $\zeta \in \mathbb{T}$ and $\alpha > 1$ define the Stolz angle

$$\Gamma_\alpha(\zeta) = \{z \in \mathbb{D} : |z - \zeta| \leq \alpha(1 - |z|^2)\}.$$  

In our considerations the value of $\alpha$ is of no importance, so we will write $\Gamma(\zeta)$ for the generic Stolz angle with fixed aperture $\alpha$. For a given $\Lambda$ consider now the non-tangential maximal function

$$M_\Lambda(\zeta) = \sup_{\lambda \in \Gamma(\zeta)} \log \frac{1}{|B_\lambda(\lambda)|}.$$  

Let $L^1_w$ denote the weak-$L^1$ space and let

$$L^1_{w,0}(\mathbb{T}) = \{f : \lim_{t \to \infty} t\sigma(\{\zeta : |f(\zeta)| > t\}) = 0\}.$$  

**Corollary 1.5.** Let $\Lambda$ be a sequence in $\mathbb{D}$.

(a) If $\Lambda \in \text{Int } \mathbb{N}^+$ then $M_\Lambda \in L^1_{w,0}(\mathbb{T})$. If $\Lambda \in \text{Int } \mathbb{N}$ then $M_\Lambda \in L^1_w(\mathbb{T})$.

(b) If $M_\Lambda \in L^1(\mathbb{T})$ then $\Lambda \in \text{Int } \mathbb{N}^+$ (and hence $\Lambda \in \text{Int } \mathbb{N}$).

**Proof.** (a) is a consequence of (c) in Theorem 1.3. Indeed, it is a general fact that the non-tangential maximal function of the Poisson transform of a positive finite measure belongs to $L^1_{w,0}(\mathbb{T})$ (see for instance [Gar81, p.28-29]). A more careful analysis of the cited result shows that if $\mu$ is absolutely continuous, then its Poisson transform is in $L^1_{w,0}(\mathbb{T})$, which yields the result for $\mathbb{N}^+$.

In order to prove (b) consider the arcs associated with $\lambda \in \mathbb{D}$, defined as

$$I_\lambda = \{\zeta \in \mathbb{T} : |\arg \zeta - \arg \lambda| \leq \pi(1 - |\lambda|)\}$$  

(6)
Take \( w = (1 + \pi^2)M_{\Lambda} \) and apply (c) in Theorem 1.2:

\[
P[w](\lambda) \geq \frac{1}{1 - |\lambda|} \int_{I_{\lambda}} M_{\Lambda}(\zeta) d\sigma(\zeta) \geq \frac{1}{1 - |\lambda|} \int_{I_{\lambda}} \log \frac{1}{|B_{\Lambda}(\lambda)|} d\sigma(\zeta) = \log \frac{1}{|B_{\Lambda}(\lambda)|}.
\]

Some Carleson-type conditions can be deduced from the implicit analytic characterisation and from Corollary 1.5.

**Corollary 1.6.** (a) If \( \Lambda \in \text{Int } N^+ \), then

\[
\lim_{|\lambda| \to 1} (1 - |\lambda|) \log \frac{1}{|B_{\Lambda}(\lambda)|} = 0.
\]

(b) If \( \Lambda \in \text{Int } N \), then

\[
\sup_{\lambda \in \Lambda} (1 - |\lambda|) \log \frac{1}{|B_{\Lambda}(\lambda)|} < \infty.
\]

**Proof.** Since

\[
I_{\lambda} \subset \{ \zeta \in \mathbb{T} : M_{\Lambda}(\zeta) \geq \log \frac{1}{|B_{\Lambda}(\lambda)|} \}, \quad \lambda \in \Lambda,
\]

it suffices to apply condition (a) of Corollary 1.5.

These are the best possible necessary conditions expressed in these terms, as reveals the next result.

**Proposition 1.7.** Assume that \( \Lambda \subset \mathbb{D} \) lies in a finite union of Stolz angles.

(a) \( \Lambda \in \text{Int } N^+ \) if and only if (7) holds.

(b) \( \Lambda \in \text{Int } N \) if and only if (8) holds.

It should be mentioned that (b) can also be derived from Naftalevič’s result [Na56, Theorem 3].

From Corollary 1.5 we can deduce as well a sufficient condition.

**Corollary 1.8.** Let \( \Lambda \subset \mathbb{D} \) be Blaschke. If

\[
\sum_{\lambda \in \Lambda} (1 - |\lambda|) \log \frac{1}{|B_{\Lambda}(\lambda)|} < \infty,
\]

then \( \Lambda \in \text{Int } N^+ \) (and so also \( \Lambda \in \text{Int } N \)).

**Proof.** Set \( a_{\lambda} = \log(1/|B_{\Lambda}(\lambda)|) \) and \( u = \sum_{\lambda} a_{\lambda} \chi_{I_{\lambda}} \). By assumption \( u \in L^1(\mathbb{T}) \) and obviously \( M_{\Lambda} \leq u \), hence the result follows from Corollary 1.5(b).

It turns out that for a certain type of sequences condition (9) is both necessary and sufficient for free interpolation in \( N \) and \( N^+ \) (see Remark 5.2), so that there is no intrinsic analogue of Corollary 1.8 for \( N \). In this direction we state here the following result.
**Proposition 1.9.** For every sequence of positive numbers \( (\varepsilon_n) \notin \ell^1 \) there exists a Blaschke sequence \( \Lambda \notin \text{Int} N \) such that \( \varepsilon_n \simeq (1 - |\lambda_n|) \log 1/|B_{\lambda_n}(\lambda_n)| \) for all \( \lambda_n \in \Lambda \).

In the comparison of the different geometric conditions we exploit to some extent the two extremal cases mentioned previously: \( \Lambda \) radial (or in a finite union of Stolz angles), in Proposition 1.7, and \( \Lambda \) “very” tangential (in Proposition 1.9).

The paper is organized as follows. In Section 2 we prove the sufficiency of the analytic conditions of Theorems 1.2 and 1.3. We essentially use a result by Garnett allowing interpolation by \( H^\infty \) functions on sequences which are denser than Carleson sequences, under some decrease assumptions on the interpolated values.

In Section 3 we study the necessity part of Theorems 1.2 and 1.3. We first observe that in the product \( B_\lambda \) appearing in (2), only the factors \( b_\lambda(\lambda') \) with \( \lambda' \) close to \( \lambda \) are relevant. Then we split the sequence into four pieces, thereby reducing the interpolation problem, in a way, to that on separated sequences.

The trace space characterisation will be discussed in Section 4.

Section 5 is devoted to the proofs of Propositions 1.7 and 1.9.

In the final section, we exploit the reasoning of Section 2 to construct non-Carleson interpolating sequences for “big” Hardy-Orlicz classes.

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### 2. Proof of the sufficient conditions

For a given Blaschke sequence \( \Lambda \subset \mathbb{D} \) set \( \delta_\lambda = |B_\lambda(\lambda)| \). The key result in the proof of the sufficient condition is the following theorem by Garnett \([\text{Gar77}] \), that we cite for our purpose in a slightly weaker form (see also \([\text{Nik02}] \) as a general source, in particular C.3.3.3(g) (Volume 2) for more results of this kind).

**Theorem.** Let \( \varphi : [0, \infty) \rightarrow [0, \infty) \) be a decreasing function such that \( \int_0^\infty \varphi(t) \, dt < \infty \). If a sequence \( (a_\lambda) \) satisfies

\[
|a_\lambda| \leq \delta_\lambda \varphi(\log e / \delta_\lambda), \quad \lambda \in \Lambda,
\]

then there exists a function \( f \in H^\infty \) such that \( f(\lambda) = a_\lambda \) for all \( \lambda \in \Lambda \).

As we have already noted in Remark 1.1, in order to have free interpolation in the Nevanlinna and Smirnov classes, it is sufficient that \( \ell^\infty \subset N|\Lambda \) and \( \ell^\infty \subset N^+|\Lambda \) respectively. Our aim will be to accommodate the decrease given in Garnett’s result by an appropriate function in \( N \) or \( N^+ \). This is the crucial step in the proof of the sufficient conditions of Theorems 1.2 and 1.3, and it occupies its main part.

**Proof of sufficiency of (2) and (4).** The proof will be presented for the more difficult case of the Nevanlinna class. So, assume that \( g \) and \( S \) are as in (4).
As in the comment after Theorem 1.3, we can assume that the function \( w \in L^1(T) \) associated with \( g \) by
\[
g(z) = \exp \left( \int_T \frac{\zeta + z}{\zeta - z} w(\zeta) \, d\sigma(\zeta) \right),
\]
is negative. Let \( \mu_S \) be the singular measure associated with \( S \). The measure \( d\mu = -w \, dm + d\mu_S \) is finite and positive, hence the function
\[
h(z) = \int_T \frac{\zeta + z}{\zeta - z} d\mu(\zeta)
\]
is holomorphic with positive real part in \( D \) (in fact \( h = -\log(gS) \)). By Smirnov’s theorem, \( h \) is an outer function in some \( H^p \), \( p < 1 \), and therefore in \( N^+ \) (see [Nik02], in particular A.4.2.3 (Volume 1)). By assumption we have
\[
\log\left(\frac{1}{\delta} \lambda \right) \leq \log\left(\frac{1}{|g(\lambda)S(\lambda)|}\right) = \Re h(\lambda), \quad \lambda \in \mathbb{D}.
\]
Take now \( \varphi(t) = (1 + t)^{-2} \), which obviously satisfies the hypothesis of Garnett’s theorem, and set \( H = (2 + h)^2 \), which is still outer in \( N^+ \). We have the estimate
\[
|H(\lambda)| = |2 + h(\lambda)|^2 \geq (2 + \Re h(\lambda))^2 \geq (1 + \log \frac{e}{\delta})^2 = \frac{1}{\varphi(\log(e/\delta))},
\]
hence the sequence \((\gamma_\lambda)\) defined by
\[
\gamma_\lambda = \frac{1}{H(\lambda)\varphi(\log(e/\delta))}, \quad \lambda \in \Lambda,
\]
is bounded by 1.

In order to interpolate \( \omega = (\omega_\lambda) \in \ell^\infty \) by a function in \( N \), split
\[
\omega_\lambda = \left(\omega_\lambda \gamma_\lambda \frac{g(\lambda)S(\lambda)}{\delta_\lambda} \delta_\lambda \varphi(\log \frac{e}{\delta_\lambda})\right) \cdot \frac{H(\lambda)}{g(\lambda)S(\lambda)}.
\]

Since by hypothesis \((\omega_\lambda \gamma_\lambda g(\lambda)S(\lambda)/\delta_\lambda)\) is bounded, we can apply Garnett’s result to interpolate the sequence
\[
a_\lambda = \omega_\lambda \gamma_\lambda \frac{g(\lambda)S(\lambda)}{\delta_\lambda} \delta_\lambda \varphi(\log \frac{e}{\delta_\lambda}), \quad \lambda \in \Lambda,
\]
by a function \( f \in H^\infty \). Now \( F = fH/gS \) is a function in \( N \) with \( F|\Lambda = \omega \).

The proof for the Smirnov case is obtained by deleting all the appearances of \( S(\lambda) \) and the singular measure \( \mu_S \).

3. Proof of the necessary conditions

We first show that in order to construct the appropriate function estimating \( |B_\lambda(\lambda)| \) from below we only need to consider the factors given by points \( \mu \in \Lambda \) which are close to \( \lambda \). This is in accordance with the results for some related spaces of functions [HaMa01, Theorem 1].
**Proposition 3.1.** Let $\Lambda$ be a Blaschke sequence. There exists an outer function $g \in N^+$ such that

$$\prod_{\mu: |b_\lambda(\mu)| \geq 1/2} |b_\lambda(\mu)| \geq |g(\lambda)|, \quad \lambda \in \Lambda.$$ 

It is clear from the proof that the constant $1/2$ can be replaced by any $\delta \in (0, 1)$. Of course this implies Corollary 1.4.

**Proof.** Consider the intervals $I_\lambda$ defined in (3). By the Blaschke condition, the function

$$w(\zeta) = -\sum_{\lambda \in \Lambda} \chi_{I_\lambda}(\zeta)$$

belongs to $L^1(\mathbb{T})$. Thus the function

$$g(z) = \exp \left( c \int_\mathbb{T} \frac{\zeta + z}{\zeta - z} w(\zeta) d\sigma(\zeta) \right), \quad c > 0,$$

is outer in $N^+$, and it is sufficient to prove that

$$cP[w](\lambda) = -c \sum_{\mu \in \Lambda} \int_{I_\mu} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} d\sigma(\zeta) \leq \log \prod_{|b_\lambda(\mu)| \geq 1/2} |b_\lambda(\mu)| = -\sum_{|b_\lambda(\mu)| \geq 1/2} \log \frac{1}{|b_\lambda(\mu)|}$$

for some constant $c > 0$.

Using that $\log \frac{1}{t} \simeq 1 - t^2$ for $t \in [1/2, 1]$, and $1 - |b_\lambda(\mu)|^2 = \frac{(1-|\lambda|^2)(1-|\mu|^2)}{1-\lambda^2\mu^2}$ we see that it suffices to prove that

$$\frac{(1 - |\lambda|^2)(1 - |\mu|^2)}{|1 - \lambda^2\mu^2|} \leq c \int_{I_\mu} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} d\sigma(\zeta).$$

We consider two situations. Let $\mu^* = \mu/|\mu|$ and define

$$D(\mu^*) = \{ z \in \mathbb{D} : |z - \mu^*| \leq 2(1 - |\mu|) \}.$$

If $\lambda \in D(\mu^*)$ and $\zeta \in I_\mu$ then $|\zeta - \lambda| \leq 3(1 - |\mu|)$, and so

$$\int_{I_\mu} \frac{d\sigma(\zeta)}{|\zeta - \lambda|^2} \geq \frac{1}{3} \int_{I_\mu} \frac{d\sigma(\zeta)}{(1 - |\mu|)^2} \geq \frac{1}{1 - |\mu|} \geq \frac{1 - |\mu|}{|1 - \lambda^2\mu^2|}.$$

If $\lambda \not\in D(\mu^*)$ and $\zeta \in I_\mu$ then $|\zeta - \lambda| \simeq |\mu - \lambda|$, and so

$$\int_{I_\mu} \frac{d\sigma(\zeta)}{|\zeta - \lambda|^2} \simeq \int_{I_\mu} \frac{d\sigma(\zeta)}{|\lambda - \mu|^2} \simeq \frac{1 - |\mu|}{|\lambda - \mu|^2} \simeq \frac{1 - |\mu|}{|1 - \lambda^2\mu^2|}.$$

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**Proof of the necessity of (2) and (4).** We split the sequence into four pieces: $\Lambda = \bigcup_{i=1}^4 \Lambda_i$ such that each piece $\Lambda_i$ lies in a union of dyadic “squares” which are uniformly separated from each other. More precisely, consider, for $n \in \mathbb{N}$ and $0 \leq k \leq 2^n - 1$ the dyadic “squares”:

$$Q_{n,k} = \{ r\zeta \in \mathbb{D} : 1 - \frac{1}{2^n} \leq r < 1 - \frac{1}{2^{n+1}}, \zeta \in \frac{2\pi}{2^n}[k, k + 1) \}.$$
Set
\[ \Lambda_1 = \bigcup_j (\Lambda \cap Q^{(j)}) , \]
where the family \( \{Q^{(j)}\}_j \) is given by \( \{Q_{2n,2k}\}_{n,k} \) (for the remaining three sequences we respectively choose \( \{Q_{2n,2k+1}\}_{n,k}, \{Q_{2n+1,2k}\}_{n,k} \) and \( \{Q_{2n+1,2k+1}\}_{n,k} \). In order to avoid technical difficulties we count only those \( Q^{(j)} \) containing points of \( \Lambda \) (in case \( \Lambda_j \) is empty there is nothing to prove). In what follows we will argue on one sequence, say \( \Lambda_1 \). The arguments are the same for the other sequences.

Our first observation is that, by construction,
\[ \rho(Q^{(j)}, Q^{(l)}) := \inf_{z \in Q^{(j)}, w \in Q^{(l)}} |b_z(w)| \geq \delta > 0, \quad j \neq l, \]
for some fixed \( \delta \). Also, the closed rectangles \( \overline{Q^{(j)}} \) are compact in \( \mathbb{D} \) so that \( \Lambda_1 \cap Q^{(j)} \subset \Lambda \cap Q^{(j)} \) can only contain a finite number of points (they contain at least one point, by assumption). Therefore
\[ 0 < m_j := \min_{\lambda \in \Lambda_1 \cap Q^{(j)}} |B_\lambda(\lambda)| \]
(note that we consider the entire Blaschke product \( B_\lambda \) associated with \( \Lambda \setminus \{\lambda\} \)). Take \( \lambda_j^{(j)} \in Q^{(j)} \) such that \( m_j = |B_{\lambda_j^{(j)}}(\lambda_j^{(j)})| \).

Assume now that \( \Lambda \in \text{Int } N \). Since \( \ell^\infty \subset N|\Lambda \), there exists a function \( f_1 \in N \) such that
\[ f_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\lambda_j^{(j)}\}_j \\ 0 & \text{if } \lambda \notin \{\lambda_j^{(j)}\}_j. \end{cases} \]

By the Riesz-Smirnov factorisation we have
\[ f_1 = B_{\Lambda \setminus \{\lambda_j^{(j)}\}} \frac{h_1}{h_2 T_2}, \]
where \( T_2 \) is singular inner, \( h_1 \) is some function in \( H^\infty \) and \( h_2 \) is outer in \( H^\infty \). Again, we can assume \( \|h_i\|_\infty \leq 1, i = 1, 2 \). Hence
\[ 1 = f_1(\lambda_k^1) \leq |B_{\Lambda \setminus \{\lambda_j^{(j)}\}}(\lambda_k^1)| \cdot \frac{1}{|h_2(\lambda_k^1) T_2(\lambda_k^1)|}; \]
and
\[ |B_{\Lambda \setminus \{\lambda_j^{(j)}\}}(\lambda_k^1)| \geq |h_2(\lambda_k^1) T_2(\lambda_k^1)|, \quad k \in \mathbb{N}. \]

Since \( h_2 T_2 \) does not vanish and is bounded above by \( 1 \), the function \( \log |h_2 T_2| \) is a negative harmonic function. By Harnack’s inequality, there exists an absolute constant \( c \geq 1 \) such that
\[ \frac{1}{c} \log |h_2(\lambda_k^1) T_2(\lambda_k^1)| \leq \log |h_2(z) T_2(z)| \leq c \log |h_2(\lambda_k^1) T_2(\lambda_k^1)|, \quad z \in Q^{(k)}, \]
hence
\[ |h_2(\lambda_k^1) T_2(\lambda_k^1)|^c \leq |h_2(z) T_2(z)| \leq |h_2(\lambda_k^1) T_2(\lambda_k^1)|^{1/c}, \quad z \in Q^{(k)}. \]
This yields
\[ |(h_2T_2)^c(\mu)| \leq |(h_2T_2)(\lambda_k^1)| \leq |B_{\Lambda\setminus\{\lambda_k^1\}}(\lambda_k^1)| \]
for every \( \mu \in \Lambda_1 \cap Q^{(k)}. \)

Let us now exploit Proposition [3,1]. By construction, the sequence \( \{\lambda_j^1\} \subset \Lambda_1 \) is separated. Therefore, there exists an outer function \( G_1 \) in the Smirnov class such that
\[ |B_{\Lambda\setminus\{\lambda_j^1\}}(\lambda_k^1)| \geq |G_1(\lambda_k^1)|, \quad k \in \mathbb{N}. \]
Again, \( G_1 \) is a quotient of two bounded outer functions and we can suppose that \( G_1 \) is outer in \( H^\infty \) with \( \|G_1\|_\infty \leq 1 \). Also, we can use Harnack’s inequality as above to get
\[ |G_1(\lambda_k^1)| \geq |G^c_1(\mu)| \]
for every \( \mu \in \Lambda_1 \cup Q^{(k)}. \) This together with (11) and our definition of \( \lambda_k^1 \) give
\[ |B_{\Lambda\setminus\{\mu\}}(\mu)| \geq |B_{\Lambda\setminus\{\lambda_k^1\}}(\lambda_k^1)| = |B_{\Lambda\setminus\{\lambda_j^1\}}(\lambda_k^1)| \cdot |B_{\Lambda\setminus\{\lambda_j^1\}}(\lambda_k^1)| \]
\[ \geq |(h_2T_2)^c(\mu)| \cdot |G^c_1(\mu)| \]
for every \( \mu \in Q^{(k)} \) and \( k \in \mathbb{N}. \) Setting now \( g_1 = (h_2G_1)^c \) and \( S_1 = T_2^c \) we get the claim for all points in \( \Lambda_1 \). Note also that, by construction, \( g_1 \) is outer with \( \|g_1\|_\infty \leq 1 \) and \( S_1 \) is singular inner.

To finish the proof, construct in a similar way functions \( g_i, S_i \) for the sequences \( \Lambda_i, \ i = 2, 3, 4 \) and define the products
\[ g = \prod_{i=1}^{4} g_i \quad \text{and} \quad S = \prod_{i=1}^{4} S_i. \]
Of course \( g \) is outer in \( H^\infty \), and \( S \) is singular inner. So, whenever \( \mu \in \Lambda \), there exists \( k \in \{1, 2, 3, 4\} \) such that \( \mu \in \Lambda_k \), and hence
\[ |B_{\Lambda}(\lambda)| \geq |g_k(\lambda)S_k(\lambda)| \geq |g(\lambda)S(\lambda)|. \]
The proof for \( N^+ \) follows the same lines, just disregarding the singular inner factors. 

4. THE TRACE SPACE

In this short section we prove the trace space characterisation of free interpolation given in Theorems [1,2] and [1,3].

In order to see that (d) of both theorems implies free interpolation it suffices to observe that \( \ell^\infty \subset l_{N^+} \subset l_N \) and to use Remark [1,7].

For the proof of the converse, we will only consider the situation in the Nevanlinna class, since the case of the Smirnov class is again obtained by removing the singular part of the measure and the singular inner factors.

Assume that \( (a_\lambda)_{\lambda} \in N|\Lambda \) and that \( f \in N \) is such that \( f(\lambda) = a_\lambda, \ \lambda \in \Lambda \). Since \( f \) can be written as \( f = f_1/S_2f_2 \), where \( f_1 \in H^\infty, \|f_1\|_\infty \leq 1, \) \( S_2 \) is singular inner with associated singular measure \( \mu_S, \) and \( f_2 \in H^\infty \) is an outer function with \( \|f_2\|_\infty \leq 1, \) we can define the
positive finite measure \( \mu = \log(1/|f_2|) \, dm + d\mu_S \), that obviously satisfies \( P[\mu](\lambda) \geq \log^+ |a_\lambda|, \lambda \in \Lambda \).

Conversely, suppose that \((a_\lambda)_\lambda \) is such that there is a positive finite measure \( \mu \) with \( P[\mu](\lambda) \geq \log^+ |a_\lambda| \). The Radon-Nikodym decomposition of \( \mu \) is given by \( d\mu = w \, dm + d\mu_S \), where \( w \in L^1(\mathbb{T}) \) is positive and \( \mu_S \) is a positive finite singular measure. Let \( S \) be the singular inner function associated with \( \mu_S \), and let \( f \) be the function defined by

\[
f(z) = \exp \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} w(\zeta) \, d\sigma(\zeta) \right), \quad z \in \mathbb{D}.
\]

By definition, \( f \) is outer in \( N^+ \) and \( F = f/S \in N \). Clearly, \( \log^+ |a_\lambda| \leq \log |F(\lambda)| \), thus \( |a_\lambda| \leq |F(\lambda)| \). Since \( N|\Lambda \) is ideal by assumption, there exists \( f_0 \in N \) interpolating \((a_\lambda)_\lambda \). \( \Box \)

5. PROOFS OF PROPOSITIONS 1.7 AND 1.9

**Proof of Proposition 1.7.** It is enough to consider the case where \( \Lambda \) is contained in only one Stolz angle. Indeed, if \( \Lambda = \bigcup_{i=1}^n \Lambda_i \) with \( \Lambda_i \subset \Gamma_{\xi_i}, l = 1, \ldots, n \), and \( \xi_i \neq \xi_j \), then \( \lim_{z \to \xi_i; z \in \Gamma_{\xi_i}} |B_{\Lambda_j}(z)| = 1, j \neq i \), so that \( \log 1/|B_{\Lambda}(\lambda)| \) behaves asymptotically like \( \log 1/|B_{\Lambda \setminus \Lambda}(\lambda)| \) in \( \Gamma_{\xi_i} \) (here \( \lambda \in \Lambda_i \)).

Also, we can assume that the sequence is radial (this means that we replace the initial sequence by one which is in a uniform pseudo-hyperbolic neighbourhood of the initial one; by Harnack’s inequality such a perturbation does not change substantially the behaviour of positive harmonic functions).

By Corollary 1.6 it remains to prove the sufficiency of the conditions. Let us first show that condition \( (\mathcal{Q}) \) implies interpolation in \( N^+ \). In order to construct a function \( w \in L^1(\mathbb{T}) \) meeting the requirement of Theorem 1.2(c) assume that \( \Lambda = \{ \lambda_n \}_n \subset [0, 1) \) is arranged in increasing order and set \( \varepsilon_n = (1 - |\lambda_n|) \log(1/|B_{\lambda_n}(\lambda_n)|) \). Clearly there exists a decreasing sequence \( (\varepsilon_n)_n \) with \( \varepsilon_n \leq \varepsilon_n, n \in \mathbb{N} \), and \( \lim_{n} \varepsilon_n = 0 \). Now, if \( I_n = I_{\lambda_n} \) are the arcs defined in \( (\mathcal{R}) \), \( J_n = I_n \setminus I_{n+1} \) and \( \beta_n = \varepsilon_n - \varepsilon_{n+1} \), set

\[
w(\zeta) = \sum_n \frac{\beta_n}{|J_n|} \chi_{J_n}(\zeta), \quad \zeta \in \mathbb{T},
\]

where \( |J_n| \) denotes the Lebesgue measure of the set \( J_n \). Then \( w \in L^1(\mathbb{T}) \), and

\[
P[w](\lambda_n) \geq \int_{I_n} P(\lambda_n, \zeta) \sum_k \frac{\beta_k}{|J_k|} \chi_{J_k}(\zeta) \, d\sigma(\zeta) \geq \sum_{k \geq n} \frac{\beta_k}{|J_k| (1 - |\lambda_n|)} \int_{J_k} d\sigma(\zeta)
\]

\[= \frac{\sum_{k \geq n} \beta_k}{(1 - |\lambda_n|)} = \frac{\varepsilon_n}{1 - |\lambda_n|} \geq \frac{\varepsilon_n}{1 - |\lambda_n|} = \log \frac{1}{|B_{\lambda_n}(\lambda_n)|}.
\]

This and Theorem 1.2 prove the assertion.

The proof for the Nevanlinna class is even simpler. Set \( d\mu_s = \delta_1 \), the Dirac mass on \( 1 \in \mathbb{T} \). From \( (\mathcal{S}) \) we get

\[
\log \frac{1}{|B_{\lambda_n}(\lambda_n)|} \leq \frac{1}{1 - |\lambda_n|} \leq P[\mu_s](\lambda),
\]
and we finish by applying Theorem 1.3.

Similar ideas are used in the proof of Proposition 1.9, which will be a consequence of the following auxiliary result. For \( \lambda \in \mathbb{D} \), denote
\[
K_\lambda = \{ \zeta \in \mathbb{T} : \arg \zeta - \arg \lambda \leq \pi \sqrt{1 - |\lambda|} \}.
\]

**Proposition 5.1.** Let \( \Lambda \subset \mathbb{D} \) be such that
\[
K_\lambda \cap K_{\lambda'} = \emptyset, \quad \lambda, \lambda' \in \Lambda, \quad \lambda \neq \lambda'.
\]
Then, for no sequence of positive numbers \( (\varepsilon_{\lambda})_\lambda \) with \( \sum_{\lambda} \varepsilon_{\lambda} = \infty \) there exists a finite positive measure \( \mu \) such that \( \varepsilon_{\lambda}/(1 - |\lambda|) \leq P[\mu](\lambda) \) for all \( \lambda \in \Lambda \).

Notice that the hypothesis implies automatically that \( \Lambda \) is a Carleson sequence.

**Proof.** Suppose on the contrary that there exists a finite positive measure \( \mu \) satisfying the above estimate. Note that
\[
P[\mu](\lambda) = \int_{K_\lambda} P(\lambda, \zeta) \, d\mu(\zeta) + \int_{\mathbb{T} \setminus K_\lambda} P(\lambda, \zeta) \, d\mu(\zeta) \leq \int_{K_\lambda} P(\lambda, \zeta) \, d\mu(\zeta) + c_\mu,
\]
where \( c_\mu = \mu(\mathbb{T}) \). So
\[
\varepsilon_{\lambda} \leq c_\mu (1 - |\lambda|) + (1 - |\lambda|) \int_{K_\lambda} P(\lambda, \zeta) \, d\mu(\zeta) \leq c_\mu (1 - |\lambda|) + 2 \int_{K_\lambda} d\mu(\zeta),
\]
and consequently
\[
\sum_{\lambda \in \Lambda} \varepsilon_{\lambda} \leq c_\mu \sum_{\lambda \in \Lambda} (1 - |\lambda|) + 2 \int_{\bigcup_{\lambda \in \Lambda} K_\lambda} \, d\mu(\zeta) \leq c_\mu \sum_{\lambda \in \Lambda} (1 - |\lambda|) + 2c_\mu.
\]
Since \( \Lambda \) is a Blaschke sequence, this is not possible for any finite measure \( \mu \).

**Proof of Proposition 1.9.** Let \( (\varepsilon_n)_n \not\in \ell^1 \). There is no restriction in assuming \( \lim_n e^{-\varepsilon_n/(1-|\lambda_n|)} = 0 \).

Take now \( \Lambda_1 \) as in the previous proposition. In the neighbourhood of each \( \lambda_n = r_n e^{i \varphi_n} \in \Lambda_1 \) we fix a point \( \lambda'_n = r'_n e^{i \varphi_n} \) such that \( r'_n \geq r_n \) and \( |b_{\lambda'_n}(\lambda_n)| = e^{-\varepsilon_n/(1-|\lambda_n|)} \) (so \( \lambda'_n \) lies on the shortest segment connecting \( \lambda_n \) to \( \mathbb{T} \)). Since \( e^{-\varepsilon_n/(1-|\lambda_n|)} \) decreases to 0, by a standard perturbation argument, the sequence \( \Lambda_2 = \{ \lambda'_n \}_n \) is also Carleson.

Define \( \Lambda = \Lambda_1 \cup \Lambda_2 \). By construction
\[
|B_\lambda(\lambda)| \asymp e^{-\varepsilon_\lambda/(1-|\lambda|)}, \quad \lambda \in \Lambda,
\]
where \( \varepsilon_\lambda = \varepsilon_n \) both for \( \lambda = \lambda_n \) and \( \lambda = \lambda'_n \).

The previous proposition and Theorem 1.3(c) show that \( \Lambda \not\in \text{Int} \ N \).

**Remark 5.2.** Let \( \Lambda = \{ \lambda_{n,k} \}_{n \in \mathbb{N}, k \leq N} \) be a sequence constructed similarly to the prove above, i.e. \( N \in \mathbb{N} \) is some fixed number, \( \{ \lambda_{n,1} \}_n \) satisfies the hypothesis of Proposition 5.1 and \( |b_{\lambda_{n,k}}(\lambda_{n,l})| \leq \delta, k, l = 1, \ldots, N, \) for some fixed \( \delta \in (0, 1) \). Then the previous results show that the following assertions are equivalent...
(i) $\sum_{\lambda \in \Lambda} (1 - |\lambda|) \log 1/|B_\lambda(\lambda)| < \infty$,
(ii) $\Lambda \in \text{Int} N$,
(iii) $\Lambda \in \text{Int} N^+$.

6. HARDY-ORLICZ CLASSES

Let $\varphi : \mathbb{R} \to [0, \infty)$ be a convex, nondecreasing function satisfying

(i) $\lim_{t \to \infty} \varphi(t)/t = \infty$
(ii) $\Delta_2$-condition: $\varphi(t+2) \leq M\varphi(t) + K$, $t \geq t_0$ for some constants $M, K \geq 0$ and $t_0 \in \mathbb{R}$.

Such a function is called strongly convex (see \[RosKov\]), and one can associate with it the corresponding Hardy-Orlicz class

$$\mathcal{H}_\varphi = \{ f \in \mathcal{C}^+ : \int_T \varphi(\log |f(\zeta)|) d\sigma(\zeta) < \infty \},$$

where $f(\zeta)$ is the non-tangential boundary value of $f$ at $\zeta \in T$, which exists almost everywhere. In \[Har99\], the following result was proved.

**Theorem.** Let $\varphi$ be a strongly convex function satisfying (i), (ii) and the $V_2$-condition:

$$2\varphi(t) \leq \varphi(t + \alpha), \quad t \geq t_1$$

where $\alpha > 0$ is a suitable constant and $t_1 \in \mathbb{R}$. Then $\Lambda \subset \mathbb{D}$ is free interpolating for $\mathcal{H}_\varphi$ if and only if $\Lambda$ is a Carleson sequence, and in this case

$$\mathcal{H}_\varphi|_{\Lambda} = \{ a = (a_\lambda)_{\lambda} : |a|_{\varphi} = \sum_{\lambda} (1 - |\lambda| ) \varphi(\log |a_\lambda|) < \infty \}.$$  

The conditions on $\varphi$ imply that for all $\mathcal{H}_\varphi$ there exist $p, q \in (0, \infty)$ such that $H^p \subset \mathcal{H}_\varphi \subset H^q$. In particular, the $V_2$-condition implies the inclusion $H^p \subset \mathcal{H}_\varphi$ for some $p > 0$. This $V_2$-condition has a strong topological impact on the spaces. In fact, it guarantees that metric bounded sets are also bounded in the topology of the space (and so the functional analysis tools still apply in this situation; see \[Har99\] for more on this and for further references). It was not clear whether this was only a technical problem or if there existed a critical growth for $\varphi$ (below exponential growth $\varphi(t) = e^{pt}$ corresponding to $H^p$ spaces) giving a breakpoint in the behaviour of interpolating sequences for $\mathcal{H}_\varphi$.

We can now affirm that this behaviour in fact changes between exponential and polynomial growth. Let $\varphi$ be a strongly convex function with associated Hardy-Orlicz space $\mathcal{H}_\varphi$. Assume moreover that $\varphi$ satisfies

$$(11) \quad \varphi(a + b) \leq c(\varphi(a) + \varphi(b)),$$

for some fixed constant $c \geq 1$ and for all $a, b \geq t_0$. The standard example in this setting is $\varphi_p(t) = t^p$ for $p > 1$. We have the following result.

**Theorem 6.1.** Let $\varphi : \mathbb{R} \to [0, \infty)$ be a strongly convex function such that (11) holds. If there exists a positive weight $w \in L^1(\mathbb{T})$ such that $\varphi \circ w \in L^1(\mathbb{T})$ and (3) holds, then $\Lambda \in \text{Int} \mathcal{H}_\varphi$. 
Proof. Note first that (1.9) implies that $\mathcal{H}_\varphi$ is an algebra contained in $N^+$, hence it is sufficient to interpolate bounded sequences (see Remark [1.1]). By assumption, the outer function defined by

$$g(z) = \exp \left\{ \int \frac{\zeta + z}{\zeta - z} (-w(\zeta)) d\sigma(\zeta) \right\}$$

satisfies $|B_\lambda(\lambda)| \geq |g(\lambda)|$, $\lambda \in \Lambda$. The reasoning carried out in Section 2 leads to an interpolating function of the form $fH/g$, with $f \in H^\infty$, and $H = (2 + h)^2$ is outer in $H^p$ for all $p < 1$ (note that the measure $\mu$ defining $h$ here is absolutely continuous, in fact $\mu = -w \, dm$). Also, $H^p \subset \mathcal{H}_\varphi$ for any $p > 0$ by our conditions on $\varphi$. By construction, $\int \varphi(\log |1/g|) = \int \varphi \circ w < \infty$ so that $1/g \in \mathcal{H}_\varphi$. Since $\mathcal{H}_\varphi$ is an algebra, we deduce that $fH/g \in \mathcal{H}_\varphi$. \hfill \blacksquare

Example 6.2. We give an example of an interpolating sequence for $\mathcal{H}_\varphi$ which is not Carleson, thus justifying our claim that there is a breakpoint between Hardy-Orlicz spaces verifying the $V_2$-condition and those that do not.

Consider the functions $\varphi_\varepsilon$ and let $\Lambda_0 = \{\lambda_n\}_n \subseteq \mathbb{D}$ be a Carleson sequence verifying $I_n \cap I_k = \emptyset$, $n \neq k$. Since $\sum (1 - |\lambda_n|) < \infty$, there exists a strictly increasing sequence of positive numbers $(\gamma_n)_n$ such that $\sum_n (1 - |\lambda_n|) \gamma_n < \infty$ and $\lim_{n \to \infty} \gamma_n = \infty$. Setting $\varepsilon_n = (1 - |\lambda_n|)^{\frac{1}{1 - p}}$ and

$$u_n = \frac{\varepsilon_n}{1 - |\lambda_n|} \chi_{I_n},$$

we obtain $\int \varphi \circ u = \sum_n \frac{\varepsilon_n}{(1 - |\lambda_n|)^{1 - p}} = \sum_n (1 - |\lambda_n|) \gamma_n < \infty$. As in the proof of Proposition [1.9], we attach a second Carleson sequence $\Lambda_1 = \{\lambda'_n\}_n$ such that the pseudo-hyperbolic distance between corresponding points satisfies $|b_{\lambda'_n}(\lambda_n)| = e^{-\varepsilon_n/(1 - |\lambda_n|)}$. Since $\gamma_n \to \infty$ we have $\varepsilon_n/(1 - |\lambda_n|) = \gamma^{1/p}_n \to \infty$, i.e. the elements of the sequence $\Lambda = \Lambda_1 \cup \Lambda_2$ are arbitrarily close and $\Lambda$ cannot be a Carleson sequence. By construction, condition (3) holds (as before, we may possibly have to multiply $u$ with some constant $c$ to have that condition also in the points $\lambda'_n$, but this operation conserves the integrability condition), and therefore $\Lambda \in \text{Int } \mathcal{H}_\varphi$.

REFERENCES

[Gar77] J.B. Garnett, Two remarks on interpolation by bounded analytic functions, Banach spaces of analytic functions (Proc. Pelczynski Conf., Kent State Univ., Kent, Ohio, 1976), pp. 32–40. Lecture Notes in Math., Vol. 604, Springer, Berlin, 1977.

[Gar81] J.B. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

[Har99] A. Hartmann, Free interpolation in Hardy-Orlicz spaces, Studia Math. 135 (1999), no. 2, 179–190.

[HaMa01] A. Hartmann & X. Massaneda, Interpolating sequences for holomorphic functions of restricted growth, to appear in Ill. J. Math.

[McC92] J. McCarthy, Topologies on the Smirnov class, J. Funct. Anal. 104 (1992), no. 1, 229–241.

[Na56] A.G. Naftalevič, On interpolation by functions of bounded characteristic (Russian), Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslo Ser. 5 (1956), 5–27.

[Nik86] N.K. Nikolski [Nikol’ski˘ı], Treatise on the shift operator, Springer-Verlag, Berlin etc., 1986.

[Nik02] N.K. Nikolski, Operators, functions, and systems: an easy reading. Vol. 1, Hardy, Hankel, and Toeplitz; Vol.2, Model Operators and Systems, Mathematical Surveys and Monographs, 92 and 93. American Mathematical Society, Providence, RI, 2002.

[RosRov] M. Rosenblum & J. Rovnyak, Hardy classes and operator theory, Oxford Mathematical Monographs, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1985.
[ShHSh] 
H.S. Shapiro & A.L. Shields, On some interpolation problems for analytic functions, Amer. J. Math., 83 (1961), 513–532.

[ShSh] 
J. Shapiro & A. Shields, Unusual topological properties of the Nevanlinna class, Amer. J. Math. 97 (1975), 915–936.

[Ya74] 
N. Yanagihara, Interpolation theorems for the class $N^+$, Illinois J. Math., 18 (1974), 427–435.

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