Dynamics of a Pair of Interacting Spins Coupled to an Environmental Sea

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We solve for the dynamics of a pair of spins, coupled to each other and also to an environmental sea of oscillators. The environment mediates an indirect interaction between the spins, causing both mutual coherence effects and dissipation. This model describes a wide variety of physical systems, ranging from 2 coupled microscopic systems (eg., magnetic impurities, bromophores, etc), to 2 coupled macroscopic quantum systems.

We obtain analytic results for 3 regimes, viz., (i) The “locked” regime, where the 2 spins lock together; (ii) The “correlated relaxation” regime (mutually correlated incoherent relaxation); and (iii) The “mutual coherence” regime, with correlated damped oscillations.

These results cover most of the parameter space of the system.

\section{I. INTRODUCTION}

In this paper we wish to establish the dynamics of a model which describes a pair of spin-1/2 systems (ie., 2-level systems), which are coupled both to each other and to an environmental sea of oscillators. Our Hamiltonian thus has the form

\[ H = H(\tau_1, \tau_2) + H_{\text{osc}}(\{x_k\}) + H_{\text{int}}(\tau_1, \tau_2; \{x_k\}) \]  \hspace{1cm} (1.1)

where \( \tau_1 \) and \( \tau_2 \) are 2 Pauli spins and the \( x_k \), with \( k = 1, 2, ...N \), are the oscillator coordinates. This model is interesting for many reasons, and we begin by listing some of them:

(i) We shall see that the dynamics of \( \tau_1 \) and \( \tau_2 \) can actually be solved (once we have integrated out the environment). Such solvable models of quantum dissipative systems are rare. Previous examples include the “spin-boson” model\textsuperscript{1} where a single 2-level system couples to an oscillator bath, the “central spin model”\textsuperscript{2}–\textsuperscript{6} (where a 2-level system couples to a bath of spins), quantum Brownian motion\textsuperscript{7,8} and the damped oscillator\textsuperscript{9,10} (where a single oscillator couples to an oscillator bath). Only the last two of these have a classical analogue. In both classical and quantum physics, an enormous amount about the physics of relaxation and decay has been learnt from such models for the study of decoherence. However, significantly absent from these studies is the generalisation to two coupled quantum systems. In classical physics such a generalisation introduces resonant coupling, “beats”, and correlates the relaxation. We shall see here that the quantum generalisation has all this, as well as quantum interference between the 2 systems. Such phenomena are of course well-known in the context of many examples, but the advantage of Eq. (1.1) is that it allows us to understand how things work when the 2 systems are strongly coupled (either to each other or to their environment).

(ii) The model Eq. (1.1) is particularly interesting when one or both spins describe the low-energy dynamics of a macroscopic system. In other papers under preparation\textsuperscript{12}, we have applied the results gained herein to understand the tunneling dynamics of coupled Josephson junctions in a SQUID\textsuperscript{11} and coupled nanomagnets.

Perhaps the most fundamental way in which Eq. (1.1) can be applied is to discuss the quantum measurement process\textsuperscript{13}. We are aware of no previous work which attempts to discuss both a measured system and the measuring apparatus on an entirely quantum level, whilst both are coupled to their environment. Previous studies (in particular those by Leggett et al.\textsuperscript{1,14,15}) deal exclusively with one or the other of these systems. In this paper we will already see some startling results relevant to this problem; we will devote an entire future paper to them\textsuperscript{16}.

(iii) Models of the type Eq. (1.1) are used widely throughout physics, particularly in condensed matter physics. Some disparate examples should suffice to illustrate this. A set of 2 coupled Anderson (or Kondo) impurities are described by Eq. (1.1) with the oscillators representing low-energy conduction electron excitations. A pair of tunneling defects coupled to phonons or electrons\textsuperscript{17} or a pair of chromophores in a large molecule, coupled to...
phonons or polaritons, is almost trivially described by (1.1), as are nuclear spins in a metal. The problem of coupled 2-level systems is also crucial to the understanding of many glasses. On a more macroscopic level, we have already mentioned coupled SQUID’s, nanomagnets, and the measurement problem. On a more microscopic level, (1.1) is an obvious generalisation of Wilson’s nucleus-meson model to 2 nucleons. All these examples except for the nuclear spins involve strong coupling to the environment. There are of course many examples where this coupling is weak (perhaps the best known is that of a pair of 2-level atoms, coupled to the EM field). Such weak-coupling examples are well understood, and do not require the methods used below for their solution.

In this paper, we shall be concerned principally with the application of Eq. (1.1) to macroscopic systems at low temperature. This will lead us to study a special case of (1.1), in which \( H = H_0 + H_{int} \), where

\[
H_0 = -\frac{1}{2}(\Delta_1 \hat{\tau}_1^z + \Delta_2 \hat{\tau}_2^z) + \frac{1}{2}K_{zz} \hat{\tau}_1^z \hat{\tau}_2^z + \frac{1}{2} \sum_{k=1}^{N} m_k (\hat{x}_k^2 + \omega_k^2 \hat{x}_k^2)
\]

(1.2)

\[
H_{int} = \frac{1}{2} \sum_{k=1}^{N} (\epsilon_k^{(1)} e^{i\hat{k} \cdot \hat{R}_1} \hat{\tau}_1^z + \epsilon_k^{(2)} e^{i\hat{k} \cdot \hat{R}_2} \hat{\tau}_2^z) \hat{x}_k
\]

(1.3)

where the \( N \) oscillator bath modes \( \{\hat{x}_k\} \) are assumed for simplicity (but without real loss of generality) to be momentum eigenstates. The Pauli matrices \( \hat{\tau}_1 \) and \( \hat{\tau}_2 \) represent the 2 different spins. The two spins are coupled through a direct interaction term \( K_{zz} \) giving ferro- or antiferromagnetic correlations, depending on whether \( K_{zz} \) is negative or positive. In the absence of the coupling to the environmental sea, the spins have their levels split by “tunneling” matrix elements \( \Delta_1 \) and \( \Delta_2 \). We work in the basis in which \( |\uparrow\rangle \) and \( |\downarrow\rangle \), eigenstates of \( \hat{\tau}_z \), are degenerate until split by the off-diagonal tunneling.

Henceforth we will refer to Hamiltonians like (1.1) as \( H_{PISCES}(\tau_1, \tau_2; \{\hat{x}_k\}) \), where “PISCES” is an abbreviation for “Pair of Interacting Spins Coupled to an Environmental Sea”. They are an obvious generalisation of the well-known spin-boson Hamiltonian, which has the form

\[
H = -\frac{1}{2} \frac{\Delta}{\tau} + \hat{\tau}_1^z \hat{\tau}_2^z + \frac{1}{2} \sum_{k=1}^{N} m_k (\hat{x}_k^2 + \omega_k^2 \hat{x}_k^2) + \frac{1}{2} \sum_{k=1}^{N} \epsilon_k \hat{x}_k
\]

(1.4)

whose dynamics was discussed in detail by Leggett et al.

The task of the present paper is to solve for the the 2-spin reduced density matrix \( \hat{\rho}(\tau_1, \tau_2; \tau_1', \tau_2'; t) \), produced by integrating out the bath oscillators; this describes the dynamics of the 2 spins. We emphasize here that such a study is an essential preliminary to any attempt to understand the dynamics of any of the strongly-coupled physical systems listed above; we devote the whole of the present paper to elucidating this dynamics. Although the application of these results is reserved mostly to later papers, we will comment on this herein, where possible.

We may (very briefly) summarize our results by remarking that there are essentially three regimes of interest for the PISCES model, depending on how strong is the coupling of each spin to the bath, and also how high is the temperature \( T \). These are

(i) The “Strong Coupling” regime, where the 2 spins are sufficiently correlated so they lock together, and essentially behave as a single spin, with a quite different tunneling matrix element and a different coupling to the bath.

(ii) The “Correlated Relaxation” regime, where the 2 spins tunnel incoherently, but nevertheless their motion is correlated.

(iii) The “Mutual Coherence” regime, where the motion of one or both spins still shows (damped) oscillations, which are partially correlated.

There is also a 4th solvable regime which we have ignored - this is the perturbative regime, where the 2 spins are so weakly coupled to the bath and to each other that their dynamics can be understood by perturbation expansion in the interaction. What we have done in the present paper is to calculate the 2-spin density matrix in analytic form in the above 3 regimes. We have concentrated on the case where the spins are coupled “Ohmically” to the bath, since this is usually the physically interesting case. In this Ohmic case, the results can be presented in the form of a “phase diagram”; see section V.

Organisation of the paper: In section II we discuss in more detail the physical origin of the PISCES model, in particular for the case where the 2 spins describe the low-energy dynamics of 2 macroscopic systems. In section III we discuss a simple “toy” model, where 2 spins couple by a direct interaction \( J_0 \hat{\tau}_1^z \hat{\tau}_2^z \) only. This model demonstrates correlations between the 2 spins in the absence of any dissipative environment; it also makes clear the physical meaning of the various density matrix elements. In section IV we set up the formal solution for \( \hat{\rho} \) for the full PISCES model, using influence functional methods. Section V is the most important in the paper; therein we give the detailed analytic
results for \( \hat{\rho} \) in the various regimes described above, and discuss their physical significance. We summarise our results in section VI. The more complicated details of the calculations are relegated to a series of Appendices.

Those readers whose primary interest is the results for the PISCES dynamics should go directly to section V, which can mostly be read independently of the rest of the paper.

II. THE MODEL

In this section we summarize the model, and sketch the way in which the truncated PISCES Hamiltonian can arise from a more microscopic model. No attempt is made at a detailed derivation, since this would require that we specify a particular physical system - only the basic ideas are outlined.

A. Effective Hamiltonians and Lagrangians

To see how a PISCES Hamiltonian can arise, we start with a very simple model, with a total Lagrangian given by

\[
L = L_0 + L_{\text{int}} + L_B
\]  

(2.1)

\[
L_B = \frac{1}{2} \sum_k m_k (\dot{x}_k^2 - \omega_k^2 x_k^2)
\]  

(2.2)

\[
L_0 = \frac{1}{2} (M_1 q_1^2 + M_2 q_2^2) - (V_1(q_1) + V_2(q_2))
\]  

(2.3)

\[
L_{\text{int}} = - \sum_k \left[ (q_1 c_k^{(1)} + q_2 c_k^{(2)}) x_k + \frac{1}{2} \left( \frac{x_k^2}{m_k \omega_k^2} \right) (|c_k^{(1)}|^2 q_1^2 + |c_k^{(2)}|^2 q_2^2) \right]
\]  

(2.4)

where it is assumed that \( V_1(q_1) \) and \( V_2(q_2) \) each describe 1-dimensional 2-well potentials, which are each symmetric under the interchange \( q_0 \rightarrow -q_0 \), with minima at \( q_1 = \pm q_0/2, q_2 = \pm q_0/2 \). Note that in (2.4), the effective couplings \( c_k^{(1)} \) and \( c_k^{(2)} \) are in general complex and out of phase with each other. Thus, for example, if the two systems in \( L_0 \) in (2.3) represent two systems at different positions \( \mathbf{R}_1 \) and \( \mathbf{R}_2 \), a commonly used form for the \( c_k^{(2)} \) will be \( c_k^{(2)} = c_k e^{i k \cdot (\mathbf{R}_2 - \mathbf{R}_1)} \), where \( k \) is a wave vector and \( \alpha = 1, 2 \). The form we have chosen for the coupling between the environmental modes \( \{x_k\} \) and the coordinates \( q_\alpha \) is linear in both, as in the usual Feynman-Vernon/Caldeira-Leggett scheme. We discuss the validity and generality of this choice below.

Included in (2.4) are two separate counterterms for each potential, quadratic in \( q_\alpha \). The purpose is, as usual, to restore the bare potential \( V(q_1, q_2) = V_1(q_1) + V_2(q_2) \), after renormalisation by the coupling to the environmental bosons, to its original value so that we may treat it as the physical potential. In Eq. (2.2), we have not yet truncated the problem to a "PISCES" problem, so that no direct interaction terms appear. This is why the counterterm involves 2 separate contributions. If in (2.3) there had been a term \( V_{12}(q_1, q_2) \) involving \( q_1 \) and \( q_2 \) together in a direct interaction, then the counterterm would no longer be separable either. Thus, in the form (2.3) the PISCES model only involves indirect interaction between system 1 and system 2, mediated by the oscillator bath.

The Lagrangian (2.1) is a simple generalization of the single 2-well model studied by Leggett et al., which they truncated to the spin-boson model. The manoeuvres required for the truncation of (2.1) are the same; we require simply that the separations \( \Omega_0^{(1)} \) and \( \Omega_0^{(2)} \), of the 2 lowest levels in wells 1 and 2 respectively, from the higher levels in these wells, be much greater than any energy scale (such as \( T \)) that we are interested in. If we impose an ultraviolet cutoff \( \omega_c \) on the bath modes, such that \( \Omega_0^{(1)}, \Omega_0^{(2)} \gg \omega_c \gg \Delta_1, \Delta_2, T \), then we may truncate (2.1) to the form

\[
H = H_0 + H_{\text{int}} + H_B
\]  

(2.5)

\[
H_B = \frac{1}{2} \sum_k m_k (\dot{x}_k^2 + \omega_k^2 x_k^2)
\]  

(2.6)

\[
H_0 = -\frac{1}{2} (\Delta_1 \dot{\hat{x}}_1^2 + \Delta_2 \dot{\hat{x}}_2^2)
\]  

(2.7)
\[ H_{\text{int}} = \frac{g_{01}}{2} \sum_{k} c^{(1)}_{k} x_{k} + \frac{g_{02}}{2} \sum_{k} c^{(2)}_{k} x_{k} \] (2.8)

This is just the PISCES model of Eq. (1.2) and (1.3) (in (1.2) and (1.3), the factors \(g_{01}\) and \(g_{02}\) are absorbed into the couplings \(c^{(1)}_{k}\) and \(c^{(2)}_{k}\)).

The splittings \(\Delta_{1}\) and \(\Delta_{2}\) can be calculated from \(L^{0}_{e}\) using WKB-instanton methods. Although this procedure is quite complicated and technically interesting for a spin-boson problem in which the wells are not all degenerate, we will not discuss it here. We will simply assume that \(\Delta_{1}\) and \(\Delta_{2}\) are given quantities, in the spirit of the low-energy effective Hamiltonian, and that they are, as usual, exponentially small compared to the energy scale of \(V(q_{1}, q_{2})\) in (2.3) (ie., exponentially smaller than \(\Omega_{0}^{(1)}\) and \(\Omega_{0}^{(2)}\) respectively).

\(H_{\text{PISCES}}\) as written in (1.2) is clearly not the most general model of this kind. A much wider range of direct couplings is possible; instead of \(K_{zz} \hat{\tau}_{1} \hat{\tau}_{2}\) we could use

\[ H_{\text{int}}^{\text{dir}} = \frac{1}{2} \sum_{\mu \nu} K_{\mu \nu} \hat{\tau}_{1}^{\mu} \hat{\tau}_{2}^{\nu} \] (2.9)

We could also use more complicated indirect couplings (ie., couplings to the bath) like \(\frac{1}{2} \sum_{\mu} c^{(1)}_{\mu} e^{i k_{\mu} \cdot R} c^{(2)}_{k} x_{k}\), with \(\mu = x, y, z\) and \(\alpha = 1, 2\). In the present paper we will stick to the diagonal coupling in (1.3) and keep only the direct coupling in \(\hat{\tau}_{1} \hat{\tau}_{2}\). Our reasoning is as follows. Just as for the single spin-boson problem, we expect diagonal couplings to dominate the low-energy dynamics of the combined system, since the spins spend almost all their time in a diagonal state (only a fraction \(\Delta_1/\Omega_0\) of their time is in a non-diagonal state, when the relevant system is tunneling under the barrier). In certain cases the diagonal couplings can be zero (usually for symmetry reasons), and then one must include non-diagonal couplings like \(\frac{1}{2} \sum_{\alpha} c^{(1)}_{\alpha} e^{i k_{\alpha} \cdot R} c^{(2)}_{k} x_{k} + H.c.\). Here we deal only with the generic case of dominant diagonal couplings; we shall see in any case that the physics generated by (1.3) is complicated enough as it is.

Our reason for keeping the longitudinal direct coupling in (2.9) is connected to our choice of diagonal couplings between the bath and the systems. The interaction in (1.3) will couple \(\hat{\tau}_{1}\) and \(\hat{\tau}_{2}\) to produce a longitudinal direct coupling of the form \(\frac{1}{2} \hat{J} \hat{\tau}_{1} \hat{\tau}_{2}\). In fact, one may quite generally observe that in a field-theoretical context, any direct interaction can be viewed as the result of a coupling between \(\hat{\tau}_{1}\) and \(\hat{\tau}_{2}\) through the high-frequency or “fast” modes of some dynamic field; “fast” in this context simply means much faster than the low-energy scales of interest, so that the interaction may be treated as quasi-instantaneous. In our case, integrating out these fast modes, those with a frequency greater than \(\Omega_0\) then produces the longitudinal static interaction in (1.2).

One further omission from (1.2) is an applied bias acting on one or both of the spins, of the form \(\epsilon_{j} \hat{\tau}_{j}\), for example. This important extra ingredient will be discussed elsewhere; our present interest is in the dynamic interaction between the spins, generated through the bath, and most of the present paper is devoted to unravelling its effects.

### B. The Oscillator Bath Representation

The “high-energy” Lagrangian of (2.1) to (2.4) already uses an oscillator bath representation of the environment, ie., it already involves a low-energy truncation of some more detailed microscopic representation of the environment. We briefly recall the arguments that justify this representation (1.3) and when they fail.

One begins by assuming a large number \(N\) of degrees of freedom in the environment, with couplings \(f_{ij}\) between the environmental excitations \(|i\rangle\) and \(|j\rangle\) of order \(1/N\). One also assumes that the coupling \(V_{j}\) of \(|j\rangle\) to an external “system” obeys \(V_{j} \sim O(N^{-1/2})\). In the PISCES model, as well as the spin-boson model, the \(V_{j}\) are just the \(q_{k}\). There are of course many canonical examples of such environments, such as \(^3\text{He}\) (normal, superfluid, solid, or quantum gas), \(^4\text{He}\) superfluid and solid, metals, semiconductors, magnets, superconductors, simple insulators, etc. Notice that any delocalised degrees of freedom will formally satisfy these requirements, simply because their wave-functions will be normalised inside a box of size \(\sim N\).

If we treat the couplings \(V_{j}\) perturbatively, up to \(2^{nd}\)-order, it is always possible to represent the environment as a bath of oscillators (c.f., Feynman and Vernon (1.3), pp. 153-159). We are then bound to end up with an effective Lagrangian of form

\[ L_{\text{eff}} = L_{0} + L_{\text{int}} + L_{B} \] (2.10)

where \(L_{B}\) takes the form (2.2), \(L_{0}(Q, \dot{Q})\) is the Lagrangian of the system and
\( L_{\text{int}} = - \sum_k (F_k(Q)x_k + G_k(\dot{Q})\dot{x}_k) \) (2.11)

(or some form related to this by a canonical transformation). It is often assumed that the excursions of the system are small, and then \( L_{\text{int}} \) is written as \( L_{\text{int}} = - \sum_k q_k Q_k \), as in the PISCES and spin-boson problems. However in many practical situations one obviously has to go to more general couplings (eg., the coupling \( L_{\text{int}} = - \sum_k (V_q e^{i q \cdot r} b_k^+ a_k + H.c.) \), arising from a potential probe \( V(r) = \sum V_q e^{i q \cdot r} \); here \( b_k^+ \) is an environmental boson operator, which could equally describe fermion particle-hole pairs via \( b_k^+ = \sum_k a_k^+ a_k \), and one is led to (2.11).

Thus the question of the validity of the oscillator bath representation of the environment boils down to the applicability of the 2nd-order perturbative treatment of the \( \{V_j\} \). It is important to understand that even if there appears to be some breakdown of the perturbative expansion, it may easily be curable by a suitable canonical transformation. Consider, eg., the Luttinger liquid with back scattering; although the \( \{V_j\} \) are formally \( \sim O(N^{-1/2}) \), they are infrared divergent (in Caldeira-Leggett language, they are subohmic). Nevertheless, a canonical transformation to a quite new set of environmental modes reveals that this subohmic form is an artefact of the original bosonisation- the coupling to the new modes is a conventional ohmic one. Another salutary example is provided by the coupling of solitons to their environment.

However, an environment which cannot in general be mapped to an oscillator bath is an environment of uncoupled (or very weakly coupled) spins, or 2-level systems. In this case one finds that the couplings of the N individual spins \( \{\sigma_k\} \) to the central quantum system are not \( \sim O(N^{-1/2}) \), but independent of \( N \). In some cases they are also very strong, so that instead of the environmental dynamics being only weakly affected by the quantum system of interest, they are in fact slaved to it.

From this very brief discussion (for a much more lengthy one see ref.), we see that provided we can ignore the effects of some “spin bath” environment (eg., nuclear or paramagnetic spins, or local defects of some kind), the oscillator bath description should be a good representation of the low-energy environmental degrees of freedom.

### III. DENSITY MATRIX FOR TWO COUPLED SPINS (NO ENVIRONMENT)

Before dealing with the full PISCES model, we summarize the results for the dynamics of a much simpler problem, in which the 2 spins couple to each other by a static longitudinal interaction \( J_0 \tau_1^z \tau_2^z \), and no environment is present. In a certain sense the PISCES model can be viewed as a dissipative version of this “toy model”.

#### A. Single Spin

First recall that a single spin in a bias field \( \epsilon \), with Hamiltonian

\[
H = -\frac{\Delta_0}{2} \tau_x + \frac{\epsilon}{2} \tau_z
\]

and with eigenvalues

\[
E_\pm = \pm |E| = \pm \frac{1}{2} \sqrt{\Delta^2 + \epsilon^2}
\]

and eigenfunctions

\[
\psi_\pm = A_\pm \left[ (E_\pm + \epsilon) |\uparrow\rangle - \Delta_0 |\downarrow\rangle \right],
\]

\[
A_\pm = \left[ \frac{1}{(E_\pm + \epsilon)^2 + (\Delta_0)^2} \right]^{1/2}
\]

has a “1-spin” density matrix

\[
\rho_{\alpha\beta}(t) = c_\alpha(t) c^*_\beta(t)
\]

in which the wave-function \( \Psi(t) \) of the system, at time \( t \), is expanded as \( \Psi(t) = \sum_\alpha c_\alpha(t) \phi_\alpha(t) \), with \( \phi_1 = |\uparrow\rangle \) and \( \phi_2 = |\downarrow\rangle \). Assuming an initial state \( |\uparrow\rangle \), \( \rho(t) \) then evolves as
\[
\rho(t) = \begin{pmatrix}
1 - \Delta_0^2 \sin^2 Et & -i \Delta_0 \sin Et \\
-i \Delta_0 \sin Et & \Delta_0^2 \sin^2 Et
\end{pmatrix}
\]  
(3.6)

In the equivalent 2-well problem, the initial “wave-packet” \(|\uparrow\rangle\) partially oscillates between the 2 wells- the diagonal elements give the occupation probability of the wells, and the off-diagonal elements describe oscillatory quantum interference between them, which is suppressed by the bias; when \(\epsilon \gg \Delta_0\), the particle stays in one well, in the absence of any coupling to the environment.

**B. Two Spins**

Now consider 2 coupled spins, with a Hamiltonian containing a simple longitudinal interaction:

\[
H = -\frac{1}{2}(\Delta_1 \hat{\tau}_1^x + \Delta_2 \hat{\tau}_2^x) + J_0 \hat{\tau}_1^z \hat{\tau}_2^z
\]  
(3.7)

The coupling creates a “dynamical bias” whereby each spin tends to force the other spin into one or other of its wells. The 2-spin density matrix takes the form

\[
\rho_{\tau_1\tau_2;\tau'_1\tau'_2}(t) = A_{\tau_1\tau_2}(t) A^*_{\tau'_1\tau'_2}(t)
\]  
(3.8)

(c.f., Eq (3.5)), in which we order the states as \(|\tau_1\tau_2\rangle\} = \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}; the amplitudes \(A_{\tau_1\tau_2}(t)\) are given by

\[
A_{\uparrow\uparrow} = \frac{1}{2} \left( \cos \Omega_+ t + \cos \Omega_- t \right) + i J_0 \frac{1}{\Omega_+} \sin \Omega_+ t + \frac{1}{\Omega_-} \sin \Omega_- t \right)
\]

\[
A_{\uparrow\downarrow} = -\frac{i}{4} \left( \frac{\Delta_1 - \Delta_2}{\Omega_-} \sin \Omega_- t - \frac{\Delta_1 + \Delta_2}{\Omega_+} \sin \Omega_+ t \right)
\]

\[
A_{\downarrow\uparrow} = -\frac{i}{4} \left( \frac{\Delta_1 - \Delta_2}{\Omega_-} \sin \Omega_- t + \frac{\Delta_1 + \Delta_2}{\Omega_+} \sin \Omega_+ t \right)
\]

\[
A_{\downarrow\downarrow} = \frac{1}{2} \left( \cos \Omega_- t - \cos \Omega_+ t \right) + i J_0 \frac{1}{\Omega_+} \sin \Omega_+ t - \frac{1}{\Omega_-} \sin \Omega_- t \right)
\]  
(3.9)

where at \(t = 0\) we have assumed an initial state \(|\tau'_1\tau'_2\rangle = |\uparrow\uparrow\rangle\); and the eigenfrequencies \(\Omega_{\pm}\) are given by

\[
\Omega_\pm^2 = J_0^2 + \frac{1}{4}(\Delta_1 \pm \Delta_2)^2
\]  
(3.10)

so that when \(J_0 = 0\), the amplitudes in (3.9) reduce to

\[
A^{(0)}_{\uparrow\uparrow} = \cos \frac{\Delta_1 t}{2} \cos \frac{\Delta_2 t}{2}
\]

\[
A^{(0)}_{\uparrow\downarrow} = -i \cos \frac{\Delta_1 t}{2} \sin \frac{\Delta_2 t}{2}
\]

\[
A^{(0)}_{\downarrow\uparrow} = -i \sin \frac{\Delta_1 t}{2} \cos \frac{\Delta_2 t}{2}
\]

\[
A^{(0)}_{\downarrow\downarrow} = \sin \frac{\Delta_1 t}{2} \sin \frac{\Delta_2 t}{2}
\]  
(3.11)
From this we obtain the probability $P_{\tau_1\tau_2}(t) = |A_{\tau_1\tau_2}(t)|^2$ of finding the 2-spin system in state $|\tau_1\tau_2\rangle$ at time $t$ (assuming a state $|\uparrow\uparrow\rangle$ when $t = 0$), and the interference between these states. When $J_0 = 0$, the 2 spins oscillate independently, with product wave function $|\tau_1\rangle|\tau_2\rangle$, and $\rho(t)$ has oscillation components corresponding to the sum and difference frequencies ($\Delta_1 \pm \Delta_2$) (see (3.11)). When $J_0$ is finite, we still have sum and difference frequencies $\Omega_\pm$ in $\rho(t)$, but the behaviour is more complicated because the wave-function is no longer a product wave-function - the coupling “entangles” the 2 spins. In fact the eigenfunctions are now

$$
\Psi_{++} = \left(\frac{\Omega_+ - J_0}{2\Omega_+}\right)^{1/2} \left(\frac{\Delta_1 + \Delta_2}{2(\Omega_+ - J_0)} |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + \frac{\Delta_1 + \Delta_2}{2(\Omega_+ - J_0)} |\downarrow\downarrow\rangle\right)
$$

$$
\Psi_{+-} = \left(\frac{\Omega_+ - J_0}{2\Omega_-}\right)^{1/2} \left(\frac{\Delta_1 - \Delta_2}{2(\Omega_- - J_0)} |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + \frac{\Delta_1 + \Delta_2}{2(\Omega_- - J_0)} |\downarrow\downarrow\rangle\right)
$$

$$
\Psi_{-+} = \left(\frac{\Omega_- + J_0}{2\Omega_+}\right)^{1/2} \left(\frac{\Delta_1 - \Delta_2}{2(\Omega_+ + J_0)} |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + \frac{\Delta_1 + \Delta_2}{2(\Omega_+ + J_0)} |\downarrow\downarrow\rangle\right)
$$

$$
\Psi_{--} = \left(\frac{\Omega_- + J_0}{2\Omega_-}\right)^{1/2} \left(\frac{\Delta_1 + \Delta_2}{2(\Omega_- + J_0)} |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + \frac{\Delta_1 + \Delta_2}{2(\Omega_- + J_0)} |\downarrow\downarrow\rangle\right)
$$

(3.12)

with energies

$$
E_{++} = -E_{--} = \Omega_+
$$

$$
E_{+-} = -E_{-+} = \Omega_-
$$

(3.13)

and the way in which the entanglement occurs depends on the magnitude of $J_0$ and its sign. For $J_0 < 0$ we have (FM) ferromagnetic coupling; when $-J_0 \gg \Delta_1, \Delta_2$, the ground state $|\Psi_{--}\rangle$ tends toward the superposition $2^{-1/2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$. For antiferromagnetic (AFM) coupling, the states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are emphasized and for $J_0 \gg \Delta_1, \Delta_2$, the ground state, which is now $|\Psi_{++}\rangle$ tends to the superposition $2^{-1/2}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$.

Consider now the probability $P_{\tau_1\tau_2}(0)$, for a system to start in $|\uparrow\uparrow\rangle$ at $t = 0$, and finish at time $t$ in state $|\tau_1\tau_2\rangle$. In Figs.3 and 4 we plot these probabilities for both strong-coupling ($|J_0| \gg \Delta_1, \Delta_2$) and weak-coupling ($|J_0| \ll \Delta_1, \Delta_2$). Consider first the strong-coupling regime. For both FM and AFM coupling, if the initial state is $|\uparrow\uparrow\rangle$, we see that the system essentially oscillates between $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ states with an effective frequency $\bar{\Delta} = \Delta_1 \Delta_2/(2|J_0|)$. The oscillations between $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are very fast, frequency $\sim J_0$, but their amplitude is reduced by a factor $\Delta/J_0$. Notice the AFM system cannot relax to some combination of the lower-energy states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ (there is no environment), a problem which does not arise for this initial state in the FM case, where the initial state $|\uparrow\uparrow\rangle$ is already a superposition of the 2 low-lying eigenstates $|\Psi_{++}\rangle$ and $|\Psi_{--}\rangle$ in (3.12). This situation is precisely analogous to the strongly biased 2-well system, where an initial state $|\uparrow\rangle$ in the biased system is stuck at high energy. In the same way, there is little coherence between the ($|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$) manifold and the ($|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$) manifold when $|J_0|$ is large, i.e., $J_0$ suppresses both coherence between singlet and triplet states, and the off-diagonal elements in $\rho(t)$.

In the opposite limit of weak coupling the density matrix $\rho(t)$ differs little from $\rho^{(0)}(t)$, and there is mixing between all four states. From Eq. (3.13) we see that for small $|J_0|$, the displacement of the energy levels $\sim O(J_0^2)$ unless $\Delta_1 = \Delta_2$, in which case the 2 central levels separate linearly with $J_0$. In this limit

$$
\Psi_{++} = \Psi_{++}^0 + \frac{1}{2} \frac{J_0}{\Delta_1 + \Delta_2} \left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)
$$

(3.14)

$$
\Psi_{+-} = \Psi_{+-}^0 + \frac{1}{2} \frac{J_0}{\Delta_1 - \Delta_2} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)
$$

(3.15)

$$
\Psi_{-+} = \Psi_{-+}^0 - \frac{1}{2} \frac{J_0}{\Delta_1 - \Delta_2} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)
$$

(3.16)

$$
\Psi_{--} = \Psi_{--}^0 - \frac{1}{2} \frac{J_0}{\Delta_1 + \Delta_2} \left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)
$$

(3.17)

This concludes our analysis of this coupled 2-spin system.
IV. DENSITY MATRIX FOR THE PISCES MODEL

In this section we set up a formal expression for the 2-spin density matrix for the PISCES system. This is done by first integrating out the oscillator bath modes to produce an expression in terms of an influence functional; then we show how the resulting expression can be summed over instanton paths. Some of the derivation is an obvious generalization of the work from Leggett et al., whereas other parts are somewhat less trivial. To help readers who are familiar with the spin-boson model, we have adopted a notation which coincides fairly closely with that of Leggett et al., with obvious generalizations.

A. The Influence Functional

In order to calculate the reduced density matrix for the system of interest, we will use the well known technique of integrating out the environmental modes in a path integral formalism. Thus, for some general Lagrangian of the form \( L \), with an associated action

\[
S(Q, \{ x_k \}) = S_0[Q] + S_{env}[\{ x_k \}] + S_{int}[Q, \{ x_k \}],
\]

(here we denote the environmental coordinates by \( \{ x_k \} \)), the reduced density matrix \( \rho \) propagates according to

\[
\rho(Q_f, Q'_f; t) = \int dQ_i dQ'_i J_\rho(Q_f, Q'_f; t; Q_i, Q'_i, 0) \rho(Q_i, Q'_i; 0)
\]

We write the propagator \( J \) as a weighted double path integral:

\[
J_\rho(Q_f, Q'_f; t; Q_i, Q'_i, 0) = \int^{q(t)=Q_f,q(0)=Q_i}_{q'(t)=Q'_f,q'(0)=Q'_i} D[q] D[q'] e^{i(S_0[q]-S_0[q'])} F[q, q']
\]

a double integral over the paths \( q(t) \) and \( q'(t) \), starting at \( t = 0 \) from the positions \( Q_i \) and \( Q'_i \) and ending at time \( t \) in \( Q_f \) and \( Q'_f \) respectively. \( F[q, q'] \) is the famous “influence functional”, whose general properties are discussed by Feynman and Vernon. It incorporates the interaction between the paths \( q \) and \( q' \) with an environment. In the case of a system coupled independently to a set of independent external “environmental coordinates”, we have

\[
F[q, q'] = \prod_k F_k[q, q']
\]

with \( F_k \) the functional of the \( k^{th} \) external system (which in this case is just the \( k^{th} \) oscillator).

Here we are interested in the representation of the PISCES problem either via a 4 well representation, or as eigenstates of \( \tau^{(1)}_1 \tau^{(2)}_2 \) (which in the truncated approximation are equivalent). The path \( q \) of the combined 2-spin system is written as a double path \( q = (q_1, q_2) \), one for each spin (and likewise for the path \( q' \) as well as for \( Q = (Q_1, Q_2) \)). The general form of the kernel \( J_\rho \) in equation (4.2) is then

\[
J_\rho = \int D[q_1] \int D[q_2] \int D[q'_1] \int D[q'_2] A_1[q_1] A_2[q_2] A'_1[q'_1] A'_2[q'_2] F[q_1, q'_1, q_2, q'_2]
\]

where the boundary conditions are defined by

\[
\int D[q_\alpha] = \int^{q_\alpha(t)=Q_{\alpha,t}}_{q_\alpha(0)=Q_{\alpha,i}} D[q_\alpha]
\]

with \( \alpha = 1, 2 \). The factors \( A_1[q_1] = i\Delta_1/2, A_2[q_2] = i\Delta_2/2 \) are the transition amplitude associated with a single flip (instanton) of the relevant spin (1 or 2). In an instanton calculation \( \Delta \sim \Omega_0 e^{-S_{cl}} \), where \( S_{cl} \) is the action of the path that minimises Lagrange’s equations in imaginary time, and \( \Omega_0 \) is the preexponential factor, of order the small oscillation frequency of the system. \( F[q, q'] = F[q_1, q_2, q'_1, q'_2] \) is again the influence functional, produced by integrating out the oscillator environment.

To determine the form of \( F[q, q'] \) we need the correlation functions of the boson bath. To illustrate the general case let us first consider a particular one, in which the oscillator modes \( \{ x_k \} \) can be classified by momentum quantum number \( k \), and the two spins are considered to be at 2 positions \( R_1 \) and \( R_2 \). Then defining \( x(r, t) = \sum_k e^{-ik\cdot r}x_k(t) \) and \( R = R_1 - R_2 \), we require the correlation function
\[ J_{\alpha\beta}(\mathbf{R}, \tau, T) = \left< x(\mathbf{R}_\alpha, \tau) x(\mathbf{R}_\beta, \tau) \right>_T \] (4.7)

which is conveniently written in the matrix form

\[ J_{\alpha\beta}(\mathbf{R}, \omega) = \begin{pmatrix} J_{11}(\omega) & J_{12}(\mathbf{R}, \omega) \\ J_{21}(\mathbf{R}, \omega) & J_{22}(\omega) \end{pmatrix} \] (4.8)

with \( J_{\alpha\beta}(\mathbf{R}, \omega) = \int d\tau e^{-i\omega\tau} J_{\alpha\beta}(\mathbf{R}, \tau) \); as before, the indices \( \alpha, \beta = 1, 2 \), and label the two different spins. The diagonal terms \( J_\alpha = J_{\alpha\alpha} \) are given directly from \( H_{PICSCE}(\text{Eq. (1.2)} \text{ and (1.3)}) \) as

\[ J_\alpha(\omega) = \frac{\pi}{2} \sum_k \frac{|c_k^{(\alpha)}|^2}{m_k \omega_k} \delta(\omega - \omega_k) \] (4.9)

whereas the inter-spin spectral function is given by

\[ J_{12}(\mathbf{R}, \omega) = \frac{\pi}{2} \sum_k e^{i\mathbf{k}\cdot\mathbf{R}} D_k(\omega) \delta(\omega - \omega_k) \] (4.10)

where \( D(\mathbf{R}, \omega_k) = \sum_k e^{i\mathbf{k}\cdot\mathbf{R}} D_k(\omega) \) is just the propagator for the bath modes (e.g., in a phonon bath it is the usual phonon propagator). The diagonal spectral functions \( J_\alpha(\omega) \) are of course equivalent to those used by Caldeira and Leggett. \( J_{12} \) is similar, but note that (a) in general the function \( J_{12}(\mathbf{R}, \omega) \) is not a separable function of \( \mathbf{R} \) and \( \omega \), and (b) it is complex (it is a retarded correlation function).

The retardation effects begin to be important once \( \omega_k \geq v_k/|\mathbf{R}| \), where \( v_k \) is the propagation velocity of the \( k \)-th bath mode. For distances \( |\mathbf{R}| \ll v_k/\omega_k \), we can ignore retardation effects, and \( J_{12}(\mathbf{R}, \omega) \) will be separable. In this paper we will always work in this limit. Typically we will be interested in interactions mediated by either phonons or electrons. In the case of an electronic bath, one has typically the Ohmic form

\[ J_{\alpha\beta}(\mathbf{R}, \omega) = \omega \eta_{\alpha\beta} e^{-\omega/\omega_c} \] (4.11)

\[ \eta_{12}(\mathbf{R}) \sim (\eta_1 \eta_2)^{1/2} \mathcal{V}_e(\mathbf{R}) \] (4.12)

where \( \eta_1 \) and \( \eta_2 \) are local friction coefficients of the kind discussed by Caldeira and Leggett, and \( \mathcal{V}_e(\mathbf{R}) \) has the form \( \mathcal{V}_e(\mathbf{R}) \sim \sin^2(k_j R)/(k_j R)^2 \), in 3 dimensions. The upper cut-off \( \omega_c \) in an electron bath must satisfy \( \omega_c \ll \Omega_0^{(1)}, \Omega_0^{(2)} \), as discussed in section II. Typical electronic velocities are \( \sim \) the Fermi velocity \( v_f \sim 10^6 m \text{s}^{-1} \); this means that if, say, \( \Omega_0 \sim 1K \), then retardation effects do not become important until \( R \sim 50 \mu m \).

For a phonon bath, one usually has the super-Ohmic form

\[ J_{\alpha\beta}(\mathbf{R}, \omega) = \tilde{g}_{\alpha\beta}(\mathbf{R}) \Theta_D \left( \frac{\omega}{\delta_D} \right)^m e^{-\omega/\omega_c} \] (4.13)

\[ \tilde{g}_{12}(\mathbf{R}) \sim (\tilde{g}_1 \tilde{g}_2)^{1/2} \mathcal{V}_\phi(\mathbf{R}) \] (4.14)

where \( m \geq 3 \) in 3 dimensions, and \( \mathcal{V}_\phi(\mathbf{R}) \sim (a_0/|\mathbf{R}|)^3 \), with \( a_0 \) a lattice constant, and \( \tilde{g}_1 \) and \( \tilde{g}_2 \) again being local coupling constants. For a phonon velocity \( \sim 5 \times 10^3 m \text{s}^{-1} \), and \( \Omega_0 \sim 1K \) again, we now find that retardation effects become important for distances \( R \sim 250 nm \) or greater. For distances \( |\mathbf{R}| \) much less than these limits, these forms can be used. Again, a more precise specification of \( J_{\alpha\beta}(\mathbf{R}, \omega) \) can be given once we fix our physical problem.

Returning now to the influence functional, we introduce the time correlation functions

\[ \Phi_\alpha(\tau - s) = \int_0^\infty d\omega J_\alpha(\omega) \sin(\omega(\tau - s)) \] (4.15)

\[ \Gamma_\alpha(\tau - s) = \int_0^\infty d\omega J_\alpha(\omega) \cos(\omega(\tau - s)) \coth(\omega/2T) \] (4.16)

for the diagonal elements \( (\alpha = 1, 2) \), and
\[ \Phi_{12}(\mathbf{R}, \tau - s) = \int_0^\infty d\omega \, J_{12}(\mathbf{R}, \omega) \sin \omega(\tau - s) \]  
\[ (4.17) \]

\[ \Gamma_{12}(\mathbf{R}, \tau - s) = \int_0^\infty d\omega \, J_{12}(\mathbf{R}, \omega) \cos \omega(\tau - s) \coth(\omega/2T) \]  
\[ (4.18) \]

for the off-diagonal elements. The general form of the influence functional is now

\[ F[q_1, q_2, q'_1, q'_2] = F_1[q_1, q'_1] F_2[q_2, q'_2] F_{12}[q_1, q_2, q'_1, q'_2], \]  
\[ (4.19) \]

with \( F_1 \) and \( F_2 \) being the influence functionals for systems 1 and 2 alone, and \( F_{12} \) the part representing their interaction through the bath. Each path \( q_\alpha(t) \), with \( \alpha = 1, 2 \), is composed of a series of jumps between wells, which we assume to be located at \( \pm q_{0\alpha}/2 \). We can then define the functions

\[ \xi_\alpha(\tau) = q_{0\alpha}^{-1}[q_\alpha(\tau) - q'_\alpha(\tau)] \]  
\[ (4.20) \]

\[ \chi_\alpha(\tau) = q_{0\alpha}^{-1}[q_\alpha(\tau) + q'_\alpha(\tau)] \]  
\[ (4.21) \]

so that the single spin functionals are given by

\[ F_\alpha[\tau, \tau'] = \exp \left( \frac{q_{0\alpha}^2}{\pi} \int_0^t dt \int_0^{\tau} ds (i\Phi_\alpha(\tau - s)\xi_\alpha(s) - \Gamma_\alpha(\tau - s)\xi_\alpha(s)) \right) \]  
\[ (4.22) \]

whereas the interaction functional \( F_{12} \) is

\[ F_{12} = \exp \left( \frac{q_{01}q_{02}}{\pi} \int_0^t dt \int_0^{\tau} ds (i\Phi_{12}(\mathbf{R}, \tau - s)\xi_1(s)\chi_2(s) + \xi_2(s)\chi_1(s)) - \Gamma_{12}(\mathbf{R}, \tau - s)[\xi_1(\tau)\xi_2(\tau) + \xi_2(\tau)\xi_1(\tau)] \right) \]  
\[ (4.23) \]

We see that in both the single spin functional \( F_\alpha \) and in the interaction functional \( F_{12} \), there is a purely reactive “phase” correction \( \Phi \), and a purely dissipative damping term \( \Gamma \). The damping operates when paths depart from each other, i.e., when both \( \xi_\alpha(\tau) \) and \( \chi_\beta(\tau) \) are non-zero. We may interpret these equations diagramatically (Fig. 3). For both the spin-boson and PISCES problems Eq. (4.22) and (4.23) massively simplify, because \( \xi_\alpha(\tau) \) and \( \chi_\alpha(\tau) \) may only take one of the 3 values 0, \( \pm 1 \); each time the spin \( \alpha \) tunnels, both \( \xi_\alpha \) and \( \chi_\alpha \) change by \( \pm 1 \). Notice that these equations give \( F_{\alpha\beta} \) solely in terms of the sum and difference variables between pairs of paths; the paths themselves have now disappeared from the formalism.

### B. Path Integral for the Density Matrix

The formal calculation of the density matrix elements (and hence all physical properties of the system) can be carried out starting from (4.2), (4.3), (4.22) and (4.23). In order to do this, we must integrate over 4 separate paths \((q_1(t), q_2(t), q'_1(t), q'_2(t))\), or, equivalently \(\xi_1(t), \xi_2(t), \chi_1(t), \chi_2(t)\).

One way of handling the path integral for \(J\) would be to write it in terms of a single path over the 16 possible states of \(q_1, q_2, q'_1\), and \(q'_2\). Here we shall use a procedure analogous to that employed by Leggett et al. for the single spin-boson problem. Taking advantage of the fact that the \(\chi_\alpha\) and \(\xi_\alpha\) are not independent (when \(\chi_\alpha = 0\), \(\xi_\alpha = \pm 1\) and vice-versa), we write \(F_{\alpha\beta}\) as an integral over 2 paths, one for each spin, in which we distinguish only between diagonal “sojourn” states \((\xi_\alpha = 0, \chi_\alpha = \pm 1)\) and off-diagonal “blip” states \((\xi_\alpha = \pm 1\) whilst \(\chi_\alpha = 0)\). This has the diagrammatic representation shown in Fig. 4. Each such path must contain an even number of transitions, with action \(-i\Delta_\alpha/2\) for each transition. The transitions occur at times \(t_j\) \((j = 1, 2, \ldots, 2n_1)\) for spin 1, and times \(t_k\) \((k = 1, 2, \ldots, 2n_2)\) for spin 2. We may then follow the standard manoeuvre of introducing a set of “charges” which label the states (blip or sojourn) for each path; calling these charges \(\eta_{1j}, \zeta_{1j}\) for spin 1 and \(\eta_{2k}, \zeta_{2k}\) for spin 2, we allow them to have values \(\pm 1\) according to

\[ \chi_1(\tau) = \sum_{j=0}^{n_1} \eta_{1j} \left[ \theta(\tau - t_{2j}) - \theta(\tau - t_{2j+1}) \right] \]  
\[ (4.24) \]
\[
\xi_1(\tau) = \sum_{j=0}^{n_1} \zeta_{1j} [\theta(\tau - t_{2j-1}) - \theta(\tau - t_{2j})] \tag{4.25}
\]

\[
\chi_2(\tau) = \sum_{k=0}^{n_2} \eta_{2k} [\theta(\tau - u_{2k}) - \theta(\tau - u_{2k+1})] \tag{4.26}
\]

\[
\xi_2(\tau) = \sum_{k=0}^{n_2} \zeta_{2k} [\theta(\tau - u_{2k-1}) - \theta(\tau - u_{2k})]. \tag{4.27}
\]

The sum over the 2 paths now becomes a sum over all possible arrangements of the charges \(\eta_{\alpha j}\) and \(\zeta_{\alpha k}\), these charges living on a set of discrete times between 0 and \(t\) (with, however, certain restrictions on the paths, and on the charges, depending on which density matrix element is being computed).

The explicit expansion of the density matrix \(\rho\) in terms of sums over the various charge and blip ordering configurations is rather messy; to make the paper easier to follow, the detailed path integral calculations are confined to a series of appendices. In Appendix A the detailed structure of the complete influence functional is given, along with explicit expressions for the blip-blip and blip-sojourn interactions, both for bath-mediated interactions between different states of the same spin, and also for interspin interactions. The results can be summarised as follows. The single spin functionals \(F_\alpha\) are those derived by Leggett et al.\[1\], they can often be treated in the “dilute blip approximation”, in which only the “blip self-energy” terms (i.e., between 2 times in the same blip), and interaction between a blip and the sojourn immediately preceding it are included. This works, provided the interactions with the bath are sufficiently strong that blips are rare; essentially it assumes that the environmental decoherence is strong enough to severely suppress the “off-diagonal” paths in the density matrix, in which the damping term \(\Gamma_\alpha(\tau - s)\) in Eq. (4.23) comes in. Consequently, during the waiting time between two such off-diagonal excursions or “blips”, the system completely “forgets” the effects of the previous blip.

The dilute blip approximation is much more problematic when an interaction term in the Hamiltonian is trying to force the density matrix into off-diagonal states. This point is rather crucial when we come to the interaction functional \(F_{12}\), since the main effect of the bath turns out to be the introduction of an extra interaction \(\tilde{c}(R) \hat{\tau}_1^z \hat{\tau}_2^-\) between the 2 spins, with \(\tilde{c}(R)\) given by

\[
\tilde{c}(R) = \frac{\eta_{12}(R) \omega_c}{\pi} \int_0^\infty d\omega \frac{J_{12}(R, \omega)}{\omega}. \tag{4.28}
\]

For Ohmic dissipation, the explicit form of \(\tilde{c}(R)\) is

\[
\tilde{c}(R) = \alpha_{12}(R) \omega_c \tag{4.29}
\]

where \(\alpha_{12}(R) = \eta_{12}(R) q_{01} q_{02} / 2\pi \hbar\). For superohmic dissipation, we get

\[
\tilde{c}(R) = 2\Gamma(s) \left(\frac{\bar{\omega}}{\omega_c}\right)^s \bar{g}_{12}(R) \omega_c \tag{4.30}
\]

In the subohmic case, \(s < 1\) so that the integral diverges and the model becomes ill-defined. It is important to notice that the magnitude of this bias depends on the upper cut-off frequency of the bath - this is simply because the total strength of the interaction involves the high-energy (adiabatic) modes. Now, this \textit{bath-induced} interaction simply adds to the original high frequency “direct” interaction to give the total adiabatic interaction between the two spins as

\[
\mathcal{J}(R) \equiv K_{zz}(R) + \tilde{c}(R) \tag{4.31}
\]

The big problem for the theory is now that \(\mathcal{J}(R)\) plays the role of a “mutual bias” between the 2 spins, just as \(J_0\) did in our toy model of section III; it turns out that most of its effects come from interactions between blips of one system and sojourns of the other. Unless both systems are overdamped (or both locked together, so that blips on one cannot overlap with sojourns of the other), then it is clear that we cannot introduce any kind of dilute blip approximation on a single system, independently of the other, to handle the interspin term \(F_{12}\). Nevertheless we will show, in the next section (and in Appendices C and D) how it is possible to evaluate \(F_{12}\) by generalising this approximation. We now turn to the detailed behaviour of the density matrix.
V. DYNAMICS OF THE PISCES SYSTEM

In this section, we obtain the dynamics of the 2-spin reduced density matrix. We explicitly calculate the 4 diagonal matrix elements \( P_{\tau_1\tau_2}(t) \equiv \rho(\tau_1\tau_2,t;\uparrow\uparrow,0) \), which are the probabilities for the system to end up in a state \(|\tau_1\tau_2\rangle\) after a time \( t \), having started in a state \(|\uparrow\uparrow\rangle\). We will concentrate in what follows on the most interesting Ohmic case, where the 2 spins are coupled Ohmically to the sea of oscillators. Since it is conventional to express the strength of this Ohmic coupling by a dimensionless constant \( \alpha \), we will henceforth use the index \( \beta = 1, 2 \) to label the 2-spins (we do not use \( \beta = 1/kT \) in what follows). We will use temperature and frequency units in which \( k_B = \hbar = 1 \), so that the thermal energy \( T \) and the tunneling splitting energies are \( \Delta_\beta \).

For this Ohmic PISCES problem, we will concentrate on those parameter regimes in which analytic results can be obtained for \( P_{\tau_1\tau_2}(t) \). Having achieved these, it is possible to build up a fairly complete picture of the behaviour of \( P_{\tau_1\tau_2}(t) \), in all temperature and coupling range of interest. We can identify 4 regimes in the dynamics, as shown in the “phase diagram” of Fig. 3.

These 4 regimes are, in the Ohmic case, defined as follows:

(i) The “Locked Phase” \((J \gg T, \Delta_\beta/\alpha_\beta)\): In this regime, the effective coupling \( J \) is so strong that the 2 spins lock together, in either the states \(|\uparrow\uparrow\rangle\) or \(|\downarrow\downarrow\rangle\) depending on the sign of \( J \). This state is very similar to the strongly-coupled state described in section 3; we shall see that the combined “locked spin” oscillates between \(|\uparrow\uparrow\rangle\) and \(|\downarrow\downarrow\rangle\) (for FM coupling), or between \(|\uparrow\downarrow\rangle\) and \(|\downarrow\uparrow\rangle\) (for AFM coupling), at a renormalised frequency \( \Delta_c = \Delta_1\Delta_2/|J| \). However these oscillations are damped - in fact the locked spin now behaves like a single spin-boson system, with a new coupling \( \alpha_c = \alpha_1 + \alpha_2 \pm 2\alpha_1\alpha_2 \) to the oscillator bath, the + (−) corresponding to ferromagnetic (antiferromagnetic) coupling between the spins.

(ii) The “Mutual Coherence” phase \((\Delta_\beta/\alpha_\beta \gg T \gg |J|, \Delta_\beta > \Delta_\beta)\): Here, the thermal energy overcomes the mutual coupling; nevertheless if the dissipative couplings \( \alpha_\beta \)'s are sufficiently small \((\alpha_\beta \ll 1)\), it is possible for the energy scale \( \Delta_\beta/\alpha_\beta \) to dominate even if \( \Delta_\beta < |J| \). In this case, even though we are dealing with a strong coupling, and the bath dissipation is still important, some coherence in the motion of each spin is maintained - moreover, the small \( J \) causes “mutual coherence” between the two spins, i.e., their damped oscillations are correlated to some extent.

(iii) The “Correlated Relaxation” or High-\( T \) phase \((T \gg \Delta_\beta/\alpha_\beta, J)\). In this regime, the bath causes each spin to relax incoherently; however, the relaxation of the two spins is still correlated (indeed each spin relaxes in the time-dependent bias generated by the other).

(iv) Finally, and much less interesting, the “perturbative regime” \((J \ll \Delta_\beta^*), \) in which the total coupling is so weak that the 2 spins relax almost independently; all correlations can be handled perturbatively, and only weakly affect the behaviour that one calculates from the standard spin-boson model.

We notice that the locked and correlated relaxation phases always exist; but the mutual coherence phase is much more delicate, and only appears if the coupling of each spin to the bath is weak (i.e., \( \alpha_\beta \ll 1 \)). To get a better intuitive feeling for the energy scales and the different regimes, it is useful to imagine how the behaviour of the PISCES system changes as one moves along various paths in the phase diagram. Imagine first that we start off in the lower right half of Fig. 3, at a very low \( T \) (much less than any other energy). Then temperature fluctuations are irrelevant, even if each spin is strongly coupled to the bath; the effect of the induced interaction \( J \) is simply to lock the 2 spins into one. This “spin complex” also couples to the bath, but now with a quite different coupling; and it can also tunnel between 2 orientations, but at a much lower frequency. Typically its motion at finite \( T \) will be completely overdamped, and it will localize as \( T \) tends to zero. However if we now raise \( T \), the thermal fluctuations increase in importance, and eventually compete with \( J \); the spins start to unlock from each other. This crossover regime cannot be handled analytically, but further increase in \( T \) causes complete unlocking; we enter the correlated relaxation phase, where the thermal fluctuations are strong enough to destroy any coherent dynamics of the system, but the relaxation of each is still correlated with that of the other. If \( T \gg J \), even these correlations become irrelevant- the 2 spins relax independently.

One may make a similar traversal, this time from correlated to locked phase, this time by starting at high \( T \) but weak coupling (top left in the Figure), and then increasing the coupling. Much more interesting is to go to low \( T \) and then increase the coupling. Two situations are then possible. One is where the coupling \( \alpha \) to the bath is not small. Then there is no mutual coherence regime, and we find ourselves initially in the lower right quadrant of the Figure. Even with large \( \alpha \), one may still envisage a very small \( J \) (e.g., if the 2 spins are very far apart). Then each behaves independently (with weak perturbative interactive corrections), as a spin-boson system. Increasing the coupling (i.e., increasing their distance apart) increases the correlations, until they eventually lock; the crossover cannot be handled analytically.

On the other hand if \( \alpha \ll 1 \), each spin is weakly coupled to the bath, one can get a situation in which their mutual coupling, although weak, still exceeds the renormalised tunneling splitting \( \Delta^* \). In this case the \( T \)-fluctuations are not strong enough to upset their mutual correlations, or to destroy coherent behaviour on the part of each spin,
even if $T > J$, provided $\alpha T < \Delta^*$. The spin is simply so weakly coupled to the bath that it hardly sees the $T$-fluctuations, but $J$ causes the oscillations of the 2 spins to correlate- we get “damped beats”. Further increase in $T$ decoheres these completely, and further increase in $J$ locks the 2 spins together. This mutual coherence phase is clearly going to be favoured if we have 2 spins close to each other (to make $J$ big enough), yet very weakly coupled to the bath.

The difficulty of setting up a PISCES model which allows a mutual coherence phase is connected with the following important general feature of the results. In the problem of a single spin coupled to an oscillator bath, it is known that one may always get some coherent behaviour provided $T$ is low enough, and also $\alpha < 1/2$. However in the present case of 2 spins, coherent behaviour is much harder to find; even if $\alpha$ is very small, coherence is destroyed if we “switch on” the coupling between the 2 spins, once $J$ reaches the very small energy scale $\Delta^*/\alpha$. This is because the “dynamic bias” on one spin, coming from the other, removes the “resonance condition” between initial and final states which is necessary for coherent oscillations. It shows, in essence, that mutual interactions increase decoherence effects. This enhanced decoherence is particularly interesting in the context of the measurement operation, or indeed for any attempt to look for macroscopic quantum coherence.

A more formal understanding of these energy scales comes out of a renormalisation group treatment, to which we proceed first. After this the analytic results for each regime are presented.

### A. Energy Scales From the Renormalisation Group

Although a renormalisation group analysis cannot give us the dynamics of the PISCES model, it gives considerable insight into the energy scales mentioned above. We shall use the “poor-man” renormalisation group procedure first applied to the Kondo problem by Anderson et al.\[28\] and then later to the single spin-boson problem\[17\]. Cardy\[17\] devised a method to treat the multisite problem, which was then applied to the case of two Anderson impurities\[28\] and two tunneling impurities in metal\[17\]. As shown in Appendix B, in the case where both spins are identical (i.e., where $\Delta_1 = \Delta_2 = \Delta$ and $\alpha_1 = \alpha_2 = \alpha$) the partition function for this problem is identical to that considered by Chakravarty and Hirsch\[3\] and we carry over directly their scaling equations for a set of fugacities

\[
\begin{align*}
\frac{dy}{d\ln \tau_c} &= y(1 - \alpha) + y(y_F + y_{AF}) \\
\frac{dy_F}{d\ln \tau_c} &= y_F(1 - \alpha/2 - \alpha_{12}/2) + y^2 \\
\frac{dy_{AF}}{d\ln \tau_c} &= y_{AF}(1 - \alpha/2 + \alpha_{12}/2) + y^2
\end{align*}
\]  

(5.1)

where the dimensionless fugacity $y = \Delta/\omega_c$ and $y_F$ and $y_{AF}$ are the fugacities associated with simultaneous transitions of the 2 spins. Since $\alpha >> \alpha_{12}$, the behaviour of the fugacities is determined solely by the single spin dissipation coefficients. These coefficients are also renormalised; their scaling is

\[
\begin{align*}
\frac{d\alpha}{d\ln \tau_c} &= -\alpha(2y^2 + y_F^2 + y_{AF}^2) - \alpha_{12}(y_F^2 - y_{AF}^2) \\
\frac{d\alpha_{12}}{d\ln \tau_c} &= -\alpha_{12}(2y^2 + y_F^2 + y_{AF}^2) - \alpha(y_F^2 - y_{AF}^2)
\end{align*}
\]  

(5.2)

Finally, there is also a scaling of the coupling energy

\[
\frac{d(\tau J)}{d\ln \tau_c} = (1 - 4y^2)(\tau J)
\]  

(5.3)

The renormalisation group analysis assumes that the dimensionless quantities $y$, $\tau_cT$ and $\tau_cJ$ are all much less than one. The scaling has to be stopped if one of the parameters renormalises to the strong coupling regime. The three important scaling parameters are thus $1/\Delta^*$, a renormalised tunneling matrix element, $1/T$ and $1/J$. Let us first set $J = 0$. The scaling equations are then very similar to the case of the single spin-boson system. At $T = 0$, the scaling of $y$ is determined by $\alpha$. For $\alpha > 1$, $y$ decreases and since $y_F$ and $y_{AF}$ grow as $y^2$, this decrease cannot be compensated and the scaling leads to $y = 0$, the well-known localisation phenomena. The dynamics will then be determined by $y_F$ or $y_{AF}$. Depending on the strongest of the two, only the ferromagnetic or antiferromagnetic states will be occupied. For $\alpha < 1$, then the scaling can stop at some $\tau = 1/\Delta^*$, a renormalised tunneling matrix element determined self-consistently as
$\Delta_\beta^* = \Delta_\beta \left( \frac{\Delta_\beta}{\Omega_0} \right)^{\alpha/(1-\alpha)}$ \hspace{1cm} (5.4)

In this case, the behaviour is determined by the single tunneling fugacity, the two spins are still correlated, but behave nearly independently. At finite temperature, the value of the renormalised tunneling matrix element is determined by the ratio between the energy scales $1/\Delta^*$ and $1/T$. If $T$ is the largest energy scale, the scaling must be terminated at $\tau_c \sim 1/T$ and the resulting tunneling matrix element is $\Delta^*(T) = \Delta_0(T/\Omega_0)^{\alpha/(1-\alpha)}$.

In the presence of an interaction, the symmetry between the levels is broken and Eq. (5.3) indicates that this difference will grow with the scaling, even if the fugacity grows as well (since $y \ll 1$ by assumption). The time it takes for $\tau_c J$ to grow to the strong coupling phase is roughly $\tau \sim 1/J$. If both $1/\Delta^*$ and $1/T$ are smaller than $1/J$, the scaling stops before the strong coupling phase is attained and the system behaves like two single spins, although coupled by the interaction $J$. However, if $J$ is the largest energy scale, the system rapidly gets into the strong coupling phase and the RG equations break down. It is clear however that the new strongly coupled phase is made of the two spins locked together by $J$. The thermal fluctuations (controlled by $T$) and the tendency to individual dynamics (determined by $\Delta^*$) cannot compensate this strong coupling.

**B. Locked Phase**

Here $J$ is so strong that a prohibitively high energy cost is incurred if a blip on one path overlaps with a sojourn on the other. In Appendix B this is discussed in more detail, and we demonstrate that for a strongly FM-coupled system which starts in initial state $| \uparrow \uparrow \rangle$, the resulting dynamics is equivalent to that of a single spin-boson system, with a new tunneling matrix element $\Delta_c = \Delta_1 \Delta_2 / |J|$, and with this “spin complex” now coupled to the environment with a new Ohmic coupling $\alpha_c = \alpha_1 + \alpha_2 + 2\alpha_{12}$. The new oscillation frequency is not surprising– we have already seen it in our toy model in the strong coupling regime. As for the new coupling to the bath, it is simply equivalent to the coefficient $K_{F,F}$ coming from the renormalisation group analysis (see Appendix B). Physically, in the case of FM coupling, the system oscillates between $| \uparrow \uparrow \rangle$ and $| \downarrow \downarrow \rangle$ at a frequency $\Delta_c$, if we ignore the environmental dissipation. Mixing with the high-energy states $| \uparrow \downarrow \rangle$ and $| \downarrow \uparrow \rangle$ is negligible (with $J \gg T$, even thermally-excited transitions, via absorption of bath quanta, will be exponentially rare), and so the system reduces to the single spin-boson problem.

A system coupled by a strong AFM interaction will have the same effective tunneling matrix element $\Delta_c$, but will now be coupled to the environment by a dissipation coefficient $\alpha_c = \alpha_1 + \alpha_2 - 2\alpha_{12}$. Once again, the new coupling can be related to a the RG analysis, in this case, it is simply the coefficient $K_{AF,AF}$ of Appendix B.

Notice that the case of an AFM-coupled system which begins in the state $| \uparrow \uparrow \rangle$ is quite different, since this is then a high energy state. Without the bath, the system is frozen in this state (c.f., section 3). The coupling to the bath allows relaxation to the low-energy manifold (now composed of the states $| \uparrow \downarrow \rangle$ and $| \downarrow \uparrow \rangle$). Thus this case needs to be treated slightly differently. Intuitively one expects that the system will first try to relax to an AFM state before simultaneous transitions take place (since a transition from $| \uparrow \uparrow \rangle$ to $| \downarrow \downarrow \rangle$ still takes the system to a high-energy state); this turns out to be correct.

Let us begin with the FM-coupled case. The equivalence to the unbiased spin-boson system means that the dilute-blip approximation can be used, and we will simply take over the results already derived by Leggett et al. Thus, defining the Laplace transform of $P_{\uparrow \uparrow}(t)$ as

$$P_{\uparrow \uparrow}(\lambda) = \int_0^{\infty} dt e^{-\lambda t} P_{\uparrow \uparrow}(t)$$ \hspace{1cm} (5.5)

one finds in this locked phase

$$P_{\uparrow \uparrow}(\lambda) = \frac{1}{2\lambda} + \frac{1}{\lambda + f(\lambda)}.$$ \hspace{1cm} (5.6)

The function $f(\lambda)$ is defined in terms of an effective tunneling matrix element $\Delta^*_{c,eff}$, viz.,

$$\Delta^*_{c,eff} = |\Gamma(1 - 2\alpha_c) \cos(\pi \alpha_c)|^{1/(1-\alpha_c)} \Delta_c^*$$ \hspace{1cm} (5.7)

$$\Delta_c^* = \Delta_c (\Delta_c / \Omega_0)^{\alpha/(1-\alpha)}$$ \hspace{1cm} (5.8)

$$\Delta_c = \frac{\Delta_1 \Delta_2}{|J|}$$ \hspace{1cm} (5.9)
The function \( f(\lambda) \), in the limits appropriate to the truncation procedure \( (T, \Delta_c \ll \Omega_0, \text{ as well as } \omega_c t \gg 1) \) is given by

\[
f(\lambda, T) = \Delta_c^{\text{eff}} \left( \frac{2\pi T}{\Delta_c^{\text{eff}}} \right)^{2\alpha_c^{-1}} \frac{\Gamma(\alpha_c + \lambda/2\pi T)}{\Gamma(1 - \alpha_c + \lambda/2\pi T)}
\]

(5.10)

where \( \Gamma(x) \) is the Gamma function and where, as noted above, \( \alpha_c = \alpha_1 + \alpha_2 + 2\alpha_{12} \). The corresponding behaviour of \( P_{\uparrow\uparrow}(t) \) is well known:

(i) At \( T = 0 \) and for \( \alpha_c < 1/2 \), the system exhibits damped oscillations of frequency \( \sim \Delta_c^{\text{eff}} \). For \( 1/2 < \alpha_c < 1 \), the dynamics is an exponential decay superposed on an incoherent background and for \( \alpha_c > 1 \), the system is localised, that is \( P_{\uparrow\uparrow}(t) = 1 \) for all times. (ii) At finite \( T \), this oscillatory behaviour persists as long as \( T < \Delta_c/\pi\alpha_c \). For \( T > \Delta_c/\pi\alpha_c \), the dynamics is overdamped, and the system relaxes exponentially to \( P_{\uparrow\uparrow} = 1/2 \).

The various regimes are shown in Fig. \( 3 \). For the present problem the system is most likely to be in the overdamped regime; because \( \Delta_c^{\text{eff}} \) is so small, extremely small values of \( \alpha_c \) and/or \( T \) will be required to see underdamped oscillations. In the overdamped regime, the damping rate is

\[
\Gamma_c = \frac{\Delta_c^2}{2\Omega_0} \left[ \frac{2\pi T}{\Omega_0} \right]^{2\alpha_c^{-1}} \frac{\Gamma(2\alpha_c)}{\Gamma(2\alpha_c)}
\]

(5.11)

and, provided \( T > \Delta_c/\pi\alpha_c \), one has the simple result

\[
P_{\uparrow\uparrow}(t) = \frac{1}{2} + \frac{1}{2} e^{-\Gamma_c t}
\]

(5.12)

If \( T < \Delta_c/\alpha_c \), numerical methods must be employed if one is interested in the exact behaviour of the system. Notice that when \( \alpha_c > 1/2 \), an increase in temperature increases \( \Gamma_c \), corresponding to faster relaxation. However, if \( \alpha_c < 1/2 \), the relaxation rate \( \Gamma_c \) (which is now much smaller) decreases with increasing \( T \). Thus the thermal environment plays a different role on either side of the “Toulouse line” \( \alpha_c = 1/2 \); for \( \alpha_c > 1/2 \), it actively retards relaxation. In the limit \( \alpha_c \ll 1 \), we can establish an upper bound on the relaxation rate \( \Gamma_c < \Delta_c/2\pi \).

Obviously, the same discussion will go through if the coupling is antiferromagnetic but with an initial \( | \uparrow\downarrow \rangle \) or \( | \downarrow\uparrow \rangle \) configuration.

Consider now the case of an AFM coupling with an initial \( | \uparrow\uparrow \rangle \) state. As mentioned before, the system must first relax to a low energy state before any simultaneous transitions can take place. In the present case of strong coupling this is obvious- the energy gap between the upper “FM-ordered” doublet and the lower AFM doublet is so large that this is by far the fastest transition, and so we do not attempt to derive this result. The initial relaxation is then simply determined by the relaxation of a single spin-boson system in the presence of a static bias \( \mathcal{J} \gg T \)

\[
\Gamma_{AFM} = \alpha\beta(\Delta_\beta^2/\mathcal{J})(\mathcal{J}/\Omega_0)^{2\alpha\beta}
\]

(5.13)

where \( \beta \) represent the spin with the fastest relaxation rate. Once the system has relaxed to the low-energy states \( \{\uparrow\downarrow\} \) or \( \{|\downarrow\uparrow\} \), the analysis for the coupled dynamics can be applied.

It has to be said that the locked phase is the one that is most likely to be encountered in physical applications. For microscopic models such as the problem of 2 coupled magnetic impurities, it represents the ordered state; for macroscopic models such as 2 coupled nanomagnets or 2 coupled SQUID’s, it will prevail unless the mutual coupling is very weak (ie., they are very far apart, since usually \( \alpha \) is not small for such systems). Many quantum measurement schemes actually require that a strong correlation be established between apparatus and measured system at some point, thereby freezing the system dynamics (even if the system is otherwise able to display coherent tunneling).

C. Correlated Relaxation Phase

As the coupling \( \mathcal{J} \) is reduced, or \( T \) is increased, temperature fluctuations break up the “locking of blips” which allowed the solution of the locked phase. When \( T \gg \mathcal{J} \), we may use instead an approximation in which blips of one path almost always overlap with with sojourns of the other. It is important that this approach only works outside of the perturbative regime - but it is the strong-coupling regime we are interested in here.

Again, the detailed evaluation of the path integrals, etc., is given in Appendix D. There we show that one can ignore both interactions between blips on the same path, and between blips on the 2 separate paths. This requires \( \mathcal{J} \gg \Delta_\beta \), and thus places us directly outside the range of a perturbative treatment. Our results therefore cannot be necessarily extended to the limit \( \mathcal{J} \to 0 \), especially if one is interested in the coherent properties of the coupled
system in this limit. However, provided that the two conditions \( T \gg J \) and \( J \gg \Delta_\beta \), then our results are valid for any values of the dissipation coefficients \( \alpha_\beta \).

In fact one finds that the same path summation technique also works for the mutual coherence phase, and so we first give a general expression for the Laplace transform of \( P_{\uparrow\uparrow}(t) \) (again we assume an initial state \( | \uparrow\uparrow \rangle \)). As shown in Appendix D, \( P_{\uparrow\uparrow}(\lambda) \) is given in terms of the 2 functions

\[
g_\beta(\lambda) = \Delta_\beta^2 \int_0^\infty dt \, e^{-\lambda - Q^{(\beta)}_1(t)} \cos[Q^{(\beta)}_1(t)] \cos[Jt]
\]

\[
= \frac{1}{2} (f_\beta(\lambda + iJ) + f_\beta(\lambda - iJ))
\]

\[
h_\beta(\lambda) = \Delta_\beta^2 \int_0^\infty dt \, e^{-\lambda - Q^{(\beta)}_1(t)} \sin[Q^{(\beta)}_1(t)] \sin[Jt]
\]

\[
= \frac{\tan \pi \alpha}{2i} (f_\beta(\lambda - iJ) - f_\beta(\lambda + iJ))
\]

where \( f_\beta(\lambda) \) is equivalent to Eq.(5.10) provided that the appropriate changes of parameters are made. We then have

\[
P_{\uparrow\uparrow}(\lambda) = \frac{1}{\lambda} - \frac{1}{4} \left[ \frac{g_1(1 + h_1)}{g_1} + g_2 \left( 1 + \frac{h_2}{g_2} \right) \right] \frac{1}{\lambda} + \frac{1}{\lambda + g_1 + g_2}
\]

\[
- \frac{1}{4} \left[ \frac{g_1(1 + h_1)}{g_1} + g_2 \left( 1 + \frac{h_2}{g_2} \right) \right] \frac{1}{\lambda^2 + \lambda(g_1 + g_2) + g_1 g_2 (1 - \frac{h_1 h_2}{g_1 g_2})}
\]

\[
- \frac{1}{2} g_1 g_2 \left[ 1 - \frac{h_1 h_2}{g_1 g_2} \right] \frac{1}{\lambda} \frac{1}{\lambda^2 + \lambda(g_1 + g_2) + g_1 g_2 (1 - \frac{h_1 h_2}{g_1 g_2})}.
\]

Analogous expressions for the other probabilities appear in Appendix D. The dynamics is controlled by the poles of \( P_{\tau_1 \tau_2}(\lambda) \). There are two types of denominator; the “global denominator”,

\[
(\lambda + g_1(\lambda) + g_2(\lambda))
\]

(5.17)

corresponds to the overall system dynamics, whereas the “individual spin” denominator,

\[
(\lambda + g_1(\lambda))(\lambda + g_2(\lambda)) - h_1(\lambda) h_2(\lambda)
\]

(5.18)

corresponds to the individual spin dynamics. Both spins are correlated through the terms in \( g_\beta/h_\beta \), which essentially behave as \( \tanh J/2T \) at long times. This correlation is present both in the denominators (which give the decay rates and oscillation frequency) and in their coefficients (controlling the weight of each term) which results in a very complex behaviour.

On the other hand the limit of \( \lambda P_{\tau_1 \tau_2}(\lambda) \) as \( \lambda \to 0 \) yields the universal long time probabilities

\[
P_{\uparrow\uparrow}(t \to \infty) = P_{\downarrow\downarrow}(t \to \infty) = \frac{1}{4} [1 - \tanh J/2T]
\]

(5.19)

\[
P_{\uparrow\downarrow}(t \to \infty) = P_{\downarrow\uparrow}(t \to \infty) = \frac{1}{4} [1 + \tanh J/2T].
\]

(5.20)

These results are of course nothing but the thermodynamic equilibrium expectation values, once all dynamic relaxation to equilibrium has ceased.

Let us now deal with the regime \( T \gg J \), and also assume that \( T > \Delta_\beta/\alpha_\beta \); this is the overdamped correlated relaxation regime (cf. Fig. (3)). Formally this regime is very easy to deal with, since we may set \( \lambda = 0 \) in \( f_\beta(\lambda) \), and the explicit dependence on \( h_1 \) and \( h_2 \) in \( P_{\tau_1 \tau_2}(\lambda) \) can be eliminated using\( h_\beta(0)/g_\beta(0) = \tanh(J/2T) \). The Laplace inversion results in a sum of decaying exponentials for each \( P_{\tau_1 \tau_2}(t) \):

\[
P_{\uparrow\uparrow}(t) = A_+ e^{-\left( \Gamma_1 + \Gamma_2 \right) t} + B_+ e^{-R_+ t} + B_- e^{-R_- t}
\]

(5.21)

\[
P_{\downarrow\downarrow}(t) = A_- e^{-\left( \Gamma_1 + \Gamma_2 \right) t} - B_+ e^{-R_+ t} - B_- e^{-R_- t}
\]

(5.22)

\[
P_{\uparrow\downarrow}(t) = A_+ [1 - e^{-\left( \Gamma_1 + \Gamma_2 \right) t}] + C_+ [e^{-R_+ t} - e^{-R_- t}]
\]

(5.23)
$$P_{↑↑}(t) = A_+ [1 - e^{-\left[(\Gamma_1 + \Gamma_2) t\right]}} - C_+ [e^{-R_+ t} - e^{-R_- t}]$$, (5.24)

with the different constants defined as

$$A_\pm = \frac{1}{4} [1 \pm \tanh(J/2T)]$$ (5.25)

$$B_\pm = \frac{1}{4} \left[1 \pm \frac{\Gamma_1 + \Gamma_2}{\sqrt{R_{12}}} \tanh(J/2T)\right]$$ (5.26)

$$R_\pm = \frac{1}{2} (\Gamma_1 + \Gamma_2) \pm \frac{1}{2} \sqrt{R_{12}}$$ (5.27)

$$R_{12}(T) = \Gamma_1^2 + \Gamma_2^2 - 2\Gamma_1\Gamma_2 [1 - 2 \tanh^2(J/2T)]$$ (5.28)

$$C_+ = A_+ \frac{\Gamma_1 - \Gamma_2}{\sqrt{R_{12}}}$$ (5.29)

The relaxation rates \(\Gamma_1\) and \(\Gamma_2\) are the same as in the previous section, but they are still not fully correlated (as in the locked phase). Physically, when \(\Gamma_1\) and \(\Gamma_2\) are not significantly different, the correlation of the two spins is only weakly felt in the relaxation rates. However, when \(\Gamma_1\) and \(\Gamma_2\) are significantly different, the correlation of the two spins is more strongly felt in the relaxation rates.

$$\Gamma_\beta = \frac{\Delta_\beta^2}{2\Omega_0} \left[\frac{2\pi T}{\Omega_0}\right]^{2\alpha_\beta - 1} \cosh(J/2T) \frac{\cosh(J/2T)}{\Gamma(2\alpha_\beta)} - |\Gamma(\alpha_\beta + iJ/2\pi T)|^2$$ (5.30)

Eq. (5.13) corresponds to the limit \(J \gg T\) of this equation. In the other limiting case, \(J \ll T\) and for \(\alpha_\beta \sim O(1)\),

$$\Gamma_\beta = \frac{\Delta_\beta^2}{2\Omega_0} \left[\frac{2\pi T}{\Omega_0}\right]^{2\alpha_\beta - 1} \frac{\Gamma^2(\alpha_\beta)}{\Gamma(2\alpha_\beta)} + O(J/T)^2 \quad (J/T \ll 1)$$ (5.31)

However, if \(\alpha_\beta \ll 1\) as well, then it is

$$\Gamma_\beta = \frac{\Delta_\beta^2}{2\Omega_0} \left[\frac{2\pi T}{\Omega_0}\right]^{2\alpha_\beta - 1} \frac{2\alpha_\beta}{\alpha_\beta^2 + (J/2\pi T)^2} \quad (J/T, \alpha_\beta \ll 1)$$ (5.32)

Thus one can get different behaviour depending on the ratio between \(\alpha_\beta\) and \(J/T\).

The relaxation times appearing in \(P_{\tau_1\tau_2}(t)\) if \(T \gg J\) can be quite different, and their contributions will in general appear in different proportions. Fig. 8 illustrates this by plotting the 4 probabilities.

Let us now look more closely at the structure of \(P_{\tau_1\tau_2}(t)\). If \(J = 0\), so that the two system are completely uncorrelated, it simply decomposes into a product of the individual spin probabilities \(P_{\tau_1\tau_2}(t) = P_{\tau_1}(t)P_{\tau_2}(t)\), with \(A_\pm = B_\pm = C_\pm = 1/4\), \(R_+ = \Gamma_1\) and \(R_- = \Gamma_2\). Turning on the interaction \(J\) causes the individual probabilities to merge together to produce a term corresponding to the relaxation of the system as a whole (with relaxation rate \(\Gamma_1 + \Gamma_2\), and weight \(A_\pm\), mixed in with correlated individual relaxation (of rates \(R_\pm\), and weight \(B_\pm\) or \(C_\pm\)). The mixing of the relaxation rate is independent of the sign of \(J\); as \(J\) increase, \(R_+\) tends to \(\Gamma_1 + \Gamma_2\) while \(R_-\) tend to 0, indicating a reduction of the individual dynamics of the spins. However, the coefficients \(A_\pm\), \(B_\pm\) and \(C_\pm\) do depend on the sign of the interaction, as we would expect - we have changed the system in the state \(|↑↑\rangle\). Notice that the correlations of the two spins is only weakly felt in the relaxation rates \(R_\pm\); it represents a correction of order \(J/T \ll 1\). The relaxation at long time will reflect this correlation more strongly, mostly through the terms \(A_\pm\).

It is important to remember that all these expressions assume \(J \ll T\) (and can thus be expanded in powers of \(J/T\)). Using them for \(J \sim T\) would be incorrect- one is then in the crossover region between the correlated and the locked phase. In this crossover, we cannot solve the problem analytically - the assumption of non-overlapping blips is breaking down, but they are still not fully correlated (as in the locked phase). Physically, when \(J \sim T\) the spins are trying to lock, but thermal fluctuations are preventing them.
D. Mutual Coherence Phase

If \( \alpha_\beta \geq O(1) \), then we have exhausted all possible phases of the PISCES system; either the 2 spins are locked, or they exhibit correlated overdamped relaxation, depending on the ratio of \( J \) to \( T \).

However if \( \alpha_\beta \ll 1 \), a new possibility emerges: for temperatures such that \( J \ll T \ll \Delta_\beta/\alpha_\beta \), the thermal fluctuations are strong enough to prevent the locking of the blips but still not capable of destroying coherence. As a result, the dynamics is composed of underdamped oscillations. In the PISCES problem the oscillations of the 2 systems will be correlated, and we can even expect decaying “beats”, between the 2 spins.

Because we are in the non-perturbative limit \( J \gg \Delta \) (even though \( \alpha_\beta \ll 1 \), the oscillation frequencies are roughly equal to \( J \), with corrections of order \( O((\Delta_\beta/J)^2) \) and \( O((\alpha_\beta T)^2/J^4) \). There are 2 characteristic relaxation rates in the system. There is a “global” decay at a rate \( \Gamma_{\text{glob}} \sim \alpha_\beta T(\Delta_\beta/J)^2 \), and a much faster damping of the mutual oscillations, at a rate \( \Gamma_{\text{osc}} \sim \alpha_\beta T \ll \Gamma_{\text{glob}} \). We give more detailed expressions below.

As noted above, expressions like \((5.16)\) for \( P_{\overline{\gamma}t\gamma}(\lambda) \) are still valid for this regime, but an expansion of \( f_\beta(\lambda, T) \) in \((5.10)\) (and hence of \( g_\beta(\lambda) \) and \( h_\beta(\lambda) \)) in powers of \( \lambda \) is invalid - instead we must expand in \( (\alpha_\beta + \lambda/2\pi T) \ll 1 \). This expansion gives

\[
f_\beta(\lambda) = \Delta_\beta^2 \left( \frac{2\pi T}{\Delta_\beta} \right)^{2\alpha_\beta} \frac{1}{\Gamma(1 - 2\alpha_\beta)} \frac{1}{2\pi T \alpha_\beta + \lambda} \tag{5.33}
\]

In general the denominators in \( P_{\overline{\gamma}t\gamma} \) are of polynomial degree 3 or 6, which renders analytic inverse Laplace transformation hopeless. We can however study 2 cases. One is the case where both spins are equivalent, i.e., \( \Delta_1 = \Delta_2 = \Delta \) and \( \alpha_1 = \alpha_2 = \alpha \); this is the Equivalent Spin case. We can also examine what happens if one spin is overdamped, and the other is underdamped; this is the Overdamped plus Underdamped case. We re- emphasize that real physical situations it will be quite difficult to see mutual coherence. One possibility for macroscopic systems involves 2 conducting nanomagnetic particles imbedded in a semiconductor substrate; the low electronic density so reduces \( \alpha \) and the mutual RKKY interactions that the equivalent spin phase is a real possibility. The “overdamped plus underdamped” system is interesting in view of the possibility of coupling an overdamped measuring apparatus to an underdamped quantum system, without destroying the coherence properties of the latter.

We start by defining three energy scales, viz., \( a \equiv \pi \alpha T, (J^2 + \Delta^2)^{1/2} \) and \( \Delta \), where \( \Delta \) is defined as

\[
\Delta^2 = \frac{\Delta^2}{\Gamma(1 - 2\alpha)} \left( \frac{2\pi T}{\Delta} \right)^{2\alpha} \tag{5.34}
\]

Of these 3 energy scales, \( a \) is the smallest. It is now convenient to define (by analogy with the biased spin- boson model)\(^{[1]}\):

\[
\begin{align*}
\gamma_{R,s}(a_\beta, J, \Delta_\beta^2) &= 2a_\beta \frac{\Delta_\beta^2}{J^2 + \Delta_\beta^2} \\
\gamma_s(a_\beta, J, \Delta_\beta^2) &= 2a_\beta - \frac{1}{2} \gamma_{R,s} = \frac{a_\beta}{J^2 + \Delta_\beta^2}(2J^2 + \Delta_\beta^2) \\
\nu_\gamma^2(a_\beta, J, \Delta_\beta^2) &= J^2 + \Delta_\beta^2 + 4a_\beta^2 - 2\gamma_{R,s} \gamma_s - \gamma_s^2 = J^2 + \Delta_\beta^2 - \frac{a_\beta^2}{(J^2 + \Delta_\beta^2)^2} \left[ 1 + \frac{J^2}{\Delta_\beta^2} \right]
\end{align*}
\tag{5.35}
\]

with \( a_\beta \equiv \pi \alpha \beta T \) and \( \Delta_\beta \) defined as in Eq. \((5.34)\) with \( \Delta \) replaced by \( \Delta_\beta \). In the limits considered in this section, \( \gamma_{R,s} \sim 2a_\beta \Delta_\beta^2/J^2, \gamma_s \sim 2a_\beta \) and \( \nu_\gamma^2 \sim J^2 \).

1. Equivalent Spin Case

For this case, defined by \( T \gg J \) and \( \Delta/\alpha \gg T \) for both spins, the equation for the roots of the denominator in \( P_{\overline{\gamma}t\gamma}(\lambda) \) decouples into 2 equations of polynomial degree 3, which we can handle by perturbation in \( a = \pi \alpha T \). The resulting forms for the “FM- correlated matrix elements” \( P_{\bar{\gamma}t}(t) \) and \( P_{\bar{t}\bar{\gamma}}(t) \) are then found to be (cf. App. E):

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\[ P_{\uparrow\downarrow}(t) = A_{-} - C_{2}e^{-\Gamma_{\uparrow} t} + C_{3}e^{-\Gamma_{\downarrow} t} \cos(\mu t) + C_{4}e^{-\Gamma_{\downarrow} t} \sin(\mu t) + \sum_{i=\pm} C_{5i}e^{-\gamma_{ni} t} + C_{6i}e^{-\gamma_{ni} t} \cos(\nu_{i} t) + C_{7i}e^{-\gamma_{ni} t} \sin(\nu_{i} t) \]  

(5.36)

\[ P_{\downarrow\uparrow}(t) = A_{-} - C_{2}e^{-\Gamma_{\downarrow} t} + C_{3}e^{-\Gamma_{\uparrow} t} \cos(\mu t) + C_{4}e^{-\Gamma_{\uparrow} t} \sin(\mu t) + \sum_{i=\pm} C_{5i}e^{-\gamma_{ni} t} + C_{6i}e^{-\gamma_{ni} t} \cos(\nu_{i} t) + C_{7i}e^{-\gamma_{ni} t} \sin(\nu_{i} t) \]  

(5.37)

where \( \Gamma_{R} = \gamma_{R,s}(a, J, 2\Delta^2) \), \( \Gamma = \gamma_{s}(a, J, 2\Delta^2) \), \( \mu^2 = \nu_{s}^2(a, J, 2\Delta^2) \), and \( A_{\pm} \) is defined by Eq. (5.24). If we now introduce \( J_{\pm} = 4\sqrt{A_{\pm}} \) and \( \Delta_{\pm}^2 = 4\Delta_{s}^2 \), then we have \( \gamma_{R,\pm} = \gamma_{R,s}(a, J_{\pm}, \Delta_{s}^2) \), \( \gamma_{\pm} = \gamma_{s}(a, J_{\pm}, \Delta_{s}^2) \) and the frequencies \( \nu_{\pm}^2 = \nu_{s}(a, J_{\pm}, \Delta_{s}^2) \). The explicit expressions for the 3 characteristic frequencies of the system are then:

\[ \mu^2 = J^2 + 2\Delta^2 - 4\frac{a^2\Delta^4}{(J^2 + 2\Delta^2)^2} \left[ 1 + 2\frac{J^2}{\Delta^2} \right] \]  

(5.38)

\[ \nu_{\pm}^2 = J^2 + \Delta^2 - \frac{a^2\Delta^4}{(J^2 + \Delta^2)^2} \left[ 4A_{\pm} + 16A_{\pm} \frac{J^2}{\Delta^2} \right] \]  

(5.39)

The frequency \( \mu \) pertains to the combined system, whereas the frequency pair \( \nu_{\pm} \) describes correlated oscillations of the individual spins ("beats"). The various constant \( C_{i} \)'s in Eq. (5.36) and Eq. (5.37) are complicated functions of the frequencies and decay rates and given in Appendix E.

On the other hand, in this equivalent spin limit, the AFM correlated probabilities \( P_{\uparrow\downarrow}(t) \) and \( P_{\downarrow\uparrow}(t) \) have a much simpler form:

\[ P_{\uparrow\downarrow}(t) = P_{\downarrow\uparrow}(t) = A_{+} + C_{2}e^{-\Gamma_{\uparrow} t} - C_{3}e^{-\Gamma_{\downarrow} t} \cos(\mu t) - C_{4}e^{-\Gamma_{\downarrow} t} \sin(\mu t) \]  

(5.40)

Let us now try to understand some of the details of this result. The appearance of oscillations in the density matrix at 3 different frequencies reflects the fact that the 2 spins are neither in a completely correlated nor in a completely uncorrelated state. The total dynamics then results from a compromise between global and individual dynamics. The global dynamics can be eliminated if one considers the probability of a spin being in a given state, independently of the state of the other. For example, the quantity \( [P_{\uparrow\downarrow}(t) + P_{\downarrow\uparrow}(t)] \) describes the dynamics of the first spin, with the second one "averaged out". Notice, incidentally, that it is not possible to recover the oscillation frequencies of our "toy-model", Eq. (5.38) and (5.39), in the limit \( \alpha = 0 \) since paths with overlapping blips do not contribute to \( P_{\uparrow\downarrow} \) in the uncorrelated phase. The lower limit on \( \alpha \) is set by \( J \sim \Delta/\alpha \).

To leading order in \( T/J \) and \( \Delta/J \), the decay rates \( \Gamma_{R} \) and \( \gamma_{R,i} \) are given by \( \gamma_{R,i} = \Gamma_{R}/2 \sim 2\pi\alpha T\Delta^2/J \), which simply corresponds to the limit \( \alpha T \ll J \) (always with \( J \gg \Delta \)) of the individual decay rate \( \Gamma_{R} \) given in Eq. (5.26). The same is true of the decay rates \( \Gamma \) and \( \gamma_{i} \), the decay rates associated to the oscillatory terms. However, they are strongly corrected by a term of order \( O((J/\Delta)^2) \), so that the oscillations are damped out very quickly compared to the general decay of the probability function.

We illustrate this case with Fig. (10), showing a special case of the Fourier transform of \( P_{\uparrow\downarrow}(t) \), namely \( Im[-\lambda P_{\uparrow\downarrow}(i\lambda)] \). Without the environment, the oscillation frequencies would show up as divergences in this function. With the environment, the divergences are smeared out by the decay rates \( \Gamma \) and \( \gamma_{\pm} \). At this scale, the differences between \( \mu \) and \( \nu_{\pm} \) are clear, but not the difference between \( \nu_{+} \) and \( \nu_{-} \). Taking a larger value of \( \alpha \) (resulting in a higher \( \alpha = \pi \alpha T \)) increases the decay rates associated with the oscillations and the width of the individual resonances becomes large enough for the two to overlap.

2. Overdamped plus Underdamped Case

We can also imagine two different systems labelled by 1 and 2, such that \( \Delta_{2}/\alpha_{2} > T \) whereas \( \Delta_{1}/\alpha_{1} < T \). In a single spin analysis, the spin-1 would be underdamped with the spin-2 overdamped. The limit \( J \gg \Delta_{1}, \Delta_{2} \) assuring the non-overlap of the blips is implicit.

To establish the dynamical behaviour, let us first look at the two denominators involved in the Laplace transform. In Appendix F, we show that the global denominator \( (\lambda + g_{1} + g_{2}) \) yields decay rates and oscillation frequencies given by \( \Gamma_{2} + \tilde{\gamma}_{R}, \Gamma_{2} + \tilde{\gamma} \) and \( \tilde{\mu}_{2}^{2} \), where \( \Gamma_{2} \) is the decay rate of the overdamped spin, Eq. (5.30) and \( \tilde{\gamma}_{R} \), \( \tilde{\gamma} \) and \( \tilde{\mu}_{2}^{2} \) are expressed with respect to Eq. (5.35) as

\[ \tilde{\gamma}_{R} = \gamma_{R,s}(a, J_{\pm}, \Delta_{s}^2) \]  

(5.41)

\[ \tilde{\gamma} = \gamma_{s}(a, J_{\pm}, \Delta_{s}^2) \]  

(5.42)

\[ \tilde{\mu}_{2}^{2} = \mu_{s}(a, J_{\pm}, \Delta_{s}^2) \]  

(5.43)
\[
\tilde{\gamma}_R = \gamma_{R,s}(\vec{a}, \mathcal{J}, \Delta_1^2)
\]
\[
\tilde{\gamma} = \gamma_s(a, \mathcal{J}, \Delta_1^2)
\]
\[
\hat{\mu}^2 = \nu_1^2(a, \mathcal{J}, \Delta_1^2)
\]

(5.41)

where \(\vec{a} \equiv a_1 - \Gamma_2/2\).

To treat the second denominator \(\left[(\lambda + g_1)(\lambda + g_2) - h_1h_2\right]\), we simply drop the terms \(h_1(\lambda)h_2(\lambda)\). We can do this because \(h_1 \sim \mathcal{J}/T\), whether the spin is over- or underdamped. Thus as far as the decay rates and oscillation frequencies are concerned, these terms will only be a small correction of order \((\mathcal{J}/2T)^2(\Delta_1/\mathcal{J})^2 \ll 1\). We are then left with the decay rates and oscillation frequencies of a single over- or underdamped spin in a static bias \(\mathcal{J}\). These are \(\Gamma_2\), as given by Eq. (5.30). \(\gamma_{1,R} = \gamma_{R,s}(a_1, \mathcal{J}, \Delta_1), \gamma_1 = \gamma_s(a_1, \mathcal{J}, \Delta_1)\) and the frequency \(\nu_1^2 = \nu_1^2(a_1, \mathcal{J}, \Delta_1)\), defined in Eq. (5.35). Notice however, that the correlations brought by \(h_1, h_2\) in the numerators of \(P_{\gamma_1\gamma_2}(\lambda)\) must be kept to determine the weight of the relaxation and oscillation terms as well as the long time limit of the probabilities.

With the decay rates and oscillation frequency identified, it is straightforward to perform the inverse Laplace transform to obtain the probability of occupation \(P_{\uparrow\uparrow}(t)\) as

\[
P_{\uparrow\uparrow}(t) = A_+ + C_2e^{-\Gamma_2 - \delta} + C_3e^{-\Gamma_2 - \delta} \cos(\hat{\mu}t) + C_4e^{-\Gamma_2 - \delta} \sin(\hat{\mu}t)
\]
\[
+ C_5e^{-\Gamma_2 t} + C_6e^{-\gamma_1}, \quad t + C_7e^{-\gamma_1 t} \cos(\nu_1 t) + C_8e^{-\gamma_1 t} \sin(\nu_1 t)
\]

(5.42)

Again, oscillations associated to the global and the individual dynamics of the spins are present in \(P_{\uparrow\uparrow}\). Notice that the corrections to the frequency \(\hat{\mu}\) are “tunable” if one has very good control over all the parameters. In fact, with \(\Gamma_2 = a_1\), the frequency \(\hat{\mu}^2\) is simply equal to \(\mathcal{J}^2 + \Delta^2\). On the other hand, once again most of the effect of the environment is felt as a damping of the oscillatory terms, much faster than the the global decay (with the oscillations of frequency \(\hat{\mu}\) being damped faster than those of frequency \(\nu_1\)).

This case is illustrated of Fig. (II). The coherence peaks are still present, but the line-width caused by the environment impedes their resolution.

VI. CONCLUSION

We have spoken very little in this article about the possible physical applications of our results, preferring first to gain a general understanding of how the model behaves. We have in fact already used these results to do a number of detailed calculations for 2 particular problems. The first concerns the dynamics of a pair of nanomagnets, each coupled to a background conducting or semiconducting substrate. To properly treat this we need to add to what we have done here the presence of an external bias acting on each nanomagnet. It is straightforward to make this generalisation, and also to calculate from microscopic theory what are the parameters entering the PISCES Hamiltonian as functions of nanomagnet size, distance between nanomagnets, electron density, etc.; and one may also incorporate the effect of nuclear spins and dipolar interactions. This problem is not only of academic interest; it is also relevant to future information storage technology.

The second problem we have looked at describes a fairly general class of physical systems in which we have a macroscopic measuring apparatus coupled to a central “measured” system; and in which at low energies the dynamics of both truncates to 2-level systems. The purpose of this work is to go beyond the usual idealised models of the measurement process, and see how the 3 important partners in the measurement (system, apparatus, and environment) work together. Again one finds that the PISCES model (with added biases) can be used to derive the full dynamics, with some rather surprising results. Both of these investigations will be the subject of detailed future papers; and they confirm that the results we have here can be used directly to give a realistic description of the full quantum dynamics of coupled macroscopic systems (as well as a large number of microscopic ones).

VII. ACKNOWLEDGEMENTS

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APPENDIX A: REDUCED DENSITY MATRIX OF THE TWO-SPIN SYSTEM

In this appendix, we derive the form of the reduced density matrix of the combined two-spin system. We will assume the system starts in the polarised state $|↑↑\rangle$, so both $\eta_{10}$ and $\eta_{20}$ must be $+1$. Now consider the density matrix element $\rho_{\tau_1\tau_2}^{↑↑}(t)$, i.e., the probability $P_{\tau_1\tau_2}(t)$ that at time $t$ the system is in the state $\tau_1\tau_2$ where $\tau_1$ and $\tau_2$ represent either $\uparrow$ or $\downarrow$. Using the decomposition of the influence functional into individual functionals and the interaction $F_{12}$, we express the complete summation for the probabilities as

$$P_{\tau_1\tau_2}(t) = \sum_{n_1,n_2} (-1)^{n_1+n_2+(d^{(1)}_{\alpha\beta}+d^{(2)}_{\alpha\beta})/2} \left( \frac{\Delta_1}{2} \right)^{2n_1+(d^{(1)}_{\alpha\beta})/2} \left( \frac{\Delta_2}{2} \right)^{2n_2+(d^{(2)}_{\alpha\beta})/2} \int_0^t D\{t_{2n_1}\} \int_0^t D\{u_{2n_2}\} \sum_{\{\zeta_1,\eta_1\}} \sum_{\{\zeta_2,\eta_2\}} \tilde{F}^{(1)}_{n_1}\{\{t_j\},\{\zeta_1\},\{\eta_1\}\} \tilde{F}^{(2)}_{n_2}\{\{u_{2k}\},\{\zeta_2\},\{\eta_2\}\} G_{n_1n_2}\{\{t_{1j}\},\{u_{2k}\},\{\zeta_1\},\{\eta_1\},\{\zeta_2\},\{\eta_2\}\}$$

(A1)

We first explain each term of this expression. The factors $d^{(i)}_{\alpha\beta}$ are different for each probability calculated and reflect the constraint on the last sojourn of each paths. If we are interested in calculating $P_{↑↑}(t)$, then at time $t$, the two paths must be in the state $|↑↑\rangle$ after having performed $2n_1$ and $2n_2$ jumps. We thus have the constraints $\eta_{1n_1} = \eta_{2n_2} = +1$ and $d^{(1)}_{\uparrow\downarrow} = d^{(2)}_{\downarrow\uparrow} = 0$. However, for $P_{↓↑}(t)$, the paths must end in the state $|↓↑\rangle$. This means that at least one blip must be present in the paths considered. They will thus be composed of $n_1 + 1$ and $n_2 + 1$ blips, both $n_1$ and $n_2$ ranging from zero to infinity, with the constraints $\eta_{1n_1+1} = \eta_{2n_2+1} = -1$ and $d^{(1)}_{\downarrow\uparrow} = d^{(2)}_{\uparrow\downarrow} = 2$. Finally, of course, the paths for $P_{↑↓}(t)$ and $P_{↑↑}(t)$ will be formed by a mix of these two cases. For $P_{↓↓}$, the constraints are $\eta_{1n_1} = +1$ and $\eta_{2n_2+1} = -1$ with $d^{(1)}_{\downarrow\downarrow} = 0$ and $d^{(2)}_{\uparrow\uparrow} = 2$, while for $P_{↑↑}$, the constraints are $\eta_{1n_1+1} = -1$ and $\eta_{2n_2} = +1$ and $d^{(1)}_{\uparrow\downarrow} = 2$, $d^{(2)}_{\downarrow\uparrow} = 0$.

The time integrations are defined as

$$\int_0^t D\{t_{2n}\} = \int_0^t dt_{2n} \int_0^t dt_{2n-1} \cdots \int_0^t dt_2 \int_0^{t_1} dt_1$$

(A2)

i.e., an integration over the times at which the jumps occur. Notice that the blips of a given path are well ordered, i.e., they occur successively; however, nothing specifies the order of the blips of one path with respect to those of the other, and so all the different different configurations must be included. For example, consider a path relevant to the calculation of $P_{↑↑}(t)$ with $n_1 = n_2 = 1$; there are four transition times $t_1$, $t_2$, $u_1$ and $u_2$, which can be arranged in 6 different configurations, as represented in Fig. 12, always with the restriction $t_1 < t_2 < t$ and $u_1 < u_2 < t$. In general, there will be

$$\left( \begin{array}{c} 2n_1 + 2n_2 \\ 2n_1 \end{array} \right) = \frac{(2n_1 + 2n_2)!}{2n_1!2n_2!}$$

(A3)

such configurations. This is simply a reflection of the fact that we cannot describe the two paths with respect to a single time where the all the jumps would be well ordered.

The terms $F^{(α)}_{n_α}\{\{t_j\},\{\zeta_α\},\{\eta_α\}\}$ are the single-spin functionals which can be decomposed as

$$F^{(α)}_{n_α}\{\{t_j\},\{\zeta_α\},\{\eta_α\}\} = F^{(α;\sigma\epsilon)}_{n_α} F^{(α;\beta-b)}_{n_α} F^{(α;\beta-s)}_{n_α}$$

(A4)

where

$$F^{(α;\sigma\epsilon)}_{n_α} = \exp \left[ -\frac{g_0^2}{\pi} \sum_{j=1}^{n_α} Q^{(α)}_2(t_{2j} - t_{2j-1}) \right]$$

(A5)

$$F^{(α;\beta-b)}_{n_α} = \exp \left[ -\frac{g_0^2}{\pi} \sum_{j>k}^{n_α} \zeta_α j \zeta_α k L^{(α)}_{jk} \right]$$

(A6)

$$F^{(α;\beta-s)}_{n_α} = \exp \left[ -\frac{g_0^2}{\pi} \sum_{j>k}^{n_α} \zeta_α j \eta_α k X^{(α)}_{jk} \right]$$

(A7)
in which the kernel matrices are

\[
\Lambda_{jk}^{(\alpha)} \equiv Q_2^{(\alpha)}(t_{2j} - t_{2k-1}) + Q_2^{(\alpha)}(t_{2j-1} - t_{2k}) - Q_2^{(\alpha)}(t_{2j} - t_{2k}) - Q_2^{(\alpha)}(t_{2j-1} - t_{2k-1}) \tag{A8}
\]

\[
X_{jk}^{(\alpha)} = Q_1^{(\alpha)}(t_{2j} - t_{2k+1}) + Q_1^{(\alpha)}(t_{2j-1} - t_{2k}) - Q_1^{(\alpha)}(t_{2j} - t_{2k}) - Q_1^{(\alpha)}(t_{2j-1} - t_{2k+1}) \tag{A9}
\]

and are defined in terms of the correlation functions

\[
Q_2^{(\alpha)}(t) = \int_0^{\infty} d\omega \frac{J_{\alpha\alpha}(\omega)}{\omega^2} \left[ 1 - \cos \omega t \right] \coth(\omega/2T). \tag{A10}
\]

\[
Q_1^{(\alpha)}(t) = \int_0^{\infty} d\omega \frac{J_{\alpha\alpha}(\omega)}{\omega^2} \sin \omega t \tag{A11}
\]

The correlation \( Q_1^{(\alpha)}(t) \) refers to the reactive (phase) term \( \Phi_\alpha(t) \) in (122), and \( Q_2^{(\alpha)}(t) \) refers to the dissipative term \( \Gamma_\alpha(t) \).

As discussed at great length by Leggett et al., (A1) can be analysed in many regimes using the “dilute blip” approximation, in which the interactions \( \Lambda_{jk}^{(\alpha)} \) are dropped entirely and only the term \( \tilde{F}_{n\alpha n\alpha}^{(\alpha)} \) and the term in \( X_{jk}^{(\alpha)} \) corresponding to the interaction of a blip with the sojourn immediately preceding it are kept. Thus the single-spin influence functionals \( \tilde{F}_{n\alpha n\alpha}^{(\alpha)} \) in (A1) can be evaluated without worrying about the precise ordering of the charges in the configuration.

Now let us turn to the interaction functional \( F_{12} \) in (123). One finds that \( F_{12} \) is a sum over all possible double paths integrals of the form

\[
\int_0^t D\{t_{2n_1}\} \int_0^t D\{u_{2n_2}\} \sum_{(\zeta_1,\eta_1)} \sum_{(\zeta_2,\eta_2)} \tilde{G}_{n_{1n_2}}(\{t_{1j}\}, \{u_{2j}\}, \{\zeta_1\}, \{\eta_1\}, \{\zeta_2\}, \{\eta_2\}) \tag{A12}
\]

with weightings \(( -i\Delta_1/2 )^{2n_1} ( -i\Delta_2/2 )^{2n_2} \) (here we have chosen a set of paths corresponding to the calculation of \( P_{12}(t) \); c.f., Eq (A1)). The integrand \( \tilde{G}_{n_{1n_2}} \) has the form

\[
\tilde{G}_{n_{1n_2}}(R) = G_{n_{1n_2}}^{(b-b)}(R) G_{n_{1n_2}}^{(b-s)}(R) \tag{A13}
\]

where

\[
G_{n_{1n_2}}^{(b-b)}(R) = \exp \left[ -\frac{q_0^1 q_0^2}{\pi} \sum_{j,k}^{n_{1n_2}} \zeta_{1j} \zeta_{2k} \Lambda_{jk}(R) \right] \tag{A14}
\]

in which the blip-blip interaction matrix is

\[
\Lambda_{jk}^{(12)}(R) = [Q_2^{(12)}(R, |t_{2j} - u_{2k-1}|) + Q_2^{(12)}(R, |t_{2j-1} - u_{2k}|) - Q_2^{(12)}(R, |t_{2j} - u_{2k-1}|) - Q_2^{(12)}(R, |t_{2j-1} - u_{2k}|)] \tag{A15}
\]

Notice the difference from (A8). The \( G_{n_{1n_2}}^{(b-s)}(R) \) term is

\[
G_{n_{1n_2}}^{(b-s)}(R) = \exp \left[ i \frac{q_0^1 q_0^2}{\pi} \sum_{j}^{n_{1}} \sum_{k}^{n_{2}} X_{jk}^{(12)}(R) (\zeta_{1j}\eta_{2k} + \eta_{1j}\zeta_{2k}) \right] \times \exp \left[ -iK_{zz}(R) \sum_{j=1}^{n_1} \zeta_{1j}\eta_{2}\eta_{2}(t_{2j} - t_{2j-1}) + \sum_{k=1}^{n_2} \zeta_{2k}\eta_{1}\eta_{1}(u_{2k} - u_{2k-1}) \right] \tag{A16}
\]

where

\[
X_{jk}^{(12)}(R) = Q_1^{(12)}(R, |t_{2j} - u_{2k-1}|) + Q_1^{(12)}(R, |t_{2j-1} - u_{2k}|) - Q_1^{(12)}(R, |t_{2j} - u_{2k-1}|) - Q_1^{(12)}(R, |t_{2j-1} - u_{2k}|) \tag{A17}
\]
We notice that there is no term in (A13) corresponding to the term $F$ where we deal with these interactions. Thus the PISCES problem is intrinsically more difficult than the single-spin problem, since we must in some blip approximation, so useful for the single spin-boson problem, we would be throwing away all spin-spin interaction terms. This point is crucial - if we try to make an approximation analogous to the non-interacting interaction functional corresponding to the the blip self-energy in the single-spin functional

$$Q^{(12)}_1(t) = \int_{0}^{\infty} d\omega \frac{J_{12}(\omega, \omega)}{\omega^2} \sin \omega t$$

(A18)

$$Q^{(12)}_2(t) = \int_{0}^{\infty} d\omega \frac{J_{12}(\omega, \omega)}{\omega^2} [1 - \cos \omega t] \coth(\omega/2T)$$

(A19)

We notice that there is no term in (A13) corresponding to the term $P^{(1)}_n$ in (A4), i.e., no “self-energy” term in the interaction functional corresponding to the the blip self-energy in the single-spin functional $F_j$; $G_{n_1, n_2}$ contains only spin-spin interaction terms. This point is crucial - if we try to make an approximation analogous to the non-interacting blip approximation, so useful for the single spin-boson problem, we would be throwing away all spin-spin interaction terms! Thus the PISCES problem is intrinsically more difficult than the single-spin problem, since we must in some way deal with these interactions.

Consider now the structure of the interaction influence functional. We need only consider three explicit configurations- the main characteristics of the analysis depend on whether or not the blips of a path overlap the sojourns of the other (well separated blips and sojourns give standard results). First, if the blip and sojourn are not overlapping, the interaction is identical to what was found before. An example of this case is the interaction of the first spin-2 blip (charge $\zeta_2$) with the first spin-1 sojourn (charge $\eta_1$) in Fig. 12-a The result is:

$$\exp i \eta_1 \zeta_2 \left[ Q^{(12)}_1(\mathbf{R}, u_1 - t_0) + Q^{(12)}_1(\mathbf{R}, u_2 - t_1) - Q^{(12)}_1(\mathbf{R}, u_2 - t_0) - Q^{(12)}_1(\mathbf{R}, u_1 - t_1) \right]$$

(A20)

However, if a blip occurs entirely within the time interval of a sojourn, complications arise. As an example, consider the interaction of the first spin-1 blip (charge $\zeta_1$) with the first spin-2 sojourn (charge $\eta_2$) in Fig. 12-b The time integration is:

$$\int_{t_1}^{t_2} \int_{t_0}^{\tau} ds \sin \omega(\tau - s) = \frac{1}{\omega}(t_2 - t_1) - \frac{1}{\omega^2} [\sin \omega(t_2 - t_0) - \sin \omega(t_1 - t_0)]$$

(A21)

which gives the result

$$\exp i \zeta_1 \eta_2 \left[ ( -Q^{(12)}_1(\mathbf{R}, t_1 - t_0) + Q^{(12)}_1(\mathbf{R}, t_2 - t_0) ) - (K_{zz}(\mathbf{R}) + \bar{\epsilon}(\mathbf{R}))(t_2 - t_1) \right]$$

(A22)

where $\bar{\epsilon}(\mathbf{R})$ is given by

$$\bar{\epsilon}(\mathbf{R}) \equiv \frac{q_0 q_{02}}{\pi} \int_{0}^{\infty} d\omega \frac{J_{12}(\omega, \omega)}{\omega}$$

(A23)

This function is discussed in the main text.

The 3rd case involves overlapping blips, and the total interaction appears as well. If the blips are overlapped as in Fig. 12-b it is convenient to consider together the $\zeta_{11}$ - $\eta_2$ and the $\zeta_21$ - $\eta_{11}$ interactions. The corresponding factor in the influence functional is then

$$\exp i \zeta_{11} \eta_{20} [ Q^{(12)}_1(\mathbf{R}, t_2 - u_1) + Q^{(12)}_1(\mathbf{R}, t_1 - t_0) - Q^{(12)}_1(\mathbf{R}, t_2 - t_0) ] \times \exp i \zeta_2 \eta_{11} Q^{(12)}_1(\mathbf{R}, u_2 - t_2)$$

(A24)

Finally, if a blip completely overlaps another blip, as in Fig. 12-c, the result of the interaction with the two neighbouring sojourns is

$$\exp i \zeta_{11} \eta_{20}[ Q^{(12)}_1(\mathbf{R}, t_1 - t_0) + Q^{(12)}_1(\mathbf{R}, t_2 - u_1) - Q^{(12)}_1(\mathbf{R}, t_2 - t_0) - i \zeta_2 \eta_{11} Q^{(12)}_1(\mathbf{R}, u_2 - t_2)$$

(A25)

All other cases in Fig. 12 are similar to these, provided one interchanges the times $t_j$ and $u_k$.

This summarises the construction of the influence functional for any configuration.
APPENDIX B: POOR MAN’S SCALING FOR THE OHMIC PISCES PROBLEM

We start from the reduced partition function of the Ohmic problem, expressed as:

\[
\frac{Z}{Z_{\text{env}}} = \int D[q_1]D[q_2] e^{-S_0[q_1]-S_0[q_2]+\mathcal{J}(\mathbf{R})} \int_0^{1/T} ds_1(q_1(s)q_2(s)) \exp \frac{1}{2\pi} \int_0^{1/T} ds \int_0^{1/T} ds' [\eta_1 \dot{q}_1(s)\dot{q}_1(s') + \eta_2 \dot{q}_2(s)\dot{q}_2(s') + \eta_12(\mathbf{R})(\dot{q}_1(s)\dot{q}_2(s') + \dot{q}_2(s)\dot{q}_1(s'))]
\]

(B1)

where \(Z_{\text{env}}\) is the partition function of the environment, \(S_0[q_\alpha]\) is the action of the \(\alpha\)-th two-level system, and the remaining terms represent their interaction with the oscillator bath. Notice that the total adiabatic interaction \(\mathcal{J}(\mathbf{R})q_1(s)q_2(s)\) is used, so that all interactions resulting from static configurations are explicitly separated from those coming from changes in the configurations (ie., transitions). Each term in the equation can easily be interpreted with respect to the single spin-boson problem.

The partition function is evaluated using the usual instanton method. The trajectories are defined by a series of tunneling jumps (the instantons), of duration smaller than \(\tau_c\), taking place at times \(t_j\) and connecting the different configurations of the system. We refer to the different configurations by the label \(z_i\), \(i = 1, 2, 3, 4\) such that \(z = \{\{\uparrow\uparrow\}, \{\uparrow\downarrow\}, \{\downarrow\downarrow\}, \{\downarrow\uparrow\}\}\). These are connected by the tunneling matrix elements \(\Delta(k, l) = \Delta(l, k)\); here \(\Delta(1, 1) = \Delta_1\) corresponds to transitions between \(z_1\) or \(z_2\) with \(z_3\) or \(z_4\), \(\Delta(2, 2) = \Delta_2\) connects \(z_1, z_4\) to \(z_2, z_3\) and we introduce \(\Delta(1, 3) = \Delta_F\) and \(\Delta(2, 4) = \Delta_{AF}\) connecting the ferro- and antiferromagnetic configurations \(z_1 - z_3\) and \(z_2 - z_4\) respectively. Even if no such transitions are present in the original problem, they will be generated through the renormalisation procedure. We also define the dimensionless fugacities \(y(k, l) = \tau_c\Delta(k, l)\). To each configuration is associated a static energy \(\epsilon_l\), and a corresponding dimensionless energy \(M_l \equiv \tau_c\epsilon_l\). For the PISCES problem considered here, \(M_1 = M_3 = \tau_c\Delta_F/2\) and \(M_2 = M_4 = -\tau_c\Delta_F/2\).

The boundary conditions on the trajectories are such that a path starting in a configuration \(z_j\) at \(t = 0\) must return to this configuration at the time \(t = 1/T\). Paths starting from the 4 different configurations must be considered. The time integrals in Eq. (B1) can now be done, and the partition function is

\[
\frac{Z}{Z_{\text{env}}} = \sum_{n=0}^{\infty} \tilde{Z}_n = \sum_n \sum_{z_i} \prod_{l=1}^n \left[ y(l, l-1) \int_0^{1/T} dt_l \Theta(t_l - t_{l-1} - \tau_c) \right] \exp \left[ -\sum_{l=0}^{n} \left( \frac{t_l - t_{l-1}}{\tau_c} \right) M_l + U(\{t_l\}) \right]
\]

(B2)

Just as for the dynamical behaviour of the PISCES problem, nothing specifies the ordering of the instantons and we must sum over all the possible set of configuration \(\{z_i\}\). The time integral accounts for all the possible positions of the instantons, and the \(\Theta\)-function insuring that none are within \(\tau_c\) of each other. The effect of the bath is contained in the interaction matrix \(U(\{t_l\})\), defined as

\[
U(\{t_l\}) = \sum_{l>j} \ln \left( \frac{t_l - t_{l-1}}{\tau_c} \right) [K_{l,j-1} + K_{l-1,j} - K_{l,j} - K_{l-1,j-1}]
\]

(B3)

with \(K_{l,j}\) representing the interaction of the \(l\)-th instanton with all those at earlier times. In terms of the parameters of the PISCES problem, these are \(K_{2,2} = \alpha_\beta\), \(K_{2,F} = \alpha_\beta + \alpha_{12}\), \(K_{\beta,AF} = \alpha_\beta - \alpha_{12}\), and, \(K_{F,F} = \alpha_1 + \alpha_2 + 2\alpha_{12}\), \(K_{AF,AF} = \alpha_1 + \alpha_2 - 2\alpha_{12}\), \(K_{F,AF} = \alpha_1 - \alpha_2\). This interaction is logarithmic, which is what allows the use of the poor man’s renormalisation group. If one suppresses the energy difference between configurations, this partition function is completely equivalent to the one treated by Cardy. In the present case where symmetry between the levels is broken, the poor man’s renormalisation procedure requires the inequalities \(y(k, l) \ll 1\), \(T \ll \Omega_0\), and \(\mathcal{J} \ll \Omega_0\). If we also assume that the 2 spins are identical, ie., that \(\Delta_F = \Delta_2\) and \(\alpha = \alpha_1 = \alpha_2\), then the partition function is equivalent to that derived by Chakravarty and Hirsch for the 2-impurity Anderson model, and we may simply take over their scaling equations, quoted in the text.

**APPENDIX C: DYNAMICS IN THE LOCKED PHASE**

When the indirect coupling is the largest energy scale, the two spins will tend to tunnel simultaneously. The time interval spent in a “unlocked” state (\(\{\uparrow\uparrow\}, \{\downarrow\downarrow\}\) if the coupling is ferromagnetic (or \(\{\uparrow\downarrow\}, \{\downarrow\uparrow\}\) if the coupling is antiferromagnetic) will be \(\sim 1/\mathcal{J}\). In terms of our path integral for the density matrix, this means that the dominant paths will be those where the blips are overlapping. This minimises the overlap between blips and sojourns, thus cancelling the fast varying factor exp \(i\mathcal{J}(t_j - u_k)\) in the influence functional which would otherwise give this
configuration a vanishing contribution to the path integral. Thus, we can restrict the summation to be only over the paths where the beginning (end) of one blip is within $J^{-1}$ of the beginning (end) of the second blip. Furthermore, the paths $q_1(\tau)$ and $q_2(\tau')$ are nearly identical to the paths $q_2(\tau)$ and $q_2(\tau')$ respectively, and this brings a restriction on the values that the charges of the blips and sojourns can take. In the ferromagnetic case, they must be equal. That is $\zeta_{1j} = \zeta_{2j} = \pm 1$ and $n_{1j} = \eta_{2j} = \pm 1$. However, in the antiferromagnetic case, they must be opposite ($\zeta_{1j} = -\zeta_{2j} = \pm 1$ and $n_{1j} = -\eta_{2j} = \pm 1$).

With these paths, the time integration over the end-points of the blips, Eq. (A11) is greatly simplified. For strong bias, such that $|J|$ is the shortest time scale of the problem, we can set $t_j - u_k = t_j - t_k$ provided that $j \neq k$. In the influence functional, we can also set $Q_2^{(12)}(t_j - u_k) = Q_2^{(12)}(0) = 0$ and $Q_1^{(12)}(t_j - u_k) = Q_1^{(12)}(0) = 0$ since they will be of order $1/|J|$. We clearly cannot do the same with the factor $e^{i\mathcal{J}(t_k - u_k)}$ but it is now easy to integrate over it. Always in the limit of strong coupling, the integration over the endpoints of the $n^{th}$ pair of overlapping blips with charge $\zeta_n$, placed between sojourns of charge $\eta_{n-1}$ and $\eta_n$ will be

$$\lim_{1/J \to 0} \frac{1}{2} \int_0^{1/J} du e^{i\zeta_n \mathcal{J}u} \int_0^{1/J} du' e^{i\eta_n \mathcal{J}u'} \sim 4 \frac{1}{\eta_n - i\zeta_n^2 \eta_n} .$$

(C1)

$u$ and $u'$ being short times over which the blips are not overlapping. The signs $\pm$ correspond to ferro- or antiferromagnetic coupling respectively and the factor 4 comes from the four possible configuration of two overlapping blips. In Eq. (C2), we have neglected a factor of $O(1)$ coming from the terms $e^{i\zeta_n \mathcal{J}u}$.

Taking into account the $(n+1)^{th}$ blip brings a equivalent factor, but with charges $\eta_n \zeta_n^2 + 1 \eta_{n+2}$ so that for a chain of $n$ overlapping blips, the total contribution from the time integration of the overlap is

$$\frac{(-1)^n 2^{2n}}{\mathcal{J}^2} \frac{1}{\eta_n \eta_n} .$$

(C2)

This shows that we now have a new effective tunneling matrix element equal to $\Delta_c = \Delta_1 \Delta_2/|\mathcal{J}|$ (compare Fig. 4): there, the problem is set up so that $J_0 = J/2$). Furthermore, within this approximation, the interaction of two overlapping blips, as given by Eq. (A15) becomes

$$\Lambda_{jk}(R) \rightarrow 2Q_2^{(12)}(t_{2k} - t_{2k-1}) .$$

(C3)

which is simply a contribution to the self-energy of each blips. Similarly, the interaction of the blip of one spin with the sojourn of the other spin becomes also simply adds to the contribution of the individual spins.

Therefore, the whole system behaves like a single spin-boson system to which is associated a tunneling matrix element $\Delta_c$ and coupled to the environment through a new coefficient $\alpha_c = \alpha_1 + \alpha_2 + 2\alpha_{12}$ for ferromagnetic coupling (since $\zeta_{1j} \zeta_{2j} = +1$) and $\alpha_c = \alpha_1 + \alpha_2 - 2\alpha_{12}$ for antiferromagnetic (in this case $\zeta_{1j} \zeta_{2j} = -1$).

Now the dissipative dynamics for this system is very well known, and all known results can be taken over directly. Assuming a ferromagnetic coupling, so that the system oscillates between the state $\{\uparrow\uparrow\}$ and $\{\downarrow\downarrow\}$, it is straightforward to use the dilute-blip approximation and to obtain Eq. (4.4), the Laplace transform of $P_\uparrow(t)$. The expression for $P_\downarrow(\lambda)$ will be equivalent, but with the + sign replaced by a − sign (coming from the constraint $\eta_0 = 0, \eta_n = -1$).

$$P_{\uparrow\downarrow}(\lambda) = \frac{1}{2\lambda} - \frac{1}{\lambda + f(\lambda)} .$$

(C4)

with $f(\lambda)$ given by Eq. (5.10).

Notice that if we had an antiferromagnetically coupled system, starting at $t = 0$ in the position $\{\uparrow\downarrow\}$ or $\{\downarrow\uparrow\}$, the structure of the summation is completely similar, one merely needs to replace $P_\uparrow$ and $P_\downarrow$ by $P_\downarrow$ and $P_\uparrow$. The case of antiferromagnetic coupling, but starting configuration $\{\uparrow\uparrow\}$ is discussed in the main body of the text.

APPENDIX D: DYNAMICS IN THE CORRELATED RELAXATION PHASE

As long as $T \gg \mathcal{J}$, the renormalisation group analysis shows that there is no great tendency of the 2 spins to lock together. In terms of the path integral, this means that the oscillating factors $e^{i\zeta_{1j} \eta_{3k} \mathcal{J}(t_j - u_k)}$ do not reduce the weight of paths with overlapping blips and sojourns as efficiently as in the locked phase. On the other hand, in the limit $\mathcal{J} \gg \Delta_\theta$, if the blip of one path overlaps with a blip of the other, a factor $\Delta_1 \Delta_2/\mathcal{J} \ll \Delta_\theta$ still appears in the path integral, which then tends to reduce drastically such overlaps. If furthermore, the blips on the individual paths are naturally dilute, then the occurrence of overlapping blips will be extremely rare and we can neglect them...
completely. The paths summation can then be done by using only paths where a blip of one path only overlaps in time with a sojourn of the other path. The two essential conditions of this approximation are thus $T \gg \mathcal{J}$, to insure that we are not in the locked phase, and $\mathcal{J} \gg \Delta_\beta$, so as to neglect paths with overlapping blips. In addition the blips of each individual spin must be in a dilute state.\footnote{We thank E. Mueller for pointing out this matrix method to us. The complete path summation was first done using a more complicated combinatorial method.} If $\alpha_\beta > 1$, this is so for all values of $\mathcal{J}$ and $T$. On the other hand, if $\alpha_\beta \ll 1$, then the blips are dilute provided that

$$
\frac{\Delta_\beta^2}{2\lambda} \sim \frac{\Delta_\beta^2}{(2\pi \alpha_\beta T)^2 + \mathcal{J}^2}
$$

(D1)

Notice that as long as $\mathcal{J} \gg \Delta_\beta$, the ratio $\alpha_\beta T/\Delta_\beta$ can be arbitrary, thus allowing the occurrence of the mutual coherence phase.

We can now proceed to explain in detail the path summations that lead to the expression given in section V-C. Each paths are formed of $n_1$ blips of the first path, and $n_2$ blips of the second path, such that no blips are overlapping. This approximation corresponds to the dilute-blip approximation for the single-spin influence functional and means that we keep only the direct interaction ($\mathcal{J} = K_{zz} + \epsilon$) in the interspin functional $\tilde{G}_{n_1 n_2}$. All the other terms vanish by the usual arguments of the dilute-blip approximation.

Blips of a given type are well ordered amongst themselves. What changes from one configuration to the other is the ordering between blips of different type. Therefore, the summation over the $\{\zeta_{ij}\}$ and $\{\zeta_{jk}\}$ in Eq. (A1) is independent of the configuration and can be performed immediately. It brings a factor

$$
2^{n_1} \prod_{j=1}^{n_1} \cos [\eta_{1j-1} Q_1^{(1)}(t_{2j} - t_{2j-1}) - \eta_{2j} \mathcal{J}(t_{2j} - t_{2j-1})]
$$

$$
\times 2^{n_2} \prod_{k=1}^{n_2} \cos [\eta_{2k-1} Q_1^{(2)}(u_{2k} - u_{2k-1}) - \eta_{1k} \mathcal{J}(u_{2k} - u_{2k-1})]
$$

(D2)

where $\eta_{1j}$ and $\eta_{2k}$ are the charges of the sojourns overlapping the blips $\zeta_{jk}$ and $\zeta_{ij}$ respectively.

With the charges of the blips removed from the problem, we will refer to the blips by the charge of the sojourns immediately preceding them. The fact that the blips are not overlapping then allows the use of the Laplace transform to perform the summation, just as in the single spin-boson problem.\footnote{We thank E. Mueller for pointing out this matrix method to us. The complete path summation was first done using a more complicated combinatorial method.} To keep track of the different indices involved in the summation, it is convenient to define a $2 \times 2$ matrix $g^{(\alpha)}(\lambda)$ with components $(g^{(\alpha)})^\sigma_\tau$ as

$$
g^{(\alpha)} = \begin{pmatrix}
(g^{(\alpha)})_+^+ & (g^{(\alpha)})_+^- \\
(g^{(\alpha)})^-_+ & (g^{(\alpha)})^-^-
\end{pmatrix}
$$

(D3)

with the definition

$$
(g^{(\alpha)}(\lambda))_\tau^\sigma = -\frac{\Delta_\beta^2}{2\lambda} \int_0^\infty dt e^{-\lambda t} e^{-Q_2^{(\alpha)}(t)} \cos [\tau \mathcal{J} t - \sigma Q_1^{(\alpha)}(t)]
$$

(D4)

The function $(g^{(\alpha)})^\sigma_\tau$ represents the contribution of a blip of type-$\alpha$. The index $\sigma$ refers to the charge of the sojourn on the path $\alpha$ immediately preceding the blip while the index $\tau$ refers to the charge of the sojourn overlapping the blip.

The contribution of the blips to a given path will now be expressed as a product of the components of $g^{(\alpha)}(\lambda)$. What remains to be done is the sum over the configurations and the set of the $\{\eta_{ij}\}$ and $\{\eta_{jk}\}$. This must be performed simultaneously since the summation over the $\eta$ gives a different result from one configuration to the other.

We will give a detailed explanation of the procedure to calculate $P_{\uparrow \uparrow}^{(\tau)}(t)$ and simply state what needs to be done differently in calculating the three other probabilities. The basic building blocks of the summation are the chains of coupled “clusters” of blips. The clusters are defined as the ensemble of successive blips of the same type $\alpha$, successive meaning that they are all overlapping the same sojourn of charge $\eta_{ij}$, $\eta_{jk}$. This corresponds to a single spin-boson system evolving in a static bias $\eta_{ij} \mathcal{J}$. All the different clusters are then linked together by the charge of the sojourn preceding the first blip of a given cluster. $P_{\uparrow \uparrow}^{(\tau)}(\lambda)$ then consists in a summation over all the possible chains of clusters,
each cluster containing all the possible number of successive blips. This is represented schematically on Fig. 13 and Fig. 14.

The complete summation over a cluster is easy since it is equivalent to the spin boson problem. The total contribution of a cluster of type \( \alpha \), whose preceding sojourn has a charge \( \sigma \) and overlapping a sojourn of charge \( \tau \) is then

\[
(g^{(\alpha)})_\tau^\sigma = \sum_{\{\sigma_j\}} \sum_{j=0}^{\infty} \left( \prod_{j=0}^{n} (g^{(\alpha)})_{\sigma_j}^{\sigma_j} \right) \frac{1}{1 - (g^{(\alpha)})^{\sigma_j} - (g^{(\alpha)})^{\sigma_j}}
\]

Notice that the expression \((\tilde{g}^{(\alpha)})_\tau^+ - (\tilde{g}^{(\alpha)})_\tau^- \) is nothing but the function \( P(\lambda) = P_1(\lambda) - P_\downarrow(\lambda) \) obtained for the single biased spin-boson system in a static bias \( \tau J \) within the dilute-blip approximation.

The summation of the chains of clusters is now straightforward. Let us refer to the contribution of a chain of clusters of type \( \alpha \) and ending with a cluster of type \( \beta \), with \( n \) clusters of type \( \alpha \) as \( C_{\alpha,\beta}^{n} \). The summation over the charges of the sojourn is included in this notation. The complete probability summation can be expressed as

\[
P_\uparrow\downarrow = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{n=1}^{\infty} C_{\alpha,\beta}^{(1-\alpha)} + \frac{1}{\lambda} \sum_{n=1}^{\infty} C_{\alpha,\beta}^{(2-\alpha)} + \frac{1}{\lambda} \sum_{n=1}^{\infty} C_{\alpha,\beta}^{(1-\alpha)} + \frac{1}{\lambda} \sum_{n=1}^{\infty} C_{\alpha,\beta}^{(2-\alpha)}
\]

where the factor \( 1/\lambda \) comes from the definition of the Laplace transformation. Let us consider in detail a particular contribution \( P^{\alpha-\beta} = \sum C_{\alpha,\beta}^{(n-\alpha)} \). It is composed of a sum over a product of \( 2n - 2 \) successive clusters (\( n \) \( \alpha \)-clusters and \( n - 2 \) \( \beta \)-clusters). No clusters of the same type are adjacent. Before the summation over the charges, the product is thus of the form \((\tilde{g}^{(\alpha)}\sigma_{1,\tau_1}(\tilde{g}^{(\beta)}\tau_{\tau_2})\sigma_{\tau_2-1}(\tilde{g}^{(\alpha)}\sigma_{\tau_2}...\tilde{g}^{(\alpha)}\sigma_{\tau_{n-1}+1}(\tilde{g}^{(\alpha)}\sigma_{\tau_{n-2}})...) \). However, since they are successive, there is the restriction \( \sigma_{\tau_{n+1}} = \tau \). It is then possible to write

\[
P^{\alpha-\beta} = \sum_{n=0}^{\infty} (\tilde{g}^{(\alpha)}\sigma_{\tau_{1,\tau_1}}(\tilde{g}^{(\beta)}\tau_{\tau_2})\sigma_{\tau_2-1}(\tilde{g}^{(\alpha)}\sigma_{\tau_2})...\tilde{g}^{(\alpha)}\sigma_{\tau_{n-1}+1}(\tilde{g}^{(\alpha)}\sigma_{\tau_{n-2}})...) \]

where for each \( n \), there is a restriction \( \sigma_{\tau_{n+1}} = 1 \). This is then nothing but a particular element of the multiplication of two matrices:

\[
P^{\alpha-\beta} = (\tilde{g}^{(\alpha)})_+ \left[ (1 - \tilde{g}^{(\alpha)}\tilde{g}^{(\beta)})^{-1} \right]_+ \]

The similar expression for a chain where the two limiting clusters are of the same type is

\[
P^{\alpha-\alpha} = (\tilde{g}^{(\alpha)})_+ \sum_{n=1}^{\infty} \prod_{j=1}^{n} (\tilde{g}^{(\beta)}\sigma_{\tau_{\tau_2-1}}(\tilde{g}^{(\alpha)}\sigma_{\tau_2})...\tilde{g}^{(\alpha)}\sigma_{\tau_{n-1}+1}(\tilde{g}^{(\alpha)}\sigma_{\tau_{n-2}})...) \]

\[
= (\tilde{g}^{(\alpha)})_+ \left[ (1 - \tilde{g}^{(\alpha)}\tilde{g}^{(\alpha)})^{-1} \right]_+ \]

The complete probability \( P_\uparrow\downarrow(\lambda) \) is thus given by

\[
P_\uparrow\downarrow(\lambda) = \frac{1}{\lambda} + \frac{1}{\lambda} (\tilde{g}^{(1)})_+ \left[ (1 - \tilde{g}^{(2)}\tilde{g}^{(1)}^{-1})^{-1} \right]_+ + \frac{1}{\lambda} (\tilde{g}^{(2)})_+ \left[ (1 - \tilde{g}^{(2)}\tilde{g}^{(1)}^{-1})^{-1} \right]_+ + \frac{1}{\lambda} (\tilde{g}^{(2)})_+ \left[ (1 - \tilde{g}^{(2)}\tilde{g}^{(1)}^{-1})^{-1} \right]_+ \]

\[
(D10)
\]

The other probabilities are then obtained in a straightforward manner, they simply correspond to taking different matrix elements in Eq. (D10). We obtain:

\[
P_{\downarrow\downarrow}(\lambda) = \frac{1}{\lambda} (\tilde{g}^{(1)})_+ \left[ \tilde{g}^{(2)}(1 - \tilde{g}^{(1)}\tilde{g}^{(2)})^{-1} \right]_+ + \frac{1}{\lambda} (\tilde{g}^{(2)})_+ \left[ \tilde{g}^{(1)}(1 - \tilde{g}^{(2)}\tilde{g}^{(1)})^{-1} \right]_+ + \frac{1}{\lambda} (\tilde{g}^{(2)})_+ \left[ \tilde{g}^{(1)}(1 - \tilde{g}^{(2)}\tilde{g}^{(1)})^{-1} \right]_+ + \frac{1}{\lambda} (\tilde{g}^{(2)})_+ \left[ \tilde{g}^{(1)}\tilde{g}^{(2)}(1 - \tilde{g}^{(1)}\tilde{g}^{(2)})^{-1} \right]_+ \]

\[
(D11)
\]

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Notice that the matrices now involve $\cdots$, since the final sojourns must have a charge of $-1$. There is also the factor $1/\lambda$ missing, since there must be at least one blip in each path, to go from the state $|\uparrow\uparrow\rangle$ to $|\downarrow\downarrow\rangle$.

Finally, for the last two remaining probabilities,

$$P_{\uparrow\downarrow}(\lambda) = \frac{1}{\lambda} (\hat{g}^{(1)})^+ \left[ g^{(2)} (1 - g^{(1)} g^{(2)})^{-1} \right]^+ + \frac{1}{\lambda} (\hat{g}^{(2)})^+ \left[ (1 - g^{(1)} g^{(2)})^{-1} \right]^- +$$

$$\frac{1}{\lambda} (\hat{g}^{(1)})^+ \left[ g^{(2)} g^{(1)} (1 - g^{(2)} g^{(1)})^{-1} \right]^+ + \frac{1}{\lambda} (\hat{g}^{(2)})^+ \left[ g^{(1)} (1 - g^{(2)} g^{(1)})^{-1} \right]^-$$

(D12)

with $P_{\downarrow\uparrow}(\lambda)$ simply related to $P_{\uparrow\downarrow}(\lambda)$ by the substitution $(1) \leftrightarrow (2)$.

These are obviously fairly complicated expressions, despite their compactness. However, the matrices $g^{(\alpha)}$ are $2 \times 2$ so that it is relatively easy to expand. Notice first that the denominator of $P_{\tau_1 \tau_2}(\lambda)$, which determines to poles in $\lambda$ and consequently the dynamics is given as

$$1 - Tr(\hat{g}^{(1)} \hat{g}^{(2)}) + Det(\hat{g}^{(1)} \hat{g}^{(2)})$$

(D13)

This denominator reduces to a product of polynomials of degree 2 in $\lambda$ and it is then possible to simplify $P_{\tau_1 \tau_2}(\lambda)$ further.

At this point, we can revert to the notation of Leggett et al. and use the functions $g_\beta(\lambda)$ and $h_\beta(\lambda)$ defined by Eq. (5.15) to simplify $P_{\uparrow\downarrow}(\lambda)$ to Eq. (5.16) of section V.

Similarly, we obtain for $P_{\downarrow\uparrow}(\lambda)$

$$P_{\downarrow\uparrow}(\lambda) = -\frac{1}{4} \left[ g_1 \left( 1 + \frac{h_1}{g_1} \right) + g_2 \left( 1 + \frac{h_2}{g_2} \right) \right] \frac{1}{\lambda} \frac{1}{\lambda + g_1 + g_2}$$

$$+ \frac{1}{4} \left[ g_1 \left( 1 + \frac{h_1}{g_1} \right) + g_2 \left( 1 + \frac{h_2}{g_2} \right) \right] \frac{1}{\lambda^2 + \lambda(g_1 + g_2) + g_1 g_2 (1 - \frac{h_1 h_2}{g_1 g_2})}$$

$$+ \frac{1}{4} \frac{g_1 g_2}{g_1 g_2} \left[ 1 - \frac{h_1 h_2}{g_1 g_2} \right] \frac{1}{\lambda} \frac{1}{\lambda^2 + \lambda(g_1 + g_2) + g_1 g_2 (1 - \frac{h_1 h_2}{g_1 g_2})}.$$ (D14)

While the last two remaining probabilities simplify to

$$P_{\uparrow\downarrow}(\lambda) = \frac{1}{4} \left[ g_1 \left( 1 + \frac{h_1}{g_1} \right) + g_2 \left( 1 + \frac{h_2}{g_2} \right) \right] \frac{1}{\lambda} \frac{1}{\lambda + g_1 + g_2}$$

$$- \frac{1}{4} \left[ g_1 \left( 1 + \frac{h_1}{g_1} \right) - g_2 \left( 1 + \frac{h_2}{g_2} \right) \right] \frac{1}{\lambda^2 + \lambda(g_1 + g_2) + g_1 g_2 (1 - \frac{h_1 h_2}{g_1 g_2})}.$$ (D15)

$$P_{\downarrow\uparrow}(\lambda) = \frac{1}{4} \left[ g_1 \left( 1 + \frac{h_1}{g_1} \right) + g_2 \left( 1 + \frac{h_2}{g_2} \right) \right] \frac{1}{\lambda} \frac{1}{\lambda + g_1 + g_2}$$

$$+ \frac{1}{4} \left[ g_1 \left( 1 + \frac{h_1}{g_1} \right) - g_2 \left( 1 + \frac{h_2}{g_2} \right) \right] \frac{1}{\lambda^2 + \lambda(g_1 + g_2) + g_1 g_2 (1 - \frac{h_1 h_2}{g_1 g_2})}.$$ (D16)

For identical systems, $P_{\uparrow\downarrow} = P_{\downarrow\uparrow}$, as should be expected, but the form of these probabilities also become quite simple, setting $g = g_1 = g_2$ and $h = h_1 = h_2$, the probabilities are

$$P_{\uparrow\downarrow} = P_{\downarrow\uparrow} = \frac{1}{2\lambda} (g + h) \frac{1}{\lambda + 2g}$$ (Identical systems) (D17)

**APPENDIX E: MUTUAL COHERENCE REGIME: IDENTICAL SYSTEMS**

In this appendix, we discuss the more tedious details pertaining to the mutual coherence case. We start with the denominators in $P_{\tau_1 \tau_2}(\lambda)$, which yields the oscillation frequencies and decay rates and then give the form of the coefficients $C_{ij}$ appearing in Eq. (5.36), Eq. (5.37) and Eq. (5.40). All the following expression are of course quite messy, although straightforward to obtain. Furthermore, they are obtained by working to lowest order in $a = \pi\alpha T$. 28
It is nevertheless important to get an idea of the value of relaxation rates and oscillation frequencies as well as the relative weight of the different terms in the overall probability function.

For two equivalent spins, the denominators of the Laplace transforms $P_{\tau_1 \tau_2}(\lambda)$, Eq. (5.17) and Eq. (5.18) are now

$$\lambda + 2g(\lambda)$$

(E1)

and

$$(\lambda + g(\lambda) + h(\lambda))(\lambda + g(\lambda) - h(\lambda))$$

(E2)

With the appropriate forms of $g(\lambda)$ and $h(\lambda)$ inserted, the equations for the poles of the Laplace transform are then

$$\lambda^3 + 4a\lambda^2 + \lambda(4a^2 + J^2 + \bar{\Delta}^2) + 2a\bar{\Delta}^2$$

(E3)

$$\lambda^3 + 4a\lambda^2 + \lambda(4a^2 + J^2 + \bar{\Delta}^2) + \bar{\Delta}^2(2a \pm J)$$

(E4)

The structure of these equations is completely similar to the equation obtained by Weiss and Wollensack for the single biased spin-boson system. The equations have two imaginary roots if $T > \Delta/\alpha$, which corresponds to oscillating terms in $P_{\tau_1 \tau_2}(t)$. The form of the decay rates and oscillation frequency can be expressed easily in terms of the relaxation rates and oscillation frequency of the single spin-boson problem, Eq. (5.33) and yield the values cited in the text.

We can now discuss the values of the coefficients $C_{ij}$. The coefficients $C_2$, $C_3$ and $C_4$ are related to the dynamics of the two systems together. $C_2$ is the coefficient of the pure relaxation term, with value

$$C_2 = \frac{\bar{\Delta}^2/2}{(\Gamma_R - \Gamma)^2 + \mu^2} \left(1 - \frac{8aA_+}{\Gamma_R}\right)$$

(E5)

In the limit $J \gg \bar{\Delta}$ this coefficient is of order $O(1)$, with limiting value

$$C_2 \sim A_- \quad (J \gg \bar{\Delta})$$

(E6)

which makes the connection with the overdamped relaxation of the correlated phase. The coefficients $C_3$ and $C_4$ corresponds to oscillating terms with frequency $\mu$. It simpler to write them directly as a sum of cosine and sine since it avoids the complication of introducing a phase in the oscillations. We obtain

$$C_3 = \frac{\bar{\Delta}^2 (\Gamma^2 - \pi^2 + 8aA_+ (\Gamma_R - 2\Gamma))}{2 ((\Gamma^2 + \pi^2)((\Gamma - \Gamma_R)^2 + \mu^2))}$$

(E7)

$$C_4 = \frac{\bar{\Delta}^2 8aA_+ (\Gamma (\Gamma_R - \Gamma) + \mu^2) - (\Gamma_R - \Gamma)(\Gamma^2 + \mu^2)}{2 ((\Gamma^2 + \pi^2)((\Gamma - \Gamma_R)^2 + \mu^2))}$$

(E8)

The main point about these coefficients is their smallness. In the limit $J \gg \bar{\Delta}$, they are

$$C_3 \sim \frac{\bar{\Delta}^2}{2J^2}$$

(E9)

$$C_4 \sim \frac{\bar{\Delta}^2(\pi \alpha T)}{J^3}$$

(E10)

We now get to the coefficients pertaining to the individual dynamics of the two spins. In this case, the Laplace transform of $P_{\tau_1 \tau_2}(t)$ includes terms with denominator of the form $(\lambda + g(\lambda))^2 - h^2(\lambda)$, giving rise to very complex coefficients. There is the apparition of two individual relaxation rate $\gamma_R \pm$, with a coefficient $C^\tau_{5\sigma}$, where the index $\tau = \pm 1$ refers to $P_{\tau \pm}(t)$ ($\tau = +1$) and $P_{\pm \sigma}(t)$ ($\tau = -1$). Its expression is

$$C^\tau_{5\sigma} = \frac{\bar{\Delta}^2}{2} \frac{\gamma_R - \gamma_{R\sigma} - 8aA_+}{(\gamma_{R\sigma} - \gamma_R)(\nu_\sigma^2 + (\gamma_{R\sigma} - \gamma_R)^2)(\nu_\sigma^2 + (\gamma_{R\sigma} - \gamma_R)^2)} \left(J^2 + (2a - \gamma_R)^2 + \tau \frac{\bar{\Delta}^2}{2\gamma_{R\sigma}}(8aA_- - \gamma_{R\sigma})\right)$$

(E11)

In the limit of interest to us, $J \gg \bar{\Delta}$, $C^\tau_{5\sigma}$ is given by
\[ C_{x\sigma} = \frac{1}{2} A_+ + \frac{1}{A_{-\sigma} - A_\sigma} [1 + \frac{\tau}{2} A_{-}] \]  

(E12)

Which is of order \( O(1) \), as is expected since we are close to the totally overdamped limit.

The global denominator Eq. (5.17) is given by

\[ \text{and by defining } \bar{\Delta} \text{ oscillations takes place against are very large "background".} \]

The different terms appearing in the numerators and denominators are given by,

\[ C_{6\sigma} = \frac{\Delta^2}{2\nu_\sigma |D|^2} \left( N_1^\sigma D_1^\sigma - D_2^\sigma N_1^\sigma + \tau \frac{\Delta^2}{\gamma_\sigma + \nu_\sigma} (N_1^\sigma D_1^\sigma - N_3^\sigma D_2^\sigma) \right) \]  

(E13)

\[ C_{7\sigma} = \frac{\Delta^2}{2\nu_\sigma |D|^2} \left( N_1^\sigma D_1^\sigma + D_2^\sigma N_2^\sigma + \tau \frac{\Delta^2}{\gamma_\sigma + \nu_\sigma} (N_3^\sigma D_1^\sigma + N_4^\sigma D_2^\sigma) \right) \]  

(E14)

The remaining terms are given by,

\[ N_1^\sigma = [\gamma_\sigma + 8aA_+] [(2a - \gamma_\sigma)^2 - \nu_\sigma^2 + J^2] - 2\nu_\sigma^2 (2a - \gamma_\sigma) \]

\[ \sim -8a\Delta^2 \sigma A_+ \]  

(E15)

\[ N_2^\sigma = 2\nu_\sigma (2a - \gamma_\sigma) [-\gamma_\sigma + 8aA_+] + \nu_\sigma [(2a - \gamma_\sigma)^2 - \nu_\sigma^2 + J^2] \]

\[ \sim -J \Delta^2 \sigma \]  

(E16)

\[ N_3^\sigma = -\gamma_\sigma [(2a - \gamma_\sigma)^2 - \nu_\sigma^2 - 4a^2 \tanh^2 (J/2T)] + 2\nu_\sigma^2 (2a - \gamma_\sigma) \]

\[ \sim -\gamma_\sigma J^2 \sim 8J^2 a A_+ \]  

(E17)

\[ N_4^\sigma = -\nu_\sigma [(2a - \gamma_\sigma)^2 - \nu_\sigma^2 - 4a^2 \tanh^2 (J/2T)] - 2\gamma_\sigma \nu_\sigma (2a - \gamma_\sigma) \]

\[ \sim J^3 \]  

(E18)

The dominant term in then \( N_{4\sigma} \), and the difference between \( N_{4+} \) and \( N_{4-} \) can be neglected at this level.

The remaining terms are given by

\[ D_1^\sigma = [(\gamma_\sigma - \gamma_{\sigma +})^2 + \nu_\sigma^2 - \nu_\sigma^2] [(\gamma_{R+} - \gamma_\sigma) (\gamma_{R-} - \gamma_\sigma) - \nu_\sigma^2] - 2\nu_\sigma^2 (\gamma_{\sigma +} - \gamma_\sigma) [\gamma_{R+} + \gamma_{R-} - 2\gamma_\sigma] \]

\[ \sim -4\Delta^2 J^2 (A_{-\sigma} - A_\sigma) \]  

(E19)

\[ D_2^\sigma = 2\nu_\sigma (\gamma_{\sigma -} - \gamma_\sigma) [(\gamma_{R+} - \gamma_\sigma) (\gamma_{R-} - \gamma_\sigma) - \nu_\sigma^2] + \nu_\sigma[\gamma_{R+} + \gamma_{R-} - 2\gamma_\sigma] [\gamma_\sigma - \gamma_{\sigma +} + \nu_\sigma^2 - \nu_\sigma^2] \]

\[ \sim -8a\Delta^2 \sigma (A_{-\sigma} - A_\sigma) \]  

(E20)

\[ |D|^2 = [(\gamma_{R+} - \gamma_\sigma)^2 + \nu_\sigma^2][(\gamma_{R-} - \gamma_\sigma)^2 + \nu_\sigma^2] [(\gamma_\sigma - \gamma_{\sigma +})^4 + 2(\nu_\sigma^2 + \nu_\sigma^2)(\gamma_\sigma - \gamma_{\sigma +})^2 + (\nu_\sigma^2 - \nu_\sigma^2)^2] \]

\[ \sim 16J^4 \Delta^4 (A_{-\sigma} - A_\sigma)^2 \]  

(E21)

Thus we see that both \( C_{6\sigma}^\sigma \) are quite small, so that in this case again, not only is the damping of the oscillations quite rapid compared to the general decay of the probabilities, but they have a very small amplitude so that the oscillations takes place against are very large “background”.

APPENDIX F: MUTUAL COHERENCE REGIME: OVERDAMPED PLUS UNDERDAMPED

The Laplace inversion in the Overdamped plus Underdamped case is very similar in spirit to the fully Underdamped case. The global denominator Eq. (5.37) is given by

\[ (\lambda + \Gamma_2)((2a_1 + \lambda)^2 + J^2) + \Delta^2_1 (2a_1 + \lambda) \]  

(F1)

and by defining \( \bar{a} \equiv a_1 - \Gamma_2/2 \), the roots for the variable \( \lambda + \Gamma_2 \) are exactly given by Eq. (5.35) with \( a \) replaced by \( \bar{a} \).
In the text, we have argued that we could separate the individual denominator, Eq. (5.18) into the individual products \((\lambda + g_2)(\lambda + g_1)\), i.e., we neglect the correlation between the two spins when calculating decay rates and oscillation frequency. It is then clear that we can directly use Eq. (5.35) for the underdamped spin.

The decay rates \(\Gamma_2 + \gamma_R\) and \(\Gamma_2 + \tilde{\gamma}\) are fully expressed as

\[
\Gamma_2 + \gamma_R = \gamma_{R,s} + \Gamma_2 \frac{J^2}{J^2 + \Delta^2} \sim \gamma_{R,s} + \Gamma_2
\]

and

\[
\Gamma_2 + \tilde{\gamma} = 2a - \frac{1}{2} \gamma_{R,s}(\bar{a}, J^2, \Delta^2) \sim 2a - (a - \Gamma_2/2) \frac{\Delta^2}{J^2}
\]

where the last forms are in the limit \(J \gg \Delta_1\). The coefficient \(C_2\) is given by

\[
C_2 = \frac{-1}{(\gamma_R + \Gamma_2)} \left[ (\gamma_R^2\frac{1}{\gamma_{R,s}}) + \nu^2 \right] \left[ \Gamma_2 A_+((2\bar{a} - \gamma_R)^2 + J^2) + \frac{\Delta^2}{4}(8a_1 A_+ - \gamma_R + \Gamma_2 J/2T) \right]
\]

which, in the limit \(J \gg \Delta_1\) reduces to

\[
C_2 \sim -\frac{\Gamma_2 A_+ \gamma_{R,s} + \Gamma_2}{J^2}
\]

which is of \(O(1)\) is \(\Gamma_2 \gg \gamma_{R,s}\) and negligible otherwise. The coefficients relating to the terms oscillating at a frequency \(\bar{\mu}\) are

\[
C_3 = \frac{1}{\bar{\mu}} \frac{1}{|D^2|} [N_1 D_2 + N_2 D_1]
\]

and

\[
C_4 = \frac{1}{\bar{\mu}} \frac{1}{|D^2|} [N_1 D_1 - N_2 D_2]
\]

given in a form similar to the coefficients in the underdamped case. The various factors are:

\[
N_1 = \Gamma_2 A_+[(2a - \gamma)^2 - \bar{\mu}^2 + J^2] + \frac{\Delta^2}{4}(8a_1 A_+ - \gamma_1 + \Gamma_2 J/2T) \sim \frac{1}{4} \Delta^2[(\bar{a}J/T - \Gamma_2]
\]

\[
N_2 = -2\Gamma_2 A_+\bar{\mu}(2a_1 - \gamma_1) + \frac{\Delta^2}{4}\nu_1 \sim \frac{1}{4} \Delta^2 \bar{J}
\]

The remaining terms are given by

\[
D_1 = (\bar{\gamma} - \Gamma_2)(\bar{\gamma}_R - \bar{\gamma}) + \bar{\mu}^2 \sim J^2
\]

\[
D_2 = \bar{\mu}(2\bar{\gamma} - \bar{\gamma}_R + \Gamma_2) \sim 2a\bar{J}
\]

\[
|D^2| = |(\bar{\gamma} + \Gamma_2)^2 + \bar{\mu}^2|[(\bar{\gamma}_R - \gamma_1)^2 + \bar{\mu}^2] \sim J^4
\]

The coefficients \(C_5 = C(\Gamma_2, \gamma_1, R)\) and \(C_6 = C(\gamma_1, R, \Gamma_2)\) where the function \(C(x, y)\) is defined as

\[
C(x, y) = \frac{1}{y - x} \left[ (\gamma_1^2 - \nu_1^2)^2 + \nu_1^2 \right] \left[ \Gamma_2 A_+((2a_1 - x)^2 + J^2) + \frac{\Delta^2}{4}(x + 8a_1 A_-) \right]
\]

The coefficients of the oscillating terms are defined similarly to the Equivalent Spin case

\[
C_7 = \frac{-1}{\nu_1} \frac{1}{\gamma_1^2 + \nu_1^2} \frac{1}{|D^2|} [N_3 D_2 + N_4 D_1] - \frac{1}{\nu_1} \frac{1}{|D^2|} [N_1 D_2 + N_2 D_1]
\]
With numerators and denominators given by

\[ |D^2| = [(\gamma_1, R - \gamma_1)^2 + \nu_2^2][(\Gamma_2 - \gamma_1)^2 + \nu_2^2] \] (F16)

\[ N_1 = \Gamma_2 A_+ [(2a_1 - \gamma_1)^2 - \nu_1^2 + J^2 + \frac{\Delta_1^2}{4}(8a_1 A_+ - \gamma_1)] \] (F17)

\[ N_2 = -2\Gamma_2 A_+ \nu_1 (2a_1 - \gamma_1) + \frac{\Delta_1^2}{4}\nu_1 \] (F18)

\[ N_3 = \frac{\Delta_1^2}{2}\Gamma_2 [\nu_1^2 - \gamma_1(2a_1 - \gamma_1)] \] (F19)

\[ N_4 = \Gamma_2 \Delta_1^2 \nu_1 a_1 \] (F20)
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Figure Captions

Fig1: The probabilities $P_{\tau_1\tau_2}(t)$ for a system of spins, coupled by $J_0$, to start in the state $|↑↑\rangle$ at time $t = 0$, and finish at time $t$ in state $|\tau_1\tau_2\rangle$. We assume $\Delta_2 = 2.5\Delta_1$, and a fairly strong coupling $|J_0| = 5\Delta_1$. $P_{\tau_1\tau_2}(t)$ is plotted as a function of $t/\Delta_1$. Oscillations between the 2 states $|↑↑\rangle$ and $|↓↓\rangle$ occur at the slow “beat” frequency $\Delta_1\Delta_2/2|J_0|$, with very weak high frequency oscillations superposed, coming from the weak mixing with the states $|↑↓\rangle$ and $|↓↑\rangle$. These high-frequency oscillations dominate $P_{\tau_1}(t)$ and $P_{\tau_2}(t)$, albeit with amplitude reduced by a factor $\sim (\Delta_1 + \Delta_2)/2J_0$.

Fig2: The same probabilities as in Fig. 2, but now for weak coupling; we have $\Delta_2 = 2.5\Delta_1$ again, but now $|J_0| = \Delta_1/10$.

Fig3: Diagrammatic interpretation of the equation of motion for the density matrix. We show equation 4.2 in Feynman diagram form; $J$ is the propagator (or Green function) for $\rho$.

Fig4: Contributions to the single spin influence functional. The label $\alpha = 1, 2$ labels one of the 2 spins, and $\gamma_\alpha \equiv \Gamma_\alpha - i\Phi_\alpha$, (see Eq. (4.15) and (4.16)) These contributions are exponentiated to give the influence functional Eq. (4.22).

Fig5: Contributions to the interaction functional $F_{12}$ (Eq. 4.23)). There are only interactions between the path associated with the spin 1 and that of the spin 2. Here, $\gamma_{12} \equiv \Gamma_{12} - i\Phi_{12}$, as given by Eq. (4.17) and (4.18).

Fig6: Example of a typical path for the 2-spin system. The top path is for the 1st spin, and shows transitions between eigenstates of $\tau_1^0$, at times $t_j$; the bottom path is for the 2nd spin, with transitions at times $u_k$.

Fig7: The various regimes in which the PISCES model dynamics are solved for analytically in the text. We assume in the figure that the 2 spins are equivalent, i.e., that $\Delta^* = \Delta_1^* = \Delta_2^*$, and $\alpha_1 = \alpha_2 = \alpha$. We also assume weak damping, so that $\alpha \ll 1$. If $\alpha$ is much larger, i.e., $\alpha \sim O(1)$, there is no mutual coherence phase, and motion in the locked phase (where the 2 spins rotate rigidly together) is overdamped at any $T$. We do not discuss the perturbative regime in this paper - in this regime correlations are very weak between the 2 spins. If the 2 spins are different (i.e., $\alpha_1 \neq \alpha_2$, and/or $\Delta_1^* \neq \Delta_2^*$), then one must draw 2 separates diagrams of this type, one for each spin - then it is possible for the 2 spins to be in different phases. The Mutual Coherence regime is strictly defined by the conditions $\Delta^*/\alpha \gg T \gg (J^2 + \Delta^*^2)^{1/2}$, but since we always assume $J \gg \Delta^*$ in this paper (i.e., well away from the perturbative regime), the inequality $T \gg (J^2 + \Delta^*^2)^{1/2}$ is equivalent to the condition $T \gg J$ used in the text.

Fig8: The various dynamical regimes which describe the motion of the PISCES system in the locked phase, as functions of $T$ and of the coupling $\alpha_c$ between the locked 2-spin complex and the sea of oscillators. The overdamped relaxation phase is divided into 2 parts by the “Toulouse line” $\alpha_c = 1/2$. On the “weak relaxation” side, the damping rate $\Gamma_c$ decreases with increasing $T$, and is small; on the “strong relaxation” side, $\Gamma_c$ is large and increases with increasing $T$. When $T = 0$, the motion is still damped for any finite $\alpha_c$ (with oscillations for $\alpha_c < 1/2$), but for $\alpha_c > 1$, the system is completely frozen by the coupling to the oscillators.

Fig9: Probabilities of occupation for a system in the "correlated relaxation" regime, with parameters $\alpha_1 = 1.5$, $\alpha_2 = 2$, at a temperature such that $T/\Delta_1 = 100$, $T/\Delta_2 = 300$ and a ferromagnetic coupling $J/2T = -0.02$; the time
is expressed in units of inverse temperature. The system starts at \( t = 0 \) in the state \(| \uparrow \uparrow \rangle \); \( P_{\alpha\beta} \) is then the probability that at time \( t \) the system has \( \tau^1_1 = \alpha \) and \( \tau^2_2 = \beta \). We plot the probabilities as a function of \( \ln(t) \) in order to clearly show the different relaxation times involved.

Fig10: Imaginary part of the Fourier transform of \( P_{\uparrow\uparrow}(t) \) in the mutual coherence case. The peaks represent the oscillation frequencies, and their width is proportional to their damping rates. We use the values \( \Delta/J = 0.6 \), \( J/T = 0.1 \), \( T = 100 \) (in arbitrary units) and \( a = \pi \alpha T = 0.25 \). The real frequency \( i\lambda \) is in the same units as the temperature.

Fig11: Imaginary part of the Fourier transform of \( P_{\uparrow\uparrow}(t) \) in the Overdamped plus Underdamped case. The parameters are \( \Delta_1/J = 0.6 \), \( \Delta_2/J = 0.1 \), \( a_1 = 0.25 \) and \( a_2 = 15 \) (where \( a_\beta = \pi \alpha_\beta T \)). Again \( J/T = 0.1 \), \( T = 100 \). Notice that we still have \( \alpha_2 \ll 1 \), although the spin 2 is overdamped. The strong broadening of the oscillations caused by the environment does not allow a clear separation of the two different oscillation frequencies.

Fig12: Paths for \( P_{\uparrow\uparrow} \) with \( n_1 = n_2 = 1 \) (cf. equation (A2)); there are 6 possible orderings of the transitions, which must be summed over. In Fig 12(a) we also show the states associated with the periods between the transitions.

Fig13: Schematic representation of the path integral for \( P_{\uparrow\uparrow} \). We sum over all possible chains of coupled clusters. The charges of the boundary sojourns are fixed at \( \eta = +1 \) by boundary conditions and there is a summation over the charges of all the internal sojourns.

Fig14: Representation of a cluster, used for the summation of \( P_{\tau_1\tau_2} \). In this graph, the charge of the lower path sojourn is fixed and constant. The charges of the boundary sojourns of the upper paths are fixed, and a summation is done over the charges of the internal sojourns. This summation then results in Eq. (D5).
FIG. 1.
FIG. 2.
\[ J_\rho \]

FIG. 3.
\[ \gamma_{\alpha}(\tau - s) \]

\[ \gamma^*(\tau - s) \]

\[ -\gamma_{\alpha}(\tau - s) \]

\[ -\gamma^*(\tau - s) \]
\[ \gamma_{12}(\tau - s) \]

\[ -\gamma_{12}(\tau - s) \]

FIG. 5.
Path 1

\[ t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t \]

Path 2

\[ t=0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \]

FIG. 6.
FIG. 7.
OVERDAMPED RELAXATION

Strong Relaxation

Weak Relaxation

UNDERDAMPED OSCILLATIONS

LOCALISED

FIG. 8.
FIG. 9.
FIG. 10.
FIG. 11.

\[ |\text{Im}(\lambda \mathcal{P}_f(\tau, \lambda))| \]

\[ \lambda \]

\[ \mathcal{P}_f(\tau, \lambda) \]
FIG. 12.
\[ P_\text{H} = \ldots + \ldots + \ldots + \ldots + \ldots + \ldots + \ldots + \ldots + \ldots + \ldots + \ldots + \ldots \]

FIG. 13.
FIG. 14.