From Type IIA Black Holes to T-dual Type IIB D-Instantons in $N=2, D=4$ Supergravity

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Abstract

We discuss the T-duality between the solutions of type IIA versus IIB superstrings compactified on Calabi-Yau threefolds. Within the context of the $N=2, D=4$ supergravity effective Lagrangian, the T-duality transformation is equivalently described by the c-map, which relates the special Kähler moduli space of the IIA $N=2$ vector multiplets to the quaternionic moduli space of the $N=2$ hyper multiplets on the type IIB side (and vice versa). Hence the T-duality, or c-map respectively, transforms the IIA black hole solutions, originating from even dimensional IIA branes, of the special Kähler effective action, into IIB D-instanton solutions of the IIB quaternionic $\sigma$-model action, where the D-instantons can be obtained by compactifying odd IIB D-branes on the internal Calabi-Yau space. We construct via this mapping a broad class of D-instanton solutions in four dimensions which are determined by a set of harmonic functions plus the underlying topological Calabi-Yau data.

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1 Introduction

Much of the recent progress in non-perturbative string and field theory is due to the insight that open strings with Dirichlet boundary conditions, which had been discovered already some time ago [1], can be used to describe type I and type II string theory in non-perturbative, solitonic or instanton backgrounds [2]. Specifically, the Dirichlet (D) $p$-branes, carrying the charges of the Ramond-Ramond (RR) gauge potentials, arise as the solutions of type IIA supergravity in ten dimensions for $p$ even, whereas the Dirichlet $p$-branes, with $p$ odd, appear as the solitons in the type IIB superstring. One of the most important results in this context, which is obtained by counting D-brane states, is the computation [3] of the statistical entropy of five- and four-dimensional extremal black holes in type II strings, which are obtained by wrapping certain configurations of higher dimensional IIA or M-theory branes around the internal cycles of the compactification space. This statistical entropy agrees with the macroscopic, Bekenstein-Hawking entropy which is proportional to the finite horizon of the extremal black hole solutions.

Recently stationary extremal solutions of four-dimensional $N = 2$ supergravity coupled to $N = 2$ vector multiplets were investigated in some detail [4]. Since the $N = 2$ vector couplings are determined by special Kähler geometry [5, 6], it follows naturally that the stationary $N = 2$ background fields, like the metric or moduli fields, can be expressed entirely by objects which appear in $N = 2$ special geometry, like the Kähler potential or the $U(1)$ Kähler connection; the corresponding solutions of the field equations are completely determined by a set of harmonic functions. Of particular interest are the $N = 2$ extremal black hole solutions. In the type IIA string compactifications on a Calabi-Yau space, $N = 2$ black holes can be obtained by wrapping three ten-dimensional 4-branes around the internal Calabi-Yau 4-cycles plus adding also a ten-dimensional 0-brane which lives in the internal space. The macroscopic Bekenstein Hawking entropy of the $N = 2$ black holes follows from the extremization of the $N = 2$ central charge of the underlying $N = 2$ super-algebra with respect to the moduli fields [7]. By means of this procedure it is possible to compute the classical entropy of static black holes [8] as well as to incorporate quantum corrections [9]. So the full entropy depends in the heterotic string framework on space-time instanton numbers, whereas in the type IIA compactifications on Calabi-Yau threefolds the entropy depends on the topological data of the Calabi-Yau space, like the intersection numbers, the Euler number and the rational world-sheet instanton numbers. In addition, it was shown in [10] that the contribution of the intersection numbers to the macroscopic entropy matches the microscopic entropy obtained by considering one internal IIA 0-brane plus three IIA 4-branes, wrapped around the internal Calabi-Yau 4-cycles.

An interesting class of non-perturbative solutions of type IIB supergravity, which break half of the supersymmetries, are given by the ten-dimensional D-instantons [11], i.e. $(-1)$-branes in Euclidean space. They are dual to the type IIB 7-branes, which have to be included in the F-theory interpretation [12] of the type IIB superstring. In addition, D-instantons were shown [13] to lead to non-perturbative corrections to some higher order curvature terms in the effective gravitational action of type IIB string compactifications.

T-duality between the type IIA and the type IIB string (on a circle) transforms the even and odd $p$-configurations into each other $[1, 14]$. Hence by T-duality the D-instanton of IIB is related to the 0-brane solution of IIA, i.e to a black hole. In this paper we will investigate the T-duality between the supersymmetric solitonic solutions of the IIA and IIB theories, compactified on the same Calabi-Yau threefold.
In particular, we will map the four-dimensional \( N = 2 \) IIA black holes on four-dimensional D-instanton solutions of the type IIB \( N = 2 \) effective actions. From the ten-dimensional p-brane point of view, the T-duality transforms the \((0, 4, 4, 4)\) IIA black hole configuration into the \((-1, 3, 3, 3)\) IIB D-instanton solution, whereas both brane configurations preserve 1/8 of the ten-dimensional supersymmetries, i.e. half of the four-dimensional \( N = 2 \) supersymmetries of the compactified type II strings.

The T-duality with respect to the time coordinate among the four-dimensional \( N = 2 \) IIA and IIB superstrings, compactified on the same Calabi-Yau space, is completely equivalent to the \( N = 2 \) c-map between these two theories, which was considered some time ago in [15, 16]. Namely, comparing the IIA and IIB spectra on a Calabi-Yau space with Hodge numbers \( h_{1,1} \) and \( h_{2,1} \), one realizes that the c-map (almost) exchanges the number of \( N = 2 \) vector and \( N = 2 \) hyper multiplets: \( N_V^{(A)} = h_{1,1} \leftrightarrow N_H^{(B)} - 1 = h_{1,1}, \quad N_H^{(A)} - 1 = h_{2,1} \leftrightarrow N_V^{(B)} = h_{2,1} \). More specifically, the \( N_V \) electric/magnetic Abelian gauge fields in the RR sector of the IIA superstring get transformed by T-duality into complex scalars in the RR sector of the type IIB theories, which are then members of \( N = 2 \) hyper multiplets. Moreover the universal IIA graviphoton vector field gets mapped into the universal RR scalar field which, together with the NS-NS dilaton-axion field, builds the universal IIB hyper multiplet. So the T-duality transformations relate the special Kähler moduli space \( K_{h_{1,1}}^{(A)} \) of complex dimension \( h_{1,1} \) of the IIA vector multiplets to the quaternionic moduli space \( Q_{h_{1,1}+1}^{(B)} \) of quaternionic dimension \( h_{1,1} + 1 \) of the hyper multiplets (and vice versa for the IIA hyper multiplets and the IIB vector multiplets). Hence, whereas the IIA black holes are the solutions of the IIA special Kähler Lagrangian, the T-dual IIB D-instantons arise as the solutions of the field equations of the IIB quaternionic non-linear \( \sigma \)-model. The aim of this paper is to explore this relation between the solutions of special Kähler IIA geometry and the solutions of the quaternionic geometry on the type IIB side. Applying the T-duality or c-map on known type IIA black hole solutions we will find a full new class of type IIB D-instanton type of solutions. In the simplest case, the Reissner-Nordstrom charged black-hole, which arises as the solution of the universal type IIA Lagrangian of the \( N = 2 \) graviphoton field, gets mapped to the D-instanton solution which solves the field equations of the quaternionic \( \sigma \)-model with the universal dilaton hyper multiplet field. Via T-duality the IIA black hole metric background gets transformed into a non-trivial background for the NS-NS dilaton-axion field on the type IIB side. More generally, we will consider the T-duality among solutions with more than one vector/hyper multiplet, starting for example with non-axionic black holes on the IIA side. Just as the IIA black holes, the IIB D-instanton solutions are then determined by a set of harmonic functions plus the topological data of the underlying Calabi-Yau space, like the intersection numbers.

Our paper is organized as follows. In the next section we will first review the construction of the D-instanton solutions of 10-dimensional type IIB superstring; we will consider the dimensional reduction of \((0, 4, 4, 4)\) IIA brane black hole brane configurations together with their dual \((-1, 3, 3, 3)\) IIB compactified D-instanton configurations. In section 3 we will discuss the effective \( N = 2 \) supergravity Lagrangian of \( N = 2 \) vector and hyper multiplets focussing on the c-map between the special Kähler geometry of the vector fields and the quaternionic geometry which is coming from the hyper multiplet fields. Though this material is known in the literature we have reviewed it with emphasis on the T-duality perspective to make the presentation self-contained; the behaviour of the universal sector under the c-map is investigated with particular care. In section 4, the main part of the paper, we will apply the T-duality, or c-map respectively, to relate the IIA black hole solutions to the IIB D-instanton solutions. Section 5 contains our conclusions.
2 Ten-dimensional black holes and D-instantons

2.1 D-branes and D-instantons in ten dimensions

The notion of a D-brane arises, when one considers open strings with Dirichlet boundary conditions along 10−p−1 directions. Then their endpoints are constrained to live on a (p+1)-dimensional submanifold, which is interpreted as the world volume of a p-dimensional extended non-perturbative object, called a D( Dirichlet)-p-brane, and which is thought of as a stringy soliton or instanton. Such D-p-branes are the carriers of R-R charges, which are predicted by string duality but absent in type I or type II perturbation theory. To be specific, the (p+1)-dimensional worldvolume can couple to (p+1)-form gauge potential, implying that they are charged under the corresponding (p+2)-form field strength. Since one can choose Neumann or Dirichlet boundary conditions along each space-time direction independently, one expects that D-branes can exist for any values −1 ≤ p ≤ 9 of p and should exist if the corresponding gauge field exists in the perturbative spectrum.

Within the low energy effective action, D-branes can be identified with BPS-saturated, R-R charged p-brane solutions. Such p-brane solutions can be characterized in terms of a function \( H \), which is harmonic with respect to the directions transversal to the world volume. To describe a single p-brane this function must be taken to depend only on the transversal radius \( r = \sqrt{x_t \cdot x_t} \), i.e. \( H = H(r) = c_1 + c_2 r^{-p} \). Then the geometry in the string frame is

\[
\begin{align*}
  ds_p^2 &= \frac{1}{\sqrt{H}} (-dt^2 + (dx^1)^2 + \cdots + (dx^p)^2) + \sqrt{H}((dx^{p+1})^2 + \cdots + (dx^9)^2) \\
  e^{-2\Phi} &= H^{\frac{p-1}{2}}. 
\end{align*}
\]

(2.1)

and the dilaton is

\[
(2.2)
\]

The gauge fields can likewise be expressed in terms of \( H \)

\[
F = \begin{cases} 
  d\frac{1}{H} \wedge dt \wedge dx^1 \wedge \cdots \wedge dx^p , & \text{for } p \leq 3 \\
  *d\frac{1}{H} \wedge dt \wedge dx^1 \wedge \cdots \wedge dx^p , & \text{for } p \geq 3
\end{cases}
\]

(2.3)

and the field strengths fall of as \( \frac{1}{r^{p+1}} \) for \( r \to \infty \), which is the characteristic power law for a charged p-brane in ten dimensions.

Since T-duality for open strings [1, 17, 14] mutually exchanges Neumann and Dirichlet boundary conditions, p-branes can be related to (p+1)-branes by dualizing over a transversal direction and to (p−1) branes by dualizing over a world volume direction. From (2.1) and (2.2) it is obvious that T-duality acts appropriately on the metric and the dilaton. The action on the gauge fields has been worked out in [14]. Note that T-dualization changes the number of transversal directions, on which \( H \) is allowed to depend. Thus, in order to go from a single localized p-brane to a single, localized (p±1)-brane we have to restrict or relax the functional dependence of \( H \), such that it depends on the new transversal radius. This is always understood when saying that a p-brane is T-dual to a (p±1)-brane. We would also like to recall that in the case of type II strings T-duality exchanges type IIA with type IIB and vice versa [1].

Let us now recall the massless perturbative spectra and the known D-p-brane solutions of ten-dimensional type II string theories. The massless perturbative NS-NS spectrum is the same for IIA and IIB. It consists of the metric \( G_{MN} \), the antisymmetric tensor \( B_{MN} \) and the dilaton \( \Phi \). The R-R spectrum of IIA contains
a 1-form $A_M$ and a 3-form $A_{MNP}$ whereas the massless R-R spectrum of IIB consists of a 0-form $A$, a 2-form $A_{MN}$ and a 4-form (with selfdual field strength) $A_{MNPQ}$. This implies that IIA (IIB) has only D-$p$-branes with even (odd) $p$. The type II D-branes with $0 \leq p \leq 6$ correspond to standard $p$-brane solitons of the IIA / IIB effective supergravity actions. For $p = 0, 1, 2$ they carry electric and for $p = 6, 5, 4$ they carry magnetic charge under the 1-, 2-, 3-form gauge potential, respectively, whereas the selfdual 3-brane couples to the selfdual 4-form. We refer to [18] for a review and more references.

The $(-1)$-brane and the 7-brane have special properties, that distinguish them from the other branes. The associated gauge field is the 0-form $A$, which has an axion like shift symmetry $A \rightarrow A + \text{constant}$. Therefore it only enters physics via its derivatives $F_M = \partial_M A$ which play the role of a field strength. A more conventional gauge theory description arises when dualizing $A$ into a 8-form $A^{(8)}$ with 9-form field strength $F^{(9)} = dA^{(8)}$ and gauge invariance $A^{(8)} \rightarrow A^{(8)} + d\Lambda^{(7)}$ where $\Lambda^{(7)}$ is an arbitrary 7-form. The $(-1)$-brane (7-brane) carries electric charge with respect to the 0-form (8-form) and magnetic charge with respect to the 8-form (0-form).

The D-7-brane solution [11] is the ten-dimensional analogue of the four-dimensional stringy cosmic string [19] and therefore it is not asymptotically flat. Moreover it plays an important role in the construction of F-theory [12]. On the other hand the $(-1)$-brane solution [11] is most naturally interpreted as an instanton, and has to be identified with the D-instanton of [20]. Thus one has to take time in (2.1) to be imaginary. We will consider this solution in more detail in the next subsection. Both the $(-1)$-brane and the 7-brane are completely non-singular in the string frame.

Finally we would like to mention that 8-branes and 9-branes, for which no gauge potential is present in the standard perturbative action have also been considered in the literature [2, 14].

### 2.2 The ten-dimensional D-instanton

Let us now recall the ten-dimensional D-instanton solution of IIB supergravity [11]. We will focus on the crucial role of boundary terms in the Euclidean domain. The discussion of other aspects of the solution is more sketchy. For more details we refer the reader to [11] and to the discussion of the four-dimensional D-instanton solution in section 4.1.

The action for the non-vanishing fields can be given either in terms of the R-R zero-form $A$

\[
S = \int d^{10}x \sqrt{G} \left\{ R - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} e^{2\Phi} (\partial A)^2 \right\},
\]

or in terms of the Hodge-$\star$-dual 9-form $F^{(9)} = e^{2\Phi} \star dA$

\[
S' = \int d^{10}x \sqrt{G} \left\{ R - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2 \cdot 9!} e^{-2\Phi} (F^{(9)})^2 \right\},
\]

where $R$ is the curvature scalar, $G$ the absolute value of the determinant of the metric and $\Phi$ is the dilaton.

The actions above refer to the Einstein frame with signature $(- + \cdots +)$. When looking for instanton solutions one has to Wick-rotate the theory to Euclidean signature. We take the Euclidean action to be positive. Then, within our conventions, it is obtained from the Minkowski action by replacing all fields by their Euclidean version and putting an overall minus sign. Note however that some subtleties arise, which are related to the fact that this Euclidean continuation does not commute with $\star$-dualization.

This has the following effect: Dualizing the Euclidean version of (2.5)

\[
S'_{\text{Euc}} = \int d^{10}x \sqrt{G} \left\{ -R + \frac{1}{2} (\partial \Phi)^2 + \frac{1}{2 \cdot 9!} e^{-2\Phi} (F^{(9)})^2 \right\},
\]

5
yields
\[ S_{Euc} = \int d^{10}x \sqrt{G} \left\{ -R + \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} e^{2\Phi} (\partial A)^2 \right\} + \int e^{2\Phi} A \wedge *dA, \]  
(2.7)
where the Euclidean 1-form field strength \( dA \) has been defined by \(*\)-dualization of the Euclidean 9-form \( F^{(9)} \). The Euclidean action \( (2.7) \) differs from a naive Wick rotation of \( (2.4) \) in two respects as pointed out in [11] and in the appendix of [21]: First the kinetic term of \( A \) has a flipped sign, which makes the sum of the scalar kinetic terms in the bulk part of action \( (2.7) \) indefinite. Second there is a boundary term, which is crucial for finding the correct instanton action. Note that the dual Euclidean 9-form action \( (2.6) \) is completely standard.

The D-instanton solution can be derived from either \( (2.6) \) or \( (2.7) \). In terms of the 0-form \( A \) it takes the following form [11]: First one requires that half of the Euclidean IIB supersymmetries remain unbroken in order to get a BPS saturated configuration. This can be achieved by relating \( A \) to the dilaton \( \Phi \) by
\[ dA = \pm e^{-\Phi} d\Phi. \]  
(2.8)
With this ansatz the equations of motion are solved by taking the Einstein metric to be flat and \( e^\Phi \) to be harmonic. Thus a single D-instanton is described by
\[ e^\Phi = e^{\Phi_\infty} + \frac{c}{r^8} = H(r). \]  
(2.9)
\( e^\Phi \) is singular at \( r = 0 \), whereas the metric in the string frame
\[ ds_{-\frac{2}{1}}^2 = e^{\Phi/2} (dt)^2 + (dx^1)^2 + \ldots + (dx^3)^2 = \sqrt{H} (dr^2 + r^2 d\Omega_{-9}^2) \]  
(2.10)
is asymptotically flat for \( r \to 0 \). In fact it describes a finite neck wormhole connecting two asymptotically flat regions at \( r \to 0, \infty \) [11].

Plugging this solution into \( (2.6) \) or \( (2.7) \) one obtains the finite instanton action
\[ S_{Inst} = -\int *d\Phi = \frac{|Q^{(-1)}|}{g}, \]  
(2.11)
where the integration is over asymptotic nine-spheres at \( r = 0 \) and \( r = \infty \), \( Q^{(-1)} \) is the electric charge of the solution with respect to \( A \), (i.e. the Noether charge associated to the 0-form gauge symmetry \( A \to A + \) constant) and \( g = e^{\Phi(\infty)} \) is the string coupling at \( \infty \). (By explicit computation one finds that the boundary at \( r = 0 \) does not contribute to the action.) The instanton action has a factor \( \frac{1}{g} \) in front, which is characteristic for its origin from the R-R sector. Note that when using action \( (2.7) \) the bulk part vanishes and the whole instanton action comes from the boundary term. This is the reason why the boundary term should be kept despite of not contributing to the equations of motion.

2.3 The \((0, 4, 4, 4)\) solution of IIA

By T-duality the D-instanton solution of IIB can be related to a (Euclideanized) 0-brane solution of IIA, i.e. to a black hole. When later studying D-instanton like solutions of \( D = 4, N = 2 \) supergravity and string theory we will make use of the fact that extremal black holes of these theories have been studied extensively in the last two years. Using T-duality on a general black hole (or even stationary) solution we can then generate a variety of instanton solutions. In order to get non-degenerate black holes and instantons in four dimensions it is not sufficient just to compactify ten dimensional 0-brane
and \((−1)\)-brane solutions, but one has to add further 4-branes and 3-branes. This will be recalled and explained in the next two subsections.

Let us consider how to construct four-dimensional black hole solutions out of ten-dimensional ones. This can be done starting from (2.1) for \(p = 0\) by toroidal compactification. Then \(H\) has to be a harmonic function with respect to the remaining three transversal spatial direction and thus behaves like \(\frac{1}{r}\) for \(r \to 0\). This implies that the asymptotic sphere at \(r = 0\) has vanishing area, and one obtains a black hole with degenerate horizon and vanishing Bekenstein-Hawking entropy. To get a finite horizon one needs to have more species of charges, leading to more harmonic functions in the solution. In four dimensions one needs precisely four charges to obtain a finite horizon.

When starting in ten dimensions such solutions can be constructed by considering a configuration of three 4-branes and one 0-brane with the following transversal intersection pattern [22]

\[
\begin{array}{cccccccc}
0 & \times & & & & & & \\
4 & \times & \times & \times & \times & \times & & \\
4 & \times & \times & \times & & & & \\
4 & \times & \times & \times & \times & \times & & \\
\end{array}
\]  
(2.12)

In this table world volume directions of the various branes are marked by a \(\times\). A BPS saturated configuration solving the IIA equations of motion and having this intersection pattern can be found and it depends on four functions \(H_0, \ldots, H^3\), which are harmonic with respect to the three overall transversal directions \(x = (x_1, x_2, x_3)\) (and independent of the others). The \(D = 10\) string frame metric is [22]

\[
ds^2 = \frac{-1}{\sqrt{H_0 H^1 H^2 H^3}} dt^2 + \sqrt{H_0 H^1 H^2 H^3} d\mathbf{x}^2 + \\
+ \sqrt{\frac{H_0 H^1}{H^2 H^3}} (dy_1^2 + dy_2^2) + \sqrt{\frac{H_0 H^2}{H^1 H^3}} (dy_3^2 + dy_4^2) + \sqrt{\frac{H_0 H^3}{H^1 H^2}} (dy_5^2 + dy_6^2)
\]  
(2.13)

and the ten-dimensional dilaton is

\[
e^{-2\Phi} = \sqrt{\frac{H^1 H^2 H^3}{H_0^3}}.
\]  
(2.14)

The gauge field strengths associated with the (electric) 0-brane and with the first of the (magnetic) 4-branes in (2.12) are

\[
F^{(2)} = \frac{1}{H_0} \wedge dt \quad \text{and} \quad F^{(4)} = *d \frac{1}{H^1} \wedge dy_3 \wedge dy_4 \wedge dy_5 \wedge dy_6 \wedge dt.
\]  
(2.15)

The expressions for the remaining 4-branes are obtained by obvious replacements of coordinates.

Compactifying to \(D = 4\) one obtains a black hole with metric

\[
ds_4^2 = \frac{-1}{\sqrt{H_0 H^1 H^2 H^3}} dt^2 + \sqrt{H_0 H^1 H^2 H^3} d\mathbf{x}^2
\]  
(2.16)

and constant four-dimensional dilaton \(\varphi\)

\[
e^{-2\varphi} = e^{-2\Phi} \sqrt{G(\text{int})} = 1.
\]  
(2.17)
Equation (2.17) implies that string and Einstein frame coincide. The black hole (2.16) is extremal, carries one electric charge \( q_0 \) and three magnetic charges \( p^1, p^2, p^3 \) and has a finite horizon. The geometry is similar to the extreme Reissner-Nordstrom black hole. The harmonic functions take the form

\[
H_0 = h_0 + \frac{q_0}{r}, \quad H^A = h^A + \frac{p^A}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}.
\] (2.18)

When taking all the harmonic functions to be equal, \( H_0 = H^1 = H^2 = H^3 \), then (2.16) precisely describes the (outer part of the) extreme Reissner Nordstrom geometry with the horizon located at \( r = 0 \). To see that the horizon is finite one looks at the behaviour of (2.16) for \( r \to 0 \), using that the asymptotic behaviour of the harmonic functions:

\[
\sqrt{H_0 H^1 H^2 H^3} \to \frac{\sqrt{q_0 p^1 p^2 p^3}}{r^2} (dr^2 + r^2 d\Omega_2^2)
\] (2.19)

This shows that one has to start with four charged objects in \( D = 10 \) in order to have a black hole with finite horizon in \( D = 4 \). The geometry of the six internal dimensions enters the four-dimensional effective action through the three moduli scalars

\[
T_1 = \sqrt{H_0 H^2 H^3}, \quad T_2 = \sqrt{H_0 H^1 H^3} \quad \text{and} \quad T_3 = \sqrt{H_0 H^1 H^2},
\]

which parametrize the size of three (pairwise orthogonal) internal 2-tori. For a generic choice of the four harmonic functions the moduli are space dependent, i.e. the geometry of the internal space varies when one moves around in four-dimensional space-time.

### 2.4 The \((-1, 3, 3, 3)\) solution of IIB

One can now rotate the \((0, 4, 4, 4)\) solution of IIA to Euclidean time and then T-dualize it over the time direction. The result is a configuration with one \((-1)\)-brane and three 3-branes with the following intersection pattern [22]:

|   | \( t \) | \( y_1 \) | \( y_2 \) | \( y_3 \) | \( y_4 \) | \( y_5 \) | \( x_1 \) | \( x_2 \) | \( x_3 \) |
|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| -1| \( 3 \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) |
| 3 | \( 3 \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) |
| 3 | \( 3 \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) |

The metric in the ten-dimensional string frame is

\[
d s^2 = \sqrt{H_0 H^1 H^2 H^3} dt^2 + \sqrt{H_0 H^1 H^2 H^3} d\mathbf{x}^2
\] (2.21)

\[
+ H_0 H^1 H^2 H^3 \left( d y_1^2 + d y_2^2 \right) + \sqrt{H_0 H^2 H^3} \left( d y_3^2 + d y_4^2 \right) + \sqrt{H_0 H^1 H^2} \left( d y_5^2 + d y_6^2 \right)
\]

and the dilaton is

\[
e^{-2\phi} = \frac{1}{H_0^2}.
\] (2.22)

Note that all world volume directions are spacelike so that the whole configuration is instanton-like. The 0-form gauge potential \( A \) is related to the dilaton by the instanton ansatz (2.8) which implies

\[
A = \mp H_0^{-1} + \text{constant}.
\] (2.23)
The selfdual 5-form field strength associated with the first 3-brane in (2.20) is
\[ F^{(5)} = d \frac{1}{H^1} \wedge d y_3 \wedge d y_4 \wedge d y_5 \wedge d y_6 + \star (d \frac{1}{H^1} \wedge d y_3 \wedge d y_4 \wedge d y_5 \wedge d y_6) \] (2.24)
and similar expressions hold for the other 3-branes.

When compactifying to four dimensions we have to introduce the four-dimensional dilaton \( \varphi \):
\[ e^{-2\varphi} = e^{-2\phi} \sqrt{G^{(int)}} = (H_0 H^1 H^2 H^3)^{-1/2}. \] (2.25)

The four-dimensional string frame and Einstein frame metrics are
\[ ds^2 = e^{2\varphi} ds^2_E = \sqrt{H_0 H^1 H^2 H^3} (dt^2 + (d\mathbf{x})^2). \] (2.26)

As we explained earlier the harmonic functions \( H_0, H^A \) should depend on the overall transversal radius in order to describe a localized D-instanton. For our four-dimensional case this means that they are functions of \( r = \sqrt{t^2 + \mathbf{x}^2} \):
\[ H_0 = h_0 + \frac{q_0}{r^2}, \quad H^A = h^A + \frac{p^A}{r^2}. \] (2.27)

The simplest solution is obtained by taking all four harmonic functions (and therefore all the charges) to be equal, implying that \( e^\varphi \) harmonic:
\[ e^\varphi = H_0 = H^1 = H^2 = H^3 = e^{\varphi(\infty)} + \frac{q_0}{r^2}, \quad A = \mp \frac{1}{e^{\varphi(\infty)} + \frac{q_0}{r^2}} + \text{constant}. \] (2.28)

This is the four-dimensional analogue of the ten-dimensional finite neck wormhole discussed in section 2.2. Note that one needs 4 harmonic functions in (2.26) in order to stabilize the neck of the wormhole. This is the same condition that one has for black holes in order to get a finite horizon. As in the black hole case the internal geometry enters the four-dimensional theory through the three non-trivial moduli fields \( T_1 = \sqrt{H_0 H^1 H^2 H^3}, \quad T_2 = \sqrt{H_0 H^2 H^3 H^4}, \quad \text{and} \quad T_3 = \sqrt{H_0 H^1 H^3 H^4}. \)

In the following we will generalize this simple setup in two ways. First, the black hole solutions obtained here by compactification from \( D = 10 \) are not generic \( D = 4 \) solutions. In four dimensions 2-form gauge fields are dual to 2-form gauge fields and therefore black hole solutions are in general dyonic. Moreover one can vary the values of various moduli scalar fields at infinity and deform the solution. This way one can for example turn on non–trivial \( \theta \)-angles, leading to axionic black holes which are not easily described from the ten-dimensional point of view. In this case it is appropriate to start with the general \( D = 4 \) action and solve the corresponding equations of motion. Secondly we have used toroidal compactification to go from ten to four dimensions. This procedure gives us solutions of \( N = 8 \) supergravity in ten dimensions. We would, however, like to have a smaller number of supersymmetries. In particular we are interested in \( N = 2 \), which corresponds to compactification on a Calabi-Yau threefold.

Both kinds of generalization make it necessary to recall certain facts about \( D = 4, N = 2 \) supergravity, which will be done in the next section before returning to explicit solutions in the following one.

### 3 \( N = 2, D = 4 \) type II string theory on Calabi-Yau manifolds

#### 3.1 Ungauged \( N = 2 \) supergravity: action, equations of motion and special geometry

The action of the effective \( N = 2 \) supergravity theory that we want to study in four dimensions will contain the following kinds of \( N = 2 \) multiplets: First there is the gravity multiplet
\[ (e^m_{\mu}, A_\mu, \psi^i_\mu) \quad m, \mu = 0, \ldots, 3, \quad i = 1, 2, \] (3.1)
which contains the graviton, the graviphoton (with field strength $T_{\mu\nu}$) and two gravitini. Second there are $N_V$ vector multiplets

$$(A^A_{\mu}, z^A_i, \lambda^A_i), \quad A = 1, \ldots, N_V, \ i = 1, 2, \quad (3.2)$$

which consist of a vector (with field strength $F^A_{\mu\nu}$), a complex scalar and two gauginos, which are Weyl spinors. Third there are $N_H$ hyper multiplets

$$(q^u, \xi_\alpha), \quad u = 1, \ldots, 4N_H, \quad \alpha = 1, \ldots, 2N_H \quad (3.3)$$

with 4 real scalars and two Weyl spinors.

The structure of $N = 2$ supergravity coupled to vector and hyper multiplets is governed by special geometry (for a review see [23, 24, 25]). This means that the action is completely specified in terms of three geometrical data:

1. The special Kähler manifold $K_{N_V}$ parametrized by the vector multiplet scalars $z^A_i$. It describes the coupling and self-interactions of $N_V$ vector multiplets to $N = 2$ supergravity [5, 6, 25].

2. The quaternionic manifold $Q_{N_H}$ parametrized by the hyper multiplet scalars $q^u$. It describes the coupling and self-interactions of $N_H$ hyper multiplets to $N = 2$ supergravity [26, 27, 28, 23].

3. The choice of the gauge group $G$.

For type II Calabi-Yau compactifications one gets at generic points in the moduli space what is called 'ungauged supergravity', i.e. all the vector multiplets are abelian and all hyper multiplets are neutral. The gauge group is $G = U(1)^{N_V+1}$, where the additional $U(1)$ is due to the graviphoton. One important issue of matter coupled $N = 2$ supergravity is, reflecting the points (1) and (2), that the moduli space is a product space

$$\mathcal{M} = K_{N_V} \otimes Q_{N_H}. \quad (3.4)$$

Vector and hyper multiplet can only couple through gravity and through gauge couplings. The later ones are absent in the ungauged case.

As already mentioned above $N = 2$ supergravity imposes further geometric constraints on the scalar manifold besides factorization, which define what is called special geometry. The vector multiplet moduli space must be a special Kähler manifold. This is a Kähler-Hodge manifold with the additional properties that (i) the Kähler potential can be obtained from a holomorphic section $(X^I(z), F_I(z))$ of a symplectic vector bundle over the manifold by $K(z, \overline{z}) = -\log(i\overline{X^I(z)}F_I(z) - X^I(z)\overline{F_I(z)})$ and (ii) that this section satisfies $X^I(z)\partial_A F_I(z) = F_I(z)\partial_A X^I(z)$. All physical quantities of the vector multiplet sector and in particular the couplings appearing in the action can be obtained from the section. In case that the matrix $(N_V + 1) \times (N_V + 1)$ matrix $M_{IJ}$, which is defined by $M_{IJ} = D_A X^I$ and $M_{0I} = X^I$ is invertible, one can obtain the section (and thus everything) from a prepotential $F(X^I)$ by $F_I = \partial F/\partial X^I$. (Here $D_A$ denotes the space-time and Kähler covariant derivative.) The prepotential is holomorphic and homogenous of degree 2 in $X^I$. One can always go to a section which comes from a prepotential by a symplectic transformation, and for the purposes of this paper such symplectic transformations will not change the physics of the model (see however: [29, 30]).

The hyper multiplet moduli space $Q_{N_H}$ is likewise subject to geometric restrictions. It was shown in [26] that the following conditions must hold: (i) the holonomy group must be contained in $SU(2) \otimes Sp(2N_H)$,
(ii) the $SU(2)$ part of the curvature must be non–trivial. Such manifolds are called quaternionic because they simultaneously admit three complex structures $J_i$, $i = 1, 2, 3$, such that the metric is hermitian with respect to all three of them and they satisfy the quaternionic algebra

$$J_i J_j = -\delta_{ij} \mathbf{1} + \epsilon_{ijk} J_k . \quad (3.5)$$

The associated hyper Kähler form, i.e. the $SU(2)$ triplet of Kähler forms corresponding to the three complex structures is proportional to the $SU(2)$ curvature. In the next section we will show in a simple example how the hyper Kähler form and the three complex structure can be obtained from the $SU(2)$ connection. It might be useful to remark that some authors, for example [24], prefer to define quaternionic manifolds in terms of the existence of three complex structures and their properties. Then the structure of the holonomy can be deduced from the definition. We have followed here reference [26] which defines a quaternionic manifold by the constraint on the holonomy group. Then the existence of three complex structures with certain properties is a consequence.

Before formulating the final condition we have to recall that quaternionic spaces are Einstein spaces, i.e. they have a constant curvature scalar $R$. (The case of one quaternionic dimension is somewhat special in that the holonomy constraint is trivial and one has to add this condition by hand [26].)Quaternionic manifolds describing hyper multiplet couplings to $N = 2$ supergravity have to obey an additional constraint: (iii) the curvature scalar must be negative and is fixed by the number of hyper multiplets (i.e. the dimension):

$$R[Q_{NH}] = -8 \left( N_H^2 + 2N_H \right). \quad (3.6)$$

We will in the following take this extra condition as part of the definition of quaternionic.

Let us now recall the bosonic part of the action of ungauged $N = 2$ supergravity with $N_V$ vector and $N_H$ hyper multiplets [24]

$$S = \int d^4 x \sqrt{G} \left( R - 2 g_{\mu \nu} \partial_\mu z^A \partial_\nu \bar{z}^A - \frac{1}{4} \left( \Im N_{IJ} F^I_{\mu \nu} F^J_{\mu \nu} + \Re N_{IJ} F^I_{\mu \nu} \ast F^J_{\mu \nu} \right) - \bar{g}_{uv} \partial_\mu q^u \partial_\nu q^v \right). \quad (3.7)$$

Here $G_{\mu \nu}$ is the space-time metric in the Einstein frame, $G = |\det G_{\mu \nu}|$ and $R$ the corresponding curvature scalar. The $N_V$ complex scalars $z^A$ coming from vector multiplets parametrize a special Kähler manifold $K_{N_V}$ with metric $g_{AB}(z, \bar{z})$, whereas the $4N_H$ real scalars $q^u$ parametrize the quaternionic manifold $Q_{NH}$ with metric $\bar{g}_{uv}(q)$. $F^I_{\mu \nu}$ are the field strength of the $N_V + 1$ vectors in a symplectic basis, $\ast F^I_{\mu \nu}$ the (Hodge-) dual field strength, $N_{IJ}(z, \bar{z})$ is the gauge kinetic matrix. Its imaginary part $\Im N_{IJ}$ is positive definite and generalized the gauge couplings $\frac{1}{\sqrt{g_{AB}}}$ of ordinary Yang-Mills theory, whereas the real part $\Re N_{IJ}$ is not restricted and acts as a generalized (field dependent) $\Theta$ angle.

The symplectic field strength $F^I_{\mu \nu}$ are related to the field strength $T_{\mu \nu}, F^A_{\mu \nu}$ of the graviphoton and of the $N_V$ vectors sitting in vector multiplets as follows: The (antiselfdual part of the) graviphoton field strength is given in terms of the section $(X^I, F_I)$ by the symplectic invariant combination $T_{\mu \nu} = F_I F^{-I}_{\mu \nu} - X^I G^-_{\mu \nu}$, where $G^-_{\mu \nu} = \overline{N}_{IJ} F^{-J}_{\mu \nu}$. On the other hand the $N_V$ field strength of vector multiplets are $F^A_{\mu \nu} = g_{AB} (D^B_F F^A_{\mu \nu} - F^B_F \overline{X^T} G^-_{\mu \nu})$. The vector multiplet couplings $g_{AB}$ and $N_{IJ}$ can be obtained from the holomorphic section $(X^I(z), F_I(z))$ this way: The Kähler metric is derived from the Kähler potential by $g_{AB} = \partial_A \partial_B K$ with $K = -\log((i(X^T(z) F_I(z) - X^I(z) F_I(z)))$, whereas the gauge kinetic matrix $N_{IJ}$ can be computed as follows: First one defines $f^I_A = D_A X^I$ and $h_{AI} = D_A F_I$, where $D_A$ is the Kähler covariant derivative (see the references quoted above for the details). Then one
defines two \((N_V + 1) \times (N_V + 1)\) matrices \(f = (f_I^J), h = (h_{IJ})\) by introducing \(f_{I0} = \mathcal{F}_I^J\) and \(h_{0I} = \mathcal{F}_I^J\). Finally the gauge kinetic matrix is given by \(\mathcal{N}_{IJ} = \mathcal{F}_{IK}^J J^{-1}_{KJ}\). In case that we choose the section such that a prepotential exists, \(\mathcal{N}_{IJ}\) can be computed by the more familiar formula

\[
\mathcal{N}_{IJ} = \frac{1}{2} \frac{\partial^2 \mathcal{F}_{IK}^J}{\partial X^I \partial X^J} - \frac{1}{8} G_{\mu\nu} \left( \partial_{\mu} \mathcal{N}_{IJ} F^I_{\mu\rho} F^J|\rho|\sigma + \partial_{\nu} \mathcal{N}_{IJ} F^I_{\nu\rho} F^J|\rho|\sigma \right)
\]

where \(F^I_{\mu\nu} = \partial_{\mu} \mathcal{F}_{IK}^J F^J|\nu|\). In this way we will next list the equations of motion for the case of vanishing fermions. The gravitational one is:

\[
R_{\mu\nu} = 2 g_{\mu\nu} \left( \partial_{\mu} \mathcal{F}_{IK}^J F^J|\nu| + \frac{1}{2} \partial_{\nu} \mathcal{N}_{IJ} F^I_{\mu\rho} F^J|\rho| + \partial_{\nu} \mathcal{N}_{IJ} F^I_{\mu\rho} F^J|\rho| \right)
\]

\[
- \frac{1}{8} G_{\mu\nu} \left( \partial_{\mu} \mathcal{N}_{IJ} F^I_{\nu\rho} F^J|\rho| + \partial_{\nu} \mathcal{N}_{IJ} F^I_{\mu\rho} F^J|\rho| \right)
\]

The equation for the scalars \(q^u\),

\[
\frac{2}{\sqrt{G}} \partial_{\mu} \left( \sqrt{G} \tilde{g}_{uv} \partial^\mu q^u \right) - \partial_{w} \tilde{g}_{uv} \partial_{\mu} q^u \partial^\mu q^v = 0
\]

(3.10)

can be rewritten using the definition of Christoffel symbols in a suggestive form:

\[
\square_G q^w + \Gamma^w_{uv} q^u \partial^\mu q^v = 0
\]

(3.11)

This looks like a generalized geodesic equation, in which 4-dimensional curved space-time is mapped by the scalar fields \(q^u\) into moduli space. (Note that \(\square_G\) is the Laplace operator in space-time, whereas \(\Gamma^w_{uv}\) are the Christoffel symbols in moduli space.)

The equations for the scalars \(z^A\) can be brought to a similar form,

\[
\square_G z^D + \Gamma^D_{AB} \partial_{\mu} z^A \partial^\mu z^B = \frac{1}{8} g^{DE} \left( \partial_E \mathcal{N}_{IJ} F^I_{\mu\nu} F^J|\mu\nu| + \partial_{E} \mathcal{N}_{IJ} F^I_{\mu\nu} F^J|\mu\nu| \right)
\]

(3.12)

where this time the equation is inhomogenous by contributions of the gauge fields. This inhomogenity could be interpreted as a potential that modifies the geodesic motion of the \(z^A\). Note that the r.h.s. is not only absent if there are no gauge fields, \(F^I_{\mu\nu} = 0\), but also for holomorphic \(\mathcal{N}\), \(\partial_E \mathcal{N} = 0\). The gauge kinetic matrix \(\mathcal{N}\) is holomorphic in \(z^A\) if and only if the prepotential is quadratic. This is the case of minimal coupling which does not occur in type II compactifications [15].

Finally we have the equations of motions for the gauge fields,

\[
\nabla_\mu (\Im \mathcal{N}_{IJ} F^J|\mu\nu| + \Re \mathcal{N}_{IJ} * F^J|\mu\nu|) = 0
\]

(3.13)

which have to be supplemented by the Bianchi identities,

\[
\nabla_\mu * F^I|\mu\nu| = 0
\]

(3.14)

### 3.2 Type II compactifications, T-duality and the c-map

In the last section we dealt with generic properties of ungauged \(N = 2\) supergravity. Let us now consider the specific case of compactifications of IIA and IIB superstrings on Calabi-Yau threefolds. (In fact all this can be formulated for general type II compactifications with general internal conformal field theories that are only constrained to give \(N = 2\) space time supersymmetry in four dimensions.) Let us recall
the spectra of these theories and their relation to Calabi-Yau geometry (see for example [31]). Since we know that we have \( N = 2 \) supersymmetry in \( D = 4 \) we only need to consider the bosonic spectrum.

Consider first the IIA theory compactified on a Calabi-Yau threefold with Hodge numbers \( h_{1,1} \) and \( h_{2,1} \), which count the numbers of independent deformations of the \( \tilde{K} \)ähler and complex structure, respectively. We split the indices into real space-time indices \( \mu, \nu = 0, \ldots , 3 \) and complex internal indices \( i, j, \ldots = 1, 2, 3 \). The resulting \( D = 4 \) bosonic spectrum is as follows: From the ten-dimensional metric \( G_{MN} \) and torsion \( B_{MN} \) we get the four-dimensional graviton \( G_{\mu \nu} \), the four-dimensional axion \( \phi \sim B_{\mu \nu} \ h_{2,1} \) complex scalars \( G^K_{ij} \), \( K = 1, \ldots , h_{2,1} \) describing deformations of the complex structure of the internal manifold and \( h_{1,1} \) complex scalars \( z^A \sim (G^A_{ij} + iB^A_{ij}) \), \( A = 1, \ldots , h_{1,1} \), related to deformations of the complexified \( \tilde{K} \)ähler structure of the internal manifold. In addition one gets the four-dimensional dilaton \( \varphi \) from the ten-dimensional one. Next we look at the R-R sector: The 1-form \( A_M \) gives a vector \( A_\mu \) and from the 3-form \( A_{MNP} \) we get \( h_{1,1} \) vectors \( A^A_{\mu ij} \) together with \( h_{2,1} + 1 \) complex scalars \( A^K_{ij} \) and \( A_{ijk} \). These fields combine with fermions into the \( \tilde{N} \)-vector multiplet, \( N_V^{(A)} = h_{1,1} \) vector multiplets and \( N^H_{(A)} = h_{2,1} + 1 \) hyper multiplets. There is one special hyper multiplet that does not contain any moduli of the Calabi-Yau but the four-dimensional coupling (dilaton) \( \varphi \) together with the axion \( \phi \) and two real R-R scalars \( \zeta^0, \zeta^0_0 \) corresponding to the complex scalar \( A_{ijk} \). It is called the universal hyper multiplet.

Let us now consider IIB supergravity on the same Calabi-Yau threefold. In \( D = 10 \) we have the same NS-NS fields as for IIA. In the R-R sector we get the following: The 0-form \( A \) gives a scalar \( \zeta^0 \). From the 2-form \( A_{MN} \) we get \( h_{1,1} + 1 \) scalars \( \zeta^A \sim A^A_{ij} \) and \( \zeta^0_0 \sim A^\mu_\nu \) and from the selfdual 4-form \( A_{MNPQ} \) we get \( h_{2,1} + 1 \) vectors \( A^K_{ij} \), \( A^K_{ijk} \) and \( h_{1,1} \) scalars \( \zeta_A \sim A^\mu_\nu_{ij} \). These fields combine with fermions into the gravity multiplet, \( N_V^{(B)} = h_{2,1} \) vector multiplets and \( N^H_{(B)} = h_{1,1} + 1 \) hyper multiplets. The singled out universal hyper multiplet this time contains besides the dilaton \( \varphi \) and the axion \( \phi \) and the R-R scalars \( \zeta^0, \zeta^0_0 \).

Comparing the two spectra one realizes that the numbers of vector multiplets and hyper multiplets are (almost) exchanged:

\[
N_V^{(A)} = h_{1,1} = N^H_{(B)} - 1, \quad N^H_{(A)} = h_{2,1} = N_V^{(B)}.
\]

Since the ten-dimensional IIA and IIB theory are related by T-duality and since we compactified both IIA and IIB on the same space one might expect that this can be explained by four-dimensional T-duality. It has been shown some time ago that this is indeed the case [32, 15, 16]. As a consequence, there exists a mapping, called the c-map which relates the two moduli spaces:

\[
c : K_{h_{1,1}}^{(A)} \times Q_{h_{2,1} + 1}^{(A)} \longleftrightarrow Q_{h_{1,1} + 1}^{(B)} \times K_{h_{2,1}}^{(B)}
\]

(3.16)

Let us sketch briefly how the T-duality transformation is performed and how the c-map is obtained. We will follow [16]. For details the reader can consult either this paper or go through the almost identical procedure used in section 4.3 for T-dualizing a stationary solution into an instanton.

Consider first the gravity and vector multiplet sector of [3.7]

\[
S_{\text{IIA}}^{G/V} = \int d^4x \sqrt{|G_{(10)}|} \left\{ R - 2g_{|\mu|\nu}\partial_\mu z^A \partial_\nu z^A - \frac{1}{4} \left( 3N_{IJ}F^I_{|\mu|\nu}F^J_{|\nu|\mu} + 3N_{IJ}F^I_{|\mu|\nu}F^J_{|\nu|\mu} \right) \right\}
\]

(3.17)

For definiteness we have called this theory the IIA theory and we will call its dual the IIB theory. Note however that there is no fundamental distinction between IIA and IIB in \( D = 4 \) since by mirror symmetry IIA on a Calabi–Yau is the same theory as IIB on the mirror manifold.
In $D = 4$ we start with the metric $G_{\mu\nu}^{IIB}$. $N_V + 1$ physical vectors $A_{\mu}^I$ (including the graviphoton) and $N_V$ physical complex scalars $z^A$. Now compactify this action to $D = 3$ along an isometry direction and decompose the four-dimensional fields into three-dimensional ones. From the metric we get a Kaluza-Klein scalar $\phi$ and a Kaluza-Klein vector $\omega_m$ $m = 1, 2, 3$. (The remaining three-dimensional metric is non–dynamical, because there are no gravitons in three dimensions.) Likewise every four-dimensional vector gives a scalar $\zeta^I \sim A_{\mu}^I$ and a vector $A_{\mu}^I$. And the scalars $z^A$ remain of course scalars. Now in three dimensions a vector can be dualized into a scalar. Replacing the Kaluza Klein vector $\omega_m$ and the vectors $A_{\mu}^I$ by scalars $\phi$, $\zeta_I$ we end up with a total of $4(N_V + 1)$ real scalars, which can be identified with the $4(N_V + 1)$ scalars that one gets by compactifying the hyper multiplet scalars of the dual IIB theory on a circle of inverse radius. After decompactification one gets the hyper multiplet part of the dual action

$$S_{IIB}^H = \int d^4x \sqrt{\text{det}(G^{IIB})} \left\{ -2g_{AB} \partial_\mu z^A \partial^\mu \tau^B - \frac{1}{2\phi^2} (\partial \phi)^2 - \frac{1}{2\phi^2} (\partial \phi + \zeta^I \partial \zeta_I - \partial \zeta^I \zeta_I)^2 \right\}$$

(3.18)

where $\mathcal{N}^{IJ}$ is the inverse of $\mathcal{N}_{IJ}$. As shown in [16] this $\sigma$-model is indeed quaternionic. We will come back to this point later.

Thus one can map every special Kähler manifold $K_{N_V^{(A)}}$ of complex dimension $N_V^{(A)}$ to a quaternionic manifold $Q_{N_V^{(A)} + 1}$ of quaternionic dimension $N_V^{(A)} + 1 = N_H^{(B)}$:

$$s_{N_V^{(A)}} : \quad K_{N_V^{(A)}} \rightarrow Q_{N_V^{(A)} + 1}$$

(3.19)

This is called the s-map. Note that one quaternionic dimension comes from the 4 bosonic degrees of freedom of the gravity multiplet. Quaternionic manifolds that can be obtained this way are called special quaternionic manifolds.

On the other hand T-duality requires that every hyper multiplet moduli space arising from a type II compactification must be special quaternionic, i.e. the corresponding scalar $\sigma$-model can be written as in (3.18). Reversing the above procedure one can map every such quaternionic manifold of quaternionic dimension $N_H^{(A)}$ to a corresponding special Kähler manifold of complex dimension $N_H^{(A)} - 1$:

$$s^{-1}_{N_H^{(A)}} : \quad Q_{N_H^{(A)}} \rightarrow K_{N_H^{(A)} - 1}$$

(3.20)

This time $2N_H^{(A)} + 4$ real scalars get mapped to the graviton and to the $\mathcal{N}_H^{(A)} = N_V^{(B)} + 1$ vector fields of the dual theory. In particular the dynamical degrees of freedom of the IIB metric in (3.18) arise from two scalars in $S_{IIB}^H$ which is dualized into $S_{III}^{G/V}$.

The full $c$-map is given by combining both sectors:

$$c = s_{N_V^{(A)}} \times s^{-1}_{N_H^{(A)}} : K_{N_V}^{(A)} \times Q_{N_H}^{(A)} \rightarrow Q_{N_V + 1}^{(B)} \times K_{N_H - 1}^{(B)}$$

(3.21)

Let us consider the s-map and the $\sigma$-model in some more detail. Following [16] the action 3.18 can rewritten in a way that makes the geometry more transparent and allows one to show that the target space is quaternionic and that certain subspaces are Kähler. This will be used later when we dualize solutions of the equations of motion.
So we combine the $2N_V^{(A)} + 4$ real scalars $\phi, \bar{\phi}, \zeta^I, \bar{\zeta}_I$ into $N_V^{(A)} + 2$ complex ones \[16\]:

\[
S' = \phi - i\zeta^I N_{IJ} \zeta^J + i\bar{\phi} - i\bar{\zeta}^I \bar{\zeta}_I \\
C_I = -3N_{IJ} \zeta^J + i(\bar{\zeta}_I + 3N_{IJ} \zeta^J)
\] (3.22)

Rewriting the action (3.18) in terms of the $2(N_V^{(A)} + 1)$ complex scalars $S', C_I, z^A$ results in

\[
S_{IIIB}^H = \int d^4x \sqrt{G_{IIIB}} \left\{ -2g_{AB} \partial z^A \bar{\partial} z^B - 2 \left( \frac{\partial S' - (C + \bar{C})_I \Im N^{IJ} \partial C - \frac{1}{2} (C + \bar{C})_I \Im N^{IJ} \partial N_{JK} \Im N^{KL}(C + \bar{C})_L}{S' + \bar{S}' - \frac{1}{2} (C + \bar{C})_I \Im N^{IJ}(C + \bar{C})_J} \right) \right\} \] (3.23)

Starting from this expression one can define a vielbein in terms of $S', C_I, z^A$ and show that the target space is quaternionic by computing the corresponding curvature which has holonomy group $SU(2) \times Sp(2(N_V^{(A)} + 1))$ and the appropriate negative curvature scalar. Likewise one can explicitly find the three complex structures. We will work this out for the case of one hyper multiplet in the next subsection and refer to \[16\] for the general case.

Moreover the action (3.23) can be used to identify certain interesting subspaces of the quaternionic manifold. First one can set $C_I$ to constant, purely imaginary values, $\partial_{\mu} C_I = 0, (C + \bar{C})_I = 0$. Then the action reduces to

\[
S[z^A, S'] = \int d^4x \sqrt{G_{IIIB}} \left\{ -2g_{AB} \partial z^A \bar{\partial} z^B - 2 \frac{\partial_{\mu} S' \partial_{\mu} S'}{(S' + \bar{S}')^2} \right\} \] (3.24)

which has the Kähler target space $K_{N_V^{(A)}}^{(A)} \times SU(1,1)/U(1)$ parametrized by $z^A$ and $S'$ respectively \[16\]. Note that $K_{N_V^{(A)}}^{(A)}$ is the dual special Kähler manifold. On the other hand one can set $z^A$ to constant values and explore the directions that have been added to $K_{N_V^{(A)}}^{(A)}$ by the s-map. After shifting

\[
S' \rightarrow S = S' - \frac{1}{2} C_I (\Im N)^{IJ} C_J ,
\] (3.25)

ones gets the action

\[
S[S, C_I] = \int d^4x \sqrt{G_{IIIB}} \left\{ -2\tilde{K}_{S\bar{S}} \partial_{\mu} S \partial_{\mu} \bar{S} - 2\tilde{K}_{S\bar{C}_I} \partial_{\mu} S \partial_{\mu} C_I - 2\tilde{K}_{C_I \bar{S}} \partial_{\mu} C_I \partial_{\mu} S - 2\tilde{K}_{C_I \bar{C}_J} \partial_{\mu} C_I \partial_{\mu} C_J \right\} ,
\] (3.26)

with $\tilde{K} = -\log(S + \bar{S} - C_I \Im N^{IJ} C_J)$. This shows that $S$ and $C_I$ parametrize for fixed $z^A$ the Kähler manifold \[16\]

\[
K(S, C_I) = \frac{SU(1, N_V^{(A)} + 2)}{U(1) \times SU(N_V^{(A)} + 2)} \cong Q_{N_V^{(A)}}^{(B)} \] (3.27)

Thus the directions added to $K_{N_V^{(A)}}^{(A)}$ by the s-map are completely universal.

By construction, special quaternionic manifolds have a lot of isometries. The gauge symmetries of the field strength $F_{\mu
u}$ and of the Kaluza Klein boson $\omega_m$ imply that the scalars $\zeta^I, \bar{\zeta}_I$ and $\bar{\phi}$ have axion-like shift symmetries. Together with a scale symmetry of $\phi$ this gives a total of $2N_V^{(A)} + 4$ isometries, that
act on the complex fields as

\[ S' \rightarrow S' + i\alpha - 2C_I \gamma^I - \gamma^I N_I \gamma^J \]
\[ S' \rightarrow \lambda S' \]
\[ C_I \rightarrow C_I + i\beta_I + N_I \gamma^J \]
\[ C_I \rightarrow \lambda^{1/2} C_I \]

with \(2N_V^{(A)} + 4\) real parameters \(\alpha, \beta_I, \gamma^I\) and \(\lambda\). Due to these isometries, the quaternionic metric does only depend on \(z^A\) and \(\Re S\) (or \(z^A\) and \(\phi\)). Note that this is only the minimal set of isometries. In addition, all isometries of the dual special Kähler manifold are isometries of the quaternionic manifold, and there may be further isometries as well.

### 3.3 The universal sector

The c-map involves both space-time quantities (metric, graviphoton, dilaton) and quantities related to the internal Calabi-Yau threefold (Kähler and complex structure moduli) and mixes them. In order to separate these two kinds of quantities and to have a simple and illustrative example it is useful to consider the minimal case in which the c-map makes sense, namely \(N_V^{(A)} = 0, N_H^{(A)} = 1\), which is called \(N = 2\) dilaton-supergravity. Before going into this let us note that this case cannot be obtained by Calabi-Yau compactification because a Calabi-Yau threefold has at least one modulus (its Kähler class, i.e. its overall radius) implying that one has at least one additional vector or hyper multiplet. Nevertheless dilaton-supergravity appears as a subsector in every Calabi-Yau compactification, characterized by setting all multiplets related to internal degrees of freedom to zero (and hence only depending on space-time quantities). Note that this way one can in particular embed all solutions of dilaton-supergravity into every Calabi-Yau compactification.

From the discussion in the last subsection it is clear that dilaton-supergravity is selfdual under the c-map, which simply exchanges the gravity multiplet (containing the graviton and the graviphoton as its bosonic part) with the single hyper multiplet (containing 4 scalars including the dilaton).

The scalar part of the action can be obtained by setting \(z^A = \text{const.}\) and \(C_A = \text{const.}\) in \(3.24\) or more simply by setting \(C_A = \text{constant}\) in \(3.24\). Defining \(C = \Im(N^{-1/2})^{0I} C_I = \Im(N^{-1/2})^{00} C_0\) we get a Kähler \(\sigma\)-model with Kähler potential \(K = -\log(S + \overline{S} - C\overline{C})\), i.e. the moduli space associated with the universal hyper multiplet is

\[ \mathcal{M}(S, C) = \frac{SU(2,1)}{SU(2) \times U(1)} \]

Thus in this case the hyper multiplet moduli space is Kähler.

In order to display the quaternionic structure it is useful to work with the shifted field \(S'\) instead of \(S\), because then the action can be written in terms of holomorphic squares of \(\partial_\mu S'\) and \(\partial_\mu C\) [16]. From \(3.23\) one obtains

\[ S'^H = \int d^4x \sqrt{g} \left\{ -2e^{2\overline{K}} |\partial_\mu S' - (C + \overline{C})\partial_\mu C|^2 - 2e^{\overline{K}} |\partial_\mu C|^2 \right\} \]

with \(\overline{K} = -\log(S' + \overline{S}' - \frac{1}{2}(C + \overline{C})^2)\). Introducing the complex 1-forms

\[ u = e^{\overline{K}/2} dC, \quad v = e^{\overline{K}} dS' - e^{\overline{K}} (C + \overline{C}) dC \]
we can rewrite (3.31) as

$$S[S', C] = \int \left\{ -u \wedge *\pi - v \wedge *\nu \right\}. \quad (3.32)$$

We introduce the quaternionic vielbein

$$V = \begin{pmatrix} u & \pi \\ v & -\pi \end{pmatrix} = i \Im(u) 1 + \Re(v) \sigma_1 + \Re(u) \sigma_2 + \Re(u) \sigma_3. \quad (3.33)$$

(Recall that the quaternionic algebra is isomorphic to the one generated by $1$ and $-i\sigma_i$.) The matrix indices labeling the rows and columns are $SU(2)$ indices and refer to the first (second) factor of the holonomy group $SU(2) \times SU(2)$ as we will see in a moment. Using

$$du = -\frac{1}{2} (v + \bar{v}) \wedge u \quad \text{and} \quad dv = v \wedge \bar{v} + u \wedge \bar{u} \quad (3.34)$$

the connection $\Omega$ can be found from the covariantly constancy condition,

$$(d + \Omega) V = 0, \quad (3.35)$$

with the result

$$\Omega = p \otimes 1_2 + 1_2 \otimes q, \quad (3.36)$$

where

$$p = \begin{pmatrix} \frac{1}{4}(v - \pi) & -u \\ \pi & -\frac{1}{4}(v - \pi) \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} -\frac{3}{4}(v - \pi) & 0 \\ 0 & \frac{3}{4}(v - \pi) \end{pmatrix}. \quad (3.37)$$

The resulting curvature is

$$R = d\Omega + \Omega \wedge \Omega = R_1 \otimes 1_2 + 1_2 \otimes R_2 \quad (3.38)$$

where $R_i$ are $SU(2)$ curvatures. The curvature of the first factor is related to the hyper Kähler form $J$:

$$R_1 = dp + p \wedge p = \begin{pmatrix} \frac{1}{2}(\pi \wedge u - \pi \wedge v) & \pi \wedge u \\ \pi \wedge v & -\frac{1}{2}(\pi \wedge u - \pi \wedge v) \end{pmatrix} = \sum_{i=1}^3 \frac{1}{2} \alpha_i \sigma^i = -iJ \quad (3.39)$$

where the expansion coefficients $\alpha_i = w^a \sigma_{ab} w^b$, $(w^a) = (u, v)^T$ are explicitly given by

$$\alpha_1 = \pi \wedge v + \pi \wedge u, \quad \alpha_2 = -i\pi \wedge v + i\pi \wedge u, \quad \alpha_3 = \pi \wedge u - \pi \wedge v. \quad (3.40)$$

The three complex structures $J_i$ are found by converting the 2-form components $\alpha_i$ of the hyper Kähler form $J$ into $(1, 1)$ tensors $J_i = -i w^a \sigma_{ab} w^b$, where $w^a$ is the vector field dual to the 1-form $w^a$. The required quaternionic algebra (3.5) is obtained with $\pi \wedge u + \bar{v} \wedge v$ as its unit element. Thus we have made the quaternionic structure obvious by identifying the hyper Kähler form $J$ and the three complex structures $J_i$, which are by construction covariantly constant with respect to the $SU(2)$ connection.

Finally we have to check the constraint on the curvature scalar. The curvature of the second $SU(2)$ factor is

$$R_2 = dq + q \wedge q = -\frac{3}{2} (v \wedge \pi + u \wedge \pi) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.41)$$
Computing the Ricci-scalar of the full curvature one finally finds $R(\Omega) = -24$, which according to (3.4) is the correct result for $N_H = 1$. This completes the proof that the scalar manifold of $N = 2$ dilaton-supergravity (3.29) is a quaternionic manifold that is consistent with local $N = 2$ supersymmetry [26].

There are two different ways to break the moduli space of the universal sector (3.29) down to the coset space $SU(1,1)/U(1)$. First, one neglects all R-R scalars, setting $S' = \phi + i\tilde{\phi}$, $C = 0$:

$$S[C] = \int d^4x \sqrt{G} \left\{ -2 \frac{|\partial_\mu S'|^2}{(S' + S')^2} \right\} = \int d^4x \sqrt{G} \left\{ -2 \frac{(\partial_\mu \phi)^2 + (\partial_\mu \tilde{\phi})^2}{4\phi^2} \right\}$$

$$= \int d^4x \sqrt{G} \left\{ -2 \left( (\partial_\mu \phi)^2 + \frac{1}{4} e^{4\phi} (\partial_\mu \tilde{\phi})^2 \right) \right\} \quad (3.42)$$

where we introduced the standard four-dimensional dilaton $\phi$ by $\phi = e^{-2\overline{\phi}}$ for comparison with section 2. This yields the standard NS-NS dilaton-axion moduli space.

Second, one keeps the dilaton together with one of the R-R scalars, which for definiteness we take to be the R-R pseudoscalar $\tilde{\zeta}_0$. Setting $S' = \phi$ and $C = i\tilde{\phi}$ gives

$$S[T] = \int d^4x \sqrt{G} \left\{ -2 \frac{|\partial_\mu T|^2}{(T + \overline{T})^2} \right\} = \int d^4x \sqrt{G} \left\{ -2 \left( \frac{(\partial_\mu \phi)^2}{4\phi^2} + \frac{(\partial_\mu \tilde{\phi})^2}{2\phi} \right) \right\}$$

$$= \int d^4x \sqrt{G} \left\{ -2 \left( (\partial_\mu \phi)^2 + \frac{1}{2} e^{2\phi} (\partial_\mu \tilde{\phi})^2 \right) \right\} \quad (3.43)$$

where $T = \phi + \alpha$ with $(\partial_\mu \alpha)^2/2\phi = (\partial_\mu \tilde{\phi})^2$. Comparing (3.43) to (3.42) we explicitly see that the R-R scalar $\tilde{\zeta}_0$ couples to the dilaton in a different way than the NS-NS scalar $\tilde{\phi}$. The action (3.43) is the natural starting point for the construction of $D = 4$, $N = 2$ D-instanton solutions.

## 4 Stationary IIA solutions and IIB D-instantons

We now turn to the construction of explicit solutions of the equations of motion (3.9) - (3.14). During the last year a lot of non-trivial static and more recently stationary solutions to the gravity and vector multiplet equations have been found while keeping the hyper multiplet scalars constant, so that (3.11) is trivially solved. These solutions describe in the static case charged black holes with various moduli fields turned on. Using T-duality and the c-map we can relate any such solution to a configuration with trivial (Euclidean, Einstein frame) metric and trivial vector multiplets, but non-trivial hyper multiplet scalars. These solutions are instantons and contain as a subclass four-dimensional D-instanton solutions, which in addition to a non-trivial dilaton have several non-trivial moduli fields.

For the stability of solutions it is crucial to have BPS saturated solutions which are invariant under at least one supersymmetry transformation. This is the case for all the configurations that we discussed in section 2. Our general strategy here will be to start with supersymmetric solutions on the IIA side and to apply T-duality in order to get a solution on the IIB side. Moreover these solutions can be interpreted as Calabi-Yau compactifications of supersymmetric D-p-brane solutions. Therefore we expect that our solutions are supersymmetric. A detailed check is postponed to future work together with a more detailed analysis of our solutions.
The plan of this section, which contains our main results, is as follows. In section 4.1 we study dilaton supergravity, i.e. the gravity multiplet together with the universal hyper multiplet. In 4.1.1 we first recall the extreme Reissner-Nordström black hole, which is a BPS solution of pure $N = 2$ supergravity. Then we find the T-dual D-instanton solution by solving the equations of motion of the T-dualized action using an instanton ansatz analogue to [11]. We explore the geometry of the D-instanton, which is a finite wormhole in the string frame and compare it to its ten-dimensional analogue and to the Reissner-Nordstrom black hole. Some related configurations, like NS-NS instantons are briefly mentioned. In 4.1.2 we discuss the D-instanton in the context of the quaternionic geometry of the hyper multiplet moduli space.

Section 4.2. is devoted to the IIB duals of non-axionic IIA black holes. If the IIA black hole is in addition double-extreme, then the non-trivial IIB fields live in a symmetric Kählerian subspace of the quaternionic hyper multiplet moduli space. In this case one can work on the IIB side with complex fields which are standard coset coordinates. This is used in 4.2.1 to explore this part of the moduli space. In particular we find that non-axionic solutions can be characterized by a reality constraint. When moving away from the double extreme limit on the IIA side in 4.2.2, it is more convenient to work in terms of real fields on the IIB side. Then the solution still looks simple and its structure is very similar to the well known non-axionic IIA black holes. In particular the solutions can be expressed in terms of harmonic functions. At the end of 4.2.2 we interpret the solutions from the ten-dimensional point of view, compare them to those obtained in section 2 by toroidal compactification and comment on the role of the geometry of the internal Calabi-Yau threefold.

Section 4.3 gives our central result, an explicit set of formulae which specifies the fact that instantons are T-dual to the most general stationary solution of the gravity/vector multiplet sector. The results of section 4.1 and 4.2 should be considered as first and relatively simple application of this formalism.

### 4.1 Dilaton supergravity

#### 4.1.1 RN black holes in pure supergravity and D-instantons in D=4

In this section, we discuss explicit solutions in $D = 4$ and the associated c-map. First, we relate the black hole solution on the type IIA side and the D-instantons of type IIB theory through the c-map in $D = 4$. We take $G_N = 1$.

The bosonic part of the action of ungauged $N = 2$ supergravity coupled to $N_V$ vector multiplets is given by,

$$e^{-1} \mathcal{L} = R - 2g_{AB} \partial_\mu z^A \partial^\mu \bar{z}^B - \frac{1}{4} \text{Im} \mathcal{N}_{IJ} F^I_{\mu\nu} F^{J\mu\nu} - \frac{1}{4} \text{Re} \mathcal{N}_{IJ} F^I_{\mu\nu} \ast F^{J\mu\nu}$$

(4.1)

Here the index $I$ runs from 0, 1, \ldots, $N_V$, where the extra index $0$ is due to the graviphoton degrees of freedom. The D-instanton of [11] contains the dilaton and the R-R scalar as non-trivial fields. They are world-sheet configurations which correspond to a target space-time event giving rise to exponentially suppressed contributions to scattering amplitudes of order $e^{-\frac{1}{g^2}}$, where $g$ denotes the string coupling constant. The dilaton has its origin in the dual type IIA metric and the RR scalar in the graviphoton. The c-map maps these two fields to the dilaton and the RR scalar on the type IIB side respectively. Thus, it is sufficient to consider pure $N = 2$ (type IIA) supergravity containing only the graviphoton and we go to the universal IIB hyper multiplet via c-map. The prepotential for pure $N = 2$ supergravity
is given by, \( F(X) = 2i(X^0)^2 \). We show here that the RN black holes in pure supergravity are related to the D-instantons via c-map. It is sufficient in the following to discuss only the electric type black hole solution on the type IIA side. It follows from the previous definition that, \( N_{IJ} = N_{00} = i \). Thus, the Lagrangian is given by

\[
e^{-1} \mathcal{L} = R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{4.2}
\]

The equations of motion are obtained as,

\[
\nabla_\mu F^{\mu\nu} = 0 \tag{4.3}
\]

\[
R_{\mu\nu} + \frac{1}{8} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} + \frac{1}{2} F_{\mu\rho} F_{\nu}^{\rho} = 0. \tag{4.4}
\]

The corresponding spherically symmetric black hole solution is given by the well-known Reissner-Nordstrom solution. In the Einstein frame, the metric has the following particular form in Minkowski space (\( x^\mu = x^0, x^m \))

\[
ds_E^2 = - e^{2U(r)} \, dt^2 + e^{-2U(r)} \, dx^2 \tag{4.5}
\]

Note that the real dilaton in the type IIA model is trivial, but the metric is not flat. What we call here \( U \) is related to the dilaton on the type IIB side (\( U = -\varphi \)) The components of the Ricci tensor are given by,

\[
R_{mn} = g_{mn} \partial^2 U - 2 \partial_m U \partial_n U \tag{4.6}
\]

\[
R_{00} = -g_{00} \partial^2 U \tag{4.7}
\]

Then the Ricci scalar reads as,

\[
R = 2\partial^2 U - 2(\partial U)^2 \tag{4.8}
\]

Restricting ourselves to the pure electric solution \( F_{\mu\nu} \sim F_{0m} \) we find three equations of motion, namely,

\[
\partial_m (e^{-2U} F^{m0}) = 0 \tag{4.9}
\]

\[
R_{00} - \frac{1}{8} g_{00} F^2 = 0 \tag{4.10}
\]

\[
R_{mn} + \frac{1}{8} g_{mn} F^2 - \frac{1}{2} F_{m0} g^{00} F_{n0} = 0 \tag{4.11}
\]

Taking \( F_{m0} = \partial_m A \) it follows that \( F^2 = -2(\hat{\partial} A)^2 \), where hatted quantities refer to flat space (\( \hat{\partial}^2 = \eta^{mn} \partial_m \partial_n \)). Using the ansatz \( A = 2e^U \), we find that the equations of motion are satisfied, provided

\[
\hat{\partial}^2 e^{-U} = 0. \tag{4.12}
\]

Thus we find that the pure electric Reissner Nordstrom black hole is specified by the harmonic function

\[
e^{-U(r)} = e^{-U(\infty) + \frac{q_0}{r}}. \tag{4.13}
\]

In the following we assume flat space at infinity \( (e^{-U(\infty) = 1}) \).

From the spatial component of the metric we can read off the entropy and the ADM mass, given by the expressions,

\[
S_{BH} = \pi (r^2 g_{rr})_{r=0} = \pi \, q_0^2, \quad \quad M_{ADM} = q_0 \tag{4.14}
\]
On the other hand, recent developments in four dimensional $N = 2$ supergravity and string theory show that the entropy and the mass of a BPS saturated black hole are completely determined by the corresponding prepotential

$$
S_{BH} = \pi |Z|^2, \quad M_{BPS}^2 = |Z|^2 = e^{K(z, \bar{z})} q_I X_I^I(z) - p_I F_I(z)^2.
$$

(4.15)

where $Z$ is the central charge, $q_I$ and $p_I$ are the electric and magnetic charges respectively. It is easy to verify that, for the case at hand, (4.14) and (4.15) agree with each other. Hence, the corresponding electric Reissner Nordstrom black hole preserves one half of the $N = 2$ supersymmetry.

Now we discuss the corresponding picture on the type IIB side. The Lagrangians in both theories are dual to each other as discussed before and are related by the c-map. Performing the c-map, the dual Lagrangian in terms of real fields is given by,

$$
e^{-1} \mathcal{L} = R - 2(\partial \varphi)^2 - e^{2\varphi}(\partial \zeta^0)^2.
$$

(4.16)

The scalar fields parametrize the $SU(1,1)/U(1)$ coset. Since the c-map exchanges the $N = 2$ supergravity sector with the type II dilaton in the universal sector, the metric is now flat while the type IIB dilaton is non-trivial. The Euclidean version of this action represents the action corresponding to D-instanton in type IIB theory. In 4-dimensions, one has the duality between the D-instanton ($p = -1$ brane) and D-string ($p = 1$ brane) analogous to the D-instanton and 7-brane duality in ten dimensions. One could also have started with a magnetically charged solution in the type IIA side, for which $F_{0m} = 0$ and $F_{mn} \sim \epsilon_{mnp}\partial \tilde{\psi},$ where the field $\tilde{\psi}$ is later identified with $\tilde{\zeta}$ in comparison with [16]. This is because of the electric-magnetic duality in 4 dimensions where the black hole in general is dyonic as the dual of a 2-form gauge field strength is also a 2-form in $D = 4$.

In order to discuss the D-instanton in type IIB theory, we perform a Wick rotation of $\zeta^0 \to i \zeta^0$ to go to the Euclidean version. The Euclidean action is given by,

$$
e^{-1} \mathcal{L}_E = R - 2(\partial \varphi)^2 + e^{2\varphi}(\partial \zeta^0)^2 + \text{boundary terms}
$$

(4.17)

The origin of the boundary term has been discussed in section 2 and we shall comment on it later. The corresponding field equations in Euclidean spacetime read as,

$$\nabla_\mu \left( e^{2\varphi} \partial^\mu \zeta^0 \right) = 0 \quad (4.18)
$$

$$2 \nabla^2 \varphi + e^{2\varphi} (\partial \zeta^0)^2 = 0 \quad (4.19)
$$

$$R_{\mu
u} - 2 \partial_\mu \partial_\nu \varphi + e^{2\varphi} \partial_\mu \zeta^0 \partial_\nu \zeta^0 = 0 \quad (4.20)
$$

Now we consider the instanton ansatz of [11] with $d\zeta^0 = \sqrt{2}e^{-\varphi}d\varphi$ and flat Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$ in our four dimensional context. Note that the fields still only depend on 3 spatial coordinates and are independent of the time $t$. With this ansatz, one gets the constraint $\partial^2 \varphi = -(\partial \varphi)^2$ and the instanton ansatz leads to the equation of motion,

$$\bar{\partial}^2 e^{\varphi} = 0 \quad (4.21)
$$

with general 3-dimensional spherically symmetric solution

$$
e^{\varphi} = e^{\varphi_0} + \frac{q_0}{r}.
$$

(4.22)

\[2\] In comparison with [16], we have $\zeta = 0 = \phi, \phi = e^{-2\varphi}.$

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The string and Einstein frame metrics associated with this solution are:

\[ ds^2_S = e^{2\varphi} ds^2_E = (e^{\varphi\infty} + \frac{q_0}{r})^2 (dt^2 + dr^2 + r^2 d\Omega_2^2) \]  

(4.23)

with \( r^2 = x^2 + y^2 + z^2 \). The \( t \) direction is an isometry, whereas the geometry in the three other directions is an infinite throat interpolating between the flat two-space and a two-sphere of radius \(|q_0|\). This configuration is T-dual to the Euclideanized extreme Reissner-Nordstrom black hole. Now we can relax the constraint that the solution has the isometry in the Euclidean time direction and allow for a more general dependence on all four coordinates. So the general spherically symmetric solution is given by,

\[ e^{\varphi} = e^{\varphi\infty} + \frac{q_0}{r^2} \]  

(4.24)

This is the D-instanton solution in four dimensions. The dilaton is singular at the origin and the Einstein metric is flat. Here \( \varphi\infty \) is the value of the dilaton at infinity, which we take to be constant. On the other hand, if we transform the instanton solution to the string frame with the metric \( G_{\mu\nu} = e^{2\varphi} g_{\mu\nu} \), then we find that the D-instanton solution

\[ ds^2_S = (e^{\varphi\infty} + \frac{q_0}{r^2})^2 (dr^2 + r^2 d\Omega_2^3) \]  

(4.25)

is invariant under the transformation,

\[ r \rightarrow \frac{q_0 e^{-\varphi\infty}}{r} \]  

(4.26)

The interpretation of this result is as follows: in the string frame the instanton solution is a wormhole solution connecting two asymptotically flat Euclidean regions by a neck. The D-instanton solution on the IIB side is expected to be supersymmetric as the corresponding black hole solution in the IIA theory is supersymmetric.

The electric Noether charge \( q_0 \) associated with the Noether current \( j_\mu = e^{2\varphi} \partial_\mu A \) of the global (Peccei-Quinn) symmetry transformation \( \zeta \rightarrow \zeta + \theta \) with \( \theta \in \mathbb{R} \) reads as,

\[ \oint_{\partial M} j_\mu d\Sigma^\mu = - \oint_{\partial M} \partial_\mu e^{\varphi} d\Sigma^\mu = 2\pi q_0 \]  

(4.27)

The corresponding D-instanton action comes from the boundary only. The action in the bulk vanishes by substituting the instanton ansatz. Contribution from the boundary is given by, \( S_{\partial M} = -4\partial^2 \varphi \). Hence the instanton action is given by,

\[ S_{\text{inst}} = -4 \int_M d^4 x \partial^2 \varphi = -4 \int_{\partial M} d\Sigma_\mu \partial^\mu \varphi = 8\pi \frac{|q_0|}{g} \]  

(4.28)

From the point of view of the dual type IIA model the type IIB D-instanton action has its origin in the four-dimensional Einstein action and T-duality. This is an interesting observation with respect to the uncompactified ten-dimensional type IIB theory. There the D-instanton action has been derived using D-instanton / D-7-brane Poincare duality [11, 21]. In our four-dimensional context this corresponds to the Poincare duality of a D-string and a D-instanton. In the context of the D-string / D-instanton duality one can start with the Minkowski action written in terms of the hodge dual three form field strength and then perform a Wick rotation to obtain the Euclidean action which is manifestly positive as the \( F^2 \) term does not change sign under Wick rotation and one does not need to introduce a boundary term. On the other hand, if we dualize the three form action to scalar(pseudoscalar) field action, then one has to take proper consideration of a boundary term, which makes the scalar action positive. To
obtain the scalar action with proper boundary term, one starts with the action in terms of a 3-form field strength and adds a Lagrange multiplier term involving the three form and the 0-form. Integrating over the 0-form (scalar field), one obtains the dual action for the 3-form. To go to the Euclidean version, one uses the properties of p-forms under Poincare duality in 4 dimensions. Integrating over the 3-form, one obtains the action for the scalar field together with the required boundary term. The presence of the boundary term is really crucial in computing the action in the instanton background. This is also true for D-instantons in ten dimensions [11, 21].

Since we started our analysis on the type IIA side, which was determined by the $N = 2$ supergravity prepotential only, the dual type IIB D-instanton action is determined by the dual type IIA model. Moreover, in terms of the central charge of the type IIA model we find $S_{\text{inst,IIB}} \sim |Z|_{IIA}$.

It is interesting to consider what happens if we replace the R-R scalar $\xi^0$ by the NS-NS scalar $\tilde{\phi}$ which has a different coupling to the dilaton. In this case, the instanton ansatz goes through with the simple modification that now $e^{2\phi}$ has to be harmonic. Taking this function to depend on the 4-radius $r$, where $r^2 = t^2 + x^2 + y^2 + z^2$, the string and Einstein frame metrics are related by,

$$ds^2_S = e^{2\phi} ds^2_E = (e^{2\phi} + \frac{q_0}{r^2}) (dr^2 + r^2 d\Omega_3^2)$$

This describes a semi-infinite wormhole which interpolates between the flat 3-space at $r \to \infty$ and a 3-sphere of radius $|q_0|$ at $r \to 0$ [34]. This time the instanton action is proportional to $\frac{1}{r^2}$. By restricting the dependence of $\phi$ such that one gets an isometry, one can further T-dualize this solution into a 0-brane type solution. Clearly this result will be different from the extreme Reissner-Nordstrom solution, since in the expression for the metric, the harmonic function appears with a first power instead of a second power. But this is to be expected because the scalar $\tilde{\phi}$ comes from dualizing the Kaluza-Klein vector $w_m$. So it is a priori clear that the T-dual 0-brane is a non-static stationary solution.

### 4.1.2 D-Instantons and quaternionic structure

In the last section, we discussed about the dualized black hole solution in type IIA side and showed how to obtain the D-instanton solution in the type IIB theory by using the c-map. Here we explore the quaternionic geometry associated with the $D$-instantons. We consider the minimal case of the type IIB dilaton hyper multiplet coupled to $N = 2$ supergravity. Here, we consider the case where there is no vector multiplet. So the four real scalars namely $\phi, \tilde{\phi}, \zeta^0$ and $\tilde{\zeta}_0$ of the quaternion can be combined into 2 complex scalars $S'$ and $C_I$ as before, but in this case, the index runs only over 0 as we are in the pure gravity sector. For the D-instanton, the $S'$ and $C$ fields in terms of non vanishing scalars are given by the expression,

$$S' = \phi + (\zeta_0)^2$$

$$C_0 = -\zeta_0$$

which follows from the general definition of the complex fields $S'$, $C_I$ and by putting $\tilde{\phi} = 0 = \zeta$. The action in terms of the complex fields looks like,

$$S_{\text{inst,IIB}} = \int d^4x \sqrt{G_{\text{IIB}}} \left\{ -2 \frac{|\partial S' - (C_0 + C_0) \partial C_0|^2}{(S' + S')^2} - 2 \frac{|\partial C_0|^2}{(S' + S')^2} \right\}$$

$$S_{\text{IIB}}^{'H} = \int d^4x \sqrt{G_{\text{IIB}}} \left\{ -2 \frac{|\partial S' - (C_0 + C_0) \partial C_0|^2}{(S' + S')^2} - 2 \frac{|\partial C_0|^2}{(S' + S')^2} \right\}$$

$$S_{\text{IIB}}^{'H} = \int d^4x \sqrt{G_{\text{IIB}}} \left\{ -2 \frac{|\partial S' - (C_0 + C_0) \partial C_0|^2}{(S' + S')^2} - 2 \frac{|\partial C_0|^2}{(S' + S')^2} \right\}$$

$$S_{\text{IIB}}^{'H} = \int d^4x \sqrt{G_{\text{IIB}}} \left\{ -2 \frac{|\partial S' - (C_0 + C_0) \partial C_0|^2}{(S' + S')^2} - 2 \frac{|\partial C_0|^2}{(S' + S')^2} \right\}$$

$$S_{\text{IIB}}^{'H} = \int d^4x \sqrt{G_{\text{IIB}}} \left\{ -2 \frac{|\partial S' - (C_0 + C_0) \partial C_0|^2}{(S' + S')^2} - 2 \frac{|\partial C_0|^2}{(S' + S')^2} \right\}$$
This is equivalent to the dualized type II A action written before. In order to display the coset structure parametrized by the complex fields, we shift the $S'$ field as

$$S' \rightarrow S = S' - \frac{1}{2} C_0^2$$

and the Lagrangian can be written as,

$$e^{-1} \mathcal{L} = -2 g_{ab} \partial_\mu z^a \partial^\mu \bar{z}^b$$

where, $g_{\dot{a}\dot{b}} = \partial_\dot{a} \partial_{\dot{b}} \tilde{K}$ and $\tilde{K}$ is the Kähler metric given by the expression,

$$\tilde{K} = - \log(S + \bar{S} - C_0 \bar{C}_0)$$

The complex fields $S$ and $C_0$ parametrize the Kähler manifold $\mathcal{M} = \frac{SU(1,2)}{U(1) \times SU(2)}$ and the coset metric is given by,

$$g = (S + \bar{S} - C_0 \bar{C}_0)^{-2} \begin{pmatrix} 1 & -C_0 \\ -\bar{C}_0 & S + \bar{S} \end{pmatrix}$$

Next, we show that the moduli space of the $N = 2$ supergravity is a quaternionic manifold, i.e. we consider the minimal case of the type II B dilaton hyper multiplet coupled to $N = 2$ supergravity. Here, the 4 real scalars $\phi$, $\tilde{\phi}$, $\zeta$ and $\tilde{\zeta}$ (which comprise the universal hyper multiplet) are again combined into the two complex fields,

$$S' = \phi + (\zeta_0)^2 + i \tilde{\phi} - \zeta_0 \tilde{\zeta}_0$$
$$C_0 = -\zeta_0 + i \tilde{\zeta}_0$$

Now we introduce the two one-forms namely,

$$u = e^{\frac{\tilde{K}}{2}} dC; \quad v = e^{\tilde{K}} (dS' - (C + \bar{C}) dC)$$

with $\tilde{K}$ as defined before. With this definition, the IIB Lagrangian becomes,

$$-e^{-1} \mathcal{L}_0 = (u, \bar{u}) + (v, \bar{v})$$

where, $(,)$ denotes contraction of components. Here, we have $dN = 0$, that is the type II A dilaton is constant. The torsion of the quaternionic vielbein is given by,

$$\left( \begin{array}{c} du \\ dv \end{array} \right) = \begin{pmatrix} -\frac{1}{2} (v + \bar{v}) \wedge u \\ u \wedge \bar{v} + v \wedge \bar{u} \end{pmatrix}$$

The curvature 2-form $R$ can be directly computed and one can obtain the expression for Hyper-Kähler2-form associated with the $SU(2)$ connection as discussed in the quaternionic geometry of the universal hyper multiplet in section 3. So one obtains the result that the scalar manifold of $N = 2$ dilaton-supergravity is quaternionic in agreement with [16].
4.2 Static solutions of ungauged $N = 2$, $D = 4$ Supergravity

Here, we discuss IIB instanton solutions, which are T-dual to specific static IIA solutions. The explicit solutions can be obtained as special cases of the general dualized stationary IIA solution, which is presented in section 4.3.

4.2.1 Type IIB dual of non-axionic, double extreme IIA black holes

In the last section, we dealt with the RN black hole solution and its T-dual version, which was shown to correspond to the D-instanton solution in type IIB theory. We discussed the simplest case of c-map involving $N_V^{(A)} = 0$ and $N_H^{(A)} = 1$. Here we consider the more general case of $N_V$ vector multiplets coupled to $N = 2$ supergravity. So in the case, we have $N_V$ physical complex scalars $z^A$, the R-R scalars $C_I (I = 0, \ldots, N_V)$ and the complex field $S'$ which are the coordinates for the dual quaternionic manifold. While working out explicitly the geometry of the quaternionic metric of the IIB hypermultiplets, one finds that the computation becomes quite complicated as the derivative of the matrix $N(z, \bar{z})$ appears there and leads to various cross terms in the metric. If the matrix $N$ becomes holomorphic, then the quaternionic manifold becomes Kählerian. This can happen for the case of minimal coupling, i.e. quadratic prepotentials. But we do not consider this here as this case does not occur in string compactifications. However, we can consider the case of type IIA solutions with constant scalar fields i.e. $z^A = \text{const}$, which corresponds to double extreme black holes, which are given by configurations where the moduli take constant values all the way from the horizon up to spatial infinity. In this case, the $Q$ manifold again becomes Kählerian and is the manifold of pure R-R scalars. So the nontrivial scalars $S'$ and $C_I (I = 0, \ldots, N_V)$ parametrize the Kähler quaternionic manifold $SU(2+2N_V,1)$ $\otimes$ $U(1)$. In this case, all the nontrivial scalars come from the IIA gravity multiplet and from the IIA field strength. The metric and the connections are sufficiently simple in order to explicitly write down the IIB scalar equation of motion, which are solved by the T-dualized double extreme black holes. On the IIB side, this corresponds to the generalizations of the D-instanton solutions discussed in the previous section.

In the following, we recall the standard parametrization of the coset, derive the metric and associated connection.

The action in terms of $2(N_V^{(A)} + 1)$ complex scalars $S'$, $C_I$ and $z^A$ has been written down in section (3). For constant moduli fields (scalars $z^A$), the action reduces to,

$$S^H = \int d^4x \sqrt{G}\{-2e^{2\tilde{K}}[\partial S' - (C + \bar{C})_J (3N)^{IJ} \partial C_I]^2 - 2e^{2\tilde{K}} \partial C_I (3N)^{IJ} \partial \bar{C}_J}\} \tag{4.42}$$

where the fields $S'$ and $C_I$ have been defined before in section 3. The coset structure becomes more transparent by rewriting the action in terms of the shifted field $S$. The Kähler potential $\tilde{K}$ is given by,

$$\tilde{K} = -\log(S + \bar{S} - C_I \bar{3N}\bar{C}_I) \tag{4.43}$$

In terms of canonical coset coordinates $S$ and $C$, the coset metric is given by,

$$g = (S + \bar{S} - C_I \bar{3N}\bar{C}_I)^{-2} \begin{pmatrix}
1 & -C_I (3N^{-\frac{1}{2}})_{JK} \\
-(3N^{-\frac{1}{2}})_{IJ} \bar{C}_J & \delta_{IK}(S + \bar{S} - C_J \bar{C}_J) + \bar{C}_I C_K
\end{pmatrix} \tag{4.44}$$
The inverse metric is given by,

\[ \mathbf{g}^{-1} = \left( S + \bar{S} - C_I \mathbb{A}^{IJ} \mathbf{C}_I \right) \begin{pmatrix}
S + \bar{S} & C_J (\mathbb{A}^{-\frac{1}{2}})_{JK} \\
(\mathbb{A}^{-\frac{1}{2}})_{JI} \mathbf{C}_J & \delta_{IK}
\end{pmatrix} \quad (4.45) \]

Defining \( C_J = C_I (\mathbb{A}^{-\frac{1}{2}})_{IJ} \), the connection as a matrix valued one-form is given by, \( \Gamma = \Gamma_S dS + \Gamma_L dC_I \). The matrices \( \Gamma_S \) and \( \Gamma_L \) are given in terms of matrices, \( \Gamma_S = \mathbf{g}^{-1} \partial_S \mathbf{g} \) and \( \Gamma_L = \mathbf{g}^{-1} \partial_L \mathbf{g} \). The explicit matrices are given by,

\[
\Gamma_S = \frac{1}{S + \bar{S} - C_J} \begin{pmatrix}
-2 & \mathcal{C}_K \\
0 & -\delta_{IK}
\end{pmatrix} \quad (4.46)
\]

and,

\[
\Gamma_L = \frac{1}{S + \bar{S} - C_J} \begin{pmatrix}
2 \mathcal{C}_L & -\mathcal{C}_L \mathcal{C}_K - \delta_{KL}(S + \bar{S} - C_J) \\
0 & \mathcal{C}_L \delta_{IK}
\end{pmatrix} \quad (4.47)
\]

The equation of motion of the real scalars \( q_u \) are written down in section 3. If we consider D-instanton type solutions, then the space-time metric is Euclidean and flat. If we further assume that the fields \( q_u \) do depend on the 4-radius \( r \), where \( r^2 = t^2 + x^2 + y^2 + z^2 \), then we get a geodesic equation in \( r \),

\[
\frac{d^2 q_u}{dr^2} + \frac{3}{r} \frac{dq_u}{dr} + \Gamma_{uv} \frac{dq^v}{dr} \frac{dq^w}{dr} = 0 \quad (4.48)
\]

We have previously combined the real scalars into complex ones, \( S \) and \( C_I \). The Christoffel connection coincides with the hermitean connection because the scalar manifold is \( K\ddot{a}hler \).

Now, we consider dualizing an axion-free double extreme black hole solution with one electric and \( N_V \) magnetic charges. For these black hole solutions, we have,

\[
\Re \mathcal{N} = 0; \quad N_{0A} = 0 \quad (4.49)
\]

Since the solution is static and has no magnetic components of \( F^0_{\mu\nu} \) and no electric components of \( F^A_{\mu\nu} \), it implies that the following dual scalars vanish:

\[
\tilde{\phi} = 0; \quad \tilde{\zeta}^0 = 0, \quad \zeta_A = 0 \quad (4.50)
\]

From the above condition, it is clear that we are associating electric type solutions with \( \zeta_0 \) and magnetic type solutions with \( \tilde{\zeta}^A \), where \( A = 1 \ldots N_V \).

After substituting this in the hyper multiplet part of the dual action, we obtain,

\[
S^H_{IIB} = \int d^4x \sqrt{G^{(IIB)}} \left\{ -\frac{1}{2\phi^2} (\partial \phi)^2 - \frac{1}{\phi} \partial \zeta^0 \mathcal{N}_{00} \partial \zeta^0 - \frac{1}{\phi} \partial \tilde{\zeta}_A \mathcal{N}^{AB} \partial \tilde{\zeta}_B \right\} \quad (4.51)
\]

Defining \( e^{-2\varphi} = \phi \) as before, we rewrite the action as,

\[
S^H_{IIB} = \int d^4x \sqrt{G^{(IIB)}} \left\{ -2(\partial \varphi)^2 - e^{2\varphi} \partial \zeta^0 \mathcal{N}_{00} \partial \zeta^0 - e^{2\varphi} \partial \tilde{\zeta}_A \mathcal{N}^{AB} \partial \tilde{\zeta}_B \right\} \quad (4.52)
\]

The Euclidean version of this action is a generalization of D-instanton action obtained previously by dualizing the black hole solution in pure gravity. The limit \( \tilde{\zeta}_A = 0 \) and \( \mathcal{N}_{00} = 1 \), corresponds to D-instanton action in type IIB theory. For the non-axionic case, the complex coordinates \( S' \) and \( C \) reduce
Note that, $S'$ and $C_0$ are real while $C_A$ is imaginary. The Kählerpotential is computed to be, $\tilde{K} = -\log(2\phi)$. The canonical coset coordinate $S$ is given by,

$$S = \phi + \frac{1}{2} \zeta^0 \Im_N \zeta^0 +\frac{1}{2} \zeta_A \Im_N^{AB} \tilde{\zeta}_B$$

Thus using the c-map, the non-axionic double extreme back holes on the IIA side are mapped directly to the generalized D-instanton solutions in the IIB side which are defined by the reality constraints. The actions in both the cases are dual to each other.

We also consider the case of double extreme black holes, where axions are also allowed. In this case, the solution on the type IIA side is that of a stationary solution and using the c-map, the action in the dual theory in terms of non zero fields is given by,

$$S^H_{II} = \int d^4x \sqrt{G^{(IIB)}} \left\{-\frac{1}{2\phi^2} (\partial \phi)^2 - \frac{1}{2\phi^2} (\partial \tilde{\phi} + \zeta^i \partial \tilde{\zeta}_i - \partial \zeta_i \tilde{\zeta}_i)^2 - \frac{1}{\phi} \partial \zeta^i \Im_N^{IJ} \partial \zeta^J \right\}$$

The coset is again given by $\frac{SU(1,N_V+2)}{U(1) \times SU(N_V+2)}$. The structure of the coset metric is as given before for the non-axionic case, but in terms of the shifted coordinates $S$ and $C$ which are now more complicated. They are given by,

$$S = \phi + \frac{1}{2} \zeta^i \Im_N^{IJ} \zeta^J +\frac{1}{2} (\tilde{\zeta}_i + \Re N_{IK} \tilde{\zeta}^K) \Im_N^{IJ} (\tilde{\zeta}_J + \Re N_{JL} \zeta^L) + i\tilde{\phi}$$

$$C_I = -\Im N_{IJ} \zeta^J + i(\tilde{\zeta}_I + \Re N_{IJ} \zeta^J)$$

In order to explore the quaternionic structure in the double extreme case, one again has to introduce the complex 1-forms as,

$$E^A = e^{(\bar{K} - K)/2} P_I^A (N^{-1})^{IJ} dC_J$$

$$u = 2e^{(\bar{K} + K)/2} z^I dC_I$$

$$v = e^{\bar{K}} (dS' - (C + \bar{C})_I \Im_N^{IJ} dC_J$$

where $P_I^A$ is a $N_V \times (N_V + 1)$ matrix. The connection $\Omega$ can be written down from the covariantly constancy condition of the quaternionic vielbein $V$, i.e. $(d + \Omega)V = 0$. The form of $\Omega$ is given by,

$$\Omega = p \otimes 1_{2N_V+1} + 1_2 \begin{pmatrix} q & t \\ -t^\dagger & -q^T \end{pmatrix}$$
where,

\[
p = \begin{pmatrix}
\frac{1}{4}(v - \bar{v}) & -u \\
\bar{u} & -\frac{1}{4}(v - \bar{v})
\end{pmatrix} \quad (4.64)
\]

\[
q = \begin{pmatrix}
-\frac{3}{4}(v - \bar{v}) & E^B \\
-E^A & -\frac{1}{4}(v - \bar{v})
\end{pmatrix} \quad (4.65)
\]

\[
t = \begin{pmatrix}
0 & 0 \\
0 & -\frac{f_{ABC}E^C}{4zNz}
\end{pmatrix} \quad (4.66)
\]

The curvature 2-form has \(SU(2) \otimes Sp(2N_V + 2)\) holonomy and can be explicitly computed. We do not give the details here.

Till now, we discussed the case of double extreme black hole solutions corresponding to constant moduli \(z^A\) and used the c-map to go to the dualized version in type IIB. If \(z^A\) are not constant, then there will be additional terms in the dualized action involving terms like derivative of the matrix \(\mathcal{N}\). The corresponding c-map becomes more complicated, but nevertheless one can again work out the details as before and explore the quaternionic structure. All the dualized solutions we have found here through a c-map will eventually satisfy the geodesic equation of motion as mentioned in the beginning. Note that, we did not have to solve these inhomogeneous second order equations explicitly to obtain the solution. The knowledge of the corresponding solution in the IIA side and the T-dualized version (c-map) is sufficient to write down the solution in the IIB theory.

4.2.2 The T-duals of static, non-axionic solutions

During the last years extremal black hole solutions of \(D = 4, N = 2\) supergravity coupled to vector multiplets have been studied extensively. Such solutions can be expressed in terms of harmonic functions, which are related to the different kinds of electric and magnetic charges that the black hole carries. The subclass of non-axionic black holes has a particular simple structure. In order to get a better idea about the physics of D-instantons we will consider in this subsection the T-duals of extreme, non-axionic black holes.

The axion-free case can be defined on the IIA side by restricting the scalar fields to purely imaginary values, i.e. the real part, which has an axion-like shift symmetry vanishes: \(\Re z^A = 0\). For simplicity we will restrict ourselves to a proper subclass where half of the charges are set to zero (in general we only get \(N_V\) relations between the \(2N_V + 2\) charges). For definiteness we choose the field strength \(F_{\mu\nu}^0\) to be purely electric, whereas the other field strength \(F_{\mu\nu}^A\) are purely magnetic. Such a non-axionic black hole carries one electric charge \(q_0\) and \(N_V\) magnetic charges \(p^A\). The prepotential and hence the black hole solution of a generic type II compactification is very complicated. In order to have a sufficiently simple configuration we go to a limit in moduli space where the prepotential reduces to the cubic form \(F = d_{ABC}X^A X^B X^C\). The static non-axionic solution for this case has been found in [35] and it depends on \(N_V + 1\) functions \(H_0, H^A\) which are harmonic with respect to the transversal 3-radius. Using the general formulæ that are derived in section 4.3 we can get the corresponding axion-free wormhole solution. As already explained we have to take the harmonic functions to depend on the transversal 4-radius \(r, r^2 = t^2 + x^2 + y^2 + z^2\).
in order to describe a localized \((-1)\)-brane. The string and Einstein frame metrics are:

\[
ds_S^2 = e^{2\varphi(r)} ds_E^2 = e^{2\varphi(r)}(dr^2 + r^2 d\Omega_3^2)
\]

with dilaton

\[
e^{2\varphi(r)} = \sqrt{4H_0 d_{ABC} H^A H^B H^C}.
\]

The remaining non-trivial scalar fields are

\[
z^A = 2iH_0 H^A e^{-2\varphi}, \quad \zeta^0 = \frac{1}{\sqrt{2H_0}}, \quad \partial_\mu \tilde{z}^A = -\frac{1}{2\sqrt{2}} \Im N_{AB} \partial_\mu H^B
\]

where the IIA gauge kinetic matrix \(N_{IJ}\) is purely imaginary, hence does not contain \(\theta\)-angles. It can explicitly be expressed in terms of harmonic functions as

\[
N_{00} = 2iH_0^2 e^{-2\varphi}, \quad N_{0A} = N_{A0} = 0, \quad N_{AB} = 4g_{AB}N_{00} = 8iH_0^2 e^{-2\varphi}g_{AB},
\]

where \(g_{AB}\) is the IIA vector multiplet metric.

In order to describe a single \((-1)\)-brane the harmonic functions have to be taken to depend on the overall radius:

\[
H_0 = h_0 + \frac{q_0}{r^2}, \quad H^A = h^A + \frac{p^A}{r^2}
\]

For \(r \to \infty\) the string frame metric becomes flat:

\[
e^{2\varphi} \to e^{2\varphi_\infty} = \sqrt{4h_0 d_{ABC} H^A H^B H^C}
\]

On the IIA side one has to put the constraint

\[
4h_0 d_{ABC} H^A H^B H^C = 1
\]

in order to ensure that the black hole metric is asymptotic to the standard Minkowski metric for \(r \to \infty\).

On the IIA side the Bekenstein-Hawking entropy \(S_{BH}\), the minimized central charge \(Z_{\text{min}}\) and area \(A\) of the event horizon can be expressed in terms of the charges

\[
\frac{1}{\pi} S_{BH} = \frac{1}{4\pi} A = |Z_{\text{min}}|^2 = \sqrt{4q_0 d_{ABC} p^A p^B p^C}
\]

It is natural to expect that this quantity will also play an important role on the IIB side. When exploring the behaviour for \(r \to 0\) one indeed finds that \(Z_{\text{min}}\) naturally appears. One can easily verify that the metric (4.67) is invariant under the transformation

\[
r \to \frac{|Z_{\text{min}}|}{r}, \quad h_0 \to \frac{q_0}{|Z_{\text{min}}|}, \quad q_0 \to h_0|Z_{\text{min}}|, \quad h^A \to \frac{p^A}{|Z_{\text{min}}|}, \quad p^A \to h^A|Z_{\text{min}}|.
\]

Note that one has to transform charges into asymptotic moduli and that the minimized central charge replaces the charge appearing in the formula (1.22) for the D-instanton of dilaton supergravity.

Let us now specialize to the double extreme case that we already discussed from the IIB side in section 4.2.1. This case is defined by setting the scalars \(z^A\) to constant values, which implies that the asymptotics of the harmonic functions is fixed by the charges, i.e. \(h_0\) and \(h^A\) can be expressed in terms of \(q_0\) and \(p^A\):

\[
h_0 = \frac{q_0}{|Z_{\text{min}}|}, \quad h^A = \frac{p^A}{|Z_{\text{min}}|}.
\]
Note that this is consistent with (4.75) because it just puts the asymptotic moduli $h_0, h^A$ to their fixed points under (4.75). In other words: The IIB fixed point values of the moduli, i.e. fixed points under the space inversion, are the same as attractor fixed points in IIA which guarantee full supersymmetry on the event horizon. The scalars $z^A$ take the constant values $z^A = i^{2q_0 p^A} \sqrt{4q_0 d_{ABC} p^B p^C} p^A p^B p^C$ and the dilaton is

$$e^{2\phi} = \left(1 + \frac{|Z_{\text{min}}|}{r^2}\right)^2$$

(4.77)

This is precisely what we got for the simple D-instanton solution with $e^{\phi}\infty = 1$ and the original $q_0$ replaced by $|Z_{\text{min}}| = (4q_0 d_{ABC} p^A p^B p^C)^{1/4}$.

The reason that we get $e^{\phi}\infty = 1$ is that we started with a black hole which was asymptotically flat thanks to the constraint (4.73).

Now the solution has the simple inversion symmetry

$$r \to \frac{|Z_{\text{min}}|}{r},$$

(4.78)

because the moduli are at their fixed point values.

The neck of the wormhole is localized at the selfdual radius $r = |Z_{\text{min}}|^{1/2}$, i.e. the IIA central charge indeed characterizes the wormhole geometry.

**Non-axionic Calabi-Yau black holes and D-instantons**

The non-axionic black hole and D-instanton solutions that we discuss in this section are similar to the four-dimensional configurations that we got in section 2 by toroidal compactification of the ten-dimensional $(0, 4, 4, 4)$ solution of IIA and the $(-1, 3, 3, 3)$ solution of IIB. We will now explain this in some more detail.

Let us recall the relation between ten- and four-dimensional theories in generic terms, without specifying at the moment whether the Ricci-flat manifold we compactify on is a torus or a Calabi-Yau threefold. For definiteness and simplicity we will assume that we study axion-free solutions that carry one electric and several magnetic charges. This is the case for all the solutions discussed in sections 2 and in sections 4.1 and 4.2.

The relation between ten- and four-dimensional R-R gauge fields is provided by decomposing the various R-R $p$-forms into a four-dimensional space-time and a six-dimensional internal part. Massless four-dimensional gauge fields are obtained if the internal part is a harmonic differential form. Thus the number of massless four-dimensional gauge fields is determined by topological data, namely the Betti numbers of the internal manifold. On the other hand the R-R charges of the four-dimensional theory come from the charges carried by the various D-$p$-branes of the ten-dimensional theory. Upon compactifications the branes are wrapped around homology cycles of the internal manifold. Poincare duality between non-trivial cycles and harmonic differential forms ensures that the number of charges equals the number of gauge fields in four dimensions, which is of course needed for consistency. Finally all parameters describing marginal deformations of the internal metric and other internal fields appear as moduli scalars in the four-dimensional theory.

Let us for definiteness consider a configuration which consists of 3-branes which are wrapped around 4-cycles of the internal manifold together with $(-1)$ branes located at points (0-cycles). The simplest case, with the internal manifold being a torus was discussed in section 2. Note that the complete worldvolume,
including the world-sheet time direction is wrapped, so that the configuration is an instanton from
the four-dimensional point of view. Taking $q_0$ $(-1)$-branes together with $p^A$ 3-branes in the $A$-th
primitive homology class, one gets a four-dimensional solution with electric charge $q_0$ and magnetic
charges $p^A$. In the case of a torus, there are three primitive 4-cycles, for which we have specified the
explicit representatives in section 2 (in general the Betti numbers of a $d$-dimensional torus are just
$b_p = d!/(p!(d-p)!))$. Therefore the solution could be expressed in terms of four harmonic functions $H_0,$
$H^1,$ $H^2$ and $H^3$.

General static black hole solutions of four-dimensional $N = 2$ supergravity can be expressed in terms
of harmonic functions by using special geometry [36]. For a Calabi-Yau compactification the number
of harmonic functions $H^A$ will depend on the number of primitive 4-cycles, which is $b_4 = h_{1,1}$. Passing
from cycles to harmonic forms by Poincare duality we can analyse the dimensional reduction of the
gauge fields. From the material recalled in section 3.2 it is obvious that all non-trivial ten-dimensional
R-R gauge fields become four-dimensional R-R scalars which sit in hyper multiplets. One can also look
what happens to the NS-NS scalars that describe deformations of the internal metric. In the toroidal
case we found in section 2 that the solution leads to three non-trivial moduli $T_1,$ $T_2$ and $T_3,$ which are
related to the size of certain 2- and 4-cycles of the internal torus. For the Calabi-Yau case we again
find that the non-trivial moduli appearing in the solution are related to 2- and 4-cycles and using the
material reviewed in section 3 one sees that they sit in hyper multiplets. Moreover one can check that
all the fields in the four-dimensional gravity and vector multiplets descend from ten-dimensional fields
that are trivial in a solution based on $(-1)$-branes and 3-branes. Thus all fields that are non-trivial in
the four-dimensional solution sit in hyper multiplets.

Let us finally consider the limit in which the Calabi-Yau compactification is described by a cubic pre-
potential on the IIA side. We found in the last subsection that the string frame geometry of the IIB
instanton depends on the combination $H_0 d_{ABC} H^A H^B H^C$ of harmonic functions. To explain the relation
with the combination $H_0 H^1 H^2 H^3$ that we found in the toroidal case, we have to recall that the param-
eters $d_{ABC}$ in the case of a Calabi-Yau compactification are related to the triple-intersection numbers
$C_{ABC}$ by $d_{ABC} = \frac{1}{3!} C_{ABC}$. A torus has three primitive homology classes of 4-cycles and its intersection
numbers are $C_{ABC} = 1$ if $A, B, C$ are all different and $C_{ABC} = 0$ otherwise. Thus, dependence of the
dilaton on harmonic functions is the same for both torus and Calabi-Yau compactification (in the limit
of a cubic prepotential).

4.3 Dualizing the general IIA solution

In this section, we will T-dualize (c-map) the general IIA solution. As a result we will find a IIB instanton
solution.

The solution

For our type IIA solution we assume that the scalar fields in the hyper multiplets are trivial, the
corresponding action is given in eq. (3.7). In order to dualize the fields we need an isometry, which is in
Next we dualize the action (3.7) by compactifying over a radius $R$ and decompactifying over $1/R$ (as usually for T-duality, see also [14]). The only subtlety in this procedure occurs from the gauge field part. For the curvature and scalar part the standard rules give (including surface terms and in order to avoid confusions, we will often suppress all space time indices.)

\[
\int d^4x \sqrt{G} \left[ R + 4(\partial \varphi)^2 - 2e^{-2\varphi} \frac{1}{\sqrt{g}} \partial (e^{-\varphi} \partial e^{-\varphi}) - \frac{1}{12}\hat{H}^2 - 2g_{AB} \partial z^A \partial z^B \right].
\]
The new fields are

\[ ds^2 = e^{2\varphi} dx^\mu dx^\nu , \quad e^{2\varphi} = e^{-2U} , \]  

\[ \hat{H}_{0mn} = \partial_m \omega_n - \partial_n \omega_m . \]  

As a next step, we have to consider the gauge field part. First one has to go to three dimensions. Using standard formulae for the dimensional reduction (see e.g. [37]) we obtain for the 3-d Lagrangian

\[ S_3 = \int d^3x \sqrt{G} e^{-2\varphi} \left[ R + 4(\partial \varphi)^2 - \frac{1}{12} \hat{H}^2 - 2e^{2\varphi} \frac{1}{2} \partial(z^A \partial \hat{z}^B) - 2g_{AB} \partial \hat{z}^A \partial \hat{z}^B - \right. \]

\[ -e^{2\varphi} \Im N_{IJ} \partial \zeta^I \partial \zeta^J - \frac{1}{12} e^{2\varphi} (\hat{H}^I - \sqrt{2} \zeta^I \hat{\bar{H}}) \Im N_{IJ} (\hat{H}^J - \sqrt{2} \zeta^J \hat{\bar{H}}) + \]

\[ +\sqrt{2} e^{2\varphi} \Re N_{IJ} \partial \zeta^I \partial \zeta^J (\hat{H}^J - \sqrt{2} \zeta^J \hat{\bar{H}}) \]  

(4.88) 

with

\[ A^I_0 = \sqrt{2} \zeta^I , \quad F^I_{mn} = \frac{1}{2} \epsilon_{mnp} \partial_p \hat{H}^I , \quad (d\omega)_{mn} = \partial_m \omega_n - \partial_n \omega_m . \]  

(4.89) 

After decompactification over the inverse radius, i.e. \( G_{00} \to 1/G_{00} \) and adding the curvature part (4.86) we find for the 4-d theory

\[ S_4 = \int d^4x \sqrt{G} e^{-2\varphi} \left[ R + 4(\partial \varphi)^2 - \frac{1}{12} \hat{H}^2 - 2e^{2\varphi} \frac{1}{2} \partial(z^A \partial \hat{z}^B) - 2g_{AB} \partial \hat{z}^A \partial \hat{z}^B - \right. \]

\[ -e^{2\varphi} \Im N_{IJ} \partial \zeta^I \partial \zeta^J - \frac{1}{12} e^{2\varphi} (\hat{H}^I - \sqrt{2} \zeta^I \hat{\bar{H}}) \Im N_{IJ} (\hat{H}^J - \sqrt{2} \zeta^J \hat{\bar{H}}) + \]

\[ +\sqrt{2} e^{2\varphi} \Re N_{IJ} \partial \zeta^I \partial \zeta^J (\hat{H}^J - \sqrt{2} \zeta^J \hat{\bar{H}}) \]  

(4.90) 

where the new antisymmetric tensors are

\[ \hat{H}^I_{mn} = F^I_{mn} , \quad (\ast \hat{H})^\mu = \frac{1}{6} \sqrt{G} \epsilon^{\mu \nu \rho \lambda} \hat{H}_{\nu \rho \lambda} . \]  

(4.91) 

This is the type IIB action in the string frame. We go to the Einstein frame with the relation,

\[ g_{E \mu \nu} = e^{-2\varphi} g_{\mu \nu} = \delta_{\mu \nu} . \]  

(4.92) 

Again, including all surface terms one obtains

\[ S_4^E = \int d^4x \left[ -4 \partial^2 \varphi - 2(\partial \varphi)^2 - 2g_{AB} \partial z^A \partial \hat{z}^B - e^{2\varphi} \Im N_{IJ} \partial \zeta^I \partial \zeta^J - \right. \]

\[ -\frac{1}{12} e^{-4\varphi} \hat{H} - \frac{1}{12} e^{-2\varphi} \left( \hat{H} - \sqrt{2} \zeta^I \hat{\bar{H}} \right) \Im N_{IJ} \left( \hat{H} - \sqrt{2} \zeta^J \hat{\bar{H}} \right) + \]

\[ +\sqrt{2} \Re N_{IJ} \partial \zeta^I \partial \zeta^J \left( \hat{H} - \sqrt{2} \zeta^J \hat{\bar{H}} \right) \]  

(4.93) 

Using our supersymmetric configuration (4.79), the IIB antisymmetric tensors can be written in a covariant way, namely,

\[ H^{\mu \nu \lambda} = \frac{1}{2} e^{\mu \nu \lambda \partial} \hat{H} , \quad \hat{H}^{\mu \nu} = \frac{1}{2} e^{\mu \nu \lambda \partial} \partial \hat{H} \]  

(4.94) 

where \( \hat{H} \) is given in (4.83). Like on the IIA side, the Bianchi identities are only fulfilled if all \( H \)’s are harmonic. This formula already suggests that on the type IIB side we can relax the isometry along the \( t \)-direction by allowing harmonic functions depending on all four coordinates. As in 10 dimensions, this localizes the solution in all transversal directions, i.e. for the instanton, also in the time direction. If the harmonic functions depend only on three coordinates we recover the expressions given in (4.87) and (4.91).
As next step we dualize the antisymmetric tensors \((\mathcal{H}^I, \hat{\mathcal{H}})\) into further scalar fields, where we use the same notation as introduced in section 3. In doing this, we have to first add a Lagrange multiplier term,

\[
\delta S = -\int \left[ (\dot{\phi} - \zeta I \dot{\zeta}_I) \partial^* \mathcal{H} + \sqrt{2} \dot{\zeta}_I \partial^* \mathcal{H}^I \right] = \int \left[ \partial(\dot{\phi} - \zeta I \dot{\zeta}_I) \partial^* \mathcal{H} + \sqrt{2} \partial \dot{\zeta}_I \partial^* \mathcal{H}^I \right] - \int \partial \left[ (\dot{\phi} - \zeta I \dot{\zeta}_I) \partial^* \mathcal{H} + \sqrt{2} \dot{\zeta}_I \partial^* \mathcal{H}^I \right].
\]

(4.95)

to the previous action. Note that the second term is a further total derivative term. Integrating \(\dot{\phi}\) and \(\dot{\zeta}_I\) yields the Bianchi identities. On the other hand taking the variations with respect to the torsions yields

\[
\mathcal{H}^I - \sqrt{2} \zeta^I \dot{\mathcal{H}} = -\sqrt{2} e^{2\varphi} \Im N^I \partial(\partial \dot{\zeta}_I + \Re N_{IL} \partial \zeta^L).
\]

(4.96)

After inserting this field into the action and varying \(\dot{\mathcal{H}}\) we find

\[
\dot{\mathcal{H}} = -e^{4\varphi} \partial(\dot{\phi} + \zeta^I \partial \zeta_I - \tilde{\zeta} \partial \zeta).
\]

(4.97)

So, we end up with the 4-d dualized Einstein action

\[
S_E^I = \int d^4x \left[ -2(\partial \varphi)^2 - 2g_{AB} \partial z^A \partial \bar{z}^B - e^{2\varphi} \partial \zeta^I \Im N_{IJ} \partial \zeta^J + e^{2\varphi}(\Re N_{IL} \partial \zeta^L + \partial \tilde{\zeta}_I) \Im N_{IJ} (\Re N_{JK} \partial \zeta^K + \partial \tilde{\zeta}_J) + \frac{1}{2}e^{4\varphi}(\partial \dot{\phi} + \zeta^I \partial \zeta_I - \tilde{\zeta} \partial \zeta)^2 \right] + S_{BM}.
\]

(4.98)

The surface terms are

\[
S_{BM} = -\int d^4x \partial \left[ 4 \partial \varphi + 2 e^{2\varphi} \tilde{\zeta}_I \Im N^I \partial z^I (\Re N_{IL} \partial \zeta^L + \partial \tilde{\zeta}_I) + e^{4\varphi}(\tilde{\phi} + \tilde{\zeta} \zeta^I)(\partial \dot{\phi} + \zeta^I \partial \zeta_I - \tilde{\zeta}^I \partial \zeta_I) \right].
\]

(4.99)

Inserting the expressions for the antisymmetric tensors as given in (4.94), we can express the IIB solution in terms of the IIA fields defined in (4.79) and (4.80)

\[
e^{2\varphi} = e^{-2U}, \quad \sqrt{2} \zeta^I = A^I_0, \quad z^A = \frac{X^A}{N^{2\varphi}},
\]

\[
2 \partial \dot{\phi} = -e^{-2\varphi} \partial \mathcal{H} + \sqrt{2} \dot{\tilde{\zeta}} \partial A^I_0 - \sqrt{2} A^I_0 \partial \dot{\tilde{\zeta}},
\]

\[
\sqrt{2} \partial \dot{\tilde{\zeta}} = -\frac{1}{2}e^{-2\varphi} \Im N_{IJ} (\partial H^I - A^I_0 \partial \tilde{\mathcal{H}}) - \Re N_{IJ} \partial A^I_0,
\]

(4.100)

where \(\mathcal{H}\) is defined in (4.83). Since \(\tilde{\phi}\) and \(\tilde{\zeta}\) are scalars that appear via dualization of antisymmetric tensors, we have only an expression for their derivatives. The integrability constraint (4.82) ensures that one can integrate these equations. Again, off-shell it is supersymmetric for any functions \((H^I, H_I)\), but on-shell they have to be harmonic, see Bianchi identity for the torsion in (4.94).

5 Conclusions

In this paper we discussed the T-duality between the extremal black hole type solutions in compactified four-dimensional IIA strings and and the D-instanton solutions in IIB superstrings compactified on the same internal Calabi-Yau space. In the context of the effective four-dimensional \(N = 2\) supergravity, the T-duality precisely acts like the c-map which exchanges the special Kähler moduli space of the type IIA
vector multiplets with the quaternionic moduli space of the IIB hyper multiplets. Hence, the IIA black holes are the BPS solutions of the special Kähler Lagrangian, whereas the D-instantons arise as the solutions of the dual quaternionic σ-model. This scenario opens room for several interesting discussions which are worth exploring further. For example we find that our wormhole solutions connect two flat regions, where one of them is strongly coupled and both regions are related by a symmetry of the solution. It would be important to clarify the precise meaning of the IIA black hole entropy in the context of the IIB D-instantons. We observed that for the double-extreme case (with extremized central charge) the black hole entropy becomes the self-dual radius in the D-instanton wormhole geometry; so we might ask what precisely happens at the self-dual wormhole radius. And the relation of the self-dual radius to the microscopic D-brane counting is still mysterious. Finally the D-instanton solutions we found will be relevant for the computation of stringy instanton effects along the lines of [21, 13].

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