Difference Map for Coded-Aperture Phasing Has Unique Fixed Point

Albert Fannjiang

1Department of Mathematics, UC Davis, CA 95616

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Difference Map (DM) for phase retrieval with a random mask is analyzed. DM is a general class of iterative phasing schemes including Hybrid-Projection-Reflection (HPR) and Douglas-Rachford (DR) algorithms. For rank-2 complex objects whose pixel values are limited to a convex sector in the complex plane, it is proved that the fixed point is unique with probability exponentially close to unity with respect to the number of nonzero pixels of the object.

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1. Introduction

Phase retrieval is the task of extracting the Fourier phase information from the Fourier magnitude data. This so called phase problem in crystallography is fundamental in optics and many imaging applications.

The main challenge of phase retrieval is to resolve the ambiguities resulting from the loss of Fourier phase information. Several difficulties arise in connection to Fourier magnitude-only measurements (the diffraction patterns in coherent diffractive imaging). First, even with oversampling of the diffraction pattern in the absence of noise [13], both trivial and nontrivial ambiguities generally exist as Fourier magnitude data contain enough information to determine only the object’s autocorrelation, but not the object itself [10]. Second, the mainstream reconstruction schemes, the iterative transform algorithms [9,12], are based on the idea of projections onto convex sets (POCS) and proceed by alternately enforcing the object domain constraints (positivity, support constraint etc) and the Fourier magnitude constraint [1]. But unfortunately the Fourier magnitude data constitute a non-convex constraint. Hence the convergence of these schemes can not be ascertained a priori.

The dual problems of non-convexity and the presence of multiple fixed points such as the twin and shifted objects (trivial ambiguities) often significantly deteriorate numerical performance of the phasing schemes, causing stagnation or non-convergence in the iterative procedure.

In the previous work [5-7] we have studied phase retrieval with masked Fourier magnitude measurement. Given an object represented by the function \( f(n) : n = (n_1, n_2) \in \mathcal{N} \) where \( \mathcal{N} = \{0 \leq n_1 \leq N_1, 0 \leq n_2 \leq N_2\} \), the effect of the mask represented by the function \( \mu(n) : n = (n_1, n_2) \in \mathcal{N} \) is multiplicative, resulting in a masked object \( \mu(n)f(n) : n \in \mathcal{N} \). Mask-aided phase retrieval is then to reconstruct \( f \), which satisfies the object domain constraints, from the Fourier magnitudes of the masked object \( \mu f \) as well as (complete or partial) knowledge of \( \mu \).

When the mask function \( \mu \) is randomly generated and perfectly known, this is a form of coded aperture imaging and results in uniqueness of phasing solution [3], rapidly convergent numerical process, stability with respect to various kinds of noise [6] as well as reduced dynamic range in the measured data (Fig. 1). Moreover, when the mask function \( \mu \) is only roughly known, we have extended the uniqueness result and introduced numerical schemes with excellent performance for mask uncertainty greater than 50% [7].

In short, the (trivial and nontrivial) ambiguities...
in standard phase retrieval are essentially removed by mask-aided phase retrieval. In the present work we go one-step further to prove the uniqueness of fixed point for a general class of iterative transform algorithms, called the Difference Map (DM), for mask-aided phase retrieval with perfect knowledge of the random mask. The method is elementary and based on the uniqueness theorems in [5, 6]. This would resolve for DM the aforementioned problem with multiple fixed points.

2. Mask-aided phase retrieval

Fig. 1: Conceptual layout of coherent lensless imaging with a random mask (left) before (for random illumination) or (right) behind (for wavefront sensing) the object (phantom). (middle) The diffraction pattern measured without a mask has a larger dynamic range. The color bar is on a logarithmic scale.

In mask-aided phase retrieval (Fig. 1) the diffraction pattern is given by

\[
|F(\omega)|^2 = \sum_{n=-[N_1,N_2]} C(n)e^{-i2\pi n \cdot \omega}
\]

which is the Fourier transform of the autocorrelation of the masked object

\[
C(n) = \sum_{m \in \mathcal{N}} (\mu f)(m+n)(\mu f)(m).
\]

Hence by sampling the diffraction pattern on the doubly refined grid \( \mathcal{L} \) defined by

\[
\{(\omega_1, \omega_2) \mid \omega_j = \frac{-N_j}{2N_j+1}, \frac{-N_j+1}{2N_j+1}, \ldots, \frac{N_j}{2N_j+1}\},
\]

we can recover the autocorrelation function \( C \). The remaining task is to recover \( f \) from its autocorrelation function, the object domain constraints and the knowledge of \( \mu \).

The number of grid points in \( \mathcal{L} \) is roughly 4 times the number of pixels in \( \mathcal{N} \), which is the standard oversampling ratio [13]. To utilize the oversampled data in iterative transform algorithms it is convenient to extend the original object \( f \), by zero-padding, to the bigger grid \( \tilde{N} = \{ -N_1 \le n_1 \le N_1, -N_2 \le n_2 \le N_2 \} \) with \( \mathcal{N} \) as the support. This is called zero-padding, a form of support constraint and part of the object domain constraints in iterative phasing procedures.

Accordingly, let \( \Phi \) be the discrete Fourier transform from the extended domain \( \tilde{N} \) to \( \mathcal{L} \). Let the mask function \( \mu \) be (arbitrarily) extended to \( \tilde{N} \) and let the matrix \( \mathcal{M} = \text{diag}(\mu) \) represent the action of mask. We assume \( \mu(n) \neq 0, \forall n \in \tilde{N} \), almost surely.

We shall consider the class of complex objects whose wrapped phases \( \angle f(n) \) (i.e. the principal values of \( \arg \{ f(n) \} \)) to a sector \([-\alpha \pi, \beta \pi], a, b \le 1, a + b \ge 0 \). For example, in the X-ray spectrum most object transmission functions \( f \) have positive real and imaginary parts and hence satisfy the sector constraint with \( a = 0, b = 1/2 \). The two-parameter family of sector constraints serves as transition between the positivity constraint \( (a = b = 0) \) and the null constraint \( (a = b = 1) \).

Uniqueness for mask-aided phase retrieval takes various forms dependent on the object constraints [5]. Below we state the uniqueness theorem for the sector condition with \( a + b < 2 \) (5, Theorem 4(i)) based on which we shall prove uniqueness of fixed point.

**Proposition 1.** Let the object \( f \) be rank-2 and satisfy the sector condition. Suppose that \( \mu \neq 0 \) almost surely and \( \angle \mu(n) \) are independent, uniform random variables on \([-\pi, \pi]\). Let \( g \) be another complex object satisfying the same sector condition and \( \| \mathcal{M} f \| = \| \mathcal{M} g \| \) on \( \mathcal{L} \). Then \( g = e^{i\theta} f \), for some real constant \( \theta \), with probability at least

\[
1 - N_1 N_2 \| S/2 \|_2 \ge (1)
\]

where \( S \) (the sparsity) is the number of the nonzero pixels in \( f \) and \( \| \cdot \|_2 \) is the Gauss symbol.

When the sector condition is tight, i.e. \( -\alpha \pi = \min_{n \in \mathcal{N}} \angle f(n), b \pi = \max_{n \in \mathcal{N}} \angle f(n), \) then \( g = f \) (i.e. \( \theta = 0 \)) with probability at least given by [7].

**Remark 1.** For real, nonnegative objects \( a = b = 0 \) the probability for uniqueness is 1. For arbitrary
complex objects \((a = b = 1)\) one needs an additional independent diffraction pattern to uniquely determine the object \([5\), Theorem 6].

**Remark 2.** The lower bound \([1]\) for the uniqueness probability suggests that success rate in phase retrieval increases (exponentially) with the object sparsity but decreases with the range of object phases.

3. Difference map

Define the data fitting operator \(\mathcal{T}\) as

\[
\mathcal{T}G(\omega) = \begin{cases} 
\Phi M f(\omega) e^{i \angle G(\omega)} & \text{if } |G(\omega)| > 0 \\
\Phi M f(\omega) & \text{if } |G(\omega)| = 0 
\end{cases}.
\]

Let

\[ P_m = \mathcal{M}^{-1} \Phi^{-1} \mathcal{T} \Phi \mathcal{M} \]

be the masked projection onto the Fourier domain constraint. \(\mathcal{M}^{-1}\) is well-defined since \(\mu\) vanishes nowhere.

Let \(P_o\) be the projection onto the object domain constraints including zero-padding and the sector condition. Below we will consider only the case of convex sector with \(a + b \leq 1\). Specifically, \(P_o\) is defined as follows.

**Zero-padding:** For \(n \in \mathbb{N} \setminus \mathbb{N}', P_o g(n) = 0\).

**Sector condition:** For \(n \in \mathbb{N}',\)

\[
P_o g(n) =
\begin{cases} 
g(n) & \text{if } \angle h(n) \in [-a \pi, b \pi] \\
\mathbb{R} \{g(n) e^{-i b \pi} e^{i b \pi} & \text{if } \angle h(n) \in [b \pi, (b + 1/2) \pi] \\
\mathbb{R} \{g(n) e^{i a \pi} e^{-i a \pi} & \text{if } \angle h(n) \in [(a + 1/2) \pi, -a \pi] \\
0 & \text{else}
\end{cases}
\]

where \(\mathbb{R}\) stands for the real part.

The Difference Map \(\mathcal{D}\) is defined as follows. Let \(\mathcal{D} = I + \beta \Delta\)

with

\[
\Delta = P_o ((1 + \alpha_2)P_m - \alpha_2 I) - P_m ((1 + \alpha_1)I - \alpha_1 P_o)
\]

where \(\beta \neq 0, \alpha_1, \alpha_2\) are three real parameters. When \(\alpha_1 = 0\) and \(\alpha_2 = 1/\beta\), the difference map reduces to the one-parameter family of Hybrid-Projection-Reflection (HPR) schemes \([2\):

\[
I + \beta (P_o ((1 + 1/\beta)P_m - 1/\beta I) - P_m)
\]

(3)

In particular, with \(\beta = 1\), HPR is the same as the Douglas-Rachford (DR) algorithm in convex optimization \(\frac{1}{2} (I + R_o R_m)\) where \(R_o = 2P_o - I\) and \(R_m = 2P_m - I\) are reflectors \([1]\).

The widely used standard algorithm Hybrid-Input-Output (HIO) \([6, 8]\) can be viewed as a member of HPR if the object domain constraints are limited to the support constraint such as zero-padding \([11]\) but do not include the positivity or sector condition. The most basic iterative transform algorithm, Error Reduction (ER), defined as iteration of \(P_o P_m\) \([8\), is definitely not a member of the Difference Map.

Now we state the main result of the paper.

**Theorem 1.** Suppose \(\alpha_1 \geq 0\) and \(\alpha_2 \neq 0\). Let \(f_\ast\) be a fixed point of the Difference Map \(\mathcal{D}\). Then, under the assumptions of Proposition \([7\) with \(a + b \leq 1\), \(f_\ast = e^{i \theta} f\) for some real constant \(\theta\) with probability at least given by \([7]\).

In the case of tight sector condition, \(f_\ast = f\) with probability at least given by \([7]\).

**Remark 3.** With \(a + b \leq 1\), the probability lower bound \([1]\) is at least \(1 - N_1 N_2 2^{-||x||/2}\).

**Proof.** Let \(f_\ast\) be a fixed point of \(\mathcal{D}\), i.e. \(\mathcal{D} f_\ast = f_\ast\) or equivalently

\[
P_o ((1 + \alpha_2)P_m - \alpha_2 I) f_\ast = P_m ((1 + \alpha_1)I - \alpha_1 P_o) f_\ast
\]

Setting

\[
g = ((1 + \alpha_1)I - \alpha_1 P_o) f_\ast
\]

(4)

\[
h = ((1 + \alpha_2)P_m - \alpha_2 I) f_\ast
\]

(5)

we obtain that \(P_o h = P_m g\), namely \(\mu P_o h\) shares the same Fourier magnitude as does \(\mu f\) and the same Fourier phase as does \(\mu g\). In other words

\[
|\Phi M P_o h| (\omega) = |\Phi M f| (\omega), \ \omega \in \mathcal{L}
\]

(6)

\[
\angle \Phi M P_o h(\omega) = \angle \Phi M g(\omega), \ \omega \in \mathcal{L}
\]

(7)

Since, by the very definition of \(P_o\), \(P_o h\) satisfies the object domain constraints, \([6]\) implies that \(P_o h = e^{i \theta} f\) for some real constant \(\theta\) with probability at least given by \([1]\) \((\text{Proposition 1})\).

Also we recall the uniqueness theorem for magnitude retrieval \([6, 10]\)

**Proposition 2.** Let the mask function \(\mu\) be as in Proposition \([7]\). If two complex objects \(h\) and \(g\) satisfy \(\angle \Phi M h = \angle \Phi M g\), then almost surely \(g = \mu h\) for some positive number \(c\).
Remark 4. The original version with $M = I$ was proved in [7] under the additional assumption that the $z$-transform of the object has no conjugate symmetric factors. With a random mask, it can be proved that the $z$-transform of the masked object almost surely has no conjugate symmetric factors ([2], Lemmas 1–3), hence Proposition 2 follows.

By Proposition 2 ([7]) implies that $g = cP_0h$ where $c$ is a positive constant. Combining the two identities, we have $e^{i\theta}f = g/c$.

Substituting $g = ce^{i\theta}f$ into (4) gives

$$(1 + \alpha_1)f_s = \alpha_1P_0f_s + ce^{i\theta}f.$$  \hspace{1cm} (8)

Since both $e^{i\theta}f$ and $P_0f_s$ satisfy the object domain constraints which are convex for $a+b \leq 1$, so does $f_s$ as $\alpha_1, c \geq 0$. Hence $f_s = P_0f_s$ implying $f_s = ce^{i\theta}f$.

We claim that $c = 1$. This can be seen as follows. Substituting $f_s = ce^{i\theta}f$ into (5), we obtain

$$h = (1 + \alpha_2)P_m(ce^{i\theta}f) - c\alpha_2e^{i\theta}f$$

$$= (1 + \alpha_2)e^{i\theta}f - c\alpha_2e^{i\theta}f$$

$$= (1 + (1-c)\alpha_2)e^{i\theta}f.$$  \hspace{1cm} (9)

Note that $1 + (1-c)\alpha_2 > 0$. Otherwise $P_0h \equiv 0$, by convexity of the sector condition, which contradicts $P_0h = e^{i\theta}f$ and the fact that $f$ is rank-2. Hence $1 + (1-c)\alpha_2 > 0$ and

$$P_0h = (1 + (1-c)\alpha_2)e^{i\theta}f = e^{i\theta}f$$

implying $c = 1$ as $\alpha_2 \neq 0$ by assumption. Therefore $f_s = e^{i\theta}f$ as claimed.

The case of tight sector condition follows the same argument. \hfill \Box

4. Conclusions

We have proved a probabilistic version of uniqueness of fixed point for the Difference Map, including HPR and DR, with the set-up of phase retrieval with a random mask. This suggests the dynamics of the iterated Difference Map for phasing with a randomly coded aperture may be relatively simple.

It is unclear if the uniqueness result can be extended to HIO and ER. Partial result for ER toward this direction was previously established in [9] (Theorem 4). Our extensive numerical experiments show global convergence to a unique fixed point for both ER and HIO with a randomly coded aperture [5, 6].

Uniqueness of fixed point is a first but important step toward full understanding of the iterative phasing process. The question of global convergence for the Difference Map likely requires a different set of techniques to resolve. By contrast, convergence for ER is relatively easy to prove because of monotonicity in the dynamics of the residual [6, 8].

In addition, the convergence property would likely impose more restriction on the parameter space than that for uniqueness of fixed point ($\alpha_1 \geq 0, \alpha_2 \neq 0$) as well as stricter condition on $a+b$.

For example, the choice of parameters $\alpha_2 = 1/\beta = 1 + \alpha_1$ has been suggested to achieve optimal performance on a heuristic ground [3, 4]. Also our previous numerical experiments [6, 7] showed global convergence to the unique fixed point for HPR (with $\beta \approx 1$) with randomly coded aperture. Fig. 2 shows the HPR, [3] with $\beta = 0.9$, reconstruction residual $\|P_mf_k - f_k\|_2/\|f\|_2$ as a function of the iteration for the nonnegative phantom $(a+b = 0)$. Monotonicity is evident in both plots. These members of the Difference Map are natural candidates for convergence analysis.

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