ENDOMORPHISM RINGS OF MODULES OVER PRIME RINGS

Mohammad Baziar* and Christian Lomp

Abstract. Endomorphism rings of modules appear as the center of a ring, as the fix ring of a ring with group action or as the subring of constants of a derivation. This note discusses the question whether certain \#-prime modules have a prime endomorphism ring. Several conditions are presented that guarantee the primeness of the endomorphism ring. The contours of a possible example of a \#-prime module whose endomorphism ring is not prime are traced.

1. INTRODUCTION

Endomorphism rings of modules appear in many ring theoretical situations. For example the center $C(R)$ of a (unital, associative) ring $R$ is isomorphic to the endomorphism ring of $R$ seen as a bimodule over itself, i.e. as a left $R \otimes R^{op}$-module. The subring $R^G$ of elements that are left invariant under the action of a group $G$ on $R$ is isomorphic to the endomorphism ring of $R$ seen as a left module over the skew group ring $R \ast G$. The subring $R^\partial$ of constants of a derivation $\partial$ of $R$ is isomorphic to the endomorphism ring of $R$ seen as a left module over its differential operator ring $R[x, \partial]$. More generally the subring $R^H$ of elements invariant under the action of a Hopf algebra $H$ acting on $R$ is isomorphic to the endomorphism ring of $R$ seen as left module over the smash product $R\#H$. This identifications motivated the use of module theory in the study of Hopf algebra actions in [4, 10, 11, 12].

Prime numbers and prime ideals are basic concepts in algebra. While the idea of a prime ideal is well established, the idea of a prime submodule of a module is not. The purely essence of a prime ideal had been destilled already by Birkhoff in the concept...
of a prime element in a partially ordered groupoid. In [3], Bican et al. introduced an operation on the lattice of submodules of a module, turning it into a partially ordered groupoid. Let \( R \) be any (associative, unital) ring and \( M \) a left \( R \)-module. For any submodules \( N, L \) we denote

\[
N \ast L = N \text{Hom}(M, L) = \sum \{(N) f \mid f : M \to L\}.
\]

Note that we will write homomorphisms opposite of scalars, i.e. on the right side of an element. A submodule \( P \) is a prime element in \( M \) if for any two submodules \( N, L \) of \( M \)

\[
N \ast L \subseteq P \Rightarrow N \subseteq P \quad \text{or} \quad L \subseteq P.
\]

Those modules whose zero submodule is a prime element had been termed \( \ast \)-prime modules, i.e. \( N \ast L \neq 0 \) for all non-zero \( N, L \subseteq M \). Of course for \( M = R \), the \( \ast \)-product equals the product of left ideals and \( R \) is a \( \ast \)-prime \( R \)-module if and only if it is a prime ring. The meaning of the module theoretic prime concept for a ring \( R \) with Hopf algebra action \( H \) seen as left \( R \# H \)-module has been studied in [12] in connection with an open question in this area, due to Miriam Cohen, asking whether \( R \# H \) is a semiprime algebra provided \( R \) is semiprime and \( H \) is semisimple (see [5]).

The main purpose of this note is to shed new light into the following question which had been left open in [12]:

**Question.** Is the endomorphism ring of a \( \ast \)-prime module a prime ring?

From [12, Proposition 4.2] it is known that the answer is yes, if the \( \ast \)-prime module \( M \) satisfies a light projectivity condition. Although we were unable to answer this question completely we will indicate various sufficient conditions for a \( \ast \)-prime module to have a prime endomorphism ring which narrows down the class of possible examples that could provide a negative answer.

Let \( M \) be a left \( R \)-module. \( S = \text{End}_R(M) \) shall always denote the endomorphism ring of \( M \). Since any \( \ast \)-prime module \( M \) has a prime annihilator ideal \( \text{Ann}(M) \) and since \( \text{Hom}_R(M, N) = \text{Hom}_{R/\text{Ann}(M)}(M, N) \) holds for any submodule \( N \) of \( M \), we will assume throughout this note that \( M \) is a faithful left module over a (unital, associative) prime ring \( R \).

### 1.1. Retractable modules

A \( \ast \)-prime module \( M \) is **retractable**, i.e. \( \text{Hom}(M, K) \neq 0 \) whenever \( 0 \neq K \subseteq M \). Note that it is always true that a retractable module with prime endomorphism ring is a \( \ast \)-prime module (see [12, Theorem 4.1]) and our question is whether this sufficient condition is also necessary. The retractability condition (called **quotient like** in [9] and **slightly compressible** in [14]) stems from the non-degeneration of the standard Morita context \((R, M, M^*, S)\) between a ring \( R \) and the endomorphism ring \( S \) of a module \( M \) via \( M^* = \text{Hom}(M, R) \) (see [17]). In the case of a group \( G \) acting on a ring \( R \), the
retractability of $R$ as $R \ast G$-module says that every non-zero $G$-stable left ideal contains a non-zero fixed element. The Bergman-Isaacs theorem [2] says that $R$ is retractable as left $R \ast G$-module if $G$ is a finite group acting on a semiprime ring $R$ such that no non-zero element of $R$ has additive $|G|$-torsion. This fact had been used by Fisher and Montgomery in [8] to prove that $R \ast G$ is semiprime provided $R$ is semiprime and has no $|G|$-torsion, which originally with [6] motivated Cohen’s question for Hopf algebra actions.

For a locally nilpotent derivation $\partial$ of a ring $R$ it had been shown in [4, Lemma 3.8] that $R$ is always retractable as $R[x, \partial]$-module. Rings $R$ that are retractable as $R \otimes R^{op}$-module are those whose non-zero ideals contain non-zero elements like for example in the case of semiprime PI-rings ([13, Theorem 2]), central Azumaya rings ([15, 26.4]) or enveloping algebras of semisimple Lie algebras ([7, 4.2.2]). The retractability condition can be expressed by saying that the function from the lattice of left $R$-submodules of the module $M$ to the lattice of left ideals of $S$ defined as $N \mapsto \text{Hom}(M, N)$ for submodules $N$ of $M$ has the property that the only submodule mapped to the zero left ideal of $S$ is the zero submodule.

1.2. **Endoprime modules**

It is known by [12, 1.3] that the endomorphism ring of a right $R$-module $M$ is prime if and only if $\text{Hom}(M/N, M) = 0$ for all non-zero fully invariant, $M$-generated submodules $N$ of $M$. With slightly different notation, Haghany and Vedadi defined a module $M$ to be endoprime if $\text{Hom}(M/K, M) = 0$ for all non-zero fully invariant submodules $K$ of $M$ (see [9]). Thus endoprime modules have a prime endomorphism ring. Note that $\text{Hom}(M, K)\text{Hom}(M/K, M) = 0$ holds for all submodules $K$ of $M$. Hence a retractable module $M$ with prime endomorphism ring $S$ is endoprime. In other words a retractable module has a prime endomorphism ring if and only if it is endoprime. Since $\ast$-prime modules are retractable, our question can be equivalently reformulated to

**Question:** Are $\ast$-prime modules endoprime in the sense of Haghany and Vedadi?

1.3. **Semi-projective modules**

As mentioned before under a light projectivity condition our question has an affirmative answer. Recall from [15] that a module $M$ is called semi-projective if any diagram

\[
\begin{array}{c}
M \\
\downarrow^g \\
M \xrightarrow{f} K \longrightarrow 0
\end{array}
\]

with $K \subseteq M$ can be extended by some endomorphism of $M$. In other words, $M$ is semi-projective if and only if for any endomorphism $f$ of $M$ we have $\text{Hom}(M, (M)f) = Sf$. 

Lemma 1.1. ([12, Proposition 4.2]). A semi-projective module is *-prime if and only if it is a retractable module with prime endomorphism ring.

Let $R$ be a ring and $B \subseteq \text{End}_\mathbb{Z}(R)$ be a subring of the ring of $\mathbb{Z}$-linear endomorphisms of $R$ such that all left multiplications $L_a : R \to R$ defined by $L_a(x) = ax$ for $a, x \in R$ belong to $B$. $R$ becomes naturally a left $B$-module by evaluating of functions. The subring $R^B = \{(1)f \mid f \in B\}$ can be seen to be a generalized subring of $R$ with respect to $B$. It is not difficult to see, that $R^B$ is isomorphic to $\text{End}_B(R)$ (see [11, Lemma 1.8]). This general situation mimics the case of $R$ considered as a bimodule or $R$ considered having a Hopf algebra $H$ acting on it. To ask that $R$ is a semi-projective as $B$-module, is to say that for each $x \in R^B$ one has $R^B x = (Rx) \cap R^B$.

Considering $R$ as a bimodule, we let $B$ to be the subring of $\text{End}_\mathbb{Z}(R)$ generated by all left and right multiplications of elements of $R$. The $B$-module structure of $R$ is identical with the bimodule structure of $R$. Then $R$ is semi-projective as $R \otimes R^{op}$-module if for example all non-zero central elements of $R$ are non-zero divisors in $R$. Because if $x$ is central and $ax$ is central for some $a \in R$, then for any $b \in R$ one has $(ab - ba)x = abx - bax = axb - axb = 0$, i.e. $ab = ba$ and $x$ is central. Thus $Rx \cap C(A) = C(A)x$. In case $R$ is *-prime as $R \otimes R^{op}$-module, $0 \neq x \in C(R)$ and $I = \text{Ann}(x) = \{a \in R \mid ax = 0\}$ is its annihilator, the *-product of $I$ and $Rx$ is given by:

$$I * (Rx) = I \text{Hom}_{R \otimes R^{op}}(R, Rx) = I((Rx) \cap C(R)) \subseteq Ix = 0.$$ 

Since we supposed that $R$ is *-prime and $x \neq 0$, we get $I = 0$. This shows that no non-zero central element of $R$ is a zero-divisor in $R$. Consequently we can state the following

**Corollary 1.2.** A ring $R$ is a *-prime $R \otimes R^{op}$-module if and only if the center of $R$ is an integral domain and large in $R$.

Here we say that a subring $R'$ of $R$ is large in $R$ if any non-zero ideal of $R$ contains a non-zero element of $R'$.

Let $G$ be a group acting on $R$. It is known that $R$ is a projective $R * G$-module if and only if $G$ is a finite group and $|G|1$ is invertible in $R$. Thus in this case $R$ is a *-prime $R * G$-module if and only if $R^G$ is a prime ring.

If $R$ is an algebra over a field $F$ and $\partial$ is a locally nilpotent derivation of $R$ and either $\text{char}(F) = 0$ or $\partial^{\text{char}(F)} = 0$, then $R$ is self-projective as left $R[x, \partial]$-module by [4, Proposition 3.10]. Hence in this situation (using also [4, Lemma 3.8]) $R$ is a *-prime left $R[x, \partial]$-module if and only if $R^G$ is a prime ring.
The purpose of this section is to gather conditions for a \(*\)-prime module to have a prime endomorphism ring. Denote by \(l\text{.ann}_S(I)\) (resp. \(r\text{.ann}_S(I)\)) the left (resp. right) annihilator in \(S\) of an ideal \(I\).

**Theorem 2.1.** The following statements are equivalent for a \(*\)-prime module \(M\) with endomorphism ring \(S\):

(a) \(S\) is prime.

(b) \(S\) is semiprime.

(c) \(l\text{.ann}_S(I) \subseteq r\text{.ann}_S(I)\) holds for any ideal \(I\) of \(S\).

(d) \(gSf = 0 \Rightarrow fSg = 0\) for all \(f, g \in S\).

**Proof.** \((a) \Rightarrow (b) \Rightarrow (c)\) is trivial since the left and right annihilator of an ideal coincide in a semiprime ring. \((c) \Rightarrow (a)\) Suppose that \(IJ = 0\) for two ideals \(I, J\) of \(S\). Then \(M\text{Hom}(M, MI)J \subseteq MIJ = 0\) implies \(\text{Hom}(M, MI)J = 0\). By \((c)\) \(J\text{Hom}(M, MI) = 0\). Hence \((MJ) * (MI) = MJ\text{Hom}(M, MI) = 0\) and since \(M\) is \(*\)-prime, we have \(MI = 0\) or \(MJ = 0\), i.e. \(I = 0\) or \(J = 0\). Thus \(S\) is prime.

Condition \((d)\) is equivalent to saying that

\[
l\text{ann}_S(SfS) = l\text{ann}_S(Sf) \subseteq r\text{.ann}_S(fS) = r\text{ann}_S(SfS)
\]

for all \(f \in S\), which is a consequence of \((c)\). On the other hand, assuming \((d)\) condition \((c)\) follows since for any non-zero ideal \(I\) we have \(l\text{.ann}_S(I) = \bigcap_{f \in I} l\text{.ann}_S(SfS)\) and the analogous statement for \(r\text{.ann}_S(I)\).

Note that \((c) \Rightarrow (a)\) needed only the primeness condition for fully invariant submodules. These modules had been investigated by R. Wisbauer and I. Wijayanti and termed fully prime modules. We deduce two corollaries from the last theorem:

**Corollary 2.2.** Let \(M\) be a left \(R\)-module with endomorphism ring \(S\). Then \(S\) is prime and \(M\) is retractable if and only if \(M\) is \(*\)-prime and \(gSf = 0\) implies \(fSg = 0\) for all \(f, g \in S\).

As a particular case we recover the characterization of \(R\) being \(*\)-prime as bimodule (see 1.2):

**Corollary 2.3.** Let \(M\) be a left \(R\)-module with commutative endomorphism ring \(S\). Then \(M\) is \(*\)-prime if and only if \(M\) is retractable and \(S\) is an integral domain.

Since semiprime PI-rings or central Azumaya rings have large center, we see that any such ring is a \(*\)-prime bimodule if and only if its center is a domain. The next result generalizes the fact that semi-projective \(*\)-prime modules have prime endomorphism.
Proposition 2.4. Assume that for any non-zero ideal \( J \) of \( S \) which is essential as left and right ideal there exists a non-zero submodule \( N \) of \( M \) such that \( \text{Hom}(M, N) \subseteq J \). Then \( S \) is prime if \( M \) is \( \ast \)-prime.

Proof. Let \( I^2 = 0 \) for an ideal \( I \) of \( S \). Then \( J = r\text{ann}_S(I) \cap l\text{ann}_S(I) \) is a non-zero ideal of \( S \) which is essential on both sides. By assumption \( \text{Hom}(M, N) \subseteq J \) for some non-zero submodule \( N \) of \( M \). Thus \( MI \ast N = M\text{Hom}(M, N) \subseteq MIJ = 0 \) and as \( M \) is \( \ast \)-prime and \( N \) non-zero we have \( I = 0 \), i.e. \( S \) is semiprime and by Theorem 2.1 \( S \) is prime.

A left \( R \)-module \( M \) is called torsionless if it is cogenerated by \( R \). A result by Amitsur says that any faithful torsionless module over a prime ring has a prime endomorphism ring (see [1, Corollary 2.8]). The following Proposition gives sufficient conditions for a \( \ast \)-prime module \( M \) to be torsionless.

Proposition 2.5. Let \( M \) be a faithful left \( R \)-module over a prime ring \( R \). In any of the following cases \( M \) is torsionless and hence has a prime endomorphism ring.

1. \( M \) is a \( \ast \)-prime module and is not a singular left \( R \)-module.
2. \( M \) is a \( \ast \)-prime module and \( R \) is a left duo ring, i.e. any left ideal is twosided.
3. \( M \) is non-singular and is cogenerated by all of its essential submodules.

Proof. Note that any non-zero submodule \( N \) of \( M \) that is not singular contains a submodule which is isomorphic to a non-zero left ideal of \( R \). To see this let \( 0 \neq x \in N \) be an element whose annihilator \( A = l\text{ann}_R(x) \) is not essential in \( R \). Let \( B \) be a complement of \( A \), i.e. a left ideal of \( R \) which is maximal with respect to \( A \cap B = 0 \). Then \( I = A \oplus B \) is an essential left ideal of \( R \) and \(Ix \neq 0 \) since \( B \) is non-zero. As \( B \simeq Ix \), we see that \( B \) is isomorphic to a submodule of \( M \).

1. As explained above, if \( M \) is not singular, then there exists a non-zero left ideal \( B \) of \( R \) which is isomorphic to a submodule of \( M \). Since \( M \) is cogenerated by any of its non-zero submodules, it is cogenerated by \( B \) and hence by \( R \) as \( B \subseteq R \). Thus \( M \) is torsionless.
2. Since \( M \) is a (faithful) prime module, every submodule is also faithful. By hypothesis \( I = 1\text{ann}_R(m) \) is two sided for any element \( m \) of \( R \) and hence \( 0 = \text{Ann}(Rm) = \text{Ann}(R/I) = I \), i.e. \( M \) is not singular and the result follows from (1).
3. Let \( M \) be any non-zero nonsingular module that cogenerated by every essential submodule of itself. By Zorn’s Lemma there exists a maximal direct sum \( \bigoplus \limits_i C_i \) of cyclic modules \( C_i = Rm_i \) non of which is singular. Let \( A_i = 1\text{ann}_R(m_i) \) for each \( i \in I \). Since \( A_i \) is not essential in \( R \), there exists a non-zero complement \( B_i \) of \( A_i \) in \( R \) such that \( K_i = A_i \oplus B_i \) is an essential left ideal of \( R \). Let
a be any element in \( R \) such that \( a \notin A_i \). Then there exists an essential left ideal \( E \) of \( R \) such that \( Ea = Ra \cap K_i \). Because \( M \) is nonsingular, we have \( 0 \neq Eam_i \subseteq K_i m_i \cap Ram_i \). Thus \( K_i m_i \) is an essential submodule of \( C_i \) and moreover \( K_i m_i \cong B_i \). Hence \( N = \bigoplus_{i \in I} K_i m_i \) is essential in \( M \) and by hypothesis cogenerates \( M \).

Since \( N \cong \bigoplus_{i \in I} B_i \subseteq R \langle I \rangle \), \( M \) is torsionless.

The Wisbauer category of a module \( M \) is the full subcategory of \( R\text{-Mod} \) consisting of submodules of quotients of direct sums of copies of \( M \). For \( M = R \), we have \( \sigma[R] = R\text{-Mod} \). A module \( N \in \sigma[M] \) is called \( M \)-singular if there are modules \( K, L \in \sigma[M] \) with \( K \) being an essential submodule of \( L \) and \( N \cong L/K \). For \( M = R \), \( R \)-singular modules are called singular. A module \( M \) is called polyform or non-\( M \)-singular if it does not contain any \( M \)-singular submodule or equivalently if \( \text{Hom}(L/K, M) = 0 \) for all essential submodules \( K \subseteq L \subseteq M \) (see [15]).

**Proposition 2.6.** The endomorphism ring of a \(*\)-prime polyform module is a prime ring.

**Proof.** Recall our general hypothesis that \( M \) is a faithful left module over a prime ring \( R \). Let \( I^2 = 0 \) for some ideal \( I \) of \( S \). Then \( MI \) is fully invariant. Note that any fully invariant submodule \( N \in \sigma[M] \) is called \( M \)-singular if there are modules \( K, L \in \sigma[M] \) with \( K \) being an essential submodule of \( L \) and \( N \cong L/K \). For \( M = R \), \( R \)-singular modules are called singular. A module \( M \) is called polyform or non-\( M \)-singular if it does not contain any \( M \)-singular submodule or equivalently if \( \text{Hom}(L/K, M) = 0 \) for all essential submodules \( K \subseteq L \subseteq M \) (see [15]).

**Proposition 2.6.** The endomorphism ring of a \(*\)-prime polyform module is a prime ring.

**Proof.** Recall our general hypothesis that \( M \) is a faithful left module over a prime ring \( R \). Let \( I^2 = 0 \) for some ideal \( I \) of \( S \). Then \( MI \) is fully invariant. Note that any fully invariant submodule \( N \in \sigma[M] \) is called \( M \)-singular if there are modules \( K, L \in \sigma[M] \) with \( K \) being an essential submodule of \( L \) and \( N \cong L/K \). For \( M = R \), \( R \)-singular modules are called singular. A module \( M \) is called polyform or non-\( M \)-singular if it does not contain any \( M \)-singular submodule or equivalently if \( \text{Hom}(L/K, M) = 0 \) for all essential submodules \( K \subseteq L \subseteq M \) (see [15]).

**Proposition 2.6.** The endomorphism ring of a \(*\)-prime polyform module is a prime ring.

**Proof.** Recall our general hypothesis that \( M \) is a faithful left module over a prime ring \( R \). Let \( I^2 = 0 \) for some ideal \( I \) of \( S \). Then \( MI \) is fully invariant. Note that any fully invariant submodule \( N \in \sigma[M] \) is called \( M \)-singular if there are modules \( K, L \in \sigma[M] \) with \( K \) being an essential submodule of \( L \) and \( N \cong L/K \). For \( M = R \), \( R \)-singular modules are called singular. A module \( M \) is called polyform or non-\( M \)-singular if it does not contain any \( M \)-singular submodule or equivalently if \( \text{Hom}(L/K, M) = 0 \) for all essential submodules \( K \subseteq L \subseteq M \) (see [15]).

**Proof.** Recall our general hypothesis that \( M \) is a faithful left module over a prime ring \( R \). Let \( I^2 = 0 \) for some ideal \( I \) of \( S \). Then \( MI \) is fully invariant. Note that any fully invariant submodule \( N \in \sigma[M] \) is called \( M \)-singular if there are modules \( K, L \in \sigma[M] \) with \( K \) being an essential submodule of \( L \) and \( N \cong L/K \). For \( M = R \), \( R \)-singular modules are called singular. A module \( M \) is called polyform or non-\( M \)-singular if it does not contain any \( M \)-singular submodule or equivalently if \( \text{Hom}(L/K, M) = 0 \) for all essential submodules \( K \subseteq L \subseteq M \) (see [15]).
$K$ is simple. By Schur’s Lemma $\text{End}(K)$ is a division ring and hence there exists an inverse $g \in \text{End}(K)$ of $f$ restricted to $K$, i.e. $gf = id_K$. Considering $g$ as a map from $K$ to $M$ we showed that $f$ splits, i.e. $K$ is a direct summand of $M$.  

Since by the last Lemma, simple modules of a $*$-prime module are direct summands, we have the following

**Corollary 3.2.** Any weakly compressible module with DCC or ACC on direct summands and non-zero socle is homogeneous semisimple.

Recall that a ring $R$ is said to be left quotient finite dimensional (qfd) if every cyclic left $R$-module has finite Goldie dimension. Any left noetherian or more general any ring with Krull dimension is qfd.

**Theorem 3.3.** Let $R$ be a semilocal or a left qfd ring, then any $*$-prime module with non-zero socle has a prime endomorphism ring.

**Proof.** If $M$ is a $*$-prime module with a non-zero socle, then $\text{Soc}(M)$ is essential and homogeneous semisimple. Any cyclic $C$ submodule of $M$ is also a $*$-prime module with non-zero essential socle and by assumption has finite Goldie dimension (in case $R$ is qfd) or finite dual Goldie dimension (in case $R$ is semilocal). In either case $C$ has ACC on direct summands and by Corollary 3.2 $C$ is homogeneous simple. Thus $M = \text{Soc}(M) \simeq E(\Lambda)$ is homogeneous semisimple and $\text{End}(M) \simeq \text{End}(E(\Lambda))$ is a prime ring.

Recall that a ring $R$ has left Krull dimension 0 if it is left artinian and left Krull dimension 1 if every proper cyclic left $R$-module $M \neq R$ is artinian.

**Proposition 3.4.** Any $*$-prime left module over a ring with left Krull dimension less or equal to 1 has a prime endomorphism ring.

**Proof.** Let $R$ be a ring with left Krull dimension $\leq 1$ and let $M$ be a $*$-prime left $R$-module. If $M$ is not singular, then it has a prime endomorphism ring by Proposition 2.5. Suppose that $M$ is singular and let $C$ be a non-zero cyclic submodule of $M$, then $C$ is also singular and hence proper, i.e. $C \simeq R/I$ with $I \neq 0$. By hypothesis $R$ has Krull dimension $\leq 1$ and thus $C$ is artinian. This shows that $M$ has a non-zero socle. By 3.3 $M$ has a prime endomorphism.

This implies that for instance any $*$-prime module over the first Weyl algebra $A_1$ has a prime endomorphism ring.

4. Conclusion

Let $C(R)$ denote the center of $R$. Faithful $*$-prime modules $M$ that are not singular have prime endomorphism ring by Proposition 2.5. This applies in particular to the following case:
(1) if $M$ has a non-zero submodule which is finitely generated over $C(R)$ or
(2) if $R$ has a non-zero left ideal which is finitely generated over $C(R)$ or
(3) if $R$ has a non-zero left socle.

In case (1), if $N = C(R)x_1 + \cdots + C(R)x_n$, then
$$0 = \text{Ann}(M) = \text{Ann}(N) = \text{Ann}(x_1) \cap \cdots \cap \text{Ann}(x_n).$$
Thus not all of the elements $x_i$ can be singular and $M$ is not a singular module. Case (2) reduces to the first case, because if $I$ is a non-zero left ideal of $R$ which is finitely generated over $C(R)$, then since $M$ is faithful, there must exist a non-zero element $m \in M$ with $N = Im \neq 0$. But then $N$ is a non-zero submodule of $M$ which is finitely generated over $C(R)$ and (1) applies.

In case (3) we also see that due to $0 = \text{Ann}(M) = \bigcap_{x \in M} \text{Ann}(x)$ not all the annihilators $\text{Ann}(x)$ can be essential left ideals, since otherwise the left socle would be contained in $\text{Ann}(M)$ and would be zero. Hence $M$ is not a singular module.

From the preceding we can conclude that if there exists a $*$-prime faithful left $R$-module $M$ whose endomorphism ring is not prime, then
- $R$ is not a left duo ring;
- $R$ has zero left socle
- $R$ does not contain any non-zero left ideal which is finitely generated over $C(R)$;
- the Krull dimension of $R$ is greater than 1;
- $\text{End}(M)$ is not commutative;
- $M$ is a singular left $R$-module which is neither torsionless nor semi-projective;
- $M$ is not polyform, i.e. $M$ is cogenerated by some $M$-singular submodule;
- no non-zero submodule of $M$ is finitely generated over the center $C(R)$ of $R$;
- if $M$ has non-zero socle, then $R$ cannot be semilocal nor can $R$ have Krull dimension.

**ACKNOWLEDGMENT**

The authors would like to thank the referee for his/her useful comments and corrections.

**REFERENCES**

1. S. A. Amitsur, Rings of quotients and Morita contexts, *J. Algebra*, **17** (1971), 273-298.
2. G. M. Bergman and I. M. Isaacs, Rings with fixed-point-free group actions, *Proc. London Math. Soc.*, **27**(3) (1973), 69-87.
3. L. Bican, P. Jambor, T. Kepka and P. Nemec, Prime and coprime modules, *Fund. Math.*, **107**(1) (1980), 33-45.
4. I. Borges and C. Lomp, Irreducible actions and compressible modules, *J. Algebra Appl.*, **10**(1) (2011), 101-117.

5. M. Cohen, Hopf algebras acting on semiprime algebras, *Contemp. Math.*, **43** (1985), 49-61.

6. M. Cohen and L. H. Rowen, Group graded rings, *Comm. Algebra*, **11** (1983), 1253-1270.

7. J. Dixmier, Enveloping algebras, Vol. 11 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 1996. Revised reprint of the 1977 translation.

8. J. W. Fisher and S. Montgomery, Semiprime skew group rings, *J. Algebra*, **52**(1) (1978), 241-247.

9. A. Haghany and M. R. Vedadi, Endoprime modules, *Acta Math. Hungar.*, **106**(1-2) (2005), 89-99.

10. C. Lomp, When is a smash product semiprime? A partial answer, *J. Algebra*, **275**(1) (2004), 339-355.

11. C. Lomp, A central closure construction for certain algebra extensions. Applications to Hopf actions, *J. Pure Appl. Algebra*, **198**(1-3) (2005), 297-316.

12. C. Lomp, Prime elements in partially ordered groupoids applied to modules and Hopf algebra actions, *J. Algebra Appl.*, **4**(1) (2005), 77-97.

13. L. Rowen, Some results on the center of a ring with polynomial identity, *Bull. Amer. Math. Soc.*, **79** (1973) 219-223.

14. P. F. Smith, Modules with many homomorphisms, *J. Pure Appl. Algebra*, **197**(1-3) (2005), 305-321.

15. R. Wisbauer, *Modules and algebras: bimodule structure and group actions on algebras*, Vol. 81 of Pitman Monographs and Surveys in Pure and Applied Mathematics, Addison Wesley Longman Ltd., Harlow, Essex, 1996.

16. J. Zelmanowitz, A class of modules with semisimple behavior, In A. Facchini and C. Menini, editors, *Abelian Groups and Modules.*, Kluwer Acad. Publ. 1995, pp. 491-500.

17. J. Zelmanowitz, Endomorphism rings of torsionless modules, *J. Algebra*, **5** (1967), 325-341.

Mohammad Baziar
Department of Mathematics
Yasouj University
Yasouj 75914, Iran
E-mail: mbaziar@yu.ac.ir

Christian Lomp
Department of Mathematics
University of Porto
Porto, Portugal
E-mail: clomp@fc.up.pt