Even dimensional general relativity from Born-Infeld gravity

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Abstract

It is an accepted fact that requiring the Lovelock theory to have the maximum possible number of degree of freedom, fixes the parameters in terms of the gravitational and the cosmological constants. In odd dimensions, the Lagrangian is a Chern-Simons form for the (A)dS group. In even dimensions, the action has a Born-Infeld-like form.

Recently was shown that standard odd-dimensional General Relativity can be obtained from Chern-Simons Gravity theory for a certain Lie algebra $B$.

Here we report on a simple model that suggests a mechanism by which standard even-dimensional General Relativity may emerge as a weak coupling constant limit of a Born-Infeld theory for a certain Lie subalgebra of the algebra $B$. Possible extension to the case of even-dimensional supergravity is briefly discussed.

1 Introduction

The most general action for the metric satisfying the criteria of general covariance and second-order field equations for $d > 4$ is a polynomial of degree $d/2$ in the curvature known as the Lanczos-Lovelock gravity theory (LL) [1],[2].

The LL lagrangian in a $d$-dimensional Riemannian manifold can be defined as a linear combination of the dimensional continuation of all the Euler classes of dimension $2p < d$ [3],[4]:

$$S = \int \sum_{p=0}^{[d/2]} \alpha_p L^{(p)}$$

(1)

where $\alpha_p$ are arbitrary constants and

$$L^{(p)} = \varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_2} R^{(q)\alpha_2 \cdots \alpha_{d}} R^{a_2 \cdots a_{2p} e_{a_{2p+1}} \cdots e_{a_{d}}}$$

(2)

with $R^{ab} = d\omega^{ab} + \omega^{ca} \omega_{cb}$. The expression (1) can be used both for even and for odd dimensions.
The large number of dimensionful constants in the $LL$ theory $\alpha_p, \ p = 0, 1, \cdots, \lfloor d/2 \rfloor$, which are not fixed from first principles, contrast with the two constants of the Einstein-Hilbert action.

In ref. [5] it was found that these parameters can be fixed in terms of the gravitational and the cosmological constants, and that the action in odd dimensions can be formulated as a Chern-Simons theory of the $AdS$ group.

The closest one can get to a Chern-Simons theory in even dimensions is with the so-called Born-Infeld theories [8], [6], [7]. The Born-Infeld lagrangian is obtained by a particular choice of the parameters in the Lovelock series, so that the lagrangian is invariant only under local Lorentz rotations in the same way as is the Einstein-Hilbert action.

If Born-Infeld theories are the appropriate evendimensional theories to provide a framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to General Relativity.

In Ref. [9] was shown that the standard, five-dimensional General Relativity (without a cosmological constant) can be obtained from Chern-Simons gravity theory for a certain Lie algebra $\mathfrak{B}$.

It is the purpose of this paper to show that standard, even-dimensional General Relativity (without a cosmological constant) emerges as a limit of a Born-Infeld theory invariant under a certain subalgebra of the Lie algebra $\mathfrak{B}$. On the other hand, the Lie subalgebra of the algebra $\mathfrak{B}$, which is denoted by $\mathfrak{L}^B$, can be also obtained from the Lorentz algebra and a particular semigroup $S$ by means of the $S$-expansion procedure introduced in Refs. [10], [11].

The Born-Infeld Lagrangian is built from the 2-form curvature $F$ for $\mathfrak{L}^B$ algebra which depends on a scale parameter $l$ which can be interpreted as a coupling constant that characterizes different regimes within the theory. The field content induced by $\mathfrak{L}^B$ includes the vielbein $e^a$, the spin connection $\omega^{ab}$ and one extra bosonic fields $k^{ab}$.

This paper is organized as follows: In Sec. II we briefly review some aspect of the Lovelock gravity theory and of the $S$-expansion procedure. An explicit action for four, six and $2n$-dimensional Born-Infeld gravity invariant under the $\mathfrak{L}^B$ Lie algebra is considered in Sec. III. The weak coupling constant limit of this action is then shown to yield the even-dimensional Einstein-Hilbert action and the corresponding Einstein field equations. Sec.IV concludes the work with a comment about possible developments.

2 The Lovelock Gravity Theory

In this section we shall review some aspects of higher dimensional gravity and of the $S$-expansion procedure. The main point of this section is to display the differences between the invariances of $LL$ action when odd and even dimensions are considered.
2.1 The local AdS Chern-Simons and Born-Infeld like gravity

The \( LL \) action is a polynomial of degree \( d/2 \) in curvature, which can be written in terms of the Riemann curvature and the vielbein \( e^a \) in the form \( \ref{eq:llaction} \), \( \ref{eq:llaction1} \). In first order formalism the \( LL \) action is regarded as a functional of the vielbein and spin connection, and the corresponding field equations obtained by varying with respect to \( e^a \) and \( \omega^{ab} \) read \( \ref{eq:fieldequations} \):

\[
\varepsilon_a = \sum_{p=0}^{(d-1)/2} \alpha_p (d - 2p) \varepsilon_a^p = 0; \quad \varepsilon_{ab} = \sum_{p=1}^{(d-1)/2} \alpha_p (d - 2p) \varepsilon_{ab}^p = 0
\]

where

\[
\varepsilon_a^p := \varepsilon_{ab_1 \ldots b_{d-1}} R^{b_1 b_2} \ldots R^{b_{2p-1} b_{2p}} e^{b_{2p+1}} \ldots e^{b_{d-1}}
\]

\[
\varepsilon_{ab}^p := \varepsilon_{ab_1 \ldots b_d} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} T^{a_{2p+1}} e^{a_{2p+2}} \ldots e^{a_d}.
\]

Here \( T^a = de^a + \omega^a_b e^b \) is the torsion 2-form. Using the Bianchi identity one finds

\[
D\varepsilon_a = \sum_{p=1}^{(d-1)/2} \alpha_{p-1} (d - 2p + 2)(d - 2p + 1) e^b \varepsilon_{ba}^p. \tag{6}
\]

Moreover

\[
e^b \varepsilon_{ba} = \sum_{p=1}^{(d-1)/2} \alpha_p (d - 2p) e^b \varepsilon_{ba}^p. \tag{7}
\]

From \( \ref{eq:fieldequations} \) and \( \ref{eq:fieldequations2} \) one finds for \( d = 2n - 1 \)

\[
\alpha_p = \alpha_0 \left( \frac{2n - 1}{2n - 2p - 1} \right) \left( \frac{n - 1}{p} \right); \tag{8}
\]

with \( \alpha_0 = \frac{\kappa}{8\pi l^2}, \gamma = -\text{sign}(\Lambda) \frac{l^2}{2}, \) where for any dimensions \( l \) is a length parameter related to the cosmological constant by \( \Lambda = \pm (d - 1)(d - 2)/2l^2 \).

With these coefficients, the \( LL \) action is a Chern-Simons \((2n - 1)\)-form invariant not only under standard local Lorentz rotations \( \delta e^a = \kappa^a_b e^b, \delta \omega^{ab} = -D\kappa^{ab} \) but also under a local AdS boost \( \ref{eq:boost} \).

For \( d = 2n \) it is necessary to write equation \( \ref{eq:fieldequations} \) in the form \( \ref{eq:fieldequations3} \):

\[
D\varepsilon_a = T^a \sum_{p=1}^{[n-1]} 2\alpha_{p-1}(n - p + 1) T^p_{ab} - \sum_{p=1}^{[n-1]} 4\alpha_{p-1}(n - p + 1)(n - p)e^b \varepsilon_{ba}^p \tag{9}
\]

with

\[
T_{ab} = \frac{\delta L}{\delta R^{ab}} = \sum_{p=1}^{(d-1)/2} \alpha_p T^p_{ab} \tag{10}
\]
where
\[ T_{ab}^p = \varepsilon_{a_1 a_2 \cdots a_4 R_{a_3 a_4} \cdots R_{a_2p-1 a_2p} T_{a_2p+1} a_{2p+2} \cdots a_{2d}. \] (11)

The comparison between (7) and (9) leads to [5]
\[ \alpha_p = \alpha_0 (2\gamma)^p \left( \frac{n}{p} \right). \] (12)

With these coefficients the LL lagrangian takes the form [5]
\[ L = \kappa \epsilon_{a_1 a_2 \cdots a_d} \bar{R}_{a_1 a_2} \cdots \bar{R}_{a_{d-1} a_d} \] (13)
which is the Pfaffian of the 2-form \( \bar{R}_{ab} = R_{ab} + \frac{1}{l^2} e^a e^b \) and can be formally written as the Born-Infeld like form [5]. The corresponding action, known as Born-Infeld action is invariant only under local Lorentz rotations.

2.2 Born-Infeld Lorentz Gravity

A Born-Infeld action for gravity in \( d = 2n \) dimensions is given by [5], [8]
\[ S = \int \sum_{p=0}^{[d/2]} \frac{\kappa}{2n} \epsilon_{a_1 a_2 \cdots a_d} R_{a_1 a_2} \cdots R_{a_{2p-1} a_{2p}} e_{a_{2p+1}} \cdots e_{a_d}. \] (14)

where \( e^a \) corresponds to the 1-form vielbein, and \( R_{ab} = \omega_{ab} + \frac{1}{l^2} e^a e^b \) to the Riemann curvature in the first order formalism.

The action (14) is off-shell invariant under the Lorentz-Lie algebra \( SO(2n-1,1) \), whose generators \( \bar{J}_{ab} \) of Lorentz transformations satisfy the commutation relationships
\[ \left[ \bar{J}_{ab}, \bar{J}_{cd} \right] = \eta_{cb} \bar{J}_{ad} - \eta_{ca} \bar{J}_{bd} + \eta_{db} \bar{J}_{ca} - \eta_{da} \bar{J}_{cb} \]
The Levi-Civita symbol \( \epsilon_{a_1 \cdots a_{2n}} \) in (14) should be regarded as the only non-vanishing component of the symmetric, \( SO(2n-1,1) \), invariant tensor of rank \( n \), namely
\[ \left\langle \bar{J}_{a_1 a_2} \cdots \bar{J}_{a_{2n-1} a_{2n}} \right\rangle = \frac{2^{n-1}}{n} \epsilon_{a_1 \cdots a_{2n}}. \] (15)

In order to interpret the gauge field as the vielbein, one is forced to introduce a length scale \( l \) in the theory. To see why this happens, consider the following argument: Given that (i) the exterior derivative operator \( d = dx^\mu \partial_\mu \) is dimensionless, and (ii) one always chooses Lie algebra generators \( T_A \) to be dimensionless as well, the one-form connection fields \( A = A_A^A T_A dx^\mu \) must also be dimensionless. However, the vielbein \( e^a = e^a_\mu dx^\mu \) must have dimensions of length if it is to be related to the spacetime metric \( g_{\mu \nu} \) through the usual equation \( g_{\mu \nu} = e^a_\mu e^b_\nu \eta_{ab} \). This means that the “true” gauge field must be of the form \( e^a / l \), with \( l \) a length parameter.
Therefore, following Refs. [13], [14], the one-form gauge field $A$ of the Chern–Simons theory is given in this case by

$$A = \frac{1}{l} e^a \tilde{P}_a + \frac{1}{2} \omega^{ab} \tilde{J}_{ab}. \quad (16)$$

It is important to notice that once the length scale $l$ is brought into the Born-Infeld theory, the lagrangian splits into several sectors, each one of them proportional to a different power of $l$, as we can see directly in eq. (14).

### 2.3 The S-expansion procedure

In this subsection we shall review the main aspects of the $S$-expansion procedure and their properties introduced in Ref. [10].

Let $S = \{\lambda_\alpha\}$ be an abelian semigroup with 2-selector $K_{\alpha\beta}^\gamma$ defined by

$$K_{\alpha\beta}^\gamma = \begin{cases} 1 & \text{when } \lambda_\alpha \lambda_\beta = \lambda_\gamma \\ 0 & \text{otherwise} \end{cases}, \quad (17)$$

and $\mathfrak{g}$ a Lie (super)algebra with basis $\{T_a\}$ and structure constant $C_{AB}^C$,

$$[T_a, T_b] = C_{AB}^C T_C. \quad (18)$$

Then it may be shown that the product $\mathfrak{g} = S \times \mathfrak{g}$ is also a Lie (super)algebra with structure constants $C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta}^\gamma C_{AB}^C$,

$$[T_{(A,\alpha)}, T_{(B,\beta)}] = C_{AB}^C T_{(C,\gamma)}. \quad (19)$$

The proof is direct and may be found in Ref. [10].

**Definition 1** Let $S$ be an abelian semigroup and $\mathfrak{g}$ a Lie algebra. The Lie algebra $\mathfrak{G}$ defined by $\mathfrak{G} = S \times \mathfrak{g}$ is called $S$-Expanded algebra of $\mathfrak{g}$.

When the semigroup has a zero element $0_S \in S$, it plays a somewhat peculiar role in the $S$-expanded algebra. The above considerations motivate the following definition:

**Definition 2** Let $S$ be an abelian semigroup with a zero element $0_S \in S$, and let $\mathfrak{G} = S \times \mathfrak{g}$ be an $S$-expanded algebra. The algebra obtained by imposing the condition $0_S T_A = 0$ on $\mathfrak{G}$ (or a subalgebra of it) is called $0_S$-reduced algebra of $\mathfrak{G}$ (or of the subalgebra).

An $S$-expanded algebra has a fairly simple structure. Interestingly, there are at least two ways of extracting smaller algebras from $S \times \mathfrak{g}$. The first one gives rise to a resonant subalgebra, while the second produces reduced algebras.

A useful property of the $S$-expansion procedure is that it provides us with an invariant tensor for the $S$-expanded algebra $\mathfrak{G} = S \times \mathfrak{g}$ in terms of an invariant tensor for $\mathfrak{g}$. As shown in Ref. [10] the theorem VII.2 provide a general expression for an invariant tensor for a $0_S$-reduced algebra.
Theorem VII.2 of Ref. [10]: Let $S$ be an abelian semigroup with nonzero elements $\lambda_i$, $i = 0, \cdots, N$ and $\lambda_{N+1} = 0$. Let $g$ be a Lie (super)algebra of basis $\{T_A\}$, and let $\langle T_{A_1} \cdots T_{A_n} \rangle$ be an invariant tensor for $g$. The expression

$$\langle T_{(A_1, i_1)} \cdots T_{(A_n, i_n)} \rangle = \alpha_j K_{i_1 \cdots i_n}^j \langle T_{A_1} \cdots T_{A_n} \rangle$$ (20)

where $\alpha_j$ are arbitrary constants, corresponds to an invariant tensor for the $0_S$-reduced algebra obtained from $\mathfrak{G} = S \times g$.

Proof: the proof may be found in section 4.5 of Ref. [10].

3 Even-dimensional Einstein-Hilbert Action from Born-Infeld gravity

In this section we show how to obtain the even-dimensional General Relativity from Born-Infeld Gravity.

The lagrangian for four, six-dimensional Born-Infeld gravity can be written as

$$L^{(4)}_{\text{BI}} = \frac{\kappa}{4} \varepsilon_{a_1a_2a_3a_4} \left( \frac{1}{l^4} e^{a_1} e^{a_2} e^{a_3} e^{a_4} + \frac{2}{l^2} R^{a_1a_2} e^{a_3} e^{a_4} + R^{a_1a_2} R^{a_3a_4} \right).$$ (21)

$$L^{(6)}_{\text{BI}} = \frac{\kappa}{6} \varepsilon_{a_1a_2a_3a_4a_5a_6} \left( \frac{1}{l^6} e^{a_1} e^{a_2} e^{a_3} e^{a_4} e^{a_5} e^{a_6} + \frac{3}{l^4} R^{a_1a_2} e^{a_3} e^{a_4} e^{a_5} e^{a_6} \right) + \frac{\kappa}{36} \varepsilon_{a_1a_2a_3a_4a_5a_6} \left( \frac{3}{l^4} R^{a_1a_2} R^{a_3a_4} e^{a_5} e^{a_6} + R^{a_1a_2} R^{a_3a_4} R^{a_5a_6} \right)$$ (22)

From this lagrangian it is apparent that neither the $l \to \infty$ nor the $l \to 0$ limit yields the Einstein–Hilbert term alone. Rescaling $\kappa$ properly, those limits will lead either to the Euler density or to the cosmological constant term by itself, respectively. In the case of large $l$ we have $\frac{1}{l^4} \ll \frac{1}{l^2} \ll \frac{1}{l^2}$ and in the case of small $l$ we have $\frac{1}{l^4} \gg \frac{1}{l^2} \gg \frac{1}{l^2}$. Since the density of Euler is a topological invariant, that is not contribute to the equations of motion, we have:

(a) For $d = 4$ dimensions and $l \to \infty$ the dominant term would be the Einstein Hilbert term.

(b) For dimensions $d > 4$ we can see that neither the $l \to \infty$ nor the $l \to 0$ limit yields the Einstein–Hilbert term.

The Lagrangian (21) is invariant under the Lorentz algebra. This algebra choice is crucial, since it permits the interpretation of the gauge field $\omega^{ab}$ as the spin connection. It is, however, not the only possible choice: as we explicitly show below, there exist other Lie algebras that also allow for a similar identification and lead to a Born-Infeld Lagrangian which leads to the Einstein-Hilbert Lagrangian in a certain limit.
Following the definitions of Ref. [10] (see subsection (2.3)), let us consider
the $S$-expansion of the Lie algebra $SO(2n-1,1)$ using as semigroup a sub-
semigroup of $S^{(3)}_E$. After performing its $0_S$-reduction, one finds a new Lie algebra,
call it $\mathfrak{g}_S$ which is a subalgebra of the so called $\mathfrak{B}$ algebra, with the desired
properties.

### 3.1 The Lagrangian in $D=4$

Following the definitions of Ref. [10] (see subsection (2.3)), let us consider the
$S$-expansion of the Lie algebra $SO(3,1)$ using as semigroup the sub-semigroup
$S^{(3)}_0 = \{\lambda_0, \lambda_2, \lambda_4\}$ of semigroup $S^{(3)}_E = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. After performing its
$0_S$-reduction, one finds a new Lie algebra, call it $\mathfrak{g}_B$ which is a subalgebra of
the so called $\mathfrak{B}_5$ algebra, whose generators $J_{ab} = \lambda_0 \tilde{J}_{ab}$, $Z_{ab} = \lambda_2 \tilde{J}_{ab}$ satisfy the
commutation relationships

\begin{align}
[J_{ab}, J_{cd}] &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{da} J_{cb}, \\
[J_{ab}, Z_{cd}] &= \eta_{cb} Z_{ad} - \eta_{ca} Z_{bd} + \eta_{db} Z_{ca} - \eta_{da} Z_{cb}, \\
[Z_{ab}, Z_{cd}] &= 0,
\end{align}

which is the expanded algebra $\mathfrak{g}_S = S^{(3)}_0 \otimes V_0$. Using Theorem VII.2 of
Ref. [10] (see subsection (2.3)), it is possible to show that the only non-vanishing
components of a invariant tensor for the $\mathfrak{g}_S$ algebra are given by

\begin{align}
\langle J_{ab} J_{cd} \rangle_{\mathfrak{g}_S} &= \alpha_0 l^2 \epsilon_{abcd}, \\
\langle J_{ab} Z_{cd} \rangle_{\mathfrak{g}_S} &= \alpha_2 l^2 \epsilon_{abcd},
\end{align}

where $\alpha_0$ and $\alpha_2$ are arbitrary independent constants of dimensions $[\text{length}]^{-2}$.

Using the dual procedure of $S$-expansion in terms of the Maurer-Cartan
forms [11], we find that the Born-Infeld Lagrangian invariant under the $\mathfrak{g}_S$
algebra is given by

\begin{align}
L_{BI}^{(4)} &= \frac{\alpha_0}{4} \epsilon_{abcd} l^2 R^{ab} R^{cd} + \frac{\alpha_2}{2} \epsilon_{abcd} R^{ab} \epsilon^{cd} + i^2 D \omega k^{ab} R^{cd}.
\end{align}

Here we can see that the Lagrangian (26) is split into two independent
pieces, one proportional to $\alpha_0$ and the other to $\alpha_2$. The term proportional to
$\alpha_0$ corresponds to the Euler invariant. The piece proportional to $\alpha_2$ contains
the Einstein-Hilbert term $\epsilon_{abcd} R^{ab} \epsilon^{cd} + i^2 D \omega k^{ab}$, besides the usual curvature $R^{ab}$, a bosonic matter field $k^{ab}$.

Unlike the Born-Infeld Lagrangian (21) the coupling constant $l^2$ does not appear explicitly in the Einstein Hilbert term but accompanies the remaining
elements of the Lagrangian. This allows recover four dimensional the Einstein-
Hilbert Lagrangian in the limit where $l$ equals to zero.

The variation of the Lagrangian, modulo boundary terms, is given by
\[ \delta L_{BI}^{(4)} = \varepsilon_{abcd} \left( \alpha_2 R^{ab} e^c \right) \delta e^d + \varepsilon_{abcd} \delta \omega^{ab} \left( \alpha_2 T^e e^d + \alpha_2 k^c e^d R^{ed} \right). \] (27)

from which we see that to recover the field equations of general relativity is not necessary to impose the limit \( l = 0 \). \( \delta L_{BI}^{(4)} = 0 \) leads to the dynamics of Relativity when considering the case of a solution without matter \( (k^{ab} = 0) \). This is possible only in 4 dimensions. However, to recover the field equations of general relativity in dimensions greater than 4, is necessary to take a limit of the coupling constant \( l \).

### 3.2 The Lagrangian in \( D = 6 \)

Following the definitions of Ref. [10] (see subsection (2.3)), let us consider the \( S \)-expansion of the Lie algebra \( \text{SO}(5,1) \) using as a semigroup the sub-semigroup \( S^{(5)}_0 = \{ \lambda_0, \lambda_2, \lambda_4, \lambda_6 \} \) of semigroup \( S^{(5)}_E = \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \} \). After performing its \( S^0(= \lambda_6) \)-reduction, one finds a new Lie algebra, call it \( \mathfrak{L}^{28}_7 \) which is a subalgebra of the so called \( \mathfrak{B}_7 \) algebra, whose generators \( J_{ab} = \lambda_0 J_{ab}, Z_{ab}^{(1)} = \lambda_2 Z_{ab}, Z_{ab}^{(2)} = \lambda_4 Z_{ab} \) satisfy the commutation relationships

\[
\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb}, \\
[J_{ab}, Z_{cd}^{(1)}] &= \eta_{cb} Z_{ad}^{(1)} - \eta_{ca} Z_{bd}^{(1)} + \eta_{db} Z_{ca}^{(1)} - \eta_{da} Z_{cb}^{(1)}, \\
[Z_{ab}^{(1)}, Z_{cd}^{(1)}] &= \eta_{cb} Z_{ad}^{(2)} - \eta_{ca} Z_{bd}^{(2)} + \eta_{db} Z_{ca}^{(2)} - \eta_{da} Z_{cb}^{(2)}, \\
[J_{ab}, Z_{cd}^{(2)}] &= \eta_{cb} Z_{ad}^{(2)} - \eta_{ca} Z_{bd}^{(2)} + \eta_{db} Z_{ca}^{(2)} - \eta_{da} Z_{cb}^{(2)}, \\
[Z_{cd}^{(1)}, Z_{cd}^{(2)}] &= [Z_{cd}^{(1)}, Z_{cd}^{(2)}] = 0.
\end{align*}
\] (28)

which is the expanded algebra \( \mathfrak{L}^{28}_7 = S^{(5)}_0 \otimes V_0 \). Using Theorem VII.2 of Ref. [10] (see subsection (2.3)), it is possible to show that the only non-vanishing components of a invariant tensor for the \( \mathfrak{L}^{28}_7 \) algebra are given by

\[
\begin{align*}
\langle J_{ab} J_{cd} J_{ef} \rangle_{\mathfrak{L}^{28}_7} &= \frac{4}{3} \alpha_0 l^4 \varepsilon_{abcdef}, \\
\langle J_{ab} J_{cd} Z_{ef}^{(1)} \rangle_{\mathfrak{L}^{28}_7} &= \frac{4}{3} \alpha_2 l^4 \varepsilon_{abcdef}, \\
\langle J_{ab} J_{cd} Z_{ef}^{(2)} \rangle_{\mathfrak{L}^{28}_7} &= \langle J_{ab} Z_{cd}^{(1)} Z_{ef}^{(2)} \rangle_{\mathfrak{L}^{28}_7} = \frac{4}{3} \alpha_4 l^4 \varepsilon_{abcdef}.
\end{align*}
\] (29-31)

where \( \alpha_0, \alpha_2 \) and \( \alpha_4 \) are arbitrary independent constants of dimensions \( \text{[length]}^{-2} \).

In order to write down a Born-Infeld Lagrangian for the \( \mathfrak{L}^{28}_7 \) algebra, we
start from the two-form curvature
\[ F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{2} \left( D_\omega k^{(ab,1)} + \frac{1}{l^2} e^a e^b \right) Z^{(1)}_{ab} \]
\[ + \frac{1}{2} \left( D_\omega k^{(ab,2)} + k^a e^{(1)} k e^{b(1)} + \frac{1}{l^2} \left[ e^a h^{(b,1)} + h^{(a,1)} e^b \right] \right) Z^{(2)}_{ab}. \] (32)

which is obtained by applying the S-expansion procedure to 2-form curvature used in the construction of the Born-Infeld action (see Appendix).

Using the dual procedure of S-expansion in terms of the Maurer-Cartan forms [11], we find that the 6-dimensional Born-Infeld Lagrangian in variant under the $L_{BI}$ algebra is given by
\[ L_{BI}^\omega = \frac{\alpha_4}{2} \varepsilon_{abcdef} R^{ab} R^{cd} R^{ef} \]
\[ + \frac{\alpha_2}{2} \varepsilon_{abcdef} \left( R^{ab} R^{cd} e^d e^f + l^2 R^{ab} R^{cd} e^d e^f \right) \]
\[ + \frac{\alpha_4}{2} \varepsilon_{abcdef} \left( R^{ab} R^{cd} e^d e^f + 4 R^{ab} R^{cd} e h^{(c,1)} e^f \right) \] (33)

where
\[ R^{(ab,1)} = D_\omega k^{(ab,1)}, \] (34)
\[ R^{(ab,2)} = D_\omega k^{(ab,2)} + k^a e^{(1)} k e^{b(1)}. \] (35)

From (33) we can see that in the limit $l = 0$ we obtain the Einstein-Hilbert Lagrangian
\[ L_{BI}^\omega = \frac{\alpha_4}{2} \varepsilon_{abcdef} R^{ab} e^d e^f. \] (36)

Note that in the limit $l = 0$, the variation of the Lagrangian (36) leads now to the field equations of general relativity
\[ \varepsilon_{abcdef} R^{ab} e^d e^f = 0 \] (37)
\[ \varepsilon_{abcdef} T^{ef} e^e = 0. \] (38)

### 3.3 The Lagrangian in D = 2n

Following the definitions of Ref. [10] (see subsection (2.3)), let us consider the S-expansion of the Lie algebra SO $(2n - 1, 1)$ using as a semigroup the sub-semigroup $S_0^{(2n-1)} = \{ \lambda_0, \lambda_2, \lambda_4, \lambda_6, \cdots, \lambda_{2n} \}$ of semigroup $S_0^{(2n-1)} = \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \cdots, \lambda_{2n} \}$. After performing its $0_5(= \lambda_{2n})$-reduction, one finds a new Lie algebra, call it $S_2n+1$ which is a subalgebra of the so called $S_2n+1$ algebra, whose generators $J_{ab} = \lambda_0 J_{ab}, Z^{(1)}_{ab} = \lambda_2 J_{ab}, Z^{(2)}_{ab} = \lambda_4 J_{ab}, \cdots, Z^{(n)}_{ab} = \lambda_{2n} J_{ab}$ satisfy the commutation relationships
which is the expanded algebra $\mathfrak{L}_{2n+1}^{38} = S_{9}^{(2n-1)} \otimes V_{0}$. Using Theorem VII.2 of Ref. [10] (see subsection (2.3)), it is possible to show that the only non-vanishing components of an invariant tensor for the $\mathfrak{L}_{2n+1}^{38}$ algebra are given by

$$\langle J_{(a_{1}a_{2},i_{1})} \cdots J_{(a_{2p-1}a_{2p},i_{k})} \rangle = \frac{2^{p-1}2^{p-2}}{p!} \alpha_{j} \delta_{i_{1}+\cdots+i_{k}}^{j} \varepsilon_{a_{1} \cdots a_{2p}},$$

(40)

where $k = 0, \cdots, 2n-2$ and $\alpha_{j}$ are arbitrary independent constants of dimension $\text{length}^{2-2p}$.

In order to write down a Born-Infeld Lagrangian for the $\mathfrak{L}_{2n+1}^{38}$ algebra, we start from the two-form curvature

$$F = \sum_{k=0}^{n-1} \frac{1}{2} F^{(ab,2k)} J_{(ab,2k)}$$

(41)

where

$$F^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd} \omega^{(ac,2i)} \omega^{(db,2j)} \delta_{i+j}^{k} + \frac{1}{l^{2}} \tilde{e}^{(a,2i+1)} e^{(b,2j+1)} \delta_{i+j+1}^{k}, \quad (42)$$

$$F^{(a,2k+1)} = de^{(a,2k+1)} + \eta_{bc} e^{(ab,2i)} e^{(c,2j)} \delta_{i+j}^{k}. \quad (43)$$

Using the dual procedure of $S$-expansion in terms of the Maurer-Cartan forms [11], we find that the 2n-dimensional Born-Infeld Lagrangian invariant under the $\mathfrak{L}_{2n+1}^{38}$ algebra is given by

$$L_{BI}^{38} (2n) = \sum_{k=1}^{n} \frac{2^{k-2}2^{n-1}}{2n} \alpha_{j} \delta_{i_{1}+\cdots+i_{k}}^{j} \varepsilon_{q_{1}+q_{2}} \cdots \varepsilon_{q_{2n-1}+q_{2n}}$$

$$R^{(a_{1}a_{2},i_{1})} + e^{(a_{1},q_{1})} e^{(a_{2},q_{2})} \cdots$$

$$\cdots \left( R^{(a_{2n-1}a_{2n},i_{k})} + e^{(a_{2n-1},q_{2n-1})} e^{(a_{2n},q_{2n})} \right). \quad (44)$$

From (44) which we can see that in the limit $l = 0$ the only nonzero term corresponds to the case $k = 1$, namely
whose only nonzero component (corresponding to the case \( p = q_1 = \cdots = q_{2n-2} = 0 \)) is proportional to the Einstein-Hilbert Lagrangian

\[
L_{BI}^n \mid_{l=0} = \frac{2^{n-1}}{2n} \alpha_j \delta^i_{l+k_1+\cdots+k_{2n-2}} \varepsilon_{a_1\cdots a_{2n}} R^{(a_1 a_2 \cdot \cdot \cdot)} e^{(a_3, k_1)} \cdot \cdot \cdot e^{(a_{2n}, k_{2n-1})} \\
= \frac{2^{n-1}}{2n} \alpha_j \delta^i_{2p+2q_1+1+\cdots+2q_{2n-2}} + 1 \varepsilon_{a_1\cdots a_{2n}} R^{(a_1 a_2 \cdot \cdot \cdot)} e^{(a_3, 2q_1+1)} \cdot \cdot \cdot e^{(a_{2n}, 2q_{2n-2}+1)} \\
= \frac{2^{n-1}}{2n} \alpha_j \delta^i_{2(p+q_1+\cdots+q_{2n-2})+2n-2} \varepsilon_{a_1\cdots a_{2n}} R^{(a_1 a_2 \cdot \cdot \cdot)} e^{(a_3, q_1+1)} \cdot \cdot \cdot e^{(a_{2n}, q_{2n-2}+1)}.
\]

\[ (45) \]

\[ (46) \]

4 Comments and Possible Developments

In the present work we have shown that standard even-dimensional general relativity, emerges as a limit of a Born-Infeld theory invariant under a certain algebra \( L^B \). This algebra can be obtained from the Lorentz algebra and a particular semigroup \( S \) by means of the \( S \)-expansion procedure introduced in Refs. [10], [11].

The toy model and procedure considered here could play an important role in the context of supergravity in higher dimensions. In fact, it seems likely that it is possible to recover the standard ten-dimensional Supergravity from a Born-Infeld gravity theory, in a way very similar to the one shown here. In this way, the procedure sketched here could provide us with valuable information of what the underlying geometric structure of Supergravity in \( d = 10 \) could be (work in progress).

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A S-expansion of the Lorentz curvature

Consider the Lorentz 2-form curvature

\[
\tilde{F} = \frac{1}{2} \left( R^{ab} + \frac{1}{l^2} e^a b \right) J_{ab},
\]

\[ (47) \]
which allows to obtain the Born-Infeld action invariant under the Lorentz group

\[ L_{BI}^{(6)} = \frac{k}{6} \varepsilon_{abcdef} \left( R^{ab} + \frac{1}{l^2} e^a e^b \right) \left( R^{cd} + \frac{1}{l^2} e^c e^d \right) \left( R^{ef} + \frac{1}{l^2} e^e e^f \right). \]  

(48)

The S-expanded 2-form curvature is obtained as follows

\[ F = \lambda_2 \tilde{F}^\alpha \]
\[ = \frac{1}{2} \lambda_0 R^{ab,0} J_{ab} + \frac{1}{2} \left( \lambda_2 R^{ab,2} + \frac{1}{l^2} \lambda_2 \left( e^a e^b \right)^2 \right) J_{ab} \]
\[ + \frac{1}{2} \left( \lambda_4 R^{ab,4} + \frac{2}{l^2} \lambda_4 \left( e^a e^b \right)^4 \right) J_{ab} \]
\[ = \frac{1}{2} R^{ab,0} J_{ab,0} + \frac{1}{2} \left( R^{ab,2} + \frac{1}{l^2} \left( e^a e^b \right)^2 \right) J_{ab,2} \]
\[ + \frac{1}{2} \left( R^{ab,4} + \frac{2}{l^2} \left( e^a e^b \right)^4 \right) J_{ab,4}. \]

The Riemann curvature is given by \( R^{ab} = d\omega^{ab} + \omega^a \omega^{cb} \). This means that the expansion of the curvature can be obtained by expanding the spin connection \( \omega^{ab} \). For example \( R^{(ab,4)} \) is obtained as follows

\[ \lambda_4 R^{(ab,4)} = \lambda_4 d\omega^{(ab,4)} + \lambda_2 \lambda_2 \omega^{a} \omega^{c} \omega^{b} \omega^{2} \omega^{cb} + \lambda_4 \lambda_4 \omega^{a} \omega^{c} \omega^{b} \omega^{0} \omega^{c} \omega^{b} \omega^{4}. \]

Defining \( \omega^{ab,4} = k^{(ab,2)} \), \( \omega^{ab,2} = k^{(ab,1)} \), \( \omega^{ab,0} = \omega^{ab} \) we have

\[ R^{(ab,4)} = dk^{(ab,2)} + k^{(ab,1)} \]
\[ = \omega^{a} \omega^{c} \omega^{b} \omega^{c} \omega^{b} \omega^{b} \omega^{c} \omega^{2} \omega^{cb} + \omega^{a} \omega^{c} \omega^{b} \omega^{c} \omega^{b} \omega^{0} \omega^{c} \omega^{4} + \omega^{a} \omega^{c} \omega^{b} \omega^{c} \omega^{b} \omega^{2} \omega^{cb} + \omega^{a} \omega^{c} \omega^{b} \omega^{c} \omega^{b} \omega^{0} \omega^{c} \omega^{4}.
\]

(49)

The field \( h^{(a,1)} \) can be obtained from the vielbein \( e^a \) as follows

\[ \lambda_1 \lambda_3 e^a \omega^{(b,3)} = \lambda_4 e^a h^{(a,1)} \]

So that the 2-form curvature

\[ F = \frac{1}{2} R^{ab,0} J_{ab,0} + \frac{1}{2} \left( R^{ab,2} + \frac{1}{l^2} \left( e^a e^b \right)^2 \right) J_{ab,2} + \frac{1}{2} \left( R^{ab,4} + \frac{2}{l^2} \left( e^a e^b \right)^4 \right) J_{ab,4} \]

can be rewritten as

\[ F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{2} \left( D_\omega k^{(ab,1)} + \frac{1}{l^2} e^a e^b \right) Z_{ab}^{(1)} \]
\[ + \frac{1}{2} \left( D_\omega k^{(ab,2)} + k^{(ab,1)} k^{(1)} \right) \left( D_\omega e^{(a),1} e^{(b),1} \right) Z_{ab}^{(2)}. \]

(50)

where

\[ R^{(ab,0)} = R^{ab}, \quad R^{(ab,2)} = D_\omega k^{(ab,1)}, \quad (e^a e^b)^2 = e^a e^b, \quad R^{(ab,4)} = D_\omega k^{(ab,2)} + k^{(ab,1)} k^{(1)}, \quad (e^a e^b)^4 = e^a h^{(b,1)}. \]

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