Quantum gravity corrections to the thermodynamics and phase transition of Schwarzschild black hole

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Abstract: In this work, we derive a new kind of rainbow functions, which has generalized uncertainty principle parameter. Then, we investigate modified thermodynamic quantities and phase transition of rainbow Schwarzschild black hole by employing this new kind of rainbow functions. Our results demonstrate that the effect of rainbow gravity and generalized uncertainty principle have a great effect on the picture of Hawking radiation. It prevents black holes from total evaporation and causes the remnant. In addition, after analyzing the the modified local thermodynamic quantities, we find that effect of rainbow gravity and generalized uncertainty principle lead to one first-order phase transition, two second-order phase transitions, and two Hawking-Page-type phase transitions in the thermodynamic system of rainbow Schwarzschild black hole.

Keywords: Black hole; Rainbow gravity; Generalized uncertainty principle parameter; Phase transition

1 Introduction

In the theoretical physics, one of the biggest challenges is to combine the quantum theory with gravity theory. In order to achieve this purpose, various quantum gravity theories, such as the string theory, loop quantum gravity, M-theory, and non-commutative geometry have been proposed [1–5]. According to those theories, it is found that the existence of a minimum measurable length, which is expected to be of the order of the Planck scale. However, under the linear Lorentz transformations, one may find that the Planck length is variable, which is conflict with the previous conclusions. To resolve this contradiction, the standard energy-momentum dispersion relation needs to be modified [6–8]. Nowadays, the modifications of standard dispersion relation are called as modified dispersion relation (MDR) [9–11]. Interestingly, it should be noted that the MDR can modify the special relativity. In Ref. [12], this generalization of special relativity is called as double special relativity (DSR), which has two invariants: the velocity of light $c$ and the Planck length $\ell_p$ (or Planck energy $E_p$). On the basis of the DSR, Magueijo and Smolin proposed the rainbow gravity (RG), which is a more general theory that is motivated by MDR [13]. In the RG, the energy of the test particle affects the geometry. In other word, the spacetime cannot be described by a single metric, but a family of metrics (rainbow of metrics) parameterized by the ratio $E/E_p$. According to the RG, the deformed energy-momentum relation is given by

$$E^2 f^2 (E/E_p) - p^2 g^2 (E/E_p) = m^2,$$

where $E_p$ represents to the Planck energy. $p$ and $m$ are the momentum and mass of a test particle, respectively. The correction terms $f (E/E_p)$ and $g (E/E_p)$ are the rainbow functions, which satisfy the limits

$$\lim_{E/E_p \to 0} f (E/E_p) = 1 \quad \text{and} \quad \lim_{E/E_p \to 0} g (E/E_p) = 1.$$

In recent years, the implications of the aspects of RG have been investigated in many contexts, such as cosmology, astrophysics, and black hole physics [14–18]. In particular, the RG has a very significant effect on the thermodynamics of black holes. In Refs. [19–27], the authors studied the modified thermodynamics of black holes by using the RG. These results demonstrated that the modifications are departed from the classical cases when the size of black holes approach the Planck scale. It indicates that the RG can stop the Hawking radiation in the last stages of black holes’ evolution and naturally leads to the remnants. Therefore, the RG can used to address the information loss and naked singularity problems of black holes. Very recently, according to the rainbow functions that were proposed by Magueijo and Smolin, we discussed the thermodynamics and phase transition of a spherically symmetric black hole. Our calculations showed that the RG changes the picture of black hole’s thermodynamic phase transition [28]. In addition, we found three Hawking-Page-type phase transitions in the framework of the RG.

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On the other hand, the minimum measurable length can also change the conventional Heisenberg uncertainty principle to the so-called generalized uncertainty principle (GUP). Since the GUP can be applied to many physical systems, it has got people’s attention now [29]. In Ref. [29], Adler, Chen and Santiago pointed out that uncertainties for position and momentum of the GUP (namely, \( \Delta x \) and \( \Delta p \)) can be considered as the temperature and radius of the black holes. With this heuristic theory, they studied the GUP corrected Hawking temperature of Schwarzschild (SC) black hole and showed that the modified Hawking temperature of SC black hole is higher than the conventional case. However, in our previous work, using the GUP together with Hamilton-Jacobi equation, we found that the GUP corrected thermodynamics of SC black hole are similar to the results obtained in the framework of RG. This indicates that there may be some connection between the GUP and RG [37]. From the above discussion, it is interesting to incorporate the GUP with RG to investigate the modified thermodynamics and phase transition of black holes. In this paper, we derive a new kind of rainbow functions, which contains the effect of GUP. After that, we use this new kind of rainbow functions to analysis the modified thermodynamics, thermodynamic criticality and the phase transition of SC black hole.

The remainders of this paper are outlined as follows. In Section 2, according to the GUP that is proposed Kempf, Mangano, and Mann, we derive a MDR. Then, comparing this MDR with the general form of RG, a new kind of rainbow functions, which contains the GUP parameter is obtained. In Section 3, according to the new kind of rainbow functions, we calculate the modified thermodynamics of SC black hole in the context of RG. In Section 4, we study the modified local thermodynamic quantities of rainbow SC black hole, and discuss the phase transition between the hot space and the states of black holes. Finally, the discussion is given in Section 5.

Throughout the paper, the natural units of \( k_B = c = 1 \) are used.

## 2 The rainbow functions with effect of GUP

To begin with, we need to recapitulate the way of getting a MDR, which influenced by the GUP. In Ref. [54], according to the GUP which is proposed Kempf, Mangano, and Mann, we derive a kind of MRD, and used it to show that the GUP parameters can be constrained by the gravitational wave event GW150914 and GW151226. The expression of the GUP is given by

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \beta (\Delta p)^2 \right],
\]

where \( \Delta x \) and \( \Delta p \) are the uncertainties for position and momentum, respectively. \( \beta = \beta_0 \epsilon^2 = \beta_0 / M_p^2 = \beta_0 / E_p^2 \) with the GUP parameters \( \beta_0 \) [39]. According to Eq. (1), one can easy obtain a nonzero minimal uncertainty \( \Delta x_{\text{min}} \approx \ell_p \sqrt{\beta_0} \). Furthermore, the operators for the position \( x_i \) and momentum \( p_i \) in Eq. (1) can be defined as

\[
x_i = x_{0i}; \quad p_i = p_{0i} (1 + \beta p^2),
\]

where \( x_{0i} \) and \( p_{0i} \) satisfying the canonical commutation relations, that is, \([x_{0i}, p_{0j}] = i \hbar \delta_{ij} \). \( \ell_p \), \( M_p \), and \( E_p \) are represent the Planck length, Planck mass, and Planck energy, respectively. When considering a general gravitational background metric \( ds^2 = g_{ab} dx^a dx^b = g_{00} dt^2 + g_{ij} dx^i dx^j \) the modified square of the four-momentum can be expressed as [40]

\[
p_a p^a = g_{ab} p^a p^b = -g_{00} \left( p^0 \right)^2 + g_{ij} p^i p^j (1 + \beta p^2)^2.
\]

It is normally set that \( g_{00} = -1 \), in which case the spacetime of MDR is flat. Next, by ignoring the higher order term \( \mathcal{O} (\beta) \), the equation above can be rewritten as

\[
p_a p^a = - \left( p^0 \right)^2 + p^2 + 2 \beta p^2 p^2.
\]

In the right side of Eq. (5), the first two terms form the standard massless dispersion relation, that is \( - \left( p^0 \right)^2 + p^2 = 0 \). Thus, Eq. (5) can be rewritten as \( p_a p^a = 2 \beta p^2 p^2 \), and its time component of the momentum becomes

\[
\left( p^0 \right)^2 = p^2 (1 - 2 \beta p^2) .
\]
Using the definition of a particle’s energy \( E = p^0 \), the energy of a particle can be expressed in terms of the three spatial momentum and mass as follows

\[
E^2 - p^2 \left(1 - 2\beta p^2\right) = 0.
\]

It should be noted that the \( p^2 \) in the parentheses of Eq. (4) still satisfying the standard massless energy-momentum dispersion relation, that is \( E^2 = p^2 \). Therefore, the Eq. (7) can be finally rewritten as [54]

\[
E^2 - p^2 \left(1 - 2\beta_0 E^2 / E_p^2\right) = 0.
\]

Next, comparing Eq. (8) with Eq. (1), it is natural obtain a new kind of rainbow functions:

\[
f \left(E/E_p\right) = 1, \quad g \left(E/E_p\right) = 1 - 2\beta_0 E^2 / E_p^2.
\]

Now, by employing the GUP Eq. (1) and a general gravitational background metric, we derive a new kind of rainbow functions (9). It is obviously that Eq. (9) contains the GUP parameter \( \beta_0 \). In addition, Eq. (9) satisfying the conditions of \( \lim_{E/E_p \to 0} f \left(E/E_p\right) = 1 \) and \( \lim_{E/E_p \to 0} g \left(E/E_p\right) = 1 \) at low energies. In the subsequent discussions, by incorporating the rainbow functions (9) with a specific line element, we investigate the modified thermodynamics of a rainbow black hole.

### 3 The modified thermodynamics of Schwarzschild black hole in the frame work of RG

In this section, by utilizing the rainbow functions (9) with the definition of surface gravity and first law of thermodynamics, we calculate the modified thermodynamics of rainbow SC black hole. In Ref. [13], the line element of rainbow SC black hole is given by

\[
ds^2 = Adt^2 + Bdr^2 + Cd\Omega^2
\]

\[
= \frac{1 - (2GM/r)}{f^2(E/E_p)} dr^2 + \frac{g^2(E/E_p)}{[1 - (2GM/r)]} dr^2 + \frac{r^2}{g^2(E/E_p)} d\Omega^2,
\]

where \( d\Omega^2 \) denotes the line elements of 2-dimensional hypersurfaces, which can be expressed as \( d\theta^2 + \sin^2 \theta d\phi^2 \). Obviously, metric (10) has only one root of \( r_H = 2GM \) satisfying \( A(r_H) = 0 \). For \( f \left(E/E_p\right) = g \left(E/E_p\right) = 1 \), Eq. (10) is reduced to the original metric of SC black hole. According to the line element of rainbow SC black hole, the surface gravity on the horizon is given by

\[
\kappa = \frac{1}{2} \lim_{r \to r_H} \left[ \frac{g^{11}}{g^{00}} \right] = \frac{g \left(E/E_p\right)}{f \left(E/E_p\right)} \frac{1}{4GM}.
\]

It is clear that the original surface gravity \( \kappa_0 = 1/4GM \) is modified by the rainbow functions. Using the modified surface gravity, one can easily obtain the modified Hawking temperature of SC black hole

\[
T_H^{\text{modified}} = \frac{\kappa}{2\pi} = \frac{g \left(E/E_p\right)}{f \left(E/E_p\right)} T_H = \sqrt{1 - 2\beta_0 \frac{E^2}{E_p^2} \frac{1}{8\pi GM}},
\]

where \( T_H = 1/8\pi GM \) is the original Hawking temperature of SC black hole. It shows that the modified Hawking temperature depends on the GUP parameter \( \beta_0 \), the ratio \( E/E_p \) and the mass of SC black hole, it means that the GUP and RG have a very significant effect on the thermodynamics of black hole. When \( \beta_0 \to 0 \) or \( E/E_p \to 0 \), the original Hawking temperature is recovered.

In Ref. [29], the authors pointed out that the Heisenberg uncertainty principle (HUP) \( \Delta x \geq 1/\Delta p \) still holds in the RG. Therefore, the HUP can be translated into a lower bound on the energy of a particle emitted in Hawking radiation, that is, \( E \geq 1/\Delta x \). On the other hand, in the vicinity of the black hole surface, the
uncertainty in the position can be taken to be the radius of the SC black hole $\Delta x \approx r_H = 1/2GM$. Thus, one has the following relation

$$E \geq \frac{1}{r_H} = \frac{1}{2GM}.$$  \hspace{1cm} (13)

Substituting Eq. (13) into Eq. (12), the modified Hawking temperature becomes

$$T_H^{\text{modified}} = \sqrt{1 - \frac{\beta_0}{2G^2M^2E_p^2}} T_H = \sqrt{1 - \frac{\beta_0 M_p^2}{2M^2}} \frac{1}{8\pi GM},$$ \hspace{1cm} (14)

where in above equation the relation $E_p^{-2} = M_p^{-2} = G$ is used. It should be noted that the expression in the square root can not be less than zero, namely, $1 - \beta_0 M_p^2/2M^2 \geq 0$. Therefore, one can obtain a minimum mass of rainbow SC black hole, which is

$$M_{\text{min}} = \sqrt{\frac{\beta_0}{2}} M_p.$$ \hspace{1cm} (15)

Eq. (15) shows that the a minimum mass of rainbow SC black hole is related to $\beta_0$ and $M_p$, it indicates that the rainbow SC black hole has the a remnant, which is of the order of Planck scale. The behaviour of the original Hawking temperature $T_H$ and modified Hawking temperature $T_H^{\text{modified}}$ of SC black hole is depicted in Fig. 1.

![Figure 1: The $T - M$ diagram of original SC black hole and rainbow SC black hole, where $\beta_0$ and $M_p$ are equal to 1.](image)

In Fig. 1 the black dotted curve represents the original Hawking temperature of SC black hole for $\beta_0 = 0$, whereas the red solid curve illustrates the modified Hawking temperature of SC black hole for $\beta_0 = 1$. This diagram shows that the original Hawking temperature tends to zero as $M \to 0$. On the contrary, the modified Hawking temperature shows another image of the black hole thermodynamics evolution. At the beginning of the evolution, the behaviour of modified Hawking temperature and that of original Hawking temperature look very much alike, since the effect of quantum gravity is week at large mass scale. However, as the evolution progresses, the mass of the black hole gradually decrease, the behaviour modified Hawking temperature deviated from the original case. The modified temperature rises first, and then is decreases rapidly after it reaches a maximum value. Finally, the modified temperature approaches zero when the mass tends to its minimum value $M_{\text{min}}$.

By using the first law of black hole thermodynamics, the modified entropy associated with the Hawking temperature Eq. (14) is given by

$$S^{\text{modified}} = \frac{2\pi G}{M \sqrt{2 - \beta_0 M_p^2}} \Xi.$$ \hspace{1cm} (16)
where \( \Xi = \sqrt{2M \left(2M^2 - \beta_0 M_p^2\right)} + \beta_0 M_p \sqrt{2M^2 - \beta_0 M_p^2 \ln \left(2M + \sqrt{4M^2 - 2\beta_0 M_p^2}\right)} \). When \( \beta_0 = 0 \), the above equation becomes to the area law of the entropy \( S = A/4 = 4\pi GM^2 \) with the area of black hole \( A \), which is consistent with the original entropy in Ref. [55]. Next, in order to determine whether the rainbow SC black hole exists a remnant, it is necessary to further analyze the specific heat of rainbow SC black. In Ref. [37], the definition of specific heat is given by
\[
C = T H \left(\frac{\partial S}{\partial M}\right) \left(\frac{\partial T_H}{\partial M}\right)^{-1}.
\]
Therefore, the modified specific heat is
\[
C_{\text{modified}} = -\frac{4\pi GM^4}{M^2 - M_p^2 \beta_0} \sqrt{4 - \frac{2M_p^2 \beta_0}{M^2}}.
\]
(17)

From the equation above, it is found that \( C_{\text{modified}} = 0 \) at \( M_{\text{min}} = M_p \sqrt{\beta_0/2} \), which is equal to Eq. (15). For \( \beta_0 = 0 \), Eq. (17) recovers the original case \( C_0 = -8\pi M^2 \). The associated \( C - M \) diagram is displayed in Fig. 2.

![Figure 2: The \( C - M \) diagram of original SC black hole and rainbow SC black hole, where we set \( \beta_0 = M_p = 1 \).](image)

One can see that the value of original specific heat (black dashed curve) is always negative, and it vanishes when \( M \to 0 \). Different from the original one, the modified specific heat (red solid curve) has a vertical asymptote at a certain point, where the temperature of rainbow SC black hole reaches its maximum value. It causes the value of specific heat to change from negative to positive, which implies a thermodynamic phase transition occurred at this certain location. The specific phase transition of rainbow SC black hole will be analyzed in the next section. In addition, it is clear that the modified specific heat vanishes when the mass of the reaches \( M_{\text{min}} \). Since the black holes do not exchange their heat with the surrounding space when the specific heat is equal to zero, we can determine the rainbow SC black hole has a remnant at \( M_{\text{min}} \). Therefore, the remnant mass is \( M_{\text{res}} = M_{\text{min}} \).

Recently, Kim et al. suited the modified black hole temperatures in rainbow gravity by employing three different types of GUP [54]. However, our results are different from theirs, the difference may may be caused by different methods we used.

4 Modified local thermodynamic quantities and phase transition of rainbow SC black hole

It is well known that the black holes have rich phase structures and critical phenomena [43–53]. The phase structures and critical phenomena of black holes have been a subject of fascination for more than forty years since they do not only offer a deep insight into the quantum properties of gravity, but also allows for a deeper understanding of the thermal properties of conformal field theories. Therefore, in this section,
for investigating the phase transition from the hot space to black hole, we need to further study the local thermodynamic quantities of the rainbow SC black hole. According to Refs. [55–57], in order to obtain the local thermodynamic quantities, the rainbow SC black hole needs to be located at the center of a spherical cavity of radius $r$. Hence, the expression of local temperature of rainbow SC black hole is given by

\[
T_{\text{modified}} = \sqrt{\frac{1 - \beta_0 \frac{M_p^2}{2M^2}}{8\pi GM\\sqrt{1 - \frac{2GM}{r}}}},
\]

(18)

It recovers the modified Hawking temperature for $r \to \infty$. When considering the cavity of radius $r$ as an invariable quantity, one can calculate the critical value of black hole’s mass $M_c$, GUP parameter $\beta_c$, and local temperature $T_c$ by the following equations [58]

\[
\left( \frac{\partial T_{\text{modified}}}{\partial M} \right)_r = 0, \quad \left( \frac{\partial^2 T_{\text{modified}}}{\partial M^2} \right)_r = 0
\]

(19)

Substituting Eq. (18) into Eq. (19), one can obtain the critical mass, critical GUP parameter, and critical local temperature as follows

\[
M_c = \frac{4r}{15G}, \quad \beta_c = \frac{16r^2}{375G^2M_p^2}, \quad T_c = \frac{15}{32\pi r} \sqrt{\frac{3}{2}}.
\]

(20)

For the convenience of discussion, here we setting $r = 10$, $G = M_p = 1$ as an example. Thus, one obtain that $M_c = 2.66667$, $\beta_c = 4.26667$ and $T_c = 0.0182741$. According to Eq. (13) and Eq. (20), the corresponding “$T_{\text{local}} - M$” diagram of rainbow SC black hole for different $\beta_0$ is depicted in Fig. 3

Figure 3: The $T_{\text{local}} - M$ diagram of rainbow SC black hole for different $\beta_0$ with $r = 10$ and $\beta_0 = M_p = 1$.
a maximum value; at the final stage \((M_0 \leq M \leq M_1)\), the mass of rainbow SC black hole cannot get smaller than \(M_0\) since the negative temperature becomes violates the laws of thermodynamics. It indicates that the RG can stop the black hole evolution and leads to a remnant. Therefore, one can define a large black hole (LBH) where \(M_1 \leq M \leq M_2\), intermediate black hole (IBH) where \(M_2 \leq M \leq M_3\), and small black hole (SBH) for \(M_0 \leq M \leq M_1\). Notedly, there is only one small black hole for \(T_0 \leq T \leq T_2\). Meanwhile, one large black hole for \(T \geq T_1\), and all the three states for \(T_2 \leq T \leq T_1\).

![Figure 4: The original and modified local temperature versus mass at \(G = M_p = 1\) and \(r = 10\).](image)

Employing the first law of thermodynamics, expression of total internal energy is given by

\[
E_{\text{modified}}^{\text{local}} = \int_{M_0}^{M} T_{\text{modified}}^{\text{local}} dS = \frac{r}{G} \left( \sqrt{1 - \frac{M_p G \sqrt{2 \beta_0}}{r}} - \sqrt{1 - \frac{2GM}{r}} \right),
\]

(21)

When \(\beta_0 \to 0\), the local energy of original SC black hole \(E_{\text{local}} = r \left( 1 - \sqrt{1 - 2GM/r} \right)/G \) is recovered. Then, using the modified local temperature Eq. (18) and the total internal energy Eq. (21), the modified heat capacity at fixed \(r\) can be expressed as

\[
C_{\text{modified}}^{\text{local}} = \left( \frac{\partial E_{\text{modified}}^{\text{local}}}{\partial T_{\text{modified}}^{\text{local}}} \right)_{r} = \frac{8\pi GM^3 (r - 2GM) \sqrt{4M^2 - 2\beta_0 M_p^2}}{M (6GM^2 - 2Mr - 5\beta_0 GM_p^2) + 2rM_p^2 \beta_0}.
\]

(22)

Clearly, when \(\beta_0\) vanishes, the Eq. (22) reduces to the local heat capacity of original SC black hole \(C_{\text{local}} = 8\pi GM^2 (r - 2GM)/(3GM - r)\). According to Eq. (22), the variation of the heat capacity \(C_{\text{modified}}^{\text{local}}\) with the mass \(M\) is plotted in Fig. 5.
As shown in Fig. 5 the rainbow heat capacity of SC black hole (red solid line) changes its sign at \( M_1 \) and \( M_2 \) for which the denominator in Eq. (14) vanishes. Meanwhile, the rainbow heat capacity \( C_{\text{rainbow}} \) goes to zero when mass approaches to \( M_0 \). When \( M \to M_3 \), the rainbow heat capacity tends to the original case (black dashed line) since the effect of GUP is negligible at this point. The rainbow heat capacity is divergent at the inflection points \( M_1 \) and \( M_2 \). Hence, the phase transitions near there are second-orders. It is well known that the positive heat capacity means the thermally stable state, while the negative heat capacity is considered to be the unstable solution. Therefore, it is easy to see that the SBH and the LBH are stable, whereas the intermediate black hole (IBH) is unstable, which means that only SBH and LBH can survive for long time within the frame of RG and GUP theories.

In order to obtain more details of the thermodynamic phase transition of the rainbow SC black hole, we use the three states of rainbow SC black hole to analyze the phase transition. In Refs. [60,61], the free energy is defined as \( F_{\text{on}} = E_{\text{local}} - T_{\text{local}}S \). Therefore, the free energy of rainbow SC black hole is given by

\[
F_{\text{on}}^{\text{modified}} = \frac{r}{G} \left( \sqrt{1 - \frac{M_p G \sqrt{2 \beta_0}}{r}} - \sqrt{1 - \frac{2GM}{r}} \right)
- \frac{\sqrt{2} M (2M^2 - M_p^2 \beta_0) + M_p^2 \beta_0 \sqrt{2M^2 - \beta_0 M_p^2} \ln \left(2M + \sqrt{4M^2 - 2\beta_0 M_p^2}\right)}{4M^2 \sqrt{2 - 4GM/r}}.
\]

(23)

Note that Eq. (23) is reduced to the original free energy for \( \beta_0 = 0 \). Meanwhile, as seen from Fig. 7 that the thermodynamic phase transition of the rainbow SC black hole occurs for \( 0 < \beta_0 < \beta_c \).
Figure 6: Variation in free energy of rainbow SC black hole with local temperature for different $\beta_0$.

From Fig. 7, we show $F_{\text{on}}$ curves with $T_{\text{local}}$ for different $\beta_0$. The value of GUP parameter $\beta_0$ decreases from top to bottom. It is clear that the swallow tail structure appears when the GUP parameter $\beta_0$ is smaller than the critical value $\beta_c$, which indicates there is a two-phase coexistence state. The results are consistent with the profile of $T_{\text{local}} - M$ in Fig. 5. For further investigate the phase transition between the black holes and the hot space, we plot Fig. 8.

Figure 7: (a) shows the original free energy of SC black hole as function of the local temperatures. (b) shows the modified free energy of SC black hole as function of the local temperatures. Here we choose $G = 1$ and $r = 10$.

As seen from Fig. 7(a), it is found that the original free energy vanishes when $M \to 0$ because the vacuum state is Minkowski spacetime. Therefore, the $F_{\text{on}}$ represents the free energy of the hot flat space (HFS). In addition, the local temperature has a minimal value $T'_1$, below which no black hole solution exists. The small-large black hole (Hawking-Page) transition occurs at $T'_c$. For $T < T'_c$, both the free energy of small and large black holes are higher than the $F_{\text{HFS}}$, it means that the HFS is more probable than the SBH and LBH. However, for $T > T'_c$, the free energy of SBH is higher than the $F_{\text{HFS}}$ while the LBH is lower than the $F_{\text{HFS}}$, which indicates that the LBH is more probable than the HFS. Therefore, one can find the radiation collapse to the LBH, and the SBH eventually decays into the LBH thermodynamically [58].
Next, let us focus on Fig. 7(b) for $G = \beta_0 = M_p = 1$ and $r = 10$. In contrast to the conventional one, the rainbow free energy becomes to zero when $M \rightarrow M_{\text{res}}$, which implies the rainbow SC black hole only exits the hot curved space (HCS) \cite{62}. Therefore, we study phase transition between various black hole states and the HCS via analyzing the free energies of the rainbow SC black hole.

(i) The SBH and IBH are degenerate at $T_1$ with the mass $M_1$; the intermediate-large black hole transition occurs at the inflection point $T_2$ corresponding to $M_2$. Interestingly, it is found two intersections points (Hawking-Page-type critical points) between the line of free energy and the $F_{\text{HFS}}$ at $T_1^{(1)}$ and $T_1^{(2)}$ in Fig. 7(b) while only one Hawking-Page phase transition in original case, as seen from Fig. 7(a).

(ii) There is a one-order phase transition at critical temperature $T^{\text{First-order}}_c$ since the images shows a characteristic swallow tail behavior.

(iii) For $0 < T < T_c^{(1)}$, the free energy of IBH and LBH are higher than the $F_{\text{HCS}}^{\text{on}}$, whereas the SBH is lower than $F_{\text{HCS}}^{\text{on}}$, it implies that the stable SBH is more probable than HFS. For $T_c^{(1)} < T < T_1^{\text{First-order}}$, both SBH and LBH are lower than the the $F_{\text{HCS}}^{\text{on}}$. It should be noted that LBH and IBH should decay into SBH since the free energy of SBH is higher than those of IBH and LBH in this region. Consequently, for $0 < T < T_1^{\text{First-order}}$, SBH should undergoes a tunneling, which indicates that the HFS can collapse to SBH. As the local temperature increases, one can find that free energies of black holes satisfy the relation $F_{\text{LBH}}^{\text{on}} < F_{\text{SBH}}^{\text{on}} < F_{\text{HCS}}^{\text{on}} < F_{\text{IBH}}^{\text{on}}$ for $T_1^{\text{First-order}} < T < T_1^{(2)}$. Then, the relation becomes $F_{\text{LBH}}^{\text{on}} < F_{\text{SBH}}^{\text{on}} < F_{\text{IBH}}^{\text{on}} < F_{\text{HCS}}^{\text{on}}$ for $T_1^{(2)} < T < T_1$, it means that HCS does not only collapse into SBH but also into IBH and LBH. However, the IBH with negative heat capacity is unstable. Therefore, it would decay into the SBH. Meanwhile, it is obvious that the free energy of SBH is always higher than free energy of the LBH, it leads to the SBH eventually decays into the LBH thermodynamically. Therefore, for $T_c^{\text{First-order}} < T < T_1$, the HCS would eventually decays into the LBH via the quantum tunneling effect.

5 Discussion and conclusion

In this paper, we first derived a new kind of rainbow functions, which contains the GUP parameter. Then, we considered this this rainbow functions to investigated the modified thermodynamics and phase transitions the rainbow SC black hole. First of all, our results showed that the rainbow functions can change the picture of Hawking radiation and stop the evaporation of a black hole as its size approaches the Planck scale. It naturally leads to a remnant of SC black holes. Then, as seen from Fig. 8, the phase transition occur when the GUP parameter is smaller than the critical points. Therefore, we have set the rainbow parameters as $\beta_0 = 1$ in order for the study to be focused and feasible. Third, since the heat capacity enjoys two divergencies in order $M_1$ and $M_2$. Fourth, according to the “$F_{\text{on}} - T_{\text{local}}$” diagram, one can find an unstable black hole (IBH) interpolating between the stable SBH and LBH, which never exists in the original case. Fifth, the $F_{\text{modified}}^{\text{on}}$ surface showed the swallow tail shape, which indicates that the thermodynamic system of rainbow SC black hole exists a first-order transition. Sixth, it is found two Hawking-Page-type critical points from Fig. 7(b) while only one Hawking-Page critical point in the original case. Finally, for $0 < T < T_1^{(1)}$, since the free energy of SBH is lower than those of the IBH and LBH, the unstable IBH and stable LBH decay into the stable SBH. However, for $T_c^{(1)} < T < T_1$, it is obviously the free energy of stable LBH is always lower than those of the SBH and IBH, this leads to the SBH and IBH eventually collapse into the LBH thermodynamically.

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