1. Introduction

In this paper, we focus on finite planar point sets in general position; that is, no three points are collinear. In 1935, Erdős and Szekeres [1] posed a problem: for any integer \( k \geq 4 \), determine the smallest positive integer \( f(k) \) such that any finite point set of at least \( f(k) \) points has a subset of \( k \) points whose convex hull contains exactly \( k \) vertices. In 1961, they [2] showed that \( f(k) \geq 2^{k-2} + 1 \) for all integer \( k \geq 3 \) and then conjectured that \( f(k) \geq 2^{k-2} + 1 \) for all integer \( k \geq 3 \). In 1974, Bonnice [3] proved that \( f(3) = 3 \) and \( f(4) = 5 \). In 1970, Kalbfleisch et al. [4] showed that \( f(5) = 9 \). In 2006, the computer solution for \( k = 6 \) was presented by Szekeres and Peters [5]; that, \( f(6) = 17 \).

In 2001, Avis et al. [6] posed an interior point problem: for any integer \( k \geq 1 \), determine the smallest positive integer \( g(k) \) such that any finite point set \( P \) of at least \( g(k) \) points has a subset \( Q \) for which the interior of the convex hull of the set \( Q \) contains exactly \( k \) points in the set \( P \). Moreover, they also showed the results that \( g(1) = 1 \) and \( g(2) = 4 \). In 1974, Bonnice [3] showed that \( g(k) \geq 3k - 1 \) for all integer \( k \geq 3 \). In 2008, Wei and Ding [7] showed that \( g(k) \geq 3k \) for all integer \( k \geq 3 \). Moreover, in 2009, they [8] also showed that \( g(3) = 9 \). In 2011, Sroysang [9] showed that \( g(k) \geq 4k \) for all integer \( k \geq 4 \). Moreover, in 2012, he [10] also showed that \( g(k) \geq k^2 \) for all integer \( k \geq 4 \).
2. Preliminaries

In this section, we list propositions and notations about the set \( P \), where \( P \) is a finite planar point set such that no three points are collinear.

An interior point of the set \( P \) is a point of the set \( P \) such that it is not on the boundary of the convex hull of the set \( P \).

We denote notations as follows:

\[
I(P) := \text{the set of interior points of the set } P,
\]

\[
i(P) := \text{the number of elements in the set } I(P),
\]

\[
CH(P) := \text{the convex hull of the set } P,
\]

\[
\text{intCH}(P) := \text{the interior of the set } CH(P),
\]

\[
V(P) := \text{the set of vertices of the set } CH(P),
\]

\[
v(P) := \text{the number of elements in the set } V(P).
\]

For \( Q \subseteq P \),

\[
I^*(Q) := I(P) \cap \text{intCH}(Q),
\]

\[
i^*(Q) := \text{the number of elements in the set } I^*(Q).
\]

For \( x, y, z \in P \),

\[
\Delta xyz := \text{the triangle with vertices } x, y, \text{ and } z.
\]

An edge of the set \( P \) is an edge in \( CH(P) \). A subset \( Q \) of the set \( P \) is called a \( k \)-int subset if \( i^*(Q) = k \).

Note that there is \( Q \subseteq P \) such that \( i^*(Q) \neq i^*(Q) \).

**Proposition 1** (see [8]). 9 is the smallest integer such that any finite point set \( P \) of at least 9 interior points has a subset \( Q \) for which the interior of the convex hull of the set \( Q \) contains exactly 3 points in the set \( P \).

For any positive integer \( k \geq 3 \), we let \( h(k) \) be the smallest integer such that every planar point set \( P \) with no three collinear points and with at least \( h(k) \) interior points has a subset \( Q \) for which the interior of the convex hull of the set \( Q \) contains exactly \( k \) or \( k + 3 \) points of the set \( P \).

For any positive integer \( k \geq 3 \),

\[
h(k) = \min \{ s : i(P) \geq s \Rightarrow \exists Q \subseteq P \text{ s.t. } i^*(Q) = k \text{ or } k + 3 \}.
\]

A finite planar point set \( P \) is called a deficient point set of type \( P(m, s, k, n) \) and denoted by \( P = P(m, s, k, n) \) if \( v(P) = m, i(P) = s, \) and \( i^*(Q) \neq \{ k, n \} \) for all \( Q \subseteq P \).

An edge \( xy \) of the set \( P(3, s, 3, 3) \) is of type \( k \) if there exists a subset \( Q \) of the set \( P \) with \( i^*(Q) = k \) such that the edge \( xy \) is an edge of the set \( Q \).

**Proposition 2** (see [8]). Every edge of a deficient point set of type \( P(3, 7, 3, 3) \) is of type 2.

3. Main Results

In this section, we will show that 8 is the smallest positive integer such that any finite point set \( P \) of at least 8 interior points has a subset \( Q \) for which the interior of the convex hull of the set \( Q \) contains exactly 3 or 6 points in the set \( P \).

**Lemma 3.** \( h(3) \leq 9 \).

**Proof.** Let \( P \) be a finite planar point set such that \( i(P) \geq 9 \). By Proposition 1, there is a subset \( Q \) of the set \( P \) such that \( i^*(Q) = 3 \). Then \( i^*(Q) \in \{ 3, 6 \} \). Hence, \( h(3) \leq 9 \). \( \square \)

**Lemma 4.** \( h(3) \geq 8 \).

**Proof.** This suffices to show the existence of a deficient point set of type \( P(3, 7, 3, 6) \). We construct a deficient point set \( P \) of type \( P(3, 7, 3, 6) \) as shown in Figure 1. Hence, \( h(3) \geq 8 \). \( \square \)

**Lemma 5.** Let \( P \) be a finite planar point set. Assume that \( v(P) = 3 \) and \( i(P) = 8 \). Then the set \( P \) has a 3-int or 6-int subset.

**Proof.** Suppose that each subset of a planar point set \( P \) is not a 3-int subset. In [8], we have only three different configurations of the type \( P(3, 8, 3, 3) \) as shown in Figures 2, 3, and 4. However, each configuration has a subset \( Q \) for which the interior of the convex hull of the set \( Q \) contains exactly 6 points of the set \( P \). Hence, the set \( P \) has a 3-int or 6-int subset. \( \square \)

**Lemma 6.** Let \( P \) be a finite planar point set. Assume that \( v(P) = 4 \) and \( i(P) = 8 \). Then the set \( P \) has a 3-int or 6-int subset.

**Proof.** Let \( V(P) = \{ x, y, z, \text{ and } w \} \) be such that vertices \( x, y, z, \text{ and } w \) are put into counterclockwise positions, respectively (see Figure 5).
Suppose that each subset of the planar point set $P$ is not a 6-int subset. Then the sets $\Delta xyz, \Delta yzw, \Delta zwx,$ and $\Delta wxy$ are not 6-int subsets.

If $\Delta xyz$ is a 2-int subset, then the set $\Delta zwx$ is a 6-int subset. Then the set $\Delta xyz$ is not a 2-int subset. Similarly, the sets $\Delta yzw, \Delta zwx,$ and $\Delta wxy$ are not 2-int subsets.

Let $T = \{\Delta xyz, \Delta yzw, \Delta zwx, \Delta wxy\}$.

To show that the set $P$ has a 3-int subset, we divide into seven cases.

Case 1. There is an element $A$ in the set $T$ such that the set $A$ is a 3-int subset.

In this case, the set $P$ has a 3-int subset.

Case 2. There is an element $A$ in the set $T$ such that the set $A$ is a 2-int subset.

Without loss of generality, we assume that $A = \Delta xyz$. Then the set $\Delta wxy$ is a 3-int subset. Thus, the set $P$ has a 3-int subset.

Case 3. There is an element $A$ in the set $T$ such that the set $A$ is a 7-int subset.

Without loss of generality, we assume that $A = \Delta xyz$. Then the set $\Delta zwx$ is a 1-int subset. If the set $A$ has a 3-int subset, then the set $P$ has a 3-int subset. Assume that the set $A$ is a deficient point set of type $P(3, 7, 3, 3)$. By Proposition 2, there is a subset $B$ of the set $CH(\Delta xyz)$ with $i^*(B) = 2$ such that the edge $xz$ is an edge of the set $B$. Let $Q = B \cup \Delta zwx$. Then $i^*(Q) = 3$. Thus, the set $P$ has a 3-int subset.

Case 4. There is an element $A$ in the set $T$ such that the set $A$ is a 1-int subset.

Without loss of generality, we assume that $A = \Delta xyz$. Then the set $\Delta wxy$ is a 7-int subset. Similar to Case 3, the set $P$ has a 3-int subset.

Case 5. There is an element $A$ in the set $T$ such that the set $A$ is an 8-int subset.

By Lemma 5, the set $P$ has a 3-int subset.

Case 6. There is an element $A$ in the set $T$ such that the set $A$ is a 0-int subset.

Without loss of generality, we assume that $A = \Delta xyz$. Then the set $\Delta zwx$ is an 8-int subset. By Lemma 5, the set $P$ has a 3-int subset.

Case 7. The sets $\Delta xyz, \Delta yzw, \Delta zwx$, and $\Delta wxy$ are 4-int subsets.

If one of them has a 3-int subset, then the set $P$ has a 3-int subset. Assume that they are deficient point sets without a 3-int subset. If the edge $xz$ of the set $\Delta xyz$ is of type 2, then we obtain that $i^*(P \setminus \{y\}) = 6$. It follows that, the edge $xz$ of the set $\Delta xyz$ is of type 0 or type 1. If the edge $xz$ of the set $\Delta xyz$ is of type 0, then the edge $xy$ of the set $\Delta xyz$ is of type 3, so the set $P$ has a 3-int subset. Next, we will assume that the edge $xz$ of the set $\Delta xyz$ is of type 1. Similarly, it suffices to assume that the edge $xz$ of the set $\Delta zwx$ is only of type 1, the edge $yw$ of the set $\Delta yzw$ is only of type 1, and the edge $yw$ of the set $\Delta yzw$ is only of type 1. Hence, we obtain only one possible configuration as shown in Figure 6.

However, there is a subset $Q$ of $P$ such that $i^*(Q) = 3$, as shown in Figure 7. Thus, the set $P$ has a 3-int subset.

Therefore, the set $P$ has a 3-int or 6-int subset. This proof is completed.

**Lemma 7.** Let $P$ be a finite planar point set. Assume that $v(P) \geq 5$ and $i(P) = 8$. Then the set $P$ has a 3-int or 6-int subset.
Figure 6: The edges $xz$ and $yw$ are only of type 1.

Figure 7: A 3-int subset of the set in Figure 6.

Proof. Let $V(P) = m$ and $V(P) = \{v_1, v_2, \ldots, v_m\}$ be such that vertices $v_1, v_2, \ldots, v_m$ are put into counterclockwise positions, respectively (see in Figure 8).

Suppose that each subset of the set $P$ is not a 6-int subset.

Then the set $\Delta v_j v_j v_{j+1}$ is not a 6-int subset for all $j$.

Let $T = \{\Delta v_j v_j v_{j+1} \mid j = 2, 3, \ldots, m\}$.

To show that the set $P$ has a 3-int subset, we divide into six cases.

**Case 1.** There is an element $A$ in the set $T$ such that the set $A$ is a 3-int subset.

In this case, the set $P$ has a 3-int subset.

**Case 2.** There is an element $A$ in the set $T$ such that the set $A$ is a 7-int subset.

Then the set $\Delta v_j v_j v_{j+1}$ is a 1-int subset for some $t$. Without loss of generality, we assume $A = \Delta v_j v_j v_{j+1}$ for some $j > t$.

If the set $A$ has a 3-int subset, then the set $P$ has a 3-int subset. Assume that the set $A$ is a deficient point set of type $P(3, 7, 3, 3)$. By Proposition 2, there is a subset $B$ of the set $CH(A)$ with $t^*(B) = 2$ such that the edge $v_j v_j$ is an edge of the set $B$. Let $Q = B \cup \Delta v_j v_j v_{j+1}$. Then $t^*(Q) = 3$. Thus, the set $P$ has a 3-int subset.

**Case 3.** There is an element $A$ in the set $T$ such that the set $A$ is a 5-int subset.

We divide into three subcases.

**Subcase 3.1.** There is an element $B$ in the set $T$ such that the set $B$ is a 3-int subset.

In this subcase, the set $P$ has a 3-int subset.

**Subcase 3.2.** There exist elements $B, C$, and $D$ in the set $T$ such that the sets $B, C$, and $D$ are 1-int subsets.

It follows that $A = \Delta v_j v_j v_{j+1}$, $B = \Delta v_j v_j v_{j+1}$, $C = \Delta v_j v_j v_{j+1}$, and $D = \Delta v_j v_j v_{j+1}$, where $j, k, r \in \{2, 3, \ldots, m\}$. Without loss of generality, we assume that $|t - j| = \min(|t - j|, |s - j|, |r - j|)$. Then the set $A \cup B$ is a 6-int subset. This is impossible.

**Subcase 3.3.** There exist elements $B, C$ in the set $T$ such that the set $B$ is a 1-int subset and the set $C$ is a 2-int subset.

It follows that $A = \Delta v_j v_j v_{j+1}$, $B = \Delta v_j v_j v_{j+1}$ and $C = \Delta v_j v_j v_{j+1}$, where $j, s, r \in \{2, 3, \ldots, m\}$. If $r$ is not between $t$ and $j$, then the set $A \cup B$ is a 6-int subset which is impossible. Thus, $r$ is between $t$ and $j$. We choose $Q = B \cup C$. Then $t^*(Q) = 3$. Hence, the set $P$ has a 3-int subset.

**Case 4.** There is an element $A$ in the set $T$ such that the set $A$ is a 4-int subset.

We divide into five subcases.

**Subcase 4.1.** There is an element $B$ in the set $T$ such that the set $B$ is a 3-int subset.

In this subcase, the set $P$ has a 3-int subset.

**Subcase 4.2.** There exist elements $B, C, D$, and $E$ in the set $T$ such that the sets $B, C, D$, and $E$ are 1-int subsets.

It follows that $A = \Delta v_j v_j v_{j+1}$, $B = \Delta v_j v_j v_{j+1}$, $C = \Delta v_j v_j v_{j+1}$, $D = \Delta v_j v_j v_{j+1}$, and $E = \Delta v_j v_j v_{j+1}$, where $j, s, r \in \{2, 3, \ldots, m\}$. Without loss of generality, we can assume that $t < s < k < r$. If $j < s$, then the set $C \cup D \cup E$ is a 3-int subset. If $k < j$, then the set $B \cup C \cup D$ is a 3-int subset. Thus, the set $P$ has a 3-int subset if $j < s < k < r$. Next, we will show that the statement "$s < k < j$" is impossible. We suppose that $s < j < k$. Then the set $A \cup C \cup D$ is a 6-int subset which is a contradiction.
**Subcase 4.3.** There exist elements $B$ and $C$ in the set $T$ such that the sets $B$ and $C$ are 2-int subsets.

It follows that $A = \Delta v_j v_{j+1}$, $B = \Delta v_j v_{j+1}$, and $C = \Delta v_j v_{j+1}$, where $j, t, r \in \{2, 3, \ldots, m\}$. Without loss of generality, we assume $|r - j| < |r - j|$. Then the set $A \cup B$ is a 6-int subset. This is impossible.

**Subcase 4.4.** There exist elements $B$, $C$, and $D$ in the set $T$ such that the sets $B$ and $C$ are 2-int subsets.

It follows that $A = \Delta v_j v_{j+1}$, $B = \Delta v_j v_{j+1}$, $C = \Delta v_j v_{j+1}$, and $D = \Delta v_j v_{j+1}$, where $j, t, s, r \in \{2, 3, \ldots, m\}$. Without loss of generality, we can assume that $j < r$. Let $k = \max\{t, s\}$ and $l = \min\{t, s\}$. If $l < r$, then the set $A \cup B \cup C$ is a 6-int subset. If $r < l$, then the set $A \cup D$ is a 6-int subset. Thus, we obtain that $l < r < k$. Then the set $D \cup \Delta v_k v_{k+1}$ is a 3-int subset. Hence, the set $P$ has a 3-int subset.

**Subcase 4.5.** There is an element $B$ in the set $T \setminus \{A\}$ such that the set $B$ is a 4-int subset.

Without loss of generality, we assume that $A = \Delta v_j v_{j+1}$ and $B = \Delta v_j v_{j+1}$, where $j < t$. Let $C = \{v_j, v_r, v_{r+1}\}$ and $D = \{v_j, v_r, v_{r+1}\}$. Then the sets $C$ and $D$ are 6-int subsets. If the set $C$ is a 6-int subset or the set $B$ is an 8-int subset then, by Lemma 6, the set $P$ has a 3-int subset. If the set $C$ is a 8-int subset, then the set $\Delta v_j v_{j+1}$ is a 3-int subset, so the set $P$ has a 3-int subset. If the set $D$ is a 7-int subset, then the set $\Delta v_j v_{j+1}$ is a 3-int subset, so the set $P$ has a 3-int subset. If the set $C$ is a 5-int subset, then the set $\Delta v_j v_{j+1}$ is a 3-int subset, so the set $P$ has a 3-int subset. If the set $D$ is a 5-int subset, then the set $\Delta v_j v_{j+1}$ is a 3-int subset, so the set $P$ has a 3-int subset. Next, we assume that the sets $C$ and $D$ are 4-int subsets. Then $\Delta v_j v_{j+1}$ is a 0-int subset. Then the set $\{v_j, v_{j+1}, v_r, v_{r+1}\}$ is an 8-int subset. By Lemma 6, the set $P$ has a 3-int subset.

**Case 5.** There is an element $A$ in the set $T$ such that the set $A$ is an 8-int subset.

By Lemma 5, the set $P$ has a 3-int subset.

**Case 6.** We have $i^+(A) \leq 2$ for all $A \in T$.

If there exist elements $A$, $B$, $C$, and $D$ in the set $T$ such that the sets $A$, $B$, $C$, and $D$ are 2-int subsets where the sets $A$, $B$, $C$, and $D$ put into anticlockwise positions, then the set $A \cup B \cup C$ is a 6-int subset. Thus, we obtain that there is an element in the set $T$ such that it is a 1-int subset. It is easy to see that $P$ has a 3-int subset.

Therefore, the set $P$ has a 3-int or 6-int subset. This proof is completed.

**Theorem 8.** One has $h(3) = 8$.

**Proof.** By Lemmas 3 and 4, it follows that $8 \leq h(3) \leq 9$. By Lemmas 5, 6, and 7, we obtain that $h(3) \leq 8$. Hence, $h(3) = 8$.

**4. Conclusion and Discussion**

In [6], 3 is the smallest positive integer such that any finite point set $P$ of at least 3 interior points has a subset $Q$ for which the interior of the convex hull of the set $Q$ contains exactly 3 or 4 points in the set $P$.

In [13], 7 is the smallest positive integer such that any finite point set $P$ of at least 7 interior points has a subset $Q$ for which the interior of the convex hull of the set $Q$ contains exactly 3 or 5 points in the set $P$.

In this paper, 8 is the smallest positive integer such that any finite point set $P$ of at least 8 interior points has a subset $Q$ for which the interior of the convex hull of the set $Q$ contains exactly 3 or 6 points in the set $P$.

In [14], 7 is the smallest positive integer such that any finite point set $P$ of at least 7 interior points has a subset $Q$ for which the interior of the convex hull of the set $Q$ contains exactly 3 or 7 points in the set $P$.

In [15], 8 is the smallest positive integer such that any finite point set $P$ of at least 8 interior points has a subset $Q$ for which the interior of the convex hull of the set $Q$ contains exactly 3 or 8 points in the set $P$.

Moreover, 9 is the smallest positive integer such that any finite point set $P$ of at least 9 interior points has a subset $Q$ for which the interior of the convex hull of the set $Q$ contains exactly 3 or 9 points in the set $P$.

For any positive integer $k \geq 4$, we let $h^*(k)$ be the smallest integer such that every planar point set $P$ with no three collinear points and with at least $h^*(k)$ interior points has a subset $Q$ for which the interior of the convex hull of the set $Q$ contains exactly $3$ or $k$ points in the set $P$.

For any positive integer $k \geq 3$,

$$h^*(k) = \min \{s : i(P) \geq s \Rightarrow \exists Q \subseteq P \text{ s.t. } i^+(Q) = 3 \text{ or } k\}.$$ (2)

Thus, we have the following formulas:

$$h^*(k) = 3 \quad \text{if } k = 4;$$

$$h^*(k) = 7 \quad \text{if } k = 5, 7;$$

$$h^*(k) = 8 \quad \text{if } k = 6, 8;$$

$$h^*(k) = 9 \quad \text{if } k \geq 9.$$ (3)

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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