ASYMPTOTIC BEHAVIOR OF POLYNOMICALLY BOUNDED SOLUTIONS OF LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

NGUYEN VAN MINH AND VU TRONG LUONG

Abstract. In this paper we study the asymptotic behavior of solutions of fractional differential equations of the form $D^\alpha C u(t) = Au(t) + f(t)$ on the half line, where $D^\alpha C u(t)$ is the derivative of the function $u$ in Caputo’s sense, $A$ is generally an unbounded closed operator, $f$ is polynomially bounded. To this end we develop a spectral theory for functions of polynomial growth on the half line. Our main result claims that if $u$ is mild solution of the Cauchy problem such that $\lim_{h \to 0} \sup_{t \geq 0} \|u(t+h) - u(t)/(1 + t)^n\| = 0$, and $\sup_{t \geq 0} \|u(t)/(1 + t)^n\| < \infty$, then, $\lim_{t \to \infty} u(t)/(1 + t)^n = 0$ provided that the spectral set $\Sigma(A, \alpha) \cap i\mathbb{R}$ is countable, where $\Sigma(A, \alpha)$ is defined to be the set of complex numbers $\xi$ such that $\lambda^{\alpha - 1}(\lambda^{\alpha} - A)^{-1}$ is analytic in a neighborhood of $\xi$, and $u$ satisfies some ergodic conditions with zero means. The obtained result extends known results on strong stability of solutions to fractional equations.

1. Introduction

In this paper we deal with the asymptotic behavior of solutions to linear fractional differential equations the form

$$D^\alpha C u(t) = Au(t) + f(t), \quad u(0) = x, \quad 0 < \alpha \leq 1,$$

where $D^\alpha C u(t)$ is the derivative of the function $u$ in the Caputo’s sense.

In recent decades fractional differential equations are of increasing interests to many researchers as this kind of equations allows us to model complex processes. In the model using these equations one can take into account of nonlocal relations in space as well as in time. Due to these properties fractional differential equations have been applied extensively in engineering. We refer the reader to the monographs [16, 19] for an account of applications in Physics and Engineering. For general results and concepts in abstract spaces the reader is referred to [4, 11, 12]. In recent years the asymptotic behavior of mild solutions of fractional differential equations are extensively studied. Among many results we would like to mention [14, 15, 20, 21] that deal with existence, uniqueness of mild solutions as well as their asymptotic behavior. In this short paper we would like to extend a famous result on stability of $C_0$-semigroups due to Sklyar-Shirman [30] in the bounded case, and Arendt-Batty [2], Lyubich-Vu [22] in the general unbounded case. Many

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extensions of this result (that is now referred to as ABLV Theorem) are given in [3, 8, 13, 25, 6, 26], see also [17, 23, 24] for related results on applications of spectral theory of functions to the study of the asymptotic behavior of solutions. One of interesting directions of extension of this theorem is individual versions for stability of bounded mild solutions of evolution equations of the form $u'(t) = Au(t) + f(t), t \geq 0$ (see e.g. [7, 8], [25, Theorem 5.3.6]). Further, stability with weight of individual orbits of $C_0$-semigroups (or more general objects as representations) is also studied in [9, 10] with general conditions on weights. In this direction, a concept of spectrum of a bounded function on the half line is introduced based on its Laplace transform (that is defined to be the subset of the imaginary axe in the complex plane where the Laplace transform of the function has no analytic extension through).

In this paper we choose a simple approach that is based on the analysis of the set of solutions of ordinary differential equations $y' = \lambda y + g(t)$ in order to examine the resolvent $R(\lambda, \mathcal{D})$ of the operator induced by the differentiation operator $d/dt$ in a quotient space. This allows us to define the spectrum of a polynomially bounded function $g$ on the half line. Our main result is Theorem 4.4 that is an extension of the ABLV Theorem for polynomially bounded mild solutions and for fractional evolution equations in Banach spaces. When Eq. (1.1) is homogeneous (that is, $f = 0$), and the homogeneous equation is well posed with $\alpha = 1$ our results have some overlaps with some results obtained in [9, 10]. Otherwise, to our best knowledge, the obtained results of this paper is new, in addition to its simple approach via the differentiation operator $d/dt$.

2. Preliminaries and Notations

Throughout this paper we will denote by $\mathbb{R}, \mathbb{R}^+, \mathbb{C}$ the real line ($-\infty, \infty$), half line $[0, +\infty)$ and the complex plane. For $J$ being either $\mathbb{R}$ or $\mathbb{R}^+$, the notation $BUC(J, X)$ stands for the function space of all bounded and uniformly continuous functions taking values in a (complex) Banach space $X$ with sup-norm. Below we denote $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t > 0, \alpha > 0$.

For a complex number $z$, $\Re z$ denotes its real part. In this paper the single valued power function $\lambda^\alpha$ of the complex variable $\lambda$ is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{i \arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$.

2.1. Fractional differentiation in Caputo’s sense. Let $\alpha > 0, t \geq a$, and $a$ is a fixed number. Then, the fractional operator

$$(2.1) \quad J^\alpha_a u(t) := (g_\alpha * u)(t) = \int_a^t g_\alpha(t-\tau)u(\tau)d\tau$$

is called fractional Riemann-Liouville integral of degree $\alpha$. The function

$$D^\alpha_{a^+} u(t) := \begin{cases} J^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, & n-1 < \alpha < n \in \mathbb{N}, \\ u^{(n)}(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

is called the fractional derivative in Caputo’s sense of degree $\alpha$. By this notation we have for $0 < \alpha \leq 1$

$$J^\alpha_a D^\alpha_{a^+} u(t) = u(t) - u(a).$$
2.2. **Cauchy Problem.** For a fixed $0 < \alpha \leq 1$, consider the Cauchy problem

\[(2.2)\]
\[D^\alpha_C u(t) = Au(t), u(0) = x,\]

where $A$ is generally an unbounded linear operator.

The well-posedness of (2.2) is equivalent to that of the problem

\[(2.3)\]
\[u(t) = x + \int_0^t g_\alpha(t-s)Au(s)\,ds.\]

The reader is referred to the monograph [27] for an extensive study of the well-posedness of this kind of equations when $A$ is generally an unbounded operator. Recent extensions for more general equations can be found in [18] and their references for more details.

Let us consider inhomogeneous linear equations of the form

\[(2.4)\]
\[D^\alpha_C u(t) = Au(t) + f(t), t \geq 0.\]

**Definition 2.1.** A mild solution $u$ of Eq. (2.4) on $\mathbb{R}^+$ is a continuous function on $\mathbb{R}^+$ such that, for each $t \in \mathbb{R}^+$, $J^\alpha u(t) \in D(A)$ and

\[u(t) = AJ^\alpha u(t) + J^\alpha f(t) + u(0).\]

For a fixed integer $n \geq 0$ we will use $BC_n(\mathbb{R}^+, X)$ to denote the space of all continuous function on $\mathbb{R}^+$ with values in $X$ such that

\[(2.5)\]
\[\sup_{t \in \mathbb{R}^+} \|f(t)\| (1 + t)^n < \infty,\]

and the norm of an element $f \in BC_n(\mathbb{R}^+, X)$ is defined to be $\|f\|_n$. Every function satisfying (2.5) is called $n$-bounded. It is easy to see that $BC_n(\mathbb{R}^+, X)$ with norm

\[(2.6)\]
\[\|f\|_n := \sup_{t \in \mathbb{R}^+} \frac{\|f(t)\|}{(1 + t)^n}\]

becomes a normed space.

**Definition 2.2.** We say that a function $f : \mathbb{R}^+ \to X$ is $n$-uniformly continuous if it is continuous and

\[(2.7)\]
\[\lim_{h \to 0} \sup_{t \in \mathbb{R}^+} \frac{\|f(t + h) - f(t)\|}{(1 + t)^n} = 0.\]

We denote by $BUC_n(\mathbb{R}^+, X)$ as the part of $BC_n(\mathbb{R}^+, X)$ consisting of all $n$-uniformly continuous functions from $\mathbb{R}^+$ to $X$.

**Lemma 2.3.** The normed space $(BUC_n(\mathbb{R}^+, X), \|\cdot\|_n)$ is complete, so it is a Banach space.

**Proof.** We will make use of the fact that is widely known in the literature that the function space $BC(\mathbb{R}^+, X)$ with sup-norm is a Banach space. Therefore, if $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in $BUC_n(\mathbb{R}, X)$, then the sequence $\{g_k\}_{k=1}^\infty$, where $g_k(t) := f_k(t)/(1 + t)^n$, is a Cauchy sequence in $BC(\mathbb{R}^+, X)$, so it is convergent to a function
$g \in BC(\mathbb{R}^+, \mathbb{X})$. We are going to show that if $f$ is defined as $f(t) = (1 + t)^n g(t)$, then $f \in BUC_n(\mathbb{R}^+, \mathbb{X})$ and $f$ is the limit of $\{f_k\}_{k=1}^\infty$. Indeed, by assumption,

$$\lim_{k \to \infty} \|f_k - f\|_n = \lim_{n \to \infty} \sup_{t \in \mathbb{R}^+} \|f_n(t) - f(t)\|_{(1 + t)^n} = \lim_{k \to \infty} \sup_{t \in \mathbb{R}^+} \|g_k(t) - g(t)\| = \lim_{k \to \infty} \|g_k - g\| = 0.$$  

(2.8)

This yields that

$$\sup_{t \in \mathbb{R}^+} \|f(t)\|_{(1 + t)^n} < \infty.$$  

Next, we will show that $f$ is $n$-uniformly continuous. In fact, since

$$\frac{\|f(t + h) - f(t)\|}{(1 + t)^n} \leq \frac{\|f(t + h) - f_n(t + h)\|}{(1 + t)^n} + \frac{\|f_n(t + h) - f_n(t)\|}{(1 + t)^n} + \frac{\|f_n(t) - f(t)\|}{(1 + t)^n},$$

for every given $\epsilon > 0$ we can find a (fixed) sufficiently large $N$ such that

$$\sup_{t \in \mathbb{R}^+} \frac{\|f_N(t) - f(t)\|}{(1 + t)^n} < \frac{\epsilon}{6}.$$  

By the $n$-uniformness of $f_N$ there exists a positive $\delta$ such that if $0 < h < \delta$, then

$$\sup_{t \in \mathbb{R}^+} \frac{\|f_n(t + h) - f_n(t)\|}{(1 + t)^n} < \frac{\epsilon}{6}.$$  

Next, for $0 < h < \delta$

$$\frac{\|f(t + h) - f_n(t + h)\|}{(1 + t)^n} = \frac{\|f(t + h) - f_n(t + h)\|}{(1 + t)^n} \cdot \frac{1 + t + h}{1 + t} \leq \sup_{t \in \mathbb{R}^+} \frac{\|f(t) - f_n(t)\|}{(1 + t)^n} \cdot \frac{1 + t + \delta}{1 + t} \leq \frac{\epsilon}{6}(1 + \delta)^n.$$  

If we choose $\delta = \delta_0 := 2^{1/n} - 1$, then $(1 + \delta)^n < 2$, so

$$\sup_{t \in \mathbb{R}^+} \frac{\|f(t + h) - f_n(t + h)\|}{(1 + t)^n} \leq \frac{\epsilon}{3}.$$  

Finally, we have proved that for each $\epsilon > 0$ there exists a $\delta$ such that for $0 < h < \delta$,

$$\sup_{t \in \mathbb{R}^+} \frac{\|f(t + h) - f_n(t + h)\|}{(1 + t)^n} < \epsilon.$$  

(2.9)

This means

$$\limsup_{h \downarrow 0, t \in \mathbb{R}^+} \frac{\|f(t + h) - f(t)\|}{(1 + t)^n} = 0,$$

or, the $n$-uniformness of $f$. Therefore, $f \in BUC_n(\mathbb{R}^+, \mathbb{X})$ and by (2.8) it is the limit of $\{f_n\}_{n=1}^\infty$. The lemma is proved. \qed
Example 2.4. Let \( f \in BC_n(\mathbb{R}^+, \mathbb{X}) \). If its derivative \( f' \) is also an element of \( BC_n(\mathbb{R}^+, \mathbb{X}) \), then \( f \in BUC_n(\mathbb{R}^+, \mathbb{X}) \).

**Proof.** For all \( t \in \mathbb{R}^+ \) and \( h > 0 \), we have

\[
\lim_{h \downarrow 0} \sup_{t \in \mathbb{R}^+} \frac{\| f(t + h) - f(t) \|}{(1 + t)^n} \leq \lim_{h \downarrow 0} \sup_{t \in \mathbb{R}^+} \frac{\| f'(\xi) \|}{(1 + t)^n} \cdot h
\]

Note that by the \( n \)-boundedness of \( f' \),

\[
\frac{\sup_{t \leq \xi \leq t + h} \| f'(\xi) \|}{(1 + t)^n} = \frac{(1 + t + h)^n}{(1 + t)^n} \sup_{t \leq \xi \leq t + h} \| f'(\xi) \| \leq \frac{(1 + t + h)^n}{(1 + t)^n} \sup_{t \leq \xi \leq t + h} (1 + \xi)^n
\]

Therefore,

\[
\lim_{h \downarrow 0} \sup_{t \in \mathbb{R}^+} \frac{\sup_{t \leq \xi \leq t + h} \| f'(\xi) \|}{(1 + t)^n} \cdot h = \lim_{h \downarrow 0} h(1 + h)^n \| f' \|_n = 0.
\]

This completes the proof of the example’s claim. \( \square \)

3. A Spectral Theory of Polynomially bounded Functions

We note that for every function \( f \in BUC_n(\mathbb{R}^+, \mathbb{X}) \) its Laplace transform

\[
\mathcal{L} f(\lambda) := \int_0^\infty e^{-\lambda t} f(t) dt
\]

exists for any \( \Re \lambda > 0 \), so the definition of the spectrum \( Sp_+(f) \) as the set of all reals \( \xi_0 \) such that its Laplace transform has no analytic extension to any neighborhood of \( i \xi_0 \) as in [2] can be formally extended to \( f \in BUC_n(\mathbb{R}^+, \mathbb{X}) \). The problem is how this spectrum can control the asymptotic behavior of the function \( f \) on the half line \( \mathbb{R}^+ \) is not clear due to the unboundedness of polynomially bounded functions \( f \). In what follows we will discuss an approach to the concept of spectrum of \( f \) and how under some further “ergodic” conditions it controls the behavior of the functions \( f \in BUC_n(\mathbb{R}^+, \mathbb{X}) \). We will begin with the translation semigroup \((S(t))_{t \geq 0}\) in \( BUC_n(\mathbb{R}^+, \mathbb{X}) \), i.e., \( S(t)f := f(t + \cdot) \) for each \( f \in BUC_n(\mathbb{R}^+, \mathbb{X}) \).

**Lemma 3.1.** For each \( t \geq 0 \), we have

\[
\| S(t) \|_n \leq (1 + t)^n.
\]

**Proof.** For each \( f \in BUC_n(\mathbb{R}^+, \mathbb{X}) \) we have

\[
\| S(t)f \|_n = \sup_{s \geq 0} \| f(t + s) \| = \sup_{s \geq 0} \left( \frac{\| f(t + s) \|}{(1 + (t + s))^n} \cdot \frac{[1 + (t + s)]^n}{(1 + s)^n} \right) \leq \| f \|_n \cdot \sup_{s \geq 0} \frac{[1 + (t + s)]^n}{(1 + s)^n} = \| f \|_n \cdot \sup_{s \geq 0} \left( 1 + \frac{t}{1 + s} \right)^n \leq (1 + t)^n \| f \|_n.
\]
Let $\mathcal{D}$ denote the differentiation operator $d/dt$ in $BUC_n(\mathbb{R}^+,\mathbb{X})$ with domain

$$D(\mathcal{D}) = \{ f \in BUC_n(\mathbb{R}^+,\mathbb{X}) : \exists f' \in BUC_n(\mathbb{R},\mathbb{X}) \}.$$ 

**Lemma 3.2.** The following assertions are valid:

i) The translation semigroup $(S(t))_{t \geq 0}$ in $BUC_n(\mathbb{R}^+,\mathbb{X})$ is strongly continuous;

ii) The infinitesimal generator $\mathcal{G}$ of $(S(t))_{t \geq 0}$ is the differentiation operator $\mathcal{D}$ in $BUC_n(\mathbb{R}^+,\mathbb{X})$.

**Proof.** (i) Let $f \in BUC_n(\mathbb{R}^+,\mathbb{X})$. Then, we have to show that

$$\lim_{h \to 0^+} \|S(t)f - f\|_n = 0.$$ 

or

$$\limsup_{h \downarrow 0} \frac{\|f(t + h) - f(t)\|}{(1 + t)^n} = 0.$$ 

This is obvious when $f$ is $n$-uniformly continuous.

(ii) If $f \in D(\mathcal{G})$, where $\mathcal{G}$ is the infinitesimal generator of $(S(t))_{t \geq 0}$, we can see easily that the right derivative of $f$ exists and is in $BUC_n(\mathbb{R}^+,\mathbb{X})$, so it is known (see, for instance, [31, pp. 239-240]) that its derivative exists and is equal to its right derivative. Therefore, $\mathcal{G}f = \mathcal{D}f$. This shows that $D(\mathcal{G}) \subset D(\mathcal{D})$ and if $f \in D(\mathcal{G})$, then, $\mathcal{G}f = \mathcal{D}f$.

Conversely, we will show that $D(\mathcal{D}) \subset D(\mathcal{G})$ and if $f \in D(\mathcal{D})$, then, $\mathcal{D}f = \mathcal{G}f$. In fact, $f \in D(\mathcal{D})$ means that $f$ and $f' = \mathcal{D}f \in BUC_n(\mathbb{R}^+,\mathbb{X})$. For each fixed $t \in \mathbb{R}^+$, set

$$g(h) = f(t + h) - f(t) - hf'(t),$$

where $h$ is positive and small. We have

$$\|g(h) - g(0)\| = \|g(h)\| \leq \sup_{0 \leq \xi \leq h} \|g'(\xi)\| \cdot h = \sup_{0 \leq \xi \leq h} \|f'(t + \xi) - f'(t)\| \cdot h$$

Since $f'$ is $n$-uniformly continuous, that is,

$$\limsup_{h \downarrow 0} \frac{\|f'(t + h) - f'(t)\|}{(1 + t)^n} = 0,$$

for given $\epsilon > 0$ there exists a positive $\delta$ such that if $0 < h < \delta$, for all $t \in \mathbb{R}^+$,

$$\frac{\|f'(t + h) - f'(t)\|}{(1 + t)^n} < \epsilon.$$ 

That means, for each given $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < h < \delta$, then

$$\frac{1}{(1 + t)^n} \| \frac{f(t + h) - f(t)}{h} - f'(t) \| \leq \frac{\|f'(t + h) - f'(t)\|}{(1 + t)^n} < \epsilon.$$

for all $t \in \mathbb{R}^+$. Therefore, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < h < \delta$, then

$$\| \frac{S(h)f - f}{h} - \mathcal{D}f \|_n = \sup_{t \in \mathbb{R}^+} \frac{\|f(t + h) - f(t)\|}{h} - f'(t) \| \cdot \frac{1}{(1 + t)^n} < \epsilon.$$
This proves $f \in D(G)$, and $Gf = Df$. \hfill \Box

3.1. Operator $\hat{D}$. Throughout the paper we will use the following notation

$$C_{0,n}(\mathbb{R}^+, \mathbb{X}) := \{ f \in BUC_n(\mathbb{R}^+, \mathbb{X}) : \lim_{t \to \infty} \|f(t)/(1 + t)^n\| = 0 \}. $$

It is easy to see that $C_{0,n}(\mathbb{R}^+, \mathbb{X})$ is a closed subspace $BUC_n(\mathbb{R}^+, \mathbb{X})$, and is invariant under the translation semigroup $(S(t))_{t \geq 0}$. In the space $BUC_n(\mathbb{R}^+, \mathbb{X})$ we introduce the following relation $R$

$$f R g \text{ if and only if } f - g \in C_{0,n}(\mathbb{R}^+, \mathbb{X}). $$

This is an equivalence relation and the quotient space $BUC_n(\mathbb{R}^+, \mathbb{X})/R$ is a Banach space. We will also denote the norm in this quotient space by $\| \cdot \|_n$ if it does not cause any confusion.

The class containing $f \in BUC_n(\mathbb{R}^+, \mathbb{X})$ will be denoted by $\hat{f}$. Define $\hat{D}$ in $BUC_n(\mathbb{R}^+, \mathbb{X})/R$ as follows:

$$D(\hat{D}) := \{ \hat{f} \in \mathbb{Y} : \exists u \in \hat{f}, u \in D(D) \}$$

$$\hat{D}\hat{f} := \hat{D}u.$$  

**Lemma 3.3.** With the above notations, $\hat{D}$ is a well defined single valued linear operator in $\mathbb{Y}$.

**Proof.** First we show that the operator is a well defined single-valued operator. In fact, assuming $\hat{f} \in D(\hat{D})$, we will prove that the definition of $\hat{D}\hat{f}$ does not depend on the choice of representatives $u$ of this class $\hat{f}$. To this end, suppose that $u, v \in \hat{f}$ such that $u, v \in D(D)$. Then, by the definition of $\hat{D}\hat{f}$, $\hat{D}\hat{f} = \hat{D}u$, and at the same time $\hat{D}\hat{f} = \hat{D}v$. We will show that $\hat{D}u = \hat{D}v$, or equivalently, $D(u - v) \in C_{0,n}(\mathbb{R}^+, \mathbb{X})$.

In fact, since $u, v \in \hat{f}$, if we set $h := u - v$, then $h \in C_{0,n}(\mathbb{R}^+, \mathbb{X})$, and $h \in D(D)$. Therefore,

$$\lim_{t \to 0^+} \frac{S(t)h - h}{t} = Dh.$$  

Note that both $S(t)h$ and $h$ are in $C_{0,n}(\mathbb{R}^+, \mathbb{X})$, so is $Dh = D(u - v)$. This proves that $\hat{D}$ is a well defined single valued operator. Its linearity is clear. The lemma is proved. \hfill \Box

For each given $f \in BUC_n(\mathbb{R}^+, \mathbb{X})$ consider the following complex function $\hat{f}(\lambda)$ in $\lambda$ defined as

$$\hat{f}(\lambda) := (\lambda - \hat{D})^{-1}\hat{f}.$$  

**Lemma 3.4.** $\hat{f}(\lambda)$ exists as an analytic function of $\lambda$ in the region $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. Moreover, the following inequality is valid

$$\| (\lambda - \hat{D})^{-1}\hat{f} \|_n \leq \frac{\|f\|_n}{|\Re\lambda|}$$

for all $\lambda \in \mathbb{C} \setminus i\mathbb{R}$.

**Proof.** Consider the equation

$$y' = \lambda y + f(t), \ t \geq 0.$$  

**The case when $\Re\lambda < 0$:** the general solution of the equation in this case is

$$y(t) = e^{\lambda t}y_0 + \int_0^t e^{\lambda(t-s)}f(s)ds, \ y_0 \in \mathbb{X}, t \geq 0.$$
Since $\Re \lambda < 0$ all functions $g(t) = e^{\lambda t}y_0$ are exponentially decay to zero, so they are all in $C_{0,n}(\mathbb{R}^+, \mathbb{X})$. We are going to show that in this case ($\Re \lambda < 0$), $\lambda \in \rho(\tilde{D})$, and the function

$$u : \mathbb{R}^+ \ni t \mapsto \int_0^t e^{\lambda(t-s)} f(s) ds$$

is a representative of the equivalence class $(\lambda - \tilde{D})^{-1} \tilde{f}$. To this purpose, let $g \in C_{0,n}(\mathbb{R}^+, \mathbb{X})$. Then, the general solution of the equation

$$y' = \lambda y + (f(t) + g(t))$$

is the following

$$y(t) = e^{\lambda t} y_0 + \int_0^t e^{\lambda(t-s)} (f(s) + g(s)) ds$$

$$= e^{\lambda t} y_0 + \int_0^t e^{\lambda(t-s)} g(s) ds + \int_0^t e^{\lambda(t-s)} f(s) ds.$$

It suffices to show that the function $h$ defined as

$$h(t) := \int_0^t e^{\lambda(t-s)} g(s) ds,$$

is in $C_{0,n}(\mathbb{R}^+, \mathbb{X})$. In fact, for a given $\epsilon' > 0$ there is a sufficiently large number $T_0$ such that for $t \geq T_0$

$$\frac{\|g(t)\|_n}{(1 + t)^n} < \epsilon'.$$

Next, for $t \geq T_0$ we have

$$\frac{\int_0^t e^{\lambda(t-s)} g(s) ds}{(1 + t)^n} \leq \frac{1}{(1 + t)^n} \left( \int_0^{T_0} e^{\Re \lambda(t-s)} \|g(s)\|_n ds + \int_{T_0}^t e^{\Re \lambda(t-s)} \|g(s)\|_n ds \right)$$

$$\leq \frac{e^{\Re \lambda t}}{(1 + t)^n} \int_0^{T_0} e^{-\Re \lambda s} \|g(s)\|_n ds + \int_{T_0}^t e^{\Re \lambda(t-s)} \|g(s)\|_n ds.$$

Since $\Re \lambda < 0$,

$$\lim_{t \to \infty} \frac{e^{\Re \lambda t}}{(1 + t)^n} \int_0^{T_0} e^{-\Re \lambda s} \|g(s)\|_n ds = 0.$$

For any given $\epsilon > 0$, there is a sufficiently large number $T_1$ such that if $t \geq T_1$, then

$$\frac{e^{\Re \lambda t}}{(1 + t)^n} \int_0^{T_0} e^{-\Re \lambda s} \|g(s)\|_n ds < \frac{\epsilon}{2}.$$

On the other hand,

$$\int_{T_0}^t e^{\Re \lambda(t-s)} \|g(s)\|_n ds \leq \frac{e^{\Re \lambda T_0}}{(1 + T_0)^n} \int_0^{T_0} e^{-\Re \lambda s} \epsilon' ds$$

$$= \frac{\epsilon'}{-\Re \lambda} \left( 1 - e^{\Re \lambda(t-T_0)} \right)$$

$$\leq \frac{\epsilon'}{-\Re \lambda}.$$
Finally, for any given $\epsilon > 0$, if we choose $T = \max(T_0, T_1)$ and $\epsilon' = -\epsilon \Re \lambda / 2$, then, for $t \geq T$

\[
\frac{\|h(t)\|}{(1 + t)^n} < \epsilon.
\]

This shows that $h \in C_{0,n}(\mathbb{R}^+, \mathcal{X})$.

We will show that the function $u$ defined as

\[
u(t) := \int_0^t e^{\lambda(t-s)} f(s) ds.
\]

is in $BUC_n(\mathbb{R}^+, \mathcal{X})$ for every $\Re \lambda < 0$, and $f \in BUC_n(\mathbb{R}^+, \mathcal{X})$. In fact, as noted in the Example (2.4) $u$ is a solution of the equation $u' = \lambda u + f$, so this claim is proved if we can show that $u$ is $n$-bounded. Since $\Re \lambda < 0$, we have

\[
\sup_{t \in \mathbb{R}^+} \left\| \frac{\nu(t)}{(1 + t)^n} \right\| \leq \sup_{t \in \mathbb{R}^+} \frac{\int_0^t e^{\Re \lambda(t-s)} \|f(s)\| ds}{(1 + t)^n} \\
\leq \sup_{t \in \mathbb{R}^+} \int_0^t e^{\Re \lambda(t-s)} \|f(s)\| (1 + s)^n ds \\
= \sup_{t \in \mathbb{R}^+} \int_0^t e^{\Re \lambda(t-s)} \|f(s)\|_n ds \\
= \sup_{t \in \mathbb{R}^+} \left\| \frac{f}{|\Re \lambda|} \right\|_n (1 - e^{\Re \lambda t}) \\
\leq \left\| \frac{f}{|\Re \lambda|} \right\|_n
\]

By the above argument, we have proved that if $\Re \lambda < 0$, $\lambda \in \rho(\tilde{D})$, and

\[
(\lambda - \tilde{D})^{-1} \hat{f} = \hat{u}
\]

and, by (3.10)

\[
\left\| (\lambda - \tilde{D})^{-1} \hat{f} \right\|_n \leq \left\| \frac{f}{|\Re \lambda|} \right\|_n.
\]

**The case when $\Re \lambda > 0$:** the general solution of the equation

(3.11) $y' = \lambda y + f(t),$

where $y(t) \in \mathcal{X}$, and $f \in BUC_n(\mathbb{R}^+, \mathcal{X})$, is

(3.12) $y(t) = e^{\lambda t} y_0 - \int_t^\infty e^{\lambda(t-s)} f(s) ds, \ y_0 \in \mathcal{X}, t \geq 0.$

Note that the function $h(t) := e^{\lambda t} \|y_0\|$ for non-zero $y_0$ grows exponentially as $t \to \infty$. Let us consider the function

\[
g(t) := \int_t^\infty e^{\lambda(t-s)} f(s) ds.
\]
We will show that \( g \) is the only \( n \) bounded solution. Indeed,
\[
\frac{\|g(t)\|}{(1 + t)^n} \leq \frac{1}{(1 + t)^n} \int_{t}^{\infty} e^{\Re \lambda (t-s)} \|f(s)\|(1 + s)^n ds
\]
\[
\leq \frac{\|f\|_n}{(1 + t)^n} \int_{t}^{\infty} e^{\Re \lambda (t-s)(1 + s)^n} ds
\]
\[
= \frac{\|f\|_n}{(1 + t)^n} \int_{t}^{\infty} e^{-\Re \lambda \xi (1 + t - \xi)^n} d\xi
\]
\[
= \|f\|_n \int_{0}^{\infty} e^{-\Re \lambda \xi (1 + t - \xi)^n} d\xi
\]
\[
\leq \frac{\|f\|_n}{\Re \lambda} \int_{0}^{\infty} e^{-\Re \lambda \xi} d\xi
\]
(3.13)

This means, \( g \) is the unique solution in \( BUC_n(\mathbb{R}^+, X) \) of (3.11), and thus \((\lambda - \mathcal{D})^{-1} f = g\). Consequently, \( \lambda \in \rho(\tilde{\mathcal{D}}) \), and by (3.13)
\[
\|(\lambda - \mathcal{D})^{-1} f\|_n \leq \|g\|_n \leq \frac{\|f\|_n}{\Re \lambda}.
\]
(3.14)
This completes the proof of the lemma.

\[\square\]

**Definition 3.5.** Let \( f \in BUC_n(\mathbb{R}^+, X) \). The set of all points \( \xi_0 \in \mathbb{R} \) such that \( \hat{f}(\lambda) \) has no analytic extension to any neighborhood of \( i \xi \) is defined to be the spectrum of \( f \), denoted by \( \sigma_n(f) \).

We will need the following result that was lemma in [3, Lemma 4.6.6], [27, Chap. 0], [23, Lemma 2.17]:

**Lemma 3.6.** Let \( f(z) \) be a complex function taking values in a Banach space \( X \) and be holomorphic in \( \mathbb{C} \setminus i \mathbb{R} \) such that there is a positive number \( M \) independent of \( z \) for which
\[
\|f(z)\| \leq \frac{M}{|\Re z|}, \quad \text{for all } \Re z \neq 0.
\]
Assume further that \( i \xi \in i \mathbb{R} \) is an isolated singular point of \( f(z) \) at which the Laurent expansion is of the form
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z - i \xi)^n;
\]
where
\[
a_n = \frac{1}{2\pi i} \int_{|z-i \xi|=r} \frac{f(z) dz}{(z - i \xi)^{n+1}}, \quad n \in \mathbb{Z}.
\]
Then,
\[
\|r^2 a_{-(n+1)} + a_{-(n+3)}\| \leq 2Mr^{n+2}, \quad n \in \mathbb{Z}.
\]

**Theorem 3.7.** Let \( g \in BUC_n(\mathbb{R}^+, X) \). Then,

i) If \( \xi_0 \) is an isolated point in \( \sigma_n(g) \), then \( i \xi_0 \) is either removable or a simple pole of \( \hat{g}(\lambda) \).
ii) If $\sigma_n(g) = \emptyset$, then $g \in C_{0,n}(\mathbb{R}^+, X)$;

iii) $\sigma_n(g)$ is a closed subset of $\mathbb{R}$.

Proof. (i) By Lemma 3.6 we consider the Laurent expansion of $\hat{g}(\lambda) := R(\lambda, \tilde{D}) \tilde{g}$ in a neighborhood of $i\xi_0$. Assuming that all notations we use are the same as in the lemma, we have

\[
\| r^2 a_{-(n+1)} + a_{-(n+3)} \| \leq 2Mr^{n+2}, \quad n \in \mathbb{Z}.
\]

Therefore, if $n + 2 \geq 1$, by letting $r$ tend to zero we should have

\[
a_{-(n+3)} = 0.
\]

This means, all coefficients $a_{-k}$ are zero with $k \geq 2$. The only possible non-zero coefficients are $a_{-1}, a_0, a_1, a_2, \cdots$. This means i$\xi_0$ is either a simple pole or a removable singular point of the complex function $\hat{f}(\lambda)$.

(ii) We choose any fixed $\xi_0 \in \mathbb{R}$. If $\sigma_n(g) = \emptyset$ we can take $r$ as large as we like. We already have $a_n = 0$ for all $n = -1, -2, \ldots$. From (3.19) if $m \geq 0$, then

\[
\| a_m + \frac{a_{m-2}}{r^2} \| \leq \frac{2M}{r^m+1}.
\]

Therefore, for $m = 0$, for any large $r$

\[
\| a_0 \| \leq \frac{2M}{r}.
\]

This shows $a_0 = 0$. Since $a_m = 0$ for all $m = 0, -1, -2, -3, \ldots$ an easy induction can yield that $a_m = 0$ for all $m$. This means, $\hat{f}(\lambda) = R(\lambda, \tilde{D}) \tilde{f} = 0$. This happens only when $\tilde{f} = 0$, that is, $f \in C_{0,n}(\mathbb{R}^+, X)$.

(iii) This is clear from the definition. □

**Corollary 3.8.** Let $g \in BUC_n(\mathbb{R}^+, X)$, and $i\xi_0$ be an isolated singular point of $\hat{g}(\lambda)$, where $\xi_0 \in \mathbb{R}$, such that

\[
\lim_{\eta \downarrow 0} \eta R(\eta + i\xi_0, \tilde{D}) \tilde{g} = 0.
\]

Then, $i\xi_0$ is a removable singular point of $\hat{g}$.

*Proof.* By (i) of Theorem 3.7, $i\xi_0$ is a simple pole of $R(\lambda, \tilde{D}) \tilde{g}$, so

\[
R(\lambda, \tilde{D}) \tilde{g} = \sum_{k=-1}^{\infty} a_k (\lambda - i\xi_0)^k
\]

Therefore, for small $|\eta|$,\[
\eta R(\eta + i\xi_0, \tilde{D}) \tilde{g} = \eta \sum_{k=-1}^{\infty} a_k (\eta)^k
\]

\[
= a_{-1} + a_0 \eta^1 + a_1 \eta^2 + a_2 \eta^3 + \cdots
\]

By (3.22), $a_{-1} = 0$, that means, $i\xi_0$ is a removable singular point of $\hat{g}$. □
4. Asymptotic behavior of solutions of fractional differential equations

We are going to apply the spectral theory of \( n \)-bounded functions in the previous section to study the asymptotic behavior of mild solutions to fractional differential equations of the form

\[
D^\alpha_C u(t) = Au(t) + f(t), 
\]

where \( \alpha \) is a fixed number, \( 0 < \alpha \leq 1 \), \( A \) is a closed linear operator in a complex Banach space \( X \), \( f \) is an element of \( C_{0,n}(\mathbb{R}^+, X) \). Recall that a mild solution \( u \) on \( \mathbb{R}^+ \) is defined to be a continuous function \( u \) on \( \mathbb{R}^+ \) such that, for each \( t \in \mathbb{R}^+ \), \( J^\alpha u(t) \in D(A) \) and

\[
u(t) = AJ^\alpha u(t) + J^\alpha f(t) + x,
\]

for all \( t \in \mathbb{R}^+ \).

4.1. Estimate of the spectrum of an \( n \)-bounded solution. Below we will denote by \( \rho(A, \alpha) \) the set of all \( \xi_0 \in \mathbb{C} \) such that \( (\lambda^\alpha - A) \) has an inverse \( (\lambda^\alpha - A)^{-1} \) that is analytic in a neighborhood of \( \xi_0 \), and by \( \Sigma(A, \alpha) := \mathbb{C} \setminus \rho(A, \alpha) \).

We first assume that \( \Re \lambda > 0 \). Then, for any \( n \)-bounded function \( h \) by the proof of Lemma 3.4 we have

\[
\hat{h}(\lambda) = (\lambda - \hat{\mathcal{D}})^{-1}\hat{h} = \hat{g},
\]

where

\[
g(t) = \int_t^\infty e^{\lambda(t-s)}h(s)ds
\]

\[
 = \int_0^\infty e^{-\lambda}\xi h(t+\xi)d\xi.
\]

Therefore, for each \( h \in BUC_n(\mathbb{R}^+, X) \),

\[
\left[\hat{h}(\lambda)\right](t) = \mathcal{L}(S(t)\hat{h})(\lambda).
\]

Next, for each \( s \in \mathbb{R}^+ \), let us denote \( u_s(t) := u(t+s), f_s(t) := f(t+s) \) for all \( t \geq 0 \). Then, we have

\[
u_s(t) = AJ^\alpha u_s(t) + J^\alpha f_s(t) + u_s(t).
\]

Taking the Laplace transforms of both sides gives

\[
\mathcal{L}u_s(\lambda) = \lambda^{-\alpha}A\mathcal{L}u_s(\lambda) + \lambda^{-\alpha}\mathcal{L}f_s(\lambda).
\]

Therefore,

\[
\lambda^{1-\alpha}(\lambda^\alpha - A)\mathcal{L}u_s(\lambda) = \lambda^{1-\alpha}\mathcal{L}f_s(\lambda) + u(s).
\]

Next, for \( \lambda \) in a neighborhood of a point \( i\xi_0 \) where \( \xi_0 \in \rho(A, \alpha) \) and \( \xi_0 \neq 0 \),

\[
\mathcal{L}u_s(\lambda) = (\lambda^\alpha - A)^{-1}\mathcal{L}f_s(\lambda) + \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}u(s).
\]

Recall that

\[
\mathcal{L}(S(s)u)(\lambda) = \mathcal{L}u_s(\lambda), \mathcal{L}(S(s)f)(\lambda) = \mathcal{L}f_s(\lambda).
\]

Therefore,

\[
\hat{u}(\lambda) = (\lambda^\alpha - A)^{-1}\hat{f}(\lambda) + \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\hat{u}.
\]
As we assume that $f \in C_{0,n}(\mathbb{R}^+, X)$, $\hat{f}(\lambda) = 0$. Hence, for $\Re \lambda > 0$,
\begin{equation}
\hat{u}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\hat{u}.
\end{equation}

Below we introduce a new notation
\begin{equation}
R_\alpha(\lambda, A) := \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}.
\end{equation}

The lemma below is actually stated in [1, Lemma 2.2]. For the reader’s convenience we restate it in the following form with an adapted proof.

**Lemma 4.1.** Let $u \in BUC_n(\mathbb{R}^+, X)$, $\xi_0 \in \mathbb{R}$, and let the function $G(\lambda)$ (in $\lambda$) be an analytic extension of the function $\hat{u}(\lambda) = (\lambda - \bar{D})^{-1}\hat{u}$ with $\Re \lambda > 0$ on the open disk $B(i\xi_0, r)$ with some positive $r$. Then, $G(\lambda) = \hat{u}(\lambda)$ for $\Re \lambda < 0$ on a disk $B(\xi_0, r)$.

**Proof.** In $B(\xi_0, r)$ the function $\lambda \mapsto (\lambda - \bar{D})R(1, \bar{D})G(\lambda)$ is an analytic function. By assumption, in $B(i\xi_0)$ for $\Re \lambda > 0$,
\begin{align*}
(\lambda - \bar{D})R(1, \bar{D})G(\lambda) &= (\lambda - \bar{D})R(1, \bar{D})R(\lambda, \bar{D})\hat{u} \\
&= R(1, \bar{D})(\lambda - \bar{D})R(\lambda, \bar{D})\hat{u} \\
&= R(1, \bar{D})\hat{u}.
\end{align*}

This yields that the function $\lambda \mapsto (\lambda - \bar{D})R(1, \bar{D})G(\lambda)$ is a constant function $R(1, \bar{D})\hat{u}$ on the whole $B(i\xi_0, r)$. Therefore, if $\Re \lambda < 0$,
\begin{align*}
R(1, \bar{D})G(\lambda) &= R(\lambda, \bar{D})R(1, \bar{D})\hat{u} = R(1, \bar{D})R(\lambda, \bar{D})\hat{u}.
\end{align*}

Finally, for $\Re \lambda < 0$, the above identity yields $G(\lambda) = R(\lambda, \bar{D})\hat{u}$.

\[ \square \]

**Corollary 4.2.** Let $u \in BUC_n(\mathbb{R}^+, X)$ be a mild solution of Eq. (4.1). Then,
\begin{equation}
i\sigma_n(u) \subset \Sigma(A, \alpha) \cap i\mathbb{R}.
\end{equation}

**Proof.** By Lemma 4.1 it suffices to find the set of the points $i\xi$ with $\xi \in \mathbb{R}$ such that $\hat{u}(\lambda)$ with $\Re \lambda > 0$ has an analytic extension to a neighborhood of $i\xi$. By (4.3) the lemma’s claim is clear. \[ \square \]

**Definition 4.3.** A function $h \in BUC_n(\mathbb{R}^+, X)$ is said to be $n$-uniformly ergodic at $i\eta$ if
\begin{equation}
M_\eta(h) := \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\eta, \bar{D})h
\end{equation}
exists as an element of $\in BUC_n(\mathbb{R}^+, X)$.

**Theorem 4.4.** Let $u \in BUC_n(\mathbb{R}^+, X)$ be a mild solution to Eq. (4.1) with $f \in C_{0,n}(\mathbb{R}^+, X)$. Assume further that
\begin{enumerate}
  \item $\Sigma(A, \alpha) \cap i\mathbb{R}$ is countable;
  \item $u$ is $n$-uniformly ergodic at each $i\eta$ of this set, and $M_\eta(u) = 0$.
\end{enumerate}

Then,
\begin{equation}
\lim_{t \to +\infty} \frac{1}{(1 + t)^n} u(t) = 0.
\end{equation}
Assume that Eq. (4.7) is well posed with a resolvent operator \( S\).

A family of operators \( \{S_\alpha(t)\}_{t \geq 0} \subset L(\mathbb{X}) \) is called a resolvent operator of (2.2) if

- \( \{S_\alpha(t)\} \) strongly continuous \( t \geq 0 \) and \( S_\alpha(0) = I \),
- \( S_\alpha(t)D(A) \subset D(A) \) and \( AS_\alpha(t)x = S_\alpha(t)A, x \in D(A), t \geq 0 \),
- \( S_\alpha(t)x \) is a solution of (4.7) for all \( x \in D(A) \).

If Eq. (4.7) has a resolvent operator \( S_\alpha(t) \), then (see [27, Proposition 1.1]) every mild solution \( u \) is of the form

\[
u(t) = S_\alpha(t)u(0).
\]

**Corollary 4.6.** Assume that Eq. (4.7) is well posed with a resolvent operator \( \{S_\alpha(t)\}_{t \geq 0} \) and the following conditions are satisfied

i) \( S_\alpha(t) \) satisfies

\[
sup_{t \geq 0} \frac{\|S_\alpha(t)\|}{(1 + t)^\alpha} < \infty;
\]

ii) \( \Sigma(A, \alpha) \cap i\mathbb{R} \) is countable;

iii) At each \( i\zeta \in \Sigma(A, \alpha) \cap i\mathbb{R}, x \in \mathbb{X} \)

\[
\lim_{\eta \to 0} \eta R_\alpha(\eta + i\zeta, A)x = 0.
\]

Then, every mild solution \( u(\cdot) = S_\alpha(\cdot)x_0 \in BUC_\alpha(\mathbb{R}^+, \mathbb{X}) \) of Eq. (4.7) satisfies

\[
\lim_{t \to \infty} \frac{1}{(1 + t)^n} \|u(t)\| = 0.
\]

**Proof.** It suffices to check the uniform ergodicity condition in Theorem 4.4. By (4.3) we have

\[
0 \leq \lim_{\eta \to 0} \|\eta R_\alpha(\eta + i\zeta, \tilde{D})\tilde{u}\|_n
= \lim_{\eta \to 0} \|\eta \tilde{u}(\eta + i\zeta)\|_n
= \lim_{\eta \to 0} \|\eta R_\alpha(\eta + i\zeta, A)S_\alpha(\cdot)x_0\|_n
\leq \|S_\alpha(\cdot)\|_n \lim_{\eta \to 0} \|\eta R_\alpha(\eta + i\zeta, A)x_0\|
= 0.
\]

\[\square\]
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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF ARKANSAS AT LITTLE ROCK,
2801 S UNIVERSITY AVE, LITTLE ROCK, AR 72204. USA
E-mail address: mvnguyen1@ualr.edu

DEPARTMENT OF MATHEMATICS, TAY BAC UNIVERSITY, SON LA CITY, SON LA, VIETNAM
E-mail address: vutrongluong@gmail.com