Characterizations of Ruled Surfaces in \( \mathbb{R}^3 \) and of Hyperquadrics in \( \mathbb{R}^{n+1} \) via Relative Geometric Invariants

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Abstract

We consider hypersurfaces in the real Euclidean space \( \mathbb{R}^{n+1} \) \((n \geq 2)\) which are relatively normalized. We give necessary and sufficient conditions a) for a surface of negative Gaussian curvature in \( \mathbb{R}^3 \) to be ruled, b) for a hypersurface of positive Gaussian curvature in \( \mathbb{R}^{n+1} \) to be a hyperquadric and c) for a relative normalization to be constantly proportional to the equiaffine normalization.

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1 Preliminaries

In this section we shall fix our notation and state some of the most important notions and formulae concerning the relative Differential Geometry of hypersurfaces in the real Euclidean space \( \mathbb{R}^{n+1} \) \((n \geq 2)\). Our presentation is mainly based on the text [3] and [5]. For a more detailed exposition of the subject the reader might read [4].

In the Euclidean space \( \mathbb{R}^{n+1} \) let \( \Phi = (M, \bar{x}), M \subset \mathbb{R}^n \), be a \( C^r \)-hypersurface defined by an \( n \)-dimensional, oriented, connected \( C^r \)-manifold \( M \) \((r \geq 3)\) and by a \( C^r \)-immersion \( \bar{x} : M \to \mathbb{R}^{n+1} \), whose Gaussian curvature \( K_1 \) never vanishes on \( M \). A \( C^s \)-mapping \( \bar{y} : M \to \mathbb{R}^{n+1} \) \((r > s \geq 1)\) is called a \( C^s \)-relative normalization, if

\[
\text{rank} \left( \{ \bar{x}/1, \bar{x}/2, \ldots, \bar{x}/n, \bar{y} \} \right) = n + 1, \tag{1a}
\]

\[
\text{rank} \left( \{ \bar{x}/1, \bar{x}/2, \ldots, \bar{x}/n, \bar{y}/i \} \right) = n, \quad \forall \; i = 1, 2, \ldots, n, \tag{1b}
\]

for all \((u^1, u^2, \ldots, u^n) \in M\), where

\[
f_{/i} := \frac{\partial f}{\partial u^i}, \; f_{/ij} := \frac{\partial^2 f}{\partial u^i \partial u^j} \quad \text{etc.}
\]

denote partial derivatives of a function (or a vector-valued function) \( f \). We will also say that the pair \((\Phi, \bar{y})\) is a relatively normalized hypersurface of \( \mathbb{R}^{n+1} \).
The covector \( \bar{X} \) of the tangent vector space is defined by

\[ \langle \bar{X}, \bar{x}_{/i} \rangle = 0 \quad \text{and} \quad \langle \bar{X}, \bar{y} \rangle = 1 \quad (i = 1, 2, \ldots, n), \]

where \( \langle \ , \ \rangle \) denotes the standard scalar product in \( \mathbb{R}^{n+1} \).

The quadratic differential form

\[ G = G_{ij} \, du^{i} \, du^{j}, \quad \text{where} \quad G_{ij} := \langle \bar{X}, \bar{x}_{/ij} \rangle, \]

is definite or indefinite, depending on whether the Gaussian curvature \( K_{I} \) of \( \Phi \) is positive or negative, and is called the relative metric of \( \Phi \). From now on we shall use \( G_{ij} \) as the fundamental tensor for “raising and lowering the indices” in the sense of classical tensor notation.

Let \( \bar{\xi} : M \to \mathbb{R}^{n+1} \) be the Euclidean normalization of \( \Phi \). By virtue of (1) the support function of the relative normalization \( \bar{y} \), which is defined by

\[ q := \langle \bar{\xi}, \bar{y} \rangle : M \to \mathbb{R}, \quad q \in C^{\infty}(M), \]

never vanishes on \( M \). In the sequel we choose \( \bar{\xi} \) and \( \bar{X} \) to have the same orientation. Then \( q \) is positive everywhere on \( M \).

Because of (2), it is

\[ \bar{X} = q^{-1} \bar{\xi}, \quad G_{ij} = q^{-1} h_{ij}, \quad G^{(ij)} = q h^{(ij)}, \]

where \( h_{ij} \) are the components of the second fundamental form \( II \) of \( \Phi \) and \( h^{(ij)} \) resp. \( G^{(ij)} \) the inverses of the tensors \( h_{ij} \) and \( G_{ij} \). Let \( \nabla_{i}^{G} \) denote the covariant derivative corresponding to \( G \). By

\[ A_{jkl} := \langle \bar{X}, \nabla_{i}^{G} \nabla_{k}^{G} \bar{x}_{/j} \rangle, \]

is the (symmetric) Darboux tensor defined. It gives occasion to define the Tchebychev-vector \( \bar{T} \) of the relative normalization \( \bar{y} \)

\[ \bar{T} := T^{m} \bar{x}_{/m}, \quad \text{where} \quad T^{m} := \frac{1}{n} A_{i}^{im}, \]

and the Pick-invariant

\[ J := \frac{1}{n (n - 1)} \, A_{jkl} A^{jkl}. \]

We mention, that when the second fundamental form \( II \) is positive definite, so does \( G \) and in this case \( J \geq 0 \) holds on \( M \) (see e.g. [2, p. 133]).

Denoting by \( H_{I} \) the Euclidean mean curvature of \( \Phi \), by \( \nabla^{II} \) resp. \( \Delta^{II} \) the first resp. the second Beltrami differential operator with respect to the fundamental form \( II \) of \( \Phi \) and by \( S_{II} \) the scalar curvature of \( II \), the Pick-invariant is computed by (see [3])

\[ J = \frac{3(n + 2)}{4n(n - 1)} \, q \nabla^{II} \left( \ln q, \ln q - \ln |K_{I}|^{\frac{1}{n+2}} \right) + q \frac{1}{n (n - 1)} P, \]

where \( P \) is the function [6, p. 231]

\[ P = n (n - 1) (S_{II} - H_{I}) + (2K_{I})^{-2} \nabla^{II} K_{I}. \]
The relative shape operator has the coefficients $B^j_i$ such that

$$\bar{y}_i = -B^j_i \bar{x}_j.$$  

The mean relative curvature, which is defined by

$$H := \frac{1}{n} \text{tr} \left( B^j_i \right),$$

is computed by (see [3])

$$H = q H_I + \frac{q}{n} \left[ \triangle^H (\ln q) + \nabla^H \left( \ln q, \ln \left( q |K_I|^{-\frac{1}{2}} \right) \right) \right].$$

The scalar curvature $S$ of the relative metric $G$, which is defined formally and is the curvature of the Riemannian or pseudo-Riemannian manifold $(\Phi, G)$, the mean relative curvature $H$ and the Pick-invariant $J$ satisfy the Theorema Egregium of the relative Differential Geometry, which states that

$$H + J - S = \frac{n}{n-1} \| \bar{T} \|_G,$$

where $\| \bar{T} \|_G := G_{ij} \bar{T}^i \bar{T}^j$ is the relative norm of the Tchebychev-vector $\bar{T}$.

## 2 The Tchebychev-function and some related formulae

We consider the function

$$\varphi := \left( \frac{q}{q_{\text{AFF}}} \right)^{\frac{n+2}{2n}},$$

where

$$q_{\text{AFF}} := |K_I|^{-\frac{1}{n+2}}$$

is the support function of the equiaffine normalization $\bar{y}_{\text{AFF}}$ and we call it the Tchebychev-function of the relative normalization $\bar{y}$. It is known, that for the components of the Tchebychev-vector holds [3, p. 199]

$$T^i = G^{(ij)} (\ln \varphi)_{/j}.$$  

Hence, by (3c), we obtain

$$\bar{T} = \nabla^G (\ln \varphi, \bar{x}) = q \nabla^H (\ln \varphi, \bar{x})$$

and

$$\| \bar{T} \|_G = \nabla^G (\ln \varphi) = q \nabla^H (\ln \varphi).$$

We notice that the Tchebychev-vector vanishes identically iff the Tchebychev-function $\varphi$ is constant, i.e., by (3), iff $q = c q_{\text{AFF}}$, $c \in \mathbb{R}^*$, which means that the relative normalization
\( \tilde{y} \) and the equiaffine normalization \( \tilde{y}_{\text{AFF}} \) are constantly proportional. From the relation (3) we obtain the Pick-invariant of the Euclidean normalization \((q = 1)\)

\[
J_{\text{EUK}} = \frac{1}{n(n-1)} P.
\]

Hence by using (5) we find

\[
(10) \quad J_{\text{EUK}} = S_{II} - H_I + \frac{(n+2)^2}{4n(n-1)} \nabla^{II} (\ln q_{\text{AFF}}).
\]

From (8) and (10) we conclude that the relation (4) can be written as

\[
(11) \quad \frac{J}{q} = \frac{3(n+2)}{4n(n-1)} \left[ \frac{4n^2}{(n+2)^2} \nabla^{II} (\ln \varphi) - \nabla^{II} (\ln q_{\text{AFF}}) \right] + J_{\text{EUK}}.
\]

For the equiaffine \((\varphi = 1)\) Pick-invariant \(J_{\text{AFF}}\) we deduce

\[
(12) \quad \frac{J_{\text{AFF}}}{q_{\text{AFF}}} = -\frac{3(n+2)}{4n(n-1)} \nabla^{II} (\ln q_{\text{AFF}}) + J_{\text{EUK}}.
\]

By subtracting (12) from (11) we obtain

\[
(13) \quad \frac{J}{q} - \frac{J_{\text{AFF}}}{q_{\text{AFF}}} = \frac{3n}{(n-1)(n+2)} \nabla^{II} (\ln \varphi).
\]

Similarly, taking account of (6) and (8), we find

\[
(14) \quad \frac{H}{q} - H_I = \frac{2}{n+2} \triangle^{II} (\ln \varphi) + \frac{4n}{(n+2)^2} \nabla^{II} (\ln \varphi)
\]

\[
- \frac{n-2}{n+2} \nabla^{II} (\ln \varphi, \ln q_{\text{AFF}}) + \frac{1}{n} \nabla^{II} (\ln q_{\text{AFF}}) - \frac{1}{2} \nabla^{II} (\ln q_{\text{AFF}}).
\]

For the mean equiaffine curvature \(H_{\text{AFF}}\) we infer

\[
(15) \quad \frac{H_{\text{AFF}}}{q_{\text{AFF}}} - H_I = \frac{1}{n} \nabla^{II} (\ln q_{\text{AFF}}) - \frac{1}{2} \nabla^{II} (\ln q_{\text{AFF}}).
\]

By subtracting (15) from (14) we obtain

\[
(16) \quad \frac{H}{q} - \frac{H_{\text{AFF}}}{q_{\text{AFF}}} = \frac{2}{n+2} \triangle^{II} (\ln \varphi) + \frac{4n}{(n+2)^2} \nabla^{II} (\ln \varphi) - \frac{n-2}{n+2} \nabla^{II} (\ln \varphi, \ln q_{\text{AFF}}).
\]

The relations (7), (9), (13) and (16) may be combined into

\[
\frac{S}{q} - \frac{J_{\text{AFF}} + H_{\text{AFF}}}{q_{\text{AFF}}} = \frac{2}{n+2} \triangle^{II} (\ln \varphi) - \frac{n(n-2)}{(n+2)^2} \nabla^{II} (\ln \varphi) - \frac{n-2}{n+2} \nabla^{II} (\ln \varphi, \ln q_{\text{AFF}})
\]

and with reference to

\[
(17) \quad S_{\text{AFF}} = J_{\text{AFF}} + H_{\text{AFF}},
\]

where \(S_{\text{AFF}}\) denotes the inner equiaffine curvature, we conclude that

\[
(18) \quad \frac{S}{q} - \frac{S_{\text{AFF}}}{q_{\text{AFF}}} = \frac{2}{n+2} \triangle^{II} (\ln \varphi) - \frac{n(n-2)}{(n+2)^2} \nabla^{II} (\ln \varphi) - \frac{n-2}{n+2} \nabla^{II} (\ln \varphi, \ln q_{\text{AFF}}).
\]
3 Characterizations of ruled surfaces of $\mathbb{R}^3$ and of hyperquadrics of $\mathbb{R}^{n+1}$

Let now $\alpha$ be any real number. By using the relations (13) and (16)–(18) we obtain

$$\frac{\alpha (S-H) + J}{q} = (\alpha + 1) \frac{J_{\text{AFF}}}{q_{\text{AFF}}} - \frac{n [\alpha (n-1) - 3]}{(n-1)(n+2)} \nabla^I \ln \varphi.$$ 

For $\alpha = \frac{3}{n-1}$ we get

$$3 (S-H) + (n-1) J = (n+2) \frac{J_{\text{AFF}}}{q_{\text{AFF}}}.$$ \hspace{1cm} (19)

This result implies the following

**Proposition 3.1.** Let $(\Phi, \bar{y})$ be a relatively normalized hypersurface of $\mathbb{R}^{n+1}$. Then the function

$$\frac{3 (S-H) + (n-1) J}{q}$$

is independent of the relative normalization and vanishes iff $J_{\text{AFF}} = 0$.

On account of the relations (7) and (19) we infer that

$$\| \bar{T} \|_G = \frac{(n-1)(n+2)}{3n} \left( J - \frac{q}{q_{\text{AFF}}} J_{\text{AFF}} \right) = \frac{n+2}{n} \left( H - S + \frac{q}{q_{\text{AFF}}} J_{\text{AFF}} \right).$$ \hspace{1cm} (20)

From (20) it follows immediately that

$$J_{\text{AFF}} = 0 \iff 3n \| \bar{T} \|_G = (n-1)(n+2) J \iff n \| \bar{T} \|_G = (n+2) (H - S).$$ \hspace{1cm} (21)

We suppose that $n = 2$ and $K_I < 0$. It is well known (see [1, p. 125]), that the vanishing of $J_{\text{AFF}}$ characterizes the ruled surfaces of $\mathbb{R}^3$ among the surfaces of negative Gaussian curvature. So, from the relations (19) and (21) we obtain the following characterizations for ruled surfaces in $\mathbb{R}^3$:

**Proposition 3.2.** Let $\Phi \subset \mathbb{R}^3$ be a surface of negative Gaussian curvature. Then $\Phi$ is a ruled surface iff there exists a relative normalization of $\Phi$, for which one of the following equivalent properties holds true:

(a) $3 (S-H) + J = 0$,

(b) $3 \| \bar{T} \|_G = 2 J$,

(c) $\| \bar{T} \|_G = 2 (H - S)$.

Let now be $n \geq 2$ and $K_I > 0$. Moreover, without loss of generality, we assume that the second fundamental form $II$ is positive definite. It is also well-known (see [5, p. 380]) that in this case the equiaffine Pick-invariant is non-negative and that it vanishes iff $\Phi$ is a hyperquadric. So, by using the relations (19) and (21), we can characterize the hyperquadrics of $\mathbb{R}^{n+1}$ among all hypersurfaces of positive Gaussian curvature as the following proposition states:
Proposition 3.3. Let $\Phi \subset \mathbb{R}^{n+1}$ be a hypersurface of positive Gaussian curvature. Then $\Phi$ is a hyperquadric iff there exists a relative normalization of $\Phi$, for which one of the following equivalent properties holds true:

(a) $3(S - H) + (n - 1)J = 0$,  
(b) $3n\|T\|_G = (n - 1)(n + 2)J$,  
(c) $n\|T\|_G = (n + 2)(H - S)$.

4 The vanishing of the Pick-invariant and some integral formulae

Another consequence of relation (13) are the following two propositions:

Proposition 4.1. Let $\Phi \subset \mathbb{R}^{n+1}$ be a hypersurface of positive Gaussian curvature. For the Pick-invariant of every relative normalization $\bar{y}$ the following relation is valid

\[ \frac{J}{q} - \frac{J_{\text{aff}}}{q_{\text{aff}}} \geq 0. \]  
The equality holds iff the relative normalization $\bar{y}$ and the equiaffine normalization $\bar{y}_{\text{aff}}$ are constantly proportional.

Proof. Because of the assumption $K_I > 0$, it is $\nabla^{II}(\ln \varphi) \geq 0$, so from (13) it follows the inequality. Furthermore it is

\[ \frac{J}{q} - \frac{J_{\text{aff}}}{q_{\text{aff}}} = 0 \iff \nabla^{II}(\ln \varphi) = 0 \iff \varphi = \text{const.} \iff q = c q_{\text{aff}}, c \in \mathbb{R}^*, \]

which proves the assertion. \qed

Proposition 4.2. Let $\Phi \subset \mathbb{R}^{n+1}$ be a hypersurface of positive Gaussian curvature. If there is a relative normalization $\bar{y}$, whose Pick-invariant vanishes identically, then $\Phi$ is a hyperquadric. Furthermore $\bar{y}$ is constantly proportional to the equiaffine normalization $\bar{y}_{\text{aff}}$.

Proof. Let $\bar{y}$ be a relative normalization of $\Phi$ with vanishing Pick-invariant. Then, from the relation (13) we obtain

\[ -\frac{J_{\text{aff}}}{q_{\text{aff}}} = \frac{3n}{(n - 1)(n + 2)} \nabla^{II}(\ln \varphi). \]

Because of $J_{\text{aff}} \geq 0$ and $\nabla^{II} \ln \varphi \geq 0$, both members of (23) vanish. But $J_{\text{aff}} \geq 0$ implies that $\Phi$ is a hyperquadric and $\nabla^{II} \ln \varphi = 0$ implies that the function $\varphi$ is constant, which means that $q = c q_{\text{aff}}, c \in \mathbb{R}^*$ and the proof is completed. \qed

We conclude the paper by considering closed surfaces of positive Gaussian curvature (ovaloids) in $\mathbb{R}^3$. For $n = 2$ relation (16) becomes

\[ \frac{H}{q} - \frac{H_{\text{aff}}}{q_{\text{aff}}} = \frac{1}{2}\nabla^{II}(\ln \varphi) + \frac{1}{2}\Delta^{II}(\ln \varphi), \]

from which we have
Proposition 4.3. Let \((\Phi, \bar{y})\) be a relatively normalized ovaloid in \(\mathbb{R}^3\). Then
\[
\iint_{M} \left( \frac{H}{q} - \frac{H_{\text{AFF}}}{q_{\text{AFF}}} \right) \, dO_{II} \geq 0,
\]
where \(dO_{II}\) is the element of area of \(\Phi\) with respect to the second fundamental form \(II\) of \(\Phi\). The equality is valid iff the relative normalization \(\bar{y}\) is constantly proportional to the equiaffine normalization \(\bar{y}_{\text{AFF}}\).

Furthermore, for \(n = 2\), relation (18) becomes
\[
(24) \quad \frac{S}{q} - \frac{S_{\text{AFF}}}{q_{\text{AFF}}} = \frac{1}{2} \Delta^H (\ln \varphi).
\]
From this equation we easily deduce:

Proposition 4.4. Let \((\Phi, \bar{y})\) be a relatively normalized ovaloid of \(\mathbb{R}^3\). If the function
\[
\frac{S}{q} - \frac{S_{\text{AFF}}}{q_{\text{AFF}}}
\]
does not change its sign on \(M\), then the relative normalization \(\bar{y}\) and the equiaffine normalization \(\bar{y}_{\text{AFF}}\) are constantly proportional.

Finally, from the relations (10), (12), (15) and (17) for \(n = 2\) we obtain
\[
(25) \quad \frac{S_{\text{AFF}}}{q_{\text{AFF}}} - S_{II} = \frac{1}{2} \Delta^H (\ln q_{\text{AFF}}).
\]
If we now integrate (24) and (25) over \(M\) we get
\[
\iint_{M} \frac{S}{q} \, dO_{II} = \iint_{M} \frac{S_{\text{AFF}}}{q_{\text{AFF}}} \, dO_{II} = \iint_{M} S_{II} \, dO_{II} = 2\pi \chi,
\]
where \(\chi\) is the Euler characteristic of \(\Phi\). Hence we arrive at

Proposition 4.5. Let \(\Phi\) be a relatively normalized ovaloid of \(\mathbb{R}^3\). Then the following integral formula is valid
\[
\iint_{M} \frac{S}{q} \, dO_{II} = 2\pi \chi,
\]
where \(\chi\) is the Euler characteristic of \(\Phi\).

Corollary 4.6. For an ovaloid \(\Phi \subset \mathbb{R}^3\) the following integral formula is valid
\[
\iint_{M} \frac{S_{\text{AFF}}}{q_{\text{AFF}}} \, dO_{II} = 2\pi \chi.
\]
References

[1] Blaschke, W.: Vorlesungen über Differentialgeometrie II, Affine Differentialgeometrie, Verlag von Julius Springer, Berlin (1923).

[2] Huck, H. et al: Beweismethoden der Differentialgeometrie im Großen, Lecture Notes in Math. 335, Springer-Verlag, Berlin 1973.

[3] Manhart, F.: Relativgeometrische Kennzeichnungen Euklidischer Hypersphären, Geom. Dedicata 29 (1989), 193–207.

[4] Schirokow, P. A. und Schirokow, A. P.: Affine Differentialgeometrie, B. G. Teubner Verlagsgesellschaft, Leipzig 1962.

[5] Schneider, R.: Zur affinen Differentialgeometrie im Großen I, Math. Z. 101 (1967), 375–406.

[6] Schneider, R.: Closed convex hypersurfaces with second fundamental form of constant curvature, Proc. Amer. Soc. 35 (1972), 230–233.