Inelastic thermoelectric transport and fluctuations in mesoscopic systems

Rongqian Wang, Chen Wang, Jincheng Lu and Jian-Hua Jiang

ABSTRACT
In the past decade, a new research frontier emerges at the interface between physics and renewable energy, termed as inelastic thermoelectric effects, where inelastic transport processes play a key role. The study of inelastic thermoelectric effects broadens our understanding of thermoelectric phenomena and provides new routes towards high-performance thermoelectric energy conversion. Here, we review the main progress in this field, with a particular focus on inelastic thermoelectric effects induced by the electron-phonon and electron–photon interactions. We introduce the motivations, the basic pictures, and prototype models, as well as the unconventional effects induced by inelastic thermoelectric transport. These unconventional effects include the separation of heat and charge transport, the cooling by heating effect, the linear thermal transistor effect, nonlinear enhancement of performance, Maxwell demons, and cooperative effects. We find that elastic and inelastic thermoelectric effects are described by significantly different microscopic mechanisms and belong to distinct linear thermodynamic classes. We also pay special attention to the unique aspect of fluctuations in small mesoscopic thermoelectric systems. Finally, we discuss the challenges and future opportunities in the field of inelastic thermoelectrics.

CONTACT Chen Wang wangchen@zju.edu.cn Department of Physics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China; Jincheng Lu jincheng.lu1993@gmail.com Jiangsu Key Laboratory of Micro and Nano Heat Fluid Flow Technology and Energy Application, School of Physical Science and Technology, Suzhou University of Science and Technology, Suzhou, Jiangsu, China; Jian-Hua Jiang jianhuajiang@suda.edu.cn Institute of Theoretical and Applied Physics, School of Physical Science and Technology & Collaborative Innovation Center of Suzhou Nano Science and Technology, Soochow University, Suzhou, Jiangsu, China

ARTICLE HISTORY
Received 07 December 2021
Accepted 15 May 2022

KEYWORDS
Mesoscopic physics; thermoelectric transport; quantum thermodynamics
1. Introduction

The study of the fundamental science of thermoelectric effects will inevitably encounter the tight connection between quantum physics and thermodynamics when describing the microscopic processes [1,2]. Starting from the 1980’s when the fundamental theory of mesoscopic transport was applied to investigate thermoelectric transport [see illustrations in Figure 1(a,b)], the basic elements of quantum mechanics appear in this field, e.g. coherent transport [3–6], dephasing and dissipation [7,8], Onsager’s reciprocal relationship [9–11], and broken time-reversal symmetry induced thermodynamic bounds [12–14]. Moreover, the quantum confinement effect efficiently tunes the density-of-states of electrons, and thus considerably modifies the thermoelectric performance using nano- and meso-structures. However, for decades long, the focus has been mainly on elastic transport that can be decomposed via Büttiker’s theory of multi-terminal transport into many pairs of two-terminal processes, which is commonly believed to be sufficient to describe all thermoelectric transport phenomena [15–30].

Starting from a decade ago, another fundamental category of nonequilibrium processes, i.e. inelastic transport processes, has attracted increasing attention [31–42]. The community gradually became aware of its oddness, which is essentially due to the fact that these inelastic processes cannot be decomposed into many pairs of two-terminal processes [see illustration in Figure 1(c)]. Note that strictly speaking, there exist inelastic processes that can be decomposed in such a way, which is not within the scope of this review. For instance, the Mott-Cutler theory of thermoelectric transport can include inelastic processes between two terminals [43]. The main focus of this review is on the inelastic processes that cannot be described by Büttiker’s theory. In fact, in many cases, the basic structure of such inelastic processes is the correlated transport among multiple terminals at the quantum mechanical level, as illustrated in Figure 1(d).
In the past decades, based on the elastic transport theory, many efforts have been devoted to improving the thermoelectric figure of merit $ZT$ by investigating and engineering the microscopic transport mechanisms to achieve enhanced electrical conductivity and reduced thermal conductivity [44–51]. However, the inevitable correlation between charge and heat transport in the conventional Mott-Cutler theory for thermoelectric effect sets a bottleneck to such an approach. Here, we focus on an alternative approach where the thermoelectric effect is induced instead by inelastic transport mechanisms. In this regime, the theory of thermoelectric figure of merit $ZT$ must be reconsidered. Indeed, the figure of merit, the optimal energy efficiency and output power for inelastic thermoelectric effect are quite different from their conventional counterparts [33,52–58]. Research in this direction indicates that inelastic thermoelectric effects could be a promising approach towards the next-generation high-performance thermoelectric energy conversion and functional devices [32,33,59].

**Figure 1.** (a) Schematic of Landauer’s theory of transport between two reservoirs. (b) A typical system that can be described by Landauer’s transport theory: resonant tunneling through quantum dots. (c) Büttiker’s theory of decomposing complex transport in multi-terminal systems into many pairs of two-terminal transport processes. (d) An inelastic transport process that cannot be decomposed into pairs of two-terminal transport processes.
Before elaborating on the various surprising properties, we first give an inspirative comparison between thermoelectric engines and solar cells. As shown in Figure 2(a), a conventional thermoelectric engine consists of two types of semiconductor materials. One of them is \( n \)-doped, while the other is \( p \)-doped. Their electrical connection and thermal contact with the heat source and sink are achieved in a bridge-like structure in the figure. If this structure is stretched to be straight, as shown in Figure 2(b), it becomes similar to a solar cell (Figure 2(c)). An interesting question arises why are solar cells more efficient than thermoelectric heat engines, despite the fact that their structures are similar? The key difference between solar cells and thermoelectric heat engines is that they rely on different transport mechanisms. Thermoelectric heat engines rely on diffusive thermoelectric transport in the \( p \)- and \( n \)-types of semiconductors which are connected by Ohmic contact via metal electrodes. In contrast, solar cells rely on photon-carrier generation and carrier splitting due to the built-in electric field in the depletion region of the \( p-n \) junction. While diffusive thermoelectric transport is based mostly on elastic transport processes, photon-carrier generation due to solar radiation is typical of inelastic transport processes in semiconductors. Another significant difference is that there are three reservoirs in solar cells, the source, the drain and the Sun. Energy exchange simultaneously takes place among these reservoirs. In contrast, there are only two reservoirs in a thermoelectric heat engine. We believe that the much higher energy efficiency in solar cells (typically \ (> 20\% \eta_c \) with \( \eta_c = 1 - T_E/T_S \) being the Carnot efficiency of solar cells where \( T_E \) is the ambient temperature on earth and \( T_S \) is the black-body radiation temperature of the Sun) \[60,61] , as compared with the lower energy efficiency of thermoelectric heat engines (typically \( < 20\% \eta'_C \) with \( \eta'_C = 1 - T_c/T_h \) denoting the Carnot efficiency of thermoelectric heat engines where \( T_c \) and \( T_h \) are the temperatures of the cold and hot reservoirs) \[62,63] is not only due to their differences in the Carnot efficiency, \( \eta_c \gg \eta'_C \), but also due to the above differences in their transport mechanisms and
thermodynamic properties. These differences may also be responsible for the much higher output power in solar cells [53,55,56]. The above thinking inspired us to study inelastic thermoelectric transport with the aim of developing an approach toward high-performance thermoelectric energy conversion beyond the conventional one. In this sense, solar cells are a particular type of inelastic thermoelectric systems that have been put into industrial applications. Solar cells also provide a prototype demonstration of how mesoscopic inelastic thermoelectric systems can be integrated into macroscopic devices. This review is dedicated to the efforts devoted to the emergent field of inelastic thermoelectric effects which seeks a deeper understanding of the underlying physics, generalization of physical mechanisms, exploration of new effects and new material systems, and investigation of new applications.

In the past decade, research on inelastic thermoelectric effects has made notable progress. There are different types of inelastic thermoelectric systems. For instance, phonon-assisted inelastic thermoelectric systems [33,52,53,55,59,65–69], photon-assisted inelastic thermoelectric systems [57,70–72], magnon-assisted inelastic thermoelectric systems [73], and those systems where inelastic transport is assisted by Coulomb interactions between electrons [74–80]. Due to the limited space, we focus in this review on phonon- and photon-assisted inelastic thermoelectric effects. In this context, we point out that Coulomb-assisted inelastic thermoelectric systems are reviewed in Ref. [32]. At this point, it is necessary to state that to have inelastic thermoelectric effects well defined, we need the phonons (or other collective excitations) to have a temperature different from the electrons. This condition often cannot be met in macroscopic systems; therefore, we discuss inelastic thermoelectric effects mainly in mesoscopic systems. However, it is possible, via microfabrication technologies, to integrate these mesoscopic inelastic thermoelectric systems into macroscopic devices. As stated above, solar cells are a successful demonstration of such integration.

In this review, we start with a general analysis of thermoelectric transport in mesoscopic systems where elastic and inelastic transport processes are formulated with equal footing. Based on this, we give the bounds on linear transport coefficients for the elastic and inelastic transport processes, respectively. In Section IV, we discuss a simple model for phonon-assisted inelastic thermoelectric transport. Unconventional thermoelectric effects induced by inelastic transport processes, such as rectification, transistor, cooling by heating, and cooling by thermal current effects in the nonlinear regime, are considered in Section V. Effects that can lead to enhancement of thermoelectric performance, such as the nonlinear transport effect, cooperative effect, and near-field effect are also introduced. The statistics of efficiency for three-terminal systems with (broken) time-reversal symmetry, thermal transistor amplification
factor, and cooling by heating energy efficiency under the Gaussian fluctuation framework are reviewed in Section X. Thermophotovoltaic systems with near-field enhancement are also reviewed as a special category of inelastic thermoelectric systems. Finally, we summarize and give outlooks in Section XII.

2. Elastic versus inelastic thermoelectric transport in mesoscopic systems

Thermoelectric transport in mesoscopic systems is driven by thermodynamics forces (e.g. temperature gradients and voltage biases). The steady-state transport is characterized by electrical currents and heat currents. The latter consists of contributions from electrons and other quasiparticles such as phonons and photons. Thermoelectric transport can generally be categorized into two main classes: (i) elastic transport and (ii) inelastic transport [59,81].

When a mesoscopic system is connected with two electronic reservoirs, the voltage bias $V$ and the temperature difference $\Delta T = T_h - T_c$ between the two reservoirs (hot and cold reservoirs, with temperatures $T_h > T_c$) drive a charge current $I_e$ and a heat current $I_Q$. In the linear-response regime, the charge and heat currents are related to the thermodynamic affinities (i.e. the voltage bias and the temperature difference) via the Onsager matrix [82]

$$
\begin{pmatrix}
I_e \\
I_Q
\end{pmatrix} =
\begin{pmatrix}
G & L \\
L & K
\end{pmatrix}
\begin{pmatrix}
V \\
\Delta T / T
\end{pmatrix},
$$

which is time-reversal symmetry [9,10]. $T$ is the average temperature of the system. $G$ and $K$ denote the charge and heat conductivities, respectively. $L$ represents thermoelectric effect and the thermopower (or Seebeck coefficient) is $S = L/(TG)$ [83]. The energy efficiency of the two-terminal thermoelectric system is limited by the second law of thermodynamics [84]. In the linear-response regime, the maximum efficiency is given by [1,2,15,85]

$$
\eta_{\text{max}} = \eta_C \sqrt{1 + ZT} - 1 \leq \eta_C,
$$

where $\eta_C = 1 - T_c / T_h$ is the Carnot efficiency. The maximum efficiency shows a monotonous increase as a function of the dimensionless figure of merit $ZT$, where $ZT = L^2/(GK - L^2)$. Clearly, the maximum efficiency $\eta_{\text{max}}$ approaches the Carnot efficiency $\eta_C$ when $ZT$ approaches $\infty$. Unfortunately, high values of $ZT$ are difficult to be achieved. In the definition of $ZT$, the heat conductivity $K$ consists of both the electronic heat conductivity and the phononic heat conductivity. In particular, Mahan and
Sofo proposed that the ‘best thermoelectrics’ can be realized in narrow-band conductors [86]. Their proposal is based on the arguments that electronic heat conductivity can be suppressed in these narrow-band conductors, while a decent Seebeck coefficient can still be achieved. However, this argument leads to a lot of debates [87], and phonon thermal transport will inevitably suppress the figure of merit in these narrow-band conductors. This reveals that there exists an intrinsic correlation between the charge and heat transport since they are both carried by electrons. Although the separation of charge and heat transport is impossible in elastic transport processes, we will show that it becomes possible in inelastic transport processes.

2.1. The elastic thermoelectric transport: from two-terminal to multiple-terminal setup

Landauer’s scattering theory is an effective description of quantum transport in a two-terminal setup [88–90], as shown in Figure 1(a). Later, Büttiker’s multi-terminal version of the scattering theory was placed on a more solid theoretical footing by Ref. [91], which derived it from the Kubo linear-response formalism [see Figure 1(c)]. The Landauer-Büttiker scattering theory is capable of describing the electrical, thermal, and thermoelectric properties of non-interacting electrons in an arbitrary potential, in terms of the probability that the electrons go from one reservoir to another.

Moreover, the Landauer-Büttiker scattering theory is only applicable to ‘elastic transport process’, with each microscopic process only involving two reservoirs. Based on the standard Landauer-Büttiker theory [92–94], the elastic electronic currents are expressed as

\[
I^i_{\text{el}} = \frac{e}{\hbar} \int_{-\infty}^{\infty} dE \sum_{i \neq j} T_{i \rightarrow j} [f_i(E) - f_j(E)],
\]

\[
I^Q_{\text{el}} = \frac{1}{\hbar} \int_{-\infty}^{\infty} dE \sum_{i \neq j} T_{i \rightarrow j} (E - \mu) [f_i(E) - f_j(E)],
\]

respectively, where \( T_{ij} \) is the transmission function from reservoir \( j \) to reservoir \( i \), \( f_i = \{ \exp[(E - \mu_i)/k_B T_i] + 1 \}^{-1} \) is the Fermi-Dirac distribution function, with temperature \( T_i \) in the \( i \)th fermion bath and \( \mu_i \) the corresponding chemical potential. \( k_B \) is the Boltzmann constant. Moreover, the probability conservation requires that \( \sum_{ij} T_{i \rightarrow j} = 1 \) [33]. From Equation (3), it is known that elastic currents are dominated by two-terminal nonequilibrium processes.
2.2. The inelastic thermoelectric transport assisted by a boson bath

For the three-terminal setup shown in Figure 3, the electronic (from reservoir $L$ and $R$) and bosonic heat currents (from boson bath) are nonlinearly coupled. Such nonlinearity mainly stems from the inelastic electron-phonon scattering process, which cooperatively involve three reservoirs. This phonon-assisted transport process in the three-terminal nanodevices is termed as ‘inelastic transport process’. We emphasize that the inelastic transport process in this work is defined for reservoirs (terminals) rather than particles. Therefore, inelastic transport processes must involve interactions between particles from at least three different terminals. While processes involving only two terminals, although they may involve interactions and energy exchange between quasiparticles, are still elastic transport processes. The inelastic transport process in the present work thus unveils a large number of processes ignored in the conventional study of the mesoscopic transport. It is interesting to note that the inelastic current densities flowing into these terminals are the same. Specifically, the inelastic heat currents flowing into the three reservoirs are expressed via the Fermi golden rule [52,65]

![Figure 3](image-url)
\[ I^L_{\text{inel}} = -I^R_{\text{inel}} = \int dE_1 d\omega_3 j_{\text{in}}(E_1, \omega_3), \]

\[ I^L_Q_{\text{inel}} = \int dE_1 d\omega_3 (E_1 - \mu_1) j_{\text{in}}(E_1, \omega_3), \]

\[ I^R_Q_{\text{inel}} = \int dE_1 d\omega_3 (E_1 - \mu_2 + \omega_3) j_{\text{in}}(E_1, \omega_3), \]

\[ I^{\text{ph}}_Q_{\text{inel}} = \int dE_1 d\omega_3 \omega_3 j_{\text{in}}(E_1, \omega_3), \]

where \( j_{\text{in}}(E_1, \omega_3) = C_{\text{in}} f_1(E_1)[1 - f_2(E_2)]N_B(\omega_3) - C_{\text{in}} f_2(E_2)[1 - f_1(E_1)] \]

\[ [1 + N_B(\omega_3)], E_2 - E_1 = \omega_3 \text{ and } N_B(\omega) = [\exp(\omega/k_B T) - 1]^{-1} \]

being the Bose–Einstein distribution function. The transition coefficient \( C_{\text{in}} \) is the probability for electrons/bosons to tunnel from the \( i \)th reservoir into the scatterer.

### 3. Bound on the linear transport coefficients for elastic and inelastic transport

Typically, for a scatterer interacting with three reservoirs, we have three corresponding heat currents. However, due to heat current conservation \( (I^L_Q + I^R_Q + I^{\text{ph}}_Q = 0) \) in the linear-response regime, two of them are independent, e.g. \( I^L_Q \) and \( I^{\text{ph}}_Q \) are two independent heat currents. The transport equation of these heat currents can be expressed as \[14\]

\[
\begin{pmatrix}
I^L_Q \\
I^{\text{ph}}_Q
\end{pmatrix} =
\begin{pmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{pmatrix}
\begin{pmatrix}
\frac{T_L - T_R}{T_R} \\
\frac{T_{\text{ph}} - T_R}{T_R}
\end{pmatrix}.
\]

\( K_{11(22)} \) and \( K_{12} \) are the diagonal and off-diagonal thermal conductances, which are originally derived based on the Onsager theory. These two coefficients are obtained by \( K_{11} = \partial I^L_Q / \partial T_L \), \( K_{12} = \partial I^L_Q / \partial T_{\text{ph}} \), and \( K_{22} = \partial I^{\text{ph}}_Q / \partial T_{\text{ph}} \) in the limit \( T_L, T_R, T_{\text{ph}} \to T \) with \( |T_{L(\text{ph})} - T_R| \ll T_R \).

Then, the bounds of Onsager coefficients based on elastic and inelastic scattering mechanisms can be described, separately. We first consider the generic elastic transport. The elastic coefficients are specified as

\[
K_{ij}^\text{el} = \langle E^2 \rangle j C_{ij}^\text{el} \quad (i = 1, 2, 3),
\]

where the average under the elastic processes is given by \[95\]
\[
\langle O(E) \rangle_{ij} = \frac{\int dE O(E) G_{ij}^{\text{el}}(E)}{\int dE G_{ij}^{\text{el}}(E)},
\]

(7)

with the probability weight

\[
G_{11}^{\text{el}}(E) = (T_{12} + T_{13}) f(E) [1 - f(E)],
\]

(8a)

\[
G_{12}^{\text{el}}(E) = (T_{13}) f(E) [1 - f(E)],
\]

(8b)

\[
G_{22}^{\text{el}}(E) = (T_{13} + T_{23}) f(E) [1 - f(E)].
\]

(8c)

As the transmission probability \( T_{ij} \geq 0 \) is positive, it is straightforward to obtain the boundary of elastic transport coefficients as follows,

\[-1 \leq K_{12}^{\text{el}} / K_{22}^{\text{el}} \leq 0, \quad -1 \leq K_{12}^{\text{el}} / K_{11}^{\text{el}} \leq 0.
\]

(9)

The above expression is presented graphically by the red shadow regime in Figure 4.

While for a typical inelastic device consisting of three terminals, the Onsager coefficients are expressed as [86]

\[
K_{11}^{\text{inel}} = \left\langle E_1^2 \right\rangle C_{11}^{\text{inel}},
\]

(10a)

\[
K_{12}^{\text{inel}} = \left\langle E_1 \omega_3 \right\rangle C_{12}^{\text{inel}},
\]

(10b)

\[
K_{22}^{\text{inel}} = \left\langle \omega_3^2 \right\rangle C_{22}^{\text{inel}}.
\]

(10c)

where the ensemble average over all inelastic processes is carried out as
\[
\langle Q(E, \omega) \rangle = \frac{\iiint dE d\omega Q(E, \omega) G^{\text{inel}}(E, \omega)}{\iiint dE d\omega G^{\text{inel}}(E, \omega)},
\]

with \( G^{\text{inel}} = C_{\text{in}} f_1(E_1)[1 - f_2(E_2)] N_B(\omega_3). \) By applying the Cauchy-Schwarz inequality \( \langle E_1^2 \rangle \langle \omega_3^2 \rangle - \langle E_1 \omega_3 \rangle^2 \geq 0, \) it is interesting to find that inelastic transport coefficients are bounded by

\[
\frac{K_{11}^{\text{inel}}}{K_{12}^{\text{inel}}} \times \frac{K_{22}^{\text{inel}}}{K_{12}^{\text{inel}}} \geq 1.
\]

We have provided a generic description of linear electronic and bosonic transport in the three-terminal geometry. Remarkably, the two simple relationships Equations (9) and (12) hold for all thermodynamic systems in the linear-response regime.

4. The basic model of inelastic thermoelectric transport: three-terminal double qd device

A typical inelastic thermoelectric device consists of three terminals: two electrodes (the source and the drain) and a boson bath (e.g. a phonon bath), which is schematically depicted in Figure 5. In phonon-assisted hopping transport, the figure of merit is limited by the average frequency and bandwidth of the phonons (rather not electrons) involved in the inelastic transport [59]. Hartke et al. [96] experimentally probes the electron–phonon interaction in a suspended InAs nanowire double QD, which consists of electric-dipole coupled to a microwave cavity [97–103].

Specifically, the system is described as the Hamiltonian

\[
\hat{H} = \hat{H}_{\text{DQD}} + \hat{H}_{\text{e-ph}} + \hat{H}_{\text{lead}} + \hat{H}_{\text{tun}} + \hat{H}_{\text{ph}},
\]

with

\[
\hat{H}_{\text{DQD}} = \sum_{i=\ell,r} E_i \hat{c}^\dagger_i \hat{c}_i + (t \hat{c}^\dagger_\ell \hat{c}_r + \text{H.c.}),
\]

\[
\hat{H}_{\text{e-ph}} = \gamma_{\text{e-ph}} \hat{c}^\dagger_\ell \hat{c}_r (\hat{a} + \hat{a}^\dagger) + \text{H.c.},
\]

\[
\hat{H}_{\text{ph}} = \omega_0 \hat{a}^\dagger \hat{a},
\]

\[
\hat{H}_{\text{lead}} = \sum_{j=L,R} \sum_k \epsilon_j k \hat{c}^\dagger_{j,k} \hat{c}_{j,k},
\]
Figure 5. Illustration of three-terminal inelastic transport. An electron left the source into the left QD (with energy $E_l$) hops to the right QD (with a different energy $E_r$) as assisted by a phonon from the phonon bath (with temperature $T_{ph}$). The electron then tunnels into the drain electrode from the right QD. Such a process gives inelastic charge transfer from the source to the drain assisted by the phonon from the phonon bath. Both the process and its time-reversal contribute to the inelastic thermoelectricity in the system. The electrochemical potential and temperature of the source (drain) are $\mu_l$ and $T_l$ ($\mu_r$ and $T_r$), respectively. $t$ is a hopping element between the QDs and $\Gamma_{l/r}$ are the hybridization energies of the dots to the source and drain electrodes, respectively.

$$\hat{H}_{\mathrm{tun}} = \sum_k V_{L,k} \hat{c}_k^{\dagger} \hat{c}_{L,k} + \sum_k V_{R,k} \hat{c}_k^{\dagger} \hat{c}_{R,k} + \text{H.c.,}$$

(14e)

where $\hat{c}_i^{\dagger}$ ($i = \ell, r$) creates an electron in the $i$-th QD with an energy $E_i$, $\gamma_{e-\text{ph}}$ is the strength of electron–phonon interaction, and $\hat{a}^{\dagger}$ ($\hat{a}$) creates (annihilates) one phonon with the frequency $\omega_0$.

For the three-terminal setup in Figure 5, nonequilibrium steady-state quantities of interest are the electric current $I_e$, the electronic heat current traversing from the left reservoir to the right reservoir $I^r_Q = \frac{1}{2} (I^L_Q - I^R_Q)$, and the phonon heat current $I^\text{ph}_Q$, with $I^i_Q$ ($i = L, R, \text{ph}$) denoting the heat current flowing from the $i$th reservoir. Specifically, the inelastic contribution to the currents is obtained from the Fermi golden rule of [52],

$$I_e = eI_N, I^r_Q = \frac{1}{2} (E_l + E_r)I_N, I^\text{ph}_Q = (E_r - E_l)I_N.$$  

(15)

The current factor is $I_N = \Gamma_{l \rightarrow r} - \Gamma_{r \rightarrow l}$, and the transition rates are $\Gamma_{l \rightarrow r} = \gamma_{e-\text{ph}} f_r (1 - f_l) N^\text{ph}_r$ and $\Gamma_{r \rightarrow l} = \gamma_{e-\text{ph}} f_l (1 - f_r) N^\text{ph}_l$, with $N^\text{ph}_p = N_B + \frac{1}{2} \pm \frac{1}{2} \text{sgn}(E_r - E_l)$ and the Bose–Einstein distribution for phonons $N_B = \exp((E_r - E_l)/T_{ph}) - 1]^{-1}$. 


The thermodynamic affinities conjugated to those three currents satisfy the following relation [9,10]

\[ \dot{S}_{\text{tot}} = I_e A_1 + I_Q^e A_2 + I_Q^{ph} A_3, \]  

where these conjugated affinities are

\[ A_1 = \frac{\mu_L - \mu_R}{e} \left( \frac{1}{2T_L} + \frac{1}{2T_R} \right), \quad A_2 = \frac{1}{T_R} - \frac{1}{T_L}, \]  

\[ A_3 = \frac{1}{2T_L} - \frac{1}{2T_R} - \frac{1}{T_{\text{ph}}}. \]  

Hence, based on the phenomenological transport equations in the linear-response regime, the currents are reexpressed as [14,52,54]

\[ \left( \begin{array}{c} I_e \\ I_Q^e \\ I_Q^{ph} \end{array} \right) = \left( \begin{array}{ccc} G & L_1 & L_2 \\ L_1 & K_0^0 & L_3 \\ L_2 & L_3 & K_{pe} \end{array} \right) \left( \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \right), \]  

And two thermopowers are defined as [52]

\[ S_1 = \frac{L_1}{TG}, \quad S_2 = \frac{L_2}{TG}. \]  

In the above Onsager matrix, \( G \) denotes the charge conductivity, \( L_1 \) and \( L_2 \) represent the longitudinal and transverse thermoelectric effects [104–106], respectively. \( K_0^0, K_{pe} \) and \( L_3 \) are the diagonal and off-diagonal thermal conductance, which are originally derived based on the Onsager theory:

\[ L_1 = G \frac{E_i + E_f}{2e}, \quad L_2 = G \frac{E_f - E_i}{e}, \]  

\[ K_0^0 = G \frac{(E_i + E_f)^2}{4e^2}, \quad K_{pe} = G \frac{(E_f - E_i)^2}{e^2}, \]  

\[ L_3 = G \frac{(E_i + E_f)(E_f - E_i)}{2e^2}. \]  

The conductance is \( G = \frac{e^2}{k_B T} \Gamma_{1 \rightarrow 2} \), with \( \Gamma_{1 \rightarrow 2} \) being the inelastic transition rate between the two QDs. We assume here that the coupling between the left QD and the source as well as that between the right QD the drain is much stronger than the coupling between the two QDs.

5. **Unconventional thermoelectric effects induced by inelastic transport**

In this section, we show how phonon-assisted inelastic transport leads to unconventional thermoelectric effects, such as the rectification effect, transistor effect, cooling by heating effect and cooling by thermal current effect.
5.1. Transistors and rectifiers

Diodes and transistors are key components of modern electronics. In recent years, the manipulation and separation of thermal and electrical currents to process information in nanoscale devices have attracted tremendous interests [107–114]. The design and experimental realization of the thermoelectric device present a striking first step in spin caloritronics, which concerns the coupling of heat, spin, and charge currents in magnetic thin films and other nanostructures [115]. Meanwhile, phononic devices, which are devoted to the only use of heat currents for information processing, have also aroused extensive discussions over the past few decades [116].

In Ref. [59], we have shown that thermoelectric rectifier and transistor can be realized in a three-terminal double QD system, in which charge current and electronic and phononic heat currents are inelastically coupled. Specifically, the coupled thermal and electrical transport allows standard rectification, i.e. charge rectification induced by a voltage bias. The magnitudes of the rectification effects are, respectively, defined by $R_e = \frac{I_e(V)+I_e(-V)}{|I_e(V)|+|I_e(-V)|}$ for charge rectification, $R_t = \frac{I_t(Q,\delta T)+I_t(Q,-\delta T)}{|I_t(Q,\delta T)|+|I_t(Q,-\delta T)|}$ for electronic heat rectification, $R_{et} = \frac{I_{et}(\delta T)+I_{et}(-\delta T)}{|I_{et}(\delta T)|+|I_{et}(-\delta T)|}$ for charge rectification induced by the temperature difference $\delta T$, and $R_{te} = \frac{I_{te}^e(V)+I_{te}^e(-V)}{|I_{te}^e(V)|+|I_{te}^e(-V)|}$ for heat rectification induced by voltage bias. The results displayed in Figure 6 including $I_e - V$, $I_t^e - \delta T$, $I_t^e - V$, and the $I_e - \delta T$ curves demonstrate significant rectification effects.

In addition to the diode effect, we further show that the three-terminal QD system is able to exhibit the thermal transistor effect [see Figure 7]. It has been proposed that negative differential thermal conductance is compulsory for the thermal transistor effect [111,116,118]. Here, we remove such restrictions on the thermal transistor effect, directly arising from the second law of thermodynamics.

From the phenomenological Onsager transport equation given by Equation (5), the average heat current amplification factor is then given by

$$\bar{\alpha} = \frac{\partial_{T_{th}} I_Q^L}{\partial_{T_{th}} I_Q^{ph}} = \frac{K_{12}}{K_{22}}.$$  \hspace{2cm} (21)

It should be noted that $\bar{\alpha}$ only relies on the general expression of the transport coefficients $K_{12}$ and $K_{22}$. Specifically, for the elastic thermal transport, $\bar{\alpha}_{el}$ is always below the unit as $-1 < K_{12}^{el}/K_{22}^{el} < 0$ (red shadow regime in Figure 4). While for the inelastic case with the constraint coefficients bound at Equation (12), the average efficiency is given by $\bar{\alpha}_{inel} < |K_{12}^{inel}/K_{12}^{inel}|$, which can be modulated in the regime $\bar{\alpha}_{inel}$. Hence, the stochastic transistor
Figure 6. Charge, heat, and cross rectification effects. Figures adapted from Jiang et al. [59].
may work as $K_{11}^{inel}/K_{12}^{inel} > 1$. Moreover, for the inelastic transport case, the Onsager coefficients are constrained by the second law of thermodynamics, $K_{11}K_{22} - K_{12}^2 \geq 0$. Therefore, the bound of amplification average efficiency is given by $0 < \bar{\alpha} < \infty$ (blue shadow regime in Figure 4).

A realistic example that achieves $\bar{\alpha} > 1$ in the linear-response regime can be found in the three-terminal double QD system [59], which is expressed as

$$\bar{\alpha} = \frac{|E_l - \mu|}{|E_l - E_r|}. \quad (22)$$

When $|E_l - \mu| > |E_l - E_r|$, $\bar{\alpha}$ can be greater than unity. Therefore, we conclude that the thermal transistor effect can then also be realized in the linear response regime, in the absence of negative differential thermal conductance.

To further explain these phenomena, we expand the currents up to second order in affinities
Table 1. Functionality of second-order coefficients.

| Terms (L_{ijk}) | Diode or Transistor effect                        |
|-----------------|--------------------------------------------------|
| L_{111}         | charge rectification                             |
| L_{222}, L_{333}| electronic and phononic heat rectification      |
| L_{233}, L_{222}| off-diagonal heat rectification                 |
| L_{122}, L_{133}| charge rectification by temperature difference  |
| L_{211}, L_{111}| heat rectification by voltage bias              |
| L_{113}, L_{123}| boson-thermoelectric transistor                 |
| L_{212}, L_{112}| other nonlinear thermoelectric effects          |

\[ I_i = \sum_j M_{ij} A_j + \sum_{jk} L_{ijk} A_j A_k + O(A^3), \quad (23) \]

where \( M_{ij} = M_{ij}^{el} + M_{ij}^{inel} \), with \( M_{ij} \) denoting the linear-response coefficients and the second-order terms \( L_{ijk} \) only show up from inelastic transport processes. Practically, \( M_{ij} \) and \( L_{ijk} \) can be calculated with realistic material parameters [59]. The first term on the right-hand side describes the linear response, whereas the second term gives the lowest-order nonlinear response. The functionalities represented by various second-order coefficients are summarized in Table 1.

The influence of the strong electron–phonon interaction on thermoelectric transport is an intriguing research topic in nonequilibrium transport [119–125]. However, the expression of currents in Equation (15) may break down as the electron–phonon interaction becomes strong, where high-order electron-phonon scattering processes should be necessarily included to properly characterize the electron current and energy current. Alternatively, the strong light–matter interaction also provides an excellent way for designing efficient thermoelectric devices. In Ref. [117], we show that significant rectification effects (including charge and Peltier rectification effects) [see Figure 8] and linear thermal transistor effects [see Figure 9] can be enhanced due to the nonlinearity induced by the large electron–photon interaction in circuit-quantum-electrodynamics systems. The above results show that the synergism of electronics and boson in open systems can provide a novel solution for seeking high-performance thermoelectric devices and information storage technology in the future.

5.2. Cooling by heating effects

According to Clausius’ second law of thermodynamics, we know that heat cannot spontaneously transfer from the cold reservoir to the hot reservoir [126]. Usually, the second law is expressed in a two-terminal system. For three-terminal systems, the second law of thermodynamics has a more complex case where some counterintuitive effects can be allowed [127–131]. For example, in Ref. [71], Cleuren et al. proposed that one cold
reservoir can be cooled by two hot reservoirs without changing the rest of the world due to the transport mechanism of inelastic scattering, which is termed as ‘cooling by heating’ effect.

As exemplified in Figure 10, to perform cooling by heating, a device must have three reservoirs (source, drain, and photon bath) and two adjoining quantum dots. Each quantum dot has a lower and upper energy level. The source is kept at ambient temperature $T_L$, photon bath is hotter with $T_{ph} > T_L$, and drain is colder with $T_R < T_L$. The device then utilizes the heat flowing from the photon bath to the source to ‘drag’ heat out of the drain even though the drain is colder than the other two hot reservoirs.

The basic mechanism is that under the influence of high-temperature photons, the electrons with energy lower than the Fermi level in the source will inelastically pass through the lower energy regime of the two quantum dots and tunnel into the drain. Similarly, electrons with energy higher than the Fermi level in the drain will be transported into the source through the

---

Figure 8. The current (a) and the heat current (c) as the function of $\Delta \mu$. (b) Charge rectification $R_e$ and (d) cross rectification $R_{te}$ as function of $E_l$ and $E_r$. Figures adapted from Lu et al. [117].
higher energy regime of the two quantum dots. Simultaneously, the electron needs to absorb one photon to complete the cyclic transition process between two quantum dots. The cooling by heating effect in quantum systems can be understood that as the quantum device is driven by the external work, the heat is extracted from the cooling reservoir and absorbed by the hot reservoir.
The efficiency of cooling by a heating device (refrigerator) is defined as the heat current flowing out of the drain (the drain being refrigerated) divided by the heat current flowing out of the photon bath, i.e.\[ \eta_{\text{CBH}} = \frac{I_Q^R}{I_Q^\text{ph}}. \] The upper bound on such a refrigerator efficiency is given by the condition that no entropy is generated. Then, the corresponding efficiency is given by\[ \eta_{\text{CBH}}^{\text{rev}} = \frac{T_R(T_{\text{ph}} - T_L)}{T_{\text{ph}}(T_L - T_R)}. \] (24)

Meanwhile, the refrigerator reaches the reversible regime with both heat currents\( I_Q^R \) and \( I_Q^\text{ph} \) vanishing simultaneously, while it contains a nonzero cooling efficiency\[ \eta_{\text{CBH}} = \frac{E_2 - E_1}{2E_g}, \] (25)

with the \( E_1 \) and \( E_2 \) being the energies of the right quantum dot, and \( E_g \) being the energy gap between the upper (down) levels. It is worth noting that the increase in the entropy rate of the whole system is not negative, and the system satisfies the second law of thermodynamics for the entropy reduction in the source and drain is compensated by the larger entropy increase in the photon bath.

### 5.3. Cooling by heat current effects

In this section, we show that a nontrivial phonon drag effect, termed by ‘cooling by heat current’ [58], can emerge in four-terminal QD thermo-electric systems with two electrodes and two phonon baths, as shown in Figure 11. The source (or the drain) can be cooled by passing a thermal current between the two phonon baths, without net heat exchange between the heat baths and the electrodes. This effect, which originates from the inelastic-scattering process, could improve the cooling efficiency and output power due to the spatial separation of charge and heat transport [132,133].

Specifically, the system consists of four quantum dots: QDs 1 and 2 with electronic energy \( E_1 \) and \( E_2 \) are coupled with the hot heat bath \( H \), while QDs 3 and 4 with energy \( E_3 \) and \( E_4 \) are coupled with the cold heat bath \( C \). There is one electrical current \( I_e \) flowing from the source to drain and four heat currents, \( I_Q^L, I_Q^R, I_Q^H, \) and \( I_Q^C \). Due to the energy conservation [9,10], i.e.\[ I_Q^L + \frac{\mu_e}{e} I_e + I_Q^H + I_Q^C + I_Q^R - \frac{\mu_R}{e} I_e = 0, \] the entropy production of the whole system is given by [14]\[ \frac{dS}{dt} = I_Q^L A_S + I_Q^R A_{\text{in}} + I_Q^H A_q + I_e A_e, \] (26)

and the affinities are defined as...
Figure 11. Schematic of quantum dots four-terminal thermoelectric devices. There are two parallel transport channels. Each channel has two quantum dots with different energies and a heat bath to enable inelastic transport. The two channels are spatially separated so that the heat bath $H$ ($C$) couples only to the upper (lower) channel. Four heat currents $I^H_Q, I^L_Q, I^R_Q, I^C_Q$ and the electric currents $I_e$ are illustrated.

\begin{align}
A_S &= \frac{1}{T_R} - \frac{1}{T_L}, \\
A_{\text{in}} &= \frac{1}{T_R} - \frac{1}{2T_H} - \frac{1}{2T_C}, \\
A_q &= \frac{1}{T_C} - \frac{1}{T_H}, \\
A_e &= \frac{\mu_L - \mu_D}{eT_R}.
\end{align}

(27)

$I_{Q}^{\text{in}} = I_{Q}^H + I_{Q}^C$ is regarded as the total heat current injected into the central quantum system from the two thermal baths. $I_{Q}^H = (I_{Q}^H - I_{Q}^C)/2$ is the exchanged heat current between the two heat baths intermediated by the central quantum system. $T_i$ ($i = L, R, H, C$) are the temperatures of the four reservoirs, respectively. Here, we restrict our discussions to situations where there is only one energy level in each QD that is relevant for the transport.

In this regime, the heat currents derived from the Fermi golden rule [59] can be written as,

\begin{align}
I_Q^L &= E_1 I_{12} + E_3 I_{34}, \\
I_Q^H &= \omega_u I_{12}, \\
I_Q^C &= \omega_d I_{34}, \\
I_{Q}^{\text{in}} &= \omega_u I_{12} + \omega_d I_{34}, \\
I_{Q}^{\text{out}} &= \frac{1}{2} (\omega_u I_{12} - \omega_d I_{34}).
\end{align}

(28)
Here $I_{12} = \Gamma_{1\rightarrow2} - \Gamma_{2\rightarrow1}$ ($I_{34} = \Gamma_{3\rightarrow4} - \Gamma_{4\rightarrow3}$) is the phonon-assisted hopping particle currents through the up (down) channel. $\Gamma_{i\rightarrow j}$ is the electron transfer rate from QD $i$ to QD $j$ [52,65], and $\omega_u = E_2 - E_1$, $\omega_d = E_4 - E_3$ denoting the QDs energy difference in the up and down channels.

We note that the source by driving a heat current between the heat baths $H$ and $C$, i.e. ‘cooling by heat current effect’, is different from the above ‘cooling by heating effect’ where cooling is driven by a finite heat current injected into the quantum system. In the cooling by heat current effect, heat injected into the quantum system is not necessary, since the driving force of the cooling is the energy exchange between the two heat baths via the central quantum system.

For convenience, we demonstrate the cooling by heat current effect in the situations with $A_e = A_{in} = 0$. The coefficient of performance (COP) in our four-terminal system can be given by [58,134]

$$\eta_{CBHC} = \frac{I_Q^L}{I_Q^R}. \quad (29)$$

The reversible COP is $\eta_{CBHC}^{rev} = -A_q/A_S$. We show how the cooling power $I_Q^L$ and COP $\eta_{CBHC}$ vary with the two energies, $\omega_u$ and $\omega_d$ in Figure 12. Both the COP $\eta_{CBHC}$ and the cooling power $I_Q^L$ favor the situations with $-\omega_u > \omega_d$. In such a regime, cooling induced by the cold terminal $C$ is more effective, for each phonon emission process provides more energy to the heat bath $C$. 

**Figure 12.** (a) COP and (b) cooling power of the cooling by transverse heat current effect as functions the two energies $\omega_u$ and $\omega_d$. The white areas represent the parameter regions where the cooling by heat current effect cannot be achieved, i.e. $I_Q^L < 0$. Figures adapted from Lu et al. [58].
As shown in Figure 13, we further find that the cooling by heat current effect can indeed exist when the total heat current injected into the quantum system vanishes (i.e. $I^\text{in}_Q = 0$), which is termed as ‘Maxwell demon’ [58,135–142]. The Maxwell demon based on two nonequilibrium baths (the cold and hot baths) can reduce the entropy of the system (the source and the drain),
without giving energy or changing the particle number of the system. More specifically, the heat current can flow from the cold bath to the hot one without external energies or changing the number of particles in the system.

6. Enhancing three-terminal thermoelectric performance using nonlinear transport effects

Nonlinear transport effects can enhance the elastic and inelastic thermoelectric efficiency and power when the voltage and/or temperature bias are large [55]. The reason is that linear-response theory usually fails when the voltage and/or temperature bias on the scale of the electrons’ relaxation length (typically given by the electron–electron or electron–phonon scattering length) is comparable to the average temperature. This point is particularly important for many thermoelectric applications. In particular, Sánchez et al. based on the seminal works [143–147], investigated nonlinear quantum transport through nanostructures and mesoscopic systems driven by thermal gradients or in combination with voltage biases. Specifically, when the temperature of the phonon bath increases, the nonlinear thermoelectric transport leads to significant improvement of both the heat-to-work energy efficiency and the output electric power. All these effects are found to be associated with inelastic and elastic thermoelectric contributions.

6.1. Effects of nonlinear transport on efficiency and power for elastic thermoelectric devices

We study the nonlinear transport effects on the performance of elastic thermoelectric devices. A simple candidate for such devices is a two-terminal QD thermoelectric device, i.e. a QD with energy $E_0$ connected to the source (of temperature $T_h$) and the drain (of temperature $T_c < T_h$) electrodes via resonant tunneling [148,149]. The electrical and heat currents can be calculated using the Landauer formula [92–94]

$$I_e = e \int \frac{dE}{2\pi} T_e(E) [f_L(E) - f_R(E)],$$

$$I_Q = \int \frac{dE}{2\pi} (E - \mu_L) T_e(E) [f_L(E) - f_R(E)],$$

with the energy-dependent transmission function $T_e(E) = \frac{\gamma^2}{(E - E_0)^2 + \gamma^2}$. Here, we consider harvesting heat from a hot reservoir to generate electricity. The energy efficiency is hence described as

$$\eta_{HE} = \frac{P_{HE}}{Q_{in}} \leq \eta_C,$$
with $T_h = T_{ph}$ and $T_c = T_L = T_R$. The output power is

$$P_{HE} = -I_eV.$$  

(32)

with $\mu_L = eV/2 = -\mu_R$. The heat injected into the system from the hot reservoir is given by

$$Q_{in} = I_Q^c + I_Q^{pr},$$  

(33)

with $I_Q^{pr}$ being the parasitic phonon heat current [55].

### 6.2. Nonlinear transport enhances efficiency and power for inelastic thermoelectric devices

We study the energy efficiency and output power of a double-QDs three-terminal thermoelectric device in the nonlinear transport regime. The device is schematically depicted in Figure 5. Here, we consider harvesting heat from the phonon bath to generate electricity. The heat injected into the system from the photon bath is given by

$$Q_{in} = I_Q^{ph} + I_Q^{pr},$$  

(34)

where $I_Q^{ph} = 2(E_r - E_i)\Gamma_{12}$ is the phononic current flowing from the phonon bath and $\Gamma_{12}$ is the rate of electron transfer from the left QD to the right QD due to the electron-phonon scattering. The electrical current is given by

$$I_e = 2e\gamma_e[f_L(E_i) - f_1] + 2e\gamma'_e[f_L(E_r) - f_2],$$  

(35)

where the factor of 2 in the above equation comes from the electron spin degeneracy. $f_i$ ($i = 1, 2$) are the probabilities of finding an electron on the $i$th QD, and they are determined by the nonequilibrium steady-state distributions on the QDs,

$$0 = \frac{df_i}{dt} = -\gamma_e[f_1 - f_L(E_i)] - \gamma'_e[f_1 - f_R(E_i)] - \Gamma_{12},$$

$$0 = \frac{df_i}{dt} = -\gamma_e[f_2 - f_R(E_r)] - \gamma'_e[f_2 - f_L(E_r)] + \Gamma_{12}.$$  

(36)

$\gamma_e/\gamma'_e$ is the tunneling between the QD and the reservoir. The linear transport coefficients are obtained by calculating the ratios between currents and affinities in the regime with very small voltage bias and temperature difference [see Equation (18)].

In Figure 14, we perform a comparative study of the nonlinear transport effect on the maximum efficiency and power of inelastic and elastic thermoelectric devices systematically. We find that the nonlinear effect can
significantly improve the performance of thermoelectric devices, e.g. thermodynamic efficiency and output power, both for elastic and inelastic cases.

7. Enhancing efficiency and power of three-terminal device by thermoelectric cooperative effects

In the following section, we discuss how the efficiency and output power of the three-terminal heat device can be enhanced by the thermoelectric cooperative effect in the linear-response regime. We consider the setup shown schematically in Figure 15, which consists of two electronic reservoirs and a phonon bath. The central cavity, which is warmed up by the phonon bath, is connected to two electrodes via two QDs at energy $E_{F(r)}$. There are two thermoelectric effects, one of which belongs to inelastic processes, while the other exists in the elastic process. These two effects
are related to two temperature gradients and correspond to the transverse and longitudinal thermoelectric effects, respectively. We show that the energy cooperation between transverse and longitudinal thermoelectric effects in three-terminal thermoelectric systems can lead to markedly improved performance of the heat device.

A full description of the thermoelectric transport in three-terminal systems is given by Equation (18). The cooperative effects in the thermoelectric engine can be elucidated by a geometric interpretation \[54,95,150\]. The two temperature differences can be parameterized as

$$
\delta T = T_A \cos \theta, \quad \Delta T = T_A \sin \theta.
$$

At given $\theta$, the figure of merit is given by

$$
ZT = \frac{G S^2_{\text{eff}} T^2}{K_{\text{eff}} - G S^2_{\text{eff}} T^2}.
$$

Here, $S_{\text{eff}} = S_1 \cos \theta + S_2 \sin \theta$ and $K_{\text{eff}} = K_e^0 \cos^2 \theta + 2L_3 \sin \theta \cos \theta + K_{pe} \sin^2 \theta$. $S_1$ and $S_2$ are given by Equation (19) denote the longitudinal and transverse thermopowers, respectively. Then, the ‘second-law efficiency’ of the thermoelectric engine is expressed as

$$
\phi = \frac{-I_e V}{I_1^L A_2 + I_1^R A_3} \leq \phi_{\text{max}} = \frac{\sqrt{ZT+1}-1}{\sqrt{ZT+1+1}},
$$
which is defined by the output-free energy divided by the input-free energy [14,151,152]. The rate of variation of the free energy associated with a current is given by the product of the current and its conjugated thermodynamic force. Hence, the denominator of the above equation consists of heat currents multiplied by temperature differences. Such free-energy efficiencies have been discussed for near-equilibrium thermodynamics (in the linear response regime) or arbitrarily far from equilibrium, ranging from biological [153] to quantum Hall system [136,151,154,155].

Upon optimizing the output power of the thermoelectric engine, one obtains $W_{\text{max}} = \frac{1}{4} P_F T_A^2$, with the power factor

$$P_F = G S_{\text{eff}}^2.$$  \hspace{1cm} (40)

When $\theta = 0$ or $\pi$, Equations (38) and (40) give the well-known figure of merit and power factor for the longitudinal thermoelectric effect

$$Z_l T = \frac{G S_{1}^2 T^2}{K_e^0 - G S_{1}^2 T^2}, \quad P_{Fl} = G S_{1}^2.$$  \hspace{1cm} (41)

While the transverse thermoelectric figure of merit and power factor, i.e. $\theta = \pi/2$ or $3\pi/2$, are given by

$$Z_t T = \frac{G S_{2}^2 T^2}{K_{pe} - G S_{2}^2 T^2}, \quad P_{Fl} = G S_{2}^2.$$  \hspace{1cm} (42)

Actually, one can maximize the figure of merit by tuning the angle $\theta$. This is achieved at $\partial_\theta (ZT) = 0$ and one finds that the maximum figure of merit is

$$Z_m T = \frac{G(K_e^0 K_{pe} - L_3^2)}{D_M} - 1.$$  \hspace{1cm} (43)

where $D_M = G K_e^0 K_{pe} - G L_3^2 - K_{pe} L_1^2 + 2 L_1 L_2 L_3 - K_e^0 L_2^2$ denotes the determinant of the $3 \times 3$ transport matrix in Equation (18). One can also tune $\theta$ to find the maximum power factor

$$P_{Fm} = G(S_1^2 + S_2^2)$$  \hspace{1cm} (44)

is greater than both $P_{Fl}$ and $P_{Fl}$ unless $S_1$ or $S_2$ is zero.

Figure 16(a) shows $ZT$ versus the angle $\theta$ in a polar plot for a specific set of transport coefficients. Remarkably for $0 < \theta < \pi/2$ and $\pi < \theta < 3\pi/2$, $ZT$ is greater than both $Z_l T$ and $Z_t T$. To understand the underlying physics, we decompose the electric current into three parts $I = I_0 + I_1 + I_2$ with $I_0 = GV$, $I_1 = L_1 \Delta T / T$, and $I_2 = L_2 \delta T / T$. The two thermoelectric effects add up constructively as $I_1$ and $I_2$ have the same sign, which takes place when $0 < \theta < \pi/2$ and $\pi < \theta < 3\pi/2$. Figure 16(b) shows the power factor versus the angle $\theta$. The power factor is also larger when the two currents $I_1$
and $I_2$ are in the same direction. Therefore, the cooperation of the two thermoelectric effects leads to an enhanced figure of merit and output power.

Besides the multilayer thermoelectric engines, where one electric current is coupled to two temperature gradients, the energy cooperation effects in quantum thermoelectric systems with multiple electric currents and only one heat current have also been studied [95,150], where the elastic tunneling through quantum dots is considered. Constructive cooperation in these quantum thermoelectric systems results in enhanced thermoelectric power and efficiency for various quantum-dot energies, tunneling rates, etc. Moreover, this cooperative enhancement, dubbed as the thermoelectric cooperative effect, is found to be universal in three-terminal thermoelectric energy harvesting [56,156].

8. Near-field three-terminal thermoelectric heat engine

Near-field thermal radiation has recently emerged as one promising route to efficiently transfer heat at the nanoscale [157–159], which dramatically stimulates the advance of thermoelectrics [85]. In Ref. [57], we proposed a near-field thermoelectric heat engine composed of two continuous spectra, e.g. narrow-bandgap semiconductor, separately interacting with a single quantum dot and inelastically coupled via near-field thermal emission. The near-field inelastic heat engine is exhibited to effectively rectify the charge flow of photon-carrier and convert near-field heat radiation into useful electrical power. Such near-field thermoelectric devices take the following advantages of near-field radiations: first, the near-field radiation can strongly enhance heat transfer across the vacuum gap and thus lead to

![Figure 16. Polar plot of (a) figure of merit $ZT$ and (b) power factor $P_F$ [in arbitrary unit (a.u.)] versus angle $\theta$. At $\theta = 0^\circ$ or $180^\circ$ $ZT$ and $P_F$ recover the values for the longitudinal thermoelectric effect (red dots), while at $\theta = 90^\circ$ and $270^\circ$ they go back to those of the transverse thermoelectric effect (green squares). The arrows in the I, II, III, IV quadrants label the direction of the currents $I_1 = L_1 \Delta T/T$ (red arrows) and $I_2 = L_2 \delta T/T$ (green arrows). Figures adapted from Jiang et al. [54].](image-url)
significant heat flux injection. Second, unlike phonon-assisted interband transitions, photon-assisted interband transition is not limited by the small phonon frequency and can work for larger band gaps due to the continuous photon spectrum.

Here, we present a microscopic theory for thermoelectric transport in the near-field inelastic heat engine. The Hamiltonian of the system is described as

$$H = H_{SD} + H_{QD} + H_C + H_{\text{tun}} + H_{\text{e-ph}}.$$  \hspace{1cm} (45)

Specifically, the Hamiltonian for the source and drain is expressed as

$$H_{SD} = \sum_{\vec{q}} (E_{S,\vec{q}} c_{S,\vec{q}}^\dagger c_{S,\vec{q}} + E_{D,\vec{q}} c_{D,\vec{q}}^\dagger c_{D,\vec{q}}),$$

where $\vec{q}$ is the wavevector of electrons. The Hamiltonian of the QDs is $H_{QD} = \sum_{j=\ell,r} E_j d_j^\dagger d_j$, where $j = \ell, r$ denotes the left and right dots, respectively. We first consider the case where only one (two if spin degeneracy is included) level in each QD is relevant for the transport. The Hamiltonian for the two central continua is

$$H_C = \sum_{\vec{q}} (E_{v,\vec{q}} c_{v,\vec{q}}^\dagger c_{v,\vec{q}} + E_{e,\vec{q}} c_{e,\vec{q}}^\dagger c_{e,\vec{q}}).$$

The tunnel coupling through the QDs is given by

$$H_{\text{tun}} = \sum_{\vec{q}} (J_{S,\vec{q}} c_{S,\vec{q}}^\dagger d_\ell + J_{D,\vec{q}} c_{D,\vec{q}}^\dagger d_r + J_{v,\vec{q}} c_{v,\vec{q}}^\dagger c_{e,\vec{q}}^\dagger a_{\vec{q},\tau} + J_{e,\vec{q}} c_{e,\vec{q}}^\dagger d_\tau) + \text{H.c.}.$$ \hspace{1cm} (46)

The coupling coefficients $J$ determine the tunnel rates $\Gamma_{ij}$ [57]. The Hamiltonian governing the photon-assisted transitions in the center is

$$H_{\text{e-ph}} = \sum_{\vec{q},\vec{k},\tau} \frac{g_{\vec{k},\tau}}{V} c_{e,\vec{q}}^\dagger c_{\vec{q}} c_{\vec{k},\tau} a_{\vec{k},\tau}^\dagger + \text{H.c.},$$ \hspace{1cm} (47)

where $g_{\vec{k}}$ is the electron–photon interaction strength, the operator $a_{\vec{k},\tau}$ ($\tau = s, p$ denotes the $s$ and $p$ polarized light) annihilates an infrared photon with polarization $\tau$. $V$ is the volume of the photonic system.

Via the Fermi golden rule, the thermoelectric transport coefficients in the linear response regime are obtained as

$$G_{ve} = \frac{e^2}{k_B T} \int d\omega \Gamma_0(\omega),$$ \hspace{1cm} (48a)

$$L_{ve} = \frac{e}{k_B T} \int d\omega \Gamma_0(\omega) \hbar \omega,$$ \hspace{1cm} (48b)
\[
K_{ve} = \frac{1}{k_B T} \int d\omega \Gamma_0(\omega) \hbar^2 \omega^2, \tag{48c}
\]

where

\[
\Gamma_0(\omega) = 2\pi v_p h F_{nf}(\omega) \sum_{\vec{q}} |g(\omega)|^2 \delta(E_{e,\vec{q}} - E_{v,\vec{q}} - \hbar \omega) \times f^0(E_{v,\vec{q}}, T)[1 - f^0(E_{e,\vec{q}}, T)] N^0(\omega, T). \tag{49}
\]

The superscript 0 in the above stands for the equilibrium distribution, \(N^0(\omega, T) = 1/\left[\exp(\frac{\hbar \omega}{k_B T}) - 1\right]\) is the equilibrium photon distribution function, and \(f^0(E_{e,\vec{q}}, T) = 1/\left[\exp\left(\frac{E_{e,\vec{q}} - \mu}{k_B T}\right) + 1\right]\) is the Fermi-Dirac distribution function. \(|g(\omega)|^2 = \frac{\hbar^2 d_x^2}{2m_\epsilon^2}\), \(v_p h\) is the photon density of states, and the factor is given by

\[
F_{nf}(\omega) = \frac{1}{4} \int_0^1 \frac{x_k dx_k}{\sqrt{1 - x_k^2}} \sum_{\tau} T_\tau(\omega, x_k n\omega/c, d), \tag{50}
\]

where \(x_k = k ||/(n \omega/c)\). It is interesting to show that the photon tunneling probability is specified as [157,160]

\[
T_\tau(\omega, k ||, d) = \begin{cases} \frac{(1-|r_{01}^*|^2)(1-|r_{02}^*|^2)}{|1-r_{01}^* r_{02}^* e^{2i\phi_d^0}|^2}, & \text{if } k || \leq \omega/c \\ \frac{4\Im(r_{01}^*)\Im(r_{02}^*) e^{-2\phi_d^0} d}{|1-r_{01}^* r_{02}^* e^{-2\phi_d^0} d|^2}, & \text{otherwise} \end{cases} \tag{51}
\]

Here \(r_{01}^* (r_{02}^*)\) is the Fresnel reflection coefficient for the interface between the vacuum (denoted as ‘0’) and the emitter (absorber) [denoted as ‘1’ (‘2’)]. \(k_z^0 = \sqrt{\left(\omega/c\right)^2 - k_z^2}\) is the wavevector perpendicular to the planar interfaces in the vacuum. For \(k_z > \omega/c\), the perpendicular wavevector in the vacuum is imaginary \(i\beta_z^0 = i\sqrt{k_z^0 - (\omega/c)^2}\), where photon tunneling is dominated by evanescent waves. For isotropic electromagnetic media, the Fresnel coefficients are given by \(r_{0j}^* = \frac{k_j^2 - k_z^2}{k_j^2 + k_z^2}\) and \(r_{0j}^p = \frac{\epsilon_j k_j^2 - k_z^2}{\epsilon_j k_j^2 + k_z^2}\) (\(j = 1, 2\)), where \(k_j^2 = \sqrt{\epsilon_j (\omega/c)^2 - k_z^2}\) and \(\epsilon_j (j = 0, 1, 2)\) are the (complex) wave vector along the z direction and the relative permittivity in the vacuum, emitter, and the absorber, respectively.

Consequently, the Seebeck coefficient of the near-field inelastic three-terminal heat engine is obtained as

\[
S = \frac{\langle h\omega \rangle}{eT}, \tag{52}
\]
and the figure of merit is given by

$$ZT = \frac{\langle \hbar \omega \rangle^2}{\alpha \langle \hbar^2 \omega^2 \rangle - \langle \hbar \omega \rangle^2 + \Lambda_{nf}},$$

(53)

where the average is defined as

$$\langle \ldots \rangle = \int \frac{d\omega \Gamma_0(\omega)\ldots}{d\omega \Gamma_0(\omega)},$$

$$\alpha = G_{ve}/G_{eff},$$

and

$$\Lambda_{nf} = e^2K_{para}/G_{ve}$$

characterizes the parasitic heat conductance $K_{para}$ that does not contribute to thermoelectric energy conversion. It is shown in Figure 17(c).

---

**Figure 17.** (a) Schematic of near-field three-terminal thermoelectric heat engine. A hot thermal reservoir of temperature $T_h$ injects heat flux into the device through near-field heat radiation. The device is held at a lower temperature $T_c$. The absorption of the heat radiation is realized by photon-assisted transitions between the two continua. As a result, the upper and lower continua have different chemical potentials, $\mu_e$ and $\mu_v$, respectively. The source and drain have different electrochemical potentials, denoted as $\mu_s$, $\mu_D$, and $\mu_{QDs}$, separately. The typical energy of QDs in the left (right) layer is $E_L$ ($E_R$). (b) A possible set-up for the three-terminal near-field heat engine. The emitter is a heat source of temperature $T_h$, which is separated from the device by a vacuum gap of thickness $d$. The device is held at a lower temperature $T_c$ which consists of the source, drain, and absorber layers. These three parts are divided by two layers of quantum dots arrays. (c) Seebeck coefficient $S$ (in unit of mV/K) and (d) thermoelectric figure of merit $ZT$ for the inelastic thermoelectric transport as functions of the chemical potential $\mu$ and the temperature $T$. Figures adapted from Jiang et al. [57].
that the Seebeck coefficient does not change significantly by tuning the chemical potential, which is a generic characteristic of the inelastic thermoelectric effect, because the average energy $\langle \hbar \omega \rangle$ is mainly limited by the band gap $E_g$ and the temperature $T$. Moreover, the figure of merit with small parasitic heat conduction, e.g. $\Lambda_{nf} = 0.2E_g^2$ in Figure 17(d), can be optimized to be as large as $ZT > 7$ around $T = 350$ K and $\mu > 0.15$ eV. Therefore, our work presents an intriguing mechanism of photon-induced inelastic thermoelectricity, which may provide physical insight for future thermoelectric technologies based on inelastic transport mechanisms, and serve as the foundation for future studies.

9. Quantum efficiency bound for continuous heat engines coupled to non-canonical reservoirs

The efficiency of heat engines is fundamentally restricted by the second law of thermodynamics to the Carnot limit [84]. This canonical bound is being challenged nowadays by quantum and classical effects. However, nonequilibrium reservoirs that are characterized by additional parameters besides their temperature are exploited to construct devices with efficiency beyond the Carnot bound [161,162].

We study the energy conversion in quantum engines absorbing heat from a non-canonical reservoir [163]. The device consists of a single qubit coupled to hot squeezed photon bath and two cold electronic reservoirs (the source and drain), as shown in Figure 18. In order to describe the system quantum mechanically, we apply the two-time measurement protocol to define the characteristic function as

$$Z(\lambda_c, \lambda_e, \lambda_{ph}) = \langle e^{i\lambda_c \hat{A}_c + i\lambda_e \hat{A}_e + i\lambda_{ph} \hat{A}_{ph}} e^{-i\lambda_c \hat{A}_c(t) - i\lambda_e \hat{A}_e(t) - i\lambda_{ph} \hat{A}_{ph}(t)} \rangle.$$ 

$\lambda_{c,e,ph}$ are counting parameters for the charge, electronic energy, and photonic energy, respectively. $\hat{A}_c$, $\hat{A}_e$ and $\hat{A}_{ph}$ are the respective operators: $\hat{A}_c$ is the number operator corresponding to the total charge in the L/R electrode, $\hat{A}_e$ is the Hamiltonian operator for the L/R electrode and $\hat{A}_{ph}$ is the Hamiltonian operator for the photon bath. $\langle \ldots \rangle$ represent an average with respect to the total initial density matrix, which takes a factorized form with respect to the system (s) and (L, R and ph) reservoirs, $\rho_s(0) = \rho_s(0) \otimes \rho_l \otimes \rho_R \otimes \rho_{ph}$. The state of the metal leads is described by a grand canonical distribution, $\rho_i = \exp[-\beta_i (\hat{H}_i - \mu_i \hat{N}_i)] / Z_i$, with $Z_i = Tr\{\exp[-\beta_i (\hat{H}_i - \mu_i \hat{N}_i)]\}$ being the partition function, $\beta_i = 1/k_B T_i$ being the inverse temperature, and $\mu_i$ the chemical potential in the $i$th reservoir, respectively.
9.1. **Equilibrium thermal photon bath**

The state of the photon bath is canonical, \( \rho_{\text{ph}} = \exp[-\beta_{\text{ph}} \hat{H}_{\text{ph}}]/Z_{\text{ph}} \), with \( Z_{\text{ph}} = \text{Tr}[\exp(-\beta_{\text{ph}} \hat{H}_{\text{ph}})] \). The fluctuation relation \( \ln[P_t(\Delta S)/P_t(-\Delta S)] = \Delta S \) translates to

\[
\frac{P_t(I_N, I_E, I_Q^{\text{ph}})}{P_t(-I_N, -I_E, -I_Q^{\text{ph}})} = e^{\beta_{\text{el}} \Delta \mu I_N + (\beta_{\text{el}} - \beta_{\text{ph}}) I_Q^{\text{ph}}}. \tag{54}
\]

Here, \( I_N \) denotes the number of electrons transferred from \( R \) to \( L \) during the time interval \( t \). Similarly, \( I_E \) is the electronic energy and \( I_Q^{\text{ph}} \) photonic heat that are exchanged between the baths during the time interval \( t \). The characteristic function thus satisfies

\[
Z(\lambda_c, \lambda_e, \lambda_{\text{ph}}) = Z(-\lambda_c + i\beta_{\text{el}}(\mu_R - \mu_L), -\lambda_e, -\lambda_{\text{ph}} - i(\beta_{\text{ph}} - \beta_{\text{el}})) \tag{55}
\]

This relation straightforwardly results in \( 1 = \langle e^{-\beta_{\text{el}} \Delta \mu I_N + (\beta_{\text{ph}} - \beta_{\text{el}}) I_Q^{\text{ph}}} \rangle \). Using Jensen’s inequality, we obtain \( [-\beta_{\text{el}} \Delta \mu \langle I_N \rangle + (\beta_{\text{ph}} - \beta_{\text{el}}) \langle I_Q^{\text{ph}} \rangle] \leq 0 \). Therefore, the efficiency, \( \langle \eta \rangle \equiv -\Delta \mu \langle I_N \rangle / \langle I_Q^{\text{ph}} \rangle \), thus obeys the Carnot bound.

---

**Figure 18.** Photoelectric quantum heat engine. Energy absorbed by the qubit from a hot squeezed thermal reservoir is converted to electrical power in the cold electronic system.
\[ \langle \eta \rangle \leq \frac{\beta_{\text{el}} - \beta_{\text{ph}}}{\beta_{\text{el}}}. \]  

(56)

9.2. Noncanonical photon bath

The squeezed thermal reservoir can be depicted as a combination of orthogonal components, which oscillate as \( \cos \omega t \) and \( \sin \omega t \) [164]. Squeezed states have reduced fluctuations in one of the quadratures – but enhanced noise in the other quadrature – to satisfy the bosonic commutation relation. Such states are defined by two parameters, the squeezing factor \( r \) and phase [164]. To restore the detailed balance relation for the \( r \neq 0 \) case, one can identify an effective temperature [165]

\[ \beta_{\text{eff}} = \beta_{\text{ph}} + \frac{1}{\hbar \omega_0} \ln \left[ \frac{1 + 1 + e^{-\beta_{\text{ph}} \hbar \omega_0}}{1 + 1 + e^{\beta_{\text{ph}} \hbar \omega_0}} \right], \]  

(57)

which is unique in the present model, with \( \hbar \omega_0 \) is the energy gap of the qubit.

Identifying the entropy production associated with the photon energy flow by \( \langle \Delta S \rangle = (\beta_{\text{el}} - \beta_{\text{eff}}) \langle I_{Q}^{\text{ph}} \rangle \), we confirm the symmetry Equation (55) by replacing \( \beta_{\text{ph}} \) with \( \beta_{\text{eff}} \)

\[ Z(\lambda_c, \lambda_e, \lambda_{\text{ph}}) = Z(-\lambda_c + i\beta_{\text{el}}(\mu_R - \mu_L), -\lambda_e, -\lambda_{\text{ph}} - i(\beta_{\text{eff}} - \beta_{\text{el}})). \]  

(58)

The fluctuation symmetry relation implies that \( 1 = \langle e^{-\beta_{\text{el}} \Delta I_N + (\beta_{\text{eff}} - \beta_{\text{el}}) I_{Q}^{\text{ph}}} \rangle \).

Thus, the averaged efficiency, \( \langle \eta \rangle = -\Delta \mu \langle I_N \rangle / \langle I_{Q}^{\text{ph}} \rangle \), is bounded by

\[ \langle \eta \rangle \leq 1 - \frac{\beta_{\text{eff}}}{\beta_{\text{el}}}. \]  

(59)

We note that this bound is universal, holding even beyond the squeezed-bath case. Explicitly, the efficiency bound for our photoelectric engine [163] is given by

\[ \langle \eta \rangle \leq 1 - \frac{\beta_{\text{eff}}}{\beta_{\text{el}}} + \frac{1}{\beta_{\text{el}} \hbar \omega_0} \ln \left[ \frac{1 + 1 + e^{\beta_{\text{ph}} \hbar \omega_0} \sinh^2 r}{1 + 1 + e^{-\beta_{\text{ph}} \hbar \omega_0} \sinh^2 r} \right], \]  

(60)

We now discuss several interesting results of Equation (60). First, we expand it close to thermal equilibrium assuming that \( \sinh^2 r \) is a small parameter. As well, we assume that the temperature of the photon bath is high, e.g. \( \beta_{\text{ph}} \hbar \omega_0 \ll 1 \). Then, Equation (60) is reduced to
\[ \langle \eta \rangle \leq 1 - \frac{\beta_{\text{ph}}}{\beta_{\text{el}}(1 + 2 \sinh^2 r)}. \]  

which agrees with Refs. [161,166]. Another interesting case is the deep quantum regime \((\beta_{\text{ph}} \hbar \omega_0 \gg 1)\). Assuming small \(r\), from Equation (60) we receive an exponential quantum enhancement in comparison to the classical case,

\[ \langle \eta \rangle \leq 1 - \frac{\beta_{\text{ph}}}{\beta_{\text{el}}} + \frac{\beta_{\text{ph}} \hbar \omega_0}{\beta_{\text{el}}^2 \hbar \omega_0} \frac{\sinh^2 r}{1 + \sinh^2 r}. \]  

Figure 19 clearly exhibits these results: (i) Squeezing enhances the efficiency beyond the Carnot limit. (ii) In the quantum regime \((\beta_{\text{ph}} \omega_0 > 1)\), the bound is greatly reinforced beyond the thermodynamical limit.

10. Thermoelectric efficiency and its statistics

Fluctuations cannot be ignored in mesoscopic systems and are particularly important for understanding quantum transport. It can also be considered as a resource for the operation of open quantum systems as functional devices. As a widely used theoretical framework, the fluctuation theorem has been applied to the statistics of the electronic currents, heat currents, and thermodynamic fluctuations [69,167–180]. In this section, from the perspective of statistical physics, we utilize the fluctuation theorem to analyze thermal fractional devices.
10.1. Efficiency statistics for three-terminal systems with broken time-reversal symmetry

By analyzing the stochastic efficiency, it was recently shown that the Carnot efficiency is the least likely stochastic efficiency [173], later found to be solely the consequence of the fluctuation theorem for time-reversal symmetric (TRS) energy transducers [182]. Breaking the time-reversal symmetry can shift the least likely efficiency away from the Carnot efficiency [182,183].

We consider a generic situation in which there are two energy output channels (‘1’ and ‘2’). Each of the channels has a thermodynamic ‘current’ and an affinity. The time-integrated currents are denoted by \( J_i \) \((i = 1, 2)\) while the time-intensive current is defined as \( I_i = J_i / t \) with \( t \) being the total time of operation. A small time-reverse broken (TRB) machine can be characterized in the linear-response regime by \( \bar{I}_i = M_{ij} A_j \) \((i, j = 1, 2)\). In this regime, the statistics of the currents at long time \( t \) can be described within the Gaussian approximation by the distribution \( P_t(\bar{I}) = \frac{\sqrt{\det((\hat{M}^{-1})_{\text{sym}})}}{4\pi} \exp(-\frac{1}{4} \delta \bar{I}^T \cdot \hat{M}^{-1} \cdot \delta \bar{I}) \) [184,185]. Here \( \det((\hat{M}^{-1})_{\text{sym}}) \) is the determinant of the symmetric part of the inverse of the Onsager response matrix \( \hat{M} \) and the superscript ‘\( T \)’ denotes transpose. The averaged quantities are represented with a bar over the symbols throughout this paper. \( \delta \bar{I} = \bar{I} - \bar{I} \) represents fluctuations of the currents. From the probability distribution of stochastic currents, we calculate the distribution of efficiency \( P_t(\eta) \). We then obtain the large deviation function (LDF) of the stochastic efficiency \( G(\eta) = -\lim_{t \to \infty} t^{-1} \ln[P_t(\eta)] \).

Consequently, the scaled LDF \( (J(\eta) = G(\eta)/\bar{S}_{\text{tot}}) \) is given by

\[
J_{\text{HE}}(\eta) = \frac{J_{\text{HE}}(\eta_C)(\eta + a^2 + aq b + aq \eta)^2}{(1 + a^2 + aq b + aq)(\eta^2 + a^2 + aq \eta + aq b \eta)},
\]

(63)

where \( \bar{S}_{\text{tot}} = \sum_i \bar{I}_i A_i \) is the average total entropy production rate and \( J_{\text{HE}}(\eta_C) = \frac{4 - q^2(1 + b)^2}{16(1 - q^2 b)} \) is the scaled LDF at Carnot efficiency. Here, \( q = \frac{M_{11}}{\sqrt{M_{22} M_{11}}} \), \( b = \frac{M_{12}}{M_{21}} \), \( a = \frac{A_1 \sqrt{M_{11}}}{A_2 \sqrt{M_{22}}} \) are dimensionless parameters that characterize the responses of the system and the applied affinities. In our scheme, the efficiency is scaled so that the Carnot (reversible) efficiency corresponds to \( \eta_C = 1 \).

In particular, the minimum \( J_{\text{HE}}(\bar{\eta}_{\text{HE}}) = 0 \) is reached at the average efficiency \( \bar{\eta}_{\text{HE}} = -a(a + qb)/(aq + 1) \), whereas the maximum value \( J_{\text{HE}}(\eta^*) = 1/4 \) is realized at the least probable efficiency.
In the TRS limit, the least likely efficiency is always identical to the Carnot efficiency, \( \eta^* = \eta_C = 1 \). For TRB systems, in contrast, we find here that \( \eta^* \) depends on the parameters \( q, a, \) and \( b \), see Figure 20(a–c).

Moreover, the width of the distribution around the average efficiency, \( \sigma_{\eta}^{\text{HE}} \), is considered as another key characteristic of efficiency fluctuations. Expanding \( J_{\text{HE}}(\eta) \) around its minimum \( \overline{\eta}_{\text{HE}} \), one writes

\[
J_{\text{HE}}(\eta) \simeq \frac{1}{2(\sigma_{\eta}^{\text{HE}})^2} (\eta - \overline{\eta}_{\text{HE}})^2 + O((\eta - \overline{\eta}_{\text{HE}})^3),
\]

to provide here...
\[ \sigma_{\eta}^{\text{HE}} = \frac{2\sqrt{2}|a|(1 - q^2b)(1 + a^2 + aq + abq)}{(1 + aq)^2 \sqrt{4 - q^2(1 + b)^2}}. \]  

(65)

We exemplify our analysis within a mesoscopic triple-QD thermoelectric device under a piercing magnetic flux, as shown in Figure 21.

### 10.2. Large-deviation function for efficiency: beyond linear-response

In the following, we study the statistics of efficiency fluctuations in the non-equilibrium regime. In a recent study, Esposito et al. analyzed the thermoelectric efficiency statistics in a purely coherent charge transport model [186]. In parallel, classical models were also examined [172]. Alternatively, the three-terminal device offers a rich opportunity to examine thermoelectric efficiency beyond linear response, explore the new concept of efficiency fluctuations, and interrogate the role of quantum effects and many-body interactions in the operation of a molecular thermoelectric engine.

Due to the stochastic nature of small systems, efficiency fluctuations are typically not bounded and can take arbitrary values. In general, it is useful to investigate the probability distribution function \( P_t(\eta) \) to obtain the fluctuating work and heat within the interval \( t \), also to observe the value \( \eta \) within time \( t \). According to the theory of large deviations, the probability function assumes an asymptotic long time form [188,189],

\[ P_t(\eta) \sim e^{-f_{\text{NL}}(\eta)} \]  

(66)
with $J_{NL}(\eta)$ being the ‘large deviation function’. The large deviation function for efficiency can be obtained from $G(\lambda_w, \lambda_q)$ by setting $\lambda_q = \eta \eta C \lambda_w$, and minimizing it with respect to $\lambda_w$, 

$$J_{NL}(\eta) = - \min_{\lambda_w} G(\lambda_w, \eta \eta C \lambda_w).$$

where $G(\lambda)$ is the cumulant generating function (CGF) of the three-terminal device. $\lambda_w$ and $\lambda_q$ are the counting fields for work and heat, respectively. Note that we do not explicitly evaluate the probability distribution function $P_t(\eta)$. It can be confirmed that $J_{NL}(\eta)$ has a single minimum, coinciding with the macroscopic efficiency of the engine, and a single maximum, corresponding to the least likely efficiency, which equals to the Carnot efficiency, i.e. $\eta = 1$.

We numerically investigate the thermoelectric efficiency and its statistics in the three-terminal device, considering the effects of mode anharmonicity [Figure 22(a)] and harmonic [Figure 22(b)] vibrational mode beyond linear-response situations. The CGFs for an anharmonic impurity and harmonic vibrational modes are given by Ref. [187]. In Figure 23, we compare the scaled LDF $J(\eta)$ for the two modes in the linear-response regime [Equation (63)] and beyond the linear-response regime [Equation (67)]. It is found that the position $\eta$ of the minimum of $J(\eta)$ can be well captured based on the Gaussian assumption in the linear-response regime. Moreover, such coincidence also persists even at finite thermodynamic bias for the anharmonic case.

10.3. Brownian linear thermal transistors

In Ref. [81], we have studied the statistical distributions of the thermal transistor amplification factor and the cooling by heating efficiency under the assumption of the Gaussian fluctuation. Particularly in the linear-response regime, the statistics of the stochastic heat currents at long time can be
described within the Gaussian approximation [183] by the distribution
\[ P_i(Q^{(i)}_L, Q^{(i)}_{ph}) = \frac{\sqrt{\det(K^{-1})}}{4\pi} \exp\left[-\frac{1}{4} \Delta Q^{(i)} \cdot \hat{K}^{-1} \cdot \Delta Q^{(i)}\right]. \]
Here \( Q^{(1)}_{L(ph)} = I^{(ph)}_Q(T + \delta T) \) and \( Q^{(2)}_{L(ph)} = I^{(ph)}_Q(T_{ph} = T) \) with \( \delta T/T \to 0 \). \( \Delta \tilde{Q} = \tilde{Q} - \bar{Q} \) represents fluctuations of the heat currents, where \( \bar{Q} \) is the average heat current, and \( \tilde{Q} \) is stochastic. Based on the probability distribution of stochastic heat currents, we obtain the LDF of stochastic thermal transistor [81]

\[ h(\alpha) = \frac{[(K_{12} - K_{22} \alpha)A_{ph}]^2}{8(K_{11} - 2K_{12} \alpha + K_{22} \alpha^2)}. \]
where $\Delta A_{ph}^{(1)} = A_{ph}^{(1)} - A_{ph}^{(2)}$ and $A_{ph(L)}^{(i)}$ ($i = 1, 2$) are the affinities for heat currents $Q_{ph(L)}^{(i)}$.

The minimum $h(\bar{\alpha}) = 0$ is located at the average transistor amplification [59]

$$\bar{\alpha} = \frac{K_{12}}{K_{22}},$$

which correspond to the maximal probability for the appearance of the amplification efficiency.

The amplification fluctuation is obtained as

$$\sigma_{\alpha} = \frac{2\sqrt{K_{22}(K_{11}K_{22} - K_{12}^2)}}{K_{22}\Delta A_{ph}},$$

which obeys the bound of the Onsager coefficients $K_{11}K_{22} - K_{12}^2 \geq 0$ and $K_{22} \geq 0$ [14]. The equality is reached as the fluctuation width completely vanishes. Obviously, when this equality is reached, the total entropy production rate of the system in the linear-response regime is $dS/dt = 0$, i.e. the system is in the equilibrium state.

### 10.4. The statistics of refrigeration efficiency

Here, we reveal the fluctuations of cooling by heating refrigerators in the linear-response regime. The scaled LDF of stochastic efficiency [173,186] can be expressed as

$$g_{CBH}(\eta) = \frac{[1 - y\eta + (x - \eta)z]^2}{4[x + \eta(-2 + y\eta)](xz^2 + 2z + y)},$$

with dimensionless parameters $x = K_{11}/K_{12}$, $y = K_{22}/K_{12}$, and $z = A_L/A_{ph}$. The thermodynamic forces are $A_L = (T_R - T_L)/T$ and $A_{ph} = (T_{ph} - T_L)/T$, respectively.

The minimum of $g_{CBH}(\bar{\eta}_{CBH}) = 0$ is reached at the average efficiency

$$\bar{\eta}_{CBH} = \frac{xz + 1}{y + z},$$

The fluctuating width of the average efficiency, $\sigma_{\eta}^{CBH}$, is obtained by expanding $h(\bar{\eta}_{CBH}) = 0$ around its minimum $\bar{\eta}_{CBH}$,

$$\sigma_{\eta}^{CBH} = \frac{(y + z)^2}{(xz^2 + 2z + y)\sqrt{2(xy - 1)}}.$$
Figure 24. (a) $\bar{\alpha}$ and (b) $\sigma_\alpha$ as functions of $K_{12}$ and $K_{22}$. The white region is forbidden by the thermodynamic bound. (c) The average efficiency $\bar{\eta}_{CBH}$ and (d) the width of cooling efficiency distribution $\sigma^\eta_{CBH}$. The white region is forbidden by the thermodynamic bound. Figures adapted from Lu et al. [81].

Figure 24(c,d) illustrate the cooling efficiency $\bar{\eta}_{CBH}$ and the behavior of the width of cooling efficiency distribution $\sigma^\eta_{CBH}$ when $z = \infty$. We can observe that the $\sigma^\eta_{CBH}$ reaches the maximum under the limit condition, i.e. $(xz + 1)(y + z) = 0$.

In summary of this section, we emphasize that the statistics of energy efficiency can reveal information on the three-terminal thermoelectric system in the linear and nonlinear regimes, and the average efficiency and its fluctuations can further characterize the properties of the system.

11. Thermophotovoltaic systems based on near-field tunneling effect

As a solid-state renewable energy resource, thermophotovoltaic systems have immense potential in a wide range of applications including solar energy harvesting and waste heat recovery [190–192]. In the thermophotovoltaic system, a photovoltaic cell is placed in the proximity of a thermal
emitter and converts the thermal radiation from the emitter into electricity via infrared photoelectric conversion. However, the thermophotovoltaic performance is significantly reduced due to the frequency mismatch between the thermal emitter and the photovoltaic cell in the thermophotovoltaic systems at moderate temperatures (i.e. 400 ~ 900 K which is the majority spectrum of the industrial waste heat). To overcome this obstacle, materials which support surface polaritons have been used to introduce a resonant near-field energy exchange between the emitter and the absorber [191,193,194]. As a consequence, near-field thermophotovoltaic systems have been proposed to achieve appealing energy efficiency and output power [195,196]. Near-field systems based on graphene, hexagonal-boron-nitride (h-BN) and their heterostructures have been shown to demonstrate excellent near-field couplings due to surface plasmon polaritons, surface phonon polaritons and their hybridizations [i.e. surface plasmon-phonon polaritons] [197–201]. In Ref. [202], we propose to use graphene- h-BN heterostructures [199–201,203,204] as the emitter and the graphene-covered InSb p-n junction as the thermophotovoltaic cell. We find that such a design leads to significantly improved performance as compared to the existing studies [198,205,206].

Figure 25 presents the proposed near-field thermophotovoltaic system. The emitter is a graphene-covered h-BN film of thickness $h$, kept at temperature $T_{\text{emit}}$. The thermophotovoltaic cell is made of an InSb p-n junction, kept at temperature $T_{\text{cell}}$, which is also covered by a layer of graphene. The thermal radiation from the emitter is absorbed by the cell and then
converted into electricity via photoelectric conversion. The performance of the near-field thermophotovoltaic system is characterized by the output electric power density \( P_e \) and energy efficiency \( \eta \).

The output electric power density \( P_e \) of the near-field thermophotovoltaic system is defined as the product of the net electric current density \( I_e^{\text{NTPV}} \) and the voltage bias \( V \) [202],

\[
P_e^{\text{NTPV}} = -I_e^{\text{NTPV}} V, \tag{74}
\]

and the energy efficiency \( \eta^{\text{NTPV}} \) is given by the ratio between the output electric power density \( P_e^{\text{NTPV}} \) and incident radiative heat flux \( Q_{\text{inc}} \),

\[
\eta^{\text{NTPV}} = \frac{P_e^{\text{NTPV}}}{Q_{\text{inc}}}. \tag{75}
\]

The incident radiative heat flux is given by

\[
Q_{\text{rad}} = Q_{\omega < \omega_{\text{gap}}} + Q_{\omega \geq \omega_{\text{gap}}} \tag{76}
\]

where \( Q_{\omega < \omega_{\text{gap}}} \) and \( Q_{\omega \geq \omega_{\text{gap}}} \) are the heat exchanges below and above the band gap of the cell, respectively [207,208].

The electric current density of a near-field thermophotovoltaic cell is calculated via the detailed balance analysis [60],

\[
I = I_{\text{ph}} - I_0[\exp(V/V_{\text{cell}}) - 1], \tag{77}
\]

where \( V_{\text{cell}} = k_B T_{\text{cell}}/e \) is a voltage which measures the temperature of the cell [60]. \( I_{\text{ph}} \) and \( I_0 \) are the photo-generation current density and the reverse saturation current density, respectively. The reverse saturation current density is determined by the diffusion of minority carriers in the InSb \( p-n \) junction and the photo-generation current density \( I_{\text{ph}} \) is contributed from the above-gap thermal heat exchange [202].

The performances of four different near-field thermophotovoltaic configurations are examined as follows: (i) the h-BN-InSb device (denoted as hBN-InSb, with the mono-structure bulk h-BN being the emitter and the uncovered InSb \( p-n \) junction being the cell), (ii) the h-BN-graphene/InSb device (denoted as fBN-G/InSb, with the bulk h-BN being the emitter and the graphene-covered InSb \( p-n \) junction as the cell), (iii) the h-BN/graphene-InSb device (denoted as fBN/G-InSb, with the h-BN/graphene heterostructure film being the emitter and the uncovered InSb \( p-n \) junction as the cell), and (iv) the h-BN/graphene-graphene/InSb device (denoted as fBN/G-G/InSb, with the h-BN/graphene heterostructure film being the emitter and the graphene-covered InSb \( p-n \) junction as the cell). We study and compare their performances for various conditions to optimize the performance of the near-field thermophotovoltaic system. As shown in Figure 26, the primitive hBN-InSb set-up has poor energy efficiency and output power.
The overall best performance comes from the $h$ BN-G/InSb (if high output power is preferred) and the fBN/G-InSb (if high energy efficiency is preferred) set-ups. The underlying physics for the different characteristics of the four different set-ups is understood to be due to the resonant coupling between the emitter and the $p$-$n$ junction, where the surface plasmon polaritons in graphene and surface phonon polaritons in $h$-BN play crucial roles [199–202].

Since semiconductor thin-films have been explored in near-field thermophotovoltaic systems, we further investigate the performance of near-field thermophotovoltaic systems based on thin-film $p$-$n$ junctions. A near-field thermophotovoltaic system based on an InAs thin-film cell with appealing performance operating at high temperatures has been recently proposed.

Figure 26. Optimal performances of the four near-field thermophotovoltaic devices. (a) and (b), Optimal (a) output power density $P_{e\text{NPV}}$ and (b) energy efficiency in unit of the Carnot efficiency ($\eta_{\text{NPV}}/\eta_C$) as functions of the vacuum gap $d$. The temperatures of the emitter and the cell are set as $T_{\text{emit}} = 450$ K and $T_{\text{cell}} = 320$ K, respectively. (c) and (d), Optimal (c) output power density $P_{e\text{NPV}}$ and (d) energy efficiency in unit of the Carnot efficiency ($\eta_{\text{NPV}}/\eta_C$) as functions of the emitter temperature $T_{\text{emit}}$ with $d = 20$ nm and $T_{\text{cell}} = 320$ K. For all these figures, the chemical potential of graphene is set as $\mu_g = 1.0$ eV. The chemical potential difference across the InSb $p$-$n$ junction $\Delta\mu$ is optimized independently for each configuration. Figures adapted from Wang et al. [202].
[191,209]. But the system suffers from low energy efficiency (below 10%) when operating at moderate temperatures due to the parasitic heat transfer induced by the phonon-polaritons of InAs. In Ref. [210], we use InSb as the near-field absorber since the bandgap energy of InSb is lower compared to InAs and its photon–phonon interaction is much weaker than InAs. In this work, we examine the performances of two near-field thermophotovoltaic devices: the graphene-\(h\)-BN-graphene-InSb cell (denoted as G-FBN-G-InSb cell, with the graphene-\(h\)-BN-graphene sandwich structure being the emitter and the InSb thin-film being the cell) and the graphene-\(h\)-BN-graphene-\(h\)-BN-InSb cell (denoted as G-FBN-G-FBN-InSb cell, with the double graphene-\(h\)-BN heterostructure being the emitter and the InSb thin-film being the cell). It is found that the G-FBN-G-InSb cell, despite having a simpler structure, performs better than the G-FBN-G-FBN-InSb cell. While both near-field thermophotovoltaic systems based on InSb thin-film cells underperform the ones based on bulk InSb cells. This is due to the exponential decay characteristic of the electromagnetic wave propagating in the InSb thin-film, which induces an actual availability of the above-gap photons in the photon-carrier generation process [210]. In general, these devices are promising for heat-to-electricity energy conversion in the common industrial waste heat regime.

12. Summary and outlooks

This paper attempts to provide a succinct review of the research frontier of inelastic thermoelectric effects. We summarized both theoretical and experimental progresses on inelastic thermoelectric transport and fluctuation in mesoscopic systems. We first give a general theoretical framework of the thermoelectric elastic and inelastic transport and revealed the unique role of the inelastic process of thermal transport in mesoscopic systems. We then show the distinct bounds on the linear transport coefficients of the elastic and inelastic thermoelectric transport from a general theoretical framework. We further summarize the unprecedented phenomena emerging from inelastic thermoelectric transport such as linear thermal transistor, cooling by heating, heat-charge cross rectification, and cooling by thermal current. Inspired or based on inelastic thermoelectric effects, several approaches to improve thermoelectric performance are summarized, including heat-charge separation, thermoelectric cooperative effects, nonlinear enhancement of performance, non-canonical reservoirs, and near-field enhancement effect. For near-field enhanced thermoelectric energy conversion, we discuss a set of examples including quantum-dot systems and graphene-\(h\)-BN-InSb systems.
Moreover, by integrating spin thermoelectric effect with the concepts from magnonics, the electron–magnon interactions for the nonequilibrium transport has been studied recently in many theoretical [211–218] and experimental works [219–222]. The asymmetric spin Seebeck effect has recently been discovered both in metal/insulating magnet interfaces and magnon tunneling junctions, which leads to many interesting effects, such as spin thermal rectifiers effect [223], spin transistor effects [224], logic gates, and negative differential spin Seebeck effects [225]. These properties could have various implications in flexible thermal and spin information control. The generalization of nanoscale metal-magnetic insulator interfaces with electron–magnon interactions for inelastic thermoelectric transport and fluctuations in mesoscopic system is an interesting future direction.

In addition to the contents reviewed above, there are still many interdisciplinary research frontiers of great curiosity, which are partially listed below:

(i) **Thermodynamic uncertainty relation.** Recently, a thermodynamic uncertainty relation has been formulated for classical Markovian systems demonstrating a trade-off between current fluctuation (precision) and dissipation (cost) in nonequilibrium steady state [226–230]. The thermodynamic uncertainty relation implies that a precise thermodynamic process with little noise requires high-entropy production. It is believed to be important in exploring the thermodynamic uncertainty bounds on the multi-terminal inelastic thermoelectric heat engine.

(ii) **Geometric-phase-induced pump.** The second law of thermodynamics indicates that heat cannot be transferred spontaneously from a low-temperature heat reservoir to a high-temperature heat reservoir. To go beyond this conventional thermoelectric energy conversion, a Berry-phase-like effect provides an additional geometric contribution [231–238] to pump electric and heat currents against the thermodynamic bias. Hence, it is intriguing to analyze the influence of the geometric-phase-induced pump in the periodically driven quantum thermal machines [239], e.g. the inelastic thermoelectric engine.

(iii) **Enhancing performance of near-field thermophotovoltaic systems via twisted bilayer two-dimensional materials.** The performance of near-field thermophotovoltaic systems can be greatly improved due to the hybridization effect of polaritons. Recently, the concept of photonic magic angles has attracted the attention of many researchers, due to the manipulation of the photonic dispersion of phonon polaritons in van der Waals bilayers [240]. The twisted two-dimensional bilayer
anisotropy materials or insulator slabs are explored in the near-field systems, and the near-field radiative heat transfer can be significantly enhanced by the twist-nonresonant surface polaritons [241–243]. Inspired by this concept, it is extraordinarily promising to enhance the output power and energy efficiency of near-field thermophotovoltaic systems by employing twisted bilayer two-dimensional materials: the near-field absorber and emitter can consist of two-dimensional anisotropic material/structure, e.g. van der Waals materials or grating structures.

(iv) Angular momentum radiation. The spin–orbit interaction, i.e. the coupling between the electron (light) orbital motion and the corresponding spin, is fundamentally important in spintronics [244], topological physics [245,246], and nano-optics [247,248]. Recently, the concept of angular momentum radiation was proposed via the spin–orbital interaction in the molecular junctions interacting with the electromagnetic waves [249]. Based on the nonequilibrium Green’s function method, the angular momentum selection rule for inelastic transport was unraveled. Hence, it should be interesting in incorporating the angular momentum selection rule into the photon-involved inelastic thermoelectric machines.

Disclosure statement
No potential conflict of interest was reported by the author(s).

Funding
This work was supported by the support from the funding for Distinguished Young Scientist from the National Natural Science Foundation of China (Grant Nos. 12125504, 12074281, 12047541, 12074279, and 11704093), the Major Program of Natural Science Research of Jiangsu Higher Education Institutions (Grant No. 18KJA140003), the Jiangsu specially appointed professor funding, the Academic Program Development of Jiangsu Higher Education (PAPD), the Opening Project of Shanghai Key Laboratory of Special Artificial Microstructure Materials and Technology, the China Postdoctoral Science Foundation (Grant No. 2020M681376), the faculty start-up funding of Suzhou University of Science and Technology, and Jiangsu Key Disciplines of the Fourteenth Five-Year Plan (Grant No. 2021135).

References
[1] Harman TC, Honig JM. Thermoelectric and thermomagnetic effects and applications. NewYork: McGraw-Hill; 1967.
[2] Imry Y. Introduction to mesoscopic Physics. London: Oxford University Press; 1997.
[3] Brandner K. Coherent transport in periodically driven mesoscopic conductors: from scattering amplitudes to quantum thermodynamics. Z Naturforsch A. 2020;75:483–500.

[4] Brandner K, Bauer M, Seifert U. Universal coherence-induced power losses of quantum heat engines in linear response. Phys Rev Lett. 2017;119:170602.

[5] Sánchez R, Gorini C, Fleury G. Extrinsic thermoelectric response of coherent conductors. Phys Rev B. 2021;104:115430.

[6] Potanina E, Flindt C, Moskalets M, et al. Thermodynamic bounds on coherent transport in periodically driven conductors. Phys Rev X. 2021;11:021013.

[7] Leggett AJ, Chakravarty S, Dorsey AT, et al. Dynamics of the dissipative two-state system. Rev Mod Phys. 1987;59:1–85.

[8] Proesmans K, Cleuren B, Van den Broeck C. Power-efficiency-dissipation relations in linear thermodynamics. Phys Rev Lett. 2016;116:220601.

[9] Onsager L. Reciprocal relations in irreversible processes. i. Phys Rev. 1931;37:405–426.

[10] Onsager L. Reciprocal relations in irreversible processes. ii. Phys Rev. 1931;38:2265–2279.

[11] Callen HB. The application of onsager’s reciprocal relations to thermoelectric, thermo-magnetic, and galvanomagnetic effects. Phys Rev. 1948;73:1349–1358.

[12] Saito K, Benenti G, Casati G, et al. Thermopower with broken time-reversal symmetry. Phys Rev B. 2011;84:201306.

[13] Benenti G, Saito K, Casati G. Thermodynamic bounds on efficiency for systems with broken time-reversal symmetry. Phys Rev Lett. 2011;106:230602.

[14] Jiang J-H. Thermodynamic bounds and general properties of optimal efficiency and power in linear responses. Phys Rev E. 2014;90:042126.

[15] Haug H, Jauho AP. Quantum kinetics in transport and optics of semiconductors. Berlin Heidelberg: Springer-Verlag; 2008.

[16] Büttiker M. Coherent and sequential tunneling in series barriers. IBM J Res Dev. 1988;32:63–75.

[17] Büttiker M. Four-terminal phase-coherent conductance. Phys Rev Lett. 1986;57:1761–1764.

[18] Büttiker M. Transport as a consequence of state-dependent diffusion. Z Phys B. 1987;68:161–167.

[19] Meir Y, Wingreen NS. Landauer formula for the current through an interacting electron region. Phys Rev Lett. 1992;68:2512–2515.

[20] Jauho A-P, Wingreen NS, Meir Y. Time-dependent transport in interacting and noninteracting resonant-tunneling systems. Phys Rev B. 1994;50:5528–5544.

[21] Blanter YM, Büttiker M. Shot noise in mesoscopic conductors. Phys Rep. 2000;336:1–166.

[22] Wang J-S, Wang J, Zeng N. Nonequilibrium green’s function approach to mesoscopic thermal transport. Phys Rev B. 2006;74:033408.

[23] Lü JT, Wang J-S. Coupled electron and phonon transport in one-dimensional atomic junctions. Phys Rev B. 2007;76:165418.

[24] Wang J-S, Wang J, Lü JT. Quantum thermal transport in nanostructures. Eur Phys J B. 2008;62:381–404.

[25] Wang J-S, Agarwalla BK, Li H, et al. Nonequilibrium green’s function method for quantum thermal transport. Front Phys. 2014;9:673–697.

[26] Lü J-T, Wang J-S, Hedegård P, et al. Electron and phonon drag in thermoelectric transport through coherent molecular conductors. Phys Rev B. 2016;93:205404.
[27] Zhang Z-Q, Lü J-T. Thermal transport through a spin-phonon interacting junction: a nonequilibrium green’s function method study. Phys Rev B. 2017;96:125432.

[28] Brandner K, Hanazato T, Saito K. Thermodynamic bounds on precision in ballistic multiterminal transport. Phys Rev Lett. 2018;120:090601.

[29] Tu Z-C. Abstract models for heat engines. Front Phys. 2021;16:1–12.

[30] Carrega M, Cantemi LM, De Filippis G, et al. Engineering dynamical couplings for quantum thermodynamic tasks. PRX Quantum. 2022;3:010323.

[31] Guo J, Lü J-T, Feng Y, et al. Nuclear quantum effects of hydrogen bonds probed by tip-enhanced inelastic electron tunneling. Science. 2016;352:321–325.

[32] Sothmann B, Sánchez R, Jordan AN. Thermoelectric energy harvesting with quantum dots. Nanotechnology. 2015;26:032001.

[33] Jiang J-H, Imry Y. Linear and nonlinear mesoscopic thermoelectric transport with coupling with heat baths. C R Phys. 2016;17:1047–1059.

[34] Thierschmann H, Sánchez R, Sothmann B, et al. Thermoelectrics with Coulomb-coupled quantum dots. C R Phys. 2016;17:1109–1122.

[35] Entin-Wohlman O, Imry Y, Aharony A. Three-terminal thermoelectric transport through a molecular junction. Phys Rev B. 2010;82:115314.

[36] Simine L, Segal D. Vibrational cooling, heating, and instability in molecular conducting junctions: full counting statistics analysis. Phys Chem Chem Phys. 2012;14:13820–13834.

[37] Arrachea N, Bode N, von Oppen F. Vibrational cooling and thermoelectric response of nanoelectromechanical systems. Phys Rev B. 2014;90:125450.

[38] Roy TR, Donald Raj J, Sen A. Inelastic tunnel transport and nanoscale junction thermoelectricity with varying electrode topology. Adv Theory Simul. 2021;4:2100054.

[39] Zhou H, Thingna J, Wang J-S, et al. Thermoelectric transport through a quantum nanoelectromechanical system and its backaction. Phys Rev B. 2015;91:045410.

[40] Henriot L, Jordan AN, Le Hur K. Electrical current from quantum vacuum fluctuations in nanomotors. Phys Rev B. 2015;92:125306.

[41] Yamamoto K, Entin-Wohlman O, Aharony A, et al. Efficiency bounds on thermoelectric transport in magnetic fields: the role of inelastic processes. Phys Rev B. 2016;94:121402.

[42] McConnell C, Nazir A. Strong coupling in thermoelectric nanojunctions: a reaction coordinate framework. New J Phys. 2022;24:025002.

[43] Cutler M, Mott NF. Observation of Anderson localization in an electron gas. Phys Rev. 1969;181:1336–1340.

[44] Hicks LD, Dresselhaus MS. Effect of quantum-well structures on the thermoelectric figure of merit. Phys Rev B. 1993;47:12727–12731.

[45] Hicks LD, Dresselhaus MS. Thermoelectric figure of merit of a one-dimensional conductor. Phys Rev B. 1993;47:16631–16634.

[46] DiSalvo FJ. Thermoelectric cooling and power generation. Science. 1999;285:703–706.

[47] Venkatasubramanian R. Lattice thermal conductivity reduction and phonon localization like behavior in superlattice structures. Phys Rev B. 2000;61:3091–3097.

[48] Lon EB. Cooling, heating, generating power, and recovering waste heat with thermoelectric systems. Science. 2008;321:1457–1461.

[49] Snyder GJ, Toberer ES. Complex thermoelectric materials. Nat Mater. 2008;7:105–114.

[50] Biswas K, He J, Blum ID, et al. High-performance bulk thermoelectrics with all-scale hierarchical architectures. Nature. 2012;489:414–418.
[51] Su L., Wang D., Wang S., et al. High thermoelectric performance realized through manipulating layered phonon-electron decoupling. Science. 2022;375:1385–1389.

[52] Jiang J-H., Entin-Wohlman O., Imry Y. Thermoelectric three-terminal hopping transport through one-dimensional nanosystems. Phys Rev B. 2012;85:075412.

[53] Jiang J-H., Entin-Wohlman O., Imry Y. Three-terminal semiconductor junction thermoelectric devices: improving performance. New J Phys. 2013;15:075021.

[54] Jiang J-H. Enhancing efficiency and power of quantum-dots resonant tunneling thermo-electrics in three-terminal geometry by cooperative effects. J Appl Phys. 2014;116:194303.

[55] Jiang J-H., Imry Y. Enhancing thermoelectric performance using nonlinear transport effects. Phys Rev Appl. 2017;7:064001.

[56] Wang R., Lu J., Wang C., et al. Nonlinear effects for three-terminal heat engine and refrigerator. Sci Rep. 2018;8:2607.

[57] Jiang J-H., Imry Y. Near-field three-terminal thermoelectric heat engine. Phys Rev B. 2018;97:125422.

[58] Lu J., Jiang J-H., Imry Y. Unconventional four-terminal thermoelectric transport due to inelastic transport: cooling by transverse heat current, transverse thermoelectric effect, and Maxwell demon. Phys Rev B. 2021;103:085429.

[59] Jiang J-H., Kulkarni M., Segal D., et al. Phonon thermoelectric transistors and rectifiers. Phys Rev B. 2015;92:045309.

[60] Shockley W., Queisser HJ. Detailed balance limit of efficiency of p-n junction solar cells. J Appl Phys. 1961;32:510–519.

[61] Scully MO. Quantum photocell: using quantum coherence to reduce radiative recombination and increase efficiency. Phys Rev Lett. 2010;104:207701.

[62] Jaziri N., Bougahmouara A., Miller J., et al. A comprehensive review of thermoelectric generators: technologies and common applications. Energy Reports. 2020;6:264–287.

[63] Tohidi F., Ghazanfari Holagh S., Chitsaz A. Thermoelectric generators: a comprehensive review of characteristics and applications. Appl Therm Eng. 2022;201:117793.

[64] Li L., Jiang J-H. Staircase quantum dots configuration in nanowires for optimized thermoelectric power. Sci Rep. 2016;6:31974.

[65] Jiang J-H., Entin-Wohlman O., Imry Y. Hopping thermoelectric transport in finite systems: boundary effects. Phys Rev B. 2013;87:205420.

[66] Sothmann B., Sánchez R., Jordan AN., et al. Powerful energy harvester based on resonant-tunneling quantum wells. New J Phys. 2013;15:095021.

[67] Agarwalla BK., Kulkarni M., Mukamel S., et al. Giant photon gain in large-scale quantum dot-circuit qed systems. Phys Rev B. 2016;94:121305.

[68] Kumar Agarwalla B., Kulkarni M., Mukamel S., et al. Tunable photonic cavity coupled to a voltage-biased double quantum dot system: diagrammatic nonequilibrium green’s function approach. Phys Rev B. 2016;94:035434.

[69] Kumar Agarwalla B., Kulkarni M., Segal D. Photon statistics of a double quantum dot micromaser: quantum treatment. Phys Rev B. 2019;100:035412.

[70] Rutten B., Esposito M., Cleuren B. Reaching optimal efficiencies using nanosized photoelectric devices. Phys Rev B. 2009;80:235122.

[71] Cleuren B., Rutten B., Van den Broeck C. Cooling by heating: refrigeration powered by photons. Phys Rev Lett. 2012;108:120603.

[72] Houck AA, Türeci HE, Koch J. On-chip quantum simulation with superconducting circuits. Nat Phys. 2012;8:292–299.

[73] Sothmann B., Büttiker M. Magnon-driven quantum-dot heat engine. Europhys Lett. 2012;99:27001.
[74] Sánchez R, Büttiker M. Optimal energy quanta to current conversion. Phys Rev B. 2011;83:085428.
[75] Sánchez R, Sothmann B, Jordan AN, et al. Correlations of heat and charge currents in quantum-dot thermoelectric engines. New J Phys. 2013;15:125001.
[76] Sothmann B, Sánchez R, Jordan AN, et al. Rectification of thermal fluctuations in a chaotic cavity heat engine. Phys Rev B. 2012;85:205301.
[77] Whitney RS, Sánchez R, Haupt F, et al. Thermoelectricity without absorbing energy from the heat sources. Physica E. 2016;75:257–265.
[78] Zhang Y, Guo J, Chen J. Thermoelectric performance of three-terminal quantum dot refrigerators in two configurations. Phys E Low Dimens Syst Nanostruct. 2020;118:113874.
[79] Zhang Y, Zhang X, Ye Z, et al. Three-terminal quantum-dot thermal management devices. Appl Phys Lett. 2017;110:153501.
[80] Mayrhofer RD, Elouard C, Spllettsstoesser J, et al. Stochastic thermodynamic cycles of a mesoscopic thermoelectric engine. Phys Rev B. 2021;103:075404.
[81] Lu J, Wang R, Wang C, et al. Brownian thermal transistors and refrigerators in mesoscopic systems. Phys Rev B. 2020;102:125405.
[82] Entin-Wohlman O, Jiang J-H, Imry Y. Efficiency and dissipation in a two-terminal thermoelectric junction, emphasizing small dissipation. Phys Rev E. 2014;89:012123.
[83] Yamamoto K, Aharony A, Entin-Wohlman O, et al. Thermoelectricity near Anderson localization transitions. Phys Rev B. 2017;96:155201.
[84] Strasberg P, Winter A. First and second law of quantum thermodynamics: a consistent derivation based on a microscopic definition of entropy. PRX Quantum. 2021;2:030202.
[85] Chen G. Nanoscale energy transport and conversion. London: Oxford University Press; 2005.
[86] Mahan GD, Sofo JO. The best thermoelectric. Proc Natl Acad Sci USA. 1996;93:7436–7439.
[87] Zhou J, Yang R, Chen G, et al. Optimal bandwidth for high efficiency thermoelectrics. Phys Rev Lett. 2011;107:226601.
[88] Landauer R. Spatial variation of currents and fields due to localized scatterers in metallic conduction. IBM J Res Dev. 1957;1:223–231.
[89] Landauer R. Electrical resistance of disordered one-dimensional lattices. Phil Mag. 1970;21:863–867.
[90] Mazza F, Bosisio R, Benenti G, et al. Thermoelectric efficiency of three-terminal quantum thermal machines. New J Phys. 2014;16:085001.
[91] Douglas Stone A, Szafer A. What is measured when you measure a resistance? The Landauer formula revisited. IBM J Res Dev. 1988;32:384–413.
[92] Sivan U, Imry Y. Multichannel Landauer formula for thermoelectric transport with application to thermopower near the mobility edge. Phys Rev B. 1986;33:551–558.
[93] Butcher PN. Thermal and electrical transport formalism for electronic microstructures with many terminals. J Phys Condens Matter. 1990;2:4869.
[94] Benenti G, Casati G, Saito K, et al. Fundamental aspects of steady-state conversion of heat to work at the nanoscale. Phys Rep. 2017;694:1–124.
[95] Lu J, Wang R, Liu Y, et al. Thermoelectric cooperative effect in three-terminal elastic transport through a quantum dot. J Appl Phys. 2017;122:044301.
[96] Hartke TR, Liu Y-Y, Gullans MJ, et al. Microwave detection of electron-phonon interactions in a cavity-coupled double quantum dot. Phys Rev Lett. 2018;120:097701.
[97] Petersson KD, Mcfaul LW, Schroer MD, et al. Circuit quantum electrodynamics with a spin qubit. Nature. 2012;490:380–383.
[98] Liu Y-Y, Petersson KD, Stehlik J, et al. Photon emission from a cavity-coupled double quantum dot. Phys Rev Lett. 2014;113:036801.

[99] Gullans MJ, Liu Y-Y, Stehlik J, et al. Phonon-assisted gain in a semiconductor double quantum dot maser. Phys Rev Lett. 2015;114:196802.

[100] Prete D, Erdman PA, Demontis V, et al. Thermoelectric conversion at 30 k in inas/inp nanowire quantum dots. Nano Lett. 2019;19:3033–3039.

[101] Dorsch S, Svilans A, Joseffson M, et al. Heat driven transport in serial double quantum dot devices. Nano lett. 2021;21:988–994.

[102] Chen B, Wang B, Cao G, et al. Enhanced readout of spin states in double quantum dot. Sci Bull. 2017;62:712–716.

[103] Chen M-B, Jiang S-L, Wang N, et al. Microwave-resonator-detected excited-state spectroscopy of a double quantum dot. Phys Rev Appl. 2021;15:044045.

[104] Chen Z, Zhang X, Ren J, et al. Leveraging bipolar effect to enhance transverse thermoelectricity in semimetal mg2pb for cryogenic heat pumping. Nat Commun. 2021;12:1–7.

[105] Zhou W, Yamamoto K, Miura A, et al. Seebeck-driven transverse thermoelectric generation. Nat Mater. 2021;20:463–467.

[106] Yamamoto K, Iguchi R, Miura A, et al. Phenomenological analysis of transverse thermoelectric generation and cooling performance in magnetic/thermoelectric hybrid systems. J Appl Phys. 2021;129:223908.

[107] Datta S. Quantum transport: atom to transistor. London: Cambridge University Press; 2005.

[108] Bergenfeldt C, Samuelsson P, Sothmann B, et al. Hybrid microwave-cavity heat engine. Phys Rev Lett. 2014;112:076803.

[109] Sánchez R, Thierschmann H, Molenkamp L.W. All-thermal transistor based on stochastic switching. Phys Rev B. 2017;95:241401.

[110] Guo B-Q, Liu T, Yu C-S. Quantum thermal transistor based on qubit-qutrit coupling. Phys Rev E. 2018;98:022118.

[111] Liu H, Wang C, Wang L-Q, et al. Strong system-bath coupling induces negative differential thermal conductance and heat amplification in nonequilibrium two-qubit systems. Phys Rev E. 2019;99:032114.

[112] Guo B-Q, Liu T, Yu C-S. Multifunctional quantum thermal device utilizing three qubits. Phys Rev E. 2019;99:032112.

[113] Wang C, Chen X-M, Sun K-W, et al. Heat amplification and negative differential thermal conductance in a strongly coupled nonequilibrium spin-boson system. Phys Rev A. 2018;97:052112.

[114] Liu Y-Q, Yu D-H, Yu C-S. Common environmental effects on quantum thermal transistor. Entropy. 2022;24: 10.3390/e24010032.

[115] Bauer G, Saitoh E, Van Wees BJ. Spin caloritronics. Nat Mater. 2012;11:391.

[116] Li N, Ren J, Wang L, et al. Colloquium: phononics: manipulating heat flow with electronic analogs and beyond. Rev Mod Phys. 2012;84:1045–1066.

[117] Lu J, Wang R, Ren J, et al. Quantum-dot circuit-QED thermoelectric diodes and transistors. Phys Rev B. 2019;99:035129.

[118] Li B, Wang L, Casati G. Negative differential thermal resistance and thermal transistor. Appl Phys Lett. 2006;88:143501.

[119] Burkard G, Gullans MJ, Mi X, et al. Superconductor–semiconductor hybrid-circuit quantum electrodynamics. Nat Rev Phys. 2020;2:129–140.

[120] Zhu J-X, Balatsky AV. Theory of current and shot-noise spectroscopy in single-molecular quantum dots with a phonon mode. Phys Rev B. 2003;67:165326.
[121] Jiang J-H, John S. Photonic crystal architecture for room-temperature equilibrium Bose–Einstein condensation of exciton polaritons. Phys Rev X. 2014;4:031025.
[122] Ren J, Zhu J-X, Gubernatis JE, et al. Thermoelectric transport with electron–phonon coupling and electron-electron interaction in molecular junctions. Phys Rev B. 2012;85:155443.
[123] Mi X, Cady JV, Zajac DM, et al. Strong coupling of a single electron in silicon to a microwave photon. Science. 2017;355:156–158.
[124] Jin P-Q, Jeske J, Greentree AD, et al. Microwave quantum optics as a direct probe of the overhauser field in a quantum dot circuit quantum electrodynamics device. Phys Rev B. 2021;103:045301.
[125] Chen Z-H, Che H-X, Chen Z-K, et al. Tuning nonequilibrium heat current and two-photon statistics via composite qubit-resonator interaction. Phys Rev Res. 2022;4:013152.
[126] Maxwell JC. Theory of heat. London: Longman; 1871.
[127] Erdman PA, Bhandari B, Fazio R, et al. Absorption refrigerators based on Coulomb-coupled single-electron systems. Phys Rev B. 2018;98:045433.
[128] Bhandari B, Chiriacò G, Erdman PA, et al. Thermal drag in electronic conductors. Phys Rev B. 2018;98:035415.
[129] Friedman HM, Segal D. Cooling condition for multilevel quantum absorption refrigerators. Phys Rev E. 2019;100:062112.
[130] Manikandan SK, Jussiau É, Jordan AN. Autonomous quantum absorption refrigerators. Phys Rev B. 2020;102:235427.
[131] Liu J, Segal D. Coherences and the thermodynamic uncertainty relation: insights from quantum absorption refrigerators. Phys Rev E. 2021;103:032138.
[132] Entin-Wohlman O, Imry Y, Aharony A. Enhanced performance of joint cooling and energy production. Phys Rev B. 2015;91:054302.
[133] Mazza F, Valentini S, Bosisio R, et al. Separation of heat and charge currents for boosted thermoelectric conversion. Phys Rev B. 2015;91:245435.
[134] Lu J, Liu Y, Wang R, et al. Optimal efficiency and power, and their trade-off in three-terminal quantum thermoelectric engines with two output electric currents. Phys Rev B. 2019;100:115438.
[135] Sánchez R, Samuelsson P, Potts PP. Autonomous conversion of information to work in quantum dots. Phys Rev Res. 2019;1:033066.
[136] Sánchez R, Splettstoesser J, Whitney RS. Nonequilibrium system as a demon. Phys Rev Lett. 2019;123:216801.
[137] Annyby-Andersson B, Samuelsson P, Maisi VF, et al. Maxwell’s demon in a double quantum dot with continuous charge detection. Phys Rev B. 2020;101:165404.
[138] Koski JV, Maisi VF, Sagawa T, et al. Experimental observation of the role of mutual information in the nonequilibrium dynamics of a Maxwell demon. Phys Rev Lett. 2014;113:030601.
[139] Koski JV, Kutvonen A, Khaymovich IM, et al. On-chip Maxwell’s demon as an information-powered refrigerator. Phys Rev Lett. 2015;115:260602.
[140] Koski JV, Maisi VF, Pekola JP, et al. Experimental realization of a Szilard engine with a single electron. Proc Natl Acad Sci USA. 2014;111:13786–13789.
[141] Chida K, Desai S, Nishiguchi K, et al. Power generator driven by Maxwell’s demon. Nat Commun. 2017;8:15310.
[142] Xi M, Wang R, Lu J, et al. Coulomb thermoelectric drag in four-terminal mesoscopic quantum transport. Chin Phys Lett. 2021;38:088801.
[143] Sánchez D, López R. Nonlinear phenomena in quantum thermoelectrics and heat. C R Phys. 2016;17:1060–1071.
[144] Sánchez D, Serra L. Thermoelectric transport of mesoscopic conductors coupled to voltage and thermal probes. Phys Rev B. 2011;84:201307.
[145] Sánchez D, López R. Scattering theory of nonlinear thermoelectric transport. Phys Rev Lett. 2013;110:026804.
[146] López R, Sánchez D. Nonlinear heat transport in mesoscopic conductors: rectification, peltier effect, and Wiedemann-Franz law. Phys Rev B. 2013;88:045129.
[147] Sánchez D, Sánchez R, López R, et al. Nonlinear chiral refrigerators. Phys Rev B. 2019;99:245304.
[148] Saryal S, Gerry M, Khait I, et al. Universal bounds on fluctuations in continuous thermal machines. Phys Rev Lett. 2021;127:190603.
[149] Liu J, Jung KA, Segal D. Periodically driven quantum thermal machines from warming up to limit cycle. Phys Rev Lett. 2021;127:200602.
[150] Liu Y, Lu J, Wang R, et al. Energy cooperation in quantum thermoelectric systems with multiple electric currents. Chin Phys B. 2020;29:40504.
[151] Hajiloo F, Sánchez R, Whitney RS, et al. Quantifying nonequilibrium thermodynamic operations in a multiterminal mesoscopic system. Phys Rev B. 2020;102:155405.
[152] Manzano G, Sánchez R, Silva R, et al. Hybrid thermal machines: generalized thermodynamic resources for multitasking. Phys Rev Res. 2020;2:043302.
[153] Caplan SR. A characteristic of self-regulated linear energy converters the hill force-velocity relation for muscle. J Theor Biol. 1966;11:63–86.
[154] Sánchez R, Sothmann B, Jordan AN. Chiral thermoelectrics with quantum hall edge states. Phys Rev Lett. 2015;114:146801.
[155] Gresta D, Real M, Arrachea L. Optimal thermoelectricity with quantum spin hall edge states. Phys Rev Lett. 2019;123:186801.
[156] Jordan AN, Sothmann B, Sánchez R, et al. Powerful and efficient energy harvester with resonant-tunneling quantum dots. Phys Rev B. 2013;87:075312.
[157] Zhang ZM. Nano/microscale heat transfer. New York: McGraw-Hill; 2007.
[158] Song B, Thompson D, Fiorino A, et al. Radiative heat conductances between dielectric and metallic parallel plates with nanoscale gaps. Nat Nanotechnol. 2016;11:509–514.
[159] Biehs S-A, Messina R, Venkataram PS, et al. Near-field radiative heat transfer in many-body systems. Rev Mod Phys. 2021;93:025009.
[160] Polder D, Van Hove M. Theory of radiative heat transfer between closely spaced bodies. Phys Rev B. 1971;4:3303–3314.
[161] Roßnagel J, Abah O, Schmidt-Kaler F, et al. Nanoscale heat engine beyond the carnot limit. Phys Rev Lett. 2014;112:030602.
[162] Klaers J, Faelt S, Imamoglu A, et al. Squeezed thermal reservoirs as a resource for a nanomechanical engine beyond the carnot limit. Phys Rev X. 2017;7:031044.
[163] Agarwalla BK, Jiang J-H, Segal D. Quantum efficiency bound for continuous heat engines coupled to noncanonical reservoirs. Phys Rev B. 2017;96:104304.
[164] Breuer HP, Petruccione F. The theory of open quantum systems. New York: Oxford University Press; 2006.
[165] Huang XL, Wang T, Yi XX. Effects of reservoir squeezing on quantum systems and work extraction. Phys Rev E. 2012;86:051105.
[166] Manzano G, Galve F, Zambrini R, et al. Entropy production and thermodynamic power of the squeezed thermal reservoir. Phys Rev E. 2016;93:052120.
[167] Blickle V, Bechinger C. Realization of a micrometre-sized stochastic heat engine. Nat Phys. 2012;8:143.
[168] Seifert U. Stochastic thermodynamics, fluctuation theorems and molecular machines. Rep Prog Phys. 2012;75:126001.
[169] Ciliberto S. Experiments in stochastic thermodynamics: short history and perspectives. Phys Rev X. 2017;7:021051.
[170] Seifert U. From stochastic thermodynamics to thermodynamic inference. Annu Rev Condens. 2019;10:171–192.
[171] Martinez IA, Roldán E, Dinis L, et al. Brownian carnot engine. Nat Phys. 2016;12:67.
[172] Verley G, Massimiliano Esposito TW, Van Den Broeck C. The unlikely carnot efficiency. Nat Commun. 2014;5:4721.
[173] Verley G, Willaert T, Van den Broeck C, et al. Universal theory of efficiency fluctuations. Phys Rev E. 2014;90:052145.
[174] Ken F, Quan HT. Path integral approach to quantum thermodynamics. Phys Rev Lett. 2018;121:040602.
[175] Liu F, Su S. Stochastic floquet quantum heat engines and stochastic efficiencies. Phys Rev E. 2020;101:062144.
[176] Fei Z, Quan HT. Nonequilibrium green’s function’s approach to the calculation of work statistics. Phys Rev Lett. 2020;124:240603.
[177] Fei Z, Freitas N, Cavina V, et al. Work statistics across a quantum phase transition. Phys Rev Lett. 2020;124:170603.
[178] Ma Y-H, Zhai R-X, Chen J, et al. Experimental test of the 1/τ - scaling entropy generation in finite-time thermodynamics. Phys Rev Lett. 2020;125:210601.
[179] Fei Z, Chen J-F, Ma Y-H. Efficiency statistics of a quantum Otto cycle. Phys Rev A. 2022;105:022609.
[180] Lin J, Li K, He J, et al. Power statistics of Otto heat engines with the mpemba effect. Phys Rev E. 2022;105:014104.
[181] Jiang J-H, Agarwalla BK, Segal D. Efficiency statistics and bounds for systems with broken time-reversal symmetry. Phys Rev Lett. 2015;115:040601.
[182] Polettini M, Verley G, Esposito M. Efficiency statistics at all times: carnot limit at finite power. Phys Rev Lett. 2015;114:050601.
[183] Gaspard P. Multivariate fluctuation relations for currents. New J Phys. 2013;15:115014.
[184] Andrieux D, Gaspard P. Fluctuation theorem and onsager reciprocity relations. J Chem Phys. 2004;121:6167–6174.
[185] Proesmans K, Dreher Y, Gavrilov M, et al. Brownian duet: a novel tale of thermodynamic efficiency. Phys Rev X. 2016;6:041010.
[186] Esposito M, Ochoa MA, Galperin M. Efficiency fluctuations in quantum thermoelectric devices. Phys Rev B. 2015;91:115417.
[187] Agarwalla BK, Jiang J-H, Segal D. Full counting statistics of vibrationally assisted electronic conduction: transport and fluctuations of thermoelectric efficiency. Phys Rev B. 2015;92:245418.
[188] Touchette H. The large deviation approach to statistical mechanics. Phys Rep. 2009;478:1–69.
[189] Agarwalla BK, Jiang J-H, Segal D. Thermoelectricity in molecular junctions with harmonic and anharmonic modes. BEILSTEIN J Nanotechnol. 2015;6:2129–2139.
[190] Liao T, Cai L, Zhao Y, et al. Efficiently exploiting the waste heat in solid oxide fuel cell by means of thermophotovoltaic cell. J Power Sources. 2016;306:666–673.
[191] Zhao B, Chen K, Buddhiraju S, et al. High-performance near-field thermophotovoltaics for waste heat recovery. Nano Energy. 2017;41:344–350.
[192] Tervo E, Bagherisereshti E, Zhang ZM. Near-field radiative thermoelectric energy converters: a review. Front Energy. 2018;12:5–21.
[193] Ilic O, Jablan M, Joannopoulos JD, et al. Overcoming the black body limit in plasmonic and graphene near-field thermopovoltaic systems. Opt Express. 2012;20:A366–A384.

[194] Svetovoy VB, Palasantzas G. Graphene-on-silicon near-field thermopovoltaic cell. Phys Rev Appl. 2014;2:034006.

[195] Laroche M, Carminati R, Greffet JJ. Near-field thermophotovoltaic energy conversion. J Appl Phys. 2006;100:063704.

[196] Molesky S, Jacob Z. Ideal near-field thermopovoltaic cells. Phys Rev B. 2015;91:205435.

[197] Svetovoy VB, Van Zwol PJ, Chevrier J. Plasmon enhanced near-field radiative heat transfer for graphene covered dielectrics. Phys Rev B. 2012;85:155418.

[198] Messina R, Ben-Abdallah P. Graphene-based photovoltaic cells for near-field thermal energy conversion. Sci Rep. 2013;3:1383.

[199] Zhao B, Zhang ZM. Enhanced photon tunneling by surface plasmonphonon polaritons in graphene/hBN heterostructures. J Heat Transfer. 2015;139:022701–022701–8.

[200] Zhao B, Guizal B, Zhang ZM, et al. Near-field heat transfer between graphene/hBN multilayers. Phys Rev B. 2017;95:245437.

[201] Shi K, Bao F, He S. Enhanced near-field thermal radiation based on multilayer graphene/hbn heterostructures. ACS Photonics. 2017;4:971–978.

[202] Wang R, Lu J, Jiang J-H. Enhancing thermopovoltaic performance using graphene-hbn-InSb near-field heterostructures. Phys Rev Appl. 2019;12:044038.

[203] Brar VW, Jang MS, Sherrott M, et al. Hybrid surface-phonon-plasmon polariton modes in graphene/monolayer h-bn heterostructures. Nano Lett. 2014;14:3876–3880.

[204] Kumar A, Low T, Fung KH, et al. Tunable light-matter interaction and the role of hyperbolicity in graphene-hBN system. Nano Lett. 2015;15:3172–3180.

[205] Heavens OS. Optical properties of thin solid films. Chicago: Courier Corporation; 1991.

[206] Knittl Z. Optics of thin films: an optical multilayer theory. London: Wiley; 1976.

[207] Polder D, Van H M. Theory of radiative heat transfer between closely spaced bodies. Phys Rev B. 1971;4:3303.

[208] Pendry JB. Radiative exchange of heat between nanostructures. J Phys Condens Matter. 1999;11:6621.

[209] Papadakis GT, Buddhiraju S, Zhao Z, et al. Broadening near-field emission for performance enhancement in thermopovoltaics. Nano Lett. 2020;20:1654–1661.

[210] Wang R, Lu J, Jiang J-H. Moderate-temperature near-field thermophotovoltaic systems with thin-film insb cells. Chin Phys Lett. 2021;38:024201.

[211] Tulapurkar AA, Suzuki Y. Contribution of electronmagnon scattering to the spin-dependent seebeck effect in a ferromagnet. Solid State Commun. 2010;150:466–470. spin Caloritronics.

[212] Bauer GEW, Saitoh E, Van Wees BJ. Spin caloritronics. Nat Mater. 2012;11:391–399.

[213] Chumak AV, Vasyuchka VI, Serga AA, et al. Magnon spintronics. Nat Phys. 2015;11:453–461.

[214] Rezende SM, Rodriguez-Suárez RL, Azevedo A. Theory of the spin seebeck effect in antiferromagnets. Phys Rev B. 2016;93:014425.

[215] Qaiumzadeh A, Ado IA, Duine RA, et al. Theory of the interfacial Dzyaloshinskii-Moriya interaction in rashba antiferromagnets. Phys Rev Lett. 2018;120:197202.

[216] Tang G, Chen X, Ren J, et al. Rectifying full-counting statistics in a spin seebeck engine. Phys Rev B. 2018;97:081407.
[217] Upadhyay V, Naseem MT, Marathe R, et al. Heat rectification by two qubits coupled with Dzyaloshinskii-Moriya interaction. Phys Rev E. 2021;104:054137.
[218] Wang L, Wang Z, Wang C, et al. Cycle flux ranking of network analysis in quantum thermal devices. Phys Rev Lett. 2022;128:067701.
[219] Uchida K, Takahashi S, Harii K, et al. Observation of the spin seebeck effect. Nature. 2008;455:778–781.
[220] Jaworski CM, Yang J, Mack S, et al. Observation of the spin-seebeck effect in a ferromagnetic semiconductor. Nat Mater. 2010;9:898–903.
[221] Slachter A, Bakker FL, Adam J-P, et al. Thermally driven spin injection from a ferromagnet into a non-magnetic metal. Nat Phys. 2010;6:879–882.
[222] Walter M, Walowski J, Zbarsky V, et al. Seebeck effect in magnetic tunnel junctions. Nat Mater. 2011;10:742–746.
[223] Ren J, Zhu J-X. Heat diode effect and negative differential thermal conductance across nanoscale metal-dielectric interfaces. Phys Rev B. 2013;87:241412.
[224] Ren J, Zhu J-X. Theory of asymmetric and negative differential magnon tunneling under temperature bias: towards a spin seebeck diode and transistor. Phys Rev B. 2013;88:094427.
[225] Ren J. Predicted rectification and negative differential spin seebeck effect at magnetic interfaces. Phys Rev B. 2013;88:220406.
[226] Barato AC, Seifert U. Thermodynamic uncertainty relation for biomolecular processes. Phys Rev Lett. 2015;114:158101.
[227] Hasegawa Y. Thermodynamic uncertainty relation for general open quantum systems. Phys Rev Lett. 2021;126:010602.
[228] Liu J, Segal D. Thermodynamic uncertainty relation in quantum thermoelectric junctions. Phys Rev E. 2019;99:062141.
[229] Horowitz JM, Gingrich TR. Thermodynamic uncertainty relations constrain non-equilibrium fluctuations. Nat Phys. 2020;16:15–20.
[230] Yan J-L, Zhang J-W, Yun M-R, et al. Experimental verification of dissipation-time uncertainty relation. Phys Rev Lett. 2022;128:050603.
[231] Sinitsyn NA, Nemenman I. Universal geometric theory of mesoscopic stochastic pumps and reversible ratchets. Phys Rev Lett. 2007;99:220408.
[232] Sinitsyn NA, Nemenman I. The berry phase and the pump flux in stochastic chemical kinetics. Europhys Lett. 2007;77:58001.
[233] Ren J, Hánggi P, Li B. Berry-phase-induced heat pumping and its impact on the fluctuation theorem. Phys Rev Lett. 2010;104:170601.
[234] Wang C, Ren J, Cao J. Unifying quantum heat transfer in a nonequilibrium spin-boson model with full counting statistics. Phys Rev A. 2017;95:023610.
[235] Wang Z, Wang L, Chen J, et al. Geometric heat pump: controlling thermal transport with time-dependent modulations. Front Phys. 2022;17:1–14.
[236] Wang Z, Chen J, Liu Z, et al. Observation of geometric heat pump effect in periodic driven thermal diffusion. arXiv:2110.10001. 2021.
[237] Terrén Alonso P, Abiuso P, Perarnau-Llobet M, et al. Geometric optimization of nonequilibrium adiabatic thermal machines and implementation in a qubit system. PRX Quantum. 2022;3:010326.
[238] Lu J, Wang Z, Peng J, et al. Geometric thermodynamic uncertainty relation in a periodically driven thermoelectric heat engine. Phys Rev B. 2022;105:115428.
[239] Bhandari B, Alonso PT, Taddei F, et al. Geometric properties of adiabatic quantum thermal machines. Phys Rev B. 2020;102:155407.
[240] Hu G, Ou Q, Si G, et al. Topological polaritons and photonic magic angles in twisted α-moo 3 bilayers. Nature. 2020;582:209–213.
[241] He M, Qi H, Ren Y, et al. Active control of near-field radiative heat transfer by a graphene-gratings coating-twisting method. Opt Lett. 2020;45:2914–2917.
[242] Tang G, Chen J, Zhang L. Twist-induced control of near-field heat radiation between magnetic Weyl semimetals. ACS Photonics. 2021;8:443–448.
[243] Peng J, Tang G, Wang L, et al. Twist-induced near-field thermal switch using nonreciprocal surface magnon-polaritons. ACS Photonics. 2021;8:2183–2189.
[244] Awschalom D, Samarth N. Spintronics without magnetism. Physics. 2009;2:50.
[245] Hasan MZ, Kane CL. Colloquium: topological insulators. Rev Mod Phys. 2010;82:3045–3067.
[246] Qi X-L, Zhang S-C. Topological insulators and superconductors. Rev Mod Phys. 2011;83:1057–1110.
[247] Bliokh KY, Rodríguez-Fortuño FJ, Nori F, et al. Spin-orbit interactions of light. Nat Photonics. 2015;9:796–808.
[248] Konstantin YB, Franco N. Transverse and longitudinal angular momenta of light. Phys Rep. 2015;592:1–38.
[249] Zhang Z-Q, Lü J-T, Wang J-S. Angular momentum radiation from current-carrying molecular junctions. Phys Rev B. 2020;101:161406.