Anderson localization in finite disordered vibrating rods

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Abstract – We study, both experimentally and numerically, the Anderson localization phenomenon in torsional waves of a disordered elastic rod, which consists of a cylinder with randomly spaced notches. We find that the normal-mode wave amplitudes are exponentially localized as occurs in disordered solids. The localization length, measured using these wave amplitudes, decreases as a function of frequency. The normal-mode spectrum is also measured as well as computed, so its level statistics can be analyzed. From the spectrum, the nearest-neighbor spacing distribution can be obtained. This distribution can be described by a phenomenological expression that depends on a parameter $\alpha$, related to the level repulsion, which is also a function of the frequency. Therefore, the localization length can be expressed in terms of the parameter $\alpha$. There exists a range in which the localization length grows linearly with $\alpha$. However, at low values of this parameter the linear dependence does not hold.

The Anderson localization phenomenon is a very important subject in condensed-matter physics since it is crucial to understand the transport properties of materials. As a matter of fact, the original work of Anderson [1] is among the most cited papers in twentieth-century physics and it is at the core of many papers, not only in solid-state studies [2–7] but also in optics [8–11], cold atomic gases [12], microwaves [13–16] and acoustics [17–22]. The theory of Anderson localization studies the alterations brought about on the localization of the electronic wave functions by disorder in the system. In a perfect lattice a band spectrum arises with extended wave functions for the allowed energy levels. However, if the system presents random imperfections, for example the presence of strange atoms in an otherwise perfect structure or when there are unit cells of different size, wave functions can be localized, affecting the transport properties of the system. Up to now experimental studies of Anderson localization are mainly related to the transmission coefficient or correlations but measurements of wave amplitudes are scarce. One should mention, however, that in microwave cavities, wave functions in chaotic and disordered billiards were studied [15]. In ref. [14] a relation between the absorption parameter and the measured wave amplitudes is obtained in a few cases. In ref. [23], on the other hand, a deep study of the correlations of the 2D wave amplitudes is performed (for a review see refs. [24–26]). In this work we go beyond these studies since we analyze many wave amplitudes and extract from them the localization length.

Anderson localization can also be studied using elastic vibrating systems, such as the rod shown in fig. 1. The system consists of $N$ rods of radius $R$ with lengths $d_i$, $i = 1, \ldots, N$, joined by smaller cylinders of length $\epsilon \ll d_i$, $\forall i$, and radius $r = \eta R$, where the coupling constant $\eta$ is such that $0 < \eta < 1$. The total length of the rod is $L = 3.65$ m. According to the nature of the family of numbers $\{d_i\}$ different phenomena are observed. We should remark that we measure not only the normal-mode frequencies but also the associated wave amplitudes. In this aspect, the analysis of elastic vibrations is easier than in the quantum-mechanical or optical cases, since for such systems wave functions are difficult to measure directly.

In this letter we shall study torsional vibrations in a disordered rod by taking the family $\{d_i = d(1 - n_i \Delta)\}$ with $n_i$ an uncorrelated random number in the interval $[-1, 1]$. Therefore the set of numbers $\{d_i\}$ are uncorrelated random numbers with a uniform distribution in the interval $[d(1 - \Delta), d(1 + \Delta)]$. Here $d = \langle d_i \rangle$ is the average of $d_i$ and $\Delta$ measures the disorder. Notice that, on average, with this disorder, the total length of the rod does not change.

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The value in the aluminum alloy we have used. A parameter spacing distribution. This distribution can be described by the other hand, we are able to study the nearest-neighbor $\xi$ then be eliminated to obtain a relation between the interaction of eddy currents in the metallic rod with excite the oscillations. The transducer operates through the interaction of eddy currents in the metallic rod with a permanent magnetic field. According to the relative position of the magnet and the coil, the EMAT can either excite or detect selectively compressional, torsional or flexural vibrations. Used as a detector, the EMAT measures acceleration. This transducer has the advantage of operating without mechanical contact with the rod. This is crucial to avoid perturbing the shape of the localized wave amplitudes. Both the detector and exciter can be moved automatically along the bar axis and then the wave amplitudes can be measured easily. The experimental setup is described in detail in ref. [27].

Since it is expected that the wave amplitudes in the disordered rod are localized, it is convenient to know in advance the approximate position of the maximum of each wave amplitude. Otherwise it would be extremely difficult to measure the normal-mode frequencies and wave functions. To perform the measurements, the rod is excited by the EMAT with a monochromatic signal of frequency $f$ and we should produce an educated guess on the position of the maximum of the amplitude. To proceed in this direction we consider a system in which the coupling parameter $\eta \ll 1$. The small cylinders of length $d_i$ are then almost independent of each other. These values of $\eta$ define what we have called the independent rod model, in the same sense of independent particle models. We have used it successfully in dealing with the localized states of the Wannier-Stark ladders [28].

Assuming $\eta \ll 1$, the rod of length $d_i$ shows resonant frequencies $f_i^r = \frac{nc}{2\pi d_i}$, where $c$ is the velocity of torsional waves and $n$ is an integer number. Therefore when $f = f_i^r$.

Fig. 1: Rod used to measure localization. The values used in the experiment are: number of rods $N = 50$, $R = 1.28$ cm, $\epsilon = 1.02$ mm, $L = 3.65$ m and $\eta = \frac{1}{R} = 0.65$; the average of $d_i$ is $d = 7.2$ cm, and the amount of disorder is $\Delta = 0.35$. The largest frequency considered is less than 100 kHz so the lowest wavelength $\lambda_{\text{min}} = \frac{c}{f_{\text{max}}} > 2R$ and the system behaves indeed as 1D. The value $c = 314.0 \frac{\text{m}}{\text{s}}$ was measured for torsional waves in the aluminum alloy we have used.

Fig. 2: (Colour on-line) Normal-mode frequencies, obtained numerically by using the Poincaré map method, as a function of the disorder strength $\Delta$. The (red) vertical dashed line corresponds to the value of $\Delta$ used in the experiment. In order to make the level repulsion evident, the inset shows a zoom of the small (blue) rectangle in which the vertical and horizontal scales have been changed.

The spectrum changes with the disorder strength $\Delta$ as shown in fig. 2 for a fixed realization of $n_i$. One can notice that for $\Delta \ll 1$ a band spectrum appears; at higher values of $\Delta$, avoided crossings are observed and the bands and gaps disappear. This allows us to study the Anderson localization phenomenon in elastic systems; on the one hand, we can observe how the wave amplitudes of the normal modes decay exponentially with a localization length $\xi$. The localization length is then measured as a function of the frequency $f$. Measuring the spectrum, on the other hand, we are able to study the nearest-neighbor spacing distribution. This distribution can be described by a parameter $\alpha$, that also depends on $f$. The frequency can then be eliminated to obtain a relation between $\xi$ and $\alpha$.

To perform the measurements we used the electromagnetic-acoustic transducer (EMAT) developed by us [27]. The EMAT consists of a permanent magnet and a coil, and can be used either to detect or excite the oscillations. The transducer operates through the interaction of eddy currents in the metallic rod with a permanent magnetic field. According to the relative position of the magnet and the coil, the EMAT can either excite or detect selectively compressional, torsional or flexural vibrations. Used as a detector, the EMAT measures acceleration. This transducer has the advantage of operating without mechanical contact with the rod. This is crucial to avoid perturbing the shape of the localized wave amplitudes. Both the detector and exciter can be moved automatically along the bar axis and then the wave amplitudes can be measured easily. The experimental setup is described in detail in ref. [27].

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The amplitude is maximum at the location of the $i$-th rod. Furthermore, since in general $d_j \neq d_i$ for $i \neq j$, the neighboring rods of the $i$-th one will be excited only with a smaller amplitude. The states are then localized as indeed we have found experimentally and can be seen in figs. 3(a) and (c). Occasionally, another $d_j$ has a value very close to $d_i$, and the wave function can show more than one maxima, as shown in figs. 3(b) and (d).

If the cylinders have random lengths, the elastic vibrations are waves on a random structure, which is analogous to what happens with the Schrödinger wave functions in a random potential, frequency playing the role of energy. The independent rod model then shows that introducing disorder in $\{d_i\}$ is a way to simulate diagonal disorder in a quantum-mechanical one-dimensional tight-binding Hamiltonian, where the coupling $\eta$ between nearest neighbors is a constant [29]. In this case Anderson localization phenomenon also occurs.

We shall now compute and measure the localization length $\xi$ for a disordered rod. This quantity can be defined in at least three ways [26]: using the exponential decay of the transmission coefficient; by the Lyapunov exponent of the transfer matrix, or by the exponential decay of the wave amplitude envelope. Because the wave functions are in general not accessible experimentally, the first two definitions are the ones normally used in the literature. However, we do have access to the wave functions, so we shall use the last definition. We should note that the tails of the distribution of the squared amplitudes, as those of ref. [30], correspond to the maxima of the wave functions. Thus the method used in our work, based on the wave amplitudes themselves, is equivalent to considering the intensity distribution.

The localization length $\xi$ as a function of frequency is given in fig. 4. One should mention the overall agreement between the numerical calculation using the Poincaré map method (squares) and experiment (dots). The numerical values were obtained from an ensemble of 5000 families $\{d_i\}$ but the experimental values were measured only for 50 eigenfunctions of a single rod. It is to be noted that $\xi$ decreases with $f$. The reflection coefficient for a single
notch of length $\epsilon$ for low frequencies is an increasing function of $f$. As a consequence, when frequency grows the reflection by the notch is more efficient and the localization length decreases with $f$. It is interesting to note that this effect is the opposite in the quantum case, since then the reflection coefficient is a decreasing function of energy. To obtain the wave amplitudes, such as those shown in fig. 3, the spectrum of the disordered rod must first be obtained. This is the case both numerically and in the laboratory. See fig. 5, in which a typical spectrum measured for a disordered rod is shown. We are then provided with an extra bonus: the statistical properties of the elastic spectra which render themselves to studies like those analyzed in spectral statistics and quantum chaos [32,33]. In what follows we shall give the nearest-neighbor spacing distribution $p(s_i)$, where $s_i = f_{i+1} - f_i$ is the normalized spacing, and show how the distribution varies as a function of frequency. Before obtaining the nearest-neighbor spacing distribution, it is necessary to eliminate secular variations of the level density.

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spreading distribution and not the level repulsion only [29] since the small spreading behavior \((s \ll 1)\) in the Anderson model strongly depends on the specific properties of the system [35–37]. For \(\alpha = 1, 2, 4\) the parameter \(\alpha\) coincides with the Dyson parameter \(\beta\) [38]. We should remark that we use the parameter \(\alpha\) to characterize the nearest-neighbor spreading distribution but it is not related to any symmetry.

Given these results one could also eliminate the frequency and obtain the relationship between \(\xi\) and \(\alpha\). For zero disorder, \(\Delta = 0\), on the one hand, the spectrum is equidistant in the center of the band. Thus the nearest-neighbor spreading distribution should be of the form \(P(s) \rightarrow \delta(s - 1)\) and the normal-mode wave amplitudes are extended. For strong disorder, on the other hand, all eigenstates are localized and the nearest-neighbor spreading distribution should be close to a Poissonian distribution \(P(s) \rightarrow \exp(-s)\). In between one has to expect an intermediate statistics which could be compared with the Wigner-Dyson distribution that is known to emerge for classically chaotic systems described by full random matrices or well in quasi-one-dimensional models described by banded random matrices [39,40] when the localization length is larger than the sample size [29]. The question is which relationship exists between the localization length \(\xi\) and the nearest-neighbor spreading distribution parameter \(\alpha\) between these two limits. The plot \(\xi\) vs. \(\alpha\) for the elastic rods is given in fig. 9. The squares were obtained for the ensemble of 5000 rods. One can see from the dashed line in this figure that indeed \(\xi\) grows linearly with \(\alpha\) for \(\alpha \gtrsim 1\); this result also appears in the one-dimensional Anderson model [29], as well as in quasi-one-dimensional disordered systems [41–43]. However, at low values of \(\alpha\), a different behavior is observed. In this regime a theoretical prediction is not available. The experimental results show large deviations from the numerics. This is so because the measurements were obtained for a single rod. Nevertheless, the experimental values are consistent with the numerical results, as can be seen in the inset of fig. 9.

To conclude, in this letter we have measured the exponential localization of elastic-wave amplitudes in disordered rods of finite length as well as the spectrum and calculated the parameter \(\alpha\) describing the nearest-neighbor spreading distribution. It was found that the localization length presents two regimes. For \(\alpha > 1\) the localization length grows linearly as a function of the parameter \(\alpha\). This means that, in this regime, the localization length is just the parameter \(\alpha > 1\) correctly scaled. This result seems to be valid not only for the rods we analyzed here but it also appears in the kicked rotor [44,45], in Wigner banded random matrices [46], in the standard Anderson model [29] as well as in the quasi-one-dimensional disordered wave systems [42,43]. However, there is a region, \(\alpha < 1\), in which the localization length does not grow linearly with \(\alpha\). It is important to note that it is often easier to measure a spectrum than the wave function. Thus, an extraction of the localization length by means of the nearest-neighbor spreading distribution might often be favourable in experiments.

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