THE FERMION MASS PROBLEM AND THE ANTI-GRAND UNIFICATION MODEL

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We describe the Anti-Grand Unification Model (AGUT) and the Multiple Point Principle (MPP) used to calculate the values of the Standard Model gauge coupling constants in the theory, from the requirement of the existence of degenerate vacua. The application of the MPP to the pure Standard Model predicts the existence of a second minimum of the Higgs potential close to the cut-off, which we take to be the Planck scale, giving our Standard Model predictions for the top quark and Higgs masses: \( M_t = 173 \pm 5 \) GeV and \( M_H = 135 \pm 9 \) GeV. We also discuss mass protection by chiral charges and present a fit to the charged fermion mass spectrum using the chiral quantum numbers of the maximal AGUT gauge group \( SMG^3 \times U(1)_f \), where \( SMG \equiv SU(3) \times SU(2) \times U(1) \).

The neutrino mass and mixing problem is then briefly discussed for models with chiral flavour charges responsible for the charged fermion mass hierarchy.

1 Introduction

One of the outstanding problems in particle physics is to explain the observed pattern of quark-lepton masses and of flavour mixing. This is the problem of the hierarchy of Yukawa coupling constants in the Standard Model (SM), which range in value from of order 1 for the top quark to of order \( 10^{-5} \) for the electron. However there is no reason in the SM for the Higgs field to prefer to couple to one fermion rather than another; in fact one would expect them all to be of order unity. We suggest that the natural resolution to this problem is the existence of some approximately conserved chiral charges beyond the SM. These charges, which we assume to be the gauge quantum numbers in the fundamental theory beyond the SM, provide selection rules forbidding the transitions between the various left-handed and right-handed quark-lepton states, except for the top quark. In order to generate mass terms for the other fermion states, we have to introduce new Higgs fields, which break the fundamental gauge symmetry group \( G \) down to the SM group. We also need suitable intermediate fermion states to mediate the forbidden transitions, which we take to be vector-like Dirac fermions with a mass of order the fundamental scale \( M_F \) of the theory. In this way effective SM Yukawa coupling constants are generated, which are suppressed by the appropriate product of Higgs field vacuum expectation values measured in units of \( M_F \).

If we want to explain the observed spectrum of quarks and leptons, it is clear that we need charges which—possibly in a complicated way—separate the generations and, at least for \( t - b \) and \( c - s \), also quarks in the same generation. Just using the usual simple \( SU(5) \) GUT charges does not help, because both \( \mu_R \) and \( e_R \) and \( \mu_L \) and \( e_L \) have the same \( SU(5) \) quantum numbers. So we prefer to keep each SM irreducible representation in a separate irreducible representation of \( G \) and introduce extra gauge quantum numbers distinguishing the generations, by adding extra cross-product factors to the SM gauge group. In this talk we consider the maximal anomaly free gauge group of this type—the anti-grand unification (AGUT) group \( SMG^3 \times U(1)_f \). In section 2 we discuss the structure of the AGUT model and the prediction of the values of the SM gauge coupling constants, using the so-called Multiple Point Principle (MPP). We apply this principle to the pure SM in section 3, assuming a desert up to the Planck scale, and obtain predictions for the top quark and SM Higgs particle masses. In section 4 we consider the Higgs fields responsible for breaking the AGUT gauge group and the structure of the resulting quark-lepton mass matrices, together with details of a fit to the observed spectrum. The problem of neutrino mass and mixing in models with approximately conserved chiral flavour charges are discussed in section 5. Finally we present our conclusions in section 6.

2 Anti-Grand Unification

In the AGUT model the SM gauge group is extended in much the same way as Grand Unified \( SU(5) \) is often assumed; it is just that we assume another non-simple gauge group \( G = SMG^3 \times U(1)_f \), where \( SMG = SU(3) \times SU(2) \times U(1) \).
SU(3) × SU(2) × U(1), becomes active near the Planck scale $M_{\text{Planck}} \approx 10^{19}$ GeV. So we have a pure SM desert, without any supersymmetry, up to an order of magnitude or so below $M_{\text{Planck}}$. The existence of the $SMG^3 \times U(1)_f$ group means that, near the Planck scale, each of the three quark-lepton generations has got its own gauge group and associated gauge particles with the same structure as the SM gauge group. There is also an extra abelian $U(1)_f$ gauge boson, giving altogether $3 \times 8 = 24$ gluons, $3 \times 3 = 9$ $W$'s and $3 \times 1 + 1 = 4$ abelian gauge bosons.

At first sight, this $SMG^3 \times U(1)_f$ group with its 37 generators seems to be just one among many possible SM gauge group extensions. However, it is actually not such an arbitrary choice, as it can be uniquely specified by postulating 4 reasonable requirements on the gauge group $G \supseteq SMG$:

1. $G$ should transform the presently known (left-handed, say) Weyl particles into each other, so that $G \subseteq U(45)$. Here $U(45)$ is the group of all unitary transformations of the 45 species of Weyl fields (3 generations with 15 in each) in the SM.

2. No anomalies, neither gauge nor mixed. We assume that only straightforward anomaly cancellation takes place and, as in the SM itself, do not allow for a Green-Schwarz type anomaly cancellation.

3. The various irreducible representations of Weyl fields for the SM group remain irreducible under $G$.

4. $G$ is the maximal group satisfying the other 3 postulates.

With these four postulates a somewhat complicated calculation shows that, modulo permutations of the various irreducible representations in the Standard Model fermion system, we are led to our gauge group $SMG^3 \times U(1)_f$. Furthermore it shows that the SM group is embedded as the diagonal subgroup of $SMG^3$, as required in our AGUT model. The AGUT group breaks down an order of magnitude or so below the Planck scale to the SM group. The anomaly cancellation constraints are so tight that, apart from various permutations of the particle names, the $U(1)_f$ charge assignments are uniquely determined up to an overall normalisation and sign convention. In fact the $U(1)_f$ group does not couple to the left-handed particles or any first generation particles, and the $U(1)_f$ quantum numbers can be chosen as follows:

$$Q_f(\tau_R) = Q_f(b_R) = Q_f(c_R) = 1$$

$$Q_f(\mu_R) = Q_f(s_R) = Q_f(t_R) = -1$$

![Figure 1: Evolution of the Standard Model fine structure constants $\alpha_i$ (in the SU(5) inspired normalisation) from the electroweak scale to the Planck scale. The anti-GUT model predictions for the values at the Planck scale, $\alpha_i^{-1}(M_{\text{Planck}})$, are shown with error bars.](image)

The situation is more complicated for the abelian groups, because it is possible to have gauge invariant cross-terms between the different $U(1)$ groups in the Lagrangian density such as:

$$\frac{1}{4g^2}F_{\mu\nu}^1(x)F_{\mu\nu}^2(x)$$

So, in first approximation, for the SM $U(1)$ fine structure constant we get:

$$\alpha_1(M_{\text{Planck}}) = \frac{\alpha_1^{\text{Multiple Point}}}{6}$$

The agreement of these AGUT predictions with the data is shown in figure 1.
3 The MPP Prediction for the Top Quark and Higgs masses in the Standard Model

The application of the MPP to the pure Standard Model (SM), with a cut-off close to $M_{\text{Planck}}$, implies that the SM parameters should be adjusted, such that there exists another vacuum state degenerate in energy density with the vacuum in which we live. This means that the effective SM Higgs potential $V_{\text{eff}}(|\phi|)$ should have a second minimum degenerate with the well-known first minimum at the electroweak scale $\langle |\phi_{\text{vac}}| \rangle = 246$ GeV. Thus we predict that our vacuum is barely stable and we just lie on the vacuum stability curve in the top quark, Higgs particle (pole) mass ($M_t, M_H$) plane. Furthermore we expect the second minimum to be within an order of magnitude or so of the fundamental scale, i.e. $\langle |\phi_{\text{vac}}| \rangle \sim M_{\text{Planck}}$. In this way, we essentially select a particular point on the SM vacuum stability curve and hence the MPP condition predicts precise values for $M_t$ and $M_H$.

For the purposes of our discussion it is sufficient to consider the renormalisation group improved tree level effective potential $V_{\text{eff}}(\phi)$. We are interested in values of the Higgs field of the order $|\phi_{\text{vac}}| \sim M_{\text{Planck}}$, which is very large compared to the electroweak scale, and for which the quartic term strongly dominates the $\phi^2$ term; so to a very good approximation we have:

$$V_{\text{eff}}(\phi) \simeq \frac{1}{8} \lambda (\mu = |\phi|)|\phi|^4$$

The running Higgs self-coupling constant $\lambda(\mu)$ and the top quark running Yukawa coupling constant $g_t(\mu)$ are readily computed by means of the renormalisation group equations, which are in practice solved numerically, using the second order expressions for the beta functions.

The vacuum degeneracy condition is imposed by requiring:

$$V_{\text{eff}}(\phi_{\text{vac}} 1) = V_{\text{eff}}(\phi_{\text{vac}} 2)$$

Now the energy density in vacuum 1 is exceedingly small compared to $\phi_{\text{vac}} 2 \sim M_{\text{Planck}}^4$. So we basically get the degeneracy condition, eq. (6), to mean that the coefficient $\lambda(\phi_{\text{vac}} 2)$ of $\phi^4$ must be zero with high accuracy. At the same $\phi$-value the derivative of the effective potential $V_{\text{eff}}(\phi)$ should be zero, because it has a minimum there. Thus at the second minimum of the effective potential the beta function $\beta_\lambda$ also vanishes:

$$\beta_\lambda(\mu = \phi_{\text{vac}} 2) = \lambda(\phi_{\text{vac}} 2) = 0$$

which gives to leading order the relationship:

$$\frac{9}{4} g_2^2 + \frac{3}{2} g_2 g_1^2 + \frac{3}{4} g_1^4 - 12 g_t^2 = 0$$

between the top quark Yukawa coupling and the electroweak gauge coupling constants $g_1(\mu)$ and $g_2(\mu)$ at the scale $\mu = \phi_{\text{vac}} 2 \sim M_{\text{Planck}}$. We use the renormalisation group equations to relate the couplings at the Planck scale to their values at the electroweak scale. Figures 2 and 3 show the running coupling constants $\lambda(\phi)$ and $g_t(\phi)$ as functions of $\log(\phi)$. Their values at the electroweak scale give our predicted combination of pole masses:

$$M_t = 173 \pm 5 \text{ GeV} \quad M_H = 135 \pm 9 \text{ GeV}$$

4 Fermion Mass Hierarchy in AGUT

The $\text{SMG}^3 \times U(1)_f$ gauge group is broken by a set of Higgs fields $S, W, T$ and $\xi$ down to the SM gauge group.
Together with the Weinberg Salam Higgs field, $\phi_{WS}$, they are responsible for breaking the quark-lepton mass protection by the chiral AGUT quantum numbers. We have the freedom of choosing the abelian quantum numbers of the Higgs fields, which we can express as charge vectors of the form:

$$\vec{Q} = \left(\frac{y_1}{2}, \frac{y_2}{2}, \frac{y_3}{2}, Q_f\right),$$

where $y_i/2 (i=1,2,3)$ are the $U(1)$, weak hypercharges. However we fix their non-abelian representations by imposing a natural generalisation of the SM charge quantisation rule

$$y_i/2 + d_i/2 + t_i/3 = 0 \mod 1 \tag{12}$$

and requiring that they be singlet or fundamental representations. The duality, $d_i$, and triality, $t_i$, here are given by $d_i = +1,0$ for the doublet and singlet representations respectively of $SU(2)_i$, and $t_i = +1, -1,0$ for the $3,\bar{3},1$ representations of $SU(3)_i$.

By requiring a realistic charged fermion spectrum (with $\phi_{WS}$ giving an unsuppressed top quark mass), we are led to the following choice:

$$\vec{Q}_{\phi_{WS}} = (0,\frac{2}{3},-\frac{1}{6},1), \quad \vec{Q}_W = (0,-\frac{1}{2},\frac{1}{2},\frac{4}{3}),$$

$$\vec{Q}_T = (0,-\frac{1}{6},\frac{1}{6},-\frac{2}{3}), \quad \vec{Q}_\xi = (\frac{1}{6},-\frac{1}{6},0,0),$$

$$\vec{Q}_S = (\frac{1}{6},-\frac{1}{6},0,-1) \tag{13}$$

The orders of magnitude for the effective SM Yukawa coupling matrix elements are then given by:

$$Y_U \approx \begin{pmatrix} SWT^2(\xi)^2 & W^2(\xi)T^3 & S^2T^2(\xi)^2 W T^2 & S^4T^4 \end{pmatrix},$$

$$Y_D \approx \begin{pmatrix} SW(T^2)^2 & W(T^2)^2 & S^2(T^4)^2 & SW^2(T^4)^2 \end{pmatrix},$$

$$Y_E \approx \begin{pmatrix} SW(T^2)^2 & W(T^2)^2 & S^2(T^4)^2 & SW^2(T^4)^2 \end{pmatrix}. \tag{16}$$

Table 1: Best fit to experimental data. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

| Fitted  | Experimental |
|--------|--------------|
| $m_u$  | 3.6 MeV      | 4 MeV          |
| $m_d$  | 7.0 MeV      | 9 MeV          |
| $m_c$  | 0.87 MeV     | 0.5 MeV        |
| $m_e$  | 1.02 GeV     | 1.4 GeV        |
| $m_s$  | 400 MeV      | 200 MeV        |
| $m_{\mu}$ | 88 MeV | 105 MeV       |
| $M_t$  | 192 GeV      | 180 GeV        |
| $m_b$  | 8.3 GeV      | 6.3 GeV        |
| $m_\tau$ | 1.27 GeV | 1.78 GeV      |
| $V_{us}$ | 0.18     | 0.22          |
| $V_{cb}$ | 0.018   | 0.041         |
| $V_{ub}$ | 0.0039 | 0.0035        |

Since the diagonals of $Y_U$, $Y_D$ and $Y_E$ are equal we expect to have the approximate relations

$$m_u \approx m_\tau, \quad m_s \approx m_\mu \tag{17}$$

at $M_{Planck}$, since these masses come from the diagonal elements. There are no such relations involving the top or charm quark masses, since they come from off-diagonal elements which dominate $Y_U$. We also note that we expect

$$m_d \gtrsim m_u \approx m_e \tag{18}$$

at $M_{Planck}$, since there are two approximately equal contributions to the down quark mass.

The VEVs of the three Higgs fields $W$, $T$ and $\xi$ are taken to be free parameters in a fit to the 12 experimentally known charged fermion masses and mixing angles. The results of the best fit, which reproduces all the experimental data to within a factor of 2, are given in table 1 and correspond to the parameters

$$\langle W \rangle = 0.179 \quad \langle T \rangle = 0.071 \quad \langle \xi \rangle = 0.099, \tag{19}$$

in Planck units. This fit is as good as we can expect in a model making order of magnitude predictions.

### 5 Neutrino Mass and Mixing Problem

There is now strong evidence that the neutrinos are not massless as they would be in the SM. Physics beyond the SM can generate an effective light neutrino mass term

$$\mathcal{L}_{\nu-mass} = \sum_{i,j} \bar{\psi}_i \psi_j \epsilon^{ij}_\alpha (M_\nu)_{\gamma i} \tag{20}$$

in the Lagrangian, where $\psi_{i,j}$ are the Weyl spinors of flavour $i$ and $j$, and $\alpha, \beta = 1,2$. Fermi-Dirac statistics mean that the mass matrix $M_\nu$ must be symmetric.
In models with chiral flavour symmetry we typically expect the elements of the mass matrices to have different orders of magnitude. The charged lepton matrix is then expected to give only a small contribution to the lepton mixing. As a result of the symmetry of the neutrino mass matrix and the hierarchy of the mass matrix elements it is natural to have an almost degenerate pair of neutrinos, with nearly maximal mixing. This occurs when an off-diagonal element dominates the mass matrix. The recent Super-Kamiokande data on the atmospheric neutrino anomaly strongly suggests large $\nu_{\mu} - \nu_{\tau}$ mixing with $\Delta m^2_{\mu\tau} \sim 10^{-3}$ eV$^2$. Large $\nu_{\mu} - \nu_{\tau}$ mixing is given by the mass matrix

$$M_\nu = \begin{pmatrix} A & \times & \times \\ \times & \times & A \\ \times & A & \times \end{pmatrix}$$

(21)

and we have

$$\Delta m^2_{23} \ll \Delta m^2_{12} \sim \Delta m^2_{13}$$

(22)

$$\sin^2 \theta_{23} \sim 1$$

(23)

However, this hierarchy in $\Delta m^2$s is inconsistent with the small angle (MSW) solution to the solar neutrino problem, which requires $\Delta m^2_{12} \sim 10^{-5}$ eV$^2$.

Hence we need extra structure for the mass matrix such as having several elements of the same order of magnitude. e.g.

$$M_\nu = \begin{pmatrix} a & A & B \\ A & \times & \times \\ B & \times & \times \end{pmatrix}$$

(24)

with $A \sim B \gg a$. This gives

$$\frac{\Delta m^2_{12}}{\Delta m^2_{23}} \sim \frac{a}{\sqrt{A^2 + B^2}}$$

(25)

The mixing is between all three flavours. and is given by the mixing matrix

$$U_\nu \sim \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \cos \theta & \frac{1}{\sqrt{2}} \sin \theta \\ \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{2}} \cos \theta - \sin \theta & 0 \\ \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{2}} \cos \theta + \sin \theta & 0 \end{pmatrix}$$

(26)

where $\theta = \tan^{-1} \frac{\Delta m^2_{12}}{\Delta m^2_{23}}$. So we have large $\nu_{\mu} - \nu_{\tau}$ mixing with $\Delta m^2 = \Delta m^2_{23}$, and nearly maximal electron neutrino mixing with $\Delta m^2 = \Delta m^2_{12}$. However the AGUT model naturally gives a structure like eq. (24) rather than eq. (21).

There is also some difficulty in obtaining the required mass scale for the neutrinos. In models such as the AGUT the neutrino masses are generated via super-heavy intermediate fermions in a see-saw type mechanism. This leads to too small neutrino masses:

$$m_\nu \lesssim \frac{(\phi W S)^2}{M_F} \sim 10^{-5} \text{ eV},$$

(27)

for $M_F = M_{Planck}$ (in general $m_\nu$ is also suppressed by the chiral charges). So we need to introduce a new mass scale into the theory. Either some intermediate particles with mass $M_F \lesssim 10^{15}$ GeV, or an $SU(2)$ triplet Higgs field $\Delta$ with $\langle \Delta^0 \rangle \sim 1$ eV is required. Without further motivation the introduction of such particles is ad hoc.

6 Conclusions

We presented two applications of the Multiple Point Principle, according to which nature should choose coupling constants such that the vacuum can exist in degenerate phases. Applied to the AGUT model, it successfully predicts the values of the three fine structure constants, as illustrated in figure 1. In the case of the pure SM, it leads to our predictions for the top quark and Higgs pole masses: $M_t = 173 \pm 5$ GeV and $M_H = 135 \pm 9$ GeV.

The maximal AGUT group $SMG^3 \times U(1)_Y$ assigns a unique set of anomaly free chiral gauge charges to the quarks and leptons. With an appropriate choice of Higgs field quantum numbers, the AGUT chiral charges naturally give a realistic charged fermion mass hierarchy. An order of magnitude fit in terms of 3 Higgs VEVs is given in table 1, which reproduces all the masses and mixing angles within a factor of two. On the other hand, the puzzle of the neutrino masses and mixing angles presents a challenge to the model.

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