We study Davis’ series of moderate deviations probabilities for \( L^p \)-bounded sequences of random variables \( p > 2 \). A certain subseries therein is convergent for the same range of parameters as in the case of martingale difference or i.i.d. sequences.

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1. Introduction and main results

Let \((X_n)_{n \geq 1}\) be a sequence of random variables on a probability space \((\Omega, \mathcal{F}, P)\) and \(p \geq 1\) a fixed real number. We say that \((X_n)_{n \geq 1}\) is \(L^p\)-bounded if it has finite \(p\)th moments, that is, \(\|X_n\|_p \leq C\) for some \(C > 0\) and any \(n \geq 1\). Let \(\epsilon > 0\); finding the rate of convergence of the moderate deviations probabilities \(P[|\sum_{k=1}^{n} X_k| > \epsilon n a_n]\) with \(a_n = (n \log n)^{1/2}\) or \((n \log \log n)^{1/2}\) is known in the literature as Davis’ problems. More precisely, let \(\delta = \delta(p) \geq 0\) be a function of \(p \geq 1\) and consider the series

\[
\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P\left[\left|\sum_{k=1}^{n} X_k\right| > \epsilon (n \log n)^{1/2}\right],
\]

\[
\sum_{n=3}^{\infty} \frac{1}{n (\log n)^{\delta}} P\left[\left|\sum_{k=1}^{n} X_k\right| > \epsilon (n \log \log n)^{1/2}\right];
\]

(1.1)

the convergence of series (2.1) has been studied by Davis (see [7, 8]) and Gut (see [10]) when \((X_n)_{n \geq 1}\) are \(L^p\)-bounded i.i.d. sequences, and by Stoica (see [14, 15]) when \((X_n)_{n \geq 1}\) are \(L^p\)-bounded martingale difference sequences.

In the sequel, we are interested in Davis’ theorems under the only assumption that \((X_n)_{n \geq 1}\) is \(L^p\)-bounded. Our results rely on the “subsequence principle,” that is, given any sequence of \(L^p\)-bounded random variables, one can find a subsequence that satisfies, together with all its further subsequences, the same type of limit laws as do i.i.d. variables (or martingale difference sequences) with similar moment bounds. This principle
was introduced by Chatterji (see [4–6]) and unifies results by Banach and Saks, Komlós, Révész, Steinhaus in the context of law of large numbers, iterated logarithm, and central limit theorem; extensions to exchangeable sequences were given by Aldous [1] and Berkes and Péter [3]. Also note that Gut [11] and Asmussen and Kurtz [2] gave necessary and sufficient requirements for subsequences to satisfy the famous Hsu-Robbins-Erdös complete convergence result related to the law of large numbers. Our results go a step further, that is, we replace the i.i.d. assumption by $L^p$-boundedness, and consider Davis normalizing factors $\sum_{n=2}^{\infty} (\log n)^{\delta}/n$ and $\sum_{n=3}^{\infty} (1/n(\log n)^{\delta})$ instead of complete convergence. We thus have the following.

**Theorem 1.1.** For any $p > 2$ and $L^p$-bounded sequence $(X_n)_{n \geq 1}$, there exist $1 \leq n_1 < n_2 \cdots$ such that the series

$$\sum_{N=2}^{\infty} \frac{(\log N)^{\delta}}{N} P\left[ \left| \sum_{k=1}^{N} X_{n_k} \right| > \varepsilon (N \log N)^{1/2} \right]$$

(1.2)

is convergent for any $0 \leq \delta < p/2 - 1$ and any $\varepsilon > 0$.

**Theorem 1.2.** For any $p \geq 2$ and $L^p$-bounded sequence $(X_n)_{n \geq 1}$, there exist $1 \leq n_1 < n_2 \cdots$ such that the series

$$\sum_{N=3}^{\infty} \frac{1}{N(\log N)^{\delta}} P\left[ \left| \sum_{k=1}^{N} X_{n_k} \right| > \varepsilon (N \log \log N)^{1/2} \right]$$

(1.3)

is convergent for any $\varepsilon > 0$ if either $\delta > 1$, or $\delta = 1$ and $p > 2$.

If $\delta = 1$, Theorem 1.1 holds under the same hypotheses (i.e., $p > 4$), as in the case of martingale difference sequences (see [15]). In the i.i.d case, Theorem 1.1 holds for $L^2$-bounded centered sequences (see [7, 10]).

Theorem 1.2 holds under the same hypotheses as in the case of martingale difference sequences (see [14]). In the i.i.d. case, slightly less than a second moment is needed: $E[X_n^2 \log^+ \log^+ |X_n|^{-\eta}] < \infty$ for some $0 < \eta < 1$ (see [8, 10]), and for necessary moment conditions, one may consult [13].

In the case of martingale difference sequences, Theorem 1.1 fails if $\delta \geq p/2 - 1$ and Theorem 1.2 fails if $0 \leq \delta < 1$ (see [14]), therefore Theorems 1.1 and 1.2 are the best results one can expect in the $L^p$-bounded case.

2. Proofs

**Proof of Theorem 1.1.** In the sequel we will make use of the so-called $c_r$-inequality (see [12, page 57]), which says that

$$E|X + Y|^p \leq 2^{p-1} (E|X|^p + E|Y|^p)$$

(2.1)

for any random variables $X, Y$ and $p > 1$. Throughout the paper, $C$ denotes a constant that depends on $p$ and $\varepsilon$ (but not on $k, n, N$), and may vary from line to line, even within the same line.
As \((X_n)_{n \geq 1}\) is bounded in \(L^p\), according to [9, Corollary IV.8.4], it is weakly sequentially compact. Denote by \((Y_n)_{n \geq 1}\) a subsequence of \((X_n)_{n \geq 1}\) that converges weakly in \(L^p\) to some \(Y \in L^p\). Subtracting \(Y\) from each element of \((Y_n)_{n \geq 1}\), we reduce the problem to a sequence \((Y_n)_{n \geq 1}\) that converges weakly in \(L^p\) to 0. Further, for any \(n \geq 1\), we choose a simple random variable \(Z_n\) (i.e., \(Z_n\) takes only a finite number of distinct values), such that

\[ ||Y_n - Z_n||_p < \frac{1}{2^n}. \]  

Using Markov’s inequality and (2.1), one has

\[
\sum_{N=2}^{\infty} \frac{(\log N)^\delta}{N} \mathbb{P} \left[ \left| \sum_{k=1}^{N} Y_{n_k} \right| > \varepsilon (N \log N)^{1/2} \right] 
\leq \varepsilon^{-p} \sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} \mathbb{E} \left[ \sum_{k=1}^{N} Y_{n_k} \right]^p 
\]

\[
\leq C \left( \sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} \mathbb{E} \sum_{k=1}^{N} Z_{n_k} \right)^p + \sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} \mathbb{E} \left[ \sum_{k=1}^{N} (Y_{n_k} - Z_{n_k}) \right]^p. \]  

According to (2.1), we have

\[ E \left[ \sum_{k=1}^{N} (Y_{n_k} - Z_{n_k}) \right]^p \leq 2^{p-1} \left( E \left[ Y_{n_1} - Z_{n_1} \right]^p + E \left[ \sum_{k=2}^{N} (Y_{n_k} - Z_{n_k}) \right]^p \right), \]  

whence by iteration

\[ E \left[ \sum_{k=1}^{N} (Y_{n_k} - Z_{n_k}) \right]^p \leq \sum_{k=1}^{N} 2^{k(p-1)} E \left[ Y_{n_k} - Z_{n_k} \right]^p, \]  

and assumption (2.2) yields

\[ E \left[ \sum_{k=1}^{N} (Y_{n_k} - Z_{n_k}) \right]^p \leq \sum_{k=1}^{N} 2^{k(p-1)-n_k} \leq \sum_{k=1}^{N} \frac{1}{2^k} \leq 1 \quad \text{for any } N \geq 1, \]  

(we used that the subsequence \((n_k)_{k \geq 1}\) is strictly increasing, so \(n_k \geq k\)), therefore the last series in (2.3) converges. To prove Theorem 1.1, it suffices to exhibit a subsequence \((n_k)_{k \geq 1}\) such that

\[ \sum_{N=2}^{\infty} \frac{(\log N)^{\delta-p/2}}{N^{1+p/2}} \mathbb{E} \left[ \sum_{k=1}^{N} Z_{n_k} \right]^p < \infty. \]  

One can see that \((Z_n)_{n \geq 1}\) also converges weakly in \(L^p\) to 0. Indeed, for any \(Q \in L^q\), where \(1/p + 1/q = 1\), we have \(E(Z_n Q) = E((Z_n - Y_n)Q) + E(Y_n Q)\), and the first term on the right-hand side tends to 0 by Hölder’s inequality and (2.1), while the second term tends to 0 because \(Y_n\) converges weakly in \(L^p\) to 0.
Moderate deviations for bounded subsequences

By induction, one may choose a subsequence of natural numbers $1 \leq n_1 < n_2 \cdots$ such that

$$E[Z_{n_k} \mid Z_i, i \in I] \leq \frac{1}{2^k} \quad \text{for each } I \subset \{n_1, \ldots, n_{k-1}\},$$

(2.8)

where $E[Z_{n_k} \mid Z_i, i \in I]$ denotes the conditional expectation of $Z_{n_k}$ given the $\sigma$-algebra $\sigma(Z_i, i \in I)$ generated by $(Z_i)_{i \in I}$. This can be done because $\sigma(Z_i, i \in I)$ consists of a finite partition of $\Omega$, and as $Z_n \to 0$ weakly in $L^p$, we have $\int_A Z_n dP \to 0$ for any $A$ in $\sigma(Z_i, i \in I)$.

We now prove that $(n_k)_{k \geq 1}$ is the required subsequence in (2.7). Indeed, one can write

$$Z_{n_k} = V_k + W_k,$$

(2.9)

where $E[Z_{n_k} \mid V_1, \ldots, V_{k-1}] = 0$ and $|W_k| \leq 1/2^k$. In particular, $(V_k)_{k \geq 1}$ is a martingale difference sequence. Using Minkowski’s inequality, we deduce that

$$\left( E \left| \sum_{k=1}^{N} Z_{n_k} \right|^{p} \right)^{1/p} \leq \left( E \left| \sum_{k=1}^{N} V_k \right|^{p/2} \right)^{1/p} + \left( E \left| \sum_{k=1}^{N} W_k \right|^{p} \right)^{1/p}.$$ (2.10)

According to Burkholder and Hölder’s inequalities, we have

$$E \left| \sum_{k=1}^{N} V_k \right|^{p} \leq CE \left( \sum_{k=1}^{N} V_k^2 \right)^{p/2} \leq CN^{p/2-1} \sum_{k=1}^{N} E|V_k|^p \leq CN^{p/2}.$$ (2.11)

Also,

$$E \left| \sum_{k=1}^{N} W_k \right|^{p} \leq \left( \sum_{k=1}^{N} \frac{1}{2^k} \right)^{p} \leq 1 \quad \text{for any } N \geq 1.$$ (2.12)

Using (2.9)–(2.12), we obtain

$$\sum_{N=2}^{\infty} \frac{\left( \log N \right)^{\delta-p/2}}{N^{1+p/2}} E \left| \sum_{k=1}^{N} Z_{n_k} \right|^{p} \leq C \sum_{N=2}^{\infty} \frac{\left( \log N \right)^{\delta-p/2}}{N^{1+p/2}} \left( N^{1/2} + 1 \right)^p \leq C \sum_{N=2}^{\infty} \frac{\left( \log N \right)^{\delta-p/2}}{N}.$$ (2.13)

The latter series in (2.13) is convergent if and only if $\delta < p/2 - 1$, thus (2.7) holds and Theorem 1.1 is proved.

**Proof of Theorem 1.2.** With the same notations and method as in the proof of Theorem 1.1, it suffices to prove the following analog of (2.7):

$$\sum_{N=3}^{\infty} \frac{1}{N^{1+p/2}(\log N)^{\delta} (\log \log N)^{p/2}} E \left| \sum_{k=1}^{N} Z_{n_k} \right|^{p} < \infty.$$ (2.14)

Using (2.9)–(2.12), the series in (2.14) is dominated by

$$C \sum_{N=3}^{\infty} \frac{\left( N^{1/2} + 1 \right)^p}{N^{1+p/2}(\log N)^{\delta} (\log \log N)^{p/2}} \leq C \sum_{N=3}^{\infty} \frac{1}{N(\log N)^{\delta} (\log \log N)^{p/2}}.$$ (2.15)
The latter series in (2.15) is convergent if and only if either $\delta > 1$, or $\delta = 1$ and $p > 2$; thus (2.14) holds and Theorem 1.2 is proved. 

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