Infinite-dimensional stochastic differential equations arising from Airy random point fields

Abstract: We identify infinite-dimensional stochastic differential equations (ISDEs) describing the stochastic dynamics related to Airy_β random point fields with β = 1, 2, 4. We prove the existence of unique strong solutions of these ISDEs. When β = 2, this solution is equal to the stochastic dynamics defined by the space-time correlation functions obtained by Spohn and Johansson among others. We develop a new method to construct a unique, strong solution of ISDEs. We expect that our approach is valid for other soft-edge scaling limits of stochastic dynamics arising from the random matrix theory.

1 Introduction

Gaussian ensembles are introduced as random matrices with independent elements of Gaussian random variables, under the constraint that the joint distribution is invariant under conjugation with appropriate unitary matrices. The ensembles are divided into classes according to whether their elements are real, complex, or real quaternion, and their invariance by orthogonal (Gaussian orthogonal ensemble, GOE), unitary (Gaussian unitary ensemble, GUE) and unitary symplectic (Gaussian symplectic ensemble, GSE) conjugation.

The distribution of eigenvalues of the ensembles with size n x n is given by

\begin{equation}
\mu_{\beta}^n(dx_n) = \frac{1}{Z} \prod_{i<j} |x_i - x_j|^\beta \exp \left\{-\frac{\beta}{4} \sum_{k=1}^{n} |x_k|^2 \right\} dx_n,
\end{equation}

where \( x_n = (x_1, \ldots, x_n) \) and \( dx_n = dx_1 \cdots dx_n \). The GOE, GUE, and GSE correspond to \( \beta = 1, 2 \) and 4, respectively. The probability density coincides with the Boltzmann factor normalized by the partition function \( Z \) for a log-gas system at three specific values of the inverse temperature \( \beta = 1, 2, \) and 4. The measures \( \mu_{\beta}^n \) still make sense for any \( 0 < \beta < \infty \), and are examples of log-gasses \[8\].

Let \( \mu_{\beta} \) be the distribution of \( n^{-1} \sum_{x_i/\sqrt{n}} \) under \( \mu_{\beta}^n(dx_n) \). Wigner’s celebrated semi-circle law states that the sequence \( \{\mu_{\beta}\} \) weakly converges to the nonrandom
\( \sigma_{\text{semi}}(x)dx \) in the space of probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). Here \( \sigma_{\text{semi}} \) is defined as

\[
\sigma_{\text{semi}}(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \chi_{[-2,2]}(x).
\]

There exist two typical thermodynamic scalings in (1.1), namely bulk and soft edges. The former (centered at the origin) is given by the correspondence \( x \mapsto \frac{x}{\sqrt{n}} \), which yields the random point field (RPF) \( \mu_{\sin, \beta}^n \) with labeled density \( m_{\sin, \beta}^n \) such that

\[
m_{\sin, \beta}^n(d\mathbf{x}_n) = \frac{1}{Z} \left\{ \prod_{i<j} |x_i - x_j|^{\beta} \right\} \exp \left\{ -\frac{\beta}{4n} \sum_{k=1}^n |x_k|^2 \right\} d\mathbf{x}_n.
\]

The latter, on the other hand, is centered at \( 2\sqrt{n} \) given by the correspondence \( x \mapsto x/n^{1/6} + 2\sqrt{n} \) with density \( m_{\text{Ai}, \beta}^n \) such that

\[
m_{\text{Ai}, \beta}^n(d\mathbf{x}_n) = \frac{1}{Z} \left\{ \prod_{i<j} |x_i - x_j|^{\beta} \right\} \exp \left\{ -\frac{\beta}{4} \sum_{k=1}^n \left| \frac{2\sqrt{n} + n^{-1/6}x_k}{2}\right|^2 \right\} d\mathbf{x}_n.
\]

Suppose \( \beta = 2 \). The limit RPF \( \mu_{\sin, 2} \) of the finite-particle system (1.3) is then the determinantal RPF with \( n \)-correlation functions \( \rho_{\sin, 2}^n \) defined as

\[
\rho_{\sin, 2}^n(x_n) = \det [K_{\sin, 2}(x_i, x_j)]_{i,j=1}^n.
\]

Here \( K_{\sin, 2} \) is a continuous kernel such that, for \( x \neq y \),

\[
K_{\sin, 2}(x,y) = \frac{\sin \{ \pi(x-y) \}}{\pi(x-y)}.
\]

The limit RPF \( \mu_{\text{Ai}, 2} \) of the finite-particle system (1.4) is also the determinantal RPF with \( n \)-correlation functions \( \rho_{\text{Ai}, 2}^n \) defined as

\[
\rho_{\text{Ai}, 2}^n(x_n) = \det [K_{\text{Ai}, 2}(x_i, x_j)]_{i,j=1}^n.
\]

Here \( K_{\text{Ai}, 2} \) is the continuous kernel given by, for \( x \neq y \),

\[
K_{\text{Ai}, 2}(x,y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y},
\]

where we set \( \text{Ai}'(x) = d\text{Ai}(x)/dx \) and denote by \( \text{Ai}(\cdot) \) the Airy function such that

\[
\text{Ai}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} dk \ e^{i(kz+k^3/3)}, \quad z \in \mathbb{C}.
\]

For \( \beta = 1, 4 \) similar expressions in terms of the quaternion determinant are obtained as shown in (2.3).
From (1.3) we obtain the associated stochastic dynamics \(X_t = (X^1_t, \ldots, X^n_t)\) from the stochastic differential equation (SDE):

\[
(1.10) \quad dX^i_t = dB^i_t + \frac{\beta}{2} \sum_{j=1, j \neq i}^{n} \frac{1}{X^j_t - X^i_t} dt - \frac{\beta}{4n} X^i_t dt \quad (i = 1, \ldots, n).
\]

For \(\beta = 1, 2\) and 4, this SDE was introduced by Dyson and is referred to as the equation for Dyson’s Brownian motions. Taking \(n \to \infty\), we can naturally obtain the infinite-dimensional SDE (ISDE)

\[
(1.11) \quad dX^i_t = dB^i_t + \frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty} \frac{1}{X^j_t - X^i_t} dt \quad (i \in \mathbb{Z}).
\]

This ISDE (with \(\beta = 2\)) is often called Dyson’s model (in infinite dimensions), and was introduced by Spohn at the heuristic level. Spohn [44] constructed the associated unlabeled dynamics as an \(L^2\) Markovian semi-group by introducing a Dirichlet form related to \(\mu_{\sin,2}\).

In [32, 31], not only for \(\beta = 2\), but also for \(\beta = 1, 4\), the first author constructed the \(\mu_{\sin,\beta}\)-reversible unlabeled diffusion \(X_t = \sum_{i \in \mathbb{Z}} \delta_{X^i_t}\). He proved that tagged particles \(X^i_t\) never collide with one another that the associated labeled system \(X_t = (X^i_t)_{i \in \mathbb{Z}}\) solves the ISDE

\[
(1.12) \quad dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r \to \infty} \sum_{j \neq i \text{ s.t. } |X^j_t - X^i_t| < r} \frac{1}{X^j_t - X^i_t} dt \quad (i \in \mathbb{Z}).
\]

We therefore have infinitely many, non-intersecting paths describing the motions of a limit particle system as a \(\mathbb{R}^\mathbb{Z}\)-valued diffusion process. We remark that since \(\mu_{\sin,\beta}\) is translation invariant, only conditional convergence is possible for the sum in (1.12). Because of the conditional convergence, the shape of the limit SDEs is quite sensitive. In fact, the ISDE describing Ginibre interacting Brownian motions, which is the two-dimensional counterpart of (1.12), has multiple expressions [31].

In soft-edge scaling, we obtain from (1.3) the \(n\)-particle dynamics:

\[
(1.13) \quad dX^i_t = dB^i_t + \frac{\beta}{2} \sum_{j=1, j \neq i}^{n} \frac{1}{X^j_t - X^i_t} dt - \frac{\beta}{2} \{n^{1/3} + \frac{1}{2n^{1/3}} X^i_t\} dt \quad (i = 1, \ldots, n).
\]

Because of the divergence of the second and third terms on the right-hand side as \(n \to \infty\), no simple guess of the limit SDE is possible.

The purpose of this paper is to detect and solve the limit ISDE at the soft-edge scaling. In fact, from (1.13), we derive the ISDE

\[
(1.14) \quad dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r \to \infty} \left\{ \left( \sum_{j \neq i, |X^j_t| < r} \frac{1}{X^j_t - X^i_t} \right) - \int_{|x| < r} \frac{\hat{\varphi}(x)}{-x} dx \right\} dt.
\]
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Here we set
\[
\hat{\rho}(x) = \frac{1_{(-\infty,0)}(x)}{\pi} \sqrt{-x}.
\]

We prove that ISDE (1.14) has a strong solution \( X = (X^i)_{i \in \mathbb{N}} \) for \( \mu_{Ai,\beta} \)-a.s. \( s = \sum_{i=1}^{\infty} X^i \) and strong uniqueness of the solutions in one of the main theorems (Theorem 2.3). In Corollary 2.4, we prove that the first \( m \)-components of the solution \( X_m = (X^1, \ldots, X^m) \) of (1.14) is the weak limit in \( C([0, \infty); \mathbb{R}^m) \) of those of the finite-dimensional equations (1.13) for \( \beta = 2 \), which may be regarded as the dynamical soft-edge scaling limit of Gaussian ensembles. In our forthcoming papers [25], this result is generalized for \( \beta = 1, 4 \) using a different method.

We prove (1.14) through an equivalent yet more refined representation of the ISDE

\[
dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r \to \infty} \left\{ \sum_{j \neq i, |X^j_t| < r} \frac{1}{X^i_t - X^j_t} - \int_{|y| < r} \frac{\rho^{1}_{Ai,\beta,X^i_t}(y)}{X^i_t - y} dy \right\} dt
\]

\[
+ \frac{\beta}{2} \lim_{r \to \infty} \left\{ \int_{|y| < r} \frac{\rho^{1}_{Ai,\beta,X^i_t}(y)}{X^i_t - y} dy - \int_{|y| < r} \frac{\hat{\rho}(y)}{-y} dy \right\} dt.
\]

Here \( \rho^{1}_{Ai,\beta,x} \) is a one-correlation function of the reduced Palm measure \( \mu^{1}_{Ai,\beta,x} \) conditioned at \( x \). Note that the second term on the right-hand side is neutral, and the coefficient of the third term can be regarded as the \(-\frac{1}{2}\) multiple of the derivative of the free potential \( \Phi_{\beta} \) in Theorem 5.6.

An outline of the derivation of (1.14) is given below. Taking into account the inverse of the soft-edge scaling \( x \mapsto xn^{-1/6} + 2\sqrt{n} = \sqrt{n}(xn^{-2/3} + 2) \), we set

\[
\tilde{\rho}^n(x) = n^{1/3} \sigma_{\text{sem}}(xn^{-2/3} + 2).
\]

From (1.17) we take \( \tilde{\rho}^n \) as the first approximation of the one-correlation function \( \rho^{1}_{Ai,\beta,x} \) of the reduced Palm measure \( \mu^{n}_{Ai,\beta,x} \) of the \( n \)-particle Airy RPF \( \mu^{n}_{Ai,\beta} \).

From (1.17) we see that

\[
\int_{\mathbb{R}} \tilde{\rho}^n(x) dx = n.
\]

The prefactor \( n^{1/3} \) in the definition of \( \tilde{\rho}^n(x) = n^{1/3} \sigma(xn^{-2/3} + 2) \) is chosen for (1.18). A simple calculation shows that

\[
\tilde{\rho}^n(x) = \frac{1_{(-4n^{2/3},0)}(x)}{\pi} \sqrt{-x} \left( 1 + \frac{x}{4n^{2/3}} \right).
\]

\[
\lim_{n \to \infty} \tilde{\rho}^n(x) = \hat{\rho}(x) \quad \text{compact uniformly.}
\]

The key point of the derivation is that

\[
n^{1/3} = \int_{\mathbb{R}} \frac{\tilde{\rho}^n(x)}{-x} dx.
\]
Equations (1.19) and (1.20) then justify the appearance of $\hat{\rho}(x)$ in the limit ISDE (1.14). Indeed, we prove that, as $n \to \infty$,

\begin{align*}
dX_i^t &\sim dB_i^t + \frac{\beta}{2} \left\{ \left( \sum_{j \neq i}^{n} \frac{1}{X_i^t - X_j^t} \right) - n^{1/3} \right\} dt \\
&\sim dB_i^t + \frac{\beta}{2} \lim_{r \to \infty} \left\{ \left( \sum_{j \neq i, |X_j^t| < r} \frac{1}{X_i^t - X_j^t} \right) - \int_{|x| < r} \frac{\hat{\rho}(x)}{-x} dx \right\} dt \quad \text{by (1.21)} \\
&\sim dB_i^t + \frac{\beta}{2} \lim_{r \to \infty} \left\{ \left( \sum_{j \neq i, |X_j^t| < r} \frac{1}{X_i^t - X_j^t} \right) - \int_{|x| < r} \frac{\hat{\rho}(x)}{-x} dx \right\} dt \quad \text{by (1.20)},
\end{align*}

thereby obtaining the ISDE (1.14). We expect that such a limit procedure is universal for RPFs appearing in soft-edge scaling, and valid for other examples such as tacnode [17].

We also prove that no particles of the solution $X = (X_i^t)_{i \in \mathbb{N}}$ collide with one another for all $t$. That is, 

\begin{equation}
(1.22) \quad P(X_i^t \neq X_j^t \text{ for all } 0 \leq t < \infty, i \neq j) = 1.
\end{equation}

Hence, we always label the particles in such a way that $X_i^t > X_j^t$ for all $i < j \in \mathbb{N}$. Therefore, the right-most particle is denoted by $X_1^t$. The solution $(X_i^t)_{i \in \mathbb{N}}$ is a $\mathbb{R}_>^N$-valued process, where $\mathbb{R}_>^N = \{(x_i) \in \mathbb{R}^N; x_i > x_j \ (i < j)\}$.

We use a general theory presented in [31] to solve (1.14) in Theorem 2.2. The solution at this stage has the usual meaning; that is, a pair of infinite-dimensional processes $(X, B)$ satisfying (1.14).

Using a result obtained in the co-paper [37], we refine the result much further in Theorem 2.3. We prove the existence of a unique, strong solution of (1.14). Here as usual a strong solution means that $X$ is a function of the (given) Brownian motion $B$, with the uniqueness explained in Theorem 2.3.

The result from [37] used in this paper is a general theory on the existence and uniqueness of strong solutions of ISDEs in $(\mathbb{R}^d)^N$. This result is valid for quite a wide range of ISDEs, and thus we have devoted a separate paper [37] to it.

The uniqueness of the solutions of the ISDE yields several significant consequences, such as the uniqueness of quasi-regular Dirichlet forms and (suitably formulated) martingale problems. The most important one is the identity between our construction and the algebraic construction based on space-time correlation functions where $\beta = 2$ (Theorem 2.4).

When $\beta = 2$, natural infinite-dimensional stochastic dynamics have been constructed using the extended Airy kernel [9] [23] [18], which we refer as the algebraic construction in the above. Such a construction has been studied by Prähofer-Spohn [41] and Johansson [16], amongst others. In these works, the construction of the dynamics of the top particle $A(t)$, in what was called the Airy process Prähofer-Spohn [41], has attracted much attention.
We prove that these stochastic dynamics also satisfy ISDE (1.14). From the uniqueness of the solutions of (1.14), we deduce that these two stochastic dynamics are the same. Hence, in particular, the right-most particle $X_t^1$ is treated with the Airy process $A(t)$.

When $\beta = 1, 4$, no algebraic construction of the stochastic dynamics is known. Our result is valid even for the cases where $\beta = 1, 4$.

We further prove in Theorem 2.5 the Cameron-Martin formula for the solution of SDE (1.14). As an application, we obtain each tagged particle in a semi-martingale manner, and the local properties of their trajectories are similar to those of Brownian motions.

The paper is organized as follows. In Section 2, we define the problems and state our main theorems (Theorem 2.1–Theorem 2.5). In Section 3 we prepare the general theory from [37] on ISDEs describing interacting Brownian motions. Various estimates of finite-particle systems approximating Airy RPFs are investigated in Section 4. In Section 5–Section 9 we prove the main theorems Theorem 2.1–Theorem 2.5 respectively. In Section 10 (Appendix 1), we recall the definition of a quaternion determinant. In Section 11 (Appendix 2) and Section 12 (Appendix 3), we give some estimates of Airy functions and Airy kernels, and Hermite polynomials, respectively.

2 Problem definition and main results.

This section defines the problem and states the main theorems.

We begin by recalling the notion of the configuration space over $\mathbb{R}$. Let

\begin{equation}
S = \{s = \sum_i \delta_{s_i} ; s(K) < \infty \text{ for all compact sets } K \subset \mathbb{R}\},
\end{equation}

where $\delta_a$ denotes the delta measure at $a$. We endow $S$ with a vague topology, under which $S$ is a Polish space. $S$ is called the configuration space over $\mathbb{R}$.

A probability measure $\mu$ on $(S, \mathcal{B}(S))$ is called the RPF on $\mathbb{R}$. To define Airy RPFs, we recall the notion of correlation functions.

A symmetric locally integrable function $\rho^n : \mathbb{R}^n \rightarrow [0, \infty)$ is called the $n$-point correlation function of an RPF $\mu$ on $S$ w.r.t. the Lebesgue measure if $\rho^n$ satisfies

\begin{equation}
\int_{A_{1}^{k_1} \times \ldots \times A_{m}^{k_m}} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_S \prod_{i=1}^m \frac{s(A_i)!}{(s(A_i) - k_i)!} d\mu
\end{equation}

for any sequence of disjoint bounded measurable subsets $A_1, \ldots, A_m \subset \mathbb{R}$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. When $s(A_i) - k_i < 0$, according to our interpretation, $s(A_i)!/(s(A_i) - k_i)! = 0$ by convention. It is known that $\{\rho^n\}_{n \in \mathbb{N}}$ determines the measure $\mu$ under a weak condition. In
particular, determinantal RPFs generated by given kernels and reference measures are unique \([13]\).

We denote by \(\mu_{\text{AI},2}\) the RPF whose correlation functions are given by \((1.7)\), and call it the Airy RPF. \(\mu_{\text{AI},1}\) and \(\mu_{\text{AI},4}\) are defined similarly using quaternions. By definition, \(\mu_{\text{AI},\beta}\) (\(\beta = 1, 4\)) are RPFs with \(n\)-correlation functions \(\rho^n_{\text{AI},\beta}\) given by

\[
(2.3) \quad \rho^n_{\text{AI},\beta}(x_1, \ldots, x_n) = \text{qdet}[K_{\text{AI},\beta}(x_i, x_j)]^n_{i,j=1}.
\]

Here \text{qdet} denotes the quaternion determinant defined by \((10.3)\), and the \(K_{\text{AI},\beta}\) are quaternion-valued kernels defined by \((11.6)\) for \(\beta = 1\), and by \((11.7)\) for \(\beta = 4\).

For a subset \(A \subset \mathbb{R}\), we define the map \(\pi_A : S \to S\) by \(\pi_A(s) = s \cdot (\cdot \cap A)\). We say a function \(f : S \to \mathbb{R}\) is local if \(f\) is \(\sigma[\pi_A]\)-measurable for some compact set \(A\).

Let \(u\) be the map defined on \(\{\sum_{k=1}^\infty \mathbb{R}^k\} \cup \mathbb{R}^\infty\) such that \(u((x_i)) = \sum_i \delta_{x_i}\). For a function \(f : S \to \mathbb{R}\), there exists a unique symmetric function \(\tilde{f}\) on \(u^{-1}(S)\) such that \(f(s) = f((s_i))\) for \(s = \sum_i \delta_{s_i}\). We say function \(f : S \to \mathbb{R}\) is smooth if \(\tilde{f}\) is smooth.

We introduce the natural square field on \(S\) and Dirichlet forms for given RPF \(\mu\). Let \(D_0\) be the set of all local, smooth functions on \(S\). For \(f, g \in D_0\), we set \(D[f, g] : S \to \mathbb{R}\) according to

\[
(2.4) \quad D[f, g](x) = \frac{1}{2} \sum_i \frac{\partial_i \tilde{f}(x)}{\partial x_i} \frac{\partial_i \tilde{g}(x)}{\partial x_i},
\]

where \(x = \sum_i \delta_{x_i}\) and \(x = (x_i)\). Let \(E^u\) be the bilinear form defined as

\[
(2.5) \quad E^u(f, g) = \int_S D[f, g] d\mu
\]

with domain \(D^u = \{ f \in D_0 ; E^u(f, f) < \infty, f \in L^2(S, \mu)\}\).

Let \(\Lambda\) denote the Poisson RPF whose intensity is the Lebesgue measure. If \(\mu = \Lambda\), then the bilinear form \((E^\Lambda, D^\Lambda_0)\) is closable on \(L(S, \Lambda)\), and the closure is of a quasi-regular Dirichlet form. The associated diffusion \(\mathbb{B}_t^\Lambda = \sum_{i \in \mathbb{N}} \delta_{B_{t+i}^\Lambda}\) is the \(S\)-valued Brownian motion starting at \(s = \sum_i \delta_{s_i}\). In fact, \{\{B_t^\Lambda\}_{i \in \mathbb{N}}\} are independent copies of one-dimensional standard Brownian motion \([28]\). It is thus natural to ask, if we replace \(\Lambda\) by \(\mu_{\text{AI},\beta}\) (\(\beta = 1, 2, 4\)), whether the forms \((E^{\mu_{\text{AI},\beta}}, D^\Lambda_{\text{AI},\beta})\) are still closable on \(L^2(S, \mu_{\text{AI},\beta})\) and associated diffusions exist.

We write \(s(x) = s(\{x\})\). Let

\[
(2.6) \quad S_{s,1} = \{ s \in S ; s(x) \leq 1 \text{ for all } x \in \mathbb{R}, s(\mathbb{R}) = \infty \}
\]

\[
S_{s,1}^+ = \{ s \in S_{s,1} ; s(\mathbb{R}^+) < \infty \}
\]

By definition, \(S_{s,1}\) is the set of configurations consisting of an infinite number of single-point measures, and \(S_{s,1}^+\) is its subset with only a finite number in \(\mathbb{R}^+\). It is well known that

\[
(2.7) \quad \mu_{\text{AI},\beta}(S_{s,1}^+) = 1 \quad \text{for } \beta = 1, 2, 4.
\]
From (2.6) and (2.7), for $\mu_{\text{Air}, \beta}$-a.s. $s = \sum_i \delta_{s_i}$, we can and do label $\{s_i\}$ in such a way that $s_i > s_j$ for all $i < j \in \mathbb{N}$. Therefore, let $\mathbb{R}_+^N = \{(x_i)_{i \in \mathbb{N}}; x_i > x_j$ for all $i < j \in \mathbb{N}\}$ and define the map $I: S_{s,i} \to \mathbb{R}_+^N$ by

$$I(s) = (s_1, s_2, \ldots), \text{ where } s = \sum_{i=1}^{\infty} \delta_{s_i}.$$  

We see that $\mu_{\text{Air}, \beta}$ can be regarded as probability measures on $\mathbb{R}_+^N$ according to $\mu_{\text{Air}, \beta} \circ I^{-1}$.

We call $I$ a label. In general, there exist infinitely many different types of labels. However, we always take the label as above because this choice is clearly the most natural for the Airy RPF. We remark that for other RPFs such as sine and Ginibre RPFs, there is generally no such canonical choice of labels.

Let $A \subset S_{s,i}$. Let $C([0, \infty); A)$ be the set of $A$-valued paths. The element $X \in C([0, \infty); A)$ can be written as $X_t = \sum_{i=1}^{\infty} \delta_{X^i_t}$, where $X^i_t \in C([t; \mathbb{R})$ with (possibly random) interval $I^i$ of the form $I^i = [0, b^i)$ or $I^i = (a^i, b^i]$ ($0 < a^i < b^i \leq \infty$). We take each interval $I$ to be the maximal one. The expression $X_t = \sum_{i=1}^{\infty} \delta_{X^i_t}$ is then unique up to labeling because of the assumption $A \subset S_{s,i}$. Let $C_{\text{ne}}([0, \infty); A)$ be the subset of $C([0, \infty); A)$ consisting of non-explosive paths. Then $I^i = [0, \infty)$ and

$$C_{\text{ne}}([0, \infty); A) = \{X \in C([0, \infty); A); X_t = \sum_{i=1}^{\infty} \delta_{X^i_t}, \ X^i_t \in C([0, \infty), \mathbb{R}) \ (\forall i)\}.$$  

We remark that generally $C_{\text{ne}}([0, \infty); A) \neq C([0, \infty); A)$ because we equip $\mathcal{S}$ with a vague topology. The advantage of considering the set of non-explosive and non-intersecting paths is that we can naturally relate the labeled path $X \in C([0, \infty); \mathbb{R}^N)$ to each $X \in C_{\text{ne}}([0, \infty); S_{s,i})$ as follows.

Let $I_{\text{path}}$ be the map from $C_{\text{ne}}([0, \infty); A)$ to $C([0, \infty]; \mathbb{R}_+^N)$ defined as

$$I_{\text{path}}(X) = \{(X^i_t)_{i \in \mathbb{N}}\}_{t \in [0, \infty)}, \text{ where } X = \{\sum_{i=1}^{\infty} \delta_{X^i_t}\}_{t \in [0, \infty)}.$$  

We write $X = I_{\text{path}}(X)$, and call $X$ (resp. $X$) a labeled (resp. unlabeled) process.

To state the main theorems, we recall the terminology for diffusion processes in a general framework. For a Polish space $S$, we say a family of probability measures $\{P_s\}_{s \in S}$ on $C([0, \infty); S)$ is a conservative diffusion with state space $S$ if, under $P_s$, the canonical process $\{(X_t, P_s)\}$ has a strong Markov property and $X_0 = s$. By construction, $\{X_t\}$ is a continuous process. We say a diffusion is $\mu$-reversible if it has an invariant probability measure $\mu$ and is symmetric with respect to $\mu$. For a given closable nonnegative form $(\mathcal{E}, \mathcal{D}_0)$ on $L^2(S, \mu)$, we say a diffusion $\{P_s\}_{s \in S_0}$ with state space $S_0$ is associated with $(\mathcal{E}, \mathcal{D}, L^2(S, \mu))$ if $\mu(S_0) = 0$ and $E_s[f(X_t)] = T_t f(s)$ $\mu$-a.e $s \in S_0$ for any $f \in L^2(S, \mu)$ and for all $t$. Here $(\mathcal{E}, \mathcal{D})$ is the closure of $(\mathcal{E}, \mathcal{D}_0)$ on $L^2(S, \mu)$, and $T_t$ is the associated $L^2$-semi group. Such a closed form becomes
automatically a local Dirichlet form and, by construction, $T_t$ is a Markovian semi-group (see [11]).

We state our first main theorem.

**Theorem 2.1.** Assume $\beta = 1, 2, 4$. Then:
1. The bilinear forms $(\mathcal{E}^\mu_{\text{Air}^{,\beta}}, \mathcal{D}^\mu_{\text{Air}^{,\beta}})$ are closable on $L^2(S, \mu_{\text{Air}^{,\beta}})$.
2. There exists a diffusion $\{P_s\}_{s \in S}$ associated with $(\mathcal{E}^{\text{Air}^{,\beta}}, \mathcal{D}^{\text{Air}^{,\beta}}, L^2(S, \mu_{\text{Air}^{,\beta}}))$. Here $(\mathcal{E}^{\text{Air}^{,\beta}}, \mathcal{D}^{\text{Air}^{,\beta}})$ is the closure of $(\mathcal{E}^\mu_{\text{Air}^{,\beta}}, \mathcal{D}^\mu_{\text{Air}^{,\beta}})$ on $L^2(S, \mu_{\text{Air}^{,\beta}})$.
3. There exists a subset $S_{\text{Air}^{,\beta}} \subset S$ satisfying the following.

$$
\text{(2.11)} \quad S_{\text{Air}^{,\beta}} \subset S^{+,\beta}, \quad \mu_{\text{Air}^{,\beta}}(S_{\text{Air}^{,\beta}}) = 1,
$$

$$
\text{(2.12)} \quad P_s(X_t \in S_{\text{Air}^{,\beta}} \text{ for all } 0 \leq t < \infty) = 1 \quad \text{for all } s \in S_{\text{Air}^{,\beta}},
$$

$$
\text{(2.13)} \quad P_s(\sup_{t \in [0,T]} |X^i_t| < \infty \text{ for all } T, i \in N) = 1 \quad \text{for all } s \in S_{\text{Air}^{,\beta}}.
$$

Here $X = (X^i)_{i \in N} = I_{\text{path}}(X)$ is the labeled process (see (2.10)).

**Remark 2.1.**

1. From (2.12) we deduce that $\{P_s\}_{s \in S_{\text{Air}^{,\beta}}}$ is a diffusion with state space $S_{\text{Air}^{,\beta}}$. Moreover, from (2.11) we see that $\{P_s\}_{s \in S_{\text{Air}^{,\beta}}}$ is reversible with invariant probability measure $\mu_{\text{Air}^{,\beta}}(\cdot \cap S_{\text{Air}^{,\beta}})$.
2. According to $S_{\text{Air}^{,\beta}} \subset S^{+,\beta}$, the conditions (2.12) and (2.13) are equivalent to

$$
\text{(2.14)} \quad P_s(X \in C_{\text{ne}}([0, \infty); S_{\text{Air}^{,\beta}})) = 1 \quad \text{for all } s \in S_{\text{Air}^{,\beta}}.
$$

We remark that, if $X \in C_{\text{ne}}([0, \infty); S^{+,\beta})$ and $X_0 \in S^{+,\beta}$, then $X \in C_{\text{ne}}([0, \infty); S^{+,\beta})$. Indeed, $X_t(\mathbb{R}^+) = X_t([0, X^1_t]) < \infty$ for each $t$. Here we interpret $X_t([0, X^1_t]) = 0$ if $X^1_t < 0$. The identity $X_t(\mathbb{R}^+) = X_t([0, X^1_t]) < \infty$ is clear because $X^1_t$ is the position of the top particle.

We next solve the ISDE (1.14). Let $I$ and $S_{\text{Air}^{,\beta}}$ be as in (2.8) and Theorem 2.1 respectively. Let $S_{\text{Air}^{,\beta}} = I(S_{\text{Air}^{,\beta}})$ and $P_s = P_s \circ I_{\text{path}}^{-1}$, where $s = I(s)$. From (2.14), we deduce that

$$
P_s(C([0, \infty); \mathbb{F}^N)) = 1.
$$

Hence $X = I_{\text{path}}(X) \in C([0, \infty); \mathbb{F}^N)$. We call $B = (B^j)_{j \in N}$ an $\mathbb{R}^N$-valued Brownian motion if $\{B^j\}_{j \in N}$ are independent copies of the standard Brownian motion on $\mathbb{R}$.

**Theorem 2.2.** Assume $\beta = 1, 2, 4$. Let $S_{\text{Air}^{,\beta}}$ and $P_s$ be as above. Then:
1. Let $s \in S_{\text{Air}^{,\beta}}$. Then, under $P_s$, the canonical process $X = \{(X^i_t)_{i \in N}\}$ is a solution of the ISDE (1.14) starting at $s = (s_i)_{i \in N}$. That is, there exist an $\mathbb{R}^N$-valued Brownian motion $B$ defined on $(C([0, \infty); \mathbb{R}^N), P_s)$ such that the pair $(X, B)$ satisfies

$$
\text{(1.14)} \quad dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r \to \infty} \left\{ \sum_{j \neq i, |X^j_t| < r} \frac{1}{X^j_t - X^i_t} - \int_{|x| < r} \frac{\hat{\theta}(x)}{-x} dx \right\} dt \quad (i \in N)
$$

$X_0 = s$. 

We state our first main theorem.
(2) \( \{P_s\}_{s \in S_{\mu_{Ai, \beta}}} \) is a \( \mu_{Ai, \beta} \circ \Gamma^{-1} \)-reversible diffusion with state space \( S_{\mu_{Ai, \beta}} \).

(3) The distribution \( X_t^1 \) under \( P_{\mu_{Ai, \beta} \circ \Gamma^{-1}} \) is the \( \beta \) Tracy-Widom distribution if \( \beta = 2 \).

**Remark 2.2.** The reversibility of the labeled dynamics Theorem 2.2 (2) is specific in Airy interacting Brownian motions. For there exist simple bijections between the support of the random point fields in configuration spaces and the labeled space \( \mathbb{R}^N \). In the case of Bessel interacting Brownian motions, there also exist the canonical bijection between the support of the Bessel RPFs and the labeled space \( [0, \infty]^N \).

In \cite{37}, we developed a general theory of the uniqueness and existence of a strong solution of the ISDEs concerning interacting Brownian motions including (1.14), Dyson’s model, and Ginibre interacting Brownian motions and others. Using this theory together with the results obtained in the present paper, we refine the meaning of the ISDE (1.14) in the next theorem.

Let \( T(S) \) be the tail \( \sigma \)-field of \( S \) defined by (3.25). We say a random point field \( \nu \) is tail trivial if \( \nu(A) \in \{0, 1\} \) for all \( A \in T(S) \). Let \( \mu_{Ai, \beta, t} = \mu_{Ai, \beta}(|T(S)) \) be the regular conditional probability. It is known that \( \mu_{Ai, \beta, t} \) is tail trivial for \( \mu_{Ai, \beta} \)-a.s. \( t \in T(S) \) (see Lemma 3.4).

We say a family of solution \((X, B)\) satisfy \( \nu \)-absolute continuity condition if

\[
(2.15) \quad P_{\nu} \circ X_t < \nu \quad \text{for all } t.
\]

Here \( P_{\nu} = \int_\mathbb{S} P_s \nu(ds) \), \( P_s = P_{u(s)} \), and \( P_s \) is the distribution of \( X \) staring at \( s \).

For an \( \mathbb{R}^N \)-valued Brownian motion \( B \) starting at the origin, we denote by \( P^B \) its distribution.

**Theorem 2.3.** Assume \( \beta = 1, 2, 4 \). Then the following holds.

1. **Existence of the strong solutions:** The ISDE (1.14) has strong solutions \( X(\cdot, s) \) starting at \( \mu_{Ai, \beta} \circ \Gamma^{-1} \)-a.e. \( s \), satisfying the \( \mu_{Ai, \beta, t} \)-absolute continuity condition (2.15) and being \( \mu_{Ai, \beta, t} \circ \Gamma^{-1} \)-reversible diffusions for \( \mu_{Ai, \beta} \)-a.s. \( t \).

2. **Strong uniqueness:** If \((X, B)\) and \((\tilde{X}, B)\) are solutions of (1.14) with the same Brownian motion \( B \) satisfying the \( \nu \)-absolute continuity condition (2.15). Suppose that \( \nu \) is tail trivial. Then \( X \) and \( \tilde{X} \) become strong solutions and

\[
(2.16) \quad P^B(X(\cdot, s) = \tilde{X}(\cdot, s)) = 1 \quad \text{for } \nu \circ \Gamma^{-1} \text{-a.e. } s.
\]

In particular, there exists the unique strong solution satisfying the \( \mu_{Ai, \beta, t} \)-absolute continuity condition for \( \mu_{Ai, \beta} \)-a.s. \( t \).

**Remark 2.3.** (1) Strong solutions in Theorem 2.3 (1) mean that, for a given Brownian motion \( B \) and initial point \( s \in S_{\mu_{Ai, \beta}} \), there exists a function \( X = X(B, s) \) of \((B, s)\) such that the pair \((X, B)\) satisfies the ISDE (1.14).

(2) Theorem 2.3 (2) implies that if \((X, B)\) are solutions of (1.14) starting at \( \nu \circ \Gamma^{-1} \text{-a.s. } s \) satisfying the \( \nu \)-absolute continuity condition (2.15), and if \( \nu \) is tail trivial, then \((X, B)\) become strong solutions for \( \nu \circ \Gamma^{-1} \text{-a.s. starting points } s \). In this sense,
the ISDE (1.14) has a strong uniqueness.

(3) We have proved the unique existence of strong solutions satisfying \( \mu_{\text{Ai}, \beta, t} \)-absolute continuity condition for \( \mu_{\text{Ai}, \beta} \)-a.s. \( t \). This result does not excludes the possibility that a solution not satisfying the absolute continuity condition exists. Such solutions, if exist, would change the tail.

When \( \beta = 2 \), one can construct the infinite-dimensional stochastic dynamics by using the space-time correlation functions [9, 41, 16, 23, 18]. Let \( \mathcal{A}(s, x, t, y) \) be the extended Airy kernel defined by

\[
\mathcal{A}(s, x, t, y) = \begin{cases} 
\int_{-\infty}^{0} du \ e^{(t-s)u/2}\text{Ai}(x-u)\text{Ai}(y-u), & \text{if } s \leq t \\
\int_{0}^{\infty} du \ e^{(t-s)u/2}\text{Ai}(x-u)\text{Ai}(y-u), & \text{if } s > t 
\end{cases}
\]

The determinantal process \( Y = \{ Y_t \} \) with the extended kernel \( \mathcal{A}(s, x, t, y) \) is a \( \mathcal{S} \)-valued process such that, for any integer \( M \in \mathbb{N} \), \( f = (f_1, f_2, \ldots, f_M) \in C_0(\mathbb{R})^M \), a sequence of times \( t = (t_1, t_2, \ldots, t_M) \) with \( 0 < t_1 < \cdots < t_M < \infty \), if we set \( \chi_{t_m}(x) = e^{f_m(x)} - 1, 1 \leq m \leq M \), the moment generating function of multi-time distribution,

\[
\Psi_t[f] = \mathbb{E} \left[ \exp \left\{ \sum_{m=1}^{M} \int_{\mathbb{R}} f_m(x) Y_{t_m}(dx) \right\} \right],
\]

is given by a Fredholm determinant

\[
\Psi_t[f] = \det_{(s,t) \in \{t_1, t_2, \ldots, t_M\}^2, (x,y) \in \mathbb{R}^2} \left[ \delta_{st} \delta(x-y) + \mathcal{A}(s, x, t, y) \chi_t(y) \right].
\]

We refer to [19] for detail.

Finite- and infinite-dimensional determinantal processes were introduced as multi-matrix models [7, 24], tiling models [13], and surface growth models [41], and they have been extensively studied in [4, 3] and others. We remark that the Markov property of infinite-dimensional determinantal processes as above is highly nontrivial unlike that of the infinite-dimensional stochastic dynamics given by the Dirichlet form approach.

Let \( Q_s \) be the distribution of the determinantal process with the extended Airy kernel starting at \( s \) [20]. It is known [21] that there exists \( S_0 \) such that \( \mu_{\text{Ai}, 2}(S_0) = 1 \) and that \( \{ Q_s \}_{s \in S_0} \) is a continuous, stationary Markov process. In [45], we refine this result in such a way that \( \{ Q_s \}_{s \in S_0} \) is a diffusion process with state space \( S_0 \).

We thus see that there exist two completely different approaches to the construction of infinite-dimensional stochastic dynamics related to the Airy random point field with \( \beta = 2 \). In the next theorem, we prove that these two infinite-dimensional stochastic dynamics are the same.
Theorem 2.4. Assume $\beta = 2$. Let $\{Q_s\}_{s \in S_0}$ be as above. Then $\{Q_s\}_{s \in S_0}$ satisfies (2.11)–(2.13). The associated labeled process $X = \mathbb{l}_{\text{path}}(X)$ is a unique strong solution of the ISDE (1.14) with initial condition $s = \mathbb{l}(s)$ for $\mu_{\text{AI},2}$-a.s. $s$. Moreover,

$$
(2.19) \quad P_s = Q_s \quad \text{for } \mu_{\text{AI},2}\text{-a.s. } s.
$$

The key point of the proof of Theorem 2.4 is the uniqueness theorem Theorem 2.3 and the fact that $\mu_{\text{AI},2}$ has a trivial tail $\sigma$-field [34].

For $s \in S_0$ and $n \in \mathbb{N}$ we put $P^n(s) = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ such that $s_i \geq s_{i+1}$ for all $i$. Let $X^n_i = (X^n_i, X^n_i, \ldots, X^n_i)$ be the solution of (1.13) with $X^0_n = \mathbb{l}(s) = (s_1, s_2, \ldots, s_n)$. Let $X^{n,m} = (X^{n,1}, \ldots, X^{n,m})$ be the first $m$-components of $X^n$. Let $\mu_{\text{AI},\beta}^n$ be the RPF whose labeled density $m_{\text{AI},\beta}^n$ is given by (1.3). Then, as a corollary of the above theorem, we have the following result.

Corollary 2.4. (1) Let $P^n_{\mu_{\text{AI},2}}$ be the distribution of the unlabeled process $\{u(X^n_i)\}$ such that $u(X^n_0) = \mu_{\text{AI},2}^n$ in law. Then $P^n_{\mu_{\text{AI},2}}$ converges to $P_{\mu_{\text{AI},2}}$ weakly as $n \to \infty$.

(2) Under $P^n_{\mu_{\text{AI},2}}$ and $P_{\mu_{\text{AI},2}}$, the first $m$-labeled processes $X^{n,m}$ converge to that of the limit $X^m = (X^1, \ldots, X^m)$ as follows:

$$
(2.20) \quad \lim_{n \to \infty} X^{n,m} = X^m \quad \text{weakly in } C([0, \infty); \mathbb{R}^m)
$$

for each $m \in \mathbb{N}$.

We next turn to a Girsanov formula. For $T \in (0, \infty)$ fixed, we denote by $W^m_s$ the distribution of $\mathbb{R}^m$-valued Brownian motion $\{(B^i_t + s_i)_{i=1}^m\}_{t \in [0,T]}$ starting at $s = (s_i) \in \mathbb{R}^m$, $m \in \mathbb{N} \cup \{\infty\}$. Let $P_s$ be the distribution of the solution $X$ starting at $s$ as in Theorem 2.2. It is then clear that the probability measure $P_s$ is not absolutely continuous with respect to $W^\infty_s$. We will therefore formulate a Girsanov-type formula for a finite number of particles $X^m = \{(X^1_t, \ldots, X^m_t)\}_{t \in [0,T]}$ for each $m \in \mathbb{N}$.

We set $X^{m*} = \{(X^{m*}_t)_{t \in [0,m*+1]}\}_{t \in [0,T]}$, and introduce the regular conditional probability $P^X^{m*}_s$ of $P_s(X^m \in \cdot)$ conditioned at $X^{m*}$:

$$
(2.21) \quad P^X^{m*}_s(\cdot) = P_s(X^m \in \cdot | X^{m*}).
$$

Let $b^m_X : \mathbb{R}^m \times [0, T] \to \mathbb{R}^m$ be the vector $b^m_X = (b^m_X, t)_{1 \leq i \leq m}$ such that

$$
(2.22) \quad b^m_X(x, t) = \frac{\beta}{2} \left\{ \sum_{j \neq i} \frac{1}{x_i - x_j} + \lim_{r \to \infty} \left( \sum_{j=m+1}^\infty \frac{1}{x_i - X^i_t} - \int_{|y| < r} \frac{\hat{\theta}(y)}{y} dy \right) \right\}.
$$

Theorem 2.5. Assume $\beta = 1, 2, 4$ and $T \in (0, \infty)$. Let $\{P_s\}$ be as in Theorem 2.2. For $h \in \mathbb{N}$, let $\tau_h : C([0, T]; \mathbb{R}^m) \to \mathbb{R} \cup \{\infty\}$ be a stopping time with respect to the canonical filtering such that

$$
(2.23) \quad \tau_h(W) = \inf\{t \wedge T ; h \leq \int_0^t |b^m_X(W_u, u)|^2 du \}.\]
Let \( s \in \mathbb{L}(S_{\mu_{A_{1}}, \beta}) \). Let \( P_{s,h}^{X^{m*}} = P_{s}^{X^{m*}} \circ (W_{\cdot \wedge \tau_{h}})^{-1} \) and \( W_{s,h}^{m} = W_{s}^{m} \circ (W_{\cdot \wedge \tau_{h}})^{-1} \). Then, for \( P_{s} \)-a.s. \( X^{m*} \), the distribution \( P_{s,h}^{X^{m*}} \) is absolutely continuous w.r.t. \( W_{s,h}^{m} \) with Radon-Nikodym density
\[
\frac{dP_{s,h}^{X^{m*}}}{dW_{s,h}^{m}} = e^{\int_{0}^{\tau_{h}} b_{X}^{\frac{\alpha}{2}}(W_{u}, u) dW_{u} - \frac{1}{2} \int_{0}^{\tau_{h}} |b_{X}^{\frac{\alpha}{2}}(W_{u}, u)|^{2} du}.
\]
Furthermore, for \( P_{s} \)-a.s. \( X^{m*} \),
\[
\lim_{h \to \infty} \tau_{h}(W) = T \text{ for } P_{s}^{X^{m*}} \text{-a.s. } W.
\]

**Remark 2.4.** (1) Theorem 2.5 implies that the local properties of the each line of \( X \) are same as Brownian motions. In particular, each lines \( X_{i} \) are non-differentiable and \( \alpha \)-Hölder continuous for any \( \alpha \leq 1/2 \).

(2) Since the unlabeled process \( X \) is reversible, one can extend the time parameter from \([0, \infty)\) to \( \mathbb{R} \) by the stationarity of the time shift. Assume \( \beta = 2 \) and consider the process given by the extended Airy kernel. Let \( A(t) \) be the top particle. So far we denote this by \( X_{1} \). \( A(t) \) is usually called the Airy process. We consider the case of the time stationary Airy process. So the distribution of \( A(t) \) is independent of \( t \). In [16] Conjecture 1.5 Johansson conjectured that \( H(t) = A(t) - t^{2} \) has a unique point of maximum in \([-T,T]\) almost surely. We have affirmative answer for this conjecture from Theorem 2.5 immediately. In fact this is the case for Brownian path, and the Airy process is absolutely continuous on the time interval \([-T,T]\) with respect to Brownian motions starting from the distribution of \( A(0) \) at time \(-T\). We remark that the conjecture referred above has already been solved by Corwin [5] and Hägg [12] by a different method.

## 3 Preliminary to the proof of Theorems 2.1–2.5

In this section we prepare three theorems for the proof of the main theorems (Theorems 2.1–2.5).

The key notion for the existence of \( \mu \)-reversible diffusions on the configuration space \( S \) and their SDE representation are the quasi-Gibbs property and the logarithmic derivative of \( \mu \) introduced in [32] and [31], which we now explain.

Let \( S = \mathbb{R} \). Although \( S \) was taken to be \( \mathbb{R}^{d} \) in [32] and [31], we take here only \( \mathbb{R} \) but keep the notation according with [32] and [31]. We introduce a Hamiltonian on a bounded Borel set \( A \). For Borel measurable functions \( \Phi : S \to \mathbb{R} \cup \{ \infty \} \) and \( \Psi : S \times S \to \mathbb{R} \cup \{ \infty \} \) with \( \Psi(x, y) = \Psi(y, x) \), let
\[
H_{A}^{\Phi, \Psi}(x) = \sum_{x_{i} \in A} \Phi(x_{i}) + \sum_{x_{i}, x_{j} \in A, i < j} \Psi(x_{i}, x_{j}), \quad \text{where } x = \sum_{i} \delta_{x_{i}}.
\]

We assume \( \Phi < \infty \) a.e. to avoid triviality. The functions \( \Phi \) and \( \Psi \) are called free and interaction potentials, respectively.
For two measures \( \nu_1 \) and \( \nu_2 \) on a measurable space \((\Omega, \mathcal{B})\) we write \( \nu_1 \preceq \nu_2 \) if \( \nu_1(A) \leq \nu_2(A) \) for all \( A \in \mathcal{B} \). We say a sequence of finite Radon measures \( \{\nu^n\} \) on a Polish space \( \Omega \) converge weakly to a finite Radon measure \( \nu \) if \( \lim_{n \to \infty} \int f d\nu^n = \int f d\nu \) for all \( f \in C_b(\Omega) \).

For an increasing sequence \( \{b_r\} \) of natural numbers we set
\[
S_r = \{ s \in S \mid |s| < b_r \}, \quad S_r^m = \{ s \in S \mid s(S_r) = m \}.
\]
Although \( S_r \) and \( S_r^m \) depend on the sequence \( \{b_r\} \), we omit it from the notation.

Let \( \Lambda_r \) be the Poisson RPF whose intensity is \( 1_{S_r} dx \). We set
\[
\Lambda_r^m = \Lambda_r(\cdot \cap S_r^m).
\]
Note that \( \Lambda_r = \sum_{m=0}^{\infty} \Lambda_r^m \).

**Definition 3.1.** A probability measure \( \mu \) is said to be a \((\Phi, \Psi)\)-quasi Gibbs measure if there exists an increasing sequence \( \{b_r\} \) of natural numbers such that, for each \( r, m \in \mathbb{N} \), \( \mu_r^m := \mu(\cdot \cap S_r^m) \) satisfy and that, for all \( r, m \in \mathbb{N} \) and for \( \mu^m_r \)-a.e. \( s \in S \),
\[
(3.4) \quad \min_{1 \leq s \leq m} e^{-\mathcal{H}_r(x)} \Lambda_r^m(dx) \leq \mu_r^m(\cdot \cap S_r^m)(dx) \leq \max_{1 \leq s \leq m} e^{-\mathcal{H}_r(x)} \Lambda_r^m(dx).
\]
Here \( \mathcal{H}_r(x) = \mathcal{H}_{S_r}^{\Phi, \Psi}(x), \ c_1 = \min_{1 \leq s \leq m} e^{-\mathcal{H}_r(x)} \Lambda_r^m(dx) \) is a positive constant and \( \mu_r^m \) is the conditional probability measure of \( \mu_r^m \) defined as
\[
(3.5) \quad \mu_r^m(\cdot \cap S_r^m)(dx) = \mu_r^m|_{\pi S_r^m} = d\xi, \quad \pi S_r^m = \sum_{s \in S_r^m} \delta_s.
\]

**Remark 3.1.** (0) The notion of quasi-Gibbs states was introduced in previous papers of the first author [31, 32, 33], where a subordinate sequence \( \{\mu_{r,k}^m\}_{k \in \mathbb{N}} \) of measures was introduced to define the quasi-Gibbs state. We note that Definition 3.1 above is equivalent to this.

(1) Recall that a probability measure \( \mu \) is said to be a \((\Phi, \Psi)\)-canonical Gibbs measure if \( \mu \) satisfies the Dobrushin-Lanford-Ruelle (DLR) equation (3.6); that is, for each \( r, m \in \mathbb{N} \), the conditional probability measure \( \mu_{r,s}^m \) satisfies
\[
(3.6) \quad \mu_{r,s}^m(dx) = \frac{1}{\beta} e^{-\mathcal{H}_r(x) + I_r(x,s)} \Lambda_r^m(dx) \quad \text{for} \ \mu_r^m \text{-a.e.} \ s.
\]
Here \( 0 < \beta < \infty \) is the normalization and, for \( x = \sum_i \delta_{x_i} \) and \( s = \sum_j \delta_{s_j} \), we set
\[
(3.7) \quad I_r(x,s) = \sum_{x_i \in S_r, s_j \in S_r^m} \Psi(x_i, s_j).
\]
By construction \((\Phi, \Psi)\)-canonical Gibbs measures are \((\Phi, \Psi)\)-quasi Gibbs measures. The converse is, however, not true. This is the common features of infinite volume RPFs appeared in random matrix theory such as Ginibre RPF and sine RPFs \((\beta = \ldots \)}
1, 2, 4). In fact, when $\Psi(x, y) = -\beta \log |x - y|$ and $\mu$ are translation invariant, $\mu$ are not $(\Phi, \Psi)$-canonical Gibbs measures because the DLR equation does not make sense. Indeed, $|L_r(x, s)| = \infty$ for $\mu$-a.s. $s$. The point is that a cancellation can be expected between $e^{2L_r(x, s)}$ and $e^{-L_r(x, s)}$ even if $|L_r(x, s)| = \infty$. The main task of the proof of the quasi-Gibbs property of Airy RPFs is to seek this cancellation.

(2) Unlike canonical Gibbs measures, the notion of quasi-Gibbs measures is quite flexible for free potentials. Indeed, if $\mu$ is a $(\Phi, \Psi)$-quasi Gibbs measure, then $\mu$ is also $(\Phi + F, \Psi)$-quasi Gibbs measure for any locally bounded measurable function $F$. So we write $\mu$ a $\Psi$-quasi Gibbs measure if $\mu$ is a $(0, \Psi)$-quasi Gibbs measure.

The significance of the quasi-Gibbs property is that it yields, combined with local boundedness of the correlation functions and the minimal regularity of potentials of a given RPF $\mu$, the construction of diffusions associated with the RPFs $\mu$. This result is a consequence of the general theory in [32, 31, and 30], involving Dirichlet forms. We will quote some of them in Theorem 3.1 and Theorem 3.2.

Let $\mu$ be a RPF over $S$ with correlation functions $\rho^k (k \in \mathbb{N})$. Let $\sigma_r^k$ be the $k$-density function of $\mu$ with respect to the Lebesgue measure on $S^k_r$. We assume:

(B1) $\rho^1 \in L^1_{\text{loc}}(S, dx)$ and $\sigma_r^2 \in L^2(S^k_r, d\mathbf{x}_k)$ for all $k, r \in \mathbb{N}$.  

(B2) $\mu$ is a $(\Phi, \Psi)$-quasi Gibbs measure with upper semi-continuous $(\Phi, \Psi)$. We quote:

**Theorem 3.1.** Assume (B1) and (B2). Then the following holds.

1. $(\mathcal{E}^\mu, \mathcal{D}_c^\mu)$ is closable on $L^2(S, \mu)$.
2. There exists a $\mu$-reversible diffusion $\{P_s\}_{s \in S^\mu}$ associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(S, \mu))$. Here $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is the closure of $(\mathcal{E}^\mu, \mathcal{D}_c^\mu)$ on $L^2(S, \mu)$.

**Proof.** If we replace $\sigma_r^k \in L^2(S^k_r, d\mathbf{x}_k)$ by $\sigma_r^k \in L^\infty(S^k_r, d\mathbf{x}_k)$ in (B1), then Theorem 3.1 was proved in [28, Theorem 2.1]. The assumption $\sigma_r^k \in L^\infty(S^k_r, d\mathbf{x}_k)$ is used only in the proof of [28, Lemma 2.4]. We can still prove [28, Lemma 2.4] under the present assumption by using Schwarz inequality on the second line in [28, 126 p].

We will use Theorem 3.1 to prove Theorem 2.1. From Theorem 3.1 we have unlabeled dynamics $X_t = \sum_i \delta_{X^i_t}$. Under additional assumptions, we can construct labeled dynamics $X = \{\text{path}(X) = \{(X^i_t)_{i \in \mathbb{N}}\}_{t \in I}, (B3), (B4),$ and (3.11) below). We next quote another general result concerning the ISDE representation of the labeled dynamics $X$. For this we recall the notion of reduced Palm and Campbell measures.

For a RPF $\mu$ over $S$, a probability measure $\mu_x$ is called the reduced Palm measure of $\mu$ conditioned at $x = (x_1, \ldots, x_k) \in S^k$ if it is defined by

$$
\mu_x = \mu(\cdot - \sum_{i=1}^k \delta_{x_i} \mid s(x_i) \geq 1 \text{ for } i = 1, \ldots, k).
$$

(3.8)
Let $\rho^k$ be the $k$-point correlation function of $\mu$ w.r.t. the Lebesgue measure. Let $S[k] = S^k \times S$ and $\mu^{[k]}$ be the measure on $S[k]$ defined by

$$\mu^{[k]}(A \times B) = \int_A \mu_x(B) \rho^k(x) dx.$$  \hspace{1cm} (3.9)

Here we set $dx = dx_1 \cdots dx_k$ for $x = (x_1, \ldots, x_k) \in S^k$. The measure $\mu^{[k]}$ is called the $k$-Campbell measure. By convention we set $\mu^{[0]} = \mu$.

We assume :

(B3) Cap$^\mu(\{S_{s,i}\}) = 0$, where Cap$^\mu$ is the capacity of $(S^\mu, D^\mu, L^2(S, \mu))$.

(B4) There exists a $T > 0$ such that for each $R > 0$

$$\liminf_{r \to \infty} N(\sqrt{r}/(r + R)^T) \{ \int_{|x| \leq r + R} \rho^1(x) dx \} = 0.$$  \hspace{1cm} (3.10)

Here $N(t) = \int_t^\infty (1/\sqrt{2\pi}) e^{-|x|^2/2} dx$.

The assumptions (B3) and (B4) have clear dynamical interpretations. Indeed, (B3) means that the particles never collide with each other:

$$P_s(\forall 0 \leq t < \infty, X_t \in S_{s,i}) = 1 \quad \text{for all } s \in S^\mu.$$  \hspace{1cm} (3.11)

And we deduce from (B4) that no labeled particle ever explodes:

$$P_s(\sup_{t \in [0,T]} |X_t^i| < \infty \quad \text{for all } T, i \in \mathbb{N}) = 1 \quad \text{for all } s \in S^\mu.$$  \hspace{1cm} (3.12)

All subsequent arguments follow from (3.11) and (3.12) instead of (B3) and (B4). We state here the assumption geometrically as much as possible. We remark that (B3) is equivalent to (3.11), and that (B4) is a sufficient condition for (3.12).

From (3.11) and (3.12) we can and do take the state space $S^\mu$ in such a way that

$$S^\mu \subset S_{s,i}.$$  \hspace{1cm} (3.13)

In addition, we can label the particles in such a way that $X_t = \sum_i \delta_{X_t^i}$, where each $\{X_t^i\}$ are continuous processes defined on the time interval $[0, \infty)$. This expression is unique up to the initial labeling $I(X_0) = (X_0^i)_{i \in \mathbb{N}}$ of the processes $\{(X_t^i)_{i \in \mathbb{N}}\}_{t \in [0,\infty)}$. In fact, once $I(X_0) = (X_0^i)_{i \in \mathbb{N}}$ is given, then each particle carry the initial label for all the time by (3.11) and (3.12). This correspondence is called a label path map, and denoted by

$$I_{\text{path}}(X) = X.$$  \hspace{1cm} (3.14)

To derive the ISDE satisfied by the labeled dynamics $X = (X^i)_{i \in \mathbb{N}}$, we introduce a notion of the logarithmic derivative of a RPF $\mu$. 



Definition 3.2. We call $d\mu = (d\mu_m)_{m=1,\ldots,d} \in \{L^1_{\text{loc}}(\mu^{[1]}_s)\}^d$ the logarithmic derivative of $\mu$ if $d\mu$ satisfies

$$
\int_{S \times S} d\mu f d\mu^{[1]} = -\int_{S \times S} \nabla_x f d\mu^{[1]} \quad \text{for all } f \in C^\infty_0(S) \otimes D_0.
$$

(3.15)

Very loosely, (3.15) can be written as $d\mu = \nabla_x \log \mu^{[1]}$. This expression is the reason why we call $d\mu$ the logarithmic derivative of $\mu$. We remark that $d\mu$ is a logarithmic derivative of the 1-Campbell measure $\mu^{[1]}$ rather than $\mu$. This choice is suitable for the description of SDE for $X$.

Indeed, $\frac{1}{2} d\mu$ expresses the force that each tagged particle $X_i$ exerted by all other infinitely many particles $\sum_{j \neq i} \delta_{X_j}$. When we treat the systems consisting of particles of $k$ species, we can define the logarithmic derivative of $k$-Campbell measures to derive the associated ISDEs. We assume:

(B5) There exists a logarithmic derivative $d\mu$ in the sense of (3.15).

The following lemma is a special case of [31][Theorem 26] with a slight modification, and will be used in the proof of Theorem 2.2.

**Theorem 3.2.** Assume (B1)–(B5). Let $\{P_s\}_{s \in S_\mu}$ be as in Theorem 3.1. Let $I$ be a label. Then there exists an $S_0$ such that

$$
\mu(S_0) = 1, \quad S_0 \subset S_\mu \subset S_{\text{st}},
$$

(3.16)

and that, for all $s = I(s) \in I(S_0)$, the labeled path $X = \text{lpath}(X)$ under $P_s \circ I^{-1}$ satisfies

$$
dX_i^t = dB_i^t + \frac{1}{2} d\mu(X_i^t, X_i^{\diamond}) dt \quad (i \in \mathbb{N}),
$$

(3.17)

$$
X_0 = s.
$$

(3.18)

Here $B = \{(B_i^t)_{i \in \mathbb{N}}\}_{0,\infty}$ is a $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion, and $X_i^{\diamond} = \sum_{j \neq i} \delta_{X_j}$ and $s = (s_i)_{i \in \mathbb{N}}$. Moreover, $X$ satisfies

$$
P(u(X_i^t) \in S_0, 0 \leq \forall t < \infty) = 1.
$$

(3.19)

Here $u: S^{\mathbb{N}} \to S$ such that $u((x_i)) = \sum_i \delta_{x_i}$.

Let $\rho^1$ and $\mathcal{N}(t) = \int_t^\infty (1/\sqrt{2\pi}) e^{-|x|^2/2} dx$ be as in (3.10). Then we assume:

(B6) For each $r, T \in \mathbb{N}$

$$
\int_{\mathbb{R}} \mathcal{N}(\frac{|x| - r}{\sqrt{T}}) \rho^1(x) dx < \infty.
$$

(3.20)

We will construct a unique strong solution of (3.17). For this we assume that the one labeled weak solution $(X_i^t, X_i^{\diamond})$ stay the set $H^{[1]}$ for all $t$ and that the set $H^{[1]}$ satisfies that the coefficient $d\mu(x, s)$ takes finite value. For a subset $A \subset S$ we set $A^{[1]} = (u^1)^{-1}(A) \subset S \times S$. We assume:
(B7) There exists $H \in \mathcal{B}(S)$ such that $d\mu(x, s)$ takes finite value for all $(x, s) \in H^{|]}$ and that
\begin{equation}
\text{Cap}\mu(H^c) = 0.
\end{equation}

Let $\mu^{[1]}$ be the one-Campbell measure of $\mu$ as before. In [37] we prove that the one-labeled process $(X^{i}_t, X^{i}_t \diamond)$ is a $\mu^{[1]}$-symmetric diffusion and, from (3.21), we can deduce that
\begin{equation}
\text{Cap}\mu^{[1]}((H^{[1]})^c) = 0,
\end{equation}
where $\text{Cap}\mu^{[1]}$ is the capacity of the one-labeled diffusion $(X^{i}_t, X^{i}_t \diamond)$.

We next introduce a system of finite-dimensional SDEs associated with the ISDE (3.17) and (3.18). For this we prepare a set of notations.

For a path $X = (X_i^t)_{i \in \mathbb{N}} \in C([0, \infty); \mathbb{R}^N)$ and $m \in \mathbb{N}$, we set $X^* = \sum_{i=m+1}^{\infty} \delta_{X_i^t}$ and $X^{m*} = (X^m)_{n=m+1}^\infty$. Let $S_0$ be as in (3.16), $S_0 = \pi(S_0)$, and $W_{sol} = \mathcal{I}_{path}(S_0)$. Then for $X \in W_{sol}$, $s = (s_i)_{i=1}^m \in S_0$, and $m \in \mathbb{N}$, we introduce the finite-dimensional SDE (3.23) of $Y^m = (Y^{m,1}, \ldots, Y^{m,m})$ such that
\begin{equation}
dY^m_{i,t} = dB^i_t + \frac{1}{2} d\mu(Y^{m,i}_t, Y^{m,i\diamond}_t + X^{m*}_t) dt \quad (i = 1, \ldots, m)
\end{equation}
\begin{equation}
Y^m_0 = (s_1, \ldots, s_m) \in S^m.
\end{equation}
Here we set $Y^{m,i\diamond}_t = \sum_{j \neq i}^m \delta_{Y^{m,j}_t}$ and $(s_1, \ldots, s_m)$ is the first $m$-components of $s$.

We remark that $X^{m*}$ is interpreted as a part of the coefficients of the SDE (3.23). We assume:

(B8) For each $s \in S_0$ and $X \in W_{sol}$ such that $X_0 = s$, the SDE (3.28) has a unique, strong solution $Y^m$ for each $m \in \mathbb{N}$. Moreover, $Y^m$ satisfies
\begin{equation}
(Y^m, X^{m*}) \in W_{sol}.
\end{equation}

Let $\mathcal{T}(S)$ be the tail $\sigma$-field of $S$ defined by
\begin{equation}
\mathcal{T}(S) = \bigcap_{r=1}^\infty \sigma[\pi_{S_r}].
\end{equation}
We say $\mu$ is tail trivial if $\mu(A) \in \{0, 1\}$ for all $A \in \mathcal{T}(S)$. We assume:

(B9) $\mu$ is tail trivial.

In [37], we clarify the significance of the tail triviality to the construction of the unique, strong solution of the ISDE (3.17), and we quote one of the main theorems in the co-paper [37], used in the proof of Theorem 2.3. The next result is a special case of [37] Theorem 2.1.
Theorem 3.3 ([37, Theorem 2.1]). Assume that $\mu$ satisfies (B1)–(B9). Then there exists a set $S_1$ satisfying $\mu(S_1) = 1$ and the following:

1. The ISDE (3.17)–(3.19) has a strong solution $(X_s, P_s)$ for each $s \in S_1$ such that $\{(X_s, P_s)\}_{s \in S_1}$ is a $S_1$-valued diffusion. The associated unlabeled processes $\{(X_s, P_s)\}_{s \in S_1}$ is a $S_1$-valued, $\mu$-reversible diffusion. Here $S_1 = u(S_1)$.

2. A family of strong solutions $\{(X_s, P_s)\}_{s \in S_1}$ of (3.23) and (3.24) satisfying the $\mu$-absolute continuity condition (2.15) is unique for $\mu$-$\ell$-a.s. $s$.

Proof. We denote by $(A^*)$ the conditions in [37, Theorem 2.1]. We will check that the conditions $(A1)$–$(A9)$ in [37] are satisfied.

The assumptions $(A1)$–$(A5)$ follow from $(B5)$, $(B2)$, $(B1)$, $(B3)$, and $(B4)$, respectively. From $(B6)$ we can apply [37, Proposition 8.4] to obtain $(A6)$. The assumptions $(A7)$–$(A9)$ follows from $(B7)$–$(B9)$, respectively.

It is known in [34] that all determinantal RPFs are tail trivial. Hence we can apply Theorem 3.3 to the case of $\beta = 2$. In general, quasi-Gibbs measures are not tail trivial. Hence we introduce the tail decomposition (3.27) of $\mu$ in the sequel.

Let $\mu_{\xi}$ be the regular conditional probability defined by

$$
\mu_{\xi}(\cdot) = \mu(\cdot | T(S))\mu(\xi).
$$

Then $\mu_{\xi}(A)$ is $T(S)$-measurable for any $A \in \mathcal{B}(S)$ by definition, and we have

$$
\mu(\cdot) = \int S \mu_{\xi}(\cdot) \mu(d\xi).
$$

The following follows from the argument of [15]

Lemma 3.4 ([37, Lemma 7.1]). Assume that $\mu$ is a quasi-regular Gibbs measure. Then, for $\mu$-$\ell$-a.s. $\xi$,

$$
\mu_{\xi}(A) = 1_A(\xi) \text{ for all } A \in T(S).
$$

From (3.28) we introduce the equivalent relation $S/T(S)$, such that

$$
t \sim t' \iff t, t' \in A \text{ or } t, t' \not\in A \text{ for all } A \in T(S).
$$

A significant property of the tail decomposition (3.27) is the stability for the assumptions (B1)–(B8) as we see in [37] Lemma 7.2. Combining this with Theorem 3.3 we can dispense with (B9) as follows. We quote a result from [37], which is a special case of [37] Theorem 2.2.

Proposition 3.5 ([37, Theorem 2.2]). Assume that $\mu$ satisfies (B1)–(B8). Let $l$ be a label. Then there exists $S_2$ such that $\mu(S_2) = 1$ satisfying the following:

1. The ISDE (3.23) and (3.24) has a strong solution $(X_s, P_s)$ for each $s \in S_2 = l(S_2)$.

2. $S_2$ can be decomposed as a disjoint sum $S_2 = \sum_{t \in S/T(S)} S_{2,t}$ such that

$$
\mu_{l}(S_{2,t}) = 1, \text{ where } S_{2,t} = u(S_{2,t}).
$$
and that the sub collection \( \{(X,P_s)\}_{s \in S_{2,t}} \) is a \( S_{2,t} \)-valued, \( \mu_t \)-reversible diffusion satisfying, for \( \mu \)-a.s. \( t \),

\[
P_{\mu_t} \circ X_t^{-1} < \mu_t \text{ for all } t \in [0, T].
\]

Here \( P_s = P_s \circ u^{-1} \) and \( s = t(s) \). Moreover, \( \{(X,P_s)\}_{s \in S_{2,t}} \) is a \( S_{2,t} \)-valued diffusion for \( \mu \)-a.s. \( t \).

3. A family of weak solutions \( \{(X,P_s)\}_{s \in S_{2,t}} \) of \( (3.23) \) and \( (3.24) \) satisfying \( (3.30) \) are unique for \( \mu \)-a.s. \( s \), and becomes a family of strong solutions.

**Remark 3.2.** (1) Proposition 3.5 (1) asserts the strong uniqueness in the sense that a family of (not necessary strong) solutions become automatically strong solutions.

(2) The uniqueness in Proposition 3.5 does not exclude the possibility of existence of solutions not satisfying \( (3.30) \).

(3) The diffusion \( \{(X,P_s)\}_{s \in S_{2,t}} \) in Proposition 3.5 conserves the tail \( \sigma \)-field of \( \mu \).

## 4 Finite particle approximation of Airy RPFs

We will use the results in Section 3 to prove the main theorems in Section 2. Hence in this section we will check the assumptions (B1)–(B8) in Section 3.

**Lemma 4.1.** Let \( \beta = 1, 2, 4 \). Then the following holds.

1. \( \mu_{\alpha,\beta} \) satisfies (B1) and (B4).

2. Suppose that \( \mu_{\alpha,\beta} \) satisfies (B2). Then \( \mu_{\alpha,\beta} \) satisfies (B3).

**Proof.** Remind that the correlation functions \( \rho^{k}_{\alpha,\beta}(x_k) \) are given by the determinant with elements \( K^{\alpha,\beta}_{2}(x_i,x_j) \) if \( \beta = 2 \), and the quaternion determinant with elements \( K^{\alpha,\beta}_{4}(x_i,x_j) \) if \( \beta = 1, 4 \). We deduce (B1) and (B4) from the locally boundedness of the kernels \( K^{\alpha,\beta}_{k}(x,y) \) and the asymptotic of \( \rho^{1}_{\alpha,\beta}(x) = O(|x|^{1/2}) \).

We deduce (B3) from [29, Theorem 2.1] and the locally Lipschitz continuity of the kernel \( K^{\alpha,\beta}_{k} \). This completes the proof.

From Theorem 3.1, Theorem 3.2, and Lemma 4.1 it only remains to prove (B2) and (B5) with the logarithm derivative given by (6.1) for proving Theorems 2.1 and 2.2. Hence we will study the quasi-Gibbs property and the logarithmic derivative of \( \mu_{\alpha,\beta} \) in Sections 5 and 6. In the rest of this section, we collect some estimates used in these sections.

Let \( m_{\alpha,\beta}^n \) be the RPF whose labeled density \( m_{\alpha,\beta}^n \) is given by (1.4) as before. The sequence \( \{m_{\alpha,\beta}^n\}_{n \in \mathbb{N}} \) of probability measures on the space \( S_{s,1}^{+} \) of unlabeled finite particles, which approximates the measure \( \mu_{\alpha,\beta} \), plays an important role in the following two sections.

The \( n \)-correlation function of \( \mu_{\alpha,\beta}^n \) is denoted by \( \rho_{\alpha,\beta}^n \). For \( \beta = 2 \), \( \rho_{\alpha,\beta}^n \) is represented by the determinant with the correlation kernel \( K_{\alpha,\beta}^n \) defined by (12.6), and for \( \beta = 1, 4 \), \( \rho_{\alpha,\beta}^n \) is represented by

\[
\rho_{\alpha,\beta}^n(x_1, \ldots, x_n) = \text{qdet}[K_{\alpha,\beta}^n(x_i, x_j)]_{i,j=1,\ldots,n}
\]
with the quaternion valued correlation kernels $K_{\alpha, \beta}^n$ defined by (12.7) for $\beta = 1$ and by (12.9) for $\beta = 4$. (See for instance [22, 2, 8].) The reduced Palm measure is also a (quaternion) determinantal point process and its kernel is represented as

\begin{equation}
K_{\alpha, \beta}^n(y, z) = K_{\alpha, \beta}^n(y, z) - \frac{K_{\alpha, \beta}^n(y, x)K_{\alpha, \beta}^n(x, z)}{K_{\alpha, \beta}^n(x, x)}.
\end{equation}

We refer to [42] for the proof. If $K_{\alpha, \beta}^n(x, x) > 0$, then we deduce from (1.2) the formula of the 1-correlation functions of the reduced Palm measures given by

\begin{equation}
\rho_{\alpha, \beta, x}^{n,1}(y) = \frac{\rho_{\alpha, \beta}^{n,2}(x, y)}{\rho_{\alpha, \beta}^{n,1}(x)}.
\end{equation}

**Lemma 4.2.** Let $\beta = 1, 2, 4$. Then the following holds.

\begin{align}
\lim_{n \to \infty} \rho_{\alpha, \beta}^{n,1} = \rho_{\alpha, \beta}^1, & \quad \lim_{n \to \infty} \partial_i \rho_{\alpha, \beta}^{n,1} = \partial_i \rho_{\alpha, \beta}^1 \quad \text{compact uniformly}, \\
\lim_{n \to \infty} \rho_{\alpha, \beta}^{n,1} = \rho_{\alpha, \beta}^1, & \quad \lim_{n \to \infty} \partial_i \rho_{\alpha, \beta}^{n,1} = \partial_i \rho_{\alpha, \beta}^1 \quad \text{compact uniformly}.
\end{align}

Here $\partial_i$ is the partial derivative in the $i$’th variable, where $i = 1, \ldots, n$.

**Proof.** Recall that $K_{\alpha, 2}^n(x, y) = K_{\alpha, 2}^n(y, x)$. It is well known that $K_{\alpha, 2}^n$ and $\partial_x K_{\alpha, 2}^n$ converge to $K_{\alpha, 2}$ and $\partial_x K_{\alpha, 2}$ compact uniformly. The same convergence also holds for $\beta = 1, 4$ from the above combined with the definitions of $K_{\alpha, \beta}^n$ and $K_{\alpha, \beta}^n$ for $\beta = 1, 4$ given by (10.6), (10.7), (12.7), and (12.9). Hence we obtain (4.3) from the definition of correlation functions given by (1.8), (2.3), and (4.1).

We deduce (1.5) from (4.3), (4.4), and $\rho_{\alpha, \beta}(x) > 0$ immediately. \hfill $\square$

**Lemma 4.3.** Let $\tilde{\varrho}^n$ and $\hat{\varrho}$ be as in (1.19) and (1.15), respectively. Then

\begin{equation}
\lim_{n \to \infty} \lim_{s \to \infty} \sup_{|x| \leq r} \left| \int_{|y| < s} \left( \tilde{\varrho}^n(y) - \hat{\varrho}(y) \right) \left( \frac{1}{x-y} - \frac{1}{-y} \right) dy \right| = 0 \quad (\forall r \in \mathbb{N}).
\end{equation}

Here and after the integral at $x$ and the origin in (4.6) are Cauchy’s principal values.

**Proof.** (4.6) follows from a direct calculation immediately. \hfill $\square$

We next quote an estimate from [21] and give its consequence. We will take the functions $\tilde{\varrho}^n$ and $\hat{\varrho}$ as the main terms of the approximations to the 1-correlation functions $\rho_{\alpha, \beta}^{n,1}$ and $\rho_{\alpha, \beta}^1$, respectively.

**Lemma 4.4.** Let $\beta = 1, 2, 4$. Then there exists a positive constant $c_3$ such that

\begin{align}
|\rho_{\alpha, \beta}^{n,1}(x) - \tilde{\varrho}^n(x)| & \leq \frac{3}{5} \left( \frac{1}{|x|} + \frac{1(\beta \neq 2)}{|x|^{1/4}} \right) \quad \text{for all } x \in [-2n^{2/3}, \infty), \ n \in \mathbb{N}, \\
|\rho_{\alpha, \beta}^1(x) - \hat{\varrho}(x)| & \leq \frac{3}{5} \left( \frac{1}{|x|} + \frac{1(\beta = 4)}{|x|^{1/4}} \right) \quad \text{for all } x \in \mathbb{R}.
\end{align}
Here \( 1(\beta \neq 2) = 1 \) if \( \beta \neq 2 \), and \( 1(\beta \neq 2) = 0 \) otherwise, \( 1(\beta = 4) \) is defined similarly. Moreover, the following asymptotic holds. For each \( r \in \mathbb{N} \)

\[
\lim_{n \to \infty} \lim_{s \to \infty} \sup_{|x| \leq r} \left| \int_{|y| < s} \frac{\rho_{\text{Ai}, \beta, x}^{n, 1}(y) - \rho_{\text{Ai}, \beta, x}^{s, 1}(y) - (\tilde{\varphi}^n(y) - \tilde{\varphi}(y))}{x - y} \, dy \right| = 0.
\]

**Proof.** If \( \beta = 2 \), then Lemma 4.4 follows from [21, Lemma 4.3]. If \( \beta = 1, 4 \), then we deduce from (4.11), (12.7) and (12.9) the relation

\[
(4.10) \quad \rho_{\text{Ai}, 1}^{n, 1}(x) = \rho_{\text{Ai}, 2}^{n, 1}(x) + \frac{1}{2} \psi_{n-1}(x) \psi_n(x), \quad n \in 2\mathbb{N},
\]

\[
(4.11) \quad \rho_{\text{Ai}, 4}^{n, 1}(x) = \frac{1}{21/3} \rho_{\text{Ai}, 2}^{n, 1}(2^{2/3} x) + \frac{\sqrt{2n + 1}}{2^{1/3} \sigma_n} \psi_{2n}(2^{2/3} x) \psi_{2n+1}(2^{2/3} x), \quad n \in \mathbb{N},
\]

with the function \( \psi_n \) defined in (12.1). Hence (4.7) follows from the case \( \beta = 2 \) by a simple calculation with the bound (12.27) and Lemma (12.1).

The correlation functions \( \rho_{\text{Ai}, \beta} \), \( \beta = 1, 4 \) are represented by the quaternion determinant with kernel (10.6) and (10.7), respectively, in particular

\[
(4.12) \quad \rho_{\text{Ai}, 1}^{1}(x) = \rho_{\text{Ai}, 2}^{1}(x) + \frac{1}{2} \text{Ai}(x)(1 - \int_{x}^{\infty} \text{Ai}(u) du)
\]

\[
(4.13) \quad \rho_{\text{Ai}, 4}^{1}(x) = \frac{1}{21/3} \rho_{\text{Ai}, 2}^{1}(2^{2/3} x) - \frac{1}{2^{2/3}} \text{Ai}(2^{2/3} x) \int_{x}^{\infty} \text{Ai}(2^{2/3} u) du.
\]

Hence (4.8) follows from the case \( \beta = 2 \) with Lemmas (11.1) and (11.2).

From (4.7) and (4.8) and the fact that \( \rho_{\text{Ai}, \beta}^{n, 1}(x + 2n^{2/3}) \) and \( \tilde{\varphi}^n(x + 2n^{2/3}) \) are symmetric function of \( x \) we deduce that for each \( l \in \mathbb{N} \)

\[
(4.14) \quad \lim_{n \to \infty} \lim_{s \to \infty} \sup_{|x| \leq r} \left| \int_{(l+1)r \leq |y| < s} \frac{\rho_{\text{Ai}, \beta}^{n, 1}(y) - \rho_{\text{Ai}, \beta}^{s, 1}(y) - (\tilde{\varphi}^n(y) - \tilde{\varphi}(y))}{x - y} \, dy \right|
\]

\[
\leq \lim_{n \to \infty} \lim_{s \to \infty} \sup_{|x| \leq r} \int_{(l+1)r \leq |y| < s} \left[ \frac{1}{y^{1/4}|x - y|} (1 + 1(-2n^{2/3} \leq y)) \right] \, dy
\]

\[
+ \lim_{n \to \infty} \sup_{|x| \leq r} \int_{-2n^{2/3}}^{2n^{2/3}} \frac{2}{|y + 4n^{2/3} \sqrt{1}^{1/4}|x - y|} \, dy = O(l^{-1/4}).
\]

From Lemma (4.2) we deduce that for each \( l \in \mathbb{N} \)

\[
(4.15) \quad \lim_{n \to \infty} \sup_{|x| \leq r} \left| \int_{|y| < (l+1)r} \frac{\rho_{\text{Ai}, \beta}^{n, 1}(y) - \rho_{\text{Ai}, \beta}^{s, 1}(y) - (\tilde{\varphi}^n(y) - \tilde{\varphi}(y))}{x - y} \, dy \right| = 0.
\]

Using (11.8) of Lemma (11.5) and (12.28) of Lemma (12.4) we obtain (4.9) from (4.14), (11.15).
We will use Lemma 4.5 in the proof of Lemma 4.6. Since the proof of Lemma 4.5 is quite long, we prolong it in Section 12.

**Lemma 4.5.** Assume $\beta = 1, 2, 4$. Let $K_{n}^{n,\beta}_{Ai,\beta}$ be as in (12.6), (12.7), and (12.9). We set

\[
I_{n,1}^{\beta}(x,s) = \int_{|x-u| > s} \frac{K_{n}^{n,\beta}_{Ai,\beta}(u,u)(u,u)}{(x-u)^2} du,
\]

\[
I_{n,2}^{\beta}(x,s) = \int_{|x-u| > s} \frac{K_{n}^{n,\beta}_{Ai,\beta}(u,x)}{(x-u)} du,
\]

\[
I_{n,3}^{\beta}(x,s) = \int_{|x-u| > s} \frac{K_{n}^{n,\beta}_{Ai,\beta}(u,v)(u,v)}{(x-u)(x-v)} du dv,
\]

\[
I_{n,4}^{\beta}(x,s) = \int_{|x-v| > s} \frac{K_{n}^{n,\beta}_{Ai,\beta}(u,v)(v,u)}{(x-u)} dv,
\]

\[
I_{n,5}^{\beta}(x,s) = \int_{|x-v| > s} \frac{K_{n}^{n,\beta}_{Ai,\beta}(u,v)(v,u)}{(x-u)(x-v)} dv du.
\]

Then

\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{|x| \leq r} I_{n,k}^{\beta}(x,s) = 0 \quad \text{for all } k = 1, 2, 3, 4, 5.
\]

**Lemma 4.6** (Key estimate). Assume $\beta = 1, 2, 4$. We set

\[
w_{n,s}^{\beta}(x,y) = \sum_{|x-y| \geq s} \frac{1}{x-y} - \int_{|x-y| \geq s} \frac{\rho_{n,1}^{\beta,\beta}(y)}{x-y} dy.
\]

Then

\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{|x| \leq r} E_{n}^{\beta,\beta,s} \left[ |w_{n,s}^{\beta}(x,\cdot)|^2 \right] = 0 \quad \text{for each } r \in \mathbb{N}.
\]

**Proof.** We first note that $E_{n}^{\beta,\beta,s} \left[ |w_{n,s}^{\beta}(x,\cdot)|^2 \right] = 0$. Hence

\[
E_{n}^{\beta,\beta,s} \left[ |w_{n,s}^{\beta}(x,\cdot)|^2 \right] = \text{Var}_{n}^{\beta,\beta,s} \left[ w_{n,s}^{\beta}(x,\cdot) \right].
\]

From the standard calculation of correlation functions and determinantal kernels, we deduce that

\[
\text{Var}_{n}^{\beta,\beta,s} \left[ w_{n,s}^{\beta}(x,\cdot) \right] = \int_{|x-u| > s} \frac{K_{n}^{n,\beta}_{Ai,\beta,x}(u,u)(u,u)}{(x-u)(x-v)} dv du + \int_{|x-v| > s} \frac{K_{n}^{n,\beta}_{Ai,\beta,x}(u,u)}{(x-u)^2} du.
\]
From the relation (4.2) we see by a direct calculation that

\[\begin{align*}
K_n^\alpha \theta_{\beta,x} (u,v) &= K_n^\alpha \theta_{\beta,x} (u,v) \\
+ \frac{K_n^\alpha \theta_{\beta,x} (u,x) K_n^\alpha \theta_{\beta,y} (x,v)}{K_n^\alpha \theta_{\beta,y} (x,x)} \\
- 2K_n^\alpha \theta_{\beta,x} (u,x) K_n^\alpha \theta_{\beta,y} (x,v) K_n^\alpha \theta_{\beta,x} (v,u) K_n^\alpha \theta_{\beta,x} (x,x).
\end{align*}\]

Here we note that all denominators and numerators on the right-hand side are scalars, and can be regarded as real numbers. Combining (4.20)–(4.21), we obtain

\[\begin{align*}
\text{Var}_{\mu} \left[ w_n^\beta ; (x, \cdot) \right] &\leq I_n^\beta (x, s) - I_n^\beta (x, s) - \left( \frac{I_n^\beta (x, s)}{K_n^\alpha \theta_{\beta,y} (x,x)} \right)^2 - 2I_n^\beta (x, s) K_n^\alpha \theta_{\beta,x} (x,x).
\end{align*}\]

Since \(K_n^\alpha\) converge \(K\) compact uniformly and \(K_n^\alpha \theta_{\beta,x} (x,x)\) are locally uniformly positive in \(x\), we obtain that

\[\liminf_{n \to \infty} \inf_{|x| \leq r} K_n^\alpha \theta_{\beta,x} (x,x) > 0 \quad \text{for each } r \in \mathbb{N}.
\]

We therefore deduce (4.18) from (4.19), (4.22) and (4.23), and Lemma 4.5.

5 Proof Theorem 2.1

In this section, we prove Theorem 2.1. We begin by the quasi-Gibbs property \((B2)\). For this we quote a result from [32, 33]. In the following, we take \(d = 1\) and \(S = \mathbb{R}\). So \(S\) is the configuration space over \(\mathbb{R}\). In addition to \((B1)\), we introduce three further conditions \((C1)–(C3)\) below for the quasi-Gibbs property. These conditions guarantee that \(\mu\) has a good finite-particle approximation \(\{\mu_n\}_{n \in \mathbb{N}}\) that enables us to prove the quasi-Gibbs property of \(\mu\).

\[(C1)\] There exists a sequence of RPFs \(\{\mu_n\}_{n \in \mathbb{N}}\) on \(S\) satisfying the following.

1. The \(n\)-correlation functions \(\rho_n^{\alpha,n}\) of \(\mu^n\) satisfy

\[\lim_{n \to \infty} \rho_n^{\alpha,n}(x_n) = \rho^n(x_n) \quad \text{a.e. for all } n \in \mathbb{N},\]

\[\sup \{ \rho_n^{\alpha,n}(x_n); n \in \mathbb{N}, x_n \in \{|x| < r\} \} \leq \{(\kappa \delta)^n\} \quad \text{for all } n, r \in \mathbb{N},\]

where \(c_4 = \frac{d + \kappa}{4}\) and \(\delta = \delta(r) < 1\) are constants depending on \(r \in \mathbb{N}.

2. \(\mu^n(s(\mathbb{R})) \leq n_n = 1\) for some \(n_n \in \mathbb{N}.

3. \(\mu^n\) is a \((\Phi^n, -\beta \log |x - y|)\)-canonical Gibbs measure, where \(0 < \beta < \infty\).
(C2) There exists a sequence \( \{m^n_x\}_{n \in \mathbb{N}} \) in \( \mathbb{R} \) such that

\[
\lim_{n \to \infty} \{ \Phi^n(x) - m^n_x \} = \Phi(x) \quad \text{for a.e. } x,
\]

\[
\inf_{n \in \mathbb{N}} \inf_{|x| < r} \{ \Phi^n(x) - m^n_x \} > -\infty \quad \text{for each } r \in \mathbb{N}.
\]

(C3) There exists \( m^\beta_x \) in \( \mathbb{R} \) such that

\[
\lim_{r \to \infty} m^\beta_x = m^n_x,
\]

\[
\sup_{n \in \mathbb{N}} |m^\beta_x| < \infty \quad \text{for all } r \in \mathbb{N},
\]

\[
\lim_{r \to \infty} \sup_{n \in \mathbb{N}} \left\| \left\{ \beta \sum_{r \leq |x_i| < \infty} \frac{1}{x_i} \right\} - (m^n_x - m^\beta_x) \right\|_{L^2(S,\mu^n)} = 0.
\]

Moreover, the sequence of 1-correlation functions \( \rho^{n,1} \) of \( \mu^n \) satisfies

\[
\sup_{n \in \mathbb{N}} \left\{ \int_{1 \leq |x| < \infty} \frac{1}{|x|^2} \rho^{n,1}(x)dx \right\} < \infty.
\]

The next lemma has been proved in [33, Theorem 2.2].

**Lemma 5.1.** Let \( \beta \in (0, \infty) \). Assume (B1) and (C1)–(C3). Then \( \mu \) is quasi-Gibbssian with potential \( (\Phi, -\beta \log |x-y|) \).

We now apply Lemma 5.1 to Airy RPFs. We see this in a sequence of lemmas, in which we assume \( \beta = 1, 2 \) or 4.

**Lemma 5.2.** Let \( \mu_{\alpha,\beta}^n \) be the RPF on \( \mathbb{R} \) whose labeled distribution is given by \( m^n_{\alpha,\beta} \) in (1.4). Set \( n_\alpha = n \) and \( \Phi^n(x) = \beta \frac{n^{-1/3}}{2} x^2 + \frac{n^{1/3}}{2} x \). Then (C1) holds.

**Proof.** (1) of (C1) is well known (see [8], for example). Moreover, (2) and (3) of (C1) are obvious from (1.4). \( \square \)

**Lemma 5.3.** (1) The following limit exists compact uniformly in \( C(\mathbb{R}) \).

\[
u_\beta(x) = \lim_{s \to \infty} \left\{ \int_{|y| < s} \frac{\rho_{\alpha,\beta,x}(y)}{x-y} dy - \int_{|y| < s} \frac{\hat{\rho}(y)}{-y} dy \right\}.
\]

(2) Let \( u^n_\beta (n \in \mathbb{N}) \) be the continuous functions defined by

\[
u_\beta(x) = \int_{\mathbb{R}} \frac{\rho_{\alpha,\beta,x}^{n,1}(y)}{x-y} dy - \frac{n^{1/3}}{2} x - \frac{n^{-1/3}}{2} x.
\]

Then \( u^n_\beta \) converge \( u_\beta \) compact uniformly in \( C(\mathbb{R}) \).
Proof. (5.9) follows from (4.8) and (11.8), immediately. Indeed,

\[
\limsup_{s \leq t < \infty} \left| \int_{s \leq |y| < t} \frac{\rho_{\Lambda_i, \beta, x}(y)}{x - y} dy - \int_{s \leq |y| < t} \frac{\hat{\varrho}(y)}{x - y} dy \right|
\]

\[
\leq \limsup_{s \leq t < \infty} \int_{s \leq |y| < t} \left| \frac{\rho_{\Lambda_i, \beta, x}(y)}{x - y} - \frac{\hat{\varrho}(y)}{x - y} \right| + \left| \frac{\hat{\varrho}(y)}{x - y} - \frac{\hat{\varrho}(y)}{|x - y|} \right| dy
\]

\[
\leq \int_{s \leq |y| < \infty} \frac{|x - y||y|^{1/4}}{|x - y| |y|^1} + \frac{\hat{\varrho}(y)|x|}{|x - y| |y|^2} dy = O(s^{-1/4}) \quad (s \to \infty).
\]

Here we used the definition \( \hat{\varrho}(y) = \frac{1}{1 - \infty \sqrt{y}} \) for the last line.

Recall that \( n^{1/3} = - \int_\mathbb{R} \hat{\varrho}(x)/x dx \) by (1.21). So we see from (5.9) and (5.10) that

\[
|u_\beta^n(x) - u_\beta(x)|
\]

\[
= \left| \int_\mathbb{R} \frac{\rho_{\Lambda_i, \beta, x}(y)}{x - y} dy - \frac{n^{-1/3}}{2} - \lim_{s \to \infty} \left\{ \int_{|y| < s} \frac{\rho_{\Lambda_i, \beta, x}(y)}{x - y} dy - \int_{|y| < s} \frac{\hat{\varrho}(y)}{-y} dy \right\} \right|
\]

\[
= \lim_{s \to \infty} \left\{ \int_{|y| < s} \frac{\rho_{\Lambda_i, \beta, x}(y)}{x - y} dy - \frac{n^{-1/3}}{2} \right\}
\]

\[
+ \lim_{s \to \infty} \left\{ \int_{|y| < s} \left( \frac{\hat{\varrho}(y)}{-y} - \frac{\hat{\varrho}(y)}{(x - y)} \right) \left\{ \frac{1}{x - y} - \frac{1}{-y} \right\} dy \right\} + \frac{n^{-1/3}}{2} |x|.
\]

Hence applying (4.6) and (4.9) to the last two lines, we obtain (2). \( \square \)

For \( r \in \mathbb{N} \cup \{ \infty \} \), we set

\begin{align*}
(5.11) \quad m_\infty^n = \beta \int_{|y| < r} \frac{\rho_{\Lambda_i, \beta, 0}(y)}{-y} dy.
\end{align*}

Lemma 5.4. Let \( \Phi^n \) and \( u_\beta \) be as in Lemma 5.2 and Lemma 5.3 respectively. Let \( m_\infty^n \) be as in (5.11). Let \( \Phi(x) = -\beta u_\beta(0)x \). Then (C2) holds.

Proof. Let \( u_\beta^n \) be the continuous function defined by (5.10). Then from Lemma 5.3 and (5.11), we deduce that

\begin{align*}
(5.12) \quad \lim_{n \to \infty} \{ \beta n^{1/3} - m_\infty^n \} = \lim_{n \to \infty} -\beta u_\beta^n(0) = -\beta u_\beta(0).
\end{align*}

So by definition we have

\begin{align*}
(5.13) \quad \lim_{n \to \infty} \{ \Phi^n(x) - m_\infty^n x \} = \lim_{n \to \infty} \{ (\frac{\beta}{4} n^{-1/3} x^2 + \beta n^{1/3} x) - m_\infty^n x \} = -\beta u_\beta(0)x.
\end{align*}

Then (C2) follows from (5.12) and (5.13) immediately. \( \square \)
Lemma 5.5. Let $m_n^p$ and $m_n^p_\infty$ be as in (5.11). Then (C3) holds with $\rho^{n,1} = \rho^{n,1}_{\text{Ai},\beta}$.

**Proof.** Recall that $\int_\mathbb{R} \rho^{n,1}_{\text{Ai},\beta,0}(y)dy = n - 1$. Hence (5.5) follows from

\[
\rho^{n,1}_{\text{Ai},\beta,0}(y) \leq \frac{\rho^{n,1}_{\text{Ai},\beta,0}}{n} \rightarrow 0 \quad (r \rightarrow \infty).
\]

Moreover, we deduce (5.5) from the compact uniform convergence of $\rho^{n,1}_{\text{Ai},\beta,0}$ to $\rho^{1}_{\text{Ai},\beta,0}$ and $\{\rho^{1}_{\text{Ai},\beta,0}\}'$. We deduce (5.7) from (4.18).

Finally, we deduce from (4.7) that

\[
\int_1^\infty |x|^{-\beta} \rho^{n,1}_{\text{Ai},\beta}(x)dx \leq \int_1^\infty \left\{ |\rho^{n,1}_{\text{Ai},\beta}(x) - \tilde{\varrho}^{n}(x)| + \tilde{\varrho}^{n}(x) \right\}dx \\
\leq \int_1^\infty \left\{ \frac{c_3}{|x|^2} + \frac{\sqrt{|x|}}{\pi |x|^2} \right\}dx.
\]

This implies (5.8) immediately.

**Theorem 5.6.** Let $\beta = 1, 2, 4$ and $u_\beta$ be in (5.9). Put

\[
\Phi_\beta(x) = \beta \int_x^0 u_\beta(y)dy, \quad \Psi_\beta(x, y) = -\beta \log |x - y|.
\]

Then Airy RPF $\mu_{\text{Ai},\beta}$ is a $(\Phi_\beta, \Psi_\beta)$-quasi Gibbs measure.

**Proof.** Let $\Phi = -\beta u_\beta(0)x$. Since $F(x) = \Phi_\beta(x) - \Phi(x)$ is a locally bounded measurable function by $u_\beta \in C(\mathbb{R})$, Theorem 5.6 is equivalent to that $\mu_{\text{Ai},\beta}$ is a $(\Phi, \Psi_\beta)$-quasi Gibbs measure, from Remark 3.1 (1). We first note that (B1) is clear from Lemma 4.1. We deduce (C1)–(C3) from Lemma 5.2, Lemma 5.4 and Lemma 5.5, respectively. Hence the claim follows from Lemma 5.1.

**Proof of Theorem 2.1.** By Lemma 4.1 we see that $\mu_{\text{Ai},\beta}^{n,1}$ satisfies (B1). The quasi-Gibbs property (B2) follows from Theorem 5.6 Hence the assumptions of Theorem 3.1 are fulfilled. In particular, (1) and (2) of Theorem 2.1 follow from Theorem 3.1 immediately.

We next prove (3) of Theorem 2.1. Recall the definition of $S_{s,i}$ and $S_{s,i}^{+g}$ given by (2.6). If one relax the claim $S_{s,i}^{+g} \subset S_{s,i}^+$ in (5.12) to $S_{\mu,\text{Ai},\beta} \subset S_{s,i}$, then (3) of Theorem 2.1 follows from (3) of Theorem 3.1.

We deduce from (2.7) that $\mu_{\text{Ai},\beta}(S_{\mu,\text{Ai},\beta} \cap S_{s,i}^{+g}) = 1$. Hence we deduce from Remark 2.1 that $S_{\mu,\text{Ai},\beta} \cap S_{s,i}^{+g}$ satisfy the requirement of (3) of Theorem 2.1 which completes the proof.
6 Proof of Theorem 2.2

We prove Theorem 2.2 by applying Theorem 3.2 to $\mu_{\chi,\beta}$. For this we check that $\mu_{\chi,\beta}$ satisfy the conditions (B1)–(B9) in Section 3. We note that, for $\mu_{\chi,\beta}$, we have already proved (B1), (B3), and (B4) in Lemma 4.1 and (B2) in Theorem 5.6.

We begin by proving the existence of the logarithmic derivative $d\mu_{\chi,\beta}$ of $\mu_{\chi,\beta}$ and by its explicit representation. We write $y = \sum \delta_{y_j}$ below.

**Theorem 6.1.** For each $\beta = 1, 2$ and 4, the logarithmic derivative $d\mu_{\chi,\beta}$ exists in $L_p(\mu_{\chi,\beta})$ for some $p > 1$ and is given by

\[ d\mu_{\chi,\beta}(x, y) = \beta \lim_{s \to \infty} \left\{ \sum_{|x - y_j| < s} \frac{1}{x - y_j} - \int_{|y| < s} \frac{\hat{\varrho}(y)}{x - y} dy \right\}. \] (6.1)

To prove Theorem 6.1 we divide $d\mu_{\chi,\beta}$ as

\[ d\mu_{\chi,\beta}(x, y) = \beta \{ u_{\beta}(x) + \lim_{s \to \infty} g_{\beta,s}(x, y) \}. \] (6.2)

Here $u_{\beta}$ is the continuous function defined by (5.9) and $g_{\beta,s}(x, y)$ is

\[ g_{\beta,s}(x, y) = \sum_{|x - y_j| < s} \frac{1}{x - y_j} - \int_{|y| < s} \frac{\rho_{\chi,\beta,x}(y)}{x - y} dy. \] (6.3)

The convergence of $g_{\beta,s}$ as $s \to \infty$ is not trivial, and will be proved in the proof of Theorem 6.1.

We calculate the logarithmic derivatives of the finite particle approximation $\{\mu_{\chi,\beta}^n\}$. From (1.4) we easily deduce that the logarithmic derivative $d\mu_{\chi,\beta}$ of $\mu_{\chi,\beta}$ becomes

\[ d\mu_{\chi,\beta}(x, y) = \beta \left\{ \sum_{j=1}^{n-1} \frac{1}{x - y_j} - n^{1/3} - \frac{n^{-1/3}}{2} x \right\}. \] (6.4)

Here $y = \sum_{j=1}^{n-1} \delta_{y_j}$ because $\mu_{\chi,\beta}^n(\{s(S) = n\}) = 1$.

Let $u_{\beta}^n(x)$ and $w_{\beta,s}^n(x, y)$ be as in (5.10) and (4.17), respectively. Let

\[ g_{\beta,s}^n(x, y) = \sum_{|x - y_j| < s} \frac{1}{x - y_j} - \int_{|y| < s} \frac{\rho_{\chi,\beta,x}^n(y)}{x - y} dy. \] (6.5)

Then from (6.4) we deduce that

\[ d\mu_{\chi,\beta}^n(x, y) = \beta \{ u_{\beta}^n(x) + g_{\beta,s}^n(x, y) + w_{\beta,s}^n(x, y) \}. \] (6.6)

Furthermore we have:
Lemma 6.2. Assume $\beta = 1, 2, 4$. Then for some $\hat{p} > 1$

\begin{align}
(6.7) \quad & \lim_{n \to \infty} u_{\beta}^n(x) = u_\beta(x) \quad \text{in} \quad L^\hat{p}_0(\mathbb{R}, dx), \\
(6.8) \quad & \lim_{n \to \infty} g_{\beta,s}^n(x, y) = g_{\beta,s}(x, y) \quad \text{in} \quad L^\hat{p}_0(\mu_{\text{Ai},\beta}) \quad \text{for any} \quad s > 0, \\
(6.9) \quad & \lim_{s \to \infty} \lim_{n \to \infty} \int_{[-r, r] \times S} |u_{\beta,s}^n(x, y)|^2 d\mu_{\text{Ai},\beta}^n = 0 \quad (\forall r \in \mathbb{N}).
\end{align}

**Proof.** We deduce (6.7) from Lemma 5.3. (6.8) follows from (4.9) in Lemma 4.4, with (1.20). (6.9) follows from (4.18) in Lemma 4.6.

**Proof of Theorem 6.1.** We use [31, Theorem 45] to prove Theorem 6.1. Indeed, from Lemma 6.2, (4.6), (5.2), and (6.6) we see that the assumptions of [31, Theorem 45] are fulfilled. Hence we deduce (6.2) from [31, Theorem 45]. Then we easily see from (6.2) that the logarithmic derivative has the expression in (6.1).

**Proof of Theorem 2.2.** At the beginning of this section we have already checked (B1)–(B4). By Theorem 6.1 we see that (B5) is satisfied with the logarithmic derivative given by (6.1). Hence Theorem 2.2 (1) follows from Theorem 3.2 immediately. Since $l$ gives the injection from the support of the $\mu_{\text{Ai},\beta}$ to $\mathbb{R}^N$, we obtain Theorem 2.2 (2). Theorem 2.2 (3) is clear because Tracy-Widom distribution is equal to the distribution of the top particle under $\mu_{\text{Ai},2}$.

7 Proof of Theorem 2.3.

In this section we prove Theorem 2.3 by applying Theorem 3.3 and Proposition 3.5. For this we will check that $\mu_{\text{Ai},\beta}$ satisfy the conditions (B1)–(B8) given in Section 3. The conditions (B1)–(B5) have been already checked in Section 6. Hence our task is to show other conditions (B6)–(B8).

**Lemma 7.1.** Let $\beta = 1, 2, 4$. Then there exists a constant $c_5$ such that

\begin{align}
(7.1) \quad & \rho_{\text{Ai},\beta}^1(x) \leq c_5 \{ \sqrt{|x|} + 1 \} \quad \text{for all} \quad x \in \mathbb{R}.
\end{align}

**Proof.** Recall that $\tilde{\rho}_1^1(x) = \frac{1}{1(1-x)} \sqrt{-x}$ as we set in (1.15). Combining this with Lemma 4.4 yields (7.1).

**Lemma 7.2.** Let $\beta = 1, 2, 4$. Then $\mu_{\text{Ai},\beta}$ satisfy (B6).

**Proof.** The bound (7.1) immediately yields (B6).
notations. Let \( a = \{a_k\}_{k \in \mathbb{N}} \) be a sequence of increasing sequences \( a_k = \{a_k(r)\}_{r \in \mathbb{N}} \) of natural numbers such that
\[
(7.2) \quad a_k(r) = kr^3.
\]
Let \( K_{k,r} = \{s \mid s(S_r) \leq a_k(r)\} \) and set
\[
(7.3) \quad K[a] = \bigcup_{k=1}^{\infty} \bigcap_{r=1}^{\infty} K_{k,r}.
\]

**Lemma 7.3.** Let \( \beta = 1, 3, 4. \) Then the sequence \( a \) satisfies
\[
(7.4) \quad \mu_{A_1,\beta}(K[a]) = 1.
\]

**Proof.** (7.4) follows from Lemma 7.1, Borel-Cantelli’s lemma, and Chebycheff’s inequality. Indeed, we easily see that
\[
K[a] = \bigcup_{k=1}^{\infty} \lim\inf_{r \to \infty} K_{k,r}.
\]
Hence from this and the monotonicity of the sets \( K_{k,r} \) in \( k \), we deduce that
\[
(7.5) \quad \mu_{A_1,\beta}(K[a]) = \mu_{A_1,\beta}(\bigcap_{k=1}^{\infty} \lim\sup_{r \to \infty} K_{k,r}) = \mu_{A_1,\beta}(\lim\sup_{k \to \infty} \bigcup_{r=1}^{\infty} K_{k,r}).
\]

From Chebycheff’s inequality and Lemma 7.1 we deduce that
\[
(7.6) \quad \mu_{A_1,\beta}(K[a]) \leq \frac{1}{a_k(r)} \mathbb{E}(A_{1,\beta}[s(S_r)]) = \frac{1}{a_k(r)} \int_{S_r} \rho_{1,\beta}(x)dx = O(r^{-3/2}).
\]

Combining (7.5) and (7.6) and applying Borel-Cantelli’s lemma, we deduce that \( \mu_{A_1,\beta}(K[a]) = 0 \), which implies (7.4).

**Lemma 7.4.** The derivative in \( x \) of \( d^{\mu_{A_1,\beta}} \) is given by
\[
(7.7) \quad \nabla_x d^{\mu_{A_1,\beta}}(x,y) = -\beta \sum_j \frac{1}{(x-y_j)^2}.
\]

Here the sum in (7.7) converge absolutely in \( L^2_{\text{loc}}(\mathbb{R} \times S, \mu_{A_1,\beta}^{[1]}) \).

**Proof.** This follows from the bound in Lemma 7.1 and the standard calculation of correlation functions immediately. Indeed, we see that
\[
(7.8) \quad \int_{S_r \times S} \left| \sum_j \frac{1}{(x-y_j)^2} \right|^2 d^{\mu_{A_1,\beta}} = \int_{S_r} \rho_{A_1,\beta}(x)dx \left\{ \int_{S_r} \left| \sum_j \frac{1}{(x-y_j)^2} \right|^2 d^{\mu_{A_1,\beta}} \right\}
\]
\[
= \int_{S_r} \rho_{A_1,\beta}^1(x)dx \left\{ \int_{\mathbb{R}^2} \rho_{A_1,\beta}^2(y, z) (x-y)^2 (x-z)^2 dydz + \int_{\mathbb{R}} \rho_{A_1,\beta}^2(x, y) (x-y)^2 dy \right\}
\]
\[
= \int_{S_r \times \mathbb{R}^2} \rho_{A_1,\beta}^3(x, y, z) (x-y)^2 (x-z)^2 dxdydz + \int_{S_r \times \mathbb{R}} \rho_{A_1,\beta}^2(x, y) (x-y)^4 dxdy < \infty.
\]
From this we see the convergence of the sum in (7.7). The equality in (7.7) is immediate by differentiating both sides of (6.1).

Proof of Theorem 2.3. We use the results in [37, Subsection 8.3]. We take \( a \) in [37, Subsection 8.3] as in (7.2) above. Then by Lemma 7.3 the condition (U1) in [37] is satisfied. Take \( \ell \) in (Z1) in [37] to be \( \ell = 1 \). Then (Z1) in [37] holds by Lemma 7.4. The condition (Z2) in [37] follows from [37, Lemma 8.10] and Lemma 7.1. Therefore we deduce (B7) and (B8) (A7 and A8 in [37]) from [37, Proposition 8.7].

8 Proof of Theorem 2.4.

In this section we will prove Theorem 2.4. For this we first prepare a general result Theorem 8.3 on the uniqueness of local, quasi-regular Dirichlet forms. We will later deduce Theorem 2.4 from Theorem 8.3.

We suppose that we have a Dirichlet form \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) \) on \( L^2(S, \mu) \) such that

\[
\hat{\mathcal{D}}^\mu \supset D^\mu, \quad \hat{\mathcal{E}}^\mu(f, f) = E^\mu(f, f) \quad \text{for all } f \in D^\mu.
\]

Lemma 8.1. Assume (B1)–(B9) and (8.1). Assume that there exists a diffusion \( \{\hat{P}_s\}_{s \in S} \) associated with \( (\hat{\mathcal{E}}, \hat{\mathcal{D}}) \) on \( L^2(S, \mu) \). Then \( \{\hat{P}_s\}_{s \in S} \) satisfies (3.11) and (3.12).

Proof. From (8.1) we see that the capacity of \( (E^\mu, D^\mu) \) dominates that of \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) \). From this we obtain (3.11). The proof of (3.12) for \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) \) is same as that for \( (E^\mu, D^\mu) \).

Remark 8.1. Since the associated diffusion \( \{\hat{P}_s\}_{s \in S} \) exists by assumption, the Dirichlet form \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) \) is quasi-regular and local by the inverse theorem due to [1]. Hence the capacity associated with the Dirichlet form \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) \) exists. Furthermore, since \( 1 \in D^\mu \subset \hat{\mathcal{D}}^\mu \), the diffusion \( \{\hat{P}_s\}_{s \in S} \) is \( \mu \)-reversible.

From (3.11) and (3.12), we can construct the labeled process \( \hat{X} = \text{l}_{\text{path}}(X) \). The next lemma is a slight generalization of Theorem 3.2.

Lemma 8.2. Under the same assumptions as Lemma 8.1, the labeled process \( \hat{X} = \text{l}_{\text{path}}(X) \) under \( \{\hat{P}_s\}_{s \in S} \) is a weak solution of (3.17)–(3.19).

Proof. This lemma follows from the analogy of [31, Theorem 26]. Since the domain of the Dirichlet form greater than that of the original one by (8.1), we can prove [31, Theorem 26] for \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) \) exactly the same fashion as for \( (E^\mu, D^\mu) \).

Theorem 8.3. Under the same assumptions as Lemma 8.1 \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) = (E^\mu, D^\mu) \).

Proof. By Proposition 3.5 there exists an unique strong solution \( X \) associated with the Dirichlet form \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) \) on \( L^2(S, \mu) \).

By Lemma 8.2 there exists a weak solution \( \hat{X} \) of the ISDE (3.17)–(3.19). The unlabeled process \( \hat{X} \) corresponding to \( \hat{X} \) is associated with the Dirichlet form \( (\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}^\mu) \).
on $L^2(S, \mu)$. From [37, Theorem 2.2], the solution becomes a unique strong solution of the ISDE (3.17–(3.19)). Hence, because of the uniqueness of the ISDE (3.17–(3.19), $\hat{X}$ equals the strong solution $X$ given by the Dirichlet form $(E^\mu, D^\mu)$. We therefore conclude $(\hat{E}^\mu, \hat{D}^\mu) = (E^\mu, D^\mu)$.

As a corollary of Theorem 8.1 we see the uniqueness of quasi-regular Dirichlet form.

**Corollary 8.3.** Let $(\hat{E}^\mu, \hat{D}^\mu)$ be a local quasi-regular Dirichlet form on $L^2(S, \mu)$ satisfying (8.1). Assume $(B1)$–$(B9)$. Then $(\hat{E}^\mu, \hat{D}^\mu) = (E^\mu, D^\mu)$.

In the sequel we devote to the proof of Theorem 2.4 through Theorem 8.3. For this we use some results from [19, 21, 45, 37].

**Lemma 8.4.** Let $\{Q_s\}$ be as in Theorem 2.4. Then $\{Q_s\}$ are diffusion processes associated with a Dirichlet form $(E^{KT}, D^{KT})$ on $L^2(S, \mu_{Ai^2})$ such that

$$D^\mu_{Ai^2, \beta} \subset D^{KT}, \quad E^{KT}(f, g) = E^{\mu_{Ai^2, \beta}}(f, g) \quad \text{for all } f, g \in D^{\mu_{Ai^2, \beta}}. \quad \text{(8.2)}$$

**Proof.** In [39], it is proved that $\{Q_s\}$ is a diffusion, and the associated local, quasi-regular Dirichlet form $(E^{KT}, D^{KT})$ on $L^2(S, \mu_{Ai^2})$ exists.

Let $P$ be the set consisting of polynomials in $S$, that is,

$$P = \{ f; f(s) = F(\langle f_1, s \rangle, \ldots, \langle f_m, s \rangle), \quad F \in \text{Poly}(\mathbb{R}^m), \quad f_i \in C_0^\infty(\mathbb{R}) \quad (i = 1, \ldots, m), \quad m \in \mathbb{N} \}, \quad \text{(8.3)}$$

where $\text{Poly}(\mathbb{R}^m)$ is the set of all polynomials in $\mathbb{R}^m$ and $\langle f, s \rangle = \sum_i f_i(s_i)$ for $s = \sum_i \delta_{s_i}$. Then it was proved in [19] that $P \subset D^{KT}$ and $E^{KT}(f, g) = E^{\mu_{Ai^2, \beta}}(f, g)$ for all $f, g \in P$.

In [30], we prove that $D^{\mu_{Ai^2, \beta}}$ is equal to the closure of $P$ with respect to the Dirichlet form $(E^{\mu_{Ai^2, \beta}}, D^{\mu_{Ai^2, \beta}})$ on $L^2(S, \mu_{Ai^2})$. Hence (8.2) follows from this and (8.3).

**Proof of Theorem 2.4.** Lemma 8.4 implies that $(E^{KT}, D^{KT})$ fulfills the assumptions of $(\hat{E}, \hat{D})$ in Theorem 8.3, which deduces Theorem 2.4 immediately.

**Proof of Corollary 2.4.** In [7], it is proved that the unlabeled diffusions $P_{\mu_{Ai^2}}^n$ converge to $\{Q_{\mu_{Ai^2}}\}$ in finite-dimensional distributions (f.d.d.). Hence from Theorem 2.4 we deduce that $P_{\mu_{Ai^2}}^n$ converge to $P_{\mu_{Ai^2}}$ in f.d.d. Applying Lyons-Zheng decomposition to each component, we can refine the convergence to the weak convergence in $C([0, \infty); S)$. This proves (1).

The second claim (2) follows from (1) and the Lyons-Zheng decomposition.
9 Proof of Theorem 2.5.

In this section we complete the proof of Theorem 2.5.

Proof of Theorem 2.5. Let \(X\) be the strong solution of (1.14) in Theorem 2.3. Then the condition (B8) is satisfied by the proof of Theorem 2.3 in Section 7. Hence, for \(P\)-a.s. fixed \(X^m = \{(X^m_{t+1}, X^m_{t+2}, \ldots)\}_{t \in [0,T]}\), the first \(m\)-components \(X^m = \{(X^m_t, X^m_{t+1}, \ldots)\}_{t \in [0,T]}\) become the unique strong solution of the (finite-dimensional) SDE (3.23) with \(\mu = \mu_{Ai,\beta}\) on the interval \([0,T]\). Namely, under \(P\), \(X^m\) satisfies

\[
\begin{align*}
\dot{X}^m_t &= dB^m_t + \beta \lim_{r \to \infty} \left\{ \sum_{j \neq i, |X^m_j|<r} \frac{1}{X^m_j - X^m_i} - \int_{|x|<r} \frac{\hat{\phi}(x)}{-x} dx \right\} dt \\
X^m_0 &= (s_1, \ldots, s_m) \in S^m.
\end{align*}
\]

We emphasize that \(X^m\) is regarded as a part of the coefficients. Then (2.24) follows from the SDE (9.1) and the standard argument of Girsanov formula.

The second statement (2.25) follows from (B7). Indeed, let \(H^{[1]}\) be the set as in (3.22). Then we see by definition that, when \(\mu = \mu_{Ai,\beta}\),

\[
H^{[1]} \subset \bigcup_{k=1}^{\infty} \left\{ (x,s) ; \lim_{r \to \infty} \left\{ \sum_{|x_j|<r} \frac{1}{x_j - x_i} - \int_{|y|<r} \frac{\hat{\phi}(y)}{-y} dy \right\} < k \right\}.
\]

Since \(\text{Cap}^{[1]}((H^{[1]})^c) = 0\) by (3.22), we immediately obtain (2.25). \(\square\)

10 Appendix 1: Quaternion determinant and kernels

We recall the standard quaternion notation for \(2 \times 2\) matrices (see [22, Ch. 2.4]),

\[
1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\]

A quaternion \(q\) is represented by \(q = q^{(0)}1 + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3\), where \(q^{(i)}\) are complex numbers. There is a natural identification between the \(2 \times 2\) complex matrices and the quaternions given by

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}(a+d)1 - \frac{i}{2}(a-d)e_1 + \frac{1}{2}(b+c)e_2 - \frac{i}{2}(b-c)e_3.
\]

For a quaternion \(q = q^{(0)}1 + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3\), we call \(q^{(0)}\) the complex scalar part of \(q\). A quaternion is called complex scalar if \(q^{(i)} = 0\) for \(i = 1, 2, 3\). We often identify a complex scalar quaternion \(q = q^{(0)}1\) with the complex number \(q^{(0)}\).
Let \( \bar{q} = q^{(0)}1 - \{q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3\} \). A quaternion matrix \( A = [a_{ij}] \) is called self-dual if \( a_{ij} = a_{ji} \) for all \( i, j \). For a self-dual \( n \times n \) quaternion matrix \( A = [a_{ij}] \) we set

\[
\text{qdet} A = \sum_{\sigma \in S_n} \text{sign}[\sigma] \prod_{i=1}^{L(\sigma)} (a_{\sigma_{i}(1)}a_{\sigma_{i}(2)}a_{\sigma_{i}(3)} \cdots a_{\sigma_{i}(\ell)}a_{\sigma_{i}(1)})^{(0)}.
\]

Here \( \sigma = \sigma_1 \cdots \sigma_{L(\sigma)} \) is a decomposition of \( \sigma \) to products of the cyclic permutations \( \{\sigma_i\} \) with disjoint indices. We write \( \sigma_i = (\sigma_i(1), \sigma_i(2), \ldots, \sigma_i(\ell)) \), where \( \ell \) is the length of the cyclic permutation \( \sigma_i \). The decomposition is unique up to the order of \( \{\sigma_i\} \). As before \([\cdot]^{(0)}\) means the complex scalar part of the quaternion \( \cdot \). It is known that the right hand side is well defined (see [22, Section 5.1]).

We now introduce the quaternion kernels \( K_{Ai,1} \) and \( K_{Ai,4} \) via the \( 2 \times 2 \) matrix representation of quaternions.

Let \( K_{Ai,2} \) and \( Ai \) be as in (1.8) and (1.9), respectively. Let

\[
J_1(x, y) = K_{Ai,2}(x, y) + \frac{1}{2} Ai(x) (1 - \int_y^\infty Ai(u) du)
\]

\[
J_4(x, y) = K_{Ai,2}(x, y) - \frac{1}{2} Ai(x) \int_y^\infty Ai(u) du.
\]

Then we define the quaternion kernels \( K_{Ai,1} \) and \( K_{Ai,4} \) by

\[
K_{Ai,1}(x, y) = \begin{bmatrix}
J_1(x, y) & -\frac{\partial}{\partial y} J_1(x, y) \\
-\frac{1}{2} \text{sign}(x - y) + \int_y^x J_1(u, y) du & J_1(y, x)
\end{bmatrix},
\]

\[
2^{-2/3} K_{Ai,4}(2^{-2/3} x, 2^{-2/3} y) = \frac{1}{2} \begin{bmatrix}
J_4(x, y) & -\frac{\partial}{\partial y} J_4(x, y) \\
\int_y^x J_4(u, y) du & J_4(y, x)
\end{bmatrix}.
\]

11 Appendix 2: Estimates of Airy functions

We use the following asymptotic expansions of the Airy functions in the classical sense of Poincaré [16, 26, 27]:

Lemma 11.1. For \( x \gg 1 \)

\[
Ai(x) = e^{-\frac{2}{3}x^{3/2}} \left( 1 + O(x^{-3/2}) \right), \quad Ai'(x) = -\frac{x^1/4 e^{-\frac{2}{3}x^{3/2}}}{2 \pi^{1/2}} \left( 1 + O(x^{-3/2}) \right),
\]

\[
Ai(-x) = \frac{1}{\pi^{1/2} x^{1/4}} \left[ \cos \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) \left( 1 + O(x^{-3/2}) \right) \right],
\]

\[
Ai'(-x) = \frac{x^{1/4}}{\pi^{1/2}} \left[ \sin \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) \left( 1 + O(x^{-3/2}) \right) \right], \quad x \to \infty.
\]
Lemma 11.2. The asymptotic behaviors
\[
\int_{0}^{x} \mathrm{Ai}(u)du = \frac{1}{3} - \frac{\exp\left\{-\frac{2}{3}x^{3/2}\right\}}{2\sqrt{\pi}x^{3/4}} (1 + o(1)), \quad x \to \infty,
\]
\[
\int_{-x}^{0} \mathrm{Ai}(u)du = \frac{2}{3} + \mathcal{O}\left(x^{-3/4}\right), \quad x \to \infty.
\]
are fulfilled. In particular,
\[
1 - \int_{-x}^{\infty} \mathrm{Ai}(u)du = \mathcal{O}\left(x^{-3/4}\right), \quad x \to \infty,
\]
(11.1)

We apply the above asymptotic behaviors to examine the one-correlation function \(\rho_{1,2}^{\mathrm{Ai}}\) and the Airy kernels \(K_{\mathrm{Ai},\beta}\) \((\beta = 1, 2, 4)\).

Lemma 11.3. As \(x \to \infty\), we have the following:
\[
\rho_{1,2}^{\mathrm{Ai}}(x) = \mathcal{O}\left(e^{-\frac{4}{3}x^{3/2}}\right),
\]
(11.2)
\[
\rho_{1,2}^{\mathrm{Ai}}(-x) = \frac{\sqrt{x}}{\pi}\{1 + \mathcal{O}(x^{-3/2})\}.
\]
(11.3)

Proof. From (1.8) and the continuity of \(K_{\mathrm{Ai},2}(x, y)\), we easily deduce that
\[
K_{\mathrm{Ai},2}(x, x) = (\mathrm{Ai}'(x))^{2} - x(\mathrm{Ai}(x))^{2}.
\]
(11.4)

Combining this with Lemma 11.1 yields Lemma 11.3 \(\square\)

Lemma 11.4. Let \(x \in \mathbb{R}\) be fixed. Then it holds that, as \(|y| \to \infty|\)
\[
\left|\int_{x}^{y} K_{\mathrm{Ai},2}(u, x)du\right| = \mathcal{O}(1),
\]
(11.5)
\[
K_{\mathrm{Ai},2}(x, y) = \mathcal{O}(|y|^{-3/4}),
\]
(11.6)
\[
\frac{\partial K_{\mathrm{Ai},2}}{\partial y}(x, y) = \mathcal{O}(|y|^{-1/4}).
\]
(11.7)

Proof. Remind the definition (1.8) of \(K_{\mathrm{Ai},2}(x, y)\). Since
\[
\int_{0}^{y} K_{\mathrm{Ai},2}(u, x)du = \int_{0}^{y} \frac{\mathrm{Ai}(u)\mathrm{Ai}'(x) - \mathrm{Ai}'(u)\mathrm{Ai}(x)}{u - x}du
\]
\[
= \mathrm{Ai}'(x) \int_{-x}^{y-x} \frac{\mathrm{Ai}(x + w)}{w}dw - \mathrm{Ai}(x) \int_{-x}^{y-x} \frac{\mathrm{Ai}'(x + w)w}{w}dw,
\]
(11.5) follows from Lemma 11.1 (11.6) follows from Lemma 11.1. Since
\[
\frac{\partial K_{\mathrm{Ai},2}}{\partial y}(x, y) = -\frac{\mathrm{Ai}'(x)\mathrm{Ai}'(y) + y\mathrm{Ai}(x)\mathrm{Ai}(y)}{x - y} + \frac{K_{\mathrm{Ai},2}(x, y)}{x - y},
\]
(11.7) follows from Lemma 11.1 \(\square\)
Lemma 11.5. Let $\beta = 1, 2, 4$ and $x \in \mathbb{R}$. There exists a positive constant $c_6$ such that, for $|y| > |x| + 1$,

$$|\rho_{\Lambda_i,\beta,x}^1(y) - \rho_{\Lambda_i,\beta}^1(y)| \leq c_6 \left\{ |y|^{-3/2} + 1 (\beta \neq 2) \right\}. \tag{11.8}$$

Here $1(\beta \neq 2) = 1$ for $(\beta = 1, 4)$ and $1(\beta \neq 2) = 0$ for $\beta = 2$.

**Proof.** If $\beta = 2$, from the estimate (11.6) we have

$$\rho_{\Lambda_i,\beta,x}^1(y) - \rho_{\Lambda_i,\beta}^1(y) = -K_{\Lambda_i,\beta}(x,y) \frac{K_{\Lambda_i,\beta}(x,y)}{\rho(x)} = O(|y|^{-3/2}).$$

If $\beta = 1$, we have

$$\rho_{\Lambda_i,1,x}^1(y) - \rho_{\Lambda_i,1}^1(y) = \frac{\partial J_1(x,y)}{\partial y} \left( \int_x^y J_1(u,x) \, du + \frac{\text{sign}(x-y)}{2} \right).$$

From Lemma 11.1, (11.5), (11.6) and (11.7),

$$\int_x^y J_1(u,x) \, du = O(1), \quad J_1(x,y)J_1(y,x) = O(|y|^{-1}), \quad |y| \to \infty,$$

and from Lemma 11.1 and (11.7)

$$\frac{\partial}{\partial y} J_1(x,y) = \frac{\partial}{\partial y} K_{\Lambda_i,2}(x,y) + \frac{\text{Ai}(x)\text{Ai}(y)}{2} = O(|y|^{-1/4}), \quad |y| \to \infty.$$

Then, the estimate (11.8) is obtained.

If $\beta = 4$, we have

$$2^{2/3} \left\{ \rho_{\Lambda_i,4,2^{-2/3}x}^1(2^{-2/3}y) - \rho_{\Lambda_i,4,2^{-2/3}x}^2(2^{-2/3}y) \right\} = 2^{2/3} K_{\Lambda_i,4}(2^{-2/3}x,2^{-2/3}y) K_{\Lambda_i,24}(2^{-2/3}y,2^{-2/3}x)$$

$$= J_4(x,y)J_4(y,x) - \frac{\partial J_4(x,y)}{\partial y} \int_x^y J_4(u,x) \, du.$$

Then the estimate (11.8) is obtained by the same argument as the case $\beta = 1$. \qed

12 Appendix 3: Estimates of Hermite polynomials

Let $\hat{H}_n(x) = (-1)^n e^{-x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$ ($n \in \{0\} \cup \mathbb{N}$) be the Hermite polynomials and $\phi_n(x)$ be the normalized oscillator functions defined by

$$\phi_n(x) = \frac{1}{\sqrt{\sqrt{2\pi}n!}} e^{-x^2/4} \hat{H}_n(x).$$
Then, for $n \in \mathbb{N}$, the distribution $\mu_{\text{bulk},2}^n$ of $n$ particles in GUE system is the determinantal point process with correlation kernel
\[
K_{\text{GUE}}^n(x, y) = \sqrt{n} \phi_n(x)\phi_{n-1}(y) - \phi_{n-1}(x)\phi_n(y)
\]

We also introduce the function defined by
\[
\psi_n(x) = n^{1/2} \phi_n \left( 2\sqrt{n} + \frac{x}{n^{1/6}} \right),
\]
which has the following properties (see for instance page 101 in [2]):
\[
\psi_n(x - 2n^{2/3}) \text{ is even or odd as } n \text{ is even or odd},
\]
\[
\int_{\mathbb{R}} \psi_{2k+1}(x) dx = 0, \quad \lim_{k \to \infty} \int_{\mathbb{R}} \psi_{2k}(x) dx = 2, \quad k \in \mathbb{N},
\]
\[
\psi''_n(x) = \left\{ \frac{x^2}{4n^{2/3}} + x + \frac{1}{2n^{1/3}} \right\} \psi_n(x)
\]
and
\[
n^{1/6} \psi'_n(x) = -\frac{\sqrt{n+1}}{2} \left( \frac{n}{n+1} \right)^{1/12} \psi_{n+1} \left( \left( \frac{n+1}{n} \right)^{1/6} x + \frac{2(n+1)^{1/6}}{\sqrt{n+1} + \sqrt{n}} \right) + \frac{\sqrt{n}}{2} \left( \frac{n}{n-1} \right)^{1/12} \psi_{n-1} \left( \left( \frac{n-1}{n} \right)^{1/6} x - \frac{2(n-1)^{1/6}}{\sqrt{n-1} + \sqrt{n}} \right).
\]

By means of the function $\psi_n$ the correlation kernel of $\mu_{\text{Ai},2}^n$ is written as
\[
K_{\text{Ai},2}^n(x, y) = \frac{n^{1/3} \psi_n(x)\psi_{n-1}(y) - \psi_{n-1}(x)\psi_n(y)}{x - y} = \frac{\psi_n(x)\psi'_n(y) - \psi'_n(x)\psi_n(y)}{x - y} - \frac{1}{2n^{1/3}} \psi_n(x)\psi_n(y).
\]

For $\beta = 1, 4$, the correlation kernel $K_{\text{Ai},\beta}^n$ of the determinantal point process $\mu_{\text{Ai},\beta}^n$ is defined as
\[
K_{\text{Ai},1}^n(x, y) = \left[ -\frac{1}{2} \text{sign}(x - y) + \int_y^x J_1^n(u, y) du \frac{-\partial}{\partial y} J_1^n(x, y) \right]
\]
with
\[
J_1^n(x, y) = K_{\text{Ai},2}^n(x, y) + \frac{1}{2} \psi_{n-1}(x)\psi_n(y) + \frac{\psi_{n-1}(x)}{\int_{\mathbb{R}} \psi_{n-1}(y) dy} \mathbf{1}(n \text{ is odd}),
\]
and by
\[
2^{-2/3} K_{\text{Ai},4}^n(2^{-2/3} x, 2^{-2/3} y) = \frac{1}{2} \left[ \int_y^x J_4^n(u, y) du \frac{-\partial}{\partial y} J_4^n(x, y) \right]
\]
with
\begin{equation}
J_n^a(x, y) = K_{A,2}^{2n+1}(x, y) + \frac{\sqrt{2n+1}}{2(2n)^{1/2}} \psi_{2n}(x) \varepsilon \psi_{2n+1}(y),
\end{equation}
respectively (refer to Section 3.9 in [2]). Here, for each integrable real-valued function \( f \) on \( \mathbb{R} \)
\[
(\varepsilon f)(x) = \int_\mathbb{R} \frac{1}{2} \text{sign}(x - y) f(y) dy = \frac{1}{2} \int_\mathbb{R} f(y) dy - \int_x^\infty f(y) dy.
\]

The asymptotic behaviors of Hermite polynomials given in [40] are summarized as follows.

**Lemma 12.1.** (i) When \( x \in (-2n^{2/3}, 0) \), for any \( L \in \mathbb{N} \)
\[
\psi_n(x) = \frac{1 + \mathcal{O}(1/n)}{\sqrt{\pi}} \left( \frac{2}{f(x, n)} \right)^{1/4} \sum_{k=0}^{L-1} \sum_{m=0}^k C_{km}(n, \theta) \cos(g_n(x) - c_{km}(\theta)) + \mathcal{O}(f(x, n)^{-10^{L+1}}),
\]
where \( \theta = \theta(x, n) \) is the value in \((0, \pi/2)\) satisfying \( x = 2n^{1/6}(\sqrt{n} + 1 \cos \theta - \sqrt{n}) \),
\[
C_{km}(n, \theta) = \frac{1 + (-1)^k}{2} \frac{\Gamma(m + \frac{k+1}{2})}{(n + 1)^{k/2}(\sin \theta)^{m+k/2}} a_{km}
\]
with constants \( a_{km} \) do not depend on \( n \) and \( \theta \), for instance \( a_{00} = 1 \), \( a_{10} = 0 \), and
\[
g_n(x) = \frac{n + 1}{2} (2\theta - \sin 2\theta), \quad c_{km}(\theta) = \frac{\theta}{2} - \left( m + \frac{k}{2} \right) \left( \frac{\pi}{2} + \theta \right),
\]
\[
f(x, n) = n^{2/3} \sin^2 \theta = -x + \frac{x}{n + 1} + \frac{n^{2/3}}{n + 1} - \frac{x^2 n^{1/3}}{4(n + 1)}.
\]

(ii) When \( x > 0 \), for any \( L \in \mathbb{N} \)
\begin{equation}
\psi_n(x) = \frac{1 + \mathcal{O}(n^{-1})}{\sqrt{4\pi \sinh \theta}} \left( \frac{1}{n} \right)^{1/4} \exp \left\{ \left( \frac{n + 1}{2} \right) (2\theta - \sinh 2\theta) \right\} \times \left[ \sum_{k=0}^{L-1} \sum_{m=0}^k \hat{C}_{km}(n, \theta) + \mathcal{O}(n^{-L/2} \left( \frac{1 - e^{-2\theta}}{2} \right)^{-3L/2}) \right].
\end{equation}
where \( \theta = \theta(x, n) \) is the value in \((0, \infty)\) satisfying \( x = 2n^{1/6}(\sqrt{n} + 1 \cosh \theta - \sqrt{n}) \),
\[
\hat{C}_{km}(n, \theta) = \frac{1 + (-1)^k}{2} \frac{\Gamma(m + \frac{k+1}{2})}{(n + 1)^{k/2}} \left( \frac{2}{1 - e^{-2\theta}} \right)^{m+k/2} a_{km}.
\]

(iii) When \( |x| = \mathcal{O}(n^\varepsilon) \) for some \( \varepsilon \in (0, 1/6) \),
\[
\psi_n(x) = B \left( x + n^{-1/3} \sqrt{2n} - \frac{x}{\sqrt{2n^{1/6}}} \right) + \mathcal{O}(n^{6\varepsilon - 1}),
\]
where
\[ B(x, u) = \text{Ai}(x) + \left( \frac{2}{u} \right)^{\frac{3}{2}} c_{12} x^2 \text{Ai}(x) + \left( \frac{2}{u} \right)^{\frac{5}{4}} \{ c_{21} x \text{Ai}(x) + c_{22} x^2 \text{Ai}'(x) + c_{24} x^4 \text{Ai}(x) \} \]
with some constants \( c_{\nu n}, c_{\nu m}, 0 \leq \nu, n \leq 2\nu \), which do not depend on \( x \) and \( u \).

**Lemma 12.2.** There is a positive constant \( c_7 \) such that
\[
\begin{align*}
(12.12) & \quad |\psi_n(x)| \leq c_7 |x|^{1/4}, \quad x \in [-2n^{2/3}, \infty), \quad n \in \mathbb{N}, \\
(12.13) & \quad |\psi_n'(x)| \leq c_7 |x|^{1/4}, \quad x \in [-2n^{2/3}, \infty), \quad n \in \mathbb{N}, \\
(12.14) & \quad |\psi_n''(x)| \leq c_7 |x|^{1/4}, \quad x \in [-2n^{2/3}, \infty), \quad n \in \mathbb{N}, \\
(12.15) & \quad \sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}} |\varepsilon \psi_n(x)| \leq c_7.
\end{align*}
\]

**Proof.** We prove this lemma by applying Lemma [12.1]. We take \( \varepsilon \in (0, 1/9) \) and consider the following three cases:

(I) \( x \in [-2n^{2/3}, -n^{\varepsilon}] \), (II) \( x \in [n^{\varepsilon}, \infty) \), and (III) \( x \in [-n^{\varepsilon}, n^{\varepsilon}] \).

For each case, the first claim \((12.12)\) is derived from Lemma [12.1] and the third claim \((12.14)\) is derived from \((12.12)\) with \((12.3)\).

For proving the second claim \((12.13)\), by virtue of \((12.5)\) with some calculation it is enough to show the following estimate:
\[
(12.16) \quad n^{1/3} |\psi_{n+1}(x + n^{-1/3}) - \psi_{n-1}(x - n^{-1/3})| \leq C'|x|^{1/4}, \quad x \in [-2n^{2/3}, \infty),
\]
for some positive constant \( C' \).

For the case (I) we use Lemma [12.1] (i) with \( L \) which is a positive integer greater than \( \frac{1}{3} \left( \frac{4}{\sin \theta} - 2 \right) \). Remind the relation \( 2\sqrt{n} + n^{-1/6} x = 2\sqrt{n} + \ell n^{-1/3} \), since \( \varepsilon \in (0, 1/9) \),
\[
(12.17) \quad |f(x, n)|^{1/4} \leq c|x|^{1/4 - 1/(3\varepsilon)} \leq cn^{-1/3}|x|^{1/4},
\]
for some positive constant \( c \). Note that for fixed \( \ell \in \mathbb{Z} \)
\[
2\sqrt{n} + n^{-1/6} x = \{2\sqrt{n} + \ell + (n + \ell)^{-1/6} \} (1 + \mathcal{O}(n^{-1})),
\]
and so
\[
2\sqrt{n} + \ell \cos \theta(x, n) = \{2\sqrt{n} + \ell \cos \theta(x + \ell n^{-1/3}, n + \ell)\} (1 + \mathcal{O}(n^{-1})).
\]

We put \( x_\ell = x + \ell n^{-1/3} \) and \( \theta_\ell = \theta(x + \ell n^{-1/3}, n + \ell) \). By the same calculation as in the proof of Lemma 5.2 of [21] we see that
\[
|\theta_1 - \theta_{-1}| = \mathcal{O} \left( \frac{1}{n \sin \theta_0} \right) = \mathcal{O}(n^{-2/3 - \varepsilon/2}),
\]
\[
|g_{n+1}(x_1) - g_{n-1}(x_{-1})| = 2\theta_0 + \mathcal{O} \left( \frac{1}{n \sin \theta_0} \right) = \mathcal{O} \left( \frac{|x|^{1/2}}{n^{1/3}} \right),
\]
\[
|C_{km}(\ell, \theta_\ell)| = \mathcal{O}(|x|^{-3k/4}), \quad \ell = \pm 1,
\]
\[
|C_{km}(n + \ell, \theta_1) - C_{km}(n, n + 1, \theta_{-1})| = \mathcal{O}(n^{-1/3}|x|^{1/2 - 3k/4}).
\]
Then we have

\[
\left(\frac{2}{f(x_1, n+1)}\right)^{1/4} \sum_{k=0}^{L-1} \sum_{m=0}^{k} C_{km}(n+1, \theta_1) \cos(g_n(x_1) - c_{km}(\theta_1)) \\
- \left(\frac{2}{f(x_{-1}, n-1)}\right)^{1/4} \sum_{k=0}^{L-1} \sum_{m=0}^{k} C_{km}(n-1, \theta_{-1}) \cos(g_{n-1}(x_{-1}) - c_{km}(\theta_{-1})) \\
= \mathcal{O}(n^{-1/3}|x|^{1/4}).
\]

By using the above estimate with (12.17) we have (12.16) from Lemma 12.1 (i).

For the case (II) we use Lemma 12.1 (ii), in which the relation \(2\sqrt{n} + n^{-1/6}x = 2\sqrt{n} + 1 \cosh \theta\) is used. Since

\[
cosh \theta = \sqrt{\frac{x}{n+1}} + \frac{x}{2n^{1/6}\sqrt{n+1}} = 1 + \frac{x}{2n^{2/3}} + \mathcal{O}(n^{-1}), \; n \to \infty,
\]

and

\[
sinh \theta = \sqrt{x} + \mathcal{O}\left(\frac{x}{n^{2/3}}\right), \; \theta = \sqrt{x} + \mathcal{O}\left(\frac{x}{n^{2/3}}\right),
\]

\[
2\theta - \sinh 2\theta = -\frac{4x^{3/2}}{3n} + \mathcal{O}\left(\frac{x^2}{n^{4/3}}\right), \; \frac{x}{n^{2/3}} \to 0,
\]

we see that when \(x = o(n^{2/3})\), \(n \to \infty\), and \(x \geq 1\),

\[
(12.18) \quad \psi_n(x) = \frac{1 + \mathcal{O}(1/n)}{2\sqrt{\pi}} \exp\left(-\left(\frac{2}{3} + o(1)\right)x^{3/2}\right), \; n \to \infty,
\]

from Lemma 12.1 (ii). If \(1/x = \mathcal{O}(n^{-2/3})\), \(n \to \infty\),

\[
(12.19) \quad |\psi_n(x)| \leq C \exp\left\{-cx^2\right\}
\]

with some constants \(c, C > 0\) from Lemma 12.1 (ii). Then we can readily obtain the desire estimate. For the case (III), let \(L\) be a positive integer greater than \(\frac{1}{3}(\frac{4}{3\varepsilon} - 2)\). Since \(1 - 6\varepsilon > 1/3\), (12.13) is derived from Lemma 12.1 (iii) and Lemma 11.1.

We prove the last claim (12.15). From the estimates (12.18) and (12.19) we have

\[
\left|\sup_{n \in \mathbb{N}} \sup_{x > 0} \int_0^x \psi_n(y)dy\right| < \infty.
\]

From (12.2), we also see that

\[
\left|\sup_{n \in \mathbb{N}} \sup_{-\infty < x < -4n^{2/3}} \int_0^x \psi_n(y)dy\right| < \infty.
\]
Then to derive (12.15), it is enough to show

\[(12.20) \quad \sup_{-2n^{2/3} \leq x \leq 0} \sup_{n \in \mathbb{N}} \int_{0}^{x} \psi_n(y) dy < \infty.\]

We use Lemma 12.4 (i) with \(L = 2\). Since \(C_{00} = \sqrt{\pi}, a_{00} = 1, a_{10} = 0\) and \(c_{00} = \frac{1}{2},\) we see that the integral of

\[\psi_n(x) - \left(\frac{2}{f(x, n)}\right)^{1/4} \cos\left(g_n(x) - \frac{\theta}{2}\right)\]

over \((-2n^{2/3}, 0]\) is bounded in \(n\). Since

\[\sup_{x < 0, n \in \mathbb{N}} \int_{x}^{0} \left(\frac{1}{f(x, n)}\right)^{1/4} \cos\left(g_n(x) - \frac{\theta}{2}\right) dx < \infty,\]

we obtain the desire result. \(\square\)

For \(a, b, c \in \mathbb{R}\) we introduce functions \(G\) and \(\hat{G}\) on \(\mathbb{R}^2\) defined by

\[G_{a,b,c}(x, y) = \frac{1}{|x|^a |y|^b |x - y|^c},\]

and

\[\hat{G}_{a,b,c}(x, y) = G_{a,b,c}(x, y) + G_{a,b,c}(y, x),\]

respectively. Note that \(G_{a,b,c}(x, y)G_{a', b', c'}(x, y) = G_{a+a', b+b', c+c'}(x, y),\)

**Lemma 12.3.** (i) There exist \(c_8 > 0\) such that for \(x, y \in [-2n^{2/3}, \infty), n \in \mathbb{N},\)

\[(12.21) \quad |K_{\text{Ai},2}^n(x, y)| \leq c_8 \hat{G}_{-1/4,1/4,1}(x, y),\]

\[(12.22) \quad \left|\frac{\partial K_{\text{Ai},2}^n(x, y)}{\partial y}\right| \leq c_8 \left\{\hat{G}_{-1/4,1/4,2}(x, y) + \hat{G}_{-3/4,1/4,1}(x, y)\right\}.\]

(ii) For each \(r > 0\) there exist \(c_9 > 0\) such that for any \(y \in [-2n^{2/3}, \infty), n \in \mathbb{N}\)

with \(|y| > r + 1,\)

\[(12.23) \quad \max_{x \in [-r, r]} |K_{\text{Ai},2}^n(x, y)| \leq c_9 |y|^{-3/4},\]

\[(12.24) \quad \max_{x \in [-r, r]} \left|\frac{\partial K_{\text{Ai},2}^n(x, y)}{\partial y}\right| \leq c_9 |y|^{-1/4}.\]

(iii) There exist \(c_{10} > 0\) such that

\[(12.25) \quad \sup_{y \in (-1, x+1)} |K_{\text{Ai},2}^n(x, y)| \leq c_{10} (|x|^{1/2} + 1), \quad x \in [-2n^{2/3}, \infty), n \in \mathbb{N},\]

\[(12.26) \quad \sup_{y \in (-1, x+1)} \left|\frac{\partial K_{\text{Ai},2}^n(x, y)}{\partial y}\right| \leq c_{10} (|x| + 1), \quad x \in [-2n^{2/3}, \infty), n \in \mathbb{N},\]

\[(12.27) \quad \sup_{y \in \mathbb{R}} \left|\int_{x}^{y} K_{\text{Ai},2}^n(u, x) du\right| \leq c_{10} \quad x \in \mathbb{R}, n \in \mathbb{N}.\]
Proof. To show the claims in (i), the identity (12.3) is used. From Lemma 12.2, the case that \(x, y \in [-2n^{2/3}, 2n^{2/3}]\) is derived by simple calculation. To derive the case that \(x\) or \(y\) is in \((2n^{2/3}, +\infty)\), (12.19) and (12.5) are used. The claims in (ii) is derived from (i).

To obtain (12.25) and (12.26) we use (12.4) in addition to Lemma 12.2 and apply Taylor’s theorem. The last claim (12.27) is derived from the same argument as that to obtain (11.5) and estimates in Lemma 12.1.

Lemma 12.4. Let \(\beta = 1, 2, 4\). For any \(x \in \mathbb{R}\) there exists \(c_{11} > 0\) such that for any \(y \in [-2n^{2/3}, +\infty), n \in \mathbb{N}\) with \(|y| > |x| + 1\)

\[
|\rho_{\beta, x}^{n}(y) - \rho_{\beta, y}^{n}(y)| \leq 11 \left\{ \frac{1}{|y|^{3/2}} + \frac{1}{|y|^{1/4}} \right\}. \tag{12.28}
\]

Proof. First note that from (4.2)

\[
|\rho_{\beta, x}^{n}(y) - \rho_{\beta, y}^{n}(y)| = \frac{K_{\beta, x}^{n}(x, y)K_{\beta, y}^{n}(y, x)}{\rho_{\beta, y}^{n}(x)}. \tag{12.29}
\]

In the case \(\beta = 2\), (12.28) is derived from (12.23). In the case that \(\beta = 1\) we assume that \(n\) is even. The other case can be treated similarly. From the definition (12.7) of the kernel \(K_{1,1}^{n}\) we have

\[
K_{1,1}^{n}(x, y)K_{1,1}^{n}(y, x) = J_{1}^{n}(x, y)J_{1}^{n}(y, x) - \frac{\partial J_{1}^{n}(x, y)}{\partial y} \left\{ \int_{x}^{y} J_{1}^{n}(u, x)du - \frac{1}{2} \text{sign}(y - x) \right\}. \tag{12.30}
\]

From Lemma 12.2 and Lemma 12.3 there exists \(C > 0\) which does not depend on \(n\) such that for \(x, y \in [-2n^{2/3}, +\infty)\)

\[
|J_{1}^{n}(x, y)| \leq C \left\{ \hat{G}_{-1/4,1/4,1}(x, y) + |x|^{-1/4} \right\}, \tag{12.30}
\]

\[
\left| \frac{\partial J_{1}^{n}(x, y)}{\partial y} \right| \leq C \left\{ \hat{G}_{-1/4,1/4,2}(x, y) + \hat{G}_{-3/4,1/4,1}(x, y) \right\}, \tag{12.31}
\]

and

\[
\int_{x}^{y} J_{1}^{n}(u, x)du \leq C, y \in \mathbb{R}. \tag{12.32}
\]

Then, we see that there exist \(C > 0\) which does not depend on \(n\), such that for \(x, y \in [-2n^{2/3}, +\infty)\)

\[
|K_{1,1}^{n}(x, y)K_{1,1}^{n}(y, x)| \leq C \left\{ \hat{G}_{-1/4,1/2,1}(x, y) + \hat{G}_{0,1/4,1}(x, y) + \hat{G}_{1/4,1/4,0}(x, y) + \hat{G}_{-3/4,1/4,1}(x, y) + \hat{G}_{-1/2,1/2,2}(x, y) + \hat{G}_{0,0,2}(x, y) \right\}. \tag{12.33}
\]
Then we obtain (12.28). In the case that \( \beta = 4 \), from the definition (12.9) of the kernel \( K_{\alpha,4}^n(x, y) \) we have

\[
2^{2/3} K_{\alpha,4}^n(x, y) K_{\alpha,4}^n(y, x) = J_{\alpha}^n(x, y) J_{\alpha}^n(y, x) - \frac{\partial J_{\alpha}^n(x, y)}{\partial y} \int_x^y J_{\alpha}^n(u, x) du.
\]

Then we can obtain (12.28) by the same argument as the case that \( \beta = 1 \). \( \square \)

Lemma 12.5. Suppose that \( g \) is a function on \([0, \infty)^2\) satisfying

\[
|g(x, y)| \leq CG_{a,b,c}(x, y), \quad \text{and} \quad \sup_{y \in (x-1, x+1)} g(x, y) \leq C(1 + |x|^\nu), \quad x, y > 0
\]

for some \( a, b, c \in \mathbb{R}, C > 0 \) and \( \nu > 0 \).

(i) Suppose that \( c \leq 1, a + b + c > 0 \) and \( \nu > 0 \). Then

\[
\lim_{s \to \infty} \int_s^\infty dx \int_0^\infty dy \frac{g(x, y)}{xy} = 0.
\]

(ii) Suppose that \( c > 1 \) and \( a + b + c > 0 \) and \( \nu \in \left(0, \frac{a+b+c}{c-1}\right) \). Then (12.36) holds.

Proof. First we prove (i). Take \( \gamma > (\nu - 1) \vee 0 \), and choose \( \kappa \in (0, a + b + c) \) such that \( \gamma < \frac{a+b+c-\kappa}{\kappa} \). Putting \( \delta = 1 - c + \kappa > 0 \), we obtain

\[
1 + a + 1 + b - \delta + c = 1 + a + b + c - \kappa > 1, \quad \text{and} \quad c + \delta = 1 + \kappa > 1.
\]

Since

\[
\frac{1}{|x|^{1+a}} \frac{1}{|x+w|^{b+1}} \frac{1}{|w|^c} \leq \frac{1}{|x|^{1+a+b+c-\kappa}} \frac{1}{|w|^{1+\kappa}},
\]

we have

\[
\int_s^\infty dx \int_0^\infty dy \frac{g(x, y)}{xy} = C \int_s^\infty dx \int_{x-\gamma}^\infty dw \frac{1}{|x|^{1+a}} \frac{1}{|x+w|^{1+b}} \frac{1}{|w|^c}
\]

\[
\leq C \int_s^\infty dx \int_{x-\gamma}^\infty dw \frac{1}{|x|^{1+a+b+c-\kappa}} \frac{1}{|w|^{1+\kappa}}
\]

\[
= \mathcal{O}(|s|^{-(a+b+c-\kappa-\nu)}) = o(1), \quad s \to \infty.
\]

From the second condition in (12.35)

\[
\int_s^\infty dx \int_0^\infty dy \frac{g(x, y)}{xy} \leq 2C \int_s^\infty dx x^{-2+\nu-\gamma}, \quad s \geq 1.
\]

Since \( \gamma > \nu - 1 \), we obtain (i).
Next we give the proof of (ii). In the case where $c > 1$
\[
\int_s^\infty dx \int_x^\infty dy \frac{g(x, y)}{xy} \leq C \int_s^\infty dx |x|^{-(2+\alpha+b)+(c-1)\gamma} = o(1), \quad s \to \infty,
\]
for any $\gamma < \frac{1+\alpha+b}{c-1}$, and
\[
\int_s^\infty dx \int_x^{x+y} dy \frac{g(x, y)}{xy} \leq C \int_s^\infty dx |x|^{-2+\nu-\gamma} = o(1), \quad s \to \infty,
\]
for any $\nu < 1 + \gamma < \frac{\alpha+\beta}{c-1}$. This completes the proof.

Proof of Lemma 4.5. Put $\Lambda_n = [-2n^{2/3}, \infty)$ and $x^* = -(4n^{2/3} + x)$ for $x \in \mathbb{R}$. Since $|\psi_n(x)| = |\psi_n(x^*)|$, $|\psi_n'(x)| = |\psi_n'(x^*)|$ and $|\psi_n''(x)| = |\psi_n''(x^*)|$ from (12.22), we can obtain the same estimates of $K_{\alpha,\beta}^n(x, y)$, $\tilde{K}_{\alpha,\beta}^n(x, y)$ and $K_{\alpha,\beta}^n(x, y^*)$ as that of $K_{\alpha,\beta}^n(x, y)$ by the same procedure. Then Lemma 1.5 is derived if
\[
\lim_{s \to \infty} \limsup_{n \to \infty} \sup_{|x| \leq r} \tilde{K}_{\alpha,\beta}^n(x, s) = 0 \quad \text{for all } k = 1, 2, 3, 4, 5.
\]
are shown, where $\tilde{K}_{\alpha,\beta}^n(x, s)$ is defined from $K_{\alpha,\beta}^n(x, s)$ by exchanging $K_{\alpha,\beta}^n(x, y)$ into $K_{\alpha,\beta}^n(x, y)1(x, y \in \Lambda_n) \equiv K_{\alpha,\beta}^n(x, y)$.

The claim (12.38) with $k = 1$ is easily derived from the estimate $\rho_{\alpha,\beta}^n(u) = \mathcal{O}(|u|^{1/2})$, $|u| \to \infty$, which is derived from Lemma 4.4.

We have
\[
\max_{x \in [-1, 1]} \left| \tilde{K}_{\alpha,\beta}^n(x, y) \tilde{K}_{\alpha,\beta}^n(y, x) \right| \leq C|y|^{-1/4}, \quad |y| > r + 1,
\]
from the estimate (12.23) in the case $\beta = 2$, and from the estimate (12.33) in the case $\beta = 1$. The case $\beta = 4$ is derived by the same argument as the case $\beta = 1$.

Then (12.33) with $k = 2, 3$ is derived from the above estimate (12.39).

In the case of $\beta = 2$, (12.38) with $k = 4, 5$ is derived by applying Lemma 12.5 with (12.21), (12.22) and (12.25). In the case that $\beta = 1$, from (12.3), (12.29) and Lemma 12.3 (iii) there exists $C > 0$ which does not depend on $n$ such that
\[
\sup_{y \in (x-1, x+1)} \max_{n \in \mathbb{N}} \left| \tilde{K}_{\alpha,\beta}^n(x, y) \tilde{K}_{\alpha,\beta}^n(y, x) \right| \leq C(1 + |x|), \quad x \in \mathbb{R}.
\]
From (12.38) and (12.40) we can obtain (12.38) with $k = 4$ by applying Lemma 12.5. The case $\beta = 4$ is derived by the same argument as the case $\beta = 1$.

Remark that $K_{\alpha,1}^n(u, x)K_{\alpha,1}^n(v, x)K_{\alpha,1}^n(v, u)$ is a scalar and equals
\[
J_\alpha^n(u, x)J_\alpha^n(v, x)J_\alpha^n(v, u) - J_\alpha^n(u, x) \frac{\partial J_\alpha^n(v, x)}{\partial v} \left\{ \int_u^v J_\alpha^n(t, u) dt - \frac{1}{2} \text{sign}(v - u) \right\}
- \frac{\partial J_\alpha^n(u, x)}{\partial x} \left\{ \int_u^x J_\alpha^n(t, v) dt - \frac{1}{2} \text{sign}(x - v) \right\} J_\alpha^n(v, u)
- \frac{\partial J_\alpha^n(u, x)}{\partial x} J_\alpha^n(x, v) \left\{ \int_u^v J_\alpha^n(t, u) dt - \frac{1}{2} \text{sign}(v - u) \right\}.
\]
Then we can obtain (12.38) with $k = 5$ by the same argument used in the above. The case $\beta = 4$ is derived by the same argument as the case $\beta = 1$.

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