Multiplicativity of Accessible Fidelity and Quantumness for Sets of Quantum States

K. M. R. Audenaert\textsuperscript{1}, C. A. Fuchs\textsuperscript{2,3}, C. King\textsuperscript{3,4}, A. Winter\textsuperscript{5}

\textsuperscript{1} University of Wales, Bangor
School of Informatics
Bangor (Gwynedd) LL57 1UT, Wales
E-mail: kauden@informatics.bangor.ac.uk

\textsuperscript{2} Bell Labs, Lucent Technologies
600-700 Mountain Ave.
Murray Hill, NJ 07974, USA
E-mail: cafuchs@research.bell-labs.com

\textsuperscript{3} Communication Networks Research Institute
Dublin Institute of Technology
Rathmines Road
Dublin 6, Ireland

\textsuperscript{4} Department of Mathematics
Northeastern University
Boston, MA 02115, USA
E-mail: king@neu.edu

\textsuperscript{5} Department of Computer Science
University of Bristol
Merchant Venturers Building
Woodland Road
Bristol BS8 1UB, England
E-mail: winter@cs.bris.ac.uk

1 August 2003

Abstract: Two measures of sensitivity to eavesdropping for alphabets of quantum states were recently introduced by Fuchs and Sasaki in \textit{quant-ph/0302092}. These are the accessible fidelity and quantumness. In this paper we prove an important property of both measures: They are multiplicative under tensor products. The proof in the case of accessible fidelity shows a connection between the measure and characteristics of entanglement-breaking quantum channels.

1. Introduction and statement of results

The security of quantum cryptography relies on the notion that any measurement on a quantum system causes a disturbance to it, thereby revealing the presence of an eavesdropper. However the idea that 'measurement causes disturbance' must be applied carefully in order to be useful. For example, given a state $|\psi\rangle$, the measurement which projects onto $|\psi\rangle$ and its orthogonal complement causes no disturbance to the state. Furthermore, if a signal is encoded using orthogonal states for different letters in an alphabet, then an eavesdropper can gain complete information by projecting onto those states, again without causing any changes in the signal. So in order to be successfully exploited for quantum cryptography (for example as in \cite{4}), an encoding scheme must use an ensemble of nonorthogonal signal states to prevent a disturbance-free measurement. In
other words, the sender cannot use a classical ensemble of states to implement quantum cryptography.

Thus for purposes of implementing quantum cryptography some ensembles are better than others. This raises the question of trying to quantify the 'amount of quantumness' in an ensemble of states. We will address one aspect of this question using the approach introduced in the paper [6]. (For a different approach, see [8].) The idea of the present approach is to consider the transmission of an ensemble of states from a sender to a receiver, and to see how easily an eavesdropper can be detected at participating in an intercept/resend strategy. Specifically, suppose that the sender draws states randomly from an ensemble 

\[ \mathcal{E} = \{ p_i, |\psi_i\rangle \} \]

After transmission the receiver obtains the ensemble 

\[ \mathcal{E}' = \{ p_i, |\psi_i\rangle' \} \]

In the absence of noise or an eavesdropper, these ensembles should have fidelity equal to 1. Recall that the fidelity is given by

\[ F = \sum_i p_i |\langle \psi_i | \psi_i' \rangle|^2 \]. \hspace{1cm} (1)

Now suppose that the eavesdropper is allowed to make any measurement on the intercepted states, that is any fixed POVM \{ \mathcal{E}_b \} can be applied. Based on the result of this measurement, the eavesdropper substitutes any other state \(|\phi_b\rangle\) in place of \(|\psi_i\rangle\) and sends this on to the receiver. The fidelity between the original ensemble and this new ensemble is

\[ F' = \sum_i \sum_b p_i \langle \psi_i | \mathcal{E}_b | \psi_i \rangle |\langle \psi_i | \phi_b \rangle|^2 \]. \hspace{1cm} (2)

In order to minimize her probability of remaining undetected, the eavesdropper should use a POVM and set of states that maximize (2). This leads to the following definition:

**Definition 1** Let \( \mathcal{E} = \{ p_i, |\psi_i\rangle \} \) be an ensemble of states. The accessible fidelity of \( \mathcal{E} \) is defined to be

\[ F(\mathcal{E}) = \sup_{\{ \mathcal{E}_b \}} \sup_{\{ |\phi_b\rangle \}} \sum_i \sum_b p_i \langle \psi_i | \mathcal{E}_b | \psi_i \rangle |\langle \psi_i | \phi_b \rangle|^2 \]. \hspace{1cm} (3)

Since \( F(\mathcal{E}) \) is the pointwise maximum of functions that are linear in the weights \( p_i \), it is a convex function of the \( p_i \). Because the set of possible weights \( \{ p_i \} \) is convex (more precisely, a simplex), the maximum value of \( F(\mathcal{E}) \) over all weights is achieved in an extreme point of the simplex [11]. These points are characterised by one of the \( p_i \) being 1 and all the others being 0. Thus

\[ \max_{\{ p_i \}} F(\mathcal{E}) = \max_i \sup_{\{ \mathcal{E}_b \}} \sup_{\{ |\phi_b\rangle \}} \sum_b p_i \langle \psi_i | \mathcal{E}_b | \psi_i \rangle |\langle \psi_i | \phi_b \rangle|^2 \]

\[ = 1 \].

The optimum is achieved by taking \( \{ \mathcal{E}_b \} = \{ I \} \) and \( |\phi_b\rangle = |\psi_i\rangle \) (for any choice of \( i \)). Hence the maximum of \( F(\mathcal{E}) \) over all ensembles is not particularly interesting.

On the other hand, there are nontrivial lower bounds for the accessible fidelity as a function of the \( \{ p_i \} \) [6]. In particular, the quantumness of a set of states \( \{|\psi_i\rangle\} \) provides an intrinsic and nontrivial character for the set itself.
Definition 2 The quantumness of a collection of states \( \{ |\psi_i\rangle \} \) is defined to be

\[
Q\left(\{ |\psi_i\rangle \} \right) = \inf_{\{p_i\}} F(\{p_i, |\psi_i\rangle\}) .
\] (4)

The quantumness specifies the best use that can be made of a set of states for revealing the existence of an eavesdropper: It is an inverted measure, the smaller the quantumness, the greater the departure from classical characteristics (since in the classical world an unconstrained eavesdropper cannot be detected at all).

The purpose of this paper is to show that both the accessible fidelity and the quantumness satisfy an important multiplicativity property for product structures. To be specific, given two ensembles \( E_1 = \{p_i, |\psi_i\rangle\} \) and \( E_2 = \{q_j, |\theta_j\rangle\} \), define the product ensemble \( E_1 \otimes E_2 \) by

\[
E_1 \otimes E_2 = \{p_i q_j, |\psi_i\rangle \otimes |\theta_j\rangle\} .
\] (5)

We prove the following two theorems:

Theorem 1 For any ensembles \( E_1 \) and \( E_2 \),

\[
F(E_1 \otimes E_2) = F(E_1) F(E_2) .
\] (6)

and

Theorem 2 For any collections \( \{ |\psi_i\rangle \} \) and \( \{ |\theta_j\rangle \} \),

\[
Q\left(\{ |\psi_i\rangle \otimes |\theta_j\rangle\} \right) = Q\left(\{ |\psi_i\rangle\} \right) Q\left(\{ |\theta_j\rangle\} \right) .
\] (7)

The significance of these theorems is the following. In the first case, imagine not a single shot through the eavesdropping channel, but rather a source that repeatedly generates states from the ensemble \( E \). One could imagine a smart eavesdropper who saves up multiple signals before performing her measurement on the chance that it will help her remain undetected. Our first theorem shows that this more complicated strategy provides no help. The second theorem makes a statement about optimal uses of an alphabet. It says, given a state preparation device that can only prepare states from a given collection \( \{ |\psi_i\rangle \} \), it is never in the sender’s interest to generate correlations between separate transmissions. In this way, quantumness is quite distinct from a channel capacity. For in contrast to channel capacity—where introducing correlation is generally necessary for achieving it—eavesdropping detection prefers uncorrelated signals. Theorem 1 and 2 together support the notion that accessible fidelity and quantumness are intrinsic properties of an ensemble and its underlying set of states.

The remainder of the paper is organized as follows. In Section 2 we lay out the basic ingredients required for proving the theorems. Following that, in Section 3 we prove Theorem 1, and in Section 4 we prove Theorem 2. We conclude in Section 5 with a small discussion about the potential implications of this work. In an Appendix we give a new proof of the multiplicativity of the maximal \( \infty \)-norm for entanglement breaking channels.
We describe here the two principal ingredients in the proof. The first ingredient is an application of the duality principal of convex analysis, which allows the accessible fidelity to be rewritten as an infimum over affine functions which majorize its value on the pure states. Similar ideas have been exploited recently in other areas of quantum information theory, in particular in the work on equivalence of additivity questions [3,13].

The second ingredient is an additivity result for a particular class of completely positive maps known as entanglement-breaking maps. This property was first established by Shor for the minimal entropy of maps [12], and later extended to the noncommutative $p$-norms for all $p \geq 1$ [9].

To describe the first ingredient, it is convenient to define the following completely positive map $\Phi$ associated with an ensemble $E = \{p_i, |\psi_i\rangle\}$:

$$\Phi(\rho) = \sum_i p_i \Pi_i \rho \Pi_i , \quad (8)$$

where $\Pi_i = |\psi_i\rangle\langle\psi_i|$. Then the accessible fidelity of an ensemble $E$ can be rewritten in terms of $\Phi$:

$$F(E) = \sup_{\{E_b\}} \sup_{\{|\phi_b\rangle\}} \sum_i \sum_b p_i \langle\psi_i | E_b |\psi_i\rangle \left| \langle\psi_i | \phi_b\rangle \right|^2$$

$$= \sup_{\{E_b\}} \sum_b \sup_{|\phi_b\rangle} \langle\phi_b | \Phi(E_b) |\phi_b\rangle$$

$$= \sup_{\{E_b\}} \sum_b \left| \left| \Phi(E_b) \right| \right| . \quad (9)$$

Furthermore, given a POVM $\{E_b\}$, we associate to it an ensemble of states $\{\alpha_b, \sigma_b\}$, given by

$$\alpha_b = \frac{1}{d} \text{Tr}(E_b) , \quad \sigma_b = \frac{1}{d\alpha_b} E_b , \quad (10)$$

where $d$ is the dimension of the state space. This defines a 1–1 correspondence between POVM’s and ensembles whose average state is $\frac{1}{d}I$. Hence (9) can be rewritten as a sup over such ensembles, that is

$$F(E) = d \sup \left\{ \sum_b \alpha_b \left| \left| \Phi(\sigma_b) \right| \right| : \sum_b \alpha_b \sigma_b = \frac{1}{d} I \right\} . \quad (11)$$

Introduce the following function on states:

$$g(\rho) = \left| \left| \Phi(\rho) \right| \right| . \quad (12)$$

This function is obviously convex. The concave closure of $g$ is defined as follows:

$$\hat{g}(\rho) = \sup_{\{\alpha_b, \sigma_b\}} \left\{ \sum_b \alpha_b g(\sigma_b) : \sum_b \alpha_b \sigma_b = \rho \right\} . \quad (13)$$
The concave closure of a function \( g \) is the smallest concave function on the set of all states that coincides with \( g \) on the pure states. Comparing with (11) we can see that

\[
F(\mathcal{E}) = d \hat{g}\left(\frac{1}{d}I\right).
\]  

(14)

By the dual formulation of the concave closure, \( \hat{g} \) can also be expressed as the infimum over all affine functions that dominate \( g \) \([11,3]\); that is

\[
\hat{g}(\rho) = \inf_X \left\{ \text{Tr}(X\rho) : \text{Tr}(X|\psi\rangle\langle\psi|) \geq g(|\psi\rangle\langle\psi|), \text{ all } |\psi\rangle \right\},
\]

(15)

where the infimum runs over all self-adjoint matrices.

Without loss of generality we can assume that the signal states span the state space, so that any \( X \) satisfying the conditions in (15) must be positive definite. We denote by \( \mathcal{F}(g) \) the collection of matrices which satisfy these conditions, and call this the feasible set for \( g \). So

\[
\hat{g}(\rho) = \inf_{X \in \mathcal{F}(g)} \text{Tr}(X\rho).
\]

(16)

For the second ingredient, recall that a completely positive map \( \Psi \) is entanglement breaking \([12]\) if it can be written in the form

\[
\Psi(\rho) = \sum_k R_k \text{Tr}(X_k\rho),
\]

(17)

where \( \{R_k\} \) and \( \{X_k\} \) are positive semidefinite. Comparing with (8) we can see that \( \Phi \) is entanglement breaking, where

\[
R_k = p_k \Pi_k, \quad X_k = \Pi_k.
\]

(18)

For any \( p \geq 1 \), the maximal \( p \)-norm of a CP map \( \Omega \) is defined by

\[
\nu_p(\Omega) = \sup_{|\psi\rangle} ||\Omega(|\psi\rangle\langle\psi|)||_p,
\]

(19)

where the \( p \)-norm of a matrix \( A \) is defined by

\[
||A||_p = \left( \text{Tr}(A^*A)^{p/2} \right)^{1/p}.
\]

(20)

The minimal output entropy of a trace preserving CP map is equal to the derivative of the maximal \( p \)-norm at \( p = 1 \):

\[
S_{\min}(\Omega) = \frac{d}{dp} \nu_p(\Omega) \bigg|_{p=1}.
\]

(21)

Shor proved that the minimal output entropy of a product channel is additive, provided that at least one of the channels is entanglement breaking \([12]\). It was later shown that the maximal \( p \)-norm of such a product channel is always multiplicative, for any \( p \geq 1 \) \([9]\). In fact, with a slight modification of the proof of \([9]\) one can show that multiplicativity also holds for general CP maps, not necessarily trace-preserving ones. In this paper we will make use of this latter
result for the case $p = \infty$. The proof presented in [9] uses the powerful Lieb-Thirring inequality [10] to derive the result for all $p$. It turns out that for the case $p = \infty$ there is a simpler method of proof which does not need this level of sophistication. Therefore we state this case as a separate Lemma below, and present its proof in the Appendix.

**Lemma 1** Let $\Phi$ be an entanglement-breaking CP map, and let $\Omega$ be any other CP map. Then

$$
\nu_{\infty}(\Phi \otimes \Omega) = \nu_{\infty}(\Phi) \nu_{\infty}(\Omega).
$$

(22)

3. Proof of Theorem 1

First we note that the inequality

$$
F(\mathcal{E}_1 \otimes \mathcal{E}_2) \geq F(\mathcal{E}_1) F(\mathcal{E}_2)
$$

(23)

follows immediately from the definition [11], since the fidelity of the product ensemble $\mathcal{E}_1 \otimes \mathcal{E}_2$ can only decrease by restricting to product POVM’s and product states $\phi_b$. So the Theorem reduces to proving the inequality

$$
F(\mathcal{E}_1 \otimes \mathcal{E}_2) \leq F(\mathcal{E}_1) F(\mathcal{E}_2).
$$

(24)

Let $\Phi_1$ and $\Phi_2$ denote the CP maps defined as in [8] for the two ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$. It follows that the corresponding CP map for the product ensemble $\mathcal{E}_1 \otimes \mathcal{E}_2$ is the product map $\Phi_1 \otimes \Phi_2$. As in (12) we define the associated functions

$$
g_i(\rho) = ||\Phi_i(\rho)||, \quad i = 1, 2, \quad g_{12}(\rho) = ||(\Phi_1 \otimes \Phi_2)(\rho)||.
$$

(25)

Now recall [14]. This implies

$$
F(\mathcal{E}_i) = d_i \hat{g}_i \left( \frac{1}{d_i} I \right), \quad i = 1, 2,
$$

(26)

where $d_i$ is the dimension of the state space for the ensemble $\mathcal{E}_i$. From [10] it follows that there are optimal self-adjoint matrices $X_1$ and $X_2$ belonging to the feasible sets for $g_1$ and $g_2$, respectively, such that

$$
F(\mathcal{E}_i) = \text{Tr} (X_i), \quad i = 1, 2,
$$

(27)

and also that

$$
F(\mathcal{E}_1 \otimes \mathcal{E}_2) = \inf_{X_{12} \in \mathcal{F}_{(g_{12})}} \text{Tr} (X_{12}).
$$

(28)

Assuming that $X_1 \otimes X_2 \in \mathcal{F}(g_{12})$, it follows that

$$
F(\mathcal{E}_1 \otimes \mathcal{E}_2) \leq \text{Tr} (X_1 \otimes X_2) = \text{Tr}(X_1) \text{Tr}(X_2)
$$

(29)

which gives the desired inequality [24].

So we are left with proving the assumption:

**Lemma 2** Let $X_1$ and $X_2$ belong to the feasible sets of $g_1$ and $g_2$ respectively. Then $X_1 \otimes X_2$ belongs to the feasible set of $g_{12}$.
Multiplicativity of accessible fidelity and quantumness

Proof: Recall that every matrix in the feasible set of $\mathcal{E}$ is positive definite. Given the two matrices $X_i \in \mathcal{F}(g_i)$, $i = 1, 2$, define the entanglement-breaking CP maps $\Omega_1$ and $\Omega_2$ by

$$
\Omega_i(\rho) = \Phi_i \left( X_i^{-1/2} \rho X_i^{-1/2} \right), \quad i = 1, 2.
$$

(30)

The feasibility of $X_i$ means that for all pure states $|\psi\rangle$:

$$
\text{Tr}[X_i |\psi\rangle \langle \psi|] \geq g_i(|\psi\rangle \langle \psi|) = ||\Phi_i(|\psi\rangle \langle \psi|)||.
$$

(31)

Substituting $|\psi\rangle = X_i^{-1/2} |\phi\rangle$,

$$
||\Omega_i(|\phi\rangle \langle \phi|)|| \leq 1, \quad i = 1, 2
$$

(32)

it follows that for any pure state $|\phi\rangle$

$$
||\Omega_i(|\phi\rangle \langle \phi|)|| \leq 1, \quad i = 1, 2
$$

(33)

and hence that

$$
\nu_\infty(\Omega_i) \leq 1.
$$

(34)

Hence from Lemma 1 we get

$$
|| (\Omega_1 \otimes \Omega_2)(|\psi_{12}\rangle \langle \psi_{12}|) || \leq 1 = \text{Tr} |\psi_{12}\rangle \langle \psi_{12}|
$$

(35)

for any pure state $|\psi_{12}\rangle$. This implies in turn that

$$
|| (\Phi_1 \otimes \Phi_2)(\rho_{12}) || \leq \text{Tr}[(X_1 \otimes X_2)\rho_{12}]
$$

(36)

for any bipartite state $\rho_{12}$. Hence $X_1 \otimes X_2$ is in the feasible set for $g_{12}$. \qed

4. Proof of Theorem 2

First, by restricting to product distributions it follows immediately from Theorem 1 that

$$
Q\left( \{ |\psi_i\rangle \otimes |\theta_j\rangle \} \right) \leq Q\left( \{ |\psi_i\rangle \} \right) Q\left( \{ |\theta_j\rangle \} \right).
$$

(37)

So it sufficient to prove the bound in the other direction.

We need to prove that for any joint distribution $\{p_{ij}\}$ on the collection of product states $\{ |\psi_i\rangle \otimes |\theta_j\rangle \}$, we have

$$
F(\{p_{ij}, |\psi_i\rangle \otimes |\theta_j\rangle \}) \geq Q\left( \{ |\psi_i\rangle \} \right) Q\left( \{ |\theta_j\rangle \} \right).
$$

(38)

Indeed, taking the infimum over all distributions $\{p_{ij}\}$ in (38) gives the inequality

$$
Q\left( \{ |\psi_i\rangle \otimes |\theta_j\rangle \} \right) \geq Q\left( \{ |\psi_i\rangle \} \right) Q\left( \{ |\theta_j\rangle \} \right),
$$

(39)

which together with (38) yields (7).
Since the accessible fidelity is a supremum over POVMs, to prove (38), it is sufficient to find a particular POVM \( \{M_{b,c}\} \) such that

\[
\sum_{b,c} \left\| \sum_{i,j} p_{ij} \left( \Pi_i \otimes \tilde{\Pi}_j \right) M_{b,c} \left( \Pi_i \otimes \tilde{\Pi}_j \right) \right\| \geq Q\left( \{|\psi_i\rangle\} \right) Q\left( \{|\theta_j\rangle\} \right),
\]

with \( \Pi_i = |\psi_i\rangle\langle\psi_i| \) and \( \tilde{\Pi}_j = |\theta_j\rangle\langle\theta_j| \). It will become clear further on why we have equipped \( M_{b,c} \) with two indices.

The POVM \( \{M_{b,c}\} \) is constructed in two steps. First, define the marginal distribution

\[
p_i = \sum_j p_{ij}.
\]

Let \( \{E_b\} \) be an optimal POVM realising \( F(\{p_i, |\psi_i\rangle\}) \), so that

\[
F(\{p_i, |\psi_i\rangle\}) = \sum_b \left\| \sum_i p_i \Pi_i E_b \Pi_i \right\|.
\]

For each \( b \), let \( |\phi_b\rangle \) be the dominating eigenvector of \( \sum_i p_i \Pi_i E_b \Pi_i \) so that

\[
\langle \phi_b | \sum_i p_i \Pi_i E_b \Pi_i | \phi_b \rangle = \left\| \sum_i p_i \Pi_i E_b \Pi_i \right\|.
\]

Define for every \( b \) a new distribution \( \{q_{b,j}\} \) by

\[
q_{b,j} = \frac{1}{N_b} \sum_i p_{ij} \langle \phi_b | \Pi_i E_b \Pi_i | \phi_b \rangle,
\]

where the normalisation constant is \( N_b = \sum_j \sum_i p_{ij} \langle \phi_b | \Pi_i E_b \Pi_i | \phi_b \rangle \). Remark that with this notation,

\[
F(\{p_i, |\psi_i\rangle\}) = \sum_b N_b.
\]

For each \( b \) let \( \{F_{b,c}\} \) be an optimal POVM realising \( F(\{q_{b,j}, |\theta_j\rangle\}) \), so that

\[
F(\{q_{b,j}, |\theta_j\rangle\}) = \sum_c \left\| \sum_j q_{b,j} \tilde{\Pi}_j F_{b,c} \tilde{\Pi}_j \right\|.
\]

and, for each \( b,c \), let \( |\chi_{b,c}\rangle \) be the dominating eigenvector of \( \sum_j q_{b,j} \tilde{\Pi}_j F_{b,c} \tilde{\Pi}_j \), so that

\[
\langle \chi_{b,c} | \sum_j q_{b,j} \tilde{\Pi}_j F_{b,c} \tilde{\Pi}_j | \chi_{b,c} \rangle = \left\| \sum_j q_{b,j} \tilde{\Pi}_j F_{b,c} \tilde{\Pi}_j \right\|.
\]
Hence,

\begin{align*}
F(\{q_{b,j}, |\theta_j\}) & = \sum_c \langle \chi_{b,c} | \sum_j q_{b,j} \tilde{\Pi}_j F_{b,c} \tilde{\Pi}_j | \chi_{b,c} \rangle \\
& = \sum_c \frac{1}{N_b} \sum_{i,j} p_{ij} \langle \phi_b | p_i \Pi_i E_b \Pi_i | \phi_b \rangle \langle \chi_{b,c} | q_{b,j} \tilde{\Pi}_j F_{b,c} \tilde{\Pi}_j | \chi_{b,c} \rangle \\
& = \frac{1}{N_b} \sum_c \langle \phi_b \otimes \chi_{b,c} | \sum_{i,j} p_{ij} (\Pi_i \otimes \tilde{\Pi}_j) E_b \otimes F_{b,c} (\Pi_i \otimes \tilde{\Pi}_j) | \phi_b \otimes \chi_{b,c} \rangle \\
& \leq \frac{1}{N_b} \sum_c \left\| \sum_{i,j} p_{ij} (\Pi_i \otimes \tilde{\Pi}_j) E_b \otimes F_{b,c} (\Pi_i \otimes \tilde{\Pi}_j) \right\| .
\end{align*}

(48)

Define yet another distribution \{r_b\} by

\begin{align*}
\sum_b r_b F(\{q_{b,j}, |\theta_j\}) & \leq \frac{1}{\sum_a N_a} \sum_{b,c} \left\| \sum_{i,j} p_{ij} (\Pi_i \otimes \tilde{\Pi}_j) E_b \otimes F_{b,c} (\Pi_i \otimes \tilde{\Pi}_j) \right\| .
\end{align*}

(49)

The POVM \{M_{b,c}\} is now defined by

\begin{align*}
M_{b,c} = E_b \otimes F_{b,c} .
\end{align*}

(50)

From combining all the above it follows that

\begin{align*}
F(\{p_i, |\psi_i\}) \sum_b r_b F(\{q_{b,j}, |\theta_j\}) & \leq \sum_{b,c} \left\| \sum_{i,j} p_{ij} (\Pi_i \otimes \tilde{\Pi}_j) M_{b,c} (\Pi_i \otimes \tilde{\Pi}_j) \right\| .
\end{align*}

(51)

Now the definition of quantumness implies that

\begin{align*}
F(\{p_i, |\psi_i\}) \geq Q(\{|\psi_i\})
\end{align*}

(52)

and

\begin{align*}
F(\{q_{b,j}, |\theta_j\}) \geq Q(\{|\theta_j\}) ,
\end{align*}

(53)

so that

\begin{align*}
\sum_b r_b F(\{q_{b,j}, |\theta_j\}) \geq Q(\{|\theta_j\}) ,
\end{align*}

(54)

and together with (51) this gives (40). \qed
5. Discussion

The present work clarifies two things. First, that both accessible fidelity and quantumness should be written in “single-letterized” expressions, as they were originally proposed. Second, Theorem 1 may lend some evidence to the idea that collective eavesdropping strategies need not be considered in a full quantum eavesdropping analysis after all—an idea that has been toyed with in the past [7]. If true, this would significantly relieve the technological requirements for operational systems in which unconditional security is sought.

Beyond this, one of the authors (CAF) is hopeful that these measures—particularly quantumness—will be useful to a certain line of attack in quantum foundations [5]. In that approach, a quantum state represents not an intrinsic property of a system, but rather an observer’s information—namely, the best information that can be had given that the components of the world have a certain fundamental sensitivity to the touch.

Acknowledgement. KA and AW thank CNRI in Dublin for its hospitality, where part of this work was performed. CF and CK were supported in part by Science Foundation Ireland under the National Development Plan. CK was also supported in part by National Science Foundation Grant DMS-0101205. AW was supported by the U.K. Engineering and Physical Sciences Research Council.

6. Appendix: Proof of Lemma 1

The proof follows the line of argument presented in [9], but replacing the Lieb-Thirring inequality with a simpler bound for the operator norm.

We show here that entanglement-breaking CP maps satisfy multiplicativity of the maximal ∞-norm. The maximal ∞-norm of a CP map $\Omega$ is defined as

$$\nu_\infty(\Omega) = \sup_{\rho} ||\Omega(\rho)|| ,$$

where the sup runs over all density matrices in the domain of $\Omega$. It is trivial to show that

$$\nu_\infty(\Psi \otimes \Omega) \geq \nu_\infty(\Psi) \nu_\infty(\Omega) .$$

Simply let $\rho_1$ and $\rho_2$ be states that achieve $\nu_\infty(\Psi)$ and $\nu_\infty(\Omega)$, respectively. Then $\rho_1 \otimes \rho_2$ is not necessarily optimal for $\nu_\infty(\Psi \otimes \Omega)$, so that

$$\nu_\infty(\Psi \otimes \Omega) \geq ||(\Psi \otimes \Omega)(\rho_1 \otimes \rho_2)|| = ||\Psi(\rho_1)|| ||\Omega(\rho_2)|| = \nu_\infty(\Psi) \nu_\infty(\Omega) .$$

Therefore, to prove the Lemma, we only need to show that

$$\nu_\infty(\Psi \otimes \Omega) \leq \nu_\infty(\Psi) \nu_\infty(\Omega) .$$

To set up the notation, consider the action of the map (17) on a bipartite state $\rho_{12}$:

$$\nu_\infty(\Psi \otimes \Omega) \leq \nu_\infty(\Psi) \nu_\infty(\Omega) .$$
and let
\[ \rho_{12} = (I \otimes \Omega)(\tau_{12}) . \] (57)
Then
\[ (\Psi \otimes I)(\rho_{12}) = (\Psi \otimes \Omega)(\tau_{12}) . \] (58)
Define
\[ x_k = \text{Tr}[(X_k \otimes I)\tau_{12}] \] (59)
\[ G_k' = \text{Tr}_1[(X_k \otimes I)\tau_{12}] / x_k \] (59)
\[ G_k = \Omega(G_k') = \text{Tr}_1[(X_k \otimes I)\rho_{12}] / x_k . \] (59)

Then (56) reads
\[ (\Psi \otimes I)(\rho_{12}) = \sum_{k=1}^K x_k R_k \otimes G_k \] (60)
\[ (\Psi \otimes I)(\tau_{12}) = \sum_{k=1}^K x_k R_k \otimes G_k' , \] (61)

where now \( \{ R_k, G_k \} \) are all positive matrices, \( G_k' \) is a density matrix and \( x_k \geq 0 \).
Writing \( \tau_1 = \text{Tr}_2(\tau_{12}) \) for the reduced density matrix it follows from (61) that
\[ \Psi(\tau_1) = \sum_{k=1}^K x_k R_k . \] (62)

Noting that for any Hermitian matrix \( X, X \leq ||X||I \), we have
\[ (\Psi \otimes \Omega)(\tau_{12}) = (\Psi \otimes I)(\rho_{12}) \]
\[ = \sum_{k=1}^K x_k R_k \otimes G_k \]
\[ \leq \sum_{k=1}^K x_k R_k \otimes ||G_k|| I \]
\[ \leq (\max_k ||G_k||) \sum_{k=1}^K x_k R_k \otimes I \]
\[ \leq (\max_k ||G_k||) \Psi(\tau_1) \otimes I . \] (63)

Now recollect that \( G_k = \Omega(G_k') \) and that \( G_k' \) is a density matrix. Therefore (63) implies that
\[ ||G_k|| \leq \nu_\infty(\Omega) \]
for any \( k \). Together with (63) and the fact that tensoring in the identity does not change the operator norm, this implies
\[ ||(\Psi \otimes \Omega)(\tau_{12})|| \leq \nu_\infty(\Omega) ||\Psi(\tau_1)|| . \] (64)
Using again (55) we get

\[ \| (\Psi \otimes \Omega)(\tau_{12}) \| \leq \nu_\infty(\Omega) \nu_\infty(\Psi). \]  

(65)

Since this bound holds for all \( \tau_{12} \) it follows that

\[ \nu_\infty(\Psi \otimes \Omega) \leq \nu_\infty(\Psi) \nu_\infty(\Omega). \]  

(66)

\[ \square \]

References

1. G. G. Amosov and A. S. Holevo, “On the multiplicativity conjecture for quantum channels”, math-ph/0103015.
2. G. G. Amosov, A. S. Holevo, and R. F. Werner, “On Some Additivity Problems in Quantum Information Theory”, Problems in Information Transmission, 36, 305–313 (2000).
3. K.M.R. Audenaert and S.L. Braunstein, “On strong superadditivity of the entanglement of formation”, quant-ph/0303045 (2003).
4. C.H. Bennett and G. Brassard, “Quantum Cryptography: Public Key Distribution and Coin Tossing”, Proceedings of IEEE International Conference on Computers Systems and Signal Processing, Bangalore India, December 1984, pp. 175-179.
5. C. A. Fuchs, “Quantum Mechanics as Quantum Information (and only a little more),” quant-ph/0205039
6. C. A. Fuchs and M. Sasaki, “Squeezing Quantum Information through a Classical Channel: Measuring the ‘Quantumness’ of a Set of Quantum States,” Quantum Information and Computation, 3, 377–404 2003. C. A. Fuchs and M. Sasaki, The Quantumness of a Set of Quantum States, in Proceedings of the Sixth International Conference on Quantum Communication, Measurement and Computing, edited by J. H. Shapiro and O. Hirota (Rinton Press, Princeton, NJ, 2003), pp. 475–480.
7. N. Gisin, G. Ribordy, W. Tittel, H. Zbinden, “Quantum Cryptography,” to appear in Rev. Mod. Phys., quant-ph/0101098
8. P. Hayden, R. Jozsa, and A. Winter, “Trading Quantum for Classical Resources in Quantum Data Compression,” J. Math. Phys. 43, 4404–4444 (2002).
9. C. King, “Maximal p-norms of entanglement-breaking channels,” Quantum Information and Computation, 3, 186–190 (2003).
10. E. Lieb and W. Thirring, “Inequalities for the Moments of the Eigenvalues of the Schrödinger Hamiltonian and Their Relation to Sobolev Inequalities,” in Studies in Mathematical Physics, E. Lieb, B. Simon, A. Wightman eds., pp. 269–303 (Princeton University Press, 1976).
11. R.T. Rockafellar, Convex Analysis, (Princeton University Press, Princeton, 1970).
12. P. W. Shor, “Additivity of the classical capacity of entanglement-breaking channels,” J. Math. Phys., 43, 4334–4340, 2002.
13. P. W. Shor, “Equivalence of additivity questions in quantum information theory,” quant-ph/0305035