 Bounds on the Growth of High Sobolev Norms of Solutions to 2D Hartree Equations

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Abstract. In this paper, we consider Hartree-type equations on the two-dimensional torus and on the plane. We prove polynomial bounds on the growth of high Sobolev norms of solutions to these equations. The proofs of our results are based on the adaptation to two dimensions of the techniques we had previously used in \cite{40, 41} to study the analogous problem in one dimension. Since we are working in two dimensions, a more detailed analysis of the resonant frequencies is needed, as in \cite{22}. As a corollary, we obtain bounds on the solutions of the cubic NLS on the plane, which improve bounds previously proved in \cite{16}.

1. Introduction.

1.1. Statement of the problem and of the main results: In this paper, we study the 2D Hartree initial value problem:

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} + \Delta u &= (V * |u|^2)u, \quad x \in \mathbb{T}^2 \text{ or } x \in \mathbb{R}^2, \quad t \in \mathbb{R} \\
u|_{t=0} &= \Phi \in H^s(\mathbb{T}^2), \text{ or } \Phi \in H^s(\mathbb{R}^2), \quad s > 1.
\end{aligned}
\end{equation}

The assumptions that we have on $V$ are the following:

(i) $V \in L^1(\mathbb{T}^2)$, or $V \in L^1(\mathbb{R}^2)$, respectively.

(ii) $V \geq 0$

(iii) $V$ is even.

The Hartree equation arises naturally in the dynamics of large quantum systems. It occurs in the context of the mean-field limit of $N$-body dynamics when we take $V$ to be the interaction potential \cite{26, 39}. It makes physical sense to consider this equation both in the periodic, and in the non-periodic setting.

The Hartree equation has the following conserved quantities:

$$M(u(t)) := \int |u(x,t)|^2 \, dx, \quad (\text{Mass})$$

$$E(u(t)) := \frac{1}{2} \int |\nabla u(x,t)|^2 \, dx + \frac{1}{4} \int (V * |u|^2)(x,t) |u(x,t)|^2 \, dx, \quad (\text{Energy}).$$

The region of integration is either $\mathbb{T}^2$ or $\mathbb{R}^2$, depending whether we are considering the periodic or the non-periodic setting. The fact that mass is conserved follows from the fact that $V$ is real-valued. The fact that energy is conserved follows from integration by parts, by using the fact that $V$ is even \cite{13}.

By using the two conservation laws, and by arguing as in \cite{29}, we can deduce global existence of (1) in $H^1$ and a priori bounds on the $H^s$ norm of a solution, in the non-periodic setting. By persistence of regularity, we obtain global existence in $H^s$, for $s > 1$. Hence, it makes sense to
analyze the behavior of \(\|u(t)\|_{H^s}\). A similar argument holds in the periodic setting, whereas here, we need to use periodic variants of Strichartz estimates [3].

Given a real number \(x\), we denote by \(x^+\) and \(x^-\) expressions of the form \(x + \epsilon\) and \(x - \epsilon\) respectively, where \(0 < \epsilon \ll 1\). With this notation, the result that we prove for (1) on \(\mathbb{T}^2\) is:

**Theorem 1.1.** (Bound for the Hartree equation on \(\mathbb{T}^2\)) Let \(u\) be the global solution of (1) on \(\mathbb{T}^2\). Then, there exists a function \(C_s\), continuous on \(H^1(\mathbb{T}^2)\) such that for all \(t \in \mathbb{R}\):

\[
\|u(t)\|_{H^s(\mathbb{T}^2)} \leq C_s(\Phi)(1 + |t|)^{s+} \|\Phi\|_{H^s(\mathbb{T}^2)}.
\]

Similarly, in the non-periodic setting one has:

**Theorem 1.2.** (Bound for the Hartree equation on \(\mathbb{R}^2\)) Let \(u\) be the global solution of (1) on \(\mathbb{R}^2\). Then, there exists a function \(C_s\), continuous on \(H^1(\mathbb{R}^2)\) such that for all \(t \in \mathbb{R}\):

\[
\|u(t)\|_{H^s(\mathbb{R}^2)} \leq C_s(\Phi)(1 + |t|)^{s+} \|\Phi\|_{H^s(\mathbb{R}^2)}.
\]

Heuristically, we expect to get a better bound in the non-periodic setting, due to the presence of stronger dispersion.

In the non-periodic setting, let us formally take \(V = \delta\). Then, (1) becomes:

\[
\begin{cases}
iu_t + \Delta u = |u|^2 u, & x \in \mathbb{R}^2, t \in \mathbb{R} \\
u|_{t=0} = \Phi \in H^s(\mathbb{R}^2), & s > 1.
\end{cases}
\]

The Cauchy problem (1) is also known to be globally well-posed in \(H^s\) [28]. We will see that the proof of Theorem 1.2 holds when we formally take \(V = \delta\). Hence, we also deduce the following:

**Corollary 1.3.** (Bound for the Cubic NLS on \(\mathbb{R}^2\)) Let \(u\) be the global solution of (1). Then, there exists a function \(C_s\), continuous on \(H^1(\mathbb{R}^2)\) such that for all \(t \in \mathbb{R}\):

\[
\|u(t)\|_{H^s(\mathbb{R}^2)} \leq C_s(\Phi)(1 + |t|)^{s+} \|\Phi\|_{H^s(\mathbb{R}^2)}.
\]

This improves the previously known bound \(\|u(t)\|_{H^s} \lesssim (1 + |t|)^{s+} \|\Phi\|_{H^s}\), for all \(s \in \mathbb{N}\). This bound was proved in [16]. Similarly, we can take \(V = \delta\) in the periodic setting. However, in this way, we obtain the bound \(\|u(t)\|_{H^s} \lesssim (1 + |t|)^{s+} \|\Phi\|_{H^s}\), which had been proved for all \(s \in \mathbb{N}\) in [10].

### 1.2. Motivation for the problem and previously known results

The growth of high Sobolev norms has a physical interpretation in the context of the *Low-to-High frequency cascade*. In other words, we see that \(\|u(t)\|_{H^s}\) weights the higher frequencies more as \(s\) becomes larger, and hence its growth gives us a quantitative estimate for how much of the support of \(|u|^2\) has transferred from the low to the high frequencies. This sort of problem also goes under the name *weak turbulence* [1] [2] [45].

By local well-posedness theory [6] [14] [29] [44], it can be observed that there exist \(C, \tau_0 > 0\), depending only on the initial data \(\Phi\) such that for all \(t\):

\[
\|u(t + \tau_0)\|_{H^s} \leq C\|u(t)\|_{H^s}.
\]

Iterating (6) yields the exponential bound:

\[
\|u(t)\|_{H^s} \leq Cte^{C_2t}.
\]

Here, \(C_1, C_2 > 0\) again depend only on \(\Phi\).
For a wide class of nonlinear dispersive equations, the analogue of (7) can be improved to a polynomial bound, as long as we take \( s \in \mathbb{N} \), or if we consider sufficiently smooth initial data. This observation was first made in the work of Bourgain [4], and was continued in [42, 43].

The crucial step in the mentioned works was to improve the iteration bound (6) to:

\[
\|u(t)\|_{H^s} \leq \|u(t_0)\|_{H^s} + C\|u(t_0)\|_{H^s}^{1-r}.
\]

As before, \( C, \tau_0 > 0 \) depend only on \( \Phi \). In this bound, \( r \in (0, 1) \) satisfies \( r \sim \frac{1}{s} \). One can show that (8) implies that for all \( t \in \mathbb{R} \):

\[
\|u(t)\|_{H^s} \leq C(\Phi)(1 + |t|)^{\frac{1}{2}}.
\]

In [4], (8) was obtained by using the Fourier multiplier method. In [42, 43], the iteration bound was obtained by using multilinear estimates in \( X^{s,b} \)-spaces. Similar estimates were used in [36] in the study of well-posedness theory. The key was to use a multilinear estimate in an \( X^{s,b} \)-space with negative first index. Such a bound was then used as a smoothing estimate. A slightly different approach, based on the analysis in [11], is used to obtain (8) in the context of compact Riemannian manifolds in [13], and [46].

An alternative iteration bound, based on the use of the upside-down I-method, which was used in our previous work [40, 11], gave better polynomial bounds for solutions of nonlinear Schrödinger equations on \( S^1 \) and \( \mathbb{R} \). The main idea was to consider the operator \( D \), related to \( D^s \) such that \( \|Du\|_{L^2} \) was slowly varying. This is the technique which we will apply in the present paper as well.

In the case of the linear Schrödinger equation with potential on \( \mathbb{T}^d \), better results are known. In [7], Bourgain studies the equation:

\[
iu_t + \Delta u = Vu.
\]

The potential \( V \) is taken to be jointly smooth in \( x \) and \( t \) with uniformly bounded partial derivatives with respect to both of the variables. It is shown that solutions to (10) satisfy for all \( \epsilon > 0 \) and all \( t \in \mathbb{R} \):

\[
\|u(t)\|_{H^s} \lesssim_{s, \Phi, \epsilon} (1 + |t|)^{\epsilon}
\]

The proof of (11) is based on separation properties of the eigenvalues of the Laplace operator on \( \mathbb{T}^d \).

Recently, a new proof of (11) was given in [24]. The argument given in this paper is based on an iterative change of variable. In addition to recovering the result (11) on any \( d \)-dimensional torus, the same bound is proved for the linear Schrödinger equation on any Zoll manifold, i.e. on any compact manifold whose geodesic flow is periodic. So far, it is an open problem to adapt any of these techniques to obtain bounds like (11) for nonlinear equations.

If we knew that (1) scattered in \( H^s \), we would immediately obtain uniform bounds on \( \|u(t)\|_{H^s} \). However, in the periodic setting, no scattering results have ever been proved, and one doesn’t expect them to hold due to limited dispersion. In the non-periodic setting, there are several known scattering results [27, 30, 31, 32, 34, 37], but none of them are strong enough to imply scattering in \( H^s \) for (1) on \( \mathbb{R}^2 \). For a detailed explanation, we refer the reader to Remark 4.6.

Let us finally mention that the problem of Sobolev norm growth was also recently studied in [29], but in the sense of bounding the growth from below. In this paper, the authors exhibit the existence of...
of smooth solutions of the cubic defocusing nonlinear Schrödinger equation on $\mathbb{T}^2$, whose $H^s$ norm is arbitrarily small at time zero, and is arbitrarily large at some large finite time. One should note that behavior at infinity is still an open problem.

1.3. Techniques of the proof. As was mentioned in the previous section, the main idea is to define $D$ to be an upside-down $I$-operator. This operator is defined as a Fourier multiplier operator. By construction, we will be able to relate $\|u(t)\|_{H^s}$ to $\|Du(t)\|_{L^2}$, so we consider the growth of the latter quantity. Following the ideas of the construction of the standard $I$-operator, as defined by Colliander, Keel, Staffilani, Takaoka, and Tao [17, 18, 19], our goal is to show that the quantity $\|Du(t)\|_{L^2}$ is slowly varying. This is done by applying a Littlewood-Paley decomposition and summing an appropriate geometric series. Let us remark that a similar technique was applied in the low-regularity context in [18].

As in our previous work [40, 41], we will use higher modified energies, i.e. quantities obtained from $\|Du(t)\|_{L^2}$ by adding an appropriate multilinear correction. In this way, we will obtain $E^2(u(t)) \sim \|Du(t)\|_{L^2}$, which is even more slowly varying. Due to more a more complicated resonance phenomenon in two dimensions, the construction of $E^2$ is going to be more involved than it was in one dimension. In the periodic setting, $E^2$ is constructed in Subsection 3.3. In the non-periodic setting, $E^2$ is constructed in Subsection 4.3.

We prove Theorem 1.1 and Theorem 1.2 for initial data $\Phi$, which we assume lies only in $H^s(\mathbb{T}^2)$ and $H^s(\mathbb{R}^2)$, respectively. We don’t assume any further regularity on the initial data. However, in the course of the proof, we work with $\Phi$ which is smooth, in order to make our formal calculations rigorous. The fact that we can do this follows from an appropriate Approximation Lemma (Proposition 3.2 and Proposition 4.2).

Organization of the paper:

In Section 2, we give some notation, and we recall some facts from Harmonic Analysis. In Section 3, we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. In the Appendix, we prove local-in-time bounds for (1) on the torus. The techniques mentioned in the Appendix apply to prove analogous bounds for (1) on the plane.

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2. Notation and known facts.

In our paper, we denote by $A \lesssim B$ an estimate of the form $A \leq CB$, for some $C > 0$. If $C$ depends on $d$, we write $A \lesssim_d B$. We also write the latter condition as $C = C(d)$.

We are taking the convention for the Fourier transform on $\mathbb{T}^2$ to be:

$$\hat{f}(n) := \int_{\mathbb{T}^2} f(x) e^{-i(x,n)} dx$$

On $\mathbb{R}^2$, we define the Fourier transform by:

$$\hat{f}(\xi) := \int_{\mathbb{R}^2} f(x) e^{-i(x,\xi)} dx$$

Here $n \in \mathbb{Z}^2$ and $\xi \in \mathbb{R}^2$. 

On $\mathbb{T}^2 \times \mathbb{R}$, we define the spacetime Fourier transform by:

$$\tilde{u}(n, \tau) := \int_{\mathbb{T}^2} \int_{\mathbb{R}} u(x, t) e^{-i\langle x, n \rangle - it\tau} dt dx.$$ 

On $\mathbb{R}^2 \times \mathbb{R}$, we define it by:

$$\tilde{u}(\xi, \tau) := \int_{\mathbb{R}^2} \int_{\mathbb{R}} u(x, t) e^{-i\langle x, \xi \rangle - it\tau} dt dx.$$ 

Let us take the following convention for the Japanese bracket $\langle \cdot \rangle$:

$$\langle x \rangle := \sqrt{1 + |x|^2}.$$ 

Let us recall that we are working in Sobolev Spaces $H^s(\mathbb{T}^2)$ on the torus, and $H^s(\mathbb{R}^2)$ on the plane, whose norms are defined for $s \in \mathbb{R}$ by:

$$\|f\|_{H^s(\mathbb{T}^2)} := \left( \sum_{n \in \mathbb{Z}^2} |\hat{f}(n)|^2 \langle n \rangle^{2s} \right)^{\frac{1}{2}}.$$ 

and

$$\|g\|_{H^s(\mathbb{R}^2)} := \left( \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \langle \xi \rangle^{2s} d\xi \right)^{\frac{1}{2}}.$$ 

Let us define:

$$H^\infty(\mathbb{T}^2) := \bigcap_{s > 0} H^s(\mathbb{T}^2)$$ 

and

$$H^\infty(\mathbb{R}^2) := \bigcap_{s > 0} H^s(\mathbb{R}^2).$$ 

An important tool in our work will also be $X^{s,b}$ spaces. We recall that these spaces come from the norm defined for $s, b \in \mathbb{R}$:

$$\|u\|_{X^{s,b}(\mathbb{T}^2 \times \mathbb{R})} := \left( \sum_{n \in \mathbb{Z}^2} \int_{\mathbb{R}} |\tilde{u}(n, \tau)|^2 \langle n \rangle^{2s} \langle \tau - |n|^2 \rangle^{2b} d\tau \right)^{\frac{1}{2}}.$$ 

and

$$\|u\|_{X^{s,b}(\mathbb{R}^2 \times \mathbb{R})} := \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}} |\tilde{u}(\xi, \tau)|^2 \langle \xi \rangle^{2s} \langle \tau - |\xi|^2 \rangle^{2b} d\xi d\tau \right)^{\frac{1}{2}}.$$ 

When there is no confusion, we write these spaces just as $H^s$ and $X^{s,b}$.

In our proofs, we will frequently have to use Littlewood-Paley decompositions. Given a function $u \in L^2(\mathbb{T}^2)$ and a dyadic integer $N$, we define by $u_N$ the function obtained from $u$ by restricting its Fourier transform to the dyadic annulus $|n| \sim N$. Hence, we have:

$$u = \sum_N u_N.$$ 

We analogously define $v_N$ for $v \in L^2(\mathbb{R}^2)$.

Having defined the spaces in which we will be working, let us recall some estimates which we will use in our analysis.
2.1. Estimates on $\mathbb{T}^2$. By Sobolev embedding on $\mathbb{T}^2$, we know that, for all $2 \leq q < \infty$, one has:

\[(12) \quad \|u\|_{L^q} \lesssim \|u\|_{H^1}\]

From \[33\], we know that on $\mathbb{T}^2$:

\[(13) \quad \|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0+\frac{1}{2}+}}\]

(A similar local-in-time estimate was earlier noted in \[3\].)

By definition, one has:

\[(14) \quad \|u\|_{L^2_{t,x}} = \|u\|_{X^{0,0}}\]

From Sobolev embedding, it follows that:

\[(15) \quad \|u\|_{L^\infty_{t,x}} \lesssim \|u\|_{X^{1+\frac{1}{2}+}}\]

If we take the $\frac{1}{2}+$ in \[13\] to be very close to $\frac{1}{2}$, we can interpolate between \[13\] and \[14\] to deduce:

\[(16) \quad \|u\|_{L^+_{t,x}} \lesssim \|u\|_{X^{0+\frac{1}{2}+}}\]

Similarly, we can interpolate between \[13\] and \[15\] to obtain:

\[(17) \quad \|u\|_{L^+_{t,x}} \lesssim \|u\|_{X^{0+\frac{1}{2}+}}\]

Let $c < d$ be real numbers, and let us denote by $\chi = \chi(t) = \chi_{[c,d]}(t)$. One then has, for all $s \in \mathbb{R}$, and for all $b < \frac{1}{2}$:

\[(18) \quad \|\chi u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b+}}\]

The proof of \[18\] is the same as the proof of Lemma 2.1. in \[10\] (see also \[12\] \[20\]). From the proof, we note that the implied constant is independent of $c$ and $d$. We omit the details.

We can interpolate between \[13\] and \[15\] to deduce that, for $M \gg 1$, one has:

\[(19) \quad \|u\|_{L^M_{t,x}} \lesssim \|u\|_{X^{1+\frac{1}{2}+}}\]

Furthermore, from Sobolev embedding in time, we know that:

\[(20) \quad \|u\|_{L^\infty_{t} L^2_{x}} \lesssim \|u\|_{X^{0+\frac{1}{2}+}}\]

We can interpolate between \[13\] and \[20\] to obtain:

\[(21) \quad \|u\|_{L^+_{t} L^2_{x}} \lesssim \|u\|_{X^{0+\frac{1}{2}+}}\]

An additional estimate we will use is:
The estimate (22) is a consequence of the following:

**Lemma 2.1.** Suppose that $Q$ is a ball in $\mathbb{Z}^2$ of radius $N$, and center $n_0$. Suppose that $u$ satisfies $\text{supp } \hat{u} \subseteq Q$. Then, one has:

\begin{equation}
\|u\|_{L^4_{t,x}} \lesssim N^\frac{1}{2} \|u\|_{X^{0,\frac{1}{4}+}}
\end{equation}

Lemma 2.1 is proved in [6] by using the Hausdorff-Young inequality and Hölder’s inequality. We omit the details.

To deduce (22), we write $u = \sum_N u_N$. By the triangle inequality and Lemma 2.1, we obtain:

\begin{equation}
\|u\|_{L^4_{t,x}} \leq \sum_N \|u_N\|_{L^4_{t,x}} \lesssim \sum_N N^{\frac{1}{2}} \|u_N\|_{X^{0,\frac{1}{4}+}}
\end{equation}

\begin{equation}
\lesssim \sum_N \frac{1}{N^{0}}\|u_N\|_{X^{0,\frac{1}{4}+}} \lesssim \|u\|_{X^{\frac{1}{8},\frac{1}{4}+}}
\end{equation}

We can now interpolate between (13) and (22) to deduce:

\begin{equation}
\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{s_1,0}}
\end{equation}

whenever $\frac{1}{4} < b_1 < \frac{1}{8} + , s_1 > 1 - 2b_1$.

By using an appropriate change of summation, as in [6], we see that (24) implies:

**Lemma 2.2.** Suppose that $u$ is as in the assumptions of Lemma 2.1 and suppose that $b_1, s_1 \in \mathbb{R}$ satisfy $\frac{1}{4} < b_1 < \frac{1}{8} + , s_1 > 1 - 2b_1$. Then, one has:

\begin{equation}
\|u\|_{L^4_{t,x}} \lesssim N^{s_1} \|u\|_{X^{0,b_1}}
\end{equation}

2.2. **Estimates on** $\mathbb{R}^2$. We note that all the mentioned estimates in the periodic setting carry over to the non-periodic setting. However, there are some estimates which hold only in the non-periodic setting, which express the fact that the dispersion phenomenon is stronger on $\mathbb{R}^2$ than on $\mathbb{T}^2$. Such estimates allow us to get a better bound in Theorem 1.2 than the one we obtained in Theorem 1.1.

The first modification is that, on the plane, (13) is improved to:

\begin{equation}
\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{4}+}}
\end{equation}

Consequently, one can improve (16) to:

\begin{equation}
\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{4}+}}
\end{equation}

On the plane, we will use the following estimate:

\begin{equation}
\|u\|_{L^{2+}_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{4}+}}
\end{equation}

follows from (20), the fact that $\|u\|_{L^{2+}_{t,x}} = \|u\|_{X^{0,0}},$ and interpolation.

Furthermore, a key fact is the following result, which was first noted by Bourgain in [5]:

(28)
Proposition 2.3. (Improved Strichartz Estimate) Suppose that $N_1, N_2$ are dyadic integers such that $N_1 \gg N_2$, and suppose that $u, v \in X^{\frac{1}{2}+}_{t,x}$ satisfy, for all $t$: $\text{supp} \hat{u}(t) \subseteq \{ |\xi| \sim N_1 \}$, and $\text{supp} \hat{v}(t) \subseteq \{ |\xi| \sim N_2 \}$. Then, one has:

\begin{equation}
\|uv\|_{L^2_{t,x}} \lesssim \frac{N_1^2}{N_2^2} \|u\|_{X^{\frac{1}{2}+}_{t,x}} \|v\|_{X^{\frac{1}{2}+}_{t,x}}.
\end{equation}

An alternative proof (in the 1D case) is given in [17].

Let us note the following corollary of Proposition 2.3.

Corollary 2.4. Let $u, v \in X^{\frac{1}{2}+}_{t,x}$ be as in the assumptions of Proposition 2.3. Then one has:

\begin{equation}
\|uv\|_{L^2_{t,x}} \lesssim \frac{N_1^2}{N_2^2} \|u\|_{X^{\frac{1}{2}+}_{t,x}} \|v\|_{X^{\frac{1}{2}+}_{t,x}}.
\end{equation}

Proof. We observe that:

\begin{equation}
\|uv\|_{L^\infty_{t,x}} \lesssim \|u\|_{L^\infty_{t,x}} \|v\|_{L^\infty_{t,x}} \lesssim \frac{N_1^2}{N_2^2} \|u\|_{L^\infty_{t,x}} \|v\|_{L^\infty_{t,x}}.
\end{equation}

In order to deduce this bound, we used Bernstein's inequality, and the non-periodic analogue of (20).

For completeness, we recall Bernstein's inequality [44]. Namely, if $1 \leq p \leq q \leq \infty$, and if $f \in L^p(\mathbb{R}^2)$ satisfies $\text{supp} \hat{f} \subseteq \{ |\xi| \sim N \}$, then one has:

\begin{equation}
\|f\|_{L^q_{t,x}} \lesssim N^{\frac{2}{q} - \frac{2}{p}} \|f\|_{L^p_{t,x}}.
\end{equation}

We interpolate between (20) and (31) and the Corollary follows.

In our analysis, we will have to work with $\chi = \chi_{[t_0, t_0 + \delta]}(t)$, the characteristic function of the time interval $[t_0, t_0 + \delta]$. It is difficult to deal with $\chi$ directly, since this function is not smooth, and since its Fourier transform doesn’t have a sign. Instead, we will decompose $\chi$ as a sum of two functions which are easier to deal with. This goal will be achieved by using an appropriate approximation to the identity. We will use the following decomposition, which is originally found in [17]:

Given $\phi \in C_0^\infty(\mathbb{R})$, such that: $0 \leq \phi \leq 1$, $\int_\mathbb{R} \phi(t) \, dt = 1$, and $\lambda > 0$, we recall that the rescaling $\phi_\lambda$ of $\phi$ is defined by:

$$
\phi_\lambda(t) := \frac{1}{\lambda} \phi\left(\frac{t}{\lambda}\right).
$$

We observe that such a rescaling preserves the $L^1$ norm:

\begin{equation}
\|\phi_\lambda\|_{L^1_t} = \|\phi\|_{L^1_t}.
\end{equation}

Having defined the rescaling, we write, for the scale $N > 1$:

\begin{equation}
\chi(t) = a(t) + b(t), \text{ for } a := \chi \ast \phi_{N^{-1}}.
\end{equation}
In Lemma 8.2. of [17], the authors note the following estimate:

\[ \|a(t)f\|_{X^{0,\frac{1}{2}+}} \lesssim N_1^{0+}\|f\|_{X^{0,\frac{1}{2}+}}. \]  

(The implied constant here is independent of \(N_1\).)

On the other hand, for any \(M \in (1, +\infty)\), one obtains:

\[ \|b\|_{L^M_t} = \|\chi - \chi \ast \phi_{N-1}\|_{L^M_t} \leq \|\chi\|_{L^M_t} + \|\chi \ast \phi_{N-1}\|_{L^M_t} \]

which is by Young’s inequality:

\[ \leq \|\chi\|_{L^M_t} + \|\chi\|_{L^M_t}\|\phi_{N-1}\|_{L^1_t} = 2\|\chi\|_{L^M_t} = C(M, \chi). \]

If we now define:

\[ \|b\|_{L^M_t} \leq C(M, \chi) = C(M, \Phi). \]

To explain the fact that \(C(M, \chi) = C(M, \Phi)\), we note that \(\chi\) is defined as the characteristic function of an interval of size \(\delta\), and \(\delta\), in turn, depends only on \(\Phi\).

We will frequently use the following consequence of Proposition 2.3

**Proposition 2.5.** (Improved Strichartz Estimate with rough cut-off in time) Let \(u, v \in X^{0,\frac{1}{2}+}(\mathbb{R}^2 \times \mathbb{R})\) satisfy the assumptions of Proposition 2.3. Suppose that \(N_1 \gtrsim N\). Let \(u_1, v_1\) be given by:

\[ \tilde{u}_1 := |(\chi u)|, \tilde{v}_1 := |v|. \]

Then one has:

\[ \|u_1v_1\|_{L^2_{t,x}} \lesssim \frac{N_1^{\frac{3}{2}}}{N_1^\frac{1}{2}}\|u\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}}. \]

The same bound holds if

\[ \tilde{u}_1 := |\tilde{u}|, \tilde{v}_1 := |(\chi v)|. \]

Proposition 2.5 follows from Proposition 2.3 Corollary 2.4 the decomposition 33, and the estimates associated to this decomposition. We omit the details of the proof. An analogous statement is proved in one dimension in [41]. The only difference is that on \(\mathbb{R}^2\), the coefficient on the right-hand side of 29 is \(\frac{N_1^{\frac{3}{2}}}{N_1^\frac{1}{2}}\), instead of \(\frac{1}{N_1^\frac{1}{2}}\), and hence we obtain the coefficient \(\frac{N_1^{\frac{3}{2}}}{N_1^\frac{1}{2}}\) on the right-hand side of 37.

We also must consider estimates on the product \(uv\), when \(u, v\) are localized in dyadic annuli as before, but when we no longer assume that \(N_1 \gg N_2\).

By using Hölder’s inequality and 28, it follows that:

\[ \|uv\|_{L^2_{t,x}} \leq \|u\|_{L^4_{t,x}}\|v\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}}. \]

We note that 31 still holds. We now interpolate between 31 and 38 to deduce:

\[ \|uv\|_{L^{2+}_{t,x}} \lesssim N_1^{0+}N_2^{0+}\|u\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}}. \]
An additional form of a bilinear Strichartz Estimate that we will have to use will be the following bound, which was first observed in \cite{22}:

**Proposition 2.6.** (Angular Improved Strichartz Estimate) Let $0 < N_1 \leq N_2$ be dyadic integers, and suppose $\theta_0 \in (0, 1)$. Suppose $v_j \in X^{0, \frac{1}{j+1}}, j = 1, 2$ satisfy: supp$v_j \subseteq \{|\xi| \sim N_j\}$. Then the function $F$ defined by:

$$F(t, x) := \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(t_1 + \tau_2 + i(x, \xi_1 + \xi_2)} |\xi|^\theta \hat{v}_1(\xi_1, \tau_1) \hat{v}_2(\xi_2, \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2$$

obeys the bound:

$$\|F\|_{L^2_x} \lesssim \theta_0^{\frac{1}{2}} \|v_1\|_{X^{0, \frac{1}{2}}} \|v_2\|_{X^{0, \frac{1}{2}}}$$

For the proof of Proposition 2.6 we refer the reader to the proof of Lemma 8.2. in \cite{22}.

Let us give some useful notation for multilinear expressions, which can also be found in \cite{17, 21}. Let us first consider the periodic setting. For $k$ and suppose $\theta \in [0, 1)$.

Finally, let us recall the following Calculus fact, which is often referred to as the *Double Mean Value Theorem*:

**Proposition 2.7.** Let $f \in C^2(\mathbb{R})$. Suppose that $x, \eta, \mu \in \mathbb{R}^2$ are such that: $|\eta|, |\mu| \ll |x|$. Then, one has:

$$|f(x + \eta + \mu) - f(x + \eta) - f(x + \mu) + f(x)| \lesssim |\eta||\mu|\|\nabla^2 f(x)\|.$$
3. The Hartree equation on \( T^2 \).

3.1. Definition of the \( \mathcal{D} \)-operator. As in our previous work \[40, 41\], we want to define an upside-down \( I \) operator. We start by defining an appropriate multiplier:

Suppose \( N > 1 \) is given. Let \( \theta : \mathbb{Z}^2 \rightarrow \mathbb{R} \) be given by:

\[
\theta(n) := \begin{cases} \left( \frac{|n|}{N} \right)^s, & \text{if } |n| \geq N \\ 1, & \text{if } |n| \leq N \end{cases}
\]

Then, if \( f : T^2 \rightarrow \mathbb{C} \), we define \( \mathcal{D}f \) by:

\[
\hat{\mathcal{D}}f(n) := \theta(n) \hat{f}(n).
\]

We observe that:

\[
\|\mathcal{D}f\|_{L^2} \lesssim s \|f\|_{H^s} \lesssim s \|\hat{\mathcal{D}}f\|_{L^2}.
\]

Our goal is to then estimate \( \|\mathcal{D}u(t)\|_{L^2} \), from which we can estimate \( \|u(t)\|_{H^s} \) by \[45\]. In order to do this, we first need to have good local-in-time bounds.

3.2. Local-in-time bounds. Let \( u \) denote the global solution to (1) on \( T^2 \). One then has:

**Proposition 3.1.** (Local-in-time bounds for the Hartree equation on \( T^2 \))

There exist \( \delta = \delta(s, E(\Phi), M(\Phi)) > 0 \), which are continuous in energy and mass, such that for all \( t_0 \in \mathbb{R} \), there exists a globally defined function \( v \) such that:

\[
v|_{[t_0, t_0 + \delta]} = u|_{[t_0, t_0 + \delta]}.
\]

\[
\|v\|_{X^{1, \frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi))
\]

\[
\|\mathcal{D}v\|_{X^{0, \frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi))\|Du(t_0)\|_{L^2}.
\]

Proposition 3.1 is similar to local-in-time bounds we had to prove in \[40, 41\]. Since we are working in two dimensions, the proof is going to be a little different. Our proof of Proposition 3.1 is similar to the proof of Theorem 2.7. in Chapter V of \[6\]. For completeness, we present it in the Appendix.

As in \[40\], Proposition 3.1 implies the following:

**Proposition 3.2.** (Approximation Lemma for the Hartree equation on \( T^2 \))

If \( \Phi \) satisfies:

\[
\begin{cases}
iu + \Delta u = (V * |u|^2)u, \\
u(x, 0) = \Phi(x).
\end{cases}
\]

and if the sequence \( (u^{(n)}) \) satisfies:

\[
\begin{cases}
iu^{(n)} + \Delta u^{(n)} = (V * |u^{(n)}|^2)u^{(n)}, \\
u^{(n)}(x, 0) = \Phi_n(x).
\end{cases}
\]

where \( \Phi_n \in C^\infty(T^2) \) and \( \Phi_n \overset{H^s}{\longrightarrow} \Phi \), then, one has for all \( t \):

\[
u^{(n)}(t) \overset{H^s}{\longrightarrow} u(t).
\]
The mentioned approximation Lemma allows us to work with smooth solutions and pass to the limit in the end. Namely, we note that if we take initial data $\Phi_n$ as earlier, then $u^{(n)}(t)$ will belong to $H^\infty(\mathbb{T}^2)$ for all $t$. This allows us to rigorously justify all of our calculations. Now, we want to argue by density. For this, we first need to know that energy and mass are continuous on $H^1$.

The fact that mass is continuous on $H^1$ is obvious. To see that energy is continuous on $H^1$, let $1 = \frac{1}{p^*} + \frac{1}{w}$. Then, by Hölder’s inequality, Young’s inequality, and (52), we obtain:

$$
\int (V * (u_1 u_2)) u_3 u_4 dx \leq ||V * (u_1 u_2)||_{L^2} ||u_3 u_4||_{L^2}
$$

$$
\leq ||V||_{L^1} ||u_1||_{L^2} ||u_2||_{L^2} ||u_3||_{L^2} ||u_4||_{L^2}
$$

$$
\lesssim ||u_1||_{H^1} ||u_2||_{H^1} ||u_3||_{H^1} ||u_4||_{H^1}
$$

(51)

Continuity of energy on $H^1$ follows from (51).

Now, by continuity of mass, energy, and the $H^s$ norm on $H^s$, it follows that:

$$
M(\Phi_n) \to M(\Phi), \ E(\Phi_n) \to E(\Phi), \ \|\Phi_n\|_{H^s} \to \|\Phi\|_{H^s}.
$$

Suppose that we knew that Theorem 1.1 were true in the case of smooth solutions. Then, for all $t \in \mathbb{R}$, it would follow that:

$$
\|u^{(n)}(t)\|_{H^s} \leq C(s,k,E(\Phi_n), M(\Phi_n))(1 + |t|)^{2s+} ||\Phi_n||_{H^s},
$$

The claim for $u$ would now follow by applying the continuity properties of $C$ and the Approximation Lemma. So, from now on, we can work with $\Phi \in C^\infty(\mathbb{T}^2)$.

3.3. A higher modified energy and an iteration bound. As in [20, 21], we let:

$$
E^1(u(t)) := \|Du(t)\|_{L^2}^2.
$$

Arguing as in [20, 21], we obtain that for some $c \in \mathbb{R}$, one has:

$$
\frac{d}{dt} E^1(u(t)) = ic \sum_{n_1+n_2+n_3+n_4=0} ((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2)
$$

$$
\hat{V}(n_3 + n_4) \hat{u}(n_1) \hat{u}(n_2) \hat{u}(n_3) \hat{u}(n_4)
$$

(52)

As in the previous works, we consider the higher modified energy:

$$
E^2(u) := E^1(u) + \lambda_4(M_4;u)
$$

The quantity $M_4$ will be determined soon.

The modified energy $E^2$ is obtained by adding a “multilinear correction” to the modified energy $E^1$ we considered earlier. In order to find $\frac{d}{dt} E^2(u)$, we need to find $\frac{d}{dt} \lambda_4(M_4; u)$. If we fix a multiplier $M_4$, we obtain:

$$
\frac{d}{dt} \lambda_4(M_4; u) =
$$

$$
- i \lambda_4(M_4(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2); u)
$$

$$
- i \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} [M_4(n_{123}, n_4, n_5, n_6) \hat{V}(n_1 + n_2)]
$$

(53)
Lemma 3.3. With notation as above, the following bound holds:

\[
-M_4(n_1, n_{234}, n_5, n_6)\nabla(n_2 + n_3) + M_4(n_1, n_2, n_{45}, n_6)\nabla(n_3 + n_4)
\]

(54)

\[
-M_4(n_1, n_2, n_3, n_{456})\nabla(n_4 + n_5)\nabla(n_5)\nabla(n_6)
\]

We can compute that for \((n_1, n_2, n_3, n_4) \in \Gamma_4\), one has:

\[
|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 2n_{12} \cdot n_{14}
\]

(55)

We notice that the numerator vanishes not only when \(n_{12} = n_{14} = 0\), but also when \(n_{12}\) and \(n_{14}\) are orthogonal. Hence, on \(\Gamma_4\), it is possible for \(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2\) to vanish, but for \((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\) to be non-zero. Consequently, unlike in our previous work on the 1D Hartree equation \([10, 41]\), we can’t cancel the whole quadrilinear term in \((52)\). We remedy this by canceling the non-resonant part of the quadrilinear term. A similar technique was used in \([22]\).

More precisely, given \(\beta_0 \ll 1\), which we determine later, we decompose:

\[
\Gamma_4 = \Omega_{nr} \cup \Omega_r.
\]

Here, the set \(\Omega_{nr}\) of non-resonant frequencies is defined by:

\[
\Omega_{nr} := \{(n_1, n_2, n_3, n_4) \in \Gamma_4 ; n_{12}, n_{14} \neq 0, |\cos \angle (n_{12}, n_{14})| > \beta_0\}
\]

(56)

and the set \(\Omega_r\) of resonant frequencies \(\Omega_r\) is defined to be its complement in \(\Gamma_4\).

We now define the multiplier \(M_4\) by:

\[
M_4(n_1, n_2, n_3, n_4) := \begin{cases} 
(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 \nabla(n_3 + n_4), & \text{if } (n_1, n_2, n_3, n_4) \in \Omega_{nr} \\
0, & \text{if } (n_1, n_2, n_3, n_4) \in \Omega_r
\end{cases}
\]

(57)

Let us now define the multiplier \(M_6\) on \(\Gamma_6\) by:

\[
M_6(n_1, n_2, n_3, n_4, n_5, n_6) := M_4(n_{123}, n_4, n_5, n_6)\nabla(n_1 + n_2) - M_4(n_1, n_{234}, n_5, n_6)\nabla(n_2 + n_3) + M_4(n_1, n_2, n_{345}, n_6)\nabla(n_3 + n_4) - M_4(n_1, n_2, n_{345}, n_6)\nabla(n_4 + n_5)
\]

(58)

We now use \([52]\) and \([51]\), and the construction of \(M_4\) and \(M_6\) to deduce that \([1]\):

\[
\frac{d}{dt}E^2(u) = \sum_{n_1 + n_2 + n_3 + n_4 = 0, |\cos \angle (n_{12}, n_{14})| \leq \beta_0} \left((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\right)\nabla(n_3 + n_4)\nabla(n_1)\nabla(n_2)\nabla(n_3)\nabla(n_4) + \sum_{n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 0} M_6(n_1, n_2, n_3, n_4, n_5, n_6)\nabla(n_1)\nabla(n_2)\nabla(n_3)\nabla(n_4)\nabla(n_5)\nabla(n_6)
\]

(59)

Before we proceed, we need to prove pointwise bounds on the multiplier \(M_4\). In order to do this, let \((n_1, n_2, n_3, n_4) \in \Gamma_4\) be given. We dyadically localize the frequencies, i.e., we find dyadic integers \(N_j\) s.t. \(|n_j| \sim N_j\). We then order the \(N_j\)'s to obtain: \(N_1^* \geq N_2^* \geq N_3^* \geq N_4^*\). We slightly abuse notation by writing \(\theta(N_j^*)\) for \(\theta(N_j^*, 0)\).

Lemma 3.3. With notation as above, the following bound holds:

\[
M_4 = O\left(\frac{1}{\beta_0(N_1^*)^2} \theta(N_1^*) \theta(N_2^*)\right).
\]

(60)

\(^1\)Since \((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 = 0\) whenever \(n_{12} = 0\) or \(n_{14} = 0\), the terms where \(n_{12} = 0\) or \(n_{14} = 0\) don’t contribute to the first sum. We henceforth don’t have to worry about defining the quantity \(\cos(0, -)\).
Proof. By construction of the set $\Omega_{n_0}$, and by the fact that $|\tilde{V}| \lesssim 1$, we note that:

\begin{equation}
|M_4| \lesssim \frac{|(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2|}{|n_{12}| |n_{14}| \beta_0}
\end{equation}

Let us assume, without loss of generality, that:

\begin{equation}
|n_1| \geq |n_2|, |n_3|, |n_4|, \text{ and } |n_{12}| \geq |n_{14}|.
\end{equation}

We now have to consider three cases:

**Case 1:** $|n_1| \sim |n_{12}| \sim |n_{14}|$

In this Case, one has:

\[M_4 = O\left(\frac{1}{\beta_0} \frac{(\theta(n_1))^2}{|n_1|^2}\right) = O\left(\frac{1}{\beta_0} \frac{1}{(N_1^*)^2} \theta(N_1^*) \theta(N_2^*)\right)\]

**Case 2:** $|n_1| \sim |n_{12}| \gg |n_{14}|$

We use the *Mean Value Theorem*, and monotonicity properties of the function $\frac{\theta(n)}{|n|}$ to deduce:

\begin{equation}
(\theta(n_1))^2 - (\theta(n_4))^2 = (\theta(n_1))^2 - (\theta(n_1 - n_{14}))^2 = O(|n_{14}| \frac{(\theta(n_1))^2}{|n_1|})
\end{equation}

\begin{equation}
(\theta(n_2))^2 - (\theta(n_3))^2 = (\theta(n_3 + n_{14}))^2 - (\theta(n_3))^2 = O\left(|n_{14}| \frac{(\theta(n_1))^2}{|n_1|}\right)
\end{equation}

Using (61), (63), (64), and the fact that $|n_{12}| \sim |n_1|$, it follows that:

\[M_4 = O\left(\frac{\theta(n_1)^2}{|n_1|^2 \beta_0}\right) = O\left(\frac{1}{\beta_0} \frac{1}{(N_1^*)^2} \theta(N_1^*) \theta(N_2^*)\right)\]

**Case 3:** $|n_1| \gg |n_{12}|, |n_{14}|$

We write:

\[(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 = (\theta(n_1))^2 - (\theta(n_1 - n_{12}))^2 + (\theta(n_1 - n_{12} - n_{14}))^2 - (\theta(n_1 - n_{14}))^2\]

By using the *Double Mean-Value Theorem* [42], it follows that this expression is $O\left(\frac{(\theta(n_1))^2}{|n_1|^2} |n_{12}| |n_{14}|\right)$. Consequently:

\[M_4 = O\left(\frac{1}{\beta_0} \frac{1}{(N_1^*)^2} \theta(N_1^*) \theta(N_2^*)\right)\]

The Lemma now follows.

\[\square\]

Let us choose:

\[\beta_0 \sim \frac{1}{N}\]

The reason why we choose such a $\beta_0$ will become clear later. For details, see Remark 5.6.

Hence Lemma 5.5 implies:

\begin{equation}
M_4 = O\left(\frac{N}{(N_1^*)^2} \theta(N_1^*) \theta(N_2^*)\right)
\end{equation}
The bound from (66) allows us to deduce the equivalence of $E^1$ and $E^2$. We have the following bound:

**Proposition 3.4.** One has that:

$$E^1(u) \sim E^2(u)$$  \hspace{1cm} (67)

Here, the constant is independent of $N$ as long as $N$ is sufficiently large.

**Proof.** We estimate $E^2(u) - E^1(u) = \lambda_4(M_4; u)$. By construction, one has:

$$|\lambda_4(M_4; u)| \lesssim \sum_{n_1+n_2+n_3+n_4=0} |M_4(n_1, n_2, n_3, n_4)||\hat{u}(n_1)||\hat{u}(n_2)||\hat{u}(n_3)||\hat{u}(n_4)|$$

Let us dyadically localize the $n_j$, i.e., we find $N_j$ dyadic integers such that $|n_j| \sim N_j$. We consider the case when $N_1 \geq N_2 \geq N_3 \geq N_4$. The other cases are analogous. We know that the nonzero contributions occur when:

$$N_1 \sim N_2 \gtrsim N$$  \hspace{1cm} (68)

Let us denote the corresponding contribution to $\lambda_4(M_4; u)$ by $I_{N_1,N_2,N_3,N_4}$. We use Parseval's identity and (66) to deduce that:

$$|I_{N_1,N_2,N_3,N_4}| \lesssim \sum_{n_1+n_2+n_3+n_4=0, |n_j| \sim N_j} \frac{N}{N_1^4} |\hat{D}u_{N_1}(n_1)||\hat{D}u_{N_2}(n_2)||\hat{u}_{N_3}(n_3)||\hat{u}_{N_4}(n_4)|$$

Let us define $F_j : j = 1, \ldots, 4$ by:

$$\hat{F}_1 := |\hat{D}u_{N_1}|, \hat{F}_2 := |\hat{D}u_{N_2}|, \hat{F}_3 := |\hat{u}_{N_3}|, \hat{F}_4 := |\hat{u}_{N_4}|.$$  

By Parseval's identity, one has:

$$|I_{N_1,N_2,N_3,N_4}| \lesssim \frac{N}{N_1^4} \int_{\mathbb{R}^2} F_1 F_2 F_3 F_4 dx$$

which by an $L^2_\sigma, L^2_\sigma, L^\infty_\sigma, L^\infty_\sigma$ Hölder's inequality is:

$$\lesssim \frac{N}{N_1^4} \|F_1\|_{L^2_\sigma} \|F_2\|_{L^2_\sigma} \|F_3\|_{L^\infty_\sigma} \|F_4\|_{L^\infty_\sigma}$$

Furthermore, we use Sobolev embedding, and the fact that taking absolute values in the Fourier transform doesn’t change Sobolev norms to deduce that this expression is:

$$\lesssim \frac{N}{N_1^4} \|F_1\|_{L^2_\sigma} \|F_2\|_{L^2_\sigma} \|F_3\|_{H^1_\sigma} \|F_4\|_{H^1_\sigma} \lesssim \frac{N}{N_1^4} \|Du_{N_1}\|_{L^2_\sigma} \|Du_{N_2}\|_{L^2_\sigma} \|u_{N_3}\|_{H^1_\sigma} \|u_{N_4}\|_{H^1_\sigma} \lesssim \frac{N}{N_1^4} \|Du\|_{L^2_\sigma} \|u\|_{H^1_\sigma} \lesssim \frac{N}{N_1^4} E^1(u)$$

Here, we used the fact that $\|u\|_{H^1} \lesssim 1$.

We now recall (68) and sum in the $N_j$ to deduce that:

$$|E^2(u) - E^1(u)| = |\lambda_4(M_4; u)| \lesssim \frac{1}{N_1} E^1(u).$$

The claim now follows.
Let \( \delta > 0 \), \( v \) be as in Proposition 3.1. For \( t_0 \in \mathbb{R} \), we are interested in estimating:

\[
E^2(u(t_0 + \delta)) - E^2(u(t_0)) = \int_{t_0}^{t_0 + \delta} \frac{d}{dt} E^2(u(t)) dt = \int_{t_0}^{t_0 + \delta} \frac{d}{dt} E^2(v(t)) dt
\]

The iteration bound that we will show is:

**Lemma 3.5.** For all \( t_0 \in \mathbb{R} \), one has:

\[
|E^2(u(t_0 + \delta)) - E^2(u(t_0))| \lesssim \frac{1}{N^{1-}} E^2(u(t_0)).
\]

Arguing similarly as in [40, 41], Theorem 1.1 will follow from Lemma 3.5. We recall the proof for completeness.

**Proof.** (of Theorem 1.1 assuming Lemma 3.5)

The point is that we can iterate the following bound (obtained from Lemma 3.5):

\[
E^2(u(t_0 + \delta)) \leq (1 + \frac{C}{N^{1-}}) E^2(u(t_0))
\]

\( \sim N^{1-} \)-times without getting any exponential growth. We hence obtain that for \( T \sim N^{1-} \), one has:

\[
\|D u(T)\|_{L^2} \lesssim \|D \Phi\|_{L^2}.
\]

By recalling (45), it follows that:

\[
\|u(T)\|_{H^s} \lesssim N^s \|\Phi\|_{H^s}
\]

and hence:

\[
\|u(T)\|_{H^s} \lesssim T^{s+} \|\Phi\|_{H^s} \lesssim (1 + T)^{s+} \|\Phi\|_{H^s}.
\]

This proves Theorem 1.1 for times \( t \geq 1 \). The claim for times \( t \in [0, 1] \) follows by local well-posedness theory. The claim for negative times holds by time-reversibility.

We now have to prove Lemma 3.5.

**Proof.** (of Lemma 3.5)

Let us WLOG consider \( t_0 = 0 \). The general claim will follow by time translation, and the fact that all of the implied constants are uniform in time. Let \( v \) be the function constructed in Proposition 3.1 corresponding to \( t_0 = 0 \).

By (69), and with notation as in this equation, we need to estimate:

\[
\int_{0}^{\delta} \left( \sum_{n_1+n_2+n_3+n_4=0, |\cos \angle(n_{12}, n_{14})| \leq \beta_0} (\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 \right) \hat{V}(n_3+n_4) \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \hat{v}(n_4) + \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_6(n_1, n_2, n_3, n_4, n_5, n_6) \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \hat{v}(n_4) \hat{v}(n_5) \hat{v}(n_6) \right) dt = \]

\[
\int_{0}^{\delta} I dt + \int_{0}^{\delta} II dt =: A + B
\]

We now have to estimate \( A \) and \( B \) separately. Throughout our calculations, let us denote by \( \chi = \chi(t) = \chi[0, \delta](t) \).
3.3.1. Estimate of A (Quadrilinear Terms). By symmetry, we can consider WLOG the contribution when:

\[ |n_1| \geq |n_2|, |n_3|, |n_4|, \text{ and } |n_2| \geq |n_4|, \]

We note that when all \( |n_j| \leq N \), one has: \( (\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 = 0 \). Hence, we need to consider the contribution in which one has:

\[ |n_1| > N, |\cos \angle(n_{12}, n_{14})| \leq \beta_0. \]

We dyadically localize the frequencies: \( |n_j| \sim N_j; j = 1, \ldots, 4 \). We order the \( N_j \) to obtain \( N_j^* \geq N_2^* \geq N_3^* \geq N_4^* \). Since \( n_1 + n_2 + n_3 + n_4 = 0 \), we know that:

\[
N_1^* \sim N_2^* \gtrsim N
\]

Let us note that \( N_1 \sim N_2 \). Namely, if it were the case that: \( N_1 \gg N_2 \), then one would also have: \( N_1 \gg N_4 \), and the vectors \( n_{12} \) and \( n_{14} \) would form a very small angle. Hence, \( |\cos \angle(n_{12}, n_{14})| \) would be close to 1, which would be a contradiction to the assumption that \( |\cos \angle(n_{12}, n_{14})| \leq \beta_0 \).

Consequently:

\[
N_1 \sim N_2 \sim N_4^* \gtrsim N
\]

We denote the corresponding contribution to \( A \) by \( A_{N_1, N_2, N_3, N_4} \). In other words:

\[
A_{N_1, N_2, N_3, N_4} := \int_0^\delta \sum_{n_1 + n_2 + n_3 + n_4 = 0, |\cos \angle(n_{12}, n_{14})| \leq \beta_0} \left((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\right) \hat{V}(n_3 + n_4) \\
\hat{v}_{N_1}(n_1)\bar{\hat{v}}_{N_2}(n_2)\bar{\hat{v}}_{N_3}(n_3)\bar{\hat{v}}_{N_4}(n_4) \, dt
\]

Arguing analogously as in the proof of Lemma 3.3 it follows that for the \( n_j \) that occur in the above sum, one has:

\[
(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 = O\left(|n_{12}| |n_{14}| \frac{\theta(N_1^*) \theta(N_2^*)}{(N_1^*)^2} \right)
\]

By (70), it follows that \( |n_3|, |n_4| \lesssim N_3^* \). Consequently:

\[ |n_{12}| = |n_{34}| \leq |n_3| + |n_4| \lesssim N_3^*. \]

One also knows that:

\[ |n_{14}| \leq |n_1| + |n_4| \lesssim N_1^*. \]

Substituting the last two inequalities into the multiplier bound (71) and using Parseval’s identity in time, it follows that:

\[
|A_{N_1, N_2, N_3, N_4}| \lesssim \sum_{n_1 + n_2 + n_3 + n_4 = 0, |\cos \angle(n_{12}, n_{14})| \leq \beta_0} \left(\int_{\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0} N_3^* N_1^* \frac{\theta(N_1^*) \theta(N_2^*)}{(N_1^*)^2} \right) \bar{v}_{N_1}(n_1, \tau_1) ||\bar{v}_{N_2}(n_2, \tau_2)|| \bar{v}_{N_3}(n_3, \tau_3) ||(\chi \bar{v})_{N_4}(n_4, \tau_4) \right| d\tau_j
\]

\[
\lesssim \frac{1}{N_1^*} \sum_{n_1 + n_2 + n_3 + n_4 = 0} \left(\int_{\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0} |(Dv)_{N_1}(n_1, \tau_1)|||(Dv)_{N_2}(n_2, \tau_2)|||(\nabla v)_{N_3}(n_3, \tau_3)|||(\chi v)_{N_4}(n_4, \tau_4) \right| d\tau_j
\]

Let us define \( F_j; j = 1, \ldots, 4 \) by:

\[
\hat{F}_1 := ||(Dv)_{N_1}||, \hat{F}_2 := ||(Dv)_{N_2}||, \hat{F}_3 := ||(\nabla v)_{N_3}||, \hat{F}_4 := ||(\chi v)_{N_4}||
\]
Consequently, by Parseval’s identity:

\[ |A_{N_1, N_2, N_3, N_4}| \lesssim \frac{1}{N_1} \int_\mathbb{R} \int_{\mathbb{T}^2} F_1 F_2 F_3 F_4 dx dt \]

By using an \( L^4_{t,x}, L^4_{t,x}, L^4_{t,x}, L^4_{t,x} \) Hölder inequality, the corresponding term is:

\[ \lesssim \frac{1}{N_1} \|F_1\|_{L^4_{t,x}} \|F_2\|_{L^4_{t,x}} \|F_3\|_{L^4_{t,x}} \|F_4\|_{L^4_{t,x}} \]

By using (13), (17), (16), and the fact that taking absolute values in the spacetime Fourier transforms doesn’t change the \( X^{α,β} \) norm, it follows that this term is:

\[ \lesssim \frac{1}{N_1} \|Dv_{N_1}\|_{X^{0,\frac{1}{2}+}} \|Dv_{N_2}\|_{X^{0,\frac{1}{2}+}} \|Dv_{N_3}\|_{X^{1+\frac{1}{2}+}} \|\chi v\|_{N_4} \|\chi v\|_{X^{0,\frac{1}{2}+}} \]

By using frequency localization and (18), this expression is:

\[ \lesssim \frac{1}{(N_1)} \|Dv\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{1+\frac{1}{2}+}} \lesssim \frac{1}{(N_1)} E^1(\Phi) \]

In the last inequality, we used Proposition 3.1. By using the previous inequality, and by recalling (67), it follows that:

\[ |A_{N_1, N_2, N_3, N_4}| \lesssim \frac{1}{(N_1)} E^2(\Phi) \]

Using (72), summing in the \( N_j \), and using (69) to deduce that:

\[ |A| \lesssim \frac{1}{N_1} E^2(\Phi) \]

3.3.2. Estimate of \( B \) (Sextilinear Terms). Let us consider just the first term in \( B \) coming from the summand \( M_4(n_{123}, n_4, n_5, n_6) \) in the definition of \( M_6 \). The other terms are bounded analogously. In other words, we want to estimate:

\[ B^{(1)} := \int_0^\delta \sum_{n_{123}+n_4+n_5+n_6=0} M_4(n_{123}, n_4, n_5, n_6) \hat{(v\bar{v})}(n_1+n_2+n_3) \hat{v}(n_4) \hat{\bar{v}}(n_5) \hat{\bar{v}}(n_6) dt \]

We now dyadically localize \( n_{123}, n_4, n_5, n_6 \), i.e., we find \( N_j; j = 1, \ldots, 4 \) such that:

\[ |n_{123}| \sim N_1, |n_4| \sim N_2, |n_5| \sim N_3, |n_6| \sim N_4. \]

Let us define:

\[ B^{(1)}_{N_1, N_2, N_3, N_4} := \int_0^\delta \sum_{n_{123}+n_4+n_5+n_6=0} M_4(n_{123}, n_4, n_5, n_6) \hat{(v\bar{v})}_{N_1}(n_1+n_2+n_3) \hat{v}_{N_2}(n_4) \hat{\bar{v}}_{N_3}(n_5) \hat{\bar{v}}_{N_4}(n_6) dt \]

We now order the \( N_j \) to obtain: \( N_1^* \geq N_2^* \geq N_3^* \geq N_4^* \). As before, we know the following localization bound:

\[ N_1^* \sim N_2^* \gtrsim N \]

In order to obtain a bound on the wanted term, we have to consider two cases.

**Case 1:** \( N_1 = N_1^* \) or \( N_1 = N_2^* \).

**Case 2:** \( N_1 = N_3^* \) or \( N_1 = N_4^* \)
Case 1:

It suffices to consider the case when $N_1 = N_1^*$, $N_2 = N_2^*$, $N_3 = N_3^*$, $N_4 = N_4^*$. The other cases are analogous. We use (16) and Parseval’s identity to obtain that:

$$|B_{N_1,N_2,N_3}^{(1)}| \lesssim \sum_{n_1+\cdots+n_6=0} \int_{\tau_1+\cdots+\tau_6=0} \frac{N}{(N_1^*)^2} \theta(n_1) \theta(n_2) ||(v \bar{v} \vec{\gamma})_N (n_1+n_2, \tau_1+\tau_2) ||V_N (n_1, \tau_1) ||(\chi v)_{N_3} (n_5, \tau_5) ||\bar{v} N_4 (n_6, \tau_6) |d\tau_j$$

Since $||(v \bar{v} \vec{\gamma})_N|| \leq ||v \bar{v} \vec{\gamma}||$, and since $\theta(n_1) \sim \theta(n_1+n_2+n_3) \lesssim \theta(n_1) + \theta(n_2) + \theta(n_3)$, by symmetry, it follows that we just have to bound:

$$K_{N_1,N_2,N_3} := \sum_{n_1+\cdots+n_6=0} \int_{\tau_1+\cdots+\tau_6=0} \frac{N}{(N_1^*)^2} ||(Dv)\bar{v}_N (n_1, \tau_1) ||V_N (n_2, \tau_2) ||\bar{v} (n_3, \tau_3) ||(Dv)\bar{v}_N (n_4, \tau_4) ||(\chi v)_{N_3} (n_5, \tau_5) ||\bar{v} N_4 (n_6, \tau_6) |d\tau_j$$

Let us define the functions $F_j$; $j = 1, \ldots, 6$ by:

$$F_1 := ||(Dv)||, F_2 := \bar{v}, F_3 := ||(Dv)\bar{v}||, F_4 := ||(\chi v)_{N_3}||, F_5 := ||\bar{v} N_4||, F_6 := ||\bar{v} N_4||$$

For $M \gg 1$, we use an $L^2_{t,x}$ Hölder inequality to deduce that:

$$K_{N_1,N_2,N_3} \lesssim \sum_{n_1+\cdots+n_6=0} \int_{\tau_1+\cdots+\tau_6=0} \frac{N}{(N_1^*)^2} ||F_1||_{L^2_{t,x}} ||F_2||_{L^2_{t,x}} ||F_3||_{L^2_{t,x}} ||F_4||_{L^2_{t,x}} ||F_5||_{L^2_{t,x}} ||F_6||_{L^2_{t,x}}$$

By using (19), (17), (16), and the fact that taking absolute values in the spacetime Fourier transform leaves the $X^{s,b}$ norm invariant, it follows that the previous expression is:

$$\lesssim \frac{N}{(N_1^*)^2} ||Dv||_{X^{0,0}} ||v||_{X^{1,1}} + ||v||_{X^{1,1}} ||Dv||_{X^{0,0}} ||v||_{X^{1,1}} + ||v||_{X^{1,1}} ||Dv||_{X^{0,0}} ||v||_{X^{1,1}} + 2 ||v||_{X^{1,1}}$$

We use frequency localization and (18) to deduce that this is:

$$\lesssim \frac{N}{(N_1^*)^2} ||Dv||_{X^{0,0}} ||v||_{X^{1,1}} + \frac{N}{(N_1^*)^2} ||Dv||_{X^{0,0}} ||v||_{X^{1,1}} + \frac{N}{(N_1^*)^2} ||Dv||_{X^{0,0}} ||v||_{X^{1,1}} + \frac{N}{(N_1^*)^2} ||Dv||_{X^{0,0}} ||v||_{X^{1,1}}$$

In the last inequality, we used Proposition 3.1.

Case 2: $N_1 = N_3^*$ or $N_1 = N_4^*$.

Let us assume that:

$$N_3 \gtrsim N_2 \gtrsim N_1 \gtrsim N_4.$$

The other cases are dealt with similarly.

Arguing similarly as in Case 1, it follows that:

$$|B_{N_1,N_2,N_3}^{(1)}| \lesssim \sum_{n_1+\cdots+n_6=0} \int_{\tau_1+\cdots+\tau_6=0} \frac{N}{(N_1^*)^2} ||\bar{v} (n_1, \tau_1) ||V_N (n_2, \tau_2) ||\bar{v} (n_3, \tau_3) ||(Dv)\bar{v}_N (n_4, \tau_4) ||(\chi v)_{N_3} (n_5, \tau_5) ||\bar{v} N_4 (n_6, \tau_6) |d\tau_j$$
We now use an \( L^M_{t,x}, L^M_{l,t,x}, L^M_{t}, L^{4+}_{t,x}, L^{2}_{t,x} \) Hölder inequality and argue as earlier to see that this term is:

\[
\lesssim \frac{N}{(N_1)^2} \|v\|_{X^1}^3 \|v_u\|_{X^0} + \frac{\|v_\nu\|_{X^0}}{2} \left( \|\nu\|_{X^0}^2 + \|v\|_{X^0}^4 \right) \lesssim \frac{N}{(N_1)^2} E^3(\Phi)
\]

(76)

From (75), (76), and (67), it follows that:

\[
|B_{N_1, N_2, N_3, N_4}| \lesssim \frac{N}{(N_1)^2} E^2(\Phi)
\]

(77)

We now use (77), sum in the \( N_j \), and recall (74) to deduce that:

\[
|B| \lesssim \frac{1}{N_1} E^2(\Phi)
\]

(78)

The Lemma now follows from (73) and (78). \( \square \)

### 3.4. Further remarks on the equation.

**Remark 3.6.** The quantity \( \beta_0 \) was chosen as in (60) in order to get the same decay factor in the quantities \( A \) and \( B \). We note that the quantity \( \beta_0 \) only occurred in the bound for \( B \), whereas in the bound for \( A \), we only used the fact that the terms corresponding to the largest two frequencies in the multiplier \( (\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 \) appear with an opposite sign. As we will see, in the non-periodic setting, the quantity \( \beta_0 \) will occur both in the bound for \( A \) and in the bound for \( B \). For details, see (111) and (119).

**Remark 3.7.** Let us observe that, when \( s \) is an integer, or when \( \Phi \) is smooth, essentially the same bound as in Theorem 1.1 can be proved by using the techniques of [46]. The approach is more complicated due to the presence of the convolution potential, but the proof for the cubic NLS can be shown to work for the Hartree equation too. The reason why one uses the fact that \( s \) is an integer is because one wants to use exact formulae for the (Fractional) Leibniz Rule for \( D^s \). By using an exact Leibniz Rule, one sees that certain terms which are difficult to estimate are in fact equal to zero. We omit the details here.

**Remark 3.8.** The equation (1) on \( \mathbb{T}^2 \) has non-trivial solutions which have all Sobolev norms uniformly bounded in time. Similarly as on \( S^1 \) [40], given \( \alpha \in \mathbb{C} \) and \( n \in \mathbb{Z}^2 \), the function:

\[
u(x, t) := \alpha e^{-i\hat{\nu}(0)\xi^2 t + i(n, x) - \xi^2 t} \]

is a solution to (1) on \( \mathbb{T}^2 \) with initial data \( \Phi = \alpha e^{i(n, x)} \). A similar construction was used in [10] to prove instability properties in Sobolev spaces of negative index.

### 4. The Hartree Equation on \( \mathbb{R}^2 \).

#### 4.1. Definition of the \( D \)-operator.

Let us now consider (1) on \( \mathbb{R}^2 \). The proof of Theorem 1.2 will be based on the adaptation of the previous techniques to the non-periodic setting. We start by defining an appropriate upside-down I-operator.

Let \( N > 1 \) be given. Similarly as in the periodic setting, we define \( \theta : \mathbb{R}^2 \to \mathbb{R} \) to be given by:

\[
\theta(\xi) := \begin{cases} (\frac{|\xi|}{N})^N, & \text{if } |\xi| \geq 2N \\ 1, & \text{if } |\xi| \leq N \end{cases}
\]

(79)
We then extend $\theta$ to all of $\mathbb{R}^2$ so that it is radial and smooth. Arguing similarly as in the 1D setting \[41\], it follows that, for all $\xi \in \mathbb{R}^2 \setminus \{0\}$, one has:

\begin{align}
\|\nabla \theta(\xi)\| &\lesssim \frac{\theta(\xi)}{|\xi|} \\
\|\nabla^2 \theta(\xi)\| &\lesssim \frac{\theta(\xi)}{|\xi|^2}
\end{align}

Then, if $f : \mathbb{R}^2 \to \mathbb{C}$, we define $D f$ by:

\begin{equation}
\hat{D} f(\xi) := \theta(\xi) \hat{f}(\xi).
\end{equation}

We also observe that:

\begin{equation}
\|D f\|_{L^2} \lesssim \|f\|_{H^s} \lesssim s^N \|D f\|_{L^2}.
\end{equation}

4.2. Local-in-time bounds. Let $u$ denote the global solution of (1) on $\mathbb{R}^2$. As in the periodic setting, our goal is to estimate $\|D u(t)\|_{L^2}$.

We start by noting:

**Proposition 4.1.** (Local-in-time bounds for the Hartree equation on $\mathbb{R}^2$) There exist $\delta = \delta(s, E(\Phi), M(\Phi))$, $C = C(s, E(\Phi), M(\Phi)) > 0$, which are continuous in energy and mass, such that for all $t_0 \in \mathbb{R}$, there exists a globally defined function $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ such that:

\begin{align}
v|_{[t_0, t_0+\delta]} &= u|_{[t_0, t_0+\delta]} \\
\|v\|_{X^1, 1^2} &\leq C(s, E(\Phi), M(\Phi)) \\
\|D v\|_{X^0, 1^2} &\leq C(s, E(\Phi), M(\Phi)) \|D u(t_0)\|_{L^2}.
\end{align}

Furthermore, we have:

**Lemma 4.2.** If $u$ satisfies:

\begin{equation}
\begin{cases}
    iu_t + \Delta u = (V * |u|^2)u, \\
    u(x, 0) = \Phi(x).
\end{cases}
\end{equation}

and if the sequence $(u^{(n)})$ satisfies:

\begin{equation}
\begin{cases}
    iu_t^{(n)} + \Delta u^{(n)} = (V * |u^{(n)}|^2)u^{(n)}, \\
    u^{(n)}(x, 0) = \Phi_n(x).
\end{cases}
\end{equation}

where $\Phi_n \in C^\infty(\mathbb{R}^2)$ and $\Phi_n \overset{H^s}{\to} \Phi$, then, one has for all $t$:

\begin{equation}
u^{(n)}(t) \overset{H^s}{\to} u(t).
\end{equation}

The proofs of Propositions 4.1 and 4.2 are analogous to the proofs of Propositions 3.1 and 3.2. The main point is that all the auxiliary estimates still hold in the non-periodic setting. As before, we can assume WLOG that $\Phi \in \mathcal{S}(\mathbb{R}^2)$. We omit the details.
4.3. A higher modified energy and an iteration bound. As in the periodic setting, we will apply the method of higher modified energies. We will see that we can obtain better estimates in the non-periodic setting due to the fact that we can apply the improved Strichartz estimate (Proposition 2.3) and the angular improved Strichartz estimate (Proposition 2.6).

We start by defining:

\[ E^1(u(t)) := \|Du(t)\|^2_{L^2}. \]

As before, we obtain that for some \( c \in \mathbb{R} \), one has:

\[
\frac{d}{dt} E^1(u(t)) = ic \int_{\xi_1, \xi_2, \xi_3, \xi_4 = 0} ((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2) \]

\[ \tilde{V}(\xi_3 + \xi_4)\tilde{u}(\xi_1)\tilde{u}(\xi_2)\tilde{u}(\xi_3)\tilde{u}(\xi_4) d\xi_j \]

As in the previous works, we consider the higher modified energy:

\[ E^2(u) := E^1(u) + \lambda_4(M_4; u) \]

The quantity \( M_4 \) will be determined soon.

For a fixed multiplier \( M_4 \), we obtain:

\[
\frac{d}{dt} \lambda_4(M_4; u) = -i \lambda_4(M_4(|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2); u)
\]

\[
-\sum_{\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_6 = 0} [M_4(\xi_{123}, \xi_4, \xi_5, \xi_6)\tilde{V}(\xi_1 + \xi_2)
\]

\[- M_4(\xi_1, \xi_{234}, \xi_5, \xi_6)\tilde{V}(\xi_2 + \xi_3) + M_4(\xi_1, \xi_2, \xi_{345}, \xi_6)\tilde{V}(\xi_3 + \xi_4)
\]

\[- M_4(\xi_1, \xi_2, \xi_3, \xi_{456})\tilde{V}(\xi_4 + \xi_5)\tilde{u}(\xi_1)\tilde{u}(\xi_2)\tilde{u}(\xi_3)\tilde{u}(\xi_4)\tilde{u}(\xi_5)\tilde{u}(\xi_6) \]

As in the periodic setting, we can compute that for \((\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4\), one has:

\[
|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2 = 2\xi_{12} \cdot \xi_{14}
\]

As before, we decompose:

\[
\Gamma_4 = \Omega_{nr} \sqcup \Omega_r.
\]

Here, the set \( \Omega_{nr} \) of non-resonant frequencies is defined by:

\[
\Omega_{nr} := \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4; \xi_{12}, \xi_{14} \neq 0, |\cos(\xi_{12}, \xi_{14})| > \beta_0 \}
\]

and the set \( \Omega_r \) of resonant frequencies \( \Omega_r \) is defined to be its complement in \( \Gamma_4 \).

We now define the multiplier \( M_4 \) by:

\[
M_4(\xi_1, \xi_2, \xi_3, \xi_4) := \begin{cases} 
\frac{(\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2}{|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2} \tilde{V}(\xi_3 + \xi_4), & \text{if } (\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega_{nr} \\
0, & \text{if } (\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega_r
\end{cases}
\]

Let us now define the multiplier \( M_6 \) on \( \Gamma_6 \) by:

\[
M_6(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) := M_4(\xi_{123}, \xi_4, \xi_5, \xi_6)\tilde{V}(\xi_1 + \xi_2) - M_4(\xi_1, \xi_{234}, \xi_5, \xi_6)\tilde{V}(\xi_2 + \xi_3) +
\]
Lemma 4.3. We then have:

\[ \text{we dyadically localize the frequencies, i.e, we find dyadic integers } \beta \text{ s.t. to obtain: } N_1^* \geq N_2^* \geq N_3^* \geq N_4^*. \]

One has that:

\[ M_4(\xi_1, \xi_2, \xi_345, \xi_6)\tilde{V}(\xi_3 + \xi_4) - M_4(\xi_1, \xi_2, \xi_3, \xi_456)\tilde{V}(\xi_4 + \xi_5) \]

We now use (89) and (91), and the construction of \( M_4 \) to deduce that \( 3 \).

\[ \int_{\xi_1+\xi_2+\xi_3+\xi_4=0,|\cos\xi(\xi_2,\xi_4)|\leq\beta_0} \left( (\theta(\xi_1))^2-(\theta(\xi_2))^2-(\theta(\xi_3))^2-(\theta(\xi_4))^2 \right) \tilde{V}(\xi_3+\xi_4)\tilde{u}(\xi_1)\tilde{u}(\xi_2)\tilde{u}(\xi_3)\tilde{u}(\xi_4)d\xi_1 + \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} M_6(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)\tilde{u}(\xi_1)\tilde{u}(\xi_2)\tilde{u}(\xi_3)\tilde{u}(\xi_4)\tilde{u}(\xi_5)\tilde{u}(\xi_6) \]

(96)

As before, we need to prove pointwise bounds on the multiplier \( M_4 \). Given \((\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4\), we dyadically localize the frequencies, i.e, we find dyadic integers \( N_j \) s.t. \(|\xi_j| \sim N_j\). We then order the \( N_j \)'s to obtain: \( N_1^* \geq N_2^* \geq N_3^* \geq N_4^* \). We again abuse notation by writing \( \theta(N_j^*) \) for \( \theta(N_j^*,0) \). One then has:

Lemma 4.3. With notation as above, the following bound holds:

\[ M_4 = O\left( \frac{1}{\beta_0 (N_1^*)^2} \theta(N_1^*) \theta(N_2^*) \right). \]

(97)

The proof of Lemma 4.3 is analogous to the proof of Lemma 3.3 and it will be omitted.

In the non-periodic setting, we will see that we can choose a larger \( \beta_0 \) from which we can get a better bound. Let us choose:

\[ \beta_0 \sim \frac{1}{N^\alpha} \]

(98)

Here, we take \( \alpha \in (0,1) \). We determine \( \alpha \) precisely later (see (123)). For now, we notice:

\[ \beta_0 \geq \frac{1}{N} \]

(99)

We observe that Lemma 4.3 and (99) imply:

\[ M_4 = O\left( \frac{N}{(N_1^*)^2} \theta(N_1^*) \theta(N_2^*) \right) \]

(100)

The bound from (100) allows us to deduce the equivalence of \( E^1 \) and \( E^2 \). We have the following bound:

Proposition 4.4. One has that:

\[ E^1(u) \sim E^2(u) \]

(101)

Here, the constant is independent of \( N \) as long as \( N \) is sufficiently large.

The proof of Proposition 4.4 is analogous to the proof of Proposition 3.4. We omit the details. Let \( \delta > 0, v \) be as in Proposition 4.1. For \( t_0 \in \mathbb{R} \), we are interested in estimating:

\[ E^2(u(t_0 + \delta)) - E^2(u(t_0)) = \int_{t_0}^{t_0 + \delta} \frac{d}{dt} E^2(u(t)) dt = \int_{t_0}^{t_0 + \delta} \frac{d}{dt} E^2(v(t)) dt \]

The iteration bound that we will show is:

\[ \text{As in the periodic setting, we recall that } (\theta(\xi_1))^2 - (\theta(\xi_2))^2 - (\theta(\xi_3))^2 = 0, \text{ whenever } \xi_{12} = 0 \text{ or } \xi_{14} = 0, \text{ hence the corresponding terms again don’t contribute to the quadrilinear term. Therefore, we don’t have to worry about defining the quantity } \cos(0, \cdot). \]
Lemma 4.5. For all $t_0 \in \mathbb{R}$, one has:

$$|E^2(u(t_0 + \delta)) - E^2(u(t_0))| \lesssim \frac{1}{N^4} E^2(u(t_0)).$$

Arguing as in the case of (1) on $T^2$, Theorem 1.2 will follow from Lemma 4.5.

We now prove Lemma 4.5.

Proof. It suffices to consider the case when $t_0 = 0$. As on $T^2$, we compute that $E^2(u(\delta)) - E^2(u(0))$ equals:

$$\int_0^\delta \left( \int_{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, |\cos \angle(\xi_{12}, \xi_{14})| \leq \beta_0} ((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2) \hat{V}(\xi_3 + \xi_4) \hat{v}(\xi_1) \hat{v}(\xi_2) \hat{v}(\xi_3) \hat{v}(\xi_4) d\xi_j + \right) dt =$$

$$= \int_0^\delta I dt + \int_0^\delta II dt =: A + B$$

We now have to estimate $A$ and $B$ separately.

4.3.1. Estimate of $A$ (Quadrilinear Terms). By symmetry, we can consider WLOG the contribution when:

$$|\xi_1| \geq |\xi_2|, |\xi_3|, |\xi_4|, \text{ and } |\xi_2| \geq |\xi_1|.$$ 

Hence, we are considering the contribution in which one has:

$$|\xi_1| > N_1, |\cos \angle(\xi_{12}, \xi_{14})| \leq \beta_0.$$ 

We dyadically localize the frequencies: $|\xi_j| \sim N_j; j = 1, \ldots, 4$. We order the $N_j$ to obtain $N^*_j \geq N^*_2 \geq N^*_3 \geq N^*_1$. As in the periodic setting, we have:

$$N_1 \sim N_2 \sim N^*_1 \gtrsim N$$

We denote the corresponding contribution to $A$ by $A_{N_1, N_2, N_3, N_4}$. In other words:

$$A_{N_1, N_2, N_3, N_4} = \int_0^\delta \int_{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, |\cos \angle(\xi_{12}, \xi_{14})| \leq \beta_0} ((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2) \hat{V}(\xi_3 + \xi_4) \hat{v}_{N_1}(\xi_1) \hat{v}_{N_2}(\xi_2) \hat{v}_{N_3}(\xi_3) \hat{v}_{N_4}(\xi_4) d\xi_j dt$$

As in the periodic setting, we have:

$$|A_{N_1, N_2, N_3, N_4}| \lesssim \int_{T_1 + T_2 + T_3 + T_4 = 0} \int_{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, |\cos \angle(\xi_{12}, \xi_{14})| \leq \beta_0} \frac{N^*_j}{N_1} \theta(N^*_1) \theta(N^*_2)$$

Using Parseval’s identity in time, it follows that:

$$|A_{N_1, N_2, N_3, N_4}| \lesssim \int_{T_1 + T_2 + T_3 + T_4 = 0} \int_{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, |\cos \angle(\xi_{12}, \xi_{14})| \leq \beta_0} \frac{N^*_j}{N_1} \theta(N^*_1) \theta(N^*_2)$$

We now consider two subcases:

**Subcase 1:** $N_4 \sim N_1$
We observe that:

\[
|A_{N_1, N_2, N_3, N_4}| \lesssim \frac{1}{N_1^4} \int_{T_4} \int_{\mathbb{R}^2} F_1 := |(Dv)N_1|, \ F_2 := |(\chi Dv)N_2|, \ F_3 := |(\nabla v)N_3|, \ F_4 := |\bar{v}N_4| d\xi \, d\tau
\]

Let us define \(F_j; j = 1, \ldots, 4\) by:

\[
(105) \quad F_1 := |(Dv)N_1|, \ F_2 := |(\chi Dv)N_2|, \ F_3 := |(\nabla v)N_3|, \ F_4 := |\bar{v}N_4|
\]

Consequently, by Parseval’s identity:

\[
|A_{N_1, N_2, N_3, N_4}| \lesssim \frac{1}{N_1^4} \int_{\mathbb{R}^2} F_1 \, F_2 \, F_3 \, F_4 \, dx \, dt
\]

We use an \(L^4_t, L^2_x, L^4_t, L^4_x\) Hölder inequality, and argue as earlier to deduce that, in this subcase:

\[
|A_{N_1, N_2, N_3, N_4}| \lesssim \frac{1}{N_1^4} ||(Dv)N_1||_{\chi^0} + ||(\chi Dv)N_2||_{\chi^0} + ||(\nabla v)N_3||_{\chi^0} + ||vN_4||_{\chi^0}
\]

\[
\lesssim \frac{1}{(N_1^4)^{\frac{1}{2}}} ||Dv||_{\chi^0} ||v||_{L^1} + \left(\frac{1}{N_4} ||v||_{L^1} \right)
\]

\[
(106) \quad \lesssim \frac{1}{(N_1^4)^{\frac{1}{2}}} ||Dv||_{\chi^0} ||v||_{L^1} \lesssim \frac{1}{(N_1^4)^{\frac{1}{2}}} E^1(\Phi)
\]

In the last step, we used Proposition 4.1.

**Subcase 2: \(N_1 \gg N_4\)**

In this subcase, we need to consider two sub-subcases. Let \(\gamma \in (0, 1)\) be fixed. We will determine \(\gamma\) later. (in equation (122))

**Sub-case 1: \(N_3 \lessapprox N_1^\gamma\)**

Let the functions \(F_j; j = 1, \ldots, 4\) be defined as in (105). We use an \(L^2_t, L^2_x\) Hölder inequality, and we argue as before to deduce that:

\[
|A_{N_1, N_2, N_3, N_4}| \lesssim \frac{1}{N_1^4} ||F_1 F_3||_{L^2_t} ||F_2 F_4||_{L^2_t}
\]

We use Proposition 2.3 and Proposition 2.5 to deduce that this expression is:

\[
\lesssim \frac{1}{N_1^4} \left(\frac{N_1^2}{N_1^4} ||DuN_1||_{\chi^0} + ||vN_3||_{\chi^0} \right) \left(\frac{N_1^2}{N_3^2} ||DuN_2||_{\chi^0} + ||vN_4||_{\chi^0} \right)
\]

\[
(107) \quad \lesssim \frac{1}{(N_1^4)^{\frac{1}{2}}} ||Dv||_{\chi^0} ||v||_{L^1} \lesssim \frac{1}{(N_1^4)^{\frac{1}{2}}} E^1(\Phi)
\]

**Sub-case 2: \(N_3 \approx N_1^\gamma\)**

In this sub-case, we have to work a little bit harder. The crucial estimate will be Proposition 2.6. We suppose that \((\xi_1, \xi_2, \xi_3, \xi_4)\) is a frequency configuration occurring in the integral defining \(A_{N_1, N_2, N_3, N_4}\). We argue as in [22]. We note the elementary trigonometry fact that in this frequency regime, one has: \(\angle(\xi_1, \xi_4) = O\left(\frac{N_1}{N_4}\right), \angle(\xi_3, \xi_4) = O\left(\frac{N_1}{N_3}\right)\). Furthermore, one can use Lipschitz properties of the cosine function to deduce that:

\[
(108) \quad |\cos \angle(\xi_1, \xi_3)| \lesssim \beta_0 + \frac{N_1}{N_3}
\]

We now define:
We now order the summands in other words, we want to estimate:

\[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(\tau_1 + \tau_2) + i(x, \xi_1 + \xi_2)} \chi_{\cos z(\xi_1, \xi_2)} \leq \beta_0 + \frac{N}{N^2} F_1(\xi_1, \tau_1) F_3(\xi_2, \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \]

We now use an \( L^2_{1,v} \) Hölder inequality, and recall \( A \) to deduce that one now has:

\[ |A_{N_1, N_2, N_3, N_4}| \leq \frac{1}{N^2} \| F \|_{L^2_{1,v}} \| F_2 F_4 \|_{L^2_{1,v}} \]

which by using Proposition 2.6 and Proposition 2.5 is:

\[
\begin{align*}
\lesssim & \frac{1}{N_1} \left( \beta_0 + \frac{N}{N_3} \right)^{\frac{1}{2}} \| F_1 \|_{X^{\frac{1}{2}, \frac{1}{2}}} \| F_4 \|_{X^{\frac{1}{2}, \frac{1}{2}}} \| (\frac{N^2}{N^2})^\frac{1}{2} \| DvN_2 \|_{X^{\frac{1}{2}, \frac{1}{2}}} \| vN_4 \|_{X^{\frac{1}{2}, \frac{1}{2}}} \\
& + \frac{1}{(N_1^*)^{\frac{1}{2}} + \frac{1}{2}} \| DvN_1 \|_{X^{\frac{1}{2}, \frac{1}{2}}} \| DvN_2 \|_{X^{\frac{1}{2}, \frac{1}{2}}} \| vN_3 \|_{X^{\frac{1}{2}, \frac{1}{2}}} \| vN_4 \|_{X^{\frac{1}{2}, \frac{1}{2}}} \\
& \leq (\frac{\beta_0}{(N_1^*)^{\frac{1}{2}}} + \frac{1}{(N_1^*)^{\frac{1}{2}} + \frac{1}{2}}) E^1(\Phi)
\end{align*}
\]

(109)

We combine (106), (107), and (109) to deduce that:

\[ |A_{N_1, N_2, N_3, N_4}| \lesssim \left( \frac{\beta_0}{(N_1^*)^{\frac{1}{2}}} + \frac{1}{(N_1^*)^{\frac{1}{2}} + \frac{1}{2}} + \frac{1}{(N_1^*)^{2 - \frac{1}{2}}} \right) E^2(\Phi)
\]

(110)

We then sum in the \( N_j \), use (103), and Proposition 4.4 to deduce that:

\[ |A| \lesssim \left( \frac{\beta_0}{N^{\frac{1}{2}}} + \frac{1}{N^{\frac{1}{2}} + \frac{1}{2}} + \frac{1}{N^{2 - \frac{1}{2}}} \right) E^2(\Phi)
\]

(111)

4.3.2. Estimate of \( B \) (Sextilinear Terms). Let us consider just the first term in \( B \) coming from the summand \( M_4(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \) in the definition of \( M_6 \). The other terms are bounded analogously. In other words, we want to estimate:

\[
B^{(1)} := \int_0^\delta \int_{\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0} M_4(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)(v(x, \xi_4)) \tilde{v}(\xi_4) \tilde{v}(\xi_5) \tilde{v}(\xi_6) d\xi_j dt
\]

The bounds that we will prove for \( B^{(1)} \) will also hold for \( B \), with different constants.

We now dyadically localize \( \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6 \), i.e., we find \( N_j; j = 1, \ldots, 4 \) such that:

\[ |\xi_1| \sim N_1, |\xi_4| \sim N_2, |\xi_5| \sim N_3, |\xi_6| \sim N_4. \]

Let us define:

\[
B_{N_1, N_2, N_3, N_4}^{(1)} := \int_0^\delta \int_{\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0} M_4(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)(v(x, \xi_4)) \tilde{v}(\xi_4) \tilde{v}(\xi_5) \tilde{v}(\xi_6) d\xi_j dt
\]

We now order the \( N_j \) to obtain: \( N_1^* \geq N_2^* \geq N_3^* \geq N_4^* \). As before, we know the following localization bound:

\[ N_1^* \sim N_2^* \gtrsim N \]

(112)

In order to obtain a bound on the wanted term, we have to consider two cases.
Case 1: $N_1 = N_1^*$ or $N_1 = N_2^*$.

Case 2: $N_1 = N_3^*$ or $N_1 = N_4^*$

Case 1: As in the periodic case, we consider the case when:

$$N_1 = N_1^*, N_2 = N_2^*, N_3 = N_3^*, N_4 = N_4^*.$$  

The other cases are analogous.

We use Parseval’s identity together with the Fractional Leibniz Rule for $D$, and argue as in the periodic case to deduce that it suffices to bound the quantity:

$$K_{N_1, N_2, N_3, N_4} :=$$

$$\int_{\tau_1 + \cdots + \tau_6 = 0} \int_{\xi_1 + \cdots + \xi_6 = 0} \frac{1}{\beta_0(N_1)^2} \left| (Dv')((\xi_1, \tau_1), \cdots, \xi_6, \tau_6) \right| d\xi d\tau$$

We must consider several subcases:

Subcase 1: $N_1 \gg N_3$

Let us define the functions $F_j; j = 1, \ldots, 6$ by:

$$F_1 := \|Dv\|, F_2 := \|v\|, F_3 := \|Dv\|, F_4 := \|v\|, F_5 := \|Dv\|, F_6 := \|v\|$$

We use localization in frequency to deduce that this is:

$$\lesssim \frac{1}{\beta_0(N_1)^2} \left( \frac{N_1}{N_3} \right)^{\frac{1}{2}} \|Dv\|_{X_0^{\frac{1}{2}+}} \|v\|_{X_0^{\frac{4}{2}+}}$$

We use Proposition 4.1 is:

$$\lesssim \frac{1}{\beta_0(N_1)^2} \|Dv\|^2_{X_0^{\frac{1}{2}+}} \|v\|^4_{X_0^{\frac{1}{2}+}}$$

Subcase 2: $N_3 \sim N_1$

We use an $L^{2, \infty}, L^{1, \infty}, L^{1, \infty}$ Hölder inequality, and we argue as in the periodic case to deduce that:

$$K_{N_1, N_2, N_3, N_4} \lesssim \frac{1}{\beta_0(N_1)^2} \left( \frac{N_1}{N_3} \right)^{\frac{1}{2}} \|Dv\|_{X_0^{\frac{1}{2}+}} \|v\|_{X_0^{\frac{4}{2}+}} \|Dv\|_{X_0^{\frac{1}{2}+}} \|v\|_{X_0^{\frac{4}{2}+}}$$

Case 2: $N_1 = N_3^*$ or $N_1 = N_4^*$. 

We assume as in the periodic case that \( N_1 = N_2^* \). Let’s also suppose that \( N_3 = N_1^+ \), \( N_2 = N_2^* \). The other contributions are bounded analogously. Arguing as in the periodic case, we have to bound:

\[
L_{N_1, N_2, N_3, N_4} := \int_{\tau_1 + \cdots + \tau_6 = 0} \int_{\xi_1 + \cdots + \xi_6 = 0} \frac{1}{\beta_0(N_1^*)^2} \left| \tilde{v}(\xi_1, \tau_1) \right| \left| \tilde{v}(\xi_2, \tau_2) \right| \left| \tilde{v}(\xi_3, \tau_3) \right| \left| (\mathcal{D} \tilde{v})^{-}_{N_2} (\xi_4, \tau_4) \right| \left| \mathcal{D}^{-}_{N_3} (\xi_5, \tau_5) \right| \left| \tilde{v}_{N_4} (\xi_6, \tau_6) \right| d\xi_j d\tau_j
\]

We consider two subcases:

**Subcase 1:** \( N_1^+ \gg N_4 \)

We know that: \( N_2 \gg N_4 \)

Let us estimate \( L_{N_1, N_2, N_3, N_4} \). We define \( F_j, j = 1, \ldots, 4 \) by:

\[
\tilde{F}_1 := |\tilde{v}|, \quad \tilde{F}_2 := |(\mathcal{D} \tilde{v})^{-}_{N_2}|, \quad \tilde{F}_3 := |(\mathcal{D} \tilde{v})^{+}_{N_3}|, \quad \tilde{F}_4 := |\tilde{v}_{N_4}|
\]

We use an adapted to the non-periodic setting, Proposition 2.3 and 2.8 to deduce that:

\[
L_{N_1, N_2, N_3, N_4} \lesssim \frac{1}{\beta_0(N_1^*)^2} \|F_1\|^3_{L^3_{t,x}} \|F_2 F_3 F_4\|_{L^1_{t,x}}^2 \lesssim \frac{1}{\beta_0(N_1^*)^2} \|v\|^3_{X^{\frac{1}{2}, \frac{1}{2}+}} \left( \frac{N_4^+}{N_2^2} \right) \|D^{-}_{N_2} v\|_{X^{0, \frac{1}{2}+}} \|v_{N_3}\|_{X^{0, \frac{1}{2}+}} \|D_{N_3} v\|_{X^{0, \frac{1}{2}+}} \lesssim \frac{1}{\beta_0(N_1^*)^2} \|Dv\|^2_{X^{0, \frac{1}{2}+}} \|v\|^4_{X^{\frac{1}{2}, \frac{1}{2}+}} \lesssim \frac{1}{\beta_0(N_1^*)^2} E^1(\Phi)
\]

For the last inequality, we used Proposition 4.1.

**Subcase 2:** \( N_4 \sim N_1^+ \)

We argue similarly as in Subcase 2 of Case 1 to deduce that:

\[
L_{N_1, N_2, N_3, N_4} \lesssim \frac{1}{\beta_0(N_1^*)^3} E^1(\Phi)
\]

We use (113), (116), (119), and (117) to deduce that:

\[
|B^{(1)}_{N_1, N_2, N_3, N_4}| \lesssim \frac{1}{\beta_0(N_1^*)^2} E^1(\Phi)
\]

We sum in \( N_j \). Using (112) and (118), it follows that:

\[
|B^{(1)}| \lesssim \frac{1}{\beta_0 N^{\frac{3}{2}}} E^1(\Phi)
\]

By Proposition 2.4 and by construction of \( B^{(1)} \), we deduce that:

\[
|B| \lesssim \frac{1}{\beta_0 N^{\frac{3}{2}}} E^2(\Phi)
\]
4.4. Choice of the optimal parameters. By (102), (111), and (119), it follows that:

\begin{equation}
|E^2(u(\delta)) - E^2(u(0))| \lesssim \left( \frac{\beta_0^2}{N^{\frac{3}{2}}} + \frac{1}{N^{\frac{3}{2}} + \frac{\gamma}{2}} + \frac{1}{N^{2 - \frac{\gamma}{2}}} + \frac{1}{\beta_0 N^{2 - \frac{\gamma}{2}}} \right) E^2(\Phi)
\end{equation}

We now choose \( \gamma \) s.t. \( \frac{3}{2} + \frac{\gamma}{2} = 2 - \frac{\gamma}{2} = \frac{7}{4} \). Hence, we choose:

\begin{equation}
\gamma := \frac{1}{2}
\end{equation}

One then has that:

\begin{equation}
\frac{3}{2} + \frac{\gamma}{2} = 2 - \frac{\gamma}{2} = \frac{7}{4}
\end{equation}

Let us now choose \( \beta_0 \). We recall that by (98), one has: \( \beta_0 \sim \frac{1}{N^{\alpha}}, \alpha \in (0, 1) \).

In order to have \( \frac{\beta_0^2}{N^{\frac{3}{2}}} \leq \frac{1}{N^{\frac{3}{2}} + \frac{\gamma}{2}} \), we should take: \( \alpha \geq \frac{1}{4} \).

In order to have \( \frac{1}{\beta_0 N^{2 - \frac{\gamma}{2}}} \leq \frac{1}{N^{2 - \frac{\gamma}{2}}} \), we should take: \( \alpha \leq \frac{3}{4} \).

Consequently, we take:

\begin{equation}
\alpha \in \left[ \frac{1}{2}, \frac{3}{4} \right]
\end{equation}

From the preceding, we may conclude that:

\begin{equation}
|E^2(u(\delta)) - E^2(u(0))| \lesssim \frac{1}{N^{\frac{3}{2}}} E^2(u(0))
\end{equation}

Lemma 4.5 now follows. \( \square \)

4.5. Further remarks on the equation.

Remark 4.6. Let us observe that Theorem 1.2 would follow immediately if we knew that the equation (1) on \( \mathbb{R}^2 \) scattered in \( H^{s} \). To the best of our knowledge, this result isn’t available, and it can’t be deduced from currently known techniques used to prove scattering. Some scattering results for the Hartree equation were previously studied in [30, 31, 32]. In [30, 31], the asymptotic completeness step was proved by using techniques from [38], which work in dimensions \( n \geq 3 \). In [32], the one and two-dimensional equations are studied. In this case, scattering results are deduced for a subset of solutions with initial data which belongs to a Gevrey class.

Further scattering results for the Hartree equation are noted in [27, 34]. In these papers, one assumes that the initial data lies in an appropriate weighted Sobolev space. The implied bounds depend on the corresponding weighted Sobolev norms of the initial data. Hence, uniform bounds on appropriate Sobolev norms of solutions whose initial data doesn’t lie in the weighted Sobolev spaces can’t be deduced by density. Finally, the techniques used to prove [37] and similar results are restricted to dimensions \( n \geq 5 \).

5. Appendix: Proof of Proposition 3.1

Proof. The proof is based on a fixed-point argument. Let us WLOG look at \( t_0 = 0 \). With notation as in [40], we consider:

\begin{equation}
Lw := \chi_\delta(t)S(t)\Phi - i\chi_\delta(t) \int_0^t S(t - t')(V * |w_\delta|^2)w_\delta(t')dt'
\end{equation}
Let $c > 0$ be the constant such that $\| \chi_{\delta} S(t) \Phi \|_{X^{s,b}} \leq c \delta^{\frac{1-2b}{2}} \| \Phi \|_{H^{s}}$. Such a constant exists by using arguments from [35, 40]. We then define:

$$B := \{ w; \| w \|_{X^{1,b}} \leq 2c \delta^{\frac{1-2b}{2}} \| \Phi \|_{H^{1}}, \| w \|_{X^{s,b}} \leq 2c \delta^{\frac{1-2b}{2}} \| \Phi \|_{H^{s}} \}$$

Arguing as in [40], $B$ is complete w.r.t $\| \cdot \|_{X^{1,b}}$. For $w \in B$, we obtain:

$$\| Lw \|_{X^{s,b}} \leq c \delta^{\frac{1-2b}{2}} \| \Phi \|_{H^{s}} + c_{1} \delta^{\frac{1-2b}{2}} \| (V \ast |w_{3}|^{2})w_{3} \|_{X^{s,b-1}}$$

Similarly, we obtain:

$$\| Dw \|_{X^{s,b}} \leq c \delta^{\frac{1-2b}{2}} \| \Phi \|_{H^{s}} + c_{1} \delta^{\frac{1-2b}{2}} \| (V \ast |w_{3}|^{2})w_{3} \|_{X^{s,b-1}}$$

We now estimate $\| (V \ast |w_{3}|^{2})w_{3} \|_{X^{s,b-1}}$ by duality. Namely, suppose that we are given $c = c(n, \tau)$ such that:

$$\sum_{n} \int d\tau |c(n, \tau)|^{2} = 1.$$ 

We want to estimate:

$$I := \sum_{n_{1}-n_{2}+n_{3}-n_{4}=0} \int_{\tau_{1}-\tau_{2}+\tau_{3}-\tau_{4}=0} \frac{|c(n_{4}, \tau_{4})|}{1 + |\tau_{4} - |n_{4}|^{2}|^{1-\beta}} (1 + |n_{4}|^{s}) \overline{w_{5}}(n_{1}, \tau_{1}) \overline{w_{5}}(n_{2}, \tau_{2}) \overline{w_{5}}(n_{3}, \tau_{3}) \overline{\hat{V}}(n_{1} + n_{2}) d\tau_{j}$$

Since $\hat{V} \in L^{\infty}(\mathbb{Z}^{2})$, this expression is:

$$\lesssim \sum_{n_{1}-n_{2}+n_{3}-n_{4}=0} \int_{\tau_{1}-\tau_{2}+\tau_{3}-\tau_{4}=0} \frac{|c(n_{4}, \tau_{4})|}{1 + |\tau_{4} - |n_{4}|^{2}|^{1-\beta}} (1 + |n_{4}|^{s}) \overline{w_{5}}(n_{1}, \tau_{1}) \overline{w_{5}}(n_{2}, \tau_{2}) \overline{w_{5}}(n_{3}, \tau_{3}) d\tau_{j}$$

Let us write:

$$\mathbb{Z}^{2} = \bigcup_{k=0}^{\infty} D_{k}; D_{k} = \{ n \in \mathbb{Z}^{2}; |n| \sim 2^{k} \}$$

Let $I_{k_{1},k_{2},k_{3}}$ denote the contribution to $I$ with $n_{j} \in D_{k_{j}}$, for $j = 1, 2, 3$. Let us consider WLOG the case when:

$$k_{1} \geq k_{2} \geq k_{3}.$$ 

The contributions from other cases are bounded analogously.

Following [40], we write:

$$D_{k_{1}} \subseteq \bigcup_{\alpha} Q_{\alpha}$$

Here, $Q_{\alpha}$ are balls of radius $2^{k_{2}}$. We can choose this cover so that each element of $D_{k_{1}}$ lies in a fixed finite number of $Q_{\alpha}$. This number is independent of $k_{1}$ and $k_{2}$.

\[\text{This time localization estimate, and all the other similar estimates that we had to use in [40] carry over to the torus.}\]
If $n_1 \in Q_\alpha$, then since $n_4 = n_1 - n_2 + n_3$, $|n_2|, |n_3| \lesssim 2^{k_2}$, it follows that $n_4$ lies in $\tilde{Q}_\alpha$, a dilate of $Q_\alpha$. Thus, the term that we want to estimate is:

$$J_{k_1, k_2, k_3} := 2^{k_1 s} \sum_{\alpha} \sum_{n_1 \in Q_\alpha, n_2 \in D_{k_2}, n_3 \in D_{k_3}}^{\sum} \int_{\tau_1 - \tau_2 + \tau_3 - \tau_4 = 0} \frac{|c(n_4, \tau_4)|}{(1 + |\tau - |n_4|^2|)^{\frac{1}{2}}} d\tau_j$$

We now define:

$$F_\alpha(x, t) := \sum_{n \in \tilde{Q}_\alpha} \int d\tau \frac{|c(n, \tau)|}{(1 + |\tau - |n|^2|)^{\frac{1}{2}}} e^{i(n,x) + \tau t}$$

$$G_\alpha(x, t) := \sum_{n \in \tilde{Q}_\alpha} \int d\tau |\tilde{w}_\delta(n, \tau)| e^{i(n,x) + \tau t}$$

$$H_j(x, t) := \sum_{n \in \tilde{D}_{k_j}} \int d\tau |\tilde{w}_\delta(n, \tau)| e^{i(n,x) + \tau t}$$

By Parseval’s identity and Hölder’s inequality, we deduce:

$$J_{k_1, k_2, k_3} \lesssim 2^{k_1 s} \sum_{\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} F_\alpha G_\alpha H_3 dx dt$$

$$\leq 2^{k_1 s} \sum_{\alpha} \| F_\alpha \|_{L^4_{t,x}} \| G_\alpha \|_{L^4_{t,x}} \| H_2 \|_{L^4_{t,x}} \| H_3 \|_{L^4_{t,x}} .$$

Now, from Lemma 2.2 with $s_1, b_1$ as in the assumptions of the Lemma, we have:

$$\| H_2 \|_{L^4_{t,x}} \lesssim 2^{k_2 s_1} (\sum_{n \in \tilde{D}_{k_2}} d\tau (1 + |\tau - |n|^2|)^{2b_1} |\tilde{w}_\delta(n, \tau)|^2)^\frac{1}{2}$$

$$\lesssim 2^{k_2 s_1} \| (w_3)_{2^{k_2}} \| \chi^{0, b_1}_0 ,$$

Here $(w_3)_M$ is defined by: $(w_3)_M = \tilde{w}_\delta \chi_M$, and we note that localization in $t$ and in $n$ commute. This is a slight abuse of notation, but we interpret $w_3$ as a localization in time if $\delta > 0$ is small, and we interpret $w_\psi$ as a localization in frequency if $\nu$ is a dyadic integer.

By interpolation, it follows that:

$$\| (w_3)_{2^{k_2}} \| \chi^{0, b_1}_0 \lesssim \| (w_3)_{2^{k_2}} \|^\theta \chi^{0, 0}_0 \| (w_3)_{2^{k_2}} \|^{1-\theta} \chi^{0, b}_0 .$$

Here:

$$\theta := 1 - \frac{b_1}{b}$$

By construction of $\psi_\delta$, we obtain:

$$\| (w_3)_{2^{k_2}} \| \chi^{0, 0}_0 = \| (w_3)_{2^{k_2}} \|_{L^2_{t,x}} = \| (w_3)_{2^{k_2}} \psi_\delta \|_{L^2_{t,x}}$$

We now Hölder’s inequality and (21) to see that this expression is:

$$\lesssim \| (w_3)_{2^{k_2}} \|_{L^1_{t} L^2} \| \psi_\delta \|_{L^4_t} \lesssim \delta^{\frac{1}{2}} \| (w_3)_{2^{k_2}} \|_{\chi^{0, \frac{1}{4}}_0} \lesssim \delta^{\frac{1}{2}} \| (w_3)_{2^{k_2}} \| \chi^{0, b}_0 .$$
Consequently:

\[ \|H_2\|_{L_t^4} \lesssim 2^{k_{2s_1}} \delta^{\frac{n}{2}} \|(w_\delta)_\delta\|_{X^{0,b}} \]

(133)

\[ \lesssim 2^{k_{2s_1}} \delta^{\frac{n}{2} + \frac{1 - 2b}{2}} \|w_{2^{k_2}}\|_{X^{0,b}} \]

In the last inequality, we used appropriate time-localization in \(X^{0,b}\).

Analogously:

\[ \|H_3\|_{L_t^4} \lesssim 2^{k_{3s_1}} \delta^{\frac{n}{2} + \frac{1 - 2b}{2}} \|w_{2^{k_3}}\|_{X^{0,b}} \]

(134)

Given an index \(\alpha\), we define \((w_\delta)_\alpha\), and \(w_\alpha\) to be the restriction to \(n \in Q_\alpha\) of \(w_\delta\) and \(w\) respectively. We note that this is a different localization than the ones we used before. Since each \(Q_\alpha\) has radius \(2^{k_2}\), Lemma 2.2 implies that:

\[ \|G_\alpha\|_{L_t^4} \lesssim 2^{k_{2s_1}} \left( \sum_{n \in Q_\alpha} d\tau (1 + |\tau - |n|^2|^{2b_1} |\tilde{w}_\delta(n, \tau)|^2) \right)^\frac{1}{2} \]

\[ \lesssim 2^{k_{2s_1}} \|(w_\delta)_\alpha\|_{X^{0,b_1}} \]

Arguing as in (133), (134), we obtain:

\[ \|G_\alpha\|_{L_t^4} \lesssim 2^{k_{2s_1}} \delta^{\frac{n}{2} + \frac{1 - 2b}{2}} \|w_\alpha\|_{X^{0,b}} \]

(135)

Furthermore, each \(Q_\alpha\) is of radius \(\sim 2^{k_2}\). Let \(c_\alpha\) be the restriction of \(c\) to \(n \in \tilde{Q}_\alpha\). Let us also choose \(b_1\) such that:

\[ b_1 \leq 1 - b. \]

From Lemma 2.2 and the previous definitions, we obtain:

\[ \|F_\alpha\|_{L_t^4} \lesssim 2^{k_{2s_1}} \|F_\alpha\|_{X^{0,b_1}} \lesssim 2^{k_{2s_1}} \|F_\alpha\|_{X^{0,1-b}} \]

(137)

\[ \lesssim 2^{k_{2s_1}} \|c_\alpha\|_{L_t^2}. \]

From (133), (134), (135), (137), it follows that:

\[ J_{k_1, k_2, k_3} \lesssim \sum_\alpha \delta^{\frac{n}{2} + \frac{3(n - 2b_1)}{2}} 2^{k_1} 8^{k_2 s_1} 2^{k_3 s_1} \|w_{2^{k_2}}\|_{X^{0,b}} \|w_{2^{k_3}}\|_{X^{0,b}} \|w_\alpha\|_{X^{0,b}} \|c_\alpha\|_{L_t^2} \]

We apply the Cauchy-Schwarz inequality in \(\alpha\) to deduce that the previous expression is

\[ \lesssim \delta^{\frac{n}{2} + \frac{3(n - 2b_1)}{2}} 2^{k_1} 8^{k_2 s_1} 2^{k_3 s_1} \|w_{2^{k_2}}\|_{X^{0,b}} \|w_{2^{k_3}}\|_{X^{0,b}} \|w_{2^{k_3}}\|_{X^{0,b}} \|c_2 s_1\|_{L_t^2} \]

We write \(8^{k_2 s_1} = (8^{k_2 s_1})^0 - (8^{k_2 s_1})^{1+}, 2^{k_3 s_1} = (2^{k_3 s_1})^0 - (2^{k_3 s_1})^{1+}\), and we sum a geometric series in \(k_2, k_3\) to deduce that:

\[ \text{(Strictly speaking, we are making the annulus } |n| \sim 2^{k_1} \text{ a little bit larger, but we write the localization in the same way as before.)} \]
Using the Cauchy-Schwarz inequality in $k_1$, this expression is:

$$\lesssim \delta^{\frac{4}{1+b}} (1) \left\| u \right\|_{H^{s+b}} \left\| \Phi \right\|_{H^{s+b}} \left\| w \right\|_{X^{s+b}} \left\| w \right\|_{X^{s+b}}$$

(138)

Let us take $s_1 = \frac{1}{2}$. Then, the assumptions of Lemma 2.2 will be satisfied if we take $b_1 = \frac{1}{2} + \frac{2}{3} \theta$. Since $b = \frac{1}{2} + \frac{2}{3} \theta$, (132) is then satisfied. By our construction in (132), one has:

$$\theta = 1 - \frac{1}{2} + \frac{1}{3} > \frac{1}{2}$$. Hence, $\rho_0 := \frac{4}{1+b} + 3(1-2b) > 0$.

Thus, by (126), and by definition of $B$ it follows that for $w \in B$:

$$\left\| Lw \right\|_{X^{s+b}} \leq c_3 \delta^{\frac{1}{b}} \left\| \Phi \right\|_{H^{s+b}} + c_2 \delta^{\frac{4}{1+b} + 2(1-2b) \left\| w \right\|_{X^{s+b}} \left\| w \right\|_{X^{s+b}}$$

$$\leq c_3 \delta^{\frac{1}{b}} \left\| \Phi \right\|_{H^{s+b}} + c_3 \delta^{\frac{1}{b} + \frac{1}{2} \left\| \Phi \right\|_{H^{s+b}} \delta^{\frac{4}{1+b} + 3(1-2b) \left\| \Phi \right\|_{H^{s+b}}$$

Similarly, for $v, w \in B$, one has:

$$\left\| Lv - Lw \right\|_{X^{1,b}} \leq c_1 \delta^{\frac{4}{1+b} + 2(1-2b) \left( \left\| v \right\|_{X^{1,b}} + \left\| w \right\|_{X^{1,b}} \right)^2 \left\| v - w \right\|_{X^{1,b}}$$

$$\leq c_2 \delta^{\frac{4}{1+b} + 3(1-2b) \left\| \Phi \right\|_{H^{s+b}} \left\| v - w \right\|_{X^{1,b}}$$

We now argue as in (40) to obtain a fixed point $v \in B$. We then take $D$’s of both sides and use (127). Now, we have to estimate:

$$\left\| D((V \ast |v|^{2})v_0) \right\|_{X^{0,b}}$$

Arguing as before, it follows that this expression is:

$$\lesssim \delta^{\rho_0 \left\| Dv \right\|_{X^{0,b}} \left\| v \right\|_{X^{1,b}}}$$

Namely, in the analogue of $J_{k_1,k_2,k_3}$, we can replace the $2^{k_1} \theta$ by $\theta^{\frac{k_1}{s}}$, which is equal to $\frac{2^{k_1}}{N}$ if $2^{k_1} \geq N$, and 1 otherwise. One then argues as in (40), and (17), (48) immediately follow.

We now check uniqueness, i.e. (40). Namely, we suppose that:

$$\begin{cases}
iv_t + \Delta u = (V \ast |u|^{2})u, x \in \mathbb{T}^2, t \in \mathbb{R} \\
v_t + \Delta v = (V \ast |v|^{2})v, x \in \mathbb{T}^2, t \in \mathbb{R} \\
u|_{t=0} = v|_{t=0} \in H^s(\mathbb{T}^2), s > 1.
\end{cases}$$

(139)

We are assuming that $u$ is a well-posed solution to (1) on $\mathbb{T}^2$, and hence $\left\| u(t) \right\|_{H^s}$ satisfies exponential bounds, as was noted in the Introduction. Furthermore, since $v \in X^{s,\frac{1}{2}+}$, by Sobolev embedding
time, it follows that $v \in L^\infty_t H^s_x$. Consequently, there exist $A, B > 0$ such that, for all $t \in \mathbb{R}$, one has:

$$
\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \leq Ae^{B|t|}
$$

We observe:

$$
u(t) - v(t) = -i \int_0^t S(t - t')( (V * |u|^2)u - (V * |v|^2)v ) (t') dt'
$$

We take $L^2$ norms in $x$ and use Minkowski’s inequality to deduce:

$$
\|u(t) - v(t)\|_{L^2_x} \leq \int_0^t \| (V * |u|^2)u - (V * |v|^2)v \|_{L^2_x} dt'
$$

In order to bound the integral, we need the two following bounds, which follow from Hölder’s inequality, Young’s inequality, and Sobolev embedding.

$$
\|(V * (u_1 u_2)) u_3\|_{L^2_x} \leq \|V * (u_1 u_2)\|_{L^\infty_x} \|u_3\|_{L^2_x} \leq \|V\|_{L^1_x} \|u_1\|_{L^\infty_x} \|u_2\|_{L^\infty_x} \|u_3\|_{L^2_x}
$$

Also:

$$
\|(V * (u_1 u_2)) u_3\|_{L^2_x} \leq \|V * (u_1 u_2)\|_{L^2_x} \|u_3\|_{L^\infty_x} \leq \|V\|_{L^2_x} \|u_1 u_2\|_{L^2_x} \|u_3\|_{L^\infty_x}
$$

Substituting (142) and (143) into (141), and using (140) it follows that:

$$
\|u(t) - v(t)\|_{L^2_x} \lesssim \int_0^t (\|u\|_{H^s} + \|v\|_{H^s})^2 \|u - v\|_{L^2_x} dt' \lesssim \int_0^t e^{2\beta t'} \|u - v\|_{L^2_x} dt'
$$

By Gronwall’s inequality, it follows that on $[0, t]$, one has $\|u - v\|_{L^2_x} = 0$, hence $u = v$. The same argument works for negative times. □

Arguing as in [40], we note that all the implied constants depend on $s$, energy, and mass, and that they are continuous in energy and mass.

This proves Proposition 3.1.

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