Examples of certain kind of
minimal orbits of Hemann actions

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Abstract
We give examples of certain kind of minimal orbits of Hermann actions and
discuss whether each of the examples is austere.

1 Introduction
Let \( N = G/K \) be a symmetric space of compact type equipped with the \( G \)-invariant
metric induced from the Killing form of the Lie algebra of \( G \). Let \( H \) be a symmetric
subgroup of \( G \) (i.e., \( (\text{Fix} \tau)_0 \subset H \subset \text{Fix} \tau \) for some involution \( \tau \) of \( G \)),
where \( \text{Fix} \tau \) is the fixed point group of \( \tau \) and \( (\text{Fix} \tau)_0 \) is the identity component of \( \text{Fix} \tau \). The
natural action of \( H \) on \( N \) is called a Hermann action (see [HPTT], [Kol]). Let \( \theta \) be
an involution of \( G \) with \( (\text{Fix} \theta)_0 \subset K \subset \text{Fix} \theta \). According to [Co], when \( G \) is simple,
we may assume that \( \theta \circ \tau = \tau \circ \theta \) by replacing \( H \) to a suitable conjugate group of
\( H \) if necessary except for the following three Hermann action:

(i) \( \text{Sp}(p + q) \lhd SU(2p + 2q)/S(U(2p - 1) \times U(2q + 1)) \quad (p \geq q + 2) \),
(ii) \( \text{U}(p + q + 1) \lhd \text{Spin}(2p + 2q + 2)/\text{Spin}(2p + 1) \times \mathbb{Z}_2 \text{Spin}(2q + 1) \quad (p \geq q + 1) \),
(iii) \( \text{Spin}(3) \times \mathbb{Z}_2 \text{Spin}(5) \lhd \text{Spin}(8)/\mu(\text{Spin}(3) \times \mathbb{Z}_2 \text{Spin}(5)) \),

where \( \mu \) is the triality automorphism of \( \text{Spin}(8) \). Here we note that we remove
transitive Hermann actions.

Assumption. In the sequel, we assume that \( \theta \circ \tau = \tau \circ \theta \). Then the Hermann
action \( H \lhd G/K \) is said to be commutative.

Let \( \mathfrak{g}, \mathfrak{k} \) and \( \mathfrak{h} \) be the Lie algebras of \( G, K \) and \( H \), respectively. Denote the in-
volutions of \( \mathfrak{g} \) induced form \( \theta \) and \( \tau \) by the same symbols \( \theta \) and \( \tau \), respectively.
Set \( \mathfrak{p} := \text{Ker}(\theta + \text{id}) \) and \( \mathfrak{q} := \text{Ker}(\tau + \text{id}) \). The vector space \( \mathfrak{p} \) is identified with
\( T_e K(G/K) \), where \( e \) is the identity element of \( G \). Denote by \( B_\mathfrak{g} \) the Killing form of \( \mathfrak{g} \).
Give $G/K$ the $G$-invariant metric arising from $B_g|_{xp}$. Take a maximal abelian subspace $b$ of $p \cap q$. For each $\beta \in b^*$, we set $p_\beta := \{ X \in p \mid \text{ad}(b)^2(X) = -\beta(b)^2X \ (\forall b \in b) \}$ and $\Delta' := \{ \beta \in b^* \setminus \{ 0 \} \mid p_\beta \neq \{ 0 \} \}$. This set $\Delta'$ is a root system. Let $\Pi' = \{ \beta_1, \ldots, \beta_\gamma \}$ be the simple root system of the positive root system $\Delta'_+ \subseteq \Delta'$ under a lexicographic ordering of $b^*$. Set $\Delta^V_+ := \{ \beta \in \Delta'_+ \mid p_\beta \cap q \neq \{ 0 \} \}$ and $\Delta^H_+ := \{ \beta \in \Delta'_+ \mid p_\beta \cap h \neq \{ 0 \} \}$. Define a subset $\tilde{C}$ of $b$ by

$$\tilde{C} := \{ b \in b \mid 0 < \beta(b) < \pi (\forall \beta \in \Delta^V_+), \frac{\pi}{2} < \beta(b) < \pi (\forall \beta \in \Delta^H_+) \}.$$

The closure $\overline{\tilde{C}}$ of $\tilde{C}$ is a simplicial complex. Set $C := \text{Exp}(\overline{\tilde{C}})$, where $\text{Exp}$ is the exponential map of $G/K$ at $eK$. Each principal $H$-orbit passes through only one point of $C$ and each singular $H$-orbit passes through only one point of $\text{Exp}(\partial \overline{\tilde{C}})$. For each simplex $\sigma$ of $\overline{\tilde{C}}$, only one minimal $H$-orbit through $\text{Exp}(\sigma)$ exists. See proofs of Theorems A and B in [K2] (also [I]) about this fact. For $\beta \in \Delta'_+$, we set $\beta = \sum_{i=1}^{r} n_i^\beta \beta_i$, $m_\beta := \dim p_\beta$, $m^V_\beta := \dim(p_\beta \cap q)$ and $m^H_\beta := \dim(p_\beta \cap h)$. Let $Z_0$ be a point of $b$. We consider the following two conditions for $Z_0$:

(I) \[ \begin{array}{l}
\beta(Z_0) \equiv 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6} \pmod{\pi} \quad (\forall \beta \in \Delta^V_+) \quad \& \quad \beta \in \Delta^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{6} \pmod{\pi} \\
\quad + \quad \sum_{\beta \in \Delta^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}} 3n_i^\beta m^V_\beta \\
\quad + \quad \sum_{\beta \in \Delta^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}} n_i^\beta m^V_\beta \\
\quad + \quad \sum_{\beta \in \Delta^H_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{6} \pmod{\pi}} 3n_i^\beta m^H_\beta \\
\quad + \quad \sum_{\beta \in \Delta^H_+ \text{ s.t. } \beta(Z_0) \equiv \frac{5\pi}{6} \pmod{\pi}} n_i^\beta m^H_\beta \\
\end{array} \]

and

(II) \[ \begin{array}{l}
\beta(Z_0) \equiv 0, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi} \quad (\forall \beta \in \Delta'_+) \quad \& \quad \beta \in \Delta'_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{4} \pmod{\pi} \\
\quad + \quad \sum_{\beta \in \Delta'_+ \text{ s.t. } \beta(Z_0) \equiv \frac{3\pi}{4} \pmod{\pi}} 3n_i^\beta m^H_\beta \\
\quad + \quad \sum_{\beta \in \Delta'_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{6} \pmod{\pi}} n_i^\beta m^H_\beta \\
\quad + \quad \sum_{\beta \in \Delta'_+ \text{ s.t. } \beta(Z_0) \equiv \frac{5\pi}{6} \pmod{\pi}} 3n_i^\beta m^H_\beta \\
\quad + \quad \sum_{\beta \in \Delta'_+ \text{ s.t. } \beta(Z_0) \equiv \frac{7\pi}{6} \pmod{\pi}} n_i^\beta m^H_\beta \\
\end{array} \]

Denote by $L$ the isotropy group of $H$ at $\text{Exp} Z_0$. Denote by $h$ (resp. $l$) the Lie algebra of $H$ (resp. $L$) and $B_g$ the Killing form of $g$. Also, denote by $g_l$ the induced metric
on the submanifold $M$ in $G/K$ and $\nabla^\perp$ the normal connection of the submanifold $M$.  

In the case where $(\mathfrak{h},\mathfrak{l})$ admits a reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$, we denote the canonical connection of the principal $L$-bundle $\pi : H \to H/L(= M)$ with respect to this reductive decomposition by $\omega_m$. Let $F^\perp(M)$ be the normal frame bundle of $M$. Define a map $\eta : H \to F^\perp(M)$ by $\eta(h) = h^* u_0$ ($h \in H$), where $u_0$ is an arbitrary fixed element of $F^\perp(M)_{\text{Exp} Z_0}$, where $F^\perp(M)_{\text{Exp} Z_0}$ is the fibre of $F^\perp(M)$ over $\text{Exp} Z_0$. This map $\eta$ is an embedding. By identifying $H$ with $\eta(H)$, we regard $\pi : H \to H/L(= M)$ as a subbundle of $F^\perp(M)$. Denote by the same symbol $\omega_m$ the connection of $F^\perp(M)$ induced from $\omega_m$ and $\nabla_{\omega_m}$ the linear connection on $T^\perp M$ associated with $\omega_m$.

In this paper, we prove the following results for the orbit $M = H(\text{Exp} Z_0)$ of the Hermann action $H \curvearrowright G/K$.

**Theorem A.** If $Z_0$ satisfies the condition (I) or (II), then the orbit $M$ is a minimal submanifold satisfying the following conditions:

(i) $(\mathfrak{h},\mathfrak{l})$ admits a reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$ such that $B_0(\mathfrak{l},\mathfrak{m}) = 0$,

(ii) $\nabla^\perp = \nabla_{\omega_m}$ holds.

Also, $\cap_{v \in T^\perp_M} \text{Ker } A_v$ is equal to

$$g_0*(\mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b})) + \sum_{\beta \in \Delta^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{2}{T} \text{ (mod } \pi)} g_0*(\mathfrak{p}_{\beta} \cap \mathfrak{q}) + \sum_{\beta \in \Delta^H_+ \text{ s.t. } \beta(Z_0) \equiv 0 \text{ (mod } \pi)} g_0*(\mathfrak{p}_{\beta} \cap \mathfrak{h}),$$

where $\mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b})$ is the centralizer of $\mathfrak{b}$ in $\mathfrak{p} \cap \mathfrak{h}$.

Let $M$ be a submanifold in a Riemannian manifold $N$. If, for any unit normal vector $v$, the spectrum of the shape operator $A_v$ is invariant with respect to the $(-1)$-multiple (with considering the multiplicities), then $M$ is called an **austere submanifold**. By using Theorem A, we can show the following fact.

**Theorem B.** Assume that $Z_0$ satisfies the condition (I) or (II). If $m^V_\beta = m^H_\beta$ for all $\beta \in \Delta^V_+$ and if $Z_0$ satisfies $\beta(Z_0) \equiv 0, \frac{\pi}{T}, \frac{2\pi}{3} \text{ (mod } \pi)$ for all $\beta \in \Delta^V_+$, then the orbit $M$ is an austere submanifold satisfying the conditions (i) and (ii) in Theorem A.

**Remark 1.1.** The austere orbits of the commutative Hermann actions were classified in [I].

Also, we can show the following facts.
Theorem C. Assume that $Z_0$ satisfies the condition (I). In particular, if $\Delta^\prime H \cap \Delta^\prime_+ = \emptyset$, if $\beta(Z_0) \equiv 0$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$ (mod $\pi$) for all $\beta \in \Delta^\prime_+$ and if $\beta(Z_0) \equiv \frac{2\pi}{3}$, $\frac{4\pi}{3}$ (mod $\pi$) for all $\beta \in \Delta^\prime_+$, then $M$ is a minimal submanifold satisfying the conditions (i), (ii) in Theorem A. Furthermore, if the cohomogeneity of the $H$-action is equal to the rank of $G/K$, then $(g_1)_{eL} = \frac{1}{2}B_{g|\mathfrak{m} \times \mathfrak{m}}$ and $\bigcap_{v \in T^\perp_x M} \text{Ker} A_v = \{0\}$ hold.

Theorem D. Assume that $Z_0$ satisfies the condition (I). In particular, if $\Delta^\prime V \cap \Delta^\prime_+ \cap \Delta^\prime H = \emptyset$, if $\beta(Z_0) \equiv 0$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$ (mod $\pi$) for all $\beta \in \Delta^\prime_+$ and if $\beta(Z_0) \equiv \frac{2\pi}{3}$, $\frac{4\pi}{3}$ (mod $\pi$) for all $\beta \in \Delta^\prime_+$, then $M$ is a minimal submanifold satisfying the conditions (i), (ii) in Theorem A. Furthermore, if the cohomogeneity of the $H$-action is equal to the rank of $G/K$, then $(g_1)_{eL} = \frac{1}{2}B_{g|\mathfrak{m} \times \mathfrak{m}}$ and $\bigcap_{v \in T^\perp_x M} \text{Ker} A_v = \{0\}$ hold.

Theorem E. Assume that $Z_0$ satisfies the condition (II). In particular, if $\Delta^\prime V \cap \Delta^\prime_+ \cap \Delta^\prime H = \emptyset$, if $\beta(Z_0) \equiv 0$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$ (mod $\pi$) for all $\beta \in \Delta^\prime_+$ and if $\beta(Z_0) \equiv \frac{2\pi}{3}$, $\frac{4\pi}{3}$ (mod $\pi$) for all $\beta \in \Delta^\prime_+$, then $M$ is a minimal submanifold satisfying the conditions (i), (ii) in Theorem A. Furthermore, if the cohomogeneity of the $H$-action is equal to the rank of $G/K$, then $(g_1)_{eL} = \frac{1}{2}B_{g|\mathfrak{m} \times \mathfrak{m}}$ and $\bigcap_{v \in T^\perp_x M} \text{Ker} A_v = \{0\}$ hold.

Theorem F. If $\Delta^\prime V \cap \Delta^\prime_+ \cap \Delta^\prime H = \emptyset$, if $\beta(Z_0) \equiv 0$, $\frac{\pi}{2}$ (mod $\pi$) for all $\beta \in \Delta^\prime_+$, then $M$ is a totally geodesic submanifold satisfying the conditions (i), (ii) in Theorem A. Furthermore, if the cohomogeneity of the $H$-action is equal to the rank of $G/K$, then $(g_1)_{eL} = B_{g|\mathfrak{m} \times \mathfrak{m}}$ holds.

Remark 1.2. (i) If $H = K$ then we have $\Delta^\prime H = \emptyset$ and hence $\Delta^\prime V \cap \Delta^\prime_+ \cap \Delta^\prime H = \emptyset$.

(ii) In Theorems C~F, when $G$ is simple, there exists an inner automorphism $\rho$ of $G$ with $\rho(K) = H$ by Proposition 4.39 of [1].

In the final section, we give examples of Hermann actions $H \curvearrowright G/K$ and $Z_0 \in \mathfrak{b}$ as in Theorems B, C and F.

2 Basic notions and facts

In this section, we recall some basic notions and facts.

Shape operators of orbits of Hermann actions

Let $H \curvearrowright G/K$ be a Hermann action and $\theta$ (resp. $\tau$) an involution of $G$ with $(\text{Fix} \theta)_0 \subset K \subset \text{Fix} \theta$ (resp. $(\text{Fix} \tau)_0 \subset H \subset \text{Fix} \tau$). Assume that $\theta \circ \tau = \tau \circ \theta$. 

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Let $\mathfrak{k}, \mathfrak{p}, \mathfrak{h}, \mathfrak{q}, \mathfrak{b}, \mathfrak{p}_\beta, \Delta', \Delta^V_+ \text{ and } \Delta^H_+$ be as in Introduction. Fix $Z_0 \in \mathfrak{b}$. Set $M := H(\text{Exp} Z_0)$ and $g_0 := \text{exp} Z_0$, where Exp is the exponential map of $G/K$ at $eK$ and exp is the exponential map of $G$. Set

$$\Delta^V_{Z_0} := \{ \beta \in \Delta^V_+ \mid \beta(Z_0) \equiv 0 \mod \pi \}$$

and

$$\Delta^H_{Z_0} := \{ \beta \in \Delta^H_+ \mid \beta(Z_0) \equiv \frac{\pi}{2} \mod \pi \}.$$

Denote by $A$ the shape tensor of $M$. The tangent space $T_{\text{Exp} Z_0} M$ of $M$ at $\text{Exp} Z_0$ is given by

$$T_{\text{Exp} Z_0} M = g_0_\ast \left( \mathfrak{h} - \mathfrak{q} + \sum_{\beta \in \Delta^V_+ \setminus \Delta^V_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{q}) + \sum_{\beta \in \Delta^H_+ \setminus \Delta^H_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{h}) \right)$$

and hence

$$T_{\text{Exp} Z_0} ^\perp M = g_0_\ast \left( \mathfrak{b} + \sum_{\beta \in \Delta^V_0} (\mathfrak{p}_\beta \cap \mathfrak{q}) + \sum_{\beta \in \Delta^H_0} (\mathfrak{p}_\beta \cap \mathfrak{h}) \right).$$

Denote by $L$ the isotropy group of the $H$-action at $\text{Exp} Z_0$. The slice representation $\rho_{Z_0}^S : L \to G(L(T_{\text{Exp} Z_0} ^\perp M)$ of the $H$-action at $\text{Exp} Z_0$ is given by $\rho_{Z_0}^S(h) = h_\ast T_{\text{Exp} Z_0} ^\perp M \ (h \in H_{Z_0})$. Then we have $\bigcup_{h \in H_{Z_0}} \rho_{Z_0}^S(h)(g_0_\ast \mathfrak{b}) = T_{\text{Exp} Z_0} ^\perp M$ and

$$A_{\rho_{Z_0}^S(h)(g_0_\ast \mathfrak{b})} \rho_{Z_0}^S(h)(g_0_\ast (\mathfrak{h} - \mathfrak{q} + \sum_{\beta \in \Delta^V_0} (\mathfrak{p}_\beta \cap \mathfrak{q}))) = 0,$$

$$A_{\rho_{Z_0}^S(h)(g_0_\ast \mathfrak{q})} \rho_{Z_0}^S(h)(g_0_\ast (\mathfrak{p}_\beta \cap \mathfrak{q})) = -\frac{\beta(v)}{\tan \beta(Z_0)} \text{id} \ (\beta \in \Delta^V_0 \setminus \Delta^V_{Z_0}),$$

$$A_{\rho_{Z_0}^S(h)(g_0_\ast \mathfrak{h})} \rho_{Z_0}^S(h)(g_0_\ast (\mathfrak{p}_\beta \cap \mathfrak{h})) = \beta(v) \tan \beta(Z_0) \text{id} \ (\beta \in \Delta^H_0 \setminus \Delta^H_{Z_0}),$$

where $h \in L$ and $v \in \mathfrak{b}$.

The canonical connection

Let $H/L$ be a reductive homogeneous space and $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$ be a reductive decomposition (i.e., $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$), where $\mathfrak{l}$ (resp. $\mathfrak{l}$) is the Lie algebra of $H$ (resp. $L$). Also, let $\pi : P \to H/L$ be a principal $G$-bundle, where $G$ is a Lie group. Assume that $H$ acts on $P$ as $\pi(h \cdot u) = h \cdot \pi(u)$ for any $u \in P$ and any $h \in H$. Then there uniquely exists a connection $\omega$ of $P$ such that, for any $X \in \mathfrak{m}$ and any $u \in P$, $t \mapsto (\text{exp} tX)(u)$ is a horizontal curve with respect to $\omega$, where exp is the exponential map of $H$. This connection $\omega$ is called the canonical connection of $P$ associated with the reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$. 

5
3 Proof of Theorems A∼F

In this section, we shall first prove Theorems A∼F. We use the notations in Introduction. Let \( H \sim G/K \) be a Hermann action and \( Z_0 \) be an element of \( b \). Set \( M := H(\text{Exp}Z_0) \).

**Proof of Theorem A.** Denote by \( \mathcal{H} \) the mean curvature vector of \( M \). From (2.1) and (2.3), we have

\[
\langle \mathcal{H}_{\text{Exp}Z_0}, \rho_{Z_0}^S(h)(g_0v) \rangle = -\sum_{i=1}^r \sum_{\beta \in \Delta_+^{\mathcal{V}} \setminus \Delta_{Z_0}^{\mathcal{V}}} \frac{n_i^\beta m_{\beta}^{\mathcal{V}}}{\tan \beta(Z_0)} \beta_i(v) + \sum_{i=1}^r \sum_{\beta \in \Delta_+^{\mathcal{H}} \setminus \Delta_{Z_0}^{\mathcal{H}}} n_i^\beta m_{\beta}^{\mathcal{H}} \tan \beta(Z_0) \beta_i(v)
\]

for any \( v \in b \) and any \( h \in L \). Hence, \( \mathcal{H}_{\text{Exp}Z_0} \) vanishes if and only if the following relations hold:

\[
(3.1) \sum_{\beta \in \Delta_+^{\mathcal{V}} \setminus \Delta_{Z_0}^{\mathcal{V}}} \frac{n_i^\beta m_{\beta}^{\mathcal{V}}}{\tan \beta(Z_0)} = \sum_{\beta \in \Delta_+^{\mathcal{H}} \setminus \Delta_{Z_0}^{\mathcal{H}}} n_i^\beta m_{\beta}^{\mathcal{H}} \tan \beta(Z_0) \quad (i = 1, \ldots, r).
\]

Since \( Z_0 \) satisfies the condition (I) or (II) in Theorem A, (3.1) holds, that is, \( \mathcal{H}_{\text{Exp}Z_0} \) vanishes. Therefore \( M \) is minimal.

Next we shall show that there exists a reductive decomposition \( \mathfrak{h} = \mathfrak{l} + \mathfrak{m} \) with \( B_0(\mathfrak{l}, \mathfrak{m}) = 0 \). Easily we have

\[
(3.2) \mathfrak{l} = \mathfrak{z}_{\mathfrak{r} \cap \mathfrak{h}}(\mathfrak{b}) + \sum_{\beta \in \Delta_{Z_0}^{\mathcal{V}}} (\mathfrak{e}_\beta \cap \mathfrak{h}) + \sum_{\beta \in \Delta_{Z_0}^{\mathcal{H}}} (\mathfrak{p}_\beta \cap \mathfrak{h}).
\]

Define a subspace \( \mathfrak{m} \) of \( \mathfrak{h} \) by

\[
(3.3) \mathfrak{m} := \mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b}) + \sum_{\beta \in \Delta_{Z_0}^{\mathcal{V}}} (\mathfrak{e}_\beta \cap \mathfrak{h}) + \sum_{\beta \in \Delta_{Z_0}^{\mathcal{H}}} (\mathfrak{p}_\beta \cap \mathfrak{h}).
\]

Easily we can show that \( \mathfrak{h} = \mathfrak{l} + \mathfrak{m} \) is a reductive decomposition and that \( B_0(\mathfrak{l}, \mathfrak{m}) = 0 \).

Next we shall show that \( \nabla_{\omega^m} = \nabla^\perp \). Take \( v \in \mathfrak{b} \subset g_0^{-1} T_{\text{Exp}Z_0}M \). Set \( g_s := \exp(1-s)Z_0 \). Let \( Z : [0,1] \to \mathfrak{b} \) be a \( C^\infty \)-curve such that \( Z(0) = Z_0 \) and that \( Z((0,1]) \) is contained in a fundamental domain of the Coxeter group associated with the principal \( H \)-orbit at an intersection point of the orbit and \( \mathfrak{b} \). Set \( M_s := H(\text{Exp}Z(1-s)) \ (0 \leq s \leq 1) \). Denote by \( A^s \) the shape tensor of \( M_s \) and \( \nabla \) the
Levi-Civita connection of $G/K$. Let $\tilde{v}^s$ be the $H$-equivariant normal vector field of $M_s$ ($0 \leq s < 1$) arising from $g_{s*}v$. Since $M_s$ ($0 \leq s < 1$) is a principal orbit of a Hermann (hence hyperpolar) action, $\tilde{v}^s$ is well-defined and it is a parallel normal vector field with respect to $\nabla^\perp$. Take $X \in \mathfrak{t}_\beta \cap \mathfrak{h} (\subset \mathfrak{m})$ ($\beta \in \Delta^H \setminus \Delta^H_{Z_0}$). Then, by using (2.3), we have

$$\tilde{\nabla}_{X_{\exp Z(1-s)}}^s = -A_\beta^s X^*_{\exp Z(1-s)} = \frac{\beta(v)}{\tan \beta(Z_0)} X^*_{\exp Z(1-s)},$$

and hence

$$\tilde{\nabla}_{X^*_{\exp Z_0}} (\exp tX)_{*\exp(Z_0)}(v) = \lim_{s \to 1-0} \tilde{\nabla}_{X^*_{\exp Z(1-s)}}^s = \frac{\beta(v)}{\tan \beta(Z_0)} X^*_{\exp Z_0} \in T_{\exp Z_0}M.$$

Hence we obtain $\nabla^\perp_{X^*_{\exp Z_0}} (\exp tX)_{*\exp(Z_0)}(v) = 0$. Take $Y \in \mathfrak{p}_\beta \cap \mathfrak{h} (\subset \mathfrak{m})$ ($\beta \in \Delta^H \setminus \Delta^H_{Z_0}$). Then, by using (2.3), we have

$$\tilde{\nabla}_{Y^*_{\exp Z(1-s)}}^s = -A_\beta^s Y^*_{\exp Z(1-s)} = -\beta(v) \tan \beta(Z_0) Y^*_{\exp Z(1-s)},$$

and hence

$$\tilde{\nabla}_{Y^*_{\exp Z_0}} (\exp tY)_{*\exp Z_0}(v) = \lim_{s \to 1-0} \tilde{\nabla}_{Y^*_{\exp Z(1-s)}}^s = -\beta(v) \tan \beta(Z_0) Y^*_{\exp Z_0} \in T_{\exp Z_0}M.$$

Hence we obtain $\nabla^\perp_{Y^*_{\exp Z_0}} (\exp tY)_{*\exp(Z_0)}(v) = 0$. Therefore, it follows from the arbitrariness of $X$, $Y$ and $\beta$ that $t \mapsto (\exp t\hat{X})_{*\exp Z_0}(v)$ is $\nabla^\perp$-parallel along $t \mapsto (\exp t\hat{X})(\exp Z_0)$ for any $\hat{X} \in \mathfrak{m}$. Take any $h \in L$. Similarly, we can show that $t \mapsto (\exp t\hat{X})_{*\exp Z_0}(\rho_{Z_0}^S(h)(g_{0*}v))$ is $\nabla^\perp$-parallel along $t \mapsto (\exp t\hat{X})(\exp Z_0)$ for any $\hat{X} \in \mathfrak{m}$. Note that this fact has been showed in [IST] in different method. On the other hand, it follows from the definition of $\omega$ that $t \mapsto (\exp t\hat{X})_{*\exp Z_0}(\rho_{Z_0}^S(h)(g_{0*}v))$ is $\nabla^\omega$-parallel along $t \mapsto (\exp t\hat{X})(\exp Z_0)$ for any $\hat{X} \in \mathfrak{m}$. Therefore we obtain $\nabla^\perp = \nabla^\omega$. The statement for $v \in T^*_L M$ follows from (2.3) directly.

$q.e.d.$

Next we prove Theorem B.

**Proof of Theorem B.** This statement of this theorem follows from (2.3) directly.

$q.e.d.$
Next we prove Theorems C~F.

Proof of Theorems C~F. Define a diffeomorphism \( \psi : H/L \to M \) by \( \psi(hL) := h \cdot \exp Z_0 \) \((h \in H)\). Next we shall show that \((\psi^* g_t)_{eL} = cB_g|_{m \times m}\), where

\[
\begin{align*}
  c &= \begin{cases} 
    3 & \text{(in case of Theorems C)} \\
    1 & \text{(in case of Theorem D)} \\
    \frac{7}{6} & \text{(in case of Theorem E)} \\
    \frac{11}{6} & \text{(in case of Theorem F)}.
  \end{cases}
\end{align*}
\]

In the sequel, we omit the notation \( \psi^* \). For each \( X \in \mathfrak{m} = T_{eL}(H/L) = T_{\exp Z_0} M \), denote by \( X^* \) the Killing field on \( M \) associated with \( X \), that is, \( X_p^* := \frac{d}{dt} |_{t=0} (\exp tX)(p) \) \((p \in M)\). From the definition of \( \psi \), we have \( \psi_{eL} X = X^*_{\exp Z_0} \). Take \( S_{\beta_1} \in f_{\beta_1} \cap \mathfrak{h} \) \((\beta_1 \in \Delta_+^H \setminus \Delta_{Z_0}^V)\) and \( \hat{S}_{\beta_2} \in p_{\beta_2} \cap \mathfrak{h} \) \((\beta_2 \in \Delta_+^V \setminus \Delta_{Z_0}^V)\). Let \( T_{\beta_1} \) be the element of \( p_{\beta_1} \cap \mathfrak{g} \) such that \( \text{ad}(b)(S_{\beta_1}) = \beta_1(b)T_{\beta_1} \) for any \( b \in \mathfrak{g} \). Then we have

\[
(3.4) \quad \psi_{eL}(S_{\beta_1}) = (S_{\beta_1}^*)_{\exp Z_0} = -\sin \beta_1(Z_0)(\exp Z_0)_*(T_{\beta_1})
\]

and

\[
(3.5) \quad \psi_{eL}(\hat{S}_{\beta_2}) = (\hat{S}_{\beta_2}^*)_{\exp Z_0} = \cos \beta_2(Z_0)(\exp Z_0)_*(\hat{T}_{\beta_2}).
\]

Hence, since \( H \) and \( Z_0 \) is as in Theorems C~F, we have \((g_t)_{eL}(S_{\beta_1}, S_{\beta_1}) = cB_g(S_{\beta_1}, S_{\beta_1}) \) and \((g_t)_{eL}(\hat{S}_{\beta_2}, \hat{S}_{\beta_2}) = cB_g(\hat{S}_{\beta_2}, \hat{S}_{\beta_2}) \). If the cohomogeneity of the \( H \)-action is equal to the rank of \( G/K \), then we have \( \text{Ker } A_v = \emptyset \). Therefore we obtain \((g_t)_{eL} = cB_g|_{m \times m}\). Also, in Theorems C~E, \( \cap_{v \in T_{\beta_1}^H M} \text{Ker } A_v = \{0\} \) follows from the statement for \( \cap_{v \in T_{\beta_1}^H M} \text{Ker } A_v \) in Theorem A directly. q.e.d.

4 Examples

In this section, we give examples of a Hermann action \( H \curvearrowright G/K \) and \( Z_0 \in \bar{C} \) as in Theorems B, C and F. We use the notations as in Introduction.

Example 1. We consider the isotropy action of \( SU(3n+3)/SO(3n+3) \). Then we have \( \Delta_+ = \Delta'_+ = \Delta''_+ \) (which is of \( (a_{3n+2}) \)-type) and \( \Delta'_+ = 0 \). Let \( \Pi = \{\beta_1, \ldots, \beta_{3n+2}\} \) be a simple root system of \( \Delta'_+ \), where we order \( \beta_1, \ldots, \beta_{3n+2} \) as the Dynkin diagram of \( \Delta'_+ \) is as in Fig. 1. \( \Delta'_+ = \{\beta_i + \cdots + \beta_j | 1 \leq i, j \leq 3n+2\} \). For any \( \beta \in \Delta'_+ \), we have \( m_{\beta} = 1 \). Let \( Z_0 \) be the point of \( \mathfrak{b} \) defined by \( \beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \frac{\pi}{2} \) and \( \beta_i(Z_0) = 0 \) \((i \in \{1, \ldots, 3n+2\} \setminus \{n+1, 2n+2\})\). Clearly we have \( m'_{\beta} = 1 \),
$m^H_\beta = 0$ and $\beta(Z_0) \equiv 0, \frac{\pi}{3}$ or $\frac{2\pi}{3}$ (mod $\pi$) for any $\beta \in \Delta'_+$. For simplicity, set $\beta_{ij} := \beta_i + \cdots + \beta_j$ ($1 \leq i \leq j \leq 3n + 2$). Easily we can show

\[
\{\beta \in \Delta^V_+ \mid \beta(Z_0) \equiv \frac{\pi}{3} \text{ (mod } \pi)\} \\
= \{\beta_{ij} \mid 1 \leq i \leq n + 1 \leq j < 2n + 2, \text{ or } n + 1 < i \leq 2n + 2 \leq j \leq 3n + 2\}
\]

and

\[
\{\beta \in \Delta^V_+ \mid \beta(Z_0) \equiv \frac{2\pi}{3} \text{ (mod } \pi)\} \\
= \{\beta_{ij} \mid 1 \leq i \leq n + 1, 2n + 2 \leq j \leq 3n + 2\}.
\]

From these facts, it follows that the condition (I) holds. Thus $Z_0$ is as in the statement of Theorem C. Also, it is easy to show that $M$ is not austere.

\[
\begin{array}{c}
\beta_1 & \beta_2 \\
\beta_{n+1} & \beta_{n+2}
\end{array}
\]

Figure 1.

Example 2. We consider the isotropy action of $SU(6n+6)/Sp(3n+3)$. Then we have $\Delta_+ = \Delta'_+ = \Delta^V_+$ (which is of $(a_{3n+2})$-type) and $\Delta^H_+ = \emptyset$. Let $\Pi = \{\beta_1, \cdots, \beta_{3n+2}\}$ be a simple root system of $\Delta'_+$, where we order $\beta_1, \cdots, \beta_{3n+2}$ as above. We have $m_\beta = 4$ for any $\beta \in \Delta'_+$. Let $Z_0$ be the point of the closure of $b$ defined by $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \frac{\pi}{3}$ and $\beta_i(Z_0) = 0$ ($i \in \{1, \cdots, 3n + 2\} \setminus \{n + 1, 2n + 2\}$).

Clearly we have $m^V_\beta = 4$, $m^H_\beta = 0$ and $\beta(Z_0) \equiv 0, \frac{\pi}{3}$ or $\frac{2\pi}{3}$ (mod $\pi$) for any $\beta \in \Delta'_+$. For simplicity, set $\beta_{ij} := \beta_i + \cdots + \beta_j$ ($1 \leq i \leq j \leq 3n + 2$). Easily we can show

\[
\{\beta \in \Delta^V_+ \mid \beta(Z_0) \equiv \frac{\pi}{3} \text{ (mod } \pi)\} \\
= \{\beta_{ij} \mid 1 \leq i \leq n + 1 \leq j < 2n + 2, \text{ or } n + 1 < i \leq 2n + 2 \leq j \leq 3n + 2\}
\]

and

\[
\{\beta \in \Delta^V_+ \mid \beta(Z_0) \equiv \frac{2\pi}{3} \text{ (mod } \pi)\} \\
= \{\beta_{ij} \mid 1 \leq i \leq n + 1, 2n + 2 \leq j \leq 3n + 2\}.
\]

From these facts, it follows that the condition (I) holds. Thus $Z_0$ is as in the statement of Theorem C. Also, it is easy to show that $M$ is not austere.

Example 3. We consider the isotropy action of $SU(3)/S(U(1) \times U(2))$ (2-dimensional complex projective space). Then we have $\Delta_+ = \Delta'_+ = \Delta^V_+ = \{\beta, 2\beta\}$ and $\Delta^H_+ = \emptyset$, $m_\beta = 2$ and $m_{2\beta} = 1$. Let $Z_0$ be the point of $b$ defined by $\beta(Z_0) = \frac{\pi}{3}$. Clearly $Z_0$ satisfies the condition (I). Thus $Z_0$ is as in the statement of Theorem C. Also, it is easy to show that $M$ is not austere.
Example 4. We consider the isotropy action of $Sp(3n + 2)/U(3n + 2)$. Then we have $\Delta_+ = \Delta'_+ = \Delta^V_+$ (which is of $(3n+2)$-type) and $\Delta^H_+ = 0$. Let $\Pi = \{\beta_1, \cdots, \beta_{3n+2}\}$ be a simple root system of $\Delta'_+$, where we order $\beta_1, \cdots, \beta_{3n+2}$ as the Dynkin diagram of $\Delta'_+$ is as in Fig. 2. We have $m_\beta = 1$ for any $\beta \in \Delta'_+$. Let $Z_0$ be the point of $b$ defined by $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \beta_{3n+2}(Z_0) = \frac{\pi}{3}$ and $\beta_i(Z_0) = 0$ ($i \in \{1, \cdots, 3n + 2\} \setminus \{n + 1, 2n + 2, 3n + 2\}$). Clearly we have $m_{\beta}^V = 1$, $m_{\beta}^H = 0$ and $\beta(Z_0) \equiv 0, \frac{\pi}{3}$ or $\frac{2\pi}{3}$ (mod $\pi$) for any $\beta \in \Delta'_+$. For simplicity, set $\beta_{ij} := \beta_i + \cdots + \beta_j$ ($1 \leq i \leq j \leq 3n + 2$), $\hat{\beta}_i := 2(\beta_i + \cdots + \beta_{3n+1}) + \beta_{3n+2}$ and $\hat{\beta}_{ij} := \beta_i + \cdots + \beta_{j-1} + 2(\beta_j + \cdots + \beta_{3n+1}) + \beta_{3n+2}$ ($1 \leq i < j \leq 3n + 1$). Easily we can show

$\{\beta \in \Delta^V_+ | \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}\}$

$= \{\beta_{ij} | 1 \leq i \leq n + 1 \leq j < 2n + 2 \text{ or } n + 1 < i \leq 2n + 2 \leq j < 3n + 2$

or $2n + 3 \leq i \leq j = 3n + 2\}$

$\cup \{\hat{\beta}_i | 2n + 3 \leq i \leq 3n + 1\}$

$\cup \{\hat{\beta}_{ij} | 2n + 3 \leq i < j \leq 3n + 1 \text{ or } 1 \leq i < n + 1 \leq j \leq 2n + 2\}$

and

$\{\beta \in \Delta^V_+ | \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}\}$

$= \{\beta_{ij} | "1 \leq i \leq n + 1 \& 2n + 2 \leq j \leq 3n + 1" \text{ or } "n + 2 \leq i \leq 2n + 2 \& j = 3n + 2"\}$

$\cup \{\hat{\beta}_i | 1 \leq i \leq n + 1\}$

$\cup \{\hat{\beta}_{ij} | 1 \leq i < j \leq n + 1 \text{ or } n + 2 \leq i \leq 2n + 2 < j \leq 3n + 1\}$.

From these facts, it follows that the condition (I) holds. Thus $Z_0$ is as in the statement of Theorem C. Also, it is easy to show that $M$ is not austere.

---

From Table 1 and 2 in [K2], we shall list up Hermann actions of cohomogeneity two on irreducible symmetric spaces of compact type and rank two satisfying

(i) $m_\beta^V = m_\beta^H$ (for $\beta \in \Delta'_+$) or (ii) $\Delta^V_+ \cap \Delta^H_+ = \emptyset$.

All of such Hermann actions satisfying (i) are as in Table 1. In Table 1, $\beta$ means

$m_\beta^V = m_\beta^H = m$. All of such Hermann actions satisfying (ii) are the dual actions (see Table 3) of Hermann actions on symmetric spaces of non-compact type as in
Table 2. In Table 3, \( \beta \) means \( m_\beta^V \) or \( m_\beta^H \) is equal to \( m \). Since the Hermann actions in Table 2 are commutative, so are also the Hermann actions in Table 3. Also, since \( \Delta^V_+ \cap \Delta^H_+ = \emptyset \) as in Table 3 and \( G/K \) is irreducible, there exists an inner automorphism \( \rho \) of \( G \) with \( \rho(K) = H \) by Proposition 4.39 in [I]. According to the proof of the proposition, \( \rho \) is given explicitly by \( \rho = \text{Ad}_G(\exp b) \), where \( \text{Ad}_G \) is the adjoint representation of \( G \) and \( b \) is the element of \( b \) satisfying

\[
(\beta_1(b), \beta_2(b)) = \begin{cases} 
(0, \frac{\pi}{2}) & \text{(in case of (1),(2),(3),(4),(6),(9),(10),(11))} \\
(\frac{\pi}{2}, 0) & \text{(in case of (5),(7))} \\
(\frac{\pi}{2}, \frac{\pi}{2}) & \text{(in case of (8)).}
\end{cases}
\]

| \( H \wr G/K \) | \( \Delta^V_+ = \Delta^H_+ \) |
|-----------------|------------------|
| \( SO(6) \wr SU(6)/Sp(3) \) | \( \{\beta_1, \beta_2, \beta_1 + \beta_2\} \) |
| \( SO(2)^2 \times SO(3)^2 \wr (SO(5) \times SO(5))/SO(5) \) | \( \{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\} \) |
| \( SU(2)^4 \wr (Sp(2) \times Sp(2))/Sp(2) \) | \( \{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\} \) |
| \( Sp(4) \wr E_6/F_4 \) | \( \{\beta_1, \beta_2, \beta_1 + \beta_2\} \) |
| \( SU(2)^4 \wr (G_2 \times G_2)/G_2 \) | \( \{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2, 3\beta_1 + 2\beta_2\} \) |

Table 1.

| \( \) | \( \) |
|---|---|
| (1) | \( SO_0(1,2) \wr SL(3,\mathbb{R})/SO(3) \) |
| (2) | \( Sp(1,2) \wr SU^*(6)/Sp(3) \) |
| (3) | \( U(2,3) \wr SO^*(10)/U(5) \) |
| (4) | \( SO_0(2,3) \wr SO(5,\mathbb{C})/SO(5) \) |
| (5) | \( U(1,1) \wr Sp(2,\mathbb{R})/U(2) \) |
| (6) | \( Sp(2,\mathbb{R}) \wr Sp(2,\mathbb{C})/Sp(2) \) |
| (7) | \( Sp(1,1) \wr Sp(2,\mathbb{C})/Sp(2) \) |
| (8) | \( SO^*(10) \cdot U(1) \wr E_6^{-14}/Spin(10) \cdot U(1) \) |
| (9) | \( F_4^{\cdot 20} \wr E_6^{\cdot 26}/F_4 \) |
| (10) | \( SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \wr G_2^2/\text{SO}(4) \) |
| (11) | \( G_2^2 \wr G_2^2/\text{SO}(4) \) |

Table 2.
Denote by \( a \) (non-totally geodesic) austere submanifold.

**Proposition 4.1.** Let \( H \ltimes G/K \) be a Hermann action in Table 1 and \( Z_0 \) an element of \( \mathfrak{b} \) satisfying \((\beta_1(Z_0),\beta_2(Z_0)) = (0, \frac{\pi}{4}), \left(\frac{\pi}{4}, 0\right) \) or \( \left(\frac{\pi}{4}, \frac{\pi}{4}\right) \). Then \( M = H(\exp Z_0) \) is a (non-totally geodesic) austere submanifold.

Denote by \( Z_{(a,b)} \) the element \( Z \) of \( \mathfrak{b} \) satisfying \((\beta_1(Z),\beta_2(Z)) = (a,b)\). In the case where \( \triangle' \) is of type \( (a_2) \), three points of \( \mathfrak{b} \) as in Proposition 4.1 are as in Figure 3.

\[
\begin{array}{|c|c|c|}
\hline
& H \ltimes G/K & \triangle' \ltimes V \\
\hline
(1) & SO_0(1,2) \ltimes SU(3)/SO(3) & \{\beta_1\} \quad \{\beta_2, \beta_1 + \beta_2\} \\
\hline
(2) & Sp(1,2) \ltimes SU(6)/Sp(3) & \{\beta_1\} \quad \{\beta_2, \beta_1 + \beta_2\} \\
\hline
(3) & U(2,3) \ltimes SO(10)/U(5) & \{\beta_1, 2\beta_1, 2\beta_1 + 2\beta_2\} \quad \{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\} \\
\hline
(4) & SO_0(2,3) \ltimes (SO(5) \times SO(5))/SO(5) & \{\beta_1\} \quad \{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\} \\
\hline
(5) & U(1,1) \ltimes Sp(2)/U(2) & \{\beta_2, 2\beta_1 + \beta_2\} \quad \{\beta_1, \beta_1 + \beta_2\} \\
\hline
(6) & Sp(2,\mathbb{R}) \ltimes (Sp(2) \times Sp(2))/Sp(2) & \{\beta_1\} \quad \{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\} \\
\hline
(7) & Sp(1,1) \ltimes (Sp(2) \times Sp(2))/Sp(2) & \{\beta_2, 2\beta_1 + \beta_2\} \quad \{\beta_1, \beta_1 + \beta_2\} \\
\hline
(8) & (SO(10) \cdot U(1)) \ltimes E_6/Spin(10) \cdot U(1) & \{\beta_1, 2\beta_1, 2\beta_1 + 2\beta_2\} \quad \{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\} \\
\hline
(9) & (F_4^{-20})^* \ltimes E_6/F_4 & \{\beta_1\} \quad \{\beta_2, \beta_1 + \beta_2\} \\
\hline
(10) & (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))^* \ltimes G_2/\mathbb{S}O(4) & \{\beta_1, 3\beta_1 + 2\beta_2\} \quad \{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2\} \\
\hline
(11) & (G_2^2)^* \ltimes (G_2 \times G_2)/G_2 & \{\beta_1, 3\beta_1 + 2\beta_2\} \quad \{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2\} \\
\hline
\end{array}
\]

**Table 3.**

According to Theorem B, we obtain the following fact.

**Proposition 4.1.** Let \( H \ltimes G/K \) be a Hermann action in Table 1 and \( Z_0 \) an element of \( \mathfrak{b} \) satisfying \((\beta_1(Z_0),\beta_2(Z_0)) = (0, \frac{\pi}{4}), \left(\frac{\pi}{4}, 0\right) \) or \( \left(\frac{\pi}{4}, \frac{\pi}{4}\right) \). Then \( M = H(\exp Z_0) \) is a (non-totally geodesic) austere submanifold.

Denote by \( Z_{(a,b)} \) the element \( Z \) of \( \mathfrak{b} \) satisfying \((\beta_1(Z),\beta_2(Z)) = (a,b)\). In the case where \( \triangle' \) is of type \((a_2)\), three points of \( \mathfrak{b} \) as in Proposition 4.1 are as in Figure 3.
Proposition 4.2. Let $H \acts G/K$ be a Hermann action in Table 3 and $Z_0$ an element of the closure of $\tilde{C}(\subset \mathfrak{b})$ such that $H(\text{Exp} \ Z_0)$ is minimal. Then, as in Tables 4 ~ 13, $Z_0$ satisfies the condition in Theorem C or F, or it does not satisfy the conditions in Theorems C~F.

Remark 4.1. There exist exactly seven elements $Z_0$ of the closure of $\tilde{C}(\subset \mathfrak{b})$ such that $H(\text{Exp} \ Z_0)$ is minimal.

| $(a, b)$                              | $Z_{(a,b)}$                          | $M = SO_0(1, 2)^* (\text{Exp} \ Z_{(a,b)})$ | $\dim M$ |
|---------------------------------------|--------------------------------------|---------------------------------------------|----------|
| $(0, -\frac{\pi}{2})$                | as in Theorem F                      | one-point set                               | 0        |
| $(0, \frac{\pi}{2})$                 | as in Theorem F                      | one-point set                               | 0        |
| $(\pi, -\frac{\pi}{2})$              | as in Theorem F                      | one-point set                               | 0        |
| $(0, 0)$                              | as in Theorem F                      | totally geodesic                            | 2        |
| $(\frac{\pi}{2}, 0)$                 | as in Theorem F                      | totally geodesic                            | 2        |
| $(\frac{\pi}{2}, -\frac{\pi}{2})$   | as in Theorem F                      | totally geodesic                            | 2        |
| $(\pi, \frac{\pi}{2})$               | as in Theorem C                      | not austere                                 | 3        |

$SO_0(1, 2)^* \acts SU(3)/SO(3)$
$(\dim SU(3)/SO(3) = 5)$

Table 4.

The positions of $Z_0$’s in Table 4 are as in Figure 4.

![Figure 4](image-url)
\[
Z_{(a,b)}(a,b) = \text{Sp}(1,2)^*(\text{Exp} Z_{(a,b)}) \\quad \text{dim} \ M
\]

\[
\begin{array}{|c|c|c|c|}
\hline
(a, b) & Z_{(a,b)} & M = \text{Sp}(1,2)^*(\text{Exp} Z_{(a,b)}) & \text{dim} \ M \\
\hline
(0, -\frac{\pi}{4}) & \text{as in Theorem F} & \text{one-point set} & 0 \\
(0, \frac{\pi}{4}) & \text{as in Theorem F} & \text{one-point set} & 0 \\
(\pi, -\frac{\pi}{4}) & \text{as in Theorem F} & \text{one-point set} & 0 \\
(0,0) & \text{as in Theorem F} & \text{totally geodesic} & 8 \\
\left(\frac{\pi}{2}, 0\right) & \text{as in Theorem F} & \text{totally geodesic} & 8 \\
\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) & \text{as in Theorem F} & \text{totally geodesic} & 8 \\
\left(\frac{\pi}{4}, -\frac{\pi}{4}\right) & \text{as in Theorem C} & \text{not austere} & 12 \\
\hline
\end{array}
\]

\[
\text{Sp}(1,2)^* \retimes SU(6)/\text{Sp}(3) \\
(\text{dim} \ SU(6)/\text{Sp}(3) = 14)
\]

**Table 5.**

The positions of \(Z_0\)'s in Table 5 are as in Figure 4.

\[
\begin{array}{|c|c|c|c|}
\hline
(a, b) & Z_{(a,b)} & M = U(2,3)^*(\text{Exp} Z_{(a,b)}) & \text{dim} \ M \\
\hline
(0, \frac{\pi}{2}) & \text{as in Theorem F} & \text{one-point set} & 0 \\
(0,0) & \text{as in Theorem F} & \text{totally geodesic} & 12 \\
\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) & \text{as in Theorem F} & \text{totally geodesic} & 8 \\
\left(\arctan\sqrt{\frac{7}{3}}, \arctan\frac{\pi}{2}\right) & \text{not as in Theorems C-F} & \text{not austere} & 14 \\
\left(0, \arctan\frac{1}{\sqrt{13}}\right) & \text{not as in Theorems C-F} & \text{not austere} & 13 \\
\left(\arctan\frac{\pi}{2}, -\arctan\frac{\pi}{3}\right) & \text{not as in Theorems C-F} & \text{not austere} & 17 \\
\left(a_0, b_0\right) & \text{not as in Theorems C-F} & \text{not austere} & 18 \\
\hline
\end{array}
\]

\[
U(2,3)^* \retimes SO(10)/U(5) \\
(\text{dim} \ SO(10)/U(5) = 20)
\]

**Table 6.**

The positions of \(Z_0\)'s in Table 6 are as in Figure 5. Also, the numbers \(a_0\) and \(b_0\) in Table 6 are real numbers such that \(a_0, b_0 \not\equiv \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi}\).
\[ (2\beta_1 + \beta_2)^{-1}(\frac{\pi}{2}) \]

\[ Z_{(\frac{\pi}{2}, -\frac{\pi}{2})} \]

\[ Z_{(\arctan \sqrt{\frac{3}{2}}, -\arctan \sqrt{\frac{3}{2}})} \]

\[ (\beta_1)^{-1}(0) \quad Z_{(0,0)} \]

\[ (\beta_1 + \beta_2)^{-1}(0) \quad Z_{(0, \arctan \frac{1}{\sqrt{13}})} \]

---

**Figure 5.**

---

| \((a, b)\) | \(Z_{(a, b)}\) | \(M = SO_0(2, 3)^*(\text{Exp } Z_{(a, b)})\) | \(\dim M\) |
|---|---|---|---|
| \((0, -\frac{a}{\pi})\) | as in Theorem F | one-point set | 0 |
| \((0, \frac{a}{\pi})\) | as in Theorem F | one-point set | 0 |
| \((\frac{\pi}{2}, -\frac{\pi}{2})\) | as in Theorem F | totally geodesic | 4 |
| \((0,0)\) | as in Theorem F | totally geodesic | 6 |
| \((\arctan \sqrt{\frac{3}{2}}, -\frac{\pi}{2})\) | not as in Theorems C~F | not austere | 6 |
| \((\arctan \sqrt{\frac{3}{2}}, \frac{a}{\pi} - 2 \arctan \sqrt{3})\) | not as in Theorems C~F | not austere | 6 |
| \((\arctan \frac{1}{\sqrt{13}}, -\arctan \frac{1}{\sqrt{13}})\) | not as in Theorems C~F | not austere | 8 |

\(SO_0(2, 3)^* \vartriangleleft (SO(5) \times SO(5))/SO(5)\)

\((\dim (SO(5) \times SO(5))/SO(5) = 10)\)

**Table 7.**

The positions of \(Z_0\)'s in Table 7 are as in Figure 6.

---

**Figure 6.**

15
\[(a, b)\]  \[Z_{(a,b)}\]  \[M = U(1, 1)^*(\text{Exp } Z_{(a,b)})\]  \[\dim M\]  
\[
\begin{array}{|c|c|c|}
\hline
(a, b) & Z_{(a,b)} & M = U(1, 1)^*(\text{Exp } Z_{(a,b)}) & \dim M \\
\hline
\left(\frac{\pi}{2}, 0\right) & \text{as in Theorem F} & \text{one-point set} & 0 \\
\left(-\frac{\pi}{2}, \pi\right) & \text{as in Theorem F} & \text{one-point set} & 0 \\
(0, 0) & \text{as in Theorem F} & \text{totally geodesic} & 2 \\
\left(\frac{\pi}{2}, 0\right) & \text{as in Theorem C} & \text{not austere} & 3 \\
\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) & \text{as in Theorem C} & \text{not austere} & 3 \\
(0, \frac{\pi}{2}) & \text{as in Theorem F} & \text{totally geodesic} & 3 \\
(0, \arctan \sqrt{2}) & \text{not as in Theorems C~F} & \text{not austere} & 4 \\
\hline
\end{array}
\]

\[U(1, 1)^* \cong Sp(2)/U(2)\]
\[(\dim Sp(2)/U(2) = 6)\]

Table 8.

The positions of \(Z_0\)’s in Table 8 are as in Figure 7.

\[\beta_2^{-1}(0)\]
\[Z(0, \arctan \sqrt{2})\]
\[(\beta_1 + \beta_2)^{-1}(\frac{\pi}{2})\]
\[Z(\frac{\pi}{2}, 0)\]
\[Z(\frac{\pi}{2}, 0)\]
\[Z(0, \frac{\pi}{2})\]
\[Z(0, 0)\]
\[\beta_2^{-1}(0)\]
\[Z(\frac{\pi}{2}, 0)\]
\[(2\beta_1 + \beta_2)^{-1}(0)\]

Figure 7.
$$Z(a,b) = \text{as in Theorem F}$$

$$M = \text{one-point set}$$

$$\dim M = 0$$

$$\rightarrow (\text{Exp } Z(a,b))$$

$$\dim (\text{Exp } Z(a,b)) = 0$$

Table 9.

The positions of $Z_0$'s in Table 9 are as in Figure 6.

Table 10.

The positions of $Z_0$'s in Table 10 are as in Figure 7.
\[
(\text{dim } E_6/\text{Spin}(10) \cdot U(1) = 32)
\]

Table 11.

The positions of \(Z_0\)'s in Table 11 are as in Figure 8. The numbers \(a_i\) \((i = 1, 2, 3, 4)\) and \(b\) in Table 11 are real numbers such that \(a_i, b \neq \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}\mod \pi\).
\[
M = (F_4^{-20} \setminus \text{Exp } Z_{(a,b)}) \, \dim M
\]

Table 12.

The positions of \(Z_0\)'s in Table 12 are as in Figure 4.

| \( (a, b) \) | \( Z_{(a,b)} \) | \( M = (F_4^{-20} \setminus \text{Exp } Z_{(a,b)}) \) | \( \dim M \) |
|---|---|---|---|
| \( (0, -\frac{\pi}{2}) \) | as in Theorem F | one-point set | 0 |
| \( (0, \frac{\pi}{4}) \) | as in Theorem F | one-point set | 0 |
| \( (\pi, -\frac{\pi}{2}) \) | as in Theorem F | one-point set | 0 |
| \( (0, 0) \) | as in Theorem F | totally geodesic | 16 |
| \( (\frac{\pi}{4}, 0) \) | as in Theorem F | totally geodesic | 16 |
| \( (\frac{\pi}{2}, -\frac{\pi}{2}) \) | as in Theorem F | totally geodesic | 16 |
| \( (\frac{\pi}{4}, -\frac{\pi}{2}) \) | as in Theorem C | not austere | 24 |

\[
(F_4^{-20})^* \, \lhd \, E_6/F_4
\]

\[(\dim E_6/F_4 = 26)\]

Table 13.

The positions of \(Z_0\)'s in Table 13 are as in Figure 9. The numbers \(a_4\) and \(b_2\) in Table 13 are real numbers such that \(a_4, b_2 \not\equiv \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi}\).

| \( (a, b) \) | \( Z_{(a,b)} \) | \( M = (SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))^* \setminus \text{Exp } Z_{(a,b)} \) | \( \dim M \) |
|---|---|---|---|
| \( (0, -\frac{\pi}{2}) \) | as in Theorem F | one-point set | 0 |
| \( (0, \frac{\pi}{4}) \) | as in Theorem F | one-point set | 0 |
| \( (\frac{\pi}{4}, -\frac{\pi}{2}) \) | as in Theorem F | totally geodesic | 4 |
| \( (\frac{\pi}{2}, -\frac{\pi}{2}) \) | as in Theorem C | not austere | 3 |
| \( \left(\arctan \sqrt{5}, \frac{\pi}{2} - 2 \arctan \sqrt{5}\right) \) | not as in Theorems C~F | not austere | 5 |
| \( (a_4, b_2) \) | not as in Theorems C~F | not austere | 6 |

\[
(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))^* \, \lhd \, G_2/SO(4)
\]

\[(\dim G_2/SO(4) = 8)\]

Table 13.
\[ (2\beta_1 + \beta_2)^{-1}\left(\frac{\pi}{2}\right) \]
\[ (\beta_2)^{-1}\left(-\frac{\pi}{2}\right) \]
\[ Z\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \]
\[ Z\left(0, \frac{\pi}{2}\right) \]
\[ Z\left(0, -\frac{\pi}{2}\right) \]
\[ Z\left(\arctan\sqrt{5}, \frac{\pi}{2} - 2\arctan\sqrt{5}\right) \]
\[ (\beta_1)^{-1}(0) \]
\[ Z_{(a_4, b_2)} \]
\[ Z_{(0, 0)} \]
\[ Z_{(0, \frac{\pi}{2})} \]

**Figure 9.**

| \((a, b)\) | \(Z_{(a,b)}\) | \(M = (G_2^2)^\ast(\text{Exp } Z_{(a,b)})\) | \(\dim M\) |
|---|---|---|---|
| \(0, -\frac{\pi}{2}\) | as in Theorem F | one-point set | 0 |
| \(0, \frac{\pi}{2}\) | as in Theorem F | one-point set | 0 |
| \(\frac{\pi}{2}, -\frac{\pi}{2}\) | as in Theorem F | totally geodesic | 8 |
| \(\frac{\pi}{2}, \frac{\pi}{2}\) | as in Theorem C | not austere | 6 |
| \((\arctan\sqrt{5}, \frac{\pi}{2} - 2\arctan\sqrt{5})\) | not as in Theorems C~F | not austere | 10 |
| \((a_5, b_3)\) | not as in Theorems C~F | not austere | 12 |

\[ (G_2^2)^\ast \succ (G_2 \times G_2)/G_2 \]
\[ \dim (G_2 \times G_2)/G_2 = 14 \]

**Table 14.**

The positions of \(Z_0\)'s in Table 14 are as in Figure 9. The numbers \(a_5\) and \(b_3\) in Table 14 are real numbers such that \(a_4, b_2 \not\equiv \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi}\).

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