Heavy quark masses from Fermilab Fermions

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Using automated perturbation theory techniques, we have computed the one-loop mass of Fermilab fermions, with an improved gluon action. We will present the results of these calculations, and the resulting predictions for the charm and bottom quark masses in the MSbar scheme. We report $m_c(m_c) = 1.22(9)$ GeV and $m_b(m_b) = 4.7(4)$ GeV. In addition we present results for the one-loop coefficients of the Fermilab action.

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In recent years there has been a lot of progress in lattice perturbation theory \([1, 2]\). In particular, the implementation of automated perturbation theory routines has allowed many problems to be tackled. Lattice perturbation theory has many applications, for example the determinations of improvement coefficients and renormalization factors. In addition, lattice perturbation theory is used in connecting non-perturbative simulations to quantities of interest to the wider high-energy community. An example of this is the recent determination of the strong coupling constant \(\alpha_s\) \([3]\).

In this report we present results of our one-loop calculations of current renormalizations, action parameters and quark masses for the Fermilab fermion action \([4]\). The lattice perturbation theory produces an estimate of the quark’s pole mass. This can be converted into the \(\overline{MS}\) scheme at some scale \(\mu\). We give a few details of this conversion and report preliminary values for the quark masses. These are compared to the PDG values.

Simulating heavy quarks on the lattice poses a special problem because the scale \(m_0a\) is not small for typical lattice spacings. For actions designed for light quarks \(O(m_0a)\) errors can be large. There are a few different approaches to this problem. One popular one is the Fermilab approach, which resums all mass dependance into the coefficients in the action.

The Fermilab action is

\[
S = a^4 \sum_x \bar{\psi}(x) \left[ m_0 + \frac{1 + \gamma_4}{2} D_4^+ - \frac{1 - \gamma_4}{2} D_4^- + \zeta \bar{\gamma} \cdot \vec{D} \right.
\]

\[
\left. - \frac{ar_s}{2} \xi^{(3)} + \frac{iacE}{2} \bar{\gamma} \cdot \vec{B} - \frac{acE}{2} \bar{\alpha} \cdot \vec{E} \right] \psi(x) \tag{1}
\]

where definitions of the various operators can be found in \([4]\). This action is designed to smoothly interpolate between the zero mass and infinite-mass limits. As such it is ideal for simulations of charm quarks, where other methods (such as NRQCD) might have problems \([5]\). All the coefficients in the Fermilab action are mass dependent, however at tree level we can use

\[
\zeta = r_s = c_E = c_B = 1. \tag{2}
\]

Using our automated perturbation theory techniques we have completed all the “basic” one-loop calculations for this action. These are the mass and wavefunction renormalizations, the renormalization of heavy-light and heavy-heavy vector and axial vector currents and the one-loop matching of the action parameters \(c_B\) and \(c_E\). In the following sections we will present results for the action parameters and the mass renormalizations.

All of the calculations presented here were carried out with using our automated perturbation theory codes. Apart from the wavefunction renormalization all these quantities are infrared finite and gauge invariant. Infrared divergences in individual diagrams were regulated by using a gluon mass.

To match the coefficients \(c_B(c_E)\) we compute the scattering of a quark off of a background chromo-magnetic(electric) field in both the lattice and continuum field theories, then tune the action parameters until the difference vanishes. The relevant diagrams are shown in figure \([4]\).

One interesting feature of our calculation is the use of lattice to lattice matching \([6]\). Rather than computing the continuum contribution using standard methods, we use a simple lattice theory, with a spacing \(a'\) that is driven very small. Figure \([3]\) illustrates this for two cases, naive fermions (which have a quadratic approach to \(a' = 0\)) and Wilson fermions (which have a linear \(a \to 0\)
behaviour). What is shown in figure 2 is actually the difference between the two sets of diagrams (Fermilab at spacing $a$ and Wilson/naive at spacing $a'$) there are additional counterterms in the matching coming from the one loop part of $\zeta$, [7] has details. One sees that the same result for the matching coefficient $\delta_B$ is obtained using the continuum limit ($a' \to 0$) of either Wilson or naive quarks for the continuum side of the matching.

The one-loop contribution to $c_B$ is plotted in figure 3. The results for $c_E$ are very similar. It is clear from the figure that the result, when tadpole improved, is nearly zero over the whole range of interesting masses ($0 < m_0a < 2$). This means that errors due to using only the tree level action parameters have likely been overestimated. This conclusion only applies if the action has been tadpole improved. The unimproved coefficients are quite large.

To date all lattice determinations of the hyperfine splittings in the $J/\phi$ system have come out too low ([8] and [9]). These splittings are quite sensitive to the coefficient of $\Sigma \cdot B$ so it was believed that the one-loop determination of $c_B$ would bring the splittings up. This is not the case, the one-loop coefficient is very small. However, there is evidence [9] that the discrepancy in the hyperfine splittings is decreasing as $a \to 0$. A determination with the fully $O(a^2)$ improved Fermilab action [10] would be very useful.

In addition to the action parameters, we have computed the quark masses for the Fermilab action. This calculation is similar to [11], however we have used the Symanzik improved gluon action. There are two masses to compute in the Fermilab formalism, the rest mass $M_1$ and the
kinetic mass $M_2$. These are defined via the small $pa$ expansion of the quark energy

$$E = M_1 + \frac{p^2}{2M_2} + O(p^4 a^4).$$ (3)

The rest mass has the perturbative expansion $M_1 = M_1^{[0]} + \alpha_V(q^*)M_1^{[1]} + \cdots$ where $M_1^{[0]} = \log(1 + m_0 a)$ and $\alpha_V(q^*)$ is the QCD coupling in the $V$ scheme evaluated at the BLM [12] scale. The kinetic mass is usually expressed as follows

$$M_2 = Z_{M_2} \exp M_1 \sinh M_1 \frac{1}{1 + \sinh M_1}$$ (4)

where $M_1$ is the all orders rest mass and

$$Z_{M_2} = 1 + \alpha_V(q^*)Z_{M_2}^{[1]} + \cdots.$$ (5)

Figures 4 and 5 show the one loop coefficients of the rest mass and the kinetic mass factor $Z_{M_2}$ over a wide range of input bare masses. In all cases the we see a smooth transition from the small to large mass limits.

The values $M_1$ and $M_2$ can be used to provide two different estimates of the pole mass of the quark [13]. The first method is to estimate the binding energy, $B_1 a = M_1^{QQ', \text{lat}} - N_Q M_1 a$, where $M_1^{QQ', \text{lat}}$ is a spin average meson mass computed on the lattice ($J/\psi/\eta_c$ for the $c$ quark, $B_s/B_s^*$ for the $b$ quark) and $N_Q$ is the number of heavy quarks in the meson (2 and 1, respectively). The pole mass is then $m_{\text{pole}} = M_1^{\bar{Q}Q', \text{expt}} - B_1$, where $M_1^{\bar{Q}Q', \text{expt}}$ is the meson mass taken from experiment. Beyond the truncation of the perturbation series this method suffers from two major sources of error. The first is that one needs to divide out the lattice spacing $a$, and the second is that it is quite sensitive to the bare mass $m_0$ used as input.

The second method for estimating the pole mass does not suffer from these errors. One begins with a perturbative determination of $M_2 a$ and takes

$$m_{\text{pole}} = (M_2 a)^{\text{PT}} \frac{M_1^{\bar{Q}Q', \text{expt}}}{M_1^{QQ', \text{lat}} a}.$$ (6)
This method avoids the errors of the method one, the dependence on the lattice spacing cancels and a small mistuning of the bare mass largely cancels in the ratio.

Once we have a value for the pole mass we can convert it into a value for the $\overline{MS}$ mass at some scale $\mu$ using

$$m_{\text{pole}} = \overline{m}(\mu) \left\{ 1 + \frac{4}{3\pi} \left[ 1 + \log \left( \frac{\mu^2}{m_0^2} \right) \right] \alpha_V(q^*) \right\}$$

We follow the conventional practice, and quote $\overline{m}(\mu)$ at the scale of the $\overline{MS}$ mass itself, $\mu = \overline{m}$. We use the BLM method to determine the scale $q^*$ in the coupling $\alpha_V(q^*)$. We estimate the relative systematic error coming from the neglected higher-orders in the perturbative matching as $\pm \alpha^2_V(q^*)$.

We obtain

$$\overline{m}_c^{\text{method 1}}(\overline{m}_c) = 1.24(1)(9)\text{ GeV} \quad \text{Fine} \quad (8)$$

$$\overline{m}_c^{\text{method 2}}(\overline{m}_c) = 1.22(0)(9)\text{ GeV} \quad \text{Fine} \quad (9)$$

$$\overline{m}_c^{\text{method 1}}(\overline{m}_c) = 1.39(1)(13)\text{ GeV} \quad \text{Coarse} \quad (10)$$
\[
\begin{align*}
\bar{m}_c^{\text{method2}}(\bar{m}_c) &= 1.30(0)(12) \text{ GeV} \quad \text{Coarse} \\
\bar{m}_b^{\text{method1}}(\bar{m}_b) &= 4.4(1)(3) \text{ GeV} \quad \text{Fine} \\
\bar{m}_b^{\text{method2}}(\bar{m}_b) &= 4.7(0)(4) \text{ GeV} \quad \text{Fine}
\end{align*}
\]

where the first error is from the determination of the lattice spacing (which only affects method one) and the second our estimate of the unknown two-loop error.

Our best values are the method two determinations on the fine lattice, which compare well with the PDG values \( \bar{m}_c^{\text{PDG}}(\bar{m}_c) = 1.25(10) \text{ GeV} \) and \( \bar{m}_b^{\text{PDG}}(\bar{m}_b) = 4.25(15) \text{ GeV} \). These results are based on preliminary values for the input masses \[\text{[14]}\] so they may change somewhat.

In this report we have presented results for the action parameters and masses of Fermilab fermions to one-loop. For truly high precision determinations two-loop precision will be needed. These calculations are in progress.

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Perturbation theory for the Fermilab action and heavy quark masses in QCD

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Outline

• The Fermilab Action
• Status of the Perturbation Theory
• Currents and Vertex Renormalization
• One-loop masses
• Results for $\overline{m}_c(\overline{m}_c)$
The Fermilab Action

\[ S = a^4 \sum_x \bar{\psi}(x) \left[ m_0 + \frac{1 + \gamma_4}{2} D^-_4 - \frac{1 - \gamma_4}{2} D^+_4 + \right. \\
\left. - \frac{a r_s \zeta}{2} \Delta^{(3)} - \frac{i a c_B \zeta}{2} \vec{\Sigma} \cdot \vec{B} - \frac{a c_E \zeta}{2} \vec{\alpha} \cdot \vec{E} \right] \psi(x) \]

- Smoothly interpolates between \( m_0 a = 0 \) and \( m_0 a \to \infty \)
- Ideal for charm quarks
- At tree level \( \zeta = r_s = c_E = c_B = 1 \)
Status of this Work

- Basic one-loop perturbation theory for this action is complete
- Coefficients in the action have been computed to one-loop
- Vector and Axial Vector Currents, with Asqtad light quark
- Mass and wavefunction renormalizations
- Conversion to $\overline{MS}$ masses
- Results for $\overline{m}_{c,b}(\overline{m}_{c,b})$
Vertex Renormalization

These diagrams give $c_{B,E}$

We evaluated them over the full mass range

At $m_0 a = 0$ and as $m_0 \to \infty$ we recover known results

Used lattice to lattice matching
Lattice to lattice matching

• In perturbation theory there is no problem with taking $a \to 0$

• Use a lattice theory with $a' \ll a$ as the “continuum” part of the matching calculation

• Tested this with two different “continuum” fermion actions
Figure 1: One loop lattice to lattice matching of $\delta_B = c_B - \text{counterterms}$ at $m_0a = 0.1$, with improved glue.
Figure 2: The one-loop coefficient of $\Sigma \cdot B$, with and without tadpole improvement
Results

• One-loop parts of $c_{E,B}$ were computed.

• Final, tadpole improved, numbers were smaller than expected.

• For charm quarks on the MILC lattices $c_{B,E} \approx 0.1 \alpha$.

• Means that there is a minimal impact on the hyperfine splitting which is 15% too low.

• These results only give 2% increase on Fermilab coarse lattice results.

• The action has $\mathcal{O}(a^2 \Lambda^2) \approx 10\%$ errors which raise the hyperfine splittings.
Current Matching

- Computed matching factors for Vector and Axial-Vector currents
- Fermilab heavy quarks with Asqtad or clover light quarks
- $O(a)$ errors in lattice current operators removed using rotated spinors $\Psi(x) = [1 + d_1 \gamma \cdot D]$
- Qualitative behaviour is the same as Kronfeld et al. calculations with Wilson glue
- These results are an important part of the determination of $f_D$ (Aubin, et al. [hep-lat/0506030])
Ratio Method

• Two-loop corrections could be $\mathcal{O}(1)\alpha^2 \approx 10\%$

• However, Fermilab ratios

$$\rho_J = \frac{Z^{h\ell}_J}{\sqrt{Z^{hh}_{V_t} Z^{\ell\ell}_{V_t}}}$$

have small $\mathcal{O}(0.1)\alpha$ one-loop corrections

• This feature should hold at two-loops, giving smaller $\mathcal{O}(0.1)\alpha^2 \approx 1\%$ errors

• Our perturbative calculations of $\rho$ are combined with non-perturbative determinations of $Z_{V_t}$
Lattice Masses

• Calculation of self-energy diagrams gives the one-loop corrections to the rest mass ($M_1$) and kinetic mass ($M_2$)

• We find $\mathcal{O}(1)$ coefficients for $M_{1,2}$ over the whole mass range

• Qualitative behaviour is the same as Wilson

• $q^*$ scales are small $\ll 1/a$

• This is expected in a series for the pole mass
Figure 3: The one-loop rest mass, with and without pole improvement
Figure 4: The one-loop kinetic mass renormalization
Perturbative Expressions

\[ M_2 = m_2 \left( M_1^{[0]} \right) \left\{ 1 + \alpha_V(q^*) M_2^{[1]} \right\} \]

\[ m_2(x) = \frac{\exp(x) \sinh(x)}{1 + \sinh(x)} \]

\[ M_2^{[1]} = \left( M_1^{[1]} + \frac{m_0}{1 + m_0} u_0^{[1]} \right) \left( 1 + \frac{\coth \left( M_1^{[0]} \right)}{1 + \sinh \left( M_1^{[0]} \right)} \right) \]

\[ M_1^{[0]} = \log(1 + m_0) \quad \text{tree level rest mass} \]
\[ M_1^{[1]} \quad \text{one loop rest mass} \]
\[ Z_M^{[1]} \quad \text{one loop kinetic mass renormalization} \]
\[ u_0^{[1]} \quad \text{one loop tadpole term} \]
Conversion to $\overline{\mathrm{MS}}$ mass

- The perturbative pole mass has a renormalon ambiguity
- Results in very poor convergence

$$\frac{m_{b}^{\text{pole}}}{m_{b}(\mu)} = 1 + 0.09 + 0.05 + 0.03 + \cdots$$

- Much better to determine $\overline{m}$ mass in terms of the lattice mass
- This uses the continuum formula

$$m_{\text{pole}} = \overline{m}(\mu) \left[ 1 + \frac{4}{3\pi} \left( 1 + \log \frac{\mu^{2}}{m_{0}^{2}} \right) \right]$$

- There are two ways to do this “Method 1” and “Method 2”
“Method 1”

- For charm quarks we have calculations of the averaged $c\bar{c}$ rest mass $M_{1,c\bar{c}}^{\text{latt}}$
- We compute the binding energy

$$B_1 = M_{1,c\bar{c}}^{\text{latt}} - 2M_1$$

- The lattice determination of the pole mass is

$$m_{\text{pole}} = \frac{1}{2}(M_{c\bar{c}}^{\text{expt}} - B_1)$$

- This is combined with the one-loop continuum formula to get a final answer for the $\overline{\text{MS}}$ mass.
“Method 1” Cont.

• For bottom quarks the $\eta_b$ has not been observed, so we cannot use $M_{b\bar{b}}$

• Instead we use the spin averaged $B_s$ mass

• The binding energy is

$$B_1 = M_{1,B_s}^{\text{latt}} - M_1$$

• The lattice determination of the pole mass is

$$m_{\text{pole}} = M_{B_s}^{\text{expt}} - B_1$$
“Method 2”

• Method 2 uses the kinetic mass $M_2$ as it’s estimate for the pole mass

• To reduce errors associated with tuning the bare quark masses and the lattice spacing we take

\[
m_{\text{pole}} = M_2^{\text{PT}} a \frac{M_{c\bar{c}}^{\text{expt}}}{M_{2,c\bar{c}}^{\text{latt}} a}
\]

where $M_{2,c\bar{c}}^{\text{latt}}$ is the spin averaged kinetic mass of charmonium

• For the bottom quark we again substitute the spin averaged $B_s$ mass
Optimal scale, and coupling evaluation

- The scale $\mu$ is arbitrary we can make it whatever we want

- For method one we pick $\mu$ such that the one loop coefficient is zero

- For method 2 we pick it to make the \( \overline{\text{MS}} \) mass as close to the tree level lattice mass as possible in the limit $m_0a \to 0$

$$\overline{m}(\mu) \approx m_2 \left( M_1^{[0]} \right)$$

- This scale is $\mu a \approx 3.6$

- This gives for $m_0a \neq 0$

$$\overline{m}(\mu) \approx m_2 \left( M_1^{[0]} \right) \left[ 1 + \alpha_V (q^*) \times \text{small} + \mathcal{O}(\alpha^2_V) \right]$$
Scale setting

- Results are not very sensitive to the one-loop correction since we’ve picked $\mu$ such that it’s small (method two) or zero (method one)

- Still need a correct scale $q^*$ to estimate $\pm \alpha_V^2(q^*)$ errors from the unknown two-loop term

- Techniques for scale setting appropriate to this problem are outlined in Horbostel, et. al., Phys. Rev. D67, 034023.

- We need the first and second log moments of the lattice and continuum series
Scale setting cont.

- For the lattice series we numerically integrate

\[
\langle f \log^n q^2 \rangle = \int \frac{d^4k}{(2\pi)^4} f(k) \log^n q^2
\]

- This is trivial to implement in our code

- For the continuum series implement the method proposed in Horbostel et. al.

\[
\int \frac{d^4k}{(2\pi)^4} \frac{f(k)}{k\delta} = \langle f \rangle - \delta \langle f \log q^2 \rangle + \frac{\delta^2}{2} \langle f \log^2 q^2 \rangle
\]

- The final result for method two is \( q^*(\mu a = 3.6) \approx \frac{1}{a} \)

- For method one, \( 1 < q^*a < 2 \)
Results

- For the bare masses we use values provided by the Fermilab group from their runs on the unquenched MILC lattices.

- We compute the quark masses using both methods and take the average as our final result.

| $a^{-1}$ GeV | $m_0 a$  | $M_{1,c\bar{c}}$ | $M_{2,c\bar{c}}$ | $\overline{m}_c$ (GeV) |
|--------------|-----------|-------------------|-------------------|-----------------------|
| 1.596(30)    | 0.6604    | 1.193(25)         | 1.350(20)         | 1.193(25)             |
| 2.258(32)    | 0.33324   | 1.6893(7)         | 2.229(49)         | 2.229(49)             |

- We also find $\overline{m}_b(\overline{m}_b) = 4.52(38)$ GeV.
Results cont.

- Our best value of the charm mass is $m_c(m_c) = 1.23(9)$ GeV from the fine lattice.
- The PDG estimates $m_c(m_c) = 1.25(10)$ GeV.
- Our lattice result is consistent with this.
- Our result $m_b(m_b) = 4.52(38)$ GeV is consistent, within errors, with the PDG value $4.25(15)$ GeV.
- To lower the errors on these determinations a two-loop calculation of the mass is needed.
- This calculation is in progress.
Conclusions

• One-loop perturbation theory for the Fermilab action is complete

• Ratio method and tadpole improvement produced very small one-loop corrections

• With the $\mathcal{O}(\alpha a)$ errors corrected, the leading errors in the Fermilab action are $\mathcal{O}(a^2)$

• Mass determinations allow a value of $\overline{m}_c$ to be determined

• $\overline{m}_c(\overline{m}_c) = 1.23(9)$ GeV agrees with PDG value