Numerical solution to Volterra Integro-Differential Equations of the second kind by hybrid one-step block method

M R Janodi1, Z A Majid2, F Ismail2 and N Senu1

1 Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia
2 Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia

E-mail: am_zana@upm.edu.my

Abstract. The hybrid block one-step method of order four is presented and implemented to solve first order Volterra Integro-Differential Equations (VIDEs). The technique is developed using Lagrange interpolation method. The numerical solutions of VIDEs will be solved at two-point concurrently using the proposed numerical method. Properties of the method such as consistency, order and region of absolute stability are explored. After that, the linear and nonlinear of VIDEs has been solved and the numerical results show that the accuracy and efficiency of the hybrid block method is better than the other methods which is RK4 and VIM.

1. Introduction

Volterra integro-differential equation are frequently used to model problems in science and engineering. It generally performed in many physical applications such as heat transfer, electromagnetic field and electrical circuit analysis. The Volterra integro-differential equation of second kind is given by

\[ y'(x) = F(x, y(x), z(x)), \quad y(0) = y_0, \quad 0 \leq x \leq a, \]

\[ z(x) = \int_0^x K(x, s, y(s))ds. \]

VIDEs are commonly challenging to solve analytically so there is a need to get an efficient imprecise solution. In this paper, hybrid block one-step method is applied to solve linear and nonlinear VIDEs. The concept was overcome by several authors who proposed off-step points in the derivation process (see [1] and [2]), it is based on the hybrid concepts for solving ordinary differential equations (ODEs). In 1982, [3] extended the theory of hybrid method for ODEs to the problem of VIDEs. The differential part of VIDEs is solved using hybrid method with two off-step points while the integral part is solved using numerical quadrature rule. The benefit of this method is that convergent hybrid method can attain higher order than a convergent linear multistep method. In 2011, [4] modified hybrid method in a one-step method. The idea of the method is by constructed in the joint of the methods of Runge-Kutta and
Adams family. Therefore, superiority of the method has sufficiently high accuracy and provides constancy of the amount of calculations at each step. Then, [5] extended the hybrid method and proposed concrete methods with orders of accuracy of \( p = 6 \) and \( p = 8 \) using information concerning to the solution of the considered problem with one and two mesh points, respectively. In 2014, [6] presented the hybrid block method with off-step point for solving first order ODEs. This hybrid method is applied simultaneously in block form to calculate the approximation results for the main and off-step points. Finally, [7] and [8] proposed one-step block method and multistep block method for solving linear and nonlinear VIDEs respectively. Here, the two-point hybrid block method will be formulated based on the Lagrange interpolating polynomial for solving first order VIDEs of second kind.

2. Formulation of hybrid block method

In this section, derivation of the block method is discussed based on numerical integration. The two approximate values of \( y_{n+1} \) and \( y_{n+2} \) will be computed concurrently in a block. The point at \( y_{n+1} \) and \( y_{n+2} \) can be developed by integrating (1) over the interval \([x_n, x_{n+1}]\) and \([x_{n+1}, x_{n+2}]\) respectively. Hence, the formulae of \( y_{n+1} \) and \( y_{n+2} \) can be obtained as,

\[
y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} F(x, y, z) \, dx,
\]

and

\[
y_{n+2} - y_{n+1} = \int_{x_{n+1}}^{x_{n+2}} F(x, y, z) \, dx.
\]

The function \( F(x, y, z) \) in (2) and (3) will be approximated using Lagrange interpolating polynomial and the interpolating points involved are four points i.e \( x_n, x_{n+1}, x_{n+1}, x_{n+2} \). Hence, at the point \( y_{n+1} \) taking \( x = x_{n+1} + sh \) and by replacing \( dx = hds \) into (2). The limit of integration from -1 to 0. Then, point \( y_{n+2} \) considering \( x = x_{n+2} + sh \) and \( ds = hds \) in (3) and changing the limit of integration from -2 to -1. The proposed method can be obtained using MAPLE software and gives the Two-Point Hybrid One-Step Block Method with One-Off Step (HB1OP) formulae as shown below. The implementation is based on the predictor and corrector formulae in the PE(CE) mode where \( P \) is the application of predictor while \( C \) denote the corrector formula and \( E \) represent the evaluation of function \( f \).

Here, the hybrid block method formulae would be as follows

predictor formulae

\[
y_{n+1} = y_n + \frac{h}{2} (F_n),
\]

\[
y_{n+2} = y_{n+1} + \frac{h}{2} (F_{n+1}),
\]

\[
y_{\frac{n+1}{2}} = y_n + \frac{h}{24} (F_n + 4F_{n+1} + F_{n+2}),
\]

and
corrector formulae

\[ y_{n+1} = y_n + \frac{h}{6} \left( F_n + 4F_{n+1} + F_{n+2} \right), \]

\[ y_{n+2} = y_{n+1} + \frac{h}{6} \left( F_n - 4F_{n+1} + 7F_{n+2} - 2F_{n+3} \right). \]  

(5)

3. Properties of the method

3.1 Order of the method

In this section, the order of the method is discussed. A linear difference operator \( L \) is defined by

\[ L[y(x); h] = \sum_{j=0}^{k} \left[ \alpha_j y(x + jh) - h\beta_j y'(x + jh) - h\beta v_j y'(x + jh) \right]. \]  

(6)

for any function \( y(x) \). Expanding the right hand side (6) as Taylor series about \( x \) gives

\[ L[y(x); h] = C_0 y(x) + C_1 hy^{(1)}(x) + \ldots + C_q h^q y^q(x) + \ldots, \]  

(7)

where the \( C_q \) are constants.

**Definition 1** [9]. The difference operator (6) and the related linear multistep method are said to be of order \( q \) if \( C_0 = C_1 = \ldots = C_q = 0 \) and \( C_{q+1} \neq 0 \).

The formula for the constants \( C_q \) is applied to determine the order of this method.

\[ C_0 = \sum_{j=0}^{k} \alpha_j, \]

\[ C_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j - \sum_{j=0}^{k} \beta v_j, \]

\[ \ldots \]

\[ C_q = \frac{1}{q!} \left[ \sum_{j=0}^{k} j^q \alpha_j - q \left( \sum_{j=0}^{k} j^{q-1} \beta_j + \sum_{j=0}^{k} j^{q-1} \beta v_j \right) \right] \]

(8)

where \( q = 2, 3, 4, \ldots \).
Therefore, it can be calculated the order and obtained error constant of the HB1OP method using equation (8).

For $q = 0$,

$$C_0 = \left[ \begin{array}{c} (-1) \\ 0 \end{array} + \begin{array}{c} 1 \\ -1 \end{array} + \begin{array}{c} 0 \\ 1 \end{array} \right] = \begin{array}{c} 0 \\ 0 \end{array}.$$  

For $q = 1$,

$$C_1 = \left[ \begin{array}{c} 1 \\ -1 \end{array} + 2 \begin{array}{c} 0 \\ 1 \end{array} \right] + \left[ \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \end{array} - \begin{array}{c} \frac{1}{6} \\ -\frac{2}{6} \end{array} - \begin{array}{c} 0 \\ \frac{4}{6} \end{array} \right] = \begin{array}{c} 0 \\ 0 \end{array}.$$  

For $q = 2$,

$$C_2 = \frac{1}{2!} \left[ \begin{array}{c} \begin{array}{c} 0 \\ 0 \end{array} + \begin{array}{c} 1 \\ -1 \end{array} + 4 \begin{array}{c} 0 \\ 1 \end{array} \right] - 2 \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \end{array} + \begin{array}{c} 0 \\ \frac{2}{6} \end{array} + \begin{array}{c} 0 \\ \frac{4}{6} \end{array} \right) = \begin{array}{c} 0 \\ 0 \end{array}.$$  

For $q = 3$,

$$C_3 = \frac{1}{3!} \left[ \begin{array}{c} \begin{array}{c} 0 \\ 0 \end{array} + \begin{array}{c} 1 \\ -1 \end{array} + 8 \begin{array}{c} 0 \\ 1 \end{array} \right] - 3 \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \end{array} + \begin{array}{c} 0 \\ \frac{2}{6} \end{array} + \begin{array}{c} 0 \\ \frac{4}{6} \end{array} \right) = \begin{array}{c} 0 \\ 0 \end{array}.$$  

For $q = 4$,

$$C_4 = \frac{1}{4!} \left[ \begin{array}{c} \begin{array}{c} 0 \\ 0 \end{array} + \begin{array}{c} 1 \\ -1 \end{array} + 16 \begin{array}{c} 0 \\ 1 \end{array} \right] - 4 \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \end{array} + \begin{array}{c} 0 \\ \frac{2}{6} \end{array} + \begin{array}{c} 0 \\ \frac{4}{6} \end{array} \right) = \begin{array}{c} 0 \\ 0 \end{array}.$$  

4
For $q = 5$, 

$$C_5 = \frac{1}{5!} \left[ \left( 0 \right) + \left( 1 \right) + 32 \left( 0 \right) \right] - 5 \left[ \left( \frac{1}{6} \right) + \frac{2}{6} + \frac{4}{6} \right] = \left[ \frac{1}{2880} \frac{31}{2880} \right] \neq \left( 0 \right).$$

Therefore, from the calculation the method in equation (5) is of the order four and the coefficient of error constant is

$$C_{q+1} = C_5 = \left[ \frac{1}{2880} \frac{31}{2880} \right].$$

**Definition 2** [9]. The local truncation error at $x_{n+k}$ of the linear multistep method is defined to be expression $L[y(x_n); h]$, when $y(x)$ is the theoretical solution of the initial value problem.

Consider the general local truncation error of hybrid method as

$$L[y(x); h] = \sum_{j=0}^{k} \alpha_j y(x_n + jh) - h\sum_{j=1}^{k} \beta_j y'(x_n + jh) - \sum_{j=1}^{k} h\beta_j y'(x_n + jh) + O(h^{p+1}).$$

The basic local truncation error at $x_{n+k}$ is then defined to be $C_{p+1} h^{p+1} y^{p+1}(x_n)$. Hence, the local truncation error of the HB1OP method is $C_4 y^4(x_n) + C_5 y^5(x_n) + O(h^5)$ where $C_4 y^4(x_n)$ is error constants and $C_4 y^4(x_n)$ is order of the method.

**Definition 3** [10]. A block method is said to be zero stable if and only if providing the roots of $R_j, j = 1(1)k$ the first characteristic polynomial, $\rho(R)$ specified as:

$$\rho(R) = \det \left[ \sum_{j=0}^{k} A_j R^{j-1} \right] = 0$$

satisfies with $|R_j| < 1$ and those roots with $|R_j| = 1$.

Consider (6) the general linear multistep method and the HB1OP method (5) can be represented as
\[ A_0 Y_m - A_1 Y_{m-1} - A_2 Y_{m-2} - h(B_0 F_m + B_1 F_{m-1} + B_2 F_{m-2}) \]  \hspace{1cm} (10)

where

\[ A_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 2/6 \\ -4/6 & 7/6 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 4/6 & 1/6 \\ 0 & 1/6 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1/6 \\ 0 & 1/6 \end{pmatrix}. \]  \hspace{1cm} (11)

\[ Y_m = \begin{bmatrix} y_{n+2} \\ y_{n+1} \end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix}, \quad Y_{m-2} = \begin{bmatrix} y_{n-1} \\ y_{n-2} \end{bmatrix}, \quad F_m = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}, \quad F_{m-1} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}, \quad F_{m-2} = \begin{bmatrix} F_n \end{bmatrix}. \]

Regarding definition (6) and (7),

\[ \rho(r) = \det[R A^0 - A^1] = \det[R - R \begin{pmatrix} 0 & 1 \\ -R & R \end{pmatrix}] = \det[R - R \begin{pmatrix} 0 & 1 \\ -R & R \end{pmatrix}] = R(R-1), \]

where \( R(R-1) = 0 \) and \( R_1 = 0, R_2 = 1 \). Therefore, the hybrid block method is zero stable.

3.2 Consistency
The block method with an order more than or equal to one is considered consistent [9]. Thus, the HB1OP method is consistent.

3.3 Stability region
In this part, the region of absolute stability of the HB1OP method together with modified Simpson’s 1/3 rule are analysed. Here, the test equation of first order VIDE is

\[ y'(x) = \xi y(x) + \eta \int_0^x y(t) dt \]  \hspace{1cm} (12)

where \( \xi = \lambda + \mu \) and \( \eta = -\lambda \mu \) are real constant. Therefore,

\[ y'(x) = (\lambda + \mu) y(x) - \lambda \mu \int_0^x y(t) dt. \]  \hspace{1cm} (13)
Definition 4 [11]. The method is said to be $A$-stable if and only if the region of the absolute stability contains at the quarter plane $h\xi < 0, h\eta < 0$.

Thus, the HB1OP method for the numerical solution to develop the characteristics polynomial $\rho(r), \sigma(r), \rho(r)$ and $\sigma(r)$ as follows

The point at $y_{n+1}$ and $y_{n+2}$ of corrector formula

\[
\rho(r) = r^2 - 1, \\
\sigma(r) = -\frac{1}{6} r^2 + \frac{4}{6} r + 1,
\]

and

\[
\rho(r) = r^4 - r^2, \\
\sigma(r) = \frac{2}{6} r^4 - \frac{7}{6} r^2 + \frac{4}{6} r + \frac{1}{6}.
\]

Modified Simpson’s 1/3 rule

\[
\rho(r) = r^2 - 1, \\
\sigma(r) = r^2 + \frac{4}{6} r + 1.
\]

The stability polynomial of the HB1OP method considered can be obtained by substituting equation (14) and equation (15) into the formula shown below

\[
\pi(r, h\xi, h\eta) = \rho(r) [\rho(r) - h\xi\sigma(r)] - h^2 n\sigma(r)\sigma(r).
\]

The region of absolute stability plotted below is the result of combinations HB1OP method after substitution into the stability polynomial. The stability region of HB1OP method in Figure 1 is $A$-stable type within the shaded region based on Definition 4.
4. Implementation of the hybrid block method

The two-point hybrid block method with one-off step together with quadrature rules is implemented for solving VIDEs. The formulae are given by

\[ y_{n+1} = y_n + \frac{h}{6} \left( F(x_n, y_n, z_n) + 4F(x_{n+1/2}, y_{n+1/2}, z_{n+1/2}) + F(x_{n+1}, y_{n+1}, z_{n+1}) \right), \]

and

\[ y_{n+2} = y_{n+1} + \frac{h}{6} \left( F(x_{n+1}, y_{n+1}, z_{n+1}) - 4F(x_{n+1/2}, y_{n+1/2}, z_{n+1/2}) + 7F(x_{n+1}, y_{n+1}, z_{n+1}) + 2F(x_{n+2}, y_{n+2}, z_{n+2}) \right). \]

Therefore, for predictor formulae of the Euler’s method and one-off step point are consider to evaluate initial values for the corrector method. Thus, we will focus on solving first order linear and nonlinear VIDEs of the second kind. Two different approaches are used to solve Volterra integro on different situation which is the Case 1 is \( K(x, s) = 1 \) and Case 2 is \( K(x, s) \neq 1 \).

i. Case 1: \( K(x, s) = 1 \),

The Simpson’s 1/3 rule will be used to calculate the integral part of VIDEs.

\[ z_{n+1} = z_n + \frac{h}{6} \left( y_n + 4y_n + \frac{1}{2} + y_{n+1} \right). \]

ii. Case 2: \( K(x, s) \neq 1 \),

The Composite Simpson’s rule will be used to calculate the integral part of VIDEs.

\[ z_{n+1} = \frac{h}{3} \sum_{i=0}^{n} \omega_i K(x_{n+1}, x_i, y_i) + \frac{h}{6} \left[ K(x_{n+1}, x_n, y_n) + 4K(x_{n+1}, x_{n+1}, y_{n+1}) + K(x_{n+1}, x_{n+1}, y_{n+1}) \right], \]

\[ z_{n+2} = \frac{h}{3} \sum_{i=0}^{n} \omega_i K(x_{n+2}, x_i, y_i), \]

where \( \omega_i \) are the Simpson’s rule weights 1, 4, 2, 4, \ldots, 2, 4, 1.
5. Numerical results and discussion

In this section, the performance of hybrid block with one-off step method are verified on four problems of VIDE and validate the accuracy of the method.

**Problem 1**

\[ y'(x) = - \int_0^x y(s) \, ds, \quad y(0) = 1, \quad 0 \leq x \leq 1. \]

Exact solution: \( y(x) = \cos(x) \) and source: [7].

**Problem 2**

\[ y'(x) = - \sin(x) - \cos(x) + \int_0^x 2 \cos(x-s) \, y(s) \, ds, \quad y(0) = 1, \quad 0 \leq x \leq 5. \]

Exact solution: \( y(x) = e^{-x} \) and source: [8].

**Problem 3**

\[ y'(x) = x e^{1-y(x)} - \frac{1}{(1 + x)^2} - x - \int_0^x \frac{x}{(1 + s)^2} e^{1-y(s)} \, ds, \quad y(0) = 1, \quad 0 \leq x \leq 4. \]

Exact solution: \( y(x) = \frac{1}{1 + x} \) and source: [8].

**Problem 4**

\[
\begin{align*}
\dot{y}_1(x) &= 2y_2(x) - \frac{1}{3} x^4 + \cos(y_1(x)) - 1 + \int_0^x 2s \sin(y_1(s)) + sx(2) \, ds, \\
\dot{y}_2(x) &= 1 - x \sin(y_2(x)) - \frac{1}{2} x^2 \sin(y_1(x)) + \int_0^x sx^2 \cos(y_1(s)) + x \cos(y_2(s)) \, ds, \\
y_1(0) &= y_2(0) = 0, \quad 0 \leq x \leq 1.
\end{align*}
\]

Exact solution: \( y_1(x) = x^2, \quad y_2(x) = x \) and source: [12].

| \( h \) | METHOD | MAXE   | TS  | TFC  | TIME(s) |
|-------|--------|--------|-----|------|---------|
| 0.025 | RK4    | 7.7609E-07 | 40  | 160  | 0.1058  |
|       | HB1OP  | 3.1279E-09 | 20  | 140  | 0.0777  |
| 0.0125| RK4    | 9.6801E-08 | 80  | 320  | 0.1924  |
|       | HB1OP  | 8.1718E-10 | 40  | 300  | 0.1701  |
| 0.00625| RK4    | 1.2087E-08 | 160 | 640  | 0.3014  |
|       | HB1OP  | 9.4976E-11 | 80  | 560  | 0.1809  |
Table 2. Numerical results for Problem 2

| h     | METHOD | MAXE     | TS  | TFC  | TIME(s) |
|-------|--------|----------|-----|------|---------|
| 0.025 | RK4    | 1.4620E-06 | 200 | 800  | 0.3072  |
|       | HB1OP  | 7.1579E-08 | 100 | 680  | 0.1851  |
| 0.0125| RK4    | 4.9899E-06 | 400 | 1600 | 0.6017  |
|       | HB1OP  | 7.7697E-08 | 200 | 1540 | 0.3106  |
| 0.00625| RK4   | 1.8096E-07 | 800 | 3200 | 0.9133  |
|       | HB1OP  | 5.7793E-09 | 400 | 2800 | 0.5121  |

Table 3. Numerical results for Problem 3

| h     | METHOD | MAXE     | TS  | TFC  | TIME(s) |
|-------|--------|----------|-----|------|---------|
| 0.025 | RK4    | 1.9714E-05 | 160 | 640  | 0.2939  |
|       | HB1OP  | 4.1593E-09 | 80  | 480  | 0.1917  |
| 0.0125| RK4    | 1.3369E-06 | 320 | 1280 | 0.5554  |
|       | HB1OP  | 1.9318E-10 | 160 | 1120 | 0.3511  |
| 0.00625| RK4   | 8.7364E-08 | 640 | 2560 | 1.1720  |
|       | HB1OP  | 8.7149E-11 | 320 | 2240 | 0.6330  |

Table 4. Numerical results for Problem 4

| x     | VIM        | HB1OP       |
|-------|------------|-------------|
|       | y_1        | y_2         | y_1         | y_2       |
| 0     | 0          | 0           | 0            | 0         |
| 0.2   | 2.9592E-05 | 1.6362E-06 | 1.1763E-08  | 1.1971E-08|
| 0.4   | 2.2480E-04 | 1.3064E-05 | 7.7549E-08  | 7.7917E-08|
| 0.6   | 7.4203E-04 | 2.7894E-05 | 9.3679E-07  | 9.3019E-07|
| 0.8   | 1.7055E-03 | 1.3135E-04 | 4.4437E-07  | 5.4959E-06|
| 1     | 3.1011E-03 | 4.8173E-04 | 3.3979E-06  | 4.4497E-06|

TIME(s) 0.0435

In problems 1 to 3, the numerical results are solved linear and nonlinear VIDEs using different step size. The numerical outcomes in tables 1 to 3 display that the HB1OP method comparing with RK4 in the same order. In all tested problems, the maximum error is better than RK4 and for the total step and execution times in seconds is still less than RK4. It is apparent that the HB1OP is very efficient and less costly in terms of total number of steps and implementation times. Table 4 display the results for problem 4 and in term of maximum error the HB1OP is slightly better compared with VIM. It can also be conclude that the accuracy of the method is improved when the off-step point is include to find the estimate of y_{n+1} and y_{n+2}. All the numerical results for Problems 1 to 4 was written in C language.
Figure 2. Assessment of solutions of problem 1 for \( h = 0.025, 0.0125 \) and 0.00625.

Figure 3. Comparison of solutions of problem 2 for \( h = 0.025, 0.0125 \) and 0.00625.

Figure 4. Comparison of solutions of problem 3 for \( h = 0.025, 0.0125 \) and 0.00625.
Figures 2-4 display the HB1OP method which manages to achieve better accuracy compared to RK4. Then, in terms of timing, the HB1OP is slightly better than the RK4. It was observed that the HB1OP gave faster results compared the RK4. Figure 5 show the accuracy of the HB1OP method in approximating the systems of nonlinear VIDE problems.

6. Conclusion
In this study, the two-point hybrid one-step block method with one-off step are proposed and together with quadrature rules for solving linear and nonlinear VIDEs. The results indicate that the constructed hybrid one-step block method is acceptable to solve VIDEs.

Acknowledgements
The author thankfully acknowledges the financial funding of Graduate Research Fund (GRF) from Universiti Putra Malaysia.

References
[1] Gear C W 1965 SIAM J. Numerical Analysis 2 69-86
[2] Gragg W and Stetter H J 1964 J. Assoc. Computational 11 188-209
[3] Makroglou A 1982 IMA Journal of Numerical Analysis 2 21-35
[4] Mehyideva G, Imanova M and Ibrahimov V 2013 World Academy of Science, Engineering and Technology International Journal of Mathematical and Computational Sciences Vol 5
[5] Mehyideva G, Imanova M and Ibrahimov V 2013 Proceedings of the World Congress on Engineering Vol I
[6] Lee K Y and Ismail F 2014 International Conference on Mathematical Sciences and Statistics DOI 10.1007/978-981-4585-33-0_28
[7] Mohamed N A and Majid Z A 2015 In AIP Conference Proceedings 020018
[8] Mohamed N A and Majid Z A 2016 Malaysian Journal of Mathematical Sciences 10 33-48
[9] Lambert J D 1973 Computational Methods in Ordinary Differential Equations, John Wiley and Sons (New York, NY, USA)
[10] Fatunla S O 1991 International Journal of Computer Mathematics 41(1) 55–63
[11] Brunner H and Lambert J D 1974 Springer- Verlag 12 75-89
[12] Berenguer M I, Guillem A I G and Galan M R 2013 Applied Numerical Mathematics 67 126-135
[13] Saberi N J and Tamamgar M 2008 Comput. Math. Applied 56 346–351