Extended mean field games - formulation, existence, uniqueness and examples

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Abstract

In this paper we consider mean-field games where the interaction of each player with the mean field takes into account the collective behavior of the players and not only their state. To do so, we develop a random variables point framework which is particularly convenient for these problems. We prove an existence result for extended mean field games and establish uniqueness conditions. In the final section we discuss some further directions in the study of these problems. We present the Master equation formulation and discuss various properties of the solutions. We also consider briefly problems which include correlations.

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1 Introduction

Mean field games is a recent area of research started by Minyi Huang, Peter E. Caines, and Roland P. Malhamé [HMC06], [HCM07] and Pierre Louis Lions and Jean Michel Lasry [LL06a, LL06b, LL07a, LL07b] which attempts to understand the limiting behavior of systems involving very large numbers of rational agents which play dynamic games under partial information and symmetry assumptions. Inspired by ideas in statistical physics, these authors introduced a class of models in which the individual player contribution is encoded in a mean field that contains only statistical properties about the ensemble. The literature on mean field games and its applications is growing fast, for a recent surveys see [LLG10b, Car11] and reference therein. Applications of mean field games arise in the study of growth theory in economics [LLG10a] or environmental policy [ALT], for instance, and it is likely that in the future they will play an important rôle in economics and population models. There is also a growing interest in numerical methods for these problems [ALT], [ADI0], [CAD10]. Also, the discrete state problem is considered both in discrete time [GMS10] and the continuous time [GMS11]. Several problems have been worked out in detail in [Gue09a, Gue09b].

This paper is structured as follows: we start in Section 2 by discussing the original formulation by Lions and Lasry of mean field games as a coupled system of transport equations with a Hamilton-Jacobi equation. Then in Section 2.1 we present a reformulation of this problem

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as coupled system of an ordinary differential equation in an $L^q$, $q > 1$ space, together with a Hamilton-Jacobi equation. This formulation is essentially the one used by Minyi Huang, Peter E. Caines, and Roland P. Malhamé, in [HMC06]. The main reason to use this random variables formulation is that it makes it very natural to consider extended problems. Indeed, in Section 3, we consider an extension of the original mean-field problem where the interaction a player and the mean-field also takes into account the collective behaviour of the players not only its state. In Section 4 we prove existence of solutions to the extended mean field game system. Then, in Section 5.1 we present some examples for which the solutions can be found explicitly. Conditions that ensure uniqueness of solutions to this system are given in Section 6. Finally, in section 7 we present the Master equation for these problems, give the definition of the solutions to this equation. At the end we discuss the Master equation for problems which include correlations.

Our main results are the following: we prove existence of solutions to extended mean field games by using a fixed point argument (Theorem 1). Even in the original mean field problem this result does not follow obviously from the corresponding result in the notes by Pierre Cardaliaguet [Car11] P.-L. Lions lectures at College de France.

We discuss conditions for the absolute continuity of the law in Theorem 2. Concerning uniqueness, we first consider a version of the Lions- Lasry monotonicity argument for classical solutions (Theorem 3) and an additional improvement for viscosity solutions (Theorem 4). Then we present second approach to uniqueness which uses the optimality nature of the solutions. Our technique is formulated in terms of Lagrangians (Theorem 5) and holds under a very general setting, without requirements on the absolute continuity of the law or regularity of the solution. Finally, in the last section we introduce "Master equation" both in the deterministic case and for mean field games with correlations. We define the solution to these problems and prove an existence result for the deterministic Master equation, as well as various additional properties.

2 Two formulations of deterministic mean field games

In this section we review the original formulation for deterministic mean-field games from Lions-Lasry [LL06a, LL06b, LL07a, LL07b]. Then we discuss a reformulation in terms of random variables. This set up is very close to the one used by Minyi Huang, Peter E. Caines, and Roland P. Malhamé, in [HMC06] (though in that paper they were considering second-order equations) and is particularly suited to the extensions we consider in this paper.

In the standard mean field game setting one considers a population of players where each individual has a state given by his position $x \in \mathbb{R}^d$. We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures in $\mathbb{R}^d$. This set is a metric space endowed with the Wasserstein metric $W_2$, see for instance [Vil03]. The population of players is described at each time $t$ by a probability measure $\theta(t) \in \mathcal{P}(\mathbb{R}^d)$. Given an individual player who, for some reason knows the $\theta(t)$ for all time, his or her objective is to minimize a certain performance criterion. For this let $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be a running cost, $\psi : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be a terminal cost. We suppose that both $L$ and $\psi$ satisfy standard hypothesis for optimal control problems, that is:

- $L$ and $\psi$ are continuous functions and bounded below, without loss of generality we can assume $L, \psi \geq 0$.
- $\psi$ is Lipschitz in the first coordinate.
- $L$ is coercive:
  $$\frac{L(v, x, \theta)}{|v|} \xrightarrow{|v| \to \infty} \infty, \text{ uniformly in } x.$$  
- $L$ is uniformly convex in $v$. 

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A typical example is
\[ L(x, v, \theta) = \frac{|v|^2}{2} - \int_{\mathbb{R}^d} V(x, y) d\theta(y), \] (1)
for some function \( V: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \).

The value function from the point of view of a player at the point \( x \) at time \( t \) is defined through the optimal control problem:
\[ u(x, t) = \inf_{\mathbb{x}} \int_t^T L(x, \mathbf{x}, \theta(s)) ds + \psi(x(T), \theta(T)). \]

For \((p, x, \theta) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)\) we define the Hamiltonian
\[ H(p, x, \theta) = \sup_{v \in \mathbb{R}^d} -v \cdot p - L(x, v, \theta). \]

Then it is well known that \( u \) is the unique viscosity solution of the Hamilton-Jacobi equation
\[ -u_t + H(D_x u, x, \theta) = 0 \] (2)
satisfying the terminal condition \( u(x, T) = \psi(x, \theta(T)) \). It is also known that if \( u \) is a classical solution to (2) then the optimal trajectories are given by
\[ \mathbf{x}(s) = -D_p H(D_x u(x(s), s), x(s), \theta(s)). \] (3)

The mean-field game hypothesis consists in assuming that all players have access to the same information and act in a rational way. Therefore each one of them follows the optimal trajectories (3). This then implies that the probability distribution of players is transported by the vector field \(-D_p H(D_x u(x, t), x, \theta(t)). \) Therefore \( \theta \) is a (weak) solution of the equation
\[ \theta_t - \text{div}(D_p H(D_x u, x, \theta)) = 0, \]
for \( \theta \in \mathbb{P}(\mathbb{R}^d), \) which encodes the initial distribution of players. This leads to the system
\[ \begin{cases} -u_t + H(D_x u, x, \theta) = 0 \\ \theta_t - \text{div}(D_p H(D_x u, x, \theta)) = 0, \end{cases} \] (4)
subjected to the initial-terminal conditions
\[ \begin{cases} u(x, T) = \psi(x, \theta(T)) \\ \theta(x, 0) = \theta_0. \end{cases} \] (5)

The problem (4) its second order analoge was first introduced and studied by Pierre Louis Lions and Jean Michel Lasry in [LL06b]. Detailed proofs of existens and uniqueness to those systems can be found in notes by Pierre Cardaliaguet from P.-L. Lions lectures at College de France.

2.1 Random variables point of view

Let \((\Omega, \mathcal{F}, P)\) be a probability space, where \( \Omega \) is an arbitrary set, \( \mathcal{F} \) is a \( \sigma \)-algebra on \( \Omega \) and \( P \) is a probability measure. We recall that a \( \mathbb{R}^d \) valued random variable \( X \) is a function \( X : \Omega \rightarrow \mathbb{R}^d \). We denote by \( L^q(\Omega, \mathbb{R}^d) \) the set of \( \mathbb{R}^d \) valued random variable whose norm is in \( L^q(\Omega) \). The law \( \mathcal{L}(X) \) of a \( \mathbb{R}^d \) valued random variable is the probability measure in \( \mathbb{R}^d \) defined by
\[ \int_{\mathbb{R}^d} \varphi(x) d\mathcal{L}(X)(x) = E \varphi(X). \]
Note that since all relevant random variables are \( \mathbb{R}^d \) valued, we will write \( L^q(\Omega) \) instead of \( L^q(\Omega, \mathbb{R}^d) \), to simplify the notation. We can reformulate the mean field game problem by replacing the probability \( \theta(t) \) encoding the distribution of players by a random variable \( \mathbf{X}(t) \in L^q(\Omega) \) such that \( \mathcal{L}(\mathbf{X}(t)) = \theta(t) \). Of course for each measure \( \theta \) there are many possible random variables with law \( \theta \), however this will not create any problem. Each outcome of the random variable \( \mathbf{X} \) represents the position of a random player chosen according to the probability \( \theta \).

We will say that a function \( f : L^q(\Omega) \to \mathbb{R} \) depends only on the law if for any \( X, \bar{X} \in L^q(\Omega) \) with the same law, i.e., \( \mathcal{L}(X) = \mathcal{L}(\bar{X}) \), we have \( f(X) = f(\bar{X}) \). Let \( \eta : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \), \( \eta(\theta) \), we can define a function, \( \tilde{\eta} : L^q(\Omega; \mathbb{R}^d) \to \mathbb{R} \), \( \tilde{\eta}(X) \), which depends only on the law of \( X \), by

\[
\tilde{\eta}(X) = \eta(\mathcal{L}(X)).
\]

This allows us to identify functions in \( \mathcal{P}(\theta) \) with functions in \( L^q(\Omega) \) which depend only on the law. Where \( \mathcal{P}(\theta) \) is the set of probability measures \( \theta \in \mathcal{P}(\Omega) \) with finite \( q \)-th moment:

\[
\int_{\mathbb{R}^d} \|x\|^q d\theta(x) < +\infty.
\]

To make the presentation more intuitive, we use the same notation for functions whether they are written in terms of the probability measure \( \theta \) or in terms of a random variable \( X \) with \( \mathcal{L}(X) = \theta \), i.e. we omit the tilde and write simply \( \eta(X) \) or \( \eta(\theta) \), according to the previous identification.

In this new setting, the Lagrangian is a function \( L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R} \) that we denote by \( L(x, v, X) \), which in the last coordinate depends only on the law. For example the Lagrangian can be now written as

\[
L(x, v, X) = \frac{|v|^2}{2} - EV(x, X).
\]

As before, suppose an individual player knows the distribution of players which is now encoded on a trajectory \( \mathbf{X}(t) \in L^q(\Omega) \) for all times. His or her objective is to minimize a certain performance criterion, determined as before by a running cost \( L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega; \mathbb{R}^d) \to \mathbb{R} \), and a terminal cost \( \psi : \mathbb{R}^d \times L^q(\Omega; \mathbb{R}^d) \to \mathbb{R} \). We assume that both \( L(x, v, X) \) and \( \psi(x, X) \) depend only on the law of \( X \).

Then the value function from the point of view of a reference player which is at a point \( x \) at time \( t \) is

\[
u(x, t) = \inf_{x} \int_{t}^{T} L(x, \dot{x}, \mathbf{X}(s)) ds + \psi(\mathbf{X}(T), \mathbf{X}(T)).
\]

As before, for \( (p, x, X) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \), the Hamiltonian is given by

\[
H(p, x, X) = \sup_{v \in \mathbb{R}^d} -v \cdot p - L(x, v, X).
\]

The Hamiltonian \( H \) which is a function \( H : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R} \), that we denote by \( H(p, x, X) \), depends only on the last coordinate through its law, i.e., if \( X, \bar{X} \in L^q(\Omega) \) have the same law, i.e., \( \mathcal{L}(X) = \mathcal{L}(\bar{X}) \) then

\[
H(p, x, X) = H(p, x, \bar{X}).
\]

Then \( u \) is the unique viscosity solution of the Hamilton-Jacobi equation

\[-u_t(x, t) + H(D_x u(x, t), x, \mathbf{X}(t)) = 0\]

with the terminal condition \( u(x, T) = \psi(x, \mathbf{X}(T)) \).

Because of the rationality hypothesis, the dynamics of a typical player at position \( \mathbf{X}(s)(\omega) \), \( \omega \in \Omega \), is then given by

\[
\dot{\mathbf{X}}(s)(\omega) = -D_p H(D_x u(\mathbf{X}(s)(\omega), s), \mathbf{X}(s)(\omega), \mathbf{X}(s)).
\]

This yields the following alternative formulation of the mean field game

\[
\begin{cases}
- u_t(x, t) + H(D_x u(x, t), x, \mathbf{X}(t)) = 0 \\
\dot{\mathbf{X}}(t) = -D_p H(D_x u(x, t), \mathbf{X}(t), \mathbf{X}(t)).
\end{cases}
\]
where the initial-terminal condition is replaced by
\[
\begin{align*}
  u(x, T) &= \psi(x, X(T)) \\
  X(0) &= X_0,
\end{align*}
\]
where \( L(X_0) = \theta_0 \).

The connection between the two formulations is an easy consequence of the following well-known result:

**Proposition 1.** Let \( b: \mathbb{R}^d \times [0, T] \to \mathbb{R}^d \) be a smooth, globally Lipschitz vector field over \( \mathbb{R}^d \) and let \( X(t) : [0, T] \times \Omega \to \mathbb{R}^d \) be a solution to
\[
\dot{X} = b(X, t).
\]
(6)

With the law \( \theta = L(X) \) which is absolutely continuous with a smooth density \( \theta(x, t) \). Then \( \theta(x, t) \) is a solution to
\[
\theta_t(x, t) + \text{div}(b(x, t)\theta(x, t)) = 0.
\]
with initial condition \( \theta(0) = L(X(0)) \).

**Proof.** We have
\[
\int_{\mathbb{R}^d} \phi(x)\theta_t(x, t)dx = \frac{d}{dt} \int_{\mathbb{R}^d} \phi(x)\theta(x, t)dx = \frac{d}{dt} E\phi(X(t)) = E\dot{\theta}(X(t))\dot{X}(t) =
\]
\[
= E\dot{\theta}(X)b(X, t) = \int_{\mathbb{R}^d} D\phi(x)b(x, t)\theta dx = -\int_{\mathbb{R}^d} \phi(x) \text{div}(b(x, t)\theta)dx
\]
for every \( \phi \in C_1^1(\mathbb{R}^d) \). Thus \( \theta_t + \text{div}(b(x, t)\theta) = 0. \)

\[
\square
\]

### 3 Extended mean-field games

In many applications one must consider mean field games where the pay-off of each player depends not only on the statistical information or state of the remaining players but also on the actions the other players take. In the random variables point of view this corresponds to the running costs that depend on \( \dot{X} \). As before we assume that the distribution of players is represented by a random variable \( X(t) \in L^q(\Omega) \), which we suppose to be differentiable with derivative \( \dot{X}(t) \in L^q(\Omega) \). Many other alternative spaces could be used here and this is not essential for this part of discussion, for instance we could consider \( X \in L^q([0, t], L^q(\Omega)) \). As before, the objective of an individual player is to minimize a certain performance criterion. For this let \( L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega) \to \mathbb{R} \) be a Lagrangian, \( \psi : \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R} \) be a terminal cost. We assume that \( L(x, v, X, Z) \) depends only on the joint law of \( (X, Z) \) i.e., if \( X, Z, \tilde{X}, \tilde{Z} \in L^q(\Omega) \) satisfy \( L(X, Z) = L(\tilde{X}, \tilde{Z}) \) then
\[
L(x, v, X, Z) = L(x, v, \tilde{X}, \tilde{Z}),
\]
and that \( \psi(x, X) \) depends only on the law of \( X \).

We suppose that both \( L \) and \( \psi \) satisfy standard hypothesis for optimal control problems, that is:

- \( L \) and \( \psi \) are continuous functions and bounded below.
- \( \psi \) is Lipschitz in the first coordinate.
• \( L \) is coercive:
\[
\lim_{|v| \to \infty} \frac{L(v, x, X, Z)}{|v|} = \infty, \quad \text{uniformly in } x.
\]

• \( L \) is convex in \( v \).

The value function from the point of view of a player at the point \( x \) at time \( t \) is
\[
u(x, t) = \inf_{x} \int_{t}^{T} L(\dot{x}, x, X(s), \dot{X}(s)) \, ds + \psi(x(T), X(T)),
\]
where the infimum is taken over absolutely continuous trajectories. As before the Hamiltonian is given by
\[
H(p, x, X, Z) = \sup_{v \in \mathbb{R}^d} -v \cdot p - L(x, v, X, Z).
\]

The Hamiltonian \( H \) is a function \( H : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega) \to \mathbb{R} \), denoted by \( H(p, x, X, Z) \), that in the last two coordinates depends only on its joint law.

An important example is the Lagrangian
\[
L(v, x, X, Z) = \frac{|v|^2}{2} + \beta v \cdot EZ - V(x, X),
\]
which corresponds to the Hamiltonian
\[
H(p, x, X, Z) = \frac{(\beta EZ + v)^2}{2} + V(x, X).
\]

Given that the trajectory \( X(t) \) is known for every player we get that the value function \( u \) satisfies the Hamilton-Jacobi equation:
\[
-u_t + H(D_x u, x, X, \dot{X}) = 0,
\]
then by mean field hypothesis each player should follow the optimal trajectories
\[
\dot{x} = -D_p H(D_x u(x, t), x, X, \dot{X}).
\]

This leads to the extended mean field system
\[
\begin{aligned}
-u_t + H(D_x u, x, X, \dot{X}) &= 0 \\
\dot{X} &= -D_p H(D_x u(X, t), X, X, \dot{X}),
\end{aligned}
\]
with
\[
\begin{cases}
u(x, T) = \psi(x, X(T)) \\
X(0) = X_0.
\end{cases}
\]

### 3.1 Examples

The second equation in (10) involves a fixed point problem for \( X \) which may pose additional problems. One possibility to deal with this issue is to assume that
\[
|D_p H(p, x, X, Z) - D_p H(p, x, X, \tilde{Z})| \leq \rho \| Z - \tilde{Z} \|,
\]
for some \( \rho \leq 1 \). In this last case the equation
\[
Z = -D_p H(P, x, X, Z),
\]
for \( P \in L^q(\Omega), x \in \mathbb{R}^d \) and \( X \in L^q(\Omega) \), has a unique solution by a standard contraction argument.
However it may not be desirable in all applications to impose such restrictive assumptions. For instance, the Hamiltonian \((9)\) does not satisfy \((11)\). However, for \(P \in L^q(\Omega)\) the equation \((12)\) is
\[
Z = -\beta E Z - P
\]
has a unique solution \(Z\) if \(1 + \beta \neq 0\). Indeed
\[
Z + \beta E Z = -P,
\]
from where we gather \(E Z = -\frac{1}{1 + \beta} EP\), and so
\[
Z = \frac{\beta}{1 + \beta} EP - P.
\]

### 3.2 Linear - Quadratic

We consider the following Hamiltonian
\[
H(p,x,X,Z) = \frac{|p + \beta E Z|^2}{2} + \frac{1}{2} x^T A(X)x + B(X) \cdot x + C(X),
\]
where \(A : L^q(\Omega) \to \mathbb{R}^{d \times d}, B : L^q(\Omega) \to \mathbb{R}^d,\) and \(C : L^q(\Omega) \to \mathbb{R}\) are Lipschitz. We assume the terminal condition is also quadratic in \(x\), i.e.
\[
\psi(x,X) = \frac{1}{2} x^T M(X)x + N(X) \cdot x + Q(X),
\]
with \(M : L^q(\Omega) \to \mathbb{R}^{d \times d}, N : L^q(\Omega) \to \mathbb{R}^d,\) and \(Q : L^q(\Omega) \to \mathbb{R}\).

Then, because of the quadratic structure we can look for solutions of the form
\[
u(x,t) = \frac{1}{2} x^T \Gamma(t)x + \Theta(t) \cdot x + \zeta(t).
\]
By separation of variables this gives rise to the following system of differential equations
\[
\begin{align*}
-\dot{\Gamma} + \frac{1}{2} \Gamma^T \Gamma + A(X) &= 0 \\
-\dot{\Theta} + \beta \Gamma E \dot{X} + \Gamma \Theta + B(X) &= 0 \\
-\dot{\zeta} + \frac{1}{2} |\Theta + \beta E \dot{X}|^2 + C(X) &= 0,
\end{align*}
\]
with terminal conditions
\[
\Gamma(T) = M(X(T)), \quad \Theta(T) = N(X(T)), \quad \zeta(T) = Q(X(T)).
\]
This system is coupled with the forward equation
\[
\begin{align*}
\dot{X} &= -\Gamma X - \Theta - \beta E \dot{X}, \\
X(0) &= X_0,
\end{align*}
\]
that is,
\[
\begin{align*}
\dot{X} &= -\Gamma X - \frac{1}{1 + \beta} \Theta + \frac{\beta}{1 + \beta} (\Gamma E X) \\
X(0) &= X_0.
\end{align*}
\]
3.3 Second example

We consider the following Lagrangian
\[
L(v, x, X, Z) = \frac{|v|^2}{2} + x^4 + U(X, Z),
\]
associated with the dynamics \( \dot{x} = f(x, v) \), for \( f(x, v) = \frac{v}{x} \). We assume the terminal cost has the form \( \psi(x, X) = A(X)x^4 + B(X) \) Then the Hamiltonian is
\[
H(p, x, X, Z) = \frac{|p|^2}{2x^2} - x^4 - U(X, Z).
\]
The mean field equations are
\[
\begin{align*}
\begin{cases}
-u_t + \frac{|Du|^2}{2x^2} - x^4 - U(X, \dot{X}) = 0 \\
\dot{X} = -\frac{Du(X, t)}{X} \\
u(x, T) = \psi(x, X(T)).
\end{cases}
\end{align*}
\]
We look for the solutions of the form \( u(x, t) = x^4 p(t) + q(t) \). Note that the power \( x^4 \) is the only power for which we are able to use this type of separation of variables method. Plugging this in the previous system we get:
\[
\begin{align*}
\begin{cases}
p' - 8p^2 + 1 = 0 \\
q' = -U(X, \dot{X}) \\
\dot{X} = -4pX \\
p(T) = A(X(T)), q(T) = B(X(T)) \\
X(0) = X_0.
\end{cases}
\end{align*}
\]
Elementary computations give
\[
p(t) = \frac{1}{2\sqrt{2}} \frac{1 + ce^{4\sqrt{2}t}}{1 - ce^{4\sqrt{2}t}},
\]
and
\[
X(t) = \left[ \frac{ce^{4\sqrt{2}t}}{c - 1} - 1 \right]^{-\frac{1}{2}} e^{\sqrt{2}t} X_0.
\]
One can now find the constant \( c \) from terminal conditions on \( p \). Then
\[
u(x, t) = \frac{x^4}{2\sqrt{2}} \frac{1 + ce^{4\sqrt{2}t}}{1 - ce^{4\sqrt{2}t}} + q(t),
\]
where \( q(t) \) solves \( \dot{q} = -U(X, \dot{X}) \) with \( q(T) = B(X(T)) \).

4 Existence of solutions to the extended mean field games

In this section we prove the existence of the solutions to (3). We start by listing Assumptions
\[
\psi : \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R},
\]
\[
L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega) \to \mathbb{R},
\]
and \( H = L^* \) is the Legendre transform of \( L \) defined by
\[
H(p, x, X, Z) = \sup_v \{ -v \cdot p - L(v, x, X, Z) \}.
\]
Suppose \( X_0, \psi \) and \( L \) satisfy following properties:
a) $X_0 \in L^q(\Omega)$ and has an absolutely continuous law,

b) $\psi$ is a Lipschitz continuous in $x$ and is bounded,

c) For any $x \in \mathbb{R}^d$ $X, Z \in L^q(\Omega)$ $L(v, x, X, Z)$ is uniformly convex in $v$ and satisfies the coercivity condition

$$
\lim_{|v| \to \infty} \frac{L(v, x, X, Z)}{|v|^p} = \infty,
$$

uniformly in $x$.

d) $L(v, x, X, Z) \geq -c_0E(|X|^q + |Z|^q + 1)$,

e) For every $X, Z \in L^q(\Omega)$ there exists a continuous function $v_0 : L^q(\Omega) \times L^q(\Omega) \to \mathbb{R}$ such that $L(x, v_0(X, Z), X, Z) \leq c_1$,

f) $|D_vL|, |D^2_vL|, |D^2_{xx}L|, |D^2_{xv}L| \leq (c_2L + c_3)E(|X|^q + |Z|^q + 1)$ and $|D_xL| \leq c_2L + c_3$,

g) For any $X, Y, P \in L^q(\Omega)$ the equation $Z = -D_pH(P, X, Y, Z)$ can be solved with respect to $Z$ as

$$
Z = G(P, X, Y)
$$

where the map $G : L^q(\Omega) \times L^q(\Omega) \times L^q(\Omega) \to L^q(\Omega)$ is Lipschitz,

h) $H$ is continuous in $X, Z$ locally uniformly in $x, p$,

i) $D_xH$ is Lipschitz in $\mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega)$,

j) For any $R > 0$ there exists a constant $C(R)$ such that

$$
|H(p_1, x, X_1, Z_1) - H(p_2, y, X_2, Z_2)| \leq C(R)(|p_1 - p_2| + |x - y| + \|X_1 - X_2\|_{L^q(\Omega)} + \|Z_1 - Z_2\|_{L^q(\Omega)})
$$

for $|p_1|, |p_2|, |x|, |y|, \|X_1\|_{L^q(\Omega)}, \|X_2\|_{L^q(\Omega)}, \|Z_1\|_{L^q(\Omega)}, \|Z_2\|_{L^q(\Omega)} \leq R$.

As an example we may take

$$
L(x, v, X, Z) = \frac{\beta EZ + v^2}{2} - V(x, X), \text{ and } L(x, v, X, Z) = \frac{|v|^2}{2} + \beta v EZ - V(x, X)
$$

with $V : \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R}^d$ is bounded, smooth in $x$ variable and Lipschitz in $X$. Then the corresponding Hamiltonians are

$$
H(p, x, X, Z) = \frac{|p|^2}{2} + \beta p EZ + V(x, X) \text{ and } H(p, x, X, Z) = \frac{\beta EZ + p^2}{2} + V(x, X).
$$

For both Lagrangians the function $G$ is given by $G(X, P) = -P + \frac{\beta}{1 + \beta}EP$.

For every $\Phi \in C^0(\mathbb{R}^n)$ consider the following system of ODEs in $L^q(\Omega)$

$$
\begin{cases}
\dot{X} = -D_pH(P, X, X, \dot{X}) \\
\dot{P} = D_xH(P, X, X, \dot{X}) \\
X(0) = X_0, P(0) = D_x\Phi(X_0).
\end{cases}
$$

(13)

This system can be rewritten as

$$
\begin{cases}
\dot{X} = G(P, X, X) \\
\dot{P} = D_xH(P, X, X, G(P, X, X)) \\
X(0) = X_0, P(0) = D_x\Phi(X_0).
\end{cases}
$$

(14)
Since \( \Phi \) is Lipschitz \( D_x \Phi \) exists almost everywhere and since \( X_0 \) has an absolutely continuous law \( \mathbb{P}(0) \) is well defined on a set of full measure. The Lipschitz conditions on \( G \) and \( D_x H \) guarantee the existence of a unique solution to (13). Standard arguments imply that \( X, P \in C^1([0,T]; L^2(\Omega)) \) and \( X \) is Lipschitz in \( t \) that is \( X \in C^{1,1}([0,T]; L^2(\Omega)) \).

Now suppose \((X, P)\) is a solution to (13). We define the function \( \tilde{u}(x,t) \) as the solution to the optimal control problem

\[
\tilde{u}(x,t) = \inf_{x} \int_{t}^{T} L(x, \dot{x}, x, \dot{X}) + \psi(x(T), X(T)),
\]

(15)

where the infimum is taken over all absolutely continuous trajectories \( x(s) \) with \( x(t) = x \).

**Lemma 1.** Let \( \Phi \in C^0(\mathbb{R}^2) \) be a bounded Lipschitz function with \( \text{Lip}(\Phi) \leq R \) and let \( X \in C^{1,1}([0,T]; L^2(\Omega)) \) be the random variable defined by (13). Then there exists a constant \( c_4 = c_4(R, L, X_0, T, q) \) such that

\[
E|X(t)|^q + |X(t)|^q \leq c_4, \quad 0 \leq t \leq T.
\]

**Proof.** From (14) since \( G \) and \( D_x H \) are Lipschitz we have

\[
\begin{align*}
E|X|^q &\leq CE(|X|^q + |P|^q) + C, \\
E|\dot{P}|^q &\leq CE(|X|^q + |P|^q) + C,
\end{align*}
\]

(16)

so

\[
E(|\dot{X}|^q + |\dot{P}|^q) \leq CE(|X|^q + |P|^q) + C,
\]

where \( C \) depends only on \( G \) and \( D_x H \). Let \( R = (X, P) \) and \( \|R\| = [E(|\dot{X}|^q + |\dot{P}|^q)]^{\frac{1}{q}} \) then the inequality above gives \( \|R\| \leq C + C\|R\| \). From Gronwall’s inequality

\[
R(t) \leq C(T)(1 + \|R(0)\|), \quad \forall t \in [0, T],
\]

thus

\[
E(|X|^q + |P|^q) \leq C(T)(1 + E(|X_0|^q + |Du(X_0)|^q)) \leq C(T)(1 + E|X_0|^q + |\text{Lip}(\Phi)|^q)
\]

which then combined with (16) yields the required result. \( \square \)

**Lemma 2.** Let \( \Phi \in C(\mathbb{R}^n) \) be any bounded, Lipschitz function. Then the function \( \tilde{u}(x,t) \) defined by (15) is uniformly bounded and Lipschitz in \( x \). Furthermore for any \( t < T, \tilde{u} \) is semi-concave in \( x \). More specifically we have

1. \( \tilde{u} \leq c_1(T - t) + \|\psi\|_{\infty} \) for all \( x \in \mathbb{R}^d, 0 \leq t \leq T. \)
2. \( |\tilde{u}(x + y, t) - \tilde{u}(x, t)| \leq c_5 |y| \) for all \( x, y \in \mathbb{R}^d, 0 \leq t \leq T. \)
3. \( \tilde{u}(x + y, t) + \tilde{u}(x - y, t) - 2\tilde{u}(x, t) \leq c_6 |y|^2 \) for all \( x, y \in \mathbb{R}^d, \quad 0 \leq t \leq t_1 < T. \)

Where \( c_5 \) depends only on \( L, \psi, T \) and \( c_6 \) depends only on \( L, \text{Lip}(\Phi), T \) and \( T - t_1 \).

**Proof.** We have

\[
\tilde{u}(x,t) \leq \int_{t}^{T} L(v_0(X, X), x, X, \dot{X}) + \psi(x(T), X(T)) \leq (T - t) c_1 + \|\psi\|_{\infty}.
\]
In order to prove that \( \tilde{u} \) is Lipschitz take, \( x, y \in \mathbb{R}^d \) with \( |y| \leq 1 \). Let \( x^* \) be the optimal trajectory at a point \((x, t)\). Such optimal trajectories exist by standard control theory arguments. We have
\[
\tilde{u}(x, t) = \int_t^T L(x^*, \dot{x}, \dot{x}, \ddot{x}) + \psi(x^*(T), \dot{x}(T)),
\]
and
\[
\tilde{u}(x + y, t) \leq \int_t^T L(x^*, \dot{x} + y, \dot{x}, \ddot{x}) + \psi(x^*(T) + y, \dot{x}(T)).
\]
Let
\[
f(\tau) = \int_t^T L(x^*, \dot{x} + \tau y, \dot{x}, \ddot{x}).
\]
We have
\[
f(0) = \tilde{u}(x, t) - \psi(x^*(T), \dot{x}(T)) \leq C(T - t) + C,
\]
where the constants \( C \) depends only on \( L \), \( \psi \), and \( T \).
\[
f'(\tau) = \int_t^T D_x L(x^*, \dot{x} + \tau y, \dot{x}, \ddot{x}) \cdot y \leq \int_t^T (c_2 L(x^*, \dot{x} + \tau y, \dot{x}, \ddot{x}) + c_3)|y| \leq (c_2 f(\tau) + T c_3)|y|.
\]
Hence by Gronwall inequality \( f(\tau) \leq C \) and \( f'(\tau) \leq (C(T - t) + C)|y| \). Therefore
\[
\tilde{u}(x + y, t) - \tilde{u}(x, t) \leq f(1) - f(0) + \psi(x^*(T) + y, \dot{x}(T)) - \psi(x^*(T), \dot{x}(T)) \leq (C(T - t) + C)|y|,
\]
and this proves that \( \tilde{u} \) is uniformly Lipschitz in \( x \).
For the semi-concavity we take any \( t_1 < T, t \leq t_1, x, y \in \mathbb{R}^d \) with \( |y| \leq 1 \) and \( x^* \) as above. Let
\[
y(s) = \frac{T - s}{T - t},
\]
and
\[
g(\tau) = \int_t^T L(x^*(s) + \tau \dot{y}(s), \dot{x}^*(s) + \tau y(s), \dot{x}(s), \ddot{x}(s)) ds.
\]
\[
g'(\tau) = \int_t^T [D_x L(x^*(s) + \tau \dot{y}(s), \dot{x}^*(s) + \tau y(s), \dot{x}(s), \ddot{x}(s)) \dot{y}]
\]
\[
+ D_x L(x^*(s) + \tau \dot{y}(s), \dot{x}^*(s) + \tau y(s), \dot{x}(s), \ddot{x}(s)) |y| ds
\]
using Lemma [1] with the bounds on \( DL \) from assumption [2]. We get \( g'(\tau) \leq C(Lip(\Phi), T - t_1)(g(\tau) + 1) \), hence by Gronwall inequality \( g(\tau) \leq C'(Lip(\Phi), T - t_1) \). Then
\[
g''(\tau) \leq \int_t^T ||D^2_{xx} L(x^* + \tau \dot{y}, \dot{x}^* + \tau y, \dot{x}, \ddot{x})|| |\dot{y}|^2 +
\]
\[
||D^2_{xx} L(x^* + \tau \dot{y}, \dot{x}^* + \tau y, \dot{x}, \ddot{x})|| |\dot{y}|^2 + 2 ||D^2_{xx} L(x^* + \tau \dot{y}, \dot{x}^* + \tau y, \dot{x}, \ddot{x})|| |\dot{y}| |y| ds \leq
\]
\[
\leq \int_t^T (1 + c_1)(c_2 L(x^* + \tau \dot{y}, \dot{x}^* + \tau y, \dot{x}, \ddot{x}) + c_3)(|\dot{y}|^2 + |y|^2) ds
\]
\[
\leq (1 + c_1)C g(\tau) + C \left( C + \frac{C}{(T - t)^2} \right) |y|^2 \leq c_0 |y|^2,
\]
where $c_6$ depends only on $L, \psi, T$, $\text{Lip}(\Phi)$ and $T - t_1$, and we used Lemma 1 with the bounds on $D^2L$ from assumption 1. We conclude

$$\tilde{u}(x + y, t) + \tilde{u}(x - y, t) - 2\tilde{u}(x, t) \leq g(1) + g(-1) - 2g(0) \leq 2 \max_{[-1,1]} g'' \leq c_6|y|^2.$$ 

\[\square\]

Let $\overline{c_5}, \overline{c_6}$ be the constants from Lemma 2 for $t_1 = 0$, and

$$\overline{c_7} = \max\{ Tc_0(c_4(L, c_5, X_0, T, q) + 1) + \|\psi\|_\infty, Tc_1 + \|\psi\|_\infty \}.$$ 

Let $A$ be the set of functions $\Phi \in C([\mathbb{R}^d])$ with $|\Phi| \leq \overline{c_7}$, $\text{Lip}(\Phi) \leq \overline{c_6}$ and $\Phi$ semi-concave with the constant $\overline{c_6}$.

**Lemma 3.** The mapping

$$F: \Phi(\cdot) \mapsto \tilde{u}(\cdot, 0),$$

is a continuous and compact mapping from $A$ into itself.

**Proof.** First we show that the mapping maps the set $A$ into itself. Using the estimate from Lemma 1 we get

$$\tilde{u}(x, 0) = \inf_{\Phi} \int_0^T L(x, \dot{x}, X, \dot{X}) + \psi(x(T), X(T) \geq -c_0 \int_0^T E(|X|^q + |\dot{X}|^q + 1) - \|\psi\|_\infty$$

$$\geq -Tc_0 (c_4(L, c_5, X_0, T, q) + 1) - \|\psi\|_\infty \geq -\overline{c_7}$$

which combined with Lemma 2 implies that $\tilde{u} \in A$. We prove the continuity of the mapping $F$ by assuming the contrary: there exist $\Phi_n \to \Phi$ locally uniformly in $\mathbb{R}^d$ such that $F(\Phi_n) \not\to F(\Phi)$ locally uniformly in $\mathbb{R}^d$. Then since $F(\Phi_n) \in A$ and $A$ is compact we can assume without loss of generality that $\tilde{u}_n = F(\Phi_n) \to \Phi \neq F(\Phi)$ locally uniformly, since $\Phi_n$ are uniformly semi-concave we can furthermore assume that $D\Phi_n \to D\Phi$ almost everywhere. We have that the corresponding $(X_n, P_n)$ solve the equation

$$\begin{cases}
\dot{X}_n = G(P_n, X_n) \\
\dot{P}_n = D_xH(P_n, X_n, X_n, G(P_n, X_n)) \\
X_n(0) = X_0, P_n(0) = D_xu_n(X_0)
\end{cases}$$

Then by Gronwall’s inequality

$$E(|X_n(t) - X(t)|^q + |P_n(t) - P(t)|^q) \leq C E|X_n(0) - X(0)|^q + |P_n(0) - P(0)|^q = CE|D\Phi_n(X_0) - D\Phi(X_0)|^q.$$ 

By the dominated convergence theorem the right hand side here goes to zero. Consequently $X_n \to X$ and $P_n \to P$ in $L^\infty([0, T]; L^q(\Omega))$, the equation 1.1 then implies that $X_n \to X$. Thus using Assumptions 3 and 4

$$H(p, x, X_n, \dot{X}_n) \to H(p, x, X, \dot{X})$$

locally uniformly in $x, p,$

and

$$\psi(x, X_n(t)) \to \psi(x, X(T))$$

locally uniformly in $x$.

Since $\tilde{u}_n \to \tilde{u}$ locally uniformly the stability of viscosity solutions ([FS06]) implies that $\tilde{u}$ is a viscosity solution to the Hamilton-Jacobi equation with the Hamiltonian

$$\tilde{H}(p, x, t) = H(p, x, X(t), \dot{X}(t))$$

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\[
\begin{aligned}
-\ddot{u}(x, t) + \ddot{H}(D_x \dot{u}(x, t), x, t) &= 0, \\
\dot{u}(x, T) &= \psi(x, X(T)).
\end{aligned}
\] (17)

On the other hand from the definition of \( \ddot{u} = F(\Phi) \) and standard optimal control theory results we have that \( \ddot{u} \) also is a viscosity solution to the (17).

Since \( X \) and \( \dot{X} \) are Lipschitz continuous in \( t \), the assumption implies

\[
|\ddot{H}(p, x, t) - \ddot{H}(p', y, s)| \leq C(R)(|p - p'| + |x - y| + |t - s|),
\]

for any \( R > 0 \) and \( t, s \in [0, T] \) and all \( x, y, p, q \in \mathbb{R}^d \) with \( |p|, |p'| \leq R \). This condition implies the uniqueness of the viscosity solutions to the equation (17) (FS06), thus \( \Phi = \ddot{u} \) which is a contradiction. Hence the mapping \( F \) is continuous, the compactness of \( \mathcal{A} \) then implies that \( F \) is also compact.

**Theorem 1.** Under the above conditions on \( H, \psi \) and \( X_0 \) there exist a continuous Lipschitz semiconcave function \( u \) on \( \mathbb{R}^n \) and a random variable \( X \in C^{1,1}([0, T], L^2(\Omega)) \), such that the couple \((u, X)\) solves the system of extended mean field equations (10) in the sense that \( u \in C([0, T] \times \mathbb{R}^d) \) is a viscosity solution to the Hamilton-Jacobi equation:

\[
\begin{aligned}
-\dot{u}(t, x) + H(D_x u(t, x), X(t), \dot{X}(t)) &= 0, \text{ in } [0, T] \times \mathbb{R}^d, \\
\dot{u}(x, T) &= \psi(x, X(T)),
\end{aligned}
\]

\( u \) is differentiable at every point \((X(t), t), t \geq 0 \), and \( X \in C^{1,1}([0, T]; L^2(\Omega)) \) is a classical solution to the ODE:

\[
\begin{aligned}
\dot{X} &= -D_p H(D_x u(X, s), X, \dot{X}), \text{ in } [0, T] \times \Omega, \\
X(0) &= X_0.
\end{aligned}
\]

**Proof.** The set \( C(\mathbb{R}^d) \) endowed with the topology of locally uniformly convergence is a topological vector space, and \( \mathcal{A} \) is its compact, convex subset. Thus by Lemma 3 and Schauder’s fixed point theorem there exists \( \Phi \in C(\mathbb{R}^d) \) such that

\[
\Phi(\cdot) = F(\Phi) = \ddot{u}(.), 0),
\]

where \( \ddot{u}(x, t) \) is defined as in (15). Let us denote it by \( u(x, t) := \ddot{u}(x, t) \). Then \( u \) solves the Hamilton-Jacobi equation

\[
\begin{aligned}
-\dot{u}(x, t) + H(D_x u(x, t), x, X(t), \dot{X}(t)) &= 0, \\
\dot{u}(x, T) &= \psi(x, X(T)).
\end{aligned}
\] (18)

From optimal control theory (see FS06) we know that for almost every \( x \) we have the existence of the optimal trajectories which are given by the Hamiltonian flow

\[
\begin{aligned}
\dot{x}(x, t) &= -D_p H(p, x, X, \dot{X}), \\
\dot{p}(x, t) &= D_x H(p, x, X, \dot{X}), \\
x(x, 0) &= x, \quad p(x, 0) = D_x u(x, 0).
\end{aligned}
\] (19)

We also know that \( p(x, t) = Du(x(x, t), t) \) and that \( Du \) exists for all points \((x(x, t), t)\) with \( t > 0 \). Now put \( Y(t) = x(X_0, t) \) and \( Q(t) = p(X_0, t) \), then from (19)

\[
\begin{aligned}
\dot{Y} &= -D_p H(Q, Y, X, X), \\
\dot{Q} &= D_x H(Q, Y, X, X), \\
Y(0) &= X_0, \quad Q(0) = D_x u(X_0).
\end{aligned}
\] (20)
Since $D_p H, D_x H$ are Lipschitz in $p, x$ the uniqueness of solutions to the system \((20)\) of ordinary differential equations in $L^q(\Omega)$ yields $X(t) = Y(t)$ and $P(t) = Q(t) = p(X_0, t) = Du(Y(t), t)$ for all $t \in [0, T]$, thus

$$
\begin{cases}
\dot{X}(t) = -D_p H(Du(X(t), t), X(t), \dot{X}(t), \dot{X}(t)), \\
X(0) = X_0.
\end{cases}
$$

It is important to notice that this existence proof does not require the absolute continuity of the law of the solution $X$ of the extended mean-field game, only the absolutely continuity of law of the initial condition $X_0$. As such, the previous result extends the results in [Car11]. We discuss in the next section additional conditions which yield absolute continuity for the law of $X$.

## 5 Absolute continuity of the law of trajectories

In this section we give conditions under which solutions to extended mean-field games have an absolutely continuous law. Our techniques are related to the ones in [Car11], where the case of quadratic Hamiltonians was discussed.

Consider a Hamiltonian $H^\# : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ and the corresponding Lagrangian $L^\#$ satisfying the following conditions:

1. There exist constants $c_0, c_1 > 0$ and $v_0 \in \mathbb{R}^d$ such that $L^\#(v, x, t) \geq -c_0$, and $L^\#(v_0, x, t) \leq c_1$.
2. $L^\#$ is coercive in $v$:
   $$
   \lim_{|v| \to \infty} \frac{L^\#(v, x, t)}{|v|} = \infty,
   $$
   uniformly in $x$.
3. There exist a constant $C > 0$ such that $|D_x L^\#|, |D_x L^\#|, |D_{vv}^2 L^\#|, |D_{vv}^2 L^\#|, |D_{xx}^2 L^\#| \leq C L^\# + C$,
4. $H^\#$ is twice differentiable in $p, x$ with bounded derivatives: $|D_{pp}^2 H^\#|, |D_{pp}^2 H^\#| \leq C$.
5. $D_{pp}^2 H^\#(p, x, t)$ is uniformly Lipschitz in $p, x, t$.

In order to obtain the absolute continuity of the law for extended mean-field games, we will apply the next theorem to

$$
L^\#(x, v, t) = L(x, v, X(t), \dot{X}(t))
$$

and

$$
H^\#(p, x, t) = H(p, x, X(t), \dot{X}(t)).
$$

We note that the first three hypothesis on $L^\#$ will be satisfied under the hypothesis required in the previous section for any solution $X$. The last two hypothesis, however, do not follow from the ones in the previous section.

**Theorem 2.** Let $u$ be a viscosity solution to the Hamilton-Jacobi equation:

$$
\begin{cases}
-u_t + H^\#(Du, x, t) = 0, \\
u(x, T) = \psi(x),
\end{cases}
$$

where $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded and Lipschitz, and let $X \in C^1([0, T], L^q(\Omega))$ solve the equation

$$
\begin{cases}
\dot{X}(t) = -D_p H^\#(Du(X(t), t), X(t), t), \\
X = X_0,
\end{cases}
$$

(21)
where $X_0$ has an absolutely continuous law with respect to the Lebesgue measure. Then under the above conditions on $H$ for every $t < T$, $X(t)$ also has an absolutely continuous law with respect to the Lebesgue measure.

Remark: We recall, that as before (21) has a well defined solution since $X_0$ has an absolutely continuous law. The key point is the absolutely continuity of the law of $X$.

Before proving the theorem we establish an auxiliary lemma:

**Lemma 4.** Let $z, q \in C^4([0, T]; \mathbb{R}^d)$ and $A, B \in C([0, T]; \mathbb{R}^{d \times d})$ be such that

$$\dot{z}(t) = -A(t)q(t) - B(t)z(t).$$

Suppose further that there exist constants $C, \theta > 0$ such that

$$|z(t)|, |q(t)| \leq C,$$

$$z(t) \cdot q(t) \leq C|z(t)|^2,$$

$$|A(t)|, |B(t)| \leq C,$$

$$A(t) \geq \theta I,$$

$$|A(t_2) - A(t_1)| \leq C|t_2 - t_1|.$$ 

Then there exists a constant $C_1$, which only depends on the constants $C, \theta$ and $T$, such that

$$|z(t)| \geq C_1|z(0)|, \quad \forall t \in [0, T].$$

**Proof.** Since $A$ is Lipschitz in $t$ with constant $C$ there exist matrices $A_n \in C^1([0, T]; \mathbb{R}^d)$ with $A_n \geq \theta I$, $\|A_n\| \leq C$ and $\|\frac{d}{dt}A_n(t)\| \leq C$ such that $A_n \to A$ uniformly in $[0, T]$. Let $z_n \in C^1([0, T]; \mathbb{R}^d)$ solve the equation

$$\begin{cases}
\dot{z}_n(t) = -A_n(t)q(t) - B(t)z_n(t), \\
z_n(0) = z(0).
\end{cases}$$

(23)

By standard arguments for ODE’s we have $|z_n - z| \leq \epsilon_n$, where $\epsilon_n \to 0$.

Since $A_n \geq \theta I$, $\|A_n\| \leq C$, $\|\frac{d}{dt}A_n(t)\| \leq C$ there exist constants $\theta', C' > 0$ such that $A_n^{-1} \geq \theta' I$, $\|A_n^{-1}\| \leq C'$ and $\|\frac{d}{dt}A_n^{-1}(t)\| \leq C'$. Thus from (23) one gets

$$z_n \cdot A_n^{-1}\dot{z}_n = -z_n \cdot q - z_n A_n^{-1}Bz_n = -(z_n - z) \cdot q - z \cdot q - z_n \cdot A_n^{-1}Bz_n \geq - (z_n - z) \cdot q - C|z|^2 - z_n \cdot A_n^{-1}Bz_n \geq -C\epsilon_n - C|z_n|^2.$$ 

(24)

Therefore

$$\frac{d}{dt}(z_n \cdot A_n^{-1}z_n) \geq -C\epsilon_n - C|z_n|^2 - z_n \cdot (\frac{d}{dt}A_n^{-1})z_n \geq -C\epsilon_n - C|z_n|^2 - \frac{C}{\theta'}z_n \cdot A_n^{-1}z_n.$$ 

Gronwall’s inequality implies that

$$z_n(t) \cdot A_n^{-1}(t)z_n(t) \geq -C'\epsilon_n + C'''e^{-C''t}z_n(0) \cdot A_n^{-1}(0)z_n(0).$$

Passing to the limit when $n \to \infty$ yields

$$z(t) \cdot A^{-1}(t)z(t) \geq C'''e^{-C''t}z(0) \cdot A^{-1}(0)z(0),$$

which together with $A_n^{-1} \geq \theta' I$, $\|A_n^{-1}\| \leq C'$ implies

$$|z(t)| \geq C_1|z(0)|,$$

for some constant $C_1 > 0.$
Now we proceed to the proof of Theorem 2.

**Proof.** From standard optimal control theory results we know that the function \( u(x, t) \) is the solution to the optimal control problem

\[
u(x, t) = \inf_x \int_0^T L^2(x, x, s) ds + \psi(x(T)).\]

Standard arguments (see the proof of Lemma 2) show that \( u \) is uniformly bounded and Lipschitz and is uniformly semi-concave in any interval \([0, T']\), \( T' < T \). It is known that the optimal trajectories at \((x, 0)\) every point of differentiability \( x \) of \( u \) are given by the Hamiltonian flow:

\[
\begin{align*}
\dot{x}(x, t) &= -D_p H^2(p, x, t) \\
\dot{p}(x, t) &= D_x H^2(p, x, t) \\
x(x, 0) &= x, \quad p(x, 0) = Du(x, 0),
\end{align*}
\]

furthermore along the trajectories \((x(x, t), t)\), \( t > 0 \) is differentiable and \( p(x, t) = Du(x(x, t), t) \). Let \( K \subset \mathbb{R}^d \) be any compact set and let \( x, y \in K \) points for which the flow is defined let \( z(t) = x(x, t) - x(y, t) \) and \( q(t) = p(x, t) - p(y, t) \) then

\[
\dot{z}(t) = -A(t)q(t) - B(t)z(t),
\]

where

\[
A(t) = \int_0^1 D^2_{pp} H^2(\tau p(x, t) + (1 - \tau)p(y, t), \tau x(x, t) + (1 - \tau)x(y, t), t) d\tau
\]

and

\[
B(t) = \int_0^1 D^2_{px} H^2(\tau p(x, t) + (1 - \tau)p(y, t), \tau x(x, t) + (1 - \tau)x(y, t), t) d\tau.
\]

Equation (25) implies that there exists a constant \( C_K \) such that the trajectories \( x(x, t), p(x, t) \) are \( C_K \)-Lipschitz in \( t \) and \( |z|, |q| \leq C_K \). The assumptions on \( H^2 \) imply that \( A(t) \) is also Lipschitz with some constant \( C_K \). Furthermore it is clear that \( A(t) \geq \theta I \). Since \( u \) is semiconcave in \( x \) uniformly for \( t \in [0, T'] \) we have

\[
z(t) \cdot q(t) = (x(x, t) - x(y, t), Du(x(x, t), t) - Du(x(x, t), t)) \leq C|x(x, t) - x(y, t)|^2 = C|z(t)|^2.
\]

Thus \( z, q \) satisfy the conditions of Lemma 3 therefore \( |x(x, t) - x(y, t)| \leq C_K(T')|x-y|, t \in [0, T'] \), that is the mapping \( x \mapsto x(x, t) \) is invertible on the set where it is defined, and for any compact \( K \subset \mathbb{R}^d \) we have that the inverse of the map \( x \in K \rightarrow x(x, t) \) is Lipschitz.

Let \( A \in \mathbb{R}^d \) be a set of Lebesgue measure zero and \( B = (x)^{-1}([t, T])(A) \). For any compact \( K \subset \mathbb{R}^d \) the inverse of the map \( x \in K \rightarrow x(x, t) \) is Lipschitz. Hence \( B \cap K \) has Lebesgue measure zero. This implies that \( B \) has Lebesgue measure zero. Since \( x \) is defined a.e. we have that \( X(t) = x(X_0, t) \) a.s.. Therefore \( X_0 = (x)^{-1}([0, T])(X(t)) \) a.s.. \( X_0 \) has an absolutely continuous law, consequently \( P(X(t) \in A) = P(X_0 \in B) = 0 \). Hence this proves that \( X(t) \) has an absolutely continuous law.

\[\Box\]

6 Uniqueness

We present here two approaches to the uniqueness problem. First we show how to adapt Lions-Lasry monotonicity argument for this setting, we observe however, that this proof applies to classical solutions (though it may be possible to extend for viscosity solutions if the mean-field trajectories \( X \) admit a absolutely continuous law). We then discuss a second uniqueness technique which gives uniqueness for more general class of problems without any additional conditions on solutions, therefore generalizing, even for classical mean-field games, the results in the literature.
6.1 Lions-Lasry monotonicity argument

In this section we consider the adaptation of Lions-Lasry monotonicity method to prove uniqueness for extended mean-field games. The original idea can be explained as follows: let \((\theta, u)\) and \((\tilde{\theta}, \tilde{u})\) be two solutions of \((\mathcal{H})\). Then one looks at the quantity

\[
\frac{d}{dt} \int (\theta - \tilde{\theta})(u - \tilde{u})
\]

and shows that it is strictly monotone in time. For more detailed arguments see the paper \([LL06a]\) by Pierre Louis Lions and Jean Michel Lasry or notes by Pierre Cardaliaguet \([Car11]\).

In our setting if have two solutions \((X, u)\) and \((\tilde{X}, \tilde{u})\) this amounts to look at

\[
\frac{d}{dt} E \left( u(X(t), t) - \tilde{u}(X(t), t) + \tilde{u}(\tilde{X}(t), t) - u(\tilde{X}(t), t) \right)
\]

and show that it is monotone. To simplify, we will assume that

1. 
   \[
   H(p, x, X, Z) = H_0(p + \beta EZ, x) + V(x, X).
   \]
   with \(\beta \geq 0\).

2. As usual we suppose \(H_0\) convex in \(p\).

3. monotonicity condition:
   \[
   E \left( V(X, X) - V(X, \tilde{X}) + V(\tilde{X}, \tilde{X}) - V(\tilde{X}, X) \right) < 0,
   \]
   if \(X \neq \tilde{X}\). For all \(X, \tilde{X} \in L^q(\Omega)\).

4. \(\psi\) satisfies
   \[
   E \left( \psi(X, X) - \psi(X, \tilde{X}) + \psi(\tilde{X}, \tilde{X}) - \psi(\tilde{X}, X) \right) \geq 0,
   \]
   for all \(X, \tilde{X} \in L^q(\Omega)\).

**Theorem 3.** Assume \(H\) satisfies conditions \((H3)\), \(\psi\) satisfies \((4)\) then there exists at most one (classical) solution \((u, X)\) to \((\mathcal{H})\).

**Proof.** Let \((u, X)\) and \((\tilde{u}, \tilde{X})\) be two solutions to \((\mathcal{H})\). Then we have

\[
\frac{d}{dt} (u(X(t), t) - \tilde{u}(X(t), t)) = H_0(D_x u(X(t), t) + \beta E\tilde{X}, X(t)) -
\]

\[
H_0(D_x \tilde{u}(X(t), t) + \beta E\tilde{\tilde{X}}, X(t)) + V(X(t), X(t)) - V(X(t), \tilde{X}(t)) +
\]

\[
\tilde{X}(D_x u(X(t), t) - D_x \tilde{u}(X(t), t)) \leq (D_p H_0(D_x u(X(t), t) + \beta E\tilde{X}, X(t)) + \dot{X} \times
\]

\[
(D_x u(X(t), t) - D_x \tilde{u}(X(t), t)) + \beta X (E\dot{\tilde{X}} - EX) + V(X(t), X(t)) - V(\tilde{X}(t), X(t)).
\]

Then if we compute in a similar way \(\frac{d}{dt}\tilde{u}(\tilde{X}(t), t) - u(\tilde{X}(t), t)\) and add we obtain

\[
\frac{d}{dt} \left( u(X(t), t) - \tilde{u}(X(t), t) + \tilde{u}(\tilde{X}(t), t) - u(\tilde{X}(t), t) \right)
\]

\[
= V(X(t), X(t)) - V(X(t), \tilde{X}(t)) + V(\tilde{X}(t), \tilde{X}(t)) - V(\tilde{X}(t), X(t)) +
\]

\[
+ \beta(\dot{X} - \dot{\tilde{X}})(E\dot{\tilde{X}} - EX).
\]
By taking the expectation and using both the monotonicity condition and that $\beta \geq 0$ we obtain that

$$\frac{d}{dt} E \left( u(X(t),t) - \bar{u}(X(t),t) + \bar{u}(\tilde{X}(t),t) - u(\tilde{X}(t),t) \right) < 0.$$  

This is a contradiction since

$$E \left( u(X(0),0) - \bar{u}(X(0),0) + \bar{u}(\tilde{X}(0),0) - u(\tilde{X}(0),0) \right) = 0,$$

and

$$E \left( u(X(T),T) - \bar{u}(X(T),T) + \bar{u}(\tilde{X}(T),t) - u(\tilde{X}(T),T) \right) =$$

$$E \left( \psi(X(T),X(T)) - \psi(X(T),\tilde{X}(T)) + \psi(\tilde{X}(T),\tilde{X}(T)) - \psi(\tilde{X}(T),X(T)) \right) \geq 0. \quad \Box$$

To be able to prove the uniqueness of solutions $(u,X)$ in case when $u$ is not everywhere differentiable we assume further

5. $X_0 \in L^p(\Omega)$ has an absolutely continuous law with respect to the Lebesgue measure;

6. $\psi$ is bounded and Lipschitz in $x$;

7. $V$ is $C^2$ bounded in $x$;

8. There exist a constant $C > 0$ such that $|D_vL_0|, |D^2_{vv}L_0| \leq CL_0 + C$, where $L_0$ is the Legendre transform of $H_0$;

9. $D^2_{pp}H_0$ is Lipschitz continuous.

**Theorem 4.** Assume $H$ satisfies conditions $[7,8]$ then there exists at most one solution $(u,X)$ to the system $[3]$ with $X \in C^1([0,T],L^q(\Omega))$ and $X$ Lipschitz in $t$.

**Proof.** Let $(u,X)$ and $(\tilde{u},\tilde{X})$ be two solutions to the system $[3]$ with $X,\tilde{X} \in C^1([0,T],L^q(\Omega))$ and $X,\tilde{X}$ Lipschitz in $t$. It is easy to check that under the conditions on $H_0$ and $V$ the Hamiltonians $\bar{H}_1(p,x,t) = H(p,x,X(t),\tilde{X}(t))$ and $\bar{H}_2(p,x,t) = H(p,x,\tilde{X}(t),\tilde{X}(t))$ satisfy the conditions of the Theorem $[2]$. Thus we conclude that $X(t)$ and $\tilde{X}(t)$, $t < T$ have absolutely continuous laws with respect to the Lebesgue measure, since $u,\tilde{u}$ are Lipschitz and thus $Du(X(t),t)$, $Du(\tilde{X}(t),t)$, $D\tilde{u}(X(t),t)$, $D\tilde{u}(\tilde{X}(t),t)$ are well defined almost surely and we can carry out the arguments from the proof of Theorem $[3]$. \Box

### 6.2 The second approach for uniqueness

In this section we discuss another approach for uniqueness of solutions to extended mean field system $[10]$. This second approach does not require classical solutions to the Hamilton-Jacobi PDE in $[13]$, and therefore gives a more general uniqueness result. Now we suppose that $L$ satisfies the monotonicity conditions

10. $E \left( L(Z,X,X,Z) - L(\tilde{Z},\tilde{X},X,Z) + L(\tilde{Z},\tilde{X},\tilde{Z},\tilde{Z}) - L(Z,X,\tilde{X},\tilde{Z}) \right) > 0$

if $(X,Z) \neq (\tilde{X},\tilde{Z})$.

**Theorem 5.** Under Assumptions $[4]$ and $[10]$ there exists at most one solution to $[10]$. 

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Proof. Suppose \((X,u)\) and \((\tilde{X}, \tilde{u})\) are two solutions to \([10]\). For a.e. \(\omega\), \(X(\omega)\) and \(\tilde{X}(\omega)\) lie on the optimal trajectories for the optimal control problems with Lagrangians \(L(v,x,X,X)\) and \(L(v,x,\tilde{X},\tilde{X})\) and terminal values \(\psi(x,X(T))\) and \(\psi(x,\tilde{X}(T))\) respectively. Since \(u, \tilde{u}\) are the respective value functions we have
\[
\begin{align*}
     &u(X(0),0) = \int_0^T L(\dot{X}(s),X(s),\dot{X}(s))ds + \psi(X(T),X(T)) \\
     &u(\tilde{X}(0),0) \leq \int_0^T L(\dot{\tilde{X}}(s),\tilde{X}(s),\dot{\tilde{X}}(s))ds + \psi(\tilde{X}(T),\tilde{X}(T)), \\
     &\tilde{u}(\tilde{X}(0),0) = \int_0^T L(\ddot{\tilde{X}}(s),\tilde{X}(s),\ddot{\tilde{X}}(s))ds + \psi(\tilde{X}(T),\tilde{X}(T)),
\end{align*}
\]

and
\[
\tilde{u}(X(0),0) \leq \int_0^T L(\dot{\tilde{X}}(s),X(s),\dot{\tilde{X}}(s))ds + \psi(X(T),\tilde{X}(T)).
\]

Summing up this inequalities with respective signs we get the inequality

\[
0 = E\left( u(X(0),0) - \tilde{u}(X(0),0) + \tilde{u}(\tilde{X}(0),0) - u(\tilde{X}(0),0) \right) \geq \\
\int_0^T E\left( L(\dot{X}(s),X(s),\dot{X}(s)) - L(\dot{\tilde{X}}(s),\tilde{X}(s),\dot{\tilde{X}}(s)) + \\
L(\ddot{\tilde{X}}(s),\tilde{X}(s),\ddot{\tilde{X}}(s)) - L(\dot{\tilde{X}}(s),X(s),\dot{\tilde{X}}(s)) \right)ds + \\
E\left( \psi(X(T),X(T)) - \psi(\tilde{X}(T),X(T)) + \psi(\tilde{X}(T),\tilde{X}(T)) - \psi(X(T),\tilde{X}(T)) \right).
\]

The conditions \([10]\) and the previous inequalities imply \(X(s) = \tilde{X}(s)\) but then the uniqueness of viscosity solutions implies that \(u = \tilde{u}\). \(\square\)

The first monotonicity condition \([10]\) can be reformulated in terms of the second derivative of \(L\), indeed we have
\[
E\left( L(Z,X,X,Z) - L(\tilde{Z},\tilde{X},X,Z) + L(\tilde{Z},\tilde{X},\tilde{X},\tilde{Z}) - L(Z,X,\tilde{X},\tilde{Z}) \right) = \\
\int_0^1 \int_0^1 E[(Z - \tilde{Z})^T D^2_{zw}L \cdot (Z - \tilde{Z}) + (X - \tilde{X})^T D^2_{zx}L \cdot (X - \tilde{X}) + \\
(Z - \tilde{Z})^T D^2_{z\tilde{z}}L \cdot (Z - \tilde{Z}) + (X - \tilde{X})^T D^2_{bz}L \cdot (Z - \tilde{Z})]d\theta d\tau,
\]

where all the derivatives of \(L\) are evaluated at the point \((Z_\tau, X_\tau, Z_\theta, X_\theta)\) where \(X_\theta = (1-\theta)X + \theta \tilde{X}\) and \(Z_\theta = (1-\theta)Z + \theta \tilde{Z}\).

So uniqueness holds if we have the inequality
\[
E[(Z^T D^2_{zw}L \cdot Z + Y^T D^2_{zx}L \cdot Y + Z^T D^2_{z\tilde{z}}L \cdot Y + Y^T D^2_{bz}L \cdot Z)] > 0
\]
if \((Y,Z) \neq 0\) where the derivatives are evaluated at an arbitrary point \((A,B,C,D)\).

It is easy to see that in the proof of Theorem \([3]\) we actually use a weaker version of condition \([10]\).
11. \[ E \left( L(Z, X, X, Z) - L(\tilde{Z}, \tilde{X}, X, Z) + L(\tilde{Z}, \tilde{X}, \tilde{Z}) - L(Z, X, \tilde{X}, \tilde{Z}) \right) \leq 0 \]

if and only if \( L(v, x, X, Z) = L(v, x, \tilde{X}, \tilde{Z}) \) for all \( v, x \in \mathbb{R}^d \).

As an example which satisfy 11 consider the Lagrangian

\[ L(v, x, X, Z) = L_0(v) + \beta vE(Z) - V(x, X) \]

where \( L_0 \) is strictly convex, \( \beta \geq 0 \) and \( V \) satisfies the monotonicity condition. The corresponding Hamiltonian for this Lagrangian is

\[ H(p, x, X, Z) = H_0(p + \beta E(Z)) + V(x, X) \]

where \( H_0 = L_0^* \), and so is an arbitrary convex function. Thus the uniqueness result in this section generalizes the result in the previous section.

Another example of Lagrangian which satisfies the condition 11 is

\[ L(v, x, X, Z) = \frac{|v + \beta E(Z)|^2}{2} - V(x, X), \]

where \( V \) satisfies the monotonicity condition.

7 Extensions and further directions

7.1 Master equation in deterministic case

In this section we introduce the so called "Master equation" for deterministic MFG's. The master equation for mean field games was introduced first by Lions in his Collège de France Course lectures. Let \((\Omega, \mathcal{F}, P)\) be a probability space, where \( \Omega \) is an arbitrary set, \( \mathcal{F} \) is a \( \sigma \)-algebra on \( \Omega \) and \( P \) is a probability measure. We start by considering the following optimal control problem:

\[ V(x, Y, t) = \inf_v \left[ \int_t^T L(v(s), x(s), Y(s), \dot{Y}(s))ds + \psi(x(T), Y(T)) \right], \quad (27) \]

where \( x \) is the trajectory of a player which starts at time \( t \) at point \( x(t) = x \), and is controlled by \( \dot{x} = v \). \( Y(\cdot) \) is the trajectory of the population of the players which move along a vector field \( b: L^q(\Omega; \mathbb{R}^d) \times [0, +\infty) \rightarrow L^q(\Omega; \mathbb{R}^d) \),

that is

\[ \dot{Y} = b(Y, t), \quad Y(0) = Y, \quad (28) \]

where \( Y \) is the initial distribution of the population. We define \( L^q_{ac}(\Omega; \mathbb{R}^d) \) to be the subspace of \( L^q(\Omega; \mathbb{R}^d) \) of random variables which have absolutely continuous laws. We assume the vector field \( b \) is such that for any \( Y \in L^q_{ac}(\Omega; \mathbb{R}^d) \) equation (28) has a unique solution in \( L^q(\Omega; \mathbb{R}^d) \). Then \( V \) is well defined and is a viscosity solution to the Hamilton-Jacobi equation

\[ -V_t - D_Y V(x, Y, t) \cdot b(Y, t) + H(D_x V(x, Y, t), x, Y, b(Y, t)) = 0, \quad (29) \]

where

\[ H(p, x, Y, Z) = \sup_v \{-p \cdot v - L(v, x, Y, Z)\}. \]

If \( V \) is a classical solution to (29), the optimal control is given in feedback form by \( v^* = -D_p H(D_x V(x, Y, t), x, Y, Y) \).
As previously, if we assume all players act rationally then they all need to follow the optimal flow, that is

\[ \dot{Y} = -D_p H(D_z V(Y,Y,t),Y,Y,\dot{Y}), \]

we assume one can solve this equation in \( \dot{Y} \) as \( \dot{Y} = G(D_z V(Y,Y,t),Y) \), hence we should take \( b(Y,t) = G(D_z V(Y,Y,t),Y) \).

Thus we arrive to the equation

\[
\begin{cases}
-V_t(x,Y,t) + D_y V(x,Y,t) \cdot G(D_z V(Y,Y,t),Y) + H(D_z V(x,Y,t),x,Y,b(Y,t)) = 0, \\
V(x,Y,T) = \psi(x,Y).
\end{cases}
\]

We call this the "Master equation".

In general first order PDEs do not admit classical solutions so we must look, for instance, for solutions which are only differentiable almost everywhere with respect to the variable \( x \). Therefore one can only make sense of \( G(D_z V(Y,Y,t),Y) \) if \( Y \) has an absolutely continuous law. This is the reason we work in the space \( L^q_{ac}(\Omega;\mathbb{R}^d) \).

**Definition 1.** Let \( V: \mathbb{R}^d \times L^q_{ac}(\Omega;\mathbb{R}^d) \times [0,T] \to \mathbb{R}^d \) be a continuous function and \( b: L^q_{ac}(\Omega;\mathbb{R}^d) \times [0,T] \to \mathbb{R}^d \) a vector field. Then we say that the couple \((V,b)\) is a solution to the equation \( (30) \) if

- \( V \) is a viscosity solution to the equation

\[
\begin{cases}
-V_t(x,Y,t) + D_y V(x,Y,t) \cdot b(Y,t) + H(D_z V(x,Y,t),x,Y,b(Y,t)) = 0, \\
V(x,Y,T) = \psi(x,Y),
\end{cases}
\]

that is, for any continuous function \( \phi: \mathbb{R}^d \times L^q(\Omega) \times [0,T] \to \mathbb{R} \) and any point \((x,Y,t)\) \in \text{argmax} V - \phi (resp. \text{argmin}) where \( \phi \) is differentiable

\[-\phi(x,Y,t) + D_y \phi(x,Y,t) \cdot b(Y,t) + H(D_z \phi(x,Y,t),x,Y,b(Y,t)) \leq 0 \text{ (resp. } \geq 0).\]

- \( b(Y,t) = G(D_z V(Y,Y,t),Y), \forall Y \in L^q_{ac}(\Omega;\mathbb{R}^d) \).

Here after we assume that \( L \) and \( \psi \) satisfy the following hypothesis

1. \( \psi \) is bounded in both variables and Lipschitz in \( x \):

\[ |\psi(x_1,X) - \psi(x_2,X)| \leq C |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^d. \]

2. There exist constants \( c_0, c_1 > 0 \) and a vector function \( v_0: L^q(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^d) \to \mathbb{R}^d \) such that

\[ L(v, x, X, Z) \geq -c_0, \]

and

\[ L(v_0(X,Z),x,X,Z) \leq c_1, \]

for all \( x, v \in \mathbb{R}^d, X, Z \in L^q(\Omega;\mathbb{R}^d) \).

3. \( L \) is twice differentiable in \( x, v \) and we have the following bounds

\[ |D_x L(v, x, X, Z)|, |D^2_{xx} L(v, x, X, Z)|, |D^2_{zv} L(v, x, X, Z)|, |D^2_{zz} L(v, x, X, Z)| \leq CL(v, x, X, Z) + C. \]

for all \( x, v \in \mathbb{R}^d, X \in L^q(\Omega;\mathbb{R}^d) \).

**Proposition 2.** Assume that \( (27) \) hold. Then the function \( V \) defined in \( (27) \) for a fixed vector field \( b \) is finite, bounded, Lipschitz and semiconcave in \( x \):
1. There exists a constant $C$ such that
\[
|V(x, Y, t)| \leq C, \quad \forall t \in [0, T], \, x, h \in \mathbb{R}^d, \, Y \in L^q(\Omega; \mathbb{R}^d).
\]

2. There exists a constant $C$ such that
\[
|V(x + h, Y, t) - V(x, Y, t)| \leq C|h|, \quad \forall t \in [0, T], \, x, h \in \mathbb{R}^d, \, Y \in L^q(\Omega; \mathbb{R}^d).
\]

3. For any $t, t < T$ there exists a constant $C(t)$ such that
\[
V(x + h, Y, t) + V(x - h, Y, t) - 2V(x, Y, t) \leq C(t)|h|^2, \quad \forall x, h \in \mathbb{R}^d, \, Y \in L^q(\Omega; \mathbb{R}^d).
\]

Where the constants are uniform in $b$.

**Proof.**

- **Bounded**

  From the bounds from below on $L$ we have
  \[
  \int_t^T L(v(s), x(s), Y(s), \dot{Y}(s))ds + \psi(x(T), Y(T)) \geq -c_0(T - t) - \|\psi\|_\infty
  \]
  thus $V(x, Y, t) \geq -c_0(T - t) - \|\psi\|_\infty$. On the other hand from the upper bound on $L$ and (27) we obtain
  \[
  V(x, Y, t) \leq \int_t^T L(v_0(Y(s), \dot{Y}(s)), x, Y(s), \dot{Y}(s))ds + \psi(x(T), Y(T)) \geq c_1(T - t) + c_1\|\psi\|_\infty
  \]
  where $x(s) = x + \int_t^s v_0(Y(s), \dot{Y}(s))ds$. This proves that $V$ is bounded and thus also finite.

- **Lipschitz in $x$**

  Let $x, h \in \mathbb{R}^d$ and $x^\varepsilon$ be an $\varepsilon$-suboptimal trajectory in (27):
  \[
  V(x, Y, t) \geq \int_t^T L(v^\varepsilon(s), x^\varepsilon(s), Y(s), \dot{Y}(s))ds + \psi(x^\varepsilon(T), Y(T)) - \varepsilon,
  \]
  where $v^\varepsilon(s) = \dot{x}^\varepsilon(s)$. We have
  \[
  V(x + h, Y, t) \leq \int_t^T L(v^\varepsilon(s), x^\varepsilon(s) + h, Y(s), \dot{Y}(s))ds + \psi(x^\varepsilon(T) + h, Y(T)),
  \]
  thus
  \[
  V(x + h, Y, t) - V(x, Y, t) \leq \int_t^T L(v^\varepsilon(s), x^\varepsilon(s) + h, Y(s), \dot{Y}(s)) -
  L(v^\varepsilon(s), x^\varepsilon(s), Y(s), \dot{Y}(s))ds + \psi(x^\varepsilon(T) + h, Y(T)) - \psi(x^\varepsilon(T), Y(T)) + \varepsilon,
  \]
  for all $\varepsilon > 0$.

  Let
  \[
  f(\tau) = \int_t^\tau L(v^\varepsilon(s), x^\varepsilon(s) + \tau h, Y(s), \dot{Y}(s))ds.
  \]
  We have
  \[
  f(0) \leq V(x, Y, t) - \psi(x^\varepsilon(T), Y(T)) + \varepsilon \leq C,
  \]
  (22)
Proposition 3. Let \( u(x, t) = V(x, X(t), t) \), then the pair \((u, X)\) solves equation (3).
Proof. Since $V$ is smooth we have

$$D_xu(x,t) = D_xV(x,X(t),t),$$

and

$$u_t(x,t) = V_t + D_yV \cdot \dot{X} = V_t(x,X(t),t) - D_yV(x,X(t),t) \cdot D_pH(D_xV(X(t),X(t),x,t),X(t),X(t),\dot{X}(t)).$$

Plugging this in (30) with $Y = X(t)$ we get

$$-u_t(x,t) + H(D_xu(x,t),x,X(t),\dot{X}(t)) = 0,$$

we have also $u(x,T) = V(x,X(T),T) = \psi(x,X(T))$.

Now we assume that $\psi$ and $L$ satisfy conditions [1][1] then by Theorem [1] for any $Y \in L^q_\text{ac}(\Omega,\mathbb{R}^d)$ there exist a solutions $(u(x,s,t),X(s,t))$ to

$$\begin{cases}
-u_s(x,s,t) + H(D_xu(x,s,t),x,X(s,t),\frac{\partial X}{\partial s}(s,t)) = 0 \\
\frac{\partial X}{\partial s}(s,t) = -D_pH(D_xu(X(s,t),s),X(s,t),X(s,t),\frac{\partial X}{\partial s}(s,t)) \\
u(x,T,t) = \psi(x,X(T)), \quad X(t,t) = Y,
\end{cases}$$

we let $\tilde{V}(x,Y,t) = u(x,t,t)$.

Proposition 4. Assume $\psi$ and $L$ satisfy conditions [7][7] then $\tilde{V}$ is a viscosity solution to

$$\begin{cases}
-V_t(x,Y,t) + D_yV(x,Y,t) \cdot b(Y,t) + H(D_xV(x,Y,t),x,Y,b(Y,t)) = 0, \\
V(x,Y,T) = \psi(x,Y),
\end{cases}$$

for $b(Y,t) = G(D_xV(Y,Y,t),Y)$.

Proof. Note that by definition of $\tilde{V}$, we have $\tilde{V}(x,X(s,t),s) = u(x,s,t)$. Let $\phi: \mathbb{R}^d \times L^q(\Omega;\mathbb{R}^d) \times \mathbb{R}$ be a continuous function which is differentiable at $(x,Y,t)$, and the function $\tilde{V} - \phi$ has a local maximum at $(x,Y,t)$. Let $\varphi(\cdot,s) = \phi(\cdot,X(s,t),s)$. Then $\varphi$ is a continuous function which is differentiable at $(x,t)$, and the function $u - \varphi$ has a local maximum at $(x,t)$. Since $u$ is a viscosity solution to the Hamilton-Jacobi equation we have

$$-\varphi_t(x,t) + H(D_x\varphi(x,t),x,X(t),t), \frac{\partial X}{\partial s}(t),t)) \leq 0.$$

Because $\varphi_t(x,t) = \phi_t(x,X(t),t),t) + D_y\phi(x,X(t),t),t) \cdot \frac{\partial X}{\partial s}(t) = \phi_t(x,Y,t) + D_y\phi(x,Y,t) \cdot b(Y,t)$ and $D_x\varphi(x,t) = D_x\phi(x,Y,t)$, we get

$$-\phi_t(x,Y,t) - D_y\phi(x,Y,t) \cdot b(Y,t) + H(D_x\phi(x,Y,t),x,Y,b(Y,t)) \leq 0.$$

This proves that $\tilde{V}$ is a viscosity subsolution. Similarly we prove that it is also a supersolution. For $t = T$ we have $X(\cdot,T) \equiv Y$ thus $\tilde{V}(x,Y,T) = u(x,T,T) = \psi(x,Y)$.

7.2 MFG’s with correlations

In this section we discuss the Master equation for extended MFG’s and its application in modeling problems which involve correlations. Let $(\Omega, F, P)$ be a probability space, where $\Omega$ is an arbitrary set, $F$ is a $\sigma$-algebra on $\Omega$ and $P$ is a probability measure. Let $W: [0,T] \times \Omega \to \mathbb{R}^d$ be a standard d-dimensional Brownian motion, with associated filtration $\{F_t\}_{t \geq 0}$. We consider a model with
correlations where a generic player and the whole population of players are subjected to the same Brownian motion:

A generic player who starts at a point \( x \) at time \( t \) is controlled by SDE

\[
dx_t = \nu_t ds + \sigma dW_s, \quad x(t) = x,
\]

where \( \nu_t \) is a progressively measurable control, and the dynamics of the population of the players is given by the SDE:

\[
dY(s) = b(Y(s), s) ds + \sigma dW_s, \quad Y(t) = Y,
\]

where \( Y \in L^2(\Omega) \) describes the initial distribution of the players. We consider the generalized value function

\[
V(x, Y, t) = \inf_{\nu} E \left[ \int_t^{T} L(\nu(s), x(s), Y(s)) ds + \psi(x(T), Y(T)) \right | F_t].
\]

(32)

Then, at least heuristically, we have the dynamic programming principle:

\[
V(x, Y, t) = \inf_{\nu} E \left[ \int_t^{t+h} L(\nu(s), x(s), Y(s)) ds + V(x(t+h), Y(t+h), t+h) | F_t] \right .
\]

Thus

\[
V(x, Y, t) \leq E \left[ \int_t^{t+h} L(\nu(s), x(s), Y(s)) ds + V(x(t+h), Y(t+h), t+h) | F_t] \right .
\]

and we have an equality for an optimal control \( \nu = \nu^* \). Formally we can use Itô’s formula:

\[
V(x(t+h), Y(t+h), t+h) - V(x, Y, t) = \int_t^{t+h} [V_t + D_x V \cdot v + D_y V \cdot b + \frac{\sigma^2}{2} (\Delta_x V + 2 \partial_x \delta c V + \delta_x^2 V)] ds + \int_t^{t+h} \sigma D_x V dW_s + \int_t^{t+h} \sigma D_y V dW_s,
\]

where \( \delta c V \) is the derivative of \( V \) with respect to constants in the direction of \( Y \), that is

\[
\delta c V(x, Y, t) \cdot a = \lim_{\varepsilon \to 0} \frac{V(x, Y + \varepsilon a, t) - V(x, Y, t)}{\varepsilon}, \quad \text{for any } a \in \mathbb{R}^d.
\]

Thus

\[
E[V(x(t+h), Y(t+h), t+h) - V(x, Y, t) | F_t] = E[\int_t^{t+h} [V_t + D_x V \cdot v + D_y V \cdot b + \frac{\sigma^2}{2} (\Delta_x V + 2 \partial_x \delta c V + \delta_x^2 V)] ds | F_t] .
\]

(33)

The equation (33) then yields

\[
0 \leq E[\int_t^{t+h} (V_t + L(\nu(s), x(s), Y(s)) + D_x V \cdot v + D_y V \cdot b + \frac{\sigma^2}{2} (\Delta_x V + 2 \partial_x \delta c V + \delta_x^2 V)) ds | F_t].
\]

If we divide by \( h \) and send \( h \to 0 \), we get

\[
0 \leq V_t + L(v, x, Y) + D_x V \cdot v + D_y V \cdot b + \frac{\sigma^2}{2} (\Delta_x V + 2 \partial_x \delta c V + \delta_x^2 V),
\]

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and we have an equality for optimal control, thus
\[
0 = -\sup_{v} \{-L(v, x, Y) - D_x V \cdot v\} + V_t + D_y V \cdot b + \frac{\sigma^2}{2}(\Delta_x V + 2\partial_x \partial_y V + \delta^2 V),
\]
then
\[
- V_t - D_y V \cdot b - \frac{\sigma^2}{2}(\Delta_x V + 2\partial_x \partial_y V + \delta^2 V) + H(D_x V, x, Y) = 0.
\]
(34)
The optimal control then is given in feedback form:
\[
v^* = -D_p H(D_x V(x, Y, t), x, Y).
\]

Since we suppose that the players act rationally, they should follow the optimal trajectories. Then the whole population evolves along the vector \(-D_p H(D_x V(x, Y, t), x, Y)\). Thus this suggests us to take \(b(Y, t) = -D_p H(D_x V(Y, Y, t), Y, Y)\). Hence we arrive at the equation
\[
- V_t - D_y V \cdot b - \frac{\sigma^2}{2}(\Delta_x V(x, Y, t) + \partial_x \partial_y V(x, Y, t) + \delta^2 V(x, Y, t)) + H(D_x V(x, Y, t), x, Y) = 0,
\]
(35)
with terminal condition \(V(x, Y, T) = \psi(x, Y)\).

1. \(\psi\) is bounded and is Lipschitz in \(x\):
\[
|\psi(x_1, X) - \psi(x_2, X)| \leq C|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^d.
\]

2. There exist constants \(c_0, c_1 > 0\) such that
\[
L(v, x, X) \geq -c_0,
\]
and
\[
L(0, x, X) \leq c_1
\]
for all \(x, v \in \mathbb{R}^d, X \in L^q(\Omega; \mathbb{R}^d)\).

3. \(L\) is twice differentiable in \(x, v\) and we have the following bounds
\[
|D_x L(v, x, X)|, |D_{xx}^2 L(v, x, X)|, |D_{v}^2 L(v, x, X)|, |D_{vx}^2 L(v, x, X)| \leq CL(v, x, X) + C.
\]
for all \(x, v \in \mathbb{R}^d, X \in L^q(\Omega; \mathbb{R}^d)\).

**Proposition 5.** Assume that (5-7) hold. Then the function \(V\) defined in (32) for a fixed vector field \(b\) is finite, bounded, Lipschitz and semiconvex in \(x\):

1. There exists a constant \(C\) such that
\[
|V(x, Y, t)| \leq C, \quad \forall t \in [0, T], x, h \in \mathbb{R}^d, Y \in L^q(\Omega; \mathbb{R}^d).
\]

2. There exists a constant \(C\) such that
\[
|V(x + h, Y, t) - V(x, Y, t)| \leq C|h|, \quad \forall t \in [0, T], x, h \in \mathbb{R}^d, Y \in L^q(\Omega; \mathbb{R}^d).
\]

3. For any \(t, t < T\) there exists a constant \(C\) such that
\[
V(x + h, Y, t) + V(x - h, Y, t) - 2V(x, Y, t) \leq C|h|^2, \quad \forall x, h \in \mathbb{R}^d, Y \in L^q(\Omega; \mathbb{R}^d).
\]

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Proof.

• **Bounded**
  From the bounds from below on $L$ we have
  \[
  E \int_t^T L(v(s), x(s), y(s)) ds + \psi(x(T), y(T)) \geq -c_0(T - t) - \|\psi\|_\infty
  \]
  thus $V(x, y, t) \geq -c_0(T - t) - \|\psi\|_\infty$. On the other hand from the upper bound on $L$ and \[32\] taking the control $v = 0$:
  \[
  dx = dW_s, \ x(t) = x
  \]
  we obtain
  \[
  V(x, y, t) \leq E \int_t^T L(0, x(s), y(s)) ds + \psi(x(T), y(T)) \geq c_1(T - t) + c_1 \|\psi\|_\infty
  \]
  This proves that $V$ is bounded and thus also finite.

• **Lipschitz**
  Let $x, h \in \mathbb{R}^d$ and $v^\varepsilon$ be an $\varepsilon$-suboptimal control in \[27\]:
  \[
  V(x, y, t) \geq E \int_t^T L(v^\varepsilon(s), x^\varepsilon(s), y(s), \dot{y}(s)) ds + \psi(x^\varepsilon(T), y(T)) - \varepsilon,
  \]
  where
  \[
  dx^\varepsilon(s) = v^\varepsilon(s) ds + dW_s, \ x^\varepsilon(t) = x.
  \]
  We have
  \[
  V(x + h, y, t) \leq E \int_t^T L(v^\varepsilon(s), x^\varepsilon(s) + h, y(s), \dot{y}(s)) ds + \psi(x^\varepsilon(T) + h, y(T)),
  \]
  thus
  \[
  V(x + h, y, t) - V(x, y, t) \leq E \int_t^T L(v^\varepsilon(s), x^\varepsilon(s) + h, y(s), \dot{y}(s)) - L(v^\varepsilon(s), x^\varepsilon(s), y(s), \dot{y}(s)) ds
  + \psi(x^\varepsilon(T) + h, y(T)) - \psi(x^\varepsilon(T), y(T)) + \varepsilon, \forall \varepsilon > 0.
  \]
  Let
  \[
  f(\tau) = E \int_t^T L(v^\varepsilon(s), x^\varepsilon(s) + \tau h, y(s), \dot{y}(s)) ds.
  \]
  We have
  \[
  f(0) \leq V(x, y, t) - \psi(x^\varepsilon(T), y(T)) + \varepsilon \leq C,
  \]
  \[
  f'(\tau) = E \int_t^T D_x L(v^\varepsilon, x^\varepsilon + \tau h, y, \dot{y}) \cdot h \leq E \int_t^T (CL(v^\varepsilon, x^\varepsilon + \tau h, y, \dot{y}) + C)|h| \leq (Cf(\tau) + C)|h|.
  \]
  Hence by Gronwall inequality $f(\tau) \leq C$ and $f'(\tau) \leq C|h|$. Therefore
  \[
  V(x + h, y, t) - V(x, y, t) \leq f(1) - f(0) + \psi(x^\varepsilon(T) + h, y(T)) - \psi(x^\varepsilon(T), y(T)) + \varepsilon \leq C|h| + \varepsilon,
  \]
  for all $x, h \in \mathbb{R}^d, \varepsilon > 0$, which implies $|V(x + h, y, t) - V(x, y, t)| \leq C|h|, \forall x, h \in \mathbb{R}^d$. 

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• **Semiconcavity** As above let \( x, h \in \mathbb{R}^d \) and \( v^\varepsilon \) be an \( \varepsilon \)-suboptimal control in \((27)\)

\[
V(x, Y, t) \geq \mathbb{E} \int_t^T L(v^\varepsilon(s), x^\varepsilon(s), Y(s)) ds + \psi(x^\varepsilon(T), Y(T)) - \varepsilon,
\]

let \( h(s) = h \frac{T-s}{T-t} \), so that \( h(T) = 0 \) and \( h(t) = h \), and consider the function

\[
g(\tau) = \mathbb{E} \int_t^T L(v^\varepsilon(s) + \tau h(s), x^\varepsilon(s) + \tau h(s), Y(s), \dot{Y}(s)) ds.
\]

Using similar arguments to ones above for function \( f \) we get \( g(\tau) \leq C \). Then

\[
g''(\tau) \leq \mathbb{E} \int_t^T (D^2_{vv} L(v^\varepsilon + \tau h, x^\varepsilon + \tau h, Y, \dot{Y})||h||^2 + D^2_{xx} L(v^\varepsilon + \tau h, x^\varepsilon + \tau h, Y, \dot{Y})||\dot{h}||^2 + D^2_{xy} L(v^\varepsilon + \tau h, x^\varepsilon + \tau h, Y, \dot{Y})||h|||\dot{h}| ds \leq \mathbb{E} \int_t^T (C L(v^\varepsilon + \tau h, x^\varepsilon + \tau h, Y, \dot{Y}) + C)||h||^2 + ||\dot{h}||^2 + (Cg(\tau) + C)||h||^2 \leq (C + \frac{C}{T-t})||h||^2,
\]

Since we have

\[
V(x \pm h, Y, t) \leq \mathbb{E} \int_t^T L(v^\varepsilon(s) + h(s), x^\varepsilon(s) + h(s), Y(s), \dot{Y}(s)) ds + \psi(x^\varepsilon(T), Y(T)),
\]

we conclude

\[
V(x+h, t)+V(x-h, t)-2V(x, Y, t) \leq g(1)+g(-1)-2g(0)+\varepsilon \leq 2 \max_{[-1,1]} g'' + \varepsilon \leq (C + \frac{C}{T-t})||h||^2 + \varepsilon.
\]

\[ \blacksquare \]

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