CONVEX RAMSEY MATRICES AND NON-AMENABILITY OF AUTOMORPHISM GROUPS OF GENERIC STRUCTURES

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Abstract. In this paper we prove that the automorphism groups of certain countable generic structures are not amenable. We first prove the existence of particular matrices that do not satisfy the convex Ramsey condition. Then for a pair of elements in a smooth class, we introduce the property of forming a “free-pseudoplane” in the generic structure. The existence of such particular matrices and the correspondence in [6] allow us to prove the non-amenability of the automorphism group of a generic structure obtained from a smooth class with a pair that forms a free-pseudoplane. As an application we show that the automorphism group of an ab-initio generic structure that is constructed using a pre-dimension function with irrational coefficients is not amenable.

1. Introduction

The study of amenable groups originated in the work of von Neumann in his analysis of the Banach-Tarski paradox. Since then, amenability, non-amenability, and paradoxicality have been studied for various groups appearing in different parts of mathematics. The definition of an amenable group can be considered for any topological group, although the original definition was first phrased only for locally compact Hausdorff groups. Let $G$ be a topological group. A $G$-flow is a continuous action of $G$ on a compact Hausdorff space. A group $G$ is amenable if every $G$-flow admits an invariant Borel probability measure. Well-known examples of amenable groups are finite groups, solvable groups, and locally compact abelian groups.

The study of amenability of topological groups benefit from various viewpoints that range from analytic to combinatorial. The groups that we are considering in this paper are the automorphism groups of countable structures. The automorphism group of a countable first-order structure equipped with the point-wise convergence topology is a Polish group, which is also a closed subgroup of the symmetric group of its underlying set. Kechris, Pestov, and Todorcevic in [8] established a very general correspondence which equates a stronger form of amenability, called extreme amenability, of the automorphism group of an ordered Fraïssé-limit structure with the Ramsey property of the class of its finite substructures. This is particularly interesting because in a suitable language any closed subgroup of the symmetric group of a countable set can be seen as the automorphism group of a Fraïssé-limit structure.

In the same spirit Moore in [11] showed a correspondence between the automorphism groups of Fraïssé-limit structures and an another structural Ramsey property, called convex Ramsey property, which englobes Følner’s existing treatment in the analytic approach. In this paper, we consider automorphism groups of generic structures. A generic structure, similar to a Fraïssé-limit structure, is constructed from a countable class of finite structures, called a smooth class, with an amalgamation property. In [6] following ideas of [8, 11] it is

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shown that the amenability and extreme amenability of the automorphism group of a generic structure corresponds to the structural Ramsey type properties of its smooth class.

It is shown in Theorem 32 in [6] that the automorphism group of a generic structure is amenable if and only if the automorphism group has the convex Ramsey property with respect to the smooth class (see Definition 4). Then the correspondence is used to show that if the smooth class of a generic structure contains a certain pair of finite closed substructures, called a tree-pair (Definition 39 in [6]), then the automorphism group of the generic structure is not amenable (see Theorem 40 in [6]). Existence of a tree-pair implies that there is an open subgroup of the automorphism group of the generic structure that acts on a tree which is a substructure of the generic structure.

The class of generic structures that are mainly considered are those that are obtained from pre-dimension functions $\delta_\alpha$ where $\alpha \in (0, 1)$ (see Subsection 3.1). They are originated in the seminal work of Hrushovski in [7] where he constructs a strongly minimal structure with a non-locally modular geometry that does not interpret an infinite group. In Theorem 40 in [6] it is shown that the generic structures that are obtained from smooth classes of finite structures with pre-dimension functions $\delta_\alpha$ where $\alpha$ is a rational number, contain tree-pairs and hence the automorphism groups of their generic structures are not amenable. However, for the generic structures that are obtained from pre-dimension functions with irrational $\alpha$’s the statement of Theorem 40 in [6] do not hold. These generic structures are of a particular interest since by [2] their theory is the zero-one law theory of graphs with the edge probability $n^{-\alpha}$ (see [10]).

In this paper, by exhibiting a combinatorial (or geometrical) criterion for a pair of elements in a smooth class, we show that the automorphism group of certain generic structures are not amenable. We first prove in Section 2 using probabilistic methods, the following theorem that guarantees the existence of certain matrices.

**Theorem.** For every $k \geq 1$, there is an $n > k$ and $n \times n$-matrix $X$ whose entries are 0 or 1 such that $X$ satisfies the $k$-configuration exhibiting condition but does not satisfy the convex Ramsey property.

We then generalize Theorem 40 in [6] and introduce another sufficient condition for a pair of elements in a smooth class, called forming a free-pseudoplane (see Definition 8) in the generic structure. The existence of such a pair implies that the automorphism group of the generic structure is not amenable. More precisely, in Section 3 we prove the following theorem.

**Theorem.** Suppose $\mathcal{M}$ is the $(\mathcal{C}, \leq)$-generic structure of a smooth class $(\mathcal{C}, \leq)$ with AP (amalgamation property). Suppose for $n \in \mathbb{N}$ there is $A, B \in \mathcal{C}$ such that $\left(\begin{array}{c} B \\ A \end{array}\right) = n$ and $(A; B)$ is a free 2-pseudoplane and moreover, assume there is an $n \times m$-matrix $X$ such that $X$ satisfies the 2-configuration exhibiting condition but does not satisfy the convex Ramsey property. Then $\text{Aut}(\mathcal{M})$ is not amenable.

Then finally using the above theorem, in Subsection 3.1 we prove the following.

**Corollary.** The automorphism group of generic structures that are obtained from pre-dimension functions $\delta_\alpha$, where $\alpha$ is irrational are not amenable.

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1 David M. Evans in an email correspondence in 2015 informed the second author that, using a different method, the automorphism groups of generic structures that are obtained from pre-dimension functions with irrational coefficients and the $\omega$-categorical generic structures are not amenable.
1.1. Setting. As we have mentioned in the introduction, there is a natural topology on the automorphism group of a countable $\mathcal{L}$-structure $M$; that is the pointwise convergence topology. This topology turns $\text{Aut}(M)$ into a topological group and indeed a Polish group (see [4]). Our focus in this paper is the automorphism groups of generic structures, a specific kind of countable structures that are constructed from a class a finite structures with an amalgamation property.

1.1.1. Generic structures.

Definition 1. Let $\mathcal{L}$ be a countable relational language and $C$ be a class of $\mathcal{L}$-structures which is closed under isomorphism and substructure. Assume $\emptyset \in C$. Let $\leq$ be a reflexive and transitive relation on elements $A, B \in C$ where $A \subseteq B$ and moreover, invariant under $\mathcal{L}$-embeddings that has the following property

- If $A, A_1, A_2 \in C$ and $A_1, A_2 \subseteq A$, then $A_1 \leq A$ implies $A_1 \cap A_2 \leq A_2$.

The class $C$ together with the relation $\leq$ is called a smooth class. For $A, B \in C$ if $A \leq B$, then we say $A$ is $\leq$-closed substructure of $B$, or simply $A$ is a $\leq$-closed in $B$. Moreover, if $N$ is an infinite $\mathcal{L}$-structure such that $A \subseteq N$, we write $A \leq N$ whenever $A \leq B$ for every finite substructure $B$ of $N$ that contains $A$. We say an embedding $\Gamma$ of $A$ into $N$ is $\leq$-embedding if $\Gamma[A] \leq N$.

Notation 1. Suppose $A, B, C$ are $\mathcal{L}$-structures with $A, B \subseteq C$. We write $AB$ for the $\mathcal{L}$-substructure of $C$ with domain $A \cup B$. For an $\mathcal{L}$-structure $N$, write $\text{Age}(N)$ for the set of all finite substructures of $N$, up to isomorphism. Write $\overline{\mathcal{C}}$ for the class of all $\mathcal{L}$-structures $N$ such that $\text{Age}(N) \subseteq C$. Suppose $A \in C$ and let $N \in \overline{\mathcal{C}}$. Write $\langle N_A \rangle$ for the set of all $\leq$-embeddings of $A$ into $N$ and write $\langle N_A \rangle$ for the set of all finitely supported probability measures on $\langle N_A \rangle$.

Suppose $X \leq Y \leq Z$ and let $p \in \langle Z_Y \rangle$. By $\langle p_X \rangle$ we mean the set

$$\left\{ q \in \langle Z_X \rangle : \exists \tau \in \langle Y_X \rangle \forall \Gamma \in \langle Y_X \rangle \forall \Lambda \in \langle Z_Y \rangle, q(\Lambda \circ \Gamma) = p(\Lambda) \cdot r(\Gamma) \right\}.$$ 

Definition 2. Let $(C, \leq)$ be a smooth class.

1. We say $(C, \leq)$ has the joint-embedding property (JEP) if for every two elements $A, B \in C$ there is $C \in C$ such that $A, B \leq C$.
2. Suppose $A, B$ and $C$ are elements of $C$ such that $A \leq B, C$. The free-amalgam of $B$ and $C$ over $A$ is a structure consisting of the disjoint union of $B$ and $C$ over $A$ whose only $\mathcal{L}$-relations are those from $B$ and $C$, denoted by $B \otimes_A C$.
3. We say $(C, \leq)$ has the $\leq$-amalgamation property (AP) if for every $A, B$ and $C$ elements of $C$ with $A \leq B, C$, there is $D \in C$ such that $B \leq D$ and $C \leq D$.
4. We say $(C, \leq)$ has the free-amalgamation property if for every $A, B$ and $C$ elements of $C$ with $A \leq B, C$, then $B \otimes_A C \in C$ and $B, C \leq B \otimes_A C$.

Fact 1. (See [13]) Let $(C, \leq)$ be a smooth class and suppose $A \in C$ and $N \in \overline{\mathcal{C}}$ are $\mathcal{L}$-structures such that $A \subseteq N$. Then, there is a unique smallest $\leq$-closed set that contains $A$ in $N$. It is called $\leq$-closure of $A$ in $N$, denoted by $\text{cl}_N(A)$.

Proposition 1. Suppose $(C, \leq)$ is a smooth class with AP. Then there is a unique countable structure $\mathcal{M}$, up to isomorphism, satisfying:

1. $\text{Age}(\mathcal{M}) = C$;
2. $\mathcal{M} = \bigcup_{i \in \omega} A_i$ where $(A_i : i \in \omega)$ is a chain of $\leq$-closed finite sets;
If \( A \leq M \) and \( A \leq B \in \mathcal{C} \), then there is an embedding \( \Lambda : B \rightarrow M \) with \( \Lambda \upharpoonright A = id_A \) and \( \Lambda [B] \leq M \).

Proof. See [13]. \( \square \)

Definition 3. The structure \( M \), that is obtained in the above proposition, is called the Fraïssé-Hrushovski \((\mathcal{C}, \leq)\)-generic structure or simply the \((\mathcal{C}, \leq)\)-generic structure.

In [6] it is shown that the amenability and extreme amenability of the automorphism group of a generic structure corresponds to structural Ramsey properties of its smooth class.

Definition 4. Suppose \( M \) is the \((\mathcal{C}, \leq)\)-generic structure of a smooth class \((\mathcal{C}, \leq)\). We say \( \text{Aut}(M) \) has the convex \( \leq \)-Ramsey property with respect to \((\mathcal{C}, \leq)\) if for every \( A, B \in \mathcal{C} \) with \( A \leq B \) and every 2-coloring function \( f : \binom{M}{A} \rightarrow \{0, 1\} \), there is \( p \in \binom{M}{B} \) such that \( |f(p_1) - f(p_2)| \leq \frac{1}{2} \) for every \( q_1, q_2 \in \binom{M}{A} \).

The following correspondence has been proved in [6].

Theorem 1. Suppose \( M \) is the \((\mathcal{C}, \leq)\)-generic structure of a smooth class \((\mathcal{C}, \leq)\). Then \( \text{Aut}(M) \) has the convex \( \leq \)-Ramsey property with respect to \((\mathcal{C}, \leq)\) if and only if \( \text{Aut}(M) \) is amenable.

Later in [6] it has been shown that the convex \( \leq \)-Ramsey property with respect to \((\mathcal{C}, \leq)\) for \( \text{Aut}(M) \) can be reformulated in the following manner: Namely the coloring matrices of all 2-coloring functions of \( M \) have the convex Ramsey property (see Section 3 in [6]) and here we work with this latter formulation.

1.1.2. Convex Ramsey matrices. Before stating the definition of the convex Ramsey property for a matrix, we need the following preliminary definitions.

Definition 5. (1) An \((m \times 1)\)-matrix \( W \) is called a Dirac-weight matrix if there is exactly one entry in \( W \) with value 1 and exactly one entry with value \(-1\) while all the other entries of \( W \) are 0.

(2) A \( 1 \times t \)-matrix \( P = (p_1 \ldots p_t)_{1 \times t} \) is called a probability matrix if \( p_i \geq 0 \) for every \( 1 \leq i \leq t \), and \( \sum_{1 \leq i \leq t} p_i = 1 \).

Here is the definition of a convex Ramsey matrix.

\[
\begin{bmatrix}
\tilde{y}_1 \\
\vdots \\
\tilde{y}_n
\end{bmatrix}
\]

Definition 6. (Definition 27. in [6]) Let \( Y = \begin{bmatrix}
\tilde{y}_1 \\
\vdots \\
\tilde{y}_n
\end{bmatrix} \) be an \((n \times m)\)-matrix whose entries are 0 or 1. Moreover, assume that no two rows of \( Y \) are the same. We say \( Y \) satisfies the convex Ramsey condition if there is a \( 1 \times t \)-probability matrix \( P \) such that for every \( m \times 1 \) Dirac-weight matrix \( W \) we have

\[
P \times Y \times W \leq \frac{1}{2}.
\]

It is easy to give examples of matrices that satisfy the convex Ramsey condition; for example a matrix with a constant row of 0 or 1 satisfies the convex Ramsey condition. A matrix with one column of constant 0 and one column of constant 1 does not satisfy the convex Ramsey condition. In order to show that the automorphism group of a generic structure is amenable we need to show that every 2-coloring matrix satisfies the convex Ramsey property.
Conversely, if we show there is a coloring matrix of the generic structure which does not satisfy the convex Ramsey condition then the automorphism group is not amenable.

2. Convex Ramsey and $k$-configuration exhibiting matrices

For a matrix to be a coloring matrix of a generic structure some extra conditions are needed to be satisfied. The following condition is inspired by Lemma 28 in [6].

**Definition 7.** Suppose $k \geq 1$ is an integer. An $n \times m$-matrix $C = \begin{bmatrix} \bar{c}_1 \\ \vdots \\ \bar{c}_n \end{bmatrix}$ whose entries are 0 or 1 and $m \geq k$. Then we say $C$ satisfies the $k$-configuration exhibition condition if for every $\sigma : \{1, \ldots, k\} \to \{0, 1\}$ and every $l_1, \ldots, l_k$ where $1 \leq l_1 < \cdots < l_k \leq m$, there is $1 \leq j \leq n$ such that $c_{jl_1} = \sigma(i)$ for $1 \leq i \leq k$.

With this new terminology, that the following matrix satisfies the 1-configuration exhibition condition but does not satisfy the convex Ramsey condition

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}_{6 \times 6}
$$

The matrix above was introduced in [6] which enabled the authors to conclude that the automorphism groups of ab-initio Hrushovski generic structures that are obtained from pre-dimension functions with rational coefficients are not amenable (Theorem 40 in [6]).

Definition 7 helps us to extend the negative result about the amenability of automorphism groups of generic structures to those that are obtained from pre-dimensions with irrational coefficients.

Here, we prove the following

**Theorem 2.** Suppose $k \geq 1$ is an integer. Then there is an $n \times n$-matrix $X$ whose entries are 0 or 1 and $n > k$ such that $X$ satisfies the $k$-configuration exhibiting condition but does not satisfy the convex Ramsey property.

**Proof.** We use the probabilistic method in combinatorics (see in [1]) in order to show that such a matrix $X$ exists. As we have mentioned before for $k = 1$ and $n = 6$ existence of $X$ is already proved in [1] (and for $n \geq 6$ a similar idea can be modified). For the sake of simplicity we only present the argument for $k = 2$. At the end we mention what kind of change is needed for $k > 2$.

Suppose $Y$ is the following $n \times n$-matrix

$$
\begin{pmatrix}
1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0
\end{pmatrix}_{n \times n}
$$
where in each row the first $\frac{p}{2}$ entries are 1 and the rest of the entries are 0. Now we construct matrix $X$ from $Y$ in the following random procedure. Independently, with probability $p$ ($0 < p < 1$ and $q := 1 - p > p$) change the entries of $Y$: if 1 to 0 and if 0 to 1.

We first check the 2-configuration condition of $X$. Write $X^j$ for the column $j$ of matrix $X$ where $1 \leq j \leq n$ and define $X^{jk} := X^j X^k$ where $1 \leq j < k \leq n$. We now calculate the probability of the failure of the 2-configuration condition for $X$. First notice that the followings are true:

- $\Pr(00 \text{ occur in row } s \text{ of } X^{jk}) = \Pr(11 \text{ occur in row } s \text{ of } X^{lm}) = \Pr(01 \text{ occur in row } s \text{ of } X^{jl})$ where $1 \leq j, k, l \leq n$ and $1 \leq s \leq n$.

Then

\[
\Pr \left( \exists j,k \text{ s.t. 2-config. fails for } X^{jk} \right) \leq \sum_{j,k} \Pr(00 \text{ does not occur in any row of } X^{jk}) \leq 4 \cdot \left( \frac{n}{2} \right) \cdot (1 - p^2)^n \leq 2 \cdot n \cdot (n - 1) \cdot e^{-p^2n}.
\]

Let $p := n^{-\frac{1}{2} + \varepsilon}$ for a fixed and small $\varepsilon > 0$. Then $\Pr \left( \exists j,k \text{ s.t. 2-config. fails for } X^{jk} \right) \to 0$ as $n \to \infty$. Therefore, for large enough $n$ the random matrix $X$ satisfies the 2-configuration condition almost surely (i.e. with probability tending to 1 as $n \to \infty$).

Now we want to prove that the convex Ramsey property eventually fails for $X$ when $n \to \infty$. Suppose on the contrary that $X$ satisfies the convex Ramsey property. Let $P := (p_1 \cdots p_n)_{1 \times n}$ be a probability matrix such that

\[
P \times X \times W \leq \frac{1}{2},
\]

for every Dirac-weight $n \times 1$-matrix $W$. Fix the following notations: Define $I^j := \{i : x_{ij} \neq y_{ij}\}$ for $1 \leq j \leq n$. For $1 \leq j \leq \frac{n}{2}$ write $\bar{j} := j + \frac{n}{2}$ and define $W^j$ be the Dirac-weight $n \times 1$-matrix that $w^j_{j,1} = 1$ and $w^j_{\bar{j},1} = -1$. Then the convex Ramsey property for $X$ implies the following for $1 \leq j \leq \frac{n}{2}$:

\[
P \times X \times W^j = \sum_{i \in I^j \cup \bar{I}^j} p_i + \sum_{i \in I^j \cup \bar{I}^j} \xi_i \cdot p_i \leq \frac{1}{2};
\]

where $\xi_i = \{1, 0, -1\}$. Then $\sum_{i \in I^j \cup \bar{I}^j} \xi_i \cdot p_i \geq \frac{1}{4}$ as $\sum_i p_i = 1$. Hence

\[
\sum_{1 \leq j \leq \frac{n}{2}} \sum_{i \in I^j \cup \bar{I}^j} \xi_i \cdot p_i \geq \frac{n}{2} \cdot \frac{1}{4} = \frac{n}{8}.
\]

Note that $\sum_{1 \leq j \leq \frac{n}{2}} \sum_{i \in I^j \cup \bar{I}^j} \xi_i \cdot p_i = \sum_{1 \leq i \leq n} n_i \cdot \xi_i \cdot p_i$ where $n_i := \{j : x_{ij} \neq y_{ij}\}$.

Moreover, notice that $n_i$ has the binomial distribution $B(n, p)$, and $p \leq \frac{1}{16}$ for large enough $n$. Thus using the Chernoff bound one obtains the following

\[
\sum_{1 \leq i \leq n} \Pr \left( n_i \geq \frac{n}{16} \right) = n \cdot \Pr \left( n_1 \geq \frac{n}{16} \right) \leq n \cdot \exp \left( -n \cdot D \left( \frac{16}{n} \| p \right) \right) \leq n \cdot \exp \left( -n \cdot D \left( \frac{16}{n} \| \frac{1}{16} \right) \right) \leq n \cdot e^{-cn} \to 0 \text{ as } n \to \infty.
\]

for some constant $c$ where $D(\cdot \| y) = x \ln \frac{x}{y} + (1 - x) \ln \left( \frac{1-x}{y} \right)$ is the Kullback-Leibler divergence. Note that $n \cdot e^{-cn} \to 0$ when $n \to \infty$. Therefore, almost surely $n_i \leq \frac{n}{16}$ for all
which is a contradiction. Hence, the randomly generated matrix $X$ almost surely does not satisfy the convex Ramsey condition for large enough $n$.

For $k > 2$ a similar argument works and we only need to consider $p = n^{-\frac{1}{k} + \epsilon}$ this time. ~

3. Generic structures with embeddings that form free-pseudoplanes

Similar to Section 4 in [3], using Theorem 32 in [3] we present a sufficient condition for two embeddings to show that the automorphism groups of certain generic structures are not amenable. The condition is called forming a free-pseudoplane and automorphism groups of generic structures with such embeddings include especially the automorphism groups of generic structures that are obtained from pre-dimensions with irrational coefficients (see Subsection 3.1).

**Definition 8.** Let $(\mathcal{C}, \leq)$ be a smooth class with AP, and let $\mathcal{M}$ be the $(\mathcal{C}, \leq)$-generic structure. Suppose $X \subseteq \mathcal{M}$ and let $A, B \in \mathcal{C}$ with $A \leq B$. Suppose $k \geq 2$ is an integer such that $k < \left| \binom{B}{A} \right|$

1. We call embeddings $\Lambda \in \binom{X}{B}$ a $B$-line in $X$ and embedding $\Gamma \in \binom{X}{A}$ an $A$-point in $X$.
2. We say two $B$-lines $\Lambda_1$ and $\Lambda_2$ in $X$ are connected via a path in $X$ if there are $d \geq 1$ and $B$-lines $\Lambda_{i_0}, \ldots, \Lambda_{i_d}$ in $X$ such that $\Lambda_{i_0} = \Lambda_1$, $\Lambda_{i_d} = \Lambda_2$, and $\Lambda_{i_j} (B) \cap \Lambda_{i_{j+1}} (B)$ contains at least one $A$-point (in $X$) for each $0 \leq j < d$. We say $\Lambda_1$ and $\Lambda_2$ have distance $d$ if $d + 1$ is the minimum number of embeddings needed to connect $\Lambda_1$ and $\Lambda_2$ via a path.
3. We say $\Lambda_1$ and $\Lambda_2$ lay on an $m$-cycle (for $m \geq 2$) of $B$-lines in $X$ if there are distinct $B$-lines $\Lambda_{i_0}, \ldots, \Lambda_{i_{m-1}}$ in $X$ where $\Lambda_{i_0} = \Lambda_1$, $\Lambda_{i_{m-1}} = \Lambda_2$ such that:
   a. $\Lambda_{i_j} (B) \cap \Lambda_{i_{m-1}} (B)$ and $\Lambda_{i_j} (B) \cap \Lambda_{i_{j+1}} (B)$ contain at least one $A$-point, for each $0 \leq j < m - 1$;
   b. The intersection of any two distinct elements of the set $\{ \Lambda_{i_0} (B) \cap \Lambda_{i_{m-1}} (B), \Lambda_{i_j} (B) \cap \Lambda_{i_{j+1}} (B) : 0 \leq j < m - 1 \}$ does not contain a common $A$-point.
4. We say $(A; B)$ forms a $k$-pseudoplane in $X$ if every two distinct $B$-lines in $X$ intersect in at most $(k - 1)$-many $A$-points.
5. Suppose $\Lambda$ is a $B$-line in $X$. Let

$$I_X (\Lambda) := \left\{ \Gamma (A) : \Gamma \in \binom{\mathcal{M}}{A}, \Gamma (A) \subseteq \Lambda (B), \exists \Lambda' \neq \Lambda, \Lambda' (B) \subseteq X, \Gamma (A) \subseteq \Lambda' (B) \right\},$$

and $I_X (\Lambda) := \left| I_X (\Lambda) \right|$ (i.e. $I_X (\Lambda)$ is the set of all $A$-points in $X$ that lay in strictly more than one $B$-point in $X$). Define $H_k^X (X) := \Lambda (B) \setminus \bigcup \mathcal{I}_X (\Lambda)$ when $I_X (\Lambda) \leq k$; otherwise let $H_k^X (X) = \emptyset$.
6. Suppose $X \subseteq \mathcal{M}$ is a finite substructure and let $(\Lambda_i : 0 \leq i < b)$ be an enumeration of all $B$-lines in $X$ where $b = \left| \binom{X}{B} \right|$. Let $X_0 := X$ and define inductively $X_{i+1} := X_i \setminus H_k^X (X_i)$ for $1 \leq i < b$. We say $(A; B)$ forms a free $k$-pseudoplane in $X$ if there is an enumeration $(\Lambda_i : i \in b)$ of all $B$-lines in $X$ such that $\binom{X_i}{A} = \emptyset$. We say $(A; B)$...
forms a free \( k \)-pseudoplane in an infinite subset \( X \subseteq \mathcal{M} \) if \((A; B)\) form a free \( k \)-pseudoplane in every finite subset of \( X \).

**Remark 1.**

(1) Using this new terminology a tree-pair (Definition 39 in [6]) is a free 2-pseudoplane. However, the converse is not true.

(2) Note that for any finite substructure \( X \subseteq \mathcal{M} \) and any enumeration \((\Lambda_i: 0 \leq i < b)\) of all \( B \)-lines in \( X \), when \( i \to \infty \) the set \( X_{i+1} := X_i \setminus H^A_{k^*}(X_i) \) eventually remains fixed where \( i^* \equiv i + b \) and \( 0 \leq i^* \leq b \). If \((A; B)\) forms a free \( k \)-pseudoplane in \( \mathcal{M} \), then for every enumeration that we choose there is \( i_0 \) such that \( \left( X_{i_0}^A \right) = \emptyset \).

Then we can prove the following which is similar to Theorem 40 in [6].

**Theorem 3.** Suppose \((C, \leq)\) is a smooth class with AP, and \( \mathcal{M} \) is the \((C, \leq)\)-generic structure. Suppose for \( n \in \mathbb{N} \) there is \( A, B \in \mathcal{C} \) such that \( \left( \begin{array}{c} B \\ A \end{array} \right) \mid = n \) and \((A; B)\) is a free 2-pseudoplane. Moreover assume there is an \( n \times m \)-matrix \( X \) such that \( X \) satisfies 2-configuration exhibiting condition but does not satisfy the convex Ramsey property. Then \( Aut(\mathcal{M}) \) is not amenable.

**Proof.** We present a coloring function \( f: \left( \begin{array}{c} \mathcal{M} \\ A \end{array} \right) \to \{0, 1\} \) such that the full-coloring matrix of \( f \) for copies of \( B \) in \( \mathcal{M} \) is the matrix \( X \). We first prove that one can assign a consistent coloring for every finite subset of \( \mathcal{M} \) using rows of \( X \).

Since \((A; B)\) forms a free 2-pseudoplane then for every finite substructure \( C \) of \( M \) there is an enumeration \((\Lambda_i: 0 \leq i < c)\) of embeddings of \( B \)-lines in \( C \) such that \( C_0 := C \) and \( \left( \begin{array}{c} C_i \\ A \end{array} \right) = \emptyset \) where \( c = \left( \begin{array}{c} C_b \\ B \end{array} \right) \) and \( C_{i+1} := C_i \setminus H^A_{k^*}(C_i) \) is inductively defined. We consistently color \( B \)-lines using rows of \( X \), inductively. Start with \( \Lambda_{c-1} \). Pick a row in the matrix \( X \) and assign a coloring for \( A \)-points of \( \Lambda_{c-1}(B) \) according to the row. Now suppose a consistent coloring for \( \Lambda_i(B) \) is already chosen according to the rows of the matrix \( X \). Then \( \Lambda_{i-1}(B) \) interests with \( A \)-lines of \( C_i \) in at most 2 many \( A \)-points. Since the matrix \( X \) satisfies the 2-configuration exhibiting condition then there is a row in \( X \) that the colorings of \( A \)-points agree with the coloring of \( A \)-points in the intersection. Hence, we can pick a coloring for \( \Lambda_{i-1} \) from rows of \( X \) in a consistent way and therefore one can use matrix \( X \) to color all \( B \)-points of \( C \).

The matrix \( X \) have only finitely many rows and by the above argument for every finite subset of \( \mathcal{M} \) there is a consistent coloring using \( X \). Then by Rado’s selection lemma in [9] there is a coloring \( f \) of \( A \)-points of \( \mathcal{M} \) that its coloring matrix of \( B \)-lines in \( \mathcal{M} \) is \( X \). Then Theorem 32 in [6] implies that \( Aut(\mathcal{M}) \) does not have the convex \( \leq \)-Ramsey property with respect to \((C, \leq)\) and hence not \( Aut(\mathcal{M}) \) is not amenable.

**Theorem 4.** Suppose \((C, \leq)\) is a smooth class with AP, and \( \mathcal{M} \) is the \((C, \leq)\)-generic structure. Assume for \( n \in \mathbb{N} \) there are \( A, B \in \mathcal{C} \) such that \( \left( \begin{array}{c} B \\ A \end{array} \right) \mid = n \) and \((A; B)\) is a free \( k \)-pseudoplane and there is an \( n \times m \)-matrix \( X \) such that \( X \) satisfies \( k \)-configuration exhibiting condition but does not satisfy the convex Ramsey property. Then \( Aut(\mathcal{M}) \) is not amenable.

### 3.1. Pre-dimensions with irrational coefficients

Let \( K \) be the class of all finite graphs and \( \alpha \in (0, 1) \setminus \mathbb{Q} \). Define \( \delta_\alpha : K \to \mathbb{R} \) as \( \delta_\alpha(A) = |A| - \alpha \cdot |\mathcal{R}(A)| \) where \( \mathcal{R}(A) \) is the set of \( \mathcal{R} \)-relations of \( A \) (set of all edges of \( A \)). For every \( A \subseteq B \in K \), define \( A \leq_\alpha B \)

The second author would like to thank Martin Ziegler for suggesting this shorter proof.
Lemma 1. if and only if $δ_α (C) − δ_α (A) ≥ 0$, for every $C$ with $A ⊆ C ⊆ B$. Finally, put $K_α := \{ A ∈ K : δ_α (A') ≥ 0, \text{ for every } A' ⊆ A \}$. The class $K_α$ contains the class of sparse graphs.

Notation 2. Suppose $A, B, C ∈ K_α$ and $A, B ⊆ C$. Then let $δ_α (A/B) := δ_α (AB) − δ_α (B)$.

Fact 2. (See [3]) The $(K_α, ≤_α)$ is a smooth class with the free-amalgamation property.

We call $(K_α, ≤_α)$ an ab-initio smooth class that is obtained from $δ_α$ and we write $M^α$ for the countable $(K_α, ≤_α)$-generic structures that is obtained from Theorem [3]. As we mentioned in the introduction, the theory $M^α$ is the zero-one law theory of graphs with the edge probability $n^{-α}$ (see [10]).

Here are some important properties of $δ_α$

Fact 3. (See [3]) Suppose $A, B, C ⊆ M^α$ are finite subsets. Then the followings hold

1. $δ_α (ABC) = δ_α (AB/C) + δ_α (C) = δ_α (A/BC) + δ_α (B/C) + δ_α (C)$.
2. $δ_α (AB/C) ≤ δ_α (A/C) + δ_α (B/C) − δ_α ((A ∩ B)/C)$.
3. Assume $A, B$ are disjoint and $−9^{M^α} (a, b)$ for every $a ∈ A$ and $b ∈ B$. Then $δ_α (AB/C) = δ_α (A/C) + δ_α (B/C)$.

Lemma 1. Suppose $(A; B)$ forms a 2-pseudoplane in $M^α$. If $δ_α (B/A) < \frac{δ_α (A)}{2}$, then $(A; B)$ forms a free 2-pseudoplane.

Proof. Put $ε := δ_α (B/A)$. Let $X$ be a finite subset of $M^α$. Let $(Λ_i : i ∈ I)$ be an enumeration. Let sets $X_i ⊆ X$ for $i ∈ N$ be defined similar to Remark [1](2). Let $i_0$ be the minimum number such that $X_{i_0} = X_{i_0 + 1}$. If $\left( \frac{X_{i_0}}{A} \right) = ∅$, then it means that $X$ is a free 2-pseudoplane. Otherwise, let $m$ be the number of $B$-lines in $X_{i_0}$. Then

$$δ_α (B) ≤ δ_α (X_{i_0}) ≤ m · δ_α (B) − \frac{3m}{2} · δ_α (A),$$

where $\frac{3m}{2}$ is the minimum number of recalculating $δ_α (A)$. Then

$$0 ≤ 2 (m - 1) : (δ_α (B/A)) − (m + 2) : δ_α (A)$$

$$\frac{(m + 2) : δ_α (A)}{2 (m - 1)} ≤ \frac{2 (m - 1) · ε}{ε}$$

It is clear that $\frac{(m + 2) : δ_α (A)}{2 (m - 1)}$ is decreasing when $m → ∞$. Hence $\frac{δ_α (A)}{2} ≤ ε$ which is a contraction with our assumption that $ε < \frac{δ_α (A)}{2}$. Therefore $\left( \frac{X_{i_0}}{A} \right) = ∅$ and $X$ is free 2-pseudoplane. □

Lemma 2. For $n ≥ 3$, there are $A, B ∈ K_α$ such that

1. $A ⊆ B$ and $\left( \frac{B}{A} \right) = n$;
2. $(A; B)$ is a free 2-pseudoplane.

Before proving Lemma [2], we prove some technical lemmas (but folklore) that are used in the proof.

Fact 4. For an irrational $α ∈ (0, 1)$, for every $N ∈ N$ there are infinitely many integers $r_i$ and $s_i$ such that $−\frac{1}{N} < r_i − α · s_i < 0$.

Write $K_n$ for the complete graph with $n$-vertices and $K_{n,m}$ for the complete $(n, m)$-bipartite graph.

Lemma 3. For every $N ∈ N$, there is $EF ∈ K_α$ such that
(1) \( E \cap F = \emptyset \) and \( -\frac{1}{m} < \delta_\alpha(E/F) < 0 \);
(2) Every \( \leq \)-closed \( K_3 \)-embedding in \( EF \) is a \( \leq \)-closed \( K_3 \)-embedding in \( F \).

Proof. Let \( m \) be an integer such that \( \frac{1}{m+1} \leq \alpha \leq \frac{1}{m} \) and using Fact \( \mathbb{H} \) choose \( r, s \in \mathbb{N} \) such that \( r > m^2 + \frac{m+1}{m} \) and \( -\frac{1}{\alpha} < r - \alpha \cdot s < 0 \). Then \( s < \frac{1}{\alpha} \cdot (r + \frac{1}{N}) \leq (m+1) \cdot (r + \frac{1}{N}) \). Let \( r_0 := r - m \) and \( r_1 := m \). Consider the complete bipartite graph \( K_{r_0,r_1} \). Then
\[
0 \leq r - \frac{r_0r_1}{m} \leq r - \alpha \cdot r_0r_1 = \delta_\alpha(K_{r_0,r_1}).
\]
Then it is easy also to see \( s - r_0r_1 = s - m(r - m) \leq (m+1) \cdot (r + \frac{1}{N}) - mr + m^2 = r + \frac{m+1}{N} + m^2 \). Now we introduce a graph \( EF \) that satisfies the properties of the lemma. Let \( E \) be a graph that \( E \cong K_{r_0,r_1} \), and take \( F \in \mathcal{K}_\alpha \) with \( 4 \leq |F| \leq s - r_0r_1 \) such that \( \delta(F') \geq \frac{1}{N} \) for every nonempty \( F' \subseteq F \) and moreover there are two pairs of vertices that are not connected (for example any graph without edges). We want to draw \( (s - r_0r_1) \)-many edges between \( E \) and \( F \) in such way that no \( K_3 \)-graphs (or 3-cycles) appears in \( EF \) apart from possibly those in \( F \). By our assumption \( F \) contains two pairs of vertices \((f_{k_1},f_{k_3})\) and \((f_{k_2},f_{k_4})\) that are not connected (i.e. \( \neg \Re^F(f_{k_1},f_{k_3}) \land \neg \Re^F(f_{k_2},f_{k_4}) \)). Note that \( E \) is a bipartite graph and let \( E_1 \) and \( E_2 \) be the bipartite partitions of \( E \) and let \( (e_i^1 : i < r_0) \) and \( (e_i^2 : i < r_1) \) be the enumeration of elements of \( E_1 \) and \( E_2 \); respectively. Now let
\[
\begin{align*}
(1) & \ EF = \bigwedge_{i < r_0} \Re(e_i^1,f_{k_1}) \land \bigwedge_{i < r_1} \Re(e_i^2,f_{k_2}); \\
(2) & \ EF = \bigwedge_{i < m_0} \Re(e_i^1,f_{k_3}) \land \bigwedge_{i < m_1} \Re(e_i^2,f_{k_4});
\end{align*}
\]
where \( m_0 + m_1 = s - r_0r_1 \) and \( m_0 \leq r_0 \) and \( m_1 \leq r_1 \). Note that valency of each vertex in \( E \) is at least \((m+1)\) and one can check that \( \delta_\alpha(E'/F) \geq \delta_\alpha(E/F) \) for every \( E' \subseteq E \). Here are more detailed calculations: suppose that \( m^* \leq m \) and \( r^* \leq r_0 \) where \( m^* + r^* = |E'| \). Then
\[
\begin{align*}
\delta_\alpha(E/E'F) & \leq (r_0 - r^*) + (m - m^*) - ((m+1) \cdot (r_0 - r^*) + (m - m^*) \cdot (r^* + 1)) \cdot \alpha \\
& \leq (r_0 - r^*) + (m - m^*) - ((m+1) \cdot (r_0 - r^*) + (m - m^*) \cdot (r^* + 1)) \cdot \frac{1}{m+1} \\
& \leq (m - m^*) \left( 1 - \frac{r^* + m^*}{m+1} \right)
\end{align*}
\]
When \( r^* > m \) then \( \delta_\alpha(E/E'F) < 0 \). Note that \( \delta_\alpha(E/F) = \delta_\alpha(E/E'F) + \delta_\alpha(E'/F) \) and therefore \( \delta_\alpha(E'/F) \geq \delta_\alpha(E/F) \). When \( r^* \leq m \) then
\[
\delta_\alpha(E'/F) = r^* + m^* - (r^* m^* + r^* + m^*) \cdot \alpha \\
\geq (m-1) \cdot (r^* m^*) \cdot \frac{m}{4m} \\
\geq (r^* m^*) \cdot (4(m-1) - (r^* m^*)) \\
\geq \frac{m}{0}
\]
The last two inequality holds because \( r^* + m^* \leq 2m \) and \( r^* m^* \leq \left( \frac{r^* + m^*}{2} \right)^2 \). Therefore, in order to check that \( EF \in \mathcal{K}_\alpha \), we only need to check \( \delta_\alpha(E/F') \geq 0 \) for every \( F' \subseteq f_{k_1} f_{k_2} f_{k_3} f_{k_4} \). By the properties of the pre-dimension \( \delta_\alpha(E/F') = \delta_\alpha(E/F') + \delta_\alpha(F') > -\frac{1}{N} + \frac{1}{N} \geq 0 \) and hence \( EF \in \mathcal{K}_\alpha \). Notice that all the \( K_3 \)-embeddings of \( EF \) have to be \( F \). \( \square \)

Proof of Lemma 2. Fix \( n \). We construct \( A, B \in \mathcal{K}_\alpha \) using the free-amalgamation property of \( \mathcal{K}_\alpha \). Let \( 2 \leq i \leq n \) and write \([n]^i := \{ u \subseteq \{0, \ldots, n-1\} : |u| = i \}\). Let \( A \) be the \( K_3 \)-complete graph; namely, \( A \) is a graph with three vertices \( a_i \) with \( 0 \leq i < 3 \) such that \( A \models \Re(a_0,a_1) \land \Re(a_1,a_2) \land \Re(a_2,a_0) \). Then \( \delta_\alpha(A) = 3 - 3 \cdot \alpha \geq 0 \) as \( \alpha \in (0, 1) \) and hence \( A \in \mathcal{K}_\alpha \). We construct \( B \in \mathcal{K}_\alpha \) such that \( B := \bigcup_{i \in [n]} \bigcup_{u \in [n]^i : 2 \leq |u| \leq n} \chi_u \) as a set, where \( \chi_u \)'s are isomorphic copies of \( A \) with the following properties:
(1) $\delta_\alpha(B) \geq \delta_\alpha(A_i)$ for $i \in n$;
(2) $\delta_\alpha(C) > \delta_\alpha(B)$ for every $C \subsetneq B$ that contains at least two copies of $A$, namely $\text{cl}(C) = B$;
(3) $\left(\frac{B}{A}\right) = n$.

It is clear if Conditions (1), (2) and (3) hold for $A$ and $B$, then $(A; B)$ is a 2-pseudoplane. For Condition (3) we need to make sure that there are no $K_3$-graphs in $B$ apart from $A_i$'s. In order to satisfy Condition (2) for every $2 \leq i \leq n$ and $u = \{u_0, \ldots, u_{i-1}\} \in [n]^i$ we construct $X_u$ such that $\delta_\alpha\left(X_u/\left(\bigcup_{j \in i} A_{u_j}\right)\right) < 0$. We now check the conditions for $X_u$ in order to get the properties that are needed. Let $\epsilon_u := -\delta_\alpha\left(X_u/\left(\bigcup_{j \in i} A_{u_j}\right)\right)$. Put $\epsilon_0 := \min_{2 \leq i \leq n} \left\{\epsilon_u : u \in [n]^i\right\}$ and $\epsilon_1 := \max_{2 \leq i \leq n} \left\{\epsilon_u : u \in [n]^i\right\}$ which are non-zero. Here are the calculations for finding sufficient conditions

$$n \cdot \delta_\alpha(A) - (2^n - n - 1) \cdot \epsilon_1 \leq \delta_\alpha(B) = n \cdot \delta_\alpha(A) - \sum_{2 \leq i \leq n} \sum_{u \in [n]^i} \epsilon_u \leq n \cdot \delta_\alpha(A) - (2^n - n - 1) \cdot \epsilon_0.$$ 

In order to have $(A; B)$ satisfying Condition (1) we demand

$$\frac{n \cdot \delta_\alpha(A) - (2^n - n - 1) \cdot \epsilon_1}{(n-1) \cdot \delta_\alpha(A)} \geq \frac{\delta_\alpha(A)}{(2^n - n - 1) \cdot \epsilon_1} \geq \frac{\epsilon_1}{\epsilon_1}$$

Suppose $2 \leq k < n$ and let $c \in [n]^k$. Then

$$k \cdot \delta_\alpha(A) - (2^k - k - 1) \cdot \epsilon_1 \leq \delta_\alpha\left(\bigcup_{i \in c} A_i \cup \bigcup_{2 \leq i \leq n} \bigcup_{u \subseteq c} X_u\right) = k \cdot \delta_\alpha(A) - \sum_{2 \leq i \leq n} \sum_{u \subseteq c} \epsilon_u.$$ 

We demand the following in order to obtain Condition (2)

$$k \cdot \delta_\alpha(A) - (2^k - k - 1) \cdot \epsilon_1 > \delta_\alpha(B).$$

Note that $k \cdot \delta_\alpha(A) - (2^k - k - 1) \cdot \epsilon_1$ is increasing as $k \to n - 1$. Hence, we only need to satisfy:

$$2 \cdot \delta_\alpha(A) - (2^2 - 2 - 1) \cdot \epsilon_1 > \delta_\alpha(B).$$

Then we demand

$$\frac{2 \cdot \delta_\alpha(A) - \epsilon_1}{\epsilon_0 - \epsilon_1} \geq \frac{n \cdot \delta_\alpha(A) - (2^n - n - 1) \cdot \epsilon_0}{(n-2) \cdot \delta_\alpha(A)} \geq \frac{\epsilon_0}{\frac{n-2}{2^n-n-2} \cdot \delta_\alpha(A) + \xi}$$

where $\xi = \frac{\epsilon_1 - n}{2^n - n - 2}$. Then all these conditions together demand

$$\xi + \frac{n-2}{2^n-n-2} \cdot \delta_\alpha(A) \leq \epsilon_0 \leq \epsilon_1 \leq \frac{n-1}{2^n-n-1} \cdot \delta_\alpha(A) \quad (*)$$

for every $u \in [n]^i$ where $2 \leq i \leq n$. One can check that when $n \geq 3$ the inequalities above are valid. We choose $\epsilon_u$'s in such a way that $\xi$ is ignorable

$$\frac{n-1}{2^n-n-1} \cdot \delta_\alpha(A) - \epsilon_u \leq \frac{n-1}{2^n-n-1} \cdot \delta_\alpha(A) - \epsilon_0 \leq \frac{1}{\xi}$$
for some $C \in \mathbb{N}$. Now we use Lemma 3 and Fact 4 we construct $X_u$ such that $\varepsilon_u = \delta_\alpha \left( X_u / \left( \bigcup_{j \in i} A_{u_j} \right) \right)$ satisfies (*). Using Fact 4 for every $u \in [n]^i$ choose $r_u, s_u$ such that $-\left( \frac{n-1}{2^n-n-1} \right) \cdot \delta_\alpha (A) < r_u - \alpha \cdot s_u \leq -\left( \frac{n-1}{2^n-n-1} \right) \cdot \delta_\alpha (A)$ and let $\varepsilon_u := - (r_u - \alpha \cdot s_u)$. Moreover, we choose $r_u$ and $s_u$’s such that $\xi$ is ignorable and the followings hold

1. $r_u - s_u \cdot \alpha = - \varepsilon_u < 0$;
2. $r_u - (s_u - r_u) \alpha = - \varepsilon_u + r_u \cdot \alpha \geq 0$.

Note that Condition (2) is satisfiable because $r_u$ can be chosen arbitrarily large. We construct $X_u$ every $u \in [n]^i$ such that $|X_u| = r_u$, $\delta_\alpha \left( X_u / \left( \bigcup_{j \in i} A_{u_j} \right) \right) = - \varepsilon_u$ and $X_u \cup \left( \bigcup_{j \in i} A_{u_j} \right)$ has only $i$-many $K_3$-subgraphs, namely only $A_{u_i}$’s. Using the same idea that is used in the proof of Lemma 3 one can find $K_3$-free graphs $E$ of arbitrary the required size in such that $\delta_\alpha \left( E / \left( \bigcup_{j \in i} A_{u_j} \right) \right) < 0$. By modifying the edges drawn between $E$ and $F = \bigcup_{j \in i} A_{u_j}$ in Lemma 3 we can choose $X_u \cong E$ to be a bipartite graph of size $r_u$ such that $X_u \in K_\alpha$ and $|\mathfrak{F}(X_u)| = s_u - r_u$. Note that $A_{u_i}$’s for $j \in i$ are disjoint. Note that $F$ satisfies the condition of Lemma 3. We modify the drawn edges between $E$ and $F$ is such a that for each $j \in i$ there is at least one edges between $X_u$ and $A_{u_j}$. It is easy draw the edges in such way that no $K_3$-graph is embeddable in $X_u \cup \left( \bigcup_{j \in i} A_{u_j} \right)$ apart from $\bigcup_{j \in i} A_{u_j}$. Hence $\left| \left( X_u \cup \left( \bigcup_{j \in i} A_{u_j} \right) \right) \right| = i$. Now let $B$ be the free-amalgamation of all these $X_u \cup \left( \bigcup_{j \in i} A_{u_j} \right)$ over $\bigcup_{j \in i} A_{u_j}$. Then $\left( B / \left( \bigcup_{j \in i} A_{u_j} \right) \right) = n$ and Conditions (1-3) hold for $B$ and $A$ hence they are 2-pseudoplane. We now want to show embeddings of $B$ and $A$ form free 2-pseudoplane. By Lemma we need to check whether $\delta_\alpha (B/A) = \delta_\alpha (B) - \delta_\alpha (A) < \frac{1}{2} \cdot \delta_\alpha (A)$ holds. As we have chosen

$$ \left( \frac{n-1}{2^n-n-1} - \frac{1}{C} \right) \cdot \delta_\alpha (A) \leq \varepsilon_u \leq \frac{n-1}{2^n-n-1} \cdot \delta_\alpha (A) $$

Then

$$ 2 \cdot \delta_\alpha (B) - 3 \cdot \delta_\alpha (A) < 2 \left( n \cdot \delta_\alpha (A) - (2^n-n-1) \cdot \left( \frac{n-1}{2^n-n-1} - \frac{1}{C} \right) \cdot \delta_\alpha (A) \right) - 3 \cdot \delta_\alpha (A) < 2 \delta_\alpha (A) - 3 \delta_\alpha (A) + 2 \frac{2^n-n-1}{C} \cdot \delta_\alpha (A) < 2 \cdot \frac{2^n-n-1}{C} \cdot \delta_\alpha (A) - \delta_\alpha (A) $$

If we choose $C > 2 \cdot (2^n-n-1)$ then we have $\delta_\alpha (B/A) < \frac{1}{2} \cdot \delta_\alpha (A)$ and therefore embeddings of $B$ and $A$ form a free 2-pseudoplane. \hfill \Box

Now using Theorem 3 we obtain the following.

**Corollary 1.** Aut $(M^n)$ is not amenable.

4. **Further remarks and questions**

In this paper we have shown the automorphism group of certain generic structures are not amenable. The following question is a natural question to ask

**Question 1.** Is there an example of a smooth class $(C, \leq)$ with HP and JEP such the notion of closedness notion of $\leq$ is strictly stronger that $\subseteq$, and the automorphism group of the $(C, \leq)$-generic structure is amenable?
Note that in the question above we asked $\subseteq$ to be strictly stronger than $\leq$ to insure the generic structure is not the usual Fraïssé-limit structure and the reason is that in Fraïssé-limit structures there is no pair that form a free-pseudoplane. It is interesting to find a border line that also grasps the geometric behavior of the algebraic closure in the generic and connects it with the amenability of the automorphism group.

It is also interesting to determine whether or not in the automorphism group of the generic structures that we proved the non amenability for one can find a copy of $F_2$ that acts freely on structure or a substructure of the generic model (see [5]). Another property to verify is Kazhdan Property (T) and its connection to amenability in these cases, although it is known there is no direct connection between amenability and Kazhdan Property (T) (see [12] for more details).

REFERENCES

1. N. Alon and J. H. Spencer, The probabilistic method, fourth ed., Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2016. MR 3524748
2. J. T. Baldwin and S. Shelah, Randomness and semigenericity, Trans. Amer. Math. Soc. 349 (1997), no. 4, 1359–1376. MR 1407480 (97j:03065)
3. J. T. Baldwin and N. Shi, Stable generic structures, Ann. Pure Appl. Logic 79 (1996), no. 1, 1–35. MR 1390325 (97c:03103)
4. P. J. Cameron, Oligomorphic permutation groups, London Mathematical Society Lecture Note Series, vol. 152, Cambridge University Press, Cambridge, 1990. MR 1066691 (92f:20002)
5. D. M. Evans and T. Tsankov, Free actions of free groups on countable structures and property (T), Fund. Math. 232 (2016), no. 1, 49–63. MR 3417738
6. Z. Ghadernezhad, H. Khalilian, and M. Pourmahdian, Automorphism groups of generic structures: Extreme amenability and amenability, ArXiv (August 2015).
7. E. Hrushovski, A new strongly minimal set, Ann. Pure Appl. Logic 62 (1993), no. 2, 147–166. Stability in model theory, III (Trento, 1991). MR 1226304 (94d:03064)
8. A. Kechris, V. Pestov, and S. Todorcevic, Fraïssé limits, ramsey theory, and topological dynamics of automorphism groups, GAFA (2005), no. 15, 106–189.
9. R. Rado, A selection lemma, J. Combinatorial Theory Ser. A 10 (1971), 176–177. MR 0270920
10. S. Shelah and J. Spencer, Zero-one laws for sparse random graphs, J. Amer. Math. Soc. 1 (1988), no. 1, 97–115. MR 924703
11. J. Tatch Moore, Amenability and ramsey theory, Fund. Math. 220 (2013), 263–280.
12. T. Tsankov, Unitary representations of oligomorphic groups, Geom. Funct. Anal. 22 (2012), no. 2, 528–555. MR 2929072
13. O. Wagner, Relational structures and dimensions, Automorphisms of first-order structures, Oxford Sci. Publ., Oxford Univ. Press, New York, 1994, pp. 153–180. MR 1325473

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