In this contribution we report on theoretical studies of $\eta$ nuclear quasi-bound states in few- and many-body systems performed recently by the Jerusalem-Prague Collaboration [1, 2, 3, 4, 5]. Underlying energy-dependent $\eta N$ interactions are derived from coupled-channel models that incorporate the $N^*(1535)$ resonance. The role of self-consistent treatment of the strong energy dependence of subthreshold $\eta N$ amplitudes is discussed. Quite large downward energy shift together with rapid decrease of the $\eta N$ amplitudes below threshold result in relatively small binding energies and widths of the calculated $\eta$ nuclear bound states. We argue that the subthreshold behavior of $\eta N$ scattering amplitudes is crucial to conclude whether $\eta$ nuclear states exist, in which nuclei the $\eta$ meson could be bound and if the corresponding widths are small enough to allow detection of these $\eta$ nuclear states in experiment.
1. Energy and model dependence of $\eta N$ scattering amplitudes

Calculations of $\eta$ nuclear quasi-bound states presented in this contribution are based on the $\eta N$ scattering amplitudes derived from coupled-channel models that incorporate the $N'(1535)$ resonance. The amplitudes near threshold are both attractive and strongly energy dependent, as illustrated in Fig. 1 for three selected meson-baryon interaction models, GW [6], CS [7], and GR [8]. Moreover, the $\eta N$ scattering amplitudes are highly model dependent; they differ considerably from each other below as well as above the $\eta N$ threshold (except common value Im$F_{\eta N} \approx 0.2 - 0.3$ fm at threshold). This suggests that the predictions for the $\eta$ nuclear states would be model dependent and that the strong energy dependence of the $\eta N$ scattering amplitudes has to be treated self-consistently.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Real (left panel) and imaginary (right panel) parts of the free $\eta N$ c.m. scattering amplitude $F_{\eta N}(\sqrt{s})$ as a function of energy in three meson–baryon interaction models: dashed, GW [6]; solid, CS [7]; dotted, GR [8]. The vertical line denotes the $\eta N$ threshold.}
\end{figure}

The crucial point is that in the nuclear medium the energy argument $\sqrt{s}$ is given by

$$\sqrt{s} = \sqrt{(\sqrt{s_{th}} - B_\eta - B_N)^2 - (\vec{p}_\eta + \vec{p}_N)^2} \leq \sqrt{s_{th}},$$

where $\sqrt{s_{th}} \equiv m_\eta + m_N$ and $B_\eta$ and $B_N$ are meson and nucleon binding energies, and the momentum dependent term generates additional substantial downward energy shift, since $(\vec{p}_\eta + \vec{p}_N)^2 \neq 0$ unlike the case of the two-body c.m. system. This has significant consequences for the calculated binding energies and widths as will be shown below.

2. The $\eta$ meson in few-body systems

Few-body calculations of $\eta$ nuclear clusters have been performed within standard few-body techniques: Faddeev-Yakubovsky equations [9] or variational methods. In ref. [3] the $\eta$ nuclear cluster wave functions were expanded in a hyperspherical basis. More recent calculations [4,8] were based on the Stochastic Variational Method (SVM) with a correlated Gaussian basis [10]. Both variational approaches showed sufficient accuracy in the description of $\eta$ nuclear quasi-bound states and provided almost identical results for $\eta d$, $\eta^3$He and $\eta^4$He systems.
In our calculations, the nucleon part is described by the Minnesota central $NN$ potential \[1\] or the Argonne AV4’ potential \[2\]. The interaction of $\eta$ with nucleons of the core is given by a complex two-body energy dependent potential derived from a full chiral coupled-channels model:

$$v_{\eta N}(\sqrt{\delta s}, r) = -\frac{4\pi}{2\mu_{\eta N}} b(\sqrt{\delta s}) \rho_A(r),$$

(2.1)

where $\delta \sqrt{s} = \sqrt{s} - \sqrt{s_h}$, $\rho_A(r) = (\frac{\Lambda}{2\sqrt{s}})^3 \exp\left(-\frac{\Lambda^2 r^2}{4}\right)$, and the amplitude $b(\sqrt{\delta s})$ is fitted to phase shifts derived from the $\eta N$ scattering amplitude $F_{\eta N}(\delta \sqrt{s})$ in the GW and CS models. The scale parameter $\Lambda$ is inversely proportional to the range of $V_{\eta N}$ potential. We consider two different values of the scale parameter, $\Lambda = 2$ and $4$ fm$^{-1}$ (the choice of the value of $\Lambda$ is discussed in ref. \[3\]). It is to be noted that in ref. \[5\], the $NN$ and $\eta N$ potentials were constructed within a pionless EFT approach.

The energy argument $\delta \sqrt{s}$ relevant for calculations of $\eta$ nuclear few-body clusters is expressed in the form \[3\]:

$$\delta \sqrt{s} = -\frac{B}{A} \frac{A - 1}{A} B_\eta - \xi_N \frac{A - 1}{A} \langle T_{NN} \rangle - \xi_\eta \left(\frac{A - 1}{A}\right)^2 \langle T_\eta \rangle,$$

(2.2)

where $B$ is the total binding energy of the system, $\xi_N(\eta) = m_N(\eta)/(m_N + m_\eta)$, $T_{NN}$ is the $NN$ kinetic energy in the total c.m. frame and $T_{NN}$ is the pairwise $NN$ kinetic energy operator in the $NN$ pair c.m. system \[3\]. The conversion widths are calculated using the expression

$$\Gamma_\eta = -2 < \Psi_{g.s.} | \text{Im} V_{\eta N} | \Psi_{g.s.} >$$

(2.3)

where $|\Psi_{g.s.}>$ stands for the ground state obtained after variation. As was stated already in \[3\], this approximation is reasonable due to small imaginary contribution $|\text{Im} V_{\eta N}| \ll |\text{Re} V_{\eta N}|$.

The results of calculations of $\eta$ nuclei with $A = 3$ and $4$ were discussed in detail in refs. \[3\] \[4\] \[5\]. To summarize, no bound $\eta NN$ system was found in the considered two-body interaction models. For $\eta NNN$, a weakly bound state (with $\eta$ separation energy below 1 MeV) was found for the Minnesota $NN$ potential and one particular variant of the $\eta N$ potential that reproduced the GW scattering amplitudes. No $\eta NNN$ bound states were found using more realistic $NN$ interaction model.

In Fig. \[3\], we demonstrate the self-consistent solution for $^4\text{He}$, calculated using the AV4’ $NN$ potential and GW $V_{\eta N}$ potential with $\Lambda = 4$ fm$^{-1}$. The $^4\text{He}$ bound state energy $E$ and the expectation value $< \delta \sqrt{s} >$ are plotted as a function of the subthreshold energy argument $\delta \sqrt{s}$ of the input potential $V_{\eta N}$. The self-consistency condition is fulfilled by requiring $\delta \sqrt{s} = < \delta \sqrt{s} >$. The corresponding value of $E(< \delta \sqrt{s} >)$ then represents the self-consistent energy of the $\eta$ nuclear cluster.

A precise self-consistent calculation of $p$-shell $\eta$ nuclear clusters, such as $^6\text{Li}$, represents highly non-trivial goal. In this report, we present our preliminary results for $^6\text{Li}$ using the central Minnesota $V_{NN}$ and GW $V_{\eta N}$ potentials. This should be regarded as the first step before doing calculations with a more realistic $NN$ potential to account for spin dependent force components in the $p$ shell. Moreover, we employed only one spin-isospin configuration in the description of the $^6\text{Li}$ nuclear core, which yielded binding energy $B(^6\text{Li}) = 34.66$ MeV. It is reasonable to expect
that taking into account all possible configurations in $^6\text{Li}$ will further increase the binding.\footnote{In ref. \cite{ref}, a value of $B(^6\text{Li}) = 36.51$ MeV was quoted for the SVM calculation with the Minnesota potential when more spin-isospin configurations were considered.} A full account will be given elsewhere in due course.

The results of the SVM calculations of $\eta$ binding energies $B_\eta$ and widths $\Gamma_\eta$ in $^7\text{H}$, $^4\text{He}$, and $^6\text{Li}$ are summarized in Fig. 3. Moreover, the figure illustrates the extent of the dependence of $B_\eta$ and $\Gamma_\eta$ on the parameter $\Lambda$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Binding energies $B_\eta$ (left) and widths $\Gamma_\eta$ (right) of $1s$ $\eta$ quasi-bound states in few-body nuclear systems calculated using the Minnesota $NN$ potential and the $\eta N$ potential GW with $\Lambda = 2$ and 4 fm$^{-1}$.}
\end{figure}

\textbf{Figure 2:} $\eta^4\text{He}$ bound state energy $E$ (red line, squares) and the expectation value $\langle \delta \sqrt{s} \rangle$ (blue line, circles), calculated using the AV4$'$ $NN$ potential (denoted here AV4p), as a function of the input energy argument $\delta \sqrt{s}$ of the $\eta N$ potential GW with $\Lambda = 4$ fm$^{-1}$. The dotted vertical line marks the self-consistent output values of $\langle \delta \sqrt{s} \rangle$ and $E$. The black dashed line denotes the $^4\text{He}$ g.s. energy which serves as threshold for bound $\eta$. The green curve shows the expectation value $<H_N>$ of the nuclear core energy. Figure adapted from ref. [4].

The results of the SVM calculations of $\eta$ binding energies $B_\eta$ and widths $\Gamma_\eta$ in $^7\text{H}$, $^4\text{He}$, and $^6\text{Li}$ are summarized in Fig. 3. Moreover, the figure illustrates the extent of the dependence of $B_\eta$ and $\Gamma_\eta$ on the parameter $\Lambda$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Binding energies $B_\eta$ (left) and widths $\Gamma_\eta$ (right) of $1s$ $\eta$ quasi-bound states in few-body nuclear systems calculated using the Minnesota $NN$ potential and the $\eta N$ potential GW with $\Lambda = 2$ and 4 fm$^{-1}$.}
\end{figure}

\textbf{Figure 3:} Binding energies $B_\eta$ (left) and widths $\Gamma_\eta$ (right) of $1s$ $\eta$ quasi-bound states in few-body nuclear systems calculated using the Minnesota $NN$ potential and the $\eta N$ potential GW with $\Lambda = 2$ and 4 fm$^{-1}$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Binding energies $B_\eta$ (left) and widths $\Gamma_\eta$ (right) of $1s$ $\eta$ quasi-bound states in few-body nuclear systems calculated using the Minnesota $NN$ potential and the $\eta N$ potential GW with $\Lambda = 2$ and 4 fm$^{-1}$.}
\end{figure}
3. The $\eta$ meson in many-body systems

The binding energies $B_\eta$ and widths $\Gamma_\eta$ of $\eta$ quasi-bound states in nuclear many-body systems are determined by solving self-consistently the Klein-Gordon equation

$$[\nabla^2 + \tilde{\omega}_\eta^2 - m_\eta^2 - \Pi_\eta(\omega_\eta, \rho)] \psi = 0,$$

where $\tilde{\omega}_\eta = \omega_\eta - i \Gamma_\eta/2$ is complex energy of $\eta$, $\omega_\eta = m_\eta - B_\eta$. The self-energy operator $\Pi_\eta(\sqrt{s}, \rho) \equiv 2\omega_\eta V_\eta = -(\sqrt{s}/E_N) 4\pi F_{\eta N}(\sqrt{s}, \rho) \rho$ is constructed self-consistently using the relevant in-medium $\eta N$ scattering amplitude $F_{\eta N}(\sqrt{s})$ and RMF density of the core nucleus.

Modifications of the free-space amplitudes $G_W$ due to Pauli blocking in the medium are accounted for by using the multiple scattering approach \[14\]. In the chirally inspired meson-baryon interaction models CS and GR, Pauli blocking restricts integration domain in the in-medium Green’s function which enters the underlying Lippmann-Schwinger (Bethe-Salpeter) equations \[7\]. Moreover, hadron self-energy insertions reflecting in-medium modifications of hadron masses could be included in the in-medium Green’s function, as well.

The energy argument in the scattering amplitude $F_{\eta N}(\sqrt{s})$ is approximated as \[1\]

$$\delta \sqrt{s} = \sqrt{s} - \sqrt{s_{th}} \approx -B_N \frac{\rho}{\bar{\rho}} - \xi_N B_\eta \frac{\rho}{\rho_0} - \xi_N T_N (\frac{\rho}{\rho_0})^{2/3} - \frac{\sqrt{s}}{\omega_\eta E_N} 2\pi \text{Re} F_{\eta N}(\sqrt{s}, \rho) \rho,$$

where $\bar{\rho}$ is the average nuclear density, $T_N = 23.0$ MeV at $\rho_0$, and $B_N \approx 8.5$ MeV is the average nucleon binding energy. It is to be stressed that all terms in Eq. 3.2 are negative definite and thus provide substantial downward energy shift. Since $\text{Re} F_{\eta N}(\sqrt{s})$ and $B_\eta$ appear as arguments in the expression for $\delta \sqrt{s}$ (Eq. 3.2), which in turn serves as an argument for the self-energy $\Pi_\eta$ in Eq. 3.1, a self-consistency scheme is required in calculations. \[2\]

\[2\] A slightly different form of $\delta \sqrt{s}$ has been used in recent calculations \[15, 16\], see the contribution of A. Gal in these proceedings.

**Figure 4:** Binding energies (left) and widths (right) of the $1s$ $\eta$ nuclear states in selected nuclei calculated using the GR $\eta N$ scattering amplitude \[8\] with different procedures for subthreshold energy shift $\delta \sqrt{s}$. 

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\[\text{Jiří Mareš}\]
Figure 5: Binding energies (left) and widths (right) of $1s\eta$ nuclear states in selected nuclei across the periodic table calculated self consistently using the GW, GR, and GR $\eta N$ scattering amplitudes.

It is instructive to compare our self-consistency procedure based on $\delta\sqrt{s}$ of Eq. 3.2, with a self-consistency requirement $\delta\sqrt{s} = -B_\eta$ applied in Ref. [17]. This comparison is presented in Fig. 4 for the in-medium GR amplitude. Our self-consistency formula in Eq. 3.2 (marked $\delta\sqrt{s}$) reduces considerably binding energies and widths of the $\eta$ meson in nuclei with respect to the calculations of ref. [17] that used $\delta\sqrt{s} = -B_\eta$ (marked $-B_\eta$). However, even the reduced GR widths are still rather large, which suggests that it would be extremely difficult to resolve $\eta$ nuclear states in this case.

The model dependence of the $\eta N$ amplitudes, shown in Fig. 1, has an impact on the calculations of $\eta$ nuclear quasi-bound states. This is illustrated in Fig. 5 were we present binding energies $B_\eta$ and widths $\Gamma_\eta$ calculated for $1s\eta$ nuclear states in selected nuclei using the GW, CS and GR models. In the left panel, the hierarchy of the three curves for the $\eta$ binding energies reflects the strength of the Re$F_{\eta N}(\sqrt{s})$ amplitudes below threshold (compare Fig. 1). For each $\eta N$ interaction model the binding energy increases with $A$ and tends to saturate for large values of $A$.

The right panel demonstrates substantial differences between the $\eta$ absorption widths $\Gamma_\eta$. While the CS and GW models produce relatively small widths (2 to 4 MeV), almost constant across the periodic table, the GR model yields much larger widths of order 20 MeV which increase with $A$.

4. Conclusions

In this contribution we briefly reviewed our calculations of $\eta$ nuclear quasi-bound states across the periodic table. We applied $\eta N$ scattering amplitudes derived from recent meson-baryon coupled-channel interaction models. We demonstrated that the strong energy dependence of scattering amplitudes calls for proper self-consistent treatment. The corresponding $\eta N$ amplitudes relevant for calculations of $\eta$ nuclear states are substantially weaker than the $\eta N$ scattering lengths. As a result our calculated $\eta$ bound states energies and widths are considerably smaller than those obtained in other comparable calculations.
In few-body calculations we explored whether the $\eta$ meson binds in light nuclei. We found no $\eta NN$ bound state. Our results suggest that the onset of $\eta^3$He binding occurs for the models providing the $\eta N$ scattering length $Re a_{\eta N} \sim 1$ fm. The binding $\eta^4$He requires $Re a_{\eta N} \geq 0.7$ fm. It is to be noted that the searches for $\eta^4$He bound states performed with the WASA-at-COSY facility have not revealed any signal for a narrow $\eta$ nuclear state \[13\].

Small conversion widths in heavier $\eta$ nuclei obtained in calculations using the CS and GW amplitudes might encourage experimental searches for $\eta$ nuclear bound states. It is to be stressed, however, that the size of the widths $\Gamma_\eta$ and binding energies $B_\eta$ is strongly model dependent. Other models produce either substantially larger widths or even do not generate any $\eta$ nuclear bound state in a given nucleus.

Acknowledgments

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Additional contributions to the widths due to $\eta N \to \pi N$ and $\eta NN \to NN$ processes, disregarded in our calculations, are estimated to add a few MeV to the total $\eta$ nuclear widths.