Abstract

A theoretical study of wave propagation in 1D metamaterial is presented. A system of nonlinear evolution equation for electromagnetic waves with both polarizations account is derived by means of projection operators method for general nonlinearity and dispersion. The system describes interaction of opposite directed waves with a given polarization. The particular case of Kerr nonlinearity and Drude dispersion is considered. In such approximation it results in the correspondent systems of nonlinear equations that generalizes the Schäfer-Wayne one. Particular solutions in case of slow-varying envelopes are found, plotted and analyzed in gigahertz range. Travelling wave solution for the system of equation of interaction of orthogonal-polarized waves is also obtained and the correspondent nonlinear dispersion relations are written in explicit form.

Introduction

On nonlinear pulse propagation theory

The nonlinear behavior of electromagnetic (EM) wave propagation depends on relations between the field and induced polarization. It is obvious that it is necessary to use either numerical scheme or approximations to obtain an analytical solution of a nonlinear problem. The first successful approach of such reduction was use the set of slowly varying envelopes. The simplest model scalar equation for a directed wave propagation, based on this approach, have a form of the nonlinear Schrödinger equation, derived by Zakharov in 1968 [1]. Its integrability [2] made the model very attractive because the rich "zoo" of the equation explicit solutions [3].

A natural step of integrable generalization of such model lies in a plane of better approximations of dispersion, dissipation [12] and nonlinearity (modified nonlinear Schrödinger (MNS) equation, see e.g. [3]), that allows to extend pulse durations down to picoseconds.

There are plenty of alternative ideas on the few-cycle pulse soliton-type description in different media [4].

The next step of this movement to the ultrashort pulses description that maintains integrability is made in works of Shäfer-Wayne [5, 6]. The Short Pulse Equation (SPE) again relates to unidirectional propagation for which a special kind of dispersion law and nonlinearity action was account in a rescaled evolution. In 2017 Z. Zhaqilao et al. derived an N-fold Darboux transformation from the Lax pair of the two-component short pulse system with loop. [22]

A generalization that allows to include description and interaction of opposite directed waves is connected with idea of joint account of the correspondent spaces of "hybrid" electric-magnetic amplitudes [7, 8]. The projecting operators (PO) method [8] works at arbitrary dispersion and nonlinearity. Similar universality demonstrates a method of [9]. The PO technique gives systematic transition to hybrid fields with simultaneous superposition of nonlinear terms, that effectively approximate weak nonlinearity, arriving at the mentioned celebrated model equations at the subspaces of directed waves [11]. The field hybridization may account ab initio dispersion and dissipation [12] and nonlinearity by iteration procedure [13].
Next, natural for electromagnetic field step accounts polarization and leads to double component vector equations [14], similar vector equation is studied in [15]. Both direction and polarization studied theoretically and, of importance, experimentally in [10].

On Drude model

One of important applications of such model relates to metamaterials that characterized by negative values of the parameters $\varepsilon$ and $\mu$, that must be anomalous dispersive, i.e., their permittivity and permeability must be frequency dependent, otherwise they would not be causal [16]. The two-time derivative Lorentz material model encompasses the metamaterial models most commonly discussed; it has the frequency domain susceptibility [17]:

$$\chi = \frac{\omega_p \chi_\alpha + i \omega_p \chi_\beta \omega - \chi_\gamma \omega}{\omega_0 + i \omega \Gamma - \omega}, \quad (0.1)$$

its particular case is the 2TDLM model, which produces a resonant response at $\omega = \omega_0$ when $\Gamma = 0$. It recovers the Drude model when this resonant frequency goes to zero, and the constants $\chi_\alpha = 1$, $\chi_\beta = \chi_\gamma = 0$. For this model Kanattakiov and Pietrzyk pointed, that the propagation of ultra-short pulses could be described by short pulse Schäfer - Wayne equation [18].

Aim and scope

It’s continued systematic application of projecting approach, originated from [8] for 1D metamaterial with account of both polarizations of the EM wave. The technique and results of our previous work [19] on nonlinear evolution equations of opposite directed waves with one polarization in Drude 1D-metamaterial has been developed.

In this paper application projecting operators method for this case is demonstrated. Obtained nonlinear equation for metamaterial with our previous results and vector SPE is compared. On base of resulting equations, the wave packets for linear and nonlinear cases are studied. Preliminary results got in [23].

For ordinary nonlinear Kerr material Kaplan in 1983 showed saving the arrangements of polarizations [20]. But for metamaterial the situation is different, as shown in our work [19] for unique polarization. We discover the change of arrangements of wave modes. Now, the questions to be answered: what happens with account of polarization? And how looks interactions of all four modes in metamaterial? The content of the paper is following

- Sec.1: Statement the boundary regime problem,
- Sec.2: The 4x4 matrix projection operators with arbitrary dispersion account are built for the case of two polarizations,
- Sec.3: Derivation the general linear system of equations for two left and two right waves with orthogonal polarizations,
- Sec.4: The general nonlinear system of equations for the left and right waves and two polarizations is obtained.
- Sec.5: For the particular case of the Kerr nonlinearity within approximate Drude dispersion got the novel system of short pulse equations that is reduced to the Schäfer-Wayne one for unique polarization.
- In the Sec 6 attention have been fixed to wave trains, starting from linear and, taking into account nonlinearity, obtain a plane wave with amplitude dependent wavelength.
1 Maxwell’s equations. Boundary regime problem

The starting point is the Maxwell equations for linear isotropic dispersive dielectric media, in the SI unit system:

\[
\begin{align*}
\text{div} \vec{D}(\vec{r},t) &= 0, \\
\text{div} \vec{B}(\vec{r},t) &= 0, \\
\text{rot} \vec{E}(\vec{r},t) &= -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}, \\
\text{rot} \vec{H}(\vec{r},t) &= \frac{\partial \vec{D}(\vec{r},t)}{\partial t}.
\end{align*}
\]

Restricting ourselves to a one-dimensional model, similarly to Shäfer, Wayne [5], where the $x$-axis is chosen as the direction of a wave propagation assuming zero longitudinal field components allows to write the Maxwell equations with arbitrary polarization account:

\[
\begin{align*}
\frac{\partial D_y}{\partial t} &= -\frac{\partial H_z}{\partial x}, \\
\frac{\partial D_z}{\partial t} &= \frac{\partial H_y}{\partial x}, \\
\frac{\partial B_y}{\partial t} &= \frac{\partial E_z}{\partial x}, \\
\frac{\partial B_z}{\partial t} &= -\frac{\partial E_y}{\partial x}.
\end{align*}
\]

To close the system (1.5) we need to add material relations: where $\hat{\mu}$ and $\hat{\varepsilon}$ are integral convolution-type operators [19]:

\[
\begin{align*}
\hat{\varepsilon}\psi(x,t) &= \int_{-\infty}^{\infty} \varepsilon(t-s)\psi(x,s)ds, \\
\hat{\mu}\psi(x,t) &= \int_{-\infty}^{\infty} \mu(t-s)\psi(x,s)ds,
\end{align*}
\]

with kernels

\[
\begin{align*}
\varepsilon(t-s) &= \frac{\varepsilon_0}{2\pi} \int_{-\infty}^{\infty} \varepsilon(\omega) \exp(i\omega(t-s))d\omega, \\
\mu(t-s) &= \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} \mu(\omega) \exp(i\omega(t-s))d\omega.
\end{align*}
\]

Hence operator form of the equation (1.5) is:

\[
\begin{align*}
\partial_t(\hat{\varepsilon}\vec{E}_{y,z}) &= \mp \partial_x(\hat{\mu}^{-1}B_{z,y}), \\
\partial_t B_{y,z} &= \pm \partial_x E_{z,y}.
\end{align*}
\]

Here it’s marked:

\[
\frac{\partial}{\partial x} \equiv \partial_x, \quad \frac{\partial}{\partial t} \equiv \partial_t.
\]

Adding boundary conditions to state the problem:

\[
E_{y,z}(0,t) = j_{y,z}(t), \quad B_{y,z}(0,t) = \ell_{y,z}(t),
\]

$j_i$ and $\ell_i$ are arbitrary functions, continued to the half space $t < 0$ antisymmetrically:

\[
j_i(-t) = -j_i(t), \quad \ell_i(-t) = -\ell_i(t), i = y, z.
\]
2 Dynamic projecting operators

Doing the Fourier transformations like in [19] and plugging them into the system of equations (1.9) one have the closed system:

\[ \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \varepsilon(\omega) \mathcal{E}_{yz}(x, \omega) \exp(i\omega t) d\omega = \pm \frac{1}{\mu_0 \varepsilon_0} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \mathcal{B}_{zy}(x, \omega) \mu(\omega) \exp(i\omega t) d\omega. \] (2.1)

The inverse Fourier transformation yields in the four equations of (1.9), written in short form:

\[ \frac{\partial}{\partial x} \mathcal{B}_{zy} = \pm i\omega \mu_0 \varepsilon_0 \varepsilon(\omega) \mathcal{E}_{yz}. \] (2.2)
\[ \frac{\partial}{\partial x} \mathcal{E}_{zy} = \pm i\omega \mathcal{B}_{yz}. \] (2.3)

Let us define the column of the field components transform and matrix operator with obvious elements from (2.2, 2.3)

\[ \tilde{\Psi} = \begin{pmatrix} \mathcal{B}_z \\ \mathcal{B}_y \\ \mathcal{E}_z \\ \mathcal{E}_y \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} \hat{0} & \mathcal{L}_1 \\ \mathcal{L}_2 & 0 \end{pmatrix}, \] (2.4)

arriving at

\[ \frac{\partial}{\partial x} \tilde{\Psi} = \hat{L} \tilde{\Psi}. \] (2.5)

Doing projection operators technique, similarly with [19], finally we obtain four matrix projecting operators in \( t \)-representation by the standard general formula:

\[ P^{(1)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -\hat{a} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\hat{a}^{-1} & 0 & 0 & 1 \end{pmatrix}, \quad P^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \hat{a} & 0 \\ 0 & \hat{a}^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \] (2.6)
\[ P^{(3)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \hat{a} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{a}^{-1} & 0 & 0 & 1 \end{pmatrix}, \quad P^{(4)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -\hat{a} & 0 \\ 0 & -\hat{a}^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (2.7)

Projectors \( P^{(1,2)} \) correspond to \( \lambda_1 \) and two other ones - to \( \lambda_2 \).

Operators \( \hat{a}, \hat{a}^{-1} \) are defined as [19]:

\[ \hat{a} \eta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta(x, \tau) \int_{-\infty}^{\infty} a(\omega) \exp(i\omega(t-\tau)) d\omega \ dr, \] (2.8)
\[ \hat{a}^{-1} \xi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi(x, \tau) \int_{-\infty}^{\infty} \frac{1}{a(\omega)} \exp(i\omega(t-\tau)) d\omega \ dr, \]

where \( a(\omega) \) is positive solution of the quadratic equation (2.9):

\[ \mu_0 \varepsilon_0 \varepsilon(\omega) \mu(\omega) \equiv c^{-2} \varepsilon(\omega) \mu(\omega) \equiv a^2(\omega), \] (2.9)

where \( c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \) is the velocity of light in vacuum.

3 Separated equations and definition for left and right waves

Let’s return to the time-domain. Let’s write the matrix equation (2.5) in this representation:

\[ \frac{\partial}{\partial x} \Psi = \hat{L} \Psi, \] (3.1)
\[ \Psi = \begin{pmatrix} B_z \\ B_y \\ E_z \\ E_y \end{pmatrix}, \quad (3.2) \]

\[ \hat{L} = \begin{pmatrix} 0 & 0 & 0 & -\partial_t \tilde{a}^2 \\ 0 & 0 & \partial_t \tilde{a}^2 & 0 \\ 0 & \partial_t & 0 & 0 \\ -\partial_t & 0 & 0 & 0 \end{pmatrix}. \quad (3.3) \]

Action of projectors \( P^{(1)} \) and \( P^{(2)} \) on (3.1) yields to hybrid waves \( \Pi_1 \) and \( \Pi_2 \):

\[ \Pi_1 = \frac{1}{2} (B_z - \tilde{a} E_y), \quad (3.4) \]

\[ \Pi_2 = \frac{1}{2} (B_y + \tilde{a} E_z). \quad (3.5) \]

Action of projectors \( P^{(3)} \) and \( P^{(4)} \) on (3.1) yields to hybrid waves \( \Lambda_1 \) and \( \Lambda_2 \):

\[ \Lambda_1 = \frac{1}{2} (B_z + \tilde{a} E_y), \quad (3.6) \]

\[ \Lambda_2 = \frac{1}{2} (B_y - \tilde{a} E_z). \quad (3.7) \]

Waves \( \Pi_1 \) and \( \Lambda_1 \) introduced for the case of unique polarization \[19\] and describe a propagation of \( y \)-polarization. The other two do the same for \( z \)-polarization.

\[ \partial_x \Pi_1 = \partial_t \tilde{a} \Pi_1, \quad \partial_x \Pi_2 = \partial_t \tilde{a} \Pi_2, \quad \partial_x \Lambda_1 = -\partial_t \tilde{a} \Lambda_1, \quad \partial_x \Lambda_2 = -\partial_t \tilde{a} \Lambda_2. \quad (3.8) \]

Using definitions of (3.4) and (3.5) and (1.11) we derive boundary regime conditions for left and right waves:

\[ \Lambda_1(0, t) = \frac{1}{2} (B_z(0, t) + \tilde{a} E_y(0, t)) = \frac{1}{2} (k_z(t) + \tilde{a} j_y(t)), \quad (3.11) \]

\[ \Lambda_2(0, t) = \frac{1}{2} (B_y(0, t) - \tilde{a} E_z(0, t)) = \frac{1}{2} (k_y(t) - \tilde{a} j_z(t)), \quad (3.12) \]

\[ \Pi_1(0, t) = \frac{1}{2} (B_z(0, t) - \tilde{a} E_y(0, t)) = \frac{1}{2} (\ell_z(t) + \tilde{a} j_y(t)), \quad (3.13) \]

\[ \Pi_2(0, t) = \frac{1}{2} (B_y(0, t) + \tilde{a} E_z(0, t)) = \frac{1}{2} (\ell_y(t) + \tilde{a} j_z(t)). \quad (3.14) \]

### 4 General nonlinearity account

Let us consider a nonlinear problem. The starting point is the Maxwell’s equations (1.5) again with generalized nonlinear material relations:

\[ D_i = \varepsilon E_i + P_i^{(NL)}, \quad (4.1) \]

\[ B_i = \mu H_i + M_i^{(NL)}, \quad i = y, z. \]

\( P_{NL} \) - nonlinear part of polarization \( (M_{NL} \) - one for magnetization). Linear parts of polarization and magnetization have already been taken into account. In time-domain, a closed nonlinear version of (1.9) is:

\[ \partial_t (\varepsilon E_{y,z}) + \partial_t P_{y,z}^{(NL)} = \mp \partial_y \mu^{-1} B_{z,y} + \mp \partial_z \mu^{-1} M_{z,y}^{(NL)}, \quad (4.2) \]

\[ \partial_t B_{z,y} = -\partial_y E_{y,z}. \quad (4.3) \]
Action of operator $\hat{\mu}$ on the first pair of equations of system (4.2) and use of the same notations $\Psi$ and $\hat{L}$ from (3.2, 3.3) once more, produce a nonlinear analogue of the matrix equation (3.1):

$$\partial_x \Psi - \hat{L} \Psi = \partial_x \begin{pmatrix} M_x^{(NL)} \\ M_y^{(NL)} \\ 0 \\ 0 \end{pmatrix} - \hat{\mu} \partial_t \begin{pmatrix} P_y^{(NL)} \\ P_z^{(NL)} \\ 0 \\ 0 \end{pmatrix}.$$  (4.4)

In the r.h.s there is a vector of nonlinearity for case of the opposite directed 1D-waves:

$$N = \begin{pmatrix} \partial_x M_x^{(NL)} - \hat{\mu} \partial_t P_y^{(NL)} \\ \partial_x M_y^{(NL)} + \hat{\mu} \partial_t P_z^{(NL)} \\ 0 \\ 0 \end{pmatrix}.$$  (4.5)

Next, acting by operators $\hat{P}^{(1,2,3,4)}$ (2.6) on the Eq. (4.4) one can find:

$$\partial_x \Pi_1 - \partial_t \hat{a} \Pi_1 = -\frac{1}{2} \left( \partial_x M_z^{(NL)} \right),$$  (4.6)

$$\partial_x \Pi_2 - \partial_t \hat{a} \Pi_2 = \frac{1}{2} \left( \partial_x M_y^{(NL)} + \hat{\mu} \partial_t P_z^{(NL)} \right),$$  (4.7)

$$\partial_x \Lambda_1 + \partial_t \hat{a} \Lambda_1 = \frac{1}{2} \left( \partial_x M_z^{(NL)} - \hat{\mu} \partial_t P_y^{(NL)} \right),$$  (4.8)

$$\partial_x \Lambda_2 + \partial_t \hat{a} \Lambda_2 = -\frac{1}{2} \left( \partial_x M_y^{(NL)} + \hat{\mu} \partial_t P_z^{(NL)} \right).$$  (4.9)

Generally the r.h.s. of each equation (4.6) depends on the field vectors $\vec{E}, \vec{B}$, that should be presented in terms of the fields $\vec{\Pi}, \vec{\Lambda}$ to close the system. The vectors components are expressed by means of the inverse transformation of (3.4 - 3.5).

5 Kerr nonlinearity account for lossless Drude metamaterials

5.1 Equations of interaction of the waves via Kerr effect

For nonlinear Kerr materials [21], the third-order nonlinear part of polarization [14, 21] has the form:

$$P_y^{(NL)} = \varepsilon_0 \chi^{(3)} (E_y^3 + E_y E_z E_y).$$

From (4.5), deleting magnetic nonlinearity, one can find the vector $N$:

$$N = -\varepsilon_0 \hat{\mu} \chi^{(3)} (E_y E_y E_y^2 + E_y E_z E_z).$$  (5.1)

Account for the definitions of the hybrid fields as (3.4) gives:

$$E_y = -\frac{1}{2} \hat{\alpha}^{-1} (\Pi_1 - \Lambda_1),$$  (5.2)

$$E_z = -\frac{1}{2} \hat{\alpha}^{-1} (\Pi_2 - \Lambda_2).$$  (5.3)

The system for left and right waves with two polarizations equations in a medium with Kerr nonlinearity:
\[
\partial_2 \Pi_1 - \partial_2 \tilde{a} \Pi_1 = -\frac{1}{16} \varepsilon_0 \chi_e^{(3)} \hat{\partial}_t [(\tilde{a}^{-1}(\Pi_1 - \Lambda_1))^3 + \tilde{a}^{-1}(\Pi_1 - \Lambda_1)(\tilde{a}^{-1}(\Pi_2 - \Lambda_2))^2], \tag{5.4}
\]
\[
\partial_2 \Pi_2 - \partial_2 \tilde{a} \Pi_2 = -\frac{1}{16} \varepsilon_0 \chi_e^{(3)} \hat{\partial}_t [(\tilde{a}^{-1}(\Pi_2 - \Lambda_2))^3 + \tilde{a}^{-1}(\Pi_2 - \Lambda_2)(\tilde{a}^{-1}(\Pi_1 - \Lambda_1))^2] \tag{5.5}
\]
\[
\partial_x \Lambda_1 + \partial_k \tilde{a} \Lambda_1 = -\frac{1}{16} \varepsilon_0 \hat{\partial}_t [(\tilde{a}^{-1}\Lambda_1)^3 + \tilde{a}^{-1}(\tilde{a}^{-1}\Lambda_1)^2], \tag{5.6}
\]
\[
\partial_x \Lambda_2 + \partial_k \tilde{a} \Lambda_2 = -\frac{1}{16} \varepsilon_0 \hat{\partial}_t [(\tilde{a}^{-1}\Lambda_2)^3 + \tilde{a}^{-1}\Lambda_2(\tilde{a}^{-1}\Lambda_1)^2]. \tag{5.7}
\]
In unidirectional case with \(\Pi_1, \Pi_2 = 0\) one can obtain the system, that describe interaction between hybrid fields with different polarizations:
\[
\partial_x \Lambda_1 + \partial_k \tilde{a} \Lambda_1 = -\frac{1}{16} \varepsilon_0 \hat{\partial}_t [(\tilde{a}^{-1}\Lambda_1)^3 + \tilde{a}^{-1}(\tilde{a}^{-1}\Lambda_1)^2], \tag{5.8}
\]
\[
\partial_x \Lambda_2 + \partial_k \tilde{a} \Lambda_2 = -\frac{1}{16} \varepsilon_0 \hat{\partial}_t [(\tilde{a}^{-1}\Lambda_2)^3 + \tilde{a}^{-1}\Lambda_2(\tilde{a}^{-1}\Lambda_1)^2]. \tag{5.9}
\]
Because of propagation in one direction, it’s useful to mark \(\Lambda_1\) and \(\Lambda_2\) as: Applying Drude model, approximately write
\[
\tilde{a}^{-1} \eta(x,t) \approx \frac{c}{pq} \partial_t^2 \eta(x,t), \tag{5.10}
\]
\[
\tilde{\mu} \eta(x,t) \approx -\mu_0 q^2 \partial_t^{-2} \eta(x,t), \tag{5.11}
\]
(see again [19] for details), plugging \(\varepsilon_0 \mu_0 = c^{-2}\),
\[
\frac{c}{pq} \partial_x \Lambda_1 + \partial_t^{-1} \Lambda_1 = \frac{1}{16} \varepsilon_0 \chi_e^{(3)} \frac{c^2}{p^2 q^2} \partial_t^{-1} [(\partial_t^2 \Lambda_1)^3 + \partial_t^2 \Lambda_1(\partial_t^2 \Lambda_2)^2], \tag{5.12}
\]
\[
\frac{c}{pq} \partial_x \Lambda_2 + \partial_t^{-1} \Lambda_2 = \frac{1}{16} \varepsilon_0 \chi_e^{(3)} \frac{c^2}{p^2 q^2} \partial_t^{-1} [(\partial_t^2 \Lambda_2)^3 + \partial_t^2 \Lambda_2(\partial_t^2 \Lambda_1)^2]. \tag{5.13}
\]
Differentiation on \(t\) three times leads to equation:
\[
\frac{c}{pq} \partial_{tttt} \Lambda_1 + \partial_t^2 \Lambda_1 = \gamma^2 \partial_t^2 [(\partial_t^2 \Lambda_1)^3 + \partial_t^2 \Lambda_1(\partial_t^2 \Lambda_2)^2], \tag{5.14}
\]
\[
\frac{c}{pq} \partial_{tttt} \Lambda_2 + \partial_t^2 \Lambda_2 = \gamma^2 \partial_t^2 [(\partial_t^2 \Lambda_2)^3 + \partial_t^2 \Lambda_2(\partial_t^2 \Lambda_1)^2], \tag{5.15}
\]
where
\[
\gamma^2 = \frac{1}{16} \varepsilon_0 \chi_e^{(3)} \frac{c^2}{p^2 q^2}. \tag{5.16}
\]
Introducing new field functions \(l_1\) and \(l_2\) and variable \(\chi\) as:
\[
l_1 \equiv \gamma^{-1} \partial_t^2 \Lambda_1, \tag{5.17}
\]
\[
l_2 \equiv \gamma^{-1} \partial_t^2 \Lambda_2, \tag{5.18}
\]
\[
\partial_x = \frac{pq}{c} \partial_\chi, \tag{5.19}
\]
one can obtain the generalized SPE system:
\[
\partial_x l_1 + l_1 = \partial_t^2 (l_1^3 + l_2^2 l_1), \tag{5.20}
\]
\[
\partial_x l_2 + l_2 = \partial_t^2 (l_2^3 + l_1^2 l_2). \tag{5.21}
\]
That’s a generalization of Schäfer-Wayne equation for case of interaction of two left waves, which is one of objectives of this work.
6 Wave trains

6.1 Linear wave packets for the right waves

We consider the system (5.13), differentiated on $t$:

\[
\frac{c}{pq} \partial_{xt} \Lambda_1 + \Lambda_1 = \frac{1}{16} \chi \left( \frac{c^2}{pq^2} \left[ (\partial_t^2 \Lambda_1)^3 + \partial_t^2 \Lambda_1 (\partial_t^2 \Lambda_2)^2 \right] \right),
\]

(6.1)

\[
\frac{c}{pq} \partial_{xt} \Lambda_2 + \Lambda_2 = \frac{1}{16} \chi \left( \frac{c^2}{pq^2} \left[ (\partial_t^2 \Lambda_2)^3 + \partial_t^2 \Lambda_2 (\partial_t^2 \Lambda_1)^2 \right] \right).
\]

(6.2)

In linear case the r.h.s. will be equal 0. These equations are identical, hence, take one of them:

\[
\frac{c}{pq} \partial_{xt} \Lambda_1 + \Lambda_1 = 0,
\]

(6.3)

plugging the wavetrain solution, that we prepare for a comparison with nonlinear case:

\[
\Lambda_1 = A(x,t) \exp[i(kx - \omega t)] + c.c.
\]

(6.4)

Differentiating

\[
\partial_{xt} \Lambda_1 = A_x \exp[i(kx - \omega t)] + i k A_t \exp[i(kx - \omega t)] - i \omega A_x \exp[i(kx - \omega t)] - i \omega (ik) A \exp[i(kx - \omega t)] + c.c.,
\]

(6.5)

Putting the result in the equation (6.3), assuming slow varying amplitude:

\[
A_x << k A, A_t << \omega A,
\]

(6.6)

to kill the zeroth order term gives the dispersion relation:

\[
k(\omega) = -\frac{pq}{c\omega},
\]

(6.7)

then, in the first order the equation arrives at

\[
A_x - \frac{\omega}{k} A_x = 0,
\]

(6.8)

Next, denoting

\[
v_g = \frac{\omega}{k},
\]

(6.9)

after conventional change the variables:

\[
\eta = t - \frac{x}{v_g}, \xi = t + \frac{x}{v_g},
\]

(6.10)

\[
\partial_x = \frac{1}{v_g} \partial_{\xi} + \frac{1}{v_g} \partial_{\eta}, \partial_x = \frac{1}{v_g} \partial_{\xi} - \frac{1}{v_g} \partial_{\eta}, A(x,t) \rightarrow \mathcal{A}(\eta, \xi),
\]

the (6.8) trivializes as

\[
2 \mathcal{A}_{\eta} = 0.
\]

(6.11)

It’s shown, the amplitude function $A$ is independent on $\eta$

\[
\mathcal{A} = f(\xi); A(x,t) = f \left( t + \frac{x}{v_g} \right).
\]

Substituting this relation into (6.8) leads to the definition of $v_g$:

\[
v_g = \frac{\omega^2 c}{pq},
\]

(6.12)
Sign "minus" means right direction of wave propagation.

To fix the unique solution, it’s necessary to add a boundary condition:

\[ A(0, t) = A_0 \exp \left( - \left( \frac{t}{\tau} \right)^2 \right), \quad (6.13) \]

\( \tau \) characterizes a width of wave packet and \( \omega \) characterizes the period of oscillation. Accounting the boundary regime \([6.13]\), for the \( \Lambda_1 \)-wave, propagated to the right the explicit formula is obtained:

\[ \Lambda_1(x, t) = A_0 \exp \left\{ - \left( \frac{t + \frac{x}{\nu}}{\tau} \right)^2 + i(kx - \omega t) \right\} + c.c. \quad (6.14) \]

The wavetrains with other polarization differs only by electric and magnetic fields components numbers as it is prescribed by \([3.7]\). The opposite directed waves are defined by \([3.4], [3.5]\), its formulas differ from \([6.14]\) only by signs by \( \frac{v}{\nu} \).

### 6.2 Dispersionless nonlinear equations for envelopes

We consider the system \([6.2]\). For \( \Lambda_1 \) and \( \Lambda_2 \) in the wavetrain form with the frequency chosen by boundary condition as in linear case:

\[ \Lambda_1 = A(x, t) \exp[i(kx - \omega t)] + c.c., \quad \Lambda_2 = B(x, t) \exp[i(kx - \omega t)] + c.c., \quad (6.15) \]

and plugging these relations in the equations together, with account of \([6.6]\), linear independence of complex conjugated parts and strong inequality \([6.6]\), leaving the nonlinear resonant terms in the r.h.s, one can obtain

\[ c(ikA_t - i\omega A_x) = -\frac{\chi(3) c^2 \omega^6}{16\rho^4 q^2} [A(3|A|^2 + 2|B|^2) + A^* B^2], \quad (6.16) \]

\[ c(ikB_t - i\omega B_x) = -\frac{\chi(3) c^2 \omega^6}{16\rho^4 q^2} [B(2|A|^2 + 3|B|^2) + A^2 B^*]. \quad (6.17) \]

The parameter of the solution \( k \) is chosen to simplify the equations as

\[ -ci\omega(ik)A + pqA = (k\omega + \frac{pq}{c})A = 0, \quad (6.18) \]

that is equivalent to the expression

\[ k = -\frac{pq}{c \omega}, \quad (6.19) \]

that fix the phase velocity of the carrier wave as in linear case. Equations \([6.16]\) and \([6.17]\) with account approximation \([6.6]\) are:

\[ \frac{k}{\omega} A_t - A_x = -i\frac{\chi(3) c^2 \omega^5}{16\rho^4 q^2} [A(3|A|^2 + 2|B|^2) + B^2 A^*], \quad (6.20) \]

\[ \frac{k}{\omega} B_t - B_x = -i\frac{\chi(3) c^2 \omega^5}{16\rho^4 q^2} [B(2|A|^2 + 3|B|^2) + A^2 B^*]. \quad (6.21) \]

This system of equations describes the interaction between orthogonal polarization modes, propagating to the left in metamaterials.

### 6.3 Particular solution of \([6.20]\)

After change the variables:

\[ \eta = x - \frac{t\omega}{k}, \xi = x + \frac{t\omega}{k}, \]

\[ \partial_x = \partial_\eta + \partial_\xi, \partial_t = \frac{\omega}{k} \partial_\xi - \frac{\omega}{k} \partial_\eta, A(x, t) \rightarrow A(\eta, \xi), B(x, t) \rightarrow B(\eta, \xi), \quad (6.22) \]
To plot the dependence of the shift on amplitude ratio, we mark
\[ C \] means the phase shift. There are two solutions (6.23) in this case. We consider only the one with phase
\[ \phi = 2 \left( \frac{A^2}{A_0} \right) \exp(i(2\varphi_2(\eta) - 2\varphi_1(\eta))) ]] \]
\[ 2\varphi_1' = \frac{\chi e^c\omega_5}{16p^4q^2} [(3|A_0|^2 + 2|B_0|^2) + B_0^2 A_0^* \exp(i(2\varphi_2(\eta) - 2\varphi_1(\eta)))], \]
\[ 2\varphi_2' = \frac{\chi e^c\omega_5}{16p^4q^2} [2|A_0|^2 + 3|B_0|^2) + A_0^2 B_0^* \exp(i(2\varphi_1(\eta) - 2\varphi_2(\eta))]. \]
Substraction the second equation from the first one and mark
\[ 2\varphi_1 - 2\varphi_2 \equiv z(\eta), \]
\[ z' = \frac{\chi e^c\omega_5}{16p^4q^2} [2|A_0|^2 - |B_0|^2 + B_0^2 A_0^* \exp(i\eta) + A_0^2 B_0^* \exp(-i\eta)]. \]
6.3.1 case \( \varphi_1(\eta) = \varphi_2 + C \)
where:
\[ C = \text{const, exp}(2iC) \equiv y \]
C means the phase shift. There are two solutions (6.23) in this case. We consider only the one with phase shift:
\[ A_2 = A_0 \exp \left\{ i \frac{\chi c^3}{32p^4q^2} (3|A_0|^2 + 4|B_0|^2)\eta - \frac{i}{2} \ln \left( \frac{B_0}{A_0^2 B_0^*} \right)^2 \right\}, \]
\[ B_2 = B_0 \exp \left( \frac{3\chi e^c\omega_5}{32p^4q^2} (|A_0|^2 + |B_0|^2)\eta. \right) \]
The phase shift is:
\[ C_2 = - \frac{1}{2} \ln \left( \frac{B_0}{A_0^2 B_0^*} \right)^2 = - \frac{1}{2} \ln \left( \frac{B_0 A_0^2}{A_0 B_0^*} B_0^* \right). \]
To plot the dependence of the shift on amplitude ratio, we mark \( \frac{B_0}{A_0} \) as \( z \). Then:
\[ C_2 = - \frac{1}{2} \ln \left( \frac{B_0 A_0^2}{A_0 B_0^*} - \frac{B_0^2}{A_0^2} \right) \rightarrow - \frac{1}{2} \ln \left( \frac{z}{z^2} - z^2 \right). \]
7 Traveling wave solution of (6.2)
We return to (6.2):
\[ \frac{c}{pq} \partial_x \Lambda_1 + \Lambda_1 = \frac{1}{16} \chi e^c \frac{c^2}{p^4q^2} [c^2 \Lambda_1^3 + \partial_t^2 \Lambda_1 (\partial_t^2 \Lambda_2)^2]. \]
\[ \frac{c}{pq} \partial_x \Lambda_2 + \Lambda_2 = \frac{1}{16} \chi e^c \frac{c^2}{p^4q^2} [c^2 \Lambda_2^3 + \partial_t^2 \Lambda_2 (\partial_t^2 \Lambda_1)^2]. \]
After transition to variables:
\[ x = \eta, \xi = x + vt, \]
Figure 1: Real parts of functions $A_2(x, t)$, $B_2(x, t)$ for $p = 10^9 \text{Hz}, q = 10^3 \eta, \omega = 1.2q, \chi = 10^{-20} \frac{m^2}{v^2}, A_0 = B_0 = 1 T, x = 2 m$

Figure 2: Phase shift $C_2$ in dependence of amplitude ratio $\frac{B_0}{A_0}$ for real or imaginary $A_0$ and $B_0$

where $v$ - parameter with dimension of velocity, and apply the approximation.

$$
\partial_t L_1 = \partial_t L_2 = 0,
$$
and then dividing all on $\frac{\kappa \lambda}{pq}$:

$$
\partial^2 L_1 + \kappa L_1 = \lambda \xi^2 [(\partial^2 L_1)^3 + \partial^2 L_1(\partial^2 L_2)^2], \quad (7.4)
$$
$$
\partial^2 L_2 + \kappa L_2 = \lambda \xi^2 [(\partial^2 L_2)^3 + \partial^2 L_2(\partial^2 L_1)^2]. \quad (7.5)
$$

$\kappa, \lambda^2$ - parameters:

$$
\kappa = \frac{pq}{vc}, \quad (7.6)
$$
$$
\lambda^2 = \frac{v^5}{16} \chi^{(3)} \frac{c}{p^3 q}. \quad (7.7)
$$

Reduction

$$
L_1 = L_2 = L. \quad (7.8)
$$
and division on \(2\lambda^2\) yield:

\[
(\partial^2_\xi L)^3 - \frac{1}{2\lambda e} \partial^2_\xi L - \frac{\kappa}{2\lambda e^2} L = 0. \tag{7.9}
\]

Solving this cubic equation by Cardano formula with respect to \(\partial^2_\xi L\), accounting only real solution and expanding it in power series on \(L\), we obtain the approximate equation:

\[
(\partial^2_\xi L)_1 + 2\lambda^3 \xi^2 L^3 + \kappa L = 0, \tag{7.10}
\]

After rescaling of variables and accounting relations (7.6, 7.7):

\[
\partial^2_\xi L + \frac{1}{8} q^2 v^2 \chi(3) c^2 \alpha^2 n^2 L^3 + \frac{pq}{vc} L = 0, \tag{7.11}
\]

Introducing amplitude parameter \(\alpha\), dimensionless function \(l\) as:

\[
L = \alpha l,
\]

and dimensionless variable \(\beta\) as:

\[
\xi = n\beta,
\]

we obtain an equation on \(l\):

\[
l_{\beta\beta} + \frac{pq}{vc} n^2 l + \frac{1}{8} q^2 v^2 \chi(3) c^2 \alpha^2 n^2 L^3 = 0, \tag{7.12}
\]

Comparison with equation for elliptic cosine \(\text{cn}(u, m)\):

\[
y'' + (1 - 2m^2)y + 2m^2 y^3 = 0,
\]

leads to the system of equation relative \(m\) and \(v\):

\[
1 - 2m^2 = \frac{pq}{vc} n^2, \tag{7.13}
\]

\[
\frac{1}{8} q^2 v^2 \chi(3) c^2 \alpha^2 n^2 = 2m^2. \tag{7.14}
\]

From this system it’s easy to find \(m\):

\[
m^2 = \frac{1}{16} \frac{q^2 v^2 \chi(3) c^2}{c^2} \alpha^2 n^2, \tag{7.15}
\]

After doing some algebra, one can easily derive the cubic equation on \(v\)

\[
v - \frac{1}{8} q^2 \chi(3) c^2 \alpha^2 n^2 v^3 - \frac{pq}{c} n^2 = 0. \tag{7.16}
\]

Solution by Cardano formula can lead to wrong results for case \(\alpha \to 0\). Hence, for small amplitudes we need to look for solution using the successive approximations method. In first approximation \(\alpha \to 0\):

\[
v_0 = \frac{pq}{c} n^2 \tag{7.17}
\]

Substitution it into (7.16) yields:

\[
v_1 \approx \frac{1}{8} q^2 \chi(3) c^2 \alpha^2 n^2 v^3 + \frac{pq}{c} n^2. \tag{7.18}
\]

In small amplitude approximation \(\Lambda_2\)-wave is:

\[
\Lambda_2 = \alpha cn\left(\frac{x + vt}{n}, \frac{1}{4} q v \sqrt{\chi(3)} \alpha n\right), \tag{7.19}
\]

where \(v\) and \(\alpha\) are velocity and amplitude of induced wave. \(\frac{v}{n}\) is a frequency of nonlinear wave on boundary \((x = 0)\).

Changing \(\alpha, n\) we can change the behaviour of the wave on boundary.
8 Conclusion

In this work the wave propagation of two polarizations in 1D-metamaterial has been studied. The general equation of directed wave propagation in 1D-metamaterial with two polarization has been obtained. It’s shown for Drude metamaterial with Kerr nonlinearity it has the form of “vector SPE”.

The system of equations (6.20) which describe the interaction between orthogonal polarization modes, propagating to the left in metamaterials for case of slow-varying envelopes has been obtained.

References

[1] V. I. Talanov, ZhETF Pis. Red. 2, 223 (1965)[JETP Lett. 2, 141 (1965)]. Zakharov V.E. J. Appl. Mech. Tech. Phys. 9, 190 (1968).
[2] Zakharov, V.E. and Shabat, A.B. (1971) Exact theory of two-dimensional self- focusing and one-dimensional modulation of waves in nonlinear media, Zhurn. Eksp. Teor. Fiz. 61, 118-134 [(1972) Sov. Phys. JETP 34, 62-69].
[3] E. Doktorov S.B. Leble Dressing method in mathematical physics. (Springer-Verlag, 2007)
[4] Sazonov S. V., Ustinov N. V. New class of extremely short electromagnetic solitons, Pis’ma v Zh. Eksp. Teoret. Fiz., 83:11 (2006), 573-578 General class of the traveling waves propagating in a nonlinear oppositely-directional coupler
[5] Schäfer T., Wayne C.E. Propagation of ultra-short optical pulses in cubic nonlinear media. Phys. D 196, 90-105 (2004)
[6] Chung Y., Jones C.K.R.T., Schäfer T., Wayne C.E. Ultra-short pulses in linear and nonlinear media. // Nonlinearity, 18. - 2005, P. 1351-1374
[7] P. Kinsler, Phys. Rev. A 81, (2010), 023808.
[8] S. Leble. Nonlinear Waves in Waveguides (Springer, Heidelberg, 1990).
[9] V.V.Belov, S.YU.Dobrokhotov, T.YA.Tudorovskiy. Operator separation of variables for adiabatic problem in quantum and wave mechanic // Journal of Engineering Mathematics (2006)
[10] S. Pitois, G. Millot, S. Wabnitz, Nonlinear polarization dynamics of counterpropagating waves in an isotropic optical fiber: theory and experiments, J. Opt. Soc. Am. B/ Vol. 18, No. 4/ April 2001
[11] M. Kuszner, S. Leble, Directed Electromagnetic Pulse Dynamics: Projecting Operators Method J. Phys. Soc. Jpn. 80 (2011) 024002.
[12] A. A. Perelomova , Projectors in nonlinear evolution problem: acoustic solitons of bubbly liquid, Applied Mathematics Letters, 13 (2000), 93-98; Nonlinear dynamics of vertically propagating acoustic waves in a stratified atmosphere, Acta Acustica, 84(6) (1998), 1002-1006.
[13] A. Perelomova, Development of linear projecting in studies of non-linear flow. Acoustic heating induced by non-periodic sound , Phys. Lett. A 357,2006, 42-47.
[14] M.Kuszner, S.Leble, Ultrashort Opposite Directed Pulses Dynamics with Kerr Effect and Polarization Account Journal of the Physical Society of Japan 83 (2014)
[15] M. Pietrzyk, I. Kanattsikov, and U. Bandelow, "On the propagation of vector ultra-short pulses", J. of Nonlin. Math. Phys. 15, 2, 2008
[16] R.W. Ziolkowski and A. Kipple. Causality and double-negative metamaterials Phys. Rev. E, vol.68, 026615, Aug. 2003
[17] R.W. Ziolkowski and F. Auzanneau. Passive artificial molecule realizations of dielectric materials, J. Appl. Phys., vol.82, pp.3195-3198, Oct. 1997
[18] M. Pietrzyk, I. Kanattsikov, On the Generalized Short Pulse Equation Describing Propagation of Few-Cycle Pulses in Metamaterial Optical Fibers // Theoretical Physics and Its Applications, Moscow, 2013.
[19] D. Ampilogov, S. Leble, General Equation for Directed Electromagnetic Wave Propagation in 1D Metamaterial: Projecting Operator Method. TASK Quarterly, V.20, No.2 (2016)
[20] A. E. Kaplan, Light-induced nonreciprocity, field invariants, and nonlinear eigenpolarizations, Opt. Lett. 8, pp.560- 562 (1983)
[21] Christos Argyropoulos, et. al. Enhanced Nonlinear Effects in Metamaterials and Plasmonics // Advanced Electromagnetics, Vol. 1, No. 1, May 2012, pp. 46-51
[22] Z.Zhaqiao, Q.Hu, Z. Qiao, Multi-soliton solutions and the Cauchy problem for a two-component short pulse system // Nonlinearity, V.30, No.10, 2017

[23] D.Ampilogov, Interaction of orthogonal-polarized waves in 1D-metamaterial // TASK QUARTERLY vol. 21, No 2, 2017, pp. 605-619