Landau Levels in the noncommutative $AdS_2$

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Abstract

We formulate the Landau problem in the context of the noncommutative analog of a surface of constant negative curvature, that is $AdS_2$ surface, and obtain the spectrum and contrast the same with the Landau levels one finds in the case of the commutative $AdS_2$ space.

I. INTRODUCTION

Noncommutative spaces have been of current interest with various motivations, in particular they arise in the framework of M-theory and in interesting settings of string and branes [1], [2], [3] (the bibliography is so vast that we do not attempt comprehensive referencing).

A sector of the study of the physics in noncommutative spaces concerns the exploration of the consequences for the quantum mechanics of one particle [4], [5], [6].

The physics in the noncommutative spaces is closely related to the problem of a charged particle moving on a surface with constant magnetic field giving rise to Landau Levels (see

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for instance [1]). Hence it is interesting to study Landau Levels by comparing the settings of commutative and noncommutative spaces. This research has been carried out for the case of the plane [7], [8], [9], the sphere [7] and the torus [10].

In this paper we consider the Landau Levels problem in the case of a surface of negative constant curvature, that is \( AdS_2 \). The commutative case has been studied in various papers, [11], [12], and also been extended to cover the case of the higher genus Riemann surfaces, which can be realized by a tessellation of \( AdS_2 \) [13], [14]. We may note in passing that, as higher genus Riemann surfaces appear as building blocks of higher orders in string perturbation theory, this provides a further link between \( AdS \) spaces and string theory, besides the celebrated relation with conformal field theories.

We first of all recall in Section II the results on the commutative \( AdS \) surface, by making an explicit derivation, using appropriate complex coordinates and giving the resulting eigenfunctions, eigenvalues and their (infinite) multiplicity. We may also recall that for higher genus Riemann surfaces one gets the same spectrum but with a finite multiplicity dictated by the Riemann-Roch theorem [14].

Then we give the algebraic formulation of the same problem, by expressing the Hamiltonian in terms of the generators of \( SO(2,1) \) and representing \( AdS_2 \) as an embedding of a surface in the flat \((2 + 1)\)-dimensional Minkowski space.

In the case of the Riemann surfaces a quantization condition for the magnetic field naturally emerges. Indeed, the wave function can be regarded as a differential form on the surface (which is in fact the proper object of the Riemann-Roch theorem) and the requirement of definite monodromy for transport along noncontractible loops implies a quantization condition. One can also require some periodicity properties of the wave function in the noncompact case of \( AdS_2 \), like it is common to use periodic boundary conditions in quantum mechanical problems on noncompact spaces, for instance on the plane. The results are actually the same, except one has to assume a quantized value for the magnetic field.

The algebraic formulation of Section II will allow us to properly define the analogous problem in the noncommutative setting, Section IIIA. The commutation relations among the
Minkowski space coordinates are taken to be the ones of the $SO(2,1)$, and the appropriate Casimir is fixed in order to define the embedding in this case, similarly to the construction for the noncommutative sphere done in ref [7]. The resulting setting is described by two commuting $SO(2,1)$ algebras. We have not attempted the construction of noncommutative higher genus Riemann surfaces.

We have first of all to define the Hamiltonian for the noncommutative case: we assume it to be formally identical to the one defined on the commutative surface.

The next issue concerns how to define the constant magnetic field. Here we have studied two options.

In the first one, we fix the two Casimirs of the two commuting $SO(2,1)$ algebras, similarly to what was done in ref [7] for the sphere. With this option the Hamiltonian for the noncommutative case may not be formally the same as for the commutative surface, and the commutative limit may require some care and adjustment of parameters appearing in the Hamiltonian, see ref [7]. Here, we see that this option can be in conflict with the requirement that a universal (commuting or noncommuting) form of the Hamiltonian makes physical sense.

In the second option, we keep fixed one observable among a complete set of mutually commuting ones. This observable is formally identical to the magnetic field defined in the commutative case. In this case, retaining the same Hamiltonian, formally identical to the one defined on the commutative surface, makes always physical sense, and the commutative limit is straightforward.

By using the representation theory we obtain the spectrum in both options, Section IIIB and Section IIIC respectively. This is done in general for all possible representations of the algebra to yield the spectrum. Requiring, in addition, quantization of the eigenvalues in order to explore the possible noncommutative generalization of the features holding for Riemann surfaces, implies retaining only a quite small subset of the levels. Actually, the construction of the Landau Levels on the noncommutative version of the Riemann surfaces remains so far a completely unsolved problem, despite announcements which may have
II. LANDAU LEVELS IN $ADS_2$

We consider a constant magnetic field on $AdS_2$, that is a magnetic field proportional to the curvature. We can describe $AdS_2$ by using complex coordinates $z, \bar{z}$ in the upper half plane $y > 0$ and taking the Poincare’ metric $g_{z\bar{z}} = 1/y^2$:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$  \hfill (2.1)

The relevant covariant derivatives are

$$\nabla = \partial + \frac{B}{z - \bar{z}} \quad \bar{\nabla} = \bar{\partial} + \frac{B}{z - \bar{z}}.$$  \hfill (2.2)

and

$$[\nabla, \bar{\nabla}] = \frac{B}{2y^2}. \hfill (2.3)$$

We take the Hamiltonian as

$$H = -2g^{z\bar{z}}(\nabla\bar{\nabla} + \bar{\nabla}\nabla) - B^2$$

$$= -4g^{z\bar{z}}\nabla\bar{\nabla} + B(1 - B)$$

$$= -4g^{z\bar{z}}\bar{\nabla}\nabla - B(1 + B). \hfill (2.4)$$

Notice that, by taking into account the appropriate measure, the operators $-g^{z\bar{z}}\nabla\bar{\nabla}$ and $-g^{z\bar{z}}\bar{\nabla}\nabla$ are both semipositive definite, and therefore

$$H \geq |B|(1 - |B|). \hfill (2.5)$$

We take $B > 0$, since the case $B < 0$ is obtained by interchanging $z$ and $\bar{z}$.

Consider the eigenvalue problem

$$H\Psi = E\Psi. \hfill (2.6)$$

The lowest eigenstate, i.e. $E = B(1 - B)$, is obtained as a solution of $\bar{\nabla}\Psi_0 = 0$. 

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By defining $\tilde{\Psi}_0 = g_{zz}^{B/2}\Psi_0$ we see that this means $\tilde{\partial}\tilde{\Psi}_0 = 0$. This state is not unique. The different states can be labeled by the eigenvalues of an operator (that commutes with $H$), which we will, in the following, identify with a generator of $SO(2,1)$:

$$J_3 = -\frac{\imath}{2}((1 + z^2)\partial + (1 + \bar{z}^2)\bar{\partial} + (z - \bar{z})B).$$

(2.6)

The explicit form of the set of lowest level eigenstates is

$$\Psi_0^{(n)} = \frac{(z - \bar{z})^B}{(i + z)^{2B}} \left(\frac{-\imath + z}{i + z}\right)^n,$$

(2.7)

corresponding to the eigenvalues $J_3 = B + n$, with $n$ any nonnegative integer.

We observe that under a holomorphic coordinate transformation $z \to z' = \frac{az + b}{cz + d}$ the wave function $\Psi$ transforms like a differential form of the kind $T^{B/2}_{B/2}$. That is, if $z'$ is another local coordinate on the surface and the domain of $z'$ intersect the domain of $z$, we first observe that $g_{zz}dzd\bar{z} = g_{z'z'}d'z'd\bar{z}'$ and that

$$H' = UHU^{-1},$$

with $U = \frac{d\bar{z}'}{dz} \cdot \frac{dz}{d\bar{z}}^{B/2}$. Therefore the wave function in the new coordinates is related to the wave function in the old ones by

$$\Psi' = U\Psi,$$

that is $\Psi(\frac{dz}{d\bar{z}})^{B/2}$ is invariant. It follows that $\tilde{\Psi} = (g_{zz})^{B/2}\Psi$ transforms like a $T_{B}$ form.

Also, the wave functions for the excited levels can be expressed as (we abbreviate, $g \equiv g_{zz}$)

$$\Psi_i^{(n)} = g^{B/2-1}\partial g^{-1} \cdot g^{B/2-2}\partial g^{-2} \cdot \ldots \cdot g^{B/2-l}\partial g^{-l}\Phi,$$

(2.8)

where $\Phi$ is a $T^{B/2}_{B/2-l}$ form. The eigenfunction equation requires $\Phi = (g_{zz})^{B/2}\Phi$, which is a $T_{B-l}$ form, to be holomorphic.

In conclusion the wave functions of the operator $J \circ J = j(1 - j)$ are in correspondence with $T_j$ holomorphic forms.

On a Riemann surface the Riemann-Roch theorem tells us that the dimensionality of the holomorphic $T_j$ forms is $(2j - 1)(h - 1)$ where $h$ is the genus of the surface and $j =$
Therefore, on the Riemann surface this quantity must be an integer. We expect this quantization to generalize to the possible noncommutative version of the Riemann surface.

Further, requiring periodicity for transport along noncontractible loops fixes $j$ to be integer (it can also be half-integer if we only require periodicity up to $\pm$ signs, giving rise to the “spin structures” of the half-forms, which are familiar from perturbative String theory). Periodic boundary conditions are very natural for eigenvalue problems and we can choose to require it even for the $AdS_2$ surface (which can be considered as a limiting case of a very large Riemann surface). Therefore we are led to consider the case of $j$ to be integer (and hence we will restrict $B$ to integer values) and further investigate the noncommutative version of this requirement.

In the language of group theory, considering $j$ integer means considering representations of the group $SO(2,1)$, rather than just of its algebra, see ref. [13] for an illuminating discussion. Thus this requirement is immediately transferred to the noncommutative case, where it implies requiring the quantum numbers labeling the representations to be integers (or half-integers, if we allow for $\pm$ signs). We will contrast the relevance of the more general (non integer) representations of the algebra and the richer set of spectrum it gives with the restricted set applicable for Riemann surfaces etc. in our analysis.

This discrete part of the spectrum, which we will call Landau Levels, comprises the eigenvalues

$$E_j = j(1-j),$$

with $j = B - l$ up to the maximal $l = B - 1$, each having a degeneracy corresponding to the eigenvalues $J_3 = (B - l) + n$, with $n$ any nonnegative integer.

The corresponding wavefunctions are

$$\Psi_l^{(n)} = (\partial - (B/2 - 1)\partial l \partial g_{zz}) \cdot (\partial - (B/2 - 2)\partial l \partial g_{zz}) \cdot \cdots (\partial - (B/2 - l)\partial l \partial g_{zz}) (i + z)^{2l} \Psi_0^{(n)}. \hspace{1cm} (2.10)$$
Besides the above discrete levels, there is a continuum spectrum with nonnegative values for $E$.

The above results can be cast in an algebraic form, by making use of the invariance group of $AdS_2$, that is $SO(2, 1)$. The $AdS_2$ manifold is conveniently described by embedding it in flat Minkowski manifold with coordinates $x_1, x_2, x_3$ with the constraint:

$$x \circ x = x_1^2 + x_2^2 - x_3^2 = -1,$$

(2.11)

where we have defined $V \circ W \equiv V_1 W_1 + V_2 W_2 - V_3 W_3$ for two vectors $V$ and $W$.

The $SO(2, 1)$ generators $J_1, J_2, J_3$ satisfy the commutation relations:

$$[J_1, J_2] = -iJ_3 \quad [J_2, J_3] = iJ_1 \quad [J_3, J_1] = iJ_2,$$

(2.12)

$$[J_1, x_2] = -ix_3 \quad [J_2, x_3] = ix_1 \quad [J_3, x_1] = ix_2.$$

(2.13)

We are considering here the standard commuting operators for $x$, therefore

$$[x_l, x_n] = 0.$$

(2.14)

The relation with the previous formalism in terms of operators in complex coordinates are:

$$x_1 = i\frac{z + \bar{z}}{z - \bar{z}} \quad x_2 = i\frac{1 - z\bar{z}}{z - \bar{z}} \quad x_3 = i\frac{1 + z\bar{z}}{z - \bar{z}},$$

(2.15)

and

$$J_1 = i(z\partial + \bar{z}\bar{\partial})$$

$$J_{3,2} = -\frac{i}{2}((1 \pm z^2)\partial + (1 \pm \bar{z}^2)\bar{\partial} \pm B(z - \bar{z})).$$

(2.16)

We can verify that $x \circ J = -B$ and that $H = J \circ J$, therefore eq.(2.5) tells us that $J \circ J + B(B - 1) \geq 0$.

It is well known [10] that the unitary representations of SO(2,1) algebra are of two kinds: the discrete ones $D_j^\pm$, in which $J \circ J = j(1 - j)$ and $J_3 = \pm(j, j + 1, \ldots, j + n, \ldots)$ with $j \geq 1/2$, (but restricted to positive integer or half integer, if we look for representation of
the group instead) and \( n \) nonnegative integer, and the continuum ones \( C_j \) in which \( J \circ J \) is real positive.

Therefore we find that the Landau Levels, we have obtained correspond to \( D_j^+ \) with \( j \leq B \).

Note that if the surface is more in general described by the constraint \( x \circ x = x_1^2 + x_2^2 - x_3^2 = -r^2 \) (previously \( r = 1 \)), then the metric is \( ds^2 = r^2 \frac{dx^2 + dy^2}{y^2} \) and the Hamiltonian (2.4) is now \( H = r^2 J \circ J \) while \( x \circ J = -rB \). Thus the discrete spectrum is \( E_j = r^2 j(1 - j) \), with the same \( j = B - l \) as for \( r = 1 \).

It is convenient to redefine the Hamiltonian to be

\[
H_0 = J \circ J, \tag{2.17}
\]

for generic \( r \), keeping the same definition \( x \circ J = -rB \), so that the discrete spectrum is always \( E_l = -(B - l)(B - l - 1) \). We will keep the Hamiltonian of eq.(2.17) in the generalization to the noncommutative surface described in the next Section.

III. LANDAU LEVELS IN THE NONCOMMUTATIVE \( ADS_2 \)

A. Definition of the problem

In order to define the noncommutative \( AdS_2 \), we introduce a set of noncommuting coordinates \( R_j \) with the \( SO(2, 1) \) algebra as relevant noncommuting rules:

\[
[R_1, R_2] = -iR_3 \quad [R_2, R_3] = iR_1 \quad [R_3, R_1] = iR_2, \tag{3.1}
\]

\[
[J_1, R_2] = -iR_3 \quad [J_2, R_3] = iR_1 \quad [J_3, R_1] = iR_2, \tag{3.2}
\]

where the \( J_i \) are the \( SO(2, 1) \) generators satisfying the algebra eq.(2.12).

Now, instead of requiring \( x_1^2 + x_2^2 - x_3^2 = -r^2 \) which describes \( AdS_2 \) in the commutative case, we require a fixed negative value for the Casimir \( R \circ R \equiv R_1^2 + R_2^2 - R_3^2 \). We know
from the $SO(2,1)$ representation theory [16] that such a negative Casimir is of the form $R \circ R = r(1 - r)$. Of the two discrete representations ($D^+_r$ distinguished by positive or negative $R_3$) we choose $D^+_r$, and then $R_3 = r, r + 1, \ldots$ and so on.

We still maintain the Hamiltonian to be:

$$H_0 = J \circ J \equiv J_1^2 + J_2^2 - J_3^2, \quad (3.3)$$

as it is formally in the commutative case.

We note that the system is described by two mutually commuting $SO(2,1)$ algebrae, $K_i = J_i - R_i$ and $R_i$.

The relevant decompositions are (see ref. [17] for a useful summary on combining $SO(2,1)$ representations):

$$D^+_k \otimes D^+_r = \sum_{m=0} D^+_j, \quad j = k + r + m, \quad m \text{ integer}, \quad (3.4)$$

and

$$D^-_k \otimes D^+_r = (\sum_{j=|r-k|} D^+_j, \quad j = |r - k| \text{ mod}(1) + n, \quad n \text{ integer}) \oplus \int C_j. \quad (3.5)$$

with $D^+_j$ for $r > k$ and $D^-_j$ for $k > r$.

The other relevant formula is:

$$C_k \otimes D^+_r = (\sum_{j=r+n} D^+_j, \quad n = 0, 1, \ldots) \oplus \int C_j. \quad (3.6)$$

In the commutative case we fixed $B = -(x \circ J)/r$ and studied the spectrum with this additional constraint. Now we must analogously decide what to fix to represent the constant magnetic field.

1) We may follow the philosophy of Nair and Polychronakos ref. [7] and fix the value of the two Casimirs. Since $R \circ R$ is fixed by definition, this amounts to parameterize the magnetic field by the choice of $K \circ K$.

Note that the approach of ref. [7] is such that it allows a redefinition of the Hamiltonian by an overall constant which can be positive or negative depending on the range of the parameters, in particular of the magnetic field.
Our approach is to keep always the definition of the Hamiltonian as in eq. (3.3).

2) Alternatively we may stick to the choice similar to the one in the commutative case, and use the commuting set of observables \( J \circ J, K \circ K, R \circ R, J_3 \) to keep fixed

\[
K \circ K - J \circ J = R \circ R - 2R \circ J.
\]

That is, like in the commutative case, we define and keep fixed \( B \equiv -(x \circ J)/r \). This, we consider, is more appropriate to our definition of the problem, in keeping with the Hamiltonian of eq. (3.3).

In this case the limit \( r \to \infty \) (at fixed \( B \)) is expected to reproduce the physics of the Landau Levels on the commutative \( AdS_2 \) surface: in fact by defining \( x_i \equiv R_i/r \) we get approximate commuting coordinates.

Let us explore the resulting spectrum for both the choices.

Following the discussion of the previous Section, we may require the eigenvalue \( j \) (and also \( r \) for consistency) to be integer or half-integer. This will be probably relevant for the setting of the Landau Levels problem on a noncommutative Riemann surface, which is still to come. Since the derivation is essentially the same, we consider \( j \) to be real as the general case. Requiring integer or half-integer \( j \) would simply imply to discard the levels in which \( j \) is not so; the allowed levels would be much sparser then in the real \( j \) case, also depending on a fine tuning of the values of \( r \) and \( B \).

### B. Case 1

Let us start with \( K \circ K > 0 \). Since \( J_i = K_i + R_i \), the resulting spectrum of the Hamiltonian is obtained from eq. (3.6). We get a continuum nonnegative part of the spectrum \( C_j \) and an unbounded discrete spectrum \( D^+_j \) with \( J \circ J = j(1 - j), j = r, r + 1, \ldots \) up to infinity; therefore the Hamiltonian (3.3) is unbounded from below and above.

If \( K \circ K < 0 \), we have to consider two cases, corresponding to the representations \( D^-_k \) and \( D^+_k \).
For $D_{k}^{-}$, the relevant decomposition is eq. (3.3). We get a nonnegative continuum spectrum $C_{j}$ as well as a finite discrete set of Landau Levels, $D_{j}^{\pm}$, with $J \circ J = -j(j-1)$; $j = |r - k|, |r - k| - 1, ... |r - k| \mod (1)$ and hence bounded from below.

For $D_{k}^{+}$ the relevant decomposition is eq. (3.4), giving thus again an unbounded negative discrete spectrum, and therefore the Hamiltonian (3.3) is unbounded from below.

C. Case 2

Here we do not choose a particular value for $K \circ K$ and therefore the spectrum is composed of various parts, which must be consistent with the magnetic constraint

$$K \circ K - J \circ J = N \equiv -r(r - 1) + 2rB. \quad (3.7)$$

We take $N$ to be positive or negative. There are still several possibilities in the parameter space.

a) Since $N = -r(r - 1) + 2rB$ and we would like to discuss in particular the case $r$ large with $B$ fixed (because in this limit we recover the commutative $AdS_{2}$), we begin by assuming $N < 0$. Let us define $M = -N > 0$: the magnetic constraint reads

$$J \circ J - K \circ K = M \equiv r(r - 1) - 2rB. \quad (3.8)$$

In particular, when both $J \circ J$ and $K \circ K$ are in the discrete part of the spectrum, the constraint reads

$$(k - \frac{1}{2})^2 - (j - \frac{1}{2})^2 = M. \quad (3.9)$$

We analyze different ranges for $M$.

ai) $\sqrt{M} \geq r$: this means $B < -\frac{1}{2}$.

We find that $D_{k}^{+}$ cannot occur because (3.4) is incompatible with the constraint (3.8) which would require $(j - \frac{1}{2})^2 - (k - \frac{1}{2})^2 = -M$ which is impossible since $j > k$. 

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The case $D_k^-$ is possible.

First of all, from (3.3) we can have the continuum $C_j$, with $j$ real positive and $j^2 = M - k(k - 1)$; since $k$ is an arbitrary positive parameter, we get the part of the continuum spectrum for $j \leq M$.

If $k > r$ we can also have a discrete part of the spectrum: in this case from (3.3) we have $D_j^-$. The constraint (3.9) is solved by:

$$j_l = (|\frac{M}{n_l} - n_l| + 1)/2$$

$$k_l = (\frac{M}{n_l} + n_l + 1)/2$$ (3.10)

We can always choose to restrict $n_l \leq \sqrt{M}$. Since from (3.3) we have $j_l = k_l - r - l$ we get $n_l = r + l$, with $l$ a nonnegative integer; this is consistent with $\sqrt{M} \geq r$. Since the minimum (maximum) possible $j_l$ is obtained for the maximum (minimum) possible $n_l$, we get the following discrete part of the spectrum $J \circ J = -j_l(j_l - 1)$:

$$\frac{1}{2} \leq j_l = (\frac{M}{r + l} - r - l + 1)/2 \leq (\frac{M}{r} - r + 1)/2.$$ (3.12)

with $l$ integer restricted by $r + l \leq \sqrt{M}$.

If $r > k$ there is no discrete spectrum for $\sqrt{M} \geq r$.

In fact we would have from (3.3) $j_l = r - k_l - l$; by using (3.10, 3.11) this gives $n_l = \frac{M}{r - 1 - l} \leq M$.

Now $n_l(min) = \frac{M}{r - 1}$ corresponding to $j_l(max) = (r - \frac{M}{r - 1})/2$, whereas $j_l(min) \geq j_l(n_l = \sqrt{M}) = \frac{1}{2}$.

Therefore, this is possible iff $\frac{1}{2} \leq (r - \frac{M}{r - 1})/2$ which implies $r \geq \sqrt{M} + 1$.

Finally, the case $C_k$ is also possible. It gives the part of the continuum spectrum for $j \geq M$, since $j^2 = M + k^2$ with $k$ arbitrary. We do not get a discrete part from it because, from (3.8), this would require $-j(j - 1) = M + k^2$, which is not possible.

Summarizing: for $\sqrt{M} > r$ we get the entire continuum spectrum ($0 \leq j^2 \leq \infty$) and the discrete part of the spectrum reported in eq. (3.12).
a) $r - 1 \leq \sqrt{M} \leq r$: this means $-\frac{1}{2} \leq B \leq \frac{1}{2} - \frac{1}{2r}$.

From the analysis of the case a) we conclude that in this case we have only the continuum spectrum.

a) $\sqrt{M} \leq r - 1$: this means $\frac{1}{2} - \frac{1}{2r} \leq B$ (and also $B < \frac{r-1}{2}$ in order to have $M > 0$).

In this case from the analysis of the case a) we conclude that we have the continuum spectrum, and the following discrete part of the spectrum $J \circ J = -j_l(j_l - 1)$, coming from the the representation $D_k^+$ and $r > k$:

$$\frac{1}{2} \leq j_l = (r - l - \frac{M}{r - 1 - l})/2 \leq (r - \frac{M}{r - 1})/2.$$  \hfill (3.13)

with $l$ integer restricted by $\frac{M}{r - 1 - l} \leq \sqrt{M}$.

Let us now consider the above results in the commutative limit $r \to \infty$ at fixed $B$, looking at the lowest part of the spectrum.

By taking the limit of eqs. (3.12) and (3.13), and keeping the integer $l$ fixed (note that $\frac{M}{r - 1 - l} \sim r - 2B + l$ and that $\frac{M}{r + l} \sim r - 1 - 2B - l$), we find for $|B| > \frac{1}{2}$ the approximate discrete spectrum $J \circ J = -j_l(j_l - 1)$ with:

$$j_l \sim |B| - l$$ \hfill (3.14)

which is indeed the same result as for the commutative AdS$_2$. In the Figs.1,2 we compare the discrete spectrum for the commutative and noncommutative case.

For $|B| < \frac{1}{2}$ we only get the continuum spectrum.
FIG. 1. Plot of the discrete levels $H_0 = -jl(jl-1)$ as a function of $l$ for the case $r = 150$, $B = -30$ (full points), see eq.3.12, compared with the levels of the commutative case (open circles).

FIG. 2. Same as Fig.1 but for the case $r = 150$, $B = 30$, see eq.3.13.

b) Let us turn now for completeness to study the case

$$N \equiv -r(r-1) + 2rB > 0$$

which implies $B$ large in the commutative limit $r \to \infty$.

The magnetic constraint now reads:

$$K \circ K - J \circ J = N > 0.$$  \hspace{1cm} (3.15)
The analysis parallels the one done for the case a). When both \( J \circ J = -j(j-1) \) and \( K \circ K = -k(k-1) \) are in the discrete spectrum this constraint reads:

\[
(j - \frac{1}{2})^2 - (k - \frac{1}{2})^2 = N,
\]

(3.16)

which can be solved by writing:

\[
\begin{align*}
    j_l &= \left( \frac{N}{n_l} + n_l + 1 \right)/2 \\
    k_l &= \left( \frac{N}{n_l} - n_l + 1 \right)/2.
\end{align*}
\]

(3.17, 3.18)

With no loss of generality we have assumed \( n_l \leq N \).

Also here we consider the three ranges of \( N \).

**bi) \( \sqrt{N} > r \).** This means \( B > r - 1/2 \).

Now we can have \( D^+ \) since the decomposition (3.4) is allowed, implying \( n_l = r + l \leq \sqrt{N} \), with \( l \) nonnegative integer.

The maximum \( j_l(max) = \left( \frac{N}{r} + r + 1 \right)/2 \) is obtained for \( n_l(min) = r \), whereas the minimum \( j_l(min) \geq \sqrt{N} + \frac{1}{2} \) corresponds to the maximum possible \( n_l \).

We thus get a discrete part of the spectrum \( J \circ J = -j_l(j_l - 1) \):

\[
\sqrt{N} + \frac{1}{2} \leq j_l = \left( \frac{N}{r + l} + r + l + 1 \right)/2 \leq \left( \frac{N}{r} + r + 1 \right)/2.
\]

(3.19)

with \( l \) nonnegative integer restricted by \( r + l \leq \sqrt{N} \).

Instead \( D^- \) is not allowed: from the decomposition (3.5) we find that all the possibilities are ruled out, since \( D^- \) would imply \( j \leq k - r \) contradicting the constraint (3.13) which gives \( j > k \), and the same constraint (3.15) would require for \( C_j \) that \( j^2 = -k(k-1) - N \) which is nonsense.

As for the last possibility \( D^+_j \), implying \( r > k \), the analysis is slightly longer: from the parametrization (3.17) and (3.18) we get \( n_l = \frac{N}{r + l} \leq \sqrt{N} \) (with nonnegative integer \( l \)) which in turn implies \( \sqrt{N} \leq r - 1 \), which is outside the range of \( N \) considered here.
We can also have $C_k$. The relevant decomposition is eq.(3.6) from which we get the continuum spectrum $C_j$, that is $J \circ J = j^2$, with any value for $j$ since $k$ is arbitrary and $j^2 = k^2 - N$.

Moreover, from eq.(3.6) we also get another part of the discrete spectrum $D_j^+$, that is $J \circ J = -j_l(j_l - 1)$, with $j_l = r + l$, with $l$ nonnegative integer.

The magnetic constraint (3.13) gives now:

$$ (j_l - \frac{1}{2})^2 = N + \frac{1}{4} - k^2 \Rightarrow l = \frac{1}{2} + \sqrt{N + \frac{1}{4} - k^2 - r}. $$

Since $k$ is arbitrary, this gives a possible range $0 \leq l \leq \frac{1}{2} + \sqrt{N + \frac{1}{4} - r}$.

Summarizing, we find for $\sqrt{N} > r$ the continuum spectrum $J \circ J = j^2$ for any $j$, and two parts of the discrete spectrum $J \circ J = -j_l(j_l - 1)$, namely the part described in eq.(3.19) and another part in which

$$ r \leq j_l = r + l \leq \frac{1}{2} + \sqrt{N + \frac{1}{4}}. \quad (3.20) $$

Since $l$ is zero or integer, the two parts do not overlap.

bii) $r - 1 < \sqrt{N} \leq r$. This means $r - 3/2 + 1/2r < B \leq r - 1/2$.

From the analysis of the case bi) we conclude that in this case we have only the continuum spectrum.

biii) $\sqrt{N} \leq r - 1$. This means $r/2 - 1/2 < B \leq r - 3/2 + 1/2r$.

From the analysis of the case bi) we see that $D_k^+$ is not allowed, whereas $D_k^-$ is allowed for $r > k$, giving the discrete spectrum $D_j^+$, while the continuum $C_j$ is forbidden by the constraint (3.13).

The eigenvalues for $j_l$ are of the form of eq.(3.17) with $n_l = \frac{N}{r - 1 - l}$ with the nonnegative integer $l$ bounded by $n_l \leq \sqrt{N}$.
The maximum $j_l$ corresponds to the minimum $n_l$, that is for $l = 0$, and the minimum $j_l$ is obtained from the maximum $n_l \leq \sqrt{N}$ implying $\sqrt{N} + \frac{1}{2} \leq j_l$.

We thus get the discrete spectrum $J \circ J = -j_l(j_l - 1)$ with

$$\sqrt{N} + \frac{1}{2} \leq j_l = \left( \frac{N}{r - 1 - l} + r - l \right)/2 \leq \left( \frac{N}{r - 1} + r \right)/2,$$

with the nonnegative integer $l$ bounded by $l \leq r - 1 - \sqrt{N}$.

Finally we can have $C_k$. From the analysis of the case bi) we see that we do not get here discrete spectrum, but only the continuum part $C_j$, with $j^2 = k^2 - N$ which is always possible for any $j$.

Summarizing, we find that for $\sqrt{N} \leq r - 1$ the continuum spectrum $J \circ J = j^2$ for any $j$, and the discrete spectrum $J \circ J = -j_l(j_l - 1)$ described in eq.$(3.21)$.

After this paper appeared on the net, we received a paper (hep-th/0201070) by B.Morariu and A.P.Polychronakos, which implied a contrast with our results. The present revised version clarifies some points raised by hep-th/0201070.

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