String duality, automorphic forms, and generalized Kac-Moody algebras

Gregory Moore

Department of Physics, Yale University, Box 208120, New Haven, CT 06520, USA

We review some recent work that has been done on the relation of BPS states, automorphic forms and the geometry of the quantum ground states of string compactifications with extended supersymmetry.

1. Introduction

In the past two years it has become clear that there are some interesting relations between string duality, the theory of generalized Kac-Moody algebras (GKM’s), and their associated automorphic forms. These connections are of three kinds:
1. The space of BPS states is an algebra, sometimes it is a GKM.
2. Black hole BPS states are counted by a denominator product for a GKM.
3. Threshold corrections involve automorphic forms similar to those associated to GKM’s.

Unfortunately, the precise relations between these three occurrences are still shrouded in mystery. This talk reviewed these three circles of ideas, emphasizing a point of view we developed in a series of papers with J. Harvey.

2. The algebra of BPS states

In theories of extended supersymmetry the space of BPS states canonically forms an algebra [29,30].

2.1. Definition

Let us recall the definition from [29,30]. We assume that there are exactly conserved charges $Q$ so that

a.) The Hilbert space splits as a sum of superselection sectors: $\mathcal{H} = \bigoplus Q \mathcal{H}^Q$

b.) In each sector there is a Bogomolnyi bound: $E \geq \| Z(Q) \|$. Here $\| Z(Q) \|$ is the invariant norm of the central charge in $\mathcal{H}^Q$.

In each sector there is a distinguished subspace of “BPS states”:

$$\mathcal{H}_{BPS} = \bigoplus Q \mathcal{H}_{BPS}^Q$$

defined by the space of states saturating the bound. Note that with this definition the space of BPS states includes multi-particle (noninteracting) states in addition to one-particle boundstates at threshold.

The algebra structure

$$R : \mathcal{H}_{BPS} \otimes \mathcal{H}_{BPS} \rightarrow \mathcal{H}_{BPS}$$

is defined as follows. We consider the $S$-matrix $S(\psi_1 + \psi_2 \rightarrow F)$ where $F$ is an arbitrary final state. Under analytic continuation of the total center of mass-squared the $S$-matrix has a distinguished pole:

$$S(\psi_1 + \psi_2 \rightarrow F) \sim \frac{Res}{s+ \| Z(Q_1 + Q_2) \|^2}$$

Since

$$E(\vec{p}) \geq \| Z(Q_1) \| + \| Z(Q_2) \| \geq \| Z(Q_1 + Q_2) \|$$

finding the pole requires analytic continuation unless there are boundstates at threshold in the sector $Q_1 + Q_2$. The residue of the pole must be expressed in terms of matrix elements with on-shell states. By charge conservation and the Bogomolnyi bound these states must be the BPS states in the superselection sector $Q_1 + Q_2$. Therefore the residue of the pole “factors through” the space of BPS states:

$$Res = \sum_{\psi_1 \in \mathcal{H}_{BPS}^{Q_1 + Q_2}} \langle F | R | \psi_3 \rangle \langle \psi_1 | R | \psi_1 \otimes \psi_2 \rangle$$

$^2$our signature is $(-, +, \ldots)$ in $d$ spacetime dimensions, and $s$ is the usual Mandelstam variable.
where the sum runs over an orthonormal basis for $\mathcal{H}_{BPS}^{Q_1+Q_2}$. By varying the final state $F$ one can extract the matrix elements $\langle \psi_2^i | R | \psi_1 \otimes \psi_2 \rangle$ and determine $R$, up to unitary rotation in $\mathcal{H}_{BPS}^{Q_1+Q_2}$.

Little is known about the general properties of the algebra. Examples show that it might or might not be commutative, associative, Lie, etc. It is also worth noting that the algebra can depend on the other kinematic parameters, in particular, on the “boost direction” encoded in $t$.

As a simple and explicit example of this procedure one can use the the exact $S$-matrices of $d = 2, N = 2$ integrable systems. We consider the $A_n$ model with superpotential $W = \frac{X^n}{n+1} - X$. The solitons $K_{j,j+r}$ are labelled by $r$ which is an integer modulo $n$. They have mass $m_r = \frac{2\pi}{n+1} \sin \mu r$, with $\mu = \pi/n$. BPS multiplets are in small representations of $N = 2$ supersymmetry and therefore form doublets $(u_r, d_r)$. The $S$-matrix is determined by $2 \rightarrow 2$ scattering and the residue of the $4 \times 4$ matrix for scattering the two-kink state

$$K_{j,j+r}K_{j+r,j+r+s} \rightarrow K_{j,j+s}K_{j+s,j+s+r} \quad (6)$$

factors as $Res = C^1 C$ where $C$ is a certain $2 \times 4$ matrix of rank $2$. For $r + s = 0 \mod n$ the residue vanishes. From this one finds the BPS algebra up to unitary rotation:

$$u_r \cdot u_s = \sqrt{\sin \mu (r+s)} u_{r+s}$$
$$d_r \cdot d_s = 0$$
$$u_r \cdot d_s = \sqrt{\sin \mu s} d_{r+s}$$
$$d_r \cdot u_s = \sqrt{\sin \mu r} d_{r+s} \quad (7)$$

for $r = 1, \ldots, n-1$ while the product vanishes for $r + s = 0 \mod n$. We give this example to illustrate that the algebra makes sense and is computable in at least one example. Note that the algebra structure does not commute with supersymmetry. (This is a virtue, not a problem. The BPS algebra might combine with supersymmetry in an interesting way.)

Two other classes of examples of infinite-dimensional BPS algebras are provided by string compactification. We discuss these in some more detail.

### 2.2. Toroidal compactification of heterotic string

Dabholkar-Harvey (DH) states are the perturbative string BPS states of toroidally compactified heterotic string. They form a subalgebra of the BPS algebra which, in the tree-level approximation, turns out to be a GKM. The residue of a pole in the tree level $S$-matrix is given by the BRST class in the operator product of two colliding on-shell vertex operators. Therefore, denoting by $\Lambda_\theta$ the action of a Lorentz boost required to tune to the BPS pole the product of two (BRST classes of) vertex operators $V_1, V_2$ is simply:

$$\mathcal{R}(V_1 \otimes V_2) \equiv \lim_{z_1 \to z_2} \Lambda_\theta(V_1(z_1, \bar{z}_1))\Lambda_\theta(V_2(z_2, \bar{z}_2)) \quad (8)$$

where the product is projected to BRST cohomology.

It is well-known that the operator product in the BRST cohomology of a chiral vertex operator algebra forms a Lie algebra. (In fact, much more is true: it is related to “BV algebras” and “Gerstenhaber algebras”.) This is “just” the basis of the construction of gauge symmetry in the heterotic string. The above construction also generalizes the construction of DDF operators.

### 2.3. Calabi-Yau algebras

The BPS states arising in compactifications of type II string theory on Calabi-Yau $d$-folds $X$ with $d = 2, 3, 4$ provide another fascinating class of examples of BPS algebras.

In addition to the perturbative $U(1)$ gauge bosons arising from Kaluza-Klein reduction of supergravity fields there are nonperturbative states associated to wrapped D-branes. The lattice of D-brane charges may be identified with the cohomology lattice of $X$. The data specifying a “wrapped Dbrane state” on a supersymmetric cycle $\Sigma \subset X$ includes the data of a “Chan-Paton” vector bundle on $\Sigma$ with connection. The Dbrane charges of the state may be expressed in terms of...

\[\text{In} \text{ } [11] \text{ R. Borchers defined a notion of “generalized Kac-Moody algebras.” The algebra of DH states is a further generalization. In order to avoid the term “generalized generalized Kac-Moody algebras” we reserve the term Borchers algebras for the objects defined in [11].}\]
the characteristic classes of the Chan-Paton bundle. The BPS states may then be associated to the supersymmetric ground states of a supersymmetric Yang-Mills theory. In this way the space of stable BPS states of RR charge $Q$ can be written in the form:

$$\mathcal{H}_{BPS}^Q = H^*(\mathcal{M}(Q)) \otimes \pi$$

(9)

where $\mathcal{M}(Q)$ is a certain moduli space and $\pi$ is a representation of the supertranslation algebra $[53, 10, 54, 31]$. Some aspects of this construction were anticipated in Kontsevich’s work on mirror symmetry $[43]$. An attempt was made in $[52, 30]$ to give a precise definition of $\mathcal{M}(Q)$ but problems remain. According to $[43]$ one should work in the bounded derived category.

2.4. The correspondence conjecture

From the general discussion above we see that BPS algebras in type II compactifications must involve some product structure on the cohomology spaces $H^*(\mathcal{M}(Q))$. An interesting example is provided by the boundstates of $(0, 2, 4)$-branes or of $(0, 2, 4, 6)$-branes in compactification of type IIA string on $K3$-folds and 3-folds, respectively. In this case it was argued in $[52, 30]$ that $\mathcal{M}(Q)$ is given by a certain moduli space of sheaves whose characteristic classes are determined by $Q$. (Essentially, $Q \sim ch(\nu E)$ where $\nu : \Sigma \to X$ is the inclusion and $E$ is the Chan-Paton bundle on $\Sigma$.) In this case there is a natural conjecture, formulated in $[80]$, for the product structure on the cohomology spaces called the correspondence conjecture.

We may motivate the correspondence conjecture as follows. Suppose two Dbranes collide at a point $P$. Their respective Chan-Paton spaces $E_1, E_2$ at $P$ must combine to form a new space $E_3$. However, there are many ways in which this can be done, each corresponds to an exact sequence: $0 \to E_1 \to E_3 \to E_2 \to 0$. The set of such sequences is a Grassmannian. When we replace the Chan-Paton spaces at a point $P$ by Chan-Paton bundles we obtain a correspondence subvariety of $\mathcal{M}(Q_1) \times \mathcal{M}(Q_2) \times \mathcal{M}(Q_3)$. Dbrane groundstates are identified with harmonic forms on $\mathcal{M}(Q)$ and it is natural to conjecture that the matrix element $(\omega_3, R(\omega_1 \otimes \omega_2))$ is simply the overlap integral over the correspondence variety. Several points in this proposal must be clarified before it can be taken to be a precisely formulated conjecture. Some of these have been addressed in ongoing work with J. Harvey and D. Morrison. Some aspects of the enhanced gauge symmetries of $M$-theory compactification on Calabi-Yau 3-folds are successfully captured by the correspondence conjecture $[12]$.

2.5. Mathematical applications

The nicest mathematical applications of BPS algebras are provided by the heterotic/type II dual pairs. As we have seen the heterotic BPS algebras involve (at least, at tree level) vertex operator constructions of affine Lie algebras, Borcherds algebras, and generalizations thereof. On the other side, the type II BPS algebras involve products related to correspondence varieties. These constructions are very closely related to the work of Nakajima $[54]$ (which, of course, provided some motivation for the correspondence conjecture). Indeed, detailed consideration of subalgebras for the dual pair of the heterotic string on $T^4$ and the type IIA string on $K3$ “explains” Nakajima’s geometric constructions of Heisenberg algebras and affine Lie algebras $[80]$. The idea that string duality would lead to a physical explanation of Nakajima’s construction of affine Lie algebras was first published in $[61]$.

The above philosophy suggests that there are extensive generalizations of Nakajima’s work. For example, to any Calabi-Yau threefold we can associate two infinite dimensional algebras exchanged by mirror symmetry. Almost nothing is known about these algebras, with the exception of the K3-fibered Calabi-Yau’s entering in dual pairs. In this case the subalgebras associated to boundstates of $(0, 2, 4)$-branes in the K3 fiber

\footnote{This follows from $[26]$ together with some corrections which will be further discussed in $[43]$. It suggests, as first noted by M. Kontsevich and G. Segal, that conceptually the correct home for D-brane charges is $K$-theory.}
form certain infinite-dimensional algebras which, by string duality, should be related to vertex operator algebras constructed from Dabholkar-Harvey states in $K3 \times T^2$ compactification of heterotic string. These algebras seem closely related to the suggestion in [27] that there are GKM algebras associated to algebraic $K3$ surfaces. A related mathematical application concerns the relation between automorphic forms and Calabi-Yau manifolds. These have been explored in [13,27,37]. As we discuss below, these forms bear some resemblance to automorphic forms associated to GKM’s.

2.6. Physical applications

There are three potential applications of BPS algebras to physics. First, it is an old and haunting idea that the infinite tower of massive string states form some kind of infinite tower of generalized massive gauge bosons for some underlying symmetry which has been spontaneously broken. Turning this intuition into precise mathematical statements has proven rather difficult. Past attempts include a search for some kind of “duality invariant string algebra” or “universal string symmetry” [24,49]. It is worth noting that the BPS algebras associated to Narain compactifications of heterotic string are certainly duality invariant, since they are defined in terms of the physical S-matrix, while the massless subalgebras are always the unbroken gauge algebras at all points on Narain moduli space. This is one reason we believe that these algebras will eventually play a deeper role in the formulation of string theory.

The dreamy musings of the previous paragraph are subject to the sharp criticism that having a strongly broken gauge symmetry is as useless as having no symmetry at all. This is correct in general, however, the systems under study are very special. (For example, they are often related to integrable systems.) Indeed, as we discuss in the remainder of the talk, the properties of BPS states determine the geometry of the moduli space of string vacua, at least for backgrounds with eight supersymmetries. The connection is made through the automorphic functions associated to infinite-dimensional algebras. It is well-known that the characters of affine Lie algebras define automorphic forms for subgroups of $SL(2, Z)$. The work of Borcherds [11,12] and of Gritsenko-Nikulin [27] shows that analogous phenomena occur for generalized Kac-Moody algebras, with $SL(2, Z)$ replaced by higher rank arithmetic groups.

Third, the automorphic forms related to GKM’s have also been proposed as counting functions for counting degeneracies of supersymmetric black holes [18].

3. Automorphic forms and special geometry

Automorphic forms of the kind associated to GKM’s appear in the low energy effective actions of compactified string theories. Particularly interesting examples are the effective couplings

$$\int \frac{1}{g^2_{\text{gauge}}} \text{Tr} F \wedge F$$

and

$$\int \frac{1}{g^2_{\text{gravity}}} \text{Tr} R \wedge R$$

in $d = 4, N = 2$ compactifications of heterotic and type II strings. These cases are accessible, yet nontrivial. The main points which we wish to make about these couplings are:

First, the quantities are determined entirely by the BPS states [29,21,7,51,5]. Indeed they should be viewed as regularized sums over the BPS multiplets:

$$\mathcal{F} \sim \sum_{vm} Z^2 \log Z^2 - \sum_{hm} Z^2 \log Z^2$$

$$\frac{1}{g^2_{\text{gauge}}} \sim \sum_{vm} Q^2 a \log m^2 - \sum_{hm} Q^2 a \log m^2$$

$$\frac{1}{g^2_{\text{gravity}}} \sim \sum_{vm} \log m^2 - \sum_{hm} \log m^2$$

(11)

where $\mathcal{F}$ is the prepotential, $Z$ is the central charge, $Q_a$ is the gauge charge, and $m$ is the mass.

Second, the resulting formulae are ideally suited for checking string duality conjectures. See below.

Third, the quantum corrections in four dimensions always take the form (for the nonholomorphic, effective couplings):

$$\frac{1}{g^2(p^2; z)} = \frac{b}{16\pi^2} \log \frac{M_{\text{Planck}}^2}{p^2} - \log \| \Phi(z) \|^2$$

(12)
where $\Phi(z)$ is an “interesting” function (or section of a line bundle $\mathcal{L}$) over moduli space, for example, an automorphic form of $O(2,n; Z)$, $\| \cdot \|$ is an invariant norm on $\mathcal{L}$, $b$ is the coefficient of the $\beta$-function, and $z$ is a point on moduli space.

3.1. Threshold corrections in heterotic string theory

Compactification of heterotic string theory on $K3$ leads to a very interesting set of automorphic functions. In particular one can consider compactification on $K3 \times T^{b_+}$ where $T^{b_+}$ is a torus of $b_+$ dimensions. The low-energy theory involves a sigma model with target space $\mathcal{M}_{b_+}$ which is the moduli space of vector multiplets. The perturbative corrections to the geometry of $\mathcal{M}_{b_+}$ are governed by integrals of the form $\int_{\mathcal{F}} \frac{d^2 \tau}{\tau^2} F(q, y) \Theta_{b_+, b_-}(\tau, \bar{\tau}; z)$:

$$\hat{F}(z) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau^2} F(q, y) \Theta_{b_+, b_-}(\tau, \bar{\tau}; z)$$

where $\tau$ is the worldsheet modular parameter, $\mathcal{F}$ is the fundamental domain for $PSL(2, \mathbb{Z})$, $y = 3\tau$, $F(q, y)$ is a finite sum of the form $\sum_{\mu>0} y^{-\mu} F_\mu(q)$ where $F_\mu(q)$ are modular forms.

$\Theta_{b_+, b_-}(\tau, \bar{\tau}; z)$ is a certain sum (depending on the threshold correction) over the Narain lattice $\Gamma_{b_+, b_-}$ associated to the flat triples (of metric, 2-form, and Wilson line, $(G, B, A)$) on $T^{b_+}$, and $z \in Gr(b_+, b_-)$ is a point in the Grassmannian determined by $(G, B, A)$.

3.2. The $\Theta$-transform

Integrals of the type (13) have been extensively studied by many authors recently in both the physics and the mathematics literature. The transform from $F(q, y)$ to $\hat{F}(z)$ is sometimes known as a $\Theta$-transform or “theta lift.” The integrals can be done rather explicitly and the formulae for $\hat{F}(z)$ are expressed in terms of a null-reduced lattice $\nu^\perp/\nu$ where $\nu \in \Gamma_{b_+, b_-}$ is a null vector. Physically, a choice of $\nu$ corresponds to a choice of decompactification to one higher dimension. Mathematically, it corresponds to a choice of cusp domain in the Narain moduli space.

As an example we consider $b_+ = 2$. Then there is a well-known tube domain realization: $\mathcal{H}^s \equiv O(s+2,2)/[O(s+2) \times O(2)] \cong R^{s+1,1} + iV^{s+1,1}_+$ where $V_+$ is the forward lightcone. If $F(q) = \sum_{n=-1}^\infty c(n) q^n = q^{-1} + c(0) + \cdots$ is a modular form of weight $-s/2$ then $\Phi(z)$:

$$\hat{F}(z) = -\log \| \Phi(z) \|^2$$

where the product is (roughly) over $r > 0$ in $\nu^\perp/\nu$. Similarly, if we take the transform of a nonholomorphic form such as $E_2 F(q)$ then the result is an infinite sum of polylogarithms:

$$\hat{E}_2 F(z) = \sum_{r>0} \frac{1}{\pi \tau(3\tau)^3} \left[ \sum_{r>0} c(r) \bar{L}_3(e^{2\pi i r z}) + \cdots \right]$$

where $L_3$ is the “Bloch-Ramakrishnan-Wigner” polylogarithm.

Moreover, these integrals behave nicely under the action of invariant differential operators on $\mathcal{H}^{s+1,1}$. For example, we have

$$\Delta_z (s+4) \hat{E}_2 F = 6(s+4) \log \| \Psi(z) \|^2 + \text{const.}$$

The method of evaluation of $\hat{F}(z)$ is straightforward, although the details are a little intricate. Choosing a null vector one uses the Poisson summation formula to convert a subsum in $\Theta$ over vectors in a hyperbolic lattice $\Gamma^{1,1}$ to a sum over a pair of integers ($u_1, u_2$) which - in the language of conformal field theory of periodic bosons on the worldsheet torus - may be regarded

$^6$In general for compactifications using rational conformal field theory one should use vector valued modular forms.

$^7$It also corresponds to a choice of elliptic fibration in the $F$-theory dual.
as winding numbers around the $a, b$-cycles. One
then separates the sum over $(w_1, w_2)$ into a sum
over $\ell = g.c.d.(w_1, w_2)$ and a sum over relatively
prime integers. The latter sum can be used to
unfold the fundamental domain to the strip. The
sum on $\ell$ then becomes an infinite sum of Bessel
functions.

Part of the interest of the integrals (13) is
that they are, manifestly, invariant under the
arithmetic group $O^{b_+ b_-}$. We can thus learn
things about automorphic forms. For example,
for $b_+ = 2$ we see that the integral is almost
holomorphically split. The nonholomorphic term
$\sim \log(3z)^2$ shows that $\Phi(z)$ is in fact an auto-
morphic form of weight $c(0)/2$. The zeroes and
poles of $\Phi(z)$ are easily found since the singulari-
ties of $\tilde{F}(z)$ always arise from the divisors where
BPS states become massless. In this way we can
reproduce the results of [12].

An important point is that the integrals show
chamber dependence. That is $\tilde{F}$ (or some deriva-
tive) is discontinuous across real codimension one
subvarieties of $Gr(b_+, b_-)$. This is the technical
source of chamber dependence of prepotentials
and gauge couplings in 4 and 5 dimensional
heterotic string compactifications.

3.3. Results for heterotic compactifica-
tions

In the case of four-dimensional compactifica-
tions one defines a vector valued modular form
from the trace in the internal superconformal field
theory:

$$i \frac{1}{2} \eta^2 \text{Tr}_R J_0 e^{i \tau J_0} q^H \tilde{q}^R = \sum_i Z_i(q, \tilde{q}) f_i(q)$$

(18)

where $f_i(q) = \sum c_i(x) q^x$ are modular forms for
$\Gamma(N) \subset SL(2, \mathbb{Z})$ and $Z_i(q, \tilde{q}) \sim \sum q^{x_1^2} \tilde{q}^{x_2^2}$ are
sums over cosets of the Narain lattice. Following

9In the example of gauge coupling threshold corrections
these Bessel functions are Green’s functions, as expected
once we identify $\Theta$ with a Schwinger parameter. In four-
dimensional case of $b_+ = 2$ the sum produces the log-
arithmic series $\log[1 - e^{2 \pi i r}]$. Thus, the logarithmic
couplings and trilogarithm prepotentials arise from sums
over the Kaluza-Klein towers. This explains - at least tech-
nically - the relation to the trilogarithms which occur in
N. Nekrasov’s work on five dimensional super Yang-Mills
theory [5].

the one-loop analysis of \[ \frac{1}{2} \sum_{w_1, w_2} \text{g.c.d.}(w_1, w_2) \]

one finds the perturbative prepotential is given by:

$$F^{\text{het}}(\tau_S, z) = \frac{1}{2} \tau_S z^2 + \frac{1}{3!} d_{ABC} z^A z^B z^C + \frac{i^c}{(2\pi)^3} \Delta n$$

$$+ \frac{1}{(2\pi)^3} \sum_{r > 0, i} c_i(-r^2/2) Li_3(e^{2\pi i r z}) + O(e^{2\pi i r z})$$

(19)

where $\tau_S$ is the heterotic axiodil, $\Delta n$ is the
net number of massless vector-multiplets minus
hyper-multiplets, $z$ is a point in the tube domain,
the sum on $r > 0$ is (roughly) a sum on vectors in $v^+ / v$
in the forward light cone, and $d_{ABC}$ is a
chamber-dependent symmetric tensor. Using this
expression and identities such as (17) one finds
expressions such as (12) for the gauge couplings.
Similar results hold for gravitational couplings.

In the case of five-dimensional compactifica-
tions ($b_+ = 1$) one derives piecewise linear gauge
 couplings and piecewise cubic prepotentials similar
to those that have been studied in [5, 20, 36].

The connection between the 4 and 5-dimensional
results can be obtained by taking an appropriate
limit of the moduli. In terms of the standard
moduli $T = 2i R_3 / R_1 U = i R_3 R_1$ for
$T^2$ in $K3 \times T^2$ compactification one finds that
the prepotential and coupling behave as: $F \rightarrow \frac{1}{3!} d_{ABC} z^A z^B z^C$ and $\frac{1}{g} \rightarrow (\tilde{p}, z)$
where $\tilde{p}$ is obtained from the Weyl vector for the infinite
product occurring in (15).

A very curious point is that the $\Theta$-transform
turns out to be closely related to Donaldson in-
variants [45], as formulated in Donaldson-Witten
theory [55]. (Such a connection had been
guessed in [14].) In particular, the linear gauge
couplings studied in [5, 20, 36] correspond to one-
point functions of a 2-observable in the theory of
a Euclidean 3-brane wrapping a Del-Pezzo base of
an elliptically fibered Calabi-Yau. Unfortunately,
it is not clear a priori why there should be a re-
lation between these two quantities.

3.4. Applications to duality

Equation (19) should be compared to the fa-
mous nonperturbatively exact expression (almost
equivalent to the multiple cover formula) for the
prepotential for special Kahler geometry of type
II compactification

\[
\mathcal{F}_{\text{TypeI}} = \frac{N_{4,0} e^{\frac{1}{2} \tau_2 + 1}}{4 \pi i \tau_1} \sum_{r>0} \sum_{n} n(r) L_{23}(e^{2 \pi i r \cdot z})
\]

where \( z \) are flat coordinates in the complexified Kähler cone, the sum \( r > 0 \) is over the subcone of the Mori cone generated by the rational curves, and \( N_{ABC} \) are classical intersection numbers.

Comparison with the previous heterotic results gives concrete and nontrivial evidence for string dual pairs such as those proposed in \([38]\). The vector multiplet \( \tau_S \) is identified with the volume of the base of the IIA K3 fibration \([44,6]\). The nontrivial fact that the heterotic expression, valid of the base of the IIA K3 fibration \([44,6]\). The vector multiplet gives concrete and nontrivial evidence for string and the family probably involves an integral of an Euler class of an “antighost” bundle.

We can turn the above reasoning around. Assuming string duality one can use heterotic computations to make very nontrivial predictions about the geometry of K3 surfaces. For example, the heterotic dual of a simple \( Z_2 \) orbifold studied in \([29]\) involves the Calabi-Yau manifold \( x_1^4 + x_2^4 + x_3 + x_4 + x_5 = 0 \) in \( P_4 \) with K3 fibers \( x_1^2 + x_2 + x_3^2 + x_4 = 0 \) in \( P_3 \). String duality predicts that the holomorphic rational curves in such K3 surfaces should be “counted” \([9]\) by the modular form \( F_6/\eta^{24} \). This should be contrasted with the formula \( 1/\eta^{24} \) which counts rational curves in K3, which are holomorphic in some complex structure compatible with a fixed hyperkähler structure \([6,8]\).

As another example of a (new) result along these lines one can combine the techniques of \([14]\) with the integral formula of \([1\) to calculate the one-loop heterotic formula for the threshold correction \( F_g \). The result is an expression of the form:

\[
F_g = \Re \left[ \sum_{r>0} \sum_{n} n(r) c(r^2/2, t) \right]
\]

\[
\left\{ \mathcal{P}_0 \log(1 - e^{2 \pi i r \cdot z}) + \mathcal{P}_1 \frac{1}{1 - e^{2 \pi i r \cdot z}} + \cdots + \mathcal{P}_{2g-2} \left( \frac{1}{1 - e^{2 \pi i r \cdot z}} \right)^{2g-2} \right\}
\]

(21)

Here \( c(r^2/2, t) \) are coefficients of certain modular forms of weight \( g \) which have the form:

\[
\frac{E_4 E_6}{\eta^{24}} \left( \frac{1}{g!} \left( \frac{\pi^2}{3} \hat{E}_2 \right)^g + \cdots + \frac{2 \zeta(2g)}{g} E_{2g} \right) = \sum c(m, t) q^m y^{-t}
\]

(22)

(\( E_{2k}(r) \) are Eisenstein series of weight \( 2k \), \( \mathcal{P} \) are Laurent polynomials in \((3z), 3(r \cdot z) \) and \( r \cdot z \equiv \Re(r \cdot z) + i \Im(r \cdot z) \)).

According to \([1\) a certain holomorphic function of \( z \) extracted from equation (21) will compute the number of holomorphic genus \( g \) curves in the K3 fiber of the Calabi-Yau in the dual type IIA compactification.

4. Statecounting for black holes

In \([1\] R. Dijkgraaf, E. Verlinde, and H. Verlinde proposed a counting formula for the degeneracies of black holes preserving 1/4 of the supersymmetry of the background IIA/K3 × \( T^2 \), or, equivalently, heterotic/\( T^6 \). Their formula involves an automorphic form of a GKM in an intriguing way.

Specifically, using a computation by T. Kawai \([1\) of a particularly interesting \( \Theta \)-transform which converts the K3 elliptic genus \( \chi_{r,z}(K3) = \sum_{h \geq 0, m \in \mathbb{Z}} c(4h - m^2) e^{2 \pi i h(r \cdot z + m z)} \) to the automorphic form \([22]\):

\[
\Phi(T, U, V) = e^{2 \pi i (T + U + V)} \cdot \Pi_{(k,l,m) \geq 0} \left( 1 - e^{2 \pi i (k T + l U + m V)} \right)^{c(4kl - m^2)}
\]

(23)
which is the denominator product for the GKM based on the Cartan matrix:

$$A = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix},$$

[83] proposed a counting formula for the supersymmetric black holes with (electric, magnetic) charge vectors \((q_c, q_m) \in \mathbb{I}^{6,22} \oplus \mathbb{I}^{6,22};\)

$$d(q_c, q_m) = \int_0^1 dTdUdV \frac{e^{i\pi(q_c^2 T + q_m^2 U + 2q_c \cdot q_m V)}}{\Phi(T, U, V)}$$

(25)

Their argument used ideas about six-dimensional string theories.

Quite generally the elliptic genera of symmetric products in more general situations. Consider the theory of a \(d = 4, N = 2\) supersymmetric vectormultiplet for a compact gauge group \(G\) with Lie algebra \(\mathfrak{g}\). In the semiclassical regime the Coulomb branch is \(\mathfrak{l} \otimes C/W\) where \(\mathfrak{l}\) is the Cartan subalgebra and \(W\) is the Weyl group. The perturbative prepotential is:

$$F_{\text{rigid,1-loop}} = -\frac{1}{8\pi^2} \sum_{\vec{a} > 0} (\vec{a} \cdot \vec{a})^2 \log \left(\frac{(\vec{a} \cdot \vec{a})^2}{\Lambda^2}\right)$$

(27)

where the sum is taken over the positive roots of \(\mathfrak{g}\). Let us compare this with the perturbative prepotential (19). Letting \(M_{\text{st}}\) denote the string mass we can take the limit from local to rigid special Kähler geometry by taking: \(z = z_0 + (\frac{T}{M_{\text{st}}}; 0, 0)\) and letting \(M_{\text{st}} \to \infty\) at fixed \(\vec{a}\), \(z_0 = (\vec{0}; T, U)\). Using

$$Li_3(1 - x) \to -\frac{1}{2} T^2 \log x + O(x^3 \log x)$$

(28)

and suitably renormalizing \(\tau_S\) we recover the expression

$$i F = \tilde{\tau}_S \vec{a}^2 - \frac{1}{8\pi^2} \sum_{\vec{a} > 0} (\vec{a} \cdot \vec{a})^2 \log \left[\frac{2\pi i \vec{a} \cdot \vec{a}}{M_{\text{st}}}\right]$$

(29)

We thus learn that the trilogarithms in (19) should be viewed as stringy deformations of the logarithms of Seiberg-Witten theory [29]. Moreover, the prepotential (20), which is the non-perturbative completion of (19), is exact.\footnote{In contrast to string theory, the exact Seiberg-Witten prepotential does not seem to be usefully expressed in terms of a sum of logarithmic functions. For a related context where this is true see [3].}

This raises the idea [29] that the nonperturbative quantum corrections are completely determined by some algebraic structure, presumably, something like the algebra of BPS states. The sum over rational curves in (20) should be viewed as a sum over positive roots. The numbers of rational curves are root degeneracies. The analog of the complexified Cartan algebra is the complexified Kähler cone. The gauge couplings \(\tau_{IJ}\) are, quite generally, logarithms of “automorphic products” constructed using the geometry of the Calabi-Yau 3-fold and generalizing the denominator products of GKM’s. Speculating further

\footnote{generalizing the famous result of [2] [3] [23]}
an algebraic foundation for determining the geometry of string moduli spaces could lead to classification of string vacua, perhaps along the lines of the finiteness results of [74].

We find the above ideas and speculations attractive, but they clearly suffer from the imprecision that no prescription is given for telling how to sum over BPS states, and with what weighting. With this in mind with J. Harvey we investigated in [31,32] certain gravitational threshold corrections which seem to be particularly closely related to known GKM’s. Unfortunately, the precise relation of these corrections to the BPS algebra remains unclear, although the result of [32] is quite suggestive. This paper shows that the gravitational correction $F_1$ for the dual pair of [23] is determined by the denominator product of a certain generalized Kac-Moody superalgebra studied in [13].

In conclusion, the work we have reviewed has lead to some interesting new results on quantum corrections and automorphic forms. It might lead to important geometrical realizations of affine Lie algebras and generalized Kac-Moody algebras. However, our main goal: showing that new algebraic structures, generalizing Kac-Moody algebras, are fundamental to string duality, remains elusive. This idea has the potential to become extremely useful, but at present there are only hints to support its validity.

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