Supersymmetric black rings and non-linear sigma models

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Abstract: In this paper we investigate the non-linear sigma model arising in the reduction of D = 5 supergravity to D = 3, and present the application of this sigma model to supersymmetric black ring solutions in five-dimensional minimal supergravity. With the ansatz of stationary solutions with $R \times U(1) \times U(1)$ isometry, we obtain a two-dimensional Lagrangian corresponding to geodesic motion of a string-like object on the coset $G_{2(+2)}/SO(4)$, and study the algebra of conserved charges and supersymmetry constraints of supersymmetric black rings. We also obtain the semi-classical wave function of supersymmetric black rings.
The study of black rings in high dimensions has been a subject of great interest over the past few years. As is well known, the horizon topology of black holes in high dimensional spacetime is not unique. The discovery of a new BH phase: a rotating black hole solution with horizon topology $S^1 \times S^2$ and carrying angular momentum along the $S^1$, is called as black ring. It was initiated by Emparan and Reall who focused on the solution of the five-dimensional vacuum Einstein equations [1], then extended to include electric charge [2]. Remarkably, the discovery of a supersymmetric black ring solution of five-dimensional minimal supergravity was introduced in [3]. Several important developments are studied more recently in [4,5]. For the review of black ring solutions, see Refs. [6].

It is known that there exists a variety of solution generation methods which were developed to derive these “black ring” solutions. These methods allow to find solutions possessing a certain number of isometries. After reducing the high-dimensional solutions to three dimensions with the corresponding Killing vectors, one can obtain non-linear sigma models in three dimensions, which are harmonic maps from a 3-dimensional base space to a $N$-dimensional target space $\mathcal{M}$. The target space is isomorphic to the coset $G/H$, with $G$ being the isometry group of $\mathcal{M}$ and $H \subseteq G$ the local isotropy group. In general, the non-linear sigma model can be considering as the effective theory of high-dimensional gravity model. In the context of five-dimensional minimal supergravity with
two commuting Killing vectors: one time-like and one space-like Killing vectors, one of which is hypersurface-orthogonal, it was shown that the dimensional reduction to three dimensions leads to three-dimensional non-linear sigma models coupled to gravity, with the moduli spaces \( G_{2(+2)}/SO(4) \) \([7–9]\) in the case of a Lorentzian three-space. Several important application of this model has been discussed in \([10–16]\).

Recently, Berkooz and Piochin in \([17]\) addressed this non-linear sigma model and further study the stationary solutions of five-dimensional minimal supergravity with the symmetry group of moduli space. Especially, by imposing the ansatz of stationary solutions with \( SU(2) \times U(1) \) isometry, one obtain a one-dimensional Lagrangian corresponding to geodesic motion on the coset \( G_{2(+2)}/SO(4) \). Analyzing the algebra of conserved charges and supersymmetry constraints, the authors also obtain the corresponding semi-classical radial wave functions of black holes. The above work is in the context of BPS black hole solutions, naturally, the authors of \([17]\) also hope to try to extend these methods to include the stationary solutions with \( U(1) \times U(1) \) isometries, which is relevant to black ring solutions.

It is the aim of this article to further study the application of the non-linear sigma model to black ring solutions in five dimensional supergravity. Inspired by \([17]\), we study the five-dimensional black rings in 5D supergravity with \( U(1) \) isometry by dimensional reduction to three dimensions. With the ansatz of stationary solutions with \( R \times U(1) \times U(1) \) isometry, we obtain a two-dimensional Lagrangian with the coordinates \( r \) and \( \theta \), which describes the geodesic motion of the string-like on coset space \( G_{2(+2)}/SO(4) \), and study the algebra of conserved charges and supersymmetry constraints of supersymmetric black rings. Furthermore, by solving the supersymmetry constraints, we also obtain the semi-classical wave function of supersymmetric black rings, which describes that the motion of a string-like object can be represented as a semi-classical wave function.

This paper is organized as follows. In the next section, we briefly review the supersymmetric solutions of \( \mathcal{N} = 1 \) supergravity and specialized to the case of black rings in the Gibbons-Hawking spaces. In section 3 we give a brief review about the non-linear sigma model arising in the reduction of \( D = 5 \) supergravity to \( D = 3 \), and present the application of this sigma model to \( R \times U(1) \times U(1) \) symmetric solutions in five-dimensional minimal supergravity. In section 4 we compute the conserved charges and supersymmetry constraints of supersymmetric black rings, and obtain the semi-classical wave function of supersymmetric black ring. The last section is devoted to discussions.

2. Supersymmetric black rings in \( D = 5 \) , \( \mathcal{N} = 1 \) supergravity

In the present section we briefly present a review of the five dimensional low energy supergravity theory. From the view of M-theory, the compactifications of M-theory on a Calabi-Yau manifold \( CY_3 \) lead to the five dimensional supergravity coupled to an arbitrary number of vector multiplets. This model is usually studied in the context of real or very special geometry. Relevant references can be found in \([18–21]\). We will adopt the same conventions as \([22]\).

At the two-derivative level, the bosonic part of \( \mathcal{N} = 1 \ D = 5 \) ungauged supergravity
coupled to \(n-1\) abelian vector multiplets with scalars \(\phi^i, i = 1, ..., n-1\), is read as

\[
S = \frac{1}{16\pi G_5} \int R \ast 1 - G_{IJ} \left( F^I \wedge \ast F^J + dX^I \wedge \ast dX^J \right) - \frac{C_{IJK}}{6} F^I \wedge F^J \wedge A^K,
\]

which the two-form field strength \(F^I = dA^I\), the real scalars \(X^I = X^I(\phi^i), I, J, K = 1, ..., n\), are defined by the cubic equation: \(\frac{1}{6} C_{IJK} X^I X^J X^K = 1\) and the constants \(C_{IJK}\) are the topological intersection numbers and symmetric on \(IJK\). The metric \(g_{ij}\) on the scalar manifold with the coordinates \(\phi^i\) is given by

\[
g_{ij} = G_{AB} \partial_i X^I \partial_j X^J,
\]

where the notation \(\partial_i = \frac{\partial}{\partial \phi^i}\). It is convenient to define

\[
X_I \equiv \frac{1}{6} C_{IJK} X^J X^K, \Rightarrow X_I X^I = 1,
\]

and so

\[
X^I \partial_i X_I = \partial_i X^I X_I = 0.
\]

Thus one can express \(G_{IJ}\) in terms of \(X_I\) through the equations:

\[
G_{IJ} = \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K.
\]

Using the above Eqs. (2.3) and (2.5), one can follows that the relation between the derivative of the special coordinate \(X_I\) and that of the dual coordinate \(X^A\) is given by

\[
X_A = \frac{2}{3} G_{AB} X^B, \quad X^A = \frac{3}{2} G^{AB} X_B,
\]

and

\[
\partial_i X_A = -\frac{2}{3} G_{AB} \partial_i X^B, \quad \partial_i X^A = -\frac{3}{2} G^{AB} \partial_i X_B.
\]

### 2.1 Time-like case and BPS equations

We are interested in solutions preserving some supersymmetry. It is well known that in general a spacetime symmetry can be represented by a generating Killing vector field. To obtain the stationary solutions, we need find the corresponding Killing vector. All supersymmetric solutions of five-dimensional supergravity \([23, 24]\) imply the existence of a non-spacelike Killing vector field. By assuming that in a region the Killing vector field \(V = \partial/\partial t\) is time-like, the line element of five dimensions and field strengths \(F^I\) can be written as

\[
ds_5^2 = -f^2 \left( dt + \omega \right)^2 + f^{-1} ds_{M_4}^2, \quad F^I = d \left[ (f X^I dt + \omega) \right] + \Theta^I,
\]

where \(M_4\) is a four-dimensional hyper-Kähler manifold. \(f\) is a scalar function harmonic on \(M_4\), \(\omega\) and \(\Theta^I\) are a 1-form and closed self-dual 2-forms, respectively. In additional of the supersymmetry transformation of the bosonic fields, the scalar \(f\), one-forms \(\omega\) and \(\Theta^I\) on \(M_4\) are given by the following equations named as “BPS” equations:

\[
\left( \Theta^I \right)^- = 0, \quad \Delta_{M_4} (f^{-1} X_I) = \frac{1}{6} C_{IJK} \Theta^J \cdot \Theta^K, \quad (d \omega)^+ = -\frac{3}{2} f^{-1} X_I \Theta^I,
\]

here, the operator \(\Delta_{M_4}\) is the Laplacian. We use the superscripts “±” to denote the self-dual and antiself-dual part with respect to the base \(M_4\), and for 2-forms \(\alpha\) and \(\beta\) on \(M_4\) we define \(\alpha \cdot \beta = \alpha^{mn} \beta_{mn}\), with indices raised by the matrix \(h^{mn}\) on \(M_4\).
2.2 Black rings solutions in Gibbons-Hawking base spaces

To further analyze the solutions of the BPS equations \(2.9\) in more detail, we will concentrate on the so-called Gibbons-Hawking base spaces. It has been shown that such base manifold admits the existence of a Killing vector which can preserves the hyper-Kähler structure. Assuming the existence of the Killing vector \(\partial_5\), the metric of the base space \(\mathcal{M}_4\) in the Gibbons-Hawking coordinates is given by

\[ ds^2_{\mathcal{M}_4} = H^{-1}(dx^5 + \chi)^2 + H \delta_{ij}dx^i dx^j, \]  

(2.10)

where \(\chi = \chi_i dx^i\) \((i,j = 1, 2, 3)\), \(H\) and \(\chi\) are independent of \(x^5\) and can be solved explicitly, \(\chi\) is determined by \(\nabla \times \chi = \nabla H\), which implies that \(H\) is a harmonic function on the Euclid space \(\mathbb{E}^3\). We introduce one-forms \(\eta^I\) and \(\Theta^I = d\eta^I\). It is convenient to set

\[
\omega = \omega_5(dx^5 + \chi) + \omega_4, \tag{2.11a}
\]

\[
\eta^I = \eta_5^I(dx^5 + \chi) + \eta_4^I, \tag{2.11b}
\]

where \(\omega_4 = \omega_4 dx^i\), \(\eta_4^I = \eta_4^I dx^i\). Solving the BPS equations, we obtain a general supersymmetric solution in terms of \(2n + 2\) harmonic functions \(H\), \(K^I\), \(L_I\) and \(B\) on \(\mathbb{E}^3\) \[4\]:

\[
\nabla \times \omega_4 = H\nabla B - B\nabla H + \frac{3}{4}(L_I \nabla K^I - K^I \nabla L_I), \tag{2.12}
\]

\[
\omega_5 = -\frac{1}{48}H^{-2}C_{IPQ}K^IK^PK^Q - \frac{3}{4}H^{-1}L_I K^I + B, \tag{2.13}
\]

\[
f^{-1}X_I = \frac{1}{24}H^{-1}C_{IPQ}K^IK^PK^Q + L_I. \tag{2.14}
\]

It is well known that \(H\) determines the Gibbons-Hawking base, such as three examples of Gibbons-Hawking metrics: flat space \((H = 1)\) or \(H = 1/|x|\), Taub-NUT space \((H = 1 + 2M/|x|)\) and the Eguchi-Hanson space \((H = 2M/|x| + 2M/|x - x_0|)\) \[23\]. In order to get the black ring solution with the horizon topology of \(S_2 \times S_1\), it is convenient to take the base space \(\mathcal{M}\) to be flat space \(\mathbb{E}^4\) with metric

\[ ds^2_{\mathbb{E}^4} = H^{-1}(d\psi + \chi)^2 + H(dx^2 + r^2[\theta^2 + \sin^2 \theta d\phi^2]), \tag{2.15} \]

where \(H = 1/|x| \equiv 1/r\) and \(\chi = \cos \theta d\phi\), which satisfies \(\nabla \times \chi = \nabla H\). The range of the angular coordinates are \(0 < \theta < \pi\), \(0 < \phi < 2\pi\) and \(0 < \psi < 4\pi\). Thus, the multi-black rings solutions is given by \[4\]

\[
K^I = \sum_{i=1}^{M} q^I_i h_i,
\]

\[
L_I = \lambda_I + \frac{1}{24} \sum_{i=1}^{M} (Q_{Ii} - C_{IJK}q^J_i q^K_i) h_i,
\]

\[
B = \frac{3}{4} \sum_{i=1}^{M} \lambda_I q^I_i - \frac{3}{4} \sum_{i=1}^{M} \lambda_I q^I_i |x_i| h_i, \tag{2.16}
\]
where $h_i$ are harmonic functions in $E^3$ centred at $x_i$, $h_i = 1/|x - x_i|$, and $Q_{Ii}$, $q^I_i$ and $\lambda_I$ are constants.

Next, let us consider the special case of minimal $\mathcal{N} = 1$ supergravity in five dimensions, which was first constructed in [3] and will be main background in our subsequent analysis. In this model, a single supersymmetric black ring sitting at $x_0 = (0, 0, -R)$, a distance $R$ along the negative $z-$axis of the three-dimensional space, has

$$f^{-1} = H^{-1}K^2 + L = 1 + \frac{Q - q^2}{4\Sigma} + \frac{r q^2}{4 \Sigma^2},$$
$$\omega_5 = H^2K^3 + \frac{3}{2} H^{-1}KL + M = -\frac{r^2 q^3}{8 \Sigma^3} - \frac{3r q}{4 \Sigma} (1 + \frac{Q - q^2}{4 \Sigma}) + \frac{3q}{4} - \frac{3q R}{4 \Sigma},$$
$$K = -\frac{q}{2 \Sigma}, \quad L = 1 + \frac{Q - q^2}{4 \Sigma}, \quad B = \frac{3q}{4} - \frac{3q R}{4 \Sigma}. \quad (2.17)$$

In addition, the gauge potential are given by

$$\nabla \times \hat{\eta}_4 = -\nabla (H \eta_5), \quad \eta_5 = -\frac{r q}{2 \Sigma} + \frac{q}{2}, \quad (2.18)$$

here $\Sigma = |x - x_0| = \sqrt{r^2 + R^2 + 2Rr \cos \theta}$. Noticed that in this special case we can drop the (sub-)superscripts “$I$” in (2.10).

3. Non-linear sigma model and $R \times U(1) \times U(1)$ symmetric solutions

In this section, first, we will give a brief review about the non-linear sigma model arising in the reduction of $D = 5$ supergravity to $D = 3$. This is the starting point of that we can present the application of this sigma model to supersymmetric black ring solutions in five-dimensional minimal supergravity. In order to obtain a non-linear sigma model, we proceed in the following two steps:

(i) Reducing the five-dimensional space-time to four-dimensional Euclidean space. Assuming the existence of a time-like Killing vector $\partial_t$, the five-dimensional metric and gauge fields can be taken the forms of (2.8). One can compactify along time direction and reduce to 4D spaces. This leads to four-dimensional Euclidean (instead of Minkowskian) $\mathcal{N} = 2$ supergravity, coupled to $n + 1$ vector multiplets, which has been studied in [25].

(ii) Further reduction to three dimensions with a space-like Killing vector $\partial_5$. With this ansatz, four-dimensional hyper-Kähler base $ds^2_{M_4}$ can be written as the form of (2.10), and the gauge fields are read as (2.11). The equations of motion for the gauge fields $\hat{\eta}_4^A$ and $\chi$ can allow to define the dual scalars $\tilde{\eta}_4^A$ and $\sigma$. The dual scalars for $\hat{\eta}_4^A$ and $\chi$ are defined by\(^1\):

$$d\tilde{\eta}_4^A = ie^{2U} (Im \mathcal{N})_{\Lambda \Sigma} \star (d\eta_4^\Sigma + \eta_5^\Sigma d\chi) + (Re \mathcal{N})_{\Lambda \Sigma} d\eta_5^\Sigma, \quad (3.1a)$$
$$d\sigma = ie^{4U} \star d\chi + (\eta_5^A d\eta_4^A - \tilde{\eta}_4^A d\eta_5^A), \quad (3.1b)$$

\(^1\)Note that here the reduction is from four-dimensional Euclidean space to three dimension. Considering analytic continuation, we need perform an wick rotation $\eta_5^A \rightarrow i\eta_5^A$. So the dual equations have some difference comparing with the equations in the appendix of [13].
where $e^{2U} = H^{-1}$, which is defined as harmonic on the Euclid space $\mathbb{E}^3$ in (2.10), $N_{\Lambda\Sigma}$ is complex symmetric matrix on symplectic sections. Thus, one reduces the five-dimensional Lagrangian to three dimensions and obtains $N = 4$ supergravity coupled to a non-linear sigma model [26]

$$\mathcal{L}_3 = -\frac{1}{2} R - \frac{1}{2} G_{ab} \partial_i \varphi^a \partial^i \varphi^b, \quad (3.2)$$

here, $i = 1, 2, 3$ and the target space with the coordinates $\varphi^a = \{U, z^I, \bar{z}^I, \eta_5^{\Lambda}, \bar{\eta}_5^{\Lambda}, \sigma\}$, which are the moduli fields, and $G_{ab}$ is the moduli space metric. 

So far we have reduce the $D = 5$ $N = 1$ supergravity to $D = 3$ gravity coupled to a non-linear sigma model with the $R \times U(1)$ isometry of the stationary solutions. In order to study the black ring solutions, one need impose an extra $U(1)$ group of isometries to supersymmetric stationary solutions, and further reduce to two dimensions with the other space-like Killing vector $\partial \phi$. Taking the base space $\mathcal{M}_4$ to be flat space $\mathbb{E}^4$, one can obtain three-dimensional metric as

$$ds_3^2 = N^2(\rho, \theta) dr^2 + r(\rho, \theta)^2 [d\theta^2 + \sin^2 \theta d\phi^2], \quad (3.3)$$

with this metric, the stationary solutions have a isometry group $R \times U(1) \times U(1)$. Upon reduction along the $\phi$ direction, the Lagrangian is written as

$$\mathcal{L}_2 = \frac{2 \sin \theta}{N^2 r^2} \left[ -N^3 r_\rho^2 + r N^3 \csc \theta \left( \cos \theta r_\theta + \sin \theta r_{\theta \theta} \right) 
+ r^2 N \left( -N^2 + N \cot \theta N_\theta + N N_{\theta \theta} + r_\rho^2 \right) 
+ r^3 \left( -2 r_\rho N_\rho + 2 N r_{\rho \rho} \right) \right] - \frac{\sin \theta r^2}{N} g_{ab} \varphi^a_{\rho} \varphi^b_{\rho} + N \sin \theta g_{ab} \varphi^a_{\theta} \varphi^b_{\theta}. \quad (3.4)$$

Thus we obtain the two-dimensional effective Lagrangian with the coordinates $\rho$ and $\theta$, which describes the geodesic motion of the string-like object on coset space. Remarkably, we can observe that effective Lagrangian (3.4) is more complicated than the case of $R \times U(1) \times SU(2)$ symmetric solutions in [17].

3.1 Algebra structure of coset space

In the following part of our paper, we will study a more interesting model: minimal $N = 1$ supergravity in five dimensions. In this background, we reduce the theory down to three dimensions following the above method, and obtain the corresponding moduli space, which is the coset $G_{2(1)+2}/SL(2,R) \times SL(2,R)$. Relevant references can be found in [7–9]. Following the conventions of [17,29], the coset space $G_{2(1)+2}/SL(2,R) \times SL(2,R)$ has metric:

$$ds_{M_{2d}}^2 = \epsilon_{a\beta} \epsilon_{AB} V^{aA} \otimes V^{\beta B} = u \bar{u} + v \bar{v} + e \bar{e} + E \bar{E}, \quad (3.5)$$

with the quaternionic vielbein [31]

$$V^{aA} = \begin{pmatrix} \bar{u} & \bar{v} \\ \bar{e} & \bar{E} \\ E & e \\ v & u \end{pmatrix}, \quad (3.6)$$
where $\epsilon_{\alpha\beta}$ and $\epsilon_{AB}$ are the anti-symmetric tensors, $\alpha = 1, 2$ and $A = 1, 2, 3, 4$, with one-form entries defined as [29]

\begin{align}
  u &= \frac{e^{-U}}{2\sqrt{2} \tau^3} \left( d\tilde{\zeta}_0 + \tau d\tilde{\zeta}_1 + 3 \tau^2 d\zeta_1 - \tau^3 d\zeta_0 \right), \\
  v &= dU - \frac{i}{2}e^{-2U} (d\sigma - \zeta_0 d\tilde{\zeta}_0 - \zeta_1 d\tilde{\zeta}_1 + \tilde{\zeta}_0 d\zeta_0 + \tilde{\zeta}_1 d\zeta_1), \\
  e &= -\frac{\sqrt{3}}{2\tau^2} d\tau, \\
  E &= -\frac{e^{-U}}{2\sqrt{6} \tau^3} \left( 3d\tilde{\zeta}_0 + d\tilde{\zeta}_1 (\tau + 2\bar{\tau}) + 3\bar{\tau} (2\tau + \bar{\tau}) d\zeta_1 - 3\tau \bar{\tau} d\zeta_0 \right),
\end{align}

where the bar denotes complex conjugate and $\tau \equiv \phi^1 + if \equiv \tau_1 + i\tau_2$.

After reduction to 3D non-linear sigma model with the coset $G_{2(+2)}/SL(2, R) \times SL(2, R)$ in the context of minimal $\mathcal{N} = 1$ supergravity, one can further perform reduction to two dimensions along the $\phi$ direction. The two-dimensional Lagrangian (3.4) is written as

\begin{align}
  \mathcal{L}_2 &= \frac{2 \sin \theta}{N^2 \tau^2} \left[ -N^3 r_\theta^2 + r N^3 \csc \theta (\cos \theta r_\theta + \sin \theta r_{\theta\theta}) \\
  &+ r^2 N (-N^2 + N \cot \theta N_\theta + N N_{\theta\theta} + r_\theta^2) \\
  &+ r^3 (-2r_\rho N_\rho + 2Nr_{\rho\rho}) - \frac{\sin \theta r^2}{N} \left( u \bar{u} + v \bar{v} + e \bar{e} + E \bar{E} \right)_\rho \\
  &+ N \sin \theta \left( u \bar{u} + v \bar{v} + e \bar{e} + E \bar{E} \right)_\theta,
\end{align}

where the terms $(u \bar{u} + v \bar{v} + e \bar{e} + E \bar{E})_\rho$ and $(u \bar{u} + v \bar{v} + e \bar{e} + E \bar{E})_\theta$ correspond to the values along $\rho$ and $\theta$ direction respectively. Thus, we obtain a two-dimensional Lagrangian with thecoordinates $\rho$ and $\theta$, which can describes the geodesic motion of the string-like object on coset space $G_{2(+2)}/SO(4)$, comparing with one-dimensional effective Lagrangian [17] describing the geodesic motion of a particle.

### 3.2 Killing vectors and conserved currents

Now, we study the algebra of conserved charges with the $R \times U(1) \times U(1)$ isometry of the stationary solutions. It is well known that in the case of the black hole solution, the conserved charges correspond to Killing vectors, which have been computed in [17]. By replacing the operator $\partial_\varphi$ by the momentum $p_\varphi$ conjugate to $\varphi^\rho$, we can obtain the conserved charges associated with the corresponding Killing vector. Such methods were first proposed in [27], and later extended to more models in [17, 28–30]. However, for the stationary solutions with the $R \times U(1) \times U(1)$ isometry, following the same method, the results associated with the corresponding Killing vectors are dependent on the angular coordinate $\theta$. We know that the conserved currents correspond to the Killing vectors, which usually depend on some angular coordinates. In order to obtain the total conserved charges of the model, we need integrate the currents over the angular coordinates.

Next, following the method in [17], we give more detail on how to compute the conserved charge of the stationary solutions with the $R \times U(1) \times U(1)$ isometry. Considering
the Lagrangian (3.4), we can obtain the conjugate momentum $p_{\varphi^0}$ of $\varphi^0$:

$$p_U = 4r^2 \sin \theta \frac{U}{N}, \quad p_{\tau_1} = 3r^2 \sin \theta \frac{\tau_1}{N}, \quad p_{\tau_2} = 3r^2 \sin \theta \frac{\tau_2}{N},$$

$$p_\sigma = \frac{r^2 e^{-4U} \sin \theta}{N}(d\sigma + \tilde{\tau}_I d\zeta_I - \zeta^I d\tilde{\tau}_I), \quad (3.9a)$$

$$p_{\zeta_0} = \frac{r^2 e^{-2U} \sin \theta}{2N\tau_2^3} \left(4d\tilde{\zeta}_0 + 2(\tau + \bar{\tau})d\bar{\zeta}_1 + 3(\tau + \bar{\tau})^2 d\zeta^1 - \frac{1}{2}(\tau + \bar{\tau})^3 d\zeta^0\right) - \zeta^0 p_\sigma,$$

$$p_{\zeta_1} = \frac{r^2 e^{-2U} \sin \theta}{2N\tau_2^3} \left(2(\tau + \bar{\tau})d\tilde{\zeta}_0 + \frac{2}{3}(\tau^2 + \bar{\tau}^2 + 4\tau \bar{\tau})d\tilde{\zeta}_1ight) + \frac{1}{2}(\tau^3 + \bar{\tau}^3 + 11\tau \bar{\tau}^2 + 11\tau^2 \bar{\tau})d\zeta^1 - \frac{1}{2}(\tau + \bar{\tau})^3 d\zeta^0 - \zeta^0 p_\sigma,$$

$$p_{\zeta_0} = \frac{r^2 e^{-2U} \sin \theta}{2N\tau_2^3} \left(-\frac{1}{2}(\tau + \bar{\tau})^3 d\tilde{\zeta}_0 - \tau \bar{\tau}^2 d\tilde{\zeta}_1\right) + 4\tau^3 \bar{\tau}^3 d\zeta^0 - 6(\tau \bar{\tau})^2 (\tau + \bar{\tau})d\zeta^1 + \tilde{\zeta}_0 p_\sigma,$$

$$p_{\zeta_1} = \frac{r^2 e^{-2U} \sin \theta}{2N\tau_2^3} \left(3(\tau + \bar{\tau})^2 d\tilde{\zeta}_0 + \frac{1}{2}(\tau^3 + \bar{\tau}^3 + 11\tau \bar{\tau}^2 + 11\tau^2 \bar{\tau})d\tilde{\zeta}_1\right) + 6(\tau^3 \bar{\tau} + \bar{\tau}^3 \tau + 4\tau^2 \bar{\tau}^2)d\zeta^1 - 6(\tau^2 \bar{\tau}^3 + \tau^2 \bar{\tau}^3)d\zeta^0 + \tilde{\zeta}_1 p_\sigma. \quad (3.9b)$$

It is well known that the Killing vectors associated to the right-action of $G_{2(+2)}$ can generate the Lie algebra of $G_2$, and the 14-dimensional Lie algebra of $G_2$ consists of the commutation relations. For details, see [29]. We replace the operator $\partial_{\varphi^0}$ by the conjugate momentum $p_{\varphi^0}$ and rewrite the fourteen generators of $G_2$ as

$$E_k = p_\sigma, \quad E_{\rho^0} = p_{\zeta_0} - \zeta^0 p_\sigma, \quad E_{\varphi^0} = -p_{\zeta_0} - \tilde{\zeta}_0 p_\sigma,$$

$$E_{\rho^1} = \sqrt{3}(p_{\zeta_1} - \zeta^1 p_\sigma), \quad E_{\varphi^1} = \frac{1}{\sqrt{3}}(-p_{\zeta_1} - \tilde{\zeta}_1 p_\sigma),$$

$$H = -p_U - 2\sigma p_\sigma - \zeta^0 p_{\zeta_0} - \tilde{\zeta}_0 p_{\tilde{\zeta}_0} - \zeta^1 p_{\zeta_1} - \tilde{\zeta}_1 p_{\tilde{\zeta}_1}, \quad (3.10a)$$

$$Y_+ = \frac{1}{\sqrt{2}}(p_{\tau_1} + \zeta^0 p_{\tilde{\zeta}_1} - 6\zeta^1 p_{\tilde{\zeta}_1} - \zeta^1 p_{\zeta_0}),$$

$$Y_0 = -\frac{1}{2}(2\tau_1 p_{\tau_1} + 2\tau_2 p_{\tau_2} - 3\zeta^0 p_{\zeta_0} + 3\zeta^0 p_{\zeta_0} - \zeta^1 p_{\zeta_1} + \tilde{\zeta}_1 p_{\tilde{\zeta}_1}),$$

$$Y_- = \frac{1}{3\sqrt{2}} \left(6p_{\tau_1 \tau_1} \tau_1 + 3p_{\tau_1} \left(\tau_1^2 - \tau_2^2\right) + 9p_{\zeta_1} \tilde{\zeta}_0 - 9p_{\zeta_0} \zeta^1 + 2p_{\zeta_1} \tilde{\zeta}_1\right). \quad (3.10b)$$

The other negative roots are too bulky to display. It is obvious to see that the above Killing vectors depend on not only the radial coordinate $r$ but also the angular coordinate $\theta$. Noticed that in the case of the solution of black holes [17], the Killing vectors are independent of $\theta$. Now, in the background of the stationary solutions with the $R \times U(1) \times U(1)$ isometry, actually, the Killing vectors we obtain in Eqs.(3.10) correspond to the conserved currents. These currents describe the density of on the two-dimensional world sheet of a string-like object. In order to obtain the conserved charges independent of the coordinates, we should integrate these currents over the angular coordinate $\theta$, i.e. a circle $S^1$ as the contour.
4. Supersymmetric black rings

In this section, we have an application of the above results to supersymmetric black ring in minimal D=5, \( \mathcal{N} = 1 \) supergravity.

4.1 Conserved charges

In the last part of the section two, we have introduced the solution of supersymmetric black ring. By making use of the result in (2.17) and (2.18), we can write the coordinates \( \varphi^a \) of the target space of \( \sigma^- \) model as:

\[
N^2 = H = \rho^{-1}, \quad a^2 = b^2 = \rho, \quad \tau_2 = -i\tau_1 = f = \left(1 + \frac{Q - q^2}{4|x - x_0|} + \frac{\rho q^2}{4|x - x_0|^2}\right)^{-1},
\]

\[
\zeta^0 = \frac{1}{\sqrt{2}} \left(-\frac{\rho^2 q^3}{8|x - x_0|^3} - \frac{3\rho q}{4|x - x_0|}(1 + \frac{Q - q^2}{4|x - x_0|}) + \frac{3q}{4} - \frac{3qR}{4|x - x_0|}\right),
\]

\[
\zeta^1 = \frac{1}{\sqrt{2}} \left(-\frac{\rho q}{2|x - x_0|} + \frac{q}{2}\right),
\]

where \( Q \) is the electric charge, equal to the ADM mass by the BPS condition, \( q \) is the dipole charge. In order to obtain the remaining coordinates of the non-linear sigma model, one should solve the dual equations (3.1). In case of the black ring solutions, we can find the \( \tilde{\zeta}_0 \) and \( \tilde{\zeta}_1 \) always take constant values by solving Eq. (3.1a). Moreover, analyzing the connection between Eqs. (3.1a) and (3.9a), we can get \( E_k = -i \sin \theta \).

Making use of the above results to solve Eq. (3.11), we can obtain the remaining coordinate of the non-linear sigma model,

\[
\sigma = -i\rho - \frac{1}{\sqrt{2}i} \zeta_0 \left(-\frac{\rho^2 q^3}{8|x - x_0|^3} - \frac{3\rho q}{4|x - x_0|}(1 + \frac{Q - q^2}{4|x - x_0|}) - \frac{3qR}{4|x - x_0|}\right)
\]

\[
- \frac{1}{\sqrt{2}} \tilde{\zeta}_1 \left(-\frac{\rho q}{2|x - x_0|}\right),
\]

It is obvious to see that the eight coordinators of the target space depend on not only the radial coordinate \( r \) but also the angular coordinate \( \theta \), which is more complex than the case of black hole. Remarkably, we compute the first column of the vielbein \( V \) and obtain:

\[
\bar{u} = \bar{e} = E = v = 0.
\]

Thus, we can observed that Eq. (4.3) takes the same result as that of black hole in [17]. It is also shown that the solutions (2.17) and (2.18) preserve the same supersymmetry, which is consistent with the known facts.

Next, we should integrate the conserved currents in (3.10) over the angular coordinate \( \theta \), \( 0 < \theta < \pi \), i.e. a circle \( S^1 \) as the contour. After a tedious calculation, we obtain the
we obtain

\[ F \]

Meanwhile, considering that a supersymmetric black ring has two independent angular dimensions with the Killing vectors and find those conserved currents which have the angular momenta \( J \). Two independent angular momenta of a supersymmetric black ring are always unequal. In general, the angular momenta of a supersymmetric black ring are always unequal. In order to identify the angular momenta \( \psi \) and \( \phi \), we choose the dipole charge \( q \) as the base dimensions, thus, we obtain

\[
[F_{q1}] = [F_{p0}] = [F_{p1}] = [F_{q0}] = [q]^3. \tag{4.5}
\]

It is obvious to show that the above results confirms the identification of \( \int S^1 Y_+ \) and \( \int S^1 F_{p0} \) as the electric charge and \( J_\psi \), one of two angular momenta \[3\). Moreover, from the conserved charges \( \int S^1 E_{\rho \sigma} \) and \( \int S^1 E_{p1} \) taking the value of zero, it is well known that the black rings have only purely electric charges.

In general, the angular momenta of a supersymmetric black ring are always unequal. Two independent angular momenta \( J_\phi \) and \( J_\psi \) correspond to the Killing vector \( \partial_\phi \) and \( \partial_\psi \). Observing the results in (4.4b), we only find a conserved charge \( \int S^1 F_{q0} \) associating with the angular momenta \( J_\psi \). In order to identify the angular momenta \( \psi \), we need know the dimensions of the fourteen Killing vectors and find those conserved currents which have the same dimensions with \( F_{q0} \). Choosing the dipole charge \( q \) as the base dimensions, thus, we obtain

\[
[F_{q1}, F_{q1}] = 0, \tag{4.6}
\]

\[
[F_{q1}, F_{p1}] = [F_{p0}] = [F_{p1}] = [F_{q0}] = [q]^3. \tag{4.5}
\]
we can consider that the conserved charge \( \int S_1 F_{q1} \) is the angular momenta \( J_\phi \). Thus, using the algebra of conserved charges, we compute the conserved charges of supersymmetric black rings in minimal \( N = 1 \) supergravity and obtain the result consistent with [3].

### 4.2 Hamilton-Jacobi equation and the semi-classical wave function of supersymmetric black rings

In this subsection we give a discussion about the semi-classical wave function of supersymmetric black rings. It is well known that the motion of a particle can be represented as a wave function in the formulation of Hamilton-Jacobi equation (HJE). We can introduce the function \( S(x_1, ..., x_N, t) \) called phase function and require that \( S \) satisfy HJE:

\[
H(x_1, ..., x_N, \frac{\partial S}{\partial x_1}, ..., \frac{\partial S}{\partial x_N}, t) + \frac{\partial S}{\partial t} = 0. \tag{4.7}
\]

The conjugate momenta correspond to the first derivatives of \( S \) with respect to the generalized coordinates:

\[
p_k = \partial_{x_k} S. \tag{4.8}
\]

Thus the corresponding wave function can be written as:

\[
\Psi \propto \exp(iS). \tag{4.9}
\]

Now, using the same method in [17], we can study the supersymmetric constraints and further obtain the semi-classical wave function of supersymmetric black rings. In the above, we know that the five constraints keep the supersymmetric properties of the solution:

\[
\bar{u} = \bar{e} = E = v = p_r + 4 \sin \theta = 0. \tag{4.10}
\]

Considering the conjugate momenta (4.8) and conserved currents

\[
Y_+ = \tilde{Q}(\rho, \theta) = \frac{3}{\sqrt{2}} (i \rho^2 \frac{dt_2}{\tau_2} + 12 \rho \zeta^1 d \zeta^1 - 6i (\zeta^1)^2), \quad E_k = \tilde{K}(\rho, \theta) = -i \sin \theta, \tag{4.11}
\]

the equations (4.10) can be solved. Meanwhile, we obtain the phase function

\[
S_{\tilde{Q}, \tilde{K}, f} = -4 \rho \sin \theta + i \tilde{K} e^{2U} + \sqrt{2} Q \tilde{r} + \tilde{K} \left[ \sigma + \zeta^0 \zeta_0 + \zeta^1 \zeta_1 + 6 \tilde{\tau} \zeta^1 - 6 \tilde{\tau}^2 \zeta^0 \zeta^1 + 2 \tilde{\tau}^3 (\zeta^0)^2 \right] + f \left( \zeta_0 + 3 \tilde{\tau}^2 \zeta^1 - \tilde{\tau}^3 \zeta^0, \zeta_1 + 6 \tilde{\tau} \zeta^1 - 3 \tilde{\tau}^2 \zeta^0 \right), \tag{4.12}
\]

with \( f \) is an arbitrary function of two variables.

Substituting the phase function (4.12) into the wave function (4.9), we find that the solutions of the semi-classical approximation can be expressed as

\[
\Psi_{\tilde{Q}, \tilde{K}, p^\rho, p^\lambda} \propto \exp \left[ i S_{\tilde{Q}, \tilde{K}, 0} + ip^0 \left( \zeta_0 + \tilde{\tau} \zeta_1 + 3 \tilde{\tau}^2 \zeta^1 - \tilde{\tau}^3 \zeta^0 \right) + ip^1 \left( \zeta_1 + 6 \tilde{\tau} \zeta^1 - 3 \tilde{\tau}^2 \zeta^0 \right) \right]. \tag{4.13}
\]

Thus, we obtain the semi-classical wave function of supersymmetric black rings, which is dependent of the coordinates \( \rho \) and \( \theta \). It is obvious to see that the motion of a string-like object can be represented as a semi-classical wave function. It is noticed that the wave function of supersymmetric black ring take the form analogous to that of black hole in five dimensions [17]. However, since the moduli fields \( \varphi^a = \{ U, z^I, \bar{z}^I, \eta^a, \bar{\eta}_\Lambda, \sigma \} \) are dependent on angular coordinate, the wave function is more complex.
5. Conclusion

In this paper, with the $R \times U(1)$ isometry of the stationary solutions, we reduce 5D supersymmetric black ring solutions to three dimensions and obtain a non-linear sigma model. Further, adding the extra $U(1)$ symmetry to the sigma model, we obtain the two-dimensional effective Lagrangian with the coordinates $r$ and $\theta$, which describes the geodesic motion of the string-like object on the target space. Because the target space of the sigma model is the symmetric space $G_{2(+2)}/SO(4)$, we can analyse the Lie algebra of $G_{2(+2)}$ and study the algebra of conserved charges and supersymmetry constraints of supersymmetric black rings. In the context of black rings, the conserved charge is dependent of coordinate $\theta$. In order to obtain the charges independent of coordinates, we integrate these charge over $S^1$. After a tedious calculation, we obtain that $\int_{S^1} Y_+$, $\int_{S^1} F_{q_1}$ and $\int_{S^1} F_{q_0}$ correspond with the electric charge $Q$ and two angular momentums $J_\phi$ and $J_\psi$ respectively. Moreover, analyzing the supersymmetric constraints (4.10), we obtain the semi-classical wave function of supersymmetric black rings, which is dependent of the coordinate $r$ and $\theta$ and describe that the motion of a string-like can be represented as a semi-classical wave function.

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