A mixed volume from the anisotropic Riesz-potential

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Abstract

As a geometrical understanding of the maximal gravitational potential in computational and mathematical physics, this paper investigates a mixed volume induced by the so-called anisotropic Riesz-potential and establishes a reverse Minkowski-type inequality. It turns out that such a mixed volume is equal to the anisotropic Riesz-capacity and has connections with the anisotropic sup-Riesz-potential space. Two restrictions on the Lorentz spaces in terms of the anisotropic Riesz-capacity are also characterized. Besides, we also prove a Minkowski-type inequality and a log-Minkowski-type inequality as well as its reverse form.

1. The first definition and a reverse Minkowski-type inequality

Our starting point is the well-known gravitational potential of $B_r(0)$ in $\mathbb{R}^3$, the Euclidean ball with center $0$ and radius $r > 0$: $B_r(0) = \{ x \in \mathbb{R}^3 : |x| < r \}$. The gravitational potential of $B_r(0)$ in the physical space of unit mass density may be formulated by (see, for example, [15])

$$
\frac{1}{4\pi} \int_{B_r(0)} \frac{dy}{|x-y|} = \begin{cases} 
\frac{r^2}{2} - \frac{|x|^2}{6}, & \text{if } x \in B_r(0); \\
\frac{r^3}{3|x|^3}, & \text{if } x \in \mathbb{R}^3 \setminus B_r(0).
\end{cases}
$$

Clearly, if $V(\cdot)$ stands for the volume then

$$
\sup_{x \in \mathbb{R}^3} \int_{B_r(0)} \frac{dy}{|x-y|} = \int_{B_r(0)} \frac{dy}{y} = 2\pi \left( \frac{V(B_r(0))}{V(B_1(0))} \right)^{2/3} = 3 \left( \frac{2}{V(B_r(0))} \right)^{2/3} V(B_1(0))^{1/3}. \quad (1.1)
$$

Such a simple but important computation leads to the following question: Is it possible to extend (1.1) to any $n$-dimensional space $(\mathbb{R}^n, \|\cdot\|)$, where $\|\cdot\|$ is a norm defined on $\mathbb{R}^n$?

To settle this question, let us agree on some conventions. It is well known that if $\|\cdot\|$ is a norm on $\mathbb{R}^n$, then $B_{\|\cdot\|} = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$ defines an origin-symmetric convex body. Hereafter, $K$ is an origin-symmetric convex body if $K$ is a convex compact subset in $\mathbb{R}^n$ with nonempty interior and satisfies

$$
K = -K = \{ -y : y \in K \}.
$$

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On the other hand, for any given origin-symmetric convex body $K$, one can define a norm $\| \cdot \|_K$ on $\mathbb{R}^n$. In fact, $\| \cdot \|_K$ can be the Minkowski functional of $K$, which is defined by: for $x \in \mathbb{R}^n$,

$$\|x\|_K = \inf \{ \lambda > 0 : x \in \lambda K \} \quad \text{and} \quad \lambda K = \{ \lambda y : y \in K \}.$$ 

It can be checked that $B_{\|\cdot\|_K} = K$. Let $B_r^K(y) = \{ x \in \mathbb{R}^n : \|x - y\|_K \leq r \}$ denote the $K$-ball centered at $y$ with radius $r$. The usual Euclidean norm $| \cdot |$ is equal to the Minkowski functional of $B_r^\mathbb{R}(0) = \{ x \in \mathbb{R}^n : |x| \leq 1 \}$, the closed unit Euclidean ball of $\mathbb{R}^n$. Denote by $B_r^n(y)$ the Euclidean ball with center at $y \in \mathbb{R}^n$ and radius $r > 0$. As $K$ is origin-symmetric, one has $\|ax\|_K = |a| \cdot \|x\|_K$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$. Moreover, $K = \{ x \in \mathbb{R}^n : \|x\|_K \leq 1 \}$, and hence for all $t > 0$,

$$V(\{ x : \|x\|_K \leq t \}) = t^n V(K).$$

A set $L$ is said to be a convex body if $L$ is a convex compact subset in $\mathbb{R}^n$ with nonempty interior. If the origin is in the interior of the convex body $L$, one can define its polar body $L^\circ$ as

$$L^\circ = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \quad \forall y \in L \}.$$ 

Clearly $L^\circ$ is a convex body with the origin in the interior. Moreover, if $L$ is origin-symmetric, then its polar $L^\circ$ is clearly origin-symmetric as well. More background about convex geometry can be found in [16].

In what follows, $K$ is always assumed to be an origin-symmetric convex body unless otherwise stated. For $E \subset \mathbb{R}^n$, $V(E)$ denotes the $n$-dimensional volume of $E$; furthermore, $E^c$, $\text{int}(E)$ and $\bar{E}$ refer to the complement of $E$ in $\mathbb{R}^n$, the interior of $E$ and the closure of $E$, respectively.

1.1. The first definition

Let $0 \leq \alpha < n$ and $E \subset \mathbb{R}^n$ be a bounded measurable set. We define the anisotropic $\alpha$-Riesz-potential of $E$ at $y$ with respect to $K$ by

$$I_\alpha(E, K; y) = \int_E \frac{dx}{\|x - y\|_K^\alpha}.$$ 

Now we can define $V_\alpha(E, K)$, the mixed volume induced by the anisotropic $\alpha$-Riesz-potential $I_\alpha(E, K; \cdot)$.

DEFINITION 1. Let $0 \leq \alpha < n$ and $E \subset \mathbb{R}^n$ be a bounded measurable set. Define

$$V_\alpha(E, K) = \begin{cases} \sup_{y \in \mathbb{R}^n} I_\alpha(E, K; y), & \text{if } 0 < \alpha < n; \\ V(E), & \text{if } \alpha = 0. \end{cases}$$

If $E$ is bounded, one has

$$\lim_{\|y\|_K \to \infty} I_\alpha(E, K; y) = 0. \quad (1.2)$$

In fact, as $E$ is bounded, there is a constant $R > 0$ such that $\|x\|_K \leq R$ for all $x \in E$. On the other hand, if $\|y\|_K > 2R$, one has $\|x\|_K \leq \|y\|_K / 2$ and hence

$$\|x - y\|_K \geq \|y\|_K - \|x\|_K \geq \|y\|_K / 2 > R.$$ 

This further implies that, for $0 < \alpha < n$,

$$I_\alpha(E, K; y) = \int_E \frac{dx}{\|x - y\|_K^\alpha} \leq \int_E R^{-\alpha} \leq R^{-\alpha} V(E).$$
Therefore, for $0 < \alpha < n$,  
\[
0 \leq \lim_{\|y\|_K \to \infty} I_\alpha(E, K; y) \leq \lim_{R \to \infty} R^{-\alpha} V(E) = 0.
\]

Moreover, $I_\alpha(E, K; y)$ is uniformly continuous on $y \in \mathbb{R}^n$ (see Theorem A.1 in Appendix). Consequently, there is a point $y_0 \in \mathbb{R}^n$, such that, for $0 < \alpha < n$,  
\[
V_\alpha(E, K) = \sup_{y \in \mathbb{R}^n} I_\alpha(E, K; y) = I_\alpha(E, K; y_0) = \int_E \frac{dx}{\|x - y_0\|_K^\alpha}.
\]

To see this, let $V_\alpha(E, K) > 0$ (as otherwise it is trivial). Then, there exists $y_1 \in \mathbb{R}^n$ such that $I_\alpha(E, K; y_1) > 0$. By formula (1.2), one can find $R_0 > 0$ (depending on $\alpha$), such that  
\[
0 \leq I_\alpha(E, K; y) < I_\alpha(E, K; y_1)/2, \quad \forall y \in (B_{R_0}(0))^c.
\]

In other words, the supremum of $I_\alpha(E, K; y)$ cannot be obtained in $(B_{R_0}(0))^c$. On the other hand, the function $I_\alpha(E, K; y)$ is continuous in $B_{R_0}(0)$, a compact set in $\mathbb{R}^n$. Hence, for $0 < \alpha < n$, there is $y_0$ (depending on $\alpha$) in $B_{R_0}(0)$, such that  
\[
V_\alpha(E, K) = \sup_{y \in B_{R_0}(0)} I_\alpha(E, K; y) = I_\alpha(E, K; y_0) = \int_E \frac{dx}{\|x - y_0\|_K^\alpha}.
\]

For $0 < \alpha < n$, denote by $M_\alpha$ the set of all $y \in \mathbb{R}^n$ such that $V_\alpha(E, K) = I_\alpha(E, K; y)$. Clearly, $M_\alpha \subseteq B_{R_0}(0)$.

Note that $(n - \alpha)V_\alpha(K, K) = nV(K)$ for $\alpha \in [0, n)$, a consequence following from the forthcoming Theorem 1(ii). In the literature, several anisotropic norms and perimeters have been introduced and investigated (see, for example, [1, 4, 5, 8, 11, 12, 17] and their references). The basic idea behind those anisotropic norms and perimeters is to substitute the Euclidean norm $\| \cdot \|$ by the Minkowski norm $\| \cdot \|_K$. This naturally brings convex geometry into consideration and greatly enhances the already existing connections between analysis and convex geometry.

### 1.2. A reverse Minkowski-type inequality

The Minkowski inequality is one of the most important inequalities in convex geometry with many applications (see, for example, [16]). For two convex bodies $L, M \subseteq \mathbb{R}^n$, the Minkowski inequality asserts that the mixed volume  
\[
V(L, M) = \lim_{\epsilon \to 0} \frac{V(L + \epsilon M) - V(L)}{\epsilon n}
\]

with $L + \epsilon M = \{x + \epsilon y : x \in L \text{ and } y \in M\}$, is bounded from below by $V(L)^{1-1/n}V(M)^{1/n}$, that is,  
\[
V(L, M) \geq V(L)^{(n-1)/n}V(M)^{1/n}
\]

(1.3) with equality if and only if $L$ and $M$ are homothetic to each other. We now establish a reverse Minkowski-type inequality for the mixed volume $V_\alpha(E, K)$, which actually provides a solution to the question mentioned above (right below formula (1.1)). Note that the characterization for equality in Theorem 1 gives, for $0 < \alpha < n$,  
\[
\sup_{y \in \mathbb{R}^n} \int_{B_r^K(0)} \frac{dx}{\|x - y\|_K^\alpha} = \frac{n}{n - \alpha} V(B_r^K(0))^{(n-\alpha)/n}V(B_1^K(0))^{\alpha/n}.
\]

In particular, if $K = B_1^K(0)$, then for $0 < \alpha < n$,  
\[
\sup_{y \in \mathbb{R}^n} \int_{B_r^K(0)} \frac{dx}{\|x - y\|^\alpha} = \frac{n}{n - \alpha} V(B_r^n(0))^{(n-\alpha)/n}V(B_1^n(0))^{\alpha/n},
\]

which is an extension of formula (1.1) to all $n$. A Minkowski-type inequality similar to inequality (1.3) for $\tilde{V}_\alpha(E, K)$ will be proved in Section 5.
Theorem 1. Let $K$ be an origin-symmetric convex body. The following reverse Minkowski-type inequalities hold.

(i) For a bounded measurable set $E \subset \mathbb{R}^n$ and for $0 \leq \alpha < n$, one has
\begin{equation}
V_{\alpha}(E, K) \leq \frac{n}{n - \alpha} V(E)^{(n - \alpha)/n} V(K)^{\alpha/n}.
\end{equation}
Equality holds trivially if $\alpha = 0$ or $V(E) = 0$. For $\alpha \in (0, n)$ and bounded measurable set $E$ with $V(E) > 0$, equality holds if and only if $E$ is almost a $K$-ball; namely, there is $y \in \mathbb{R}^n$, such that
\begin{equation}
V(E^c \cap B_r^K(y)) = V\left((B_r^K(y))^c \cap E\right) = 0, \quad \text{with} \quad r = \left(\frac{V(E)}{V(K)}\right)^{1/n}.
\end{equation}

(ii) For convex body $L \subset \mathbb{R}^n$ and $0 < \alpha < n$, one has
\begin{equation}
V_{\alpha}(L, K) \leq \frac{n}{n - \alpha} V(L)^{(n - \alpha)/n} V(K)^{\alpha/n},
\end{equation}
with equality if and only if $K$ and $L$ are homothetic to each other. In particular,
\begin{equation}
\frac{(V_{n/2}(K^\circ, K))^2}{4} \leq V(K^\circ) V(K)
\end{equation}
with equality if and only if $K$ is an Euclidean ball.

Remark. Note that $V(K^\circ)V(K)$ is known as the Mahler volume product of $K$ and its polar body $K^\circ$. The well-known Blaschke–Santaló inequality states that, for all origin-symmetric convex body $K$ in $\mathbb{R}^n$,
\begin{equation}
V(K^\circ)V(K) \leq [V(B_1^n(0))]^2,
\end{equation}
with equality if and only if $K$ is an ellipsoid (that is, $TB_1^n(0)$ for some invertible linear transform $T$ defined on $\mathbb{R}^n$). Regarding the lower bound of $V(K^\circ)V(K)$, the famous Mahler conjecture asks whether
\begin{equation}
V(K^\circ)V(K) \geq \frac{4^n}{n!}
\end{equation}
holds for all origin-symmetric convex body $K$ in $\mathbb{R}^n$. Inequality (1.6) provides a lower bound for $V(K^\circ)V(K)$ and may be useful in improving well-known results for the isomorphic solutions of the Mahler conjecture: there is a universal constant $c > 0$ (independent of $n$ and $K$), such that
\begin{equation}
V(K^\circ)V(K) \geq c^n [V(B_1^n(0))]^2
\end{equation}
holds for all origin-symmetric convex body $K$ in $\mathbb{R}^n$ (see [3, 9, 13]).

Proof. (i) The desired inequality (1.4) holds trivially if $V(E) = 0$ or $\alpha = 0$. We only need to consider the case $\alpha \in (0, n)$ and $0 < V(E) < \infty$. Let $y \in \mathbb{R}^n$ be fixed and $B_r^K(y)$ be the $K$-ball with center $y$ and radius
\begin{equation}
r = \left(\frac{V(E)}{V(K)}\right)^{1/n} > 0.
\end{equation}
Note that $V(\{x : \|x\|_K \leq t\}) = t^n V(K)$ for all $t > 0$. Thus $V(B_r^K(y)) = V(E)$, which further implies
\begin{equation}
V(E^c \cap B_r^K(y)) = V\left((B_r^K(y))^c \cap E\right).
\end{equation}
Moreover, the following integral can be calculated by Fubini’s theorem:

\[
\int_{B^K_r(y)} \frac{dx}{\|x - y\|_K^\alpha} = \int \left\{ x : \|x - y\|_K \leq r \right\} \left( \int_0^\infty \alpha t^{-\alpha - 1} dt \right) \|x - y\|_K^\alpha \, dt + \int_r^\infty \alpha t^{-\alpha - 1} \left( \int \left\{ x : \|x - y\|_K \leq r \right\} \frac{dx}{\|x - y\|_K^\alpha} \right) \, dt
\]

\[
= V(K) \int_0^r \alpha t^{-\alpha + n - 1} \, dt + r^n V(K) \int_r^\infty \alpha t^{-\alpha - 1} \, dt
\]

\[
= \frac{n}{n - \alpha} V(E)^{(n - \alpha)/n} V(K)^{\alpha/n}.
\]  

(1.8)

Formula (1.7) together with the fact

\[
\left\{ \begin{array}{ll}
\|x - y\|_K \leq r, & \forall x \in E^c \cap B^K_r(y); \\
\|x - y\|_K > r, & \forall x \in (B^K_r(y))^c \cap E,
\end{array} \right.
\]

implies

\[
\int_{E^c \cap B^K_r(y)} \frac{dx}{\|x - y\|_K^\alpha} \geq \frac{V(B^K_r(y) \cap E^c)}{r^n} = \frac{V((B^K_r(y))^c \cap E)}{r^n} \geq \int_{(B^K_r(y))^c \cap E} \frac{dx}{\|x - y\|_K^\alpha}.
\]  

(1.9)

Consequently, one has

\[
\int_E \frac{dx}{\|x - y\|_K^\alpha} = \int_{E \cap B^K_r(y)} \frac{dx}{\|x - y\|_K^\alpha} + \int_{E \cap (B^K_r(y))^c} \frac{dx}{\|x - y\|_K^\alpha} \leq \int_{E \cap B^K_r(y)} \frac{dx}{\|x - y\|_K^\alpha} + \int_{E \cap (B^K_r(y))^c} \frac{dx}{\|x - y\|_K^\alpha} = \int_{B^K_r(y)} \frac{dx}{\|x - y\|_K^\alpha}.
\]

By formula (1.8), one has, for \(0 < \alpha < n\),

\[
V_\alpha(E, K) = \sup_{y \in \mathbb{R}^n} \int_E \frac{dx}{\|x - y\|_K^\alpha} \leq \sup_{y \in \mathbb{R}^n} \int_{B^K_r(y)} \frac{dx}{\|x - y\|_K^\alpha} \leq \frac{n}{n - \alpha} V(E)^{(n - \alpha)/n} V(K)^{\alpha/n}.
\]

To check the equality situation of inequality (1.4), let us make the following consideration. On the one hand, if \(E\) is almost a \(K\)-ball, that is, there is \(y_1 \in \mathbb{R}^n\) and \(r_0 > 0\), such that

\[
V(E^c \cap B^K_{r_0}(y_1)) = V((B^K_{r_0}(y_1))^c \cap E) = 0,
\]
then the following equality in (1.9) holds:

$$\int_{E \cap B_{r_0}^K(y)} \frac{dx}{\|x - y\|_K^2} = \int_{(B_{r_0}^K(y))' \cap E} \frac{dx}{\|x - y\|_K^2} = 0,$$

which, together with formula (1.8) and $r_0 = (V(E)/V(K))^{1/n}$, implies that

$$\int_{E} \frac{dx}{\|x - y\|_K^2} = \int_{B_{r_0}^K(y)} \frac{dx}{\|x - y\|_K^2} = \frac{n}{n - \alpha} V(E)^{(n-\alpha)/n} V(K)^{\alpha/n}.$$

Consequently, the equality in (1.4) holds.

On the other hand, assume that $E$ is not a $K$-ball with center at $y \in M_\alpha$ (indeed we can assume $y \in M_\alpha$ due to translation invariance, see Theorem 2), where $M_\alpha$ is as above. Then, for $r = (V(E)/V(K))^{1/n} > 0$ and for $y \in M_\alpha$, one has

$$V(E^c \cap B_r^K(y)) \neq 0 \quad \text{and} \quad V(B_r^K(y)^c \cap E) \neq 0.$$

Consequently, inequality (1.9) is strict and cannot have equality. Namely,

$$\int_{E^c \cap B_r^K(y)} \frac{dx}{\|x - y\|_K^2} > \int_{(B_r^K(y))^c \cap E} \frac{dx}{\|x - y\|_K^2}.$$

Thus, equality in inequality (1.4) cannot hold, because for $y \in M_\alpha$,

$$V_\alpha(E, K) = \int_{E} \frac{dx}{\|x - y\|_K^2} < \int_{B_r^K(y)} \frac{dx}{\|x - y\|_K^2} = \frac{n}{n - \alpha} V(E)^{(n-\alpha)/n} V(K)^{\alpha/n}.$$

In conclusion, to have equality in inequality (1.4), $E$ must be almost a $K$-ball.

(ii) Inequality (1.5) follows immediately from inequality (1.4). Note that $V(L) > 0$ and $0 < \alpha < n$. Thus, equality holds in inequality (1.5) if and only if $L$ is almost a $K$-ball. Then there is $y \in \mathbb{R}^n$, such that, for $r = (V(L)/V(K))^{1/n} > 0$

$$V(L^c \cap B_r^K(y)) = V(B_r^K(y) \cap L) = 0.$$

A simple argument by separation shows that $L = B_r^K(y) = y + rK$, and hence $K$ and $L$ are homothetic to each other.

Inequality (1.6) follows immediately from inequality (1.5) if we let $L = K^\circ$ and $\alpha = n/2$. Equality holds if and only if $K$ and $K^\circ$ are homothetic to each other; namely, $K^\circ = aK$ for some constant $a > 0$. Consequently, $\langle x, ax \rangle \leq 1$ for all $x \in K$, which is equivalent to $K \subseteq a^{-1/2}B_1(0)$ or $a^{1/2}K \subseteq B_1(0)$. This further implies that $B_1^\circ(0) \subseteq a^{-1/2}K^\circ = a^{1/2}K$, and hence $K = a^{-1/2}B_1^\circ(0)$. \hfill \Box

2. Metric properties

2.1. Basic properties

The following theorem establishes the fundamental properties of the newly defined mixed volume $V_\alpha(E, K)$.

**Theorem 2.** Let $E, E_1, E_2, E_3, \ldots \subset \mathbb{R}^n$ be bounded measurable sets and $\alpha \in [0, n)$. The set-function $E \mapsto V_\alpha(E, K)$ is nonnegative and has the following metric properties.
(i) Homogeneity and translation invariance: \( \forall r, s > 0 \) and \( \forall x_0 \in \mathbb{R}^n \), one has

\[
V_\alpha(sE, rK) = s^{n-\alpha}r^\alpha V_\alpha(E, K) \quad \text{and} \quad V_\alpha(x_0 + E, K) = V_\alpha(E, K),
\]

where \( x_0 + E = \{x_0 + y : y \in E\} \).

(ii) Monotonicity: if \( E_1 \subseteq E_2 \), then \( V_\alpha(E_1, K) \leq V_\alpha(E_2, K) \). On the other hand, if \( K_1 \subseteq K_2 \), then \( V_\alpha(E, K_1) \leq V_\alpha(E, K_2) \).

(iii) Sub-additivity: \( V_\alpha(E_1 \cup E_2, K) \leq V_\alpha(E_1, K) + V_\alpha(E_2, K) \).

(iv) Downward-monotone-convexity: if \( \{E_j\}_{j=1}^\infty \) is a decreasing sequence, that is, \( E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots \), then

\[
\lim_{j \to \infty} V_\alpha(E_j, K) = V_\alpha(\cap_{j=1}^\infty E_j, K).
\]

(v) Upward-monotone-convexity: if \( \{E_j\}_{j=1}^\infty \) is an increasing sequence, that is, \( E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots \), and \( \cup_{j=1}^\infty E_j \) is bounded, then

\[
\lim_{j \to \infty} V_\alpha(E_j, K) = V_\alpha(\cup_{j=1}^\infty E_j, K).
\]

(vi) Interpolation: if \( 0 \leq \alpha < \beta < \gamma < n \), then

\[
\left\{ \begin{array}{l}
[V_\beta(E, K)]^{\gamma - \alpha} \leq [V_\alpha(E, K)]^{\gamma - \beta} [V_\gamma(E, K)]^{\beta - \alpha};

\ln[V_\beta(E, K)] \leq \frac{\gamma - \beta}{\gamma - \alpha} \ln[V_\alpha(E, K)] + \frac{\beta - \alpha}{\gamma - \alpha} \ln[V_\gamma(E, K)].
\end{array} \right.
\]

In particular,

\[
\beta \mapsto \left[ \frac{V_\beta(E, K)}{V(E)} \right]^{1/\beta}
\]

is an increasing function on \((0, n)\).

**Proof.** (i) The formula \( V_\alpha(E, rK) = r^\alpha V_\alpha(E, K) \) follows immediately from

\[
\|x - y\|_{rK} = r^{-1}\|x - y\|_K, \quad \forall x, y \in \mathbb{R}^n.
\]

The formula \( V_\alpha(rE, K) = r^{n-\alpha}V_\alpha(E, K) \) follows from Definition 1 and the following calculation:

\[
V_\alpha(rE, K) = \sup_{y \in \mathbb{R}^n} \int_{rE} \frac{1}{\|x - y\|_K^\alpha} \, dx
\]

\[
= \sup_{y \in \mathbb{R}^n} \int_{rE} \frac{1}{\frac{\|x - y\|_K^\alpha}{r}} r^{n-\alpha} \, dx
\]

\[
= \sup_{y \in \mathbb{R}^n} \int_{E} \frac{1}{\|x - y\|_K^\alpha} r^{n-\alpha} \, dx
\]

\[
= r^{n-\alpha} V_\alpha(E, K).
\]

Combining the above two formulas, one can easily get the desired homogeneity result:

\[
V_\alpha(sE, rK) = r^\alpha V_\alpha(sE, K) = s^{n-\alpha}r^\alpha V_\alpha(E, K).
\]

Now let us prove the translation invariance. For \( x_0 \in \mathbb{R}^n \), one has

\[
V_\alpha(x_0 + E, K) = \sup_{y \in \mathbb{R}^n} \int_{x_0 + E} \frac{1}{\|x - y\|_K^\alpha} \, dx
\]

\[
= \sup_{y \in \mathbb{R}^n} \int_{E} \frac{1}{\|z + x_0 - y\|_K^\alpha} \, dz
\]
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\[ V_\alpha (E, K) = \sup_{x \in \mathbb{R}^n} \int_E \frac{1}{\|z - w\|_K^\alpha} dz \]

where we have let \( x = x_0 + z \) and \( y = w + x_0 \).

(ii) For the other side, one has, for all \( y \in \mathbb{R}^n \),

\[ \int_{E_1} \frac{dx}{\|x - y\|_K^\alpha} \leq \int_{E_2} \frac{dx}{\|x - y\|_K^\alpha}. \]

Hence

\[ V_\alpha (E_1, K) = \sup_{y \in \mathbb{R}^n} \int_{E_1} \frac{dx}{\|x - y\|_K^\alpha} \leq \sup_{y \in \mathbb{R}^n} \int_{E_2} \frac{dx}{\|x - y\|_K^\alpha} = V_\alpha (E_2, K). \]

On the other hand, if \( K_1 \subseteq K_2 \), one can check that \( \|x\|_{K_1} \geq \|x\|_{K_2} \) for all \( x \in \mathbb{R}^n \). Hence,

\[ V_\alpha (E, K_1) = \sup_{y \in \mathbb{R}^n} \int_E \frac{dx}{\|x - y\|_{K_1}^\alpha} \leq \sup_{y \in \mathbb{R}^n} \int_E \frac{dx}{\|x - y\|_{K_2}^\alpha} = V_\alpha (E, K_2). \]

(iii) By Definition 1, one has

\[ V_\alpha (E_1 \cup E_2, K) = \sup_{y \in \mathbb{R}^n} \int_{E_1 \cup E_2} \frac{dx}{\|x - y\|_K^\alpha} \]

\[ \leq \sup_{y \in \mathbb{R}^n} \int_{E_1} \frac{dx}{\|x - y\|_K^\alpha} + \sup_{y \in \mathbb{R}^n} \int_{E_2} \frac{dx}{\|x - y\|_K^\alpha} \]

\[ = V_\alpha (E_1, K) + V_\alpha (E_2, K). \]

(iv) If \( \{E_j\}_{j=1}^\infty \) is a decreasing sequence of measurable sets with \( E_1 \) bounded, then

\[ \lim_{j \to \infty} V(E_j) = V(\cap_{j=1}^\infty E_j). \]

Hence, \( \forall \varepsilon > 0, \exists i_0 \in \mathbb{N} \) (the set of all natural numbers) such that

\[ V(E_{i_0} \setminus \cap_{j=1}^\infty E_j) \leq \left[ \frac{(n - \alpha) \varepsilon}{nV(K)^\alpha/n} \right]^{n/(n - \alpha)}. \]

Thus, by Theorem 1 and Properties (ii) and (iii) above, one has

\[ \lim_{j \to \infty} V_\alpha (E_j, K) \leq V_\alpha (E_{i_0}, K) \]

\[ \leq V_\alpha (\cap_{j=1}^\infty E_j, K) + V_\alpha (E_{i_0} \setminus \cap_{j=1}^\infty E_j, K) \]

\[ \leq V_\alpha (\cap_{j=1}^\infty E_j, K) + \frac{n}{n - \alpha} V(E_{i_0} \setminus \cap_{j=1}^\infty E_j)^{(n - \alpha)/n} V(K)^{\alpha/n} \]

\[ \leq V_\alpha (\cap_{j=1}^\infty E_j, K) + \varepsilon. \]

Letting \( \varepsilon \to 0 \) in the above inequality, one has

\[ \lim_{j \to \infty} V_\alpha (E_j, K) \leq V_\alpha (\cap_{j=1}^\infty E_j, K). \]

For the other side, one has, for all \( j \in \mathbb{N} \),

\[ V_\alpha (\cap_{j=1}^\infty E_j, K) \leq V_\alpha (E_j, K). \]

Letting \( j \to \infty \) gives

\[ V_\alpha (\cap_{j=1}^\infty E_j, K) \leq \lim_{j \to \infty} V_\alpha (E_j, K), \]
which leads to the desired equality
\[ \lim_{j \to \infty} V_\alpha(E_j, K) = V_\alpha(\bigcap_{j=1}^\infty E_j, K). \]

(v) Let \( \{E_j\}_{j=1}^\infty \) be increasing such that \( \bigcup_{j=1}^\infty E_j \) is bounded. The monotonicity in (ii) implies that

\[ V_\alpha(E_k, K) \leq \lim_{i \to \infty} V_\alpha(E_i, K) \leq V_\alpha(\bigcup_{j=1}^\infty E_j, K), \quad \forall k \in \mathbb{N}. \]

On the other hand, \( \forall \varepsilon > 0, \exists i_0 \in \mathbb{N} \) such that

\[ V(\bigcup_{j=1}^\infty E_j \setminus E_{i_0}) \leq \left[ \frac{(n-\alpha)\varepsilon}{nV(K)^{\alpha/n}} \right]^{n/(n-\alpha)}. \]

By Theorem 1 and (ii) and (iii) above, one has

\[ V_\alpha(\bigcup_{j=1}^\infty E_j, K) \leq V_\alpha(E_{i_0}, K) + V_\alpha(\bigcup_{j=1}^\infty E_j \setminus E_{i_0}, K) \]
\[ \leq \lim_{i \to \infty} V_\alpha(E_i, K) + \frac{n}{n-\alpha}V(\bigcup_{j=1}^\infty E_j \setminus E_{i_0})^{(n-\alpha)/n}V(K)^{\alpha/n} \]
\[ \leq \lim_{i \to \infty} V_\alpha(E_i, K) + \varepsilon. \]

Letting \( \varepsilon \to 0 \), one has

\[ V_\alpha(\bigcup_{j=1}^\infty E_j, K) \leq \lim_{j \to \infty} V_\alpha(E_j, K), \]

which leads to the desired equality

\[ \lim_{j \to \infty} V_\alpha(E_j, K) = V_\alpha(\bigcup_{j=1}^\infty E_j, K). \]

(vi) Under the assumption on \( \alpha, \beta, \gamma \) one has \( 0 < \frac{\beta-\alpha}{\gamma-\alpha} < 1 \). By Hölder’s inequality, it follows that

\[ V_\beta(E, K) = \sup_{y \in \mathbb{R}^n} \int_E \frac{dx}{\|x-y\|_K^\beta} \]
\[ = \sup_{y \in \mathbb{R}^n} \int_E \left( \frac{1}{\|x-y\|_K^\gamma} \right)^{(\gamma-\beta)/(\gamma-\alpha)} \left( \frac{1}{\|x-y\|_K} \right)^{(\beta-\alpha)/(\gamma-\alpha)} dx \]
\[ \leq \left( \sup_{y \in \mathbb{R}^n} \int_E \frac{dx}{\|x-y\|_K^\gamma} \right)^{(\gamma-\beta)/(\gamma-\alpha)} \left( \sup_{y \in \mathbb{R}^n} \int_E \frac{dx}{\|x-y\|_K} \right)^{(\beta-\alpha)/(\gamma-\alpha)} = (V_\alpha(E, K))^{(\gamma-\beta)/(\gamma-\alpha)} (V_\gamma(E, K))^{(\beta-\alpha)/(\gamma-\alpha)}. \]

The desired inequality follows by taking power \( \gamma - \alpha \) from both sides.

Of course, a rearrangement of the interpolation inequality with \( \alpha = 0 \) derives the desired monotonicity right away. \( \square \)

2.2. Regularity results

From Theorem 2, one sees that \( V_\alpha(\cdot, K) \) has many properties similar to the Lebesgue measure. This can be further strengthened by the following regularity result for \( V_\alpha(\cdot, K) \). Denote by \( G \Delta E \) the difference set of two sets \( E \) and \( G \) in \( \mathbb{R}^n \).

**Theorem 3.** Let \( 0 \leq \alpha < n \) and \( E \subset \mathbb{R}^n \) be a bounded measurable set.
(i) If $G \subset \mathbb{R}^n$ is bounded and measurable with $V(G \Delta E) = 0$, then
$$V_\alpha(G, K) = V_\alpha(E, K).$$

(ii) If $G \subset \mathbb{R}^n$ is bounded and measurable with $E \subseteq G$ and $V(G \setminus E) = 0$, then
$$V_\alpha(G, K) = V_\alpha(E, K).$$

In particular, $V_\alpha(E, K) = V_\alpha(E, K)$ if $V(E \setminus E) = 0$.

(iii) The mixed volume $V_\alpha(\cdot, K)$ is outer regular: for any bounded measurable set $E \subset \mathbb{R}^n$, one has
$$V_\alpha(E, K) = \inf_{\text{open } O \supseteq E} V_\alpha(O, K).$$

The mixed volume $V_\alpha(\cdot, K)$ is also inner regular: for any bounded measurable set $E$,
$$V_\alpha(E, K) = \sup_{\text{compact } L \subseteq E} V_\alpha(L, K).$$

Proof. (i) The monotonicity and sub-additivity in Theorem 2 imply that
$$V_\alpha(G, K) \leq V_\alpha(G \cap E, K) + V_\alpha(G \setminus E, K) \leq V_\alpha(E, K) + V_\alpha(G \Delta E, K).$$

Similarly, one has
$$V_\alpha(E, K) \leq V_\alpha(G, K) + V_\alpha(G \Delta E, K).$$

Suppose $V(G \Delta E) = 0$. By Theorem 1, one has
$$0 \leq |V_\alpha(G, K) - V_\alpha(E, K)| \leq V_\alpha(G \Delta E, K) \leq \frac{n}{n-\alpha} V(G \Delta E)^{(n-\alpha)/n} V(K)^{\alpha/n} = 0,$$
which implies $V_\alpha(G, K) = V_\alpha(E, K)$ as desired.

(ii) This is an immediate consequence of (i).

(iii) First of all, monotonicity in Theorem 2 implies that
$$V_\alpha(E, K) \leq V_\alpha(O, K)$$
for any open set $O$ with $E \subseteq O$. Taking the infimum over $O$, one gets
$$V_\alpha(E, K) = \inf_{\text{open } O \supseteq E} V_\alpha(O, K).$$

On the other hand, similar to the calculation in (i), one has, for all open sets $O$ such that $E \subseteq O$,
$$0 \leq V_\alpha(O, K) - V_\alpha(E, K) \leq \frac{n}{n-\alpha} V(O \setminus E)^{(n-\alpha)/n} V(K)^{\alpha/n}. $$

As $E$ is measurable, for any $\varepsilon > 0$, one can select an open set $O_\varepsilon$ such that $E \subseteq O_\varepsilon$ and
$$V(O_\varepsilon \setminus E) < \left[ \frac{(n-\alpha)\varepsilon}{nV(K)^{\alpha/n}} \right]^{n/(n-\alpha)}.$$ 

This in turns implies
$$V_\alpha(O_\varepsilon, K) < V_\alpha(E, K) + \varepsilon,$$
and consequently $V_\alpha(E, K) = \inf_{\text{open } O \supseteq E} V_\alpha(O, K)$ as desired.

For the inner regularity, the monotonicity in Theorem 2 implies that $V_\alpha(E, K) \geq V_\alpha(L, K)$ for any compact set $L$ with $L \subseteq E$. This implies
$$V_\alpha(E, K) \geq \sup_{\text{compact } L \subseteq E} V_\alpha(L, K).$$
On the other hand, similar to the calculation in (i), one has, for all compact sets \( L \) such that \( L \subseteq E \),

\[
0 \leq V_\alpha(E, K) - V_\alpha(L, K) \leq \frac{n}{n - \alpha} V(E \setminus L)^{(n-\alpha)/n} V(K)^{\alpha/n}.
\]

As \( E \) is measurable, for any \( \varepsilon > 0 \), one can select a compact set \( L_\varepsilon \) such that \( L_\varepsilon \subseteq E \) and

\[
V(E \setminus L_\varepsilon) < \left[ \frac{(n - \alpha)\varepsilon}{nV(K)^{\alpha/n}} \right]^{n/(n-\alpha)}.
\]

This in turn implies

\[
V_\alpha(L_\varepsilon, K) > V_\alpha(E, K) - \varepsilon,
\]

and consequently \( V_\alpha(E, K) = \sup_{\text{compact } L \subseteq E} V_\alpha(L, K) \) as desired. \( \square \)

3. An anisotropic Riesz-capacity and the second definition

In this section, we will provide another definition for \( V_\alpha(E, K) \). To this end, let us first point out

\[
V_\alpha(E, K) = \| \chi_E \|_{S_\alpha K},
\]

where \( \chi_E \) is the characteristic function of \( E \) (that is, \( \chi_E(x) = 1 \) for \( x \in E \) and \( \chi_E(x) = 0 \) for \( x \in E^c \)) and

\[
\| f \|_{S_\alpha K} = \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\| x - y \|_K} \, dx.
\]

In fact, for \( \alpha \in [0, n) \), \( \| \cdot \|_{S_\alpha K} \) defines a norm on the space of functions \( S_\alpha^0 \), the anisotropic sup-Riesz-potential space:

\[
S_\alpha^0 = \{ f : f \in L^1_{\text{loc}}(\mathbb{R}^n) \text{ with } \| f \|_{S_\alpha K} < \infty \},
\]

where \( L^1_{\text{loc}}(\mathbb{R}^n) \) stands for the space of all locally integrable functions on \( \mathbb{R}^n \). Note that the space \( S_\alpha^0 \) is related to the weighted Morrey’s space, see, for example, [10, 14].

We now define the anisotropic Riesz-capacity \( \text{cap}(\cdot; S_\alpha^0) \) induced by the norm \( \| \cdot \|_{S_\alpha K} \). Let \( C_0(\mathbb{R}^n) \) be the class of all continuous functions with compact support in \( \mathbb{R}^n \).

**DEFINITION 2.** Let \( \alpha \in [0, n) \).

(i) The anisotropic Riesz-capacity of a compact set \( L \subset \mathbb{R}^n \) is defined by

\[
\text{cap}(L; S_\alpha^0) = \inf \{ \| f \|_{S_\alpha^0} : f \in C_0(\mathbb{R}^n) \text{ and } f \geq \chi_L \}.
\]

(ii) The anisotropic Riesz-capacity of any open set \( O \subset \mathbb{R}^n \) is defined by

\[
\text{cap}(O; S_\alpha^0) = \sup_{\text{compact } L \subseteq O} \text{cap}(L; S_\alpha^0).
\]

(iii) The anisotropic Riesz-capacity of an arbitrary measurable set \( E \subset \mathbb{R}^n \) is defined by

\[
\text{cap}(E; S_\alpha^0) = \inf_{\text{open } O \supseteq E} \text{cap}(O; S_\alpha^0).
\]

The following result reveals that \( V_\alpha(E, K) \) is an alternative of the anisotropic Riesz-capacity \( \text{cap}(E; S_\alpha^0) \).

**THEOREM 4.** If \( E \subset \mathbb{R}^n \) is a bounded measurable set and \( 0 \leq \alpha < n \), then

\[
V_\alpha(E, K) = \text{cap}(E; S_\alpha^0).
\]
Proof. According to Definition 2 and Theorem 3(iii), it is enough to prove the theorem for compact sets.

Let \( L \) be a compact subset of \( \mathbb{R}^n \). For \( f \in C_0(\mathbb{R}^n) \) with \( f \geq \chi_L \), one has

\[
V_\alpha(L, K) = \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_L(x)}{\|x - y\|_K^\alpha} \, dx \leq \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\|x - y\|_K^\alpha} \, dx.
\]

Taking the infimum over all \( f \in C_0(\mathbb{R}^n) \) with \( f \geq \chi_L \), one gets

\[
V_\alpha(L, K) \leq \operatorname{cap}(L; S^\alpha_K).
\]

On the other hand, \( \forall \varepsilon > 0 \), there is an open set \( O_\varepsilon \supseteq L \) such that

\[
0 < V(O_\varepsilon \setminus L) \leq \left[ \frac{(n - \alpha)\varepsilon}{nV(K)^{\alpha/n}} \right]^{n/(n - \alpha)}.
\]

Moreover, one can find a function \( g \), such that, \( g = 1 \) on \( L \), \( g \in C_0(\mathbb{R}^n) \), \( 0 \leq g \leq 1 \), and the support of \( g \) (denoted by \( \operatorname{supp}(g) \)) is contained in \( O_\varepsilon \). Note that \( g \geq \chi_L \). From Theorem 1, it follows that

\[
\operatorname{cap}(L; S^\alpha_K) \leq \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(x)|}{\|x - y\|_K^\alpha} \, dx \leq \sup_{y \in \mathbb{R}^n} \int_{L} \frac{|g(x)|}{\|x - y\|_K^\alpha} \, dx + \sup_{y \in \mathbb{R}^n} \int_{L^c} \frac{|g(x)|}{\|x - y\|_K^\alpha} \, dx \leq \sup_{y \in \mathbb{R}^n} \int_{L} \frac{dx}{\|x - y\|_K^\alpha} + \sup_{y \in \mathbb{R}^n} \int_{O_\varepsilon \setminus L} \frac{dx}{\|x - y\|_K^\alpha} \leq V_\alpha(L, K) + \frac{n}{n - \alpha} V(O_\varepsilon \setminus L)^{(n - \alpha)/n} V(K)^{\alpha/n} \leq V_\alpha(L, K) + \varepsilon.
\]

This in turn yields

\[
\operatorname{cap}(L; S^\alpha_K) \leq V_\alpha(L, K)
\]

by letting \( \varepsilon \to 0 \). Hence, one has \( \operatorname{cap}(L; S^\alpha_K) = V_\alpha(L, K) \), as desired. \( \square \)

4. Two restrictions on the Lorentz spaces

4.1. Restriction on \( L^p_\mu \)

To begin with, we have the following restriction result.

**Theorem 5.** Let \( \mu \) be a Radon measure on \( \mathbb{R}^n \) and \( 0 < p < \infty \). The following two inequalities are equivalent.

(i) The analytic inequality: there is a constant \( c_{\alpha, p} > 0 \) such that

\[
\|f\|_{L^p_\mu} \leq c_{\alpha, p} \left( \int_0^\infty (V_\alpha(\{x \in \mathbb{R}^n : |f(x)| \geq t\}, K))^p \, dt \right)^{1/p} \tag{4.1}
\]

holds for any \( f \in C_0(\mathbb{R}^n) \).
(ii) The anisotropic isoperimetric inequality: there is a constant $c_{\alpha,p} > 0$ such that

$$\left(\mu(O)\right)^{1/p} \leq c_{\alpha,p} V_{\alpha}(O,K)$$

(4.2)

holds for any bounded domain $O \subset \mathbb{R}^n$.

**Proof.** (ii) $\Rightarrow$ (i). Suppose that (ii) holds. Note that if $f \in C_0(\mathbb{R}^n)$ then

$$O_t(f) = \{x \in \mathbb{R}^n : |f(x)| > t\}$$

is a bounded domain. So, by (4.2), Fubini’s theorem and Theorem 3(i) we get the desired inequality (4.1) as follows:

$$\|f\|_{L_p^\mu} = \left( \int_{\mathbb{R}^n} |f(x)|^p d\mu(x) \right)^{1/p}$$

$$= \left( \int_{\mathbb{R}^n} \left[ \int_0^{|f(x)|} pt^{p-1} dt \right] d\mu(x) \right)^{1/p}$$

$$= \left( \int_0^\infty \left[ \int_{O_t(f)} pt^{p-1} d\mu(x) \right] dt \right)^{1/p}$$

$$= \left( \int_0^\infty \mu(O_t(f)) dt^p \right)^{1/p}$$

$$\leq \left( \int_0^\infty \left( V_{\alpha}(O_t(f),K) \right)^p dt^p \right)^{1/p}$$

Consequently, (i) holds.

(i) $\Rightarrow$ (ii). Suppose that (i) holds. For any bounded domain $O \subset \mathbb{R}^n$ and $0 < \epsilon < 1$, let

$$f_\epsilon(x) = \begin{cases} 1 - \epsilon^{-1}\text{dist}(x,\overline{O}), & \text{if dist}(x,\overline{O}) < \epsilon, \\ 0, & \text{if dist}(x,\overline{O}) \geq \epsilon, \end{cases}$$

where dist$(x,E)$ denotes the Euclidean distance of a point $x$ to a set $E$. Then $f_\epsilon \in C_0(\mathbb{R}^n)$ and hence inequality (4.1) holds for $f_\epsilon$. Moreover, one can check, by the dominated convergence theorem, that

$$\left(\mu(\overline{O})\right)^{1/p} = \lim_{\epsilon \to 0^+} \|f_\epsilon\|_{L_p^\mu}.$$  

(4.3)

Let

$$O_\epsilon = \{x \in \mathbb{R}^n : \text{dist}(x,\overline{O}) < \epsilon\}.$$  

Inequality (4.1) implies that for any $\epsilon \in (0,1)$,

$$\|f_\epsilon\|_{L_p^\mu} \leq c_{\alpha,p} \left( \int_0^\infty \left( V_{\alpha}(O_t(f_\epsilon),K) \right)^p dt^p \right)^{1/p}$$

$$= c_{\alpha,p} \left( \int_0^1 \left( V_{\alpha}(O_t(f_\epsilon),K) \right)^p dt^p \right)^{1/p}$$

$$\leq c_{\alpha,p} V_{\alpha}(\overline{O}_\epsilon,K),$$
where the last inequality is due to the monotonicity in Theorem 2(ii) and \( \overline{O}_\epsilon(f) \subseteq \overline{O}_\epsilon \).

Theorem 2(iv) and formula (4.3) imply inequality (4.2), if we let \( \epsilon \to 0^+ \).

4.2. Restriction on \( L^{p,\infty}_\mu \)

Coming up next is a result on how \( S^K_\alpha \) is contained in the weak Lebesgue \( p \)-space based on a Radon measure \( \mu \) on \( \mathbb{R}^n \):

\[
L^{p,\infty}_\mu = \left\{ f : \text{measurable function on } \mathbb{R}^n \text{ with } \| f \|_{L^{p,\infty}_\mu} < \infty \right\},
\]

where

\[
\| f \|_{L^{p,\infty}_\mu} = \sup_{t > 0} t\mu(O_t(f))^{1/p} \quad \text{and} \quad O_t(f) = \left\{ x \in \mathbb{R}^n : |f(x)| > t \right\}
\]

are set for any measurable function \( f \) on \( \mathbb{R}^n \). Below is an imbedding of \( S^K_\alpha \) into \( L^{p,\infty}_\mu \).

**Theorem 6.** Let \( \mu \) be a Radon measure on \( \mathbb{R}^n \) and \( 0 < p < \infty \). The following two inequalities are equivalent.

(i) The analytic inequality: there is a constant \( c_{\alpha,p} > 0 \) such that

\[
\| f \|_{L^{p,\infty}_\mu} \leq c_{\alpha,p} \| f \|_{S^K_\alpha}
\]

holds for any \( f \in C_0(\mathbb{R}^n) \).

(ii) The anisotropic isoperimetric inequality: there is a constant \( c_{\alpha,p} > 0 \) such that

\[
\left( \mu(O) \right)^{1/p} \leq c_{\alpha,p} V_\alpha(O, K) \tag{4.4}
\]

holds for any bounded domain \( O \subset \mathbb{R}^n \).

**Proof.** (ii) \( \Rightarrow \) (i). Assume that (ii) holds. If \( f \in C_0(\mathbb{R}^n) \) and \( t > 0 \), then

\[ O_t(f) = \left\{ x \in \mathbb{R}^n : |f(x)| > t \right\} \]

is a bounded domain, and hence

\[
\| f \|_{L^{p,\infty}_\mu} \leq \sup_{t > 0} t\mu(O_t(f))^{1/p} = c_{\alpha,p} \sup_{t > 0} tV_\alpha(O_t(f), K)
\]

\[
\leq c_{\alpha,p} \sup_{t > 0} \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} t \frac{\chi_{O_t(f)}(x)}{\| x - y \|^\alpha_K} dx
\]

\[
= c_{\alpha,p} \sup_{t > 0} \sup_{y \in \mathbb{R}^n} \int_{O_t(f)} t \frac{\chi_{O_t(f)}(x)}{\| x - y \|^\alpha_K} dx
\]

\[
\leq c_{\alpha,p} \sup_{t > 0} \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)| \chi_{O_t(f)}(x)}{\| x - y \|^\alpha_K} dx
\]

\[
\leq c_{\alpha,p} \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\| x - y \|^\alpha_K} dx = c_{\alpha,p} \| f \|_{S^K_\alpha}
\]

and so (i) holds.
(i) ⇒ (ii). Suppose that (i) holds. Let \( O \subset \mathbb{R}^n \) be a bounded domain. As in Theorem 5, for any \( \epsilon \in (0,1) \) we choose
\[
O_\epsilon = \{ x \in \mathbb{R}^n : \text{dist}(x, \overline{O}) < \epsilon \}
\]
and the following \( C_0(\mathbb{R}^n) \)-function
\[
f_\epsilon(x) = \begin{cases} 
1 - \epsilon^{-1}\text{dist}(x, \overline{O}), & \text{if dist}(x, \overline{O}) < \epsilon, \\
0, & \text{if dist}(x, \overline{O}) \geq \epsilon,
\end{cases}
\]
thereby using both Fubini’s Theorem and Theorem 4 to derive
\[
(1 - \epsilon) \cdot (\mu(\overline{O}))^{1/p} \leq \|f_\epsilon\|_{L^p,\infty}
\]
\[
= c_{\alpha,p} \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_\epsilon(x)|}{\|x - y\|^\alpha_K} \, dx
\]
\[
= c_{\alpha,p} \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \left( \int_0^{\text{dist}(x, \overline{O})} \frac{1}{\|x - y\|^\alpha_K} \, dt \right) dx
\]
\[
= c_{\alpha,p} \sup_{y \in \mathbb{R}^n} \int_0^\infty \left( \int_{O_\epsilon(f_\epsilon)} \frac{dx}{\|x - y\|^\alpha_K} \right) dt
\]
\[
\leq c_{\alpha,p} \int_0^\infty \alpha_{O_\epsilon(f_\epsilon),K} \, dt
\]
\[
= c_{\alpha,p} \int_0^1 \alpha_{O_\epsilon(f_\epsilon),K} \, dt
\]
\[
\leq c_{\alpha,p} \alpha_{\overline{O}_\epsilon,K},
\]
where the last inequality follows from the monotonicity in Theorem 2(ii) and \( O_\epsilon(f_\epsilon) \subseteq \overline{O}_\epsilon \). By taking \( \epsilon \to 0^+ \), inequality (4.4) follows from Theorem 2(iv) so (ii) is valid.

5. A Minkowski-type inequality

Note that Theorem 1 depends essentially on the hypothesis \( \alpha \in [0, n) \). A natural question to ask is: can we obtain appropriate results for \( \alpha \geq n \)? In this section, we will establish a Minkowski-type inequality for \( \alpha > n \) which provide a solution to this question for \( \alpha > n \). In Section 6, a log-Minkowski inequality as well as a reverse log-Minkowski inequality for \( \alpha = n \) will be proved.

Observe that for \( \alpha \geq n \), the function \( I_\alpha(E, K; y) \) may not be well defined and so is \( \alpha_{E, K} \). We will modify the function \( I_\alpha(E, K; y) \) as
\[
\tilde{I}_\alpha(E, K; y) = \int_{E^c} \frac{dx}{\|x - y\|^\alpha_K}
\]
and define the analogous mixed volume \( \tilde{\alpha}_{E, K} \) induced by \( \tilde{I}_\alpha(E, K; y) \) as
\[
\tilde{\alpha}_{E, K} = \inf_{y \in \mathbb{R}^n} \int_{E^c} \frac{dx}{\|x - y\|^\alpha_K}.
\]
A similar result to Theorem 2 (such as translation invariance) can be established for \( \tilde{\alpha}_{E, K} \), and we leave this to the readers.
Note that \( \tilde{V}_\alpha(E, K) < \infty \) if \( E \) has nonempty interior. To see this, let \( y_0 \) be an interior point of \( E \). Then there is \( r_0 > 0 \) such that \( \|x - y_0\|_K \geq r_0 \) for all \( x \in E^c \). Therefore, \( E^c \subseteq (B_{r_0}^K(y_0))^c \) and

\[
\tilde{V}_\alpha(E, K) \leq \int_{E^c} \frac{dx}{\|x - y_0\|_K^\alpha} \\
= \frac{n}{\alpha - n} \int_{(B_{r_0}^K(y_0))^c} \frac{dx}{\|x - y_0\|_K^\alpha} \\
= \frac{n}{\alpha - n} r_0^{\alpha - \alpha} V(K),
\]

as calculated in formula (5.4).

We say that a measurable set \( E \subset \mathbb{R}^n \) with \( \text{int}(E) \neq \emptyset \) is regular if for all (small enough) \( \epsilon > 0 \) and \( y \notin \text{int}(E) \), one has

\[ V(B^n_\epsilon(y) \cap E^c) \sim \epsilon^n. \]

A simple argument by separation shows that if \( E \) is a convex body, then \( E \) is regular. Note that, such a regularity condition may also be called type (A) condition (see, for example, [7]) and is common in analysis, especially in the study of partial differential equations.

When \( E \) is regular, one has, for all \( y \notin \text{int}(E) \) and for all (small enough) \( \epsilon > 0 \),

\[
\tilde{V}_\alpha(E, K; y) = \int_{E^c} \frac{dx}{\|x - y\|_K^\alpha} \geq \int_{E^c \cap B^n_\epsilon(y)} \frac{dx}{\|x - y\|_K^\alpha} \geq \int_{E^c \cap B^n_\epsilon(y)} \epsilon^{-\alpha} dx \geq \epsilon^{n-\alpha},
\]

where we have used the equivalence between \( \| \cdot \|_K \) and \( | \cdot | \) in the last inequality. Letting \( \epsilon \to 0 \), one gets, for \( \alpha > n \) and for all \( y \notin \text{int}(E) \),

\[ \tilde{I}_\alpha(E, K; y) = \infty. \]  

We now prove the following Minkowski-type inequality.

**Theorem 7.** Let \( \alpha > n \) be a constant. For all bounded measurable set \( E \subset \mathbb{R}^n \), one has

\[ \tilde{V}_\alpha(E, K) \geq \frac{n}{\alpha - n} V(E)^{(n-\alpha)/n} V(K)^{\alpha/n}. \]  

(5.3)

If in addition \( E \) is regular and has nonempty interior, equality holds in inequality (5.3) if and only if \( E \) is almost a \( K \)-ball, namely, there is \( y \in \mathbb{R}^n \) such that

\[ V(E^c \cap B^n_r(y)) = V((B^n_r(y))^c \cap E) = 0 \quad \text{with} \quad r = \left( \frac{V(E)}{V(K)} \right)^{1/n}. \]

In particular, if \( E \) is a convex body, then inequality (5.3) holds with equality if and only if \( K \) and \( E \) are homothetic to each other.

**Proof.** The proof of Theorem 7 is similar to that for Theorem 1 and is essentially identical to that for [17, Theorem 3] (which corresponds to \( \alpha \in (n, n + 1) \)). Here we include a brief proof for completeness.

It is enough to consider \( 0 < V(E) < \infty \). Let \( r = (V(E)/V(K))^{1/n} > 0 \). For any fixed \( y \in \mathbb{R}^n \), one has \( V(E^c \cap B^n_r(y)) = V((B^n_r(y))^c \cap E) \). Note that \( \|x - y\|_K \leq r \) for \( x \in E^c \cap B^n_r(y) \) and \( \|x - y\|_K \geq r \) for \( x \in (B^n_r(y))^c \cap E \). Thus,
\[
\int_{E^c} \frac{dx}{\|x - y\|_K^n} = \int_{E^c \cap B_K^c(y)} \frac{dx}{\|x - y\|_K^n} + \int_{E^c \cap (B_K^c(y))^c} \frac{dx}{\|x - y\|_K^n} \\
\geq \int_{(B_K^c(y))^c \cap E} \frac{dx}{\|x - y\|_K^n} + \int_{E^c \cap (B_K^c(y))^c} \frac{dx}{\|x - y\|_K^n} \\
= \int_{(B_K^c(y))^c} \frac{dx}{\|x - y\|_K^n},
\]
where the last integral can be calculated by Fubini’s theorem as follows:
\[
\int_{(B_K^c(y))^c} \frac{dx}{\|x - y\|_K^n} = \int_{\{x : \|x - y\|_K > r\}} \left( \int_0^\infty \alpha t^{-\alpha - 1} dt \right) dx \\
= \int_r^\infty \alpha t^{-\alpha - 1} \left( \int_{\{x : \alpha \|x - y\|_K \leq t\}} dx \right) dt \\
= V(K) \int_r^\infty \alpha t^{-\alpha - 1} \left( t^n - r^n \right) dt \\
= \frac{n}{\alpha - n} V(E)^{(n-\alpha)/n} V(K)^{\alpha/n}. \tag{5.4}
\]
Hence, we can conclude that, for any \( y \in \mathbb{R}^n \),
\[
\int_{E^c} \frac{dx}{\|x - y\|_K^n} \geq \frac{n}{\alpha - n} V(E)^{(n-\alpha)/n} V(K)^{\alpha/n}.
\]
The desired inequality follows by taking the infimum over all \( y \in \mathbb{R}^n \).

Now let us check equality situation of inequality (5.3). Assume that \( E \) is regular and has nonempty interior. On the one hand, if \( E \) is almost a \( K \)-ball, there is \( y_0 \in \mathbb{R}^n \) and \( r_0 > 0 \), such that
\[
V(E \setminus B_{r_0}^K(y_0)) = V(B_{r_0}^K(y_0) \setminus E) = 0.
\]
Hence, we have
\[
\int_{E^c \cap B_{r_0}^K(y_0)} \frac{dx}{\|x - y_0\|_K^n} = \int_{(B_{r_0}^K(y_0))^c \cap E} \frac{dx}{\|x - y_0\|_K^n} = 0.
\]
This further implies that
\[
\int_{E^c} \frac{dx}{\|x - y_0\|_K^n} = \int_{(B_{r_0}^K(y_0))^c} \frac{dx}{\|x - y_0\|_K^n} = \frac{n}{\alpha - n} V(E)^{(n-\alpha)/n} V(K)^{\alpha/n}.
\]
Consequently, equality in inequality (5.3) holds.

On the other hand, Theorem A.1 in the Appendix implies that \( \tilde{I}_\alpha(E, K; y) \) is continuous in \( \text{int}(E) \). Note that \( \tilde{V}_\alpha(E, K) < \infty \) by inequality (5.1) as \( \text{int}(E) \neq \emptyset \). Moreover, there is a sequence \( \{y_m\}_{m \geq 1} \subset \mathbb{R}^n \) such that
\[
\tilde{V}_\alpha(E, K) = \lim_{m \to \infty} \tilde{I}_\alpha(E, K; y_m) < \infty.
\]
Formula (5.2) implies that \( y_m \in \text{int}(E) \subset \overline{E} \) for all \( m > m_0 \) with \( m_0 \) some fixed integer. As \( E \) is bounded, one can find a convergent subsequence of \( \{y_m\} \) (without loss of generality, denote by \( \{y_m\} \) this subsequence) such that \( y_m \to y_0 \in \overline{E} \) and
\[
\infty > \tilde{V}_\alpha(E, K) = \lim_{m \to \infty} \tilde{I}_\alpha(E, K; y_m) = \tilde{I}_\alpha(E, K; y_0).
\]
This, plus (5.2) under \( \alpha > n \), implies \( y_0 \in \text{int}(E) \).
For $\alpha > n$ denote by $\mathcal{N}_\alpha$ the set of all $y \in \mathbb{R}^n$ such that
$$\tilde{V}_\alpha(E, K) = \tilde{I}_\alpha(E, K; y),$$
and hence $\mathcal{N}_\alpha \subset \text{int}(E)$. If $E$ is not a $K$-ball, then for $r = (V(E)/V(K))^{1/n} > 0$ and for all $y \in \mathbb{R}^n$, one has
$$V(E^c \cap B^K_r(y)) \neq 0 \quad \text{and} \quad V(B^K_r(y)^c \cap E) \neq 0.$$
Without loss of generality (due to translation invariance), one can assume $y \in \mathcal{N}_\alpha$. Hence,
$$\int_{E^c \cap B^K_r(y)} dx/\|x-y\|_K^n > \int_{(B^K_r(y))^c \cap E} dx/\|x-y\|_K^n.$$
Thus, equality in inequality (5.3) cannot hold, because for $y \in \mathcal{N}_\alpha$,
$$\tilde{V}_\alpha(E, K) = \int_{E^c} \frac{dx}{\|x-y\|_K^n} > \int_{(B^K_r(y))^c} \frac{dx}{\|x-y\|_K^n} = \frac{n}{\alpha - n} V(E)^{(n-\alpha)/n} V(K)^{\alpha/n}.$$
In conclusion, in order to have equality in inequality (5.3), $E$ must be almost a $K$-ball.

**Remark.** The anisotropic fractional $\alpha$-perimeter of $E$ with respect to $K$ [12] is defined by, for $\alpha \in (n, n+1)$,
$$P_\alpha(E, K) = \int_E \int_{E^c} \frac{1}{\|x-y\|_K^n} \ dx dy.$$
It has been proved in [17] that, for $\alpha \in (n, n+1)$,
$$P_\alpha(E, K) \geq \frac{n}{\alpha - n} V(K)^{\alpha/n} V(E)^{(2n-\alpha)/n}. \quad (5.5)$$
It is clear that
$$P_\alpha(E, K) \geq V(E) \cdot \tilde{V}_\alpha(E, K) \geq \frac{n}{\alpha - n} V(K)^{\alpha/n} V(E)^{(2n-\alpha)/n}.$$
Therefore, to have equality in inequality (5.5) for $E$ being a convex body, one must have
$$\tilde{V}_\alpha(E, K) = \frac{n}{\alpha - n} V(E)^{(n-\alpha)/n} V(K)^{\alpha/n}.$$
In other words, $E$ being homothetic to $K$ is the only possibility to have equality in inequality (5.5). A more detailed discussion on the sharpness of inequality (5.5) can be found in [17].

6. Two log-Minkowski type inequalities

6.1. A reverse log-Minkowski inequality

As promised, we now establish a reverse log-Minkowski inequality for $\alpha = n$. To deal with the case $\alpha = n$, we must bring the logarithm into play. In fact, we have the following reverse log-Minkowski inequality.
Theorem 8. Let \( E \subset \mathbb{R}^n \) be a bounded measurable set. For all \( A > 0 \), one has
\[
\sup_{y \in \mathbb{R}^n} \int_{E \cap \{ x \in \mathbb{R}^n : \| x - y \|_K \geq A \}} \frac{dx}{\| x - y \|_K^n} \leq \sup_{y \in \mathbb{R}^n} V(K) \ln \left( \frac{V(E \cap \{ x \in \mathbb{R}^n : \| x - y \|_K \geq A \})}{A^n V(K)} + 1 \right). \tag{6.1}
\]

Proof. It is enough to verify the following inequality: for any \( A > 0 \) and any \( y \in \mathbb{R}^n \), one has
\[
\int_{E \cap \{ x \in \mathbb{R}^n : \| x - y \|_K \geq A \}} \frac{1}{\| x - y \|_K^n} \, dx \leq V(K) \ln \left( \frac{V(E \cap \{ x \in \mathbb{R}^n : \| x - y \|_K \geq A \})}{A^n V(K)} + 1 \right). \tag{6.2}
\]

We only consider \( 0 < V(E) < \infty \), as otherwise inequality (6.2) holds trivially if \( V(E) = 0 \). The calculation is similar to that in the proof of Theorem 1, so we will keep our calculation short with concentration on the main modification. Let \( A > 0 \) and \( y \in \mathbb{R}^n \) be fixed. For simplicity, we let
\[
E_y = E \cap \{ x \in \mathbb{R}^n : \| x - y \|_K \geq A \}.
\]
Let \( R_r(y) \) be the \( K \)-ring centered at \( y \) with in-radius \( A \) and out-radius \( r_y \):
\[
R_r(y) = \{ x \in \mathbb{R}^n : A \leq \| x - y \|_K \leq r_y \},
\]
where \( r_y = \left( \frac{V(E_y)}{V(K)} + A^n \right)^{1/n} \). A simple calculation shows that \( V(R_r(y)) = V(E_y) \). This in turn implies
\[
V(E_y^c \cap R_r(y)) = V((R_r(y))^c \cap E_y).
\]
Together with the facts \( \| x - y \|_K \leq r_y \) if \( x \in E_y^c \cap R_r(y) \) and \( \| x - y \|_K \geq r_y \) if \( x \in R_r(y)^c \cap E_y \), one gets
\[
\int_{E_y^c \cap R_r(y)} \frac{dx}{\| x - y \|_K^n} \geq \frac{V(E_y^c \cap R_r(y))}{r_y^n} = \frac{V(R_r(y)^c \cap E_y)}{r_y^n} \geq \int_{R_r(y)^c \cap E_y} \frac{dx}{\| x - y \|_K^n}.
\]
This further implies
\[
\int_{E_y} \frac{dx}{\| x - y \|_K^n} \leq \int_{R_r(y)} \frac{dx}{\| x - y \|_K^n}.
\]
The last integral can be calculated by Fubini’s Theorem as follows:
\[
\int_{R_r(y)} \frac{dx}{\| x - y \|_K^n} = \int_{\{ x : A \leq \| x - y \|_K \leq r_y \}} \left( \int_{\| x - y \|_K}^{\infty} nt^{-n-1} \, dt \right) \, dx = \int_{A}^{r_y} nt^{-n-1} \left( \int_{\{ x : A \leq \| x - y \|_K \leq t \}} dx \right) \, dt + \int_{r_y}^{\infty} nt^{-n-1} \left( \int_{\{ x : A \leq \| x - y \|_K \leq r_y \}} dx \right) \, dt = V(K) \cdot n \cdot \ln \left( \frac{r_y}{A} \right) = V(K) \ln \left( \frac{V(E \cap \{ x \in \mathbb{R}^n : \| x - y \|_K \geq A \})}{A^n V(K)} + 1 \right), \tag{6.3}
\]
which gives the desired inequality (6.2). Consequently, inequality (6.1) follows from inequality (6.2) by taking the supremum over $y \in \mathbb{R}^n$.

**Remark.** Since $E \cap \{ x \in \mathbb{R}^n : ||x - y||_K \geq A \} \subseteq E$, one can easily have

$$\sup_{y \in \mathbb{R}^n} \int_{E \cap \{ x \in \mathbb{R}^n : ||x - y||_K \geq A \}} \frac{dx}{||x - y||_K^n} \leq V(K) \ln \left( \frac{V(E)}{A^n V(K)} + 1 \right).$$

**6.2. A log-Minkowski inequality**

The conjecture of the log-Minkowski inequality can be stated as follows (see [2, p. 1976]): for origin-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$, does the following inequality hold true

$$\int_{S^{n-1}} \ln \left( \frac{h_L(u)}{h_K(u)} \right) d\overline{V}_K(u) \geq \ln \left( \left( \frac{V(L)}{V(K)} \right)^{1/n} \right).$$

Here $h_K$ and $h_L$ are the support functions of $K$ and $L$, $S(K, \cdot)$ is the surface area measure of $K$ defined on $S^{n-1}$ (the unit sphere of $\mathbb{R}^n$), and

$$d\overline{V}_K(u) = \frac{h_K(u)}{n V(K)} dS(K, u),$$

is the normalized cone measure of $K$. The conjecture has been confirmed only for $n = 2$, but it is still open in general. A dual log-Minkowski inequality has been established in [6] for any dimension $n$. Here, we will prove a log-Minkowski-type inequality. For convenience, we let $\ln(\infty) = \infty$.

**Theorem 9.** Let $0 < B < \infty$ be a constant. For a bounded measurable set $E$ with $V(E) > 0$, one has:

$$\inf_{y \in \mathbb{R}^n} \int_{E \cap \{ x \in \mathbb{R}^n : ||x - y||_K \leq B \}} \frac{dx}{||x - y||_K^n} \geq \inf_{y \in \mathbb{R}^n} V(K) \ln \left( \frac{B^n V(K)}{V(E \cap \{ x \in \mathbb{R}^n : ||x - y||_K \leq B \})} \right).$$

(6.4)

**Proof.** Let $0 < B < \infty$ be a constant and $y \in \mathbb{R}^n$ be a fixed point. For simplicity, we let

$$\tilde{E}_y = E^c \cap \{ x \in \mathbb{R}^n : ||x - y||_K \leq B \}.$$ Denote by $\tilde{R}_\rho(y)$ the $K$-ring centered at $y$ with in-radius $\rho$ and out-radius $B$:

$$\tilde{R}_\rho(y) = \{ x \in \mathbb{R}^n : \rho \leq ||x - y||_K \leq B \},$$

where the in-radius $\rho$ is defined by

$$\rho^n = B^n - \frac{V(\tilde{E}_y)}{V(K)}.$$ Note that we can rewrite $\rho$ as

$$V(K) \rho^n = V(E \cap \{ x \in \mathbb{R}^n : ||x - y||_K \leq B \}),$$ which follows from

$$V(K) B^n = V \left( \{ x \in \mathbb{R}^n : ||x - y||_K \leq B \} \right)$$

$$= V \left( \tilde{E}_y \cup (E \cap \{ x \in \mathbb{R}^n : ||x - y||_K \leq B \}) \right)$$

$$= V(\tilde{E}_y) + V \left( (E \cap \{ x \in \mathbb{R}^n : ||x - y||_K \leq B \}) \right).
A simple calculation shows that \( V(\tilde{R}_\rho(y)) = V(\tilde{E}_y) \) and \( V(\tilde{E}_y \setminus \tilde{R}_\rho(y)) = V(\tilde{R}_\rho(y) \setminus \tilde{E}_y) \). These lead to, by a similar calculation to formula (6.3),
\[
\int_{\tilde{E}_y} \frac{dx}{\|x - y\|_K^n} \geq \int_{\tilde{R}_\rho(y)} \frac{dx}{\|x - y\|_K^n} = V(K) \cdot n \cdot \ln \left( \frac{B}{\rho} \right)
\]
and while if \( \rho = 0 \),
\[
\int_{\tilde{E}_y} \frac{dx}{\|x - y\|_K^n} \geq \int_{\tilde{R}_\rho(y)} \frac{dx}{\|x - y\|_K^n} \geq \lim_{y \to 0} V(K) \cdot n \cdot \ln \left( \frac{B}{\eta} \right) = \infty.
\]
The desired inequality (6.4) holds if we take the infimum over \( y \in \mathbb{R}^n \), that is,
\[
\inf_{y \in \mathbb{R}^n} \int_{E \cap \{x \in \mathbb{R}^n : \|x - y\|_K \leq B\}} \frac{dx}{\|x - y\|_K^n} \geq \inf_{y \in \mathbb{R}^n} V(K) \ln \left( \frac{B^n V(K)}{V(E \cap \{x \in \mathbb{R}^n : \|x - y\|_K \leq B\})} \right).
\]

\[\square\]

**Remark.** If \( E \) is a bounded measurable set with \( V(E) > 0 \), the right-hand side of inequality (6.4) is bounded from above by a finite number. In fact, \( \overline{E} \) is a bounded closed set, hence \( \overline{E} \) is a compact set in \( \mathbb{R}^n \). Therefore, there is a finite number of open covering \( \{x \in \mathbb{R}^n : \|x - y_j\|_K < B\}, \quad j = 1, 2, \ldots, m \), such that
\[
E \subseteq \overline{E} \subseteq \bigcup_{j=1}^m \{x \in \mathbb{R}^n : \|x - y_j\|_K < B\} \subseteq \bigcup_{j=1}^m \{x \in \mathbb{R}^n : \|x - y_j\|_K \leq B\}.
\]
By sub-additivity of Lebesgue measure, one has
\[
0 < V(E) \leq \sum_{j=1}^m V(E \cap \{x \in \mathbb{R}^n : \|x - y_j\|_K \leq B\}).
\]
Thus, there must have (at least) one \( y_0 \in \mathbb{R}^n \) such that
\[
V(E \cap \{x \in \mathbb{R}^n : \|x - y_0\|_K \leq B\}) > 0.
\]
Consequently, one has
\[
\inf_{y \in \mathbb{R}^n} V(K) \ln \left( \frac{B^n V(K)}{V(E \cap \{x \in \mathbb{R}^n : \|x - y\|_K \leq B\})} \right) < \infty.
\]
One also has, as \( E \cap \{x \in \mathbb{R}^n : \|x - y\|_K \leq B\} \subseteq E \),
\[
\inf_{y \in \mathbb{R}^n} \int_{E \cap \{x \in \mathbb{R}^n : \|x - y\|_K \leq B\}} \frac{dx}{\|x - y\|_K^n} \geq V(K) \ln \left( \frac{B^n V(K)}{V(E)} \right).
\]

**Appendix**

A.1. **Continuity for \( \alpha \neq n \)**

In what follows, we will prove the continuity of the potential functions.
Theorem A.1. Let $E$ be a bounded measurable set in $\mathbb{R}^n$. Then the function $I_\alpha(E, K; y)$ for $0 < \alpha < n$ is uniformly continuous on $\mathbb{R}^n$. Moreover, if $\text{int}(E) \neq \emptyset$, then the function $I_\alpha(E, K; y)$ for $\alpha > n$ is continuous in $\text{int}(E)$.

Proof. Let $y \in \mathbb{R}^n$ be a fixed point and $\varepsilon > 0$ be given. Firstly, we consider $\alpha \in (0, 1)$. Given $z \in \mathbb{R}^n$ let

$$E_1 = \{x \in E : \|x - y\|_K \geq \|x - z\|_K\}.$$ 

By the triangle inequality, one can check that

$$\|y - z\|_K \leq 2\|x - y\|_K \quad \forall x \in E_1.$$ 

Consequently, if $x \in E_1$ and $0 < \beta < \alpha$ then

$$\left(\frac{\|y - z\|_K}{\|x - y\|_K}\right)^\alpha = 2^{\alpha - \beta},$$ 

and hence

$$\frac{\|y - z\|_K^\alpha}{\|x - y\|_K^\alpha \cdot \|x - z\|_K^\alpha} \leq \frac{2^{\alpha - \beta} \cdot \|y - z\|_K^\beta}{\|x - y\|_K^\beta \cdot \|x - z\|_K^\beta} \leq 2^{\alpha - \beta} \cdot \frac{\|y - z\|_K^\beta}{\|x - z\|_K^{\alpha + \beta}}.$$ 

Together with Theorem 1, we have that if $\alpha + \beta < n$ then

$$\int_{E_1} \frac{\|y - z\|_K^\alpha}{\|x - y\|_K^\alpha \cdot \|x - z\|_K^\alpha} dx \leq 2^{\alpha - \beta} \cdot \|y - z\|_K^\beta \int_{E_1} \frac{1}{\|x - z\|_K^{\alpha + \beta}} dx$$

$$\leq 2^{\alpha - \beta} \cdot \|y - z\|_K^\beta \cdot V_{\alpha + \beta}(E_1, K) \leq C_\beta \cdot \|y - z\|_K^\beta,$$

where

$$C_\beta = 2^{\alpha - \beta} \cdot \frac{n}{n - \alpha - \beta} \cdot V(E) \cdot V(K)^{\frac{\alpha + \beta}{n}}.$$

Along the same line, one can obtain

$$\int_{E_1} \frac{\|y - z\|_K^\alpha}{\|x - y\|_K^\alpha \cdot \|x - z\|_K^\alpha} dx \leq C_\beta \cdot \|y - z\|_K^\beta.$$ 

It can be checked that, by the triangle inequality and the fact $(a + b)^\alpha \leq a^\alpha + b^\alpha$ for $a, b \geq 0$ and for $\alpha \in (0, 1)$,

$$\|x - y\|_K^\alpha - \|x - z\|_K^\alpha \leq \|y - z\|_K^\alpha.$$ 

Hence, for $0 < \alpha < 1$,

$$|I_\alpha(E, K; y) - I_\alpha(E, K; z)|$$

$$= \left|\int_E \frac{dx}{\|x - y\|_K^\alpha} - \int_E \frac{dx}{\|x - z\|_K^\alpha}\right|$$

$$\leq \int_E \frac{\|x - y\|_K^\alpha - \|x - z\|_K^\alpha}{\|x - y\|_K^\alpha \cdot \|x - z\|_K^\alpha} dx$$

$$\leq \int_E \frac{\|y - z\|_K^\alpha}{\|x - y\|_K^\alpha \cdot \|x - z\|_K^\alpha} dx.$$
\[ \leq \int_{E_1} \frac{\|y - z\|^\alpha_K}{\|x - y\|^\alpha_K \cdot \|x - z\|^\alpha_K} dx + \int_{E \setminus E_1} \frac{\|y - z\|^\alpha_K}{\|x - y\|^\alpha_K \cdot \|x - z\|^\alpha_K} dx \]
\[ \leq 2C_\beta \cdot \|y - z\|^\beta_K. \]

Hence, for any \( \varepsilon > 0 \), there is \( \delta = (\varepsilon/2C_\beta)^{1/\beta} \) (depending on \( \varepsilon \) only), such that, if \( \|y - z\|_K < \delta \), then
\[ |I_\alpha(E, K; y) - I_\alpha(E, K; z)| \leq 2C_\beta \cdot \|y - z\|^{\beta}_K < \varepsilon. \]

This concludes the uniform continuity of \( I_\alpha(E, K; y) \) on \( \mathbb{R}^n \) for \( \alpha \in (0, 1) \). Note that the above proof also covers the entire case for \( n = 1 \) under which \( \alpha \in (0, n) \) is just \( \alpha \in (0, 1) \).

Now we only need to consider \( n \geq 2 \) and \( \alpha \in [1, n) \). Note that \( 1 - (1 - t)^{\alpha} \leq at \) if \( \alpha \geq 1 \) and \( 0 < t \leq 1 \). On the set \( E_1 \), let \( t = (\|x - y\|_K - \|x - z\|_K)/(|x - y| \in E_1) \leq 0 \), then
\[ \int_{E_1} \frac{\|x - y\|^\alpha_K - \|x - z\|^\alpha_K}{\|x - y\|^\alpha_K \cdot \|x - z\|^\alpha_K} dx = \int_{E_1} \frac{1 - (1 - t)^\alpha}{\|x - y\|^\alpha_K} dx \]
\[ \leq \alpha \int_{E_1} \frac{\|x - y\|^\alpha_K - \|x - z\|^\alpha_K}{\|x - y\|^\alpha_K \cdot \|x - z\|^\alpha_K} dx \]
\[ \leq \alpha \int_{E_1} \frac{\|y - z\|_K}{\|x - y\|^\alpha_K \cdot \|x - z\|^\alpha_K} dx \]
\[ \leq \alpha \cdot 2^{1 - \beta} \cdot \|y - z\|_K^{\beta} \int_{E_1} \frac{1}{\|x - z\|^\alpha + \beta} dx \]
\[ \leq C_\beta' \cdot \|y - z\|^{\beta}_K, \]

where \( \beta \) is a constant such that \( 0 < \beta < \min\{1, n - \alpha\} \), and \( C_\beta' \) is a constant given by
\[ C_\beta' = \alpha \cdot 2^{1 - \beta} \cdot \frac{n}{n - \alpha - \beta} \cdot V(E)^{(n - \alpha - \beta)/n} V(K)^{(\alpha + \beta)/n}. \]

The uniform continuity of \( I_\alpha(E, K; y) \) then follows along the same line as the case \( \alpha \in (0, 1) \).

The argument for the ‘moreover-part’ is similar to that for the former part of Theorem A.1. Here, we include a brief proof with modifications emphasized.

Let \( y \in \text{int}(E) \neq \emptyset \). Then, there is a constant (depending on \( y \)) \( \delta_0 > 0 \), such that \( \{ x : \|x - y\|_K < 3\delta_0 \} \subseteq E \). Assume that \( \|y - z\|_K < \delta_0 \), then \( z \in \text{int}(E) \) and \( \{ x : \|x - z\|_K < 2\delta_0 \} \subseteq E \) by the triangle inequality. Let \( E_2 = \{ x \in E^c : \|x - y\|_K \geq \|x - z\|_K \} \). One has \( \|x - y\|_K \geq 3\delta_0 \) and \( \|x - z\|_K \geq 2\delta_0 \) for all \( x \in E_2 \). Consequently, for \( \alpha > n \),
\[ \int_{E_2} \frac{\|x - y\|^\alpha_K - \|x - z\|^\alpha_K}{\|x - y\|^\alpha_K \cdot \|x - z\|^\alpha_K} dx \leq \alpha \cdot \int_{E_2} \frac{\|y - z\|_K}{\|x - y\| \cdot \|x - z\|^\alpha_K} dx \]
\[ \leq \frac{\alpha \cdot \|y - z\|_K}{2\delta_0} \int_{\{ x : \|x - z\|_K \geq 2\delta_0 \}} \frac{1}{\|x - z\|^\alpha_K} dx \]
\[ = C \cdot \|y - z\|_K, \]

where we have used inequality (5.1) and \( C \) is a constant given by
\[ C = \frac{\alpha \cdot n \cdot 2^{n - \alpha - 1} \cdot \delta_0^{n - \alpha - 1} \cdot V(K)}{(\alpha - n)} > 0. \]
Along the same line, one can check
\[
\int_{E \cap E_2} \frac{\|x-z\|_K^n - \|x-y\|_K^n}{\|x-y\|_K \cdot \|x-z\|_K^n} \, dx \leq \tilde{C} \cdot \|y-z\|_K.
\]
This leads to
\[
|\tilde{I}_\alpha(E;K;y) - \tilde{I}_\alpha(E;K;z)| \\
\leq \int_{E_2} \frac{\|x-y\|_K^n - \|x-z\|_K^n}{\|x-y\|_K \cdot \|x-z\|_K^n} \, dx + \int_{E \cap E_2} \frac{\|x-z\|_K^n - \|x-y\|_K^n}{\|x-y\|_K \cdot \|x-z\|_K^n} \, dx \\
\leq 2\tilde{C} \cdot \|y-z\|_K.
\]
For any \( \varepsilon > 0 \), let
\[
\delta = \min \{ \delta_0, \varepsilon/(2\tilde{C}) \}
\]
(usually depending on \( y \)). Then, for all \( z \) such that \( \|y-z\|_K < \delta \), one has
\[
|\tilde{I}_\alpha(E;K;y) - \tilde{I}_\alpha(E;K;z)| < \varepsilon.
\]
This concludes the continuity of \( \tilde{I}_\alpha(E;K;\cdot) \) at \( y \in \text{int}(E) \), and the desired argument follows. \( \square \)

A.2. Continuity for \( \alpha = n \)

**Theorem A.2.** Let \( E \) be a bounded measurable set in \( \mathbb{R}^n \) and \( 0 < A, B < \infty \) be constants. Then, the function
\[
\int_{E \cap \{ x \in \mathbb{R}^n : \|x-y\|_K \geq A \}} \frac{dx}{\|x-y\|_K^n}
\]
is uniformly continuous on \( \mathbb{R}^n \). Moreover, if \( \text{int}(E) \neq \emptyset \), then
\[
\int_{E \cap \{ x \in \mathbb{R}^n : \|x-y\|_K \leq B \}} \frac{dx}{\|x-y\|_K^n}
\]
is continuous in \( \text{int}(E) \).

**Proof.** For \( y, z \in \mathbb{R}^n \), let
\[
E_3 = E \cap \{ x \in \mathbb{R}^n : \|x-z\|_K \geq A \} \cap \{ x \in \mathbb{R}^n : \|x-y\|_K \geq A \} \cap \{ x : \|x-y\|_K \geq \|x-z\|_K \}.
\]
As in the proof of the ‘moreover-part’ of Theorem A.1, one has
\[
I_1 = \int_{E_3} \frac{\|x-y\|_K^n - \|x-z\|_K^n}{\|x-y\|_K \cdot \|x-z\|_K^n} \, dx \\
\leq n \cdot \int_{E_3} \frac{\|y-z\|_K}{\|x-y\|_K \cdot \|x-z\|_K} \, dx \\
\leq \frac{n \cdot \|y-z\|_K}{A} \int_{E_3} \frac{1}{\|x-z\|_K^n} \, dx \\
\leq n \cdot A^{-1} \cdot V(K) \cdot \ln \left( \frac{V(E)}{A^n V(K)} + 1 \right) \cdot \|y-z\|_K.
\]
where the last inequality follows from Theorem 8 and its remark. Similarly,
\[
I_2 = \int_{(E \cap \{x \in \mathbb{R}^n : \|x - z\|_K < A\} \cap \{x \in \mathbb{R}^n : \|x - y\|_K \geq A\}) \backslash E} \frac{dx}{\|x - y\|_K^n} \leq n \cdot A^{-1} \cdot V(K) \cdot \ln \left( \frac{V(E)}{A^n V(K)} + 1 \right) \cdot \|y - z\|_K.
\]
By the triangle inequality, one has
\[
\{x \in \mathbb{R}^n : \|x - z\|_K < A\} \cap \{x \in \mathbb{R}^n : \|x - y\|_K \geq A\} \subseteq \{x : A \leq \|x - y\| < A + \|y - z\|_K\},
\]
which implies that, by Theorem 8 and its remark,
\[
I_3 = \int_{E \cap \{x \in \mathbb{R}^n : \|x - y\| < A\} \cap \{x \in \mathbb{R}^n : \|x - z\|_K \geq A\}} \frac{dx}{\|x - y\|_K^n} \leq V(K) \cdot \ln \left( \frac{V(\{x : A \leq \|x - y\| < A + \|y - z\|_K\})}{A^n V(K)} + 1 \right) \leq \frac{V(K) [V(\{x : A \leq \|x - y\| < A + \|y - z\|_K\})^n - A^n]}{A^n} \leq \frac{2n \cdot V(K) \cdot \|y - z\|_K}{A},
\]
as long as \(\|y - z\|_K < \tau A\) for some constant \(\tau > 0\) (in fact, one can let \(\tau = 1\) if \(n = 1\), and \(\tau = 2^{1/(n-1)} - 1 > 0\) if \(n > 1\)). Similarly, as long as \(\|y - z\|_K < \tau A\), one has
\[
I_4 = \int_{E \cap \{x \in \mathbb{R}^n : \|x - y\|_K < A\} \cap \{x \in \mathbb{R}^n : \|x - z\|_K \geq A\}} \frac{dx}{\|x - y\|_K^n} \leq \frac{2n \cdot V(K) \cdot \|y - z\|_K}{A}.
\]
In conclusion, we have, as long as \(\|y - z\|_K < \tau A\),
\[
\left| \int_{E \cap \{x \in \mathbb{R}^n : \|x - y\|_K < A\}} \frac{dx}{\|x - y\|_K^n} - \int_{E \cap \{x \in \mathbb{R}^n : \|x - z\|_K \geq A\}} \frac{dx}{\|x - z\|_K^n} \right| \leq I_1 + I_2 + I_3 + I_4 \leq C_n \|y - z\|_K,
\]
where \(C_n\) is a constant independent of \(y\) and \(z\):
\[
C_n = \frac{2n \cdot V(K)}{A} \cdot \left[ \ln \left( \frac{V(E)}{A^n V(K)} + 1 \right) + 2 \right].
\]
The uniform continuity follows from a routine argument.

The 'moreover-part' can be proved similarly and we leave the details to the readers. □

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