VANISHING OF COHOMOLOGY GROUPS OF RANDOM SIMPLICIAL COMPLEXES

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Abstract. We consider $k$-dimensional random simplicial complexes that are generated from the binomial random $(k + 1)$-uniform hypergraph by taking the downward-closure, where $k \geq 2$. For each $1 \leq j \leq k - 1$, we determine when all cohomology groups with coefficients in $\mathbb{F}_2$ from dimension one up to $j$ vanish and the zero-th cohomology group is isomorphic to $\mathbb{F}_2$. This property is not deterministically monotone for this model of random complexes, but nevertheless we show that it has a single sharp threshold. Moreover we prove a hitting time result, relating the vanishing of these cohomology groups to the disappearance of the last minimal obstruction. We also study the asymptotic distribution of the dimension of the $j$-th cohomology group inside the critical window. As a corollary, we deduce a hitting time result for a different model of random simplicial complexes introduced in [Linial and Meshulam, Combinatorica, 2006], a result which was previously only known for dimension two [Kahle and Pittel, Random Structures Algorithms, 2016].

1. Introduction

1.1. Motivation. In their seminal paper [18], Erdős and Rényi introduced the uniform random graph and, among other results, addressed the problem of determining the probability of this graph being connected. This classical result is usually stated for the binomial random graph $G(n, p)$ on $n$ vertices, in which each edge is present with a given probability $p$ independently: the property of $G(n, p)$ being connected undergoes a phase transition around the sharp threshold $p = \frac{\log n}{n}$ [41]. Throughout the paper, we denote the natural logarithm by log and we say that an event holds with high probability (whp for short) if it holds with probability tending to 1 as $n$ tends to infinity.

Theorem 1.1 ([18] [41]). Let $\omega$ be any function of $n$ which tends to infinity as $n \to \infty$. Then with high probability the following holds.

(i) If $p = \frac{\log n - \omega}{n}$, then $G(n, p)$ is not connected.
(ii) If $p = \frac{\log n + \omega}{n}$, then $G(n, p)$ is connected.

As an even stronger result, Erdős and Rényi [18] determined the limiting probability of $G(n, p)$ being connected around the point of the phase transition. More precisely, this result can be stated for $G(n, p)$ as follows.
Theorem 1.2 (see e.g. [20 Theorem 4.1]). Let \( c \in \mathbb{R} \) be a constant and suppose that \((c_n)_{n \geq 1}\) is a sequence of real numbers that converges to \( c \) as \( n \to \infty \). If
\[
p = \frac{\log n + c_n}{n},
\]
then
\[
P (G(n, p) \text{ is connected}) \xrightarrow{n \to \infty} e^{-e^{-c}}.
\]

We note that while [20 Theorem 4.1] is stated for the uniform random graph, it is actually proved via the binomial model \( G(n, p) \) and thus immediately translates into Theorem 1.2.

Subsequently, Bollobás and Thomason [9] proved a hitting time result for the random graph process, in which edges are added one at a time uniformly at random. This result relates the connectedness of the random graph process to the disappearance of the last smallest obstruction, an isolated vertex.

Theorem 1.3 (19). With high probability, the random graph process becomes connected at exactly the moment when the last isolated vertex disappears.

Since then, many higher-dimensional analogues of both random graphs and connectedness have been analysed and in particular two different approaches have received considerable attention. A first natural generalisation for dimension \( n \geq 1 \) have been analysed and in particular two different approaches have received particular attention is generalisations of the \((\ell, \omega)\)-th cohomology group with coefficients in the two-element field \( \mathbb{F}_2 \) of the random \((\ell, \omega)\)-set forms a \((\ell + 1, \omega)\)-simplex with probability \( p \) independently. They showed that the property of the vanishing of the \((k-1)\)-th cohomology group \( H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) \) with coefficients in \( \mathbb{F}_2 \) has a sharp threshold at
\[
p = \frac{k \log n}{n}.
\]

Theorem 1.4 ([30, 35]). Let \( \omega \) be any function of \( n \) which tends to infinity as \( n \to \infty \). Then with high probability,
\begin{enumerate}
\item if \( p = \frac{k \log n - \omega}{n} \), then \( H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) \neq 0 \);
\item if \( p = \frac{k \log n + \omega}{n} \), then \( H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) = 0 \).
\end{enumerate}

Meshulam and Wallach [35] further proved that the same statement remains true if the coefficients of the cohomology group are taken from any finite abelian group.

Later, Kahle and Pittel [28] derived a hitting time result for \( \mathcal{Y}_p \) (analogous to Theorem 1.3) in the case \( k = 2 \). Moreover, they determined the limiting distribution of \( \dim (H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2)) \) for general \( k \geq 2 \) and for \( p \) inside the critical window.
Theorem 1.5 ([28] Theorem 1.10). Let $k \geq 2$ and $c \in \mathbb{R}$ be a constant. If $$p = \frac{k \log n + c}{n},$$ then $\dim \left( H^{k-1}(Y_p; \mathbb{F}_2) \right)$ converges in distribution to a Poisson random variable with expectation $e^{-c/k!}$. In particular, we have $$\mathbb{P} \left( H^{k-1}(Y_p; \mathbb{F}_2) = 0 \right) \xrightarrow{n \to \infty} e^{-c/k!}.$$ 

Observe that Theorem 1.5 can be generalised to hold for $p = (k \log n + c_n)/n$, where $(c_n)_{n \geq 1}$ is a sequence of real numbers that converges to $c$ as $n \to \infty$ (cf. Theorem 1.2), because $\dim \left( H^{k-1}(Y_p; \mathbb{F}_2) \right)$ is a monotone function in $p$.

In this paper, we aim to bridge the gap between random hypergraphs and random simplicial complexes, considering random simplicial $k$-complexes that arise as the downward-closure of random $(k+1)$-uniform hypergraphs (Definition 1.7). Unlike $Y_p$, in this model the presence of the full $(k-1)$-dimensional skeleton is not guaranteed, thus the vanishing of the cohomology groups of dimensions lower than $k - 1$ does not hold trivially. Therefore, for each $1 \leq j \leq k - 1$, we introduce $\mathbb{F}_2$-cohomological $j$-connectedness of a $k$-dimensional simplicial complex (Definition 1.8) as the vanishing of all cohomology groups with coefficients in $\mathbb{F}_2$ from dimension one up to $j$ and the zero-th cohomology group being isomorphic to $\mathbb{F}_2$.

Although this notion of connectedness is not deterministically monotone for our model, we prove that $\mathbb{F}_2$-cohomological $j$-connectedness has a sharp threshold. Furthermore, we derive a hitting time result and determine the limiting probability for $\mathbb{F}_2$-cohomological $j$-connectedness inside the critical window. As a corollary, we deduce a hitting time result for $Y_p$ in general dimension, thus extending the hitting time result of Kahle and Pittel [28].

1.2. Model. Throughout the paper let $k \geq 2$ be a fixed integer. For positive integers $\ell$ and $1 \leq i \leq \ell$, write $[\ell] := \{1, \ldots, \ell\}$ and denote by $\binom{[\ell]}{i}$ the family of $i$-element subsets of $[\ell]$.

Definition 1.6. A family $\mathcal{G}$ of non-empty finite subsets of a vertex set $V$ is called a simplicial complex if it is downward-closed, i.e. if every non-empty set $A$ that is contained in a set $B \in \mathcal{G}$ also lies in $\mathcal{G}$, and if furthermore the singleton $\{v\}$ is in $\mathcal{G}$ for every $v \in V$.

The elements of a simplicial complex $\mathcal{G}$ of cardinality $k+1$ are called $k$-simplices of $\mathcal{G}$. If $\mathcal{G}$ has no $(k+1)$-simplices, then we call it $k$-dimensional, or $k$-complex. If $\mathcal{G}$ is a $k$-complex, then for each $j = 0, \ldots, k - 1$ the $j$-skeleton of $\mathcal{G}$ is the $j$-complex formed by all $i$-simplices in $\mathcal{G}$ with $0 \leq i \leq j$.

We define a model of random $k$-complexes starting from the binomial random $(k+1)$-uniform hypergraph $G_p$, on vertex set $[n]$: the $0$-simplices are the vertices of $G_p$, the $k$-simplices are the hyperedges of $G_p$, but there is more than one way to guarantee the downward-closure property to obtain a simplicial complex. In the model $Y_p$ considered by Meshulam and Wallach in [35], the full $(k - 1)$-skeleton on $[n]$ is always included. In contrast, we only include those simplices that are necessary to ensure the downward-closure property.

Definition 1.7. We denote by $\mathcal{G}_p = \mathcal{G}(k; n, p)$ the random $k$-dimensional simplicial complex on vertex set $[n]$ such that:

- the $0$-simplices are the singletons of $[n]$;
- the $k$-simplices are the hyperedges of the binomial random $(k+1)$-uniform hypergraph $G_p$;
- for each $j \in [k - 1]$, the $j$-simplices are exactly the $(j+1)$-subsets of hyperedges of $G_p$. 

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In other words, $G_p$ is the random $k$-complex on $[n]$ obtained from $G_p$ by taking the downward-closure of each hyperedge. For instance, denote by $F_p$ the set of hyperedges of the binomial random 4-uniform hypergraph $G_p = G(3; n, p)$. Then the corresponding two models of random 3-dimensional simplicial complexes are given by

$$
\mathcal{Y}_p = \mathcal{Y}(3; n, p) = \binom{[n]}{1} \cup \binom{[n]}{2} \cup \binom{[n]}{3} \cup F_p \quad \text{and}
$$

$$
\mathcal{G}_p = \mathcal{G}(3; n, p) = \binom{[n]}{1} \cup \partial(F_p) \cup \partial F_p \cup F_p,
$$

where $\partial E$ for a set $E$ of $j$-simplices, $j \geq 1$, denotes the set of all $(j - 1)$-simplices that are contained in elements of $E$.

Given a simplicial complex $\mathcal{G}$, let $H^i(\mathcal{G}; \mathbb{F}_2)$ be its $i$-th cohomology group with coefficients in $\mathbb{F}_2$ (see (14) in Section 2.3 for the definition). We define a notion of connectedness for a simplicial complex via the vanishing of its cohomology groups. Since the 0-th cohomology group $H^0(\mathcal{G}; \mathbb{F}_2)$ cannot vanish, we require this group to be “as small as possible”.

**Definition 1.8.** Given a positive integer $j$, a simplicial complex $\mathcal{G}$ is called $\mathbb{F}_2$-cohomologically $j$-connected (short for $j$-cohom-connected) if

- $H^0(\mathcal{G}; \mathbb{F}_2) = \mathbb{F}_2$;
- $H^i(\mathcal{G}; \mathbb{F}_2) = 0$ for all $i \in [j]$.

Observe that $H^0(\mathcal{G}; \mathbb{F}_2)$ being isomorphic to $\mathbb{F}_2$ is equivalent to connectedness of $\mathcal{G}$ in the topological sense, which we call topological connectedness in order to distinguish it from other notions of connectedness. For $\mathcal{G} = \mathcal{G}_p$, this is also equivalent to vertex-connectedness of the associated $(k + 1)$-uniform hypergraph.

Moreover, one might define an analogous version of connectedness via the vanishing of homology groups, which would be equivalent to our definition of $\mathbb{F}_2$-cohomological $j$-connectedness by the Universal Coefficient Theorem (see e.g. [37]).

A significant difference between $\mathcal{G}_p$ and $\mathcal{Y}_p$ is that for $\mathcal{Y}_p$, the only requirement for $\mathbb{F}_2$-cohomological $(k - 1)$-connectedness is the vanishing of the $(k - 1)$-th cohomology group, since the presence of the full $(k - 1)$-skeleton guarantees topological connectedness and the vanishing of the $j$-th cohomology groups for all $j \in [k - 2]$.

Moreover, it is important to observe that $\mathbb{F}_2$-cohomological $j$-connectedness is not necessarily a monotone increasing property of $\mathcal{G}_p$: adding a $k$-simplex to a $j$-cohom-connected complex might yield a complex without this property (see Example 3.2). Thus, the existence of a single threshold for $j$-cohom-connectedness is not guaranteed, but one of our main results shows that such a threshold indeed exists (Theorem 1.11).

1.3. **Main results.** The main contributions of this paper are fourfold. Firstly, we prove (Theorem 1.11) that for each $j \in [k - 1]$, the probability

$$
p_j := \frac{(j + 1) \log n + \log \log n}{(k - j - 1)n^{k-j}}(k-j)!
$$

is a sharp threshold for $\mathbb{F}_2$-cohomological $j$-connectedness. Secondly, we prove a hitting time result (also Theorem 1.11), relating the $j$-cohom-connected threshold to the disappearance of all copies of the minimal obstruction $M_j$ (Definition 1.10). Thirdly, our results directly imply an analogous hitting time result for $\mathcal{Y}_p$ (Corollary 1.12), which Kahle and Pittel [28] proved for $k = 2$. Lastly, we analyse the critical window given by the threshold $p_j$, showing that inside the window the dimension of the $j$-th cohomology group converges in distribution to a Poisson random variable (Theorem 1.13).
Proving that \( p_j \) is indeed a (sharp) threshold turns out to be considerably more challenging than might be expected, largely because \( \mathbb{F}_2 \)-cohomological \( j \)-connectedness of \( G_p \) is not a monotone increasing property. In particular, the subcritical case is much more involved than it would be for a monotone property, where often a simple second moment argument suffices. In order to circumvent the difficulties arising from the non-monotonicity, we introduce auxiliary structures called *local obstacles* (Definition 1.9), showing that whp there are no more obstructions to \( j \)-cohom-connectedness. In order to bound the number of potential “large” obstructions, basic calculations are not sufficient and therefore we define a suitable search process, which gives us more precise bounds on their number (Lemma 5.7).

Before defining the minimal obstruction \( M_j \) (Definition 1.10), we introduce the following necessary concepts.

**Definition 1.9.** Given a \( k \)-simplex \( K \) in a \( k \)-dimensional simplicial complex \( G \), a collection \( \mathcal{F} = \{ P_0, \ldots, P_{k-j} \} \) of \( j \)-simplices forms a *\( j \)-flower in \( K \)* (see Figure 1) if \( K = \bigcup_{i=0}^{k-j} P_i \) and \( C := \bigcap_{i=0}^{k-j} P_i \) satisfies \(|C| = j\). We call the \( j \)-simplices \( P_i \) the *petals* and the set \( C \) the *centre* of the \( j \)-flower \( \mathcal{F} \).

![Figure 1. Examples of \( j \)-flowers in a \( k \)-simplex \( K \), for \( k = 4 \) and \( j = 1, 2, 3 \).](image)

(i) The 1-flower in \( K \) with centre \( C = \{ c_1 \} \) (bold black) and petals \( P_i = C \cup \{ w_i \}, \quad i = 0, 1, 2, 3 \) (grey).

(ii) The 2-flower in \( K \) with centre \( C = \{ c_1, c_2 \} \) (bold black) and petals \( P_i = C \cup \{ w_i \}, \quad i = 0, 1, 2 \) (grey).

(iii) The 3-flower in \( K \) with centre \( C = \{ c_1, c_2, c_3 \} \) (bold black) and petals \( P_i = C \cup \{ w_i \}, \quad i = 0, 1 \) (grey).

Observe that for each \( k \)-simplex \( K \) and each \( (j - 1) \)-simplex \( C \subseteq K \), there is a unique \( j \)-flower in \( K \) with centre \( C \), namely

\[
\mathcal{F}(K, C) := \{ C \cup \{ w \} \mid w \in K \setminus C \}.
\] (2)

When \( j \) is clear from the context, we simply refer to a \( j \)-flower as a *flower*.

A *\( j \)-cycle* is a set \( J \) of \( j \)-simplices such that every \( (j - 1) \)-simplex is contained in an even number of \( j \)-simplices in \( J \).

**Definition 1.10.** A *copy of \( M_j \)* (see Figure 2) in a \( k \)-complex \( G \) is a triple \((K, C, J)\) where

(M1) \( K \) is a \( k \)-simplex in \( G \);

(M2) \( C \) is a \( (j - 1) \)-simplex in \( K \) such that each petal of the flower \( \mathcal{F} = \mathcal{F}(K, C) \) is contained in no other \( k \)-simplex of \( G \);

(M3) \( J \) is a \( j \)-cycle in \( G \) that contains exactly one petal of the flower \( \mathcal{F} \), i.e. there exists a vertex \( w_0 \in K \setminus C \) such that

\[
J \cap \mathcal{F} = \left\{ C \cup \{ w_0 \} \right\}.
\]
obstruction for $F$ is gradually increased from 0 to 1, we may interpret $k$ can define a sharp threshold for $F$.

We will see in Section 3.1 that a copy of $M_j$ is exactly the complex generated by $G_j$; more precisely, for each $(k+1)$-set of vertices in $[n]$ independently, sample a birth time uniformly at random from $[0,1]$. (With probability 1 no two $(k+1)$-sets have the same birth time.) Then $G_j$ is exactly the complex generated by the $(k+1)$-sets with birth times at most $p$, by taking the downward-closure. If $p$ is gradually increased from 0 to 1, we may interpret $G_j$ as a process. Thus, we can define $p_{M_j}$ as the birth time of the $k$-simplex whose appearance causes the last copy of $M_j$ to disappear. More formally, let

$$p_{M_j} := \sup\{p \in [0,1] \mid G_j \text{ contains a copy of } M_j\}. \quad (3)$$

Our first main result states that the value $p_{M_j}$ is the hitting time for $j$-cohom-connectedness of $G_j$ and is “close” to $p_j$ defined in (1), implying that $p_j$ is in fact a sharp threshold for $\mathbb{F}_2$-cohomological $j$-connectedness.

**Theorem 1.11.** Let $k \geq 2$ be an integer and let $\omega$ be any function of $n$ which tends to infinity as $n \to \infty$. For each $j \in [k-1]$, with high probability the following statements hold.

(i) \(\frac{(j+1)\log n + \log \log n - \omega}{(k-j-1)n^{k-j}}(k-j)! < p_{M_j} < \frac{(j+1)\log n + \log \log n + \omega}{(k-j+1)n^{k-j}}(k-j)!\).

(ii) For all $p < p_{M_j}$, $G_j$ is not $\mathbb{F}_2$-cohomologically $j$-connected, i.e.

\[ H^0(G_j; \mathbb{F}_2) \neq \mathbb{F}_2 \quad \text{or} \quad H^i(G_j; \mathbb{F}_2) \neq 0 \text{ for some } i \in [j]. \]

(iii) For all $p \geq p_{M_j}$, $G_j$ is $\mathbb{F}_2$-cohomologically $j$-connected, i.e.

\[ H^0(G_j; \mathbb{F}_2) = \mathbb{F}_2 \quad \text{and} \quad H^i(G_j; \mathbb{F}_2) = 0 \text{ for all } i \in [j]. \]

For the case $j = k - 1$, Theorem 1.11 gives a threshold $p_{k-1} = \frac{k\log n + \log \log n}{2n}$ for $\mathbb{F}_2$-cohomological $(k-1)$-connectedness, which is about half as large as the threshold $\frac{k\log n}{n}$ in Theorem 1.13 for $Y_p$. The reason for this is that the minimal obstructions are different: in $Y_p$, the minimal obstruction is a $(k-1)$-simplex which is not contained in any $k$-simplex of the complex (such a $(k-1)$-simplex is called

![Figure 2](image-url)
isolated). By definition, isolated \((k - 1)\)-simplices do not exist in \(\mathcal{G}_p\), because \(\mathcal{G}_p\) contains only those \((k - 1)\)-simplices that lie in some \(k\)-simplex.

Observe that Theorems 1.11 and 1.13 provide a hitting time result for the process described above. A similar result was proved by Kahle and Pitel [28] for \(\mathcal{Y}_p\), but only for the two-dimensional case. They considered the random complex process associated with \(\mathcal{Y}_p\) and related the vanishing of the first cohomology group to the disappearance of the last isolated edge (i.e. 1-simplex). As a corollary of Theorem 1.11, we obtain a hitting time result for \(\mathcal{Y}_p\) for general \(k \geq 2\). To this end, let

\[
\pi_{\text{isol}} := \sup\{p \in [0,1] \mid \mathcal{Y}_p \text{ contains isolated } (k - 1)\text{-simplices}\}
\]

be the birth time of the \(k\)-simplex whose appearance causes the last isolated \((k - 1)\)-simplex in \(\mathcal{Y}_p\) to disappear and let

\[
\pi_{\text{conn}} := \sup\{p \in [0,1] \mid H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) \neq 0\}
\]

be the time when \(\mathcal{Y}_p\) becomes \(\mathbb{F}_2\)-cohomologically \((k - 1)\)-connected. Then, with high probability

\[
\pi_{\text{conn}} = \pi_{\text{isol}}.
\]

In other words, with high probability the random process associated with \(\mathcal{Y}_p\) becomes \(\mathbb{F}_2\)-cohomologically \((k - 1)\)-connected at exactly the moment when the last isolated \((k - 1)\)-simplex disappears.

Our last main result gives an explicit expression for the limiting probability of the random complex \(\mathcal{G}_p\) being \(\mathbb{F}_2\)-cohomologically \(j\)-connected inside the critical window given by the threshold \(p_j\) (cf. Theorems 1.2 and 1.5). More generally, we prove that the dimension of the \(j\)-th cohomology group with coefficients in \(\mathbb{F}_2\) converges in distribution to a Poisson random variable.

**Theorem 1.13.** Let \(k \geq 2\) be an integer, \(j \in [k - 1]\) and \(c \in \mathbb{R}\) be a constant. Suppose that \(c_n\) is a sequence of real numbers that converges to \(c\) as \(n \to \infty\). If

\[
p = \frac{(j + 1) \log n + \log \log n + c_n (k - j)!}{(k - j + 1)^{2j}}
\]

then \(\dim (H^j(\mathcal{G}_p; \mathbb{F}_2))\) converges in distribution to a Poisson random variable with expectation

\[
\lambda_j := \frac{(j + 1)e^{-c}}{(k - j + 1)^{2j}}.
\]

while whp \(H^0(\mathcal{G}_p; \mathbb{F}_2) = \mathbb{F}_2\) and \(H^i(\mathcal{G}_p; \mathbb{F}_2) = 0\) for all \(i \in [j - 1]\). In particular,

\[
\mathbb{P}(\mathcal{G}_p \text{ is } j\text{-cohom-connected}) \xrightarrow{n \to \infty} e^{-\lambda_j}.
\]

Indeed, in the proof we will see that whp \(\dim (H^j(\mathcal{G}_p; \mathbb{F}_2))\) equals the number of pairs \((K, C)\) for which there exists a \(j\)-cycle \(J\) such that \((K, C, J)\) is a copy of \(M_j\) in \(\mathcal{G}_p\).

### 1.4. Related work

This paper draws inspiration from [30] and [35], but the proof techniques are considerably different. We first note that in \(\mathcal{Y}_p\) the presence of the full \((k - 1)\)-dimensional skeleton trivially yields the topological connectedness of \(\mathcal{Y}_p\) and the vanishing of all the \(i\)-th cohomology groups with \(i \in [k - 2]\). This is not true in \(\mathcal{G}_p\) and therefore we need to consider all cohomology groups up to dimension \(j\), for each \(j \in [k - 1]\).

Moreover, in [30] and [35] one standard application of the second moment method is sufficient for the analysis of the subcritical case (i.e. statement (i)) of Theorem 1.4. By contrast, \(\mathbb{F}_2\)-cohomological \(j\)-connectedness of \(\mathcal{G}_p\) is not a monotone increasing
property (see Example 3.2). This makes the subcritical case far from trivial. More precisely, it does not suffice to prove that \( \mathcal{G}_p \) is not \( j \)-cohom-connected at \( p_\cdot = (j+1)\log n + \log \log n - (k-j)! \); rather we need to show that whp the property is not satisfied for any \( p \) up to and including \( p_\cdot \). Also observe that in terms of our hitting time result, it is not enough to show that for each “small” \( p \) whp \( \mathcal{G}_p \) is not \( j \)-cohom-connected. Rather, we need to know that \( \mathcal{G}_p \) is not \( j \)-cohom-connected whp for all such \( p \) simultaneously.

The proof of the supercritical case \( p \geq p_{M_j} \) is also more challenging than for \( \mathcal{Y}_p \); we are forced to derive stronger bounds for the number of bad functions (see Definition 2.4), due to the fact that for \( j = k-1 \), the threshold in Theorem 1.11 is about half as large as the corresponding threshold in \( \mathcal{Y}_p \). To this end, we define a breadth-first search process that makes use of the new notion of traversability (Definition 5.3). Moreover, non-monotonicity of \( j \)-cohom-connectedness forces us to prove that for all \( p \geq p_{M_j} \), the probability of \( \mathcal{G}_p \) not being \( j \)-cohom-connected is small enough that we can apply a union bound over all relevant values of \( p \).

1.5. Paper overview. This paper is structured as follows.

In Section 2 we present some preliminary results that we will use throughout the paper and we provide an overview of cohomology theory, which will allow us to define the concept of a bad function (see Definition 2.4), a configuration in a complex \( \mathcal{G} \) that is a witness for \( H^j(\mathcal{G}; \mathbb{F}_2) \) not vanishing. Section 3 is devoted to the main concepts and the proof ideas used in this paper. After explaining why a copy of \( M_j \) is a minimal obstruction to \( j \)-cohom-connectedness, we heuristically show why the value \( p_j \) defined in (1) should be the threshold for \( j \)-cohom-connectedness and give an outline of the proofs of our main theorems.

In Section 4, we provide auxiliary results needed for the proofs of Theorem 1.11 (i) and (ii). We analyse the subcritical case when \( p < p_{M_j} \) and determine the approximate value of \( p_{M_j} \), i.e. when the last minimal obstruction disappears. In Section 5 we define a breadth-first search process which will allow us to examine the supercritical case when \( p \geq p_{M_j} \) and to obtain results necessary for the proofs of Theorem 1.11 (iii) and Theorem 1.13.

We prove the main results Theorems 1.11 and 1.13 and Corollary 1.12 in Section 6 using the auxiliary results from Sections 4 and 5. Finally, in Section 7 we discuss some open problems.

2. Preliminaries

2.1. Birth times. We mentioned in Section 1.3 how to use the standard birth times interpretation to describe the binomial model \( \mathcal{G}_p \) as a process. In this setting, it is useful to introduce the operation of “adding a simplex”.

**Definition 2.1.** Given a complex \( \mathcal{G} \) on vertex set \( V \) and a non-empty set \( B \subseteq V \), we define \( \mathcal{G} + B \) to be the complex obtained by adding the set \( B \) and its downward-closure to \( \mathcal{G} \), i.e.

\[
\mathcal{G} + B := \mathcal{G} \cup \{2^B \setminus \emptyset\}.
\]

Observe that if \( B \) is already a simplex of \( \mathcal{G} \), then \( \mathcal{G} + B = \mathcal{G} \). With this operation, \( \mathcal{G}_p \) (interpreted as a process) may also be described in the following way. If \( p_K \) is the smallest birth time larger than \( p \) of any \( k \)-simplex \( K \), then \( \mathcal{G}_{p_K} = \mathcal{G}_p + K \).

A property \( \mathcal{P} \) of \( k \)-complexes is called monotone increasing if \( \mathcal{P} \) is closed under adding \( k \)-simplices. The complement of a monotone increasing property is called monotone decreasing. Finally, \( \mathcal{P} \) is monotone if it is monotone increasing or decreasing.

Considering the birth times interpretation, we shall take union bounds over finite sets of birth times. With a slight abuse of terminology, sometimes we will
talk about taking “union bounds over p” in some interval, which makes little sense if we think of p as being able to take any value within the interval, but indeed we are conditioning on the set of birth times and taking the union bound over all birth times in the relevant interval.

We also note that conditioned on a k-simplex not being present at time \( p = q_1 \), the probability that it is present at time \( q_2 \) is \( \frac{q_2 - q_1}{1 - q_1} \). Thus we may obtain \( G_{q_2} \) from \( G_{q_1} \) by exposing an additional probability of \( \frac{q_2 - q_1}{1 - q_1} \). Since we will only ever want to consider such a situation with \( q_1 = o(1) \), we often simply take \( q_2 - q_1 \) as an approximation (and lower bound) for \( \frac{q_2 - q_1}{1 - q_1} \), or use \( q_2 \) as an upper bound.

2.2. Probabilistic tools. We frequently use the following Chernoff bound.

**Lemma 2.2** (see e.g. [23 Theorem 2.1]). Given a binomial random variable \( X \) with expectation \( \mu \) and a real number \( a > 0 \),

\[
\mathbb{P}(X \geq \mu + a) \leq \exp \left( -\frac{a^2}{2(\mu + a/3)} \right);
\]
\[
\mathbb{P}(X \leq \mu - a) \leq \exp \left( -\frac{a^2}{2\mu} \right).
\]

For the analysis of the critical window (cf. Theorem 1.12), we will need the method of moments, as presented in the following lemma.

**Lemma 2.3** (see e.g. [20 Theorem 20.11]). Let \( (S_n)_{n \geq 1} \) be a sequence of sums of indicator random variables. Suppose that there exists \( \lambda > 0 \) such that for every fixed integer \( t \geq 1 \)

\[
\lim_{n \to \infty} \mathbb{E} \left( \frac{S_n}{t} \right) = \frac{\lambda^t}{t!}.
\]

Then, for every integer \( s \geq 0 \),

\[
\lim_{n \to \infty} \mathbb{P}(S_n = s) = e^{-\lambda} \frac{\lambda^s}{s!},
\]
i.e. \( S_n \) converges in distribution to a Poisson random variable with expectation \( \lambda \). We write \( S_n \overset{d}{\to} Po(\lambda) \).

2.3. Cohomology terminology. We formally introduce cohomology groups with coefficients in \( F_2 \) for a simplicial complex. The following notions are all standard, except the definition of a bad function (Definition 2.4).

Given a k-complex \( G \), for each \( j \in \{0, \ldots, k\} \) denote by \( C^j(G) \) the set of \( j \)-cochains, that is, the set of 0-1 functions on the \( j \)-simplices. The support of a function in \( C^j(G) \) is the set of \( j \)-simplices mapped to 1. Each \( C^j(G) \) forms a group with respect to point-wise addition modulo 2. We define the coboundary operators \( \delta^j : C^j(G) \to C^{j+1}(G) \) for \( j = 0, \ldots, k - 1 \) as follows: for \( f \in C^j(G) \), the \((j + 1)\)-cochain \( \delta^j f \) assigns to each \((j + 1)\)-simplex \( \sigma \) the value

\[
\delta^j f(\sigma) := \sum_{\tau \subset \sigma, \; |\tau| = j+1} f(\tau) \quad (\text{mod} \; 2).
\]

In addition, we denote by \( \delta^{-1} \) the unique group homomorphism \( \delta^{-1} : \{0\} \to C^0(G) \). The \( j \)-cochains in \( \text{im}\delta^{-1} \) are called \( j \)-coboundaries, and the \( j \)-cochains in \( \text{ker}\delta^j \) are called \( j \)-cochains. A straightforward calculation shows that each coboundary operator is a group homomorphism and that every \( j \)-coboundary is also a \( j \)-cocycle, i.e. \( \text{im}\delta^{-1} \) is a subgroup of \( \text{ker}\delta^j \). Therefore, we can define the \( j \)-th cohomology group of \( G \) with coefficients in \( F_2 \) as the quotient group

\[
H^j(G; F_2) := \text{ker}\delta^j / \text{im}\delta^{-1}.
\]


By definition, \( H^j(\mathcal{G}; \mathbb{F}_2) \) vanishes if and only if every \( j \)-cocycle is a \( j \)-coboundary. This motivates the following definition of a bad function.

**Definition 2.4.** For a \( k \)-complex \( \mathcal{G} \) and \( j \in [k-1] \), we say that a function \( f \in C^j(\mathcal{G}) \) is **bad** if

(i) \( f \) is a \( j \)-cocycle, i.e. it assigns an even number of 1’s to the \( j \)-simplices on the boundary of each \( (j+1) \)-simplex;

(ii) \( f \) is not a \( j \)-coboundary, i.e. it is not induced by a \( (j-1) \)-cochain.

Thus, \( H^j(\mathcal{G}; \mathbb{F}_2) \) vanishes if and only if no bad function in \( C^j(\mathcal{G}) \) exists.

Recall that a set of \( j \)-simplices is a \( j \)-cycle if every \( (j-1) \)-simplex is contained in an even number of \( j \)-simplices of the set. It is easy to see that if \( f \) is a \( j \)-cocycle and \( J \) is a \( j \)-cycle such that the restriction \( f|_J \) has support of odd size, then \( f \) is not a \( j \)-coboundary and thus is a bad function.

### 3. Intuition and outline of proofs

For the rest of the paper, let \( j \in [k-1] \) be fixed.

#### 3.1. Minimal obstructions

Let us explain why \( M_j \) (Definition 1.10) can be interpreted as the (unique) minimal obstruction to \( j \)-cohom-connectedness. Given a triple \((K, C, J)\) which forms a copy of \( M_j \) in a \( k \)-complex \( \mathcal{G} \), it is easy to define a bad function \( f \in C^j(\mathcal{G}) \) (see Definition 2.4): let \( f \) take value 1 on the petals of the flower \( F(K, C) \) (see (2)) and 0 everywhere else. Since the petals are all in the \( k \)-simplex \( K \) but in no further \( k \)-simplices, every \((j+1)\)-simplex \( L \) in \( \mathcal{G} \) is even, because \( L \) contains either two petals (if \( C \subseteq L \subseteq K \)) or none (otherwise). However, \( J \) would be a \( j \)-cycle containing precisely one \( j \)-simplex, namely the petal \( C \cup \{w_0\} \), on which \( f \) takes value 1, ensuring that \( f \) is not a \( j \)-coboundary. Thus \( f \) is bad and has support of size \( k-j+1 \), which is the number of petals of \( F(K, C) \).

In the following lemma we show that in fact such a bad function is the only possibility for an obstruction which is minimal with respect to the size of the support. Given a \( k \)-simplex \( K \) and a collection \( S \) of \( j \)-simplices, define \( S_K \) to be the set of \( j \)-simplices of \( S \) contained in \( K \).

**Lemma 3.1.** Let \( \mathcal{G} \) be a \( k \)-complex and let \( S \) be the support of a \( j \)-cocycle. Then for each \( k \)-simplex \( K \),

(i) either \( S_K = \emptyset \) or both \( |S_K| \geq k-j+1 \) and \( \bigcup_{\sigma \in S_K} \sigma = K \);

(ii) if \( |S_K| = k-j+1 \), then \( S_K \) forms a \( j \)-flower in \( K \).

**Proof.** \[(i)\] Suppose \( S_K \neq \emptyset \) and let \( \sigma_0 \in S_K \). Let the vertices of \( K \setminus \sigma_0 \) be denoted by \( v_1, \ldots, v_{k-j} \). Each \((j+1)\)-simplex \( \sigma_0 \cup \{v_i\} \) has to be even with respect to \( f \) and thus contains some \( j \)-simplex \( \sigma_i \in S_K \setminus \{\sigma_0\} \), which therefore contains \( v_i \). The simplices \( \sigma_0, \ldots, \sigma_{k-j} \) are distinct, because each \( v_i \) lies in \( \sigma_i \) but in no other \( \sigma_j \).

Therefore \( |S_K| \geq k-j+1 \) and

\[
K \supseteq \bigcup_{\sigma \in S_K} \sigma \supseteq \sigma_0 \cup \{v_1, \ldots, v_{k-j}\} = K.
\]

\[(ii)\] Suppose now that \( S_K = \{\sigma_0, \ldots, \sigma_{k-j}\} \), with \( \sigma_0, \ldots, \sigma_{k-j} \) defined as above. For \( 2 \leq i \leq k-j \), the \((j+1)\)-simplex \( \tau := \sigma_1 \cup \{v_i\} \) contains \( \sigma_1 \), but no \( \sigma_i \) with \( \ell \notin \{1, i\} \). By the choice of \( S \) as the support of a \( j \)-cocycle, \( \tau \) is even and thus \( \sigma_i \subset \tau \). This means that

\[
\sigma_1 \cap \sigma_i = \tau \setminus \{v_1, v_i\} = \sigma_0 \cap \sigma_1.
\]

As this holds for all \( i \), \( S_K \) forms a flower in \( K \) with centre \( \sigma_0 \cap \sigma_1 \). \( \square \)
Both the presence of a copy of $M_j$ and $j$-cohom-connectedness in $G_p$ are not monotone properties, as the following example shows.

**Example 3.2.** Let $G$ be the 2-complex on vertex set $\{1, 2, 3, 4, 5\}$ generated by the 3-uniform hypergraph with hyperedges $\{1, 2, 3\}$ and $\{1, 4, 5\}$, see Figure 3. Then $G$ is 1-cohom-connected and thus contains no copies of $M_1$. Adding to $G$ the 2-simplex $\{2, 3, 4\}$ (and its downward-closure) creates several copies of $M_1$ and thus yields a complex $G'$ which is not 1-cohom-connected. If we further add the 2-simplex $\{1, 3, 4\}$ to $G'$, we obtain a 2-complex $G''$ which is 1-cohom-connected and thus contains no copies of $M_1$.

![Figure 3. Adding simplices might create new copies of $M_j$ or destroy existing ones.](image)

3.2. **Finding the threshold.** In this section we provide a heuristic argument for why the threshold for the disappearance of the last copy of $M_j$ should be around $p_j$. To do this, we will make use of a simplified version of the obstruction $M_j$.

**Definition 3.3.** A copy of $M_j^-$ (see Figure 4) in a $k$-complex $G$ is a pair $(K, C)$ where

- (M1) $K$ is a $k$-simplex in $G$;
- (M2) $C$ is a $(j - 1)$-simplex in $K$ such that each petal of the flower $F(K, C)$ is contained in no other $k$-simplex of $G$.

![Figure 4. A copy of $M_j^-$, for $k = 5$ and $j = 2$. The $k$-simplex $K$ contains the flower $F(K, C)$ with centre $C = \{c_1, c_2\}$ and petals $P_i = C \cup \{w_i\}$, for $i = 0, 1, 2, 3$. Each petal $P_i$ is contained in no other $k$-simplex except $K$.](image)

In other words, a copy of $M_j^-$ can be viewed as a copy of $M_j$ without the condition [M3] i.e. without the $j$-cycle $J$ containing one of the petals (see Figures 2 and 4). Therefore,

$$M_j^- \not\subset G_p \implies M_j \not\subset G_p.$$
Moreover, we will show (Lemma 4.6) that, for \( p \) approaching the value \( p_j \), the \( j \)-cycle \( J \) needed to extend a copy of \( M_j^- \) to a copy of \( M_j \) is very likely to exist. Hence in this range the existence of \( M_j^- \) and \( M_j \) are essentially equivalent events.

Let us estimate the expected number of copies of \( M_j^- \) in \( G_p \). The probability of \( k + 1 \) arbitrary vertices with a fixed centre \( C \) forming a copy of \( M_j^- \) is about \( p(1 - p)^{(k-j+1)C_j} \), which we can approximate by

\[
p c^{-(k-j+1)C_j} p^k,
\]

so the expected number of copies of \( M_j^- \) is of order \( n^{k+1}p c^{-(k-j+1)C_j} p^k \). We seek \( p \) such that

\[
n^{k+1}p c^{-(k-j+1)C_j} p^k = 1.
\]

This holds when

\[
(k + 1) \log n + \log p - \frac{(k - j + 1)n^{k-j}}{(k - j)!} p = 0,
\]

which implies

\[
p = \frac{(k + 1) \log n + \log p}{(k - j + 1)n^{k-j} (k - j)!} (k - j)! = \frac{(j + 1) \log n + \log n + O(1)}{(k - j + 1)n^{k-j} (k - j)!},
\]

which corresponds to the stated threshold \( p_j \) defined in \( \text{(1)} \).

3.3. Outline of the proofs. We now give an outline of the proofs of our main theorems. Let us begin with Theorem 1.11. To analyse the zero-th cohomology group, we define the probabilities

- \( p_0 := \frac{\log n}{n} \cdot p \);
- \( p_T := \sup \{ p \in [0, 1] \mid G_p \text{ is not topologically connected} \} \).

In other words, \( p_T \) is the birth time of the \( k \)-simplex whose appearance causes the complex \( G_p \) to become topologically connected. Recall that topological connectedness is equivalent to the random hypergraph \( G_p \) becoming vertex-connected. It is known (see e.g. \( \text{[12, 35, 39]} \)) that \( p_0 \) is the threshold for vertex-connectedness of the random \( (k + 1) \)-uniform hypergraph, that is whp \( p_T = (1 + o(1))p_0 \) (Lemma \( \text{(2)} \)).

Recall from \( \text{(4)} \) and \( \text{(5)} \) that for each \( j \in [k - 1] \) we have

- \( p_j := \frac{(j + 1) \log n + \log n}{(k - j + 1)n^{k-j} (k - j)!}; \)
- \( p_{M_j} := \sup \{ p \in [0, 1] \mid G_p \text{ contains a copy of } M_j \}. \)

In other words, \( p_{M_j} \) is the birth time of the \( k \)-simplex whose appearance causes the last copy of \( M_j \) to disappear.

In Section \( \text{(4)} \) we study the subcritical case when \( p < p_{M_j} \), providing results needed for the proof of Theorem 1.11(ii). Moreover, we show that whp the value of \( p_{M_j} \) is “close” to \( p_j \) (Corollary \( \text{(4.1)} \)), thus proving Theorem 1.11(i)

In order to prove Theorem 1.11(ii) we aim to show that whp \( H^2(G_p; \mathbb{F}_2) \neq 0 \) throughout the interval \( [p_{M_{j-1}}, p_{M_j}] \). A direct argument based on determining the dimensions of \( C^j(G_p) \), \( C^{j+1}(G_p) \) and \( C^{j+1}(G_p) \) may be considered, but it would work only for some values of \( j \) and some ranges of \( p \) (see Section 7.1). We actually prove a stronger result (Lemma \( \text{(5.3)} \)), for which we define the following probabilities: for each \( j \in [k - 1] \), set
This completely covers the subcritical case (Theorem 1.11 (ii)).

G from copies of any smallest j supports of size we call traversability. This will in particular imply that whp and show that for each of these subintervals, whp there is one copy of G. We will also need the value

\[ p_0^- := \log \frac{n}{n^k} = \frac{p_0}{k!}. \]

The motivation behind these seemingly arbitrary definitions will become clear as the argument develops. We will prove that three copies of M_j suffice to cover the interval \([p_{j-1}, p_{M_j}]\), which whp contains the interval \([p_{M_j-1}, p_{M_j}]\) by Theorem 1.11 (i).

Lemma 3.4. Let \(j \in [k - 1]\). With high probability, there exist three triples \((K_\ell, C_\ell, J_\ell)\), \(\ell = 1, 2, 3\), such that for all \(p \in [p_{j-1}, p_{M_j}]\), \((K_\ell, C_\ell, J_\ell)\) forms a copy of \(M_j\) in \(G_p\) for some \(\ell\). In particular, whp \(H^1(G_p; \mathbb{F}_2) \neq 0\) for all \(p \in [p_{j-1}, p_{M_j}]\).

This will in particular imply that whp \(G_p\) is not \(j\)-cohom-connected in the interval \([p_{j-1}, p_{M_j}]\). By Lemma 3.3 applied with \(j\) replaced by \(i\) for each \(i \in [j]\) and by the fact that \(G_p\) is not topologically connected in \([0, p_T)\) by definition, whp \(G_p\) is not \(j\)-cohom-connected in the range

\[ [0, p_T) \cup \bigcup_{i=1}^{j} [p_{i-1}, p_{M_i}) \] (whp) \[= [0, p_{M_j}). \]

This completely covers the subcritical case (Theorem 1.11 (ii)).

In order to prove Lemma 3.4 we divide the interval \([p_{j-1}, p_{M_j}]\) into smaller subintervals

\[ [p_{j-1}, p_{M_j}] = [p_{j-1}, p_{j}^{(1)}] \cup [p_{j}^{(1)}, p_{j}] \cup [p_{j}, p_{M_j}] \]

and show that for each of these subintervals, whp there is one copy of \(M_j\) which exists in \(G_p\) throughout this interval, using the following strategy.

(I) At around \(p_{j-1}\), whp there exist “many” copies of \(M_j\) (Lemma 1.4) and whp at least one of these survives until probability \(p_{j}^{(1)}\) (Lemma 1.12).

(II) For any \(p \geq p_{j}^{(1)}\), whp all copies of \(M_j^-\) give rise to copies of \(M_j\), thus the existence of \(M_j^-\) and \(M_j\) are essentially equivalent events (Lemma 4.10). In particular, the last \(M_j\) to disappear corresponds to the last \(M_j^-\) (Corollary 4.10).

(III) At around \(p_j^-\), whp there are “many” copies of \(M_j^-\) (Lemma 5.7) and whp one of these already existed at \(p_{j}^{(1)}\) (Lemma 4.13).

(IV) The last \(M_j^-\) to disappear whp already existed at \(p_j^-\) (Lemma 4.14).

In Section 5 we study the supercritical case, i.e. the case \(p \geq p_{M_j}\), and derive auxiliary results, necessary to prove Theorem 1.11 (iii). By the definition of \(p_{M_j}\), we know that \(G_p\) contains no \(M_j\) in this range, so by Lemma 3.1 it remains to show that whp there are no bad functions with support of size \(s > k - j + 1\). In other words, we need to prove that each \(j\)-cocycle with support of size \(s\) is also a \(j\)-coboundary.

To this end, we prove (Corollaries 5.8 and 5.10 that from slightly before the threshold \(p_j\) onwards, every \(j\)-cocycle can be written as the sum of functions arising from copies of \(M_j^-\) (see Definition 5.1). We first show (Lemma 5.4) that the support of any smallest \(j\)-cocycle not generated by copies of \(M_j^-\) satisfies a property which we call traversability (Definition 5.3). We then bound the probability that such a support of size \(s\) exists. For constant \(s\), simple bounds will suffice (Lemma 5.5); for larger values of \(s\), traversability will allow us to define a breadth-first search process.
that we use to track the construction of a traversable support and thus count the
number of such supports much more accurately (Lemma 5.7).

Combining the results from Sections 4 and 5 we prove Theorem 1.11 in Section 6. We then apply Theorem 1.11 to derive Corollary 1.12 which provides a hitting time result for $Y_p$, relating the vanishing of $H^{k-1}(Y_p; \mathbb{F}_2)$ to the disappearance of the last isolated $(k - 1)$-simplex.

Finally, we prove Theorem 1.13 in Section 6.3. We analyse $\mathbb{F}_2$-cohomological $j$-connectedness of $G_p$ within the critical window given by the threshold for this property, i.e. we consider $p = \frac{(j + 1) \log n + \log \log n + O(1)}{(k - j + 1)n^k}$. In this range, whp all $j$-cocycles arise from copies of $M_j^{-}$ (Corollary 5.8). Using the method of moments (Lemma 2.3), we will show that the number of copies of $M_j^{-}$ converges in distribution to a Poisson random variable and that whp this number equals the dimension of the $j$-th cohomology group of $G_p$. Thus, in particular we derive an explicit expression for the limiting probability of $G_p$ being $j$-cohom-connected.

4. Subcritical regime

In this section we study the subcritical case $p < p_M$ and derive the necessary results for the proofs of statements (i) and (ii) of Theorem 1.11.

4.1. Topological connectedness. We begin with a result stating that $p_0 = \frac{\log n}{n^k} k!$ is a sharp threshold for topological connectedness of $G_p$. Recall that $p_T$ is the birth time of the $k$-simplex whose appearance causes the complex $G_p$ to become topologically connected.

Lemma 4.1. Let $\omega$ be any function of $n$ which tends to infinity as $n \to \infty$. Then with high probability

$$\log n - \omega \frac{n^k}{k!} < p_T < \log n + \omega \frac{n^k}{k!}$$

and thus in particular $p_T > p_0$.

Observe that Lemma 4.1 is equivalent to $p_0$ being a sharp threshold for vertex-connectedness of the random $(k + 1)$-uniform hypergraph, which follows for instance from [12] or [39] as a special case of each (see also [38] for a stronger result). The proof relies on standard applications of the first and second moment methods and is an easy generalisation of the graph case (see e.g. [29]).

4.2. Counting obstructions. In this section we provide several results concerning the number of minimal obstructions that exist in $G_p$ whp. First we define a special case of $M_j$ (Definition 4.3), which will be useful in the subsequent arguments.

Definition 4.2. For any $(j + 2)$-set $A$ in a complex $G$, the collection of all $(j + 1)$-subsets of $A$ is called a $j$-shell if each of them forms a $j$-simplex in $G$. The $j$-shell is called hollow if $A$ does not form a $(j + 1)$-simplex in $G$.

If the collection of all $(j + 1)$-subsets of a $(j + 2)$-set $A$ forms a $j$-shell, with a slight abuse of terminology we also refer to the set $A$ itself as a $j$-shell.

Definition 4.3. Given a $k$-complex $G$ on vertex set $[n]$, a $(k + 1)$-set $K$ in $G$, a $j$-set $C \subseteq K$, and two vertices $w \in K \setminus C$ and $a \in [n] \setminus K$, we say that the 4-tuple $(K, C, w, a)$ forms a copy of $M_j^{*}$ (see Figure 5) if

- (M1) $K$ is a $k$-simplex in $G$;
- (M2) $C$ is a $(j - 1)$-simplex in $K$ such that each petal of the flower $F(K, C)$ is contained in no other $k$-simplex of $G$;
Recall that \([M1]\) and \([M2]\) mean that \((K, C)\) forms a copy of \(M_j^-\) (see Definition \(5.3\)). We call the \(j\)-simplex \(C \cup \{w\}\) the base and \(a\) the apex vertex of the \(j\)-shell \(C \cup \{w\} \cup \{a\}\). Every other \(j\)-simplex in \(C \cup \{w\} \cup \{a\}\) is called a side of the \(j\)-shell.

\[
\text{Figure 5. A copy of } M_j^*, \text{ for } k = 5 \text{ and } j = 2. \text{ The pair } (K, C), \text{ with } K \text{ a } k\text{-simplex and } C = \{c_1, c_2\}, \text{ forms a copy of } M_j^-_. \text{ The } (j + 2)\text{-set } C \cup \{w\} \cup \{a\} \text{ is a } j\text{-shell with base } C \cup \{w\} \text{ and apex vertex } a.
\]

Observe that given a 4-tuple \((K, C, w, a)\) which forms a copy of \(M_j^*\) in \(G_p\), the \(j\)-shell \(C \cup \{w\} \cup \{a\}\) is hollow by \([M2]\) and the fact that every \((j + 1)\)-simplex in \(G_p\) is contained in a \(k\)-simplex. Moreover, since the \(j\)-simplices of a \(j\)-shell form a \(j\)-cycle, a copy of \(M_j^*\) is in particular a copy of \(M_j\) (Definition \(1.10\)). Therefore, the following implications hold.

\[
M_j^* \subset G_p \Rightarrow M_j \subset G_p \Rightarrow M_j^- \subset G_p. \tag{5}
\]

We will see later (Lemma \(4.6\)) that for “large” \(p\), whp every copy of \(M_j^-\) is extendable to several copies of \(M_j^*\). Therefore, the existence of copies of \(M_j^-\), \(M_j^*\) and \(M_j\) in \(G_p\) are essentially equivalent events in that range.

Define \(X_*\) to be the number of copies of \(M_j^*\) in \(G_p\). We need a general expression for its expectation for certain possible values of the probability \(p\). To this end, consider the family \(T^*\) of 4-tuples \(T^* = (K, C, w, a)\), where \(K \subseteq [n]\) with \(|K| = k + 1\), where \(C\) is a \(j\)-subset of \(K\), where \(w \in K \setminus C\), and where \(a \in [n] \setminus K\). Each of these tuples may form a copy of \(M_j^*\) with \(K\) as \(k\)-simplex, \(C\) as the centre, and \(C \cup \{w\} \cup \{a\}\) as the \(j\)-shell with base \(C \cup \{w\}\) and apex vertex \(a\). For each such tuple \(T^*\), let \(X_{T^*}\) be the indicator random variable of the event that \(T^*\) forms a copy of \(M_j\).

We next show that at probability \(p_{j-1}^-\) the number of copies of \(M_j^*\) is concentrated around its expectation, whose order we also determine.

**Lemma 4.4.** If \(p = p_{j-1}^-\), then \(E(X_*) = \Theta((\log n)^j/\beta^j)\). Furthermore, with high probability \(X_* = (1 + o(1))E(X_*)\).

**Proof.** Let \(T^* = (K, C, w, a) \in T^*\) be a fixed 4-tuple. Recall that \(T^*\) forms a copy of \(M_j^*\) in \(G_p\) if conditions \([M1]\), \([M2]\) and \([M3*]\) of Definition \(4.3\) hold.

Clearly, \([M1]\) holds with probability \(p\). In order to determine the probability that \([M2]\) holds, consider a fixed petal. The probability that this petal lies in no other \(k\)-simplex is

\[
r = r(p, n, k, j) := (1 - p)^{(n - j - 1)\cdot r(k, j)}. \tag{6}
\]
For $p = p_{j-1} = \Theta \left( \frac{\log n}{n^{k-j+1}} \right)$, we have
\[
r \geq 1 - \left( \frac{n - j - 1}{k - j} \right) p = 1 - o(1),
\]
and thus each petal lies in no other $k$-simplices whp. Therefore, taking a union bound, $[M2]$ holds with probability at least $1 - (k - j + 1)(1 - r) = 1 - o(1)$.

Now consider $[M3^*]$ conditioned on the event that both $[M1]$ and $[M2]$ hold. The base $C \cup \{ w \}$ of $C \cup \{ w \} \cup \{ a \}$ already lies in $K$, so it remains to prove that all other $(j + 1)$-sets in $C \cup \{ w \} \cup \{ a \}$, i.e. the sides of this (potential) $j$-shell, are $j$-simplices in $G_p$. Denote the sides of $C \cup \{ w \} \cup \{ a \}$ by $L_1, \ldots, L_{j+1}$. The number of $(k + 1)$-sets containing $L_i$ is ${n - j - 1 \choose k - j}$, but some of these $(k + 1)$-sets might not be allowed to be $k$-simplices because they contain a petal of the flower $F(K, C)$ (see (2)). However, the number of $(k + 1)$-sets for which this is the case is $O(n^{k-j-1})$. All other $(k + 1)$-sets meet $C \cup \{ w \} \cup \{ a \}$ only in $L_i$, which in particular implies that the events that $L_1, \ldots, L_{j+1}$ lie in $k$-simplices (conditional on $[M1]$ and $[M2]$) are independent. Thus, each $L_i$ lies in a $k$-simplex independently with probability
\[
1 - (1 - p)^{n - j - 1 + O(n^{k-j-1})} = (1 + o(1))q,
\]
where
\[
q := \frac{pn^{k-j}}{(k-j)!} = \Theta \left( \frac{\log n}{n} \right). \tag{8}
\]

Therefore, conditional on $[M1]$ and $[M2]$ holding, $[M3^*]$ holds with probability $(1 + o(1))q^{j+1}$. The probability that $T^*_\ast$ forms a copy of $M^*_j$ is thus
\[
(1 + o(1))pq^{j+1}.
\]

The number of $4$-tuples $(K, C, w, a) \in T^*$ is
\[
\binom{n}{k+1} \binom{k+1}{j} (k-j+1)(n-k-1) = (1 + o(1)) \frac{n^{k+2}}{j!(k-j)!}
\]
and thus we have
\[
E(X^*_\ast) = (1 + o(1)) \frac{pq^{j+1}n^{k+2}}{j!(k-j)!} = \Theta \left( \frac{(\log n)^{j+2}}{j!(k-j)!} \right), \tag{9}
\]
as required.

In order to prove the second statement of the lemma, we will show that $E(X^*_2) = (1 + o(1))E(X^*_\ast)^2$ and then apply Chebyshev’s inequality. We have
\[
E(X^*_2) = \sum_{T^*_1, T^*_2 \in T^*} \mathbb{P} \left( \{ X_{T^*_1} = 1 \} \cap \{ X_{T^*_2} = 1 \} \right).
\]

Given two $4$-tuples $T^*_1 = (K_1, C_1, w_1, a_1)$ and $T^*_2 = (K_2, C_2, w_2, a_2)$, we define
- $I = I(T^*_1, T^*_2) := (K_1 \cup \{ a_1 \}) \cap (K_2 \cup \{ a_2 \})$ and $i := |I|$;
- $s = s(T^*_1, T^*_2) := \begin{cases} 1 & \text{if } K_1 = K_2, \\ 2 & \text{otherwise}; \end{cases}$
- $L_\ell$ to be the set of all $(j+1)$-subsets of $\{ C_\ell \cup \{ a_\ell \} \cup \{ w_\ell \} \}$ for $\ell = 1, 2$ and $t = t(T^*_1, T^*_2) := |(L_1 \cup L_2) \setminus \{ C_1 \cup \{ w_1 \}, C_2 \cup \{ w_2 \})|$, i.e. the number of $(j+1)$-sets that are sides of the (potential) $j$-shells of $T^*_1$ and $T^*_2$, but not a base of either $j$-shell.

If $s = 2$ and the intersection of the two simplices contains a petal, then $T^*_1$ and $T^*_2$ cannot both form an $M^*_j$, because $[M2]$ would be violated. In the following, we therefore assume that this is not the case.
The probability that both $T_1^*$ and $T_2^*$ satisfy $\text{[M1]}$ is $p^*$. As before, $\text{[M2]}$ holds whp. Conditioned on $\text{[M1]}$ and $\text{[M2]}$ holding, we claim that $\text{[M3*]}$ holds for both tuples simultaneously with probability $(1 + o(1))q^t$. In order to prove this, denote the relevant sides of the two $j$-shells by $L_1, \ldots, L_t$. No $k$-simplex can contain more than one side of the same $j$-shell, because otherwise it would also contain the base of the $j$-shell, which would contradict $\text{[M2]}$. In particular, no $k$-simplex contains all the summands, distinguishing the possible values of $s$, all the summands, distinguishing the possible values of $s$, and thus the sets of sides of the two $j$-shells are disjoint, i.e. $t = 2j + 2$. Therefore we get a contribution of order

$$O \left( p q^{2j+2} n^{2k+4-(k+1)} \right) = O \left( \frac{(X_s)^2}{pn^{k+1}} \right) = o \left( \mathbb{E}(X_s)^2 \right).$$

Observe that $|\mathcal{T}^2(i, s, t)| = O(n^{2k+4-i})$. We can now estimate the contributions of all the summands, distinguishing the possible values of $s$ and $i$.

**Case 1:** $s=1$. This means that $K_1 = K_2$ and thus $i \geq k+1$.

- $i = k+1$. In this case $a_1 \neq a_2$ and thus the sets of sides of the two $j$-shells would be disjoint, i.e. $t = 2j + 2$. Therefore we get a contribution of order

$$O \left( p q^{2j+2} n^{2k+4-(k+1)} \right) = O \left( \frac{(X_s)^2}{pn^{k+1}} \right) = o \left( \mathbb{E}(X_s)^2 \right).$$

- $i = k+2$. The two $j$-shells have the same apex vertex and thus the $j$-shells coincide if and only if they have the same base. This means that $t \geq j+1$, which gives a contribution of order

$$O \left( pq^{2j+2} n^{2k+4-(k+2)} \right) = O \left( \mathbb{E}(X_s) \right) = o \left( \mathbb{E}(X_s)^2 \right).$$

**Case 2:** $s=2$.

- $i = 0$. We show that this case represents the dominant contribution to $\mathbb{E}(X_s^2)$. The two $j$-shells are disjoint, hence $t = 2j + 2$. Recall that we have

$$(1 + o(1)) \frac{n^{k+2}}{j!(k-j)!}$$

choices for $T_1^*$. For any fixed $T_1^*$, the number of choices for $T_2^*$ that yield $i = 0$ is

$$\binom{n-k-1}{k+1} \binom{k+1}{j} (k-j+1)(n-2k-3) = (1 + o(1)) \frac{n^{k+2}}{j!(k-j)!}.$$

Thus, the contribution of all such pairs is

$$O \left( (1 + o(1)) \frac{pq^{2j+2} n^{2k+4}}{(j!(k-j)!)^2} \right) = O \left( (1 + o(1)) \mathbb{E}(X_s)^2 \right).$$
\textbf{Proof.} Recall that

\begin{equation}
\Theta \left( pq^{2j+2}n^{k+3} \right) \leq \Theta \left( \frac{(\log n)^{2j+3}}{n^j} \right) = o(1),
\end{equation}

by Markov’s inequality we deduce that whp in \( G_{p_{i-1}} \) each copy of \( M_j^- \) can be extended to at most one copy of \( M_j^+ \). We will make use of this observation in Lemma 4.12.

In contrast to Remark 4.5, the following lemma ensures that at around \( p = p_j^{(1)} \), whp every \( j \)-simplex in \( G_{p} \) is the base of “many” \( j \)-shells. Thus it is very likely that each copy of \( M_j^- \) gives rise to several copies of \( M_j^+ \), allowing us to consider just copies of \( M_j^- \) as obstructions to \( j \)-cohom-connectedness. In other words, whp for each \( p \geq p_j^{(1)} \),

\[ M_j^- \subset G_p \implies M_j^+ \subset G_p. \]

Combining this with (3), the existence of copies of \( M_j^- \), \( M_j^+ \) and \( M_j \) are essentially equivalent for \( p \geq p_j^{(1)} \). Recall from Definition 4.11 that for a complex \( \mathcal{G} \) and a set \( B, \mathcal{G} + B \) is the complex obtained by adding the set \( B \) and its downward-closure to \( \mathcal{G} \).

\textbf{Lemma 4.6.} Let \( p = p_j^{(1)} \). Then there exists a positive constant \( \gamma \) such that with high probability for every \((j + 1)\)-set \( B \) the complex \( \mathcal{G}_p + B \) contains at least \( \gamma n \) many \( j \)-shells that contain \( B \).

\textbf{Proof.} Recall that

\[ p = p_j^{(1)} = \frac{1}{10(j+1)(j+1)!} n^{k-j}. \]

Let \( L_1, \ldots, L_{j+1} \) denote the \((j - 1)\)-simplices contained in \( B \). We are interested in the number of vertices \( a \) such that \( B \cup \{a\} \) forms a \( j \)-shell, i.e. the number of \( a \notin B \) such that \( L_i \cup \{a\} \) is a \( j \)-simplex in \( \mathcal{G}_p + B \) for all \( i \in [j+1] \). To ensure independence in the following calculations, we will only consider a certain type of such \( j \)-shells, giving us a lower bound on their total number. Pick two disjoint sets \( A \) and \( D \) both of size \( \lceil n/3 \rceil \) such that \( A \cap B = D \cap B = \emptyset \). We will consider only (potential) \( j \)-shells formed in the following way.
The vertex $a$ is in $A$;  
for each $i = 1, \ldots, j + 1$, the $j$-simplex $L_i \cup \{a\}$ is present in $G_p$ (and thus also in $G_p + B$) as a subset of the $k$-simplex $R_i \cup L_i \cup \{a\}$, for some (not necessarily distinct) $(k - j)$-sets $R_1, \ldots, R_{j+1}$ in $D$.

In this way all the required $j$-simplices would come from different $k$-simplices, ensuring independence.

Fix $a \in A$ and let $E_a$ be the event that $B \cup \{a\}$ is a $j$-shell. Observe that for each $L_i$, the probability that there is no suitable set $R_i \subseteq D$ is
\[
(1 - p)^{\binom{|D|}{k-j}} \leq (1 - p)^{\frac{n^{k-j}}{3(k-j)!}}.
\]

Therefore, setting $\beta := 10(j + 1) \binom{k+1}{j+1} 4^{k-j}(k-j)!$, by independence we have
\[
\mathbb{P}(E_a) \geq \left(1 - (1 - p)^{\frac{n^{k-j}}{3(k-j)!}}\right)^{j+1} \geq \left(\frac{n^{k-j}}{4^{k-j}(k-j)!} - \frac{1}{2} \left(\frac{n^{k-j}}{4^{k-j}(k-j)!}\right)^2\right)^{j+1} = \left(\frac{1}{\beta} - \frac{1}{2\beta^2}\right)^{j+1} = : \lambda > 0.
\]

The events $E_a$ are independent for distinct $a$, so the number of $j$-shells we count in this way dominates $\text{Bi}(\lceil n/3 \rceil, \lambda)$. Fixing a constant $0 < \gamma < \lambda/3$, we can apply the Chernoff bound (Lemma 2.2) to deduce that
\[
\mathbb{P}(\text{Bi}(\lceil n/3 \rceil, \lambda) < \gamma n) \leq \exp\left(-\frac{(n\lambda/3 - \gamma n)^2}{2n\lambda/3}\right) = \exp\left(-\frac{n(\lambda/3 - \gamma)^2}{2\lambda/3}\right).
\]

Finally, taking a union bound over all $\binom{n}{j+1}$ possible choices for the set $B$, we can bound the probability that the desired property does not hold by
\[
\left(\frac{n}{j+1}\right) \exp\left(-\frac{n(\lambda/3 - \gamma)^2}{2\lambda/3}\right) = o(1),
\]
as required. \hfill \Box

We now also prove that shortly before the (claimed) critical threshold for $\mathbb{F}_2$-cohomological $j$-connectedness, the number of copies of $M^-_j$ is concentrated around its expectation, using similar techniques as in Lemma 4.3.

**Lemma 4.7.** Let $\omega = o(\log n)$ be a function of $n$ which tends to infinity as $n \to \infty$. Let
\[
p \in \left[\frac{p^-}{j+1}, \frac{(j + 1) \log n + \log \log n - \omega}{(k - j + 1)n^{k-j}}\right],
\]
and let $X_-$ be the number of copies of $M^-_j$ in $G_p$. Then $\mathbb{E}(X_-) = \Omega(c^\omega)$ and with high probability $X_- = (1 + o(1))\mathbb{E}(X_-)$.

**Proof.** Let $K$ be a $(k+1)$-set and let $C$ be a $j$-set in $K$. In order for $(K, C)$ to form a copy of $M^-_j$, we need $K$ to be a $k$-simplex and each petal of the flower $F(K, C) = \{C \cup \{w\} \mid w \in K \setminus C\}$ to lie in no other $k$-simplex. For a fixed petal, the probability of this event is equal to $r = (1 - p)^{\binom{k-j}{j+1} - 1}$ defined in (9). Moreover, there are $O(n^{k-j-1})$ many $(k+1)$-sets that contain more than one petal. Now since
\[
(1 - p)^{\binom{n^{k-j}}{j+1}} = 1 - o(1),
\]

whp there are no $k$-simplices containing more than one petal. Thus,

$$
\mathbb{E}(X_{-}) = (1 + o(1))\left(\frac{n}{k+1}\right)\left(\begin{array}{c} k+1 \\ j \end{array}\right)pr^{k-j+1}
= (1 + o(1))\left(\frac{n}{k+1}\right)\left(\begin{array}{c} k+1 \\ j \end{array}\right)p(1-p)^{(k-j+1)(n-j)}.
$$

(11)

The derivative of the right hand side of (11) with respect to $p$ is negative throughout the considered interval. Therefore the upper extreme of $p$ gives the smallest expectation, which is of order

$$
\Theta(n^{k+1})\Theta\left(\frac{\log n}{n^{k-j}}\right)\Theta(\exp(-j+1)\log n - \log \log n + \omega)) = \Theta(e^\omega) \to \infty.
$$

In order to apply a second moment argument, we will now show that

$$
\mathbb{E}(X_2^2) = (1 + o(1))\mathbb{E}(X_-)^2,
$$

implying that whp $X_-$ is concentrated around its expectation. Let $T^-$ denote the family of pairs $T^- = (K, C)$, where $K \subseteq [n]$ with $|K| = k+1$ and $C$ is a $j$-subset of $K$. Each of these pairs may form a copy of $M_j$ with $K$ as $k$-simplex and $C$ as centre of the flower $\mathcal{F}(K, C)$.

Given two pairs $T^-_1 = (K_1, C_1)$ and $T^-_2 = (K_2, C_2)$, we define

- $s = s(T^-_1, T^-_2) := \begin{cases} 1 & \text{if } K_1 = K_2, \\ 2 & \text{otherwise}; \end{cases}$
- $\mathcal{F}_\ell := \mathcal{F}(K, C)$ for $\ell = 1, 2$;
- $t = t(T^-_1, T^-_2) := |\mathcal{F}_1 \cup \mathcal{F}_2|$, i.e. the total number of (potential) petals.

The probability of two pairs in $T^-$ both forming a copy of $M_j$ is $(1 + o(1))pr^t$. With this observation, we can determine the contribution to $\mathbb{E}(X_2^2)$ of the pairs with a fixed value of $s$.

- $s = 1$. Petals can be shared, but certainly $t \geq k-j+1$ and the contribution is at most of order

$$
O\left(n^{k+1}pr^{k-j+1}\right) \leq O(\mathbb{E}(X_-)) = o(\mathbb{E}(X_-)^2).
$$

- $s = 2$. By definition, a petal cannot lie in any other $k$-simplex and thus only the pairs with $t = 2(k-j+1)$ have a positive probability of both forming a copy of $M_j$. The number of such pairs is

$$
\left(\frac{n}{k+1}\right)\left(\begin{array}{c} n-k-1 \\ k+1 \end{array}\right)\left(\begin{array}{c} k+1 \\ j \end{array}\right)^2 + O(n^{2k+1}) = (1 + o(1))\left(\frac{n}{k+1}\right)^2\left(\begin{array}{c} k+1 \\ j \end{array}\right)^2.
$$

Thus these pairs provide a contribution of

$$
(1 + o(1))\left(\frac{n}{k+1}\right)^2\left(\begin{array}{c} k+1 \\ j \end{array}\right)^2p^t_r2^{2(k-j+1)} \leq (1 + o(1))\mathbb{E}(X_-)^2.
$$

In total, we have $\mathbb{E}(X_2^2) = (1 + o(1))\mathbb{E}(X_-)^2$, and Chebyshev’s inequality implies that $X_- = (1 + o(1))\mathbb{E}(X_-)$ whp. \hfill \Box

4.3. Excluding obstructions and determining the hitting time. The goal of this section is to determine when there are no more copies of $M_j$ in $G_p$ whp. This result, together with Lemmas 4.6 and 4.7, will enable us to prove that whp the birth time $p_{M_j}$ is close to $p_j$, the (claimed) threshold for $j$-cohom-connectedness (Corollary 4.11).
Consider the probability
\[
\bar{p}_j := \frac{(j+1) \log n + \frac{1}{2} \log \log n}{(k-j+1)n^{k-j}}(k-j)!. 
\] (12)

Define \( \bar{p}_{M_j} \) as the first birth time \( p \) larger than \( \bar{p}_j \) such that there are no copies of \( M_j \) in \( \mathcal{G}_p \). By Lemmas 4.6 and 4.7, whp \( \mathcal{G}_{\bar{p}_j} \) contains a growing number of copies of \( M_j \). By definition of \( p_{M_j} \), conditioned on this high probability event we have \( \bar{p}_{M_j} \leq p_{M_j} \). In the next lemma we show that in fact they are equal whp. To do so, we need the following definition.

**Definition 4.8.** Given a \( k \)-complex \( \mathcal{G} \), a \( k \)-simplex \( K \) is a local obstacle if \( K \) contains at least \( k-j+1 \) many \( j \)-simplices which are not contained in any other \( k \)-simplex of \( \mathcal{G} \).

Note that this definition is similar to that of \( M_j^- \) (Definition 3.3), but without the restriction that the \( k-j+1 \) many \( j \)-simplices must form a flower.

**Lemma 4.9.** With high probability, for all \( p \geq \bar{p}_j \) every local obstacle that exists in \( \mathcal{G}_p \) also exists in \( \mathcal{G}_{\bar{p}_j} \). In particular, we have \( p_{M_j} = \bar{p}_{M_j} \) whp.

**Proof.** Suppose that \( \mathcal{G}_p \) contains a local obstacle which is not present in \( \mathcal{G}_{\bar{p}_j} \), and let \( K \) be the \((k+1)\)-set realising this obstacle. Then its birth time \( p_K \) satisfies \( p_K \in (\bar{p}_j, p] \). The set \( K \) can become a local obstacle only if

(i) \( K \) contains a collection \( \mathcal{L} \) of (at least) \( k-j+1 \) many \((j+1)\)-sets which are not yet \( j \)-simplices in \( \mathcal{G}_{\bar{p}_j} \);

(ii) \( p_K \) is smaller than the birth time of any other \((k+1)\)-set containing at least one of the \((j+1)\)-sets in \( \mathcal{L} \).

If \( K \) satisfies [i], then for any \((j+1)\)-set \( L \in \mathcal{L} \), no \((k+1)\)-set intersecting \( K \) precisely in \( L \) is allowed to be a \( k \)-simplex in \( \mathcal{G}_{\bar{p}_j} \); and thus there are at least \( \binom{n-k-1}{k-j} \binom{k-j+1}{k-j+1} \) many \((k+1)\)-sets which are not \( k \)-simplices. Hence, given \( K \) and \( k-j+1 \) fixed \((j+1)\)-sets within \( K \), the probability of satisfying property [ii] in \( \mathcal{G}_{\bar{p}_j} \) is bounded from above by

\[
(1-\bar{p}_j)^{\binom{n-k-1}{k-j+1}(k-j+1)} = (1 + o(1)) \exp\left(-\log\left(n^{j+1}\right) - \log\left(\log n\right)^{1/2}\right)
= O\left(\frac{1}{n^{j+1}\sqrt{\log n}}\right).
\]

On the other hand, each \((j+1)\)-set in \( \mathcal{L} \) is contained in \( \binom{n-j-1}{k-j} \) potential \( k \)-simplices. In order for \( K \) to satisfy [ii], all those \( k \)-simplices would need to have larger birth time than \( K \), which happens with probability \( O\left(\frac{1}{n^{k-j}}\right) \). Thus, the expected number of sets \( K \) satisfying [i] and [ii] is at most

\[
\left(\frac{n}{k+1}\right)^{\binom{n-j}{k-j+1}}O\left(\frac{1}{n^{j+1}\sqrt{\log n}}\right)O\left(\frac{1}{n^{k-j}}\right) = O\left(\frac{1}{\sqrt{\log n}}\right) = o(1)
\]
and the conclusion follows by Markov’s inequality. \( \square \)

Observe that in particular each copy of \( M_j^- \) is a local obstacle. Thus, we derive the following corollary.

**Corollary 4.10.** Whp for all \( p \geq p_{M_j} \), there are no copies of \( M_j^- \) in \( \mathcal{G}_p \).

We can now easily deduce that the birth time \( p_{M_j} \) at which the last copy of \( M_j \) disappears is close to \( \bar{p}_j \). Observe that the following corollary is exactly Theorem 1.1 [i].
Corollary 4.11. Let $\omega$ be any function of $n$ which tends to infinity as $n$ tends to infinity. Then whp
\[
\frac{(j+1) \log n + \log \log n - \omega}{(k-j+1)n^{k-j}} < \frac{(j+1) \log n + \log \log n + \omega}{(k-j+1)n^{k-j}}(k-j)!.
\]

Proof. We may assume without loss of generality that $\omega = o(\log n)$. By Lemmas 4.6 and 4.7 $p_{M_j} > \frac{(j+1) \log n + \log \log n - \omega}{(k-j+1)n^{k-j}}(k-j)!$ whp. On the other hand, setting $p = \frac{(j+1) \log n + \log \log n + \omega}{(k-j+1)n^{k-j}}(k-j)!$ and arguing as in Lemma 4.7 (see (11)), the expected number of copies of $M_j^-$ is bounded from above by
\[
(1 + o(1))n^{k+1}p \exp \left(-\frac{(n-j-1)^{k-j}}{n^{k-j}}(k-j)! \right) = \Theta \left(n^{j+1} \log n \exp(-j+1) \log n - \log n - \omega) \right) = \Theta (e^{-\omega}) = o(1).
\]

So by Markov's inequality, whp there are no copies of $M_j^-$ and thus also no copies of $M_j$ in $G_p$, i.e.
\[
\tilde{p}_{M_j} < \frac{(j+1) \log n + \log \log n + \omega}{(k-j+1)n^{k-j}}(k-j)!
\]
and by Lemma 4.9 we have $\tilde{p}_{M_j} = p_{M_j}$ whp. \qed

4.4. Covering the intervals: proof of Lemma 3.4. In order to prove Lemma 3.4 we show that for each $j \in [k-1]$, whp there exist three minimal obstructions which survive throughout each of the intervals $[p_{j-1}^+, p_j^{(1)}]$, $[p_j^{(1)}, p_j^-]$ and $[p_j^{(1)}, p_{M_j}]$, respectively.

Recall that
\[
p_j^{(1)} = \frac{1}{10(j+1)(j+1)! n^{k-j}}.
\]

The first step is to show that at least one of the $X_\ast = \Theta((\log n)^{j+2})$ copies of $M_j^*$ which are present whp at probability $p_{j-1}^+$ (Lemma 4.3) survives until time $p_j^{(1)}$.

To do so, we will count the number of dangerous sets, that is $(k+1)$-sets which, if they are selected as $k$-simplices, make one or more of copies of $M_j^*$ disappear. Then we show that whp up to probability $p_j^{(1)}$ the number of copies of $M_j^*$ destroyed by dangerous sets which become $k$-simplices is less than $X_\ast$.

Lemma 4.12. With high probability one copy of $M_j^*$ exists in $G_p$ throughout the range $[p_{j-1}^+, p_j^{(1)}]$.

Proof. Define $x = 2\mathbb{E}(X_\ast)$ at time $p_{j-1}^-$. By Lemma 4.4 we know that whp
\[
x \frac{3}{2} \leq X_\ast \leq x,
\]
so let us condition on this high probability event occurring.

We know that we can generate $G_{p_j^{(1)}}$ from $G_{p_{j-1}}$ by exposing an additional probability of $\frac{p_j^{(1)} - p_{j-1}}{1 - p_{j-1}} \leq p_j^{(1)}$, therefore we will use the upper bound $p_j^{(1)}$ in the following calculations. Set $p = p_j^{(1)}$ and let $Y$ be the number of dangerous sets selected as $k$-simplices in $G_p$. A $(k+1)$-set can contain at most $(j+1)!$ petals, each of which can be part of at most $j + 1$ different copies of $M_j^-$, since by definition a petal belongs to exactly one $k$-simplex and within this petal we have $j+1 = j + 1$ choices for the centre which then uniquely defines the copy of $M_j^-$. So each of the $k$-simplices
counted by \( Y \) can destroy at most \( c := \binom{k+1}{j+1}(j + 1) \) copies of \( M_j^- \). Moreover, by Remark 4.5, whp \( c \) is also the maximum number of copies of \( M_j^* \) that can disappear by adding a dangerous set to the complex. Therefore, we now show that
\[
P \left( cY \geq \frac{x}{3} \right) = o(1).
\]
This will imply that whp \( cY < X_* \), so at least one of the copies of \( M_j^* \) counted by \( X_* \) will survive throughout the considered probability interval.

A dangerous \((k+1)\)-set makes one or more copies of \( M_j^* \) disappear if it becomes a \( k \)-simplex and contains at least one petal of each of their flowers. For a copy of \( M_j^* \), the number of \((k+1)\)-sets that intersect it in at least one petal is at most
\[
\binom{k-j+1}{k-j} n^{k-j} x \leq 2n^{k-j} x =: N.
\]
Due to the independence of the chosen \( k \)-simplices, \( Y \) is dominated by \( \text{Bi}(N, p) \).

Since
\[
\mathbb{E}(\text{Bi}(N, p)) = Np = \frac{x}{5c},
\]
by the Chernoff bound (Lemma 2.2) we have
\[
P \left( Y \geq \frac{x}{3c} \right) \leq P \left( \text{Bi}(N, p) \geq \frac{x}{3c} \right)
\leq \exp \left( -\frac{\left( \frac{x}{3c} - Np \right)^2}{2(Np + \left( \frac{x}{3c} - Np \right)/3)} \right)
\leq \exp \left( -\frac{2x}{50c} \right) = o(1),
\]
because \( x \xrightarrow{n \to \infty} \infty \) by Lemma 4.4. \( \square \)

We now consider the second subinterval \([p_j^{(1)}, p_j^-]\). In this range, we will show that whp one of the “many” copies of \( M_j^- \) which exist whp at time \( p_j^- \) (Lemma 4.7) was already present at the beginning of the interval. Together with the fact that whp each \( M_j^- \) gives rise to a copy of \( M_j^* \) (Lemma 4.6), this will imply that whp one copy of \( M_j^- \) exists throughout this interval.

**Lemma 4.13.** With high probability one copy of \( M_j^- \) exists in \( G_p \) throughout the range \([p_j^{(1)}, p_j^-]\).

**Proof.** Set
\[
p = p_j^- = \left( 1 - \frac{1}{\sqrt{\log n}} \right) \frac{(j + 1) \log n}{(k-j+1)n^{k-j} (k-j)!}.
\]
By Lemma 4.7 at probability \( p \) the number \( X_- \) of copies of \( M_j^- \) is concentrated around its expectation
\[
\mathbb{E}(X_-) = \Theta \left( n^{k+1} p(1-p)(k-j+1)n^{k-j} \right) = \Theta \left( \frac{n^{k+1} \log n}{n^{k-j} \log n} \right),
\]
which is growing with \( n \). Note that a fixed \( k \)-simplex can give rise to only \( \binom{k+1}{j} = \Theta(1) \) different copies of \( M_j^- \). Therefore whp there are \( \Theta \left( \frac{n^{k+1} \log n}{n^{k-j} \log n} \right) \) many copies of \( M_j^- \) that arise from different \( k \)-simplices, and whose birth times are thus independent. Given that these copies exist at time \( p_j^- \), the birth times of the corresponding
$k$-simplices are uniformly distributed in the interval $[0, p_j^{-1}]$. The probability that any fixed such copy already existed at time $p_j^{(1)}$ is therefore

$$\frac{p_j^{(1)}}{p_j^{-1}} = \Theta \left( \frac{1}{\log n} \right).$$

Thus, because of the independence, the probability that none of them was present at $p_j^{(1)}$ is at most

$$\left( 1 - \Theta \left( \frac{1}{\log n} \right) \right)^{p_j^{-1} (1 + o(1)) \log n} \leq \exp \left( -\Theta \left( \frac{1 + o(1)}{n^{1/3} \log n} \right) \right) = o(1),$$

as required. □

We now conclude the argument by covering the third interval $[p_j^{-1}, p_M]$ of the subcritical range.

**Lemma 4.14.** With high probability one copy of $M_j^-$ exists in $G_p$ throughout the range $[p_j^{-1}, p_M]$.

**Proof.** By the definition of $p_j^{-1}$ and Corollary 4.11, we know that whp $p_j^{-1} = (1 - o(1))p_M$. So, conditioning on this high probability event and arguing as in the proof of Lemma 4.13 the final minimal obstruction to disappear at time $p_M$ already existed at time $p_j^{-1}$ with probability at least

$$\frac{p_j^{-1}}{p_M} = 1 - o(1),$$

as required. □

**Proof of Lemma 3.4.** By Lemma 4.6, the copies of $M_j^-$ from Lemmas 4.13 and 4.14 whp give rise to copies of $M_j^*$, and thus in particular to copies of $M_j$. Therefore, Lemmas 4.12, 4.13 and 4.14 together imply Lemma 3.4. □

5. Critical window and supercritical regime

5.1. Overview. In this section, we study obstructions around the point of the claimed phase transition and in the supercritical regime, that is, for $p = (1+o(1))p_j$ and $p \geq p_M$, respectively. The results of this section will form the foundation of the proof of Theorem 1.11 (iii). Furthermore, they will be an essential ingredient in the proof of Theorem 1.13.

By the definition of $p_M$, there are no copies of $M_j$ in $G_p$ (and whp also no copies of $M_j^-$ by Corollary 4.10) for any $p \geq p_M$. It remains to show that there are no other obstructions either. In fact, we shall even prove (Corollary 5.10) that from slightly before $p_M$ onwards, all $j$-cocycles are generated by copies of $M_j^-$ (recall that a $j$-cocycle is a $j$-cochain in $\ker \delta^j$, see Section 2.3). To make this more precise, we need the following terminology.

**Definition 5.1.** Let $(K, C)$ be a copy of $M_j^-$ in a $k$-complex $G$. We say that a $j$-cochain $f_{K,C}$ arises from $(K, C)$ if its support is the $j$-flower $F(K, C)$. (Observe that $f_{K,C}$ is then a $j$-cocycle.)

We say that a $j$-cocycle $f$ in $G$ is generated by copies of $M_j^-$ if it lies in the same cohomology class as a sum of $j$-cocycles that arise from copies of $M_j^-$. We denote by $\mathcal{N}G$ the set of $j$-cocycles that are not generated by copies of $M_j^-$. 
Our goal is to show that whp $\mathcal{N}_{G_p} = \emptyset$ for $p \geq p_j^-$ (Corollaries 5.8 and 5.10), which in particular will imply that whp each $j$-cocycle in $G_p$ is also a $j$-coboundary (i.e. there are no bad functions, see Definition 2.4) for all $p \geq p_M$. Furthermore, it will enable us to directly relate the number of copies of $M_j^-$ to the dimension of $H^j(G_p; F_2)$ (cf. Theorem 1.13).

**Definition 5.2.** For each $p \in [0, 1]$, let $f_p$ be a function in $\mathcal{N}_{G_p}$ with smallest support $S_p$, if such a function exists.

In order to prove that whp $\mathcal{N}_{G_p}$ is empty, we show (Lemma 5.4) that for any $k$-complex $\mathcal{G}$, a smallest support of elements of $\mathcal{N}_G$ (and so in particular $S_p$ in $G_p$) would have to be traversable (see Definition 5.3). We then show that whp no $G_p$ with $p \geq p_j^- - 1$ can contain a traversable support $S_p$. For “small” sizes of $S_p$ and $p = (1 + o(1))p_j$, basic estimates and a union bound argument will suffice (Lemma 5.5); for larger size, we will make use of traversability to define a breadth-first search process that finds all possible supports. In this way, we can bound the number of possibilities for $S_p$ more carefully, thus allowing us to prove that whp for all relevant $p$ simultaneously, $S_p$ cannot be “large” (Lemma 5.7). Finally, we complete the argument proving that whp no new elements of $\mathcal{N}_{G_p}$ with “small” support size can appear if we increase $p$ (Lemma 5.9).

5.2. **Traversability.**

**Definition 5.3.** Let $\mathcal{G}$ be a $k$-complex in which each simplex is contained in a $k$-simplex, and let $S$ be a collection of $j$-simplices of $\mathcal{G}$. For $\sigma_1, \sigma_2 \in S$, we set

$$\sigma_1 \sim \sigma_2 \quad \text{if} \quad \sigma_1 \text{ and } \sigma_2 \text{ lie in a common } k \text{-simplex.}$$

We say that the set $S$ is traversable if the transitive closure of $\sim$ is $S \times S$.

In other words, a set of $j$-simplices in such a $k$-complex is traversable if it cannot be partitioned into two non-empty subsets such that each $k$-simplex (and thus also each $(j + 1)$-simplex) contains $j$-simplices in at most one of the two subsets.

**Lemma 5.4.** Let $\mathcal{G}$ be a $k$-complex in which each simplex is contained in a $k$-simplex, and let $f$ be an element of $\mathcal{N}_G$ with smallest support $S$. Then $S$ is traversable. In particular, $S_p$ is traversable in $G_p$, if it exists, for each $p \in [0, 1]$.

**Proof.** Suppose $S$ is not traversable. Then we can find a partition $S = T_1 \cup T_2$, with $T_1$ and $T_2$ non-empty such that each $(j + 1)$-simplex of $\mathcal{G}$ contains $j$-simplices in at most one of the two parts. Define $g_1$ and $g_2$ to be $j$-cochains with supports $T_1$ and $T_2$, respectively. By the choice of $T_1$ and $T_2$, both $g_1$ and $g_2$ are $j$-cocycles. Moreover, neither of them lies in $\mathcal{N}_G$ by the minimality of $S$. As the property of being generated by copies of $M_j^-$ is closed under summation, $f = g_1 + g_2$ is generated by copies of $M_j^-$, a contradiction to $f \in \mathcal{N}_G$. $\square$

5.3. **Small supports.** The following counting argument shows that whp, at around $p_j$ traversable supports of $j$-cocycles of constant size do not exist. This implies in particular that $S_p$ (if it exists) has to be “large”.

**Lemma 5.5.** For $p = (1 + o(1))p_j$ and for any constant $d \geq k - j + 2$, with high probability there is no $j$-cocycle in $G_p$ with traversable support of size $s$ with $k - j + 2 \leq s \leq d$. In particular, with high probability either $S_p$ does not exist or $|S_p| > d$.

**Proof.** Consider a traversable support $S$ of a $j$-cocycle of size $s$ with $k - j + 2 \leq s \leq d$. Suppose that $S$ covers $v$ vertices and denote by $\ell$ the number of $k$-simplices that make $S$ traversable. These quantities are easily bounded by

$$\frac{s}{(j+1)} \leq \ell \leq s \leq d,$$

(14)
and
\[ v \leq j + 1 + (k - j)\ell. \]  
(15)

We know by Lemma 3.1 that if a \( k \)-simplex contains a \( j \)-simplex in \( S \), then all its \( k + 1 \) vertices are covered by \( S \). Therefore, all \( s\binom{n-j}{k-j} \) many \( (k+1) \)-sets consisting of the vertices of one \( j \)-simplex in \( S \) and \( k-j \) vertices not covered by \( S \) cannot be \( k \)-simplices in \( G_p \). Thus, the probability that a fixed such \( S \) exists is at most
\[
p^f(1-p)^{s\binom{n-j}{k-j} + O(n^{k-j-1})}
= O\left(\left(\frac{\log n}{n^{k-j}}\right)^{\ell} \exp\left(-\frac{s(j+1)}{k-j+1} \log n + o(\log n)\right)\right)
= O\left(n^{-\ell(k-j) - \frac{\log\log n}{\log n}} + o(1) (\log n)\ell\right).
\]

Denote by \( E_{s,v,\ell} \) the event that a traversable support \( S \) with parameters \( s, v \), and \( \ell \) exists. There are \( O(n^v) \) different ways of choosing \( S \), thus
\[
\mathbb{P}(E_{s,v,\ell}) = O\left(n^{-\ell(k-j) - \frac{\log\log n}{\log n}} + o(1) (\log n)\ell\right).
\]

Using (15) and the fact that \( s \geq k - j + 2 \), we obtain
\[
v - \ell(k-j) - \frac{s(j+1)}{k-j+1} + o(1) \leq -\frac{j+1}{k-j+1} + o(1) \leq -\frac{j}{k-j+1}
\]
and thus
\[
\mathbb{P}(E_{s,v,\ell}) = o(1).
\]

Finally, observe that by (14) and (15), there is only a constant number of possible values for \( s, v \), and \( \ell \). Therefore, the probability that any such support \( S \) exists is \( o(1) \), as required.

Note that a similar argument also works for \( s \) up to \( O\left(\frac{\log n}{\log\log n}\right) \), but we only need it for constant size, since we will cover the range between constant size and size \( O\left(\frac{\log n}{\log\log n}\right) \) with a different argument that we use for all large \( s \).

5.4. Large supports. For larger support sizes, the previous calculations do not work anymore and we will need a more careful technique for bounding the number of possible supports, namely a breadth-first search process. We will also make use of the following proposition due to Meshulam and Wallach 35.

**Proposition 5.6** (35 Proposition 3.1). Let \( \Delta \) be the downward-closure of the \((n-1)\)-simplex on vertex set \([n]\), where \( n \geq j + 2 \). For \( f \in C^j(\Delta) \), define \( w(f) \) to be the smallest size of a support of a \( j \)-cochain of the type \( f + \delta^{j-1}g \), where \( g \in C^{j-1}(\Delta) \). Furthermore, denote by \( b(f) \) the size of the support of \( \delta^jf \), i.e. the number of \((j+1)\)-simplices in \( \Delta \) containing an odd number of \( j \)-simplices of the support in \( f \). Then
\[ b(f) \geq \frac{w(f)n}{j+2}. \]

In the next lemma we show that whp in the supercritical range, a smallest support of elements of \( N_{G_p} \) cannot be “large”.

**Lemma 5.7.** There exists a positive constant \( \bar{d} \) such that with high probability for all \( p \geq p_j \), either \( S_p \) does not exist or \( |S_p| < \bar{d} \).
Proof. Write $s := |S_p|$. By Lemma 5.4 $S_p$ (if it exists) is traversable and thus we can discover it via the following breadth-first search process: start from any $j$-simplex in $S_p$ and query all $(k + 1)$-sets containing it. Since $S_p$ is the support of the $j$-cocycle $f_j$, any of these sets which forms a $k$-simplex must contain at least one other $j$-simplex in $S_p$. From all $j$-simplices in $S_p$, found in this way, we can continue the process according to some pre-determined order of $j$-simplices, but we explore only $(k + 1)$-sets which would give us some previously undiscovered $j$-simplex in $S_p$. By the traversability of $S_p$, we discover all of $S_p$ in this process.

Let us bound the number of traversable supports of size $s$ which are contained in $\ell \leq s$ many $k$-simplices (recall (14)), which we can find via the described search process. Define the sequence $b_i = (b_1, \ldots, b_s)$, where $b_i \geq 0$ is the number of $k$-simplices we discover from the $i$-th $j$-simplex in this process. From the $i$-th $j$-simplex we may query up to $\binom{n}{k-j}$ many $(k+1)$-sets and for each of the $b_i$ discovered $k$-simplices we can find at most $\binom{k-1}{j-1} - 1$ new $j$-simplices of the support, so this can happen in at most $\binom{n}{k-j}2^{\binom{k+1}{j+1}b_i}$ different ways. Thus, if we condition on the sequence $b_i$ the number of supports of size $s$ we can find is bounded from above by

$$
\left( \binom{n}{j+1} \prod_{i=1}^{s} \binom{n}{k-j} \right) 2^{\binom{k+1}{j+1}b_i} \leq n^{j+1} \binom{n}{k-j} 2^{\binom{k+1}{j+1}} \frac{\ell!}{\prod_{i=1}^{s} b_i!}
$$

where we are using that $\sum_{i=1}^{s} b_i = \ell$.

In order to apply Proposition 5.6 to $f_j$ (which is possible, because $G_p$ is a subcomplex of $\Delta$), let us determine the value $w(f_j)$. First observe that for $p \geq p_j$, whp $G_p$ has a complete $(j-1)$-dimensional skeleton, which can be proved by a simple first moment calculation. Thus, if we consider $f_j + \delta^{j-1}g$ with $g \in C^{j-1}(\Delta)$, then whp also $g \in C^{j-1}(G_p)$ and thus $f_j + \delta^{j-1}g$ lies in the same cohomology class of $H^j(G_p; \mathbb{F}_2)$ as $f_j$. By the minimality of $S_p$, this implies that $w(f_j) = |S_p| = s$ whp. For the rest of the proof, let us condition on this high probability event.

Now Proposition 5.6 tells us that at least $\frac{sn}{(k-j-1)}$ many $(j+2)$-sets would form odd $(j+1)$-simplices if they were present in $G_p$. The fact that $f_j$ is a $j$-cocycle implies that no such $(j+2)$-set is allowed to be in a $k$-simplex. Each $(j+2)$-set is contained in $\binom{n-j-2}{k-j-1}$ many $(k+1)$-sets, each of which contains $\binom{k+1}{j+1}$ many $(j+2)$-sets. Therefore the number of $(k+1)$-sets that cannot be chosen as $k$-simplices in $G_p$ is at least

$$
\frac{sn\binom{n-j-2}{k-j-1}}{(j+2)\binom{k+1}{j+1}} \geq \alpha_0 s n^{k-j} \geq \alpha_0 \ell n^{k-j},
$$

for some constant $\alpha_0 = \alpha_0(k,j) > 0$. Thus, the probability that a fixed support exists together with the $\ell$ many $k$-simplices that make it traversable, but that no odd $(j+1)$-simplices are present is at most

$$
(p(1-p)^{\alpha_0 n^{k-j}})^\ell.
$$

The derivative of this expression with respect to $p$ is negative throughout the range $p \geq p_j$, therefore in the following calculations involving $p$ we can use the lower bound $p_j$. Given the sequence $b_i$ the probability $q_s$ that some such support exists
and that the connecting $k$-simplices have no odd $(j + 1)$-simplices satisfies
\[
q_b \prod_{i=1}^{s} b_i! \leq n^{j+1} \left( 2^{(k+1)} \binom{n}{k-j} p(1-p)^{\alpha_0 n^{k-j}} \right)^\ell \\
\leq n^{j+1} \left( 2^{(k+1)} \frac{(j+1)!}{k-j+1} \frac{\log n}{\alpha_0 n^{j+1}(k-j)!} \log n \right)^\ell \\
\leq n^{j+1} \left( \frac{n^{-\alpha_0 s}}{n^{\alpha_0 s}(k-j)!} \right)^\ell \\
\leq n^{j+1} n^{-\alpha_0 \ell} \leq n^{-\alpha_1 \ell},
\]
where $\alpha_1 = \alpha_1(k,j) > 0$ and the last inequality holds for $\ell \geq \frac{2(j+1)}{\alpha_1}$. Moreover, since $\ell \geq \frac{s}{\alpha_1}$, we can find another positive constant $\alpha_2$ such that
\[
q_b \prod_{i=1}^{s} b_i! \leq n^{-\alpha_2 s}.
\]

For each sequence $b = (b_1, \ldots, b_s)$ define
\[
t(b) := |\{i : b_i \geq n^{\alpha_2/2}\}|
\]
and let $B_t$ be the set of all sequences $b$ such that $t(b) = t$. We can crudely bound $|B_t|$, the number of sequences in $B_t$, by
\[
s^t \left( \frac{n}{k-j} \right)^t n^{-(\alpha_2/2)s-t}.
\]
On the other hand, if $b \in B_t$, then
\[
\prod_{i=1}^{s} b_i! \geq \left( n^{\alpha_2/2} \right)^t \geq \left( n^{\alpha_2/2} n^{\alpha_2/3} \right)^t \geq n^t n^{\alpha_2/4}.
\]
Summing over all possible sequences $b$, we obtain
\[
\sum_b \frac{1}{\prod_{i=1}^{s} b_i!} = \sum_{t=0}^{s} \frac{1}{\prod_{i=1}^{s} b_i!} \\
\leq \sum_{t=0}^{s} \frac{s^t \left( \frac{n}{k-j} \right)^t n^{\alpha_2/2} n^{\alpha_2/4}}{n^t n^{\alpha_2/4}} \\
= \frac{s^t \left( \frac{n}{k-j} \right)}{n^{\alpha_2/2} n^{\alpha_2/4}} \\
\leq (s+1) n^{\alpha_2 s/2}.
\]
Combining (16) and (17), the probability that some support of fixed size $s$ exists is at most
\[
(s+1) n^{\alpha_2 s/2} n^{-\alpha_2 s} \leq n^{-\alpha_2 s/3}.
\]
Let $\tilde{d} > \frac{4(k+1)}{\alpha_2}$ be a constant. If we sum over all $s \geq \tilde{d}$, we see that the probability that $S_p$ exists and $|S_p| \geq \tilde{d}$ is at most $n^{-\alpha_2 \tilde{d}/4}$. This holds for every $p \geq p_\gamma$ and thus, taking a union bound over all $O(n^{k+1})$ birth times in this range, the probability for $S_p$ of size at least $\tilde{d}$ to exist for any $p \geq p_\gamma$ is $O\left(n^{k+1-(\alpha_2 \tilde{d}/4)}\right)$, which tends to zero for our choice of $\tilde{d}$.
\[
\square
\]
We can now show that whp for $p$ “close” to $p_j$ each $j$-cocycle in $G_p$ arises from copies of $M_j$. 

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Corollary 5.8. For every $p = (1 + o(1))p_j$ with $p \geq p_j^-$, we have $\mathcal{N}_{\mathcal{G}_p} = \emptyset$ with high probability.

Proof. By Lemma 5.1 and the definition of $\mathcal{N}_{\mathcal{G}_p}$ (Definition 5.1), whp either $S_p$ does not exist or $|S_p| \geq k - j + 2$. Furthermore, Lemma 5.7 tells us that whp for all $p \geq p_j^-$, either $S_p$ does not exist or it must be of constant size. For $p = (1 + o(1))p_j$, Lemma 5.5 implies that whp $S_p$ does not have constant size, and thus whp $S_p$ does not exist, meaning that whp $\mathcal{N}_{\mathcal{G}_p}$ is empty. \hfill \Box

5.5. Monotonicity with high probability. Although the existence of bad functions in $\mathcal{G}_p$ is not intrinsically a monotone property, in this section we show that in fact, from time $p_{M_j}$ on, whp this property behaves in a monotone way.

By Corollary 4.11 whp we can apply Corollary 5.8 with $p = p_{M_j}$, therefore whp $\mathcal{N}_{\mathcal{G}_{p_{M_j}}}$ is empty. In other words, whp there are no bad functions in $\mathcal{G}_{p_{M_j}}$, i.e. $H^1(\mathcal{G}_{p_{M_j}}; F_2) = 0$. However, we still need to prove that $\mathcal{G}_p$ does not lose this property for any larger $p$. More precisely, we already know by Lemma 5.7 that whp no $\mathcal{G}_p$ for $p \geq p_{M_j}$ contains a $j$-cocycle with “large” support, but “small” supports have been excluded by Lemma 5.5 only in the range $p = (1 + o(1))p_j$. In the next lemma we show that if a new obstruction appears, then the $k$-simplex whose birth causes this appearance must be a local obstacle (Definition 5.8). But we already know by Lemma 4.10 that whp no new local obstacles appear, which will complete the argument.

Lemma 5.9. Whp either $\mathcal{N}_{\mathcal{G}_p} = \emptyset$ for all $p \geq p_{M_j}$, or the $k$-simplex $K$ with smallest birth time $p K \geq p_{M_j}$, for which $\mathcal{N}_{\mathcal{G}_{p_{M_j}}} \neq \emptyset$, forms a local obstacle in $\mathcal{G}_{p_{M_j}}$.

Proof. The lemma is trivially true if whp $\mathcal{N}_{\mathcal{G}_p} = \emptyset$ for all $p \geq p_{M_j}$, we may thus assume that $K$ exists with positive probability. Let $p < p_{M_j}$ be such that $\mathcal{G}_{p_{M_j}} = \mathcal{G}_p + K$.

Suppose first that $S_{p_{M_j}} \cap \mathcal{G}_p \neq \emptyset$. Let $S$ be a maximal subset of $S_{p_{M_j}}$ which is traversable in $\mathcal{G}_p$ and let $f$ be the $j$-cochain in $\mathcal{G}_p$ with support $S$. Every $k$-simplex of $\mathcal{G}_p$ containing some $j$-simplex in $S$ cannot contain $j$-simplices in $S_{p_{M_j}} \setminus S$ by the maximality of $S$. Therefore, every $(j + 1)$-simplex of $\mathcal{G}_p$ is even with respect to $f$, because it is even with respect to $f_{p_{M_j}}$. This means that $f$ is a $j$-cocycle in $\mathcal{G}_p$.

Lemma 5.7 implies that there exists a constant $d$ such that whp $|S_{p_{M_j}}| < d$ and thus also $|S| < d$. But Lemma 5.6 together with the fact that $p > p_{M_j} > p_j^{(1)}$, whp implies that whp each $j$-simplex in $S$ lies in linearly many $j$-shells in $\mathcal{G}_p$, at most $|S| - 1$ of which can contain other elements of $S$. Thus, whp there are $j$-shells in $\mathcal{G}_p$ that contain precisely one element of $S$, which means that $f$ is not a $j$-coboundary, i.e. $f$ is a bad function in $\mathcal{G}_p$. Now recall that whp there are no copies of $M_j^+$ in $\mathcal{G}_p$ by Corollary 4.10 and thus all bad functions lie in $\mathcal{N}_{\mathcal{G}_p}$. This means that $\mathcal{N}_{\mathcal{G}_p} \neq \emptyset$, a contradiction to the choice of $K$.

Thus, whp $S_{p_{M_j}}$ is entirely contained in $K$ and its simplices are not in other $k$-simplices of $\mathcal{G}_{p_{M_j}}$. Moreover, it follows from Lemma 5.4 that $|S_{p_{M_j}}| \geq k - j + 1$, implying that whp $K$ forms a local obstacle in $\mathcal{G}_{p_{M_j}}$. \hfill \Box

The following corollary shows that in the supercritical regime $p \geq p_{M_j}$, whp no $j$-cocycle arises from copies of $M_j^-$. 

Corollary 5.10. With high probability $\mathcal{N}_{\mathcal{G}_p} = \emptyset$ for all $p \geq p_{M_j}$ simultaneously.

Proof. Recall that by Corollaries 4.11 and 5.8 $\mathcal{N}_{\mathcal{G}_{p_{M_j}}} = \emptyset$ whp. If $\mathcal{N}_{\mathcal{G}_p} \neq \emptyset$ for some $p > p_{M_j}$, then whp the $k$-simplex whose birth creates a $j$-cocycle that is not generated by copies of $M_j^-$ would form a local obstacle by Lemma 5.9. But Lemma 4.10 tells us that whp no new local obstacles appear after time $p_j$, which whp is smaller than $p_{M_j}$ by (12) and Corollary 4.11. \hfill \Box
6. Proofs of main results

6.1. Proof of Theorem 1.11

Corollary 4.4 states that for any function \( \omega \) of \( n \) which tends to infinity as \( n \to \infty \), whp we have

\[
\frac{(j + 1) \log n + \log \log n - \omega}{(k - j + 1)n^{k-j}} < p_{M_j} < \frac{(j + 1) \log n + \log \log n + \omega}{(k - j + 1)n^{k-j}},
\]

which is precisely Theorem 1.11 (i).

To prove (ii) recall that Lemma 4.1 states that for all \( i \in [j] \), whp \( H^i(\mathcal{G}_p; \mathbb{F}_2) \neq 0 \) for all \( p \in [p_{i-1}, p_{M_i}) \). By (i) whp for all \( i \in [j-1] \)

\[
p_{M_i} > \left(1 - \frac{1}{\sqrt{\log n}}\right) \frac{(i + 1) \log n}{(k - i + 1)n^{k-i}}(k - i)! = p_i^*,
\]

and thus whp \( \mathcal{G}_p \) is not \( j \)-cohom-connected throughout \( \bigcup_{i=1}^j [p_{i-1}, p_{M_i}) = [p_0, p_{M_j}) \). 

Now observe that by Lemma 4.1 whp \( p_T > p_0 \) and that \( \mathcal{G}_p \) is not topologically connected in \( [0, p_T) \) by the definition of \( p_T \). Therefore, whp \( \mathcal{G}_p \) is not \( j \)-cohom-connected in

\[
[0, p_{M_j}) = [0, p_T) \cup [p_0, p_{M_j}),
\]

as required.

It remains to prove (iii). We have to show that whp there are no bad functions in \( \mathcal{G}_p \) for every \( p \geq p_{M_j} \). By Corollary 4.4 whp for all \( p \geq p_{M_j} \), there are no copies of \( M_j \) in \( \mathcal{G}_p \). Thus, if \( H^j(\mathcal{G}_p; \mathbb{F}_2) \neq 0 \), then any representative of a non-zero cohomology class cannot arise from copies of \( M_j \) and therefore lies in \( \mathcal{N}_{\mathcal{G}_p} \). (Definition 5.1). But by Corollary 5.10 whp each such \( \mathcal{N}_{\mathcal{G}_p} \) is empty and thus whp \( H^j(\mathcal{G}_p; \mathbb{F}_2) = 0 \) for all \( p \geq p_{M_j} \). Analogously, whp all cohomology groups \( H^i(\mathcal{G}_p; \mathbb{F}_2) \) for \( i \in [j-1] \) vanish, because whp \( p_{M_i} < p_{M_j} \) by (i). Finally, by (ii) and Lemma 4.1 whp \( p_T < p_{M_j} \), meaning that whp \( \mathcal{G}_p \) is topologically connected for all \( p \geq p_{M_j} \). This implies that whp each such \( \mathcal{G}_p \) is \( F_2 \)-cohomologically \( j \)-connected. \( \square \)

6.2. Proof of Corollary 1.12

Let \( \omega \) be any function of \( n \) which tends to infinity as \( n \to \infty \). It is known (see e.g. [35]) that whp

\[
\frac{k \log n - \omega}{n} < p_{\text{sol}} < \frac{k \log n + \omega}{n}.
\]

The proof is an easy application of the first and second moment methods.

In order to prove that \( p_{\text{conn}} = p_{\text{sol}} \) whp, suppose that a \( (k-1) \)-simplex \( \sigma \) is isolated in \( \mathcal{Y}_p \) for some \( p \). The indicator function \( f_\sigma \) of \( \sigma \) is a \( (k-1) \)-cocycle, because \( \sigma \) is isolated. But \( f_\sigma \) is not a \( (k-1) \)-coboundary, because \( \sigma \) lies in \( (n-k) \)-shells. In particular, \( H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) \neq 0 \). The definitions of \( p_{\text{conn}} \) and \( p_{\text{sol}} \), this implies that \( p_{\text{conn}} \geq p_{\text{sol}} \).

For the opposite direction, fix the birth times of all \( k \)-simplices. Then for all \( p \geq p_{\text{sol}} \), we have \( \mathcal{Y}_p = \mathcal{G}_p \) and therefore \( \mathcal{Y}_p \) is \( \mathbb{F}_2 \)-cohomologically \( (k-1) \)-connected whp for every \( p \geq \max(p_{\text{sol}}, p_{M_k-1}) \) by Theorem 1.11 (iii). By (18) and Theorem 1.11 (i) whp for any (slowly) growing function \( \omega \)

\[
p_{\text{sol}} > \frac{k \log n - \omega}{n} > \frac{k \log n + \log \log n + \omega}{2n} > p_{M_k-1},
\]

hence whp for all \( p \geq p_{\text{sol}} \) we have \( H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) = H^{k-1}(\mathcal{G}_p; \mathbb{F}_2) = 0 \). This means that whp \( p_{\text{conn}} \leq p_{\text{sol}} \) and thus \( p_{\text{conn}} = p_{\text{sol}} \), as required. \( \square \)
6.3. Proof of Theorem 1.13 We are interested in the asymptotic distribution of $D_j := \dim \{H^j(G_p; \mathbb{F}_2)\}$ for
\[
p = \frac{(j + 1) \log n + \log \log n + c_n (k - j)!(k - j + 1)!}{(k - j + 1)!},
\]
where $c_n \xrightarrow{n \to \infty} c \in \mathbb{R}$.

Recall that $X_-$ is the random variable defined in Lemma 1.7 which counts the number of copies of $M^-_j$. We apply the method of moments (Lemma 2.3) to $X_-$, showing that it converges in distribution to a Poisson random variable with expectation
\[
\lambda_j = \frac{1}{(k - j + 1)2^j j!}.
\]

Subsequently, we will prove that whp $X_- = D_j$. In particular this will imply that
\[
D_j \xrightarrow{d} \text{Po}(\lambda_j),
\]
as required.

In order to determine the expectation of $X_-$, let $K \subset [n]$ be a $(k + 1)$-set and let $C$ be a $j$-subset of $K$. Recall that the probability that a (potential) petal $C \cup \{w\}$ with $w \in K \setminus C$ lies in no other $k$-simplex is given by
\[
r = (1 - p)^{(n - j - 1)(k - j + 1) - 1}
\]
(see (9)). Arguing as in Lemma 4.7 we see that dependencies between the petals are negligible and thus
\[
E(X_-) = (1 + o(1))\left(\frac{n}{k + 1}\right)\left(\frac{k + 1}{j}\right)p^k p^{j+1}.
\]

We observe that
\[
n^k (k - j + 1)! p \left(\frac{k - j + 1}{p}\right)
\]
\[
= \exp \left(\frac{n^k}{(k - j + 1)!} \log (k - j + 1) + O \left(\frac{n^k}{(k - j + 1)!} \log n\right)\right)
\]
\[
= \exp \left(-\frac{(j + 1) \log n + \log \log n + c_n + o(1)}{n^k} \right)
\]
\[
= (1 + o(1))\frac{e^{-c_n}}{n^k}.
\]

Therefore, we have
\[
E(X_-) = (1 + o(1))\left(\frac{n^k}{k - j + 1)!}\right)\left(\frac{(j + 1) \log n + \log \log n + c_n (k - j)!e^{-c_n}}{n^k}\right)
\]
\[
= (1 + o(1))\frac{e^{-c_n}}{n^k} \lambda_j.
\]

Denote by $T^-$ the set of all pairs $(K, C)$ that can form a copy of $M^-_j$ in $G_p$. For each $T^- \in T^-$, denote by $X_{T^-}$ the indicator random variable of the event that $T^-$ forms a copy of $M^-_j$ in $G_p$. For each fixed integer $t \geq 1$, we now determine the binomial moments
\[
E\left(X_-^t\right) = \sum_{s \in (T^-)} P\left(\bigcap_{T^- \in S} \{X_{T^-} = 1\}\right).
\]

Suppose first that all $T^- \in S$ have different $(k + 1)$-sets. In this case, if all $T^- \in S$ form copies of $M^-_j$, none of the petals are shared (by property [M2] of $M^-_j$, see Definition 3.3). If we choose $t$ distinct $(k + 1)$-sets uniformly at random,
whp they will be disjoint and in particular no two $T^+_1, T^-_2 \in \mathcal{S}$ will share a petal.

To choose $t$ distinct $(k+1)$-sets, there are
\[
\binom{\frac{n}{k+1}}{t} = \binom{n}{t} \left( \frac{k+1}{t} \right)^t
\]
choices. Therefore, the contribution to $\mathbb{E}(X^-_t)$ made by the sets $\mathcal{S}$ for which all $T^- \in \mathcal{S}$ have distinct $(k+1)$-set is
\[
(1 + o(1))\left( \binom{n}{t} \left( \frac{k+1}{t} \right)^t \right) p^t r^{(k-j+1)} \overset{(19)}{=} (1 + o(1)) \frac{\mathbb{E}(X^-_t)^t}{t!}
\]
which is the desired asymptotic value.

We now show that the contribution coming from sets $\mathcal{S}$ whose elements use $u < t$ different $(k+1)$-sets is negligible. We have $\binom{\frac{n}{k+1}}{u}$ ways to select the $(k+1)$-sets and at most $u^t - u^{(k+1)^t}$ different ways to locate the $t$ potential $M^-_j$ in them. Moreover, observe that two different copies of $M^-_j$ in the same $k$-simplex share at most one petal (otherwise they would have the same centre and thus be identical) and in that case these two copies have $(k-j+1) + (k-j)$ petals in total. This means that each of the $u$ many $(k+1)$-sets contains at least $k-j+1$ petals, and at least one $(k+1)$-set contains at least $(k-j+1) + (k-j)$ petals. Therefore the total number of petals required for such a set $\mathcal{S}$ is bounded from below by $u(k-j+1) + (k-j)$.

In total, the contribution of such sets $\mathcal{S}$ to the binomial moment is at most
\[
\binom{\frac{n}{k+1}}{u} u^t - u^{(k+1)^t} p^{u(k-j+1)} r^{k-j}.
\]
Replacing $t$ by $u$ in (22), we deduce that
\[
\binom{\frac{n}{k+1}}{u} u^t - u^{(k+1)^t} p^{u(k-j+1)} = (1 + o(1)) \frac{\lambda^u}{u!} \left( u \left( \frac{k+1}{j} \right) \right)^{t-u} = \Theta(1).
\]
Furthermore, (20) yields $r^{k-j} = o(1)$. Together with (22), we deduce that
\[
\mathbb{E}(X^-_t) = (1 + o(1)) \frac{\lambda^t}{t!}
\]
for each fixed integer $t \geq 1$. Now Lemma 2.23 yields $X^- \overset{d}{\to} \text{Po}(\lambda_j)$.

It remains to show that $X^- \leq D_j$ whp. To this end, denote by $f_1, \ldots, f_{X^-}$ the $j$-cocycles arising from the copies of $M^-_j$ in $\mathcal{G}_p$. Corollary 5.8 in particular implies that $X^- = \text{Po}(\lambda_j)$.

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In order to prove the opposite direction, we show that the cohomology classes of $f_1, \ldots, f_{X^-}$ are linearly independent. Observe first that whp $X^- = o(n)$ by Markov’s inequality, because $X^-$ has bounded expectation. Let $I \subseteq [X^-]$ be non-empty and let $S$ be the support of $\sum_{i \in I} f_i$. By the arguments above for $t = 2$ and $u = 1$, whp there are no two $M^-_j$ that share the same $k$-simplex. Thus, whp the $f_i$’s have disjoint support by property (M2) of an $M^-_j$ (Definition 3.3), and in particular $S \neq \emptyset$. Pick $L \in S$. Lemma 1.6 and the fact that $p > p_j^{(1)}$ tell us that whp there are $\Theta(n)$ many $j$-shells in $\mathcal{G}_p$ that contain $L$. All these $j$-shells meet only in $L$, thus at most $|S| \leq (k-j+1)|I| \sim o(n)$ of them can contain another $j$-simplex in $S$. Thus, there are $j$-shells that meet $S$ only in $L$, showing that $\sum_{i \in I} f_i$ is not a $j$-coboundary. Therefore the cohomology classes of $f_1, \ldots, f_{X^-}$ are linearly independent whp. This shows that whp $X^- \leq D_j$ and thus $X^- = D_j$, as desired.
Together with \( X \xrightarrow{d} \Pr(\lambda_j) \), this proves that \( D_j \xrightarrow{d} \Pr(\lambda_j) \). By Theorem 1.11 (for \( j - 1 \) instead of \( j \)) whp \( H^j(G_p; \mathbb{F}_2) = \mathbb{F}_2 \) and \( H^i(G_p; \mathbb{F}_2) = 0 \) for all \( i \in [j - 1] \).

In particular,

\[
\mathbb{P}(G_p \text{ is } j\text{-cohom-connected}) = \mathbb{P}(H^j(G_p; \mathbb{F}_2) = 0) + o(1) = (1 + o(1))\Pr(\lambda_j) = 0
\]

This concludes the proof of Theorem 1.13.

\[\square\]

7. Concluding remarks

7.1. Comparison of proof methods. Let us note that for the subcritical regime (Theorem 1.11 (iii)), one might try to use a different approach in order to prove that \( H^j(G_p; \mathbb{F}_2) \) does not vanish in the interval \([p_{j-1}, p_M]\). If the dimension of \( C^j(G_p) \) (viewed as an \( \mathbb{F}_2 \)-vector space) is larger than the sum of the dimensions of \( C^{j-1}(G_p) \) and \( C^{j+1}(G_p) \), then \( H^j(G_p; \mathbb{F}_2) \neq 0 \) would follow. However, this behaviour only happens for “small” \( p \in [p_{j-1}, p_M] \) and, more importantly, only for \( j \geq \frac{k-1}{2} \). In contrast, our proof method works for all values of \( j \). Moreover, our result that \([p_{j-1}, p_M]\) whp is covered by three copies of \( M_j \) (Lemma 3.3), together with the fact that whp \( G_{p_{k-1}} \) has isolated vertices (this can be proved using an easy second moment argument), implies the following slightly stronger statement.

Proposition 7.1. With high probability for every \( p < p_M \), the complex \( G_p \) contains an isolated vertex or a copy of \( M_i \) for some \( i \in [j] \).

In the supercritical regime (Theorem 1.11 (iii)), the counting methods used in [30] [33] for \( Y_p \) are not sufficient to prove the non-existence of \( j \)-cocycles in \( G_p \). This is due to the fact that these methods have been designed for the special case \( j = k - 1 \) and for a threshold which is about twice as large as \( p_{k-1} \). For this reason, the more careful arguments used in Lemmas 5.4 to 6.3 become necessary.

7.2. Alternative models. There are several ways to define random \( k \)-complexes. If the \( k \)-simplices are chosen independently with probability \( p \), then the models \( Y_p \) and \( G_p \) are somewhat extremal constructions, in the sense that \( Y_p \) contains all simplices of lower dimension, while \( G_p \) only comprises those simplices that are necessary in order to be a complex. What happens in between, i.e. when the complex contains all simplices in \( G_p \), but in addition, some simplices of dimensions \( 1, \ldots, k-1 \) might be added in a random fashion? Depending on the choice of probabilities, such a complex might show behaviour that is different from both \( Y_p \) and \( G_p \).

Random complexes also arise naturally from random graphs. For instance, the random clique complex \( \mathcal{X}_p(n) \) (also known as flag complex) on vertex set \([n]\) can be defined as the maximal complex whose 1-skeleton is the binomial random graph. Equivalently, a non-empty set \( U \subseteq [n] \) forms a simplex in \( \mathcal{X}_p(n) \) if and only if \( U \) is a clique in the binomial random graph. Topological properties of \( \mathcal{X}_p(n) \) have been studied in [16] [25] [20]. Another example is the random neighbourhood complex arising from the binomial random graph by letting each non-empty set of vertices that have a common neighbour form a simplex [24]. See [27] for an overview of these and other models of random complexes.

7.3. Other notions of connectedness. The vanishing of cohomology groups with coefficients in \( \mathbb{F}_2 \) is just one possible way of defining the concept of “connectedness” of \( G_p \). An obvious alternative would be to consider coefficients from other groups or fields. For \( Y_p \), such notions of connectedness have been studied for coefficients
in any finite abelian group, in $\mathbb{Z}$, or in any field \cite{11 21 31 33 35}. In particular, the threshold for the vanishing of $H^{k-1}(Y_p; R)$ for a finite abelian group $R$ is independent of the choice of $R$ \cite{35}. For $G_p$, it is not obvious whether the threshold for $j$-cohom-connectedness depends on the choice of the group of coefficients. An indication that it might indeed depend on the group, even if we restrict attention only to finite abelian groups, is the observation that $M_j$ only remains an obstruction when the coefficients are taken from a group of even order. For groups of odd order, the minimal obstruction becomes larger, and thus one would expect the threshold for $j$-cohom-connectedness to decrease.

A rather strong notion of connectedness would be to require the homotopy groups $\pi_1(G_p), \ldots, \pi_1(G_p)$ to vanish. For the 2-dimensional case, the vanishing of $\pi_1(Y_p)$ was studied by Babson, Hoffman and Kahle \cite{3}. In particular, they showed that whp $\pi_1(Y_p) \neq 0$ at the time that $H^1(Y_p; \mathbb{F}_2)$ becomes zero. From that time on, the models $Y_p$ and $G_p$ coincide. As $\pi_1(G_p) \neq 0$ follows immediately from $H^1(G_p; \mathbb{F}_2) \neq 0$, the range that should be of particular interest with respect to $\pi_1(G_p)$ in the 2-dimensional case is

$$\log n + \frac{1}{2} \log \log n \leq p \leq 2 \log n + \omega.$$ 

A natural conjecture would be that whp $\pi_1(G_p) \neq 0$ in this range.

Theorem\cite{11 13} provides a limit result for the dimension $D_j = \dim(H^j(G_p; \mathbb{F}_2))$ of the $j$-th cohomology group of $G_p$ around the point of the phase transition. It would be interesting to know the behaviour of $D_j$ also for earlier regimes. More precisely, how large is $D_j$ in the interval $[p_{j-1}, p_M]$? How far below $p_{j-1}$ do we have $D_j > 0$ whp?

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