Blackbody radiation in $\kappa$–Minkowski spacetime

Hyeong-Chan Kim$^1$, Chaiho Rim$^2$, and Jae Hyung Yee$^1$

$^1$Department of Physics, Yonsei University, Seoul 120-749, Republic of Korea
and

$^2$Department of Physics and Research Institute of Physics and Chemistry, Chonbuk National University, Jeonju 561-756, Korea.

We have computed the black body radiation spectra in $\kappa$–Minkowski space-time, using the quantum mechanical picture of massless scalar particles as well as effective quantum field theory picture. The black body radiation depends on how the field theory (and thus how the $\kappa$–Poincaré algebra) handles the ordering effect of the noncommutative space-time. In addition, there exists a natural momentum cut-off of the order $\kappa$, beyond which a new real mode takes its shape from a complex mode and the old real mode flows out to be a new complex mode. However, the new high momentum real mode should not be physical since its contributions to the black-body radiation spoils the commutative limit.

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I. INTRODUCTION

In the last decade, there have been attempts to explain the cosmic observational data \cite{1} as a quantum gravitational effect, and several non-commutative theories based on the uncertainty of Planck scale position \cite{2}, on the twist formulation \cite{3, 4} and on the Lie algebraic deformation \cite{5}, have been proposed as a theoretical framework to study such effects. The theories, however, show some unsatisfactory points, such as a strong fine-tuning problem at one-loop level \cite{6} in Moyal deformed theory when applied to the high energy scattering theory, and complex metric problem when applied to gravity theory directly \cite{7}. When applied to space-time noncommutative field theories, unitarity of the theory is in question \cite{8, 9}. This suggests that one needs to understand the structure of the quantum field theories in noncommutative space-times more clearly, find the limits and differences of the results from the usual commutative field theory approaches, and investigate the possibility of explaining the observational data.

One of the candidates to study the quantum gravity effect on the cosmic observational results is the $\kappa$–Minkowski space-time \cite{10} since this deformation respects rotational sym-
metry. The space-time coordinates do not commute with each other and satisfy the commutation relations,

\[ [\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa} \hat{x}^i, \quad [\hat{x}^i, \hat{x}^j] = 0, \quad i, j = 1, 2, 3 \]  \hspace{1cm} (1)

where \( \kappa \) is a positive parameter which represents the deformation of the space-time, whose natural choice is to put \( \kappa = M_P \), the Planck mass \([11, 12, 13]\). This \( \kappa \)-Minkowski space-time gives the dual picture in terms of the \( \kappa \)-deformed Poincaré algebra \([14]\). When the dimensional parameter \( 1/\kappa \) is suppressed, the deformed Poincaré symmetry reduces to the commutative limit.

One may construct its differential calculus where arises a fifth dimension naturally as shown in \([13, 16]\). In momentum space this turns out to give the 4-dimensional de Sitter space:

\[ (P_0)^2 - (P_i)^2 - (P_5)^2 = -\kappa^2, \]

where \( P_A \) denotes appropriately normalized momentum in 5-dimensional momentum space. (See Eq. (25) below). Equipped with the differential structure, scalar field theory has been studied in \( \kappa \)-Minkowski space-time \([11, 17, 18]\). The \( \kappa \)-deformation is extended to the curved space with \( \kappa \)-Robertson-Walker metric and is applied to cosmic microwave background radiation in \([19]\).

The black-body radiation effect is another physical example to test the theory. The case with Moyal deformed space has appeared in \([20]\) and the deformed effect is shown to be of the order of \( T^6 \). In this paper, we present the black-body radiation formula in \( \kappa \)-Minkowski space-time using the complex (free massless) scalar field theory. It is to be noted that the scalar field theories constructed in the \( \kappa \)-Minkowski space-time are not unique. Depending on the ordering of the kernel of the Fourier transformation, the multiplication law and the equations of motion for field operators are differently realized \([21, 22]\). And the different ordering results in different dispersion relations for particles and antiparticles. Here, we consider the ordering effect on the dispersion relation of the particle and anti-particle spectrum. Another effect to take into consideration is the measure in the momentum integral which affects the counting of massless modes. We derive the black-body radiation formula in two ways. Quantum mechanical approach is the one, where the dispersion relation for the stable massless mode is used, and an effective thermal field theory approach using the free massless scalar field theory is the other.

This paper is organized as follows. In section II, we calculate the thermal energy density based on the statistical mechanics of one particle quantum mechanics in \( \kappa \)-Minkowski space-time. In section III we investigate the ordering effect on the free field action of a scalar field, and second-quantize the theories. The case of asymmetric ordering is presented in subsection III A and the symmetric ordering case in subsection III B. In the asymmetric case the particle and antiparticle have different dispersion relations, but there is a regularity in the energy spectra of the complex modes. On the other hand, the symmetric ordering has the same particle and antiparticle spectra but the regularity in the energy spectra of the complex modes is lost. In both cases, when the momentum reaches a certain value of order \( \kappa \), the energy dispersion becomes infinite. When the momentum exceeds this value, then some of complex modes become real modes and vice versa. It turns out that the contribution of these high momentum modes to the black-body radiation spectra does not have the correct commutative limits. In Section IV we propose an effective thermal field
theory in $\kappa$-Minkowski space-time using an effective Hamiltonian, which reproduces the black-body radiation spectra obtained in Section II. This is possible even in this infinite derivative theory since the theory we are considering is the diagonalized one and is decoupled from the infinite complex modes. It is noted that in this formalism, one naturally puts the integration measure as the $\kappa$-deformed Poincaré invariant one. In section V, our results are summarized and some considerations on the high momentum mode which exceeds the momentum cut-off are given.

II. BLACK-BODY RADIATION

The dispersion relation of the massless particle in $\kappa$-Minkowski space-time is known to be different from that of the commutative one. For example, in the case of the asymmetric deformation, the energies of the stable modes with momentum $k$ are given by

$$\Omega_k^+ = -\kappa \log(1 - \frac{|k|}{\kappa}), \quad \Omega_k^- = \kappa \log(1 + \frac{|k|}{\kappa}).$$

For the positive mode ($\Omega_k^+$), the momentum is restricted to $|k| < \kappa$. This dispersion relation is derived from the asymmetric ordering of the kernel function of the Fourier transformation (see the details in section III A).

These massless particles, in a thermal equilibrium at temperature $1/\beta$, will give internal thermal energy density $\rho_+(\beta)$,

$$\rho_+(\beta) = \int_{|k| \leq \kappa} \frac{d^3k}{(2\pi)^3} \frac{e^{\beta \Omega_k^+}}{e^{\beta \Omega_k^+} - 1},$$

if one assumes that the particles do not interact with each other. Here we introduce the measure factor $e^{\alpha \Omega_k^+}/(e^{\alpha \Omega_k^+} - 1)$ for the positive mode in the momentum integral. This measure factor is 1 (i.e. $\alpha = 0$) if one simply counts the modes in momentum space. On the other hand, if one requires the integration measure to be invariant under the $\kappa$-Poincaré transformation, one needs to set $\alpha = 3$. It is not clear at this point which one is the correct choice. (More details are given in section IV and V).

Explicit form of $\rho_+$ is

$$\rho_+(\beta) = \beta^{-4} \frac{4\pi}{(2\pi)^3} f(\beta \kappa), \quad f(x) = x^4 \int_0^1 dp \frac{p^2(1 - p)^{-\alpha} \log(1 - p)^{-1}}{(1 - p)^{-x} - 1}.$$ (4)

One may expand $f(x)$ as a series in a inverse power of $x$ (by setting $1 - p = e^{-t}$),

$$f(x) = x^4 \int_0^\infty dt \frac{t^3(a_0 + a_1 t + a_2 t^2 \cdots)}{e^{xt} - 1} = \sum_{i=0}^\infty \frac{\Gamma(4 + i) \zeta(4 + i) a_i}{x^i} = \frac{\pi^4}{15} + \frac{24 \zeta(5)}{x} a_1 + \frac{8\pi^6}{63} a_2 x^2 + \cdots.$$ (5)

The first few coefficients are given by $a_0 = 1, a_1 = \alpha - 2, a_2 = 25/12 - 2\alpha + \alpha^2/2, a_3 = -3/2 + 25\alpha/12 - \alpha^2 + \alpha^3/6$.

The negative mode contribution to the energy density is given by

$$\rho_-(\beta) = \beta^{-4} \frac{4\pi}{(2\pi)^3} g(\beta \kappa), \quad g(x) = x^4 \int_0^\infty dp \frac{p^2(1 + p)^{-\alpha} \log(1 + p)}{(1 + p)^x - 1}.$$ (6)
One may again expand $g(x)$ as a series in a inverse power of $x$ (by setting $1 + p = e^t$),

$$g(x) = x^4 \int_0^\infty dt \frac{t^3 (1 + b_1 t + b_2 t^2 + \cdots)}{e^{xt} - 1} = \sum_{i=0} \Gamma(4 + i) \zeta(4 + i) \frac{b_i}{x^i}$$

$$= \frac{\pi^4}{15} + 24 \zeta(5) \frac{b_1}{x} + \frac{8 \pi^6}{63} \frac{b_2}{x^2} + \cdots. \quad (7)$$

where the coefficients are given by $b_1 = 2 - \alpha$, $b_2 = 25/12 - 2\alpha + \alpha^2/2$, $b_3 = 3/2 - \frac{5\alpha}{12} + \alpha^2 + \alpha^3/6$. It is clear that the positive mode contribution is not the same as the negative mode contribution since $b_i(\alpha) = (-1)^i a_i(\alpha)$ in Eqs. (5) and (7).

If the system consists of positive and negative modes together, then the two contributions are added

$$\rho(\beta) = \rho_+ (\beta) + \rho_- (\beta)$$

$$= \frac{T^4}{2\pi^2} \sum_{i=0} \Gamma(4 + i) \zeta(4 + i) \frac{(a_i + b_i)}{x^i}$$

$$= T^4 \left( \frac{\pi^2}{15} + \frac{2\pi^4 (25 - 24\alpha + 6\alpha^2) T^2}{189 \kappa^2} + O \left( \frac{T^4}{\kappa^4} \right) \right).$$

This shows that to the lowest order, the internal energy density agrees with the commutative result. The deviation from the commutative result comes at $O(T^4/\kappa^4)$.

The dispersion relation is different when derived from the time-symmetric ordering of the kernel function of the Fourier transformation as shown in Sec. III B. In this case the dispersion relations for the stable positive and negative modes are the same and is given by

$$\omega_k = \kappa \log \left( 1 + \frac{k^2}{2\kappa^2} + \frac{|k|}{\kappa} \sqrt{1 + \frac{k^2}{4\kappa^2}} \right). \quad (9)$$

The thermal energy density for the positive mode is given by

$$\rho_s^{(+)}(\beta) = \int \frac{d^3 k}{(2\pi)^3} \frac{e^{\beta \omega_k/\kappa}}{e^{\beta \omega_k} - 1} \frac{\omega_k}{e^{\beta \omega_k/\kappa} - 1}. \quad (10)$$

The $\kappa$–Poincaré invariant measure factor is achieved if one puts $\bar{\kappa} = 3/2$. The explicit expression for the energy density is given by

$$\rho_s^{(+)}(\beta) = \beta^{-4} \frac{4\pi}{(2\pi)^3} h(\beta \kappa), \quad h(x) = x^4 \int_0^\infty dp \frac{p^2 A^{\bar{\alpha}} \log A}{A^x - 1}. \quad (11)$$

where $A = 1 + \frac{p^2}{2} + \sqrt{(1 + p^2/2)^2 - 1}$. Putting $A(p) = e^t$, one has

$$h(x) = x^4 \int_0^\infty dt \frac{(1 - e^{-t})^2 e^{(\bar{\alpha}+1)t} \cosh(t/2)}{e^{xt} - 1} = x^4 \int_0^\infty dt \frac{t^3 (c_0 + c_1 t + c_2 t^2 + \cdots)}{e^{xt} - 1}. \quad (12)$$

where $c_0 = 1$, $c_1 = \bar{\alpha}$, $c_2 = 5/24 + \bar{\alpha}^2/2$, $c_3 = 5\bar{\alpha}/24 + \bar{\alpha}^3/6$. Comparing $h(x)$ with $g(x)$ in Eq. (7), we have

$$h(x) = \sum_{i=0} \Gamma(4 + i) \zeta(4 + i) \frac{c_i}{x^i}. \quad (13)$$
Obviously, the negative mode contribution \( \rho_s^{(-)}(\beta) \) is given if one replaces \( \bar{\alpha} \) to \(-\bar{\alpha}\). If the system has particle and anti-particle together, then the energy density becomes

\[
\rho_s^{(+)}(\beta) + \rho_s^{(-)}(\beta) = T^4 \left( \frac{\pi^2}{15} + \frac{\pi^4}{189} \frac{(5 + 12\bar{\alpha}^2)T^2}{\kappa^2} + O \left( \frac{T^4}{\kappa^4} \right) \right). \quad (13)
\]

The energy density coincides with the commutative one at the lowest order but the next order effect is not same as the the previous asymmetric result Eq. (8). Thus, the ordering effect results in the different blackbody radiation formula.

III. \( \kappa \)-DEFORMED FREE FIELD ACTION

The construction of field theories in \( \kappa \)-Minkowski space-time is not simple since, for example, the deformed Poincaré transformation is not realized linearly in non-commutative coordinate spacetime in so-called bi-crossproduct basis\[14]\ (see Appendix for summary). On the other hand, the \( \kappa \)-deformed Poincaré transformation can be described confidently in the momentum space as a dual description of the \( \kappa \)-Minkowski spacetime, \( \langle p^\nu, x^\mu \rangle = i\eta^{\mu\nu} \) with \( \text{diag}(\eta^{\mu\nu}) = (1, -1, -1, -1) \). Thus, we will construct the action for the scalar field theory in the momentum space first and convert it into the one in coordinate space by using the Fourier transformation.

The 4-momenta are required to commute with each other:

\[
[p^\mu, p^\nu] = 0. \quad (14)
\]

In addition, the Lorentz algebra remains un-deformed from that of the commutative case and the rotational invariance is maintained\[14]\:

\[
[M_i, M_j] = i\epsilon_{ijk}M_k, \quad [M_i, N_j] = i\epsilon_{ijk}N_k, \quad [N_i, N_j] = -i\epsilon_{ijk}M_k
\]

where \( M_i \) and \( N_i \) are rotation and boost generator respectively.

\( \kappa \)-deformed Poincaré algebra deforms the commutators between the boost and the translation generators, which reflects the non-commutative nature of the space-time. The explicit form of the commutators is not unique but is related to the Hopf algebra of the co-product.

A. Asymmetric deformation

The co-product of four momenta is given by\[11]\:

\[
\Delta(p^0) = p^0 \otimes 1 + 1 \otimes p^0, \quad \Delta(p^i) = p^i \otimes e^{-p^0/\kappa} + 1 \otimes p^i. \quad (16)
\]

This is related to a composition rule of the exponential operator \( e^{-ip^0\hat{x}} \), which is the kernel of the Fourier transformation and transforms the theory in space-time coordinates to the one in momentum space. Here \( \hat{x} = (\hat{x^0}, \hat{x}) \), \( p = (p^0, p) \) and \( p \cdot \hat{x} \equiv p^0\hat{x^0} - p \cdot \hat{x} \). Its ordering is defined as

\[
: e^{-ip^0\hat{x}}: \equiv e^{-ip^0\hat{x^0}} e^{ip \cdot \hat{x}}. \quad (17)
\]
Multiplication of two ordered exponential functions follows from Eqs. (1) and (17):

\[ e^{-ip\cdot\hat{x}} \cdot e^{-iq\cdot\hat{x}} = e^{-i(p^0+q^0)\hat{x}^0 + i(p e^{-q^0/\kappa} + q)\cdot\hat{x}}. \] (18)

whose four momentum addition rule indicates the co-product (16). This co-product changes the commutation relations between the boost and translation generators:

\[ [N_i, p_j] = i\delta_{ij} \left( \frac{\kappa}{2} (1 - e^{-2p^0/\kappa}) + \frac{p^2}{2\kappa^2} e^{p^0/\kappa} \right) - \frac{i}{\kappa} p_ip_j, \quad [N_i, p_0] = ip_i. \] (19)

To construct field theory, one employs the \( \ast \)-product formalism where all the coordinate variables \( x \) commute with each other and the homomorphism of the exponential non-commutative operator product is maintained so that the non-commuting effect is encoded in the definition of the \( \ast \)-product. From now on, instead of using the non-commutative space-time coordinates directly, we will use the \( \ast \)-product formalism of the exponential function defined as

\[ e^{-ip\cdot\hat{x} \ast e^{-iq\cdot\hat{x}}} = e^{-iv(p,q)\cdot\hat{x}} \] where \( v(p,q) = (p^0 + q^0, pe^{-q^0/\kappa} + q) \). (20)

An arbitrary function of coordinates is expressed through the Fourier transformation of a momentum function,

\[ F(x) = \int_p e^{-ip\cdot\hat{x}} f(p), \] (21)

where \( \int_p \) denotes the integration over 4-momentum, \( \int d^4p/(2\pi)^4 \).

One needs \( \ast \)-derivatives of exponential function to construct an action for a field theory. The exterior derivative of deformed algebra is obtained in (11, 13):

\[ \partial_{\mu} \ast e^{-ip\cdot\hat{x}} = -i\chi_{\mu}(p)e^{-ip\cdot\hat{x}}, \] (22)

where

\[ \chi_0(p) = \kappa \left[ \sinh \frac{p_0}{\kappa} + \frac{p^2}{2\kappa^2} e^{p^0/\kappa} \right], \quad \chi_i(p) = p_i e^{p^0/\kappa}. \] (23)

The derivative function \( \chi_{\mu} \) satisfies the simple commutation relations with boost and rotation generators:

\[ [N_i, \chi_j] = i\delta_{ij}\chi_0, \quad [N_i, \chi_0] = i\chi_i, \quad [M_i, \chi_j] = i\epsilon_{ijk}\chi_k, \quad [M_i, \chi_0] = 0. \] (24)

Therefore, \( \chi_{\mu}(p) \) is \( \kappa \)-deformed Poincaré invariant. The explicit expression for this function is given by

\[ \chi_{\mu}(p) = M_\kappa^2(p) \left( 1 + \frac{M_\kappa^2(p)}{4\kappa^2} \right), \] (25)

where \( M_\kappa^2(p) \) is the first Casimir invariant (\( [M_{\mu\nu}, M_\kappa^2(p)] = 0 \))

\[ M_\kappa^2(p) = \left( 2\kappa \sinh \frac{p_0}{2\kappa} \right)^2 - p^2 e^{p^0/\kappa}. \] (26)

The invariant volume form \( d^4x = d\chi_0 d\chi_1 d\chi_2 d\chi_3 \) is related to the momentum integration;

\[ d^4\chi(p) = \frac{M_\kappa^2(p)}{2\kappa^2} d^4p e^{3p^0/\kappa}. \] (27)
Thus $d^4 p e^{3p^0/\kappa}$ is the invariant volume element. Employing the notation $\tilde{p}$ where

$$
\tilde{p}^0 = p^0, \quad \tilde{p} = e^{p^0/\kappa} p,
$$

the invariant measure is written as

$$
d^4 p e^{3p^0/\kappa} = d^4 \tilde{p},
$$

and its integration is denoted as $\int_{\tilde{p}} = \int d^4 \tilde{p}/(2\pi)^4$.

The $\ast$-derivative has the conjugate counter-part $\partial^\dagger_\mu$ through the defining relation

$$
\int_x \phi_1(x) \ast (\partial_\mu \phi_2(x)) \equiv \int_x (\partial^\dagger_\mu \phi_1(x)) \ast \phi_2(x).
$$

Its explicit representation is given by

$$
\partial^\dagger_\mu \ast e^{-i p \cdot x} = -i \chi_\mu (-\tilde{p}) e^{-i p \cdot x}.
$$

We will call $p \to -\tilde{p}$ the conjugate transformation ($\tilde{p}$ is given in Eq. (28)). It is noted that the Casimir invariant $M^2_\kappa(p)$ and invariant volume element do not change their form under the conjugate transformation;

$$
M^2_\kappa(p) = M^2_\kappa(-\tilde{p}), \quad d^4 \chi(p) = d^4 \chi(-\tilde{p}).
$$

Before constructing the scalar field action, let us first consider the equation of motion for a free scalar field. The classical field equation will be given by

$$
\left[ \partial_\mu \ast \partial^\ast_\mu + m^2 \right] \phi(x) = 0.
$$

Using the Fourier transformed form of the field

$$
\phi(x) = \int_p e^{-i p \cdot x} \varphi(p),
$$

one finds the equation in momentum space $\mathcal{E}(p) \varphi(p) = 0$ where

$$
\mathcal{E}(p) = \chi_\mu \chi_\mu(p) - m^2
$$

is the $\kappa$-deformed invariant. The solution of Eq. (32) has the form in coordinate space

$$
\Phi(x) = \int_p e^{-i p \cdot x} f(p) \delta(\mathcal{E}(p)),
$$

where $f(p)$ is an arbitrary regular function.

It is obvious that the solution also satisfies the conjugate equation of motion,

$$
\left[ \partial^\dagger_\mu \ast \partial^{\dagger\mu} + m^2 \right] \Phi(x) = 0,
$$

due to the invariant property of $\chi_\mu$ in Eqs. (25) and (31). On the other hand, noting

$$
\left(: e^{-ip \cdot x} : \right)^{\dagger} = e^{-i \tilde{p} \cdot \tilde{x}} e^{i p^0 \tilde{x}^0} = e^{i(p^0 \tilde{x}^0 - e^{p^0/\kappa} p \cdot \tilde{x})} = : e^{i \tilde{p} \cdot x}:
$$

(37)
one may define a conjugate scalar field of $\phi(x)$ in accordance with the conjugation of the ordered exponential operator,

$$\phi^c(x) \equiv \int_{\bar{p}} e^{i\bar{p}\cdot x} \varphi^\dagger(p) = \int_{\bar{p}} e^{-i\bar{p}\cdot x} \varphi^\dagger(-\bar{p}).$$  \hspace{1cm} (38)

Here $\varphi^\dagger(p)$ denotes the ordinary complex conjugate of $\varphi(p)$ in the momentum space and invariant measure $\int_{\bar{p}}$ is used. Using $\phi^c(x)$ one may construct other solution of the equation of motion

$$\Phi^c(x) = \int_{\bar{p}} e^{i\bar{p}\cdot x} f^\dagger(p)\delta(E(p)).$$  \hspace{1cm} (39)

One concludes that if $\Phi(x)$ is the classical solution of the equation of motion, then so is $\Phi^c(x)$. It is to be noted that if we start with the exponential operator Eq. (37) (with $p$ replaced by $-\bar{p}$) instead of the original one in Eq. (17), then the role of $\phi(x)$ and $\phi^c(x)$ are switched.

Thus, $\phi^c(x)$ is the analogue of the complex conjugate in coordinate space. To avoid the confusion with the ordinary complex conjugate in commutative space, we put the superscript $c$ instead of $\dagger$. The definition of the conjugation leads to the following properties,

$$\left(\phi^c(x)\right)^c = \phi(x), \quad \left(\phi_1(x) \ast \phi_2(x)\right)^c = \phi_2^c(x) \ast \phi_1^c(x), \quad \left(\partial_\mu \phi(x)\right)^c = -\partial^c_\mu \phi^c(x).$$  \hspace{1cm} (40)

It is natural to have the delta-function

$$\int_{\bar{p}} e^{-i\bar{p}\cdot x} = (2\pi)^4\delta^4(x)$$  \hspace{1cm} (41)

and its inverse $\int_{x} e^{-i\bar{p}\cdot x} = (2\pi)^4\delta^4(p)$ where $\int_{x} = \int d^4x$. Using this delta function, one has

$$\int_{x} \phi_1^c(x) \ast \phi_2(x) = \int_{\bar{p}} \varphi_1^\dagger(p) \varphi_2(p) = \int_{\bar{p}} \varphi_1^\dagger(-\bar{p}) \varphi_2(-\bar{p}),$$  \hspace{1cm} (42)

which is invariant under the deformed Poincaré transformation if $\varphi_1^\dagger(p) \varphi_2(p)$ is. It should be noted that the Poincaré transformation property of the space-time coordinates is not defined clearly, but one can always check if the integrated value is Poincaré invariant in momentum space.

The classical equation can be obtained from action,

$$S = \int_{x} \phi^c(x) \ast \left[-\partial_\mu \ast \partial^\mu \ast -m^2\right] \phi(x) = \int_{\bar{p}} \varphi^\dagger(p)\Delta^{-1}_F(p) \varphi(p),$$  \hspace{1cm} (43)

where

$$\Delta^{-1}_F(p) = E(p) + i\epsilon$$  \hspace{1cm} (44)

will be called the Feynman propagator and $\epsilon$, the small positive real number, is added to avoid the singularity on the real axis of $p^0$. The scalar fields in coordinate space, $\phi(x)$ and $\phi^c(x)$ are defined in Eq. (33) and (38). It is noted that the action is $\kappa$-Poincaré invariant if $\varphi^\dagger(p)\varphi(p)$ is. Thus we will require $\varphi(p)$ and $\varphi^\dagger(p)$ be $\kappa$-Poincaré invariant.
The Feynman propagator is factorized as $\Delta^{-1}_F(p) = F_+(p)F_-(p)$ where

$$F_+(p) = \frac{\kappa}{2} e^{p_0/\kappa} (e^{-p_0/\kappa} + \alpha_+) (e^{-p_0/\kappa} + \alpha_-),$$

$$F_-(p) = \frac{\kappa}{2} e^{p_0/\kappa} (e^{-p_0/\kappa} - \alpha_+) (e^{-p_0/\kappa} - \alpha_-),$$

with

$$\alpha_\pm = \sqrt{1 + \frac{m^2 - i\epsilon}{\kappa^2}} \pm \sqrt{\frac{m^2 + p^2 - i\epsilon}{\kappa^2}}.$$

$F_\pm(p)$ is invariant under the conjugate transformation $F_\pm(p) = F_\pm(-\bar{p})$. It is noted that $\Delta^{-1}_F(p)$ is periodic in $p_0$ with a period $i\kappa\pi$;

$$\Delta^{-1}_F(p_0 + i\kappa\pi, p) = \Delta^{-1}_F(p_0, p).$$

Thus, the on-shell condition is given by

$$p_0 = \Omega^+_p + in\pi\kappa, \quad -\Omega^-_p + in\pi\kappa$$

(47)

where $n$ is an integer and

$$\Omega^+_p = -\kappa \ln \alpha_-, \quad \Omega^-_p = +\kappa \ln \alpha_+.$$

The existence of the infinite number of on-shell values is due to the infinite number of time derivatives in the $*$-equation of motion.

Here $\Omega^-_p$ is always positive for real $p$, but $\Omega^+_p$ is positive only when $p^2 < \kappa^2$. If $p^2 > \kappa^2$, $\Omega^+_p$ becomes complex but $-\kappa \ln (-\alpha_-)$ becomes real. The meaning of this mode change is not clear at this moment. In this section we proceed to second-quantize the positive and negative on-shell modes of this scalar field using the momentum cut-off. Some more consideration is given in section $\Box$

Suppose one try to second-quantize the field. To do this one may introduce the source terms to the action

$$\int_x \left( F^c(x)\dot{\phi}(x) + \dot{\phi}^c(x)J(x) \right) = \int_{\bar{p}} \left( j^\dag(p)\varphi(p) + \varphi^\dag(p)j(p) \right)$$

and identify the field correlation as

$$\langle \varphi(p)\varphi^\dag(q) \rangle e^{3q^0/\kappa} = \langle \varphi^\dag(q)\varphi(p) \rangle e^{3p^0/\kappa} = i \Delta_F(p)(2\pi)^4 \delta^4(p - q).$$

In analogy with the commutative field theory, one may introduce the second-quantized field operator with a prescription when integrated over the momentum space. Putting $\Phi_{op}(\mathbf{p}) = \Phi^{(+)}_{op}(\mathbf{p}) + \Phi^{(-)}_{op}(\mathbf{p})$ (superscript $+/-$ refers to the positive/negative frequency part)

$$[\Phi^{(+)}_{op}(\mathbf{p}), \Phi^{(-)}_{op}(\mathbf{q})] = \langle \Phi^{(+)}_{op}(\mathbf{p})\Phi^{(-)}_{op}(\mathbf{q}) \rangle = i \oint_{lower} \frac{dp^0}{2\pi} \Delta_F(p)(2\pi)^3 \delta^3(p - q).$$

$\oint_{lower}$ is the prescription of the integral of $p^0$ so that the contour is taken along the lower half-plane of the complex $p^0$ plane to include the real pole $\Omega^+_p$. Explicit evaluation results in the quantization rule,

$$[\Phi^{(+)}_{op}(\mathbf{p}), \Phi^{(-)}_{op}(\mathbf{q})] = \sum_{\text{mode}} \frac{(2\pi)^3 \delta^3(p - q)}{2E(\mathbf{p})}.$$

(49)
\[ E(p) = \sqrt{(m^2 + p^2)(1 + m^2/\kappa^2)} \] comes from the residue
\[
\left| \frac{\delta \Delta_F^{-1}(p)}{\delta p^0} \right|_{p^0 = \Omega_p^\pm + in\pi \kappa} = 2E(p) .
\]

Note that the summation in (49) is due to the periodic properties of the pole structure in the Feynman propagator (46). Each pole in the complex plane contributes to each mode sum. Likewise,
\[
\left[ \Phi_{op}^{(-)}(p), \Phi_{op}^{(+)}(q) \right] = - \left\langle \Phi_{op}^{(+\dagger)}(q) \Phi_{op}^{(+)}(p) \right\rangle = -i \oint_{\text{upper}} \frac{dp^0}{2\pi} \Delta_F(p)(2\pi)^3 \delta^3(p - q)
\]
\[
= \sum_{\text{mode}} \frac{(2\pi)^3 \delta^3(p - q)}{2E(p)} .
\]
\( \oint_{\text{upper}} \) is the contour taken along the upper half-plane to include the pole \(-\Omega_p^\pm\). This identification results in the second-quantized field
\[
\Phi_{op}^{(+)}(p) = \frac{1}{2E(p)} \left( a(p) + \sum_{n>0} (a_n(p) + b_n(-p)) \right),
\]
\[
\Phi_{op}^{(-)}(p) = \frac{1}{2E(p)} \left( b^\dagger(p) + \sum_{n<0} (a_n^\dagger(p) + b_n^\dagger(-p)) \right),
\]
\[
\Phi_{op}^{(+\dagger)}(p) = \frac{1}{2E(p)} \left( a^\dagger(p) + \sum_{n>0} (a_n^\dagger(p) + b_n^\dagger(-p)) \right),
\]
\[
\Phi_{op}^{(-\dagger)}(p) = \frac{1}{2E(p)} \left( b(p) + \sum_{n<0} (a_n(p) + b_n(-p)) \right),
\]
where the creation and annihilation operators satisfy the commutation relations
\[
\left[ a(p), a^\dagger(q) \right] = 2E(p) (2\pi)^3 \delta^3(p - q) , \quad \left[ a_n(p), a_m^\dagger(q) \right] = 2E(p) (2\pi)^3 \delta^3(p - q) \delta_{m,n} ,
\]
\[
\left[ b(p), b^\dagger(q) \right] = 2E(p) (2\pi)^3 \delta^3(p - q) , \quad \left[ b_n(p), b_m^\dagger(q) \right] = 2E(p) (2\pi)^3 \delta^3(p - q) \delta_{m,n} .
\]

The quantum field in coordinate space may be defined as
\[
\Phi_{op}(x) = \int \frac{d^3p}{(2\pi)^3 2E(p)} \left[ e^{-ip \cdot x} a(p) + \sum_{n>0} e^{-n\pi\kappa|x|^0} e^{-ip \cdot x} \left( a_n(p) + a^\dagger_n(-p) \right) \right] \]
\[
+ \int \frac{d^3p}{(2\pi)^3 2E(p)} \left[ e^{ip \cdot x} b(p) + \sum_{n<0} e^{-n\pi\kappa|x|^0} e^{ip \cdot x} \left( b_n(p) + b^\dagger_n(-p) \right) \right] ,
\]
\[
\Phi_{op}^c(x) = \int \frac{d^3p e^{3i\pi p / \kappa}}{(2\pi)^3 2E(p)} \left[ e^{ip \cdot x} a(p) + \sum_{n>0} e^{-n\pi\kappa|x|^0} e^{ip \cdot x} \left( e^{3i\pi \kappa |x|^0} a^\dagger_n(p) + e^{-3i\pi \kappa |x|^0} a_n(p) \right) \right] \]
\[
+ \int \frac{d^3p e^{-3i\pi p / \kappa}}{(2\pi)^3 2E(p)} \left[ e^{-ip \cdot x} b(p) + \sum_{n<0} e^{-n\pi\kappa|x|^0} e^{-ip \cdot x} \left( e^{-3i\pi \kappa |x|^0} b^\dagger_n(p) + e^{3i\pi \kappa |x|^0} b_n(p) \right) \right] ,
\]
where \( p_+ = (\Omega^+_p, \mathbf{p}) \) and \( p_- = (\Omega^-_p, \mathbf{p}) \). This quantized field satisfies the equation of motion (32) and its conjugate one. \( a^{\dagger}(\mathbf{p}) \) creates a positive mode of energy \( \Omega^+_p \) and momentum \( \mathbf{p} \) from the vacuum \( |0\rangle \) with momentum \( |\mathbf{p}| < \kappa \). (The momentum integration of \( a^{\dagger}(\mathbf{p}) \) and \( a(\mathbf{p}) \) mode in (32) is assumed to be limited by \( \kappa \).) Likewise, \( b^{\dagger}(\mathbf{p}) \) creates a negative mode of energy \( \Omega^-_p \) and momentum \( \mathbf{p} \). (One may identify the positive mode with a particle and the negative mode with an anti-particle or vice versa.)

It should be remarked that the quantum field theory in \( \kappa \)-Minkowski spacetime is not well established yet, even though the above procedure is one of plausible proposals. For example, the complex mode with nonzero imaginary energy comes from the periodicity of the Feynman propagator Eq. (16) which originates from the non-trivial form of the Casimir invariant Eq. (25). The presence of the complex mode is not of the usual form of the canonical field theory and may raise the unitarity problem. Nevertheless, the complex mode lives only for a short period of time of the order \( 1/(n\kappa) \) and for this reason, we do not include the complex modes in the calculation of the black body radiation in Sec. II. In a similar spirit, one may find some authors neglect this complex modes from the beginning [23].

Another concern is that the relation between the propagator in coordinate space and one in momentum space is not clearly defined yet. One might obtain the Feynman propagator in coordinate space directly from the given action. In this process, however, one needs to evaluate the functional derivative inside the \(*\)-product and also define an appropriate \(*\)-product between different spacetime points. This non-trivial questions are to be treated carefully for a consistent and complete field theory, whose analysis is beyond the scope of our paper. This issue will be treated in a seperate paper along with interacting theory.

Let us turn to the real field case. The dispersion relations of the positive mode and negative mode energy parts do not coincide when \( \mathbf{p} \neq 0 \):

\[
\Omega^+_p - \Omega^-_p = -\kappa \ln(\alpha_+\alpha_-) = -\kappa \ln(1 - \mathbf{p}^2/\kappa^2),
\]

which is positive when \(|\mathbf{p}| < \kappa\). This mismatch leads to the difficulty in defining the real field as a self conjugate operator. As a consequence, one may use either of the following expansions as a candidate for real field representation:

\[
\Phi_1(x) = \int \frac{d^3p}{(2\pi)^32E(p)} \left[ e^{-ip_+x}a(p) + \sum_{n>0} e^{-n\pi\kappa|x^0|} e^{-ip_+x} \left( a_n(p) + a^{\dagger}_{-n}(p) \right) \right]
\]

\[
+ \int \frac{d^3p}{(2\pi)^32E(p)} \left[ e^{ip_+x}a^{\dagger}(p) + \sum_{n>0} e^{-n\pi\kappa|x^0|} e^{ip_+x} \left( a^{\dagger}_n(p) + a_{-n}(p) \right) \right],
\]

\[
\Phi_2(x) = \int \frac{d^3p}{(2\pi)^32E(p)} \left[ e^{-ip_-x}b(p) + \sum_{n>0} e^{-n\pi\kappa|x^0|} e^{-ip_-x} \left( b_n(p) + b^{\dagger}_{-n}(p) \right) \right]
\]

\[
+ \int \frac{d^3p}{(2\pi)^32E(p)} \left[ e^{ip_-x}b^{\dagger}(p) + \sum_{n>0} e^{-n\pi\kappa|x^0|} e^{ip_-x} \left( b^{\dagger}_n(p) + b_{-n}(p) \right) \right].
\]

This raises a question: Can we choose a vacuum or an energy operator such that the positive and negative spectra are symmetric? Or is it possible to shift the on-shell value \( p^0 \rightarrow p^0 - (\Omega^+_p - \Omega^-_p)/2 \), and redefine the on-shell value so that positive and negative dispersion relations are the same, \( \Omega^+_p = \Omega^-_p = \frac{\kappa}{2} \ln(\alpha_+/\alpha_-) \)? To answer this question, we use a freedom to choose the definition of the \(*\)-product by redefining the ordering of the exponential function in the following subsection.
B. Symmetric deformation

Let us newly define the ordering of exponential function in a symmetric way [21]:

\[ e^{-ip^0 z^0/2} e^{ip \hat{x} \cdot \hat{x}} \equiv e^{-ip^0 z^0/2} e^{i\frac{p_0}{2\kappa} + i(\mathbf{p} e^{-\frac{p_0}{2\kappa}} + \mathbf{q} e^{\frac{p_0}{2\kappa}})} \].

The multiplication of the two symmetric exponential functions is given by

\[ e^{-ip^0 z^0/2} \cdot e^{-iq^0 z^0/2} = e^{-i(p^0 + q^0)z^0 + i(\mathbf{p} e^{-\frac{p_0}{2\kappa}} + \mathbf{q} e^{\frac{p_0}{2\kappa}}) \cdot \hat{x}}. \]

Therefore, the new multiplication rule is given as

\[ e^{-i p \cdot \hat{x}} \ast_s e^{-i q \cdot \hat{x}} = e^{-iv(p,q) \cdot \hat{x}} \]

where

\[ v(p, q) = (p^0 + q^0, \mathbf{p} e^{-\frac{p_0}{2\kappa}} + \mathbf{q} e^{\frac{p_0}{2\kappa}}). \]

This is equivalent to changing the co-product of \( p_\mu \) as

\[ \Delta(p^0) = 1 \otimes p^0 + p^0 \otimes 1, \quad \Delta(p^i) = p^i \otimes e^{-\frac{p_0}{2\kappa}} + e^{\frac{p_0}{2\kappa}} \otimes p^i, \]

and the commutation relation involving the boost generator in \( \kappa \)-deformed Poincaré algebra as

\[ [N_i, p_j] = i\delta_{ij} e^{\frac{p_0}{2\kappa}} \left( \frac{\kappa}{2}(1 - e^{-2p^0/\kappa}) + \frac{p^2}{2\kappa} e^{\frac{p_0}{\kappa}} \right) - \frac{i}{2\kappa} p_i p_j e^{-\frac{p_0}{2\kappa}}, \quad [N_i, p_0] = ip_i e^{-\frac{p_0}{2\kappa}}. \]

Other algebraic relations involving translation and rotation are not affected. This change is effectively summarized as the shift of the 3-momenta \( p \rightarrow p e^{\frac{p_0}{2\kappa}} \) in the asymmetric definition, Eq. (16) and (19).

The \( \ast_s \)-derivative of exponential function now takes the form

\[ \partial_\mu \ast_s e^{-ip \cdot \hat{x}} = -i\xi_\mu(p) e^{-ip \cdot \hat{x}}, \]

\[ \xi_0(p) = \kappa \left[ \sinh \frac{p_0}{\kappa} + \frac{p^2}{2\kappa^2} \right], \quad \xi_i(p) = p_i e^{\frac{p_0}{2\kappa}}, \]

and its conjugate partial derivative is simply given by

\[ \partial^\mu \ast_s e^{-ip \cdot \hat{x}} = -i\xi_\mu(-p) e^{-ip \cdot \hat{x}}. \]

Under the conjugate transformation, \( p \) simply changes to \( -p \).

The Poincaré symmetry of the derivative functions \( \xi_\mu \) remains the same;

\[ [N_i, \xi_j] = i\delta_{ij} \xi_0, \quad [N_i, \xi_0] = i\xi_i, \quad [M_i, \xi_j] = i\epsilon_{ijk} \xi_k, \quad [M_i, \xi_0] = 0, \]

and, as in the case of the asymmetric ordering, \( \xi^\mu \xi_\mu(p) \) is invariant. The explicit expression for the invariant is

\[ \xi^\mu \xi_\mu(p) = M_2^2(p) \left( 1 + \frac{M_2^2(p)}{4\kappa^2} \right) = \xi^\mu \xi_\mu(p) = \xi^\mu \xi_\mu(-p), \]
where the Casimir invariant \( M_s^2(p) \) has a different form:

\[
M_s^2(p) = \left( 2\kappa \sinh \frac{p_0}{2\kappa} \right)^2 - p^2 = M_s^2(-p) .
\] (64)

In addition, the invariant volume form \( d^4\xi = d\xi_0 d\xi_1 d\xi_2 d\xi_3 \) is given in terms of the momentum variables by

\[
d^4\xi(p) = \frac{M_s^2(p)}{2\kappa^2} d^4p e^{\frac{3p_0}{2\kappa}} .
\] (65)

Thus the invariant measure for the momentum integral becomes

\[
d^4p_s \equiv d^4p e^{\frac{3p_0}{2\kappa}} .
\] (66)

Now the equation of motion for a free scalar field will be given by

\[
[\partial_\mu \ast_s \partial^{\mu} \ast_s + m^2] \phi_s(x) = 0 .
\] (67)

One may use the Fourier transformed field

\[
\phi_s(x) = \int_p e^{-ip\cdot x} \varphi(p) .
\] (68)

The equation of motion becomes in momentum space

\[
\mathcal{H}(p) \varphi(p) = 0 , \quad \mathcal{H}(p) = \xi^\mu \xi_\mu(p) - m^2
\] (69)

and \( \mathcal{H}(p) \) is the \( \kappa \)-deformed Poincaré invariant.

The solution of the classical equation of motion, Eq. (67), has the form

\[
\Phi_s(x) = \int_p e^{-ip\cdot x} f(p)\delta(\mathcal{H}(p)) ,
\] (70)

with \( f(p) \) an arbitrary regular function. This solution also satisfies the conjugate equation,

\[
[\partial^\dagger_\mu \ast \partial^{\mu} \ast + m^2] \Phi_s(x) = 0 .
\] (71)

On the other hand, using the complex conjugation property

\[
e^{-ip\cdot x} \dagger = e^{ip\cdot x} ,
\] (72)

one may define the conjugate of \( \phi_s(x) \) as

\[
\phi_s^c(x) = \int_p e^{ip\cdot x} \phi^\dagger(p) ,
\] (73)

just the complex conjugate of \( \phi_s(x) \) Thus, \( \Phi_s^c(x) = \int_p e^{ip\cdot x} f^\dagger(p)\delta(\mathcal{H}(p)) \) is again a solution of the conjugate equation of motion since

\[
\partial^\dagger_\mu e^{ip\cdot x} = -i\xi_\mu(p) e^{ip\cdot x} ,
\] (74)
and $\mathcal{H}(-p) = \mathcal{H}(p)$. This is similar to the ordinary complex conjugation except for the change of measure. It should be noted that

$$\int_x \phi_s^c(x) * \psi_s(x) = \int_p \phi^\dagger(p) \psi(p)$$

is $\kappa$-deformed Poincaré invariant if the Fourier-transformed $\phi^\dagger(p) \psi(p)$ is since $\int_p \equiv \int d^4p_s/(2\pi)^4$ is invariant.

The equation of motion can be obtained from the $\kappa$-deformed Poincaré invariant action,

$$S_2 = \int_x \phi_s^c(x) *_s [ -\partial_{\mu} *_s \partial^\mu *_s -m^2 ] \phi_s(x) = \int_p \phi^\dagger(p) \Delta_\mathcal{S}^{-1}(p) \varphi(p)$$

where $\Delta_\mathcal{S}^{-1}(p)$ is modified as

$$\Delta_\mathcal{S}^{-1}(p) = \mathcal{H}(p) + i\epsilon = \Delta_\mathcal{S}^{-1}(-p)$$

with $\epsilon$ small real number to avoid the singularity on the real axis of $p^0$.

The one-shell condition ($\Delta_\mathcal{S}^{-1}(p) = 0$) is given by

$$\cosh \frac{p^0}{\kappa} = \frac{p^2}{2\kappa^2} \pm \sqrt{1 + \frac{m^2 - i\epsilon}{\kappa^2}}.$$
and its on-shell value of the energy $\pm \kappa (\ln \gamma)$ is complex. However, when $p^2$ exceeds the above limit Eq. (81), the pair of the complex valued energies becomes a pair of real positive and negative energies.

Second-quantized scalar field can be obtained similarly as in the asymmetric case in Eq. (52). However, the residue for the $p^0$ integration is not simple as in the asymmetric case due to the periodicity in Eq. (80). Nevertheless, defining the residue at $\omega_p$ as

$$D(p) = \left| \frac{\delta \Delta^{-1}_S(p)}{\delta p^0} \right|_{p^0 = \pm \omega_p}$$

one may write the scalar field as

$$\Phi_s(x) = \int \frac{d^3p}{(2\pi)^3 2D(p)} \left( e^{-ip \cdot x} a(p) + e^{ip \cdot x} b^{\dagger}(p) \right) + \cdots, \quad (82)$$

$$\Phi^c_s(x) = \int \frac{d^3p}{(2\pi)^3 2D(p)} \left( e^{\frac{3 \omega_p}{2\pi}} e^{ip \cdot x} a^{\dagger}(p) + e^{-\frac{3 \omega_p}{2\pi}} e^{-ip \cdot x} b(p) \right) + \cdots,$$

where $\cdots$ denotes the contributions from the complex modes and higher energy modes at the Planck scale and the momentum is limited as in Eq. (81). The commutation relations become

$$[a(p), a^{\dagger}(q)] = 2D(q) (2\pi)^3 \delta^3(p - q), \quad [b(p), b^{\dagger}(q)] = 2D(q) (2\pi)^3 \delta^3(p - q). \quad (83)$$

Both $a^{\dagger}(p)$ and $b^{\dagger}(p)$ create modes with energy $\omega_p$ and momentum $p$. This shows that the real field can be self-conjugate if one demands

$$a(p) = b(p) \quad \text{or} \quad \Phi^c_s(x) = \Phi_s(x). \quad (84)$$

**IV. EFFECTIVE THERMAL FIELD THEORY**

The free scalar field theory constructed in $\kappa$-Minkowski space-time in the last section leads to the massless dispersion relation $\pm \Omega_k$ of Eq. (2) in the asymmetric ordering case, and $\pm \omega_k$ of Eq. (9) in the symmetric ordering case. Based on these dispersion relations, we have used the quantum mechanical particle picture to evaluate the black-body radiation spectra for the massless scalar particles living in the $\kappa$-Minkowski space-time.

In the case of the conventional commutative field theories, the black-body radiation spectrum evaluated by quantum mechanical method, agrees with that computed by using the thermal field theory, which is formulated based on the observation that the time evolution operator in quantum field theory can be analytically continued to the Boltzmann factor of the quantum statistical mechanics.

However, the field theory in the $\kappa$-Minkowski space-time involves infinite order time derivatives and one can not define the Hamiltonian operator as in the conventional field theory. Due to this difficulty we do not have a reliable field theoretic method to evaluate the quantum partition function. Nevertheless, for this diagonalized theory of the scalar field considered in the last section, one can proceed with the individual modes. Note that there are two stable modes $\pm \Omega^\pm_k$ in the asymmetric case and $\pm \omega_k$ in the symmetric case. If there are two stable modes only, we may start with an effective “Hamiltonian”

$$H[\phi] = \int_x \phi^{\dagger} \left( \frac{\partial}{\partial \tau} + \tilde{\omega}^+( -i \nabla) \right) \left( \frac{\partial}{\partial \tau} + \tilde{\omega}^-( -i \nabla) \right) \phi(x) \quad (85)$$
with Euclidean “time” $\tau$. $\tilde{\omega}^{\pm}(-i\nabla)$ reduces to $\tilde{\omega}^{\pm}(k)$ in momentum space and refers to the $\Omega_{k}^{\pm}$ or $\omega_{k}$, decoupled stable one-particle dispersion relation of $\phi$. It is assumed that the real part of $\tilde{\omega}^{\pm}(k)$ is positive. This consideration is similar to the one considered in section II and thus, should yield the same black-body radiation formula through the relation,

$$\langle H \rangle_{\beta} = -\frac{\partial}{\partial \beta} \log Z(\beta), \quad Z(\beta) = \int [d\phi] e^{-\beta H}. \quad (86)$$

One may expand the field operator as

$$\phi(x, \tau) = \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \int_{p} e^{i(p \cdot x + \omega_{n} \tau)} \phi_{n}(p), \quad (87)$$

where $\beta \omega_{n} = 2\pi n$ with $n$ integer so that $\phi(x, \tau = 0) = \phi(x, \tau = \beta)$ and $\int_{p}$ denotes the momentum integral with an appropriate measure. Then the partition function is written in terms of the modes as,

$$\log Z = -\text{Tr} \log \left(\frac{\partial}{\partial \tau} + \tilde{\omega}^{+}(p) + \tilde{\omega}^{-}(p)\right)$$

$$= -\sum_{n=-\infty}^{\infty} \int_{p} \log \left((2\pi ni + \beta \tilde{\omega}^{+}_{p})(2\pi ni + \beta \tilde{\omega}^{-}_{p})\right) \quad (88)$$

where $\beta \omega_{n} = 2\pi n$ is used. The thermal energy density is given as

$$\langle H \rangle_{\beta} = \sum_{n} \int_{p} \left(\frac{\tilde{\omega}^{+}_{p}}{2\pi ni + \beta \tilde{\omega}^{+}_{p}} + \frac{\tilde{\omega}^{-}_{p}}{2\pi ni + \beta \tilde{\omega}^{-}_{p}}\right)$$

$$= \int_{p} \left(\frac{\tilde{\omega}^{+}_{p} + \tilde{\omega}^{-}_{p}}{2} + \frac{\tilde{\omega}^{+}_{p}}{e^{\tilde{\omega}^{+}_{p}} - 1} + \frac{\tilde{\omega}^{-}_{p}}{e^{\tilde{\omega}^{-}_{p}} - 1}\right) \quad (89)$$

where the summation over the discrete $n$ is done using the identities,

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi ni + x} = x \sum_{n=-\infty}^{\infty} \frac{1}{(2n\pi)^{2} + x^{2}}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2\pi n)^{2} + x^{2}} = \frac{1}{|x|} \left(\frac{1}{2} + \frac{1}{e^{|x|} - 1}\right).$$

Apart from the zero point energy (the first term in the thermal energy), the thermal energy reproduces the the quantum mechanical result given in section II.

In the above, we have constructed an effective thermal field theory in the $\kappa-Minkowski$ space-time, which reproduces the quantum mechanical results given in Sec. II, by singling out the stable real energy modes. This was possible since the quadratic part of the action in momentum space is factorized into the product of each mode factors, and thus the field operator is expanded as a linear combination of these mode contributions. Following the same procedure one may include the unstable modes also, as far as the dispersion relation of the decoupled mode leads to the positive real energy, whose effective “Hamiltonian” is of the form,

$$H[\phi] = \int_{x} \phi^{\dagger}(x) \prod_{m} \left(\frac{\partial}{\partial \tau} + \tilde{\omega}_{m}(-i\nabla)\right) \phi(x) \quad (90)$$
where $\tilde{\omega}_m(k)$ denotes the energy for the diagonalized on-shell mode we are considering. One may reproduce this effective “Hamiltonian” from the expression of $\Delta_F^{-1}(p)$ in Eq. (15) or $\Delta_S^{-1}(p)$ in Eq. (79) noting that

$$\sinh x = x \prod_{n=1}^{\infty} \left( 1 + \frac{x^2}{n^2\pi^2} \right), \quad \cosh x = \prod_{n=0}^{\infty} \left( 1 + \frac{4x^2}{(2n+1)^2\pi^2} \right).$$

Therefore, one may view Eq. (86) as the truncated version of full spectra. By-product of this consideration is the solution to the ambiguity of the integration measure in the momentum integral: The momentum integration in Eq. (88) with the $\kappa$-deformed Poincaré invariant measure is preferred even in this truncated version. This is reasonable since the field theories in the $\kappa$-Minkowski space-time and the related discussions of doubly special relativity [21, 24, 25, 26, 27, 28], are based on the symmetry of the system, namely, the $\kappa$-Poincaré invariance. As we have discussed earlier, the field theory action and related quantities are $\kappa$-Poincaré invariant in momentum space. In this sense, the invariant measure in the momentum integral is needed.

V. SUMMARY AND DISCUSSION

We have calculated the black body radiation spectra for massless scalar particles in $\kappa$-Minkowski space-time, for particle systems of two different dispersion relations corresponding to different orderings of the kernel function of the Fourier transformation. We have shown that the radiation spectrum depends on the form of the dispersion relations, and thus on the ordering of the exponential kernel function, or on how the $\kappa$-Poincaré algebra is realized in constructing field theories in $\kappa$-Minkowski space-time. This implies that the differently realized scalar field theories in $\kappa$-Minkowski space-time are not physically equivalent.

The deviations in the radiation spectra from those of the commutative case have two different origins. One is from the $\kappa$-dependent modification of dispersion relations and the other is due to the presence of the nontrivial measure factor in the momentum integral, all of which give the $O(T^6/\kappa^2)$ contribution to the thermal energy density. As explained in the last section, if we require the $\kappa$-deformed Poincaré invariance in momentum space, this measure contribution constitutes the dominant correction to the radiation spectrum.

In Sec.III we have shown how different orderings of the exponential kernel function lead to different realizations of the scalar field theories in $\kappa$-Minkowski space-time. Noting that the scalar field action in momentum space may be factorized into the product of each mode factors, we have selected out the stable real energy modes to second-quantize the scalar field theory for both the asymmetric and symmetric ordering cases. We have used this fact in the last section, to propose an effective thermal field theory, which reproduces the black-body radiation spectra obtained by using the quantum mechanical picture in Sec.II.

As noted in the last two sections, one of the most peculiar features of the field theories in $\kappa$-Minkowski space-time, is the real and complex mode changing phenomenon. That is, when momentum exceeds certain value, $\kappa$ in the asymmetric ordering and $2\kappa$ in the symmetric ordering case, a real mode becomes complex and a complex mode becomes real. To understand the role of this new real mode, we estimate the contribution to the black-body radiation spectra.
For asymmetric ordering case, the complex modes described by Eq. (51) become real when \(|k| \geq \kappa\), satisfying the dispersion relation, \(\tilde{\Omega}_k = -\kappa \log(|k|/\kappa - 1)\) for \(\kappa < |k| \leq 2\kappa\) and \(\tilde{\Omega}_k = \kappa \log(|k|/\kappa - 1)\) for \(|k| \geq 2\kappa\). These modes, in thermal equilibrium at temperature \(1/\beta\), will give internal thermal energy density,

\[
\rho_a^1(\beta) = \int_{2\kappa \geq |k| \geq \kappa} \frac{d^3k}{(2\pi)^3} \frac{\hat{\Omega}_k^+}{e^{\alpha \hat{\Omega}_k/\kappa} - 1}, \quad \rho_a^2(\beta) = \int_{|k| \geq 2\kappa} \frac{d^3k}{(2\pi)^3} \frac{\hat{\Omega}_k^+}{e^{\beta \hat{\Omega}_k} - 1}.
\]

Following the line of calculations in Sec. II, we find that the contribution of these new high momentum modes to the energy density is

\[
\rho_a = \rho_a^1 + \rho_a^2 = \frac{\kappa^2 T^2}{\pi^2} \left[ 4\zeta(2) + \frac{\alpha \zeta(3)T}{\kappa} + \frac{6(2\alpha^2 + 9)\zeta(4)T^2}{\kappa^2} + \ldots \right].
\]

Note that the dominant contribution of this high momentum mode is the term of \(O(\kappa^2 T^2)\), and does not reduces to the commutative theory limit.

In the symmetric ordering case, the new high energy modes with \(|k| \geq 2\kappa\) satisfies the dispersion relation \(\omega_{ak} = \kappa \log\left(\frac{k^2}{4\kappa^2} - 1 + \frac{|k|}{\kappa} \sqrt{\frac{k^2}{4\kappa^2} - 1}\right)\). The contribution of these high momentum modes to the black-body radiation spectrum turns out to be,

\[
\rho_{sa}^+(\beta) + \rho_{sa}^-(\beta) = \frac{2\kappa T^3}{\pi^2} \left[ 2\zeta(3) - \frac{6\zeta(4)T}{\kappa} + \frac{(12\alpha^2 + 19)\zeta(5)T^2}{\kappa^2} + O\left(\frac{T^3}{\kappa^3}\right) \right],
\]

whose dominant contribution is of the order \(O(\kappa T^3)\) and does not have the correct commutative limit.

Field theories in \(\kappa\)-Minkowski space-time and the \(\kappa\)-Poincaré algebra are constructed in such a way that the formulations reduce to those in commutative space-time as \(\kappa\) becomes infinite. Since the new high momentum modes do not satisfy this criterion, one must impose a condition to suppress these modes from appearing in the physical processes. One of the simplest ways to do this is to restrict the momenta by \(|k| \leq \kappa\), which reminds us of similar restriction needed in the formulations of doubly special relativity [24, 28]. In addition, the cut-off in the momentum space affects the completeness of the energy spectra and the effects on the coordinate space representation is to be seen.

Finally, in usual approach, the black body radiation is defined from the radiation of a gauge field. Therefore, to complete the study, we need to know how to construct the \(U(1)\) gauge field theory and then use the theory to calculate the black body radiation. Even though there exist some studies [29] on the \(U(1)\) gauge field theory in \(\kappa\)-Minkowski space-time, the study is still incomplete and there are many things to be known. We leave these subjects for later studies.

**APPENDIX A: BI-CROSSPRODUCT BASIS OF \(\kappa\)-MINKOWSKI SPACETIME**

In this appendix we summarize the bi-crossproduct basis of the \(\kappa\)-Poincaré algebra and its noncommutative Hopf algebra used in Sec. III [14]. The ordinary Poincaré algebra is
given as

\[[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - (\mu \leftrightarrow \nu)\right)\]

\[[M_{\mu\nu}, p_\rho] = -i\left(\eta_{\mu\rho} p_\nu - \eta_{\nu\rho} p_{\mu}\right)\]

\[[p_\mu, p_\nu] = 0 \quad (A1)\]

where \(\eta_{\mu\nu} = (+, -, -, -)\) is used. In Eq. (15), \(M_i = \epsilon_{ijk} M_{jk}/2\) and \(N_i = M_{i0}\) are used.

The \(\kappa\)-deformed Poincaré algebra is given by the modified form of the boost part of Eq. (A1), which depends on the ordering of exponential operator \(e^{-ip \cdot x}\), Eq. (17). For the asymmetric ordering in section III A, the boost part is given as Eq. (19);

\[[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - (\mu \leftrightarrow \nu)\right)\]

\[[M_i, p_j] = i\epsilon_{ijk} p_k, \quad [M_i, p_0] = 0\]

\[[N_i, p_j] = i\delta_j^i \left(\frac{\kappa}{2} (1 - e^{-2p_0/\kappa}) + \frac{p^2}{2\kappa}\right) - \frac{i}{\kappa} p_i p_j, \quad [N_i, p_0] = ip_i\]

\[[p_\mu, p_\nu] = 0 \quad (A2)\]

The corresponding noncocommutative Hopf algebra (coalgebra, antipode, and counit) is

\(\triangle(M_i) = M_i \otimes 1 + 1 \otimes M_i, \quad \triangle(N_i) = N_i \otimes e^{-p_0/\kappa} + 1 \otimes N_i - \frac{1}{\kappa} \epsilon_{ijk} M_j \otimes p_k, \quad \triangle(p_i) = p_i \otimes e^{-p_0/\kappa} + 1 \otimes p_i, \quad \triangle(p_0) = p_0 \otimes 1 + 1 \otimes p_0, \quad S(M_i) = -M_i, \quad S(N_i) = -\left(N_i + \frac{1}{\kappa} \epsilon_{ijk} M_j p_k\right) e^{p_0/\kappa}, \quad S(p_i) = -p_i e^{p_0/\kappa}, \quad S(p_0) = -p_0, \quad \epsilon(p_\mu, M_i, N_i) = 0.\)

The \(\kappa\)-Minkowski spacetime \(T^*\) is defined as a dual Hopf algebra to the algebra of translations \((T, \text{momenta})\) with the use of pairing,

\[\langle p_\mu, x_\nu \rangle = \eta_{\mu\nu}, \quad (A4)\]

together with the axiom of Hopf algebra duality,

\[\left\{\langle t, x_\mu x_\nu \rangle = \left\langle t(1), x_\mu \right\rangle \left\langle t(2), x_\nu \right\rangle, \langle p_\mu p_\nu, x \rangle = \langle p_\mu, x(1) \rangle \langle p_\nu, x(2) \rangle\right\} \quad \forall p_\mu \in T, \quad x_\nu \in T^*. \quad (A5)\]

From this we get

\[[x_i, x_j] = 0, \quad [x_0, x_i] = \frac{i}{\kappa} x_i, \quad (A6)\]
and
\[ \Delta x_\mu = x_\mu \otimes 1 + 1 \otimes x_\mu. \quad (A7) \]

Using the Hopf algebra of spacetime coordinates one writes down the covariant action of \( T \) on \( T^* \) (module algebra):
\[ t \triangleright x = x(t_1)\langle t, x(t_2)\rangle, \quad \forall x \in T^*, t \in T \quad \text{and} \quad t \triangleright (xy) = (t_1 \triangleright x)(t_2 \triangleright y), \quad 1 \triangleright x = x. \quad (A8) \]

The action \( \triangleright \) of \( h \in U(so(1, 3)) \) on momenta \( p_\mu \) is defined by \( p_\mu \triangleright h = [h, p_\mu] \). The duality relation of \( \kappa \)-Minkowski space given as
\[ \langle t, h \triangleright x \rangle \equiv \langle t \triangleright h, x \rangle, \quad \forall t \in T, \quad h \in U(so(1, 3)), \quad x \in T^*. \]

Then, Eq. (A2) is translated into
\[ M_i \triangleright x_0 = 0, \quad M_i \triangleright x_j = i\epsilon_{ijk}x_k, \quad \text{and} \quad N_i \triangleright x_0 = ix_i, \quad N_i \triangleright x_j = i\delta_{ij}x_0. \quad (A10) \]

From these, one has the relations,
\[ N_i \triangleright x_0^2 = (N_i(1) \triangleright x_0)(N_i(2) \triangleright x_0) = i(x_0x_i + x_ix_0) + \frac{x_i^2}{\kappa}, \quad (A11) \]
\[ N_i \triangleright x_jx_k = (N_i(1) \triangleright x_j)(N_i(2) \triangleright x_k) = i(\delta_{ij}x_0x_k + \delta_{ik}x_jx_0) + \frac{1}{\kappa}(\delta_{ij}x_k - \delta_{jk}x_i). \]

Therefore, one has \( N_i \triangleright (x_0^2 - x^2 + 3ix_0/\kappa) = 0 \) and \( x_0^2 - x^2 + 3ix_0/\kappa \) is a Lorentz-invariant. Likewise, the \( \kappa \)-deformed Poincaré algebra is given for symmetric ordering given in section [IIIIB 21]:
\[ [M_\mu, M_\rho] = i\left(\eta_{\mu\sigma}M_\nu + \eta_{\nu\rho}M_\mu - (\mu \leftrightarrow \nu)\right), \]
\[ [M_i, p_j] = i\epsilon_{ijk}p_k, \quad [M_i, p_0] = 0 \]
\[ [N_i, p_j] = i\delta_{ij}e^{\frac{\kappa}{2}(1 - e^{-2p_0/\kappa})} \left(\frac{P^2}{2\kappa}e^{-\frac{p_0}{\kappa}}\right) - \frac{i}{2\kappa}p_ip_je^{-\frac{p_0}{\kappa}}, \quad [N_i, p_0] = ip_ie^{-\frac{p_0}{\kappa}} \quad (A12) \]

and its Hopf-algebraic structure,
\[ \Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i, \quad (A13) \]
\[ \Delta(N_i) = N_i \otimes e^{-p_0/\kappa} + 1 \otimes N_i - \frac{1}{\kappa}\epsilon_{ijk}M_j \otimes p_k e^{-p_0/(2\kappa)}, \]
\[ \Delta(p_i) = p_i \otimes e^{-p_0/(2\kappa)} + e^{p_0/(2\kappa)} \otimes p_i \]
\[ \Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0, \]
\[ S(M_i) = -M_i, \]
\[ S(N_i) = -\left(\frac{1}{\kappa}\epsilon_{ijk}M_jp_ke^{-p_0/(2\kappa)}\right)e^{p_0/\kappa} \]
\[ S(p_i) = -p_i \]
\[ \epsilon(p_\mu, M_i, N_i) = 0. \]

From this algebra one may find the duality relation in coordinate space through the pairing \( (A4) \) and duality Eqs. \((A5,A8,A9)\). It turns out that the deformed algebra in the \( \kappa \)-Minkowski spacetime does not change and has the same forms Eqs. \((A6,A7,A10,A11)\) so that \( x_0^2 - x^2 + 3ix_0/\kappa \) is Lorentz-invariant.
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