KASHAEV INVARIANTS OF TWICE-ITERATED TORUS KNOTS

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Abstract. We calculate the asymptotic behavior of the Kashaev invariant of a twice-iterated torus knot and obtain topological interpretation of the formula in terms of the Chern–Simons invariant and the twisted Reidemeister torsion.

1. Introduction

For a knot $K$ in the three-sphere $S^3$ and an integer $N \geq 2$, let $\langle K \rangle_N \in \mathbb{C}$ be the Kashaev invariant [6]. In [7], he conjectured that $|\langle K \rangle_N|$ grows exponentially with growth rate $\text{Vol}(S^3 \setminus K)/(2\pi)$ for $N \to \infty$ when $K$ is hyperbolic, where $\text{Vol}(S^3 \setminus K)$ is the hyperbolic volume of $S^3 \setminus K$.

J. Murakami and the first author proved that the Kashaev invariant coincides with $J_N(K; \exp(2\pi i/N))$, where $J_N(K; q)$ is the colored Jones polynomial of a knot $K$ in the three-sphere $S^3$ associated with the $N$-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, normalized so that $J_N(U; q) = 1$ for the unknot $U$ [13]. They also proposed the following conjecture:

Conjecture 1.1 (Volume Conjecture). For any knot $K$, we have

$$\log \left| J_N(K; \exp(2\pi i/N)) \right| \sim \frac{\text{Vol}(S^3 \setminus K)}{2\pi},$$

where $\text{Vol}(S^3 \setminus K)$ is the simplicial volume of $S^3 \setminus K$ normalized so that the simplicial volume of a hyperbolic knot complement equals its hyperbolic volume. In particular, when $K$ is hyperbolic, Kashaev’s conjecture holds.

This conjecture was first proved for torus knots by Kashaev and O. Tirkkonen [8]. Note that since the complement of a torus knot is a Seifert fibred space, its simplicial volume is zero. A proof of the conjecture for the figure-eight knot was given by T. Ekholm (see for example [14] for the proof).

J. Andersen and S. Hansen proved for the figure-eight knot $4_1$ the following asymptotic equivalence holds [1].

$$J_N(4_1; \exp(2\pi i/N)) \sim \frac{2^{3/2}}{\sqrt{\pi}} N^{3/2} \left( \frac{N}{2\pi i} \right)^{3/2} \tau(4_1) \exp \left( \frac{N}{2\pi i} S(4_1) \right),$$

where $\tau(4_1) := \left( \frac{2}{\sqrt{\pi}} \right)^{1/2}$ and $S(4_1) := \sqrt{-1} \text{Vol}(S^3 \setminus 4_1)$. It is known that $\tau(4_1)^{-2}$ is the homological Reidemeister torsion twisted by the holonomy representation of $\pi_1(S^3 \setminus 4_1)$ associated with the meridian, and $S(4_1)$ is the $\text{SL}(2; \mathbb{C})$ Reidemeister torsion of the holonomy representation. It is also conjectured that for any hyperbolic knot $K$, we have

$$J_N(K; \exp(2\pi i/N)) \sim \frac{2^{3/2}}{\sqrt{\pi}} N^{3/2} \left( \frac{N}{2\pi i} \right)^{3/2} \tau(K) \exp \left( \frac{N}{2\pi i} S(K) \right),$$
where \( \tau(K) \) and \( S(K) \) are defined as above. See [4] and [2]. See also [15, 16].

Let \( T(c, d) \) be the torus knot of type \((c, d)\) for coprime integers \(c\) and \(d\). J. Dubois and Kashaev [3] obtained the following formula:

\[
J_N(T(c, d); e^{2\pi i/N}) \sim \frac{\pi^{3/2}}{2cd} \left( \frac{N}{2\pi i} \right)^{3/2} \frac{cd}{\sum_{k=1}^{cd-1} (-1)^{k+1} k^2 \tau(k) \exp \left( \frac{S(k)N}{2\pi i} \right)} + O(1),
\]

where

\[
S(k) := \frac{(k-cd)^2\pi^2}{cd},
\]

\[
\tau(k) := \frac{4\sin(k\pi/c)\sin(k\pi/d)}{\sqrt{cd}}.
\]

See also [5] for the formulation above. They also show that \( S(k) \) is the Chern–Simons invariant and \( \tau(k)^{-2} \) is the homological twisted Reidemeister torsion both associated with suitable irreducible representation from \( \pi_1(S^3 \setminus T(c, d)) \to \text{SL}(2, \mathbb{C}) \).

See also [8].

**Remark 1.2.** If we define

\[
\tilde{S}(k) := \frac{k^2 \pi^2}{cd},
\]

the right hand side of (1.1) becomes

\[
\frac{1}{2\sqrt{\pi}} \left( \frac{N}{2\pi i} \right)^{3/2} \frac{cd}{\sum_{k=1}^{cd-1} (-1)^{k(N+1)} i^{-cdN} \tau(k) \tilde{S}(k) \exp \left( \frac{\tilde{S}(k)N}{2\pi i} \right)} + O(1).
\]

Note that the Chern–Simons invariant is defined modulo \( \pi^2\mathbb{Z} \) and that \( S(k) \equiv \tilde{S}(k) \) (mod \( \pi^2\mathbb{Z} \)).

In [11] and [12], the first author obtained a similar asymptotic formula for

\[
J_N(T(2, 2a+1)^{(2, 2b+1)}; \exp(\xi/N)),
\]

with \( \xi \neq 2\pi i \), where \( T(2, 2a+1)^{(2, 2b+1)} \) is the \((2, 2b+1)\)-cable of \( T(2, 2a+1) \). The purpose of this paper is to give an asymptotic formula for \( J_N(T(2, 2a+1)^{(2, 2b+1)}; \exp(\xi/N)) \).

**Theorem 1.3.** If \( 2b + 1 > 4(2a + 1) > 0 \), then we have

\[
J_N(T(2, 2a+1)^{(2, 2b+1)}; e^{2\pi i/N}) \sim \frac{1}{2\sqrt{\pi}} \left( \frac{N}{2\pi i} \right)^{3/2} \sum_{l=0}^{2b} \tau_1(l) S_1(l) e^{\frac{N}{2\pi i} S_1(l)}
\]

\[
+ (-1)^N \frac{1}{2\sqrt{\pi}} \left( \frac{N}{2\pi i} \right)^{3/2} \frac{4a+1}{\sum_{m=0}^{4a+1} \tau_2(m) S_2(m) e^{\frac{N}{2\pi i} S_2(m)}}
\]

\[
- \frac{1}{2} \left( \frac{N}{2\pi i} \right)^2 \sum_{(j, k) \in \mathcal{B}} \tau_3(j, k) S_3(j, k) e^{\frac{N}{2\pi i} S_3(j, k)} + O(N^{1/2}),
\]

where

\[
\tau_1(l) = (-1)^l \sqrt{2} \frac{2b+1}{2b+1} \frac{\sin \left( \frac{(2+2l)(\pi)}{2b+1} \right)}{\cos \left( \frac{(2+2l)(2+2l)(\pi)}{2b+1} \right)},
\]

\[
S_1(l) = \frac{(2l+1)^2 \pi^2}{2(2b+1)},
\]

\[
\tau_2(m) = (-1)^m \sqrt{2} \frac{2}{2a+1} \frac{\sin \left( \frac{(2m+1)(\pi)}{2a+1} \right)}{\cos \left( \frac{(2m+1)(2m+1)(\pi)}{2a+1} \right)},
\]

\[
\tau_3(j, k) = \frac{1}{2} \sqrt{2} \frac{2b+1}{2b+1} \frac{\sin \left( \frac{(j+k)(\pi)}{2b+1} \right)}{\cos \left( \frac{(j+k)(j+k)(\pi)}{2b+1} \right)}.
\]
\[ S_2(m) = \frac{(2m + 1)^2 \pi^2}{2(2a + 1)}, \]
\[ \tau_3(j, k) = (-1)^{j+k} \frac{4}{\sqrt{(2a + 1)(2b + 1 - 4(2a + 1))}} \sin \left( \frac{(2k + 1)\pi}{2a + 1} \right), \]
\[ S_3(j, k) = \left( \frac{2(2k + 1)^2}{2(2a + 1) + 2(2b + 1 - 4(2a + 1))} \right) \pi^2, \]

and \( B \) is the set of all pairs of integers \((j, k)\) such that \(0 \leq k \leq 4a + 1, 0 \leq j \leq 2b - 4(2a + 1), \) and \((2b + 1 - 4(2a + 1))(2k + 1) < 2(2a + 1)(2j + 1).\)

We can also prove that \( \tau_2(l)^{-2}, \tau_3(m)^{-2}, \) and \( \tau_3(l, m)^{-2} \) are the homological twisted Reidemeister torsions of certain representations of the fundamental group to \( \text{SL}(2, \mathbb{C}) \), and that \( S_1(l), S_2(m), \) and \( S_3(l, m) \) are the Chern–Simons invariants of these representations.

2. Proof of the asymptotic formula

We will follow [10, Section 5.2]. Let \( e_n \) be the \( n \)-th Jones–Wenzl idempotent in the Kauffman bracket skein algebra of an annulus defined by \( z \) with \( e_0 = 1, e_1 = z_1 \), where \( z_1 \) is the circle around the annulus. Let \( \langle e_n \rangle_{K^\sigma} \) be the Kauffman bracket of the element obtained from a framed knot \( K^\sigma \) by replacing a diagram of \( K^\sigma \) with \( e_n \), where \( \sigma \) is the framing. If \( T^{2(2b+1)} \) is the \((2, 2b+1)\)-cable of the torus knot \( T(2, 2a+1) \) with framing \( 2(2b+1) \), then in [10, Proposition 4] Q. Liu proved

\[
(-1)^n \exp \left( \frac{(2b + 1 - 3(2a + 1))\pi i}{4(n + 1)} \right) (-n - 1) \left( \frac{\langle e_n \rangle_{T^{2(2a+1)}}}{n + 1} \right) \bigg|_{A = \exp(\pi i/(2(n+1))} = \frac{2(n + 1)^3 i}{\pi^4 \sqrt{(2a + 1)(2b + 1 - 4(2a + 1))}} \exp \left( \frac{-\pi i}{(2a + 1)(n + 1)} \right) \times \int_{C_{n/4}} \int_{C_{n/4}} dz_1 dz_2 \psi_1(z_1) \psi_2(z_2) \delta(z_1, z_2) e^{(n+1)\theta(z_1, z_2)},
\]

where

\[
[k] := \frac{A^{2k} - A^{-2k}}{A^2 - A^{-2}},
\]
\[
\theta(z_1, z_2) := -\frac{e^2}{2a + 1} - \frac{4(z_1 - z_2)^2}{2b + 1 - 4(2a + 1)} \pi i + 4z_1,
\]
\[
\delta(z_1, z_2) := \frac{e^2}{2a + 1} + \frac{4(z_1 - z_2)^2}{2b + 1 - 4(2a + 1)},
\]
\[
\psi_1(z_1) := \frac{1}{2\cosh(2z_1)},
\]
\[
\psi_2(z_2) := \frac{\sinh(\frac{2z_2}{2a + 1})}{2\cosh(z_2)}.
\]

Since the colored Jones polynomial \( J_N(K; q) \) is normalized so that \( J_N(U; q) = 1 \) with \( U \) the unknot, if \( K \) is the knot obtained from a framed knot \( K^\sigma \) by forgetting the framing, we have

\[
J_N(K; q) = \frac{\langle e_{N-1} \rangle_{K^\sigma}}{\langle e_{N-1} \rangle_U} = \left( -1 \right)^{N-1} A^{N^2-1} - \sigma \left( z_{N-1} \right)_{K^\sigma} (1)^{N+1} [N],
\]
where $\mathcal{K}^0$ is the 0-framed knot obtained from $\mathcal{K}^r$ by changing the framing and $U^0$ is the framed unknot with framing 0. Therefore we have

$$J_N(T(2, 2a + 1); e^{2\pi i/N}) = (-1)^{N-1} \frac{1}{-N} \exp \left( -\frac{(2b + 1 - 3(2a + 1))\pi i}{4N} \right) \exp \left( -\frac{-\pi i}{(2a + 1)N} \right) \times \frac{(-1)^{N-1} \exp \left( \frac{(-N^2 - 1)\pi i}{2N} \right)}{2N^3i} \times \frac{2(2b+1)}{(2a+1)(2b + 1 - 4(2a + 1)) \pi^4} \times \int_{C_{\pi/4}} \int_{C_{\pi/4}} dz_1 dz_2 \psi_1(z_1) \psi_2(z_2) \delta(z_1, z_2) e^{N\theta(z_1, z_2)} $$

where

$$\theta(z_1, z_2) := \frac{z_1^2}{2(2a + 1)\pi i} - \frac{4(z_1 - z_2)^2}{(2b + 1 - 4(2a + 1))\pi i} + 4z_1, $$

$$\delta(z_1, z_2) := \frac{z_2^2}{2a + 1} + \frac{4(z_1 - z_2)^2}{2b + 1 - 4(2a + 1)}, $$

$$\psi_1(z_1) := \frac{1}{2 \cosh(2z_1)}, $$

$$\psi_2(z_2) := \frac{\sinh(\frac{2z_2}{2a+1})}{2 \cosh(2z_2)}. $$

**Remark 2.1.** We need to multiply by $(-1)^{N-1} \left( -\frac{(-N^2 - 1)\pi i}{2N} \right)^{-2(2b+1)}$. The first $(-1)^{N-1}$ is because Liu normalized the colored Jones polynomial by dividing the Kauffman bracket by $q^{N^2/2 - N/2}$ but we need to divide it by $(-1)^{N-1} q^{N^2/2 - N/2}$, which is the Kauffman bracket for the unknot. The second one is because Liu’s formula is for a framed knot with framing $2(2b + 1)$.

**Remark 2.2.**

1. $\theta(z_1, z_2)$ has a unique critical point

$$(w_1, w_2) = \left( \frac{(2b + 1)\pi i}{2}, 2(2a + 1)\pi i \right).$$

2. The poles of $\psi_1(z_1)$ between $C_{\pi/4}$ and $C_{\pi/4} + w_1$ are $\xi_l = \frac{2l+1}{2} \pi i$ ($0 \leq l \leq 2b$), where $C_{\pi/4} + w_1$ is the line passing through $w_1$ that is parallel to $C_{\pi/4}$. Moreover, we have

$$\text{Res}(\psi_1, \xi_l) = (-1)^{l-1} \frac{i}{4}.$$

3. The poles of $\psi_2(z_2)$ between $C_{\pi/4}$ and $C_{\pi/4} + w_2$ are $\eta_m = \frac{2m+1}{2} \pi i$ ($0 \leq m \leq 4a+1$), where $C_{\pi/4} + w_2$ is the line passing through $w_2$ that is parallel to $C_{\pi/4}$. Moreover, we have

$$\text{Res}(\psi_2, \eta_m) = (-1)^{m} \frac{\sin((2m + 1)\pi)}{2a + 1}. $$
Put
\[
F_N(z_1, z_2) := \delta(z_1, z_2) e^{N\theta(z_1, z_2)},
\]
\[
I_N := \int_{C_{\pi/4}} \int_{C_{\pi/4}} dz_1 dz_2 \psi_1(z_1) \psi_2(z_2) F_N(z_1, z_2),
\]
\[
\mathcal{I}_N := (-1)^{N-1} \left( \frac{2N^2i}{\sqrt{(2a + 1)(2b + 1 - 4(2a + 1))\pi^4}} \right) I_N.
\]

Then \( e^{\frac{2\pi i}{2\pi} (2b+1-3(2a+1))\pi^{-1}} J_N(K, e^{2\pi i/4}) = \mathcal{I}_N. \)

By shifting the paths of integrations from \( C_{\pi/4} \) to \( C_{\pi/4} + w_1 \) for the integration with respect to \( z_1 \), and from \( C_{\pi/4} \) to \( C_{\pi/4} + w_2 \) for the integration with respect to \( z_2 \), we have
\[
I_N = I_{0,N} + I_{1,N} + I_{2,N} + I_{3,N}
\]
with
\[
I_{0,N} := \int_{C_{\pi/4} + w_2} \int_{C_{\pi/4} + w_2} dz_1 dz_2 \psi_1(z_1) \psi_2(z_2) F_N(z_1, z_2),
\]
\[
I_{1,N} := \sum_{l=0}^{2b} (2\pi i) \text{Res}(\psi_1, \xi_l) \int_{C_{\pi/4} + w_2} dz_2 \psi_2(z_2) F_N(\xi_l, z_2),
\]
\[
I_{2,N} := \sum_{m=0}^{2b} \sum_{l=0}^{4m+1} (2\pi i) \text{Res}(\psi_2, \eta_m) \int_{C_{\pi/4} + w_2} dz_1 \psi_1(z_1) F_N(z_1, \eta_m),
\]
\[
I_{3,N} := \sum_{l=0}^{2b} \sum_{m=0}^{4m+1} (2\pi i)^2 \text{Res}(\psi_1, \xi_l) \text{Res}(\psi_2, \eta_m) F_N(\xi_l, \eta_m).
\]
Put \( \mathcal{I}_{k,N} := (-1)^{N-1} \left( \frac{2N^2i}{\sqrt{(2a+1)(2b+1-4(2a+1))\pi^4}} \right) I_{k,N} \) so that
\[
\mathcal{I}_N = \mathcal{I}_{0,N} + \mathcal{I}_{1,N} + \mathcal{I}_{2,N} + \mathcal{I}_{3,N}.
\]

2.1. \( I_{3,N} \). In this subsection we calculate \( I_{3,N} \).

Since \( \xi_l = \frac{2l+1}{2\pi} i \) and \( \eta_m = \frac{2m+1}{2\pi} i \), we have
\[
\delta(\xi_l, \eta_m) = \frac{\eta_m^2}{2a + 1} + \frac{4(\xi_l - \eta_m)^2}{2b + 1 - 4(2a + 1)}
\]
\[
= (\pi i)^2 \left( \frac{(2m + 1)^2}{4(2a + 1)} + \frac{(2l+1) - (2m + 1)^2}{2b + 1 - 4(2a + 1)} \right) = -\frac{1}{2} \tilde{S}_3(l, m),
\]
where we put
\[
\tilde{S}_3(l, m) := 2\pi^2 \left( \frac{(2m + 1)^2}{4(2a + 1)} + \frac{(2l+1) - (2m + 1)^2}{4(2b + 1 - 4(2a + 1))} \right).
\]
Hence \( \theta(\xi_l, \eta_m) = - (\pi i)^{-1} \delta(\xi_l, \eta_m) + 4\xi_l = \frac{1}{2\pi i} \tilde{S}_3(l, m) + (2l+1)\pi i \) and
\[
F_N(\xi_l, \eta_m) = \delta(\xi_l, \eta_m) e^{N\theta(\xi_l, \eta_m)} = \frac{(-1)^{N-1}}{2} \tilde{S}_3(l, m) e^{\frac{iN}{2\pi i} \tilde{S}_3(l, m)}.
\]
Since \( \text{Res}(\psi_1, \xi_l) \text{Res}(\psi_2, \eta_m) = (-1)^{l+m-1} \frac{i}{8} \sin \left( \frac{(2m+1)\pi}{2a+1} \right) \), we obtain
\[
I_{3,N} = \sum_{l=0}^{2b} \sum_{m=0}^{4m+1} (\pi i)^{l+m-1} \frac{i}{4} \sin \left( \frac{(2m+1)\pi}{2a+1} \right) \tilde{S}_3(l, m) e^{\frac{iN}{2\pi i} \tilde{S}_3(l, m)}.
\]
Hence

\[ J_{3, N} = - \frac{N^2}{2\pi^2} \sum_{l = 0}^{2b} \sum_{m = 0}^{4a + 1} \frac{(-1)^{l+m}}{\sqrt{(2a + 1)(2b + 1 - 4(2a + 1))}} \times \sin \left( \frac{(2m + 1)\pi}{2a + 1} \right) \tilde{S}_3(l, m) e^{\frac{iN}{2\pi} \tilde{S}_3(l, m)} \]

\[ = - \frac{N^2}{8\pi^2} \sum_{l = 0}^{2b} \sum_{m = 0}^{4a + 1} \Re(\psi(z_1, \eta_m)) \times \sin \left( \frac{(2m + 1)\pi}{2a + 1} \right) \tilde{S}_3(l, m) e^{\frac{iN}{2\pi} \tilde{S}_3(l, m)}. \]

2.2. \( I_{2, N} \). Next we calculate \( I_{2, N} \).

Consider the integral

\[ \int_{C_{\pi/4} + w_1} dz_1 \psi(z_1) F_N(z_1, \eta_m). \]

The polynomial \( \theta(z_1, \eta_m) = -\frac{\eta_m^2}{(2a + 1)\pi i} - \frac{4(z_1 - \eta_m)^2}{(2b + 1 - 4(2a + 1))\pi i} + 4z_1 \) has a unique critical point

\[ \zeta_m := \frac{2b + 1 - 4(2a + 1)}{2} \pi i + \eta_m = \frac{(2b + 1) - 4(2a + 1) + (2m + 1)}{2} \pi i, \]

and

\[ \theta(z_1, \eta_m) = \theta(\zeta_m, \eta_m) = \frac{4}{(2b + 1 - 4(2a + 1))\pi i} (z_1 - \zeta_m)^2. \]

Note that \( \zeta_m \) is not a pole of \( \psi(z_1) \) for any integer \( m \).

We have

\[ \int_{C_{\pi/4} + \zeta_m} dz_1 \psi(z_1) F_N(z_1, \eta_m) \]

\[ = \int_{C_{\pi/4} + \zeta_m} dz_1 \psi(z_1) F_N(z_1, \eta_m) \]

\[ - \sum_{l'' = (2b + 1) - 4(2a + 1) + (2m + 1)} (2\pi i) \Re(\psi_1(z_1, \eta_m)) F_N(z_1, \eta_m) \]

Note that the poles of \( \psi(z_1) \) between \( C_{\pi/4} + w_1 \) and \( C_{\pi/4} + \zeta_m \) are \( \xi_{l''} = \frac{(2b + 1) - 4(2a + 1) + (2m + 1) + l''}{2} \).

Since \( F_N(z_1, \eta_m) = e^{\theta(z_1, \eta_m)} \delta(z_1, \eta_m) e^{-\frac{4N}{(2b + 1 - 4(2a + 1))\pi i} (z_1 - \zeta_m)^2} \) we have

\[ \int_{C_{\pi/4} + \zeta_m} dz_1 \psi(z_1) F_N(z_1, \eta_m) \]

\[ = e^{\theta(z_1, \eta_m)} \int_{C_{\pi/4} + \zeta_m} dz_1 \psi(z_1) \delta(z_1, \eta_m) e^{-\frac{4N}{(2b + 1 - 4(2a + 1))\pi i} (z_1 - \zeta_m)^2} \]

\[ = e^{\theta(z_1, \eta_m)} \int_{C_{\pi/4}} dz_1 \psi(z_1 + \zeta_m) \delta(z_1 + \zeta_m, \eta_m) e^{-\frac{4N}{(2b + 1 - 4(2a + 1))\pi i} (z_1 + \zeta_m)^2}. \]

By the saddle point method (see, for example, [10, Lemma 1]) we have

\[ \int_{C_{\pi/4}} dz_1 \psi(z_1 + \zeta_m) \delta(z_1 + \zeta_m, \eta_m) e^{-\frac{4N}{(2b + 1 - 4(2a + 1))\pi i} (z_1 + \zeta_m)^2} \]

\[ \sim \sqrt{\frac{(2b + 1 - 4(2a + 1))\pi^2}{4N}} \psi_1(z_1) \delta(z_1, \eta_m) + O(N^{-3/2}). \]
Hence
\[
\int_{C_{\pi/4} + \zeta_m} dz_1 \psi_1(z_1) F_N(z_1, \eta_m) = \sqrt{\frac{(2b + 1 - 4(2a + 1))\pi^2}{4N}} \psi_1(\zeta_m) F_N(\zeta_m, \eta_m) + O(N^{-3/2}).
\]
Since \(\zeta_m = \frac{(2b+1)-4(2a+1)+(2m+1)}{2}\pi i\) and \(\eta_m = \frac{2m+1}{2}\pi i\) we have
\[
\delta(\zeta_m, \eta_m) = (\pi i)^2 \left( \frac{(2m+1)^2}{4(2a+1)} + 2b + 1 - 4(2a+1) \right)
= -\frac{1}{2} S_2(m) - \pi^2 (2b + 1 - 4(2a + 1)).
\]
Hence
\[
\theta(\zeta_m, \eta_m) = -(\pi i)^{-1} \delta(\zeta_m, \eta_m) + 4\zeta_m = \frac{1}{2\pi i} S_2(m) + ((2b+1)-4(2a+1)+(2m+1))\pi i
\]
and
\[
F_N(\zeta_m, \eta_m) = -\frac{1}{2} S_2(m) e^{\pi i S_2(m)} - \pi^2 (2b + 1 - 4(2a + 1)) e^{\pi i S_2(m)}.
\]
Since \(\psi_1(\zeta_m) = \frac{1}{2\cos(2\zeta_m)} = \frac{1}{2}\), we obtain
\[
\int_{C_{\pi/4} + \zeta_m} dz_1 \psi_1(z_1) F_N(z_1, \eta_m)
= -\frac{\pi}{8} \sqrt{\frac{(2b + 1 - 4(2a + 1))i}{N}} S_2(m) e^{\pi i S_2(m)}
- \frac{\pi^3}{4} \sqrt{\frac{(2b + 1 - 4(2a + 1))i}{N}} ((2b + 1) - 4(2a + 1)) e^{\pi i S_2(m)}
+ O(N^{-3/2}).
\]
From (2.1) we have
\[
I_{2,N}
= -\sum_{m=0}^{4a+1} \sum_{l'=(2b+1)-4(2a+1)+(2m+1)} (2\pi i)^2 \text{Res}(\psi_1, \xi_{l'}) \text{Res}(\psi_2, \eta_{m}) F_N(\xi_{l'}, \eta_{m})
+ \sum_{m=0}^{4a+1} (2\pi i) \text{Res}(\psi_2, \eta_{m}) \int_{C_{\pi/4} + \zeta_m} dz_1 \psi_1(z_1) F_N(z_1, \eta_m)
= (-1)^N + \frac{\pi^2 i}{4} \sum_{m=0}^{4a+1} \sum_{l'=(2b+1)-4(2a+1)+(2m+1)} (-1)^l' + m - 1 \sin \left( \frac{(2m+1)\pi}{2a+1} \right)
\times S_3(l', m) e^{\pi i S_2(l', m)}
- \frac{\pi^2 i}{8} \sqrt{\frac{(2b + 1 - 4(2a + 1))i}{N}} \sum_{m=0}^{4a+1} (-1)^m \sin \left( \frac{(2m+1)\pi}{2a+1} \right) S_2(m) e^{\pi i S_2(m)}
- \frac{\pi^4 i}{4} \sqrt{\frac{(2b + 1 - 4(2a + 1))i}{N}} ((2b + 1) - 4(2a + 1))
\times \sum_{m=0}^{4a+1} (-1)^m \sin \left( \frac{(2m+1)\pi}{2a+1} \right) e^{\pi i S_2(m)}
+ O(N^{-3/2}).
\]
Since \( S_2(m + 2a + 1) = S_2(m) + 4(a + m + 1)\pi^2 \), we have
\[
\sum_{m=0}^{4a+1} (-1)^m \sin \left( \frac{(2m + 1)\pi}{2a + 1} \right) e^{\eta_N S_2(m)} = \sum_{m=0}^{2a} (-1)^m \sin \left( \frac{(2m + 1)\pi}{2a + 1} \right) e^{\eta_N S_2(m)} + \sum_{m=2a+1}^{4a+1} (-1)^m \sin \left( \frac{(2m + 1)\pi}{2a + 1} \right) e^{\eta_N S_2(m)}
\]
(put \( m' := m - 2a - 1 \) in the second term)
\[
= \sum_{m=0}^{2a} (-1)^m \sin \left( \frac{(2m + 1)\pi}{2a + 1} \right) e^{\eta_N S_2(m)} + \sum_{m'=0}^{2a} (-1)^{m'+1} \sin \left( \frac{(2m' + 1)\pi}{2a + 1} \right) e^{\eta_N S_2(m')}
= 0.
\]
Therefore we have
\[
I_{2,N} = (-1)^{N+1} \frac{\tau^2}{4} \sum_{m=0}^{4a+1} \sum_{l'=0}^{2b} (-1)^{l'+m-1} \sin \left( \frac{(2m + 1)\pi}{2a + 1} \right)
\]
\[
\times \tilde{S}_3(l', m)e^{\eta_N S_3(l', m)} - \pi^2 i \frac{N}{8} \sum_{m=0}^{4a+1} (-1)^m \sin \left( \frac{(2m + 1)\pi}{2a + 1} \right) S_2(m)e^{\eta_N S_2(m)} + O(N^{-3/2}).
\]
and so
\[
J_{2,N} = (-1)^{N-1} \left( \frac{2N^2 i}{\sqrt{(2a + 1)(2b + 1) - 4(2a + 1)}} \right) I_{2,N}
= \frac{N^2}{8\pi^2} \sum_{m=0}^{4a+1} \sum_{l'=0}^{2b} \tau_3(l', m)\tilde{S}_3(l', m) e^{\eta_N S_3(l', m)} + \frac{N^3/2}{2} \left( -1 \right)^{N-1} \sqrt{\frac{2}{\pi}} \sum_{m=0}^{2a} \tau_2(m)S_2(m)e^{\eta_N S_2(m)} + O(N^{1/2}).
\]

2.3. \( I_{1,N} \). Now we calculate \( I_{1,N} \).

Consider the integral
\[
\int_{C_{\gamma/4+\mu_2}} dz z^2 \psi(z)F_N(\xi, z).
\]
The polynomial \( \theta(\xi, z) = -\frac{z^2}{(2a + 1)\pi i} - \frac{4(\xi - z)^2}{(2b + 1 - 4(2a + 1))\pi i} + 4\xi \) has a unique critical point
\[
\zeta' := \frac{4(2a + 1)}{2b + 1} \xi = \frac{(2a + 1)(2l + 1)}{2b + 1} \pi i
\]
and

\[ \theta(\xi_l, z_2) = \theta(\xi_l, \zeta_l') - \frac{2b + 1}{(2a + 1)(2b + 1 - 4(2a + 1))\pi i}(z_2 - \zeta_l')^2. \]

Note that \( \zeta_l' \) is not a pole of \( \psi_2(z_2) \) for any \( l \).

We have

\[
\int_{C_{\pi/4} + \zeta_l'} dz_2 \psi_2(z_2) F_N(\xi_l, z_2) = \int_{C_{\pi/4} + \zeta_l'} dz_2 \psi_2(z_2) F_N(\xi_l, z_2) - \sum_{m' = \left\lfloor \frac{2a + 1}{2b + 1} + \frac{1}{2} \right\rfloor}^{4a + 1} (2\pi i) \text{Res}(\psi_2, \eta_{m'}) F_N(\xi_l, \eta_{m'}). \]

Note that the pole of \( \psi_2(z_2) \) between \( C_{\pi/4} + w_2 \) and \( C_{\pi/4} + \zeta_l' \) are \( \eta_{m'} \left( \frac{2a + 1)(2(2a + 1))}{2b + 1} - \frac{1}{2} \right) < m' < 4a + \frac{3}{2} \), that is, \( \left\lfloor \frac{2a + 1)(2(2a + 1))}{2b + 1} + \frac{1}{2} \right\rfloor \leq m' \leq 4a + 1 \).

By the same argument as in the previous case, we have

\[
\int_{C_{\pi/4} + \zeta_l'} dz_2 \psi_2(z_2) F_N(\xi_l, z_2) \xrightarrow{N \to \infty} \frac{(2a + 1)(2b + 1 - 4(2a + 1))\pi^2 i}{(2b + 1)N} \psi_2(\zeta_l') F_N(\xi_l, \zeta_l') + O(N^{-3/2}). \]

Since \( \xi_l = \frac{2l + 1}{4}\pi i \) and \( \zeta_l' = \frac{(2a + 1)(2(2a + 1))}{2b + 1} \pi i \) we have

\[
\delta(\xi_l, \zeta_l') = (\pi i)^2 \frac{(2l + 1)^2}{4(2b + 1)} = -\frac{1}{2} S_1(l). \]

Hence \( \theta(\xi_l, \zeta_l') = -(\pi i)^{-1} \delta(\xi_l, \zeta_l') + 4\xi_l = \frac{1}{2\pi i} S_1(l) + (2l + 1)\pi i \) and

\[
F_N(\xi_l, \zeta_l') = \frac{(-1)^{N - 1}}{2} S_1(l) e^{\frac{N}{\pi i} S_1(l)}. \]

Since

\[
\psi_2(\zeta_l') = \frac{\sinh(\frac{2(2l + 1)}{2b + 1})}{2 \cosh(\zeta_l')} = \frac{i \sin(\frac{2(2l + 1)}{2b + 1})}{2 \cos(\frac{2(2l + 1)}{2b + 1}) \pi}, \]

we obtain

\[
\int_{C_{\pi/4} + \zeta_l'} dz_1 \psi_1(z_1) F_N(z_1, \eta_m) = (-1)^{N - 1} \frac{\pi i}{4} \sqrt\frac{(2a + 1)(2b + 1 - 4(2a + 1))i}{(2b + 1)N} \frac{\sin(\frac{2(2l + 1)}{2b + 1})}{\cos(\frac{2(2l + 1)}{2b + 1}) \pi} S_1(l) e^{\frac{N}{\pi i} S_1(l)} + O(N^{-3/2}). \]
From (2.1) we have

\[ I_{1,N} \]

\[ = - \sum_{l=0}^{2b} \sum_{m'=[\frac{2b+1}{2b+1}] + \frac{1}{2}}^{4a+1} (2\pi i)^2 \text{Res}(\psi_1, \xi) \text{Res}(\psi_2, \eta_{m'}) F_N(\xi, \eta_{m'}) \]

\[ + \sum_{l=0}^{2b} (2\pi i) \text{Res}(\psi_1, \xi) \int_{C_{\pi/4} + \xi_i} dz_2 \psi_2(z_2) F_N(\xi, z_2) \]

\[ = (-1)^{N+1} \frac{\pi}{8} \sum_{l=0}^{2b} \sum_{m'=[\frac{2b+1}{2b+1}] + \frac{1}{2}}^{4a+1} (-1)^{l+m'} \sin\left(\frac{(2m' + 1)\pi}{2a + 1}\right) \]

\[ \times \tilde{S}_3(l, m') e^{\frac{\pi}{4} b} S_3(l, m') \]

\[ + (2\pi i) S_1(l) e^{\frac{\pi}{4} b} S_1(l) + O(N^{-3/2}). \]

Hence

\[ J_{1,N} = (-1)^{N-1} \left( \frac{2N^2 i}{\sqrt{(2a + 1)(2b + 1 - 4(2a + 1) \pi)}} \pi \right) I_{1,N} \]

\[ = \frac{N^2}{8\pi} \sum_{l=0}^{2b} \sum_{m'=[\frac{2b+1}{2b+1}] + \frac{1}{2}}^{4a+1} \tau_{l}(l, m') \tilde{S}_3(l, m') e^{\frac{\pi}{4} b} S_3(l, m') \]

\[ - N^{3/2} \sum_{l=0}^{2b} \frac{1}{4\pi^2} \tau_{l}(l) S_1(l) e^{\frac{\pi}{4} b} S_1(l) + O(N^{1/2}). \]

2.4. \( I_{0,N}. \) In this subsection we calculate \( I_{0,N}. \)

We have

\[ I_{0,N} = \int_{C_{\pi/4} + w_1} \int_{C_{\pi/4} + w_2} dz_1 d^2 z_2 \psi_1(z_1) \psi_2(z_2) \delta(z_1, z_2) e^{N\theta(z_1, z_2)} \]

\[ = \int_{C_{\pi/4}} d^2 z_2 \psi_1(z_1 + w_1) \psi_2(z_2 + w_2) \delta(z_1 + w_1, z_2 + w_2) e^{N\theta(z_1 + w_1, z_2 + w_2)}. \]

Note that \( \psi_1(z_1 + w_1) = -\psi_1(z_1) \) and \( \psi_2(z_2 + w_2) = \psi_2(z_2). \) Moreover, we have

\[ \delta(z_1 + w_1, z_2 + w_2) = \delta(z_1, z_2) + \delta(w_1, w_2) + \frac{2zw_2}{2a + 1} + \frac{8(z_1 - z_2)(w_1 - w_2)}{2b + 1 - 4(2a + 1)} \]

\[ = \delta(z_1, z_2) - \pi^2 (2b + 1) + 4(2a + 1) \]

and

\[ \theta(z_1 + w_1, z_2 + w_2) = -(\pi i)^{-1} \delta(z_1 + w_1, z_2 + w_2) + 4(2a + 1) \]

\[ = -(\pi i)^{-1} \delta(z_1, z_2) - (2b + 1)\pi i + 4w_1 \]

\[ = -(\pi i)^{-1} \delta(z_1, z_2) + (2b + 1)\pi i. \]
Hence
\[ I_{0,N} = (-1)^{N-1} \times \int_{C_{\pi/4}} \int_{C_{\pi/4}} dz_1 \, dz_2 \psi_1(z_1) \psi_2(z_2) \left( \delta(z_1, z_2) - \pi^2 (2b + 1) + (4\pi i)z_1 \right) e^{-\frac{\sqrt{N}}{\pi} \delta(z_1, z_2)}. \]

Since \( \psi_1(z_1) \psi_2(z_2) \) is an odd function in \( (z_1, z_2) \) and \( \delta(z_1, z_2) \) is even, we obtain
\[ I_{0,N} = (-1)^{N-1} \int_{C_{\pi/4}} \int_{C_{\pi/4}} dz_1 \, dz_2 \psi_1(z_1) \psi_2(z_2) (4\pi i)z_1 e^{-\frac{\sqrt{N}}{\pi} \delta(z_1, z_2)}. \]

From the saddle point method (see, for example, [10, Lemma 3]), we have
\[ I_{0,N} \sim O(N^{-2}). \]

Hence
\[ J_{0,N} = (-1)^{N-1} \left( \frac{2N^2 i}{(2a + 1)(2b + 1)} \frac{\pi^4}{(2a + 1)^2} \right) I_{0,N} \sim O(1). \]

Finally we have
\[ J_N(K, e^{2\pi i/N}) = e^{\frac{\pi i}{4N}(2b+1-3(2a+1)+\pi^2)} (J_{0,N} + J_{1,N} + J_{2,N} + J_{3,N}) \]
\[ \sim N^{3/2} \cdot \sqrt{\frac{1}{2}} \cdot \frac{1}{4\pi^2} \sum_{l=0}^{2b} \tau_1(l) S_1(l) e^{\frac{\pi i}{N} S_2(l)} \]
\[ + N^{3/2} (-1)^{N-1} \cdot \sqrt{\frac{1}{2}} \cdot \frac{1}{4\pi^2} \sum_{m=0}^{4a+1} \tau_2(m) S_2(m) e^{\frac{\pi i}{N} S_2(m)} \]
\[ - \frac{N^2}{8\pi^2} \sum_{l=0}^{2b} \sum_{m=0}^{4a+1} \tau_3(l, m) S_3(l, m) e^{\frac{\pi i}{N} S_3(l, m)} \]
\[ + \frac{N^2}{8\pi^2} \sum_{m=0}^{2b} \sum_{l=0}^{(2b+1)-4(2a+1)+(2m+1)} \tau_3(l, m) S_3(l, m) e^{\frac{\pi i}{N} S_3(l, m)} \]
\[ + O(N^{1/2}). \]

The sum of the three double summations becomes
\[ \sum_{(l,m) \in \mathcal{A}} \tau_3(l, m) S_3(l, m) e^{\frac{\pi i}{N} S_3(l, m)}, \]
where
\[ \mathcal{A} := \{(l, m) | 0 \leq m \leq 4a + 1, 0 \leq l \leq 2b, l \leq 2b + 1 \} \]
\[ + 4(2a + 1) + 2m + 1, \]
\[ (2b + 1)(2m + 1) < 2(2a + 1)(2l + 1). \]

Putting \( j := l - (2m + 1) \) and replacing \( m \) with \( k \), the summation above becomes
\[ \sum_{(j,k) \in \mathcal{B}} \tau_3(j + 2k + 1, k) S_3(j + 2k + 1, j) e^{\frac{\pi i}{N} S_3(j + 2k + 1, k)} \]
\[ = - \sum_{(j,k) \in \mathcal{B}} \tau_3(j, k) S_3(j, k) e^{\frac{\pi i}{N} S_3(j, k)}, \]
where

\[ B := \{(j, k) \mid 0 \leq k \leq 4a + 1, 0 \leq j \leq 2b - 4(2a + 1), \]
\[ (2b + 1 - 4(2a + 1))(2k + 1) < 2(2a + 1)(2j + 1) \} \]

and

\[ S_3(j, k) := 2\pi^2 \left( \frac{(2k + 1)^2}{4(2a + 1)} + \frac{(2j + 1)^2}{4(2b + 1 - 4(2a + 1))} \right). \]

The theorem follows.

3. Topological interpretation

In this section we give a topological interpretation of the right hand side of (1.2).

3.1. Fundamental group. We calculate the fundamental group of the complement of the twice-iterated torus knot \( T(2, 2a + 1) \).

Put \( E := S^3 \setminus \text{Int } N(T(2, 2a + 1)) \). Then \( E \) can be decomposed into \( C := S^3 \setminus \text{Int } N(T(2, 2a + 1)) \) (Figure 1) and \( D \), where \( N \) means the regular neighborhood, \( \text{Int} \) means the interior, and \( D \) is the complement of \((2, 2b + 1)\) torus knot in the solid torus (Figure 2). Note that \( D \) is homeomorphic to \( S^3 \setminus \text{Int } N(T(2, 4)) \) (see

![Figure 1. Torus knot \( T(2, 2a + 1) \)]
If we choose \( x \) and \( y \) as generators of \( \pi_1(C) \) as indicated in Figure 1, we have
\[
\pi_1(D) = \langle x, y \mid (xy)^a x = y(xy)^a \rangle,
\]
where we choose the basepoint on the boundary of \( N(T(2, 2a + 1)) \). Let \( p \) and \( t \) be generators of \( \pi_1(S^3 \setminus \text{Int} N(T(2, 4))) \) as in Figure 3. Then we have
\[
\pi_1(S^3 \setminus \text{Int} N(T(2, 4))) = \langle p, t \mid ptpt = tptp \rangle,
\]
where the basepoint is at the bottom-right of the torus. Let \( q \in \pi_1(D) \) be the element as indicated in Figure 4. Then we see that \( q = tpt^{-1} \). From van Kampen’s theorem we have
\[
\pi_1(E) = \langle x, y, p, t \mid (xy)^a x = y(xy)^a, y(xy)^{2a} x^{-4a-1} = tx^{-b}, x = ptpt^{-1} \rangle.
\]
Moreover we can see that the meridian \( \mu \) and the preferred longitude \( \lambda \) of \( T(2, 2a + 1)(2, 2b+1) \) are given as follows:
\[
\begin{align*}
\mu &= p, \\
\lambda &= y(xy)^{2a} x^{b-4a-1} p(tpt^{-1})^{-b} y(xy)^{2a} x^{b-4a-1} p^{-3b-1}.
\end{align*}
\]

3.2. Representation. In this subsection, we construct non-Abelian representations from \( \pi_1(E) \) to \( \text{SL}(2; \mathbb{C}) \). Note that we do not know whether we exhaust all such representations.
Put $\omega_1 := \exp\left(\frac{\pi i}{2b+1}\right)$, $\omega_2 := \exp\left(\frac{\pi i}{2a+1}\right)$, and $\omega_3 := \exp\left(\frac{\pi i}{2b+1-4(2a+1)}\right)$. Let $\rho_{AN}^{u,l}$ be the representation $\pi_1(E) \to \text{SL}(2; \mathbb{C})$ defined by

$$
\begin{align*}
\rho_{AN}^{u,l}(x) &:= \rho_{AN}^{u,l}(y) = T_1^{-1} \begin{pmatrix} \omega_1^{2l+1} & e^{-u/2} \\ 0 & \omega_1^{-(2l+1)} \end{pmatrix} T_1, \\
\rho_{AN}^{u,l}(p) &:= \begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix}, \\
\rho_{AN}^{u,l}(t) &:= T_1^{-1} \begin{pmatrix} \omega_1^{(2l+1)b} & \omega_1^{(2l+1)b} - \omega_1^{-(2l+1)b} \\ 0 & \omega_1^{-(2l+1)b} \end{pmatrix} T_1,
\end{align*}
$$

where

$$
T_1 := \begin{pmatrix} 1 & 0 \\ \omega_1^{-(2l+1)} e^{u/2} - e^{-u/2} & 1 \end{pmatrix}
$$

and $l = 0, 1, \ldots, b-1$. Note that $\rho_{AN}^{u,l+2b+1} = \rho_{AN}^{u,l}$
and that $\rho^\text{AN}_{u,2b-1}$ is conjugate to $\rho^\text{AN}_{u,l}$ by
\[
R_{1,l} := \begin{pmatrix} e^{-u/2} - \omega^{2l+1} & e^{-u/2} \omega^{2l+1} \\ 0 & 1 \end{pmatrix}.
\]
The longitude $\lambda$ is sent to
\[
\rho^\text{AN}_{u,l}(\lambda) = \begin{pmatrix} -e^{-(2b+1)u} & \frac{e^{(2b+1)u} - e^{-(2b+1)u}}{e^{u/2} - e^{-u/2}} \\ 0 & -e^{(2b+1)u} \end{pmatrix}.
\]
Let $\rho^\text{NA}_{u,m}$ be the representation with
\[
\rho^\text{NA}_{u,m}(x) := \begin{pmatrix} e^u & e^{u/2} + e^{-u/2} \\ 0 & e^{-u} \end{pmatrix},
\]
\[
\rho^\text{NA}_{u,m}(y) := \begin{pmatrix} e^u & 0 \\ (2m+1) & e^{u/2} + e^{-u/2} \end{pmatrix},
\]
\[
\rho^\text{NA}_{u,m}(p) := \begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix},
\]
\[
\rho^\text{NA}_{u,m}(t) := \begin{pmatrix} -e^{-2m+1} & \frac{e^{-2m+1} - e^{2m+1}}{e^{u/2} - e^{-u/2}} \\ 0 & 1 \end{pmatrix}
\]
with $m = 0, 1, \ldots, a-1$. The longitude $\lambda$ is sent to
\[
\rho^\text{NA}_{u,m}(\lambda) = \begin{pmatrix} -e^{-4(2n+1)u} & \frac{e^{-4(2n+1)u} - e^{4(2n+1)u}}{e^{u/2} - e^{-u/2}} \\ 0 & e^{4(2n+1)u} \end{pmatrix} = \rho^\text{NA}_{u,m}(p)^{-8(2n+1)}.
\]
Note the following symmetries:
\[
\rho^\text{NA}_{u,k+2a+1} = \rho^\text{NA}_{u,k},
\]
\[
\rho^\text{NA}_{u,2a-k} = \rho^\text{NA}_{u,k}.
\]
Let $\rho^\text{NN}_{u,j,k}$ be the representation with
\[
\rho^\text{NN}_{u,j,k}(x) := T^{-1}_2 \begin{pmatrix} \omega^{2j+1} & 1 \\ 0 & \omega^{-(2j+1)} \end{pmatrix} T_2,
\]
\[
\rho^\text{NN}_{u,j,k}(y) := T^{-1}_2 \begin{pmatrix} \omega^{2k+1} & \omega^{2j+1} \\ \omega^{2j+1} & \omega^{-(2k+1)} \end{pmatrix} T_2,
\]
\[
\rho^\text{NN}_{u,j,k}(p) := \begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix},
\]
\[
\rho^\text{NN}_{u,j,k}(t) := T^{-1}_2 \begin{pmatrix} -e^{-2m+1} & \frac{e^{-2m+1} - e^{2m+1}}{e^{u/2} - e^{-u/2}} \\ 0 & 1 \end{pmatrix} T_2,
\]
where
\[
T_2 := \begin{pmatrix} e^{u/4} & 0 \\ \omega^{-(2j+1)} e^{u/4} - e^{-3u/4} & e^{-u/4} \end{pmatrix},
\]
$= 0, 1, \ldots, 2b - 4(2a + 1)$, and $k = 0, 1, \ldots, a - 1$. The longitude $\lambda$ is sent to
\[
\rho^\text{NN}_{u,j,k}(\lambda) = \begin{pmatrix} -e^{-(2b+1)u} & \frac{e^{(2b+1)u} - e^{-(2b+1)u}}{e^{u/2} - e^{-u/2}} \\ 0 & -e^{(2b+1)u} \end{pmatrix}.
\]
Note the following symmetries:
\[
\rho^\text{NN}_{u,j,k+2a+1} = \rho^\text{NN}_{u,j,k},
\]
\[
\rho^\text{NN}_{u,j,2a-k} = \rho^\text{NN}_{u,j,k}.$$
the corresponding chain complex
\( C \)
the homological twisted Reidemeister torsion of
\( Z \)
need to specify bases of
\( H \)
\( \rho \)
representation of
\( 16 \) HITOSHI MURAKAMI AND ANH T. TRAN
3.1 Remark
is the
\( C \)
chain group
\( Tr \rho \)
d for an odd integer
3.3. Chern–Simons invariant. For a knot \( K \) in \( S^3 \), let \( M \) be the complement of the interior of the regular neighborhood of \( K \). Denote by \( \mu \) and \( \lambda \) the meridian and the preferred longitude of \( M \), respectively. By a conjugation we may assume that a representation \( \rho : M \to \text{SL}(2; \mathbb{C}) \) sends \( \mu \) and \( \lambda \) to
\[
\rho(\mu) = \begin{pmatrix} e^{u/2} & * \\ 0 & e^{-u/2} \end{pmatrix},
\]
\[
\rho(\lambda) = \begin{pmatrix} e^{v/2} & * \\ 0 & e^{-v/2} \end{pmatrix},
\]
respectively. The \( \text{SL}(2; \mathbb{C}) \) Chern–Simons invariant is a map from \( X(M) \) to \( \mathbb{C} \) modulo \( \pi^2 \mathbb{Z} \), where \( X(M) \) is the \( \text{SL}(2; \mathbb{C}) \) character variety of \( M \). Note that we need to fix log branches of the eigenvalues \( e^{u/2} \) and \( e^{v/2} \). See [9] for details.

In [11, § 2.5] the first author proved that the Chern–Simons invariants of \( \rho_{u;j}^\text{AN} \), \( \rho_{u;m}^\text{NA} \), and \( \rho_{u;j,k}^\text{NN} \) are given as follows.

**Theorem 3.2** ([11]). Let \( \rho_{u;j}^\text{AN} \), \( \rho_{u;m}^\text{NA} \), and \( \rho_{u;j,k}^\text{NN} \) be representations given in Subsection 3.2. Then we have
\[
\text{CS}(\rho_{u;j}^\text{AN}) = \frac{(2l + 1)^2 \pi^2}{2(2b + 1)} + \frac{1}{2} d_1 u \pi i,
\]
\[
\text{CS}(\rho_{u;m}^\text{NA}) = \frac{(2m + 1)^2 \pi^2}{2(2a + 1)} + d_2 \mu \pi i,
\]
\[
\text{CS}(\rho_{u;j,k}^\text{NN}) = \frac{(2k + 1)^2 \pi^2}{2(2a + 1)} + \frac{(2j + 1)^2 \pi^2}{2(2b + 1 - 4(2a + 1))} + \frac{1}{2} d_3 u \pi i,
\]
for an odd integer \( d_1 \), and integers \( d_2 \) and \( d_3 \). Here we choose \(-2(2b + 1)u + 2d_1 \pi i, -2(2a + 1)u + 2d_2 \pi i, -8(2a + 1)u + 4d_3 \pi i, \) and \(-2(2b + 1)u + 2d_3 \pi i \) as lifts of the eigenvalues of \( \rho_{u;j}(\lambda) \), \( \rho_{u;m}^\text{NA}(\lambda) \), and \( \rho_{u;j,k}^\text{NN}(\lambda) \), respectively.

Therefore we conclude
\[
S_1(l) = \text{CS}(\rho_{u;j}^\text{AN}),
\]
\[
S_2(m) = \text{CS}(\rho_{u;m}^\text{NA}),
\]
\[
S_3(l, m) = \text{CS}(\rho_{u;j,k}^\text{NN}).
\]
3.4. Reidemeister torsion. Let \( M \) be the complement of the interior of the regular neighborhood of a knot, and \( \tilde{M} \) be its universal covering space. Then the chain group \( C_j(M; \mathbb{Z}) \) can be regarded as a \( \mathbb{Z}[\pi_1(M)] \)-module. Given a representation \( \rho : \pi_1(M) \to \text{SL}(2; \mathbb{C}) \), we can also regard the Lie algebra \( \mathfrak{s}\mathfrak{l}(2; \mathbb{C}) \) as a \( \mathbb{Z}[\pi_1(M)] \)-module by the adjoint action. So we can define the tensor product \( C_j(M; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathfrak{s}\mathfrak{l}(2; \mathbb{C}) \) and denote it by \( C_j(M; \rho) \). The Reidemeister torsion of the corresponding chain complex \( \{C_j(M; \rho), \partial_j\} \) is denoted by \( \tau(M; \rho) \) and called the homological twisted Reidemeister torsion of \( M \) associated with \( \rho \). Note that we need to specify bases of \( H_j(M; \rho) \) unless \( \{C_j(M; \rho), \partial_j\} \) is acyclic, where \( H_j(M; \rho) \) is the \( j \)-th homology group of the chain complex \( \{C_j(M; \rho), \partial_j\} \).

In [12], the first author calculated the homological twisted Reidemeister torsions of \( S^3 \setminus \text{Int } N(T(2, 2a + 1)^{2b+1}) \) associated with \( \rho_{u;j}^\text{AN}, \rho_{u;m}^\text{NA} \), and \( \rho_{u;j,k}^\text{NN} \).
Theorem 3.3 ([12]). Put $M := S^3 \setminus \text{Int} N (T(2, 2a + 1)(2b+1))$. The homological twisted Reidemeister torsions of $M$ associated with the representations defined in Subsection 3.2 are given as follows.

\[
T(M; \rho_{AN}^{}\omega, l) = \frac{(2b + 1) \cos^2 \left( \frac{(2a+1)(2b+1)\pi}{2b+1} \right)}{2 \sin^2 \left( \frac{2(2b+1)\pi}{2b+1} \right)},
\]

\[
T(M; \rho_{NA}^{}\omega, m) = \frac{(2a + 1)}{2 \sin^2 \left( \frac{(2m+1)\pi}{2a+1} \right)},
\]

\[
T(M; \rho_{NN}^{}\omega, j,k) = \frac{(2a + 1)(2b+1 - 4(2a + 1))}{16 \sin^2 \left( \frac{(2k+1)\pi}{2a+1} \right)}.
\]

Note that since we have

\[
\dim H_j(M; \rho_{AN}^{}) = \begin{cases} 
1 & \text{if } j = 1, 2 \\
0 & \text{otherwise}, 
\end{cases}
\]

\[
\dim H_j(M; \rho_{NA}^{}) = \begin{cases} 
1 & \text{if } j = 1, 2 \\
0 & \text{otherwise}, 
\end{cases}
\]

\[
\dim H_j(M; \rho_{NN}^{}) = \begin{cases} 
2 & \text{if } j = 1, 2 \\
0 & \text{otherwise}, 
\end{cases}
\]

we need to specify bases of non-trivial homology groups. See [12] for details.

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