BILL VEECH'S CONTRIBUTIONS TO DYNAMICAL SYSTEMS

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Bill Veech died suddenly on August 30, 2016 at the age of 77. He was a major figure in the development of dynamical systems in the past 50 years with fundamental contributions to topological dynamics, Interval Exchange Transformations, and more generally to the field now called Teichmüller dynamics, of which he was one of the founders.

According to his obituary on the Statesboro Herald, William Austin Veech “was born on Christmas Eve in 1938 in Detroit, Michigan, and obtained his BA from Dartmouth College in 1960. He earned his Ph.D. in 1963 under the supervision of Salomon Bochner at Princeton University (with a dissertation on Almost Automorphic Functions). He joined the faculty of Rice University in 1969. He served as department chair for three years between 1982 and 1986 and held an endowed chair since 1988, Milton Brockett Porter Chair, 1988-2003; Edgar Odell Lovett Chair, since 2003.”

During his career Veech authored approximately 60 papers and one book on complex analysis. All of his papers are single authored. According to his obituary “he believed in the importance of developing one’s own unique perspective”. Any reader of his papers might add that he also had his own personal, idiosyncratic writing style, exacting and deep, not always easily accessible.

Veech had few students, the Mathematical Genealogy Project lists five: J. Martin (Ph. D. 1971), M. Stewart (Ph. D. 1978), C. Ward (Ph. D. 1996), Y. Wu (Ph. D. 2006) and J. Fickenscher (Ph. D. 2011), all at Rice University, and we are not aware of any others. Despite the small number of students, he had broad personal influence, as he was always ready to discuss mathematics and was very generous with his time, his ideas, as well as praise and encouragement for younger researchers. He also generously gave credit to others for originating ideas and for motivating his own research, sometimes acknowledging his intellectual debt in the very title of his paper (“Boshernitzan’s criterion” [87], “Bufetov’s question” [92], . . .).

It seems only fair that several of the groundbreaking results or concepts that he introduced bear his name: in topological dynamics the Veech relation and
the Veech structure theorem; in Teichmüller dynamics, the Rauzy–Veech induction, the Veech zippered rectangles flow, the Masur–Veech measures, the Veech dichotomy and Veech surfaces, the Veech group, the Siegel–Veech transform and constants. Several of these results and concepts will be examined more in detail below.

In Veech’s research activity it is possible to distinguish quite clearly and unambiguously two phases: the first, focused on topological dynamics and uniform distribution, goes from his Ph. D. thesis on Almost Automorphic Functions with Bochner in 1963 to until the late 70’s when he became interested in Interval Exchange Transformations. After that his research was almost entirely focused on Interval Exchange Transformations, and their renormalization dynamics, Teichmüller dynamics and flat geometry.

In later years, he become interested in Sarnak’s Möbius orthogonality conjecture. His last papers [94], [95], which he submitted shortly before his death and appeared posthumous in 2017 and 2018, are on this topic.

1. VEECH’S WORK IN TOPOLOGICAL DYNAMICS

In the first part of his scientific career, Veech was interested in questions of minimality, unique ergodicity for transformations and flows, and related questions on the uniform distribution of sequences. Two of his important contributions, which are still named after him, are the Veech relation and the Veech structure theorem. The Veech relation is introduced in the paper [72], which is devoted to the study of the equicontinuous structure relation of a continuous transformation group \( (X, T) \), with group \( T \) acting on a compact Hausdorff space \( X \). The equicontinuous structure relation \( S_{eq} \) is the minimal closed relation on \( X \) such that the system \( (X/S_{eq}, T) \) is equicontinuous. The equicontinuous structure relation was already known explicitly in some cases, for instance for distal systems on metric spaces, for locally almost periodic systems, for minimal almost automorphic systems (the latter case established by Veech himself in [71], his third published paper). Veech’s goal in introducing his relation was to describe the equicontinuous relation for minimal actions of Abelian groups.

Let us describe the significance of the Veech relation in some detail. The regionally proximal relation (RP) is defined as follows: \( (x, y) \in \text{RP} \) if given neighborhoods \( U \) and \( V \) of \( x \) and \( y \) respectively, and \( W \) a neighborhood of the diagonal \( \Delta \subset X \times X \), there are \( x' \in U \), \( y' \in V \) and \( t \in T \) such that \( (tx', ty') \in W \). The flow \( (X, T) \) is equicontinuous if and only if \( \text{RP} = \Delta \), and the equicontinuous structure relation \( S_{eq} \) is the closed equivalence relation generated by RP. If the flow \( (X, T) \) is minimal, then in many cases (including when \( T \) is abelian) \( \text{RP} \) is an equivalence relation and coincides with \( S_{eq} \). The Veech relation \( V \) is defined by the condition that \( (x, y) \in V \) if there is a \( z \in X \) and a net \( t_n \) in \( T \) with \( t_n x \to z \) and \( t_n^{-1} z \to y \). It is immediate to see that \( V \subset \text{RP} \subset S_{eq} \). In [72] Veech introduced the relation and proved that if \( X \) is metric and \( T \) is abelian then \( V = S_{eq} \). His proof is based on harmonic analysis. A dynamical proof of this result (in fact of the identity \( V = \text{RP} \)) for minimal systems on a metric space with an invariant
measure was given only many years later by J. Auslander, G. Greschonig and A. Nagar [1].

A major contribution of Veech to topological dynamics is the structure theorem named after him. This theorem belonged to a flurry of activity which led to a general understanding of the structure of minimal flows, initiated by the structure theorem of Furstenberg [38] (see also [22]). Furstenberg's theorem states that any minimal distal system $(X, T)$, with “phase” group $T$ acting on a metrizable space $X$, is isomorphic to a system obtained through a transfinite sequence of almost periodic extensions from a point system. We recall that a pair points $x, y$ are called distal from one another if there exists a continuous pseudo-metric $\rho$ on $X$ and $\epsilon > 0$ such that $\rho(t x, t y) > \epsilon$ for all $t \in T$. The point $x$ is a distal point if $x$ and $y$ are distal for all $y \neq x$. The system $(X, T)$ is point distal if it has a distal point and distal if every point is a distal point. In the paper [74] Veech generalized Furstenberg's structure theorem to all systems with a residual set of distal points. A few years later the Veech's structure theorem was extended further by R. Ellis in a paper titled The Veech structure theorem [21]. Ellis proved the result under the hypothesis that the system has one distal point. By Ellis' generalization [21] and by a remark of Veech in [74], it follows that the existence of a single distal point implies that of a residual set. Veech's structure theorem was followed by a general structure theorem for minimal flows by Ellis, Glasner and Shapiro, MacMahon and Wu. In his 1977 Bulletin of AMS article on Topological Dynamics [75] Veech went even further. He proved a general structure theorem which says roughly that the class of minimal flows (with fixed phase group) is the smallest class of flows containing the trivial flow and closed under (a) homomorphisms, (b) projective limits, and (c) three “building blocks”. These building block are given by different kind of extensions of a systems, which involve several extension constructions and notions: almost periodic, almost automorphic, proximal and highly proximal and weak mixing extensions. In particular, the notion of almost automorphic “flow” had already appeared in Veech's early work [71] on “almost automorphic functions”, a notion introduced by S. Bochner, Veech's thesis advisor.

While the second part of the Bulletin of AMS article [75] summarizes and completes Veech's contributions to the structure theory of minimal flows, the first part surveys developments in ergodic theory, in particular in applications of ergodic theory to the study of statistical properties of sequences and to the theory of uniform distributions, which played an important part in Veech's research in the 1970's. The first part ends with a subsection (§§1.14) on Interval Exchange Transformations, where Veech listed several early results and questions around the problem of minimality and unique ergodicity for Interval Exchange Transformations (IET's), and in particular the "Keane conjecture" that almost all irreducible IET's are uniquely ergodic. Keane [51] had by then proved his theorem on minimality of IET's, and had initially conjectured that all minimal IET's are uniquely ergodic. Counterexamples to this conjecture, in the case of at least
four intervals, were soon found by Keynes and Newton [56] who gave an example of a minimal IET on 5 intervals, which is not ergodic (with respect to the Lebesgue measure). This example certainly resonated with Veech since, as he remarks in [75], it is based on the construction of a *minimal non-ergodic skew-shift over a rotation of the circle* he himself had constructed several years earlier [73]. Keynes and Newton proposed another, most economical, conjecture, that ergodicity implies unique ergodicity, but the new conjecture was proved incorrect by Keane [52], who then stated the conjecture on unique ergodicity which bears his name. Keane’s example also had rationally independent lengths which dispelled any hope of reducing the general problem to familiar Diophantine conditions.

Veech’s 1977 brief survey on IET’s is interesting from a historical perspective. On one hand, it appears that the contributions of the Russian school to the study of IET’s were not readily available in the West. In particular there is no mention of the bounds of V. Oseledets and A. Katok [46] on the number of independent ergodic measures, and the applications of minimality of IET’s to billiard flow in a rational polygon is attributed to preprints of Keane (the Katok-Zemlyakov paper [50], which had already appeared, is also quoted). Keane’s work appeared the following year in [10].

On the other hand, it is interesting to realize that Veech had already been thinking of a proof of the Keane conjecture to the point of having formed a complete strategy which indeed will lead him to the result a few years later. He wrote: *The construction he (Keane) employs in [52] suggests strongly that an auxiliary transformation on the “body” of interval exchanges will be of use in settling his conjecture.* Veech then introduced transformations $Q_i$ on the simplex of IET’s on $n$-intervals given by inducing on the $i$-th sub-interval and states problems concerning the ergodicity and the existence of an absolutely continuous invariant measure for these transformations. After observing that the answer to these questions is affirmative for $n = 2$ and $n = 3$ (although the measure has infinite mass at least in the case $n = 2$), he concluded that *Keane’s conjecture would be true, for example, if the measure exists and is finite for $n \geq 4$.* It would take him several years of work to fully realize this program.

## 2. VEECH’S WORK ON IET’S AND TEICHMÜLLER DYNAMICS

The Bulletin of the AMS article [75] was received by the editors on August 5, 1976 and appeared in September 1977. In the Spring 1977 Veech traveled to Rennes, presumably to visit Keane. He briefly described his visit in an e-mail to Luc Pirio (and Selim Ghazouani)\(^1\) on May 9th, 2016: *A bit of Rennes trivia. I visited Rennes, April-June, 1977. On June 21, I met Gérard Rauzy for the first time. He gave a talk, in French, but after the talk Gérard, Mike Keane and I went to Mike’s office, and Mike helped with translation for a language-challenged*
2.1. **The Keane conjecture.** In his first paper devoted to Interval Exchange Transformations [76], Veech developed the comments in his Bulletin of the AMS article [75] mentioned above. He proceeded to a *formal study of the inducing process* (for induction on “admissible” subintervals) with the hope of establishing a *usable criterion for unique ergodicity* which could lead to a proof of the Keane conjecture. He characterized the cone of invariant measures, hence in particular the property of unique ergodicity, in terms of the induction and derived bounds on its dimension, by relating the set of invariant probability measures of a minimal IET to the set of IET’s topologically conjugate to it. (Apparently, he was still not aware of Katok’s [46] bound.) The main conclusion of
the paper is a reduction of the problem to constructing recurrent absolutely continuous invariant measures for the induction map. However, the main criterion established in the paper [76] assumes the existence of a finite absolutely continuous invariant measure for the induction map. The work does not consider the special induction introduced by Rauzy [66] (although by the reception date of January 12, 1978, the paper follows Veech’s visit to Rennes). In the papers [77] and in the fundamental [79] (see also the announcement [78]) Veech was able to prove the Keane conjecture, for IET’s on 4 intervals in the first paper, then for any number of intervals. The argument followed the program he had outlined, with the important difference that it was based on the Rauzy induction and on the construction of infinite, but conservative, absolutely continuous “Gauss” measures for the Rauzy induction.

The result achieved in [77] was limited to 4 intervals by the difficulty of deriving the conservative property of “Gauss measures” from explicit formulas for their density. Finally, Veech derived the property from the Poincaré recurrence theorem for a measure of finite mass for a suspension flow over the Rauzy induction he introduced, now called Veech zippered rectangles flow [78], [79].

Indeed, in his work on the “Keane conjecture” Veech built on Rauzy’s work [66] in several ways. The Rauzy induction indeed provides a very effective “auxiliary” transformation on the “body” of interval exchanges. The key idea of the Rauzy induction is to induce on a sub-interval obtained by cutting out the smaller between the last sub-intervals in the domain and range of the IET. In this way the induced transformation is an IET on exactly the same number of intervals as the original one, and an auxiliary transformation is indeed well-defined almost everywhere. The formulas for the Rauzy induction are especially simple and are given by matrices with non-negative entries. The Rauzy induction generates an equivalence relation on permutations, whose equivalence classes are called Rauzy classes.

Veech introduced a geometrically defined natural extension of Rauzy induction. The first step was to construct special suspensions of IET’s with piece-wise constant roof functions which yields linear flows (with singularities) on a compact surface equipped with a natural euclidean metric. The classical suspension construction of an IET with roof function constant on subintervals naturally leads to a surface with “holes” which is the union of finitely many euclidean rectangles with horizontal sides glued to sub-intervals of the domain of the IET. Veech found a condition on the heights which allows one to “zip” the vertical sides to construct a closed surface of genus determined by the permutation (assumed irreducible) of the IET. This is an outline of Veech’s zippered rectangles construction. The Rauzy induction algorithm naturally extends to zippered rectangles, and defines an invertible map, called the Rauzy–Veech induction, which is isomorphic to the “natural extension” of the Rauzy induction. Then, on the space of all the zippered rectangles, Veech considered the flow consisting of scaling the length in the horizontal and vertical directions by reciprocal factors $e^t$ and $e^{-t}$. The scaling flow is easily seen to commute with the Rauzy-Veech
induction map, hence it defines a flow on the space of zippered rectangles quotiented out by the action of Rauzy-Veech induction: the Veech zippered rectangles flow. By construction, the return map of the zippered rectangles flow to an appropriate transverse hypersurface equals the Rauzy-Veech induction map. Veech finally proved that, for each Rauzy class, the zippered rectangles flow has a finite absolutely continuous invariant probability measure. The recurrence property of the measure then follows from Poincaré recurrence theorem, and extends to the “Gauss” measure induced on the transverse section, invariant for the Rauzy–Veech induction.

Thus the program was fully carried out and the Keane conjecture was finally proved [79]. In fact, in the same issue of the journal *Annals of Mathematics*, H. Masur [60] published an independent proof of the Keane conjecture based on the dynamics of a flow generated by the Teichmüller deformation, which he introduced: the Teichmüller (geodesic) flow. In the introduction to his own paper [79], Veech explicitly acknowledged Masur’s priority in the full proof of the conjecture in a statement that speaks by itself about his attitude and ethics:

*The Keane conjecture has also been proved, independently and earlier, by H. Masur [60]. Masur’s preprint in fact arrived as we were writing up the results of this paper [79].*

Veech also remarked that the outlines of Masur’s and his proofs of the Keane conjecture are remarkably similar. Indeed, the main strategy of the proof, which is to derive unique ergodicity of IET’s from (Poincaré) recurrence for a renormalization dynamical system, is common to both approaches. In fact, the space of Veech’s zippered rectangles surfaces can be identified with a finite cover of the moduli space of Abelian (holomorphic) differentials on Riemann surfaces, so that the Veech zippered rectangles flow projects onto Masur’s Teichmüller flow. For each Rauzy class, the unique absolutely continuous probability measure invariant under the zippered rectangles flow projects onto the unique absolutely continuous probability measure invariant under the Teichmüller flow on a “stratum” of the moduli space of Abelian differentials. Such measures are now called Masur–Veech measures. The zippered rectangles flow therefore provides a finite-to-one symbolic coding for the Teichmüller flow on a full measure set of Abelian differentials in every connected component of every stratum of the moduli space (in fact, on the set of all Abelian differentials without vertical saddle connections). This coding has been applied successfully to the study of finer dynamical properties of the Teichmüller flow (for instance, Avila, Gouëzel and Yoccoz’s proof [6] of the exponential mixing of the Teichmüller flow with respect to Masur–Veech measures works with the zippered rectangles flow).

The papers [60] of Masur and [79] established the foundation for the field of study of the dynamics of IET’s, flows on surfaces, billiards in rational polygons, Teichmüller geodesic and horocycle flows, and related systems, which nowadays goes under the (rather unfortunate) name of *Teichmüller dynamics*.

A few years later M. Boshernitzan, Veech’s colleague at Rice, found a different, more direct, unique ergodicity criterion for Interval Exchange Transformations,
not based on renormalization but on a Diophantine property, which he called ‘Property P’, and was able to give a new, fundamentally different proof of the Keane conjecture [11]. He conjectured that a natural weaker condition that the minimum length of continuity intervals of the iterates of a minimal IET decay no faster then linearly with the number of iterates implies unique ergodicity. Veech [87] confirmed Boshernitzan’s conjecture, and claimed the right to name it Boshernitzan’s criterion in the title of his papers. As a “simple corollary” of Veech’s “Boshernitzan’s criterion”, Boshernitzan [12] then derived that all minimal IET’s with a length vector of rank 2 over the rationals are uniquely ergodic. As remarked by Y. Vorobets [96], the “Boshernitzan’s criterion” implies a slightly weaker version of a unique ergodicity criterion of H. Masur from [60] (often called, the “Masur criterion”).

Over the years other interesting proofs of the Keane conjecture have been given. M. Rees [67] derived the result from a direct proof of ergodicity of the action of the mapping class group on the space of projective measured foliations (Masur [60] had already proven the ergodicity of the square of the action). S. Kerckhoff’s proof [54] followed basically Veech’s strategy but established recurrence via combinatorial properties of the algorithm and Veech’s Jacobian computation from [76]. More recently, a proof in the spirit of Masur's [60], based on the hyperbolicity of the Teichmüller flow (equivalently, on the “spectral gap” of the Rauzy-Veech or Kontsevich–Zorich cocycle) and on the Katok’s fundamental class [46] was outlined in [35]. Along the same lines, a quantitative ergodicity criterion in terms of the speed of divergence of the Teichmüller trajectory has been given by R. Treviño [70] and exploited by J. Chaika and R. Treviño [17].

2.2. The metric theory of IET’s. In his landmark paper [79] proving the Keane conjecture, Veech also established the ergodicity of the zippered rectangles flows (for each Rauzy class). (In his paper, Masur had also proved ergodicity of the Teichmüller flow, but only on the maximal stratum of quadratic differentials with simple zeroes.) In three consecutive papers in the American Journal of Mathematics [81], [82], [83] Veech proceeded to derive from the ergodicity of the Rauzy induction a wealth of results on the dynamics of the typical IET’s, with respect to the Lebesgue measure on the length parameters. In the first paper [81], Veech proved that Lebesgue almost all IET’s are rank one, hence have simple spectrum, and are rigid, and are totally ergodic, that is, all of the powers are ergodic. A transformation $T$ of the unit interval $[0,1]$ is called rank-one if any partition of the unit interval can be approximated (in measure sense with respect to the Lebesgue measure) by the partition generated by a Rokhlin tower, that is, by a partition of the form $\{E, T(E), \ldots, T^{h-1}(E)\}$ into disjoint images of a measurable subset $E \subset [0,1]$. It follows from the definition that the towers have to be arbitrarily high (the integer $h$ is called the height of the tower) and have to fill an arbitrarily large fraction of the unit interval. A transformation $T$ is called rigid if it has a sequence of iterates $\{T^n\}$ which converge to the identity, in the sense that for every subset $E \subset [0,1]$ the measure $\text{Leb}(T^{n_k}(E)\Delta E)$ converges to zero.
The basic idea behind these results is that at macroscopic scale IET’s with one “very large” subinterval, occupying almost all the domain, are by definition approximately rank-one and rigid at time one. Such IET’s form a shrinking family of open sets. By the ergodicity of the Rauzy–Veech induction, for almost all IET’s the orbit visits any sequence of sets in the family infinitely often, thereby producing a sequence of rank-one and rigidity times. Indeed, for the rank-one property the set $E$ in the above definition (the base of the Rokhlin tower) is given by a small interval.

In the same paper [81] Veech also proves that almost all IET’s are weakly mixing, that is, have continuous spectrum, under the condition that the suspension under the (constant) roof function $(1, \ldots, 1)$ does NOT result in a closed “zippered rectangles” surface. The condition holds for IET’s defined by certain permutations, for instance for all permutations of the form $(d, d-1, \ldots, 2, 1)$ for any odd $d \geq 3$ and for almost all length vector. This result extends to an arbitrary number of intervals the theorem of A. Katok and A. M. Stepin [49]. A few years earlier A. Katok had proved the only “universal” spectral result on IET’s: they are neither mixing nor do they have mixing factors [47].

In his papers on the ergodic theory of IET’s, Veech also proved that (not renormalized) Rauzy induction is an ergodic (non-conservative) map, which established that a useful invariant (the Sah-Arnoux-Fathi (SAF) invariant) is not measurable [83], and approximation results for typical IET’s by periodic IET’s [82], and counting results for periodic IET’s.

Veech was well aware that his weak mixing result was unlikely to be the definitive statement on the genericity of weak mixing for interval exchanges [81]. In the paper he asked whether weak mixing holds for almost all length vectors if the permutation $\pi \in S_m$ verifies the condition $\pi(j+1) \neq \pi(j) + 1$ for at least 2 values of $j \in \{1, \ldots, m-1\}$, but did not venture further. A complete solution of the weak mixing problem required tools that were not available at that time. In fact, only after results on the non-vanishing of Lyapunov exponents of the Rauzy-Veech cocycle [32], A. Avila and G. Forni [4] were able to prove that almost all IET’s are weakly mixing for all irreducible permutations which are not rotations. The argument in [4] is based on the same criterion, which the authors called Veech’s weak mixing criterion (see [81], Prop. 6.5), on which Veech based his own above-mentioned weak mixing result.

The theory of Lyapunov exponents of the Rauzy–Veech cocycle was initiated by A. Zorich to explain a phenomenon of deviation of ergodic averages of Interval Exchange Transformations, which he had discovered in numerical simulations. Zorich [98], [99] proved the existence of Lyapunov exponents by introducing an acceleration of the Rauzy–Veech induction, and derived a bound on the speed of convergence of ergodic averages of generic IET’s from Veech’s hyperbolicity theorem [85] for the Teichmüller geodesic flow. Zorich [99] and Kontsevich [57], [58] proposed a series of conjectures about the Lyapunov spectrum, and Kontsevich outlined the proof of a remarkable formula for the sum
of the non-negative exponent, based on methods of Hodge theory. The Kontsevich formula \[57\], \[58\] was generalized in \[32\] to partial sums of exponents. A proof of the Kontsevich–Zorich conjectures on the Lyapunov spectrum and on deviations of ergodic averages was given in \[32\] with the exception of the simplicity of the spectrum, which was later proved by A. Avila and M. Viana \[7\] (for the Masur–Veech measures) by a different method. Finally, a question of Veech motivated the discovery of examples of measures with zero Lyapunov exponents (the Eierlegende Wollmilchsau \[33\], \[41\] and the Platypus \[34\]) and a general program to understand their appearance \[36\], \[37\], which culminated in S. Filip \[31\] structure theorem. Filip’s theorem establishes that zero exponents are always the result of symmetries in the cocycle, and gives a representation-theoretic list of all possible symmetry groups (up to compact factors).

2.3. Property S and prime. Another question related to Veech’s work at the beginning of the 80’s, which is still open, is whether almost all IET’s which are not rotations are prime, in the sense that they do not have non-trivial factors. In \[80\] he introduced a criterion for a transformation to be prime, based on a notion of property S, that is, all of its ergodic self-joinings are either the product measure or supported on a graph. A transformation with Veech’s property S is nowadays called 2-simple. In \[80\] he classified factors of 2-simple transformations: they come from compact subgroups of the centralizer. A. del Junco and D. Rudolph \[45\] used this to construct the first rigid and prime transformation. Also, while Glasner and Weiss \[39\] constructed a prime transformation that was not 2-simple, their example is still a factor of a simple system and uses Veech’s classification of factors of 2-simple systems to establish that it is prime. Additionally, del Junco constructed a 2-simple system with no prime factors \[43\] and A. Danilenko and A. Solomko \[19\] used the classification of factors to build a transformation with uncountably many prime factors. The notion of 2-simple was coined and subsequently generalized by del Junco and Rudolph \[44\] to higher order simplicity and other group actions. They classified the joinings of these systems and arbitrary ergodic transformations.

Returning to IET’s, Veech asked in the paper whether the typical IET is 2-simple. Very recently, J. Chaika and A. Eskin \[16\] have answered Veech’s question in the negative: IET’s in 3-intervals are typically not 2-simple. Their work however does not yet answer whether they are prime. In addition, Veech asked whether the typical IET has any non-trivial compact subgroups in its centralizer or even commutes with transformations of finite order. D. Bernazzani \[9\] answered this latter question in the negative when one considers the intersection of the centralizer with the group of IET’s, but in general both questions are wide open.

2.4. The Teichmüller flow. Although the Teichmüller flow had been introduced by H. Masur \[60\], who was the first to study it \[60\], \[61\], it was Veech who initiated, in \[84\], \[85\], a systematic investigation of its dynamical properties. In the long and difficult paper \[85\], Veech advanced the dynamical description of
the Teichmüller flow and introduced several important ideas. In summary he proved that, on any stratum of the space of quadratic differentials on Riemann surfaces with punctures, the Teichmüller flow is a (locally uniformly) measurably Anosov, mixing flow, in fact, Bernoulli flow with respect to a “canonical” absolutely continuous invariant probability (Masur–Veech) measure. He computed its entropy (from the Pesin entropy formula) in terms of the dimension of the stratum, and derived (from a closing lemma) a lower bound on the growth of the length spectrum of periodic orbits (which implies a lower bound on conjugacy classes of pseudo-Anosov maps).

Among the important, far-reaching new ideas that Veech introduced in this work, we would like to mention in particular the introduction of Lyapunov exponents and Pesin theory to Teichmüller dynamics, and the dynamical application of the action of the full group $SL(2,\mathbb{R})$. The core of Veech’s study is indeed the proof that the Teichmüller flow is “measurably Anosov”, that is, it has non-zero Lyapunov exponents. The argument is based on estimates of the contraction of tangent vectors to the natural stable foliation with respect to the Teichmüller metric. Veech’s results on Lyapunov exponents were later generalized and refined in [32] replacing the Teichmüller metric with the Hodge metric. From the measurable Anosov property, ergodicity follows from the Hopf argument, since the Teichmüller flow has geometrically defined global invariant (stable and unstable) foliations. Pesin theory then implies that the flow has the $K$ property, while the Bernoulli property follows from the $K$ property by an argument of Ornstein and Weiss. The Pesin entropy formula gives the value of the entropy.

The introduction of the action of the full group $SL(2,\mathbb{R})$ is of course of fundamental importance for later advances in Teichmüller dynamics, from Veech’s own later work on lattice surfaces and Siegel measures [88], [91] all the way to the work of Eskin, Mirzakhani and Mohammadi [27], [28] on the measure rigidity of the $SL(2,\mathbb{R})$-action, and subsequent developments.

The Teichmüller geodesic flow is given by the sub-action of the diagonal subgroup of $SL(2,\mathbb{R})$, while the Teichmüller horocycle flow corresponds to the sub-action of a unipotent subgroup. The Teichmüller horocycle flow was introduced by H. Masur [61] in his proof of ergodicity of the action of the modular group on the space of measured foliations. The horocycle flow is of great importance in many applications of Teichmüller dynamics beyond Lebesgue generic points.

The $SL(2,\mathbb{R})$ action was introduced independently and around the same time by Veech [84], [85] and by S. Kerckhoff, H. Masur and J. Smillie [55]. In [55] it was applied to a proof of the unique ergodicity in almost every direction of the foliation of every quadratic differential, and in particular in almost every direction of the flow of rational billiards, which form in every stratum a set of zero Masur–Veech measure.

Veech [84], [85] gave the first application of the $SL(2,\mathbb{R})$ action to the dynamics of the Teichmüller geodesic flow when he derived its mixing property from
the ergodicity of the $SL(2, \mathbb{R})$ action (which in turns follows from that of its diagonal subgroup, the Teichmüller flow itself) by Moore’s theorem, a fundamental result in the ergodic theory of $SL(2, \mathbb{R})$ actions.

The work of Veech established that the Teichmüller geodesic flow can be studied to a great extent as an Anosov flow, uniformly hyperbolic on compact sets, on a non-compact space, which could be expected from the analogy with the genus one case of the modular flow. It left open two main questions: the questions of the exponential decay of correlations, or exponential mixing of the Teichmüller flow, and the question on the upper bound in the growth of the length spectrum of periodic orbits. The first question was settled (in the affirmative) by Avila, Gouëzel and Yoccoz [6] for Masur–Veech measures, and later by Avila and Gouëzel [5] for all $SL(2, \mathbb{R})$ invariant measures. The second questions was answered, in the context of a broader program [2], [26] modeled on Margulis’ thesis for Anosov flows, by A. Eskin and M. Mirzakhani [26] for the moduli space, and by A. Eskin and M. Mirzakhani and K. Rafi [29] for strata. The analogous, but easier, question for Veech’s zippered rectangles flow was solved earlier by methods of symbolic dynamics by A. Bufetov [14].

2.5. **Cohomological equations.** In the same year (1986), Veech published a paper on the cohomological equation for toral automorphisms [86] which contains the most important study to date for partially hyperbolic diffeomorphisms in the essentially accessible, non-accessible, case. The paper is based on methods of Fourier analysis and of number theory, which are the only ones available in the non-accessible case. For accessible partially hyperbolic systems, the study of the cohomological equation was initiated many years later by A. Katok and A. Kononenko [48], and finally completed in the definitive work of A. Wilkinson [97] (albeit under a “bunching condition”). In contrast with the accessible case, in the Veech examples there is a definite loss of regularity between coboundaries and the corresponding solutions of the cohomological equation, the “transfer functions”. In addition, the obstructions to the existence of solutions are given by sums over periodic orbits, like in the classical Livšic theory, while in the work on the accessible case [48], [97] the obstructions are given by the so-called “periodic cycle functionals” (PCF), which are integrals along loops in the stable and unstable distributions. Finally, in his paper [86] Veech gave examples of $C^1$ functions, in the kernel of all periodic obstructions, with no Sobolev $H^1$ solutions.

We are not aware of any developments in the non-accessible case after Veech’s work, with the exception of an example of D. Dolgopyat [20] of an essentially accessible, not accessible, partially hyperbolic diffeomorphism with a $C^\infty$ coboundary whose transfer map is continuous, but not $C^1$.

Veech never came back to the topic of partially hyperbolic dynamics in his later work, but returned to the topic of cohomological equations with a paper [93], one of his last, on invariant distributions for IET’s. In this paper he extended bounds on the dimension of the cone of ergodic measures to invariant distributions in (dual) Hölder classes.
2.6. **Lattice surfaces.** Veech wrote the fundamental paper [88] on lattice surfaces, now often called **Veech surfaces**, to answer the question, attributed in the introduction of the paper to M. Gromov, on the existence of periodic $SL(2,\mathbb{R})$ orbits in moduli space, in the sense that the isotropy group of any point in the orbit is a lattice in $SL(2,\mathbb{R})$. After noticing that the answer is certainly affirmative as a consequence of a construction of Thurston, which gives plenty of examples for which the isotropy subgroup is a finite index subgroup of the lattice $SL(2,\mathbb{Z})$, Veech sets out to find examples of lattice surfaces whose isotropy group is not commensurable to any conjugate of $SL(2,\mathbb{Z})$. His famous examples, the **Veech lattice surfaces**, are given by the unfolding of rational billiards in isosceles triangles with base angles equal to $\pi/n$ for all even integers $n > 2$. In the paper Veech proves several general results for lattice surfaces, namely the **Veech Dichotomy** according to which the directional flow on a lattice surface is either uniquely ergodic or completely periodic (every regular orbit is closed), as well as results on the asymptotics of the counting function for the number of cylinders of periodic orbits.

These results have been extremely influential in the field. For instance, despite the fact that in his paper Veech already wonders about classifying all lattice surfaces, the classification problem has proved to be very difficult, even in low genus. It is known that lattice surfaces are rare, in the sense that they form a zero measure set with respect to the Masur–Veech measure in every stratum. However, in every stratum there are countably many **arithmetic** lattice surfaces, that is, surfaces with isotropy group commensurable with a conjugate of $SL(2,\mathbb{Z})$. By a theorem of E. Gutkin and C. Judge [40] a surface is arithmetic if and only if it is a branched cover of a torus with a single branch point (or, equivalently, if and only if it is tiled by parallelograms). Non-arithmetic examples are much harder to find. Over the years new ones have been found by several authors: C. Ward, Y. Vorobets [96], R. Kenyon and J. Smillie [53], J.-Ch. Puchta [65], K. Calta [15], C. McMullen [63] (Kenyon and Smillie found all Veech surfaces given by acute, **non-isosceles**, rational-angled triangles and established that an acute, isosceles, rational-angled triangle gives a Veech surface if and only if the smallest angle is of the form $\pi/n$. Calta and McMullen in fact found of all Veech surfaces in the genus 2 stratum $H(2)$ of Abelian differentials with a single double zero, and McMullen obtained a complete classification. According to [53] and [65], there are exactly three acute, non-isosceles, rational-angled triangles with angles $(\pi/4,\pi/3,5\pi/12)$, $(\pi/5,\pi/3,7\pi/15)$, $(2\pi/9,\pi/3,4\pi/9)$. Of these, the first was known to Veech (see [88]) who attributes its discovery to numerical experiments by his colleague M. Boshernitzan.

A vast literature has developed on characterization and classification of Veech surfaces, on their isotropy groups and the related affine groups, now called **Veech groups**, on their geometry and on related dynamical properties. Indeed, the study of Veech surfaces is an important part of Teichmüller dynamics and has motivated the introduction of methods of algebraic geometry into the field.
In particular these methods have led to the discovery of the Bouw-Möller family of Veech surfaces [13], which includes Veech’s and Ward’s examples.

It was known to Veech [88] that the isotropy group of a surface is never co-compact. However, whenever it is a lattice, the corresponding $SL(2, \mathbb{R})$ orbits carries a natural finite $SL(2, \mathbb{R})$ invariant measure. Veech [90] proved that this condition is equivalent to the statement that the $SL(2, \mathbb{R})$ orbit is topologically closed in the moduli space, but only under the condition that the isotropy (Veech) group of the quadratic differential is finitely generated. The proof of the unconditional statement is attributed by Veech (in [90]) to J. Smillie and the paper [90] ends with an outline of Smillie’s argument. Several years later J. Smillie and B. Weiss [69] gave another proof by a somewhat different approach. It should be noted that in [90, Question 6.3] Veech asks whether the isotropy group of a quadratic differential is always finitely generated. This question was answered in the negative by P. Hubert and T. Schmidt, and independently by C. McMullen [64].

In the general program of understanding measure-generic ergodic properties for directional flows of Abelian and quadratic differentials on Riemann surfaces with respect to all $SL(2, \mathbb{R})$ invariant measures, those measures supported on closed orbits of Veech surfaces are often a prime example to consider. For instance, A. Avila and V. Delecroix [3] have proved the weak mixing property in the Lebesgue generic direction for non-arithmetic Veech surfaces (and the relative weak mixing property in the arithmetic case). Another example is the question on the generic rigidity of directional flows and IET’s, which Veech answered in the affirmative in [81] with respect to Masur–Veech measures, as mentioned above, which is largely open with respect to measures supported on non-arithmetic Veech surfaces.

In his groundbreaking paper [88], Veech also addressed the question of the asymptotic growth of the number of cylinders of periodic orbits on lattice surfaces as their length increases. H. Masur had recently proved quadratic upper and lower bounds on the growth of the counting function (as function of the length of the cylinders) for an arbitrary quadratic differential. Veech [88] proved quadratic asymptotics for any lattice surface. In fact, he was able to relate the generating function of the “length spectrum” of the flat lattice surface to the Eisenstein series associated to the corresponding lattice. The constant in the asymptotics can then be computed from knowledge of the lattice and of its fundamental domain. In a follow-up paper [89] Veech computes formulæ for the constants appearing in the quadratic asymptotics of all lattice surfaces coming from billiards in regular polygons. He remarks on the non-trivial number theoretical nature of these numbers, in particular their relations to values of the Riemann zeta function. He asks what (other) geometric significance the constant(s) might have.

About the counting function of the length spectrum Veech wrote in [88]:

In this regard we note that Masur [62] has obtained a “Tchebychev theorem” (for arbitrary quadratic differentials) exhibiting upper and lower bounds for the...
quadratic growth of the length spectrum. It would be of great interest to identify a larger class of differentials for which the “Tchebychev theorem” can be replaced by a “prime geodesics Theorem”

This question would provide ten years later an important motivation for Veech’s last great work [91], and for a large part of the more recent research in Teichmüller dynamics, from the paper [24] of A. Eskin and H. Masur to the work of A. Eskin, M. Mirzakhani and A. Mohammadi [27], [28].

2.7. Siegel measures. Veech’s breakthrough in the study of the asymptotic behavior of the counting function for general quadratic differentials came from the intuition that work on analogous problems in homogeneous dynamics might be relevant. As he wrote at the end of the Introduction of his paper [91]:

Indeed, the thought that Siegel’s Theorem might be relevant, at least in spirit, to the study of periodic trajectories for quadratic differentials was provoked by a lecture on [30] and, at Luminy, by G. A. Margulis.

This intuition would prove extremely fruitful for later developments in Teichmüller dynamics. After the groundbreaking exchange between Eskin, Margulis and Mozes [30], Veech [91] and Eskin and Masur [24], many authors have adapted ideas and methods from homogeneous dynamics to the moduli space of Abelian or quadratic differentials on Riemann surfaces, or vice versa.

The Siegel integral formula, or Siegel mean value theorem, is a fundamental identity in the geometry of numbers. It states that the integral of any compactly supported function on the Euclidean space \( \mathbb{R}^n \) equals the integral of the Siegel transform of the function over the space of lattices \( SL(n, \mathbb{R})/SL(n, \mathbb{Z}) \) up to multiplicative constant (equal to the value at \( n \) of the Riemann zeta function).

In [91] Veech proved a far-reaching ergodic theoretic generalization of the Siegel formula. The main notion involved in Veech’s generalization is the notion of a Siegel measure (nowadays called as Siegel–Veech measure). A Siegel measure is a probability measure on the space \( \mathcal{M}_n \) of Borel measures \( \nu \) on \( \mathbb{R}^n \) such that the \( \nu \)-measure of the unit ball \( B(0, R) \subset \mathbb{R}^n \) grows at most like \( R^n \) (more precisely, such that the supremum of the ratio \( \nu(B(0, R))/R^n \) is finite). In addition, a Siegel measure is invariant and ergodic with respect to the action of the group \( SL(n, \mathbb{R}) \) on the above space of measures, induced by natural action of the group on \( \mathbb{R}^n \). For any non-negative function \( \psi \) on \( \mathbb{R}^n \), Veech defined, by duality, a transform function (now called the Siegel-Veech transform) \( \hat{\psi} \) as function on \( \mathcal{M}_n \) defined for each Borel measure \( \nu \in \mathcal{M}_n \) as the integral of the function \( \psi \) over \( \mathbb{R}^n \). He then proved that for any Siegel measure \( \mu \) there exists a constant \( c(\mu) \) such that for every \( \psi \geq 0 \) the following integral formula holds:

\[
\int_{\mathcal{M}_n} \hat{\psi}(\nu) \mu(d\nu) = c(\mu) \int_{\mathbb{R}^n} \psi(u) du.
\]

On the left hand side of the formula is the integral (of the Siegel-Veech transform) over the space of measures \( \mathcal{M}_n \), endowed with the weak topology of uniform convergence over compact subsets of \( \mathbb{R}^n \), with respect to the Siegel measure \( \mu \) on it. On the right hand side, is the integral of the non-negative function \( \psi \) over \( \mathbb{R}^n \).
with respect to the Lebesgue measure on $\mathbb{R}^n$. The constant $c(\mu)$ is now called a Siegel-Veech constant.

Veech also proved a general $L^1$ ergodic theorem, which states that the ratio $\nu(B(0,R))/R^n$ (as a function of $\nu \in \mathcal{M}_n$) converges to $c(\mu)\text{Vol}_n(B(0,1))/n$ in mean (for $R \to +\infty$) with respect to the Siegel measure $\mu$ on $\mathcal{M}_n$. For $n > 2$ he proved point-wise convergence under the condition that all transforms of compactly supported functions are square-integrable with respect to the Siegel measure.

In the case of the group $SL(2,\mathbb{R})$ which is relevant for applications to moduli spaces, Veech was able to prove the point-wise convergence only under the assumption that the action has a “spectral gap” (a property which always hold for $n > 2$, by Kazhdan property $T$).

From the general theorem, Veech derived a result on the counting function of cylinders of quadratic differentials. He proved that for every ergodic $SL(2,\mathbb{R})$ invariant measure $\mu$ on the moduli space of quadratic differentials, there exists a (Siegel-Veech) constant $c(\mu)$ such that the normalized counting function $N(x,R)/R^2$ converges to $\pi c(\mu)$ in mean, with respect to the measure $\mu$, over all quadratic differentials $x$ in the moduli space. The derivation of this result from the general theorems follows from a crucial insight which motivates the notion of a Siegel measure: any quadratic differential $x$ determines the set $\Pi(x)$ of “holonomy vectors” of its regular periodic trajectories, a subset of $\mathbb{R}^2$, and this correspondence is equivariant with respect to the action of the group $SL(2,\mathbb{R})$; there exists therefore a map from the moduli space of quadratic differentials to the space $\mathcal{M}_2$ of Borel measures on $\mathbb{R}^2$ given by taking the counting measure $\nu_x$ supported on $\Pi(x)$ for any quadratic differential $x$ (the counting measure $\nu_x$ belongs to the space $\mathcal{M}_2$ by the above-mentioned theorem of H. Masur [62]). The push-forward of any $SL(2,\mathbb{R})$-invariant measure $\mu$ on the moduli space under the map $x \to \nu_x \in \mathcal{M}_2$ is a Siegel measure, that is, an $SL(2,\mathbb{R})$-invariant ergodic probability measure on $\mathcal{M}_2$.

Inspired by Veech’s paper, A. Eskin and H. Masur [24] improved Veech’s theorem from mean convergence to point-wise almost everywhere convergence by applying an ergodic theorem of A. Nevo, that is, they proved the point-wise almost everywhere convergence of the normalized counting functions of periodic orbits and saddle connections. It is interesting to remark that after the result of A. Avila and S. Gouëzel [5] who proved that all $SL(2,\mathbb{R})$-invariant measures have a spectral gap, it is possible to derive a proof of the above mentioned theorem of Eskin and Masur function directly from Veech’s results.

The work of Veech on Siegel measures [91] has had a formidable impact on Teichmüller dynamics and has motivated a large part of the research in the field. A complete account of these later developments would be well beyond the scope of this essay. We indicate here three main lines of developments which have guided research in the field: Eskin, Masur and Zorich [25] derived from the Siegel–Veech formula the result that the Siegel–Veech constants of Masur–Veech measures can be computed in terms of the volumes of the strata of the moduli space and their “principal boundary”; the Siegel–Veech constants appear (as a
boundary term) in the Eskin–Kontsevich–Zorich formula [23] for the sum of the non-negative Lyapunov exponents of the so-called Kontsevich–Zorich cocycle; the asymptotics of the counting functions which motivated Veech’s work, and Eskin and Masur’s [24], was an important motivation behind Eskin, Mirzakhani and Mohammadi’s work [27], [28] on the rigidity of the \( SL(2, \mathbb{R}) \) action on moduli space. They were able to prove (see [27, Th. 1.7]) that the quadratic asymptotics for the counting function holds for all quadratic differentials in Cesaro mean over time. It is conjectured that the asymptotics holds of all quadratic differentials, but a proof would require a better understanding of invariant measures for the Teichmüller horocycle flow, a difficult problem which has so far resisted, despite important recent advances (in particular [8] and results announced by J. Chaika, J. Smillie and B. Weiss).

A recent development in the study of moduli spaces is related to the large genus asymptotics of Masur–Veech volumes and of Siegel–Veech constants. D. Chen, M. Möller and D. Zagier [18] have proved that counting functions of torus coverings weighted with Siegel–Veech constants have a classical property called quasi-modularity. From these results the authors derive explicit generating functions for the volumes and Siegel–Veech constants in the case of the principal stratum of Abelian differentials. An analysis of the generating functions leads to a proof (for principal strata) of Eskin-Zorich conjectures on large genus asymptotics for the volume and the Siegel-Veech constants. A. Sauvaget [68] has proved similar results in the case of strata of Abelian differentials with a single zero.

Finally, the Siegel–Veech integral formula has in principle a much wider range than counting problems for quadratic differentials. For instance, J. Marklof and A. Strombergsson [59] rediscovered a special case of the formula in their work on free path lengths in quasi-crystals (their Siegel integral formula for quasi-crystal is indeed an instance of Veech’s formula).

We conclude this essay paraphrasing a great philosopher:

“Bill Veech may have not been in a class by himself, but if he wasn’t, it doesn’t take long to call the roll”.

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