Every longest circuit of a 3-connected, $K_{3,3}$-minor free graph has a chord

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Abstract

Carsten Thomassen conjectured that every longest circuit in a 3-connected graph has a chord. We prove the conjecture for graphs having no $K_{3,3}$ minor, and consequently for planar graphs.

Carsten Thomassen made the following conjecture [1, 7], where a circuit denotes a connected 2-regular graph:

Conjecture 1 (Thomassen) Every longest circuit of a 3-connected graph has a chord.

That conjecture has been proved for planar graphs with minimum degree at least four [9], cubic graphs [8] and graphs embeddable in several surfaces [4, 5, 6]. In this paper, we prove it for planar graphs in general. In fact, our result concerns a class of graphs which contains the planar graphs.

Let us denote by $K_{3,3}$ the complete bipartite graph drawn in Figure 1. A minor of a graph $G$ is a graph $H$ which can be obtained from $G$ by a sequence of vertex deletions, edge deletions and edge contractions. For more details about graph minors or the classical notations in graph theory used throughout the paper, the reader can refer to any general book on graph theory, for instance [2, 3].

The main result of this paper is the following:

Theorem 2 Let $G$ be a 3-connected graph with no $K_{3,3}$ minor. Then every longest circuit of $G$ has a chord.
Denoting by $K^5$ the complete graph on five vertices, Kuratowski’s theorem [3] states that a graph is planar if and only if it contains neither $K^5$ nor $K_{3,3}$ as a minor. Therefore, Theorem 2 immediately yields:

**Corollary 3** Every longest circuit in a planar 3-connected graph has a chord.

**Proof of Theorem 2:**
Let $G$ be a 3-connected graph that does not contain $K_{3,3}$ as a minor and let $C$ be a longest circuit in $G$. Suppose that $C$ has no chord. We denote by $v_0, \ldots, v_p$ the vertices of $C$ in cyclic order.

Let $H_1, \ldots, H_r$ be the connected components of $G \setminus C$. We denote by $N(i)$ the set of vertices of attachment of the component $H_i$, that is the set of vertices of $C$ that are adjacent to a vertex of $H_i$.

Let $P$ be an arc of $C$, that is a connected subgraph of $C$, and let $i \in \{1, \ldots, r\}$. We say that $P$ is a support of $H_i$ if $N(i) \subseteq V(P)$.

Let us first state some straightforward observations about the vertices of attachment of the components of $G \setminus C$:

**Lemma 4** i) For every $i \in \{1, \ldots, r\}$, $|N(i)| \geq 3$.

ii) For every $k \in \{0, \ldots, p\}$, there exist $i \in \{1, \ldots, r\}$ such that $v_k \in N(i)$.

iii) Two consecutive vertices on $C$ cannot belong to the same set $N(i)$.

iv) There are no integers $(k, l) \in \{0, \ldots, p\}^2$ and integers $(i, j) \in \{1, \ldots, r\}^2$ such that $\{v_k, v_l\} \subseteq N(i)$ and $\{v_{k+1}, v_{l+1}\} \subseteq N(j)$. (By convention, $v_{p+1} = v_0$)

**Proof**

i) This is a direct consequence of the 3-connectedness of $G$.

ii) As $G$ is 3-connected, $v_k$ has degree at least three. As $C$ has no chord, one of the neighbours of $v_k$ therefore does not belong to $V(C)$.

iii) and iv) Figure 2 show that both cases would contradict the fact that $C$ is a longest circuit of $G$. Indeed, the bold circuits are longer, as their subpaths in the connected components of $G \setminus C$ contain at least one vertex.
Consider an integer \( i \in \{1, \ldots, r\} \) and a support \( P \) of \( H_i \) such that \( P \) is minimal with respect to inclusion among all the supports; that is, there exists no pair \((j, Q)\) such that \( Q \) is a support of \( H_j \) and \( Q \) is a proper subpath of \( P \).

By reordering the vertices of \( C \), we may assume that \( P = v_0, \ldots, v_q, 2 \leq q \leq p \).

We denote by \( \hat{P} \) the path \( v_1 \ldots v_{q-1} \). By Lemma 4(ii) and iii), there exists at least one index \( j \neq i \) such that \( N(j) \cap \hat{P} \neq \emptyset \).

Let \( j \neq i \) and \( Q \) a support of \( H_j \) be such that:

- \( N(j) \cap \hat{P} \neq \emptyset \)
- Subject to this condition, \( P \cup Q \) is minimal (that is, there exists no pair \((k, R)\) satisfying it and such that \( P \cup R \) is a proper subpath of \( P \cup Q \)).
- Subject to the two former conditions, \( Q \) is minimal (that is, there exists no pair \((k, R)\) with \( N(k) \cap \hat{P} \neq \emptyset \), \( P \cup R = P \cup Q \) and such that \( R \) is a proper subpath of \( Q \)).

We can then choose six vertices, which will be denoted as the vertices of interest:

- Three distinct vertices of \( N(i) \) in \( P \), including both ends of \( P \).
- Three distinct vertices of \( N(j) \) in \( Q \), including both ends of \( Q \) and such that one of them is on \( \hat{P} \).

Let us first assume that \( P \) is a subpath of \( Q \). This case leads to the configuration shown in Figure 3. This figure has to been read in the following way:

- The left hand side figure shows the repartition of the vertices of interest along \( C \), where the bold arcs are possibly just single vertices. To make the figure clearer, each vertex of interest is indicated by a label \( i \) or \( j \), meaning that it belongs to \( N(i) \) or \( N(j) \), respectively.
Figure 3: The configuration when $P$ is a subpath of $Q$

- The right hand side figure shows that in that configuration, $G$ admits a $K_{3,3}$ minor.

Indeed, there exists an index $k \in \{1, \ldots, r\}$ different from $i$ and $j$ such that some vertex of $N(k)$ lies on a sub-arc of $P$ which is not bold: it is straightforward to verify that the absence of such a vertex would contradict part iii) or part iv) of Lemma 4.

We can choose that vertex on the sub-arc shown on Figure 3 because if not, there has to be a vertex labelled $j$ on that sub-arc and we obtain the same figure by symmetry.

Moreover, by minimality of $P \cup Q$, $N(k)$ either contains both ends of $P \cup Q$ or intersects the arc of $C$ with vertex set $V(C) \setminus V(P \cup Q)$. In any case, one of the two dashed lines exists.

Let then $A_1$, $A_2$ and $A_3$ be the three bold arcs of $C$. Let $B_1 = H_i$, $B_2 = H_j$ and $B_3$ be the union of $H_k$ and the arcs of $C \setminus (A_1 \cup A_2 \cup A_3)$ intersecting $N(k)$. The above six sets are disjoint and connected, and each $A_i$ is linked to each $B_j$ by an edge. Therefore, contracting each set to a single vertex, we obtain a $K_{3,3}$ minor of $G$.

We use the same technique to deal with the remaining cases, that is when $P$ is not a subpath of $Q$.

The possible orderings (up to symmetry) of the six vertices of interest are shown on the left side of Figure 4. The right side has to be read as in Figure 3 and shows that in each case, one can exhibit a $K_{3,3}$ minor.

The three following remarks are to fully understand that Figure:

- To obtain the second dashed line on the top right figure, we use the minimality of $Q$ instead of the minimality of $P \cup Q$.

- In case (b), we can restrict ourselves to the case where the vertex labelled $k$ is between the two vertices of interest labelled $j$. Indeed, if not, there
has to be a vertex labelled $i$ on that place and we can use case (a) to conclude.

- In case (c), the leftmost vertex labelled $j$ is distinct from the leftmost vertex labelled $i$ by minimality of $P$.

The fact that all possible configurations lead to a contradiction ensures that $C$ has a chord.

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Figure 4: The three possible configurations when $P$ is not a subpath of $Q$. 

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