Entropic uncertainty relation in Garfinkle-Horowitz-Strominger dilation black hole

Fariba Shahbazi,\textsuperscript{a} Soroush Haseli,\textsuperscript{b,1} Hazhir Dolatkhah\textsuperscript{a} and Shahriar Salimi\textsuperscript{a}

\textsuperscript{a}Department of Physics, University of Kurdistan, P.O. Box 66177-15175, Sanandaj, Iran
\textsuperscript{b}Faculty of Physics, Urmia University of Technology, Urmia, Iran

E-mail: shahbazi.fariba62@gmail.com, soroush.haseli@uut.ac.ir, h.dolatkhah@gmail.com, shsalimi@uok.ac.ir

Received June 21, 2020
Revised August 24, 2020
Accepted September 13, 2020
Published October 20, 2020

Abstract. Heisenberg’s uncertainty principle is a fundamental element in quantum mechanics. It sets a bound on our ability to predict the measurement outcomes of two incompatible observables simultaneously. In quantum information theory, the uncertainty principle can be expressed using entropic measures. The entropic uncertainty relation can be improved by considering an additional particle as a memory particle. The presence of quantum correlation between the memory particle and the measured particle reduces the uncertainty. In a curved space-time, the presence of the Hawking radiation can reduce quantum correlation. Therefore, concerning the relationship between the quantum correlation and entropic uncertainty lower bound, we expect that the Hawking radiation increases the entropic uncertainty lower bound. In this work, we investigate the entropic uncertainty relation in Garfinkle-Horowitz-Strominger (GHS) dilation black hole. We consider a model in which the memory particle is located near the event horizon outside the black hole, while the measured particle is free falling. To study the proposed model, we will consider examples with Dirac fields. We also explore the effect of the Hawking radiation on the quantum secret key rate.

Keywords: astrophysical black holes, quantum black holes

ArXiv ePrint: 2006.03387

\textsuperscript{1}Corresponding author.
1 Introduction

The uncertainty principle is one of the distinguishing features between quantum and classical theory. The Heisenberg uncertainty principle states that it is not possible to measure the location and momentum of a particle accurately \[1\]. Accurate measurement of one observable reduces the accuracy of another observable measurement. So far, this principle has been expressed in various ways. One of the most fundamental forms of expressing the uncertainty principle was provided by Schrödinger \[2\] and Robertson \[3\]. They showed that for any arbitrary pairs of noncommuting observables \(Q\) and \(R\) the following relation is established for the uncertainty principle

\[\Delta Q \Delta R \geq \frac{1}{2}|\langle[Q,R]\rangle|,\] \[(1.1)\]

where \(\Delta X = \sqrt{(X^2) - \langle X \rangle^2}\) with \(X \in \{Q,R\}\) is the standard deviation of the associated observable \(X\), \(\langle X \rangle\) shows the expectation value of operator \(X\) and \([Q,R] = QR - RQ\). The lower bound in eq. (1.1) is state dependent which leads to a trivial bound if \(|\langle[Q,R]\rangle| = 0\). In quantum information theory, it has been shown that the most appropriate quantity to show the uncertainty is entropy. Uncertainty relations that are defined in terms of entropy are called entropic uncertainty relation (EUR). The first EUR was speculated by Deutsch \[4\], then improved by Kraus \[5\], and finally proved by Maassen and Uffink \[6\]. They showed that for any arbitrary pairs of noncommuting observables \(Q\) and \(R\) EUR can be written as

\[H(Q) + H(R) \geq \log_2 \frac{1}{c}\] \[(1.2)\]

where \(H(Q) = \sum_i p_i \log_2 p_i\) and \(H(R) = \sum_j m_j \log_2 m_j\) are the Shannon entropy, \(p_i = \langle q_i | \rho | q_i \rangle\), \(m_j = \langle r_j | \rho | r_j \rangle\) and \(c = \max_{i,j} \{|\langle q_i | r_j \rangle|^2\}\) where \(|q_i\rangle\) and \(|r_j\rangle\) are eigenstates of observables \(Q\) and \(R\), respectively. This statement of uncertainty principle can be described by an interesting game between Alice and Bob. At the beginning of the game, Bob prepares the particle in a quantum state \(\rho\) and sends it to Alice. In the second step, Alice and Bob agree on the measurement of two observables \(Q\) and \(R\) by Alice on the particle. Then Alice measures one of the two observable \(Q\) or \(R\) on her state and sends her measurement choice...
to Bob via a classical communication channel. If Bob guesses Alice’s measurement correctly, he will win the game. Bob’s uncertainty about Alice’s measurement outcomes is bounded by eq. (1.2). In this statement of the uncertainty principle, there was only one particle. But when Bob prepares a correlated bipartite state \( \rho_{AB} \) for a two particle quantum system and sends one of the particles to Alice and keeps the other part as a quantum memory, he can guess the result of Alice’s measurement more accurately. Based on this uncertainty game, Berta et al. have presented the EUR in the presence of quantum memory (EUR-QM) as

\[
S(Q|B) + S(R|B) \geq \log_2 \frac{1}{c} + S(A|B),
\]

where \( S(Q|B) = S(\rho^{QB}) - S(\rho^B) \) and \( S(R|B) = S(\rho^{RB}) - S(\rho^B) \) are the conditional von-Neumann entropies of the post measurement states

\[
\rho^{QB} = \sum_i (|q_i\rangle\langle q_i| \otimes I) \rho^{AB} (|q_i\rangle\langle q_i| \otimes I),
\]

\[
\rho^{RB} = \sum_j (|r_j\rangle\langle r_j| \otimes I) \rho^{AB} (|r_j\rangle\langle r_j| \otimes I),
\]

and \( S(A|B) = S(\rho^{AB}) - S(\rho^B) \) is the conditional von Neumann entropy. Let’s take a look at some special cases: at first, If particles \( A \) and \( B \) are entangled, the conditional von-Neumann entropy is negative, and Bob can guess the result of Alice’s measurement with better accuracy. Second, If Bob prepares the maximally entangled state in uncertainty game then Bob can guess the result of Alice’s measurement perfectly [7]. Third, If there is no memory particle, then from eq. (1.3), the EUR is obtained as

\[
H(Q) + H(R) \geq \log_2 \frac{1}{c} + S(A).
\]

Due to the presence of an additional term \( S(A) \), the above EUR is tighter than Maassen and Uffink uncertainty relation. So far, several works have been done to improve the EUR [8–66]. In ref. [62], the authors introduced a new bound for the EUR-QM. They showed that Bob’s uncertainty about the results of Alice’s measurement is bounded by

\[
S(Q|B) + S(R|B) = H(Q) - I(Q; B) + H(R) - I(R; B)
\geq \log_2 \frac{1}{c} + S(A) - [I(Q; B) + I(R; B)]
= \log_2 \frac{1}{c} + S(A|B) +
+ \{I(A; B) - [I(Q; B) + I(R; B)]\}.
\]

Based on their results the EUR-QM can be written as

\[
S(Q|B) + S(R|B) \geq \log_2 \frac{1}{c} + S(A|B) + \max\{0, \delta\}
\]

where

\[
\delta = I(A; B) - (I(Q; B) + I(R; B))
\]

and

\[
I(X; B) = S(\rho^B) - \sum_x p_x S(\rho^B_x), \quad X \in \{Q, R\},
\]
Eq. (1.9) is known as Holevo quantity, \( p_x = tr_A B(Π_x^A ρ^{AB} Π_x^A) \) is the probability of x-th outcome and \( ρ_x^B = tr_A (Π_x^A ρ^{AB} Π_x^A) \) is Bob’s state after the measurement of X by Alice. It is worth noting that the EUR is tighter than other EURs in the presence of quantum memory.

The EUR has a variety of applications in quantum information theory such as entanglement detection [67–70], and quantum cryptography [71, 72]. The security of quantum key distribution protocols can be verified using the EURs [73, 74]. It has been shown that the bound of EUR-QM is directly related to the quantum secret key (QSK) rate [66, 75]. In ref. [66], the authors have shown that the amount of key that can be extracted by Alice and Bob \( K \) is lower bounded as

\[
K \geq \log_2 \frac{1}{c} + \max\{0, \delta\} - S(R|B) - S(Q|B),
\]  

(1.10)

The study of the EUR-QM from a relativistic point of view has been the subject of some recent works [76–79]. What is clear is that the entropic uncertainty bound decreases with increasing quantum correlation between the measured particle A and the quantum memory B. In refs. [80–84], the authors have shown that the quantum correlation decreases under the influence of Hawking radiation and so the entropic uncertainty increases. In order to study the effects of Hawking radiation on entropic uncertainty bound, we consider the most straightforward black hole: Garfinkle-Horowitz-Strominger (GHS) dilation black hole. We also consider the Dirac fields states as examples. In this situation, a quantum state is a combination of vacuum state and excited states of Dirac fields. In GHS dilation black hole space-time, we will consider a model in which the memory particle B is located somewhere near the event horizon outside the black hole, and the measured particle A is free falling. When the memory particle gets closer to the event horizon, the entanglement between particle memory B and measured particle A decreases due to the Unruh effect. In such a situation, the lower bound of EUR-QM increases with decreasing entanglement. This article is organized as follows. In section 2, the Quantum channel interpretation of the vacuum structure for Dirac fields in the GHS dilation black hole will be reviewed. In section 3 the entropic uncertainty lower bound (EULB) in Garfinkle-Horowitz-Strominger dilation black hole is investigated by considering some examples. Finally, conclusions are presented in section 4.

2 Quantum channel interpretation of the vacuum structure for Dirac fields in the GHS dilation black hole

In refs. [85, 86], the thermal Fermi-Dirac distribution of particles with the Hawking temperature \( T = \frac{1}{8\pi(M-D)} \) has been investigated in GHS dilation black hole [87]. The existence of such radiation has been described as the Hawking effect. The cosmological parameters \( M \) and \( D \) represent the mass of the black hole and dilation field, respectively. Here, according to the Dirac vacuum field in GHS dilation black hole, the global coordinates \( (t, r, \theta, \phi) \) is used to represent the spherically symmetric line element of the GHS dilation black hole [88–90]

\[
ds^2 = -\left(\frac{r-2M}{r-2D}\right)dt^2 + \left(\frac{r-2M}{r-2D}\right)^{-1}dr^2 + r(r-2D)(d\theta^2 + \sin^2 \theta d\phi^2).
\]  

(2.1)

Throughout this paper the natural units are set as \( \hbar = G = c = k_B = 1 \). The massless Dirac equation can be written as \( γ^a \bar{c}_μ^a (\partial_μ + Γ_μ) \psi = 0 \), where \( γ^α \) is the Dirac matrix, \( \bar{c}_μ^a \) corresponds
Figure 1. The Penrose diagrams for the GHS dilation black hole which shows the world-line of Bob and Anti-Bob. $i_0$ denotes the spatial infinities, $i^- (i^+)$ represents time-like past (future) infinity. $I_- (I_+)$ shows light-like past (future) infinity.

to the inverse of the tetrad and $\Gamma^\mu$ is the spin connection coefficient. Solving massless Dirac equation near the event horizon leads to positive frequency outgoing solutions outside the region $I$ and inside regions $II$ as

$$\psi^{\nu^+}_k = \xi e^{\pm i\omega u},$$

where $\nu \in \{I, II\}$ shows the regions, $k$ represents the field mode, $\xi$ is a 4-component Dirac spinor, and $\omega$ is a monochromatic frequency of the Dirac field. In eq. (2.2), $u$ represents the retarded time and is defined as follows

$$u = t - 2(M - D) \ln \left[ \frac{r - 2M}{2M - 2D} \right].$$

To show the general form of the GHS dilation black hole, the Carter-Penrose diagrams for this space-times is plotted in figure 1. In diagram $r = 2M$ shows the event horizons and $r = 2D$ represents the singularity of the black hole. $I$ and $II$ show the two general disconnected regions. The Dirac field can be quantized by using the complete orthogonal basis $\psi^{\nu^+}_k$ as

$$\psi_{\text{out}} = \sum_{\nu = I, II} \int dk \left( a^{\nu^+}_k \psi^{\nu^+}_k + b^{\nu^-}_k \psi^{\nu^-}_k \right)$$

where $a^{\nu^+}_k$ and $b^{\nu^-}_k$ are the fermion annihilation and anti-fermion creation operators respectively. One can use the generalized Kruskal coordinates to introduce the new orthogonal basis for positive energy mode as

$$\chi^{I+}_k = e^{2(M - D)\pi \omega} \psi^{I+}_k + e^{-2(M - D)\pi \omega} \psi^{II-}_k,$$

$$\chi^{II+}_k = e^{-2(M - D)\pi \omega} \psi^{I-}_k + e^{2(M - D)\pi \omega} \psi^{II+}_k.$$  

These new bases can be used to expand the Dirac fields in the Kruskal coordinates as

$$\psi_{\text{out}} = \sum_{\nu = I, II} \int dk \frac{1}{\sqrt{2 \cosh[4(M - D)\pi \omega]}} \times \left( c^{\nu^+}_k \chi^{\nu^+}_k + d^{\nu^-}_k \chi^{\nu^-}_k \right),$$

where $c^{\nu^+}_k$ and $d^{\nu^-}_k$ are the fermion annihilation and antifermion creation operators which act on the Kruskal vacuum. Eq. (2.4) is the expansion of the Dirac field in GHS dilation, while
where the partial trace is done over the state of the interior region. In figure 2a, the entropic uncertainty lower bound is plotted in terms of probability parameter \( p \) for different values of Hawking temperature. As can be seen, the entropic uncertainty lower bound increases with increasing Hawking temperature. This is what we expected, the quantum correlation decreases under the influence of Hawking radiation and so the entropic uncertainty increases with increasing Hawking temperature [80–84]. Figure 2b shows the contour plot of entropic uncertainty lower bound in terms of Hawking temperature \( T \) and probability parameter \( p \).

As can be seen, from figure 2b, at different Hawking temperatures the entropic uncertainty lower bound has its lowest value for the case in which \( p = 1 \) and the state is maximally entangled. It is also observed that for different values of \( p \) this entropic uncertainty lower bound increases with increasing Hawking temperature.
Figure 2. (a) Entropic uncertainty lower bound when Bob prepares a correlated bipartite state in a special class of state: \( \rho_{AB} = p |\Psi^-\rangle \langle \Psi^-| + \frac{1-p^2}{2} (|\Psi^+\rangle \langle \Psi^+| + |\Phi^+\rangle \langle \Phi^+|) \) in terms of probability parameter \( p \) for different values of Hawking temperature when \( \omega = 1 \). (b) The contour plot of entropic uncertainty lower bound when Bob prepares a correlated bipartite state in a special class of state: \( \rho_{AB} = p |\Psi^-\rangle \langle \Psi^-| + \frac{1-p^2}{2} (|\Psi^+\rangle \langle \Psi^+| + |\Phi^+\rangle \langle \Phi^+|) \) in terms of Hawking temperature \( T \) and probability parameter \( p \) when \( \omega = 1 \).

Figure 3. (a) QSK rate bound when Bob prepares a correlated bipartite state in a special class of state: \( \rho_{AB} = p |\Psi^-\rangle \langle \Psi^-| + \frac{1-p^2}{2} (|\Psi^+\rangle \langle \Psi^+| + |\Phi^+\rangle \langle \Phi^+|) \) in terms of probability parameter \( p \) for different values of Hawking temperature when \( \omega = 1 \). (b) The contour plot of QSK rate bound when Bob prepares a correlated bipartite state in a special class of state: \( \rho_{AB} = p |\Psi^-\rangle \langle \Psi^-| + \frac{1-p^2}{2} (|\Psi^+\rangle \langle \Psi^+| + |\Phi^+\rangle \langle \Phi^+|) \) in terms of Hawking temperature \( T \) and probability parameter \( p \) when \( \omega = 1 \).

3 Entropic uncertainty lower bound in Garfinkle-Horowitz-Strominger dilatation black hole

In this section the uncertainty game between Alice and Bob in Garfinkle-Horowitz-Strominger dilatation black hole is investigated. At the beginning of the game, Bob prepares a correlated bipartite state \( \rho_{AB} \) then sends the first part to Alice and keeps the other part as a quantum memory. In this step of the game, both of them free falling towards the black hole. In the next step, Alice remains free falling into the black hole while Bob is in a fixed position outside
the black hole. Then Alice measures one of the two observable \( Q \) or \( R \) on her state and sends her measurement choice to Bob via a classical communication channel. The main purpose of this game for Bob is to reduce his uncertainty about the result of Alice’s measurement. If Bob can correctly guess the result of Alice’s measurement in this situation, he will win this game. Due to the fact that the resident observer cannot access modes beyond the event horizon, the lost information reduces the entanglement between Alice and Bob. So, it changes the uncertainty bound. By reducing Bob’s distance from the black hole’s event horizon, his uncertainty about the result of Alice’s measurement will increase.

3.1 Examples

3.1.1 Bell diagonal state

As a first example, let us consider the case in which Alice and Bob share the set of two-qubit states with the maximally mixed marginal states. This state is defined as follows

\[
\rho_{AB} = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} w_{ij} \sigma_i \otimes \sigma_j \right)
\]  

(3.1)

where \( \sigma_i (i = 1, 2, 3) \) are the Pauli matrices. Using the singular value decomposition, the matrix \( W = \{ w_{ij} \} \) can be diagonalized by a local unitary transformation. So, eq. (3.1) can be rewritten as

\[
\rho_{AB} = \frac{1}{4} \left( I \otimes I + \sum_{i} r_i \sigma_i \otimes \sigma_i \right),
\]  

(3.2)

where \( r = (r_1, r_2, r_3) \) is limited to a tetrahedron defined by the set of vertices \((-1, -1, -1), (-1, 1, 1), (1, -1, 1) \) and \((1, 1, -1)\). We consider the case in which \( r_1 = 1 - 2p \), \( r_2 = -p \) and \( r_3 = -p \) where \( 0 \leq p \leq 1 \). So the state in eq. (3.3) can be written as

\[
\rho_{AB} = p |\Psi^-\rangle \langle \Psi^-| + \frac{1 - p}{2} \left( |\Psi^+\rangle \langle \Psi^+| + |\Phi^+\rangle \langle \Phi^+| \right),
\]  

(3.3)

where \( |\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) \) and \( |\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) \). We consider a model in which the memory particle \( B \) (At the disposal of Bob) is located near the event horizon outside the black hole, while the measured particle \( A \) (At the disposal of Alice) is free falling. The effect of Hawking radiation can be defined by applying the Unruh channel in eq. (2.9) on Bob’s state.

In figure 3a, the QSK rate bound is plotted in terms of probability parameter \( p \) for different values of Hawking temperature. As can be seen, the QSK rate bound decreases with increasing Hawking temperature. This is what we expected, the quantum correlation decreases under the influence of Hawking radiation and so the QSK rate bound decreases with increasing Hawking temperature. Figure 3b shows the contour plot of QSK rate bound in terms of Hawking temperature \( T \) and probability parameter \( p \). As can be seen, from figure 3b, at different Hawking temperatures the QSK rate bound has its highest value for the case in which \( p = 1 \) and the state is maximally entangled. It is also observed that for different values of \( p \) the QSK rate bound decreases with increasing Hawking temperature. As can be seen, from figures 3a and 3b, the QSK rate bound is negative for some values of \( p \) and \( T \). So, one can conclude that for these values of \( p \) and \( T \) the states are not good enough to support quantum key distribution protocols.
As a second example, let us consider the case in which Alice and Bob initially share a two-qubit Werner state
\[
\rho^{AB} = \frac{1}{4} I \otimes I + p|\psi^{-}\rangle\langle\Psi^-|, 
\] (3.4)
where \(0 \leq p \leq 1\).

In figure 4a, the entropic uncertainty lower bound is plotted in terms of probability parameter \(p\) for different values of Hawking temperature. As can be seen, the entropic uncertainty lower bound increases with increasing Hawking temperature. Figure 4b shows the contour plot of entropic uncertainty lower bound in terms of Hawking temperature \(T\) and probability parameter \(p\) when \(\omega = 1\).

**3.1.2 Werner state**

As a second example, let us consider the case in which Alice and Bob initially share a two-qubit Werner state
\[
\rho^{AB} = \frac{1}{4} I \otimes I + p|\psi^{-}\rangle\langle\Psi^-|, 
\] (3.4)
where \(0 \leq p \leq 1\).

In figure 4a, the entropic uncertainty lower bound is plotted in terms of probability parameter \(p\) for different values of Hawking temperature. As can be seen, the entropic uncertainty lower bound increases with increasing Hawking temperature. Figure 4b shows the contour plot of entropic uncertainty lower bound in terms of Hawking temperature \(T\) and probability parameter \(p\) when \(\omega = 1\).
Figure 6. (a) Entropic uncertainty lower bound when Bob prepares a correlated bipartite state in a special class of state: $\rho^{AB} = p|\Psi^+\rangle\langle\Psi^+| + (1-p)|11\rangle\langle11|$ in terms of probability parameter $p$ for different values of Hawking temperature when $\omega = 1$. (b) The contour plot of entropic uncertainty lower bound when Bob prepares a correlated bipartite state in a special class of state: $\rho^{AB} = p|\Psi^+\rangle\langle\Psi^+| + (1-p)|11\rangle\langle11|$ in terms of Hawking temperature $T$ and probability parameter $p$ when $\omega = 1$.

Figure 7. (a) QSK rate bound when Bob prepares a correlated bipartite state in a special class of state: $\rho^{AB} = \frac{1-p}{4}I \otimes I + p|\psi^-\rangle\langle\psi^-|$ in terms of probability parameter $p$ for different values of Hawking temperature when $\omega = 1$. (b) The contour plot of QSK rate bound when Bob prepares a correlated bipartite state in a special class of state: $\rho^{AB} = \frac{1-p}{4}I \otimes I + p|\psi^-\rangle\langle\psi^-|$ in terms of Hawking temperature $T$ and probability parameter $p$ when $\omega = 1$.

probability parameter $p$. As can be seen, from figure 4b, at different Hawking temperatures the entropic uncertainty lower bound has its lowest value for the case in which $p = 1$ and the state is maximally entangled. It is observed that for different values of $p$ this entropic uncertainty lower bound increases with increasing Hawking temperature. It is also observed that for the case in which $p = 0$, the entropic uncertainty lower bound not affected by Hawking radiation.

In figure 5a, the QSK rate bound is plotted in terms of probability parameter $p$ for different values of Hawking temperature. As can be seen, the QSK rate bound decreases with increasing Hawking temperature. This is what we expected, the quantum correlation decreases under the influence of Hawking radiation and so the QSK rate bound decreases
with increasing Hawking temperature. Figure 5b shows the contour plot of QSK rate bound in terms of Hawking temperature $T$ and probability parameter $p$. As can be seen, from figure 5b, at different Hawking temperatures the QSK rate bound has its highest value for the case in which $p = 1$ and the state is maximally entangled. It is also observed that for different values of $p$ the QSK rate bound decreases with increasing Hawking temperature. As can be seen, from figures 5a and 5b, the QSK rate bounds is negative for some values of $p$ and $T$. So, one can conclude that for these values of $p$ and $T$ the states are not good enough to support quantum key distribution protocols. It is also observed that for the case in which $p = 0$, the QSK rate bound not affected by Hawking radiation.

### 3.1.3 Two-qubit X states

As the last example, let us consider the case in which Alice and Bob share a special class of two qubit X states

$$\rho^{AB} = p|\Psi^+\rangle\langle\Psi^+| + (1 - p)|11\rangle\langle11|,$$  

where $0 \leq p \leq 1$.

In figure 6a, the entropic uncertainty lower bound is plotted in terms of probability parameter $p$ for different values of Hawking temperature. The entropic uncertainty lower bound increases with increasing Hawking temperature. Figure 6b shows the contour plot of entropic uncertainty lower bound in terms of Hawking temperature $T$ and probability parameter $p$. As can be seen, from figure 6b, at different Hawking temperatures the entropic uncertainty lower bound has its lowest value for the case in which $p = 1$ and the state is maximally entangled. It is observed that for different values of $p$ this entropic uncertainty lower bound increases with increasing Hawking temperature. It is also observed that for the case in which $p = 0$, the entropic uncertainty lower bound not affected by Hawking radiation.

In figure 7a, the QSK rate bound is plotted in terms of probability parameter $p$ for different values of Hawking temperature. As can be seen, the QSK rate bound decreases with increasing Hawking temperature. This is what we expected, the quantum correlation decreases under the influence of Hawking radiation and so the QSK rate bound decreases with increasing Hawking temperature. Figure 7b shows the contour plot of QSK rate bound in terms of Hawking temperature $T$ and probability parameter $p$. As can be seen, from figure 7b, at different Hawking temperatures the QSK rate bound has its highest value for the case in which $p = 1$ and the state is maximally entangled. It is also observed that for different values of $p$ the QSK rate bound decreases with increasing Hawking temperature. As can be seen, from figures 7a and 7b, the QSK rate bound is negative for some values of $p$ and $T$. So, one can conclude that for these values of $p$ and $T$ the states are not good enough to support quantum key distribution protocols. It is also observed that for the case in which $p = 0$, the QSK rate bound not affected by Hawking radiation.

### 4 Conclusion

In this work we studied the entropic uncertainty relation in Garfinkle-Horowitz-Strominger dilation black hole. For this purpose, we consider the uncertainty game between Alice and Bob. At first Bob prepares a correlated bipartite state $\rho_{AB}$ then he sends the first part $A$ to Alice and keeps the other part as a quantum memory $B$. In this step of the game, both of them free falling towards the black hole. In the next step, Alice remains free falling into the black hole while Bob is in a fixed position outside the black hole. Then Alice measures one of the two observable $Q$ or $R$ on her state and sends her measurement choice to Bob via
a classical communication channel. The main purpose of this game for Bob is to reduce his uncertainty about the result of Alice’s measurement. As mentioned before, quantum correlations are reduced by the effect of Hawking radiation. Therefore, due to the inverse relation between quantum correlation and uncertainty the uncertainty bound will increase as a result of Hawking effect. We also investigated the effects of the Hawking radiation on QSK rate bound. It was shown that the QSK rate bound decreases by increasing Hawking temperature.

References

[1] W. a Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik (in German), Z. Phys. 43 (1927) 172 [INSPIRE].

[2] E. Schrödinger, About Heisenberg uncertainty relation, Sitzungsber. Preuss. Akad. Wiss. Berlin 19 (1930) 296 [Bulg. J. Phys. 26 (1999) 193] [ quant-ph/9903100 ] [INSPIRE].

[3] H.P. Robertson, The uncertainty principle, Phys. Rev. 34 (1929) 163 [INSPIRE].

[4] D. Deutsch, Uncertainty in quantum measurements, Phys. Rev. Lett. 50 (1983) 631.

[5] K. Kraus, Complementary observables and uncertainty relations, Phys. Rev. D 35 (1987) 3070.

[6] H. Maassen and J.B.M. Uffink, Generalized entropic uncertainty relations, Phys. Rev. Lett. 60 (1988) 1103 [INSPIRE].

[7] M. Berta, M. Christandl, R. Colbeck, J.M. Renes and R. Renner, The uncertainty principle in the presence of quantum memory, Nature Phys. 6 (2010) 659 [arXiv:0909.0950].

[8] I. Bialynicki-Birula, Formulation of the uncertainty relations in terms of the Rényi entropies, Phys. Rev. A 74 (2006) 052101.

[9] S. Wehner and A. Winter, Entropic uncertainty relations — a survey, New J. Phys. 12 (2010) 025009.

[10] M. Berta, M. Christandl, R. Colbeck, J.M. Renes and R. Renner, The uncertainty principle in the presence of quantum memory, Nature Phys. 6 (2010) 659 [arXiv:0909.0950].

[11] A.K. Pati, M.M. Wilde, A.R.U. Devi, A.K. Rajagopal and Sudha, Quantum discord and classical correlation can tighten the uncertainty principle in the presence of quantum memory, Phys. Rev. A 86 (2012) 042105.

[12] M.A. Ballester and S. Wehner, Entropic uncertainty relations and locking: tight bounds for mutually unbiased bases, Phys. Rev. A 75 (2007) 022319.

[13] J.I. de Vicente and J. Sánchez-Ruiz, Improved bounds on entropic uncertainty relations, Phys. Rev. A 77 (2008) 042110.

[14] S. Wu, S. Yu and K. Molmer, Entropic uncertainty relation for mutually unbiased bases, Phys. Rev. A 79 (2009) 022104.

[15] L. Rudnicki, S.P. Walborn and F. Toscano, Optimal uncertainty relations for extremely coarse-grained measurements, Phys. Rev. A 85 (2012) 042115.

[16] T. Pramanik, P. Chowdhury and A.S. Majumdar, Fine-grained lower limit of entropic uncertainty in the presence of quantum memory, Phys. Rev. Lett. 110 (2013) 020402.

[17] L. Maccone and A.K. Pati, Stronger uncertainty relations for all incompatible observables, Phys. Rev. Lett. 113 (2014) 260401.

[18] P.J. Coles and M. Piani, Improved entropic uncertainty relations and information exclusion relations, Phys. Rev. A 89 (2014) 022112.

[19] S. Zozor, G.M. Bosyk and M. Portesi, General entropy-like uncertainty relations in finite dimensions, J. Phys. A 47 (2014) 495302.
[20] L. Rudnicki, Z. Puchała and K. Życzkowski, Strong majorization entropic uncertainty relations, *Phys. Rev. A* 89 (2014) 052115.

[21] S. Liu, L.-Z. Mu and H. Fan, Entropic uncertainty relations for multiple measurements, *Phys. Rev. A* 91 (2015) 042133.

[22] K. Korzekwa, M. Lostaglio, D. Jennings and T. Rudolph, Quantum and classical entropic uncertainty relations, *Phys. Rev. A* 89 (2014) 042122.

[23] L. Rudnicki, Majorization approach to entropic uncertainty relations for coarse-grained observables, *Phys. Rev. A* 91 (2015) 032123.

[24] J. Zhang, Y. Zhang and C.S. Yu, Entropic uncertainty relation and information exclusion relation for multiple measurements in the presence of quantum memory, *Sci. Rept.* 5 (2015) 11701.

[25] T. Pramanik, S. Mal and A.S. Majumdar, Lower bound of quantum uncertainty from extractable classical information, *Quantum Inf. Process.* 15 (2015) 981.

[26] W.-M. Lv et al., Experimental test of fine-grained entropic uncertainty relation in the presence of quantum memory, *Sci. Rept.* 9 (2019) 8748.

[27] M.-L. Hu and H. Fan, Upper bound and shareability of quantum discord based on entropic uncertainty relations, *Phys. Rev. A* 88 (2013) 014105.

[28] M.-L. Hu and H. Fan, Quantum-memory-assisted entropic uncertainty principle, teleportation, and entanglement witness in structured reservoirs, *Phys. Rev. A* 86 (2012) 032338.

[29] M.-L. Hu and H. Fan, Competition between quantum correlations in the quantum-memory-assisted entropic uncertainty relation, *Phys. Rev. A* 87 (2013) 022314.

[30] D. Wang et al., Quantum-memory-assisted entropic uncertainty relation in a Heisenberg XYZ chain with an inhomogeneous magnetic field, *Laser Phys. Lett.* 14 (2017) 065203.

[31] D. Wang, F. Ming, A.-J. Huang, W.-Y. Sun and L. Ye, Entropic uncertainty for spin-1/2 XXX chains in the presence of inhomogeneous magnetic fields and its steering via weak measurement reversals, *Laser Phys. Lett.* 14 (2017) 095204.

[32] D. Wang, F. Ming, A.-J. Huang, W.-Y. Sun, J.-D. Shi and L. Ye, Exploration of quantum-memory-assisted entropic uncertainty relations in a noninertial frame, *Laser Phys. Lett.* 14 (2017) 055205.

[33] A.-J. Huang, J.-D. Shi, D. Wang and L. Ye, Steering quantum-memory-assisted entropic uncertainty under unital and nonunital noises via filtering operations, *Quantum Inf. Process.* 16 (2016) 46.

[34] A.-J. Huang, D. Wang, J.-M. Wang, J.-D. Shi, W.-Y. Sun and L. Ye, Exploring entropic uncertainty relation in the Heisenberg XX model with inhomogeneous magnetic field, *Quantum Inf. Process.* 16 (2017) 204.

[35] D. Wang et al., Entropic uncertainty relations for Markovian and non-Markovian processes under a structured bosonic reservoir, *Sci. Rept.* 7 (2017) 1066.

[36] P.-F. Chen, W.-Y. Sun, F. Ming, A.-J. Huang, D. Wang and L. Ye, Observation of quantum-memory-assisted entropic uncertainty relation under open systems and its steering, *Laser Phys. Lett.* 15 (2017) 015206.

[37] M.-N. Chen, W.-Y. Sun, A.-J. Huang, F. Ming, D. Wang and L. Ye, Unveiling the decoherence effect of noise on the entropic uncertainty relation and its control by partially collapsed operations, *Laser Phys. Lett.* 15 (2017) 015207.

[38] D. Wang et al., Effects of Hawking radiation on the entropic uncertainty in a Schwarzschild space-time, *Annalen Phys.* 530 (2018) 1800080.
[39] F. Ming, D. Wang, A.-J. Huang, W.-Y. Sun and L. Ye, Decoherence effect on quantum-memory-assisted entropic uncertainty relations, *Quantum Inf. Process.* **17** (2017) 9.

[40] Y. Zhang, M. Fang, G. Kang and Q. Zhou, Controlling quantum memory-assisted entropic uncertainty in non-Markovian environments, *Quantum Inf. Process.* **17** (2018) 62.

[41] F. Ming, D. Wang, W.-N. Shi, A.-J. Huang, W.-Y. Sun and L. Ye, Entropic uncertainty relations in the Heisenberg XXZ model and its controlling via filtering operations, *Quantum Inf. Process.* **17** (2018) 89.

[42] Y.N. Guo, M.F. Fang and K. Zeng, Entropic uncertainty relation in a two-qutrit system with external magnetic field and Dzyaloshinskii-Moriya interaction under intrinsic decoherence, *Quantum Inf. Process.* **17** (2018) 187.

[43] J.-Q. Li, L. Bai and J.-Q. Liang, Entropic uncertainty relation under multiple bosonic reservoirs with filtering operator, *Quantum Inf. Process.* **17** (2018) 206.

[44] F. Ming et al., Exploring uncertainty relation and its connection with coherence under the Heisenberg spin model with the Dzyaloshinskii-Moriya interaction, *Quantum Inf. Process.* **17** (2018) 267.

[45] Y. Zhang, Q. Zhou, M. Fang, G. Kang and X. Li, Quantum-memory-assisted entropic uncertainty in two-qubit Heisenberg XYZ chain with Dzyaloshinskii-Moriya interactions and effects of intrinsic decoherence, *Quantum Inf. Process.* **17** (2018) 326.

[46] D. Wang et al., Probing entropic uncertainty relations under a two-atom system coupled with structured bosonic reservoirs, *Quantum Inf. Process.* **17** (2018) 335.

[47] P.-F. Chen, L. Ye and D. Wang, The effect of non-Markovianity on the measurement-based uncertainty, *Eur. Phys. J. D* **73** (2019) 108.

[48] W.-N. Shi, F. Ming, D. Wang and L. Ye, Entropic uncertainty relations in the spin-1 Heisenberg model, *Quantum Inf. Process.* **18** (2019) 70.

[49] Y.-Y. Yang, W.-Y. Sun, W.-N. Shi, F. Ming, D. Wang and L. Ye, Dynamical characteristic of measurement uncertainty under Heisenberg spin models with Dzyaloshinskii-Moriya interactions, *Front. Phys.* **14** (2019) 31601.

[50] M.-N. Chen, D. Wang and L. Ye, Characterization of dynamical measurement’s uncertainty in a two-qubit system coupled with bosonic reservoirs, *Phys. Lett. A* **383** (2019) 977.

[51] F. Ming, D. Wang and L. Ye, Dynamical measurement’s uncertainty in the curved space-time, *Annalen Phys.* **531** (2019) 1900014.

[52] Y.-B. Yao, D. Wang, F. Ming and L. Ye, Dynamics of the measurement uncertainty in an open system and its controlling, *J. Phys. B* **53** (2020) 035501.

[53] Z.-Y. Ding, H. Yang, H. Yuan, D. Wang, J. Yang and L. Ye, Experimental investigation of linear-entropy-based uncertainty relations in all-optical systems, *Phys. Rev. A* **101** (2020) 022116.

[54] H. Yang et al., Experimental certification of the steering criterion based on a general entropic uncertainty relation, *Phys. Rev. A* **101** (2020) 022324.

[55] Z.-Y. Ding, H. Yang, D. Wang, H. Yuan, J. Yang and L. Ye, Experimental investigation of entropic uncertainty relations and coherence uncertainty relations, *Phys. Rev. A* **101** (2020) 032101.

[56] M.R. Pourkarimi, Quantum correlations and entropic uncertainty relation in a three-qubit spin chain under the effect of magnetic field and DM interaction, *Int. J. Quantum Inform.* **16** (2018) 1850057.
[57] M.R. Pourkarimi, Time evolution of quantum-memory-assisted entropic uncertainty relation and quantum correlations under dissipative environment, *Int. J. Quantum Inform.* 17 (2019) 1950008.

[58] Z. Huang, Quantum-memory-assisted entropic uncertainty in spin models with Dzyaloshinskii-Moriya interaction, *Laser Phys. Lett.* 15 (2018) 025203.

[59] B.-L. Fang, J. Shi and T. Wu, Quantum-memory-assisted entropic uncertainty relation and quantum coherence in structured reservoir, *Int. J. Theor. Phys.* 59 (2020) 763.

[60] S. Haddadi, M.R. Pourkarimi, A. Akhound and M. Ghominejad, Quantum correlations and quantum-memory-assisted entropic uncertainty relation in two kinds of spin squeezing models, *Laser Phys. Lett.* 16 (2019) 095202.

[61] M.R. Pourkarimi and S. Haddadi, Quantum-memory-assisted entropic uncertainty, teleportation, and quantum discord under decohering environments, *Laser Phys. Lett.* 17 (2020) 025206.

[62] F. Adabi, S. Salimi and S. Haseli, Tightening the entropic uncertainty bound in the presence of quantum memory, *Phys. Rev. A* 93 (2016) 062123.

[63] F. Adabi, S. Haseli and S. Salimi, Reducing the entropic uncertainty lower bound in the presence of quantum memory via LOCC, *EPL (Europhys. Lett.)* 115 (2016) 60004.

[64] H. Dolatkhah, S. Haseli, S. Salimi and A.S. Khorashad, Tightening the entropic uncertainty relations for multiple measurements and applying it to quantum coherence, *Quantum Inf. Process.* 18 (2018) 13.

[65] S. Haseli, H. Dolatkhah, S. Salimi and A.S. Khorashad, Controlling the entropic uncertainty lower bound in two-qubit systems under decoherence, *Laser Phys. Lett.* 16 (2019) 045207.

[66] S. Haseli, H. Dolatkhah, H.R. Jahromi, S. Salimi and A. Khorashad, The lower bound of quantum memory-assisted entropic uncertainty and secret rate for two topological qubits under environments, *Opt. Commun.* 461 (2020) 125287.

[67] M.H. Partovi, Entanglement detection using majorization uncertainty bounds, *Phys. Rev. A* 86 (2012) 022309.

[68] Y. Huang, Entanglement criteria via concave-function uncertainty relations, *Phys. Rev. A* 82 (2010) 012335.

[69] R. Prevedel, D.R. Hamel, R. Colbeck, K. Fisher and K.J. Resch, Experimental investigation of the uncertainty principle in the presence of quantum memory and its application to witnessing entanglement, *Nature Phys.* 7 (2011) 757.

[70] C.-F. Li, J.-S. Xu, X.-Y. Xu, K. Li and G.-C. Guo, Experimental investigation of the entanglement-assisted entropic uncertainty principle, *Nature Phys.* 7 (2011) 752.

[71] M. Tomamichel, C.C.W. Lim, N. Gisin and R. Renner, Tight finite-key analysis for quantum cryptography, *Nature Commun.* 3 (2012) 634.

[72] N.H.Y. Ng, M. Berta and S. Wehner, Min-entropy uncertainty relation for finite-size cryptography, *Phys. Rev. A* 86 (2012) 042315.

[73] A.K. Ekert, Quantum cryptography based on Bell’s theorem, *Phys. Rev. Lett.* 67 (1991) 661 [inSPIRE].

[74] J.M. Renes and J.-C. Boileau, Conjectured strong complementary information tradeoff, *Phys. Rev. Lett.* 103 (2009) 020402.

[75] I. Devetak and A. Winter, Distillation of secret key and entanglement from quantum states, *Proc. Roy. Soc. A* 461 (2005) 207.

[76] J. Feng, Y.-Z. Zhang, M.D. Gould and H. Fan, Entropic uncertainty relations under the relativistic motion, *Phys. Lett. B* 726 (2013) 527 [arXiv:1309.7443] [inSPIRE].
[77] J. Feng, Y.-Z. Zhang, M.D. Gould and H. Fan, *Uncertainty relation in Schwarzschild spacetime*, Phys. Lett. B 743 (2015) 198 [arXiv:1501.01700] [INSPIRE].

[78] J.-L. Huang, W.-C. Gan, Y. Xiao, F.-W. Shu and M.-H. Yung, *Holevo bound of entropic uncertainty in Schwarzschild spacetime*, Eur. Phys. J. C 78 (2018) 545 [arXiv:1712.04287] [INSPIRE].

[79] L. Jia, Z. Tian and J. Jing, *Entropic uncertainty relation in de Sitter space*, Ann. Phys. 353 (2015) 37.

[80] I. Fuentes-Schuller and R.B. Mann, *Alice falls into a black hole: entanglement in non-inertial frames*, Phys. Rev. Lett. 95 (2005) 120404 [quant-ph/0410172] [INSPIRE].

[81] Q. Pan and J. Jing, *Hawking radiation, entanglement and teleportation in background of an asymptotically flat static black hole*, Phys. Rev. D 78 (2008) 065015 [arXiv:0809.0811] [INSPIRE].

[82] J. Wang, Q. Pan and J. Jing, *Entanglement redistribution in the Schwarzschild spacetime*, Phys. Lett. B 692 (2010) 202 [arXiv:1007.3331] [INSPIRE].

[83] E. Martin-Martinez, L.J. Garay and J. Leon, *Unveiling quantum entanglement degradation near a Schwarzschild black hole*, Phys. Rev. D 82 (2010) 064006 [arXiv:1006.1394] [INSPIRE].

[84] E. Martin-Martinez and J. Leon, *Quantum correlations through event horizons: fermionic versus bosonic entanglement*, Phys. Rev. A 81 (2010) 032320 [arXiv:1001.4302] [INSPIRE].

[85] G.W. Gibbons and S.W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev. D 15 (1977) 2738 [INSPIRE].

[86] S.W. Hawking, *Particle creation by black holes*, Commun. Math. Phys. 43 (1975) 199 [Erratum ibid. 46 (1976) 206] [INSPIRE].

[87] T. Damour and R. Ruffini, *Black hole evaporation in the Klein-Sauter-Heisenberg-Euler formalism*, Phys. Rev. D 14 (1976) 332 [INSPIRE].

[88] J. Wang, Q. Pan and J. Jing, *Projective measurements and generation of entangled Dirac particles in Schwarzschild spacetime*, Annals Phys. 325 (2010) 1190 [arXiv:0905.3430] [INSPIRE].

[89] E. Martin-Martinez, L.J. Garay and J. Leon, *Unveiling quantum entanglement degradation near a Schwarzschild black hole*, Phys. Rev. D 82 (2010) 064006 [arXiv:1006.1394] [INSPIRE].

[90] D.E. Bruschi, J. Louko, E. Martin-Martinez, A. Dragan and I. Fuentes, *The Unruh effect in quantum information beyond the single-mode approximation*, Phys. Rev. A 82 (2010) 042332 [arXiv:1007.4670] [INSPIRE].