GENERALIZED FRACTIONAL COUNTING PROCESS

KULDEEP KUMAR KATARIA AND MOSTAFIZAR KHANDAKAR

Abstract. In this paper, we obtain additional results for a fractional counting process introduced and studied by Di Crescenzo et al. (2016). For convenience, we call it the generalized fractional counting process (GFCP). It is shown that the one-dimensional distributions of the GFCP are not infinitely divisible. Its covariance structure is studied using which its long-range dependence property is established. It is shown that the increments of GFCP exhibits the short-range dependence property. Also, we prove that the GFCP is a scaling limit of some continuous time random walk. A particular case of the GFCP, namely, the generalized counting process (GCP) is discussed for which we obtain a limiting result, a martingale result and establish a recurrence relation for its probability mass function. We have shown that many known counting processes such as the Poisson process of order $k$, the Pólya-Aeppli process of order $k$, the negative binomial process and their fractional versions etc. are other special cases of the GFCP. An application of the GCP to risk theory is discussed.

1. Introduction

The time fractional Poisson process (TFPP) is a renewal process with heavy-tailed distributed waiting times (see Laskin (2003), Beghin and Orsingher (2009)). Biard and Saussereau (2014) showed that its increment process exhibits the long-range dependence (LRD) property and discussed its applications in risk theory. The processes that exhibit the LRD property has applications in several areas, for example, finance (see Ding et al. (1993)), hydrology (see Doukhan et al. (2003), pp. 461-472), internet data traffic modeling (see Karagiannis et al. (2004)) etc. For other fractional versions of the Poisson process we refer the reader to Beghin (2012), Orsingher and Polito (2012), Kataria and Vellaisamy (2017) etc. and references therein.

Let $\{M^\alpha(t)\}_{t \geq 0}$, $0 < \alpha \leq 1$ be a fractional counting process which performs $k$ kinds of jumps of amplitude $1, 2, \ldots, k$ with positive rates $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively, where $k \geq 1$ is a fixed integer and whose state probabilities $p^\alpha(n, t) = \Pr\{M^\alpha(t) = n\}$ satisfy

$$\frac{d^\alpha}{dt^\alpha} p^\alpha(n, t) = -\Lambda p^\alpha(n, t) + \sum_{j=1}^{\min\{n,k\}} \lambda_j p^\alpha(n-j, t), \quad n \geq 0, \quad (1.1)$$

with

$$p^\alpha(n, 0) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases}$$
Here, $\Lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ and $\frac{d^\alpha}{dt^\alpha}$ is the Caputo fractional derivative which is defined as

$$
\frac{d^\alpha}{dt^\alpha}f(t) := \begin{cases} 
\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) \, ds, & 0 < \alpha < 1, \\
f'(t), & \alpha = 1.
\end{cases}
$$

The process $\{M^\alpha(t)\}_{t \geq 0}$ is introduced and studied by Di Crescenzo et al. (2016). Throughout this paper, we call it the generalized fractional counting process (GFCP). Its probability mass function (pmf) is given by (see Di Crescenzo et al. (2016))

$$
p^\alpha(n,t) = \sum_{r=0}^{n} \sum_{i_1+i_2+\cdots+i_k=r} \frac{\lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_k^{i_k} t^{r\alpha}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) \, ds, \quad n \geq 0, \quad (1.2)
$$

where $i_1, i_2, \ldots, i_k$ are non-negative integers and $E_{\alpha,\alpha+1}^{r+1}(\cdot)$ is the three-parameter Mittag-Leffler function defined in (2.1). Its mean and variance are given by

$$
\mathbb{E}(M^\alpha(t)) = St^\alpha, \quad \text{Var}(M^\alpha(t)) = R t^{2\alpha} + T t^\alpha, \quad (1.3)
$$

where

$$
S = \sum_{j=1}^k j \lambda_j \frac{1}{\Gamma(\alpha+1)}, \quad R = \left( \frac{2}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma^2(\alpha+1)} \right) \left( \sum_{j=1}^k j \lambda_j \right)^2, \quad T = \sum_{j=1}^k j^2 \lambda_j \frac{1}{\Gamma(\alpha+1)}.
$$

For $k = 1$, the GFCP reduces to TFPP. A limiting case of the GFCP, namely, the convoluted fractional Poisson process (CFPP) which is obtained by taking suitable $\lambda_j$’s and letting $k \to \infty$ is studied by Kataria and Khandakar (2021). For $\alpha = 1$, the GFCP reduces to a special case, namely, the generalized counting process (GCP). We discuss the GCP in detail later in the paper.

Di Crescenzo et al. (2016) showed that

$$
M^\alpha(t) \overset{d}{=} \sum_{i=1}^{N^\alpha(t)} X_i, \quad t \geq 0, \quad (1.4)
$$

where $\overset{d}{=}$ denotes equal in distribution. Here, $\{N^\alpha(t)\}_{t \geq 0}$ is the TFPP with intensity parameter $\Lambda$ which is independent of the sequence of independent and identically distributed (iid) random variables $\{X_i\}_{i \geq 1}$ such that

$$
\Pr\{X_1 = j\} = \frac{\lambda_j}{\Lambda}, \quad j = 1, 2, \ldots, k. \quad (1.5)
$$

It is also known that

$$
M^\alpha(t) \overset{d}{=} M(Y^\alpha(t)), \quad (1.6)
$$

where the inverse $\alpha$-stable subordinator $\{Y^\alpha(t)\}_{t \geq 0}$ is independent of the GCP $\{M(t)\}_{t \geq 0}$.

In this paper, we study some additional results for the GFCP and for its special case, the GCP. In Section 2 some known results on the Mittag-Leffler function and the inverse $\alpha$-stable subordinator are provided. In Section 3 we obtain the characteristic function and the Lévy measure of GCP. It is shown that the process $\{M(t) - \sum_{j=1}^k j \lambda_j t\}_{t \geq 0}$ is a
martingale with respect to a suitable filtration. The following recurrence relation for the
probability mass function (pmf) $p(n, t) = \Pr\{M(t) = n\}$ of GCP is obtained:
\[
p(n, t) = \frac{t}{n} \min(n, k) \sum_{j=1}^{\min(n, k)} j \lambda_j p(n - j, t), \quad n \geq 1.
\]
Also, we have shown that
\[
\lim_{t \to \infty} \frac{M(t)}{t} = \sum_{j=1}^{k} j \lambda_j, \quad \text{in probability.}
\]
The above limiting result is used to show that the one-dimensional distributions of GFCP
are not infinitely divisible. The explicit expressions for the probability generating function
(pgf) and the $r$th factorial moment of GFCP are obtained. Its LRD property is established
by utilizing its covariance. Also, it is shown that the increments of GFCP has the short-
range dependence (SRD) property. We discuss a continuous time random walk (CTRW)
whose scaling limit is the GFCP.

In Section 4, it is shown that some known counting processes such as the Poisson process
of order $k$, the Pólya-Aeppli process of order $k$, the negative binomial process and their
fractional versions etc. are special cases of the GFCP. Some results for these particular as
well as limiting cases are obtained.

In Section 3 we considered a risk model in which the GCP is used to model the number
of claims received. The governing differential equation for the joint probability of the time
to ruin and the deficit at the time of ruin is derived for the introduced risk model. The
closed form expression for its ruin probability with no initial capital is obtained.

2. Preliminaries

Here, we provide some known results related to Mittag-Leffler function and inverse $\alpha$-
stable subordinator. These results will be required later.

2.1. Mittag-Leffler function. The three-parameter Mittag-Leffler function is defined as
\[
E_{\delta, \beta, \gamma}(x) := \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{\Gamma(\delta + k) x^k}{k! \Gamma(k \beta + \gamma)}, \quad x \in \mathbb{R},
\]
where $\beta > 0$, $\gamma > 0$ and $\delta > 0$. It reduces to the two-parameter and the one-parameter
Mittag-Leffler function for $\delta = 1$ and $\delta = \gamma = 1$, respectively. The following holds true (see
Kilbas et al. (2006), Eq. (1.8.22)):
\[
E_{\beta, \gamma}^{(n)}(x) = n! E_{\beta, \beta+\gamma}^{n+1}(x), \quad n \geq 0.
\]
where $E_{\beta, \gamma}^{(n)}(\cdot)$ denotes the $n$th derivative of two-parameter Mittag-Leffler function.

2.2. Inverse $\alpha$-stable subordinator. A $\alpha$-stable subordinator $\{D_\alpha(t)\}_{t \geq 0}, 0 < \alpha < 1$
is a non-decreasing Lévy process. Its Laplace transform is given by $\mathbb{E}\left(e^{-sD_\alpha(t)}\right) = e^{-ts^\alpha}$,
$s > 0$. Its first passage time $\{Y_\alpha(t)\}_{t \geq 0}$ is called the inverse $\alpha$-stable subordinator and it is defined as
\[
Y_\alpha(t) = \inf\{x > 0 : D_\alpha(x) > t\}.
\]
The mean of \( Y_\alpha(t) \) is given by (see Leonenko et al. (2014))

\[
\mathbb{E}(Y_\alpha(t)) = \frac{t^\alpha}{\Gamma(\alpha + 1)}.
\] (2.3)

Let \( B(\alpha, \alpha + 1) \) and \( B(\alpha, \alpha + 1; s/t) \) denote the beta function and the incomplete beta function, respectively. It is known that (see Leonenko et al. (2014), Eq. (10))

\[
\mathrm{Cov}(Y_\alpha(s), Y_\alpha(t)) = \frac{1}{\Gamma^2(\alpha + 1)} \left( \alpha s^{2\alpha} B(\alpha, \alpha + 1) + F(\alpha; s, t) \right),
\] (2.4)

where \( 0 < s \leq t \) and \( F(\alpha; s, t) = \alpha t^{2\alpha} B(\alpha, \alpha + 1; s/t) - (ts)^\alpha \). On using the following asymptotic result (see Maheshwari and Vellaisamy (2016), Eq. (8)):

\[
F(\alpha; s, t) \sim -\alpha^2 \frac{s^{\alpha+1}}{(\alpha + 1) t^{1-\alpha}}, \quad \text{as } t \to \infty,
\] in (2.4), we get the following result for fixed \( s \) and large \( t \):

\[
\mathrm{Cov}(Y_\alpha(s), Y_\alpha(t)) \sim \frac{1}{\Gamma^2(\alpha + 1)} \left( \alpha s^{2\alpha} B(\alpha, \alpha + 1) - \frac{\alpha^2}{(\alpha + 1) t^{1-\alpha}} \right).
\] (2.5)

3. Generalized fractional counting process

In this section, we obtain some additional results for the GFCP and its special case, the GCP.

Di Crescenzo et al. (2016) showed that the GFCP \( \{M_\alpha(t)\}_{t \geq 0} \) is not a Lévy process. However, it can be seen from (1.4) that for the case \( \alpha = 1 \), i.e., the GCP \( \{M(t)\}_{t \geq 0} \) is equal in distribution to a compound Poisson process which is a Lévy process. Thus, the GCP is a Lévy process and its characteristic function is given by

\[
\mathbb{E} \left( e^{i\xi M(t)} \right) = \mathbb{E} \left( e^{i\xi \sum_{i=1}^{N(t)} X_i} \right), \quad \xi \in \mathbb{R}
\]

\[
= \exp \left( -\mathbb{E} (N(t)) \left( 1 - \mathbb{E} (e^{i\xi X_1}) \right) \right)
\]

\[
= \exp \left( -t \sum_{j=1}^{k} (1 - e^{i\xi j}) \lambda_j \right),
\] (3.1)

where the last step follows from (1.5). Also, its Lévy measure is given by

\[
\Pi(dx) = \sum_{j=1}^{k} \lambda_j \delta_j dx,
\] (3.2)

where \( \delta_j \)'s are Dirac measures.

The pgf \( G^\alpha(u, t) = \mathbb{E} (u^{M_\alpha(t)}) \) of GFCP is given by

\[
G^\alpha(u, t) = E_{\alpha, 1} \left( \sum_{j=1}^{k} \lambda_j (u^j - 1)t^\alpha \right), \quad |u| \leq 1,
\] (3.3)

whose proof follows similar lines to that of Proposition 2.1, Di Crescenzo et al. (2016).

Next, we obtain a recurrence relation for the state probabilities of GCP.
Proposition 3.1. The state probabilities $p(n,t)$ of GCP satisfy

$$p(n,t) = \frac{t}{n} \sum_{j=1}^{\min(n,k)} j \lambda_j p(n - j, t), \quad n \geq 1.$$ 

Proof. From the definition of pgf, we get

$$\frac{d}{du} G(u,t) = \sum_{i=0}^{\infty} (i+1)p(i+1, t)u^i.$$ 

On substituting $\alpha = 1$ in (3.3) and taking derivative, we get

$$\frac{d}{du} G(u,t) = t \sum_{j=1}^{k} j \lambda_j u^{j-1} G(u,t).$$

On equating the above two equations, we get

$$\sum_{i=0}^{\infty} (i+1)p(i+1, t)u^i = t \sum_{j=1}^{k} j \lambda_j u^{j-1} \sum_{i=0}^{\infty} p(i, t)u^i$$

$$= t \sum_{j=1}^{k} j \lambda_j \sum_{i=j-1}^{\infty} p(i - j + 1, t)u^i$$

$$= t \sum_{j=1}^{k} j \lambda_j \left( \sum_{i=j-1}^{k-2} p(i - j + 1, t)u^i + \sum_{i=k-1}^{\infty} p(i - j + 1, t)u^i \right)$$

$$= t \sum_{i=0}^{k+1} j \lambda_j p(i - j + 1, t)u^i + t \sum_{i=k-1}^{\infty} \sum_{j=1}^{k} j \lambda_j p(i - j + 1, t)u^i.$$ 

On equating the coefficients of $u^i$ for $0 \leq i \leq k-2$, we get

$$(i+1)p(i+1, t) = t \sum_{j=1}^{i+1} j \lambda_j p(i - j + 1, t)$$

which reduces to

$$p(n,t) = \frac{t}{n} \sum_{j=1}^{n} j \lambda_j p(n - j, t), \quad 1 \leq n \leq k - 1.$$ (3.4)

Again on equating the coefficients of $u^i$ for $i \geq k - 1$, we get

$$(i+1)p(i+1, t) = t \sum_{j=1}^{k} j \lambda_j p(i - j + 1, t)$$

which reduces to

$$p(n,t) = \frac{t}{n} \sum_{j=1}^{k} j \lambda_j p(n - j, t), \quad n \geq k.$$ (3.5)

Finally, the result follows on combining (3.4) and (3.5). □

Proposition 3.2. The process $\{M(t) - \sum_{j=1}^{k} j \lambda_j t\}_{t \geq 0}$ is a martingale with respect to natural filtration $\mathcal{F}_t = \sigma(M(s), s \leq t)$.
Proof. Let \( Q(t) = M(t) - \sum_{j=1}^{k} j \lambda_j t \). Note that \( M(t) \) has independent increments as it’s a Lévy process. Hence, for \( s \leq t \), we have

\[
\mathbb{E} \left( Q(t) - Q(s) \mid \mathcal{F}_s \right) = \mathbb{E} \left( M(t) - M(s) \mid \mathcal{F}_s \right) - \sum_{j=1}^{k} j \lambda_j (t - s) = 0.
\]

This completes the proof. \( \square \)

**Lemma 3.1.** The following limiting result holds for GCP:

\[
\lim_{t \to \infty} \frac{M(t)}{t} = \sum_{j=1}^{k} j \lambda_j, \quad \text{in probability.} \tag{3.6}
\]

**Proof.** On substituting \( \alpha = 1 \) in (3.3), we get the pgf of GCP as

\[
G(u, t) = \prod_{j=1}^{k} e^{t \lambda_j (u^j - 1)}. \tag{3.7}
\]

Thus, the GCP is equal in distribution to the following weighted sum of \( k \) independent Poisson process:

\[
M(t) = \sum_{j=1}^{k} j N_j(t).
\]

The weighted Poisson process \( N_1(t) + 2N_2(t) + \cdots + kN_k(t) \) is studied by Zuo et al. (2021). Here, \( \{N_j(t)\}_{t \geq 0} \) is a Poisson process with intensity \( \lambda_j \). Thus,

\[
\lim_{t \to \infty} \frac{M(t)}{t} = \sum_{j=1}^{k} j \lim_{t \to \infty} \frac{N_j(t)}{t}
\]

\[
= \sum_{j=1}^{k} j \lambda_j, \quad \text{in probability},
\]

where we have used \( \lim_{t \to \infty} N_j(t)/t = \lambda_j \) almost surely. This completes the proof. \( \square \)

**Remark 3.1.** Kataria and Khandakar (2021) studied a limiting case of the GCP, namely, the convoluted Poisson process (CPP). It is denoted by \( \{N_c(t)\}_{t \geq 0} \). Let \( \{\beta_j\}_{j \in \mathbb{Z}} \) be a sequence of intensity parameters such that \( \beta_j = 0 \) for all \( j < 0 \) and \( \beta_j > \beta_{j+1} > 0 \) for all \( j \geq 0 \) with \( \lim_{j \to \infty} \beta_{j+1}/\beta_j < 1 \). On taking \( \lambda_j = \beta_{j-1} - \beta_j, \ j \geq 1 \) and letting \( k \to \infty \) in (1.1), with \( \alpha = 1 \), the GCP reduces to the CPP. Thus, from (3.6), the following holds for the CPP:

\[
\lim_{t \to \infty} \frac{N_c(t)}{t} = \sum_{j=0}^{\infty} \beta_j, \quad \text{in probability.}
\]

**Remark 3.2.** From (3.7), we note that the GCP can be represented as a sum of \( k \) independent compound Poisson processes \( \{C_j(t)\}_{t \geq 0}, \ j = 1, 2, \ldots, k \) where

\[
C_j(t) = \sum_{i=1}^{N_j(t)} X_i.
\]

Here, \( X_i = j \) with probability 1 and \( \{N_j(t)\}_{t \geq 0} \) is the Poisson process with intensity \( \lambda_j \).
Proposition 3.3. The one-dimensional distributions of GFCP are not infinitely divisible.

Proof. On using the self-similarity property of inverse $\alpha$-stable subordinator $\{ Y_\alpha(t) \}_{t \geq 0}$ in (1.6), we get

$$ M^\alpha(t) \overset{d}{=} M(t^\alpha Y_\alpha(1)). $$

Thus,

$$ \lim_{t \to \infty} \frac{M^\alpha(t)}{t^\alpha} = \lim_{t \to \infty} \frac{M(t^\alpha Y_\alpha(1))}{t^\alpha} = Y_\alpha(1) \sum_{j=1}^{k} j \lambda_j, $$

where we have used Lemma 3.1 in the last step. Now, let us assume that $M^\alpha(t)$ is infinitely divisible. Thus, $M^\alpha(t)/t^\alpha$ is infinitely divisible. It follows that $Y_\alpha(1)$ is infinitely divisible as $\lim_{t \to \infty} M^\alpha(t)/t^\alpha$ is infinitely divisible which follows by using a result on p. 94 of Steutel and van Harn (2004). This leads to a contradiction as $Y_\alpha(1)$ is not infinitely divisible (see Vellaisamy and Kumar (2018)). \qed

Next we obtain the factorial moments of GFCP by using its pgf.

Proposition 3.4. The $r$th factorial moment of GFCP, that is, $\psi^\alpha(r,t) = \mathbb{E}(M^\alpha(t)(M^\alpha(t) - 1) \ldots (M^\alpha(t) - r + 1))$, $r \geq 1$, is given by

$$ \psi^\alpha(r,t) = r! \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{n!} \sum_{m_1, \ldots, m_r \in \mathbb{N}} \left( \frac{1}{m_1! \ldots m_r!} \sum_{j=1}^{k} (j)_{m_j} \lambda_j \right), $$

where $(j)_{m_j} = j(j-1) \ldots (j-m_j+1)$ denotes the falling factorial.

Proof. On using the $r$th derivative of composition of two functions (see Johnson (2002), Eq. (3.3)) in (3.3), we get

$$ \psi^\alpha(r,t) = \frac{\partial^r G^\alpha(u,t)}{\partial u^r} \bigg|_{u=1} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( t^\alpha \sum_{j=1}^{k} \lambda_j (u^j - 1) \right) B_{r,n} \left( t^\alpha \sum_{j=1}^{k} \lambda_j (u^j - 1) \right) \bigg|_{u=1}, $$

where

$$ B_{r,n} \left( t^\alpha \sum_{j=1}^{k} \lambda_j (u^j - 1) \right) \bigg|_{u=1} = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \left( -t^\alpha \sum_{j=1}^{k} \lambda_j (u^j - 1) \right)^{n-m} \frac{d^r}{du^r} \left( t^\alpha \sum_{j=1}^{k} \lambda_j (u^j - 1) \right)^{m} \bigg|_{u=1} = t^\alpha \frac{d^r}{du^r} \left( \sum_{j=1}^{k} \lambda_j (u^j - 1) \right)^n \bigg|_{u=1}. $$

(3.10)
From (2.2), we get
\[ E_{\alpha,1}^{(n)} \left( t^\alpha \sum_{j=1}^{k} \lambda_j (u^j - 1) \right) \bigg|_{u=1} = n! E_{\alpha,n\alpha+1}^{(n+1)} \left( t^\alpha \sum_{j=1}^{k} \lambda_j (u^j - 1) \right) \bigg|_{u=1} = \frac{n!}{\Gamma(n\alpha + 1)}. \quad (3.11) \]

Now, by using the following result (see Johnson (2002), Eq. (3.6))
\[ \frac{d^r}{dw^r} \left( g(w) \right)^n = \sum_{m_1+m_2+\ldots+m_n=r} \frac{r!}{m_1!m_2!\ldots m_n!} g^{(m_1)}(w)g^{(m_2)}(w) \ldots g^{(m_n)}(w), \]
we get
\[ \frac{d^r}{du^r} \left( \sum_{j=1}^{k} \lambda_j (u^j - 1) \right)^n \bigg|_{u=1} = r! \sum_{\substack{\lambda \in N \\lambda_{\ell} \neq 0}} \prod_{\ell=1}^{n} \frac{1}{m_{\ell}!} \frac{d^{m_{\ell}}}{du^{m_{\ell}}} \left( \sum_{j=1}^{k} \lambda_j (u^j - 1) \right) \bigg|_{u=1} = r! \sum_{\substack{\lambda \in N \\lambda_{\ell} \neq 0}} \prod_{\ell=1}^{n} \frac{1}{m_{\ell}!} \sum_{j=1}^{k} \lambda_j. \quad (3.12) \]

The right hand side of (3.12) vanishes for \( n = 0 \). The proof follows on substituting (3.10) and (3.11) in (3.9) and then using (3.12). \( \square \)

**Remark 3.3.** On substituting \( k = 1 \) in (3.8), we get
\[ \psi^{\alpha}(r, t) = r! \sum_{n=1}^{r} \frac{t^{\alpha n}}{\Gamma(n\alpha + 1)} \sum_{\substack{\lambda \in N \\lambda_{\ell} \neq 0}} \prod_{\ell=1}^{n} \frac{1}{m_{\ell}!} \left( 1 \right)_{m_{\ell}} = r! \frac{(\lambda_1 t^{\alpha})}{\Gamma(r\alpha + 1)}, \]
which is the \( r \)th factorial moment of TFPP (see Beghin and Orsingher (2009), Eq. (2.9)).

**Remark 3.4.** Di Crescenzo et al. (2016) give an expression for the \( r \)th moment \( \mu^{r}(r, t) = \mathbb{E} \left( (M^{\alpha}(t))^{r} \right) \), \( r \geq 1 \) of GFCP. Here, we give an alternate expression for it as follows:
\[ \mu^{r}(r, t) = r! \sum_{n=1}^{r} \frac{t^{\alpha n}}{\Gamma(n\alpha + 1)} \sum_{\substack{\lambda \in N \\lambda_{\ell} \neq 0}} \prod_{\ell=1}^{n} \frac{1}{m_{\ell}!} \left( \sum_{j=1}^{k} \lambda_j \right), \]
whose proof follows along the similar lines to that of Proposition 3.4.

### 3.1. Dependence structure of the GFCP and its increments

Here, we show that the GFCP has LRD property whereas its increments exhibits the SRD property.

The LRD and SRD properties for a non-stationary stochastic process \( \{X(t)\}_{t \geq 0} \) are defined as follows (see Maheshwari and Vellaisamy (2016)):

**Definition 3.1.** Let \( s > 0 \) be fixed and \( \{X(t)\}_{t \geq 0} \) be a stochastic process whose correlation function satisfies
\[ \text{Corr}(X(s), X(t)) \sim c(s)t^{-\theta}, \quad \text{as } t \to \infty, \]
for some \( c(s) > 0 \). The process \( \{X(t)\}_{t \geq 0} \) is said to exhibit the LRD property if \( \theta \in (0, 1) \) and the SRD property if \( \theta \in (1, 2) \).
We use Theorem 2.1 of Leonenko et al. (2014) to obtain the covariance of GFPC as follows: Let $0 < s \leq t$. Then,

$$\text{Cov} (M^\alpha(s), M^\alpha(t)) = \text{Var} (M(1)) E (Y_\alpha(s)) + (E(M(1)))^2 \text{Cov} (Y_\alpha(s), Y_\alpha(t))$$

$$= Ts^\alpha + \left( \sum_{j=1}^{k} j \lambda_j \right)^2 \text{Cov} (Y_\alpha(s), Y_\alpha(t)), \quad (3.13)$$

where we have used (1.3) with $\alpha = 1$ and (2.3) in the last step.

Now, on using (2.5) in (3.13), we obtain

$$\text{Cov} (M^\alpha(s), M^\alpha(t)) \sim Ts^\alpha + S^2 \left( \alpha s^{2\alpha} B(\alpha, \alpha + 1) - \frac{\alpha^2}{(\alpha + 1)^{1-\alpha}} s^{\alpha+1} \right) \text{ as } t \to \infty. \quad (3.14)$$

**Remark 3.5.** The mean and variance of the GFPC are obtained by Di Crescenzo et al. (2014). Alternatively, these can be obtained from Theorem 2.1 of Leonenko et al. (2014).

**Theorem 3.1.** The GFPC exhibits the LRD property.

**Proof.** Using (1.3) and (3.14), we get the following for fixed $s > 0$ and large $t$:

$$\text{Corr} (M^\alpha(s), M^\alpha(t)) \sim \frac{Ts^\alpha + S^2 \left( \alpha s^{2\alpha} B(\alpha, \alpha + 1) - \frac{\alpha^2}{(\alpha + 1)^{1-\alpha}} \right)}{\sqrt{\text{Var} (M^\alpha(s)) \sqrt{T^2 t^{2\alpha} + T t^\alpha}}} \sim c_0(s) t^{-\alpha},$$

where

$$c_0(s) = \frac{\Gamma(2\alpha + 1)Ts^\alpha + \left( \sum_{j=1}^{k} j \lambda_j \right)^2 s^{2\alpha}}{\Gamma(2\alpha + 1)\sqrt{\text{Var} (M^\alpha(s)) R}}.$$  

As $0 < \alpha < 1$, the result follows. \hfill \square

Similarly, it can be shown that the GCP exhibits the LRD property.

For a fixed $h > 0$, the increment process of GFPC is defined as

$$Z_h^\alpha(t) := M^\alpha(t + h) - M^\alpha(t), \quad t \geq 0.$$

**Theorem 3.2.** The increment process $\{Z_h^\alpha(t)\}_{t \geq 0}$ has the SRD property.

**Proof.** The proof follows similar lines to that of Theorem 1, Maheshwari and Vellaisamy (2016) and Theorem 5.5, Kataria and Khandakar (2021). For the sake of completeness, we give a brief outline of the proof.

Let $s > 0$ be fixed such that $0 < s + h \leq t$. Then,

$$\text{Cov}(Z_h^\alpha(s), Z_h^\alpha(t)) = \text{Cov}(M^\alpha(s + h), M^\alpha(t + h)) + \text{Cov}(M^\alpha(s), M^\alpha(t))$$

$$- \text{Cov}(M^\alpha(s + h), M^\alpha(t)) - \text{Cov}(M^\alpha(s), M^\alpha(t + h)). \quad (3.15)$$

From (3.14) and (3.15), we get the following for large $t$:

$$\text{Cov}(Z_h^\alpha(s), Z_h^\alpha(t)) \sim \frac{\alpha^2 h(1 - \alpha)}{\alpha + 1} (s + h)^{\alpha+1} S^2 t^{\alpha-2}. \quad (3.16)$$

On using (2.4) in (3.13), we get

$$\text{Cov}(M^\alpha(t), M^\alpha(t + h)) = T t^\alpha + S^2 \left( \alpha t^{2\alpha} B(\alpha, \alpha + 1) + F(\alpha; t, t + h) \right), \quad (3.17)$$

where $F(\alpha; t, t + h) = \alpha(t + h)^{2\alpha} B(\alpha, \alpha + 1; t/(t + h)) - (t(t + h))^\alpha$. 

Thus, the process \( \{ \alpha \} \) where we have used (1.3), (3.17) and the result \( B(\alpha, \alpha + 1; t(t + h)) \sim B(\alpha, \alpha + 1) \) for large \( t \) in the last step. Finally, from (3.16) and (3.18), we get
\[
\text{Corr}(Z_h^\alpha(s), Z_h^\alpha(t)) \sim c_1(s)t^{-(3-\alpha)/2}, \quad \text{as } t \to \infty,
\]
where
\[
c_1(s) = \frac{\alpha^2 h^2 (1 - \alpha)((s + h)^{\alpha+1} - s^\alpha)}{(\alpha + 1)\sqrt{\text{Var}(Z_h^\alpha(s))\sqrt{\alpha h^T}}}.
\]
Thus, the process \( \{ Z_h^\alpha(t) \}_{t \geq 0} \) exhibits the SRD property as \( 1 < (3 - \alpha)/2 < 3/2 \). \( \square \)

3.2. GFPC as a scaling limit of a CTRW. Consider a renewal process
\[
R(t) = \max\{n \geq 0 : W_1 + W_2 + \cdots + W_n \leq t\},
\]
where \( W_1, W_2, \ldots, W_n \) are iid waiting times such that \( \Pr\{W_n > t\} = t^{-\alpha}L(t), \ 0 < \alpha < 1 \) and \( L \) is a slowly varying function. Then, there exist \( b_n > 0 \) such that
\[
b_n(W_1 + W_2 + \cdots + W_n) \Rightarrow D_\alpha(1),
\]
where \( \Rightarrow \) denotes convergence in distribution. It means that \( W_1 \) belongs to the strict domain of attraction of some stable law \( D_\alpha(1) \). Let \( b(t) = b(t) \). It can be shown that there exists a regularly varying function \( \tilde{b} \) with index \( \alpha \) such that \( 1/b(\tilde{b}(c)) \sim c, \text{ as } c \to \infty \) (see Meerschaert et al. (2011)).

Let \( S^{(\nu)}(n) = \sum_{i=1}^n Z_i \) where \( Z_i = X_iV^{(\nu)}, \ i \geq 1 \). Here, \( \{X_i\}_{i \geq 1} \) is a sequence of iid random variables whose distribution is given by (1.3) and \( V^{(\nu)} \) is a Bernoulli random variable independent of \( \{X_i\}_{i \geq 1} \) such that \( \Pr\{V^{(\nu)} = 1\} = p \) and \( \Pr\{V^{(\nu)} = 0\} = 1 - p \). Note that \( S^{(\nu)}(R(t)) \) is a CTRW with heavy-tailed waiting times and jumps distributed accordingly to the law of \( Z_1 \).

The following result holds for the GFPC:
\[
\left\{ S^{(1/b(c))}([\Lambda R(ct)]) \right\}_{t \geq 0} \Rightarrow \{ M^\alpha(t) \}_{t \geq 0}
\]
as \( c \to \infty \) in the \( M_1 \) topology on \( D([0, \infty), \mathbb{R}) \). That is, the GFPC is the scaling limit of a CTRW. The result given in (3.19) can be proved along the similar lines to that of Theorem 4.8 of Kataria and Khandakar (2021). Thus, the proof is omitted.

4. Some special cases of the GFPC

In this section, we discuss few special cases of the GFPC. It is known that the TFPP and the CFPP are particular and limiting cases of the GFPC, respectively (see Di Crescenzo et al. (2016), Kataria and Khandakar (2021)). Its other special cases are as follow:
4.1. Poisson process of order \( k \) and its fractional version. The Poisson process of order \( k \) (PPoK) \( \{N^k(t)\}_{t \geq 0} \) is a compound Poisson process introduced and studied by Kostadinova and Minkova (2013). It is defined as

\[
N^k(t) := \sum_{i=1}^{N(t)} X_i,
\]

where \( \{X_i\}_{i \geq 1} \) is a sequence of iid discrete uniform random variables such that

\[
\Pr\{X_1 = j\} = \frac{1}{k}, \quad j = 1, 2, \ldots, k.
\]

The sequence \( \{X_i\}_{i \geq 1} \) is independent of the Poisson process \( \{N(t)\}_{t \geq 0} \) whose intensity parameter is \( k\lambda \). For \( k = 1 \), the PPoK reduces to the Poisson process. Recently, a fractional version of the PPoK, namely, the time fractional Poisson process of order \( k \) (TFPPoK) \( \{N^k_\alpha(t)\}_{t \geq 0} \) is studied by Gupta and Kumar (2021), Kadankova et al. (2021). It is defined as

\[
N^k_\alpha(t) := N^k(Y_\alpha(t)), \quad 0 < \alpha < 1,
\]

where the PPoK \( \{N^k(t)\}_{t \geq 0} \) and the inverse \( \alpha \)-stable subordinator \( \{Y_\alpha(t)\}_{t \geq 0} \) are independent of each other.

On substituting \( \lambda_j = \lambda \) for all \( j = 1, 2, \ldots, k \) in (1.1), we get the governing system of fractional differential equations for the state probabilities of TFPPoK (see Gupta and Kumar (2021), Eq. (30); Kadankova et al. (2021), Eqs. (18)-(19)). Further, on taking \( \alpha = 1 \), we get the governing system of differential equations for the state probabilities of PPoK (see Kostadinova and Minkova (2013), Eq. (9)). Here, \( \Lambda = k\lambda \). Thus, the PPoK and its fractional version TFPPoK are particular cases of the GFCP.

The pmf \( p^k_\alpha(n, t) = \Pr\{N^k_\alpha(t) = n\} \) of TFPPoK is obtained by Gupta and Kumar (2021), Kadankova et al. (2021). Its alternate form can be obtained by substituting \( \lambda_j = \lambda \) for \( j = 1, 2, \ldots, k \) in (1.2), and it is given by

\[
p^k_\alpha(n, t) = \sum_{r=0}^{n} \sum_{i_1+i_2+\ldots+i_k=r} \left( \frac{r}{i_1+i_2+\ldots+i_k} \right) (\lambda t^\alpha)^r E_{\alpha,\alpha+1}(-k\lambda t^\alpha), \quad n \geq 0.
\]

Similarly, the \( r \)th factorial moment of TFPPoK can be obtained from Proposition 3.4. Moreover, the characteristic function of PPoK (see Gupta et al. (2020), Eq. (10)) and a limiting result for PPoK (see Sengar et al. (2020), Eq. (9)) follow from (3.1) and (3.6), respectively.

4.2. Pólya-Aeppli process of order \( k \) and its fractional version. The Pólya-Aeppli process of order \( k \) (PAPoK) \( \{\hat{N}^k(t)\}_{t \geq 0} \) is a compound Poisson process studied by Chukova and Minkova (2015). It is defined as

\[
\hat{N}^k(t) := \sum_{i=1}^{N(t)} X_i,
\]

where \( \{X_i\}_{i \geq 1} \) is a sequence of iid truncated geometrically distributed random variables with the following pmf:

\[
\Pr\{X_1 = j\} = \frac{1-\rho}{1-\rho^k}, \quad j = 1, 2, \ldots, k,
\]
where \( 0 \leq \rho < 1 \). The sequence \( \{X_i\}_{i \geq 1} \) is independent of the Poisson process \( \{N(t)\}_{t \geq 0} \) whose intensity parameter is \( \lambda \). Recently, a fractional version of the PAPoK, namely, the fractional Pólya-Aeppli process of order \( k \) (FPAPoK) \( \{\hat{N}^k(t)\}_{t \geq 0} \) is introduced by Kadankova et al. (2021). It is defined as

\[
\hat{N}^k(t) := \hat{N}^k(Y_\alpha(t)), \quad 0 < \alpha < 1,
\]

where \( \{\hat{N}^k(t)\}_{t \geq 0} \) and \( \{Y_\alpha(t)\}_{t \geq 0} \) are independent of each other.

On substituting \( \lambda_j = \lambda (1 - \rho) \rho^{j-1} / (1 - \rho^k) \) for all \( j = 1, 2, \ldots, k \) in (1.1), we get the governing system of fractional differential equations for the state probabilities of FPAPoK (see Kadankova et al. (2021), Eqs. (37)-(38)). Further, on taking \( \alpha = 1 \), we get the governing system of differential equations for the state probabilities of PAPoK (see Chukova and Minkova (2015), Eq. (9)). Here, \( \Lambda = \lambda \). Thus, the PAPoK and its fractional version FPAPoK are particular cases of the GFCP. From Lemma 3.1, we get the following limiting result for the PAPoK:

\[
\lim_{t \to \infty} \frac{\hat{N}^k(t)}{t} = \frac{\lambda}{1 - \rho^k} \left( 1 + \rho + \cdots + \rho^{k-1} - k\rho^k \right), \quad \text{in probability.}
\]

It is important to note that the pmf of FPAPoK is not known. On substituting \( \lambda_j = \lambda (1 - \rho) \rho^{j-1} / (1 - \rho^k) \) for \( j = 1, 2, \ldots, k \) in (1.2), the pmf \( \hat{p}^k(n, t) = \Pr\{\hat{N}^k(t) = n\} \) of FPAPoK can be obtained as follows:

\[
\hat{p}^k(n, t) = \sum_{r=0}^{n} \sum_{i_1 + i_2 + \cdots + i_k = r} \left( \frac{\lambda (1 - \rho) t^\alpha}{\rho (1 - \rho^k)} \right)^r \rho^n E_{\alpha, \alpha+1}(\lambda t^\alpha), \quad n \geq 0
\]

and its pgf can be obtained from (3.3) in the following form:

\[
\hat{G}^k(u, t) = E_{\alpha, 1} \left( \frac{\lambda (1 - \rho)}{1 - \rho^k} \sum_{j=1}^{k} \rho^{j-1} (u^j - 1) t^\alpha \right), \quad |u| \leq 1.
\]

Further, \( \alpha = 1 \) gives the pmf \( \hat{p}^k(n, t) = \Pr\{\hat{N}^k(t) = n\} \) of PAPoK as follows:

\[
\hat{p}^k(n, t) = \sum_{r=0}^{n} \sum_{i_1 + i_2 + \cdots + i_k = r} \left( \frac{\lambda (1 - \rho) t^\alpha}{\rho (1 - \rho^k)} \right)^r \frac{\rho^n e^{-\lambda t}}{i_1 ! i_2 ! \cdots i_k !}, \quad n \geq 0.
\]

From (3.2), it follows that its Lévy measure is

\[
\Pi(dx) = \frac{\lambda (1 - \rho)}{(1 - \rho^k)} \sum_{j=1}^{k} \rho^{j-1} \delta_j dx.
\]

### 4.3. Pólya-Aeppli process and its fractional version.

The Pólya-Aeppli process (PAP) \( \{\hat{N}(t)\}_{t \geq 0} \) is a compound Poisson process studied by Chukova and Minkova (2013). It is defined as

\[
\hat{N}(t) := \sum_{i=1}^{N(t)} X_i,
\]

where \( \{X_i\}_{i \geq 1} \) is a sequence of iid geometrically distributed random variables such that

\[
\Pr\{X_1 = j\} = (1 - \rho) \rho^{j-1}, \quad j \geq 1.
\]
where \( 0 \leq \rho < 1 \). The sequence \( \{X_i\}_{i \geq 1} \) is independent of the Poisson process \( \{N(t)\}_{t \geq 0} \) whose intensity parameter is \( \lambda \). Beghin and Macci (2014) introduced a fractional version of the PAP, namely, the fractional Pólya-Aeppli process (FPAP) \( \{\hat{N}_\alpha(t)\}_{t \geq 0} \). It is defined as

\[
\hat{N}_\alpha(t) := \hat{N}(Y_\alpha(t)), \quad 0 < \alpha < 1,
\]

where \( \{\hat{N}(t)\}_{t \geq 0} \) and \( \{Y_\alpha(t)\}_{t \geq 0} \) are independent of each other.

On letting \( k \to \infty \) in (1.1) with \( \lambda_j = \lambda(1 - \rho)\rho^{j-1} \) for all \( j \geq 1 \) the system (1.1) reduces to the governing system of differential equations for the state probabilities of FPAP (see Beghin and Macci (2014), Eq. (19)). Further, on taking \( \alpha = 1 \), we get the governing system of differential equations for the state probabilities of PAP (see Chukova and Minkova (2013), Eq. (10)). Here, \( \Lambda = \lambda \). Thus, the PAP and its fractional version FPAP are obtained as a limiting process of GFCP. From Lemma 3.1, we get the following limiting result for the PAP:

\[
\lim_{t \to \infty} \frac{\hat{N}(t)}{t} = \frac{\lambda}{1 - \rho}, \quad \text{in probability.}
\]

4.4. Negative binomial process and its fractional version. The negative binomial process (NBP) \( \{\bar{N}(t)\}_{t \geq 0} \) is a compound Poisson process studied by Kozubowski and Podgórski (2009). It is defined as

\[
\bar{N}(t) := \sum_{i=1}^{N(t)} X_i,
\]

where \( \{X_i\}_{i \geq 1} \) is a sequence of iid random variables with discrete logarithmic distribution such that

\[
\Pr\{X_1 = j\} = \frac{(1 - p)^j}{j \ln(1/p)}, \quad j \geq 1,
\]

where \( 0 < p < 1 \). The sequence \( \{X_i\}_{i \geq 1} \) is independent of the Poisson process \( \{N(t)\}_{t \geq 0} \) whose intensity parameter is \( \ln(1/p) \). Beghin and Macci (2014) studied a fractional version of the NBP, namely, the fractional negative binomial process (FNBP) which we denote by \( \{\bar{N}_\alpha(t)\}_{t \geq 0} \). It is defined as

\[
\bar{N}_\alpha(t) := \bar{N}(Y_\alpha(t)), \quad 0 < \alpha < 1,
\]

where \( \{\bar{N}(t)\}_{t \geq 0} \) and \( \{Y_\alpha(t)\}_{t \geq 0} \) are independent of each other.

On letting \( k \to \infty \) in (1.1) with \( \lambda_j = (1 - p)^j/j \) for all \( j \geq 1 \) the system (1.1) reduces to the governing system of differential equations for the state probabilities of FNBP (see Beghin (2015), Eq. (66)). Here, \( \Lambda = \ln(1/p) \). Thus, the FNBP is obtained as a limiting process of GFCP. From Lemma 3.1, we get the following limiting result for the NBP:

\[
\lim_{t \to \infty} \frac{\bar{N}(t)}{t} = \frac{1 - p}{p}, \quad \text{in probability.}
\]

5. An application to risk theory

Consider the following risk model with GCP as the counting process:

\[
X(t) = ct - \sum_{j=1}^{M(t)} Z_j, \quad t \geq 0,
\]
where \( c > 0 \) denotes the constant premium rate. Here, \( \{Z_j\}_{j \geq 1} \) is the sequence of positive iid random variables with common distribution \( F \). The \( Z_j \)'s represent the claim sizes and these are independent of the GCP.

Let \( \mu = \mathbb{E}(Z_j) \). The relative safety loading factor \( \eta \) for the risk model (5.1) is given by

\[
\eta = \frac{\mathbb{E}(X(t))}{\mathbb{E}\left( \sum_{j=1}^{M(t)} Z_j \right)} = \frac{c}{\mu \sum_{j=1}^{k} j \lambda_j} - 1.
\]

Hence, \( c > \mu \sum_{j=1}^{k} j \lambda_j \) holds when the safety loading factor is positive. Let \( u \geq 0 \) denote the initial capital and \( \{U(t)\}_{t \geq 0} \) be the surplus process where \( U(t) = u + X(t) \).

Let \( \tau \) denote the time to ruin of an insurance company. So,

\[
\tau = \inf\{t > 0 : U(t) < 0\},
\]

where \( \inf \phi = \infty \) and the ruin probability is given by \( \psi(u) = \Pr\{\tau < \infty\} \). Let \( G(u, y) \) be the joint probability that the ruin occurs in finite time and the deficit \( D = |U(\tau)| \) at the time of ruin is not more than \( y \geq 0 \), that is,

\[
G(u, y) = \Pr\{\tau < \infty, D \leq y\}.
\]

Note that

\[
\lim_{y \to \infty} G(u, y) = \psi(u).
\]

The transition probabilities of GCP are given by (see Di Crescenzo et al. (2016), Section 2)

\[
\Pr\{M(t + h) = n|M(t) = m\} = \begin{cases} 1 - \Lambda h + o(h), & n = m, \\ \lambda_j h + o(h), & n = m + j, \ j = 1, 2, \ldots, k, \\ o(h), & n > m + k, \end{cases} \tag{5.3}
\]

where \( \Lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k \).

Let \( \tilde{u} = u + ch \) and \( F^{*j}(\cdot) \) be the distribution of \( Z_1 + Z_2 + \cdots + Z_j \) for all \( j = 1, 2, \ldots, k \). Then, from (5.2) and (5.3), we get

\[
G(u, y) = (1 - \Lambda h)G(\tilde{u}, y) + o(h) + \sum_{j=1}^{k} \lambda_j h \left( F^{*j}(\tilde{u} + y) - F^{*j}(\tilde{u}) + \int_{0}^{\tilde{u}} G(\tilde{u} - x, y) dF^{*j}(x) \right)
\]

which can be rewritten as follows:

\[
\frac{G(\tilde{u}, y) - G(u, y)}{ch} = \frac{\Lambda}{c} G(\tilde{u}, y) + \frac{o(h)}{h} - \frac{1}{c} \sum_{j=1}^{k} \lambda_j \left( F^{*j}(\tilde{u} + y) - F^{*j}(\tilde{u}) + \int_{0}^{\tilde{u}} G(\tilde{u} - x, y) dF^{*j}(x) \right). \tag{5.4}
\]

Let

\[
H(x) = \frac{1}{\Lambda} \sum_{j=1}^{k} \lambda_j F^{*j}(x)
\]

be the mixture distribution whose mixture components are the distributions \( F^{*j}(\cdot) \)'s of the aggregated claims \( Z_1 + Z_2 + \cdots + Z_j \). It follows that \( H(0) = 0, \ H(\infty) = 1 \). On letting \( h \to 0 \) in (5.4), we get

\[
\frac{\partial G(u, y)}{\partial u} = \frac{\Lambda}{c} \left( G(u, y) + H(u) - H(u + y) - \int_{0}^{u} G(u - x, y) dH(x) \right). \tag{5.5}
\]
Further, on letting $y \to \infty$, we get
\[
\frac{d}{du}\psi(u) = \frac{\Lambda}{c} \left( \psi(u) + H(u) - 1 - \int_0^u \psi(u-x)dH(x) \right).
\]

**Theorem 5.1.** The function $G(0, y)$ is given by
\[
G(0, y) = \frac{\Lambda}{c} \int_0^y (1 - H(u))du. \tag{5.6}
\]

**Proof.** Integrating (5.5) with respect to $u$ from 0 to $\infty$ and using $G(\infty, y) = 0$, we get
\[
-G(0, y) = \frac{\Lambda}{c} \left( \int_0^\infty G(u, y)du + \int_0^\infty (H(u) - H(u+y))du - \int_0^\infty \int_0^u G(u - x, y)dH(x)du \right).
\]

Thus, the change of variable yields
\[
G(0, y) = \frac{\Lambda}{c} \int_0^\infty (H(u + y) - H(u))du = \frac{\Lambda}{c} \int_0^y (1 - H(u))du.
\]
This completes the proof. \qed

**Corollary 5.1.** The ruin probability for zero initial capital is given by
\[
\psi(0) = \frac{\mu}{\Lambda} \sum_{j=1}^k j\lambda_j.
\]

**Proof.** On taking limit $y \to \infty$ in (5.6), we get
\[
\psi(0) = \frac{\Lambda}{c} \int_0^\infty (1 - H(u))du.
\]

Let $X$ be a random variable with distribution function $H(x)$. Then,
\[
\mathbb{E}(X) = \frac{\mu}{\Lambda} \sum_{j=1}^k j\lambda_j.
\]
Using the fact that $\mathbb{E}(X) = \int_0^\infty (1 - H(u))du$, we get the required result. \qed

6. Concluding remarks

We obtain some additional results and study new properties for the GFCP, a fractional counting process introduced by Di Crescenzo et al. (2016). Its $r$th factorial moment and the covariance are derived. We establish the LRD and SRD properties for it and its increments, respectively. It is shown that the GFCP is a scaling limit of some CTRW. A particular case of the GFCP, namely, the GCP is discussed in detail for which we obtain a martingale result and establish a recurrence relation for its pmf. We obtain a limiting result for the GCP using which we prove that the one-dimensional distributions of GFCP are not infinitely divisible. It is shown that many known counting processes recently introduced and studied by several researchers such as the PPoK, PAP, PAPoK, NBP and their fractional versions are special cases of the GFCP. An application of the GCP to ruin theory is discussed where it is used as a counting process.
References

[1] Beghin, L. (2012). Random-time processes governed by differential equations of fractional distributed order. *Chaos Solitons Fractals* **45** (11), 1314-1327.

[2] Beghin, L. (2015). Fractional gamma and gamma-subordinated processes. *Stoch. Anal. Appl.* **33**(5), 903-926.

[3] Beghin, L. and Macci, C. (2014). Fractional discrete processes: compound and mixed Poisson representations. *J. Appl. Probab.* **51**(1), 19-36.

[4] Beghin, L. and Orsingher, E. (2009). Fractional Poisson processes and related planar random motions. *Electron. J. Probab.* **14**(61), 1790-1827.

[5] Biard, R. and Saussereau, B. (2014). Fractional Poisson process: long-range dependence and applications in ruin theory. *J. Appl. Probab.* **51**(3), 727-740.

[6] Chukova, S. and Minkova, L. D. (2013). Characterization of the Pólya-Aeppli process. *Stoch. Anal. Appl.* **31**(4), 590-599.

[7] Chukova, S. and Minkova, L. D. (2015). Pólya-Aeppli of order \( k \) risk model. *Comm. Statist. Simulation Comput.* **44**(3), 551-564.

[8] Di Crescenzo, A., Martinucci, B. and Meoli, A. (2016). A fractional counting process and its connection with the Poisson process. *ALEA Lat. Am. J. Probab. Math. Stat.* **13**(1), 291-307.

[9] Ding, Z., Granger, C. W. J. and Engle, R. F. (1993). A long memory property of stock market returns and a new model. *J. Empirical Finance* **1**(1), 83-106.

[10] Doukhan, P., Oppenheim, G. and Taqqu, M. S. (Eds) (2003). *The Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston.

[11] Gupta, N. and Kumar, A. (2021). Fractional Poisson processes of order \( k \) beyond. *arXiv:2008.06022v3*.

[12] Gupta, N., Kumar, A. and Leonenko, N. (2020). Skellam type processes of order \( k \) beyond. *Entropy* **22**(11), 21 pp.

[13] Johnson, W. P. (2002). The curious history of Fà di Bruno’s formula. *Amer. Math. Monthly* **109**(3), 217-234.

[14] Kadankova, T., Leonenko, N. and Scalas, E. (2021). Fractional non-homogeneous Poisson and Pólya-Aeppli processes of order \( k \) and beyond. *arXiv:2008.09421v4*.

[15] Karagiannis, T., Molle, M. and Faloutsos, M. (2004). Long-range dependence ten years of internet traffic modeling. *IEEE Internet Computing* **8**(5), 57-64.

[16] Katariya, K. K. and Khandakar, M. (2021). Convoluted fractional Poisson process. To appear in *ALEA Lat. Am. J. Probab. Math. Stat*.

[17] Katariya, K. K. and Vellaisamy, P. (2017). Saigo space-time fractional Poisson process via Adomian decomposition method. *Statist. Probab. Lett.* **129**, 69-80.

[18] Kilbas, A. A., Srivastava, H. M. and Trujillo J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam.

[19] Kostadinova, K. Y. and Minkova, L. D. (2013). On the Poisson process of order \( k \). *Pliska Stud. Math. Bulgar.* **22**, 117-128.

[20] Kozubowski, T. J. and Podgórski, K. (2009). Distributional properties of the negative binomial Lévy process. *Probab. Math. Statist.* **29**(1), 43-71.

[21] Laskin, N. (2003). Fractional Poisson process. *Commun. Nonlinear Sci. Numer. Simul.* **8**(3-4), 201-213.

[22] Leonenko, N. N., Meerschaert, M. M., Schilling, R. L. and Sikorskii, A. (2014). Correlation structure of time-changed Lévy processes. *Commun. Appl. Ind. Math.* **6**(1), e-483, 22 pp.

[23] Maheshwari, A. and Vellaisamy, P. (2016). On the long-range dependence of fractional Poisson and negative binomial processes. *J. Appl. Probab.* **53**(4), 989-1000.

[24] Meerschaert, M. M., Nane, E. and Vellaisamy, P. (2011). The fractional Poisson process and the inverse stable subordinator. *Electron. J. Probab.* **16**(59), 1600-1620.

[25] Orsingher, E. and Polito, F. (2012). The space-fractional Poisson process. *Statist. Probab. Lett.* **82**(4), 852-858.

[26] Sengar, A. S., Maheshwari, A. and Upadhye, N. S. (2020). Time-changed Poisson processes of order \( k \). *Stoch. Anal. Appl.* **38**(1), 124-148.

[27] Steutel, F. W. and van Harn, K. (2004). *Infinite Divisibility of Probability Distributions on the Real Line*. Marcel Dekker, New York.
[28] Vellaisamy, P. and Kumar, A. (2018). First-exit times of an inverse Gaussian process. *Stochastics* **90**(1), 29-48.

[29] Zuo, H., Shen, Z. and Rang, G. (2021). Hitting probabilities of weighted Poisson processes with different intensities and their subordinations. *Acta Math Sci* **41**, 67-84.

**Kuldeep Kumar Kataria, Department of Mathematics, Indian Institute of Technology Bhilai, Raipur 492015, India.**

*Email address: kuldeepk@iitbilai.ac.in*

**Mostafizar Khandakar, Department of Mathematics, Indian Institute of Technology Bhilai, Raipur 492015, India.**

*Email address: mostafizark@iitbilai.ac.in*