LECTURES ON CHIRAL DISORDER IN QCD

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Abstract
I explain the concept that light quarks diffuse in the QCD vacuum following
the spontaneous breakdown of chiral symmetry. I exploit the striking analogy
with disordered electrons in metals, identifying, among others, the universal
regime described by random matrix theory, diffusive regime described by chiral
perturbation theory and the crossover between these two domains.

Keywords: Chiral disorder, spectral fluctuations, random matrix theory, diffusion, QCD.

Introduction
In these lectures I review spectral aspects of the mechanism of the sponta-
neous breakdown of the chiral symmetry in Quantum Chromodynamics. Most
probably, the spontaneous breakdown of the chiral symmetry is a collective
phenomenon caused by the microscopic disorder, in striking resemblance to
the diffusive phenomena appearing in disordered metals. Despite the micro-
scopic, detailed nature of this disorder is still unknown, the constraints arising
from realizations of chiral symmetry in QCD are so strong, that allow us to
predict several non-trivial consequences of this phenomenon. Among most
profound, are the predictions on the spectral properties of the Dirac operator.
As such, these are amenable to test using the lattice calculations.

The lectures are intended to be elementary and self-contained. The outline
of the lectures is as follows:

In Part 1, I introduce the basic facts on symmetries and anomalies of the QCD,
leading to fundamental low energy constraints.

In Part 2, I explore the analogy between the metal viewed as a complex quantum
system sharing universal properties with so-called chaotic systems from
one side, and Euclidean QCD with light quarks diffusing in disordered medium
built from the lumps of the gauge field, on the other side. In particular, I identify
the relevant scales and determine the “diffusion constant” of the QCD vacuum.

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Then I comment on the close relation of this picture to several known descriptions of chiral symmetry breaking.

In Part 3, I demonstrate how the hierarchy of spectral scales explains the origin of the appearance of random matrix ensembles in QCD. Then I show how random matrix regime breaks down at a certain spectral scale (analog of Thouless energy), leading to various versions of chiral perturbation theory. I also point at some explicit confirmations of this picture coming from the recent lattice calculations. Next, I suggest how this picture can be modified/generalized in the presence of external parameters of the QCD. I mention the possibility of (multi-critical) scaling. I conclude with an (incomplete) list of works on disorder and QCD, that could be used as a guide for further reading and also as a good starting point for the original research.

Part 1

Let me refresh here some low energy theorems of the QCD, which will be needed for the second part of these lectures. Then I switch to the Euclidean regime, and I recall the Banks-Casher relation.

QCD

Quantum Chromodynamics is a Yang Mills theory based on the local gauge group $SU(3)_{color}$. This means that all six quarks of different flavors interact with gluons, as well gluons interact with themselves, with the same universal coupling $g$. Looking at the typical masses of the quarks, we see that they cover several orders of magnitude – $u,d,s,c,b,t$, \(5,7,150,1400,4400,175000\) MeV, – respectively. The origin of this hierarchy goes beyond the strong interaction sector and is unknown. The fact, that three masses are less or at most comparable to the scale of the strong interaction, and three others are well beyond the scale, suggests a simplification, in which we put three heavy masses equal to infinity, and three light equal to zero. The heavy sector decouples from the light, and the light one reveals a series of essential symmetries of the theory. We denote this idealization of the QCD as QCD\(\iota\). First, let us note that such classical theory lacks any dimension-full parameter, therefore it is scale invariant. Second, if we introduce the notion of left and right-handed quarks,

\[
q_L = \frac{1}{2}(\bar{1} + \gamma_5)q \quad q_R = \frac{1}{2}(\bar{1} - \gamma_5)q
\]

we see that in the presence of massless quarks the left and the right handed quarks interact independently with the gluons, i.e. we have two decoupled
chiral copies of the initial theory:

$$S_{QCD} = \int d^3x \sum_f \left[ q_R^{(f)} \bar{\psi}(A) c_R^{(f)} + (R \leftrightarrow L) + S_{\text{glue}}(A) \right]$$  \hspace{1cm} (2)

Constructing the standard set of the Noether currents corresponding to the symmetries yields:

$$V_a = q \ t_a q$$
$$V_0 = q \ q$$
$$A_a = q \ s t_a q$$
$$A_0 = q \ s q$$
$$J_{\text{scale}} = x$$  \hspace{1cm} (3)

where $a = 1; 2; 3; \ldots$ for SU (3), $t_a$ are corresponding generators (here Gell-Mann matrices, modulo normalization), $q = (u; d; s)$, $\bar{\psi}$ denotes symmetric traceless energy momentum tensor, and we used the linear transformation for the original left and right currents ($V = R + L$, $A = R - L$).

For three light flavors, we are left with a nonet of the conserved vector currents, a nonet of the conserved axial currents, and conserved currents corresponding to the scale invariance.

$$SU(3)_V \quad SU(3)_A \quad U(1)_V \quad U(1)_A \quad \text{SCALE INV}$$  \hspace{1cm} (4)

This classical picture is however strongly distorted at the quantum level. First, QCD has anomalies - i.e. certain classical symmetries are violated at the quantum level. The scale-invariance is broken (scale anomaly),

$$@ J_{\text{scale}} = \frac{2}{g} \left( \frac{g}{4} \right) g^i \ G^i \ \text{renorm}$$  \hspace{1cm} (5)

and the appearance of beta function introduces the scale of the strong interactions, $\Lambda_{QCD} \approx 200 \text{ MeV}$.

Second, the singlet component of the axial current is also anomalous

$$@ A_0 = \frac{N_f}{2} \ \frac{g^2}{8 \pi^2} G_i G^i$$  \hspace{1cm} (6)

A similar anomaly appears in QED, with $N_f = 2$ ! $1$, $g ! e$, and non-Abelian $G$ replaced by Abelian electromagnetic $F$ tensor. These are the all anomalies of the QCD, if no external (e.g. electromagnetic) currents are added. The appearance of the Gell-Mann function in the scale anomaly reminds about another feature of QCD: the interactions between quarks and gluons get stronger at smaller energy scales, invalidating the perturbative calculation in low energy domain, contrary to the precisely opposite behavior in QED. Quarks and
gluons interact strongly forming colorless states, and the unraveled nature of long-wavelength limit of these interactions is usually coded under the name “confinement”.

The closer look at the experimental spectrum of elementary excitations (hadrons) of the QCD vacuum shows that the picture is more complicated, and another, on top of confinement, nonperturbative phenomenon has to take place in QCD. If we look at the remaining symmetries after quantizing the QCD, we see that they are

$$ SU(3)_A \quad SU(3)_V \quad U(1)_V $$

where the last one corresponds to the baryon number conservation ($3V = B$). The interaction preserves baryon number and is invariant under $SU(3)_V$ and $SU(3)_A$ symmetries. Since the masses of up and down quarks are 20 times smaller than the scale of the QCD, the lightest particles in real QCD should show the traces of the exact symmetry (7). This is at odds with the experiment, which shows that the light vector-like particles differ from axial-like particles, despite similar flavor content - e.g. $(770)$ is much lighter than axial $(1200)$. This asymmetry holds also for baryons of opposite parity, (nucleon $940$ versus $1535$ vector) and manifests dramatically at the level of the scalars - the lowest pseudoscalar (pion) $(140)$ seems to not have a narrow chiral partner at all - when comparing pion to $f_0 (400 \ 1200)$ with full width $600 \ 1000$ MeV, an interpretation of $f_0$ as a particle is controversial...

This suggests that the vacuum state of the QCD is not respecting all the symmetries of the QCD. This phenomenon is called a spontaneous breakdown of the symmetry. Then, chirally invariant interactions of the QCD, acting on chirally non-invariant vacuum, can indeed produce such an asymmetry in the hadronic spectrum.

The important hint comes from the observation by Vafa and Witten [1], that theories with vector-like couplings (e.g. QCD) cannot break spontaneously vector symmetries. We are therefore left with the alternative:

$$ Q_A^{a} \, \mathcal{D} > \quad \mathcal{P} S^{a} > \ 0 $$

where $Q_A^{a}$ are axial charges corresponding to the currents $A_a$. Vacuum state respects therefore only vector symmetries. Since the Hamiltonian of the QCD still commutes with all the 16 generators $Q_V^{a}$, $Q_A^{a}$ we see that the vacuum state ($H \, \mathcal{D} = E_0 \mathcal{D}$) is degenerated with the octet of the states $Q_A^{a} \, \mathcal{D}$, ($a = 1; 2; \ldots 8$).

Indeed

$$ H \, \mathcal{P} S^{a} = H \, Q_A^{a} \, \mathcal{D} = Q_A^{a} H \, \mathcal{D} = E_0 Q_A^{a} \, \mathcal{D} = E_0 \mathcal{P} S^{a} > $$

This is a basic message of the Goldstone theorem [2]. To each broken generator of the axial current corresponds a massless, spinless excitation corresponding
to quantum number of the generator. In the case of three flavors, the Goldstone theorem predicts the appearance of an octet of massless, pseudoscalar mesons. They correspond to massless pions (isotriplet), kaons (two isodoublets) and an eta (isosinglet).

A priori, from the point of view of chiral symmetry, QCD allows (at least) two phases: asymmetric (Nambu-Goldstone phase), described above, and the symmetric one (Wigner-Weyl phase), where vacuum is respecting all symmetries, \( Q^a_0 \mathcal{D} > = Q^a_0 \mathcal{D} > = 0 \). Hence, a phase transition may happen between these two phases. Such phase transition can be characterized by the appearance of an order parameter. The lowest-dimensional order parameter is the expectation value of \( q_R^a q_L^a + q_L^a q_R^a \). It carries zero baryon number, is a scalar (vacuum respects space reflection), and is diagonal in flavor. Note that e.g. \( < u d > \) is not invariant under rotations generated by isospin matrices, therefore it is not invariant under the vector, unbroken group of the vacuum. There are infinitely many other operators, which are order parameters, e.g. \( q_L^a q_R^b + q_R^a q_L^b \), but they carry higher canonical dimension. The lowest dimensional order parameter \( q q \) is called "quark condensate". It is believed that under the action of external parameters, like e.g. temperature and/or density, the crossover to other phases of QCD is possible (see lectures by Rob Pisarski and Krishna Rajagopal), including the phase with restored (approximate) chiral symmetry. It is important to mention, that despite some experimental signals point at the restoration of the chiral symmetry (cf.[3]), there is, in my opinion, no "smoking gun" evidence for this phenomenon and most of our understanding of chiral phase transition comes from the lattice studies (see lectures by Frithjof Karsch). We should also remember, that in real QCD, the masses of u,d,s quarks are light, but non zero, therefore on top of the phenomenon of spontaneous breakdown of the symmetry we have also an explicit, albeit small, explicit breakdown of the chiral symmetry due to the explicit presence of mass terms \( m q q \) in the Hamiltonian. Therefore the chiral restoration is not really a phase transition, but rather a crossover process.

**Condensate, pion, GMOR**

We still consider a QCD. Let us come back to the Goldstone theorem, and show that indeed the presence of the condensate forces the presence of massless excitations in the spectrum. To demonstrate this, we use the Ward identity, relating the correlator of the axial SU(3) current \( A_a \) and pseudoscalar SU(3) current \( P^b = i q^b \) to scalar densities,

\[
\theta \ < \ 0 \mathcal{F} A^i (0) \mathcal{F} > = i \theta \ < 0 \mathcal{F} P^i (0) \mathcal{F} > = i \theta \ < 0 \mathcal{F} [\tau^i \tau^j \tau^k]_q \mathcal{F} > \quad (10)
\]

**Exercise.** Justify (10), making use of equal-time commutation relations (cf. the lecture of Jean-Paul Blaizot).
Since Lorentz invariance implies that Fourier transformation of the correlator has to take a form
\[ \int dx e^{i k x} <0\mathcal{A}^a(x)\mathcal{P}(0)\mathcal{D}> = p^i j^a(k^2) \] (11)
the Ward identity says that \( p^2 j^a(k^2) = i j^a <\mathcal{Q}\mathcal{Q}> \). (Taking a derivative corresponds to multiplying the matrix element by \( ip \). Note that the vacuum expectation value of the second term in the second term in (10) has to vanish due to the invariance under vector transformations). Since the condensate (expectation value of r.h.s. of (10)) is non-zero in the broken phase, the spectral function \( j^a(k^2) \) has to contain a massless particle corresponding to the pole at \( k^2 = 0 \).

Now we can make use of a powerful condition, (formulated by 't Hooft), relating fundamental theories with theories in which particles are bound states of the fundamental constituents. This condition, known as anomaly matching condition, states that a composite particle has to reproduce exactly the anomaly present in the fundamental theory.

To see how this condition works let us consider the axial \( SU(3)_A \) current corresponding to \( t^3 \), i.e. \( A^3 = q t^3 s q \) with \( q = (u; d; s) \) and \( t^3 = \text{diag}(1; 1; 0) \). This current is anomaly free in QCD, but if we allow quarks couple to photons, the matrix element
\[ <0\mathcal{A}^a j^b> = \frac{p^2}{p^2} 2 N_c (e_u^2 e_d^2) <0\mathcal{F}\mathcal{F} j^b> \] (12)
It is easy to understand the r.h.s. of (12). This is simply the difference of two electromagnetic anomalous \( U(1) \) currents, corresponding to the charges with fractions \( e_u = 2/3 \) and \( e_d = 1/3 \), respectively. This equation represents the contribution from the celebrated triangle anomaly.

From the above identity we infer immediately the form of the matrix element
\[ <0\mathcal{A}^a j^b> = \frac{p^2}{p^2} 2 N_c (e_u^2 e_d^2) <0\mathcal{F}\mathcal{F} j^b> + \ldots \] (13)
where \( p \) is the momentum of both photons and the dots stand for less singular terms. The pole here is simply the remnant of the massless quark circulating in a triangle graph. If we now assume confinement, anomaly matching condition says that in the hadronic (composite) world, there must exist a massless, colorless object which couples to the axial current and photons and which reproduces exactly the pole in the anomaly above. Since massless baryons are excluded (cf. Vafa-Witten theorem), this object has to be a meson. Mesons fulfill the anomaly condition in an easy way. We do not need to circulate them inside the triangle loops like quarks, it is enough if there is a relation between the axial current and the pion. This well-known relation reads
\[ <0\mathcal{A}^a j^b> = i e^a e^b \] (14)
where $F$ is pion decay constant. So anomaly matching means, that we read out the r.h.s. of (13) as a product of (14), massless pion propagator $\frac{i}{p^2}$ and the amplitude for $^0\pi^\ast$ decay (rest of the r.h.s. of (13). The similar matching of the trace anomaly between the quark and hadronic worlds was discussed in Dima Kharzeev lectures). Let us come back to Ward identity (10). In the $p^2 \rightarrow 0$ limit the correlator is fully determined by the condensate,

$$\begin{align*}
Z \int dx e^{ipx} <0\not\!F A \not\!x (x) \not\!F (0) \not\!\not\!F > = \frac{p}{p^2} <\bar{q}q> \not\!F
\end{align*}$$

(15)

Saturating the correlator with pion states, and using (14), we read out comparing left and right h.s., the value of the another matrix element

$$
<0\not\!F \not\!\not\!F > = \frac{jk<\bar{q}q>}{F}
$$

(16)

With the help of the above relations we can now prove the Gell-Mann Oakes Renner relation [4]. Introducing the small quark masses, we see that Dirac equation implies

$$0 (\not\!x (x) \not\!s d (x)) = (\not\!\not\!m_u + \not\!m_d) \not\!x (x) \not\!s d (x)
$$

(17)

Calculating the vacuum to pion matrix element of the above identity, and using formulae (14) and (16), we get

$$M^2 F^2 = (\not\!m_u + \not\!m_d) <\bar{q}q>
$$

(18)

This relation shows how the gap in the Goldstone mode appears due to the presence of the nonzero quark mass.

**Euclidean world**

In this subsection we present the formulae that allow the transcription from Minkowski to Euclidean space. The advantages of working in Euclidean space are two-fold: first, several mathematical operations are well defined, second, the formulation is comparable to the lattice simulations. The following set of rules defines the transition (l.h.s. denotes Minkowski, r.h.s. denote Euclidean)

- **Space time:** $i x_0, x_i, x_i$
- **Vector potentials:** $A_0, i A_4, A_\perp, A_\perp$
- **Gamma matrices:** $\not\!0, \not\!4, \not\!i, \not\!4, \not\!\not\!i, \not\!5, \not\!5$
- **Fermi fields:** $\not\!i \not\!q, \not\!q, \not\!q$
- **Action:** $iS, S$
Then, Euclidean Dirac matrices obey $+ = 2$, all four matrices are hermitian, as well as the Dirac operator $i \partial$. Finally, the Euclidean action for QCD reads:

$$S = \int d^4x \frac{1}{4} G^a G^a q (i \mathcal{D} (A) + im) q$$

(19)

and partition function $Z = \exp \left( - S \right)$.

**Banks Casher relation**

We will now demonstrate that the chiral condensate is related to the Dirac operator spectrum in Euclidean space-time. Consider a Dirac propagator in a Euclidan box $V = L^4$ in the presence of the gluonic background field $A$. From the action above, we read that propagator is $S_F = \left( \mathcal{D} = \partial + m \right)$. We can write then:

$$\langle 0 \mid \bar{q}(0) q(0) \mid A \rangle = \text{Tr} S_F (\kappa; x) = \frac{1}{Z} \lim_{V \rightarrow 1} \int_d \frac{1}{im} \langle \rangle$$

(20)

where $\text{Tr}$ denotes trace over coordinates, color, spin and Dirac indices, and spectral density is defined as $\langle \rangle = \int_k \langle \rangle$ where $\mathcal{D} q_k = k q_k$. Now, we average the above equation over the gluonic configuration weighted with the full QCD measure (including standard gauge fixing etc.)

$$\langle \rangle = \langle \rangle_{\text{QCD}} [\mathcal{D} A] [\cdots] \text{det} [\mathcal{D} (A) + m] e^{S_{\text{glue}}}$$

(21)

As a result, we get

$$\langle \rangle = \frac{1}{Z} \int_{\text{im}} \frac{1}{im} \langle \rangle$$

(22)

where $\langle \rangle_{\text{QCD}} = \langle \rangle_{\text{QCD}}$ is the average spectral density, i.e. the density $\langle \rangle$ averaged over the full QCD measure. As a final step, we take a chiral limit and use the relation

$$\lim_{m \rightarrow 0} \frac{1}{im} = PV \frac{1}{im} \langle \rangle$$

(23)

As a result, we relate the chiral condensate to the average spectral density around zero eigenvalues

$$\langle \rangle = \langle \rangle_{\text{QCD}}$$

(24)

Note that the contribution from principle value (PV) part drops, since due to the chiral property $[\mathcal{D}, \gamma_5]$, the eigenvalues come in pairs (corresponding to
eigenfunctions $q_k$ and $\bar{q}_k$), so the average spectral density is an even function 
$\langle \rho(q) \rangle = \langle \rho(-q) \rangle$. Note that the same property guarantees the positivity of 
the QCD measure, despite at the first look $D + m$ seems to create complex measure.

The relation (24) was first suggested by Banks and Casher [5]. (For the 
careful discussion of the UV part of the spectrum, see [6]). It is very important 
that the chiral limit $m \to 0$ is taken after the thermodynamical limit $V = L^4 \to L$ , for otherwise the average spectral density would be zero. The result 
states that in a vector like theory with chirally symmetric spectra, the quark 
condensate is related to mean spectral density at zero virtuality (i.e. at $q = 0$).

We can now ask the crucial question, what kind of mechanism can cause 
spectral density to be non-vanishing at zero virtuality? Note that the levels of 
free particle closed in the box scale like $2^n L$, with $n$ integer, so the mean 
level spacing goes like $L^{\frac{1}{2}}$, and the average spectral density (proportional to 
the inverse of the mean level spacing) for free particle scales like $L = V^{\frac{1}{1-d}}$, 
therefore will never be able to balance in the thermodynamical limit the l.h.s. of 
(24) provided the condensate is non zero. The only solution is that spontaneous 
symmetry breakdown requires enormous accumulation of the eigenvalues in the 
vicinity of zero, with the condensation rate scaling like $V$, so $\langle 0 \rangle \sim V$. This 
the first look obvious fact was first emphasized and exploited by Leutwyler 
and Smilga [6] and forms the cornerstone of the spectral analysis of the Dirac 
operator.

We will devote the next lecture to unravel the most plausible microscopic 
mechanism responsible for such a spectral behavior.

**Part 2**

In this part, we outline the basic concepts of diffusion and translate them 
to the Euclidean QCD language, identifying in this way the hierarchy of the 
spectral scales of the Dirac operator.

**Primer on the diffusion**

The Banks-Casher relation is reminiscent of the Einstein relation describing 
the conductivity of degenerate gas of electrons,

$$\sigma = 2\pi e^2 D \langle \mathcal{E}_F \rangle$$

(25)

where $D$ is a diffusion constant, $\langle \mathcal{E}_F \rangle$ is the density of states (per spin direc-
tion) at the Fermi surface. Is it possible that the spontaneous breakdown of 
the chiral symmetry has also the diffusive nature, with the Fermi surface replaced 
by the zero virtuality band? We will show explicitly in this chapter, that this 
indeed is the case.

In order to prove this conjecture, we have to remind some basic facts on the 
diffusion [7, 8]. The diffusion is a process, in which a typical distance covered
by the diffusing particle in a time $t$ varies as

$$x^2(t) = D \cdot t \tag{26}$$

where $D$ is the diffusion constant characterizing the medium. If we consider the diffusive motion in a cube $L^3$ of the linear size $L$, it is natural to define the time scale $t_c = L^2/D$, characterizing the time during which the particle can probe the whole system. The energy scale corresponding to this time, known as a Thouless energy [9] (in units where $\hbar = 1$) is $E_c = 1/t_c = D = L^2$. For times shorter than $t_c$, the diffusing particle can probe only part of the volume. For even shorter times (shorter then the time of mean free path between the “dirt” causing the diffusion, $t_e$) the diffusion concept is meaningless. On the other side, for very large times ($t_H = 1$), the diffusive particle will always leave the volume (see Edmond Iancu’s lectures). When such times start corresponding to the inverse of mean quantum spacing of the quantum mechanical levels, the classical concept of the diffusion also becomes meaningless. The above hierarchy of scales could be pictured by a cartoon, where we introduced also the names of four different regimes using the terminology borrowed from the mesoscopic physics [7, 8]: Quantum ($t > t_H$), Ergodic (universal) ($t_H < t < t_c$), Diffusive ($t_c < t < t_e$) and Ballistic ($t < t_e$).

![Figure 1. Schematic ordering of the diffusion time-scales.](image)

In metallic grains, in the ergodic window the spectral properties of the metals are universal, and described by the random matrix theory (universal conductance fluctuations), hence the second name of this regime.

Let us try to make this description more quantitative. Let us introduce the retarded Green’s function $G_R(x; y; E) = \langle x | \mathbb{1} + i\omega_n \mathcal{H} + i\mathcal{A} | y \rangle$. The time-Fourier transformation $G_R(x; y; t)$ describes the amplitude, that the diffusing particle propagates from $x$ to $y$ in a time $t$, under the influence of the dynamics governed by some Hamiltonian $\mathcal{H}$. The Hamiltonian is the microscopic source of the disorder. A classic example is the Anderson Hamiltonian [10]

$$H_A = \frac{1}{2m} (x - i\omega_n)^2 + V(x) \tag{27}$$

where $\langle V(x) \rangle = 0$ and $\langle V(x) V(y) \rangle = (x - y) = \mathcal{O} \cdot \mathcal{O} \cdot t_c$. Note that the disorder is static, i.e. time independent. We can now define a crucial concept of the return probability. This is simply the square of the amplitude
of returning to the same point $x$ in a time $t$, averaged over the disorder. For a particle at a Fermi surface, it reads

$$P(t) = \lim_{x,y \to} \mathcal{V} \int_{-\infty}^{\infty} \mathcal{D} g(x, y) \mathcal{G}_R(x, y; E_F + 2)^\lambda$$

The prefactor $\mathcal{V}$ comes from the translational invariance of the return probability, and the denominator $2$ ($E_F$) guarantees the normalization of the probability to 1.

For a static random potential of the type above one can perform the averaging and then the integration, with the following result:

$$P(x, y; t) = \sum_{q} \frac{e^{D q^2 t} e^{i q(x-y)}}{4 \pi d^d}$$

Hence

$$P(t) = \sum_{q} e^{D q^2 t}$$

or, equivalently, in Fourier space

$$P(\omega) = \sum_{q} \frac{1}{1 + D q^2}$$

In the above, while performing the averaging over the microscopic disorder, we integrated out fast degrees of freedom, getting the effective description in terms soft modes $q_i$. Details of fast degrees of freedom are now hidden in effective parameters like the diffusion constant $D$, and the obvious name "diffuson" for a soft modes $q_i$ is natural in the light of the form $P(x, y; t)$, being the Green's function of a diffusion operator ($\partial_t + D \nabla^2$).

Formally, for very large times ($t \to 0$), the return probability develops a pole (diffuson pole at $q^2 = 0$). However, for such times, quantum effects become relevant, introducing the natural cutoff at the energy scale of the average quantum spacing. This cutoff suppresses infinitely long diffusive orbits, and regulates the pole

$$P(t) = e^{D q^2 t}$$

or, after Fourier transformation

$$P(\omega) = \sum_{q} \frac{1}{1 + D q^2}$$

In the next section, we will demonstrate, that all the above listed concepts are directly applicable to QCD.
Euclidean QCD is diffusive in \( d=4 \)

Let us look at the Dirac operator in the background of some Euclidean gluonic configuration \( A \)

\[
(i D_A + i m) q_k = \kappa (A) q_k
\]

(34)

as a "Hamiltonian" corresponding to the "eigenenergy" \( \kappa \). The imaginary constant shift \( i m \) does not spoil the analogy. For static Hamiltonians in \( d = 1; 2; 3 \) we know from the quantum mechanics, that the evolution in time \( t \) is governed by \( q_k(t) = \exp(i E_k t) q_k(0) \). Time \( t \) is dual to the energy \( E \). We will parallel this construction here, introducing dual "time" to the virtuality ("energy") \( i m \) is by definition independent on this "time" (static in ), we can write down the eigenmode evolution in this "time"

\[
q_k(\tau) = e^{i(\kappa + i m)} q_k(0)
\]

(35)

corresponding to \( 4 + 1 \) dimensional Schrödinger-like equation with static (independent) potential

\[
i \partial_\tau q(\tau) = (i D_A + i m) q(\tau)
\]

(36)

Note that this construction does not modify any properties of the Dirac operator, and is basically equivalent to introducing the Schwinger proper time \( \tau \).

We will prove now, following [14], that the dynamics of the Euclidean \( d = 4 \) Dirac operator in Schwinger time \( \tau \) parallels the dynamics of the usual diffusion \( d = 1; 2; 3 \) in the real time \( t \). This means, that we will identify, in the spectrum of the Dirac operator, four distinct regimes corresponding to quantum, ergodic, diffusive and ballistic regimes of the mesoscopic physics.

From now on, we follow, step by step, the construction outlined in the previous chapter. Instead of Green’s function for a diffusing electron we take a Green’s function for a quark in some unknown gluonic background, which is constant in Schwinger time

\[
G^R(x; y; E) = S_F(x; y; i)
\]

(37)

We can now define by analogy to (28) the return probability in time for a quark at the zero virtuality surface

\[
P(\tau) = \frac{V}{2} \left< \frac{1}{\text{Tr} \langle S_F^\dagger S_F^\gamma \rangle_{\text{QCD}}} \right>(0)
\]

(38)

Here \( S_F S_F^\dagger (x; y; m \ i = 2) \), trace appear due to the matrix structure of the Dirac operator and the averaging is done over the full QCD measure (21), representing the analog of Anderson Hamiltonian representing the disorder of
the system. The parameter $m$ appears here due to the uniform shift $\imath m$ of the original spectrum of the massless Dirac operator.

Till now the analogy was exact. But now we have to perform the averaging over the a priori unknown measure of the QCD. The first alternative is to choose some model of the disorder. Actually the analytical instanton model of Diakonov and Petrov [11] and the numerical simulation by Shuryak [12] are the first realizations of this scenario. We will return to this point later.

Surprisingly, we can calculate $P(\ldots)$ without assuming any model of the disorder, but making use of almost exact low energy theorems of the QCD, introduced in Part 1. First, let us note, that due to the chiral properties of the Dirac operator

$$< \text{Tr} S_F S_F^\gamma > = < \text{Tr} S_F (x; y; z) S_F (y; x; z) >$$

where $z = m = 2$. We may introduce also the isospin sources. One then immediately recognizes the similarity to the pion correlation function structure

$$C^{ab}(x; y; m) = < S_F (x; y; m) i_5 a S_F (y; x; m) i_5 b >_{QCD}$$

where we used the pion dominance formulae (16).

Second, we use another low-energy theorem (18), to replace the $M$ on the r.h.s. of the above equation by the current quark mass $m$. Third, we analytically continue the $C^{ab}(x; y; m)$, by replacing the mass $m$ by $z = m i = 2$. We recognize then that the integrand in the return probability involves the analytically continued pion correlation function, so averaging is equivalent to the analytical continuation of the r.h.s. of $C$

$$P(x; y; z) = \frac{1}{Z} \int_x Z \text{d}^Q \text{e}^{i \langle Q \rangle} C(x; y; z(\ldots))$$

Finally, after integrating over by the residue method, we see already the diffusive structure of $P(x; y; z(\ldots))$, and the return probability (limit $x \to y$) reads

$$P(\ldots) = \text{e}^{2m} \frac{1}{\sqrt{\text{det} Q^2 + 2m + 1}}$$

or, after Fourier transforming

$$P(\ldots) = \text{e}^{DQ^2} \frac{1}{\sqrt{i + 2m + DQ^2}}$$
We constructed in this way the precise analog of the diffusive return probability, with diffusion constant \( D = F^2 = j \langle \sigma \sigma < j \rangle \) \( 0.22 \) fm and the slow "diffusion modes" \( Q = 2^n = L \), with \( n = 1; 2; 3; 4 \) and \( n \) integers.

In fact, using the low energy theorems and the above construction, we implicitly integrated out the fast (gluonic) degrees of freedom, getting in this way the diffusion constant. The remaining dynamics of slow modes (diffusons) is the dynamics of the pions - longest wave excitations of the QCD. The Goldstone nature of the pion manifest itself as a pole in the chiral limit and \( \pi \) limit. We see easily that the very heavy quarks do not diffuse at all (due to the exponential damping). Note also that for light but massive quarks we can read GOR as an expression of the coherence length of the QCD vacuum. Indeed, by definition, coherence length is

\[
L_{\text{coh}} = \frac{1}{D} = \frac{q}{\pi} = \frac{1}{m}\n
\]

where we used that for QCD \( m = 2 \) and the expression for the diffusion constant calculated above. Coherence length of the QCD vacuum is related to the pion mass, so in the massless limit, pions are indeed much more vacuum modes than the pairs of bounded constituent quark and antiquark.

Now we can identify the different regimes in the spectrum of the Dirac operator. Since the Schwinger time is dual to the eigenvalue of the Dirac operator, we immediately identify the analog of Thouless energy, here the Thouless virtuality \([14, 15]\)

\[
\tau_c = D = \frac{L^2}{F^2} = \left( \frac{D}{\mu} \right)
\]

where \( j \langle \sigma \sigma < j \rangle \). Therefore the eigenvalues smaller than \( \tau_c \) are expected to belong to the ergodic (universal) regime, whereas larger than \( \tau_c \) to the diffusive. Since the Banks Casher relation gives us an estimate of the mean level spacing, we know also the borderline of the quantum and ergodic regime. Finally, for very short times (large eigenvalues) the concept of the diffusion becomes meaningless. It is not very difficult to argue, that the borderline between the diffusive and ballistic regime is given by twice the mass of the constituent quark, basically the mass of the meson. Indeed, for times shorter than the required to travel one mean-free path between lumps of the gauge field the concept of dressing (via multiple scatterings) of the current mass is void.

The above hierarchy of scales could be summarized again by the cartoon, where we also remind about the thermodynamical ordering of the scales. On the basis of the analogy with the condensed matter systems, we may suspect, that similarly to the universal conductance fluctuation in the ergodic regime, we may see universal spectral fluctuations in the ergodic window of the QCD, hopefully described by some sort of random matrix theory. In the next chapters, we will see that this indeed is the case.
Ergodic (universal) regime of the QCD

Let us consider in more detail the ergodic regime. We expand the sum over the diffuson (pion) modes valid in diffusive regime, i.e. we sum over the quadruples of integers \( Q = (n_1, n_2, n_3, n_4) \) such that

\[
P(\ ) = e^{2m} \left( 1 + 8e^{4} \epsilon^{2D=4} + \cdots \right)
\]

where the first term comes from all \( n_1 = 0 \), second from all combinations of \( n_1 \) of the type \( (1; 0; 0; 0) \) etc. Since \( D = L^2 = c^1 \), the return probability reads

\[
P(\ ) = e^{2m} \left( 1 + 8e^{4} \epsilon^{2} c^1 + \cdots \right)
\]

For the times larger than \( t_c \), all terms except of the first one vanish exponentially, so we obtain, that in the ergodic regime the return probability is simply

\[
P(\ ) = e^{2m}
\]

approaching constant for very large \( \epsilon \). Only the softest modes (zero modes \( n_1 = n_2 = n_3 = n_4 = 0 \)) determine the return probability in the ergodic regime. This points that the properties of the ergodic regime are universal—they are independent on the details of space-time interactions, since the Goldstone bosons interaction involves derivative terms, which vanish in long range limit.

This result was known since long ago, although rephrased in a different way. In the usual chiral perturbation theory, pion momenta are of order \( 1 = L \), the mass of the pion scales therefore like \( 1 = L^2 \), and since \( m = M^2 \), the combination \( m^2 \frac{V}{\rho} = (1) \) is fixed. The systematic expansion based on this counting is the clue of the Weinberg [13] chiral perturbation expansion and was practically realized by Gasser and Leutwyler [16]. What is less known, that Gasser and Leutwyler looked also at finite volume, and what happens, when one keeps only zero modes of the pion propagator. Since the propagator in the final volume reads

\[
G = \frac{1}{V} \sum_{Q} e^{i Q_n \times} \frac{1}{Q_n^2 + M^2}
\]

\[
= \frac{1}{VM^2} + \frac{1}{V} \sum_{Q} e^{i Q_n \times} \frac{1}{Q_n^2 + M^2}
\]

Figure 2. Disorder regimes in the eigenvalue spectrum of the Dirac operator. Note the thermodynamical ordering.
the counting based on the concept of having $mV$ fixed \cite{17}(equivalent to $M^2V \circ (1)$) kills all the terms in the primed sum (non-zero modes) and leaves the zero mode. This is very different to the counting based on before-mentioned $m^2V = 1$ principle (chiral perturbation theory).

Gasser and Leutwyler managed to re-sum the contribution of all the zero modes based on $mV \circ (1)$ counting, and obtained the exact partition function

$$Z(m) = \int \mathcal{D}U e^{V \text{Tr} m(U + U^y)}$$

with $U = \exp(i\sigma)$ and $\sigma = \langle \vec{q}\vec{q} \rangle$. This formula is universal: it depends only on the ways how chiral symmetry is to be broken spontaneously (choice of the measure $\mathcal{D}U$ of the Goldstone modes) and how chiral symmetry is being broken explicitly, by the $(N_f;N_f) + (N_f;N_f)$ representation in the exponent.

In the ergodic regime any theory/model sharing the same global symmetries as QCD belongs to the same universality class, i.e. any theory/model leading to the return probability $P(\cdot) = e^{2m}$ will exhibit the same universal spectral properties in the ergodic window. Since the information about the dimensionality is being lost in the ergodic regime, this could be even a zero-dimensional version of the field theory. But field-theory in zero-dimensions is a matrix model. So in the ergodic window QCD is equivalent to a certain matrix model. Before we will unravel the details of this model, let us remind once more that this equivalence between QCD and Random Matrix Model (RMM) happened only for eigenvalues below $c$. Since the edges of the ergodic window scale with volume as $1=V$ (Banks-Casher) and as $c=\sqrt{V}$ (Thouless virtuality in four dimensions) the ergodic, universal window shrinks to a point in the infinite volume limit.

**Where does the "color dirt" come from?**

At this moment we may start to worry, what plays the role of the diffusive dirt in the case of the QCD? QCD is a fundamental theory, so the "dirt" has to be an immanent feature of the QCD itself. Whereas we do not have an exact answer what is the color dirt in QCD, let us observe, that most of the localized, Euclidean (i.e. static from the point of view of Schwinger time) gluonic configurations do the job. Typical and perhaps the most natural are instantons \cite{18}. First, they are the classical, localized stable Euclidean solutions of the QCD. Second, Dirac equations in the presence of instanton background possesses the chiral zero modes, therefore the instanton vacuum immediately provides a microscopic "hopping" mechanism from one instanton field to another and guarantees flipping the chirality at each "scattering" in "time". Quantitatively, each instanton provides a seed of non-conservation of the chiral charge, by integrated form of the anomaly:

$$Q_A = 2N_fQ$$
where $Q$ is the topological charge of the instanton. The instanton model involves basically two parameters, the average size of the instanton ($1\sim 3$ fm) and the concentration $n \sim \text{fm}^{-4}$ (the typical density of instantons). Therefore every dimension-full quantity in the instanton model depends parametrically on the combination of the two scales. In particular, the smallness of the diffusion constant calculated in the previous chapter, finds a natural explanation as a diluteness of the instanton medium. Note that the finite value of the condensation requires that thermodynamical limit $V \to 1$, $N \to 1$, with $N = V$ fixed, where $N$ is the number of instantons and antiinstantons.

Each instanton vacuum configuration is a snapshot in a time $\tau$. Each snapshot violates the Lorentz invariance ("static", particular distribution of instantons and antiinstantons in four volume) and gauge invariance (each instanton freezes a direction in color). But the averaging over collective coordinates of the instantons (here over the centers of instantons and over their color measure) restores the gauge and Lorentz invariance. We would like to stress, that instantons are sufficient, but not necessary configurations to realize the diffusive scenario. Several other models may also provide chiral disorder, e.g. family of stochastic vacuum models [19]. Each of these models comes with a certain correlation length scale, which corresponds quantitatively to the diffusion constant.

In other ways, the spontaneous breakdown of the chiral symmetry in QCD is a very robust phenomenon, comparing to the confinement. Usually, the models of color dirt either ignore confinement, or introduce it by hand, or as instantons, seem to be (at least naively) not related to confinement at all. Mysteriously, lattice evidence suggests strongly that both phenomena (confinement and chiral symmetry breakdown) are correlated and vanish at the same temperature. Since the confining configurations are the topic of vivid speculations, the fundamental understanding of the "color dirt" and chiral disorder is still missing.

Part 3

I analyze in more detail two regimes of the QCD, the ergodic one and the diffusive one. Then I confront few sample predictions with the "experimental data" obtained from lattice simulations.

Random matrices - field theory in 0 dimension

As stated before, in the ergodic regime any model obeying the global symmetries of the QCD belongs to same universality class as QCD. Since this is the regime where only the constant pionic modes matter, we can ignore space-time dependence and stick to the field theory in zero dimensions, i.e. the theory where fields are numbers and do not have any space-time dependence. The exact form of such theory in the QCD can be inferred in numerous ways. Here we follow the historical route [11, 12, 20]. Imagine that we have two lumps of
the "dirt", e.g. instanton and anti-instanton, separated at the very large distance. Then, the Dirac equation in the field of the instanton has an exact fermionic right-handed zero mode \( R \), and the Dirac equation for the anti-instanton has a similar, but left-handed zero mode \( L \). When we decrease the distance between instantons, the degenerated pair of zero modes is replaced by the pair of eigenvalues \( (T; \bar{T}) \), where the overlap \( T = \int d^4x \frac{\gamma_5}{R} \otimes L \) depends on the distance and the mutual orientation of the instantons. Let us now add more and more instantons into this medium, and still work in a dilute gas approximation. The infinite fermion determinant, when calculated in the basis of left and right handed quark zero modes, is now approximated by the matrix of overlaps between the \( I \)-th instanton and \( J \)-th anti-instanton \( T_{IJ} \). The off-diagonal block structure comes from the chirality flipping mechanism. The diagonal blocks are zero in the chiral limit. In this way, instanton picture trades the a priori unknown QCD measure (21) into the approximate measure

\[
< (::) > = \frac{Z}{\det T} \int d[ \mathfrak{m}(::) \mathfrak{e}^{S_{\text{glue}}(\mathfrak{m})} \prod_{N_f} T (\mathfrak{m})^{T (\mathfrak{m})}} (52)
\]

where \( \mathfrak{m} \) is the set of collective coordinates of the instantons (positions and color orientations), and \( S_{\text{glue}} \) is the gluonic part of the QCD action saturated with the initial instanton vacuum ansatz. Let us truncate now all the space time dependence in the above action. We are left with the model of the type

\[
Z = \int dT \mathfrak{e}^{N_f T T} \prod_{N_f} \det T (\mathfrak{m})^{N_f} (53)
\]

Here \( T \) is an \( N \times N \) matrix built out of complex numbers (one could generalize the matrices to the rectangular ones as well). The gluonic measure is replaced by the polynomial measure with some potential \( \mathfrak{v} (\mathfrak{m}) \), here being the simplest - the harmonic potential. The factor \( N_f \) in front of the potential guarantees, that each integration over \( T \) is appropriately weighted. The model has one scale, i.e. the "width" of the Gaussian matrix measure. The partition function \( Z \) defines the chiral Gaussian random matrix model [21]. The name chiral comes from the off-diagonal block structure, which mimics the original chiral structure of the Dirac operator, \( [A(\mathfrak{m}) ; \mathfrak{m}] = 0 \).

We will show now, that such defined model is exactly solvable for any finite \( N \). The main problem is the integration over the measure \( dT \). We will use here the trick, similar to the change of the variables from Cartesian coordinates to the spherical ones. The simplest way to change usual coordinate variables is to look at the infinitesimal interval

\[
ds^2 = dx^2 + dy^2 + dz^2 = dx^2 + r^2 dr^2 + r^2 \sin^2 \theta d\theta^2\]

so the metric tensor is \( g_{\mathfrak{m}} = \mathfrak{e}^{\mathfrak{m} (::) \mathfrak{e}} \), \( 1 ; t^2 ; r^2 \sin^2 \theta \). Hence the Jacobian (the square root of the determinant of the metric tensor) reads \( J = \mathfrak{e}^{\mathfrak{m}g_{\mathfrak{m}} / \mathfrak{m}g_{\mathfrak{m}}} \).
\[ p = r^2 \sin \theta. \] We observe now that the integrand of the partition function (53) depends only on the “radial” combination \( X = T^2 \), and we follow the trivial example above. Every hermitian matrix can be diagonalized by a unitary transformation. Since \( X \) is hermitian, introducing \( X = U R U^T \), where the unitary matrix \( U = \exp(iH) \), \( H \) hermitian, and \( R_{ij} = \delta_{ij} r_i \) is positive diagonal, we calculate

\[
\text{tr}(dX^2) = \text{tr}(U (dR + iR; dH) U^T)^2 = \text{tr}(dR + iR; dH)^2
\]

\[
= \sum_k dX_k^2 + \sum_{i<k} (r_i^2 - r_j^2) \delta_{ij} H_{ij}
\]

(55)

Hence the metric tensor reads

\[
g_{lk} = \text{diag}(1; \ldots; 1; r_1^2; \ldots; r_N^2; 1; \ldots; 1)
\]

(56)

so \( \det g_{lk} = \prod_{i<j} (r_i^2 - r_j^2) \) and Jacobian \( J = p g = \prod_{i<j} (r_i^2 - r_j^2) \).

Since the integrand and the Jacobian do not depend on the angles parameterizing the unitary matrices, we are left with \( N \) integrations over the diagonal eigenvalues ("radial" variables) \( r_i \)

\[
Z = \prod_i \int dX_i \exp(-N \sum_i r_i^2 - N \sum_i r_i^2)
\]

(57)

Let us note, that the expression \( \langle \mathcal{R} \rangle \) can be rewritten as a determinant (Vandermonde determinant).

\[
\langle \mathcal{R} \rangle = \prod_{i<j} (r_i - r_j)
\]

\[
= \begin{vmatrix}
1 & r_1 & r_1^2 & \cdots & r_1^N \\
1 & r_2 & r_2^2 & \cdots & r_2^N \\
& & \ddots & \ddots & \ddots \\
1 & r_N & r_N^2 & \cdots & r_N^N
\end{vmatrix}
\]

(58)

\textbf{[Exercise. Prove (58).]} Since we can add to each row of the determinant an arbitrary combination of the other rows (without changing the value of the determinant), we can replace the original Vandermondian by the determinant build out of polynomials.

\[
\langle \mathcal{R} \rangle = \begin{vmatrix}
P_0 (r_1) & P_1 (r_1) & P_2 (r_1) & \cdots & P_N (r_1) \\
P_0 (r_2) & P_1 (r_2) & P_2 (r_2) & \cdots & P_N (r_2) \\
& & \ddots & \ddots & \ddots \\
P_0 (r_N) & P_1 (r_N) & P_2 (r_N) & \cdots & P_N (r_N)
\end{vmatrix}
\]

(59)

These polynomials are a priori arbitrary, modulo the fact that the coefficient at the highest power is always 1. We choose now the polynomials in such a way,
that they are orthogonal with respect to the measure. In our case we require, that
\[ Z_1 \int \mathcal{D}r \mathbb{E} \prod_{k=1}^N \mathcal{P}_k \mathcal{P}_k (r) = 1 \]  
(60)

It is obvious, that in the case of \( N_f = 0 \) such polynomials are Laguerre polynomials (modulo trivial rescaling of the coefficients, so the highest power is multiplied by 1) and in case of \( N_f \neq 0 \) are associated Laguerre polynomials \( L_{nk}^{(n_f)} \). Due to the orthogonality of the polynomials all integrations in the partition function factorize, and we can write down arbitrary \( k \)-correlation function between the \( k \) eigenvalues. Integrating over all eigenvalues except of the last one, we arrive at the average spectral density

\[
\mathcal{M}_{\text{micro}} (s) \frac{1}{N} \left( \frac{1}{x = N^{1/2}} \right) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{k!}{(N_f + k + 1)} L_k^{(N_f)} L_k^{(N_f)} z^{N_f + 1} e^{z} \]  
(61)

where we used (in the second line) the Christoffel-Darboux summation formula and where \( z = N^{1/2} \) is an argument of the polynomials, with \( z \) being the original eigenvalue of the matrix \( T \).

Let us now come back to physics. In the previous section we stressed, that the ergodic regime corresponds to the limit, where \( \sqrt{m} \) is fixed (in units of condensate \( \phi \)). This suggests, that the relevant limit for ergodic QCD follows from the chiral random matrix model with \( N \) fixed (in units of the width \( \nu \))\(^1\). Technically this is achieved by so called unfolding procedure. We define the microscopic spectral density \( \mathcal{M}_{\text{micro}} (s) \frac{1}{N} \left( \frac{1}{x = N^{1/2}} \right) \). Using the asymptotic property of Laguerre polynomials

\[
\lim_{N \to \infty} \frac{1}{N} \frac{1}{a} L_N^{(a)} (x = N) = \frac{1}{x^{a-2}} J_a (Z \sqrt{x}) \]  
(62)

we rewrite (61) in the microscopic limit in terms of the following combination of the Bessel functions

\[
\mathcal{M}_{\text{micro}} (s) = \frac{1}{2} 2s (\mathcal{J}_{N_f}^2 (s) \mathcal{J}_{N_f + 1} (s) \mathcal{J}_{N_f + 1} (s)) \]  
(63)

This formula [23] represents the universal fluctuations (universality class of chiral Gaussian unitary ensemble) of the spectral density in the QCD. The parameter \( a \) is fixed by tuning the dimensionless quantity \( N \) to dimensionless quantity \( \sqrt{m} \). It is important to mention, that this formula is universal, i.e. similar for any polynomial potential \( \nu (T^\dagger T) \) of the random matrix model. If we would not impose the limit \( N = O (1) \) while
Instead of \( m_{\text{zero}}(s) \) given by (63) we would have obtained the non-universal spectral density of the Gaussian random models, i.e. Wigner semi-circle \( \frac{p}{4} \). Colloquially speaking, all the universal fluctuations are hidden at a single point of the Wigner semicircle, corresponding to point with coordinates \( (s, \pi) \). The "zooming" of this micro-dot by factor \( N \) unravels the fluctuating, universal pattern of microscopic spectral density of the chiral ensemble (Fig. 3).

**Crossover to the diffusive regime - chiral perturbation**

The universal (ergodic) regime ends at the scale corresponding to the Thouless virtuality. For the scales larger than Thouless virtuality neglecting non-zero modes is no longer justified – all modes have to be taken into account. Let us look at the scaling properties of the eigenvalues. The first eigenvalue which falls off from the ergodic window (i.e. by definition, Thouless virtuality) has to scale \([14, 15]\) (in units of \( z = F^2 \)) as

\[
< c = \frac{D}{L^2} = \frac{\langle \xi \rangle}{\langle \xi \rangle} \frac{F^2}{L^2} = \frac{F^2 V}{L^2} = F^2 L^2
\]

where we used the Banks-Casher relation and the definitions of the Thouless virtuality and the diffusion constant.

It is remarkable, that the analogy with the diffusion can lead to a quantitative description of the spectral correlations in the diffusive regime of the QCD as well. Let us define two-point correlation density of states

\[
R(s) = \frac{1}{\langle (c) \rangle^2} \hbar (c = 2) (c + = 2)i 1
\]
where \( s = 1 \) and, as usual, \( 1 = V \). Then we define the spectral formfactor

\[
K(t) = \frac{1}{2} e^{i t} \int dR(\sigma) e^{\sigma t}
\]

(66)

For times shorter than the Heisenberg time \( t_H \), the standard semiclassical argument developed for mesoscopic systems by [25] allows to relate spectral formfactor to the return probability \( P(t) \)

\[
K(t) = \frac{1}{4} t^2 P(t)
\]

(67)

By definition, the two-level spectral correlation function, integrated from 0 to measures the fluctuation \( \langle \rangle \) of the number of levels \( N \) in a strip of a width

\[
\langle Z < N > Z < N > \rangle = \int_0^\infty d1d2R(12)
\]

(68)

In the ergodic regime (for \( t \to 1 \) ) \( P(t) \) is a constant, hence \( K(t) \to t \). The variance \( \sigma^2 \) grows as logarithm of \( t \), as expected in the random matrix theory. However, for times smaller than \( t_H \) (eigenvalues greater then Thouless energy), the return probability is given by the classical diffusion result

\[
P(t) = \frac{V}{(4Dt)^{d-2}}
\]

(69)

Then \( K(t) = t^1 d^{-2} \), which corresponds, after Fourier transforming, to

\[
R(s) = \sum_{Q} \frac{1}{(s+2DQ)^2}
\]

(70)

For mesoscopic systems (e.g. for disordered electrons in \( d = 3 \) metallic grains), this is the seminal result obtained by Altshuler and Shklovskii [26], demonstrating, that the two-point correlation function comes diagrammatically from the two-diffusion exchange. The variance \( \sigma^2(E) \) reads then

\[
\sigma^2(E) = \frac{E}{E_c} d^{-2} \frac{L}{L(E)}
\]

(71)

The second equation comes from definition of the Thouless scale and from \( E = D = L^2(E) \). The power behavior reflects the fact, that in the diffusive regime diffusion of a particle with energy \( E \) is non-homogeneous, and takes place independently in the number of sub-blocks \( (L=L(E))^3 \) of the original block \( V = L^d \).
The above mentioned crossover from universal regime (logarithmic behavior of the variance) to diffusive regime (power behavior $E = E_c^{3-2}$) was confirmed by [27] in numerical simulation of the metallic regime for Anderson model with $20^3$ sites.

On the basis of the diffusive scenario presented in these notes, we expect similar behavior for QCD, but with $d = 4$ and $t!$. In our case, the square denominator in (70) follows from the exchange of two pions in the double ring diagram corresponding to density-density correlation function, reflecting on disconnected quark susceptibility. Note that we have obtained this result on the basic of spectral analysis, and not by using a standard diagrammatics of the chiral perturbation. The diffusive regime of Altschuler-Shklovskii in mesoscopic systems is equivalent to chiral perturbation theory. The effective models of diffusons (sigma models) correspond to sigma models of the pions, and supersymmetric formulation of the diffuson models by Efetov [28] resembles the family of so-called (partially) quenched chiral perturbation theories [29]. Finite volume QCD plays a role of a 4-dimensional quantum dot.

For QCD, the diffusive regime should show the power behavior $2^2(\ldots)$, since the diffusion takes place in $d = 4$ ([14]). Recent lattice simulation [30] of the $SU(2)$ QCD has confirmed this scenario, demonstrating clearly the expected crossover from ergodic regime (logarithmic behavior) to diffusive regime (power-like behavior) predicted for the QCD. It is illuminating to compare the character of the crossover in metallic grains (e.g. Fig. 3 in [27]) to crossover regime of the disconnected quark susceptibility (e.g. Fig. 8 in [30]).

Let me mention another intriguing analogy. In the case of Hamiltonians invariant under the time reflection, one does not differentiate between the direction, in which the loop contributing to return probability is traversed. Hence the naive contribution to the return probability is doubled, which corresponds to additional diffuson-like contribution in the formula for return probability. This quantum interference of the identical orbits traversed in opposite directions is called in condensed matter coherent backscattering (weak localization), and the collective diffusion like-excitation corresponding to this effect is called a cooperon, since the charges add during this interference. This phenomenon has also an analogy in QCD [14, 31]. In case of a real color group (e.g. $SU(2)$), we do not distinguish between the quark and antiquark, and on top of the usual pions (diffusons) we can form the collective states corresponding to $q\bar{q}$ pairs, so called baryonic pions - hence QCD-cooperons. The above mentioned lattice analysis [30] clearly identified the baryonic pions in quenched lattice simulation, in agreement with theoretical predictions based on the diffusive scenario of the QCD.

In the light of the above results, it is tempting to speculate, that the ideas borrowed from mesoscopic systems may have much broader domain of applications in the Euclidean QCD. Some preliminary analysis of the chiral disorder
influenced by external sources seems to show, that this is indeed the case [31]. The effects on disorder of quark chemical potential resemble a complex electric Aharonov-Bohm effect, breaking particle-antiparticle symmetry, accumulating two flux lines and leading to rupture of "baryonic" quark-antiquark pair at $\mu = M = 2$. This is observed in quenched lattice simulations and in random matrix models [32]. The phenomenon of persistence currents in disordered media finds an analogy to diffusion of light quarks in the presence of several Abelian Aharonov-Bohm fluxes. Some magnetic properties of chiral condensate can be explained by replacing the diffuson/cooperon trajectories by the four-dimensional Landau orbits. Low energy theorems obtained in [33] can be interpreted as negative magnetoresistance of the QCD vacuum (quark condensate grows with the magnetic field). Low temperatures correspond to the replacement of initial cube $L^4$ by asymmetric box $L^2$, and the diffuson modes reproduce the lowest temperature corrections for the chiral observables.

An extremely interesting case corresponds to the situation, when the critical temperature is reached. In the case of a second order phase transition, pion wavefunction, susceptibility and the condensate undergo the following scaling

$$Z = \gamma; \quad \beta = 1; \quad \mu = m^{1-\alpha}$$

(72)

where I used the standard notation for the critical exponents. These effects have an obvious quantitative effects on the diffuson (pion), modifying the return probability. In particular, the probability of return at the critical point tends to universal behavior $P(\gamma) = (\mu = \gamma^2)$. This has to be contrasted with the vacuum result $P(\gamma) \propto \text{const}$ in the ergodic regime and vacuum result $P(\gamma) \propto \text{const}$ in the diffusive regime. Finally, let me point that the concept of the return probability at the Schwinger time allows to check scenarios beyond the rigorous theory of phase transitions. For metal-insulator type transition, $P(\gamma) \propto m^{1-\alpha}$. The multifractal exponent estimated on the basis of chiral disorder is $\alpha = 4 = 0.057$ [31]. The smallness of $\alpha$ makes the return probability to look "diffusive" with $\alpha = 2$. Finally, the asymmetry of the box may cause the appearance of two diffusion processes: "temporal" one and the "spatial" one. Since the spatial diffusion constant $D_s$ is small in the high temperature phase (typically, $D_s = D_t \propto \text{at } T = 180 \text{ MeV}$), the asymmetry in conduction properties may cause a "percolation" from $\alpha = 4$ to $\alpha = 1$, with one-dimensional diffusive behavior $P(\gamma) \propto m^{1-\alpha}$. These speculations are interesting from the point of the view of lattice spectral analysis in the vicinity of the critical temperature, since the return probability in the Schwinger time can be explicitly expressed in terms of the eigenfunctions and eigenvalues of the Dirac operator.
Threefold ways and lattice spectra

Let me finish these lectures reminding the classification of the global symmetries of the Dirac operator, hence the classification of the chiral universality classes.

For complex quarks, the Dirac operator has no symmetries. For real quarks (two colors), Dirac has an anti-unitary symmetry, $[ 2 \ 4 \ 2K \ ; iD_F ] = 0$, where $K$ denotes complex conjugation. For adjoint quarks Dirac operator has another anti-unitary symmetry, $[ 2 \ 4K \ ; iD_A ] = 0$, where subscripts $F; A$ denote the fundamental and adjoint representations, respectively. This completes the list, forming the threefold way, how global symmetries can be realized for the Dirac Hamiltonian [34]:

I. For complex quarks with $N_f$ flavors, the pattern of spontaneous breakdown of the chiral symmetry corresponds to $SU(2N_f)$! $SU(2N_f)$! $SU(N_f)$, and the random matrix realization is obtained by filling the matrix with complex numbers (chiral (denoted by $\chi$) Gaussian unitary ensemble ($G_{\text{ue}}$)). This is the case of the QCD, analyzed in these lectures.

II. For real quarks with $N_f$ flavors, the pattern of spontaneous breakdown of the chiral symmetry corresponds to $SU(2N_f)$! $Sp(2N_f)$, and the random matrix realization is obtained by filling the matrix with real numbers (chiral Gaussian orthogonal ensemble ($G_{\text{oe}}$)). This is the case of QCD with two colors.

III. For adjoint quarks with even $N_f$, the pattern of spontaneous breakdown of the chiral symmetry corresponds to $SU(N_f)$! $O(2N_f)$! $O(N_f)$, and the random matrix realization is obtained by filling the matrix with quaternions (chiral Gaussian symplectic ensemble ($G_{\text{spe}}$)). This is the case of QCD with adjoint fermions, for any number of colors.

For each of these ensembles, one can obtain exact spectral formulae. Historically, the first one was (63), obtained in [23]. Technically, the complex case is the easiest one.

This triad is distorted on the lattice, due to the known problems of incompatibility of having chiral fermions on the discrete lattice with local action $^2$.

For example, the additional symmetries of Kogut-Susskind Hamiltonian cause, that for two colors the lattice universality class is $G_{\text{spe}}$, and not, as expected from II, $G_{\text{oe}}$. Historically, the microscopic spectrum for this ensemble was the first one explicitly confirmed by lattice, in [35]. The formula (63) was confirmed successfully on the lattice ($N_f = 0$ case) only recently [36].

Nowadays, there exist an impressive list of theoretical predictions for several spectral (also higher-point) correlations and plethora of lattice evidence for various combinations of gauge groups, fermion representations, topological sectors, with quenched and unquenched and "double scaled" determinants etc.
We refer the curious reader to original literature. In all cases, the agreement with RMM is excellent.

There is also a growing interest in moving toward the diffusive regime. First, the scaling (64) was confirmed by lattice study [37]. In the previous subsection we mentioned the measurements of the crossover between the ergodic and diffusive regime. They are also first measurements on the chiral parameters, motivated by the spectral properties of the Dirac operator.

**Further reading**

An excellent review on chiral dynamics is [38], where the reader will find deeper justification of several statements made in Part I. The original arguments on diffusion in QCD, based on rather compact paper [14] and the sequels [31], will become more obvious after reading condensed matter literature on this subject [7, 8]. The consequences of Banks-Casher relation are discussed in [6]. Basic informations on the phenomenology of the instanton vacuum could be found in [18]. Mehta’s book [39] is a classics in random matrix theory, for the reviews on the chiral ensembles we refer to [40, 41]. Readers interested in broader aspects of RMM we refer to [42, 43]. I did not discuss here at all the schematic use of random models in QCD phenomenology, concentrating here on exact, quantitative predictions in the ergodic window. Sample applications could be found in a review [44] and references therein. Finally, a guidebook to advanced details (supersymmetric technique, replicas etc.) of the partially quenched perturbation techniques could be found in recent reviews [41, 45] and references therein.

**Conclusions**

In these lectures, I tried to demonstrate in a physical way how much the concepts of the spontaneous breakdown of the chiral symmetry share with the concepts of the disorder in mesoscopic systems. This should comply with the introductory character of these lectures. I perhaps omitted several important works to quote, and I apologize for this. The main idea of these lectures was to introduce a non-expert to the fascinating field of chiral disorder in the QCD. I therefore sacrificed several advanced technical details and mathematical aspects, which are often necessary to prove the analogies presented here in a strict way. Luckily, there exist reviews, some quoted above, which exhaustively can guide the reader through the technicalities of random matrices and various versions of chiral perturbation theories adapted for finite volume systems. They also provide the more complete bibliography.

I pointed, that the ergodic (universal) domain, is to large extent understood and documented in an impressive way. Universality of the ergodic regime helped lattice QCD to check to what extend chiral properties (including the
global features e.g. zero modes) of the quarks are reflected by simulation. They also allowed to quantify the discrepancies, in the light of the exact analytical predictions of the RMM. I tried to emphasize, how challenging is to look at the spectra of the diffusive regime. The lattice analogs of the diffuson theories open a way to systematic extraction of the effective parameters of the chiral Lagrangian, following step by step the Weinberg expansion. (The first example is a pion decay constant). It is also possible to study the dependence of these constants of the chiral Lagrangian on the external parameters and it is important to see what happens to them when the critical regime is approached. It is also plausible, that the spectral analysis in the diffusive regime may serve as an important tool for studying the nonperturbative nature of the gluon fields and their quantitative rearrangements in the vicinity of the phase transition, hopefully shading some more light on the nature of the lumps of the "color dirt".

A more general message, which the reader should infer from these lectures and from several other lectures at this school is, how broadly and how fruitfully the theory of strong interactions can borrow from the concepts and ideas of the condensed matter theory.

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Notes

1. The rigorous proof of the equivalence of RMM limit and chiral partition function is non-trivial [22].
2. The original classification can be however reproduced using nonlocal actions.

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