ORLICZ-BESOV EXTENSION AND AHLFORS $n$-REGULAR DOMAINS

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Abstract Let $n \geq 2$ and $\phi : [0, \infty) \to [0, \infty)$ be a Young’s function satisfying
\[ \sup_{t \in (0, \infty)} \int_0^1 \frac{\phi(tx)}{\phi(t)} \frac{dx}{t} < \infty. \]
We show that Ahlfors $n$-regular domains are Besov-Orlicz $B^\phi$ extension domains, which is necessary to guarantee the nontriviality of $\dot{B}^\phi$. On the other hand, assume that $\phi$ grows sub-exponentially at $\infty$ additionally. If $\Omega$ is a Besov-Orlicz $B^\phi$ extension domain, then it must be Ahlfors $n$-regular.

1. Introduction

Let $\phi : [0, \infty) \to [0, \infty)$ be a Young function, that is, $\phi$ is a convex, $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ and $\lim_{t \to \infty} \phi(t) = +\infty$. Given any domain $\Omega \subset \mathbb{R}^n$, the Orlicz-Besov space $\dot{B}^\phi(\Omega)$ consists of all measurable functions $u$ in $\Omega$ whose (semi-)norms
\[ \|u\|_{\dot{B}^\phi(\Omega)} := \inf \left\{ \alpha > 0 : \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(x) - u(y)|}{\alpha} \right) \frac{dxdy}{|x-y|^{2n}} \leq 1 \right\} \]
is finite. Modulo constant functions, $\dot{B}^\phi(\Omega)$ is a Banach space. We refer to [14] for the applications of Orlicz-Besov spaces in quasi-conformal geometry. Note that, in the case of $\phi(t) = t^p$ with $p \geq 1$, the $\dot{B}^\phi(\Omega)$-norms are written as
\[ \|u\|_{\dot{B}^\phi(\Omega)} = \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{2n}} \frac{dxdy}{|x-y|^{2n}} \right)^{1/p}. \]
By this, when $\phi(t) = t^p$ with $p > n$, $\dot{B}^\phi(\Omega)$ is exactly the Besov spaces $B^{n/p}_p(\Omega)$ (or fractional Sobolev spaces $W^{n/p,p}(\Omega)$). However, when $\phi(t) = t^p$ with $p \leq n$, thanks to (1.1) and [4], the space $\dot{B}^\phi(\Omega)$ is trivial, that is, only contains constant functions.

In general, to guarantee the nontrivially of $\dot{B}^\phi(\Omega)$, we always assume
\[ \|u\|_{B^{n/p}_p(\Omega)} := \sup_{x \in \Omega} \int_0^1 \frac{\phi(t)}{t^n} \frac{ds}{s^n} < \infty. \]
Indeed, (1.2) does imply that $\dot{B}^\phi(\Omega)$ contains smooth functions with compact supports, and hence nontrivial; see Lemma 2.2 below. If $\phi(t) = t^p$, observe that $\phi$ satisfies (1.2) if and only if $p > n$, and $\dot{B}^\phi(\Omega)$ is nontrivial if and only if $p > n$. In this sense, we see that (1.2) is optimal to guarantee the nontrivially of $\dot{B}^\phi(\Omega)$. There are some other interesting Young functions satisfying (1.2), for example, $t^\alpha \log(1 + t)$ with $\alpha > 1$, $t^\alpha \log(1 + t)^\alpha$ with $p > n$ and $\alpha \geq 1$, $t^\alpha e^{ct^2}$ with $p > n$, $c > 0$ and $\alpha > 0$, and $e^{ct^2} - \sum_{j=0}^{[n/\alpha]} (ct^2)^j/j!$ where $c > 0$ with $\alpha > 0$, where $[n/\alpha]$ is the maximum of integers no bigger than $n/\alpha$.

In this paper, we obtain the following results for the Orlicz-Besov extension in Ahlfors $n$-regular domains. Recall that a domain $\Omega$ is Ahlfors $n$-regular if there exists a constant $C_A(\Omega) > 0$ such that
\[ |B(x, r) \cap \Omega| \geq C_A(\Omega) r^n \quad \forall x \in \Omega, 0 < r < 2 \operatorname{diam} \Omega. \]
A domain $\Omega$ is called a $B^\phi$-extension domain if any function $u \in B^\phi(\Omega)$ can be extended to as a function $\overline{u} \in B^\phi(\mathbb{R}^n)$ continuously and linearly; in other words, there exists a bounded linear operator $E : B^\phi(\Omega) \to B^\phi(\mathbb{R}^n)$ with $E|_{\Omega} = u$ for any $u \in B^\phi(\Omega)$.

\textbf{Theorem 1.1.} Let $\phi$ be a Young function satisfying (1.2).

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\(\text{lim sup} \phi(x)e^{-cx} = 0 \quad \forall c > 0.\)

(i) If \(\Omega \subset \mathbb{R}^n\) is Ahlfors \(n\)-regular domain, then \(\Omega\) is a \(B^\phi\)-extension domain.

(ii) Assume that \(\phi\) additionally satisfies

\[
\text{subexp} \begin{align*}
\lim_{x \to \infty} \phi(x)e^{-cx} = 0 & \quad \forall c > 0. \\
\end{align*}
\]

If \(\Omega \subset \mathbb{R}^n\) is a \(B^\phi\)-extension domain, then \(\Omega\) is Ahlfors \(n\)-regular.

Note that the condition (1.3) in Theorem 1.1 (ii) allows a large class of Young functions, including \(t^\alpha[\ln(1+t)]^\alpha\) with \(\alpha > 0\), \(t^\alpha[\ln(1+t)]^\alpha\) with \(p > n\) and \(\alpha \geq 1\), \(t^\alpha e^{ct}\) with \(p > n\) and \(\alpha \in (0, 1)\), and \(e^{ct} - \sum_{j=0}^{[n/\alpha]} (ct^j)/j!\) where \(c > 0\) with \(\alpha \geq 1\),

Theorem 1.1 extends the known results for fractional Sobolev spaces \(W^{1/p,p}(\Omega)\) or Besov space \(B^{1/p,p}(\Omega)\).

Recall that the extension problem for function spaces (including Sobolev, fractional Sobolev, Hajłasz-Sobolev, Besov and Triebel-Lizorkin spaces) have been widely studied in the literature, see [7, 8, 20, 9, 10, 2, 11, 16, 21, 22, 5, 17, 19] and the references therein. Given function spaces \(X(U)\) defined in any domain \(U \subset \mathbb{R}^n\) in the same manner, define \(X\)-extension domains similarly to \(B^\phi\)-extension domains. It turns out that the extendability of functions in \(X(\Omega)\) not only relies on the geometry of the domain but also on the analytic properties of \(X\). In particular, it was essentially known that Ahlfors \(n\)-regular domains are fractional Sobolev \(W^{s,p}\)-extension domains for any \(s \in (0, 1)\) and \(p \geq 1\); see Jonsson-Wallin [9] (also Shvartsman [17]). Here \(W^{s,p}(\Omega)\) is the set of all functions with

\[
||u||_{W^{s,p}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p} < \infty.
\]

Moreover, by Shvartsman [18] and Hajłasz et al [5, 6], Ahlfors \(n\)-regular domains also are Hajłasz-Sobolev \(M^{1/p}\)-extension domain with \(p \geq 1\). Recall that for a given function \(u\) in \(\Omega\), we say \(g\) is a Hajłasz gradient of \(u\) (for short \(g \in D(u, \Omega)\)) if

\[
|u(x) - u(y)| \leq |x - y| |\nabla g(x) + g(y)| \quad \text{for almost all } (x, y) \in \Omega \times \Omega.
\]

The Hajłasz Sobolev space \(M^{1/p}(\Omega)\) is the set of all functions \(u\) in \(\Omega\) with

\[
||u||_{M^{1/p}(\Omega)} := \inf_{g \in D(u, \Omega)} ||g||_{L^p(\Omega)} < \infty.
\]

Conversely, Hajłasz [5, 6] essentially proved that Hajłasz-Sobolev \(M^{1/p}\)-extension domains must be Ahlfors \(n\)-regular; and by [24], similar results hold true for fractional Sobolev \(W^{s,p}\)-extension for any \(s \in (0, 1)\) and \(p \geq 1\). Note that \(W^{n/p,p}(\Omega) = B^{n/p,p}(\Omega) = B^\phi(\Omega)\) for any \(p > n\) and \(\phi(t) = tp\).

To prove Theorem 1.1 (i), it suffices to define a suitable linear extension operator and prove its boundedness. Following Jones [8], to define the extension operator we have to find suitable reflecting cubes of Whitney cubes for \(\mathbb{R}^n \setminus \Omega\). If we use the reflecting cubes the same as in [11, 5, 23, 24] which may have unbounded overlaps, we cannot prove the boundedness of the extension operator in general since the Young function may grow exponentially at \(\infty\). For details, see Remark 4.1. Instead, we use the reflecting cubes of Shvartsman [17, 18], which have bounded overlaps (see Lemma 2.2), to define extension operator. The bounded overlaps of reflecting cubes allow us to use the convexity of \(\phi\), and also avoid using maximal functions. With some careful analysis, we finally obtain the boundedness of extension operator.

Theorem 1.1 (ii) is proved in section 5 by borrowing some ideas from [5, 24]. Precisely, we first prove \(\Omega\) supports the following imbedding: there exists positive constants \(C_I(\Omega)\) and \(C(\alpha)\) such that

\[
\lim_{|B| \to 0} \inf_{c \in \mathbb{R}} \int_{B\setminus\Omega} \exp \left( \frac{|u - c|}{\alpha} \right) \, dx \leq C(\alpha)|B| \quad \text{for any ball } B \subset \mathbb{R}^n.
\]

whenever \(u \in B^\phi(\Omega)\) and \(\alpha > C_I(\Omega)||u||_{B^\phi(\Omega)} > 0\). Then we calculate the precise \(||u||_{B^\phi(\Omega)}\)-norm of some cut-off functions. Using this and the sub-exponential growth of \(\phi\) following the idea from [5] (see also [6, 24]), we are able to prove \(\Omega\) is Ahlfors \(n\)-regular.
Corollary 1.2. Suppose that \( \phi \) is a Young function satisfying (1.2) and (1.3). Let \( \Omega \subset \mathbb{R}^n \) be any domain. The following are equivalent:

(i) \( \Omega \) is Ahlfors \( n \)-regular;
(ii) \( \Omega \) is a \( \dot{B}^\phi \)-extension domain;
(iii) \( \Omega \) supports the imbedding (1.5).

Remark 1.3. We conjecture that Theorem 1.1 (ii) holds without the additional assumption (1.3). The difficult to remove (1.3) is to find a suitable imbedding properties of \( \dot{B}^\phi(\mathbb{R}^n) \) better than (1.5) when \( \phi \) does not satisfies (1.3).

Note that (1.5) is always true when \( \Omega \) is a \( \dot{B}^\phi \)-extension domain, but it is not enough to prove that \( \Omega \) is Ahlfors \( n \)-regular in general. If \( \phi(t) = e^t - \sum_{j=0}^{\lfloor n/\alpha \rfloor} t^j/j! \) for \( t \geq 0 \), by Lemma 2.5, \( \Omega \) supports the imbedding

\[
\int_B \exp \left( \frac{|u(x) - u_B|}{\alpha} \right)^\gamma \, dx \leq C(n)
\]

whenever \( \alpha > C(\gamma, n)\|u\|_{\dot{B}^\phi(\mathbb{R}^n)} \). However, when \( \dot{B}^\phi \)-extension domain, such a imbedding is also not enough to prove \( \Omega \) is Ahlfors \( n \)-regular.

Notation used in the following is standard. The constant \( C(n, \alpha, \phi) \) would vary from line to line and is independent of parameters depending only on \( n, \alpha, \phi \). Constants with subscripts would not change in different occurrences, like \( C_\phi \). Given a domain, set \( B_\Omega(x, r) = B(x, r) \cap \Omega \) for convenience. We denote by \( u_X \) the average of \( u \) on \( X \), namely, \( u_X = \frac{1}{|X|} \int_X u \, dx \). For a domain \( \Omega \) and \( x \in \mathbb{R}^n \), we use \( d(x, \Omega) \) to describe the distance from \( x \) to \( \Omega \).

2. Some basic properties

We list several basic properties of Orlicz-Besov spaces.

Lemma 2.1. Suppose that \( \phi \) is a Young function. Let \( \Omega \subset \mathbb{R}^n \) be any domain. Then \( \dot{B}^\phi(\Omega) \subset L^1(B \cap \Omega) \) for any ball \( B \subset \mathbb{R}^n \), in particular, \( \dot{B}^\phi(\Omega) \subset L^1(\Omega) \) when \( \Omega \) is bounded.

Proof. For any \( \alpha > \|u\|_{\dot{B}^\phi(\Omega)} \), we have

\[
\int_\Omega \int_\Omega \phi \left( \frac{|u(x) - u(y)|}{\alpha} \right) \frac{dy \, dx}{|x - y|^{2n}} \leq 1.
\]

By Fubini’s theorem, for almost all \( x \in \Omega \) we have

\[
\int_\Omega \phi \left( \frac{|u(x) - u(y)|}{\alpha} \right) \frac{dy}{|x - y|^{2n}} < \infty.
\]

Fix such a point \( x \). For any \( B = B(z, r) \subset \mathbb{R}^n \) with \( B \cap \Omega \neq \emptyset \), we have \( |x - y| \leq |x| + |z| + r \) for all \( y \in B \cap \Omega \), and hence

\[
\int_{B \cap \Omega} \phi \left( \frac{|u(x) - u(y)|}{\alpha} \right) \, dy < \infty.
\]

By Jessen’s inequality, we have

\[
\phi \left( \frac{1}{|B \cap \Omega|} \int_{B \cap \Omega} \frac{|u(x) - u(y)|}{\alpha} \, dy \right) < \infty,
\]

which implies that

\[
\int_{B \cap \Omega} |u(y)| \, dy \leq |u(x)| + \int_{B \cap \Omega} |u(x) - u(y)| \, dy < \infty,
\]

that is, \( u \in L^1(B \cap \Omega) \) as desired. \( \square \)
Lemma 2.2. Suppose that \( \phi \) is a Young function satisfying (1.2). Let \( \Omega \subset \mathbb{R}^n \) be any domain. Then \( \mathcal{C}_c^1(\Omega) \subset \mathbf{B}^\phi(\Omega) \).

Proof. Assume that \( L = \|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)} > 0 \). Let \( V = \text{supp} \ u \in W \ominus \Omega \). Then

\[
H := \int_\Omega \int_\Omega \phi \left( \frac{|u(z) - u(w)|}{\alpha} \right) \frac{dzdw}{|z-w|^{2n}} \leq \int_\Omega \int_\Omega \phi \left( \frac{|z-w|}{\alpha/L} \right) \frac{dz}{|z-w|^{2n}} + 2 \int_\Omega \int_\Omega \phi \left( \frac{L}{\alpha} \right) \frac{dzdw}{|z-w|^{2n}}.
\]

By (1.2), we have

\[
\int_\Omega \int_\Omega \phi \left( \frac{|z-w|}{\alpha/L} \right) \frac{dz}{|z-w|^{2n}} \leq \int_\Omega \int_{B(w,2|\text{diam } W|/\alpha)} \phi \left( \frac{t}{\alpha/L} \right) \frac{dt}{|t|^{n+1}} \frac{dw}{|w|^{2n}}
\]

\[
= n\omega_n \int_\Omega \int_0^{2|\text{diam } W|/\alpha} \phi \left( \frac{s}{\alpha^2} \right) \frac{ds}{s^{n+1}} \frac{dw}{|w|^{2n}}
\]

\[
= n\omega_n |W|^{2-n} |\text{diam } W|^{-n} \phi \left( \frac{2L}{\alpha} \frac{|W|}{|\text{dist } (V, W^c)|^{2-n}} \right).
\]

Moreover,

\[
2 \int_\Omega \int_V \phi \left( \frac{1}{\alpha/L} \right) \frac{dzdw}{|z-w|^{2n}} \leq 2 \phi \left( \frac{L}{\alpha} \right) \int_\Omega \int_{B(z, \text{dist } (V, W^c))} \frac{dwdz}{|z-w|^{2n}} \leq 2 \omega_n \phi \left( \frac{L}{\alpha} \right) |V| \text{dist } (V, W^c)^{-n}.
\]

Obviously, letting \( \alpha \) sufficiently enough and using the convexity of \( \phi \), we have \( H \leq 1 \). That is, \( u \in \mathbf{B}^\phi(\Omega) \) as desired.

The following Poincaré type inequality is needed in Section 4. Below denote by \( \omega_n \) the area of the unit sphere \( S^{n-1} \).

Lemma 2.3. Suppose that \( \phi \) is a Young function. For any ball \( B \subset \mathbb{R}^n \) and \( u \in \mathbf{B}^\phi(B) \), we have

\[
\int_B \phi \left( \frac{|u(x) - u_B|}{\alpha} \right) dx \leq \omega_n^2
\]

when \( \alpha > \|u\|_{\mathbf{B}^\phi(B)} \), and

\[
\int_B |u(x) - u_B| dx \leq \phi^{-1}(\omega_n^2) \|u\|_{\mathbf{B}^\phi(B)}.
\]

Proof. Let \( u \in \mathbf{B}^\phi(B) \). For any \( \alpha > \|u\|_{\mathbf{B}^\phi(B)} \), by Jensen’s inequality, we have

\[
\phi \left( \int_B \frac{|u(x) - u_B|}{\alpha} dx \right) \leq \int_B \phi \left( \frac{|u(x) - u_B|}{\alpha} \right) dx
\]

\[
\leq \int_B \int_B \phi \left( \frac{|u(x) - u_B|}{\alpha} \right) \frac{dydx}{|x-y|^{2n}} \leq \omega_n^2 \int_B \int_B \phi \left( \frac{|u(x) - u_B|}{\alpha} \right) \frac{dydx}{|x-y|^{2n}} \leq \omega_n^2,
\]

that is,

\[
\int_B |u(x) - u_B| dx \leq \phi^{-1}(\omega_n^2).
\]

Letting \( \alpha \to \|u\|_{\mathbf{B}^\phi(B)} \), we obtain

\[
\int_B |u(x) - u_B| dx \leq \phi^{-1}(\omega_n^2) \|u\|_{\mathbf{B}^\phi(B)}
\]

as desired. \( \square \)
As a consequence of Lemma 2.3, we have the following imbedding. Denote by \(BMO(\Omega)\) the space of functions with bounded mean oscillations, that is, the collection of \(u \in L^1_{\text{loc}}(\Omega)\) such that
\[
\|u\|_{BMO(\Omega)} = \sup_{B \subset \Omega} \int_B |u(x) - u_B| \, dx < \infty.
\]

**Corollary 2.4.** Suppose that \(\phi\) is a Young function. Let \(\Omega \subset \mathbb{R}^n\) be any domain. We have \(\dot{B}^\phi(\Omega) \subset BMO(\Omega)\) and \(\|u\|_{BMO(\Omega)} \leq \phi^{-1}(\omega_n^2)\|u\|_{\dot{B}^\phi(\Omega)}\) for all \(u \in \dot{B}^\phi(\Omega)\).

Note that if
\[
\phi_o(t) = e^t - \sum_{j=0}^{[n/\gamma]} \frac{t^j}{j!}
\]
with \(\gamma \geq 1\), then \(\phi_o\) a Young’s function satisfying (1.2). Denote by \(\dot{B}^{\phi_o}(\mathbb{R}^n)\) the associated Orlicz-Besov space.

**Lemma 2.5.** For \(\gamma \geq 1\), there exists constant \(C(\gamma, n) \geq 1\) such that for any \(u \in \dot{B}^{\phi_o}(\mathbb{R}^n)\) and ball \(B \subset \mathbb{R}^n\), we have
\[
\int_B \exp \left( \frac{|u(x) - u_B|}{\alpha} \right)^\gamma \, dx \leq C(n)
\]
whenever \(\alpha > C(\gamma, n)\|u\|_{\dot{B}^{\phi_o}(\mathbb{R}^n)}\).

**Proof.** By Corollary 2.4, we have \(u \in BMO(\mathbb{R}^n)\) and \(\|u\|_{BMO(\mathbb{R}^n)} \leq \phi^{-1}(\omega_n^2)\|u\|_{\dot{B}^\phi(\mathbb{R}^n)}\). Thus by the John-Nirenberg inequality, we have
\[
\int_B |u(x) - u_B|^{[n/\gamma]} \, dx \leq C(\gamma, n)\|u\|_{BMO(\mathbb{R}^n)}^{[n/\gamma]} \leq C(\gamma, n)\|u\|_{\dot{B}^\phi(\mathbb{R}^n)}^{[n/\gamma]}.
\]
Thus for all \(1 \leq j \leq [n/\gamma]\), we have
\[
\int_B \left( \frac{|u(x) - u_B|}{\alpha} \right)^{\gamma j} \, dx \leq 1/n
\]
when \(\alpha \geq C(\gamma, n)\|u\|_{\dot{B}^{\phi_o}(\mathbb{R}^n)}\) for some constant \(C(\gamma, n)\). Note that by Lemma 2.3, one has
\[
\int_B \phi_o \left( \frac{|u(x) - u_B|}{\alpha} \right) \, dx \leq 1/n
\]
when \(\alpha > n\omega_n^2\|u\|_{\dot{B}^{\phi_o}(\mathbb{R}^n)}\). Since
\[
e^t = \phi_o(t) + 1 + \sum_{j=1}^{[n/\gamma]} \frac{t^j}{j!}
\]
we obtain
\[
\int_B \exp \left( \frac{|u(x) - u_B|}{\alpha} \right)^\gamma \, dx \leq 3.
\]
when \(\alpha > [n\omega_n^2 + C(\gamma, n)]\|u\|_{\dot{B}^{\phi_o}(\mathbb{R}^n)}\). \(\square\)

3. Whitney’s decomposition and the reflected quasi-cubes

In this section, we always let \(\Omega\) be an Ahlfors \(n\)-regular domain. Observe that \(|\partial \Omega| = 0\); see [17, Lemma 2.1] and also [24, 5]. Moreover, \(\text{diam } \Omega = \infty\) if and only if \(|\Omega| = \infty\). Write \(U := \mathbb{R}^n \setminus \overline{\Omega}\). Without loss of generality, we assume \(U \neq \emptyset\). It’s well know that \(U\) admits a Whitney decomposition.

**Lemma 3.1.** There exists a collection \(W = \{Q_i\}_{i \in \mathbb{N}}\) of (closed) cubes satisfying
(i) \(U = \bigcup_{i \in \mathbb{N}} Q_i\), and \(Q^c_i \cap Q^c_k = \emptyset\) for all \(i, k \in \mathbb{N}\) with \(i \neq k\);
(ii) \(l(Q_k) \leq \text{dist}(Q_k, \partial \Omega) \leq 4 \sqrt{n} l(Q_k)\);
(iii) \(\frac{1}{4} l(Q_k) \leq l(Q_i) \leq 4 l(Q_k)\) whenever \(Q_k \cap Q_i \neq \emptyset\).
The following basic properties of Whitney’s decomposition are used quite often in Section 4. For any $Q \in \mathcal{W}$, denote by $N(Q)$ the neighbor cubes of $Q$ in $\mathcal{W}$, that is,

$$N(Q) := \{ P \in \mathcal{W} : P \cap Q \neq \emptyset \}.$$

Then, by (iii) there exists an integer $\gamma_0$ depending only on $n$ such that

$$\#N(Q) \leq \gamma_0 \quad \text{for all} \quad Q \in \mathcal{W}. \quad (3.1)$$

By (iii) again, for any $P, Q \in \mathcal{W}$ we know that

$$P \in N(Q) \quad \text{if and only if} \quad Q \in N(P), \quad \text{if and only if} \quad \frac{Q}{2} \cap \frac{Q}{2} \neq \emptyset. \quad (3.2)$$

It then follows that

$$\frac{1}{|Q|} \int_{U} \chi_{\frac{Q}{2}}(x) \, dx \leq 4^{n} \gamma_0 \quad \text{for all} \quad Q \in \mathcal{W}. \quad (3.3)$$

Indeed, by (3.2) we write

$$\frac{1}{|Q|} \int_{U} \chi_{\frac{Q}{2}}(x) \, dx = \sum_{P \in N(Q)} \frac{1}{|Q|} \int_{P} \chi_{\frac{Q}{2}}(x) \, dx.$$

By $l_{Q} \leq 4l_{P}$ given in (iii), and (3.1), we arrive at

$$\frac{1}{|Q|} \int_{U} \chi_{\frac{Q}{2}}(x) \, dx \leq \sum_{P \in N(Q)} \frac{|P|}{|Q|} \leq 4^{n} \gamma_0$$

as desired.

Below we recall the reflected quasi-cubes of Whitney’s cubes as given by Shvartsman [17, Theorem 2.4]. For any $\epsilon > 0$, set

$$\mathcal{W}_{\epsilon} := \{ Q \in \mathcal{W} : l_{Q} < \frac{1}{\epsilon} \text{diam} \Omega \}.$$

Obviously, $\mathcal{W} = \mathcal{W}_{\epsilon}$ for all $\epsilon > 0$ if $\text{diam} \Omega = \infty$, and $\mathcal{W}_{\epsilon} \subseteq \mathcal{W}$ for any $\epsilon > 0$ if $\text{diam} \Omega < \infty$.

For any $Q = Q(x_{0}, l_{Q}) \in \mathcal{W}_{\epsilon}$, fix any $x_{\epsilon}^{*} \in \Omega$ so that dist $(Q, \Omega) = \text{dist} (x_{\epsilon}, Q)$. By Lemma 3.1 (ii), one has

$$\tilde{Q}^{\epsilon} := Q(x_{\epsilon}^{*}, l_{Q}) \subset 10 \sqrt{n} Q.$$

Set

$$\tilde{Q}^{\epsilon} := (\epsilon Q^{\star} \cap \Omega) \setminus \left( \bigcup \{ \epsilon P^{\star} : P \in \mathcal{A}^{\epsilon}_{\Omega} \} \right),$$

where

$$\mathcal{A}^{\epsilon}_{\Omega} := \left\{ P \in \mathcal{W}_{\epsilon} : \epsilon P^{\star} \cap \epsilon Q^{\star} \neq \emptyset, l_{P} \leq \epsilon l_{Q} \right\}.$$

Below, when $\epsilon$ is small enough, we define $\tilde{Q}^{\epsilon}$ as reflected quasi-cubes of $Q \in \mathcal{W}_{\epsilon}$ so that they enjoy some nice properties; see [17, Theorem 2.4] for the proof, here we omit the details.

**Lemma 3.2.** Let $\epsilon_{0} = [C_{A}(\Omega) / 2 \gamma_{0}]^{1/n} / (30 \sqrt{n})$. Denote by $Q^{\star} = \tilde{Q}^{\epsilon_{0}}$ as quasi-cubes of any cube $Q \in \mathcal{W}_{\epsilon_{0}}$. Then the following hold:

(i) $Q^{\star} \subset (10 \sqrt{n} Q) \cap \Omega$ for any $Q \in \mathcal{W}_{\epsilon_{0}}$;

(ii) $|Q^{\star}| \leq \gamma_{1} |Q^{\star}|$ whenever $Q \in \mathcal{W}_{\epsilon_{0}}$;

(iii) $\sum_{Q \in \mathcal{W}_{\epsilon_{0}}} |Q^{\star}| \leq \gamma_{2}$.

Above $\gamma_{1}$ and $\gamma_{2}$ are positive constants depending only on $n$ and $C_{A}(\Omega)$.

If $\Omega$ is bounded, we let $Q^{\star} = \Omega$ as the reflected quasi-cube of any cube $Q \in \mathcal{W} \setminus \mathcal{W}_{\epsilon_{0}} \neq \emptyset$. Write

$$\mathcal{W}^{(k)}_{\epsilon_{0}} = \{ Q \in N(P) : P \in \mathcal{W}^{(k-1)}_{\epsilon_{0}} \} \quad \forall k \geq 1,$$

where $\mathcal{W}^{(0)}_{\epsilon_{0}} = \mathcal{W}_{\epsilon_{0}}$. That is, $\mathcal{W}^{(k)}_{\epsilon_{0}}$ is the $k$-th-neighbors of $\mathcal{W}_{\epsilon_{0}}$.

$$V^{(k)} := \bigcup \{ x \in \mathcal{Q} : Q \in \mathcal{W}^{(k)}_{\epsilon_{0}} \} \quad \forall k \geq 0. \quad (3.4)$$
Since $Q^* = \Omega$ for $Q \notin \mathcal{W}_{\ell_0}$, by Lemma 3.3 (iii) we have
\[
\sum_{Q \in \mathcal{W}_{\ell_0}} \chi_{Q^*} \leq \sum_{Q \in \mathcal{W}_{\ell_0}} \chi_{Q^*} + \#(\mathcal{W}_{\ell_0}^{l_0} \setminus \mathcal{W}_{\ell_0}) \chi_{\Omega} \leq [\gamma_2 + \#(\mathcal{W}_{\ell_0}^{l_0} \setminus \mathcal{W}_{\ell_0})] \chi_{\Omega} \quad \forall k \geq 1.
\]

For $Q \in \mathcal{W}_{\ell_0}^{l_0} \setminus \mathcal{W}_{\ell_0}$, observe that $l_0 \geq \frac{1}{\ell_0} \text{diam } \Omega$ and $l_0 \leq 4^k l_p \leq \frac{4^k}{\ell_0} \text{diam } \Omega$ for some $P \in \mathcal{W}_{\ell_0}$. Thus, by Lemma 3.1 (ii), we have
\[
Q \subset Q(\tilde{x}, \text{diam } \Omega + 8 \sqrt{n} \frac{4^k}{\ell_0} \text{diam } \Omega)
\]
for any fixed $\tilde{x} \in \Omega$, and hence
\[
\#(\mathcal{W}_{\ell_0}^{l_0} \setminus \mathcal{W}_{\ell_0}) \leq (1 + 8 \sqrt{n} \frac{4^k}{\ell_0}) \epsilon_0 \leq (\epsilon_0 + 4^{k+2} \sqrt{n})^n.
\]
This yields that
\[
\sum_{Q \in \mathcal{W}_{\ell_0}^{l_0}} \chi_{Q^*} \leq \gamma_2 + (\epsilon_0 + 4^{k+2} \sqrt{n})^n \quad \forall k \geq 1.
\]

Finally, associated to $\mathcal{W}$, one has the following partition of unit of $U$.

**Lemma 3.3.** There exists a family $\{\varphi_Q : Q \in \mathcal{W}\}$ of functions such that
(i) for each $Q \in \mathcal{W}$, $0 \leq \varphi_Q \in C^\infty(\Omega \setminus \frac{1}{16} \mathcal{Q})$;
(ii) for each $Q \in \mathcal{W}$, $|\nabla \varphi_Q| \leq L/|Q|$;
(iii) $\sum_{Q \in \mathcal{W}} \varphi = 1$ on $U$.

4. Proof of Theorem 1.1(i)

It suffices to prove the existence of a bounded linear operator $E : \mathcal{B}^\phi(\Omega) \to \mathcal{B}^\phi(\mathbb{R}^n)$ such that $Eu|_{\Omega} = u$ for all $u \in \mathcal{B}^\phi(\Omega)$. Define the operator $E$ by
\[
Eu(x) = \begin{cases} 
  u(x), & x \in \Omega, \\
  0, & x \in \partial \Omega, \\
  \sum_{Q \in \mathcal{W}} \varphi_Q(x) u_{Q^*}, & x \in U
\end{cases}
\]
for any $u \in \mathcal{B}^\phi(\Omega)$. Recall that $\mathcal{W}$ is the Whitney cubes of $U$ as in Lemma 3.1 and $\{\varphi_Q\}_{Q \in \mathcal{W}}$ as in Lemma 3.3; that $Q^*$ is the reflected quasi-cube of $Q \in \mathcal{W}_{\ell_0}$ as given in Lemma 3.2, and $Q^* = \Omega$ if $Q \in \mathcal{W}_{\ell_0} \setminus \mathcal{W}_{\ell_0}$ (when $\Omega$ is bounded). By Lemma 2.1, $u_{Q^*} = \frac{1}{|Q^*|} \int_{Q^*} u \, dx$ is always finite.

Obviously, $E$ is linear, $Eu|_{\Omega} = u$ in $\Omega$, and moreover, if $\|u\|_{\mathcal{B}^\phi(\Omega)} = 0$, then $u$ and hence $Eu$ must be a constant function essentially. Thus, to prove the boundedness of $E : \mathcal{B}^\phi(\Omega) \to \mathcal{B}^\phi(\mathbb{R}^n)$, by the definition of the norm $\| \cdot \|_{\mathcal{B}^\phi(\mathbb{R}^n)}$, we only need to find a constant $M > 0$ depending only on $n$, $C_A(\Omega)$ and $\phi$ such that
\[
H(\alpha) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi \left( \frac{|Eu(x) - Eu(y)|}{\alpha} \right) \frac{dxdy}{|x-y|^{2n}} \leq 1.
\]
whenever $\|u\|_{\mathcal{B}^\phi(\Omega)} = 1$ and $\alpha > M$. Below we assume that $\|u\|_{\mathcal{B}^\phi(\Omega)} = 1$. Since $|\partial \Omega| = 0$, one writes
\[
H(\alpha) = \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(x) - u(y)|}{\alpha} \right) \frac{dxdy}{|x-y|^{2n}} + 2 \int_{U} \int_{U} \phi \left( \frac{|Eu(x) - u(y)|}{\alpha} \right) \frac{dxdy}{|x-y|^{2n}}
\]
\[
=: H_1(\alpha) + 2H_2(\alpha) + H_3(\alpha).
\]
To get (4.1), it suffices to find constants $M_i \geq 1$ depending only on $n$, $C_A(\Omega)$ and $\phi$ such that $H_i(\alpha) \leq 1/4$ whenever $\alpha \geq M_i$ for $i = 1, 2, 3$. Indeed, by taking $M = M_1 + M_2 + M_3$, we have $H(\alpha) \leq 1$ whenever $\alpha \geq M$. 

Firstly, we may let $M_1 = 4$. Indeed, if $\alpha > 4$ that is, $\alpha/4 > 1$, by the convexity of $\phi$ and $\|u\|_{B^{\phi}(\Omega)} = 1$, we have

$$H_1(\alpha) \leq \frac{1}{4} \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(x) - u(y)|}{\alpha/4} \right) \frac{dy}{|x - y|^{2\alpha}} \leq \frac{1}{4}.$$  

To find $M_2$ and $M_3$, we consider two cases: $\text{diam} \, \Omega = \infty$ and $\text{diam} \, \Omega < \infty$.

**Case** $\text{diam} \, \Omega = \infty$. To find $M_2$, for any $x \in U$ and $y \in \Omega$, since $\sum_{Q \in W} \phi_Q(x) = 1$ by Lemma 3.3, one has

$$Eu(x) - u(y) = \sum_{Q \in W} \phi_Q(x)[u_Q - u(y)],$$

and hence, by the convexity of $\phi$ and Jensen’s inequality,

$$\phi \left( \frac{|Eu(x) - u(y)|}{\alpha} \right) \leq \phi \left( \sum_{Q \in W} \phi_Q(x) \frac{|u_Q - u(y)|}{\alpha} \right) \leq \sum_{Q \in W} \phi_Q(x) \phi \left( \int_{Q^*} \frac{|u(z) - u(y)|}{\alpha} dz \right) \leq \sum_{Q \in W} \phi_Q(x) \int_{Q^*} \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) dz.$$  

If $\phi_Q(x) \neq 0$, then $x \in \frac{17}{16} Q$. For $z \in Q^*$, by $Q^* \subset 10 \sqrt{\pi} Q$, we have $|x - z| \leq 40 n l(Q)$. Since $|x - y| \geq d(x, \Omega) \geq l(Q)$, we have $|x - z| \leq 20 n |x - y|$, that is,

$$|y - z| \leq |x - y| + |x - z| \leq 21 n |x - y|$$

So we have

$$\int_{\Omega} \phi \left( \frac{|Eu(x) - u(y)|}{\alpha} \right) \frac{dy}{|x - y|^{2\alpha}} \leq (21 n)^{2n} \sum_{Q \in W} \phi_Q(x) \int_{Q^*} \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) \frac{dz}{|z - y|^{2\alpha}}.$$  

Thus, by Lemma 3.2 (ii) we write

$$H_2(\alpha) \leq 2(21 n)^{2n} \int_U \sum_{Q \in W} \phi_Q(x) \int_{Q^*} \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) \frac{dz}{|y - z|^{2\alpha}} dx \leq 2\gamma_1 (21 n)^{2n} \sum_{Q \in W} \left( \frac{1}{|Q|} \int_U \phi_Q(x) dx \right) \int_{Q^*} \int_{\Omega} \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) \frac{dz}{|y - z|^{2\alpha}}.$$  

Since $\phi \leq \chi_{\frac{1}{\alpha} Q}$ as given in Lemma 3.3, by (3.3) we have

$$\frac{1}{|Q|} \int_U \phi_Q(x) dx \leq \frac{1}{|Q|} \int_U \chi_{\frac{1}{\alpha} Q}(x) dx \leq 4^n \gamma_0,$$

which implies that

$$H_2(\alpha) \leq 2\gamma_1 4^n \gamma_0 (21 n)^{2n} \sum_{Q \in W} \int_{Q^*} \int_{\Omega} \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) \frac{dz}{|y - z|^{2\alpha}}.$$  

By $\sum_{Q \in W} \chi_{Q^*} \leq \gamma_2$ as in Lemma 3.2 (iii), we obtain

$$H_2(\alpha) \leq 2\gamma_1 4^n \gamma_0 \gamma_2 (21 n)^{2n} \int_{\Omega} \int_{Q^*} \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) \frac{dz}{|y - z|^{2\alpha}}.$$  

Take $M_2 = 8\gamma_1 4^n \gamma_0 \gamma_2 (21 n)^{2n}$. By the convexity of $\phi$ again, if $\alpha > M_2$, we have $H_2(\alpha) \leq 1/4$. 

equ4.w2
Thus we have
\[ X_1(x) := \left\{ y \in U : |x - y| \geq \frac{1}{132n} \max\{d(x, \Omega), d(y, \Omega)\} \right\} \quad \text{and} \]
\[ X_2(x) := U \setminus X_1(x) = \left\{ y \in U : |x - y| < \frac{1}{132n} \max\{d(x, \Omega), d(y, \Omega)\} \right\}. \]

Write
\[ H_3(\alpha) = \int_U \int_{X_1(x)} \phi \left( \frac{|Eu(x) - Eu(y)|}{\alpha} \right) \frac{dydx}{|x - y|^{2n}} + \int_U \int_{X_2(x)} \phi \left( \frac{|Eu(x) - Eu(y)|}{\alpha} \right) \frac{dydx}{|x - y|^{2n}} \]
\[ = H_{31}(\alpha) + H_{32}(\alpha). \]

Below, we show that there exists \( M_{3i} \geq 1 \) such that if \( \alpha > M_{3i} \), then \( H_{3i} \leq 1/8 \) for \( i = 1, 2 \). If this is true, then letting \( M_3 := \max\{M_{31}, M_{32}\} \), for \( \alpha > M_3 \) we have \( H_3 \leq \frac{1}{4} \) as desired.

To find \( M_{31} \), for \( x \in U \) and \( y \in X_1(x) \), since
\[ \sum_{Q \in \mathcal{W}} \varphi_Q(x) = \sum_{P \in \mathcal{W}} \varphi_P(y) = 1, \]
we have
\[ Eu(x) - Eu(y) = \sum_{P \in \mathcal{W}} \sum_{Q \in \mathcal{W}} \varphi_Q(x) \varphi_P(y) [u_Q - u_P] \]
\[ = \sum_{P \in \mathcal{W}} \sum_{Q \in \mathcal{W}} \varphi_Q(x) \varphi_P(y) \int_Q \int_{P^*} [u(z) - u(w)] dwdz. \]

Applying the convexity of \( \phi \) and Jensen’s inequality, one obtains
\[ \phi \left( \frac{|Eu(x) - Eu(y)|}{\alpha} \right) \leq \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{W}} \varphi_Q(x) \varphi_P(y) \phi \left( \frac{\int_Q \int_{P^*} |u(z) - u(w)|}{\alpha} \right) dwdz. \]

For \( x \in Q \) and \( z \in Q^* \), by \( Q^* \subset 10 \sqrt{n}Q \), we have \(|z - x| \leq 10n\sqrt{n} \leq 10nd(x, \Omega)\). Similarly, for \( y \in P \), and \( w \in P^* \), we have \(|y - w| \leq 10nd(y, \Omega)\). If \( y \in X(x) \), that is, \( 132n|x - y| \geq \max\{d(x, \Omega), d(y, \Omega)\} \), we further have
\[ |z - w| \leq |x - z| + |x - y| + |y - w| \leq 2641n|x - y|. \]
Thus
\[ H_{31}(\alpha) \leq (2641n)^{2n} \int_U \int_{X_1(x)} \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{W}} \varphi_Q(x) \varphi_P(y) \int_Q \int_{P^*} \phi \left( \frac{|u(z) - u(w)|}{\alpha} \right) dwdz. \]

By \( |Q| \leq \gamma_1|Q^*| \) and \( |P| \leq \gamma_1|P^*| \) as given in Lemma 2.2 (ii), we have
\[ H_{31}(\alpha) \leq (2641n)^{2n} \gamma_1 \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{W}} \left( \frac{1}{|Q|} \int_U \varphi_Q(x) dx \right) \frac{1}{|P|} \int_U \varphi_P(y) dy \int_Q \int_{P^*} \phi \left( \frac{|u(z) - u(w)|}{\alpha} \right) dwdz. \]

By Lemma 3.3 and (3.3) we have
\[ \frac{1}{|Q|} \int_U \varphi_Q(x) dx \frac{1}{|P|} \int_U \varphi_P(y) dy \leq (4^n \gamma_0)^2. \]
Thus
\[ H_{31}(\alpha) \leq (2641n)^{2n} \gamma_1 (4^n \gamma_0)^2 \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{W}} \int_Q \int_{P^*} \phi \left( \frac{|u(z) - u(w)|}{\alpha} \right) dwdz. \]
Observing $\sum_{Q \in \mathcal{W}} x_{Q^r} \leq \gamma_2$ as in Lemma 3.2 (iii), we arrive at

$$H_{31}(\alpha) \leq (2641n)^{2n} \gamma_1^2 \gamma_2^2 (4^n \gamma_0)^2 \int_U \int_U \phi \left( \frac{|u(z) - u(w)|}{\alpha} \right) \frac{dwdz}{|z - w|^{2n}}.$$  

Letting $M_{31} = 8(2641n)^{2n} \gamma_1^2 \gamma_2^2 (4^n \gamma_0)^2$. If $\alpha > M_{31}$, by the convexity of $\phi$ again we have $H_{31}(\alpha) \leq 1/8.$

To find $M_{32}$, write

$$H_{32}(\alpha) = \int_U \sum_{P \in \mathcal{W}} \sum_{P \cap X_2(x)} \phi \left( \frac{1}{\alpha} |Eu(x) - Eu(y)| \right) \frac{dy}{|x - y|^{2n}} dx.$$  

Let $x \in U$ and $y \in X_2(x) \cap P$ for some $P \in \mathcal{W}$. Since

$$\sum_{Q \in \mathcal{W}} [\varphi Q(x) - \varphi Q(y)] = 0,$$

we write

$$Eu(x) - Eu(y) = \sum_{Q \in \mathcal{W}} [\varphi Q(x) - \varphi Q(y)] u_{Q^r} = \sum_{Q \in \mathcal{W}} [\varphi Q(x) - \varphi Q(y)] [u_{Q^r} - u_P].$$

Note that by Lemma 3.3,

$$|\nabla \varphi| \leq \frac{L}{l_Q} \chi_{\frac{\alpha}{n}Q}^r.$$  

One gets

$$|Eu(x) - Eu(y)| \leq L \sum_{Q \in \mathcal{W}} \frac{|x - y|}{l_Q} \left[ \chi_{\frac{4}{n}Q}^r(x) + \chi_{\frac{4}{n}Q}^r(y) \right] |u_{Q^r} - u_P|.$$  

Moreover, we have

$$|Eu(x) - Eu(y)| \leq 2L \sum_{Q \in \mathcal{N}(P)} \frac{|x - y|}{l_Q} \chi_{\frac{4}{n}Q}^r(x) |u_{Q^r} - u_P|.$$  

Indeed, since $y \in X_2(x)$, that is, $|x - y| \leq \frac{1}{132n} \max\{d(x, \Omega), d(y, \Omega)\}$, taking $\bar{y} \in \bar{\Omega}$ with $|y - \bar{y}| = d(y, \Omega)$ we have

$$d(x, \Omega) \leq |x - \bar{y}| \leq |x - y| + |y - \bar{y}| \leq \frac{1}{132n} d(x, \Omega) + \frac{1 + 132n}{132n} d(y, \Omega),$$

which implies

$$d(x, \Omega) \leq \frac{132n + 1}{132n - 1} d(y, \Omega).$$

Similarly, we have

$$d(y, \Omega) \leq \frac{132n + 1}{132n - 1} d(x, \Omega).$$

Thus,

$$|x - y| \leq \frac{1}{132n} \frac{132n + 1}{132n - 1} d(x, \Omega).$$

If $y \in \frac{15}{16} Q$, by $y \in P$ we have $Q \in \mathcal{N}(P)$, and hence

$$d(y, \Omega) \leq d(y, Q) + \max_{z \in Q} d(z, \Omega) \leq \frac{1}{16} \sqrt{n} l_Q + 4 \sqrt{n} l_Q \leq \frac{65}{16} \sqrt{n} l_Q.$$  

Thus

$$|x - y| \leq \frac{1}{132n} \frac{132n + 1}{132n - 1} \times \frac{65}{16} \sqrt{n} l_Q \leq \frac{1}{32} \sqrt{n} l_Q,$$

which implies that $x \in \frac{9}{8} Q$. Moreover, if $x \in \frac{17}{16} Q$, similarly we have $y \in \frac{9}{8} Q$, and hence $Q \in \mathcal{N}(P)$. We conclude that

$$\chi_{\frac{4}{n}Q}^r(x) + \chi_{\frac{4}{n}Q}^r(y) = 0.$$
when \( Q \notin N(P) \), and
\[
\chi_{\frac{x}{2}Q}(x) + \chi_{\frac{x}{2}Q}(y) \leq 2\chi_{\frac{x}{2}Q}(x)
\]
when \( Q \in N(P) \). This gives (4.3).

Note that by Lemma 3.1(iii), \( \sum_{Q \in \mathcal{W}} \chi_{\frac{x}{2}Q}(x) \leq \gamma_0 \). From the convexity of \( \phi \) and (4.3) it follows that
\[
\phi \left( \frac{|Eu(x) - Eu(y)|}{\alpha} \right) \leq \phi \left( \sum_{Q \in N(P)} \frac{|x-y|}{l_Q} 2\chi_{\frac{x}{2}Q}(x) \frac{|u_Q - u_P|}{\alpha/L} \right)
\]
\[
\leq \frac{1}{\gamma_0} \sum_{Q \in N(P)} \chi_{\frac{x}{2}Q}(x) \phi \left( \frac{|x-y|}{l_Q} \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right).
\]
Therefore, we obtain
\[
H_{32}(\alpha) \leq \frac{1}{\gamma_0} \int_U \int_{D_2(x)} \int_{P \cap X_2(x)} \sum_{Q \in N(P)} \chi_{\frac{x}{2}Q}(x) \phi \left( \frac{|x-y|}{l_Q} \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) dy dx
\]
\[
= \frac{1}{\gamma_0} \int_U \int_{D_2(x)} \int_{P \cap X_2(x)} \chi_{\frac{x}{2}Q}(x) \int_{P \cap X_2(x)} \phi \left( \frac{|x-y|}{l_Q} \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) dy dx dy
\]
Observe that for \( x \in \frac{a}{e}Q \) and \( y \in P \cap X_2(x) \), by \( d(x, \Omega) \leq 4 \sqrt{n}l_Q \) we have
\[
|x - y| \leq \frac{1}{132n} \frac{132n + 1}{132n - 1} d(x, \Omega) \leq l_Q.
\]
By the assumption (1.2) for \( \phi \), we have
\[
\int_{P \cap X_2(x)} \phi \left( \frac{|x-y|}{l_Q} \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) dy \leq n\omega_n \int_0^{\frac{1}{132n} d(x, \Omega)} \phi \left( \frac{t}{l_Q} \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) dt
\]
\[
\leq n\omega_n (l_Q)^{-n} \left( \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right)^n \int_0^{\frac{1}{132n} d(x, \Omega)} \phi \left( \frac{t}{l_Q} \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) dt
\]
\[
\leq nC\omega_n (l_Q)^{-n} \left( \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right)^n.
\]
Using the above inequality and (3.3), one has
\[
H_{32}(\alpha) \leq nC\frac{1}{\gamma_0} \int_U \int_{D_2(x)} \int_{P \cap X_2(x)} \chi_{\frac{x}{2}Q}(x) \phi \left( \frac{|u_Q - u_P|}{4\alpha/2L\gamma_0} \right) dx
\]
\[
\leq nC\frac{1}{\gamma_0} \int_U \int_{D_2(x)} \int_{P \cap X_2(x)} \phi \left( \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) dx
\]
\[
\leq C\frac{1}{\gamma_0} \int_U \int_{D_2(x)} \phi \left( \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) dx.
\]
For each \( P \in \mathcal{W} \) and \( Q \in N(P) \), by Jessen’s inequality
\[
\phi \left( \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) \leq \int_{Q^*} \int_{P^*} \phi \left( \frac{|u(z) - u(w)|}{\alpha/2L\gamma_0} \right) dz dw
\]
Note that by Lemma 3.2 (i), \( P^* \subset 10 \sqrt{n}P \) and \( Q^* \subset 10 \sqrt{n}Q \). Thus for any \( z \in P^* \) and \( w \in Q^* \), by \( Q \in N(P) \), we have
\[
|z - w| \leq 10 \sqrt{n}(l_Q + l_P) \leq 50n \min(l_Q, l_P).
\]
Since \( |Q| \leq \gamma_1|Q^*| \) and \( |P| \leq \gamma_1|P^*| \) as given in Lemma 3.2 (ii), one gets
\[
|z - w| \leq (50n)^{3n} (\gamma_1)^2 |Q| |P^*|.
\]
Thus, \[ \phi \left( \frac{|u_Q - u_P|}{\alpha/2L\gamma_0} \right) \leq (50n)^{2n}(\gamma_1)^2 \int_Q \int_P \phi \left( \frac{|u(z) - u(w)|}{\alpha/2L\gamma_0} \right) \frac{dz\, dw}{|z - w|^{2n}} \]

and hence
\[ H_{32}(\alpha) \leq C_{\phi} n \omega_n (50n)^{2n}(\gamma_1)^2 \sum_{P \in W} \sum_{Q \in N(P)} \int_Q \int_P \phi \left( \frac{|u(z) - u(w)|}{\alpha/2L\gamma_0} \right) \frac{dz\, dw}{|z - w|^{2n}}. \]

With \( \sum_{Q \in W} \chi_Q \leq \gamma_2 \) as given in Lemma 2.2 (iii), we obtain
\[ H_{32}(\alpha) \leq C_{\phi} n \omega_n (50n)^{2n}(\gamma_1)^2(\gamma_2)^2 \int_\Omega \int_\Omega \phi \left( \frac{|u(x) - u(y)|}{\alpha/2L\gamma_0} \right) \frac{dx\, dy}{|x - y|^{2n}}. \]

Letting \( M_{32} = 8L\gamma_0 C_{\phi} n \omega_n (50n)^{2n}(\gamma_1)^2(\gamma_2)^2 \). If \( \alpha > M_{32} \), we have \( H_{32}(\alpha) \leq 1/8 \) as desired.

**Case** \( \text{diam} \Omega < \infty \).

To find \( M_2 \), write
\[ H_2(\alpha) = \int_{V(2)} \int_\Omega \phi \left( \frac{|Eu(x) - u(y)|}{\alpha} \right) \frac{dy\, dx}{|x - y|^{2n}} + \int_{U \setminus V(2)} \int_\Omega \phi \left( \frac{|u(x) - u(y)|}{\alpha} \right) \frac{dy\, dx}{|x - y|^{2n}} = H_{21}(\alpha) + H_{22}(\alpha). \]

Recall that \( V(2) \) is defined by (3.4) in Section 3. It suffices to find \( M_{2i} \) such that \( H_{2i} \leq 1/8 \) for \( i = 1, 2 \).

Regards of \( H_{22}(\alpha) \), observe that for any \( Q \in W' \setminus W'_{e_0} \), we have \( N(Q) \cap W'_{e_0} = \emptyset \), and hence \( P^* = \Omega \) for all \( P \in N(Q) \). Thus, for any \( u \in U \setminus V(2) \), by Lemma 3.3 and Lemma 3.1 we have
\[ Eu(x) = \sum_{P \in W} \phi_P(x)u_P = \sum_{P \in N(Q)} \phi_P(x)u_P = u_Q. \]

Thus
\[ H_{22}(\alpha) = \int_{U \setminus V(2)} \int_\Omega \phi \left( \frac{|u_Q - u(y)|}{\alpha} \right) \frac{dy\, dx}{|x - y|^{2n}}. \]

By Jensen’s inequality, one gets
\[ H_{22}(\alpha) \leq \int_{U \setminus V(2)} \int_\Omega \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) \frac{dz\, dy}{|x - y|^{2n}} \]
\[ = \int_\Omega \int_{U \setminus V(2)} \frac{dx}{|x - y|^{2n}} \int_\Omega \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) \frac{dz\, dy}{|x - y|^{2n}} \]
\[ = \int_\Omega \left( \frac{|\text{diam} \Omega|^{2n}}{|\Omega|} \int_{U \setminus V(2)} \frac{dx}{|x - y|^{2n}} \right) \int_\Omega \phi \left( \frac{|u(z) - u(y)|}{\alpha} \right) \frac{dz}{|z - y|^{2n}}. \]

For any \( x \in U \setminus V(2) \) and \( y \in \Omega \), since there exists \( Q \in W' \setminus W'_{e_0} \) so that \( x \in Q \), one always has
\[ |x - y| \geq d(x, \Omega) \geq l_Q \geq \frac{1}{e_0} \text{diam} \Omega. \]

Moreover, by the Ahlfors \( n \)-regular assumption, it holds that \( |\Omega| \geq C_A(\Omega) |\text{diam} \Omega|^n \). Thus,
\[ \frac{|\text{diam} \Omega|^{2n}}{|\Omega|} \int_{U \setminus V(2)} \frac{dx}{|x - y|^{2n}} \leq \frac{1}{C_A(\Omega)} \text{diam} \Omega^n \int_{|x - y|}^{\frac{1}{e_0} \text{diam} \Omega} \frac{dx}{|x - y|^{2n}} \]
\[ \leq |\text{diam} \Omega|^n n \omega_n \int_{\frac{1}{e_0} \text{diam} \Omega}^{\infty} \frac{1}{r^{n+1}} dr \]
\[ \leq \omega_n \frac{1}{C_A(\Omega)^n e_0^n}. \]
from which, we conclude that
\[ H_{22}(\alpha) \leq \omega_n \frac{1}{C_A(\Omega)} \epsilon \int_{\Omega} \int_{\Omega} \frac{|u(z) - u(y)|}{|y - z|^{2n}} \, dz \, dy. \]

Letting \( M_{22} = 8 \omega_n \frac{1}{C_A(\Omega)} \epsilon \), by the convexity of \( \phi \) again, for \( \alpha > M_{22} \) we have \( H_{22}(\alpha) \leq 1/8. \)

Regard of \( H_{21}(\alpha) \), observe that
\[ \sum_{Q \in W} \varphi_Q(x) = \sum_{Q \in W_{\infty}} \varphi_Q(x) = 1 \]
whenever \( x \in V(2) \). With aid of this and following, line by line, the argument to get (4.2) for \( H_2(\alpha) \) in the case \( \text{diam} \, \Omega = \infty \), one has
\[ H_{21}(\alpha) \leq 2 \gamma_1 4^n \gamma_0 (21n)^{2n} \sum_{Q \in \mathcal{W}_{\infty}} \int_{\Omega} \int_{\Omega} \frac{|u(z) - u(y)|}{|y - z|^{2n}} \, dz \, dy. \]

Here we omit the details. Since
\[ \sum_{Q \in \mathcal{W}_{\infty}} \chi_Q \leq \gamma_2 + (\epsilon_0 + 64 \sqrt{n})^n, \]
we have
\[ H_{21}(\alpha) \leq 2 \gamma_1 4^n \gamma_0 [\gamma_2 + (\epsilon_0 + 64 \sqrt{n})^n](21n)^{2n} \int_{\Omega} \int_{\Omega} \frac{|u(z) - u(y)|}{|y - z|^{2n}} \, dz \, dy. \]

Set \( M_{21} = 16 \gamma_1 4^n \gamma_0 [\gamma_2 + (\epsilon_0 + 64 \sqrt{n})^n](21n)^{2n} \). By the convexity of \( \phi \) again, if \( \alpha > M_{21} \), we have \( H_{21}(\alpha) \leq 1/8 \) as desired.

To find \( M_3 \), notice that
\[ U \times U \subset [V(3) \times V(3)] \cup [V(2) \times (U \setminus V(3))] \cup [(U \setminus V(3)) \times V(2)] \cup [(U \setminus V(2)) \times (U \setminus V(2))]. \]

We write
\[ \begin{align*}
H_3 &= \int_{V(3)} \int_{V(3)} \frac{|E u(x) - E u(y)|}{|x - y|^{2n}} \, dy \, dx + 2 \int_{V(2)} \int_{U \setminus V(3)} \frac{|E u(x) - E u(y)|}{|x - y|^{2n}} \, dy \, dx \\
&\quad + \int_{U \setminus V(2)} \int_{U \setminus V(2)} \phi \frac{|E u(x) - E u(y)|}{|x - y|^{2n}} \, dy \, dx \\
&=: H_{31}(\alpha) + 2H_{32}(\alpha) + H_{33}(\alpha). \end{align*} \]

Since \( E u(x) = E u(y) = u_\Omega \) for \( x, y \in U \setminus V(2) \), we have \( H_{33}(\alpha) = 0 \). It suffices to find \( M_3 \) such that \( H_3(\alpha) \) for all \( \alpha > M_3 \) and \( i = 1, 2 \).

Regard of \( H_{31}(\alpha) \), similarly to \( H_3(\alpha) \) in the case \( \text{diam} \, \Omega = \infty \) and taking \( M_{31} \) as \( M_3 \) there with \( \gamma_2 \) replaced by \( \gamma_2 + (\epsilon_0 + 4^5 \sqrt{n})^n \), we can show that if \( \alpha \geq M_{31} \), then \( H_{31}(\alpha) \leq 1/8 \). Here we omit the details.

For \( H_{32}(\alpha) \), note that for \( y \in U \setminus V(2) \), we have \( E u(y) = u_\Omega \). Thus
\[ H_{32}(\alpha) = 2 \int_{V(2)} \int_{U \setminus V(3)} \phi \frac{|E u(x) - u_\Omega|}{|x - y|^{2n}} \, dy \, dx. \]

By Jessen’s inequality, one has
\[ H_{32}(\alpha) \leq \int_{V(2)} \int_{U \setminus V(3)} \frac{dy}{|x - y|^{2n}} \int_{\Omega} \phi \left( \frac{|E u(x) - u(z)|}{\alpha} \right) \, dx \, dz. \]

For any \( x \in V(2) \) and \( y \in U \setminus V(3) \) note that \( |x - y| \geq h(Q) \geq \frac{1}{\epsilon_0} \text{diam} \, \Omega \), where \( Q \in \mathcal{W}_{\infty} \setminus \mathcal{W}_{\epsilon_0} \) and \( y \in Q \). Thus
\[ \int_{U \setminus V(3)} \frac{dy}{|x - y|^{2n}} \leq \epsilon_0^n (\text{diam} \, \Omega)^{-n}. \]
Since \(|\Omega| \geq C_\alpha(\Omega) \text{diam } \Omega\), one has

\[ H_{32}(\alpha) \leq \frac{1}{C_\alpha(\Omega)} e_0^{p} \left( \text{diam } \Omega \right)^{-2n} \int_{V^{(2)}} \int_{\Omega} \phi \left( \frac{|E u(x) - E u(z)|}{\alpha} \right) \, dx \, dz. \]

Note that for any \(x \in V^{(2)}\) there exists a \(P_i \in \mathcal{P}^{(i)}\) such that \(x \in P_2\) and \(P_i \in N(P_{i-1})\) for \(i = 1, 2\). Since \(l(P_0) \leq \frac{1}{e_0} \text{diam } \Omega\), by Lemma 3.1 we know that \(l(P_2) \leq 4^2 \frac{1}{e_0} \text{diam } \Omega\). Thus for \(y \in \Omega\), one has

\[ |x - y| \leq \text{dist } (x, \Omega) + \text{diam } \Omega \leq \text{diam } P_2 + \text{dist } (P_2, \Omega) + \text{diam } \Omega \leq 4^4 \frac{1}{e_0} \sqrt{n} \text{diam } \Omega. \]

Therefore,

\[ H_{32}(\alpha) \leq \frac{1}{C_\alpha(\Omega)} e_0^{p} \left( 4^4 \frac{1}{e_0} \sqrt{n} \right)^{2n} \int_{V^{(2)}} \int_{\Omega} \phi \left( \frac{|E u(x) - E u(z)|}{\alpha} \right) \, dx \, dz \leq \frac{1}{C_\alpha(\Omega)} 4^{8n} e_0^{-n} H_{21}(\alpha). \]

If \(\alpha > M_{32} = \frac{8}{C_\alpha(\Omega)} 4^{8n} e_0^{-n} M_{21}\), we have \(H_{32}(\alpha) \leq 1/8\). This completes the proof of Theorem 1.1 (i).

**Remark 4.1.** We emphasize that the bounded overlaps of reflecting cubes \(Q^*\) in Lemma 3.2 (iii) play central roles in the proof of the boundedness of extension operator \(E : B^\phi(\Omega) \to B^\phi(\mathbb{R}^n)\).

Similarly to [24, 5, 11] and the reference therein, one may define the extension operator \(E u\) similarly to \(E u\) but replacing \(Q^*\) in \(E u\) with \(Q(x_Q^*, l_Q) \cap \Omega\), where \(x_Q^*\) is the nearest point in \(\Omega\) of \(Q^* \in \mathcal{W}\). Note that \(\{Q(x_Q^*, l_Q) \cap \Omega, Q \in \mathcal{W}\}\) does not have bounded overlap property as in Lemma 3.2 (iii) in general.

In the case \(p(\Theta) = t^p\) with \(p > n\), similarly to [24], one may prove that \(\tilde{E}\) is bounded from \(B^{\mu/p}(\Omega)\) to \(B^{\mu/p}(\mathbb{R}^n)\). The point is prove that

\[ \frac{|E u(x) - \tilde{E} u(y)|}{|x - y|^{2n}} \leq M \left( \frac{|u(z) - u(w)|}{|z - w|^{2n}} \chi_{Q \cap \Omega} \right) (x, y) \]

where \(M\) is certain Hardy-Littlewood maximal operator. See page 968 in the proof of [24, Theorem 1.1].

For general \(\phi\) in Theorem 1.1, some appropriate estimates of \(\phi(\frac{E u(x) - E u(y)}{\alpha})\) via certain maximal functions are not available for us. We do not know if it is possible to obtain the boundedness of \(\tilde{E}\) from \(B^\phi(\Omega)\) to \(B^\phi(\mathbb{R}^n)\). Note the our proof of the boundedness of \(E\) does not work for \(\tilde{E}\) since \(\{Q(x_Q^*, l_Q) \cap \Omega, Q \in \mathcal{W}\}\) does not have the bounded overlap property.

5. Proof of Theorem 1.1 (ii)

We divide the proof into 3 steps.

**Step 1.** Since \(\Omega\) is a \(B^\phi\)-extension domain, there exists a bounded linear extension operator \(E : B^\phi(\Omega) \to B^\phi(\mathbb{R}^n)\). For any \(u \in B^\phi(\Omega)\), we have \(E u \in B^\phi(\mathbb{R}^n)\) with \(E u = u\) in \(\Omega\), \(\|E u\|_{B^\phi(\mathbb{R}^n)} \leq \|E u\|_{B^\phi(\Omega)}\). By Lemma 2.3, we have \(u \in BMO(\mathbb{R}^n)\) and \(\|E u\|_{BMO(\mathbb{R}^n)} \leq \|u\|_{B^\phi(\mathbb{R}^n)}\). From the John-Nirenberg inequality, it follows that

\[ \int_B \exp \left( \frac{|E u - (E u)_B|}{C_J n \|E u\|_{B^\phi(\mathbb{R}^n)}} \right) \, dx \leq C(n) |B| \quad \text{for all balls} \ B \subset \mathbb{R}^n. \]

Thus,

\[ \inf_{c \in \mathbb{R}} \int_{B \cap \Omega} \exp \left( \frac{|u - c|}{C(\phi, n, \Omega) \|u\|_{B^\phi(\Omega)}} \right) \, dx \leq C(n) |B| \quad \text{for all balls} \ B \subset \mathbb{R}^n. \]  \hspace{2cm} (5.1)

**Step 2.** For \(x \in \Omega\) and \(0 < r < t < \text{diam } \Omega\), set the function

\[ u_{x,r}(z) = \begin{cases} 1 & z \in B(x, r) \cap \Omega \\ \frac{|z - x|}{r} & z \in (B(x, t) \setminus B(x, r)) \cap \Omega \\ 0 & z \in \Omega \setminus B(x, t) \end{cases} \]

We have the following.
Lemma 5.1. Suppose that $\phi$ is a Young function satisfying (1.2). For $x \in \Omega$ and $0 < r < t < \text{diam} \Omega$, we have $u_{x,t,r} \in \mathcal{B}^\phi(\Omega)$ with
\[
\|u_{x,t,r}\|_{\mathcal{B}^\phi(\Omega)} \leq 8\omega_n \left[ C_\phi 4^n + 1 \right] \left[ \frac{(t-r)^n}{|B(x,t) \cap \Omega|} \right]^{-1}.
\]

Proof of Lemma 5.1. Write
\[
\int_{\Omega} \int_{\Omega} \phi \left( \frac{|u_{x,t,r}(z) - u_{x,t,r}(w)|}{\alpha} \right) \frac{dz}{|z-w|^{2n}} = \int_{B(x,t) \cap \Omega} \int_{B(x,t) \cap \Omega} \phi \left( \frac{|u_{x,t,r}(z) - u_{x,t,r}(w)|}{\alpha} \right) \frac{dz}{|z-w|^{2n}} = H_1(\alpha) + H_2(\alpha).
\]
If suffices to find a constant $M$ depending only on $n$ such that for $\alpha = M \left[ \frac{(t-r)^n}{|B(x,t) \cap \Omega|} \right]^{-1}$, we have $H_1 \leq \frac{1}{2}$ and $H_2(\alpha) \leq \frac{1}{2}$.

Write
\[
H_1(\alpha) \leq \int_{B(x,t) \cap \Omega} \int_{B(x,t) \cap \Omega} \phi \left( \frac{|z-w|}{\alpha(t-r)} \right) \frac{dz}{|z-w|^{2n}} \leq n\omega_n \int_0^{t-r} \phi \left( \frac{s}{\alpha(t-r)} \right) \frac{ds}{s^{n+1}} \leq n\omega_n (t-r)^{-n} \alpha^{-n} \int_0^{1/\alpha} \phi(s) \frac{ds}{s^{n+1}}.
\]

Applying (1.2), we have
\[
H_1(\alpha) \leq n\omega_n \phi(t-r)^{-n} \frac{1}{\alpha}.
\]

On the other hand,
\[
\int_{B(x,t) \cap \Omega} \frac{dz}{|z-w|^{2n}} \leq \int_{\mathbb{R}^n \setminus B(x,t)} \frac{dz}{|z-w|^{2n}} = \omega_n (t-r)^{-n}.
\]

Thus
\[
H_1(\alpha) \leq n\omega_n (n\phi + 1) \frac{|B(x,t) \cap \Omega|}{(t-r)^n} \frac{1}{\alpha}.
\]

If $\alpha = M \frac{(t-r)^n}{|B(x,t) \cap \Omega|}$ and $M \geq 2(n\phi + 1)\omega_n$, we have
\[
H_1(\alpha) \leq \frac{(n\phi + 1)\omega_n}{M} \leq 1/2.
\]

Write
\[
H_2(\alpha) \leq \int_{B(x,t) \cap \Omega} \int_{B(x,t) \cap \Omega} \phi \left( \frac{|t-z-x|}{\alpha(t-r)} \right) \frac{dw}{|z-w|^{2n}} \leq \int_{\mathbb{R}^n \setminus B(x,t) \setminus |z-x|} \frac{dw}{|z-w|^{2n}} \leq \omega_n (t-|z-x|)^{-n}.
\]

Note that $\Omega \setminus B(x,t) \subset \Omega \setminus B(z,t - |z-x|)$, we have
\[
\int_{\Omega \setminus B(x,t)} \frac{dw}{|z-w|^{2n}} \leq \int_{\mathbb{R}^n \setminus B(z,t-|z-x|)} \frac{dw}{|z-w|^{2n}} \leq \omega_n (t-|z-x|)^{-n}.
\]

Hence,
\[
H_2(\alpha) \leq \int_{B(x,t) \cap \Omega} \int_{B(x,t) \cap \Omega} \phi \left( \frac{|t-z-x|}{\alpha(t-r)} \right) \omega_n (t-|z-x|)^{-n} \frac{dz}{|z-w|^{2n}} \leq 2\omega_n \frac{|B(x,t) \cap \Omega|}{(t-r)^n} \left[ \sup_{s \in (0,1)} \phi \left( \frac{s}{\alpha} \right) \frac{1}{s^n} + \phi \left( \frac{1}{\alpha} \right) \right].
\]
Notice that
\[
\sup_{s \in (2^{-j}, 2^{-j+1})} \phi \left( \frac{s}{\alpha} \right) \frac{1}{s^n} \leq 2^n \int_{2^{-j}}^{2^{-j+1}} \phi \left( \frac{s}{\alpha} \right) \frac{ds}{s^{n+1}}
\]
an hence
\[
\sup_{s \in (0, 1)} \phi \left( \frac{s}{\alpha} \right) \frac{1}{s^n} \leq 2^n \int_{0}^{2} \phi \left( \frac{s}{\alpha} \right) \frac{ds}{s^{n+1}} \leq 2^n \int_{0}^{2} \phi(s) \frac{ds}{s^{n+1}} \leq C_{0} 4^n \phi \left( \frac{1}{\alpha/2} \right)
\]
Therefore,
\[
H_2(\alpha) \leq 2(C_{0} 4^n + 1) \omega_n \frac{|B(x, t) \cap \Omega|}{(t - r)^n} \phi \left( \frac{2}{\alpha} \right).
\]
If \( \alpha = M[\phi^{-1} \left( \frac{(t - r)^n}{|B(x, r)|} \right)]^{-1} \) and \( M \geq 8(C_{0} 4^n + 1) \omega_n \), we have
\[
H_2(\alpha) \leq \frac{2(C_{0} 4^n + 1) \omega_n}{M/2} \leq \frac{1}{2},
\]
as desired. \( \Box \)

**Step 3.** Let \( x \in \Omega \) and \( 0 < r < 2 \text{ diam } \Omega \). Let \( b_0 = 1 \) and \( b_j \in (0, 1) \) for \( j \in N \) such that
\[
|B(x, b_j r) \cap \Omega| = 2^{-j} |B(x, b_{j-1} r) \cap \Omega| = 2^{-j} |B(x, r) \cap \Omega|.
\]
Let \( u_j = u_{x, b_j+1 r, b_j r} \) for \( j \geq 1 \) be as in Lemma 5.1. By (5.1), we have
\[
\inf_{c \in \mathbb{R}} \int_{B(x, b_{j+1} r) \cap \Omega} \exp \left( \frac{|u_j - c|}{C(\phi, n, \Omega)||u_j||_{B^\alpha(\Omega)}} \right) \, dy \leq C(n)r^n;
\]
for any \( c \in \mathbb{R} \), we know that \( |u_j - c| \geq 1/2 \) either on \( B(x, b_{j+1} r) \cap \Omega \) or on \( [B(x, b_{j-1} r) \setminus B(x, b_j r)] \cap \Omega \), and note that, by (5.2),
\[
|B(x, b_{j+1} r) \cap \Omega| = |B(x, b_{j-1} r) \setminus B(x, b_j r)] \cap \Omega| = 2^{-j+1} |B(x, r) \cap \Omega|.
\]
Thus, for any \( j \geq 1 \), we have
\[
2^{-j+1} |B(x, r) \cap \Omega| \exp \left( \frac{|u_j - c|}{C(\phi, n, \Omega)||u_j||_{B^\alpha(\Omega)}} \right) \leq C(n)r^n
\]
that is,
\[
\frac{|u_j - c|}{C(\phi, n, \Omega)||u_j||_{B^\alpha(\Omega)}} \leq \ln \left( 2^j \frac{C(n)r^n}{|B(x, r) \cap \Omega|} \right).
\]
Since
\[
||u_j||_{B^\alpha(\Omega)} \leq C(\phi, n) \phi^{-1} \left( \frac{(b_{j+1} r)^{n}}{|B(x, b_j r) \cap \Omega|} \right)^{-1} \leq C(\phi, n) \phi^{-1} \left( 2^j \frac{(b_{j+1} r)^{n}}{|B(x, r) \cap \Omega|} \right)^{-1},
\]
we have
\[
\frac{1}{C(\phi, n, \Omega)} \phi^{-1} \left( 2^j \frac{(b_{j+1} r)^{n}}{|B(x, r) \cap \Omega|} \right) \leq \ln \left( 2^j \frac{C(n)r^n}{|B(x, r) \cap \Omega|} \right),
\]
and hence
\[
(b_j - b_{j+1})^n \leq 2^{-j} |B(x, r) \cap \Omega| r^n \phi \left[ C(\phi, n, \Omega) \ln \left( 2^j \frac{2C(n)r^n}{|B(x, r) \cap \Omega|} \right) \right]
\]
By (1.3), for any \( \delta > 0 \), we have \( \phi(t) \leq C(\delta) e^{\delta t} \) for all \( t \geq 0 \). Taking \( \delta_0 = 1/2C(\phi, n, \Omega) \), that is, \( C(\phi, n, \Omega)\delta_0 = 1/2 \), we obtain
\[
(b_j - b_{j+1})^n \leq C(\delta_0)(2C(n))^{\delta_0} C(\phi, n, \Omega) \left( 2^{-j} \frac{|B(x, r) \cap \Omega|}{r^n} \right)^{1-\delta_0} C(\phi, n, \Omega)
\]
\[ b_1 = \sum_{j=1}^{\infty} (b_j - b_{j+1}) \leq C(\phi, n, \Omega) \left( \frac{|B(x, r) \cap \Omega|}{r^n} \right)^{1/2} 2^{-j/2}. \]

Thus

\[ |B(x, r) \cap \Omega| \geq C(\phi, n, \Omega) r^n \]

as desired.

If \( b_1 \geq 1/10 \), we get

\[ |B(x, r) \cap \Omega| \geq C(\phi, n, \Omega) r^n \]

as desired. This completes the proof of Theorem 1.1 (ii).

REFERENCES

[1] S.M. Buckley and P. Koskela, Criteria for imbeddings of Sobolev-Poincaré type, Internat. Math. Res. Notices 18(1996), 881-902.
[2] R. A. DeVore, R. C. Sharpley, Besov spaces on domains in \( \mathbb{R}^d \), Trans. Amer. Math. Soc. 335 (1993), 843-864.
[3] L. Grafakos, Classical and Modern Fourier Analysis, Pearson Education Inc. 2008.
[4] A. Gogatishvili, P. Koskela and Y. Zhou, Characterizations of Besov and Triebel-Lizorkin spaces on metric measure spaces, Forum Math. 25(2013), 787-819.
[5] P. Hajłasz, P. Koskela and H. Tuominen, Sobolev imbeddings, extensions and measure density condition, J. Funct. Anal. 254 (2008), 1217-1234.
[6] P. Hajlasz, P. Koskela and H. Tuominen, Measure density and extendability of Sobolev functions, Rev. Mat. Iberoam. 24 (2008), 645-669.
[7] P. W. Jones, Extension theorems for BMO, Indiana Univ. Math. J. 29 (1980), 41-66.
[8] P. W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. 147 (1981), 71-88.
[9] A. Jonsson, H. Wallin, A Whitney extension theorem in \( L^p \) and Besov spaces, Ann. Inst. Fourier (Grenoble) 29 (1979), 103-119.
[10] Jonsson A, Wallin H. Function spaces on subsets of \( \mathbb{R}^n \), J. Mathematical Reports, 1984(1).
[11] P. Koskela, Extensions and imbeddings, J. Funct. Anal. 159 (1998), 369-384.
[12] P. Koskela, D. Yang and Y. Zhou, Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings, Adv. Math. 226 (2011), 3579-3621.
[13] P. Koskela, Y. Zhang and Y. Zhou, Morrey-Sobolev Extension Domains, 27(2017), 1413-1434.
[14] Matias Carrasco Piaggio, Orlicz spaces and the large scale geometry of Heintze groups, Mathematische Annalen 368(6) (2017), 433-481.
[15] Nzza, Eleonora Di, G.Palatucci, and E. Valdinoci. Hatchikier’s guide to the fractionalSobolev spaces, Buletin Des Science Mathematiques 1136.5(2012), 521-573.
[16] V. S. Rychkov, On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains, J. London Math. Soc. (2) 60 (1999), 237-257.
[17] P. Shvartsman, Local approximations and intrinsic characterizations of spaces of smooth functions on regular subsets of \( \mathbb{R}^n \), Math. Nachr. 279 (2006), 1212-1241.
[18] P. Shvartsman, On extensions of Sobolev functions defined on regular subsets of metric measure spaces, Journal of Approximation Theory. 2144 (2007), 139-161.
[19] P. Shvartsman, On Sobolev extension domains in \( \mathbb{R}^n \), J. Funct. Anal. 258 (2010), 2205-2245.
[20] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J. 1970

[21] H. Triebel, Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers, Rev. Mat. Complut. 15 (2002), no. 2, 475-524.

[22] H. Triebel, Function spaces and wavelets on domains, EMS Tracts in Mathematics, 7. European Mathematical Society (EMS), Zürich, 2008. x+256 pp.

[23] Z. Wang, J. Xiao and Y. Zhou, A solution to the $Q$-restriction-extension problem in uniform domains, submitted.

[24] Y. Zhou, Fractional Sobolev extension and imbedding, Trans. Amer. Math. Soc. 367 (2015), 959-979.

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