Unidirectional Reflectionless Transmission for Two-Dimensional \(\mathcal{PT}\)-symmetric Periodic Structures

Lijun Yuan

College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing, China

Ya Yan Lu

Department of Mathematics, City University of Hong Kong, Hong Kong

Unidirectional reflectionless propagation (or transmission) is an interesting wave phenomenon observed in many \(\mathcal{PT}\)-symmetric optical structures. Theoretical studies on unidirectional reflectionless transmission often use simple coupled-mode models. The coupled mode theory can reveal the most important physical mechanism for this wave phenomenon, but it is only an approximate theory, and it does not provide accurate quantitative predictions with respect to geometric and material parameters of the structure. In this paper, we rigorously study unidirectional reflectionless transmission for two-dimensional (2D) \(\mathcal{PT}\)-symmetric periodic structures sandwiched between two homogeneous media. Using a scattering matrix formalism and a perturbation method, we show that real zero-reflection frequencies are robust under \(\mathcal{PT}\)-symmetric perturbations, and unidirectional reflectionless transmission is guaranteed to occur if the perturbation (of the dielectric function) satisfies a simple condition. Numerical examples are presented to validate the analytical results, and to demonstrate unidirectional invisibility by tuning the amplitude of balanced gain and loss.

I. INTRODUCTION

In recent years, \(\mathcal{PT}\)-symmetry has attracted considerable attention in the optics and photonics community \([1-3]\). A \(\mathcal{PT}\)-symmetric optical structure is usually realized by a complex dielectric function with a symmetric real part and an anti-symmetric imaginary part (i.e. a balanced gain and loss). The \(\mathcal{PT}\)-symmetry provides a fertile and feasible tool to manipulate lightwaves. Many interesting wave phenomena have been observed on \(\mathcal{PT}\)-symmetric optical structures. Noticeable examples include unidirectional reflectionless propagation \([5-17]\), single-mode lasing \([18-19]\), and simultaneous lasing and coherent perfect absorption \([20, 21]\).

Unidirectional reflectionless propagation (or transmission) is the phenomenon wherein the reflection is zero for an incident wave coming from one side and nonzero for an incident wave coming from the other side. For lossless dielectric structures with certain symmetry, it is well known that zero reflection and zero transmission can really occur \([22, 24]\), but unidirectional reflectionless transmission is impossible, because the unitarity of the scattering matrix implies that zero reflections for left and right incident waves must occur at the same frequency. This is not the case for \(\mathcal{PT}\)-symmetric structures, since the scattering matrix is no longer unitary \([21, 25]\). A particularly interesting case of unidirectional reflectionless transmission is unidirectional invisibility, for which the transmitted wave is identical to that without the local structure \([5]\). Unidirectional reflectionless transmission has been studied on various \(\mathcal{PT}\)-symmetric optical structures, including one-dimensional (1D) structures \([5, 6]\), planar layered structures \([8, 12]\), planar inhomogeneous structure \([13]\), two-dimensional (2D) closed waveguides \([14]\), and 2D coupled waveguide resonator systems \([15]\). Experimental demonstrations have been reported for \(\mathcal{PT}\)-symmetric photonic lattices \([16]\) and microscale SOI waveguides \([17]\). It should be mentioned that unidirectional reflectionless transmission can also occur in non-\(\mathcal{PT}\)-symmetric optical structures \([20, 30]\). Existing studies on unidirectional reflectionless transmission typically employ 1D Helmholtz equations or coupled-mode models.

In this paper, we consider 2D \(\mathcal{PT}\)-symmetric periodic structures sandwiched between two homogeneous media, and find exact conditions under which unidirectional reflectionless transmission is guaranteed to occur. More specifically, assuming the \(\mathcal{PT}\)-symmetric structure is a small perturbation of a lossless dielectric structure and the dielectric structure has a simple (i.e. nondegenerate) real zero-reflection frequency, we show that real zero-reflection frequencies continue to exist, and they are different for left and right incident waves if the perturbation satisfies a simple condition. The continual existence (i.e. robustness) of real zero-reflection frequencies is proved using properties of the scattering matrix. A perturbation method is used to estimate the real zero-reflection frequencies and show that unidirectional reflectionless transmission occurs at arbitrarily small perturbations. Numerical examples are presented to validate our analytical results, and show that unidirectional invisibility can be obtained by tuning the amplitude of balanced gain and loss.

The rest of this paper is organized as follows. In Sec. [I], we recall some properties of the scattering matrix for general structures and discuss zero reflections for lossless dielectric structures with different symmetries. In Sec. [II], we show that real zero-reflection frequencies are robust under \(\mathcal{PT}\)-symmetric perturbations. In Sec. [IV]...
we use a perturbation method to estimate the shifts of the zero-reflection frequencies for left and right incident waves and derive a condition to guarantee unidirectional reflectionless transmission. In Sec. [V] numerical examples are presented to illustrate unidirectional reflectionless transmission and unidirectional invisibility.

II. SCATTERING MATRIX

We consider two-dimensional (2D) structures that are invariant in $z$, periodic in $y$ with period $L$, bounded in the $x$ direction, and surrounded by vacuum, where $\{x, y, z\}$ is a Cartesian coordinate system. The dielectric function for such a structure and the surrounding media satisfies

$$\epsilon(x, y + L) = \epsilon(x, y)$$

for all $(x, y)$ and $\epsilon(x, y) = 1$ for $|x| > D$, where $D$ is a given constant. For the $E$-polarization, the $z$-component of the electric field, denoted by $u$, satisfies the following 2D Helmholtz equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k_0^2 \epsilon u = 0,$$  \hspace{1cm} (1)

where $k_0 = \omega/c$ is the free space wavenumber, $\omega$ is the angular frequency, and $c$ is the speed of light in vacuum.

In the left and right homogeneous media, we specify two incident plane waves

$$u^{(i)}_l(x, y) = a_l e^{i(\alpha(x+D)+\beta y)} \hspace{1cm} \text{for} \hspace{1cm} x < -D$$

and

$$u^{(i)}_r(x, y) = a_r e^{-i(\alpha(x-D)-\beta y)} \hspace{1cm} \text{for} \hspace{1cm} x > D,$$  \hspace{1cm} (2)

where $a_l$ and $a_r$ are the amplitudes of the incident waves, $(\pm \alpha, \beta)$ are the incident wave vectors, $\beta$ is real, and $\alpha^2 + \beta^2 = k_0^2$. Since the structure is periodic and the medium is homogeneous for $|x| > D$, the solution of Eq. (1) can be written as

$$u(x, y) = a_l e^{i(\alpha(x+D)+\beta y)} + \sum_{j=-\infty}^{+\infty} b_j e^{-i(\alpha_j(x-D)-\beta_j y)},$$

for $x < -D$ and

$$u(x, y) = a_r e^{-i(\alpha(x-D)-\beta y)} + \sum_{j=-\infty}^{+\infty} b_j e^{i(\alpha_j(x-D)+\beta_j y)},$$

for $x > D$, where $\{b_j\}$ and $\{b_j\}$ are the amplitudes of the out-going plane waves, and

$$\beta_j = \beta + 2j\pi/L, \hspace{1cm} \alpha_j = \sqrt{k_0^2 - \beta_j^2},$$

for $j = 0, \pm 1, \pm 2, \ldots$. Notice that $\alpha_0 = \alpha$ and $\beta_0 = \beta$.

If $\beta$ is real, $\beta \in [-\pi/L, \pi/L]$, $\omega$ is real, and $k_0$ satisfies

$$|\beta| < k_0 < 2\pi/L - |\beta|,$$

then $\alpha_0$ is real and all $\alpha_j$ for $j \neq 0$ are pure imaginary with positive imaginary parts. In that case, all out-going plane waves for $j \neq 0$ decay to zero exponentially as $|x| \to \infty$, and the only out-going propagating plane waves are those for $j \neq 0$, i.e. the plane waves in Eqs. (4) and (5) with coefficients $b_0$ and $b_0$.

Let $S = S(\omega, \beta)$ be the $2 \times 2$ scattering matrix satisfying

$$\begin{bmatrix} b_0 \\ b_0 \end{bmatrix} = S(\omega, \beta) \begin{bmatrix} a_l \\ a_r \end{bmatrix} = r_L(\omega, \beta) t_L(\omega, \beta) \begin{bmatrix} a_l \\ a_r \end{bmatrix} + t_R(\omega, \beta) r_R(\omega, \beta) \begin{bmatrix} a_l \\ a_r \end{bmatrix}, \hspace{1cm} (8)$$

where $r_L$ and $r_R$ ($t_L$ and $t_R$) are the reflection (transmission) coefficients for left and right incident waves, respectively, and $R_L = |r_L|^2$, $R_R = |r_R|^2$, $T_L = |t_L|^2$ and $T_R = |t_R|^2$ are the corresponding reflectance and transmittance.

Although $\omega$ is generally real, it is possible to study the diffraction problem for a complex frequency. If $\omega$ is allowed to have a small (positive or negative) imaginary part, and the real part of $k_0 = \omega/c$ satisfies Eq. (7), then $k_0^2 - \beta^2$ is close to the positive real axis, and $k_0^2 - \beta_j^2$ (for $j \neq 0$) are close to the negative real axis. In order to define $\alpha_j = \sqrt{k_0^2 - \beta_j^2}$ that depends continuously on the imaginary part of $\omega$, we can use a complex square root with a branch cut on the negative imaginary axis. That is, if $\eta = |\eta| e^{i\psi}$ for $-\pi/2 < \psi \leq 3\pi/2$, then $\sqrt{\eta} = |\eta|^{1/2} e^{i\psi/2}$. Using this square root, Eqs. (4) and (5) are still valid for $|x| > D$, the out-going waves are still dominated by plane waves with coefficients $b_0$ and $b_0$, and the scattering matrix can be defined as in Eq. (8).

If for a real frequency $\omega$ and a real wavenumber $\beta$, we have $r_L = 0$ and $r_R \neq 0$ (or $r_L \neq 0$ and $r_R = 0$), then we say unidirectional reflectionless transmission occurs at $(\omega, \beta)$ for a left (or right) incident wave. The scattering matrix has some important properties. The reciprocity gives rise to

$$S(\omega, -\beta) = S^T(\omega, \beta),$$

(9)

where the superscript “$T$” denotes matrix transpose, and $S(\omega, -\beta)$ is the scattering matrix for incident waves with $e^{-i\beta y}$ dependence. Equation (9) is a general result valid for complex dielectric function $\epsilon$ and complex $\omega$. For easy reference, we give a proof in Appendix A. Equation (9) gives rise to $r_L(\omega, \beta) = r_L(\omega, -\beta)$, $r_R(\omega, \beta) = r_R(\omega, -\beta)$ and $t_L(\omega, -\beta) = t_R(\omega, \beta)$. It is clear that if unidirectional reflectionless transmission occurs at a pair $(\omega, \beta)$, then it also occurs at $(\omega, -\beta)$. If $\beta = 0$, i.e. for normal incident waves, then $t_L = t_R$ for any $\omega$.

For structures with some symmetries, the scattering matrix can be simplified. If the structure has an inversion symmetry, i.e.

$$\epsilon(x, y) = \epsilon(-x, -y) \hspace{1cm} \text{for all} \hspace{1cm} (x, y),$$

then the mapping $(x, y) \to (-x, -y)$ changes $\beta$ to $-\beta$, swaps $a_r$ with $a_l$ and $b_0$ with $b_0$. This leads to

$$\begin{bmatrix} b_0 \\ b_0 \end{bmatrix} = S(\omega, -\beta) \begin{bmatrix} a_l \\ a_r \end{bmatrix}.$$
Therefore,
\[ S(\omega, \beta) = PS(\omega, -\beta)P, \]
where
\[ P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

From Eq. (9), we obtain
\[ S(\omega, \beta) = PS^T(\omega, \beta)P. \] (10)

This implies that \( r_L = r_R \) for all \( \omega \) and \( \beta \). Therefore, unidirectional reflectionless transmission is impossible for structures with the inversion symmetry.

If the structure has a reflection symmetry in the \( y \) direction, i.e.
\[ \epsilon(x, y) = \epsilon(x, -y) \quad \text{for all} \quad (x, y), \]
then the mapping \( y \to -y \) changes \( \beta \) to \(-\beta\). Thus
\[ S(\omega, -\beta) = S(\omega, \beta). \]

From Eq. (9), we have
\[ S^T(\omega, \beta) = S(\omega, \beta). \] (11)

This implies that \( t_R = t_L \) for all \( \omega \) and \( \beta \).

If the structure has a reflection symmetry in the \( x \) direction, i.e.
\[ \epsilon(x, y) = \epsilon(-x, y) \quad \text{for all} \quad (x, y), \]
then the mapping \( x \to -x \) swaps \( a_r \) with \( a_l \) and \( b_{0r} \) with \( b_{0l} \). Thus
\[ PS(\omega, \beta)P^T = S(\omega, \beta). \] (12)

It implies that \( r_L = r_R \) and \( t_L = t_R \) for all \( \omega \) and \( \beta \). Clearly, unidirectional reflectionless is again impossible in this case. Notice that the symmetry in the \( x \) direction gives more constraints than each of the other two symmetries studied above.

When \( \epsilon \) and \( \omega \) are real, the power (per period) carried by the incident wave must equal to the power radiated out by the out-going waves. This leads to the condition that \( S \) must be unitary. A more general result for real \( \epsilon \) and complex \( \omega \) is
\[ S^*(\bar{\omega}, \beta)S(\omega, \beta) = I, \] (13)
where the superscript "*" denotes conjugate transpose, \( \bar{\omega} \) is the complex conjugate of \( \omega \), and \( I \) is the identity matrix. A proof of Eq. (13) is given in Appendix B. For a real \( \omega \), the unitarity of \( S \) gives
\[ |r_L|^2 + |t_L|^2 = 1 \quad \text{and} \quad \bar{r}_Lt_R + t_LR = 0. \]

Thus, if \( r_L = 0 \), then \( |t_L| = 1 \) and \( r_R = 0 \). Similarly, if \( r_R = 0 \), we must have \( r_L = 0 \). Therefore, unidirectional reflectionless transmission is impossible for lossless dielectric structures.

If \( \epsilon \) is real and symmetric in the \( x \) direction, it is known that there could be real frequencies with zero reflections and zero transmissions. Popov et al. [22] first studied a variant of this problem based on the scattering matrix. Shipman and Tu [23] analyzed this problem assuming the structure supports a bound state in the continuum for a nearby \( \beta \) and a nearby \( \omega \). If one assumes that \( \omega(r) \) is a simple (i.e. nondegenerate) zero of \( r_L \) (as an analytic function of \( \omega \)), and it is the only zero in a domain \( \mathcal{W} \) (of the complex \( \omega \) plane) containing both \( \omega(r) \) and it complex conjugate, then Eq. (13) allows us to show that \( \bar{\omega}(r) \) is a zero of \( r_R \). Since \( r_L = r_R \) when \( \epsilon \) is symmetric in \( x \), and \( \mathcal{W} \) contains only one zero of \( r_L \), we conclude that \( \omega(r) \) must be real.

\section{III. \( \mathcal{PT} \)-Symmetric Structures}

We are interested in a class of \( \mathcal{PT} \)-symmetric structures for which the dielectric function \( \epsilon \) is complex and satisfies
\[ \epsilon(x, y) = \bar{\epsilon}(-x, y). \] (14)

In other words, the real part of \( \epsilon \) is symmetric and the imaginary part is anti-symmetric in the \( x \) direction. In Appendix C, we show that the scattering matrix satisfies
\[ PS^*(\bar{\omega}, \beta)PS(\omega, \beta) = I, \] (15)
where \( P \) is the \( 2 \times 2 \) matrix given in Sec. II and \( I \) is the identity matrix. A more general result (without the parameter \( \beta \)) on the scattering matrix of \( \mathcal{PT} \)-symmetric structures is given in [25].

For a real \( \omega \), Eq. (15) gives
\[ r_LR + \bar{t}_R t_L = 1, \] (16)
\[ t_R R + \bar{t}_R^2 R = 0, \] (17)
\[ t_L^2 R + \bar{t}_L t_R^2 = 0. \] (18)

Let \( \phi_1 \) and \( \phi_2 \) be the phases of \( r_L \) and \( t_L \), respectively. Equation (17) implies that the phase of \( r_L \) is either \( \phi_1 + \pi/2 \) or \( \phi_1 - \pi/2 \). Equation (18) implies that the phase of \( r_R \) is either \( \phi_2 + \pi/2 \) or \( \phi_2 - \pi/2 \). For all cases, Eq. (16) leads to \( \phi_1 = \phi_2 \). Therefore \( r_LR \) and \( r_R \bar{r}_L \) are real. It can be verified that \( \lambda_1 \lambda_2 = \text{det}(S) = -e^{2i \phi_1} \), where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of \( S \).

If \( t_R \bar{t}_L < 1 \), \( r_R \bar{r}_L \) is positive, thus the phases of \( r_L \) and \( r_R \) are identical, and Eq. (16) leads to
\[ \sqrt{r_L} R R = 1 - \sqrt{T_L} T_R. \]

The above is the generalized energy conservation law [25]. If \( t_R \bar{t}_L > 1 \), \( r_R \bar{r}_L \) is negative, there is a \( \pi \) difference between the phases of \( r_L \) and \( r_R \), and Eq. (16) leads to
\[ \sqrt{r_L} R R = \sqrt{T_L} T_R - 1. \]

If \( t_R \bar{t}_L = 1 \), then at least one of \( r_L \) and \( r_R \) is zero. If only one of them is zero, we have unidirectional reflectionless
transmission. For \( \beta = 0 \) or \( \beta \neq 0 \) but the structure has an additional reflection symmetry in the \( y \) direction, then it is easy to show that \( t_L = t_R \). This special case has been extensively studied before [25].

We are interested in \( \mathcal{PT} \)-symmetric structures that are perturbations of a lossless dielectric structure (with a real dielectric function). If the unperturbed structure has a simple real zero-reflection frequency \( \omega_L \), which is the only zero of \( r_L \) contained in a domain \( W \) (\( W \) can be chosen as a small disk of the complex \( \omega \) plane and centered at \( \omega_L \)), we expect \( r_L \) of the perturbed \( \mathcal{PT} \)-symmetric structure still has only one simple zero \( \tilde{\omega}_L \) in \( W \). We show that \( \tilde{\omega}_L \) must still be real. Equation (15) can be written down explicitly as

\[
\begin{align*}
& t_R(\omega, \beta)\tilde{f}_L(\tilde{\omega}, \beta) + r_R(\omega, \beta)\tilde{f}_L(\tilde{\omega}, \beta) = 1, \\
& r_L(\omega, \beta)\tilde{f}_R(\tilde{\omega}, \beta) + t_L(\omega, \beta)\tilde{f}_L(\tilde{\omega}, \beta) = 1, \\
& t_R(\omega, \beta)\tilde{f}_R(\tilde{\omega}, \beta) + r_R(\omega, \beta)\tilde{f}_R(\tilde{\omega}, \beta) = 0, \\
& r_L(\omega, \beta)\tilde{f}_L(\tilde{\omega}, \beta) + t_L(\omega, \beta)\tilde{f}_R(\tilde{\omega}, \beta) = 0.
\end{align*}
\]

Since \( r_L = 0 \) at \( \omega = \tilde{\omega}_L \), the last equation above gives either \( r_L(\tilde{\omega}_L, \beta) = 0 \) or \( \tilde{f}_L(\tilde{\omega}_L, \beta) = 0 \). But the second equation above indicates that \( \tilde{f}_L(\tilde{\omega}_L, \beta) \) can not be zero, thus \( r_L(\tilde{\omega}_L, \beta) = 0 \). Since the domain \( W \) only contains only one zero of \( r_L \), we must have \( \omega_L = \tilde{\omega}_L \), i.e. \( \tilde{\omega}_L \) is real. Similarly, if we make proper assumptions about a right real zero-reflection frequency, \( \omega_R \), of the unperturbed structure, we can show that there must be a real zero, \( \tilde{\omega}_R \), of the right reflection coefficient \( r_R \) for the perturbed \( \mathcal{PT} \)-symmetric structure.

\section*{IV. Perturbation Analysis}

The previous section establishes the continual existence of real zero-reflection frequencies for both the left and right incident waves under \( \mathcal{PT} \)-symmetric perturbations. In this section, we use a perturbation method to show that these two frequencies are different in general, and thus unidirectional reflectionless transmission indeed occur. First, we consider a lossless dielectric structure with a real dielectric function \( \epsilon \) that is also symmetric in the \( x \) direction, and assume that there is a real wavenumber \( \beta \) and real zero-reflection frequency \( \omega_L = \omega_R \) for both left and right incident waves. Let \( u_L \) and \( u_R \) be the corresponding diffraction solutions for left and right incident plane waves with unit amplitude, respectively. The symmetry of the structure in \( x \) implies that \( u_R(x, y) = u_L(-x, y) \). At \( x = \pm D \), these two solutions can be written down as

\[
\begin{align*}
& u_R(D, y) = u_L(-D, y) = e^{i\beta y} + \sum_{j=-\infty}^{\infty} b_j^- e^{i\beta y}, \quad (19) \\
& u_R(-D, y) = u_L(D, y) = \sum_{j=-\infty}^{\infty} b_j^+ e^{i\beta y}, \quad (20)
\end{align*}
\]

where \( \{b_j^-\} \) are the Fourier coefficients of \( u_L(D, y) \), and \( \{b_j^+\} \) are the Fourier coefficients of \( u_L(-D, y) - e^{i\beta y} \). Since the reflection is zero and energy is conserved, we have

\[
b_0^- = 0, \quad |b_0^+| = 1.
\]

Next, we consider a perturbed \( \mathcal{PT} \)-symmetric structure with a dielectric function

\[
\epsilon = \epsilon + \delta F(x, y),
\]

where \( \delta \) is a small real number, \( F \) is a complex \( O(1) \) function satisfying

\[
F(x, y) = \tilde{F}(-x, y),
\]

and \( F(x, y) = 0 \) for \( |x| > D \). According to Sec. III there must be real frequencies \( \tilde{\omega}_L \) and \( \tilde{\omega}_R \) such that \( r_L = 0 \) and \( r_R = 0 \), respectively, for the perturbed structure \( \epsilon + \delta F \) and the fixed \( \beta \).

Let \( \tilde{k}_L = \tilde{\omega}_L/c \) and \( \tilde{k}_R = \tilde{\omega}_R/c \), where \( c \) is the speed of light in vacuum. We expand \( \tilde{k}_L \) and \( \tilde{k}_R \) in power series of \( \delta \):

\[
\begin{align*}
& \tilde{k}_L = k_0 + k_1^{(L)} \delta + k_2^{(L)} \delta^2 + \ldots, \quad (22) \\
& \tilde{k}_R = k_0 + k_1^{(R)} \delta + k_2^{(R)} \delta^2 + \ldots, \quad (23)
\end{align*}
\]

where \( k_0 = \omega_L/c = \omega_R/c \). In Appendix D, we show that the coefficients \( k_1^{(L)} \) and \( k_1^{(R)} \) are given by

\[
\begin{align*}
k_1^{(L)} &= \frac{iL}{\alpha} (b_0^+ - \bar{b}_0^+) + iL \sum_{j \neq 0} \frac{b_j^+ \bar{b}_j^+ + b_j^- \bar{b}_j^-}{\alpha_j} + 2 \int_{\Omega} \epsilon u_L \bar{u}_R dxdy, \quad (24) \\
k_1^{(R)} &= \frac{iL}{\alpha} (b_0^- - \bar{b}_0^+) + iL \sum_{j \neq -\infty} \frac{b_j^+ \bar{b}_j^+ + b_j^- \bar{b}_j^-}{\alpha_j} + 2 \int_{\Omega} \epsilon u_L u_R dxdy, \quad (25)
\end{align*}
\]
where $\Omega_D$ is the rectangle given by $|x| < D$ and $|y| < L/2$. The coefficients $\{a_j\}$ are defined in Sec. IV, $\omega_L$ and $\beta$ are assumed such that $a = a_0$ is real, and all other $a_j$ for $j \neq 0$ are pure imaginary. It is easy to verify that $\tilde{u}_R u_L$ is a $\mathcal{PT}$-symmetric function satisfying the same Eq. (14). Using this result and the symmetry of $\epsilon$ and $F$, it is straightforward to show that $k_1^{(L)}$ and $k_1^{(R)}$ are real, and the denominators of Eqs. (24) and (25) are identical. Therefore, if $F(x, y)$ satisfies

$$\int_{\Omega_D} [F(x, y) - F(-x, y)] u_L \tilde{u}_R dxdy \neq 0,$$

then $k_1^{(L)} \neq k_1^{(R)}$. This implies that as far as $F$ satisfies Eq. (26), $\omega_L \neq \omega_R$ for arbitrarily small $\delta$, and unidirectional reflectionless transmission occurs at frequencies $\omega_L$ and $\omega_R$ for left and right incident waves, respectively.

V. NUMERICAL EXAMPLES

In this section, we present numerical results to illustrate the unidirectional reflectionless transmission phenomenon, validate the perturbation results of Sec. IV, and show examples of unidirectional invisibility. We consider a periodic array of identical circular cylinders with period $L$ in the $y$ direction and surrounded by air as shown in Fig. 1(a). The coordinates are chosen so that the centers of the cylinders are on the $y$ axis and the center of one cylinder is at the origin. The first example is for cylinders with a $y$-independent dielectric function given by

$$\epsilon(x, y) = \epsilon_1 + i\delta \sin \left(\frac{\pi x}{2a}\right),$$

where $a = 0.3L$ is the radius of the cylinders, $\epsilon_1 = 10$, and $\delta$ is a real parameter. If $\delta \neq 0$, the structure is $\mathcal{PT}$-symmetric with respect to the reflection in the $x$ direction. The diffraction problem for a given incident wave can be solved by many different numerical methods. We use a mixed Fourier-Chebyshev pseudospectral method to discretize the Helmholtz equation inside the disk of a radius $a$ (corresponding to the cross section of the central cylinder), and use cylindrical and plane wave expansions outside the cylinders. In Fig. 1(b), we show the reflection and transmission spectra in logarithmic scale for left and right normal incident waves. For $\delta = 0$, $\epsilon$ is real and symmetric in both $x$ and $y$ directions, thus the reflection and transmission spectra are identical for left and right incident waves, i.e. $R_L = R_R$ and $T_L = T_R$. In Fig. 1(b), a dip can be observed in the reflection spectrum for $\delta = 0$. Presumably, the reflection coefficient is exactly zero at normalized frequency $\omega L/(2\pi c) = 0.5882$. To use the perturbation results of Sec. IV, we assume $D = L/2$, then $\Omega_D$ is a square of side length $L$ centered at the origin. Using the numerical solutions to evaluate Eqs. (24) and (25), we obtain $k_1^{(L)} \approx 0.018L^{-1}$ and $k_1^{(R)} \approx -0.018L^{-1}$. This implies that when $\delta$ is increased from zero, the zero-reflection frequencies for left and right incident waves will increase and decrease, respectively. The numerical results for $\delta = 0.1$ are also shown in Fig. 1 and they indicate that the normalized zero-reflection frequency $\omega L/(2\pi c)$ is increased to 0.5885 for left incident waves, and is decreased to 0.5879 for right incident waves. According to our perturbation theory, the normalized zero-reflection frequency to the left incident waves is approximately $0.5882 + \delta k_1^{(L)} L/(2\pi)$. Keeping four significant digits, this value is also 0.5885. Therefore, the numerical and perturbation results agree very well with each other. At the zero-reflection frequency for left (or right) incident waves, the reflection coefficient for right (or left) incident waves is non-zero, therefore, we have unidirectional reflectionless transmissions. Since the structure is symmetric in $y$, the transmission coefficients for left and right incident waves are identical. From Eq. (16), it is clear that the transmittance is exactly 1 at the zero-reflection frequencies.

![FIG. 1: (a) A periodic array of identical circular cylinders with incident waves from left and right. (b) Example 1: reflection and transmission spectra for normal incident waves.](image)
For $\delta = 0$, the zero-reflection frequencies for left and right incident waves are identical. For $\delta \neq 0$, they are different in general, thus unidirectional reflectionless transmission can occur. Overall, the curves for left and right zero-reflection frequencies are mirror images of each other. At a zero-reflection frequency for a left incident wave, the transmission coefficient $r_L$ has unit magnitude, and we can define a phase $\theta$ relative to the incident wave (extended to the whole space) by

$$e^{i\theta} = t_L e^{-2i\alpha D}.$$  

In Fig. 2(b), we show the relative phase $\theta$ for all zero-reflection frequencies on the blue solid curves in Fig. 2(a). Point A in Figs. 2(a) and (b) is a point with a zero relative phase. It is obtained at $\omega L / (2\pi c) = 0.5069$ for $\delta = 2.8$. In Figs. 3(a) and (c), we show the diffraction solutions for left and right normal incident waves for point A. For a left incident wave (i.e. $a_l = e^{-i\alpha D}, a_r = 0$), the total wave is identical to the plane wave $e^{i\alpha y}$ (shown in Fig. 3(b)) away from the cylinders. This implies that the cylinders are invisible to left incident waves. For a right incident wave (i.e. $a_l = 0, a_r = e^{i\alpha D}$), the transmitted wave also has unit amplitude and zero relative phase, the reflected wave has a magnitude about 6.93. Although the $\mathcal{PT}$-symmetric structure has a balanced gain and loss profile, energy does not need to be conserved. For this case, a strong reflected wave is produced thanks to the gain medium.

In order to study $\mathcal{PT}$-symmetric structures without the reflection symmetry in $y$, we consider another example. The structure is again a periodic array of identical circular cylinders with their centers on the $y$ axis, but the dielectric function of the cylinder centered at the origin is given by

$$\epsilon(x, y) = \epsilon_1 + 2 \sin \left( \frac{\pi y}{2a} \right) + 2i \sin \left( \frac{\pi x}{2a} \right), \quad (28)$$

where $\epsilon_1$ and $a$ are the same as the first example. The structure is $\mathcal{PT}$-symmetric with respect to a reflection in the $x$ direction, but it is not symmetric in $y$. For $\beta \neq 0$, the transmission coefficients for left and right incident waves are different in general. In Fig. 4 we show the reflection and transmission spectra for left and right incident waves with $\beta L / (2\pi) = 0.2$. At $\omega L / (2\pi c) = 0.3569$, the reflection is zero for left incident waves and nonzero for right incident waves, thus unidirectional reflectionless transmission occurs. The corresponding transmittance are $T_L = 0.9488$ and $T_R = 1.0539$. From Sec. III, we know that $t_R l_L$ is always real. To verify this, we show the value of $t_R l_L - 1$ as a function of the frequency in Fig. 5(a). Clearly, $t_R l_L = 1$ at zero-reflection frequency $\omega L / (2\pi c) = 0.3569$. In Fig. 5(b), we show the phases of $r_L$ and $r_R$. Notice that the phase of $r_L$ has a jump discontinuity of $\pi$ at the zero-reflection frequency.

FIG. 2: (a) Zero-reflection frequencies for different $\delta$ in the first example. Blue solid line: left incident wave, red dashed line: right incident waves. (b) Relative phases $\theta$ of the transmitted waves at zero-reflection frequencies for left incident waves.

FIG. 3: Real part of the diffraction solutions, i.e. Re($u$), corresponding to point A in Fig. 2(a). (a) Left incident wave with $a_l = e^{-i\alpha D}$ and $a_r = 0$. (b) Plane wave $e^{i\alpha x}$ propagating in air. (c) Right incident wave with $a_l = 0$ and $a_r = e^{i\alpha D}$. Red circles denote the cylinders. The wave fields are capped from $-1$ to $1$. 

FIG. 4: Reflection and transmission spectra for (a) left incident waves, (b) right incident waves.
In this paper, we studied 2D \( \mathcal{PT} \)-symmetric periodic structures sandwiched between two homogeneous media. Using a scattering matrix formalism we showed that the real zero-reflection frequencies are robust under \( \mathcal{PT} \)-symmetric perturbations. A simple condition on the perturbed dielectric function was derived by a perturbation method to guarantee that the real zero-reflection frequencies for left and right incident waves are different. Therefore, as far as the original unperturbed dielectric structure is symmetric and has a real non-degenerate zero-reflection frequency, unidirectional reflectionless transmission is certain to occur for almost any \( \mathcal{PT} \)-symmetric perturbations. Numerical examples are presented for periodic arrays of circular cylinders with \( \mathcal{PT} \)-symmetric dielectric functions. The numerical results confirmed the perturbation theory, and illustrated unidirectional invisibility and other interesting wave phenomena.

Both the scattering matrix formalism and the perturbation analysis can be easily extended to unperturbed structures that are themselves \( \mathcal{PT} \)-symmetric. The scattering matrix formalism allows us to conclude that real non-degenerate zero-reflection frequencies are protected by the \( \mathcal{PT} \)-symmetry, in the sense that these frequencies remain real for any \( \mathcal{PT} \)-symmetric perturbations. The perturbation theory gives quantitative results on the changes of the real zero-reflection frequencies caused by perturbations. It is also straightforward to consider non-\( \mathcal{PT} \)-symmetric perturbation that could move the real zero-reflection frequencies to the complex plane. Our study enhances the theoretical understanding on the zero-reflection frequencies, unidirectional reflectionless transmission, and unidirectional invisibility for \( \mathcal{PT} \)-symmetric structures, and provides a solid foundation for further studies on these wave phenomena and for exploring their potential applications.

ACKNOWLEDGMENTS

The authors acknowledge support from the Science and Technology Research Program of Chongqing Municipal Education Commission, China (Grant No. KJ1706155), and the Research Grants Council of Hong Kong Special Administrative Region, China (Grant No. CityU 11304117).

APPENDIX

Appendix A: Reciprocity

To derive Eq. (9), we consider the diffraction problem for incident waves \( \tilde{a}_l e^{i[\alpha(x+D)-\beta y]} \) and \( \tilde{a}_r e^{-i[\alpha(x-D)+\beta y]} \) with frequency \( \omega \) and wavenumber \( -\beta \). The diffraction solution \( \tilde{u} \) can be written as

\[
\tilde{u}(x, y) = \tilde{a}_l e^{i[\alpha(x+D)-\beta y]} + \sum_{j=-\infty}^{+\infty} \tilde{b}_{lj} e^{-i[\tilde{\alpha}_j(x+D)-\beta_j y]},
\]

for \( x < -D \) and

\[
\tilde{u}(x, y) = \tilde{a}_r e^{-i[\alpha(x-D)+\beta y]} + \sum_{j=-\infty}^{+\infty} \tilde{b}_{lj} e^{i[\tilde{\alpha}_j(x-D)+\beta_j y]},
\]

for \( x > D \), where

\[
\beta_j = -\beta + 2j \pi/L, \quad \tilde{\alpha}_j = \sqrt{k_0^2 - \beta_j^2},
\]
for $j = 0, \pm 1, \pm 2, \ldots$. Notice that
\[
\tilde{\beta}_j = -\beta_{-j}, \quad \tilde{\alpha}_j = \alpha_{-j}, \quad \text{for} \quad j = 0, \pm 1, \pm 2, \ldots.
\]
The coefficients $\hat{b}_{l0}$ and $\hat{b}_{r0}$ are related to the incident coefficients $\hat{a}_l$ and $\hat{a}_r$, by scattering matrix $S(\omega, \beta)$ as
\[
\begin{bmatrix}
\hat{b}_{l0} \\
\hat{b}_{r0}
\end{bmatrix} = S(\omega, -\beta) \begin{bmatrix}
\hat{a}_l \\
\hat{a}_r
\end{bmatrix}.
\]
From the governing equations of $u$ and $\hat{u}$, we have
\[
0 = \hat{u}(\Delta u + k_0^2 \epsilon u) - u(\Delta \hat{u} + k_0^2 \epsilon \hat{u}) = \nabla \cdot (\hat{u} \nabla u) - \nabla \cdot (u \nabla \hat{u}),
\]
where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator. Integrating the above equation on domain $\Omega_D$, we have
\[
\int_{\partial \Omega_D} \left( \hat{u} \frac{\partial u}{\partial \nu} - u \frac{\partial \hat{u}}{\partial \nu} \right) \, ds = 0.
\]
The line integrals on the two edges of $\Omega_D$ at $y = \pm L/2$ cancel out. From the expressions of $u$ (i.e. Eqs. [1] and [3]) and $\hat{u}$ at $x = \pm D$, and the relations of $\beta_j$ and $\beta_{-j}$, we obtain
\[
\hat{a}_l \hat{b}_{l0} + \hat{a}_r \hat{b}_{r0} = a_l \hat{b}_{l0} + a_r \hat{b}_{r0}.
\]
Therefore,
\[
[a_l, a_r] \left[ S^T(\omega, \beta) - S(\omega, -\beta) \right] \begin{bmatrix}
\hat{a}_l \\
\hat{a}_r
\end{bmatrix} = 0
\]
for any complex $a_l, a_r, \hat{a}_l$ and $\hat{a}_r$. This leads to Eq. [9].

**Appendix B: Unitarity for lossless dielectric structures**

Assume $\epsilon$ is real and $\omega$ is complex with a small (positive and negative) imaginary part. We consider the diffraction problem for two incident waves $\hat{a}_l e^{i[\alpha(\omega)(x+D) + \beta y]}$ and $\hat{a}_r e^{-i[\alpha(\omega)(x-D) - \beta y]}$ with frequency $\alpha$ and wavenumber $\beta$, where $\alpha(\omega) = \sqrt{k_0^2 - \beta^2}$ and $k_0 = \omega/c$. Here the square root is defined with a branch cut on the negative imaginary axis as shown in Sec. [1].

The diffraction problem is governed by the Helmholtz equation
\[
\Delta w + k_0^2 \epsilon w = 0.
\]
The solution $w$ can be written as
\[
w(x, y) = \hat{a}_l e^{i[\alpha(\omega)(x+D) + \beta y]} + \sum_{j=-\infty}^{+\infty} \hat{b}_{lj} e^{-i[\alpha(\omega)(x+D) - \beta y]},
\]
for $x < -D$ and
\[
w(x, y) = \hat{a}_r e^{-i[\alpha(\omega)(x-D) - \beta y]} + \sum_{j=-\infty}^{+\infty} \hat{b}_{rj} e^{i[\alpha(\omega)(x-D) + \beta y]},
\]
for $x > D$, where
\[
\alpha_j(\omega) = \sqrt{k_0^2 - \beta_j^2}, \quad \text{for} \quad j = 0, \pm 1, \pm 2, \ldots.
\]
The relations between $\alpha_j(\omega)$ and $\alpha_j(\omega)$ are
\[
\alpha_j(\omega) = \begin{cases} 
\hat{a}_j(\omega), & j = 0 \\
-\hat{a}_j(\omega), & j = \pm 1, \pm 2, \ldots.
\end{cases}
\]
The coefficients $\hat{b}_{l0}$ and $\hat{b}_{r0}$ are related to the incident coefficients $\hat{a}_l$ and $\hat{a}_r$ by scattering matrix $S(\omega, \beta)$ as
\[
\begin{bmatrix}
\hat{b}_{l0} \\
\hat{b}_{r0}
\end{bmatrix} = S(\omega, \beta) \begin{bmatrix}
\hat{a}_l \\
\hat{a}_r
\end{bmatrix}.
\]
From the governing equations of $u$ and $w$, we have
\[
0 = \hat{w}(\Delta u + k_0^2 \epsilon u) - u(\Delta \hat{w} + k_0^2 \epsilon \hat{w}) = \nabla \cdot (\hat{w} \nabla u) - \nabla \cdot (u \nabla \hat{w}).
\]
Integrating the above equation on domain $\Omega_D$, we have
\[
\int_{\partial \Omega_D} \left( \hat{w} \frac{\partial u}{\partial \nu} - u \frac{\partial \hat{w}}{\partial \nu} \right) \, ds = 0.
\]
The line integrals on the two edges of $\Omega_D$ at $y = \pm L/2$ cancel out. From the expressions of $u$ and $w$ at $x = \pm D$, and the relations of $\alpha_j$ and $\alpha_j(\omega)$, we obtain
\[
a_l \hat{a}_l + a_r \hat{a}_r = b_{l0} \hat{b}_{l0} + b_{r0} \hat{b}_{r0}.
\]
Therefore,
\[
[a_l, a_r] \left[ S^* (\omega, \beta) S(\omega, \beta) - I \right] \begin{bmatrix}
\hat{a}_l \\
\hat{a}_r
\end{bmatrix} = 0
\]
for any complex $a_l, a_r, \hat{a}_l$ and $\hat{a}_r$. This leads to Eq. [13].

**Appendix C: Scattering matrix for PT-symmetric structures**

Let $\epsilon$ satisfy the $PT$-symmetric condition Eq. [14], $\omega$ be complex with a small (positive or negative) imaginary part, $w$ be the diffraction solution defined in Appendix B and $v = \hat{w}(-x, y)$, then
\[
\Delta v + k_0^2 \epsilon v = 0.
\]
From the governing equations of $u$ and $v$, we have
\[
0 = v(\Delta u + k_0^2 \epsilon u) - u(\Delta v + k_0^2 \epsilon v) = \nabla \cdot (v \nabla u) - \nabla \cdot (u \nabla v).
\]
Integrating the above equation on domain $\Omega_D$, we have
\[
\int_{\partial \Omega_D} \left( v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) \, ds = 0.
\]
The line integrals on the two edges of $\Omega_D$ at $y = \pm L/2$ cancel out. Using the expressions of $u$ and $v$ (notice that $v = \hat{w}(-x, y)$), we have
\[
a_l \hat{a}_r + a_r \hat{a}_l = b_{l0} \hat{b}_{l0} + b_{r0} \hat{b}_{r0}.
\]
From Eqs. [8] and [29], we obtain
\[
[a_l, a_r] \left[ PS^* (\omega, \beta) PS(\omega, \beta) - I \right] \begin{bmatrix}
a_l \\
a_r
\end{bmatrix} = 0
\]
for any complex $a_l, a_r, \hat{a}_l$ and $\hat{a}_r$. This leads to Eq. [15].
Appendix D: Perturbation analysis

To carry out the perturbation analysis, we first formulate a boundary value problem for the diffraction problem. Notice that a diffraction solution for incident waves given in Eqs. (2) and (3) has the general expansions given in Eqs. (4) and (5) for $|x| > D$. If we define a linear operator $T$ such that

$$T e^{j\beta y} = io_j e^{j\beta y}, \quad j = 0, \pm 1, \pm 2, \ldots,$$

then the diffraction solution $u$ satisfies the following boundary conditions [31]

$$\begin{cases}
\frac{\partial u}{\partial x} = -Tu + 2ioa_te^{j\beta y}, & x = -D, \\
\frac{\partial u}{\partial x} = Tu - 2ioa_te^{j\beta y}, & x = D.
\end{cases}$$

(31)

In the $y$-direction, $u$ satisfies the quasi-periodic conditions

$$u(x, L/2) = e^{i\beta L}u(x, -L/2),$$

(32)

$$\frac{\partial u}{\partial y}(x, L/2) = e^{i\beta L}\frac{\partial u}{\partial y}(x, -L/2).$$

(33)

Thus the diffraction problem is a boundary value problem of Helmholtz equation (1) with boundary conditions Eqs. (31) and (32). Function $u_L(u_R)$ is a solution of the boundary value problem with $a_l = 1$ and $a_r = 0$ ($a_l = 0$ and $a_r = 1$).

To derive Eq. (24), we let $\tilde{u}_L$ be the diffraction solution of the perturbed structure $\tilde{\xi}$ at zero-reflection frequency $\tilde{\omega}_L$ (i.e. $\tilde{k}_L = \tilde{\omega}_L/c$) for a left incident wave with unit amplitude. Then $\tilde{u}_L$ is a solution of the boundary value problem with $k_0$ replaced by $k_L$. $a_l = 1$ and $a_r = 0$. We expand $\tilde{u}_L$, $\tilde{T}$ and $\tilde{\alpha}_j$ for $\tilde{k}_L$ in power series of $\delta$:

$$\tilde{u}_L = u_L + u_1 \delta + u_2 \delta^2 + \ldots,$$

(34)

$$\tilde{T} = T + T_1 \delta + T_2 \delta^2 + \ldots,$$

(35)

$$\tilde{\alpha}_j = \sqrt{k_L^2 - \beta_j^2} = \alpha_j + \gamma_1 \delta + \gamma_2 \delta^2 + \ldots,$$

(36)

for $j = 0, \pm 1, \pm 2, \ldots$. Similar to the definition of $T$ in Eq. (30), the actions of $T_1$ and $T_2$ on $e^{j\beta y}$ are simply $e^{j\beta y}$ multiplied by $i\alpha_j$, $i\gamma_1 j$ and $i\gamma_2 j$, respectively. Using expansion Eq. (22), we have

$$\gamma_1 j = \frac{k_0 k_1}{\alpha_j}, \quad j = 0, \pm 1, \pm 2, \ldots.$$

(37)

Substituting expansions (21), (22) and (34)-(36) into the boundary value problem for $u_L$, and comparing the coefficient of $\delta$, we have

$$\begin{cases}
\Delta u_1 + k_0^2 u_1 = -(2k_0 k_1 \epsilon + k_0^2 F)u_L, \quad (x, y) \in \Omega_D \\
\frac{\partial u_1}{\partial x} = -(Tu_1 + T_1 u_L) + 2i\gamma_1 e^{j\beta y}, \quad x = -D, \\
\frac{\partial u_1}{\partial x} = Tu_1 + T_1 u_L, \quad x = D,
\end{cases}$$

(38)

and $u_1$ satisfies the quasi-periodic condition Eq. (32) in the $y$ direction. Let

$$\hat{u}_L(-D, y) = e^{j\beta y} + \sum_{j=-\infty}^{\infty} \hat{b}_j^- e^{j\beta y},$$

$$\hat{u}_L(D, y) = \sum_{j=-\infty}^{\infty} \hat{b}_j^+ e^{j\beta y},$$

where $\{\hat{b}_j^+\}$ are the Fourier coefficients of $\hat{u}(D, y)$ and $\{\hat{b}_j^-\}$ are the Fourier coefficients of $\hat{u}(-D, y) - e^{i\beta y}$, then

$$\hat{b}_0^- = 0, \quad |\hat{b}_0^+| = 1.$$

Let

$$u_1(\pm D, y) = \sum_{j=-\infty}^{\infty} c_j^\pm e^{j\beta y},$$

where $\{c_j^\pm\}$ are the Fourier coefficients of $u_1(\pm D, y)$, then we must have $c_0^- = 0$.

From the governing equations of $u_R$ and $u_1$, we have

$$-(2k_0 k_1 \epsilon + k_0^2 F)u_L u_R = u_R (\Delta u_1 + k_0^2 \epsilon u_1)$$

$$-u_1(\Delta \hat{u}_R + k_0^2 \hat{u}_R) = \nabla \cdot (u_R \nabla u_1) - \nabla \cdot (u_1 \nabla \hat{u}_R).$$

Integrating the above equation on domain $\Omega_D$, we obtain

$$\int_{\partial \Omega_D} \left( \frac{\partial u_R}{\partial \nu} \frac{\partial u_1}{\partial \nu} - u_1 \frac{\partial u_R}{\partial \nu} \right) ds$$

$$= - \int_{\Omega_D} (2k_0 k_1 \epsilon + k_0^2 F) u_L \hat{u}_R dxdy.$$  (39)

Due to the quasi-periodic condition (32), the line integrals on the two edges of $\Omega_D$ at $y = \pm L/2$ cancel out. Furthermore, by using the boundary conditions of $u_1$ at $x = \pm D$ and expansions of $u_L, u_R$ and $u_1$ at $x = \pm D$, the left-hand side of Eq. (39) can be reduced to

$$iL \gamma_1 \sum_{j=-\infty}^{\infty} (\hat{b}_j^- b_j^+ + \hat{b}_j^+ b_j^-).$$

In the above, the conditions $\hat{b}_0^- = c_0^- = 0$ are used. Substituting the above into Eq. (39) and noticing the formula of $\gamma_1 j$ (i.e. Eq. (37)), we obtain Eq. (24). Equation (25) can be similarly derived.
[1] C. M. Bender and S. Boettcher, “Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$-symmetry,” Phys. Rev. Lett. 80, 5243-5246 (1998).

[2] L. Feng, R. El-Ganainy, and L. Ge, “Non-Hermitian photonics based on parity-time symmetry,” Nature Photonics 11, 752-762 (2017).

[3] S. Longhi, “Parity-time symmetry meets photonics: A new twist in non-Hermitian optics,” EPL 120, 64001 (2017).

[4] M.-A. Miri and A. Alu, “Exceptional points in optics and photonics,” Science 363, 7709 (2019).

[5] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, “Unidirectional invisibility induced by $\mathcal{PT}$-symmetric periodic structures,” Phys. Rev. Lett. 106, 213901 (2011).

[6] S. Longhi, “Invisibility in $\mathcal{PT}$-symmetric complex crystals,” J. Phys. A: Math. Theor. 44, 485302 (2011).

[7] Y. Huang, Y. Shen, C. Min, S. Fan, and G. Veronis, “Unidirectional reflectionless light propagation at exceptional points,” Nanophotonics 584, 192 (2017).

[8] S. Kalish, Z. Lin, and T. Kottos, “Light transport in random media with $\mathcal{PT}$ symmetry,” Phys. Rev. A 85, 055802 (2012).

[9] A. Mostafazadeh, “Invisibility and $\mathcal{PT}$-symmetry,” Phys. Rev. A 87, 012103 (2013).

[10] E. Yang, T. Lu, Y. Wang, Y. Dai, and P. Wang, “Unidirectional reflectionless phenomenon in periodic ternary layered material,” Opt. Express 24, 14311-14321 (2016).

[11] M. Sarisaman, “Unidirectional reflectionless and invisibility in the TE and TM modes of a $\mathcal{PT}$-symmetric slab system,” Phys. Rev. A 95, 013806 (2017).

[12] M. Sarisaman and M. Tas, “Unidirectional invisibility and $\mathcal{PT}$ symmetry with graphene,” Phys. Rev. B 97, 045409 (2018).

[13] S. A. R Horsley, M. Artoni, and G. C. La Rocca, “Spatial Kramers-Kronig relations and the reflections of waves,” Nat. Photon. 9, 436-439 (2015).

[14] Y. Fu, Y. Xu, and H. Chen, “Zero index metamaterials with $\mathcal{PT}$ symmetry in a waveguide system,” Opt. Express 24, 1648-1657 (2016).

[15] N. X. A. Rivolta, B. Maes, “Side-coupled resonators with parity-time symmetry for broadband unidirectional invisibility,” Phys. Rev. A 94, 053854 (2016).

[16] A. Regensburger, C. Bersch, et al, “Parity-time synthetic photonic lattices,” Nature 488, 167-171 (2012).

[17] L. Feng, Y. Xu, W. S. Fegadolli, et al, “Experimental demonstration of a unidirectional reflectionless parity-time metamaterial at optical frequencies,” Nat. Mater. 12, 108-113 (2013).

[18] M.-A. Miri, P. Likanwa, and D. N. Christodoulides, “Large area single-mode parity-time-symmetric laser amplifiers,” Opt. Lett. 37, 764-766 (2012).

[19] W. Liu et al., “An integrated parity-time symmetric wavelength-tunable single-mode microring laser,” Nat. Commun. 8, 15389 (2017).

[20] S. Longhi, “$\mathcal{PT}$-symmetric laser absorber,” Phys. Rev. A 82, 031801(R) (2010).

[21] Y. D. Chong, L. Ge, and A. D. Stone, “$\mathcal{PT}$-symmetry breaking and laser-absorber modes in optical scattering systems,” Phys. Rev. Lett. 106, 093902 (2011).

[22] E. Popov, L. Mashev, and D. Maystre, “Theoretical study of the anomalies of coated dielectric gratings,” Optica Acta 33(5), 607-619 (1986).

[23] S. P. Shipman and H. Tu, “Total Resonant Transmission and Reflection by Periodic Structures,” SIAM J. Appl. Math. 72, 216-239 (2012).

[24] L. Chesnel and S. A. Nazarov, “Non reflection and perfect reflection via Fano resonances in waveguides,” Commun. Math. Sci. 16, 1779-1800 (2018).

[25] L. Ge, Y. D. Chong, and A. D. Stone, “Conservation relations and anisotropic transmission resonances in one-dimensional $\mathcal{PT}$-symmetric photonic heterostructures,” Phys. Rev. A 85, 023802 (2012).

[26] Y. Shen, X. Deng, L. Chen, “Unidirectional invisibility in a two-layer non-$\mathcal{PT}$-symmetric slab,” Opt. Express 22, 19440-19447 (2014).

[27] Y. Huang, G. Veronis, and C. Min, “Unidirectional reflectionless propagation in plasmonic waveguide-cavity systems at exceptional points,” Opt. Express 23, 29882-29895 (2015).

[28] X. Gu, R. Bai, et al, “Unidirectional reflectionless propagation in a non-ideal parity-time metasurface based on far field coupling,” Opt. Express 25, 11778-11787 (2017).

[29] F. Zhao, T. Dai, C. Zhang, R. Bai, Y. Q. Zhang, X. R. Jin, and Y. Lee, “Dual-band unidirectional reflectionless at exceptional points in a plasmonic waveguide system based on near-field coupling between two resonators,” Nanotechnology 30, 045205 (2019).

[30] L. Feng, X. Zhu, S. Yang, et al., “Demonstration of a large-scale optical exceptional point structure,” Opt. Express 22, 1760-1767 (2014).

[31] G. Bao, D. C. Dobson, and J. A. Cox, “Mathematical studies in rigorous grating theory,” J. Opt. Soc. Am. A 12, 1029 (1995).

[32] L. N. Trefethen, “Spectral methods in MATLAB,” Society for Industrial and Applied Mathematics (2000).

[33] Y. Huang and Y. Y. Lu, “Scattering from periodic arrays of cylinders by Dirichlet-to-Neumann maps,” J. Lightwave Technol. 24, 3448-3453 (2006).