A NECESSARY AND SUFFICIENT CONDITION FOR
LOCAL CONTROLLABILITY AROUND CLOSED ORBITS.

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Abstract. In this paper we give a necessary and sufficient condition
for local controllability around closed orbits for general smooth control
systems. We also prove that any such system on a compact manifold
has a closed orbit.

1. Introduction.

1.1. Motivation. The aim of this note is to formulate and prove a necessary
and sufficient condition for local controllability of general control systems
around a closed orbit. Let $M$ be a smooth (or real analytic) manifold, and let
$U$ be a subset of $\mathbb{R}^k$. Consider a smooth (or real analytic) control system $(\Sigma)$
$\dot{x} = f(x, u)$, $u \in U$, where controls $u : [0, T] \rightarrow U$ are bounded measurable,
and the final time $T = T(u) \geq 0$ is not fixed and depends on a control $u$.
If $u : [0, T(u)] \rightarrow U$ is a control then a solution of the ordinary differential
equation $\dot{x}(t) = f(x(t), u(t))$ is called a trajectory (or an admissible curve,
or an orbit) of $(\Sigma)$ generated by $u$. The system $(\Sigma)$ is said to be controllable
if for every $x, y \in M$ there exists a control $u$ defined on $[0, T(u)]$ such that
if $\gamma$ is the trajectory of $(\Sigma)$ generated by $u$ and satisfying $\gamma(0) = x$, then
$\gamma(T(u)) = y$. The system $(\Sigma)$ is controllable at a point $x$ if there exists a
neighbourhood $U$ of $x$ such that the restriction of $(\Sigma)$ to $U$ is a controllable
system. A neighbourhood $U$ as above is called a controllable neighbourhood.

There are a lot of results devoted to controllability question for control
systems in connection with the existence of closed or ‘almost closed’ orbits,
for instance: [2], [3], [5], [6], [10], [11]. Before we cite a few of them, we will
fix some notation. If $Z_1, ..., Z_l$ are vector fields on a manifold $M$ then denote
by $\text{Lie}\{Z_1, ..., Z_l\}$ the Lie algebra generated by $Z_1, ..., Z_l$. For an $x \in M$,
let $\text{Lie}_x\{Z_1, ..., Z_l\}$ stand for the subspace in $T_x M$ spanned by all vectors $v$
of the form $v = W(x)$ where $W \in \text{Lie}\{Z_1, ..., Z_l\}$. Recall that a point $x$ is
Poisson stable for a vector field $X$ if for every neighbourhood $V$ of $x$, and
for every $T > 0$ there exist $t_1, t_2 > T$ such that $g_{X}^{t_1}(x) \in V$, $g_{X}^{-t_2}(x) \in V$.
Also, a vector field $X$ defined on a Riemannian manifold is conservative if
$g_{X}^{t}$ preserves the natural measure on $M$. In both cases $g_{X}^{t}$ stands for the
flow of $X$.

Let us start from citing two results on global controllability.

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Theorem (Bonnard [2]): Consider an affine control system \( \dot{x} = X + \sum_{i=1}^{k} u_i Y_i \) on an analytic manifold \( M \), where \( \sum_{i=1}^{k} |u_i| \leq 1 \) and the fields \( X, Y_i, i = 1, ..., k \), are supposed to be analytic. Assume that the set of points which are Poisson stable for \( X \) is dense in \( M \). Then the system in question is controllable if and only if \( \dim \text{Lie}_x \{ X, Y_1, ..., Y_k \} = \dim M \) for every \( x \in M \).

In particular, controllability holds if all orbits of \( X \) are closed.

Theorem (Lobry [11]): Consider an affine control system \( \dot{x} = X + \sum_{i=1}^{k} u_i Y_i \) on a compact analytic manifold \( M \), where \( \sum_{i=1}^{k} |u_i| \leq 1 \) and the fields \( X, Y_i, i = 1, ..., k \), are supposed to be analytic and conservative. Then the system in question is controllable if and only if \( \dim \text{Lie}_x \{ X, Y_1, ..., Y_k \} = \dim M \) for every \( x \in M \).

Two last theorems are not exact quotations but can be deduced respectively from [2] and [11].

There are also results concerning local controllability. The result which is closest to our interests is as follows.

Theorem (Nam, Arapostathis [12]): Consider a smooth control system \( \dot{x} = X + \sum_{i=1}^{k} u_i Y_i, u \in U \), where \( U \) is an arbitrary subset of \( \mathbb{R}^k \), \( f \) is a continuous mapping \( M \times U \rightarrow TM \), and \( f_u \) is a smooth vector field on \( M \) for every \( u \in U \). Our main assumption is

\[
\text{(1.2)} \quad \dim \text{Lie}_x \{ f_u : u \in U \} = n = \dim M
\]

for every \( x \in M \). Similarly as above, our controls are bounded measurable and the final time is not fixed. It follows from known results for ODE’s with measurable right hand side (see e.g. [4]) that under such assumptions, to every control \( u : [0, T] \rightarrow U \) there corresponds an admissible trajectory of \( (\Sigma) \) (defined maybe on a smaller interval).
The first result that we will prove is the following

**Theorem 1.1.** Consider the control system \((\Sigma)\) for which (1.2) holds, and suppose that \(M\) is compact. Then the system \((\Sigma)\) has closed orbits.

Let \(x \in M\) and take its neighbourhood \(U\). Denote by \(A^+(x, U)\) the reachable set from a point \(x\) in \(U\) for the system \((\Sigma)\), i.e. the set of endpoints of all trajectories of \((\Sigma)\) that start from \(x\), are generated by measurable controls (final time is not fixed), and are contained in \(U\). The sets \(A^+(x, M)\) will be denoted simply by \(A^+(x)\). Let us remark that controllability of \((\Sigma)\) means that \(A^+(x) = M\) for every \(x \in M\).

Suppose now that \(\Gamma\) is a closed orbit for \((\Sigma)\). If a point \(x\) belongs to \(\Gamma\) then \(\Gamma_x\) will stand for the set \(\Gamma \setminus \{x\}\).

**Definition 1.1.** We say that the closed orbit \(\Gamma\) is regular, if there exists a point \(x \in \Gamma\) and a neighbourhood \(U\) of \(x\) such that
\[
\Gamma_x \cap A^+(x, U) \subset \text{int } A^+(x, U).
\]

Our second result can be stated as follows.

**Theorem 1.2.** Suppose that \(\Gamma\) is a closed orbit for the system \((\Sigma)\) for which (1.2) holds. Then the necessary and sufficient condition for \((\Sigma)\) to be locally controllable at every point of \(\Gamma\) is that \(\Gamma\) be a regular closed orbit. More precisely, a closed orbit \(\Gamma\) of \((\Sigma)\) is regular if and only if \(\Gamma\) possesses a controllable neighbourhood.

Note that in theorem 1.2 \(M\) is not supposed to be compact. Let us also note that the curve \(\Gamma\) need not be smooth. Theorem 1.2 generalizes slightly results from [12] as it will be clarified at the end of this paper.

**2. Proofs of Theorems.**

Along with the system \((\Sigma)\) we will consider the system
\[
(\Sigma^-) \quad \dot{x} = -f(x, u), \quad u \in \mathcal{U}.
\]

Let us note a simple observation which will be useful later.

**Lemma 2.1.** \(\gamma(t)\) is a trajectory of the system \((\Sigma)\) generated by a control \(u(t)\) if and only if \(\tilde{\gamma}(t) = \gamma(-t)\) is a trajectory of the system \((\Sigma^-)\) generated by a control \(\tilde{u}(t) = u(-t)\).

Denote by \(A^-(x, U)\) the corresponding reachable set from \(x\) for the system \((\Sigma^-)\). At the same time let \(A^+_0(x)\), \(A^-_0(x)\) be the reachable sets for \((\Sigma)\) and \((\Sigma^-)\), respectively, generated by piecewise constant controls. Recall now [9] Krener’s theorem which states that under the assumption (1.2) the inclusion \(A^+_0(x) \subset \text{int } A^+(x)\) (and the same for \(A^-_0(x)\)) holds true. Therefore \(\text{int } A^+(x)\) and \(\text{int } A^-(x)\) are non-empty for every \(x \in M\). Notice also that
\[
x \in \text{int } A^+(x) \cap \text{int } A^-(x)
\]
for any \( x \in M \). Indeed, by Krener's theorem
\[
x \in A_0^+(x) \subseteq \operatorname{int} A_0^+(x) \subseteq \operatorname{int} A^+(x),
\]
and the same for \( A^- \).

**Lemma 2.2.** \( y \in \operatorname{int} A^+(x) \) if and only if \( x \in \operatorname{int} A^-(y) \).

**Proof.** Suppose that \( y \in \operatorname{int} A^+(x) \). Since \( y \in \operatorname{int} A^-(y) \) it follows that \( \operatorname{int} A^+(x) \cap \operatorname{int} A^-(y) \neq \emptyset \). Taking a \( z \in \operatorname{int} A^+(x) \cap \operatorname{int} A^-(y) \) we see that there exist admissible curves for the system \((\Sigma)\): \( \sigma_1 \) joining \( x \) to \( z \), and \( \sigma_2 \) joining \( z \) to \( y \). Reversing time in \( \sigma_1 \cup \sigma_2 \) we obtain an admissible curve \( \tilde{\sigma} \) for the system \((\Sigma^-)\) that joins \( y \) to \( x \), and which belongs to the interior \( \operatorname{int} A^-(y) \) starting from a certain time \( t_0 > 0 \) (for instance \( t_0 \) corresponds to a point \( z \)). But this means that \( \tilde{\sigma} \) stays in \( \operatorname{int} A^-(y) \) for all \( t > t_0 \), therefore \( x \in \operatorname{int} A^-(y) \). \( \square \)

We come to the proof of theorem 1.1 now. First we need to establish the following proposition.

**Proposition 2.1.** The family \( \{ \operatorname{int} A^+(x) \}_{x \in M} \) forms an open covering of \( M \).

**Proof.** Fix a point \( x \in M \). Send through it a trajectory \( \gamma \), \( \gamma(0) = x \), of \((\Sigma^-)\) such that \( \gamma(t) \in \operatorname{int} A^-(x) \) for a \( t > 0 \); by our assumptions such a curve exists. Now, the above lemmas imply that \( x \in \operatorname{int} A^+(\gamma(t)) \), proving the assertion. \( \square \)

Suppose that \( M \) is compact. By proposition 2.1 there are points \( x_1, \ldots, x_m \in M \) such that \( M = \bigcup_{i=1}^m \operatorname{int} A^+(x_i) \). Now \( x_1 \in \operatorname{int} A^+(x_{i_1}) \), for an \( i_1 \in \{1, \ldots, m\} \), \( x_{i_1} \in \operatorname{int} A^+(x_{i_2}) \) for \( i_2 \in \{1, \ldots, m\} \) etc. In this way we are led to an infinite sequence \( \{x_{i_k}\}_{k=1,2,\ldots} \) with \( x_{i_k} \in \operatorname{int} A^+(x_{i_{k+1}}) \) and \( i_k \in \{1, \ldots, m\} \). Therefore we can find positive integers \( l \) and \( p \) such that \( x_{i_l} \in \operatorname{int} A^+(x_{i_{l+p}}) \), \( x_{i_l+1} \in \operatorname{int} A^+(x_{i_{l+2}}) \), \ldots , \( x_{i_l+p} \in \operatorname{int} A^+(x_{i_l}) \). This ends the proof of theorem 1.1.

Now we move on to the proof of theorem 1.2. First of all let us list immediate properties of closed orbits. If \( \Gamma \) is a closed orbit for \((\Sigma)\) then \( A^+(x_1) = A^+(x_2) \) for every \( x_1, x_2 \in \Gamma \). Moreover, \( A^+(x) = A^+(\Gamma) \) for \( x \in \Gamma \), where by \( A^+(\Gamma) \) we mean \( \bigcup_{x \in \Gamma} A^+(x) \). Since, \( \Gamma \), under suitable parameterization, is a closed orbit also for \((\Sigma^-)\), we have \( A^-(x_1) = A^-(x_2) = A^-(\Gamma) \) for any \( x_1, x_2 \in \Gamma \). Let us also recall a standard fact from control theory asserting that the reachable set \( A^+(x) \) is open if and only if \( x \in \operatorname{int} A^+(x) \).

Next we prove

**Lemma 2.3.** If \( \Gamma \) is a regular closed orbit for \((\Sigma)\) then the set \( A^+(\Gamma) \) is open.

**Proof.** Take an \( x \in \Gamma \) and \( U \) such that \([1.3]\) is satisfied, i.e. \( \Gamma_x \cap A^+(x, U) \subseteq \operatorname{int} A^+(x, U) \). Clearly \( \operatorname{int} A^+(x, U) \subseteq \operatorname{int} A^+(x) \). Take a point \( y \in \Gamma_x \cap A^+(x, U) \) and an open set \( V \) such that \( y \in V \subseteq A^+(x) \). For any \( z \in V \)...
one can construct a trajectory of $(\Sigma)$ joining $y$ to $z$: we connect $y$ to $x$ by a suitable segment of $\Gamma$, and then $x$ to $z$ ($z \in \mathcal{A}^+(x)$). In this way we proved that $V \subset \mathcal{A}^+(y)$, i.e. $y \in \text{int} \mathcal{A}^+(y)$. This proves that $\mathcal{A}^+(y) = \mathcal{A}^+(\Gamma)$ is open. \(\square\)

The last stage in proving theorem 1.2 is the following observation.

**Lemma 2.4.** Let $\Gamma$ be a closed orbit for $(\Sigma)$. $\Gamma$ is regular for $(\Sigma)$ if and only if it is regular for $(\Sigma^-)$ (under suitable parameterization).

**Proof.** Because of symmetry, it is enough to prove one implication. Suppose that $\Gamma$ is regular for $(\Sigma)$ and choose $x_1$ and $U$ such that $\Gamma_{x_1} \cap \mathcal{A}^+(x_1, U) \subset \text{int} \mathcal{A}^+(x_1)$. Take a point $x_2 \in \Gamma_{x_1} \cap \mathcal{A}^+(x_1, U)$ and denote by $[x_1, x_2]$ the segment of $\Gamma$ bounded by points $x_1$ and $x_2$. By lemma 2.3 for every $z \in [x_1, x_2]$, $x_2 \in \text{int} \mathcal{A}^+(z)$ which, by lemma 2.2, means that $z \in \text{int} \mathcal{A}^-(x_2)$. Thus $[x_1, x_2] \subset \text{int} \mathcal{A}^-(x_2)$, and consequently $\Gamma_{x_2} \cap \mathcal{A}^-(x_2, W) \subset \text{int} \mathcal{A}^-(x_2, W)$ for suitably chosen neighbourhood $W$ of $x_2$, proving that $\Gamma$ is regular for $(\Sigma^-)$. \(\square\)

**Corollary 2.1.** If $\Gamma$ is a regular closed orbit for $(\Sigma)$ then the set $\mathcal{A}^-(\Gamma)$ is open.

In order to finish the proof of theorem 1.2 it is enough to notice that if $\Gamma$ is a regular orbit for $(\Sigma)$ then $U = \mathcal{A}^+(\Gamma) \cap \mathcal{A}^-(\Gamma)$ is a controllable neighbourhood. Indeed, take arbitrary $x, y \in U$. Since $x \in \mathcal{A}^-(\Gamma)$ there exists a trajectory of $(\Sigma)$ joining $x$ to a point of $\Gamma$. Similarly, since $y \in \mathcal{A}^+(\Gamma)$ there exists a trajectory of $(\Sigma)$ joining a point of $\Gamma$ to $y$. Finally, it is clear that any two points belonging to $\Gamma$ can be joined by a trajectory of $(\Sigma)$. Evidently, any admissible trajectory joining $x$ to $y$ obtained in this way does not leave $U$ by the very definition of $U$.

### 3. One example.

Before we state our example let us recall a concept of geometric optimality and so-called singular extremals for the system $(\Sigma)$. So fix a trajectory $\gamma : [0, T] \rightarrow U$ of $(\Sigma)$, $U$ being an open subset of $M$, which is generated by a control $\tilde{u} : [0, T] \rightarrow U$. We say that $\gamma$ (or $\tilde{u}$) is geometrically optimal in $U$ if $\gamma(t) \in \partial_U \mathcal{A}^+(\gamma(t), U)$; $\partial_U$ denotes here the boundary operator with respect to $U$. On the other hand, $\gamma : [0, T] \rightarrow M$ is called an extremal, if there exists an absolutely continuous $p : [0, T] \rightarrow T^*M$ (called an extremal lift) such that $p(t) \in T^*_{\gamma(t)}M \setminus \{0\}$ for every $t$, and such that if we set $\mathcal{H}_u(x, p) = \langle p, f_u(x) \rangle$, then

(i) $\langle \dot{\gamma}(t), \dot{p}(t) \rangle = \overrightarrow{\mathcal{H}_{\dot{u}}(\gamma(t), p(t))}$ a.e. on $[0, T]$ (\(\overrightarrow{\mathcal{H}_u}\) is the Hamiltonian vector field on $T^*M$ corresponding to the function $(x, p) \mapsto \mathcal{H}_u(x, p)$),

(ii) $\mathcal{H}_{\dot{u}}(\gamma(t), p(t)) = 0$ on $[0, T]$, and

(iii) $\mathcal{H}_{\dot{u}}(\gamma(t), p(t)) = \max_{u \in U} \mathcal{H}_u(\gamma(t), p(t))$ a.e. on $[0, T]$. 

It is proved \[1\] that a necessary condition for \( \gamma \) to be geometrically optimal is that \( \gamma \) be an extremal. Now, an extremal \( \gamma(t) \) generated by a control \( \tilde{u} \) with values in \( \text{int} \ U \) is called a singular extremal if there exists an extremal lift \( p(t) \) such that additionally

\[
(iv) \quad \partial H_u(\gamma(t), p(t)) \bigg|_{u=\tilde{u}(t)} = 0 \quad \text{for every } t.
\]

It is a standard fact that if \( \gamma \) is a geometrically optimal trajectory of \((\Sigma)\) generated by a steering \( u : [0, T] \rightarrow \text{int} \ U \) with values in \( \text{int} \ U \), then \( \gamma \) is a singular trajectory of \((\Sigma)\).

Consider now a control affine system

\[(3.1) \quad \dot{x} = X + uY, \quad |u| \leq 1,
\]

defined on a manifold \( M \). Fix a point \( x_0 \) and a time interval \([0, T]\). Let \( \gamma \) be the trajectory of \( X \) initiating at a point \( x_0 \); in other words \( \gamma \) is a trajectory of our control system generated by the control \( u^0(t) \equiv 0 \). Next, consider the so-called endpoint map \( \Phi^{T, x_0} \), i.e. the mapping which to each control \( u : [0, T] \rightarrow [-1, 1] \) assigns the point \( \Phi^{T, x_0}(u) = \gamma_u(T) \), where \( \gamma_u \) is the trajectory of \((3.1)\) that starts from \( x_0 \) and is generated by \( u \). It can be proved (see e.g. \[3\]) that

\[
\text{im} \ d_{x_0} \Phi^{T, x_0} = \text{Span}\{Y(\gamma(T)), \left(ad^k X.Y\right)(\gamma(T)) : k = 1, 2, \ldots\},
\]

where \( adX.Y = [X, Y] \), and \( ad^{k+1}X.Y = [X, ad^k X.Y] \), \( k = 1, 2, \ldots \). It is known (see again e.g. \[3\]) that \( \gamma \) is not a singular trajectory for \((3.1)\) if and only if

\[(3.2) \quad \dim \text{Span}\{Y(\gamma(T)), \left(ad^k X.Y\right)(\gamma(T)) : k = 1, 2, \ldots\} = \dim M.
\]

Now let us take a closer look at the result from \[12\] cited in the introduction, applied to the system \((3.1)\). Suppose that \( \Gamma \) is a closed orbit of \( X \) and fix an \( x \in \Gamma \). If \((1.1)\) is satisfied at \( x \) then \((3.2)\) does not have to be satisfied, as it is explained in \[12\]. On the other hand assume that \((3.2)\) is satisfied at \( x \). Then of course \((1.1)\) is also satisfied and, by the above remark, \( \Gamma \) is not a singular trajectory. Consequently, it is not geometrically optimal from \( x \) and, what follows, it is a regular closed orbit for \((3.1)\). Thus the satisfaction of \((3.2)\) implies that \( \Gamma \) is a regular closed orbit.

Now, we are going to present a simple construction of a closed trajectory \( \Gamma \) which does not satisfy neither \((3.2)\) nor \((1.1)\) but anyway is a regular closed orbits.

To this end consider \( W = \{(x_1, x_2, x_3) : x_2^2 + x_3^2 < 1, \ 0 \leq x_1 \leq 2\pi \} \subset \mathbb{R}^3 \). Let us introduce the following equivalence relation on \( W \): \((x_1, x_2, x_3) \sim (x'_1, x'_2, x'_3)\) if and only if \( x_2 = x'_2, \ x_3 = x'_3, \ x_1 = 0, \ x'_1 = 2\pi \) or \( x_1 = 2\pi, \ x'_1 = 0 \). Consider the factorization \( p : W \rightarrow M = W/ \sim \). The space \( M \) is a 3-dimensional manifold which in an obvious way is embedded in \( \mathbb{R}^3 \). Let \( \tilde{X} = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \tilde{Y} = \frac{\partial}{\partial x_2}, \ k \geq 3 \), be vector fields on \( \mathbb{R}^3 \). After factorization...
they are transformed to vector fields

\[(3.3) \quad X = p_\ast \tilde{X}, \quad Y = p_\ast \tilde{Y}\]

on \(M\). Now denote by \((\Sigma)\) the control system \((3.1)\) on \(M\) where \(X\) and \(Y\) are defined by \((3.3)\). It is easily seen that the image under \(p\) of the \(x_1\)-axis, denoted by \(\Gamma\), is a closed and singular trajectory for \((\Sigma)\). Indeed, its extremal lift is given by \(\lambda(t) = (t \mod 2\pi, 0, 0, 0, 0, 1)\).

Define a rank 2 distribution \(H\) on \(M\) by letting \(H = \text{Span}\{X, Y\}\). If \(x\) is a point in \(M\) and \(l\) is a positive integer, then we will write \(H_x^l\) for the span of all vectors of the form

\[\left[ X_1, [X_2, ..., [X_{i-1}, X_i]] \right](x), \]

where \(X_1, ..., X_i\) are smooth local sections of \(H\) defined near \(x, i \leq l\). Now it is not difficult to see that if \(S = \{x_2 = 0\}\), then \(H\) is a contact distribution on \(M \setminus S\), i.e. \(H_x^2 = T_xM\) whenever \(x \in M \setminus S\). It can also be seen that \(H\) has the following bracket properties on \(S\): \(H_x^l \subset H_x, 1 \leq l \leq k,\) and \(H_x^{k+1} = T_xM\) whenever \(x \in S\). All this permits us to conclude that, as it is explained in \([7]\), \((\Sigma)\) is an affine control system induced by the generalized Martinet sub-Lorentzian structure of Hamiltonian type of order \(k\). Suppose that \(k\) is odd. It follows \([7]\) that for every \(x_0 \in \Gamma\) there exists a neighbourhood \(U\) of \(x_0\) and coordinates \(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\) on \(U\), \(\tilde{x}_1(x_0) = \tilde{x}_2(x_0) = \tilde{x}_3(x_0) = 0\), such that \(S \cap U = \{\tilde{x}_2 = 0\}\), \(\Gamma \cap U = \{\tilde{x}_2 = \tilde{x}_3 = 0\}\), and \(A^+(x_0, U) = A_1 \cup A_2\), where

\[A_1 = \{x \in U : \eta_1(\tilde{x}_1(x), \tilde{x}_2(x), \tilde{x}_3(x)) \leq 0\} \cap \{\tilde{x}_1(x) \geq 0, \tilde{x}_3(x) \geq 0\},\]

\[A_2 = \{x \in U : \eta_2(\tilde{x}_1(x), \tilde{x}_2(x), \tilde{x}_3(x)) \leq 0\} \cap \{\tilde{x}_1(x) \geq 0, \tilde{x}_3(x) \leq 0\},\]

with

\[\eta_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \tilde{x}_3 + \frac{1}{2k}(\tilde{x}_1 + \tilde{x}_2)(\tilde{x}_2^k - \frac{1}{2k}(\tilde{x}_1 + \tilde{x}_2)^k) + O(r^{k+2}),\]

\[\eta_2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = -\tilde{x}_3 - \frac{1}{2k}(\tilde{x}_1 - \tilde{x}_2)(\tilde{x}_2^k + \frac{1}{2k}(\tilde{x}_1 - \tilde{x}_2)^k) + O(r^{k+2});\]

here \(r = (\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2)^{1/2}\).

Since \(\eta_1(\tilde{x}_1, 0, 0) < 0\) and \(\eta_2(\tilde{x}_1, 0, 0) < 0\) (we choose \(U\) to be sufficiently small), it is seen that \(\Gamma_{x_0} \cap U \subset \text{int } A^+(x_0, U)\), and \(\Gamma\) is a regular closed orbit. At the same time one easily sees that \([\tilde{X}, \tilde{Y}] = -k\tilde{x}_2^{-1}\frac{\partial}{\partial \tilde{x}_3}\) which yields \(ad^l \tilde{X}, \tilde{Y} = 0\) for all \(l \geq 2\), meaning that \((1.1)\) does not hold at any point of \(\Gamma\).

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