Chordality of locally semicomplete and weakly quasi-transitive digraphs

Jing Huang and Ying Ying Ye

Abstract

Chordal graphs are important in the structural and algorithmic graph theory. A digraph analogue of chordal graphs was introduced by Haskin and Rose in 1973 but has not been a subject of active studies until recently when a characterization of semicomplete chordal digraphs in terms of forbidden subdigraphs was found by Meister and Telle.

Locally semicomplete digraphs, quasi-transitive digraphs, and extended semicomplete digraphs are amongst the most popular generalizations of semicomplete digraphs. We extend the forbidden subdigraph characterization of semicomplete chordal digraphs to locally semicomplete chordal digraphs. We introduce a new class of digraphs, called weakly quasi-transitive digraphs, which contains quasi-transitive digraphs, symmetric digraphs, and extended semicomplete digraphs, but is incomparable to the class of locally semicomplete digraphs. We show that weakly quasi-transitive digraphs can be recursively constructed by simple substitutions from transitive oriented graphs, semicomplete digraphs, and symmetric digraphs. This recursive construction of weakly quasi-transitive digraphs, similar to the one for quasi-transitive digraphs, demonstrates the naturalness of the new digraph class. As a by-product, we prove that the forbidden subdigraphs for semicomplete chordal digraphs are the same for weakly quasi-transitive chordal digraphs.

1 Introduction

We consider digraphs which do not contain loops or multiple arcs but may contain digons (i.e., pairs of arcs joining vertices in opposite directions). If an arc is contained in a digon then it is called a symmetric arc. A digraph which does not contain a symmetric arc is called an oriented graph. A digraph which contains only symmetric arcs is called a symmetric digraph. Graphs may be viewed as symmetric digraphs.

Two vertices in a digraph $D$ are adjacent and referred to as neighbours of each other if there is at least one arc between them. When $uv$ is an arc in $D$ (symmetric or not), we
say that \( u \) is an \textit{in-neighbour} of \( v \) and \( v \) an \textit{out-neighbour} of \( u \). The set of all in-neighbours of a vertex \( v \) is denoted by \( N^-(v) \) and the set of all out-neighbours of \( v \) is denoted by \( N^+(v) \). We use \( S(D) \) to denote the spanning subdigraph of \( D \) whose arc set consists of all symmetric arcs in \( D \).

A vertex \( v \) of in a digraph \( D \) is di-simplicial if for every \( u \in N^-(v) \) and \( w \in N^+(v) \) with \( u \neq w \), \( uw \) is an arc of \( D \). A digraph \( D \) is chordal if every induced subdigraph of \( D \) contains a di-simplicial vertex. It follows that every chordal digraph \( D \) has a vertex ordering \( v_1, v_2, \ldots, v_n \) such that \( v_i \) is a di-simplicial vertex in the subdigraph of \( D \) induced by \( v_i, v_{i+1}, \ldots, v_n \) for each \( i \geq i \). Such an ordering is called a \textit{perfect elimination ordering} of \( D \).

Perfect elimination orderings of digraphs arise in the study of sparse linear systems by Gaussian elimination, cf. [9]. When a digraph is symmetric, di-simplicial vertices coincide with simplicial vertices of its underlying graph. Thus, a symmetric digraph is chordal if and only if its underlying graph is chordal. It is well-known that chordal graphs are precisely the graphs which do not contain an induced cycle of length \( \geq 4 \), cf. [8].

Little is known about the forbidden structure of chordal digraph. In particular, there is no known characterization of chordal digraphs by forbidden subdigraphs. Recently, Meister and Telle [11] found a forbidden subdigraph characterization for semicomplete chordal digraphs. A digraph \( D \) is \textit{semicomplete} if between any two vertices there is an arc. The following theorem is proved in [11].

**Theorem 1.1.** [11] A semicomplete digraph \( D \) is chordal if and only if \( S(D) \) is chordal and \( D \) does not contain any of the digraphs in Figure 1 as an induced subdigraph.

![Figure 1: Semicomplete digraphs which are not chordal](image)

A digraph \( D \) is called \textit{locally semicomplete} if for every vertex \( v \), \( N^-(v) \) and \( N^+(v) \) each induces a semicomplete subdigraph in \( D \). Locally semicomplete digraphs are a popular generalization of semicomplete digraphs and have been extensively studied, cf. [11, 2, 3, 10]. Many properties for semicomplete digraphs hold for locally semicomplete digraphs, cf. [11]. However, there are locally semicomplete digraphs which are neither semicomplete nor chordal. Any directed cycle consisting of non-symmetric arcs is locally semicomplete but not chordal, and is not semicomplete if it has four or more vertices. We will prove (see Theorem 2.4) that directed cycles with four or more vertices consisting of non-symmetric arcs are the only minimal locally semicomplete digraphs which are not chordal and which are not semicomplete.
Quasi-transitive digraphs are another well-studied class of digraphs generalizing semi-complete digraphs, cf. [4, 5, 6, 7]. A digraph $D = (V, A)$ is called quasi-transitive if for any three vertices $u, v, w$, $uv \in A$ and $vw \in A$ imply $uw \in A$ or $wu \in A$ (or both), cf. [4]. The class of quasi-transitive digraphs contains all transitive oriented graphs. These are the oriented graphs which satisfy the property that for any three vertices $u, v, w$, $uv \in A$ and $vw \in A$ imply $uw \in A$. Equivalently, they are the oriented graphs in which every vertex is a di-simplicial vertex. Quasi-transitive chordal digraphs are studied recently in [12], where it is proved that they have the same forbidden subdigraphs as for semicomplete chordal digraphs as stated in Theorem 1.1.

Let $v$ be a vertex and $u, w$ be neighbours of $v$ in a digraph $D$. Then $u, w$ are called synchronous neighbours of $v$ if $u, w$ are both in $N^-(v) \setminus N^+(v)$, or in $N^+(v) \setminus N^-(v)$, or in $N^-(v) \cap N^+(v)$; otherwise they are called asynchronous neighbours of $v$. We call a digraph $D$ weakly quasi-transitive if for each vertex $v$ of $D$, any two asynchronous neighbours of $v$ are adjacent.

The class of weakly quasi-transitive digraphs contains all quasi-transitive digraphs (and hence contains all semicomplete digraphs as well as all transitive oriented digraphs). Indeed, suppose $D$ is not weakly quasi-transitive. Then some vertex $v$ has two non-adjacent asynchronous neighbours $u, w$. Since $u, w$ are asynchronous neighbours of $v$, one of of $u, w$ is in $N^-(v)$ and the other is in $N^+(v)$. Hence $D$ is not a quasi-transitive digraph. Clearly every symmetric digraph is weakly quasi-transitive. Symmetric digraphs have the property that the neighbours of each vertex are synchronous and any digraph having this property is weakly quasi-transitive. If a digraph $D$ is weakly quasi-transitive then any digraph obtained from $D$ by substituting an independent set for each vertex of $D$ is also weakly quasi-transitive. Extended semicomplete digraphs are the digraphs obtained this way from semicomplete digraphs so they are all weakly quasi-transitive. Therefore the class of weakly quasi-transitive digraphs simultaneously contains quasi-transitive digraphs, symmetric digraphs, and extended semicomplete digraphs. Figure 2 depicts a containment hierarchy of the digraph classes relevant to this paper.

![Figure 2: A containment hierarchy](image)

Let $D$ be a digraph with vertices $v_1, v_2, \ldots, v_n$ and let $H_1, H_2, \ldots, H_n$ be vertex-disjoint digraphs. A substitution of the digraphs $H_i$ for the vertices $v_i$ in $D$ for each $i$ is a new
digraph $D^*$ obtained from $H_1, H_2, \ldots, H_n$ by adding all possible arcs $xy$ where $x \in V(H_i)$ and $y \in V(H_j)$ for each arc $v_iv_j$ in $D$. We use $D[H_1, H_2, \ldots, H_n]$ to denote the new digraph $D^*$ and also say that it is obtained from $D$ by substituting $H_i$ for $v_i$ for each $i$.

**Theorem 1.2.** [4] Let $D$ be a quasi-transitive digraph. Then the following statements hold:

1. If $D$ is non-strong, then $D = T[H_1, H_2, \ldots, H_n]$ where $T$ is a transitive oriented graph and each $H_i$ is a strong quasi-transitive digraph.

2. If $D$ is strong, then $D = S[H_1, H_2, \ldots, H_n]$ where $S$ is a strong semicomplete digraph and each $H_i$ is either a single-vertex digraph or a non-strong quasi-transitive digraph.

Thus every quasi-transitive digraph can be obtained from transitive oriented graphs and semicomplete digraphs recursively by substitutions. Weakly quasi-transitive digraphs admit a similar construction. We will show (see Theorem 3.2) that weakly quasi-transitive digraphs can be constructed recursively from transitive oriented graphs, symmetric digraphs, and semicomplete digraphs by substitutions. As a by-product of this recursive construction, we prove that the forbidden subdigraphs for weakly quasi-transitive chordal digraphs are exactly those for semicomplete chordal digraphs. This generalizes the results of [12] on quasi-transitive chordal digraphs and extended semicomplete chordal digraphs.

### 2 Locally semicomplete chordal digraphs

Let $D$ be a digraph and $C : v_1v_2 \ldots v_kv_1$ be a directed cycle in $D$. If there is no arc between $v_i$ and $v_j$ for all $i, j$ with $|i - j| \notin \{1, k - 1\}$, then the cycle $C$ is called induced in $D$.

**Lemma 2.1.** If $D$ is a chordal digraph, then $D$ does not contain an induced directed cycle consisting of non-symmetric arcs and $S(D)$ does not contain an induced directed cycle of length $\geq 4$.

**Proof:** Suppose that $C$ is either an induced directed cycle in $D$ consisting of non-symmetric arcs or an induced directed cycle of length $\geq 4$ in $S(D)$. Then the subdigraph of $D$ induced by the vertices of $C$ has no di-simplicial vertex and hence is not a chordal digraph. Therefore $D$ is not a chordal digraph.

When $S(D)$ contains no an induced directed cycle of length $\geq 4$, $S(D)$ is a chordal digraph and hence has di-simplicial vertices. The di-simplicial vertices of $S(D)$ necessarily contain the di-simplicial vertices of $D$, as observed in [11].

**Lemma 2.2.** [11] Every di-simplicial vertex of a digraph $D$ is a di-simplicial vertex of $S(D)$.  

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Suppose that \( v \) is a di-simplicial vertex of \( S(D) \) but not a di-simplicial vertex of \( D \). Since \( v \) is not a di-simplicial vertex of \( D \), there exist \( u \in N^-(v) \) and \( w \in N^+(v) \) such that \( uw \) is not an arc of \( D \) and we shall call such an ordered triple \((u, v, w)\) of vertices a \textit{violating triple} for \( v \). We remark that a violating triple \((u, v, w)\) exists only if \( v \) is a di-simplicial vertex of \( S(D) \) and it certifies that \( v \) is not a di-simplicial vertex of \( D \). Since \( v \) is a di-simplicial vertex of \( S(D) \) and \( uw \) is not an arc of \( D \), at least one of \( uv, vw \) is non-symmetric. We call \( v \) \textit{type 1} if for every violating triple \((u, v, w)\), both \( uv, vw \) are non-symmetric and \textit{type 2} otherwise. The following lemma allows us to streamline the selection of violating triples.

**Lemma 2.3.** Let \( D \) be a locally semicomplete digraph such that \( S(D) \) is chordal and \( D \) does not contain an induced directed cycle consisting of non-symmetric arcs or any digraph in Figure \[\text{Figure 1}\] as an induced subdigraph. Suppose that \((u, v, w)\) is a violating triple. Then the following statements hold:

1. If \( uv \) is a non-symmetric arc then there exists a di-simplicial vertex \( u' \) of \( S(D) \) (possibly \( u' = u \)) such that \((u', v, w)\) is a violating triple and \( u'v \) is a non-symmetric arc.

2. If \( vw \) is a non-symmetric arc then there exists a di-simplicial vertex \( w' \) of \( S(D) \) (possibly \( w' = w \)) such that \((u, v, w')\) is a violating triple and \( vw' \) is a non-symmetric arc.

**Proof:** The two statements can be obtained from each other by reversing the arcs of \( D \). Thus we only prove the first one. Assume \( uv \) is non-symmetric. Consider \( S(D) - (N^-[w] \cap N^+[w]) \) where \( N^-[w] = N^-(w) \cup \{w\} \) and \( N^+[w] = N^+(w) \cup \{w\} \). Since \( uw \) is not an arc of \( D \), \( u \) is not a vertex in \( N^-[w] \cap N^+[w] \) and hence a vertex of \( S(D) - (N^-[w] \cap N^+[w]) \).

Let \( u_1u_2 \ldots u_k \) where \( u_1 = u \) be a directed path in \( S(D) - (N^-[w] \cap N^+[w]) \). We prove by induction on \( k \) that \((u_k, v, w)\) is a violating triple and \( u_kv \) is a non-symmetric arc of \( D \). This is true when \( k = 1 \). So assume \( k > 1 \), \((u_{k-1}, v, w)\) is a violating triple and \( u_{k-1}v \) is a non-symmetric arc of \( D \). Suppose that \( vw \) is a symmetric arc. Then there is an arc between \( u_{k-1} \) and \( w \) as they are both in-neighbours of \( v \). Since \( u_{k-1}w \) is not an arc of \( D \), \( u_{k-1}w \) is a non-symmetric arc. Thus both \( w \) and \( u_k \) are in-neighbours of \( u_{k-1} \) so there is an arc between them. Since \( u_k \not\in N^-[w] \cap N^+[w] \), \( w \) and \( u_k \) are joined by a non-symmetric arc. Since \( u_k \) and \( v \) are out-neighbours of \( u_{k-1} \), they are adjacent. If \( u_k \) and \( v \) are joined by symmetric arcs, then \( u_k \) and \( w \) are in \( N^-(v) \cap N^+(v) \). Since \( v \) is a di-simplicial vertex, \( u_k \) and \( w \) are joined by symmetric arcs, which contradicts the fact \( u_k \not\in (N^-[w] \cap N^+[w]) \). If \( vu_k \) or \( u_kw \) is an arc of \( D \), then the subdigraph of \( D \) induced by \( v, w, u_{k-1}, u_k \) is Figure \[\text{Figure 1}(a), (b) or (c)\], contradicting to our assumption. Hence \( vu_k \) is not an arc (i.e., \( u_kv \) is a non-symmetric arc) and \( u_kw \) is not an arc of \( D \), that is, \((u_k, v, w)\) is a violating triple. On the other hand, suppose that \( vw \) is a non-symmetric arc. Since \( u_{k-1}w \) is not an arc and \( D \) does not contain an induced directed cycle consisting of non-symmetric arcs, there is no arc between \( u_{k-1} \) and \( w \). This implies there is no arc between \( u_k \) and \( w \) as otherwise \( u_{k-1}, w \) are non-adjacent vertices in \( N^-(u_k) \) or in \( N^+(u_k) \), which contradicts that \( D \) is locally semicomplete. The vertices \( u_k, v \) are adjacent because they are out-neighbours of
**Theorem 2.4.** A locally semicomplete digraph $D$ is chordal if and only if $S(D)$ is chordal and does not contain as an induced subdigraph a directed cycle consisting of non-symmetric arcs or a digraph in Figure 4.

**Proof:** The necessity follows from Theorems [1.1] and Lemma 2.3. For the other direction assume that $S(D)$ is chordal and $D$ contains neither a directed cycle consisting of non-symmetric arcs nor a digraph in Figure 1 as an induced subdigraph. To prove $D$ is chordal it suffices to show it has a di-simplicial vertex. Since $S(D)$ is chordal, $S(D)$ has di-simplicial vertices. If some di-simplicial vertex of $S(D)$ is a di-simplicial vertex of $D$ then we are done. Hence we also assume that none of the di-simplicial vertices of $S(D)$ is a di-simplicial vertex of $D$.

Suppose first that $S(D)$ has di-simplicial vertices of type 1. Let $v$ be such a vertex. Then there is a violating triple for $v$ and thus by Lemma 2.3 there is a canonical triple $(u, v, w)$ for $v$. Note that $u, v$ are both di-simplicial vertices of $S(D)$ and $uw$ is not an arc. Since $D$ contains no directed cycles consisting of non-symmetric arcs, $uw$ is not an arc and so $u, w$ are not adjacent. We claim that the triple $(u, v, w)$ can be chosen so that $u$ is type 1. We prove this by contradiction. So assume $u$ is type 2. Then there is a canonical violating triple $(u_1, u, w_1)$ for $u$ such that exactly one of $u_1u, uw_1$ is a non-symmetric arc. Suppose first that $u_1u$ is non-symmetric and $uw_1$ is symmetric. Since $u_1, w_1$ are both in-neighbours of $u$ and $D$ is locally semicomplete, they are adjacent. But $u_1w_1$ is not an arc so $uw_1$ is a non-symmetric arc. There is no arc between $w_1$ and $w$ as otherwise $w, u$ are in-neighbours or out-neighbours of $w_1$, which contradicts the fact that they are not adjacent. Since $w$ is an out-neighbour of $v$ but not adjacent to $w_1$, $w_1$ cannot be an out-neighbour of $v$. But $u_1$ and $v$ are adjacent as they are out-neighbours of $u$ so $w_1v$ is a non-symmetric arc. Since $u$ is an out-neighbour of $u_1$ but not adjacent to $w$, $w$ cannot be an out-neighbour of $v$. Similarly, $w_1$ is an in-neighbour of $u_1$ but not adjacent to $w$, $w$ cannot be an in-neighbour of $u_1$. Hence $w$ is not adjacent to $u_1$. There must be an arc between $v$ and $u_1$ as they are out-neighbours of $w_1$. But $u_1$ cannot be an out-neighbour of $v$ because it is not adjacent to $w$ which is an out-neighbour of $v$. Hence $uw$ is a non-symmetric arc and $(u_1, v, w)$ is a violating triples. Since $(u_1, u, w_1)$ is a canonical violating triple...
triple and \( u_1u \) is non-symmetric, \( u_1 \) is a di-simplicial vertex of \( S(D) \) and hence \((u_1, v, w)\) is a canonical violating triple. A similar proof shows that if \( u_1u \) is symmetric and \( uw_1 \) is non-symmetric then \((w_1, v, w)\) is a canonical violating triple. Therefore we have proved that in the case when \( u \) is not type 1 there exists a vertex \( x \) (which is \( u_1 \) or \( w_1 \)) such that \((x, v, w)\) is a canonical violating triple and the arc between \( x \) and \( u \) is non-symmetric. If \( x \) is type 1 then it is a desired vertex. Otherwise \( x \) is type 2. Repeating the same argument as above with \( x \) replacing of \( u \) we find the next vertex \( x' \) which either a desired vertex or a type 2 vertex such that \((x', v, w)\) is a canonical violating triple. Continuing this way in a finite number of steps we either find a desired vertex \( u \) (i.e., \( u \) is type 1 and \((u, v, w)\) is a canonical violating triple) or a ‘circuit’ \( x_1, x_2, \ldots, x_k \), along with vertices \( y_1, y_2, \ldots, y_k \), such that for each \( i = 1, 2, \ldots, k \),

- \( x_i \) is di-simplicial vertex of \( S(D) \) of type 2,
- \((x_i, v, w)\) and \((y_i, v, w)\) are canonical violating triples, and
- there is a non-symmetric arc between \( x_i \) and \( x_{i+1} \) and either \((x_{i+1}, x_i, y_i)\) or \((y_i, x_i, x_{i+1})\) is a canonical violating triple (subscripts are modulo \( k \)).

Assume the latter occurs and the circuit has the minimum length. Note that the vertices \( x_1, \ldots, x_k, y_1, \ldots, y_k \) are in-neighbours of \( v \) so they are pairwise adjacent. Since \( D \) does not contain any digraph in Figure 1 as an induced subdigraph, the circuit is not a directed cycle (consisting of non-symmetric arcs). Hence we may assume without loss of generality that \( x_1x_2, x_1x_k \) are non-symmetric arcs. Then \((y_1, x_1, x_2)\) and \((x_1, x_k, y_k)\) are canonical violating triples. If \( x_2y_k \) is a non-symmetric arc then \( x_1, x_2, y_k \) induce Figure 1(d), a contradiction to assumption. If \( x_2y_k \) is symmetric then \( x_1, x_2, y_1, y_k \) induce Figure 1(a), (b) or (c), also a contradiction. So \( y_kx_2 \) is a non-symmetric arc. Since \( x_k \) is a di-simplicial vertex of \( S(D) \) and \( y_k \) is a adjacent to \( x_k \) but not to \( x_2 \) in \( S(D) \), \( x_2 \) is not adjacent to \( x_k \) in \( S(D) \), that is, the arc between \( x_2 \) and \( x_k \) is non-symmetric. If \( x_2x_k \) is an arc then \( x_2, \ldots, x_k \) would be a shorter circuit, a contradiction to our choice of circuit. So \( x_kx_2 \) is a non-symmetric arc. There is an arc between \( y_1 \) and \( x_k \) as they are out-neighbours of \( x_1 \). If \( y_1x_k \) is non-symmetric, then \( y_1, x_k, x_2 \) induce Figure 1(d) and if \( x_ky_1 \) is non-symmetric, then \( x_1, x_k, y_1, y_k \) induce Figure 1(a), (b) or (c), a contradiction to assumption. Hence \( y_1x_k \) is a symmetric arc and \( x_2, \ldots, x_k \) is a shorter circuit, which is also a contradiction. Therefore for every type 1 vertex \( v \) there exists a canonical violating triple \((u, v, w)\) such that \( u \) is a type 1 vertex. This implies that there exists a directed cycle on type 1 vertices consisting of non-symmetric arcs. Assume that \( v_1, v_2, \ldots, v_t \) is the shortest such cycle. Since \( D \) does not contain an induced directed cycle consisting of non-symmetric arcs, \( t > 3 \) and there is a symmetric arc joining a pair of non-consecutive vertices of the cycle. Without loss of generality assume \( v_1v_s \) is a symmetric arc of the shortest distance along the cycle, that is, \( v_i, v_j \) are not adjacent for all \( 1 \leq i < j \leq s - 1 \) except \( i = 1 \) and \( j = s \). Since \( v_2 \) and \( v_s \) are out-neighbour of \( v_1 \), they are adjacent. This implies that \( s = 3 \) and so \( v_1v_3 \) is a symmetric arc. Hence \((v_2, v_3, v_1)\) is a violating triple in which \( v_3v_1 \) is a symmetric arc, which contradicts the assumption that \( v_3 \) is a type 1 vertex. Therefore \( S(D) \) has no type 1 di-simplicial vertex, that is, every di-simplicial vertex of \( S(D) \) is type 2.
Let \( v \) be a di-simplicial vertex of \( S(D) \). Since \( v \) is type 2, there is a canonical violating triple \((u, v, w)\) such that exactly one of \( uv, vw \) is a non-symmetric arc. If \( uv \) is non-symmetric then \( u \) is a di-simplicial vertex of \( S(D) \). If \( vw \) is non-symmetric then \( w \) is a di-simplicial. This implies that for each di-simplicial vertex of \( S(D) \) there is a di-simplicial vertex \( z \) of \( S(D) \) such that \( z, v \) are part of a canonical violating triple for \( v \) and the arc between \( v \) and \( z \) is non-symmetric. It follows that there exists a ‘circuit’ \( z_1, z_2, \ldots, z_r \), along with vertices \( w_1, w_2, \ldots, w_r \), such that for each \( i = 1, 2, \ldots, r \),

- \( z_i \) is a di-simplicial vertex of \( S(D) \) of type 2,
- either \((z_{i+1}, z_i, w_i)\) or \((w_i, z_i, z_{i+1})\) is a canonical violating triple,
- the arc between \( z_i \) and \( z_{i+1} \) is non-symmetric and the arcs between \( w_i \) and \( z_i \) are symmetric (subscripts are modulo \( r \)).

We again assume that the circuit is chosen to have the minimum length. Suppose \( r = 2 \). If the non-symmetric arc between \( z_1 \) and \( z_2 \) is \( z_1 z_2 \), then \((w_1, z_1, z_2)\) and \((z_1, z_2, w_2)\) are the canonical violating triples where \( w_1 z_1 \) and \( z_2 w_2 \) are symmetric arcs. Neither \( w_1 z_2 \) nor \( z_1 w_2 \) is an arc. Since \( w_1 \) and \( z_2 \) are out-neighbours of \( z_1 \), they are adjacent so \( z_2 w_1 \) is a non-symmetric arc. Similarly, \( w_2 z_1 \) is a non-symmetric arc. There is an arc between \( w_1 \) and \( w_2 \) as they are in-neighbours of \( z_1 \). Depending the arcs between \( w_1 \) and \( w_2 \), the subdigraph induced by \( z_1, z_2, w_1, w_2 \) is Figure I(a), (b) or (c), which contradicts the assumption. The same conclusion holds if the non-symmetric arc between \( z_1 \) and \( z_2 \) is \( z_2 z_1 \). So \( r \geq 3 \).

Suppose that \( z_1 z_2 \ldots z_r z_1 \) is a directed cycle. Since \( D \) does not contain an induced directed cycle consisting of non-symmetric arcs, \( r > 3 \) and there is a symmetric arc between a pair of non-consecutive vertices of the cycle. Without loss of generality assume \( z_1 z_s \) is a symmetric arc of the shortest distance along the cycle, that is, \( z_i, z_j \) are not adjacent for all \( 1 \leq i < j - 1 \leq s - 1 \) except \( i = 1 \) and \( j = s \). Since \( z_2 \) and \( z_s \) are out-neighbour of \( z_1 \), they are adjacent. This implies that \( s = 3 \) and so \( z_1 z_2 \) is a symmetric arc. Since \( z_3 \) and \( z_r \) are in-neighbours of \( z_1 \), they are adjacent. Since \( z_3 \) and \( w_r \) are out-neighbours of \( z_1 \), they are adjacent. The arcs between \( z_3 \) and \( w_r \) cannot be symmetric as otherwise \( z_1 \) and \( w_r \) are both neighbours of \( z_3 \) in \( S(D) \) but \( z_1 w_r \) is a non-symmetric arc, which contradicts the fact that \( z_3 \) is a di-simplicial vertex of \( S(D) \). So \( z_3 \) and \( w_r \) are joined by a non-symmetric arc. If \( w_r z_3 \) is a non-symmetric arc, then the subdigraph induced by \( z_1, z_3, z_r, w_r \) is Figure II(a), (b) or (c), a contradiction. Hence \( z_3 w_r \) is a non-symmetric arc. The arc between \( z_3 \) and \( z_r \) must be non-symmetric as otherwise \( z_3 \) and \( w_r \) are non-adjacent neighbours of \( z_r \) in \( S(D) \), which contradicts the fact that \( z_r \) is a di-simplicial vertex of \( S(D) \). If \( z_3 z_r \) is a non-symmetric arc then \( z_1, z_3, z_r, w_r \) induce Figure II(c), a contradiction. On the other hand, if \( z_r z_3 \) is a non-symmetric arc, then \( z_3, \ldots, z_r \) would be a directed cycle of length shorter than \( r \) consisting of non-symmetric arcs, which contradicts the choice of circuit. Therefore \( z_1 z_2 \ldots z_r z_1 \) is not a directed cycle. Hence we may assume without loss of generality that \( z_1 z_2 \) and \( z_1 z_r \) are non-symmetric arcs.

Since \( z_1 z_2 \) and \( z_1 z_r \) are non-symmetric arcs, \((w_1, z_1, z_2)\) and \((z_1, z_r, w_r)\) are canonical violating triples. Since \( z_2 \) and \( z_r \) are out-neighbours of \( z_1 \), they are adjacent. So \( z_2 \)
is an in-neighbour or an out-neighbour of \( z_r \). Combining this with the fact that \( w_r \) is both an in-neighbour and an out-neighbour of \( z_r \) we see that \( z_2 \) and \( w_r \) are adjacent. If \( z_2 \) and \( w_r \) are joined by symmetric arcs then \( z_1, z_2, w_1, w_r \) induced Figure 1(a), (b) or (c), a contradiction. So \( z_2 \) and \( w_r \) are joined by a non-symmetric arc. If \( z_2 w_r \) is a non-symmetric arc, then \( z_1, z_2, w_r \) induce Figure 1(d), a contradiction. Hence \( w_r z_2 \) is a non-symmetric arc. This means that \( w_r \) is not adjacent to \( z_2 \) in \( S(D) \). However, \( w_r \) is adjacent to \( z_r \) in \( S(D) \) and \( z_r \) is a di-simplicial vertex of \( S(D) \). It follows that \( z_2 \) and \( z_r \) are joined by a non-symmetric arc. If \( z_2 z_r \) is a non-symmetric arc, then \( z_2, \ldots , z_r \) would be a shorter circuit, a contradiction to our choice. So \( z_r z_2 \) is a non-symmetric arc. Since \( w_1 \) and \( z_r \) are out-neighbours of \( z_1 \), they are adjacent. If \( w_1 \) and \( z_r \) are joined by symmetric arcs, then again \( z_2, \ldots , z_r \) would be a shorter circuit, a contradiction. So \( w_1 \) and \( z_r \) are joined by a non-symmetric arc. It cannot be \( w_1 z_r \) as otherwise \( w_1, z_r, z_2 \) induce Figure 1(d), a contradiction. Hence \( z_r w_1 \) is a non-symmetric arc. The subdigraph induced by \( z_1, z_r, w_1, w_r \) is Figure 1(a), (b) or (c), a contradiction. Therefore, \( D \) has a di-simplicial vertex. This completes the proof.

### 3 Weakly quasi-transitive digraphs

According to Theorem 1.2, transitive oriented graphs and semicomplete digraphs are basic building blocks for quasi-transitive digraphs. Using these blocks one can form a class \( Q \) of digraphs as follows:

1. Each transitive oriented graph is in \( Q \).
2. Each semicomplete digraph is in \( Q \).
3. If \( D, H_1, H_2, \ldots , H_n \in Q \), then \( D[H_1, H_2, \ldots , H_n] \in Q \), provided that \( H_i \) is a single-vertex digraph when the vertex \( v_i \) for which \( H_i \) is substituted is incident with a symmetric arc for each \( i \).

Transitive oriented graphs and semicomplete digraphs are quasi-transitive. Moreover, the substitution operation for defining \( Q \) maintain the property of being quasi-transitive. Hence the digraphs in \( Q \) are all quasi-transitive. Theorem 1.2 ensures that every quasi-transitive digraph can be obtained from transitive oriented graphs and semicomplete digraphs by substitutions. Therefore we have the following:

**Corollary 3.1.** The class \( Q \) consists of quasi-transitive digraphs.

Interestingly, weakly quasi-transitive digraphs can also be constructed in a similar way from transitive oriented graphs, semicomplete digraphs and symmetric digraphs.

Let \( W \) be the class of digraphs defined as follows:

1. each transitive oriented graph is in \( W \);
2. each semicomplete digraph is in \( W \);

3. each symmetric digraph is in \( W \);

4. if \( D \) is in \( W \) then any digraph obtained from \( D \) by substituting digraphs of \( W \) for the vertices of \( D \) is in \( W \).

A module in a digraph \( D \) is an induced subgraph \( H \) of \( D \) such that for any vertex \( x \) not in \( H \), either \( x \) is adjacent to no vertex in \( H \) or the vertices in \( H \) are synchronous neighbours of \( x \). A module is called trivial if it has only one vertex or is the entire digraph \( D \) and non-trivial otherwise. An oriented path in \( D \) is a sequence of vertices \( v_1, v_2, \ldots, v_k \) such that \( v_i \) and \( v_{i+1} \) are joined by a non-symmetric arc for each

**Theorem 3.2.** The class \( W \) consists of weakly quasi-transitive digraphs.

**Proof:** Transitive oriented graphs and semicomplete digraphs are quasi-transitive, so they are weakly quasi-transitive. Symmetric digraphs are also weakly quasi-transitive because any vertex in a symmetric digraph has only synchronous neighbours. To prove the rest of digraphs in \( W \) are all weakly quasi-transitive, let \( D^* = D[H_1, H_2, \ldots, H_n] \) where \( D, H_1, H_2, \ldots, H_n \) are weakly quasi-transitive. Consider three vertices \( u, v, w \) where \( u, w \) are asynchronous neighbours of \( v \). Assume \( u \in V(H_i), v \in V(H_j) \) and \( w \in V(H_k) \). If \( i = j = k \) then \( u, w \) are adjacent as \( H_i \) is weakly quasi-transitive. Suppose \( i = j \neq k \). Since \( v \) and \( w \) are adjacent, each vertex of \( H_i \) is adjacent to all vertices of \( H_k \) and in particular, \( u \) is adjacent to \( w \). Similarly, if \( i \neq j = k \), then \( u \) and \( w \) are adjacent. Suppose that \( i \neq j \neq k \). Then \( i \neq k \) because \( u \) and \( w \) are asynchronous neighbours of \( v \). Since \( D \) is weakly quasi-transitive, the two vertices of \( D \) corresponding to \( H_i \) and \( H_k \) are adjacent so \( u \) and \( w \) are adjacent. Hence all digraphs in \( W \) are weakly quasi-transitive.

We prove by induction on number of vertices that every weakly quasi-transitive digraph is in \( W \). Let \( D \) be a weakly quasi-transitive with \( n \) vertices. Assume that every weakly quasi-transitive digraph with fewer than \( n \) vertices is in \( W \). If \( D \) is quasi-transitive or symmetric then it is in \( W \). So assume that \( D \) is neither quasi-transitive nor symmetric. Since \( D \) is not quasi-transitive, there exist vertices \( u, v, w \) with \( u \in N^-(v) \) and \( w \in N^+(v) \) such that \( u \) and \( w \) are not adjacent in \( D \). Thus \( u \) and \( w \) are non-adjacent neighbours of \( v \). Since \( D \) is weakly quasi-transitive, any two asynchronous neighbours of \( v \) are adjacent. Hence \( u \) and \( w \) are synchronous neighbours of \( v \), which implies \( u \) and \( w \) are both in \( N^+(v) \cap N^-(v) \).

Suppose \( H \) is a non-trivial module in \( D \). Let \( D' \) be the digraph obtained from \( D \) by deleting all vertices of \( H \) except one. Then \( D = D'[H_1, H_2, \ldots, H_k] \) where \( H_1 = H \) and each \( H_i \) with \( i \geq 2 \) is a single-vertex digraph. The digraphs \( D', H_1, \ldots, H_k \) each has fewer than \( n \) vertices and is weakly quasi-transitive and hence they are in \( W \). This means that \( D \) is obtained from digraphs in \( W \) by substitution and by definition \( D \) is in \( W \). Thus, it suffices to show that there is a non-trivial module in \( D \).

Let \( R \) be the subdigraph of \( D \) induced by \( N^+(v) \cap N^-(v) \). Then \( u \) and \( w \) are a pair of non-adjacent vertices in \( R \). Let \( M_1 \) be the subdigraph of \( R \) induced by the vertices which are connected to \( u \) by paths in \( U(R) \). Clearly, \( M_1 \) contains \( u \) and \( w \) but not \( v \). Suppose
$x$ is a vertex in $N^+[v] \cup N^-[v]$ but not in $M_1$. We claim that $x$ is completely adjacent to $M_1$. Indeed, if $x \in N^+[v] \cap N^-[v]$, then the definition of $M_1$ implies that $x$ is completely adjacent to $M_1$. On the other hand, if $x \in N^+(v) \oplus N^-(v)$, then $x$ and any vertex of $M_1$ are asynchronous neighbours of $v$ so $x$ is also completely adjacent to $M_1$. By definition any two vertices of $M_1$ are connected by a path in $\overline{U(M_1)}$. In such a path any two consecutive vertices are not adjacent in $D$ and hence are synchronous neighbours of $x$. It follows that the vertice of $M_1$ are synchronous neighbours of $x$. Suppose $x \notin N^+[v] \cup N^-[v]$. If $x$ is adjacent to some vertex $y$ in $M_1$, then $x$ and $v$ are non-adjacent neighbours of $y$ and hence they must be synchronous neighbours of $y$. The fact that $v$ is joined to $y$ by symmetric arcs implies $x$ is joined to $y$ by symmetric arcs. Thus if $x$ is completely adjacent to $M_1$ then the vertices of $M_1$ are synchronous neighbours of $x$. It follows that $M_1$ is a module if for each $x \notin N^+[v] \cup N^-[v]$, either $x$ is adjacent to no vertex in $M_1$ or completely adjacent to $M_1$. We may assume $M_1$ is not a module as otherwise we are done. This means that there exist vertices $x, y, y'$ with $x \notin N^+[v] \cup N^-[v]$ and $y, y' \in M_1$ such that $x$ is adjacent to $y$ but not to $y'$. These three vertices $x, y, y'$ along with $M_1$ will be refered to in the rest of proof.

Suppose $N^+(v) \oplus N^-(v) \neq \emptyset$. Any vertex in $N^+(v) \oplus N^-(v)$ is a neighbour of $v$ asynchronous to those of $v$ in $N^+(v) \cap N^-(v)$. Hence every vertex in $N^+(v) \oplus N^-(v)$ is completely adjacent to $N^+(v) \cap N^-(v)$ and in particular to $M_1$. Suppose that the arcs between $N^+(v) \oplus N^-(v)$ and $M_1$ are all symmetric. Let $M_2$ be the subdigraph of $D$ induced by vertices which are connected to $v$ by oriented paths. Clearly, $M_2$ contains $v$ and all vertices in $N^+(v) \oplus N^-(v)$. We show that $x$ is not a vertex in $M_2$. Suppose not; there is an oriented path connecting $x$ and a vertex in $N^+(v) \oplus N^-(v)$. Let $a_1 \sim a_2 \sim \cdots \sim a_s$ be such a path where $a_1 = x$ and $a_s \in N^+(v) \oplus N^-(v)$. Note that $a_s$ is joined to each vertex of $M_1$ by symmetric arcs and $a_1$ ($= x$) is not adjacent to $y'$ (in $M_1$). Let $j$ be the largest subscript such that $a_j$ is not adjacent to some vertex $y''$ of $M_1$. Then $j < k$ and $a_j \notin N^+[v] \cup N^-[v]$. Since $a_j$ and $a_{j+1}$ are joined by a non-symmetric arc, $a_{j+1} \notin N^+[v] \cap N^-[v]$. Either $a_{j+1} \in N^+(v) \oplus N^-(v)$ or $a_{j+1} \notin N^+[v] \cup N^-[v]$. In either case $a_{j+1}$ is joined to each vertex of $M_1$ by symmetric arcs. Thus $a_j$ and $y''$ are non-adjacent asynchronous neighbours of $a_{j+1}$, contradicting the assumption that $D$ is weakly quasi-transitive. So $x$ is not a vertex of $M_2$. We show that $M_2$ is a module. Let $z$ be a vertex not in $M_2$. By definition $z$ cannot be joined to any vertex of $M_2$ by a non-symmetric arc. Suppose $z$ is joined to some vertex $h$ of $M_2$ by symmetric arcs. Since $h$ can reach every other vertex of $M_2$ by an oriented path, following such a path we see that $z$ is joined to every vertex in the path by symmetric arcs. Hence the vertices of $M_2$ are synchronous neighbours of $z$. Therefore $M_2$ is a non-trivial module in $D$.

Suppose now that the arcs between $N^+(v) \oplus N^-(v)$ and $M_1$ are not all symmetric. Let $M_3$ be a subdigraph of $D$ induced by the vertices defined recursively as follows:

- $u$ is a vertex in $M_3$;
- if $h$ is a vertex in $N^+(v) \cap N^-(v)$ that is not adjacent to a vertex in $M_3$ then $h$ is a vertex in $M_3$;

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• if \( h \) is not in \( N^+(v) \cap N^-(v) \) that is joined to a vertex in \( M_3 \) by symmetric arcs then \( h \) is a vertex in \( M_3 \).

It is easy to see that \( M_3 \) contains \( u, v, w, x \) and all vertices of \( M_1 \). Let \( b \) be a vertex in \( N^+(v) \oplus N^-(v) \) which is joined to a vertex in \( M_1 \) by a non-symmetric arc. Assume that \( b \in N^-(v) \setminus N^+(v) \). From the above we know that the vertices of \( M_1 \) are synchronous neighbours of \( b \). In particular, \( y, y' \) are synchronous neighbours of \( b \). The vertex \( y \) is joined to \( b \) by a non-symmetric arc and joined to \( x \) by symmetric arcs. Thus \( b \) and \( x \) are asynchronous neighbours of \( y \) and hence they must be adjacent. So \( x \) and \( v \) are neighbours of \( b \). Since \( x \) and \( v \) are not adjacent, they are synchronous neighbours of \( b \). Since \( b \in N^-(v) \setminus N^+(v) \), \( bv \) is a non-symmetric arc, so \( bx \) is also a non-symmetric arc. Since \( bx \) is a non-symmetric arc and \( x, y' \) are non-adjacent neighbours of \( b \), \( by' \) is also a non-symmetric arc. The fact that the vertices of \( M_1 \) are synchronous neighbours of \( b \) so there is a non-symmetric arc from \( b \) to every vertex in \( M_1 \). Similarly, if \( b \in N^+(v) \setminus N^-(v) \) is joined to a vertex in \( M_1 \) by a non-symmetric arc then \( xb \) is a non-symmetric arc and there is a non-symmetric arc from every vertex of \( M_1 \) to \( b \).

We claim that \( b \) is not a vertex in \( M_3 \). Suppose not; \( b \) is in \( M_3 \). By the definition of \( M_3 \) there exists a sequence of vertices \( h_0, h_1, \ldots, h_t \) where \( h_0 = y \) and \( h_t = b \) such that for each \( i > 0, h_i \in N^+(v) \cap N^-(v) \) implies that \( h_i \) is not adjacent to \( h_{i-1} \), and \( h_i \notin N^+(v) \cap N^-(v) \) implies \( h_i \) is joined to \( h_{i-1} \) by symmetric arcs. We choose such a vertex \( b \) so that the sequence is as short as possible. Assume \( b \in N^-(v) \setminus N^+(v) \). Then \( b (= h_t) \) is joined to \( h_{t-1} \) by symmetric arcs. We claim \( h_{t-1} \in N^+(v) \cap N^-(v) \). Indeed, since \( b \) is joined to \( h_{t-1} \) by symmetric arcs, \( h_{t-1} \in N^+(v) \cup N^-(v) \). Suppose \( h_{t-1} \in N^-(v) \setminus N^+(v) \). The choice of \( b \) implies that there can only be symmetric arcs between \( h_{t-1} \) and \( M_1 \). Since \( h_{t-1} \) and \( x \) are asynchronous neighbours of \( b \), they are adjacent. In particular, \( h_{t-1}x \) is a non-symmetric arc. Thus \( x, y' \) are non-adjacent asynchronous neighbours of \( h_{t-1} \), a contradiction. So \( h_{t-1} \notin N^-(v) \setminus N^+(v) \). A similar proof shows \( h_{t-1} \notin N^+(v) \setminus N^-(v) \). So \( h_{t-1} \in N^+(v) \cap N^-(v) \). Since \( b \) is joined to \( h_{t-1} \) by symmetric arcs and joined to each vertex of \( M_1 \) by a non-symmetric arc, \( h_{t-1} \notin M_1 \) and thus \( t > 2 \). Hence \( h_{t-1} \) is not adjacent to \( h_{t-2} \) and is completely adjacent to \( M_1 \). If \( h_{t-2} \in N^+[v] \cup N^-[v] \), then \( h_{t-2} \) must be in \( N^+(v) \cup N^-(v) \) and hence adjacent to \( b \). Thus \( h_{t-1}, h_{t-2} \) are neighbours of \( b \). Since \( h_{t-1}, h_{t-2} \) are not adjacent, they are synchronous neighbours of \( b \), which implies \( b \) is joined to \( h_{t-2} \) by symmetric arcs. This contradicts the choice of the sequence as \( h_0, h_1, \ldots, h_{t-2}, b \) is a shorter sequence. So \( h_{t-2} \notin N^+[v] \cup N^-[v] \). Let \( \ell \) be the largest integer such that \( h_{t-2}, \ldots, h_{t-\ell} \) are not in \( N^+[v] \cup N^-[v] \). Then \( h_{t-\ell} \) is joined to \( h_{t-\ell-1} \) by symmetric arcs for each \( i = 2, \ldots, \ell \). We must have \( h_{t-\ell-1} \in N^+(v) \cap N^-(v) \). The vertex \( b \) is not adjacent to \( h_{t-\ell} \) as otherwise \( h_{t-1}, h_{t-\ell} \) are non-adjacent asynchronous neighbours of \( b \), a contradiction. For the same reason, we see that \( b \) is not adjacent to \( h_{t-\ell} \) for each \( i = 2, \ldots, \ell \). Since \( b, h_{t-\ell-1} \) are asynchronous neighbours of \( v \), they are adjacent. They must be joined by symmetric arcs, as otherwise \( b, h_{t-\ell} \) are non-adjacent asynchronous neighbours of \( h_{t-\ell-1} \), a contradiction. But this contradicts the choice of the sequence because \( h_0, h_1, \ldots, h_{t-\ell-1}, b \) is a shorter sequence. Therefore \( b \) is not a vertex in \( M_3 \). So if \( b \in N^-(v) \setminus N^+(v) \) is joined to a vertex in \( M_1 \) with a non-symmetric arc then \( b \notin M_3 \) and there is a non-symmetric arc from \( b \) to every vertex in \( M_1 \). A similar proof shows that if \( b \in N^+(v) \setminus N^-(v) \) is joined to a vertex in \( M_1 \) with a non-symmetric arc then \( b \notin M_3 \) and
there is a non-symmetric arc from each vertex of $M_1$ to $b$.

We show that $M_3$ is a module. Let $z$ be a vertex that is not in $M_3$. For each vertex $h \in M_3$, there is a sequence of vertices $h_0, h_1, \ldots, h_t$ where $h_0 = y$ and $h_t = h$ such that for each $i > 0$, $h_i \in N^+(v) \cap N^-(v)$ implies that $h_i$ is not adjacent to $h_{i-1}$, and $h_i \notin N^+(v) \cap N^-(v)$ implies $h_i$ is joined to $h_{i-1}$ by symmetric arcs. Suppose first that $z \in N^-(v) \setminus N^+(v)$. We know from the above that $zx$ is a non-symmetric arc and $zh$ is a non-symmetric arc for all $h \in M_1$. In particular, $zy = zh_0$ is a non-symmetric arc. Suppose $k > 0$ and $zh_{k-1}$ is a non-symmetric arc. If $h_k \notin N^+(v) \cap N^-(v)$, then $h_{k-1}, h_k$ are non-adjacent neighbours of $z$ so $zh_k$ is a non-symmetric arc. If $h_k \notin N^+(v) \cap N^-(v)$, then $z, h_k$ are asynchronous neighbours of $h_{k-1}$ so they are adjacent. There are two cases. Either $h_k \in N^+(v) \oplus N^-(v)$ or $h_k \notin N^+(v) \cup N^-(v)$. If $h_k \notin N^+(v) \cup N^-(v)$, then clearly $zh_k$ is a non-symmetric arc. Assume $h_k \in N^+(v) \oplus N^-(v)$. Since $h_k \notin M_3$, $h_k$ is joined to each vertex in $M_1$ by symmetric arcs. In particular, $h_k$ is joined to $y'$ by symmetric arcs. Since $y'$ is not adjacent to $x$, $h_k$ and $x$ cannot be adjacent as otherwise $y'$ and $x$ are non-adjacent asynchronous neighbours of $h_k$, a contradiction. Hence $h_k$ and $x$ are synchronous neighbours of $z$. Since $zx$ is a non-symmetric arc, $zh_k$ is a non-symmetric arc. Therefore $zh$ is a non-symmetric arc for all $h \in M_3$. A similar proof shows that if $z \in N^+(v) \setminus N^-(v)$ then $hz$ is a non-symmetric arc for all $h \in M_3$. Suppose next that $z \in N^+(v) \cap N^-(v)$. Since $z$ is not in $M_3$, $z$ is adjacent to every vertex in $M_3$. In particular, $z$ is adjacent to $x$. Note that $z$ and $x$ are joined by symmetric arcs. Since $x$ and $y'$ are not adjacent, $z$ is adjacent to $y'$ by symmetric arcs. This implies $z$ is also joined to $y$ by symmetric arcs. Suppose that $k > 0$ and $z$ is joined to $h_{k-1}$ by symmetric arcs. If $h_k \notin N^+(v) \cup N^-(v)$ then clearly $z$ is joined to $h_k$ by symmetric arcs. If $h_k \in N^+(v) \cap N^-(v)$, then $h_k$ is not adjacent to $h_{k-1}$ and thus $h_k, h_{k-1}$ are non-adjacent neighbours of $z$. Since $z$ is joined to $h_{k-1}$ by symmetric arcs, $z$ is joined to $h_k$ by symmetric arcs. If $h_k \in N^+(v) \oplus N^-(v)$, then $h_k$ is joined to $y'$ by symmetric arcs. Since $y'$ and $x$ are not adjacent, $h_k$ and $x$ are not adjacent. Thus $h_k$ and $x$ are non-adjacent neighbours of $z$, which implies $z$ is joined to $h_k$ by symmetric arcs. Suppose now that $z \notin N^+\setminus[v] \cup N^-\setminus[v]$. Since $z$ is not in $M_3$, it is not adjacent to any vertex in $M_1$. In particular, $z$ is not adjacent to $y$. Suppose that $k > 0$ and $z$ is not adjacent to $h_{k-1}$. If $h_k \notin N^+(v) \cap N^-(v)$, then $z$ is not adjacent to $h_k$ as otherwise $z$ is joined to $h_k$ by symmetric arcs, which implies $z \in M_3$, a contradiction to assumption. If $h_k \notin N^+(v) \cap N^-(v)$, then $h_k$ is joined to $h_{k-1}$ by symmetric arcs. Since $z$ is not adjacent to $h_{k-1}$, $z$ cannot be joined to $h_k$ by a non-symmetric arc. Since $z \notin M_3$ and $h_k \in M_3$, $z$ cannot be joined to $h_k$ by symmetric arcs. Hence $z$ is not adjacent to $h_k$.

The only case remaining is that $N^+(v) \oplus N^-(v) = \emptyset$. Since $D$ is not a symmetric digraph, it has a non-symmetric arc. Suppose $fg$ is a non-symmetric arc in $D$. Let $M_4$ be the subdigraph induced by the vertices which are connected to $f$ by oriented paths. Then any two vertices in $M_4$ are connected by an oriented path. Since $N^+(v) \oplus N^-(v) = \emptyset$, there is no oriented path connecting $f$ and $v$. So $v$ is not a vertex in $M_4$. Suppose $z$ is not in $M_4$ but is adjacent to a vertex $h$ in $M_4$. Then $z$ is joined to $h$ by symmetric arcs. Each vertex of $M_4$ is connected to $h$ by an oriented path. Following these oriented paths we see that $z$ is joined to each vertex of $M_4$ by symmetric arcs and hence the vertices of $M_4$ are synchronous neighbours of $z$. Therefore $M_4$ is a non-trivial module.}

The class of weakly quasi-transitive digraphs strictly contains quasi-transitive digraphs.
and extended semicomplete digraphs, which in turn as classes strictly contain all semicomplete digraphs. Surprisingly, these four classes of digraphs share the same forbidden subdigraphs for being chordal.

**Theorem 3.3.** A weakly quasi-transitive digraph $D$ is chordal if and only if $S(D)$ is chordal and $D$ does not contain any digraph in Figure 1 as an induced subdigraph.

**Proof:** If $D$ is chordal then it does not contain any digraph in Figure 1 as an induced subdigraph. Suppose $D$ does not contain any digraph in Figure 1 as an induced subdigraph. We prove by induction on the number of vertices that $D$ is chordal. It suffice to show that $D$ has a di-simplicial vertex. This is true if $D$ is a transitive oriented graph, a semicomplete digraph, or a symmetric digraph. Assume $D$ is a weakly quasi-transitive digraph but not a transitive oriented graph, a semicomplete digraph, or a symmetric digraph. For the inductive hypothesis, assume that any induced subdigraph of $D$ with fewer vertices than $D$ has a di-simplicial vertex. By Theorem 3.2, $D = D'[H_1, H_2, \ldots, H_n]$ where $D'$ and one of $H_i$’s have at least two vertices. Then $D'$ and each $H_i$ is an induced subdigraph of $D$ with fewer vertices than $D$ and by the inductive hypothesis each of them has a di-simplicial vertex. Suppose that $v$ is a di-simplicial vertex of $D'$ and $H_j$ is substituted for $v$. Then it is easy to verify that a di-simplicial vertex of $H_j$ is a di-simplicial vertex of $D$.

**Corollary 3.4.** [12] Let $D$ be a quasi-transitive digraph or an extended semicomplete digraph. Then $D$ is chordal if and only if $S(D)$ is chordal and $D$ does not contain any digraph in Figure 1 as an induced subdigraph.

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