Weakly driven anomalous diffusion in non-ergodic regime: an analytical solution

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December 5, 2013

Abstract. We derive the probability density of a diffusion process generated by nonergodic velocity fluctuations in presence of a weak potential, using the Liouville equation approach. The velocity of the diffusing particle undergoes dichotomic fluctuations with a given distribution $\psi(\tau)$ of residence times in each velocity state. We obtain analytical solutions for the diffusion process in a generic external potential and for a generic statistics of residence times, including the non-ergodic regime in which the mean residence time diverges. We show that these analytical solutions are in agreement with numerical simulations.

PACS. 02.50.Ey Stochastic processes – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

1 Introduction

A theoretical study of diffusive transport phenomena has been a fundamental subject of research since the seminal work on Brownian motion published by Einstein \cite{Einstein1905} in 1905, the first of four papers of his \textit{annus mirabilis}. The problem was soon afterwards tackled by Smoluchowski \cite{Smoluchowski1906} within a more microscopic picture not based on the assumption of a Stokes-law for the fluid. Both derivations produce a theoretical description of ordinary diffusion leading to the well known linear relation between mean squared displacement and time, $\langle x^2 \rangle = D t$, with $D$ being the diffusion coefficient. More recently, several relevant physical and biological conditions have been discovered where diffusion is characterized by a non-linear “anomalous” relationship between mean squared displacement and time, $\langle x^2 \rangle = D_s t^{\alpha}$. Diffusion through porous media or within a cellular crowded environment are two important examples where such anomalous diffusion is detected. In the past twenty years anomalous diffusion has therefore become the subject of intense research work \cite{BarkaiBenson2003,Bologna2007,Bologna2010}. Herein we aim at developing a theoretical description for anomalous diffusion generated by dichotomic velocity fluctuations based on the following non-linear equation \cite{Bologna2009}

$$\frac{dx}{dt} = -\nabla U(x) + g(x)\xi(t)$$

(1)

where $\xi(t)$ can be interpreted as a stochastic force, that we shall consider to be a dichotomic stochastic variable, while $-\nabla U(x)$ is the deterministic force. A dependence $g(x)$ of the amplitude of the stochastic component is also introduced for sake of generality, but can be eliminated via a simple change of variables (see following section). We adopt this choice since the dichotomic process, by taking two well defined finite values (as opposed to a continuous Gaussian process), is suitable to describe a wide range of physical processes. Furthermore, it can be shown that, by an appropriate limit procedure, the dichotomic noise converges to Gaussian white noise and white shot noise \cite{Benzi1984}.

In spite of significant progress reached in the field, the general solution of Eq. (1) with an arbitrary potential function $U(x)$ is still an unsettled problem. Finding such a solution is the main focus of this paper. In the following we derive an analytical solution for the probability density function (pdf) $P(x,t)$, valid for small potential $U(x)$. Importantly our solution extends to the case of non-ergodic fluctuations, leading to an elegant description of a regime which is central in very recent theoretical and experimental explorations, e.g. dynamics of nanocrystals \cite{Bachtold2003}, dynamics of single molecules in cells \cite{Huse2007}, non-markovian stochastic resonance in physical and biological systems \cite{Bollermann2006}. We confirm all our results by comparison with numerical simulations of the diffusion process.

2 Liouville equation

In this section we consider the one-dimensional version of Eq. (1) which, with a simple change of variables $y = h(x)$ with $h(x) = \int^x dz/g(z)$ and consequent redefinition of the potential as $U(y) = \int^y dx \frac{1}{g(x)} \frac{\partial U}{\partial x}$,
A formal solution of Eq. (5) is given by \( \xi \) with time evolution of general form:

\[
\frac{dy}{dt} = -\frac{\partial}{\partial y} \hat{U}(y) + \xi(t).
\]

We therefore focus in the rest of the paper on the process with time evolution of general form:

\[
\frac{dx}{dt} = -\frac{\partial}{\partial x} U(x) + \xi(t),
\]

and on the corresponding Liouville equation for the stochastic density \( \rho(x,t) \), i.e. the probability density for the process to get the value \( x \) at time \( t \) for a given realisation of the fluctuations \( \xi(t) \):

\[
\frac{\partial}{\partial t} \rho(x,t) = \frac{\partial}{\partial x} [U'(x) - \xi(t)] \rho(x,t),
\]

with \( U'(x) \equiv \frac{\partial}{\partial x} U(x) \). Depending on the sign of \( \xi(t) \) which, without loss of generality, is set to be \( \xi(t) = \pm 1 \), a formal solution of Eq. (5) is given by

\[
\rho^{(\pm)}(x,t) = \frac{1}{1 \mp U'(x)} \left( t + \int_0^x \frac{1}{U'(z) \pm 1} dz \right).
\]

Assuming that the two values of \( \xi \) have the same probability, i.e. \( 1/2 \), using Van Kampen’s lemma we can connect the stochastic density \( \rho \) to the probability density \( P(x,t) \) via the relation

\[
P(x,t) = \langle \rho(x,t) \rangle = \frac{1}{2} \langle \rho^{(+)}(x,t) \rangle + \frac{1}{2} \langle \rho^{(-)}(x,t) \rangle =
\]

\[
\frac{1}{1 - U'(x)} \left\{ \frac{1}{2} \left[ t + \int_0^x \frac{1}{U'(z) - 1} dz \right] \right\} + \frac{1}{1 + U'(x)} \left\{ \frac{1}{2} \left[ t + \int_0^x \frac{1}{U'(z) + 1} dz \right] \right\}.
\]

Applying Eq. (9) to the case of a linear potential \( U(x) = kx \), i.e. a constant force, it follows:

\[
\rho^{(\pm)}(x,t) = \frac{1}{1 \mp k} \left( t \pm \frac{x}{1 \mp k} \right) \delta (t \pm \frac{x}{1 \mp k}).
\]

Using the following property of the Dirac delta function

\[
\delta [f(z)] = \sum_i \frac{1}{|f'(z_i)|} \delta (z - z_i),
\]

where \( z_i \) are the roots of the function \( f(z) = 0 \), from Eq. (8) we obtain the following exact solution

\[
\rho^{(\pm)}(x,t) = \delta [x \pm t(1 \pm k)] = \delta [x + kt \pm t] = \delta [\bar{x} \pm t]
\]

with \( \bar{x} = x + kt \). While in the case of linear potential an exact solution can be found, in order to find a solution for a generic potential we consider the case where the deterministic force is a perturbation, i.e. \( |U'(x)| \ll 1 \). We can rewrite Eq. (9) as

\[
\rho^{(\pm)}(x,t) \approx \frac{1}{1 \mp U'(x)} \delta (t - U(x) \mp x).
\]

where \( U(x) = \int_0^x U'(z) dz \). This approximation remains valid at any time if the force is limited, i.e. \( |U'(x)| \ll 1 \), condition that is necessary to avoid space confinement which would block the diffusive transport induced by the stochastic fluctuations \( \xi \). Replacing the variable \( x \) with its functional dependence on \( t \), we may write at the same order in \( U(x) \)

\[
\rho^{(\pm)}(x,t) \approx \delta (x \mp t \pm U(\mp t)).
\]

Our main effort is to evaluate the average of the above quantities over the fluctuations \( \xi \) of the dichotomous random process. Let us focus on the basic solution for which at \( t = 0 \) the stochastic variable is \( \xi = 1 \)

\[
\rho^{(+)}_R(x,t) = \delta (x - t + U(t)).
\]

After a random time \( \tau_1 \) generated with a waiting time distribution \( \psi(\tau) \) the sign of the random variable changes and we have to consider the second solution, corresponding to \( \xi = -1 \), that is to say

\[
\rho^{(-)}_R(x,t) = \delta (x + t - U(-t) + c_1(\tau_1)).
\]

Imposing the continuity condition at the time \( t = \tau_1 \) of the two solutions, (13) and (14), we obtain for the constant \( c_1(\tau_1) \) the expression

\[
c_1(\tau_1) = -2\tau_1 + U(\tau_1) + U(-\tau_1).
\]

In the present paper we shall consider the case where the potential is an odd function, \( U(z) = -U(-z) \), since this condition leads to cancellation of the potential terms on the righthand side of Eq. (15), and therefore to a simple continuity condition of the solutions at each change of \( \xi \). We leave the general solution for a totally arbitrary potential for a separate study. From the chosen potential symmetry it follows that the constant \( c_1 \) is

\[
c_1(\tau_1) = -2\tau_1.
\]

Starting with the positive value of the random variable \( \xi(t) = +1 \), we have, up to time \( \tau_1 \), when the random variable changes sign,

\[
\rho^{(+)}_{R,1}(x,t) = \delta(t - \bar{x})\theta(t)\theta(\tau_1 - t),
\]

\[
\rho^{(+)}_{R,2}(x,t) = \delta(t + \bar{x} - 2\tau_1)\theta(t - \tau_1)
\]

\[
\rho^{(-)}_{R,1}(x,t) = \delta(t - \bar{x})\theta(t)\theta(\tau_1 - t),
\]

\[
\rho^{(-)}_{R,2}(x,t) = \delta(t + \bar{x} - 2\tau_1)\theta(t - \tau_1)
\]
where, for sake of compactness, we defined $\bar{x} = x + U(t)$. Iterating the procedure for the general case of $n$ changes we may write the generic terms in the following form \cite{15}

$$\rho_{R,2n}(x,t) = \delta \left( t + \bar{x} - 2 \sum_{k=1}^{2n-1} \tau_k \sin^2 \frac{k\pi}{2} \right) \times \theta \left( t - \sum_{k=1}^{2n} \tau_k \right) \left( \sum_{k=1}^{2n} \tau_k - t \right), \quad (19)$$

and

$$\rho_{R,2n+1}(x,t) = \delta \left( t - \bar{x} - 2 \sum_{k=1}^{2n} \tau_k \cos^2 \frac{k\pi}{2} \right) \times \theta \left( t - \sum_{k=1}^{2n+1} \tau_k \right) \left( \sum_{k=1}^{2n+1} \tau_k - t \right). \quad (20)$$

Due to the form of the potential, to first order in $U(x)$, the solutions coincide with the case of null potential, where we have to replace the variable $x$ with $\bar{x} = x + U(t)$. We now take the average of the total $\rho_R$ defined as $\rho_R(x,t) = \sum \rho_{R,2n}(x,t) + \sum \rho_{R,2n+1}(x,t).$ Considering the symmetric case where the waiting time distribution, $\psi(\tau)$, for the state $\xi = 1$ is the same function of the waiting time distribution for the state $\xi = -1$, we may perform an average through the multiple integral

$$\int_0^{\infty} \prod \int_0^{\infty} dt \rho_R(x,t) \psi(\tau_i).$$

The statistical average generates the time convoluted expression

$$\langle \rho_R(x,t) \rangle = \frac{1}{2} \left[ \psi_n \left( \frac{t - \bar{x}}{2} \right) \times \int_0^{\infty} \psi_n \left( \frac{t + \bar{x} - z}{2} \right) \psi(z) dz + \psi_n \left( \frac{t + \bar{x}}{2} \right) \times \int_0^{\infty} \psi_{n-1} \left( \frac{t - \bar{x} - z}{2} \right) \psi(z) dz \right] \theta(t - |\bar{x}|), \quad (21)$$

where by definition

$$\psi_0(z) \equiv \delta(z), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
3 Poissonian velocity fluctuations

Here we consider first, the case where the stochastic dichotomous process generating the velocity fluctuations is exponentially correlated. A dichotomous renewal process with an exponential correlation is characterized by an an exponential waiting time distribution density and therefore corresponds to a Poissonian process. The process is driven by an equation for the probability density known as telegrapher’s equation and it is widely studied including its generalizations [17,18].

To make the paper as self-contained as possible, let us review briefly hereby the known results based on the correlation function technique. The correlation function approach has been successfully used, in particular, for Poisson processes. The oth correlation function of such a processes fulfills the condition [19]

\[
\frac{\partial}{\partial t} \langle \xi(t) \xi(t_1) \cdots \xi(t_n) \rangle = -\gamma \langle \xi(t) \xi(t_1) \cdots \xi(t_n) \rangle
\] (31)

where the average is performed on the \( \xi \) realizations. The above formula allows us to write a system of coupled partial differential equations for the distribution in the Poisson case. Averaging Eq. (31) over the fluctuations of \( \xi \) we may write

\[
\frac{\partial}{\partial t} \langle \rho(x,t) \rangle = \frac{\partial}{\partial x} \left[ U'(x) \langle \rho(x,t) \rangle - \langle \xi(t) \rho(x,t) \rangle \right].
\] (32)

Introducing the function \( P_1(x,t) = \langle \xi(t) \rho(x,t) \rangle \) and using the differentiation formula for Poisson processes [19], it follows that

\[
\frac{\partial}{\partial t} \langle \xi(t) \phi(t) \rangle = -\gamma \langle \xi(t) \phi(t) \rangle + \langle \xi(t) \frac{\partial}{\partial t} \phi(t) \rangle.
\] (33)

Eqs. (32) and (33) form a system of coupled partial differential equations with coefficients depending on the spatial variable \( x \)

\[
\begin{align*}
\partial_t P(t,x) &= \partial_x \left[ U'(x) P(t,x) - P_1(x,t) \right] \\
\partial_t P_1(x,t) &= -\gamma P_1(x,t) + \partial_x \left[ U'(x) P_1(x,t) - P(t,x) \right].
\end{align*}
\] (34)

It is straightforward to obtain the equilibrium distribution for Eqs. (34) i.e.

\[
P_{eq}(x) = \frac{c}{1 - U'(x)^2} \exp \left[ -\gamma \int \frac{U'(x)}{1 - U'(x)^2} dx \right]
\] (35)

From Eq. (35) we can infer that \( P_{eq}(x) \) is not always defined and it depends explicitly on the behavior of the function \( U'(x) \).

The laplace transform with respect to the variable \( t \) of the coupled system of equations (34) can be split in two second order differential equations with variable coefficients for \( \hat{P}(s,u) \) and \( \hat{P}_1(s,u) \). It is well known that second order differential equations with variable coefficients can be solved exactly only for a restrict class of coefficients. The main goal of this section is to show that the solution given by Eq. (27) describes the processes under study and consequently provides as well a solution of the system (34) for small potential \( U \). The central part of the function \( P(x,t) \), is given by [14]

\[
\hat{P}_C(s,u) = \frac{1}{4} \frac{\gamma [2su + 3(s + u)\gamma + 4\gamma^2]}{(s + \gamma)(u + \gamma)[su + (s + u)\gamma]}
\] (36)

where we used for the waiting time distribution the expression \( v(t) = \gamma \exp[-\gamma t] \). The double inverse Laplace transform with respect to \( u \) and \( v \) gives the well known result (see Ref. [20] for a review)

\[
P_C(x,t) = \frac{1}{2} \exp[-\gamma t] \gamma \left[ I_0 \left( \gamma \sqrt{t^2 - x^2} \right) + \frac{tI_1 \left( \gamma \sqrt{t^2 - x^2} \right)}{\sqrt{t^2 - x^2}} \right] \theta(t - |x|),
\] (37)

where \( I_n(z) \) is the modified Bessel function of the first kind. Thus the total pdf is:

\[
P(x,t) = \frac{1}{2} \exp[-\gamma t] \delta(t - |x|) + P_C(x,t).
\] (38)

Replacing \( x \to x + U(t) \), the solution for the case with potential is

\[
P_{1u}^{(P)}(x,t) \approx P(x + U(t),t).
\] (39)

\[\text{Fig. 1. Poissonian regime: Density profile for the diffusion process with fluctuations with exponential waiting times distribution } v(t) = \frac{1}{\gamma} \exp[-\gamma t/2] \text{ with } \gamma = 1. \text{ Plotted are the numerical simulation of the diffusion equation for the case without potential (black empty circles) and with potential } U(x) = x \frac{e^2}{2 + x^2} \text{ (black filled circles) and plots of their respective analytical solution (red lines) at time } t = 100.5 \text{ with } \varepsilon = 0.1 \text{ (arbitrary units).}\]
This function has an asymptotic power-law behavior, namely $\psi(t) \sim t^{-\mu}$ for $t \to \infty$ with $\mu = \alpha + 1$ and its Laplace transform is

$$\hat{\psi}(s) = \frac{1}{1 + (sT)^{\alpha}}$$

(43)

The central part of the distribution has an analytical expression given by [28,13]

$$P_C(x,t) = \frac{2 \left(1 - \varepsilon^2\right)^{\alpha-1} \sin \pi \alpha}{\left(1 - \frac{s}{sT}\right)^{2\alpha} + \left(1 + \frac{s}{sT}\right)^{2\alpha} + 2 \left(1 - \frac{s}{sT}\right)^{\alpha} \cos \pi \alpha}$$

(44)

Note that $P_C(x,t)$ is normalized. Similarly to the Poissonian case, we may write the expression for the probability density as

$$P_U^{(NP)}(x,t) \approx P(x + U(t),t).$$

(45)

Fig. 2 shows comparison of this solution to numerical simulations. Also we may evaluate the first and the second moment that, as for the poissonian case, are

$$\langle x \rangle = -U(t), \quad \langle x^2 \rangle \approx \langle x^2 \rangle_0 + U(t)^2,$n

(46)

with the quadratic correction in the potential negligible for limited non-divergent potentials.

Fig. 3. First moment of the distribution as a function of time in the case of potential $U(x) = \varepsilon x^\gamma$. Numerical simulation (dashed and continuous line) vs. expression Eqs. (10) and (19) (filled triangles and circles) for an exponential waiting times distribution $\psi(t) = \frac{2}{\Gamma(\gamma)} \varepsilon^{-\gamma/2}$ ($\gamma = 1$) and Mittag-Leffler distribution ($\alpha = 0.5, T = 1$) respectively.

### 5 Numerical simulation of the diffusion process

The diffusion process can be simulated by implementing numerically Eq. (2) and then deriving the correspond-
for an exponential waiting times distribution \( \psi(t) = \frac{\gamma e^{-\gamma t/2}}{2} \) with \( \gamma = 1 \) (triangles and squares) and for a Mittag-Leffler distribution with \( \alpha = 0.5, T = 1 \) (circles) are plotted and compared to numerical simulations (continuous lines). The bottom continuous line and square symbols refer to the analytical and numerical case with \( \varepsilon = 0.01 \) while the other curves correspond to the value \( \varepsilon = 0.1 \) used for Fig.1 and 2. Since the potential is not limited, the quadratic correction in Eqs. (41) and (10) is included in the middle curve (triangles), while for the bottom curves (smaller \( \varepsilon \), squares) and uppermost curve (circles) this correction is negligible compared to the unperturbed value for the second moment.

Fig. 4. Second moment of the distribution as a function of time in the case of potential \( U(x) = \frac{\varepsilon x^2}{2} \). Eqs. (41) and (10) using probability distribution as in the following. The dichotomic fluctuation \( \xi(t) = \pm w \) is obtained by generating the time intervals between each change of value of the variable \( \xi \) randomly with the specified waiting times distribution density \( \psi(\tau) \). Eq. (2) can be numerically integrated between two subsequent changes of value of the variable \( \xi(t) \) and the variable \( x \) accordingly updated. This procedure leads to a diffusion in the variable \( x \) and average over many realizations of the fluctuations of \( \xi \) allows one to evaluate the probability density at each time and position on the real axis. In the case of a Poissonian process, \( \psi(\tau) = \gamma e^{-\gamma \tau} \), the comparison of the numerical results with the solution given by Eq. (39) is shown in Fig. Fig. 2 shows the case of non-ergodic fluctuations, i.e. the comparison of the numerical with the analytical solution provided by Eq. (15) for a Mittag-Leffler distribution of waiting times \( \psi(\tau) \). In order to produce random numbers distributed according to a Mittag-Leffler function with parameters \( T \) and \( \alpha \) we adopt the same procedure as described in [32]: we first generate two numbers \( n_0, n_1 \) uniformly distributed between 0 and 1 and then convert them via the following transformation:

\[
\tau = -T \log(n_0) \left[ \frac{\sin(\pi \alpha)}{\tan(\pi \alpha_1)} - \cos(\pi \alpha) \right]^{\frac{1}{\alpha}}, \tag{47}
\]

the numbers so generated can be proved to be distributed exactly as a Mittag-Leffler function of parameters \( T \) and \( \alpha \).

Fig. 2 shows that also in this case the agreement is excellent confirming the validity of the approach followed. Finally we evaluated numerically the first and the second moment of the distribution, i.e. Eq. (10) and Eq. (11). Also in this case the agreement between theoretical and numerical is remarkable, even for the case of non-limited potential as shown in Figs. 3 and 4.

6 Concluding Remarks

We have studied a stochastic diffusion equation in presence of a potential \( U(x) \). Considering the case of an odd potential, \( U(-x) = -U(x) \) we showed that a first order approximation solution can be obtained by the unperturbed solution simply replacing the spatial variable \( x \) with \( x + U(t) \). We provided an analytical solution for the case of Poissonian fluctuations and, remarkably, for non-Poissonian fluctuations in the non-ergodic regime where the time scale of the fluctuations diverges. These solutions for the probability density of the diffusion are exact for small perturbation and can be generalized to provide a theoretical tool that can be employed in many fields of growing interest where anomalous diffusion emerges in physical [31] and biological systems [11] and compared to other approaches based e.g. on fractional calculus and Continuous Time Random Walk [32]. Interestingly, in the Poissonian case, our solution is an approximate solution of a system of coupled partial differential equations, Eqs. (34), with coefficient depending on the spatial variable \( x \). We supported our conclusions showing that the analytical solutions are in good agreement with numerical simulations, confirming that the derived solutions can be applied in all relevant cases of weakly driven anomalous diffusion produced by dichotomic fluctuations.

7 Acknowledgements

M.B. acknowledges financial support from FONDECYT project no 1110231.

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