The basis of the physical Hilbert space of lattice gauge theories

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Abstract

Non-linear Fourier analysis on compact groups is used to construct an orthonormal basis of the physical (gauge invariant) Hilbert space of Hamiltonian lattice gauge theories. In particular, the matrix elements of the Hamiltonian operator involved are explicitly computed. Finally, some applications and possible developments of the formalism are discussed.
1 Introduction

It is generally accepted that the most powerful non-perturbative method for the study of gauge theories is the lattice approach \[1\]. Although most work is performed using Lagrangian methods, the Hamiltonian point of view offers some advantages \[4\]. For instance, concepts like the Wilson-Polyakov confinement test \[3\], redefinitions of the action \[4\] and the vacuum structure are more easily and clearly formulated within the Hamiltonian, real-time, language.

In spite of the many advances along the lines of the Hamiltonian formulation, an important problem that remains is that the resulting basis functions, generated by the Wilson loops, are in general nonorthogonal and overcomplete. This overcompleteness arises from the well known Mandelstam identities \[5\]. This problem has been handled in different ways. For example, in reference \[6\] the inner product induced by the Haar measure (using Creutz’s algorithm \[7\]), has been used to find a loop basis of linearly independent loop states. More recently, such difficulty was tackled by exploiting the Cayley-Hamilton relations between matrices \[8\].

Here, we propose a different approach to describe the Hilbert space of lattice gauge theories. This is done by using harmonic analysis on a compact Lie group \(G\), as well as the formalism of intertwining operators. It is important to emphasize that such construction can be carried out in an abstract way using the properties of tensor categories, namely, the tensor category of the representations of \(G\). Consequently, the formalism can be generalized to the tensor categories of the representations of the quantum group \(SU_q(N)\). Since when \(q\) is an \(n^{th}\)-root of unity the number of the irreducible representation is finite, this could provide a natural way to obtain a finite dimensional model \(3\).

Very similar techniques have been used in the path integral approach \[10\], and to construct a q-deformed lattice gauge theory \[11\]. Our work generalizes the results obtained in \[12\] for the case of \(SU(2)\) lattice gauge theory in 2 + 1 dimensions, to arbitrary dimensions and compact gauge groups. Moreover, our procedure allows the explicit calculation of the matrix elements of the Hamiltonian.

One of the technical difficulties in manipulating the derived formulas is the proliferation of indexes, which turns out in quite cumbersome expressions. The best way to gain control of the formalism is to take advantage of the graphical methods available for performing recoupling calculus. The usefulness of graphical methods \[13\] in computations involving complicated (representation theory) tensor expressions has been particularly stressed by Citanović \[14\] (who was the first one that emphasized the non relevance of the particular basis, as well as the particular normalization chosen in the definition of Clebsh-Gordan coefficients and Wigner 3nJ-Symbol) and by Yutsis-Levinson-Vanagas \[13\]. These techniques have extensively been used in the context of loop quantum gravity (see for example \[15, 16, 17\] and reference therein).

The plan of the work is the following. Section 2 summarizes the specific theory we will be dealing with to calculate the matrix elements, although our results on the basis are completely general. Section 3 gives the basis for the gauge invariant states of the theory and also reviews the group theoretical methods needed to achieve such results. In section 4 we determine the matrix elements of the Hamiltonian operator in terms of contraction of intertwining operators for arbitrary dimension and arbitrary gauge group. Moreover, algebraic close form are explicitly derived for the \(SU(2)\) case in 2+1 and 3+1 dimensions. Section 5 discusses possible applications and developments, as well as some conclusions.

2 Hamiltonian lattice gauge theories

The Hamiltonian formalism for lattice gauge theories in \((d + 1)\) dimensions was introduced in \[2\]. To fix the notations, we are considering a lattice gauge theory defined on a \(Z^d\) periodic cubic \(d\)

\(^1\)We use the conventions of [3].
dimensional lattice with a continuum time. To make contact with the continuum theory, the lattice can
be visualized as a finite subset of points of a d-dimensional torus $T^d$ of principal radius $aL$ ($a \in \mathbb{R}^+$),
$Z^d = \{ \mathbf{x} \in T^d | \mathbf{x} = a(n_1 \mathbf{e}_1 + \ldots + n_d \mathbf{e}_d) \}$, defined by the versors $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ and $(n_1, \ldots, n_d) \in Z^d_l$. This implies that we have $N_v = L^d$ vertices, and $N_{lk} = d \cdot L^d$ links and $N_P = \frac{(d-1)}{2} \cdot L^d$ plaquettes.

For each lattice link $(\mathbf{x}, \mathbf{x} + a \mathbf{e}_k)$, one has a gauge field variable $U_k(\mathbf{x}) \in G$, where $G$ is a (compact) group (e.g., $SU(N)$). The variables conjugated to the link variables are the outgoing (ingoing) electric fields $E_\alpha^{\alpha}_{+k}(\mathbf{x})$. More precisely, $E_\alpha^{\alpha}_{+k}(\mathbf{x})$ denotes the electric fields outgoing (ingoing) from the lattice point $\mathbf{x}$ in the directions $\mathbf{e}_k$. They fulfill the commutation relations

\[ [E^{\alpha}_{+k}(\mathbf{x}), U_j(\mathbf{y})] = \delta_{\alpha j} \delta^3_{x,y} U_j(\mathbf{y}) T^\alpha , \]
\[ [E^{\alpha}_{-k}(\mathbf{x} + a \mathbf{e}_k), U_j(\mathbf{y})] = -\delta_{\alpha j} \delta^3_{x,y} T^\alpha U_j(\mathbf{y}) , \]

where $T^\alpha$ are the hermitian generators of $G$ in the fundamental representation.

The generators of the local gauge transformations are:

\[ \mathcal{G}^\alpha(\mathbf{x}) = D_k E^\alpha_k(\mathbf{x}) = \sum_{k=1}^{d} (E^{\alpha}_{+k}(\mathbf{x}) + E^{\alpha}_{-k}(\mathbf{x})) . \]

Notice that each particular gauge transformation acts on $U_k(\mathbf{x}) \in G$ according to $(\gamma(\mathbf{x}) \in G)$

\[ U_k(\mathbf{x}) \rightarrow U_k^\gamma(\mathbf{x}) = \gamma^{-1}(\mathbf{x} + a \mathbf{e}_k) U_k(\mathbf{x}) \gamma(\mathbf{x}) . \]

In discussing Hamiltonian lattice gauge theories, it is important to consider two Hilbert spaces. The
first one is the Hilbert space $H_{aux}$ composed by the square integrable functions $\psi(U) = \psi(\{U_k(\mathbf{x})\})$, with respect to the unique normalized Haar measure on the group $G$

\[ \int [dU] |\psi(U)|^2 < \infty , \quad [dU] = \prod_{\mathbf{x}, k} dU_k(\mathbf{x}) , \]

while the second is the physical Hilbert space of the theory $\mathcal{H} \subset H_{aux}$ composed by the gauge invariant square integrable functions

\[ \psi(U) = \psi(U^\gamma) = \psi(\{U_k^\gamma(\mathbf{x})\}) . \]

The Hamiltonian (defined on the two Hilbert spaces $\mathcal{H}$ and $H_{aux}$) is

\[ \hat{H} = \frac{1}{a} \left\{ \frac{g^2}{2} \sum_{\mathbf{x}, k} \sum_{\alpha \beta} q_{\alpha \beta} E^\alpha_k(\mathbf{x}) E^{\beta_k}(\mathbf{x}) + \sum_{P} V(U_P) \right\} = H_E + H_B \]

where $q_{\alpha \beta}$ is the Cartan metric, the sum over $P$ in equation (3) ranges over all unoriented plaquettes in the lattice space, and

\[ V(U_P) = \frac{1}{g^2} \left[ 1 - U_P + \frac{U_P^*}{2 \dim(U)} \right] \]

where $U_P$ is the usual plaquette variable defined by

\[ \begin{align*}
U_p &= U_{x, k, l} = \text{Tr}[U^{-1}_l(\mathbf{x}) U^{-1}_k(\mathbf{x} + a \mathbf{e}_l) U_l(\mathbf{x} + a \mathbf{e}_k) U_k(\mathbf{x})] \\
&= \text{Tr}[U^{-1}_l(\mathbf{x}) U^{-1}_k(\mathbf{x} + l) U_l(\mathbf{x} + k) U_k(\mathbf{x})] .
\end{align*} \]

\[ ^2\text{The link with the gauge fields variable} \ A^\alpha_i(x) \text{ is as follow. Consider the segment} \ x'(\lambda) = (1 - \lambda)x' + \lambda(x' + a \mathbf{e}_k), \text{then} \ U_k(\mathbf{x}) = \mathcal{P} \exp \left[ \int_0^1 d\lambda A^\alpha_i(\gamma(\mathbf{x})) \frac{d^3x}{dx} \right] \text{where} \ \mathcal{P} \ \text{exp} \text{ is the path order exponential.} \]

\[ ^3\text{We introduce the notations} \ x \pm k = x \pm a \mathbf{e}_k . \text{We will also use} \ x \pm k \pm l = x \pm a \mathbf{e}_k \pm a \mathbf{e}_l. \]
In this work, we will limit the analysis to the standard form of the magnetic term derived from the Wilson action \[8\]. Such choice is not unique. The only condition on the magnetic term potential \(V(U_P)\) is that it has to fulfill the requirement \(V(U_P) \simeq \frac{g^2}{2} \text{Tr}[F_{\alpha}^{\beta}]\). For example, a very interesting alternative definition is the use of the so called Heat-Kernel potential \[7\]. The most general one plaquette potential \(V(U)\) can be written as

\[
V(U) = \sum_{f} c_f (\chi_f[U] + \chi_f[U]^*),
\]

where the sum spans over all the irreducible representations of the group and \(\chi_f\) are the associated character functions\[\text{[1]}\]. To deal with the general case one has to determine the matrix elements of \(\chi_f[U]\). In our formalism such generalization is straightforward, it simply amounts to replace the 1 (that denotes the fundamental representation) in equation (26) with \(f\).

3 The gauge invariant basis of the physical Hilbert space

3.1 Fourier analysis on compact groups: Peter-Weyl theorem

A classical result of group theory is that the set \(\mathcal{RG} = \{\mathcal{R}^j \mid j \in J[G]\}\) of all the matrix elements of irreducible inequivalent unitary representations of \(j\) of the group \(G\) is a complete orthogonal basis on the Hilbert space \(L^2[G, dU]\). This result, known as the Peter-Weyl theorem \[19\] implies that any \(f(U) \in L^2[G, dU]\) can be expanded as

\[
f(U) = \sum_{j \in J[G]} \sum_{\alpha \beta} c_{j\alpha\beta}^j D_{j\alpha\beta}^{j}(U),
\]

where there is the following orthogonality property between the matrix functions of the irreducible representation of the group:

\[
\int dU \overline{D_{j\alpha\beta}^{j}(U)} D_{j'\alpha'\beta'}^{j'}(U) = \begin{cases} 1 & \text{if } j = j' \text{ and } \alpha = \alpha', \beta = \beta' \\ 0 & \text{otherwise} \end{cases},
\]

where \(\mathcal{H}^j\) is the Hilbert space on which the representation is defined. The general rule for performing integration over the group yields zero unless the integrand transforms as the trivial representation. Clearly this construction straightforward generalizes to functions on the Cartesian product of \(N_{lk}\) copies of the gauge group. In this way, the most general vector of \(\mathcal{H}_{aux}\) can be written as:

\[
\psi(U) = \prod_{k=1}^{d} \prod_{j_k \in J[G]} \sum_{\alpha_k, \beta_k}^{\dim(j_k)} D_{j_k\alpha_k\beta_k}^{j_k}(U) \times c_{(\alpha_{1} \ldots \alpha_{N_{lk}} \beta_{1} \ldots \beta_{N_{lk}})} \]

At this point the Peter-Weyl theorem has given us a complete characterization of \(\mathcal{H}_{aux}\). Notice that we are not interested in \(\mathcal{H}_{aux}\), but in its gauge invariant subspace. The implementation of the gauge invariance \[\text{[2]}\] turns out to be “simply” a restriction of the possible forms of \(c\)'s. In the next subsection we will see that the condition of gauge invariance imposes the condition that the \(c\)'s must be group invariant tensors.

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\(4\) The character of the unitary representation \(D^{j}_{\alpha\beta}(U)\) of a group is its trace \(\chi_f[U] = D^{\alpha\beta}_{\alpha\beta}(U)\).

\(5\) We denote by \(\mathcal{H}'\) the Hilbert space in which the irreducible representation is realized in terms of unitary operator \(T^{j}(U)\) and we suppose that a preferred orthonormal basis has been chosen. We will denoted by \(D^{\alpha\beta}_{\alpha\beta}(U), (\alpha, \beta = 1, \ldots, \dim(\mathcal{H}))\) the matrix elements of \(T^{j}(U)\) in this preferred basis. Moreover we will use \(\mathcal{R}'\) to denote the adjoint representation, defined on the same Hilbert space \(\mathcal{H}'\), of matrix elements \(D^{j}_{\alpha\beta}(U) = D^{\alpha\beta}_{\alpha\beta}(U^{-1}) = D^{\alpha\beta}_{\alpha\beta}(U)\)
3.2 Invariant tensors: the basis of the gauge invariant Hilbert space

The concept of invariant tensor is better expressed by the notions of intertwining operators. By definition, an operator \( I \) between the Hilbert space of two representations, \( \mathcal{R} \) and \( \mathcal{R}' \) of \( G \), is an intertwining operator if \( I \) is a bounded operator from \( \mathcal{H} \) to \( \mathcal{H}' \) such that \( I \cdot T(U) = T'(U) \cdot I \), \( \forall U \in G \). Now, the set of all the intertwining operators \( I(\mathcal{R}, \mathcal{R}') \) is a vector subspace of the space of bounded linear operators.

Moreover, we have the following properties between the spaces of intertwining operators: \( I(\mathcal{R} \otimes \mathcal{R}'', \mathcal{R}'') = I(\mathcal{R}, \mathcal{R}' \otimes \mathcal{R}'') \) and \( I(\mathcal{R}, \mathcal{R}'') \) and \( I(\mathcal{R}', \mathcal{R}) \) are anti-isomorphic vector spaces. Using this duality it is natural to use the trace function to induce an Hilbert space structure on \( I(\mathcal{R}, \mathcal{R}') \). In fact, \( \text{Tr}[I_1 I_2] \) makes perfect sense when \( I_1 \in I(\mathcal{R}, \mathcal{R}') \) and \( I_2 \in I(\mathcal{R}', \mathcal{R}) \). In terms of intertwiners, we have the following integration formula for the direct product of \( K \) representations:

\[
\int dU \prod_{k=1}^{K} D(\hat{\rho}_{i_k}^{\alpha_k} \beta_k)(U) = \sum_{\pi} \frac{1}{|\pi|} \prod_{k=1}^{K} |\pi(1,j_{\hat{\rho}_k})\rangle |\pi(\beta_k,\gamma_k)\rangle \int dU \prod_{k=1}^{K} D(\hat{\rho}_{i_k}^{\alpha_k} \beta_k)(U)
\]

(14)

where \( |\pi(1,j_{\hat{\rho}_k})\rangle \in I(\mathcal{R} \otimes \mathcal{R}' \otimes \mathcal{R} \otimes \mathcal{R}) \) and \( |\pi(1,j_{\hat{\rho}_k})\rangle \in I(\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R}) \) is its adjoint intertwiner.

Having fixed the notations we are now ready to impose the condition of gauge invariance to a generic vector of \( \mathcal{H}_{aux} \):

\[
\psi(U) = \psi(U^{-1}(x)) = \psi\{U_k^{-1}(x) = \psi(\{\gamma^{-1}(x + a\mathbf{e}_k)U_k(x)\gamma(x)\})
\]

(15)

This gives conditions on the possible form of the generalized Fourier transform coefficients \( c(j_{\hat{\rho}_0} \cdots j_{\hat{\rho}_{N_{lk}}} \alpha_0 \cdots \alpha_{N_{lk}}) \). It implies that the coefficients of the generalized Fourier transform of a gauge invariant function are invariant tensors under the transformation of the gauge group associated to the vertex of the lattice. To see this we can rewrite equation \( (15) \) by collecting all the terms depending on the gauge transformation at the vertex \( x \). The condition of gauge invariance implies that at each vertex \( x \), the following equation must be satisfied

\[
c(j_{\hat{\rho}_0} \cdots j_{\hat{\rho}_{N_{lk}}} \alpha_0 \cdots \alpha_{N_{lk}}) = \prod_{k=1}^{d} \sum_{a_k} \sum_{\beta_k} \left[ D(\hat{\rho}_{i_k}^{a_k} \beta_k)(x) \right] \times c(j_{\hat{\rho}_0} \cdots j_{\hat{\rho}_{N_{lk}}} \alpha_0 \cdots \alpha_{N_{lk}})
\]

The previous expression involves a large number of indices (like the ones involved in the previous decomposition of our Hilbert space). They are very cumbersome to write down, even though the concept they express is simple. For example, previous equation expresses that in the case of gauge invariant Hilbert space the \( c \)’s should be proportional to the generalized Clebsh-Gordan coefficients of all the representations associated to the links connected to the lattice site \( x \). That means,

\[\text{The generalized Clebsh-Gordan coefficients of Yutis-Levinson-Vanagas are just the matrix elements of these operators on the privileged basis (introduced in the Hilbert space of the representations).}\]

\[\text{We are assuming that this space is finite dimensional. This is exactly the case when the two representations \( R \) and \( R' \) are finite dimensional.}\]
that they must be proportional to the matrix elements of an intertwining operator. Since this apply
all the lattice site we have that

\[
\xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} (\psi_{\{\beta_0, \ldots, \beta_{\mathcal{K}}\}} | \xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} ) = \prod_{x} \sum_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} \xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} \left( \prod_{x} \sum_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} \xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} \right)
\]

where with \( \xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} \) we have denoted the (a choice of a possible) basis vectors, labeled by the index \( \pi_\mathcal{K} \), of the space of the intertwiner operators \( \mathcal{I}(\otimes_{k=1}^{d} \mathcal{A}^{j_{\mathcal{K}}}_{k}, \otimes_{k=1}^{d} \mathcal{A}^{j_{\mathcal{K}}}_{k}) \) at each lattice site \( x \). More specifically we will use the notation

\[
\xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} | \xi_{\{\beta_0, \ldots, \beta_{\mathcal{K}}\}} \rangle = \langle \xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} | \xi_{\{\beta_0, \ldots, \beta_{\mathcal{K}}\}} \rangle
\]

for their explicit matrix elements and we will denote with

\[
\xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} (\xi_{\{\beta_0, \ldots, \beta_{\mathcal{K}}\}} ) = \langle \xi_{\{\alpha_0, \ldots, \alpha_{\mathcal{K}}\}} | \xi_{\{\beta_0, \ldots, \beta_{\mathcal{K}}\}} \rangle
\]

the complex conjugate (adjoint) intertwining operator.

This result implies that the \( c \)'s coefficients factorize in a product of \( c \)'s, one for each vertex of the lattice. Summarizing, the application of the Peter and Weyl theorem and the imposition of gauge invariance gives the following description of the Hilbert space \( \mathcal{H} \) in terms of the orthogonal, so called spin-network, basis:

\[
\psi(U) = \sum_{\mathcal{J}_\pi} c_{\mathcal{J}_\pi} \psi_{\mathcal{J}_\pi}(U) = \prod_{x} \prod_{j=1}^{d} \sum_{\pi_x} \psi_{\mathcal{J}_\pi}(U)
\]

where the sum over the \( \pi_x \) ranges over a complete labeling of the basis of the intertwiners \( \mathcal{I}(\otimes_{k=1}^{d} \mathcal{A}^{j_{\mathcal{K}}}_{k}, \otimes_{k=1}^{d} \mathcal{A}^{j_{\mathcal{K}}}_{k}) \).

The spin network basis elements are the following gauge invariant functions:

\[
\psi_{\mathcal{J}_\pi}(U) = \prod_{x} \prod_{j=1}^{d} \sum_{\pi_x} D^{\hat{\alpha}_j}_{\hat{\beta}_j}(U^j_k(x)) \cdot \xi_{\{\alpha_{\mathcal{K}}, \ldots, \alpha_{\mathcal{K}}\}} (\xi_{\{\beta_{\mathcal{K}}, \ldots, \beta_{\mathcal{K}}\}} )
\]

Using the unitarity of the representations \( D^{\hat{\alpha}_j}_{\hat{\beta}_j}(U) \) we have that the complex conjugated elements is given by:

\[
\psi_{\mathcal{J}_\pi}(U) = \prod_{x} \prod_{j=1}^{d} \sum_{\pi_x} D^{\hat{\alpha}_j}_{\hat{\beta}_j}(U^{\dagger}_k(x)) \cdot \xi_{\{\alpha_{\mathcal{K}}, \ldots, \alpha_{\mathcal{K}}\}} (\xi_{\{\beta_{\mathcal{K}}, \ldots, \beta_{\mathcal{K}}\}} )
\]

For the computation of the magnetic field term, we need the following integrals (with \( j = 1 \) we denote the defining representation, i.e., \( D^{0}_{\alpha}(U) = U^{0}_{\alpha} \))

\[
\int dU \ D^{\hat{\alpha}_j}_{\hat{\beta}_j}(U) D^{0}_{\alpha}(U) D^{\hat{\gamma}_{\alpha}}_{\hat{\beta}_{\alpha}}(U^{-1}) = \sum_{\pi} | \pi \rangle \langle \pi | \ D^{\hat{\alpha}_j}_{\hat{\beta}_j}(U) D^{\hat{\gamma}_{\alpha}}_{\hat{\beta}_{\alpha}}(U^{-1})
\]

\[
\int dU \ D^{\hat{\alpha}_j}_{\hat{\beta}_j}(U) D^{0}_{\alpha}(U^{-1}) D^{\hat{\gamma}_{\alpha}}_{\hat{\beta}_{\alpha}}(U) = \sum_{\pi} | \pi \rangle \langle \pi | \ D^{\hat{\alpha}_j}_{\hat{\beta}_j}(U) D^{\hat{\gamma}_{\alpha}}_{\hat{\beta}_{\alpha}}(U^{-1})
\]
The presence of the $\sum_{\pi}$ over a basis of possible three valent intertwiners denotes the fact that for groups of ranks greater than 1 a given representation can appear more than once in the tensor product of two representations. This is not the case, as it is well known, for the $SU(2)$ group. In this case no sum over $\pi$ appears.

Summarizing, our characterization of the Hilbert space of Lattice Gauge Theories require:

1. The determination and description of the set of all the unitary inequivalent representations of the group $G$. By this, we mean their explicit form and the construction of a unique indexing of them, i.e., the complete knowledge of the set $\mathcal{RG} = \{R^j \mid j \in J[G]\}$.

2. The description of the space of all the intertwining operators $I = I(R^{j_1} \otimes \ldots \otimes R^{j_n}, R^{i_1} \otimes \ldots \otimes R^{i_m})$, and the determination of a basis $|\psi\rangle$ on it ($l \in I \leftrightarrow l = \sum c_{ij} |\psi\rangle$). This involves the decomposition of an arbitrary representation in the tensor product of irreducible representations.

In the follow we will see that in order to compute the matrix element the only explicit function we will need are: (a) the value of the quadratic Casimir invariant on the $R^j$ representation: $C_2[j]$; (b) the explicit values of the contraction of arbitrary intertwiner matrix. Without entering in details, extensively treated in literature [13, 14, 20, 21], we want to emphasize that the computation of the trace of intertwiner matrices can be always reduced to the computation of the sum of product of Wigner’s $6J$-symbols. Such elements, in particular, are very well known for the $SU(2)$ group (see for example [13]), and a quite extensive bibliography and collection of results exist for other compact groups. See for example: [22] for $SU(3)$, [23] for $SU(N)$ and [21, 24] for general overviews of known results.

4 The matrix elements of the Hamiltonian operator

We can perform the computation of the action of the Hamiltonian operator (7) on the spin-networks basis (19). In fact, the basis vectors (20) are eigenstate of the kinetic term $H_E$, while the potential (magnetic) term is realized as a multiplicative operator, i.e.:

$$\langle \vec{j}', \vec{\pi}' | \hat{H} | \vec{j}, \vec{\pi} \rangle = \left( \frac{g^2}{2a} \sum_{x} \sum_{x=1}^{d} C_2[j_x^2] + \frac{1}{ag^2} N_P \right) \langle \vec{j}', \vec{\pi}' | \vec{j}, \vec{\pi} \rangle$$

$$- \frac{1}{ag^2} \frac{2 \dim(U)}{\sum_{y} \sum_{r,s=1}^{d} (\langle \vec{j}', \vec{\pi}' | U_{y,r,s} | \vec{j}, \vec{\pi} \rangle + \langle \vec{j}, \vec{\pi} | U_{y,r,s} | \vec{j}', \vec{\pi}' \rangle)}$$

where the only non diagonal terms is given by the expectation values of the plaquette operator.

4.1 The plaquette operator

Using the integrals (22-23), it is straightforward to compute the matrix elements of the plaquette operators in the spin network basis. The final result, expressed as traces in the intertwiner spaces
associated to each vertex, is

\[
\langle \mathcal{J}', \pi' | U_{y+r,s} | \mathcal{J}, \pi \rangle = \int \prod_{x,k} dU_k(x) \psi_{\mathcal{J}', \pi'}(U) \psi_{\mathcal{J}, \pi}(U) U^{-1}_{y+r_1} U^{-1}_{y+r_3} (y + a e_s) U^{-1}_{y+r_2} (y + a e_r) U_{y+r} (y)
\]

\[
= \int \prod_{x,k=1}^{d} \left[ dU_k(x) \sum_{\alpha_k^x} \sum_{\beta_k^x} D(\alpha_k^x, \beta_k^x) (U^{-1}(x)) D(\alpha_k^x, \beta_k^x) (U_k(x)) \right] \cdot U_{y+r_1} (y + a e_s) U_{y+r_2} (y + a e_r) U_{y+r} (y)
\]

\[
\cdot U_{y+r_1} (y + a e_s) U_{y+r_2} (y + a e_r) U_{y+r} (y) [\prod_{x,k}^{d} \sum_{\alpha_k^x, \beta_k^x} D(\alpha_k^x, \beta_k^x) (U^{-1}(x)) D(\alpha_k^x, \beta_k^x) (U_k(x))] \cdot \left[ \prod_{x,k=1}^{d} \sum_{\alpha_k^x, \beta_k^x} D(\alpha_k^x, \beta_k^x) (U^{-1}(x)) D(\alpha_k^x, \beta_k^x) (U_k(x)) \right]
\]

\[
(25)
\]

Therefore, the result reduces to the computation of the trace of the intertwiner matrix. This can be done by using equations (12, 23) and (23). This result shows that the choice of an explicit basis is irrelevant, as expected. All the indices of the intertwiner matrix elements are traced over their complex conjugate, except the contractor in the lattice points \( y, y + r, y + s \) and \( y + r + s \). The corresponding matrix elements are given by:

\[
\langle \mathcal{J}', \pi' | U_{y+r,s} | \mathcal{J}, \pi \rangle = \prod_{x,k=1}^{d} \sum_{\alpha_k^x, \beta_k^x} D(\alpha_k^x, \beta_k^x) (U^{-1}(x)) D(\alpha_k^x, \beta_k^x) (U_k(x)) \cdot \left[ \prod_{x,k=1}^{d} \sum_{\alpha_k^x, \beta_k^x} D(\alpha_k^x, \beta_k^x) (U^{-1}(x)) D(\alpha_k^x, \beta_k^x) (U_k(x)) \right]
\]

\[
\cdot \left[ \prod_{x,k=1}^{d} \sum_{\alpha_k^x, \beta_k^x} D(\alpha_k^x, \beta_k^x) (U^{-1}(x)) D(\alpha_k^x, \beta_k^x) (U_k(x)) \right]
\]

\[
(26)
\]
where the quantities inside the square brackets [...] come from the group integrations. Notice that once the intertwining matrices are specified, i.e., when the Clebsh-Gordan coefficients are explicitly given, the matrix elements are known. The only non trivial part in this computation is the choice of a convenient basis for the intertwining matrices. A natural choice is to use an orthonormal basis, i.e.,

$$\sum_{\alpha, k_1, k_2 = 1}^{\dim(j_k)} |[x] \langle j_{k-1}^1, \ldots, j_{k-d}^d | \beta_{k_1}^1, \ldots, \beta_{k_2}^d \rangle \cdot |[x]^\prime \langle j_{k-1}^1, \ldots, j_{k-d}^d | \alpha_{k_1}^1, \ldots, \alpha_{k_2}^d \rangle |^2 = \delta_{|[x],[x]^\prime}.$$ (27)

In this way we have reduced the problem of the computation of the matrix elements of the plaquette operator to the computation of the trace of intertwining operators, i.e., of the trace of generalized Clebsh-Gordan coefficients. Now, it is well known that this is nothing more than the evaluation of specific Wigner’s $nJ$-symbols and that the evaluation of a Wigner’s $nJ$-symbols can be always reduced to the computation of a Wigner’s $6J$-symbol.

This means that what we really need to explicitly compute the matrix elements of the plaquette operator is just the knowledge of all the representations of the group $G$ and of the associated Wigner’s $6J$-symbol. In the case of the $SU(2)$ group these elements are known and standard references for this kind of computation are [13] and [20]. A useful convention for doing this is given by Penrose’s binor calculus, where the distinction between the representations $R^j$ and $\overline{R}^j$ completely disappears [16].

### 4.2 Matrix elements of the plaquette operator for $SU(2)$ theory in $2+1$ and $3+1$ dimensions

When the gauge group involved is $SU(2)$ the set of all unitary irreducible representations (labelled by spins) and the space of all the intertwining operators (generalized Clebsh Gordan) are well known.
In general, a basis on the space of intertwiners can be specified by $2d - 3$ additional virtual spins (see [13] chapter II or [16]). That means that in 2+1 dimensions it is necessary to specify three spins $j_x^1, j_x^2, \pi_x^1 \in J[SU(2)]$ to each lattice point $x$, while in 3+1 dimensions it is necessary to specify six spins $j_x^1, j_x^2, j_x^3, \pi_x^1, \pi_x^2, \pi_x^3 \in J[SU(2)]$, to each $x$. Consequently, the spin network basis in dimension $d = 2$ is

$$|\vec{j}; \vec{\pi}\rangle = \prod_x D^{(j_1^1 \alpha_1^1_- \beta_1^1_-)}_{x}(U_1(x)) D^{(j_2^2 \alpha_2^2_- \beta_2^2_-)}_{x}(U_2(x)) \cdot g^{j_1^1 \beta_1^1}_x g^{j_2^2 \beta_2^2}_x \cdot \left( \begin{array}{c} j_x^1 \pi_x^1 \\ \alpha_x^1 \beta_x^1 \gamma_x^1 \end{array} \right)$$

while in dimension $d = 3$ is given by:

$$|\vec{j}; \vec{\pi}\rangle = \prod_x D^{(j_1 \alpha_1 \beta_1)}_{x}(U_1(x)) D^{(j_2 \alpha_2 \beta_2)}_{x}(U_2(x)) D^{(j_3 \alpha_3 \beta_3)}_{x}(U_3(x)) \cdot g^{j_1 \alpha_1 \beta_1}_x g^{j_2 \alpha_2 \beta_2}_x g^{j_3 \alpha_3 \beta_3}_x \cdot \left( \begin{array}{c} j_x^1 \pi_x^1 \\ \alpha_x^1 \beta_x^1 \gamma_x^1 \end{array} \right) \left( \begin{array}{c} j_x^2 \pi_x^2 \\ \alpha_x^2 \beta_x^2 \gamma_x^2 \end{array} \right) \left( \begin{array}{c} j_x^3 \pi_x^3 \\ \alpha_x^3 \beta_x^3 \gamma_x^3 \end{array} \right)$$

where $\begin{pmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{pmatrix}$ and $g^{m_1 m_1}_{j}$ are the standard Wigner 3J symbol and the group metric on the irreducible representation $j$. A straightforward direct computation shows that the norm of this states is given by:

$$\sqrt{\langle \vec{j}; \vec{\pi} | \vec{j}; \vec{\pi} \rangle} = \prod_x \left( \prod_{k=1}^{d-3} \frac{1}{\dim(j_x^k)} \right) \left( \prod_{k=1}^{2d-3} \frac{1}{\dim(\pi_x^k)} \right)$$

Using the explicit values of the $SU(2)$ Clebsh-Gordan coefficients, the matrix elements of the Hamiltonian in equation (27) can be computed. The calculation and the resulting expressions in two and three dimensions are almost identical. The three dimensional case has simply more indices to sum upon. The prototype of the generic computation is indeed the two dimensional one. The expression
\[ \langle j', \pi' | U_{Y,1,2} | j, \pi \rangle = \prod_{(x,k) \neq (y,2), (y,2) \neq \{y+1,2\}, (y+2,1), (y,1), (y+1,2)} \frac{\delta_{j', k'}^{j, k}}{\text{dim}(j'_{x,y})} \cdot \prod_{x \neq y+1, y+2} \frac{\delta_{\pi', \pi}^{\pi_{x,y}}}{\text{dim}(\pi_{x,y})} \]  

(31)

where

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = 6 \prod_{k=1}^{j_k} \left( \sum_{m_k, n_k = -j_k}^{j_k} g_{(j_1)}^{m_1 n_1} g_{(j_2)}^{m_2 n_2} g_{(j_3)}^{m_3 n_3} g_{(j_4)}^{m_4 n_4} g_{(j_5)}^{m_5 n_5} g_{(j_6)}^{m_6 n_6} \right) \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} j_4 & j_5 & j_6 \\ n_1 & n_5 & n_6 \end{array} \right) \left( \begin{array}{ccc} j_4 & j_2 & l_6 \\ n_4 & n_3 & n_6 \end{array} \right) \]  

(32)

is the Wigner 6J-symbol. From the previous expression follows that the only elements different from zero are those with equal \( j \) and \( \pi \) and unequal \( X \) and \( Y \) (see figure 2). Indeed they must differ exactly by \( \epsilon_i = X^i - Y^i = \pm \frac{1}{2} \) for \( i = 1, ..., n \) (\( n = 6 \) for \( d = 2 \) and \( n = 12 \) for \( d = 3 \)). The explicit values of the matrix elements are

\[
\frac{\langle j', \pi' | U_{Y,1,2} | j, \pi \rangle}{\sqrt{\langle j' | \pi' \rangle < j | \pi \rangle}} = (-1)^{\sum_{i=1}^{n} (|\epsilon_i - \epsilon_{i+1}| + \frac{C_i}{2})} \prod_{i=1}^{n} \left[ \frac{X^i_{Y^i} X^{i+1}_{Y^{i+1}}}{Y^i_{Y^i} Y^{i+1}_{Y^{i+1}}} , C^i_{Y^i} \right] \]  

(33)

where

\[
R \left[ \frac{X^i_{Y^i} X^{i+1}_{Y^{i+1}}}{Y^i_{Y^i} Y^{i+1}_{Y^{i+1}}} , C^i_{Y^i} \right] = \frac{1 - 2C^i_{Y^i} + X^i_{Y^i} + Y^{i+1}_{Y^{i+1}}} {2} Y^i_{Y^i} + \frac{3 + 2C^i_{Y^i} + X^i_{Y^i} + Y^{i+1}_{Y^{i+1}}} {2} Y^i_{Y^i} + \frac{Y^{i+1}_{Y^{i+1}}}{2} \]  

\[
1 + 2C^i_{Y^i} + X^i_{Y^i} - Y^{i+1}_{Y^{i+1}} + Y^i_{Y^i} - Y^{i+1}_{Y^{i+1}} \]  

if \( |\epsilon_i - \epsilon_{i+1}| = 0 \)

\[
\frac{1}{2} C^i_{Y^i} + X^i_{Y^i} - Y^{i+1}_{Y^{i+1}} - Y^i_{Y^i} + Y^{i+1}_{Y^{i+1}} \]  

if \( |\epsilon_i - \epsilon_{i+1}| = 1 \)

(34)

5 Numerical vs. Analytic solutions

In the previous sections we have shown how to map Lattice Gauge Theories (based on arbitrary compact group) onto a well defined “classical” problem of quantum mechanics. The problem of the determination of the particle content of the theory and of the \( \beta \)-function (imposing scaling conditions on the energy gaps) is thus solved once the eigenvalues-eigenvectors of the theory are known. Of course, this doesn’t mean that we can straightforwardly give such solution.

The main difficulties are two. First, the imposition of the gauge invariance conditions, i.e., the labeling of the links with inequivalent irreducible representations and of the vertices with a basis of intertwining operators, give constraints which become harder to deal with increasing volume. Since one has to simultaneously satisfy all of them, and their interplay depends on the boundary conditions, the determination of the explicit basis in infinite (physical) volume is, although possible, not an easy task. Second, the associated spectral problem is not of obvious solution, and there is no general algorithm which can diagonalize a given hermitian operator in a reasonable time.
One can then follow two main guidelines. The first is to exploit “brute-force” numerical solutions in finite volume $V$ (IR-cutoff), considering a finite dimensional subspace of the full Hilbert space, e.g., restricting the analysis to a maximum allowed spin $\Lambda$ on each lattice site (UV-cutoff). This procedure has the drawback not to respect group symmetries. A similar, group invariant, cutoff can be implemented considering a $q$-deformed gauge group \cite{9}. In fact, when $q$ is chosen such that $q^n = 1$, the number of irreducible representations is finite ($n = \Lambda + 2$) \cite{25}. The main problem with such approaches is that the dimension of the Hilbert space grows very rapidly with $\Lambda$ and $V$. A rough estimation gives, in $d$ dimensions, \( \text{dim}(\Lambda, V) \approx k \Lambda^{2d-3} V \), thus making extrapolations to physical volume unlikely to be obtained. The second approach is to study the general properties of the problem at hand, implementing symmetries and determining other conserved quantities (i.e. observables commuting with the Hamiltonian). This in order to give at least constraints on the spectrum and on the form of the eigenstates, thus reducing the set of states to be considered in the diagonalization procedure. Of course, a combination of analytical and numerical techniques seems the most sensible thing to do.

As a final remark, let us show, for the sake of consistency, that we correctly recover the known solutions for the vacuum state in the two limits $g \to \infty$ and $g \to 0$. In fact, can be straightforwardly checked that the vectors
\[
|0; g, a, L\rangle = \sum_{\vec{j}, \vec{\pi}} c_{\vec{j}, \vec{\pi}}^{(0; g, a, L)} |\vec{j}; \vec{\pi}\rangle \sqrt{\langle \vec{j}; \vec{\pi}|\vec{j}; \vec{\pi}\rangle}, \tag{35}
\]
where
\[
c_{\vec{j}, \vec{\pi}}^{(0; \infty, a, L)} = \prod_x \prod_k \delta_{j^k_x 0} \delta_{\pi^k_x 0} \tag{36}
\]
\[
c_{\vec{j}, \vec{\pi}}^{(0; 0, a, L)} = \prod_x \prod_k \sqrt{\text{dim}(j^k_x)} \text{dim}(\pi^k_x) \tag{37}
\]
are eigenvectors of eigenvalues zero of the positive defined Hamiltonian operator i.e., are proportional to the vacuum. Notice that the first result is trivial. The second, on the other hand, is a consequence of the character expansion of the Dirac-$\delta$ function on the group and of the Bidenharm-Eliot identity.

6 Outlook and conclusion

In this work we showed that Peter-Weyl theorem gives a complete characterization of the physical (gauge invariant) Hilbert space of pure lattice gauge theories for any compact gauge group. The characterization is made in term of the the full set of irreducible representations of the group and of a complete basis of the space of intertwining operators. Notice that such formalism can be straightforwardly generalized to full gauge theories, where one can associate a particle transforming according to the local gauge group at each vertex \cite{3}. In particular, we showed that, once a particular basis on the spaces of intertwiner operator is selected, the matrix elements of the Hamiltonian operator \cite{24, 26} are well defined intertwiner contractions, i.e, can be expressed in terms of the the evaluation of Wigner’s $nJ$-symbol of the group $G$. Moreover, in the case of $SU(2)$ gauge group in 2+1 and 3+1 dimension we derived a complete algebraic expression for such matrix elements \cite{33}. These are our main results.

Recent results in group theory \cite{21, 24} allow us to think that the case of $SU(3)$ gauge group is not hopeless.

Extensions of such formalism to other theories, such as 2-dimensional $\sigma$-models and $CP^N$ models are under investigations \cite{25}.
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