Existence of closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$

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Abstract In this paper, we prove there exist at least $\lfloor \frac{n+1}{2} \rfloor + 1$ geometrically distinct closed characteristics on every compact convex hypersurface $\Sigma$ in $\mathbb{R}^{2n}$, where $n \geq 2$. In particular, this gives a new proof in the case $n = 3$ to a long standing conjecture in Hamiltonian analysis. Moreover, there exist at least $\lfloor \frac{n}{2} \rfloor + 1$ geometrically distinct non-hyperbolic closed characteristics on $\Sigma$ provided the number of geometrically distinct closed characteristics on $\Sigma$ is finite.

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1 Introduction and main results

Let $\Sigma$ be a fixed $C^3$ compact convex hypersurface in $\mathbb{R}^{2n}$, i.e., $\Sigma$ is the boundary of a compact and strictly convex region $U$ in $\mathbb{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose $U$ contains the origin. We consider closed characteristics $(\tau, y)$ on $\Sigma$, which are solutions of the following problem

$$
\begin{cases}
\dot{y} = J N_{\Sigma}(y), \\
y(\tau) = y(0),
\end{cases}
$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix in $\mathbb{R}^n$, $\tau > 0$, $N_{\Sigma}(y)$ is the outward normal vector of $\Sigma$ at $y$ normalized by the condition $N_{\Sigma}(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard
inner product of \( a, b \in \mathbb{R}^{2n} \). A closed characteristic \((\tau, y)\) is prime, if \( \tau \) is the minimal period of \( y \). Two closed characteristics \((\tau, y)\) and \((\sigma, z)\) are geometrically distinct, if \( y(\mathbb{R}) \neq z(\mathbb{R}) \).

We denote by \( \mathcal{T}(\Sigma) \) the set of all geometrically distinct closed characteristics on \( \Sigma \). A closed characteristic \((\tau, y)\) is non-degenerate, if 1 is a Floquet multiplier of \( y \) of precisely algebraic multiplicity 2, and is elliptic, if all the Floquet multipliers of \( y \) locate on \( \mathbb{U} = \{ z \in \mathbb{C} | |z| = 1 \} \), i.e., the unit circle in the complex plane. It is hyperbolic, if 1 is a double Floquet multiplier of it and all the other Floquet multipliers of \( y \) are away from \( \mathbb{U} \).

Liapunov [13] in 1892 and Horn [11] in 1903 proved the following result: suppose \( H : \mathbb{R}^{2n} \to \mathbb{R} \) is analytic, \( \sigma(\mathcal{J}H''(0)) = \{ \pm \sqrt{-1} \omega_1, \ldots, \pm \sqrt{-1} \omega_n \} \) are purly imaginary and satisfy \( \frac{\omega_i}{\omega_j} \notin \mathbb{Z} \) for all \( i, j \). Then there exists \( \epsilon_0 > 0 \) so small that

\[
\# \mathcal{T}(H^{-1}(\epsilon)) \geq n, \quad \forall 0 < \epsilon \leq \epsilon_0. \tag{1.2}
\]

This deep result was greatly improved by Weinstein [23]. He proved that if \( H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \) and the Hessian \( H''(0) \) is positive definite, then there exists \( \epsilon_0 > 0 \) small such that (1.2) still holds. In [6], Ekeland and Lasry proved that if there exists \( x_0 \in \mathbb{R}^{2n} \) such that

\[
r \leq |x - x_0| \leq R, \quad \forall x \in \Sigma
\]

and \( \frac{R}{r} < \sqrt{2} \), then \( \Sigma \) carries at least \( n \) geometrically distinct closed characteristics.

Note that we have the following example of weakly non-resonant ellipsoid: let \( r = (r_1, \ldots, r_n) \) with \( r_i > 0 \) for \( 1 \leq i \leq n \). Define

\[
\mathcal{E}_n(r) = \left\{ z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \middle| \frac{1}{2} \sum_{i=1}^n \frac{x_i^2 + y_i^2}{r_i^2} = 1 \right\}
\]

where \( \frac{x_i}{r_i} \neq Q \) whenever \( i \neq j \). In this case, the corresponding Hamiltonian system is linear and all the solutions can be computed explicitly. Thus it is easy to verify that \( \# \mathcal{T}(\mathcal{E}_n(r)) = n \) and all the closed characteristics on \( \mathcal{E}_n(r) \) are elliptic and non-degenerate, i.e., their linearized Poincaré map splits into \( n - 1 \) two dimensional rotation matrix \( \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix} \) with \( \frac{\theta_i}{\pi} \notin \mathbb{Q} \) for \( 1 \leq i \leq n - 1 \) and one \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) in appropriate coordinates.

Based on the above facts, there is a long standing conjecture on the number of closed characteristics on compact convex hypersurfaces in \( \mathbb{R}^{2n} \):

\[
\# \mathcal{T}(\Sigma) \geq n, \quad \forall \Sigma \in \mathcal{H}(2n). \tag{1.3}
\]

Since the pioneering works [18] of Rabinowitz and [24] of Weinstein on the existence of at least one closed characteristic on every hypersurface in \( \mathcal{H}(2n) \), the existence of multiple closed characteristics on \( \Sigma \in \mathcal{H}(2n) \) has been deeply studied by many mathematicians.

When \( n \geq 2 \), Ekeland, Lassoued, Ekeland, Hofer, and Szulkin (cf. [5, 7, 19]) proved

\[
\# \mathcal{T}(\Sigma) \geq 2, \quad \forall \Sigma \in \mathcal{H}(2n).
\]

In [10], Hofer et al. proved that \( \# \mathcal{T}(\Sigma) = 2 \) or \( \infty \) holds for every \( \Sigma \in \mathcal{H}(4) \). In [17], Long and Zhu further proved

\[
\# \mathcal{T}(\Sigma) \geq \left[ \frac{n}{2} \right] + 1, \quad \forall \Sigma \in \mathcal{H}(2n), \tag{1.4}
\]

where we denote by \( \lfloor a \rfloor \equiv \max\{k \in \mathbb{Z} | k \leq a \} \). In [20], Wang et al. proved \( \# \mathcal{T}(\Sigma) \geq 3 \) for every \( \Sigma \in \mathcal{H}(6) \), which gave a confirmed answer to the above conjecture for \( n = 3 \).
There are some related results considering the stability problem. Ekeland [3] and Long [14] proved that if $\Sigma \in \mathcal{H}(2n)$ satisfies $\# T(\Sigma) < \infty$, then there exists at least one non-hyperbolic closed characteristic on $\Sigma$. In the same paper Ekeland showed the existence of at least one elliptic closed characteristic on $\Sigma$ provided $\Sigma$ is $\sqrt{2}$-pinched. In [1], Dell’Antonio et al. proved the existence of at least one elliptic closed characteristic on $\Sigma$ provided $\Sigma \in \mathcal{H}(2n)$ satisfies $\Sigma = -\Sigma$. In [15], Long proved that $\Sigma \in \mathcal{H}(4)$ and $\# T(\Sigma) = 2$ imply that both of the closed characteristics must be elliptic. In [20], the authors proved further that $\Sigma \in \mathcal{H}(4)$ and $\# T(\Sigma) = 2$ imply that both of the closed characteristics must be non-degenerate and elliptic. Long and Zhu [17] proved that if $\# T(\Sigma) < +\infty$, then there exists at least one elliptic closed characteristic and there are at least $\lceil \frac{n}{2} \rceil$ geometrically distinct closed characteristics on $\Sigma$ possessing irrational mean indices, which are then non-hyperbolic. In [21], the author proved that if $\# T(\Sigma) < +\infty$, then at least two closed characteristics on $\Sigma$ must possess irrational mean indices; if $\# T(\Sigma) = 3$, then there are at least two elliptic closed characteristics on $\Sigma$, where $\Sigma \in \mathcal{H}(6)$. In [22], the author further proved that if $\Sigma \in \mathcal{H}(6)$ and $\# T(\Sigma) = 3$ then there are at least two non-degenerate and elliptic closed characteristics on $\Sigma$. In [12], Hu and Ou proved that there are at least two geometrically distinct elliptic closed characteristics, and moreover, there exist at least $\rho_n(\Sigma)$ ($\rho_n(\Sigma) \geq \lceil \frac{n}{2} \rceil + 1$) geometrically distinct closed characteristics such that for any two elements among them, the ratio of their mean indices is an irrational number provided $\Sigma \in \mathcal{H}(2n)$ and $\# T(\Sigma) < \infty$.

Motivated by these results, we prove the following results in this paper.

**Theorem 1.1** There exist at least $\lceil \frac{n+1}{2} \rceil + 1$ geometrically distinct closed characteristics on every compact convex hypersurface $\Sigma$ in $\mathbb{R}^{2n}$, i.e., we have $\# T(\Sigma) \geq \lceil \frac{n+1}{2} \rceil + 1$ for any $\Sigma \in \mathcal{H}(2n)$, where $n \geq 2$.

**Theorem 1.2** There exist at least $\lceil \frac{n}{2} \rceil + 1$ geometrically distinct non-hyperbolic closed characteristics on $\Sigma \in \mathcal{H}(2n)$ provided $\# T(\Sigma) < \infty$, i.e., the number of geometrically distinct closed characteristics on $\Sigma$ is finite, where $n \geq 2$.

**Remark 1.3** Note that by Theorem 1.1 we obtain one more closed characteristic than the result (1.4) of Long and Zhu for $n$ being odd. In particular, Theorem 1.1 gives a new proof in the case $n = 3$ to the conjecture (1.3). In fact, by the original method in [17], if there is no closed characteristic $(\tau, y)$ on $\Sigma$ satisfying $i(y, 1) = n$ and $\gamma_\tau(\tau)$ can be connected within $\Omega^0(\gamma_\tau(\tau))$ to $N_1(1, 1) \cup N_1(1, -1)^{(n-1)}$, then we have $\# T(\Sigma) \geq \lceil \frac{n+1}{2} \rceil + 1$. Thus the difficulty in the proof of Theorem 1.1 is to handle the case that if such a closed characteristic appears. Actually, we overcome this difficulty by using the periodic property for critical modules of iterated closed characteristics proved in [20] and a new property for closed characteristics in different common index jump intervals discovered by the author recently (cf. Lemma 3.3 below). By Theorem 1.2 we obtain one more non-hyperbolic closed characteristic than the result of Long and Zhu described above.

The proof of Theorems 1.1 and 1.2 is given in Sect. 3. The main ingredients of the proof are: the critical point theory for closed characteristics established in [20], Morse theory and the index iteration theory developed by Long and his coworkers.

Here we give the outline of the proof of Theorem 1.1. By Theorem 1.1 of [17], we have $\# T(\Sigma) \geq \lceil \frac{n}{2} \rceil + 1$. Thus in order to prove Theorem 1.1, we only need to consider the case that $n$ being odd. We prove by contradiction, i.e., assume $\# T(\Sigma) = \lceil \frac{n}{2} \rceil + 1$. Applying the Fadell–Rabinowitz index theory to the Clarke–Ekeland dual actional functional $\Phi$ (cf. (2.4) below), we obtain a sequence of critical values $-\infty < c_1 < c_2 < \cdots < c_k < c_{k+1} < \cdots < 0$ of $\Phi$ and non-zero critical points of $\Phi$ correspond exactly to closed characteristics on $\Sigma$. 

\[
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\]
Since Φ is not defined on a Hilbert space, in order to apply Gromoll–Meyer theory, we construct a functional \( \Psi_a \) (cf. (2.16) below) which have isomorphic critical modules as \( \Phi \) at corresponding critical points. Thus for each \( i \in \mathbb{N} \) we can find a critical point \( u_i \) of \( \Phi \) satisfying \( \Phi(u_i) = c_i \) and \( C_{S^1,2(i-1)}(\Phi, S^1 \cdot u_i) \neq 0 \). Applying the common index jump theorem of Long and Zhu, we obtain infinitely many tuples \((T, m_1, m_2, \ldots, m_{[\frac{n}{2}]+1})\) such that

\[
\Phi'(u_{2m_j}) = 0, \quad \Phi'(u_{2m_j}) = c_{T+1-k}, \quad C_{S^1,2T-2k}(\Phi, S^1 \cdot u_{2m_j}) \neq 0, \quad (1.5)
\]

for \( 1 \leq k \leq [\frac{n}{2}] + 1 \), where \( u_{2m_j} \) denotes the critical point of \( \Phi \) corresponding to the \( m \)th iteration \((m \tau_j, y_j)\) of a prime closed characteristic \((\tau_j, y_j)\). Moreover, we can show that \( j_1, j_2, \ldots, j_{[\frac{n}{2}]+1} \) are pairwise distinct.

Fix a tuple \((T^*, m_1^*, m_2^*, \ldots, m_{[\frac{n}{2}]+1}^*)\) found by the common index jump theorem satisfying (1.5). By the assumption \#\( T(\Sigma) = [\frac{n}{2}] + 1 \), we can show there must be a closed characteristic (denote it by \( y_1 \) without loss of generality) such that \( i(y_1, 1) = n \) and \( y_{\gamma_1}(\tau_1) \) can be connected within \( \Omega^0(\gamma_\gamma_1(\tau_1)) \) to \( N_1(1, 1) \circ N_1(1, -1)^{c(n-1)} \) (cf. “Appendix” for notations) together with \( C_{S^1,2T^*+2}(\Phi, S^1 \cdot u_1^{2m_1^*}) \neq 0 \) (cf. (3.30) below).

Next suppose \((T, m_1, m_2, \ldots, m_{[\frac{n}{2}]+1})\) is any tuple found by the common index jump theorem satisfying (1.5). By the periodic property for critical modules of iterated closed characteristics (cf. Proposition 2.8 below), we have (cf. (3.31) below)

\[
C_{S^1,2T-2}(\Phi, S^1 \cdot u_1^{2m_1}) \cong C_{S^1,2T^*+2}(\Phi, S^1 \cdot u_1^{2m_1^*}) \neq 0. \quad (1.6)
\]

Thus by the critical point theory (cf. Proposition 2.9 below), we have (cf. (3.32) below)

\[
C_{S^1,2T-2-l}(\Phi, S^1 \cdot u_1^{2m_1}) = 0, \quad \forall l \neq 0,
\]

since \( u_1^{2m_1} \) is a local maximum in the local characteristic manifold in the Gromoll–Meyer theory.

Hence by (1.5), we have \( \Phi(u_1^{2m_1}) = c_T \) and then \( \Phi(u_i^{2m_1}) > \Phi(u_i^{2m_1}) \) for \( i \in \{2, \ldots, [\frac{n}{2}] + 1\} \). Thus by Lemma 3.2, we have \( m_1 \hat{i}(y_i) > m_i \hat{i}(y_i) \) for \( i \in \{2, \ldots, [\frac{n}{2}] + 1\} \), where \( \hat{i}(y) \) denotes the mean index of \((\tau, y)\). This contradicts the new property of closed characteristics in different common index jump intervals in Lemma 3.3: in fact, Lemma 3.3 says that for any fixed \( i \in \{2, \ldots, [\frac{n}{2}] + 1\} \), it is impossible that for any tuple \((T, m_1, m_2, \ldots, m_{[\frac{n}{2}]+1})\) found by the common index jump theorem satisfying (1.5) we always have \( m_1 \hat{i}(y_i) > m_i \hat{i}(y_i) \). This proves Theorem 1.1.

In Sect. 2, we review briefly the equivariant Morse theory for closed characteristics on compact convex hypersurfaces in \( \mathbb{R}^{2n} \) developed in [20]. In the “Appendix”, we review the index iteration theory developed by Long and his coworkers.

In this paper, let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) and \( \mathbb{R}^+ \) denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, complex numbers and positive real numbers respectively. We denote by \( a \cdot b \) and \(|a|\) the norm in \( \mathbb{R}^{2n} \), and by \((\cdot, \cdot)\) and \(\|\cdot\|\) the standard \( L^2 \)-inner product and \( L^2 \)-norm. For a \( S^1 \)-space \( X \), we denote by \( X_{S^1} \) the homotopy quotient of \( X \) module the \( S^1 \)-action, i.e., \( X_{S^1} = S^\infty \times_{S^1} X \). We define the functions

\[
\begin{align*}
[a] &= \max\{k \in \mathbb{Z} | k \leq a\}, \ E(a) = \min\{k \in \mathbb{Z} | k \geq a\}, \ \varphi(a) = E(a) - [a].
\end{align*}
\quad (1.7)
\]
Specially, $\varphi(a) = 0$ if $a \in \mathbb{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbb{Z}$. In this paper we use only $\mathbb{Q}$-coefficients for all homological modules. For a $\mathbb{Z}_m$-space pair $(A, B)$, let $H_*(A, B)_{\pm Z_m} = \{\sigma \in H_*(A, B)|L_*\sigma = \pm \sigma\}$, where $L$ is a generator of the $\mathbb{Z}_m$-action.

2 A variational structure for closed characteristics

In the rest of this paper, we fix $\Sigma \in \mathcal{H}(2n)$, i.e., $\Sigma$ is a compact convex hypersurface in $\mathbb{R}^{2n}$ of class $C^3$ and assume the following condition on $\Sigma$:

• (F) There exist only finitely many geometrically distinct closed characteristics of $\Sigma$, all the details of proofs can be found in [20,21].

In this section, we review briefly the equivariant Morse theory for closed characteristics on $\Sigma$, and then we prove that the minimal period problem $(1.1)$ has solutions if and only if the fixed period problem $(2.1)$ has solutions. Hence we use the same symbol $(\tau, y)$ to denote solutions of both $(1.1)$ and $(2.1)$.

In order to use the variational method to solve $(2.1)$, we consider the fixed period problem

$$\begin{align*}
\dot{x}(t) &= JH'_a(x(t)), \\
x(t) &= x(0),
\end{align*}$$

(2.1)

It is well-known that the problems $(1.1)$ and $(2.1)$ are equivalent; especially, their solutions are one-to-one correspond to each other. Hence we use the same symbol $(\tau, y)$ to denote solutions of both $(1.1)$ and $(2.1)$.

Now we consider the following given energy problem of the Hamiltonian system:

$$\begin{align*}
\dot{x}(t) &= JH'_a(x(t)), \\
x(0) &= x(0),
\end{align*}$$

$$(2.2)$$

and define the Banach space

$$L^{\frac{2}{\alpha-1}}_0(S^1, \mathbb{R}^{2n}) = \left\{ u \in L^{\frac{2}{\alpha-1}}(S^1, \mathbb{R}^{2n}) \left| \int_0^1 u dt = 0 \right. \right\}.$$  

(2.3)

The corresponding Clarke–Ekeland dual action functional is defined by

$$\Phi(u) = \int_0^1 \left( \frac{1}{2} J u \cdot M u + H^*_a(-J u) \right) dt, \quad \forall u \in L^{\frac{2}{\alpha-1}}_0(S^1, \mathbb{R}^{2n})$$

(2.4)

where $M u$ is defined by $\frac{d}{dt} \! Mu(t) = u(t)$ and $\int_0^1 \! Mu(t)dt = 0$, $H^*_a$ is the Fenchel transform of $H_a$ defined $H^*_a(y) = \sup\{x \cdot y - H_a(x) | x \in \mathbb{R}^{2n}\}$.

Suppose $u \in L^{\frac{2}{\alpha-1}}_0(S^1, \mathbb{R}^{2n}) \setminus \{0\}$ is a critical point of $\Phi$, then by §5 of [4], there exists a unique $\xi_u \in \mathbb{R}^{2n}$ such that $z_u(t) = Mu(t) + \xi_u$ is a solution of $(2.2)$. Let $h = H_a(z_u(t))$ and $1/m$ be the minimal period of $z_u$ for some $m \in \mathbb{N}$ and

$$x_u(t) = h^{-1/\alpha}z_u(h^{(2-\alpha)/\alpha}t), \quad \tau = \frac{1}{m}h^{(\alpha-2)/\alpha}.$$

(2.5)

Then $(\tau, x_u)$ is a solution of $(2.1)$ with minimal period $\tau$, and then it is a prime closed characteristic on $\Sigma$. Conversely, suppose $(\tau, x)$ is a prime closed characteristic on $\Sigma$, then
it gives rise to a sequence \( \{z_m^\kappa\}_{m \in \mathbb{N}} \) of solutions of (2.2) and a sequence \( \{u_m^\kappa\}_{m \in \mathbb{N}} \) of critical points of \( \Phi \):

\[
z_m^\kappa(t) = (m\tau)^{-1/(2-\alpha)}x(m\tau t), \quad u_m^\kappa(t) = (m\tau)^{(\alpha-1)/(2-\alpha)}\dot{x}(m\tau t). \tag{2.6}
\]

Now we apply Morse theory to the functional \( \Phi \). For any \( \kappa \in \mathbb{R} \), we define the level set

\[
\Phi^\kappa_\underleftarrow = \left\{ u \in L_0^{\frac{\alpha}{2-\alpha}}(S^1, \mathbb{R}^{2n}) | \Phi(u) < \kappa \right\}. \tag{2.7}
\]

Firstly we want to find a sequence of critical values of \( \Phi \). Recall that for a principal \( U(1) \)-bundle \( E \to B \), the Fadell–Rabinowitz index (cf. [8]) of \( E \) is defined to be \( \text{sup}\{k \mid c_1(E)^{k-1} \neq 0\} \), where \( c_1(E) \in H^2(B, \mathbb{Q}) \) is the first rational Chern class. For a \( U(1) \)-space, i.e., a topological space \( X \) with a \( U(1) \)-action, the Fadell–Rabinowitz index is defined to be the index of the bundle \( X \times S^{\infty} \to X \times_{U(1)} S^{\infty} \), where \( S^{\infty} \to CP^{\infty} \) is the universal \( U(1) \)-bundle.

Then as in p. 218 of [4], for any \( i \in \mathbb{N}_0 \), we define

\[
c_i = \inf\{\delta \in \mathbb{R} | \hat{I}(\Phi^\delta_{\underleftarrow}) \geq i\}, \tag{2.8}
\]

where \( \hat{I} \) is the Fadell–Rabinowitz index.

Then by Proposition 3 in p. 218 of [4], we have

**Proposition 2.1** Every \( c_i \) is a critical value of \( \Phi \). If \( c_i = c_j \) for some \( i < j \), then there are infinitely many geometrically distinct closed characteristics on \( \Sigma \).

Next we define the local critical module of a critical point of \( \Phi \). Note that we have a natural \( S^1 \)-action on \( L_0^{\frac{\alpha}{2-\alpha}}(S^1, \mathbb{R}^{2n}) \) defined by \( \theta \cdot u(t) = u(\theta + t) \) for all \( \theta \in S^1 \) and \( t \in \mathbb{R} \). Clearly \( \Phi \) is \( S^1 \)-invariant. For any \( \kappa \in \mathbb{R} \), we denote by

\[
\Lambda^\kappa = \left\{ w \in L_0^{\frac{\alpha}{2-\alpha}}(S^1, \mathbb{R}^{2n}) | \Phi(w) \leq \kappa \right\}. \tag{2.9}
\]

For a critical point \( u \) of \( \Phi \), we denote by

\[
\Lambda(u) = \Lambda^{\Phi(u)} = \left\{ w \in L_0^{\frac{\alpha}{2-\alpha}}(S^1, \mathbb{R}^{2n}) | \Phi(w) \leq \Phi(u) \right\}. \tag{2.10}
\]

Clearly, both sets are \( S^1 \)-invariant. Since the \( S^1 \)-action preserves \( \Phi \), if \( u \) is a critical point of \( \Phi \), then the whole orbit \( S^1 \cdot u \) consists of critical points of \( \Phi \). Denote by \( \text{crit}(\Phi) \) the set of critical points of \( \Phi \). Note that by the condition (F), the number of critical orbits of \( \Phi \) is finite. Hence as usual we can make the following definition.

**Definition 2.2** Suppose \( u \) is a non-zero critical point of \( \Phi \), and \( \mathcal{N} \) is a \( S^1 \)-invariant open neighborhood of \( S^1 \cdot u \) such that \( \text{crit}(\Phi) \cap (\Lambda(u) \cap \mathcal{N}) = S^1 \cdot u \). Then the \( S^1 \)-critical modules of \( S^1 \cdot u \) is defined by

\[
C_{S^1, \cdot u}(\Phi, S^1 \cdot u) = H_i((\Lambda(u) \cap \mathcal{N})_{S^1}, ((\Lambda(u) \setminus S^1 \cdot u) \cap \mathcal{N})_{S^1}). \tag{2.11}
\]

Note that since we do not assume the critical point \( u \) to be non-degenerate, the critical modules may not vanish in several dimensions.

Comparing with Theorem 4 on p. 219 in [4], we have the following:
Proposition 2.3 (Proposition 3.5 of [21]) For every \( i \in \mathbb{N} \), there exists a point \( u \in L_{0}^{2n-1}(S^{1}, \mathbb{R}^{2n}) \) such that
\[
\Phi'(u) = 0, \quad \Phi(u) = c_{i}, \quad (2.12)
\]
\[
C_{S^{1}, 2(i-1)}(\Phi, S^{1} \cdot u) \neq 0. \quad (2.13)
\]

Now we want to study further properties for critical modules. We do this by constructing another functional \( \Psi_{a} \) defined on the Hilbert space
\[
L_{0}^{2}(S^{1}, \mathbb{R}^{2n}) = \left\{ u \in L^{2}([0, 1], \mathbb{R}^{2n}) \mid \int_{0}^{1} u(t)dt = 0 \right\} \quad (2.14)
\]
such that critical points of \( \Phi \) and \( \Psi_{a} \) are one-to-one correspond to each other. Moreover, \( \Psi_{a} \) and \( \Phi \) have isomorphic critical modules at corresponding critical points, and hence we can compute the critical modules of \( \Phi \) via that of \( \Psi_{a} \). In fact, since \( \Phi \) is not defined on a Hilbert space, we can not apply the tools (e.g. Gromoll–Meyer theory) in critical point theory directly to \( \Phi \).

Now we describe briefly how to construct \( \Psi_{a} \). Let \( \hat{\tau} = \inf\{\tau_{j} \mid 1 \leq j \leq q \} \). In view of Propositions 2.2–2.4 in [20], given \( a > \hat{\tau} \), there exists a function \( \varphi_{a} \in C^{\infty}(\mathbb{R}, \mathbb{R}^{+}) \) which has 0 as its unique critical point in \([0, +\infty)\) and is strictly convex for \( t \geq 0 \). Moreover, \( \varphi_{a}'(t) \) is strictly decreasing and satisfies \( \lim_{t \to 0^{+}} \varphi_{a}'(t) = 1 \) and \( \varphi_{a}(0) = 0 = \varphi_{a}'(0) \).

Define the Hamiltonian function \( H_{a}(x) = a\varphi_{a}(x(t)) \) and consider the fixed period problem
\[
\begin{align*}
\dot{x}(t) &= JH'_{a}(x(t)), \\
x(1) &= x(0).
\end{align*}
\]
Then \( H_{a} \in C^{3}(\mathbb{R}^{2n}\backslash\{0\}, \mathbb{R}) \cap C^{1}(\mathbb{R}^{2n}, \mathbb{R}) \) and is strictly convex. Solutions of (2.15) are \( x \equiv 0 \) and \( x = \rho y(\tau t) \) with \( \varphi_{a}(\rho) = \frac{\tau}{a} \), where \( (\tau, y) \) is a solution of (1.1). In particular, nonzero solutions of (2.15) are one-to-one correspond to solutions of (1.1) with period \( \tau < a \).

Similar as above we use the Clarke–Ekeland dual action principle to transform (2.12) to a variational problem and use variational methods to study the problem. Let \( H^{*}_{a} \) be the Fenchel transform of \( H_{a} \). Then \( H^{*}_{a} \in C^{2}(\mathbb{R}^{2n}\backslash\{0\}, \mathbb{R}) \cap C^{1}(\mathbb{R}^{2n}, \mathbb{R}) \) is strictly convex. Define a linear operator \( M : L_{0}^{2}(S^{1}, \mathbb{R}^{2n}) \to L_{0}^{2}(S^{1}, \mathbb{R}^{2n}) \) by \( \frac{d}{dt}Mu(t) = u(t) \). Then the dual action functional on \( L_{0}^{2}(S^{1}, \mathbb{R}^{2n}) \) is defined by
\[
\Psi_{a}(u) = \int_{0}^{1} \left( -\frac{1}{2}Ju \cdot Mu + H^{*}_{a}(-Ju) \right)dt. \quad (2.16)
\]
One can show that the functional \( \Psi_{a} \in C^{1, 1}(L_{0}^{2}(S^{1}, \mathbb{R}^{2n}), \mathbb{R}) \) is bounded from below and satisfies the Palais-Smale condition. Suppose \( x \) is a solution of (2.15). Then \( u = \dot{x} \) is a critical point of \( \Psi_{a} \). Conversely, suppose \( u \) is a critical point of \( \Psi_{a} \). Then there exists a unique \( \xi \in \mathbb{R}^{2n} \) such that \( Mu - \xi \) is a solution of (2.15). In particular, solutions of (2.15) are in one-to-one correspondence with critical points of \( \Psi_{a} \). Moreover, \( \Psi_{a}(u) < 0 \) for every critical point \( u \neq 0 \) of \( \Psi_{a} \) (cf. §2 of [20]).

Suppose \( u \) is a nonzero critical point of \( \Psi_{a} \). Then following [4] the formal Hessian of \( \Psi_{a} \) at \( u \) is defined by
\[
Q_{a}(v, v) = \int_{0}^{1} \left( Ju \cdot Mv + (H^{*}_{a})''(-Ju)Ju \cdot Jv \right)dt,
\]
which defines an orthogonal splitting \( L_0^2 = E_- \oplus E_0 \oplus E_+ \) of \( L_0^2(S^1, \mathbb{R}^{2n}) \) into negative, zero and positive subspaces. The index of \( u \) is defined by \( i(u) = \dim E_- \) and the nullity of \( u \) is defined by \( \nu(u) = \dim E_0 \). Let \( u = \dot{x} \) be the critical point of \( \Psi_u \) such that \( x \) corresponds to the closed characteristic \((r, y)\) on \( \Sigma \). Then the index \( i(u) \) and the nullity \( \nu(u) \) defined here coincide with the Ekeland indices defined by Ekeland [2–4]. In particular, \( 1 \leq \nu(u) \leq 2n - 1 \) always holds.

As above, for any \( \kappa \in \mathbb{R} \), we denote by

\[
\Lambda^\kappa = \{ w \in L_0^2(S^1, \mathbb{R}^{2n}) | \Psi_u(w) \leq \kappa \}. \tag{2.17}
\]

For a critical point \( u \) of \( \Psi_u \), we denote by

\[
\Lambda_a(u) = \Lambda^\nu(a) \cap \{ w \in L_0^2(S^1, \mathbb{R}^{2n}) | \Psi_u(w) \leq \nu(a) \}. \tag{2.18}
\]

Similar to Definition 2.2, we make the following definition.

**Definition 2.4** Suppose \( u \) is a non-zero critical point of \( \Psi_u \) and \( N \) is a \( S^1 \)-invariant open neighborhood of \( S^1 \cdot u \) such that \( \text{crit}(\Psi_u) \cap (\Lambda_a(u) \cap N) = S^1 \cdot u \). Then the \( S^1 \)-critical modules of \( S^1 \cdot u \) are defined by

\[
C_{S^1 \cdot j}(\Psi_u, S^1 \cdot u) = H_j((\Lambda_a(u) \cap N)_{S^1 \cdot j}, ((\Lambda_a(u) \cap S^1 \cdot u) \cap N)_{S^1 \cdot j}).
\]

The next proposition shows that \( \Psi_u \) and \( \Phi \) have isomorphic critical modules at corresponding critical points. Its proof is the same as Proposition 3.6 of [21].

**Proposition 2.5** Suppose \( u \) is the critical point of \( \Phi \) found in Proposition 2.3. Then we have

\[
C_{S^1 \cdot j}(\Psi_u, S^1 \cdot u) \cong C_{S^1 \cdot j}(\Phi, S^1 \cdot u), \quad \forall j \in \mathbb{Z}, \tag{2.19}
\]

where \( u_a \subseteq L_0^2(S^1, \mathbb{R}^{2n}) \) is the critical point of \( \Psi_u \) corresponding to \( u \) in the natural sense.

Now we want to compute the critical modules of \( \Psi_u \) by using Gromoll–Meyer theory. Let \( u \neq 0 \) be a critical point of \( \Psi_u \) with multiplicity \( \text{mul}(u) = m \), i.e., \( u \) corresponds to a closed characteristic \((m \tau, y) \subset \Sigma \) with \((\tau, y)\) being prime. Hence \( u(t + \frac{1}{m}) = u(t) \) holds for all \( t \in \mathbb{R} \) and the orbit of \( u \), namely, \( S^1 \cdot u \cong S^1 / \mathbb{Z}_m \cong S^1 \). Let \( f : N(S^1 \cdot u) \to S^1 \cdot u \) be the normal bundle of \( S^1 \cdot u \) in \( L_0^2(S^1, \mathbb{R}^{2n}) \) and let \( f^{-1}(\theta \cdot u) = N(\theta \cdot u) \) be the fibre over \( \theta \cdot u \), where \( \theta \in S^1 \). Let \( DN(S^1 \cdot u) \) be the \( \mathbb{R} \)-disk bundle of \( N(S^1 \cdot u) \) for some \( \mathbb{R} \) sufficiently small, i.e., \( DN(S^1 \cdot u) = \{ \xi \in N(S^1 \cdot u) | \| \xi \| < \rho \} \) and let \( DN(\theta \cdot u) = f^{-1}(\theta \cdot u) \cap DN(S^1 \cdot u) \) be the disk over \( \theta \cdot u \). Clearly, \( DN(\theta \cdot u) \) is \( \mathbb{Z}_m \)-invariant and we have \( DN(S^1 \cdot u) = DN(u) \times \mathbb{Z}_m \cdot S^1 \), where the \( \mathbb{Z}_m \)-action is given by

\[
(\theta, v, t) \in \mathbb{Z}_m \times DN(u) \times S^1 \mapsto (\theta \cdot v, \theta^{-1} t) \in DN(u) \times S^1.
\]

Thus for a \( S^1 \)-invariant subset \( \Gamma \) of \( DN(S^1 \cdot u) \), we have \( \Gamma / S^1 = (\Gamma / \mathbb{Z}_m \cdot S^1) / S^1 = \Gamma / \mathbb{Z}_m \), where \( \Gamma = \Gamma \cap DN(u) \). Since \( \Psi_u \) is only \( C^{1,1} \) on the whole space \( L_0^2(S^1, \mathbb{R}^{2n}) \), we cannot use the standard method in critical point theory to study \( \Psi_u \) directly. While we can use a finite dimensional approximation introduced by Ekeland [2] and apply critical point theory to the obtained finite dimensional submanifold. More precisely, we can construct a finite dimensional submanifold \( \Gamma(i) \) of \( L_0^2(S^1, \mathbb{R}^{2n}) \) which admits a \( \mathbb{Z}_m \)-action with \( m(u) \). Moreover \( \Psi_u \) and \( \Psi_u | \Gamma(i) \) have the same critical points. \( \Psi_u | \Gamma(i) \) is \( C^2 \) in a small tubular neighborhood of the critical orbit \( S^1 \cdot u \) and the Morse index and nullity of its critical points coincide with those of the corresponding critical points of \( \Psi_u \). Let

\[
D_i N(S^1 \cdot u) = DN(S^1 \cdot u) \cap \Gamma(i), \quad D_i N(\theta \cdot u) = DN(\theta \cdot u) \cap \Gamma(i). \tag{2.20}
\]
Then we can reduce the computation of the original critical modules to those of the corresponding critical point on the finite dimensional sub-manifold $\Gamma(u)$, i.e., we have

$$C_{S^1, \ast}(\Psi, S^1 \cdot u) \cong H_{a}(\Lambda(u) \cap D, N(u), (\Lambda(u) \setminus [u]) \cap D, N(u))^{\mathbb{Z}_m}. \quad (2.21)$$

Now we can apply the results of Gromoll and Meyer [9] to the finite dimensional manifold $D, N(u)$ with $u$ as its isolated critical point. Note that the isotropy group $\mathbb{Z}_m \subseteq S^1$ of $u$ acts on $D, N(u)$ by isometries. According to Lemma 1 of [9], we can construct a $\mathbb{Z}_m$-invariant decomposition of $T_u(D, N(u))$:

$$T_u(D, N(u)) = V^+ \oplus V^- \oplus V^0 = \{(x_+, x_-, x_0)\}$$

with $\dim V^- = i(u)$, $\dim V^0 = v(u) - 1$ and a $\mathbb{Z}_m$-invariant neighborhood $B = B_+ \times B_- \times B_0$ for $0$ in $T_u(D, N(u))$ together with two $\mathbb{Z}_m$-invariant diffeomorphisms

$$\Psi : B = B_+ \times B_- \times B_0 \to \Psi(B_+ \times B_- \times B_0) \subseteq D, N(u)$$

and

$$\eta : B_0 \to W(u) \equiv \eta(B_0) \subseteq D, N(u)$$

such that $\Psi(0) = \eta(0) = u$ and

$$\Psi_a \circ \Psi(x_+, x_-, x_0) = |x_+|^2 - |x_-|^2 + \Psi_a \circ \eta(x_0), \quad (2.22)$$

with $d(\Psi_a \circ \eta)(0) = d^2(\Psi_a \circ \eta)(0) = 0$. As [9], we call $W(u)$ a local characteristic manifold and $U(u) = B_-$ a local negative disk at $u$. By the proof of Lemma 1 of [9], $W(u)$ and $U(u)$ are $\mathbb{Z}_m$-invariant.

For a fixed closed characteristic $(\tau, y)$ on $\Sigma$, we denote by $y^m \equiv (m \tau, y)$ the $m$th iteration of $y$ for $m \in \mathbb{N}$. In order to simplify notations, in the following, we denote by $u^m$ the critical point of $\Phi$ (or $\Psi_a$) corresponding to $y^m$ as described above. Then we define the index $i(y^m)$ and nullity $v(y^m)$ by

$$i(y^m) = i(u^m), \quad v(y^m) = v(u^m).$$

The mean index of $(\tau, y)$ is defined by

$$\hat{i}(y) = \lim_{m \to \infty} \frac{i(y^m)}{m}. \quad (2.23)$$

Ekeland and Hofer [5] proved that $\hat{i}(y) > 2$ always holds (cf. Corollary 8.3.2 and Lemma 15.3.2 of [16] for a different proof).

Now we can reduce the computation of critical modules of $\Phi$ at a critical point to those of the corresponding critical point on the local characteristic manifold:

**Proposition 2.6** Suppose $(\tau, y)$ is a prime closed characteristic on $\Sigma$ and denote by $u^p$ the critical point of $\Phi$ (or $\Psi_a$) corresponding to $y^p$, where $p \in \mathbb{N}$. Then for all $j \in \mathbb{Z}$, we have

$$C_{S^1, j}(\Phi, S^1 \cdot u^p) \cong (H_{j-i(u^p)}(W(u^p) \cap \Lambda(u^p), (W(u^p) \setminus \{u^p\}) \cap \Lambda(u^p)))^{\beta(u^p)\mathbb{Z}_p}, \quad (2.24)$$

where $\beta(u^p) = (-1)^{i(u^p) - i(u)}$. Thus

$$C_{S^1, j}(\Phi, S^1 \cdot u^p) = 0, \quad \text{for } j < i(u^p) \text{ or } j > i(u^p) + v(u^p) - 1, \quad (2.25)$$

where $W(u^p)$ is a local characteristic at $u^p$ constructed as above.
9.3.4 of [16], we have

\[
W / \Phi_1
\]

is a prime closed characteristic on \( \Sigma \) and denote by \( u^p \) the critical point of \( \Phi \) (or \( \Psi_a \)) corresponding to \( y^p \), where \( p \in \mathbb{N} \). Then for all \( l \in \mathbb{Z} \), let

\[
k_l, \pm 1(u^p) = \dim( H_l(W(u^p) \cap \Lambda_a(u^p)), (W(u^p) \setminus \{u^p\}) \cap \Lambda_a(u^p))) \pm Z_p,
\]

\[
k_l(u^p) = \dim( H_l(W(u^p) \cap \Lambda_a(u^p)), (W(u^p) \setminus \{u^p\}) \cap \Lambda_a(u^p))) \beta(u^p) Z_p.
\]

\( k_l(u^p) \)'s are called critical type numbers of \( u^p \).

The following result implies that critical type numbers have the periodic property. This is a crucial point in our proof of Theorem 1.1 below.

**Proposition 2.8** (Proposition 3.13 of [20]) Suppose \( (\tau, y) \) is a prime closed characteristic on \( \Sigma \) and denote by \( u^p \) the critical point of \( \Phi \) (or \( \Psi_a \)) corresponding to \( y^p \), where \( p \in \mathbb{N} \). Then there exists a minimal \( K(u) \in \mathbb{N} \) such that

\[ v(u^{p+K(u)}) = v(u^p), \quad i(u^{p+K(u)}) - i(u^p) \in 2\mathbb{Z}. \]

Moreover, we have \( k_l(u^{p+K(u)}) = k_l(u^p) \) for all \( p \in \mathbb{N} \) and \( l \in \mathbb{Z} \).

In fact, denote by \( \gamma_\tau \) the associated symplectic path of \( (\tau, y) \). Suppose \( \lambda_i = e^{\frac{2\pi \sqrt{1}}{s_i} \tau} \) the eigenvalues of \( \gamma_\tau(\tau) \) possessing rotation angles which are rational multiple of \( 2\pi \) with \( r_i, s_i \in \mathbb{N} \) and \( (r_i, s_i) = 1 \) for \( 1 \leq i \leq k \). Let \( K'(u) \) be the least common multiple of \( s_1, \ldots, s_k \). Then we have \( v(u^{p+K'(u)}) = v(u^p) \) for all \( p \in \mathbb{N} \). By Theorem 4.6 below and Theorem 9.3.4 of [16], we have \( i(u^{m+2}) - i(u^m) \in 2\mathbb{Z} \) for any \( m \in \mathbb{N} \). Hence we have

\[
K(u) = \begin{cases} 2K'(u) & \text{if } i(u^2) - i(u) \in 2\mathbb{Z} + 1 \text{ and } K'(u) \in 2\mathbb{N} - 1, \\ K'(u) & \text{otherwise}. \end{cases}
\]

Using notations in Proposition 2.8, we define the critical type numbers \( k_l(y^m) \) of \( y^m \) to be \( k_l(u^m) \), we also define \( K(y) = K(u) \).

The next proposition tells us that the possible values of critical type numbers.

**Proposition 2.9** (Proposition 2.6 of [21]) We have \( k_l(y^m) = 0 \) for \( l \notin [0, \nu(y^m) - 1] \) and it can take only values 0 or 1 when \( l = 0 \) or \( l = \nu(y^m) + 1 \). Moreover, the following properties hold:

(i) \( k_0(y^m) = 1 \) implies \( k_l(y^m) = 0 \) for \( 1 \leq l \leq \nu(y^m) - 1 \).
(ii) \( k_{\nu(y^m) - 1}(y^m) = 1 \) implies \( k_l(y^m) = 0 \) for \( 0 \leq l \leq \nu(y^m) - 2 \).
(iii) \( k_l(y^m) \geq 1 \) for some \( 1 \leq l \leq \nu(y^m) - 2 \) implies \( k_{\nu(y^m) - 1}(y^m) = k_{\nu(y^m) - 1}(y^m) = 0 \).

**Remark 2.10** Note that in Proposition 2.9, (i) is equivalent to that \( u^m \) is a local minimum in the local characteristic manifold \( W(u^m) \); (ii) is equivalent to that \( u^m \) is a local maximum in \( W(u^m) \); (iii) is equivalent to that \( u^m \) is neither a local maximum nor a local minimum in \( W(u^m) \).
3 Proof of the main theorems

In this section, we give the proof of the main theorems by using critical point theory, the index iteration theory developed by Long and his coworkers, and especially a new property for closed characteristics in different common index jump intervals discovered by the author of this paper.

Firstly as Definition 1.1 of [17], we make the following definition.

Definition 3.1 For \(\alpha \in (1, 2)\), we define a map \(\varrho_\alpha: \mathcal{H}(2n) \to \mathbb{N} \cup \{+\infty\}\)

\[
\varrho_\alpha(\Sigma) = \begin{cases} 
+\infty, & \text{if } \# \mathcal{V}(\Sigma, \alpha) = +\infty, \\
\min \left\{ \left[ \frac{i(y, 1) + 2S^+(y) - v(y, 1) + n}{\alpha} \right] \middle| (\tau, y) \in \mathcal{V}_\infty(\Sigma, \alpha) \right\}, & \text{if } \# \mathcal{V}(\Sigma, \alpha) < +\infty,
\end{cases}
\]

where \(\mathcal{V}(\Sigma, \alpha)\) and \(\mathcal{V}_\infty(\Sigma, \alpha)\) are variationally visible and infinite variationally visible sets respectively defined in Definition 1.4 of [17].

In order to simplify notations, for a prime closed characteristic \((\tau_j, y_j)\) and \(m \in \mathbb{N}\), we denote by \(u_{j}^m\) the unique critical point of \(\Phi\) (or \(\Psi_\alpha\)) corresponding to the closed characteristic \(y_j^m \equiv (m \tau_j, y_j)\).

Lemma 3.2 There exists a large integer \(T_0 \in \mathbb{N}\) such that the following hold. For every integer \(i > T_0\), there exists a prime closed characteristic \((\tau_{j(i)}, y_{j(i)})\) and \(m(i) \in \mathbb{N}\) such that

\[
\Phi' \left( u_{j(i)}^{m(i)} \right) = 0, \quad \Phi \left( u_{j(i)}^{m(i)} \right) = c_i, \quad C_{S^1, 2(i-1)} \left( \Phi, S^1 \cdot u_{j(i)}^{m(i)} \right) \neq 0,
\]

where \(c_i\) is the critical value of \(\Phi\) found by Proposition 2.1. Moreover, for any \(i_1 > i_2 > T_0\) we have \(m(i_1) \tilde{i} (y_{j(i_1)}) > m(i_2) \tilde{i} (y_{j(i_2)})\).

Proof The lemma follows directly from Lemma 3.1 of [17], Proposition 2.3, Theorem 4.6, (2.23) and (4.8).

Note that by Theorem 1.1 of [17], we have \(#T(\Sigma) \geq \varrho_\alpha(\Sigma) \geq \left[ \frac{\alpha}{2} \right] + 1\). Thus in order to prove Theorem 1.1, we only need to consider the case that \(n\) is odd. We prove Theorem 1.1 by contradiction and assume for a fixed \(\Sigma \in \mathcal{H}(2n)\) that the following holds:

- (C) \(\#T(\Sigma) = \left[ \frac{\alpha}{2} \right] + 1\), i.e., there are exactly \(\left[ \frac{\alpha}{2} \right] + 1\) geometrically distinct closed characteristics \(\{(\tau_j, y_j)\}_{1 \leq j \leq \left[ \frac{\alpha}{2} \right] + 1} \) on \(\Sigma\).

Denote by \(\gamma_j \equiv y_j\) the associated symplectic path of \((\tau_j, y_j)\) for \(1 \leq j \leq \left[ \frac{\alpha}{2} \right] + 1\). Then by Lemma 1.3 of [17], there exist \(P_j \in \text{Sp}(2n)\) and \(M_j \in \text{Sp}(2n - 2)\) such that

\[
\gamma_j(\tau_j) = P_j^{-1}(N_1(1, 1) \diamond M_j) P_j, \quad 1 \leq j \leq \left[ \frac{\alpha}{2} \right] + 1.
\]

By Theorem 4.9 we obtain infinitely many tuples \((T, m_1, m_2, \ldots, m_{\left[ \frac{\alpha}{2} \right] + 1}) \in \mathbb{N}^{\left[ \frac{\alpha}{2} \right] + 2}\) such that the following hold

\[
i(y_j, 2m_j) \geq 2T - \frac{e(\gamma_j(\tau_j))}{2}, \quad (3.4)
i(y_j, 2m_j) + v(y_j, 2m_j) \leq 2T + \frac{e(\gamma_j(\tau_j))}{2} - 1, \quad (3.5)i(y_j, 2m_j + 1) = 2T + i(y_j, 1), \quad (3.6)i(y_j, 2m_j - 1) + v(y_j, 2m_j - 1) = 2T - \left( i(y_j, 1) + 2S^+_{y_j(\tau_j)}(1) - v(y_j, 1) \right). \quad (3.7)
\]
By Corollary 1.2 of [17], we have \( i(y_j, 1) \geq n \) for \( 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil + 1 \). Note that \( e(y_j(\tau_j)) \leq 2n \) for \( 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil + 1 \). Hence Theorem 4.8 implies

\[
i(y_j, m) + v(y_j, m) \leq i(y_j, m + 1) - i(y_j, 1) + \frac{e(y_j(\tau_j))}{2} - 1
\leq i(y_j, m + 1) - 1, \quad \forall m \in \mathbb{N}, \quad 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil + 1.
\tag{3.8}
\]

By Theorem 4.7, the matrix \( M_j \) can be connected with \( \Omega^0(M_j) \) to \( N_1(1, 1)^{\sigma p_j}\circ I_{2p_j} \circ N_1(1, -1)^{p_j}\circ M_j \), where \( p_{j-}, p_{j0}, p_{j+} \in \mathbb{N}_0 \) and \( 1 \notin \sigma(M_j) \) for \( 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil + 1 \). By Lemma 4.4, we have

\[
2S_j(y_j(\tau_j)) - v(y_j, 1) = 2S_{N_1(1, 1)} - v(N_1(1, 1)) + 2S_j(M_j) - v(M_j)
= 1 + p_{j-} - p_{j+}, \quad 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil + 1.
\tag{3.9}
\]

Note that by (3.1), (3.9) and (4.21), we have \( g_n(\Sigma) \geq \left\lceil \frac{n+1}{2} \right\rceil + 1 \) if there is no closed characteristic \((\tau_j, y_j)\) on \( \Sigma \) satisfying \( p_{j+} = n - 1 \) together with \( i(y_j, 1) = n \). Hence in order to prove Theorem 1.1, it is sufficient to consider the case that there exists some closed characteristic \((\tau_j, y_j)\) satisfying \( p_{j+} = n - 1 \) together with \( i(y_j, 1) = n \) for some \( j \in \{1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil + 1\}, \) i.e., the matrix \( M_j \) can be connected within \( \Omega^0(M_j) \) to \( N_1(1, 1)^{\sigma(n-1)} \). By a permutation of \( \{1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil + 1\} \) if necessary, we may assume without loss of generality that \( i(y_j, 1) = n \) and \( M_j \) can be connected within \( \Omega^0(M_j) \) to \( N_1(1, 1)^{\sigma(n-1)} \) whenever \( 1 \leq j \leq K \); \( i(y_j, 1) > n \) or \( M_j \) cannot be connected within \( \Omega^0(M_j) \) to \( N_1(1, 1)^{\sigma(n-1)} \) whenever \( K < j \leq \left\lceil \frac{n}{2} \right\rceil + 1 \), where \( K \) is some integer satisfying \( 1 \leq K \leq \left\lceil \frac{n}{2} \right\rceil + 1 \).

By Theorem 4.6, (3.8) and (3.9), (3.4)–(3.7) become

\[
i\left( y_j^{2m_j} \right) \geq 2T - 2n, \quad 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil + 1,
\tag{3.10}
i\left( y_j^{2m_j} \right) + v\left( y_j^{2m_j} \right) - 1 \leq 2T - 2, \quad 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil + 1,
\tag{3.11}
i\left( y_j^{2m_j+m} \right) \geq 2T, \quad \forall m \geq 1, \quad 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil + 1,
\tag{3.12}
i\left( y_j^{2m_j-1} \right) + v\left( y_j^{2m_j-1} \right) - 1 = 2T - n - 3, \quad 1 \leq j \leq K,
\tag{3.13}
i\left( y_j^{2m_j-m} \right) + v\left( y_j^{2m_j-m} \right) - 1 < 2T - n - 3, \quad \forall m \geq 2, \quad 1 \leq j \leq K,
\tag{3.14}
i\left( y_j^{2m_j-m} \right) + v\left( y_j^{2m_j-m} \right) - 1 < 2T - n - 3, \quad \forall m \geq 1, \quad K < j \leq \left\lceil \frac{n}{2} \right\rceil + 1.
\tag{3.15}
\]

Thus by Lemma 3.2, we can find \((j_k, l_{jk})\) \( 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 2 \) such that

\[
\Phi'\left( u_{jk}^{l_{jk}} \right) = 0, \quad \Phi\left( u_{jk}^{l_{jk}} \right) = c_{T+1-k}, \quad S_{c_{2T-2k}}\left( \Phi, \ S_{1} \cdot u_{jk}^{l_{jk}} \right) \neq 0,
\tag{3.16}
\]

where \( j_k = j(T + 1 - k) \) and \( l_{jk} = m(T + 1 - k) \). Note that by Proposition 2.1, the numbers \( c_{T+1-k} \) for \( 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 2 \) are pairwise distinct critical values of \( \Phi \), thus if \( j_k = j_{k'} \) for some \( 1 \leq k, k' \leq \left\lceil \frac{n}{2} \right\rceil + 2 \), we have \( l_{jk} \neq l_{j_{k'}} \). Hence we have \((j_k, l_{jk}) = (j_k, 2m_{jk}) \) for \( 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 1 \) and \( j_1, j_2, j_3, \ldots, j_{\left\lceil \frac{n}{2} \right\rceil + 1} \) are pairwise distinct, and then \( \{j_1, j_2, \ldots, j_{\left\lceil \frac{n}{2} \right\rceil + 1}\} = \{1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil + 1\} \). In fact, by Proposition 2.6 and (3.10)–(3.15), we
have $C_{S^1, 2T-2k}(\Phi, S^1 \cdot u_j^m) = 0$ for $1 \leq k, j \leq \lfloor \frac{n}{2} \rfloor + 1$ and any integer $m \neq 2m_j$. Thus $l_{jk} = 2m_{jk}$, and then $\Phi \left( u_{2m_{jk}} \right)_T = c_{T+1-k}$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$ by (3.16), hence $j_1, j_2, \ldots, j_{\lfloor \frac{n}{2} \rfloor + 1}$ are pairwise distinct.

The next property shows that for different choices of the tuples $(T, m_1, m_2, \ldots, m_{\lfloor \frac{n}{2} \rfloor + 1})$, the corresponding critical points can not stay in a fixed order in the common index jump intervals.

**Lemma 3.3** Let $1 \leq \alpha, \beta \leq \lfloor \frac{n}{2} \rfloor + 1$ and $\alpha \neq \beta$ be fixed. Then it is impossible that for any tuple $(T, m_1, m_2, \ldots, m_{\lfloor \frac{n}{2} \rfloor + 1})$ found by Theorem 4.9 satisfying (3.10)–(3.16) that we always have $m_\alpha i(\hat{y}_\alpha) > m_\beta i(\hat{y}_\beta)$.

**Proof** The proof is motivated by Theorem 5.4 of [17]. We argue by contradiction.

Let $v \in R^{12}$ be the vector associated with the symplectic paths $\{\gamma_1, \gamma_2, \ldots, \gamma_{\lfloor \frac{n}{2} \rfloor + 1}\}$ defined by (4.37), and let $A(v)$ be defined by (4.39). By Theorem 1.3 of [17], there are at least $q_n(S) - 1 \geq 1$ geometrically distinct closed characteristics on $\Sigma$ possessing irrational mean indices. Hence $v \in R^{12} \setminus Q^{12}$. If $V$ denotes the subspace associated with $v$ as in Theorem 4.10, then Theorem 4.11 implies that $\dim V \geq 1$.

Thus by Theorem 4.9, for any tuple $(T, \chi)$ satisfying (4.38), we always have

$$m_\alpha i(\hat{y}_\alpha) > m_\beta i(\hat{y}_\beta) \quad (3.17)$$

where $m_k = \left[ \left( \frac{T}{M_i(y_k)} \right) + \chi_k \right] M$ is given by (4.36); especially, (3.17) holds for any tuple $\chi(a) \equiv (\psi(a_1), \ldots, \psi(a_h))$, where $a = (a_1, \ldots, a_h) \in A(v)$ by Theorem 4.10.

Now we fix any $a = (a_1, \ldots, a_h) \in A(v)$ and let $\chi(a) \equiv (\psi(a_1), \ldots, \psi(a_h))$. By Theorems 4.10 and 4.11, we have $-a \in A(v)$ and $\chi(a) \neq \chi(-a)$. For any tuples $(T, \chi(a))$ and $(T', \chi(-a))$ satisfying (4.38), denote by $m_k = \left[ \left( \frac{T}{M_i(y_k)} \right) + \chi(a)_k \right] M$ and $m'_k = \left[ \left( \frac{T'}{M_i(y_k)} \right) + \chi(-a)_k \right] M$.

Let $\delta_1 > 0$ be small enough, $\Lambda = \max_{1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1} \hat{i}(y_j)$ and

$$t_0 = \frac{\delta_1}{6(|a| + 1)(M\Lambda + 1)}. \quad (3.18)$$

Note that in the proof of Theorem 4.9 (cf. Step 2 of Theorem 4.1 of [17]), we can further require $T$ such that the vector $\{T v\} - \chi(a)$ are located in a sufficiently small neighborhood inside the open ball in $V$ with radius $\delta_1/(6M\Lambda + 1)$ and centered at $a t_0$ (cf. p. 360 of [17]), i.e.,

$$\{T v\} - \chi(a) \in V, \quad ||\{T v\} - \chi(a) - a t_0|| < \frac{\delta_1}{6M\Lambda + 1}. \quad (3.19)$$

Then $||T v\} - \chi(a)|| < |a t_0| + \frac{\delta_1}{6M\Lambda + 1} \leq \frac{\delta_1}{2}$, hence we still have (4.38) for $\delta_1$ small enough.

**Claim 1** We have $a_\alpha i(\hat{y}_\alpha) - a_\beta i(\hat{y}_\beta) = 0$.

We prove the claim by contradiction. In fact, we can further require $T \in N$ so that the following holds:

$$\left| \left\{ \frac{T}{M_i(y_k)} \right\} - \chi(a)_k - a k t_0 \right| < \frac{t_0}{3\Lambda} \min_{a_\alpha \hat{i}(y_i) - a_\beta \hat{i}(y_j) \neq 0} |a_\alpha \hat{i}(y_i) - a_\beta \hat{i}(y_j)| \quad (3.20)$$
for $1 \leq k \leq \left\lceil \frac{d}{2} \right\rceil + 1$. Then we have
\[
m_{\alpha} \hat{i}(y_{\alpha}) - m_{\beta} \hat{i}(y_{\beta})
\]
\[
= \left( \left[ \frac{T}{M \hat{i}(y_{\alpha})} \right] + \chi(a)_{\alpha} \right) M \hat{i}(y_{\alpha}) - \left( \left[ \frac{T}{M \hat{i}(y_{\beta})} \right] + \chi(a)_{\beta} \right) M \hat{i}(y_{\beta})
\]
\[
= \left( \chi(a)_{\alpha} + \frac{T}{M \hat{i}(y_{\alpha})} - \left[ \frac{T}{M \hat{i}(y_{\alpha})} \right] \right) M \hat{i}(y_{\alpha})
\]
\[
- \left( \chi(a)_{\beta} + \frac{T}{M \hat{i}(y_{\beta})} - \left[ \frac{T}{M \hat{i}(y_{\beta})} \right] \right) M \hat{i}(y_{\beta})
\]
\[
= \left( \chi(a)_{\alpha} - \left[ \frac{T}{M \hat{i}(y_{\alpha})} \right] \right) M \hat{i}(y_{\alpha}) - \left( \chi(a)_{\beta} - \left[ \frac{T}{M \hat{i}(y_{\beta})} \right] \right) M \hat{i}(y_{\beta})
\]
\[
= \left( \chi(a)_{\alpha} - \left[ \frac{T}{M \hat{i}(y_{\alpha})} \right] \right) M \hat{i}(y_{\alpha}) - \left( \chi(a)_{\beta} - \left[ \frac{T}{M \hat{i}(y_{\beta})} \right] \right) M \hat{i}(y_{\beta})
\]
\[
= -M_{t_0}(a_\alpha \hat{i}(y_{\alpha}) - a_\beta \hat{i}(y_{\beta})) + \left( \chi(a)_{\alpha} - \left[ \frac{T}{M \hat{i}(y_{\alpha})} \right] + a_\alpha t_0 \right) M \hat{i}(y_{\alpha})
\]
\[
- \left( \chi(a)_{\beta} - \left[ \frac{T}{M \hat{i}(y_{\beta})} \right] + a_\beta t_0 \right) M \hat{i}(y_{\beta}).
\]

Thus by (3.17) and (3.20), we have
\[
a_\alpha \hat{i}(y_{\alpha}) \leq a_\beta \hat{i}(y_{\beta}).
\]

On the other hand, repeating this argument for $(T', \chi(-a))$, we obtain
\[
- a_\alpha \hat{i}(y_{\alpha}) \leq -a_\beta \hat{i}(y_{\beta}).
\]

Combining (3.22) and (3.23) we obtain Claim 1.

Let $V_{\alpha, \beta} = \{ a \in V | a_\alpha \hat{i}(y_{\alpha}) = a_\beta \hat{i}(y_{\beta}) \}$. We claim that $V_{\alpha, \beta} = V$. If $V_{\alpha, \beta} \neq V$, we define
\[
B(v) = A(v) \setminus V_{\alpha, \beta}.
\]

The set $B(v)$ is nonempty since $\dim V \geq 1$ and $A(v)$ is obtained from $V$ by deleting finitely many proper subspaces of $V$ by (ii) and (iii) of Theorem 4.11 and $V_{\alpha, \beta}$ is a proper subspace of $V$. Now choosing $a \in B(v)$, we have $a_\alpha \hat{i}(y_{\alpha}) = a_\beta \hat{i}(y_{\beta})$. This implies $a \in V_{\alpha, \beta}$, a contradiction. This contradiction shows that $V_{\alpha, \beta} = V$. On the other hand, the definition of $B(v)$ shows that $a \notin V_{\alpha, \beta}$. This contradiction implies that $V_{\alpha, \beta} = V$.

By (3.19), the vector $\{Tv\} - \chi(a)$ belongs to $V$, and thus belongs to $V_{\alpha, \beta}$. Then by the definition of $V_{\alpha, \beta}$, this implies
\[
(Tv_{\alpha}) - \chi(a)_{\alpha} \hat{i}(y_{\alpha}) = (Tv_{\beta}) - \chi(a)_{\beta} \hat{i}(y_{\beta}).
\]

By (4.37), this implies
\[
\left( \left[ \frac{T}{M \hat{i}(y_{\alpha})} \right] - \chi(a)_{\alpha} \right) \hat{i}(y_{\alpha}) = \left( \left[ \frac{T}{M \hat{i}(y_{\beta})} \right] - \chi(a)_{\beta} \right) \hat{i}(y_{\beta}).
\]

Thus by the third equality of (3.21), we have
\[
m_{\alpha} \hat{i}(y_{\alpha}) = m_{\beta} \hat{i}(y_{\beta}).
\]

This contradicts (3.17) and proves the lemma. \qed

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Now we can give the proof of Theorem 1.1.

**Proof of Theorem 1.1** We continue to use the notations as in (3.10)–(3.16). First note that we have \( l_{[\frac{n}{2}] + 1} = 2m_{[\frac{n}{2}] + 2} - 1 \) and moreover

\[
\Phi'(u_{[\frac{n}{2}] + 2}) = 0, \quad \Phi(u_{[\frac{n}{2}] + 2}) = cT - \{\pm\}_{1} - 1.
\]

(3.28)

for some \( j_{[\frac{n}{2}] + 2} \in \{1, \ldots, K\} \subset \{1, 2, \ldots, [\frac{n}{2}] + 1\} \). In fact, we have \( l_{[\frac{n}{2}] + 2} \in \{2m_{[\frac{n}{2}] + 2} - 1, 2m_{[\frac{n}{2}] + 2}\} \equiv \Delta \) since \( C_{S^{1}, 2T - [\frac{n}{2}] - 4} \left( \Phi, S^{1} \cdot u_{l_{[\frac{n}{2}] + 2}} \right) \neq 0 \) by (3.16), while we have \( C_{S^{1}, 2T - [\frac{n}{2}] - 4} \left( \Phi, S^{1} \cdot u_{m_{j_{[\frac{n}{2}] + 2}}} \right) = 0 \) for \( m \notin \Delta \) by (3.10)–(3.15), Proposition 2.6 and the fact that \( 2T - n - 3 = 2T - 2[\frac{n}{2}] - 4 \). On the other hand, since \( \{j_{1}, j_{2}, \ldots, j_{[\frac{n}{2}] + 1}\} = \{1, 2, \ldots, [\frac{n}{2}] + 1\} \) by the paragraph below (3.16), we have \( j_{[\frac{n}{2}] + 2} = j_{k} \) for some \( k \in \{1, 2, \ldots, [\frac{n}{2}] + 1\} \). Thus we have \( \Phi(u_{[\frac{n}{2}] + 2}) = \Phi(u_{j_{k}}) = cT + 1 - k \) and \( \Phi(u_{m_{j_{k}}}) = cT + 1 - ([\frac{n}{2}] + 2) \) by (3.16). By Proposition 2.1, we have \( cT + 1 - k \neq cT + 1 - ([\frac{n}{2}] + 2) \). Hence \( l_{[\frac{n}{2}] + 2} \neq 2m_{[\frac{n}{2}] + 2} \), which together with \( l_{[\frac{n}{2}] + 2} \in \Delta \) implies \( l_{[\frac{n}{2}] + 2} = 2m_{[\frac{n}{2}] + 2} - 1 \). Thus (3.28) follows from (3.16). Since

\[
C_{S^{1}, 2T - n - 3} \left( \Phi, S^{1} \cdot u_{m_{j_{[\frac{n}{2}] + 2}}} \right) = C_{S^{1}, 2T - [\frac{n}{2}] - 4} \left( \Phi, S^{1} \cdot u_{l_{[\frac{n}{2}] + 2}} \right) \neq 0,
\]

we must have \( j_{[\frac{n}{2}] + 2} \in \{1, \ldots, K\} \) by (3.10), (3.15) and Proposition 2.6.

Now we fix a tuple \((T^{*}, m_{1}^{*}, m_{2}^{*}, \ldots, m_{[\frac{n}{2}] + 2}^{*}) \) and \((y_{1}^{*}, I_{1}^{*})\) for \( 1 \leq k \leq [\frac{n}{2}] + 2 \) satisfying (3.10)–(3.16). Since \( \{j_{[\frac{n}{2}] + 2} \in \{1, \ldots, K\}, \) by a permutation of \([1, \ldots, K]\) if necessary, we may assume without loss of generality that \( j_{[\frac{n}{2}] + 2} = 1 \). Thus we have \( i(y_{1}) = 1 \) and \( M_{1} \) can be connected within \( \Omega^{1}(M_{1}) \) to \( N_{1}(1, -1)^{\infty(n-1)} \) by the definition of \( K \) (cf. the paragraph below (3.9)). Hence by Theorems 4.6, 4.7 and (3.3), we have

\[
i(y_{1}^{m}) = m(i(y_{1}, 1) + 1) - 1 - n = (n + 1)(m - 1),
\]

\[
\nu(y_{1}^{m}) = n, \quad \forall m \in N.
\]

(3.29)

By (3.13), we have \( i(y_{1}^{2m_{1}^{*} - 1}) + \nu(y_{1}^{2m_{1}^{*} - 1}) - 1 = 2T^{*} - n - 3 \), thus we have \( i(y_{1}^{2m_{1}^{*}}) + \nu(y_{1}^{2m_{1}^{*} - 1}) - 1 = 2T^{*} - 2 \). Hence by Propositions 2.6 and 2.8, we have \( K(u_{1}) = 1 \) and then

\[
\text{rank} C_{S^{1}, 2T^{*} - 2} \left( \Phi, S^{1} \cdot u_{1}^{2m_{1}^{*}} \right) = k_{\nu(u_{1}) - 1} \left( u_{1}^{2m_{1}^{*}} \right) = k_{\nu(u_{1}) - 1} \left( u_{1}^{2m_{1}^{*} - 1} \right) \neq 0,
\]

(3.30)

where the last inequality follows from (3.28) and the fact that \( 2T^{*} - n - 3 = 2T^{*} - 2([\frac{n}{2}] + 2) \).
Now suppose \((T, m_1, m_2, \ldots, m_{\lfloor \frac{n}{2} \rfloor + 1})\) is any tuple found by Theorem 4.9 and \((j_k, l_k)\) satisfy (3.10)–(3.16). Then by (3.13) and (3.29), we have \(i(y_1^{2m_1-1}) + v(y_1^{2m_1-1}) - 1 = 2T - n - 3\), and then we have \(i(y_1^{2m_1}) + v(y_1^{2m_1}) - 1 = 2T - 2\). Hence by Propositions 2.6 and 2.8, we have

\[
\begin{align*}
\text{rank} C_{S^1, 2T-2} \left( \Phi, S^1 \cdot u_1^{2m_1} \right) & = k_v(u_1^{2m_1}) - 1(u_1^{2m_1}) = k_v(u_1^{2m_1}) - 1(u_1^{2m_1}) = 0, \\
\text{rank} C_{S^1, 2T-2} \left( \Phi, S^1 \cdot u_1^{2m_1} \right) & \neq 0,
\end{align*}
\]

where the last inequality follows from (3.30). Hence

\[
\text{rank} C_{S^1, 2T-2-l} \left( \Phi, S^1 \cdot u_1^{2m_1} \right) = k_v(u_1^{2m_1}) - 1(l) = 0 \quad (3.32)
\]

for \(l \neq 0\) by (ii) of Proposition 2.9, i.e., \(u_1^{2m_1}\) is a local maximum in the local characteristic manifold \(W(u_1^{2m_1}). \) By (3.16), we have

\[
\Phi \left( u_{j_k}^{2m_1} \right) = c_T + 1 - k, \quad C_{S^1, 2T-2k} \left( \Phi, S^1 \cdot u_{j_k}^{2m_1} \right) \neq 0
\]

for \(1 \leq k \leq \left[ \frac{n}{2} \right] + 1\). Thus we must have \(j_1 = 1\) by (3.32). This implies

\[
\Phi \left( u_1^{2m_1} \right) = c_T > c_T + 1 - k_1 = \Phi \left( u_{j_k}^{2m_1} \right), \quad 2 \leq i \leq \left[ \frac{n}{2} \right] + 1
\]

for some \(k_i \in \{2, \ldots, \left[ \frac{n}{2} \right] + 1\}\) satisfying \(j_k = i\). Then by Lemma 3.2, we have \(m_1 \hat{i}(y_i) > m_j \hat{i}(y_i)\) for \(i \in \{2, \ldots, \left[ \frac{n}{2} \right] + 1\}\). This contradicts Lemma 3.3 and proves Theorem 1.1.

**Proof of Theorem 1.2** Denote the prime closed characteristics on \(\Sigma\) by \(\{(\tau_j, y_j)\}_{1 \leq j \leq q}\). Let \(y_j \equiv y_j\) be the associated symplectic path of \((\tau_j, y_j)\) for \(1 \leq j \leq q\). Then by Lemma 1.3 of [17], there exist \(P_j \in \text{Sp}(2n)\) and \(M_j \in \text{Sp}(2n-2)\) such that

\[
y_j(\tau_j) = P_j^{-1}(N_1(1, 1) \diamond M_j) P_j, \quad 1 \leq j \leq q.
\]

As in the proof of Theorem 1.1, using Theorems 4.6, 4.9, (3.8) and (3.9), we obtain some tuple \((T, m_1, m_2, \ldots, m_q) \in N^{q+1}\) such that the following hold

\[
\begin{align*}
i(y_j^{2m_j}) & \geq 2T - \frac{e(y_j^{2m_j})}{2} - n \geq 2T - 2n, \\
i(y_j^{2m_j}) + v(y_j^{2m_j}) - 1 & \leq 2T + \frac{e(y_j^{2m_j})}{2} - 2 - n \leq 2T - 2, \\
i(y_j^{2m_j+1}) & \geq 2T, \quad \forall m \geq 1, \\
i(y_j^{2m_j-1}) + v(y_j^{2m_j-1}) - 1 & \leq 2T - n - 3, \\
i(y_j^{2m_j-m}) + v(y_j^{2m_j-m}) - 1 & < 2T - n - 3, \quad \forall m \geq 2,
\end{align*}
\]

for \(1 \leq j \leq q\). Moreover, the equality in (3.37) holds if and only if \(i(y_j, 1) = n\) and \(M_j\) can be connected within \(\Omega_0^0(M_j)\) to \(N_1(1, -1)^{(n-1)}\).
As in the proof of Theorem 1.1, we can find pairwise distinct tuples \((j_k, l_{j_k})_{1 \leq k \leq \lceil \frac{q}{2} \rceil + 2}\) such that

\[
\Phi' \left( u_{j_k}^{l_{j_k}} \right) = 0, \quad \Phi \left( u_{j_k}^{l_{j_k}} \right) = c_T + 1 - k, \quad C_{S^1, 2T - 2k} \left( \Phi, S^1 \cdot u_{j_k}^{l_{j_k}} \right) \neq 0. \quad (3.39)
\]

Moreover, \((j_k, l_{j_k}) = (j_k, 2m_{j_k})\) for \(1 \leq k \leq \lceil \frac{q}{2} \rceil + 1\), and \(j_k\)s for \(1 \leq k \leq \lceil \frac{q}{2} \rceil + 1\) are pairwise distinct (cf. the paragraph below (3.16)).

\[\Box\]

**Claim 2** If \((\tau_j, y_j)\) is a hyperbolic closed characteristic for some \(j \in \{1, \ldots, q\}\), then \(C_{S^1, i}(\Phi, S^1 \cdot u_{j_k}^{2m_j}) = 0\) for \(i \neq 2T - n - 1\).

In fact, by (3.33), Theorems 4.6 and 4.7, a hyperbolic closed characteristic \((\tau_j, y_j)\) must satisfy \(e(y_j(\tau_j)) = 2\) and \(v(y_j^m) = 1\) for any \(m \in \mathbb{N}\); and then we have \(i(y_j^{2m}) = 2T - n - 1 = i(y_j^{2m_j}) + v(y_j^{2m_j}) - 1\) by (3.34) and (3.35). Thus, by Proposition 2.6, \(C_{S^1, i}(\Phi, S^1 \cdot u_{j_k}^{2m_j}) = 0\) for \(i \neq 2T - n - 1\), so that Claim 2 holds.

By (3.39) and Claim 2, the closed characteristics \((\tau_{j_k}, y_{j_k})\) for \(1 \leq k \leq \lceil \frac{q}{2} \rceil + 1\) must be non-hyperbolic if \(n\) is even. Hence Theorem 1.2 holds for \(n\) being even.

It remains to consider the case that \(n\) being odd. By (3.39) and Claim 2, the closed characteristics \((\tau_{j_k}, y_{j_k})\) for \(1 \leq k \leq \lceil \frac{q}{2} \rceil \) must be non-hyperbolic. Since already we have \(\lceil \frac{q}{2} \rceil\) non-hyperbolic closed characteristics, in order to prove Theorem 1.2, we need to find one more non-hyperbolic closed characteristic. We have the following two cases.

**Case 1** We have \(j_1 = j_{\lceil \frac{q}{2} \rceil + 2}\).

In this case, we must have \(i(y_{j_1}, 1) = n\) and \(M_{j_1}\) can be connected within \(\Omega_0(M_{j_1})\) to \(N_1(1, -1)^{o(n-1)}\). If this is not the case, then the strict inequality in (3.37) holds for \(j_1\), and then, by (3.34)–(3.38) and Proposition 2.6, we have

\[
C_{2T - n - 3} \left( \Phi, S^1 \cdot u_{j_1}^m \right) = 0, \quad \forall m \neq 2m_{j_1} \quad (3.40)
\]

By (3.39), we have

\[
\Phi \left( u_{j_1}^{2m_{j_1}} \right) = c_T \neq c_T - 1, \quad \Phi \left( u_{j_1}^{l_{\lceil \frac{q}{2} \rceil + 2}} \right) = \Phi \left( u_{j_1}^{l_{\lceil \frac{q}{2} \rceil + 2}} \right). \quad (3.41)
\]

Hence we have \(l_{\lceil \frac{q}{2} \rceil + 2} \neq 2m_{j_1}\), and this together with (3.40) contradicts to \(C_{2T - n - 3}(\Phi, S^1 \cdot u_{j_1}^{l_{\lceil \frac{q}{2} \rceil + 2}}) \neq 0\).

Hence by (3.33), Theorems 4.6 and 4.7, we have \(\hat{i}(y_{j_1}) \in \mathbb{N}\). Then by the proof of Theorem 5.3 of [17], we have \(\hat{i}(y_{j_{\lceil \frac{q}{2} \rceil + 1}}) \notin \mathbb{Q}\). In fact, suppose \(\hat{i}(y_{j_{\lceil \frac{q}{2} \rceil + 1}}) \in \mathbb{Q}\). Then by Theorem 4.9 we may require \(T \in \mathbb{N}\) further satisfies

\[
\frac{T}{\hat{M}(y_{j_{\lceil \frac{q}{2} \rceil + 1}})} \in \mathbb{N}, \quad \frac{T}{\hat{M}(y_{j_{\lceil \frac{q}{2} \rceil + 1}})} \in \mathbb{N}. \quad (3.42)
\]

Thus if we choose \(\chi\) to be a vertex of \([0, 1]^h\) with \(\chi_{j_1} = 0\) and \(\chi_{j_{\lceil \frac{q}{2} \rceil + 1}} = 0\), then (4.38) still holds for infinitely many \(T \in \mathbb{N}\) (cf. p. 359 of [17]). Hence we have...
\[ m_j \hat{i}(y_{j1}) = \left( \frac{T}{M \hat{i}(y_{j1})} + \chi_{j1} \right) \hat{M}(y_{j1}) \]
\[ = \frac{T}{M \hat{i}(y_{j1})} \hat{M}(y_{j1}) = T = \frac{T}{M \hat{i}\left(y_{j1+1}^2\right)} \hat{M}\left(y_{j1+1}^2\right) \]
\[ = \left( \frac{T}{M \hat{i}\left(y_{j1+1}^2\right)} + \chi_{j1} \right) \hat{M}\left(y_{j1+1}^2\right) = m_j \hat{i}(y_{j1+1}^2). \] (3.43)

On the other hand, (3.39) and Lemma 3.2 imply that \( 2m_j \hat{i}(y_{j1}) > 2m_j \hat{i}(y_{j1+1}^2) \). This contradiction proves that \( \hat{i}(y_{j1+1}^2) \notin \mathbb{Q} \), and then \( (\tau_{j1+1}, y_{j1+1}) \) must be non-hyperbolic. Hence the closed characteristics \( (\tau_k, y_k) \) for \( 1 \leq k \leq [\frac{n}{2}] + 1 \) are the desired \([\frac{n}{2}] + 1 \) non-hyperbolic closed characteristics. This proves Theorem 1.2 in this case.

**Case 2** We have \( j_1 \neq j_1^{\frac{n}{2}+2} \). We first prove the following claim:

**Claim 3** We have \( j_1^{\frac{n}{2}+2} \neq j_k \) for \( 2 \leq k \leq [\frac{n}{2}] + 1 \).

Suppose \( j_1^{\frac{n}{2}+2} = j_k \) for some \( 2 \leq k \leq [\frac{n}{2}] + 1 \). Then, \( i(y_{j_k}, 1) = n \) and \( M_{j_k} \) can be connected within \( \Omega^0(M_{j_k}) \) to \( N_1(1, -1)^{(n-1)} \); since otherwise the strict inequality in (3.37) holds for \( j_k \) and then we have \( C_{2T-n-3}(\Phi, S^1 \cdot u_{j_k}^m) = 0 \) for any \( m \neq 2m_{j_k} \) by (3.34)–(3.38) and Proposition 2.6, while by (3.39) we have \( \Phi(u_{j_k}^{2m_{j_k}}) = c_{T+1-k} \neq c_{T-1-\left[\frac{n}{2}\right]} = \Phi(u_{j_1^{\frac{n}{2}+2}}^{l_{j_1^{\frac{n}{2}+2}}}) \), hence we have \( l_{j_1^{\frac{n}{2}+2}} \neq 2m_{j_k} \); this contradicts \( C_{2T-n-3}(\Phi, S^1 \cdot u_{j_1^{\frac{n}{2}+2}}^{l_{j_1^{\frac{n}{2}+2}}}) \neq 0 \) in (3.39).

Then by (3.34)–(3.38) and Proposition 2.6, we have \( l_{j_1^{\frac{n}{2}+2}} \in \{2m_{j_k} - 1, 2m_{j_k}\} \). While by the above paragraph, we have \( l_{j_1^{\frac{n}{2}+2}} \neq 2m_{j_k} \). Hence we have \( l_{j_1^{\frac{n}{2}+2}} = 2m_{j_k} - 1 \) and then from (3.39) and the fact that \( 2T-n-3 = 2T-2(\left[\frac{n}{2}\right]+2) \) we conclude that

\[ C_{S^1, 2T-n-3}(\Phi, S^1 \cdot u_{j_k}^{2m_{j_k}}) \neq 0. \] (3.44)

By (3.33), Theorems 4.6 and 4.7, \( i(y_{j_k}^m) = (n+1)(m-1) \) and \( \nu(y_{j_k}^m) = n \) for \( m \in \mathbb{N} \). By (3.37), (3.44) and Proposition 2.6, \( i(y_{j_k}^{2m_{j_k}}) + \nu(y_{j_k}^{2m_{j_k}}) = 2T-n-3 \), and then \( i(y_{j_k}^{2m_{j_k}})) + \nu(y_{j_k}^{2m_{j_k}}) - 1 = 2T-2 \). Hence by Propositions 2.6, 2.8 with \( K(u_{j_k}) = 1 \) and (3.44), we have

\[ \text{rank}C_{S^1, 2T-2-l}(\Phi, S^1 \cdot u_{j_k}^{2m_{j_k}}) = k \]
\[ = \nu(u_{j_k}^{2m_{j_k}} - 1) \left( u_{j_k}^{2m_{j_k}} \right) = \nu(u_{j_k}^{2m_{j_k}} - 1) \left( u_{j_k}^{2m_{j_k}} - 1 \right) \]
\[ = \text{rank}C_{S^1, 2T-n-3}(\Phi, S^1 \cdot u_{j_k}^{2m_{j_k}-1}) \neq 0. \] (3.45)

Hence

\[ \text{rank}C_{S^1, 2T-2-l}(\Phi, S^1 \cdot u_{j_k}^{2m_{j_k}}) = k \]
\[ = \nu(u_{j_k}^{2m_{j_k}} - 1) \left( u_{j_k}^{2m_{j_k}} - 1 \right) = 0 \] (3.46)

for \( l \in \mathbb{Z} \backslash \{0\} \) by (ii) of Proposition 2.9. This contradicts \( C_{S^1, 2T-2k}(\Phi, S^1 \cdot u_{j_k}^{2m_{j_k}}) \neq 0 \) since \( k \neq 1 \). This proves the Claim 3.
Next we show that the closed characteristic \((\tau_{\lfloor \frac{n}{2} \rfloor + 2}, y_{\lfloor \frac{n}{2} \rfloor + 2})\) must be non-hyperbolic. In fact, suppose the contrary. Then by (3.34)–(3.38) and Proposition 2.6, we have

\[
C_{2T-n-3} \left( \Phi, S^1 \cdot u^m_{j_{\lfloor \frac{n}{2} \rfloor + 2}} \right) = 0, \quad \forall m \neq 2m_{j_{\lfloor \frac{n}{2} \rfloor + 2}}. \tag{3.47}
\]

On the other hand, by Claim 2, we have

\[
C_{2T-n-3} \left( \Phi, S^1 \cdot u^{2m_{j_{\lfloor \frac{n}{2} \rfloor + 2}}} \right) = 0. \tag{3.48}
\]

Now (3.47) and (3.48) contradicts \(C_{2T-n-3}(\Phi, S^1 \cdot u_{j_{\lfloor \frac{n}{2} \rfloor + 2}^0}) \neq 0\) in (3.39). Hence the closed characteristic \((\tau_{\lfloor \frac{n}{2} \rfloor + 2}, y_{\lfloor \frac{n}{2} \rfloor + 2})\) must be non-hyperbolic, and then \((\tau_j, y_j)\) for \(1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 2\) are the desired \([ \frac{n}{2} ] + 1\) non-hyperbolic closed characteristics. This proves Theorem 1.2 in this case. The proof of Theorem 1.2 is complete.

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Appendix: Index iteration theory for closed characteristics

In this section, we review briefly the index theory for symplectic paths developed by Long and his coworkers. All the details can be found in [16].

The symplectic group \(\text{Sp}(2n)\) is defined by

\[
\text{Sp}(2n) = \{ M \in \text{GL}(2n, \mathbb{R}) | M^T J M = J \}.
\]

For \(\tau > 0\) we consider paths in \(\text{Sp}(2n)\):

\[
\mathcal{P}_\tau(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) | \gamma(0) = I_{2n} \}.
\]

As in [15] we define

\[
D_\omega(M) = (-1)^{n-1} \omega^n \det(M - \omega I_{2n}), \quad \forall \omega \in \mathbf{U}, \ M \in \text{Sp}(2n).
\]

For \(\omega \in \mathbf{U}\), we define the following codimension 1 hypersurface in \(\text{Sp}(2n)\) as in [15]:

\[
\text{Sp}(2n)_{\omega}^0 = \{ M \in \text{Sp}(2n) | D_\omega(M) = 0 \}.
\]

For any \(M \in \text{Sp}(2n)_{\omega}^0\), we define the co-orientation of \(\text{Sp}(2n)_{\omega}^0\) at \(M\) by the positive direction \(d/dt Me^{ij}_\omega |_{t=0}\) of the path \(Me^{ij}_\omega\) with \(0 \leq t \leq \epsilon\) and \(\epsilon > 0\) being small enough. Denote by

\[
\text{Sp}(2n)_{\omega}^* = \text{Sp}(2n) \setminus \text{Sp}(2n)_{\omega}^0,
\]

\[
\mathcal{P}_{\tau, \omega}^*(2n) = \{ \gamma \in \mathcal{P}_\tau(2n) | \gamma(\tau) \in \text{Sp}(2n)_{\omega}^* \},
\]

\[
\mathcal{P}_{\tau, \omega}^0(2n) = \mathcal{P}_\tau(2n) \setminus \mathcal{P}_{\tau, \omega}^* (2n).
\]

For any two paths \(\xi\) and \(\eta: [0, \tau] \to \text{Sp}(2n)\) with \(\xi(\tau) = \eta(0)\), we define as usual:

\[
\eta \ast \xi(t) = \begin{cases} 
\xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\
\eta(2t - \tau), & \text{if } \tau/2 \leq t \leq \tau.
\end{cases}
\]
As in [16], for any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, we define the $\diamond$-product of $M_1$ and $M_2$ to be the following $2(m_1+m_2) \times 2(m_1+m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$ 

Denote by $M^{\circ k}$ the $k$-fold $\diamond$-product $M \diamond \cdots \diamond M$. One can easily check that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \circ \gamma_1(t)$ for $t \in [0, \tau]$.

We define a special path $\xi_n$:

$$\xi_n(t) = \left(2 - \frac{t}{\tau}, 0, (2 - \frac{t}{\tau})^{-1}\right) \diamond^n \text{ for } 0 \leq t \leq \tau. \quad (4.1)$$

**Definition 4.1** (cf. [15, 16]) For $\omega \in U$ and $M \in \text{Sp}(2n)$, define

$$v_\omega(M) = \dim \ker C(M - \omega I_{2n}). \quad (4.2)$$

For $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$v_\omega(\gamma) = v_\omega(\gamma(\tau)). \quad (4.3)$$

If $\gamma \in \mathcal{P}^*_{\tau, \omega}(2n)$, define

$$i_\omega(\gamma) = [\text{Sp}(2n)_{\omega}^0 : \gamma \ast \xi_n], \quad (4.4)$$

where the right hand side of (4.4) is the usual homotopy intersection number, and the orientation of $\gamma \ast \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}^0_{\tau, \omega}(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of $\gamma$ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{i_\omega(\beta) | \beta \in U \cap \mathcal{P}^*_{\tau, \omega}(2n)\}. \quad (4.5)$$

Then the tuple

$$(i_\omega(\gamma), v_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},$$

is called the index function of $\gamma$ at $\omega$.

For a symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbb{N}$, we define its $m$th iteration $\gamma^m : [0, m\tau] \to \text{Sp}(2n)$ to be

$$\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for } j\tau \leq t \leq (j + 1)\tau, \quad j = 0, 1, \ldots, m - 1. \quad (4.6)$$

We also denote the extended path on $[0, +\infty)$ by $\gamma$.

**Definition 4.2** (cf. [15, 16]) For $\gamma \in \mathcal{P}_\tau(2n)$, define

$$(i(\gamma, m), v(\gamma, m)) = (i_1(\gamma^m), v_1(\gamma^m)), \quad \forall m \in \mathbb{N}. \quad (4.7)$$

We define the mean index $\hat{i}(\gamma, m)$ per $m\tau$ for $m \in \mathbb{N}$ to be

$$\hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k}. \quad (4.8)$$
For $M \in \text{Sp}(2n)$ and $\omega \in \mathbf{U}$, we define the splitting numbers $S_{M}^{\pm}(\omega)$ of $M$ at $\omega$ to be

$$S_{M}^{\pm}(\omega) = \lim_{\epsilon \to 0^{+}} i_{\omega} \exp(\pm \sqrt{-1} \epsilon)(\gamma) - i_{\omega}(\gamma),$$

(4.9)

where $\gamma \in \mathcal{P}_{\tau}(2n)$ is any path satisfying $\gamma(\tau) = M$.

Given a path $\gamma \in \mathcal{P}_{\tau}(2n)$, we want to deform it to a new path $\eta$ in $\mathcal{P}_{\tau}(2n)$ such that

$$i_{1}(\gamma^{m}) = i_{1}(\eta^{m}), \quad v_{1}(\gamma^{m}) = v_{1}(\eta^{m}), \quad \forall m \in \mathbf{N},$$

(4.10)

and that $(i_{1}(\eta^{m}), v_{1}(\eta^{m}))$ is easy enough to compute. This leads to finding homotopies $\delta : [0, 1] \times [0, \tau] \to \text{Sp}(2n)$ starting from $\gamma$ in $\mathcal{P}_{\tau}(2n)$ and keeping the end points of the homotopy always stay in a certain suitably chosen maximal subset of $\text{Sp}(2n)$ so that (4.10) holds. Actually this set was first discovered in [15] as the path connected component $\Omega^0(M)$ containing $M = \gamma(\tau)$ of the set

$$\Omega(M) = \{N \in \text{Sp}(2n) | \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and } v_{2}(N) = v_{2}(M), \quad \forall \lambda \in \sigma(M) \cap \mathbf{U}\}.$$  

(4.11)

Here $\Omega^0(M)$ is called the homotopy component of $M$ in $\text{Sp}(2n)$.

Note that we have the following $2 \times 2$ and $4 \times 4$ symplectic matrix as basic normal forms (cf. [15, 16]):

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda = \pm 2,$$

(4.12)

$$N_{1}(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1, b = \pm 1, 0,$$

(4.13)

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi),$$

(4.14)

$$N_{2}(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi),$$

(4.15)

where $b = \begin{pmatrix} b_{1} & b_{2} \\ b_{3} & b_{4} \end{pmatrix}$ with $b_{i} \in \mathbf{R}$ and $b_{2} \neq b_{3}$. Moreover, we call $N_{2}(\omega, b)$ trivial if $(b_{2} - b_{3}) \sin \theta > 0$; and non-trivial if $(b_{2} - b_{3}) \sin \theta < 0$.

We have the following properties for splitting numbers:

**Lemma 4.3** (cf. [15] and Lemma 9.1.5 of [16]) Splitting numbers $S_{M}^{\pm}(\omega)$ are independent of the choice of the path $\gamma \in \mathcal{P}_{\tau}(2n)$ satisfying $\gamma(\tau) = M$ appeared in (4.9). Moreover, for $\omega \in \mathbf{U}$ and $M \in \text{Sp}(2n)$, splitting numbers $S_{N}^{\pm}(\omega)$ are constant for all $N \in \Omega^0(M)$.

**Lemma 4.4** (cf. [15], Lemma 9.1.5 and List 9.1.12 of [16]) For $M \in \text{Sp}(2n)$ and $\omega \in \mathbf{U}$, we have

$$S_{M}^{\pm}(\omega) = 0, \quad \text{if } \omega \notin \sigma(M).$$

(4.16)

$$S_{N_{1}(1, \omega)}^{\pm}(1) = \begin{cases} 1, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases}$$

(4.17)

For $M_{i} \in \text{Sp}(2n_{i})$ with $i = 0$ and $1$, there holds

$$S_{M_{0} \circ M_{1}}^{\pm}(\omega) = S_{M_{0}}^{\pm}(\omega) + S_{M_{1}}^{\pm}(\omega), \quad \forall \omega \in \mathbf{U}.$$  

(4.18)

We have the following symplectic additivity property for index functions:
Theorem 4.5 (cf. Theorem 6.1 of [17] or Theorem 6.2.7 of [16]) For any $\gamma_j \in \mathcal{P}_\tau (2n_j)$ with $j = 0, 1$, we have

$$i_\omega (\gamma_0 \diamond \gamma_1) = i_\omega (\gamma_0) + i_\omega (\gamma_1).$$

(4.19)

Now we use the above defined index function to study the Morse indices of the Clarke–Ekeland dual functional $\Phi$ at a critical point. Let $\Sigma \in \mathcal{H}(2n)$. Using notations in §1, for any closed characteristic $(\tau, y)$ and $m \in \mathbb{N}$, we define its $m$th iteration $y^m : \mathbb{R}/(m \tau \mathbb{Z}) \to \mathbb{R}^{2n}$ by

$$y^m(t) = y(t - j \tau), \quad \text{for } j \tau \leq t \leq (j + 1) \tau, \quad j = 0, 1, 2, \ldots, m - 1.$$  

(4.20)

We still denote by $y$ its extension to $[0, +\infty)$. We define via Definition 4.2 the following

$$S^+ (y) = S^+_{\tau \gamma_j (\tau)} (1),$$

$$i(y, m) = i(\gamma_j, m),$$

(4.21)

(4.22)

for all $m \in \mathbb{N}$, where $\gamma_j$ is the associated symplectic path of $(\tau, y)$. The following theorem describe the relation between the above defined indices and the indices defined by Ekeland [2–4]. Thus we can compute the Morse indices of $\Phi$ at a critical point by those of its associated symplectic paths.

Theorem 4.6 (cf. Lemma 1.1 of [17], Theorem 15.1.1 of [16]) Suppose $(\tau, y)$ is a closed characteristic on $\Sigma$. Then we have

$$i (y^m) \equiv i (m \tau, y) = i (y, m) - n, \quad v(y^m) \equiv v(m \tau, y) = v(y, m), \quad \forall m \in \mathbb{N},$$

(4.24)

where $i (y^m)$ and $v(y^m)$ are the index and nullity defined by Ekeland [2–4]. In particular, we have $\hat{i}(y) = \hat{i}(y, 1)$, where $\hat{i}(y)$ is the mean index of $(\tau, y)$ defined by Ekeland.

The following is the precise index iteration formulae for symplectic paths, which is due to Long (cf. Theorem 8.3.1 and Corollary 8.3.2 of [16]). Using this formula, we can compute the indices of any iteration of a fixed symplectic path whenever we know its initial index and end point in $\text{Sp}(2n)$. Theorem 4.7 Let $\gamma \in \mathcal{P}_\tau (2n)$, Then there exists a path $f \in C([0, 1], \Omega^0(\gamma(\tau)))$ such that $f(0) = \gamma(\tau)$ and

$$f(1) = N_2(\lambda, v_1) \cdots N_2(\lambda_{r_0}, v_{r_0}) \circ M_0$$

(4.25)

where $N_2(\omega_j, u_2)$s are non-trivial and $N_2(\lambda_j, v_j)$s are trivial basic normal forms; $\sigma(M_0) \cap U = \emptyset$; $p_-, p_0, p_+, q_-, q_0, q_+, r, r_*$ and $r_0$ are non-negative integers; $\omega_j = e^{\sqrt{-1} \alpha_j}$, $\lambda_j = e^{\sqrt{-1} \beta_j}$; $\alpha_j, \beta_j, \gamma_j \in (0, \pi) \cup (\pi, 2\pi)$; these integers and real numbers are uniquely determined by $\gamma(\tau)$. Then using the functions defined in (1.7).

$$i(y, m) = m(i(y, 1) + p_+ + p_0 - r) + 2 \sum_{j=1}^{r} \frac{m \theta_j}{2\pi} - r - p_- - p_0 - \frac{1}{2}(q_0 + q_+) + 2 \left( \sum_{j=1}^{r_0} q_j \frac{m \alpha_j}{2\pi} - r_* \right).$$

(4.26)
\[ v(\gamma, m) = v(\gamma, 1) + \frac{1 + (-1)^m}{2}(q_- + 2q_0 + q_+) + 2(r + r_* + r_0) \]
\[ -2 \left( \sum_{j=1}^{r} \varphi \left( \frac{m\theta_j}{2\pi} \right) + \sum_{j=1}^{r_*} \varphi \left( \frac{m\alpha_j}{2\pi} \right) + \sum_{j=1}^{r_0} \varphi \left( \frac{m\beta_j}{2\pi} \right) \right) \tag{4.27} \]
\[ \hat{i}(\gamma, 1) = i(\gamma, 1) + p_- + p_0 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi}. \tag{4.28} \]

Moreover, we have \( i(\gamma, 1) \) is odd if \( f(1) = N_1(1, 1), I_2, N_1(-1, 1), -I_2, N_1(-1, -1) \) and \( R(\theta); i(\gamma, 1) \) is even if \( f(1) = N_1(1, -1) \) and \( N_2(\omega, b); i(\gamma, 1) \) can be any integer if \( \sigma(f(1)) \cap U = \emptyset. \)

We have the following iteration inequalities for the index function:

**Theorem 4.8** (cf. Theorem 2.3 of [17]) Let \( \gamma \in \mathcal{P}_\tau(2n) \) and \( M = \gamma(\tau) \). Suppose that there exist \( P \in \text{Sp}(2n) \) and \( Q \in \text{Sp}(2n - 2) \) such that \( M = P^{-1}(N_1(1, 1) \circ Q)P \). Then for any \( m \in \mathbb{N} \), there holds
\[ v(\gamma, m) - \frac{e(M)}{2} + 1 \leq i(\gamma, m + 1) - i(\gamma, m) - i(\gamma, 1) \leq v(\gamma, 1) - v(\gamma, m + 1) + \frac{e(M)}{2}, \]
where \( e(M) \) is the total algebraic multiplicity of all eigenvalues of \( M \) on the unit circle in the complex plane \( C. \)

The following is the common index jump theorem of Long and Zhu.

**Theorem 4.9** (cf. Theorems 4.1–4.4 of [17]) Let \( \gamma_k \in \mathcal{P}_\tau(2n) \) for \( k = 1, \ldots, q \) be a finite collection of symplectic paths. Let \( M_k = \gamma_k(\tau_k) \). Suppose that there exist \( P_k \in \text{Sp}(2n) \) and \( Q_k \in \text{Sp}(2n - 2) \) such that \( M_k = P_k^{-1}(N_1(1, 1) \circ Q_k)P_k \) and \( \hat{i}(\gamma_k, 1) > 0 \), for all \( k = 1, \ldots, q \). Then there exist infinitely many \( (T, m_1, \ldots, m_q) \in \mathbb{N}^{q+1} \) such that
\[ v(\gamma_k, 2m_k - 1) = v(\gamma_k, 1), \tag{4.29} \]
\[ v(\gamma_k, 2m_k + 1) = v(\gamma_k, 1), \tag{4.30} \]
\[ i(\gamma_k, 2m_k - 1) + v(\gamma_k, 2m_k - 1) = 2T - (i(\gamma, 1) + 2S_{M_k}^+(1) - v(\gamma_k, 1)), \tag{4.31} \]
\[ i(\gamma_k, 2m_k + 1) = 2T + i(\gamma_k, 1), \tag{4.32} \]
\[ i(\gamma_k, 2m_k) \geq 2T - \frac{e(M_k)}{2} \geq 2T - n, \tag{4.33} \]
\[ i(\gamma_k, 2m_k) + v(\gamma_k, 2m_k) \leq 2T + \frac{e(M_k)}{2} - 1 \leq 2T + n - 1, \tag{4.34} \]
for every \( k = 1, \ldots, q \). Moreover we have
\[ \min \left\{ \left\lfloor \frac{m_k\theta}{\pi} \right\rfloor, 1 - \left\lfloor \frac{m_k\theta}{\pi} \right\rfloor \right\} < \delta, \tag{4.35} \]
whenever \( e^{\sqrt{-1}\theta} \in \sigma(M_k) \) and \( \delta \) can be chosen as small as we want (cf. (4.43) of [17]). More precisely, by (4.10) and (4.40) in [17], we have
\[ m_k = \left( \left\lfloor \frac{T}{\hat{M}\hat{i}(\gamma_k, 1)} \right\rfloor + \chi_k \right) M, \quad 1 \leq k \leq q. \tag{4.36} \]
where \( \chi_k = 0 \) or 1 for \( 1 \leq k \leq q \) and \( \frac{M_0}{\sqrt{\pi}} \in \mathbb{Z} \) whenever \( e^{\sqrt{-1} \theta} \in \sigma(M_k) \) and \( \frac{\theta}{\sqrt{\pi}} \in \mathbb{Q} \) for some \( 1 \leq k \leq q \). Furthermore, given \( M_0 \in \mathbb{N} \), by the proof of Theorem 4.1 of [17], we may further require \( M_0 \mid T \) (since the closure of the set \( \{Tv : T \in \mathbb{N}, \ M_0 \mid T \} \) is still a closed additive subgroup of \( T^h \) for some \( h \in \mathbb{N} \), where we use notations as (4.21) in [17]. Then we can use the proof of Step 2 in Theorem 4.1 of [17] to get \( T \).

In fact, let \( \mu_i = \sum_{\theta \in (0, 2\pi]} S_{M_i}^{-1}(e^{\sqrt{-1} \theta}) \) for \( 1 \leq i \leq q \) and \( \alpha_{i,j} = \frac{\theta_j}{\pi} \) where \( e^{\sqrt{-1} \theta_j} \in \sigma(M_j) \) for \( 1 \leq j \leq \mu_i \) and \( 1 \leq i \leq q \). Let \( h = q + \sum_{1 \leq i \leq q} \mu_i \) and

\[
v = \left( \frac{1}{M^i(\gamma_1, 1)}, \ldots, \frac{1}{M^i(\gamma_q, 1)}, \frac{\alpha_{1,1}}{i(\gamma_1, 1)}, \frac{\alpha_{1,2}}{i(\gamma_1, 1)}, \ldots, \frac{\alpha_{1,\mu_1}}{i(\gamma_1, 1)}, \frac{\alpha_{2,1}}{i(\gamma_2, 1)}, \ldots, \frac{\alpha_{q,\mu_q}}{i(\gamma_q, 1)} \right)
\]

\( \in \mathbb{R}^h \).

Then the above theorem is equivalent to finding a vertex of the unit cube \( [0, 1]^h \) and infinitely many \( T \in \mathbb{N} \) such that

\[
|\{Tv\} - \chi| < \epsilon
\]

for any given \( \epsilon \) small enough (cf. pp. 346 and 349 of [17]).

The next theorem describe how to choose a vertex satisfying (4.38).

**Theorem 4.10** (cf. Theorem 4.2 of [17]) Let \( H \) be the closure of the subset \( \{mv\}|m \in \mathbb{N} \) in \( T^h = (\mathbb{R}/\mathbb{Z})^h \) and \( V = T_0 \pi^{-1} H \) be the tangent space of \( \pi^{-1} H \) at the origin in \( \mathbb{R}^h \), where \( \pi : \mathbb{R}^h \to T^h \) is the projection map. Define

\[
A(v) = V \setminus \cup_{x \in \mathbb{R} \setminus \mathbb{Q}} \{x = (x_1, \ldots, x_h) \in V | x_k = 0\}.
\]

Define \( \psi(x) = 0 \) when \( x \geq 0 \) and \( \psi(x) = 1 \) when \( x < 0 \). Then for any \( a = (a_1, \ldots, a_h) \in A(V) \), the vector

\[
\chi = (\psi(a_1), \ldots, \psi(a_h))
\]

makes (4.38) hold for infinitely many \( T \in \mathbb{N} \).

Moreover, we have the following property for the set \( A(v) \):

**Theorem 4.11** (cf. Theorem 4.2 of [17])

(i) If \( v \in \mathbb{R}^h \setminus \mathbb{Q}^h \), then \( \dim V \geq 1 \), \( 0 \not\in A(v) \subset V \), \( A(v) = -A(v) \) and \( A(v) \) is open in \( V \).
(ii) If \( \dim V = 1 \), then \( A(v) = V \setminus \{0\} \).
(iii) If \( \dim V \geq 2 \), then \( A(v) \) is obtained from \( V \) by deleting all the coordinate hyperplanes with dimension strictly smaller than \( \dim V \).

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