Torsion-free hyperbolic groups and the finite cover property

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Abstract

We prove that the first order theory of non abelian free groups eliminates the $\exists^{\infty}$-quantifier (in $eq$). Equivalently, since the theory of non abelian free groups is stable, it does not have the finite cover property.

We extend our results to torsion-free hyperbolic groups under some conditions.

1 Introduction

In [Sel13] Sela proved that the theory of any torsion-free hyperbolic group is stable. This astonishing result renewed the interest of model theorists in the first order theories of such groups. Other classes of groups for which the theory of any of their elements is known to be stable, is the class of abelian groups and the class of algebraic groups over algebraically closed fields. It is quite interesting that model theory has the power to study such diverse classes of groups under a common perspective, the perspective of stability.

In this paper we strengthen the above mentioned result. Our main theorem is:

Theorem 1: Let $\mathbb{F}$ be a non abelian free group. Then $Th(\mathbb{F})^{eq}$ eliminates the $\exists^{\infty}$-quantifier.

An immediate corollary, since the first order theory of non abelian free groups is stable, is the following:

Corollary 1: The theory of non abelian free groups does not have the finite cover property (nfcp).

As a matter of fact the two statements are equivalent assuming stability. For a discussion of these notions we refer the reader to section 2.

Our main theorem provides model theorists with strong tools to study the model theory of the free group. We expect many applications esoteric to model theory but we also believe that this will be the first step towards applications of model theory to geometric group theory.

The paper is structured as follows. Sections 2 and 3 serve as introductions to model theory and geometric group theory respectively. The purpose is to quickly introduce readers lacking some of these backgrounds to the notions and tools that we will use. Needless to say the treatment is by no means complete. The central notions in section 2 will be the model theoretic imaginaries and the finite cover property. In section 3 we will start by explaining Bass-Serre theory, which will be also useful for fixing the vocabulary for later use. Then we

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will pass to generalizations of Bass-Serre theory to actions of groups on real trees and Rips’ machine. These notions will dominate the core of our proofs.

In section 4, we define limit groups using the Bestvina-Paulin method (as they were originally defined in [Sel01]), a tool that will prove useful throughout the paper. We will continue with the presentation of solid limit groups and we will finish this section with the theorem of Sela bounding the number of “strictly solid families” of morphisms associated to a solid limit group. The previously mentioned theorem, which is Theorem 4.10 in our paper, lies behind our main idea.

In section 5, we define towers and test sequences on them. The material here is an alternative presentation of the content in [Sel03]. With the notion of a tower we try to “catch” in a more abstract form the notion of a “completed well-structured MR resolution” for a limit group. We are influenced by the definition of “hyperbolic towers” in [Per11]. We then refer to the work of Sela for the definition of a test sequence (for a tower) and we finish this section with abstracting some properties of test sequences that will be enough for our purposes.

Section 6 is the core of our paper. Here is where all the technical results are proved. In brief, the idea is that we are able to say something about an element that lives in a tower provided that we know the behavior of its images under a test sequence for the tower.

In section 7, we present the graded Diophantine envelope. A strong tool whose existence has been proved by Sela in [Sel]. Graded Diophantine envelopes together with Theorem 4.10 are the main pylons of our proof.

In section 8, we bring everything together in order to prove the main results of this paper.

In the final section we add some remarks on how to extend our proof to the context of torsion-free hyperbolic groups.

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2 Some model theory

In this section we give the model theoretic background needed for the rest of the paper and we place our results into context. We will concentrate on the finite cover property and the notion of imaginaries. Since our result might be interesting for a wider audience, our exposition is meant to benefit the reader that has no prior knowledge of model theory.

In the final subsection we specify the above model theoretic notions to the first order theories of torsion-free hyperbolic groups.
2.1 Imaginaries

We fix a first order structure $\mathcal{M}$ and we are interested in the collection of definable sets in $\mathcal{M}$, i.e. all subsets of some Cartesian power of $\mathcal{M}$ (the domain of $\mathcal{M}$) which are the solution sets of first order formulas (in $\mathcal{M}$). In some cases one can easily describe this collection usually thanks to some quantifier elimination result. For example, as algebraically closed fields admit (full) quantifier elimination (in the language of rings) all definable sets are boolean combinations of algebraic sets, i.e. sets defined by polynomial equations. On the other hand, although free groups admit quantifier elimination down to boolean combinations of $\forall \exists$ formulas (see [Sel05b, Sel06]), the "basic" definable sets are not so easy to describe.

Suppose $X$ is a definable set in $\mathcal{M}$. One might ask whether there is a canonical way to define $X$, i.e. is there a tuple $\bar{b}$ and a formula $\psi(\bar{x}, \bar{y})$ such that $\mathcal{M} | = \psi(M, \bar{b}) = X$ but for any other $\bar{b}' \neq \bar{b}$, $\psi(\mathcal{M}, \bar{b}') \neq X$?

To give a positive answer to the above mentioned question one has to move to a mild expansion of $\mathcal{M}$ called $\mathcal{M}^{eq}$. Very briefly $\mathcal{M}^{eq}$ is constructed from $\mathcal{M}$ by adding a new sort for each $\emptyset$-definable equivalence relation, $E(\bar{x}, \bar{y})$, together with a class function $f_E : M^n \rightarrow S_E(M)$, where $S_E(M)$ (the domain of the new sort corresponding to $E$) is the set of all $E$-equivalence classes. The elements in these new sorts are called imaginaries. In $\mathcal{M}^{eq}$, it is not hard to see that one can assign to each definable set a canonical parameter in the sense discussed above. Let us also record the following useful lemma.

**Lemma 2.1:** Let $\phi(x_1, \ldots, x_l)$ be a first order $L_\mathcal{M}$-formula in the expanded language $L_\mathcal{M}^{eq}$, i.e. the language $L_\mathcal{M}$ expanded with function symbols and variables corresponding to the imaginary sorts. For each $i \leq l$, suppose $x_i$ belongs to the sort corresponding to some $\emptyset$-definable equivalence relation $E_i$.

Then there exists a first order $L_\mathcal{M}$-formula $\psi(\bar{z}_1, \ldots, \bar{z}_l)$ such that

$\mathcal{M}^{eq} | = \forall \bar{z}_1, \ldots, \bar{z}_l(\phi(f_{E_1}(\bar{z}_1), \ldots, f_{E_l}(\bar{z}_l)) \leftrightarrow \psi(\bar{z}_1, \ldots, \bar{z}_l))$

2.2 Stability

For a thorough introduction to stability we refer the reader to [Pil96]. We will now quickly record the notions and results we are interested in.

**Definition 2.2:** Let $\phi(\bar{x}, \bar{y})$ be a first order $L$-formula. Then $\phi(\bar{x}, \bar{y})$ has the order property in an $L$-structure $\mathcal{M}$, if there exist infinite sequences $(\bar{a}_n)_{n<\omega}$, $(\bar{b}_n)_{n<\omega}$ such that $\mathcal{M} | = \phi(\bar{a}_i, \bar{b}_j)$ if and only if $i < j$.

**Definition 2.3:** A first order theory $T$ is stable if no formula has the order property in any model of $T$.

Stable theories can also be introduced as theories that support a good independence relation, for a discussion towards this end we refer the reader to [PS13, Section 2].

The following property has been introduced in [Kei67] by Keisler.

**Definition 2.4:** Let $\phi(\bar{x}, \bar{y})$ be a first order $L$-formula. Then $\phi(\bar{x}, \bar{y})$ has the finite cover property in an $L$-structure $\mathcal{M}$, if for arbitrarily large $n$ there exist $\bar{b}_n$ such that:

$\mathcal{M} | = \neg \exists \bar{x} \bigwedge_{i \leq n} \phi(\bar{x}, \bar{b}_i)$
But for each \( j \leq n \):
\[
M \models \exists \bar{x} \bigwedge_{i \leq n, i \neq j} \phi(\bar{x}, \bar{b}_i)
\]

As before the property passes to a first order theory as follows:

**Definition 2.5:** A first order theory \( T \) does not have the finite cover property (or \( T \) has nfcp), if no formula has the finite cover property in any model of \( T \).

The following lemma is almost immediate:

**Lemma 2.6:** If a first order theory is unstable then it has the finite cover property.

There is an equivalent point of view towards nfcp if we assume stability. Towards this end we define:

**Definition 2.7:** A first order theory \( T \) eliminates the “there exists infinitely many” quantifier (or \( T \) eliminates the \( \exists^\infty \)-quantifier) if to each first order formula \( \phi(\bar{x}, \bar{y}) \) we can assign a natural number \( n_\phi \) such that for any model \( M \) of \( T \), for any \( \bar{b} \in M \), if \( |\phi(M, \bar{b})| < \infty \), then \( |\phi(M, \bar{b})| < n_\phi \).

**Theorem 2.8:** A stable first order theory \( T \) does not have the finite cover property if and only if \( T^{eq} \) eliminates the “there exists infinitely many” quantifier.

### 2.3 Some model theory of torsion-free hyperbolic groups

We will first concentrate on the notion of imaginaries in torsion-free hyperbolic groups.

We start by defining some “basic” families of imaginaries.

**Definition 2.9:** Let \( \Gamma \) be a torsion-free hyperbolic group. The following equivalence relations in \( \Gamma \) are called basic.

\( (i) \) \( E_1(a, b) \) if and only if there is \( g \in \Gamma \) such that \( a^g = b \). (conjugation)

\( (ii) \) \( E_2_m((a_1, b_1), (a_2, b_2)) \) if and only if either \( b_1 = b_2 = 1 \) or \( b_1 \neq 1 \) and \( C_\Gamma(b_1) = C_\Gamma(b_2) = (b) \) and \( a_1^{-1}a_2 \in \langle b^m \rangle \). (\( m \)-left-coset)

\( (iii) \) \( E_3_m((a_1, b_1), (a_2, b_2)) \) if and only if either \( b_1 = b_2 = 1 \) or \( b_1 \neq 1 \) and \( C_\Gamma(b_1) = C_\Gamma(b_2) = (b) \) and \( a_1a_2^{-1} \in \langle b^m \rangle \). (\( m \)-right-coset)

\( (iv) \) \( E_4_{m,n}((a_1, b_1, c_1), (a_2, b_2, c_2)) \) if and only if either \( a_1 = a_2 = 1 \) or \( c_1 = c_2 = 1 \) or \( a_1, c_1 \neq 1 \) and \( C_\Gamma(a_1) = C_\Gamma(a_2) = \langle a \rangle \) and \( C_\Gamma(c_1) = C_\Gamma(c_2) = \langle c \rangle \) and there is \( \gamma \in \langle a^m \rangle \) and \( \epsilon \in \langle c^n \rangle \) such that \( \gamma b_1 \epsilon = b_2 \). (\( m, n \)-double-coset)

It is almost immediate that \( m \)-left cosets eliminate \( m \)-right cosets (and vice versa), so from now on we are economic and forget about the \( m \)-right-cosets (this observation, which does not use any special property of torsion-free hyperbolic groups, has been pointed out to us by Ehud Hrushovski).

Sela proved the following theorem concerning imaginaries in torsion-free hyperbolic groups (see [Sel] Theorem 4.4).

**Theorem 2.10:** Let \( \Gamma \) be a torsion-free hyperbolic group. Let \( E(\bar{x}, \bar{y}) \) be a definable equivalence relation in \( \Gamma \), with \( |\bar{x}| = m \). Then there exist \( k, l < \omega \) and a definable relation:

\[
R_E \subseteq \Gamma^m \times \Gamma^k \times S_1(\Gamma) \times \ldots \times S_l(\Gamma)
\]

such that:
each $S_i(\Gamma)$ is one of the basic sorts;
(ii) for each $\bar{a} \in \Gamma^m$, $|R_E(\bar{a}, \bar{z})|$ is uniformly bounded (i.e. the bound does not depend on $\bar{a}$);
(iii) $\Gamma \models \forall \bar{z}(R_E(\bar{a}, \bar{z}) \leftrightarrow R_E(\bar{b}, \bar{z}))$ if and only if $E(\bar{a}, \bar{b})$.

The following result has a deep impact on the model theory of groups and how stable groups fit in.

**Theorem 2.11** (Sela [Sel13]): Let $\Gamma$ be a torsion free hyperbolic group. Then $Th(\Gamma)$ is stable.

### 3 Some geometry

In this section we record some useful material from geometric group theory. Our choice is biased towards our needs in this paper. In particular we will start by giving some quick introduction to Bass-Serre theory. We will then continue by defining $\mathbb{R}$-trees and as an analogue to Bass-Serre theory we will give a structure theorem, known as the Rips’ machine, for “tame” group actions on them.

#### 3.1 Bass-Serre theory

Bass-Serre theory gives a structure theorem for groups acting on (simplicial) trees, i.e. 1-dimensional CW complexes. It describes a group (that acts on a tree) as a series of amalgamated free products and HNN extensions. The mathematical notion that contains these instructions is called a graph of groups. For a complete treatment we refer the reader to [Ser83].

We start with the definition of a graph.

**Definition 3.1**: A graph $G(V,E)$ is a collection of data that consists of two sets $V$ (the set of vertices) and $E$ (the set of edges) together with three maps:

- an involution $\bar{\cdot} : E \to E$, where $\bar{e}$ is called the inverse of $e$;
- $\alpha : E \to V$, where $\alpha(e)$ is called the initial vertex of $e$; and
- $\tau : E \to V$, where $\tau(e)$ is called the terminal vertex of $e$.

so that $\bar{\bar{e}} = e$, and $\alpha(e) = \tau(\bar{e})$ for every $e \in E$.

An orientation of a graph $G(V,E)$ is a choice of one edge in the couple $(e, \bar{e})$ for every $e \in E$. We denote an oriented graph by $G^+(V,E)$.

For our purposes simplicial trees can also be viewed as combinatorial objects: a tree is a connected graph without a circuit.

**Definition 3.2** (Graph of Groups): A graph of groups $G := (G(V,E), \{G_u\}_{u \in V}, \{G_e\}_{e \in E}, \{f_e\}_{e \in E})$ consists of the following data:

- a graph $G(V,E)$;
- a family of groups $\{G_u\}_{u \in V}$, i.e. a group is attached to each vertex of the graph;
- a family of groups $\{G_e\}_{e \in E}$, i.e. a group is attached to each edge of the graph. Moreover, $G_e = G_{\bar{e}}$.
Definition 3.3: Let \( G := (G(V, E), \{G_u\}_{u \in V}, \{G_e\}_{e \in E}, \{f_e\}_{e \in E}) \) be a graph of groups. Let \( T \) be a maximal subtree of \( G(V, E) \). Then the fundamental group, \( \pi_1(G, T) \), of \( G \) with respect to \( T \) is the group given by the following presentation:

\[
\langle \{G_u\}_{u \in V}, \{t_e\}_{e \in E} \mid t_e^{-1} = t_e \text{ for } e \in E, t_e = 1 \text{ for } e \in T, f_e(a) = t_e f_e(t_e) t_e \text{ for } e \in E \text{ a } G_e \rangle
\]

Remark 3.4: It is not hard to see that the fundamental group of a graph of groups does not depend on the choice of the maximal subtree up to isomorphism (see [Ser83, Proposition 20, p.44]).

In order to give the main theorem of Bass-Serre theory we need the following definition.

Definition 3.5: Let \( G \) be a group acting on a simplicial tree \( T \) without inversions, denote by \( \Lambda \) the corresponding quotient graph and by \( p \) the quotient map \( T \to \Lambda \). A Bass-Serre presentation for the action of \( G \) on \( T \) is a triple \((T^1, T^0, \{\gamma_e\}_{e \in E(T^1) \setminus E(T^0)})\) consisting of

- a subtree \( T^1 \) of \( T \) which contains exactly one edge of \( p^{-1}(e) \) for each edge \( e \) of \( \Lambda \);
- a subtree \( T^0 \) of \( T^1 \) which is mapped injectively by \( p \) onto a maximal subtree of \( \Lambda \);
- a collection of elements of \( G \), \( \{\gamma_e\}_{e \in E(T^1) \setminus E(T^0)} \), such that if \( e = (u, v) \) with \( v \in T^1 \setminus T^0 \), then \( \gamma_e \cdot v \) belongs to \( T^0 \).

Theorem 3.6: Suppose \( G \) acts on a simplicial tree \( T \) without inversions. Let \((T^1, T^0, \{\gamma_e\})\) be a Bass-Serre presentation for the action. Let \( G := (G(V, E), \{G_u\}_{u \in V}, \{G_e\}_{e \in E}, \{f_e\}_{e \in E}) \) be the following graph of groups:

- \( G(V, E) \) is the quotient graph given by \( p : T \to \Gamma \);
- if \( u \) is a vertex in \( T^0 \), then \( G_{p(u)} = Stab_G(u) \);
- if \( e \) is an edge in \( T^1 \), then \( G_{p(e)} = Stab_G(e) \);
- if \( e \) is an edge in \( T^1 \), then \( f_{p(e)} : G_{p(e)} \to G_{\tau(p(e))} \) is given by the identity if \( e \in T^0 \) and by conjugation by \( \gamma_e \) if not.

Then \( G \) is isomorphic to \( \pi_1(G) \).

Among splittings of groups we will distinguish those with some special type vertex groups called surface type vertex groups.

Definition 3.7: Let \( G \) be a group acting on a tree \( T \) without inversions and \((T_1, T_0, \{\gamma_e\})\) be a Bass-Serre presentation for this action. Then a vertex \( v \in T^0 \) is called a surface type vertex if the following conditions hold:

- \( Stab_G(v) = \pi_1(\Sigma) \) for a connected compact surface \( \Sigma \) with non-empty boundary;
- For every edge \( e \in T_1 \) adjacent to \( v \), \( Stab_G(e) \) embeds onto a maximal boundary subgroup of \( \pi_1(\Sigma) \), and this induces a one-to-one correspondence between the set of edges (in \( T^1 \)) adjacent to \( v \) and the set of boundary components of \( \Sigma \).
3.2 Real trees

Real trees (or $\mathbb{R}$-trees) generalize simplicial trees and occur naturally in mathematics (see [Bes01]).

**Definition 3.8:** A real tree is a geodesic metric space in which for any two points there is a unique arc that connects them.

When we say that a group $G$ acts on a real tree $T$ we will always mean an action by isometries.

Moreover, an action $G \actson T$ of a group $G$ on a real tree $T$ is called non-trivial if there is no globally fixed point and minimal if there is no proper $G$-invariant subtree. Lastly, an action is called free if for any $x \in T$ and any non trivial $g \in G$ we have that $g \cdot x \neq x$.

One could ask if there is an analogue of Bass-Serre theory for group actions on real trees. If we restrict ourselves to group actions satisfying some tameness conditions the answer is positive. Let us start by recalling some families of group actions on real trees that will turn out to be the building blocks for the general analysis.

**Definition 3.9:** Let $G \actson T$ be a minimal action of a finitely generated group $G$ on a real tree $T$. Then we say:

(i) $\lambda$ is of discrete (or simplicial) type, if every orbit $G.x$ is discrete in $T$. In this case $T$ is simplicial and the action can be analyzed using Bass-Serre theory;

(ii) $\lambda$ is of axial (or toral) type, if $T$ is isometric to the real line $\mathbb{R}$ and $G$ acts with dense orbits, i.e. $G.x = T$ for every $x \in T$;

(iii) $\lambda$ is of surface (or IET) type, if $G = \pi_1(\Sigma)$ where $\Sigma$ is a surface with (possibly empty) boundary carrying an arational measured foliation and $T$ is dual to $\tilde{\Sigma}$, i.e. $T$ is the lifted leaf space in $\tilde{\Sigma}$ after identifying leaves of distance 0 (with respect to the pseudo-metric induced by the measure);

(iv) $\lambda$ is of Levitt (or thin, or exotic) type. For the definition of this type of action we refer the reader to [Lev93].

**Fact 3.10:** Suppose $\pi_1(\Sigma)$ acts on a real tree $T$ by a surface type action. Then the action is “almost free”, i.e. only elements that belong to subgroups that correspond to the boundary components fix points in $T$ and segment stabilizers are trivial. In particular when $\Sigma$ has empty boundary the action is free (see [MS91]).

We will use the notion of a graph of actions in order to glue real trees equivariantly. This notion will be useful in neatly stating the output of Rips’ machine in the next subsection. We follow the exposition in [Gui08, Section 1.3].

**Definition 3.11** (Graph of actions): A graph of actions $(G \actson T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)})$ consists of the following data:

- A simplicial type action $G \actson T$;
- for each vertex $u$ in $T$ a real tree $Y_u$;
- for each edge $e$ in $T$, an attaching point $p_e$ in $Y_{T(e)}$.

Moreover:
1. $G$ acts on $R := \{ \prod Y_u : u \in V(T) \}$ so that $q : R \to V(T)$ with $q(Y_u) = u$ is $G$-equivariant;

2. for every $g \in G$ and $e \in E(T)$, $p_{g,e} = g \cdot p_e$.

To a graph of actions $A := (G \acts T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)})$ we can assign an $\mathbb{R}$-tree $Y_A$ endowed with a $G$-action. Roughly speaking this tree will be $\prod_{u \in V(T)} Y_u / \sim$, where the equivalence relation $\sim$ identifies $p_e$ with $p_e$ for every $e \in E(T^+)$. We say that a real $G$-tree $Y$ decomposes as a graph of actions $A$, if there is an equivariant isometry between $Y$ and $Y_A$.

Assume a real $G$-tree $Y$ decomposes as a graph of actions. Then a useful property is that $Y$ is covered by $(Y_u)_{u \in V(T)}$ and moreover these trees intersect “transversally”.

**Definition 3.12:** Let $Y$ be an $\mathbb{R}$-tree and $(Y_i)_{i \in I}$ be a family of subtrees that cover $Y$. Then we call this covering a transverse covering if the following conditions hold:

- for every $i \in I$, $Y_i$ is a closed subtree;
- for every $i, j \in I$ with $i \neq j$, $Y_i \cap Y_j$ is either empty or a point;
- every segment in $Y$ is covered by finitely many $Y_i$’s.

The next lemma is by no means hard to prove (see [Gui04, Lemma 4.7].

**Lemma 3.13:** Let $A := (G \acts T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)})$ be a graph of actions. Suppose $G \acts Y$ decomposes as the graph of actions $A$. Then $(Y_u)_{u \in V(T)}$ is a transverse covering of $Y$.

### 3.3 Rips’ machine

Group actions on real trees played a significant role in Sela’s approach to the Tarski problem. The first important result in analyzing these actions came from Rips (unpublished) when he proved that if a group acts freely on a real tree then it is a free product of surface groups and free abelian groups (see [GLP94]).

Requiring an action to be free is a rather extreme condition. One could still get a structure theorem, known as Rips’ machine, by imposing some milder conditions. Recall that an action of a group on a real tree is called super-stable if for any arc $I$ with non-trivial (pointwise) stabilizer and $J$ a subarc of $I$ we have that $\text{Stab}_{G}(I) = \text{Stab}_{G}(J)$.

**Theorem 3.14** (Rips’ Machine): Let $G$ be a finitely generated group. Suppose $G$ acts non-trivially on an $\mathbb{R}$-tree $Y$. Moreover, assume that the action is minimal, super-stable and tripod stabilizers are trivial. Then $G \acts Y$ decomposes as a graph of actions $A := (G \acts T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)})$ where each of the vertex actions, $\text{Stab}_{G}(u) \acts Y_u$, is of either simplicial or surface or axial or exotic type.

Let us note that the content of this subsection will not be explicitly used in this paper. On the other hand, Rips’ machine is an essential tool for acquiring the results of subsection about test sequences and we believe that our exposition on group actions on $\mathbb{R}$-trees would be incomplete if we did not include it.

### 4 Limit groups

In this section we define the basic objects of this paper, namely graded $\Gamma$-limit groups, where $\Gamma$ is a torsion-free hyperbolic group.
We will start by defining $\Gamma$-limit groups in a geometric way: a $\Gamma$-limit group is a quotient of a finitely generated group by the kernel of an action (obtained in a specific way) of the group on a real tree. The method used to obtain such an action will prove very useful for the rest of the paper as well.

In the second subsection we consider the class of graded $\Gamma$-limit groups, which can be thought of as a class of group couples, $(G, H)$, where $G$ is a $\Gamma$-limit group and $H$ a finitely generated subgroup of $G$. We record some theorems about solid $\Gamma$-limit groups (i.e. graded $\Gamma$-limit groups that satisfy some extra conditions) that lie behind our main idea for proving the main theorem of this paper.

4.1 The Bestvina-Paulin method

We first show how an action on an $\mathbb{R}$-tree may arise as a limit of actions on $\delta$-hyperbolic spaces. The construction is credited to Bestvina [Bes88] and Paulin [Pau88] independently.

We fix a finitely generated group $G$ and we consider the set of non-trivial equivariant pseudometrics $d : G \times G \to \mathbb{R}_{\geq 0}$, denoted by $\mathcal{ED}(G)$. We equip $\mathcal{ED}(G)$ with the compact-open topology (where $G$ is given the discrete topology). Note that convergence in this topology is given by:

$$(d_i)_{i<\omega} \to d \text{ if and only if } d_i(1, g) \to d(1, g) \text{ (in } \mathbb{R}) \text{ for any } g \in G$$

It is not hard to see that $\mathbb{R}^+$ acts cocompactly on $\mathcal{ED}(G)$ by rescaling, thus the space of projectivised equivariant pseudometrics on $G$ is compact.

We also note that any based $G$-space $(X, \ast)$ (i.e. a metric space with a distinguished point equipped with an action of $G$ by isometries) gives rise to an equivariant pseudometric on $G$ as follows: $d(g, h) = d_X(g \cdot \ast , h \cdot \ast )$. We say that a sequence of $G$-spaces $(X_i, \ast_i)_{i<\omega}$ converges to a $G$-space $(X, \ast)$, if the corresponding pseudometrics induced by $(X_i, \ast_i)$ converge to the pseudometric induced by $(X, \ast)$ in $\mathcal{PED}(G)$.

A morphism $h : G \to H$ where $H$ is a finitely generated group induces an action of $G$ on $X_H$ (the Cayley graph of $H$) in the obvious way, thus making $X_H$ a $G$-space. We have:

**Lemma 4.1:** Let $\Gamma$ be a torsion-free hyperbolic group. Let $(h_n)_{n<\omega} : G \to \Gamma$ be a sequence of pairwise non-conjugate morphisms. Then for each $n < \omega$ there exists a base point $\ast_n$ in $X_\Gamma$ such that the sequence of $G$-spaces $(X_\Gamma, \ast_n)_{n<\omega}$ has a convergent subsequence to a real $G$-tree $(T, \ast)$, where the action of $G$ on $T$, $G \acts^\lambda T$, satisfies the following properties:

1. $\lambda$ is non-trivial
2. if $L := G / \text{Ker}\lambda$, where $\text{Ker}\lambda := \{ g \in G | \lambda(g, x) = x \text{ for all } x \in T \}$, then $L$ acts on $T$ as follows:

   (i) tripod stabilizers are trivial;
   (ii) arc stabilizers are abelian;
   (iii) the action is super-stable.

More importantly given the situation of Lemma 4.1 one can approximate every point in the limiting tree $T$ by a sequence of points of the converging subsequence.

**Lemma 4.2:** Assume $(X_\Gamma, \ast_n)_{n<\omega}$ converges to $(T, \ast)$ as in Lemma 4.1. Then for any $x \in T$, the following hold:
there exists a sequence \( (x_n)_{n<\omega} \) such that \( \hat{d}_n(x_n, g \cdot x_n') \to d_T(x, g \cdot x) \) for any \( g \in G \), where \( \hat{d}_n \) denotes the rescaled metric of \( X_\Gamma \), we call such a sequence an approximating sequence;

- if \( (x_n)_{n<\omega}, (x'_n)_{n<\omega} \) are two approximating sequences for \( x \in T \), then \( \hat{d}_n(x_n, x'_n) \to 0 \);

- if \( (x_n)_{n<\omega} \) is an approximating sequence for \( x \), then \( (g \cdot x_n)_{n<\omega} \) is an approximating sequence for \( g \cdot x \);

- if \( (x_n)_{n<\omega}, (y_n)_{n<\omega} \) are approximating sequences for \( x, y \) respectively, then \( \hat{d}_n(x_n, y_n) \to d_T(x, y) \).

**Definition 4.3** (Sela): Let \( \Gamma \) be a torsion-free hyperbolic group. A group \( L \) is a \( \Gamma \)-limit group if it can be obtained as the quotient of a finitely generated group \( G \) by \( \text{Ker}\lambda \) where \( \lambda \) is an action of \( G \) on a real tree obtained as in Lemma 4.1.

Note that \( \mathbb{F} \)-limit groups for \( \mathbb{F} \) a free group are traditionally called just limit groups.

Among other properties of \( \Gamma \)-limit groups Sela proved (see [Sel09, Lemma 1.4(ii)]).

**Proposition 4.4**: Let \( L \) be a \( \Gamma \)-limit group. Then \( L \) is CSA (i.e. every maximal abelian subgroup of \( L \) is malnormal).

### 4.2 Graded limit groups

Graded \( \Gamma \)-limit groups (for \( \Gamma \) a torsion-free hyperbolic group) occur naturally in the understanding of how a solution set of a system of equations in \( \Gamma \) changes as the parameters of the system change (see [Sel01, Section 10]). We will be particularly interested in a subclass of graded \( \Gamma \)-limit groups, namely solid \( \Gamma \)-limit groups.

A graded \( \Gamma \)-limit group is a \( \Gamma \)-limit group with a choice of a fixed finitely generated subgroup.

**Definition 4.5** (Graded \( \Gamma \)-limit group): Let \( L \) be a \( \Gamma \)-limit group and \( H \) a finitely generated subgroup of \( L \). Then \( L_H \) is a graded \( \Gamma \)-limit group relative to \( H \).

Any \( \Gamma \)-limit group can be seen as graded \( \Gamma \)-limit group after choosing a finitely generated subgroup of it. As a matter of fact, one can see \( \Gamma \)-limit groups as special cases of graded \( \Gamma \)-limit groups where the chosen subgroup is the trivial one.

In order to define solid \( \Gamma \)-limit groups, we first need to present the class of (graded) short morphisms.

**Definition 4.6**: (cf. [Per11, Definition 4.14]) Let \( L \) be a \( \Gamma \)-limit group and \( \Sigma_L \) be a generating set for \( L \). Let \( L \) be freely indecomposable with respect to a finitely generated subgroup \( H \).

Suppose \( \text{Mod}_H(L) \) is the graded modular group with respect to \( H \) (see [Per11, Definition 4.12]).

Then a morphism \( h : L \to \Gamma \) is called short with respect to \( H \) if:

\[
\max_{s \in \Sigma_L} |h(s)|_{\Gamma} \leq \max_{s \in \Sigma_L} |h(\sigma(s))|_{\Gamma}, \forall \sigma \in \text{Mod}_H(L)
\]

We now pass to the notion of a shortening quotient. Shortening quotients are the basic objects for constructing Makanin-Razborov diagrams. Although we will not follow this line of thought we recall that Makanin-Razborov diagrams encode the whole solution set of a system of equations in a torsion-free hyperbolic group.
Definition 4.7: Let \( L \) be a graded \( \Gamma \)-limit group. A quotient \( Q \) of \( L \) is called a graded shortening quotient with respect to \( H \) if \( Q \cong L/\Ker\lambda \) where \( \lambda \) is an action of \( L \) on a real tree obtained by a sequence of short morphisms (with respect to \( H \)) as in Lemma 4.1.

Now we are finally ready to define solid \( \Gamma \)-limit groups.

Definition 4.8 (Solid limit group): Let \( S_H \) be a graded \( \Gamma \)-limit group. Suppose \( S \) is freely indecomposable with respect to \( H \). Then \( S_H \) is a solid \( \Gamma \)-limit group relative to \( H \), if \( S \) admits a graded shortening quotient (with respect to \( H \)) isomorphic to \( S \).

Remark 4.9:

- By a graded (solid) limit group we mean a graded (solid) \( \mathbb{F} \)-limit group, where \( \mathbb{F} \) is a non-abelian free group;
- Sela [Sel01, Definition 10.2] defines also rigid limit groups. Rigid limit groups are solid limit groups that have a trivial JSJ decomposition relative to the chosen subgroup. Since rigid limit groups are special cases of solid limit groups we prefer not to treat them separately.

In [Sel09, Theorem 3.6] it was proved:

Theorem 4.10: Suppose \( L_H \) is a solid \( \Gamma \)-limit group. Then there exists a natural number \( n \) such that any morphism from \( H \) to \( \Gamma \) has at most \( n \) extensions to strictly solid morphisms from \( L \) to \( \Gamma \) that belong to different strictly solid families.

For the definitions of a strictly solid morphism and a strictly solid family of morphisms we refer the reader to [Sel01, Definition 1.5]. In the special case of a rigid limit group the above notions both “degenerate” to the notion of a rigid morphism (see [Sel01, Definition 10.5]). Thus we have:

Theorem 4.11: Suppose \( L_H \) is a rigid \( \Gamma \)-limit group. Then there exists a natural number \( n \) such that any morphism from \( H \) to \( \Gamma \) has at most \( n \) extensions to rigid morphisms from \( L \) to \( \Gamma \).

Again the distinction between the solid case and the rigid case will not be important to us.

5 Towers and test sequences

Towers or more precisely groups that have the structure of an \( \omega \)-residually free tower (in Sela’s terminology), were introduced in [Sel01, Definition 6.1] and they provide examples of groups for which Merzlyakov’s theorem (see [Mer66]) naturally generalizes (up to some fine tuning). In practice, towers appear as completions of well-structured MR resolutions of limit groups (see Definitions 1.11 and 1.12 in [Sel03]).

After defining towers we will be interested in “test sequences” defined on them. These are sequences of morphisms from groups that have the structure of a tower to some specific target group. Test sequences have been used extensively through out the work of Sela on Tarski’s problem. In this paper we will not use the full strength of the results that come with test sequences, thus for the formal definitions we will refer the reader to the work of Sela. Instead we abstract, in a series of facts, all the properties of a test sequence that we will use towards our goal.
5.1 Towers

We will start this subsection by defining the main building blocks of a tower, namely free abelian flats and surface flats.

**Definition 5.1** (Free abelian flat): Let $G$ be a group and $H$ be a subgroup of $G$. Then $G$ has the structure of a free abelian flat over $H$, if $G$ is the amalgamated free product $H *_{A} (A \oplus \mathbb{Z})$ where $A$ is a maximal abelian subgroup of $H$.

Before moving to the definition of a hyperbolic floor we recall that if $H$ is a subgroup of a group $G$ then a morphism $r : G \to H$ is called a retraction if $r$ is the identity on $H$.

**Definition 5.2** (Hyperbolic floor): Let $G$ be a group and $H$ be a subgroup of $G$. Then $G$ has the structure of a hyperbolic floor over $H$, if $G$ acts minimally on a tree $T$ and the action admits a Bass-Serre presentation $(T^1, T^0, \{\gamma_e\})$ such that:

- the set of vertices of $T^0$ is partitioned in two sets, $V_1$ and $V_2$, where all the vertices in $V_1$ are surface type vertices;
- $T^1$ is bipartite between $V_1$ and $V(T^1) \setminus V_1$;
- $H$ is the free product of the stabilizers of vertices in $V_2$;
- either there exists a retraction $r : G \to H$ that, for every $v \in V_1$, sends $\text{Stab}_G(v)$ to a non abelian image or $H$ is cyclic and there exists a retraction $r' : G * \mathbb{Z} \to H * \mathbb{Z}$ which, for every $v \in V_1$, sends $\text{Stab}_G(v)$ to a non abelian image.

![Figure 1: A graph of groups corresponding to a hyperbolic floor](image)

If a group has the structure of a hyperbolic floor (over some subgroup), and the corresponding Bass-Serre presentation contains just one surface type vertex then we call the hyperbolic floor a surface flat.

The following lemma is an easy exercise in Bass-Serre theory.

**Lemma 5.3:** Suppose $G$ has a hyperbolic floor structure over $H$. Then there exists a finite sequence $G := G^m > G^{m-1} > \ldots > G^0 := H$ such that, for each $i < m$, $G^{i+1}$ has a surface flat structure over $G^i$.

We use surface and free abelian flats in order to define towers.
Definition 5.4: A group $G$ has the structure of a tower (of height $m$) over a subgroup $H$ if there exists a sequence $G = G^m > G^{m-1} > \ldots > G^0 = H$, where for each $i$, $0 \leq i < m$, one of the following holds:

(i) $G^{i+1}$ is the free product of $G^i$ with either a free group or with the fundamental group of a closed surface of Euler characteristic at most $-2$;

(ii) $G^{i+1}$ has the structure of a surface flat over $G^i$;

(iii) $G^{i+1}$ has the structure of a free abelian flat over $G^i$.

Figure 2: A tower over $H$.

It will be useful to collect the information witnessing that a group admits the structure of a tower. Thus, we define:

Definition 5.5: Suppose $G$ has the structure of a tower over $H$ (of height $m$). Then the tower (over $H$) corresponding to $G$, denoted by $T(G,H)$, is the following collection of data:

$$(G, A(G^m, G^{m-1}), A(G^{m-1}, G^{m-2}), \ldots, A(G^1, G^0), H)$$

where by $A(G^{i+1}, G^i)$ we denote the action of $G^{i+1}$ on a tree (together with the Bass-Serre presentation) that witnesses that $G^{i+1}$ has one of the forms of Definition 5.4 over $G^i$.

The height of an element of a group that has the structure of a tower is defined as follows:
Definition 5.6: Suppose $G$ has the structure of a tower $T := \mathcal{T}(G, H)$ and $g$ is an element in $G$. Then:

- $\text{height}_T(g) = 0$, if $g \in G^0$;
- $\text{height}_T(g) = l + 1$, if $g \in G^{l+1} \setminus G^l$;

If we restrict the subgroup $H$ in certain classes of groups we have:

Definition 5.7: Suppose $G$ has the structure of a tower over $H$. Then, we say that:

- $G$ has the structure of an $H$-limit tower, if $H$ is a torsion-free hyperbolic group. Moreover if $H$ is a non abelian free group we just say that $G$ has the structure of a limit tower (this notion is the same as an $\omega$-residually free tower in \cite[Definition 6.1]{Sel01});
- $G$ has the structure of a graded $\Gamma$-limit tower over $A$, for $A$ some finitely generated subgroup of $H$, if $HA$ is a solid $\Gamma$-limit group.

Lemma 5.8: Suppose $G$ has the structure of a tower over a limit group $L$. Then $G$ is a limit group.

Sketch. It follows from the fact that the class of limit groups coincides with the class of Constructible Limit Groups (CLG) as defined in \cite[Definition 1.25]{BF09} (see Theorem 1.30 of the same paper).

Let $(G, \mathcal{A}(G^m, G^{m-1}), \mathcal{A}(G^{m-1}, G^{m-2}), \ldots, \mathcal{A}(G^1, G^0), L)$ be the tower of our hypothesis. It is enough to check that for all the possibilities for obtaining the group $G^{i+1}$ from $G^i$ according to Definition 5.4, the resulting group belongs to CLG.

At this point we should draw the attention of the reader to the following connections with the material in \cite{Sel03}. For the definition of a strict MR resolution we refer the reader to \cite[Definition 5.8]{Sel01}, a further refinement of this notion, called a well-structured MR resolution, is defined in \cite[Definition 1.11]{Sel03}.

Lemma 5.9: A completion of a well-structured MR resolution of a limit group (see \cite[Definition 1.12]{Sel03}) is a tower over a free group.

Proof. The proof is by inspection of the construction of the completion. The base floor of the completion is a free group. Each “completed decomposition” can be constructed from the previous one by a series of free products (with free abelian or surface groups), followed by a series of additions of free abelian flats and finally followed by a series of additions of surface flats.

We discuss how the above notions fit in in the work of Sela, although this discussion is not strictly needed for the continuation of this paper.

In the core of Sela’s approach to Tarski’s question is a method called formal solutions. Formal solutions is a way to obtain a validation of a true (in some non abelian free group $F$) $\forall \exists$ sentence, which is independent of $F$. This method has been first used by Merzlyakov in \cite{Mer66}, where he obtained the equality of the positive theories of non abelian free groups.

Theorem 5.10 (Merzlyakov’s theorem): Let $\Sigma(x, y) \subseteq \langle \bar{x}, \bar{y} \rangle$ be a finite set of elements in $\langle \bar{x}, \bar{y} \rangle$. Let $F$ be a non abelian free group and $F \models \forall \bar{x} \exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1)$. Then there exists a retraction $r : G_{\Sigma} \to \langle \bar{x} \rangle$, where $G_{\Sigma} := \langle \bar{x}, \bar{y} \mid \Sigma(\bar{x}, \bar{y}) \rangle$. 


The image of $\bar{y}$ under the retraction of the previous theorem gives us a “formal” witness of the truthfulness of the sentence $\forall \bar{x}\exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1)$ and this witness works for any non abelian free group.

The picture gets much more complicated when inequalities come into the scene. Consider the following sentence $\forall \bar{x}\exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1 \land \Psi(\bar{x}, \bar{y}) \neq 1)$, where by $\Psi(\bar{x}, \bar{y}) \neq 1$ we mean the following system of inequalities \{ψ₁(\bar{x}, \bar{y}) ≠ 1 \∧ \ldots \∧ ψ_k(\bar{x}, \bar{y}) ≠ 1}\. If we assume that the above sentence is true in some non abelian free group $F$, then a “formal solution” still exists (see [Sel03, Theorem 1.2]) but it only “validates” the sentence when the variables $\bar{x}$ belong to the complement (with respect to $F^k$, where $|\bar{x}| = k$) of a proper variety of $F$, where by a variety we mean the solution set of a system of equations in $F$. Thus, if one wants to “validate” the above sentence (independently of $F$) they have to produce “formal solutions” but now “over” a variety.

It can be shown that in principle this is not possible. One can only prove the existence of formal solutions “over” varieties when the variety has a special form. In particular if for a system of equations, $\Sigma(\bar{x})$, the corresponding group $G_\Sigma$ is the “completed limit group” (see p.213 [Sel03]) of some well-structured MR resolution of a limit group (and thus has the structure of a tower over a non abelian free group $F$), then one can prove the existence of formal solutions “over” the variety which is the solution set of $\Sigma(\bar{x})$ in $F$ (see [Sel03, Theorem 1.18]). To be totally precise the formal solutions exist over “closures” of the completed limit group, but we prefer to keep the discussion at the intuitive level.

The tool for obtaining formal solutions over varieties of the above special form is called test sequences and is the topic of the next subsection.

5.2 Test sequences

Convention:

(i) For the rest of the paper when we say that a group $G$ has the structure of a limit tower we will always mean that $G$ is the completed limit group of some completed well-structured MR resolution of a limit group.

(ii) When we refer to Lemma 4.1 we will always choose the sequence of base points to be the sequence of trivial elements.

If a group $G$ has the structure of a limit tower (over $F$) then a test sequence is a sequence of morphisms from $G$ to $F$ that satisfy certain combinatorial conditions. These conditions depend on the structure of the tower and their description is extremely complicated. Thus, we prefer to give a few special cases of test sequences (i.e. when the tower has very simple form) and refer to the work of Sela for the general definition.

Definition 5.11: Let $F$ be a non abelian free group. A sequence of morphisms $(h_n)_{n<\omega}: \langle x_1, \ldots, x_k \rangle \to F$, is a test sequence for $\langle \bar{x} \rangle$, if $(h_n(x_1), \ldots, h_n(x_k))$ satisfies $C'(1/n)$ for each $n < \omega$.

Remark 5.12: Without loss of generality we will assume that all $x_i$ have similar growth under $(h_n)_{n<\omega}$, i.e. for each $i, j < k$ there are $c_{i,j}, c'_{i,j} \in \mathbb{R}^+$ such that $c_{i,j} < \frac{|h_n(x_i)|}{|h_n(x_j)|} < c'_{i,j}$.

Definition 5.13: Let $F$ be a non abelian free group and $\Sigma$ be a closed surface of Euler characteristic at most $-2$. Then a sequence of morphisms $(h_n)_{n<\omega}: \pi_1(\Sigma) \to F$ is a test sequence for $\pi_1(\Sigma)$ if it satisfies the combinatorial conditions (i) – (ix) in [Sel03] p. 182].
If \((h_n)_{n<\omega} : G \to H\) is a sequence of morphisms from \(G\) to a finitely generated group \(H\) (with a fixed generating set) and \(g_1, g_2\) are in \(G\), then we say that the growth of \(g_1\) dominates the growth of \(g_2\) (under \((h_n)_{n<\omega}\)) if \(\frac{|h_n(g_2)|_H}{|h_n(g_1)|_H} \to 0\) as \(n \to \infty\).

If \(G\) has the structure of a limit tower where no surface flat occurs, then we define a test sequence recursively as follows:

**Definition 5.14:** Let \(T(G, F)\) be a limit tower. Suppose \(T(G, F)\) does not contain any surface flats. Then a sequence of morphisms \((h_n)_{n<\omega} : G \to F\) is called a test sequence for \(T(G, F)\) if the following conditions hold:

- \(h_n \upharpoonright G^0\) is the identity morphism for every \(n < \omega\);
- We define the conditions of the restriction of \((h_n)_{n<\omega}\) to the \(i+1\)-th flat by taking cases according to whether \(G^{i+1}\) has a structure of a free product or a free abelian flat over \(G^i\):

  1. Suppose \(G^{i+1}\) is the free product of \(G^i\) with \(\pi_1(\Sigma)\), for \(\Sigma\) a closed surface with \(\chi(\Sigma) \leq -2\). Then \((h_n \upharpoonright \pi_1(\Sigma))_{n<\omega}\) satisfies the requirements of Definitions 5.13. Moreover, the growth of any element in \(\pi_1(\Sigma)\) (under \(h_n\)) dominates the growth of every element in \(G^i\) (under \(h_n\));

  2. Suppose \(G^{i+1}\) is the free product of \(G^i\) with \(F_l\), then \(h_n \upharpoonright F_l\) satisfies the requirements of Definition 5.11. Moreover, the growth of any element in \(F_l\) (under \(h_n\)) dominates the growth of every element in \(G^i\) (under \(h_n\));

  3. Suppose \(G^{i+1} = G^i \ast_A (A \oplus \mathbb{Z})\), is obtained from \(G^i\) by gluing a free abelian flat along \(A\) (where \(A\) is maximal abelian in \(G^i\)). Let \(\gamma_n\) be the generator of the cyclic group (in \(F\)) that \(A\) is mapped into by \(h_n \upharpoonright G^i\). Then if we denote the generator of \(\mathbb{Z}\) by \(z\) we have \(h_n(z) = \gamma_n^m\). Moreover the growth of \(z\) (under \(h_n\)) dominates the growth of every element in \(G^i\) (under \(h_n\)).

We say that a sequence of morphisms \((h_n)_{n<\omega} : G \to F\), where \(G\) has the structure of a limit tower is a **test sequence** (for this tower) if it satisfies the combinatorial conditions (i) – (xiv) in [Sel03, p.222]. The existence of a test sequence for a group that has the structure of a limit tower (without any conditions on the occurring flats) has been proved in [Sel03, Lemma 1.21].

**Proposition 5.15:** Suppose \(G\) has the structure of a limit tower \(T(G, F)\). Then a test sequence for \(T(G, F)\) exists.

In this paper we will not use the full strength of the results connected with a test sequence. So, for our purposes the following facts, used extensively in [Sel03] (cf. Theorem 1.3, Proposition 1.8, Theorem 1.18), about test sequences assigned to groups that have the structure of a limit tower will be enough.

**Fact 5.16** (Free product limit action): Let \(T(G, F)\) be a limit tower and \((h_n)_{n<\omega} : G \to F\) be a test sequence for \(T(G, F)\).

Suppose \(G^{i+1}\) is the free product of \(G^i\) with a group \(B\) and \((h_n \upharpoonright G^{i+1})_{n<\omega}\) is the restriction of \((h_n)_{n<\omega}\) to \(G^{i+1}\). Then, any subsequence of \((h_n \upharpoonright G^{i+1})_{n<\omega}\) that converges, as in Lemma 4.1, induces a faithful action of \(G^{i+1}\) on a based real tree \((Y, \ast)\), with the following properties:

1. the action \(G^{i+1} \acts Y\) decomposes as a graph of actions \((G^{i+1} \acts T; \{Y_u\}_{u \in V(T)}\), \(\{p_e\}_{e \in E(T)}\);
2. the Bass-Serre presentation for $G^{i+1} \curvearrowright T$, $(T_1 = T_0, T_0)$, is a segment $(u,v)$;
3. $\text{Stab}_G(u) \curvearrowright Y_u$ is a surface type action coming from $\Sigma$;
4. $Y_u$ is a point and $\text{Stab}_G(v)$ is $G^i$;
5. the edge $(u,v)$ is trivially stabilized.

Figure 3: A Bass-Serre presentation in the case of a surface group.

(ii) if $B$ is $\mathbb{F}_l$, then:

1. the action $G^{i+1} \curvearrowright Y$ decomposes as a graph of actions $(G^{i+1} \curvearrowright T, \{Y_u\}_{u \in V(T)}$, $\{p_e\}_{e \in E(T)}$);
2. the Bass-Serre presentation for $G^{i+1} \curvearrowright T$, $(T_1 = T_0, T_0)$, is a segment $(u,v)$;
3. $\text{Stab}_G(u) := F_l \curvearrowright Y_u$ is a simplicial type action, its Bass-Serre presentation, $(Y^1_u, Y^0_u, t_1, \ldots, t_l)$ consists of a “star graph” $Y^1_u := \{(x,b_1), \ldots, (x,b_l)\}$ with all of its edges trivially stabilized, a point $Y^0_u = x$ which is trivially stabilized and Bass-Serre elements $t_i = e_i$, for $i \leq l$;
4. $Y_v$ is a point and $\text{Stab}_G(v)$ is $G^i$;
5. the edge $(u,v)$ is trivially stabilized.

Figure 4: A Bass-Serre presentation in the case of a free group.

**Fact 5.17** (Surface flat limit action): Let $T(G, F)$ be a limit tower and $(h_n)_{n<\omega} : G \rightarrow F$ be a test sequence with respect to $T(G, F)$.

Suppose $G^{i+1}$ is a surface flat over $G^i$, witnessed by $A(G^{i+1}, G^i)$, and $(h_n \mid G^{i+1})_{n<\omega}$ is the restriction of $(h_n)_{n<\omega}$ to the $i+1$-flat of $T(G, F)$.

Then, any subsequence of $(h_n \mid G^{i+1})_{n<\omega}$ that converges, as in Lemma 4.1, induces a faithful action of $G^{i+1}$ on a based real tree $(Y, *)$, with the following properties:

1. $G^{i+1} \curvearrowright Y$ decomposes as a graph of actions $(G^{i+1} \curvearrowright T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)}$, with the action $G^{i+1} \curvearrowright T$ being identical to $A(G^{i+1}, G^i)$;
2. the Bass-Serre presentation, \((T^1, T^0, \{t_e\})\), for the action of \(G^{i+1}\) on \(T\), is identical with the Bass-Serre presentation of the surface flat splitting \(A(G^{i+1}, G^i)\);

3. if \(v\) is not a surface type vertex then \(Y_v\) is a point stabilized by the corresponding \(G_j^i\) for some \(j \leq m\);

4. if \(u\) is the surface type vertex, then \(\text{Stab}_{G^i}(u) = \pi_1(\Sigma_{g,l})\) and the action \(\text{Stab}_{G^i}(u) \cdot Y_u\) is a surface type action coming from \(\pi_1(\Sigma_{g,l})\);

5. edge stabilizers and Bass-Serre elements in \((T^1, T^0, \{t_e\})\) are as in the Bass-Serre presentation for \(A(G^{i+1}, G^i)\).

**Fact 5.18 (Abelian flat limit action):** Let \(\mathcal{T}(G, F)\) be a limit tower and \((h_n)_{n<\omega} : G \to F\) be a test sequence with respect to \(\mathcal{T}(G, F)\).

Suppose \(G^{i+1} = G^i \ast_A (A \oplus \mathbb{Z})\) is obtained from \(G^i\) by gluing a free abelian flat along \(A\) (where \(A\) is a maximal abelian subgroup of \(G^i\)) and \((h_n \upharpoonright G^{i+1})_{n<\omega}\) is the restriction of \((h_n)_{n<\omega}\) to the \(i+1\)-flat of \(\mathcal{T}(G, F)\).

Then any subsequence of \((h_n \upharpoonright G^{i+1})_{n<\omega}\) that converges, as in Lemma 4.1, induces a faithful action of \(G^{i+1}\) on a based real tree \((Y, \ast)\), with the following properties:

1. the action of \(G^{i+1}\) on \(Y\), \(G^{i+1} \cdot Y\), decomposes as a graph of actions \((G^{i+1} \cdot Y)\) on \(T\), \(\{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)}\);

2. the Bass-Serre presentation for \(G^{i+1} \cdot Y\), \((T^1 = T_0, T_0)\), is a segment \((u, v)\);

3. \(\text{Stab}_{G^i}(u) := A \oplus \mathbb{Z} \cdot Y_u\) is a simplicial type action, its Bass-Serre presentation, \((Y_u^1, Y_u^0, t_e)\) consists of a segment \(Y_u^1 := (a, b)\) whose stabilizer is \(A\), a point \(Y_u^0 = a\) whose stabilizer is \(A\) and a Bass-Serre element \(t_e\) which is \(z\);

4. \(Y_v\) is a point and \(\text{Stab}_{G^i}(v)\) is \(G_i\);

5. the edge \((u, v)\) is stabilized by \(A\).
6 An infinitude of images

In this section we prove all the technical results that are in the core of this paper.

**Proposition 6.1:** Let $G$ be a group that has the structure of a limit tower $T(G, F)$. Suppose $g \in G \setminus F$. Then $\{h_n(g) \mid n < \omega\}$ is infinite for any test sequence $(h_n)_{n<\omega} : G \to F$ for $T(G, F)$.

**Proof.** We show that if $\{h_n(g) \mid n < \omega\}$ is finite, then $g \in F$. The proof is by induction on the height of $g$.

The base step is trivial since if $\text{height}_T(g) = 0$, then $g \in F$. Assume the claim is true for every element of height less than or equal to $i$, we prove it for elements of height $i + 1$. Since $\{h_n(g) : n < \omega\}$ is finite, there exists $n_0$ such that $h_{n_0}(g) = h_s(g)$ for arbitrarily large $s$. Thus, we can use Bestvina-Paulin method to obtain, from the (refined) sequence still denoted $(h_n)_{n<\omega}$ restricted to $G^{i+1}$, an action of $G^{i+1}$ on a based real tree $(Y, *)$ so that $g$ fixes the base point. We take cases with respect to the form of the $i+1$-flat of the tower $T(G, F)$.

- Assume that $G^{i+1}$ is a free product of $G^i$ with $B$. Let $g_1 g_2 \ldots g_m$ be the normal form of $g$ with respect to the free product $G^i * B$. We take cases according to the possibilities for $B$.

  1. Suppose $B$ is a surface group $\pi_1(\Sigma)$. The action of $G^{i+1}$ on $(Y, *)$ decomposes as a graph of actions $(G^{i+1} \curvearrowright T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)})$ with the properties listed in Fact 5.16(i) (see Figure 6). We will show that $g$ cannot fix any point $q$ in $Y_u$. We consider the following path in $T$ from $v$ to $g_1 g_2 \ldots g_m \cdot v$:

\[
(v, g_1 \cdot u, g_1 g_2 \cdot v, g_1 g_2 g_3 \cdot u, \ldots, g_1 \ldots g_m \cdot v)
\]

![Figure 6: The limit action of $G^{i+1}$ in the surface case when $g_1 \in \pi_1(\Sigma)$.](image)

It is not hard to see that $Y_u \cap g_1 \cdot Y_u$ is either $Y_u$ (in the case $g_1 \in \pi_1(\Sigma)$) or $\{\ast\}$, and $g_1 g_2 \ldots g_{2l-1} \cdot Y_u \cap g_1 g_2 \ldots g_{2l+1} \cdot Y_u = \{g_1 g_2 \ldots g_{2l} \cdot \ast\}$ for any suitable $l \geq 1$. By the freeness of the action of $\pi_1(\Sigma)$ on $Y_u$ the points of $Y$ in the following set $\{\ast, g_1 g_2 \cdot \ast, \ldots, g_1 g_2 g_3 \cdot \ast\}$ are pairwise distinct. Thus, if $m > 2$ we have that $Y_u \cap g \cdot Y_u$ is empty. Since $g \cdot q$ lives in $g \cdot Y_u$ we have that $g \cdot q \neq q$.

So, we may assume that $m \leq 2$. In the case $m = 2$ and $g_1 \in \pi_1(\Sigma)$, we have that $Y_u \cap g_1 g_2 Y_u = \{g_1 g_2 \cdot \ast\}$ (note that in the real tree $Y$, $g_1 \cdot \ast$ has been identified with
Therefore $g_1g_2 \cdot q$ must be $g_1g_2 \cdot \ast$, but then $q = \ast$ and $\ast \neq g_1g_2 \cdot \ast$. In the case $m = 2$ and $g_1 \in G^2$, we have that $Y_u \cap g_1g_2 \cdot Y_u = \{g_1 \cdot \ast\}$. Therefore, $g_1g_2 \cdot q$ must be $g_1 \cdot \ast$, but then $q$ must be $g_2^{-1} \cdot \ast$ and $g_1 \cdot \ast \neq g_2^{-1} \cdot \ast$ (note that in the real tree $Y$, $g_1 \cdot \ast$ has been identified with $\ast$). Finally, if $m = 1$ then $g_1 \in \pi_1(\Sigma)$ and by the freeness of the action $g_1$ cannot fix $q$.

2. Suppose $B$ is the free group $F_l$. Then the proof is identical to the proof of the previous case (using Fact 5.16(ii)) since the only property of the limit action we used is that it is a free action on the real tree $Y_u$.

- Suppose $G^{i+1}$ is a surface flat over $G^i = G^i_1 \ast \ldots \ast G^i_l$. Recall that $\mathcal{A}(G^{i+1}, G^i)$ is the action of $G^{i+1}$ on a (simplicial) tree together with a Bass-Serre presentation $(T^1, T^0, \{\gamma_e\})$ for the action that witnesses that $G^{i+1}$ is a surface flat over $G^i$. The limit action of $G^{i+1}$ decomposes as a graph of actions $(G^{i+1} \rtimes T, \{Y_u\}_{u \in \mathcal{V}(T)}, \{p_e\}_{e \in \mathcal{E}(T)})$ that has the properties listed in Fact 5.17. In particular the action $G^{i+1} \rtimes T$ is exactly the action $\mathcal{A}(G^{i+1}, G^i)$. Let $u$ be a surface type vertex, $v_1, \ldots, v_l$ be the non surface type vertices in $T^0$ and $e_1, \ldots, e_r$ be the edges in $T^1$ with $p_i$ be the attaching point of $e_i$ in $Y_u$ for every $i \leq r$. We will show that $g$ does not fix any point $q$ in $Y_u$. Let $g_1g_2 \ldots g_m$ be a reduced form for $g$ with respect to the above splitting. Is not hard to see, following the same argument as in the free product case, that if $m > 2$, then $Y_u \cap g \cdot Y_u$ are disjoint. Thus, since $g \cdot q$ belongs to $g \cdot Y_u$ we have that $g \cdot q \neq q$.

If $m = 1$, then either $g_1 \in \pi_1(\Sigma)$ (where $\pi_1(\Sigma)$ is the surface group of the surface flat) or $g_1$ is a Bass-Serre element $\gamma_{e_i}^\pm$ for some edge $e_i \in T^1 \setminus T^0$. The first case is trivial since $\pi_1(\Sigma)$ acts “almost freely” on $Y_u$. In the latter case we have that $Y_u \cap \gamma_e \cdot Y_u = \{p_j\}$, where $p_j$ is the attaching point of the edge $(\gamma_{e_i} \cdot \alpha(e_i), u)$ (the case for $\gamma_{e_i}^-$ is similar), thus $q$ must be $p_j$. But since $p_j \neq p_i$ we have that $\gamma_e \cdot p_j \neq p_j$.

If $m = 2$ there are seven cases to consider, according to whether $g_1$ belongs to $\pi_1(\Sigma)$ or to $\text{Stab}_{G^{i+1}}(v_j)$ for some $j \leq r$ or it is a Bass-Serre element and the combinations for the same choices for $g_2$ (with the obvious restrictions of $g_1g_2$ being in reduced form).

We will do a characteristic case and leave the rest for the reader. Let $g_1 \in \pi_1(\Sigma)$ and $g_2 \in \text{Stab}_{G^{i+1}}(v_j)$ for some $j \leq l$. Then $Y_u \cap g_1g_2 \cdot Y_u = \{g_1g_2 \cdot p_j\}$, where $p_j$ is the attaching point of the edge $(v_j, u)$. But then $q$ must be $p_j$ and since in $Y$ we have that $g_1g_2 \cdot p_j$ has been identified with $g_1 \cdot p_j$, we get $q \neq g_1g_2 \cdot q$.

![Figure 7: The limit action of $G^{i+1}$ in the free abelian flat case when $g_1 \in A \oplus \mathbb{Z}$.](image-url)
- Suppose $G^{i+1} = G^i \ast_A (A \oplus \mathbb{Z})$ has an abelian flat structure. Then the limit action of $G^{i+1}$ on $Y$ decomposes as a graph of actions that has the properties listed in Fact 5.18 (see Figure 7). The proof is identical to the free product case.

As an immediate corollary of the above proposition we get:

**Theorem 6.2:** Let $G$ be a group that has the structure of a limit tower $T(G, \mathbb{F})$. Suppose $g \in G$ and $\{h_n(g) \mid n < \omega\}$ is finite for some test sequence $(h_n)_{n<\omega}$ for $T(G, \mathbb{F})$. Then there is $\gamma \in \mathbb{F}$ such that for any test sequence $(f_n)_{n<\omega}$ for $T(G, \mathbb{F})$, for any $n < \omega$ we have that $f_n(g) = \gamma$.

We get the same result for conjugacy classes of elements that live in limit towers.

**Proposition 6.3:** Let $G$ be a group that has the structure of a limit tower $T(G, \mathbb{F})$. Suppose $g \in G$ cannot be conjugated in $\mathbb{F}$. If $b \in \mathbb{F}$ denote by $[b]$ the conjugacy class of $b$ in $\mathbb{F}$.

Then $\{[h_n(g)] \mid n < \omega\}$ is infinite, for any test sequence $(h_n)_{n<\omega} : G \to \mathbb{F}$ for $T(G, \mathbb{F})$.

**Proof.** We show that if $\{[h_n(g)] \mid n < \omega\}$ is finite, then $g$ can be conjugated in $\mathbb{F}$. The proof is by induction on the height of $g$.

The base step is trivial since if $\text{height}_T(g) = 0$, then $g \in \mathbb{F}$. Assume the claim is true for every element of height less than or equal to $i$, we prove it for elements of height $i + 1$. Since $\{[h_n(g)] : n < \omega\}$ is finite, there exists $n_0$ such that $[h_{n_0}(g)] = [h_s(g)]$ for arbitrarily large $s$. Thus, for the refined sequence, still denoted $(h_n)_{n<\omega}$, we have that $h_n(g) = c^p_n$ for some sequence of elements $(p_n)_{n<\omega}$ in $\mathbb{F}$ and $c \in \mathbb{F}$. Thus, we can use Bestvina-Paulin method to obtain, from the sequence $(h_n)_{n<\omega}$, restricted to $G^{i+1}$, an action of $G^{i+1}$ on a based real tree $(Y, \ast)$ so that $g$ fixes the point $p$ that is approximated by the sequence $(p_n)_{n<\omega}$ (seen as a sequence of points in the Cayley graph $X_{\mathbb{F}}$). To see this, one can use Lemma 4.2.

Now the proof is straightforward as in each case the real tree $(Y, \ast)$ is covered by translates of the subtree $Y_u$ where $u$ is the vertex stabilized by the group corresponding to the added floor or the added factor. But then $p$ can be translated by an element $\gamma \in G^{i+1}$ in $Y_u$, and a conjugate of $g$ fixes this point. By the same arguments as in the proof of Proposition 6.3 this conjugate of $g$ must lie in $\mathbb{F}$, and the result is proved.

Again we record the following corollary.

**Theorem 6.4:** Let $G$ be a group that has the structure of a limit tower $T(G, \mathbb{F})$. Suppose $g \in G$ and $\{[h_n(g)] \mid n < \omega\}$ is finite, for some test sequence $(h_n)_{n<\omega}$ for $T(G, \mathbb{F})$. Then there is $\gamma \in \mathbb{F}$ such that for any test sequence $(f_n)_{n<\omega}$ for $T(G, \mathbb{F})$, for any $n < \omega$ we have that $[f_n(g)] = [\gamma]$.

For couples (respectively triples) in the sort for left-$m$-cosets (respectively $(m,n)$-double cosets) we get a slightly different result. We split the proofs in three lemmata.

**Lemma 6.5:** Let $G$ be a group that has the structure of a limit tower $T(G, \mathbb{F})$. Let $(h_n)_{n<\omega} : G \to \mathbb{F}$ be a test sequence for $T(G, \mathbb{F})$ and $g \in G$ such that $\{C_{\mathbb{F}}(h_n(g)) \mid n < \omega\}$ is finite.

Then either $g \in \mathbb{F}$ or there exists a finite sequence of free abelian flats $G^{i_1} \ast_{A_{i_1}} (A_{i_1} \oplus \mathbb{Z}), G^{i_2} \ast_{A_{i_2}} (A_{i_2} \oplus \mathbb{Z}), \ldots, G^{i_m} \ast_{A_{i_m}} (A_{i_m} \oplus \mathbb{Z})$ (with $i_1 > i_2 > \ldots > i_m$), a finite sequence of couples $(a_1, b_1, c_1), (a_2, b_2, c_2), \ldots, (a_m, b_m, c_m)$, and an element $\gamma \in \mathbb{F} \setminus \{1\}$, so that the following properties hold:

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• for every $j \leq m$, the product $a_j b_j c_j$ is in normal form with respect to the splitting $G^{i_j} \ast_{A_{i_j}} (A_{i_j} \oplus \mathbb{Z})$ (with $b_j \in (A_{i_j} \oplus \mathbb{Z}) \setminus A_{i_j}$ and $a_j, c_j$ possibly trivial);

• for every $j \leq m$, the product $a_j^{-1} \gamma a_j$ belongs to $A_{i_j}$;

• for every $j < m$, we have that $a_j c_j = a_{j+1} b_{j+1} c_{j+1}$;

• $a_m c_m$ belongs to $C_F(\gamma)$;

• $g = a_1 b_1 c_1$.

Proof. The proof is by induction on the height of $g$. The base case is trivial since $\text{height}(g) = 0$, then $g \in F$.

Assume the result is true for any element $g$ of height less or equal to $i$. We show it for elements of height $i + 1$. Since $\{C_F(h_n(g)) \mid n < \omega\}$ is finite, there exists $n_0$ such that $C_F(h_{n_0}(g)) = C_F(h_s(g))$ for arbitrarily large $s$. Thus, for the refined sequence, still denoted $(h_n)_{n<\omega}$, we have that $h_n(g) = \gamma^{k_n}$ for some sequence of integers $(k_n)_{n<\omega}$ and $\gamma$ an element in $F \setminus \{1\}$. Thus, we can use Bestvina-Paulin method to obtain, from the sequence $(h_n)_{n<\omega}$, restricted to $G^{i+1}$, an action of $G^{i+1}$ on a based real tree $(Y, \ast)$ so that either $g$ fixes the base point or the segment $[\ast, g \cdot \ast]$ is fixed by $C_F(\gamma)$. To see this, one can use Lemma 4.2. In the first case $g$ belongs to $G^i$, thus the latter case occurs. We take cases with respect to the form of the $i + 1$-flat of the tower $T(G, F)$.

• Suppose $G^{i+1}$ is the free product of $G^i$ with $B$, where $B$ is either a free group or a surface group as in Definition 5.4. Then the limit tree $(Y, \ast)$ decomposes as a graph of actions as in Fact 5.16 and in each case it is covered by translates of the real tree $Y_u$ that corresponds to the vertex stabilized by $B$. Every segment in $Y_u$ is trivially stabilized, thus $g$ cannot be an element of $G^{i+1} \setminus G^i$.

• Suppose $G^{i+1}$ has a surface flat structure over $G^i$. By the same argument as in the previous case, every segment in $(Y, \ast)$ is trivially stabilized. Therefore, $g$ cannot be an element of $G^{i+1} \setminus G^i$.

Figure 8: A free abelian flat limit action.

• Suppose $G^{i+1} = G^i \ast_A (A \oplus \mathbb{Z})$ has a free abelian flat structure over $G_i$. The action of $G^{i+1}$ on $(Y, \ast)$ decomposes as a graph of actions as in Fact 5.18 (see Figure 8). Let $a_1 b_1 \ldots b_n a_m$ be the normal form for $g$ with respect to the above amalgamated free product. We first claim that exactly one $b_i$ occurs (thus $g$ has the form $a_1 b_1 a_2$). Suppose
not, then the segment \([*, g \cdot *]\) contains the subsegments \(I_1 := [*, a_1 b_1 \cdot *]\) and \(I_2 := [a_1 b_1 \cdot *, ab b_2 \cdot *]\). By the properties of the action \(\text{Stab}_{G^{n+1}}(I_1) = a_1 A a_1^{-1}\) and \(\text{Stab}_{G^{n+1}}(I_2) = a_1 b_1 a_2 A (a_1 b_1 a_2)^{-1}\). Since \(G^2\) is CSA we have that \(\text{Stab}_{G^{n+1}}(I_1) \cap \text{Stab}_{G^{n+1}}(I_2) = \{1\}\). By the superstability of the action we must have \(\text{Stab}_{G^{n+1}}([*, g \cdot *]) = \{1\}\), a contradiction.

We conclude that \(g = a_1 b_1 a_2\) and the stabilizer of the segment \([*, a_1 b_1 \cdot *]\) contains \(C_{\mathbb{F}}(\gamma)\). Now \(\text{Stab}_{G^{n+1}}([*, a_1 b_1 \cdot *]) = a_1 A a_1^{-1}\), thus \(a_1^{-1} \gamma a_1\) belongs to \(A\). So, \(h_n(a_1) \cdot h_n(b_1) \cdot h_n(a_2) = \gamma^{k_n}\) and since \(b_1\) commutes with every element in \(A\) we have that \(h_n(b_1) = h_n(a_1)^{-1} \cdot \gamma^{k_n} \cdot h_n(a_1)\). Thus \(h_n(a_1 a_2) = \gamma^{k_n-l_n}\). Since \(a_1 \cdot a_2\) belongs to \(G^2\), the inductive hypothesis gives the result.

\[\square\]

**Corollary 6.6:** Let \(G\) be a group that has the structure of a limit tower \(T(G, \mathbb{F})\). Let \((h_n)_{n<\omega}: G \rightarrow \mathbb{F}\) be a test sequence for \(T(G, \mathbb{F})\) and \(g\) be an element in \(G\) for which \(\{C_{\mathbb{F}}(h_n(g)) \mid n < \omega\}\) is finite. Then there exists \(\gamma \in \mathbb{F}\) such that for any test sequence \((f_n)_{n<\omega}\) for \(T(G, \mathbb{F})\) we have that \(C_{\mathbb{F}}(f_n(g)) = C_{\mathbb{F}}(\gamma)\).

**Proof.** Since \(\{C_{\mathbb{F}}(h_n(g)) \mid n < \omega\}\) is finite we have that \(g\) satisfies the conclusion of Lemma 6.5. If \(g \in \mathbb{F}\), then the result is trivial. Thus we have that there exists a finite sequence of free abelian flats, a sequence of triples and a non trivial element \(\gamma \in \mathbb{F}\) that satisfy the properties in Lemma 6.5.

We fix a test sequence \((f_n)_{n<\omega}\) for \(T(G, \mathbb{F})\) and we proceed by induction on the number of free abelian flats that occur in the analysis of \(g\). In the base case we have that \(g = a_1 b_1 c_1\), the product \(a_1 c_1\) belongs to \(C_{\mathbb{F}}(\gamma)\) and \(b_1\) commutes with \(a_1^{-1} \gamma a_1\). Thus, \(f_n(g) = f_n(a_1) f_n(b_1) f_n(c_1) = f_n(a_1) f_n(a_1)^{-1} \gamma^{k_n} f_n(a_1) f_n(c_1) = \gamma^{k_n} f_n(a_1) f_n(c_1)\) and since \(f_n(a_1) f_n(c_1)\) belongs to \(C_{\mathbb{F}}(\gamma)\) we have that \(f_n(g)\) belongs to \(C_{\mathbb{F}}(\gamma)\).

Assume the result holds for any element that at most \(m\) free abelian flats occur in its analysis. We prove it for \(m+1\). We have that \(g = a_1 b_1 c_1\), the product \(a_1 c_1\) equals \(a_2 b_2 c_2\) and \(b_1\) commutes with \(a_1^{-1} \gamma a_1\). Now by the inductive hypothesis we have that \(a_2 b_2 c_2\) belongs to \(C_{\mathbb{F}}(\gamma)\) and the result follows as in the base case.

\[\square\]

**Lemma 6.7:** Let \(G\) be a group that has the structure of a limit tower \(T(G, \mathbb{F})\). Let \((h_n)_{n<\omega}: G \rightarrow \mathbb{F}\) be a test sequence for \(T(G, \mathbb{F})\) and \(\gamma \in \mathbb{F}\ \backslash \{1\}\).

Let \(k\) be a natural number and denote by \(C_{\mathbb{F}}^k(\gamma)\) the subgroup of \(k\)-powers of \(C_{\mathbb{F}}(\gamma)\). Let \(g \in G\) such that \(\{h_n(g) \cdot C_{\mathbb{F}}^k(\gamma) \mid n < \omega\}\) is finite. Then either \(g \in \mathbb{F}\) or there exists a finite sequence of free abelian flats \(G^{i_1} *_{A_{i_1}} (A_{i_1} \oplus \mathbb{Z}), G^{i_2} *_{A_{i_2}} (A_{i_2} \oplus \mathbb{Z}), \ldots, G^{i_m} *_{A_{i_m}} (A_{i_m} \oplus \mathbb{Z})\) (with \(i_1 > i_2 > \ldots > i_m\)), a finite sequence of triples \((a_1, b_1, c_1), (a_2, b_2, c_2), \ldots, (a_m, b_m, c_m)\), and an element \(\delta \in \mathbb{F}\), so that the following properties hold:

- for every \(j \leq m\), the product \(a_j b_j c_j\) is in normal form with respect to the splitting \(G^{i_j} *_{A_{i_j}} (A_{i_j} \oplus \mathbb{Z})\) (with \(b_j \in (A_{i_j} \oplus \mathbb{Z}) \backslash A_{i_j}\) and \(a_j, c_j\) possibly trivial);
- for every \(j \leq m\), the product \(c_j \gamma c_j^{-1}\) belongs to \(A_{i_j}\);
- for every \(1 < j < m\), we have that \(a_j c_j = a_{j+1} b_{j+1} c_{j+1}\);
- \(a_m c_m\) belongs to \(\delta \cdot C_{\mathbb{F}}(\gamma)\);
- \(g = a_1 b_1 c_1\).
Proof. The proof is identical to the proof of Lemma 6.5. Note that we can refine the given sequence so that \( h_n(g) \cdot C^p_F(\gamma) = \delta \cdot C^p_F(\gamma) \) for some \( \delta \in F \). Thus, \( h_n(g) = \delta \cdot \gamma^k \) for some sequence \( (k_n)_{n<\omega} \) of elements in \( m \cdot Z \). Now use Lemma \( 4.2 \) to conclude that in the limit action either \( g \) fixes the base point or the stabilizer of the segment \( [*,g^{-1} \cdot *) \) contains \( C_F(\gamma) \).

Lemmata 6.5 and 6.7 give the following corollary.

**Theorem 6.8**: Let \( G \) be a group that has the structure of a limit tower \( T(G,F) \). Let \( (h_n)_{n<\omega} : G \to F \) be a test sequence for \( T(G,F) \). Let \( g_1, g_2 \) be elements in \( G \), \( k \) be a natural number and \( E_{2k} \) be the equivalence relation of \( k \)-left-cosets (see Definition 2.9).

Suppose \( \{(h_n(g_1), h_n(g_2)) | n < \omega \} \) is finite. Then for any test sequence \( (f_n)_{n<\omega} \) for \( T(G,F) \), the set \( \{(f_n(g_1), f_n(g_2)) | n < \omega \} \) is bounded by \( k \).

**Proof.** Since the set \( \{(h_n(g_1), h_n(g_2)) | n < \omega \} \) is finite we have that the set \( \{C_F(h_n(g_2)) | n < \omega \} \) is finite. Thus, by Corollary 6.6 we have that there exists a non-trivial element \( \gamma \in F \) so that, for any test sequence \( (f_n)_{n<\omega} \) for \( T(G,F) \), \( C_F(f_n(g_2)) = C_F(\gamma) \).

On the other hand, again by the finiteness of \( \{(h_n(g_1), h_n(g_2)) | n < \omega \} \) we have that \( \{h_n(g_1) \cdot C^p_F(\gamma) | n < \omega \} \) is finite. Thus, \( g_1 \) satisfies the conclusion of Lemma 6.7. In the case where \( g_1 \) belongs to \( F \) we have trivially that \( \{(f_n(g_1), f_n(g_2)) | n < \omega \} = 1 \) for any test sequence \( (f_n)_{n<\omega} \) for \( T(G,F) \). Therefore we may assume that there exists a finite sequence of free abelian flats, a sequence of triples and an element \( \delta \in F \) satisfying the properties in Lemma 6.7.

We fix a test sequence \( (f_n)_{n<\omega} \) for \( T(G,F) \) and we proceed by induction on the number of free abelian flats that occur in the analysis of \( g_1 \). It is enough to show that \( f_n(g_1) \) belongs to \( \delta \cdot C^p_F(\gamma) \). In the base case we have that \( g_1 = a_1b_1c_1 \), the product \( a_1c_1 \) belongs to \( \delta C^p_F(\gamma) \) and \( b_1 \) commutes with \( c_1^{-1} \). Therefore \( f_n(g_1) = f_n(a_1) \cdot f_n(b_1) \cdot f_n(c_1) = f_n(a_1) \cdot f_n(c_1) \gamma^k \cdot f_n(c_1^{-1}) \cdot f_n(c_1) = f_n(a_1)f_n(c_1)\gamma^k \) and since \( a_1 \cdot c_1 \) belongs to \( \delta \cdot C^p_F(\gamma) \) we have that \( f_n(g_1) = \delta \cdot C^p_F(\gamma) \).

Assume the result holds for every element that at most \( m \) free abelian flats occur in its analysis. We prove it for \( m + 1 \). We have that \( g_1 = a_1b_1c_1 \), the product \( a_1c_1 \) equals \( a_2b_2c_2 \) and \( b_1 \) commutes with \( c_1^{-1} \). Now by the inductive hypothesis we have that \( a_2b_2c_2 \) belongs to \( \delta \cdot C^p_F(\gamma) \) and the result follows as in the base case.

Similar results hold for double cosets.

**Lemma 6.9**: Let \( G \) be a group that has the structure of a limit tower \( T(G,F) \). Let \( (h_n)_{n<\omega} : G \to F \) be a test sequence for \( T(G,F) \) and \( \gamma, \delta \in F \setminus \{1\} \).

Let \( p, q \) be natural numbers and \( g \in G \) such that \( \{C^p_F(\delta) \cdot h_n(g) \cdot C^p_F(\gamma) | n < \omega \} \) is finite. Then either \( g \in F \) or there exists a finite sequence of free abelian flats \( G^{i_1} *_{A_{i_1}} (A_{i_1} \oplus Z), G^{i_2} *_{A_{i_2}} (A_{i_2} \oplus Z), \ldots, G^{i_m} *_{A_{i_m}} (A_{i_m} \oplus Z) \) (with \( i_1 > i_2 > \ldots > i_m \)), a finite sequence of tuples \( (a_{11}, b_{11}, a_{12}, \ldots, a_{1k_1}, b_{1k_1}, a_{1k_1+1}), \ldots, (a_{m1}, b_{m1}, a_{m2}, \ldots, a_{mk_m}, b_{mk_m}, a_{mk_m+1}) \), and an element \( \beta \in F \), so that the following properties hold:

- for every \( j \leq m \), the product \( a_{j1}b_{j1} \ldots a_{jk_j}b_{jk_j}a_{jk_j+1} \) is in normal form with respect to the splitting \( G^{i_j} *_{A_{i_j}} (A_{i_j} \oplus Z) \) (with \( b_{j1} \in (A_{i_j} \oplus Z) \setminus A_{i_j} \) and \( a_{j1}, a_{jk_j+1} \) possibly trivial);

- for every \( j \leq m \), the product \( (a_{j1}a_{j2} \ldots a_{jk_j-1})^{-1}b_{a_{j1}a_{j2} \ldots a_{jk_j-1}} \) belongs to \( A_{i_j} \). Moreover, either \( (a_{j1}a_{j2} \ldots a_{jk_j})^{-1} \beta a_{j1}a_{j2} \ldots a_{jk_j} \) belongs to \( A_{i_j} \) or \( a_{jk_j+1} \gamma a_{jk_j+1} \) belongs to \( A_{i_j} \).
for every $1 < j < m$, we have that
$$a_{j1}a_{j2}\ldots a_{jk_j+1} = a_{(j+1)1}b_{(j+1)1}a_{(j+1)2}\ldots a_{(j+1)k_j+1}\ b_{(j+1)k_j+1}a_{(j+1)k_j+1};$$

- $a_{m1}a_{m2}\ldots a_{mk_m+1}$ belongs to $C_F(\delta) \cdot \beta \cdot C_F(\gamma)$;

- $g = a_{11}b_{11}a_{12}\ldots a_{1k_1}b_{1k_1}a_{1k_1+1}$.

**Proof.** The proof is by induction on the height of $g$. The base case is trivial since if $\text{height}(g) = 0$, then $g \in F$.

Assume the result is true for any element $g$ of height less or equal to $i$. We show it for elements of height $i + 1$. Since $\{C_F^g(\delta) \cdot h_n(g) \cdot C_F^g(\gamma) \mid n < \omega\}$ is finite, there exists $n_0$ such that $C_F^g(\delta) \cdot C_F^g(h_n(g)) \cdot C_F^g(\gamma) = C_F^g(\delta) \cdot C_F^g(h_s(g)) \cdot C_F^g(\gamma)$ for arbitrarily large $s$. Thus, for the refined sequence, still denoted $(h_n)_{n<\omega}$, we have that $h_n(g) = \delta^nh\gamma^kh$ for some sequence of integers $(k_n)_{n<\omega}$ (respectively $(l_n)_{n<\omega}$) with $k_n \in q\mathbb{Z}$ (respectively $l_n \in p\mathbb{Z}$) for every $n < \omega$, and $\beta$ an element in $F$. Thus, we can use Bestvina-Paulin method to obtain, from the sequence $(h_n)_{n<\omega}$, restricted to $G^{i+1}$, an action of $G^{i+1}$ on a based real tree $(Y, \ast)$ so that either $g$ fixes the base point or the segment $[\ast, g \ast]$ is the union of two segments $I_1 := [\ast, x_1], I_2 := [x_1, g \ast]$ with a common endpoint and the stabilizer of $I_1$ contains $C_F(\delta)$. To see this, one can use Lemma \[4.2\].

As a matter of fact the point $x_1$ in $Y$ will be the point approximated by the sequence $(\delta^i\ast)_{n<\omega}$ (see as points in the Cayley graph $\mathcal{A}_F$). Note that we may assume that at least one of the segments is non trivial otherwise $g$ fixes the base point. We take cases according to whether or not $I_1$ is trivial.

- Assume that $I_1$ is trivial. Then by Lemma \[4.2\] we have that the stabilizer of $[\ast, g^{-1} \ast]$ contains $C_F(\gamma)$. Thus as in the proof of Lemma \[6.3\] we have that the $i + 1$ flat is a free abelian flat $G^i \ast A_i \cdot (A_i \oplus \mathbb{Z})$ and $g$ admits a normal form $a_1b_1a_2$ where $b_1 \in (A_i \oplus \mathbb{Z}) \setminus A_i$ and so that $a_2\gamma a_2'$ belongs to $A_i$. Note that in this case $h_n(a_1a_2) = \delta^nh\gamma^kh$.

- Assume that $I_1$ is not trivial. Then similar to the proof of Lemma \[6.3\] we have that the $i + 1$ flat is a free abelian flat $G^i \ast A_i \cdot (A_i \oplus \mathbb{Z})$ and $g$ admits a normal form $a_1b_1a_2\ldots a_mb_m a_{m+1}$. Note that in this case we may not assume that the normal form contains exactly one $b_i$, as the segment $[\ast, g \ast]$ might be trivially stabilized. Since $C_F(\delta)$ fixes $I_1$, we have that the segment $[\ast, a_1b_1 \ast]$ is fixed by $C_F(\delta)$. But then we have that $a_1^{-1} \cdot \delta \cdot a_1$ belongs to $A_i$ and $h_n(a_1a_2b_2\ldots b_m a_{m+1}) = \delta^nh\gamma^kh$.

In both cases $g$ is an element that belongs to a free abelian flat. Let $a_1b_1\ldots b_m a_{m+1}$ be the normal form of $g$ with respect to this flat. Now proceed inductively on the length of $a_1b_1\ldots b_m a_{m+1}$ and consider at each step the above two cases.

**Theorem 6.10:** Let $G$ be a group that has the structure of a limit tower $\mathcal{T}(G,F)$. Let $(h_n)_{n<\omega}: G \to F$ be a test sequence for $\mathcal{T}(G,F)$. Let $g_1, g_2, g_3$ be elements in $G$. Let $k, l$ be natural numbers and $E_{4k,l}$ be the equivalence relation of $k,l$-double-cosets (see Definition \[2.9\]).

Suppose $\{(h_n(g_1), h_n(g_2), h_n(g_3)) \mid E_{4k,l} \mid n < \omega\}$ is finite. Then for any test sequence $(f_n)_{n<\omega}$ for $\mathcal{T}(G,F)$, the set $\{(f_n(g_1), f_n(g_2), f_n(g_3)) \mid E_{4k,l} \mid n < \omega\}$ is bounded by $k \cdot l$.

**Proof.** The proof is left to the reader. \[\square\]
7 Diophantine envelopes

In this section we define the notion of the \textit{graded Diophantine envelope}, which will be the main tool for proving the elimination of the \(\exists^\infty\)-quantifier. Roughly speaking the graded Diophantine envelope of a first order formula \(\phi(x, y)\) over a torsion-free hyperbolic group, is a finite set of graded towers with the special property of deciding whether a “graded test sequence” is (eventually) in the solution set of \(\phi\). Note that one cannot hope for a Diophantine envelope deciding whether a single solution is in the solution set of some formula, as this would imply quantifier elimination down to Boolean combinations of \(\exists\)-formulas and an (unpublished) result of Bestvina and Feighn states that this is not possible.

Before moving to the statement of the main theorem of this section we explain the term “graded test sequence” of the previous paragraph. Let \(\mathcal{G}T(G, Sld_H)\) be a graded \(\Gamma\)-limit tower over the solid \(\Gamma\)-limit group \(\text{Sld}\) (with respect to the subgroup \(H\)). \((f_n)_{n<\omega} : H \to \Gamma\) be a sequence of morphisms and \((s_n)_{n<\omega} : Sld \to \Gamma\) be a sequence of strictly solid families of morphisms extending \(f_n\), for each \(n < \omega\). To each strictly solid family \([s_i]\) one can assign an (ungraded) \(\Gamma\)-limit tower \(T_i(G, \Gamma)\) in a way that each element of \(G\) is assigned to an element of \(G_i\) (cf. [Sel03 Section 3]). Then we say that a sequence, \((s_n, (h^m_n)_{m<\omega})_{n<\omega}\), is a graded test sequence if for every \(n < \omega\), \(s_n : Sld \to \Gamma\) is a strictly solid morphism and \((h^m_n)_{m<\omega} : G_i \to \Gamma\) is a test sequence for \(T_i(G_i, \Gamma)\) that moreover satisfy the conditions of Definition 3.1 in [Sel03]. The existence of a graded Diophantine envelope has been proved by Sela (see [Sel Theorem 1.3]) as a useful tool for obtaining elimination of imaginaries in the first order theory of a torsion-free hyperbolic group.

Let us fix some notation for the following theorem. When we want to be explicit about a generating set, \(\bar{u}\), of a group \(G\) that has the structure of a tower \(T(G, H)\), then we denote the tower by \(T(G, H)[\bar{u}]\). Moreover, if we are interested in the images (under morphisms from \(G\)) of a certain tuple of elements of \(G\), \(\bar{x}\) for example, then we denote the tower by \(T(G, H)[\bar{u}, \bar{x}]\).

\textbf{Theorem 7.1} (Graded Diophantine envelope): Suppose \(\phi(\bar{x}, \bar{y})\) is a first order formula over a torsion-free hyperbolic group \(\Gamma\). Then there exist finitely many graded \(\Gamma\)-limit towers \(\mathcal{G}T^1(G_1, Sld^1_{(\bar{y})})[(\bar{u}, \bar{x}, \bar{y})], \ldots, \mathcal{G}T^k(G_k, Sld^k_{(\bar{y})})[(\bar{u}, \bar{x}, \bar{y})]\), such that:

(i) For each \(i \leq k\), there exists \(f : (\bar{y}) \to \Gamma\), a strictly solid family of morphisms \([s] : Sld^i \to \Gamma\) (extending \(f\)) and a test sequence, \((h_n)_{n<\omega}\), for the ungraded tower \(T^i_{[s]}\) that corresponds to \([s]\), so that \(\Gamma \models \phi(h_n(\bar{x}), f(\bar{y}))\) for every \(n < \omega\);

(ii) If \(\Gamma \models \phi(\bar{b}, \bar{c})\), then there exist some \(i \leq k\) and:

1. a morphism \(f : (\bar{y}) \to \Gamma\) (where \(\bar{y}\) is the subgroup of \(Sld^i\) generated by \(\bar{y}\)), given by the rule \(\bar{y} \mapsto \bar{c}\); 
2. a strictly solid family of morphisms \([s] : Sld^i \to \Gamma\) extending \(f\);
3. a test sequence \((h_n)_{n<\omega}\) for the ungraded tower \(T^i_{[s]}(G, \Gamma)\) that corresponds to \([s]\); and
4. a morphism \(h : G \to \Gamma\) with \(h \mid \Gamma = Id\).

such that \(h(\bar{x}, \bar{y}) = (\bar{b}, \bar{c})\) and \(\Gamma \models \phi(h_n(\bar{x}), \bar{c})\) for every \(n < \omega\).

\textbf{Remark 7.2}:
• In the previous theorem, by an abuse of language, we refer to $\bar{x}$ (respectively $\bar{y}$) and the tuple it is assigned to in the ungraded tower by the same name;

• If we want to refer to the second part of this theorem with respect to some solution $(\bar{b}, \bar{c})$ of $\phi$ in $\Gamma$, we will say that $(\bar{b}, \bar{c})$ factors through the graded limit tower $\mathcal{GT}^i$.

8 The finite cover property

In this section we prove the main result of the paper. We start by proving that the “there exists infinitely many” quantifier can be eliminated in the real sort.

**Theorem 8.1:** Let $\phi(x, \bar{y})$ be a first order formula over a non abelian free group $\mathbb{F}$. Then there exists $n < \omega$ such that, for any $\bar{c} \in \mathbb{F}$, if $\phi(x, \bar{c})$ is finite, then $|\phi(x, \bar{c})| \leq n$.

**Proof.** Suppose not, and $(\bar{c}_i)_{i < \omega}$ be a sequence of tuples such that, for each $l$, $l < |\phi(\bar{x}, \bar{c}_i)| < \infty$.

We first consider the graded Diophantine envelope, $\mathcal{GT}^i(G_1, \text{Sld}_{\bar{y}}^1)[(\bar{u}, x, \bar{y})], \ldots, \mathcal{GT}^i(G_k, \text{Sld}_{\bar{y}}^k)[(\bar{u}, x, \bar{y})]$, of $\phi(x, \bar{y})$ as given by Theorem 7.1. Let $r_i$ be the bound on the number of strictly solid families of morphisms for the solid limit group $\text{Sld}_{\bar{y}}^i$ in the ground floor of $\mathcal{GT}^i$ as given by Theorem 6.10.

Let $n$ be a natural number with $n > r_1 + \ldots + r_k$. By Theorem 7.1(ii), each solution of $\phi(x, \bar{c}_n)$ factors through some graded tower in the graded Diophantine envelope for $\phi$. Consider the strictly solid family of morphisms and the ungraded tower that corresponds to this family. Since $\phi(x, \bar{c}_n)$ is finite we must have that $x$ takes finitely many values under the test sequence of the ungraded tower. Thus, by Theorem 6.2, for each tower $\mathcal{GT}^i$ we can have at most $r_i$ solutions that factor through it, a contradiction. □

We continue with the sort for conjugacy classes.

**Theorem 8.2:** Let $\phi(x, \bar{y})$ be a first order formula over a non abelian free group $\mathbb{F}$. Then there exists $n < \omega$ such that, for any $\bar{c} \in \mathbb{F}$, if $\phi(x, \bar{c})$ has finitely many solutions up to conjugation, then the number of conjugacy classes $|\phi(x, \bar{c})|_{E_1}$ is bounded by $n$.

**Proof.** The proof is identical to the proof of Theorem 8.1 using Theorem 6.4 instead of Theorem 6.2. □

**Theorem 8.3:** Let $\phi(x_1, x_2, \bar{y})$ be a first order formula over a non abelian free group $\mathbb{F}$. Then there exists $n < \omega$ such that, for any $\bar{c} \in \mathbb{F}$, if $\phi(\bar{x}, \bar{c})$ has finitely many solutions up to the equivalence relation $E_{2,m}$, then the number of $m$-left-cosets $|\phi(\bar{x}, \bar{c})|_{E_{2,m}}$ is bounded by $n$.

**Proof.** The proof is identical to the proof of Theorem 8.1 by choosing $n > m \cdot (r_1 + \ldots + r_k)$ and using Theorem 6.8 instead of Theorem 6.2. □

**Theorem 8.4:** Let $\phi(x_1, x_2, x_3, \bar{y})$ be a first order formula over a non abelian free group $\mathbb{F}$. Then there exists $n < \omega$ such that, for any $\bar{c} \in \mathbb{F}$, if $\phi(\bar{x}, \bar{c})$ has finitely many solutions up to the equivalence relation $E_{4_{p,q}}$, then the number of $p, q$-double-cosets $|\phi(\bar{x}, \bar{c})|_{E_{4_{p,q}}}$ is bounded by $n$.

**Proof.** The proof is identical to the proof of Theorem 8.1 by choosing $n > p \cdot q \cdot (r_1 + \ldots + r_k)$ and using Theorem 6.10 instead of Theorem 6.2. □
Finally combining Theorems 8.1 8.2 8.3 and 8.4 we get:

**Theorem 8.5:** Let $\mathbb{F}$ be a non abelian free group. Then $\mathcal{T}h(\mathbb{F})^{\mathcal{Q}}$ eliminates the $\exists^{\infty}$-quantifier.

**Proof.** Suppose, for the sake of contradiction, that there is a first order formula $\phi(x_E, \bar{y})$, where $x_E$ is a variable of the sort $S_E$ for some $\emptyset$-definable equivalence relation $E$ and $\bar{y}$ is a tuple of variables in the real sort, and a sequence of tuples $(\bar{c}_n)_{n<\omega}$ in $\Gamma$ such that $n < |\phi(x_E, \bar{y})| < \infty$ for each $n < \omega$.

We consider the formula:

$$
\psi(x_1, \ldots, x_k, x_{E_1}, \ldots, x_{E_l}, \bar{y}, \bar{a}) := \exists x_E(\phi(x_E, \bar{y}) \land R_E(x_E, x_1, \ldots, x_k, x_{E_1}, \ldots, x_{E_l}, \bar{a}))
$$

Where $R_E$ is the definable relation assigned to the equivalence relation $E$ by Theorem 2.10. $x_1, \ldots, x_k$ are variables from the real sort and $x_{E_j}$ are variables of the sort $S_{E_j}$ for some basic equivalence relations $E_j$ (see Definition 2.9). Then, by Theorem 2.10, the sequence of tuples $(\bar{c}_n, \bar{a})_{n<\omega}$ in $\mathbb{F}$ witnesses that $n < |\psi(x_1, \ldots, x_k, x_{E_1}, \ldots, x_{E_l}, \bar{c}_n, \bar{a})| < \infty$ for each $n < \omega$.

Finally, by Lemma 2.1 there exists a first order formula with all variables in the real sort $\theta(x_1, \ldots, x_k, \bar{z}_1, \ldots, \bar{z}_l, \bar{y}, \bar{w})$ such that

$$
\mathbb{F}^{\mathcal{Q}} \models \forall x_1, \ldots, x_k, \bar{z}_1, \ldots, \bar{z}_l, \bar{y}, \bar{w}(\psi(x_1, \ldots, x_k, f_{E_1}(\bar{z}_1), \ldots, f_{E_l}(\bar{z}_l), \bar{y}, \bar{w}) \leftrightarrow \\
\theta(x_1, \ldots, x_k, \bar{z}_1, \ldots, \bar{z}_l, \bar{y}, \bar{w}))
$$

Thus, after refining, re-enumerating and possibly expanding the sequence (that we still denote by) $(\bar{c}_n, \bar{a})$, one of the following holds:

- there exists a first order formula $\theta_1(z, \bar{y}, \bar{w})$ such that $n < |\theta_1(z, \bar{c}_n, \bar{a})| < \infty$;

- for some natural number $m$, there exists a first order formula $\theta_{2m}(z_1, z_2, \bar{y}, \bar{w})$ such that $n < |\theta_{2m}(z_1, z_2, \bar{c}_n, \bar{a})|_{E_{2m}} < \infty$;

- for some couple of natural numbers $(m, k)$, there exists a first order formula $\theta_{4m,k}(z_1, z_2, z_3, \bar{y}, \bar{w})$ such that $n < |\theta_{4m,k}(z_1, z_2, z_3, \bar{c}_n, \bar{a})|_{E_{4m,k}} < \infty$;

In each case a contradiction is reached. $\square$

Together with the stability of the theory of non abelian free groups we get:

**Corollary 8.6:** Let $\mathbb{F}$ be a non abelian free group. Then $\mathcal{T}h(\mathbb{F})$ does not have the finite cover property.

9 Further remarks

As our proofs did not use any special property of free groups or $\mathbb{F}$-limit groups we believe that a straightforward generalization to torsion-free hyperbolic groups is possible. We formally could not find a reference for the existence of test sequences with respect to towers (over a torsion-free hyperbolic group) with the properties stated in Facts 5.16 5.17 and 5.18. And it is not hard to check that this is the only essential part that is missing if one wants to generalize our result.
On the other hand, since the properties of a test sequence mentioned above are essential to many parts of the proof of Sela and given the generalization of these methods in [Sel09], our feeling is that the above basic facts extend to towers over torsion-free hyperbolic groups. In any case we state without a proof the following claim:

**Claim:** Let $\Gamma$ be a (non-cyclic) torsion-free hyperbolic group. Let $\mathcal{T}(G, \Gamma)$ be a $\Gamma$-limit tower. Then a test sequence for $\mathcal{T}(G, \Gamma)$ exists. Moreover any test sequence for $\mathcal{T}(G, \Gamma)$ satisfies Facts 3.10, 5.17, 5.18.

If the above claim is true, we get:

**Theorem 9.1:** Let $\Gamma$ be a (non cyclic) torsion-free hyperbolic group. Then $\text{Th}(\Gamma)^{eq}$ eliminates the $\exists^\infty$-quantifier.

As in the case of non abelian free groups, since any torsion-free hyperbolic group is stable, we have:

**Corollary 9.2:** Let $\Gamma$ be a (non cyclic) torsion-free hyperbolic group. Then $\text{Th}(\Gamma)$ does not have the finite cover property.

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