I. INTRODUCTION

Quantum sealing is a young member of quantum cryptography. But classical sealing has entered our everyday life for centuries. For example, people sometimes put important letters or documents inside an envelope, and seal the envelope by melting wax over the cover flap. Additionally, the wax can be impressed with an image to indicate authenticity, such as a family crest. If the wax was broken or the image looks unmatched, then someone may have opened the envelope and read the document. Thus it provides a method to check whether the secret document remains secure or not. Obviously it is useful to expand the idea to digital sealing to the classical world. However, just as other classical cryptographic protocols, no classical sealing protocol can be unconditionally secure. This is because there is no no-cloning theorem of classical data to prevent a cheater from copying and reading all the data without being detected.

On the other hand, quantum cryptography has made significant progress in the last two decades. Many quantum protocols, e.g. the conjugate coding [1] and the well-known quantum key distribution [2, 3, 4], surpassed their classical counterparts as their security is rested on the basic laws of quantum mechanics and can be unconditionally secure. Therefore it is natural to ask whether the quantum no-cloning theorem [5] can make secure quantum sealing possible. Bechmann-Pasquinucci [6] proposed the first quantum sealing protocol in 2003 which seals a single classical bit with a three-qubit state, shortly followed by Chau [7] with a protocol which seals quantum data with quantum error correcting code. In 2005, Singh and Srikanth [8] extended the idea of Ref. [6] into a many-qubit majority voting scheme, and associated it with secret sharing. That is, its security has been secured by its classical counterpart in Ref. [5].

This is not true. Though secure quantum sealing of a single classical bit is impossible, it will be shown in this paper that quantum sealing of a classical bit string can be unconditionally secure. At the first glance, this result seems odd since classical reasoning suggests that secure bit string sealing implies secure single bit sealing and hence conflicts with the above no-go proofs. But as pointed out by Kent [12], reductions and relations between classical cryptographic tasks need not necessarily apply to their quantum equivalents. Similar situation happened before in quantum cryptography. Though the possibility of unconditionally secure quantum bit commitment was excluded by the Mayers-Lo-Chau no-go theorem [13, 14], Kent found that secure quantum bit string commitment protocol exists [12]. In this paper, a secure quantum bit string sealing protocol will be proposed. It achieves the following goal: each bit of the string can be obtained by the reader with an arbitrarily small error rate, while reading the string can be detected except with an exponentially small probability. The significance of the protocol lies in three major aspects. (1) In practical, it is obvious that sealing a long bit string is much more useful than sealing a single bit. (2) As shown by Gordon Worley III [15], secure quantum sealing, if exists, has wide applications in many different fields, e.g. loose bit commitment, binary semaphores, eavesdropping detection and protective packaging. Therefore our protocol can re-open new venues for these applications that once closed by the no-go proofs. (3) The protocol is very simple to be implemented with the techniques available nowadays. In additional, the protocol also makes it possible to implement “computationally” secure quantum bit sealing in practical.
II. THE PROTOCOL

Quantum bit string sealing can be summarized as the following two-party cryptographic problem. A sender Alice encodes an n-bit string with quantum states. Any reader Bob can obtain the string from these states, while reading the string should be detectable. Clearly quantum bit sealing can be viewed as the special case where \( n = 1 \).

Consider the ideal case without transmission error. Let \( \Theta (0 < \Theta \ll \pi/4) \) and \( \alpha (0 < \alpha < 1/2) \) be two fixed constants. We propose the following quantum string sealing protocol:

Sealing: To seal a classical \( n \)-bit string \( b = b_1b_2...b_n \) \( (b_i \in \{0, 1\}) \), Alice randomly chooses \( \theta_i (-\Theta/n^\alpha \leq \theta_i \leq \Theta/n^\alpha) \) and encodes each bit \( b_i \) in a qubit state \( |\psi_i\rangle = \cos \theta_i |b_i\rangle + \sin \theta_i |\bar{b}_i\rangle \). She makes these \( n \) qubits publicly accessible to the reader, while keeping all \( \theta_i \) (\( i = 1, ..., n \)) secret.

Reading: When Bob wants to read the string \( b \), he simply measures each qubit in the computational basis \( \{ |0\rangle, |1\rangle \} \), and denotes the outcome as \( |b'_i\rangle \). He takes the string \( b' = b'_1b'_2...b'_n \) as \( b \).

Checking: At any time, Alice can check whether the sealed string \( b \) has been read by trying to project the \( i \)-th qubit into \( \cos \theta_i |b_i\rangle + \sin \theta_i |\bar{b}_i\rangle \). If all the \( n \) qubits can be projected successfully, she concludes that the string \( b \) is still unread. Otherwise if any of the qubits fails, she knows that \( b \) is read.

Obviously this new protocol achieves the following goal: each bit sealed by Alice can be read successfully by Bob, except with a probability not greater than \( \varepsilon \equiv \sin^2(\Theta/n^\alpha) \). Thus by increasing \( n \), the reading error rate \( \varepsilon \) can be made arbitrarily small.

In the next section it will be proven that the protocol also guarantees that: if the string was read, it will be detected by the checking process except with an exponentially small probability.

III. PROOF OF SECURITY

When the reader Bob measures the \( i \)-th qubit in the basis \( \{ |0\rangle, |1\rangle \} \), the qubit collapses to \( |b'_i\rangle \). We may even assume Bob to be malevolent, that he replaces each qubit with another quantum state \( |\psi'_i\rangle \) after reading it, so that his chance to pass the checking without being detected might be increased. Due to the no-cloning theorem of quantum states, Bob cannot determine and copy the state \( |\psi_i\rangle \) exactly since he does not know \( \theta_i \). He cannot even be sure whether \( b'_i = b_i \) or not after he read \( |\psi_i\rangle \). Thus he has to pick another \( \theta'_i \) \( (-\Theta/n^\alpha \leq \theta'_i \leq \Theta/n^\alpha) \) himself and prepares the fake state as \( |\psi'_i\rangle = \cos \theta'_i |b_i\rangle + \sin \theta'_i |\bar{b}_i\rangle \). Since the case \( b'_i = b_i \) (or \( b'_i = \bar{b}_i \)) will occur with the probability \( \cos^2 \theta_i \) (or \( \sin^2 \theta_i \)), the fake state \( |\psi'_i\rangle \) can be projected to

\[ |\psi_i\rangle = \cos \theta_i |b_i\rangle + \sin \theta_i |\bar{b}_i\rangle \] successfully in the checking process with the probability

\[ p_i = \cos^2 \theta_i \cos^2 (\theta_i - \theta'_i) + \sin^2 \theta_i \sin^2 (\theta_i + \theta'_i). \] (1)

When \( \theta_i \) is evenly distributed among the range \([-\Theta/n^\alpha, \Theta/n^\alpha]\), the average of \( p_i \) is

\[ \bar{p}_i = \int_{-\Theta/n^\alpha}^{\Theta/n^\alpha} p_i \, d\theta_i. \] (2)

Its maximum can be reached when \( \theta'_i = 0 \). That is, it is better for Bob not to fake the state, but simply leaves the \( i \)-th qubit as it is after measuring it. In this case

\[ p_i = 1 - \frac{1}{2} \sin^2 2\theta_i. \] (3)

Therefore the total probability for Bob to read \( k = \beta n \) \( (0 \leq \beta \leq 1) \) bits without being detected is

\[ P = \prod_{i=1}^{k} (1 - \frac{1}{2} \sin^2 2\theta_i), \] (4)

which drops exponentially as \( k \to n \), and vanishes when \( n \to \infty \) as long as \( 0 < \alpha < 1/2 \).

However, in the more general case Bob may not read the string with the measurement suggested by the protocol. He may not even want to learn each \( b_i \) individually, but tries to perform collective measurement on the whole system \( \Psi = |\psi_1\psi_2...\psi_n\rangle \) so that he can obtain some global properties of the string (e.g. parity, weight etc.). In this case, let \( H \) denotes the \( 2^n \) dimensional Hilbert space where \( \Psi \) lives in. Suppose that \( \Psi \) finally collapses into a subspace \( V \) after Bob performs certain POVMs. Let \( \{|v\}\) and \( m \) be the computational basis and the dimensionality of \( V \) respectively. Then no matter how the details of Bob’s cheating strategy could be, the amount of information Bob obtained is bound by

\[ k = \log_2 2^n - \log_2 m. \] (5)

Meanwhile, the final state of \( \Psi \) is

\[ |\Psi'\rangle = \frac{1}{N} \sum_{v \in V} |v\rangle \langle v| \Psi\rangle, \] (6)

where the normalization constant

\[ N = \left( \sum_{v \in V} |\langle v| \Psi\rangle|^2 \right)^{1/2}. \] (7)

Again, it can be shown that it is better for Bob not to fake the state. Then \(|\Psi'\rangle\) can be projected to the initial state \(|\Psi\rangle = |\psi_1\rangle |\psi_2\rangle ... |\psi_n\rangle \) successfully in the checking process with the probability

\[ P = |\langle \Psi| \Psi'\rangle|^2 = \sum_{v \in V} |\langle v| \Psi\rangle|^2. \] (8)
Since $|\psi_i\rangle = \cos \theta_i |b_i\rangle + \sin \theta_i |\bar{b}_i\rangle$ and $\{v\}$ is the computational basis, for any $v$ we have

$$|\langle \psi | \Psi \rangle|^2 \leq \prod_{i=1}^{n} \cos^2 \theta_i.$$  \hfill (9)

Therefore the total probability for Bob to obtain $k$ bits of information without being detected is

$$P \leq m \cdot \prod_{i=1}^{n} \cos^2 \theta_i = 2^{-k} \prod_{i=1}^{n} 2 \cos^2 \theta_i.$$  \hfill (10)

which also drops exponentially as $k \to n$, and vanishes when $n \to \infty$ as long as $0 < \alpha < 1/2$.

As a result, no matter Bob reads the string with individual or collective measurements, the probability for him to avoid from being detected will always be exponentially small as the amount of information he obtained increases. Thus the protocol is unconditionally secure.

IV. DISCUSSIONS

A. Relationship with the no-go proofs

The existence of this secure quantum bit string sealing protocol does not conflict with the no-go proofs of quantum single bit sealing \cite{9, 10, 11}. In fact, our quantum bit string sealing protocol can be viewed as the assembly of $n$ imperfect quantum single bit sealing process. From the security proof in the above section, we can see that if Bob reads only few bits, the disturbance on the quantum states is small that it is almost undetectable. In this sense, the sealing of each single bit of the string is insecure. Also, it is insecure to use the global properties of the string (e.g. parity, weight etc.) to implement single bit sealing. But if Bob reads a large number of bits, the small disturbance on every single qubit will be piled up together so that the detecting probability will increase dramatically. Hence the sealing of the whole string can be secure.

B. The protocol is an imperfect sealing one

The sealed string can only be read with a non-zero error rate $\varepsilon \equiv \sin^2(\Theta/n^2)$. We cannot associate the protocol with classical error-correcting codes or any other method to make the string perfectly retrievable. This is because the above security proof is based on the fact that Bob has no pre-knowledge on the string and the quantum states. If he is provided with a certain classical error-correcting code or anything relevant with the sealed string, he may have other methods to construct his collective measurements so that the security proof may not be valid any more. In fact, it is trivial to show that the no-go proof of perfect quantum bit sealing \cite{9} can be generalized to the case of perfect quantum bit string sealing.

If the whole string becomes perfectly retrievable, then each bit of the string is perfectly retrievable too. From the proof in Ref. \cite{9} we can see that if Bob reads every single bit with collective measurements, the disturbance will rigorously equal to zero. Thus the total detecting probability will not be piled up but still equal to zero. Therefore, though the error rate $\varepsilon$ can be made arbitrarily small, we cannot expect to find methods to make it completely vanished.

On the other hand, as pointed out in Ref. \cite{10}, the no-go proof of imperfect quantum single bit sealing does not cover the case of string sealing. More rigorously, in our protocol by expanding the state $|\Psi\rangle$ in the computational basis of the global Hilbert space $H$, we can see that $|\Psi\rangle$ covers all eigenvectors of $H$. That is, the “sub”-space supported by $|\Psi\rangle$ is exactly the space $H$ itself. Therefore the spaces supported by different states which encode different strings completely overlap with each other. No measurement can distinguish them apart without disturbing the states seriously. Thus the cheating strategy in the no-go proofs of imperfect quantum single bit sealing \cite{10, 11} does not apply here.

C. Implementability

Our protocol can be executed as long as Alice has the probability to prepare each single qubit in a pure state, while Bob can perform individual measurement. No entanglement or collective measurement required. Therefore the protocol can be demonstrated and verified with the techniques available nowadays. Of course for practical uses, storing quantum states for a long period of time is still a technical challenge today. But this is a problem which all quantum sealing protocols have to face. Our protocol may already be one of the simplest in all possible quantum bit string sealing protocols.

V. SEALING A SINGLE BIT IN PRACTICAL

Though secure sealing of a single bit is impossible in principle, if a protocol can be found in which reading the bit dishonestly is much more difficult than doing so honestly, it will still be valuable in practical. The no-go proof of imperfect quantum bit sealing \cite{11} leaves a clue on how to construct such a protocol. As pointed out in that reference, quantum sealing protocol generally contains the following feature: Bob knows an operation $P$ and two sets $G_0, G_1$, such that if he applies $P$ on the quantum system that seals the bit and the outcome is $g \in G_0$ (or $g \in G_1$), he should take the value of the sealed bit as 0 (or 1). Though in principle we cannot exclude this feature from the protocol (otherwise the sealed bit becomes irretrievable), we can keep the dishonest reader from knowing $P$, $G_0$ and $G_1$ too easily. The method is: Alice can seal the description of $P$, $G_0$ and $G_1$ with the quantum string sealing protocol. If Bob wants to decode
the sealed bit correctly, he should read this sealed string first. Thus the status of the sealed bit can be checked by detecting whether the sealed string has been read.

For example, Alice first encodes the following sentence into a classical binary bit string

\[ \text{"Measure the last two qubits in the basis } \{ \cos 15^\circ \ket{0} + \sin 15^\circ \ket{1}, -\sin 15^\circ \ket{0} + \cos 15^\circ \ket{1} \} \text{ and you will know the value of the sealed bit from their parity. Other qubits following this sentence are all dummy qubits. You can simply leave them alone."} \]

Then she seals it with our quantum string sealing protocol, and provides Bob the qubits encoding this sentence, followed by a large number of qubits where only the last two are actually useful.

However, this bit sealing method is still insecure in principle. This is because any given classical n-bit string can be decoded into one sentence only, and the meaning of the sentence will reveal the value of the sealed bit unambiguously. As long as a dishonest Bob knows the length n of the sealed string, he can study all the 2^n possible classical n-bit strings, decode them into sentences, and divide these sentences into \( G_0 \) and \( G_1 \) (of course there will also be tons of meaningless sentences. Bob can simply leave them alone). Then as described in Ref. [10], he needs not to know the content of the sealed sentence exactly. He simply constructs a proper collective measurement to determine whether the sentence belongs to \( G_0 \) or \( G_1 \). Thus he will know the sealed bit from the n qubits without disturbing them too much.

But if n is sufficiently large, the number of possible sentences will be enormous. There could be sentences as simple as

\[ \text{"It is 0. Ignore the rest qubits."} \]

But there are also sentences like

\[ \text{"Decode the bits following this sentence as a bitmap image and you will find clues to the value of the sealed bit"} \]

or

\[ \text{"Go to the main library. Find the book on the top left of the last shelf. Turn to the last page, and count how many times the letter K occurs in the 3rd line....."} \]

or even

\[ \text{"Dig my backyard until you find water. Count how many feet you digged. Then divide it by the height of the tree in the north corner....."} \]

In this case, even if a dishonest Bob has the technique to perform collective measurements on the n-qubit system, in practical it is nearly impossible to check all these sentences and find out the bit value they are corresponding to. On the other hand, an honest Bob needs not to worry about this. He can simply read the sealed string honestly and then follows the instruction to decode the sealed bit. In this sense, such sealing can be viewed as a kind of “computationally” secure quantum single bit sealing in practical.

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