Wave Packet in Quantum Cosmology and Definition of Semiclassical Time

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Abstract

We consider a quantum cosmology with a massless background scalar field $\phi_B$ and adopt a wave packet as the wave function. This wave packet is a superposition of the WKB form wave functions, each of which has a definite momentum of the scalar field $\phi_B$. In this model it is shown that to trace the formalism of the WKB time is seriously difficult without introducing a complex value for a time. We define a semiclassical real time variable $T_P$ from the phase of the wave packet and calculate it explicitly. We find that, when a quantum matter field $\phi_Q$ is coupled to the system, an approximate Schrödinger equation for $\phi_Q$ holds with respect to $T_P$ in a region where the size $a$ of the universe is large and $|\phi_B|$ is small.

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1 Introduction

The notion of time is one of the most serious problems in quantum cosmology.\textsuperscript{[1]} Though this is still controversial, many attempts have been done recently. One of them is to utilize the semiclassical approximation. Banks and others assumed that the solution to the Wheeler-DeWitt equation has the form of the WKB approximation, namely $Ψ^{WKB} = Φe^{i\kappa S_0}$, where $S_0$ is the Hamilton’s principal function, and they introduced a time variable $T_W$ using $S_0$.\textsuperscript{[2]–[13]} They showed that, when a quantum matter field $φ_Q$ is coupled to the system, its wave function satisfies the Schrödinger equation with respect to $T_W$ in the region where the semiclassical approximation is well justified.

It has not yet been clarified how the classical universe emerged from the quantum universe. However, it seems probable to assume that the wave function of the universe forms a narrow wave packet in the classical region. In this paper we will consider a quantum cosmology with a massless background scalar field $φ_B$ and adopt a wave packet with respect to $φ_B$ as the wave function. This wave packet is a superposition of the WKB form wave functions, each of which has a definite momentum $κ$ of the scalar field $φ_B$. Thus, it is expected that the packet tends to a classical orbital motion in the classical region. In this model first we will show that it is seriously difficult to trace the formalism of the WKB time and to define a time variable for the Schrödinger equation of $φ_Q$ without introducing a complex value for a time.

Several years ago Greensite and Padmanabhan advocated a time variable $T_E$ by requiring that the Ehrenfest principle holds with respect to $T_E$.\textsuperscript{[14]–[15]} If this Ehrenfest time $T_E$ exists, this is proportional to another time variable, a phase time, which is derived from the phase of an arbitrary solution to the Wheeler-DeWitt equation. However, recently Brotz and Kiefer showed that the Ehrenfest time $T_E$ does not always exist, since the constraints on $T_E$ is considerably severe.\textsuperscript{[16]} On the contrary it is possible to define a phase time which is real, as long as the phase of the wave function is not constant.

We will introduce a real background phase time $T_P$ and calculate it explicitly
when the width $\sigma$ of the wave packet is narrow and the size $a$ of the universe is large. The phase time $T_p$ will be compared to $T_w$ which is derived by a WKB form wave function with a momentum $\kappa_0$ where the wave packet has a peak. It will be shown that $T_p$ is a smooth extension of $T_w$ and they become identical in the narrow limit of the wave packet. It is important to examine whether an expected dynamical equation that is a Schrödinger equation for $\phi_Q$ with respect to $T_p$ can be derived. We will find that an approximate Schrödinger equation for $\phi_Q$ holds with respect to $T_p$ in a region where the size $a$ of the universe is large and $|\phi_B|$ is small.

In §2 we first review the WKB approximation and $T_w$, and we construct a wave packet from WKB wave functions. Next we show that it is seriously difficult to trace the formalism of the WKB time and to define a time variable.

In §3 we introduce a background phase time $T_p$, calculate its explicit form and compare it with $T_w$.

In §4 we examine whether an approximate Schrödinger equation for $\phi_Q$ can be derived.

We summarize in §5, and the appendix is devoted for the detailed estimation of the approximations used in §2 and §3.

2 Wave Packet and Difficulty in Definition of Time

We consider the following minisuperspace model in $(n+1)$-dimensional space-time. Though $n = 3$ in reality, we calculate in the more general case. The metric is assumed to be $ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_n^2$, where $d\Omega_n^2$ is the flat metric. We take a massless background scalar field $\phi_B(t)$ and a quantum matter field $\phi_Q(t)$. The Wheeler-DeWitt equation for a wave function $\Psi(a, \phi_B, \phi_Q)$ reads

\[ \mathcal{H}\Psi = \left(\mathcal{H}_B + \mathcal{H}_Q\right)\Psi = 0, \]

\[ \mathcal{H}_B = \frac{\hbar^2}{2v_n a^{n-2}} \left( \frac{c_n^2}{a^2} \frac{\partial}{\partial a} a^p \frac{\partial}{\partial a} - \frac{1}{a^2} \frac{\partial^2}{\partial \phi_B^2} \right) + U(a), \]
Here $\mathcal{H}_Q(a, \phi_B, \phi_Q)$ is the Hamiltonian constraint for the quantum matter field $\phi_Q$, $p_{oo}$ is a parameter of operator ordering, $\Lambda$ is a cosmological constant, $v_n$ is the spatial volume, and we assume that $v_n$ is some properly fixed finite constant.

In order to look for a WKB solution to Eqs. (1), write

$$\Psi(a, \phi_B, \phi_Q) = \Phi(a, \phi_B, \phi_Q)e^{\frac{i}{\hbar}S_0(a, \phi_B)} .$$

(2)

Substituting Eq. (2) to Eqs. (1) and equating powers of $\hbar$, we obtain

$$- c_n^2 \left( \frac{\partial S_0}{\partial a} \right)^2 + \frac{1}{2v_n a^n} \left( \frac{\partial S_0}{\partial \phi_B} \right)^2 + U(a) = 0 ,$$

(3)

and

$$i\hbar \left[ c_n \left( \frac{\partial^2 S_0}{\partial a^2} \Phi + 2 \frac{\partial S_0}{\partial a} \frac{\partial \Phi}{\partial a} + \frac{p_{oo} \partial S_0}{a} \frac{\partial \Phi}{\partial a} \right) - \frac{1}{2v_n a^n} \left( \frac{\partial^2 S_0}{\partial \phi_B^2} \Phi + 2 \frac{\partial S_0}{\partial \phi_B} \frac{\partial \Phi}{\partial \phi_B} \right) \right] + \mathcal{H}_Q \Phi = 0 ,$$

(4)

where we have regarded that $\mathcal{H}_Q$ is the order of $\hbar$ and ignored the terms in the order of $\hbar^2$. The equation (3) is the Hamilton-Jacobi equation for the gravity coupled with a background scalar field, and $S_0$ is the Hamilton’s principal function.

Let us write the solution $\Phi$ of Eq. (4) as $\Phi_0$, when there is not the quantum matter field $\phi_Q$, that is

$$c_n^2 \left( \frac{\partial^2 S_0}{\partial a^2} \Phi_0 + 2 \frac{\partial S_0}{\partial a} \frac{\partial \Phi_0}{\partial a} + \frac{p_{oo} \partial S_0}{a} \frac{\partial \Phi_0}{\partial a} \right) - \frac{1}{a^2} \left( \frac{\partial^2 S_0}{\partial \phi_B^2} \Phi_0 + 2 \frac{\partial S_0}{\partial \phi_B} \frac{\partial \Phi_0}{\partial \phi_B} \right) = 0 .$$

(5)

Now we write

$$\Psi(a, \phi_B, \phi_Q) = \Phi_0(a, \phi_B) e^{\frac{i}{\hbar}S_0(a, \phi_B)} \psi(a, \phi_B, \phi_Q) .$$

(6)

Then from Eqs. (4)-(6) we obtain

$$i\hbar \left[ \frac{c_n^2}{v_n a^{n-2}} \frac{\partial S_0}{\partial a} \frac{\partial \psi}{\partial a} - \frac{1}{v_n a^n} \frac{\partial S_0}{\partial \phi_B} \frac{\partial \psi}{\partial \phi_B} \right] + \mathcal{H}_Q \psi = 0 .$$

(7)

If we define a time variable $T_W$ as

$$- \frac{c_n^2}{v_n a^{n-2}} \frac{\partial S_0}{\partial a} \frac{\partial T_W}{\partial a} + \frac{1}{v_n a^n} \frac{\partial S_0}{\partial \phi_B} \frac{\partial T_W}{\partial \phi_B} = 1 ,$$

(8)
Eq. (7) can be written as

$$i\hbar \frac{\partial \psi}{\partial T_W} = \mathcal{H}_Q \psi .$$

(9)

This is a Schrödinger equation, so $T_W$ is a semiclassical time variable in the WKB approximation.\cite{2-13}

As in Ref. \cite{17} Eqs. (3),(5),(8) can be solved by the separation of variables, and solutions are as follows.

$$S_0 = \frac{\epsilon_a}{c_n} I_S + \kappa \phi_B + \text{const}. ,$$

(10)

$$I_S = \frac{1}{n} \sqrt{\frac{\epsilon_v a^{2n} + \kappa^2}{\kappa + 2 \ln \left( \frac{\sqrt{\epsilon_v a^{2n} + \kappa^2} + \kappa}{\sqrt{\epsilon_v a^{2n} + \kappa^2} - \kappa} \right) }} ,$$

$$\Phi_0 = c_\phi a^{\frac{1 - \rho_{oo}}{2}} (\epsilon_v a^{2n} + \kappa^2)^{-\frac{1}{4}} (\gamma = 0) ,$$

(11)

$$= a^{\frac{1 - \rho_{oo}}{2}} (\epsilon_v a^{2n} + \kappa^2)^{-\frac{1}{4}} \times$$

$$\times \left\{ c_\phi - \frac{(n - 1 + \rho_{oo}) \gamma}{4 \kappa} \phi_B I_\Phi [\frac{1}{4}] + \frac{n \gamma \kappa}{4} \phi_B I_\Phi [\frac{1}{4}] - \frac{\epsilon_a \gamma}{2 c_n} I_\Phi [\frac{1}{4}] \right\}$$

$$+ \frac{c_\phi}{2 \kappa} \phi_B \left( \gamma \neq 0 \right) ,$$

(12)

$$I_\Phi [x] = \int d a \ a^{\frac{1 - \rho_{oo}}{2}} (\epsilon_v a^{2n} + \kappa^2)^x ,$$

$$T_W = - \frac{\epsilon_a v_n}{c_n \sqrt{\epsilon_v}} \ln a + \tau_W (\phi_B) \left( \kappa = 0 \right) ,$$

(13)

$$= \frac{\epsilon_a}{c_n} (\xi I_{W1} - v_n I_{W2}) + \frac{\xi}{\kappa} \phi_B + \text{const}. \left( \kappa \neq 0 \right) ,$$

(14)

$$I_{W1} = \frac{1}{2 n \kappa} \ln \left( \frac{\sqrt{\epsilon_v a^{2n} + \kappa^2} - \kappa}{\sqrt{\epsilon_v a^{2n} + \kappa^2} + \kappa} \right) ,$$

$$I_{W2} = - \frac{1}{2 n \sqrt{\epsilon_v}} \ln \left( \frac{\sqrt{\epsilon_v a^{2n} + \kappa^2} + \sqrt{\epsilon_v a^{2n} + \kappa^2}}{\sqrt{\epsilon_v a^{2n} + \kappa^2} - \sqrt{\epsilon_v a^{2n} + \kappa^2}} \right) ,$$

where $\epsilon_a = \pm 1, \epsilon_v = 4 \hbar^2 \Lambda / 16 \pi G , \tau_W (\phi_B)$ is any function of $\phi_B$ and $\kappa, \gamma, \xi, c_\phi$ are arbitrary constants. We can identify $\kappa$ as the momentum of $\phi_B$. \cite{11-13}

This WKB time $T_W$ is very natural as a semiclassical time variable in the region where the WKB approximation is well justified.\cite{11-13} However, this formalism crucially depends on the assumption that the wave function has the WKB form (2). How can we define a time variable, if a wave function $\Psi$ is a superposition of the WKB form wave functions, for example, $\Psi$ is the following Gaussian wave packet?
\[ \Psi = \Psi_0 \psi, \quad (15) \]
\[ \Psi_0 = \int d\kappa A(\kappa) \Psi_0^{WKB}(\kappa), \quad (16) \]
\[ A(\kappa) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[ -\frac{(\kappa - \kappa_0)^2}{2\sigma^2} \right], \quad \Psi_0^{WKB}(\kappa) = \Phi_0(\kappa)e^{iS_0(\kappa)}, \]

where \( \kappa_0 \) and \( \sigma \) are arbitrary constants, \( \Phi_0(\kappa), S_0(\kappa) \) are given in Eqs. (10) - (12), and \( \psi = \psi(a, \phi_B, \phi_Q) \) is a wave function for the quantum matter field \( \phi_Q \). Note that, in the limit \( \sigma \to 0 \), \( \Psi_0 \) becomes identical to \( \Psi_0^{WKB} \).

First let us try to trace the formalism of the WKB time and to define a time variable for the wave packet. Substituting Eqs. (15), (16) to Eqs. (1) and using Eqs. (3), (5), we obtain
\[ \int d\kappa A \Phi_0 e^{iS_0} \left[ i\hbar \left( \frac{c_n^2}{v_n a^{n-2}} \frac{\partial S_0}{\partial a} \frac{\partial \psi}{\partial a} - \frac{1}{v_n a^n} \frac{\partial S_0}{\partial \phi_B} \frac{\partial \psi}{\partial \phi_B} \right) + \mathcal{H}_Q \psi \right] = 0, \quad (17) \]
where we have neglected the higher order terms in \( \hbar \). This gives
\[ i\hbar \frac{\partial \psi}{\partial T_S} = \mathcal{H}_Q \psi, \quad (18) \]
\[ \frac{\partial}{\partial T_S} = \frac{\int d\kappa A \Psi_0^{WKB} \left( -\frac{c_n^2}{v_n a^{n-2}} \frac{\partial S_0}{\partial a} \frac{\partial}{\partial a} + \frac{1}{v_n a^n} \frac{\partial S_0}{\partial \phi_B} \frac{\partial}{\partial \phi_B} \right)}{\int d\kappa A \Psi_0^{WKB}}. \quad (19) \]
At a glance this seems a Schrödinger equation with a time \( T_S \). However, \( T_S \) can not be a real variable in general. Therefore it is difficult to define a time variable in this way.

### 3 Phase Time

Next let us introduce a background phase time \( T_p \). We write the background wave packet \( \Psi_0 \) in Eqs. (16) as
\[ \Psi_0 = \rho e^{i\theta}, \quad (20) \]
where \( \rho \) and \( \theta \) are real, that is \( \rho \) and \( \theta \) are the absolute value and the phase of \( \Psi_0 \), respectively. If we replace \( S_0 \) in Eq. (8) with \( \theta \) and define a background phase time \( T_p \) as

\[
- \frac{e^2}{v_n a^n} \frac{\partial \theta \partial T_p}{\partial a \partial a} + \frac{1}{v_n a^n} \frac{\partial \theta}{\partial \Phi_B} \frac{\partial T_p}{\partial \Phi_B} = 1 ,
\]

we can obtain a real time variable \( T_p \).

However, the problem is whether an expected dynamical equation that is a Schrödinger equation for \( \phi_Q \) with respect to \( T_p \) can be derived or not. Note that our background phase time \( T_p \) is a little different from the phase time which is proportional to the Ehrenfest time \( T_E \). The latter phase time should be defined by the all over phase of a solution \( \Psi \) to the Wheeler-DeWitt equation and depend on not only background fields \( a, \Phi_B \) but also the quantum matter field \( \phi_Q \). Before we examine the Schrödinger equation, let us calculate \( T_p \) explicitly and compare it with \( T_W \).

For simplicity we choose \( p_{oo} = 1 \, , \, \gamma = 0 \). Then Eqs. (16) can be written as

\[
\Psi_0 = c_\phi \int d\kappa A(\kappa) e^{i f(\kappa)} ,
\]

\[
f(\kappa) = \varphi(\kappa) + \frac{i}{\hbar} S_0(\kappa) \, , \, \quad \varphi(\kappa) = -\frac{1}{4} \ln(e_\phi a^{2n} + \kappa^2) ,
\]

where the last equation means \( \Phi_0 = c_\phi e^{\varphi} \). If we assume that \( A(\kappa) \) has a narrow peak at \( \kappa_0 \), namely \( \sigma \) is small, then \( S_0 \) and \( \varphi \) can be expanded around \( \kappa_0 \). We neglect higher terms than \((\kappa - \kappa_0)^2\) and integrate with respect to \( \kappa \), and we obtain

\[
\Psi_0 = \frac{c_\phi}{\sqrt{1 - \sigma^2 f''}} \exp \left[ f + \frac{\sigma^2 f'^2}{2(1 - \sigma^2 f'')} \right] ,
\]

where \( f \) means \( f(\kappa_0) \) and prime means a partial derivative with respect to \( \kappa \) namely \( f' = \frac{\partial f}{\partial \kappa}(\kappa_0) \).

Suppose we assume \( \sigma^2 \) is small enough to satisfy

\[
1 \gg \sigma^2 |f''| \quad (24)
\]

and neglect higher terms than \( \sigma^2 \), we have

\[
\Psi_0 = c_\phi \exp \left[ f + \frac{\sigma^2}{2}(f'^2 + f'') \right] .
\]

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Let us consider the case when the size $a$ of the universe is large enough to satisfy following conditions,

$$|S_0''| \gg |\varphi'S_0'|, \quad (26)$$

$$S_0'^2 \gg \hbar^2|\varphi'^2 + \varphi''|. \quad (27)$$

Detailed estimation of these conditions is given in the appendix. In this case $\rho$ and $\theta$ in Eq. (20) can be written as

$$\rho = \Phi_0(\kappa_0)\exp\left[-\frac{\sigma^2}{2\hbar^2}S_0'(\kappa_0)^2\right], \quad (28)$$

$$\theta = S_0(\kappa_0) + \frac{\sigma^2}{2}S_0''(\kappa_0), \quad (29)$$

where we have ignored higher order infinitesimals.

At this stage it is interesting to mention a relation between the trace of the wave packet peak and the classical path. When $S_0$ in Eq. (10) is written as $S_0 = \frac{e_n}{c_n}I_S(a, \kappa) + \kappa (\phi_B - \alpha)$ with an arbitrary constant $\alpha$, the classical path is derived from

$$S_0' = \frac{\partial S_0}{\partial \kappa} = \beta \quad (30)$$

according to the Hamilton-Jacobi theory. Since the parameter $\beta$ is additional and can be absorbed by $\alpha$, we can set as

$$S_0' = 0 \quad (31)$$

in this case. The peak of the wave function is obtainable from Eq. (28). If we take the lowest order WKB approximation, we arrive at the classical path equation (31) by setting $S_0'(\kappa_0) = 0$ as a peak of the amplitude $\rho$.

From Eqs. (10), (29) we find Eq. (21) becomes

$$-\frac{c_n e_n}{v_n \alpha^{n-1}} \left[ e_v^2 a^{4n} + e_v (2\kappa_0^2 + \frac{\sigma^2}{2})a^{2n} + \kappa_0^4 \right] \frac{\partial T_P}{\partial a} + \frac{\kappa_0}{v_n a^n \partial \phi_B} \frac{\partial T_P}{\partial \phi_B} = 1, \quad (32)$$

and this can be solved by the separation of variables. After some calculation the result of $T_p$ is

$$T_p = -\frac{c_n v_n}{2nc_n \sqrt{e_v}} \ln\left(\alpha^{2n} + \frac{\sigma^2}{2e_v}\right) + \tau_p(\phi_B) \quad (\kappa_0 = 0), \quad (33)$$
We will now examine a Schrödinger equation for \( \phi_B \). \( \xi \) is any constant in the separation of variables, \( \alpha_1 = \sqrt{e_v a^2 n + \kappa_0^2} \), \( \alpha_2 = \sqrt{e_v + \kappa_0^2 a^{-2 n}} \), \( \Sigma^+_1 = (-\sigma^2 \pm \sqrt{\sigma^4 + 8 \sigma^2 \kappa_0^2 e_v})/4 \), \( \Sigma^+_2 = (-\sigma^2 e_v \pm \sqrt{\sigma^4 + 8 \sigma^2 \kappa_0^2 e_v})/4 \kappa_0^2 \), and we have assumed \( \Lambda > 0 \). We find from Eqs. (13),(14) and (33),(34) that, when \( \kappa \) in Eqs. (13),(14) is replaced by \( \kappa_0 \), \( \xi = \zeta \), \( \tau_W = \tau_P \) and \( \sigma \rightarrow 0 \), \( T_P \) becomes equal to \( T_W \), which may be expected from the fact that \( \Psi_0 \) becomes identical to \( \Psi_0^{WKB} \) when \( \sigma \rightarrow 0 \).

4 Schrödinger Equation

We will now examine a Schrödinger equation for \( \phi_Q \). Suppose we substitute Eq. (15) into Eqs. (1), we obtain

\[
\frac{i\hbar}{v_n a^{n-2}} \left[ c_n^2 \left( \frac{1}{\Psi_0} \frac{\partial \Psi_0}{\partial a} + \frac{1}{2a} \frac{\partial \psi}{\partial a} \right) - \frac{1}{a^2} \frac{\partial \Psi_0}{\partial a} \frac{\partial \phi_B}{\partial \phi_B} \right. \\
\left. + \frac{1}{2} \left( c_n^2 \frac{\partial^2 \psi}{\partial a^2} - \frac{1}{a^2} \frac{\partial^2 \phi_B}{\partial a^2} \right) \right] + \mathcal{H}_Q \psi = 0 , \quad (35)
\]

where we have used the WKB approximation for the background wave function, that is \( \mathcal{H}_B \Psi_0 = 0 \). Using Eqs. (20) and (21), we can derive

\[
\frac{i \hbar \frac{\partial \psi}{\partial T_P}}{2v_n a^{n-2}} = \mathcal{H}_Q \psi + \left( F_a \frac{\partial}{\partial a} + F_\phi \frac{\partial}{\partial \phi_B} \right) \psi + \frac{\hbar^2}{2v_n a^{n-2}} \left( c_n^2 \frac{\partial^2 \psi}{\partial a^2} - \frac{1}{a^2} \frac{\partial^2 \phi_B}{\partial a^2} \right) , \quad (36)
\]

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$$\frac{\partial}{\partial T} P = - c_n^2 \frac{\partial \theta}{\partial a} \frac{\partial}{\partial a} + \frac{1}{v_n a^n} \frac{\partial \theta}{\partial \phi_B} \frac{\partial}{\partial \phi_B},$$

$$F_a = \frac{\hbar^2 c_n^2}{v_n a^{n-2}} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial a} + \frac{1}{2a} \right),$$

$$F_\phi = - \frac{\hbar^2}{v_n a^n} \frac{1}{\rho} \frac{\partial \rho}{\partial \phi_B}.$$ 

With Eqs. (22), (28) and (29), the last three equations become

$$\frac{\partial}{\partial T} P = - c_n^2 \frac{\partial \theta}{\partial a} \frac{\partial}{\partial a} + \frac{1}{v_n a^n} \frac{\partial \theta}{\partial \phi_B} \frac{\partial}{\partial \phi_B},$$

$$F_a = \frac{\hbar^2 c_n^2}{v_n a^{n-2}} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial a} + \frac{1}{2a} \right),$$

$$F_\phi = - \frac{\hbar^2}{v_n a^n} \frac{1}{\rho} \frac{\partial \rho}{\partial \phi_B}.$$ 

In order to compare Eqs. (37), we need following explicit expressions:

$$S'_0 = \frac{\epsilon_a}{2 n c_n} \ln \left( \frac{\sqrt{e_v a^{2n} + \kappa_0^2}}{\sqrt{e_v a^{2n} + \kappa_0^2 + \kappa_0}} \right) + \phi_B,$$

$$\frac{\partial S_0}{\partial a} = \frac{\epsilon_a}{c_n} \sqrt{e_v a^{2n} + \kappa_0^2},$$

$$\frac{\partial S'_0}{\partial a} = \frac{\epsilon_a \kappa_0}{c_n \sqrt{e_v a^{2n} + \kappa_0^2}},$$

$$\frac{\partial S''_0}{\partial a} = \frac{\epsilon_a e_v a^{2n-1}}{c_n (e_v a^{2n} + \kappa_0^2 + \kappa_0^2)^{3/2}},$$

$$\frac{\partial \phi}{\partial a} = - \frac{\epsilon_a}{2 (e_v a^{2n} + \kappa_0^2)}.$$ 

The WKB approximation for the background wave function is estimated in the appendix, and it requires (A2), which means

$$\hbar \left| \frac{\partial S_0}{\partial a} \right| \gg \hbar^2 \left| \frac{\partial \phi}{\partial a} \right|.$$ 

When a is large enough to satisfy (A4), the last equation in (38) gives $$\frac{\partial \phi}{\partial a} \simeq - \frac{n}{2a}.$$ 

Hence we can obtain

$$\hbar \left| \frac{\partial S_0}{\partial a} \right| \gg \hbar^2 \left| \frac{\partial \phi}{\partial a} + \frac{1}{2a} \right|,$$

and we can neglect the first term in $$F_a$$ compared to the first term in $$\frac{\partial}{\partial T} P.$$ 

Next let us examine the condition that the $$\sigma^2$$-term in $$F_a$$ can be neglected than the first term in $$\frac{\partial}{\partial T} P.$$ We must require

$$\hbar \left| \frac{\partial S_0}{\partial a} \right| \gg \sigma^2 \left| \frac{S'_0 \partial S'_0}{\partial a} \right|,$$ 

(37)
and this means when $a$ is large as in (A4)

$$\hbar e_o a^{2n} \gg \sigma^2 |\kappa_0| \cdot \left| -\frac{\epsilon_o \kappa_0}{n_c n \sqrt{\epsilon_o a^n}} + \phi_B \right|,$$

(42)

where we have used Eqs. (38). If $|\phi_B|$ is small as in (A9), this condition (42) yields

$$a \gg \left( \frac{\sigma^2 \kappa_0^2}{\hbar c n \epsilon_v} \right)^{\frac{1}{n}} \sim \left( \frac{\sigma^2 |\kappa_0|}{\bar{\epsilon}_v} \frac{\epsilon_o^{n-1}}{\nu_n} \right)^{\frac{1}{n}}.$$

(43)

Here we have used the definition of $l_P$ and $l_H$ in the appendix. Unless $|\phi_B|$ is small as in (A10), we need

$$a \gg \left( \frac{\sigma}{\sqrt{\bar{\epsilon}_v}} \right)^{\frac{1}{n}} \sim \left( \frac{\sigma}{\sqrt{\bar{\epsilon}_v}} \frac{\epsilon_o^{n-1}}{\nu_n} \right)^{\frac{1}{n}}.$$

(44)

Finally consider the condition

$$\hbar |\kappa_0| \gg \sigma^2 |S_0'|,$$

(45)

so that we can neglect $F_\phi$ compared to the last term in $\frac{\partial}{\partial \bar{T}_P}$. Suppose $a$ is large as in (A4), the relation (45) means

$$\hbar |\kappa_0| \gg \sigma^2 \frac{\epsilon_o \kappa_0}{n_c n \sqrt{\epsilon_o a^n}} + \phi_B.$$

(46)

When $|\phi_B|$ is small as in (A9), this requires $\hbar |\kappa_0| \gg \frac{\sigma^2 |\kappa_0|}{\epsilon_v a^n}$, which is satisfied in the region of (A7). When $|\phi_B|$ is not small as in (A10), we need

$$\hbar |\kappa_0| \gg \sigma^2 |\phi_B|.$$

(47)

This is consistent with (A10) in the region of (A7).

If $a$ is large enough to satisfy (A4) - (A7), (A11), (A14), (43) and (44) and if $|\phi_B|$ is small enough to satisfy (47), $F_a, F_\phi$ can be neglected than $\frac{\partial}{\partial \bar{T}_P}$. In the case when the WKB approximation for the total wave function with $\psi$ is well justified, the last terms in the first equation of (36) can be also neglected, and we obtain an approximate Schrödinger equation for $\phi_Q$, namely

$$i\hbar \frac{\partial \psi}{\partial \bar{T}_P} \approx \mathcal{H}_Q \psi.$$

(48)
5 Summary

We considered a wave packet in quantum cosmology that is a superposition of the WKB wave functions, each of which has a definite momentum of a background scalar field $\phi_B$. We showed that it is seriously difficult to trace the formalism of the WKB time and to define a time variable for the Schrödinger equation of a quantum matter field $\phi_Q$ without introducing a complex value for a time. Then we introduced a background phase time $T_P$ which is real and calculated its explicit expression when $a$ is large and $\sigma$ is small. It has been shown that $T_P$ is a smooth extension of $T_w$ which is derived by a WKB form wave function and they become identical in the narrow limit of the wave packet. We found that, when a quantum matter field $\phi_Q$ is coupled to the system, an approximate Schrödinger equation for $\phi_Q$ holds with respect to $T_P$ in a region where the size $a$ of the universe is large and $|\phi_B|$ is small.
Appendix : Estimation of Approximations

We start from the action

\[ S = \int d^{n+1}x \sqrt{-g} \left[ \frac{R - 2\Lambda}{16\pi G} - \frac{g^\mu\nu}{2} \partial_\mu \phi_B \partial_\nu \phi_B + \mathcal{L}_Q \right], \quad (A1) \]

where \( \Lambda \) is a cosmological constant and \( \mathcal{L}_Q \) is a Lagrangian density for \( \phi_Q \). This action yields the Wheeler-DeWitt equation (1). From Eq. (A1) we can see that the dimension of the scalar field is \( \phi_B \sim \sqrt{\bar{\hbar}l_n^2} \), where \( l \) has a dimension of length. Since \( \kappa \) is the momentum for \( \phi_B \), \( \kappa \sim \sqrt{\bar{\hbar}l_n^2} \), and \( \kappa \) and \( \sigma \) have the same dimension. In our model the Planck length and the Hubble length are defined as \( l_P = (G\bar{\hbar})^{1/n} \) and \( l_H = \frac{1}{\sqrt{\Lambda}} \), \( \hat{\Lambda} = \frac{2\Lambda}{n(n-1)} \), respectively. Then we have \( \bar{\hbar}c_n \sim \sqrt{\bar{\hbar}l_P^{-2}} \) and \( \sqrt{\bar{\hbar}v_n} \sim \sqrt{\bar{\hbar}v_n l_H^{-1}l_P^{-1}} \).

Now let us begin to estimate approximations used in §2 and §3. In order for the WKB approximation to be well justified, \( \Phi_0 \) should vary slower than \( S_0 \), which means

\[ \frac{1}{\bar{\hbar}} \left| \frac{\partial S_0}{\partial a} \right| \gg \frac{1}{\Phi_0} \left| \frac{\partial \Phi_0}{\partial a} \right|, \quad \frac{1}{\bar{\hbar}} \left| \frac{\partial S_0}{\partial \phi_B} \right| \gg \frac{1}{\Phi_0} \left| \frac{\partial \Phi_0}{\partial \phi_B} \right|. \quad (A2) \]

Since \( \frac{\partial \Phi_0}{\partial \phi_B} = 0 \) when \( \gamma = 0 \), the second condition in (A2) is satisfied automatically. Using Eqs. (22) and (38), we find that the first condition of (A2) is

\[ (e_v a^{2n} + \kappa_0^2)^{\frac{3}{2}} \gg \bar{\hbar}c_n e_v a^{2n}. \quad (A3) \]

To make discussion easy let us consider the case when \( a \) is large enough to satisfy

\[ a \gg \left( \frac{|\kappa_0|}{\sqrt{e_v}} \right)^{\frac{1}{2n}} \sim \left( \frac{|\kappa_0| l_P^{-2} l_H}{v_n} \right)^{\frac{1}{n}}. \quad (A4) \]

In this case (A3) becomes

\[ a \gg \left( \frac{\bar{\hbar}c_n}{\sqrt{e_v}} \right)^{\frac{1}{n}} \sim \left( \frac{l_P^{-2} l_H}{v_n} \right)^{\frac{1}{n}}. \quad (A5) \]

So this condition is necessary for the WKB approximation.
Second we consider the neglect of the higher terms than \((\kappa - \kappa_0)^2\) in the expansion of \(f(\kappa)\). This requires \(|f''(\kappa_0)(\kappa - \kappa_0)^2| \gg |f'''(\kappa_0)(\kappa - \kappa_0)^3|\) when \(|\kappa - \kappa_0| \lesssim \sigma\). This condition is satisfied when

\[
\frac{|f''(\kappa_0)|}{f'''(\kappa_0)} \gg |\kappa - \kappa_0| \simeq \sigma ,
\]

which means in the case of (A4) that

\[
a \gg \left( \frac{\sigma|\kappa_0|}{\sqrt{c_v}} \right)^{\frac{1}{n}} \sim \left( \frac{\sqrt{|\kappa_0|}}{\hbar v_n} \right)^{\frac{1}{n}} \frac{(\frac{\sigma}{\sqrt{\hbar}})^2 l_H^n}{v_n} . \tag{A6}\]

Here we have used Eqs. (22) and (38).

Next let us examine the condition (24). In the case of (A4) we obtain from Eqs. (22) and (38) that \(\sigma^2|f''| \simeq \frac{\sigma^2}{\ell^2} |S_0''| \simeq \frac{\sigma^2}{\hbar c_n \sqrt{c_v a^n}}\). Therefore (24) requires

\[
a \gg \left( \frac{\sigma^2 h c_n}{\sqrt{c_v a^n}} \right)^{\frac{1}{n}} \sim \left( \frac{(\frac{\sigma}{\sqrt{\hbar}})^2 l_H^n}{v_n} \right)^{\frac{1}{n}} . \tag{A7}\]

Finally we estimate conditions (26), (27). In the case of (A4) the condition (26) gives

\[
\sqrt{c_v a^n} \gg \left| -\frac{\epsilon_a \kappa_0}{n c_n \sqrt{c_v a^n}} + \phi_B \right| . \tag{A8}\]

When \(|\phi_B|\) is small and satisfies

\[
|\phi_B| \lesssim \frac{|\kappa_0|}{n c_n \sqrt{c_v a^n}} , \tag{A9}\]

(A8) is satisfied by (A4). When \(|\phi_B|\) is not small, that is

\[
|\phi_B| \gg \frac{|\kappa_0|}{n c_n \sqrt{c_v a^n}} , \tag{A10}\]

(A8) means

\[
a \gg \left( \frac{c_n |\kappa_0||\phi_B|}{\sqrt{c_v}} \right)^{\frac{1}{n}} \sim \left( \frac{(\frac{\kappa_0|\phi_B|}{\hbar})^{n-1} l_H}{v_n} \right)^{\frac{1}{n}} . \tag{A11}\]

In the case of (A4) the condition (27) requires

\[
\left| -\frac{\epsilon_a \kappa_0}{n c_n \sqrt{c_v a^n}} + \phi_B \right| \gg \frac{\hbar}{\sqrt{c_v a^n}} . \tag{A12}\]
When $|\phi_B|$ is small as in (A9), the condition (A12) becomes

$$\frac{|\kappa_0|}{\sqrt{\hbar}} \gg \sqrt{\hbar} c_n \sim l_p^{\frac{n-1}{2}}, \quad (A13)$$

and when $|\phi_B|$ satisfies (A10), the condition (A12) yields

$$a \gg \left( \frac{\hbar}{\sqrt{v_n|\phi_B|}} \right)^{\frac{1}{n}} \sim \left( \frac{l_p^{\frac{n}{2}} \bar{h} n}{v_n|\phi_B|} \right)^{\frac{1}{n}}. \quad (A14)$$

Since $v_n a^n$ is the spatial volume of our model, (A5) may be satisfied in the semiclassical region. If $\sigma$ is small and $|\kappa_0|/\sqrt{\hbar}$ is bigger than $l_p^{\frac{n-1}{2}}$ as in (A13) but is not too big, (A4),(A6) and (A7) do not impose too severe restriction on $a$. When $|\phi_B|,|\kappa_0| \simeq \hbar$, which is allowed by the uncertainty principle, the conditions (A11) and (A14) are equivalent to (A5) and (A4), respectively. So we can think that, when $a$ is large and $\sigma$ is small, all the conditions in this appendix can be satisfied consistently.
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