Influence of an Electric Field on the Propagation of a Photon in a Magnetic field

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Abstract. In this work, a constant and uniform magnetic field is less than the Schwinger critical value. In turn, an additional constant and uniform electric field is taken much smaller than the magnetic field value. The propagation of a photon in this electromagnetic field is investigated. In particular, in the presence of a weak electric field, the root divergence is absent in the photon effective mass near the thresholds of pair creation. The effective mass of a real photon with a preset polarization is considered in the quantum energy region as well as in the quasiclassical one.

1. Introduction
The photon propagation in electromagnetic fields and the dispersive properties of the space region with magnetic fields is of very much interest. This propagation is accompanied by the photon conversion into a pair of charged particles when the transverse photon momentum is larger than the process threshold value $k_\perp > 2m$ (the system of units $\hbar = c = 1$ is used). In 1971 Adler [1] had calculated the photon polarization operator in a magnetic field using the proper-time technique developed by Schwinger [2] and Batalin and Shabad [3] had calculated this operator in an electromagnetic field using the Green function found by Schwinger [2]. In 1971 Batalin and Shabad [3] had calculated polarization operator in an electromagnetic field using the Green function found by Schwinger [2]. In 1975 the contribution of charged-particles loop in an electromagnetic field with $n$ external photon lines had been calculated in [4]. For $n = 2$ the explicit expressions for the contribution of scalar and spinor particles to the polarization operator of photon were given in [4]. For the contribution of spinor particles obtained expressions coincide with the result of [3], but another form is used.

In this work, a constant and uniform magnetic field is less than the Schwinger critical value $H_0 = m^2/e = 4.41 \cdot 10^{13} \text{ G}$. In turn, an additional constant and uniform electric field is taken much smaller than the magnetic field value. In these fields, we consider the polarization operator on mass shell ($k^2 = 0$, the metric $ab = a^0b^0 - ab$ is used) at arbitrary value of the photon energy $\omega$.

2. General expressions
Our analysis is based on the general expression for the contribution of spinor particles to the polarization operator obtained in a diagonal form in [4] (see equations (3.19) and (3.33)).
eigenvalue \( \kappa_i \) of this operator on the mass shell \((k^2 = 0)\) determines the effective mass of the real photon with the polarization \( e_i \) directed along the corresponding eigenvector:

\[
\Pi^{\mu\nu} = -\sum_{i=2,3} \kappa_i \beta_i \beta_i^\nu, \quad \beta_i \beta_j = -\delta_{ij}, \quad \beta_i k = 0; \tag{1}
\]

\[
e_i^\mu = \frac{b_i^\mu}{\sqrt{-b_i^2}}, \quad b_2^\mu = (Bk)^\mu + \frac{2\Omega_4}{\Omega} (Ck)^\mu, \tag{2}
\]

\[
b_3^\mu = (Ck)^\mu - \frac{2\Omega_4}{\Omega} (Bk)^\mu;
\]

\[
\kappa_2 = r \left( \Omega_2 - \frac{2\Omega_4^2}{\Omega} \right), \quad \kappa_3 = r \left( \Omega_3 + \frac{2\Omega_4^2}{\Omega} \right), \tag{3}
\]

\[
\Omega = \Omega_3 - \Omega_2 + \sqrt{(\Omega_3 - \Omega_2)^2 + 4\Omega_4^2}, \quad r = \frac{\omega^2 - k_3^2}{4m^2}.
\]

The consideration realizes in the frame where electric \( E \) and magnetic \( H \) fields are parallel and directed along the axis 3. In this frame the tensor of electromagnetic field \( F_{\mu\nu} \) and tensors \( F^*_{\mu\nu}, B_{\mu\nu} \) and \( C_{\mu\nu} \) have a form

\[
F_{\mu\nu} = C_{\mu\nu} E + B_{\mu\nu} H, \quad F^*_{\mu\nu} = C_{\mu\nu} H - B_{\mu\nu} E, \quad C_{\mu\nu} = g_\mu^0 g_\nu^0 - g_\mu^3 g_\nu^3, \]

\[
B_{\mu\nu} = g_\mu^2 g_\nu^2 - g_\mu^3 g_\nu^3, \quad eE/m^2 = E/E_0 \equiv \nu, \quad eH/m^2 = H/H_0 \equiv \mu; \tag{4}
\]

\[
\Omega_i = \frac{\alpha m^2}{\pi} \int_{-1}^{1} dv \int_{0}^{\infty} f_i(v, x) \exp(i\psi(v, x)) \, dx. \tag{5}
\]

Here

\[
\frac{1}{\nu x} f_1 = \frac{\cos(\mu x) \cosh(\nu x)}{\sin(\mu x) \sinh(\nu x)} - \frac{\cos(\mu x) \cosh(\mu x) \sinh(\nu x)}{\sin^2(\mu x) \sinh^2(\nu x)},
\]

\[
\frac{1}{\nu x} f_2 = 2 \frac{\cosh(\nu x)(\cos(\mu x) - \cos(\mu x))}{\sinh(\nu x) \sin^3(\mu x)} + f_1,
\]

\[
\frac{1}{\nu x} f_3 = 2 \frac{\cos(\mu x)(\cosh(\mu x) - \cosh(\nu x))}{\sin(\mu x) \sinh^3(\nu x)} - f_1,
\]

\[
\frac{1}{\nu x} f_4 = \frac{\cos(\mu x) \cos(\mu x) - 1 \cos(\mu x) \cosh(\nu x) - 1}{\sin^2(\mu x) \sinh(\nu x)} + \frac{\sin(\mu x) \sinh(\nu x)}{\sin(\mu x) \sinh(\nu x)};
\tag{6}
\]

\[
\psi(v, x) = 2r \left( \frac{\cosh(\nu x) - \cosh(\nu x)}{\nu \sinh(\nu x)} + \frac{\cos(\mu x) - \cos(\mu x)}{\mu \sin(\mu x)} \right) - x. \tag{7}
\]

Let us note that the integration contour in the equation (5) is passing slightly below the real axis.

After all calculations have been fulfilled we can return to a covariant form of the process description using the following expressions

\[
E^2, H^2 = (F^2 + G^2)^{1/2} \pm F, \quad F = (E^2 - H^2)/2, \quad G = EH, \quad C_{\mu\nu} = (F_{\mu\nu}^2 + H^2 g_{\mu\nu}) / (E^2 + H^2),
\]

\[
(C^2)_{\mu\nu} = (F^2_{\mu\nu} + H^2 g_{\mu\nu}) / (E^2 + H^2), \quad (C^2)_{\mu\nu} - (B^2)_{\mu\nu} = g_{\mu\nu}. \tag{8}
\]

The real part of \( \kappa_i \) determines the refractive index \( n_i \) of photon with the polarization \( e_i^\mu = \beta_i^\mu \), \( i = 1, 2 \):

\[
\kappa_i = (m_i^c)^2, \quad n_i = 1 - \frac{\Re \kappa_i}{2\omega^2}. \tag{9}
\]
At \( r > 1 \), the proper value of polarization operator \( \kappa_i \) includes the imaginary part, which determines the probability per unit length of pair production:

\[
W_i = -\frac{1}{\omega} \text{Im} \kappa_i.
\]

Let us note that in considered case, we have \( \mu \ll 1, \nu \ll \mu \). The high energy region, \( r \gtrsim 1/\mu^2 \), is contained in the region of the standard quasiclassical approximation (SQA) [5]. In SQA, the effective mass of a photon depends on the parameter \( \kappa \) only:

\[
\kappa^2 = 4r(\mu^2 + \nu^2) = -(Fk)^2/m^2H_0^2.
\]

For \( \kappa \gtrsim 1 \) the influence of weak electric field on the polarization operator is small. Because of this we consider now the case of energies \( r \ll 1/\mu^2 \).

### 3. Region of intermediate photon energies

Choose a point \( x_0 \) in the following way:

\[
\varphi'(x_0) = 0, \quad \varphi(x) = -\psi(0, x) = 2r\left(\frac{1}{\mu} \tan \frac{\mu x}{2} - \frac{1}{\nu} \tanh \frac{\nu x}{2}\right) + x.
\]

Then, we have the following equation for \( x_0 \):

\[
\tan^2 \frac{\nu s}{2} + \tanh^2 \frac{\mu s}{2} = \frac{1}{r}, \quad x_0 = -is.
\]

We represent the integral for \( \Omega_i \) as

\[
\Omega_i = \frac{\alpha m^2}{\pi} (a_i + b_i),
\]

where

\[
a_i = \mu \int_{-1}^{1} \frac{dv}{x_0} \int_{0}^{x_0} f_i(v, x) \exp(i\psi(v, x)) \, dx,
\]

\[
b_i = \mu \int_{-1}^{1} \frac{dv}{x_0} \int_{0}^{\infty} f_i(v, x) \exp(i\psi(v, x)) \, dx.
\]

In the integral \( a_i \) in the equation (13), the small values \( x \sim 1 \) contribute. We calculate this integral expanding the entering functions over \( x \), take into account that in the region under consideration, the condition \( r\mu^2 \ll 1 \) is fulfilled. Then in exponent, we keep the term \( -x \) only and extend the integration over \( x \) to infinity. In the result of not complicated integration over \( v \), we have:

\[
a_2 = -\frac{16}{45} (\mu^2 + \nu^2), \quad a_3 = -\frac{28}{45} (\mu^2 + \nu^2),
\]

\[
\kappa_2 = -\frac{4\alpha m^2 \nu^2}{45\pi}, \quad \kappa_3 = -\frac{7\alpha m^2 \nu^2}{45\pi}, \quad \kappa = -\frac{(Fk)^2}{H_0^2 m^2}.
\]

These asymptotics are well known.

In the integral \( b_i \) equation (15), small values \( v \) contribute. Expanding entering functions over \( v \) and extending the integration over \( v \) to infinity, we get

\[
b_i = \mu \int_{-\infty}^{\infty} \frac{dv}{x_0} \int_{x_0}^{\infty} f_i(0, x) \exp \left[-i \left(\varphi(x) + v^2 \chi(x)\right)\right] \, dx,
\]

\[
\chi(x) = rx^2 \left(\frac{\nu}{\sinh(\nu x)} - \frac{\mu}{\sin(\mu x)}\right).
\]
After the integration over $v$, one has

$$b_i = \mu \sqrt{\pi} \exp \left( -\frac{\pi}{4} \right) \int_{x_0}^{\infty} f_i(0,x) \exp (-i\varphi(x)) \, dx. \quad (19)$$

The case $\mu^{-2} \gg r \gg \mu^{-2/5}$ was considered in [6] (see an equation (31)). The result has the following invariant form:

$$\kappa_2 = \frac{\alpha m^2 \kappa}{8} \left( -\frac{8}{3\kappa} + \frac{64\tilde{F}}{15\kappa^2} \right), \quad W_3 = 2W_2, \quad (20)$$

$$\kappa = 4r \sqrt{\mu^2 + nu^2}, \quad \tilde{F} = \frac{\nu^2 - \mu^2}{2}. \quad (21)$$

Leaving the main terms of the expansion in the parameter $\xi = \nu/\mu \ll 1$, we have

$$\kappa_2 (\xi) / \kappa_2 (0) = \exp \left[ \frac{2\Delta}{3\sqrt{r}} \left( 1 + \frac{1}{r} \right) \right]. \quad (22)$$

4. The quantum energy region near the lower thresholds

We consider now the energy region where $r - 1 \ll 1$ when the moving of created particles is nonrelativistic. In this case, the equation (12) and its solutions are given by the following approximate equations

$$\frac{\xi^2 l^2}{16} \simeq \exp (-l) + \frac{1 - r}{4}, \quad l = \mu s, \quad \xi = \frac{\nu}{\mu}; \quad (23)$$

$$l \simeq 2 \ln \frac{4}{\xi \ln (4/\xi)}, \quad |r - 1| \ll \xi^2 l^2; \quad (24)$$

$$l \simeq \ln \frac{4}{r - 1}, \quad r - 1 \gg \xi^2 l^2; \quad (25)$$

Leaving the main terms of the expansion, we obtain

$$i\varphi(x) \simeq \beta(r) - \gamma e^{-iz} - izq + i\Delta x^3/12, \quad (26)$$

$$x = (z - il)/\mu, \quad q = (r - 1)/\mu, \quad \Delta = \xi^2/\mu, \quad (27)$$

$$\gamma = q + \Delta l^2/4, \quad (28)$$

$$\chi(x) \simeq x, \quad f_{1,2,4}(0,x) \simeq 0, \quad f_3(0,x) \simeq -i. \quad (29)$$

In the region of lower thresholds, where the particles occupy not very high energy levels, we present the equation (19) for $b_3$ in the form

$$b_3 = -i \sqrt{\pi \mu} \exp \left( -\frac{\pi}{4} \right) \exp (-\beta(r)) \int_0^\infty \frac{dz}{\sqrt{z - il}} \sum_{k=0}^\infty \frac{\gamma^k}{k!} \exp \left\{ i \left[ (q - k) z - \frac{\Delta (z - il)^3}{12} \right] \right\}. \quad (30)$$

If $|\delta| \ll 1, \delta = q - n, \Delta \ll 1, \gamma \simeq n$, the large $z$ contributes and we have after the change of variables

$$b_3 \simeq -i \exp \left( -\frac{\pi}{4} \right) \sqrt{\pi \mu} \exp (-\beta(r)) \frac{n^n}{n!} \int_0^\infty \frac{dz}{\sqrt{z}} \exp \left[ i (\delta z - \Delta z^3/12) \right]. \quad (31)$$
After integration over \( z \), we have the following approximate expressions

\[
b_3 = 2\sqrt{\mu \pi \frac{n^n}{n!}} \exp (-\beta(r)) d(\delta, \Delta),
\]

where

\[
d(\delta, \Delta) = -\exp \left( i \pi \frac{\vartheta(\delta)}{2} \right) \Gamma \left( \frac{1}{6} \right) \left( \frac{12}{\Delta} \right)^{1/6}, \quad |\delta| \ll \Delta^{1/3},
\]

\[
d(\delta, \Delta) = -\exp \left( i \pi \frac{\vartheta(\delta)}{2} \right) \sqrt{\pi \delta}, \quad |\delta| \ll \Delta^{1/3};
\]

\[
d(\delta, \Delta) = -\frac{1}{6} \exp \left( \frac{\pi}{6} \vartheta(\delta) \right) \left( \frac{12}{\Delta} \right)^{1/6}, \quad |\delta| \ll \Delta^{1/3};
\]

\[
(32)
\]

were

\[
|\vartheta(z)| \text{ is Heaviside function: } \vartheta(z) = 1 \text{ for } z \geq 0, \quad \vartheta(z) = 0 \text{ for } z < 0.
\]

The expression for \( \kappa_3 \) with the accepted accuracy can be rewritten in the following form:

\[
\kappa_3^b \simeq -i \alpha m^2 \sqrt{\frac{\mu}{|\delta|}} \exp \left( i \pi \frac{\vartheta(\delta)}{2} \right) e^{-\zeta(2 \zeta)^n} n!, \quad \Delta \ll |\delta|^{3/2}, \quad \zeta = 2r/\mu.
\]

\[
(35)
\]

\[
5. \text{ Low energy region } r - 1 < 0
\]

The case \( 1 - r \ll \nu^{2/3} \) was discussed above (see the equation (36)) where we have to put \( n = 0 \).

For the case \( 1 \gg 1 - r \gg \nu^{2/3} \), the following expression

\[
l \simeq \frac{2}{\xi} \sqrt{1 - r},
\]

\[
(38)
\]

is valid and one can use the method of stationary phase. Then we have

\[
\kappa_3^b \simeq -i \alpha m^2 \sqrt{\frac{\mu}{|l|}} \exp \left( i \pi \frac{\vartheta(\delta)}{2} \right) e^{-\zeta(2 \zeta)^n} n!, \quad \Delta \ll |\delta|^{3/2}, \quad \zeta = 2r/\mu.
\]

\[
(39)
\]

This method is valid for lower photon energy when \( \nu^{-2} \gg 1/r - 1 \gg \nu^{2/3} \). Then the approximate solution of the equation (12) has form

\[
s \simeq \frac{2}{\nu} \arctan \left( \frac{1}{\nu} - \frac{1}{r} \right), \quad \sqrt{-\frac{\chi(x_0) \varphi''(x_0)}{2}} \simeq \frac{\nu s}{2},
\]

\[
f_3(x_0) \simeq 1 \frac{\nu s}{r \sqrt{r(1 - r)}}, \quad f_{1,2,4}(0, x) \simeq 0,
\]

\[
(40)
\]

\[
\kappa_3^b \simeq -i \alpha m^2 \mu \sqrt{\frac{\nu}{r(1 - r)}} \exp \left( -\frac{2r}{\mu} - \frac{2}{\nu} \arctan \sqrt{\frac{1}{r} - 1} + 2r \frac{1}{\nu} \sqrt{\frac{1}{r} - 1} \right).
\]

\[
(41)
\]
For \(1 - r \ll 1\), expanding the incoming in the equation (41) functions we have the equation (39). When the value \(r\) small enough \(\nu^3 \ll r^{5/2} \ll \nu\), leaving the leading term of decomposition over \(r\), we get

\[
\kappa^b_3 \simeq -\frac{i \alpha m^2 \mu}{\sqrt{r}} \exp \left( -\frac{2r}{\mu} - \frac{\pi}{\nu} + \frac{4\sqrt{r}}{\nu} \right). \tag{42}
\]

The imaginary part of this expression coincides with the equation (40) \([6]\). The ratio of this expression to the corresponding formula in a pure electric field has the form

\[
\frac{\kappa^b_3}{\kappa_3} (\mu = 0) = \frac{\pi}{\xi} \exp \left[ r \left( \frac{\nu}{\mu} - \frac{2}{\nu} \right) \right]. \tag{43}
\]

The equations (42) and (43) are valid for the photon energy \(r \gg \nu^2\). For the region of the photon energy \(r \lesssim \nu^2\), we can use the results of [6].

6. Conclusion
We have considered the polarization operator of a photon in a constant magnetic fields in the presence of a weak electric field. The effective mass of a photon was calculated using three different overlapping approximation. In the region of SQA applicability, the created by a photon particles have ultrarelativistic energies. The role of fields in this case is to transfer the required transverse momentum and the electric field actions less than that of the magnetic field. At lower energies, the role of the electric field increases. It is necessary to note a special significance of a weak electric field \(E = \xi \hbar H (\xi \ll 1)\) in the removal of the root divergence of the probability when the particles of pair are created on the Landau levels with the electron and positron momentum \(p_3 = 0\) \([5]\). The frame is used where \(k_3 = 0\).

Generally speaking, at \(\xi \ll 1\) the formation time \(t_c\) of the process under consideration is \(1/\mu\). Here we use units \(\hbar = c = m = 1\). At this time the particles of creating pair gets the momentum \(\delta p_3 \sim \xi\). If the value \(\xi^2\) becomes larger than the distance apart Landau levels \(\mu (\Delta = \xi^2/\mu \gg 1)\) all levels overlapped. Under this condition the divergence of the probability is vanished and the method of stationary phase is valid even in the energy region \(r - 1 \lesssim \mu\), whereas that is inapplicable in the absence of electric field \([5]\). In the opposite case \(\nu^2 \ll \mu^3\) for the small value of \(p_3 \ll \sqrt{\mu}\), in the region where the influence of electric field is negligible, the formation time of the process \(t_f = 1/p_3^2\) and \(\delta p_3 \sim \nu/p_3^2 \ll p_3\). It is follows from above that \(\nu^{1/3} \ll p_3 \ll \sqrt{\mu}\).

At this condition the value of discontinuity is \(\sqrt{t_f/t_c} \sim \sqrt{\mu}/p_3\). For \(\nu^{1/3} \gg p_3\) the time \(t_f\) is determined by the self-consistent equation \(\delta \xi^2 \sim 1/t_f \sim \nu^2 t_f^2\), \(t_f \sim \nu^{-2/3}\) and the value of discontinuity becomes \(\sqrt{\mu t_f} \sim (\mu^3/\nu^2)^{1/6}\) instead of \(\sqrt{\mu}/p_3\).

In the region \(\omega \lesssim 2m\) \((r \lesssim 1)\) the energy transfer from electric field to the created particles becomes appreciable and for \(\omega \ll m\) it determines the probability of the process mainly. At \(\omega \ll eE/m\) \((\sqrt{r} \ll \nu)\), the photon assistance in the pair creation comes to the end and the probability under consideration defines the probability of photon absorption by the particles created by electromagnetic fields. The influence of a magnetic field on the process is connected with the interaction of the magnetic moment of the created particles and magnetic field. This interaction, in particular, has appeared in the distinction of the pair creation probability by field for scalar and spinor particles \([2]\).

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