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Ternary $Z_3$-graded generalization of Heisenberg’s algebra

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Abstract. We investigate a ternary, $Z_3$-graded generalization of the Heisenberg algebra. It turns out that introducing a non-trivial cubic root of unity, $j = e^{2\pi i/3}$, one can define two types of creation operators instead of one, accompanying the usual annihilation operator. The two creation operators are non-hermitian, but they are mutually conjugate. Together, the three operators form a ternary algebra, and some of their cubic combinations generate the usual Heisenberg algebra.

A cubic analogue of Hamiltonian operator is constructed by analogy with the usual harmonic oscillator. A set of eigenstates in coordinate representation is constructed in terms of functions satisfying linear differential equation of third order.

1. Introduction
Our goal being a ternary generalization of Heisenberg’s algebra, let us start with recalling basic facts about ternary algebras [1], [2]. The usual definition of an algebra involves a linear space $A$ (over real or complex numbers) endowed with a binary constitutive relations

$$ A \times A \to A $$

In a finite dimensional case, $\dim A = N$, in a chosen basis $e_1, e_2, \ldots, e_N$, the constitutive relations (1) can be encoded in structure constants $f_{ij}^k$ as follows:

$$ e_i e_j = f_{ij}^k e_k. $$

(2)

With the help of these structure constants all essential properties of a given algebra can be expressed, e.g. they will define a Lie algebra if they are antisymmetric and satisfy the Jacobi identity:

$$ f_{ij}^k = - f_{ji}^k, \quad f_{im}^k f_{jl}^m + f_{jm}^k f_{li}^m + f_{lm}^k f_{ij}^m = 0, $$

(3)

whereas an abelian algebra will have its structure constants symmetric, $f_{ij}^k = f_{ji}^k$.

Usually, when we speak of algebras, we mean binary algebras, understanding that they are defined via quadratic constitutive relations (2). On such algebras the notion of $Z_2$-grading can be naturally introduced. An algebra $A$ is called a $Z_2$-graded algebra if it is a direct sum of two parts, with symmetric (abelian) and anti-symmetric product respectively,

$$ A = A_0 \oplus A_1, $$

(4)
with grade of an element being 0 if it belongs to \( A_0 \), and 1 if it belongs to \( A_1 \). Under the multiplication in a \( Z_2 \)-graded algebra the grades add up reproducing the composition law of the \( Z_2 \) permutation group: if the grade of an element \( A \) is \( a \), and that of the element \( B \) is \( b \), then the grade of their product will be \( a + b \) modulo 2:

\[
\text{grade}(AB) = \text{grade}(A) + \text{grade}(B), \quad \text{so that} \quad AB = (-1)^{ab} BA. \tag{5}
\]

It is worthwhile to notice at this point that the above relationship can be written in an alternative form, with all the expressions on the left side as follows:

\[
AB - (-1)^{\alpha\beta} BA = 0, \quad \text{or} \quad AB + (-1)^{(\alpha\beta+1)} BA = 0 \tag{6}
\]

The equivalence between these two alternative definitions of commutation (anticommutation) relations inside a \( Z_2 \)-graded algebra is no more possible if by analogy we want to impose cubic relations on algebras with \( Z_3 \)-symmetry properties, in which the non-trivial cubic root of unity, \( j = e^{\frac{2\pi i}{3}} \), plays the role similar to that of \(-1\) in the binary relations displaying a \( Z_2 \)-symmetry.

The \( Z_3 \) cyclic group is an abelian subgroup of the \( S_3 \) symmetry group of permutations of three objects. The \( S_3 \) groups contain six elements, including the group unit \( e \) (the identity permutation, leaving all objects in place: \( (abc) \to (abc) \)), the two cyclic permutations

\[
(abc) \to (bca) \quad \text{and} \quad (abc) \to (cab),
\]

and three odd permutations,

\[
(abc) \to (cba), \quad (abc) \to (bac) \quad \text{and} \quad (abc) \to (acb).
\]

There was a unique definition of commutative binary algebras given in two equivalent forms,

\[
xy + (-1)yx = 0 \quad \text{or} \quad xy = yx. \tag{7}
\]

In the case of cubic algebras [2] we have the following four generalizations of the notion of commutative algebras:

a) Generalizing the first form of the commutativity relation (7), which amounts to replacing the \(-1\) generator of \( Z_2 \) by \( j \)-generator of \( Z_3 \) and binary products by products of three elements, we get

\[
S : \quad x^\mu x^\nu x^\lambda + j \ x^\nu x^\lambda x^\mu + j^2 \ x^\lambda x^\mu x^\nu = 0, \tag{8}
\]

where \( j = e^{\frac{2\pi i}{3}} \) is a primitive third root of unity.

b) Another primitive third root, \( j^2 = e^{\frac{4\pi i}{3}} \) can be used in place of the former one; this will define the conjugate algebra \( \bar{S} \), satisfying the following cubic constitutive relations:

\[
\bar{S} : \quad x^\mu x^\nu x^\lambda + j^2 \ x^\nu x^\lambda x^\mu + j \ x^\lambda x^\mu x^\nu = 0. \tag{9}
\]

Clearly enough, both algebras are infinitely-dimensional and have the same structure. Each of them is a possible generalization of infinitely-dimensional algebra of usual commuting variables with a finite number of generators. In the usual \( Z_2 \)-graded case such algebras are just polynomials in variables \( x^1, x^2, \ldots, x^N \); the algebras \( S \) and \( \bar{S} \) defined above are also spanned by polynomials, but with different symmetry properties, and as a consequence, with different dimensions corresponding to a given power.

c) Then we can impose the following “weak” commutation, valid only for cyclic permutations of factors:

\[
S_1 : \quad x^\mu x^\nu x^\lambda = x^\nu x^\lambda x^\mu \neq x^\nu x^\mu x^\lambda, \tag{10}
\]
d) Finally, we can impose the following “strong” commutation, valid for arbitrary (even or odd) permutations of three factors:

\[ S_0 : \ x^\mu x^\nu x^\lambda = x^\nu x^\lambda x^\mu = x^\lambda x^\mu x^\nu \]  

(11)

The four different associative algebras with cubic commutation relations can be represented in the following diagram, in which all arrows correspond to surjective homomorphisms. The commuting generators can be given the common grade 0.

\[ \begin{array}{ccc}
S & \xrightarrow{\theta^A} & \bar{S} \\
\Downarrow S_1 & & \Downarrow \bar{S}_1 \\
S_0 & & \bar{S}_0
\end{array} \]

Let us turn now to the \( Z_3 \) generalization of anti-commuting generators, which in the usual homogeneous case with \( Z_2 \)-grading define Grassmann algebras. Here, too, we have four different choices:

a) The “strong” cubic anti-commutation,

\[ \Lambda_0 : \ \Sigma_{\pi \in S_3} \ \theta^{\pi(A)} \theta^{\pi(B)} \theta^{\pi(C)} = 0, \]  

(12)

i.e. the sum of all permutations of three factors, even and odd ones, must vanish.

b) The somewhat weaker “cyclic” anti-commutation relation,

\[ \Lambda_1 : \ \theta^A \theta^B \theta^C + \theta^B \theta^C \theta^A + \theta^C \theta^A \theta^B = 0, \]  

(13)

i.e. the sum of cyclic permutations of three elements must vanish. The same independent relation for the odd combination \( \theta^C \theta^B \theta^A \) holds separately.

c) The \( j \)-skew-symmetric algebra:

\[ \Lambda : \ \theta^A \theta^B \theta^C = j \ \theta^B \theta^C \theta^A. \]  

(14)

and its conjugate algebra \( \bar{\Lambda} \), isomorphic with \( \Lambda \), which we distinguish by putting a bar on the generators and using dotted indices:

d) The \( j^2 \)-skew-symmetric algebra:

\[ \bar{\Lambda} : \ \bar{\theta}^A \bar{\theta}^B \bar{\theta}^C = j^2 \bar{\theta}^B \bar{\theta}^C \bar{\theta}^A. \]  

(15)

Both these algebras are finite dimensional. For \( j \) or \( j^2 \)-skew-symmetric algebras with \( N \) generators the dimensions of their subspaces of given polynomial order are given by the following generating function:

\[ H(t) = 1 + Nt + N^2 t^2 + \frac{N(N-1)(N+1)}{3} t^3, \]  

(16)

where we include pure numbers (dimension 1), the \( N \) generators \( \theta^A \) (or \( \bar{\theta}^B \)), the \( N^2 \) independent quadratic combinations \( \theta^A \theta^B \) and \( N(N-1)(N+1)/3 \) products of three generators \( \theta^A \theta^B \theta^C \).

The above four cubic generalization of Grassmann algebra are represented in the following diagram, in which all the arrows are surjective homomorphisms.

\[ \begin{array}{ccc}
\Lambda_0 & & \Lambda_1 \\
\Downarrow & & \Downarrow \\
\Lambda & & \bar{\Lambda}
\end{array} \]
2. Examples of $Z_3$-graded ternary algebras

2.1. The $Z_3$-graded analog of Grassman algebra

Let us introduce $N$ generators spanning a linear space over complex numbers, satisfying the following cubic relations [3], [4]:

$$\theta A \theta B \theta C = j \theta B \theta C \theta A = j^2 \theta C \theta A \theta B,$$

with $j = e^{2i\pi/3}$, the primitive root of 1. We have $1 + j + j^2 = 0$ and $\bar{j} = j^2$.

Let us denote the algebra spanned by the $\theta A$ generators by $A$ [3], [4].

We shall also introduce a similar set of conjugate generators, $\bar{\theta} A, \bar{A}, \bar{B},... = 1, 2, ..., N$, satisfying similar condition with $j^2$ replacing $j$:

$$\bar{\theta} A \bar{\theta} B \bar{\theta} C = j^2 \bar{\theta} B \bar{\theta} C \bar{\theta} A = \bar{j} \bar{\theta} C \bar{\theta} A \bar{\theta} B,$$

Let us denote this algebra by $\bar{A}$.

We shall endow the algebra $A \oplus \bar{A}$ it with a natural $Z_3$ grading, considering the generators $\theta A$ as grade 1 elements, their conjugates $\bar{\theta} A$ being of grade 2.

The grades add up modulo 3, so that the products $\theta A \theta B$ span a linear subspace of grade 2, and the cubic products $\theta A \theta B \theta C$ being of grade 0. Similarly, all quadratic expressions in conjugate generators, $\bar{\theta} A \bar{\theta} B$ are of grade 2 + 2 = 4

$mod_3 = 1$, whereas their cubic products are again of grade 0, like the cubic products of $\theta A$'s. [7]

Combined with the associativity, these cubic relations impose finite dimension on the algebra generated by the $Z_3$ graded generators. As a matter of fact, cubic expressions are the highest order that does not vanish identically. The proof is immediate:

$$\theta A \theta B \theta C \theta D = j \theta B \theta C \theta A \theta D = j^2 \theta B \theta A \theta D \theta C = j^3 \theta B \theta A \theta D \theta C = j^4 \theta A \theta B \theta C \theta D,$$

and because $j^4 = j \neq 1$, the only solution is $\theta A \theta B \theta C \theta D = 0$.

2.2. Ternary Clifford algebra

Let us introduce the following three $3 \times 3$ matrices:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (20)$$

and their hermitian conjugates

$$Q_1^\dagger = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \quad Q_2^\dagger = \begin{pmatrix} 0 & 0 & j \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (21)$$

These matrices can be allowed natural $Z_3$ grading,

$$\text{grade}(Q_k) = 1, \quad \text{grade}(Q_k^\dagger) = 2, \quad (22)$$

The above matrices span a very interesting ternary algebra. Out of three independent $Z_3$-graded ternary combinations, only one leads to a non-vanishing result. One can check without much effort that both $j$ and $j^2$ skew ternary commutators do vanish:

$$\{Q_1, Q_2, Q_3\}_j = Q_1 Q_2 Q_3 + j Q_2 Q_3 Q_1 + j^2 Q_3 Q_1 Q_2 = 0,$$

$$\{Q_1, Q_2, Q_3\}_{j^2} = Q_1 Q_2 Q_3 + j^2 Q_2 Q_3 Q_1 + j Q_3 Q_1 Q_2 = 0,$$
and similarly for the odd permutation, $Q_2 Q_1 Q_3$. On the contrary, the totally symmetric combination does not vanish; it is proportional to the $3 \times 3$ identity matrix $1$:

$$Q_a Q_b Q_c + Q_b Q_c Q_a + Q_c Q_a Q_b = \eta_{abc} 1, \quad a, b, \ldots = 1, 2, 3. \quad (23)$$

with $\eta_{abc}$ given by the following non-zero components:

$$\eta_{111} = \eta_{222} = \eta_{333} = 1, \quad \eta_{123} = \eta_{231} = \eta_{312} = 1, \quad \eta_{213} = \eta_{321} = \eta_{132} = j^2. \quad (24)$$

all other components vanishing. The relation (23) may serve as the definition of ternary Clifford algebra.

Another set of three matrices is formed by the hermitian conjugates of $Q_a$, which we shall endow with dotted indeces $\dot{a}, \dot{b}, \ldots = 1, 2, 3$: satisfying conjugate identities

$$Q_a Q_b Q_c + Q_b Q_c Q_a + Q_c Q_a Q_b = \eta_{\dot{a}\dot{b}\dot{c}} 1, \quad \dot{a}, \dot{b}, \ldots = 1, 2, 3. \quad (25)$$

with $\eta_{\dot{a}\dot{b}\dot{c}} = \bar{\eta}_{abc}$.

It is obvious that any similarity transformation of the generators $Q_a$ will keep the ternary anti-commutator (24) invariant. As a matter of fact, if we define $\tilde{Q}_a = P^{-1} Q_a P$, with $P$ a non-singular $3 \times 3$ matrix, the new set of generators will satisfy the same ternary relations, because

$$\tilde{Q}_a \tilde{Q}_b \tilde{Q}_c = P^{-1} Q_a P P^{-1} Q_b PP^{-1} Q_c P = P^{-1} (Q_a Q_b Q_c) P,$$

and on the right-hand side we have the unit matrix which commutes with all other matrices, so that $P^{-1} 1 P = 1$.

2.3. Ternary $Z_3$-graded commutator

In any associative algebra $\mathcal{A}$ one can introduce a new binary operation, the commutator, using the generator of the $Z_2$ group in form of multiplication by $-1$:

$$X, Y \in \mathcal{A} \rightarrow [X, Y] = XY + (-1) YX = XY - YX. \quad (26)$$

In the case of the $Z_2$ group the generator of its representation on complex numbers was equal to $-1$; note that $-1 + (-1)^2 = 0$ In the case of the $Z_3$ group, the generator of its complex representation can be chosen to be $j = e^{\frac{2\pi i}{3}}$, with $j + j^2 + j^3 = 0$.

Consider the following cubic combination defined on an associative algebra $\mathcal{A}$:

$$X, Y, Z \in \mathcal{A}, \quad \{X, Y, Z\} := XYZ + j YZX + j^2 ZXY.$$

One obviously has:

$$\{X, Y, Z\} = j \{Y, Z, X\} = j^2 \{Z, X, Y\}, \quad \text{and consequently} \quad \{X, X, X\} = 0.$$

In the case when $\mathcal{A}$ is a unital algebra, i.e. it contains a unit element $1$ such that $1X = X1 = X$ for any $X \in \mathcal{A}$, a unique Lie algebra is naturally generated by the cubic commutator:

$$\{X, 1, Y\} = X \cdot 1 \cdot Y + j 1 \cdot Y \cdot X + j^2 Y \cdot X \cdot 1 =$$

$$XY + j YX + j^2 YX = XY + (j + j^2) YX = XY - YX = [X, Y].$$

The following example of a $Z_3$ cubic algebra can be constructed with $2 \times 2$ complex matrices. Consider the Lie algebra spanned by three Pauli’s matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
satisfying the well known commutation relations:

\[ [\sigma_i, \sigma_j] = C_{ij}^k \sigma_k, \quad i, j, k = 1, 2, 3, \quad C_{ij}^k = 2i \epsilon_{kij} . \]

The enveloping algebra \( A_{\sigma} \) contains the unit matrix \( 1 \):

\[ \sigma_i \sigma_j = i \epsilon_{ijk} + \delta_{ij} 1 \]

Let us define the \textit{cubic j-commutator} on the algebra \( A_{\sigma} \):

\[ \{\sigma_i, \sigma_j, \sigma_k\} = \sigma_i \sigma_j \sigma_k + j \sigma_j \sigma_k \sigma_i + j^2 \sigma_k \sigma_i \sigma_j, \quad ; j = e^{2\pi i} . \]

This cubic algebra contains three cubic subalgebras generated by two \( \sigma \)-matrices out of three: for example

\[ \{\sigma_1, \sigma_2, \sigma_1\} = -2 \sigma_2, \quad \{\sigma_2, \sigma_1 \sigma_2\} = -2 \sigma_1, \]

and similarly for the couples \( \sigma_2, \sigma_3 \) and \( \sigma_3, \sigma_1 \). We have also \( \{\sigma_1, \sigma_2, \sigma_3\} = 0 \).

3. Ternary Heisenberg algebra

3.1. The \( \mathbb{Z}_2 \) Heisenberg algebra

Let us first remind the original construction of the Heisenberg algebra used in the analysis of \textit{quantum harmonic oscillator}. The Hamiltonian of classical harmonic oscillator, expressed in reduced dimensionless variables reads:

\[ H = \frac{p^2}{2} + \frac{x^2}{2} \]

after first quantization which attributes to canonical variables \( (p, x) \) corresponding quantum operators acting on the Hilbert space \( L^2(\mathbb{R}) \) of square-integrable functions

\[ p \rightarrow \hat{p} = -i\hbar \frac{d}{dx}, \quad x \rightarrow \hat{x} \]

The quantum version of the Hamiltonian is the hermitian operator

\[ \hat{H} = -\frac{d^2}{dx^2} + x^2 \]

The classical Heisenberg algebra [9] is generated by the following two operators:

\[ a = \frac{1}{\sqrt{2}} (\bar{x} + i\bar{p}), \quad a^\dagger = \frac{1}{\sqrt{2}} (\bar{x} - i\bar{p}) , \]

(in terms of dimensionless operators \( \bar{x} \) and \( \bar{p} \) defined as follows:

\[ \bar{x} = \frac{x}{\lambda}, \quad \bar{p} = \frac{\lambda p}{\hbar} . \]

with \( \lambda \) some unit of length) satisfying the well known commutation relations

\[ [\bar{x}, \bar{p}] = i. \quad [a, a^\dagger] = 1 . \]

Now the quantum version of the Hamiltonian becomes a hermitian operator, which can be expressed by means of the operators \( a \) and \( a^\dagger \):

\[ \hat{H} = -\frac{d^2}{dx^2} + x^2 = aa^\dagger + a^\dagger a, \]
The eigenfunctions of this operator, corresponding to fixed values of energy \( E \), are obtained by defining first the lowest energy state corresponding to the zero-eigenvalue state of the operator \( a, a \mid 0 >= 0 \). In coordinate representation the zero state is represented by a Gaussian function:

\[
a \mid 0 >= 0 \rightarrow \left[ \frac{d}{dx} + x \right] f(x) = 0 \rightarrow f(x) = e^{-\frac{x^2}{2}}.
\]  \hspace{1cm} (27)

Positive eigenvalue functions of the Hamiltonian \( \hat{H} \) are then obtained by acting on \( f(x) \) with consecutive powers of \( a^\dagger \). [9]

### 3.2. The \( Z_3 \) ternary Heisenberg algebra

In the \( Z_2 \)-graded case the two independent combinations of the operators \( \frac{d}{dx} \) and \( x \) are given by the following non-singular matrix:

\[
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}.
\]

By analogy, in the \( Z_3 \)-graded case, the following transformation matrix should be used, producing three independent combinations of the operators \( \frac{d}{dx} \), \( x \) and \( 1 \):

\[
\begin{pmatrix}
1 & 1 & 1 \\
\lambda j & \lambda j^2 & 1 \\
\lambda j^2 & \lambda j & 1
\end{pmatrix}
\]

A natural ternary generalization of Heisenberg’s algebra should be spanned by three generators [10] instead of two. By analogy with the \( Z_2 \) case, the \( Z_3 \)-generalization will be generated by the following three operators:

\[
c_1 = \lambda \frac{d}{dx} + jx + j^21, \quad c_2 = \lambda \frac{d}{dx} + j^2x + j1, \quad c_3 = \lambda \frac{d}{dx} + x + 1,
\]

which span the following binary Lie algebra:

\[
[c_1, c_2] = \lambda (j^2 - j) \ 1, \quad [c_2, c_3] = \lambda (1 - j^2) \ 1, \quad [c_3, c_1] = \lambda (j - 1) \ 1,
\]

The following linear relations hold:

\[
\frac{1}{3} (c_1 + c_2 + c_3) = \lambda \frac{d}{dx}, \quad \frac{1}{3} (j c_1 + j^2 c_2 + c_3) = 1, \quad \frac{1}{3} (j^2 c_1 + j c_2 + c_3) = x.
\]

Here are the independent 3-commutators between the generators of our algebra:

\[
\{c_1, c_2, c_1\} = -3\lambda c_1, \quad \{c_2, c_1, c_2\} = 3\lambda c_2,
\]

\[
\{c_2, c_3, c_2\} = -3 j \lambda c_2, \quad \{c_3, c_2, c_3\} = 3 j \lambda c_3,
\]

\[
\{c_3, c_1, c_3\} = -3 j^2 \lambda c_3, \quad \{c_1, c_3, c_1\} = 3 j^2 \lambda c_1,
\]

\[
\{c_2, c_3, c_1\} = \lambda \left[ (1 - j) c_1 + (j^2 - 1) c_2 + (j - j^2) c_3 \right],
\]

\[
\{c_1, c_3, c_2\} = \lambda \left[ (j^2 - j) c_1 + (j^2 - j) c_2 + (j^2 - j) c_3 \right].
\]

If we add the unit operator \( 1 \) to the three generators \( c_1 \), \( c_2 \) and \( c_3 \), the ordinary commutators between the generators can be interpreted as ternary commutators involving the unit operator \( 1 \) in the middle:
\[ [c_1, c_2] = \{c_1, 1, c_2\} = \lambda(j^2 - j) 1, \]
\[ [c_2, c_3] = \{c_2, 1, c_3\} = \lambda(1 - j^2) 1, \]
\[ [c_3, c_1] = \{c_3, 1, c_1\} = \lambda(j - 1) 1, \]

\[ (c_1c_3c_2 + c_3c_2c_1 + c_2c_1c_3) - (c_2c_3c_1 + c_3c_1c_2 + c_1c_2c_3) = 3\lambda(j - j^2). \]

After renormalization \( c_k \to \frac{1}{\sqrt{3}}c_k \) and setting \( \lambda = -i \), the right-hand side of this cubic commutation relation will become equal to \( \hbar 1 \). Normalized operators are as follows

\[ c_1 = \frac{1}{\sqrt{3}} \left[ \sqrt{\hbar} \frac{d}{i \ dx} + jx + j^2 1 \right], \quad c_2 = \frac{1}{\sqrt{3}} \left[ \sqrt{\hbar} \frac{d}{i \ dx} + j^2 x + j 1 \right], \quad c_3 = \frac{1}{\sqrt{3}} \left[ \sqrt{\hbar} \frac{d}{i \ dx} + x + 1 \right]. \]

3.3. Invariance group of Heisenberg’s algebra

Let us consider a general linear transformation

\[ a \to A = \alpha a + \beta a^\dagger, \quad a^\dagger \to B = \gamma a + \delta a^\dagger, \]

imposing the same constitutive relations on new generators \( \tilde{a} \) and \( \tilde{a}^\dagger \),

\[ AB - BA = 1, \]

leads to the following conditions on the coefficients \( \alpha, \beta, \gamma, \delta: \alpha\delta - \gamma\beta = 1 \), which defines the \( SL(2, C) \) group.

Keeping invariant just the commutation relations does not preserve the particular structure of the Heisenberg algebra. If we want to make sure that the new generators define a Heisenberg algebra isomorphic with the original one, we should impose supplementary condition, making sure that \( \tilde{a}^\dagger \) is indeed a hermitian conjugate of \( \tilde{a} \):

\[ a \to \tilde{a} = \alpha a + \beta a^\dagger, \quad a^\dagger \to \tilde{a}^\dagger = \gamma a + \delta a^\dagger, \]

imposing the same constitutive relations on new generators \( \tilde{a} \) and \( \tilde{a}^\dagger \),

\[ \tilde{a}\tilde{a}^\dagger - \tilde{a}^\dagger \tilde{a} = 1, \]

leads to the following conditions on the coefficients \( \alpha, \beta, \gamma, \delta: \alpha\delta - \gamma\beta = 1, \quad \bar{\alpha} = \delta, \quad \bar{\beta} = \gamma \), which defines the Bogolyubov group.

The Bogolyubov group, isomorphic with \( SU(1, 1) \) can be represented by \( 2 \times 2 \) matrices constructed as follows:

\[
\begin{pmatrix}
\cosh \psi e^{i\theta_1} & \sinh \psi e^{i\theta_2} \\
\sinh \psi e^{-i\theta_2} & \cosh \psi e^{-i\theta_1}
\end{pmatrix}
\]

It has three real parameters, \( \psi, \theta_1 \) and \( \theta_2 \).

Now we would like to determine the group of linear transformations preserving the algebra \( A_{3H} \). The \( 3 \times 3 \) matrices \( U^j_{j'} \) spanning such group transform the set of three generators \( c_i \) into a new set \( \tilde{c}_{j'} \)

\[ c_i \to \tilde{c}_{j'} = U^j_{j'} c_i. \]

\[
\{c_i, c_j, c_k\} = \rho^{m}_{ijk} c_m, \quad \{\tilde{c}_{j'}, \tilde{c}_{j''}, \tilde{c}_{k'}\} = \rho^{m'}_{j'j''k'} \tilde{c}_{m'},
\]

\[ U^j_{j'} \rho^{m'}_{j'j''k'} = \rho^{m}_{jkm} U^j_{j'} U^k_{k'} U^m_{m'}. \]
with supplementary condition
\[ \tilde{c}_1^1 = \tilde{c}_2^2, \quad \tilde{c}_2^1 = \tilde{c}_1^3, \quad \tilde{c}_3^1 = \tilde{c}_3^3. \]

It is easy to see that there are three subgroups of this group of transformations, each of the isomorphic with \( SL(2, C) \):
\[
\begin{pmatrix} U_{1'}^{1} & U_{2'}^{1} & 0 \\ U_{2'}^{1} & U_{2'}^{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} U_{1'}^{1} & 0 & U_{3'}^{1} \\ 0 & 1 & 0 \\ U_{1'}^{3} & 0 & U_{3'}^{3} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_{2'}^{2} & U_{2'}^{3} \\ 0 & U_{3'}^{2} & U_{3'}^{3} \end{pmatrix}.
\]

Each of these matrices would belong to an \( SL(2, C) \) group if we do not impose hermiticity conditions. But if we want to conserve the relations between the new generators as they were between the old ones, we shall restrain the three subgroups to corresponding Bogolyubov groups. The products of these matrices, belonging now to \( SU(1, 1) \), span the analogue of the Bogolyubov group preserving ternary \( Z_3 \)-graded Heisenberg algebra constructed above.

4. Ternary analogue of quantum oscillator

The \( Z_3 \)-graded analogues of creation and annihilation operators \( a \) and \( a^\dagger \) are the operators \( c_3 = c_3^1, \quad c_1 = c_3^1, \quad c_2 = c_1^3 \). Imposing hermiticity means that the factor \( \lambda \) in front of the derivation must be pure imaginary, so we shall set \( \lambda = -i \), as in the usual quantization scheme.

The expression \( aa^\dagger + a^\dagger a \) can be interpreted as a sum of all \( Z_2 \) permutations, or all \( S_2 \) permutations, because here all permutations are cyclic, which is not the case of \( Z_3 \) and \( S_3 \) groups. The sum of all cyclic permutations of three operators \( c_1 c_2 c_3 \), or even all (six) permutations does not lead to any simple expression. Moreover, in such combinations the complex representation of \( Z_3 \) does not appear.

However, the quadratic harmonic oscillator Hamiltonian \( aa^\dagger + a^\dagger a \) can be represented in an alternative manner, in which the representation of \( Z_2 \) by multiplications by 1 and \(-1\) does appear explicitly:
\[
\hat{H} = \frac{1}{2} \left[ (a + a^\dagger)^2 + (1)(a - a^\dagger)^2 \right] = aa^\dagger + a^\dagger a. \tag{28}
\]

Now the \( Z_3 \) generalization becomes obvious: we should form the sum of cubes of similar expressions with three generators each, multiplied by \( j \) and \( j^2 \) in all possible combinations so as to ensure the hermiticity of the resulting expression, like in (28) above. Two expressions of this type can be formed with the generators \( c_1, c_2 \) and \( c_3 \),
\[
(c_3 + c_1 + c_2)^3 + j(c_3 + jc_1 + j^2 c_2)^3 + j^2(c_3 + j^2 c_1 + jc_2)^3,
\]
and
\[
(c_3 + c_1 + c_2)^3 + j^2(c_3 + jc_1 + j^2 c_2)^3 + j(c_3 + j^2 c_1 + jc_2)^3.
\]
Neither of these two combinations is hermitian, but their sum \( \hat{H}_{Z_3} \) is (if we set \( \lambda \) pure imaginary, e.g. \(-i\)):
\[
\hat{H}_{Z_3} = 2\lambda^3 \frac{d^3}{dx^3} - x^3 - 1 = 2i \frac{d^3}{dx^3} - x^3 - 1. \tag{29}
\]
This operator is the \( Z_3 \)-graded generalization of quantum harmonic oscillator Hamiltonian. The expression \( x^3 + 1 \) appearing as the "potential" part factorizes as follows:
\[
x^3 + 1 = (x + 1)(x + j)(x + j^2).
\]
The eigenvalue equation
\[
\hat{H}_{Z_3} f(x) = Ef(x)
\]
can be solved explicitly for the particular value of $E = -1$, which amounts to solve the following differential equation of third order:

$$2i \frac{d^3 f}{dx^3} - x^3 f(x) = 0. \quad (30)$$

This equation can be solved by expanding $f(x)$ into a power series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$ 

The solution is given in terms of the generalized hypergeometric function $F([ ]; p, q; \xi)$. In the convention used in “Maple” computing programme, this symbol defines the following power series:

$$F([ ]; p, q; \xi) = \sum_{k=0}^{\infty} c_k x^k, \quad (31)$$

with the coefficients $c_k$ defined as follows using gamma-functions:

$$c_k = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+k)\Gamma(q+k)} \quad (32)$$

The eigenfunction satisfying (30) is a linear combination of three independent solutions:

$$F([ ]; \frac{2}{3}, \frac{5}{6}; \frac{ix}{432}), \quad x F([ ]; \frac{5}{6}, \frac{7}{6}; \frac{ix}{432}), \quad \text{and} \quad x^2 F([ ]; \frac{7}{3}, \frac{4}{3}; \frac{ix}{432})$$

corresponding to three different initial conditions, as it should be for a third order differential equation. All these eigenfunctions are complex.

In the classical harmonic oscillator case the operators $a$ and $a^\dagger$ were not hermitian; nevertheless their combination $aa^\dagger + a^\dagger a$ became hermitian. We might as well abandon the requirement of hermiticity, i.e. admit real value for the factor $\lambda$ in the definition of our operators $c_k$, and still get real eigenvalues for the non-hermitian third order operator, in the spirit of C. Bender’s work on the subject [11].

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