Realization of the Dirac bracket algebras through first class functions and quantization of constrained systems.

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Abstract

It is shown that a Dirac bracket algebra is isomorphic to the original Poisson bracket algebra of first class functions subject to first class constraints. The isomorphic image of the Dirac bracket algebra in the star-product commutator algebra is found.

1. Almost all the schemes of quantization of dynamical systems with second class constraints are based on conversion of original constraints into first class ones. In the Dirac bracket approach this conversion is achieved by modification of Poisson brackets. In the BRST method one introduces auxiliary variables and constructs the BRST charge which is first class. Auxiliary variables are also used in the conversion scheme of [2].

In this paper we present the approach where second class constraints are converted into first class ones without using modification of Poisson brackets or auxiliary variables. We construct the algebra with respect to the original Poisson bracket which is isomorphic to the Dirac bracket algebra. It is a quotient of the Poisson bracket algebra of first class
functions. This algebra is singled out by covariant conditions and includes only first class constraints.

A Dirac bracket algebra can be treated as the original Poisson bracket algebra of the functions on constraint surface \[3, 4, 5\] (see also [3] and references therein). However to apply this realization one needs to know a solution to constraint equations.

The new realization enables us to replace quantization of the Dirac bracket algebras by quantization of first class functions subject to first class constraints. The latter seems to be more preferable. An operator version of this approach was used for quantization of two-dimensional coset conformal field theories [7].

It is known (see [8] for a review) that the Poisson bracket algebra is isomorphic to a quotient of the corresponding star-product commutator algebra. Using this fact and the connection between the Dirac and Poisson bracket algebras we obtain realization of the Dirac bracket algebra in terms of the star-product commutator algebra.

We consider only systems with second class constraints bearing in mind that a system with first class constraints and gauge fixing is second class (see e.g. [9]).

2. In this section we review the Dirac bracket approach \([1]\) and introduce notations. Let \(M\) be a phase space with the phase variables \(\eta_n, n = 1...2N\), and the Poisson bracket

\[
[\eta_m, \eta_n] = \omega_{mn}(\eta).
\]

Let \(H(\eta)\) be the original hamiltonian and \(\varphi_j(\eta), j = 1...2J\), the second class constraints

\[
det[\varphi_j, \varphi_k]|_{\varphi=0} \neq 0.
\]

The dynamic of the system under consideration is described by the Hamilton equations

\[
\frac{d}{dt} \eta_n = [\eta_n, H^T], \quad \varphi_j = 0.
\] (1)

Here \(H^T = H + \lambda_j \varphi_j\). Functions \(\lambda_j = \lambda_j(\eta)\) are defined by the equation

\[
[H^T, \varphi_j]|_{\varphi=0} = 0.
\] (2)

Using (2) one can write equations (1) as

\[
\frac{d}{dt} \eta_n = [\eta_n, H^T]_D, \quad \varphi_j = 0.
\] (3)
Here the Dirac bracket was introduced

\[ [g, f]_D = [g, f] - [g, \varphi_j] c_{jk}[\varphi_k, f], \quad c_{jk}[\varphi_k, \varphi_l] = \delta_{jl}. \]

Let \( A = C^\infty(M) \) be the algebra of smooth functions on \( M \). One can associate with \( A \) the Lie algebras \( A_P \) and \( A_D \) with the Poisson and Dirac brackets respectively. Let \( \Phi \subset A \) be the subspace of the functions which vanish on constraint surface

\[ \Phi = \{ u \in A | u = u_j(\eta)\varphi_j \}. \]

For \( g \in A \)

\[ [g, \varphi_j]_D = 0. \]

From this it follows that for \( u \in \Phi, g \in A \)

\[ [u, g]_D \in \Phi. \]

Hence \( \Phi \) is an ideal of \( A_D \) and the quotient \( A_D/\Phi \) is an algebra.

It was observed by Dirac [1] that equations (3) can be written in the form

\[ \frac{d}{dt}\eta_n \approx [\eta_n, H^T]_D \]

(4)

where \( f \approx g \) means that \( f - g \in \Phi \).

Let \( \{g\} \in A_D/\Phi \) be the coset represented by function \( g \). Then equations (4) can be rewritten as

\[ \frac{d}{dt}\{\eta_n\} = \{[\eta_n, H^T]_D\}. \]

In \( A_D/\Phi \)

\[ \{[g, f]_D\} = \{[g], [f]_D\} \]

(5)

and we get

\[ \frac{d}{dt}\{\eta_n\} = \{[\eta_n], [H]_D\}. \]

(6)

In what follows we shall use first class functions. \( R(\eta) \) is called a first class function if

\[ [R, \varphi_j]|_{\varphi=0} = 0 \]
or, equivalently,

\[ [R, \varphi_j] = r_{jk}(\eta)\varphi_k. \]  

(7)

From equation (2) it follows that \( H^T \) is first class. It is known \cite{1} that first class functions form an algebra with respect to the Poisson bracket. We shall denote this algebra by \( \Omega \).

3. Let \( \Upsilon \) be the space of the first class functions which vanish on constraint surface

\[ \Upsilon = \{ u \in \Omega \mid u = u_j(\eta)\varphi_j \}. \]  

(8)

Using (7) and (8) for \( u \in \Upsilon, g \in \Omega \), we get

\[ [u, g]_{\varphi=0} = 0. \]

From this and definition (8) it follows

\[ [u, g] \in \Upsilon. \]  

(9)

Hence \( \Upsilon \) is an ideal of \( \Omega \) \cite{4} and \( \Omega/\Upsilon \) is an algebra.

At this stage we have two quotient algebras \( A_D/\Phi \) and \( \Omega/\Upsilon \). Our aim is to show that they are isomorphic.

Let us define the linear function \( T : \Omega/\Upsilon \to A_D/\Phi \)

\[ T(\{g\}^\bullet) = \{g\}. \]  

(10)

Here \( \{g\}^\bullet \in \Omega/\Upsilon \) is the coset represented by \( g \). Since \( \{g\}^\bullet \subset \{g\} \), \( T \) does not depend on the choice of \( g \in \{g\}^\bullet \).

To each function \( g \in A \) one can put into correspondence the first class function

\[ \tilde{g} = g - [g, \varphi_j]c_{jk}\varphi_k. \]

Let \( g' \in \{g\} \) be a first class function. It can be written in the form

\[ g' = g - [g, \varphi_j]c_{jk}\varphi_k + v_j\varphi_j \]  

(11)

where \( v_j = v_j(\eta) \) are some functions. Substituting \( g' \) into the equation

\[ [g', \varphi_k]_{\varphi=0} = 0, \]
we obtain
\[ [v_j \varphi_j, \varphi_k]_{\varphi=0} = 0. \]

From this and equation (11) it follows that each first class function \( g' \) from \( \{g\} \) is in \( \{g - [g, \varphi_j]\varphi_j, \varphi_k\} \). This allows us to define the inverse function \( T^{-1} : A_D/\Phi \to \Omega/\Upsilon \)
\[ T^{-1}(\{g\}) = \{g - [g, \varphi_j]\varphi_j, \varphi_k\}. \] (12)

Thus we have shown that \( T \) defines the one-to-one correspondence between elements of \( \Omega/\Upsilon \) and \( A_D/\Phi \).

In order to show that \( T \) is a homomorphism let us compute \( T([\{g\}^\bullet, \{f\}^\bullet]) \). Using definitions of \( \Omega/\Upsilon \) and \( T \) we have
\[ T([\{g\}^\bullet, \{f\}^\bullet]) = T([\{g, f\}^\bullet]) = [\{g, f\}]. \] (13)

Since \( g \) and \( f \) are first class,
\[ [g, \varphi_j] = g_{jj'}(\eta)\varphi_{j'}, \quad [f, \varphi_k] = f_{kk'}(\eta)\varphi_{k'} \]
and
\[ [g, f]_D = [g, f] + g_{jj'}\varphi_{j'}c_{jk}\varphi_{k'}. \]
From this it follows that the right hand side of (13) can be written as
\[ \{[g, f]\} = \{[g, f]_D\}. \] (14)
Using equation (10) and definition (10) we get
\[ \{[g, f]_D\} = \{[g], \{f\}\}_D = [T([\{g\}^\bullet]), T([\{f\}^\bullet])]_D. \] (15)
From equations (13)–(15) it follows that
\[ T([\{g\}^\bullet, \{f\}^\bullet]) = [T([\{g\}^\bullet]), T([\{f\}^\bullet])]_D \] (16)
and hence \( T \) is a homomorphism. Since \( T \) is an invertible function, algebras \( A_D/\Phi \) and \( \Omega/\Upsilon \) are isomorphic.

Elements of \( \Upsilon \) act in \( \Omega \) as constraints. Due to (8), for \( u, v \in \Upsilon \)
\[ [u, v] \in \Upsilon. \]
Hence constraints $\Upsilon$ are first class. Thus we have shown that the Dirac bracket algebra $A_D/\Phi$ is equivalent to the algebra of first class functions $\Omega$ subject to first class constraints $\Upsilon$.

Equation (9) tells us that all the functions of $\Omega$ are first class with respect to constraints $\Upsilon$. From this it follows that the system under consideration is invariant with respect to the gauge transformations

$$\delta_u g = [u, g]$$

where $u \in \Upsilon$, $g \in \Omega$.

According to (12) and (16) the image of the Hamilton equations (6) in $\Omega/\Upsilon$ is

$$\frac{d}{dt}(\tilde{\eta}_n)^\bullet = [\{\tilde{\eta}_n\}^\bullet, \{H^T\}^\bullet].$$

Here $\tilde{\eta}_n = \eta_n - [\eta_n, \varphi_j]c_{jk}\varphi_k$.

Let $\tilde{A} = A[[\lambda]]$ be the space of formal power series in a variable $\lambda$ with coefficients in $A$. Let $\ast$ be the star-product

$$g(\lambda) \ast f(\lambda) = gf + \sum_{k=1}^{\infty} \lambda^k c_k,$$  \hspace{1cm} (17)

where $g(\lambda) = g + \sum_{k=1}^{\infty} \lambda^k g_k$, $f(\lambda) = f + \sum_{k=1}^{\infty} \lambda^k f_k$ and $g, f, c_k, g_k, f_k \in A$.

Using the $\ast$-product one defines the $\ast$-commutator algebra $\tilde{A}_c$

$$[g, f]_c = \frac{1}{2\lambda}(g(\lambda) \ast f(\lambda) - f(\lambda) \ast g(\lambda)) = [g, f] + \sum_{k=1}^{\infty} \lambda^k b_k,$$ \hspace{1cm} (18)

where $b_k \in A$. From the definition it follows that $\lambda \tilde{A}_c$ is an ideal of $\tilde{A}_c$ and $\tilde{A}_c/\lambda \tilde{A}_c$ is the algebra which is isomorphic to the Poisson bracket algebra $A_P$. Let $L : A_P \rightarrow \tilde{A}_c/\lambda \tilde{A}_c$ be the corresponding map. Then the isomorphic image of $\Omega/\Upsilon$ (and $A_D/\Phi$ in $\tilde{A}_c/\lambda \tilde{A}_c$ is $L(\Omega)/L(\Upsilon)$. Thus we have obtained the realization of the Dirac bracket algebra $A_D/\Phi$ as a subalgebra of the $\ast$-commutator algebra $\tilde{A}_c/\lambda \tilde{A}_c$.

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