Martingale representations for diffusion processes and backward stochastic differential equations

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Abstract. In this paper we explain that the natural filtration of a continuous Hunt process is continuous, and show that martingales over such a filtration are continuous. We further establish a martingale representation theorem for a class of continuous Hunt processes under certain technical conditions. In particular we establish the martingale representation theorem for the martingale parts of (reflecting) symmetric diffusions in a bounded domain with a continuous boundary. Together with an approach put forward in [21], our martingale representation theorem is then applied to the study of initial and boundary problems for quasi-linear parabolic equations by using solutions to backward stochastic differential equations over the filtered probability space determined by reflecting diffusions in a bounded domain with only continuous boundary.

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1 Introduction

The Brownian motion, an important example of martingales, diffusions and Gaussian processes, possesses some remarkable properties which have been the inspiration for research in areas from probability, statistics to mathematical finance.

Let $B = (B_t)_{t \geq 0}$ be the Brownian motion in $\mathbb{R}^d$ started with an initial distribution $\mu$. The natural filtration $(\mathcal{F}_t^\mu)_{t \geq 0}$ (called the Brownian filtration, for a definition, see [5]) is continuous, and all martingales on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t^\mu, P^\mu)$ are continuous. More importantly the Brownian motion has the martingale representation property: any martingale over $(\Omega, \mathcal{F}, \mathcal{F}_t^\mu, P^\mu)$ can be expressed as an Itô integral against Brownian motion.

The (predictable) martingale representation property of a family of martingales has been studied in a more extended setting, and several general results have been obtained. For example, Jacod and Yor [18] have discovered the equivalence between the martingale representation property and the extremal property of martingale measures. Jacod [19] also obtained the martingale representation property in terms of the uniqueness of some martingale problems, and further present criteria in terms of predictable characteristics. These results have greatly illuminated the subject matter. When applied to specific situations, further work and indeed hard estimates are often required. As a matter of fact, we still have very limited examples of martingales and filtrations which possess martingale representation property (see, e.g. [1], [19], [27], [33]).

The renewed interest in recent years in the martingale presentation property has been motivated not only by its own right, but also by its important applications in the mathematical finance ([13], [31]), backward stochastic differential equations ([1], [28]) and their applications in some non-linear partial differential equations, see also for example [3], [6], [20], [24], [29], [32] and the reference therein.

An intimate question is the continuity of natural filtrations generated by semimartingales and diffusion processes. A great knowledge about them has been obtained in the past. For example, a complete characterization of the natural filtrations generated by simple jump processes and Lévy processes in terms of their sample paths is known. Much information has been obtained for a class of Markov processes. We know, from the fundamental work by Blumenthal, Chung, Dynkin, Getoor, Hunt, Meyer etc., (see for example [5], [9], [12], [26], in particular Hunt [14]), that the natural filtration of a Feller process is right continuous and quasi-left continuous. On the other hand, to the best knowledge of the present authors, there are no general conditions in literature to guarantee the martingales over the natural filtration of a Markov process to be continuous. A reasonable conjecture is that the natural filtration of a diffusion process (a continuous strong Markov process) should be continuous, so are the martingales over the natural filtration. Such a result is plausible but remains to prove.

In this paper, we show that all martingales over the natural filtration of a continuous Hunt process are continuous. Our proof follows a key idea originated from Blumenthal [4], formulated carefully in Meyer [26].
The main result of the present article is a martingale representation theorem for a class of continuous Hunt processes which satisfy a technical condition called the Fukushima representation property.

As a consequence of our main result, we establish the martingale representation theorem for symmetric diffusion processes on a domain, with Dirichlet or Neumann boundary condition. More precisely, let $D \subset \mathbb{R}^d$ be a bounded domain with a continuous boundary $\partial D$. Consider the symmetric diffusion in $D$ with a formal infinitesimal generator

$$L = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j} a^{ij}(x) \frac{\partial}{\partial x^i}$$

in $D$

subject to the Dirichlet or Neumann boundary condition (for precise meaning, see §4 and §5 below), where $(a^{ij})$ are only Borel measurable and satisfies the uniform elliptic condition.

To make it clear, let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$

be the symmetric diffusion associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ where

$$\mathcal{E}(u, v) = \frac{1}{2} \int_D \sum_{i,j=1}^{d} a^{ij} \frac{\partial u}{\partial x^j} \frac{\partial v}{\partial x^i} dx$$

and the Dirichlet space $\mathcal{F} = H_0^1(D)$ or $H^1(D)$ depending on the Dirichlet or Neumann boundary condition. Here we use the same letter $\mathcal{F}$ to denote the filtration as well as the Dirichlet space: we hope it should be clear from the context which one $\mathcal{F}$ stands for. $\mathcal{F}_t$ are the natural filtrations generated by the symmetric diffusion $(X_t)_{t \geq 0}$. It happens in this case that the coordinate functions $u^j(x) = x^j$ belong to the local Dirichlet space $\mathcal{F}_{\text{loc}}$, and $X_t = (X_t^1, \cdots, X_t^d)$ has the Fukushima's decomposition

$$X_t^j - X_0^j = M_t^j + A_t^j \quad P^x\text{-a.e. } j = 1, \cdots, d$$

for all $x \in D$ (or $\overline{D}$ in the Neumann boundary condition case) except for a zero capacity set with respect to the Dirichlet form $(\mathcal{E}, \mathcal{F})$, where $M^1, \cdots, M^d$ are continuous martingales additive functionals and $A^1, \cdots, A^d$ are continuous additive functionals with zero energy.

The following martingale representation theorem follows from our main result.

**Theorem 1.1** Under the above assumptions, for any initial distribution $\mu$ which has no charge on capacity zero sets, the family $(M^1, \cdots, M^d)$ of martingales over $(\Omega, \mathcal{F}^\mu, \mathcal{F}_t^\mu, P^\mu)$ has the martingale representation property: for any square integrable martingale $N = (N_t)_{t \geq 0}$ over $(\Omega, \mathcal{F}^\mu, \mathcal{F}_t^\mu, P^\mu)$ there are unique $(\mathcal{F}_t^\mu)$-predictable processes $F^1, \cdots, F^d$ such that

$$N_t - N_0 = \sum_{j=1}^{d} \int_0^t F_s^j dM_s^j.$$
Theorem 1.1 may be applied to the symmetric diffusions in $\mathbb{R}^d$ with Dirichlet form $(\mathcal{E}, \mathcal{F})$ where $\mathcal{F} = H^1_0(\mathbb{R}^d)$. This special case has been proved in [33] and [1].

In [33], Zheng has pointed out that the martingale part of symmetric diffusion $(X_t)_{t \geq 0}$ in $\mathbb{R}^d$ with infinitesimal generator being a uniform second order elliptic operator in divergence form has the martingale predictable representation property and described a proof based on the results on the Dirichlet process $p(t, X_t)$ obtained in Lyons and Zheng [22] and [23], where $p(t, x)$ is the probability density function of $X_t$ under the stationary distribution.

More precisely, Lyons and Zheng [23] have extended Fukushima's representation theorem for martingale additive functionals to a class of processes which has a form $f(t, X_t)$, where $f$ has finite space-time energy. Their results in particular yield that $p(t, X_t)$ is a Dirichlet process in the sense of Föllmer [15], and its martingale part can be expressed as an Itô integral against $(M_1, \cdots, M_d)$, which, together with the Markov property, allows to show that for $\xi = f_1(X_{t_1}) \cdots f_n(X_{t_n})$ the conditional expectation $E(\xi | \mathcal{F}_t)$ is again a Dirichlet process which can be expressed as an Itô’s integral against $(M^1, \cdots, M^d)$, where the expectation is taken against the stationary distribution $P^{\mu}(\cdot) = \int_{\mathbb{R}^d} P^x(\cdot) dx$. A routine procedure based on the Doob’s maximal inequality allows to prove the martingale representation theorem for the symmetric diffusion in $\mathbb{R}^d$ with the generator $L$ an elliptic operator of second order.

Apparently not knowing the work [33], in an independent work Bally, Pardoux and Stoica [1], among other things, a detailed proof has been provided.

The technical difficulty with the proof described above lies in the fact that even for a smooth function $f$, $f(X_t)$ may not be a semimartingale, so that Itô’s calculus can not be applied. Instead of considering random variables with product form such as $\xi = f_1(X_{t_1}) \cdots f_n(X_{t_n})$ which linearly span a vector space dense in $L^p(\Omega, \mathcal{F}, P^{\mu})$, we utilize a linear vector $C$ spanned by those random variables which have a product form $\xi = \xi_1 \cdots \xi_n$ with

$$\xi_j = \int_0^\infty e^{-\alpha_j t} f_j(X_t) dt$$

where $f_j$ are bounded Borel measurable functions, and $\alpha_j > 0$ for $j = 1, \cdots, n$. According to Meyer [26], $C$ is dense in $L^p(\Omega, \mathcal{F}, P^{\mu})$. The important feature is that $U^\alpha f(X_t)$ is a semimartingale for any $\alpha > 0$ and a bounded Borel function $f$, where $U^\alpha$ is the resolvent of the transition semigroup. Moreover, in the symmetric case, $U^\alpha f$ always belongs to the Dirichlet space (when $f$ is bounded and square integrable) and thus Fukushima’s representation theorem for martingale additive functionals can be applied to extend the representation to any martingales.

The martingale representation theorem for the symmetric diffusion process $(X_t)_{t \geq 0}$ in a domain $D$ allows us to study the following type of backward stochastic differential equation (BSDE)

$$dY_t = -f(t, Y_t, Z_t) dt + \sum_{j=1}^d Z^j_t dM^j_t$$

with a terminal condition that $Y_T = \xi \in L^2(\Omega, \mathcal{F}^\mu_T, P^{\mu})$, and thus gives a probability representation for weak solutions to the initial and boundary value problem for non-linear
parabolic equations. The existence and uniqueness of solutions to the BSDE follows from exactly the same approach as for the Brownian motion case, which is the pioneering work in BSDE done by Pardoux and Peng in [28]. We however describe an approach put forward in [21], which allows us to devise an alternative probability representation for the initial and boundary problem of the corresponding semi-linear parabolic equation

$$\frac{\partial}{\partial t} u - \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x^i} a^{ij} \frac{\partial}{\partial x^j} u + f(t, u, \nabla u) = 0$$

subject to $\frac{\partial}{\partial \nu} \bigg|_{\partial D} u(t, \cdot) = 0$ in a bounded domain with only continuous boundary.

The paper is organized as follows. In Section 2, we develop further an idea put forward in Meyer [26], and show both the natural filtrations and martingales over the natural filtrations for a continuous Hunt process are continuous. This result is hardly new but it seems not appear in the literature yet. In order to prove this result, we devise an important while elementary formula for $E_{\mu}(\xi | F^\mu_t)$, which may be considered as a refined version of a classical formula devised firstly by Hunt and Blumenthal for potentials and multiple potential case by Meyer. In Section 3, we establish the main result of the paper: a martingale representation theorem for a continuous Hunt process under technical assumptions called the Fukushima representation property, and give some examples in which our result may apply. In Section 4, we outline the existence and uniqueness of solutions to backward stochastic differential equations over the natural filtered probability space over a reflecting symmetric diffusion in a bounded domain with non-smooth diffusion coefficients and non-smooth boundary, and finally we apply the theory of BSDE to the study of the initial and (Neumann) boundary problem of a non-linear parabolic equation in a bounded domain with only continuous boundary. We believe these results are new even for reflecting Brownian motion in a domain with non-smooth boundary.

2 Martingales over the filtrations of continuous Hunt processes

Consider a Markov process $(\Omega, \mathcal{F}^0, \mathcal{F}^0_t, X_t, \theta_t, P^x)$ in a state space $E' = E \cup \{\partial\}$, where $E$ is a locally compact separable metric space $E$, with transition probability function $\{P_t(x, \cdot) : t \geq 0\}$, i.e.,

$$E^x \{f(X_{t+s}) | \mathcal{F}^0_s \} = \int_E f(z) P_t(X_s, dz). \quad (2.1)$$

In (2.1), $E^x$ on the left-hand side stands for the (conditional) expectation with respect to the probability measure $P^x$, and the right-hand side may be abbreviated as $P_t f(X_s)$ where

$$P_t f(x) = \int_E f(z) P_t(x, dz)$$

which is well defined for a bounded or non-negative Borel measurable function $f$. The family of kernels $(P_t)_{t \geq 0}$ is called the transition semigroup associated with the Markov process.
\( (X_t)_{t \geq 0} \). Without specification, \((\mathcal{F}_t^0)_{t \geq 0}\) is the filtration generated by \((X_t)_{t \geq 0}\); that is \(\mathcal{F}_t^0 = \sigma \{ X_s : s \leq t \} \) for \(t \geq 0\) and \(\mathcal{F}_0^0 = \sigma \{ X_s : s \geq 0 \} \).

For a \(\sigma\)-finite measure \(\mu\) on \((E, \mathcal{B}(E))\) (where \(\mathcal{B}(M)\) always represents the Borel \(\sigma\)-algebra on a topological space \(M\))

\[
P^\mu(\Lambda) = \int_E P^x(\Lambda) \mu(dx), \quad \Lambda \in \mathcal{F}_t^0,
\]
defines a measure on \((\Omega, \mathcal{F}_t^0)\). If \(\mu\) is a probability, then \((X_t)_{t \geq 0}\) is Markovian under \(P^\mu\) with transition semigroup \((P_t)_{t \geq 0}\) and initial distribution \(\mu\), in the sense that

\[
E^\mu \{ f(X_{t+s}) | \mathcal{F}_s^0 \} = \int_E f(z) P_t(X_s, dz) \quad P^\mu\text{-a.e.}
\]

and \(P^\mu \{ X_0 \in A \} = \mu(A)\) for any \(A \in \mathcal{B}(E)\), where \(E^\mu\) is the (conditional) expectation against the probability \(P^\mu\).

Denote by \(\mathcal{P}(E)\) the space of all probability measures on \((E, \mathcal{B}(E))\). If \(\mu \in \mathcal{P}(E)\), \(\mathcal{F}_t^\mu\) denotes the completion of \(\mathcal{F}_t^0\) under \(P^\mu\), and \(\mathcal{F}_t^\mu\) is the smallest \(\sigma\)-algebra containing \(\mathcal{F}_t^0\) and all sets in \(\mathcal{F}_t^\mu\) with zero probability. \((\mathcal{F}_t^\mu)_{t \geq 0}\) is called the natural filtration of the Markov process \((X_t)_{t \geq 0}\) with initial distribution \(\mu\). Let

\[
\mathcal{F}_t = \bigcap_{\mu \in \mathcal{P}(E)} \mathcal{F}_t^\mu
\]

and \((\mathcal{F}_t)_{t \geq 0}\) is called the natural filtration determined by the Markov process

\[
X = (\Omega, \mathcal{F}_t^0, X_t, \theta_t, \mathcal{P}^x).
\]

If \((\mathcal{G}_t)_{t \geq 0}\) is a filtration, i.e. an increasing family of \(\sigma\)-algebras on a common sample space, then \(\mathcal{G}_{t+} = \bigcap_{s \geq t} \mathcal{G}_s\) for \(t \geq 0\) and \(\mathcal{G}_{t-} = \sigma \{ \mathcal{G}_s : s < t \} \) for \(t > 0\). The filtration is called right (resp. left) continuous if \(\mathcal{G}_{t+} = \mathcal{G}_t\) for all \(t \geq 0\) (resp. \(\mathcal{G}_{t-} = \mathcal{G}_t\) for all \(t > 0\)). The sample function properties of a Markov process \((X_t)_{t \geq 0}\) and the continuity properties of its natural filtration had been studied by Blumenthal, Dynkin, Getoor, Hunt, Meyer etc. The fundamental results have been established via the regularity of the transition probability function \(\{ P_t(x, \cdot) : t > 0 \}\). Their work achieved the climax for Markov processes with Feller transition semigroups.

As matter of fact, the continuity of the filtration \((\mathcal{F}_t^\mu)\) (or \((\mathcal{F}_t)\)) does not follow that of sample function \((X_t)_{t \geq 0}\). For example, a right continuous Markov process does not necessarily lead to the right continuity of its natural filtration \((\mathcal{F}_t^\mu)\) (or \((\mathcal{F}_t)\)). The same claim applies to the left continuity. In fact, the regularity of natural filtrations is much to do with the nature of the Markov property, such as strong Markov property.

Let \(C_\infty(E)\) (resp. \(C_0(E)\)) denote the space of all continuous functions \(f\) on \(E\) which vanish at infinity \(\partial\), i.e. \(\lim_{x \to \partial} f(x) = 0\) (resp. with compact support). Recall that a transition semigroup \((P_t)_{t \geq 0}\) on \((E, \mathcal{B}(E))\) is Feller, if for each \(t > 0\), \(P_t\) preserves \(C_\infty(E)\) and \(\lim_{t \to 0} P_t f(x) = f(x)\) for each \(x \in E\) and \(f \in C_\infty(E)\).
For a given Feller semigroup \((P_t)_{t \geq 0}\) on \((E, \mathcal{B}(E))\), there is a Markov process

\[ X = (\Omega, \mathcal{F}^0, \mathcal{F}^0_t, X_t, \theta_t, P^x) \]

with the Feller transition semigroup \((P_t)_{t \geq 0}\) such that the sample function \(t \to X_t\) is right continuous on \([0, \infty)\) with left hand limits on \((0, \infty)\). In this case, we call \((\Omega, \mathcal{F}^0, \mathcal{F}^0, X_t, \theta_t, P^x)\) a Feller process on \(E\).

For a Feller process \((\Omega, \mathcal{F}^0, \mathcal{F}^0_t, X_t, \theta_t, P^x)\), the natural filtration \((\mathcal{F}^\mu_t)_{t \geq 0}\) for any \(\mu \in \mathcal{P}(E)\) and as well as \((\mathcal{F}_t)\) are right continuous. \((X_t)_{t \geq 0}\) and \((\mathcal{F}^\mu_t)_{t \geq 0}\) are also quasi-left continuous, that is, if \(T_n\) is an increasing family of \((\mathcal{F}^\mu_t)\)-stopping times, and \(T_n \uparrow T\), then \(\lim_{n \to \infty} X_{T_n} = X_T\) on \(\{T < \infty\}\) and \(\mathcal{F}^\mu_t = \sigma\{\mathcal{F}^\mu_{T_n} : n \in \mathbb{N}\}\). Therefore accessible \((\mathcal{F}^\mu_t)\)-stopping times are predictable. An \((\mathcal{F}^\mu_t)\)-stopping time \(T\) is totally inaccessible if and only if \(P^\mu\{T < \infty\} > 0\) and \(X_T \neq X_{T^-}\) on \(\{T < \infty\}\) \(P^\mu\)-a.e. Similarly, \(T\) is accessible if and only if \(X_T = X_{T^-}\) \(P^\mu\)-a.e. on \(\{T < \infty\}\). Hence \(X\) has only inaccessible jump times.

What we are mainly concerned in this article is Hunt processes. Hunt processes are right continuous, strong Markov processes which are quasi-left continuous. These processes are defined in terms of sample functions, rather than transition semigroups, see [5] and [9] for details. It is well-known that Feller processes are stereotype of Hunt processes or the later is an abstraction of the former.

We are interested in the martingales over the filtered probability space \((\Omega, \mathcal{F}^\mu, \mathcal{F}^\mu_t, P^x)\), and we are going to show that, if \((\Omega, \mathcal{F}^0, \mathcal{F}^0_t, X_t, \theta_t, P^x)\) is a Hunt process which has continuous sample function, then any martingale on this filtered probability space is continuous, a result one could expect for the natural filtration of a diffusion process. Indeed, this result was proved more or less by Meyer in his Lecture Notes in Mathematics 26, “Processus de Markov”. Meyer himself credited his proof to Blumenthal and Getoor, more precisely a calculation done by Blumenthal [4]. However it is surprising that the full computation, which yields more information about martingales over the natural filtration of a Hunt process, was not reproduced either in the new edition of Meyer’s “Probabilités et Potentiels” or Chung’s “Lectures from Markov Processes to Brownian Motion”, although it was mentioned in [10] where Chung and Walsh gave an alternative proof of Meyer’s predictability result, so that Blumenthal’s computation is no longer needed. However, it is fortunate that Blumenthal’s calculation indeed leads to a proof of a martingale representation theorem we are going to establish for certain Hunt processes, see section §3.

Let us first describe an elementary calculation, originally according to Meyer [26] due to Blumenthal. Let

\[ X = (\Omega, \mathcal{F}^0, \mathcal{F}^0_t, X_t, \theta_t, P^x) \]

be a Hunt process in a state space \(E' = E \cup \{\partial\}\) with the transition semigroup \((P_t)_{t \geq 0}\), where \(\partial\) plays a role of cemetery. Let \(U^\alpha = \{U^\alpha : \alpha > 0\}\) be the resolvent of the transition semigroup \((P_t)_{t \geq 0}\):

\[ U^\alpha(x, A) = \int_0^\infty e^{-\alpha t} P_t(x, A) dt \]
and \((U^\alpha)_{\alpha > 0}\) the corresponding resolvent (operators), i.e.

\[
U^\alpha f(x) = \int_E f(z)U^\alpha(x,dz) = \int_0^\infty e^{-\alpha t}P_t f(x)dt
\]

for bounded or nonnegative Borel measurable function \(f\) on \(E\). To save words, we use \(\mathcal{B}_b(E)\) to denote the algebra of all bounded Borel measurable functions on \(E\). Obviously, \(C_\infty(E) \subset \mathcal{B}_b(E)\).

Let \(K(E) \subset \mathcal{B}_b(E)\) be a vector space which generates the Borel \(\sigma\)-algebra \(\mathcal{B}(E)\). Let \(C \subset L^1(\Omega, \mathcal{F}, \mu)\) (for any initial distribution \(\mu\)) be the vector space spanned by all \(\xi = \xi_1 \cdots \xi_n\) for some \(n \in \mathbb{N}\),

\[
\xi_j = \int_0^\infty \alpha_j e^{-\alpha_j t}f_j(X_t)dt
\]

where \(\alpha_j\) are positive numbers, \(f_j \in K(E), j = 1, \cdots, n\). Meyer [26] proved that \(C\) is dense in \(L^1(\Omega, \mathcal{F}, \mu)\) for a Hunt process. Since this density result will play a crucial role in what follows, we include Meyer’s a proof for completeness and for the convenience of the reader. The key observation in the proof is the following result from real analysis.

\section*{Lemma 2.1}

Let \(T > 0\). Let \(\mathbb{K}\) denote the vector space spanned by all functions \(e_\alpha(t) = e^{-\alpha t}\), where \(\alpha > 0\), then \(\mathbb{K}\) is dense in \(C[0,T]\) equipped with the uniform norm.

The lemma follows from Stone-Weierstrass’ theorem.

\section*{Lemma 2.2 (P. A. Meyer)}

For any initial distribution \(\mu\) and \(p \in [1, \infty)\), \(C\) is dense in \(L^p(\Omega, \mathcal{F}, \mu)\).

\textbf{Proof.} First, by utilizing Doob’s martingale convergence theorem, it is easy to show that the collection \(\mathcal{A}\) of all random variables which have the following form

\[
g_1(X_{t_1}) \cdots g_n(X_{t_n}),
\]

where \(n \in \mathbb{N}, 0 < t_1 < \cdots < t_n < \infty\) and \(g_j \in K(E)\), is dense in \(L^p(\Omega, \mathcal{F}, \mu)\). Let \(\mathcal{H}\) be the linear space spanned by all \(\xi = \eta_1 \cdots \eta_n\), where

\[
\eta_j = \int_0^\infty g_j(X_t)\varphi_j(t)dt,
\]

where \(g_j \in K(E)\) and \(\varphi_j \in C[0,\infty)\) with compact supports. According to the previous lemma, for every \(\varepsilon > 0\) we may choose \(\psi_j \in \mathbb{K}\) such that

\[
|\varphi_j(t) - \psi_j(t)| < \varepsilon e^{-\lambda t} \quad \text{for all } t \geq 0
\]
for some $\lambda > 0$. Let
\[ \xi_j = \int_0^\infty g_j(X_t)\psi_j(t)dt. \]

Then $\xi = \xi_1 \cdots \xi_n \in \mathcal{G}$, and
\[ |\eta_j - \xi_j| \leq \frac{1}{\lambda} ||g_j||_\infty \varepsilon \]
where $|| \cdot ||_\infty$ is the supremum norm. It follows that
\[ E|\xi - \tilde{\xi}|^p \leq \frac{n^p}{\lambda^p} \max_j ||g_j||_\infty^p \varepsilon^p \]
and thus $\xi$ belongs to the closure of $\mathcal{G}$. Finally it is clear that any element
\[ g_1(X_{t_1}) \cdots g_n(X_{t_n}) = \lim_{k \to \infty} \eta_1^k \cdots \eta_n^k \]
where
\[ \eta_j^k = \int_0^\infty g_j(X_t)\varphi_j^k(t)dt \]
and $\varphi_j^k$ has compact support and $\varphi_j^k \to \delta_{t_j}$ weakly. We thus have completed the proof. ■

Let $\mu$ be any fixed initial distribution. If $f$ is a bounded Borel measurable function on $E$ and $\alpha > 0$, then
\[ \xi = \int_0^\infty e^{-\alpha t} f(X_t)dt \in L^1(\Omega, \mathcal{F}, P^\mu). \]

Consider the martingale $M_t = E^\mu \{ \xi | \mathcal{F}_t^\mu \}$ where $t \geq 0$. According to an elementary computation in the theory of Markov processes,
\[ M_t = E^\mu \left\{ \int_0^\infty e^{-\alpha s} f(X_s)ds | \mathcal{F}_t^\mu \right\} \]
\[ = \int_0^t e^{-\alpha s} f(X_s)ds + E^\mu \left\{ \int_t^\infty e^{-\alpha s} f(X_s)ds | \mathcal{F}_t^\mu \right\} \]
\[ = \int_0^t e^{-\alpha s} f(X_s)ds + e^{-\alpha t} \int_0^\infty e^{-\alpha s} P_s f(X_t)ds \]
\[ = \int_0^t e^{-\alpha s} f(X_s)ds + e^{-\alpha t} U^\alpha f(X_t). \]

It is known that if $X = (X_t)_{t \geq 0}$ is a Hunt process, then for any $\alpha > 0$ and bounded Borel measurable function $f$, $U^\alpha f$ is finely continuous, i.e., $t \to U^\alpha f(X_t)$ is right continuous. Moreover if $X$ is a continuous Hunt process, it follows from a result proved by Meyer that $t \to U^\alpha f(X_t)$ is continuous, and therefore, the martingale $M_t = E^\mu \{ \xi | \mathcal{F}_t^\mu \}$ is continuous. We record Meyer’s result as a lemma here. This result was proved in [26] for Hunt processes (see T15 THEOREME, page 89, [26]). A simpler proof for Feller processes may be found on page 168, [11].
Lemma 2.3 (Meyer) Let \((X_t)_{t \geq 0}\) be a Hunt process, \(f \in \mathcal{B}(E)\), \(\alpha > 0\) and \(h = U^\alpha f\) be a potential. Then
\[
h(X_{t-}) = h(X)_{t-} \quad \forall t > 0 \quad P^\mu\text{-a.e.}
\]
for any initial distribution \(\mu\).

P. A. Meyer pointed out that the previous computation can be carried out equally for random variables on \((\Omega, \mathcal{F}^0)\) which have a product form \(\xi = \xi_1 \cdots \xi_n\) where each \(\xi_j = \int_0^\infty e^{-\alpha j s} f_j(X_s) ds\). Let \(\pi_n\) denote the permutation group of \(\{1, \cdots, n\}\).

Lemma 2.4 (Blumenthal and Meyer) Let \(\xi = \xi_1 \cdots \xi_n\) where \(\xi_j = \int_0^\infty e^{-\alpha j s} f_j(X_s) ds\), \(\alpha_j > 0\) and \(f_j \in \mathcal{B}(E)\), and \(M_t = E^\mu \{\xi | \mathcal{F}^\mu_t\}\). Then
\[
M_t = \sum_{k=0}^n \sum_{(j_1, \cdots, j_k, \cdots, j_n) \in \pi_n} \left( \prod_{i=1}^k \int_0^t e^{-\alpha_j j_i s} f_{j_i}(X_s) ds \right) \cdot F_{(j_1, \cdots, j_k, \cdots, j_n)}(X_t) \quad (2.2)
\]
where
\[
F_{(j_1, \cdots, j_k, \cdots, j_n)}(x) = E^x \left\{ \left( \prod_{l=k+1}^n e^{-\alpha_j j_l t} \int_0^\infty e^{-\alpha_j j_l s} f_{j_l}(X_s) ds \right) \right\}. \quad (2.3)
\]

Proof. The task is to calculate the conditional expectation \(M_t = E^\mu \{\xi | \mathcal{F}^\mu_t\}\). The idea is very simple: splitting each \(\xi_j\) into
\[
\xi_j = \int_0^t e^{-\alpha_j j_i s} f_{j_i}(X_s) ds + e^{-\alpha_j j_i t} \int_0^\infty e^{-\alpha_j j_i s} f_{j_i}(X_s) ds
\]
so that
\[
\xi = \sum_{k=0}^n \sum_{(j_1, \cdots, j_k, \cdots, j_n) \in \pi \{1, \cdots, n\}} \left( \prod_{i=1}^k \int_0^t e^{-\alpha_j j_i s} f_{j_i}(X_s) ds \right)
\cdot \left( \prod_{l=k+1}^n e^{-\alpha_j j_l t} \int_0^\infty e^{-\alpha_j j_l s} f_{j_l}(X_s) ds \right).
\]
By using the Markov property one thus obtains
\[
M_t = E^\mu \{\xi | \mathcal{F}^\mu_t\}
= \sum_{k=1}^n \sum_{(j_1, \cdots, j_k, \cdots, j_n) \in \pi \{1, \cdots, n\}} \left( \prod_{i=1}^k \int_0^t e^{-\alpha_j j_i s} f_{j_i}(X_s) ds \right)
\cdot E^\mu \left\{ \left( \prod_{l=k+1}^n e^{-\alpha_j j_l t} \int_0^\infty e^{-\alpha_j j_l s} f_{j_l}(X_s) ds \right) \mid \mathcal{F}^\mu_t \right\}
= \sum_{k=0}^n \sum_{(j_1, \cdots, j_k, \cdots, j_n) \in \pi \{1, \cdots, n\}} \left( \prod_{i=1}^k \int_0^t e^{-\alpha_j j_i s} f_{j_i}(X_s) ds \right)
\cdot E^{X_t} \left\{ \left( \prod_{l=k+1}^n e^{-\alpha_j j_l t} \int_0^\infty e^{-\alpha_j j_l s} f_{j_l}(X_s) ds \right) \right\}. \quad (2.4)
\]
Our only contribution in this aspect is the following formula, which allows to prove not only that all martingales over the natural filtration of a continuous Hunt process are continuous, but also a martingale representation theorem in the next section.

**Lemma 2.5** Let $\alpha_j$ be positive numbers and $f_j \in \mathcal{B}_b(E)$ for $j = 1, \cdots, k$. Consider

$$F(x) = \int \cdots \int_{0 < s_1 < \cdots < s_k < \infty} e^{-\sum_{j=1}^k \alpha_j s_j} \int_{E^\otimes k} f_1(z_1) \cdots f_k(z_k) P_{s_1}(x, dz_1) \cdot P_{s_2 - s_1}(z_1, dz_2) \cdots P_{s_k - s_{k-1}}(z_{k-1}, dz_k) ds_1 \cdots ds_k.$$  

Then

$$F = U^{\alpha_1 + \cdots + \alpha_k} \left( f_1 (U^{\alpha_2 + \cdots + \alpha_k} f_2) \cdots (U^{\alpha_k} f_k) \cdots \right). \quad (2.5)$$

**Proof.** To see why it is true, we begin with the case that $k = 1$. In this case $F = \int_0^\infty e^{-\alpha s} P_s f ds = U^\alpha f$. If $k = 2$, then

$$F = \int \int \int_{0 < s_1 < s_2 < \infty} e^{-\alpha_2 s_2} e^{-\alpha_1 s_1} P_{s_1} (f_1 P_{s_2 - s_1} f_2) ds_1 ds_2$$

$$= \int_0^\infty \int_t^\infty e^{-\alpha_1 t} e^{-\alpha_2 s} P_t (f_1 P_{s-t} f_2) ds dt$$

$$= \int_0^\infty e^{-\alpha_1 t} P_t \left( \int_t^\infty e^{-\alpha_2 s} f_1 P_{s-t} f_2 ds \right) dt$$

$$= \int_0^\infty e^{-\alpha_1 t} e^{-\alpha_2 t} P_t \left( f_1 \int_0^\infty e^{-\alpha_2 s} P_s f_2 ds \right) dt$$

$$= \int_0^\infty e^{-(\alpha_1 + \alpha_2) t} P_t \left( f_1 U^{\alpha_2} f_2 \right) dt$$

$$= U^{\alpha_1 + \alpha_2} (f_1 U^{\alpha_2} f_2)$$
and by an induction argument, for a general case. Indeed, if \( k > 2 \), then

\[
F(x) = \int \cdots \int_{0 < s_1 < \cdots < s_k < \infty} e^{-\sum_{j=1}^{k} \alpha_j s_j} e^{-\alpha_{k+1} s_{k+1}} \times \int_{E^{\otimes k}} \cdots (f_k(z_k) P_{s_{k+1} - s_k} f_{k+1}(z_k)) \\
\times P_{s_1}(x, dz_1) P_{s_2 - s_1}(z_1, dz_2) \cdots P_{s_k - s_{k-1}}(z_{k-1}, d(z_k) ds_1 \cdots ds_k ds_{k+1})
\]

\[
= \int \cdots \int_{0 < s_1 < \cdots < s_k < \infty} e^{-\sum_{j=1}^{k} \alpha_j s_j} \int_{E^{\otimes k}} f_1(z_1) \cdots \times \left( f_k(z_k) \int_{s_k}^{\infty} \int e^{-\alpha_{k+1} s_{k+1}} P_{s_{k+1} - s_k} f_{k+1}(z_k) ds_{k+1} \right) \\
\times P_{s_1}(x, dz_1) P_{s_2 - s_1}(z_1, dz_2) \cdots P_{s_k - s_{k-1}}(z_{k-1}, d(z_k) ds_1 \cdots ds_k)
\]

\[
= \int \cdots \int_{0 < s_1 < \cdots < s_k < \infty} e^{-\sum_{j=1}^{k} \alpha_j s_j} \int_{E^{\otimes k}} f_1(z_1) \cdots \times \left( f_k(z_k) e^{-\alpha_{k+1} s_k} \int_{0}^{\infty} e^{-\alpha_{k+1} t} P_{1} f_{k+1}(z_k) dt \right) \\
\times P_{s_1}(x, dz_1) P_{s_2 - s_1}(z_1, dz_2) \cdots P_{s_k - s_{k-1}}(z_{k-1}, d(z_k) ds_1 \cdots ds_k)
\]

\[
= \int \cdots \int_{0 < s_1 < \cdots < s_k < \infty} e^{-\sum_{j=1}^{k} \alpha_j s_j - (\alpha_{k+1} + \alpha_k) s_k} \times \int_{E^{\otimes k}} f_1(z_1) \cdots f_k(z_k) U^{\alpha_{k+1}} f_{k+1}(z_k) dt
\]

and the formula follows the induction assumption. \( \blacksquare \)

**Lemma 2.6** Let \( f_1, \cdots, f_k \in \mathcal{B}_b(E) \), \( \alpha_j \) positive numbers, and

\[
F(x) = E^{x} \left\{ \left( \prod_{j=1}^{k} \int_{0}^{\infty} e^{-\alpha_j s} f_j(X_s) ds \right) \right\}.
\]

Then

\[
F = \sum_{\{j_1, \cdots, j_k\} \in \pi_k} U^{\alpha_{j_1} + \cdots + \alpha_{j_k}} (f_{j_1}(U^{\alpha_{j_2} + \cdots + \alpha_{j_k}} f_{j_2} \cdots (U^{\alpha_{j_k}} f_{j_k}) \cdots)
\]

(2.6)

where \( \pi_k \) is the permutation group of \( \{1, \cdots, k\} \).
\textbf{Proof.} We have
\begin{align*}
F(x) &= E^x \left\{ \int_0^\infty \cdots \int_0^\infty e^{-\alpha_{1}s_1} \cdots e^{-\alpha_{k}s_k} f_1(X_{s_1}) \cdots f_k(X_{s_k}) ds_1 \cdots ds_k \right\} \\
&= \int_0^\infty \cdots \int_0^\infty e^{-\alpha_{1}s_1} \cdots e^{-\alpha_{k}s_k} E^x \left\{ f_1(X_{s_1}) \cdots f_k(X_{s_k}) \right\} ds_1 \cdots ds_k \\
&= \sum_{\{j_1, \ldots, j_k\} \in \pi_k} \int \cdots \int_{0 < s_{j_1} < \cdots < s_{j_k} < \infty} e^{-\alpha_{1}s_1} \cdots e^{-\alpha_{k}s_k} \\
&\quad \times E^x \left\{ f_{j_1}(X_{s_1}) \cdots f_{j_k}(X_{s_k}) \right\} ds_1 \cdots ds_k \\
&= \sum_{\{j_1, \ldots, j_k\} \in \pi_k} \int \cdots \int_{0 < s_1 < \cdots < s_k < \infty} e^{-\alpha_{j_1}s_1} \cdots e^{-\alpha_{j_k}s_k} \\
&\quad \times E^x \left\{ f_{j_1}(X_{s_1}) \cdots f_{j_k}(X_{s_k}) \right\} ds_1 \cdots ds_k \\
&= \sum_{\{j_1, \ldots, j_k\} \in \pi_k} \int \cdots \int_{0 < s_1 < \cdots < s_k < \infty} \int_{E_{s_k}} e^{-\alpha_{j_1}s_1} \cdots e^{-\alpha_{j_k}s_k} f_{j_1}(z_1) \cdots f_{j_k}(z_k) \\
&\quad \times P_{s_1}(x, dz_1) P_{s_2-s_1}(z_1, dz_2) \cdots P_{s_k-s_{k-1}}(z_{k-1}, dz_k) ds_1 \cdots ds_k
\end{align*}
and (2.6) follows from Lemma 2.5. \hfill \blacksquare

From now on, we assume that
\[ X = (\Omega, \mathcal{F}^0, \mathcal{F}^0_t, X_t, \theta_t, P^x) \]
is a continuous Hunt process in \( E' = E \cup \{ \emptyset \} \) with the transition semigroup \( (P_t)_{t \geq 0} \). In other words, it is a Hunt process with continuous sample paths. Therefore, \( (X_t)_{t \geq 0} \) is a diffusion process in \( E \), i.e. \( (X_t)_{t \geq 0} \) possesses the strong Markov property with continuous sample function. Under our assumptions, any finite \( (\mathcal{F}^\mu_t) \)-stopping time is accessible and thus predictable, and therefore \( \mathcal{F}^\mu_T = \mathcal{F}^\mu_{T-} \). In particular, \( (\mathcal{F}^\mu_t) \) is left continuous, and thus the filtration \( (\mathcal{F}^\mu_t) \) is continuous for any initial distribution \( \mu \).

Since any martingale on \( (\Omega, \mathcal{F}^\mu, \mathcal{F}^\mu_t, P^\mu) \) has a right continuous modification, by a martingale we always mean a martingale with right continuous sample function.

\textbf{Lemma 2.7} Suppose \( \xi = \xi_1 \cdots \xi_n \) where each \( \xi_j \) has the following form
\[ \xi_j = \int_0^\infty e^{-\alpha_j s} f_j(X_s) ds \]
where \( \alpha_j > 0 \) and \( f_j \in \mathcal{B}_b(E) \). Let \( M_t = E^\mu \{ \xi | \mathcal{F}^\mu_t \} \). Then \( (M_t)_{t \geq 0} \) is a bounded continuous martingale on \( (\Omega, \mathcal{F}^\mu, \mathcal{F}^\mu_t, P^\mu) \).

\textbf{Proof.} According to Lemma 2.4 we need only to show that for function of the following type
\[ F(x) = E^x \left\{ \left( \prod_{l=k+1}^n e^{-\alpha_{j_l} t} \int_0^\infty e^{-\alpha_{j_l} s} f_{j_l}(X_s) ds \right) \right\} , \]
\[ t \mapsto F(X_t) \] is continuous. By Lemma \[ \text{Lemma 2.6} \], \( F \) is an \( \alpha \)-potential, so that it is finely continuous, and together with Lemma \[ \text{Lemma 2.3} \] it implies that \( t \to F(X_t) \) is continuous, which completes the proof.

We now state the main result of this section. For simplicity, a square integrable martingale \( (M_t)_{t \geq 0} \) over \( (\Omega, \mathcal{F}, \mathcal{M}_t, P) \) means \( M_t = E(\xi | \mathcal{F}_t) \) with \( \xi \in L^2(\Omega, \mathcal{F}, P) \). This is equivalent to say \( \sup_{t > 0} E[M_t^2] < \infty \).

**Theorem 2.8** Let
\[ X = (\Omega, \mathcal{F}, \mathcal{M}_t, X_t, \theta_t, P^x) \]
be a continuous Hunt process in \( E \), and \( \mu \in \mathcal{P}(E) \). If \( \xi \in L^2(\Omega, \mathcal{F}, \mu) \), then the martingale \( M_t = E^\mu \{ \xi | \mathcal{F}_t^\mu \} \) is continuous, that is, square-integrable martingales on \( (\Omega, \mathcal{F}, \mathcal{M}_t, \mu) \) are continuous. Therefore local martingales on \( (\Omega, \mathcal{F}, \mathcal{M}_t, \mu) \) are continuous.

**Proof.** We can choose a sequence \( \xi_n \in \mathcal{C} \) such that \( \xi_n \to \xi \) in \( L^2 \). Doob’s maximal inequality implies that, if necessary by considering a subsequence, the martingales \( \{ E^\mu(\xi_n | \mathcal{F}_t^\mu) : t \geq 0 \} \) converges (almost surely at least along a subsequence) to \( \{ E^\mu(\xi | \mathcal{F}_t^\mu) : t \geq 0 \} \) uniformly on any finite interval of \( t \geq 0 \). It is shown in Lemma \[ \text{Lemma 2.7} \] that for each \( n \), the martingale \( E^\mu(\xi_n | \mathcal{F}_t^\mu) \) is continuous and thus the square integrable martingale \( \{ E^\mu(\xi | \mathcal{F}_t^\mu) : t \geq 0 \} \) must be continuous.

By the localization technique, it follows thus that local martingales on \( (\Omega, \mathcal{F}, \mathcal{M}_t, \mu) \) are continuous. \( \blacksquare \)

### 3 Martingale representation for continuous Hunt process

In this section we assume that
\[ X = (\Omega, \mathcal{F}, \mathcal{M}_t, X_t, \theta_t, P^x) \]
is a continuous Hunt process in the state space \( E' = E \cup \{ \partial \} \) with transition semigroup \( \{ P_t(x, dy) : t \geq 0 \} \), where \( E \) is a locally compact separable metric space. Let \( \mu \in \mathcal{P}(E) \) be an initial distribution.

If \( \alpha > 0 \) and \( f \in \mathcal{B}(E) \) then \( M^{\alpha, f}_t \) denotes the continuous martingale
\[ M^{\alpha, f}_t = E^\mu \left\{ \int_0^{\infty} e^{-\alpha s} f(X_s) ds | \mathcal{F}_t^\mu \right\}. \]

Recall that, if \( u \) is an \( \alpha \)-potential, i.e., \( u = U^\alpha f \) where \( f \in \mathcal{B}(E) \), then \( u(X_t) - u(X_0) \) is a continuous semimartingale on \( (\mathcal{F}, \mathcal{F}_t^\mu, P^\mu) \), and possesses Doob-Meyer’s decomposition
\[ u(X_t) - u(X_0) = M^{[u]}_t + A^{[u]}_t. \]
where
\[ M_t^{[u]} = \int_0^t e^{\alpha s} dM_s^{[u]}, \quad A_t^{[u]} = \int_0^t Lu(X_s) ds \]
and \( Lu = \alpha u - f \).

We make the following assumptions on the continuous Hunt process \( X \) started with an initial distribution \( \mu \in \mathcal{P}(E) \), and we call these assumptions the Fukushima representation property.

**Assumptions.** There is an algebra (a vector space which is closed under the multiplication of functions) \( K(E) \subset \mathcal{B}(E) \) which generates the Borel \( \sigma \)-algebra \( \mathcal{B}(E) \) and is invariant under \( U_\alpha \) for \( \alpha > 0 \), and there are finite many continuous martingales \( M^1, \ldots, M^d \) over \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \) such that the following conditions are satisfied:

1. For any potential \( u = U_\alpha f \) where \( \alpha > 0 \) and \( f \in K(E) \), the martingale part \( M^{[u]} \) of the semimartingale \( u(X_t) - u(X_0) \) has the martingale representation in terms of \( (M^1, \ldots, M^d) \), that is, there are predictable processes \( F^1, \ldots, F^d \) on \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \) such that
   \[ M_t^{[u]} = \sum_{j=1}^d \int_0^t F^j_s dM^j_s \quad P\text{-a.e.} \quad (3.1) \]

2. \( \langle M^j, M^i \rangle_t \) is strictly positive definite.

The first assumption means that the martingale \( M^{[u]} \) with \( u \) being a potential may be represented. The second condition ensures that the representation (3.1) is unique.

The Fukushima representation property is mainly an abstraction of the chain role for the martingale part of \( u(X_t) \). Indeed, if \( X_t = (X^1_t, \ldots, X^d_t) \) is a \( d \)-dimensional Brownian motion and \( u \) is an \( \alpha \)-potential with \( \alpha > 0 \), then \( u \) is smooth and by Itô’s formula
\[
u(X_t) - u(X_0) = \sum_{j=1}^d \int_0^t \frac{\partial u}{\partial x^j}(X_s) dX^j_s + \int_0^t \frac{1}{2} \Delta u(X_s) ds
\]
so that
\[ M_t^{[u]} = \sum_{j=1}^d \int_0^t \frac{\partial u}{\partial x^j}(X_s) dX^j_s. \]

One can easily see that the Brownian motion satisfies the Fukushima representation property.

**Theorem 3.1 (Martingale representation)** Let \( \mu \in \mathcal{P}(E) \). Suppose that the Fukushima representation property holds for \( X \) with a finite set of martingales \( (M^1, \ldots, M^d) \). For any square-integrable martingale \( N = (N_t)_{t \geq 0} \) on \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \), there are unique predictable processes \( (F^i_t) \) such that
\[ N_t - N_0 = \sum_{i=1}^d \int_0^t F^i_s dM^i_s \quad P\text{-a.e.} \]
Proof. The uniqueness follows from condition 2) in the Fukushima representation. We prove the existence. Take \( \xi \in L^2(\Omega, \mathcal{F}^\mu, P^\mu) \) such that \( N_t = E^\mu \{ \xi | \mathcal{F}_t^\mu \} \). Since \( \mathcal{C} \) is dense in \( L^2(\Omega, \mathcal{F}^\mu, P^\mu) \), so we first prove the martingale representation for \( \xi \in \mathcal{C} \). By the linearity, we only need to consider the case that \( \xi = \xi_1 \cdots \xi_n \) where \( \xi_j = \int_0^\infty e^{-\alpha_j s} f_j(X_s) ds \) for \( \alpha_j > 0 \) and \( f_j \in K(E) \). In this case, according to 2.4, Lemma 2.5 and Lemma 2.6

\[
N_t = E^\mu \{ \xi | \mathcal{F}_t^\mu \} = \sum_m Z_t^m
\]

where the sum is a finite one, and for each \( m \), \( Z_t^m = Z_t \) has the following form

\[
Z_t^m = V_t^m u_t^m(X_t)
\]

(the superscript \( m \) will be dropped if no confusion may arise), where

\[
V_t = \prod_{i=1}^{k'} \int_0^t e^{-\beta_i s} g_i(X_s) ds
\]

and

\[
u(x) = \int \cdots \int_{0<s_1<\cdots<s_k<\infty} e^{-\sum_{j=1}^k \beta_j s_j} \int_{E^{\otimes k}} h_1(z_1) \cdots h_k(z_k) P_{s_1}(x, dz_1) \]

\[
\times P_{s_2-s_1}(z_1, dz_2) \cdots P_{s_{k-1}-s_{k-2}}(z_{k-1}, dz_k) ds_1 \cdots ds_k
\]

for some \( k' \) and \( k, \beta_i > 0 \) and functions \( g_i, h_j \) are bounded and continuous. According to Lemma 2.5

\[
u = U^{\beta_1 + \cdots + \beta_k} \left( h_1(U^{\beta_2 + \cdots + \beta_k} h_2 \cdots (U^{\beta_k} h_k) \cdots) \right).
\]

In particular, \( u \) is again a potential which has a form \( u = U^\alpha g \) for

\[
g = h_1(U^{\beta_2 + \cdots + \beta_k} h_2 \cdots (U^{\beta_k} h_k) \cdots) \in K(E)
\]

and \( \alpha = \beta_1 + \cdots + \beta_k \). Hence \( u(X_t) \) is a continuous semimartingale with decomposition

\[
u(X_t) - u(X_0) = M_t^{[u]} + A_t^{[u]}
\]

where \( A_t^{[u]} \) is continuous with finite variation, and due to the Fukushima representation property

\[
M_t^{[u]} = \sum_j^{d} \int_0^t G^j_s dM^j_s
\]

for some predictable processes \( G^j \). In particular, each \( Z_t^m \) is a continuous semimartingale. Since, by Theorem 2.8, \( N \) is a continuous martingale, so that

\[
N_t = \sum_m \text{the continuous martingale part of } V_t^m u_t^m(X_t).
\]
Therefore we are interested in the martingale part of $Z_t = V_t u(X_t)$. Since $V$ is a finite variation process, so according to Itô’s formula

$$Z_t = Z_0 + \int_0^t u(X_s) dV_s + \int_0^t V_s dA_t^u$$

so that the martingale part of $Z_t$ is

$$\sum_{i=1}^d \int_0^t V_s \cdot G^i_s dM^i_s.$$

Therefore

$$N_t = E^\mu \{ \xi | \mathcal{F}_t^\mu \} = \sum_{i=1}^d \int_0^t \sum_m \gamma^m_i \cdot G^m_i dM^i_s$$

which shows the martingale representation.

Suppose now $\xi \in L^2(\Omega, \mathcal{F}, P)$. Choose a sequence $\xi_n \in \mathcal{C}$ such that $\xi_n \rightarrow \xi$ in $L^2(\Omega, \mathcal{F}, P)$. Let $N_t^{(n)} = E^\mu(\xi_n | \mathcal{F}_t^\mu)$ and $N_t = E^\mu(\xi | \mathcal{F}_t^\mu)$. According to Doob’s maximal inequality, if necessary by passing to a subsequence, we can assume that $N_t^{(n)}$ converges to $N_t$ uniformly on any finite interval. $N_t^{(n)}$ has the martingale representation

$$N_t^{(n)} - N_0^{(n)} = \sum_{j=1}^d \int_0^t F(n)^j_s dM^j_s$$

so that

$$\langle N_t^{(n)} - N_t^{(m)}, N_t^{(n)} - N_t^{(m)} \rangle = \sum_{i,j=1}^d \int_0^t (F(n)^i_s - F(m)^i_s)(F(n)^j_s - F(m)^j_s) d\langle M^i, M^j \rangle_s.$$

Since $\langle M^i, M^j \rangle_t$ is positive, it follows that $(F(n)^1, \cdots, F(n)^d)$ converges to predictable processes $(F^1, \cdots, F^d)$ under the norm

$$\|(F^1, \cdots, F^d)\| = \sum_{N=1}^\infty \sum_{i,j=1}^d E^\mu \left[ \int_0^N F^i_s F^j_s d\langle M^i, M^j \rangle_s \right].$$

Then

$$N_t - N_0 = \sum_{j=1}^d \int_0^t F^j_s dM^j_s.$$
This theorem claims that as long as every martingale of resolvent type is representable, so is any martingale. When is the Fukushima representation property satisfied? There are many examples. In the remain of this section, we shall give three interesting examples in symmetric situation.

Brownian motion with any initial distribution is certainly an example. Indeed, for Brownian motion in $\mathbb{R}^d$, we may choose $K(E) = C_0^\infty(\mathbb{R}^d)$ (the space of smooth functions which vanish at infinity), then for $f \in K(E)$, $U^\alpha f$ is smooth, and (3.1) follows from Itô’s formula applying to $U^\alpha f$. Theorem 3.1 gives a new proof for classical martingale representation theorem.

The second example is the reflecting Brownian motion. As Example 1.6.1 in [17], we consider Dirichlet form $(\frac{1}{2}D, H^1(D))$ on $L^2(D)$ where $D$ is the classical Dirichlet integral and $D$ is a bounded domain on $\mathbb{R}^d$. We further assume that any $x \in \partial D$ has a neighborhood $U$ such that $D \cap U = \{(x_i) \in \mathbb{R}^d : x_d > F(x_1, \ldots, x_{d-1})\} \cap U$ for some continuous function $F$. Then $C_0^\infty(\overline{D})$ (the space of restriction to $\overline{D}$ of functions in $C_0^\infty(\mathbb{R}^d)$) is dense in $H^1(D)$ (see [25] for details), i.e., $(\frac{1}{2}D, H^1(D))$ is a regular Dirichlet form on $L^2(\overline{D})$. The corresponding continuous Hunt process $X = (X_t, P^x)$ is called the reflecting Brownian motion. For $x = (x^i) \in \mathbb{R}^d$, we use $u_i(x) = x^i, 1 \leq i \leq d$, to denote the coordinate functions. Then $u_i \in \mathcal{F}$ and we denote by $M^i = M[u^i]$ the martingale part in Fukushima’s decomposition. It can be seen from Corollary 5.6.2 [17] that for any $u \in C_0^\infty(\overline{D})$,

$$M_t^{[u]} = \sum_{i=1}^d \int_0^t \frac{\partial u}{\partial x_i}(X_s) dM^i_s, \quad P^x\text{-a.s. for } x \in \overline{D},$$

where q.e. means ‘quasi-everywhere’, i.e., except a set of zero-capacity. Then a routine approximation procedure shows that for any $u \in H^1(D)$, there exist Borel measurable functions $\{f_i : 1 \leq i \leq d\}$ on $\overline{D}$ such that

$$M_t^{[u]} = \sum_{i=1}^d \int_0^t f_i(X_s) dM^i_s, \quad P^x\text{-a.s. for } x \in \mathbb{R}^d.$$

Therefore the reflecting Brownian motion has Fukushima representation property, by choosing $K(\overline{D})$ to be the space of bounded measurable functions and any initial distribution $\mu$ charging no set of zero capacity, i.e., a smooth distribution, because an exceptional set exists in above representation as is always when the process is constructed through a Dirichlet form. If the boundary is Lipschitz, then the transition function has density ([2]) and in this case, the exceptional set may be erased. Notice that under the current condition, the reflecting Brownian motion $X$ itself is not necessarily a semimartingale. The readers who are interested may refer to [2], [7] and [8] about when a reflecting BM is a semimartingale and the corresponding Skorohod decomposition. It should be pointed out that, although the martingale part of the reflected Brownian motion is a Brownian motion, but the martingale
representation property does not follow from the classical representation property for Brownian motion. The reason is that, as long as the boundary is not sufficiently smooth, the natural filtration \((\mathcal{F}_t^\mu)_{t \geq 0}\) is much bigger in general than the natural filtration generated by the martingale part \((M^1, \cdots, M^d)\) of \(X\).

Another example our main result may apply is symmetric diffusions in a domain killed at boundary. Actually Theorem 6.2.2 in [17] tells us that every continuous symmetric Hunt process with a smooth core enjoys the Fukushima representation property. More precisely let \(D\) be a domain of \(\mathbb{R}^d\) with continuous boundary \(\partial D\) and \(m\) a Radon measure on \(D\). Let \(X\) be a continuous Hunt process which is symmetric with respect to \(m\) and \((\mathcal{E}, \mathcal{F})\) the associated Dirichlet form on \(L^2(D, m)\), which has \(C_0^1(D)\) as a core. For \(x = (x^i) \in \mathbb{R}^d\), we use \(u_i(x) = x^i, 1 \leq i \leq d\), to denote the coordinate functions. Then \(u_i \in \mathcal{F}_{\text{loc}}\) and we denote by \(M^i = M^i[u_i]\) the martingale part in Fukushima’s decomposition. Let \(\mu_{i,j} = \mu_{\langle M^i, M^j \rangle}\), \(1 \leq i, j \leq d\), the smooth measure associated with CAF \(\langle M^i, M^j \rangle\). Then \(\mathcal{E}\) is expressed as

\[
\mathcal{E}(u, v) = \sum_{i,j=1}^d \int_D \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} d\mu_{i,j}(x), \ u, v \in C_0^1(D).
\]

As asserted in Theorem 6.2.2 [17], for any initial smooth distribution \(\mu\) (i.e. a probability on \((D, \mathcal{B}(D))\) having no charge on capacity zero sets) and \(u \in \mathcal{F}\), the martingale part \(M^u\) in Fukushima’s decomposition of \(u\) may be represented as

\[
M^u_t = \sum_{i=1}^d \int_0^t f_i(X_s) dM^i_s \quad \mathbb{P}^\mu\text{-a.e.}
\]

where \(f_1, \cdots, f_d \in \mathcal{B}(D)\). If we take \(K(E) = L^2(E, m) \cap \mathcal{B}(D)\), \(X\) satisfies the Fukushima representation property. In these examples, \(\{M^i\}\) are the martingales corresponding to coordinate functions so we call them coordinate martingales.

To have the uniqueness, some kind of non-degenerateness is needed. We say that \(X\) is non-degenerate if the condition (2) in Fukushima representation property is satisfied: \((\langle M^i, M^j \rangle)_{1 \leq i, j \leq d}\) is positive.

**Corollary 3.2** Assume that \(X\) is either the reflecting Brownian motion on a bounded domain or a non-degenerate symmetric Hunt diffusion on a domain \(D \subset \mathbb{R}^d\) as stated above. Then the Fukushima representation property is satisfied and therefore the martingale representation holds in the sense of Theorem 3.1 with coordinate martingales and for a given initial distribution \(\mu\) charging no sets of zero capacity.

From this result, we may recover the martingale representation established in [1] and [33], where \(X\) is a diffusion process corresponding to non-degenerate symmetric elliptic operator on \(\mathbb{R}^d\).

Without essential difference, the conclusion holds also for reflecting diffusions on such domain with generator being a symmetric uniformly elliptic differential operator of second order as introduced in the beginning of next section.
4 Backward stochastic differential equations

In this section we consider backward stochastic differential equations which can be used to provide probability representations for weak solutions of the initial and boundary value problem of a quasi-linear parabolic equation.

Let $D \subset \mathbb{R}^d$ be a bounded domain with a continuous boundary $\partial D$, $\overline{D} = D \cup \partial D$ the closure of $D$. Let

$$L = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j} a^{ij} \frac{\partial}{\partial x^i}$$

be an elliptic differential operator of second order, where $a = (a^{ij})$ is a positive-definite, symmetric, matrix-valued function on $D$, $a = (a^{ij})$ is Borel measurable, and satisfies the elliptic condition:

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2 \quad \forall \xi = (\xi_i) \in \mathbb{R}^d$$

for all $x \in D$ for some constant $\lambda > 0$. Consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D, dx)$, where

$$\mathcal{E}(u,v) = \frac{1}{2} \int_D \sum_{i,j=1}^{d} a^{ij} \frac{\partial u}{\partial x^j} \frac{\partial v}{\partial x^i}$$

and $\mathcal{F} = H^1(D)$.

Let $\Omega$ be a space of all continuous paths in $\overline{D}$, $(X_t)_{t \geq 0}$ the coordinate process on $\Omega$, $\mathcal{F}^0 = \sigma\{X_s : s \geq 0\}$, $\mathcal{F}^t_0 = \sigma\{X_s : s \leq t\}$ for each $t \geq 0$, and $(\theta_t)_{t \geq 0}$ shift operators on $\Omega$. Let $X = (\Omega, \mathcal{F}^0, \mathcal{F}^t_0, X_t, \theta_t, P^x)$ be the canonical realization of the symmetric diffusion process in the state space $\overline{D}$ associated with the Dirichlet space $(\mathcal{E}, \mathcal{F})$, which is called a reflecting symmetric diffusion in $D$.

The coordinate functions $u_j(x) = x^j$ ($j = 1, \cdots, d$) belong to the local Dirichlet space $\mathcal{F}_{loc}$, so that

$$X_t^j - X_0^j = M_t^j + A_t^j \quad P^x\text{-a.e.} \quad j = 1, \cdots, d \quad (4.2)$$

for all $x \in \overline{D}$ except for a capacity zero set, where $M^j = M_{[j]}$ etc.

Let $\mathcal{J}_1(\overline{D})$ denote the space of all probability $\mu \in \mathcal{P}(\overline{D})$ which has no charge on zero capacity sets (with respect to the Dirichlet form $(\mathcal{E}, H^1(\overline{D}))$ defined by (4.1)). According to Theorem 3.1, for any initial distribution $\mu \in \mathcal{J}_1(\overline{D})$, the family of martingales $\{M^j : j = 1, \cdots, d\}$ over $(\Omega, \mathcal{F}, \mathcal{F}_t^\mu, P^\mu)$ has the martingale representation property: for any square-integrable martingale $N = (N_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}_t^\mu, P^\mu)$, there are unique predictable processes $(F_t^i)$ such that

$$N_t - N_0 = \sum_{i=1}^{d} \int_0^t F_s^i dM_s^i \quad P^\mu\text{-a.e.}$$
Let us work with a fixed smooth initial distribution \( \mu \in S_1(D) \) and the filtered probability space \((\Omega, \mathcal{F}^\mu, \mathcal{F}^\mu_t, P^\mu)\).

Consider the following backward stochastic differential equation

\[
dY^i_t = -f^i(t, Y^i_t, Z^i_t)dt + \sum_{i,j=1}^d Z^{ij}_t dM^j_t, \quad Y^i_T = \xi^i (4.3)
\]

\(i = 1, \cdots, d', \) where \(T > 0, \xi^i \in L^2(\Omega, \mathcal{F}^\mu_T, P^\mu) \) are given terminal values, and \(f^i \) are Lipschitz functions: there is a constant \(C_1 \geq 0\)

\[
|f^i(t, y, z)| \leq C_1 (1 + t + |y| + |z|)
\]

and

\[
|f^i(t, y, z) - f^i(t, \tilde{y}, \tilde{z})| \leq C_1 (|y - \tilde{y}| + |z - \tilde{z}|)
\]

for all \(t \geq 0, y, \tilde{y} \in \mathbb{R}^{d'}, z, \tilde{z} \in \mathbb{R}^{d' \times d}. \) One seeks for a solution pair \((Y, Z)\) which solves the following integral equation

\[
Y^i_t - \xi^i = \int_t^T f^i(s, Y^i_s, Z^i_s)ds - \sum_{j=1}^d \int_t^T Z^{ij}_s dM^j_s \quad (4.4)
\]

for \(t \in [0, T]. \) The integral equation (4.4) has a unique solution pair \((Y, Z)\) such that \(Y^i\) is a continuous semimartingale, and \(Z^{ij}\) are predictable processes satisfying

\[
E^\mu \int_0^T \sum_{k,l=1}^d a^{kl}(X_s) Z^{ij}_s Z^{ik}_s ds < \infty.
\]

This can be demonstrated by employing the Picard iteration for \((Y, Z)\) as in the case of Brownian motion (see [28]). Another approach, proposed in a paper by Lyons, Liang and Qian [21] which applies to a general filtered probability space, may be described as follows. The idea is to rewrite the integral equation (4.4) into a functional differential equation for the variation process part \(V\) of \(Y.\) Let \(Y = N - V\) where \(V\) is a finite variation process, and

\[
N^i_t - N^i_0 = \sum_{j=1}^d \int_0^t Z^{ij}_s dM^j_s.
\]

On the other hand

\[
N_t = E^\mu \{\xi + V_T | \mathcal{F}^\mu_t\}.
\]

Since \(Y\) is a continuous semimartingale, its decomposition is unique up to an initial value. The integral equation (4.4) leads to that

\[
V_t = -\int_t^T f^i(s, Y^i_s, Z^i_s)ds + N_T - \xi
\]
conditioned on $\mathcal{F}_t^\mu$ and we obtain
\begin{align*}
V_t &= -E^\mu \left\{ \int_t^T f^i(s, Y_s, Z_s)ds \mid \mathcal{F}_t^\mu \right\} + N_t - E^\mu \{ \xi \mid \mathcal{F}_t^\mu \} \\
&= -E^\mu \left\{ \int_t^T f^i(s, Y_s, Z_s)ds \mid \mathcal{F}_t^\mu \right\} + E^\mu \{ V_T \mid \mathcal{F}_t^\mu \} \\
&= -E^\mu \left\{ \int_0^T f^i(s, Y_s, Z_s)ds - V_T \mid \mathcal{F}_t^\mu \right\} + \int_0^t f^i(s, Y_s, Z_s)ds.
\end{align*}

Therefore the integral equation (4.4) is equivalent to
\begin{equation}
V_t - V_0 = \int_0^t f^i(s, Y_s, Z_s)ds \tag{4.5}
\end{equation}

where
\begin{align*}
Y_t &= Y(V)_t = N(V)_t - V_t, \quad N(V)_t = E^\mu \{ \xi + V_T \mid \mathcal{F}_t^\mu \}
\end{align*}

and $Z_t = Z(V)_t$ is determined by the martingale representation theorem
\[ N(V)_t = E^\mu \{ \xi + V_T \mid \mathcal{F}_t^\mu \} \]

Equation (4.5) thus may be written as a functional equation
\begin{equation}
V_t - V_0 = \int_0^t f^i(s, Y(V)_s, Z(V)_s)ds \tag{4.6}
\end{equation}

where $Y(V)$ and $Z(V)$ are considered as functionals of $V$. The Picard iteration applies to (4.6) we have

**Theorem 4.1** If $\xi \in L^2(\Omega, \mathcal{F}_T^\mu, P^\mu)$ and $f^i$ are Lipschitz continuous, then there is a unique pair $(Y, Z)$ such that $Y$ is a continuous semimartingale which solves BSDE (4.3).

For a complete proof of Theorem 4.1, the reader may refer to [21].

5 Non-linear parabolic equations

We are under the same setting as in the previous section, and use the notations established therein.

To motivate our approach, let us begin with the case that $a$ is smooth, and $D$ is bounded domain with a smooth boundary.

In this case $X^j = (X^j_t)_{t \geq 0}$ in (4.2) are continuous semimartingales, thus $A^j$ are finite variation processes. For any $h \in C^1_b([0, \infty) \times \overline{D})$ satisfying the Neumann boundary condition

\[ \frac{\partial h}{\partial n} = h \]
that \( \frac{\partial h}{\partial \nu} \big|_{\partial D} = 0 \), where \( \frac{\partial}{\partial \nu} \) denotes the normal derivative with respect to the Riemann metric \((a^{ij}) = (a_{ij})^{-1}\), we have

\[
h(t, X_t) - h(0, X_0) = M_t^h + A_t^h
\]

where

\[
M_t^h = h(t, X_t) - h(0, X_0) - \int_0^t \left( \frac{\partial}{\partial s} + L \right) h(s, X_s) ds
\]

is a martingale under \( P^x \), and

\[
A_t^h = \int_0^t \left( \frac{\partial}{\partial s} + L \right) h(s, X_s) ds.
\]

On the other hand, applying Itô’s formula

\[
h(t, X_t) - h(0, X_0) = \int_0^t \left( \frac{\partial}{\partial s} + \frac{1}{2} \sum_{i,j=1}^d a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \right) h(s, X_s) ds
\]

\[
+ \int_0^t \sum_{j=1}^d \frac{\partial}{\partial x^j} h(s, X_s) d(M^j_s + A^j_s)
\]

it thus follows that

\[
M_t^h = \sum_{j=1}^d \int_0^t \frac{\partial}{\partial x^j} h(s, X_s) dM^j_s
\]

and

\[
A_t^i = \frac{1}{2} \int_0^t \sum_{j=1}^d \frac{\partial}{\partial x^j} a^{ij}(X_s) ds.
\]

Consider a solution \( u(x, t) \) to the initial boundary problem to the non-linear parabolic equation

\[
\left\{
\begin{array}{l}
\left( \frac{\partial}{\partial t} - L \right) u + f(t, u, \nabla u) = 0, \\
u(0, x) = \varphi(x), \\
\frac{\partial u(t, \cdot)}{\partial \nu} \big|_{\partial D} = 0,
\end{array}
\right. \quad x \in \mathbb{R}^d,
\]

Then, by (5.1) and (5.3)

\[
h(T, X_T) = h(t, X_t) + \int_t^T \left( \frac{\partial}{\partial s} + L \right) h(s, X_s) ds
\]

\[
+ \sum_{j=1}^d \int_t^T \frac{\partial}{\partial x^j} h(s, X_s) dM^j_s
\]
together with the PDE (5.5) we deduce that
\[
h(t, X_t) - h(T, X_T) = \int_t^T f(T - s, h(s, X_s), \nabla h(s, X_s)) ds \\
- \sum_{j=1}^d \int_t^T \frac{\partial}{\partial x^j} h(s, X_s) dM_s^j.
\] (5.6)

Let \( Y_t = u(T - t, X_t) \) and \( Z_t^j = \frac{\partial}{\partial x^j} h(t, X_t) \). Then the previous equation may be written as
\[
Y_t - Y_T = \int_t^T f(T - s, Y_s, Z_s) ds - \sum_{j=1}^d \int_t^T Z_s^j dM_s^j
\] (5.7)
and \( Y_T = u(0, X_T) = \varphi(X_T) \). That is to say that \( Y_t = u(T - t, X_t) \) solves the scalar BSDE
\[
dY_t = -f_T(t, Y_t, Z_t) dt + \sum_{j=1}^d Z_t^j dM_t^j, \quad Y_T = \varphi(X_T)
\] (5.8)

where \( f_T = f(T - t, y, z) \).

For any fixed \( T > 0 \), let \( Y^T = \{Y_t^T: t \in [0, T]\} \) be the unique solution to the BSDE (5.8) on \( (\Omega, \mathcal{F}^\mu, \mathcal{F}_t^\mu, P^\mu) \). Since the solution to BSDE is unique, \( Y_t^T = u(T - t, X_t) \). In particular \( u(T, X_0) = Y_0^T \), and therefore
\[
\int_{\mathbb{R}^d} u(T, x) \mu(dx) = E^\mu (Y_0^T).
\] (5.9)

The above argument leading to the probabilistic representation (5.9) can not be justified in the case that \( a = (a^{ij}) \) is only Borel measurable or the boundary \( \partial D \) is only continuous, as in this case, \( (X_t)_{t \geq 0} \) is no longer a semimartingale, both (5.2) and (5.4) no longer make sense. While, in this case, boundary problem (strong or weak solutions) to the non-linear PDE (5.5) also need to be interpreted. On the other hand, the BSDE (5.8), which relies on only the martingale representation, still make sense, thus the representation theorems stated in §3 can be made as the definition of a solution to (5.5). This is the approach we will carry out.

Consider the initial value problem of the following non-linear parabolic equation in a bounded domain \( D \) with a continuous boundary \( \partial D \)
\[
\left( \frac{\partial}{\partial t} - \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x^i} a^{ij}(x) \frac{\partial}{\partial x^j} \right) u + f(t, u, \nabla u) = 0
\] (5.10)
subject to the initial and boundary conditions
\[
u(x, 0) = \varphi(x), \quad \left. \frac{\partial}{\partial \nu} u(t, \cdot) \right|_{\partial D} = 0 \text{ for } t > 0
\]
where \( a = (a^{ij}) \) is Borel measurable, satisfying the uniform ellipticity condition:

\[
\lambda \sum_{i=1}^{d} |\xi^i|^2 \leq \sum_{i,j} \xi^i \xi^j a^{ij}(x) \leq \lambda^{-1} \sum_{i=1}^{d} |\xi^i|^2 \quad \forall (\xi^i) \in \mathbb{R}^d,
\]

for some constant \( \lambda > 0 \).

**Definition 5.1** The functional on \( \mathcal{S}_1(\overline{D}) \) defined by \( \mu \to E^\mu \{ Y(t, \mu) \} \), denoted by \( u(t, \mu) \), is called the stochastic solution of the initial and boundary problem of (5.10), where for each \( t > 0 \) and \( \mu \in \mathcal{S}_1(\overline{D}) \), \( Y(t, \mu) = (Y_s)_{s \leq t} \) is the unique solution to the BSDE

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
dY_s &= -f(t-s, Y_s, Z_s)ds + \sum_{j=1}^{d} Z_j^s dM_j^s, \\
Y_t &= \varphi(X_t)
\end{array}
\right. 
\end{aligned}
\]

(5.11) on \((\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^\mu)\).

As a consequence we have

**Theorem 5.2** If \( \varphi \) is bounded and Borel measurable on \( \overline{D} \), and \( f \) is Lipschitz continuous, then there is a unique stochastic solution to the non-linear parabolic equation (5.10).

We will study the regularity theory of the stochastic solutions in a separate paper. On the other hand we would like to derive an alternative probability representation of the stochastic solution.

Let us apply the approach outlined in [21]. Let \( Y_s = N_s - V_s \) where

\[
N_s - N_0 = \sum_{j=1}^{d} \int_0^s Z_j^r dM_j^r.
\]

Then \( V = (V_s)_{s \in [0,t]} \) is the unique solution to the functional differential equation

\[
V_s = \int_0^s f(t-r, Y(V)_r, Z(V)_r)dr, \quad V_0 = 0
\]

(5.12)

where \( N(V)_s = -E^\mu \{ \varphi(X_t) + V_t | \mathcal{F}_s^\mu \} \),

\[
Y(V)_s = E^\mu \{ \varphi(X_t) + V_t | \mathcal{F}_s^\mu \} - V_s
\]

for \( s \in [0,t] \), and \( Z(V) \) is given as the density process of \( N(V) \) in the martingale representation. In particular

\[
Y(V)_0 = E^{\mu} \{ \varphi(X_t) + V_t | \mathcal{F}_0^\mu \}.
\]

We therefore have the following
**Theorem 5.3** Let $\varphi$ be bounded and measurable. For $t > 0$, let $V(t)$ be the unique solution to the functional differential equation (5.12). Then the stochastic solution to the Neumann boundary problem of the non-linear PDE (5.10) is given by

$$u(t, \mu) = E^{\mu} \{ \varphi(X_t) + V(t) \} \quad \forall \mu \in \mathcal{F}_1(\mathcal{D}).$$

(5.13)

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