Quasi-energy-independent solar neutrino transitions

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Abstract

Current solar, atmospheric, and reactor neutrino data still allow oscillation scenarios where the squared mass differences are all close to $10^{-3}\ eV^2$, rather than being hierarchically separated. For solar neutrinos, this situation (realized in the upper part of the so-called large-mixing angle solution) implies adiabatic transitions which depend weakly on the neutrino energy and on the matter density, as well as on the “atmospheric” squared mass difference. In such a regime of “quasi-energy-independent” (QEI) transitions, intermediate between the more familiar “Mikheyev-Smirnov-Wolfenstein” (MSW) and energy-independent (EI) regimes, we first perform analytical calculations of the solar $\nu_e$ survival probability at first order in the matter density, beyond the usual hierarchical approximations. We then provide accurate, generalized expressions for the solar neutrino mixing angles in matter, which reduce to those valid in the MSW, QEI and EI regimes in appropriate limits. Finally, a representative QEI scenario is discussed in some detail.

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I. INTRODUCTION

The evidence in favor of neutrino mass and mixing [1] coming from the atmospheric anomaly [2–5] and from the solar neutrino deficit [6–13], as well as the constraints on $\nu_e$ mixing from recent reactor searches [14,15], are being actively investigated both experimentally and theoretically. From the theoretical point of view, in the absence of new neutrino interactions or new (sterile) states, the mass-mixing parameters are characterized by the (unitary) mixing matrix $U$ between the flavor states $\nu_{\alpha}$ ($\alpha = 1, 2, 3$) and the mass states $\nu_i$ ($i = 1, 2, 3$),

$$\nu_{\alpha} = \sum_i U_{\alpha i} \nu_i ,$$

and by two independent squared mass differences, that we define as

$$\delta m^2 = m_2^2 - m_1^2 ,$$
$$m^2 = m_3^2 - \frac{m_1^2 + m_2^2}{2} ,$$

as graphically shown in Fig. 1. The two possible independent spectra in Fig. 1 are formally distinguished by the sign of $m^2$, while the sign of $\delta m^2$ can always be taken positive without loss of generality (see also Sec. III).

From the experimental point of view, combined solar and reactor data analyses [16–18] imply an upper limit on the smallest squared mass gap $\delta m^2$,

$$\text{solar+reactor } \nu : \delta m^2 \lesssim 0.7 \times 10^{-3} \text{ eV}^2 .$$

Conversely, atmospheric data analyses [3,19,16] imply a lower limit on the remaining (larger) squared mass gap $m^2$,

$$\text{atmospheric } \nu : |m^2| \gtrsim 1.5 \times 10^{-3} \text{ eV}^2 .$$

Equations (4) and (5) thus imply the current phenomenological limit

$$\delta m^2 \lesssim \frac{1}{2} |m^2| .$$

The above constraint is largely fulfilled in several “hierarchical” oscillation scenarios, where

$$\text{hierarchical cases } \longleftrightarrow \delta m^2 \ll |m^2| ,$$

and the zeroth order approximation in the ratio $\delta m^2/|m^2|$ is usually very accurate for both solar [20] and laboratory [21] neutrino calculations. At present, however, one cannot exclude that the limit in Eq. (6) is saturated, namely, that $\delta m^2$ and $|m^2|$ differ by less than an order of magnitude. This case, recently considered in [22] for atmospheric neutrinos, is of great interest for laboratory oscillations searches (e.g., at future neutrino factories), where it might lead to detectable CP-violating effects.
Concerning solar neutrinos, the situation characterized by both $\delta m^2$ and $m^2$ approaching $10^{-3} \text{ eV}^2$ (from below and from above, respectively) can affect the upper part of the so-called large mixing angle (LMA) solution of the solar neutrino problem \cite{20,16,23}. Such a situation, being relatively close to the regime of “energy-independent” (EI) transitions in vacuum \cite{24} (established for both $\delta m^2$ and $|m^2|$ hypothetically above $\sim 10^{-3} \text{ eV}^2$) can be called of “quasi-energy-independent” (QEI) transitions

$$\text{QEI} \longleftrightarrow \delta m^2 \sim |m^2| \sim O(10^{-3}) \text{ eV}^2,$$

and is characterized by a mild dependence on matter effects (“low density” regime) and on neutrino energy. Matter effects become increasingly larger for lower values of $\delta m^2$ ($|m^2|$ being fixed by atmospheric neutrino data), in the familiar regime of Mikheyev-Smirnov-Wolfenstein (MSW) transitions \cite{25}, where the hierarchical approximation \cite{7} can be applied.\footnote{For even lower values of $\delta m^2$, matter effects play again a subdominant role (quasivacuum oscillation regime \cite{28}), but for opposite reasons (relatively high matter density).}

At the relatively high values of $\delta m^2$ and $m^2$ implied by the QEI regime [Eq. (8)], oscillation phases are large and unobservable, flavor transitions are adiabatic, and the calculation of the $\nu_e$ survival probability is reduced to the calculation of the mixing matrix elements in matter $\tilde{U}_{ei}$. Although some analytical approximations for the $\tilde{U}_{ei}$’s at low density have been studied in early works on three-flavor oscillations\footnote{An exact analytical diagonalization of the three-flavor neutrino Hamiltonian in matter is also possible \cite{27,28} but, unfortunately, the results are not particularly transparent.} (see, e.g., \cite{27,29,30} and refs. therein), we think it useful to revisit and complete such studies, especially in order to remove restrictive hypotheses that have often been used (e.g., the assumption of sizable hierarchy $\delta m^2 < m^2$ \cite{29} or of small mixing angles \cite{30}).

Our paper is structured as follows: In Sec II we derive analytical expressions for the $\tilde{U}_{ei}$’s in the QEI regime at first order in the matter density, with no restrictive assumptions about the neutrino mass hierarchy or mixing, and without using a specific parametrization. In Sec. III we show how to embed such results in generalized expressions for the mixing angles in matter, which smoothly connect the familiar MSW regime (for $\delta m^2 \ll |m^2|$) and the QEI regime (where $\delta m^2 \sim |m^2|$), up to the EI regime. Such expressions (written in standard parametrization) may be used to improve the calculation of the solar neutrino oscillation probability in the high-$\delta m^2$ fraction of the LMA solution. Finally, we discuss in Sec. IV a specific QEI scenario compatible with present reactor data, and draw our conclusions in Sec.V.

II. PARAMETRIZATION-INDEPENDENT CALCULATIONS

Oscillations in matter are affected by the $\nu_e$ interaction potential $v$ at the position $x$,

$$v(x) = \sqrt{2} G_F N_e(x),$$

(9)
where \( N_e \) is the local electron density \([25]\). Matter effects are strong (MSW regime) when \( v \) is of the order of (at least) one of the wavenumber differences \(|k_i - k_j|\), where

\[
k_i = \frac{m_i^2}{2E},
\]

\( E \) being the neutrino energy. In the QEI regime for solar neutrinos [Eq. (8)], for typical neutrino energies, the ratio \( v(x)/|k_i - k_j| \) is instead (often much) smaller than unity for \( x \) even in the solar core (regime of “low density”). Moreover, variations of \( v \) along one oscillation wavelength are extremely small, and the three-family \( \nu_e \) survival probability takes the adiabatic form

\[
P^{3\nu}_{ee} = \tilde{U}^2_{e1} U^2_{e1} + \tilde{U}^2_{e2} U^2_{e2} + \tilde{U}^2_{e3} U^2_{e3}
\]

(see \([29]\) and refs. therein), where the \( \tilde{U}_{ei} \)'s represent the mixing matrix elements at the production point \([4]\) and we have taken the three \( \nu_e \) mixing matrix elements as real \([4]\).

The goal of this Section is to calculate the elements \( \tilde{U}_{ei} \) at first order in the small parameters \( v/|k_i - k_j| \). In Sec. IIA we discuss in some detail the spectral decomposition of the Hamiltonian (only rarely used \([27,31]\) in the neutrino literature), and in Sec. IIB we apply it to the calculation of the \( \tilde{U}_{ei} \)'s at first order in the matter potential \([5]\). No specific parametrization for \( \nu \) masses and mixing is used in Sections IIA and IIB.

**A. Spectral decomposition of the Hamiltonian**

The neutrino Hamiltonian \( H \) in the flavor basis \((\nu_e, \nu_\mu, \nu_\tau)\) can be defined as

\[
H = UKU^T + V,
\]

\[
= \tilde{U}\tilde{K}\tilde{U}^T,
\]

where \( U (\tilde{U}) \) is the mixing matrix in vacuum (matter), and

\[
V = \text{diag}(v, 0, 0),
\]

\[
K = \text{diag}(k_1, k_2, k_3),
\]

\[
\tilde{K} = \text{diag}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3),
\]

\[3\] As far as the QEI regime is concerned, the (very low) Earth matter density at the detection point can be neglected (vacuum approximation).

\[4\] This convention does not imply a loss of generality. For complex \( U \), the only difference in our QEI results would be the replacement of \( U_{ei}^2 \) with \( |U_{ei}|^2 \) (and analogously for \( \tilde{U} \)). However, one can always choose a parametrization in which the \( |U_{ei}|^2 \) do not depend explicitly on the CP violating phase. In order to avoid unnecessary book-keeping of \( U^* \) terms in the text, we prefer then to take \( U \) real from the beginning.

\[5\] We use the more compact notation \( O(v^n) \) as a substitute of \( O(v^n/|k_i - k_j|^n) \).
where the $k_\alpha$’s are given in Eq. (10), while the neutrino wavenumbers in matter $\tilde{k}_i$ represent the eigenvalues of $H$.

In flavor components, Eq. (13) reads

$$H_{\alpha\beta} = \sum_i \tilde{k}_i U_{\alpha i} \tilde{U}_{\beta i},$$

(17)

and thus the products $\tilde{U}_{\alpha i} \tilde{U}_{\beta i}$ can be identified with the matrix elements of the projector operators $Q^i$ acting on the one-dimensional space spanned by the $i$-th eigenvector,

$$Q^i_{\alpha\beta} = \tilde{U}_{\alpha i} \tilde{U}_{\beta i},$$

(18)

which admit the following factorization (also called spectral decomposition theorem of linear algebra):

$$Q^i_{\alpha\beta} = \prod_{j \neq i} \frac{\tilde{k}_j \delta_{\alpha\beta} - H_{\alpha\beta}}{\tilde{k}_j - \tilde{k}_i}. $$

(19)

After algebraic manipulations (omitted), which make use of the following two invariants of the $H$ matrix:

$$\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 = H_{ee} + H_{\mu\mu} + H_{\tau\tau},$$

(20)

$$\tilde{k}_1 \tilde{k}_2 + \tilde{k}_2 \tilde{k}_3 + \tilde{k}_3 \tilde{k}_1 = H_{ee} H_{\mu\mu} + H_{\mu\mu} H_{\tau\tau} + H_{\tau\tau} H_{ee} - (H_{\mu\tau}^2 + H_{\tau\mu}^2 + H_{\mu\mu}^2),$$

(21)

Eq. (19) can be cast in the form [31]

$$Q^i_{\alpha\alpha} = \frac{(\tilde{k}_i - H_{\beta\beta})(\tilde{k}_i - H_{\gamma\gamma}) - H_{\beta\gamma}^2}{(\tilde{k}_i - \tilde{k}_j)(\tilde{k}_i - \tilde{k}_n)}.$$ 

(22)

for the diagonal elements, and to

$$Q^i_{\alpha\beta} = \frac{H_{\alpha\beta}(\tilde{k}_i - H_{\gamma\gamma}) + H_{\alpha\gamma} H_{\beta\gamma}}{(\tilde{k}_i - \tilde{k}_j)(\tilde{k}_i - \tilde{k}_n)}$$

(23)

for the off-diagonal elements, where $(\alpha, \beta, \gamma)$ are permutations of $(e, \mu, \tau)$ and $(i, j, n)$ are permutations of $(1, 2, 3)$.7 By comparing Eq. (18) with Eq. (22), one finally gets an explicit expression for the $\tilde{U}_{ei}^2$’s as a function of the eigenvalues of $H$ and of its $(\mu, \tau)$ submatrix elements,

$$\tilde{U}_{ei}^2 = \frac{(\tilde{k}_i - H_{\mu\mu})(\tilde{k}_i - H_{\tau\tau}) - H_{\mu\tau}^2}{(\tilde{k}_i - \tilde{k}_j)(\tilde{k}_i - \tilde{k}_n)}.$$ 

(24)

6The third (unused) invariant is $\tilde{k}_1 \tilde{k}_2 \tilde{k}_3 = \det H$.

7Notice that, in the first of Ref. [31], there is a sign misprint in the expression of $Q^i_{\alpha\beta}$. 

5
B. Expressions for $\bar{U}$ valid at first order in $v$

At first order in $v$, the eigenvalues $\bar{k}_i$ of $H$ are most easily calculated in the vacuum mass basis $\nu_i$, where the Hamiltonian is

$$ H' = K + U^T V U , $$

(25)
and the eigenvalue equation $[\det(H' - \bar{k} I) = 0]$ turns out to be already factorized,

$$ \prod_i (k_i - \bar{k}_i + v U_{ei}^2) + O(v^2) = 0 , $$

(26)
leading to the (known, see [27]) result

$$ \bar{k}_i = k_i + v U_{ei}^2 + O(v^2) . $$

(27)

By means of Eq. (27) and Eq.(24) we finally obtain, after somewhat lengthy but straightforward algebra,

$$ \bar{U}_{ei}^2 = U_{ei}^2 \left[ 1 + 2v \left( \frac{U_{ej}^2}{k_i - k_j} + \frac{U_{en}^2}{k_i - k_n} \right) \right] + O(v^2) , $$

(28)
where $(i,j,n)$ are permutations of $(1,2,3)$. This is our basic result in the QEI regime.

Notice that Eq. (28) depends explicitly on the squared mass differences $m_i^2 - m_j^2$, as it should, without restrictive (i.e., hierarchical) assumptions about their relative magnitude. Notice also that Eq. (28) holds for generic values of the elements $U_{ei}^2$ (within the unitarity constraint $\sum_i U_{ei}^2 = 1$).

III. QEI RESULTS IN STANDARD PARAMETRIZATION

In the standard notation for the neutrino mixing matrix $U$ [29],

$$ U = U(\theta_{12}, \theta_{13}, \theta_{23}) = U(\omega, \varphi, \psi) . $$

(29)
the mixing matrix elements relevant to solar neutrinos read

$$ U_{e1}^2 = \cos^2 \omega \cos^2 \varphi , $$

(30)
$$ U_{e2}^2 = \cos^2 \varphi \sin^2 \omega , $$

(31)
$$ U_{e3}^2 = \sin^2 \varphi . $$

(32)

By making the further assumption of small $U_{e2}^2$ and $U_{e3}^2$, and by applying the parametrization adopted in [30] for the matrix $U$, Eq. (28) reproduces the results found at first order in $(v, U_{e2}^2, U_{e3}^2)$ in [30] for the low-density case.
and thus depend only the angles $\varphi$ and $\omega$.

By using an analogous notation for the mixing matrix and angles in matter (denoted by a tilde) one has

$$\tan^2 \tilde{\omega} = \frac{U_{e2}^2}{U_{e1}^2}, \quad (33)$$

$$\sin^2 \tilde{\varphi} = U_{e3}^2, \quad (34)$$

and Eq. (11) for the QEI probability reads

$$P_{\nu ee}^{3\nu} = \cos^2 \tilde{\varphi} \cos^2 \varphi (\cos^2 \tilde{\omega} \cos^2 \omega + \sin^2 \tilde{\omega} \sin^2 \omega) + \sin^2 \tilde{\varphi} \sin^2 \varphi. \quad (35)$$

In order to study the symmetry properties of $P_{\nu ee}^{3\nu}$, we introduce three (commuting) transformations $T$,

$$T_\omega \iff \omega \rightarrow \pi/2 - \omega, \quad (36)$$

$$T_{\delta m^2} \iff \delta m^2 \rightarrow -\delta m^2, \quad (37)$$

$$T_{m^2} \iff m^2 \rightarrow -m^2, \quad (38)$$

and also define two independent neutrino wavenumbers [associated to the squared mass gaps in Eqs. (2,3)]

$$\delta k = \frac{\delta m^2}{2E}, \quad (39)$$

$$k = \frac{m^2}{2E}. \quad (40)$$

In terms of the previous notation, Eqs. (28) and (33) imply the following $O(v)$ expression for $\tan^2 \tilde{\omega}$

$$\tan^2 \tilde{\omega} = \tan^2 \omega \left(1 + 2v \cos^2 \varphi \frac{f_\varphi}{\delta k} \right) + O(v^2), \quad (41)$$

where

$$f_\varphi = 1 - \frac{\left(\frac{\delta k}{k}\right)^2 \tan^2 \varphi}{1 - \left(\frac{\delta k}{2k}\right)^2} \quad (42)$$

The above two equations are invariant under the combined symmetry operation $T_{\delta m^2}T_\omega$, which simply means that $\delta m^2$ can always be taken positive, as far as $\omega$ is taken in its full

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9 The angle $\psi$ corresponds to a rotation in the $(\nu_\mu, \nu_\tau)$ subspace, which is unobservable in solar neutrino experiments. The possible CP violating phase can also be put in such subspace and rotated away, as far as solar $\nu$’s are concerned. This justifies the choice of real $U$ for solar neutrinos.
range \([0, \pi/2]\). The expression for \(\tilde{\omega}\) is also invariant under the operation \(T_{m^2}\), which means that \(\tilde{\omega}\) carries no information about the difference between the two spectra in Fig. 1. Such information is carried instead by \(\tilde{\varphi}\).

In fact, the angle \(\tilde{\varphi}\) can be expressed, through Eqs. (28) and (34), as
\[
\sin^2 \tilde{\varphi} = \sin^2 \varphi \left(1 + 2 \frac{v \cos^2 \varphi}{k} g_\omega \right) + O(v^2),
\]
(43)

where
\[
g_\omega = \frac{1 - \frac{\delta k \cos 2\omega}{2k}}{1 - \left(\frac{\delta k}{2k}\right)^2}.
\]
(44)

The above two equations are symmetric under \(T_{\delta m^2} T_\omega\), but not under \(T_{m^2}\). Therefore, if \(\varphi\) is nonzero and if oscillations take place in the QEI regime, the two spectra in Fig. 1 can be distinguished—in principle—by solar neutrino data. In practice, the uncertainties affecting the solar neutrino phenomenology currently prevent such discrimination \([17,32]\).

However, the sensitivity to \(m^2\) (and to its sign) might be enhanced in the near future if the solar neutrino parameters \((\delta m^2, \omega)\) were confirmed (and accurately measured) in the LMA region by reactor oscillation searches \([33]\).

Notice that the previous expressions for the mixing angles allow to rewrite Eq. (35) in the form
\[
P^{3\nu}_{ee} = \cos^2 \tilde{\varphi} \cos^2 \varphi \left(P^{2\nu}_{ee} \right)_{\nu \rightarrow \nu f_\varphi \cos^2 \varphi} + \sin^2 \tilde{\varphi} \sin^2 \varphi + O(v^2),
\]
(45)

where \(P^{2\nu}_{ee}\) represents the adiabatic survival probability for the two-flavor subcase (namely, \(\varphi = 0\) and \(P^{2\nu}_{ee} = \cos^2 \tilde{\omega} \cos^2 \omega + \sin^2 \tilde{\omega} \sin^2 \omega\), provided that the effective electron density is taken as \(f_\varphi \cos^2 \varphi N_e\) in the calculation of \(\tilde{\omega}\) [see Eq. (11)]. The above recipe for the QEI probability in three families is formally equivalent the one applicable in the MSW regime \([34,29,35,20]\), modulo the additional factor \(f_\varphi\) multiplying the effective density.

**IV. GENERALIZED EXPRESSIONS FOR MIXING ANGLES IN MATTER**

In this section we provide accurate expressions for the mixing angles in matter, generalizing those valid in the MSW, QEI, and EI regimes, which are recovered as specific subcases. Therefore, such expressions can be particularly useful for solar neutrino calculations spanning the three aforementioned regimes.

Let us express the QEI results [Eqs. (11) and (13)] in terms of the variables \(\sin^2 2\tilde{\omega}\) and \(\sin^2 2\tilde{\varphi}\),
\[
\sin^2 2\tilde{\omega}(\text{QEI}) \simeq \sin^2 2\omega \left(1 + 2 \cos 2\omega \frac{v \cos^2 \varphi}{\delta k} f_\varphi \right),
\]
(46)
\[
\sin^2 2\tilde{\varphi}(\text{QEI}) \simeq \sin^2 2\varphi \left(1 + 2 \cos 2\varphi \frac{v}{k} g_\omega \right),
\]
(47)
and write again the three-flavor QEI probability [Eq. (45)],

\[ P_{ee}^{3\nu}(\text{QEI}) \simeq \cos^2 \varphi \cos^2 \varphi \left( P_{ee}^{2\nu} \right)_{v \rightarrow v} \cos^2 \varphi + \sin^2 \varphi \sin^2 \varphi . \]  

(48)

We remind that Eqs. (46)–(48) have been derived at first order in \( v/k \) and \( v/\delta k \) (implying small matter effects), with no (hierarchical) restriction on the relative magnitude of \( k \) and \( \delta k \).

The EI transition regime is recovered from the QEI case in the limit of very large squared mass differences, i.e., at zeroth order in \( v/k \) and \( v/\delta k \) (so that \( \sin^2 \tilde{\omega} = \sin^2 \omega \) and \( \sin^2 \tilde{\varphi} = \sin^2 \varphi \)). The EI probability reads then

\[ P_{ee}^{3\nu}(\text{EI}) \simeq \cos^4 \varphi (\cos^4 \omega + \sin^4 \omega) + \sin^4 \varphi \]  

(49)

\[ = U_{e1}^4 + U_{e2}^4 + U_{e3}^4 . \]  

(50)

The MSW regime is instead characterized by \( \delta k \sim O(v) \) (strong matter effects). In this case, the expressions for \( \tilde{\omega} \), \( \tilde{\varphi} \), and \( P_{ee}^{3\nu} \) are often derived under a strictly hierarchical hypothesis (i.e., at zeroth order in both \( \delta k/k \) and \( v/k \)), but with no further restriction on the value of \( v/\delta k \), giving the well-known results

\[ \sin^2 2\tilde{\omega}(\text{MSW}) \simeq \frac{\sin^2 2\omega}{\left( \cos 2\omega - \frac{v \cos^2 \varphi}{\delta k} \right)^2 + \sin^2 2\omega} , \]  

(51)

\[ \sin^2 2\tilde{\varphi}(\text{MSW}) \simeq \sin^2 2\varphi , \]  

(52)

and

\[ P_{ee}^{3\nu}(\text{MSW}) \simeq \cos^4 \varphi \left( P_{ee}^{2\nu} \right)_{v \rightarrow v} \cos^2 \varphi + \sin^4 \varphi , \]  

(53)

where \( P_{ee}^{2\nu} \) is the probability for the two-family subcase, containing the so-called crossing probability \( P_c \) in the nonadiabatic MSW case (see [34,29,35,20] and references therein).

Less often, the above MSW expressions are improved by including the lowest-order effects of nonzero \( v/k \) and \( \delta k/k \) [36], described by the (primed) expressions [29]

\[ \sin^2 2\tilde{\omega}(\text{MSW}') \simeq \frac{\sin^2 2\omega}{\left( \cos 2\omega - \frac{v \cos^2 \varphi}{\delta k} \right)^2 + \sin^2 2\omega} , \]  

(54)

\[ \sin^2 2\tilde{\varphi}(\text{MSW}') \simeq \frac{\sin^2 2\varphi}{\left( \cos 2\varphi - \frac{v}{k + \frac{\delta k}{2} \cos 2\omega} \right)^2 + \sin^2 2\varphi} . \]  

(55)

and

\[ P_{ee}^{3\nu}(\text{MSW}') \simeq \cos^2 \tilde{\varphi} \cos^2 \varphi \left( P_{ee}^{2\nu} \right)_{v \rightarrow v} \cos^2 \varphi + \sin^2 \tilde{\varphi} \sin^2 \varphi . \]  

(56)

By comparing all the previous expressions, derived under different approximations in the QEI, EI, and MSW(′) cases, we find that they can be considered as appropriate subcases of the following generalized expressions for the mixing angles in matter,
\[ \sin^2 2\tilde{\omega} \simeq \frac{\sin^2 2\omega}{\left( \cos 2\omega - \frac{v \cos^2 \varphi}{\delta k} f_\varphi \right)^2 + \sin^2 2\omega}, \]  
\[ (57) \]

\[ \sin^2 2\tilde{\varphi} \simeq \frac{\sin^2 2\varphi}{\left( \cos 2\varphi - \frac{v}{k_\omega} \right)^2 + \sin^2 2\varphi}, \]  
\[ (58) \]

and for three-family oscillation probability (in terms of the two-family one)

\[ P_{\nu e}^{3\nu} \simeq \cos^2 \tilde{\varphi} \cos^2 \varphi \left( P_{\nu e}^{2\nu} \right)_{v \to v_f} \cos^2 \varphi + \sin^2 \tilde{\varphi} \sin^2 \varphi. \]  
\[ (59) \]

In particular, using the above generalized expressions (57)–(59), it turns out that: (i) The QEI approximation [Eqs. (46)–(48)] is recovered through a first-order expansion in \( v/k \) and \( v/\delta k \); (ii) The EI case [Eq. (49)] is reproduced in the limit \( v/\delta k, v/k \rightarrow 0 \); (iii) The usual MSW expressions [Eqs. (11)–(13)] are recovered in the (strictly hierarchical) limit \( k \rightarrow \infty \); and (iv) The improved MSW' expressions [Eqs. (54)–(56)] are recovered through a first-order expansion in \( \delta k/k \). Therefore, Eqs. (57), (58) and (59), together with definitions in Eqs. (14) and (12), provide useful generalizations of \( P_{\nu e}^{3\nu} \) and of the mixing angles in matter \( \tilde{\omega} \) and \( \tilde{\varphi} \), smoothly interpolating from the familiar MSW regime (where \( P_{\nu e}^{3\nu} \) depends only on \( \delta m^2 \)) to the QEI regime (where \( P_{\nu e}^{3\nu} \) depends on both \( \delta m^2 \) and \( m^2 \)) and further up to the EI regime (independent on both \( \delta m^2 \) and \( m^2 \)).

We have also performed the following numerical check: For many representative \((\delta m^2, m^2, \omega, \varphi, E)\) values of phenomenological interest in the QEI regime, we have computed \( P_{\nu e}^{3\nu}(E) \) both through the analytical expressions in Eqs. (57)–(59) and through Eq. (11) with \( U \) obtained by numerical diagonalization. We find differences (often much) smaller than \( 10^{-3} \) in \( P_{\nu e}^{3\nu} \).

Analogously, for the same previous \((\delta m^2, m^2, \omega, \varphi, E)\) values, we have tested the accuracy of the MSW' approximations [Eqs. (54)–(56)] which, being derived under the hierarchical assumption of small \( \delta k/k \), might not properly work in the QEI case. We typically find only a slight worsening of the accuracy (no more than a factor of two), as compared with the previous check. Such a slight worsening is maximized for the highest values of \( \delta m^2 \) and \( \sin^2 \varphi \) allowed by current neutrino phenomenology (about \( 0.7 \times 10^{-3} \) eV² and 0.05, respectively, see later). Notice that, for \( \varphi \rightarrow 0 \), Eqs. (54)–(56) and Eqs. (57)–(59) tend to the same 2\( \nu \) limit, where genuine \( m^2 \)-induced QEI effects disappear.

In conclusion, Eqs. (57)–(59) provide a general and accurate prescription to calculate the mixing angles and the \( \nu_e \) survival probability in matter, smoothly interpolating between the MSW, QEI and EI regimes. The prescription is most useful for nonvanishing \( \varphi \). Preliminary applications of this computing recipe in the analysis of the current solar neutrino phenomenology have been presented in [17,18] and will be discussed in a separate work [32]. In the next section, we just focus on a representative QEI case compatible with present reactor bounds.
V. DISCUSSION OF A REPRESENTATIVE QEI SCENARIO

In this Section we discuss a representative spectrum of squared mass differences leading to the QEI case for solar neutrinos, namely

\[(\delta m^2, m^2) = (0.6, \pm 1.5) \times 10^{-3} \text{ eV}^2,\]  

(60)

where the sign of \(m^2\) discriminates the two options in Fig. 1. The above value for \(\delta m^2\) is marginally allowed in the upper part of the LMA solution to the solar neutrino problem, while the value of \(|m^2|\) is allowed in the lower range of the oscillation solution to the atmospheric neutrino anomaly [3]. Concerning neutrino mixing, we choose a small (but nonzero) value for \(\varphi\), and a value for \(\omega\) within the LMA solution (as well as its octant-symmetric value \(\pi/2 - \omega\)),

\[\tan^2 \varphi = 0.04,\]  

(61)

\[\tan^2 \omega = 0.5 (2.0).\]  

(62)

With the above choice for the mass-mixing parameters, it turns that two different squared mass gaps \((m^2 \pm \delta m^2/2)\) are in the sensitivity range of reactor experiments such as CHOOZ [14] and Palo Verde [15] \((\gtrsim 0.7 \times 10^{-3} \text{ eV}^2)\), so that the usual bounds derived for the 2\(\nu\) case or for 3\(\nu\) case with \(\delta m^2 \simeq 0\) [37] are not immediately applicable, and require a dedicated study [13].

A. CHOOZ constraints

The general 3\(\nu\) survival probability for electron antineutrinos at reactors (in vacuum) reads

\[P_{\text{reac}}^{\text{ee}} = 1 - 4 \cos^4 \varphi \sin^2 \omega \cos^2 \omega \sin^2 \left(\frac{\delta k}{2x}\right)\]

\[- 4 \sin^2 \varphi \cos^2 \varphi \sin^2 \omega \sin^2 \left(\frac{k - \delta k/2}{2x}\right),\]

\[- 4 \sin^2 \varphi \cos^2 \varphi \cos^2 \omega \sin^2 \left(\frac{k + \delta k/2}{2x}\right),\]  

(63)

where \(x\) is the baseline. The above expression is invariant under the symmetry transformations \(T_{\delta m^2} T_{\omega} T_{\delta m^2} T_{m^2}\), and \(T_{m^2} T_{\omega}\) [defined in Eqs. (36)–(38)], implying that the two spectra in Fig. 1 cannot be distinguished by reactor neutrino data (while they can be by solar \(\nu\) data in the QEI regime, at least in principle).

In [16], Eq. (63) has been used in global 3\(\nu\) oscillation fits by using the total CHOOZ rate [14]. However, since the low-energy part of the CHOOZ spectrum is more sensitive to

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\[^{10}\text{In such a study it is sufficient to consider only CHOOZ data, the Palo Verde data being slightly less restrictive on the same mass-mixing parameters.}\]
relatively low values of $\delta m^2$, we prefer to use the full CHOOZ data set (i.e., the binned spectra from the two reactors) rather than the total rate only. In particular, we have accurately reproduced the so-called “$\chi^2$ analysis A” of [14], by using two 7-bin positron spectra and one constrained normalization parameter, for a total of $14 + 1 - 1 = 14$ independent (but correlated) data. We obtain very good agreement with Fig. 9 in [14] for the two-flavor subcase (not shown).  

Using our binned $\chi^2$ analysis for CHOOZ, and setting $\tan^2 \varphi = 0.04$, $\tan^2 \omega = 0.5$, and $\delta m^2 = 0.6 \times 10^{-3}$ eV$^2$, we obtain $\chi^2/N_{DF} = 15.5/14$ for $m^2 = +1.5 \times 10^{-3}$ eV$^2$ and $\chi^2/N_{DF} = 13.4/14$ for $m^2 = -1.5 \times 10^{-3}$ eV$^2$. Due to the symmetry $T_{m^2}T\omega$, the previous $\chi^2$ values also apply for $\tan^2 \omega = 2$ by replacing $\pm m^2$ with $\mp m^2$. In any case, the choice of parameters adopted in Eqs. (60)–(62) gives $\chi^2/N_{DF} \simeq 1$, and thus passes the goodness-of-fit test.

### B. The QEI probability

Figure 2 shows the solar $\nu_e$ survival probability derived from Eqs. (57)–(59) (and averaged over the $^8\text{B}$ production region for definiteness) as a function of neutrino energy. The QEI cases in Eqs. (60)–(62) are represented by either dot-dashed lines ($m^2 > 0$) or dashed lines ($m^2 < 0$). Such lines collapse to a single (solid) line for $m^2 \to \infty$, which provides the usual ($m^2$-independent) hierarchical limit. If one also takes $\delta m^2 \to \infty$, the energy dependence is averaged out (EI regime) and the probability becomes constant (dotted, horizontal line).

Notice that the sizable QEI deviation from the constant EI case, induced by the finite value of $\delta m^2$, changes sign from the first to the second octant of $\omega$. Such octant asymmetry, which asymptotically disappears for increasing values of $\delta m^2$, is still effective in the upper part of the LMA solution ($\delta m^2 \lesssim 10^{-3}$ eV$^2$), where it becomes manifest as a slight local preference of current solar $\nu$ data fits for $\tan^2 \omega < 1$. This preference is mainly driven by the low Chlorine rate [7], which favors relatively low values of $P_{ee}$ (realized in Fig. 2 by the lines at $\tan^2 \omega = 1/2$). The subleading QEI deviation due to the finite value and sign of $m^2$ (which splits the solid line into the dashed and dot-dashed curves in Fig. 2) plays instead a minor role in the current solar $\nu$ phenomenology [17,18], which does not show a statistically significant preference for one of the two spectra in Fig. 1. However, the situation might be improved in the near future [22], should the $(\delta m^2, \omega)$ parameters be confirmed and narrowed in the upper part of the LMA region by KamLand [33,38] and by the second-generation solar neutrino experiments SNO [13,39] and BOREXINO [40]. In this case (provided that $\varphi$ is nonvanishing), residual $m^2$-induced QEI corrections might play a role in accurate calculations of $P_{ee}^{3\nu}$.

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\[11\] The analysis A in [14] also introduces a constrained energy scale shift, that we omit for lack of published information. Its effect seems to be small a posteriori, given that our bounds reproduce those of [14] anyway.
VI. CONCLUSIONS

We have performed (perturbative) analytical calculations of the solar 3ν survival probability $P_{ee}^{3\nu}$ in the regime of quasi-energy-independent (QEI) transitions, intermediate between the more familiar MSW and energy-independent (EI) regimes, and characterized by squared mass differences all close to $\sim 10^{-3} \text{ eV}^2$. We have generalized well-known MSW expressions for $P_{ee}^{3\nu}$ and for the mixing angles $\omega = \theta_{12}$ and $\varphi = \theta_{13}$ (valid for hierarchical mass differences) in a form which smoothly matches the corresponding expressions for the (nonhierarchical) QEI regime. Our main results, summarized in Eqs. (57)–(59) [together with the definitions in Eqs. (42) and (44)], represent an accurate and simple recipe that can be used to improve current calculations of $P_{ee}^{3\nu}$ in the upper part of the LMA solution, where the QEI regime is effective and, for nonvanishing $\varphi$, there might be a residual sensitivity of solar neutrino transitions to the “atmospheric” squared mass difference.

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FIG. 1. Neutrino mass spectrum, together with our notation for the squared mass differences $\delta m^2$ and $m^2$. When both such parameters approach $10^{-3} \text{eV}^2$, solar neutrinos undergo quasi-energy-independent oscillations. In the QEI regime, both $m^2$ and the relative sign between $\delta m^2$ and $m^2$ become observable in matter and can, in principle, discriminate the two possible options in figure.
FIG. 2. QEI effects on $P_{ee}^{3\nu}$, averaged over the $^8$B production region, for $(\Delta m^2, m^2) = (0.6, \pm 1.5) \times 10^{-3} \text{ eV}^2$, together with the asymptotic behavior for $|m^2| \to \infty$ (hierarchical case) and for both $\Delta m^2$ and $|m^2|$ large (averaged oscillations). The mixing parameter $\tan^2 \phi$ is set at the value 0.04, while $\tan^2 \omega$ is taken to be either 1/2 or 2 (corresponding to octant-symmetric values of $\omega$). The QEI cases in this figure are allowed by CHOOZ spectral data.