Exhaustive search of convex pentagons which tile the plane

Michaël Rao

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Abstract

We present an exhaustive search of all families of convex pentagons which tile the plane. This research shows that there are no more than the already 15 known families. In particular, this implies that there is no convex polygon which allows only non-periodic tilings.

1 Introduction

If one asks which convex polygon can tile the plane (allowing translations, rotations and mirrors), the case of pentagons is the only opened case: every triangle and quadrilateral tiles the plane, there are 3 families of hexagons which tile the plane, and no convex polygon with more than six sides can tile the plane (see for example [4]).

The research of families of pentagons which tile the plane has an intriguing history. The first families of pentagons were presented by Reinhardt in 1918 [3]. Kershner presented new families, and announced that the list was complete in 1968 [1]. But new families were found afterwards, one by R. James in 1975, three by an amateur mathematician M. Rice in 1977, and one by R. Stein in 1985. Finally, the fifteenth (and last) family was found by Mann, McLoud and Von Derau in 2015 (see [2]).

We present here an exhaustive search of all families of pentagons which tile the plane. This search is not restricted to periodic tilings, and does not find any new family. The key point is that there are only finitely many, 371, families of angle conditions to consider.

In Section 2 we introduce the notations, and we show that if a pentagon tiles the plane, then there is a tiling such that every vertex type has positive density. In Section 3 we show that there are only finitely possible sets of vertex types in a positive density tiling by a pentagon. Then, in Section 4
we present a backtracking technique to search a tiling, when we fix the set of vertex types. This backtracking algorithm does not find any new family of pentagons.

2 Positive density tilings

Throughout this section, we fix a convex pentagon $P \subset \mathbb{R}^2$. Let $s_1, \ldots, s_5 \in \mathbb{R}^2$ be its 5 vertices in clockwise order. For $i \in \{1, \ldots, 5\}$, let $\alpha_i \times \pi$ be the angle at vertex $s_i$ and let $\alpha = (\alpha_1, \ldots, \alpha_5)$. We recall that $\sum_{i=1}^5 \alpha_i = 3$.

A tiling $T$ of $S$ by $P$ is a set of subset of $S$ such that:

- $\cup_{P \in T} P = S$,
- for every $P \in T$, there is an isometry of the plane $h_P$ such that $h_P(P) = P$,
- for every $P, Q \in T$ with $P \neq Q$, $\hat{P} \cap \hat{Q} = \emptyset$ (where $\hat{P}$ is the interior of $P$ with the usual topology on $\mathbb{R}^2$).

Given a tiling of $S$, we fix an isometry $h_P$ for every $P \in T$ such that $P = h_P(P)$. Elements of $T$ are called a tiles. A point $s$ in $S$ is a vertex of $T$ if it is a vertex of at least one tile in $T$ (that is, there is a $P \in T$ and an $i \in \{1, \ldots, 5\}$ such that $s = h_P(s_i)$). The set of vertices of $T$ is denoted $\mathcal{V}(T)$. A tiling of the plane is a tiling of $\mathbb{R}^2$.

From now on, we fix a tiling $T$ of the plane by $P$. Let $P, Q \in T$ and $s \in \mathcal{V}(T)$. We say that $Q$ follows $P$ around $s$ if there are $i$ and $j$ such that $h_P(s_i) = h_Q(s_j) = s$, $P \cap Q \neq \{s\}$, and $Q$ are just after $P$ if we turn around $s$ in clockwise order.

We distinguish two types of vertices: full and half. Let $s$ be a vertex of a tiling $T$. If there is a circular sequence $s_s = (P_1, \ldots, P_k)$ of tiles such that for every $i \in \{1, \ldots, k\}$, $P_{i+1}$ follows $P_i$ around $s$ (where the indices are taken modulo $k$) we say that $s$ is full.

Otherwise, we say that $s$ is half. In this case, there is a tile $Q$ for which $s$ is on the border, but $s$ is not a vertex of $Q$. That is, $s \in Q \setminus (\hat{Q} \cup \{s_1, \ldots, s_5\})$, and $s$ is on the line segment from $h_Q(s_i)$ to $h_Q(s_{i+1})$ for a $i \in \{1, \ldots, 5\}$ (modulo 5). Then there is a maximal sequence $s_s = (P_1, \ldots, P_k)$ of tiles such that for every $i \in \{1, \ldots, k - 1\}$, $P_{i+1}$ follows $P_i$ around $s$. (There is no tile $Q$ such that $Q$ follows $P_k$ around $s$, or $P_1$ follows $Q$ around $s$.) Note that for any vertex $s$, if $s$ is full, then the circular sequence $s_s$ is unique, and if $s$ is half, the maximal sequence $s_s$ is also unique.
The vector type of \( s \), denoted \( V(s) \), is \( (c_1, \ldots, c_5) \in \mathbb{N}^5 \), where for every \( i \in \{1, \ldots, 5\} \), \( c_i \) is the number of tiles \( P \) in \( s \) such that \( h_P(s_i) = s \). The corrected vector type of \( s \), denoted \( V^c(s) \), is either \( V(s) \) if \( s \) is full, or \( 2 \times V(s) \) if \( s \) is half. We have in any case \( V^c(s) \cdot \alpha = 2 \).

Let \( G = (T, A) \) be the oriented graph, called the underlying graph of \( T \), such that there is an arc between \( P \) and \( Q \) if \( P \) follows \( Q \) around \( s \) for a vertex \( s \) in \( T \). Moreover, we label each tile \( P \) in the graph by “+” if \( h_P \) is a translation or a rotation, or by “−” if \( h_P \) is a glide reflection. We label each arc \((P, Q)\) by \((i, j)\) with \( h_P(s_i) = h_Q(s_j) = s \), where \( s \) is such that \( P \) follows \( Q \) around \( s \).

Let \( T' \subset T \). The subgraph induced by \( T' \) is the graph \( G[T'] = (T', A') \), where \( A' = A \cap T^2 \). A tile \( P \) in \( T' \) is a frontier tile if there is a \( Q \in T \setminus T' \) such that either \((P, Q) \in A \) or \((Q, P) \in A \). The set of frontier tiles of a subgraph \( H \) is denoted \( FT(H) \).

Given an induced subgraph \( H \) of \( G \), we denote by \( S(H) = \bigcup_{P \in T(H)} P \), where \( T(H) \) is the set of tiles in \( H \). (Note that \( T(H) \) is then a tiling of \( S(H) \).)

The set of vertices of \( H \) is denoted \( V(H) = V(T(H)) \). A vertex \( s \) is an interior vertex of a subgraph \( H \) of \( G \) if for every \( P \in T \) such that \( s \) is a vertex of \( P \), then \( P \in T(H) \). The set of interior vertices of \( H \) is denoted \( TV(H) \). Moreover, the set of interior and half (resp. full) vertices of \( H \) is denoted \( TV_H(H) \) (resp. \( TV_F(H) \)). Note that an interior half vertex of \( H \) can be on the boundary of \( S(H) \).

Let \( W(T) = \{ V(s) : s \in V(T) \} \) and \( W^c(T) = \{ V^c(s) : s \in V(T) \} \). If \( T \) is clear in the context, we may write \( W \) or \( W^c \). Note that \( W(T) \) and \( W^c(T) \) are finite.

Let \( o \in \mathbb{R}^2 \) and \( r \in \mathbb{R}^+ \). Let \( T_{o, r} \) be the set of tiles \( P \in T \) such that \( P \cap D(o, r) \neq \emptyset \), where \( D(o, r) \) is the closed disk of radius \( r \) and center \( o \). Let \( G_{o, r} = G[T_{o, r}] \) be the graph induced by \( T_{o, r} \).

**Proposition 1.** There are constants \( c \) and \( c' \) in \( \mathbb{R}^+ \) such that for every \( r \in \mathbb{R}^+ \), \( \left| FT(G_{o, r}) \right| \leq c \times r + c' \).

**Proof.** This follows from the fact that for every \( r \in \mathbb{R}^+ \), there is a \( n_r \in \mathbb{N} \) such that for every \( o \in \mathbb{R}^2 \), \( \left| T_{o, r} \right| \leq n_r \).

For \( v \in W(T) \), \( o \in \mathbb{R}^2 \) and \( r \in \mathbb{R}^+ \), let:

\[
\forall o, r \in \mathbb{R}^2: \frac{|\{s \in TV(G_{o, r}) : V(s) = v\}|}{|T_{o, r}|}.
\]
We say that the tiling $T$ has \textit{positive density} if for every $v \in W(T)$ and $o \in \mathbb{R}^2$, we have $\liminf_{r \to \infty} f_{o,r}(v) > 0$. (Note that if it is true for one $o \in \mathbb{R}^2$, then it is true for every $o \in \mathbb{R}^2$.)

\textbf{Lemma 2.} If $\liminf_{r \to \infty} f_{o,r}(v) = 0$, then there is a tiling $T'$ of the plane by $\mathcal{P}$ such that $W(T') \subseteq W(T) \setminus \{v\}$.

\textit{Proof.} Let $d \in \mathbb{R}^+$. We divide $\mathbb{R}^2$ into a grid of $d \times d$ squares $S_{(i,j)}$, with $(i, j) \in \mathbb{Z}^2$. Then we decompose $T$ into a disjoint union of sets of tiles $T_{(i,j)}$ such that a tile $P$ is in $T_{(i,j)}$ if $P \cap S_{(i,j)} \neq \emptyset$ (if there are several possible choices for a tile, we chose arbitrarily). If for every $(i, j) \in \mathbb{Z}^2$, $T_{(i,j)}$ has one vertex $s$ with $V(s) = v$, then $\liminf_{r \to \infty} f_{o,r}(v) > 0$.

Thus, if $\liminf_{r \to \infty} f_{o,r}(v) = 0$, then for every $d \in \mathbb{R}^+$ there is a subgraph $H_d$ such that $S(H_d)$ contains a $d \times d$ square, and $W(H_d) \subseteq W(T) \setminus \{v\}$. We keep a connected component $H'_d$ of $H_d$ such that $S(H'_d)$ contains the center of the square. Then one can construct by compactness an infinite graph $G'$ in which every vertex is an interior vertex, and with $W(G') \subseteq W(T) \setminus \{v\}$. There are three cases: either $G'$ corresponds to a tiling of the plane, of a half-plane or of a stripe. In all cases, one can construct a tiling of the plane without vertex of vector type $v$, and no new vertex type. \hfill \Box

\textbf{Good subsets.} We say that a subset $\mathcal{X} \subseteq \mathbb{N}^5$ is \textit{good} if for every $u \in \mathbb{R}^5$ with $\sum u = 0$, either $u \cdot v = 0$ for every $v \in \mathcal{X}$, or there are $v, v' \in \mathcal{X}$ such that $u \cdot v < 0 < u \cdot v'$.

Suppose that $T$ is a tiling by the pentagon $\mathcal{P}$. By Lemma 2 we assume that $T$ has positive density.

\textbf{Proposition 3.} Let $u \in \mathbb{R}^5$ such that $\sum_{i=1}^{5} u_i = 0$, and there is a $v^+ \in W^c$ with $u \cdot v^+ > 0$. Then there is a $v^- \in W^c$ such that $u \cdot v^- < 0$.

\textit{Proof.} For every $i \in \{1, \ldots, 5\}$, if we count the number of angles $i$ in $T(G_{o,r})$, we have:

$$0 \leq |T(G_{o,r})| - \sum_{v \in W} v_i \times |\{s \in \mathcal{IV}(G_{o,r}) : V(s) = v\}| \leq |\mathcal{F}(G_{o,r})|.$$  

Thus, by Proposition 1

$$\lim_{r \to \infty} \sum_{v \in W} v \times f_{o,r}(v) = (1, 1, 1, 1).$$
Since \( \mathcal{T} \) has positive density, \( \liminf_{r \to \infty} f'_{o,r}(v) > 0 \) for every \( v \in \mathcal{W} \). Let

\[
f'_{o,r}(v) = \frac{1}{2} \left\lfloor \frac{1}{\# \{ s \in \mathcal{I} \mathcal{V}_H(G_{o,r}) : V(s) = \frac{1}{2}v \}} \right\rfloor + \left\lfloor \frac{1}{\# \{ s \in \mathcal{I} \mathcal{V}_F(G_{o,r}) : V(s) = v \}} \right\rfloor.
\]

Since \( v \in \mathbb{N}^5 \) cannot be the vector type of both a half vertex, and a full vertex in the same tiling, we have also \( \liminf_{r \to \infty} f'_{o,r}(v) > 0 \) for every \( v \in \mathcal{W}^c \). Moreover

\[
\lim_{r \to \infty} \sum_{v \in \mathcal{W}^c} v \times f'_{o,r}(v) = (1, 1, 1, 1, 1).
\]

Let \( u \in \mathbb{R}^5 \) and \( v^+ \in \mathcal{W}^c \) such that \( u \cdot (1, 1, 1, 1, 1) = 0 \) and \( u \cdot v^+ > 0 \). Suppose for the sake of contradiction that for every \( v' \in \mathcal{W}^c \), \( u \cdot v' \geq 0 \). Then \( \lim_{r \to \infty} \sum_{v \in \mathcal{W}^c} (u \cdot v) \times f'_{o,r}(v) = 0 \). We have a contradiction since \( \lim_{r \to \infty} \sum_{v \in \mathcal{W}^c} (u \cdot v) \times f'_{o,r}(v) \geq (u \cdot v^+) \times \liminf_{r \to \infty} f'_{o,r}(v^+) > 0 \).

By Proposition 3, if \( \mathcal{T} \) has positive density, then \( \mathcal{W}^c \) is good.

**Compatible vectors.** Let \( \text{span}(V) \) be the set of vectors which are linear combinations of vectors in \( V \). Let

\[
\text{Compat}(V) = \{ w \in \mathbb{N}^5 : (w, 2) \in \text{span}(\{(1, 1, 1, 1, 3) \} \cup \{(v, 2) : v \in V \}) \}.
\]

Note that if \( \mathcal{X} \) is good, then \( \text{Compat}(\mathcal{X}) \) is also good (but the converse is not necessarily true).

Given a subset \( \mathcal{X} \subseteq \mathbb{N}^5 \), we define by \( \mathfrak{P}_\mathcal{X} \) the subset of \( \mathbb{R}^5 \) such that \( \alpha = (\alpha_1, \ldots, \alpha_5) \in \mathfrak{P}_\mathcal{X} \) if and only if for every \( i \in \{1, \ldots, 5\} \), \( 0 < \alpha_i \leq 1 \), \( \sum_{i=1}^{5} \alpha_i = 3 \) and for every \( v \in \mathcal{X} \), \( \alpha \cdot v = 2 \). One has \( \mathfrak{P}_\mathcal{X} = \mathfrak{P}_{\text{Compat}(\mathcal{X})} \).

If \( \mathcal{T} \) is a tiling by a convex pentagon \( \mathcal{P} \) of angles \( (\alpha_1 \cdot \pi, \ldots, \alpha_5 \cdot \pi) \), then \( (\alpha_1, \ldots, \alpha_5) \in \mathfrak{P}_{\mathcal{W}^c(\mathcal{T})} \cap [0, 1]^5 \). Moreover, if \( \mathfrak{P}_{\mathcal{X} \cap [0, 1]^5} \neq \emptyset \), then the set \( \text{Compat}(\mathcal{X}) \) is finite.

In the next section, we compute all good sets \( \mathcal{X} \) with \( \mathfrak{P}_{\mathcal{X} \cap [0, 1]^5} \neq \emptyset \). We show in particular that there are finitely many such sets.

### 3 Computation of all good subsets

We say that the permutation in \( S_5 \) is a rotation/mirror if it can be generated by the permutations \((12345)\) and \((3)(24)(15)\). Given a permutation \( p \in S_5 \) and a vector \( v \in \mathbb{N}^5 \), let \( p(v) \) be the vector \((v_{p(1)}, \ldots, v_{p(5)})\). Let \( p(V) \) for \( V \subseteq \mathbb{N}^5 \), be \( \{ p(v) : v \in V \} \). In this section, we show the following:
Table 1a: \(\dim(\mathcal{P}) = 3\). Striked out numbers correspond to families of pentagons of Type 1.

| 1 | \(B_{1}\) | 1 | \(B_{2}\) |
|---|---|---|---|
| 0 | 11100 | 2 | 11010 |

Table 1b: \(\dim(\mathcal{P}) = 2\)

**Lemma 4.** If \(X\) is a non empty good set such that \(\mathcal{P}_X \cap ]0, 1[^5 \neq \emptyset\) then \(\mathcal{P}_X = \mathcal{P}_r(\text{Compat}(B_i))\) for an integer \(i \in \{1, \ldots, 371\}\) and a rotation/mirror \(r\). (\(B_i\) is given in Tables 1.)

The remaining of this section is devoted to the proof of Lemma 4, which is algorithmic. In this section, the order of the angles is not important, so we suppose w.l.o.g. that there is an \(\alpha = (\alpha_1, \ldots, \alpha_5)\) such that \(1 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 > 0\).

Let \(\mathcal{P}_X^\alpha\) be the set of vectors \((\alpha_1, \ldots, \alpha_5)\) such that \(1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0\), \(\sum_i \alpha_i = 3\) and for every \(v \in X\), \(v \cdot \alpha = 2\). Clearly, \(\mathcal{P}_X^\alpha\) is a convex polytope. Moreover, one has \(\mathcal{P}_X^\geq = \mathcal{P}_X \cap \mathcal{P}_0^\geq\) and \(\mathcal{P}_X^\geq = \mathcal{P}_\text{Compat}(\mathcal{X})^\geq\). If \(\mathcal{P}_X^\geq\) is non empty, let \(m_X \in [0, 1[^5\) be such that \((m_X)_i = \min\{\alpha_i : \alpha \in \mathcal{P}_X^\geq\}\). Note that \((m_X)_1 \geq \frac{2}{3}\), \((m_X)_2 \geq \frac{1}{2}\), \((m_X)_3 \geq \frac{1}{3}\), and \((m_X)_i \geq (m_X)_{i+1}\) for every \(i \in \{1, \ldots, 4\}\).

We say that a set \(\mathcal{X}\) is **maximal** if \(\mathcal{X} = \text{Compat}(\mathcal{X})\). To prove Lemma 4, it suffices to prove it for every maximal good set.

The procedure **RECURSE** (Algorithm 1) computes all maximal good sets \(\mathcal{Y} \supseteq \mathcal{X}\) with \(\mathcal{P}_X^\geq \cap ]0, 1[^5 \neq \emptyset\).

For all maximal good set \(\mathcal{Y} \supseteq \mathcal{X}\), one has \(\text{Compat}(\mathcal{X}) \subseteq \mathcal{Y}\) (line 2). For \(u\) defined line 7 since \(u \cdot (1, 1, 1, 1, 1) = 0\), for every \(v \in \mathcal{X}\), \(v \cdot u = 0\), and by definition of good subsets, we know that if there is another maximal good set \(\mathcal{Y} \supseteq \mathcal{X}\), then there is a \(w \in \mathbb{N}^5 \setminus \mathcal{X}\) such that \(w \cdot u \geq 0\). Moreover, we must
Algorithm 1 Exhaustive search of goods sets $\mathcal{Y} \supseteq \mathcal{X}$

1: procedure RECURSE($\mathcal{X}$)
2: $\mathcal{X} \leftarrow \text{Compat}(\mathcal{X})$
3: if $\mathcal{X} \cap [0,1]^5 = \emptyset$ then return end if
4: if $\mathcal{X}$ is good then
5: Add $\mathcal{X}$ to the list of good sets
6: end if
7: Let $u \in \mathbb{R}^5$ such that:
   - $u \cdot (1,1,1,1,1) = 0$
   - $\forall v \in \mathcal{X}, u \cdot v = 0$ and
   - $\forall i \in \{4,5\}, (m_{\mathcal{X}})_i = 0 \Rightarrow u_i < 0$
8: $V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \text{ and } v \cdot m_{\mathcal{X}} \leq 2\}$
9: for every $w \in V \setminus \mathcal{X}$ do
10: RECURSE($\mathcal{X} \cup \{w\}$)
11: end for
12: end procedure
| i | $S_i$ | i | $S_i$ | i | $S_i$ |
|---|---|---|---|---|---|
| 00003 00012 00010 00001 | 00000 00012 00102 01000 | 00003 00012 00102 01000 | 00003 00012 00102 01000 |
| 00003 00012 11001 20100 | 00003 00012 20100 12000 | 00003 00012 20100 12000 | 00003 00012 20100 12000 |
| 00003 00012 02010 21000 | 00003 00012 21000 31000 | 00003 00012 21000 31000 | 00003 00012 21000 31000 |
| 00003 00012 21000 03100 | 00003 00012 12000 10000 | 00003 00012 12000 10000 | 00003 00012 12000 10000 |
| 00003 00012 21000 13000 | 00003 00012 12000 21000 | 00003 00012 12000 21000 | 00003 00012 12000 21000 |
| 00003 00012 21000 10000 | 00003 00012 12000 01000 | 00003 00012 12000 01000 | 00003 00012 12000 01000 |
| 00003 00012 21000 10010 | 00003 00012 12000 11000 | 00003 00012 12000 11000 | 00003 00012 12000 11000 |
| 00003 00012 21000 10100 | 00003 00012 12000 21100 | 00003 00012 12000 21100 | 00003 00012 12000 21100 |
| 00003 00012 21000 10200 | 00003 00012 12000 02100 | 00003 00012 12000 02100 | 00003 00012 12000 02100 |
| 00003 00012 21000 10300 | 00003 00012 12000 12100 | 00003 00012 12000 12100 | 00003 00012 12000 12100 |
| 00003 00012 21000 10400 | 00003 00012 12000 22100 | 00003 00012 12000 22100 | 00003 00012 12000 22100 |
| 00003 00012 21000 10500 | 00003 00012 12000 32100 | 00003 00012 12000 32100 | 00003 00012 12000 32100 |
| 00003 00012 21000 10600 | 00003 00012 12000 03100 | 00003 00012 12000 03100 | 00003 00012 12000 03100 |
| 00003 00012 21000 10700 | 00003 00012 12000 13100 | 00003 00012 12000 13100 | 00003 00012 12000 13100 |
| 00003 00012 21000 10800 | 00003 00012 12000 23100 | 00003 00012 12000 23100 | 00003 00012 12000 23100 |
| 00003 00012 21000 10900 | 00003 00012 12000 33100 | 00003 00012 12000 33100 | 00003 00012 12000 33100 |

Table 1d: $\dim(\mathcal{P}) = 0$ (part 1/2)
have $w \cdot m_X \leq 2$, otherwise $\mathcal{P}_Y \cap [0, 1]^5$ would be empty. Thus, $w$ is in the set $V$ computed line 8. We try every possibility for $w$ at line 10. An important point of the algorithm is that $V$ is finite: if $v \in V$, then for every $i$ such that $(m_X)_i > 0$, $v_i$ is bounded by $\frac{2}{(m_X)_i}$, and thus for every $i$ with $(m_X)_i = 0$, $v_i$ is bounded by $-\frac{1}{m_i} \sum_{j: (m_X)_j > 0} (v_j \max(0, u_j))$.

The computation of a $u$ (line 7) with the required property is done using a linear program. If no such $u$ exists, then the algorithm would fail. But even it is not necessary for the proof (it suffices that one possible execution terminates), one can show that this case never happens.

**Proposition 5.** In line 7, such a $u$ always exists.

**Proof.** Note that if $\alpha, \alpha' \in \mathcal{P}_X$, then $(\alpha - \alpha') \cdot (1, 1, 1, 1, 1) = 0$ and for every $v \in X$, $(\alpha - \alpha') \cdot v = 0$. If $(m_X)_4 > 0$ and $(m_X)_5 > 0$, one can take $u = (0, 0, 0, 0, 0)$. If $(m_X)_4 > 0$ and $(m_X)_5 = 0$, there is $\alpha \in \mathcal{P}^\geq_X$ such that $\alpha_5 = 0$. Otherwise $(m_X)_4 = (m_X)_5 = 0$, and there is $\alpha \in \mathcal{P}^\geq_X$ such that $\alpha_4 = \alpha_5 = 0$. In all cases, $u = \alpha - \alpha'$, with $\alpha' \in \mathcal{P}_X^\geq \cap [0, 1]^5 \neq \emptyset$, has the desired properties. \qed

The procedure RECURSE is non deterministic, and a good choice for $u$ can reduce the size of the research tree and the computation time. But since the dimension of the subspace spanned by $X$ strictly increase at every recursive call, there is at most 5 nested calls of RECURSE, and this procedure always terminates.

**Computation.** In order to reduce the computation time, we also track vectors $w$ which are not in $Y$. Our implementation takes approximately 40 seconds to explore all the cases (1354 calls of RECURSE). There are 193
non-empty maximal goods sets $\mathcal{X}$ with $\mathcal{P}_{\mathcal{X} \cap [0,1]} \neq \emptyset$, and (taking all permutations) 3495 sets with $\mathcal{P}_{\mathcal{X} \cap [0,1]} \neq \emptyset$. If we keep only one representative for each class up to rotation/mirror, one has the 371 sets of Tables 1.

4 Testing all 371 cases

For each of the 371 cases, we do an exhaustive search by backtracking, to try to construct a tiling of an arbitrarily large region. If this backtracking is finite, then we know that there is no pentagon with these angles condition which tiles the plane.

Throughout this section one fix $\mathcal{X} = \text{Compat}(B_i)$ for an $i \in \{1, \ldots, 371\}$. One supposes that $\mathcal{P}$ is a pentagon which tiles the plane with a tiling $\mathcal{T}$ of positive density, such that $\text{Compat}(\mathcal{W}^c(\mathcal{T})) = \mathcal{X}$. Let $\mathcal{P} = \mathcal{P}_X$.

Suppose that the vertices of $\mathcal{P}$ are $s_i$ (with angle $\alpha_i \times \pi$) and the lengths of the sides are $\ell_i$, $i \in \{1, \ldots, 5\}$, in clockwise order, and such that $\ell_1$ is the length between $s_1$ and $s_2$. Moreover, we suppose w.l.o.g. that $\sum_i \ell_i = 1$.

The backtracking is done on a pair of two data-structures: a tiling graph which represents the geometric information we have for the part of the tiling, and a linear program $Q$ with represent conditions we have on $\ell$.

**Tiling graph.** The *tiling graph* is an embedded planar graph, with additional information: labels on angles, edge and faces. (Note that this graph differs significantly from the graph defined in Section 2.)

Each vertex of the graph corresponds to a vertex of the tiling. (This mapping is not necessarily injective.) Each angle has a type: either 1, 2, 3, 4, 5, $\emptyset$, $\pi$ or unknown. Each edge in the graph has also a type: either 1, 2, 3, 4 or 5. The planar graph has two types of faces. A face is either a *normal face* or is a *special face*. Each edge is adjacent to one special face, and one normal face.

There is a bijection between the normal faces and the tiles of the tiling. Thus a normal face has degree 5, and the types of its angles and edges are either (in clockwise order) 1, 2, 3, 4, 5 or 5, 4, 3, 2, 1. Moreover the type of the edge between the angles 1 and 2 is 1.

A special face corresponds either to the frontier between tiles or an unknown area of the plane. Its angles are $\emptyset$, $\pi$ or unknown. An angle of type $\emptyset$ (resp. $\pi$) corresponds to an angle of $0$ (resp. $\pi$) in the tiling. A special face is complete if it has no unknown angles. A complete special face has exactly two $\emptyset$ angles. In this case, it corresponds to a segment which is a frontier between two or more tiles.
Figure 1: Example of a tiling graph (Type 15). Unmarked angles are labeled “unknown”.

A vertex $v$ is complete if there is no unknown angle adjacent to it. Similarly to Section 2 let $V^c(v) \in \mathbb{N}^5$ be such that, for every $i \in \{1, \ldots, 5\}$, $(V^c(v))_i = c \times (V(v))_i$, where $(V(v))_i$ is the number of angles $i$ adjacent to $v$, and $c = 1$ if there is no $\pi$ adjacent to $v$ (i.e. $v$ is full) and $c = 2$ if there is one angle $\pi$ adjacent to $v$ (i.e. $v$ is half).

An example of a tiling graph is given in Figure 1. The tiling graph on the right corresponds to the tiling on the left. Note that other tiling graphs are possible to represent the same tiling.

A run on a special face is a succession of consecutive $\emptyset$ and $\pi$ angles. Each run corresponds to aligned points in the tiling. For example, on Figure 1 $(s, t, s')$, $(u, t, y, u')$ are (maximal) runs. The complete face $(y, z)$ induces also a run.

Since we want to generate a tiling graph corresponding to a tiling by $\mathcal{P}$, we keep the following conditions on the tiling graph (that is, we backtrack if one of these conditions are not fulfilled)

- for every vertex $v$, there is a $w \in \mathcal{X}$ such that $V^c(v) \leq w$,
- for every complete vertex $v$, $V^c(v) \in \mathcal{X}$,
- there is no run with more than two $\emptyset$ angles,
- there is no vertex with two $\pi$ angles adjacent to it.
Note that every finite subset $T'$ of $T$ can be represented by a tiling graph with the previous properties (but the representation is not unique). We will make some “completion” operations on it, which guarantee that the new tiling graph is also a tiling graph of the same tile set.

Moreover, during the exploration, all the operations we do on the tiling graph keep the additional following conditions: the graph is connected, has exactly one non-complete face, and has no vertices with more than one unknown angles adjacent to it.

**Completing vertices.** At every time, as soon as there is a non-complete vertex $v$ such that $V^c(v) \in W^c$, we relabel the angle labeled unknown adjacent to $v$ with the label $\emptyset$. Moreover, for every non-complete vertex $v$ such that $2 \times V^c(v) \in W^c$, we relabel the angle labeled unknown adjacent to $v$ with the label $\pi$.

**Length suppositions and completing faces.** If there is a pair of vertices $(v, v')$ on a same run, and the linear program $Q$ imply that $v$ and $v'$ are the same point in the tiling, then we merge $v$ and $v'$ in the graph.

If $Q$ does not permit to decide among the following 3 possibilities:

- $v$ and $v'$ are the same point in the tiling,
- $v$ is on the right of $v'$ (with an arbitrary orientation of the line corresponding to the run),
- $v$ is on the left of $v$,

then we branch on the 3 possibilities: we add the corresponding condition on $Q$, and recurse.

**Example.** In Figure 12 $(w, t, w')$ is a run, and the length between $t$ and $w$ (which is $\ell_3$) is the same as the distance between $t$ and $w'$. Thus we merge $w$ and $w'$, and we relabel the angle $t, w, t$ into $\emptyset$, and we create a new special complete face $(t, w)$.

We have also to consider the run $(u, t, y, u')$. We have either to choose if $u$ and $u'$ is the same point (that is, add the condition $\ell_3 = \ell_4 + \ell_5$) or not (and in this case, explore with the additional condition $\ell_3 > \ell_4 + \ell_5$).

Suppose now we explore the first case (merge of $u$ and $u'$). We create an angle $\pi$ adjacent to $u$ to complete the special face $(u, t, y)$. Since $2\alpha_3 = \pi$ in the type 15 (i=303), the vertex $u$ is now complete, and we can label the angle $r, u, r'$ as $\emptyset$. We have a new run $(r, u, r')$. Since we have already the
condition $\ell_4 = \ell_5$ in $Q$ (by the complete special face $y, z$), we know that $r$ and $r'$ must be the same vertex in the tiling, and we can merge $r$ and $r'$ without branching.

Existence of a solution which respects $Q$ Let $s(\alpha)$ be the vector such that $s(\alpha)_i = (i-1) - \sum_{j=1}^{i-1} \alpha_j$. Note that if $\alpha$ (resp. $\ell$) is the vector of angles (resp. lengths) of a pentagon, we have

$$\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0. \quad (1)$$

Given a linear program $Q$, we denote by $Q$ the set of $\ell \in \mathbb{R}^5$ such that $\ell \geq 0$, $\sum \ell = 1$ and $\ell$ respects the conditions in $Q$.

If the following condition is not fulfilled, then no convex pentagon exists with the conditions, and one backtrack.

$$\exists \ell \in Q \cap [0,1]^5, \exists \alpha \in P \cap [0,1]^5, \sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0 \quad (2)$$

If $\dim(P) = 0$ (cases from 121 to 371), since all conditions for $P$ are rational, there is a $p \in \mathbb{Z}^5$ and $q \in \mathbb{N}^+$ such that $s(\alpha) = p/q$, where $\{\alpha\} = P$, and thus the condition $\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0$ can be turned into $\ell \cdot \cos = 0$ and $\ell \cdot \sin = 0$, where $\cos_i = \cos(p_i \times \pi / q)$ and $\sin_i = \sin(p_i \times \pi / q)$. The one can decide with computations on an algebraic extension of $\mathbb{Q}$ (for example $\mathbb{Q}[\cos(\pi / q)]$).

If $\dim(P) > 0$, the verification of (2) is more complicated, and we do not try to verify it at each recursion. However, if one has a certificate that (2) is false, then we backtrack. If there is a family of polytopes $P_l$, $l \in \{1, \ldots, L\}$ such that $P \subseteq \bigcup_{l=1}^L P_l$, and for every $l \in \{1, \ldots, L\}$, $\{x \in \Omega : x \cdot \sin^+ \geq 0, x \cdot \sin^- \leq 0, x \cdot \cos^+ \geq 0$ and $x \cdot \cos^- \leq 0\} = \emptyset$, where $\sin^+_i$ (resp. $\sin^-_i$) is an upper (resp. lower) bound of $\{\sin(s(\alpha)_i) : \alpha \in P_l\}$ (and similarly for $\cos^+$ and $\cos^-$), then we know that (2) is false. This can be done using rational numbers.

This procedure cannot decide, for example, if (2) is false, but (1) has a degenerate solution $(\ell, \alpha)$ is on the boundary of $\Omega \times P$. To resolve these cases, we also backtrack in some degenerate solutions (types 20 to 24 in Table 2).

Branching. If we are not in any case of backtracking, then we add a new normal face to the tiling graph.
| Type | Condition | Type | Condition | Type | Condition |
|------|-----------|------|-----------|------|-----------|
| 1    | \(a+b+c = 2\pi\) | 2    | \(a+b+d = 2\pi\) | 0    | \(C = E\) |
| (i=1)|           | (i=2)|           | (i=6)|           |
| 3    | \(3e = 2\pi\) | 4    | \(a+b+d = 2\pi\) | 6    | \(D = E\) |
| (i=31)| \(d+2e = 2\pi\) | (i=6)|           | (i=13)|           |
|      | \(b+2e = 2\pi\) |     | \(2e = \pi\) |     |           |
| 5    | \(3e = 2\pi\) | 7    | \(A = B\) | 8    | \(A = B\) |
| (i=4)| \(d+2e = 2\pi\) | (i=17)|           | (i=14)|           |
|      | \(a+2e = 2\pi\) |     | \(A = B\) |     |           |
| 9    | \(d+2e = 2\pi\) | 10   | \(A = C\) | 11   | \(A = C\) |
| (i=15)| \(a+2c = 2\pi\) | (i=69)|           | (i=67)|           |
|      | \(2a+c = 2\pi\) |     | \(A = B\) |     |           |
| 13   | \(b+d+e = 2\pi\) | 14   | \(A = B\) | 15   | \(A = B\) |
| (i=63)| \(a+2d = 2\pi\) | (i=67)|           | (i=303)|           |
|      | \(2e = \pi\) |     | \(A = C\) |     |           |
| 17   | \(c+2e = 2\pi\) | 18   | \(D = E\) | 19   | \(B = E\) |
| (i=25)| \(2b+d+e = 2\pi\) | (i=73)|           | (i=23)|           |
|      | \(c+2e = 2\pi\) |     | \(A = B\) |     |           |
| 19   | \(c+2e = 2\pi\) | 20   | \(A = C\) | 21   | \(A = B\) |
| (i=23)| \(b+d+e = 2\pi\) | (i=27)|           | (i=12)|           |
|      | \(2e = \pi\) |     | \(A = B\) |     |           |
| 23   | \(2b+d = 2\pi\) | 24   | \(2E = A + C\) | 25   | \(2E = A + C\) |
| (i=64)| \(a+b+d = 2\pi\) | (i=69)|           | (i=64)|           |
|      | \(2e = \pi\) |     |           |     |           |

Table 2: Conditions for tilings of types 1 to 24, with \((a, \ldots, e) = (\alpha_1, \ldots, \alpha_5)\) and \((A, \ldots, E) = (\ell_1, \ldots, \ell_5)\).

We take a non-complete vertex \(w\) in the graph. We know that, if the tiling graph corresponds to a sub-tiling \(T'\) of a tiling \(T\) by \(P\), there is a tile \(P \in T \setminus T'\) such that \(w\) is a vertex of \(P\), and \(P\) shares a line segment with \(T'\). Then we branch on on all these possibilities of face addition.

**Results.** If we also backtrack if we are in one the 24 types presented in Table 2, the exhaustive search terminates, for all 371 cases for angle conditions. That is, if a convex pentagon \(P\) tiles the plane, then \(P\) is in one the 24 families.

Types 1 to 15 are the already known families of pentagons which tiles the plane. Types 16 to 19 are special cases of the 15 already known families,
i.e. every non-degenerate solution of (2) is in a known family. Type 16 is a special case of Type 10: the only solution of (2) has $\alpha_1 = \alpha_4 = \alpha_5 = \pi/2$ and $\alpha_2 = \alpha_3 = 3\pi/4$. Types 17 and 18 are special cases of Type 2: in every solution, all lengths are equal and $\alpha_1 + \alpha_3 = \pi$. Type 19 is a special case of Type 1: conditions imply that for every solution, $\alpha_3 + \alpha_4 = \pi$. Finally, types 20 to 24 are degenerate, i.e. (2) has no solution. These observations can be done using a computer algebra system, turning linear conditions on angles, lengths and (1) into a system of polynomial equations.

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