\textbf{CPT-conserved effective mass Hamiltonians through first and higher order charge operator $\mathcal{C}$ in a supersymmetric framework}

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This paper examines the features of a generalized position-dependent mass Hamiltonian $H_m$ in a supersymmetric framework in which the constraints of pseudo-Hermiticity and CPT are naturally embedded. Different representations of the charge operator are considered that lead to new mass-deformed superpotentials $W_m(x)$ which are inherently PT-symmetric. The qualitative spectral behavior of $H_m$ is studied and several interesting consequences are noted.

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\section{I. INTRODUCTION}

Non-Hermitian systems admitting $\mathcal{PT}$-symmetry (i.e. invariance under a combined action of parity $\mathcal{P}$ and time-reversal $\mathcal{T}$) have been a subject matter of intense interest \cite{1,2}. $\mathcal{PT}$-symmetry has an interesting implication that the whole class of Schrödinger Hamiltonians coming under its assignment namely, $H = p^2/(2m) + V(x)$ defined on the real line $x \in \mathbb{R}$, where the potential is typically $V(x) = V^*(-x)$, may possess real or conjugate pairs of energy eigenvalues under certain conditions related to $\mathcal{PT}$ being unbroken (i.e. exact) or spontaneously broken. It has also been realized that the concept of $\mathcal{PT}$-symmetry has its roots in the theory of pseudo-Hermitian operators and that pseudo-Hermiticity serves as one of the plausible necessary and sufficient conditions for the reality of the spectrum \cite{3}.

In \cite{4} a set of intertwining relations

$$H\zeta = \zeta H^\dagger \quad (1.1)$$

was studied in which a Hermitian operator $\zeta$ was proposed to be expressed as a product of the charge operator $\mathcal{C}$ and parity operator $\mathcal{P}$

$$\zeta = \mathcal{CP} \quad (\zeta = \zeta^\dagger). \quad (1.2)$$

It is straightforward to see that equations (1.1) and (1.2) together imply the $\mathcal{CPT}$ conservation of the Hamiltonian $H$, $\mathcal{T}$ being the time reversal operator

$$\mathcal{CPT}H = H\mathcal{CPT}. \quad (1.3)$$

Interestingly, it also follows from (1.1) that the operator $\zeta^{-1}$, if it exists, also fulfills the intertwining relations

$$H^\dagger\zeta^{-1} = \zeta^{-1}H \quad (1.4)$$

implying that $H$ is pseudo-Hermitian with respect to $\zeta^{-1}$. This can be verified as follows:

$$<\psi, H\phi, \zeta^{-1} = <\psi, \zeta^{-1}H\phi, \zeta^{-1} > = <\psi, H^\dagger\zeta^{-1}\phi, \zeta^{-1} > = <\psi, \zeta^{-1}\phi > = <H\psi, \phi >, \zeta^{-1}. \quad (1.5)$$

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Differential realizations for $\mathcal{C}$ have been considered in the literature such as $\mathcal{C} = d/dx + W(x)$ for the first-order \cite{4} and $\mathcal{C} = d^2/dx^2 + W(x)d/dx + \nu_0(x)$ for the second-order case \cite{5}. The aims of such models have been to search for closed-form solutions and to work out the solvability criterion of the embedded Hamiltonian.

In this article we intend to investigate these and related aspects of pseudo-Hermiticity and $\mathcal{CPT}$-conservation for extended versions of Schrödinger equation admitting SUSY in a position-dependent (effective) mass (PDM) framework. The 1-D effective mass Hamiltonian $H \to H_m$ obeys (in the atomic unit defined by $\hbar^2 = 2$) in real spatial coordinate \cite{6}:

\begin{equation}
H_m(x)\psi_n(x) \equiv \left( -\partial \left[ \frac{1}{m(x)} \partial \right] + \tilde{V}_m(x) \right) \psi_n(x) = E_n \psi_n(x), \quad \tilde{V}_m(x) = V_m(x) + \rho(m),
\end{equation}

where $m(x)$ is a real valued mass function in the presence of a complex potential $\tilde{V}_m(x)$:

\begin{equation}
V_m(x) = V_m^R(x) + iV_m^I(x).
\end{equation}

In equation (1.0), the mass-dependent function $\rho(m)$ has the form \cite{2}

\begin{equation}
\rho(m) = \frac{1 + b m''(x)}{2 m^2(x)} - \frac{m'^2(x)}{m^3(x)}, \quad c = 1 + b + a(a + b + 1),
\end{equation}

where $a$ and $b$ are the usual ambiguity parameters \cite{8} typical to the effective-mass models. Position dependence in mass shows up in different areas of physics - semiconductors \cite{9}, quantum dots \cite{10}, $^3$He clusters \cite{11} and many more. A number of papers have been written on the issue of PDM in this rapidly expanding literature \cite{12,13,14}.

Note that the question of boundedness and invertibility of the operator $\zeta$, assuming an explicit representation for it was addressed in \cite{4} for the constant-mass case. In the PDM scenario, the problem is trickier and will be taken up elsewhere.

**II. PSEUDO-HERMITICITY AND $\mathcal{CPT}$-SYMMETRY IN A SUPERSYMMETRIC FRAMEWORK**

In the framework of supersymmetric (SUSY) quantum mechanics \cite{33,35}, an underlying anticommutator $K$ of the supercharges $Q$ and $\overline{Q}$ can be explicitly constructed by specifying the following representation

\begin{equation}
K = \{Q, \overline{Q} \} = \begin{pmatrix} \zeta \zeta^* & 0 \\ 0 & \zeta^* \zeta \end{pmatrix},
\end{equation}

where $Q$ and $\overline{Q}$ are defined in terms of the operator $\zeta$ and its complex conjugate $\zeta^*$

\begin{equation}
Q = \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix}, \quad \overline{Q} = \begin{pmatrix} 0 & 0 \\ \zeta^* & 0 \end{pmatrix}.
\end{equation}

Assuming polynomial expansions

\begin{equation}
\zeta \zeta^* = \sum_{k=0}^{N} l_k (H_m)^{N-k}, \quad \zeta^* \zeta = \sum_{k=0}^{N} l_k (H_m^*)^{N-k}, \quad [l_0 \equiv 1, H_m^0 \equiv I_2]
\end{equation}

we get by post-multiplying the first relation and pre-multiplying the second relation above by $\zeta$ and subtracting

\begin{equation}
0 = \sum_{k=0}^{N-1} l_k (H_m)^{N-k} \zeta - \sum_{k=0}^{N-1} l_k (H_m^*)^{N-k} \zeta.
\end{equation}

Similarly, by pre-multiplying the first relation and post-multiplying the second relation by $\zeta^*$ and subtracting

\begin{equation}
0 = \sum_{k=0}^{N-1} l_k \zeta^* (H_m)^{N-k} - \sum_{k=0}^{N-1} l_k (H_m^*)^{N-k} \zeta^*.
\end{equation}

(2.4) and (2.5) lead to the intertwining relations

\begin{equation}
H_m \zeta = \zeta H_m^*, \quad H_m^* \zeta^* = \zeta^* H_m.
\end{equation}
At play are also the following constraints

Pseudo-hermiticity constraint: \( \zeta^\dagger = \zeta \Rightarrow \mathcal{C}^\dagger [-x] = \mathcal{C}[x] \), \hfill (2.7)

\( \mathcal{CPT} \) constraint: \( \mathcal{CPT} H_m = H_m \mathcal{CPT} \Rightarrow \mathcal{C}[x] H^*_m [-x] = H_m [x] \mathcal{C}[x] \), \hfill (2.8)

SUSY constraint: \( \zeta \zeta^* = \sum_{k=0}^{N} l_k H_{m}^{N-k} \Rightarrow \mathcal{C}[x] \mathcal{C}^*[x] = \sum_{k=0}^{N} l_k H_{m}^{N-k} \). \hfill (2.9)

Finally, in the context of the \( N \)-th order SUSY, a mass-deformed superpotential \( W_m(x) \) can be introduced which is given by the form \[28\] \[
W_m(x) = W(x) - \frac{N}{2} \left[ \frac{1}{\sqrt{m(x)}} \right]' . \] \hfill (2.10)

where \( W(x) \) corresponds to the superpotential of the constant mass case. A natural consequence of (2.10) is that unlike \( W(x) \) as in the constant-mass case \( W_m(x) \) turns out to be \( \mathcal{PT} \)-symmetric from the pseudo-Hermiticity constraint (2.7) as will be revealed below.

### III. FIRST ORDER CHARGE OPERATOR

The first order representation of the charge operator \( \mathcal{C} \) in a PDM scheme is given by

\[
\mathcal{C} = \frac{1}{\sqrt{m(x)}} \frac{d}{dx} + W(x) . \] \hfill (3.1)

From (2.6) we have for \( N = 1 \) the projections

\[
\zeta^* \zeta = H_m + l_1 , \quad \zeta^* \zeta = H^*_m + l_1 . \] \hfill (3.2)

Imposing the pseudo-hermiticity restriction (2.7), we have the solutions:

\[
m(x) = m(-x) , \quad W(x) = W^*(-x) - \frac{1}{2} \frac{m'(x)}{m^3/2(x)} . \] \hfill (3.3)

It is evident from (2.10) and (3.3) that

\[
W_m(x) = \mathcal{PT} W_m(x) . \] \hfill (3.4)

implying \( W_m(x) \) to be \( \mathcal{PT} \)-symmetric and the mass function to be parity-invariant. As remarked earlier, \( W(x) \) ceases to be \( \mathcal{PT} \)-symmetric. A \( \mathcal{PT} \)-symmetric \( W_m(x) \) can be implemented by choosing for \( W(x) \) the form say, \( W(x) = \exp(i\alpha x) + h(x) \) where \( \alpha \in \mathbb{R} \) and a non-\( \mathcal{PT} \) \( h(x) \) can be confronted with a suitable parity-invariant mass function leaving \( W_m(x) \) to be \( \mathcal{PT} \)-symmetric. The following concrete example is one we have in mind

\[
W(x) = \exp(i\alpha x) - \sin(x) , \quad m(x) = \frac{1}{4} \sec^2(x) \Rightarrow W_m(x) = \exp(i\alpha x) \quad 0 < x < \frac{\pi}{2} \] \hfill (3.5)

where \( W_m(x) \) is a periodic potential.

Turning to the \( \mathcal{CPT} \)-constraint (2.8) and using (3.1) we get two relations. While comparison of the \( \partial \)-term yields the difference

\[
\tilde{V}_m(x) - \mathcal{PT} \tilde{V}_m(x) = 2 \frac{W'_m(x)}{\sqrt{m(x)}} , \] \hfill (3.6)

the remaining part results in a first-order differential equation which can be readily integrated to provide for \( \tilde{V}_m(x) \) the expression

\[
\tilde{V}_m(x) = W_m^2(x) + \frac{W_m'(x)}{\sqrt{m(x)}} + \frac{1}{4} \frac{m''}{m^2} - \frac{7}{16} \frac{m'^2}{m^3} + \Lambda , \] \hfill (3.7)
where \( \Lambda \) is an arbitrary constant of integration.

A non-trivial form for \( \tilde{V}_m(x) \) is also obtained on employing the SUSY constraint \( \{ \mathcal{C}, \psi \} = \mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi - \mathcal{E} \mathcal{P} \mathcal{T} \mathcal{E} \psi = E^* \mathcal{P} \mathcal{T} \mathcal{E} \psi \), for \( N = 1 \) namely
\[
\mathcal{C}[x] \tilde{V}_m(x) = \mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi = \mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi = E^* \mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi = E^* \mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi.
\] (3.8)

A comparison between (3.7) and (3.8) fixes \( \Lambda = -l_1 \). (3.8) is our final form of \( \tilde{V}_m(x) \) for the \( N = 1 \) case. Note that the underlying \( \mathcal{C} \mathcal{P} \mathcal{T} \)-invariance has the implication
\[
\mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi(x) = \mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi(x) = \mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi(x) = E^* \mathcal{C} \mathcal{P} \mathcal{T} \mathcal{E} \psi(x).
\] (3.9)

Thus if \( (\psi, E) \) is an eigenpair of a \( \mathcal{C} \mathcal{P} \mathcal{T} \)-invariant PDM Hamiltonian \( \mathcal{H}_m \), then \( (\mathcal{C} \mathcal{P} \mathcal{T} \psi, E^*) \) must form another eigenpair provided \( \mathcal{C} \mathcal{P} \mathcal{T} \psi \neq 0 \). Thus \( \mathcal{C} \mathcal{P} \mathcal{T} \)-invariance of \( \psi \) leads to the corresponding Hamiltonian having real eigenvalues.

From the first relation of (2.3) and (1.6), the ground state \( \psi_0 \) in the \( N = 1 \) case has to obey
\[
\zeta^* \psi_0(x) = 0 \Rightarrow \left[ \frac{1}{m(x)} \partial + \mathcal{W}^*(x) \right] \psi_0(-x) = 0,
\] (3.10)
with the lowest eigenvalue \(-l_1\). From (3.10) we find
\[
\psi_0(x) = N_0 \frac{m(x)}{4} \exp \left[ \int^x \sqrt{m(y) \mathcal{W}_m(y)} \, dy \right],
\] (3.11)
\( N_0 \) being the normalization constant. Note that \( \psi_0(x) \) is non-\( \mathcal{P} \mathcal{T} \)-symmetric.

**IV. SECOND ORDER CHARGE OPERATOR**

We now look at the following mass-dependent second order representation of the charge operator \( \mathcal{C} \)
\[
\mathcal{C} = \frac{1}{m(x)} \frac{d^2}{dx^2} + \mathcal{W}(x) \frac{d}{dx} + \mathcal{U}_0(x),
\] (4.1)
accompanied by the \( N = 2 \) SUSY representations
\[
\zeta \zeta^* = H_m^2 + l_1 H_m + l_2 I_2.
\] (4.2)
as follows from (2.3). For the literature on \( N = 2 \) SUSY in the constant-mass case we refer to the readers [39–44].

Employing the pseudo-Hermiticity requirement (2.7) gives the following solutions
\[
m(-x) = m(x), \quad \mathcal{W}_m(x) = \mathcal{P} \mathcal{T} \mathcal{W}_m(x)
\] (4.3)
which are similar to the \( N = 1 \) case i.e. \( \mathcal{W}_m(x) \) is \( \mathcal{P} \mathcal{T} \)-symmetric and the mass function \( m(x) \) is parity-invariant. Note that according to (2.10), \( \mathcal{W}_m(x) \) is related to the constant-mass superpotential \( \mathcal{W}(x) \) by
\[
\mathcal{W}_m(x) = \mathcal{W}(x) - \left( \frac{1}{m} \right)'.
\] (4.4)

As an illustrative example we can take this time
\[
\mathcal{W}(x) = \exp(i\alpha x) - \sin(x), \quad m(x) = \sec(x) \quad \alpha \in \mathbb{R}, \quad 0 < x < \frac{\pi}{2}
\] (4.5)
leading again to a periodic \( \mathcal{P} \mathcal{T} \)-symmetric \( \mathcal{W}_m(x) = \exp(i\alpha x) \).

Apart from (4.3), the pseudo-Hermiticity condition also furnishes another relation namely
\[
\triangle \mathcal{U}_0(x) \equiv \mathcal{U}_0(x) - \mathcal{P} \mathcal{T} \mathcal{U}_0(x) = \mathcal{W}_m(x),
\] (4.6)
which reflects the non-\( \mathcal{P} \mathcal{T} \)-symmetric character of the function \( \mathcal{U}_0(x) \) present in (4.1).

Next, consideration of the \( \mathcal{C} \mathcal{P} \mathcal{T} \) requirement (2.3) furnishes
\[
\triangle \tilde{V}_m(x) \equiv \tilde{V}_m(x) - \mathcal{P} \mathcal{T} \tilde{V}_m(x) = 2\mathcal{W}_m(x) - \frac{m'}{m} \mathcal{W}_m(x)
\] (4.7)
which is slightly different in form from the $N = 1$ result \[3.7\]. In \[4.7\] $\bar{V}_m(x)$ is restricted by

$$\bar{V}_m(x) = \triangle \bar{V}_m(x) + f(x) - \mathcal{U}_0(x) + \Lambda$$

(4.8)

where

$$f(x) = \frac{1}{2} \left[ \frac{m \mathcal{W}_m^2(x) - m' m}{m} \mathcal{W}_m(x) - \mathcal{W}_m'(x) \right]$$

(4.9)

on making use of \[4.3\] and \[4.6\]. The constant $\Lambda$ appears in \[4.8\] through the process of integration and is left arbitrary at this stage.

In addition to \[4.7\] and \[4.8\], the non-$\mathcal{PT}$-function $\mathcal{U}_0(x)$ has to satisfy the differential equation

$$\left[ \frac{\mathcal{U}_0'(x)}{m(x)} \right]' - \mathcal{U}_0(x) \triangle \bar{V}_m(x) + \mathcal{W}_m(x) \mathcal{P} \mathcal{T} \bar{V}_m(x) \right] + \left[ \mathcal{P} \mathcal{T} \bar{V}_m(x) \right]' = 0.$$  

(4.10)

Substitution of \[4.8\] into \[4.10\] converts it to the form

$$\mathcal{U}_0'(x) \mathcal{W}_m(x) + \mathcal{U}_0(x) \left[ 2 \mathcal{W}_m'(x) + \frac{m'}{m} \mathcal{W}_m(x) \right] = \frac{f''(x)}{m} + f'(x) \left[ \mathcal{W}_m(x) - \frac{m'}{m^2} \right],$$

(4.11)

which may be solved to arrive at

$$\mathcal{U}_0(x) = \frac{f'(x)}{m(x) \mathcal{W}_m(x)} + \frac{f^2(x)}{m(x) \mathcal{W}_m^2(x)} + \frac{\Theta}{m \mathcal{W}_m^2(x)}.$$  

(4.12)

$\Theta$ being an arbitrary constant of integration.

We now attend to the SUSY constraint \[2.9\]. Here we need to compare the five coefficients of $\partial^l$, $l = 0, 1, 2, 3, 4$. While the first two produce solutions similar to \[4.3\], the last three respectively yields the following three relations:

$$2 \frac{1}{m(x)} \frac{\mathcal{U}_0(x) + \mathcal{PT} \mathcal{U}_0(x)}{2} - 2 \frac{1}{m(x)} \mathcal{W}_m'(x) = \mathcal{W}_m^2(x) - \frac{1}{m(x)} \mathcal{W}_m(x) - \frac{1}{m(x)} \mathcal{W}_m'(x),$$

(4.13)

$$\frac{1}{m(x)} \left[ \mathcal{U}_0'(x) + \mathcal{W}_m(x) \mathcal{U}_0''(x) \right] = \frac{1}{m(x)} \mathcal{W}_m''(x) + \left[ \mathcal{W}_m'(x)/2 \right]',$$

(4.14)

$$\left[ \frac{1}{m(x)} \left( \mathcal{PT} \mathcal{U}_0(x) \right)' \right]' + \mathcal{W}_m(x) \left( \mathcal{PT} \mathcal{U}_0(x) \right)' + \mathcal{U}_0(x) \mathcal{PT} \mathcal{U}_0(x) = \bar{V}_m'(x) + \mathcal{U}_0(x) + l_2 - \frac{1}{m(x)} \bar{V}_m''(x).$$

(4.15)

To tackle the set of equations \[4.13\]–\[4.15\], we observe that the second equation here can be integrated out entirely to have

$$\triangle \mathcal{U}_0(x) = \mathcal{W}_m''(x) + C \exp \left[- \int^y m(y) \mathcal{W}_m(y) \, dy \right],$$

(4.16)

where $C$ is a constant of integration. But $C$ has to be set equal to zero to be consistent with \[4.6\]. So we are left with \[4.6\] only. Incorporating it along with \[4.7\],\[4.9\] and \[4.13\] $\bar{V}_m(x)$ reads

$$\bar{V}_m(x) = \triangle \bar{V}_m(x) + f(x) - \mathcal{U}_0(x) - l_1/2,$$

(4.17)

Then looking at \[4.8\] prompts us to identify $\Lambda = -\frac{i}{4}$ and recast $\bar{V}_m(x)$ as

$$\bar{V}_m(x) = \frac{3}{2} \mathcal{W}_m'(x) + \frac{m'}{2m} \mathcal{W}_m(x) + \frac{m}{2} \mathcal{W}_m^2(x) - \mathcal{U}_0(x) - \frac{l_1}{2}.$$  

(4.18)

We now focus on the remaining SUSY constraint \[4.15\]. This can be converted to a second-order differential equation

$$\left[ \frac{1}{m(x)} \left( \mathcal{U}_0(x) + \bar{V}_m^*(x) \right) \right]' - \mathcal{U}_0'(x) \mathcal{W}_m(x) + \left[ \mathcal{U}_0(x) - \mathcal{W}_m'(x) \right] \mathcal{U}_0(x) = \left[ \bar{V}_m^*(x) + \frac{l_1}{2} \right]^2 + \left( l_2 - \frac{l_1^2}{4} \right).$$  

(4.19)
by applying the $\mathcal{PT}$-operator on both sides and rearranging. Note that the action of $\mathcal{PT}$ on any function $g(x)$ is to be understood in the usual sense: $\mathcal{PT}g(x) = g^*(-x), \mathcal{PT}g'(x) = -g^*(-x)$ and so on. The nonlinear term $U_0^m$ in (4.19) is redundant and can be eliminated in the following way. Using the relation $V_m^*(-x) \equiv \mathcal{PT} \bar{V}_m(x) = \bar{V}_m(x) - \Delta V_m(x)$, (4.18) results in

$$\bar{V}_m^*(-x) + U_0(x) = f(x) - \frac{l_1}{2}. \quad (4.20)$$

Employing (4.20), (4.19) can be reduced to the first order form

$$W_m(x)U_0'(x) + [W_m'(x) - 2f(x)]U_0(x) = \left[ \frac{1}{m(x)}(x)f'(x) \right]' - f^2(x) + \left[ \frac{l_1^2}{4} - l_2 \right]. \quad (4.21)$$

Equation (4.21) which essentially results from the SUSY constraint (4.15) is consistent with the $\mathcal{CPT}$ equation (4.11) for $U_0(x)$ given by (4.12) should we identify $\Theta = \left[ l_2 - \frac{l_1^2}{4} \right]$. In terms of $\delta = +\sqrt{l_1^2 - 4l_2}$ we express $U_0(x)$ as

$$U_0(x) = \frac{m(x)W_m^2(x)}{4} + \frac{W_m'(x)}{2} - \frac{W_m''(x)}{2m(x)W_m(x)} + \frac{1}{m(x)} \left( \frac{W_m'(x)}{2W_m(x)} \right)^2$$

$$- \frac{3m^2}{4m^3(x)} - \frac{m''(x)}{2m^2(x)} - \frac{1}{m(x)} \left( \frac{\delta}{2W_m(x)} \right)^2. \quad (4.22)$$

As a specific example we can go for the choice (4.20) which would give

$$U_0(x) = \frac{1}{4} \sec(x) \exp(2i\alpha x) - \frac{\delta^2}{4} \cos(x) \exp(-2i\alpha x) + \frac{i\alpha}{2} \exp(i\alpha x)$$

$$+ \frac{\alpha^2}{4} \cos(x) + \frac{1}{4} \sin^2(x) \sec(x) - \frac{1}{2} \sec(x). \quad (4.23)$$

Evidently $U_0(x)$ is non-$\mathcal{PT}$-symmetric.

Let us now analyze the solution of the zero-mode equation

$$\zeta^* \psi(x) = 0 \Rightarrow \left[ \frac{1}{m} \frac{d^2}{dx^2} + W^*(x) \partial + U_0^*(x) \right] \psi(-x) = 0. \quad (4.24)$$

Two linearly independent solutions of zero-mode equation (4.24) may be expressed in the following compact form (see for details [28]):

$$\psi_j(x) = N_j \sqrt{m(x)W_m(x)} \exp \left[ \int^x F_j(y) \, dy \right], \quad (4.25)$$

where

$$F_j(x) = \frac{m(x)W_m^2(x) + (-1)^j \delta}{2W_m(x)}, \quad j = 1, 2. \quad (4.26)$$

These solutions will correspond the ground and first excited states of $H_m$, a feature known in the quadratic SUSY algebra.

Now it follows from the quadratic SUSY algebra (4.2) that the lowest eigenvalues of $H_m$ are roots of the following quadratic equation

$$E^2 + l_1E + l_2 = 0 \Rightarrow E_0 = -\frac{l_1 + \delta}{2}, E_1 = -\frac{l_1 - \delta}{2}. \quad (4.27)$$

It is clear that the lowest two eigenvalues $E_{0,1}$ will be purely real if and only if the SUSY constants $l_1, l_2$ satisfy following inequality

$$l_1^2 \geq 4l_2. \quad (4.28)$$

It may be pointed out that the condition $l_1^2 \geq 4l_2$ was identified with the reducibility of the second-order SUSY construction [40] in the context of Hermitian QM. In non-Hermitian QM, we have shown that the same condition is related with the reality of the spectra.
V. N-TH ORDER CHARGE OPERATOR

The charge conjugate operator $\mathcal{C}$ may be represented as $N$-th order differential operator with $N$ coefficient functions

$$\mathcal{C} = \frac{1}{\sqrt{m^N}} \partial^N + W(x) \partial^{N-1} + \sum_{j=0}^{N-2} U_j(x) \partial^j, \quad N = 1, 2, \ldots, $$ \hfill (5.1)

Some of the previous results are possible to generalize. Firstly, the pseudo-Hermiticity constraint (2.7) need to be compared order by order from both sides for $N$-th order representation (5.1) of $\mathcal{C}$. To do this, we note that the contributions from the adjoint operation on the term $g(x) \partial^\ell$ may be computed using the Libneitz rule as follows

$$[g(x) \partial^\ell] = (-1)^\ell \partial^\ell [g(x)] = (-1)^\ell \sum_{r=0}^\ell [C_r \partial^r (g(x))] \partial^{\ell-r} \] . \hfill (5.2)$$

Then order by order comparison gives the following restrictions on the coefficient functions in the charge operator $\mathcal{C}$ given by (5.1)

$$\ell = N : \quad \mathcal{P} [m(x)] = m(x), \quad \text{(mass is parity-invariant)}$$
$$\ell = N - 1 : \quad \mathcal{P} \mathcal{T} [W_m(x)] = W_m(x), \quad \text{(superpotential is $\mathcal{P} \mathcal{T}$-invariant)} \} , \quad N = 1, 2, 3, \ldots . \hfill (5.3)$$

One may compare the general result derived above with the corresponding results for $N = 1$ [ see (3.3) and (3.4)] and for $N = 2$ [ see (4.3)]. Note that the second condition means as usual that the Re $W_m(x)$ is an even function while its imaginary part Im $W_m(x)$ is an odd function.

In contrast to the superpotential $W_m$, the functions $U_j, j = 0$ to $N - 2$, are not $\mathcal{P} \mathcal{T}$-symmetric. For instance, for $\ell = N - 2$ and $\ell = N - 3$ we have

$$U_{N-2}(x) - \mathcal{P} \mathcal{T} [U_{N-2}(x)] = (N-1)W_m(x), \quad N = 2, 3, \ldots $$ \hfill (5.4)

and so on. More generally,

$$\Delta U_{N-s}(x) = U_{N-s}(x) - \mathcal{P} \mathcal{T} U_{N-s}(x) \frac{N}{2} \left[ \frac{1}{\sqrt{m^N}} \right] \right] + N^{-1} \left[ \mathcal{P} \mathcal{T} W(x) \right] + \sum_{j=1}^{s-2} N^{-s+j} C_j \partial^j \left[ \mathcal{P} \mathcal{T} [U_{N-s+j}(x)] \right] , \hfill (5.6)$$

From the results, it is clear that the pseudo-Hermiticity constraints measure the amount of $\mathcal{P} \mathcal{T}$-asymmetry in the coefficient functions. In particular, the measure is zero for first coefficient $m(x)$ and mass-deformed superpotential $W_m(x)$.

Next comparing the coefficients of each derivative $\partial^\ell$ for $\ell = 0, 1, \ldots, N + 2$ from both sides of the $\mathcal{C} \mathcal{P} \mathcal{T}$-constraint (2.8), a straightforward calculation shows

$$\Delta \tilde{V}_m(x) = \tilde{V}_m(x) - \mathcal{P} \mathcal{T} [\tilde{V}_m(x)] = \frac{\sqrt{m^N}}{m^2} \left[ 2mW_m(x) + (N-1)m'W_m(x) \right] , N = 1, 2, 3, \ldots . \hfill (5.7)$$

Comparison for $\ell = N - 1$ gives a closed expression for the potential due to the integrability of the equation

$$N \tilde{V}_m(x) = \frac{\sqrt{m^N}}{m^2} \left[ N^2 C_2 m \right] W_m(x) + m \left\{ \sqrt{m^N} W_m(x) + (2N-1)W_m(x) - 2U_{N-2}(x) \right\} , \hfill (5.8)$$

where we set a convention that $N^2 C_j = 0, U_{N-j} = 0$ for $N < j$. Continuing this comparison up to the term $\partial^0$, we find that for all order $N$, only two coefficient functions in the representation of charge operator $\mathcal{C}$ remain independent,
which are the mass function $m(x)$ and the superpotential $W_m$. As for instance, comparing $\partial^{N-2}$ from both sides of (2.8), one obtains for $N \geq 3$

$$
\left[ \frac{U_{N-2}(x)}{m} \right]' - \left[ \frac{U_{N-2}(x)}{m} \right]'' \left[ \frac{U_{N-2}(x)}{m} \right]' - \frac{1}{m} \Delta \tilde{V}_m(x) + \frac{N-1}{m} = \frac{C_1}{m} \frac{1}{m} \Delta \tilde{V}_m(x) + \frac{2}{m} U_{N-3}(x) - (N-3) \left( \frac{1}{m} \right)' U_{N-3}(x)
$$

$$
= N+1 \frac{1}{mN} \left( \frac{1}{m} \right)'' - N C_2 \frac{\tilde{V}_m''(x)}{m^N} + W(x) \left[ N C_2 \left( \frac{1}{m} \right)'' - (N-1)(\tilde{V}_m')(x) \right], \quad N = 2, 3, \ldots, (5.9)
$$

Similar to the first and second order cases, a general nonlinear SUSY algebra can be set up. The energy in such an algebra are zeros of the same $N$-th degree polynomial

$$
E^N + l_1 E^{N-1} + l_2 E^{N-2} + \cdots + l_{N-2} E^2 + l_{N-1} E + l_N = 0
$$

from which we conclude that for an odd-order charge operator, the Hamiltonian $H_m$ possesses at least one real energy eigenvalue.

VI. CONCLUSION

In this article we have studied a generalized PDM Schrödinger equation in a non-Hermitian framework. We have proposed new differential realization for the charge operator and sought for the solvability of the model. Several interesting consequences due to PDM and non-Hermiticity of the Hamiltonian are derived. It should be noted that not all the results of the constant-mass non-Hermitian system are carried over to the PDM case. In constant-mass case, we showed that the superpotential $W(x)$ had to be $\mathcal{PT}$-symmetric to preserve $\mathcal{PT}$-symmetry and pseudo-hermiticity. In contrast, in the present work we have shown that the superpotential $W(x)$ loses its $\mathcal{PT}$-symmetric property. Instead a new mass-deformed superpotential $W_m(x)$ can be defined which turns out to be $\mathcal{PT}$-symmetric. Our work uncovers a new class of potentials $\tilde{V}_m(x)$ admitting $\mathcal{PT}$-symmetry in PDM non-Hermitian systems. We have also obtained extension of some of our results to a general $N$-th order charge operator wherein the mass function remains even and the mass-deformed superpotential $\mathcal{PT}$-symmetric.

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