THE TILTING-COTILTING CORRESPONDENCE

LEONID POSITSELSKI AND JAN ŠTOVÍČEK

Abstract. To a big $n$-tilting object in a complete, cocomplete abelian category $A$ with an injective cogenerator we assign a big $n$-cotilting object in a complete, cocomplete abelian category $B$ with a projective generator, and vice versa. Then we construct an equivalence between the (conventional or absolute) derived categories of $A$ and $B$. Under various assumptions on $A$, which cover a wide range of examples (for instance, if $A$ is a module category or, more generally, a locally finitely presentable Grothendieck abelian category), we show that its counterpart $B$ is the abelian category of contramodules over a topological ring and that the derived equivalences are realized by a contramodule-valued variant of the usual derived Hom-functor.

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INTRODUCTION

Tilting theory has its roots in representation theory of finite-dimensional algebras and has evolved into a powerful derived Morita theory with numerous applications in various fields of algebra and algebraic geometry \[3\]. Our motivation in this paper stems from a beautiful correspondence between finite-dimensional tilting and finite-dimensional cotilting modules which goes back to Brenner and Butler \[15\] and Miyashita \[39\].

We exhibit an equally symmetric and easy to state correspondence (which we call the tilting-cotilting correspondence) in the context of very general abelian categories. This puts several recent generalizations of the Brenner–Butler correspondence (see for instance \[11\], \[13\], \[41\], \[56\], \[62\]) into a unified framework.

However, we also follow another important motive in this paper. As we illustrate on various examples, derived equivalences induced by tilting and cotilting objects sometimes coincide with a comodule-contramodule correspondence introduced by the first-named author in \[42\], §0.2 and Chapter 5].

The second half of our paper is devoted to analyzing this connection in detail. It turns out that one side of our tilting-cotilting correspondence can be rather often described as a category of models of an additive infinitary algebraic theory in the sense of \[64\] (see also \[54\], Introduction]) and, in this context, a derived equivalence with the other end of the tilting-cotilting correspondence is provided by a usual derived Hom-functor. In a wide range of algebraic examples, the infinitary aspect is simply captured by a topology on a ring and the models of the corresponding algebraic theory are so-called contramodules over that topological ring (a concept related to but different from a topological module).

Further aspects of the connection between the tilting theory and comodule-contramodule correspondence are also studied in \[53\], \[55\].

Let us explain our results and their context more in detail. The nowadays classical theorem of Brenner and Butler \[15\], viewed from the perspective of tilting derived equivalences by Happel \[29\] and Cline, Parshall and Scott \[18\], can be stated as follows. If \(A\) is a finite-dimensional algebra, \(T \in A\text{-mod}_{\text{fdim}}\) is a finite-dimensional tilting module in the sense of Miyashita \[39\] and \(B = \text{Hom}_A(T, T)^{\text{op}}\) is its (finite-dimensional) endomorphism algebra, there is a triangle equivalence \(D^b(A\text{-mod}_{\text{fdim}}) \simeq D^b(B\text{-mod}_{\text{fdim}})\) which sends \(T \in A\text{-mod}_{\text{fdim}}\) to the projective generator \(B \in B\text{-mod}_{\text{fdim}}\). Moreover, the vector-space dual \(W = A^* \in A\text{-mod}_{\text{fdim}}\), which is an injective cogenerator there, is sent to a cotilting module in \(B\text{-mod}_{\text{fdim}}\) (defined formally dually to tilting modules). Hence the situation is completely self-dual and is illustrated in Figure 1.

In the last decade, there have been several attempts to recover a part of this picture outside the realm of finite-dimensional algebras. Based on an existing theory of infinitely generated (co)tilting modules, similar results were obtained in \[62\] in the case where \(A\) is a Grothendieck abelian category and \(W \in B\) is a big (i.e., infinitely generated) \(n\)-cotilting module. The heart of a t-structure associated with a
Figure 1. The tilting-cotilting correspondence: $T \in A$ is a tilting object and $W \in A$ an injective cogenerator, while $T \in B$ is a projective generator and $W \in B$ is a cotilting object.

big $n$-tilting module is considered in the recent preprint [12]. A general discussion of (co)tilting objects in triangulated categories can be found in the preprints [56, 41], and of big $n$-(co)tilting objects in abelian categories, in [41, Section 6]. In particular, the introduction to [56] emphasizes the importance of the abelian hearts of the (co)tilting $t$-structures, as opposed to simply the module categories over the endomorphism rings of the (co)tilting objects, in the context of the (co)tilting derived equivalences.

In this paper, we construct a one-to-one correspondence between the two dual settings of complete, cocomplete abelian categories $A$ with an injective cogenerator $W$ and a big $n$-tilting object $T$, and complete, cocomplete abelian categories $B$ with a projective generator $T$ and a big $n$-cotilting object $W$. The correspondence assigns to an abelian category $A$ with a tilting object $T$ the heart $\mathcal{B} = T^*D^b_{\leq 0} \cap T^*D^b_{\geq 0}$ of the tilting $t$-structure on the derived category $D^b(A)$, and to an abelian category $B$ with a cotilting object $W$ the heart $\mathcal{A} = W^*D^b_{\leq 0} \cap W^*D^b_{\geq 0}$ of the cotilting $t$-structure on $D^b(B)$. In addition, we proceed to construct triangulated equivalences $D^*(A) \simeq D^*(B)$ between the (bounded or unbounded, conventional or absolute) derived categories of the abelian categories $A$ and $B$. See again Figure 1.

Furthermore, we consider various restrictions that can be imposed on the abelian category $A$ and discuss the related properties of the abelian category $B$. In particular, whenever the category $A$ is locally presentable, the category $B$ is locally presentable, too. Locally presentable abelian categories with a projective generator can be described as the categories of models of additive $\kappa$-ary algebraic theories for some cardinal numbers $\kappa$ [53]. Formally, $B$ is up to equivalence the category of algebras/modules over an additive monad $T$ on the category of sets, where $T$ is the endomorphism monad of the tilting object $T \in A$ (as opposed to a mere endomorphism ring). When $A$ is a Grothendieck abelian category, $T$ has an additional property that the map $T(X) \to \prod_{x \in X} T(\{x\})$ is injective for all sets $X$.

In particular, when $A = A\text{-mod}$ is the abelian category of modules over an associative ring $A$, we show that the tilting heart $B$ is equivalent to the abelian category $\mathcal{R}\text{-contra}$ of contramodules, in the sense of [47, §2.1], over the topological ring $\mathcal{R} = \text{Hom}_A(T, T)^{op}$ of endomorphisms of the tilting module $T$. The same conclusion
holds in some other cases, e. g., when $A$ is a full subcategory closed under
infinite direct sums in $A\text{-mod}$, or when $A$ is a locally finitely presentable
Grothendieck category. These assertions are deduced from a series of much more
general results claiming that for any object $M \in A$ the full additive subcategory $\text{Add}(M) \subset A$
consisting of all the direct summands of infinite direct sums of copies of $M$ in $A$
is equivalent to the additive category of projective objects $\mathcal{R}\text{-contra}_{\text{proj}}$ in the abelian
category of contramodules $\mathcal{R}\text{-contra}$ over the topological ring $\mathcal{R} = \text{Hom}_A(M, M)^{\text{op}}$,
i. e., $\text{Add}(M) \simeq \mathcal{R}\text{-contra}_{\text{proj}}$.

We also show that the categories of models of additive $\kappa$-ary algebraic theories
relate to module categories in a formally dual way as compared to the Popescu–
Gabriel Theorem for Grothendieck abelian categories. Indeed, Grothendieck abelian
categories can be described as reflective full subcategories in the categories of mod-
ules over associative rings with exact reflection functors. On the other hand, the
categories of models of additive $\kappa$-ary algebraic theories can be presented as reflec-
tive full subcategories in the categories of modules over associative rings with exact
embedding functors. In particular, when $T \in A = A\text{-mod}$ is a “good” $n$-tilting
module in the sense of [11, 13], the exact forgetful functor $\mathcal{R}\text{-contra} \to \mathcal{R}\text{-mod}$ is fully
faithful; so the category $B = \mathcal{R}\text{-contra}$ is a full subcategory in $\mathcal{R}\text{-mod}$. When $n = 1$,
this is what was called the full subcategory of “costatic” modules in the paper [28]
(specifically, in the context of [28 Theorems 5.7 and 6.3]).

In the general case of a good $n$-tilting module $T$, a triangulated equivalence between
the (unbounded) derived category $\mathcal{D}(A\text{-mod})$ and a certain full subcategory or Verdier
quotient category of $\mathcal{D}(\mathcal{R}\text{-mod})$ was constructed in the paper [13]. As a particular
case of our results, we obtain a triangulated equivalence between the (bounded or
unbounded) derived categories of two abelian categories $\mathcal{D}(A\text{-mod})$ and $B = \mathcal{R}\text{-contra} \subset \mathcal{R}\text{-mod}$. This generalizes much further: for any $n$-tilting
object $T$ in a locally presentable abelian category $A$ with an injective cogenerator $W$,
replacing $T$ by the coproduct of its copies $T^{(Y)}$ indexed over a large enough set
$Y$ identifies the related abelian category $B$ with a reflective full subcategory in the
category of modules $B\text{-mod}$ over the associative ring $B = \text{Hom}_A(T^{(Y)}, T^{(Y)})^{\text{op}}$, with
an exact embedding functor $B \to B\text{-mod}$.

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1. Tilting $t$-Structures

Given an abelian category $C$, we denote by $\mathcal{D}(C)$ the derived category of bounded
complexes over $C$. Let $(\mathcal{D}^{b, \leq 0}(C), \mathcal{D}^{b, \geq 0}(C))$ denote the standard $t$-structure on $\mathcal{D}(C)$,
i. e., $\mathcal{D}^{b, \leq 0}(C) \subset \mathcal{D}(C)$ is the full subcategory of complexes with the cohomology
objects concentrated in the nonpositive cohomological degrees and $\mathcal{D}^{b, \geq 0}(C) \subset \mathcal{D}(C)$
is the full subcategory of complexes with the cohomology objects concentrated in
the nonnegative cohomological degrees. A similar notation is used for the standard t-structures on the derived categories of bounded below, bounded above, and unbounded complexes $D^+(C)$, $D^-(C)$, and $D(C)$ [14, n° 1.3].

The constructions of derived categories (and, more generally, Verdier quotient categories) involve a well-known difficulty in that the collection of all morphisms between a fixed pair of objects in the derived category may turn out to be a proper class rather than a set (see [17] for an example). We will say that a certain derived category $D^+(C)$ has Hom sets if this complication does not arise, that is morphisms between any two fixed objects in $D^+(C)$ form a set. Even when this is not the case, one can still work with $D^+(C)$ as a “very large category”, but one has to be cautious. In any event, all the abelian categories in this paper will be presumed or proved to have Hom sets.

The following useful lemma can be found in [41, Lemma 10]. We include a proof for the reader’s convenience.

**Lemma 1.1.** Let $C$ be an abelian category and $T \in C$ an object of projective dimension $\leq n$. Then for every complex $X^* \in D_{\leq -n-1}(C)$ one has $\text{Hom}_{D(C)}(T, X^*) = 0$.

**Proof.** Any morphism $T \rightarrow X^*$ in $D(C)$ can be represented as a fraction formed by a morphism $T \rightarrow Y^*$ and a quasi-isomorphism $X^* \rightarrow Y^*$ of complexes over $C$. Applying the canonical truncation, we can assume that the terms of the complex $Y^*$ are concentrated in the degrees $\leq 0$. Let $Z$ denote the kernel of the differential $Y^{n-1} \rightarrow Y^{-n}$ and let $\sigma_{\geq -n-1} Y^*$ be the subcomplex of the silly filtration of the complex $Y^*$; so one has $H^i(\sigma_{\geq -n-1} Y^*) = 0$ for $i \neq -n-1$ and $H^{-n-1}(\sigma_{\geq -n-1} Y^*) = Z$. Then the morphism of complexes $T \rightarrow Y^*$ factorizes as $T \rightarrow \sigma_{\geq -n-1} Y^* \rightarrow Y^*$ and the morphism $T \rightarrow \sigma_{\geq -n-1} Y^*$ represents an extension class in $\text{Ext}^{n+1}_C(T, Z)$, which vanishes by the assumption. \hfill $\Box$

Let $A$ be an abelian category with set-indexed products and an injective cogenerator. It follows from Freyd’s adjoint functor existence theorem that any complete abelian category with a cogenerator is cocomplete [24, Proposition 6.4]. Furthermore, in any abelian category with enough injective objects the coproducts are exact [38, Exercise III.2]. Thus set-indexed coproducts exist and are exact in $A$.

It follows that both the unbounded derived category $D(A)$ and the unbounded homotopy category $\text{Hot}(A)$ of complexes over $A$ have set-indexed coproducts, and the canonical Verdier localization functor $\text{Hot}(A) \rightarrow D(A)$ preserves set-indexed coproducts. Therefore, so do the cohomology functors $H^i: D(A) \rightarrow A$. Notice also that the bounded below derived category $D^+(A)$ has Hom sets, because it is equivalent to the bounded below homotopy category $\text{Hot}^+(A_{\text{inj}})$ of complexes of injective objects in $A$ (see e. g. [30, Proposition I.4.7]).

Let $n \geq 0$ be an integer. Let us say that an object $T \in A$ is (big) $n$-tilting if the following three conditions are satisfied:

(i) the projective dimension of $T$ in $A$ does not exceed $n$, that is $\text{Ext}^i_A(T, X) = 0$ for all $i > n$ and all $X \in A$;

(ii) $\text{Ext}^i_A(T, T^{(I)}) = 0$ for all $i > 0$ and all sets $I$, where $T^{(I)}$ denotes the coproduct of $I$ copies of $T$ in $A$;
(iii) every complex $X^\bullet \in D(A)$ such that $\text{Hom}_{D(A)}(T, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$ is acyclic.

Denote by $\text{Add}(T) \subset A$ the full subcategory formed by the direct summands of infinite coproducts of copies of the object $T \in A$. Condition (ii) of the above definition allows us to show that the triangulated subcategory of $D^b(A)$ generated by $\text{Add}(T)$ is equivalent to a full subcategory of $\text{Hot}(A)$.

**Lemma 1.2.** Suppose that $T \in A$ satisfies $\text{Ext}_A^i(T, T^{(I)}) = 0$ for all $i > 0$ and all sets $I$. Then the composition

$$\text{Hot}^b(\text{Add}(T)) \xrightarrow{i} \text{Hot}^b(A) \rightarrow D^b(A)$$

is fully faithful and the essential image is the triangulated subcategory of $D^b(A)$ generated by $\text{Add}(T)$.

**Proof.** This is completely analogous to [29, Lemma 1.1]. Since $A$ has exact coproducts, we have $\text{Ext}_A^i(T^{(I)}, T^{(J)}) = 0$ for all $i > 0$ and all sets $I, J$.

If now $X^\bullet, Y^\bullet \in \text{Hot}^b(\text{Add}(T))$, then $\text{Hom}_{\text{Hot}^b(A)}(X^\bullet, Y^\bullet) \cong \text{Hom}_{D^b(A)}(X^\bullet, Y^\bullet)$ is shown by induction on the sum of the widths of $X^\bullet$ and $Y^\bullet$. If $X^\bullet, Y^\bullet$ have widths one, then $X^\bullet \cong X'[i]$ and $Y^\bullet \cong Y'[j]$ for some $X', Y' \in \text{Add}(T)$ and $i, j \in \mathbb{Z}$. If $i \neq j$, then $\text{Hom}_{\text{Hot}^b(A)}(X^\bullet, Y^\bullet) = 0 = \text{Hom}_{D^b(A)}(X^\bullet, Y^\bullet)$ by the assumption on $T$, and if $i = j$, then $\text{Hom}_{\text{Hot}^b(A)}(X^\bullet, Y^\bullet) \cong \text{Hom}_A(X', Y') \cong \text{Hom}_{D^b(A)}(X^\bullet, Y^\bullet)$.

If $X^\bullet$ has width greater than one, we use the silly truncation to find a triangle $X_1^\bullet \rightarrow X^\bullet \rightarrow X_2^\bullet \rightarrow X_1^\bullet[1]$ such that $X_1^\bullet, X_2^\bullet \in \text{Hot}^b(\text{Add}(T))$ have widths smaller than the width of $X^\bullet$. Applying $\text{Hom}(-, Y^\bullet)$ to this triangle and using the 5-lemma, we deduce that $\text{Hom}_{\text{Hot}^b(A)}(X^\bullet, Y^\bullet) \cong \text{Hom}_{D^b(A)}(X^\bullet, Y^\bullet)$. If the width of $Y^\bullet$ is greater than one, we proceed similarly.

The following theorem is the main result of this section.

**Theorem 1.3.** Let $T \in A$ be an $n$-tilting object. Then the pair of full subcategories

$$T_{D^\leq 0} = \{ X^\bullet \in D(A) \mid \text{Hom}_{D(A)}(T, X^\bullet[i]) = 0 \text{ for all } i > 0 \},$$

$$T_{D^\geq 0} = \{ X^\bullet \in D(A) \mid \text{Hom}_{D(A)}(T, X^\bullet[i]) = 0 \text{ for all } i < 0 \}$$

is a $t$-structure on the unbounded derived category $D(A)$.

**Proof.** Let $X^\bullet \in D(A)$ be a complex. We start with constructing an approximation triangle

$$\tau_{\leq 0}^T X^\bullet \rightarrow X^\bullet \rightarrow \tau_{\geq 1}^T X^\bullet \rightarrow (\tau_{\leq 0}^T X^\bullet)[1]$$

with $\tau_{\leq 0}^T X^\bullet \in T_{D^\leq 0}$ and $\tau_{\geq 1}^T X^\bullet \in T_{D^\geq 1}$.

Notice that for every complex $Y^\bullet \in D(A)$ the collection of all morphisms $T \rightarrow Y^\bullet$ in $D(A)$ is a set, because $\text{Hom}_{D(A)}(T, Y^\bullet) = \text{Hom}_{D^+(A)}(T, \tau_{\geq -n} Y^\bullet)$ by Lemma [1.1] and the category $D^+(A)$ has Hom sets. Proceeding by induction, put $X_0^\bullet = X^\bullet$ and for every $i \geq 0$ consider a distinguished triangle

$$T[i]^{(H_i)} \rightarrow X_i^\bullet \rightarrow X_{i+1}^\bullet \rightarrow T[i + 1]^{(H_i)}.$$
where $H_i = \text{Hom}_{D(A)}(T[i], X_i^*)$ and the first map in the triangle is the canonical one. Using the condition (ii), one easily proves by induction on $i$ that

$$\text{(1)} \quad \text{Hom}_{D(A)}(T[j], X_{i+1}^*) = 0 \quad \text{for } j = 0, \ldots, i.$$  

The complexes $X_i^*$ form an inductive system $X_0^* \to X_1^* \to X_2^* \to \cdots$. Since countable coproducts exist in $D$, we can construct a homotopy colimit of this system, defined as the third object in a distinguished triangle

$$\prod_{i=0}^{\infty} X_i^* \to \coprod_{i=0}^{\infty} X_i^* \to \text{hocolim}_{i \geq 0} X_i^* \to \prod_{i=0}^{\infty} X_i^*[1],$$

where the first map in the triangle is id - shift: $\prod_{i=0}^{\infty} X_i^* \to \prod_{i=0}^{\infty} X_i^*$. Put

$$\tau_{\geq 1}^T X^* = \text{hocolim}_{i \geq 0} X_i^*.$$  

By the octahedron axiom (or more precisely, by [14, Proposition 1.1.11]), a cone of the natural morphism $X_i^* \to X_i^*$ is at the same time a cone of a certain morphism $\coprod_{i \geq k} S_{ki}^* \to \prod_{i \geq k} S_{ki}^*$.

Since the cohomology functors $D(A) \to A$ preserve countable coproducts, we can conclude that a cone of the morphism $X_i^* \to \tau_{\geq 1}^T X^*$ belongs to $D_{\leq -k-1}(A)$. Taking $j \geq 0$ and $k > n + j$, by Lemma 1.7.1 and (1) we have

$$\text{Hom}_{D(A)}(T[j], \tau_{\geq 1}^T X^*) = \text{Hom}_{D(A)}(T[j], X_i^*) = 0.$$  

Hence $\text{Hom}_{D(A)}(T[j], \tau_{\geq 1}^T X^*) = 0$ for all $j \geq 0$ and $\tau_{\geq 1}^T X^* \in T D_{\geq 1}$.

On the other hand, a cocone $\tau_{\leq 0}^T X^*$ of the morphism $X^* \to \tau_{\geq 1}^T X^*$ is at the same time a cocone of a morphism $\coprod_{i \geq 0} S_{0,i}^* \to \prod_{i \geq 0} S_{0,i}^*$. Using Lemma 1.2, one shows that the object $S_{0,i}^* \in D(A)$ can be represented by a complex of the form

$$T^{(H_{-1})} \to \cdots \to T^{(H_1)} \to T^{(H_0)}$$

in the abelian category $A$, sitting in the cohomological gradings from $-i$ to $1$. Hence the object $\coprod_{i \geq 0} S_{0,i}^*$ can be represented by a complex sitting in the cohomological degrees $\leq -1$ whose terms are copowers of the object $T$. Applying again the condition (ii) and Lemma 1.1, we conclude that

$$\text{Hom}_{D(A)}(T[j], \coprod_{i \geq 0} S_{0,i}^*) = 0 \quad \text{for } j \leq 0,$$

hence $\text{Hom}_{D(A)}(T[j], \tau_{\leq 0}^T X^*) = 0$ for $j \leq -1$ and $\tau_{\leq 0}^T X^* \in T D_{\leq 0}$.

Now let $X^*$ be a complex belonging to $T D_{\leq 0}$, so $\text{Hom}_{D(A)}(T[j], X^*) = 0$ for $j \leq -1$. Then, similarly to (1), we have

$$\text{Hom}_{D(A)}(T[j], X_{i+1}^*) = 0 \quad \text{for all } j \leq i, \quad j \in \mathbb{Z}.$$
Arguing as above, we can conclude that $\text{Hom}_{D(A)}(T[j], \tau^T_{\geq 1}X^*) = 0$ for all $j \in \mathbb{Z}$. By the condition (iii), it follows that $\tau^T_{\geq 1}X^* = 0$ and $X^* \simeq \tau^T_{\leq 0}X^*$ in $D(A)$.

Therefore, every object $X^* \in T D^{\leq 0}$ can be obtained from the objects $T, T[1], T[2], T[3], \ldots$ using extensions and coproducts in $D(A)$, or in other words, $T D^{\leq 0} \subset D(A)$ is the suspended subcategory generated by $T$. Hence $\text{Hom}_{D(A)}(X^*, Y^*) = 0$ for all $X^* \in T D^{\leq 0}$ and $Y^* \in T D^{\geq 1}$. The theorem is proved. 

\textbf{Corollary 1.4.} Let $T \in A$ be an n-tilting object. Then

(a) the pair of full subcategories

\[ T D^{\leq 0} = \{ X^* \in D^{\leq 0}(A) \mid \text{Hom}_{D^{\leq 0}(A)}(T, X^*[i]) = 0 \text{ for all } i > 0 \} , \]

\[ T D^{\geq 0} = \{ X^* \in D^{\geq 0}(A) \mid \text{Hom}_{D^{\geq 0}(A)}(T, X^*[i]) = 0 \text{ for all } i < 0 \} \]

is a t-structure on the bounded above derived category $D^{\leq 0}(A)$;

(b) the pair of full subcategories

\[ T D^{+} = \{ X^* \in D^{+}(A) \mid \text{Hom}_{D^{+}(A)}(T, X^*[i]) = 0 \text{ for all } i > 0 \} , \]

\[ T D^{-} = \{ X^* \in D^{-}(A) \mid \text{Hom}_{D^{-}(A)}(T, X^*[i]) = 0 \text{ for all } i < 0 \} \]

is a t-structure on the bounded below derived category $D^{-}(A)$;

(c) the pair of full subcategories

\[ T D^{b, \leq 0} = \{ X^* \in D^{b, \leq 0}(A) \mid \text{Hom}_{D^{b, \leq 0}(A)}(T, X^*[i]) = 0 \text{ for all } i > 0 \} , \]

\[ T D^{b, \geq 0} = \{ X^* \in D^{b, \geq 0}(A) \mid \text{Hom}_{D^{b, \geq 0}(A)}(T, X^*[i]) = 0 \text{ for all } i < 0 \} \]

is a t-structure on the bounded derived category $D^{b}(A)$.

\textbf{Proof.} By the definition of $T D^{\geq 0}$, we have $D^{\geq 0}(A) \subset T D^{\geq 0}$. Since $(T D^{\leq 0}, T D^{\geq 0})$ is a t-structure on $D(A)$ by Theorem 1.3 and $(D^{\leq 0}(A), D^{\geq 0}(A))$ is also a t-structure on $D(A)$, it follows that $D^{\leq 0}(A) \supset T D^{\leq 0}$. By Lemma 1.4, we have $D^{\leq 0}(A) \subset T D^{\leq 0}$, and consequently $D^{\geq 0}(A) \subset T D^{\geq 0}$. To sum up, the inclusions of full subcategories

\[ D^{\leq 0}(A) \subset T D^{\leq 0} \subset D^{\leq 0}(A) , \]

\[ D^{\geq 0}(A) \subset T D^{\geq 0} \subset D^{\geq 0}(A) \]

hold in the derived category $D(A)$.

Now the assertions (a–c) are easily deduced. For example, let us prove (c). Notice that we have $T D^{b, \leq 0} = D^{b, \leq 0}(A) \cap T D^{\leq 0} \subset T D^{\leq 0}$ and $T D^{b, \geq 0} = D^{b, \geq 0}(A) \cap T D^{\geq 0} \subset T D^{\geq 0}$, so $\text{Hom}_{D^{b}(A)}(X^*, Y^*) = \text{Hom}_{D(A)}(X^*, Y^*) = 0$ for all $X^* \in T D^{b, \leq 0}$ and $Y^* \in T D^{b, \geq 1}$.

Furthermore, let $X^* \in D^{b}(A) \subset D(A)$ be a bounded complex and $\tau^T_{\leq 0}X^*$, $\tau^T_{\geq 1}X^*$ be its truncations with respect to the t-structure $(T D^{\leq 0}, T D^{\geq 0})$ on $D(A)$. Then $\tau^T_{\leq 0}X^* \in T D^{\leq 0} \subset D^{\leq 0}(A) \subset D^{b}(A)$ by (2), and $\tau^T_{\geq 1}X^* \in T D^{\geq 1} \subset D^{\geq 1}(A) \subset D^{b}(A)$ by (3). It follows that the complexes $\tau^T_{\leq 0}X^*$, $\tau^T_{\geq 1}X^*$ belong to $D^{b}(A)$.

The following proposition provides the converse implication to Corollary 1.4(c). In fact, it shows that an object $T \in A$ of finite projective dimension is tilting in our sense if and only if it is a tilting object in $D^{b}(A)$ in the sense of [50, Definition 4.1].
Proposition 1.5. Let $T \in A$ be an object satisfying the conditions (i–ii). Suppose that the pair of full subcategories $(T \mathcal{D}^{b\leq 0}, T \mathcal{D}^{b\geq 0})$ is a t-structure on the bounded derived category $\mathcal{D}^b(A)$. Then the condition (iii) is satisfied.

Proof. Consider an unbounded complex $X^* \in \mathcal{D}(A)$ such that $\text{Hom}_{\mathcal{D}(A)}(T, X^*[i]) = 0$ for $0 \leq i \leq n$. We will show that $H^0(X^*) = 0$.

Let $\tau_{\leq 0}, \tau_{\geq 0}$ denote the truncation functors in the standard t-structure on $\mathcal{D}(A)$. By Lemma 1.1, we have $\text{Hom}_{\mathcal{D}(A)}(T, (\tau_{\leq 0}X^*)[i]) = 0$ and therefore

$$\text{Hom}_{\mathcal{D}(A)}(T, (\tau_{\geq 0}X^*)[i]) = \text{Hom}_{\mathcal{D}(A)}(T, X^*[i])$$

for $i \geq 0$, hence $\text{Hom}_{\mathcal{D}(A)}(T, (\tau_{\geq 0}X^*)[i]) = 0$ for $0 \leq i \leq n$. Similarly, we have

$$\text{Hom}_{\mathcal{D}(A)}(T, (\tau_{\leq 0}X^*)[i]) = \text{Hom}_{\mathcal{D}(A)}(T, (\tau_{\leq 0}X^*)[i]) = 0$$

for $0 \leq i \leq n$. Clearly, $H^0(X^*) = H^0(\tau_{\leq 0}X^*)$, so it suffices to show that $H^0(\tau_{\leq 0}X^*) = 0$.

This reduces the question to the case of a bounded complex $\tau_{\leq 0}X^*$. Thus it will be sufficient if we assume that $X^* \in \mathcal{D}^b(A)$ is a bounded complex satisfying $\text{Hom}_{\mathcal{D}^b(A)}(T, X^*) = 0$ for $0 \leq i \leq n$, and deduce that $H^0(X^*) = 0$.

By the definition of $T \mathcal{D}^{b\geq 0}$, we have $\mathcal{D}^{b\geq 0}(A) \subset T \mathcal{D}^{b\geq 0}$. By the condition (i), we have $\mathcal{D}^{b\leq 0}(A) \subset T \mathcal{D}^{b\leq 0}$. Since $(T \mathcal{D}^{b\leq 0}, T \mathcal{D}^{b\geq 0})$ is presumed to be a t-structure on $\mathcal{D}^b(A)$, we come to the inclusions of full subcategories

\begin{align*}
(4) & \quad \mathcal{D}^{b\leq 0}(A) \subset T \mathcal{D}^{b\leq 0} \subset \mathcal{D}^{b\leq 0}(A), \\
(5) & \quad \mathcal{D}^{b\geq 0}(A) \subset T \mathcal{D}^{b\geq 0} \subset \mathcal{D}^{b\geq 0}(A)
\end{align*}

in the bounded derived category $\mathcal{D}^b(A)$.

Let $\tau_{\geq 0}^T, \tau_{\leq 0}^T$ denote the truncation functors with respect to the t-structure $(T \mathcal{D}^{b\leq 0}, T \mathcal{D}^{b\geq 0})$ on $\mathcal{D}^b(A)$. Then for any complex $X^* \in \mathcal{D}^b(A)$ the natural maps

$$\text{Hom}_{\mathcal{D}^b(A)}(T, (\tau_{\geq 0}^TX^*)[i]) \longrightarrow \text{Hom}_{\mathcal{D}^b(A)}(T, X^*[i])$$

are isomorphisms for all $i \leq 0$, while the natural maps

$$\text{Hom}_{\mathcal{D}^b(A)}(T, X^*[i]) \longrightarrow \text{Hom}_{\mathcal{D}^b(A)}(T, (\tau_{\leq 0}^TX^*)[i])$$

are isomorphisms for all $i \geq 0$.

Let $X^* \in \mathcal{D}^b(A)$ be a complex such that $\text{Hom}_{\mathcal{D}^b(A)}(T, X^*[i]) = 0$ for $0 \leq i \leq n$. Then, on the one hand,

$$\text{Hom}_{\mathcal{D}^b(A)}(T, (\tau_{\geq 0}^TX^*)[i]) = \text{Hom}_{\mathcal{D}^b(A)}(T, X^*[i]) = 0 \quad \text{for } 0 \leq i \leq n$$

and, on the other hand,

$$\text{Hom}_{\mathcal{D}^b(A)}(T, (\tau_{\leq 0}^TX^*)[i]) = 0 \quad \text{for } i < 0,$$

since $\tau_{\geq 0}^TX^* \in T \mathcal{D}^{b\geq 0}$. Therefore, $\text{Hom}_{\mathcal{D}^b(A)}(T, (\tau_{\leq 0}^TX^*)[i]) = 0$ for all $i \leq n$, and it follows that

$$\tau_{\leq 0}^TX^* \in T \mathcal{D}^{b\geq 0} \subset \mathcal{D}^{b\geq 1}(A).$$

Besides, $\tau_{\leq 0}^TX^* \in T \mathcal{D}^{b\leq -1} \subset \mathcal{D}^{b\leq -1}(A)$. Thus $H^0(\tau_{\leq 0}^TX^*) = 0$ and $H^0(\tau_{\geq 0}^TX^*) = 0$, implying that $H^0(X^*) = 0$. \qed
The t-structures on the derived categories $D(A)$, $D^+(A)$, $D^-(A)$, and $D^b(A)$ provided by Theorem 1.3 and Corollary 1.4 are called the tilting t-structures associated with an $n$-tilting object $T \in A$.

We denote by $B = T D^b_{\leq 0} \cap T D^b_{\geq 0}$ the heart of the related tilting t-structure on $D^b(A)$. According to (2-3), $B$ coincides with the heart $T D^0 \cap T D^{\geq 0}$ of the tilting t-structure on $D(A)$. The category $B$ has Hom sets, since $B \subset D^b(A) \subset D^+(A)$. By the definition, $B$ is an abelian category.

The following proposition shows that the category $B$ has some properties dual to those of the category $A$.

**Proposition 1.6.** The object $T \in A \subset D(A)$ belongs to $B$ and is a projective generator of the abelian category $B$. Coproducts of copies of $T$ in $B$ coincide with such coproducts in $D(A)$ and in $A$. The projective objects of $B$ are precisely the direct summands of these coproducts. Set-indexed coproducts of arbitrary objects exist in $B$.

**Proof.** The coproduct $T^{(I)}$ of $I$ copies of $T$ in $A$ is also the coproduct of $I$ copies of $T$ in $D(A)$. The object $T^{(I)}$ belongs to $T D^{\leq 0}$ by the definition and to $T D^{\geq 0}$ by the condition (ii). Thus $T^{(I)} \in B \subset D(A)$. Being the coproduct of $I$ copies of $T$ in $D(A)$, this object is also the coproduct of $I$ copies of $T$ in $B \subset D^b(A) \subset D(A)$.

The object $T$ is projective in $B$, because $\text{Ext}^1_B(T, X) = \text{Hom}_{D^b(A)}(T, X[1]) = 0$ for every $X \in B$. The object $T$ is a projective generator of $B$, since $\text{Hom}_B(T, X) = 0$ for some $X \in B$ implies $\text{Hom}_{D^b(A)}(T, X[i]) = 0$ for all $i \in \mathbb{Z}$, so $X = 0$ by the condition (iii). It follows that the projective objects of $B$ are precisely the direct summands of the objects $T^{(I)}$.

Finally, we have shown that set-indexed coproducts of projective objects exist in $B$, and that there are enough projective objects. Set-indexed coproducts of arbitrary objects can be constructed in terms of the coproducts of projective objects. More explicitly, the category $B$ is equivalent to the additive quotient of the category of morphisms in $\text{Add}(T)$ modulo an ideal which is closed under coproducts of morphisms; see [8, Proposition IV.1.2].

\[ \square \]

2. Tilting Classes and Tilting Cotorsion Pairs

The aim of this section is to work out elementary homological algebra related to tilting and cotilting objects. This in particular allows us in Theorem 2.4 to characterize tilting objects $T \in A$ directly in the category $A$, without using the unbounded derived category as in condition (iii) in the previous section. The arguments are of purely homological nature in the spirit of [7] and so they easily dualize to the setting of cotilting modules. Our exposition is also informed by that in [41, Section 6].

Let $A$ be a complete, cocomplete abelian category with an injective cogenerator $W$, and let $T \in A$ be an object of projective dimension not exceeding $n$. Denote by $E$ the following full subcategory in $A$:

$$E = \{ E \in A \mid \text{Ext}^i_A(T, E) = 0 \text{ for all } i > 0 \}.$$
Lemma 2.1. (a) The full subcategory $E \subset A$ is closed under direct summands, extensions, and cokernels of monomorphisms in the abelian category $A$. In addition, $E \subset A$ contains the full subcategory of injective objects $A_{inj} \subset A$.

(b) Each object $X \in A$ admits an exact sequence in $A$ of the form

$$0 \longrightarrow X \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0,$$

where $E^0, E^1, \ldots, E^n \in E$.

Proof. Part (a): the closedness under direct summands is obvious, as is the assertion that $A_{inj} \subset E$. Furthermore, given a short exact sequence $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$ in $A$ with $E \in E$, one immediately concludes from the corresponding long exact sequence of $\text{Ext}^*_A(T, -)$ that $F \in E$ if and only if $G \in E$.

Part (b): by the definition, for any object $X \in A$, we use the fact that $A$ has enough injectives and consider a short exact sequence $0 \longrightarrow X \longrightarrow W^i \longrightarrow X' \longrightarrow 0$. Applying the same to $X'$, etc., we obtain an exact sequence $0 \longrightarrow X \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0$, where $E^1, \ldots, E^{n-1}$ are direct powers of $W$.

Denoting by $X^k$ the image of the morphism $E^{k-1} \longrightarrow E^k$, we have short exact sequences $0 \longrightarrow X^k \longrightarrow E^k \longrightarrow X^{k+1} \longrightarrow 0$, $0 \leq k \leq n-1$, where $X^0 = X$ and $X^n = E^n$. Then $\text{Ext}_A^i(T, E^n) = \text{Ext}_A^{i+1}(T, X^{n-1}) = \cdots = \text{Ext}_A^{i+n}(T, X) = 0$ for $i > 0$. Hence $E^n \in E$. \hfill $\Box$

The assertion of Lemma 2.1 (a) can be rephrased by saying that the full subcategory $E \subset A$ is coresolving in the sense of [62, Section 2]. Then Lemma 2.1 (b) says that the coresolution dimension of $A$ with respect to $E$ is bounded by the number $n$.

For each integer $m \geq 0$, we denote by $L_m$ the full subcategory of all objects $L \in A$ for which there exists an exact sequence in $A$ of the form

$$0 \longrightarrow L \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^m \longrightarrow 0,$$

where $T^0, T^1, \ldots, T^m \in \text{Add}(T)$. Clearly, one has $\text{Add}(T) = L_0 \subset L_1 \subset L_2 \subset \cdots \subset A$.

Lemma 2.2. Assume that the object $T \in A$ satisfies the conditions (i–ii). Then

(a) for any objects $L \in L_m$ and $E \in E$, one has $\text{Ext}_A^i(L, E) = 0$ for all $i > 0$;

(b) the intersection $L_m \cap E$ coincides with the full subcategory $\text{Add}(T) \subset A$;

(c) for each integer $m \geq n$, one has $L_m = L_{m+1}$.

Proof. Part (a): by the definition, for any object $L \in L_m$ there exists a short exact sequence $0 \longrightarrow L \longrightarrow T' \longrightarrow M \longrightarrow 0$ in $A$ with $T' \in \text{Add}(T)$ and $M \in L_{m-1}$. Since coproducts are exact in $A$, we have $\text{Ext}_A^i(T', E) = 0$ for all $E \in E$ and $i > 0$. Arguing by induction on $m$, we can assume that $\text{Ext}_A^i(M, E) = 0$ for $i > 0$. Applying the long exact sequence of $\text{Ext}_A^i(-, E)$, we obtain the desired $\text{Ext}$ vanishing.

Part (b): by the definition we have $\text{Add}(T) \subset L_m$, and by the condition (ii) we have $\text{Add}(T) \subset E$. Conversely, given an object $K \in L_m$, there exists a short exact sequence $0 \longrightarrow K \longrightarrow T' \longrightarrow M \longrightarrow 0$ with $T' \in \text{Add}(T)$ and $M \in L_{m-1}$. Now if $K \in E$, then by part (a) we have $\text{Ext}_A^i(M, K) = 0$, hence $K$ is a direct summand of $T'$.
Part (c): let \( L \in L_{m+1} \), where \( m \geq n \). Then there exists an exact sequence
\[
0 \to L \to T^0 \to \cdots \to T^m \to T^{m+1} \to 0
\]
in \( A \) with \( T^k \in \text{Add}(T) \). Denoting the image of the morphism \( T^{k-1} \to T^k \) by \( M_k \) and using the condition (ii), we compute that \( \text{Ext}^0_A(T, M^m) = \text{Ext}^2_A(T, M^{m-1}) = \cdots = \text{Ext}^{m+1}_A(T, L) \). The latter Ext group vanishes by the condition (i), so we have \( \text{Ext}^1_A(T, M^m) = 0 \). It follows that \( \text{Ext}^1_A(T^{m+1}, M^m) = 0 \), hence the short exact sequence
\[
0 \to M^m \to T^m \to T^{m+1} \to 0
\]
splits and \( M^m \in \text{Add}(T) \). Now the exact sequence
\[
0 \to L \to T^0 \to \cdots \to T^{m-1} \to M^m \to 0
\]
shows that \( L \in L_m \). \( \square \)

Assuming that the object \( T \in A \) satisfies the conditions (i–ii), we will denote the full subcategory \( L_n = L_{n+1} = L_{n+2} = \cdots \) simply by \( L \subset A \).

Now let us discuss the situation when the object \( T \in A \) is big \( n \)-tilting. In this case, using the notation \( B = T^0 \to \cdots \to T^n \to \text{D}^b(A) \) for the heart of the tilting \( n \)-structure on \( \text{D}^b(A) \), the full subcategory \( E \subset A \) can be described as the intersection \( E = A \cap B \) of the hearts of the standard and tilting \( n \)-structures on \( \text{D}^b(A) \). Hence the same category \( E \) can be also considered as a full subcategory in the abelian category \( B \).

It turns out that \( E \) as a subcategory of \( B \) satisfies dual properties to those which \( E \) has as a subcategory of \( A \). In the terminology of [62, Section 2], the next lemma says that the full subcategory \( E \subset B \) is resolving and the resolution dimension of (the objects of) \( B \) with respect to \( E \) does not exceed \( n \).

**Lemma 2.3.** (a) The full subcategory \( E \subset B \) is closed under direct summands, extensions, and kernels of epimorphisms in the abelian category \( B \). In addition, \( E \subset B \) contains the full subcategory of projective objects \( B_{\text{proj}} \subset B \).

(b) Each object \( Y \in B \) admits an exact sequence in \( B \) of the form
\[
0 \to E_n \to \cdots \to E_1 \to E_0 \to Y \to 0,
\]
where \( E_0, E_1, \ldots, E_n \in E \).

**Proof.** Part (a): the full subcategory \( E \) is closed under direct summands and extensions in \( B \), since both the full subcategories \( A \subset \text{D}^b(A) \) are closed under direct summands and extensions (in the triangulated category sense) in \( \text{D}^b(A) \). According to Proposition [1.6], we have \( B_{\text{proj}} = \text{Add}(T) \subset A \cap B \).

To show that \( E \) is closed under kernels of epimorphisms in \( B \), consider a short exact sequence \( 0 \to G \to F \to E \to 0 \) in \( B \) with \( E \) and \( F \in E \). Then there is a distinguished triangle \( G \to F \to E \to G[1] \) in \( \text{D}^b(A) \). Now we have \( E \subset A \) and \( F \in A \), hence \( G \in \text{D}^b_{\geq 0}(A) \). On the other hand, \( G \in B \subset T^0 \to \cdots \to \text{D}^b_{\leq 0} \) according to [1]. Hence \( G \in A \cap B \).

For the proof of part (b) we use essentially the same argument as in [62, Proposition 5.20]. We denote for this proof for each \( j \in \{0, 1, \ldots, n\} \) by \( E_j \) the class \( E_j = B \cap \text{D}^b_{\geq -j}(A) \subset \text{D}^b(A) \). By [1.5], we have \( E = E_0 \subset E_1 \subset \cdots \subset E_n = B \).

For any \( j \in \{1, \ldots, n\} \) and \( Y \in E_j \), by Proposition [1.5], there is a short exact sequence \( 0 \to Y' \to T^{(l)} \to Y \to 0 \) in \( B \); hence a related distinguished triangle \( Y' \to T^{(l)} \to Y \to Y'[1] \) in \( \text{D}^b(A) \). Then \( \text{Hom}_{\text{D}^b(A)}(Y', W[i-1]) \simeq \text{Hom}_{\text{D}^b(A)}(Y, W[i]) \) for all \( i > 1 \), and therefore \( Y' \in E_{j-1} \).
Starting from an arbitrary $Y \in B$, we construct by induction the desired exact sequence, even with $E_0, \ldots, E_{n-1}$ being copowers of $T$. □

The following definitions are standard; see for instance [60], [63, §5] or [54, §§3 and 4] and the references there (the concept of a cotorsion pair goes back to [59] and have been extensively used in representation and module theory [35, 27]). A pair of full subcategories $K$ and $F \subset A$ in an abelian category $A$ is called a cotorsion pair if $K$ consists precisely of all the objects $K \in K$ such that $\text{Ext}_A^1(K, F) = 0$ for all $F \in F$, and $F$ consists precisely of all the objects $F \in F$ such that $\text{Ext}_A^1(K, F) = 0$ for all $K \in K$.

A cotorsion pair $(K, F) \subset A$ is called hereditary if $\text{Ext}_A^i(K, F) = 0$ for all $K \in K$, $F \in F$, and $i \geq 1$. In a hereditary cotorsion pair, the class $K$ is also closed under the kernels of epimorphisms, and the class $F$ is closed under the cokernels of monomorphisms. Under mild conditions, these closure properties in fact characterize hereditary cotorsion pairs, see [63, Lemma 6.17] or [60, Lemma 4.25].

A cotorsion pair $(K, F)$ is called complete if for every object $X \in A$ there exist short exact sequences in $A$ (called the approximation sequences) of the form

\begin{align}
0 &\longrightarrow F' \longrightarrow K \longrightarrow X \longrightarrow 0 \\
0 &\longrightarrow X \longrightarrow F \longrightarrow K' \longrightarrow 0
\end{align}

with $K$, $K' \in K$ and $F$, $F' \in F$.

For any cotorsion pair $(K, F)$ in $A$, the full subcategory $K \subset A$ is closed under coproducts, direct summands, and extensions, while the full subcategory $F \subset A$ is closed under products, direct summands, and extensions. This is true even if products or coproducts are not exact, see [19, Proposition 8.1]. The following partial converse assertion to these observations holds: if $(K, F) \subset A$ is a pair of full subcategories such that $\text{Ext}_A^1(K, F) = 0$ for all $K \in K$ and $F \in F$, the approximation sequences (6–7) exist for all objects $X \in A$, and the full subcategories $K$, $F \subset A$ are closed under direct summands, then $(K, F)$ is a (complete) cotorsion pair in $A$.

The following theorem, which provides the promised characterization of tilting objects without using the unbounded derived category, is the main result of this section.

**Theorem 2.4.** Let $A$ be an abelian category with set-indexed products and an injective cogenerator, and let $T \in A$ be an object satisfying the conditions (i–ii) of Section 7. Then the following three conditions are equivalent:

1. the object $T \in A$ satisfies the condition (iii) as well;
2. for every object $E \in E = \bigcap_{i \geq 0} \ker \text{Ext}_A^i(T, -)$ there exists an object $T' \in \text{Add}(T)$ together with an epimorphism $T' \longrightarrow E$ in the category $A$;
3. for every object $X \in A$ there exists an object $L \in L$ (i. e., $L$ has a finite $\text{Add}(T)$-coresolution) together with an epimorphism $L \longrightarrow X$ in the category $A$.

If one of the conditions (1–3) is satisfied, then the pair of full subcategories $(L, E)$ is a hereditary complete cotorsion pair in the abelian category $A$. 

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Proof. (1) \(\implies\) (2): according to Proposition 1.4 for every object \(Y \in \mathcal{B}\) there exists an object \(T' \in \mathcal{B}_{\text{proj}} = \text{Add}(T)\) together with an epimorphism \(T' \to Y\) in the category \(\mathcal{B}\). In particular, this applies to the object \(Y = E \in \mathcal{E}\). By Lemma 2.3(a), the kernel \(E'\) of the epimorphism \(T' \to E\) belongs to \(\mathcal{E} \subset \mathcal{B}\). Now the short exact sequence \(0 \to E' \to T' \to E \to 0\) in \(\mathcal{B}\) corresponds to a distinguished triangle \(E' \to T' \to E \to E'[1]\) in \(\mathbf{D}^b(\mathcal{A})\). Since \(E' \in \mathcal{E}\) and \(E \in \mathcal{E} \subset \mathcal{A}\), it follows that the short sequence \(0 \to E' \to T' \to E \to 0\) is exact in \(\mathcal{A}\).

(2) \(\implies\) (3): first of all, let us show that the epimorphism \(T' \to E\) in (2) can be chosen in such a way that its kernel belongs to \(\mathcal{E}\). Indeed, set \(T'' = T^{(1)}\) to be the coproduct of copies of \(T\) indexed by the set \(I = \text{Hom}_\mathcal{A}(T, E)\) of all morphisms \(T \to E\), and let \(T'' \to E\) be the natural morphism. Then existence of an epimorphism \(T'' \to E\) with \(T' \in \text{Add}(T)\) implies that the morphism \(T'' \to E\) is an epimorphism, and surjectivity of the map \(\text{Hom}_\mathcal{A}(T, T'') \to \text{Hom}_\mathcal{A}(T, E)\) together with (ii) implies that the kernel \(E''\) of the morphism \(T'' \to E\) satisfies \(\operatorname{Ext}_i^\mathcal{A}(T, E'') = 0\) for all \(i > 0\).

Now we apply the dual version of \([6, \text{Theorem } 1.1]\), using the facts that the full subcategory \(\mathcal{E} \subset \mathcal{A}\) is closed under extensions (Lemma 2.1(a)) and every object of \(\mathcal{A}\) has a coresolution of length at most \(n\) by objects from \(\mathcal{E}\) (Lemma 2.1(b)). Proceeding by induction on the minimal length of such a coresolution for a given object \(X \in \mathcal{A}\), one constructs for it the approximation exact sequences \(0 \to X \to E \to L' \to 0\) and \(0 \to E' \to L \to X \to 0\) with \(E, E' \in \mathcal{E}\) and \(L, L' \in \mathcal{L}\). In particular, this proves (3). (Cf. \([11, \text{proof of Theorem } 3]\).)

(3) \(\implies\) (1): let \(X^i \in \mathcal{D}(\mathcal{A})\) be a complex such that \(\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(T, X^i[i]) = 0\) for all \(i \in \mathbb{Z}\). Since infinite coproducts are exact in \(\mathcal{A}\), it follows that \(\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(T', X^i[i]) = 0\) for all \(T' \in \text{Add}(T)\). Consequently, one has \(\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(L, X^i[i]) = 0\) for all \(L \in \mathcal{L}\).

Now let \(Z^i \in \mathcal{A}\) denote the kernel of the differential \(X^i \to X^i+1\). According to (3), there exists an object \(L \in \mathcal{L}\) together with an epimorphism \(L \to Z^i\). Since the related morphism of complexes \(L \to X^i[i]\) vanishes in \(\mathcal{D}(\mathcal{A})\), it must induce a zero morphism on the cohomology objects. But the induced morphism \(L \to H^i(X^i)\) is an epimorphism by construction, so the object \(H^i(X^i)\) has to vanish.

In addition to the equivalence of the three conditions (1–3), we have already shown that the approximation sequences \([6,7]\) exist under the assumption of these conditions. In view of Lemmas 2.2(a) and 2.1(a), in order to prove that \((\mathcal{L}, \mathcal{E})\) is a hereditary complete cotorsion pair in \(\mathcal{A}\), it only remains to check that the full subcategory \(\mathcal{L} \subset \mathcal{A}\) is closed under direct summands. The following Lemma 2.5 implies that. \(\square\)

In order to state and prove Lemma 2.5, we need some terminology from \([27, \text{§5.1}]\) (the same concepts were studied in \([17]\), even in the context of tilting theory, but using different terminology). A morphism \(f : X \to F\) from an object \(X \in \mathcal{A}\) to an object \(F \in \mathcal{E}\) is said to be an \(\mathcal{E}\)-preenvelope if the map of abelian groups \(\operatorname{Hom}_\mathcal{A}(f, E) : \operatorname{Hom}_\mathcal{A}(F, E) \to \operatorname{Hom}_\mathcal{A}(X, E)\) is surjective for all \(E \in \mathcal{E}\). Since any object of \(\mathcal{A}\) is a subobject of an object of \(\mathcal{A}_{\text{proj}} \subset \mathcal{E}\), any \(\mathcal{E}\)-preenvelope in \(\mathcal{A}\) is a monomorphism. A monomorphism \(f\) is said to be a special \(\mathcal{E}\)-preenvelope if its cokernel \(M\) has the property that \(\operatorname{Ext}^1_\mathcal{A}(M, E) = 0\) for all \(E \in \mathcal{E}\). It is clear from the long exact sequence of \(\operatorname{Ext}^1_\mathcal{A}(\_, E)\) that any special \(\mathcal{E}\)-preenvelope is an \(\mathcal{E}\)-preenvelope.
Conversely, any $E$-preenvelope of the form $f : X \to T'$ with the object $T'$ belonging to $\text{Add}(T)$ is special, because $\text{Ext}_A^1(T', E) = 0$ for all $E \in E$.

**Lemma 2.5.** Let $T \in A$ be a big $n$-tilting object. Then the following conditions are equivalent for $L \in A$: 
1. $L \in L$; 
2. $L$ is a direct summand of an object from $L$; 
3. there exists an $E$-preenvelope $L \to T'$ with $T' \in \text{Add}(T)$.

**Proof.** (1) $\implies$ (2): This is obvious.

(2) $\implies$ (3): Suppose that $L$ is a summand of $N \in L$. Let $0 \to N \to T^0 \to \cdots \to T^n \to 0$ be an exact sequence in $A$ with $T^k \in \text{Add}(T)$. Set $T'' = T^0$, and denote by $M$ the cokernel of the morphism $N \to T'$. Then $M \in L$, hence by Lemma 2.5, we have $\text{Ext}_A^1(M, E) = 0$, and therefore $N \to T'$ is an (even special) $E$-preenvelope. The composition $L \to N \to T'$ is clearly an $E$-preenvelope too.

(3) $\implies$ (1): If $f : L \to T'$ is an $E$-preenvelope with $T' \in \text{Add}(T)$ and $\text{coker}(f) = M$, then, as mentioned above, $\text{Ext}_A^1(M, E) = 0$ for all $E \in E$. From an approximation sequence $0 \to E \to E' \to M \to 0$ with $E \in E$ and $E' \in L$, whose existence was established in the proof of (2) $\implies$ (3) in Theorem 2.4, we see that $M$ is a direct summand of $E'$. Hence $M$ is a $E$-preenvelope $M \to T''$ with $T'' \in \text{Add}(T)$ by the previous implication.

Setting $T^0 = T'$, $T^1 = T''$, and proceeding further in this way, we construct an exact sequence $0 \to L \to T^0 \to T^1 \to \cdots \to T^n \to 0$ and an $E$-preenvelope $g : K \to T^n$ with $T^j \in \text{Add}(T)$ for all $0 \leq j \leq n$. Denoting by $M^j$ the image of the morphism $T^{j-1} \to T^j$ and using the conditions (i–ii), we have $\text{Ext}_A^1(T, K) = \text{Ext}_A^{i+1}(T, M^{n-1}) = \cdots = \text{Ext}_A^{n+i}(T, L) = 0$ for $i > 0$. So $K \in E$.

Now the map $\text{Hom}_A(g, K)$ surjective, hence $K$ is a direct summand of $T^n$. Thus $K \in \text{Add}(T)$ and $L \in L$. \qed

Using Theorem 2.4, we can compare our definition of a (big) $n$-tilting object with the traditional definition of an (infinitely generated) $n$-tilting module.

Let $A$ be an associative ring and $A = \text{mod} A$ be the category of left $A$-modules. According to the definition going back to the papers [2, 10], a left $A$-module $T$ is called $n$-tilting if it satisfies the conditions (i–ii) from Section 1 as an object of the category $A = \text{mod} A$, and in addition, the following condition holds:

(iii) the free left $A$-module $A$ has a finite coresolution $0 \to X \to T^0 \to \cdots \to T^n \to 0$ by $A$-modules $T^i$ belonging to the subcategory $\text{Add}(T) \subseteq \text{mod} A$.

According to [10, Proposition 3.5], when the conditions (i–ii) and (iii) are satisfied, one can have $r = n$, but $r$ cannot be made smaller than that (or more precisely, than the projective dimension of $T$).

**Corollary 2.6.** Let $T$ be a left $A$-module satisfying, as an object of the category $A = \text{mod} A$, the conditions (i) and (ii). Then $T$ satisfies the condition (iii) if and only if it satisfies (iii).
Proof. It is obvious that (iii\(_m\)) implies (iii) (cf. the proof of (3) \(\implies\) (1) in Theorem 2.4).

To prove the converse implication, note that Theorem 2.4(3) provides us with a surjection \(L \twoheadrightarrow A\). As such a surjection must split, we have \(A \in L\) by Lemma 2.5. The existence of (iii\(_m\)) follows from the very definition of \(L\). (Alternatively, one could compare the result of Lemma 4.1(a) below with the characterization of \(n\)-tilting modules provided by [2, Theorem 4.4] or [10, Proposition 3.10] in order to show that (iii) implies (iii\(_m\)).) □

Remark 2.7. In the literature, the full subcategory \(E \subset A\) is known as the \(n\)-tilting class associated with an \(n\)-tilting object \(T \in A\) [27, Section 13.1]. The cotorsion pair \((L, E)\) in \(A\) is called the \(n\)-tilting cotorsion pair.

3. The Tilting-Cotilting Correspondence

The aim of this section is to show that the assignment of the tilting heart \(B\) to an abelian category \(A\) with an \(n\)-tilting object \(T\) extends to a bijective correspondence between certain natural classes of abelian categories \(A\) with \(n\)-tilting objects \(T\) and abelian categories \(B\) with \(n\)-cotilting objects \(W\).

We start with some standard observations on tilting \(t\)-structures along the lines of [14, Remarque 3.1.17], [44, Corollary A.17]. For any \(t\)-structure \((\text{D}^{\leq 0}, \text{D}^{\geq 0})\) on a triangulated category \(D\) with an abelian heart \(C = \text{D}^{\leq 0} \cap \text{D}^{\geq 0}\), there are natural maps from the Ext groups in the abelian category \(C\) to the Hom groups in the triangulated category \(D\):

\[
\theta^i_{C, D}(X, Y) : \text{Ext}^i_C(X, Y) \rightarrow \text{Hom}_D(X, Y[i]), \quad \text{for all } X, Y \in C, \ i \geq 0.
\]

The maps \(\theta^i_{C, D}(X, Y)\) are functorial in \(X\) and \(Y\) and transform the Yoneda multiplication of Ext classes into the composition of morphisms in \(D\). The maps \(\theta^0_{C, D}\) and \(\theta^i_{C, D}\) are always isomorphisms, and the maps \(\theta^1_{C, D}\) are monomorphisms. The maps \(\theta^{i+1}_{C, D}\) are monomorphisms whenever the maps \(\theta^i_{C, D}\) are isomorphisms.

A \(t\)-structure \((\text{D}^{\leq 0}, \text{D}^{\geq 0})\) on \(D\) is said to be of the derived type if the maps \(\theta^i_{C, D}\) are isomorphisms for all \(i \geq 0\). A \(t\)-structure is of the derived type if and only if for every morphism \(\xi : X \rightarrow Y[i] \in D\) with \(X, Y \in C\) and \(i > 0\) there exists an epimorphism \(\pi : X' \rightarrow X\) in the abelian category \(C\) such that \(\xi \pi = 0\) in \(D\), and if and only if for every \(\xi\) there exists a monomorphism \(\iota : Y \rightarrow Y'\) in \(C\) such that \(\iota[i] \circ \xi = 0\).

Let \(A\) be an abelian category with set-indexed products and an injective cogenerator, and let \(T \in A\) be an \(n\)-tilting object. Let \((\text{T}D^{\leq 0}, \text{T}D^{\geq 0})\) be the tilting \(t\)-structure on \(D^b(A)\), and let \(B\) be the heart of this \(t\)-structure.

Lemma 3.1. The tilting \(t\)-structure \((\text{T}D^{\leq 0}, \text{T}D^{\geq 0})\) on \(D^b(A)\) associated with any \(n\)-tilting object \(T \in A\) is a \(t\)-structure of the derived type. Equivalently, the tilting \(t\)-structure \((\text{TD}^{\leq 0}, \text{TD}^{\geq 0})\) on \(D(A)\) is of the derived type.
Proof. Let us check that for every pair of objects \( X, Y \in \mathcal{B} \) and a morphism \( \xi : X \rightarrow Y[i] \), \( i > 0 \) in \( \mathcal{D}^b(\mathcal{A}) \) there exists an epimorphism \( \pi : X' \rightarrow X \) in \( \mathcal{B} \) such that \( \xi \pi = 0 \). Set \( I = \text{Hom}_\mathcal{B}(T, X) \) and \( X' = T(I) \); let \( \pi : X' \rightarrow X \) be the natural epimorphism. Then \( \xi \pi = 0 \) in \( \mathcal{D}^b(\mathcal{A}) \) because \( \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(T, Y[i]) = 0 \) by the definition of \( T \mathcal{D}^{b, \leq 0}_{\mathcal{B}} \) and \( \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(T(I), Y[i]) = \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(T, Y[i])^I \).

This allows us to prove that the categories \( \mathcal{A} \) and \( \mathcal{B} \) are derived equivalent (at the bounded level, see Section 4 for a more thorough discussion of derived equivalences). In fact, we provide two proofs: the first one is short and elegant, but it relies on a non-trivial technical construction of a so-called realization functor, while the second one is elementary and only uses the results of our Section 2.

**Proposition 3.2.** There is an equivalence of triangulated categories \( \mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}^b(\mathcal{A}) \) identifying the standard t-structure \( \mathcal{D}^b(\mathcal{B}), \mathcal{D}^b(\mathcal{A}) \) with the tilting t-structure \( T \mathcal{D}^{b, \leq 0}_{\mathcal{B}}, T \mathcal{D}^{b, > 0}_{\mathcal{B}} \). The restriction of this triangulated equivalence to the full subcategory \( \mathcal{B} \subset \mathcal{D}^b(\mathcal{A}) \) is the identity embedding \( \mathcal{B} \rightarrow \mathcal{D}^b(\mathcal{B}) \).

First proof. According to [114, n° 3.1] and [56, Appendix A] (for another approach, see [33, §3]), for any t-structure on the derived category \( \mathcal{D}^b(\mathcal{A}) \) with the abelian heart \( \mathcal{B} \) there exists a triangulated “realization” functor \( \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A}) \) whose restriction to \( \mathcal{B} \) is the identity embedding \( \mathcal{B} \rightarrow \mathcal{D}^b(\mathcal{A}) \). Furthermore, the inclusions \( (135) \) show that the tilting t-structure on \( \mathcal{D}^b(\mathcal{A}) \) is bounded, so the triangulated category \( \mathcal{D}^b(\mathcal{A}) \) is classically generated by its full subcategory \( \mathcal{B} \). That is, the smallest triangulated subcategory of \( \mathcal{D}^b(\mathcal{A}) \) containing \( \mathcal{B} \) is \( \mathcal{D}^b(\mathcal{A}) \) itself. As the tilting t-structure is also of the derived type by Lemma 3.1, the functor \( \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A}) \) is a triangulated equivalence (cf. [114, Proposition 3.1.16]). Since a bounded t-structure on a triangulated category is determined by its heart, the equivalence of categories \( \mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}^b(\mathcal{A}) \) identifies the standard t-structure on \( \mathcal{D}^b(\mathcal{B}) \) with the tilting t-structure on \( \mathcal{D}^b(\mathcal{A}) \).

Second proof. According to Lemmas 2.1 and 2.3, the full subcategory \( \mathcal{E} = \mathcal{A} \cap \mathcal{B} \subset \mathcal{D}^b(\mathcal{A}) \) is coresolving in \( \mathcal{A} \) and resolving in \( \mathcal{B} \). The (co)resolution dimension is bounded by \( n \) in both cases. The exact category structures inherited by \( \mathcal{E} \) from the abelian categories \( \mathcal{A} \) and \( \mathcal{B} \) coincide (e.g., because they can be inherited directly from the triangulated category structure on \( \mathcal{D}^b(\mathcal{A}) \); see [114, Section A.8]). Hence it follows that both the triangulated functors \( \mathcal{D}^b(\mathcal{E}) \rightarrow \mathcal{D}^b(\mathcal{A}) \) and \( \mathcal{D}^b(\mathcal{E}) \rightarrow \mathcal{D}^b(\mathcal{B}) \) induced by the identity embeddings \( \mathcal{E} \rightarrow \mathcal{A} \) and \( \mathcal{E} \rightarrow \mathcal{B} \) are triangulated equivalences between the bounded derived category \( \mathcal{D}^b(\mathcal{E}) \) of the exact category \( \mathcal{E} \) and the bounded derived categories of the abelian categories \( \mathcal{A} \) and \( \mathcal{B} \) (see [32, Proposition 13.2.2(ii)] or [46, Proposition A.5.6]; cf. the discussion in Section 4 and in the proof of Theorem 4.5 below). Hence we obtain the triangulated equivalence \( \mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}^b(\mathcal{A}) \).

It remains to show that the identity embeddings \( \mathcal{B} \rightarrow \mathcal{D}^b(\mathcal{B}) \) and \( \mathcal{B} \rightarrow \mathcal{D}^b(\mathcal{A}) \) form a commutative diagram with the equivalence \( \mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}^b(\mathcal{A}) \). Indeed, let \( \mathcal{Y} \in \mathcal{B} \) be an object and \( 0 \rightarrow E^{-n} \rightarrow \cdots \rightarrow E^{-1} \rightarrow E^0 \rightarrow \mathcal{Y} \rightarrow 0 \).
be an exact sequence in $B$ with $E^{-k} \in E$, as in Lemma 2.3(b). We have to construct a natural isomorphism in $D^b(A)$ between the object represented by the complex $E^\bullet$ in $D^b(A)$ and the object $Y \in B \subset D^b(A)$. For this purpose, one considers the distinguished triangles of silly filtration $E^0 \to E^\bullet \to \sigma_{\leq -1} E^\bullet \to E^0[1]$ and $E^{-1}[−1] \to \sigma_{\leq -2} E^\bullet \to E^{-1}$ in $D^b(A)$. Since $E^{-k} \in E \subset T\text{D}^{b \leq 0}$ for all $1 \leq k \leq n$, we have $\sigma_{\leq -1} E^\bullet \in T\text{D}^{b \leq -1}$ and $\sigma_{\leq -2} E^\bullet \in T\text{D}^{b \leq -2}$, so $\text{Hom}_{D^b(A)}(\sigma_{\leq -1} E^\bullet, B) = 0 = \text{Hom}_{D^b(A)}(\sigma_{\leq -2} E^\bullet[−1], B)$. It follows that the group $\text{Hom}_{D^b(A)}(E^\bullet, Y)$ is isomorphic to the group of all morphisms $E^0 \to Y$ in $B$ for which the composition $E^{-1} \to E^0 \to Y$ vanishes. In particular, from the exact sequence $0 \to E^{-n} \to \cdots \to E^0 \to Y \to 0$ in $B$ we get a natural morphism $E^\bullet \to Y$ in $D^b(A)$. To check that this morphism is an isomorphism, one computes that it induces an isomorphism $\text{Hom}_{D^b(A)}(T, E^\bullet[i]) \simeq \text{Hom}_{D^b(A)}(T, Y[i])$ for all $i \in \mathbb{Z}$. Indeed, one has $\text{Hom}_{D^b(A)}(T, E^\bullet[i]) = H^i \text{Hom}_A(T, E^\bullet) = H^i \text{Hom}_B(T, E^\bullet)$, since $\text{Ext}_A^i(T, E^{-k}) = 0$ for all $0 \leq k \leq n$ and $i > 0$, and applying $\text{Hom}_B(T, −)$ preserves exactness of the sequence $0 \to E^{-n} \to \cdots \to E^0 \to Y \to 0$, because the object $T$ is projective in $B$. □

Now we proceed to define the cotilting objects. The setting for these is completely dual to the tilting one. Let $B$ be an abelian category with set-indexed coproducts and a projective generator. It follows from these conditions that set-indexed products exist and are exact in $B$.

Let us say that an object $W \in B$ is (big) $n$-cotilting if the following three conditions are satisfied:

(i*) the injective dimension of $W$ in $B$ does not exceed $n$, that is $\text{Ext}_B^i(Y, W) = 0$ for all $i > n$ and all $Y \in B$;

(ii*) $\text{Ext}_A^i(W^I, W) = 0$ for all $i > 0$ and all sets $I$, where $W^I$ denotes the product of $I$ copies of $W$ in $B$;

(iii*) every complex $Y^\bullet \in D(B)$ such that $\text{Hom}_{D(A)}(Y^\bullet, W[i]) = 0$ for all $i \in \mathbb{Z}$ is acyclic.

**Theorem 3.3.** Let $W \in B$ be an $n$-cotilting object. Then the pair of full subcategories

\[
W^n \text{D}^{\leq} = \{ Y^\bullet \in D(B) \mid \text{Hom}_{D(B)}(Y^\bullet, W[i]) = 0 \text{ for all } i < 0 \},
\]

\[
W^n \text{D}^{\geq} = \{ Y^\bullet \in D(B) \mid \text{Hom}_{D(B)}(Y^\bullet, W[i]) = 0 \text{ for all } i > 0 \}
\]

is a t-structure on the unbounded derived category $D(B)$.

**Proof.** Dual to Theorem 1.3. □

**Corollary 3.4.** (a) Let $W \in B$ be an $n$-cotilting object. Then the pair of full subcategories

\[
W^n \text{D}^{b \leq} = \{ Y^\bullet \in D^b(B) \mid \text{Hom}_{D^b(B)}(Y^\bullet, W[i]) = 0 \text{ for all } i < 0 \},
\]

\[
W^n \text{D}^{b \geq} = \{ Y^\bullet \in D^b(B) \mid \text{Hom}_{D^b(B)}(Y^\bullet, W[i]) = 0 \text{ for all } i > 0 \}
\]

is a t-structure on the bounded derived category $D^b(B)$.
Conversely, if $W \in B$ is an object satisfying (i*) and (ii*), and the pair of full subcategories $(W D^b_{\leq 0}, W D^b_{\geq 0})$ is a t-structure on $D^b(B)$, then the object $W$ also satisfies the condition (iii*).

**Proof.** Part (a) is dual to Corollary 1.4(c). Part (b) is dual to Proposition 1.5. □

The t-structures on the derived categories $D(B)$ and $D^b(B)$ provided by Theorem 3.3 and Corollary 3.4(a), as well as the similar t-structures on $D^+(B)$ and $D^-(B)$, are called the cotilting t-structures associated with an $n$-cotilting object $W \in B$.

More generally, let $W \in B$ be an object of injective dimension not exceeding $n$. Denote by $E$ the following full subcategory in $B$:

$$E = \{ E \in B \mid \text{Ext}_B^i(E, W) = 0 \text{ for all } i > 0 \}.$$ 

In the way dual to Lemma 2.1, one shows that the full subcategory $E \subset B$ is resolving and the corresponding resolution dimension of $B$ is bounded by $n$.

Denote by $\text{Prod}(W) \subset B$ the full subcategory formed by the direct summands of infinite products of copies of the object $W \in B$. For any integer $m \geq 0$, we denote by $R_m$ the full subcategory of all objects $R \in B$ for which there exists an exact sequence in $B$ of the form

$$0 \longrightarrow W_m \longrightarrow \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow R \longrightarrow 0,$$

where $W_0, W_1, \ldots, W_m \in \text{Prod}(W)$. Clearly, one has $\text{Prod}(W) = R_0 \subset R_1 \subset \cdots \subset B$.

**Lemma 3.5.** Assume that the object $W \in B$ satisfies the conditions (i*–ii*). Then

(a) for any objects $E \in E$ and $R \in R_m$, one has $\text{Ext}_B^i(E, R) = 0$ for all $i > 0$;
(b) the intersection $R_m \cap E$ coincides with the full subcategory $\text{Prod}(W) \subset B$;
(c) for each integer $m \geq n$, one has $R_m = R_{m+1}$.

**Proof.** Dual to Lemma 2.2. □

Assuming that the object $W \in B$ satisfies the conditions (i*–ii*), we will denote the full subcategory $R_n = R_{n+1} = R_{n+2} = \cdots$ simply by $R \subset B$.

**Theorem 3.6.** Let $B$ be an abelian category with set-indexed coproducts and a projective generator, and let $W \in B$ be an object satisfying the conditions (i*–ii*). Then the following three conditions are equivalent:

(1*) the object $W \in B$ satisfies the condition (iii*);
(2*) for every object $E \in E$ there exists an object $W' \in \text{Prod}(W)$ together with a monomorphism $E \longrightarrow W'$ in the category $B$;
(3*) for every object $Y \in B$ there exists an object $R \in R$ together with a monomorphism $Y \longrightarrow R$ in the category $B$.

If one of the conditions (1*–3*) is satisfied, then the pair of full subcategories $(E, R)$ is a hereditary complete cotorsion pair in the abelian category $B$.

**Proof.** Dual to Theorem 2.4. □
Lemma 3.9. The cotilting t-structure has summands of these products. Set-indexed products of arbitrary objects exist in category

Proof. Dual to Proposition 1.6. □

Remark 3.7. Similarly to Corollary 2.6 one can show, using Theorem 1.10 (or alternatively, using Lemma 4.1(b) below), that the specialization of our definition of an n-cotilting object in an abelian category to the case of the category of modules over an associative ring \( B = B_{mod} \) is equivalent to the definition of an n-cotilting module studied in the papers [2, 10].

The full subcategory \( E \subset B \) is known as the n-cotilting class associated with an n-cotilting object \( W \in B \) [27, Section 15.1]. The cotorsion pair \((E, R)\) in \( B \) is called the n-cotilting cotorsion pair.

Let \( W \in B \) be an n-cotilting object. Set \( A = W D^{b, \leq 0} \cap W D^{b, \geq 0} \) to be the heart of the cotilting t-structure on \( D^b(B) \). By the definition, \( A \) is an abelian category.

Proposition 3.8. The object \( W \in B \subset D(B) \) belongs to \( A \) and is an injective cogenerator of the abelian category \( A \). Products of copies of \( W \) in \( A \) coincide with such products in \( D(B) \) and in \( B \). The injective objects of \( A \) are precisely the direct summands of these products. Set-indexed products of arbitrary objects exist in \( A \).

Proof. Dual to Proposition 1.6. □

Lemma 3.9. The cotilting t-structure \((W D^{b, \leq 0}, W D^{b, \geq 0})\) on \( D^b(B) \) associated with any n-cotilting object \( W \in B \) is a t-structure of the derived type. Equivalently, the cotilting t-structure \((W D^{\leq 0}, W D^{\geq 0})\) on \( D(B) \) is a t-structure of the derived type.

Proof. Dual to Lemma 3.1. □

Let \( A \) be an abelian category with set-indexed products and an injective cogenerator, and let \( T \in A \) be an n-tilting object. Choose an injective cogenerator \( W \in A \). Let \((T D^{b, \leq 0}, T D^{b, \geq 0})\) be the tilting t-structure on \( D^b(A) \) corresponding to the tilting object \( T \), and let \( B \) be the heart of this t-structure. For any set \( I \), we have \( W^I \in B \subset D^b(A) \). Furthermore, one has

\[
\text{Hom}_{D^b(A)}(X^\bullet, W^I) = \text{Hom}_{\text{Hot}(A)}(X^\bullet, W^I)
\]

for every complex \( X^\bullet \in \text{Hot}(A) \). It follows that \( W^I \in A \subset D(A) \) is the product of \( I \) copies of \( W \) in \( D(A) \), and therefore also in \( D^b(A) \subset D(A) \) and in \( B \subset D^b(A) \).

The following theorem is the main result of this section.

Theorem 3.10. Let \( A \) be a complete, cocomplete abelian category with an injective cogenerator \( W \in A \), let \( T \in A \) be an n-tilting object, and let \( B = T D^{b, \leq 0} \cap T D^{b, \geq 0} \) be the heart of the tilting t-structure on \( D^b(A) \). Then \( W \in B \subset D^b(A) \) is an n-cotilting object in the abelian category \( B \).

First proof. According to Proposition 1.6 \( B \) is a complete, cocomplete abelian category with a projective generator \( T \). We have explained that the objects \( W^I \in A \subset D^b(A) \) belong to \( B \subset D^b(A) \) and are the products of \( I \) copies of \( W \) in \( B \). By Proposition 3.2 we have an equivalence of triangulated categories \( D^b(A) \simeq D^b(B) \) which agrees with the identity embedding \( B \rightarrow D^b(A) \) and transforms the tilting t-structure on \( D^b(A) \) into the standard t-structure on \( D^b(B) \).
Let us check that the conditions (i*–iii*) hold for the object $W \in B$. We have 
\[
\Ext^i_B(W^I, W) = \Hom_{D^b(B)}(W^I, W[i]) = \Hom_{D^b(A)}(W^I, W[i]) = \Ext^i_A(W^I, W) = 0
\]
for $i > 0$, so (ii*) is satisfied.

We have 
\[
D^{b, \leq 0}(A) = \{ X^* \in D^b(A) \mid \Hom_{D^b(A)}(X^*, W[i]) = 0 \text{ for all } i < 0 \},
\]
\[
D^{b, \geq 0}(A) = \{ X^* \in D^b(A) \mid \Hom_{D^b(A)}(X^*, W[i]) = 0 \text{ for all } i > 0 \},
\]
since $W$ is an injective cogenerator of $A$. So the pair of full subcategories $(D^{b, \leq 0}(A), D^{b, \geq 0}(A))$ in $D^b(A)$ is transformed by the triangulated equivalence $D^b(A) \simeq D^b(B)$ into the pair of full subcategories $(\mathcal{W} D^{b, \leq 0}, \mathcal{W} D^{b, \geq 0})$ in $D^b(B)$. The former is a t-structure on $D^b(A)$, hence it follows that the latter is a t-structure on $D^b(B)$.

Given our identifications of the t-structures on $D^b(A)$ and $D^b(B)$, the inclusions (9) of full subcategories in $D^b(A)$ imply the inclusions
\[
D^{b, \leq 0}(B) \subset \mathcal{W} D^{b, \leq 0} \subset D^{b, \leq n}(B),
\]
\[
D^{b, \geq n}(B) \subset \mathcal{W} D^{b, \geq 0} \subset D^{b, \geq 0}(B)
\]
of full subcategories in $D^b(B)$. In particular, according to (10) for every object $Y \in B$ we have $Y \in D^{b, \leq 0}(B) \subset \mathcal{W} D^{b, \leq n}$, implying that
\[
\Ext^i_B(Y, W) = \Hom_{D^b(B)}(Y, W[i]) = 0 \quad \text{for } i > n.
\]
Thus (i*) is satisfied. Finally, it remains to apply Corollary 3.4(b) in order to deduce
\[\square\]
the condition (iii*).

**Second proof.** The following argument avoids the use of Proposition 3.2 using Lemma 3.1 and Theorem 3.6 instead.

By Lemma 3.1 we have 
\[
\Ext^i_B(W^I, W) = \Hom_{D^b(B)}(W^I, W[i]) = \Ext^i_A(W^I, W[i]) = 0
\]
for $i > 0$, so the condition (ii*) holds. To prove (i*), we notice that 
\[
\Ext^i_B(Y, W) = \Hom_{D^b(B)}(Y, W[i]) = 0 \quad \text{for all } Y \in B \text{ and } i > n,
\]
since $Y \in \mathcal{W} D^{b, \leq 0} \subset D^{b, \geq n}(A)$ by (5) and $W \in \text{A inj}$.

Finally, let us check the condition (2*) of Theorem 3.6. Since $\Ext^i_B(E, W) = \Hom_{D^b(B)}(E, W[i])$ for all $E \in B$ and $i \geq 0$ by Lemma 3.1, the full subcategory $E \subset B$ consisting of all objects $E \in B$ such that $\Ext^i_B(E, W) = 0$ for $i > 0$ can be described as the intersection $B \cap A \subset D^b(A)$. As a full subcategory of $A$, this intersection coincides with the $n$-tilting class $E = A \cap B \subset A$ discussed in Section 2.

Given an object $E \in E \subset B$, we first consider it as an object of $A$. Since $W$ is an injective cogenerator of $A$, there is a short exact sequence $0 \rightarrow E \rightarrow W^I \rightarrow E' \rightarrow 0$ in $A$. According to Lemma 2.1(a), we have $E' \in E$. Now there is a distinguished triangle $E \rightarrow W^I \rightarrow E' \rightarrow E'[1]$ in $D^b(A)$ with the objects $E, W^I$, and $E'$ belonging to $B$, hence a short exact sequence $0 \rightarrow E \rightarrow W^I \rightarrow E' \rightarrow 0$ in $B$. So $E \rightarrow W' = W^I$ is a monomorphism in $B$, as desired.
Alternatively, now that we have seen that the definition of the full subcategory $E \subset B$ discussed in Section 2 agrees with the one in the present section, we can deduce (i*) from Lemma 2.3(b). Given an object $Y \in B$, we have an exact sequence $0 \rightarrow E_n \rightarrow \cdots \rightarrow E_0 \rightarrow Y \rightarrow 0$ with $E_k \in E$. Then $\text{Ext}_B^i(E_k, W) = 0$ for all $0 \leq k \leq n$ and $i > 0$, hence $\text{Ext}_B^i(Y, W) = 0$ for all $i > n$. □

**Theorem 3.11.** Let $B$ be a complete, cocomplete abelian category with a projective generator $T \in B$, let $W \in B$ be an $n$-cotilting object, and let $A = \text{W}^\text{b}_{\leq 0} \cap \text{W}^\text{b}_{\geq 0}$ be the heart of the cotilting t-structure on $D^\text{b}(B)$. Then $T \in A \subset D^\text{b}(B)$ is an $n$-tilting object in the abelian category $A$.

*Proof.* Dual to Theorem 3.10. □

**Corollary 3.12.** The constructions of Theorems 3.10 and 3.11 establish a one-to-one correspondence between the equivalence classes of

(a) complete, cocomplete abelian categories $A$ with an injective cogenerator $W \in A$ and an $n$-tilting object $T \in A$, and

(b) complete, cocomplete abelian categories $B$ with a projective generator $T \in B$ and an $n$-cotilting object $W \in B$. □

### 4. Derived Equivalences

Let $A$ be a complete, cocomplete abelian category with an injective cogenerator $W$ and an $n$-tilting object $T$, and let $B$ be the corresponding complete, cocomplete abelian category with a projective generator $T$ and an $n$-cotilting object $W$. The aim of this section is to construct, for any conventional or absolute derived category symbol $\star = b, +, -, \emptyset, \text{abs}+, \text{abs}-,$ or $\text{abs}$, an equivalence of derived categories

\[ D^\star(A) \simeq D^\star(B). \]

Here the absolute derived category $D^\text{abs}(C)$ of an abelian category $C$ is defined as the Verdier quotient of $\text{Hot}(C)$ by the thick subcategory generated by all totalizations of short exact sequences of complexes. In other words, one forces short exact sequences of complexes to induce triangles and nothing more. The bounded versions of the absolute derived category are defined similarly (we refer to [46, Section A.1] or [48, Appendix A] and the references therein for a detailed discussion and the definitions of other exotic derived categories).

In particular, it will follow that there are two t-structures on both $D^\star(A)$ on $D^\star(B)$: the standard t-structure on $D^\star(B)$ is transformed by the equivalence (11) into (what will be called) the tilting t-structure on $D^\star(A)$, while the standard t-structure on $D^\star(A)$ is transformed into the cotilting t-structure on $D^\star(B)$.

Notice that for the bounded derived categories $D^b(A)$ and $D^b(B)$ we already have the desired picture. According to Proposition 3.2 there is a triangulated equivalence $D^b(A) \simeq D^b(B)$ transforming the tilting t-structure $(T D^b_{\leq 0}, T D^b_{\geq 0})$ on $D^b(A)$ into
the standard t-structure on $D^b(B)$. As it was explained in the first proof of Theorem 3.10, the same triangulated equivalence also transforms the cotilting t-structure $(W^b_{\leq 0}, W^b_{\geq 0})$ on $D^b(B)$ into the standard t-structure on $D^b(A)$.

As above, we denote by $E = A \cap B$ the intersection of the two t-structure hearts in the triangulated category $D^b(A) \simeq D^b(B)$. The full subcategory $E$ is coresolving in $A$ and resolving in $B$. Being, in particular, closed under extensions in both $A$ and $B$, the additive category $E$ inherits the exact category structure from either of the abelian categories $A$ and $B$; one can easily see that the two exact category structures on $E$ obtained in this way coincide (as it was mentioned in the first paragraph of the second proof of Proposition 3.2).

Next we will give an alternative characterization of the $n$-tilting class $E \subset A$ and the $n$-cotilting class $E \subset B$, but it is convenient to introduce some notation first. We consider the restriction $\Psi = T^0|_A : A \to B$ of the zero cohomology functor with respect to the tilting t-structure $(T^b_{\leq 0}, T^b_{\geq 0})$ on the derived category $D^b(A)$. Since $A \subset T^b_{\geq 0} \subset [3]$, we in fact have $\Psi = \tau^T_{\leq 0}|_A$ and the functor $\Psi : A \to B$ is left exact. Moreover, the restriction of $\Psi$ to the full subcategory $E = A \cap B$ coincides with the inclusion $E \subset B$.

Similarly, we consider an extension of the embedding $E \subset A$ to a right exact functor $\Phi : B \to A$ defined as $\Phi = W^b^0|_B = \tau^W_{\geq 0}|_B$. Here $\tau^W_{\geq 0}$ is the truncation functor and $W^b^0$ is the zero cohomology functor, respectively, with respect to the cotilting t-structure $(W^b_{\leq 0}, W^b_{\geq 0})$ on the derived category $D^b(B)$.

The functor $\Psi : A \to B$ is right adjoint to the functor $\Phi : B \to A$. Indeed, we may in view of Proposition 3.2 express $\Psi$ as the restriction to $A$ of the following composition of right adjoint functors

$$D^b_{\geq 0}(A) \xrightarrow{c} D^b(A) \simeq D^b(B) \xrightarrow{\tau^0} D^b_{\leq 0}(B).$$

Dually, the restriction to $B$ of the composition of the corresponding left adjoints coincides with $\Phi$.

As above, we denote by $\text{Add}(T) \subset A$ the full subcategory formed by the direct summands of infinite coproducts of copies of the object $T \in A$. Similarly, $\text{Prod}(W) \subset B$ denotes the full subcategory formed by the direct summands of infinite products of copies of the object $W \in B$. Then $\Psi$ restricts to a category equivalence $A_{\text{inj}} \simeq \text{Prod}(W)$, where $A_{\text{inj}} \subset A$ is the full subcategory of injective objects in $A$. One can in fact construct the functor $\Psi : A \to B$ as the unique left exact extension of the additive embedding functor $A_{\text{inj}} \simeq \text{Prod}(W) \to B$. Similarly, $\Phi : B \to A$ restricts to an equivalence $B_{\text{proj}} \simeq \text{Add}(T)$, where $B_{\text{proj}} \subset B$ is the full subcategory of projective objects in $B$. Moreover, $\Phi$ is the unique right exact extension of the additive embedding functor $B_{\text{proj}} \simeq \text{Add}(T) \to A$.

The following lemma is inspired by, and should be compared with, the results of the theory developed in the papers [2] [10] (cf. Corollary 2.6 and Remark 3.7).

**Lemma 4.1.** (a) The full subcategory $E \subset A$ consists precisely of all the objects $E \in A$ for which there exists an exact sequence of the form

$$T^{(l_n)} \to \cdots \to T^{(l_2)} \to T^{(l_1)} \to E \to 0$$

(b) The full subcategory $E \subset B$ consists precisely of all the objects $E \in B$ for which there exists an exact sequence of the form

$$E \to T^{(l_1)} \to \cdots \to T^{(l_2)} \to T^{(l_n)}$$

In particular, $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$. This also means that $E \subset A$ is the $n$-th right adjoint to the functor $\text{Add}(T)$, and $E \subset B$ is the $n$-th left adjoint to the functor $\text{Add}(T)$.
in the abelian category $A$, where $I_1, \ldots, I_n$ are some sets.

(b) The full subcategory $E \subset B$ consists precisely of all the objects $E \in B$ for which there exists an exact sequence of the form
\[
0 \longrightarrow E \longrightarrow W^{I_1} \longrightarrow W^{I_2} \longrightarrow \cdots \longrightarrow W^{I_n}
\]
in the abelian category $B$, where $I_1, \ldots, I_n$ are some sets.

Proof. It suffices to prove part (a). Since the full subcategory $E \subset A$ is coresolving, the coresolution dimension of the object $A = \ker(T(I_n) \to T(I_{n-1})) \in A$ does not exceed $n$ by Lemma 2.1, and the objects $T(I) \in A$ belong to $E$ by the condition (ii), it follows from (the dual of) [62, Proposition 2.3(1)] or [10, Corollary A.5.2] that the object $E$ in the exact sequence (12) belongs to $E$.

Conversely, given an object $E \in E \subset A$, consider the object $\Psi(E) \in E \subset B$. Since the object $T \in B$ is a projective generator, an exact sequence of the form
\[
0 \longrightarrow B \longrightarrow T(I_n) \longrightarrow \cdots \longrightarrow T(I_2) \longrightarrow T(I_1) \longrightarrow \Psi(E) \longrightarrow 0
\]
exists in the abelian category $B$. By Lemma 2.3(a), we have $T(I) \in E$ and $B \in E$, and moreover, all the objects of cocycles in the exact sequence (14) belong to $E$, so (14) is an exact sequence in the exact category $E \subset B$. Applying to (14) the exact functor $\Phi: E \to A$, we obtain the desired exact sequence (12) in the abelian category $A$ (and in fact, even in the exact category $E \subset A$; so the exact sequence (12) can be chosen in such a way that the object $A = \ker(T(I_n) \to T(I_{n-1}))$ belongs to $E$).

Remark 4.2. The above argument also shows that any object $E$ of the exact subcategory $E \subset A$ admits an arbitrarily long, and even an infinite left resolution
\[
\cdots \longrightarrow T(I_i) \longrightarrow \cdots \longrightarrow T(I_2) \longrightarrow T(I_1) \longrightarrow E \longrightarrow 0
\]
by copowers of the tilting object $T$ in the abelian category $A$.

Lemma 4.3. (a) The full subcategory $E \subset A$ is closed under infinite coproducts in the abelian category $A$.

(b) The full subcategory $E \subset B$ is closed under infinite products in the abelian category $B$.

Proof. It suffices to prove part (a), as part (b) is the dual assertion. In fact, the assertion of part (a) follows immediately from Lemma 4.1(a), but we prefer to give a direct argument as well.

Notice that for any t-structure $(D^{\leq 0}, D^{\geq 0})$ on a triangulated category $D$ the full subcategory $D^{\leq 0} \subset D$ is closed under infinite coproducts in $D$ (those infinite coproducts of objects from $D^{\leq 0}$ that exist in $D$). Furthermore, infinite coproducts of objects of $A$ taken in $A$ are at the same time their coproducts in $D^b(A)$, as it was explained in the beginning of the proof of Proposition 1.6. Finally, one has
\[
E = A \cap B = A \cap T D^{b, \leq 0} \subset D^b(A),
\]
because $A \subset D^{b, \geq 0}(A) \subset T D^{b, \geq 0}$ according to (5). Since the full subcategory $T D^{b, \leq 0}$ is closed under infinite coproducts in $D^b(A)$, we are done.
Remark 4.4. The proof of Lemma 4.3 reveals in fact more: \( E \) is closed under infinite coproducts in \( \mathcal{D}^b(A) \). Dually, \( E \) is closed under infinite products in \( \mathcal{D}^b(B) \cong \mathcal{D}^b(A) \).

Thus, in particular, both infinite coproducts and infinite products of exact sequences in \( E \) remain exact (since the same is true for triangles by [10, Proposition 1.2.1]), despite the fact that products may not be exact in \( A \) and coproducts may not be exact in \( B \).

Now we can prove the main results of the section. The case of the conventional unbounded derived categories (\( \ast = \emptyset \), i.e., \( \mathcal{D}(A) \cong \mathcal{D}(E) \cong \mathcal{D}(B) \)) can be also found in [25, Theorem 1.7 and Proposition 2.3].

Theorem 4.5. (a) For every derived category symbol \( \ast = b, +, -, \emptyset, \text{abs}+, \text{abs}-, \text{co}, \text{or abs} \), the triangulated functor \( \mathcal{D}^\ast(E) \to \mathcal{D}^\ast(A) \) induced by the exact embedding functor \( E \to A \) is an equivalence of triangulated categories.

(b) For every derived category symbol \( \ast = b, +, -, \emptyset, \text{abs}+, \text{abs}-, \text{ctr}, \text{or abs} \), the triangulated functor \( \mathcal{D}^\ast(E) \to \mathcal{D}^\ast(B) \) induced by the exact embedding functor \( E \to B \) is an equivalence of triangulated categories.

Proof. The result of part (b) is provided by [46, Proposition A.5.6], and part (a) is a dual assertion. (The case \( \ast = \emptyset \) can be found in [42, Proposition 13.2.6].) However, we provide a more detailed argument concerning part (b) for the reader’s convenience.

All the conventional or exotic derived categories of \( B \) are defined as Verdier quotients \( \mathcal{D}^\ast(B) = \text{Hot}^\ast(B) / \text{Acycl}^\ast(B) \), where \( \text{Hot}^\ast(B) \) is the correspondingly bounded homotopy category of complexes over \( B \) and \( \text{Acycl}^\ast(B) \) is a thick subcategory of acyclic complexes which contains totalizations of all short exact sequences of complexes. The derived categories for \( E \) are defined analogously. \( \mathcal{D}^\ast(E) = \text{Hot}^\ast(E) / \text{Acycl}^\ast(E) \). In order to prove that the inclusion \( \text{Hot}^\ast(E) \subset \text{Hot}^\ast(B) \) induces an equivalence \( \mathcal{D}^\ast(E) \cong \mathcal{D}^\ast(B) \), it suffices to show (see [41, Lemma 1.6]) that

(\( \alpha \)) each \( B^\ast \in \text{Hot}^\ast(B) \) admits a morphism \( f : E^\ast \to B^\ast \) with \( E^\ast \in \text{Hot}^\ast(E) \) and such that the cone of \( f \) is in \( \text{Acycl}^\ast(B) \);

(\( \beta \)) the equality \( \text{Acycl}^\ast(E) = \text{Hot}^\ast(E) \cap \text{Acycl}^\ast(B) \) holds.

Condition (\( \alpha \)) is satisfied by [46, Lemma A.3.3]. Indeed, since the \( E \)-resolution dimension of objects of \( B \) is uniformly bounded by Lemma 2.3, each \( B^\ast \in \text{Hot}^\ast(B) \) admits \( f : E^\ast \to B^\ast \) whose cone is in the smallest thick subcategory of \( \text{Hot}^\ast(B) \) generated by totalizations of suitably bounded short exact sequences of complexes over \( B \).

Regarding (\( \beta \)), for \( \ast = b, +, -, \) or \( \emptyset \), this is a direct consequence of the fact that the full subcategory \( E \) is resolving in \( B \), and that the resolution dimension is bounded by a finite constant again. The remaining cases \( \text{abs}+, \text{abs}-, \text{ctr} \), and \( \text{abs} \) are more involved. Denoting by \( E_d \subset B \) the full subcategory of all objects of resolution dimension \( \leq d \) with respect to \( E \) (so that \( E_0 = E \) and \( E_n = B \)), one proves that \( \text{Hot}^\ast(E_{d-1}) \cap \text{Acycl}^\ast(E_d) = \text{Acycl}^\ast(E_{d-1}) \) for all \( 1 \leq d \leq n \). For this purpose, one shows that every complex from \( \text{Acycl}^\ast(E_d) \) is the cokernel of an admissible monomorphism of complexes from \( \text{Acycl}^\ast(E_{d-1}) \). The key observation is that a short exact sequence (of complexes) in \( B \) has a finite resolution by short exact sequences (of complexes) in \( E \); see the proof of [42, Proposition 2.3]. The assertion involving the contraderived
category $D^{cr}(E)$ also uses Lemma [4.3](b). We refer to [21, Theorem 1.4] and [42, Theorem 7.2.2] for further technical details.

**Corollary 4.6.** For every derived category symbol $\star = b, +, -, \varnothing, \text{abs} +, \text{abs}-$, or \text{abs}, there is a natural triangulated equivalence of (conventional or absolute) derived categories $D^\star(A) \simeq D^\star(B)$ as in (11) provided by the mutually inverse derived functors $R\Psi : D^\star(A) \to D^\star(B)$ and $L\Phi : D^\star(B) \to D^\star(A)$.

**Proof.** Compare Theorem 4.5(a) with Theorem 4.5(b). \qed

5. Examples of Tilting and Cotilting Objects

Many examples of big $n$-tilting and $n$-cotilting modules are known in the literature; see [27, Chapters 13–17]. Some examples of big 1-tilting objects in locally Noetherian Grothendieck abelian categories are discussed in the recent paper [5]. To this long list, we add a few further of examples of our own. Our aim is to illustrate the observation that tilting equivalences often arise as derived equivalences between comodules and contramodules, in some sense of the word. This phenomenon will be then discussed in detail in the remaining part of the paper.

**Example 5.1.** Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset such that all elements of $S$ are nonzero-divisors in $R$ and the projective dimension of the $R$-module $S^{-1}R$ does not exceed 1. The latter condition is satisfied for example if $S = \{1, s, s^2, \ldots \}$ for $s \in R$, or, by [57, Corollaire II.3.2.7], for an arbitrary multiplicative set $S$ if $R$ is Noetherian of Krull dimension at most one.

An $R$-module $M$ is said to be $S$-torsion if for every element $x \in M$ there exists $s \in S$ such that $sx = 0$ in $M$. The maximal $S$-torsion submodule of an $R$-module $M$ is denoted by $\Gamma_S(M)$; and the full subcategory of all $S$-torsion $R$-modules is denoted by $R\mod_{S\text{-tor}} \subset R\mod$. The category $A = R\mod_{S\text{-tor}}$ is a Grothendieck abelian category with an injective cogenerator $W = \Gamma_S(\text{Hom}_R(R, Q/Z))$.

We claim that the $R$-module $T = S^{-1}R$ is a 1-tilting object in the abelian category $A = R\mod_{S\text{-tor}}$. One way to see that is using the fact that $S^{-1}R \oplus S^{-1}R/R$ is a 1-tilting $R$-module by [27, Theorem 14.59], but we can also give a more direct argument. Indeed, $T$ has projective dimension at most $1$ in $R\mod$ and hence also in $A$ (since $\text{Ext}^1_R(T, -)$ is right exact on $A$). This implies condition (i). Condition (ii), i. e., the equality $\text{Ext}^1_R(S^{-1}R, (S^{-1}R/R)^{\langle I \rangle}) = 0$ for every set $I$, follows quickly from the fact that $\text{Ext}^1_R(S^{-1}R, S^{-1}R^{\langle I \rangle}) = \text{Ext}^1_{S^{-1}R}(S^{-1}R, S^{-1}R^{\langle I \rangle}) = 0$ (see also [51, exact sequence (III)]). In order to prove condition (iii), we use the fact that, by [51, Theorem 6.6(a)], the canonical functor $D(A) \to D(R\mod)$ induced by the embedding $A \subset R\mod$ is fully faithful and its essential image is $D_A(R\mod)$, the full subcategory of complexes with $S$-torsion cohomology. Suppose now that $X^\bullet \in D(A)$ is such that $\text{Hom}_{D(A)}(T, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$ or, equivalently, such that
\( \mathbb{R} \text{Hom}_R(T, X^\bullet) = 0 \). An application of \( \mathbb{R} \text{Hom}_R(-, X^\bullet) \) to the short exact sequence \( 0 \rightarrow R \rightarrow S^{-1}R \rightarrow T \rightarrow 0 \) reveals that

\[
X^\bullet \cong \text{Hom}_R(R, X^\bullet) \cong \mathbb{R} \text{Hom}_R(S^{-1}R, X^\bullet).
\]

In particular, the cohomology of \( X^\bullet \) consists of \( S^{-1}R \)-modules. As the cohomology of \( X^\bullet \) was assumed to be \( S \)-torsion, \( X^\bullet \) is acyclic, as required. This proves the claim.

The main object of interest for us is the heart \( \mathcal{B} = T \mathcal{D}^{b, \leq 0} \cap T \mathcal{D}^{b, \geq 0} \subset \mathcal{D}^b(A) \) of the tilting \( t \)-structure for \( T \). We know so far that \( \mathcal{B}_{\text{proj}} = \text{Add}(S^{-1}R/R) \subset \mathcal{A} \subset R \text{-mod} \). It turns out that \( \mathcal{B} \) is equivalent to the category of so-called \( S \)-contramodule \( R \)-modules, which is easiest defined as

\[
R \text{-mod}_{S, \text{contra}} = \{ C \in R \text{-mod} \mid \text{Hom}_R(S^{-1}R, C) = 0 = \text{Ext}^1_R(S^{-1}R, C) \} \subset R \text{-mod}.
\]

This is a full abelian subcategory of \( R \text{-mod} \) (see, e.g., [26, Proposition 1.1]). A more intrinsic way to view this category, which will be discussed in detail in Sections 6 and 7 (see in particular Theorem 7.1 and Corollary 7.7), is as follows. The ring

\[
\mathfrak{R} = \text{Hom}_R(S^{-1}R/R, S^{-1}R/R) = \varprojlim R/sR
\]

is naturally a complete topological ring with a base \( \{ s\mathfrak{R} \mid s \in S \} \) of neighborhoods of 0 (see [51, Proposition 3.2]). As such, infinite families \( (r_i)_{i \in I} \) of elements of \( \mathfrak{R} \) which converge to zero in this topology are naturally summable, i.e., \( \sum_{i \in I} r_i \in \mathfrak{R} \) is naturally defined. The \( S \)-contramodule \( R \)-modules are, roughly speaking, precisely those \( R \)-modules \( C \) for which there is a similar and naturally unique way to define all infinite summations \( \sum_{i \in I} r_ic_i \in C \), where \( c_i \in C \) for all \( i \in I \).

Using results from [51], one quickly sees that \( \text{Hom}_R(T, -) \) induces an equivalence from \( \mathcal{B} \) to \( R \text{-mod}_{S, \text{contra}} \). Indeed, as both \( \mathcal{B} \) and \( R \text{-mod}_{S, \text{contra}} \) have enough projective objects (see the discussion at the very end of [51, Section 3]), it suffices to show that \( \text{Hom}_R(T, -) \) induces an equivalence between the full subcategories of projective objects. Since \( \text{Hom}_R(T, S^{-1}R(I)) = 0 = \text{Ext}^1_R(T, S^{-1}R(I)) \) for each set \( I \), we have

\[
\text{Hom}_R(T, T(I)) \cong \text{Ext}^1_R(T, R(I)),
\]

and retracts of such \( R \)-modules are precisely the projective objects of \( R \text{-mod}_{S, \text{contra}} \) by [51, Section 3]. Hence the restriction

\[
\text{Hom}_R(T, -) \colon \mathcal{B}_{\text{proj}} = \text{Add}(T) \rightarrow (R \text{-mod}_{S, \text{contra}})_{\text{proj}}
\]

is essentially surjective. To see that it is also fully faithful, one applies \( - \otimes_R T \) to the morphisms \( \delta_{S,R(I)} : R(I) \rightarrow \text{Ext}^1_R(T, R(I)) \cong \text{Hom}_R(T, T(I)) \) from [51, Lemma 1.6]. Since \( X \otimes_R T = 0 = \text{Tor}^1_R(X, T) \) for each \( S^{-1}R \)-module \( X \), one obtains an isomorphism

\[
T(I) \cong \text{Hom}_R(T, T(I)) \otimes_R T,
\]

which can be directly checked to be the inverse of the canonical evaluation morphism. It follows that \( - \otimes_R T : (R \text{-mod}_{S, \text{contra}})_{\text{proj}} \rightarrow \mathcal{B}_{\text{proj}} \) is an inverse equivalence to \( \text{Hom}_R(T, -) \).
Hence, we can directly apply Corollary 4.6 to recover the derived equivalences from [51, Theorem 4.6 and Corollary 6.7],

\[ \mathbb{R} \text{Hom}_R(T, -) : D^+(R\text{-mod}_{S\text{-tors}}) \rightleftarrows D^+(R\text{-mod}_{S\text{-ctra}}) : - \otimes^L_R T \]

as a special case of our tilting equivalences. The corresponding 1-cotilting object \( W \in R\text{-mod}_{S\text{-ctra}} \) is computed as

\[ W = \text{Hom}_R(S^{-1}R/R, \Gamma_S(\text{Hom}_\mathbb{Z}(R, \mathbb{Q}/\mathbb{Z}))) = \text{Hom}_\mathbb{Z}(S^{-1}R/R, \mathbb{Q}/\mathbb{Z}). \]

The tilting equivalence from Example 5.1 will be extended later in Example 7.12 in the case of commutative Noetherian rings of Krull dimension one. Let us also remark that, besides the above-mentioned [27, Theorem 14.59], there exists a more general way to assign a 1-tilting module to a multiplicative subset \( S \) in a commutative domain; see [27, Example 13.4].

**Example 5.2.** Let \( \mathcal{C} \) be a coassociative, counital coalgebra over a field \( k \). Then the category of (coassociative and counital) left \( \mathcal{C} \)-modules \( \mathcal{A} = \mathcal{C}\text{-comod} \) is a locally Noetherian (in fact, locally finite) Grothendieck abelian category. The forgetful functor \( \mathcal{C}\text{-comod} \longrightarrow k\text{-mod} \) is exact and preserves coproducts.

The coproducts of copies of the left \( \mathcal{C} \)-comodule \( \mathcal{C} \) are called the cofree left \( \mathcal{C} \)-comodules. The cofree \( \mathcal{C} \)-comodules are injective objects in \( \mathcal{C}\text{-comod} \), and every injective \( \mathcal{C} \)-comodule is a direct summand of a cofree one. Hence the cofree \( \mathcal{C} \)-comodule with one cogenerator \( T = \mathcal{C} \in \mathcal{C}\text{-comod} \) satisfies the condition (ii) of the definition of a tilting object.

Of course, the projective dimension of the left \( \mathcal{C} \)-comodule \( \mathcal{C} \) does not have to be finite. Generally speaking, one can say that the injective cogenerator \( \mathcal{C} \) is an “\( \infty \)-tilting object” in the abelian category \( \mathcal{A} = \mathcal{C}\text{-comod} \) (this is made precise in the paper [55]; see [55, Example 6.9]). A version of the tilting t-structure exists on the coderived category \( D^{co}(\mathcal{C}\text{-comod}) \) of left \( \mathcal{C} \)-comodules, in place of the conventional (bounded or unbounded) derived category. The heart of this t-structure is the abelian category of left \( \mathcal{C}\text{-contramodules} \mathcal{C}\text{-contra} \) [47, Sections 1.1–1.2], [42, Section 0.2].

Under the standing assumptions, we always have the so-called comodule-contramodule correspondence. The category \( \mathcal{C}\text{-contra} \) is isomorphic to the category of left contramodules over the topological ring \( \mathcal{R} = \mathcal{C}^* \) dual to the coalgebra \( \mathcal{C} \) [47, Section 2.3] (see also Section 6 below for the definition of contramodules over topological rings). Using this identification, the contramodule \( \text{Hom}_\mathcal{C}(\mathcal{C}, \mathcal{C})^{op} = \mathcal{C}^* \) is a projective generator of \( \mathcal{B} = \mathcal{C}\text{-contra} \) and the derived comodule-contramodule correspondence says that there is a triangulated equivalence

\[ \mathbb{R} \text{Hom}_\mathcal{C}(\mathcal{C}, -) : D^{co}(\mathcal{C}\text{-comod}) \rightleftarrows D^{cr}(\mathcal{C}\text{-contra}) : \mathcal{C} \otimes^L_\mathcal{C} - , \]

i. e., the coderived category \( D^{co}(\mathcal{C}\text{-comod}) \) is equivalent to the contraderived category \( D^{cr}(\mathcal{C}\text{-contra}) \) [42, Sections 0.2.6–0.2.7]. The symbol \( \otimes\mathcal{C} \) stands for the contratensor product and will be also discussed in Section 7.

Under suitable homological assumptions, the latter equivalence descends to the tilting equivalences from Section 4. We say that the coalgebra \( \mathcal{C} \) is left Gorenstein if

(a) the left \( \mathcal{C} \)-comodule \( \mathcal{C} \) has finite projective dimension in \( \mathcal{C}\text{-comod} \);
(b) the left $\mathcal{C}$-contramodule $\mathcal{C}^*$ has finite injective dimension in $\mathcal{C}$–contra.

The latter condition can be equivalently reformulated in terms of functors $\text{Ctrtor}^\mathcal{C}_n$, the left derived functors of the contratensor product. Since one has a natural isomorphism $\text{Ctrtor}^\mathcal{C}_n(\mathcal{C}, \mathcal{P})^* \cong \text{Ext}^n_{\mathcal{C}–\text{contra}}(\mathcal{P}, \mathcal{C}^*)$ for any left $\mathcal{C}$-contramodule $\mathcal{P}$, condition (b) simply says that the right $\mathcal{C}$-comodule $\mathcal{C}$ has finite contraflat dimension in the sense that $\text{Ctrtor}^\mathcal{C}_n(\mathcal{C}, -) \equiv 0$ for $n \gg 0$. We refer to [49, Section 3.1] for more details.

When the coalgebra $\mathcal{C}$ is left Gorenstein, the derived functors $M \mapsto \mathbb{R}\text{Hom}_\mathcal{C}(\mathcal{C}, M)$ and $\mathcal{P} \mapsto \mathbb{L}\Phi_\mathcal{C}(\mathcal{P}) = \mathcal{C}\circ_\mathcal{C}^{\mathcal{P}}$ establish the triangulated equivalence $D^\mathcal{C}(\mathcal{C}–\text{comod}) \simeq D^{\mathcal{C}–\text{contra}}$ have finite homological dimension and therefore take acyclic complexes to acyclic complexes. So they descend to the conventional derived categories, providing a triangulated equivalence $D(\mathcal{C}–\text{comod}) \simeq D(\mathcal{C}–\text{contra})$. They also take bounded complexes to bounded complexes, hence an equivalence of the bounded derived categories $D^b(\mathcal{C}–\text{comod}) \simeq D^b(\mathcal{C}–\text{contra})$.

This implies the existence of a tilting t-structure on $D^b(\mathcal{C}–\text{comod})$. By Proposition 1.5 we conclude that $T = \mathcal{C} \in \mathcal{A} = \mathcal{C}–\text{comod}$ is an $n$-tilting object, and the category $\mathcal{B} = \mathcal{C}–\text{contra}$ is the tilting heart. The related $n$-cotilting object in $\mathcal{C}–\text{contra}$ is $\mathcal{C}^*$. It follows that the projective dimension of the left $\mathcal{C}$-comodule $\mathcal{C}$ is equal to thecontraflat dimension of the right $\mathcal{C}$-comodule $\mathcal{C}$.

In fact, we will later generalize Example 9.2 to arbitrary locally Noetherian Grothendieck categories (not only those of the form $\mathcal{C}–\text{comod}$) in §9.2.

Example 5.3. Let $A$ be an associative ring and $\mathcal{C}$ be an $A$-$A$-bimodule endowed with a coassociative, counital coering structure with the comultiplication map $\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$ and the counit map $\mathcal{C} \longrightarrow A$ [10, 12, Section 1.1], [17, Section 2.5]. Assume that $\mathcal{C}$ is a flat right $A$-module; then the category of left $\mathcal{C}$-comodules $\mathcal{A} = \mathcal{C}–\text{comod}$ is a Grothendieck abelian category with an injective cogenerator $W = \mathcal{C} \otimes_A J$, where $J$ is an injective cogenerator of the category of left $A$-modules.

Assume further that $\mathcal{C}$ is a projective left $A$-module. Then the cofree left $\mathcal{C}$-comodule $T = \mathcal{C}$ satisfies the condition (ii) from the definition of a tilting object. Indeed, one has

$$\text{Ext}^i_{\mathcal{A}}(\mathcal{C}, \mathcal{C}^{(i)}) = \text{Ext}^i_{\mathcal{A}}(\mathcal{C}, A^{(i)}) = 0 \quad \text{for} \quad i > 0.$$
If we restrict to Gorenstein corings in a very similar fashion as in Example 5.2, we obtain tilting equivalences even without any additional restrictions on A. We say that the coring C is left Gorenstein if

(a) the comodule T = C has finite projective dimension in C-comod;
(b) the contramodule W = \text{Hom}_C(C, C \otimes_A J) = \text{Hom}_A(C, J) has finite injective dimension in C-\text{contra}.

Denote by R^i\Psi_C the right derived functors of the left exact functor \Psi_C = \text{Hom}_C(C, -) (constructed by applying \Psi_C to a right injective resolution of a C-comodule M) and by L^i\Phi_C the left derived functors of the right exact functor \Phi_C = C \otimes_C - (constructed by applying \Phi_C to a left projective resolution of a C-contramodule \mathcal{P}). We can again reformulate (a) and (b) equivalently by requiring that both the derived functors R^n\Psi_C and L^n\Phi_C have finite homological dimensions. The argument immediately below will show that these two homological dimensions again coincide; for the time being, let us denote the larger of them by n.

Let C be a left Gorenstein coring. Denote by E_A \subset C-comod the full subcategory formed by all the C-comodules M such that R^i\Psi_C(M) = 0 for all i > 0, and by E_B \subset C-\text{contra} the full subcategory formed by all the C-contramodules \mathcal{P} such that L^i\Phi_C(\mathcal{P}) = 0 for all i > 0. Then E_A is a coresolving subcategory in A = C-comod with every object of A having coresolution dimension \leq n, while E_B is a resolving subcategory in B = C-\text{contra} with every object of B having resolution dimension \leq n.

The composition of the two adjoint functors \Phi_C \Psi_C restricted to the full subcategory of injective left C-comodules \text{A}_{\text{inj}} \subset A is the identity functor, while the composition \Psi_C \Phi_C restricted to the full subcategory of projective left C-contramodules is the identity functor [12, Section 5.1.3], [47, Section 3.4]. Arguing as in [12, proof of Theorem 5.3] (cf. [25, Example 6.1]), one easily shows that the functors \Psi_C and \Phi_C take E_A into E_B and E_B into E_A, and establish an equivalence between these two exact categories.

Hence we obtain a derived equivalence D^*(C-comod) \simeq D^*(C-\text{contra}) for every symbol \ast = b, +, -, \varnothing, \text{abs+}, \text{abs-}, or \text{abs}. Applying Proposition 1.5 one can conclude that T = C is an n-tilting object in the Grothendieck abelian category A = C-comod. The tilting heart is the abelian category B = C-\text{contra}, and the related n-cotilting object is the left C-contramodule W = \text{Hom}_A(C, J).

An interesting special case of the latter equivalence was studied in representation theory of finite-dimensional algebras. The corresponding corings are called bocses in this context; see the next example, and for more details and references, e.g., [36] §4.

**Example 5.4.** Let N be an associative ring and A \subset N be a subring such that the factor N/A is a finitely generated projective left A-module. Then the A-A-bimodule of left A-module homomorphisms C = \text{Hom}_A(N, A) is a coassociative coring for which the counit \varepsilon: C \to A, f \mapsto f(1_N), is surjective. If \mu: N \otimes N \to N is the multiplication
on $N$, the comultiplication on $\mathcal{C}$ is given by the composition

$$\Delta : \mathcal{C} = \text{Hom}_A(N, A) \xrightarrow{\text{Hom}_A(\mu, A)} \text{Hom}_A(N \otimes_A N, A) \simeq \text{Hom}_A(N, A) \otimes_A \text{Hom}_A(N, A) = \mathcal{C} \otimes_A \mathcal{C},$$

where the Hom-groups are homomorphisms of left $A$-modules. In more pedestrian terms, the value $\Delta(f) = \sum_{i=1}^m f_{1,i} \otimes f_{2,i}$ for a given $f : N \to A$ is determined by the equality

$$f(n_1 n_2) = \sum_{i=1}^m f_{2,i}(n_1 f_{1,i}(n_2))$$

for each pair $n_1, n_2 \in N$.

The category of left $\mathcal{C}$-comodules is isomorphic to the category of left $N$-modules. Indeed, given a left comodule structure $M \to \mathcal{C} \otimes_A M \simeq \text{Hom}_A(N, M)$, a left $N$-module structure on $M$ is given by the adjoint morphism $N \otimes_A M \to M$.

By construction, $\mathcal{C}$ is a finitely generated projective right $A$-module. Assume further that $\mathcal{C}$ is also a finitely generated projective left $A$-module. Then the $A$-$A$-bimodule of left $A$-module homomorphisms $N' = \text{Hom}_A(\mathcal{C}, A)$ is a new associative algebra containing $A$ as a subalgebra. The category of left $\mathcal{C}$-contramodules is isomorphic to the category of left $N'$-modules [42, Section 10.1.1].

Under the assumptions of the previous example, i.e., that $\mathcal{C}$ is left Gorenstein, we obtain equivalences

$$\text{D}^+(N\text{-mod}) \simeq \text{D}^+(\mathcal{C}\text{-comod}) \simeq \text{D}^+(\mathcal{C}\text{-contra}) \simeq \text{D}^+(N'\text{-mod}).$$

This situation arises among others in the context of Burt–Butler’s theory, where the coring $\mathcal{C}$ is called a bocs, $N$ is the left algebra of $\mathcal{C}$ and $N'$ is the right algebra of $\mathcal{C}$, and generalizes Ringel’s duality for finite-dimensional quasi-hereditary algebras. We refer to [30, §4.3 and 4.4] and the references therein for more information.

The last case which we shall consider in this section is the one where the roles of the algebra and coalgebra are swapped.

**Example 5.5.** Let $\mathcal{C}$ be a coassociative, counital coalgebra over a field $k$. The category of $\mathcal{C}$-$\mathcal{C}$-bicomodules with the functor $- \boxtimes \mathcal{C}$ of cotensor product over $\mathcal{C}$ is a monoidal category and we can consider a monoid $(S, S \boxtimes \mathcal{C} \to S, \mathcal{C} \to S)$ in this category. Such $S$ is by definition a semiassociative, semunital *semialgebra* over the coalgebra $\mathcal{C}$, [42, Section 0.3], [47, Section 2.6].

A *left semimodule* $M$ over $\mathcal{S}$ is a left module object over the monoid $\mathcal{S}$ in the left module category of left $\mathcal{C}$-comodules over the monoidal category of $\mathcal{C}$-$\mathcal{C}$-bicomodules (i.e., $M$ is simply a left $\mathcal{C}$-comodule endowed with an associative, unital left action $\mathcal{S} \boxtimes \mathcal{C} \to M$). Assume that the semialgebra $\mathcal{S}$ is an injective left and right comodule over the coalgebra $\mathcal{C}$. Then the category of left $\mathcal{S}$-semimodules $\mathcal{S}\text{-simod}$ is a Grothendieck abelian category.

For any semialgebra $\mathcal{S}$ over a coalgebra $\mathcal{C}$ over a field $k$ satisfying the above left and right injectivity assumption, the semifree left $\mathcal{S}$-semimodule $T = \mathcal{S}$ satisfies the
condition (ii) from the definition of a tilting object. Indeed, one has
\[ \text{Ext}^i_S(S, S^{(i)}) \simeq \text{Ext}^i_{\mathcal{C}}(\mathcal{C}, S^{(i)}) = 0 \quad \text{for } i > 0, \]
because the class of injective left \( \mathcal{C} \)-comodules is closed under infinite direct sums, so \( S^{(i)} \) is an injective left \( \mathcal{C} \)-comodule.

Of course, the projective dimension of the left \( S \)-semimodule \( S \) does not have to be finite; generally speaking, one can say that \( S \) is an “\( \infty \)-tilting object” in the abelian category \( \mathcal{S} \text{-semimod} \), exactly as \( \mathcal{C} \) was in Example 5.2 (see \[53\] Example 6.10). A version of the tilting t-structure exists on the semiderived category \( \text{D}^b(\mathcal{S} \text{-simod}) \) of left \( \mathcal{S} \)-semimodules, in place of the conventional (bounded or unbounded) derived category; the heart of this t-structure is the abelian category \( \mathcal{S} \text{-sicntr} \) of so-called \( \mathcal{S} \)-semicontramodules, \[12\] Sections 0.3.4. There is an equivalence of the semiderived categories \( \text{D}^b(\mathcal{S} \text{-simod}) \simeq \text{D}^b(\mathcal{S} \text{-sicntr}) \), which is called the derived semimodule-semicontramodule correspondence in \[12\] Sections 0.3.7 and 6.3.

The category \( \mathcal{S} \text{-sicntr} \) has a projective generator \( T = \text{Hom}_R(\mathcal{S}, \mathcal{S}) = \text{Hom}_C(\mathcal{C}, \mathcal{S}) \) corresponding to the object \( T = \mathcal{S} \in \mathcal{S} \text{-simod} \) under the semimodule-semicontramodule correspondence \[17\] Section 3.5). Moreover, the abelian category \( \mathcal{S} \text{-sicntr} \) is isomorphic to the category \( \mathcal{R} \text{-contra} \) of left contramodules over the topological ring \( \mathcal{R} = \text{Hom}_R(\mathcal{S}, \mathcal{S})^{\text{op}} \) (see Example 9.15 below).

If the coalgebra \( \mathcal{C} \) has finite global dimension, or, more generally, the coalgebra \( \mathcal{C} \) is left Gorenstein in the sense of Example 5.2 then \( \mathcal{S} \) has finite projective dimension in \( \mathcal{S} \text{-simod} \), i.e., condition (i) of the definition of tilting object is satisfied by \( T = \mathcal{S} \). Moreover, the equivalence of the semiderived categories in this case takes acyclic complexes to acyclic complexes, and therefore descends to an equivalence of the conventional unbounded derived categories \( \text{D}(\mathcal{S} \text{-simod}) \simeq \text{D}(\mathcal{S} \text{-sicntr}) \), which further restricts to an equivalence of the bounded derived categories \( \text{D}^b(\mathcal{S} \text{-simod}) \simeq \text{D}^b(\mathcal{S} \text{-sicntr}) \).

Thus, if the coalgebra \( \mathcal{C} \) is left Gorenstein, \( \mathcal{S} \) is an \( n \)-tilting object in the abelian category \( \mathcal{A} = \mathcal{S} \text{-simod} \) by Proposition 1.5. Similarly, the left \( \mathcal{S} \)-semicontramodule \( W = \mathcal{S}^\ast = \text{Hom}_k(\mathcal{S}, k) \) is an \( n \)-cotilting object in the abelian category \( \mathcal{B} = \mathcal{S} \text{-sicntr} \) in this case; it corresponds to a certain natural choice of an injective cogenerator \( W \in \mathcal{S} \text{-simod} \).

If the coalgebra in the last example is finite-dimensional, the procedure provides us with a derived equivalence of usual rings in a similar fashion as in Example 5.4.

**Example 5.6.** Let \( R \) be an associative algebra over a field \( k \) and \( N \subset R \) be a finite-dimensional subalgebra. Then the dual vector space \( \mathcal{C} = N^* \) is a coassociative coalgebra. Assume that the factor \( R/N \) is a flat right \( N \)-module and set \( S = R \otimes_N \mathcal{C} \). Then \( S \) is a semiassociative semialgebra over \( \mathcal{C} \) by \[12\] §10 and the semiunit \( \eta : \mathcal{C} \to S, f \mapsto 1_R \otimes f \), is injective. To be more specific, \( S \) is canonically an \( N \)-\( N \)-bimodule, hence a \( \mathcal{C} \)-\( \mathcal{C} \)-bicomodule by the same argument as in Example 5.4. The semimultiplication is then given by the composition
\[
S \boxtimes \mathcal{C} = (R \otimes_N \mathcal{C}) \boxtimes (R \otimes_N \mathcal{C}) \simeq R \otimes_N (\mathcal{C} \boxtimes (R \otimes_N \mathcal{C})) \simeq R \otimes_N R \otimes_N \mathcal{C} \to S,
\]
where the first isomorphism uses that \( R \) is right flat over \( N \) and the rightmost map is induced by the multiplication on \( R \). The category of left \( S \)-semimodules is isomorphic to the category of left \( R' \)-modules, see [12, §10.2].

Assume further that \( S \) is an injective left \( \mathcal{C} \)-comodule (i.e., an injective left \( N \)-module) and set \( R' = N \square e S \). Then \( - \square e S \) is an exact functor and \( R' \) is another associative algebra containing \( N \) as a subalgebra [12, Section B.2]. More specifically, the multiplication on \( R' \) is given by

\[
R' \otimes_N R' = (N \square e S) \otimes_N (N \square e S) \simeq ((N \square e S) \otimes_N N) \square e S \simeq N \square e S \square e S \rightarrow R',
\]

where the last map is induced by the semimultiplication on \( S \). The category of left \( S \)-semicontramodules is isomorphic to the category of left \( R' \)-modules. Thus, if \( \mathcal{C} \) is Gorenstein, we obtain tilting equivalences

\[
D^\ast(R\text{-mod}) \simeq D^\ast(S\text{-simod}) \simeq D^\ast(S\text{-sicntr}) \simeq D^\ast(R'\text{-mod}).
\]

However, there are instances of Example 5.5 where the full strength of the formalism of semialgebras is necessary. One such class of semialgebras is related to locally profinite groups (otherwise known as locally compact, totally disconnected topological groups).

**Example 5.7.** Let \( H \) be a profinite group and \( k \) be a field (of possibly finite characteristic). Then the \( k \)-vector space \( \mathcal{C} = k(H) \) of locally constant functions \( H \rightarrow k \) has a natural structure of coassociative, counital coalgebra over \( k \). It can be constructed as the inductive limit \( k(H) = \lim_{\rightarrow U} k(H/U) \) over the open normal subgroups \( U \subset H \) of the coalgebras \( k(F) = k[F]^\ast \) dual to the group algebras of the finite quotient groups \( F = H/U \) of the group \( H \). The (left or right) \( k(H) \)-comodules are the **discrete \( H \)-modules** over \( k \), that is \( k \)-vector spaces endowed with an action of \( H \) such that the stabilizer of every vector is an open subgroup in \( H \).

Furthermore, let \( G \) be a locally profinite group and \( H \subset G \) be an compact open subgroup. Then the \( k \)-vector space \( S = k(G) \) of compactly supported locally constant functions \( G \rightarrow k \) has a natural structure of semiassociative, semiunital semialgebra over \( \mathcal{C} \). The (left or right) \( S \)-semimodules are the **smooth \( G \)-modules** over \( k \), which means, once again, \( k \)-vector spaces endowed with an action of \( G \) such that the stabilizer of every vector is an open subgroup in \( G \). One should be careful: the vector space \( S \) does not depend on the choice of a subgroup \( H \) in the given group \( G \), the semialgebra structure on \( S \) depends on this choice, and the category of \( S \)-semimodules again does not depend on it [12, Sections E.1.2–E.1.3], [17, Example 2.6].

The category of (left or right) \( S \)-semicontramodules does not depend on the choice of a compact open subgroup \( H \) in \( G \) either. Another name for \( S \)-semicontramodules is **\( G \)-contramodules** over \( k \). These can be described as \( k \)-vector spaces \( \mathfrak{P} \) endowed with a map assigning an element of \( \mathfrak{P} \) to every \( \mathfrak{P} \)-valued measure of a certain kind on the group \( G \) [17, Section 1.8], [52, Section 2]. Let us denote the abelian category of smooth \( G \)-modules over \( k \) by \( G-\text{smooth}_k = S\text{-simod} \) and the abelian category of \( G \)-contramodules over \( k \) by \( G-\text{contra}_k = S\text{-sicntr} \).

The special case when \( k \) is a field of characteristic \( p \) and \( G \) is a \( p \)-adic Lie group (such as, e.g., the group \( \text{GL}_N(\mathbb{Q}_p) \)) of invertible \( N \times N \) matrices with rational \( p \)-adic
entries) is of particular interest. In this case, for a small enough compact open subgroup \( H \subset G \) (in fact, for any compact \( p \)-adic Lie group \( H \) without \( p \)-torsion elements), the coalgebra \( C = k(H) \) has finite homological dimension (equal to the dimension \( n \) of the \( p \)-adic Lie group \( G \) or \( H \), e.g., \( n = N^2 \) for \( G = GL_N(\mathbb{Q}_p) \); see [34, Section 3] or [52, Section 0.11] and the references therein). Thus, in this case the discrete \( G \)-module \( T = S \) of compactly supported locally constant \( k \)-valued functions on \( G \) is an \( n \)-tilting object in \( G \)-smooth \( k \) and the \( G \)-contramodule \( W = S^* \) is an \( n \)-cotilting object in \( G \)-contra \( k \). We refer to [10, Example 4.2] and the paper [52] for further details.

To summarize, in all the above Examples [5.1, 5.7], the proof that the objects in question were tilting was based on having an independent construction of the tilting derived equivalence (which in turn was based on a thorough understanding of both the abelian categories \( A \) and \( B \) and the functors \( \Psi \) and \( \Phi \) between them). These equivalences all belong to the family of comodule-contramodule correspondences studied by the first-named author in the sources referred to above. In Examples [5.2, 5.7] the Gorenstein condition along the coalgebra variables was important for obtaining actual \( n \)-tilting objects with a finite \( n \) (while no condition along the ring/algebra variables was needed).

6. ABELIAN CATEGORIES WITH A PROJECTIVE GENERATOR

In this section we discuss in detail cocomplete abelian categories with a projective generator, as they form one end of our tilting-cotilting correspondence from Section 3 and their description does not seem to be widely known. We build on results from [54] and the references there.

6.1. ADDITIVE MONADS. Let \( \mathcal{B} \) be an abelian category with set-indexed coproducts and a projective generator \( P \in \mathcal{B} \). Then the functor \( T = T_P : \text{Sets} \rightarrow \text{Sets} \) assigning to a set \( X \) the set \( \text{Hom}_\mathcal{B}(P, P^{(X)}) \) has a natural structure of a monad on the category of sets, where the monadic composition \( \mu : T \circ T \rightarrow T \) comes from the composition of endomorphisms of copowers of \( P \) (see the introduction to [54] and the references therein). The category \( \mathcal{B} \) can be recovered as the category of all algebras over this monad. We will often use the term \( T \)-modules for what one usually calls \( T \)-algebras as we really view them as a generalization of modules (see Example 6.1 below).

For any set \( X \), elements of the set \( T(X) \) are interpreted as \( X \)-ary operations on \( M \). More specifically, if \( (M, \alpha_M: T(M) \rightarrow M) \) is a \( T \)-module and \( t \in T(X) \), then \( t \) acts on \( M \) via

\[
m = (m_x)_{x \in X} \in M^X \quad \mapsto \quad \alpha_M(T(m)(t)) \in M,
\]

where \( m \) is viewed as a map of sets \( X \rightarrow M \) and \( T(m) \) is the induced map \( T(X) \rightarrow T(M) \).
Example 6.1. Consider the trivial situation where $\mathcal{B} = R\text{-mod}$ is an ordinary module category and $P = R$. If $X$ is a finite set with $n = |X|$ and $t = (r_1, \ldots, r_n) \in \mathbb{T}(X) = R^n$, then $t$ acts on $M$ as $M^n \to M$, $(m_1, \ldots, m_n) \mapsto \sum_{i=1}^n r_i m_i$.

For general $P \in \mathcal{B}$, the $X$-ary operations $t \in \mathbb{T}(X)$ are additive, but possibly non-trivially infinitary operations which equip $\mathbb{T}$-modules $M$ with a module-like algebraic structure. Morally they can be viewed as infinite $\text{Hom}(P, P)$-linear combinations of elements of $M$. The equivalence between $\mathcal{B}$ and the category of $\mathbb{T}$-modules assigns to an object $B \in \mathcal{B}$ the set $\text{Hom}_\mathcal{B}(P, B)$ endowed for each $t \in \mathbb{T}(X) = \text{Hom}_\mathcal{B}(P, P^{(X)})$ with the operation

$$\text{Hom}_\mathcal{B}(P, B)^X = \text{Hom}_\mathcal{B}(P^{(X)}, B) \to \text{Hom}_\mathcal{B}(P, B),$$

$$(m: P^{(X)} \to B) \mapsto m \circ t.$$

Conversely, for any monad $T: \text{Sets} \to \text{Sets}$, the category of $\mathbb{T}$-modules $\mathcal{B} = T\text{-mod}$ has set-indexed coproducts and a natural generator $P$, which is $P = \mathbb{T}(\{0\})$, the free $\mathbb{T}$-module with one generator. The category $\mathcal{B}$ is abelian if and only if it is additive, if and only if there is a binary operation $+ \in \mathbb{T}(\{0, 1\})$, a unary operation $- \in \mathbb{T}(\{0\})$ and a constant $0 \in \mathbb{T}(\emptyset)$ in the monad $\mathbb{T}$ satisfying the usual axioms of an abelian group and commuting with all the other operations in $\mathbb{T}$, [53, §10]. A monad $T$ on $\text{Sets}$ with such properties is called additive. If this is the case, the object $P \in \mathcal{B}$ is a projective generator.

6.2. Tilting equivalences. Generalizing the above discussion, let us consider an arbitrary category $\mathcal{A}$ with set-indexed coproducts and an object $M \in \mathcal{A}$. Then, once again, the functor $T = T_M: X \mapsto \text{Hom}_\mathcal{A}(M, M^{(X)})$ is a monad on the category of sets. The following lemma is standard.

Lemma 6.2. For any category $\mathcal{A}$ with set-indexed coproducts and an object $M \in \mathcal{A}$, the full subcategory consisting of all the objects $M^{(X)}, X \in \text{Sets}$ in the category $\mathcal{A}$ is equivalent to the full subcategory consisting of all the free $\mathbb{T}$-modules $\mathbb{T}(X), X \in \text{Sets}$ in the category of $\mathbb{T}$-modules $\mathcal{B}$.

Proof. The functor $T_M: \text{Sets} \to \mathcal{A}$ assigning the object $M^{(X)}$ to a set $X$ is left adjoint to the functor $\Psi_M: \mathcal{A} \to \text{Sets}$ assigning the set $\text{Hom}_\mathcal{A}(M, N)$ to an object $N \in \mathcal{A}$. The monad $T_M: \text{Sets} \to \text{Sets}$ is the composition of these two adjoint functors, $T = \Psi_M \circ T_M$. Hence the functor $\Psi_M$ lifts naturally to a functor taking values in the category of $\mathbb{T}$-modules $\mathcal{B}$. The functor $\Psi_M: \mathcal{A} \to \mathcal{B}$, $\Psi_M(N) = \text{Hom}_\mathcal{A}(M, N)$ takes the object $M^{(X)}$ to the free $\mathbb{T}$-module $\mathbb{T}(X)$. It remains to compute the sets of morphisms in the two categories in order to conclude that the restriction of the functor $\Psi_M$ is an equivalence between the full subcategory of all the copowers of $M$ in $\mathcal{A}$ and the full subcategory of all the free $\mathbb{T}$-modules in $\mathcal{B}$,

$$\text{Hom}_\mathcal{A}(M^{(Y)}, M^{(X)}) = \text{Hom}_\mathcal{A}(M, M^{(X)}) = \mathbb{T}(X)^Y = \text{Hom}_\mathcal{B}(\mathbb{T}(Y), \mathbb{T}(X))$$

for any two sets $X$ and $Y$. \qed
In particular, let \( A \) be an idempotent-complete additive category with set-indexed coproducts. Denote, as above, by \( \text{Add}(M) \subset A \) the full additive subcategory formed by the direct summands of all coproducts of copies of \( M \) in \( A \). Then it follows from Lemma 6.2 that the category \( \text{Add}(M) \) is equivalent to the full subcategory \( B_{\text{proj}} \subset B \) of projective objects in the abelian category \( B \). For any object \( N \in A \), the construction described in §6.1 and in the proof of Lemma 6.2 endows the set \( \text{Hom}_A(M, N) \) with a \( T \)-module structure. The functor
\[
\text{Hom}_A(M, -) : A \rightarrow B
\]
extends the equivalence \( \text{Add}(M) \simeq B_{\text{proj}} \).

With the terminology above, we can conveniently restate the equivalences from Section 4. Let \( A \) be an abelian category with products and an injective cogenerator and \( T \) be an \( n \)-tilting object. Set \( T = T_T \) and \( B = T - \text{mod} \).

Then the functor \( \text{Hom}_A(T, -) \) identifies with the functor \( \Psi : A \rightarrow B \) of Section 4. Indeed, both the functors \( \Psi : A \rightarrow B \) and \( \text{Hom}_A(T, -) : A \rightarrow B \) are left exact, so it suffices to show that they coincide on the full subcategory of injective objects \( A_{\text{inj}} \subset A \). For this purpose, it suffices to construct an isomorphism between the restrictions of the two functors to the exact subcategory \( E \subset A \). Both the functors are exact on this subcategory, so the question reduces to checking that they coincide on the full subcategory of projective objects \( \text{Add}(T) \subset E \), which we know from the construction.

The derived equivalences from Corollary 4.6 then take the form
\[
R \text{Hom}_A(T, -) : D^*(A) \rightarrow D^*(B).
\]

6.3. Categories of models of algebraic theories. Under a mild technical condition, we obtain a connection between categories of modules over additive monads on one hand and ordinary module categories on the other hand, which is in some sense dual to the usual Gabriel–Popescu theorem for Grothendieck categories.

Let \( \kappa \) be a regular cardinal. A projective generator \( P \in B \) is called \( \kappa \)-small if every morphism \( P \rightarrow P^{(X)} \) in \( B \) factorizes through the natural embedding \( P^{(Z)} \rightarrow P^{(X)} \) of the coproduct of copies of \( P \) over some subset \( Z \subset X \) of cardinality less than \( \kappa \). In this case, the functor \( T \) preserves \( \kappa \)-filtered colimits, the object \( P \in B \) is \( \kappa \)-presentable, and the category \( B \) is locally \( \kappa \)-presentable. The categories of modules over monads on \( \text{Sets} \) preserving \( \kappa \)-filtered colimits are also called the categories of models of \( \kappa \)-ary algebraic theories [64] (cf. [37], where finitary algebraic theories are discussed). We call such a theory \( \text{additive} \) if the monad \( T \) is additive.

Example 6.3. The existence of an abstractly \( \kappa \)-small generator is not for free, however. Let \( k \) be a field and consider \( B = (k - \text{mod})^{op} \). This is an abelian category with a projective generator \( k \) and the corresponding additive monad \( T : \text{Sets} \rightarrow \text{Sets} \) is given by \( T(X) = \text{Hom}_k(k^X, k) \). However, \( k \) is certainly not abstractly \( \kappa \)-small in \( B \).

In order to stress the parallel between Grothendieck categories and the categories of models of \( \kappa \)-ary algebraic theories, we recall the classical Gabriel–Popescu theorem [61, §X.4].
Theorem 6.4. Let $A$ be a Grothendieck abelian category and $G \in A$ be a generator in $A$. Denote by $S$ the ring $\text{Hom}_A(G, G)^{\text{op}}$. Then the functor $A \to S^{\text{-mod}}$ assigning to an object $A \in A$ the left $S$-module $\text{Hom}_A(G, A)$ is fully faithful, and has an exact left adjoint functor.

Corollary 6.5. Any Grothendieck category $A$ can be presented as a reflective full subcategory $A \subset S^{\text{-mod}}$ in the category of modules over an associative ring $S$ such that the reflection functor $S^{\text{-mod}} \to A$ is exact.

Conversely, any reflective full subcategory in $S^{\text{-mod}}$ which is abelian as a category and for which the reflection functor is exact is a Grothendieck category.

The following theorem is a “dual-analogous” version of Theorem 6.4 for the categories of models of additive $\kappa$-ary algebraic theories. In the nonadditive context, its result goes back to Isbell [31, §2.2] (see also [58, Remark 1.3]).

Theorem 6.6. Let $B$ be a cocomplete abelian category with an abstractly $\kappa$-small projective generator $P$. Let $Y$ be a set such that the successor cardinal of the cardinality of $Y$ is greater or equal to $\kappa$. Set $Q = P^{(Y)}$, and denote by $S$ the ring $\text{Hom}_B(Q, Q)^{\text{op}}$. Then the functor $B \to S^{\text{-mod}}$ assigning to an object $B \in B$ the left $S$-module $\text{Hom}_B(Q, B)$ is exact, fully faithful, and has a left adjoint functor $\Delta$.

Corollary 6.7. The category of models of any additive $\kappa$-ary algebraic theory can be presented as a reflective full abelian subcategory in the category of modules over an associative ring $S$ such that the embedding functor $B \to S^{\text{-mod}}$ is exact.

Conversely, any reflective full abelian subcategory in $S^{\text{-mod}}$ with an exact embedding functor is a cocomplete abelian category with a projective generator (and hence the category of models of an additive $\kappa$-ary algebraic theory if it has an abstractly $\kappa$-small such generator).

Remark 6.8. If we assume Vopěnka’s principle, the latter corollary get sharper. This is because then a full subcategory in a locally presentable category is reflective if and only if it is closed under limits, and every such full subcategory is locally presentable and accessibly embedded [1, Corollary 6.24 and Theorem 6.9].

Hence then an additive category $B$ is the category of models of an additive $\kappa$-ary algebraic theory for some $\kappa$ if and only if it is a full exact abelian subcategory of $S^{\text{-mod}}$ for some ring $S$ and it is closed under products.

Proof of Theorem 6.6. One may view the functor $\text{Hom}_B(Q, -): B \to S^{\text{-mod}}$ as the restricted Yoneda functor which sends $X \in B$ to $\text{Hom}_B(\cdot, X)$ restricted to the one-object full subcategory $\{Q\} \subset B$. Then the fact that $\text{Hom}_B(Q, -)$ has a left adjoint $\Delta: S^{\text{-mod}} \to B$ is just an additive version of [1, Proposition 1.27]. To obtain a more concrete description of $\Delta$, one computes that

$$
\text{Hom}_S(S^{(X)}, \text{Hom}_B(Q, B)) = \text{Hom}_B(Q, B)^X = \text{Hom}_B(Q^{(X)}, B)
$$

for any set $X$, and hence the functor $\Delta$ is defined on the full subcategory of free $S$-modules in $S^{\text{-mod}}$ by the rule $\Delta(S^{(X)}) = Q^{(X)}$. To compute the functor $\Delta$ on an
arbitrary $S$-module, one presents it as the cokernel of a morphism of free $S$-modules and uses the fact that left adjoint functors preserve cokernels.

Obviously, $B \mapsto \text{Hom}_B(Q, B)$ is exact since $Q$ is projective. To prove that the functor is fully faithful, let us identify the category $\mathcal{B}$ with the category of modules over the monad $T$: $\text{Sets} \to \text{Sets}$, $X \mapsto \text{Hom}_B(P, P(X))$. Hence $P$ identifies with the free $T$-module $T\{0\}$ with one generator. Consider two objects $C, D \in \mathcal{B}$, whose underlying sets then admit canonical identifications $C = \text{Hom}_B(P, C)$ and $D = \text{Hom}_B(P, D)$. Since then $Q = P^{(Y)} = T(Y)$ is a free $T$-module, we also have identifications $\text{Hom}_B(Q, C) = C^Y$ and $\text{Hom}_B(Q, D) = D^Y$. (Notice that the forgetful functor $\text{Hom}_B(P, -): \mathcal{B} \to \text{Sets}$ preserves products, so our notation is unambiguous.)

Let $f: C^Y \to D^Y$ be a morphism in the category $S\text{-mod}$ (recall that $S = \text{Hom}_B(Q, Q)^{\text{op}}$). For every element $y \in Y$, denote by $p_y \in S$ the idempotent morphism $P^{(Y)} \to P^{(Y)}$ acting as the identity on the $y$-indexed component $P$ in $P^{(Y)} = Q$ and by zero on the $y'$-indexed components for all $y \neq y' \in Y$. Let $p^*_y: C^Y \to C^Y$ and $p^{*}_y: D^Y \to D^Y$ denote the induced maps on the underlying sets $C^Y = \text{Hom}_B(Q, C)$ and $D^Y = \text{Hom}_B(Q, D)$. These maps are also the idempotent projectors for the $y$-indexed components in $C^Y$ and $D^Y$, and they coincide with the $S$-module action of $p_y$ on $C^Y$ and $D^Y$, respectively. Since $f$ is an $S$-module map, we have equalities $fp_{y'} = p^*_y f$ for all $y \in Y$, which in turn means that there exist maps $g_y: C \to D$, one for each element $y \in Y$, such that the map $f: C^Y \to D^Y$ is the product of the family of maps $g_y$, that is $f = \prod_{y \in Y} g_y$.

For every pair of elements $y' \neq y'' \in Y$, denote by $s_{y'y''} \in S$ the involutive automorphism $P^{(Y)} \to P^{(Y)}$ permuting the $y'$-indexed component $P$ in $P^{(Y)}$ with the $y''$-indexed component and acting by the identity on all the other components. Let $s_{y'y''}^*: C^Y \to C^Y$ and $s_{y'y''}^*: D^Y \to D^Y$ denote the induced maps describing the $S$-module action of $s_{y'y''}$. These are also simply the maps permuting the $y'$-indexed component with the $y''$-indexed one in the Cartesian powers. We again have an equality $f s_{y'y''}^* = s_{y'y''}^* f$, which in turn means that the maps $g_{y'}$ and $g_{y''}: C \to D$ are equal. Hence our morphism $f: C^Y \to D^Y$ is the direct power of some map $g: C \to D$, i.e., $f = g^Y$.

It remains to show that the map of sets $g: C \to D$, which we obtained from the $S$-module morphism $f: C^Y \to D^Y$, is a morphism in the category $\mathcal{B}$. For this purpose, it suffices to check that $g$ commutes with all the operations in the monad $T$, that is for every set $X$ and every element $t \in T(X) = \text{Hom}_B(P, P(X))$ the induced maps $t^*: C^X \to C$ and $t^*: D^X \to D$ form a commutative square with the maps $g^X: C^X \to D^X$ and $g: C \to D$. Since the object $P$ is abstractly $\kappa$-small, one can assume that the cardinality of the set $X$ is smaller than $\kappa$. By assumption on the cardinality of the set $Y$, this means that $X$ can be embedded into $Y$. Choosing such an embedding, we can assume that $X = Y$.

Choose an element $y \in Y$; and consider the composition $q: P^{(Y)} \to P^{(Y)}$ of the projection onto the $y$-indexed component $P^{(Y)} \to P$ with the morphism $t: P \to P^{(Y)}$. The induced map $q^*: C^Y \to C^Y$ is the composition of the map $t^*: C^Y \to C^Y$...
with the embedding of the $y$-indexed component $C \rightarrow C^Y$; and the induced map $q^*: D^Y \rightarrow D^Y$ is described similarly. Thus the equation $fq^* = q^*f$ for the $S$-module map $f = g^Y$ implies the desired equation $gt^* = t^*g^Y$.

Proof of Corollary 6.4. The first part is a direct consequence of Theorem 6.6.

Conversely, let $B$ be a full exact abelian subcategory in $S\text{-mod}$ with the reflection functor $\Delta: S\text{-mod} \rightarrow B$. Then $B$ is cocomplete, as one can compute colimits in $B$ by applying the functor $\Delta$ to the colimit of the same diagram computed in $S\text{-mod}$. Besides, the category $B$ has a generator $P = \Delta(S)$ which is projective since $\Delta$ is left adjoint to an exact functor.

The following property of the exact fully faithful functor $B \rightarrow S\text{-mod}$ is also worth noting.

Lemma 6.9. In the setting of Theorem 6.6, the functor $\text{Hom}_B(Q, -): B \rightarrow S\text{-mod}$ takes the projective objects of $B$ to flat $S$-modules.

Proof. Any projective object in $B$ is a direct summand of a copower of the projective generator $P$, so it suffices to show that the $S$-modules $\text{Hom}_B(Q, P^{(X)})$ are flat for all sets $X$. Let $\lambda$ denote the cardinality of the set $Y$; by assumption, the successor cardinality $\lambda^+$ of the cardinal $\lambda$ is greater or equal to $\kappa$. Without loss of generality, we can assume that the cardinality of $X$ is not smaller than $\lambda$. Then we have

$$P^{(X)} = \lim_{Z \subset X, |Z| = \lambda} P^{(Z)},$$

where the $\lambda^+$-filtered colimit is taken over all the subsets $Z \subset X$ of the cardinality equal to $\lambda$. The object $Q \in B$ is $\lambda^+$-presentable, so we have

$$\text{Hom}_B(Q, P^{(X)}) = \lim_{Z \subset X} \text{Hom}_B(Q, P^{(Z)}).$$

Now all the objects $P^{(Z)} \in B$ are isomorphic to $Q$, so the $S$-module $\text{Hom}_B(Q, P^{(Z)})$ is free (with one generator) and the $S$-module $\text{Hom}_B(Q, P^{(X)})$ is a filtered colimit of free $S$-modules. □

6.4. Contramodules over topological rings. Important examples of categories of models of $\kappa$-ary algebraic theories come from topological rings [47, 54]. Let $R$ be a complete and separated topological associative ring with a base of neighborhoods of zero formed by open right ideals (a collection $\{U_j\}_{j \in J}$ of right ideals of $R$ makes $R$ a topological ring if and only if for each $j \in J$ and $r \in R$, there exists $j' \in J$ such that $r \cdot U_{j'} \subset U_j$). Then one defines an additive monad $T_R$ on the category of sets by putting $T_R(X) = R[[X]]$, where $R[[X]]$ is the set of all the infinite formal linear combinations $t = \sum_{x \in X} r_x x$ of elements of $X$ with the coefficients $r_x \in R$ for which the family of elements $(r_x)_{x \in X}$ converges to zero in the topology on $R$. The latter condition means that for every open subset $U \subset R$ one must have $r_x \in U$ for all but a finite subset of indices $x \in X$.

Given a map of sets $X \rightarrow Y$, one defines the induced map $R[[X]] \rightarrow R[[Y]]$ using infinite sums of converging families of elements taken in the topology of $R$. The monadic unit map $X \rightarrow R[[X]]$ takes an element $x_0 \in X$ to the formal linear
combination $\sum_x r_x x$ with $r_x = 1$ for $x = x_0$ and $r_x = 0$ for $x \neq x_0$. The monadic multiplication $\mathfrak{R}[[\mathfrak{R}[[X]]]] \to \mathfrak{R}[[X]]$ is the “opening of parentheses” map, producing a formal linear combination out of a formal linear combination of formal linear combinations. It is constructed using the multiplication in the ring $\mathfrak{R}$ and the infinite summation in the topology of $\mathfrak{R}$; the condition that open right ideals form a base of the topology of $\mathfrak{R}$ guarantees convergence.

The abelian category of modules over the monad $\mathbb{T}_\mathfrak{R}$ is called the category of left $\mathfrak{R}$-contramodules and denoted by $\mathfrak{R}$-contra (this is since if $\mathcal{C}$ is a coalgebra over a field $k$ and $\mathfrak{R} = \mathcal{C}^\ast$ is the vector space dual, then the notion of $\mathcal{C}$-contramodules from [42, 47] coincides with the one of $\mathfrak{R}$-contramodules). The free $\mathbb{T}_\mathfrak{R}$-modules $\mathfrak{R}[[X]]$ are called the free left $\mathfrak{R}$-contramodules. Assuming that the ring $\mathfrak{R}$ has a base of neighborhoods of zero of cardinality less than $\kappa$, the category $\mathfrak{R}$-contra is a locally $\kappa$-presentable abelian category. The free left $\mathfrak{R}$-contramodule with one generator $\mathfrak{R}[[\{0\}]]$ is a $\kappa$-presentable (or, equivalently, abstractly $\kappa$-small) projective generator of $\mathfrak{R}$-contra. The projective objects in $\mathfrak{R}$-contra are precisely the direct summands of free $\mathfrak{R}$-contramodules.

In addition to all the properties listed above (which are essentially common to all the categories of models of additive $\kappa$-ary algebraic theories), the category $\mathfrak{R}$-contra has a more special property that, for every family of projective objects $P_\alpha \in \mathfrak{R}$-contra, the natural map $\prod_\alpha P_\alpha \to \prod_\alpha P_\alpha$ is a monomorphism in this category. This follows from the observation that the map $P^{(X)} \to P^X$, where $P = \mathfrak{R}$ denotes the standard projective generator of $\mathfrak{R}$-contra, is a monomorphism for every set $X$, because the obvious map of sets $\mathfrak{R}[[X]] \to \mathfrak{R}^X$ is injective. We are not aware of any example of the category of models of an additive $\kappa$-ary algebraic theory having this property of the (co)products of projective objects that would not come from a topological ring.

If $\mathfrak{R}$ is a topological ring as above, we of course have the forgetful functor

$$\mathfrak{R}$-contra \to \mathfrak{R}$-mod

from $\mathfrak{R}$-contramodules to the category of ordinary $\mathfrak{R}$-modules. A natural question, closely related to Theorem 6.6, is when this functor is fully faithful. Although a complete answer does not seem to be known, it holds for the adic completions of Noetherian rings by centrally generated ideals [45, Theorem B.1.1], [46, Theorem C.5.1]. More general results of this kind can be found in [52, Theorem 1.1] and [53, Section 3]. Further classes of examples coming from tilting theory are provided below in Propositions 7.10 and 7.13 and Theorems 9.7 and 9.12.

7. Big Tilting Modules

The aim of this section is to discuss $n$-tilting objects and equivalences for abelian subcategories $A \subset A$-mod of categories of modules over associative rings $A$. The equivalences become more concrete in this situation and it allows us to connect our results to the existing theory of infinitely generated $n$-tilting $A$-modules.
In fact, we mostly work more generally with tilting objects in full subcategories $A \subset A\text{-mod}$ which are abelian as categories and closed under coproducts in $A\text{-mod}$. This is to cover cases like the one studied in Example 5.1 and [50, 51], but also with Section 9 in mind where this level of generality is needed too.

### 7.1. Tilting equivalences for categories of modules.

We start with a key result which says that the endomorphism ring of a module always carries a natural structure of a topological ring and hence we can consider contramodules over it.

**Theorem 7.1.** Let $A$ be an associative ring, $M$ be a left $A$-module, and $\text{Add}(M) \subset A\text{-mod}$ be the full additive subcategory formed by the direct summands of infinite direct sums of copies of the object $M$ in the category of left $A$-modules. Then there exists a complete, separated topological associative ring $R$ with a base of neighborhoods of zero formed by open right ideals such that the category $\text{Add}(M)$ is equivalent to the category of projective left $R$-contramodules.

**Proof.** Consider the ring $\text{Hom}_A(M, M)$ and endow it with the topology in which the base of neighborhoods of zero is formed by the annihilator ideals $\text{Ann}(E) \subset \text{Hom}_A(M, M)$ of finitely generated $A$-submodules $E \subset M$. To be precise, the annihilator

$$\text{Ann}(E) = \text{Hom}_A(M/E, M) = \{ f \in \text{Hom}_A(M, M) \mid f(E) = 0 \}$$

is a left ideal in the ring $\text{Hom}_A(M, M)$. In view of the exact sequence

$$0 \longrightarrow \text{Hom}_A(M/E, M) \longrightarrow \text{Hom}_A(M, M) \longrightarrow \text{Hom}_A(E, M),$$

the quotient module $\text{Hom}_A(M, M)/\text{Ann}(E)$ is identified with the set of all $A$-module morphisms $E \longrightarrow M$ that can be extended to $A$-module morphisms $M \longrightarrow M$. Since the datum of an $R$-module morphism $M \longrightarrow M$ is equivalent to that of a compatible system of $A$-module morphisms $E \longrightarrow M$ defined for all the finitely generated submodules $E \subset M$, we have an isomorphism

$$\text{Hom}_A(M, M) \simeq \lim_{\leftarrow E \subset M} \text{Hom}_A(M, M)/\text{Ann}(E),$$

that is $\text{Hom}_A(M, M)$ is a complete and separated topological ring.

Let $R = \text{Hom}_A(M, M)^{\text{op}}$ be the topological ring opposite to $\text{Hom}_A(M, M)$; so $R$ is a complete, separated topological ring with a base of the topology formed by open right ideals. Consider the functor $\mathbb{T} : \text{Sets} \longrightarrow \text{Sets}$ assigning to every set $X$ the set $\text{Hom}_A(M, M^{(X)})$ of all $A$-module morphisms from $M$ to the direct sum of $X$ copies of $M$. The functor $\mathbb{T}$ has a natural structure of a monad on the category of sets. We claim that this monad is isomorphic to the monad $\mathbb{T}_R$.

Indeed, $\text{Hom}_A(M, M^{(X)})$ is a subset in $\text{Hom}_A(M, M)^{X}$ and $R[[X]]$ is a subset in $R^X$; first of all, it is claimed that this is the same subset. In other words, an $X$-indexed family of $A$-module morphisms $f_x : M \longrightarrow M$ corresponds to an $R$-module morphism $M \longrightarrow M^{(X)}$ if and only if it converges to zero in the topology of $\text{Hom}_R(M, M)$, that is if and only if for every finitely-generated submodule $E \subset M$ one has $f_x(E) = 0$ for all but a finite subset of indices $x \in X$. This is obvious; and checking that the
unit and composition operations in our two monads T: X ↦ Hom_R(M, M^X) and X ↦ R[[X]] are the same is straightforward (cf. the proof of Lemma 7.3 below).

We have shown that the assignment of the R-module M^X to a free R-contra-module R[[X]] establishes an isomorphism between the category of direct sums of copies of M in A-mod and the category of free left R-contramodules. Adjoining direct summands to the categories on both sides, we obtain an equivalence between Add(M) and the category of projective left R-contramodules. □

Now we can specialize tilting equivalences to the case where A ⊂ A-mod is a full exact abelian subcategory which is closed under coproducts in A-mod. This not only covers equivalences studied in Example 5.1 and [50, 51], but we also need to understand such a situation because of Section 9.

Corollary 7.2. Let A be an associative ring and A ⊂ A-mod be a full subcategory closed under coproducts. Suppose that A is an abelian category with products and that it has an injective cogenerator. Then, for any n-tilting object T ∈ A, the abelian category B in the heart of the tilting t-structure on D(A) associated with T is equivalent to the abelian category of left contramodules R-contra over the topological ring R = Hom_A(T, T)^op.

Proof. Both the abelian categories B and R-contra have enough projectives. The category of projective objects in B is equivalent to Add(T) ⊂ A ⊂ A-mod, which is equivalent to the category of projective objects in R-contra by Theorem 7.1. Since an abelian category with enough projective objects is determined by its full subcategory of projective objects, it follows that the categories B and R-contra are equivalent. □

In particular, the combination of Corollaries 4.6 and 7.2 yields derived equivalences

\[ R\Psi: D^+(A) \rightleftarrows D^+(R-contra): L\Phi \]

We would like to describe more explicitly the pair of adjoint functors \( \Psi: A \to B \) and \( \Phi: B \to A \). The point is that we obtain a rather explicit Hom-tensor adjunction. The description of the functor \( \Psi \) follows from Section 6.2.

Lemma 7.3. For any associative ring A and A-modules M and N, the set/group Hom_A(M, N) has a natural structure of a left contramodule over the topological ring R = Hom_A(M, M)^op. The left exact functor

\[ \text{Hom}_A(M, -): \text{A-mod} \to R\text{-contra} \]

extends the additive embedding functor Add(M) ≃ R-contra_{proj} → R-contra from the full subcategory Add(M) ⊂ A-mod to the whole abelian category A-mod.

Proof. Given a set X, a family of elements \( r_x \in R \) corresponding to a family of morphisms \( g_x \in \text{Hom}_A(M, M) \) converging to zero in the topology of \( \text{Hom}_A(M, M) \), and an arbitrary family of morphisms \( f_x \in \text{Hom}_A(M, N) \), one defines the morphism \( \sum_{x \in X} r_x f_x \in \text{Hom}_A(M, N) \) by the rule

\[ \left( \sum_{x \in X} r_x f_x \right)(m) = \sum_{x \in X} f_x(g_x(m)) \]
for every $m \in M$, where the sum on the right-hand side has only a finite number of nonzero summands due to the condition on the family of morphisms $g_x$. Checking the unit and associativity equations for this structure of a module over the monad $X \mapsto \mathcal{R}[[X]]$ is straightforward. □

Corollary 7.4. Let $A \subset A\text{-mod}$ be a full subcategory closed under coproducts. Suppose that as a category, $A$ is abelian, has products, and has an injective cogenerator $W$ and an $n$-tilting object $T$. Let $B \subset D^b(A)$ be the tilting heart; so $B \simeq \mathcal{R}\text{-contra}$, where $\mathcal{R} = \text{Hom}_A(T,T)^{\text{op}}$. Then the left exact functor $\Psi : A \longrightarrow B$ can be computed as the restriction of the functor $\text{Hom}_A(T,-) : A\text{-mod} \longrightarrow \mathcal{R}\text{-contra}$ to the full subcategory $A \subset A\text{-mod}$.

Proof. We use the same argument as in Section 6.2. It suffices to observe that the composition of functors $\text{Add}(T) \simeq \text{B}_{\text{proj}} \longrightarrow B \simeq \mathcal{R}\text{-contra}$ takes the $A$-module $T(X)$ to the $\mathcal{R}$-contramodule $\mathcal{R}[[X]] \simeq \text{Hom}_A(T,T(X))$ for every set $X$ (cf. the proof of Theorem 7.1). □

7.2. Contratensor product. In order to describe the right exact functor $\Phi : B \longrightarrow A$, we need to recall one general construction related to contramodules. Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. A right $\mathcal{R}$-module $L$ is called discrete if action map $L \times \mathcal{R} \longrightarrow L$ is continuous in the given topology of $\mathcal{R}$ and the discrete topology of $L$, or in other words, if the annihilator ideal of every element in $L$ is open in $\mathcal{R}$. Discrete right $\mathcal{R}$-modules form a Grothendieck abelian category $\text{discr}\mathcal{R}$.

For any discrete right $\mathcal{R}$-module $L$ and every abelian group $V$ the group $C = \text{Hom}_{\mathbb{Z}}(L,V)$ has a natural left $\mathcal{R}$-contramodule structure with the monadic action map $\mathcal{R}[[C]] \longrightarrow C$ defined by the rule

$$
(\sum_{x \in X} r_x f_x)(l) = \sum_{x \in X} f_x(l r_x),
$$

where the family of elements $r_x \in \mathcal{R}$ converges to zero in the topology of $\mathcal{R}$ and $l \in L$. The infinite sum in the left-hand side (representing an element of $C$) is a symbolic notation for the monadic action map that we are defining, while the sum on the right-hand side is finite because the right $\mathcal{R}$-module $L$ is discrete.

Let $L$ be a discrete right $\mathcal{R}$-module and $C$ be a left $\mathcal{R}$-contramodule. The contratensor product $L \odot_{\mathcal{R}} C$ is the abelian group constructed as the quotient group of the group $L \otimes_{\mathbb{Z}} C$ by the subgroup generated by all elements of the form

$$
\sum_{x \in X} l r_x \otimes c_x - l \otimes \sum_{x \in X} r_x c_x,
$$

where $l \in L$ is an element, $X$ is a set, $r_x \in \mathcal{R}$ is a family of elements converging to zero in the topology of $\mathcal{R}$, and $c_x \in C$ is an arbitrary family of elements. The sum in the left-hand summand is finite because the right $\mathcal{R}$-module $L$ is discrete, while the infinite sum in the right-hand summand is a symbolic notation for the monad action map $\mathcal{R}[[C]] \longrightarrow C$. 43
The contratensor product is a right exact functor of two arguments

\( \odot_{\mathcal{R}} : \text{discr-}\mathcal{R} \times \mathcal{R}\text{-contra} \longrightarrow \mathbb{Z}\text{-mod}. \)

For any discrete right \( \mathcal{R} \)-module \( L \) and any set \( X \), there is a natural isomorphism of abelian groups

(19)

\[ L \odot_{\mathcal{R}} \mathcal{R}[[X]] \cong L^{(X)}. \]

For any discrete right \( \mathcal{R} \)-module \( L \), any left \( \mathcal{R} \)-contramodule \( C \), and any abelian group \( V \) there is a natural isomorphism

(20)

\[ \text{Hom}_\mathbb{Z}(L \odot_{\mathcal{R}} C, V) \cong \text{Hom}^\mathcal{R}(\mathcal{C}, \text{Hom}_\mathbb{Z}(L, V)), \]

where \( \text{Hom}^\mathcal{R} = \text{Hom}_{\mathcal{R}\text{-contra}} \) denotes the group of morphisms in the category of left \( \mathcal{R} \)-contramodules \( \mathcal{R}\text{-contra} \).

More generally, let \( A \) be an associative ring and \( L \) be a discrete right \( \mathcal{R} \)-module endowed with a left \( A \)-module structure making it an \( A\mathcal{R} \)-bimodule. Let \( V \) be a left \( A \)-module and \( \mathcal{C} \) be a left \( \mathcal{R} \)-contramodule. Then \( \text{Hom}_A(L, V) \) is a subcontramodule of the left \( \mathcal{R} \)-contramodule \( \text{Hom}_\mathbb{Z}(L, V) \), so the group \( \text{Hom}_A(L, V) \) has a natural left \( \mathcal{R} \)-contramodule structure. Furthermore, the left \( A \)-module structure on \( L \) induces a left \( A \)-module structure on the contratensor product \( L \odot_{\mathcal{R}} \mathcal{C} \). There is a natural isomorphism of abelian groups

(21)

\[ \text{Hom}_A(L \odot_{\mathcal{R}} \mathcal{C}, V) \cong \text{Hom}^\mathcal{R}(\mathcal{C}, \text{Hom}_A(L, V)). \]

In other words, the contratensor product functor

\[ L \odot_{\mathcal{R}} - : \mathcal{R}\text{-contra} \longrightarrow A\text{-mod} \]

is left adjoint to the Hom functor

\[ \text{Hom}_A(L, -) : A\text{-mod} \longrightarrow \mathcal{R}\text{-contra}. \]

**Lemma 7.5.** Let \( A \) be an associative ring, \( M \) be a left \( A \)-module, and \( \mathcal{R} \) be the topological ring \( \text{Hom}_A(M, M)^{\text{op}} \). Then \( M \) is a discrete right \( \mathcal{R} \)-module and an \( A\mathcal{R} \)-bimodule. The right exact functor

\[ M \odot_{\mathcal{R}} - : \mathcal{R}\text{-contra} \longrightarrow A\text{-mod} \]

extends the additive embedding functor \( \mathcal{R}\text{-contra}_{\text{proj}} \cong \text{Add}(M) \longrightarrow A\text{-mod} \) from the full subcategory of projective objects \( \mathcal{R}\text{-contra}_{\text{proj}} \subset \mathcal{R}\text{-contra} \) to the whole abelian category of left \( \mathcal{R} \)-contramodules. The functor \( M \odot_{\mathcal{R}} - \) is left adjoint to the functor \( \text{Hom}_A(M, -) \) from Lemma 7.3.

**Proof.** The right \( \mathcal{R} \)-module \( M \) is discrete by the definition. Moreover, it should be noticed that the construction of the \( \mathcal{R} \)-contramodule structure in the proof of Lemma 7.3 is a particular case of the construction of the \( \mathcal{R} \)-contramodule structure in the formula (17) (or more precisely, of the generalization of the latter to the case of the \( A \)-module homomorphisms, as discussed above). The isomorphism (19) shows that the functor \( M \odot_{\mathcal{R}} - \) takes \( \mathcal{R}[[X]] \) to \( M^{(X)} \). The adjunction isomorphism (21) shows that our functors \( M \odot_{\mathcal{R}} - \) and \( \text{Hom}_A(M, -) \) are adjoint. \( \square \)
Corollary 7.6. If in the setting of Corollary 7.4 we further assume that $A$ is closed under cokernels in $A$–mod, then the right exact functor $\Phi : B \to A$ can be computed as the functor $T \odot \mathfrak{R} \to A$–mod, whose image is contained in the full subcategory $A \subset A$–mod.

Proof. For any left $\mathfrak{R}$-contramodule $C$, the left $A$-module $T \odot \mathfrak{R} C$ is the cokernel of the $A$-module morphism $T \otimes \mathfrak{R}[C] \to T \otimes \mathfrak{R} C$ defined by formula (18). Since the full subcategory $A \subset A$–mod is closed under cokernels and coproducts, and $T \in A$, it follows that the functor $T \odot \mathfrak{R} -$ is left adjoint to $\Psi$, the contratensor product functor $T \odot \mathfrak{R} -$ is left adjoint to $\text{Hom}_A(T, -)$, and the functor $\Psi$ is isomorphic to $\text{Hom}_A(T, -)$ by Corollary 7.4.

Alternatively, one can argue in the way similar to the proof of Corollary 7.4. Both the functors $\Phi$ and $T \odot \mathfrak{R} -$ are right exact, so it suffices to show that they coincide on the full subcategory of projective objects $B_{\text{proj}} \subset B$. It remains to refer to the second assertion of Lemma 7.5. □

Corollary 7.7. Let $A \subset A$–mod be a full subcategory closed under all colimits. Suppose that as a category, $A$ is abelian, has products, and has an injective cogenerator $W$ and an $n$-tilting object $T$. If we denote $\mathfrak{R} = \text{Hom}_A(T, T)^{\text{op}}$, then the derived equivalences of Corollary 7.4 (where the symbol $*$ stands for one of $\mathfrak{b}$, $+$, $-$, $\emptyset$, $\text{abs}+$, $\text{abs}-$, or $\text{abs}$) can be expressed in the form

$$
\mathfrak{R} \text{Hom}_A(T, -) : D^*(A) \rightleftarrows D^*(\mathfrak{R} \text{-contra}) : - \odot \mathfrak{R} T.
$$

Remark 7.8. In Section 9.1 we will encounter a situation where $A \subset A$–mod is as in Corollary 7.4, but not necessarily closed under cokernels. However, in that situation the embedding $A \subset A$–mod will have a left adjoint $r : A$–mod $\to A$ and one adjusts the conclusion of Corollary 7.6 to $\Phi \simeq r(T \odot \mathfrak{R} -)$.

Next we address the natural question on how the contratensor product $\odot \mathfrak{R}$ is related to the ordinary tensor product $\otimes \mathfrak{R}$. This turns out to be closely connected to the problem when the forgetful functor $\mathfrak{R} \text{-contra} \to \mathfrak{R} \text{-mod}$ is fully faithful and also to the notion of good tilting module studied in [11, 13] and mentioned below in §7.3.

Lemma 7.9. Let $\mathcal{S}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Suppose that the forgetful functor $\mathcal{S} \text{-contra} \to \mathcal{S} \text{-mod}$ is fully faithful. Then the natural surjection of abelian groups

$$
N \otimes \mathcal{S} \mathfrak{C} \twoheadrightarrow N \odot \mathcal{S} \mathfrak{C}
$$

is an isomorphism for every discrete right $\mathcal{S}$-module $N$ and left $\mathcal{S}$-contramodule $\mathfrak{C}$.

Proof. Compare the two natural isomorphisms

$$
\text{Hom}_\mathbb{Z}(N \odot \mathcal{S} \mathfrak{C}, V) \simeq \text{Hom}^\mathcal{S}(\mathfrak{C}, \text{Hom}_\mathbb{Z}(N, V))
$$
and

$$\text{Hom}_\mathbb{Z}(N \otimes \mathfrak{C}, V) \simeq \text{Hom}_\mathfrak{C}(\mathfrak{C}, \text{Hom}_\mathbb{Z}(N, V)),$$

which hold for every abelian group $V$ (see (20)). □

The following proposition, which essentially is a particular case of Theorem 6.6, says that at least for theoretical purposes the assumption of Lemma 7.9 imposes no substantial restriction.

**Proposition 7.10.** Let $A$ be an associative ring, $M$ be a left $A$-module, and $Y$ be an infinite set of cardinality greater or equal to the minimal cardinality of a set of generators of $M$. Put $L = M^Y$, and denote by $\mathcal{S}$ the topological ring $\text{Hom}_A(L, L)$. Then the forgetful functor $\mathcal{S} \text{-contra} \to \mathcal{S} \text{-mod}$ is fully faithful. Hence the category $\text{Add}(M) \subset A \text{-mod}$ is equivalent to a full subcategory in the category $\mathcal{S} \text{-mod}$ of left modules over the ring $\mathcal{S}$ viewed as an ordinary (nontopological) ring.

**Proof.** Let $\mathfrak{R}$ denote the topological ring $\text{Hom}_A(M, M)$. According to Theorem 7.1 applied to the $A$-modules $M$ and $L$, we have equivalences of additive categories $\text{Add}(M) \simeq \mathfrak{R} \text{-contra}_{\text{proj}}$ and $\text{Add}(L) \simeq \mathcal{S} \text{-contra}_{\text{proj}}$. Obviously, $\text{Add}(M) \subset A \text{-mod}$ and $\text{Add}(L) \subset A \text{-mod}$ is one and the same subcategory. The resulting equivalence of the additive categories of projective objects $\mathfrak{R} \text{-contra}_{\text{proj}} \simeq \mathcal{S} \text{-contra}_{\text{proj}}$ extends to an equivalence of the abelian categories $\mathfrak{R} \text{-contra} \simeq \mathcal{S} \text{-contra}$.

The equivalence of additive categories $\text{Add}(M) \simeq \mathfrak{R} \text{-contra}_{\text{proj}}$ takes the $A$-module $M \in \text{Add}(M)$ to the free $\mathfrak{R}$-contramodule with one generator $\mathfrak{R} \in \mathfrak{R} \text{-contra}$ and the $A$-module $L \in \text{Add}(M)$ to the free $\mathfrak{R}$-contramodule $\mathfrak{R}[[Y]]$ spanned by the set $Y$. By (the proof of) Theorem 6.6 we infer that, as a ring, $\mathcal{S} = \text{Hom}^\mathfrak{R}(\mathfrak{R}[[Y]], \mathfrak{R}[[Y]])$, and if we compose the above mentioned equivalence $\mathfrak{R} \text{-contra} \simeq \mathcal{S} \text{-contra}$ with the forgetful functor $\mathcal{S} \text{-contra}_{\text{proj}} \to \mathcal{S} \text{-mod}$, we simply obtain the functor

$$\text{Hom}^\mathfrak{R}(\mathfrak{R}[[X]], -) : \mathfrak{R} \text{-contra} \to \mathcal{S} \text{-mod}.$$  

We must prove that the latter functor is fully faithful. To this end, it remains to notice that the topological ring $\mathfrak{R}$ has a base of neighborhoods of zero of the cardinality not exceeding the cardinality of $Y$ (formed by the annihilators of the submodules of $M$ generated by finite subsets of the chosen set of generators). So $\mathfrak{R}$ is an abstractly $\kappa$-small projective generator of the category $\mathfrak{R} \text{-contra}$, where $\kappa$ is the successor cardinal of the cardinality of $Y$. By Theorem 6.6 the functor $\text{Hom}^\mathfrak{R}(\mathfrak{R}[[X]], -) : \mathfrak{R} \text{-contra} \to \mathcal{S} \text{-mod}$ is fully faithful. □

**Corollary 7.11.** Let $A \subset A \text{-mod}$ be a full subcategory closed under all colimits. Suppose that the abelian category $A$ has products and an injective cogenerator $W$, and that $T \in A$ is an $n$-tilting object. Choose a set $Y$ of cardinality greater than or equal to the minimal cardinality of a set of generators of $T$ and put $N = T^Y$. Then $N \in A$ is an $n$-tilting object inducing the same tilting $t$-structures on $\text{D}^+(A)$ as $T$. Let $B \subset \text{D}^+(A)$ be the tilting heart; so $B \simeq \mathfrak{R} \text{-contra} \simeq \mathcal{S} \text{-contra}$, where $\mathfrak{R} = \text{Hom}_A(T, T)^{\text{op}}$ and $\mathcal{S} = \text{Hom}_A(N, N)$.  

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In this situation, the forgetful functor $B \simeq \mathcal{G} \rightarrow \mathcal{G} \rightarrow A \rightarrow \mathcal{G} \rightarrow \mathcal{G} \rightarrow \mathcal{G}$ is fully faithful and, if we view it as an inclusion $B \subset \mathcal{G} \rightarrow \mathcal{G}$, the adjunction $\Psi: A \Rightarrow B : \Phi$ from Section 3 arises as the restriction of the adjunction

$$\text{Hom}_A(N, -): A \rightarrow \mathcal{G} \rightarrow N \otimes A.$$ 

7.3. Big tilting modules. We finish the section by looking closer at the case where $A = A \rightarrow \mathcal{G}$. Recall from Corollary 2.6 that a module $T$ is $n$-tilting if and only if it satisfies the following three conditions:

(i) the projective dimension of $T$ in $A \rightarrow \mathcal{G}$ does not exceed $n$,

(ii) $\text{Ext}^i_A(T, T^{(I)}) = 0$ for all $i > 0$ and all sets $I$, and

(iii) the free left $A$-module $A$ has a finite right resolution

$$0 \rightarrow A \rightarrow T^0 \rightarrow \cdots \rightarrow T^r \rightarrow 0.$$ 

This definition goes back to the papers 2, 10. We can illustrate the concept rather explicitly on the following example, which is closely related to our previous Example 5.1.

**Example 7.12.** Let $R$ be a commutative Noetherian ring of Krull dimension 1 and let $S \subset R$ be a multiplicative subset consisting of nonzero-divisors. Then $S^{-1}R$ is a flat $R$-module, hence its projective dimension is at most 1 by 57 Corollaire II.3.2.7. In particular $T = S^{-1}R \oplus S^{-1}R/R$ is a 1-tilting module by 27. Theorem 14.59, analogously to Example 5.1.

Let now $P = \bigcup_{s \in S} V(s) = \{p \mid p \cap S \neq \emptyset\}$ be the set of all primes that intersect $S$. Since $S$ contains nonzero-divisors only, $P$ must consist of maximal ideals only (if $p \in \text{Spec } R$ is not maximal, it is an associated prime of the $R$-module $R$ and hence all its elements are zero-divisors). Moreover, it follows that an $R$-module is $S$-torsion (i.e., $S^{-1}M = 0$) if and only if the support of $M$ is contained in $P$, and in such a case

$$M \simeq \bigoplus_{p \in P} M_p$$

(see for example 50 Lemma 13.1). In particular, $S^{-1}R/R \simeq \bigoplus_{p \in P} S^{-1}R_p/R_p$ and, by 50. Lemma 13.5, for each $p \in P$ there exists $s_p \in S$ such that $S^{-1}R_p = R_p[s_p^{-1}]$.

We know from Corollary 7.7 that we have derived equivalences

$$(22) \quad \mathfrak{R} \text{Hom}_R(T, -): D^*(R \rightarrow \mathcal{G}) \rightarrow D^*(\mathfrak{R} \rightarrow \mathcal{G})$$

for the topological ring $\mathfrak{R} = \text{Hom}_R(T, T)^{op}$ and various choices of the symbol $\star$. It turns out that $\mathfrak{R}$ has a very nice description as a certain $2 \times 2$ triangular matrix ring. To this end, we have $\text{Hom}_R(S^{-1}R, S^{-1}R) = S^{-1}R$, $\text{Hom}_R(S^{-1}R/R, S^{-1}R) = 0$, and

$$\text{Hom}_R(S^{-1}R/R, S^{-1}R/R) = \prod_{p \in P} \text{Hom}_R(S^{-1}R_p/R_p, S^{-1}R_p/R_p) = \prod_{p \in P} \hat{R}_p,$$

since for each $p \in P$, we have $\text{Hom}_R(R_p[s_p^{-1}]/R_p, R_p[s_p^{-1}]/R_p) = \lim R/s_p^n R$, which is the usual adic completion of $R_p$ (see Example 5.1 and 51 Proposition 3.2). If we
denote $\mathfrak{A}_S = \text{Hom}_R(S^{-1}R, S^{-1}R/R)$, then as rings

$$\mathfrak{A} = \left( \begin{array}{c} S^{-1}R \\ 0 \\ \prod_{p \in P} \hat{R}_p \end{array} \right).$$

(23)

It is also interesting to have a closer look at $\mathfrak{A}_S$, as this is a version of the ring of finite adeles. To see this, let us apply $\text{Hom}_R(-, S^{-1}R/R)$ to the short exact sequence $0 \to R \to S^{-1}R \to S^{-1}R/R \to 0$. We obtain a short exact sequence

$$0 \to \prod_{p \in P} \hat{R}_p \to \mathfrak{A}_S \to S^{-1}R/R \to 0.$$ 

We claim that $\mathfrak{A}_S \simeq (\prod_{p \in P} \hat{R}_p) \otimes_R S^{-1}R$, which in fact gives $\mathfrak{A}_S$ a natural ring structure. To see this, express $S^{-1}R = \lim_{s \in S} R_s$, where $R_s = R$ for each $s$ and the maps in the direct system are given by $t \cdot s_r \mapsto R_s \to R_{st}$ for $s, t \in S$. Then

$$\mathfrak{A}_S = \text{Hom}_R(S^{-1}R, S^{-1}R/R) = \lim_{s \in S} \text{Hom}_R(R_s, S^{-1}R/R).$$

The latter inverse system can be identified with the inverse system $(\mathfrak{A}/s \prod_{p \in P} \hat{R}_p)_{s \in S}$, where $\mathfrak{A}' = S^{-1}\prod_{p \in P} \hat{R}_p$ and the maps are simply projections. It follows that $\mathfrak{A}_S \simeq \mathfrak{A}'$, which proves the claim.

To summarize, given a commutative Noetherian ring $R$ of Krull dimension 1 and a multiplicative set of nonzero-divisors in it, we have derived equivalences (22), where $\ast$ can be one of $b, +, -, \varnothing, \text{abs+}, \text{abs-}$, or $\text{abs}$. The ring structure of the topological ring $\mathfrak{A}$ has an explicit description via (23), and the topology is also the obvious one: $S^{-1}R$ is discrete, $\prod_{p \in P} \hat{R}_p$ carries the product topology of the adic topologies, and the adelic ring $\mathfrak{A}_S = S^{-1}\prod_{p \in P} \hat{R}_p$ has a canonical locally compact topology.

Finally, we will prove a stronger version of Corollary 7.11 under an assumption which was recently employed in [11, 13]. An (infinitely generated) $n$-tilting module $T \in A\text{-mod}$ is called good [27 Section 13.1] if the $A$-modules $T^i$ in the condition (iii_m) can be chosen to be direct summands of finite direct sums of copies of $T$. Obviously, replacing an arbitrary $n$-tilting $A$-module $T$ with the $A$-module $T^\vee$ for a large enough set $Y$ produces a good $n$-tilting module with the same associated tilting $t$-structure.

**Proposition 7.13.** Let $T \in A\text{-mod}$ be a good $n$-tilting $A$-module and $B \subseteq D^b(A\text{-mod})$ be the tilting heart, so $B \simeq \mathcal{S}\text{-contra}$, where $\mathcal{S} = \text{Hom}_A(T, T)^{\text{op}}$. Then the forgetful functor $\mathcal{S}\text{-contra} \to \mathcal{S}\text{-mod}$ induces a fully faithful functor $D^b(\mathcal{S}\text{-contra}) \to D^b(\mathcal{S}\text{-mod})$. In particular, $\mathcal{R} \text{Hom}_A(T, -) : D^b(A\text{-mod}) \to D^b(\mathcal{S}\text{-mod})$ is fully faithful and the adjunction

$$\text{Hom}_A(T, -) : D^b(A\text{-mod}) \rightleftarrows D^b(\mathcal{S}\text{-mod}) : T \otimes_{\mathcal{S}} -$$

restricts to an equivalence

$$\text{Hom}_A(T, -) : D^b(A\text{-mod}) \rightleftarrows D^b(\mathcal{S}\text{-contra}) : T \otimes_{\mathcal{S}} -.$$
Proof. The assertion can be deduced from a stronger result that the forgetful functor $D(S\text{-contra}) \to D(S\text{-mod})$ is fully faithful, which is a restatement of [13, Theorem 2.2 (2)] (compared with our Corollary 4.6).

Here is a more direct argument based on the assertions of [13, Lemma 1.5 (1–2)]. Since the triangulated category $D^b(B) = D^b(S\text{-contra})$ is generated by its full subcategory $E \subset B \subset D^b(B)$, in order to show that the functor $D^b(S\text{-contra}) \to D^b(S\text{-mod})$ is fully faithful it suffices to check that the maps

$$\text{Ext}^i_S(C, D) \to \text{Ext}^i_S(C, D)$$

are isomorphisms for all $i \geq 0$ and all $C, D \in E \subset B = S\text{-contra}$ (where the notation in the left-hand side stands for the Ext groups in the abelian category of left $S$-contramodules, while the Ext groups in the right-hand side are computed in the abelian category of left $S$-modules).

Replacing the $S$-contramodule $C$ by its projective left resolution in the category $S\text{-contra}$, one can assume that $C = S[[X]]$ (which corresponds to the object $T(X) \in E \subset B$). Replacing the object $D \in E \subset B$ by its right resolution by powers of the cotilting object $W$ (cf. [13] and [15]), one can assume that $D = W^I$ for some set $I$.

Since the products are exact in $S\text{-contra}$ and $S\text{-mod}$ and preserved by the forgetful functor $S\text{-contra} \to S\text{-mod}$, it suffices to consider the case $D = W$.

When $i > 0$, the left-hand side of (24) vanishes, since $C$ is a projective object in $S\text{-contra}$. To compute the right-hand side, use the vanishing assertion

$$\text{Tor}_i^S(T, \text{Hom}_A(T, E)) = 0$$

for all $E \in E \subset A\text{-mod}$ [13, Lemma 1.5 (1)]. Substituting $E = T(X)$ and applying the functor $\text{Hom}_A$ into the injective cogenerator $W \in A\text{-mod}$, one obtains

$$\text{Ext}_S^i(S[[X]], \Psi(W)) = \text{Ext}_S^i(\text{Hom}_A(T, T(X)), \text{Hom}_A(T, W))$$

$$= \text{Hom}_A(\text{Tor}_i^S(T, \text{Hom}_A(T, T(X))), W) = 0.$$ 

When $i = 0$, one can use the isomorphism

$$T \otimes_E \text{Hom}_A(T, E) \simeq E$$

[13, Lemma 1.5 (2)]. Taking again $E = T(X)$ and applying the functor $\text{Hom}_A(\_ , W)$, we get the isomorphism

$$\text{Hom}_S(S[[X]], \Psi(W)) = \text{Hom}_S(\text{Hom}_A(T, T(X)), \text{Hom}_A(T, W))$$

$$= \text{Hom}_A(T \otimes_E \text{Hom}_A(T, T(X)), W) \simeq \text{Hom}_A(T(X), W)$$

$$= \Psi(W)^X = \text{Hom}^S(S[[X]], \Psi(W)),$$

as desired. \qed
8. Properties of Categories Transferred by the Correspondence

The two dual classes of arbitrary complete, cocomplete abelian categories $A$ with an injective cogenerator and complete, cocomplete abelian categories $B$ with a projective generator appearing in the theory developed in Section 3 are often too broad and abstract. It would be interesting to know which specific subclasses of abelian categories $A$ or $B$ on one of the sides does the tilting-cotilting correspondence assign to various natural subclasses of abelian categories on the other side. Are there any natural ways to strengthen the conditions on both the categories $A$ and $B$ in Corollary 3.12 such that the correspondence remains one-to-one?

Assume that $A$ is a Grothendieck abelian category; what, precisely, can one then say about the abelian category $B$? Assume that $A$ is a locally presentable abelian category; what is, precisely, the class of abelian categories $B$ corresponding to such categories $A$ under the tilting-cotilting correspondence?

We do not know the answers to these questions (one of the reasons for this may be that we do not yet understand what are the natural subclasses of the class of all the categories of models of additive $\kappa$-ary algebraic theories). Some partial results in this direction are presented in this section.

**Proposition 8.1.** Let $A$ be a locally presentable abelian category, $M \in A$ be an object, and $\text{Add}(M) \subset A$ be the full additive subcategory formed by the direct summands of all coproducts of copies of the object $M$ in the category $A$. Then there exists a (naturally defined) category of models $B$ of an additive $\kappa$-ary algebraic theory such that the category $\text{Add}(M)$ is equivalent to the full subcategory $B_{\text{proj}} \subset B$ of projective objects in $B$.

**Proof.** Notice first of all that any locally presentable category is complete and cocomplete [1, Remark 1.56]. Consider the additive monad $T: X \mapsto \text{Hom}_A(M, M^{(X)})$ on the category of sets. Let $B$ be the category of $T$-modules; then $B$ is an abelian category with a natural projective generator $P$ corresponding to the free $T$-module with one generator. By construction, there is a natural equivalence of additive categories $\text{Add}(M) \simeq B_{\text{proj}}$ taking the object $M \in \text{Add}(M)$ to the object $P \in B_{\text{proj}}$ and the object $M^{(X)} \in \text{Add}(M)$ to the object $P^{(X)} \in B_{\text{proj}}$.

It remains to show that the object $P \in B$ is abstractly $\kappa$-small for some cardinal $\kappa$. Indeed, let $\kappa$ be the presentability rank of the object $M \in A$. Then, in particular, the object $M \in A$ is abstractly $\kappa$-small, i.e., every morphism $M \to M^{(X)}$ in $A$ factorizes through the natural embedding $M^{(Z)} \to M^{(X)}$ for some subset of indices $Z \subset X$ of the cardinality smaller than $\kappa$. In view of the equivalence of categories $\text{Add}(M) \simeq B_{\text{proj}}$, the desired assertion follows. \qed

**Corollary 8.2.** Suppose that $A$ is a locally presentable abelian category with set-indexed products and an injective cogenerator. If $T \in A$ is an $n$-tilting object and $B$ is the heart of the tilting t-structure on $\mathbf{D}^b(A)$, then $B$ is the category of models of an additive $\kappa$-ary algebraic theory for some $\kappa$. The left exact functor $\Psi: A \to B$ from Section 4 can be computed as the functor $\text{Hom}_A(T, -)$ as in (6.2).
Proof. There are enough projective objects in the tilting heart \( B \), and the full subcategory \( B_{\text{proj}} \) of projective objects in \( B \) is equivalent to the category \( \text{Add}(T) \subset A \). So the abelian category \( B \) is nothing but the category of modules over the monad \( T : X \mapsto \text{Hom}_A(T, T(X)) \) described in the proof of Proposition 8.1. The second assertion follows from \( \S 6.2 \). \( \square \)

In other words, we have shown that \( B \) is a locally presentable category whenever the category \( A \) is locally presentable. We do not know whether the converse assertion is true.

**Proposition 8.3.** Suppose that \( A \) is a Grothendieck abelian category. If \( T \in A \) is an \( n \)-tilting object and \( B \) is the heart of the tilting \( t \)-structure, then for every family of projective objects \( P_\alpha \in B \) the canonical morphism \( \prod_\alpha P_\alpha \rightarrow \bigoplus_\alpha P_\alpha \) is a monomorphism in \( B \).

**Proof.** Consider the category \( E = A \cap B \) studied in Section 4. Note that the map \( i : \prod_\alpha P_\alpha \rightarrow \bigoplus_\alpha P_\alpha \) is a filtered colimit of split inclusions (see [38, Corollary III.1.3]), hence a monomorphism in \( A \). Since \( P_\alpha \in B_{\text{proj}} = \text{Add}(T) \) for each \( \alpha \), both ends of \( i \) actually belong to \( E \) by Lemma 4.3(a) and hence we have a short exact sequence
\[
0 \rightarrow \prod_\alpha P_\alpha \rightarrow \bigoplus_\alpha P_\alpha \rightarrow \prod_\alpha P_\alpha / \bigoplus_\alpha P_\alpha \rightarrow 0
\]
in \( E \) by Lemma 2.1(a). Thus, the same exact sequence must exist in \( B \). The conclusion follows from Remark 4.4 as the coproduct and the product displayed is actually also the coproduct and product in \( B \), respectively. \( \square \)

**Corollary 8.4.** Under the assumptions of Proposition 8.3, \( B \) is the category of models of an additive \( \kappa \)-ary algebraic theory for some \( \kappa \) having the additional property that for every family of projective objects \( P_\alpha \in B \) the natural morphism \( \prod_\alpha P_\alpha \rightarrow \bigoplus_\alpha P_\alpha \) is a monomorphism in \( B \).

**Proof.** This immediately follows from Proposition 8.3 and Corollary 8.2. \( \square \)

The situation where the tilting object in \( A \), or equivalently the projective generator of \( B \), is compact in \( D(A) \simeq D(B) \), is covered by [62].

**Theorem 8.5.** Let \( A \) be an abelian category with set-indexed products, an injective cogenerator, and an \( n \)-tilting object. Let \( B \) be the heart of the corresponding tilting \( t \)-structure.

If \( B = B-\text{mod} \) is the category of modules over an associative ring \( B \), then \( A \) is a Grothendieck abelian category.

Conversely, if \( A \) is a Grothendieck abelian category and the tilting object \( T \in A \) is a compact object of the derived category \( D(A) \), then \( B \simeq B-\text{mod} \) is the category of modules over the ring \( B = \text{Hom}_A(T, T)^{\text{op}} \).

**Proof.** These are the main results of the paper [62]. The first assertion is [62, Theorem 6.2]; the second one follows from the discussion in [62, Section 1]. \( \square \)
9. Tilting Hearts which are Categories of Contramodules

Our aim in the final section is to find conditions on an abelian category $A$ with a tilting object $T$ under which the tilted category is the category of contramodules over a topological ring. This is a much more concrete situation than tilting to modules over an additive monad (cf. Example 6.3). It turns out that we do obtain a topological ring in several natural situations.

9.1. **Locally weakly finitely generated abelian categories.** Let $C$ be a cocomplete abelian category and $\lambda$ be a regular cardinal. Let us call an object $C \in C$ weakly $\lambda$-generated if every morphism $C \to \coprod_{x \in X} D_x$ from $C$ into the coproduct of a family of objects $D_x$, $x \in X$ in $C$ factorizes through the natural embedding $\coprod_{z \in Z} D_z \to \coprod_{x \in X} D_x$ of the coproduct of a subfamily indexed by a subset $Z \subset X$ of the cardinality smaller than $\lambda$. Any quotient object of a weakly $\lambda$-generated object in $C$ is weakly $\lambda$-generated. The class of all weakly $\lambda$-generated objects in $C$ is also closed under extensions and $\lambda$-small coproducts. Any $\lambda$-generated object in the sense of [H, Definition 1.67] is weakly $\lambda$-generated.

A cocomplete abelian category $C$ is called locally weakly $\lambda$-generated if it is generated by its weakly $\lambda$-generated objects, that is, for any object $M \in C$, the minimal subobject of $M$ containing the images of all morphisms into $M$ from weakly $\lambda$-generated objects of $C$ coincides with $M$. Since the class of all weakly $\lambda$-generated objects of $C$ is closed under quotients (and also $\lambda$-small coproducts), this condition simply means that every object in $C$ is the ($\lambda$-directed) union of its weakly $\lambda$-generated subobjects. (One could add the condition that $C$ has a set of generators to this definition; and then it would follow that $C$ has a set of weakly $\lambda$-generated generators; cf. Remark 4.3 below; but we will not need this.) A weakly $\omega$-generated object (where $\omega$ denotes the countable cardinal) is called weakly finitely generated. A locally weakly $\omega$-generated category is called locally weakly finitely generated.

Our next aim is to show that locally weakly finitely generated abelian categories satisfy Grothendieck’s axiom Ab5, i. e., have exact functors of filtered colimits.

**Proposition 9.1.** Let $(E_i)_{i \in I}$ be a direct system of subobjects of $M \in C$, where $C$ is a locally weakly finitely generated category. Then the colimit $\text{colim}_{i \in I} E_i$ is again a subobject of $M$, i. e., the colimit map $\text{colim}_{i \in I} E_i \to M$ is monic.

**Proof.** Put $N = \text{colim}_{i \in I} E_i$ and consider the cocone $(f_{E_i}: E_i \to N)$. We denote by $f: \coprod_{i \in I} E_i \to N$ and $p: \coprod_{i \in I} E_i \to M$ the canonical morphisms. We have to show that the epimorphism $f$ annihilates the kernel of the morphism $p$. Since $p$ factors through $f$ by the universal property of the colimit, this will mean that the kernels of $p$ and $f$ are equal and the induced map $N \to M$ is injective, as desired.

Let $b: B \to \coprod_{i \in I} E_i$ be a morphism from a weakly finitely generated object $B$ with the image lying in the kernel of $p$. It suffices to show that $fb = 0$ for every such $b$. The morphism $b$ factorizes through the coproduct of a finite subset of objects $\coprod_{j=1}^m E_j \subset \coprod_{i \in I} E_i$. Denote by $b'$ the related morphism $B \to \coprod_{j=1}^m E_j$. Choose $k \in I$ such that $E_j \subset E_k$ for all $1 \leq j \leq m$, and denote by $q: \coprod_{j=1}^m E_j \to E_k$
the natural morphism. Then \( q b' = 0 \), because \( p b = 0 \) and \( E_k \) is a subobject in \( M \). Let \( g: \coprod_{j=1}^m E_j \to N \) denote the morphism with the components \( f_{E_j}: E_j \to N \). Then \( g = f_{E_k} q \), since the system of morphisms \((f_{E_i}: E_i \to N)_{i \in I}\) is compatible. Thus \( f b = g b' = f_{E_k} q b' = 0 \). □

Corollary 9.2. Let \( C \) be a locally weakly finitely generated abelian category. Then

(a) the functors of filtered colimit are exact in \( C \);

(b) if \( C \) is complete, then for any family of objects \( C_x \in C \) the natural morphism \( \coprod_x C_x \to \prod_x C_x \) is a monomorphism.

Proof. According to [38, Proposition III.1.2 and Theorem III.1.9], part (a) is an equivalent reformulation of Proposition 9.1. Part (b) is provided by [38, Corollary III.1.3]. □

Remark 9.3. Suppose that \( C \) is a locally weakly finitely generated category with a generator \( G \) and let \( R \) be the full subcategory of \( C \) formed by all the quotient objects of finite coproducts of weakly finitely generated subobjects of \( G \). Then \( R \) is essentially small by [61, Proposition IV.6.6] and the restricted Yoneda functor \( h_R: C \to R^{op}\mod \),

\[ X \mapsto \text{Hom}_C(-, X)|_R, \]

has a left adjoint \( \Delta: R\mod \to C \) by [1 Proposition 1.27]. Here, \( R^{op}\mod \) stands for the category of \( R^{op}\)-modules, i.e., of additive functors \( R^{op} \to \mathbb{Z}\mod \).

It turns out that the counit of adjunction \( \varepsilon: \Delta \circ h_R \to 1_C \) is a natural equivalence and hence \( h_R \) is fully faithful. Indeed, for any object \( M \in C \), the morphism \( \varepsilon_M \) admits an explicit description. If \( D_M: R/M \to C \) is the canonical diagram of \( M \) (see [1 Definition 0.4]), then \( \varepsilon_M \) is simply the colimit morphism \( \text{colim}_{(g, E \to M) \in R/M} E \to M \).

Since any morphism \( E \to M \) with \( E \in R \) factors through its image \( F \in R \) as \( E \to F \subseteq M \), the direct system of all subobjects of \( M \) belonging to \( R \) is cofinal in \( D_M \). As \( M \) is the union of its subobjects belonging to \( R \), the morphism \( \varepsilon_M \) is an isomorphism by Proposition 9.1, as required.

It follows that \( C \) identifies with a coproduct-closed full reflective subcategory of \( R^{op}\mod \). Thus, we can (with obvious modifications for modules over small preadditive categories rather than rings) apply results from Section 7. Moreover, \( C \) is automatically complete.

Examples 9.4. Any locally finitely presentable (Grothendieck) abelian category is locally weakly finitely generated. In particular, any locally Noetherian or locally coherent Grothendieck category is locally weakly finitely generated.

On the other hand, a Grothendieck abelian category in general does not need to be locally weakly finitely generated. Here is a counterexample: let \( \mathbb{Q} \) be the set of all rational numbers, viewed as a topological space with the topology induced from its embedding into the real line. Let \( k \) be a (discrete) field. Then the category of sheaves of \( k \)-vector spaces over \( \mathbb{Q} \) is not locally weakly finitely generated.

Indeed, let us show that the constant sheaf \( k_{\mathbb{Q}} \) with the stalk \( k \) over \( \mathbb{Q} \) has no nonzero weakly finitely generated subobjects. For any nonzero subsheaf \( F \subset k_{\mathbb{Q}} \),
there exists a nonempty open subset $U \subset \mathbb{Q}$ such that $F$ contains the constant section $\frac{1}{U}$ of $k_{\mathbb{Q}}$ over $U$. Let $D \subset U$ be an infinite discrete subset that is closed in $\mathbb{Q}$ (e. g., a sequence of elements of $U$ converging to an irrational number). Then the composition $F \to k_D \to k_D$ is a sheaf epimorphism (where the constant sheaf $k_D$ on $D$ is viewed as a sheaf over $\mathbb{Q}$ using the extension by zero). It remains to observe that the sheaf $k_D$ is the coproduct of an infinite collection of nonzero objects (namely, the skyscraper sheaves at the points of $D$).

Let $C$ be a locally weakly finitely generated abelian category and $M \in C$ be a fixed object. We will endow the ring $\mathcal{R} = \text{Hom}_C(M, M)^{op}$ with the topology in which a base of neighborhoods of zero is formed by the annihilator ideals $\text{Ann}(E) \subset \mathcal{R}$ of weakly finitely generated subobjects $E \subset M$. These are left ideals in $\text{Hom}_C(M, M)$ and right ideals in $\mathcal{R}$.

**Lemma 9.5.** Assume that $M$ is the direct union of a set of its weakly finitely generated subobjects. Then $\mathcal{R} = \text{Hom}_A(M, M)^{op}$ is a complete and separated topological ring.

**Proof.** This is completely analogous to Theorem [7.1]. In order to show that the multiplication in a topological ring $\mathcal{R}$ is continuous with respect to a topology with a base of neighborhoods of zero formed by some right ideals, it suffices to check that for every open right ideal $\mathfrak{J} \subset \mathcal{R}$ and every element $r \in \mathcal{R}$ there exists an open right ideal $\mathfrak{J}' \subset \mathcal{R}$ such that $r\mathfrak{J}' \subset \mathfrak{J}$. In the situation at hand, let $\mathfrak{J} = \text{Ann}(E)$ be the annihilator of a weakly finitely generated subobject $E \subset M$ and $r: M \to M$ be a morphism in the category $C$. Let $F = rE \subset M$ be the image of the composition $E \to M \to M$. Then $F$ is weakly finitely generated as a quotient object of $E$. Put $\mathfrak{J} = \text{Ann}(F)$; then $\mathfrak{J} \subset \mathfrak{J}$ in $\mathcal{R}$.

The topology is separated, that is $\bigcap_E \text{Ann}(E) = 0$ in $\mathcal{R}$, because $M$ is the union of its weakly finitely generated subobjects $E$. Let us show that the topology is complete, that is the map $\mathcal{R} \to \varprojlim_E \mathcal{R} / \text{Ann}(E)$ is surjective. The group $\text{Ann}(E)$ is nothing but the group of morphisms $\text{Hom}_C(M/E, M)$. More generally, we will show that for any pair of objects $M$ and $N \in C$ the natural map

$$\text{Hom}_C(M, N) \to \varprojlim_E \text{Hom}_C(M, N) / \text{Hom}_C(M/E, N),$$

where the projective limit is taken over all the weakly finitely generated subobjects $E \subset M$, is an isomorphism. We have already explained the injectivity. Regarding the surjectivity, the exact sequence

$$0 \to \text{Hom}_C(M/E, N) \to \text{Hom}_C(M, N) \to \text{Hom}_C(E, N)$$

shows that the quotient group $\text{Hom}_C(M, N) / \text{Hom}_C(M/E, N)$ is isomorphic to the subgroup in $\text{Hom}_C(E, N)$ consisting of all the morphisms $E \to N$ that can be extended to a morphism $M \to N$.

Let $\mathcal{F}'$ be a set of weakly finitely generated subobjects in $M$ such that no proper subobject in $M$ contains all $F \in \mathcal{F}'$. There is only a set of subsets in the set $\text{Hom}_A(M, N)$ (recall that we assume all our abelian categories to have $\text{Hom}$ sets).
So we can form a set $\mathcal{F}''$ of weakly finitely generated subobjects in $M$ such that for each weakly finitely generated subobject $E \subset M$ there exists $F'' \in \mathcal{F}''$ for which the two subgroups $\text{Hom}_{\mathcal{A}}(M/E, N)$ and $\text{Hom}_{\mathcal{A}}(M/F'', N)$ in $\text{Hom}_{\mathcal{A}}(M, N)$ coincide. Let $\mathcal{F}$ denote the closure of $\mathcal{F} \cup \mathcal{F}''$ with respect to the operation of the passage to a finite sum of subobjects in $M$. Then $\mathcal{F}$ is still a set of weakly finitely generated subobjects in $M$, and $M$ is the direct union of all $F \in \mathcal{F}$.

Now any element of the projective limit in question specifies a compatible system of morphisms $f_E: E \to N$ defined for all weakly finitely generated subobjects $E$ in $M$. Since $M = \text{colim}_{F \in \mathcal{F}} F$ by Proposition 9.1, such a system of morphisms can be extended to a morphism $h: M \to N$ such that $h|_F = f_F$ for all $F \in \mathcal{F}$. For any weakly finitely generated subobject $E$ in $M$, consider a related subobject $F'' \in \mathcal{F}$ as above. Then $E + F''$ is also a weakly finitely generated subobject in $M$, and the three subgroups $\text{Hom}_{\mathcal{A}}(M/E, N)$, $\text{Hom}_{\mathcal{A}}(M/(E + F''), N)$, and $\text{Hom}_{\mathcal{A}}(M/F'', N)$ in $\text{Hom}_{\mathcal{A}}(M, N)$ coincide. Hence it follows from the compatibility of the morphisms $f_E$, $f_{F''}$, and $f_{E + F''}$ with respect to the restriction of morphisms to subobjects that the equality $h|_{F''} = f_{F''}$ implies $h|_{E + F''} = f_{E + F''}$ and $h|_E = f_E$.

**Remark 9.6.** From this point on, we will tacitly assume our locally weakly finitely generated categories to satisfy the assumption of Lemma 9.5 for all of their objects, that is, each $M \in \mathcal{C}$ is the direct union of a set of its weakly finitely generated subobjects. This will make it possible to describe morphisms $M \to N$ in $\mathcal{C}$ in terms of their restrictions to weakly finitely generated subobjects $E \subset M$, as in the argument above.

**Theorem 9.7.** Let $\mathcal{C}$ be a locally weakly finitely generated abelian category, $M \in \mathcal{C}$ be an object, and $\text{Add}(M) \subset \mathcal{C}$ be the full additive subcategory formed by the direct summands of infinite coproducts of copies of $M$. Then the category $\text{Add}(M)$ is equivalent to the category $\mathcal{S} - \text{contra}_{\text{proj}}$ of projective left contramodules over the topological ring $\mathcal{R} = \text{Hom}_{\mathcal{C}}(M, M)^{\text{op}}$.

Moreover, let $Y$ be a set of the cardinality greater or equal to the cardinality of some base of neighborhoods of zero in $\mathcal{R}$. Put $L = M^{(Y)}$ and consider the topological ring $\mathcal{S} = \text{Hom}_{\mathcal{C}}(L, L)^{\text{op}}$. Then the category $\text{Add}(M)$ is equivalent to the additive category $\mathcal{S} - \text{contra}_{\text{proj}}$ and the forgetful functor between the abelian categories $\mathcal{S} - \text{contra} \to \mathcal{S} - \text{mod}$ is fully faithful.

**Proof.** We have to show that the monad $\mathcal{T}: X \mapsto \text{Hom}_{\mathcal{C}}(M, M^{(X)})$ on the category of sets is isomorphic to the monad $\mathcal{T}_{\mathcal{R}}$. The natural morphisms $M^{(X)} \to M^{X}$ are monomorphisms in $\mathcal{C}$ by Corollary 9.2(b), so the induced maps of sets $\mathcal{T}(X) \to \prod_{x \in X} \mathcal{T}(\{x\})$ are injective. Let us describe the image of this map.

If a morphism $M \to M^{X}$ factorizes through $M^{(X)}$, then for every weakly finitely generated subobject $E \subset M$ the composition $E \to M^{X}$ factorizes through the natural split monomorphism $M^{Z} \to M^{X}$ for some finite subset $Z \subset X$. Conversely, let $M \to M^{X}$ be a morphism having this factorization property with respect to all the weakly finitely generated subobjects $E \subset M$. Then the composition of morphisms $E \to M \to M^{X}$ factorizes through the monomorphism $M^{(X)} \to M^{X}$. Let $\mathcal{E}$ be a
set of weakly finitely generated subobjects of $M$ such that $M$ is the direct union of $E \in \mathcal{E}$. Then $\prod_{E \in \mathcal{E}} E \to M$ is an epimorphism, and it follows that the morphism $M \to M^X$ also factorizes through $M^{(X)}$.

We have shown that $\mathbb{T}(X)$ as a subset in $\prod_{x \in X} \mathbb{T}(\{x\}) = \mathfrak{R}^X$ consists precisely of all the $X$-indexed families of elements in $\mathfrak{R}$ that converge to zero in the topology of $\mathfrak{R}$. According to the discussion in [54, Section 1.2], it remains to check that the “summation map” $\Sigma_X : \mathbb{T}(X) \to \mathfrak{R}$ induced by the natural morphism $M^{(X)} \to M$ is nothing but the map of summation of converging to zero $X$-indexed families of elements in the topology of $\mathfrak{R}$ (in the sense of the topological limit of finite partial sums). This is easily demonstrated by restricting a morphism $M \to M^X$ in question to weakly finitely generated subobjects $E \subseteq M$.

The last assertion follows from Theorem 6.6 and is provable in the same way as Proposition 7.10.

□

9.2. Gorenstein locally Noetherian Grothendieck categories. A wide class of categories where the latter theorem provides us with tilting equivalences are the Gorenstein locally Noetherian Grothendieck categories. Here, we call a locally Noetherian Grothendieck category $A$ Gorenstein if

\((g1)\) all injective objects in $A$ have finite projective dimension;
\((g2)\) $A$ has a generator of finite injective dimension.

Note that for any locally Noetherian Grothendieck category $A$ there exist an injective object $J \in A$ such that the full additive subcategory $\text{Add}(J) \subseteq A$ coincides with the full subcategory of injective objects $A_{\text{inj}} \subseteq A$. Equivalently, this means that $J$ contains every indecomposable injective in $A$ as a direct summand. Clearly, condition \((g1)\) is equivalent to requiring that $J$ has finite projective dimension.

**Example 9.8.** If $A$ is a two-sided Noetherian ring, then the category $A = A \text{-mod}$ is Gorenstein if and only if $A$ is an Iwanaga-Gorenstein ring [20, Example 2.3].

**Remark 9.9.** One can also prove (by a variation of the argument for [20, Lemma 2.6]) that a locally Noetherian Grothendieck category is Gorenstein if and only if it is Gorenstein in the sense of [23, Definition 2.18], introduced by Enochs, Estrada and García-Rozas.

Given any locally Noetherian Grothendieck category $A$, the additive category $A_{\text{inj}}$ is according to Theorem 9.7 or [17, Theorem 3.6] equivalent to the full subcategory of projective objects $B_{\text{proj}} \subseteq B$ in the abelian category $B = \mathfrak{R} \text{-contra}$ of left contramodules over the topological ring $\mathfrak{R} = \text{Hom}_{\mathcal{A}}(J, J)^{op}$. This equivalence assigns the free left $\mathfrak{R}$-contramodule with one generator $\mathfrak{R} \in \mathfrak{R} \text{-contra}_{\text{proj}}$ to our chosen injective object $J \in A_{\text{inj}}$. There are enough injective objects in $A$ and projective objects in $B$.

Moreover, both the full subcategories $A_{\text{inj}} \subseteq A$ and $B_{\text{proj}} \subseteq B$ are closed under both the infinite products and coproducts in the abelian categories $A$ and $B$ [17, Theorem 3.6]. According to [16, Proposition A.3.1(b)] and the assertion dual to it, we have triangulated equivalences

$$D^{\text{er}}(A) \simeq \text{Hot}(A_{\text{inj}}) \simeq \text{Hot}(B_{\text{proj}}) \simeq D^{\text{er}}(B).$$
In this sense, one can say that the object \( T = W = J \) is an “\( \infty \)-tilting object” in \( A \) and the object \( W = T = R \) is the related “\( \infty \)-cotilting object” in \( B \) (cf. Examples 5.2 and 5.5 above).

The following theorem characterizes the situation where \( T \in A \) is actually a tilting object of finite projective dimension. In view of Example 9.8, it can be viewed as a generalization of results in [41, Section 3], and part (3) generalizes the characterization of Gorenstein coalgebras from Example 5.2.

**Theorem 9.10.** Let \( A \) be a locally Noetherian Grothendieck category and \( J \in A \) be such that \( \text{Add}(J) = A_{\text{inj}} \). Then the following are equivalent:

1. \( A \) is Gorenstein;
2. \( J \) is a tilting object of \( A \);
3. \( J \) has finite projective dimension in \( A \) and the topological ring \( R = \text{Hom}_A(J, J)^{\text{op}} \) has finite injective dimension in \( B = R^\text{-contra} \).

**Proof.** (1) \( \iff \) (2) follows directly from the equivalence between (1) and (3) in Theorem 2.4 applied to \( T = J \in A \).

(2) \( \implies \) (3): If \( T \) is tilting, \( R \) is cotilting according to Corollary 3.12. By the very definition, \( J \) has finite projective dimension and \( R \) finite injective dimension.

(3) \( \implies \) (2): The additive embedding functor \( A_{\text{inj}} \simeq B_{\text{proj}} \to B \) can be uniquely extended to a left exact functor \( \Psi : A \to B \), which can be computed as taking an object \( N \in A \) to the left \( R \)-contramodule \( \text{Hom}_A(J, N) \) (see §6.2). When the projective dimension of the object \( J \in A \) is finite, \( \text{Hom}_A(J, J) \) has finite injective dimension in \( B = R^\text{-contra} \).

Similarly, the additive embedding functor \( B_{\text{proj}} \simeq A_{\text{inj}} \to A \) can be uniquely extended to a right exact functor \( \Phi : B \to A \). The functor taking every left \( R \)-contramodule \( C \) to the abelian group \( \text{Hom}_A(\Phi(C), J) \) takes, in particular, the free \( R \)-contramodule \( R[[X]] \) to the abelian group \( \text{Hom}_A(J^{(X)}, J) = R^X \), so this is nothing but the functor of homomorphisms \( \text{Hom}_R(\cdot, R) \) in the category of left \( R \)-contramodules. Both the functors \( \text{Hom}_A(\Phi(\cdot), J) \) and \( \text{Hom}_R(\cdot, R) \) are left exact, so they are isomorphic as functors on the whole abelian category \( B = R^\text{-contra} \).

It follows that when the injective dimension of the object \( R \in B \) is finite, \( \text{Hom}_R(\cdot, R) \) is left exact on the whole abelian category \( B = R^\text{-contra} \). By Proposition 1.5 and Corollary 3.4(b), it follows that \( T = J \) is an \( n \)-tilting object in \( A \) and \( W = R \) is an \( n \)-cotilting object in \( B \). (Cf. the discussion of the injective tilting module over a Noetherian Gorenstein ring in [4] and [27, Example 13.8 and Theorem 17.12].)

9.3. **Closed functors.** The result of Theorem 9.7 has a further generalization that we will now discuss. Let \( A \) be an idempotent-complete additive category, \( C \) be an...
abelian category, and \( F: A \rightarrow C \) be an additive functor. We will say that \( F \) is a \textit{closed} functor if the following conditions are satisfied:

(I) set-indexed coproducts exist in the categories \( A \) and \( C \), and the functor \( F \) preserves coproducts;

(II) the abelian category \( C \) is locally weakly finitely generated;

(III) the functor \( F \) is faithful;

(IV) for any two objects \( K \) and \( L \in A \) and any morphism \( g: F(K) \rightarrow F(L) \) in \( C \) such that for every weakly finitely generated subobject \( E \subset F(K) \) there exists a morphism \( h: K \rightarrow L \) in \( A \) for which the morphisms \( g \) and \( F(h) \) coincide in the restriction to \( E \), there exists a morphism \( f: K \rightarrow L \) in \( A \) such that \( g = F(f) \).

Notice that when the functor \( F \) is fully faithful, the complicated condition (IV) automatically holds.

**Lemma 9.11.** Let \( A \) be an additive category endowed with a closed additive functor \( F: A \rightarrow C \). Then for any object \( N \in A \) the ring \( \Omega = \text{Hom}_A(N, N)^{op} \) is a closed subring of the topological ring \( \mathfrak{R} = \text{Hom}_C(F(N), F(N))^{op} \).

**Proof.** Let \( g: F(N) \rightarrow F(N) \) be a morphism in \( C \) not belonging to the image of the map \( F: \text{Hom}_A(N, N) \rightarrow \text{Hom}_C(F(N), F(N)) \), that is \( g \in \mathfrak{R} \setminus \Omega \). Then, according to the condition (IV), there exists a weakly finitely generated subobject \( E \subset F(N) \) such that in the category \( A \) there are no morphisms \( N \rightarrow N \) whose image under \( F \) coincides with \( g \) in the restriction to \( E \). Now the set \( \mathfrak{V} \subset \mathfrak{R} \) of all morphisms \( F(N) \rightarrow F(N) \) in \( C \) coinciding with \( g \) in the restriction to \( E \) is an open neighborhood of the element \( g \in \mathfrak{R} \) that does not intersect the subring \( \Omega \subset \mathfrak{R} \). \( \square \)

We will endow the ring \( \Omega = \text{Hom}_A(N, N)^{op} \) with the induced topology of a subring in \( \mathfrak{R} = \text{Hom}_C(F(N), F(N))^{op} \). This makes \( \Omega \) a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals.

**Theorem 9.12.** Let \( A \) be an idempotent-complete additive category endowed with a closed additive functor \( F: A \rightarrow C \) into a locally weakly finitely generated abelian category \( C \). Let \( N \in A \) be an object and \( \text{Add}(N) \subset A \) be the full additive subcategory formed by the direct summands of infinite coproducts of copies of \( N \) in \( A \). Then the category \( \text{Add}(N) \) is equivalent to the category \( \Omega-\text{contra}_{\text{proj}} \) of projective left contramodules over the topological ring \( \Omega = \text{Hom}_A(N, N)^{op} \).

Moreover, let \( Y \) be a set of the cardinality greater or equal to the cardinality of some base of neighborhoods of zero in \( \Omega \). Put \( L = N^{(Y)} \) and consider the topological ring \( \mathcal{G} = \text{Hom}_A(L, L)^{op} \). Then the category \( \text{Add}(N) \) is equivalent to the additive category \( \mathcal{G}-\text{contra}_{\text{proj}} \) and the forgetful functor between the abelian categories \( \mathcal{G}-\text{contra} \rightarrow \mathcal{G}-\text{mod} \) is fully faithful.

**Proof.** We have to check that the monad \( T: X \mapsto \text{Hom}_A(N, N^{(X)}) \) on the category of sets is isomorphic to the monad \( T_{\mathcal{D}} \). For any set \( X \), the map

\[
(25) \quad \text{Hom}_A(N, N^{(X)}) \rightarrow \text{Hom}_A(N, N)^X
\]

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is injective, because $F(N^X) = F(N)^X$, the functor $F$ is faithful, and the map
\( \text{Hom}(F(N), F(N)^X) \to \text{Hom}(F(N), F(N))^X \) is injective. In view of the arguments in the proof of Theorem 9.7, the question reduces to showing that the image of the map (25) consists precisely of all the $X$-indexed families of elements of the ring $\mathfrak{R}$ converging to zero in the topology of $\Omega$.

Indeed, for any morphism $N \to N^X$, the related $X$-indexed family of elements in $\text{Hom}_A(N, N)$ converges to zero in the topology of $\Omega$ since, viewed as a family of elements of the ring $\mathfrak{R}$, it comes from a certain morphism $F(N) \to F(N)^X$, and therefore converges to zero in the topology of $\mathfrak{R}$.

To prove the converse implication, we have to check that every morphism $g: F(N) \to F(N)^X$ in $\mathfrak{C}$ whose compositions with the projection morphisms $F(N)^X \to F(N)$ are images of some morphisms $h_x: N \to N$ under the functor $F$, is itself the image of a certain morphism $f: N \to N^X$ under the functor $F$. Here we need to use the condition (IV) again.

Let $E \subset F(N)$ be a weakly finitely generated subobject. Then the restriction of the morphism $g$ to $E$ factorizes through the natural embedding $F(N)^{(Z)} \to F(N)^X$ corresponding to some finite subset $Z \subset X$. The related morphism $E \to F(N)^{(Z)}$ is the restriction of the morphism $f(h_Z)$ to $E$, where $h_Z: N \to N^{(Z)} = N(Z)$ is the morphism with the components $h_z$, $z \in Z$. Composing the morphism $h_Z$ with the embedding $N^{(Z)} \to N^X$, we obtain a morphism $h: N \to N^X$ in the category $A$ whose image under the functor $F$ coincides with the morphism $g$ in the restriction to the subobject $E \subset F(N)$.

This proves the first assertion of the theorem; and the remaining assertions are provable as above.

\[\square\]

**Corollary 9.13.** Let $A$ be an abelian category such that either

(a) $A$ is locally weakly finitely generated, or

(b) $A$ is endowed with a closed additive functor $F: A \to \mathfrak{C}$ to a locally weakly finitely generated abelian category $\mathfrak{C}$.

Suppose further that $A$ has infinite products and an injective cogenerator. Then, for any $n$-tilting object $T \in A$, the abelian category $\mathfrak{B}$ in the heart of the tilting $t$-structure on $D^b(A)$ associated with $T$ is equivalent to the abelian category of left contramodules $\mathfrak{B-contr}$ over the topological ring $\mathfrak{R} = \text{Hom}_A(T, T)^{op}$. The left exact functor $\Psi: A \to \mathfrak{B}$ can be computed as the functor $\text{Hom}_A(T, -): A \to \mathfrak{R-contr}$.

Moreover, let $Y$ be a set of the cardinality greater or equal to the cardinality of some base of neighborhoods of zero in $\mathfrak{R}$. Put $L = T(Y)$ and consider the topological ring $\mathfrak{G} = \text{Hom}_A(L, L)^{op}$. Then the abelian category $\mathfrak{B}$ is equivalent to $\mathfrak{G-contr}$ and the forgetful functor $\mathfrak{G-contr} \to \mathfrak{G-mod}$ is fully faithful.

\[\square\]

**Proof.** Part (a) follows from Theorem 9.7 and part (b) (which is more general) from Theorem 9.12. The assertion about the functor $\Psi$ was explained in \{6.2\}.

\[\square\]

**Example 9.14.** Let $A$ be an associative ring and $\mathfrak{C}$ be a right $A$-module. Denote by $\mathfrak{C-ncmod}$ the category of noncoassociative left $\mathfrak{C}$-comodules, that is left $A$-modules $M$ endowed with an abelian group homomorphism $M \to \mathfrak{C} \otimes_A M$. A morphism $M$ →
$N$ in the category $\mathcal{C}-\text{ncomod}$ is a morphism of left $A$-modules such that the square diagram $M \longrightarrow \mathcal{C} \otimes_A M \longrightarrow \mathcal{C} \otimes_A N$, $M \longrightarrow N \longrightarrow \mathcal{C} \otimes_A N$ is commutative. Then $\mathcal{C}-\text{ncomod}$ is a cocomplete, idempotent-complete additive category and the forgetful functor $\mathcal{C}-\text{ncomod} \longrightarrow A\text{-mod}$ is closed (as is the forgetful functor $\mathcal{C}-\text{ncomod} \longrightarrow \mathbb{Z}\text{-mod}$).

To check the condition (IV), it suffices to notice that for every object $\mathcal{K} \in \mathcal{C}-\text{ncomod}$ and every element $k \in \mathcal{K}$ there exists a finite set of elements $k'_1, \ldots, k'_m \in K$ such that the image of $k$ under the coaction map $\mathcal{K} \longrightarrow \mathcal{C} \otimes_A \mathcal{K}$ can be presented in the form of a tensor $\sum_{i=1}^m c_i \otimes k'_i$ with some elements $c_i \in \mathcal{C}$. Let $\mathcal{L} \in \mathcal{C}-\text{ncomod}$ be another object and $g: \mathcal{K} \longrightarrow \mathcal{L}$ be a left $A$-module homomorphism. Suppose that for every element $k \in \mathcal{K}$ and the related elements $k'_1, \ldots, k'_m \in \mathcal{K}$ there exists a morphism $h: \mathcal{K} \longrightarrow \mathcal{L}$ in the category $\mathcal{C}-\text{ncomod}$ such that $g(k) = h(k)$ and $g(k'_i) = h(k'_i)$ for all $1 \leq i \leq m$. Then $g: \mathcal{K} \longrightarrow \mathcal{L}$ is a morphism in the category $\mathcal{C}-\text{ncomod}$.

Therefore, Theorem 9.12 applies, and for every object $\mathcal{N} \in \mathcal{C}-\text{ncomod}$ the full additive subcategory $\text{Add}(\mathcal{N}) \subset \mathcal{C}-\text{ncomod}$ is equivalent to the category $\Omega-\text{contra}_\text{proj}$ of projective comodules over the topological ring $\Omega = \text{Hom}_c(\mathcal{N}, \mathcal{N})^{\text{op}}$ of endomorphism of the object $\mathcal{N}$ in the category $\mathcal{C}-\text{ncomod}$. The equivalence is provided by the functor $\text{Hom}_c(\mathcal{N}, -): \mathcal{C}-\text{ncomod} \longrightarrow \Omega-\text{contra}$ (see [6,2]).

In particular, let $\mathcal{C}$ be a counital, coassociative coring over the ring $A$ (cf. Example 5.3). Then the category $\mathcal{C}-\text{comod} \subset \mathcal{C}-\text{ncomod}$ of (conventional coassociative and counital) left $\mathcal{C}$-comodules is a full additive subcategory closed under coproducts and the images of idempotent endomorphisms in $\mathcal{C}-\text{ncomod}$. Hence for any comodule $\mathcal{N} \in \mathcal{C}-\text{comod}$ the additive categories $\text{Add}(\mathcal{N}) \subset \mathcal{C}-\text{comod}$ and $\text{Add}(\mathcal{N}) \subset \mathcal{C}-\text{ncomod}$ coincide, and the above description of this category applies. In particular, the coring $\mathcal{C}$ can be viewed as a left comodule over itself, and the category $\text{Add}(\mathcal{C}) \subset \mathcal{C}-\text{comod}$ can be described as the category of projective comodules $\mathcal{R}-\text{contra}_\text{proj}$ over the topological ring $\mathcal{R} = \text{Hom}_c(\mathcal{C}, \mathcal{C})^{\text{op}}$. The right action of $A$ in $\mathcal{C}$ defines a ring homomorphism $A \longrightarrow \mathcal{R}$.

Besides the structure of a topological ring, the abelian group $A$-$A$-bimodule $\text{Hom}_c(\mathcal{C}, \mathcal{C}) = \text{Hom}_A(\mathcal{C}, A)$ has a natural structure of left comodule over the coring $\mathcal{C}$ [42, Section 3.1.2]. In fact, the three full additive subcategories formed by

- the cofree left $\mathcal{C}$-comodules $\mathcal{C}^{(\mathcal{X})}$ in the category $\mathcal{C}-\text{comod}$,
- the free left $\mathcal{C}$-comodules $\text{Hom}_c(\mathcal{C}, \mathcal{C}^{(\mathcal{X})}) = \text{Hom}_A(\mathcal{C}, A^{(\mathcal{X})})$ in the category of left $\mathcal{C}$-comodules $\mathcal{C}-\text{comod}$, and
- the free left $\mathcal{R}$-comodules $\mathcal{R}[[\mathcal{X}]]$ in the category $\mathcal{R}-\text{contra}$

are naturally equivalent. The equivalence between the first and the second full subcategories is a well-known simple form of the co-contra correspondence [42, Section 5.1.3], [47, Section 3.4], and an equivalence between the first and the third ones we have just constructed in the previous paragraph.

Calling a $\mathcal{C}$-comodule projective if it is a direct summand of a free one, we can have an equivalence between the three additive categories $\text{Add}(\mathcal{C})$, $\mathcal{C}-\text{contra}_\text{proj}$, and $\mathcal{R}-\text{contra}_\text{proj}$. Moreover, both the equivalences $\text{Add}(\mathcal{C}) \longrightarrow \mathcal{C}-\text{contra}_\text{proj}$ and
Add(\mathcal{C}) \rightarrow \mathcal{R} \text{--contra} \text{proj} are provided by the functor Hom_\mathcal{C}(\mathcal{C}, -) (with one or another additional structure on the Hom group). Hence the equivalence \( \mathcal{C} \text{--contra} \text{proj} \cong \mathcal{R} \text{--contra} \text{proj} \) forms a commutative diagram with the forgetful functors \( \mathcal{C} \text{--contra} \text{proj} \rightarrow A\text{--mod} \) and \( \mathcal{R} \text{--contra} \text{proj} \rightarrow A\text{--mod} \).

The category \( \mathcal{C} \text{--comod} \) of left comodules over a coring \( \mathcal{C} \) is not abelian in general. In fact, \( \mathcal{C} \text{--comod} \) is an abelian category and the forgetful functor \( \mathcal{C} \text{--comod} \rightarrow A\text{--mod} \) is exact if and only if \( \mathcal{C} \) is a flat right \( A \)-module [42, Section 1.1.2]. Similarly, the category \( \mathcal{C} \text{--contra} \) is not abelian in general [46, Example B.1.1]. In fact, \( \mathcal{C} \text{--contra} \) is an abelian category and the forgetful functor \( \mathcal{C} \text{--contra} \rightarrow A\text{--mod} \) is exact if and only if \( \mathcal{C} \) is a projective left \( A \)-module [42, Section 3.1.2], [47, Section 2.5]. (For comparison, let us recall that the categories of discrete modules and contramodules over a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals are always abelian, and the forgetful functors from them to the categories of modules/abelian groups are always exact.)

Assuming that \( \mathcal{C} \) is a projective left \( A \)-module, a left \( \mathcal{C} \)-contramodule is a direct summand of a free left \( \mathcal{C} \)-contramodule if and only if it is a projective object in \( \mathcal{C} \text{--contra} \). Besides, the abelian category of left \( \mathcal{C} \)-contramodules has enough projective objects. Thus the equivalence of the additive categories of projective objects \( \mathcal{C} \text{--contra} \text{proj} \cong \mathcal{R} \text{--contra} \text{proj} \) can be uniquely extended to an equivalence of abelian categories \( \mathcal{C} \text{--contra} \cong \mathcal{R} \text{--contra} \) forming a commutative diagram with the forgetful functors \( \mathcal{C} \text{--contra} \rightarrow A\text{--mod} \) and \( \mathcal{R} \text{--contra} \rightarrow A\text{--mod} \).

Furthermore, in the same assumptions one can prove that the category of discrete right \( \mathcal{R} \)-modules \( \text{discr} \mathcal{R} \) is equivalent to the category of right \( \mathcal{C} \)-comodules \( \text{comod} \mathcal{C} \), with the equivalence forming a commutative diagram with forgetful functors to right \( A \)-modules. Both these abelian categories are also equivalent to the category of colimit-preserving functors \( \mathcal{R} \text{--contra} \rightarrow \text{Z--mod} \) (with the forgetful functor corresponding to the right comodule/discrete module \( \mathcal{C} \)).

When \( \mathcal{C} \) is a projective left \( A \)-module, the objects of the full subcategory \( \text{Add}(\mathcal{C}) \subset \mathcal{C} \text{--comod} \) are called coprojective left \( \mathcal{C} \)-comodules [42, Section 3.2.2 and Lemma 5.2(a)]. In the trivial case of a coalgebra \( \mathcal{C} \) over a field \( k \), these are simply the injective objects of the abelian category \( \mathcal{C} \text{--comod} \) [42, Section 0.2.9], [47, Lemma 3.1].

**Example 9.15.** Let \( k \) be a field and \( \mathcal{C} \) be a \( k \)-vector space. Changing the terminology from the previous example slightly, let us say that a noncoassociative left \( \mathcal{C} \)-comodule \( M \) is a \( k \)-vector space endowed with a \( k \)-linear (coaction) map \( M \rightarrow \mathcal{C} \otimes_k M \). A morphism of noncoassociative left \( \mathcal{C} \)-comodules is a \( k \)-linear map forming a commutative square diagram with the coaction maps. The cotensor product \( N \square_\mathcal{C} M \) of a noncoassociative right \( \mathcal{C} \)-comodule \( N \) and a noncoassociative left \( \mathcal{C} \)-comodule \( M \) is a \( k \)-vector space constructed as the kernel of the difference of the two maps \( N \otimes_k M \rightrightarrows N \otimes_k \mathcal{C} \otimes_k M \) induced by the coaction maps.

Let \( S \) be a fixed noncoassociative right \( \mathcal{C} \)-comodule. A nonsemiassociative left \( S \)-semimodule \( M \) is a noncoassociative left \( \mathcal{C} \)-comodule endowed with a \( k \)-linear map \( S \square_\mathcal{C} M \rightarrow M \). A morphism of nonsemiassociative left \( S \)-semimodules \( M \rightarrow N \) is a morphism of noncoassociative left \( \mathcal{C} \)-comodules such that the square diagram
The category $S \text{-nsimod}$ of nonsemiassociative left $S$-semimodules is a cocomplete, idempotent-complete additive category. We claim that the forgetful functor $S \text{-nsimod} \rightarrow k\text{-mod}$ is closed.

Checking the condition (I) is easy, and the conditions (II–III) are obvious. To prove (IV), notice that for every object $K \in S \text{-nsimod}$ there exists a finite set of elements $v_1, \ldots, v_m \in K$ such that the image of $t$ under the natural embedding $S \Box_e K \rightarrow S \otimes_k K$ can be presented in the form of a tensor $\sum_{i=1}^m s_i \otimes v_i$ with some elements $s_i \in S$.

Let $L \in S \text{-nsimod}$ be another object and $g : K \rightarrow L$ be a $k$-linear map such that for every finite-dimensional $k$-vector subspace $V \subset K$ there exists a morphism $h : K \rightarrow L$ in the category $S \text{-nsimod}$ coinciding with $g$ in the restriction to $V$. It was shown in Example 9.14 that $g$ is a morphism in the category $C \text{-ncomod}$ in this case. Let $t \in S \Box_e K$ be an element, $v_1, \ldots, v_m \in K$ be the related elements as above, and $w \in K$ be the image of $t$ under the semiaction map $S \Box_e K \rightarrow K$. Denote by $V$ the vector subspace spanned by $v_1, \ldots, v_m$, and $w$ in $K$. Then existence, for every $t$, of a morphism $h : K \rightarrow L$ in $S \text{-nsimod}$ coinciding with the $k$-linear map $g$ in the restriction to $V$ implies that $g$ is also a morphism in $S \text{-nsimod}$.

Applying Theorem 9.12, we conclude that for every object $N \in S \text{-nsimod}$ the full additive subcategory $\text{Add}(N) \subset S \text{-nsimod}$ is equivalent to the category $\Omega \text{-contra}_\text{proj}$ of projective contramodules over the topological ring $\Omega = \text{Hom}_S(N, N)^{\text{op}}$ of endomorphisms of the object $N$ in the category $S \text{-nsimod}$. The equivalence is provided by the functor $\text{Hom}_S(N, -) : S \text{-nsimod} \rightarrow \Omega \text{-contra}$.

In particular, let $S$ be a semiunital, semiassociative semialgebra over a counital, coassociative coalgebra $C$ over the field $k$ (cf. Example 5.5). Then the category $S \text{-simod}$ of (conventional semiassociative and semiunital) left $S$-semimodules is a full additive subcategory closed under coproducts and the images of idempotent endomorphisms in $S \text{-nsimod}$. Hence for any semimodule $N \in S \text{-simod}$ the additive category $\text{Add}(N) \subset S \text{-simod}$ can be described as above. In particular, the semialgebra $S$ is naturally a left semimodule over itself, and the category $\text{Add}(S) \subset S \text{-simod}$ is equivalent to the category of projective contramodules $\Omega \text{-contra}_\text{proj}$ over the topological ring $\Omega = \text{Hom}_S(S, S)^{\text{op}}$.

Assume that $S$ is an injective left $C$-comodule. Then the category $S \text{-sicntr}$ of left $S$-semicontramodules is abelian, and the forgetful functor from it to the category of $k$-vector spaces is exact [42, Section 0.3.5]. The same applies to the category $S \text{-simod}$ of right $S$-semimodules [42, Sections 0.3.2]. The objects of the category $\text{Add}(S) \subset S \text{-simod}$ are called semiprojective left $S$-semimodules [42, Sections 3.4.3 and 6.2, and Proposition 6.2.3(a)].

In the above assumption, besides the structure of a topological ring, the $k$-vector space $\text{Hom}_S(S, S) = \text{Hom}_k(C, S)$ has a natural structure of a left semicontramodule over the semialgebra $S$ [42, Section 6.1.3]. In fact, the three full additive subcategories of semiprojective left $S$-semimodules in the additive category $S \text{-simod}$, projective objects in the abelian category $S \text{-sicntr}$, and projective objects in the abelian category
\( \mathcal{R} \)-contra are naturally equivalent. The equivalence between the first and the second full subcategories is a form of the semimodule-semicontramodule correspondence [12, Sections 0.3.7 and 6.2], [47, Proposition 3.5(b)], and an equivalence between the first and the third ones was constructed above.

Moreover, both the equivalences \( \text{Add}(\mathcal{S}) \to \mathcal{S} \)-sicntr\(_{\text{proj}} \) and \( \text{Add}(\mathcal{R}) \to \mathcal{R} \)-contra\(_{\text{proj}} \) are provided by the functor \( \text{Hom}_\mathcal{S}(\mathcal{S}, -) \) (with the respective additional structure on the Hom group). Hence the equivalence \( \mathcal{S} \)-sicntr\(_{\text{proj}} \simeq \mathcal{R} \)-contra\(_{\text{proj}} \) forms a commutative diagram with the forgetful functors \( \mathcal{S} \)-sicntr\(_{\text{proj}} \to k\text{-mod} \) and \( \mathcal{R} \)-contra\(_{\text{proj}} \to k\text{-mod} \). Besides, the abelian category of left \( \mathcal{S} \)-semicontramodules has enough projective objects. Thus the equivalence of the additive categories of projective objects can be uniquely extended to an equivalence of the abelian categories \( \mathcal{S} \)-sicntr \( \simeq \mathcal{R} \)-contra forming a commutative diagram with the forgetful functors \( \mathcal{S} \)-sicntr \( \to k\text{-mod} \) and \( \mathcal{R} \)-contra \( \to k\text{-mod} \).

Finally, in the same assumptions one can prove that the category of discrete right \( \mathcal{R} \)-modules discr-\( \mathcal{R} \) is equivalent to the category of right \( \mathcal{S} \)-semimodules simod-\( \mathcal{S} \), with the equivalence forming a commutative diagram with the forgetful functors to \( k \)-vector spaces. Both these abelian categories are also equivalent to the category of \( k \)-linear colimit-preserving functors \( \mathcal{R} \)-contra \( \to k\text{-mod} \) (with the forgetful functor corresponding to the right semimodule/discrete module \( \mathcal{S} \)).

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Leonid Positselski, Department of Mathematics, Faculty of Natural Sciences, University of Haifa, Mount Carmel, Haifa 31905, Israel; and
Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 119048; and
Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127051, Russia
E-mail address: posic@mccme.ru

Jan Šťovíček, Charles University in Prague, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 75 Praha, Czech Republic
E-mail address: stovicek@karlin.mff.cuni.cz