Pluriclosed and Strominger Kähler–like metrics compatible with abelian complex structures

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Abstract
We show that the existence of a left-invariant pluriclosed Hermitian metric on a unimodular Lie group with a left-invariant abelian complex structure forces the group to be 2-step nilpotent. Moreover, we prove that the pluriclosed flow starting from a left-invariant Hermitian metric on a 2-step nilpotent Lie group preserves the Strominger Kähler–like condition.

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1 | INTRODUCTION

A Hermitian metric $g$ on a complex manifold $(M, J)$ is called pluriclosed (or SKT) if its fundamental form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ satisfies

$$dJd\omega = 0.$$ (1)
The pluriclosed condition (1) can be characterized in terms of the torsion of the Bismut (or Strominger) connection $\nabla^B$. Indeed, in [10] Bismut proved that on a Hermitian manifold $(M,J, g)$ there is a unique Hermitian connection $\nabla^B$ whose torsion $T^B$, once regarded as a $(3,0)$-tensor via $g$, is skew-symmetric. The pluriclosed condition is equivalent to $dT^B = 0$. If $T^B = 0$, the Bismut connection $\nabla^B$ coincides with the Levi–Civita condition and the metric $g$ is Kähler.

By [30] a Hermitian metric $g$ is pluriclosed and satisfies the condition $\nabla^B T^B = 0$ if and only if its Bismut curvature $R^B$ satisfies the first Bianchi identity

$$\sigma_{x,y,z} R^B(x, y, z) = 0$$

and the type condition

$$R^B(x, y, z) = R^B(Jx, Jy, z),$$

for any tangent vectors $x, y, z$ in $M$. Hermitian metrics satisfying (2) and (3) are called in literature Strominger Kähler–like and have been studied recently in [5, 17, 29, 30].

An important tool in the geometry of pluriclosed metrics is the so-called pluriclosed flow, defined by the equation

$$\frac{\partial}{\partial t} \omega(t) = -\left(\rho^B\right)^{1,1}, \quad \omega(0) = \omega_0,$$

where $(\rho^B)^{1,1}$ denotes the $(1,1)$-part of the Ricci form of the Bismut connection and $\omega_0$ is a fixed Hermitian metric. This is a parabolic flow of Hermitian metrics which preserves the pluriclosed condition [26, 27]. A natural question is to see if the Strominger Kähler–like condition is preserved by the flow.

Every conformal class of any Hermitian metric on a compact complex surface admits a pluriclosed metric, but in higher dimensions, the existence of a pluriclosed metric is not automatically guaranteed anymore. Looking at the existence of left-invariant pluriclosed metrics on 6-dimensional nilpotent Lie groups endowed with a left-invariant complex structure, only 4 out of the 34 isomorphism classes admit pluriclosed metrics and they are all 2-step nilpotent, leading to the question whether this is a general feature in arbitrary dimensions [18]. It turns out that 2 of the 4 classes in dimension six admit Strominger Kähler–like metrics [5] and that the complex structure is abelian. More in general, a characterization of 2-step nilpotent Lie algebras admitting Strominger Kähler–like metrics have been obtained in [31], showing in particular that the left-invariant complex structure has to be abelian.

We recall that a left-invariant complex structure on a real Lie group $G$ of real dimension $2n$ is completely determined by a complex structure $J$ on the Lie algebra $\mathfrak{g}$ of $G$, that is, by an endomorphism satisfying $J^2 = -\text{Id}$ and the integrability condition

$$J[x, y] - [Jx, y] - [x, Jy] - J[Jx, Jy] = 0, \quad \forall x, y \in \mathfrak{g}.$$ 

The complex structure $J$ is called abelian if

$$[Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g},$$

or equivalently if the $i$-eigenspace of $J$, denoted with $\mathfrak{g}^{1,0}$, is an abelian subalgebra of $\mathfrak{g}^C := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ (that motivates the terminology introduced in [8]). By [23] a Lie algebra admitting an abelian complex structure has abelian commutator, thus, it is 2-step solvable.
Recent results about the existence of pluriclosed metrics on solvable Lie groups have been obtained in [7, 14, 15, 19, 22].

The purpose of this paper is twofold. On one hand we study the existence of a pluriclosed metric on a unimodular Lie group with an abelian complex structure and on the other hand we investigate the interplay between the Strominger Kähler–like condition and the pluriclosed flow. We recall that a Lie group $G$ is unimodular if and only if $|\det(Ad_g)| = 1$, for every $g \in G$, where $Ad$ is the adjoint representation. For a connected Lie group $G$ this is equivalent to requiring that $tr(ad_X) = 0$, for every $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$.

The existence of other types of Hermitian inner products compatible with abelian complex structures, like for instance Kähler [3], balanced [4] and locally conformally Kähler inner products [4], has been already studied in literature. In [13] the second author and the third author, in collaboration with H. Kasuya, proved that on non-abelian Lie algebras with an abelian complex structure there are no Hermitian-symplectic structures. The latter can be regarded as special pluriclosed inner products and the natural follow-up is focusing on the existence of pluriclosed metrics compatible with abelian complex structures.

Our first result is the following

**Theorem 1.1.** Let $\mathfrak{g}$ be a unimodular Lie algebra with an abelian complex structure $J$. If $(\mathfrak{g}, J)$ admits a pluriclosed inner product, then $\mathfrak{g}$ is $2$-step nilpotent.

In the particular case when the commutator of $\mathfrak{g}$ is totally real the result follows from [19, Corollary 5.7], but our proof does not make use of the argument in [19]. Moreover, Theorem 1.1 generalizes [13, Proposition 6.1].

Next we focus on the existence of Strominger Kähler–like metrics in relation to the pluriclosed flow. By using the characterization in [31] of left-invariant Strominger Kähler–like metrics on 2-step nilpotent Lie groups, we prove the following

**Theorem 1.2.** Let $(G, J, g_0)$ be a 2-step nilpotent Lie group with a left-invariant Strominger Kähler–like Hermitian structure and let $g_t$ be the solution to the pluriclosed flow starting from $g_0$. Then $g_t$ is Strominger Kähler–like for every $t$.

## 2 PROOF OF THEOREM 1.1

We first need the following

**Lemma 2.1.** Let $\mathfrak{g}$ be a Lie algebra with an abelian complex structure $J$ and an Hermitian inner product $g$. Then, the torsion 3-form $T^B$ of the Bismut connection of $(\mathfrak{g}, J, g)$ satisfies

$$T^B(x, y, z) = -g([x, y], z) - g([y, z], x) - g([z, x], y),$$

for every $x, y, z \in \mathfrak{g}$.

**Proof.** Let $\omega$ be the fundamental form of $g$. Let $x, y, z, w \in \mathfrak{g}$, then $T^B(x, y, z) = -d\omega(Jx, Jy, Jz)$ and we directly compute
\[ d\omega(Jx, Jy, Jz) = -\omega([Jx, Jy], Jz) - \omega([Jy, Jz], Jx) - \omega([Jz, Jx], Jy) \]
\[ = -\omega([x, y], Jz) - \omega([y, z], Jx) - \omega([z, x], Jy). \]

Hence the claim follows. \qed

As a consequence we have the following:

**Proposition 2.2.** Let \((\mathfrak{g}, J)\) be a Lie algebra with an abelian complex structure. A Hermitian inner product \(g\) on \((\mathfrak{g}, J)\) is pluriclosed if and only if

\[ g([y, z], [w, x]) - g([x, z], [w, y]) + g([x, y], [w, z]) = 0 \quad (5) \]

for every \(x, y, z, w \in \mathfrak{g}\).

**Proof.** We recall that \(g\) is pluriclosed if and only if \(dT^B = 0\). Let \(x, y, z, w \in \mathfrak{g}\), then, by the previous Lemma,

\[ dT^B(w, x, y, z) = -T^B([w, x], y, z) + T^B([w, y], x, z) - T^B([w, z], x, y) \]
\[ - T^B([x, y], w, z) + T^B([x, z], w, y) - T^B([y, z], w, x) \]
\[ = g([[w, x], y], z) + g([y, z], [w, x]) + g([z, [w, x]], y) \]
\[ - g([[w, y], x], z) - g([x, z], [w, y]) - g([z, [w, y]], x) \]
\[ + g([[w, z], x], y) + g([x, y], [w, z]) + g([y, [w, z]], x) \]
\[ + g([[x, y], w], z) + g([w, z], [x, y]) + g([z, [x, y]], w) \]
\[ - g([[x, z], w], y) - g([w, y], [x, z]) - g([y, [x, z]], w) \]
\[ + g([[y, z], w], x) + g([w, x], [y, z]) + g([x, [y, z]], w) \]
\[ = 2(g([y, z], [w, x]) - g([x, z], [w, y]) + g([x, y], [w, z]). \]

where, in the last equality, we used the Jacobi identity.

Therefore, \(g\) is pluriclosed if and only if

\[ g([y, z], [w, x]) - g([x, z], [w, y]) + g([x, y], [w, z]) = 0, \]

as required. \qed

**Remark 2.3.** Note that from the complex point of view, condition (5) is equivalent to

\[ g([z_1, \bar{z}_2], [z_3, \bar{z}_4]) = g([z_1, \bar{z}_4], [z_3, \bar{z}_2]), \]

for every \(z_1, z_2, z_3, z_4 \in \mathfrak{g}^{1,0}\).
From now on, for a Lie algebra \( \mathfrak{g} \) with an abelian complex structure \( J \) we will denote by \( \zeta \) the center of \( \mathfrak{g} \) and by \( \mathfrak{g}_J^1 \) the ideal
\[
\mathfrak{g}_J^1 = \mathfrak{g}^1 + J\mathfrak{g}^1,
\]
where \( \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}] \). Note that \( \mathfrak{g}_J^1 \) is a \( J \)-invariant Lie subalgebra of \( \mathfrak{g} \).

Under the hypothesis of Proposition 2.2 we obtain the following characterization in terms of the center \( \zeta \) of \( \mathfrak{g} \).

**Corollary 2.4.** Let \( (\mathfrak{g}, J, g) \) be a Lie algebra with an abelian complex structure and a pluriclosed inner product. Then
\[
\| [x, y] \|^2 + \| [x, Jy] \|^2 = g([x, Jx], [y, Jy])
\]
for every \( x, y \in \mathfrak{g} \). In particular, \( x \in \mathfrak{g} \) lies in the center of \( \mathfrak{g} \) if and only if
\[
[x, Jx] = 0,
\]
that is,
\[
\zeta = \{ x \in \mathfrak{g} : [x, Jx] = 0 \}.
\]

**Proof.** By using (5) for \( x, y \in \mathfrak{g} \) we have
\[
\| [x, y] \|^2 + \| [x, Jy] \|^2 = g([x, y], [x, y]) + g([x, Jy], [x, Jy]) = g([Jx, Jy], [x, y]) - g([Jx, y], [x, Jy]) = g([y, Jy], [x, Jx]),
\]
and the claim follows. \( \square \)

We will need the following:

**Lemma 2.5.** Let \( \mathfrak{g} \) be a unimodular Lie algebra with an abelian complex structure \( J \). Then,
\[
\mathfrak{g}_J^1 \neq \mathfrak{g}.
\]

**Proof.** By contradiction, assume that \( \mathfrak{g}_J^1 = \mathfrak{g} \). Then, since by hypothesis \( \mathfrak{g}^1 \) is an abelian ideal in \( \mathfrak{g} \), by [9, Proposition 4.1] \( (\mathfrak{g}/\zeta, J) \) is holomorphically isomorphic to \( \text{aff}(A) \) for some commutative algebra \( A \). Since, \( \mathfrak{g} \) is unimodular, also \( \mathfrak{g}/\zeta \) is unimodular, and so \( \text{aff}(A) \) is unimodular. So, by [4, Lemma 2.6], \( A \) is nilpotent and \( \text{aff}(A) \) is a nilpotent Lie algebra. As a consequence, we have that \( \mathfrak{g}/\zeta \) is also nilpotent implying that \( \mathfrak{g} \) is nilpotent too. But, this is absurd since by [25] for a nilpotent Lie algebra \( \mathfrak{g} \) we have \( \mathfrak{g}_J^1 \neq \mathfrak{g} \). \( \square \)

**Proposition 2.6.** Let \( \mathfrak{g} \) be a Lie algebra with an abelian complex structure \( J \). Assume that \( (\mathfrak{g}, J) \) has a pluriclosed inner product \( g \) and \( \mathfrak{g}_J^1 \) is 2-step nilpotent. Then \( \mathfrak{g} \) is 2-step nilpotent.
Proof. Write
\[ \mathfrak{g} = (\mathfrak{g}_J^1)^{\perp} \oplus \mathfrak{g}_J^1 \]
with respect to the inner product \( g \). Since \( \mathfrak{g}_J^1 \) is nilpotent and has a pluriclosed inner product, its center \( \mathfrak{u} \) is \( J \)-invariant. We write
\[ \mathfrak{g}_J^1 = \mathfrak{u}^{\perp} \oplus \mathfrak{u}. \]

The key observation is that \( \mathfrak{u} \) is contained in the center of \( \mathfrak{g} \). Indeed, if \( x \in \mathfrak{u} \), then in particular we have \([x, Jx] = 0\) and Corollary 2.4 implies that \( x \) belongs to the center of \( \mathfrak{g} \).

Now let \( f \in (\mathfrak{g}_J^1)^{\perp} \). We show that \([f, x] \) lies in the center of \( \mathfrak{g} \), for every \( x \in \mathfrak{g} \).

Set \( D := \text{ad}_f : \mathfrak{g}_J^1 \to \mathfrak{g}_J^1 \). Since \( J \) is abelian, \( \text{ad}_f J = -\text{ad}_f f \) and therefore, \( D \) and \( DJ \) are both derivations. Moreover, we observe that
\[ D[x, y] = 0 \quad \text{for every} \quad x, y \in \mathfrak{g}_J^1; \]
indeed from the 2-step nilpotency of \( \mathfrak{g}_J^1 \), we have that \([x, y] \in \mathfrak{u} \), for every \( x, y \in \mathfrak{g}_J^1 \), and that \( \mathfrak{u} \subset \zeta \).

Let \( x \in \mathfrak{g}_J^1 \) and \( y \in \mathfrak{g} \). We first show that
\[ [Dx, J Dx] = [DJx, Dx]. \]
Since \( D \) is a derivation,
\[ [Dx, J Dx] = D[x, J Dx] - [x, DJ Dx], \]
now, \( x, J Dx \in \mathfrak{g}_J^1 \) because \( \mathfrak{g}_J^1 \) is \( J \)-invariant, and so, by the previous observation, \( D[x, J Dx] = 0 \). Hence, now using that also \( DJ \) is a derivation we get
\[ [Dx, J Dx] = -[x, DJ Dx] = -DJ[x, Dx] + [DJx, Dx]. \]
Similarly, \( x, Dx \in \mathfrak{g}_J^1 \), and by the 2-step nilpotency of \( \mathfrak{g}_J^1 \), \([x, Dx] \in \mathfrak{u} \). Since \( \mathfrak{u} \) is \( J \)-invariant, \( J[x, Dx] \in \mathfrak{u} \subset \zeta \), therefore \( DJ[x, Dx] = 0 \), showing the claim.

Then, taking into account that \( \mathfrak{g} \) is 2-step solvable, Corollary 2.4 yields that
\[ \|[Dx, y]||^2 + \|[Dx, Jy]||^2 = g([Dx, J Dx], [y, Jy]) = g([DJx, Dx], [y, Jy]) = 0, \]
from which we deduce that \([f, x] \) is in the center of \( \mathfrak{g} \) for all \( x \in \mathfrak{g}_J^1 \).

Now let \( f_1, f_2 \in (\mathfrak{g}_J^1)^{\perp} \). By Jacobi identity
\[ [[f_1, f_2], x] = 0 \]
for every \( x \in \mathfrak{g}_J^1 \). Hence \([f_1, f_2] \in \mathfrak{u} \) and so in the center of \( \mathfrak{g} \), as required. \( \square \)

Now we are ready to prove Theorem 1.1.
Proof of Theorem 1.1. We work by induction on the complex dimension $n$ of $\mathfrak{g}$. The base case $n = 1$ is trivial and we assume that the statement holds up to complex dimension $n - 1$. Let $(\mathfrak{g}, J, g)$ be a Lie algebra of complex dimension $n$ with an abelian complex structure and a pluriclosed inner product. In view of Lemma 2.5, $\mathfrak{g}_1$ is a proper Lie subalgebra and inherits an abelian complex structure and a pluriclosed inner product. By induction, assumption $\mathfrak{g}_1$ is 2-step nilpotent. Hence Proposition 2.6 implies that $\mathfrak{g}$ is 2-step nilpotent and the claim follows.

Remark 2.7. By Theorem 1.1, if $\mathfrak{g}$ is a unimodular Lie algebra with an abelian complex structure $J$ and a pluriclosed inner product, then $\mathfrak{g}$ is 2-step nilpotent. In particular, notice that $\mathfrak{g}_1$ is abelian. Indeed, since $J$ is abelian, $\mathfrak{g}$ is 2-step solvable and

$$[\mathfrak{g}, \mathfrak{g}] = [J\mathfrak{g}, J\mathfrak{g}] = 0.$$  

Moreover from the 2-step nilpotency of $\mathfrak{g}$ we infer that also

$$[\mathfrak{g}_1, J\mathfrak{g}_1] = 0.$$  

As a consequence, if $X = \Gamma \backslash G$ is a nilmanifold endowed with an invariant abelian complex structure $J$ and a pluriclosed metric $g$, then by [18, Theorem A], $X$ is a total space of a principal holomorphic torus bundle over a torus.

Notice that from Theorem 1.1 in particular follows that a nilpotent Lie algebra with an abelian complex structure and admitting a pluriclosed inner product is necessarily 2-step. This partially confirms the conjecture that the existence of a pluriclosed inner product on a nilpotent Lie algebra $\mathfrak{g}$ with a complex structure forces $\mathfrak{g}$ to be 2-step.

Moreover, it is quite natural to wonder how rigid is the existence of another kind of special inner products on a Lie algebra with a complex structure. In particular, the so-called astheno-Kähler metrics introduced by Jost and Yau in [20], which are characterized by the condition

$$\partial \overline{\partial} \omega^{n-2} = 0.$$  

Clearly, on a complex surface any Hermitian metric is astheno-Kähler and in complex dimension 3 the notion of astheno-Kähler metric coincides with that of pluriclosed. Here we observe that in the nilpotent case the existence of a astheno-Kähler inner product on a Lie algebra compatible with an abelian complex structure does not force the 2-step condition in contrast to Theorem 1.1 for the pluriclosed case.

Example 2.8. In view of [21, Corollary 5.1.9] we consider the 8-dimensional 3-step nilpotent Lie algebra $\mathfrak{g}$ with complex structure equations

$$d\varphi^1 = d\varphi^2 = 0, \quad d\varphi^3 = \varphi^{1\bar{1}}, \quad d\varphi^4 = B_{11}\varphi^{1\bar{1}} + B_{13}(\varphi^{1\bar{2}} + \varphi^{1\bar{3}}) + D_{3\bar{1}}(\varphi^{2\bar{1}} + \varphi^{3\bar{1}}),$$  

with $D_{3\bar{1}} \neq 0$. In particular, the complex structure $J$ is abelian.
Let
\[ \omega = \sum_{k=1}^{3} ix_{kk} \varphi^{kk} + \sum_{1 \leq k < l \leq 3} (x_{kl} \varphi^{kl} - \bar{x}_{kl} \varphi^{lk}) + \frac{i}{2} \varphi^{44}. \]

If \( i x_{22} + i x_{33} + 2 \Im m (x_{23}) = 0 \), then \( \omega \) defines an astheno Kähler metric on \( (\mathfrak{g}, J) \).

3 | PROOF OF THEOREM 1.2

Let \( G \) be a 2-step nilpotent Lie group with a left-invariant Hermitian structure \( (\mathfrak{g}, J) \) and denote by \( \mathfrak{g} \) its Lie algebra. Assume further that \( g \) is pluriclosed. In view of [31], the metric \( g \) is Strominger Kähler–like if and only if there exists an orthonormal basis \( \{x_i\}_{i=1}^{s} \) of \( \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \) and an orthonormal basis \( \{\epsilon_i\}_{i=1}^{2n} \) of \( \mathfrak{g} \) such that

1. \( J \epsilon_i = \epsilon_{i+n}, \quad i = 1, ..., n; \)
2. \( \mathfrak{g}^1 + J \mathfrak{g}^1 = \text{span}\{\epsilon_{r+1}, ..., \epsilon_n, \epsilon_{n+r+1}, ..., \epsilon_{2n}\}; \)
3. the only non-trivial brackets under \( \{\epsilon_i\} \) are
   \[ [\epsilon_i, \epsilon_{n+i}] = \lambda_i x_i, \quad i = 1, ..., s, \]
   for some positive numbers \( \{\lambda_i\}_{i=1}^{s} \) and \( n - r \leq s \leq \min\{r, 2(n-r)\} \).

Note, that in particular \( J \) has to be abelian.

If \( \{e^i\} \) is the dual basis to \( \{\epsilon_i\} \), then the metric \( g \) writes as
\[ g = \sum_{k=1}^{2n} (e^k)^2. \]

From [31] it follows that every other left-invariant pluriclosed metric \( h \) taking the diagonal form
\[ h = \sum_{k=1}^{2n} a_k \epsilon^k \epsilon^k, \quad a_k > 0, a_k = a_{n+k}, \quad \text{for every } k = 1, ..., n, \]
is Strominger Kähler–like since we can modify the basis \( \{\epsilon_k\} \) to
\[ \tilde{\epsilon}_k = \frac{1}{\sqrt{a_k}} \epsilon_k \]
which still satisfies items 1–3.

Moreover, in view of [12], the Ricci form of the Bismut connection of \( h \) takes the following expression:
\[ \rho^B_h(x, y) = \frac{1}{2} \sum_{k=1}^{s} \frac{1}{a_k} h([\epsilon_k, \epsilon_{k+n}], [x, y]), \]
which implies that $\rho^B_h$ takes the diagonal form

$$\rho^B_h = \sum_{k=1}^{s} b_k \epsilon^k \wedge \epsilon^{n+k}.$$ 

It follows that, by uniqueness, the solution to the pluriclosed flow starting from $g_0$ is diagonal for every $t$ and the claim of Theorem 1.2 follows.

**Remark 3.1.** Notice that we can give a more explicit expression for the Ricci form $\rho^B_{g_t}$ of the Bismut connection $\nabla^B$ of the metric $g_t$. Let $\{\epsilon_i\}$ be a basis satisfying items 1–3 and

$$g_0 = \sum (\epsilon^k)^2.$$

Consider the solution to the pluriclosed flow

$$g_t = \sum a^t_k (\epsilon^k)^2.$$

If $\{\epsilon^t_k\}$ is a $g_t$-orthonormal basis satisfying items 1–3, namely

$$\epsilon^t_k = \frac{1}{\sqrt{a^t_k}} \epsilon_k,$$

then

$$\rho^B_{g_t}(x, y) = \frac{1}{2} \sum g_t([\epsilon^t_k, \epsilon^t_{n+k}], [x, y]).$$

We have

$$[\epsilon^t_k, \epsilon^t_{n+k}] = \frac{1}{\sqrt{a^t_k}} \sqrt{a^t_{n+k}} [\epsilon_k, \epsilon_{n+k}] = \frac{1}{a^t_k} \lambda^t_k x^t_k$$

and

$$[\epsilon^t_k, \epsilon^t_{n+k}] = \lambda^t_{k, n+k}$$

with $\{x^t_k\}$ $g_t$-orthonormal. Hence

$$\rho^B_{g_t}(x, y) = \frac{1}{2} \sum \lambda^t_k g_t(x^t_k, [x, y]).$$

Now

$$\rho^B_{g_t}(\epsilon_i, \epsilon_{n+i}) = \frac{1}{2} \sum_{k=1}^{s} \lambda^t_k g_t(x^t_k, [\epsilon_i, \epsilon_{n+i}]).$$
Since
\[ [\varepsilon_i, \varepsilon_{n+i}] = a^l_i [\varepsilon^l_i, \varepsilon^l_{n+i}] = a^l_i \lambda^l_i x^l_i \]
we get
\[ \rho^B_{g_t}(\varepsilon_i, \varepsilon_{n+i}) = \frac{1}{2} \sum_{k=1}^s \lambda^l_k g_k(x^l_k, [\varepsilon_i, \varepsilon_{n+i}]) = \frac{1}{2} \sum_{k=1}^s \lambda^l_k a^l_i g_k(x^l_k, \lambda^l_i x^l_i) = \frac{1}{2} (\lambda^l_i)^2 a^l_i. \]

Therefore,
\[ \rho^B_{g_t} = \frac{1}{2} \sum_{k=1}^s (\lambda^l_k)^2 a^l_k \varepsilon^k \wedge \varepsilon^{n+k}. \]

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