THE STRUCTURE OF TIME AND INERTIAL FORCES IN LAGRANGIAN MECHANICS

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ABSTRACT

Classically time is kept fixed for infinitesimal variations in problems in mechanics. Apparently, there appears to be no mathematical justification in the literature for this standard procedure. This can be explained canonically by unveiling the intrinsic mathematical structure of time in Lagrangian mechanics. Moreover, this structure also offers a general method to deal with inertial forces.

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This article studies two related questions in Lagrangian mechanics: the mathematical nature of time and the one of reference frames and inertial forces.

An isolated mechanical system without constraints corresponds to a Riemannian manifold \((M, T_2)\) (the configuration space) and a 1-form \(\alpha\) on \(TM\) (the work form). The general abstract version of Newton law postulates the existence of an associated tangent field \(D\) on \(TM\), that is, a second order differential equation, which is the motion law for \((M, T_2, \alpha)\). There is no function \(f\) on \(M\) that could be used as parameter for all solutions of \(D\). Classically a new function \(t\) is added to the system. This is “the time”, which is measured by some “clock” out of the system. Then \(M\) is replaced by \(\mathbb{R} \times M\) and to the collection of differential equations, \(\dot{t} = 1\) is added. When constraints (holonomic or not) are introduced in the extended manifold \(\mathbb{R} \times M\), the classical procedure to derive the equations of motion is to apply the principle of virtual works using infinitesimal displacements in which the time remains unchanged.

This is usually done without justification, or providing only “physical” arguments; see e.g. [3], pag. 3, [4], pag. 48-49, [5], pag. 522, [6], pag. 65, [8], pag. 215.

The addition of an external time to the configuration space is somewhat artificial. In an isolated system there is no “time” function, but there is a canonical “class of time” consisting of all the 1-forms on \(TM\), out of the zero section, that contracted with second order differential equations give 1. When these forms are specialized to each solution of a second order differential equation their primitive functions serve as time parameter.

The admissible infinitesimal variations in mechanics are described as those tangent fields \(\delta\) on \(TM\) which are infinitesimal contact transformations, projecting to \(M\), and preserving the class of time. When properly formulated, this last condition appears to be equivalent to the classical commutation formula \(d \circ \delta = \delta \circ d\).

Hypersurfaces on \(TM\) equipped with local time functions are “time constraints” considered in Section 5. When free mechanical system with such constraint are considered, the equations of motion are modified in a similar way as for ordinary holonomic constraints. In a precise sense time constraints can be proved to be deformable to ordinary holonomic constraints. The precise mathematical justification for keeping the time fixed in infinitesimal displacements in D’Alembert principle is given by Theorem 6.7.

Our presentation of Lagrangian mechanics provides a natural framework for the understanding of general inertial forces. The nature of such forces seems to be an obscure point in the classical literature.

Given a manifold isomorphism \(\varphi: \mathcal{R} \to \mathcal{M}\), where \(\mathcal{R}, \mathcal{M}\) are provided with respective pseudo-Riemannian metrics \(T_2, T_2\), for each mechanical structure \((\mathcal{M}, \overline{T}_2, \overline{\alpha})\) on
M, there correspond two mechanical structures \((R, T_2, \alpha), (R, T_2, \alpha_1)\) canonically associated by \(\varphi\) to the one in \(M\). Their difference can be read in \(R\) as the inertial force caused in \((M, T_2, \alpha)\) by the reference frame \(\varphi\). This gives a precise mathematical meaning to the expression “\(\varphi\) preserves the equations of motion” that spreads across the literature without a proper previous definition. The property of “preserving the equations of motion” is interesting only when time constraints are present, as is the case for uniparametric automorphism groups, because for free systems, \(\varphi\) preserves the equation of motion only when it is an isometry, and therefore a simple change of coordinates.

Sections 0, 1, 2, 3 have been included to make the article self-contained, although the material therein is classical, except for language.

0. Notations and definitions

We start with a brief overview of notations and definitions used in the article.

Let \(M\) be a smooth manifold of dimension \(n\), and \(TM\) be its tangent bundle.

Each differential 1-form \(\alpha\) on \(M\) can be considered as a function on \(TM\), denoted by \(\dot{\alpha}\), which assigns to each \(v_a \in T_aM\) the value \(\dot{\alpha}(v_a) = \langle \alpha_a, v_a \rangle\) obtained by duality. In particular, a function \(f \in C^\infty(M)\) defines the function on \(TM\) associated to \(df\) that we denote in short by \(\dot{f}\). This definition also applies to differential forms \(\alpha\) on \(TM\) that are at each point the pull-back of a form on \(M\). In the sequel we call these forms horizontal forms.

The map \(f \mapsto \dot{f}\) from \(C^\infty(M)\) to \(C^\infty(TM)\) is a derivation of the ring \(C^\infty(M)\) taking values in the \(C^\infty(M)\)-module \(C^\infty_a(TM)\). We denote it by \(\dot{d}\) since it is essentially the differential. For each horizontal form \(\alpha\), we have \(\dot{\alpha} = \langle \alpha, \dot{\alpha} \rangle\) as functions on \(TM\).

Any derivation \(\delta\) of \(C^\infty(M)\) to the \(C^\infty(M)\)-module \(C^\infty_a(TM)\) can be viewed as a field in \(TM\) taking values in \(TM\), that is, as a rule assigning to each point \(v_a \in T_aM\) a tangent vector \(\delta_{va} \in T_aM\). More precisely, \(\delta_{va} f = (\delta f)(v_a)\), for each \(f \in C^\infty(M)\). In particular, the derivation \(\dot{\delta}\) is the identity vector field in \(TM\), \(v_a \mapsto v_a\), since \(\dot{\delta}_{va} f = \delta_{va} f = \langle \delta f, v_a \rangle = df(v_a) = v_a(f)\).

Using the vector space structure of the fibers of \(TM\) we can associate to each \(v_a \in T_aM\) a tangent vector to \(T_aM\) at each point as the derivative along \(v_a\) in \(T_aM\). Denoting by \(V_a\) this derivation, we have for \(f \in C^\infty(M)\) and a point \(w_a \in T_aM\):

\[
V_a(\dot{f})(w_a) = \lim_{t \to 0} \frac{\dot{f}(w_a + tv_a) - \dot{f}(w_a)}{t} = \dot{f}(v_a) = v_a(f) .
\]

At each \(w_a \in T_aM\), \(V_a \in T_{w_a}(T_aM)\) is called the vertical representative of \(v_a \in T_aM\) and \(v_a\) the geometric representative of \(V_a\).
This canonical association between tangent vectors to $M$ at the point $a \in M$ and tangent vectors to the fiber $T_a M$ at each one of its points, establish an isomorphism between fields on $TM$ valued on $TM$, and vertical tangent fields on $TM$. Under this isomorphism, the field $\dot{d}$ corresponds to the vertical tangent field $\dot{D}$ on $TM$ such that $\dot{D}f = \dot{f}$, for $f \in C^\infty(M)$. This field is the infinitesimal generator for the group of homotheties of the fibers of $TM$. To avoid any confusion with the notation for second order differential equations defined below, we will no longer use the capital notation $D$.

**Definition 0.1. (Second Order Differential Equation).** A vector field $D$ on $TM$ is a second order differential equation when its restriction (as derivation) to the subring $C^\infty(M)$ of $C^\infty(TM)$ is $\dot{d}$.

This is equivalent to have $\pi_*(D_{va}) = v_a$ at each point $v_a \in T_a M$ (where $\pi: TM \to M$ denotes the canonical projection).

**Remark 0.2.** The difference between two tangent vector fields on $TM$ which are second order differential equations, is a vertical vector field. Thus the second order differential equations on $TM$ are sections of an affine bundle modeled on the fiber bundle over $TM$ of the vertical tangent fields. This last one is isomorphic to the bundle of fields on $TM$ taking values on $TM$.

**Definition 0.3. (Contact System).** The contact system on $TM$ is the Pfaff system in $TM$ which consists of all the 1-forms annihilating all the second order differential equations. It will be denoted by $\Omega$.

**Remark 0.4.** The forms in the contact system also annihilate the differences of second order differential equations, i.e. all vertical fields. Therefore, they are horizontal forms; each $\omega_{va} \in \Omega_{va}$ is the pull-back to $T^*_a TM$ of a form in $T^*_a M$. Now, a horizontal 1-form kills a second order differential equation if and only if it kills the field $\dot{d}$. Thus the contact system on $TM$ consists of the horizontal 1-forms which annihilate $\dot{d}$.

**0.5. Local coordinate expressions.**

We take local coordinates $(q^1, \ldots, q^n)$ in $M$ and corresponding $(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ in $TM$. We have, using Einstein summation convention,

$$\dot{d} = \dot{q}^i \frac{\partial}{\partial q^i}.$$ 

A vertical field has the expression

$$V = f^i(q, \dot{q}) \frac{\partial}{\partial q^i}.$$
And the one for a second order differential equation is

\[ D = \dot{q}^i \frac{\partial}{\partial q^i} + f^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}. \]

Usually we will denote \( f^i \) by \( \ddot{q}^i \) understanding that it is a given function of the \( q \)'s and \( \dot{q} \)'s.

A local system of generators for the contact system \( \Omega \), out of the zero section, is given by

\[ \dot{q}^i dq^j - \dot{q}^j dq^i \quad (i, j = 1, \ldots, n). \]

1. Structure of a Second Order Differential Equation Relative to a Metric

Let \( T^*M \) be the cotangent bundle of \( M \) and \( \pi: T^*M \to M \) the canonical projection. Recall that the Liouville form \( \theta \) on \( T^*M \) is defined by \( \theta_{\alpha_a} = \pi^*(\alpha_a) \) for \( \alpha_a \in T^*_aM \). Abusing the notation we can write \( \theta_{\alpha_a} = \alpha_a \).

The 2-form \( \omega_2 = d\theta \) is the natural symplectic form associated to \( T^*M \).

In local coordinates \( (q^1, \ldots, q^n) \) in \( M \), and corresponding \( (q^1, \ldots, q^n, p_1, \ldots, p_n) \) for \( T^*M \), we have

\[ \theta = p_i dq^i, \quad \omega_2 = dp_i \wedge dq^i. \]

Let \( T_2 \) be a (non-degenerate) pseudo-Riemannian metric in \( M \). Then we have an isomorphism of vector fiber bundles

\[ TM \to T^*M \]
\[ v_a \mapsto v_a T_2 \]

\( (i_{v_a} T_2 \) is the inner contraction of \( v_a \) with \( T_2 \)).

Using the above isomorphism we can transport to \( TM \) all structures on \( T^*M \). In particular, we work with the Liouville form \( \theta \) and the symplectic form \( \omega_2 \) transported in \( TM \) with the same notation.

From the definitions we have for the Liouville form in \( TM \), at each \( v_a \in T_aM \),

\[ \theta_{v_a} = i_{v_a} T_2, \]

where the form of the right hand side is to be understood pulled-back from \( M \) to \( TM \).

Equation (1) is the point expression for:

\[ \theta = i_{\dot{v}} T_2 \]
On the other hand, from (1) we obtain, for each \( \lambda \in \mathbb{R} \):

(3) \[ \theta_{\lambda v_a} = \lambda \theta v_a \]

(equality as 1-forms in \( T^*M \)).

Let \( V \) denote the vertical field corresponding to \( \dot{d} \); \( V \) is the infinitesimal generator of the group of homotheties in the fibers of \( TM \), and from (3) we get

(4) \[ L_{V} \theta = \theta , \]

which, by using Cartan’s formula (see [2], p. 36) and \( i_{V} \theta = 0 \) (\( \theta \) being horizontal), gives

(5) \[ i_{V} d\theta = \theta \text{ or } i_{V} \omega_{2} = \theta . \]

Taking the values at each \( v_a \in T_a M \) and if we put together (1) and (5) it results the following key lemma.

**Lemma 1.1.** If \( v_a \in T_a M \) and \( V_a \) is its vertical representative at each point of \( T_a M \), we have

(6) \[ i_{V_a} \omega_{2} = i_{v_a} \omega_{2} . \]

If \( V \) is a vertical tangent field on \( TM \) and \( v \) is the corresponding field on \( TM \) taking values in \( TM \), then we have

(7) \[ i_{V} \omega_{2} = i_{v} \omega_{2} \]

(equality of horizontal forms in \( TM \)).

**Definition 1.2.** (Kinetic Energy). The function \( T = \frac{1}{2} \dot{\theta} \) on \( TM \) is the kinetic energy associated to the metric \( T_{2} \). So, for each \( v_a \in TM \), we have \( T(v_a) = \frac{1}{2} \dot{\theta}(v_a) = \frac{1}{2} T_{2}(v_a, v_a) \) or, as a function on \( TM \), \( T = \frac{1}{2} T_{2}(\dot{d}, \dot{d}) \).

**Theorem 1.3.** The metric \( T_{2} \) establishes a one-to-one correspondence between second order differential equations and horizontal 1-forms in \( TM \).

The second order differential equation \( D \) and the horizontal 1-form \( \alpha \) that correspond to each other are related by

(8) \[ i_{D} \omega_{2} + dT + \alpha = 0 . \]

**Proof.** Given a second order differential equation \( D \), we define the 1-form \( \alpha \) by (8). Now we check that \( \alpha \) is horizontal. For any vertical field \( V \) we must prove that \( \langle \alpha, V \rangle = 0 \). Using Cartan’s formula,

\[ \langle i_{D} \omega_{2}, V \rangle = \langle i_{D} d\theta, V \rangle = D(\theta, V) - V(\theta, D) - \langle \theta, [D, V] \rangle . \]
Now we have $\langle \theta, V \rangle = 0$ since $\theta$ is horizontal. Also we have $\langle \theta, D \rangle = \langle \theta, \dot{d} \rangle = \dot{\theta}$, and $V \dot{\theta} = 2 \langle \theta, v \rangle$, where $v$ is the geometric representative of $V$ and using that $\dot{\theta}$ is homogeneous of the second degree in the variables $\dot{q}$. Therefore we have
\[ V \langle \theta, D \rangle = 2 \langle \theta, v \rangle . \]
Also for $f \in C^\infty(M)$ we have
\[ [D, V]f = -V(Df) = -V(\dot{f}) = -vf , \]
thus the bracket $[D, V]$ is equal to $-v$ up to a vertical field. It follows that
\[ \langle \theta, [D, V] \rangle = -\langle \theta, v \rangle . \]
And putting all together, we have
\[ \langle i_D \omega_2, V \rangle = 0 - 2 \langle \theta, v \rangle + \langle \theta, v \rangle = -\langle \theta, v \rangle . \]
Also
\[ \langle dT, V \rangle = VT = V\left(\frac{1}{2} \dot{\theta}\right) = \langle \theta, v \rangle , \]
therefore $\alpha$ is horizontal.

Lemma (1.1) establishes a linear isomorphism, $V \mapsto i_V \omega_2$, between vertical vector fields and horizontal 1-forms. So, given $\alpha$, adding a suitable vertical vector field $V$ to a given second order differential equation $D_0$ we obtain $D = D_0 + V$ that corresponds to $\alpha$ by (8). □

**Definition 1.4. (Geodesic Field).** The geodesic field of the metric $T_2$ is the second order differential equation, $D_G$, corresponding to $\alpha = 0$:
\[ i_{D_G} \omega_2 + dT = 0 . \]

The projection to $M$ of the curves solution of $D_G$ in $TM$ are the geodesics of $T_2$.

The geodesic field $D_G$ is chosen as the origin in the affine bundle of second order differential equations. With this choice we establish a one-to-one correspondence between second order differential equations and vertical tangent fields.

\[ D \leftrightarrow D - D_G = V. \]

And recalling that to a vertical field $V$ there canonically corresponds a field $v$ on $TM$ (the geometric representative of $V$), we define:

**Definition 1.5. (Covariant Value).** We define the covariant value of the second order differential equation $D$, denoted by $D^\nabla$, as the field in $TM$ taking values in $TM$ corresponding canonically to $D - D_G$.

**Remark 1.6.** The covariant value $D^\nabla$ yields at each point $v_a \in TM$ the acceleration at the point $a \in M$ of a particle with a trajectory solution of $D$ and having $v_a$ as tangent vector at the point $a \in M$:
\[ D^\nabla_{v_a} = \nabla_{v_a} v , \]
where $v$ is the vector field tangent along the trajectory.

From this definition, Lemma (1.1), and the definition of $D_G$ it is straightforward to prove:

**Theorem 1.7.** The horizontal 1-form corresponding to the second order differential equation $D$ is related to its covariant value by

$$i_{D^\nabla} T_2 = -\alpha .$$

**Definition 1.8.** Given a horizontal 1-form $\alpha$, $\text{grad} \alpha$ is the field on $TM$ taking values in $TM$ such that

$$i_{\text{grad} \alpha} T_2 = \alpha .$$

In particular, when $\alpha = dU$ for $U \in C^\infty(M)$, $\text{grad} \alpha$ equals $\text{grad} U$.

So, Theorem (1.7) gives

$$D^\nabla = -\text{grad} \alpha .$$

**Remark 1.9.** When $T_2$ is a second order covariant symmetric tensor on $M$, we can define on $TM$ a “Liouville form” $\theta$ by (1), and a closed 2-form $\omega_2 = d\theta$. Formulae (2), (3), (4), (5) remain valid, as also Lemma (1.1). In Theorem (1.3) is still true that (8) assigns a horizontal 1-form $\alpha$ to each second order differential equation $D$, although not in a one to one way, as a rule.

1.10. Local coordinate expressions.

Consider an open set of $M$ with coordinates $q^1, \ldots, q^n$ and the corresponding open set in $TM$ with coordinates $q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$.

If the expression in local coordinates of $T_2$ is

$$T_2 = g_{jk}(q) \ dq^j dq^k$$

then the local equations for the isomorphism $TM \approx T^*M$ are

$$p_j = g_{jk}(q) \dot{q}^k .$$

The Liouville form in $TM$ is given by

$$\theta = g_{jk}(q) \dot{q}^k dq^j$$

and the symplectic form in $TM$ by

$$\omega_2 = g_{jk} dq^j \wedge dq^k + \frac{\partial g_{jk}}{\partial q^l} \dot{q}^j dq^l \wedge dq^k .$$

For the kinetic energy we have, locally,

$$T = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j .$$
so that

\[ p_j = \frac{\partial T}{\partial \dot{q}^j}. \]

Let the second order differential equation \( D \) be given by

\[ D = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}, \]

where the \( \ddot{q}^i \)'s are given function of \( q^i \)'s and \( \dot{q}^i \)'s. The local expression \([16]\) for \( \omega_2 \) gives

\[ i_D \omega_2 = -g_{ij} \dot{q}^i dq^j + g_{ij} \ddot{q}^i dq^j + \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \left( \dot{q}^k dq^j - \dot{q}^j dq^k \right). \]

On the other hand, we have

\[ dT = g_{ij} \dot{q}^i dq^j + \frac{1}{2} \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \dot{q}^j dq^k. \]

Replacing this equality into the precedent expression we get

\[ i_D \omega_2 = -dT + \frac{1}{2} \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \dot{q}^j dq^k + g_{ij} \ddot{q}^i dq^j + \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \left( \dot{q}^k dq^j - \dot{q}^j dq^k \right) \]

\[ = -dT + g_{ik} \ddot{q}^i dq^k + \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} \right) \dot{q}^i \dot{q}^j dq^k \]

\[ = -dT + (g_{ik} \ddot{q}^i + \Gamma_{ij,k} \dot{q}^i \dot{q}^j) dq^k, \]

where the \( \Gamma \)'s are the Christoffel symbols of \( T_2 \).

Now

\[ (20) \quad \alpha = -g_{ik} (\dot{q}^i + \Gamma_{ij,k} \dot{q}^i \dot{q}^j) dq^k \]

is the horizontal 1-form related to \( D \) by formula \([8]\).

For the geodesic field we have,

\[ (21) \quad D_G = \dot{q}^i \frac{\partial}{\partial q^i} - \Gamma_{ij,k} \dot{q}^i \dot{q}^j \frac{\partial}{\partial q^i}, \]

and, finally, the covariant value of \( D \) is

\[ (22) \quad D^\nabla = (\dot{q}^i + \Gamma_{ij,k} \dot{q}^i \dot{q}^j) \frac{\partial}{\partial q^i}. \]
2. **Newton-Lagrange mechanics of a free system**

**Definition 2.1. (Mechanical System).** A mechanical system \((M, T, \alpha)\) is a manifold \(M\) (the configuration space) provided with a pseudo-Riemannian metric \(T\) (to which corresponds a kinetic energy \(T\) by (1.2)), and a 1-form \(\alpha\) on \(T M\) (the work-form or force form).

**Postulate 2.2. (Newton Law).** The tangent field \(D\) on \(T M\) such that

\[
i_D \omega_2 + dT + \alpha = 0
\]

is a second order differential equation.

From Theorem (1.3), we have

**Theorem 2.3.** The force-form \(\alpha\) of a mechanical system obeying Newton’s law is horizontal.

**Definition 2.4. (Work).** A trajectory of the mechanical system is a curve in \(T M\) solution of \(D\). The integral of \(\alpha\) along a given trajectory is called work done by the system.

Contracting with \(D\) in (23), we get

\[
DT + i_D \alpha = 0,
\]

and, by integrating along a trajectory \(c\),

\[
\int_c (dT + \alpha) = 0
\]

which is an expression for the law of conservation of energy: the work done by the system equals the loss of kinetic energy.

**Definition 2.5. (Conservative System).** When \(\alpha\) is an exact differential form, the mechanical system \((M, T, \alpha)\) is called conservative. The function \(U\) (determined up to an additive constant) such that \(\alpha = dU\) is called the potential energy of the system.

**Theorem 2.6.** In a conservative system \((M, T, dU)\), the potential energy \(U\) is a function on \(M\) (lifted to \(T M\)).

**Proof.** \(dU\) is horizontal. \(\Box\)

**Definition 2.7. (Hamiltonian).** In a conservative system \((M, T, dU)\), the function \(H = T + U\) in \(T M\) is called the total energy or hamiltonian of the system.

Equation (23) for a conservative system is

\[
i_D \omega_2 + dH = 0,
\]
which, read in $T^*M$, is the set of Hamilton’s canonical equations.

Equation (12) corresponds to the classical Newton’s Law $\vec{F} = m \cdot \ddot{\vec{a}}$. For a conservative system:

\begin{equation}
D^\nabla = -\text{grad} U.
\end{equation}

or, in local coordinates,

\begin{equation}
\frac{d^2q^i}{dt^2} + \Gamma^i_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} = -g_{ih} \frac{\partial U}{\partial q^h}.
\end{equation}

For the geodesic field ($\alpha = 0$), we obtain the equations of geodesics for $T_2$

\begin{equation}
\frac{d^2q^i}{dt^2} + \Gamma^i_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} = 0
\end{equation}

as a generalization of the classical Euler’s theorem (see [1]) for the movement of a point on a surface in absence of external forces.

For a general 1-form of force $\alpha$, the corresponding field $D$ differs of the geodesic one, $D_G$, by the vertical field $D - D_G$; it is natural to consider this field as the “cause” bringing the system to move out the geodesics. So, we give the following

**Definition 2.8. (Force).** A force on $M$ is a vertical field $V$ on $TM$. The field $v$ on $TM$ which takes values in $TM$ corresponding canonically to $V$, will be called the geometric expression of $V$, or the geometric representative of $V$. The force associated to a second order differential equation $D$ by the metric $T_2$ is $D - D_G$. Its geometric expression is $D^\nabla$.

3. Constrained systems

**Definition 3.1. (Constraints).** A constrained mechanical system is a mechanical system $(M, T_2, \alpha)$ together with a Pfaff system $\Lambda$ on $TM$. Each 1-form in $\Lambda$ is a constraint.

**Postulate 3.2. (Newton-Lagrange Law).** Given the constrained mechanical system $(M, T_2, \alpha, \Lambda)$, there exists a tangent vector field $\overline{D}$ on $TM$, which is a second order differential equation, and satisfies the congruence

\begin{equation}
i_{\overline{D}} \omega_2 + dT + \alpha \equiv 0 \mod \Lambda
\end{equation}

and also the principle of virtual works (stated below (3,4)).

By Theorem (2.3) $\alpha$ is horizontal. Since $\overline{D}$ is a second order differential equation, the $\beta$’s in $\Lambda$ transforming (30) into an equality, have to be horizontal. Thus, we can always replace $\Lambda$ by its intersection with the space of horizontal forms on $TM$. In the sequel, we will assume that $\Lambda$ is a Pfaff system of horizontal forms on $TM$. Therefore, for each $\beta$ in $\Lambda$, it makes sense to consider the function $\beta$ on $TM$.
Definition 3.3. (Admissible State.) An admissible state for the constrained mechanical system \((M, T_2, \alpha, \Lambda)\) is a point \(v_a \in TM\) such that \(\dot{\beta}(v_a) = 0\) for each \(\beta\) in \(\Lambda\). The set consisting of all admissible states is denoted by \(L\).

Principle of virtual works 3.4. A curve in \(TM\) which is a solution of \(\mathcal{D}\) and passes through a point \(v_a \in L\), remains entirely in \(L\). When \(L\) is a submanifold of \(TM\) this is equivalent to \(\mathcal{D}\) being tangent to \(L\).

Remark 3.5. Congruence (30) is a form of D’Alembert principle of equilibrium between applied, inertial, and constraint forces. Definition (3.3) selects as admissible velocities those for which, in the corresponding “infinitesimal displacements”, the constraint forces do not work. The principle (3.4) means that the system remains in admissible states: the constraint forces never work. (See, e.g. [6], Sect.10).

3.6. Local computation for \(\mathcal{D}\).

We take local coordinates in \(M\) and a local basis \(\{\beta_1, \ldots, \beta_r\}\) for \(\Lambda\) in \(TM\):

\[
\beta_k = B_{kj}(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)\, dq^j \quad (k = 1, \ldots, r)
\]

Each \(\beta_k\) corresponds to a vertical tangent field \(V_k\) (a constraint force) by

\[
i_{V_k}\omega_2 = \beta_k
\]

Let \(D\) be the field for the free system \((M, T_2, \alpha)\):

\[
i_D\omega_2 + dT + \alpha = 0
\]

Then, the congruence (30) can be written as:

\[
\mathcal{D} = D + \lambda^1 V_1 + \cdots + \lambda^r V_r,
\]

where the \(\lambda^k\)’s are certain (local) functions on \(TM\), called Lagrange multipliers. Our problem is to compute the \(\lambda\)’s such that \(\mathcal{D}\) satisfies the principle of virtual works (3.4).

We assume that \(L\) is a submanifold of \(TM\). In this situation we compute the \(\lambda\)’s by requiring \(\mathcal{D}\) to be tangent to \(L\).

We have the local expressions

\[
V_k = a_k^h(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)\frac{\partial}{\partial \dot{q}^h} \quad (k = 1, \ldots, r),
\]

The local equations of \(L\) are \(\dot{\beta}_1 = 0, \ldots, \dot{\beta}_r = 0\). Then, the conditions for \(\mathcal{D}\) being tangent to \(L\) are:

\[
\lambda^k a_k^h \frac{\partial(B_{ij}\dot{q}^j)}{\partial \dot{q}^h} \equiv -D(B_{ij}\dot{q}^j) \mod (\dot{\beta}_1, \ldots, \dot{\beta}_r)
\]
Using coordinates, (32) is:

\[ (36) \quad g_{jh}a_k^h = B_{kj} \]

and (35) becomes:

\[ (37) \quad \lambda^k a^h_k \frac{\partial (g_{js} a^s_l \dot{q}^j)}{\partial \dot{q}^h} \equiv -D(g_{jh} a^h_k \dot{q}^j) \mod (\dot{\beta}_1, \ldots, \dot{\beta}_r). \]

The discussion of (37) is carried out according to the supplementary hypothesis in each particular case. We start with the most common one:

**Definition 3.7. (Linear Constraints).** \( \Lambda \) is a system of linear constraints when it is generated, as a Pfaff system on \( TM \), by a Pfaff system \( \Lambda_M \) on \( M \) (lifted to \( TM \)).

Let \( r \) be the rank of \( \Lambda_M \). For each \( a \in M \), the vectors \( v_a \in T_a M \) annihilated by the forms in \( (\Lambda_M)_a \) form a \( (n-r) \)-dimensional subspace \( \mathcal{L}_a \subset T_a M \). The collection of all \( \mathcal{L}_a \), when \( a \) varies in \( M \), is the set \( \mathcal{L} \) of admissible states. Therefore, in the case of linear constraints, \( \mathcal{L} \) is the vector subbundle of \( TM \) corresponding to the distribution of vector fields incident with (i.e. annihilated by) \( \Lambda_M \).

In this case, the \( a_k^h \) in (34) can be chosen free from the \( \dot{q} \)'s, and the left member in (37) becomes

\[ (38) \quad \lambda^k a^h_k g_{js} a^s_l \delta_{jh} = \lambda^k a^h_k g_{hs} a^s_l = \lambda^k \langle v_k, v_l \rangle, \]

where \( v_k, v_l \) are the geometric representatives of \( V_k, V_l \) (see Definition (2.8)), and \( \langle \, , \, \rangle \) is the scalar product with respect to \( T_2 \).

When \( T_2 \) is positive definite, the matrix with entries \( \langle v_k, v_l \rangle \) is non degenerate, because the \( v \)'s are linearly independent. It follows that (37) can be solved when we replace congruence by equality giving unique solutions for the \( \lambda \)'s by:

\[ (39) \quad \lambda^k \langle v_k, v_l \rangle = -D\dot{\beta}_l. \]

With these values for the \( \lambda \)'s, the \( \overline{D} \) in (33) gives:

\[ (40) \quad \overline{D}\dot{\beta}_l = D\dot{\beta}_l + \lambda^k V_k \dot{\beta}_l = D\dot{\beta}_l + \lambda^k \langle \beta_l, v_k \rangle = D\dot{\beta}_l + \lambda^k \langle v_l, v_k \rangle = 0 \]

In the open set of \( TM \) where we work, the field \( \overline{D} \) satisfies the conditions imposed by (3.2). Also \( \overline{D} \) depends upon the basis \( \{ \beta_1, \ldots, \beta_r \} \), but its restriction to \( \mathcal{L} \) does not depend.

The local fields \( \overline{D} \) can be used to build, by means of an appropriate partition of unity, a global field \( \overline{D} \) on \( TM \) satisfying (3.4). The uniqueness of \( \overline{D} |_\mathcal{L} \) follows from the local uniqueness.

We have proved:
Theorem 3.8. When $T_2$ is positive definite and $\Lambda$ is a linear system of constraints, then there exists on $TM$ a second order differential equation $\overline{D}$ that satisfies the Newton-Lagrange law (3.2). The restriction of $\overline{D}$ as tangent field to the bundle $L$ of admissible states is fixed by the condition settled in (3.4).

Remark 3.9. (About non linear constraints). In the general case of a Pfaff system $\Lambda$ of horizontal forms on $TM$, and assuming $T_2$ to be positive definite, the left member of (37) restricted to the 0 section of $TM$ is (38). Thus, in a neighborhood of the 0 section, there exists a unique solution for the $\lambda$’s when in (37) we replace $\equiv$ by $=$. And, in the analytic case, there exists a global $\overline{D}$, with possible singularities out of the 0 section (i.e. “for large velocities”).

The most important case of linear constraints is that of holonomic constraints:

Definition 3.10. (Holonomic constraints). A system $\Lambda$ of constraints is called holonomic when it is linear and generated by a completely integrable Pfaff system $\Lambda_M$ on $M$.

In this case, let $\{dB_1, \ldots, dB_r\}$ be a local basis for $\Lambda_M$. The local equations for $\Lambda$ are $\dot{B}_1 = 0, \ldots, \dot{B}_r = 0$, and (locally) $\Lambda$ is the union of the tangent bundles to the submanifolds of $M$ given by equations $B_1 = b_1, \ldots, B_r = b_r$ ($b_i$ constants). Let $N \subset M$ be one of these submanifolds. In the embedding $N \subset M$, the metric $T_2$ on $M$ specializes as $T_{2N}$. In the embedding $TN \subset TM$, the Liouville form $\theta$ on $TM$, specializes as the Liouville form on $TN$ corresponding to the metric $T_{2N}$ (this is a consequence of (1)). Then, $\omega_2$ in $TM$ specializes as $\omega_{2N}$ in $TN$, which is also the symplectic form corresponding to $T_{2N}$.

The field $\overline{D}$ on $TM$ corresponding to the mechanical system $(M, T_2, \alpha, \Lambda)$ is tangent to $TN$, because the local equations for $TN$ in $TM$ are:

$$B_1 = b_1, \ldots, B_r = b_r, \dot{B}_1 = 0, \ldots, \dot{B}_r = 0,$$

and, for each $v_a \in T_a N$ we have $\overline{D}_{v_a}(B_k) = v_a(B_k) = 0$ and $\overline{D}_{v_a} \dot{B}_k = 0$, by (3.4).

Now, in the Newton-Lagrange equation

$$i_{\overline{D}} \omega_2 + dT + \alpha = \sum_{k=1}^r \lambda^k dB_k$$

everything specializes to $TN$, and with such a specialization, we obtain

$$i_{\overline{D}_N} \omega_{2N} + dT_N + \alpha_N = 0.$$

Theorem 3.11. When the constraints are holonomic, the field $\overline{D}$ corresponding to the system $(M, T_2, \alpha, \Lambda)$, tangent to the submanifold $L$ of admissible states, is also tangent to the submanifolds $TN \subset L$, for each solution $N \subset M$ of the Pfaff system $\Lambda_M$. The restriction of $\overline{D}$ to $TN$ is the field corresponding to the free system $(N, T_{2N}, \alpha_N)$. 
3.12. Analysis of general linear constraints.

Let $D$ be a second order differential equation, $\beta$ a 1-form on $M$. Working in local coordinates we have

$$\beta = B_l(q^1, \ldots, q^n) dq^l \quad \text{and} \quad \dot{\beta} = B_l(q^1, \ldots, q^n) \dot{q}^l,$$

and

$$D\dot{\beta} = B_l\dot{q}^l + D(B_l) \dot{q}^l = \langle \beta, D^2 - \Gamma^l_{hk} \dot{q}^h \dot{q}^k \frac{\partial}{\partial q^l} \rangle + \dot{d}(B_l) \dot{q}^l$$

$$= \langle \beta, D^2 \rangle + \dot{q}^h \dot{q}^k \left( \frac{\partial B_h}{\partial q^k} - B_l \Gamma^l_{hk} \right)$$

$$= \langle \beta, D^2 \rangle + \langle \nabla_\dot{d}\beta, \dot{d} \rangle,$$

where $\nabla_\dot{d}\beta$ is an horizontal 1-form on $TM$ and can be paired by duality with $\dot{d}$.

For the geodesic field, we obtain

$$D_G\dot{\beta} = \langle \nabla_\dot{d}\beta, \dot{d} \rangle.$$

Let $v$ be the field such that $i_v T_2 = \beta$. We have

$$\langle \nabla_\dot{d}\beta, \dot{d} \rangle = \langle \nabla_\dot{d}(i_v T_2), \dot{d} \rangle = T_2(\nabla_\dot{d}v, \dot{d}) = \Pi_v(\dot{d}, \dot{d}),$$

where $\Pi_v$ is the second fundamental form associated to the field $v$.

Putting all together, we obtain

$$D\dot{\beta} = \langle \beta, D^2 \rangle + D_G\dot{\beta} = \langle \beta, D^2 \rangle + \Pi_v(\dot{d}, \dot{d}).$$

This formula will be used to clarify the behaviour of a mechanical system with linear constraints.

Let $(M, T_2, \alpha, \Lambda)$ be such a system, let $\{\beta_1, \ldots, \beta_r\}$ be a local basis for $\Lambda_M$, and $v_k = \text{grad} \beta_k$, $V_k$ the vertical representative of $v_k$, so that $i_{V_k} \omega_2 = \beta_k$ (Lemma (1.1)). We write $\Pi_k$ for the second fundamental form corresponding to $v_k$.

By a suitable choice of the basis, we set

$$\langle v_h, v_k \rangle = \delta_{hk}, \quad (h, k = 1, \ldots, r).$$

Then, equations (39) for the Lagrange multipliers are

$$\lambda^l = -D\dot{\beta}_l,$$

when $D$ is the field for the free system $(M, T_2, \alpha)$. And the field $\overline{D}$ for the constrained system is, according (33)

$$\overline{D} = D - \sum_{k=1}^r (D\dot{\beta}_k)V_k,$$

(42)
From (41) and (42), we obtain for the covariant values,

\[
\nabla D = \nabla D - \sum_{k=1}^{r} (D \dot{\beta}_k) v_k = \nabla D - \sum_{k=1}^{r} \langle v_k, D \nabla \rangle v_k - \sum_{k=1}^{r} \Pi_k (\dot{\alpha}, \dot{d}) v_k
\]

The sum \( D \nabla - \sum_{k=1}^{m} \langle v_k, D \nabla \rangle v_k \) is the orthogonal projection of \( D \nabla \) on the distribution \( \mathcal{L} \) (in this case, the orthogonal distribution to \( v_1, \ldots, v_r \)). The sum \( -\sum_{k=1}^{r} \Pi_k (\dot{\alpha}, \dot{d}) v_k \) is the only term surviving when \( D \) is the geodesic field \( (\alpha = 0) \); then, it is \( \nabla G \), the field corresponding to the constrained system \((M, T_2, 0, \Lambda)\).

The formula \( \nabla G = -\sum_{k=1}^{r} \Pi_k (\dot{\alpha}, \dot{d}) v_k \), in the particular case of holonomic constraints, express a well known geometric theorem: when \( N \) is a submanifold of \( M \), the acceleration in \( M \) of a geodesic of \( N \) is orthogonal to \( N \).

Formula (43) is a related to classical results; see e.g. Prange [5], pag. 558.

4. The class of time and the calculus of variations

As usual, differential 1-forms on \( M \) are also considered as 1-forms on \( TM \), by means of the natural pull-back. A 1-form \( \alpha \) on \( M \) defines the function \( \dot{\alpha} \) on \( TM \) and, in the open set where \( \dot{\alpha} \) does not vanish, it defines the 1-form \( \frac{\alpha}{\dot{\alpha}} \). For each second order differential equation \( D \) we have \( \langle \frac{\alpha}{\dot{\alpha}}, D \rangle = 1 \). So, for two 1-forms \( \alpha, \beta \) on \( M \) we have \( \langle \frac{\alpha}{\dot{\alpha}} - \frac{\beta}{\dot{\beta}}, D \rangle = 0 \) in the open set of \( TM \) where neither \( \dot{\alpha} \) nor \( \dot{\beta} \) vanish. From the definition of the contact system (0.3) it follows that we have in that open set

\[
\frac{\alpha}{\dot{\alpha}} \equiv \frac{\beta}{\dot{\beta}} \mod \Omega.
\]

**Definition 4.1. (Class of Time).** Let \( U \) be an open set of \( TM \) non intersecting the zero section. A 1-form \( \tau \) defined on \( U \) belongs to the class of time when at each point \( v_a \in U \) it satisfies

\[
\tau_{v_a} \equiv \frac{\alpha_{v_a}}{\dot{\alpha}(v_a)} \mod \Omega_{v_a}
\]

for each 1-form \( \alpha \) on \( M \) such that \( \dot{\alpha}(v_a) \neq 0 \).

The 1-forms belonging to the class of time are horizontal. Therefore we can apply them to the field \( \dot{d} \). The following proposition is obvious

**Proposition 4.2.** A 1-form \( \tau \) on an open set \( U \subset TM \) non intersecting the zero section belongs to the class of time if and only if it is horizontal and satisfies \( \langle \tau, \dot{d} \rangle = 1 \). Therefore \( \langle \tau, D \rangle = 1 \) for each second order differential equation \( D \).

**Definition 4.3. (Infinitesimal contact transformation).** A tangent field \( \delta \) on \( TM \) is an infinitesimal contact transformation if \( L_\delta \Omega \subseteq \Omega \) (i.e., for each 1-form \( \sigma \) in \( \Omega \), \( L_\delta \sigma \) is in \( \Omega \)).
When $\delta$ generates an uniparametric group of automorphisms of $TM$, the condition $L_\delta \Omega \subseteq \Omega$ means that this group transforms solutions of $\Omega$ into solutions of $\Omega$, i.e., maps into each other curves which are tangent to second order differential equations or vertical fields.

**Theorem 4.4.** Let $\delta$ be an infinitesimal contact transformation on $TM$, which is projectable to $M$ as a tangent field. The following two properties are equivalent:

1. $\delta$ leaves invariant the class of time: for each 1-form $\alpha$ on $M$, in the open set of $TM$ where $\dot{\alpha} \neq 0$, we have:

   $$L_\delta \left( \frac{\alpha}{\dot{\alpha}} \right) \equiv 0 \mod \Omega .$$

2. $\delta$ commutes with $\dot{\mathcal{D}}$

   $$\dot{\mathcal{D}} \circ \delta = \delta \circ \dot{\mathcal{D}}$$

   as derivations from $\mathcal{C}^\infty(M)$ to $\mathcal{C}^\infty(TM)$.

(Not that $\dot{\mathcal{D}} \circ \delta$ does make sense because $\delta$ is projectable to $M$, thus maps $\mathcal{C}^\infty(M)$ into $\mathcal{C}^\infty(M)$.)

**Proof.** Let $D$ be any second order differential equation and let $\alpha$ be a 1-form on $M$. In the open set of $TM$ where $\dot{\alpha} \neq 0$, we have

$$0 = \delta(1) = \delta\left( \frac{\alpha}{\dot{\alpha}} , D \right) = \left( L_\delta \left( \frac{\alpha}{\dot{\alpha}} \right) , D \right) + \left( \frac{\alpha}{\dot{\alpha}} , [\delta , D] \right) .$$

The first term in the last sum is 0 for arbitrary $D$ if and only if $\delta$ satisfies (1). The second term is 0 for arbitrary $\alpha$ if and only if $[\delta , D]$ is vertical, and this is equivalent to (2). □

**Definition 4.5. (Infinitesimal Variation).** An infinitesimal contact transformation that is projectable to $M$ and satisfies the equivalent conditions of Theorem (4.4) is called an infinitesimal variation.

**Theorem 4.6.** For each vector field $v$ on $M$, there exists a unique infinitesimal variation $\delta_v$ on $TM$ which projects to $M$ as $v$.

**Proof.** When $\delta_v$ exists, (2) of (4.4) gives, for each $f \in \mathcal{C}^\infty(M)$,

$$\delta_v \dot{f} = \dot{\mathcal{D}}(vf) .$$

Thus, $\delta_v$ is uniquely determined by $v$.

Given a coordinate system $q^1, \ldots, q^n$ on an open set $U \subseteq M$, let $v = a^i(q^1, \ldots, q^n) \frac{\partial}{\partial q^i}$ be the local expression for $v$. Defining $\delta_v = a^i \frac{\partial}{\partial q^i} + \dot{a}^i(q^1, \ldots, q^n) \frac{\partial}{\partial \dot{q}^i}$, we check that $\delta_v$ is an infinitesimal contact transformation by computing the Lie derivatives of the forms $\dot{q}^i dq^j - \dot{q}^j dq^i$ (local generators of $\Omega$ on $TM$ out of the zero section). Then, from
its actual definition, \( \delta_v \) fulfills (2) of (4.4) in \( U \). The uniqueness shows that the local \( \delta_v \)‘s patch together giving a \( \delta_v \) globally defined on the whole of \( TM \).

Definition 4.7. (Prolongation of a Field). \( \delta_v \) is called the prolongation of \( v \) to \( TM \).

Remark 4.8. The tangent bundle \( TM \) is the space of contact elements \( M_1 \) in the sense of Weil [7]. Formula (2) of (4.4) means that \( \delta_v \) is the canonical prolongation of \( v \) from \( M \) to \( M_1 \). Also, \( \delta_v \) is the infinitesimal generator of the prolongation to \( TM \) of the group (or local group) of automorphisms of \( M \) generated by \( v \).

Remark 4.9. (2) of (4.4) is the ‘\( d \circ \delta = \delta \circ d \)’ in the classical texts of Mechanics; e.g. in Sommerfeld [6], formulae (9) and (9a) of page 175. Our formula (1) of (4.4) is reminiscent of “keeping fixed the time” in the reasoning from Sommerfeld in pages 175, 176. Sommerfeld attributes to Euler the formula \( d \delta = \delta d \). For this reason, we shall call (2) of (4.4) Euler’s commutation formula.

The starting point for the applications of variational methods in Mechanics is the following theorem (see Prange [5], “Zentralgleichung von Lagrange”, in page 531).

Theorem 4.10. Let \( M \) be a manifold provided with a pseudo-Riemannian metric \( T^2 \), \( \theta \) the corresponding Liouville form on \( TM \), \( T \) the kinetic energy. For any second order differential equation \( D \) and any infinitesimal variation \( \delta \), we have

\[
D \langle \theta, \delta \rangle = \delta T - \langle \alpha, \delta \rangle ,
\]

where \( \alpha \) is the work form corresponding to \( D \) by Theorem (1.3).

Proof. The Cartan formula for the Lie derivative, with Definition (1.2) for \( T \) and Theorem (1.3) gives

\[
L_D \theta = i_D d\theta + d\langle \theta, D \rangle = i_D \omega_2 + 2dT = -dT - \alpha + 2dT = dT - \alpha ,
\]

thus,

\[
D \langle \theta, \delta \rangle = \langle dT - \alpha, \delta \rangle + \langle \theta, [D, \delta] \rangle = \delta T - \langle \alpha, \delta \rangle ,
\]

because \([D, \delta]\) is vertical as follows from Euler’s commutation formula.

Definition 4.11. (Lagrangian function). In a conservative system \((M, T^2, dU)\), the function \( L = T - U \) is called the Lagrangian function of the system.

Applying Theorem (4.10) to a conservative system, we obtain:

Theorem 4.12. (Hamilton’s Principle). Let \((M, T^2, dU)\) be a conservative mechanical system, \( D \) the corresponding second order differential equation and \( L \) the lagrangian function. For each infinitesimal variation \( \delta \), we have

\[
D \langle \theta, \delta \rangle = \delta L .
\]
The classical integral version of Hamilton’s principle follows from (4.12) as we show now.

Let \( c: [t_0, t_1] \rightarrow M \) be a parameterized curve such that its canonical lift to \( TM \) is a solution of \( D \). Let \( v \) be a tangent field on \( M \) vanishing at the points \( c(t_0) \) and \( c(t_1) \); let \( \delta_v \) be the prolongation of \( v \).

Along the lifting of \( c \) to \( TM \), \( D \) is \( \frac{d}{dt} \). Then, by integrating (45) and using that \( \langle \theta, \delta_v \rangle = \langle \theta, v \rangle \) vanishes at \( t_0 \) and \( t_1 \), we find

\[
\int_{t_0}^{t_1} (\delta_v L) \, dt = 0 ,
\]

which is the classical integral form of Hamilton’s principle.

Newton equation (27) is the “Euler-Lagrange” system for the variational principle (46).

Remarks 4.13. (About constrained systems). Let \((M, T_2, \alpha, \Lambda)\) be a mechanical system with linear constraints (3.7). Let \( L \subset TM \) be the linear bundle of admissible states, (3.3). A vector field \( v \) on \( M \) is an admissible virtual displacement when \( v_a \in L \) for each \( a \in M \); then, the infinitesimal variation \( \delta_v \) is an admissible infinitesimal variation.

Such a \( \delta_v \) is not tangent to \( L \) in general. The local uniparametric group generated by \( \delta_v \) may transform curves in \( L \) into curves out of \( L \), i.e. kinematically possible paths into kinematically impossible paths.

Let \( \overline{D} \) be the second order differential equation corresponding to \((M, T_2, \alpha, \Lambda)\). According (4.2), there exists a 1-form \( \beta \) in \( \Lambda \) such that

\[
i_{\overline{D}} \omega_2 + dT + \alpha + \beta = 0 .
\]

Then (44) gives

\[
\overline{D}(\theta, \delta) = \delta T - \langle \alpha, \delta \rangle - \langle \beta, \delta \rangle
\]

for each infinitesimal variation \( \delta \). When \( \delta = \delta_v \) for an admissible virtual displacement \( v \), we have \( \langle \beta, \delta \rangle = \langle \beta, v \rangle = 0 \). When, on top of the above, the system is also conservative \((\alpha = dU)\), we obtain

\[
\overline{D}(\theta, \delta) = \delta_v L ,
\]

and by the same argument given for (46), we have

\[
\int_{t_0}^{t_1} (\delta_v L) \, dt = 0
\]

when we integrate along a curve \( c \) solution of \( \overline{D} \) and \( v \) is any admissible infinitesimal displacement null at \( c(t_0), c(t_1) \).

But now, we cannot in general reach all the kinematically possible paths close to the given \( c \) by means of deformations along solutions of admissible \( \delta_v \)'s. So, we cannot
derive the extremality of real trajectories among all kinematically possible paths. See e.g. Whittaker [8], pag. 250, for a discussion.

In the case of holonomic constraints, \( \Lambda \) admits local basis of exact 1-forms \( \{ dB_1, \ldots, dB_r \} \). When \( v \) is an admissible virtual displacement, we have \( v(B_k) = 0 \) and, from Euler commutation formula, \( \delta_v \dot{B}_k = 0 \). Then, the local equations for \( \mathcal{L} \) in \( TM: B_k = \text{const}, \dot{B}_k = 0 \) \( (k = 1, \ldots, r) \) are preserved by \( \delta_v \). So each admissible infinitesimal variation is tangent to \( \mathcal{L} \); and, indeed, tangent to the tangent bundle \( TN \) for each one of the solutions \( N \) of the distribution \( \mathcal{L} \). In this way, Hamilton’s variational principle specializes, like the Lagrange equations, in the case of holonomic constraints.

5. Time constraints

**Definition 5.1. (Time constraint).** A time constraint for \( M \) is a connected closed hypersurface \( \mathcal{U} \) of \( TM \), non intersecting the 0-section, projecting regularly onto \( M \) and such that the class of time is locally represented in \( \mathcal{U} \) by exact differential forms.

**Remarks 5.2.** The last condition means that for each \( v_a \in \mathcal{U} \) there exists an open neighborhood \( \mathcal{U}_1 \) of \( v_a \) in \( \mathcal{U} \) and an exact 1-form \( \tau_1 \) on \( \mathcal{U}_1 \) that belongs to the specialization to \( \mathcal{U}_1 \) of the class of time.

Let \( f \in C^\infty(\mathcal{U}_1) \) be such that \( \tau_1 = df \). Since the forms in the class of time are horizontal, so is \( df \). Therefore, by restricting \( \mathcal{U}_1 \) to a smaller neighborhood of \( v_a \) if necessary, \( f \) is the lift to \( \mathcal{U}_1 \) of a function on \( M \). So the local representatives of the class of time in \( \mathcal{U} \) are differentials of functions on \( M \) (lifted to \( TM \)).

**Lemma 5.3.** Let \( \mathcal{U} \) be a locally closed submanifold of \( TM \), \( \dim \mathcal{U} = n + r \) \( (r \geq 1) \), projecting regularly onto an open set of \( M \). For each \( v_a \in \mathcal{U} \) there exists a neighborhood of \( v_a \) in \( TM \) and \( r \) linearly independent tangent vector fields on this neighborhood, which are second order differential equations tangent to \( \mathcal{U} \).

**Proof.** Let \( (q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n) \) be local coordinates on a neighborhood of \( v_a \) in \( TM \). The projection of \( \mathcal{U} \) to \( M \) being regular, and reordering the coordinates if necessary, we can find local equations for \( \mathcal{U} \) of the form:

\[
\dot{q}^{r+k} = F^{r+k}(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^r) \quad (k = 1, \ldots, n - r).
\]

Then, the fields

\[
D_h = \dot{q}^i \frac{\partial}{\partial q^i} + \frac{\partial}{\partial \dot{q}^h} + \sum_{k=1}^{n-r} \left( \dot{q}^i \frac{\partial}{\partial q^i} + \frac{\partial}{\partial \dot{q}^k} \right) \frac{\partial}{\partial \dot{q}^{r+k}}, \quad (h = 1, \ldots, r)
\]

satisfie the requested conditions. \( \square \)
Lemma 5.4. Let \( \mathcal{U} \) be as in (5.3), and suppose that \( \mathcal{U} \) does not intersect \( 0 \)-section. Given \( f \in C^\infty(M) \), the differential \( df \) represents in \( \mathcal{U} \) the class of time if and only if \( \dot{f} = 1 \) identically on \( \mathcal{U} \).

Proof. When \( \dot{f} = 1 \) identically on \( \mathcal{U} \), we have \( df = \dot{f} \), so \( df \) represents the class of time.

Conversely, let \( df \) represent the class of time in \( \mathcal{U} \). From (5.3), for \( v_a \in \mathcal{U} \) there exists a second order differential equation \( D \) tangent to \( \mathcal{U} \) in a neighborhood of \( v_a \) in \( TM \). Because \( df \) belongs to the class of time in \( \mathcal{U} \), we have \( \langle df, D \rangle = 1 \) in a neighborhood of \( v_a \) in \( \mathcal{U} \). In particular, we get \( \langle df, v_a \rangle = 1 \) and \( \dot{f}(v_a) = 1 \).

Lemma 5.5. Assume \( n \geq 2 \). Let \( \mathcal{U} \) be a time constraint for \( M \). Then, for \( v_a \in \mathcal{U} \) there exists a neighborhood \( U(v_a) \) in \( TM \) and \( f \in C^\infty(M) \) such that the equation for \( \mathcal{U} \cap U(v_a) \) in \( U(v_a) \) is \( \dot{f} = 1 \).

Proof. From (5.2), there exists \( f \in C^\infty(M) \) such that \( df \) represents the class of time in a neighborhood of \( v_a \) in \( \mathcal{U} \). From (5.4) it follows that \( \dot{f} = 1 \) in this neighborhood. We have \( d_v \dot{f} \neq 0 \) because \( d_v f \neq 0 \). Thus \( \dot{f} = 1 \) is the local equation for an hypersurface of \( TM \) in a neighborhood of \( v_a \). This hypersurface has to coincide with \( \mathcal{U} \) in a neighborhood of \( v_a \).

Lemma 5.6. With the same conditions as in (5.5), let \( g \in C^\infty(M) \) be such that \( \dot{g} = 1 \) on a neighborhood of \( v_a \) in \( \mathcal{U} \). Then \( f \) and \( g \) differ by an additive constant on a neighborhood of \( a \) in \( M \).

Proof. Let \( q^1, \ldots, q^n \) be local coordinates for \( M \) on a neighborhood of \( a \), taken such that \( f = q^1 \). Then, in the neighborhood of \( v_a \) in \( TM \), where the local equation of \( \mathcal{U} \) is \( \dot{q}^1 = 1 \), \( \mathcal{U} \) consists of the tangent vectors to \( M \) of the form
\[
\left( \frac{\partial}{\partial q^1} \right)_b + \sum_{j=2}^{n} \lambda^j \left( \frac{\partial}{\partial q^j} \right)_b, \quad (\lambda^j \in \mathbb{R}).
\]
The equation \( \dot{g} = 1 \) on \( \mathcal{U} \cap U(v_a) \) gives
\[
\frac{\partial g}{\partial q^1}(b) + \sum_{j=2}^{n} \lambda^j \frac{\partial g}{\partial q^j}(b) = 1
\]
for arbitrary \( \lambda^j \in \mathbb{R} \) and \( b \) in a neighborhood of \( a \). This implies \( g = q^1 + \text{const} \) in this neighborhood.

Lemma 5.7. Let \( \mathcal{U} \) be a time constraint for \( M \), and \( n \geq 2 \). For each \( a \in M \), the fiber \( \mathcal{U}_a = \mathcal{U} \cap T_a M \) is an affine hypersurface of \( T_a M \).
Proof. As $U$ is a closed hypersurface of $TM$ that projects regularly onto $M$, each fiber $U_a$ is a closed hypersurface of $T_aM$. From Lemma (5.5), it follows that $U_a$ is locally affine, thus affine. □

**Theorem 5.8.** There exists a canonical one to one correspondence between closed 1-forms without zeros on $M$ and time constraints for $M$. The 1-form $\tau$ defines the time constraint $U$ with equation $\dot{\tau} = 1$ in $TM$, and for each time constraint $U$ there is a unique $\tau$ which defines $U$ by $\dot{\tau} = 1$.

Proof. First suppose that $n = \dim M \geq 2$. Let $U$ be a time constraint for $M$. Lemmas (5.5) and (5.7) show that for each $a \in M$ there exists a neighborhood $M_1$ of $a$ in $M$ and a function $f \in C^\infty(M)$ such that the equation of $U \cap TM_1$ in $TM_1$ is $\dot{f} = 1$. Lemma (5.6) shows that $df$ is uniquely determined by this condition (reducing $M_1$ if necessary). Patching together the $df$'s we obtain a closed 1-form $\tau$ on $M$ such that $U$ is defined in $TM$ by the equation $\dot{\tau} = 1$. And this $\tau$ is unique. Conversely, given $\tau$ closed, without zeros on $M$, the equation $\dot{\tau} = 1$ defines a time constraint for $M$.

Now, let $\dim M$ be 1. Then, a closed hypersurface $U$ of $TM$ which projects regularly onto $M$ and does not intersect the 0-section is the same that a vector field $v$ without zeros on $M$. The 1-form $\tau$ defined by $\langle \tau, v \rangle = 1$ defines $U$ by $\dot{\tau} = 1$.

**Remarks 5.9.** According to Theorem (5.8), when $M$ is compact and simply connected, there are no time constraints for $M$, because a closed 1-form on $M$ is exact, and necessarily has some zeros. In that case, time constraints can be considered for open submanifolds of $M$ and non-vanishing exact forms therein.

When $M$ is compact, there is no “time function”, i.e., a function $t \in C^\infty(M)$ such that $dt$ defines a time constraint, because $dt$ has always zeros. So, if $M$ is compact, “time has always periods”.

### 6. Modification caused by a time constraint in a free system

Let $(M, T_2, \alpha)$ be a free mechanical system, and $D$ the corresponding second order differential equation, so that

$$i_D\omega_2 + d\tau + \alpha = 0$$

Let $U \subset TM$ be the time constraint defined by $\dot{\tau} = 1$, where $\tau$ is a closed 1-form without zeros on $M$.

Let $\text{grad} \tau$ be the tangent field on $M$ such that $i_{\text{grad} \tau} T_2 = \tau$ (1.8), and $\text{Grad} \tau$ the vertical field canonically associated to $\text{grad} \tau$, so that (Lemma (1.1)) $i_{\text{Grad} \tau} \omega_2 = \tau$.

**Theorem 6.1.** When $\text{grad} \tau$ is non isotropic for $T_2$ at any point of $M$ (in particular when $T_2$ is positive definite), then there exists a tangent vector field $\overline{D}$ on $TM$, satisfying the conditions
(1) \( i_D \omega_2 + dT + \alpha \equiv 0 \; \text{mod}(\tau) \)

(2) \( \overline{D} \) is tangent to the hypersurface \( U \).

Such field is a second order differential equation and its restriction as tangent field to \( U \) is uniquely defined by the conditions (1), (2).

Proof. Let \( \overline{D} \) be defined by

\[
\overline{D} = D - \frac{D\dot{\tau}}{\langle \text{grad} \tau, \text{grad} \tau \rangle} \text{Grad} \tau,
\]

where \( \langle \text{grad} \tau, \text{grad} \tau \rangle \) is the scalar product defined by \( T_2 \). Immediately \( \overline{D} \) satisfies (1), because \( D \) satisfies (49).

Also \( \overline{D} \) satisfies (2) because we have

\[
(\text{Grad} \tau)(\dot{\tau}) = \langle \text{grad} \tau, \tau \rangle = \langle \text{grad} \tau, \text{grad} \tau \rangle,
\]

thus \( \overline{D}\dot{\tau} = 0 \).

Like \( D \), \( \overline{D} \) is a second order differential equation, because \( \text{Grad} \tau \) is vertical.

Any tangent field on \( TM \) that satisfies (1) should have the form \( D + \lambda \text{Grad} \tau \) (\( \lambda \in \mathcal{C}^\infty(TM) \)), and the condition (2) determines uniquely the value of \( \lambda \) on \( U \). That proves the uniqueness of \( \overline{D} \) on \( U \). \( \square \)

Indeed the same proof shows:

**Theorem 6.2.** When \( \text{grad} \tau \) is non isotropic for \( T_2 \) at any point of \( M \) (in particular when \( T_2 \) is positive definite), there exists a tangent vector field \( \overline{D} \) on \( TM \), which satisfies the conditions

(1) \( i_D \omega_2 + dT + \alpha \equiv 0 \; \text{mod}(\tau) \)

(2) \( \overline{D} \) is tangent to each hypersurface \( \dot{\tau} = c \) (\( c \in \mathbb{R} \)) of \( TM \).

Such a field is a second order differential equation and is given by formula (50).

**Remark 6.3.** According to (6.2), the vector field \( \overline{D} \), when restricted to the hypersurface \( \dot{\tau} = c \) (\( c \in \mathbb{R} \)) gives the evolution equations of the mechanical system with time constraint defined by \( \frac{1}{c} \dot{\tau} \), when \( c \neq 0 \), or holonomic constraint \( \dot{\tau} = 0 \) in the limit case \( c = 0 \). In this sense, ordinary holonomic constraints result from “freezing the evolution of the system with respect to \( \tau \”).

**An elementary example.** Let \( M = \mathbb{R}^3 \), \( T_2 = dx^2 + dy^2 + dz^2 \), \( \alpha = 0 \) and the time constraint \( \dot{r} = 1 \) (\( r = \sqrt{x^2 + y^2 + z^2} \)).

**Physical Interpretation:** a moving point is constrained to be at a distance \( r = t \) from the origin, at each instant \( t \), with no other external force acting upon it.
We have
\[\text{grad} \, r = \frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} + \frac{z}{r} \frac{\partial}{\partial z}, \quad \|\text{grad} \, r\| = 1.\]

Thus, the geodesic field, modified by the time constraint is
\[\overline{D} = D - (Dr)\text{Grad} \, r, \quad \text{where} \quad D = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z},\]
(the geodesic field for the Euclidean metric on \(\mathbb{R}\)).

A direct calculation gives
\[\overline{D} = D + \left[\frac{(x\dot{x} + y\dot{y} + z\dot{z})^2}{r^4} - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{r^2}\right] \left(x \frac{\partial}{\partial \dot{x}} + y \frac{\partial}{\partial \dot{y}} + z \frac{\partial}{\partial \dot{z}}\right),\]
\(\overline{D}\) is tangent to the manifolds \(\dot{r} = \text{const}\) in \(T\mathbb{R}^3\).

i) On \(\dot{r} = 0\) (ordinary holonomic constraint \(r = \text{const}\)), we have
\[\overline{D}_0 = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{r^2} \left(x \frac{\partial}{\partial \dot{x}} + y \frac{\partial}{\partial \dot{y}} + z \frac{\partial}{\partial \dot{z}}\right),\]
which is the geodesic field when restricted to each \(TS \subset T\mathbb{R}^3, S = \text{sphere of radius } r_0\) with the center at the origin.

In the submanifold \(\dot{r} = 0\) of \(T\mathbb{R}^3\), to which \(\overline{D}_0\) is tangent, the functions \(r\) and \(v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\) are first integrals for \(\overline{D}_0\), and the system of differential equations defined by \(\overline{D}_0\) is
\[\frac{d^2x}{dt^2} = -\left(\frac{v}{r}\right)^2 \dot{x}, \quad \frac{d^2y}{dt^2} = -\left(\frac{v}{r}\right)^2 \dot{y}, \quad \frac{d^2z}{dt^2} = -\left(\frac{v}{r}\right)^2 \dot{z},\]
which correspond to the classical centripetal force which generates the motion along geodesics of the sphere.

ii) On \(\dot{r} = 1\) we have
\[\overline{D}_1 = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} + \frac{1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{r^2} \left(x \frac{\partial}{\partial \dot{x}} + y \frac{\partial}{\partial \dot{y}} + z \frac{\partial}{\partial \dot{z}}\right),\]
that corresponds to a force depending on position and velocity, which points to or out of the origin according to the value of the velocity.

**Remark 6.4.** Later on, we show that, in local coordinates, (50) and \(\dot{\tau} = 1\) give the differential equations for a mechanical system with kinetic energy and force depending on time, as they are found in the classical literature, e.g. Prange [5], page 556. These equations are usually derived from the “Zentralgleichung” in (44) when applied to arbitrary \(\delta\)'s that keep fixed the time, i.e. such that \(\langle \tau, \delta \rangle = 0\). But, apparently, there appears to be no mathematical justification for this choice of \(\delta\)'s, although some authors try to give a physical one; e.g. Sommerfeld [6], page 65, Nordheim [4], pages...
48, 49. In that follows, we give a precise mathematical justification, starting from the consideration of time as a constraint.

**Lemma 6.5.** The prolongation \( \delta_v \) of a vector field \( v \) on \( M \) to \( TM \) is tangent to a time constraint \( U \subset TM \) defined by \( \dot{\tau} = 1 \) if and only if \( \langle \tau, v \rangle \) is a constant function on \( M \).

**Proof.** The problem is local in \( M \), so we can take \( \tau = df \) for some \( f \in C^\infty(M) \).

Euler commutation formula gives

\[
\delta_v \dot{\tau} = \delta_v \dot{f} = \dot{d}(vf) = \dot{d}(\langle \tau, v \rangle).
\]

So, \( \delta_v \) tangent to \( U \) if and only if \( \dot{d}(\langle \tau, v \rangle) = 0 \) on \( U \).

Thus when \( \langle \tau, v \rangle \) is a constant function on \( M \), \( \delta_v \) annihilates \( \dot{\tau} \), and is tangent to \( U \).

Conversely, when \( \delta_v \) is tangent to \( U \), by taking in (51) the value of \( \dot{d} \) at each \( w_a \in U \), we obtain \( w_a \langle \tau, v \rangle = 0 \). But, for each \( a \in M \), the set consisting of \( w_a \in U \) is the affine submanifold of \( T_a M \) defined by \( \langle w_a, \tau \rangle = 1 \), and such submanifold generates the whole of \( T_a M \). Therefore, \( \langle \tau, v \rangle \) is constant on \( M \). \( \square \)

**Definition 6.6.** (Admissible infinitesimal displacement). A vector field \( v \) on \( M \) is an admissible infinitesimal displacement for the time constraint \( U \) when \( \delta_v \) is tangent to \( U \). In that case, \( \delta_v \) is called an admissible infinitesimal variation.

From Lemma (6.5) it follows

**Theorem 6.7.** Let \((v)\) be the \( C^\infty(M)\)-module generated by the vector field \( v \) on \( M \). The necessary and sufficient condition for every field in \((v)\) to be an admissible infinitesimal displacement for the time constraint \( U \) (\( \dot{\tau} = 1 \)) is \( \langle \tau, v \rangle = 0 \).

For variational problems with fixed end points, the admissible infinitesimal displacements has to admit multiplication by arbitrary functions on \( M \). Thus, Theorem (6.7) offers a precise mathematical justification for the procedure of “keeping fixed the time”. *Time must be considered as a constraint.*

### 6.8. Local expressions

Let us take local coordinates on \( M \), \( (q^0, q^1, \ldots, q^m) (n = m + 1) \) with time constraint \( \dot{q}^0 = 1 \); so, \( q^0 \) is time. We have \( \text{grad} \, \tau = \text{grad} \, q^0 \), and \( \|\text{grad} \, \tau\|^2 = g^{00} \) (coefficient of the dual metric). The differential equations for \((M, T_2, \alpha)\) with no time constraint, with \( \alpha = A_i dq^i \), are

\[
g_{ij} \ddot{q}^j + \Gamma_{hk,i} \dot{q}^h \dot{q}^k + A_i = 0 \quad (i = 0, 1, \ldots, m).
\]
When we impose the constraint $\dot{q}^0 = 1$, the field in (52) changes to $D$, given by (50), to which corresponds a force form $\alpha$ that satisfies
\[
\alpha \equiv \alpha \mod (dq^0).
\]

Therefore changing $D$ by $\overline{D}$ does not modify Equations (52) for indexes $i \neq 0$ (the “spatial indexes”). By separating spatial indexes $\mu = 1, \ldots, m$ from the temporal one $0$, and putting $\dot{q}^0 = 1$, we obtain the differential equations for the system with $q^0$ as time:

\[
\begin{cases}
\dot{q}^0 = 1 \\
g_{\mu\nu} \ddot{q}^\nu + \Gamma_{\sigma\nu,\mu} \dot{q}^\sigma \dot{q}^\nu + 2 \Gamma_{\nu 0,\mu} \dot{q}^\nu + \Gamma_{00,\mu} + A_\mu = 0,
\end{cases}
\]

see e.g. Prange [5], page 556, Eq. (64.a).

The “contravariant” form of the equations results from writing (50) in coordinates. The equations are

\[
\begin{cases}
\dot{q}^0 = 1 \\
\ddot{q}^\mu + \Gamma^\mu_{\sigma\tau} \dot{q}^\sigma \dot{q}^\tau + 2 \Gamma^\mu_{\rho 0} \dot{q}^\rho + \Gamma^\mu_{00} + (A^\mu + \frac{Dq^0}{g_{00}} g^0{}^\mu) = 0.
\end{cases}
\]

Notice that the difference with the equations given by Prange [5], page 556, Eq. (64.b) is only formal. In Prange’s equations the translation from subindices to superindices is done by using only the “spatial” matrix $(g_{\mu\nu})$ instead of the whole matrix $(g_{ij})$ that we use.

When we use as time $\frac{1}{c} q^0$ instead of $q^0$, the equations in the constrained manifold $\dot{q}^0 = c$ are

\[
\begin{cases}
\dot{q}^0 = c \\
g_{\mu\nu} \ddot{q}^\nu + \Gamma_{\sigma\nu,\mu} \dot{q}^\sigma \dot{q}^\nu + 2c \Gamma_{\nu 0,\mu} \dot{q}^\nu + c^2 \Gamma_{00,\mu} + A_\mu = 0.
\end{cases}
\]

When $c = 1$, these give (53). And when $c = 0$, (55) are the equations for the holonomic constrained system $(M, T_2, \alpha, q^0 = \text{const})$.

7. Linear constraints depending on time

**Definition 7.1. (Linear Constraints Depending on Time).** Let $(M, T_2, \alpha)$ be a mechanical system. A system of linear constraints depending on time is a Pfaff system $\Lambda_M$ on $M$ together with a closed 1-form $\tau$, without zeros on $M$, and such that, for each $a \in M$, $\tau_a \notin (\Lambda_M)_a$.

A mechanical system with this type of constraints is denoted by $(M, T_2, \alpha, \Lambda_M, \tau)$.

**Definition 7.2. (Admissible State).** An admissible state for the constrained mechanical system $(M, T_2, \alpha, \Lambda_M, \tau)$ is a point $v_a \in TM$ such that, for each $\beta_a \in (\Lambda_M)_a$, we have $\langle \beta_a, v_a \rangle = 0$, and $\langle \tau_a, v_a \rangle = 1$. 
The set consisting of all the admissible states is a submanifold \( V \subset TM \), where \( L \) is the vector distribution annihilated by \( \Lambda_M \), and \( U \) is the time constraint defined by \( \dot{\tau} = 1 \). For each \( a \in M \), the fiber \( V_a \) is the affine hypersurface of \( L_a \) defined by equation \( \langle \tau_a, v_a \rangle = 1 \).

**Theorem 7.3. (Newton-Lagrange Law).** Let \((M, T_2, \alpha, \Lambda_M, \tau)\) be a mechanical system with constraints depending on time, and let \( T_2 \) be positive definite. There exists a vector field \( \overline{D} \) on \( TM \), which is a second order differential equation, satisfying the congruence
\[
i_{\overline{D}}\omega_2 + dT + \alpha \equiv 0 \mod (\Lambda_M, \tau)
\]
and the Principle of Virtual Works:
\[
\overline{D} \text{ is tangent to the manifold } V \text{ of admissible states.}
\]
The restriction of \( \overline{D} \) to \( V \) is uniquely defined by these conditions.

**Proof.** Denote by \( D \) the second order differential equation corresponding to the free system \((M, T_2, \alpha)\) such that
\[
i_D\omega_2 + dT + \alpha = 0.
\]

Let \( U \) be a coordinate open set in \( M \) such that \( \Lambda_M \) admits on \( U \) a basis \( \{\beta_1, \ldots, \beta_r\} \); denote \( \tau = \beta_0 \), and \( v_k = \text{grad} \beta_k \). Let \( V_k \) be the vertical representative of \( v_k \), so that \( i_{V_k}\omega_2 = \beta_k (k = 0, 1, \ldots, r) \). Similarly as in Theorem (3.8), we can find functions \( \lambda_k \) on \( TM \) (the “Lagrange multipliers”) such that the vector field \( \overline{D} \) defined on \( TU \) by
\[
\overline{D} = D + \sum_{k=0}^{r} \lambda^k V_k
\]
satisfies \( \overline{D}\beta_0 = \overline{D}\beta_1 = \cdots = \overline{D}\beta_r = 0 \) on \( U \).

\( \overline{D} \) is a second order differential equation and satisfies both \( (56) \) and \( (57) \) on \( U \). The field \( \overline{D} \) built in this way depends on the choice of the local basis \( \{\beta_1, \ldots, \beta_r\} \) for \( \Lambda_M \), but its restriction to the submanifold \( V \cap TU \) does not. By using an appropriate partition of the unity in \( M \) (lifted to \( TM \)) we find a \( \overline{D} \) fulfilling \( (56) \) and \( (57) \) and uniquely defined on \( V \) by these conditions.

**7.4. Holonomic constraints depending on time.**

Suppose \( \Lambda_M \) to be completely integrable. Let \( U \) be as in the proof of (7.3) and \( \beta_k = d\beta_k (k = 0, 1, \ldots, r) \).

The equations for \( V \cap TU \) in \( TU \) are
\[
\dot{B}_0 = 1, \dot{B}_1 = \cdots = \dot{B}_r = 0
\]
The field $\mathcal{D}$ in (7.3) satisfies

$$\mathcal{D} \dot{B}_0 = \mathcal{D} \dot{B}_1 = \cdots = \mathcal{D} \dot{B}_r = 0 \quad \text{on } V \cap TU. \quad (61)$$

Let $N \subset U$ a solution of $\Lambda_M$, given by equations

$$B_1 = b_1, \ldots, B_r = b_r \quad (b_k \in \mathbb{R}) \quad (62)$$

The equations for $V \cap TN$ are (62) and (60). Applying $\mathcal{D}$ to equations (62) we obtain equalities on $V \cap TN$ because of (60); and applying $\mathcal{D}$ to (60) also yields equalities on $V \cap TN$ because of (61). Therefore, $\mathcal{D}$ is tangent to $V \cap TN$.

For this reason, in the Newton-Lagrange equation (56) we can specialize to the submanifold $V \cap TN$ and obtain

$$i_{\mathcal{D}_N} \omega_{2N} + dT_N + \alpha_N \equiv 0 \mod (\tau_N) \quad (63)$$

and

$$\mathcal{D} \dot{\tau}_N = 0 \quad (64).$$

Equations (63) and (64) show that $\mathcal{D}_N$ is the second order differential equation for the mechanical system $(N, T_{2N}, \alpha_N)$ with time constraint $\dot{\tau}_N = 1$. We have proved the following:

**Theorem 7.5.** Let $(M, T_2, \alpha)$ be a mechanical system with $T_2$ positive definite. Let $\Lambda_M$ be a completely integrable Pfaff system on $M$. Let $\tau$ be a closed 1-form without zeros on $M$, $\dot{\tau} = 1$ the time constraint defined by $\tau$. Let $\mathcal{D}$ be the second order differential equation corresponding to the system $(M, T_2, \alpha, \Lambda_M, \tau)$. Then, for each submanifold $N \subset M$ solution of $\Lambda_M$, $\mathcal{D}$ is tangent to the time constraint $\dot{\tau}_N = 1$ in $TN$, and the restriction of $\mathcal{D}$ to this submanifold is the second order differential equation corresponding to $(N, T_{2N}, \alpha_N, \tau_N)$ by the Newton-Lagrange law (6.1).

**Remark 7.6.** Theorem (7.5) means that time constraints specialize properly in the presence of holonomic constraints: we can introduce the time constraint after specializing $T_2$ and $\alpha$ to the submanifolds solutions of $\Lambda_M$.

**Remark 7.7.** It is clear that the precedent methods can be applied in some cases in which $T_2$ is not positive definite, under appropriate hypothesis (e.g. when grad $\tau$ is a field of the distribution $\mathcal{L}$ and the restrictions of $T_2$ to $\mathcal{L}$ and to the orthogonal complement of $\mathcal{L}$ are positive or negative definite).

8. Reference frames and inertial forces

8.1. General principles.
In the context of Lagrangian Mechanics it is natural to consider as reference frame of a manifold $\mathcal{M}$ a manifold isomorphism $\varphi: \mathcal{R} \rightarrow \mathcal{M}$ that transports mechanical structures from $\mathcal{M}$ to $\mathcal{R}$, where we “read them”. When $\mathcal{R}$ and $\mathcal{M}$ are provided with pseudo-Riemannian metrics, the geodesic field of $\mathcal{M}$, once transported to $\mathcal{R}$, is a second order differential equation that, by subtraction of the geodesic field of $\mathcal{R}$, gives a vertical field on $T\mathcal{R}$: the inertial force caused by $\varphi$. The most relevant dynamical elements appear when $\mathcal{R}$ and $\mathcal{M}$ are endowed with time constraints which correspond to each other by $\varphi$.

Uniparametric automorphisms groups of a given manifold $M$ are a particular case when we consider one such group as an automorphism of the manifold $\mathcal{M} = \mathbb{R} \times M$. Classical examples are uniparametric groups of isometries of $\mathbb{R}^3$, which generate centrifugal and Coriolis forces. The same method allow us to deal with more general groups, like the one of dilatations that are apparently ignored in the literature.

**Example 8.2. (Inertial forces caused by automorphism groups)**

Let $\varphi: \mathcal{R} \rightarrow \mathcal{M}$ be an isomorphism of manifolds. Let $T_2$ be a metric on $\mathcal{M}$, and $\theta$ the corresponding Liouville form (see (1.9)). It is easy to prove that $\varphi^*\theta$ is the Liouville form for $\varphi^*T_2$ on $\mathcal{R}$; thus, if $\omega_2 = d\theta$, $\varphi^*\omega_2$ is the 2-form associated to $\varphi^*T_2$ on $\mathcal{R}$.

In particular, let $T_2$ be non degenerate (i.e., a pseudo-Riemannian metric) and $\varphi$ be an isomorphism. Then, $\varphi_*$ applies the second order differential equation $D$ into $\overline{D}$, related by:

$$i_D(\varphi^*\omega_2) + d(\varphi^*T) + \varphi^*(\overline{\alpha}) = 0$$

when

$$i_{\overline{D}}\omega_2 + dT + \overline{\alpha} = 0 .$$

When $\overline{\alpha} = 0$, we observe that the geodesic field for $\varphi^*T_2$ on $\mathcal{R}$ is transformed into the geodesic field for $T_2$ on $\mathcal{M}$. When $\mathcal{R}$ is also equipped with a metric, the vertical tangent field on $T\mathcal{R}$, difference between the geodesic field for $\varphi^*T_2$ and the one of the metric given on $\mathcal{R}$, is the inertial force produced by $\varphi$.

Let us consider the case $\mathcal{R} = \mathcal{M} = \mathbb{R} \times M$, where $M$ is an $n$-dimensional manifold. And let $\varphi: \mathbb{R} \times M \rightarrow \mathbb{R} \times M$ an uniparametric automorphism group:

$$\varphi(t, a) = (t, \varphi_t(a)) ,$$

where, for each $t \in \mathbb{R}$, $\varphi_t: M \rightarrow M$ and $\varphi_0 = \text{Id}$, $\varphi_t \circ \varphi_s = \varphi_{t+s}$, as usual. Denote by $u$ the infinitesimal generator of $\varphi$.

Given a metric $T_2$ on $M$, we endow $\mathbb{R} \times M$ with the metric $\tilde{T}_2 = dt^2 \oplus T_2$. We give a formula for $\varphi^*\tilde{T}_2$. 
We have $\varphi_*(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} + u$ (we use the product structure on $\mathbb{R} \times M$ to extend $u$ from $M$ to $\mathbb{R} \times M$, and $\frac{\partial}{\partial t}$ from $\mathbb{R}$ to $\mathbb{R} \times M$). Thus

\[
\left(\varphi^*\tilde{T}_2\right)_{(t,a)}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 1 + \|u\|^2_{\varphi_1(a)}.
\]

For a field $v$ on $M$,

\[
\left(\varphi^*\tilde{T}_2\right)_{(t,a)}\left(\frac{\partial}{\partial t}, v\right) = \tilde{T}_2\left(\frac{\partial}{\partial t}, u, \varphi_*v\right) = T_2(u_{\varphi_1(a)}, \varphi_{t*}v) = \varphi^*_t T_2(u_a, v_a)
\]

\[
= \langle i_{u_a}(\varphi^*_t T_2), v_a \rangle = (dt \otimes i_{u_a}(\varphi^*_t T_2))\left(\frac{\partial}{\partial t}, v_a\right).
\]

For two fields $v, w$ on $M$:

\[
\left(\varphi^*\tilde{T}_2\right)_{(t,a)}(v, w) = \tilde{T}_2(\varphi_{t*}v_a, \varphi_{t*}w_a) = \left(\varphi^*_t \tilde{T}_2\right)_a(v, w).
\]

We obtain:

\[
\left(\varphi^*\tilde{T}_2\right)_{(t,a)} = (1 + \|u\|^2_{\varphi_1(a)}) dt^2 + 2 dt i_{u_a}(\varphi^*_t T_2) + (\varphi^*_t T_2)_a.
\]

To illustrate with simple numerical examples, let us take $M = \mathbb{R}^2$, $T_2 = dx^2 + dy^2$; then $\tilde{T}_2 = dt^2 + dx^2 + dy^2$. And the most classical groups:

**Translations:** $\varphi: (t, x, y) \mapsto (t, x + t, y)$.

In this case, we have $u = \partial/\partial x$ and

\[
\varphi^*\tilde{T}_2 = 2 dt^2 + 2 dt dx + dx^2 + dy^2.
\]

**Rotations:** $\varphi: (t, x, y) \mapsto (t, x \cos t - y \sin t, x \sin t + y \cos t)$.

\[
u = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad \|u\|^2 = x^2 + y^2, \quad \varphi^*_t T_2 = T_2,
\]

thus, from (65), we obtain

\[
\varphi^*\tilde{T}_2 = (1 + x^2 + y^2) dt^2 + 2 dt(-ydx + xdy) + dx^2 + dy^2,
\]

that can also be calculated directly.

**Dilatations:** $\varphi: (t, x, y) \mapsto (t, e^t x, e^t y)$,

\[
u = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad \|u\|^2 = x^2 + y^2, \quad \varphi^*_t T_2 = e^{2t} T_2,
\]
\[ \|u\|_{v_t(x,y)}^2 = e^{2t}(x^2 + y^2), \quad i_u(\varphi^*T_2) = e^{2t}(x \, dx + y \, dy), \]

thus

\[ \varphi^*\hat{T}_2 = [1 + e^{2t}(x^2 + y^2)] \, dt^2 + 2 e^{2t} \, dt \, dx \, dy + e^{2t}(dx^2 + dy^2). \]  

We can use these formulas in order to calculate the inertial forces associated to each of these groups. In each case, we consider in \( \mathbb{R} \times M \) the time constraint \( \dot{t} = 1 \) (\( \mathbb{R} \) is the clock for the system). The geodesic field for the given metric \( \hat{T}_2 \) is \( D_G = i \partial/\partial t + \dot{x} \partial/\partial x + \dot{y} \partial/\partial y \), which is tangent to the given constraint.

**Translations.** The transformed metric \( \varphi^*\hat{T}_2 \) has constant coefficients. Then all Christoffel symbols are zero, and the geodesic field for \( \varphi^*\hat{T}_2 \) is the same as the one for \( \hat{T}_2 \). The inertial force is 0.

**Rotations.** A routine tedious computation for the field \( D \) such that

\[ i_D(\varphi^*\tilde{\omega}_2) + d(\varphi^*\hat{T}) = 0 \]

(\( D \) transforms by \( \varphi_* \) to the geodesic field on \( T(\mathbb{R} \times \mathbb{R}^2) \)) gives

\[ D = i \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + (x \dot{t}^2 + 2 i \dot{y}) \frac{\partial}{\partial \dot{x}} + (y \dot{t}^2 - 2 i \dot{x}) \frac{\partial}{\partial \dot{y}} \]

which, in the locus of the time constraint \( \dot{t} = 1 \) is

\[ D|_{\dot{t}=1} = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + (x + 2 \dot{y}) \frac{\partial}{\partial \dot{x}} + (y - 2 \dot{x}) \frac{\partial}{\partial \dot{y}}. \]

The difference with the geodesic field is the inertial force corresponding to the group of rotations. The differential equations for this force are

\[ \frac{d^2x}{dt^2} = x + 2 \dot{y}, \quad \frac{d^2y}{dt^2} = y - 2 \dot{x}, \]

where we observe the sum of a centrifugal and a Coriolis force.

**Dilatations.** A similar computation gives for the field \( D \) which transforms by \( \varphi_* \) into the geodesic field, when restricted to time constraint \( \dot{t} = 1 \):

\[ D|_{\dot{t}=1} = D_G - (x + 2 \dot{x}) \frac{\partial}{\partial x} - (y + 2 \dot{y}) \frac{\partial}{\partial y}. \]

To which corresponds the differential equations for the inertial force caused by \( \varphi \):

\[ \frac{d^2x}{dt^2} = -(x + 2 \dot{x}), \quad \frac{d^2y}{dt^2} = -(y + 2 \dot{y}). \]

We do not give a physical interpretation of this formal result.
Definition 8.3. (Inertial frames). Let \( R, M \) be pseudo-Riemannian manifolds. A reference frame \( \varphi: R \to M \) is called inertial when the force caused by \( \varphi \) is zero. This applies in particular to uniparametric automorphism groups.

Equivalent characterizations of inertial frames are: \( \varphi \) is inertial if and only if \( \varphi_* \) transforms the geodesic field of \( T R \) into the geodesic field of \( T M \). And also, if and only if it transforms one to another the Levi-Civita connections on \( R \) and \( M \).

Example 8.4. (Inertial groups on \( \mathbb{R}^n \)).

The metric we consider in \( \mathbb{R}^n \) is the usual Euclidean one. We prove that the only inertial uniparametric groups on \( \mathbb{R}^n \) are the translation ones.

Retaining the notations of (8.2), for \( \varphi \) to be inertial, \( \tilde{T}_2 \) and \( \varphi^* \tilde{T}_2 \) must define the same Levi-Civita connection on \( \mathbb{R} \times \mathbb{R}^n \). Thus, when written in cartesian coordinates, all the Christoffel symbols for \( \varphi^* \tilde{T}_2 \) must to be zero; and then, the \( g_{ij} \) for \( \varphi^* \tilde{T}_2 \) are constants. Looking at the last term in (65), this gives \( \varphi^*_0 T_2 = \varphi^*_0 T_2 = T_2 \). Then, since the coefficient in the second term in (65) are constant, we obtain that \( u \) has constant coefficients. We conclude that \( \varphi \) is the translation group generated in \( \mathbb{R}^n \) by \( u \), q.e.d.

8.5. Preservation of the equations of motion.

In the physical literature, inertial frames are frequently referred as those which “preserve the form of the equations of motion”, without defining a precise meaning to “preserve”. We treat this question in this section.

Let \( (R, T_2) \) and \( (M, \overline{T}_2) \) be pseudo-Riemannian manifolds, and \( \varphi: R \to M \) a reference frame (= manifold isomorphism). Let \( \overline{D} \) be a second order differential equation on \( T M \), and \( \overline{\alpha} \) its canonically associated 1-form:

\[
\begin{align*}
    i_{\overline{\overline{D}}} \overline{\omega}_2 + d\overline{T} + \overline{\alpha} &= 0
\end{align*}
\]

There are two different natural ways to define a second order differential equation on \( T R \) corresponding by \( \varphi \) to \( \overline{D} \):

a) The second order differential equation \( D \) on \( T R \) canonically associated to \( \varphi^*(\overline{\alpha}) \):

\[
\begin{align*}
    i_D \omega_2 + dT + \varphi^*(\overline{\alpha}) &= 0 .
\end{align*}
\]

b) The field \( D_1 \) on \( T R \) such that \( \varphi_* D_1 = \overline{D} \):

\[
\begin{align*}
    i_{D_1} (\varphi^* \omega_2) + d(\varphi^* T) + \varphi^*(\overline{\alpha}) &= 0 .
\end{align*}
\]
Classically, \( D_1 \) is the field obtained from \( \overline{D} \) by change of coordinates.

The difference \( D_1 - D \) is a vertical field on \( T\mathcal{R} \), i.e., a force; when we think of \( D_1 \) as \( \overline{D} \) "read in the coordinates of \( \mathcal{R} \), \( D_1 - D \) is the force caused by \( \varphi \) in the mechanical system \((\mathcal{M}, \overline{T}_2, \overline{\alpha})\). The frame may be inertial in the sense of Definition (8.3), but causes a non-zero force when \( \overline{\alpha} \neq 0 \), as shown in the following

**Proposition 8.6.** Let \((\mathcal{R}, T_2)\) and \((\mathcal{M}, \overline{T}_2)\) be pseudo-Riemannian manifolds, and \( \varphi: \mathcal{R} \to \mathcal{M} \) a reference frame. The force caused by \( \varphi \) in \((\mathcal{M}, \overline{T}_2, \overline{\alpha})\) is zero for all choices of \( \overline{\alpha} \), if and only if \( \varphi \) is an isometry.

**Proof.** The "if" part is obvious. For the converse, let us take \( \overline{\alpha} = 0 \); then, for the geodesic field \( D_G \) on \( T\mathcal{R} \), we have

\[
(76) \quad i_{D_G} \omega_2 + d\tau = 0, \quad i_{D_G}(\varphi^*\omega_2) + d(\varphi^*\overline{\tau}) = 0.
\]

Then, let \( \overline{\alpha} \) be any horizontal 1-form on \( TM \), and \( \alpha = \varphi^*(\overline{\alpha}) \); let \( D \) be given by (74), and define the vertical field \( V \) by \( D = D_G + V \), so that, from (74, 76),

\[
i_{\nu} \omega_2 + \alpha = 0.
\]

By taking the field \( D_1 \) in (75), the hypothesis \( D_1 = D \) gives, from (75, 76),

\[
i_{\nu}(\varphi^*\omega_2) + \alpha = 0.
\]

Thus \( i_{\nu} \omega_2 = i_{\nu}(\varphi^*\omega_2) \); then, the geometric representative \( \nu \) of \( V \) satisfies

\[
i_{\nu} T_2 = i_{\nu}(\varphi^*\overline{T}_2).
\]

Since \( \nu \) is arbitrary, this implies \( T_2 = \varphi^*\overline{T}_2 \). \( \square \)

Thus it is not true that the uniparametric groups of translations considered in (8.2) do not produce forces; they do, but these forces may be compensated by time constraints, as we will show.

**Definition 8.7. (Preservation of the equations of motion).** Let \((\mathcal{R}, T_2)\) and \((\mathcal{M}, \overline{T}_2)\) be pseudo-Riemannian manifolds with respective time constraints \( \tau = 1 \) and \( \overline{\tau} = 1 \). Let \( \varphi: \mathcal{R} \to \mathcal{M} \) be a reference frame. We say that \( \varphi \) preserves the equations of motion when \( \varphi^*(\tau) = \tau \) and, for each second order differential equation \( D \) on \( TM \) such that \( \overline{D} \tau = 0 \), and for the corresponding \( \overline{\alpha} \) by (73), and for the field \( D_1 \) on \( T\mathcal{R} \) such that \( \varphi^* D_1 = \overline{D} \), we have

\[
(77) \quad i_{D_1} \omega_2 + d\tau + \varphi^*(\overline{\alpha}) \equiv 0 \mod(\tau).
\]

\( D_1 \) is tangent to the time constraint \( \tau = 1 \), and the congruence (77) means that \( D_1 \) is the modification of \( D \) in (74) caused by the time constraint, in the sense of Theorem (6.1).
Theorem 8.8. An uniparametric automorphism group \( \varphi : \mathbb{R} \times M \to \mathbb{R} \times M \) preserves the equations of motion if and only if it is inertial and also each \( \varphi_t : M \to M \) is an isometry.

Proof. The problem reduces to prove that for an inertial \( \varphi \), the condition of preserving the equations of motion is equivalent to be an isometry group of \( M \).

For any uniparametric automorphism group \( \varphi \) the geodesic field \( D_G \) satisfies \( D_G \dot{t} = 0 \).

Let \( \overline{D} = D_G + V \) be a second order differential equation on \( T(\mathbb{R} \times M) \) such that \( \overline{D} \dot{t} = 0 \), i.e. \( \nabla \dot{t} = 0 \). We keep notations from (8.2) and define \( \tilde{\alpha} \) by

\[
i_{\overline{D}}\tilde{\omega}_2 + d\tilde{T} + \tilde{\alpha} = 0, \quad \text{or} \quad i_V\tilde{\omega}_2 + \tilde{\alpha} = 0.
\]

Let \( D_1 \) be the field defined by \( \varphi_*D_1 = \overline{D} \), and put \( D_1 = D_G + V_1 \). As \( \varphi \) is inertial, we have \( \varphi_*V_1 = V \), thus

\[
i_{V_1}(\varphi^*\tilde{\omega}_2) + \varphi^*(\tilde{\alpha}) = 0.
\]

Let \( D = D_G + V \) be defined by

\[
i_{D}\tilde{\omega}_2 + d\tilde{T} + \varphi^*(\tilde{\alpha}) = 0,
\]

thus

\[
i_{V}\tilde{\omega}_2 + \varphi^*(\tilde{\alpha}) = 0
\]

and from (78),

\[
i_{V}\tilde{\omega}_2 = i_{V_1}(\varphi^*\tilde{\omega}_2).
\]

The condition for \( \varphi \) to preserve the equations of motion is

\[
i_{D_1}(\tilde{\omega}_2) + d\tilde{T} + \varphi^*(\tilde{\alpha}) \equiv 0 \mod (dt)
\]

Subtracting (79) from (82), this condition is

\[
i_{V_1}(\tilde{\omega}_2) - i_{V}(\tilde{\omega}_2) \equiv 0 \mod (dt)
\]

or, from (81)

\[
i_{V_1}(\tilde{\omega}_2) \equiv i_{V_1}(\varphi^*\tilde{\omega}_2) \mod (dt).
\]

By taking geometric representatives, this condition is

\[
i_{v_1}(\tilde{T}_2) \equiv i_{v_1}(\varphi^*\tilde{T}_2) \mod (dt).
\]

Thus the necessary and sufficient condition for \( \varphi \) to preserve the equations of motion is that (83) holds for each tangent field \( v_1 \) on \( T(\mathbb{R} \times M) \) taking values in \( T(\mathbb{R} \times M) \) and satisfying \( v_1 t = 0 \) (this is \( V_1 \dot{t} = 0 \)).

Looking at (65) we see that this condition for \( \varphi \) is equivalent to have \( \varphi_t^*T_2 = T_2 \) for each \( t \in \mathbb{R} \). \( \square \)
Corollary 8.9. The uniparametric automorphism groups of $\mathbb{R}^n$ which preserve the equations of motion are only the inertial groups, i.e., the translation uniparametric groups.

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