Confinement in the $q$-state Potts field theory

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Abstract

The $q$-state Potts field theory describes the universality class associated to the spontaneous breaking of the permutation symmetry of $q$ colors. In two dimensions it is defined up to $q = 4$ and exhibits duality and integrability away from critical temperature in absence of magnetic field. We show how, when a magnetic field is switched on, it provides the simplest model of confinement allowing for both mesons and baryons. Deconfined quarks (kinks) exist in a phase bounded by a first order transition on one side, and a second order transition on the other. The evolution of the mass spectrum with temperature and magnetic field is discussed.
1 Introduction

Confinement is that property of quantum field theory for which excitations which are genuine asymptotic particles in a region of coupling space become unobservable in isolation in another region, where they leave the place to new asymptotic particles (mesons, baryons, ...) of which they can be seen as "constituents" (quarks). The case of two-dimensional space-time provides the framework in which this, as other general properties of quantum field theory, can be studied in their simplest form. It is known that the gauge theory setting in which confinement is ordinarily discussed in four dimensions becomes somehow redundant in two dimensions. Indeed, due to the absence of transverse spatial dimensions, massless gauge fields do not carry particle degrees of freedom in $d = 2$, so that an alternative description of the theory exists which relies on physical excitations only. For example, two-dimensional quantum electrodynamics can be exactly mapped onto the theory of a self-interacting neutral boson which makes quite transparent the presence of quark confinement [1, 2]. In absence of electromagnetic interaction the quarks correspond to the solitons interpolating between the vacua of a periodic bosonic potential. Quark interaction destroys the degeneracy of the bosonic vacua and removes the topologically charged excitations from the spectrum of asymptotic states. What remains is a spectrum of mesons originating from confinement of soliton-antisoliton pairs\(^1\).

This mechanism of confinement through breaking of degeneracy of discrete vacua is quite general in two dimensions and exhibits its most essential features in the case of a finite number of vacua originating from the spontaneous breaking of a discrete symmetry, with the kinks interpolating between these vacua playing the role of the quarks. Then it is not surprising that two-dimensional Ising field theory (i.e. the field theory describing the scaling limit of the two-dimensional Ising model) provides the simplest model of confinement (only two vacua). The associated mesonic spectrum was first studied in [3].

In theories, like Ising, with a one-component order parameter, the confined particles are made of an even number of quarks. Indeed, in this case the vacua are located along a line in order parameter space\(^2\), so that the kink sequences starting from and going back to the true vacuum (the only ones generating bound states via confinement) consist of a number $2j$ of kinks. The lightest bound states are mesons (kink-antikink composites) corresponding to $j = 1$.

On the other hand, when the order parameter has more than one component we can find three vacua located on a plane in order parameter space. Now we can have a three-kink sequence making a loop through the three vacua. Such a sequence is confined into baryons, a finite number of which must be stable sufficiently close to the deconfining point. Indeed, if $m$ is the mass of the kink, the lightest baryons will have mass $\sim 3m$ and will not be able to decay into two mesons with mass $\sim 2m$ each.

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\(^1\)We refer to the generic case in which the $\theta$-angle is not fine-tuned to the specific value which partially preserves vacuum degeneracy.

\(^2\)This applies also to two-dimensional quantum electrodynamics, with the notion of one-dimensional order parameter referred to the bosonic version. See [4] for a discussion of confinement in theories with more general bosonic potentials.
In this paper we consider the simplest model of confinement allowing for baryons, i.e. the field theory describing the scaling limit of the two-dimensional $q$-state Potts model. The latter generalizes the Ising model to the case in which each site on the lattice can take $q$ different colors, and has an order parameter with $q - 1$ components [5, 6]. In absence of magnetic field, the ferromagnetic $q$-state Potts model undergoes an ordering transition which is continuous up to a number of colors $q_c$, which in two dimensions equals 4 [7]. The scaling limit then can be taken up to this value of $q$ and produces a field theory in which a magnetic field confines the kinks and allows for baryons at $q = 3, 4$.

In absence of the confining field, the quarks (kinks) behave as free neutral fermions in the Ising case ($q = 2$), but become interacting for $q = 3, 4$. For $q = 4$ the zero-field theory is equivalent to the sine-Gordon model at a specific value of the coupling ($\beta^2 = 2\pi$, see e.g. [8]), while for $q = 3$ the ultraviolet fixed point is non-trivial. In any case the confining interaction is non-local with respect to the quarks. The possibility of a quantitative study in weak field comes from integrability of the $q$-state Potts field theory in zero-field [9]. Hence, form factor perturbation theory [10] allows to express mass corrections in terms of the matrix elements of the magnetic operator computed in [11].

Here, however, our main interest will be in a qualitative characterization of the evolution of the mass spectrum for generic values of the temperature and of a magnetic field chosen to act on a single color. For $q = 3, 4$ this choice allows for an extended phase on the parameter plane in which the quarks are deconfined. Such a phase is bounded by the confining (first order) transition on one side, and by a spontaneous breaking (second order) transition on the other. Outside this region, the spectrum of asymptotic particles is made of mesons (everywhere) and baryons (at least sufficiently close to the deconfining transition).

The paper is organized as follows. In the next section we discuss the $q$-state Potts model with our choice of magnetic field, starting with the lattice definition and then switching to the field theoretical description of the scaling limit. In section 3 we focus on the two-dimensional case and recall the exact scattering solution for the low-temperature phase in zero-field, before showing how this is related by duality to the scattering solution for the high-temperature phase. Section 4 is devoted to the weak field analysis, while the spectrum evolution as a function of temperature and magnetic field is discussed in section 5. Few final remarks are collected in section 6.

2 Potts model with magnetic field

In this section we discuss the lattice definition of the $q$-state Potts model and the field theoretical description of the scaling limit. Some remarks about $d > 2$, and about $q > 4$ in $d = 2$ are included although they are not used in the rest of the paper.
2.1 Lattice model

The $q$-state Potts model [5, 6] is a generalization of the Ising model in which each site variable $s(x)$ at site $x$ on the lattice can assume $q$ different values (colors). In absence of magnetic field the interaction only distinguishes whether nearest neighbor sites have equal or different color, so that the Hamiltonian is invariant under the group $S_q$ of permutation of the colors. If we add a magnetic field $H$ acting only on the sites with a specific color (say $s = q$), the reduced Hamiltonian can be written as

$$\mathcal{H} = -\frac{1}{T} \sum_{(x,y)} \delta_{s(x),s(y)} - H \sum_x \delta_{s(x),q},$$

and is invariant under the group $S_{q-1}$ of permutations of the first $q-1$ colors (the first sum is over nearest neighbors).

In the ferromagnetic case at $H = 0$, the $q$ configurations in which all the sites have the same color minimize the energy and the system exhibits spontaneous magnetisation for sufficiently low values of the temperature $T$. Above a critical temperature $T_c$ the thermal fluctuations become dominant and the system is in a disordered phase. If we introduce the variables

$$\sigma_\alpha(x) = \delta_{s(x),\alpha} - \frac{1}{q}, \quad \alpha = 1, 2, \ldots, q$$

satisfying the condition

$$\sum_{\alpha=1}^q \sigma_\alpha(x) = 0,$$

the expectation values $\langle \sigma_\alpha \rangle$ differ from zero only in the low-temperature phase and can be used as order parameters.

When the magnetic field is switched on with a positive value, the ground state at $T = 0$ is unique (all sites have color $q$), and there can be no phase transition as the temperature is increased.

Different is the situation for $H < 0$, $q > 2$. As $H \to -\infty$ the color $q$ becomes forbidden and a zero-field $(q-1)$-Potts model is obtained. Then the critical points at $H = 0$ and $H = -\infty$ are the endpoints of a phase transition line which in the $T-H$ plane separates a low-temperature, spontaneously magnetized phase with $q - 1$ degenerate ground states from a high-temperature, disordered phase (Fig. 1).

The nature of the transition at $T = T_c(H)$, $H \leq 0$, depends on $q$ and on the dimensionality $d$. It is well known (see [6]) that in the zero-field $q$-state Potts model there exists a value $q_c(d)$ (not necessarily integer$^3$) such that the transition is continuous for $q \leq q_c$ and first order for $q > q_c$. Accordingly, three cases can be distinguished for the transition going from $C_q$ to $C_{q-1}$ in Fig. 1:

$^3$One can make sense of the Potts model for non-integer values of $q$ through the mapping onto the random cluster model [12]. Although we will have mainly in mind integer values, most of our discussion can be done treating $q$ as a continuous parameter, and this will be understood in the following.
Figure 1: Phase diagram of the model (1) for $q > 2$. The values $H = 0$ and $H = -\infty$ correspond to the zero-field $q$- and $(q - 1)$-state Potts model, respectively. The ordered phase possesses $q - 1$ degenerate ground states and is separated from the disordered region by two phase transition lines (dashed and thick line). The dashed line is a first order transition line along which $q$ ground states are degenerate. The nature of the transition along the thick line depends on $q$ and $d$.

i) $q - 1 > q_c$. The transition is first order with $q$ phases coexisting along the transition line. Only at $C_q$ $q + 1$ phases coexist.

ii) $2 < q \leq q_c$. The transition is continuous.

iii) $q - 1 \leq q_c < q$. The correlation length is finite at $C_q$ and infinite at $C_{q-1}$. The nature of the transition depends on $d$.

The value $q_c (2)$ is exactly known to be 4 [7], while $q_c (3)$ lies in between 2 and 3. Hence, in $d = 2$ the transition induced by the field is continuous for $q = 3, 4$ and first order for $q > 5$; in $d = 3$ it is first order for $q \geq 4$. The cases $q = 5$ in $d = 2$ and $q = 3$ in $d = 3$ are of the type iii) above and will be discussed in a moment.

2.2 Field theory description

A continuous, field theoretical description at scales much larger than the lattice spacing is possible at and around those points of the phase diagram where the correlation length diverges. For $q \leq q_c$, the transition point at $H = 0$ ($C_q$ in Fig. 1) corresponds in the scaling limit to a fixed point of the renormalization group, i.e. to a conformal field theory with action $A_{CFT}^{(q)}$. The scaling limit of (1) around $C_q$ is described by the action

$$A = A_{CFT}^{(q)} - \tau \int d^d x \varepsilon(x) - h \int d^d x \sigma_q(x),$$

where $\varepsilon(x)$ is the leading $S_q$-invariant operator in $A_{CFT}^{(q)}$, and $\sigma_q(x)$ is the leading $S_{q-1}$-preserving magnetic operator. If $X_{\Phi}^{(q)}$ denotes the scaling dimension of an operator $\Phi(x)$ at the $S_q$-invariant fixed point and $m$ a mass scale, the couplings $\tau$ and $h$ behave dimensionally as

$$\tau \sim m^{d - X_{\varepsilon}^{(q)}}, \quad h \sim m^{d - X_{\sigma}^{(q)}},$$

(5)
Figure 2: Phase diagrams of the model (1) for $d = q = 3$ (a), and for $d = 2$, $q = 5$ (b). The correlation length is infinite along $PC_2$ and at $C_4$; $q$ phases coexist at generic points along the dashed lines, $q + 1$ at the points $C_3$ and $C_5$.

and measure the deviation from critical temperature and the magnetic field, respectively.

For $2 < q \leq q_c$ the transition along the line joining $C_q$ to $C_{q-1}$ in Fig. 1 is continuous and the scaling action (4) with $\tau = 0$, $h < 0$ describes a massless flow from $A^{(q)}_{CFT}$ to $A^{(q-1)}_{CFT}$. The scaling limit around the infrared fixed point $C_{q-1}$ is described by the action

$$A_{IR} = A^{(q-1)}_{CFT} - \tilde{\tau} \int d^d x \, \varepsilon(x) + \lambda \int d^d x \, \phi(x) + \ldots ,$$

where $\tilde{\tau} \sim m^{d - X_{\varepsilon}^{(q-1)}}$ is proportional to $T - T_c$ and $\lambda \sim m^{d - X_{\phi}^{(q-1)}}$ is proportional to $1/H$. All the operators in the r.h.s. of (6) are $S_{q-1}$-invariant: $\varepsilon$ is relevant, while $\phi$ is the most relevant of the infinitely many irrelevant operators (dots) which specify the massless flow at $\tau = 0$.

In $d = 3$ the condition $2 < q \leq q_c$ is not satisfied for integer values of $q$ and the transition at $H < 0$ is first order for $q > 3$. For $q = 3$ the correlation length is finite at $C_q$ and infinite at $C_{q-1}$. Since the latter is an Ising critical point for which $\varepsilon$ is the only symmetry-preserving relevant operator, $C_2$ must be an infrared fixed point whose scaling region is described by the action (6). The ultraviolet endpoint of the massless flow at $\tilde{\tau} = 0$ must be a fixed point $P$ located on the transition line (Fig. 2a). The nature of the transition then requires that $P$ is an Ising tricritical point\(^4\).

In $d = 2$ the field theoretical description can rely on a number of exact results. Baxter solved the zero-field $q$-state Potts model at $T = T_c$ on the square lattice and found that $q_c = 4$ [7]. The scaling limit of the critical point up to $q_c$ was later identified [13] to correspond to the conformal field theory with central charge [14]

$$c(q) = 1 - \frac{6}{t(t+1)} ,$$

where the parameter $t$ is related to $q$ by the formula

$$\sqrt{q} = 2 \sin \frac{\pi (t - 1)}{2(t + 1)}.$$\(^8\)

\(^4\)Since Ising tricriticality is described by a $\Phi^6$ Landau-Ginzburg potential for which $d = 3$ is the upper critical dimension, $P$ is a Gaussian fixed point.
The scaling dimensions of the leading thermal and magnetic operators coincide with those of the operators $\phi_{2,1}$ and $\phi_{(t-1)/2,(t+1)/2}$ in the conformal theory, and read [13, 15]

\[ X^{(q)}_{\varepsilon} = \frac{1}{2} \left( 1 + \frac{3}{t} \right), \quad X^{(q)}_{\sigma} = \frac{(t - 1)(t + 3)}{8t(t + 1)}. \] (9)

The action (4) with $d = 2$ describes the scaling region around $C_q$ for $q \leq 4$ and is known to correspond to an integrable quantum field theory for $h = 0$ [9] (see next section). For $\tau = 0$, $h < 0$ and $2 < q \leq 4$ it describes a massless flow between an ultraviolet fixed point with central charge $c(q)$ and an infrared fixed point with central charge $c(q - 1)$. Around this latter fixed point we can use the action (6) with the irrelevant operator $\phi$ identified with the operator $\phi_{3,1}$ of the conformal classification. Its scaling dimension at an $S_q$-invariant critical point is

\[ X^{(q)}_{\phi} = 2 \left( 1 + \frac{2}{t} \right). \] (10)

A field theory description around $C_{q-1}$ is still possible at $q = 5$. Since $X^{(4)}_{\phi} = 2$, $\phi$ is in this case a marginal operator and the scaling region around $C_4$ is described by the action

\[ A^{(4)}_{CFT} - \tau \int d^2 x \varepsilon(x) + \lambda \int d^2 x \phi(x). \] (11)

Since the two-dimensional 4-state Potts model does not admit a tricritical point [16, 17], $\phi$ acts as a marginally relevant perturbation and the transition is first order (Fig. 2b). The action (11) is integrable also for $\tau = 0$ [18, 19] and can be used to describe exactly the transition close to $C_4$.

3 Exact scaling solution in zero field

In the two-dimensional case, to which we restrict our attention from now on, the action (4) is integrable for $h = 0$ and the solution can be found in the form of an exact, elastic and factorized $S$-matrix for the relativistic particles of the associated $(1 + 1)$-dimensional theory [20]. The scattering theories above and below $T_c$ must describe two different physical situations and, at the same time, must reflect the existence of a duality transformation [5, 6] relating the ordered and disordered phases.

3.1 Ordered phase

The $S$-matrix in the case of spontaneously broken symmetry ($\tau < 0$) was determined [6] in [9]. The $q$ ferromagnetic ground states correspond in the field theory to degenerate vacua labelled

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5 A phase diagram analogous to that of Fig. 2a, with $P$ corresponding to a $(q - 1)$-Potts tricritical point, should be obtained for $4 < q < 5$. $P$ tends to $C_q$ as $q \to 4$ and to $C_{q-1}$ as $q \to 5$.

6 The first exact $S$-matrix for $\phi_{2,1}$-perturbed conformal field theories was determined in [21]. The relation between this solution, which does not exploit $S_q$ invariance and relies on a different particle basis, and the one of [9] is explained in [22].
by an index $\alpha = 1, 2, \ldots, q$. The elementary excitations are then provided by kinks\(^7\) $K_{\alpha \beta}(\theta)$ interpolating between the vacua $\alpha$ and $\beta$ ($\alpha \neq \beta$). The space of asymptotic states consists of multi-kink configurations of the type $K_{\alpha_0 \alpha_1}(\theta_1)K_{\alpha_1 \alpha_2}(\theta_2)\ldots K_{\alpha_{n-1} \alpha_n}(\theta_n)$ ($\alpha_i \neq \alpha_{i+1}$) interpolating between the vacua $\alpha_0$ and $\alpha_n$. As a consequence of $S_q$-invariance, all the $n$-kink states fall into two topological sectors: the neutral sector, corresponding to $\alpha_0 = \alpha_n$, and the charged sector, corresponding to $\alpha_0 \neq \alpha_n$.

Integrability implies that the scattering processes are completely elastic and factorised into the product of two-kink interactions. An outgoing two-kink state can only differ from the ingoing one by the vacuum state between the kinks. Hence, the two-kink scattering can formally be described through the Faddeev-Zamolodchikov commutation relation

$$K_{\alpha \gamma}(\theta_1)K_{\gamma \beta}(\theta_2) = \sum_{\delta \neq \alpha, \beta} S_{\alpha \beta}^{\gamma \delta}(\theta_{12})K_{\alpha \delta}(\theta_2)K_{\delta \beta}(\theta_1),$$

(12)

where $\theta_{12} \equiv \theta_1 - \theta_2$, and $S_{\alpha \beta}^{\gamma \delta}(\theta_{12})$ denote the two-body scattering amplitudes (Fig. 3a). $S_q$ invariance reduces to four the number of independent amplitudes, two for the charged and two for the neutral topological sector

$$K_{\alpha \gamma}(\theta_1)K_{\gamma \beta}(\theta_2) = S_0(\theta_{12})\sum_{\delta \neq \gamma} K_{\alpha \delta}(\theta_2)K_{\delta \beta}(\theta_1) + S_1(\theta_{12})K_{\alpha \gamma}(\theta_2)K_{\gamma \beta}(\theta_1), \quad \alpha \neq \beta$$

$$K_{\alpha \gamma}(\theta_1)K_{\gamma \alpha}(\theta_2) = S_2(\theta_{12})\sum_{\delta \neq \gamma} K_{\alpha \delta}(\theta_2)K_{\delta \alpha}(\theta_1) + S_3(\theta_{12})K_{\alpha \gamma}(\theta_2)K_{\gamma \alpha}(\theta_1).$$

(13)

Using the commutation relation (12) twice one obtains the unitarity constraint

$$\sum_{\varepsilon \neq \alpha, \beta} S_{\alpha \beta}^{\gamma \varepsilon}(\theta)S_{\alpha \beta}^{\varepsilon \delta}(-\theta) = \delta_{\gamma \delta},$$

(14)

which amounts to the set of equations

$$(q - 3)S_0(\theta)S_0(-\theta) + S_1(\theta)S_1(-\theta) = 1,$$

(15)

$$(q - 4)S_0(\theta)S_0(-\theta) + S_0(\theta)S_1(-\theta) + S_1(\theta)S_0(-\theta) = 0,$$

(16)

$$(q - 2)S_2(\theta)S_2(-\theta) + S_3(\theta)S_3(-\theta) = 1,$$

(17)

$$(q - 3)S_2(\theta)S_2(-\theta) + S_3(\theta)S_2(-\theta) + S_2(\theta)S_3(-\theta) = 0.$$

(18)

Crossing symmetry provides the relations

$$S_0(\theta) = S_0(i\pi - \theta)$$

(19)

$$S_1(\theta) = S_2(i\pi - \theta)$$

(20)

$$S_3(\theta) = S_3(i\pi - \theta).$$

(21)

Using these constraints together with the Yang-Baxter and bootstrap equations (that we do not reproduce here) the following expressions for the four elementary amplitudes were determined

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\(^7\)We parameterise on-shell momenta as $p^\mu = (m \cosh \theta, m \sinh \theta)$, $m$ being the mass of the kink.
in Ref. [9]

\[ S_0(\theta) = \frac{\sinh \lambda \theta \sinh \lambda (\theta - i\pi)}{\sinh \lambda (\theta - \frac{2\pi i}{3}) \sinh \lambda (\theta - \frac{i\pi}{3})} \Pi \left( \frac{\lambda \theta}{i\pi} \right), \]  

(22)

\[ S_1(\theta) = \frac{\sin \frac{2\pi \lambda}{3} \sinh \lambda (\theta - i\pi)}{\sin \frac{2\pi}{3} \sinh \lambda (\theta - \frac{2\pi i}{3})} \Pi \left( \frac{\lambda \theta}{i\pi} \right), \]  

(23)

\[ S_2(\theta) = \frac{\sin \frac{2\pi \lambda}{3} \sinh \lambda \theta}{\sin \frac{2\pi i}{3} \sinh \lambda (\theta - \frac{i\pi}{3})} \Pi \left( \frac{\lambda \theta}{i\pi} \right), \]  

(24)

\[ S_3(\theta) = \frac{\sin \lambda \pi}{\sin \frac{2\pi}{3}} \Pi \left( \frac{\lambda \theta}{i\pi} \right), \]  

(25)

where \( \lambda \) is related to \( q \) as

\[ \sqrt{q} = 2 \sin \frac{\pi \lambda}{3}, \]  

(26)

and

\[ \Pi \left( \frac{\lambda \theta}{i\pi} \right) = \frac{\sinh \lambda (\theta + i\frac{\pi}{2})}{\sinh \lambda (\theta - i\pi)} e^{A(\theta)}, \]  

(27)

\[ A(\theta) = \int_{0}^{\infty} \frac{dx}{x} \frac{\sinh \frac{x}{2} (1 - \frac{1}{\lambda}) - \sinh \frac{x}{2} (\frac{1}{\lambda} - \frac{1}{2})}{\sinh \frac{x}{2} \cosh \frac{x}{2}} \sin \frac{x \theta}{i\pi}. \]  

(28)

The above solution is well defined for real values of \( \lambda \), and one sees that (26) implies \( 0 \leq q \leq 4 \). The pure scaling Potts model in this range of \( q \) is identified with the values of \( \lambda \) in the range going from 0 to 3/2.

The function \( \Pi(\lambda \theta/i\pi) \) is free of poles in the physical strip \( \text{Im} \theta \in (0, \pi) \) for \( q < 3 \) (i.e. \( \lambda < 1 \)). Hence, in this range of \( q \) the only poles of the scattering amplitudes in the physical strip are those located at \( \theta = 2i\pi/3 \) and \( \theta = i\pi/3 \) and correspond to the appearance of the elementary kink itself as a bound state in the direct and crossed channel, respectively.

For \( q > 3 \) (\( \lambda > 1 \)) a direct channel (positive residue) pole located at \( \theta = 2i\kappa \),

\[ \kappa = \frac{\pi}{2} \left( 1 - \frac{1}{\lambda} \right), \]  

(29)

enters the physical strip in the amplitudes \( S_2(\theta) \) and \( S_3(\theta) \). Such a pole must be accordingly associated to a (topologically neutral) kink-antikink bound state \( B \) with mass

\[ m_B = 2m \cos \kappa. \]  

(30)

The amplitudes \( S_1(\theta) \) and \( S_3(\theta) \) exhibit the corresponding crossed channel (negative residue) pole at \( \theta = i\pi - 2i\kappa \). The amplitudes \( S_{KB}(\theta) \) and \( S_{BB}(\theta) \) describing the kink-bound state scattering and the bound state self-interaction are determined by the bootstrap equations

\[ S_{KB}(\theta) = (q - 2)S_2(\theta - i\kappa)S_1(\theta + i\kappa) + S_3(\theta - i\kappa)S_3(\theta + i\kappa), \]  

\[ S_{BB}(\theta) = S_{BK}(\theta - i\kappa)S_{BK}(\theta + i\kappa), \]  

(31)

\*The range \( 3/2 < \lambda < 3 \) corresponds to the thermal perturbation of the tricritical \( q \)-state Potts model [9].
Figure 3: Pictorial representation of the kink scattering amplitudes $S_{\alpha\beta}^{\gamma\delta}$ (a), and of the high-temperature amplitudes $\tilde{S}_{{k, p-k}}^{p-j,j}$ (b).

and read

$$S_{BK}(\theta) = t_{1-\kappa/\pi}(\theta)t_{2/3-\kappa/\pi}(\theta),$$

$$S_{BB}(\theta) = t_{2/3}(\theta)t_{1-2\kappa/\pi}(\theta)t_{2/3-2\kappa/\pi}(\theta),$$

in terms of the functions

$$t_a(\theta) = \frac{\tanh \frac{1}{2}(\theta + i\pi a)}{\tanh \frac{1}{2}(\theta - i\pi a)}.$$  (34)

The poles located at $\theta = i(\pi - \kappa)$ in $S_{BK}$ and at $\theta = 2\pi/3$ in $S_{BB}$ are bound state poles corresponding to $K$ and $B$, respectively.

It has been shown [23] that the remaining poles in the amplitudes $S_{KB}$ and $S_{BB}$ are associated to multi-scattering processes rather than to new particles. Hence, the elementary kinks and their neutral bound state $B$ are the only particles in the spectrum of the field theory describing the scaling zero-field Potts model below the critical temperature.

3.2 Disordered phase

We now show how the amplitudes (22)–(25) determine also the scattering theory in the disordered phase.

Above the critical temperature there is a unique ground state and the excitations of the scaling limit must be ordinary particles rather than kinks. Observing that the group $Z_q$ of cyclic permutations is a subgroup of $S_q$, we can take these particles to have a well defined $Z_q$ charge. The simplest possibility is then to conjecture a basis of elementary excitations $A_k(\theta)$, $k = 1, \ldots, q - 1$, each carrying $k$ units of $Z_q$ charge. A multi-particle state $A_{k_1}(\theta_1) \ldots A_{k_n}(\theta_n)$ will carry a charge $k_1 + \ldots + k_n (\text{mod} \ q)$. The integrable scattering theory is characterized by the two-particle amplitudes defined through the Faddeev-Zamolodchikov algebra (Fig. 3b)

$$A_k(\theta_1)A_{p-k}(\theta_2) = \sum_{j \neq p}^{\tilde{S}_{{k, p-k}}^{p-j,j}(\theta_12)}A_{p-j}(\theta_2)A_j(\theta_1),$$

where all the particle indices are taken mod $q$. The neutral channel corresponds to $p = q$, while $p = 1, \ldots q - 1$ yields charged channels. Full $S_q$ invariance is then recovered requiring that the...
interaction does not distinguish between the charged channels; within the neutral channel and
the charged channel, the need to have well defined crossing properties forces to distinguish the
case $j = k$ from the case $j \neq k$. This leaves us with a total number of four different amplitudes

$$A_k(\theta_1)A_{p-k}(\theta_2) = \tilde{S}_0(\theta_{12}) \sum_{j \neq k,p} A_j(\theta_2)A_{p-j}(\theta_1) + \tilde{S}_1(\theta_{12})A_k(\theta_2)A_{p-k}(\theta_1), \quad p \neq q$$

$$A_k(\theta_1)A_{q-k}(\theta_2) = \tilde{S}_2(\theta_{12}) \sum_{j \neq k} A_j(\theta_2)A_{q-j}(\theta_1) + \tilde{S}_3(\theta_{12})A_k(\theta_2)A_{q-k}(\theta_1). \quad (36)$$

These identifications require that all the particles $A_k$ have the same mass $\tilde{m}$.

The unitarity equations

$$\sum_{l \neq p} \tilde{\xi}_{k,p-l}^\nu \tilde{S}_{l,p-l}^\nu(\theta) \tilde{\xi}_{l,k}^{\nu-j}(\theta) = \delta_{k}^{\nu-j} \delta_{p-k}^\nu \quad (37)$$

are obtained iterating (35) and take a form identical to (15)–(18) with the substitution $S_i \rightarrow \tilde{S}_i$. The crossing relations coincide with (19)–(21) under the same substitution.

This correspondence between the scattering theories above and below the critical temperature extends to the factorization and bootstrap equations and simply expresses the fact that the two phases are related by duality. The elementary excitations of the disordered phase are particles $A_k, k = 1, \ldots, q - 1$, with the same mass of the kinks of the low-temperature phase and whose scattering is expressed in terms of the same amplitudes which specify the kink $S$-matrix, i.e.

$$\tilde{m} = m, \quad \tilde{S}_i(\theta) = S_i(\theta), \quad i = 0, 1, 2, 3. \quad (38)$$

For $3 < q \leq 4$, the high-temperature theory contains a neutral bound state $B$ with the mass

$$\tilde{m} = m, \quad \tilde{S}_i(\theta) = S_i(\theta), \quad i = 0, 1, 2, 3. \quad (38)$$

and whose scattering is specified by amplitudes $S_{BA_k}$ and $S_{BB}$ coinciding with (32) and

$$\tilde{m} = m, \quad \tilde{S}_i(\theta) = S_i(\theta), \quad i = 0, 1, 2, 3. \quad (38)$$

Duality allows to compute correlation functions above and below $T_c$ within the form factor
approach which relies on the knowledge of the $S$-matrix. This was done in [11] using the kink
$S$-matrix. It can be checked that the same results are obtained through the high-temperature scattering theory.

4 Particle spectrum in weak magnetic field

The action (4) describes the renormalization group trajectories flowing out of the fixed point
located at the origin of the $\tau$–$h$ plane. Such trajectories can be labelled by the dimensionless parameters

$$\eta_{\pm} = \frac{\tau}{(\pm h)^{(d-X_e)/(d-X_\mu)}}, \quad (39)$$

where the upper and lower signs are used for $h > 0$ and $h < 0$, respectively, in such a way that
$\eta_+$ parameterizes the trajectories in the upper half–plane and $\eta_-$ those in the lower half–plane; the two trajectories at $h = 0$ correspond to $\eta_{\pm} = +\infty$ and $\eta_{\pm} = -\infty$ (see Fig. 4).
In this section we discuss the evolution of the mass spectrum in $d = 2$ starting from the case of small magnetic field for which perturbation theory around integrable quantum field theories [10] can be used.

4.1 Weak field above critical temperature

For small $h$ at $\tau > 0$ (i.e. for $\eta_{\pm} \to +\infty$) the corrections to the mass spectrum are determined by the matrix elements of the magnetic operator $\sigma_q(x)$ on the asymptotic states of the unperturbed ($\eta_{\pm} = +\infty$) theory. At leading order, the correction to the mass matrix is [10] (Fig. 5a)

$$ (\delta m^2)_{j,k} \simeq -2h \langle A_j(0)|\sigma_q(0)|A_k(0)\rangle, \quad j, k = 1, \ldots, q - 1 $$

(here and below the matrix element on the particles $A_k$ are intended at $h = 0$).

Let us introduce the operators

$$ \tau_k(x) = \sum_{\alpha=1}^{q} \omega_{q}^{-\alpha k} \sigma_\alpha(x), \quad k = 1, \ldots, q - 1 $$

(41)

where $\omega_n \equiv \exp(2i\pi/n)$. The generator $\Omega_q$ of cyclic permutations acts as

$$ \Omega_q \sigma_\alpha(x) = \sigma_{\alpha + 1 \pmod q}(x), \quad \Omega_q \tau_k(x) = \omega_q^k \tau_k(x), $$

(42)

showing that $\tau_k$ carries $k$ units of $\mathbb{Z}_q$ charge and can be taken as interpolating operator of particle $A_k$ at $h = 0$. Using

$$ \sum_{j=1}^{n} \omega_{n}^{\pm jk} = n \delta_{k,n} $$

(43)

and (3), one also have

$$ \sigma_q(x) = \frac{1}{q} \sum_{k=1}^{q-1} \tau_k(x). $$

(44)
Denoting
\[ M_{j,k}^{\Psi} = \langle A_j(0)|\Psi(0)|A_k(0) \rangle, \]
and taking into account conservation of \( Z_q \) charge as well as full \( S_q \) symmetry at \( h = 0 \), we have
\[ M_{j,k}^{\tau_l} = f_q \delta_{j,k+l} \text{ (mod } q) \]
and
\[ M_{j,k}^{\sigma_q} = \frac{1}{q} \sum_{l=1}^{q-1} M_{j,k}^{\tau_l} = \frac{f_q}{q} (1 - \delta_{j,k}) \]
for the matrix which determines the mass corrections (40). Diagonalization of this matrix gives
\[ A_0 = \frac{1}{\sqrt{q-1}} \sum_{k=1}^{q-1} A_k \]
with square mass
\[ m_0^2 \simeq m^2 - 2f_q \frac{q-2}{q} h, \]
and a degenerate multiplet
\[ A'_k = \frac{1}{\sqrt{2}} (A_k - A_{q-1}) , \quad k = 1, \ldots, q-2 \]
with square mass
\[ m'^2 \simeq m^2 + 2f_q \frac{q}{q} h. \]
Comparison with (44) shows that \( A_0 \) is interpolated by \( \sigma_q \) and is a singlet of the \( S_{q-1} \) symmetry surviving at \( h \neq 0 \); suitable linear combinations of the \( A'_k \) yield a multiplet in which each of the \( q-2 \) particles carries a definite (non-zero) \( Z_{q-1} \) charge.

For \( 3 < q \leq 4 \) the theory also contains the additional \( S_{q-1} \)-singlet \( B \) with a mass correction with respect to (30) given by
\[ \delta m_B^2 \simeq -2h \langle B(0)|\sigma_q(0)|B(0) \rangle . \]
This matrix element, as well as \( f_q \) above, can be obtained from the form factor results of [11] (for \( q = 4 \) see also [8]). Notice that the first order correction to the mass of \( A_0 \) vanishes for \( q = 2 \); the leading correction in this case is of order \( h^2 \) and was computed in [24].
4.2 Weak field below critical temperature

A small magnetic field acting on the sites with color $q$ affects the ground state degeneracy at $\tau < 0$. Denoting by $|0_\alpha\rangle$, $\alpha = 1, \ldots, q$ the ferromagnetic vacua, $S_q$ symmetry gives at $\eta_\pm = -\infty$

$$\langle \sigma_\gamma \rangle_\alpha \equiv \langle 0_\alpha | \sigma_\gamma (x) | 0_\alpha \rangle = \frac{v}{q-1} (q \delta_{\gamma,\alpha} - 1),$$

with $v$ positive. At first order in $h$, the energy density difference between a vacuum $|0_{\alpha \neq q}\rangle$ and $|0_q\rangle$ is then

$$\Delta E = \delta E_\alpha - \delta E_q \simeq -h \langle \langle \sigma_\gamma \rangle_\alpha - \langle \sigma_\gamma \rangle_q \rangle = \frac{v q}{q-1} h,$$

so that $|0_q\rangle$ is the unique true vacuum at $h > 0$, and the unique false vacuum at $h < 0$.

Since no finite-energy topological excitation can begin or end on a false vacuum, the space of asymptotic states of the theory does not contain kinks for $h > 0$. For $\eta_-$ large and negative, instead, the kinks $K_{\alpha\beta}$ with $\alpha, \beta \neq q$ survive as the elementary excitations of the theory. The first order correction to their mass is (Fig 5b)

$$\delta m^2_{K_{\alpha\beta}} \simeq -2h \langle K_{\alpha\beta}(0)|\sigma_q(0)|K_{\beta\alpha}(0)\rangle_{\text{conn}} = -2h F_{\alpha\beta\alpha}(i\pi),$$

where

$$F_{\alpha\beta\alpha}(\theta_1 - \theta_2) = \langle 0_\alpha | \sigma_q(0) | K_{\alpha\beta}(\theta_1) K_{\beta\alpha}(\theta_2) \rangle, \quad \alpha \neq \beta$$

is the two-kink form factor at $\eta_\pm = -\infty$ [11]. In general this function has an annihilation pole whose residue

$$-i \text{Res}_{\theta = \pi} F_{\alpha\beta\alpha}(\theta) = \langle \sigma_q \rangle_\alpha - \langle \sigma_q \rangle_\beta = \frac{v q}{q-1} (\delta_{\alpha,q} - \delta_{\beta,q})$$

vanishes precisely when both $\alpha$ and $\beta$ differ from $q$, giving a finite mass correction$^9$ (55) (which does not depend on $\alpha, \beta = 1, \ldots, q - 1$). The divergence of (55) when $\alpha$ or $\beta$ equal $q$ simply reflects the decoupling of a kink interpolating between vacua which are no longer degenerate.

Although for $h$ positive $|0_q\rangle$ is the only true vacuum and no single-kink state survives, $n$-kink states beginning and ending on the true vacuum, i.e. $K_{q\alpha_1}(\theta_1) K_{\alpha_1\alpha_2}(\theta_2) \ldots K_{\alpha_{n-1}q}(\theta_n)$, do not decouple. Here we consider all the intermediate vacua to be false, since insertion of a true intermediate vacuum would simply amount to breaking the sequence into two states of the first type. Then we have $n \leq q$.

In the non-relativistic limit valid for small rapidities, the total energy of such a configuration consist of the rest mass term $n m$, the kinetic term, a contribution coming from kink interaction which decays exponentially with interkink distance, and the false vacuum contribution

$$V(x_1, \ldots, x_n) = \Delta E \sum_{i=1}^{n-1} (x_{i+1} - x_i) = (x_n - x_1) \Delta E,$$

$^9$In this case (55) is finite irrespectively of the sign of $h$. We should recall, however, that for $h > 0$ we are computing a mass gap above a false vacuum whose energy separation from the true vacuum in a system of spatial size $L$ is $\Delta E L$. Hence, all the single-kink states decouple in the infinite system with a positive, however small, magnetic field.
where $x_1 < x_2 < \ldots < x_n$ are the spatial positions of the kinks. The positive linear potential (58) confines the $n$ kinks into a finite spatial interval and prevents the observation of isolated kinks. In this sense the kinks of the $\eta_{\pm} = -\infty$ theory play the role of “quarks” at $h > 0$. The asymptotic particles are instead the topologically neutral bound states produced by the confinement of the $n$-kink state. When $\eta_+ \to -\infty$ the confining potential is extremely shallow and the kinks are very loosely bound; the average interkink distance is large and kink interaction is negligible in first approximation. The $n$-kink bound states form an infinite tower of levels which are dense above the value $nm$ as $\eta_+ \to -\infty$. It is natural to call “mesons” the $n = 2$ (kink-antikink) bound states (Fig. 6a), and “baryons” the bound states with $n = 3$ (Fig. 6b). Recalling the conditions $n \leq q$ and $q \leq q_c = 4$, we have that the mesons can occur for $q = 2, 3, 4$, and the baryons for $q = 3, 4$; tetraquark confined states are allowed for $q = 4$.

These particles organize themselves into multiplets of the residual $S_{q-1}$ symmetry. The number of different $n$-kink sequences coincides with the possible ways of coloring the intermediate vacua. The $q-1$ mesonic sequences can be combined into the states

$$\pi_k^{(j)}(0) \sim \sum_{\alpha=1}^{q-1} \omega_{q-1}^{-k\alpha} K_{q\alpha}(\theta) K_{q\alpha}(-\theta), \quad k = 0, 1, \ldots, q - 2$$

with $j = 1, 2, \ldots$ labelling in order of increasing energy the levels originated by the confinement of the kink-antikink superposition. Since

$$\Omega_{q-1} \pi_k^{(j)} = \omega_{q-1}^k \pi_k^{(j)},$$

the mesons (59) are eigenstates under cyclic permutations of the first $q - 1$ colors with $Z_{q-1}$ charge $k$. In the non-relativistic limit valid for the lowest levels in weak field, we can think of the kink and antikink inside a meson as experiencing elastic reflection on the walls of the confining potential and elastic scattering among themselves. As the effect of the latter, the intermediate vacuum can either remain unchanged with a probability amplitude $\Sigma_3(2\theta)$, or switch to a different color with a probability amplitude $\Sigma_2(2\theta)$ which, by $S_{q-1}$-invariance, does not depend on the new color\textsuperscript{10}. The superpositions in the right hand side of (59) are eigenstates of the $S$-matrix with scattering amplitudes

$$\Sigma_3(2\theta) + [(q-1)\delta_{k,0} - 1]\Sigma_2(2\theta).$$

\textsuperscript{10}At $\eta_{\pm} = -\infty$ the scattering amplitudes $\Sigma_2$ and $\Sigma_3$ coincide with (24) and (25), respectively. Corrections in weak field are determined by the matrix elements of the magnetic operator $\sigma_q$ [10].
Figure 7: Seven-kink matrix element responsible for the decay of baryons above threshold into two mesons in positive magnetic field.

Hence, the quark interaction is different for the neutral mesons \((k = 0)\) and for the charged mesons \((k \neq 0)\). For fixed \(j\) this will lead to a singlet \(\pi_{0}^{(j)}\) and a multiplet \(\pi_{1}^{(j)}, \ldots, \pi_{q-2}^{(j)}\) with energies which differ very slightly in weak field.

Only a finite number of these confined states are stable. For \(q = 2, 3\) all those states with energy larger than twice the mass of the lightest mesons (i.e. larger than a value close to \(4m\) for weak field) lie in the continuum and can decay. This means that for \(\eta_{+}\) sufficiently large and negative we certainly have stable mesons in the energy interval \((2m, 4m)\) for \(q = 2, 3\), and stable baryons in the energy interval \((3m, 4m)\) for \(q = 3\). At \(q = 4\) the decay thresholds are lowered by the presence of the neutral particle \(B\) with unperturbed mass \(m_{B} = \sqrt{3}m\). Hence the stability intervals in weak field are \((2m, 2\sqrt{3}m)\) for the neutral mesons, \((2m, (2 + \sqrt{3})m)\) for the charged mesons, \((3m, 2\sqrt{3}m)\) for the neutral baryons and \((3m, (2 + \sqrt{3})m)\) for the charged baryons; as for the tetraquark states, they all lie in the continuum and are expected to be unstable.

The number of 3-kink sequences giving rise to baryons is \((q-1)(q-2)\). For \(q = 3\) we then have the two series of baryons

\[
p_{\pm}^{(j)}(\theta) \sim K_{31}(\theta_1)K_{12}(\theta_2)K_{23}(\theta_3) \pm K_{32}(\theta_1)K_{21}(\theta_2)K_{13}(\theta_3),
\]

with even or odd parity with respect the residual \(Z_{2}\) symmetry which interchanges the colors 1 and 2. For \(q = 4\) the residual \(S_{3}\) symmetry can be seen as the product of the group \(Z_{3}\) of cyclic permutations times the topological charge conjugation \(C_{T}\) which transforms the kink \(K_{\alpha\beta}\), \(\alpha, \beta \neq 4\), into its antikink \(K_{\beta\alpha}\). Then the six series of baryonic states

\[
p_{k, \pm}^{(j)}(\theta) \sim \sum_{\alpha=1}^{3} \omega_{3}^{-k\alpha} \left[ K_{4, \alpha}(\theta_1) K_{\alpha, \alpha+1(\text{mod } 3)}(\theta_2) K_{\alpha+1(\text{mod } 3), \alpha}(\theta_3) \pm K_{4, \alpha+1(\text{mod } 3)}(\theta_1) K_{\alpha+1(\text{mod } 3), \alpha}(\theta_2) K_{\alpha, \alpha}(\theta_3) \right],
\]

\(k = 0, 1, 2\) \(63\)

are \(S_{3}\)-eigenstates:

\[
\Omega_{3} p_{k, \pm}^{(j)} = \omega_{3}^{k} p_{k, \pm}^{(j)}, \quad C_{T} p_{k, \pm}^{(j)} = \pm p_{k, \pm}^{(j)}.
\]

\(11\)The first order correction to \(m_{B}^{2}\) coincides with \((52)\).
For a given $j$ they give rise to two singlets ($k = 0$) and two doublets ($k \neq 0$) with opposite $C_T$–parity.

As already said, a small negative magnetic field confines only the kinks $K_{\alpha\beta}$ with $\alpha$ or $\beta$ equal $q$. Since now the false vacuum is unique, the only confined states are mesons of type $K_{\alpha q}K_{\beta q}$. For $q > 2$, however, these can decay into two asymptotic kinks through expansion of bubbles of true vacuum into the false vacuum.

5 Evolution of the spectrum with temperature and magnetic field

5.1 Positive magnetic field

We discussed in the previous section how the particle spectrum of the scaling two-dimensional $q$-state Potts model changes when a small magnetic field acting on the sites with color $q$ is switched on. For positive field, below critical temperature we have a complete confinement of kinks and the generation of a dense spectrum of mesons and (for $q > 2$) baryons; the number of such particles which are stable tends to infinity as $\eta_+ \to -\infty$. Above critical temperature, on the other hand, the effect of the magnetic field is much less dramatic, simply amounting to a partial removal of the degeneracy of the mass spectrum of the $h = 0$ particles.

It must be possible to interpolate continuously between these two limiting cases following the evolution of the spectrum as $\eta_+$ grows from $-\infty$ to $+\infty$ on the plane of Fig. 4. The simplest scenario is that, as $\eta_+$ increases, more and more mesons and baryons cross the decay thresholds and become unstable. This process of depletion of the spectrum of stable excitations would continue until the only stable particles surviving as $\eta_+ \to +\infty$ are those of the high-temperature theory in zero field.

We expect that, during the evolution of the spectrum as a function of the parameters $\eta_{\pm}$, energy levels corresponding to states originating from the confinement of a same number of kinks do not cross for $h \neq 0$. Indeed, the additional degeneracy at a crossing point of this nature should normally be related to a symmetry enhancement. In the field theory (4) the only symmetry enhancement (from $S_{q-1}$ to $S_q$) occurs at $h = 0$. If we add to this that the baryons should decay more easily than the lightest mesons, it seems natural to expect that the particles surviving in the limit $\eta_+ \to +\infty$ should be identified with the lightest among the mesons produced by kink confinement at $\eta_+$ very large and negative. In particular, the lightest meson multiplet $\pi^{(1)}_1, \ldots, \pi^{(1)}_{q-2}$ should evolve for increasing $\eta_+$ into the multiplet (50). As for the particle which evolves into the singlet (48), it should be identified with $\pi^{(1)}_0$ for $q = 2, 3$. At $q = 4$ the zero-field spectrum also includes the neutral particle $B$ with mass $\sqrt{3}m$, which is the lightest particle for $\eta_+ \to -\infty$. If one assumes that it does not cross mesonic levels, then it should evolve into the neutral particle (48) for $\eta_+ \to +\infty$; then the meson $\pi^{(1)}_0$ could evolve into the particle with mass $\sqrt{3}m$ in the same limit.

According to this scenario, all the mesons (59) with $j > 1$ and all the baryons must have
become unstable by the time $\eta_+$ approaches $+\infty$. For each of this particles there should exist a finite “critical” value of $\eta_+$ for which they reach the lowest decay threshold compatible with their charge, and above which they become unstable. Such critical value is expected to decrease as the mass of the particle increases, so that the “critical trajectories” accumulate in the limit $\eta_+ \to -\infty$. Notice that, due to the non-locality of the magnetic operator with respect to the kinks, the magnetic term of the action has infinitely many matrix elements on kinks which are non-zero in zero-field. In particular, the vertex responsible for the decay of the baryons which reach the two-meson threshold is shown in (Fig. 7).

Also in view of considerations to be made below about the spectrum evolution at $h < 0$, we expect the quantity $f_q$ determining the first order mass corrections (49) and (51) to be positive. Together with the previous identifications, this leads to the expectation that for any $h > 0$ the different quark interactions inside the neutral and the charged mesons (see (61)) induce a mass splitting between $\pi^{(1)}_1, \ldots, \pi^{(1)}_{q-2}$ and $\pi^{(1)}_0$ which is positive for $q = 3$. If the previous speculations about $q = 4$ should turn out to be correct, the mass splitting would be negative in this case.

When specialized to $q = 2$ this scenario coincides with that originally proposed for the scaling Ising model by McCoy and Wu [3]. In this case the pattern is simplified by the absence of baryons and charged mesons, as well as by the absence of interaction at $h = 0$. Moreover, integrability at $\tau = 0$ [25] also allows analytic investigation for strong magnetic field. Several studies, both analytic and numerical, have confirmed the McCoy-Wu scenario and provide us with a detailed description of the mass spectrum of Ising field theory in the full $\tau-h$ plane [3, 25, 10, 24, 27, 28, 29, 30, 31] (see also [32] for an introductory review of Ising field theory).

5.2 Negative magnetic field

We now discuss the evolution of the spectrum as a function of $\eta_-$ for $q = 3, 4$ (for $q = 2$ the spectrum does not depend on the sign of the magnetic field). We saw in the previous section that the elementary excitations for $\eta_- \to -\infty$ are the kinks $K_{\alpha\beta}$ interpolating between the $q - 1$ residual vacua. For $\eta_- \to +\infty$, on the other hand, due to the negative sign of $h$ in (49) and (51), the lightest excitations are the charged particles that in the previous subsection we identified with the mesons $\pi^{(1)}_1, \ldots, \pi^{(1)}_{q-2}$. Passing from these topologically neutral particles to the topologically charged kinks requires a phase transition at an intermediate value of $\eta_-$. We already know that, for $q = 3, 4$, a second order phase transition takes place at $\eta_- = 0$. The mass of the kinks decreases as $\eta_-$ goes from $-\infty$ to zero, and vanishes at $\eta_- = 0$. Similarly, the mass of the mesons $\pi^{(1)}_1, \ldots, \pi^{(1)}_{q-2}$ decreases as $\eta_-$ goes from $+\infty$ to zero, and vanishes at $\eta_- = 0$. Deconfined kinks exist only for $\eta_- \in [-\infty, 0)$; for $\eta_- \in (0, +\infty]$ and for all positive values of $h$ the spectrum can be seen as consisting of kink bound states. The trajectory $\eta_- = 0$ is a massless flow between the $S_q$-invariant ultraviolet fixed point at $h = 0$ and the $S_{q-1}$-invariant

\footnote{This quantity can be exactly computed within the integrable field theory at $h = 0$ (see [11]).}

\footnote{An analogous transition from ordinary particles to kinks, involving the spontaneous breaking of the symmetry $S_q$ rather than $S_{q-1}$, takes place when going through $\tau = 0$ at $h = 0$.}
Figure 8: Conjectured qualitative evolution of the mass spectrum for $q = 3$. The left-hand half corresponds to the evolution in $\eta_+$, the right-hand half to the evolution in $\eta_-$. The values $\eta_\pm = \pm \infty$, as well as the values $\eta_\pm = -\infty$, describe the same renormalization group trajectory. Only three lightest mesons and the lightest neutral baryon are shown for positive magnetic field. The dashed line is the lowest decay threshold (twice the mass of the lightest particle). Particles whose mass reaches the threshold become unstable. Deconfined kinks exist for negative $\eta_-$ only.

The ratio between the mass of the neutral meson $\pi_0^{(1)}$ and that of the charged mesons $\pi_1^{(1)}, \ldots, \pi_{q-2}^{(1)}$, which is 1 at $\eta_- = +\infty$, increases as $\eta_-$ decreases until it reaches the value 2 for some positive value of $\eta_-$. Beyond this point the opening of the decay channel $\pi_0^{(1)} \to \pi_1^{(1)} \pi_{q-2}^{(1)}$ makes the neutral meson unstable. The second neutral particle present at $q = 4$ becomes unstable in the same way. We then expect that the charged mesons are the only stable particles surviving for $\eta_-$ sufficiently small and positive. Similarly for the kinks at $\eta_-$ sufficiently small and negative.

The conjectured evolution of the mass spectrum with the parameters $\eta_+$ and $\eta_-$ is illustrated in Fig. 8 for $q = 3$.

6 Conclusion

In this paper we considered the field theory describing the scaling limit of the two-dimensional $q$-state Potts model in a magnetic field acting on one of the $q$ colors. This field breaks the $S_q$ symmetry of color permutations down to $S_{q-1}$ and, for $2 < q \leq q_c = 4$ allows for an extended region in the plane of temperature and magnetic field in which the quarks (kinks interpolating between degenerate vacua) are deconfined. If the analysis is extended to a more general magnetic term $\sum_{\alpha} h_{\alpha} \sigma_{\alpha}$, a renormalization group trajectory corresponding to generic
values of the components $h_\alpha$ will possess no internal symmetry and will contain only particles made of confined quarks with unequal masses. Of course, suitable relations among the magnetic parameters identify regions in parameter space with a residual $S_{q-1}$ or (for $q = 4$) $S_{q-2}$ symmetry, which in turn contain phases with deconfined quarks.

We saw how form factor perturbation theory allows to compute mass corrections in weak field above critical temperature, and below critical temperature for the quarks in the deconfined phase. As for the mass spectrum of the particles originating from confinement, the Bethe-Salpeter approach proved to be remarkably effective for the Ising mesons [26, 29], but needs to be generalized to the case of quarks which interact already in absence of field in order to deal with $q \neq 2$. Hopefully, it will become possible to study also the baryonic spectrum along similar lines.

Numerical methods will be eventually needed for a quantitative study of the mass spectrum in strong field. Particularly promising in this respect seems the truncated conformal space approach [33], which amounts to the numerical diagonalization of the Hamiltonian on a finite-dimensional subspace of the conformal basis of states of the ultraviolet fixed point. This approach has been successfully used for the Ising case\textsuperscript{14} [10, 26, 29, 34] and can be used also for $q \neq 2$.

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\textsuperscript{14}In [26, 29] the free nature of the zero-field Ising model is exploited to diagonalize the Hamiltonian on a truncated basis of massive fermionic states.
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