Nonuniform Berry-Esseen bound for self-normalized martingales

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Abstract

We give a nonuniform Berry-Esseen bound for self-normalized martingales, which bridges the gap between the results of Haeusler (1988) and Fan and Shao (2018). The bound coincides with the nonuniform Berry-Esseen bound of Haeusler and Joos (1988) for standardized martingales. As a consequence, a Berry-Esseen bound is obtained.

1. Introduction

Let \((X_i, \mathcal{F}_i)_{i=0, \ldots, n}\) be a martingale differences defined on a probability space \((\Omega, \mathcal{F}, P)\), where \(X_0 = 0\) and \(\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}\) are increasing \(\sigma\)-fields. Let

\[
S_0 = 0, \quad S_k = \sum_{i=1}^{k} X_i, \quad k = 1, \ldots, n.
\]

Then \((S_k, \mathcal{F}_k)_{k=0, \ldots, n}\) is a martingale. Without loss generality, assume that \(E S_n^2 = 1\), which means \(S_n\) is standardized. Let \([S]\) and \(\langle S \rangle\) be, respectively, the squared variance and the conditional variance of the martingale, that is

\[
[S]_0 = 0, \quad [S]_k = \sum_{i=1}^{k} X_i^2
\]

and

\[
\langle S \rangle_0 = 0, \quad \langle S \rangle_k = \sum_{i=1}^{k} E[X_i^2|\mathcal{F}_{i-1}], \quad k = 1, \ldots, n.
\]

The absolute errors of normal approximations for martingales have been intensively studied; see, for instance, Heyde and Brown \[7\], Haeusler \[4\], Haeusler

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and Joos [5], El Machkouri and Ouchti [1], Fan and Shao [3], Fan [2] and [8]. Suppose that $E|X_i|^{2p} < \infty$ for some $p > 1$ and all $i = 1, \ldots, n$. Define

$$N_n = \sum_{i=1}^{n} E|X_i|^{2p} + E|(S)_n - 1|^p. \quad (1)$$

Heyde and Brown [7] proved the following Berry-Esseen bound

$$\sup_{x \in \mathbb{R}} |P(S_n \leq x) - \Phi(x)| \leq C_p N_n^{1/(2p+1)}, \quad (2)$$

where $p \in (1, 2]$ and $C_p$ is a positive constant depending only on $p$. As a major advance in this direction, Haeusler [4] extended the result of Heyde and Brown [7] from $p \in (1, 2]$ to $p \in (1, \infty)$. Additionally, to justify that the bound (2) is asymptotically the best possible, he showed that there exists a martingale difference sequence $(\xi_k, \mathcal{F}_k)_{k=0,\ldots,n}$, such that for $n$ large enough,

$$\sup_{x \in \mathbb{R}} |P(S_n \leq x) - \Phi(x)| N_n^{-1/(2p+1)} \geq c_p,$$

where $c_p$ is a positive constant and does not depend on $n$. Later, Fan and Shao [3] obtained that the following Berry-Esseen bounds for self-normalized and normalized martingales: for $p > 1$,

$$\sup_{x \in \mathbb{R}} |P(S_n/\sqrt{|S|} \leq x) - \Phi(x)| \leq C_p N_n^{1/(2p+1)} \quad (3)$$

and

$$\sup_{x \in \mathbb{R}} |P(S_n/\sqrt{(S)}_n \leq x) - \Phi(x)| \leq C_p N_n^{1/(2p+1)}. \quad (4)$$

Notice that the last two bounds coincide with Haeusler’s bound [4]. They also showed that the last two Berry-Esseen bounds are also the best possible in the spirit of Haeusler [4]. The first goal of this paper is to bridge the Berry-Esseen bounds of Haeusler [4], Fan and Shao [3].

On the other hand and Haeusler and Joos [5] obtained the following nonuniform version of (2), that is for $p > 1$,

$$|P(S_n \leq x) - \Phi(x)| \leq C_p N_n^{1/(2p+1)} \quad (5)$$

Clearly, the last bound improved the Berry-Esseen bounds by adding a factor decaying at a rate of polynomial. Inspired by the result of Haeusler and Joos [5], we are interested in giving a nonuniform version of Berry-Esseen bound for Fan and Shao [3], which is the second goal of this paper.

All over the paper, $c$ and $C_p$, probably enabled with some indices, denote a common positive constant and a common positive constant depending only on $p$ respectively.
2. Main results

The following theorem bridges the Berry-Esseen bounds of Haeusler [4] and Fan and Shao [3].

**Theorem 2.1.** Assume that \( E|X_i|^{2p} < \infty \) for some \( p > 1 \) and all \( i = 1, \ldots, n \).
Then there exists a constant \( C_p \) depending only on \( p \) such that

\[
\sup_{x \in \mathbb{R}} \left| \frac{S_n}{\sqrt{t_1(S)^n + t_2|S|^n + (1 - t_1 - t_2)}} \right| - \Phi(x) \leq C_p N_n^{1/(2p+1)}, \tag{6}
\]

where \( t_1, t_2, (t_1 + t_2) \in [0, 1] \).

When \( t_1 = 1 \), the result \(^{[6]}\) is exactly the Berry-Esseen bound for normalized martingale. When \( t_2 = 1 \), the result \(^{[6]}\) reduces to the Berry-Esseen bound for self-normalized martingale. When \( t_1 = t_2 = 0 \), the result \(^{[6]}\) becomes the Berry-Esseen bound of Haeusler \(^{[4]}\). Thus, the result bridges the Berry-Esseen bounds of Haeusler \(^{[4]}\) and Fan and Shao \(^{[3]}\).

Moreover, we also have the following nonuniform Berry-Esseen bound.

**Theorem 2.2.** Assume that \( E|X_i|^{2p} < \infty \) for some \( p > 1 \) and all \( i = 1, \ldots, n \).
Then there exists a constant \( C_p \) depending only on \( p \) such that

\[
\left| \frac{S_n}{\sqrt{t_1(S)^n + t_2|S|^n + (1 - t_1 - t_2)}} \right| - \Phi(x) \leq C_{t_2,p} N_n^{1/(2p+1)} \frac{1 + |x|^{2p}}{1 + |x|}, \tag{7}
\]

where \( t_2 \in [0, 1) \) and \( t_1, (t_1 + t_2) \in [0, 1] \).

Notice that the bound \(^{[7]}\) coincides with the nonuniform Berry-Esseen bound of Haeusler and Joos \(^{[3]}\).

3. Proofs of theorems

3.1. Proof of Theorem 2.1

When \( N_n > 1 \), the result is trivial, so we suppose \( N_n \leq 1 \). First, we give an upper bound for

\[
P(\{ S_n \leq x \sqrt{t_1(S)^n + t_2|S|^n + t_3} \} - \Phi(x), x \leq 0. \]

Let \( \epsilon_n \in (0, \frac{1}{4}) \) be a positive number and let \( t_3 = 1 - t_1 - t_2 \). It is easy to see that for \( x \leq 0 \),

\[
P(\{ S_n \leq x \sqrt{t_1(S)^n + t_2|S|^n + t_3} \} - \Phi(x) \leq P(\{ S_n \leq x \sqrt{t_1(1 - \epsilon_n) + t_2|S|^n + t_3} \} - \Phi(x) + P(\{ S_n \leq 1 - \epsilon_n \}) \leq P(\{ S_n \leq x \sqrt{t_1(1 - \epsilon_n) + t_2(1 - \epsilon_n) + (1 - t_1 - t_2)} \) \]

\[
- \Phi(x) + P(\{ S_n \leq 1 - \epsilon_n \}) + P(\{ |S|^n \leq 1 - \epsilon_n \})
\]

\[
= P(\{ S_n \leq x \sqrt{1 - (t_1 + t_2)\epsilon_n} \}

- \Phi(x) + P(\{ S_n \leq 1 - \epsilon_n \}) + P(\{ |S|^n \leq 1 - \epsilon_n \})
\]

\[
= J_1 + J_2 + J_3 + J_4, \tag{8}
\]
where
\[ J_1 = P\left( S_n \leq x\sqrt{1 - (t_1 + t_2)e_n} \right) - \Phi\left( x\sqrt{1 - (t_1 + t_2)e_n} \right), \]
\[ J_2 = \Phi\left( x\sqrt{1 - (t_1 + t_2)e_n} \right) - \Phi(x), \]
\[ J_3 = P\left( \langle S \rangle_n \leq 1 - \epsilon_n \right), \]
\[ J_4 = P\left( |S|_n \leq 1 - \epsilon_n \right). \]

Using Haeusler’s inequality \[4\], we obtain
\[ J_1 \leq C_{p,1}\left( \sum_{i=1}^{n} E|X_i|^{2p} + E|\langle S \rangle_n - 1|^p \right)^{1/(2p+1)}. \] \[ (9) \]

Taking one-term Taylor’s expansion and using the fact that \( e^{-x^2/2|x|} \leq 1 \), we get
\[ J_2 \leq c_1 e^{-x^2/2|x|} \left( 1 - \sqrt{1 - (t_1 + t_2)e_n} \right) \leq c_1 e^{-x^2/2|x|} \left( 1 - \sqrt{1 - (t_1 + t_2)e_n} \right) \left( 1 + \sqrt{1 - (t_1 + t_2)e_n} \right) \leq c_2 \epsilon_n. \] \[ (10) \]

Using Markov’s inequality \[4\], we have
\[ J_3 \leq P\left( |\langle S \rangle_n - 1| \geq \epsilon_n \right) \leq \epsilon_n^{-p} E|\langle S \rangle_n - 1|^p. \] \[ (11) \]

By the inequality (4.11) of Fan and Shao \[3\], we obtain for \( p > 1 \)
\[ J_4 \leq C_{p,2}\epsilon_n^{-p} \left[ \sum_{i=1}^{n} E|X_i|^{2p} + \left( \sum_{i=1}^{n} E|X_i|^{2p} \right)^{p/(2p-2)} + E|\langle S \rangle_n - 1|^p \right]. \] \[ (12) \]

Returning to \( S \), we have for \( p > 1 \),
\[ P\left( S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S|_n + t_3} \right) - \Phi(x) \leq C_{p,1}\left[ \sum_{i=1}^{n} E|X_i|^{2p} + E|\langle S \rangle_n - 1|^p \right]^{1/(2p+1)} + c_2 \epsilon_n + \epsilon_n^{-p} E|\langle S \rangle_n - 1|^p \]
\[ + C_{p,2}\epsilon_n^{-p} \left[ \sum_{i=1}^{n} E|X_i|^{2p} + \left( \sum_{i=1}^{n} E|X_i|^{2p} \right)^{p/(2p-2)} + E|\langle S \rangle_n - 1|^p \right] \leq C_{p,1}\left[ \sum_{i=1}^{n} E|X_i|^{2p} + E|\langle S \rangle_n - 1|^p \right]^{1/(2p+1)} + c_2 \epsilon_n \]
\[ + (C_{p,2} + 1)\epsilon_n^{-p} \left[ \sum_{i=1}^{n} E|X_i|^{2p} + \left( \sum_{i=1}^{n} E|X_i|^{2p} \right)^{p/(2p-2)} + E|\langle S \rangle_n - 1|^p \right]. \]
Taking
\[
\epsilon_n = \left[ \sum_{i=1}^{n} \mathbb{E}[X_i]^{2p} + \left( \sum_{i=1}^{n} \mathbb{E}[X_i]^{2p} \right)^{p/(2p-2)} + \mathbb{E}[\langle S \rangle_n - 1|S\rangle]^2 \right]^{1/(p+1)}
\]  
(13)

and using the fact that \( p/(2p-2)(p+1) \geq 1/(2p+1) \), we observe for \( x \leq 0 \) and \( p > 1 \),
\[
\mathbb{P}\left(S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S\rangle_n + t_3} \right) - \Phi(x) 
\leq C_{p,1}\left[\sum_{i=1}^{n} \mathbb{E}[X_i]^{2p} + \mathbb{E}[\langle S \rangle_n - 1|S\rangle]^2 \right]^{1/(2p+1)} 
+ C_{p,2}\sum_{i=1}^{n} \mathbb{E}[X_i]^{2p} + \left( \sum_{i=1}^{n} \mathbb{E}[X_i]^{2p} \right)^{p/(2p-2)} + \mathbb{E}[\langle S \rangle_n - 1|S\rangle]^2 \right]^{1/(p+1)} 
\leq C_{p,1}N_n^{1/(2p+1)} + C_{p,2}N_n^{1/(p+1)} + C_{p,4}\sum_{i=1}^{n} \mathbb{E}[X_i]^{2p} \right]^{1/(2p-2)(p+1)} 
\leq C_{p,5}N_n^{1/(2p+1)} + C_{p,4}\sum_{i=1}^{n} \mathbb{E}[X_i]^{2p} \right]^{1/(2p+1)} 
\leq C_{p,6}N_n^{1/(2p+1)}. 
\]  
(14)

Next, we give a lower bound for \( \mathbb{P}\left(S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S\rangle_n + t_3} \right) - \Phi(x), x \leq 0 \). Similarly, let \( \epsilon_n \in (0, \frac{1}{2}) \) be a positive number and let \( t_3 = 1 - t_1 - t_2 \),
\[
\mathbb{P}\left(S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S\rangle_n + t_3} \right) - \Phi(x) 
\geq \mathbb{P}\left(S_n \leq x\sqrt{t_1(1 + \epsilon_n) + t_2|S\rangle_n + t_3}, \langle S \rangle_n < 1 + \epsilon_n \right) - \Phi(x) 
\geq \mathbb{P}\left(S_n \leq x\sqrt{t_1(1 + \epsilon_n) + t_2|S\rangle_n + t_3} \right) - \Phi(x) - \mathbb{P}\left(\langle S \rangle_n \geq 1 + \epsilon_n \right) 
\geq \mathbb{P}\left(S_n \leq x\sqrt{t_1(1 + \epsilon_n) + t_2(1 + \epsilon_n) + (1 - t_1 - t_2)} \right) - \Phi(x) - \mathbb{P}\left(\langle S \rangle_n \geq 1 + \epsilon_n \right) 
\geq \mathbb{P}\left(S_n \leq x\sqrt{(1 + (t_1 + t_2)\epsilon_n)} \right) - \Phi(x) - \mathbb{P}\left(\langle S \rangle_n \geq 1 + \epsilon_n \right) 
\geq \mathbb{P}\left(\langle S \rangle_n \geq 1 + \epsilon_n \right) - \mathbb{P}\left(\langle S \rangle_n \geq 1 + \epsilon_n \right) 
= J_5 + J_6 - J_7 - J_8,
where
\[ J_5 = P\left(S_n \leq x\sqrt{1 + (t_1 + t_2)\epsilon_n}\right) - \Phi\left(x\sqrt{1 + (t_1 + t_2)\epsilon_n}\right), \]
\[ J_6 = \Phi\left(x\sqrt{1 + (t_1 + t_2)\epsilon_n}\right) - \Phi(x), \]
\[ J_7 = P\left(S_n \geq 1 + \epsilon_n\right), \]
\[ J_8 = P\left(S_n \geq 1 + \epsilon_n\right). \]

Similarly, we can prove that for \( x \leq 0 \) and \( p > 1 \),
\[ P\left(S_n \leq x\sqrt{t_1 \langle S \rangle_n + t_2[S]_n + t_3}\right) - \Phi(x) \geq -C_{p,10}N_n^{-p/2 + 1}. \] (15)

Therefore, it follows that
\[ \sup_{x \leq 0} P\left(S_n \leq x\sqrt{t_1 \langle S \rangle_n + t_2[S]_n + t_3}\right) - \Phi(x) \leq C_{p,9}N_n^{-p/2 + 1}. \] (16)

Note that \((-S_k, F_k)_{k=0,\ldots,n}\) is also a martingale. Applying the last inequality to \((-S_k, F_k)_{k=0,\ldots,n}\), we have
\[
\sup_{x>0} \left| P\left(S_n \leq x\sqrt{t_1 \langle S \rangle_n + t_2[S]_n + t_3}\right) - \Phi(x) \right| = \sup_{x>0} \left| \Phi(-x) - P\left(-S_n \leq -x\sqrt{t_1 \langle S \rangle_n + t_2[S]_n + t_3}\right) \right|
\leq C_{p,11}N_n^{-p/2 + 1}. \] (17)

Thus,
\[ \sup_{x \in \mathbb{R}} \left| P\left(S_n \leq x\sqrt{t_1 \langle S \rangle_n + t_2[S]_n + t_3}\right) - \Phi(x) \right| \leq C_{p,12}N_n^{-p/2 + 1}. \] (18)

### 3.2. Proof of Theorem 2.2

When \( N_n > 1 \), the result is trivial, so we suppose \( N_n \leq 1 \). First, we give a lower bound for \( P\left(S_n \leq x\sqrt{t_1 \langle S \rangle_n + t_2[S]_n + t_3}\right) - \Phi(x), x \leq 0 \). Let \( \epsilon_n =\)
$\epsilon_n(x) \geq 0$ be a positive number and let $t_3 = 1 - t_1 - t_2$. It is easy to see that
\[
P\left(S_n \leq x\sqrt{t_1(1 + \epsilon_n) + t_2(1 + \epsilon_n) + (1 - t_1 - t_2)}\right)
- \Phi(x) - P\left(\langle S \rangle_n \geq 1 + \epsilon_n\right)
= P\left(S_n \leq x\sqrt{1 + (t_1 + t_2)\epsilon_n}\right)
- \Phi(x) - P\left(\langle S \rangle_n \geq 1 + \epsilon_n\right)
= K_1(x) + K_2(x) - K_3(x) - K_4(x),
\]
where
\[
K_1(x) = P\left(S_n \leq x\sqrt{1 + (t_1 + t_2)\epsilon_n}\right) - \Phi\left(x\sqrt{1 + (t_1 + t_2)\epsilon_n}\right),
K_2(x) = \Phi\left(x\sqrt{1 + (t_1 + t_2)\epsilon_n}\right) - \Phi(x),
K_3(x) = P\left(\langle S \rangle_n \geq 1 + \epsilon_n\right),
K_4(x) = P\left(|S| \geq 1 + \epsilon_n\right).
\]
Using (5), we get that
\[
K_1(x) \geq - \frac{C_{p,1}N_n^{1/(2p+1)}}{1 + |x|^{2p}(1 + (t_1 + t_2)\epsilon_n)^p}
\geq - \frac{C_{t_1,p,1}N_n^{1/(2p+1)}}{1 + |x|^{2p}}.
\tag{19}
\]
Taking one-term Taylor’s expansion, we have
\[
K_2(x) \geq -c_1e^{-x^2/2|\epsilon_n^1(\sqrt{1 + (t_1 + t_2)\epsilon_n} - 1)}
\geq -c_1e^{-x^2/2|\epsilon_n^1(\sqrt{1 + (t_1 + t_2)\epsilon_n} - 1)(\sqrt{1 + (t_1 + t_2)\epsilon_n} + 1)}
\geq -c_2e^{-x^2/2|\epsilon_n}.\tag{20}
\]
By Markov’s inequality, we obtain
\[
K_3(x) \leq P\left(|\langle S \rangle_n - 1| \geq \epsilon_n\right)
\leq \epsilon_n^{-p}E|\langle S \rangle_n - 1|^p
\leq \epsilon_n^{-p}N_n
\leq \epsilon_n^{-p}N_n^{p+(p+1)/(2p+1)}.\tag{21}
\]
Using the inequality (4.11) of Fan and Shao \[3\] and the fact that \( p/(2p - 2) \geq (p + 1)/(2p + 1) \) and \( N_n < 1 \), we get for \( p > 1 \),

\[
K_4(x) \leq C_{p,2} \epsilon_n^p \left( \sum_{i=1}^{n} E|X_i|^{2p} + \left( \sum_{i=1}^{n} E|X_i|^{2p} \right)^{p/(2p-2)} + E|\langle S \rangle_n - 1|^p \right)
\]

\[
\leq C_{p,2} \epsilon_n^p \left( N_n + N_n^{p/(2p-2)} \right)
\]

\[
\leq C_{p,2} \epsilon_n^p \left( N_n + N_n^{(p+1)/(2p+1)} \right)
\]

\[
\leq 2C_{p,2} \epsilon_n^p N_n^{(p+1)/(2p+1)}.
\]

Then, combining (19), (20), (21) and (22) together, we have for all \( x \leq 0, p > 1 \) and \( \epsilon_n = \epsilon_n(x) \geq 0 \),

\[
P \left( S_n \leq x \sqrt{t_1 \langle S \rangle_n + t_2 |S|_n + t_3} - \Phi(x) \right)
\]

\[
\geq -C_{t_1,p,1} N_n^{1/(2p+1)} \frac{1}{1 + |x|^{2p}} - c_2 e^{-x^2/2} |x| \epsilon_n - C_{p,2} \epsilon_n^p \frac{N_n^{(p+1)/(2p+1)}}{(1 + x^2)^p}.
\]

Taking

\[
\epsilon_n = \epsilon_n(x) = (1 + x^2) N_n^{1/(2p+1)},
\]

we have for all \( x \leq 0, \)

\[
P \left( S_n \leq x \sqrt{t_1 \langle S \rangle_n + t_2 |S|_n + t_3} - \Phi(x) \right)
\]

\[
\geq -C_{t_1,p,1} N_n^{1/(2p+1)} \frac{1}{1 + |x|^{2p}} - c_2 e^{-x^2/2} |x| (1 + x^2) N_n^{1/(2p+1)} - C_{p,2} \epsilon_n^p \frac{N_n^{(p+1)/(2p+1)}}{(1 + x^2)^p}.
\]

\[
\geq -C_{t_1,p,3} N_n^{1/(2p+1)} \frac{1}{1 + |x|^{2p}}.
\]

Next, we give an upper bound for \( P \left( S_n \leq x \sqrt{t_1 \langle S \rangle_n + t_2 |S|_n + t_3} - \Phi(x), x \leq 0 \). Let \( 1 > \epsilon_n \geq 0 \) and let \( t_3 = 1 - t_1 - t_2 \). We deduce that

\[
P \left( S_n \leq x \sqrt{t_1 \langle S \rangle_n + t_2 |S|_n + t_3} - \Phi(x) \right)
\]

\[
\leq P \left( S_n \leq x \sqrt{t_1 (1 - \epsilon_n) + t_2 |S|_n + t_3} - \Phi(x) + P \left( \langle S \rangle_n \leq 1 - \epsilon_n \right) \right)
\]

\[
\leq P \left( S_n \leq x \sqrt{t_1 (1 - \epsilon_n) + t_2 (1 - \epsilon_n) + (1 - t_1 - t_2)} - \Phi(x) + P \left( \langle S \rangle_n \leq 1 - \epsilon_n \right) + P \left( |S|_n \leq 1 - \epsilon_n \right) \right)
\]

\[
= P \left( S_n \leq x \sqrt{1 - (t_1 + t_2) \epsilon_n} - \Phi(x) + P \left( \langle S \rangle_n \leq 1 - \epsilon_n \right) + P \left( |S|_n \leq 1 - \epsilon_n \right) \right)
\]

\[
= K_5(x) + K_6(x) + K_7(x) + K_8(x),
\]
where

\[
\begin{align*}
K_5(x) &= P\left(S_n \leq x\sqrt{1-(t_1+t_2)\epsilon_n}\right) - \Phi\left(x\sqrt{1-(t_1+t_2)\epsilon_n}\right), \\
K_6(x) &= \Phi\left(x\sqrt{1-(t_1+t_2)\epsilon_n}\right) - \Phi(x), \\
K_7(x) &= P\left(\langle S \rangle_n \leq 1 - \epsilon_n\right), \\
K_8(x) &= P\left(|S|_n \leq 1 - \epsilon_n\right).
\end{align*}
\]

Using an argument similar to the proof of (21), we know for all \( x \leq 0 \) satisfying \( \epsilon_n(x) \leq 1 \),

\[
P\left(S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S|_n + t_3}\right) - \Phi(x) \leq \frac{C_{t_5,4}N_n^{1/(2p+1)}}{1+|x|^{2p}}. \tag{25}
\]

When \( x \leq 0 \) satisfies \( \epsilon_n(x) > 1 \), that is \( x < -\sqrt{N_n^{-1/(2p+1)} - 1} \), it is easy to see that

\[
\begin{align*}
P\left(S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S|_n + t_3}\right) - \Phi(x) &\leq P\left(S_n \leq x\sqrt{t_3}\right) - \Phi(x\sqrt{t_3}) + \Phi(x) - \Phi(x) \\
&\leq \frac{C_{t_5}N_n^{1/(2p+1)}}{1+|x|^{2p}} + \frac{c_3e^{-x^2/2}|x|^3(1-\sqrt{t_3})}{1+|x|^{2p}} \\
&\leq \frac{C_{t_5}N_n^{1/(2p+1)}}{1+|x|^{2p}} + \frac{c_4e^{-x^2/2}|x|^3(1+|x|^{2p})N_n^{1/(2p+1)}}{1+|x|^{2p}} \\
&\leq \frac{C_{t_5}N_n^{1/(2p+1)}}{1+|x|^{2p}}. \tag{26}
\end{align*}
\]

Therefore, it follows that, for all \( x \leq 0 \),

\[
\left|P\left(S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S|_n + t_3}\right) - \Phi(x)\right| \leq \frac{C_{t_5,5}N_n^{1/(2p+1)}}{1+|x|^{2p}}. \tag{27}
\]

By an argument similar to that of (17), we get for all \( x > 0 \),

\[
\begin{align*}
&\left|P\left(S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S|_n + t_3}\right) - \Phi(x)\right| \\
&= \left|P\left(S_n \leq x\sqrt{t_1\langle S \rangle_n + t_2|S|_n + t_3}\right) - 1 + 1 - \Phi(x)\right| \\
&= \left|\Phi(-x) - P\left(-S_n \leq -x\sqrt{t_1\langle S \rangle_n + t_2|S|_n + t_3}\right)\right| \\
&\leq \frac{C_{t_5,5}N_n^{1/(2p+1)}}{1+|x|^{2p}}. \tag{28}
\end{align*}
\]
Thus,

\[
P\left( S_n \leq x \sqrt{t_1(S)n + t_2[S]n + t_3} \right) - \Phi(x) \leq \frac{C_{t_3,p,q}N^{1/(2p+1)}_n}{1 + |x|^{2p}}.
\] (29)

This completes the proof of Theorem 2.2

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