THE SPECTRUM OF $SL(2, R)/U(1)$ BLACK HOLE CONFORMAL FIELD THEORY

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ABSTRACT

We study string theory in the background of a two-dimensional black hole which is described by an $SL(2, R)/U(1)$ coset conformal field theory. We determine the spectrum of this conformal field theory using supersymmetric quantum mechanics and give an explicit form of the vertex operators in terms of the Jacobi functions. We also discuss the applicability of SUSY quantum mechanics techniques to non-linear $\sigma$-models.

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1. Introduction

Matrix models remain by far the most successful approach for studying string theories in non-critical space-time dimensions [1 – 3]. We expect to gain some insight into ‘realistic’ string theories by studying these toy models. Of all the models, $c = 1$ conformal field theory (CFT) coupled to two dimensional gravity is the most interesting and intriguing [4 – 6]. This model can be considered as a string theory with a two dimensional target space.[7, 8] In the language of matrix models, these two dimensions correspond to the time and eigenvalue of the matrix. From the continuum field theory point of view they correspond to the $c = 1$ scalar field and the Liouville field. Using the $\sigma$-model representation of this two dimensional theory, it was shown by Mandal et al.[9] that the solution of $\beta$-function equations in the graviton-dilaton sector described the space-time exterior to the horizon of the black hole. Witten observed that this solution possesses all the features of the black hole geometry. He also showed that it is possible to construct an exact conformal field theory [10] based on an $SL(2, R)/U(1)$ gauged WZW model whose target space has black hole geometry.(For related work see refs.[11 – 14] ) It was possible to get black holes with both Euclidean and Minkowski signatures. In the gauged WZW model, this signature depends upon which subgroup is gauged. There has been a lot of activity since then and both Euclidean and Minkowski black hole conformal field theories have been studied in great detail [15 – 25].

In this paper, we will study the spectrum of an $SL(2, R)/U(1)$ coset model. We will reduce the problem of finding the spectrum to a quantum mechanical problem in the target space. We shall determine the spectrum of this conformal field theory for a generic value of the current algebra level $k$, - i.e., at a generic value of central charge $c$. Even in the case of $c = 26$ conformal field theory, we shall study the full spectrum of the conformal field theory without restricting ourselves to $(1, 1)$ operators. In the language of string theory, this means that we
will study the off-shell states as well. In general, coset models of non-compact symmetry groups are non-unitary. But there were some suggestions that unitary CFT’s can be obtained if the spectrum is truncated by restricting the values of $l$ to $-1/2 > l > -k/2$ [26–28].

For orientation and motivation, we shall review the $SL(2, \mathbb{R})/U(1)$ coset model in section 2, and show that the problem of determining the spectrum of Virasoro primary fields can be reduced to a quantum mechanical problem in the target space. We shall then introduce the techniques of supersymmetric (SUSY) quantum mechanics in section 3, and briefly discuss the concept of shape invariance and its relation with SUSY quantum mechanics. In section 4, we shall apply these techniques to the problem at hand. We shall show, using shape invariance and SUSY, that the black hole quantum mechanics problem is exactly soluble. We give the exact bound state as well as scattering spectrum and write down the wavefunctions explicitly. Utilising the relation of this quantum mechanical problem with black hole CFT, we show that the bound state spectrum gives the conformal dimensions of the vertex operators which are the eigenfunctions themselves. Some of these results were obtained by Dijkgraaf, Verlinde and Verlinde [16] using different techniques. In section 5, we shall study the possibility of wider applicability of this technique. In particular, we shall show that this method can be applied to CFT with $\sigma$-model representation.

2. Review of $SL(2, R)/U(1)$ coset model

The black hole CFT, as shown by Witten [10], can be formulated as a gauged WZW model. The WZW model is based on a non-compact symmetry group $SL(2, R)$. The symmetry that is gauged in this model corresponds to some abelian subgroup $H$ of $SL(2, R)$. When $H$ is compact, we get a Euclidean black hole target space whereas for a non-compact $H$, we get a Minkowski black hole. Since we shall concentrate on Euclidean black hole quantum mechanics, we shall only review the
Euclidean version of the black hole CFT\textsuperscript{*}.

Let us parametrize the $SL(2,\mathbb{R})$ group manifold by three real coordinates $r$, $\theta_L$ and $\theta_R$. $\theta_L$ and $\theta_R$ are periodic coordinates - i.e. they are compact-, whereas $r$ is non-compact and takes values on the non-negative real axis. We write the field on the group manifold as

$$g = \exp\left(\frac{i}{2} \theta_L \sigma_2\right) \exp\left(\frac{1}{2} r \sigma_1\right) \exp\left(\frac{i}{2} \theta_R \sigma_2\right)$$  \hspace{1cm} (2.1)$$

where $\sigma_i$ are the Pauli matrices. The abelian subgroup $H$ is generated by $\sigma_2$, and the gauge transformation is a shift symmetry in $\theta_L$ and $\theta_R(\theta_{L,R} \rightarrow \theta_{L,R} + \alpha)$. With this parametrization of $g$, the gauged WZW action is given by

$$S = S_{WZW}[r, \theta_L, \theta_R] + \frac{k}{2\pi} \int d^2z [A(z, \bar{z})(\partial \theta_R + \cosh r \partial \theta_L) \hspace{1cm} (2.2)$$

where

$$S_{WZW}[r, \theta_L, \theta_R] = \frac{k}{4\pi} \int d^2z (\partial r \partial r - \bar{\partial} \theta_L \partial \theta_L - \partial \theta_R \partial \theta_R - 2 \cosh r \partial \theta_L \partial \theta_R).$$  \hspace{1cm} (2.3)$$

Gauge fixing can be done by parametrizing the gauge field as

$$A = \partial \phi_L$$
$$\bar{A} = \partial \phi_R$$  \hspace{1cm} (2.4)$$

where $\phi_L$ and $\phi_R$ are complex fields with the condition $\phi_L = (\phi_R)^\ast$. We are assuming a trivial world sheet topology while writing eq.(2.4). If we shift $\theta_L \rightarrow \theta_L + \phi_L$ and $\theta_R \rightarrow \theta_R + \phi_R$ in the action (2.2) and use the gauge invariance, we

\textsuperscript{*} In this section we shall follow the analysis of ref.[16].
see that the action depends only on the difference $\phi = \phi_L - \phi_R$. The gauge fixed action is given by

$$S_{gf} = S_{WZW}[r, \theta_L, \theta_R] + S[\phi] + S[b, c]$$

where $S[\phi]$ is the action of a time-like free scalar field and $S[b, c]$ describes the spin $(1, 0)$ ghost system. The stress energy tensor of the coset model can be written as

$$T(z) = \frac{1}{k - 2} \eta_{ab} J^a J^b + \frac{k}{4} (\partial \phi)^2 + b \partial c$$

where $\eta_{ab}$ is the metric on the $SL(2, R)$ Lie algebra and $J^a$ are $SL(2, R)$ currents. The scalar field $\phi$ is compactified in the case of a Euclidean black hole. Since the gauge fixed theory contains a free scalar field theory and an ungauged $SL(2, R)$ WZW model, the vertex operators of the coset model are products of the vertex operators of the free scalar field theory and the $SL(2, R)$ WZW model -i.e.,

$$V(z, \bar{z}) = T(r(z, \bar{z}), \theta_L(z, \bar{z}), \theta_R(z, \bar{z})) \exp (iq_L \varphi + iq_R \bar{\varphi})$$

where $\phi = \varphi + \bar{\varphi}$. Vertex operators of the gauged WZW model should satisfy the constraint $J^3 - \bar{J}^3 = 0$. In the formulation given above, this constraint is imposed by the BRST charge

$$Q_B = \int dz \, c(J^3 + \frac{i}{2} k \partial \phi) + c.c$$

The zero modes of the $SL(2, R)$ currents act as differential operators on the vertex operators

$$J^3 = -i \frac{\partial}{\partial \theta_L}$$

$$J^\pm = \exp (\pm \theta_L) (\frac{\partial}{\partial r} \mp \frac{i}{\sinh r}(\frac{\partial}{\partial \theta_R} - \cosh r \frac{\partial}{\partial \theta_L})).$$

Using the Sugawara relation, the Virasoro generators can be written in terms of the modes of $SL(2, R)$ currents and the abelian current. We wish to find the
full spectrum of the Virasoro primary fields. But we shall first concentrate on the primary fields of the current algebra. They correspond to vertex operators without any oscillator excitations. To determine the spectrum of these primary fields using the Virasoro generator $L_0$, it suffices to concentrate only on the zero mode bilinears of currents in the Sugawara relation

$$L_0 = \frac{1}{k-2} \eta_{ab} J^a_0 J^b_0 - \frac{1}{k} p^2_0$$  \hspace{1cm} (2.10)$$

where $p_0$ is the momentum conjugate to the zero mode $\phi_0$. The Virasoro generator $L_0$ expressed in terms of zero modes of the coordinates is

$$L_0 = -\frac{\Delta_0}{k-2} - \frac{1}{k} \frac{\partial^2}{\partial \theta^2_0}$$ \hspace{1cm} (2.11)$$

where

$$\Delta_0 = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \left( \frac{\partial^2}{\partial \theta^2_L} - 2 \cosh r \frac{\partial}{\partial \theta_L} \frac{\partial}{\partial \theta_R} + \frac{\partial^2}{\partial \theta^2_R} \right)$$ \hspace{1cm} (2.12)$$

is the $SL(2, R)$ Casimir operator. Thus the equation

$$L_0 V = \hbar V$$ \hspace{1cm} (2.13)$$

becomes a Schrödinger-like equation with $V$ being the eigenfunctions of the operator $L_0$ with eigenvalues $\hbar$. As was mentioned earlier, we shall consider off-shell string modes but with the constraint $L_0 - \tilde{L}_0 = 0$. This constraint enables us to decompose $T(r, \theta_L, \theta_R)$ into $T(r, \theta)$ and $T(r, \tilde{\theta})$ (where $\theta = (\theta_L + \theta_R)/2$ and $\tilde{\theta} = (\theta_L - \theta_R)/2$) which are the momentum and winding tachyons. In terms of the new variables, the $L_0$ operator for the tachyon field $T(r, \theta)$ is

$$L_0 = -\frac{1}{k-2} \left[ \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \left( \coth^2 \left( \frac{r}{2} \right) - \frac{2}{k} \right) \frac{\partial^2}{\partial \theta^2} \right].$$ \hspace{1cm} (2.14)$$

On the other hand, $L_0$ for $T(r, \tilde{\theta})$ becomes

$$L_0 = -\frac{1}{k-2} \left[ \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \left( \tanh^2 \left( \frac{r}{2} \right) - \frac{2}{k} \right) \frac{\partial^2}{\partial \tilde{\theta}^2} \right].$$ \hspace{1cm} (2.15)$$

The group invariant measure inherited by the coset model from $SL(2, R)$ group
manifold is
\[ dg = \frac{1}{4\pi^2} \sinh \, r \, dr \, d\theta. \] (2.16)

Therefore, the inner product of the tachyons \( T_1 \) and \( T_2 \) on the target space is defined as
\[ \langle T_1 | T_2 \rangle = \int dg T_1(r, \theta) T_2(r, \theta). \] (2.17)

We linearise the integration measure by absorbing \( \sinh^{1/2}(r) \) in \( T(r, \theta) \). This redefinition further simplifies the form of the \( L_0 \) operator and reduces it to the Schrödinger operator given by
\[ \frac{1}{k - 2} \left( -\frac{\partial^2}{\partial r^2} + V(r) \right), \] (2.18)

where \( V(r) \) for the winding states is given by
\[ V(r) = (\omega^2 - \frac{1}{16}) \tanh^2(\frac{r}{2}) - \frac{1}{16} \coth^2(\frac{r}{2}) - \frac{2\omega^2}{k} + \frac{3}{8} \] (2.19)

and for the momentum states is given by
\[ V(r) = (\omega^2 - \frac{1}{16}) \coth^2(\frac{r}{2}) - \frac{1}{16} \tanh^2(\frac{r}{2}) - \frac{2\omega^2}{k} + \frac{3}{8}. \] (2.20)

We have replaced \( \frac{\partial^2}{\partial \theta^2} \) by \( -\omega^2 \). We shall carry out our analysis for an arbitrary value of \( k \). The spectrum of the black hole problem can be obtained by putting \( k = 9/4 \). Thus we see that solving the problem \( L_0 T = \hbar T \) is reduced to solving a quantum mechanical problem with a specific potential. This way we can determine only those Virasoro primary fields which are also the current algebra primary fields. The remaining Virasoro primaries can be determined from the spectrum of these primary fields by attaching a string of currents and its derivatives on the left. The condition that these new vertex operators are physical fields of the coset model gives constraints on the string of currents. As a result of
this constraint, only those combinations of the currents and its derivatives which commute with the BRST charge are allowed. In the next section we will discuss the techniques of SUSY quantum mechanics and shape invariance which will be used to show that the problem stated above is exactly solvable.

3. SUSY Quantum Mechanics and Shape Invariance

The concept of supersymmetry in quantum mechanics was introduced by Witten [29]. Supersymmetry, as is well known, relates bosons to fermions. In quantum mechanics two Hamiltonians which are related to each other by supersymmetry are called partner Hamiltonians. These Hamiltonians, due to SUSY, are isospectral except for the ground state. In short, SUSY in quantum mechanics can be described as follows. Given a potential $V_-(x)$, SUSY allows us to construct a partner potential $V_+(x)$ which has an identical energy spectrum except for the ground state. Such a pair of Hamiltonians is given by

$$H_\pm = -\frac{d^2}{dx^2} + V_\pm(x)$$

(3.1)

Using two component notation, a Hamiltonian $\mathcal{H}$ can be written as an anticommutator of SUSY charges as follows

$$\mathcal{H} = \left( \begin{array}{cc} H_- & 0 \\ 0 & H_+ \end{array} \right) = \frac{1}{2}\{Q, Q\}.$$  

(3.2)

The supersymmetry charge $Q$ commutes with the Hamiltonian $\mathcal{H}$. If we parametrize the SUSY charge $Q$ as

$$Q = \left( \begin{array}{cc} 0 & \mathcal{A}^\dagger \\ \mathcal{A} & 0 \end{array} \right)$$

(3.3)

then both $H_+$ and $H_-$ can be written in the following factorised forms

$$H_+ = \mathcal{A}\mathcal{A}^\dagger \quad \text{and} \quad H_- = \mathcal{A}^\dagger\mathcal{A}.$$  

(3.4)
For the choice
\[ A = \frac{d}{dx} + W(x) \] (3.5)

the partner potentials can be written in terms of the superpotential \( W(x) \)
\[ V_{\pm}(x) = W^2(x) \pm \frac{dW(x)}{dx} \] (3.6)

Now, it trivially follows that when SUSY is unbroken, the ground state of \( H_- \) has zero energy and the ground state eigenfunction is
\[ \Phi_{0}^{-}(x) = N_0 \exp \left( - \int W(x') dx' \right). \] (3.7)

The partner Hamiltonians have an identical bound state spectrum except for the ground state of \( H_- \) so that
\[ E_{n+1}^- = E_{n}^+ \quad n = 0, 1, 2, \cdots \] (3.8)

The eigenfunctions corresponding to any given eigenvalue are related to each other through \( A \) and \( A^\dagger \) as follows
\[ A\Phi_{n+1}^-(x) = (E_{n}^+)^{1/2}\Phi_{n}^+(x) \]
\[ A^\dagger\Phi_{n}^+(x) = (E_{n}^+)^{1/2}\Phi_{n+1}^-(x). \] (3.9)

Since SUSY does not give the eigenvalues and eigenfunctions corresponding to excited states we need some additional information to actually determine the spectrum. To determine the eigenvalues and eigenfunctions we need to invoke the concept of shape invariance. The term shape invariance was introduced by Gen-denstein [30]. If the potentials \( V_+ \) and \( V_- \) have a similar shape, they are called shape invariant potentials. In mathematical terms, the shape invariance condition
can be written as the Ricatti equation

\[ W^2(x, a_i^{[0]}) + \frac{dW(x, a_i^{[0]})}{dx} = \tilde{W}^2(x, a_i^{[1]}) - \frac{d\tilde{W}(x, a_i^{[1]})}{dx} + c(a_i^{[1]}) \quad (3.10) \]

where \( a_i^{[n]} \) are the parameters appearing in the potential. If the an analytical solution to this Ricatti equation exists, \( i.e., \) if \( \tilde{W} \) can be expressed in a closed form \( * \), then by using SUSY, we can completely determine the spectrum of eigenvalues and eigenfunctions by purely algebraic means. This can easily be seen as follows. On the left hand side of the Ricatti equation we have the potential \( V^+ \). As we saw earlier, SUSY does not give us any information about the ground state of this potential. But, since right hand side of the Ricatti equation is written in terms of \( \tilde{V}^− \), we can find the ground state eigenvalue and eigenfunction using SUSY. From the Ricatti equation it is easy to see that the spectrum of \( V^+ \) and \( \tilde{V}^− \) are identical up to an overall shift in the eigenvalues. Thus the zero energy SUSY ground state of \( \tilde{V}^− \) is the ground state of \( V^+ \) with the energy \( c(a_i^{[1]}) \). Now, recall that \( V^+ \) has one state less than \( \tilde{V}^− \) and that is the ground state of \( V^− \). In other words, the ground state eigenvalue of \( V^+ \) is equal to the first excited state eigenvalue of \( V^− \). Now if we can find the parameters \( a_i^{[n]} \) as a function \( f(a_i^{[n-1]}) \) of \( a_i^{[n-1]} \) such that the Ricatti equation (3.10) is satisfied, then following the earlier discussion, the ground state of \( V^−(x, a_i^{[n]}) \) has the same energy as the first excited state of \( V^−(x, a_i^{[n-1]}) \). This state in turn has the same energy as the second excited state of \( V^−(x, a_i^{[n-2]}) \) and so on. Thus we see that \( V^−(x, a_i^{[n]}) \) gives a potential for each parameter set \( a_i^{[n]} \) and it is easy to see that \( V^−(x, a_i^{[n]}) \) has the same spectrum as that of \( V^−(x, a_i^{[0]}) \) except that the lowest \( n \) states of \( V^−(x, a_i^{[0]}) \) are missing. (For details and specific examples see ref.[31].) Since SUSY intertwines these potentials it is possible to determine the excited state eigenfunctions for any of these problems using eq.(3.9). Thus we can determine the full spectrum of eigenvalues and eigenfunctions. Thus using

\( * \) Throughout this paper, SUSY implies that the superpotential \( W \) can be determined analytically and can be written in a closed form. Similarly, shape invariance means that the Ricatti equation can be solved analytically.
shape invariance we can show that the energy spectrum of $H_-$ is given by

$$E_{n}^{(-)} = \sum_{k=1}^{n} c(a_i^{[k]}).$$

(3.11)

The eigenfunctions can be determined by using eq.(3.9) for successive values of parameters $a_i^{[n]}$, i.e.

$$\Phi_{n}^{(-)}(x, a_i^{[0]}) = A^\dagger(x, a_i^{[0]}).A^\dagger(x, a_i^{[1]})\cdots A^\dagger(x, a_i^{[n-1]})\Phi_{0}^{(-)}(x, a_i^{[n]}).$$

(3.12)

4. Black Hole Quantum Mechanics

In this section we shall apply the techniques developed in the previous section to the black hole problem. Recall that in sec. 2, we reduced the black hole CFT problem to a quantum mechanics problem on a half line.

First, let us note that the spectrum of the potential in eq.(2.19) contains both bound states and scattering states, whereas the potential in eq.(2.20) does not have any bound state for $\omega^2 > 1/16$. We shall see in our analysis that even for $\omega^2 < 1/16$ and $\omega$ real, this potential does not have any bound state.

4.1. Winding Sector

In this subsection we shall study only the winding sector i.e., the potential given in eq.(2.19). We shall first show that this potential has supersymmetry and use it to determine the superpotential. Then we shall invoke the shape invariance property of this potential to find out the complete spectrum of bound state eigenvalues and eigenfunctions. The scattering spectrum is determined by analytically continuing the bound state spectrum. From the asymptotic behaviour of these analytically continued wave-functions, we shall derive the scattering matrix
and subsequently the density of states. The potential in eq.(2.19) can be obtained from the superpotential

\[ W(r) = \left(-\frac{1}{4} \pm \omega\right) \tanh\left(\frac{r}{2}\right) - \frac{1}{4} \coth\left(\frac{r}{2}\right). \]  

(4.1)

Therefore, the Schrödinger equation written in terms of the superpotential is given by

\[ \left(-\frac{d^2}{dr^2} + W^2(r) - \frac{dW(r)}{dr} \pm \omega - \frac{2\omega^2}{k}\right)\Phi = E\Phi. \]  

(4.2)

Since from SUSY quantum mechanics we know that

\[ \left(-\frac{d^2}{dr^2} + W^2(r) - \frac{dW(r)}{dr}\right)\Phi_0 = 0 \]  

(4.3)

where \( \Phi_0 \) is the ground state eigenfunction, the ground state energy in eq.(4.2) is

\[ E_0 = \pm \omega - \frac{2\omega^2}{k}. \]  

(4.4)

The ground state wave-function is given by

\[ \Phi_0(r) = \exp\left(-\int W(r')dr'\right) = \exp\left(-\int \left(\frac{1}{4} \pm \omega\right) \tanh\left(\frac{r}{2}\right)dr + \int \frac{1}{4} \coth\left(\frac{r}{2}\right)dr\right) \]

\[ = \frac{\sinh^{1/2}(r)}{\sqrt{2}(\cosh\left(\frac{r}{2}\right))^{\pm 2\omega}}. \]  

(4.5)

The condition for this wave-function* to be a bound state wave-function (i.e. square integrable) is \( \pm \omega > 1/2 \). Since \( \omega \) takes both positive and negative values we can write this condition as \( |\omega| > 1/2 \). We shall come back to this point later in this section when we will discuss square integrability of the full spectrum of bound state eigenfunctions.

* Recall that \( \sinh^{1/2}(r) \) is precisely the prefactor that we absorbed in the wave-function to linearise the integration measure. So the wave-function corresponding to the original problem is without the \( \sinh^{1/2}(r) \) term. In addition, it will have a \( \theta \) dependent part as well. But this will not affect any of our conclusions.
Now we shall invoke the shape invariance property to determine the complete spectrum of eigenvalues and eigenfunctions. Let us show that the potential $V(r)$ given in eq.(2.19) is shape invariant. In fact, this can be shown for any general potential of the type given in eq. (2.19), i.e., we shall consider a potential with the same functional form but with arbitrary parameters in front of the $\tanh^2(r/2)$ and the $\coth^2(r/2)$ instead of $(\omega^2 - 1/16)$ and $(-1/16)$. Also instead of $r/2$ we shall use $\alpha r$. Let us choose \[ W(r,a_i^{[0]}) = A \tanh(\alpha r) - B \coth(\alpha r) \] (4.6) with the parameter set $a_i^{[0]} = (A, B)$. Then it follows that

$$ V_-(r,a_i^{[0]}) = W^2(r,a_i^{[0]}) - \frac{dW(r,a_i^{[0]})}{dr} $$
$$ = (A^2 + A\alpha) \tanh^2(\alpha r) + (B^2 - B\alpha) \coth^2(\alpha r) - 2AB + (A - B)\alpha $$
(4.7)

and

$$ V_+(r,a_i^{[0]}) = W^2(r,a_i^{[0]}) + \frac{dW(r,a_i^{[0]})}{dr} $$
$$ = (A^2 - A\alpha) \tanh^2(\alpha r) + (B^2 + B\alpha) \coth^2(\alpha r) - 2AB - (A - B)\alpha. $$
(4.8)

As we have seen earlier in sec. 3, to determine the first excited state eigenvalue and eigenfunction of $V_-$, we need to know the ground state eigenvalue and eigenfunction of $V_+$ which can be determined using the Ricatti equation. Now if we choose $a_i^{[1]} = (A - \alpha, B + \alpha)$, the new superpotential is given by

$$ \tilde{W}(r,a_i^{[1]}) = (A - \alpha) \tanh(\alpha r) - (B + \alpha) \coth(\alpha r). $$
(4.9)

With this choice of superpotential it is easy to see that eq.(3.10) is satisfied and the constant $c(\{a_1\}) = 4\alpha(A - B - \alpha)$. Thus we see that energy of the first excited state is $4\alpha(A - B - \alpha)$. We can also determine the ground state wavefunction of the partner potential $V_+(r)$ and as a consequence of eq.(3.12), the first
excited state wave-function of $V_-(r)$. In fact the Ricatti equation (3.10) in this case can be solved by doing successive transformations and choosing appropriate $a_{i}^{[n]} = f(a_{i}^{[n-1]})$. The form of the function remains the same throughout. The set $a_{i}^{[n]}$ in the above example is $(A - n\alpha , B + n\alpha)$, and therefore the energy spectrum is

$$E_n = 4n\alpha(A - B) - 4n^2\alpha^2. \quad (4.10)$$

The eigenfunctions can be deduced by successively using eq.(3.12) and are given by

$$\Phi_n(y) = (y - 1)^{B/2\alpha}(y + 1)^{-A/2\alpha}P_n^{(\frac{B}{\alpha} - \frac{1}{2}, -\frac{A}{\alpha} - \frac{1}{2})}(y) \quad (4.11)$$

where $y = \cosh 2\alpha r$ and $P_n^{(\beta, \gamma)}(y)$ are Jacobi functions.

Comparing eq.(4.7) and (2.19), we see that for the winding mode sector, $A = (|\omega| - 1/4)$, $B = 1/4$ and $\alpha = 1/2$. Since in our problem the ground state energy is not zero, the whole spectrum gets shifted. Substituting the values of $A$, $B$ and $\alpha$ and after adding the ground state energy in eq.(4.4) to the spectrum (4.10), the energy eigenvalues are

$$E_n = (2n + 1)|\omega| - n(n + 1) - \frac{2\omega^2}{k} \quad (4.12)$$

and the eigenfunctions are

$$\Phi_n(y) = (y - 1)^{1/4}(y + 1)^{-|\omega|+1/4}P_n^{(0,-2|\omega|)}(y) \quad (4.13)$$

where $y = \cosh(r)$. The square integrability of these eigenfunctions, by virtue of eq.(3.12), is determined by the square integrability of the ground state eigenfunction of the $n$th problem (i.e. the ground state eigenfunction of $V_-(r, \{a_n\})$). For the potential with parameters $a_{i}^{[n]} = (A - n\alpha , B + n\alpha)$, the condition for square integrability of the ground state wave-function for any $n$, as can be deduced from eq.(4.5), depends on the difference of these two parameters, i.e. it reduces to
$A - B > 0$. Substituting the values of $A$ and $B$ we get the relation $|\omega| - n > 1/2$. Thus we see that for a fixed value of $\omega$, there always exist a finite number of bound states.

Now let us turn our attention to the $SL(2, R)$ Casimir operator. In the winding sector, the Casimir operator $\Delta$ takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + (\tanh^2 \frac{r}{2} - 1) \frac{\partial^2}{\partial \theta^2}. \quad (4.14)$$

Comparing the spectra of the Casimir operator $\Delta$ and that of the Virasoro generator $L_0$, we see that

$$\text{Spec}(\Delta) = \text{Spec}[(k - 2)(-L_0 - \frac{2\omega^2}{k} + \omega^2)]$$

$$= (-2n + 1)|\omega| + n(n + 1) + \omega^2. \quad (4.15)$$

Equating the right hand side of eq.(4.15) with the usual eigenvalue of $l(l + 1)$ the Casimir, we get the relation $l + |\omega| = n$. Thus, from this relation we see that the number of nodes of the bound state wave-function are related to the sum $l + |\omega|$. Using this relation along with the square integrability condition, we find that bound state spectrum corresponds to $l < -1/2$ representations. But $l < -1/2$ and $l + |\omega|$ being a non-negative integer are the properties of the discrete representations of $SL(2, R)$. Hence, the bound state spectrum forms a discrete representation of $SL(2, R)$.

The scattering matrix can be determined by analytically continuing the bound state spectrum. We, therefore, express the wave-function in terms of the hypergeometric function

$$\Phi_n(y) = (y - 1)^{1/4}(y + 1)^{-\omega+1/4} \frac{1}{\Gamma(n)} F(-n, 1 - n - |\omega|, 1; \frac{1-y}{2}). \quad (4.16)$$

We then analytically continue $n$ to $|\omega| + i\lambda - 1/2$. Substituting this value of $n$ and taking large $y$ asymptotics of the hypergeometric function, the asymptotic form of
the wave-function can be written as
\[
\Phi \sim \left( \frac{1}{2} \right)^{-\frac{1}{2} + |\omega| + i\lambda} \frac{\Gamma(2i\lambda)}{\Gamma(\frac{1}{2} + |\omega| + i\lambda)\Gamma(\frac{1}{2} - |\omega| + i\lambda)} \exp(i\lambda r)
\]
\[+ \left( \frac{1}{2} \right)^{-\frac{1}{2} + |\omega| - i\lambda} \frac{\Gamma(-2i\lambda)}{\Gamma(\frac{1}{2} + |\omega| - i\lambda)\Gamma(\frac{1}{2} - |\omega| - i\lambda)} \exp(-i\lambda r). \] \quad (4.17)

It is now easy to extract from this expression, the scattering amplitude
\[
S = 2^{-2i\lambda} \frac{\Gamma(2i\lambda)\Gamma(\frac{1}{2} + |\omega| - i\lambda)\Gamma(\frac{1}{2} - |\omega| - i\lambda)}{\Gamma(-2i\lambda)\Gamma(\frac{1}{2} + |\omega| + i\lambda)\Gamma(\frac{1}{2} - |\omega| + i\lambda)}. \] \quad (4.18)

The density of states, therefore, is given by
\[
\rho(\lambda) = 2Re[2\psi(2i\lambda) - \psi\left(\frac{1}{2} + |\omega| + i\lambda\right) - \psi\left(\frac{1}{2} - |\omega| + i\lambda\right)] \] \quad (4.19)
where, \(\psi(z) = \Gamma'(z)/\Gamma(z)\).

4.2. Momentum Sector

Let us consider the potential given in eq.(2.20). As we argued earlier this potential blows up at \(r = 0\) for \(|\omega| > 1/4\). On the other hand, when \(|\omega| < 1/4\) the potential goes to \(-\infty\) at \(r = 0\) and can in principle have bound states. But as it turns out, the ground state wave-function
\[
\Phi_0 = \cosh^{-1/2} r \frac{\sinh^{2\omega - 1/2} r}{2} \] \quad (4.20)
is not square integrable and hence, this potential does not have any bound states for \(|\omega| < 1/4\) either.

The scattering wave-function can be written in terms of a hypergeometric function as
\[
\tilde{\Phi} = (y-1)^{1/4+\omega}(y+1)^{1/4} \frac{\Gamma\left(\frac{1}{2} \pm \omega + i\lambda\right)}{\Gamma\left(-\frac{1}{2} + i\lambda\right)\Gamma\left(\frac{1}{2} + i\lambda\right)} F\left(\frac{1}{2} - i\lambda, \frac{1}{2} \pm 2\omega + i\lambda; \frac{1 - y}{2}\right). \] \quad (4.21)

We look at the asymptotic behaviour of this wave-function to determine the scat-
tering amplitude. The asymptotic form of the wave-function is given by
\[
\tilde{\Phi}(r \to \infty) \sim \left(\frac{1}{2}\right)^{-\frac{1}{2} \pm \omega + i\lambda} \frac{\Gamma(2i\lambda)\Gamma(1 \pm 2\omega)}{\Gamma(1 \frac{1}{2} \mp \omega + i\lambda)\Gamma(\frac{1}{2} \pm \omega + i\lambda)} \exp(i\lambda r)
\]
\[
+ \left(\frac{1}{2}\right)^{-\frac{1}{2} \mp \omega - i\lambda} \frac{\Gamma(-2i\lambda)\Gamma(1 \pm 2\omega)\Gamma(\frac{1}{2} \pm \omega + i\lambda)}{\Gamma(\frac{1}{2} \pm \omega - i\lambda)\Gamma(-\frac{1}{2} \pm \omega + i\lambda)\Gamma(\frac{1}{2} \mp \omega + i\lambda)\Gamma(\frac{1}{2} \pm \omega - i\lambda)} \exp(-i\lambda r)
\]
(4.22)

The ratio of the coefficients of the incoming and outgoing waves gives the scattering matrix
\[
S = 2^{-2i\lambda} \frac{\Gamma(2i\lambda)\Gamma(\frac{1}{2} \pm \omega + i\lambda)\Gamma(\frac{1}{2} \pm \omega - i\lambda)}{\Gamma(\frac{1}{2} \pm \omega + i\lambda)\Gamma(-2i\lambda)\Gamma(\frac{1}{2} \pm \omega + i\lambda)}.\]
(4.23)

The density of states is given by
\[
\rho(\lambda) = 4\text{Re}[\psi(2i\lambda) - \psi(\frac{1}{2} \pm \omega + i\lambda)]
\]
(4.24)

where \(\psi(z) = \Gamma'(z)/\Gamma(z)\).

In this section, we studied the spectrum of the chiral algebra primary fields. This covers all the chiral algebra primary fields but as far as primary conformal fields, \textit{i.e.}, Virasoro primary fields are concerned, this set is far from complete. The Virasoro primary fields which are not the chiral algebra primary fields can be obtained by attaching a string of chiral algebra currents and its derivatives to the left of the chiral algebra primary field. The states corresponding to these new fields will be in the cohomology of \(Q_B\) provided the string of chiral currents commutes with \(Q_B\). Apart from this, they have to satisfy additional constraints to be the Virasoro primary fields [24]. Their conformal dimension, however, can be easily read out from their composition. In the language of states, this means that the oscillator creation operators contribute a specific integer in addition to the conformal dimension of the basic chiral primary field.

These fields along with the original set give the complete set of Virasoro primary fields. Though we identify the wave-functions with the vertex operators, it is important to recognise, at this stage, that the wave-functions are expressed in
terms of the zero modes of the fields \( r(z, \bar{z}) \) and \( \theta(z, \bar{z}) \) (or \( \tilde{\theta}(z, \bar{z}) \)). On the other hand the vertex operators are expressed in terms of the fields \( r(z, \bar{z}) \) and \( \theta(z, \bar{z}) \) (or \( \tilde{\theta}(z, \bar{z}) \)). Therefore to write correct expressions for the vertex operators, we need to replace the zero modes in the wave-functions by the fields \( r \) and \( \theta \) and then regularise them.

5. Application to \( \sigma \)-models

In the previous section, we used the techniques of SUSY quantum mechanics to solve the black hole problem. Here we will show that this method is quite general and is applicable to conformal field theories with \( \sigma \)-model representations. To illustrate this, let us consider a non-linear \( \sigma \)-model in the background of the graviton \( G_{\mu \nu} \), the dilaton \( D \) and the tachyon \( T \) \([33-35]\). The \( \sigma \)-model action with \( d \) scalar fields is given by

\[
S_{\sigma} = \frac{1}{4\pi \alpha'} \int d^2z \sqrt{g} \left( \frac{1}{2} g^{ab} G_{\mu \nu} \partial_a x^\mu \partial_b x^\nu - \alpha' R^{(2)} D(x) + T(x) \right). \tag{5.1}
\]

The condition of conformal invariance of this \( \sigma \)-model is implemented by setting the \( \beta \)-functions to zero. This condition gives the equations of motion of \( G_{\mu \nu} \), \( D \) and \( T \) as

\[
R_{\mu \nu} - 2 \nabla_\mu \nabla_\nu D + \nabla_\mu T \nabla_\nu T = 0
\]
\[
R + 4(\nabla D)^2 - 4 \nabla^2 D + (\nabla T)^2 + V(T) + c = 0
\]
\[
-2 \nabla^2 T + 4 \nabla D \nabla T + V'(T) = 0
\]

where \( c = (d - 26)/3\alpha' \). These equations can be derived from the target space action

\[
S = \int d^d x \exp(-2D) \sqrt{G}[R - 4(\nabla D)^2 + (\nabla T)^2 + V(T) + \frac{26 - d}{3}]. \tag{5.3}
\]

It is well known that, in \( \sigma \)-model representation, the \( L_0 \) operator on the world
sheet is identified with the target space Laplacian $\Delta$. It is given by

$$\Delta = \frac{1}{\exp(-2D)\sqrt{G}} \nabla_\mu \exp(-2D)\sqrt{G}G^{\mu\nu}\nabla_\nu$$

(5.4)

where $\nabla_\mu$ is the covariant derivative, $G_{\mu\nu}$ is the metric on the target space and $D$ is the dilaton field. Thus we see that the problem of determining the spectrum of Virasoro vertex operators is equivalent to finding all the solutions of the Laplacian $\Delta$. For a fixed background $G_{\mu\nu}$ and $D$, the Laplacian can be simplified. This reduces the problem to a second order differential equation. We can absorb the factor $G^{1/4}\exp(-D)$ in the solution, exactly in the same way as we did in the black hole conformal field theory in sec.2. In the case of a fixed background, it is always possible to absorb such a factor. Since this transformation linearises the integration measure we can write the Laplacian as a Schrödinger operator. It is for this Schrödinger equation that the techniques of SUSY quantum mechanics can be used, although for an arbitrary background, this problem may not have supersymmetry.

Now let us consider an exactly solvable quantum mechanics problem given by

$$\left(\frac{d^2}{dr^2} + V(r)\right)\Psi(r) = E\Psi(r)$$

(5.5)

where $V(r)$ is a potential whose full spectrum of eigenvalues and eigenfunctions is known. Let these wave-functions be characterised by a set of quantum numbers $\{b_i\}$. Then it is always possible to decompose any eigenfunction $\Psi$ into a product of two functions $\phi$ and $\chi$. The function $\phi$ carries all the information about $\{b_i\}$ while $\chi$ is independent of $\{b_i\}$, i.e.

$$\left(\frac{d^2}{dr^2} + V(r)\right)\chi(r)\phi(r, \{b_i\}) = E\chi(r)\phi(r, \{b_i\}).$$

(5.6)

If we eliminate the function $\chi(r)$ from the above equation, we get

$$\left(\frac{1}{\eta(r)} \frac{\partial}{\partial r}\eta(r)\frac{\partial}{\partial r} + \tilde{V}(r)\right)\phi(r, \{b_i\}) = E\phi(r, \{b_i\})$$

(5.7)

where $\eta(r)$ is determined by $\chi(r)$. Let us consider, as an example, the target space
to be two dimensional. Let it be parametrized by \( r \) and \( \theta \). Then we can interpret of the operator on the L.H.S. of eq.(5.7) as a Laplacian in this two dimensional target space. With the identification

\[
\tilde{V}(r) = \bar{V}(r) \frac{\partial^2}{\partial \theta^2}
\]

and

\[
\Phi(r) = \phi(r, \{b_i\}) e^{im\theta},
\]

eq(5.8)
eq(5.9)
eq(5.7) can be written as

\[
\left( \frac{1}{\eta(r)} \frac{\partial}{\partial r} \eta(r) \frac{\partial}{\partial r} + \bar{V}(r) \frac{\partial^2}{\partial \theta^2} \right) \Phi(r, \theta) = E \Phi(r, \theta).
\]

Now it is easy to read out the metric \( G_{\mu\nu} \) and the dilaton \( D \) from the Laplacian occurring in eq.(5.10). The metric is given by

\[
ds^2 = dr^2 + \frac{1}{\bar{V}(r)} d\theta^2
\]

and the dilaton is

\[
D = -\frac{1}{2} \log(\eta(r) \sqrt{\bar{V}(r)}).
\]

Thus it is possible to write down a non-linear \( \sigma \)-model starting from an exactly solvable quantum mechanics problem. But there are a few subtleties involved in showing this correspondence which are worth pointing out. Firstly, one non-linear \( \sigma \)-model corresponds to a set of quantum mechanical problems and secondly, the graviton and the dilaton derived from the quantum mechanical problem should satisfy the \( \beta \)-function equations for the \( \sigma \)-model to be conformally invariant. This is an important constraint because the target space action, which is an effective action derived from the \( \beta \)-function equations, otherwise would not make sense.
6. Summary and Discussion

Once we have the full spectrum we can ask whether all the representations are allowed. In unitary current algebra theories, only a finite number of representations occur at a given level $k$. In unitary theories, $k$ is always an integer, whereas, in our case, it can take any real value because we are considering the coset model based on a non-compact group. Allowed representations in unitary theories are called integrable representations. The notion of integrable representations can also be extended to the fractional levels of the current algebra [36] [37]. In the case of a fractional level $k = t/u$ of the SU(2) current algebra, where $u$ is a positive integer and $t$ is a non-zero integer with $u$ and $t$ coprime, we get integrable representations if

$$2u + t - 2 \geq 0.$$  \hfill (6.1)

The black hole problem corresponds to the level $k = -9/4$ of the SU(2) current algebra. (It is related to $k = 9/4$ of the SL(2,R) by Wick rotation of one of the coordinates.) It is easy to see that the black hole problem does not satisfy the condition given in eq.(6.1). Therefore all the representations are non-integrable. Hence, we have no a priori reason to rule out any representation.

We studied the black hole CFT by mapping the problem into a quantum mechanical problem. We showed that this quantum mechanics problem can be solved exactly. To show this, we used the techniques of SUSY quantum mechanics and shape invariance. We determined both the bound state and the scattering spectrum and identified it with the spectrum of the vertex operators in the black hole CFT. These vertex operators are actually the chiral algebra primary fields. We indicated how the remaining Virasoro primary fields can be determined.

We also showed that the techniques of SUSY quantum mechanics and shape invariance can be used for some CFTs with a $\sigma$-model representation. Conversely, it is possible to show that for an exactly solvable quantum mechanical problem we can associate a $\sigma$-model, i.e., it is possible to determine the background fields of
the $\sigma$-model starting from a specific solvable quantum mechanical problem. But
this correspondence between the quantum mechanical problem and the non-linear
$\sigma$-model involves a few subtle points. We hope to resolve them in future.

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