Topical Review

$\mathcal{N} = 2$ SUSY gauge theories on $S^4$

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Abstract
We review exact results in $\mathcal{N} = 2$ supersymmetric gauge theories defined on $S^4$ and its deformation. We first summarize the construction of rigid SUSY theories on curved backgrounds based on off-shell supergravity, then explain how to apply the localization principle to supersymmetric path integrals. Closed formulae for partition function as well as expectation values of non-local BPS observables are presented.

Keywords: SUSY, $\mathcal{N} = 2$ supersymmetric gauge, off-shell supergravity

1. Introduction

Four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories are known to be mathematically highly constrained, and yet they can accommodate a variety of interesting physical phenomena. One can therefore ask general questions about the physics of strong gauge interactions in these theories and expect a rather precise answer. The first non-trivial result was obtained by Seiberg and Witten [2, 3] for the structure of Coulomb branch moduli space as well as the mass of BPS particles. By combining the constraints from $\mathcal{N} = 2$ supersymmetry together with electro-magnetic duality, they determined the exact prepotential which encodes the full low-energy effective Lagrangian, including the contribution of instantons which were otherwise very difficult to evaluate at that time.

Another powerful approach to 4D $\mathcal{N} = 2$ theories is localization, which makes use of supersymmetry to reduce the difficult problem of the infinite-dimensional path integral to a much simpler problem. There is a class of 4D topological field theories, called topologically twisted theories [4], which are obtained from $\mathcal{N} = 2$ theories by changing the spin of fields according to their quantum numbers under the internal symmetry $SU(2)_R$. Once the SUSY localization is applied to those theories, the path integral can be shown to reduce to a finite-dimensional integral on instanton moduli spaces. Nekrasov later proposed the so-called Omega-deformation [5–8] of the topologically twisted theories, which further simplifies the

* This is a contribution to the review issue ‘Localization techniques in quantum field theories’ (ed V Pestun and M Zabzine) which contains 17 chapters available at [1].
integrals on moduli spaces by using the rotational symmetry of $\mathbb{R}^4$. The resulting path integral is called Nekrasov’s instanton partition function, and is expressed as a sum over point-like instanton configurations localized at the origin. Nekrasov’s partition function was shown to reproduce the prepotential of $\mathcal{N} = 2$ theories in the limit of small Omega-deformation. Moreover, it has given us a new insight into the connection between $\mathcal{N} = 2$ gauge theories and other branches of physics and mathematics, such as topological strings or integrable systems.

1.1. Pestun’s pioneering work

Application of the localization principle to quantum field theories has been long restricted to topological field theories with scalar supersymmetry. A major breakthrough was made by Pestun [9] who constructed $\mathcal{N} = 2$ supersymmetric gauge theories on the four-sphere $S^4$ and derived closed formulae for partition function as well as expectation values of certain Wilson loops [9]. This article reviews his result and some of the subsequent work on exact supersymmetric observables in $\mathcal{N} = 2$ supersymmetric gauge theories on $S^4$ and its deformations.

The original motivation of the work [9] was to prove a conjecture which arose from the study of AdS/CFT correspondence, that the expectation values of supersymmetric circular Wilson loops in $\mathcal{N} = 4$ super Yang–Mills theory are given by Gaussian matrix integral [10, 11]. Instead of topological field theories with scalar SUSY, Pestun constructed physical $\mathcal{N} = 2$ SUSY theories on $S^4$ via conformal map from flat $\mathbb{R}^4$. By a successful application of the SUSY localization principle, the path integral was shown to reduce to a finite-dimensional integral. A one-parameter (mass) deformation of the $\mathcal{N} = 4$ SYM called $\mathcal{N} = 2^*$ theory was studied in detail, and it was found that the integrand simplifies dramatically at a special value of the mass. In this way, it was analytically shown that the $S^4$ partition function is precisely given by a Gaussian matrix integral [9, 12]. See contribution [13] for more detail on the application of localization to the problems in AdS/CFT.

Pestun’s work is the first nontrivial example in which a coherent and fully explicit prescription was given for physical supersymmetric models on curved spaces, from the construction of theories to the evaluation of supersymmetric observables. Exact formulae were obtained later for partition functions of supersymmetric gauge theories on $S^3$ [14–16], $S^2$ [17, 18] and $S^5$ [19–21] by following basically the same program. Together with the supersymmetric partition functions on $S^1 \times S^d$ called superconformal indices, the sphere partition functions are now regarded as powerful analytic tools to explore non-perturbative aspects of SUSY gauge theories. In particular, for CFTs with the right number of supersymmetry in even dimensions, it was shown that the sphere partition function is protected from regularization ambiguity and computes the Kähler potential for the space of marginal couplings [22, 23].

Important applications of Pestun’s result have been made for a family of 4D $\mathcal{N} = 2$ theories of ‘class S’ [24], that are known to show up on the worldvolume of multiple M5-branes (5 + 1-dimensional object in M-theory) wrapped on punctured Riemann surfaces. In particular, Alday, Gaiotto and Tachikawa (AGT) discovered a surprising correspondence between exact $S^3$ partition functions of the class S superconformal theories for two M5-branes and correlation functions of 2D Liouville conformal field theory [25] (see contribution [26]). Generalization to gauge groups of higher rank and Toda conformal field theories was soon proposed by Wyllard [27]. This discovery brought us another new insight into the mathematical structure underlying 4D $\mathcal{N} = 2$ gauge theories. It also triggered an extensive study of similar correspondences between quantum field theories in different dimensions that follow from compactifications of multiple M5-branes.
1.2. Squashing

Supersymmetric gauge theories and exact physical observables have also been studied on manifolds which are less symmetric than sphere. One motivation for this generalization arose from the AGT relation, since the partition function on the round $S^4$ was shown to correspond to Toda CFTs at a special (self-dual) value of the coupling, $b = 1$. Nontrivial results along this line of generalization were first obtained in [28] and [29] for 3D $\mathcal{N} = 2$ supersymmetric theories on certain squashed spheres with a background vector field turned on. The supersymmetry there is characterized by generalized Killing spinors with a specific coupling to the vector field.

For theories with a different amount of SUSY and in other dimensions, the most natural framework to explore supersymmetric curved backgrounds is off-shell supergravity [30], See contribution [31]. For 4D $\mathcal{N} = 2$ theories this idea was employed in [32] to construct supersymmetric ellipsoid backgrounds, which depend on a squashing parameter $b$ measuring the deformation from the round sphere geometry. The partition function on this background was shown to reproduce the correlators of Toda CFTs at general values of the coupling. The rigid supersymmetric backgrounds were systematically classified and deformations of $S^4$ were studied within $\mathcal{N} = 1$ off-shell supergravity in [33–37], and in [38] within $\mathcal{N} = 2$ supergravity. For Lorentzian theories with $\mathcal{N} = 2$ SUSY, general supersymmetric backgrounds were studied in relation to BPS black holes in earlier work [39, 40]. Different versions of deformations of the round $S^4$ have been studied in [41, 42], while the localization principle on backgrounds of other topologies, such as products of spheres and $AdS$ spaces, have been studied in [43–47], where the results have been used to study the loop correction to the entropy of certain charged black holes. Supersymmetric deformations of the round sphere geometry have also been applied to the computation of Rényi entropy in gauge theories in $D = 3, 4, 5$; see [48–52].

1.3. Supersymmetric observables

Localization techniques have also been applied to compute expectation values of various supersymmetric observables. An important class of observables in 4D $\mathcal{N} = 2$ theories are supersymmetric Wilson and ‘t Hooft loop operators, defined from the worldlines of electrically or magnetically charged particles. It is a remarkable feature of $\mathcal{N} = 2$ supersymmetric theories that one can make quantitative statements about properties of these particles, in particular, how they are exchanged among each other under S-duality [24]. Also, a number of nontrivial conjectures on the expectation values of loop operators have been proposed from AGT relation and checked explicitly [53–55]. The effect of deformations of the theories on Wilson loop and local observables were studied in [56].

Another important class of nonlocal operators are surface operators, which have two dimensional worldvolume inside four dimensions. See [57] for a review. They are defined either by introducing two-dimensional field theory degrees of freedom on the surface or by imposing singular behavior on gauge and other fields along the surface. They were first introduced in [58] in the study of geometric Langlands program within the framework of 4D $\mathcal{N} = 4$ SYM theory. Interesting progress has been made for surface operators in $\mathcal{N} = 2$ supersymmetric theories through the comparison of the gauge theory analysis with the results from topological string or predictions from AGT relation [53, 59–67].

1.4. Conventions

Throughout this article, we use the indices $\alpha, \beta, \cdots$ and $\dot{\alpha}, \dot{\beta}, \cdots$ for 4D chiral and anti-chiral spinors. The indices are raised and lowered by the antisymmetric invariant tensors $\epsilon^{\alpha \beta}, \epsilon^{\dot{\alpha} \dot{\beta}}$, $\epsilon_{\alpha \beta}, \epsilon_{\dot{\alpha} \dot{\beta}}$ with nonzero elements.
Following Wess–Bagger [68] we suppress the pairs of undotted indices contracted in the up-left, down-right order, or pairs of dotted indices contracted in the down-left, up-right order. We also use the set of $2 \times 2$ matrices $(\sigma^a)_{\alpha\dot{\alpha}}$ and $(\bar{\sigma}^a)^{\alpha\dot{\alpha}}$ with $a = 1, \ldots, 4$ satisfying standard algebras. In terms of Pauli’s matrices $\tau^a$ they are given by
\[
\sigma^a = -i\tau^a, \quad \bar{\sigma}^a = i\tau^a, \quad (a = 1, 2, 3)
\]
\[
\sigma^4 = 1, \quad \bar{\sigma}^4 = 1.
\]
We also use $\sigma_{ab} = \frac{1}{2}(\sigma_a\sigma_b - \sigma_b\sigma_a)$ and $\bar{\sigma}_{ab} = \frac{1}{2}(\bar{\sigma}_a\sigma_b - \bar{\sigma}_b\sigma_a)$. Note that $\sigma_{ab}$ is anti-self-dual, i.e. $\sigma_{ab} = -\frac{1}{2}\epsilon_{abcd}\sigma_{cd}$, while $\bar{\sigma}_{ab}$ is self-dual.

For 4D $\mathcal{N} = 2$ theories on flat space, supersymmetry is parametrized by constant spinors $\xi^A_\alpha$ and $\xi^A_{\alpha}$. The index $A = 1, 2$ indicates that they transform as doublet under $SU(2)$ R-symmetry which commutes with the generators of Poincaré symmetry but rotates the supercharges. In addition, $\xi^A_\alpha$ and $\xi^A_{\dot{\alpha}}$ carry $U(1)$ R-charges $+1$ and $-1$. Throughout this article, these SUSY parameters are Grassmann-even quantities.

2. Construction of theories

Here we review the construction of $\mathcal{N} = 2$ supersymmetric gauge theories on $S^4$ using off-shell supergravity. We then present a number of nontrivial supergravity backgrounds with rigid supersymmetry, including the supersymmetric deformation of $S^4$ into ellipsoids.

2.1. Conformal Killing spinors on $S^4$

As the round $S^4$ is conformally flat, 4D $\mathcal{N} = 2$ superconformal theories can be constructed on $S^4$ by a conformal map from flat $\mathbb{R}^4$. Let $\ell$ be the radius of $S^4$. The superconformal symmetry is then described by conformal Killing spinors satisfying
\[
D_m \xi_A = \left( \partial_m + \frac{1}{4} \Omega^a_m \sigma_{ab} \right) \xi_A = -i\sigma_m \xi_A, \quad D_m \bar{\xi}_A = -\frac{i}{4\ell^2} \bar{\sigma}_m \bar{\xi}_A, \\
D_m \bar{\xi}_A = \left( \partial_m + \frac{1}{4} \Omega^a_m \bar{\sigma}_{ab} \right) \bar{\xi}_A = -i\bar{\sigma}_m \bar{\xi}_A, \quad D_m \xi_A = -\frac{i}{4\ell^2} \sigma_m \xi_A.
\]

This is a coupled first-order differential equation for 16 spinor components, and therefore has 16 independent solutions corresponding to the fermionic generators of the 4D $\mathcal{N} = 2$ superconformal algebra. Lagrangian theories of vector multiplets and massless hypermultiplets are all superconformal at the classical level, so they can be unambiguously defined on the round $S^4$ in this way. For massive theories on $S^4$, the superconformal symmetry is broken to a subgroup $OSp(2|4)$. This means that the mass terms are constructed in such a way that a subset of supercharges corresponding to the Killing spinors
\[
D_m \xi_A = -\frac{i}{2\ell} \sigma_m \xi_B \cdot \gamma^B_A, \quad D_m \bar{\xi}_A = -\frac{i}{2\ell} \bar{\sigma}_m \bar{\xi}_B \cdot \gamma^B_A
\]
is preserved. Here $\gamma^B_A$ are constant traceless $U(2)$ matrices satisfying $\gamma t = i t = 1$. They can be brought into a standard form, say $t = \bar{t} = \tau_3$, using R-symmetry.
2.2. Generalized Killing spinors and $\mathcal{N} = 2$ Supergravity

Off-shell supergravity allows one to construct supersymmetric field theories on more general curved backgrounds [30]. The independent fields in the standard gravity multiplet (also called Weyl multiplet) in 4D $\mathcal{N} = 2$ supergravity [69–72] (see also reviews [73, 74]) are listed in Table 1. Supergravity backgrounds are specified by the classical values of all the bosonic fields, while the fermionic fields are all taken to vanish. A background is supersymmetric if the local SUSY variation of fermions, (we quote the formula from [74] with certain rescalings of fields)

$$Q\psi_{mA} = D_m\xi_A + T^{kl}\sigma_{kl}\sigma_{m}\xi_A + i\sigma_{m}\xi_A,$$

$$Q\bar{\psi}_{mA} = D_m\bar{\xi}_A + \bar{T}^{\bar{kl}}\bar{\sigma}_{\bar{kl}}\sigma_{m}\bar{\xi}_A + i\sigma_{m}\bar{\xi}_A,$$

$$Q\bar{\eta}_A = 8\sigma^{m}\sigma\bar{\xi}_A D_mT_{mn} + 16i T^{\bar{kl}}\sigma\bar{\xi}_A = 3\bar{M}\xi_A + 2i\sigma^{mn}\xi_B(V_{mn})_A + 4i\sigma^{mn}\bar{\xi}_A\bar{V}_{mn},$$

$$Q\bar{\eta}_A = 8\sigma^{m}\sigma\bar{\xi}_A D_mT_{mn} + 16i T^{\bar{kl}}\sigma\bar{\xi}_A = 3\bar{M}\xi_A + 2i\sigma^{mn}\xi_B(V_{mn})_A - 4i\sigma^{mn}\bar{\xi}_A\bar{V}_{mn}.$$

all vanish for a suitable choice of spinor fields $(\xi_A, \bar{\xi}_A)$ and $(\xi_A', \bar{\xi}_A')$. Here the covariant derivatives are with respect to the local Lorentz as well as $SU(2) \times U(1)$ R-symmetries. For example,

$$D_m\xi_A \equiv \left( \partial_m + \frac{1}{4} \Omega^{ab}_{m} \sigma_{ab} \right) \xi_A + i\xi_B(V_{mn})_A - i\bar{V}_m\xi_A. \tag{2.4}$$

We also denoted the $U(1)_R$ gauge field strength by $\partial_m\bar{V}_n - \partial_n\bar{V}_m \equiv \bar{V}_{mn}$ and similarly for the $SU(2)_R$ field strength $(V_{mn})_A$. With the simplifying assumption

$$\bar{V}_m = 0. \tag{2.5}$$

the above BPS condition can be transformed into the form presented in [32],

$$D_m\xi_A + T^{kl}\sigma_{kl}\sigma_{m}\xi_A = -i\sigma_{m}\xi_A,$$

$$D_m\bar{\xi}_A + \bar{T}^{\bar{kl}}\bar{\sigma}_{\bar{kl}}\sigma_{m}\bar{\xi}_A = -i\sigma_{m}\bar{\xi}_A,$$

$$\sigma^{m}\bar{\sigma}D_mD_n\xi_A + 4\bar{D}_{m}T_{mn}\sigma^{m}\sigma^{l}\bar{\xi}_A = M\xi_A,$$

$$\bar{\sigma}^{m}\sigma^{n}D_mD_n\bar{\xi}_A + 4\bar{D}_{m}\bar{T}_{mn}\bar{\sigma}^{m}\bar{\sigma}^{l}\xi_A = \bar{M}\bar{\xi}_A, \tag{2.6}$$

where $M \equiv \bar{M} = -\frac{1}{2}R$. This gives a consistent generalization of the conformal Killing spinor equation (2.1) on $S^4$. Hereafter we use $M$ rather than $\bar{M}$ in accordance with [32], but note that the latter has a better transformation property under Weyl rescaling. The equations (2.6) are invariant under $g_{mn} \to e^{\phi}g_{mn}$ if accompanied by
\[ \xi_A \rightarrow e^{\frac{1}{2}\mu} \xi_A, \quad \xi_A^i \rightarrow e^{-\frac{1}{2}\mu} \xi_A^i, \quad T_{mn} \rightarrow e^{-\rho} T_{mn}, \quad \tilde{M} \rightarrow e^{-2\rho} \tilde{M}, \]
\[ \tilde{\xi}_A \rightarrow e^{\frac{1}{2}\mu} \tilde{\xi}_A, \quad \tilde{\xi}_A^i \rightarrow e^{-\frac{1}{2}\mu} \tilde{\xi}_A^i, \quad \tilde{T}_{mn} \rightarrow e^{-\rho} \tilde{T}_{mn}. \]  
(2.7)

2.3. Transformation laws and Lagrangians

Supergravity also gives a description of local SUSY-invariant couplings of matter systems to gravity. By sending the Newton constant to zero in such a description, one can decouple gravity from the matter and treat the fields in gravity multiplet as classical background fields. In this way one can systematically construct rigid SUSY theories on various curved backgrounds.

Vector multiplet consists of a gauge field \( A_m \), scalars \( \phi, \tilde{\phi} \), gauginos \( \lambda_{\alpha A}, \tilde{\lambda}_{\alpha A} \) and an \( SU(2)_R \)-triplet auxiliary scalar \( D_{AB} \). They transform under supersymmetry as
\[
\begin{align*}
Q A_m &= i \xi^A \sigma_m \lambda_A - i \xi^A \sigma_m \lambda_A, \\
Q \phi &= -i \xi^A \lambda_A, \\
Q \tilde{\phi} &= +i \xi^A \lambda_A, \\
Q \lambda_A &= \frac{1}{2} \sigma^{mn} \xi_A (F_{mn} + 8 \tilde{\phi} T_{mn}) + 2 \sigma^m D_m \phi + \sigma^m D_m \xi_A \phi + 2 i \xi_A [\phi, \tilde{\phi}] + D_{AB} \xi^B, \\
Q \bar{\lambda}_A &= \frac{1}{2} \sigma^{mn} \xi_A (F_{mn} + 8 \tilde{\phi} T_{mn}) + 2 \sigma^m D_m \phi - \sigma^m D_m \xi_A \phi - 2 i \xi_A [\phi, \tilde{\phi}] + D_{AB} \xi^B, \\
Q D_{AB} &= -i \xi_A \sigma^m D_m \lambda_B - i \xi_B \sigma^m D_m \lambda_A + i \xi_A \sigma^m D_m \tilde{\lambda}_B + i \xi_B \sigma^m D_m \tilde{\lambda}_A - 2 [\phi, \tilde{\xi}_A \lambda_B + \xi_B \lambda_A] + 2 [\phi, \tilde{\xi}_A \lambda_B + \xi_B \lambda_A].
\end{align*}
\]  
(2.8)

Note that the following combination of vector and scalar fields is \( Q \)-invariant,
\[ \hat{\phi} \equiv 2 i \xi^A \xi_A \phi - 2 i \xi^A \sigma^m \xi_A A_m, \]  
(2.9)

which will become important later. SUSY invariant Yang–Mills kinetic Lagrangian reads
\[ L_{YM} = \frac{1}{8} \text{Tr} \left( \frac{1}{2} F_{mn} F^{mn} + 16 F_{mn} (\tilde{\phi} T^{mn} + \phi T^{mn}) + 64 \tilde{\phi}^2 T_{mn} T^{mn} + 64 \phi^2 \tilde{T}_{mn} \tilde{T}^{mn} \right) \\
- 4 D_m \phi D^m \phi + 2 M \phi \phi - 2 i \lambda^A \sigma^m D_m \tilde{\lambda}_A - 2 \lambda^A [\phi, \lambda_A] + 2 \bar{\lambda}^A [\phi, \tilde{\lambda}_A] \\
+ 4 [\phi, \tilde{\phi}]^2 - \frac{i}{2} D^{AB} D_{AB} + \frac{i \theta}{32 \pi^2} \text{Tr} \left( \xi^{klm} F_{kl} F_{mn} \right), \]  
(2.10)

One instanton factor is \( q = e^{2 \pi i \tau} \) with \( \tau = \frac{\theta + 4 \pi i}{e^2} \).

For \( U(1) \) vector multiplets one can also construct a Feyet–Iliopoulos type invariant. Let \( w^{AB} = \theta^{BA} \) be an \( SU(2)_R \)-triplet background field satisfying
\[
\begin{align*}
w^{AB} \xi_B &= \frac{1}{2} \sigma^p D_p \xi^A + 2 T_{kl} \phi^{kl} \xi^A, \\
w^{AB} \tilde{\xi}_B &= \frac{1}{2} \sigma^p D_p \tilde{\xi}^A + 2 T_{kl} \phi^{kl} \tilde{\xi}^A.
\end{align*}
\]  
(2.11)

Then the following is SUSY-invariant.
\[ L_{FI} = \zeta \left\{ w^{AB} D_{AB} - M (\phi + \tilde{\phi}) - 64 \phi T_{kl} T^{kl} - 64 \tilde{\phi} \tilde{T}_{kl} \tilde{T}^{kl} - 8 F^{kl} (T_{kl} + \tilde{T}_{kl}) \right\}. \]  
(2.12)
Note that this term breaks the conformal invariance. By comparing with the Killing spinor equation (2.2), one finds \( \epsilon_{AB} = \epsilon_{BA} = i \epsilon_{AB} \) on the round \( S^3 \) of radius \( \ell \).

The system of \( r \) hypermultiplets consists of scalars \( q_{IA} \) and fermions \( \psi_{\alpha I}, \bar{\psi}_{\dot{\alpha} I} \), with \( I = 1, \cdots, 2r \). The scalars obey the reality condition
\[
(q_{IA})^\dagger = q^I = \epsilon^{AB} \Omega^I_j q_{JB},
\]
where \( \Omega^I_j \) is the real antisymmetric \( Sp(r) \)-invariant tensor satisfying
\[
(\Omega^I_j)^* = -\Omega_{IJ}, \quad \Omega^I_j \Omega_{JK} = \delta^K_L.
\]
The tensor \( \Omega^I_j \) and its inverse are used to raise or lower the \( Sp(r) \) indices. The pair of \( Sp(r) \) indices will be suppressed in the following when contracted in the top-left, bottom-right order, like \( q^I q_{IA} = q^I q_A \). The hypermultiplet fields can be coupled to vector multiplet by embedding the gauge group into \( Sp(r) \). The covariant derivative of \( q_{IA} \), for example, is then given by
\[
D_m q_{IA} \equiv \partial_m q_{IA} - i(A_m)^I_j q_{JA} + i q_{IB}(V_m)^B_A.
\]

It is a little intricate to write down an off-shell SUSY transformation rule for hypermultiplet fields explicitly. As is well known, for rigid \( N = 2 \) SUSY theories with hypermultiplets on flat space, there is no formalism which realizes all the eight supercharges at once with finite number of auxiliary fields. However, when applying the localization method, one always picks up one of the supercharges corresponding to a particular choice of Killing spinor \( \xi_A, \bar{\xi}_A \), and requires that particular supercharge be realized off-shell. What we will present here is an off-shell realization of just one supercharge.

To balance the number of bosons and fermions in hypermultiplet, we need to introduce the auxiliary scalar fields \( F_{I\dot{A}} \), where \( I \) is the \( Sp(r) \) index and \( \dot{A} = 1, 2 \) is a new auxiliary index. We also introduce [32] the spinor fields \( \bar{\xi}_A, \xi_{\dot{A}} \) satisfying
\[
\begin{align*}
\bar{\xi}_A \xi_{\dot{B}} - \bar{\xi}_{\dot{A}} \xi_B &= 0, \\
\xi^A \xi_A + \bar{\xi}^A \bar{\xi}_{\dot{A}} &= 0, \\
\bar{\xi}^A \bar{\xi}_A + \xi^A \xi_{\dot{A}} &= 0, \\
\xi^A \sigma^m \bar{\xi}_A + \bar{\xi}^A \sigma^m \xi_{\dot{A}} &= 0.
\end{align*}
\]
A solution to the above conditions is given by
\[
\bar{\xi}_{\dot{A}} = c^{+} \xi_{\dot{A}}, \quad \xi_{\dot{A}} = -c^{-1} \bar{\xi}_{\dot{A}} \quad (A = \dot{A} = 1, 2) \quad \text{where} \quad c = \frac{\bar{\xi}^A \xi_{\dot{A}}}{\xi^B \xi_{\dot{B}}}. \tag{2.16}
\]

There are more solutions since the equations (2.15) are invariant under local \( SL(2) \) transformations acting \( \bar{\xi}_{\dot{A}} \) and \( \xi_A \) through the index \( \dot{A} \), but one can show the solution is unique up to this \( SL(2) \). Using them, the SUSY transformation rule for hypermultiplet can be expressed as follows,
\[
\begin{align*}
Q q_{IA} &= -i \xi_A \psi + i \bar{\xi}_{\dot{A}} \bar{\psi}, \\
Q \psi &= 2 \sigma^m \xi_A D_m q^A + \sigma^m D_m \xi_A q^A - 4 i \xi_A \phi q^A + 2 \bar{\xi}_{\dot{A}} F_{\dot{A}}^A, \\
Q \bar{\psi} &= 2 \sigma^m \bar{\xi}_{\dot{A}} D_m q^A + \sigma^m D_m \bar{\xi}_{\dot{A}} q^A - 4 i \bar{\xi}_{\dot{A}} \phi \bar{q}^A + 2 \xi_A F_A^A, \\
Q F_{\dot{A}} &= i \sigma^m \sigma^m D_m \psi - 2 \xi_A \phi \psi - 2 \bar{\xi}_{\dot{A}} \bar{\phi} \bar{q}^B + 2 i \bar{\xi}_{\dot{A}} (\sigma^{CB} T_{AB}) \psi \\
&\quad - i \xi_A \bar{\sigma}^m \sigma^m D_m \bar{\psi} + 2 \bar{\xi}_{\dot{A}} \bar{\phi} \bar{\psi} + 2 \xi_A \phi \bar{q}^B - 2 i \xi_A (\sigma^{CB} T_{AB}) \bar{\psi} \tag{2.17}.
\end{align*}
\]
A similar off-shell transformation rule was given in [9] for 4D $\mathcal{N} = 4$ SYM theory using Berkovits construction of 10D $\mathcal{N} = 1$ SYM theory [75]. The SUSY invariant kinetic Lagrangian is

$$
\mathcal{L}_{\text{kin}} = \frac{1}{2} D_m q^A D^n q_A - q^A \{ \phi, \bar{\phi} \} q_A + \frac{i}{2} q^A D_{AB} q^B + \frac{1}{8} (M + R) q^A q_A
$$

The auxiliary symmetry transforms $F_{\hat{I}A}$, $\xi_{\hat{I}}$ and $\bar{\xi}_{\hat{I}}$ all as doublets, and it is actually $SU(2)$ since we need to impose on $F_{\hat{I}A}$ a reality condition similar to (2.13). To complete the off-shell formalism for hypermultiplets, one needs to specify the background gauge field $(\bar{V}_m)_B^A$ for this auxiliary symmetry which we call $SU(2)_B$.

The commutant of the gauge group within $Sp(r)$ gives the global symmetry. One can introduce the mass for hypermultiplets by coupling an abelian subgroup of the global symmetry to background vector multiplets. Mass parameters are identified with the constant value of their scalar components $\phi, \bar{\phi}$. They have to be chosen not to break supersymmetry, so the fermion components of the background vector multiplet must have vanishing SUSY variation. The classical values

$$
\phi = \bar{\phi} = \text{constant}, \quad D_{AB} = 2w_{AB}\phi
$$

preserve the supersymmetry if the corresponding Killing spinor satisfies (2.11).

The square of supersymmetry $Q$ yields a sum of bosonic symmetry transformations including the translation by $v^m \equiv 2\xi^A \sigma^m \xi_A$,

$$
Q^2 = i\mathcal{L}_v + \text{Gauge}[2\phi \xi^A \xi_A - 2\bar{\phi} \xi^A \xi_A + v^m A_m] + \text{Lorentz}[D_{[a}V_{b]} + v^m \Omega_{mab}] + \text{Scale}[ - \frac{1}{2} \xi^A \sigma^m D_m \xi_A - \frac{1}{2} D_m \xi^A \sigma^m \xi_A] + \text{R}_{U(1)}[ - \frac{i}{2} \xi^A \sigma^m D_m \xi_A + \frac{1}{4} D_m \xi^A \sigma^m \xi_A] + \text{R}_{SU(2)}[ - i\xi_A (\sigma^m D_m \bar{\xi}_B) + iD_m \xi_A (\sigma^m \bar{\xi}_B)] + v^m V_{mAB} + \text{R}_{SU(2)}[2i\bar{\xi}_B (\sigma^m D_m \bar{\xi}_B) - 2iD_m \xi_A (\sigma^m \bar{\xi}_B)] + 4i\xi_A (\sigma^m T_{AB} \bar{\xi}_B) + 4i\bar{\xi}_B (\sigma^m T_{AB} \xi_A) + v^m \bar{v}_{mAB}].
$$

Note that the Killing spinor $(\xi_A, \bar{\xi}_A)$, the auxiliary spinor $(\bar{\xi}_A, \xi_A)$ as well as all the background fields belonging to the gravity multiplet have to be invariant under $Q^2$. This can be used to determine the form of $(\bar{V}_m)_B^A$. Note also that, if one wants to introduce the mass or FI terms into the theory, the Killing spinor has to satisfy an extra condition (2.11). This implies

$$
\xi^A \sigma^m D_m \bar{\xi}_A = \bar{\xi}^A \sigma^m D_m \xi_A = 0,
$$

so that $Q^2$ does not yield scale or $U(1)_R$ transformations.
2.4. Examples of SUSY backgrounds

Let us review here some important examples of classical supergravity backgrounds with rigid SUSY.

2.4.1. Topological twist. It is known that 4D \( \mathcal{N} = 2 \) theories can be put on any 4D spaces preserving a single scalar supercharge by a procedure called the Donaldson–Witten topological twist [4]. In the supergravity framework, the topological twist corresponds to turning on a background \( SU(2)_R \) gauge field which equals the self-dual part of spin connection,

\[
\frac{1}{4} \Omega^{ab}_{\mu}(\partial_{ab})^A_B + i (V_m)^A_B = 0, \tag{2.22}
\]

so that the constant spinor \( \xi_{\alpha A} = 0, \dot{\xi}^\alpha_A = \delta^\alpha_A \) satisfies the Killing spinor equation (2.3). The supersymmetry \( Q \) is nilpotent up to gauge transformations, so that one can define physical observables by cohomology of \( Q \) acting on gauge-invariant operators.

The choice of the background \( SU(2)_R \) gauge field allows one to identify the indices \( A, B, \cdots \) with the dotted spinor indices. The chiral gaugino \( \lambda^A_{\alpha} \) then turns into a vector \( \psi_m \) which is the superpartner of \( A_m \) under \( Q \), whereas the anti-chiral gaugino \( \bar{\lambda}_A^{\dot{\alpha}} \) gives rise to a scalar \( \eta \) and a self-dual tensor \( \chi_{mn}^{+} \). The fermion \( \chi_{mn}^{+} \) and its superpartner play the role of the Lagrange multiplier which reduces the path integral over the gauge field to a finite-dimensional moduli space of instanton configurations satisfying

\[
\frac{1}{2} \varepsilon_{klmn} F_{kl} = - F_{mn}. \tag{2.23}
\]

Similarly, by setting the \( SU(2)_R \) gauge field equal to the anti-self-dual part of spin connection, one obtains a supersymmetric background corresponding to anti-twisted theory for which the path integrals localize to the moduli space of anti-instantons.

2.4.2. Omega backgrounds. Omega background is a deformation of topologically twisted theory such that \( Q \) is not nilpotent but squares to an isometry of the background metric. The simplest example is the Omega-deformation of flat space often denoted as \( \mathbb{R}^4_{\epsilon_1, \epsilon_2} \). Path integrals of gauge theories on such a background reduce to equivariant integrals on instanton moduli space, that is the problem of counting the configurations of point-like instantons localized at the origin, and gives the definition of Nekrasov’s instanton partition function [5–8]. See appendix A for a brief introduction to it.

To be a little more explicit, the Omega background \( \mathbb{R}^4_{\epsilon_1, \epsilon_2} \) is characterized by a scalar supercharge which squares to a rotation,

\[
Q^2 = iL_m + (\cdots), \quad \nu \equiv \epsilon_1 \left( x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) + \epsilon_2 \left( x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3} \right). \tag{2.24}
\]

To realize it within the supergravity framework, one chooses the Killing spinor with constant \( \xi_A \) as before, and also a nonvanishing \( \bar{\xi}_A \) so that \( 2\xi^A \sigma^m \xi_A = \nu^m \) holds. More explicitly,

\[
\bar{\xi}^\alpha_A = \frac{1}{\sqrt{2}} \delta^\alpha_A, \quad \xi_{\alpha A} = - \frac{1}{2} \nu_m (\sigma^m)_{\alpha A} \bar{\xi}^\alpha. \tag{2.25}
\]

In order for this to satisfy the equation (2.6) one needs to put
\[ M = T_{11} = 0, \quad T_{1i} = -\frac{1}{8} D_i \varpi_i^- \quad \left( \text{or } \frac{1}{2} T_{1i} \varpi_i^- = \frac{\epsilon_2 - \epsilon_1}{16} (\varpi_1^2 \varpi_2^2 - \varpi_3^2 \varpi_4^2) \right) \quad (2.26) \]

Note that for \( \epsilon_1 = \epsilon_2 \) no background auxiliary fields need to be turned on. A related remark is that the orientation reversal of one of the coordinate axes (‘parity’) leads to the sign flip of either \( \epsilon_1 \) or \( \epsilon_2 \), but at the same time flips the definition of chirality for spinors. Therefore, twisted theory on \( \mathbb{R}^4_{\epsilon_1, \epsilon_2} \) and anti-twisted theory on \( \mathbb{R}^4_{-\epsilon_1, -\epsilon_2} \) are related by parity.

For the choice of Killing spinor \( (2.25) \), the simplest solution to the equation \( (2.15) \) is

\[ \xi^A = \frac{1}{\sqrt{2}} \delta^A, \quad \bar{\xi}^\dot{A} = \frac{1}{2} \varpi_m (\tilde{\sigma}^m)^{\dot{a}}_a \xi^A. \quad (2.27) \]

Therefore the \( SU(2)_R \) indices are identified with dotted spinor indices as before, whereas the \( SU(2)_L \) indices are identified with undotted spinor indices.

More generally, starting from a topologically twisted theory on a manifold with an isometry generated by a Killing vector field \( v \), one can introduce Omega-deformation by choosing the Killing spinor as \( (2.25) \) and the background fields as in \( (2.26) \).

### 2.4.3. The sphere and ellipsoids.

Here we review the construction of a supersymmetric ellipsoid background following [32]. The ellipsoid of our interest is defined as a hypersurface embedded in the flat \( \mathbb{R}^5 \),

\[ \frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\ell^2} = 1, \quad (2.28) \]

with \( U(1) \times U(1) \) isometry. Note that here we are interested in the ‘physical’ SUSY and not the SUSY of topologically twisted theories, so that the observables should depend non-trivially on some of the axis-length parameters \( \ell, \tilde{\ell}, r \). The square of the SUSY will include a linear combinations of the two \( U(1) \) isometries rotating the 12- and 34-planes about the origin.

A convenient set of coordinates is the polar angles \( (\rho, \theta, \varphi, \chi) \) which are related to the Cartesian coordinates on \( \mathbb{R}^5 \) as

\[
\begin{align*}
x_0 &= r \cos \rho, \\
x_1 &= \ell \sin \rho \cos \theta \cos \varphi, \\
x_2 &= \ell \sin \rho \cos \theta \sin \varphi, \\
x_3 &= \ell \sin \rho \sin \theta \cos \chi, \\
x_4 &= \ell \sin \rho \sin \theta \sin \chi. 
\end{align*}
\quad (2.29)
\]

The two \( U(1) \) isometries of the ellipsoid are generated by Killing vectors \( \partial_\varphi \) and \( \partial_\chi \). The north pole \( (\rho = 0) \) and the south pole \( (\rho = \pi) \) are the two fixed points of the isometry. Using the above polar angle coordinate system, we see the ellipsoid as a squashed \( S^3 \) (with coordinates \( \theta, \varphi, \chi \)) fibred over a segment \( 0 \leq \rho \leq \pi \).

For the round \( S^3 \) with \( \ell = \tilde{\ell} = r \) the metric becomes

\[ ds^2 = \ell^2 (d\rho^2 + \sin^2 \rho \cdot ds^2_{(S^3)}) = E^1 E^1 + \cdots + E^4 E^4. \quad (2.30) \]

A standard choice for the vielbein one-forms \( E^a \) is

\[ E^1 = \ell \sin \rho \cos \theta d\varphi, \quad E^2 = \ell \sin \rho \sin \theta d\chi, \quad E^3 = \ell \sin \rho d\theta, \quad E^4 = \ell d\rho. \quad (2.31) \]

Note that \( E^1, E^2, E^3 \) are proportional to the vielbein on the round \( S^3 \). A nice fact about this choice of frames is that one can relate part of the Killing spinor equation on \( S^3 \) to that on \( S^3 \).
so that the independent Killing spinors on $S^4$ are all given by those on $S^3$ multiplied by some functions of $\rho$. Let us choose the following particular solution,

$$\xi_{A=1} = \sin \frac{\ell}{2} \cdot \kappa_+, \quad \xi_{A=2} = \sin \frac{\ell}{2} \cdot \kappa_-,$$

$$\xi_{A=1} = +i \cos \frac{\ell}{2} \cdot \kappa_+, \quad \xi_{A=2} = -i \cos \frac{\ell}{2} \cdot \kappa_-,$$

where we used the standard solution for $S^3$.

The square of the corresponding SUSY includes a rotation $v = \epsilon (\partial_\varphi + \partial_\chi)$ with $\epsilon = \ell^{-1}$. The theory near the north pole is thus approximately the topologically twisted theory on $\mathbb{R}^4_{+,+}$, whereas the theory near the south pole is the anti-twisted theory on $\mathbb{R}^4_{-,+}$, where the minus sign accounts for the relative orientation flip between the two polar regions. It then follows from the SUSY localization that, as long as we are interested in supersymmetric observables, the instantons and anti-instantons have to be localized at the north and south poles respectively. Their contributions are thus expressed by products of two Nekrasov partition functions with $\epsilon_1 = \epsilon_2 = \ell^{-1}$.

It is natural to ask whether there are supersymmetric deformations of the round sphere geometry which approach the general Omega background, with $\epsilon_1$ and $\epsilon_2$ independent, near the two poles. A reasonable guess would be that there should be a supersymmetric ellipsoid background with nonzero auxiliary fields in a gravity multiplet, such that (2.32) remains a two poles. A reasonable guess would be that there should be a supersymmetric ellipsoid background with nonzero auxiliary fields in a gravity multiplet, such that (2.32) remains a Killing spinor. If that is the case, the Killing vector field $v$ appearing in the square of supersymmetry is

$$v = 2 \xi^A \xi^B \sigma^m \bar{\xi}_B \partial_m = \epsilon_1 \partial_\varphi + \epsilon_2 \partial_\chi,$$

which indeed approach the desired rotation generator near the poles.

It was shown in [32] that the above naive guess is actually right. The generalized Killing spinor equation (2.6), with the above form of $\xi_A$ and $\bar{\xi}_A$ assumed, can be regarded as a linear algebraic equation for the auxiliary fields $T_{mn}, T_{mn}, (V_m)_A^B, M$ in gravity multiplet. Though the set of equations looks highly over-determined, it was shown to have a family of solutions. The explicit form of the background fields was obtained in [32] and is summarized in the appendix B. The square of the background fields was shown to be given by

$$Q^2 = i L_v + \text{Gauge}[\hat{\Phi}] + R_{SU(2)}[\Theta^A_B] + \bar{R}_{SU(2)}[\bar{\Theta}^A_B], \quad \Theta = \bar{\Theta} = -\frac{\epsilon_1 + \epsilon_2}{2} \tau^3$$

where we used the standard solution for $\xi_A, \bar{\xi}_A$ (2.16) to fix the gauge for local $SU(2)_R$ symmetry, and $\hat{\Phi}$ was defined in (2.9).

It is an interesting exercise to study the behavior of the supersymmetric ellipsoid background near the poles. Near the north pole one can use the Cartesian coordinates

$$x_1 = \ell \rho \cos \theta \cos \varphi, \quad x_2 = \ell \rho \cos \theta \sin \varphi, \quad x_3 = \ell \rho \sin \theta \cos \chi, \quad x_4 = \ell \rho \sin \theta \sin \chi,$$

assuming $|x| \ll \ell \sim \hat{\ell}$. There the chiral component $\xi_A$ of the Killing spinor (2.6) vanishes linearly in $\rho$, whereas the anti-chiral component $\bar{\xi}_A$ stays finite. Therefore, by a suitable linear Lorentz and $SU(2)_R$ rotations it can be transformed into the form (2.25). Using $\epsilon_1 = \ell^{-1}$ and $\epsilon_2 = \hat{\ell}^{-1}$, one can show the auxiliary field $T_{mn}$ agrees with (2.26) and $T_{mn} = 0$ to the leading order in small $\epsilon_i |x|$. However, the ellipsoids have nonvanishing curvature tensor, and accordingly the $SU(2)_R$ gauge field is also non-vanishing. The nonzero components of the Riemann tensor $R_{mn}^{ab}$ and the $SU(2)_R$ gauge field strength $(V_m)_A^B$, measured in Cartesian coordinates, are of the order $\epsilon_i^2$. See [76] for the full details.
2.4.4. Local $T^2$-bundle fibrations. It was shown in [76] that the ellipsoid backgrounds of [32] can be regarded as an example of supersymmetric local $T^2$-bundle fibrations, for which one can apply the same procedure as explained above to determine the necessary background auxiliary fields for general squashing parameters.

3. Partition function

Let us review here the application of the localization principle to $\mathcal{N} = 2$ supersymmetric path integrals on $S^4$, with a close look into the use of the index theorem and the fixed point formula. We also present a closed form for the partition function, and review how it simplifies to a Gaussian matrix integral for $\mathcal{N} = 2^*$ theories for special choices of mass parameter.

3.1. Localization principle

Let us recall how the SUSY localization principle simplifies the problems of path integration. Suppose a quantum field theory with an action $S$ and a path-integral measure $\int$ has a supersymmetry $Q$, which means that the expectation values of $Q$-exact observables all vanish.

$$\langle Q O \rangle = \int e^{-S} \cdot Q O = 0.$$ (3.1)

In such a theory, expectation values of $Q$-invariant observable are invariant under any deformation of the action of the form $S \rightarrow S + t Q V$, where the parameter $t$ is arbitrary and $Q^2 V = 0$. It is standard to construct $V$ as the bilinear of all the fermions $\Psi$ and their $Q$-variations, because $Q V$ will then have a manifestly positive-definite bosonic part.

$$V = \int d^4x \sqrt{g} \sum_\Psi (Q \Psi)^\dagger \Psi,$$

$$Q V = \int d^4x \sqrt{g} \left[ \sum_\Psi (Q \Psi)^\dagger Q \Psi + \cdots \right].$$ (3.2)

The values of supersymmetric observables should be $t$-independent, so one may evaluate them at a very large $t$. There the deformed action is dominated by the term $Q V$, and nonzero contribution to the path integral arise only from the vicinity of saddle points characterized by $Q \Psi = 0$ for all the fermions $\Psi$.

$$Q \Psi = 0 \text{ for all the fermions } \Psi.$$ (3.3)

Let us apply this to the general $\mathcal{N} = 2$ gauge theories of vector and hypermultiplets on ellipsoids. It is easy to check that the saddle point equation (3.3) is solved by

vector multiplet : $A_m = 0, \quad \phi = \bar{\phi} = -\frac{i}{2} a_0 \text{ (constant)}, \quad D_{AB} = -i a_0 w_{AB},$

hypermultiplet : $q_A = F_A = 0,$

(3.4)

where $w_{AB}$ was introduced in (2.11). What is more non-trivial is to prove there are no other saddle points: this has been done explicitly only for the case of round $S^4$ [9]. Assuming that it continues to be the case for more general ellipsoid backgrounds, one can argue that the path integral reduces to a finite-dimensional integral over the space of saddle points parametrized by a Lie algebra-valued constant $a_0$.

An important subtlety in solving the saddle point equation is that, if one relaxes the condition that the solution be smooth everywhere, the gauge field is allowed to take nonzero singular values localized at the two poles [9]. The field strength must be anti-self-dual at the north
pole and self-dual at the south pole. This is how the (anti-)instanton can make a nonperturbative contribution to supersymmetric observables. As was explained in the previous section, their contribution is precisely given by Nekrasov’s partition function, with argument \( q \) for instantons at the north pole and \( \bar{q} \) for the anti-instantons at the south pole.

The localization principle thus leads to the following formula for partition function,

\[
Z = \int d' a_0 e^{-S_{1-loop}(a_0)} Z_{1-loop} \left( a_0, m, \epsilon_1, \epsilon_2 \right) Z_{\text{inst}} \left( a_0, m, q, \epsilon_1, \epsilon_2 \right) Z_{\text{mat}} \left( a_0, \bar{m}, \bar{q}, \epsilon_1, \epsilon_2 \right). \tag{3.5}
\]

Here the identification \( \epsilon_1 = 1/\ell, \epsilon_2 = 1/\tilde{\ell} \) was used. \( S_{1}(a_0) \) is the original action evaluated at saddle points, and the product of Nekrasov’s partition function \( Z_{\text{inst}} \) expresses the contribution of (anti-)instantons at the poles. The one-loop factor \( Z_{1-loop} \) arises from path integration over all the modes orthogonal to the saddle point locus, for which Gaussian approximation gives an exact answer thanks to the localization principle. Finally, although the saddle points are all the modes orthogonal to the saddle point locus, for which Gaussian approximation gives an exact answer thanks to the localization principle. Finally, although the saddle points are labeled by a Lie-algebra valued parameter \( a_0 \), the integral can be reduced to its Cartan subalgebra. As is well known, the invariant measure \([da_0] \) on a Lie algebra is related to the measure \( d' a_0 \) on its Cartan subalgebra by

\[
[d_{a_0}] = d' a_0 \cdot \prod_{\alpha \in A_{+}} (a_0 \cdot \alpha)^2. \tag{3.6}
\]

In the formula (3.5), the Vandermonde factor is understood to be contained in \( Z_{1-loop} \).

The SUSY invariant action in general consists of the Yang–Mills term (2.10), the Feyet–Iliopoulos term (2.12) and the hypermultiplet kinetic term (2.18). Its classical value at the saddle point \( a_0 \) is therefore given by the sum of the following,

\[
S_{\text{YM}} = \frac{8 \pi^2}{g^2} \tilde{\ell} \tilde{\ell} \text{Tr}(a_0^2), \quad S_{\text{FI}} = -16 i \pi^2 \tilde{\ell} \tilde{\ell} \zeta a_0, \quad S_{\text{mat}} = 0. \tag{3.7}
\]

In fact, one can show that \( S_{\text{mat}} \) is exact under the supersymmetry corresponding to Killing spinors satisfying \( \xi^A \lambda_A = \xi^A \bar{\lambda}_A = 1 \).

### 3.2. Gauge fixing

We now turn to the explicit path integration. The first thing we have to do is to fix a gauge. Following the standard prescription, we introduce the ghost \( c \), anti-ghost \( \bar{c} \) and a Lagrange multiplier boson \( B \). We also introduce a nilpotent symmetry \( Q_B \) which acts on every physical field \( X \) as a gauge transformation by parameter \( c \),

\[
Q_B X = \text{Gauge}[c] X. \quad \text{(example: } Q_B a_m = D_m c, \quad Q_B \lambda_A = i \{ c, \lambda_A \} \text{)} \tag{3.8}
\]

To achieve nilpotency, the ghost fields should transform by \( Q_B \) as follows,

\[
Q_B c = i c, \quad Q_B \bar{c} = B, \quad Q_B B = 0. \tag{3.9}
\]

Here we decide not to fix the coordinate-independent part of the gauge symmetry by this procedure. Therefore the fields \( c, \bar{c}, B \) are assumed to have no constant modes. (One could alternatively eliminate the constant modes of those fields by introducing constant ‘ghost-for-ghost’ fields [9, 32].) We also define the action of \( Q \) on the ghost fields

\[
Q c = a_0 - \hat{a}, \quad Q \bar{c} = 0, \quad Q B = i L_c \bar{c} + i \{ a_0, \bar{c} \}, \tag{3.10}
\]

so that the square of the total supercharge \( \hat{Q} \equiv Q + Q_B \) acts on all the fields as follows (compare with the formula (2.34) for \( Q^2 \)).
\[ Q^2 = i \mathcal{L}_v + \text{Gauge} [a_0] + R_{SU(2)} \left[ - \frac{1}{2} (\epsilon_1 + \epsilon_2) \tau^3 \right] + R_{SU(2)} \left[ - \frac{1}{2} (\epsilon_1 + \epsilon_2) \tau^3 \right] \]  

(3.11)

Usual gauge fixing proceeds by choosing an arbitrary gauge-fixing functional \( G \), for example the Lorentz gauge \( G = \partial \mu A^\mu \), and modifying the action by the addition of gauge-fixing term \( Q_0 (\epsilon G) \). As was shown in [9], one can replace the gauge-fixing term by \( Q (\epsilon G) \) without changing the value of partition function. The total supersymmetry \( Q \) is then preserved and can be used for localization argument.

3.3. Determinants and index

We now turn to the gauge-fixed path integral with respect to fluctuations around saddle points. We take the \( Q \)-exact deformation term (including the gauge-fixing term)

\[ \hat{Q} \hat{V} = \hat{Q} (V + \epsilon G), \]  

(3.12)

to be very large, so that Gaussian approximation becomes exact and the path integral simply gives rise to determinants. We also notice that, after the introduction of ghost fields, the number of bosons and fermions agree off-shell: a vector multiplet consists of ten bosons \( (A_m, \phi, \phi, D_{AB}, B) \) and ten fermions \( (\lambda_m, \lambda^A, c, \bar{c}) \), likewise a hypermultiplet consists of four bosons \( (q_A, F_A) \) and four fermions \( (\psi^\alpha, \psi^\alpha) \). This is of course important for the localization principle to work.

We move to a new set of fields which is particularly useful for evaluating the fluctuation determinant. We first define fermions without spinor indices from gauginos,

\[ \Psi \equiv -i \xi^A \lambda_A - i \bar{\xi}^A \bar{\lambda}_A, \quad \Psi_m \equiv i \xi^A \sigma_m \bar{\lambda}_A - i \bar{\xi}^A \bar{\sigma}_m \lambda_A, \quad \Xi_{AB} \equiv 2 \xi_{(A} \lambda_{B)} - 2 \xi_{(A} \bar{\lambda}_{B)}, \]  

(3.13)

so that the supersymmetry transformation rule simplifies.

\[ Q \phi_2 = \Psi, \quad QA_m = \Psi_m, \quad Q \Xi_{AB} = D_{AB} + (\cdots). \]  

(3.14)

Likewise, from the fermion in hypermultiplet we define

\[ \Psi_A \equiv -i \xi_A \psi + i \bar{\xi}_A \bar{\psi}, \quad Q q_A = \Psi_A, \]  

\[ \Xi_A \equiv \xi_A \bar{\psi} - \bar{\xi}_A \psi, \quad Q \Xi_A = F_A + (\cdots). \]  

(3.15)

It is then convenient to take five bosons \( X = (A_m, \phi_2 \equiv \phi - \phi) \), five fermions \( \Xi = (\Xi_{AB}, \bar{c}, \bar{c}) \) and their \( Q \)-superpartners as independent variables for vector multiplet. Similarly, for hypermultiplet we take two bosons \( X = q_A \), two fermions \( \Xi = \Xi_A \) and their \( Q \)-superpartners as independent variables.

In quadratic approximation, the \( Q \)-exact deformation term (3.12) decomposes into vectormultiplet and hypermultiplet parts, and each term has the structure

\[ \hat{V} \bigg|_{\text{quad.}} = (\hat{Q} X, \Xi) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X \\ \hat{Q} \Xi \end{pmatrix} \]  

\[ - \langle \hat{Q} X, \Xi \rangle \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} \hat{Q} X \\ \Xi \end{pmatrix}, \]  

(3.16)
where we denoted \( H = \hat{Q}^2 \). The Gaussian integral thus gives the square root of the following ratio of determinants,

\[
\frac{\det K_\mu}{\det K_\nu} = \frac{\det_{\Xi} H}{\det_{\mathbf{X}} H} = \frac{\det_{\ker D_{10}} H}{\det_{\ker D_{\nu}} H}.
\]  

The last equality follows from the fact that the fields \( \mathbf{X} \) and \( \Xi \) take values on the spaces related by the operator \( D_{10} \), and that \( H \) commutes with \( D_{10} \). The ratio of determinants is closely related to the index defined by

\[
\text{Ind}(D_{10}) \equiv \text{Tr}_{\ker D_{10}}(e^{-iH_{10}}) - \text{Tr}_{\ker D_{\nu}}(e^{-iH_{10}}). 
\]  

The index can be evaluated using the fixed-point formula, which is based on the following simple idea. We are interested in the trace of \( e^{-iH} \) acting on fields \( \mathbf{X} \) and \( \Xi \) at fixed points. For vector multiplet fields at the north pole it becomes

\[
\text{Tr}_{\mathbf{X}}(e^{-iH_{10}})|_{\text{NP}} - \text{Tr}_{\Xi}(e^{-iH_{10}})|_{\text{NP}} = \text{Tr}_{\mathbf{ad}}(e^{i\phi}) \times \left\{ (q_1 + q_2 + \bar{q}_1 + \bar{q}_2 + 1) - (q_1q_2 + 1 + \bar{q}_1\bar{q}_2 + 1 + 1) \right\}
\]  

(3.22)

The contribution to the index from the North pole is therefore

\[
\text{Ind}(D_{10})|_{\text{NP}} = \text{Tr}_{\mathbf{ad}}(e^{i\phi}) \times \frac{(q_1 + q_2 + \bar{q}_1 + \bar{q}_2 + 1) - (q_1q_2 + 1 + \bar{q}_1\bar{q}_2 + 1 + 1)}{(1 - q_1)(1 - q_2)(1 - \bar{q}_1)(1 - \bar{q}_2)}
\]  

(3.23)

Neglecting some fields whose contribution is trivial, one can identify the above result with the index of the self-dual complex \( (D_{\text{SD}} : \Omega^0 \xrightarrow{d^+} \Omega^1 \xrightarrow{d^+} \Omega^{2+}) \) valued in the adjoint representation of the gauge group, defined by the instanton equation and gauge equivalence.

If the four factors in the denominator were all expanded into geometric series, the result would be interpreted as the trace of \( e^{-iH} \) evaluated by expanding all the fields into the basis of monomial functions \( x_1^m x_2^n \). However, such a trace would not make sense because there would be infinitely many degenerate eigenmodes for each eigenvalue of \( H \). The index does not suffer from the problem of infinite degeneracy, because the fraction on the right hand side
of (3.23) is reducible reflecting the cancellation between the fields $X$ and $\Xi$. But there remains another more subtle issue which requires a careful regularization, as we will see below.

After simplifying the fraction, combining the contributions from the two poles and recalling that the fields $c, \bar{c}$ do not have constant modes, the index is given by

$$\text{Ind}(D_{10})_{\text{vec}} = \text{Tr}_{\text{adj}}(e^{a_\mu}) \times \left\{ - \frac{1 + q_1 q_2}{(1 - q_1)(1 - q_2)} \right\}_{\text{NP}} + \left\{ - \frac{1 + q_1 q_2}{(1 - q_1)(1 - q_2)} \right\}_{\text{SP}} + 2 \right\}$$

for the vector multiplet. Similarly, for the hypermultiplet in the representation $R$ of the gauge group the index becomes

$$\text{Ind}(D_{10})_{\text{hyp}} = \text{Tr}_{R+\bar{R}}(e^{a_\mu}) \times \left\{ - \frac{(q_1 q_2)^{\frac{1}{2}}}{(1 - q_1)(1 - q_2)} \right\}_{\text{NP}} + \left\{ - \frac{(q_1 q_2)^{\frac{1}{2}}}{(1 - q_1)(1 - q_2)} \right\}_{\text{SP}} \right\}.$$  \hspace{1cm} (3.24)

To read from the index the spectrum of $H$ which is necessary for the computation of one-loop determinant, one needs to expand the above expressions into series in $q_1, q_2$. But a priori there is no natural choice whether to expand in positive or negative series in $q$’s. We have seen above that a fixed point formula allows one to express the index as a sum of pole contributions, but it does not give us any further information about which eigenmode of $H$ is supported around which pole. Indeed, although the index of a differential operator $D_{10}$ depends only on the term of highest order in the derivative, the detailed behavior of its zeromodes depends on the subleading terms as well. One can choose the subleading term in any convenient manner so that each eigenmode of $H$ has localized support near one of the poles. The index should of course be independent of such regularizations.

Let us look into this point in more detail, taking the hypermultiplet index as an example. To the leading order in the derivatives, the differential operator $D_{10}$ is given by

$$\Xi^4(D_{10})_{\alpha\beta}q^B = \Xi^4(i^4 A - A^\alpha \xi_{\beta}^\alpha - i A^\alpha \xi^\alpha_{\beta} D_{m} q^B.$$  \hspace{1cm} (3.25)

We are interested in how the zeromode wavefunctions get localized near the poles depending on the choice of the non-derivative terms. Since $D_{10}$ has to commute with $H$, we follow the suggestion in [9] and introduce a non-derivative term in $D_{10}$ through the modification $D_m \rightarrow D_m - 2i s v_m$, where $s$ is an arbitrary real parameter. Similar modification of differential operators was considered in the study of Morse theory [77] and in particular the derivation of holomorphic Morse inequality in [78].

Near the north pole one may identify $\Xi^4$ as a chiral spinor and $q^A$ as an anti-chiral spinor, and $(D_{10})_{\alpha\beta}$ is then simply the Dirac operator

$$D_{10} = \frac{1}{2} \sigma^m (i \partial_m + s v_m) = \left( \begin{array}{cc} \partial_{\bar{z}_2} + s \epsilon_{\bar{z}_1} & \partial_{\bar{z}_1} - s \epsilon_{\bar{z}_2} \\ \partial_{\bar{z}_1} + s \epsilon_{\bar{z}_2} & -\partial_{\bar{z}_2} + s \epsilon_{\bar{z}_1} \end{array} \right).$$  \hspace{1cm} (3.27)

For a vector multiplet, the relevant differential operator near the north pole has the index structure $(D_{10})_{\alpha\beta\gamma\delta}$, and is the adjoint of the operator above twisted by an anti-chiral spinor bundle. Assuming that $\epsilon_1 = \ell^{-1}$ and $\epsilon_2 = \ell^{-1}$ are both positive, the operator $D_{10}$ can be shown to have no $\Xi$-zeromodes, but it has $q$-zeromodes of the following form,

$$s > 0 \implies q^A = \begin{pmatrix} q_{m+\frac{1}{2},q_{m+\frac{1}{2}}} \exp(-\epsilon_1 q_{1} |z|^2 + \epsilon_2 q_{2} |z|^2) \\ 0 \end{pmatrix}, \quad e^{-i H U} = e^{a_\mu} \cdot q_{1}^{m+\frac{1}{2}} q_{2}^{n+\frac{1}{2}},$$

$$s < 0 \implies q^A = \begin{pmatrix} 0 \\ \exp(\epsilon_1 q_{1} |z|^2 + \epsilon_2 q_{2} |z|^2) \end{pmatrix}, \quad e^{-i H U} = e^{a_\mu} \cdot q_{1}^{-m-\frac{1}{2}} q_{2}^{-n-\frac{1}{2}}.$$  \hspace{1cm} (3.28)
This indicates one should expand the north-pole contribution to the index into positive (negative) series in $q_1, q_2$ if $s > 0$ (resp. $s < 0$). The analysis is similar near the south pole, with the result that one has to series-expand in the opposite way. We thus arrive at the formula for the normalized saddle-point parameter,

$$\text{Ind}(D_{10})_{\text{vec}} = \text{Tr}_{\text{adj}}(e^{\mu r}) \left\{ 2 - \sum_{m,n \geq 0} \left( q_1^m q_2^n + q_1^{m+1} q_2^{n+1} + q_1^{-m} q_2^{-n} + q_1^{-m-1} q_2^{-n-1} \right) \right\},$$

$$\text{Ind}(D_{10})_{\text{hyp}} = \text{Tr}_{R+R}(e^{\mu r}) \sum_{m,n \geq 0} \left( q_1^m q_2^n + q_1^{m+1} q_2^{n+1} + q_1^{-m} q_2^{-n} + q_1^{-m-1} q_2^{-n-1} \right).$$

(3.29)

Note the operator $D_{10}$ has infinitely many zeromodes, owing to the fact that it is not elliptic but only transversely elliptic [79].

The one-loop determinant factor $Z_{1-\text{loop}}$ in (3.5) can be easily obtained from the above formula for the index. We assume $a_0$ to be in Cartan subalgebra and neglect $a_0$-independent factors. One then finds that $Z_{1-\text{loop}}$ is a product of contributions from vector and hypermultiplets,

$$Z_{1-\text{loop}}^{\text{vec}} = \prod_{\alpha \in \Delta_+} \frac{\Upsilon(i\hat{a}_0 \cdot \alpha) \Upsilon(-i\hat{a}_0 \cdot \alpha)}{(\hat{a}_0 \cdot \alpha)^2} \times (\hat{a}_0 \cdot \alpha)^2 = \prod_{\alpha \in \Delta} \Upsilon(i\hat{a}_0 \cdot \alpha),$$

$$Z_{1-\text{loop}}^{\text{hyp}} = \prod_{\rho \in \mathcal{R}} \Upsilon\left(\frac{m}{2} + i\hat{a}_0 \cdot \rho\right)^{-1},$$

(3.30)

where we included the Vandermonde determinant (3.6) into $Z_{1-\text{loop}}^{\text{vec}}$. Here $\hat{a}_0 = \sqrt{\ell/\hat{e}}a_0$ is the normalized saddle-point parameter, $\alpha \in \Delta_+$ runs over positive roots of the gauge Lie algebra and $\rho \in \mathcal{R}$ runs over weights of the representation $\mathcal{R}$. The function $\Upsilon(x)$ is defined as an infinite product,

$$\Upsilon(x) = \text{const} \cdot \prod_{m,n \neq 0} (x + mb + nb^{-1})(Q - x + mb + nb^{-1}), \quad \left( Q = b + \frac{1}{b} \right)$$

(3.31)

where the parameter $b$ is related to the ellipsoid geometry by $b = (\ell/\hat{e})^\frac{1}{2}$. It appears frequently in observables of Liouville or Toda CFTs with coupling $b$. See for example [80], where some important properties of $\Upsilon(x)$ are also summarized.

Note that one can read off information on one-loop running of the gauge coupling from the behavior of $Z_{1-\text{loop}}$, for $\ell \hat{e} \ll a_0^2$,

$$S_{\text{YM}} = \frac{8\pi^2}{g^2} \text{Tr}(\hat{a}_0^2), \quad -\ln Z_{1-\text{loop}} \sim \ln(\ell \hat{e})^{-\frac{1}{2}} \cdot \left\{ \text{Tr}_{\text{adj}}(\hat{a}_0^2) - \text{Tr}_{\mathcal{R}}(\hat{a}_0^2) \right\},$$

(3.32)

Here we used the asymptotic behavior of $\Upsilon(x)$ at large $|x|$,

$$\ln \Upsilon(x) \sim \left(x - \frac{Q}{2}\right)^2 \ln x + \left(\frac{1}{6} - \frac{Q^2}{12}\right) \ln x - \frac{3}{2} \left(x - \frac{Q}{2}\right)^2 + \cdots.$$ 

(3.33)

3.4. $\mathcal{N} = 4$ SYM and Gaussian matrix model

$\mathcal{N} = 2$ gauge theory with a massless adjoint hypermultiplet has an enhanced supersymmetry and is called $\mathcal{N} = 4$ SYM. Application of the localization principle to this model is particularly interesting since one can expect to obtain nontrivial and precise evidences for the AdS/CFT correspondence. In this respect, there was a long standing conjecture that the
expectation value of circular Wilson loops in $\mathcal{N} = 4$ SYM is given by a simple Gaussian matrix integral [10, 11]. Pestun’s work [9] gave an analytic proof of this conjecture.

The $\mathcal{N} = 4$ SYM can be deformed to the so-called $\mathcal{N} = 2^*$ theory by making the adjoint hypermultiplet massive. The measure and the one-loop determinant part of the ellipsoid partition function for this theory read

$$Z_{\text{1-loop}} = \prod_{\alpha \in \Delta_+} \frac{\Upsilon(i \hat{a}_0 \cdot \alpha) \Upsilon(-i \hat{a}_0 \cdot \alpha)}{\Upsilon(\frac{Q}{2} + i \hat{m} + i \hat{a}_0 \cdot \alpha) \Upsilon(\frac{Q}{2} + i \hat{m} - i \hat{a}_0 \cdot \alpha)}, \quad (3.34)$$

where $\hat{m} = (\hat{\ell})^T \hat{m}$ is the normalized (dimensionless) hypermultiplet mass. Note that it is invariant under the sign-flip of $\hat{m}$ since $\Upsilon(x) = \Upsilon(Q - x)$.

An obvious special value of the mass is $\hat{m} = \pm iQ/2$, for which the $\Upsilon$ functions in the denominator and enumerator cancel precisely. Similar simplification happens also to Nekrasov’s partition function as discussed briefly in appendix A. For example, for $U(N)$ gauge group, $Z_{\text{inst}}$ is simply given by a sum over the sets of $N$ Young diagrams weighted by $q^k$, where $k$ is the total number of boxes in the $N$ diagrams. Therefore

$$Z_{\text{inst}} = \prod_{k \geq 1} (1 - q^k)^{-N}. \quad (3.35)$$

The only $a_0$-dependence remaining in the integrand is the classical action $S_{\text{YM}}$. The $a_0$ integral can be easily performed and gives $(\text{Im} \tau)^{-N/2}$. The result agrees with the torus partition function of the 2D CFT of $N$ massless scalars, but is different from the Gaussian matrix integral.

Another special value of the mass is $\hat{m} = \pm \frac{1}{2}(b^{-1} - b)$, for which the measure and the determinant become

$$Z_{\text{1-loop}} = \prod_{\alpha \in \Delta_+} \frac{\Upsilon(i \hat{a}_0 \cdot \alpha) \Upsilon(-i \hat{a}_0 \cdot \alpha)}{\Upsilon(b^{\pm 1} + i \hat{a}_0 \cdot \alpha) \Upsilon(b^{\pm 1} - i \hat{a}_0 \cdot \alpha)} = \prod_{\alpha \in \Delta_+} (\hat{a}_0 \cdot \alpha)^2, \quad (3.36)$$

which is the natural measure for the matrix integral. At the same time, the Nekrasov partition function becomes trivial for this special value of $\hat{m}$, namely $Z_{\text{inst}} = 1$, due to the emergence of fermionic zeromodes in the moduli space of $k \geq 1$ instantons. As was argued in [12], the additional fermion zeromode is the consequence of supersymmetry enhancement. Thus the SUSY path integral reduces to the Gaussian matrix integral for this special choice of $\hat{m}$.

### 4. Supersymmetric observables

We review here the application of the localization principle to the evaluation of supersymmetric non-local observables—Wilson loops, ’t Hooft loops and surface operators.

#### 4.1. Wilson loops

Having understood how to compute the partition function using the localization principle, it is straightforward to include Wilson loop operators. Wilson loops are defined as usual by holonomy integrals along closed paths, but in supersymmetric Wilson loops the gauge field is accompanied by scalar fields in a vector multiplet. Also, the loops have to be aligned with the direction of the isometry generated by $Q$. For generic mutually incommensurable choice of $\ell, \hat{\ell}$, there are only two types of supersymmetric closed paths:
\[ S_\rho^i (\rho) : (x_0, x_1, x_2, x_3, x_4) = (r \cos \rho, \ell \sin \rho \cos \varphi, \ell \sin \rho \sin \varphi, 0, 0), \]
\[ S_\chi^i (\rho) : (x_0, x_1, x_2, x_3, x_4) = (r \cos \rho, 0, 0, \tilde{\ell} \sin \rho \cos \chi, \tilde{\ell} \sin \rho \sin \chi). \] (4.1)

Namely, \( S_\rho^i (\rho) \) is a circle within an \((x_1, x_2)\)-plane at a fixed \( x_0 \) and \( x_3 = x_4 = 0 \), and similarly \( S_\chi^i (\rho) \) is a circle within an \((x_3, x_4)\)-plane. The corresponding Wilson loop operators are
\[ W_{\varphi}(R) \equiv \text{Tr}_R \exp i \int_{S_\rho^i (\rho)} d\varphi \left( A_\varphi - 2\ell \left( \phi \cos^2 \frac{P}{2} + \tilde{\phi} \sin^2 \frac{P}{2} \right) \right), \]
\[ W_\chi(R) \equiv \text{Tr}_R \exp i \int_{S_\chi^i (\rho)} d\varphi \left( A_\chi - 2\ell \left( \phi \cos^2 \frac{P}{2} + \tilde{\phi} \sin^2 \frac{P}{2} \right) \right). \] (4.2)

Note that the integrand is proportional to \( \hat{\Phi} \) of (2.9) evaluated along the path, so the SUSY invariance is very easy to check. The expectation values of these operators can thus be evaluated by just inserting their classical values
\[ W_{\varphi}(R) = \text{Tr}_R \exp \left( -2\pi b \hat{a}_0 \right), \quad W_\chi(R) = \text{Tr}_R \exp \left( -2\pi b^{-1} \hat{a}_0 \right). \] (4.3)
into the integrand of (3.5).

4.2. ‘t Hooft loops

‘t Hooft loops play an equally important role as Wilson loops. They were originally introduced in [81] as a probe to distinguish different phases of gauge theories. Also, in 4D \( \mathcal{N} = 2 \) SUSY gauge theories, the Wilson and ‘t Hooft loop operators are known to transform among one another under duality.

4.2.1. Definition of ‘t Hooft operator. A ‘t Hooft operator introduces a Dirac monopole singularity along a path in a 4D space, and its charge is specified by a coweight \( B \) of the gauge group. Insertions of ‘t Hooft operators therefore not only changes the classical SYM action \( S_{\text{SYM}} \), but also affects the one-loop and instanton parts of the formula (3.5) since it changes the boundary condition for the path integration variables. This problem was analyzed in detail in [55] for a single ‘t Hooft operator inserted along a great circle in the equator \( S^3 \) of the round sphere.

Let us first study the operator lying along the \( x^1 \)-axis \((x^2 = x^3 = x^4 = 0)\) in the flat \( \mathbb{R}^4 \). The behavior of the magnetic field around it is
\[ F \sim -\frac{B}{4 \epsilon_{ijk}} \frac{x^i dx^j dx^k}{|x|^3} \quad (i, j, k = 2, 3, 4). \] (4.4)

When the \( \theta \)-angle is nonzero, the presence of magnetic charge changes the quantization condition of electric charge [82]. This implies that the ‘t Hooft operator also induces nonzero electric field proportional to \( \theta \),
\[ F_{li} \sim \frac{i\theta g^2 B}{16\pi^2} \frac{x^i}{|x|^3}. \] (4.5)

If we require the ‘t Hooft operator to be half-BPS, the scalars are also required to take nonzero values around it. If the unbroken supersymmetry is characterized by \( \xi_A = \sigma_1 \xi_A e^{i\alpha} \), the scalars have to behave near the ‘t Hooft operator as follows,
\[ \phi \sim e^{i\alpha} \left( \frac{1}{4} - \frac{i\theta g^2}{32\pi^2} \right) \frac{B}{|x|}, \quad \tilde{\phi} \sim e^{-i\alpha} \left( \frac{1}{4} - \frac{i\theta g^2}{32\pi^2} \right) \frac{B}{|x|}. \] (4.6)
Consider now general \( \mathcal{N} = 2 \) SUSY theories on the round \( S^4 \) with radius \( \ell \), and put a 't Hooft operator with charge \( B \) along the circle \( S^1_\rho \) at \( \rho = \pi / 2 \), namely the intersection of the sphere (2.28) with \( x_0 = x_3 = x_4 = 0 \). Our Killing spinor (2.32) satisfies \( \xi_A = -\sigma_1 \xi_A \) there, so we substitute \( \epsilon^a = -1 \) into the above expressions for fields on \( \mathbb{R}^4 \) and then map to \( S^4 \). Using the Cartesian coordinates \( x_0, \cdots , x_4 \) introduced in (2.28), the value of gauge and scalar fields is

\[
F = -B \frac{\epsilon_{ijk}}{4|x|^4} x_i dx_j dx_k + \frac{i g^2 B}{16 \pi^2} \frac{\ell dx_1 dx_2}{|x|^4}, \quad (i, j, k = 0, 3, 4)
\]

\[
\phi = \left( -\frac{1}{4} + \frac{i g^2}{32 \pi^2} \right) \frac{B}{|x|} - \frac{i a_0}{2}, \quad \tilde{\phi} = \left( \frac{1}{4} + \frac{i g^2}{32 \pi^2} \right) \frac{B}{|x|} - \frac{i a_0}{2},
\]

(4.7)

Here we used \( |x| \equiv \sqrt{x_0^2 + x_3^2 + x_4^2} \), and we also included the constant terms for the scalars.

It was shown in [55] that the above expression with \( [B, a_0] = 0 \) exhausts all the saddle point configurations with the correct singular behavior of fields around the loop.

4.2.2. Localization computation. To compute the expectation values of 't Hooft operators, one needs to work out the classical action on the saddle-point configuration (4.7), one-loop determinant and instanton contribution. All of them receive nontrivial modification from 't Hooft operator, as we will now review.

The classical SYM action integral diverges near the 't Hooft loop since it corresponds to the self-energy of monopole. It can be regularized by removing a neighborhood of the loop \( B_3 \times S^1 \) from the integration domain, and adding the boundary term

\[
S_{\text{boundary}} = i \ell \int_{\partial S^4} \frac{d\varphi}{2\pi} \text{Tr} \left( e^{-i\tau \phi} + e^{i\tau \tilde{\phi}} \right) F.
\]

(4.8)

Here \( \varphi \) is the coordinate along the loop and \( \tau \) is the complexified gauge coupling. The total classical action evaluated on the saddle point (4.7) is thus finite,

\[
(S_{\text{SYM}} + S_{\text{boundary}}) |_{\text{cl}} = -i \tau \text{Tr}(\hat{a}_N^2) + i \tau \text{Tr}(\hat{a}_S^2),
\]

\[
\hat{a}_N \equiv a_0 \ell - \frac{\theta g^2 B}{16 \pi^2} + \frac{i B}{2}, \quad \hat{a}_S \equiv a_0 \ell - \frac{\theta g^2 B}{16 \pi^2} - \frac{i B}{2}.
\]

(4.9)

We notice here that \( \hat{a}_N \) and \( \hat{a}_S \) are the values of the scalar \( \hat{\Phi} \) (2.9) at the two poles, which are relevant in the evaluation of equivariant integrals over the instanton moduli spaces there. Therefore the argument of Nekrasov’s partition functions representing the effect of instantons at the north pole (anti-instantons at the south pole) should be changed from \( a_0 \) to \( \hat{a}_N \) (resp. \( \hat{a}_S \)).

Actually there is a subtlety in identifying \( \hat{a}_N, \hat{a}_S \) with the value of \( \hat{\Phi} \), since the latter contains the gauge potential \( A_m \) and there is no globally well-defined expression for it in the presence of the 't Hooft operator. By integrating the expression for the field strength (4.7) one finds

\[
A = -\frac{B}{2} \left( \frac{x_0}{|x|} - C \right) dx + \frac{i g^2 B}{16 \pi^2} \left( \ell \frac{dx_1 dx_2}{|x|^4} - 1 \right) d\varphi,
\]

\[
\left( dx = \frac{x_3 dx_4 - x_4 dx_3}{x_3^2 + x_4^2}, \quad d\varphi = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}, \quad |x| = \sqrt{x_0^2 + x_3^2 + x_4^2} \right)
\]

(4.10)
Near the north and south poles, we choose the integration constant $C$ as $+1$ or $-1$ to avoid Dirac string singularity and find $\hat{\Phi} = \hat{a}_\kappa$ or $\hat{\Phi} = \hat{a}_\Phi$, respectively. Near the equator, the natural choice $C = 0$ leads to
\[
\hat{\Phi} = \hat{a}_\Xi \equiv a_0 \ell - \frac{\theta y^2 B}{16\pi^2}.
\] (4.11)

Let us next turn to the evaluation of the one-loop Gaussian integral over fluctuations from the above saddle points. As in the previous section one can relate it to an index and express it as a sum over contributions from fixed points. In addition to the north and south poles, this time there is a nontrivial contribution from the equator in the vicinity of the loop, due to the change in the boundary condition of fields there. We introduce the coordinates $\varphi \sim \varphi + 2\pi$ and $y_1, y_2, y_3$ to parametrize the local geometry $S^1 \times \mathbb{R}^3$ near the loop, assuming the loop is at the origin of $\mathbb{R}^3$. The coordinate $\varphi$ here is the same as the angle coordinate $\varphi$ in (2.29), while the other angle coordinate $\chi$ there corresponds to the rotation angle in $(y_1, y_2)$-plane here.

Our Killing spinor $\xi_+ \xi_\pm$ and $\hat{\xi}_\pm$ are anti-periodic in $\varphi$. It is convenient to use a local $J_3$ transformations in $SU(2)_R$ and $SU(2)_R$ to make them all independent of $\varphi$. Then vector multiplet fields become all periodic in $\varphi$, while hypermultiplet fields are all antiperiodic. The index involves the trace of $e^{-i\Phi}$, where
\[
H = \hat{Q}^2 = \frac{1}{\ell} \left( i\partial_\varphi + i\partial_\chi + \text{Gauge}[\hat{a}_\Xi] \right) \equiv \frac{1}{\ell} \partial_\varphi + H_{(3)}.
\] (4.12)

The Kaluza–Klein expansion with respect to $\varphi$ thus relates the equatorial contribution to the index of our interest to a 3D index. The reduction takes the following schematic form
\[
\text{Ind}(D_{(4)}) \bigg|_{\text{eq}} = \sum_n e^{2\pi i n} \text{Ind}(D_{(3)}).
\] (4.13)

The sum with respect to $n$ is over integers for the vector multiplet index and half-odd integers for hypermultiplets. For the vector multiplet, the natural choice for the operator $D_{(3)}^{\text{vec}}$ is the one associated with the gauge equivalence classes of small fluctuations around the singular solution to Bogomolny equation $F + *D_\phi_2 = 0$,
\[
F = -\frac{B}{4|y|^3} \varepsilon_{ijk} y dy dz, \quad \phi_2 = -\frac{B}{2 |y|}.
\] (4.14)

For the hypermultiplet, the natural choice is the 3D Dirac operator $D_{(3)}^{\text{hyp}} = i\tau^i (\partial_\kappa - iA_\kappa) + \phi_2$.

In [55] the 3D indices were evaluated by using Kronheimer’s construction of $U(1)$-invariant instantons [83]. Consider the Gibbons–Hawking parametrization of flat $\mathbb{C}^2$,
\[
\begin{align*}
 dz_1 d\bar{z}_1 + d_2 d\bar{z}_2 &= \frac{1}{4r} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\chi^2 \right) + r d\psi - \frac{1}{2} \cos \theta d\chi^2 \\
 &= \frac{1}{4|y|} dy_{1} dy_{2} + |y| (d\psi + \omega)^2.
\end{align*}
\]
\[
\left( z_1 = \sqrt{r} \cos \frac{\psi}{2} e^{i\chi}, \quad z_2 = \sqrt{r} \sin \frac{\psi}{2} e^{i\chi + i\omega} \right).
\] (4.15)

An important fact here is that, if $(A, \phi_2)$ satisfies the Bogomolny equation on $\mathbb{R}^3$, then
\[
A = A - 2|y|\phi_2 (d\psi + \omega),
\] (4.16)
is an anti-self-dual and $\psi$-translation invariant gauge field configuration on $S^2$. Note also that the singular monopole solution (4.14) corresponds to a pure gauge $A \equiv B \delta \psi$ under this map. This map also relates the 3D indices of interest to the restricted 4D indices, where the trace is taken only over the space of $\psi$-independent wave functions. For example, the index of $D^{\mathrm{hyp}}_{(3)}$ can be computed from the index of 4D self-dual complex $D_{3D}$ associated to the gauge equivalence class of fluctuations from an anti-self-dual connection $A$, restricted to $\psi$-independent wave functions. The 3D index is thus obtained by averaging the 4D index over $\psi$-translational

\[
\text{Ind}(D^{\mathrm{rec}}_{(3)}) = \int_0^{2\pi} \frac{d\nu}{2\pi} \left\{ \text{Tr}_{\text{Ker}D_{3D}}(e^{-i\nu H_{(3)}} + \nu \partial_{\nu} B) - \text{Tr}_{\text{Coker}D_{3D}}(e^{-i\nu H_{(3)}} + \nu \partial_{\nu} B) \right\}_{A = B \delta \psi}
\]

\[
= \int_0^{2\pi} \frac{d\nu}{2\pi} \left\{ \text{Tr}_{\text{Ker}D_{3D}}(e^{-i\nu H_{(3)}} + \nu \partial_{\nu} B) - \text{Tr}_{\text{Coker}D_{3D}}(e^{-i\nu H_{(3)}} + \nu \partial_{\nu} B) \right\}_{A = 0}
\]

\[
= \int_0^{2\pi} \frac{d\nu}{2\pi} \text{Tr}_{\text{adj}(e^{i\nu H_{(3)} + \nu B})} \times (1 + e^{-\frac{\pi}{2}})(1 - e^{\frac{\pi}{2} + i\nu})(1 - e^{\frac{\pi}{2} - i\nu})(1 - \delta e^{\frac{i B}{\lambda}})(1 - \delta e^{\frac{-i B}{\lambda}}).
\]

(4.17)

Here in the second line we similarity-transform all the operators involved by a gauge rotation, and in the last line we introduced a parameter $\delta$ ($0 < \delta < 1$ and $\delta \to 1$) to indicate expansions into geometric series. The index of $D^{\mathrm{hyp}}_{(3)}$ is related to the 4D Dirac index in the same way. The final result is

\[
\text{Ind}(D^{\mathrm{rec}}_{(3)}) = -\frac{1}{2} (u + u^{-1}) \sum_{\alpha \in \Delta_+} (e^{\alpha \cdot \hat{a}_N} + e^{-\alpha \cdot \hat{a}_N}) u^{\frac{1}{2} |\alpha \cdot B|} - u^{-\frac{1}{2} |\alpha \cdot B|},
\]

\[
\text{Ind}(D^{\mathrm{hyp}}_{(3)}) = \frac{1}{2} \sum_{\rho \in \mathbb{R}} (e^{\rho \iint H_{(3)} + \rho \iint B} + e^{-\rho \iint H_{(3)} + \rho \iint B}) u^{\frac{1}{2} |\rho \cdot B|} - u^{-\frac{1}{2} |\rho \cdot B|}.
\]

(4.18)

Here we used $u \equiv e^{i2\ell}$, and $\tilde{m}$ is the normalized mass parameter for the hypermultiplet. Note also that by definition of coweight $B$ the inner products $\alpha \cdot B$ and $\rho \cdot B$ are always integers.

Let us now present the formula for the expectation value of a ’t Hooft operator $T_B$. Without loss of generality we can choose the charge $B$ to be the highest weight vector of an irreducible representation of $\mathfrak{g}$ (Langlands dual of the gauge group). For ’small’ charge $B$, all the weight vectors of the corresponding representation are Weyl images of $B$. In such cases, the expectation value of the ’t Hooft operator can be expressed by combining all the arguments reviewed above,

\[
\langle T_B \rangle = \int [d\hat{a}_E] \hat{q}^{\iint H_{(3)}} Z_{1\text{-loop}}(\hat{a}_N, \hat{m}) \hat{Z}_{\text{inst}}(\hat{a}_N, \hat{m}, q) \cdot Z_{1\text{-loop}}^{(eq)}(\hat{a}_E, \hat{m}, B)
\]

\[
\cdot \hat{q}^{\iint H_{(3)}} Z_{1\text{-loop}}(\hat{a}_S, \hat{m}) \hat{Z}_{\text{inst}}(\hat{a}_S, \hat{m}, \bar{q})
\]

\[
\hat{a}_N = \hat{a}_E + \frac{IB}{2}, \quad \hat{a}_S = \hat{a}_E - \frac{IB}{2}.
\]

(4.19)

This can be rewritten further as a sum over Weyl images of $B$. As an example, consider $SU(N)$ $\mathcal{N} = 2^*$ theory on $S^4$ and take as $B$ the highest weight vector for fundamental representation, namely $B = h_1 = (\frac{N-1}{N}, -\frac{1}{N}, \cdots, -\frac{1}{N})$. Then the expectation value is expressed as a sum over weight vectors $h_i$. 

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\[ (T_B) = \frac{1}{N} \sum_{k=1}^{N} \int \cdots q^{\frac{1}{2} \text{Tr} (\hat{a} + \hat{q})} Z_{1-loop}(\hat{a} + \frac{i}{2} h_k, \hat{m}) \cdot Z_{\text{inst}}(\hat{a} + \frac{i}{2} h_k, \hat{m}, \hat{q}), \]

where the one-loop determinant factor from the equator is

\[ Z_{\text{eq}}(\hat{a}, \hat{m}, h_k) = \prod_{j \neq k} \left[ \frac{\cosh \pi (\hat{a}_k - \hat{a}_j + \hat{m}) \cosh \pi (\hat{a}_k - \hat{a}_j - \hat{m})}{\cosh \pi (\hat{a}_k - \hat{a}_j)} \right]. \]

This result was shown to agree with the expectation value of Verlinde’s loop operators in \( A_{N-1} \) Toda CFT.

**4.2.3. Monopole screening.** As in (4.20), the expectation value of a ‘t Hooft operator \( \langle T_B \rangle \) in \( S^4 \) for general magnetic charge \( B \) involves the sum over weight vectors \( h \) of the highest weight representation \( B \) of the group \( \hat{G} \). The weight vector \( h \) appearing in the argument of \( Z_{1-loop} \) and \( Z_{\text{inst}} \) has an interpretation as the value of magnetic charge measured at the polar regions. Now for a ‘large’ charge \( B \), the corresponding representation has more weight vectors than just the Weyl images of \( B \). Some of the weight vectors will therefore have reduced length as compared to the length of \( B \). This is interpreted as monopole screening: smooth monopoles can surround the ‘t Hooft operator inserted at the equator and screen its magnetic charge, so that the magnetic charge \( h \) observed at the polar region is ‘smaller’ than the charge \( B \) of the monopole inserted.

There should be solutions to the Bogomolny equation describing monopole screening, which are therefore labeled by \( B \) and \( h \) and form a finite dimensional moduli space. Via Kronheimer’s construction, such solutions should be mapped to ASD connections on \( \mathbb{C}^2 \) which are invariant under \( \psi \)-translation symmetry \( U(1)_\psi \). Therefore, for the \( U(N) \) gauge group the moduli space of monopoles is parametrized by the ADHM data

\[ \{B_1 (k \times k), B_2 (k \times k), I (k \times N), J (N \times k) \} \]

satisfying \([B_1, B_2] + I = 0\) and also the condition of \( U(1)_\psi \) invariance. The number \( k \) and the action of \( U(1)_\psi \) are determined in the following way. Consider solutions to the Bogomolny equation in which the charge of a singular monopole \( M \) is reduced to \( M' \) by a screening effect.

The charges \( M, M' \) here are regarded as \( N \times N \) diagonal matrices. Then there should be a diagonal matrix \( K \), whose size \( k \) and elements are determined by the formula

\[ \text{Tr}(x^M) = \text{Tr}(x^{M'}) + (x + x^{-1} - 2) \text{Tr}(K). \]

Then the condition of \( U(1)_\psi \) invariance is given by

\[ [K, B_1] + B_1 = [K, B_2] - B_2 = K I - IM' = M' J - JK = 0. \]

An equivariant integral on this moduli space contributes another factor to the integrand of (4.19). The detail of the analysis is presented in [55] for the example of ‘t Hooft operators of higher spin representations of \( SU(2) \).

**4.3. Surface operators**

Another important example of supersymmetric observables are surface operators, which are non-local operators supported on two-dimensional submanifolds. It will be a challenging problem to give a complete classification of surface operators for general 4D gauge theories,
but a major progress have been made for BPS surface operators in \( \mathcal{N} = 2 \) supersymmetric theories, as we review here.

For \( \mathcal{N} = 2 \) theories of class S where M5-brane interpretation is available, a natural question is how to identify the surface operators describing other M-branes ending on or intersecting the M5-branes [53, 59]. For those surface operators, the calculations in gauge theories can be checked against the prediction from AGT correspondence. Another approach is to realize \( \mathcal{N} = 2 \) SUSY theories geometrically using Calabi–Yau compactification of type IIA string, where D4-branes wrapping Lagrangian submanifolds give rise to surface operators [60]. In this setting, the results of gauge theory analysis can be compared with topological string amplitudes for which there are powerful formalisms known such as the refined topological vertex.

For \( \mathcal{N} = 2 \) SUSY theories on Omega-background \( \mathbb{R}^4_{x_1,x_2} \) with coordinates \( z_1 = x_1 + ix_2, z_2 = x_3 + ix_4, \) one can introduce surface operators along the surfaces \( z_2 = 0 \) or \( z_1 = 0 \) without braking supersymmetry. For theories on the ellipsoid (2.28), one can introduce BPS surface operators along the \( S^2 \) defined by \( x_3 = x_4 = 0 \) or \( x_1 = x_2 = 0 \).

### 4.3.1 Coupled 2D–4D systems

One way to describe surface operators is in terms of 2D quantum field theories on its worldvolume. For 4D \( \mathcal{N} = 2 \) theories realized on \( S^2_b \), the objects of interest are the half-BPS surface operators which support \( \mathcal{N} = (2, 2) \) field theories on a squashed \( S^2 \). The supersymmetry for the coupled 2D–4D system is such that the \( S^2_b \) and \( S^2 \) have the north and south poles in common, that is where the instantons of 4D gauge theory and vortices of 2D theory get localized.

If the 4D theory has a Lagrangian description, one can simplify the problem by turning off the 4D gauge coupling. The system is then reduced to a 2D interacting theory and 4D free multiplets. One can still learn a great deal about surface operators from this simplified system [66]. The partition function is then a product of the 4D free hypermultiplet path integral, \( Z_{\text{hyp}}^\text{loop} \), and the \( S^2 \) partition function [17, 18] of the 2D theory. The classical value of the frozen 4D vector multiplet enters the formula as the common mass for 2D and 4D fields.

As an example, take a system of \( N^2 \) free hypermultiplets. One can regard it as a bifundamental of the group \( SU(N) \times U(N) \) and turn on the masses \( (m_1, \cdots, m_N; \bar{m}_1, \cdots, \bar{m}_N) \). The \( S^2_b \) partition function is then

\[
\prod_{i,j=1}^{N} \Upsilon\left( \frac{Q}{2} + i(m_i - \bar{m}_j) \right)^{-1}.
\]  

(4.25)

This simple theory is known to correspond to \( N \) M5-branes wrapped on a sphere with three (one simple and two full) punctures. AGT relation identifies (4.25) with the corresponding three-point function in Toda conformal field theory. Now introduce a \( \mathcal{N} = (2, 2) \) theory with the same global symmetry \( SU(N) \times U(N) \) on the surface operator. The simplest class of examples is a 2D \( U(K) \) gauge theory with \( N \) fundamental and \( N \) anti-fundamental chiral multiplets. A systematic study and detailed comparison with Toda CFT correlators were made in [66]. It was shown that, if a suitable mass is turned on for the 2D theory, which is related to \( (m_i; \bar{m}_i) \) by a suitable rescaling and imaginary shift, then the 2D–4D combined partition function reproduces the Toda four-point functions with various degenerate insertions [53].

### 4.3.2 Singularity along a surface

Another way to define surface operators is to require that the gauge field and possibly other fields develop singularities along the surface. As an example, take an \( SU(N) \) gauge theory on \( \mathbb{C}^2 \) with coordinate \( z_1 = x_1 + ix_2 \) and \( z_2 = x_3 + ix_4 \). One can then introduce a surface operator along \( z_2 = 0 \) by imposing the singular boundary condition...
\[ A \simeq A_\chi \cdot d\chi \quad (\chi \equiv \arg(z_2), A_\chi \equiv \text{diag}(\nu_1, \cdots, \nu_{n_1}, \nu_2, \cdots, \nu_{n_2}, \cdots, \nu_1, \cdots, \nu_{n_1})) \].

This breaks the gauge symmetry SU(N) to a Levi subgroup
\[ \mathbb{L} = S[U(n_1) \times \cdots \times U(n_s)], \quad \sum_i n_i = N \]
on the surface. The parameters \( \nu_i \) satisfy \( \nu_1 > \cdots > \nu_s > \nu_1 - 1 \), which in turn set the order of \( n_i \) appearing in the partition of \( N \). For half-BPS surface operators in \( \mathcal{N} = 2 \) supersymmetric theories, one needs to turn on the auxiliary field \( D_{AB} \) to ensure the SUSY variation of gaugino to vanish. For a suitable choice of unbroken supersymmetry one finds
\[ D_{11} = D_{22} = 0, \quad D_{12} = iF_{34} = 2\pi i A_\chi \cdot \delta(x_3) \delta(x_4). \]

We have seen two different descriptions of surface operators, but there are some surface operators described in both ways. For example, the surface operators of type (4.26) in pure \( \mathcal{N} = 2 \) SYM theory can also be described by a 2D \( \mathcal{N} = (2, 2) \) supersymmetric quiver gauge theory which flows to a sigma model on a flag manifold \( SU(N)/\mathbb{L} \). Here the ordering of \( n_i \) makes a subtle effect: different orderings leads to different ultraviolet gauge theory descriptions, which flow to a non-linear sigma model on the same flag manifold but with different complex structures [65].

### 4.3.3. Localization computation.

Let us consider the surface operator of the type (4.26) introduced along the \( S^2 \) inside the ellipsoid (2.28) defined by \( x_3 = x_4 = 0 \). In terms of the polar coordinates \((\rho, \theta, \varphi, \chi)\) the surface operator is at \( \theta = 0 \). The singular behavior of the gauge field is then expressed as follows,
\[ A = A_\chi \cdot d\chi. \quad (\text{near } \theta = 0) \]

At supersymmetric saddle points the gauge field takes precisely this form. The value of classical action at the saddle points labeled by \( \phi = \bar{\phi} = -ia_0/2 \) is
\[ S_{\text{YM}} = \frac{8\pi^2}{g^2} \text{Tr} \left( \ell \bar{a}_0^2 - 2i\ell a_0 A_\chi \right), \quad S_{\text{Pl}} = -16i\pi^2 \zeta \left( \ell \bar{a}_0 - iA_\chi \ell \right), \quad S_{\text{mat}} = 0. \]

The saddle points can also have point-like instantons or anti-instantons localized at the north or south poles. Due to the presence of a surface operator, the topology of gauge field configuration near the north pole is characterized by instanton number \( k \) as well as magnetic flux \( m_i \) defined by
\[ \frac{1}{2\pi} \int_{\text{surface op}} \text{Tr}(A_\chi \cdot F) = \sum_{i=1}^s \nu_i m_i, \quad \left( \sum_{i=1}^s m_i = 0 \right) \]

Such topologically non-trivial gauge field configurations are called ramified instantons. The saddle points with point-like ramified instantons labeled by \( k, m_i \) are thus weighted by a factor \( q^{k - \nu_i m_i} \) in the path integral. Similarly, anti-instantons localized at the south pole are labeled by \( \bar{k}, \bar{m}_i \) and make contributions proportional to \( q^{\bar{k} - \nu_i \bar{m}_i} \). Those contributions are organized into a generalization of Nekrasov’s instanton partition function.

Nekrasov’s partition function for ramified instantons is a generating function of equivariant integrals over the moduli spaces \( \mathcal{M}_{\text{ram}}^{k, \bar{k}, \vec{m}} \). In mathematics literature these spaces are called
Affine Laumon spaces. The equivariant parameters are \( \epsilon_1 = \frac{\ell - 1}{\ell} \), \( \epsilon_2 = \frac{\tilde{\ell} - 1}{\tilde{\ell}} \) and the constant value of the field \( \Phi \) at saddle points

\[
\Phi = a_0 - \frac{i A_x}{\ell}.
\]  

(4.32)

Actually, this space \( \mathcal{M}_{k,\vec{m},\vec{n}}^{\text{ram}} \) is known to be mathematically equivalent to another space which should be more familiar to physicists, that is the moduli space of \( U(N) \) instantons in orbifold \( \mathbb{C} \times (\mathbb{C}/\mathbb{Z}_s) \) [84]. Here the \( \mathbb{Z}_s \) is understood to act on fields through spacetime rotation as well as gauge transformation: it acts on the fundamental representation of \( U(N) \) as the multiplication by the diagonal matrix

\[
\Omega_{\vec{k}} \equiv \text{diag}(\omega, \cdots, \omega, \cdots, \omega), \quad \omega \equiv e^{i \pi}. 
\]  

(4.33)

Each instanton is assigned a \( \mathbb{Z}_s \) charge, and the moduli space is denoted as \( \mathcal{M}_{k,\vec{m},\vec{n}}^{\text{orb}} \) with \( k_i \) the number of instantons with \( \mathbb{Z}_s \) charge \( i \) (we work with the convention \( k_i = k_{i+1} \)). The two moduli spaces are related as follows,

\[
\mathcal{M}_{k,\vec{m},\vec{n}}^{\text{ram}} = \mathcal{M}_{k,\vec{m},\vec{n}}^{\text{orb}} \quad \text{if} \quad k_s = k, \ k_{i+1} = k_i + m_i. 
\]  

(4.34)

For more explanation, see [65] and references therein.

The moduli space \( \mathcal{M}_{k,\vec{m},\vec{n}}^{\text{orb}} \) can be parametrized by ADHM matrices. Let us denote \( \vec{k} \equiv \sum_{i=1}^{s} k_i \), then the set of matrices \( \{B_1, B_2, I, J\} \) satisfying (4.22) which is also subject to the \( \mathbb{Z}_s \) orbifold projection,

\[
\begin{align*}
\Omega_{\vec{k}} B_1 \Omega_{\vec{k}}^{-1} &= B_1, \\
\Omega_{\vec{k}} B_2 \Omega_{\vec{k}}^{-1} &= \omega B_2, \\
\Omega_{\vec{k}} I \Omega_{\vec{k}}^{-1} &= I, \\
\Omega_{\vec{k}} J \Omega_{\vec{k}}^{-1} &= \omega J.
\end{align*}
\]  

(4.35)

gives a parametrization of the moduli space \( \mathcal{M}_{k,\vec{m},\vec{n}}^{\text{orb}} \). Here \( \Omega_{\vec{k}} \) is a diagonal matrix defined similarly to (4.33), with eigenvalue \( \omega^i \) appearing \( k_i \) times. The chain-saw quiver describes the components of ADHM matrices which survive the orbifold projection.

![Figure 1. The chain-saw quiver diagram.](image)

Ramified instanton partition functions and their correspondence with conformal blocks for general \( W_N \) algebra were studied in [59, 61–65].

The correspondence between ramified instantons and instantons in orbifolds will be key to fully understand how to define and compute observables in the surface defect backgrounds.
This was used in [67] for surface operators in $\mathcal{N} = 2$ pure SYM and $\mathcal{N} = 2^*$ SYM theories on $S^4$, and should be extended to more general cases. The exact formulae for observables obtained this way will also help clarifying how various descriptions of surface operators are related with each other.

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**Appendix A. Instanton partition functions**

Here we quote some formulae regarding Nekrasov’s partition functions [5–8] which are relevant in this article. For a more detailed explanation see, for example, [85] and the references therein.

As was reviewed in section 2.4, topological twist of 4D $\mathcal{N} = 2$ SUSY gauge theories reduces the path integral to an integral over moduli space of instantons. For the theory with gauge group $G$, the $k$-instanton moduli space has complex dimension $2h^\vee(G)k$. In addition, the matter hypermultiplets give rise to fermionic coordinates parametrizing the bundle of Dirac zeromodes over the moduli space. The hypermultiplet in representation $R$ therefore contributes $T(R)k$ fermionic dimensions, with $T(R)$ the quadratic Casimir in the representation $R$ normalized so that $T(\text{adj}) = 2h^\vee(G)$. Due to Omega deformation, the integral with respect to these variables are further localized to the fixed points under Lorentz rotation of $\mathbb{R}^4$ and constant gauge transformations. The Nekrasov’s partition function is the generating function of the integrals thus defined,

$$Z_{\text{inst}}(a, m, q, \epsilon_1, \epsilon_2) = \sum_{k \geq 0} q^k Z_k(a, m, \epsilon_1, \epsilon_2),$$

(A.1)

with $q = e^{2\pi i \tau}$ the instanton-counting parameter.

In the following discussion we focus on the case with $U(N)$ gauge group. The $k$-instanton moduli space is known to be parametrized by the ADHM data, that is the set of matrices $\{B_1, B_2, I, J\}$ (4.22) satisfying the ADHM equation,

$$\mu_\mathbb{C} \equiv [B_1, B_2] + II = 0,$$

$$\mu_\mathbb{R} \equiv [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0,$$

(A.2)

modulo the $U(k)$ equivalence

$$\{B_1, B_2, I, J\} \sim \{UB_1U^{-1}, UB_2U^{-1}, UI, JU^{-1}\}.\]$$

(A.3)

The singularities of the moduli space due to small instantons can be conveniently resolved by turning on nonzero moment maps, namely setting $\mu_\mathbb{C} = 0$ and $\mu_\mathbb{R} = \zeta \cdot 1_{(k \times k)}$. Let $\epsilon_1, \epsilon_2$ be the parameter of Lorentz rotation about the $(x_1, x_2)$ and $(x_3, x_4)$-planes in $\mathbb{R}^4$ and $a = \text{diag}(a_1, \cdots, a_N)$ the parameter of constant gauge rotation. Then the condition for the fixed points is that there is a $k \times k$ matrix $\phi$ such that

$$[\phi, B_1] = \epsilon_1 B_1, \quad [\phi, B_2] = \epsilon_2 B_2, \quad \phi I = Ia = 0, \quad aJ - J\phi = (\epsilon_1 + \epsilon_2)J.$$

(A.4)

Different fixed points are labeled by different sets of $k$ eigenvalues of $\phi$, which turn out to be described by $k$ boxes distributed to $N$ Young diagrams $\bar{Y} = (Y_1, \cdots, Y_N)$. See [86, 87] for
pedagogical derivations. If the \(i\)th diagram \(Y_i\) has a box at the site \((i,j)\) (the \(i\)th column, the \(j\)th row), it means one of the eigenvalues of \(\phi\) is \(a_i + (i-1)e_1 + (j-1)e_2\).

For pure Yang–Mills theory, the Duistermaat–Heckman’s formula (reviewed in contribution [88]) applied to the moduli space volume gives

\[
Z_k(a, \epsilon_1, \epsilon_2) = \sum_{\bar{Y}} \frac{1}{\prod_{\alpha=1}^{2kN} w_{\alpha}(a, \epsilon_1, \epsilon_2; \bar{Y})},
\]

(A.5)

where \(w_{\alpha}(a, \epsilon_1, \epsilon_2; \bar{Y})\) are the weights of the action of the Lorentz and gauge transformations on local coordinates at the fixed point labeled by \(\bar{Y}\). It is known that \(w_{\alpha}\) are given by

\[
\sum_{\alpha=1}^{2kN} e^{w_{\alpha}} = \text{Tr}(e^{-\phi})\text{Tr}(e^{\phi}) + e^{e_1+e_2}\text{Tr}(e^{-a\phi})\text{Tr}(e^{a\phi}) - (1-e^{e_1})(1-e^{e_2})\text{Tr}(e^{-\phi})\text{Tr}(e^{\phi})
\]

\[
= \sum_{I,J=1}^{N} \left\{ \sum_{\ell \in Y_{IJ}} e^{a_{I+J}+\epsilon_1(I+J)} + \sum_{\ell \in T_{J}} e^{a_{J}\epsilon_1(J) - e_1A_{J}(J)} \right\}.
\]

(A.6)

Here the arm-length \(A_{Y}(s)\) and the leg-length \(L_{Y}(s)\) of a box \(s\) at the site \((i,j)\) with respect to the diagram \(Y\) are defined by

\[
A_{Y}(s) \equiv (\text{height of the } i\text{th column of } Y) - j,
\]

\[
L_{Y}(s) \equiv (\text{width of the } j\text{th row of } Y) - i.
\]

(A.7)

Note that the weights appear in pairs related by the ‘reflection’, \(w \rightarrow \epsilon_1 + \epsilon_2 - w\).

For the theory with matters with mass \(m\), the weights \(v_{\alpha}(a, m, \epsilon_1, \epsilon_2; \bar{Y})\) of the symmetry action on the local fermionic variables appear in the numerator of each fixed point contribution,

\[
Z_k(a, m, \epsilon_1, \epsilon_2) = \sum_{\bar{Y}} \frac{\prod_{\alpha=1}^{2kN} v_{\alpha}(a, m, \epsilon_1, \epsilon_2; \bar{Y})}{\prod_{\alpha=1}^{2kN} w_{\alpha}(a, \epsilon_1, \epsilon_2; \bar{Y})}.
\]

(A.8)

The explicit form of the fermionic weights is known for various representations of classical gauge groups; see [89] for an extensive collection of formulae. For a single adjoint representation corresponding to \(\mathcal{N} = 2^*\) theory, the Dirac zeromode bundle can be identified with the tangent bundle, so we have simply

\[
v_{\alpha}(a, m, \epsilon_1, \epsilon_2; \bar{Y}) = w_{\alpha}(a, \epsilon_1, \epsilon_2; \bar{Y}) - m. \quad (\alpha = 1, \ldots, 2kN)
\]

(A.9)

As was discussed in section 3.4, some special values of \(m\) are of particular interest. First, for \(m = 0\) or \(m = \epsilon_1 + \epsilon_2\) one has complete cancellation of the denominator and numerator for each term in the sum (A.8). \(Z_k\) then simply counts the number of ways to distribute \(k\) boxes to \(N\) Young diagrams, so that

\[
Z_{\text{inst}}(a, m, q, \epsilon_1, \epsilon_2) \bigg|_{m=0 \text{ or } \epsilon_1+\epsilon_2} = \prod_{n \geq 1} (1-q^n)^{-N}.
\]

(A.10)

The other special choices are \(m = \epsilon_1\) or \(\epsilon_2\), which lead to \(Z_k = 0\) for all \(k > 0\) because the set of weights \(\{w_{\alpha}\}\) always contains at least one pair of \(\{\epsilon_1, \epsilon_2\}\). As a consequence, one has

\[
Z_{\text{inst}}(a, m, q, \epsilon_1, \epsilon_2) \bigg|_{m=\epsilon_1 \text{ or } \epsilon_2} = 1.
\]

(A.11)
The formula for the Nekrasov’s partition functions here were presented with the convention for $a$ and $m$ that are common in the literature. To match them with the formulae in the main text, one needs to relate the parameters $a, m$ here and in the main text as follows [12],

\[ a_{\text{(here)}} = i a_{\text{(main text)}}, \quad m_{\text{(here)}} = \frac{\epsilon_1 + \epsilon_2}{2} + i m_{\text{(main text)}} \cdot \begin{pmatrix} \epsilon_1 = \frac{1}{\ell}, & \epsilon_2 = \frac{1}{\ell} \end{pmatrix} \quad (A.12) \]

### Appendix B. Supersymmetric ellipsoid background

Here we record some details of the supersymmetric ellipsoid background found in [32].

We use the polar angle coordinates $(\rho, \theta, \varphi, \chi)$ introduced in (2.29). Let $f, g, h$ be the following functions,

\[ f \equiv \sqrt{\ell^2 \sin^2 \theta + \ell^2 \cos^2 \theta}, \quad g \equiv \sqrt{r^2 \sin^2 \rho + \frac{\ell^2 \ell^2}{f^2} \cos^2 \rho}, \quad h \equiv \frac{\ell^2 - \ell^2}{f} \cos \rho \sin \theta \cos \theta. \]

Then the vielbein one-form $E^a = E^a_m dx^m$ on the ellipsoid is given by

\[ E^1 = \ell \sin \rho \cos \theta d\varphi, \quad E^2 = \ell \sin \rho \sin \theta d\chi, \quad E^3 = f \sin \theta d\rho, \quad E^4 = g \sin \theta \cos \theta d\rho, \quad E^5 = h \sin \theta \cos \theta d\rho. \]

and the spin connection $\Omega^{ab} = \Omega_{m}^{ab} dx^m$ is

\[
\begin{align*}
\Omega^{12} &= 0, \\
\Omega^{13} &= -\frac{\ell}{f} \sin \theta \sin \varphi, \\
\Omega^{23} &= \frac{\ell}{f} \cos \theta d\chi, \\
\Omega^{14} &= \frac{\ell \ell^2 \cos \rho \cos \theta}{g f^2} \sin \varphi d\varphi, \\
\Omega^{24} &= \frac{\ell \ell^2 \cos \rho \sin \theta}{g f^2} d\chi, \\
\Omega^{34} &= \frac{\ell \ell^2 \cos \rho \cos \theta}{g f^3} \sin \theta d\varphi.
\end{align*}
\]

In [32], the auxiliary fields $T_{mn}, T_{mn}, (V_m)^A_B$ and $M$ were determined by solving the Killing spinor equation (2.6) with the specific choice of Killing spinor (2.32). These equations can be viewed as $2 \times 2$ matrix equations by regarding $\xi_A, \xi'_A$ as the element of $2 \times 2$ matrices $\xi, \xi'$.

The matrices

\[ T \equiv -i \sigma_M T^M, \quad T^M \equiv -i \sigma_M T^M \quad (B.4) \]

act on their left, whereas the $SU(2)$ gauge field $V = (V^A_B)$ multiplies from their right. For later convenience we introduce $\bar{V} = \bar{V}_A E^A$ by

\[ V = \bar{V} + \frac{1}{2} \left( \left( \frac{\ell}{f} - 1 \right) d\varphi + \left( \frac{\ell}{f} - 1 \right) d\chi \right) \tau^3. \quad (B.5) \]

Finally, if one similarly defines the matrices $\xi, \xi'$ from $\xi'_A, \xi'_A$, the relation (2.21) implies there are matrices $S, S$ such that

\[ \xi' = S \xi = -i \sigma_b \delta^{[b} \xi, \quad \xi' = \bar{S} \xi = -i \sigma_b \delta^{[b} \bar{\xi}. \quad (B.6) \]

With these preparations one can show that the first two of the equations (2.6) can be rewritten into the following inhomogeneous linear equations on $T, T, S, S$ and $\bar{V}_A$:
\[ \begin{align*}
\eta & = \cosh \theta \\
\tau & = \sinh \theta \\
\Delta & = \eta + \bar{\eta} = 2 \cosh \theta \\
\Omega & = \frac{\eta^2 \cos \rho}{g f^2 \sin \rho}, \\
\Omega_1 & = \frac{\eta^2 \cos \rho}{g f^2 \sin \rho}, \\
\Omega_2 & = \frac{\eta^2 \cos \rho}{g f^2 \sin \rho}, \\
\Omega_3 & = \frac{\eta^2 \cos \rho}{g f^2 \sin \rho}.
\end{align*} \]

A particularly simple solution to the above equation is

\[ \begin{align*}
\eta_1 & = \frac{\cos \theta \eta}{2 \sin \rho} \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \cos \rho \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \xi_1 \eta_1 = \frac{\cos \theta \cos \rho}{2 \sin \rho} \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \xi_1 \eta_1, \\
\eta_2 & = \frac{\cos \theta \eta}{2 \sin \rho} \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \cos \rho \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \xi_2 \eta_2 = \frac{\cos \theta \cos \rho}{2 \sin \rho} \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \xi_2 \eta_2, \\
\eta_3 & = \frac{\cos \theta \eta}{2 \sin \rho} \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \cos \rho \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \xi_3 \eta_3 = \frac{\cos \theta \cos \rho}{2 \sin \rho} \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \xi_3 \eta_3.
\end{align*} \]

where we used \( \tau^1 \equiv \cos \theta \tau^1 + \sin \theta \tau^2 \) and \( \tau^2 \equiv i \tau^3 \). This can be shifted by the solution to the homogeneous equation (the equation (B.7) with the RHS set to zero), which are parametrized by arbitrary functions \( c_1, c_2, c_3 \) as follows:

\[ \begin{align*}
\eta & = \Delta \eta_1 = -2 \sin \theta \left( c_2 \tau^1 \eta - c_1 \tau^2 \eta \right), \\
\eta & = \Delta \eta_2 = +2 \cos \theta \left( c_2 \tau^1 \eta - c_1 \tau^2 \eta \right), \\
\eta & = \Delta \eta_3 = -2 c_1 \tau^1 \eta + 2 c_2 \tau^2 \eta, \\
\eta & = \Delta \eta_4 = +2 c_2 \tau^1 \eta - 2 c_1 \tau^2 \eta.
\end{align*} \]

The remaining two of the equations (2.6) are also satisfied if \( M \) is chosen as

\[ \begin{align*}
M & = \frac{1}{g^2} - \frac{1}{g^2} + \frac{h^2}{g^2} - \frac{4}{f g} + \Delta M, \\
\Delta M & = \frac{1}{2 g} \left( \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) \cosh \theta \eta_1 + \frac{\cosh \theta \cos \rho}{g f^2 \sin \rho} \frac{1}{g} \frac{1}{g} \frac{1}{g} \xi_1 \\
+ & 8 \left( \frac{1}{f g \sin \rho} \eta_1 + \frac{h \cosh \theta \cos \rho}{g f^2 \sin \rho} \frac{1}{g} \frac{1}{g} \frac{1}{g} \right) c_2 - 16 \left( c_1^2 + c_2^2 + c_3^2 \right).
\end{align*} \]
and the functions $c_i$ depend only on $\rho, \theta$. As was shown in [32], these functions have to be chosen carefully because the simple solution with $c_i \equiv 0$ is in fact singular at the poles.

The geometry of SUSY ellipsoid backgrounds were revisited in [38], in which they found a generalized solution containing eight free parameters.

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