UPPER AND LOWER BOUNDS FOR THE BLOW-UP TIME IN QUASILINEAR REACTION DIFFUSION PROBLEMS

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Abstract. In this paper, we consider a quasilinear reaction diffusion equation with Neumann boundary conditions in a bounded domain. Basing on Sobolev inequality and differential inequality technique, we obtain upper and lower bounds for the blow-up time of the solution. An example is also given to illustrate the abstract results obtained of this paper.

1. Introduction. Blow-up problems related to reaction diffusion equations have been widely studied by many authors and numerous interesting results have been obtained in [2,11]. Studies of these references are often concerned with the conditions of global existence and blow-up in finite time, bounds for the blow-up rate, structure of blow-up set, and the asymptotic behavior of the solutions.

In the present paper, we investigate the blow-up problem of the following quasilinear reaction diffusion equation:

\[
\begin{align*}
\frac{\partial (h(u))}{\partial t} &= \nabla \cdot (\rho (|\nabla u|^2) \nabla u) + k(t)f(u) \quad \text{in } \Omega \times (0,t^*), \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0,t^*), \\
u(x,0) &= u_0(x) \geq 0 \quad \text{in } \Omega.
\end{align*}
\]

In (1), $\Omega$ is a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial \Omega$, $\nu$ represents the unit normal vector to $\partial \Omega$, $u_0(x) \in C^1(\Omega)$ is a nonnegative function satisfying the compatibility condition, $t^*$ is the blow-up time if blow-up occurs. Set $\mathbb{R}^+ = (0,\infty)$. In this paper, we assume that $h$ is a $C^2(\mathbb{R}^+)$ function with $h'(s) > 0$ for $s > 0$, $\rho$ is a positive $C^2(\mathbb{R}^+)$ function satisfying $\rho(s) + 2s\rho'(s) > 0$ for $s > 0$, $k$ is a positive $C^1(\mathbb{R}^+)$ function, and $f$ is a nonnegative $C^1(\mathbb{R}^+)$ function. According to maximum principles [20], we know that the classical solution $u$ of problem (1) is nonnegative in $\Omega \times [0,t^*)$.

Recently, there are many papers to study the bounds of the blow-up time, especially on the lower bound for the blow-up time. As we all know, lower bounds for blow-up time seem to be more important in applications, due to the explosive nature of the solution. However, the works mentioned above usually derived an upper bound for blow-up time. At meanwhile, we note that lower bounds for blow-up
time seem harder to be determined. Since Payne and Schaefer in [17] used a first-order differential technique and derived a lower bound for blow-up time, the similar idea is also applied in more generalized problems (see [1,3-4,6-8,12-16,18-19]). The direct motivation of this paper comes from [5]. When $\Omega \subset \mathbb{R}^3$ is a bounded convex domain, Ding and Hu [5] dealt with problem (1) and derived the lower and upper bounds for blow-up time when blow-up occurs. Their main method in [5] is to use a first-order differential inequality technique. Naturally, in this paper, we hope to obtain a lower bound for blow-up time when $\Omega \subset \mathbb{R}^n (n \geq 3)$ is a bounded domain.

In addition, we also get an upper bound of blow-up time by restricting bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$. Our study is based on the use of Sobolev inequality and differential inequality technique.

The rest of this paper is organized as follows. In the second section, we establish conditions on data to ensure that the solution blows up at some finite time in measure $B(t)$ defined in (2). Moreover, an upper bound for $t^*$ is derived. In the third section, under suitable hypotheses on data, we get a lower bound for $t^*$. In the last section, an example is presented to demonstrate the results of this paper.

2. Upper bound for $t^*$. In this section, we restrict $\Omega \subset \mathbb{R}^n (n \geq 2)$ and define auxiliary functions as follows

$$B(t) = \int_\Omega H(u(x,t))dx, \quad D(t) = -\int_\Omega P(|\nabla u|^2)dx + 2k(t)\int_\Omega F(u)dx, \quad t \geq 0 \quad (2)$$

with

$$H(u) = 2\int_0^u sh'(s)ds, \quad P(|\nabla u|^2) = \int_0^{\nabla u^2} \rho(s)ds, \quad F(u) = \int_0^u f(s)ds. \quad (3)$$

With the aid of these auxiliary functions, we get an upper bound for blow-up time $t^*$. More precisely we establish the following result.

**Theorem 2.1.** Let $u$ be a classical solution of (1). Moreover, we suppose the functions $k, h, \rho, f$ to satisfy

$$k'(t) \geq 0, \quad t \geq 0, \quad (4)$$

$$h''(s) < 0, \quad sf(s) \geq 2(1 + d)F(s), \quad s\rho(s) \leq (1 + d)P(s), \quad s > 0, \quad (5)$$

where $d$ is a nonnegative constant. Furthermore, initial data are assumed to satisfy

$$D(0) = -\int_\Omega P(|\nabla u_0|^2)dx + 2k(0)\int_\Omega F(u_0)dx > 0. \quad (6)$$

Then $u$ must blow up at $t^* \leq T$ in measure $B(t)$ with

$$T = \begin{cases} \frac{B(0)}{2d(1 + d)D(0)}, & d > 0, \\ \infty, & d = 0. \end{cases}$$

**Proof.** We use Green’s formula and (5) to derive

$$B'(t) = \int_\Omega H'(u)u_tdx = 2\int_\Omega uh'(u)u_tdx$$

$$= 2\int_\Omega u \left[ \nabla \cdot \left( \rho(|\nabla u|^2)\nabla u \right) + k(t)f(u) \right] dx$$
It is easy to see that

\[ \text{(9)-(10)} \]

\[ \frac{\partial u}{\partial t} \text{ involves } \frac{\partial u}{\partial t} \text{ to derive } \]

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\[ \text{In view of (7), we deduce } \]

\[ \text{Differentiating } D \text{ and using (4), we get } \]

\[ D'(t) = -2 \int_{\Omega} \rho(|\nabla u|^2) (\nabla u \cdot \nabla u_t) \, dx + 2k(t) \int_{\Omega} F(u) \, dx + 2k(t) \int_{\Omega} f(u) u_t \, dx \]

\[ \geq -2 \int_{\Omega} \rho(|\nabla u|^2) (\nabla u \cdot \nabla u_t) \, dx + 2k(t) \int_{\Omega} f(u) u_t \, dx \]

\[ = 2 \int_{\Omega} \nabla \cdot (\rho(|\nabla u|^2) u_t \nabla u) \, dx - 2 \int_{\Omega} \rho(|\nabla u|^2) (\nabla u \cdot \nabla u_t) \, dx \]

\[ + 2k(t) \int_{\Omega} f(u) u_t \, dx \]

\[ = 2 \int_{\Omega} u_t \left[ \nabla \cdot \left( \rho(|\nabla u|^2) \nabla u \right) + k(t)f(u) \right] \, dx \]

\[ = 2 \int_{\Omega} h'(u) u_t^2 \, dx \geq 0. \]

It is easy to see that \( D(t) \) is a nondecreasing function. In view of (6)-(7), we have

\[ B'(t) > 0. \]

By Schwarz’s inequality, (7) and \( h'(s) > 0 \) for \( s > 0 \), we obtain

\[ 2(1 + d)D(t)B'(t) \leq (B'(t))^2 = \left( 2 \int_{\Omega} h'(u) u u_t \, dx \right)^2 \]

\[ = 4 \left( \int_{\Omega} h'(u) u^2 \, dx \right) \left( \int_{\Omega} h'(u) u_t^2 \, dx \right). \]

Integrating by part and using (5), we have

\[ H(u) = 2 \int_0^u sh'(s) \, ds = \int_0^u h'(s) ds^2 = h'(u) u^2 - \int_0^u s^2 h''(s) \, ds \geq h'(u) u^2. \]

We combine (9)-(10) to derive

\[ (1 + d)D(t)B'(t) \leq 2 \left( \int_{\Omega} H(u) \, dx \right) \left( \int_{\Omega} h'(u) u_t^2 \, dx \right) \leq B(t)D'(t); \]

that is

\[ \left( D(t)B^{-(1+d)}(t) \right)' \geq 0. \]

Integrating (11) from 0 to \( t \), we obtain

\[ D(t)B^{-(1+d)}(t) \geq D(0)B^{-(1+d)}(0). \]

In view of (7), we deduce

\[ B'(t)B^{-(1+d)}(t) \geq 2(1 + d)D(0)B^{-(1+d)}(0). \]

(12)
When $d > 0$, we integrate (12) from 0 to $t$ to get
\[ B^{-d}(t) \leq B^{-d}(0) - 2d(1 + d)D(0)B^{-(1+d)}(0)t. \] (13)

It is obviously that (13) cannot hold for all time. Hence, $u$ must blow up at some finite time $t^*$ in measure $B(t)$. Letting $t \to t^*$ in (13), we have
\[ t^* \leq T = \frac{B(0)}{2d(1 + d)D(0)}. \]

When $d = 0$, it follows from (12) that
\[ B(t) \geq B(0)e^{2D(0)B^{-1}(0)t}, \]
which implies $T = \infty$. □

3. Lower bound for $t^*$. In this section, we derive a lower bound for $t^*$ by restricting $\Omega \subset \mathbb{R}^n$ ($n \geq 3$). Here we impose the following constraints on data
\[ f(s) \leq as^m, \quad \rho(s) \geq b_1 + b_2s^l, \quad h'(s) > \gamma, \quad s > 0, \] (14)
\[ k(t) \geq M, \quad \frac{k'(t)}{k(t)} \leq \eta, \quad t \geq 0, \] (15)
where $a, b_2, m, l, \gamma, \eta, M$ are positive constants, $b_1$ is a nonnegative constant, and $m > 2l + 1$. Two auxiliary functions are defined as follows
\[ A(t) = k^\alpha(t) \int_{\Omega} G(u) dx, \quad t \geq 0, \quad G(u) = \beta \int_0^u s^{\beta - 1}h'(s) ds, \]
where
\[ \alpha = \frac{2r(l + 1) + l(n - 2)}{m - 1}, \quad \beta = 2r(l + 1) - 2l, \]
and $r$ is a parameter to satisfy
\[ r > \max \left\{ 1, \frac{n(m - 2l - 1) + 4l}{4(l + 1)} \right\}. \] (16)

In this section, we also need to use the following Sobolev inequality (see [9]) for $n \geq 3$
\[ \left( \int_{\Omega} (u^{l+1})^{\frac{2n}{l+1}} dx \right)^{\frac{n-2}{n}} \leq C(n, \Omega) \left( \int_{\Omega} u^{2(l+1)} dx + \int_{\Omega} |\nabla u^{l+1}|^2 dx \right)^{\frac{1}{2}}, \] (17)
where $C(n, \Omega)$ is an embedding constant. We state the main result of this section as follows.

**Theorem 3.1.** Let $u$ be a classical solution to (1) and assume that (14)-(16) hold. If $u$ blows up at finite time $t^*$ in measure $A(t)$, then blow-up time $t^*$ is bounded from below by
\[ t^* \geq \int_{A(0)}^{\infty} \frac{d\tau}{C_1 + \alpha \eta \tau + C_2 \tau} \]
with
\[ C_1 = \frac{2\beta(m - 2l - 1)}{2r(l + 1) + m - 2l - 1} \left( \frac{2b_2l(l + 1)(\beta - 1)}{r^{2l+1} [2r(l + 1) + m - 2l - 1]} + a \right) \]
\begin{align*}
\times & \left(2r(l + 1) + m - 2l - 1\right) \frac{2r(l + 1)}{4r(l + 1)} \left[\varepsilon(C(n, \Omega))^2\right]^{2r(l + 1) + m - 2l - 1} \\
& \times M^\alpha \frac{2r(l + 1)}{m - 2l - 1} |\Omega| + \frac{b_1 \beta(\beta - 1)(m - 2l - 1)}{r^2(l + 1)[2r(l + 1) + m - 2l - 1]} M^\alpha \frac{2r(l + 1)}{m - 2l - 1} |\Omega|,
\end{align*}
(18)

\begin{align*}
C_2 = & \frac{\beta [4r(l + 1) + 2l(n - 2) - n(m - 1)]}{2r(l + 1) + l(n - 2)} \left(\frac{2b_2 (l + 1)(\beta - 1)}{r^{2l+1} \left[2r(l + 1) + m - 2l - 1\right]} + a\right) \\
& \times \gamma \frac{4r(l + 1) + 2l(n - 2) - n(m - 1)}{4r(l + 1) + 2l(n - 2) - n(m - 1)} \left(\frac{4r(l + 1) + 2l(n - 2)}{n(m - 1)}\right) \varepsilon^{\frac{1}{2}}. 
\end{align*}
(19)

and

\begin{align*}
\varepsilon = & \frac{b_2 (\beta - 1)}{2[C(n, \Omega)]^{2(l + 1)} \gamma^{2(l + 1)}} \left(\frac{2b_2 (l + 1)(\beta - 1)}{r^{2l+1} \left[2r(l + 1) + m - 2l - 1\right]} + a\right)^{-1}.
\end{align*}
(20)

**Proof.** By (16), we have \( \beta > 2 \). Applying Green's formula and (14)-(15), we obtain

\begin{align*}
A'(t) = & \alpha k^\alpha(t) k'(t) \int_\Omega G(u)dx + k^\alpha(t) \int_\Omega G'(u)u_t dx \\
= & \alpha \frac{k'(t)}{k(t)} k^\alpha(t) \int_\Omega G(u)dx + \beta k^\alpha(t) \int_\Omega u^{\beta-1} h'(u)u_t dx \\
\leq & \alpha \eta A(t) + \beta k^\alpha(t) \int_\Omega \nabla \cdot (\rho (|\nabla u|^2) \nabla u) + k(t)f(u) \right) dx \\
= & \alpha \eta A(t) + \beta k^\alpha(t) \int_\Omega \nabla \cdot (u^{\beta-1} \rho (|\nabla u|^2) \nabla u) dx \\
& - \beta(\beta - 1) k^{\alpha(t)} \int_\Omega u^{\beta-2} \rho (|\nabla u|^2) |\nabla u|^2 dx + \beta k^{\alpha+1(t)} \int_\Omega u^{\beta-1} f(u) dx \\
= & \alpha \eta A(t) + \beta k^\alpha(t) \int_\partial\Omega u^{\beta-1} \rho (|\nabla u|^2) \frac{\partial u}{\partial \nu} dS \\
& - \beta(\beta - 1) k^{\alpha(t)} \int_\Omega u^{\beta-2} \rho (|\nabla u|^2) |\nabla u|^2 dx + \beta k^{\alpha+1(t)} \int_\Omega u^{\beta-1} f(u) dx \\
= & \alpha \eta A(t) - \beta(\beta - 1) k^{\alpha(t)} \int_\Omega u^{\beta-2} \rho (|\nabla u|^2) |\nabla u|^2 dx \\
& + \beta k^{\alpha+1(t)} \int_\Omega u^{\beta-1} f(u) dx \\
\leq & \alpha \eta A(t) - \beta(\beta - 1) k^{\alpha(t)} \int_\Omega u^{\beta-2} \left(b_1 + b_2 |\nabla u|^{2l}\right) |\nabla u|^2 dx \\
& + a \beta k^{\alpha+1(t)} \int_\Omega u^{\beta+m-1} dx \\
\leq & \alpha \eta A(t) - b_2 \beta(\beta - 1) k^{\alpha(t)} \int_\Omega u^{\beta-2} |\nabla u|^{2(l+1)} dx + a \beta k^{\alpha+1(t)} \int_\Omega u^{\beta+m-1} dx.
\end{align*}

Denoting \( v = u^r \) and using

\[ |\nabla u^r|^{2(l+1)} = r^{2(l+1)} u^{2(r-1)(l+1)} |\nabla u|^{2(l+1)}, \]
we can rewrite (21) as

\[ A'(t) \leq \alpha \eta A(t) - \frac{b_2 \beta (\beta - 1)}{r^2(\beta + 1)} k^\alpha(t) \int_\Omega |\nabla u'|^{2l+1} dx \\
+ a \beta k^{\alpha+1}(t) \int_\Omega \nabla^m \nabla^{-m-1} \nabla u' dx \]

\[ = \alpha \eta A(t) - \frac{b_2 \beta (\beta - 1)}{r^2(\beta + 1)} k^\alpha(t) \int_\Omega |\nabla v|^{2l+1} dx \\
+ a \beta k^{\alpha+1}(t) \int_\Omega v^{2l+1+m-2l-1} dx. \]  

(22)

By Hölder inequality and Young inequality, we have

\[
\int_\Omega |\nabla v|^{2l+1} dx = (l+1)^2 \int_\Omega v^{2l} |\nabla v|^2 dx \\
\leq (l+1)^2 \left( \int_\Omega v^{2l+1} dx \right)^{\frac{l}{l+1}} \left( \int_\Omega |\nabla v|^{2l+1} dx \right)^{\frac{1}{l+1}} \\
\leq l(l+1) \int_\Omega v^{2l+1} dx + (l+1) \int_\Omega |\nabla v|^{2l+1} dx;
\]

that is

\[
\int_\Omega |\nabla v|^{2l+1} dx \geq \frac{1}{l+1} \int_\Omega |\nabla v|^{2l+1} dx - l \int_\Omega v^{2l+1} dx.
\]

(23)

Inserting (23) into (22), we obtain

\[
A'(t) \leq \alpha \eta A(t) - \frac{b_2 \beta (\beta - 1)}{r^2(\beta + 1)} k^\alpha(t) \left( \frac{1}{l+1} \int_\Omega |\nabla v|^{2l+1} dx - l \int_\Omega v^{2l+1} dx \right) \\
+ a \beta k^{\alpha+1}(t) \int_\Omega v^{2l+1+m-2l-1} dx \\
= \alpha \eta A(t) + \frac{b_2 l \beta (\beta - 1)}{r^2(l+1)} k^\alpha(t) \int_\Omega v^{2l+1} dx \\
- \frac{b_2 \beta (\beta - 1)}{(l+1)r^2(l+1)} k^\alpha(t) \int_\Omega |\nabla v|^{2l+1} dx + a \beta k^{\alpha+1}(t) \int_\Omega v^{2l+1+m-2l-1} dx.
\]

(24)

Next, we deal with the second term of (24). In view of the fact that \( m - 2l - 1 > 0 \) and (16), we have

\[
0 < \frac{2r(l+1)}{2r(l+1) + m - 2l - 1} \leq 1, \quad \alpha - \frac{2r(l+1)}{m - 2l - 1} < 0.
\]

(25)

It follows from (15), (25), Hölder inequality and Young inequality that

\[
k^\alpha(t) \int_\Omega v^{2l+1} dx \leq \left( k^{\alpha+1}(t) \int_\Omega v^{2l+1+m-2l-1} dx \right)^{\frac{2r(l+1)}{(2r(l+1)+m-2l-1)}} \\
\times \left( k^\alpha \frac{2r(l+1)}{m-2l-1} \Omega \right)^{\frac{m-2l-1}{(2r(l+1)+m-2l-1)}} \\
\leq \frac{2r(l+1)}{2r(l+1) + m - 2l - 1} k^{\alpha+1}(t) \int_\Omega v^{2l+1+m-2l-1} dx \\
+ \frac{2r(l+1)}{2r(l+1) + m - 2l - 1} M^{\alpha - \frac{2r(l+1)}{m-2l-1}} \Omega.
\]

(26)
Substituting (26) into (24), we derive

$$A'(t) \leq \alpha \eta A(t) + \left( \frac{2b_2 l \beta (\beta - 1)(l + 1)}{r^{2(l+1)}[2r(l+1) + m - 2l - 1]} + a \beta \right)$$

$$\times k^{\alpha+1}(t) \int_{\Omega} \nu^{2(l+1)+m-2l-1} dx$$

$$+ \frac{b_2 l \beta (\beta - 1)(m - 2l - 1)}{r^{2(l+1)}[2r(l+1) + m - 2l - 1]} M^{\alpha+1} \frac{2r(l+1)}{m-2l-1} |\Omega|$$

$$- \frac{b_2 \beta (\beta - 1)}{(l+1)r^{2(l+1)}} k^{\alpha}(t) \int_{\Omega} |\nabla \nu|^{2} dx. \quad (27)$$

Now, we focus on the second term of (27). In view of (16), we have

$$0 < \frac{(n - 2)(m - 1)}{4r(l+1) + 2(n - 2)} < 1, \quad 0 < \frac{n(m - 1)}{4r(l+1) + 2(n - 2)} < 1.$$ 

By Hölder inequality and Young inequality, we deduce

$$k^{\alpha+1}(t) \int_{\Omega} \nu^{2(l+1)+m-2l-1} dx$$

$$\leq \left( k^{\alpha}(t) \int_{\Omega} \nu^{\frac{\beta}{\alpha}} dx \right)^{\frac{\beta}{\alpha}} \frac{4r(l+1)+2l(n-2)-(n-2)(m-1)}{4r(l+1)+2l(n-2)-n(m-1)}$$

$$\times \left[ \left( k^{\alpha}(t) \int_{\Omega} \nu^{\frac{\beta}{\alpha}} dx \right)^{\frac{2\alpha}{\beta}} \frac{2n}{n-1} \frac{n(m-1)}{4r(l+1)+2l(n-2)} \right]$$

$$= \left( \frac{4r(l+1)+2l(n-2)}{n(m-1)} \right)^{\frac{n(m-1)}{4r(l+1)+2l(n-2)}} \left( \frac{4r(l+1)+2l(n-2)}{n(m-1)} \right)^{-\frac{n(m-1)}{4r(l+1)+2l(n-2)}}$$

$$\leq \frac{4r(l+1)+2l(n-2)-n(m-1)}{4r(l+1)+2l(n-2)-n(m-1)} \left( \frac{4r(l+1)+2l(n-2)}{n(m-1)} \right)^{-\frac{n(m-1)}{4r(l+1)+2l(n-2)}} \int_{\Omega} \nu^{\frac{2n}{n-1}} \frac{2n}{n-1}$$

$$\leq \frac{4r(l+1)+2l(n-2)-n(m-1)}{4r(l+1)+2l(n-2)-n(m-1)} \left( \frac{4r(l+1)+2l(n-2)}{n(m-1)} \right)^{-\frac{n(m-1)}{4r(l+1)+2l(n-2)}} \int_{\Omega} \nu^{\frac{2n}{n-1}} \frac{2n}{n-1}, \quad \text{where } \varepsilon \text{ is given in (20).}$$

Using Sobolev inequality (17) to the second term of (28), we derive

$$k^{\alpha+1}(t) \int_{\Omega} \nu^{2(l+1)+m-2l-1} dx$$

$$\leq \frac{4r(l+1)+2l(n-2)-n(m-1)}{4r(l+1)+2l(n-2)}.$$
\[ \frac{4r(l + 1) + 2l(n - 2) - n(m - 1)}{4r(l + 1) + 2l(n - 2) - n(m - 1)} \varepsilon \]
\[ \times \left( k^{\alpha(t)} \int_{\Omega} v^2(t) \, dx \right) \]
\[ = \frac{4r(l + 1) + 2l(n - 2) - n(m - 1)}{4r(l + 1) + 2l(n - 2) - n(m - 1)} \varepsilon \]
\[ \times \left( k^{\alpha(t)} \int_{\Omega} \frac{\partial}{\partial t} v^2(t) \, dx \right) \]
\[ + \frac{\varepsilon}{C(n, \Omega)^{2}} \left( k^{\alpha(t)} \int_{\Omega} \frac{\partial v^{(l+1)}}{\partial t} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left[ \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right] \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \leq \frac{1}{2} k^{\alpha+1(t)} \int_{\Omega} v^{2(l+1)} + \frac{m - 2l - 1}{2r(l + 1) + m - 2l - 1} \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left[ \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right] \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \leq \frac{1}{2} k^{\alpha+1(t)} \int_{\Omega} v^{2(l+1)} + \frac{m - 2l - 1}{2r(l + 1) + m - 2l - 1} \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \leq \frac{1}{2} k^{\alpha+1(t)} \int_{\Omega} v^{2(l+1)} + \frac{m - 2l - 1}{2r(l + 1) + m - 2l - 1} \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \leq \frac{1}{2} k^{\alpha+1(t)} \int_{\Omega} v^{2(l+1)} + \frac{m - 2l - 1}{2r(l + 1) + m - 2l - 1} \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \leq \frac{1}{2} k^{\alpha+1(t)} \int_{\Omega} v^{2(l+1)} + \frac{m - 2l - 1}{2r(l + 1) + m - 2l - 1} \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \leq \frac{1}{2} k^{\alpha+1(t)} \int_{\Omega} v^{2(l+1)} + \frac{m - 2l - 1}{2r(l + 1) + m - 2l - 1} \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \leq \frac{1}{2} k^{\alpha+1(t)} \int_{\Omega} v^{2(l+1)} + \frac{m - 2l - 1}{2r(l + 1) + m - 2l - 1} \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
\[ \times \left( \frac{2r(l + 1) + m - 2l - 1}{2r(l + 1) + m - 2l - 1} \right) \]
From (14), it follows that
\[ C \]
where
\[ \text{Substituting (33) into (32), we get} \]

\[ \int_{\Omega} |\nabla v|^{l+1}^2 \, dx \]

\[ \int_{\Omega} \left[ \varepsilon (C(n, \Omega))^2 \right] \frac{2r(l+1)+m-2l-1}{2(r(y)+1)+m-2l-1} \frac{2r(l+1)+m-2l-1}{M^{\alpha-2(r(y)+1)}} |\Omega|. \]

Inserting (31) into (27), we obtain
\[ A'(t) \leq C_1 + \alpha \eta A(t) + C_3 \left( k^\alpha(t) \int_{\Omega} v^\beta \, dx \right) \]
where \( C_1 \) is given in (18) and
\[ C_3 = \frac{\beta [4r(l+1)+2l(n-2)-n(m-1)]}{2r(l+1)+l(n-2)} \left( \frac{2b_2(l+1)(\beta-1)}{2r(l+1)+m-2l-1} + \varepsilon \right) \]
\[ \times \left( \frac{4r(l+1)+2l(n-2)}{n(m-1)} \right)^{\frac{n(m-1)}{2(r(l+1)+2l(n-2)-m(m-1))}}. \]

From (14), it follows that
\[ G(u) = \beta \int_0^u h'(s)s^{\beta-1} \, ds \geq \beta \gamma \int_0^u s^{\beta-1} \, ds = \gamma u^\beta; \]
that is
\[ v^\beta = u^\beta \leq \frac{1}{\gamma} G(u). \] (33)

Substituting (33) into (32), we get
\[ A'(t) \leq C_1 + \alpha \eta A(t) + C_3 \gamma \left( \frac{4r(l+1)+2l(n-2)-n(m-1)}{2r(l+1)+2l(n-2)-m(m-1)} \right) \]
\[ \times \left( k^\alpha(t) \int_{\Omega} G(u) \, dx \right) \]
\[ = C_1 + \alpha \eta A(t) + C_2 A(t) \]
where \( C_2 \) is defined in (19). We integrate (34) from 0 to \( t \) to obtain
\[ \int_{A(0)}^{A(t)} \frac{d\tau}{C_1 + \alpha \eta A(t) + C_2 \tau} \leq t. \]

Letting \( t \to t^* \), we get a lower bound for \( t^* \)
\[ t^* \geq \int_{A(0)}^{\infty} \frac{d\tau}{C_1 + \alpha \eta A(t) + C_2 \tau}, \]
where \( \frac{4r(l+1)+2l(n-2)-n(m-1)}{2r(l+1)+2l(n-2)-m(m-1)} > 1 \) in view of (16). \( \square \)
4. Application. An example is given in this section to illustrate the abstract results of this paper.

**Example 4.1.** Let \( u \) be a classical solution of the following problem:

\[
\begin{align*}
(u + \ln(1 + u))_t &= \nabla \cdot \left( \frac{1}{200} (1 + |\nabla u|^2) \nabla u \right) + e^t u^4 \quad \text{in } \Omega \times (0, t^*), \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0, t^*), \\
u(x, 0) &= \left( 1 - \sum_{i=1}^{3} x_i^3 \right)^2 \quad \text{in } \Omega,
\end{align*}
\]

where \( \Omega = \{ x = (x_1, x_2, x_3) \mid \sum_{i=1}^{3} x_i^2 < 1 \} \subset \mathbb{R}^3 \) is a unit ball. Now, we have

\[
\begin{align*}
h(u) &= u + \ln(1 + u), \quad \rho(|\nabla u|^2) = \frac{1}{200} (1 + |\nabla u|^2), \quad k(t) = e^t, \\
f(u) &= u^4, \quad u_0(x) = \left( 1 - \sum_{i=1}^{3} x_i^3 \right)^2.
\end{align*}
\]

Choosing \( d = 1 \), it is easy to verify that (4)-(5) hold. It follows from (2)-(3) that

\[
H(u) = 2 \int_{0}^{u} sh'(s) ds = \int_{0}^{u} 2s \left( 1 + \frac{1}{1 + s} \right) ds = u^2 + 2u - 2 \ln(1 + u),
\]

\[
B(t) = \int_{\Omega} H(u) dx = \int_{\Omega} [u^2 + 2u - 2 \ln(1 + u)] dx
\]

and

\[
B(0) = \int_{\Omega} [u_0^2 + 2u_0 - 2 \ln(1 + u_0)] dx
\]

\[
= \int_{\Omega} \left[ \left( 1 - \sum_{i=1}^{3} x_i^3 \right)^4 + 2 \left( 1 - \sum_{i=1}^{3} x_i^3 \right)^2 \\
-2 \ln \left( 2 - 2 \sum_{i=1}^{3} x_i^3 + \left( \sum_{i=1}^{3} x_i^3 \right)^2 \right) \right] dx
\]

\[= 0.8009.\]

In addition, from (2)-(3), we also have

\[
P(|\nabla u|^2) = \int_{0}^{\|\nabla u\|^2} \rho(s) ds = \frac{1}{200} |\nabla u|^2 + \frac{1}{400} |\nabla u|^4,
\]

\[
F(u) = \int_{0}^{u} f(s) ds = \int_{0}^{u} s^4 ds = \frac{1}{5} u^5,
\]

\[
D(t) = -\int_{\Omega} P(|\nabla u|^2) dx + 2k(t) \int_{\Omega} F(u) dx
\]

\[
= -\int_{\Omega} \frac{1}{400} (2 |\nabla u|^2 + |\nabla u|^4) dx + \frac{2}{5} e^t \int_{\Omega} u^5 dx
\]
and

\[ D(0) = - \int_{\Omega} \frac{1}{400} \left( 2|\nabla u_0|^2 + |\nabla u_0|^4 \right) \, dx + \frac{2}{5} \int_{\Omega} u_0^5 \, dx \]

\[ = - \int_{\Omega} 0.08 \left( \sum_{i=1}^{3} x_i^2 \right) \left( 1 - \sum_{i=1}^{3} x_i^2 \right)^2 \, dx - 0.64 \left( \sum_{i=1}^{3} x_i^2 \right)^2 \left( 1 - \sum_{i=1}^{3} x_i^2 \right) \, dx \quad (36) \]

\[ + \frac{2}{5} \int_{\Omega} \left( 1 - \sum_{i=1}^{3} x_i^2 \right)^{10} \, dx = 0.0107. \]

From Theorem 2.1, we know that \( u \) must blow up at some finite time \( t^* \) in the measure \( B(t) \). Hence, \( u \) also blows up at \( t^* \). By (35)-(36), we have

\[ t^* \leq T = \frac{B(0)}{2d(1 + d)D(0)} = 18.7493. \quad (37) \]

Next selecting \( a = 1, b_1 = b_2 = \frac{1}{200}, l = 1, m = 4, M = 1, n = 3, \gamma = 1, \eta = 1, r = \frac{3}{2}, \) we can check that (14)-(16) hold. After direct computation, we get

\[ \alpha = \frac{7}{3}, \beta = 4. \]

Using the fact that the embedding constant \( C(n, \Omega) = 4 - \frac{1}{3} \pi - \frac{2}{3} \) given in [10], we compute (18)-(20) to get

\[ C_1 = 0.0071, \quad C_2 = 2.6788 \times 10^4, \quad \varepsilon = 0.0040. \]

Since

\[ A(t) = k^a(t) \int_{\Omega} G(u) \, dx = e^{\frac{7}{3}t} \int_{\Omega} \left( u^4 + \frac{4}{3} u^3 - 2u^2 - 4\ln(1 + u) + 4u \right) \, dx, \]

\( u \) blows up at \( t^* \) in measure \( A(t) \). Here, we have

\[ A(0) = \int_{\Omega} \left( u_0^4 + \frac{4}{3} u_0^3 - 2u_0^2 - 4\ln(1 + u_0) + 4u_0 \right) \, dx \]

\[ = \int_{\Omega} \left( 1 - x^2 \right)^8 + \frac{4}{3} \left( 1 - x^2 \right)^6 - 2 \left( 1 - x^2 \right)^4 \]

\[ - 4\ln \left[ 1 + \left( 1 - x^2 \right)^2 \right] + 4 \left( 1 - x^2 \right)^2 \, dx \]

\[ = 0.3239. \]

It follows from Theorem 3.1 that

\[ t^* \geq \int_{A(0)}^{\infty} \frac{d\tau}{C_1 + \alpha \eta \tau + C_2 \tau^{\frac{n-1}{(n-1)+2(n-2)}}}, \quad \text{with} \]

\[ = \int_{A(0)}^{\infty} \frac{d\tau}{0.0071 + \frac{7}{3} \tau + + 2.6788 \times 10^4 \tau^{\frac{3}{2}}} = 1.2031 \times 10^{-4}. \quad (38) \]

Combining (37) and (38), we obtain

\[ 1.2031 \times 10^{-4} \leq t^* \leq 18.7493. \]

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