SECTIONS ALONG MAPS IN GEOMETRY AND PHYSICS.

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1. Introduction.

Geometric techniques have been applied to physics in many different ways and they have provided powerful methods of dealing with classical problems from a new geometric perspective. Vector fields, forms, exterior differential calculus, Lie groups, fibre bundles, connections, etc..., are now well established tools in modern physics. One of the main contributions of Prof. W.M. Tulczyjew is the alternative way of using the geometric concepts in a more algebraic approach, using, for instance, derivations of algebras and related concepts. The aim of this paper is to point out some of my contributions in this direction [1–3,6], which have received a big influence of Tulczyjew’s works [7,8,9]. Moreover, it will be shown how this approach allows us to translate the usual concepts arising in Geometrical Mechanics to the framework of Supermechanics.

2. Hamiltonian dynamical systems.

The geometric framework for the description of classical (and even quantum) systems is the theory of Hamiltonian dynamical systems. They are triplets \((M, \omega, H)\) where \(M\) is a differentiable manifold, \(\omega \in \mathbb{Z}^2(M)\) is a regular closed 2-form in \(M\) and \(H \in C^\infty(M)\) is a function called Hamiltonian. The dynamical vector field \(X_H\) is then the solution of the equation \(i(X_H)\omega = dH\).

We recall that an infinitesimal symmetry of a Hamiltonian dynamical systems \((M, \omega, H)\) is given by a Hamiltonian vector field \(X \in \mathfrak{X}(M)\) (i.e., \(i(X)\omega \in B^1(M)\)) such that \(XH = 0\). Noether’s theorem establishes a one-to-one correspondence between infinitesimal symmetries and constants of motion: \(XH = 0\) and \(i(X)\omega = df\) if and only if \(f\) is a constant of motion.

Very interesting examples of HDS are those defined by regular Lagrangians, \((TQ, \omega_L, E_L)\), with \(\omega_L = -d\theta_L = -dL \circ S\), \(E_L = \Delta L - L\). More accurately, the geometric approach to the Lagrangian description makes use of the geometry of the tangent bundle of the configuration space.

2.1. The geometry of tangent bundles.

The tangent bundle \(\tau_Q : TQ \to Q\) is characterized by the existence of a vector field generating dilations along the fibres, called Liouville vector field, \(\Delta \in \mathfrak{X}(TQ)\), and the vertical endomorphism which is a \((1,1)\)-tensor field \(S\) in \(TQ\) defined by \(S_{(q,v)}U = \xi^{(q,v)}(\tau_u)U\), \(\forall U \in T_{(q,v)}(TQ), v \in T_qQ\), where \(\xi^{(q,v)} : T_qQ \to T_{(q,v)}(TQ)\) denotes the vertical lift defined by \(\xi^{(q,v)}(w)f = d/dt [f(v + tw)]_{t=0}\). In a natural coordinate system for \(TQ\), induced from a chart in \(Q\), \(\Delta = v^i \partial / \partial v^i\), and \(S = (\partial / \partial v^i) \otimes dv^i\).

The image under \(S\) of a section for \(T\tau_Q : T(TQ) \to TQ\), a vector field in \(TQ\), is a new vector field in \(TQ\), again. This correspondence will also be denoted \(S\).

There is another vector bundle structure on \(T(TQ)\) given by \(T\tau_Q : T(TQ) \to TQ\). Vector fields on \(TQ\) that are also sections for \(T\tau_Q\) are called SODE (second order differential equations). They are characterized by \(S(X) = \Delta\) and in tangent bundle coordinates look like \(X = v^i (\partial / \partial q^i) + f^i (\partial / \partial v^i)\).

A map \(\varphi : Q \to Q\) induces a map \(\Phi = T\varphi : TQ \to TQ\) such that \([T\Phi, S] = 0\) and \((T\Phi)(\Delta) = \Delta\), and conversely, if \(\Phi : TQ \to TQ\) satisfies these two properties, then there exists a map \(\varphi : Q \to Q\) such that \(\Phi = T\varphi\). These transformations of the velocity phase space are called point transformations. Another characterization of these point transformations is that if \(X\) is a SODE, then \((T\Phi)(X)\) is also a SODE. Notice that if \(X\) is a SODE, then \(S(T\Phi(X)) = T\Phi(S(X)) = T\Phi(\Delta) = \Delta\).
2.2. The Lagrangian formalism.

Given a function \( L \in C^\infty(TQ) \), we define the 1–form \( \theta_L \in \wedge^1(TQ) \) by \( \theta_L = dL \circ S \). When the exact 2–form \( \omega_L = -d\theta_L \) is nondegenerate the Lagrangian \( L \) is called regular and then \( (TQ, \omega_L) \) is a symplectic manifold. The energy function \( E_L \) is given by \( E_L = \Delta(L) - L \). The coordinate expressions are \( \theta_L = (\partial L/\partial v^i)dq^i \) and \( E_L = v^i(\partial L/\partial v^i) - L \).

A remarkable fact is that if \( \Phi \) is a point transformation, then \( \Phi^*\omega_L = \omega_{\Phi^*L} \) and \( \Phi^*E_L = E_{\Phi^*L} \), because \( \Phi^*\theta_L = \phi^*(dL \circ S) = (dL \circ S) \circ (T\Phi) = \Phi^*\theta_L \) and \( \Phi^*E_L = \Phi^*(\Delta L - L) = \Delta(\Phi^*L) - \Phi^*L = E_{\Phi^*L} \). However, for a general transformation \( \Phi : TQ \to TQ \), \( \Phi^*\omega_L \neq \omega_{\Phi^*L} \) and \( \Phi^*E_L \neq E_{\Phi^*L} \). Therefore, the important point in the search of symmetries is that if \( X = Y^c \) where \( Y \in \mathfrak{X}(Q) \), then \( X \) is a symmetry of \( (TQ, \omega_L, E_L) \) if \( X \) is a symmetry of \( L \) (up to a gauge term). On the contrary, for \( X \) that are not complete lifts, symmetries of \( L \) have nothing to do with symmetries of the HDS \( (TQ, \omega_L, E_L) \). Complete lifts correspond to point transformations.

In order to establish a one-to-one correspondence between symmetries of \( L \) and constants of motion we shall generalize the concept of symmetry in order to include “non-point transformations”. In the physicist’s language point transformations are written \( \delta q^i = \epsilon f^i(q) \) and \( \delta v^i = \epsilon v^j (\partial f^i/\partial q^j) \), corresponding to the flow of \( X = Y^c \) with \( Y = f^i(q) (\partial/\partial q^i) \in \mathfrak{X}(Q) \). Non-point transformations cannot be completed, \( \delta q^i = \epsilon f^i(q, v) \) and \( \delta v^i = ? \), and this fact leads to consider “objects” like \( Y = f^i(q, v) (\partial/\partial q^i) \), which are NOT vector fields but vector fields along the tangent bundle projection, a concept that we will introduce in next section.

2.3. Newtonoid vector fields.

Marmo and Mukunda characterization of symmetries is as follows [5]: Let \( D \) be a SODE and \( \mathfrak{X}_D \) denote \( \mathfrak{X}_D = \{ X \in \mathfrak{X}(TQ) \mid S([X, D]) = 0 \} \). There is a projection \( \pi_D : \mathfrak{X}(TQ) \to \mathfrak{X}_D \), given by \( \pi_D(X) = X(D) = X + S([D, X]) \). In coordinates, if \( X \) is written \( X = \eta^i \partial/\partial q^i + \xi^i \partial/\partial v^i \), then

\[
X(D) = \eta^i \partial/\partial q^i + (D\eta^i) \partial/\partial v^i.
\]

**Theorem 1.** Let \( L \in C^\infty(TQ) \) be a regular Lagrangian. If \( X \in \mathfrak{X}(TQ) \) is such that \( \exists F \in C^\infty(TM) \) satisfying

\[
\mathcal{L}_{X(D)}L = \mathcal{L}_DF \quad \text{for any} \quad \text{SODE} \ D,
\]

then \( G = i_X \theta_L - F \) is a constant of motion. Moreover, if \( \Gamma \) is the dynamical vector field, then \( \mathcal{L}_{X(\Gamma)}\Omega_L = 0 \) and \( \mathcal{L}_{X(\Gamma)}E_L = 0 \), i.e., \( X(\Gamma) \) is a symmetry of \( (TQ, \omega_L, E_L) \).

Conversely, if \( X \) is a symmetry of \( (TQ, \omega_L, E_L) \), then \( X = X(\Gamma) \) and there exists a function \( F \in C^\infty(TQ) \) such that (2.2) holds.

When \( X \) is a complete lift \( X = Y^c \) with \( Y \in \mathfrak{X}(Q) \), i.e., the flow of \( X \) is the differential of the flow of \( Y \), then \( X(D) = X \) for any SODE \( D \), and \( F \) reduces to the pullback \( \tau^*h \) of a function \( h \) on the base \( Q \).

If a vector field \( X \) of \( \mathfrak{X}(\Gamma) \) is a symmetry of \( (TQ, \omega_L, E_L) \), then its vertical components \( \xi^i \) are determined by the other ones: \( \xi^i = \Gamma \eta^i \). Essentially we should consider equivalence classes of vector fields

\[
[X] = X + \text{Vert}.
\]
3. Sections along maps.
Let $\pi : E \to M$ be a fibre bundle and $\phi : N \to M$ a differentiable map. A section along $\phi$ is a map $\sigma : N \to E$ such that $\pi \circ \sigma = \phi$. They are in a 1-1 correspondence with sections of the induced bundle $\phi^* E$,

\[(3.1) \quad \phi^* E = \{ (n, e) \in N \times E \mid \phi(n) = \pi(e) \} \subset N \times E.\]

The set of sections along $\phi$ will be noted $\Sigma_\phi(\pi)$. When $E$ is a vector bundle the set $\Sigma_\phi(\pi)$ is endowed with a $C^\infty(N)$–module structure. For more details see [1,2].

In particular we will be interested in the vector bundles $\tau_M : TM \to M$, $\pi^p_M : (T^*M)^p \to M$ and $\rho_M^p : (T^*M)^p \otimes TM \to M$, and in these cases we will denote $\mathfrak{X}(\phi) = \Sigma_\phi(\tau_M)$, $\wedge^p(\phi) = \Sigma_\phi(\pi^p_M)$ and $V^p(\phi) = \Sigma_\phi\rho_M^p$, respectively. When $N = M$ and $\phi = \text{id}$ the set $\mathfrak{X}(\text{id})$ coincides with $\mathfrak{X}(M)$ and the set $\wedge^p(\text{id})$ reduces to $\wedge^p(M)$.

Examples:
Let $\gamma : \mathbb{R} \to M$ be a curve in $M$. The tangent vectors $\dot{\gamma}$ define a section $\dot{\gamma} : \mathbb{R} \to TM$ of $\tau_M$ along $\gamma$. The restriction of $X \in \mathfrak{X}(M)$ on the curve $\gamma$ is also a vector field along $\gamma$.

The generalization of these examples is: Let $\phi$ be a map from $N$ to $M$. A vector field $Y \in \mathfrak{X}(N)$ defines a vector field along $\phi$ by $T\phi \circ Y \in \mathfrak{X}(\phi)$. Similarly, when $X \in \mathfrak{X}(M)$ the restriction $X \circ \phi$ of $X$ on the image by $\phi$ is a vector field along $\phi$. The above vector fields $X$ and $Y$ are said to be $\phi$-related when $X \circ \phi$ and $T\phi \circ Y$ coincide along $\phi$. Similarly, if $\beta$ is a $p$-form in $M$, the restriction $\beta \circ \phi$ of $\beta$ on the image by $\phi$ is a $p$-form along $\phi$.

Given $\alpha \in \wedge^p(\phi)$, $T^* \phi \circ \alpha$ is a $p$-form in $N$. The pull-back by $\phi$ of $\beta \in \wedge^p(M)$ is obtained by iteration of both processes $\phi^*(\beta) = T^* \phi \circ \beta \circ \phi$.

When $E$ is a vector bundle and $\{\sigma_\alpha\}$ is a local basis of $\Sigma(\pi)$, then $\{\sigma_\alpha \circ \phi\}$ is a local basis of $\Sigma_\phi(\pi)$, and $\sigma \in \Sigma_\pi$ can be written as $\sigma = \zeta^\alpha(\sigma_\alpha \circ \phi)$ with $\zeta^\alpha \in C^\infty(N)$.

In the above case, taking local coordinates $(z^A)$ in $N$ and $(x^i)$ in $M$ we have

\[(3.2) \quad X \in \mathfrak{X}(\phi) \quad \quad X = X^i \left( \frac{\partial}{\partial x^i} \circ \phi \right) \quad \quad \alpha \in \wedge^p(\phi) \quad \quad \alpha = \alpha_{i_1 \ldots i_p}(dx^{i_1} \circ \phi) \wedge \ldots \wedge (dx^{i_p} \circ \phi)\]

where $X^i$ and $\alpha_{i_1 \ldots i_p}$ are functions in $N$. When $N = TM$ and $\phi$ is the projection $\tau_M$ the vector fields and forms along $\tau_M$ are written

\[(3.3) \quad X = X^i(x,v) \left( \frac{\partial}{\partial x^i} \circ \tau_M \right), \quad \alpha = \alpha_{i_1 \ldots i_p}(x,v)(dx^{i_1} \circ \tau_M) \wedge \ldots \wedge (dx^{i_p} \circ \tau_M)\]

Vector fields along $\phi$ act on functions on $M$ giving rise to functions on $N$.

If $X \in \mathfrak{X}(\phi)$ and $n \in N$ then $X(n)$ is a tangent vector to $M$ at the point $\phi(n)$ which acts on a function $h \in C^\infty(M)$ by $(Xh)(n) = X(n)h$. The Leibnitz rule for tangent vectors implies that

\[(3.4) \quad X(hl) = \phi^* h Xl + \phi^* l Xh.\]
A map satisfying this property is called a $\phi^*$-derivation (of degree 0) [7].

**Definition 1.** Let $\phi : N \rightarrow M$ a differentiable map. A $\phi^*$-derivation of degree $r$ of scalar on $M$ is a $\mathbb{R}$-linear map $D : \bigwedge(M) \rightarrow \bigwedge(N)$ satisfying

\[
D\left(\bigwedge^p(M)\right) \subset \bigwedge^{p+r}(N), \quad D(\alpha \wedge \beta) = D\alpha \wedge \phi^*\beta + (-1)^{pr} \phi^*\alpha \wedge D\beta
\]

for $\beta \in \bigwedge^q(M)$ and $\alpha \in \bigwedge^p(M)$. It is said to be of type $i_*$ when $Dg = 0$, $\forall g \in C^\infty(M)$.

For instance, given a vector field along $\phi : N \rightarrow M$, $X$, a type $i_*$ $\phi$-derivation $i_X : \bigwedge^p(M) \rightarrow \bigwedge^{p-1}(N)$ of degree $-1$ is defined by $i_X g = 0 \forall g \in C^\infty(M)$ and

\[
(i_X \omega)_z(v_1, \ldots, v_{p-1}) = \omega_{\phi(z)}(X_z, \phi_{*z}v_1, \ldots, \phi_{*z}v_{p-1})
\]

where $v_1, \ldots, v_{p-1} \in T_z(N)$.

By a type $d_*$ $\phi$-derivation of degree $r$ we mean that $D \circ d_{(M)} = (-1)^r d_{(N)} \circ D$. An example of such a type $\phi$-derivation, $d_X$, is defined by

\[
d_X = i_X \circ d_{(M)} + d_{(N)} \circ i_X,
\]

where $d_{(M)}$ stands for the operator of exterior differentiation in $M$. This is of type $d_*$, i.e., $d_X \circ d_{(M)} = d_{(N)} \circ d_X$.

Note that when $X \in \mathfrak{X}(\text{id}_M) \equiv \mathfrak{X}(M)$ the $\text{id}_M$-derivations $i_X$ and $d_X$ are but the contraction or inner product $i(X)$ (or $i_X$) and the Lie derivative $L_X$, respectively. For this reason, $i_X$ and $d_X$ will be called *contraction* and *Lie derivative*, respectively.

There exists a section along $\pi$ in each vector bundle $\pi : E \rightarrow M$, which is given by the identity map in $E$. When choosing local coordinates $(x^i, y^\alpha)$ in $E$ and a local basis $\{\sigma_\alpha\}$ of sections for $\pi : E \rightarrow M$ such that $y^\alpha(e) = \sigma_\alpha(\pi(e))$, for $e \in E$, then the local expression of $C$ is $C = y^\alpha(\sigma_\alpha \circ \pi)$.

The most important cases in Classical Mechanics are those of $E = TM$ or $E = T^*M$. Then $C$ reduces in these cases to the "total time derivative" $T$ (in the time-independent formalism) and the Liouville 1-form, to be denoted $\bar{\theta}_0$, up to an identification of $\pi$-semibasic forms with forms along $\pi$ (If $\phi$ is a submersion, every $p$-form along $\phi$ may be identified to a $\phi$-semibasic $p$-form in $M$).

The coordinate expressions for $T$ and $\bar{\theta}_0$ are

\[
T = v^i \left( \frac{\partial}{\partial x^i} \circ \tau_M \right) \quad \text{and} \quad \bar{\theta}_0 = p_\pi (dx^i \circ \pi_M).
\]
4. Applications in Geometry.

4.1. The geometry of $TQ$ revisited.

The fundamental objects in tangent bundles can be introduced in an alternative way that can be generalized for the case of Supergeometry and Supermechanics.

If $f \in C^\infty(Q)$, let $f^V \in C^\infty(TQ)$ be defined by

$$f^V = df := \sum_{i=1}^m \frac{\partial F}{\partial q^i} v^i,$$

where $F := \tau^*(f)$.

A vector field $Y$ on $TQ$ is determined by its action on the functions $f^V$: if $Y \in \mathfrak{X}(TQ)$ satisfies $Y(f^V) = 0$, $\forall f \in \mathfrak{X}(TQ)$, then $Y \equiv 0$. This property allows us to define the vertical lift: If $X \in \mathfrak{X}(Q)$, then $X^V \in \mathfrak{X}(TQ)$ is defined by $X^V(f^V) = \tau^*(X(f))$, $\forall f \in C^\infty(Q)$.

Similarly, we can define the vertical lift $X^V \in \mathfrak{X}(TQ)$ of a vector field along $\tau$, $X$, by the relations $X^V(f^V) = X(f)$, $\forall f \in C^\infty(Q)$.

The Liouville vector field $\Delta \in \mathfrak{X}(TQ)$ can then be defined as the vertical lift of the total time derivative $T$: $\Delta = T^V$. On the other hand, the vertical endomorphism is the $(1,1)$–tensor field $S: \mathfrak{X}(TQ) \rightarrow \mathfrak{X}(TQ)$ defined by $S(Y) := T\tau(Y)^V$.

4.2. Covariant derivative of a vector field along a curve.

A curve $\gamma: \mathbb{R} \rightarrow M$ has associated a vector field along $\gamma$, the tangent vector field $\dot{\gamma}$. In particular, if $X: \mathbb{R} \rightarrow TQ$ is a vector field along the curve $\sigma: \mathbb{R} \rightarrow Q$, it can be considered as a curve in $TQ$. The associated vector field along $X$, $\dot{X}$, can be composed with the natural isomorphism $\Psi: T(TQ) \rightarrow T(TQ)$, that in local coordinates is given by $\Psi(q, v, \dot{q}, \dot{v}) = (q, \dot{q}, v, \dot{v})$, giving rise to a vector field along $\dot{\sigma}$, $X^1 = \Psi \circ \dot{X}$.

Given a connection on $Q$, we can also lift horizontally $X$ and we obtain the vector field along $\dot{\sigma}$, $X^H$, given by $X^H(t) = \xi^H_{\dot{\sigma}(t)}(X(t))$. The difference $X^1 - X^H$ is a vertical vector field along $\dot{\sigma}$, and therefore, there exists a vector field along $\sigma$, to be denoted $DX/Dt$, such that

$$\left(X^1 - X^H\right)(t) = \xi^V_{\dot{\sigma}(t)} \left(\frac{DX}{Dt}\right).$$

The vector field along $\sigma$ $DX/Dt$ is called total covariant derivative of $X \in \mathfrak{X}(\sigma)$. So, if $X(t) = \eta^i(t)(\partial/\partial x^i)|_{\sigma(t)}$, then

$$X^1(t) = \eta^i(t)(\partial/\partial x^i)|_{\dot{\sigma}(t)} + \frac{d\eta^i}{dt}(t)(\partial/\partial v^i)|_{\dot{\sigma}(t)},$$

and

$$X^H(t) = \eta^i(t)(\partial/\partial x^i)|_{\dot{\sigma}(t)} - \Gamma^i_j(\dot{\sigma}(t))\eta^j(t)(\partial/\partial v^i)|_{\dot{\sigma}(t)}.$$

Therefore,

$$\frac{DX}{Dt}(t) = \left[\frac{d\eta^i}{dt}(t) + \Gamma^i_j(\dot{\sigma}(t))\eta^j(t)\right] \frac{\partial}{\partial x^i}|_{\dot{\sigma}(t)}.$$

This allows us to define the concepts of parallelism and geodesic curves in the well known way. The equation for geodesic curves is:

$$\frac{d^2\sigma^i}{dt^2} + \Gamma^i_j(\dot{\sigma}(t))\frac{d\sigma^j}{dt} = 0.$$
5. Applications in Physics.

5.1. Generalized symmetries.

If \( k, l \in \mathbb{N} \), there is a natural immersion \( i_{k,l} : T^{k+l}Q \to T^l(T^kQ) \), given by \( [\rho]^{k+l} \mapsto [\rho]^l \). We remark that for \( k = 0, l = 1 \) we obtain the identity in \( TQ \), i.e., \( i_{0,1} = \mathcal{T} \). In general \( i_{k,1} : T^{k+1} \to T(T^kQ) \) is such that \( \tau_{T^kQ} \circ i_{k,1} = \tau_{k+1,k} \), and therefore \( i_{k,1} \) is a vector field along \( \tau_{k+1,k} \) which will be denoted \( \mathbf{T}^{(k)} \).

If \( X \) is a vector field along \( \tau_{1,0} : TQ \to Q \), then there exists one vector field along \( \tau_{2,1} : T^2Q \to TQ \), denoted \( X^{(1)} \), such that \( X \circ \tau_{2,1} = \tau_{1,0*} \circ X^{(1)} \) and satisfying the commutation property \( d_X \circ d_T = d_T \circ d_X \).

Second order differential equations \( \ddot{q} = F(q, \dot{q}) \) can be seen not only as vector fields \( \Gamma \in \mathfrak{X}(TQ) \), but as sections \( \gamma \) of \( \tau_{2,1} \), i.e., \( \tau_{2,1} \circ \gamma = \text{id}_{TQ} \), given by \( a = F(q, \dot{q}) \).

The relation between the two alternatives is given through the time derivative vector field along \( \tau_{2,1} : \Gamma = T^{(1)} \circ \gamma \), or in other words, \( \mathcal{L}_\Gamma = \gamma^* \circ d_{T^{(1)}} \).

In a similar way there exists a one-to-one correspondence \( I_\Gamma : \mathfrak{X}(\tau_{1,0}) \to \mathfrak{X}(\Gamma) \), given by \( X \mapsto X^{(1)} \circ \gamma \), which is but the inverse of the restriction of \( \tau_{1,0*} \) onto \( \mathfrak{X}(\Gamma) \).

Moreover, it is then possible to follow the following theorem [1]:

**Theorem 2.** Let \( L \) be a regular Lagrangian, \( \Gamma \) the dynamical vector field satisfying \( i(\Gamma)\omega_L = dE_L \), and \( \gamma : TQ \to T^2Q \) the corresponding section. If \( X \) is a vector field along \( \tau_{1,0} \) and there exists a function \( F \in C^\infty(TM) \) such that \( X^{(1)}L = d_{T^{(1)}}F \), then the function \( G = F - \dot{\theta}_L(X) \) is a constant of motion. The vector field \( X^{(1)} \circ \gamma = X(\Gamma) \) is a symmetry of \( \Gamma \).

Conversely if \( G \) is a first integral of the motion given by \( L \), then there exist \( X \in \mathfrak{X}(\tau_{1,0}) \) and \( F \in C^\infty(TM) \) such that the above relation holds.

Therefore if we call symmetries of \( L \) to those vector fields \( X \in \mathfrak{X}(\tau_{1,0}) \) satisfying the preceding condition, then there will be a one-to-one correspondence between generalized infinitesimal symmetries of \( L \) and constants of motion. Point symmetries correspond to \( X = Y \circ \tau_{1,0} \), with \( Y \in \mathfrak{X}(Q) \) and then \( F \) is a basic function.

5.2. The evolution \( K_L \)-operator.

The Legendre map \( \mathcal{F}L : TM \to T^*M \), as well as the time evolution operator \( K_L : C^\infty(T^*M) \to C^\infty(TM) \), can also be defined as sections along maps [4,2]: The 1-form \( \theta_L \) is semibasic and can be seen as a 1-form along \( \tau, \theta_L \in \mathcal{L}(\tau) \), \( \theta_L = \frac{\partial L}{\partial \dot{v}_i} \circ (dq^i \circ \tau) \). It is identified in this way with the Legendre map. Let \( \chi \) be the natural diffeomorphism between \( T^*M \) and \( T(T^*M) \), with coordinate expresion \( \chi(x, v, p_x, p_v) = (x, p_v, v, p_x) \). Then \( K = \chi \circ dL \) maps \( TM \) in \( T(T^*M) \) in such a way that \( \tau_{T^*M} \circ K = \mathcal{F}L \), say, \( K_L \) is a vector field along \( \mathcal{F}L \).

In coordinates, \( K_L \) is the vector field along \( \mathcal{F}L \) given by

\[
K_L = v^i \left( \frac{\partial}{\partial x^i} \circ \mathcal{F}L \right) + \frac{\partial L}{\partial p_i} \circ \mathcal{F}L
\]

and it is very useful to relate constraint functions arising in the Hamiltonian and Lagrangian formulations respectively. It is determined by the following two equations:

\[
i_{K_L} \omega_0 = dE_L, \quad T\pi_Q \circ K = \text{id}_{TQ}.
\]
The main properties are that when applied to a constraint function in the Hamiltonian formalism, it produces a constraint in the Lagrangian formalism, either a dynamical constraint or even a SODE constraint. More specifically, when applied to a first class constraint function in the Hamiltonian formalism produces a dynamical constraint that is $\mathcal{FL}$-projectable, while when it is applied to a second class constraint function in the Hamiltonian formalism produces a SODE constraint that is not $\mathcal{FL}$-projectable.

5.3. Applications in degenerate systems.
Let $(N,\omega)$ be a symplectic manifold and $\phi: P \to N$ of constant rank. Given a 1-form $\alpha$ in $P$, we look for the set of points in which a solution of $i_{\Gamma}(\phi^*\omega) = \alpha$ exists, where $\Gamma$ is a vector field in $P$, i.e., we are interested in the submanifold $i_C: C \to P$ of $P$ in which such a solution $\Gamma' \in \mathcal{X}(C)$ does exist, namely $i_{\Gamma'}((\phi \circ i_C)^*\omega) = i_C^*\alpha$.

The above problem may be splitted in two. First we study the conditions for the existence of $X \in \mathcal{X}(\phi)$ such that $i_X\omega = \alpha$, and then we determine the conditions for $X$ to be image under $T\phi$ of a vector field in $P$. This is equivalent to the original problem because of the relation $i_{T\phi\Gamma}\omega = i_{\Gamma}(\phi^*\omega)$. The second step is but the condition for the solution to be tangent to $P$. Using a well-known result of Linear Algebra we obtain that the equation $i_X\omega = \alpha$ has a solution with $X \in \mathcal{X}(\phi)$ iff $p \in P$ satisfies

$$\langle z, \alpha(p) \rangle = 0 \quad \text{for all } z \in T_pP \text{ such that } T_p\phi(z) = 0.$$  

If $X$ is a solution and $Z \in \mathcal{X}(\phi)$ is such that $\dot{\omega}(\phi(p))(Z(p)) \in \ker T_p^*(\phi) \quad \forall p \in P$, then $X + Z$ is a solution too.

When $\alpha$ is exact, $\alpha = dF$, if for $n \in \text{Im } \phi \subset N$ the submanifold $\phi^{-1}(n)$ is connected, then the above condition is equivalent to $F$ to be $\phi$-projectable, namely there exists $\tilde{F} \in C^\infty(N)$ such that $\phi^*(\tilde{F}) = F$.

The generalization to the case of a presymplectic manifold is: the equation $i_X\omega = \alpha$ admits a solution iff $\langle z, \alpha(p) \rangle = 0$ for any $z \in T_pP$ such that $T_p\phi(z) \in \text{rad}(\omega)$, where $\text{rad}(\omega) = \{ v \in TN \mid \omega(v, w) = 0 \quad \forall \omega \in TN \}$.

Once the condition holds in $P$, we look for the existence of a vector field $\Gamma$ in $P$ such that $T\phi \circ \Gamma = X$, which has a solution iff the equation $T_p\phi(\Gamma(p)) = X(p)$ has solution for any $p \in P$.

This is equivalent to $\langle X(p), \lambda \rangle = 0$, $\forall \lambda \in T_{\phi(p)}^*N$ such that $T_p^*\phi(\lambda) = 0$, or in other words, iff $\langle \delta, X \rangle = 0$, for all $\delta \in \Lambda^1(\phi)$ such that $T^*(\phi) \circ \delta = 0$. This gives rise to an immersed submanifold $i_1: P_1 \to P$ of $P$ and we repeat the preceding steps.

If the image by $\phi$ is an immersed submanifold $j: N_0 \to N$ of $N$, then a similar algorithm is used for finding a solution in $N_0$. If $\zeta$ is a constraint function for $N$, then $\phi^*\zeta$ is a constraint function for $P$.

This is a generalization of what happens with the theory defined by a singular Lagrangian when $\phi$ is the Legendre transformation.

5.4. Control systems.

**Definition 2.** Given a vector field along $\phi: N \to M$, we will say that a curve $\gamma: \mathbb{R} \to N$ is an integral curve for $X \in \mathcal{X}(\phi)$ if

$$X \circ \gamma = T\phi \circ \gamma = (\phi \circ \gamma)'.
If we choose coordinates \( \{z^\alpha\} \) in \( N \) and \( \{x^i\} \) in \( M \), the vector field is \( X = X^i(z^\alpha)[(\partial/\partial x^i) \circ \phi] \), and the integral curves are to be determined by solutions of the equation

\[
\frac{\partial \phi^i}{\partial z^\alpha}(\gamma(t)) \frac{d\gamma^\alpha}{dt} = X^i((\gamma(t)).
\]

This system is not in normal form and the theorems of existence and uniqueness of solution do not apply.

Let us consider the differential equation system

\[
\frac{dx^i}{dt} = F(x^i, u^\alpha), \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m.
\]

From the geometrical viewpoint this system can be seen as the one determining the integral curves of the vector field along the projection \( \pi : M \times \mathbb{R}^m \to M \), where \( \{x^i, i = 1, \ldots, x^n\} \) are the local coordinates of a point in \( M \), and \( \{u^\alpha, \alpha = 1, \ldots, m\} \) are the so called control functions. As a straightforward generalizations of this, given a fibre bundle \( \pi : B \to M \), a control system in \( B \) is a vector field along \( \pi \), \( X \in \mathfrak{X}(\pi) \). A solution of the control system is an integral curve of \( X \in \mathfrak{X}(\pi) \).

One of the main problems in the theory of control systems is to investigate the set of points accessible from one given point \( p \in M \). More specifically, a control system is said to be controllable if for any given initial point \( p \) there exists an integral curve of the corresponding vector field along \( \pi \) such that \( (\pi \circ \gamma)(0) = p \) and a value \( t_1 \) of the parameter of the curve \( \gamma \) such that \( (\pi \circ \gamma)(t_1) = q \).

The simplest example is when \( M = \mathbb{R}^n \), \( B = \mathbb{R}^n \times \mathbb{R}^m \) and the equations describing the system are of the linear type

\[
\dot{x}^i = A^i_j x^j + B^i_\alpha u^\alpha,
\]

where \( A^i_j \) and \( B^i_\alpha \) are constant matrices. In this case the Kalman rank controllability condition is that \( \text{rank}(B, AB, \ldots, A^{n-1}) = n \).

Another interesting example is when \( X = u^1 X_1(x) + \cdots + u^r X_r(x) \). In this case, if the distribution \( D \) generated by the vector fields \( X_1, \ldots, X_r \) is integrable, then the only accessible points from a given point \( x_0 \) are those of its leaf. Otherwise, we should consider the minimal integrable distribution containing \( D \) and then the system is controllable if and only if \( \dim D = n \).

6. Supermechanics.

The algebraic tools considered in the previous sections can be translated to the framework of Supergeometry, the theory of supermanifolds, and Supermechanics [3].

We recall that a graded manifold \( M \) is a pair \((M, \mathcal{A})\), where \( M \) is a nice topological space and \( \mathcal{A} \) is a sheaf of superalgebras over \( M \) such that there are open sets \( \mathcal{U} \) that cover \( M \) such that \( \mathcal{A}(\mathcal{U}) \cong C^\infty(\mathcal{U}) \otimes \wedge(\mathbb{R}^n) \) and satisfying glueing conditions on the overlaps.

Given \( \Phi = (\phi, \phi^\ast) : (\mathcal{N}, \mathcal{B}) \to (M, \mathcal{A}) \) a supervector field along \( \Phi \) is a morphism of sheaves over \( M \), \( X : \mathcal{A} \to \Phi^\ast \mathcal{B} \) such that

\[
X(fg) = X(f) \phi^\ast_u(g) + (-1)^{|X||f|} \phi^\ast_u(f) X(g).
\]
As an example, if \( X \in \mathfrak{X}(A) \), then \( \phi^* \circ X \in \mathfrak{X}(\Phi) \), and if \( Y \in \mathfrak{X}(B) \) then \( Y \circ \phi^* \in \mathfrak{X}(\Phi) \).

The supertangent bundle \( T^{k+1}M \) is defined in a recursive way as follows. One has two morphisms: first, \( \mathcal{T}_{k,k-1}: T^{k}M \to T^{k-1}M \), and the corresponding tangent map \( \mathcal{T}_{k,k-1}: T(T^{k}M) \to T(T^{k-1}M) \); second, \( I_k: T^{k}M \to T(T^{k-1}M) \) and \( \mathcal{T}_{k}: T(T^{k}M) \to T^{k+1}M \) give rise to \( I_k \circ \mathcal{T}_{k}: T(T^{k}M) \to T(T^{k-1}M) \).

Then \( T^{k+1}M \) is the subsupermanifold of \( T(T^{k}M) \) associated to the superideal

\[
I_{k+1} = \{ \tau_k^* \circ i^*_k(F) - (T\tau_{k-1,k})^*(F) : F \in T(T^{k-1}A) \}
\]

If \( q^{i,(0,k)}, \ldots, q^{i,(k,k)}, v^{i,(0,k)}, \ldots, v^{i,(k,k)}, \theta^{(0,k)}, \ldots, \theta^{(k,k)}, \zeta^{(0,k)}, \ldots, \zeta^{(k,k)} \) are supercoordinates on \( T(T^{k}M) \) then

\[
I_{k+1} = \{ q^{i,(1,k)} - v^{i,(0,k)}, \ldots, q^{i,(k+1,k)} - v^{i,(k,k)}, \theta^{(1,k)} - \zeta^{(0,k)}, \ldots, \theta^{(k+1,k)} - \zeta^{(k-1,k)} \}
\]

and the supercoordinates in \( T^{k+1}M \) are \( \{ q^{i,(0,k+1)}, \ldots, q^{i,(k+1,k+1)}, \theta^{(0,k+1)}, \ldots, \theta^{(k+1,k+1)} \} \).

The total derivative with respect to time \( T^{(k)}: T^{k}A \to T^{k+1}A \) is the element of \( \mathfrak{X}(T^{k+1}A) \) such that \( T^{(k)}(q^{i,(j-1,k)}) = q^{i,(j,k+1)} \), and \( T^{(k)}(\theta^{(j,k)}) = \theta^{(j+1,k)} \), for \( j = 1, \ldots, k+1 \). So, if \( f \in A \) then \( f^k = \tau^*_k \circ T^{(j-1)} \circ \ldots \circ T^{(1)} \circ T(f) \in T^{k}A \), and if \( Y \in \mathfrak{X}(A) \), the complete lift of \( Y \) is \( Y^{(k)}(f^k) = (Y(f))^{k,j}_j \), \( (\forall f)(\forall j) \).

When \( X \in \mathfrak{X}(T_{k,0}) \), the \( l \)-prolongation of \( X \) is

\[
X^{(l)}(f^j) := i_{k,l}(f^j) \quad (\forall f)(\forall j).
\]

If \( f \in T^{k-1}A \) and \( F = \tau^*_k \circ f \)

\[
f^V := \sum_{\alpha=1}^n \sum_{j=0}^{k-1} \frac{1}{j+1} \frac{\partial F}{\partial q^{i,(j,k)}_\alpha} q^{i,(j+1,k)} + \sum_{\alpha=1}^n \sum_{j=0}^{k-1} \frac{1}{j+1} \frac{\partial F}{\partial \theta^{(j,k)}_\alpha} \theta^{(j+1,k)}
\]

is a superfunction in \( T^{k}A \). If \( Y \in \mathfrak{X}(A) \), the vertical lift of \( Y \) is

\[
Y^V(f^V) = \tau^*_k \circ (Y(f)) \quad \forall f \in T^{k-1}A.
\]

Moreover, if \( X \in \mathfrak{X}(T_{k,k-1}) \), the vertical lift of \( X \) is given by \( X^V(f^V) = X(f) \), \( \forall f \in T^{k-1}A \), and the vertical superendomorphism is the graded tensor of type \( (1,1) \) \( S_k: \mathfrak{X}(T^{k}A) \to \mathfrak{X}(T^{k}A) \) defined by \( S_k(Y) := (Y \circ \tau^*_k \circ f)^V \).

On the other hand, he Liouville supervector field is \( \Delta_k := (T^{(k-1)})^V \), and the Cartan operator \( S^{(k)} : \Omega^{l}(T^{k}A) \to \Omega^{l}(T^{2k-1}A) \) is defined by

\[
S^{(k)} := \sum_{l=1}^{k} \frac{(-1)^{l+1}}{l!} T^{(2k-1,k+l-1)} \circ d^{l-1}_{T^{(k)}} \circ S^{(k)}.
\]

A superdifferential equation of order \( k+1 \) is a \( \Gamma \in \mathfrak{X}(T^{k}A) \) such that \( \Gamma \circ \tau^*_k \circ f = T^{(k-1)} \), or in other words, \( S_k(\Gamma) = \Delta_k \).
There is a 1–1 correspondence between the $\Gamma$’s and the morphisms $\gamma: T^{k+1}A(U) \to T^kA(U)$ such that $\gamma \circ \tau_{k,k-1}^* = \text{id}_{T^kA(U)}$.

The Cartan 1–form defined by a super–Lagrangian $L \in T^kA$ is $\Theta_L := S^{(k)}(dL)$ and the Cartan 2–form is $\Omega_L := -d\Theta_L$.

If $k = 1$, $E_L := \Delta L - L$, and then we say that $\Gamma \in \mathfrak{X}(T^2k^{-1}A)$ is Lagrangian if $\Gamma$ is a superdifferential equation of order $2k$ such that $i_\Gamma \Omega_L = dE_L$.

We will say that $X \in \mathfrak{X}(T^2k^{-1},0)$ is a generalized infinitesimal supersymmetry of the dynamical system $(TM, \Omega_L, E_L)$ if there exists $F \in T^{3k-2}A$ such that

$$X^{(k)}L = T^{(3k-2)}F.$$  

The main theorem is the following:

**Theorem 3.** Let $L$ be a regular Lagrangian. Then, if $X$ is a generalized supersymmetry there exists a constant of motion $G$ such that

$$\tau_{3k-1,2k-1}^*G = \langle X^{(k-1)}, (\mathcal{T}_{3k-2,2k-1}^*\Theta_L)^\vee \rangle - F,$$

and conversely, if $G$ is a constant of motion, there exists $X \in \mathfrak{X}(T_{2k-1,0})$ such that

$$F := \langle X^{(k-1)}, (\mathcal{T}_{3k-2,2k-1}^*\Theta_L)^\vee \rangle - \tau_{3k-1,2k-1}^*G$$

satisfies (6.9).

**References**

[1] Carriñena J.F., López C. and Martínez E., A new approach to the converse of Noether’s theorem, J. Phys. A: Math. Gen. 22 (1989), pp. 4777–87.

[2] Carriñena J.F., López C. and Martínez E., Sections along a map applied to higher-order Lagrangian Mechanics. Noether’s theorem, Acta Applicandae Mathematicae 25 (1991), pp. 127–51.

[3] Carriñena J.F. and Figueroa H., Hamiltonian versus Lagrangian formulation of supermechanics, J. Phys. A: Math. Gen. 30 (1997), (to appear).

[4] Gràcia X. and Pons J.M., On an evolution operator connecting Lagrangian and Hamiltonian formalisms, Lett. Math. Phys. 17 (1989), pp. 175–180.

[5] Marmo G. and Mukunda N., Symmetries and constants of the motion in Lagrangian mechanics: beyond point transformations, Nuovo Cim. A 92 (1986), pp. 1–12.

[6] Martínez E., Carriñena J.F., and W. Sarlet, Derivations of differential forms along the tangent bundle projection, Diff. Geom. and Appl. 2 (1992), pp. 17–43.

[7] Pidello G. and Tulczyjew W.M., Derivations of differential forms in jet bundles, Ann. Math. Pura ed Aplicata 147 (1987), pp. 249–265.

[8] Tulczyjew W.M., The Lagrange Differential, Bul. Acad. Pol. Sc., vol. XXIV (12) (1976), pp. 1089–1096.

[9] Tulczyjew W.M., The Legendre transformation, Ann. Inst. Henri Poincaré, vol. XXVII (1) (1977), pp. 101–114.