On Freudenthal theorem, Kahn-Priddy Theorem, and Curits conjecture

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Abstract

We verify Curtis conjecture on a class of elements of $2\pi_s^*$ that satisfy a certain factorisation property. To be more precise, suppose $f \in 2\pi_n^s$ pulls back to $g \in 2\pi_n^s P$ through the Kahn-Priddy map $\lambda : QP \to Q_0S^0$ such that $g$ projects nontrivially to an element $g' \in 2\pi_n^s P_{t(n)}$ with $h(g') = 0$ where $h : 2\pi_s^* QP_k \to H_* QP_k$ is the unstable Hurewicz map, and $t(n) = \lceil n/2 \rceil$. Then, mod out by elements of $2\pi_s^* \simeq 2\pi_s^* QS^0$ satisfying this property, the Curtis conjecture on the image of $h : 2\pi_s^* QS^0 \to H_* QS^0$ holds.

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1 Introduction and statement of results

Let $QS^0 = \text{colim} \, \Omega^i S^i$ be the infinite loop space associated to the sphere spectrum. Curtis conjecture then reads as follows (see [3, Proposition 7.1] and [11] for more discussions).

**Conjecture 1.1** (Curtis Conjecture). The image of the unstable Hurewicz homomorphism $h : 2\pi_s^* \simeq 2\pi_s^* QS^0 \to H_* QS^0$ is given by

$$\mathbb{Z}/2 \{h(\eta), h(\nu), h(\sigma), h(\theta_i)\}$$

where $\eta, \nu, \sigma$ are the Hopf invariant one elements and $\theta_i \in 2\pi_{2i+1}^s$ are Kervaire invariant elements.
After work of Hill, Hopkins and Ravenel [5] we know that the elements \( \theta_i \) are not to exist for \( i > 6 \). So, as a consequence the above conjecture implies that the image of \( h \) is finite.

Here and throughout, we write \( p^i \pi_*^s \) and \( p^i \pi_* \) for the \( p \)-components of \( \pi_*^s \) and \( \pi_* \), respectively. We also write \( H_* \) for \( H_*(-; k) \) where the coefficient ring will be clear from the context with an interest in \( k = \mathbb{Z}/p \); although some of our results such as Theorem 2.1 holds when \( k \) is an arbitrary \( p \)-local commutative coefficient ring. We work \( p \)-locally or \( p \)-complete which again will be clear from the context.

2 Preliminaries

We wish to examine Conjecture [2] in the light of Freudenthal theorem and Kahn Priddy theorem. Suppose \( E \) is a connected and connective spectrum, i.e. \( \pi_i E \simeq 0 \) for \( i < 1 \). Write \( \Omega^\infty E = \text{colim} \Omega^i E_i \) for the infinite loop space associated to \( E \), \( QX \) for \( \Omega^\infty(\Sigma^\infty X) \) when \( X \) is a space, and \( \epsilon : \Sigma^\infty \Omega^\infty E \to E \) for the evaluation map which is the stable adjoint to the identity \( \Omega^\infty E \to \Omega^\infty E \). The map \( \epsilon \) induces stable homology suspension \( H_* \Omega^\infty E \to H_* E \). For a spectrum \( E \), we refer to the Hurewicz homomorphism \( h^s : \pi_* E \to H_* E \) as the stable Hurewicz homomorphism whereas, in positive degrees, we refer to \( h : \pi_* E \simeq \pi_* \Omega^\infty E \to H_* \Omega^\infty E \) as the unstable Hurewicz homomorphism. The following was proved in [12, Theorem 1.3].

**Theorem 2.1.** Suppose \( f : S^n \to QS^0 \) is given so that for some spectrum \( E \) there is a factorisation \( S^n \xrightarrow{f_E} \Omega^\infty E \to QS^0 \). Then, the following statements hold.

(i) If \( E \) is \( r \)-connected, \( n \leq 2r + 1 \), and \( h^s(f_E) = 0 \), then \( h(f) = 0 \). In particular, if \( E \) is a CW -spectrum and \( f_E \) maps trivially under \( p \)-local stable Hurewicz map \( h^s_{(p)} : p\pi_* E \to H_* (E; \mathbb{Z}/p) \) only for some prime \( p \) then \( f \) maps trivially under the \( p \)-local unstable Hurewicz map \( p\pi_* QS^0 \to H_* QS^0 \). A similar statement holds in the \( p \)-complete setting.

(ii) If the above factorisation is induced by a factorisation of a map of spectra \( S^n \xrightarrow{E} E \xrightarrow{S^0} \) where \( E \) is a finite CW -spectrum of dimension \( r \), \( n \geq 2r + 1 \), and \( c_* = 0 \) then \( h(f) = 0 \). In particular, if \( c_* = 0 \) holds \( p \)-locally only for some prime \( p \) then \( f \) maps trivially under the \( p \)-local unstable Hurewicz map \( p\pi_* QS^0 \to H_* QS^0 \). A similar statement holds in the \( p \)-complete setting.

We also recall that as a corollary of the above theorem, we have proved that [12, Theorem 1.8]

**Theorem 2.2.** The image of \( h : \pi_*^s \simeq \pi_* QS^0 \to H_* (QS^0; \mathbb{Z}) \) when restricted to the submodule of decomposable elements is given by \( \mathbb{Z}\{h(\eta^2), h(\nu^2), h(\sigma^2)\} \).

Next, recall that by Kahn-Priddy theorem [6, Proposition 3.1] the map \( \lambda : QP \to Q0S^0 \) induces an epimorphism

\[ \lambda_* : 2\pi_*^s P \simeq 2\pi_* QP \to 2\pi_* Q0S^0 \simeq 2\pi_*^s. \]

Moreover, the map \( \lambda \) is induced by an infinite loop extension of a map of suspension spectra \( t : P \to S^0 \) where \( P \) is the infinite dimensional real projective space with its 0-skeleton as its base point. Note that by dimensional reasons \( t_* = 0 \).

3 Preparatory lemmata

We begin with fixing some notation. We use \( P \) for the infinite dimensional projective space and \( P^n \) for its \( n \)-skeleton, and for \( 0 < k < n \leq +\infty \) we set \( P^n_k = P^n / P^{k-1} \) as well as \( P^n_n = S^n \). For
$f \in 2\pi_n Q_0 S^0$, we write $g \in 2\pi_* QP$ for any element with $\lambda g = f$ where $\lambda$ is the Kahn-Priddy map. It is known that $\lambda$ is a split epimorphism whose inverse is provided by a map $t : Q_0 S^0 \to QP$ with $t = \Omega j_2$ where $j_2 : QS^1 \to QΣP$ is the second stable James-Hopf map [7 Corollary 2.14]. For a map of spaces $f : X \to QY$, $QY = \text{colim } Ω^s Σ^i Y$ we write for $f^s : X \to Y$ where we identify a based space $X$ with its suspensions spectrum $Σ^∞ X$.

For $g \in 2\pi_* P$, by cellular approximation, we may consider $g$ as $g : S^n \to P^n$. We ask what is the least $k$ so that $g$ also factors as $S^n \to P^k \to P^n$. Given $g^s \in 2\pi_* P^n$ and $k > 0$ we shall write $g^k : S^n \to P^n$ and by abuse of notation for the composition $S^n \to P^k \to P_k$.

For the sake of studying Conjecture 1.1 applying Theorem 2.1 provides us with the following lower bound result.

**Lemma 3.1.** Let $k > 0$. We have the followings.

(i) Suppose $f \in 2\pi_{2k+1} Q_0 S^0$ is given with $h(f) \neq 0$. Then, $g$ does not factor through $QP^k$.

(ii) Suppose $f \in 2\pi_{2k} Q_0 S^0$ is given with $h(f) \neq 0$. Then, $g$ does not factor through $QP^{k-1}$.

**Proof.** (i) In this case we have a factorisation of $f$ as

$$S^{2k+1} \to QP^k \to Q_0 S^0$$

where the map $QP^k \to Q_0 S^0$ is induced by applying $Ω^∞$ to $P^k \to P \to S^0$ with $t_0 = 0$. Theorem 2.1 implies that $f_0 = 0$.

(ii) This is similar to (i). □

Next, suppose $g^s : S^n \to P$ does not factor through $P^k$. Consider the cofibre sequence $P^k \to P \to P/P^k$ where $q$ is the projection. In this case, the composition $g^k_{k+1} = qg^s : S^{2k+1} \to Σ^∞ P_{k+1}$ represents a nontrivial element in $2\pi_* P_{k+1}$. Note that $q_*$ is an isomorphism in $H_*(-; \mathbb{Z}/2)$ for $* \geq k + 1$. We may ask, knowing $h(f) \neq 0$ and $h(g_k) \neq 0$, what can be said about $f$? We have the following.

**Theorem 3.2.** (i) Suppose $f \in 2\pi_{2k+1} Q_0 S^0$ is given with $h(f) \neq 0$ so that $h(g_{k+1}) \neq 0$. Then, $f$ is a Hopf invariant one element.

(ii) Suppose $f \in 2\pi_{2k} Q_0 S^0$ is given with $h(f) \neq 0$ so that $h(g_k) \neq 0$. Then, $f$ is a Kervaire invariant one element.

The proof is based on using knowledge on $H_* QX$ and $H_* Q_0 S^0$ to which we refer the reader to [1]. In particular, once and for all, we fix that $a_i \in H_i P^n$, likewise in $H_i P^n$, denotes a generator.

We shall write $x_i \in H_i Q_0 S^0$ for a generator which satisfies $λ^i a_i = x_i$.

**Proof.** (i) For $g_{k+1} = (Ω^∞ q)g$, if $(g_{k+1})_* \neq 0$ then by dimensional reasons

$$h(g_{k+1}) = a_{2k+1}$$

where $a_{2k+1} \in H_{2k+1} P_{k+1}$ is a generator. This implies that $h(g) = a_{2k+1}$ modulo $\ker((Ωq)_*)$. Consequently,

$$h(f) = x_{2k+1} + \text{other terms}.$$ 

It is known that in this case $f$ has to be a Hopf invariant one element.

(ii) The effect of the Hurewicz homomorphism on $2\pi^s_2 \simeq \mathbb{Z}/2\{θ_1\}$ is known. So, we assume
Note 3.3. We could eliminate $k$ being even case by geometric means as well. If $\epsilon_1 = 0$ then the stable map $g_k^* : S^{2k} \to P_k$ which factors through $P_{k}^{2k}$ by cellular approximation acts trivially in homology. Hence, the map $g_k^* : S^{2k} \to P_{k}^{2k}$ factors through $P_{k}^{2k-1}$. By Freudenthal theorem, $P_{k}^{2k-1}$ admits at least one suspension, so $P_{k}^{2k-1} = \Sigma Y_{k-1}^{2k-2}$ for some path connected CW-complex $Y_{k-1}^{2k-2}$ with its bottom cell in dimension $k - 1$. On the other hand, as $\epsilon_1 = 0$, hence $h(g_k) = a_k^2$ which together with $P_{k}^{2k-1} = \Sigma Y_{k-1}^{2k-2}$ reads as $h(g_k) = Q^k\Sigma y_{k-1}$. For $\alpha : S^{2k-1} \to QY_{k-1}^{2k-2}$, being the adjoint of $g_k : S^{2k} \to Q\Sigma Y_{k-1}^{2k-2}$, we have

$$h(\alpha) = Q^k y_{k-1} + D$$

where $D$ is a sum of decomposable terms, appearing as being the kernel of homology suspension. Note that $y_{k-1}$ being coming from the bottom cell is primitive, so $Q^k y_{k-1}$ is primitive. So, $D$ has to be a primitive, consequently it must be a square in dimension $2k - 1$. Hence, $D = 0$. The class $Q^k y_{k-1}$ has to be $\lambda_1$-annihilated. Hence, applying $Sq^*_\lambda$ eliminates the cases with $k$ being even, so $k$ is odd.

4 Some unstable results

It is possible to provide some conditions for the vanishing of $h(f)$ in terms of unstable homotopy groups of truncated projective spaces. Of course, the conditions are very strong in terms of homotopy groups, and we don’t have a complete understanding of these conditions. But, they might be of special interest, and hopefully applicable some day! We proceed as follows. Suppose $f : S^n \to Q_0 S^0$ is given with $h(f) \neq 0$. For the pull back $g$, after Theorem 3.2 we have the following observation.

Lemma 4.1. (i) Let $n = 2k + 1$. For $g$ as above, $g_{k+1} = q_{k+1} g$ pulls back to an element of $\pi_{2k+1} P_{k+1}$.

(ii) Let $n = 2k$. For $g$ as above, if $q_k g : S^{2k} \to QP_k$ is trivial in homology, then $g_k = q_k g$ pulls back to an element of $\pi_{2k} P_k$. 


Proof. (i) This follows from Freudenthal suspension theorem.
(ii) By Freudenthal’s theorem, there is an epimorphism \( \pi_{2k} \Omega \Sigma P_k \to \pi_{2k} Q P_k \). Since, we are in the meta-stable range, so we may consider the EHP sequence

\[
\pi_{2k} P_k \to \pi_{2k} \Omega \Sigma P_k \to \pi_{2k} \Omega \Sigma (P_k \wedge P_k) \simeq H_{2k} \Omega \Sigma (P_k \wedge P_k).
\]

Now, the latter isomorphism on the right is just given by the Hurewicz homomorphism. So, if \((g_k)_* = 0\) then it is in the image of the suspension homomorphism \( \pi_{2k} P_k \to \pi_{2k} \Omega \Sigma P_k \). This completes the proof. \(\square\)

By the above theorem, the existence of spherical classes in \( H_* QS^0 \) would give rise to nontrivial elements in \( \pi_{2k} P_k \) or \( \pi_{2k+1} P_{k+1} \) which survive under the stabilisation. Moreover, by computations of Section 6, if we role out spherical classes in \( H_* Q_0 S^0 \) which arise from Hopf invariant one elements, then we may well assume that \( g_{k+1} \) and \( g_k \) with \((g_k)_* = 0\) pull back to some elements of \( \pi_{2k+1} P_{k+1} \) and \( \pi_{2k} P_{k-1} \), respectively. At first, it looks that Lemma 4.1 provides a theoretical tool to show that in dimensions \( H_* QS^0 \) cannot host a spherical class. The sufficient vanishing condition in this direction might be stated as follows.

**Proposition 4.2.** After ruling out Hopf invariant one elements, the following statements hold.
(i) Suppose the image of stabilisation map \( \pi_{2k+1} P_{k+1}^{2k} \to \pi_{2k+1}^s P_{k+1}^{2k} \simeq \pi_{2k+1} Q P_{k+1}^{2k} \) is trivial. Then, the image of the unstable Hurewicz map \( h : \pi_{2k+1} QS^0 \to H_{2k+1} QS^0 \) is trivial.
(ii) Suppose the image of stabilisation map \( \pi_{2k} P_{k}^{2k-1} \to \pi_{2k}^s P_{k}^{2k-1} \simeq \pi_{2k} Q P_{k}^{2k-1} \) is trivial. Then, the image of the unstable Hurewicz map \( h : \pi_{2k} QS^0 \to H_{2k} QS^0 \) is trivial.

We note that apart from small values of \( k \) for which spherical classes in \( H_* QS^0 \) with \( * = 2k, 2k + 1 \) are completely known, the range in which the above theorem applies is the metastable range where we have the EHP-sequence. This allows to proceed with further computations.

**Case of \( \pi_{2k+1} P_{k+1}^{2k} \).** Consider the following commutative homotopy-homology ladder where the first row is the EHP-sequence

\[
\begin{array}{ccccccccc}
\pi_{2k+1}^s P_{k+1}^{2k} & \to & \pi_{2k+1} \Omega \Sigma P_{k+1}^{2k} & \to & \pi_{2k+1} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) & \to & \pi_{2k+1} P_{k+1}^{2k} & \to & \pi_{2k+1} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\simeq & & \simeq & & (\simeq) & & \simeq & & h & & \simeq \\
0 & & H_{2k+3} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) & & H_{2k+3} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) & & H_{2k+3} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) & & 2/2 \\
\end{array}
\]

where the vanishing of \( \pi_{2k+1} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) \) as well as \( H_{2k+3} \Omega \Sigma P_{k+1}^{2k} \) occurs for dimensional reasons. The commutativity of the squares on the left together with the exactness of the EHP sequence leaves us with the following short exact sequence

\[
\begin{array}{ccccccccc}
\pi_{2k+1}^s P_{k+1}^{2k} & \to & \pi_{2k+1} P_{k+1}^{2k} & \to & \pi_{2k+1} \Omega \Sigma P_{k+1}^{2k} & \to & \pi_{2k+1} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\simeq & & \simeq & & \simeq & & \simeq & & \simeq & & \\
0 & & \pi_{2k+1} P_{k+1}^{2k} & & \pi_{2k+1} \Omega \Sigma P_{k+1}^{2k} & & H_{2k+3} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) & & H_{2k+3} \Omega \Sigma (P_{k+1}^{2k} \wedge P_{k+1}^{2k}) & & 2/2 \\
\end{array}
\]
By similar reasons, for $\Sigma P_{k-1}^{2k-1}$ we have the following short exact sequence

$$
\begin{array}{cccccc}
\pi_{2k}^{s} P_{k-1}^{2k-1} \\
\rightarrow \\
0 \rightarrow \mathbb{Z}/2 \rightarrow \pi_{2k+1} \Sigma P_{k-1}^{2k-1} \rightarrow \pi_{2k+1} \Omega \Sigma^{2} P_{k-1}^{2k-1} \rightarrow H \rightarrow 0.
\end{array}
$$

The outcome of these computations is that the kernel of the stabilisation is too small it is unlikely that the map, say $\mathbb{Z}/2 \rightarrow \pi_{2k+1} P_{k-1}^{2k}$, becomes an isomorphism. Hence, it is very unlikely that this would allows to eliminate spherical classes in $H_{*}Q_{0}S^{0}$ using these classes.

5 More unstable computations

1. For $fS^{n} \rightarrow QS^{0}$, by Freudenthal theorem, it factors as $f: S^{n} \xrightarrow{f^{n}} \Omega^{n+1} S^{n+1} \rightarrow QS^{0}$. By Kahn-Priddy theorem, the mapping $f^{n}$ also lift to $\Omega^{n+1} \Sigma^{n+1} P^{n}$ through the Kahn-Priddy map $\lambda_{n}: \Omega^{n+1} \Sigma^{n+1} P^{n} \rightarrow \Omega^{n+1} \Sigma^{n+1} P^{n}$ resulting in an appropriate factorisation of $f^{n}$ which fits into a commutative diagram as

$$
\begin{array}{ccc}
\Omega^{n+1} \Sigma^{n+1} P^{n} & \xrightarrow{\lambda} & QS^{0} \\
\downarrow^{g^{n}} & & \downarrow^{\lambda} \\
S^{n} & \xrightarrow{f^{n}} & \Omega^{n+1} \Sigma^{n+1} P^{n} \\
\end{array}
$$

It follows that there is definitely a further factorisation of $f$ as

$$
S^{n} \rightarrow \Omega^{n+1} \Sigma^{n+1} P^{n} \rightarrow QP^{n} \rightarrow QP \rightarrow QS^{0}.
$$

2. Consider the homotopy equivalences $\Omega^{n+1} S^{n+1} \rightarrow \Omega^{n+1} P^{n+1}$ and $\Omega^{2n+1} S^{2n+1} \rightarrow \Omega^{2n+1} C P^{n}$. Through these equivalences, Curtis conjecture what Curtis conjecture implies about spherical classes in $H_{*}Q_{0}P^{n+1}$? What is special about the collection of spaces $\{\Omega_{n} P^{n} : n > 0\}$?

6 Some reduction results

According to Theorem 3.2 in order to a complete verification of Curtis conjecture, we must consider situations in which $f \in 2 \pi_{n} Q_{0}S^{0}$ is given with $h(f) \neq 0$ and $h(g_{i}) = 0$ where $i = [n/2]$. This does not seem a very easy point to start. Instead, we provide some reduction results which reduces the dimension of the projective space. we suggest an inductive argument which really depends on stable homotopy groups of spheres. But, with the aid of the existing knowledge on these groups together with some well known facts on cohmology operations, some reduction results could be obtained.

Suppose $h(f) \neq 0$ is given with $g^{s}: S^{n} \rightarrow P^{n}$ factoring through $P^{n-i}$ but not $P^{n-i-1}$. In this case, consider the cofibre sequence $P^{n-i} \rightarrow P^{n-i} \rightarrow S^{n-i}$. Since $g^{s}$ does not factor through $P^{n-i-1}$ then the composition $p_{i} g^{s}: S^{n} \rightarrow S^{n-i}$ is essential, representing a nontrivial element in $2 \pi_{i}^{s}$.
There are two cases: one where we can find a map \( S^{n-i} \to S^0 \) making the following commutative diagram:

\[
\begin{array}{ccc}
P_{n-i-1} & \xrightarrow{f} & P_{n-i} \\
\downarrow \quad \pi & \quad \downarrow \lambda & \quad \downarrow \theta \\
S^n & \xrightarrow{g} & S^0 \\
\downarrow \pi & \quad \downarrow \theta & \quad \downarrow \theta \\
S^{n-1} & \xrightarrow{\pi} & \Lambda \\
\end{array}
\]

and the other case when such a map does not exists. First, we deal with case when such a decomposition exists.

**Lemma 6.1.** Suppose there exists \( \lambda^s : S^{n-i} \to S^0 \) making the above diagram commutative. Then, \( f \in \{ \eta^2, \nu^2, \sigma^2 \} \).

**Proof.** In this case \( f \) is a decomposable element in \( 2\pi_i^s \) with \( f^s = \lambda^s \pi_i^g^s \). By [12, Theorem 1.8] \( f \) lives in \( 2\pi_i^s \) for \( i = 2, 6, 14 \) given by \( \mathbb{Z}/2\{\eta^2\}, \mathbb{Z}/2\{\nu^2\} \), or \( \mathbb{Z}/2\{\sigma^2, \kappa\} \), respectively. For \( i = 14 \), \( h(f) \neq 0 \) implies that \( f = \sigma^2 + \epsilon \kappa \) for some \( \epsilon \in \mathbb{Z}/2 \). However, \( \kappa \) is not a decomposable element, so \( f = \sigma^2 \). This verifies our claim.

Note that, in the case of above proof, it is known \( h(\kappa) = 0 \) [13], so even in the cases with \( f = \sigma^2 + \kappa \), we do not get any contradiction to Curtis conjecture. According to the above lemma, we focus on the cases in which \( f^s \) is not a decomposable element of \( 2\pi_i^s \).

First, we focus on the \( \theta_i \) elements. It is known that \( \theta_i = \eta^2, \nu^2, \sigma^2 \) for \( i = 1, 2, 3 \), respectively, i.e. for \( i = 12, 3 \) the \( \theta_i \) elements are decomposable. It is known that if there exists \( \theta_i \), a Kervaire invariant one element then \( h(\theta_i) \neq 0 \). It then follows from Theorem 2.2 that for \( i > 3 \), the \( \theta_i \) elements are not decomposable. Note that by work of Eccles [4, Proposition 4.1] \( \theta_i \) exists if and only if for some element \( \theta_i' \in 2\pi_i^2 \) with \( \theta_i' = \lambda \theta_i' \) we have \( h(\theta_i') = a_{2i-1} + 2 \mod 0 \) modulo other terms. It then implies that for the composition \( q_{2i-1} \theta_i' : S^{2i+1-2} \to P \to P_{2i-1} \) we have \( h(q_{2i-1} \theta_i') = a_{2i-1}^2 \mod 0 \). And, as shown below, we may approximate \( \theta_i' \) by a map \( S^{2i+1-2} \to P^{2i+1-4} \). We have obvious lower bound for the top dimension as follows.

**Lemma 6.2.** For \( i > 3 \), if \( \theta_i' \) factors as \( S^{2i+1-2} \to P^n \) then \( n > 2i - 1 \).

**Proof.** Suppose there is such a factorisation. In this case the composition \( q_{2i-1} \theta_i' : S^{2i+1-2} \to P_{2i-1}^n = S^{2i-1} \) satisfies \( h(q_{2i-1} \theta_i') = a_{2i-1}^2 \in H_* Q S^{2i-1} \). Consequently, \( q_{2i-1} \theta_i' \in 2\pi_i^s \) is a Hopf invariant one element with \( i > 3 \). This is a contradiction.

Let’s note that for a path connected space \( X \), the stable homology suspension \( \sigma_* : H_* Q X \to \tilde{H}_* X \) which is induced by the evaluation map \( \Sigma_* Q X \to X \), given \( g : S^n \to Q X \) we have \( \sigma_* h(g) = h(g^s) \).

**Case i = 0.** Suppose \( g^s \) does not factor through \( P^{n-1} \). In this case, the composition \( p_0 g^s : S^n \to S^n \) is nontrivial \( \mod 2 \). Since, the elements of \( 2\pi_0^s \) are detected by degree mod 2, this implies that \( p_0 g^s \) is nontrivial in homology, hence \( h(g^s) = a_n \) for some \( n > 0 \), implying that \( h(g) = a_n \mod 0 \). As noted before, this is equivalent to \( f = \lambda g \) being a Hopf invariant one element. So, \( n = 1, 3, 7 \).
The following cases, Case 1, Case 2, Case 3, can be dealt with by using the available description of the $E_2$ term of the ASS for $P$ [2, Theorems 1.1 and 1.3].

**Case $i = 1$.** Suppose $h(g^i) = 0$. For $g^i : S^n \to P^n$ the composition $p_n g^i : S^n \to S^n$ is trivial in homology. Since $\pi_n S^n \simeq H_n S^n$, hence $p_n g$ is null, so $g^i$ factor through $P^{n-1}$. We write $g^i : S^n \to P^{n-1}$ for the resulting map. Suppose $g^i$ does not factor through $P^{n-2}$. In this case, the composition $p_{n-1} g : S^n \to S^{n-1}$ represents a nontrivial element in $2\pi_1^{s} \simeq \mathbb{Z}/2\{\eta\}$. Since $h(\lambda g) = h(f) \neq 0$, hence $\lambda^i g^i$ is essential, where the composition $p_1 g^i$ is detected by $h_1$ in the ASS. The class $f^i = \lambda^i g^i$ can only correspond to $h_1 h_i$ of the families of permanent cycles in the ASS with $i \geq 3$ listed in [2, Theorem 1.3], or to one of the classes $h_1 h_i$ with $i = 1, 2, 3$ [2, Page 3]. In the former, $f^i = \eta_i$, an element of Mahowald family [9], for which it is known that $h(\eta_i) = 0$ for all $i \geq 3$ [12, Theorem 1.4], contradicting $h(f) \neq 0$. So this case cannot arise. For the latter cases corresponding to $h_1 h_i$ with $i = 1, 2, 3$ we have $f = \eta^2, \eta \nu = 0, \eta \sigma$. It is known that $h(\eta \sigma) = 0$ by [12, Theorem 1.8], contradicting $h(f) \neq 0$. So, we cannot have this choice either. We are then only left with $f = \eta^2$ for which it is known that $h(\eta^2) \neq 0$ with $\eta^2$ being a Kervaire invariant one element. Consequently, in this case, if $h(f) \neq 0$ then $f = \eta^2$.

**Case $i = 2$.** Suppose $g^i$ pulls back to $P^{n-2}$ but not to $P^{n-3}$. In this case, $p_2 g^i \in 2\pi_2^s \simeq \mathbb{Z}/2\{\eta^2\}$, i.e. $p_2 g^i = \eta^2$. In this case, the composition $f^i = \lambda^i g^i$ is detected by $h_2^i h_i$ which detects $\eta_2 \in 2\pi_2^s \simeq 2\mathbb{Z}$, with $\eta_i$ being an element of Mahowald family, which is a decomposable term. That is $f^i = \eta_2 \eta_i$ that consequently shows that $h(f) = 0$, whenever $j \neq 2$. The case $j = 2$ is also similar, showing that $f^i = \eta^2 \nu = 0$. This shows that it is not possible to have $f$ with this property and $h(f) \neq 0$. Therefore, $g^i$ pulls back to $P^{n-3}$.

**Case $i = 3$.** We have $p_{n-3} g^i \in 2\pi_3^s \simeq \mathbb{Z}/2\{\nu\}$, hence detected by $h_2$ in the ASS. Consequently, $f^i$ is to be detected by $h_2 h_i$ in the ASS. But, this implies that either $f^i = \eta \nu$ (if $j = 1$) or $f^i = \nu^2$ (if $j = 2$) or $f^i = \nu \sigma$ (if $j = 3$) (compare to the families listed in [2, Theorem 1.3] and the six finite families of [2, Page 3]). If $h(f) \neq 0$ then $f = \nu^2$ showing that $f$ factors as $S^6 \to P^3 \to S^0$. Note that, stably $P^3 \simeq P^2 \vee S^3$ and $g^i$ can be constructed by being $\nu$ on $S^3$ and trivial in $P^2$. In fact in this case, we can take $\lambda_{|pi} : S^3 \to S^0$ to be $\nu$. This verifies the conjecture in this case.

**Case 4,5.** Suppose $g^i$ pulls back to $P^{n-4}$. Since $2\pi_4^s \simeq 0$ as well as $2\pi_5^s \simeq 0$, by a similar reasoning as above, $g^i$ factors through $P^{n-6}$ if it factors through $P^{n-4}$.

**Case $i = 6$.** Suppose $g^i$ factors through $P^{n-6}$ but not $P^{n-7}$. In this case, $p_6 g^i \in 2\pi_6^s \simeq \mathbb{Z}/2\{\nu^2\}$, so $p_6 g^i$ is detected by $h_2^i$ in the ASS. According to [2, Theorem 1.3] we have a family which maps to $h_2^3$ in the Adams spectral sequence, showing that $f^i = \nu^3$. Consequently, $h(\nu^3) = 0$. So, it is not possible to have $f$ with $h(f) \neq 0$. Hence, $g^i$ factors through $P^{n-7}$.

**Case $i = 7$.** Suppose $g^i$ factors through $P^{n-7}$ but not $P^{n-8}$. In this case, $p_7 g^i \in 2\pi_7^s \simeq \mathbb{Z}/16\{\sigma\}$. Since $h(g^i) \neq 0$, so $p_7 g^i$ is an odd multiple of $\sigma$. We may multiply this by an add number, and without loss of generality, we may assume $p_7 g^i = \sigma$. Consequently, and using computations of [2] Families of Theorem 1.3 and Page 3, we conclude that $f^i$ is possibly detected by one of the following permanent cycles: $h_3^2, h_3 h_4, h_3 h_1, h_3 h_2$. The fist one corresponds to $f = \theta_3 = \sigma^2$. For the last two, $f^i = \sigma \eta$ or $f = \sigma \nu$ which by Theorem [2,2] we have $h(f) = 0$. So, these cases cannot arise. If $f^i$ corresponds to $h_3 h_4$ in the Adams spectral sequence, then $f^i \in 2\pi_2^2$ which is generated by $\nu \sigma$ and $\eta^2 \kappa$ [10, Table A3.3] which are decomposable elements and map trivially under $h$. Of course, for filtration reasons, it is not possible that $f \in \{\nu \sigma, \eta^2 \kappa\}$ as the generators of
$2\pi^s_{22}$ are detected in higher lines of the $E_2$-term of the ASS, and not the 2-line.

**Case** $i = 8$. Suppose $g^s$ through $P^{n-8}$ but not $P^{n-9}$. Then, $p_8g^s \in 2\pi^s_8 \simeq \mathbb{Z}/2\{\eta\sigma, \varepsilon\}$ where $\varepsilon$ is represented by a Toda bracket $\langle \nu^2, 2, \eta \rangle$. If $p_8g^s = \eta\sigma$ then, as above, we may use [2] to show that $h(f) = 0$. If $p_8g^s = \varepsilon$ then $g^s$ does not correspond to any of the families in the ASS for $P$ as $\varepsilon$ is detected by $c_0$ in the ASS. So, this case cannot arise at all. Consequently, we cannot have $f$ with $h(f) \neq$ as above which does factor through $P^{n-8}$ but not $P^{n-9}$.

**Case** $i = 9$. Suppose $g^s$ through $P^{n-9}$ but not $P^{n-10}$. Then, $p_9g^s \in 2\pi^s_9 \simeq \mathbb{Z}/2\{\nu^3, \eta^2\sigma, \mu_9\}$ where $\mu_9$ is an element coming from $2\pi_9J$ where $J$ is the fibre of the Adams operation $\psi^3 - 1 : BSO \to BSO$.

**References**

[1] Frederick R. Cohen, Thomas J. Lada, and J.Peter May., The homology of iterated loop spaces. *Lecture Notes in Mathematics. 533. Berlin-Heidelberg-New York: Springer-Verlag. VII*, 490 p. (1976).

[2] Ralph L. Cohen, Wen-Hsiung Lin, and Mark E. Mahowald. The Adams spectral sequence of the real projective spaces., *Pac. J. Math.*, 134(1):27–55, 1988.

[3] Edward B. Curtis., The Dyer-Lashof algebra and the $\Lambda$-algebra., *Ill. J. Math.*, 19:231–246, 1975.

[4] Peter John Eccles., Codimension one immersions and the Kervaire invariant one problem., *Math. Proc. Camb. Philos. Soc.*, 90:483–493, 1981.

[5] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, On the nonexistence of elements of Kervaire invariant one., *Ann. of Math.*, Ann. of Math. (2) 184 (2016), no. 1, 1262.

[6] Daniel S. Kahn and Stewart B. Priddy, Applications of the transfer to stable homotopy theory., *Bull. Am. Math. Soc.*, Vol. 78, 981–987, 1972.

[7] Nicholas J. Kuhn.. The homology of the James-Hopf maps., *Ill. J. Math.*, 27:315–333, 1983.

[8] J. Lannes., Sur les immersions de Boy. In Algebraic topology, Aarhus 1982 (Aarhus, 1982), volume 1051 of *Lecture Notes in Math.*, pages 263–270. Springer, Berlin, 1984.

[9] Mark Mahowald., A new in finite family in $2\pi^s_9$. *Topology.*, 16:249–256, 1977.

[10] Douglas C. Ravenel., Complex cobordism and stable homotopy groups of spheres. 2nd ed. Providence, RI: AMS Chelsea Publishing, 2004.

[11] Robert J. Wellington., The unstable Adams spectral sequence for free iterated loop spaces., *Mem. Am. Math. Soc.*, 36(258):225, 1982.

[12] Hadi Zare., Freudenthal theorem and spherical classes in $H_4QS^0$. submitted.

[13] Hadi Zare., On the Hurewicz homomorphism on the extensions of ideals in $2\pi^s_9$ and spherical classes in $H_4Q_0S^0$. arXiv:1504.06752.