Applications of the Backus–Gilbert method to linear and some non-linear equations

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Abstract

We investigate the use of a functional analytical version of the Backus–Gilbert Method as a reconstruction strategy to get specific information about the solution of linear and slightly non-linear systems with Frechét derivable operators. Some a priori error estimates are shown and tested for two classes of problems: a non-linear moment problem and a linear elliptic Cauchy problem. For this second class of problems a special version of the Green-formula is developed (see Theorem 4), in order to analyze the involved adjoint equations.

1 Introduction

1.1 Main results

The functional analytical approach of the Backus–Gilbert method in §1.3 was already used by other authors (see [Ch], [Ki] or [LM1,2]). In this paper we use the differentiability of the involved non-linear operator in order to develop the error estimative (9) and (13) for this reconstruction schema. If the operator is linear, we obtain the estimative (17).

In order to test this reconstruction strategy, we choose the same non-linear operator in §3.1 as Louis does in [Lo]. The numerical tests in §4.1 show that one can get good results even for noisy data.

For the second test in §3.2, we choose a linear operator, which is highly ill-posed. The results are again satisfactory provided one uses appropriated sentinels to define the reconstruction strategy (see [Ch] or [Le]).

The results presented in this paper constitute part of the author’s PhD research and they can be found with a bit more detail in [Le].

1.2 Historical overview

This reconstruction method was first proposed in 1967 by G. Backus and J. Gilbert [BG1,2,3]. They were interested in the pointwise reconstruction of a function $f \in X = L^2(\Omega)$, were $\Omega \subset \mathbb{R}^n$ is supposed to be open and bounded. The motivation of their problems was geophysical and the mathematical problem
involved in the model is known in the literature as the *moment problem*. It can be formulated as follows: find a function \( f \in X \) such that
\[
\int_{\Omega} K_i(x) f(x) \, dx = g_i, \quad i = 1, \ldots, N, \tag{1}
\]
were the kernels \( K_i \) are known real functions, which are well defined at \( \Omega \) and the right hand side \( g = \{g_i\}_{i=1}^N \in Y = \mathbb{R}^N \) correspond to the measured data of the physical problem. In order to determine the value of the solution \( f \) at some point \( x_0 \in \Omega \), they suggested a linear reconstruction schema, which is defined by a functional of the right hand side of the linear system (1). One defines the linear functional \( R_N \in Y' \) by
\[
R_N(g) := \langle \varphi, g \rangle_Y = \int_{\Omega} \left( \sum_{i=1}^N \varphi_i K_i(x) \right) f(x) \, dx = \langle \phi_N, f \rangle_X, \tag{2}
\]
were \( \varphi \in Y' \) and \( \phi_N \in X \). It is easy to observe that \( f_N(x_0) := R_N(g) \) will be a good approximation for \( f(x_0) \) if the condition \( \phi_N(\cdot) \approx \delta(x_0 - \cdot) \) is satisfied. The Backus–Gilbert idea is to force this condition by defining the quadratic functional
\[
J(\phi) := \int_{\Omega} |x_0 - x|^2 \phi^2(x) \, dx \tag{3}
\]
on \( X \) and choosing \( \phi_N \) such that
\[
J(\phi_N) = \min_{\phi \in \text{Span}\{K_i\}} J(\phi). \tag{4}
\]
The linear constraint
\[
\int_{\Omega} \phi(x) \, dx = 1
\]
is imposed in order to avoid the trivial solution in (4). Once one has evaluated the function \( \phi_N(x) = \sum \varphi_i K_i(x) \), the approximation \( f_N(x_0) \) is determined by the inner product
\[
f_N(x_0) = \langle \phi_N, f \rangle_X = \langle \varphi, g \rangle_Y, \tag{5}
\]
were \( \varphi = \{\varphi_i\}_{i=1}^N \). One great advantage of using the Backus–Gilbert method which can be recognized in (4) is that the evaluation of the reconstruction operator \( R_N(\cdot) \) does not depend on the system data. For different sets of data \( g \) is possible to reconstruct the value of the respective \( f(x_0) \) only by evaluating an inner product in \( Y \).

### 1.3 Functional analytical formulation

Let \( V \hookrightarrow X \hookrightarrow V' \) be a Hilbert triple, \( Y \) a Hilbert space, \( y \in Y \) and \( A : X \to Y \) a bounded linear operator. We analyze the problem of finding the value \( \langle \mu, x^* \rangle \)
for \( \mu \in V' \), were \( x^* \) is the generalized solution obtained by the Moore-Penrose inverse of

\[
A x = y. \quad \text{(6)}
\]

It is obvious that the expression \( \langle \mu, x^* \rangle \) does not need to be well defined, if we do not make any further regularity assumptions about \( x^* \). Depending on the physical situation involved, it is possible to guarantee that the expression \( \langle \mu, x^* \rangle \) is well defined for some \( \mu \)'s or even that \( x^* \in V \). As we suppose \( y \) is obtained by measurements, it is to be expected that only a \( y_\epsilon \) with \( ||y - y_\epsilon||_Y \leq \epsilon \) is available, with \( \epsilon > 0 \) small.

We use the Backus–Gilbert strategy and try to reconstruct the value \( f := \langle \mu, x^* \rangle_X \) using a linear functional evaluated in \( y_\epsilon \). For \( \varphi \in Y' \) we define \( f_{\epsilon, \varphi} := \langle \varphi, y_\epsilon \rangle_Y \) and estimate the error \( |f - f_{\epsilon, \varphi}| \) by

\[
|f - f_{\epsilon, \varphi}| = \left| \langle \mu, x^* \rangle_X - \langle \varphi, y_\epsilon \rangle_Y \right| \\
\leq \left| \langle \varphi, y - y_\epsilon \rangle_Y \right| + \left| \langle \mu, x^* \rangle_X - \langle \varphi, A x^* \rangle_Y \right| \\
\leq \epsilon ||\varphi||_{Y'} + |\langle \mu - A^* \varphi, x^* \rangle_X|, \quad \text{(7)}
\]

were \( A^* : Y' \to X' \) is the adjoint operator of \( A \). If we succeed in finding a solution \( \varphi \in Y' \) for the equation \( A^* \varphi = \mu \) we can write

\[
f = \langle A^* \varphi, x^* \rangle_X = \langle \varphi, y \rangle_Y \simeq \langle \varphi, y_\epsilon \rangle_Y = f_{\epsilon, \varphi}, \quad \text{(8)}
\]

and the error \( |f - f_{\epsilon, \varphi}| \) behaves like \( O(\epsilon) \). Another consequence is that the approximation \( f_{\epsilon, \varphi} \) is exact if there are no errors in the measurements (\( y_\epsilon = y \)).

In the special case of \( X \) and \( Y \) being spaces of functions defined over a region \( \Omega \), the Backus–Gilbert strategy suggests a pointwise reconstruction of \( x^* \). In order to reconstruct the value of \( f(\cdot) \) at the point \( t \in \Omega \) we should take \( \mu(\cdot) = \delta(t - \cdot) \) in \( \text{[8]} \) and solve the adjoint equation \( A^* \varphi = \delta[8] \).

We may have difficulties if \( \delta \notin \text{Rg}(A^*) \). In this case we can use the projection of \( \delta \) over \( \text{Ker}(A^*)^1 \) instead of \( \delta \) itself. This is equivalent to minimizing the error \( ||A^* \varphi - \delta||_{Y'}^2 \) or to find a solution \( \varphi \in Y' \) of the normal equation

\[
(AA^*) \varphi = A \delta. \quad \text{(9)}
\]

Louis and Maaß propose a similar approach in \([\text{LM}2]\) and use the projection of \( \delta \) over special Sobolev spaces of negative index. In \([\text{LM}1]\) (see also \([\text{Lo}]\)) the equation \( (AA^*) \varphi = Ae_h \) is considered, were \( e_h \) is a mollifier, i.e., a smooth approximation for the Dirac-distribution \( \delta \).

Another alternative for the case \( \delta \notin \text{Rg}(A^*) \) was proposed by Chavent in \([\text{Ch}]\). He tried to regularize the normal equations using the Tikhonov-strategy: \( \varphi \) is chosen as the minimum over \( Y' \) of the functional \( \langle ||A^* \varphi - \delta||_{Y'}^2 + \alpha ||\varphi||_{X'}^2 \rangle \), were \( \alpha > 0 \) is a small regularization parameter.

2 Analysis of the method

We are interested in applying the Backus–Gilbert strategy for operators of the form \( A = A_0 + \gamma A_1 \), were \( A_0 \in L(X,Y), \ A_1 : X \mapsto Y \) is continuously

\(^1\text{The Hilbert space } V \text{ must be chosen, so that } \delta \text{ belongs to } V'.\)
Lemma 1. If \( x^0 \in V \) is an approximation to a solution \( x^* \) of (6), the expression \( f_{e,\varphi} := \langle \varphi, y_e \rangle_Y \) gives an approximation for \( f := \langle \mu, x^* \rangle_X \) and the error \( |f - f_{e,\varphi}| \) is estimated by

\[
|f - f_{e,\varphi}| \leq |\langle \varphi, y - y_e \rangle_Y| + |\langle \varphi, A x^* - A x^0 - dA(x^0)(x^* - x^0) \rangle_Y| + |\langle \varphi, A x^0 - dA(x^0)x^0 \rangle_Y| + |\langle dA(x^0)^* \varphi - \mu, x^* \rangle_X|.
\]

Proof: Estimate (9) follows promptly from the following equality

\[
|f - f_{e,\varphi}| = |\langle \mu, x^* \rangle_X - \langle \varphi, y_e \rangle_Y + \pm \langle \varphi, y \rangle_Y \pm \langle \varphi, A x^0 - dA(x^0)(x^* - x^0) \rangle_Y|.
\]

Before analyzing the right hand side of (9), let us discretizate the spaces involved. We define the finite dimensional space \( Y_h = \text{Span}\{y_j\}_{j=1}^N \) by

\[
Y_h \subset \{ \varphi \in Y / \langle \varphi, A x^0 - dA(x^0)x^0 \rangle_Y = 0 \}. \tag{10}
\]

Further we let \( P_h : Y \to Y_h \) be the orthogonal projector over \( Y_h \) and choose the finite dimensional space \( X_h = \text{Span}\{x_j\}_{j=1}^N \subset D(A) \cap V \) such that the property

\[
\det (\langle dA(v)^* P_h^* x_j >)_{1 \leq i,j \leq N} \neq 0. \tag{11}
\]

is satisfied.

Theorem 2. Define \( y_h := P_h y, \ y_{e,h} := P_h y_e \) and \( f_{e,\varphi,h} := \langle \varphi, y_{e,h} \rangle_Y \). For every \( \varphi \in Y \) the following estimate holds:

\[
|f - f_{e,\varphi,h}| \leq \epsilon \|P_h\| \|\varphi\|_Y \|y + \gamma \|\varphi\|_Y \|P_h\| \|x^* - x^0\|_X^2 \]

\[
+ |\langle dA(x^0)^* P_h^* \varphi - \mu, x^* \rangle_X|. \tag{12}
\]

Proof: By an argument analogous to that used in (9) we obtain that for each \( \varphi \in Y \)

\[
|f - f_{e,\varphi,h}| \leq |\langle \varphi, y \rangle_Y - \langle \varphi, y_{e,h} \rangle_Y|
\]

\[
+ \gamma |\langle \varphi, y \rangle_Y |P_h A_1 x^* - P_h A_1 x^0 - P_h dA_1(x^0)(x^* - x^0)|_Y
\]

\[
+ |\langle \varphi, P_h A x^0 - P_h dA(x^0)x^0 \rangle_Y|
\]

\[
+ |\langle dA(x^0)^* P_h^* \varphi - \mu, x^* \rangle_X|. \tag{13}
\]

The first term in (13) can be estimated by

\[
|\langle \varphi, y \rangle_Y - \langle \varphi, y_{e,h} \rangle_Y| \leq \epsilon \|P_h\| \|\varphi\|_Y.
\]

\(^2\)The Fréchet derivative of \( A_1 \) will be denoted by \( dA_1 \).

\(^3\)For convenience we will identify the spaces \( X \) with \( X' \) and \( Y \) with \( Y' \).
For the second term we have
\[ \gamma \| \varphi \|_Y \| P_h A_1 x^* - P_h A_1 x^0 - P_h dA_1(x^0 - x^0) \|_Y \leq \gamma \| \varphi \|_Y \| P_h \| O(\|x^* - x^0\|^2_X). \]

The third term in (13) disappears because of our choice of \( Y_h \). Putting these inequalities together we obtain (12).

The last term in (12) gives us a rule for choosing our \( \varphi \in Y \). This is actually
\[ < dA^*(x^0) P_h^* \varphi, x_j >_X = < \mu, x_j >_X, \ j = 1 \ldots N. \] (14)

That means we can evaluate the coefficients of \( P_h \varphi \) in \( Y_h \) by solving the N-dimensional linear system (14). Solving this system is a well defined problem, as can be seen from the determinant condition (11).

Next we interpret the system (14) in a different way. Let us assume that the space \( X_h \) can be written as \( B^* Y_h \), where \( B \) is a linear bounded operator \( B : V' \mapsto Y \) with \( B^* : Y \mapsto V \). We are then able to write (14) as
\[ < dA^*(x^0) P_h^* \varphi - \mu, B^* w >_X = 0, \ \forall w \in Y_h, \]
i.e.,
\[ < BG_h \varphi, w >_Y = < B\mu, w >_Y, \ \forall w \in Y_h, \] (15)
where \( G_h = dA^*(x^0) P_h^* \). If \( \mu = \delta \) and we are in the special case \( \mu \in \text{Ker} B \), it follows from (15) that
\[ < BG_h \varphi, w >_Y = 0, \ \forall w \in Y_h. \] (16)

Further if it is possible to decompose the product \( BG_h \) as a square \( B^2 \) of a symmetric matrix \( B \), it follows from (16) that \( \| B\varphi \|_Y^2 = 0 \). Instead of solving system (16), we can consider the minimization problem:
\[
\begin{align*}
\| B \varphi \|_Y^2 &= \min_{\varphi \in Y_h} \| B \varphi \|_Y^2 \\
\text{under the linear constraint} & \quad < dA^*(x^0) \varphi, 1 >_X = 1
\end{align*}
\]

The extra linear constraint is motivated by the original Backus–Gilbert formulation in §1.2 and introduced in order to avoid the trivial solution in the minimization problem. The constrained minimization problem above can be interpreted as an extended Backus–Gilbert method.

We proceed to develop an error estimate for the linear case when noisy data is considered.

**Theorem 3.** Let \( A \) be a linear operator. Take \( B^* = A^* \) and \( X_h = A^* Y_h \). If we choose \( \varphi_h \in Y_h \) to be the solution of (14), i.e., \( < A^* \varphi_h - \mu, w >_X = 0, \ \forall w \in X_h \), we obtain the error estimate
\[ |f - f_{\epsilon, \varphi, h}| \leq \text{dist}(\mu, A^* Y_h)_V (1 + ||P_h||) \text{dist}(x^*, X_h)_V + O(\epsilon). \] (17)

**Proof:** Using (7), for each \( \varphi_h = P_h \varphi \in Y_h \) we obtain the equality
\[ |f - f_{\epsilon, \varphi, h}| = O(\epsilon) + |< A^* \varphi_h - \mu, x^* >_X|. \]
Define $x_h := \mathcal{P}_h x^*$, were $\mathcal{P}_h : X \mapsto X_h$ is the orthogonal projector over $X_h$. Now choosing $\varphi_h \in Y_h$ as the solution of (14), for every $\psi \in Y_h$ we have
$$<A^* \varphi_h - \mu, x^*>_X = <A^* \varphi_h - A^* \psi, x^* - x_h>_X + <A^* \psi - \mu, x^* - x_h>_X. \quad (18)$$

The first term on the right hand side of (18) disappears by the definition of $x_h$. For the second term we have
$$|<A^* \varphi_h - \mu, x^* - x_h>_X| \leq ||A^* \psi_h - \mu||_{V'} ||x^* - x_h||_V.$$

In order to estimate $||x^* - x_h||_V$ we define $\tilde{x} \in X_h$ as the solution of the minimization problem
$$||x^* - \tilde{x}||^2_V = \min_{x \in X_h} ||x^* - x||^2_V.$$

From this definition follows
$$||x^* - x_h||_V \leq ||x^* - \tilde{x}||_V + ||\tilde{x} - P_h x^*||_V \leq \text{dist}(X_h, x^*)_V + ||P_h(\tilde{x} - x^*)||_V \leq (1 + ||P_h||) \text{dist}(X_h, x^*)_V,$$

and the theorem is proved.

It is easy to conclude from (17) that the error in the approximation $f_{\epsilon, \varphi, h}$ will converge to zero with $h$ and $\epsilon$ only when we have $\mu \in \text{Rg} A^*.$

## 3 Applications

### 3.1 A non-linear moment problem

We start this discussion with a special class of non-linear moment problems. Quadratic moment problems were also analyzed by Louis in [Lo]. Let $X = Y = L^2(0, 1)$ and $A : X \mapsto Y$ the operator defined by
$$\left( Ax \right)(t) = \int_0^t x^0(t-s) x(s) \ ds + \nu \int_0^t x(t-s) x(s) \ ds, \ t \in [0, 1], \quad (19)$$

were the kernel $x^0$ of the linear component of $A$ is a $L^2(0, 1)$-function and $\nu > 0$ is a small parameter, that controls the non-linear component of $A$. Just like in §2 we will analyze the system $Ax = (A_0 + \nu A_1) x = y$.

The right hand side of this system consists of measured data, so we assume we know only a finite number of $y_i = y(t_i), t_i \in (0, 1)$. If we define the projection operator $P_h : Y \mapsto Y_h = \mathbb{R}^N$, it is possible to define a discrete version of $A$ in (19) by setting
$$\left( P_h A \right)(x) := [Ax(t_i)]^t = \left[ \int_0^{t_i} x^0(t_i - s) x(s) \ ds + \nu \int_0^{t_i} x(t_i - s) x(s) \ ds \right]^t.$$ 

If we further assume that our measurements are inexact, then we actually have a $y_{h, \epsilon} \in Y_h$ with $||P_h y - y_{h, \epsilon}|| \leq \epsilon$, were $\epsilon > 0$ is small. We will be interested in finding the solution $x^*$ of the discrete non-linear system
$$\left( P_h A \right)x^* = y_{h, \epsilon}.$$
We saw in §2 that an approximation for the solution \( x^* \) is needed. For this propose we will choose the kernel \( x_0 \) in (19). We also need the operators \( P_h dA \) and its adjoint \( (P_h dA)^* : Y_h \mapsto X \). One can easily see that for \( f \in L^2(0, 1) \) and \( w \in \mathbb{R}^N \) the equalities

\[
(P_h dA(x))(f) = \left[ \int_0^{t_i} x^0(t_i - s) f(s) \ s + 2\nu \int_0^{t_i} x(t_i - s) f(s) \ s \right]_{1 \leq i \leq N}
\]

and

\[
(P_h dA(x))^*(w) = \sum_{i=1}^{N} w_i \left[ x^0(t_i - s) + 2\nu x(t_i - s) \right] \chi_{[0,t_i]}(s).
\]

are valid.

Now we have to choose the space \( X_h = \text{Span}\{x_i\}_{i=1}^{N} \). This choice must reflect the expected regularity of the solution \( x^* \) and should be such that the system in (14) has nice properties. We choose a cubic B–spline basis for \( X_h \) for the numerical experiments. Given \( \mu \in X' \) we will have to solve the system

\[
\left[ \langle (P_h dA(x^0))^* e_i, x_j \rangle \right]_{i,j=1}^{N} [\varphi]_i^t = [\langle \mu, x_j \rangle]_i^t,
\]

where the matrix of \( \langle \varphi \rangle \) will have almost upper triangular form if the \( x_j \)'s are B–splines. We assume the points \( t_j \) are uniformly placed on the interval \( [0, 1] \) and define for \( j = 0, \ldots, N - 1 \) the cubic B–splines

\[
S_j(t) = \frac{1}{4h^3} \begin{cases}
(t - t_j)^3, & t \in [t_{j-2}, t_{j-1}] \\
h^3 + 3h^2(t - t_{j-1}) + 3h(t - t_{j-1})^2 - 3(t - t_{j-1})^3, & t \in [t_{j-1}, t_j] \\
h^3 + 3h^2(t_{j+1} - t) + 3h(t_{j+1} - t)^2 - 3(t_{j+1} - t)^3, & t \in [t_j, t_{j+1}] \\
(t_{j+2} - t)^3, & t \in [t_{j+1}, t_{j+2}]
\end{cases}
\]

In the formulation of our strategy we assumed the space \( Y_h \) satisfy the condition in (10). In order to rescue our choice of \( Y_h \), we add to the system (20) the following linear restriction to \( \varphi \)

\[
\langle \varphi, P_h A_1(x^0) - dP_h A_1(x^0) x^0 \rangle_{Y} = 0.
\]

Joining the equations in (20) and (21), we have an overdetermined system with \( N + 1 \) equations to solve, in order to determine the \( N \) coefficients of \( \varphi \). We can observe, that the matrix coefficients \( a_{i,j} \) of this system vanish for \( i > j + 2 \) and \( i \neq N + 1 \).

### 3.2 A linear elliptic Cauchy–problem

We begin with the definition of the linear operator \( A : H^{1/2}(\Gamma_r) \mapsto H^{1/2}(\Gamma_l) \)
were $H^s$ are the Sobolev spaces\footnote{For details see [Ad] or [DaLi].} of index $s \in \mathbb{R}$ and $w$ is the $H^1(\Omega; \Delta)$–solution of the mixed boundary value problem on the left hand side.

Note that solving the equation $A \varphi = f$ is equivalent of finding the trace $\varphi = w|_{\Gamma_r}$ of the $H^1(\Omega; \Delta)$–solution of following elliptic Cauchy–problem:

\[
\begin{aligned}
\Delta w &= 0, \quad \text{in } \Omega \\
 w &= \varphi, \quad \text{at } \Gamma_r \\
 w_\nu &= 0, \quad \text{at } \Gamma_l \\
 w_\nu &= 0, \quad \text{at } \Gamma_i
\end{aligned}
\]

Given a Distribution $\mu \in H^{-1/2}(\Gamma_r)$ we will use the Backus–Gilbert strategy to approximate the value $<\mu, \varphi>$ by $<\psi, f>$, were $\psi$ is the solution of

\[
A^*\psi = \mu.
\] (22)

We can see, using integration by parts, that the adjoint operator of the restriction of $A$ to $H^{1/2}_{00}(\Gamma_r)$ is the operator $A^\sharp : H^{1/2}_{00}(\Gamma_l)' \mapsto H^{1/2}_{00}(\Gamma_r)'$ defined by

\[
\begin{aligned}
\Delta v &= 0, \quad \text{in } \Omega \\
v &= 0, \quad \text{at } \Gamma_r \\
v_\nu &= \psi, \quad \text{at } \Gamma_l \\
v_\nu &= 0, \quad \text{at } \Gamma_i
\end{aligned}
\]

For $\psi \in H^{1/2}_{00}(\Gamma_l)'$, if $\varphi \in H^{1/2}(\Gamma_r) \setminus H^{1/2}_{00}(\Gamma_r)$, it’s not true that

\[
\int_{\Gamma_l} A(\varphi) \psi \, d\Gamma = -\int_{\Gamma_r} \varphi A^\sharp(\psi) \, d\Gamma.
\]

To correct this problem we need the following theorem.

**Theorem 4.** For $a, b \in \mathbb{R}$ let $\eta_{a,b} \in C^\infty(\Gamma_r)$ be a function with $\eta_{a,b}(P_1) = a$ and $\eta_{a,b}(P_2) = b$, were $P_1$ and $P_2$ are the contact points between $\Gamma_r$ and $\Gamma_l$. If $V_{a,b}$ is the subspace of $H^{1/2}(\Gamma_r)$ defined by

\[
V_{a,b} := \{ \varphi \in H^{1/2}(\Gamma_r) \mid \eta_{a,b} - \varphi \in H^{1/2}_{00}(\Gamma_r) \},
\]
then for $\varphi_1, \varphi_2 \in V_{a,b}$ we have

$$
\int_{\Gamma_l} A\varphi_1 \psi \, d\Gamma + \int_{\Gamma_r} \varphi_1 A^\sharp \psi \, d\Gamma = \int_{\Gamma_l} A\varphi_2 \psi \, d\Gamma + \int_{\Gamma_r} \varphi_2 A^\sharp \psi \, d\Gamma
$$

for every $\psi$ in $H^{-1/2}(\Gamma_l)$.

A complete proof of this Theorem can be found in [Le]. A direct consequence of Theorem 4 is that for $a,b \in \mathbb{R}$ one can defines over $H^{-1/2}(\Gamma_l)$ the linear functional

$$
r_{a,b}(\psi) := \langle A\eta_{a,b}, \psi \rangle + \langle \eta_{a,b}, A^\sharp \psi \rangle
$$

and obtain

$$
\langle A\varphi, \psi \rangle = -\langle \varphi, A^\sharp \psi \rangle + r_{a,b}(\psi),
$$

for every $\varphi \in V_{a,b}$ and $\psi \in H^{-1/2}(\Gamma_l)$.

If we are able to find a $\psi \in H^{-1/2}(\Gamma_l)$ that solves the equation

$$
-A^\sharp \psi = \mu,
$$

we can solve our reconstruction problem as before, using

$$
\langle \mu, \varphi \rangle = -\langle A^\sharp \psi, \varphi \rangle = \langle \psi, A\varphi \rangle - r_{a,b}(\psi) = \langle \psi, f \rangle - r_{a,b}(\psi).
$$

We should observe that, if $\varphi \in H^{1/2}_{00}(\Gamma_r)$, than $a = b = 0$ and $r_{a,b} \equiv 0$. In this case we have

$$
\langle \mu, \varphi \rangle = \langle \psi, f \rangle.
$$

4 Numerical results

4.1 The moment-problem

In this section we study the operator $A : L^2(0,1) \to L^2(0,1)$ defined in (19) for $x^0(t) = t$. Let us start with the linear case, i.e., taking $\nu = 0$ in (19).

We generate different right hand sides by solving the direct problem $y = Ax$ for three functions

$$
x_a(t) = \begin{cases} 
t/2, & t \leq 1/2 \\
1/4 - t, & t \geq 1/2
\end{cases}, \quad x_b(t) = \begin{cases} 
2t, & t \leq 1/2 \\
2 - 2t, & t \geq 1/2
\end{cases}
$$

and

$$
x_c(t) = \begin{cases} 
1, & 1/4 \leq t \leq 3/4 \\
0, & \text{otherwise}
\end{cases}.
$$
Our grid is defined by $t_j = j/N$, $0 \leq j \leq N$. For the space $X_h$ we choose the B–spline basis corresponding to this grid. Our objective is to reconstruct the values of the different solutions $x_a$, $x_b$ and $x_c$ at the grid points $t_j$ and at the points $(t_{j+1} + t_j)/2$, $0 \leq j \leq N - 1$. In Figure 1 we give the results for $N = 25$ and $N = 50$, when the exact right hand side $y$ is used.

In Figure 2, we show the reconstruction results for the linear operator and perturbed data. The system is solved for a right hand side $y_e$ generated by adding a $1\%$ random noise to the original $y$, i.e., $|y_j - y_{e,j}| \leq \frac{1}{100} y_j$.

Next we analyze the reconstruction error at the point $t = \frac{1}{2}$ for exact data and the functions

$$x_a(t) = 2t \quad \text{and} \quad x_b(t) = \begin{cases} 2t & , t \leq 1/2 \\ 2 - 2t & , t \geq 1/2 \end{cases}.$$ 

Analyzing Figure 3 we observe that the reconstruction is somehow better for even values of $N$. This can be explained by the existence of a B–spline
centered at the point \( t = \frac{1}{2} \) in the \( X_h \)-basis. A consequence of this is that the functional \( \delta(\cdot - \frac{1}{2}) \) will be better approximated in \( X_h \) if \( N \) is even.

Next we will analyze the operator \( A \) in (19) for small values of \( \nu \). We use the same grid as before with \( N = 25 \) and try to reconstruct the polynomial \( x^2 \) at the points \( t_j \) and \( (t_{j+1} + t_j)/2 \) using exact data. The results are shown in Figure 4.

The next example in Figure 5 shows a reconstruction for \( \nu = 0.01 \) and exact data of the functions

\[
x_b(t) = \begin{cases} 
2t & , \ t \leq 1/2 \\
2 - 2t & , \ t \geq 1/2
\end{cases}
\]

and

\[
x_c(t) = \begin{cases} 
1 & , \ \frac{1}{4} \leq t \leq \frac{3}{4} \\
0 & , \ \text{otherwise}
\end{cases}
\]

4.2 The elliptic Cauchy–problem

We will analyze the elliptic Cauchy–problem in an annulus \( \Omega \) with inner radius \( \frac{1}{2} \) and outer radius 1. Let us take the linear operator \( A \) defined in §3.2. The problem we want to solve is, given a \( \mu \in H^{-\nu/2}(\Gamma_r) \), reconstruct the value of \( <\mu, \varphi> \), where \( \varphi \in H^{\nu/2}(\Gamma_r) \) is the solution of the equation \( A\varphi = f \). In order to generate consistent data \( f \), we solve the direct problems for \( \varphi_1(t) = (t - \frac{\pi}{2})^2 \) and \( \varphi_2(t) = \pi - 2|t - \frac{\pi}{2}| \), were \( t \in [0, \pi] \).

The formulation of this elliptic Cauchy–problem in \( \Omega \) involves an extra difficulty: we are not able to characterize the space \( Rg(A^\sharp) \). As we do want to have an element \( \mu \in Rg(A^\sharp) \), we solve first the direct problem \( \mu = A^\sharp \psi \) for a
\[ \psi \in H^{-1/2}(\Gamma_l). \] For this propose we chose \( \psi \equiv 1 \), solve the mixed boundary value problem

\[
\begin{cases}
\Delta v = 0, & \text{in } \Omega \\
v = 0, & \text{at } \Gamma_r \\
v_\nu = 0, & \text{at } \Gamma_i \\
v_\nu = \psi, & \text{at } \Gamma_l
\end{cases}
\]

and set \( \mu = v_\nu|_{\Gamma_r} \in H^{-1/2}(\Gamma_r) \).

According to the Backus–Gilbert strategy discussed in §3.2, the first thing to do is to solve the equation \(-A^*\psi = \mu\). To approximate the solution \( \psi \), we use the iterative method described in [MaKo] (this iterative method is also extensively discussed in [Le]). The approximations \( \psi_k \) are shown in Figure 6, were \( k \) represents the iteration index. The grid node 0 represents the point \((0, -1)\) and the grid node 32 the point \((0, 1)\) of \( \Gamma_r \).

What we do next is to compare the values \( \langle \mu, \varphi \rangle \) with \( \langle \psi, f \rangle - r_{a,b}(\psi) \). The results are shown in Table 1 (note that \( r_{a,b} \equiv 0 \) for \( \varphi = \varphi_2 \)).

| \( \varphi \) | \( \langle \mu, \varphi \rangle \) | \( \langle \psi, f \rangle \) | \( \langle \psi, f \rangle - r_{a,b}(\psi) \) | relative error |
|---|---|---|---|---|
| \( \varphi_1 \) | 3.99577 | 5.84496 | 3.89191 | 0.01776 |
| \( \varphi_2 \) | 0.52309 | 0.49903 | 0.49903 | 0.04599 |
Other numerical tests related to this specific Cauchy problem and to the validation of Theorem 4 can be found in [Le].

5 Final remarks and Conclusions

- The numerical experiments show that one can obtain good approximations for \( \mu = \delta \) in \( Rg(dA^*P_h^*) \) if \( A \) is the integral operator defined in §3.1 and the non-linearity in \( A \) is small. In the non-linear case we can always improve an approximation defining a new \( \tilde{x}^0 \) as the B–spline interpolation of the evaluated values \( x(t_j) \) and solving the new system

\[
<dA^*(\tilde{x}^0)P_h^*\varphi, x_j>_X = <\mu, x_j>_X.
\]

Comparable and related results can be found in [Ch], [Hu], [Ki], [Lo], [LM1,2], [ScBe] and [Sn].

- We should observe an unwanted Gibb’s phenomenon in Figures 1a, 2a and 4. An explanation for this fact is that \( \delta(\cdot) \in H^{-1/2-\varepsilon} \) for \( \varepsilon > 0 \) but \( H^s_0([0,1]) \not\subseteq H^s([0,1]) \) for \( s > 1/2 \). Thus the inner product \( <\delta, x>_{L^2} \) will be in duality only if the boundary conditions \( x(0) = x(1) = 0 \) are satisfied. The same phenomenon can also be observed in Figure 5b, were the lack of regularity of the solution \( x_c \) is now responsible for the effect.

- If the operator \( A \) is defined by the elliptic Cauchy problem in §3.2, we do not know, for an arbitrary set \( \Omega \), how to characterize the space \( Rg(A^\sharp) \). But if some argument guarantees that the \( \mu_i \)'s are in \( Rg(A^\sharp) \), we can proceed as in §4.2 and solve the Cauchy problems \( A^\sharp \psi_i = \mu_i \) once for each \( \mu_i \), in order to obtain the observations

\[
<\mu_i, \varphi> = <\psi_i, f>
\]

of \( \varphi \), every time we have a different set of data \( f \). Such \( \mu_i \)'s are also known in the literature as sentinels (see [Ch]).

- When we analyzed the Cauchy–problem, we tried first to evaluate the reconstruction with \( \mu = \delta \) and \( \mu \) a \( C^\infty \)–mollifier. Using classical arguments (see [GiTr]) one can prove that no analytical solution exists in such cases when \( \Omega \) has an analytical boundary. Our numerical results showed, that in this cases the equation \( A^\sharp \psi = \mu \) has no solutions.

- It is important to point out here the ill-posed nature of the involved reconstruction problems. Fredholm operators of the first kind are typically ill-posed [Gro]. What concern the elliptic Cauchy–problems, Hadamard elaborated an example with Cauchy data that converge uniformly to zero but the respective solutions become unbounded. The example follows:

\[
\begin{cases}
\Delta u_k = 0, & (x,y) \in \Omega = (0,1) \times (0,1) \\
u_k(x,0) = 0, & x \in (0,1) \\
\frac{\partial}{\partial y}u_k(x,0) = \varphi_k(x), & x \in (0,1)
\end{cases}
\]

were \( \varphi_k = (\pi k)^{-1} \sin(\pi k x) \). The respective solutions are

\[
u_k(x,y) = (\pi k)^{-2} \sinh(\pi k y) \sin(\pi k x).
\]
• Our numerical experiments were realized on an IBM RISC 6000/250 Work Station. It took some seconds to generate and solve the systems in §4.1 for \( N = 50 \). To evaluate the first 100 steps of the iterative method, in order to solve the Cauchy–Problem in §4.2, we needed about 30 minutes CPU-time (we used the finite element method on a grid with \( \simeq 8000 \) nodes to solve each mixed BVP involved on the iterative method).

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