HITTING PROBABILITIES OF GAUSSIAN RANDOM FIELDS AND
COLLISION OF EIGENVALUES OF RANDOM MATRICES

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Abstract. Let \( X = \{X(t), t \in \mathbb{R}^N\} \) be a centered Gaussian random field with values in \( \mathbb{R}^d \) satisfying certain conditions and let \( F \subset \mathbb{R}^d \) be a Borel set. In our main theorem, we provide a sufficient condition for \( F \) to be polar for \( X \), i.e. \( \mathbb{P}(X(t) \in F \text{ for some } t \in \mathbb{R}^N) = 0 \), which improves significantly the main result in Dalang et al [7], where the case of \( F \) being a singleton was considered. We provide a variety of examples of Gaussian random field for which our result is applicable. Moreover, by using our main theorem, we solve a problem on the existence of collisions of the eigenvalues of random matrices with Gaussian random field entries that was left open in Jaramillo and Nualart [14] and Song et al [21].

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1. Introduction

This paper is motivated by a problem on the existence of collision of the eigenvalues of a random matrix with Gaussian random field entries that has been left open by Jaramillo and Nualart [14] and Song et al [21]. We start by describing briefly some history and existing results on the aforementioned problem. In the celebrated work [11], Dyson introduced independent Ornstein-Uhlenbeck processes to a Hermitian matrix as its entries and showed that the system of eigenvalue processes models the so-called time-dependent Coulomb gas. Later on, it was shown that the eigenvalue processes never collide almost surely (see, e.g., [20]). For a symmetric matrix with independent Brownian motion entries, its eigenvalues do not collide for almost all trajectories and satisfy a system of the Itô stochastic differential equations with

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non-smooth diffusion coefficients. The process formed by the ordered eigenvalues of the symmetric Brownian motion matrix is now known as Dyson’s non-colliding Brownian motion (see, e.g., [1, 18, 20] for more information). If the matrix entries are fractional Brownian motions with Hurst parameter $H$, Nualart and Pérez-Abreu [19] proved that, when $H \in (\frac{1}{2}, 1)$, the eigenvalue processes do not collide by using the stochastic calculus with respect to Young integrals. When $H \in (0, \frac{1}{2})$, Jaramillo and Nualart [14] identified the collision probability of the eigenvalues with the hitting probability of Gaussian fields and, as a consequence, obtained a sufficient condition and a necessary condition for the positivity of the collision probability.

The result of [14] was recently extended to the collision probability of multiple eigenvalues by Song et al [21] who also obtained the Hausdorff dimension of the set of collision times. The methodology based on hitting probabilities used in [14, 21], which first appeared in McKean [18, Section 4.9] in the proof of the non-collision property for Dyson’s Brownian motion, can deal with the collision problem for the eigenvalues of random matrices with more general Gaussian random field entries including fractional Brownian motion with multidimensional indices and the Brownian sheet. One of the key ingredients in [14, 21] is the result in [2] on hitting probability of Gaussian random fields. While [2] is useful for determining whether $k$ eigenvalues of a real symmetric random matrix with Gaussian random field entries may collide or not for the cases $\sum_{j=1}^{N} \frac{1}{H_{j}} > (k+2)(k-1)/2$ and $\sum_{j=1}^{N} \frac{1}{H_{j}} < (k+2)(k-1)/2$ under the setting of [14, 21] (see Section 4 below), it does not provide any useful information when $\sum_{j=1}^{N} \frac{1}{H_{j}} = (k+2)(k-1)/2$, which is referred to as the critical dimension case for the collision problem. In this case, the problem on the existence of collision of the eigenvalues of a real symmetric (or complex Hermitian) random matrix with Gaussian random field entries has been left open by Jaramillo and Nualart [14] and Song et al [21].

In this paper, we solve this problem by first establishing a hitting probability result that is stronger than that in [2] for a large class of Gaussian random fields. More specifically, let $X = \{X(t), t \in \mathbb{R}^{N}\}$ be a centered Gaussian random field with values in $\mathbb{R}^{d}$ (for brevity, $X$ is called an $(N, d)$-random field) that satisfies the general assumptions in Dalang et al [7]. We derive in Theorem 2.3 a sufficient condition for a Borel set $F \subset \mathbb{R}^{d}$ to be polar for $X$, i.e., $\mathbb{P}(X(t) \in F \text{ for some } t \in \mathbb{R}^{N}) = 0$ in terms of a condition related to the upper Minkowski dimension of $F$. This theorem improves significantly Theorem 2.6 in Dalang et al [7], where the case of $F$ being a singleton was considered, and is applicable to solutions of stochastic partial differential equations (SPDEs).

The method for proving Theorem 2.3 is based on a refined covering argument. Compared with [2] and other related references for hitting probabilities of Gaussian random fields and solutions to SPDEs such as [3–5, 9, 10, 13, 27], the method for constructing the covering sets in this paper is significantly different. In [2] and the other references, the authors covered the inverse image $\{t \in I : X(t) \in F\}$, where $I \subset \mathbb{R}^{N}$ is a compact interval, by balls whose sizes are determined by $F$ and the largest global oscillation of $X$ on $I$. Consequently, these coverings are quite coarse and the covering argument fails if the dimension of $F$ is critical for the polarity problem for $X$ (e.g., $\dim F = d - \frac{N}{2}$ when $X$ is an $(N, d)$-fractional Brownian motion of index $H$). In the present paper, we construct a random covering for $\{t \in I : X(t) \in F\}$ by using balls whose sizes match the smallest local oscillation of $X$ with very large probability, see Proposition 2.6 below and the proof of Theorem 2.3. Our covering argument is originated from Talagrand [22, 23] and extends the method in [7].

The rest of this paper is organized as follows. In Section 2, we study the hitting probability of Gaussian random fields under the general setting of Dalang et al [7]. The main result is
Theorem 2.3, which provides a sufficient condition related to the upper Minkowski dimension for a Borel set \( F \subset \mathbb{R}^d \) to be polar for \( X \). In Section 3, we give some examples of Gaussian random fields that satisfy the conditions imposed in Section 2. In particular, we show that the solutions of the systems of linear stochastic heat and wave equations with a Gaussian noise that is white in time and colored in space satisfy the conditions of Theorem 2.3. This allows us to strengthen the results in [7] and prove the polarity of a class of sets with critical dimension for the solutions of these SPDEs. In Section 4, we apply the main result Theorem 2.3 to study the collision problem for the eigenvalues of random matrices with Gaussian random field entries and prove that there is no collision of \( k \) eigenvalues of the real symmetric random matrices when \( \sum_{j=1}^{N} \frac{1}{\pi_j} = (k + 2)(k - 1)/2 \). This solves a problem that was left open in [14, 21].

2. Hitting probabilities in critical dimension

Let \( X := \{X(t) = (X_1(t), \ldots, X_d(t)), t \in \mathbb{R}^N\} \) be a centered continuous \( \mathbb{R}^d \)-valued Gaussian random field defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). In this section, we study the hitting probabilities of \( X \) in critical dimension in a general setting of Dalang et al. [7].

First we recall the following definition of an \( \mathbb{R}^d \)-valued Gaussian noise on \( \mathbb{R}_+ \).

**Definition 2.1.** Let \( \nu \) be a Borel measure on \( \mathbb{R}_+ \), and let \( A \mapsto W(A) \) be a set function defined on \( \mathcal{B}(\mathbb{R}_+) \) with values in \( L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \) such that for each \( A, W(A) \) is a centered normal random vector with values in \( \mathbb{R}^d \) and covariance matrix \( \nu(A)I_d \). Assume that \( W(A \cup B) = W(A) + W(B), \) and \( W(A) \) and \( W(B) \) are independent whenever \( A \cap B = \emptyset \). Then the set function \( A \mapsto W(A) \) is called an \( \mathbb{R}^d \)-valued Gaussian noise with control measure \( \nu \).

As in [7], we assume that the component processes \( X_1, \ldots, X_d \) of the \((N, d)\)-random field \( X \) are i.i.d.\(^1\) For finite constants \( c_j < d_j \ (j = 1, \ldots, N) \), let

\[
I := \prod_{j=1}^{N} [c_j, d_j]
\]

be a compact interval in \( \mathbb{R}^N \). Denote \( I(\epsilon) := \prod_{j=1}^{N} (c_j - \epsilon, d_j + \epsilon) \).

We impose the following assumption on the Gaussian random field \( X \), which is the same as [7, Assumption 2.1].

(A1) There is a Gaussian random field \( \{W(A, t) : A \in \mathcal{B}(\mathbb{R}_+), t \in \mathbb{R}^N\} \) and \( \epsilon_0 > 0 \) satisfying the following two conditions.

(a1) For all \( t \in I(\epsilon_0) \), \( A \mapsto W(A, t) \) is an \( \mathbb{R}^d \)-valued Gaussian noise with a control measure \( \nu_t \) such that \( W(\mathbb{R}_+, t) = X(t) \) and when \( A \cap B = \emptyset \), \( W(A, \cdot) \) and \( W(B, \cdot) \) are independent.

(a2) There exist constants \( a_0 \geq 0, c_0 > 0, \gamma_j > 0, j = 1, \ldots, N \), such that for all \( a_0 \leq a < b \leq +\infty \) and all \( s := (s_1, \ldots, s_N) \), \( t := (t_1, \ldots, t_N) \in I(\epsilon_0) \),

\[
\|W((a, b), s) - X(s) - W([a, b), t) + X(t)\|_{L^2} \leq c_0 \left[ \sum_{j=1}^{N} a^{\gamma_j} |s_j - t_j| + b^{-1} \right],
\]

\(^1\)While the independence of the component processes of \( X \) plays an important role in this paper, the condition for them to be identically distributed can be relaxed, see Section 4 for an example.
The following lemma is from [7, Proposition 2.2]. It enables us to bound the canonical metric on \( L^{(\epsilon_0)} \) induced by \( \|X(s) - X(t)\|_{L^2} \) by using the metric \( \Delta \).

**Lemma 2.2.** Under Assumption (A1), for all \( s, t \in L^{(\epsilon_0)} \) with \( \Delta(s, t) \leq \min\{a_0^{-1}, 1\} \), we have

\[
\|X(s) - X(t)\|_{L^2} \leq 4c_0 \Delta(s, t).
\]

We further impose the following two assumptions on \( X \), which are aslo stated in [7, Assumption 2.4].

(A2) There exists a constant \( d_0 > 0 \), such that \( \|X_i(t)\|_{L^2} \geq d_0 \) for all \( t \in L^{(\epsilon_0)} \) and all \( 1 \leq i \leq N \).

(A3) There exists a constant \( \rho_0 > 0 \) with the following property. For \( t \in I \), there exist \( t' = t'(t) \in L^{(\epsilon_0)} \), \( \delta_j = \delta_j(t) \in (\alpha_j, 1] \) for \( 1 \leq i \leq N \) (recalling that \( \alpha_j \)'s are given in (2.1)), and \( C = C(t) > 0 \), such that

\[
\left| \mathbb{E} \left[ X_i(t') (X_i(s) - X_i(\bar{s})) \right] \right| \leq C \sum_{j=1}^{N} |s_j - \bar{s}_j|^{\delta_j},
\]

for all \( 1 \leq i \leq N \) and all \( s, \bar{s} \in L^{(\epsilon_0)} \) with \( \max\{\Delta(t, s), \Delta(t, \bar{s})\} \leq 2\rho_0 \).

Notice that (A2) is a non-degeneracy condition on \( X \) and that (A3) is a regularity condition which yields better path regularity of \( X \) than Lemma 2.2.

We now introduce the following important parameter:

\[
Q = \sum_{j=1}^{N} \frac{1}{\alpha_j}.
\]

It follows from [2, 27] that if Assumption (A1) holds, \( d \geq Q \), and \( F \subset \mathbb{R}^d \) has \((d - Q)\)-dimensional Hausdorff measure 0, then \( F \) is polar for \( X \). However, if the \((d - Q)\)-dimensional Hausdorff measure of \( F \) is not 0 (this is always the case if \( d = Q \) and \( F \neq \emptyset \)), it is in general not known whether \( F \) is polar for \( X \) or not. The special case of \( F = \{x\} \) when \( d = Q \) was solved by Dalang et al [7]. (For completeness, we mention that if \( d < Q \), then for every \( x \in \mathbb{R}^d \), \( X^{-1}(x) \neq \emptyset \) with positive probability, see [27, Theorem 7.1].)
The following is the main result of this section which provides a sufficient condition on \( F \subset \mathbb{R}^d \) such that \( X^{-1}(F) \cap I = \emptyset \) a.s. This result improves Theorem 2.6 in Dalang et al [7] and the results in [2, 27]. For a general Gaussian random field \( X \) (except the Brownian motion and the Brownian sheet which were completely solved by Kakutani [15] and by Khoshnevisan and Shi [16], respectively) the condition (2.4) on \( F \) is the weakest general condition so far for the polarity of \( F \).

**Theorem 2.3.** Let Assumptions (A1)-(A3) hold and suppose \( d \geq Q \), where \( Q \) is given in (2.3). Let \( F \subset \mathbb{R}^d \) be a bounded set that satisfies the following condition: There exist constants \( \theta \in [0, d - Q] \), \( C_F \in (0, \infty) \), and \( \kappa \in (0, (d - \theta)/Q) \) such that

\[
\lambda_d(F^{(r)}) \leq C_F r^{d-\theta} \left( \log \log(1/r) \right)^\kappa \quad (2.4)
\]

for all \( r > 0 \) small, where \( \lambda_d \) is the Lebesgue measure on \( \mathbb{R}^d \) and

\[
F^{(r)} = \left\{ x \in \mathbb{R}^d : \inf_{y \in F} |x - y| \leq r \right\}
\]

is the (closed) \( r \)-neighborhood of \( F \). Then \( X^{-1}(F) \cap I = \emptyset \) a.s.

Observe that (2.4) implies that the upper Minkowski (or box-counting) dimension of \( F \) is at most \( \theta \) (see, e.g., [12, Proposition 2.4]) and is satisfied by many bounded sets \( F \). The following corollary of Theorem 2.3 shows two cases that could not be handled by the hitting probability result in [2].

**Corollary 2.4.** Let Assumptions (A1)-(A3) hold and let \( F \subset \mathbb{R}^d \) be a bounded set.

(i) If \( d > Q \), the Hausdorff dimension of \( F \) equals \( d - Q \), and (2.4) holds with \( \theta = d - Q \) and a constant \( \kappa < 1 \), then \( X^{-1}(F) \cap I = \emptyset \) a.s.

(ii) If \( d = Q \) and \( F \) satisfies (2.4) with \( \theta = 0 \) and a constant \( \kappa < 1 \), then \( X^{-1}(F) \cap I = \emptyset \) a.s.

For any \( t_o \in \mathbb{R}^N \) and constant \( \eta > 0 \), denote by \( B_\eta(t_o) \) the closed ball in \( \mathbb{R}^N \) centered at \( t_o \) with radius \( \eta \) in the metric \( \Delta \), i.e.,

\[
B_\eta(t_o) = \left\{ t \in \mathbb{R}^N : \Delta(t, t_o) \leq \eta \right\}.
\]

(2.5)

For proving Theorem 2.3, it suffices to show \( X^{-1}(F) \cap B_\eta(t_o) = \emptyset \) a.s. for all \( t_o \in I \), where \( \eta > 0 \) is a small constant. Hence, we assume that \( t_o \in I \) is fixed throughout the rest of this paper. Let \( t'_o \) be the corresponding point given in Assumption (A3) which is also fixed, and let \( \eta \) be a fixed small positive number satisfying

\[
B_\eta(t_o) \subset I^{(\epsilon_0)} \quad \text{and} \quad \eta < \min \left\{ \frac{1}{2} \min\{1, a^{-1}_0\}, \rho_0 \right\},
\]

where we recall that the parameters \( \epsilon_0, a_0 \) are given in (A1) and \( \rho_0 \) in (A3).

For any \( t \in B_\eta(t_o) \), denote

\[
X^1(t) = X(t) - X^2(t), \quad X^2(t) = \mathbb{E}[X(t)|X(t'_o)].
\]

(2.6)

Since \( X \) is Gaussian, \( X^1 \) and \( X^2 \) are independent and for \( 1 \leq j \leq d \),

\[
X^2_j(t) = \frac{\mathbb{E}[X_j(t)X_j(t'_o)]}{\mathbb{E}[X_j(t'_o)^2]} X_j(t'_o).
\]

(2.7)
The following result, which is a generalization of [25, Lemma 4.2], implies that $X^2$ has better path regularity than $X$ noting that $\delta_j > \alpha_j$ by Assumption (A3). Therefore, $X^1$ can be viewed as a small perturbation of $X$.

**Lemma 2.5.** Let Assumptions (A2) and (A3) hold. Then, for any $s, t \in B_\eta(t_o)$,

$$|X^2(s) - X^2(t)| \leq K_1 |X(t'_o)| \sum_{j=1}^{N} |s_j - t'_j|^{\delta_j},$$

where $K_1 > 0$ is a finite constant only depending on $d_0$ and $t'_o$.

**Proof.** For $1 \leq i \leq N$, we have

$$|X^2_i(s) - X^2_i(t)| = \left| \frac{\mathbb{E}[X_i(s)X_i(t'_o)] - \mathbb{E}[X_i(t)X_i(t'_o)]}{\mathbb{E}[X_i(t'_o)^2]} X_i(t'_o) \right|$$

$$= \frac{\left| \mathbb{E}[X_i(s) - X_i(t)]X_i(t'_o) \right|}{\mathbb{E}[X_i(t'_o)^2]} |X_i(t'_o)|$$

$$\leq \frac{C}{d_0} \sum_{j=1}^{N} |s_j - t'_j|^{\delta_j} |X_i(t'_o)|,$$

where the first equality follows from (2.7) and the inequality follows from (A2) and (A3). $\square$

We recall the following [7, Proposition 2.3] which is analogous to [22, Proposition 4.1] and is the key ingredient for the construction of a random covering for the set $X^{-1}(F) \cap B_\eta(t_o)$.

**Proposition 2.6.** Let Assumption (A1) hold. Then there exist constants $K_2 \in (0, \infty)$ and $\delta_0 \in (0, 1]$ such that for any $r_0 \in (0, \delta_0)$ and $t \in I$,

$$\mathbb{P}\left\{ \exists r \in [r_0^2, r], \sup_{s \in I^{(r.o)}; \Delta(s,t) < r} |X(s) - X(t)| \leq K_2 r \left( \log \log \frac{1}{r} \right)^{-1/Q} \right\}$$

$$\geq 1 - \exp\left( -\frac{1}{\log \frac{1}{r_0}} \right).$$

The following lemma will also be used to construct the random covering in the sequel. More specifically, it will be used to control the size of the covering ball centered at $X(t)$ where the local oscillation of $X$ around $t$ is larger than what is given in Proposition 2.6.

**Lemma 2.7.** Let Assumption (A1) hold. Then there exists a constant $K_4 \in (0, \infty)$ such that

$$\mathbb{P}\left( \sup_{s,t \in I^{(r_0.o)}; \Delta(s,t) \leq \varepsilon} |X(s) - X(t)| \leq K_4 \varepsilon \sqrt{\log \frac{1}{\varepsilon}} \right) \geq 1 - \varepsilon,$$

for all $\varepsilon \in (0, \frac{1}{2})$.

Lemma 2.7 follows from Lemma 2.1 in Talagrand [22] (there is a misprint in Lemma 2.1: on the right-hand side of (2.1), $D$ should be $D^2$). For completeness, we provide a proof of Lemma 2.7 by invoking a useful inequality presented in [17, Chapter 11] for general stochastic processes. Recall that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *Young function* if it is convex and increasing such that $\psi(0) = 0, \lim_{x \to \infty} \psi(x) = \infty$. The Orlicz space $L_\psi = L_\psi(\Omega, \mathcal{A}, \mathbb{P})$ associated to a
Young function $\psi$ is the space of all real valued random variables $Y$ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathbb{E}[\psi(|Y|/c)] < \infty$ for some $c > 0$, and it is a Banach space under the norm
\[
\|Y\|_\psi = \inf \left\{ c > 0 : \mathbb{E}[\psi(|Y|/c)] \leq 1 \right\}.
\]
In particular, it is easy to verify that if we choose $\psi(x) = e^{x^2} - 1$, then for a centered Gaussian random variable $Y$, $\|Y\|_\psi = C\|Y\|_{L^2}$ for some universal constant $C > 0$.

Let $T$ be an index set and $d$ be a pseudo-metric on $T$. Consider a general stochastic process $Z = \{Z_t, t \in T\}$ such that $\|Z_t\|_\psi < \infty$ for all $t \in T$ and
\[
\|Z_s - Z_t\|_\psi \leq d(s, t), \text{ for all } s, t \in T. \tag{2.9}
\]
If we assume that the inverse function $\psi^{-1}$ of the Young function $\psi$ satisfies
\[
\psi^{-1}(xy) \leq C_\psi \left( \psi^{-1}(x) + \psi^{-1}(y) \right) \tag{2.10}
\]
for some constant $C_\psi$ depending only on $\psi$, then we have (see inequality (11.4) in [17]), for all $u > 0$,
\[
\mathbb{P} \left( \sup_{s, t \in T} |Z_s - Z_t| > 8C_\psi \left( u + \int_0^D \psi^{-1}(N(T, d; \varepsilon))d\varepsilon \right) \right) \leq \left( \psi(u/D) \right)^{-1}, \tag{2.11}
\]
where $N(T, d; \varepsilon)$ is the smallest number of open balls of radius $\varepsilon$ in the pseudo-metric $d$ which form a covering of $T$, and $D = \sup_{s, t \in T} d(s, t)$ is the diameter of $T$ in the pseudo-metric $d$.

**Proof of Lemma 2.7.** Denote the index set
\[
T_\varepsilon = \left\{ (t, \bar{t}) \in I^\varepsilon : \Delta(t, \bar{t}) \leq \varepsilon \right\}.
\]
To prove the desired result, we shall apply (2.11) with $\psi(x) = e^{x^2} - 1$ to the Gaussian random field $Z$ on $T_\varepsilon$ defined by
\[
Z(t) = Z(t, \bar{t}) = X(t) - X(\bar{t}), \ t = (t, \bar{t}) \in T_\varepsilon.
\]
Let $d_Z$ be the canonical metric on $T_\varepsilon$ induced by $Z$, i.e., for $s = (s, \bar{s}) \in T_\varepsilon$ and $t = (t, \bar{t}) \in T_\varepsilon$,
\[
d_Z(s, t) := \|Z(s) - Z(t)\|_{L^2} = \sqrt{\mathbb{E}\left[ (X(s) - X(\bar{s})) - (X(t) - X(\bar{t})) \right]^2}. \tag{2.12}
\]
Then by the triangle inequality, we have
\[
d_Z(s, t) \leq \left\{ \begin{array}{ll}
\|X(s) - X(\bar{s})\|_{L^2} + \|X(t) - X(\bar{t})\|_{L^2} \\
\|X(s) - X(t)\|_{L^2} + \|X(\bar{s}) - X(\bar{t})\|_{L^2}.
\end{array} \right. \tag{2.13}
\]
Thus by (2.13) and Lemma 2.2, the diameter $D$ of $T_\varepsilon$ in the metric $d_Z$ is at most $8c_\varepsilon$.

Next, for $\delta \in (0, \varepsilon)$, we count the number of balls of radius $\delta$ in $d_Z$ that are needed to cover $T_\varepsilon$. Recalling $Q = \sum_{j=1}^N \frac{1}{\alpha_j}$, we note that $T_\varepsilon$ in $\mathbb{R}^N$ can be covered by,
\[
(8c_\varepsilon N)^{2Q} \prod_{j=1}^N \frac{d_j - c_j}{\delta^{1/\alpha_j}} \prod_{j=1}^N \frac{\varepsilon}{\delta^{1/\alpha_j}}. \tag{2.15}
\]
rectangles in $\mathbb{R}^{2N}$ of the form $J_1 \times J_2$, where $J_1, J_2 \subset I^{(}\epsilon)\) are rectangles in $\mathbb{R}^N$ of the form $\prod_{j=1}^{N}[f_j, g_j]$ with $|f_j - g_j|^{o_j} = \delta/(8c_0N)$. Thus, for all $s = (s, \bar{s})$, $t = (t, \bar{t}) \in J_1 \times J_2$, we have by (2.14) and Lemma 2.2,
\[ d_Z(s, t) \leq \|X(s) - X(t)\|_{L^2} + \|X(\bar{s}) - X(\bar{t})\|_{L^2} \leq \delta, \]
and hence the diameters of such $J_1, J_2$ are not bigger than $\delta$ in the metric $d_Z$. Therefore, by (2.15) we have
\[ N(T_\delta, d_Z; \delta) \leq C \left( \frac{1}{\delta} \right)^Q \left( \frac{1}{\delta} \right)^Q, \]
where $C$ is a constant depending only on $c_0, Q, N$ and $I$.

Now, we apply (2.11). By choosing $\psi(x) = e^{x^2} - 1$ and the metric $d$ defined by (2.12), the conditions (2.9) and (2.10) are satisfied, and we have for any $u > 0$,
\[ \mathbb{P} \left( \sup_{s, t \in T_\delta} |Z(s) - Z(t)| > 8C \psi \left( u + \int_0^{8C\psi} \sqrt{\log(1 + N(T_\delta, d_Z; \delta))} \, d\delta \right) \right) \leq \frac{1}{\exp(u^2/64C^2\psi^2)} - 1. \]  
(2.16)

Note that
\[ \int_0^{8C\psi} \sqrt{\log N(T_\delta, d_Z; \delta)} \, d\delta \leq C\varepsilon \sqrt{\log \frac{1}{\varepsilon}}. \]
Then by choosing $t = (t, t)$ and $u = K\varepsilon \sqrt{\log \frac{1}{\varepsilon}}$ for some proper positive constant $K$, the desired inequality (2.8) can be obtained by (2.16).

We introduce some notations before constructing a random covering for $X^{-1}(F) \cap B_{\eta}(t_o)$.

For $k \in \mathbb{N}_+$, define the random subset $R_k$ of $B_{\eta}(t_o)$ as follows
\[ R_k = \left\{ t \in B_{\eta}(t_o) : \exists r \in [2^{-2k}, 2^{-k}], \sup_{s \in I^{(\eta)}} |X(s) - X(t)| \leq K_2r \left( \log \log \frac{1}{r} \right)^{-1/Q} \right\}. \]  
(2.17)

Proposition 2.6 implies that for sufficiently large $k$, $\mathbb{P}(t \in R_k) \geq 1 - e^{-\sqrt{k}}$ for all $t \in I$. This and Fubini’s theorem yield
\[ \mathbb{E}[\lambda_N(R_k)] = \mathbb{E} \left[ \int_{B_{\eta}(t_o)} 1_{t \in R_k} \lambda_N(dt) \right] = \int_{B_{\eta}(t_o)} \mathbb{P}(t \in R_k) \lambda_N(dt) \]
\[ \geq \lambda_N(B_{\eta}(t_o))(1 - e^{-\sqrt{k}}), \]  
(2.18)

where $\lambda_N$ is the Lebesgue measure on $\mathbb{R}^N$. Also noting that $\lambda_N(R_k) \leq \lambda_N(B_{\eta}(t_o))$ a.s., by the Markov inequality and (2.18), one can derive that
\[ \mathbb{P} \left( \lambda_N(R_k) < \lambda_N(B_{\eta}(t_o))(1 - e^{-\sqrt{k}}/2) \right) = \mathbb{P} \left( \lambda_N(B_{\eta}(t_o)) - \lambda_N(R_k) > \lambda_N(B_{\eta}(t_o))e^{-\sqrt{k}/2} \right) \]
\[ \leq \frac{\mathbb{E}[\lambda_N(B_{\eta}(t_o)) - \lambda_N(R_k)]}{\lambda_N(B_{\eta}(t_o))e^{-\sqrt{k}/2}} \leq e^{-\sqrt{k}/2}. \]

Thus, denoting
\[ \Omega_{k,1} := \left\{ \omega : \lambda_N(R_k) \geq \lambda_N(B_{\eta}(t_o))(1 - e^{-\sqrt{k}/2}) \right\}, \]
we have
\[\sum_{k=1}^{\infty} \mathbb{P}(\Omega_{k,1}^c) < \infty. \tag{2.19}\]

Recalling that \(\delta_j > \alpha_j\) for \(1 \leq j \leq N\), we can choose \(\beta \in (0, 1)\) such that \(\beta < \delta_j^{-1} - 1\) for all \(1 \leq j \leq N\). Let
\[\Omega_{k,2} = \{\omega : |X(t'_0)| \leq 2^k\beta\}.
\]
Since \(X(t'_0)\) is a Gaussian random variable, we have
\[\sum_{k=1}^{\infty} \mathbb{P}(\Omega_{k,2}^c) < \infty. \tag{2.20}\]

For \(k \in \mathbb{N}_+\), similar to \(R_k\) given in (2.17), we define the random set
\[R_k = \left\{t \in \mathcal{B}_\eta(t_o) : \exists r \in [2^{-2k}, 2^{-k}], \sup_{s \in \mathcal{I}(s) : \Delta(s,t) < r} |X^1(s) - X^1(t)| \leq K_3 r \left(\log \log \frac{1}{r}\right)^{-1/Q} \right\}, \tag{2.21}\]
where \(K_3 = NK_1 + K_2\). Note that on the event \(\Omega_{k,2}\), by the triangle inequality and Lemma 2.5, we have for \(t \in \mathcal{B}_\eta(t_o)\) and \(r \in [2^{-2k}, 2^{-k}]\) with \(k\) being sufficiently large,
\[
\begin{align*}
&\sup_{s \in \mathcal{I}(s) : \Delta(s,t) < r} |X(s) - X(t)| + \sup_{s \in \mathcal{I}(s) : \Delta(s,t) < r} |X^2(s) - X^2(t)| \\
&\leq \sup_{s \in \mathcal{I}(s) : \Delta(s,t) < r} |X(s) - X(t)| + K_1 |X(t'_0)| \sup_{s \in \mathcal{I}(s) : \Delta(s,t) < r} \sum_{j=1}^{N} |s_j - t_j|^{\delta_j} \\
&= \sup_{s \in \mathcal{I}(s) : \Delta(s,t) < r} |X(s) - X(t)| + K_1 N 2^k \beta \lambda_j \min_j \{\delta_j^{-1}\} \\
&\leq \sup_{s \in \mathcal{I}(s) : \Delta(s,t) < r} |X(s) - X(t)| + K_1 N 2^k \beta \lambda_j \min_j \{\delta_j^{-1}\} \beta_q - \beta. \tag{2.22}\end{align*}
\]
Thus, by (2.22) and (2.17), noting \(2^k r \leq 1\) and \(\min_j \{\delta_j^{-1}\} \beta > 1\), we have that for sufficiently large \(k\), \(R_k(\omega) \subset R^t_k(\omega)\) for all \(\omega \in \Omega_{k,2}\). Hence, \(\Omega_{k,1} \cap \Omega_{k,2} \subset \Omega_{k,3}\) for sufficiently large \(k\), where
\[\Omega_{k,3} := \left\{\omega : \lambda_N(R^t_k) \geq \lambda_N(\mathcal{B}_\eta(t_o))(1 - e^{-\sqrt{k}/2})\right\}.
\]
Thus, by (2.19) and (2.20), we have
\[\sum_{k=1}^{\infty} \mathbb{P}(\Omega_{k,3}^c) \leq \sum_{k=1}^{\infty} \mathbb{P}(\Omega_{k,1}^c) + \sum_{k=1}^{\infty} \mathbb{P}(\Omega_{k,2}^c) < \infty. \tag{2.23}\]

The following lemma will be needed in the construction of a covering of the inverse image. It provides a nested family of subsets that shares similar properties with dyadic cubes in
the Euclidean spaces, but is adapted to the anisotropic metric $\Delta$ in our setting. For ease of description, for every $q \in \mathbb{N}_+$ we will call the sets $\{I_{q,l}\}$ in Lemma 2.8 dyadic cubes of order $q$ in the metric $\Delta$.

**Lemma 2.8.** [6, Lemma 3.9] Let $T$ be a set in $\mathbb{R}^N$ equipped with the metric $\Delta$. There exist a constant $c_1 \in (0,1)$, a sequence $\{m_q : q \in \mathbb{N}_+\}$ of positive numbers, and a family $\{I_{q,l} : 1 \leq l \leq m_q, q \in \mathbb{N}_+\}$ of Borel subsets of $T$, such that

(i) For all $q \in \mathbb{N}_+$, $T = \bigcup_{l=1}^{m_q} I_{q,l}$.

(ii) For $q_1 \geq q_2$, $1 \leq l_1 \leq m_{q_1}$, $1 \leq l_2 \leq m_{q_2}$, either $I_{q_1,l_1} \cap I_{q_2,l_2} = \emptyset$ or $I_{q_1,l_1} \subset I_{q_2,l_2}$ holds.

(iii) For each $q,l$, there exists $x_{q,l} \in T$ such that $B_{c_1 2^{-q-1}}(x_{q,l}) \subset I_{q,l} \subset B_{2^{-q-1}}(x_{q,l})$ and $\{x_{q,l} : 1 \leq l \leq m_q\} \subset \{x_{q+1,l} : 1 \leq l \leq m_{q+1}\}$ for $q \in \mathbb{N}_+$.

Now we are ready to prove Theorem 2.3.

**Proof of Theorem 2.3.** As mentioned earlier, it is sufficient to prove $X^{-1}(F) \cap B_{\eta}(t_o) = \emptyset$ a.s., where $t_o \in I$ is fixed and $\eta > 0$ is a small constant. To this end, we construct a random covering for $X^{-1}(F) \cap B_{\eta}(t_o)$ by modifying the approach used in [7, 22, 25]. We choose $T = B_{\eta}(t_o)$ in Lemma 2.8, then there exists a family $\{I_{q,l} : 1 \leq l \leq m_q, q \in \mathbb{N}_+\}$ of dyadic cubes in the metric $\Delta$ such that $\{I_{q,l} : 1 \leq l \leq m_q\}$ forms a covering of $B_{\eta}(t_o)$. For every $t \in T$ and $n \geq 1$, let $C_n(t)$ be the unique dyadic cube of order $n$ which contains $t$. Then by (iii) of Lemma 2.8, for all $u,v \in C_n(t),$

$$\Delta(u,v) < 2^{-n}. \quad (2.24)$$

We call $C_n(t)$ a good dyadic cube of order $n$ if

$$\sup_{u,v \in C_n(t)} |X^1(u) - X^1(v)| \leq 8K_32^{-n}(\log \log 2^n)^{-1/Q}. \quad (2.25)$$

By Definition $(2.21)$ of $R'_k$, we see that for each $t \in R'_k$, there exists $r \in [2^{-2k}, 2^{-k}]$ such that

$$\sup_{s \in t : \Delta(s,t) \subset r} |X^1(s) - X^1(t)| \leq K_3r \left(\log \log \frac{1}{r}\right)^{-1/Q}. \quad (2.26)$$

Assume $2^{-n} \leq r < 2^{-n+1}$, and it is easy to verify that $k \leq n \leq 2k$. By the triangle inequality and (2.26), we have

$$\sup_{u,v \in C_n(t)} |X^1(u) - X^1(v)| \leq 2 \sup_{u \in C_n(t)} |X^1(u) - X^1(t)| \leq 2 \sup_{u : \Delta(u,t) \subset r} |X^1(u) - X^1(t)| \leq 2K_3r \left(\log \log \frac{1}{r}\right)^{-1/Q} \leq 4K_32^{-n} \left(\log \left(\log 2^n - \log 2\right)\right)^{-1/Q} \leq 8K_32^{-n} \left(\log \log 2^n\right)^{-1/Q}.$$ 

This implies that for $t \in R'_k$, $C_n(t)$ is a good dyadic cube of order $n$ for some $n \in [k, 2k]$. 
Denote by $V_n$ the union of good dyadic cubes of order $n$, and let $U_k = \bigcup_{n=k}^{2k} V_n$. Then clearly $R' \subseteq U_k$, and hence $(B \setminus U_k) \cap R' = \emptyset$. We also denote by $\mathcal{H}_1(k)$ the family of dyadic cubes contained in $U_k$. Note that $B \setminus U_k$ is contained in a union of dyadic cubes of order $2k$, none of which meets $R'$, and let $\mathcal{H}_2(k)$ denote the smallest family of such dyadic cubes. Recalling the definition of the dyadic cube in Lemma 2.8, the volume of the dyadic cube of order $2k$ is at least $C_{\text{Vol}}2^{-2kQ}$, where $C_{\text{Vol}}$ is a positive constant that only depends on $c_1, N, \alpha_1, \ldots, \alpha_N$. As the event $\Omega_{k,3}$ occurs, $\lambda_N(B) \leq \lambda_N(B) - \lambda_N(R') \leq \lambda_N(B) e^{-\sqrt{k}/2}$, and thus the number of cubes in $\mathcal{H}_2(k)$ is at most

$$C_1 e^{-\sqrt{k}/2} \lambda_N(B) \prod_{j=1}^N 2^{2k\alpha_j^{-1}} = C_1 2^{2kQ} e^{-\sqrt{k}/2} \lambda_N(B).$$

(2.27)

Here, $C_1$ is a positive constant depending on $C_{\text{Vol}}, N$ and $\alpha_1, \ldots, \alpha_N$.

Denote

$$\mathcal{H}(k) = \mathcal{H}_1(k) \cup \mathcal{H}_2(k).$$

Then $\mathcal{H}(k)$ is a random family of dyadic cubes of order $n$ for $k \leq n \leq 2k$ and clearly it only depends on $\{X^1(t), t \in B_\eta(t_\eta)\}$. Let

$$\mathcal{H} = \bigcup_{k=1}^{\infty} \mathcal{H}(k).$$

Then $\mathcal{H}$ is also $\Sigma_1$-measurable, where $\Sigma_1$ is the $\sigma$-field generated by $\{X^1(t), t \in B_\eta(t_\eta)\}$.

Now for any $A \in \mathcal{H}$, where $A$ is a dyadic cube of order $n$, define

$$r_A = \begin{cases} 8K_32^{-n} (\log \log 2)^{-1/Q}, & \text{if } A \in \mathcal{H}_1(k), \ k \leq n \leq 2k; \\ \frac{1}{2}K_42^{-n}\sqrt{n}, & \text{if } A \in \mathcal{H}_2(k), \ n = 2k, \end{cases}$$

(2.28)

where $K_4$ is the constant given by Lemma 2.7. For every $A \in \mathcal{H}$, we pick a distinguished point $t_A$ in $A$. Let

$$\Omega_A = \{d(X(t_A), F) \leq 2r_A\},$$

(2.29)

where $d(x, F) = \inf_{y \in F} |x - y|$. Denote

$$\mathcal{F}(k) = \{A \in \mathcal{H}(k) : \Omega_A \text{ occurs}\}.$$  

(2.30)

Define the event

$$\Omega_{k,4} = \left\{\omega : \text{for every dyadic cube } C_k \text{ of order } k, \sup_{s,t \in C_k} |X(s) - X(t)| \leq K_42^{-k}\sqrt{k}\right\}.$$  

For $s, t \in C_k$, $\Delta(s, t) \leq 2^{-k}$ by (2.24). Then we have

$$\sum_{k=1}^{\infty} \mathbb{P}(\Omega_{k,4}) = \sum_{k=1}^{\infty} \mathbb{P}\left(\omega : \exists \text{ dyadic cube } C_k \text{ of order } k, \sup_{s,t \in C_k} |X(s) - X(t)| > K_42^{-k}\sqrt{k}\right)$$

$$\leq \sum_{k=1}^{\infty} \mathbb{P}\left(\omega : \sup_{s,t \in C_k \cap \Delta(s, t) \leq 2^{-k}} |X(s) - X(t)| > K_42^{-k}\sqrt{k}\right) < \infty,$$

(2.31)

where the last inequality follows from Lemma 2.7.

Now define

$$\Omega_k = \Omega_{k,2} \cap \Omega_{k,3} \cap \Omega_{k,4}.$$  

(2.32)
Then by (2.20), (2.23) and (2.31), we have

\[ \sum_{k=1}^{\infty} P(\Omega_k^c) < \infty, \]

which, together with the Borel-Cantelli lemma, implies

\[ P \left( \liminf_{k \to \infty} \Omega_k \right) = 1. \]

We make the following two claims:

**Claim 1.** For \( k \) large enough, on the event \( \Omega_k \) defined in (2.32), \( F(k) \) covers \( X^{-1}(F) \cap B_\eta(t_o) \), recalling that \( F(k) \) is given in (2.30). That is, for \( k \) large enough, \( F(k) \) is a random covering of \( X^{-1}(F) \cap B_\eta(t_o) \) on \( \Omega_k \).

**Proof of Claim 1.** For any \( t \in X^{-1}(F) \cap B_\eta(t_o) \), \( t \) lies in a cube in \( \mathcal{H}_1(k) \cup \mathcal{H}_2(k) \). If \( t \in A \in \mathcal{H}_2(k) \), it follows directly from the definitions of \( \Omega_{2k_A} \) and \( r_A \) that

\[ d(X(t_A), F) \leq d(X(t_A), X(t)) \leq K_2 2^{-2k} \sqrt{2k} = 2r_A. \]

If \( t \in A \in \mathcal{H}_1(k) \) for some good dyadic cube \( A \) of order \( n \) with \( k \leq n \leq 2k \), then for \( s, r \in A \), \( |s_j - r_j| \leq 2^{-\alpha_j^{-1} n} \). Recalling that \( \beta < \min_{1 \leq j \leq N} \{ \delta_j \alpha_j^{-1} \} - 1 \), by the triangle inequality, the definition of good dyadic cubes and Lemma 2.5, for \( k \) large enough,

\[
\begin{align*}
    d(X(t_A), F) &\leq |X(t_A) - X(t)| \\
    &\leq |X^1(t_A) - X^1(t)| + |X^2(t_A) - X^2(t)| \\
    &\leq 8K_4 2^{-n} (\log 2^n)^{-1/Q} + \sum_{s,r \in A} |s_j - r_j| \delta_j \\
    &\leq r_A + K_1 2^{k\beta} \sum_{j=1}^{N} 2^{-\delta_j \alpha_j^{-1} n} \leq r_A + K_1 N 2^{k\beta} 2^{-1+\beta+\beta'} n \\
    &\leq r_A + K_1 N 2^{-n} 2^{-\beta'} n \leq 2r_A,
\end{align*}
\]

where \( \beta' = \min_{1 \leq j \leq N} \{ \alpha_j^{-1} \delta_j \} - 1 - \beta > 0 \). Thus, in both cases \( t \in A \in F(k) \).

**Claim 2.** If there exist constants \( \theta \in (0, d] \), \( \kappa \geq 0 \), and some positive constant \( C_F \) depending on \( F \) only, such that \( \lambda_\mathcal{A}(F\cap r) \leq C_F r^{d-\theta} (\log \log \frac{1}{r})^\kappa \) for all \( r > 0 \) small, then for any \( A \in \mathcal{H} = \bigcup_{k=1}^{\infty} \mathcal{H}(k) \),

\[ P(\Omega_A | \Sigma_1) \leq K_5 r_a^{d-\theta} \left( \log \log \frac{1}{r_a} \right)^\kappa \quad (2.33) \]

for some finite constant \( K_5 \). We remark that for each \( A \in \mathcal{H} \), there exists an integer \( k \) such that \( A \in \mathcal{H}(k) \) and, in this case, \( P(\Omega_A | \Sigma_1) = P(A \in F(k) | \Sigma_1) \) by the definition (2.30) of \( F(k) \).

**Proof of Claim 2.** For \( t \in B_\eta(t_o) \) and \( 1 \leq i \leq N \), it follows from (2.7) that

\[ X_i^2(t) = \left( 1 + \frac{E[(X_i(t) - X_i(t_o))X_i(t_o)]}{E[X_i(t_o)^2]} \right) X_i(t_o) = g_i(t) X_i(t_o). \]

Note that Assumptions (A2) and (A3) guarantee that \( g_i(t) \) is bounded away from 0 uniformly in \( t \in B_\eta(t_o) \) and \( 1 \leq i \leq N \). Besides, \( X_i(t_o) \) is a normal random variable. Hence, the joint probability density function of \( X^2(t) \) is uniformly bounded in \( t \in T \).
Therefore, for each $A \in \mathcal{H}$, by the independence of $X^1$ and $X^2$,
\[
\mathbb{P}(\Omega_A|\Sigma_1) = \mathbb{P}\left(\inf_{y \in F} |X(t_A) - y| \leq 2r_A \Big| \Sigma_1 \right)
\]
\[
= \mathbb{P}\left(\inf_{y \in F - X^1(t_A)} |X^2(t_A) - y| \leq 2r_A \Big| \Sigma_1 \right)
\]
\[
= \mathbb{P}\left(X^2(t_A) \in (F - X^1(t_A))^{(2r_A)} \Big| \Sigma_1 \right)
\]
\[
\leq K_5 r_A^{d-\theta} \left( \log \log \frac{1}{r_A} \right)^\kappa ,
\]
for some finite constant $K_5$. In deriving the last inequality, we have used the facts that the joint density of $X^2(t)$ is uniformly bounded in $t \in B_y(t_o)$ and
\[
\lambda_d\left((F - X^1(t_A))^{(2r_A)}\right) = \lambda_d\left(F^{(2r_A)}\right) \leq 2^{d-\theta} C_F r_A^{d-\theta} \left( \log \log \frac{1}{r_A} \right)^\kappa .
\]
This verifies (2.33).

Define the function $\phi$ on $(0, \infty)$ by
\[
\phi(s) = s^{Q-d+\theta} \left( \log \log \frac{1}{s} \right)^{\frac{d-\theta}{d} - \kappa} . \tag{2.34}
\]
Notice that, under the assumption $\theta \in [0, d-Q]$ and $\kappa \in [0, (d-\theta)/Q)$, we have $\lim_{s \to 0^+} \phi(s) = \infty$.

For any $A \in \mathcal{H}$, denote $D(A) = 2^{-n}$ if $A \in C_n$. Since $\Omega_{k,3}$ is $\Sigma_1$-measurable, for $k$ large enough, it follows from (2.33) in Claim 2 that
\[
\mathbb{E}\left[1_{\Omega_{k,3}} \sum_{A \in \mathcal{F}(k)} \phi(D(A)) \right] = \mathbb{E}\left[1_{\Omega_{k,3}} \sum_{A \in \mathcal{F}(k)} \phi(D(A)) \big| \Sigma_1 \right]
\]
\[
= \mathbb{E}\left[1_{\Omega_{k,3}} \sum_{A \in \mathcal{H}(k)} \mathbb{E}\left[1_{A \in \mathcal{F}(k)} \big| \Sigma_1 \right] \phi(D(A)) \right]
\]
\[
\leq K_5 \mathbb{E}\left[1_{\Omega_{k,3}} \sum_{A \in \mathcal{H}(k)} r_A^{d-\theta} \left( \log \log \frac{1}{r_A} \right)^\kappa \phi(D(A)) \right]
\]
\[
\leq K_5 \mathbb{E}\left[1_{\Omega_{k,3}} \sum_{A \in \mathcal{H}(k)} r_A^{d-\theta} \left( \log \log \frac{1}{r_A} \right)^\kappa D(A)^{Q-d+\theta} \left( \log \log \frac{1}{D(A)} \right)^{\frac{d-\theta}{d} - \kappa} \right]. \tag{2.35}
\]
Recalling the definition (2.28) of $r_A$, one can write
\[
r_A = \begin{cases} 
8K_3 D(A) \left( \log \log \frac{1}{D(A)} \right)^{-1/Q} , & \text{if } A \in \mathcal{H}_1(k), \\
\frac{1}{2} K_4 D(A) \left( \log \frac{1}{D(A)} \right)^{1/2} , & \text{if } A \in \mathcal{H}_2(k).
\end{cases}
\]
To deal with the sum $\sum_{A \in \mathcal{H}(k)}$, we will split it into the sum $\sum_{A \in \mathcal{H}_1(k)} + \sum_{A \in \mathcal{H}_2(k)}$. In order to avoid duplication, $\sum_{A \in \mathcal{H}_1(k)}$ only sums over all good dyadic cubes $A$ that are not included
in another good dyadic cube, and this is where we use the nested property of the cubes given by Lemma 2.8. For every $A \in \mathcal{H}_1(k)$, one can verify that
\[
 r_A^{-d-\theta} \left( \log \log \frac{1}{r_A} \right)^\kappa \frac{D(A)^{Q-d-\theta}}{\log \log \frac{1}{D(A)}} \leq K_6 D(A)^Q, \tag{2.36}
\]
where $K_6$ is a constant depending only on $K_4, d, \theta,$ and $\kappa$. Besides, recalling the definition of the dyadic cube in Lemma 2.8, the volume of the dyadic cube $A$ is at least $C_{Vol} D(A)^Q$. For every $A \in \mathcal{H}_2(k)$, one can verify that
\[
 r_A^{-d-\theta} \left( \log \log \frac{1}{r_A} \right)^\kappa \frac{D(A)^{Q-d-\theta}}{\log \log \frac{1}{D(A)}} \leq K_7 D(A)^Q \left( \log \log \frac{1}{D(A)} \right)^{(d-\theta)/2} \left( \log \log \frac{1}{D(A)} \right)^{(d-\theta)/Q}, \tag{2.37}
\]
where $K_7$ is a constant depending only on $K_4, d, \theta,$ and $\kappa$.

It follows from (2.35), (2.36), and (2.37) that for sufficiently large $k$ we have
\[
 \mathbb{E} \left[ \Omega_{k,3} \sum_{A \in \mathcal{F}(k)} \phi(D(A)) \right]
 \leq K_5 K_6 \mathbb{E} \left[ \sum_{A \in \mathcal{H}_1(k)} \frac{D(A)^Q}{A} \right]
 + K_5 K_7 \mathbb{E} \left[ \Omega_{k,3} \sum_{A \in \mathcal{H}_2(k)} D(A)^Q \left( \log \log \frac{1}{D(A)} \right)^{(d-\theta)/2} \left( \log \log \frac{1}{D(A)} \right)^{(d-\theta)/Q} \right]
 \leq K_8 \lambda_N(\mathcal{B}_\eta(t_o)) + K_5 K_7 \mathbb{E} \left[ \Omega_{k,3} \sum_{A \in \mathcal{H}_2(k)} 2^{-2kQ} (2k)^{(d-\theta)/2} (\log(2k))^{(d-\theta)/Q} \right]
 \leq K_9 \lambda_N(\mathcal{B}_\eta(t_o)),
\]
where $K_9$ is a constant and the last inequality follows from the upper bound (2.27) of the size of $\mathcal{H}_2(k)$ on $\Omega_{k,3}$.

Now, let $\Omega = \lim_{k \to \infty} \Omega_k$ and notice that $\mathbb{P}(\Omega) = 1$. We consider the following quantity related to $X^{-1}(F) \cap \mathcal{B}_\eta(t_o)$:
\[
 \phi-m(X^{-1}(F) \cap \mathcal{B}_\eta(t_o)) := \lim_{k \to \infty} \sum_{A \in \mathcal{F}(k)} \phi(D(A)).
\]

To put this quantity in perspective, we mention that, when $Q - d + \theta > 0$ (we do not consider this case in the present paper), it follows from Claim 1 that $\phi-m(X^{-1}(F) \cap \mathcal{B}_\eta(t_o))$ gives an upper bound for the $\phi$-Hausdorff measure of $X^{-1}(F) \cap \mathcal{B}_\eta(t_o)$.

By Fatou’s lemma and (2.38), we have
\[
 \mathbb{E} \left[ \phi-m \left( X^{-1}(F) \cap \mathcal{B}_\eta(t_o) \right) \right] \leq \mathbb{E} \left[ \lim_{k \to \infty} \sum_{A \in \mathcal{F}(k)} \phi(D(A)) \Omega_k \right]
 \leq \lim_{k \to \infty} \mathbb{E} \left[ \sum_{A \in \mathcal{F}(k)} \phi(D(A)) \Omega_{k,3} \right] < \infty.
\]
Thus, we have shown that $\phi(m(X^{-1}(F) \cap B_n(t_o))) < \infty$ almost surely. Observe that, if $\theta \leq d - Q$, we have $\lim_{s \to +\infty} \phi(s) = \infty$. This together with the finiteness of $\phi(m(X^{-1}(F) \cap B_n(t_o)))$ forces $X^{-1}(F) \cap B_n(t_o)$ to be an empty set a.s. This finishes the proof of Theorem 2.3.

3. Examples of Gaussian random fields

Theorem 2.3 obtained in Section 2 is applicable to a broad class of Gaussian random fields, including multiparameter fractional Brownian motions, fractional Brownian sheets, and the solutions of systems of linear stochastic heat and wave equations. In this section, we verify that these examples satisfy Assumptions (A1)-(A3) imposed in Theorem 2.3.

3.1. Multiparameter fractional Brownian motions. A multiparameter fractional Brownian motion (or fractional Brownian field) with Hurst parameter $H \in (0, 1)$ is a centered $\mathbb{R}^d$-valued Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with continuous sample paths and covariance given by

$$E[X_j(s)X_k(t)] = \delta_{j,k} \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s - t|^{2H}),$$

where $| \cdot |$ is the Euclidean norm in $\mathbb{R}^N$ and $\delta_{j,k}$ is the Kronecker symbol.

Regarding the hitting probabilities of $X$, Testard [24] and Xiao [26] have proved the following results:

$$C_{d-N/H}(F) > 0 \Rightarrow \mathbb{P}\{X(I) \cap F \neq \emptyset\} > 0 \Rightarrow \mathcal{H}_{d-N/H}(F) > 0,$$

where $C_{\alpha}$ denotes the Bessel-Riesz capacity of order $\alpha$ and $\mathcal{H}_\alpha$ denotes the $\alpha$-dimensional Hausdorff measure. Dalang, Mueller and Xiao [7] have also discussed the polarity of points and proved that $X$ does not hit points in the critical dimension $d = N/H$.

Recall that the fractional Brownian motion $X$ admits the following integral representation (see [7, 23]):

$$X(t) = C \int_{\mathbb{R}^N} \frac{1 - \cos(t \cdot \xi)}{||\xi||^{H+N/2}} M_1(d\xi) + C \int_{\mathbb{R}^N} \frac{\sin(t \cdot \xi)}{||\xi||^{H+N/2}} M_2(d\xi),$$

where $M_1$ and $M_2$ are independent Gaussian white noises on $\mathbb{R}^N$ with Lebesgue control measure, and $C$ is a suitable constant. With this representation, we can define

$$W(A, t) = C \int_{||\xi||^H \in A} \frac{1 - \cos(t \cdot \xi)}{||\xi||^{H+N/2}} M_1(d\xi) + C \int_{||\xi||^H \in A} \frac{\sin(t \cdot \xi)}{||\xi||^{H+N/2}} M_2(d\xi)$$

for $A \in \mathcal{B}(\mathbb{R}_+)$ and $t \in \mathbb{R}^N$. In [7], it is shown in the proof of Theorem 6.1 that our condition (A1) is satisfied with $a_0 = 0$ and $\gamma_j = H^{-1} - 1$ for $j = 1, \ldots, N$. For every $t \in \mathbb{R}^N \setminus \{0\}$, the control measure $\nu_t$ of $W(\cdot, t)$ is given by

$$\nu_t(A) = 2C^2 \int_{||\xi||^H \in A} \left(1 - \cos(t \cdot \xi)\right) \frac{d\xi}{||\xi||^{2H+N}}.$$ 

Also, on any compact rectangle $I \subset \mathbb{R}^N \setminus \{0\}$, (A2) and (A3) are satisfied with $\delta_j = 1$ for all $j$. Therefore, our Theorem 2.3 applies to the fractional Brownian motion with $Q = N/H$ and improves Theorem 6.1 of [7].
3.2. Fractional Brownian sheets. A fractional Brownian sheet with Hurst parameters $H_1, \ldots, H_N \in (0, 1)$ is a centered, continuous, $\mathbb{R}^d$-valued Gaussian random field $\{X(t), t \in \mathbb{R}_+^N\}$ with covariance

$$E[X_j(s)X_k(t)] = \delta_{j,k} \prod_{i=1}^N \frac{1}{2} \left( s_i^{2H_i} + t_i^{2H_i} - |s_i - t_i|^{2H_i} \right).$$

When $H_i = 1/2$ for all $i$, $X$ is the Brownian sheet. In this case, the result of Khoshnevisan and Shi [16] provides a complete characterization for the polar sets: $F \subset \mathbb{R}^d$ is polar if and only if $C_{d-2N}(F) = 0$. It has been an open problem whether this result extends to fractional Brownian sheets.

In [6, Section 5.1], it is shown that the fractional Brownian sheet $X$ has the following representation:

$$X(t) = C \sum_{p \in \{0, 1\}^N} \int_{\mathbb{R}^N} \prod_{j=1}^N f_{p_j}(t_j \xi_j) \prod_{j=1}^N [\xi_j]^{H_j + 1/2} M_p(d\xi),$$

where $f_0(x) = 1 - \cos(x)$, $f_1(x) = \sin(x)$, $M_p, p \in \{0, 1\}^N$, are i.i.d. $\mathbb{R}^d$-valued Gaussian white noises on $\mathbb{R}^N$, and $C$ is a suitable constant.

Let $I = \prod_{j=1}^N [c_j, d_j]$ be a compact rectangle, where $0 < c_j < d_j < \infty (j = 1, \ldots, N)$. Set

$$W(A, t) = C \sum_{p \in \{0, 1\}^N} \int_{\max_j |\xi_j|^{H_j} \leq A} \prod_{j=1}^N f_{p_j}(t_j \xi_j) \prod_{j=1}^N [\xi_j]^{H_j + 1/2} M_p(d\xi).$$

By Lemma 5.1 of [6], our condition (A1) is satisfied with $a_0 = 0$ and $\gamma_j = H_j^{-1} - 1$ for $j = 1, \ldots, N$. It is clear that (A2) is satisfied with $d_0 = \prod_{j=1}^N c_j^{2H_j} > 0$. Also, Lemma 5.2 of [6] implies that (A3) is satisfied with $\delta_j = \min\{2H_j, 1\}$ for $j = 1, \ldots, N$. Therefore, our Theorem 2.3 and Corollary 2.4 apply to the fractional Brownian sheet with $Q = \sum_{j=1}^N H_j^{-1}$.

3.3. Systems of linear stochastic heat equations. For systems of linear and nonlinear stochastic heat equations, upper and lower bounds for hitting probabilities have been obtained by Dalang, Khoshnevisan and Nualart [3-5]. Those bounds allow us to determine the polarity of $F \subset \mathbb{R}^d$ in non-critical dimensions. For the solution of the linear stochastic heat equation (3.1) below, Dalang, Mueller and Xiao [7, Theorem 7.1] have proved that points are polar in its critical dimension, which is $d = (4 + 2N)/(2 - \beta)$, where $\beta \in (0, 2 \wedge N)$ is the constant in (3.2). We can now use our main theorem to extend the latter result for a class of non-singleton sets $F$.

Let $u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ be the solution of the following system of linear stochastic heat equations on $\mathbb{R}_+ \times \mathbb{R}^N$:

$$\begin{align*}
\frac{\partial}{\partial t} u_j(t, x) &= \Delta u_j(t, x) + \hat{M}_j(t, x), \quad j = 1, \ldots, d, \\
u_j(0, x) &= 0.
\end{align*}$$

We assume that $\hat{M}_1, \ldots, \hat{M}_d$ are i.i.d. Gaussian noises that are white in time and spatially homogeneous with spatial covariance given by the Riesz kernel, i.e., formally,

$$E[\hat{M}_j(t, x) \hat{M}_j(s, y)] = \delta(t - s) |x - y|^{-\beta}, \quad 0 < \beta < 2 \wedge N. \tag{3.2}$$

If $N = 1$, it is also possible to take $\hat{M}_1, \ldots, \hat{M}_d$ to be i.i.d. space-time white noises (and set $\beta = 1$ in this case).
Let $\tilde{M}(dt, d\xi)$ be a $\mathbb{C}^d$-valued space-time white noise, i.e., $\text{Re}\, \tilde{M}$ and $\text{Im}\, \tilde{M}$ are independent space-time white noises. Define the Gaussian random field $X = \{X(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N\}$ by

$$X(t, x) = \text{Re} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \frac{e^{-i\xi \cdot x} e^{-i\tau t} - e^{-i\|\xi\|^2}}{|\xi|^2 - i\tau} \, \tilde{M}(dt, d\xi).$$

In [7, Section 7], it is shown that $X$ has the same law as the solution $u = \{u(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N\}$ of (3.1). Also, it is shown that, for any compact rectangle $I$ in $(0, \infty) \times \mathbb{R}^N$, by setting

$$W(A, t, x) = \text{Re} \int_{|\tau| = 1} \int_{|\xi| = 1} \frac{e^{-i\xi \cdot x} e^{-i\tau t} - e^{-i|\xi|^2}}{|\xi|^2 - i\tau} \, \tilde{M}(dt, d\xi),$$

our condition (A1) is satisfied with $\alpha_j = \alpha_j - 1$ for $j = 1, \ldots, 1 + N$, where

$$\alpha_1 = \frac{2 - \beta}{4} \quad \text{and} \quad \alpha_2 = \cdots = \alpha_{1+N} = \frac{2 - \beta}{2},$$

and (A2) and (A3) are satisfied with $\delta_j = 1$ for all $j = 1, \ldots, 1 + N$. Therefore, our Theorem 2.3 is applicable to the solution $u$ of (3.1) with $Q^2 = (4 + 2N)/(2 - \beta)$.

The theorem can also be applied to systems of linear stochastic heat equations with non-constant coefficients. Let $v(t, x) = (v_1(t, x), \ldots, v_d(t, x))$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$, be the solution of

$$\begin{cases}
\frac{\partial}{\partial t} v_j(t, x) = \Delta v_j(t, x) + \sigma_j(t, x) \tilde{M}_j(t, x), & j = 1, \ldots, d, \\
v_j(0, x) = 0,
\end{cases}$$

(3.3)

where $\tilde{M}_1, \ldots, \tilde{M}_d$ are Gaussian noises as in (3.1), and for each $j = 1, \ldots, N$, $\sigma_j : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ is a non-random continuous function such that for all $T > 0$, there exist $0 < c_T < C_T < \infty$ such that $c_T \leq \sigma_j(t, x) \leq C_T$ for all $(t, x) \in [0, T] \times \mathbb{R}^N$. Define the Gaussian random field $X = \{X(t, x) = (X_1(t, x), \ldots, X_d(t, x)), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N\}$ by

$$X_j(t, x) = \text{Re} \int_{\mathbb{R}} \int_{\mathbb{R}^N} (\Phi_{t,x} * \tilde{\sigma}_j)(\tau, \xi) \frac{\tilde{M}(d\tau, d\xi)}{|\xi|^{(N-\beta)/2}},$$

where $\tilde{\sigma}_j$ is the Fourier transform of $\sigma_j$ in the variables $(t, x)$ and

$$\Phi_{t,x}(\tau, \xi) = e^{-i\xi \cdot x} e^{-i\tau t} - e^{-i|\xi|^2}/|\xi|^2 - i\tau.$$

In [7, Section 8], it is shown that $X$ has the same law as the solution $v = \{v(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N\}$ of (3.3), and, in addition, if the functions $\sigma_j$ satisfy Assumption 8.1 in [7], then our conditions (A1)-(A3) are satisfied on any compact rectangle $I \subset (0, \infty) \times \mathbb{R}^N$. In this case, our Theorem 2.3 is applicable to the solution $v$ of (3.3).

### 3.4. Systems of linear stochastic wave equations

For a class of nonlinear hyperbolic SPDEs driven by space-time white noise, Dalang and Nualart [8] have given a complete characterization for a set to be polar. For systems of linear and nonlinear stochastic wave equations driven by white noise or colored noise, the polarity of sets in non-critical dimensions have been studied by Dalang and Sanz-Solé [9, 10]. The polarity of points in the critical dimension for the solution of the linear stochastic wave equation (3.4) below has been solved by Dalang et al. [7, Theorem 9.1], and we can now improve their results for non-singleton sets.
Consider the solution \( u(t, x) = (u_1(t, x), \ldots, u_d(t, x)) \) of the following system of linear stochastic wave equations on \( \mathbb{R}_+ \times \mathbb{R}^N \):

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} u_j(t, x) &= \Delta u_j(t, x) + \dot{M}_j(t, x), \quad j = 1, \ldots, d, \\
u_j(0, x) &= 0, \quad \frac{\partial}{\partial t} u_j(0, x) = 0,
\end{aligned}
\tag{3.4}
\]

where \( \dot{M}_1, \ldots, \dot{M}_d \) are Gaussian noises as in (3.1) with \( N = 1 = \beta \), or \( N \geq 2 \) and \( 1 \leq \beta < 2 \wedge N \).

Let \( \dot{M}(d\tau, d\xi) \) be a \( \mathbb{C}^d \)-valued space-time white noise, and \( \{X(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N\} \) be the Gaussian random field defined by

\[
X(t, x) = \text{Re} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \frac{e^{-i\xi \cdot x - i\tau t}}{2|\xi|} \left( \frac{1 - e^{i\tau(\tau + |\xi|)}}{\tau + |\xi|} - \frac{1 - e^{i\tau(\tau - |\xi|)}}{\tau - |\xi|} \right) \dot{M}(d\tau, d\xi) |\xi|^{(N-\beta)/2}.
\]

In [7, Section 9], it is shown that \( X \) has the same law as the solution \( u = \{u(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N\} \) of (3.4). It is also shown that, for any compact rectangle \( I \) in \( (0, \infty) \times \mathbb{R}^N \), by setting

\[
W(A, t, x) = \text{Re} \int_{|\tau| = \nu|\xi| \in A} \frac{e^{-i\xi \cdot x - i\tau t}}{2|\xi|} \left( \frac{1 - e^{i\tau(\tau + |\xi|)}}{\tau + |\xi|} - \frac{1 - e^{i\tau(\tau - |\xi|)}}{\tau - |\xi|} \right) \dot{M}(d\tau, d\xi) |\xi|^{(N-\beta)/2},
\]

our condition \( (A1) \) is satisfied with \( \gamma_j = \alpha^{-1} - 1 \) for \( j = 1, \ldots, 1 + N \), where

\[
\alpha = \frac{2 - \beta}{2},
\]

and \( (A2) \) and \( (A3) \) are satisfied with \( \delta_j = 2 - \beta \) for all \( j = 1, \ldots, 1 + N \). Therefore, our Theorem 2.3 is applicable to the solution \( u \) of (3.4) with \( Q = (2 + 2N)/(2 - \beta) \).

3.5. Rescaled Gaussian processes. We prove the following Proposition 3.1 for rescaled Gaussian processes. As an example of application, it implies that the Ornstein-Uhlenbeck process also satisfies \( (A1)-(A3) \).

**Proposition 3.1.** Let \( X \) be the Gaussian random field that satisfies \( (A1)-(A3) \) on \( I^{(\alpha)} \) for some positive constant \( \epsilon_0 \). Let \( f > 0 \) and \( g_1, \ldots, g_N \) be locally Lipschitz continuous functions mapping \( \mathbb{R}^N \) to \( \mathbb{R} \). Denote \( g(t) = (g_1(t_1), \ldots, g_N(t_N)) \). Assume that there exist a compact interval \( \tilde{I} \) and a positive constant \( \bar{\epsilon}_0 \), such that \( g(\tilde{I}^{(\bar{\epsilon}_0)}) \) is contained in \( I^{(\alpha)} \). Then the Gaussian random field \( \tilde{X}(t) := f(t)X(g(t)) \) satisfies \( (A1)-(A3) \) on \( \tilde{I}^{(\bar{\epsilon}_0)} \).

**Proof.** Let \( \{W(A, t) : A \in \mathcal{B}(\mathbb{R}_+), t \in \mathbb{R}^N\} \) be the Gaussian random field associated with \( X \). Define the random field \( \tilde{W} \) by

\[
\tilde{W}(A, t) = f(t)W(A, g(t)), \quad \forall A \in \mathcal{B}(\mathbb{R}_+), t \in \mathbb{R}^N.
\]

It is obvious that \( \{\tilde{W}(A, t) : A \in \mathcal{B}(\mathbb{R}_+), t \in \mathbb{R}^N\} \) is a Gaussian random field satisfying (a1) in the assumption \( (A1) \). Next, we verify condition (a2) in \( (A1) \). Let \( \tilde{a}_0 = 1 + \alpha_0 \), then for \( s, t \in \tilde{I}^{(\epsilon_0)} \), \( \tilde{a}_0 \leq a < b \leq +\infty \), by the triangle inequality and the assumption that \( X \) satisfies \( (A1) \), we have

\[
\|\tilde{W}([a, b], s) - \tilde{X}(s) - \tilde{W}([a, b], t) + \tilde{X}(t)\|_{L^2} \leq |f(s)||W([a, b], g(s)) - W([a, b], g(t)) + f(t)X(g(t))|_{L^2}.
\]

\[
\|f(s)||W([a, b], g(s)) - W([a, b], g(t)) + f(t)X(g(t))|_{L^2}
\]
\[
+ |f(s) - f(t)||W([a, b), g(t)) - X(g(t))|_{L^2} \\
\leq c_0 |f(s)| \left[ \sum_{j=1}^{N} a^{\gamma_j} |g_j(s_j) - g_j(t_j)| + b^{-1} \right] + |f(s) - f(t)||X(g(t))|_{L^2} \\
\leq c_0 L \sup_{r \in I^{(\alpha)}} |f(r)| \left[ \sum_{j=1}^{N} a^{\gamma_j} |s_j - t_j| + b^{-1} \right] + L \sum_{j=1}^{N} |s_j - t_j| \sup_{t \in I^{(\alpha)}} ||X(g(t))||_{L^2} \\
\leq C \left[ \sum_{j=1}^{N} a^{\gamma_j} |s_j - t_j| + b^{-1} \right],
\]

where we have used the Lipschitz continuity of \(f, g, \ldots, g_N\), the boundedness of \(\tilde{f}^{(\alpha)}\), Lemma 2.2 and the fact that \(a \geq \tilde{a}_0 \geq 1\) in the last inequality. For the second inequality in (a2), by the triangle inequality, we have

\[
\left| \tilde{W}([0, \tilde{a}_0], s) - \tilde{W}([0, \tilde{a}_0], t) \right|_{L^2} \\
= |f(s)W([0, \tilde{a}_0], g(s)) - f(t)W([0, \tilde{a}_0], g(t))|_{L^2} \\
\leq |f(s)||W([0, \tilde{a}_0], g(s)) - W([0, \tilde{a}_0], g(t))|_{L^2} + |f(s) - f(t)||W([0, \tilde{a}_0], g(t))|_{L^2}.
\]

For the second term, by Lemma 2.2, we have

\[
|f(s) - f(t)||W([0, \tilde{a}_0], g(t))|_{L^2} \leq L \sum_{j=1}^{N} |s_j - t_j| \sup_{t \in I^{(\alpha)}} ||X(g(t))||_{L^2} \leq C \sum_{j=1}^{N} |s_j - t_j|. \tag{3.6}
\]

For the first term, since \(X\) satisfies (A1), we have

\[
\left| W([0, \tilde{a}_0], g(s)) - W([0, \tilde{a}_0], g(t)) \right|_{L^2} \\
= \left| X(g(s)) - W([\tilde{a}_0, \infty), g(s)) - X(g(t)) + W([\tilde{a}_0, \infty), g(t)) \right|_{L^2} \\
\leq c_0 \sum_{j=1}^{N} \tilde{a}_0^{\gamma_j} |s_j - t_j| \leq C \sum_{j=1}^{N} |s_j - t_j|. \tag{3.7}
\]

The second inequality in (a2) is verified by substituting (3.6) and (3.7) to (3.5).

Noting that the continuity and the positivity of \(f\) together with the compactness of \(\tilde{f}^{(\alpha)}\) imply that \(f\) is bounded away from 0 on \(I^{(\alpha)}\). Hence, \(\tilde{X}(t)\) satisfies (A2).

It remains to verify (A3). Noting that \(f, g_1, \ldots, g_N\) are Lipschitz continuous, by the triangle inequality and the Cauchy-Schwarz inequality, we have

\[
\left| \mathbb{E} \left[ \tilde{X}_i(t')(\tilde{X}_i(s) - \tilde{X}_i(s)) \right] \right| \\
= |f(t')| \left| \mathbb{E} \left[ X_i(g(t'))(f(s)X_i(g(s)) - f(s)X_i(g(s))) \right] \right| \\
\leq |f(t')| \left| \mathbb{E} \left[ X_i(g(t'))(X_i(g(s)) - X_i(g(s))) \right] \right| + |f(t')| |f(s) - f(s)| \left| \mathbb{E} \left[ X_i(g(t'))X_i(g(s)) \right] \right| \\
\leq C \sup_{r \in I^{(\alpha)}} |f(r)|^2 \sum_{j=1}^{N} |g_j(s_j) - g_j(\tilde{s}_j)|^{\delta_j} + \sup_{r \in I^{(\alpha)}} |f(r)| L \sum_{j=1}^{N} |s_j - \tilde{s}_j| \left| X_i(g(t')) \right|_{L^2} \left| X_i(g(s)) \right|_{L^2}
\]
\[ \sum_{j=1}^{N} |s_j - \bar{s}_j| + C \sum_{j=1}^{N} |s_j - \bar{s}_j| \leq C \sum_{j=1}^{N} |s_j - \bar{s}_j|, \]

where the last inequality follows from \( \delta_j \leq 1 \) for \( 1 \leq j \leq N \). The proof is concluded. \( \square \)

As an example, we consider the Ornstein-Uhlenbeck process \( X(t) \) defined by
\[
dX(t) = -\theta X(t) dt + \sigma dB(t),
\]
where \( \theta \) and \( \sigma \) are positive constants, and \( B(t) \) is a 1-dimensional standard Brownian motion. It is well known that \( X(t) \) can be represented as a time-space-rescaled Brownian motion, i.e.,
\[
\{X(t), t \in \mathbb{R}_+\} = \left\{ \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} B(e^{2\theta t}), t \in \mathbb{R}_+ \right\},
\]
where “\( \overset{d}{=} \)” means equality in distribution. Noting that \( B(t) \) satisfies (A1)-(A3) on any compact interval on \( \mathbb{R} \setminus \{0\} \) with \( \gamma_1 = \delta_1 = 1 \), one can show that the Ornstein-Uhlenbeck process \( X(t) \) also satisfies (A1)-(A3) with the same parameters by applying Proposition 3.1 with \( f(t) = \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} \) and \( g_1(t) = e^{2\theta t} \).

4. Collision of eigenvalues of random matrices

In this last section, we aim to apply our main result Theorem 2.3 to solve the problem on the collision of eigenvalues of random matrices that was left open in [14, 21].

Let \( N \in \mathbb{N} \) be fixed and consider a centered Gaussian random field \( \xi = \{\xi(t) : t \in \mathbb{R}_+^N\} \) defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with covariance given by
\[
\mathbb{E} [\xi(s)\xi(t)] = C(s, t),
\]
for some non-negative definite function \( C : \mathbb{R}_+^N \times \mathbb{R}_+^N \rightarrow \mathbb{R} \). Let \( \{\xi_{i,j}, \eta_{i,j} : i, j \in \mathbb{N}\} \) be a family of independent copies of \( \xi \). For \( \beta \in \{1, 2\} \), and \( d \in \mathbb{N} \) with \( d \geq 2 \) fixed, consider the following \( d \times d \) matrix-valued process \( X^\beta = \{X_{i,j}^\beta(t) : t \in \mathbb{R}_+^N, 1 \leq i, j \leq d\} \) with entries given by
\[
X_{i,j}^\beta(t) = \begin{cases} 
\xi_{i,j}(t) + \iota \chi_{[\beta=2]} \eta_{i,j}(t), & i < j; \\
\sqrt{2} \xi_{i,i}(t), & i = j; \\
\xi_{j,i}(t) - \iota \chi_{[\beta=2]} \eta_{j,i}(t), & i > j,
\end{cases}
\]

where \( \iota := \sqrt{-1} \) is the imaginary unit. Clearly, for every \( t \in \mathbb{R}_+^N \), \( X^\beta(t) \) is a real symmetric matrix for \( \beta = 1 \) and a complex Hermitian matrix for \( \beta = 2 \). In particular, \( X^1(t) / \sqrt{C(t,t)} \) belongs to GOE and \( X^2(t) / \sqrt{2C(t,t)} \) belongs to GUE, respectively.

By the canonical identification, the matrix-valued process \( X^\beta \) can be regarded as a Gaussian random field, still denoted by \( X^\beta \), with values in \( \mathbb{R}^{d(d+1)/2} \) for \( \beta = 1 \) and in \( \mathbb{R}^{d^2} \) for \( \beta = 2 \), respectively. The component processes of \( X^\beta \) are independent, but are not identically distributed due to the constant factor of \( \sqrt{2} \) in the diagonal entries. We denote by \( D^\beta \) the invertible, diagonal matrix such that the Gaussian random field \( \tilde{X}^\beta = D^\beta X^\beta \) has i.i.d. components.

Let \( A^1 \) be a real symmetric deterministic matrix and \( A^2 \) be a complex Hermitian deterministic matrix. Suppose that \( \{\lambda^\beta_1(t), \ldots, \lambda^\beta_d(t)\} \) is the set of eigenvalues of
\[
Y^\beta(t) = A^\beta + X^\beta(t), \quad (\beta = 1, 2).
\]


Jaramillo and Nualart [14] provided a necessary condition and a sufficient condition for the collision of eigenvalues of $Y^\beta$. The results were generalized by Song et al [21] for the case where $k$ eigenvalues collide with $2 \leq k \leq d$. More precisely, assuming that the associated Gaussian random field $\xi = \{\xi(t) : t \in \mathbb{R}^N\}$ satisfies (A1) and (A2) in [21], we have, for the real case $\beta = 1$:

(i) if $\sum_{j=1}^N \frac{1}{H_j} < (k+2)(k-1)/2$, then
$$
\mathbb{P}\left(\lambda^\beta_{i_1}(t) = \cdots = \lambda^\beta_{i_k}(t) \text{ for some } t \in I \text{ and } 1 \leq i_1 < \cdots < i_k \leq d\right) = 0; \quad (4.3)
$$

(ii) if $\sum_{j=1}^N \frac{1}{H_j} > (k+2)(k-1)/2$, then
$$
\mathbb{P}\left(\lambda^\beta_{i_1}(t) = \cdots = \lambda^\beta_{i_k}(t) \text{ for some } t \in I \text{ and } 1 \leq i_1 < \cdots < i_k \leq d\right) > 0;
$$

for the complex case $\beta = 2$:

(i) if $\sum_{j=1}^N \frac{1}{H_j} < k^2 - 1$, then
$$
\mathbb{P}\left(\lambda^\beta_{i_1}(t) = \cdots = \lambda^\beta_{i_k}(t) \text{ for some } t \in I \text{ and } 1 \leq i_1 < \cdots < i_k \leq d\right) = 0; \quad (4.4)
$$

(ii) if $\sum_{j=1}^N \frac{1}{H_j} > k^2 - 1$, then
$$
\mathbb{P}\left(\lambda^\beta_{i_1}(t) = \cdots = \lambda^\beta_{i_k}(t) \text{ for some } t \in I \text{ and } 1 \leq i_1 < \cdots < i_k \leq d\right) > 0.
$$

When $\sum_{j=1}^N \frac{1}{H_j} = (k+2)(k-1)/2$ for the real case and $\sum_{j=1}^N \frac{1}{H_j} = k^2 - 1$ for the complex case, the collision problems were left open by [14] and [21].

Before studying the collision problem at the critical dimension, we first introduce some notations. We denote by $S(d)$ and $H(d)$ the set of real symmetric $d \times d$ matrices and the set of complex Hermitian $d \times d$ matrices, respectively. By the canonical identification, we have $S(d) \simeq \mathbb{R}^{d(d+1)/2}$ and $H(d) \simeq \mathbb{C}^{d^2}$. For $k \in \{1, \ldots, d\}$, let $S(d; k)$ (resp. $H(d; k)$) be the set of real symmetric (resp. complex Hermitian) $d \times d$ matrices with at least $k$ identical eigenvalues.

The following theorem solves the collision problem at the critical dimension for the real case $\beta = 1$.

**Theorem 4.1.** Let $Y^\beta$ ($\beta = 1$) be the matrix-valued process defined by (4.2) with eigenvalues $\{\lambda^\beta_{i_1}(t), \ldots, \lambda^\beta_{i_k}(t)\}$. Assume the associated Gaussian random field $\xi = \{\xi(t) : t \in \mathbb{R}^N\}$ satisfies (A1)-(A3). For any $k \in \{2, \ldots, d\}$, if $\sum_{j=1}^N \frac{1}{H_j} = (k+2)(k-1)/2$, then (4.3) holds.

**Proof.** It follows from Lemma 2.1 and Lemma 2.3 in [21] that for any $M > 0$, the set
$$
S(d; k) \cap [-M, M]^{d(d+1)/2} \subseteq \text{Im}(G) \cap [-M, M]^{d(d+1)/2},
$$
where $G : \mathbb{R}^{d+k-1} \times \mathbb{R}^{\frac{1}{2}[d(d-1)-k(k-1)]} \to S(d)$ is a smooth function and the set $\text{Im}(G) \cap [-M, M]^{d(d+1)/2}$ has positive and finite $(\frac{1}{2}[d(d+1) - k(k+1)] + 1)$-dimensional Lebesgue measure. In the case of critical dimension (i.e., $\sum_{j=1}^N \frac{1}{H_j} = (k+2)(k-1)/2$),
$$
\frac{d(d+1)}{2} - \sum_{j=1}^N \frac{1}{H_j} = \frac{1}{2}[d(d+1) - k(k+1)] + 1,
$$
we can verify that \((\text{Im}(G) - A^\beta) \cap [-M, M]^{d(d+1)/2}\) and its image under the linear operator \(D^\beta\) satisfy condition (2.4) of Theorem 2.3 with \(\theta = \frac{d(d+1)}{2} - Q\) and \(\kappa = 0\). Applying Theorem 2.3 to the Gaussian random field \(\tilde{X}^\beta = D^\beta X^\beta\) and \(F = D^\beta ((\text{Im}(G) - A^\beta) \cap [-M, M]^{d(d+1)/2})\), we obtain
\[
P \left( X^\beta(I) \cap (\text{Im}(G) - A^\beta) \cap [-M, M]^{d(d+1)/2} \neq \emptyset \right) \\
= P \left( \tilde{X}^\beta(I) \cap D^\beta \left( (\text{Im}(G) - A^\beta) \cap [-M, M]^{d(d+1)/2} \right) \neq \emptyset \right) = 0.
\]

Therefore
\[
P \left( \lambda_i^\beta(t) = \cdots = \lambda_k^\beta(t) \text{ for some } t \in I \text{ and } 1 \leq i_1 < \cdots < i_k \leq d \right) \\
= P \left( Y^\beta(t) \in S(d; k) \text{ for some } t \in I \right) \\
= P \left( X^\beta(t) \in (S(d; k) - A^\beta) \text{ for some } t \in I \right) \\
\leq P \left( X^\beta(t) \in (\text{Im}(G) - A^\beta) \text{ for some } t \in I \right) \\
= P \left( X^\beta(I) \cap (\text{Im}(G) - A^\beta) \neq \emptyset \right) \\
= \lim_{M \to \infty} P \left( X^\beta(I) \cap (\text{Im}(G) - A^\beta) \cap [-M, M]^{d(d+1)/2} \neq \emptyset \right) = 0.
\]

This proves the non-existence of the \(k\)-collision of the eigenvalues. \(\square\)

In Section 3, we have seen that the multiparameter fractional Brownian motions, fractional Brownian sheets, solutions to linear stochastic heat equations, and the Ornstein-Uhlenbeck processes satisfy the conditions (A1)-(A3). Hence, Theorem 4.1 is applicable to these models.

**Corollary 4.2.** Let \(Y^\beta (\beta = 1)\) be the matrix-valued process defined by (4.2) with eigenvalues \(\{\lambda_1^\beta(t), \ldots, \lambda_k^\beta(t)\}\). The associated Gaussian random field \(\xi = \{\xi(t) : t \in \mathbb{R}_+^N\}\) is multiparameter fractional Brownian motion, fractional Brownian sheet, solution to linear stochastic heat equation, or the Ornstein-Uhlenbeck process. For any \(k \in \{2, \ldots, d\}\), if \(\sum_{j=1}^{N} \frac{1}{\gamma_j} = (k+2)(k-1)/2\), then (4.3) holds.

**Remark 4.3.** When \(k = 2\) and the associated Gaussian random field \(\xi\) is (fractional) Brownian motion, Theorem 4.1 recovers the non-collision property for the symmetric matrix Brownian motion (see e.g. [1]). Similarly, when \(k = 2\) and \(\xi\) is Ornstein-Uhlenbeck process, Theorem 4.1 recovers the non-collision property obtained in [18] for the real symmetric matrix Ornstein-Uhlenbeck process.

The following is the analogue of Theorem 4.1 for the complex case \(\beta = 2\).

**Theorem 4.4.** Let \(Y^\beta (\beta = 2)\) be the matrix-valued process defined by (4.2) with eigenvalues \(\{\lambda_1^\beta(t), \ldots, \lambda_k^\beta(t)\}\). Assume the associated Gaussian random field \(\xi = \{\xi(t) : t \in \mathbb{C}_+^N\}\) satisfies (A1)-(A3). For any \(k \in \{2, \ldots, d\}\), if \(\sum_{j=1}^{N} \frac{1}{\gamma_j} = k^2 - 1\), then (4.4) holds.

**Proof.** It follows from Lemma 3.2 and Lemma 3.4 in [21] that for any \(M > 0\), the set \(H(d; k) \cap [-M, M]^{d^2} \subseteq \text{Im}(\tilde{G}) \cap [-M, M]^{d^2}\), which has finite \((d^2-k^2+1)\)-dimensional Lebesgue measure. Here \(\tilde{G} : \mathbb{R}^{d-k+1} \times \mathbb{R}^{d^2-d^2+k+2} \to H(d)\) is a smooth function defined in [21, Lemma 3.2].
In the case of critical dimension (i.e., \( \sum_{j=1}^{N} \frac{1}{H_j} = k^2 - 1 \)),

\[
d^2 - \sum_{j=1}^{N} \frac{1}{H_j} = d^2 - k^2 + 1,
\]

we see that \((\text{Im}(\hat{G}) - A^\beta) \cap [-M, M]^{d(d+1)/2}\) satisfies condition (2.4) of Theorem 2.3 with \( \theta = d^2 - Q \) and \( \kappa = 0 \). As in the proof of Theorem 4.1, We apply Theorem 2.3 to the Gaussian random field \( \tilde{X}^\beta \) to obtain

\[
P \left( X^\beta(I) \cap (\text{Im}(\hat{G}) - A^\beta) \cap [-M, M]^{d(d+1)/2} \neq \emptyset \right) = 0.
\]

Therefore

\[
P \left( \lambda_{i_1}^\beta(t) = \cdots = \lambda_{i_k}^\beta(t) \text{ for some } t \in I \text{ and } 1 \leq i_1 < \cdots < i_k \leq d \right) = P \left( Y^\beta(t) \in H(d; k) \text{ for some } t \in I \right) = P \left( X^\beta(t) \in (H(d; k) - A^\beta) \text{ for some } t \in I \right) \leq P \left( X^\beta(t) \in (\text{Im}(\hat{G}) - A^\beta) \text{ for some } t \in I \right) = P \left( X^\beta(I) \cap (\text{Im}(\hat{G}) - A^\beta) \neq \emptyset \right) = \lim_{M \to \infty} P \left( X^\beta(I) \cap (\text{Im}(\hat{G}) - A^\beta) \cap [-M, M]^{d^2} \neq \emptyset \right) = 0.
\]

This proves the non-existence of the \( k \)-collision of the eigenvalues. \( \square \)

Similar to the real case, we have the following result as a corollary of Theorem 4.4.

**Corollary 4.5.** Let \( Y^\beta \) (\( \beta = 2 \)) be the matrix-valued process defined by (4.2) with eigenvalues \( \{\lambda_1^\beta(t), \ldots, \lambda_d^\beta(t)\} \). The associated Gaussian random field \( \xi = \{\xi(t) : t \in \mathbb{R}^N_+\} \) is multiparameter fractional Brownian motion, fractional Brownian sheet, solution of linear stochastic heat equation, or Ornstein-Uhlenbeck process. For any \( k \in \{2, \ldots, d\} \), if \( \sum_{j=1}^{N} \frac{1}{H_j} = k^2 - 1 \), then (4.4) holds.

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