Abstract In this article, we describe the propagation properties of the one-dimensional wave and transport equations with variable coefficients semi-discretized in space by finite difference schemes on non-uniform meshes obtained as diffeomorphic transformations of uniform ones. In particular, we introduce and give a rigorous meaning to notions like the principal symbol of the discrete wave operator and the corresponding bi-characteristic rays. The main mathematical tool we employ is the discrete Wigner transform, which, in the limit as the mesh size parameter tends to
zero, yields the so-called Wigner (semiclassical) measure. This measure provides the dynamics of the bi-characteristic rays, i.e., the solutions of the Hamiltonian system describing the propagation, in both physical and Fourier spaces, of the energy of the solution to the wave equation. We show that, due to dispersion phenomena, the high-frequency numerical dynamics does not coincide with the continuous one. Our analysis holds for the class $C^{0,1}(\mathbb{R})$ of globally Lipschitz coefficients and non-uniform grids obtained by means of $C^{1,1}(\mathbb{R})$-diffeomorphic transformations of a uniform one. We also present several numerical simulations that confirm the predicted paths of the space–time projections of the bi-characteristic rays. Based on the theoretical analysis and simulations, we describe some of the pathological phenomena that these rays might exhibit as, for example, their reflection before touching the boundary of the space domain. This leads, in particular, to the failure of the classical properties of boundary observability of continuous waves, arising in control and inverse problems theory.

**Keywords** Variable coefficients transport/wave equations · Finite difference approximations · Non-uniform mesh · Wigner transform/measure · Bi-characteristic rays · Umklapp/internal reflection

**Mathematics Subject Classification** 81S30 · 35A21 · 37C05 · 65M06 · 70K05

1 Introduction and Main Results

1.1 Motivation

This paper is devoted to analyze the propagation properties of discrete waves solving the finite difference numerical approximations for two basic 1D wave propagation models: the first-order transport equation and the second-order wave equation. This is done on non-uniform meshes. The main contribution of the paper was to introduce a suitable notion of principal symbol allowing to construct bi-characteristic and characteristic rays propagating the energy of solutions.

The work developed in this article is motivated by control and inverse problems. Indeed, the well-known boundary controllability and identifiability properties of solutions of the wave equation hold because of the fact that the energy of solutions is driven
by characteristics that reach the boundary where the controllers or observers are placed. This property is well known to generally fail for numerical schemes because of the pathological behavior of high-frequency numerical spurious solutions [16,17,51], but this analysis has been developed, so far, only in the context of numerical approximations on uniform meshes. The goal of this paper was to develop the fundamental notions and tools of microlocal analysis allowing to handle numerical discretization schemes on non-uniform meshes. As we shall see, the non-uniformity of the mesh adds further dispersion properties to the numerical solutions.

1.1.1 The Continuous Transport Equation

The first basic model analyzed in this paper is the scalar transport equation

$$\rho(y) \partial_t u(y, t) + \partial_y u(y, t) = 0, \quad y \in \mathbb{R}, \quad t > 0, \quad u(y, 0) = u^0(y). \tag{1.1}$$

Here and in what follows, $\partial_t$ and $\partial_y$ are the first-order time and space derivatives and $\rho$ is an $L^\infty(\mathbb{R})$-function satisfying the hyperbolicity condition $\rho(y) \geq \rho^- > 0$ (for all $y \in \mathbb{R}$). Under these assumptions, the following energy (coinciding with the weighted $L^2(\mathbb{R}; \rho)$-norm of the solution $u(y, t)$) is conserved in time:

$$E_\rho(u(\cdot, t)) = \int_\mathbb{R} |u(y, t)|^2 \rho(y) \, dy := E_\rho(u^0). \tag{1.2}$$

For all strictly positive density functions $\rho \in C^{0,1}(\mathbb{R})$, the transport Eq. (1.1) can be uniquely solved by the method of characteristics as $u(y(t), t) = u^0(y_0)$, where $y(t)$ are the so-called characteristic curves solving the first-order ordinary differential equation (ODE)

$$y'(t) = c(y(t)), \quad t > 0, \quad y(0) = y_0 \in \mathbb{R}, \quad \text{with} \quad c(y) := 1/\rho(y). \tag{1.3}$$

By $'$, we denote the derivative of a function depending on only one variable. When $\rho \equiv 1$, the solutions of (1.3) are straight lines of the form $y(t) = y_0 + t$ for all $y_0 \in \mathbb{R}$ and $t \geq 0$. Accordingly, the solutions of the transport Eq. (1.1) take the form $u(y, t) = u^0(y - t)$.

1.1.2 The Continuous Wave Equation

The second main model analyzed in this paper is the one-dimensional wave equation

$$\rho(y) \partial_t^2 u(y, t) - \partial_y (\sigma(y) \partial_y u)(y, t) = 0, \quad y \in \mathbb{R}, \quad t > 0, \quad u(y, 0) = u^0(y), \quad \partial_t u(y, 0) = u^1(y), \quad y \in \mathbb{R}, \tag{1.4}$$

where $\partial_t^2$ is the second-order time derivative operator and $\rho, \sigma$ are $L^\infty(\mathbb{R})$-functions with the strict hyperbolicity assumption $\rho(y) \geq \rho^-$ and $\sigma(y) \geq \sigma^-$. The total energy below is conserved in time:
\[ E_{\rho,\sigma}(u^0, u^1) := \frac{1}{2} \int_{\mathbb{R}} (\rho(y)|\partial_t u(y, t)|^2 + \sigma(y)|\partial_y u(y, t)|^2) \, dy. \] (1.5)

When the coefficients of the wave Eq. (1.4) are constant \((\rho = \sigma \equiv 1)\), for simplicity), the corresponding solution is given by the d’Alembert formula below, stating that the solution of the wave equation can be uniquely decomposed into two components, each one propagating along one of the characteristics \(y \pm t\):

\[ u(y, t) = \frac{1}{2} (u^0(y + t) + u^0(y - t)) + \frac{1}{2} \int_{y-t}^{y+t} u^1(z) \, dz. \]

For the variable coefficients case, there is no explicit formula for solutions. However, it is well known (cf. [4]) that the energy of initial data presenting high-frequency oscillation and/or concentration effects propagates along the characteristic rays. The role of these rays can be illustrated by considering highly concentrated and oscillatory initial data in (1.1) or (1.4) leading to Gaussian wave packet-type solutions concentrated, precisely, along one of the bi-characteristic lines and for which the energy localized outside any neighborhood vanishes as the wavelength parameter tends to zero [3,33,34,36,41,42].

1.1.3 Wave Propagation in the Continuous Setting and Its Applications

As mentioned above, an important application of the propagation of waves along characteristics comes from Control Theory. One of the most typical problems in control is that of exact controllability [32]. It consists in driving the solutions of the PDE under consideration (or of its numerical approximation schemes) to the equilibrium by means of an applied force, the control, localized on some subset of the domain where waves propagate as, for instance, a part or a neighborhood of the boundary or the complementary set of a bounded domain. This control problem is equivalent to the observability one, consisting in the possibility to obtain estimates of the total energy of the solutions of the uncontrolled system (1.4) in terms of the energy concentrated on the support of the control along time. For the linear wave equation with smooth coefficients, the observability problem has a positive answer if and only if the geometric control condition (GCC) holds. This GCC requires all rays of Geometric Optics to enter the control/observability region during the control time (cf. [4]).

The extension of the GCC in [4] to some numerical approximation schemes for the transport and wave equations has been developed in the particular case of constant coefficients and uniform grids in [16,17] and [51] by means of Fourier analysis.

The main goal of this paper was to rigorously construct characteristic and bi-characteristic rays for the finite difference numerical approximations of the transport and wave Eqs. (1.1) and (1.4) on non-uniform meshes in order to describe how the energy of numerical solutions propagates.
1.2 Main Results

In order to better describe the propagation properties of waves on non-uniform meshes, let us first introduce the grids and the numerical schemes we deal with. Let $h > 0$ be the mesh size parameter, $g : \mathbb{R} \to \mathbb{R}$ be an increasing function on $\mathbb{R}$, $\mathcal{G}^h := \{x_j := jh, j \in \mathbb{Z}\}$ be the uniform grid of size $h$ of $\mathbb{R}$ and $\mathcal{G}_g^h := \{g_j := g(x_j), j \in \mathbb{Z}\}$ the non-uniform grid obtained by transforming the uniform one through the map $g$ (see Fig. 1). We also set $g_{j+1/2} := g(x_{j+1/2})$ to be the image through the map $g$ of the midpoints $x_{j+1/2} := (j + 1/2)h$, $j \in \mathbb{Z}$. Along this paper, we focus on the case in which $g$ is regular. More precisely, we require $c_g := c(g)/g'$ to belong to the space $C^{0,1}(\mathbb{R})$ of Lipschitz continuous functions (cf. [18], pp. 124, 240–241), where $c(y) := 1/\varrho(y)$ and $c(y) := \sqrt{\sigma(y)/\rho(y)}$ for the transport/wave equations.

We also denote by \( \partial_h f^h := (\partial_h f_j)_{j \in \mathbb{Z}} \) and $\partial^{h,\pm}_h f^h := (\partial^{h,\pm}_h f_j)_{j \in \mathbb{Z}}$ the first-order finite difference discrete derivatives on the uniform mesh, where $\partial_h f_j := (f_{j+1} - f_{j-1})/2h$, $\partial^{h,\pm}_h f_j := \pm(f_{j+1} - f_j)/h$ are the centered and the forward (+) / backward (−) finite differences.

1.2.1 The Discrete Transport Equation

Along this paper, we consider the following finite difference semi-discretization of the transport Eq. (1.1) on the non-uniform grid $\mathcal{G}_g^h$:

$$\varrho(g_j) \partial_t u_j(t) + \frac{\partial_h u_j(t)}{\partial_h g_j} = 0, \quad j \in \mathbb{Z}, \quad t > 0, \quad u_j(0) = u^0_j, \quad j \in \mathbb{Z}. \quad (1.6)$$

Here, $u_j(t)$ is an approximation of $u(g_j, t)$, where $u$ is the solution of the transport Eq. (1.1). The total energy $E_{\varrho, g}^h(u^h(t))$ of the solution $u^h(t)$ of (1.6) is conserved in time, i.e.,

$$E_{\varrho, g}^h(u^h(t)) := h \sum_{j \in \mathbb{R}} \varrho(g_j) |\partial_h g_j| u_j(t)|^2 = E_{\varrho, g}^h(u^{h,0}).$$

1.2.2 The Discrete Wave Equation

We also consider the finite difference approximation of the wave Eq. (1.4) on the non-uniform grid $\mathcal{G}_g^h$.

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**Fig. 1** The smooth grid application $g$ transforms the nodes $x_j$ of the uniform grid $\mathcal{G}^h$ into nodes $g_j = g(x_j)$ of the non-uniform grid $\mathcal{G}_g^h$.
\[ \rho(g_j) \partial_t^2 u_j(t) - \frac{\sigma(g_{j+1/2}) \partial^h_{g_j} u_j(t) - \sigma(g_{j-1/2}) \partial^h_{-g_j} u_j(t)}{h \partial^h g_j} = 0, \]

\[ u_j(0) = u_j^0, \ \partial_t u_j(0) = u_j^1, \ j \in \mathbb{Z}, \ t > 0. \]  

(1.7)

Here, \( u_j(t) \) is an approximation of \( u(g_j, t) \), where \( u \) is the solution of the wave Eq. (1.4). The total energy \( E^h_{\rho, \sigma, g}(u^h(t), \partial_t u^h(t)) \) of the solution \( u^h(t) \) of (1.7) is also time conservative, i.e.,

\[ E^h_{\rho, \sigma, g}(u^h(t), \partial_t u^h(t)) := \frac{h}{2} \sum_{j \in \mathbb{Z}} \left[ \rho(g_j) \partial^h g_j |\partial_t u_j(t)|^2 + \frac{\sigma(g_{j+1/2})}{\partial^h_{g_j}} |\partial^h_{g_j} u_j(t)|^2 \right] \]

\[ = E^h_{\rho, \sigma, g}(u^h_0, u^h_1). \]

1.2.3 The Wigner Transform

The main tool for our theoretical analysis in both continuous and discrete settings is the so-called Wigner transform in pseudo-differential calculus introduced in 1932 by Eugene Wigner (cf. [50]). For a precise definition of this notion, see [22,33] and also Sect. 2 of this article. Roughly speaking, this is a quadratic transform used when the coefficients of the hyperbolic PDE under consideration vary at unit scale, while the wavelength of the initial data is asymptotically smaller. It allows to describe the propagation properties by capturing the characteristic rays of the model. A time-dependent wave process like the solutions of the transport and of the wave Eqs. (1.1), (1.4), and their finite difference semi-discretizations can be described either in terms of a wave function \( u(y, t) \) at the point \( y \) of the physical space or in terms of the bilinear product \( u(y_1, t) \bar{u}(y_2, t) \) of the wave function and its conjugate at two points \( y_1 \) and \( y_2 \). In quantum dynamics, the scaled Fourier transform of this product with respect to the distance between the two points \( y_1 \) and \( y_2 \) is called Wigner transform. In the limit as the wavelength of the wave packet tends to zero, the Wigner transform becomes the so-called Wigner measure, which can be seen as a particle density depending on the phase-space variables \( y \) (denoting the position) and \( \xi \) (denoting the momentum). The corresponding Liouville equation describing the propagation of the Wigner measure is similar to a kinetic equation for real particles.

1.2.4 Short Description of Our Main Results

We now present in an itemized manner the key contributions of this paper.

- The principal symbols of the finite difference approximations on non-uniform meshes. It is well known (cf. [4]) that the principal symbols of the variable coefficients transport/wave Eqs. (1.1) and (1.4) are given by

\[ \varphi(y, t, \xi, \tau) := -\rho(y)\tau - \xi \]  

(1.8)
and, respectively,

\[ \varphi(y, t, \xi, \tau) := -\rho(y)\tau^2 + \sigma(y)\xi^2. \]  

(1.9)

These symbols are obtained by considering solutions of (1.1) or (1.4) arbitrarily concentrated in space around some point \( y \in \mathbb{R} \) and oscillating at some frequencies \( \tau \) and \( \xi \) in time/space.

We prove that, for the discrete systems (1.6) and (1.7), the corresponding principal symbols are

\[ \varphi(x, t, \xi, \tau) := -g'(x)\varphi(g(x))\tau - \sin(\xi) \]  

(1.10)

and

\[ \varphi(x, t, \xi, \tau) := -g'(x)\rho(g(x))\tau^2 + 4\sin^2\left(\frac{\xi}{2}\right)\frac{\sigma(g(x))}{g'(x)}. \]  

(1.11)

We note some changes in these discrete principal symbols with respect to the continuous ones (1.8) and (1.9). Firstly, they depend on the space variable \( x = g^{-1}(y) \) corresponding to the uniform grid. As a consequence, in this new space variable \( x \), all the variable coefficients \( \varrho, \rho, \) and \( \sigma \) have to be composed with \( g \). Note also the appearance of the factor \( 1/g'(x) \) accompanying each space derivative, which is also due to the grid transformation \( y = g(x) \). Moreover, note that, in (1.10) and (1.11), the Fourier symbols \( \xi \) and \( \xi^2 \) of the first- and second-order space derivatives in the continuous symbols (1.8) and (1.9) have been replaced by the corresponding symbols \( \sin(\xi) \) and \( 4\sin^2(\xi/2) \) of the centered first-order finite difference \( \partial^h \) and of the three-point finite difference approximation of the Laplacian.

- **The propagation of the discrete Wigner measures.** In Theorems 3.3 and 3.5, we prove indeed that the expressions (1.10) and (1.11) are the appropriate ones. More precisely, we rigorously obtain the dynamics of the Wigner measures \( \mathcal{W} \) for the solutions of the discrete transport and wave equations governed by Liouville equations of the form

\[ \mathcal{W}_t = \pm (c_g(x)\omega'(\xi)\mathcal{W}_x - c'_g(x)\omega(\xi)\mathcal{W}_\xi), \]

where \( c_g := c(g)/g' \), \( c(y) := 1/\varrho(y) \) and \( c(y) := \sqrt{\sigma(y)/\rho(y)} \) are the velocities corresponding to the continuous transport/wave equations and \( \omega(\xi) := \sin(\xi) \) and \( \omega(\xi) := 2\sin(\xi/2) \) are the dispersion relations for the same discrete transport/wave equations on uniform meshes. These Liouville equations, being in particular transport equations in the phase-space variables \((x, \xi)\), can be solved by the method of characteristics, yielding our precise definition of characteristic rays. Moreover, these characteristic rays coincide with those obtained by the resolution of the Hamiltonian systems associated with the principal symbols (1.10) and (1.11).

Our results extend the ones by Macià in [34] dealing with numerical approximations of the wave equation with variable coefficients on uniform meshes and the
ones by Markowich–Pietra–Pohl in [38] analyzing several numerical schemes for the Schrödinger equation on uniform meshes.

- The qualitative analysis of the Hamiltonian systems describing the discrete characteristic rays. In Sect. 5, we present several numerical simulations aimed to graphically confirm our theoretical results. More precisely, we consider approximations of both constant coefficients transport/wave equations on the bounded interval \((-1, 1)\) and on two non-uniform meshes produced by the transformations 
\[ g^1(x) = \tan(\pi x/4) \quad \text{and} \quad g^2(x) = 2 \sin(\pi x/6), \]
yielding a gradual refinement of the grid at the center \(x = 0\) of the space interval and, respectively, at the two endpoints. We also consider high-frequency oscillatory and concentrated Gaussian initial data. At low frequencies, the numerical solutions behave basically like the solution of the continuous model, i.e., they propagate along straight characteristic lines and reflect following the Snell law when they touch one of the two endpoints (see Fig. 2). Nevertheless, when increasing the frequency, we encounter interesting phenomena like (a) the curvature of the characteristic rays; (b) the so-called umklapp or U-process (see [28], pp. 125) or internal reflection (cf. [48]), i.e., the reflection of waves without touching any endpoint of the space interval, due to the fact that the wave number of the numerical solution crosses the value \(\pi\) and the group velocity of the numerical approximation changes the sign (Fig. 3, top-left), or without touching only one of the endpoints (Fig. 3, top-right) or c) the existence of stationary waves (see Fig. 3, bottom). We will show that these pathologies can be rigorously explained by the behavior of the corresponding phase portraits of the Hamiltonian systems yielding the characteristic rays. For example, the subfigures on the left/right columns in Fig. 3 correspond to the grid transformations \(g^1\) and \(g^2\) and to the center/saddle character of the fixed points of the corresponding Hamiltonian systems. Note the analogy between the umklapp phenomenon and the lack of transmission of high-frequency numerical wave packets at the interface between two homogeneous media when the velocity on the left of the interface is larger than the one on the right (see [11], Chapter 10).

The internal reflection phenomenon has been also found by WKB techniques in [48] in the context of the centered scheme to approximate the space derivative in the constant coefficients 1D transport equation on a non-uniform grid of variable size \(h = h(x)\). Additionally, to our work, the case of fully discrete schemes and the
behavior of the wavelength inside the numerical wave packet are analyzed in [48]. Complementing the analysis in [48], we consider a more general context, including both transport and wave equations in which both the mesh and the coefficients of the continuous model are heterogeneous, paying more attention to the regularity issues on the Hamiltonian systems of bi-characteristic rays and to the nature of the fixed points of the Hamiltonian systems.

The analysis in this paper is limited to the 1D case, but the techniques we develop can also be used to analyze multidimensional problems.

1.3 On the Regularity of the Coefficients and of the Grid Transformation

The fulfillment of the observability inequality for the continuous wave equation strongly depends on the regularity of its coefficients. Classical observability estimates, in which the total energy of waves is the same as the observed one, require the velocity coefficient $c = \sqrt{\rho/\sigma}$ in (1.4) to belong to the $BV$ space of functions of bounded variation [9] or to the Zygmund class [19]. In [19], it was also proved that observability holds with a loss of a finite number of derivatives on the observed energy if the coefficients belong to logarithmic spaces (log-Lipschitz or log-Zygmund). However, below this logarithmic regularity, for instance, for Hölder coefficients $C^{0, \alpha}$ with $\alpha < 1$, observability does not hold due to a loss of an infinite number of derivatives for highly oscillatory velocity coefficients leading to exponentially concentrated eigenfunctions. All these results are based on the sideways energy estimates method.
[12], which is intrinsically $1D$ and consists in inverting the role of the space and time derivatives.

In the multidimensional setting, the control/observability problem for waves becomes more delicate due to the more complex behavior of the bi-characteristic rays and their interaction with the boundary of the spatial domain $\Omega$. To prove observability in the multidimensional case, one can use microlocal techniques (cf. [8] and [40], with coefficients $\sigma$ of the D’Alembertian $\partial^2_t - \text{div}(\sigma(x)\nabla)$ in $C^{1,1}$, or Carleman estimates [20], for $C^1$-coefficients. Carleman estimates involve exponential weights and do not require the determination of the bi-characteristic rays of Geometric Optics. As a counterpart do not provide sharp results in what concerns the region of observation. Using Carleman estimates [23], one can also link the observability properties of the variable coefficients transport equation with its viscous counterpart.

Microlocally, wave operators can be viewed as a composition of two transport operators. There is an extensive literature concerning the relationship between the well posedness of the transport equation

$$w_t(z, t) + n(z) \cdot \nabla_z w(z, t) = 0, \quad w(z, 0) = w^0(z), \quad z \in \mathbb{R}^N$$  \hspace{1cm} (1.12)

and that of the corresponding nonlinear ODE system of characteristic rays

$$z'(t) = n(z(t)), \quad z(0) = z^0,$$  \hspace{1cm} (1.13)

with the state $z(t)$, the nonlinearity $n$, and the initial datum $z^0$ being vectors in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

The classical Cauchy–Lipschitz theorem ensures the well posedness of (1.13) for globally Lipschitz nonlinearities $n \in C^{0,1}$ and any initial datum $z^0 \in \mathbb{R}^N$. The seminal paper of DiPerna–Lions [13] considers the flow $z(t)$ as a function of both initial datum $z^0$ and time and shows that if $n \in W^{1,1}_1(\mathbb{R}^N)$ and $\text{div}_z n \in L^\infty(\mathbb{R}^N)$, then there exists a unique flow $z$ in $C([0, T]; L^p_{loc}(\mathbb{R}^N))$. In [1], this result has been extended to $BV_{loc}$ nonlinearities $n$. Moreover, in [1], imposing the additional assumption that $|n| \in L^\infty(\mathbb{R}^N)$, the well posedness of (1.13) is shown for a.e. initial datum $z^0 \in \mathbb{R}^N$ and of the transport Eq. (1.12) with rough initial data of the form $\delta_{\varphi^0}$ for all $z^0 \in \mathbb{R}^N$.

The regularity of the nonlinearity $n$ can be lowered in some cases when $n = (n_1, n_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $z = (z_1, z_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ has a particular form. For instance, when $n_1 = n_1(z_1)$, well posedness has been obtained in [30] by asking $n_1 \in W^{1,1}_{loc}(\mathbb{R}^{N_1})$, $n_2 \in L^1_{loc}(\mathbb{R}^{N_1}; W^{1,1}_{loc}(\mathbb{R}^{N_2}))$, $\text{div}_{z_1} n_1 \in L^\infty(\mathbb{R}^{N_1})$ and $\text{div}_{z_2} n_2 \in L^\infty(\mathbb{R}^{N_1})$, with $N = N_1 + N_2$ (instead of the condition $n_1 \in W^{1,1}_{loc}(\mathbb{R}^{N_1})$ in [13]). This result has been extended to the $BV$ context in [31]. Note that when $N_1 = N_2 = 1$, the ODE systems (2.2) and (2.4) arising as phase-space projection of the Hamiltonian systems of null bi-characteristic rays associated with the symbols (1.8) or (1.9) fit precisely into this decomposition and then the $W^{1,1}$ or $BV$ regularity of the velocity $c$ is enough for their well posedness.

When $N_1 = N_2 = 1$ and the vector $n$ is divergence free, i.e., there exists a Hamiltonian $H$ such that $n_1(z_1, z_2) = \nabla_{z_2} H(z_1, z_2)$ and $n_2(z_1, z_2) = -\nabla_{z_1} H(z_1, z_2)$, one can prove that the regularity $n \in L^2_{loc}(\mathbb{R}^2)$ guarantees the well posedness of the flow
The Hamiltonian systems (3.8) and (3.21) associated with the discrete symbols fit precisely in this particular form, so that, taking into account the form $H(z) = H(x, \xi) = c_g(x)\omega(\xi)$ in separated variables of $H$, the results in [25] require $c_g \in H^1_{loc}(\mathbb{R})$.

When $N_1 = N_2 = d$ and $d \geq 1$, the only known particular well-posedness result holds for $n_1 = n_1(z_2) = z_2$ and $n_2 = n_2(z_1)$ (in this case, (1.12) is the so-called Vlasov equation) with $n_2 \in H^{3/4} \cap L^\infty(\mathbb{R}^d)$ [10] or with $n_2 \in BV_{loc}(\mathbb{R}^d)$ [6]. Note also that, since the inclusion relationship between $BV$ and $H^{3/4}$ is not clear, one cannot say whether the results in [10] are a particular case or an extension of the ones in [6]. In [24], the well posedness for the Vlasov equation associated with a nonlinearity $n_2 \in BV_{loc}(\mathbb{R}^d \setminus \{0\})$ having a Coulomb singularity at the origin is shown. In [2], one considers the Vlasov equation with $n_2 \in BV_{loc} \cap L^\infty(\mathbb{R}^d)$ and shows well posedness of (1.12) with initial datum $w^0$ in the space $P(\mathbb{R}^{2d})$ of probability measures on $\mathbb{R}^{2d}$.

Until now, we saw that the Cauchy–Lipschitz theorem requires $c_g \in C^{1,1}$, while the DiPerna–Lions [13] or the Hauray [25] results require $c_g \in W^{2,1}$ or $c_g \in H^1$ in order to guarantee the well posedness of (3.8) and (3.21) as two-dimensional Hamiltonian systems describing the characteristic rays for the discrete transport/wave equations and to solve the corresponding Liouville equations by the method of characteristics. Our approach reduces even more this regularity of the coefficients and grid applications with respect to the one in the literature and is based on an intrinsically one-dimensional argument. It takes advantage of the fact that the Hamiltonian $H(x, \xi) = c_g(x)\omega(\xi)$ is written in separate variables and $\omega$ is analytic. Since $H(x(t), \xi(t))$ is conserved in time along the solutions of the corresponding Hamiltonian system, the phase component $\xi(t)$ can be obtained directly from this conservation law as

$$\xi(t) = \omega^{-1}(\frac{c_g(x_0)\omega(\xi_0)}{c_g(x(t))}).$$

Using this formula for $\xi(t)$, it is enough to solve the equation of $x(t)$ which becomes

$$x'(t) = \pm c_g(x(t))\omega'(\omega^{-1}(\frac{c_g(x_0)\omega(\xi_0)}{c_g(x(t))})), \quad x(0) = x_0.$$

This last nonlinear ODE is well posed by the classical Cauchy–Lipschitz theorem for $c_g \in C^{0,1}$ which is the regularity we use along this paper.

1.4 Outline of the Paper

In Sect. 2, we present some basic facts on the continuous Wigner transform, state the main existing results concerning the limit process in the Wigner transforms of the solutions of the continuous transport and wave Eqs. (1.1) and (1.4), and describe the main ideas of the proofs that will help us to better understand the corresponding proofs in the discrete case. In Sect. 3, we introduce the numerical approximation schemes under consideration, the notion of discrete Wigner transform, some of its main properties, and our main results, Theorems 3.3 and 3.5. Section 4 is devoted to the proof
of our two main results, Theorems 3.3 and 3.5. In Sect. 5, we present some numerical simulations of high-frequency wave packets and compare the numerical results to the ones predicted by our theoretical analysis. In Sect. 6, we present the conclusions of the paper and list some related open problems. For the sake of completeness, in “Appendix,” we give the main steps for the proofs of the two convergence results, Propositions 3.1 and 3.4.

2 Preliminaries on Continuous Models and Wigner Transforms

2.1 Bi-characteristic Rays for the Continuous Transport and Wave Equations

(a) Transport equation. Eq. (1.3) can be also obtained by solving the Hamiltonian system of null bi-characteristic rays associated with the principal symbol (1.8)

\[
\begin{aligned}
Y'(s) &= \partial_\xi \varphi = -1, \\
\Xi'(s) &= -\partial_\xi \varphi = \rho'(Y(s))\tau(s), \\
\end{aligned}
\]

subjected to initial data \((Y(0), \tau(0), \Xi(0), \tau(0)) = (y_0, 0, \xi_0, \tau_0)\) such that \(\varphi(0, 0, \xi_0, \tau_0) = 0\). Let us denote by \((y(t), \xi(t)) = (Y(s), \Xi(s))\) which solves the Hamiltonian system

\[
y'(t) = c(y(t)), \quad \xi'(t) = -c'(y(t))\xi(t), \quad y(0) = y_0, \quad \xi(0) = \xi_0. \quad (2.2)
\]

Note that the first equation in (2.2) is precisely (1.3).

(b) Wave equation. For the wave Eq. (1.4), the null bi-characteristic rays associated with the principal symbol (1.9) are solutions of the following Hamiltonian system

\[
\begin{aligned}
Y'(s) &= \partial_\xi \varphi = 2\sigma(Y(s))\Xi(s), \\
\Xi'(s) &= -\partial_\xi \varphi = \rho'(Y(s))\tau^2(s) - \sigma'(Y(s))\Xi^2(s), \\
\end{aligned}
\]

subjected to initial data \((Y(0), \tau(0), \Xi(0), \tau(0)) = (y_0, 0, \xi_0, \tau_0)\) such that \(\varphi(0, 0, \xi_0, \tau_0) = 0\). Then, \(\tau(s) = \tau_0\) and \(\varphi(Y(s), \tau(s), \Xi(s), \tau(s)) = 0\) are constant in \(s\). Let us denote by \((y^\pm(t), \xi^\pm(t)) = (Y(s), \Xi(s))\) the two families of solutions of (2.3) corresponding to one of the two possible roots \(\tau_0^\pm\) of \(\varphi(Y(s), \tau(s), \Xi(s), \tau(s)) = 0\) given by \(\tau_0^\pm := \pm\Xi(s)c(Y(s))\), with \(c := \sqrt{\sigma/\rho}\). Then, \((y^\pm(t), \xi^\pm(t))\) is the solution of the following Hamiltonian system of first-order ODEs in the time variable \(t\):

\[
(y^\pm)'(t) = \mp c(y^\pm(t)), \quad (\xi^\pm)'(t) = \pm c'(y^\pm(t))\xi^\pm(t), \quad y^\pm(0) = y_0, \quad \xi^\pm(0) = \xi_0. \quad (2.4)
\]

Note that the first equation in (2.2) or (2.4) can be solved independently of the second one. As we will see, this is not the case for Hamiltonian systems corresponding to the numerical approximations for the wave and transport equations.

When \(c \equiv 1\), one can observe from (2.4) that, for each \(\xi_0\), there are two characteristics going out from each point \(y_0, y^\pm(t) = y_0 \mp t\). Moreover, if we consider
the homogeneous Dirichlet boundary value problem for the wave equation (say, on the interval \((-1, 1)\)), each characteristic reaches the boundary in an uniform time not depending on the frequency \(\xi_0\). This coincides with our intuition on the wave propagation and has several applications in control and inverse problems.

2.2 Continuous Pseudo-differential Operators and Wigner Transforms

In the continuous setting, given a \(d \times d\) matrix-valued function \(\Theta(y, \xi)\), we define the associated pseudo-differential operator \(\Theta(y, \epsilon \partial_y)\) by (cf. [22])

\[
\Theta(y, \epsilon \partial_y) f(y) := \frac{1}{2\pi} \int_{\mathbb{R}} \Theta(y, \epsilon \xi) \hat{f}(\xi) \exp(i\xi y) \, d\xi,
\]

where \(f(y) = (f_1(y), \ldots, f_d(y))\) is a (column) vector function and \(\hat{f}(\xi) = (\hat{f}_1(\xi), \ldots, \hat{f}_d(\xi))\) is the column vector containing the Fourier transforms of its components.

For \(\epsilon > 0\) and \(f^1(x) = (f^1_1(x), \ldots, f^1_d(x))\) and \(f^2(x) = (f^2_1(x), \ldots, f^2_d(x))\) being two (column) vector-valued functions, we define the so-called Wigner transform matrix at scale \(\epsilon\) of \(f^1\) and \(f^2\) as follows (\(A \otimes B\) is the tensor product of the two matrices \(A\) and \(B\), while \(A^*\) is the conjugate transpose of the matrix \(A\)):

\[
W^\epsilon[f^1, f^2](x, \xi) := \frac{1}{2\pi} \int_{\mathbb{R}} f^1(x - \frac{\epsilon z}{2}) \otimes f^{2,*}(x + \frac{\epsilon z}{2}) \exp(i \xi z) \, dz
\]

\[
= \frac{1}{\epsilon} \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \hat{f}^1\left(\frac{\xi}{\epsilon} + \frac{\eta}{2}\right) \otimes \hat{f}^{2,*}\left(\frac{\xi}{\epsilon} - \frac{\eta}{2}\right) \exp(i \eta x) \, d\eta.
\]

Set \(\mathcal{W}^\epsilon[f^1] := \mathcal{W}^\epsilon[f^1, f^1]\). When \(d = 1\), the tensor product \(\otimes\) becomes simply the multiplication of functions. In that case, we will remove the bold character of the arguments and of \(\mathcal{W}\), and we will write \(\mathcal{W}^\epsilon[f^1, f^2](x, \xi)\) for the scalar Wigner transform of two functions \(f^1(x)\) and \(f^2(x)\). Moreover, when the functions \(f^1\) and \(f^2\) depend also on the time variable \(t\), we denote by \(\mathcal{W}^\epsilon[f^1, f^2](x, t, \xi)\) the corresponding Wigner transform of \(f^1\) and \(f^2\).

Note the following two properties of the Wigner transform:

\[
\int_{\mathbb{R}} \mathcal{W}^\epsilon[f^1, f^2](x, \xi) \, d\xi = f^1(x) \otimes f^{2,*}(x),
\]

\[
\int_{\mathbb{R}} \mathcal{W}^\epsilon[f^1, f^2](x, \xi) \, dx = \frac{1}{2\pi \epsilon} \hat{f}^1\left(\frac{\xi}{\epsilon}\right) \otimes \hat{f}^{2,*}\left(\frac{\xi}{\epsilon}\right).
\]

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2.3 Existing Results on Wigner Measures for Continuous Transport Equations

Set \( w(y, t) := \sqrt{\varrho(y)}u(y, t) \), where \( u \) is the solution of (1.1). Then, \( w \) satisfies the transport equation

\[
\partial_t w(y, t) = -c(y)\partial_y w(y, t) - d(y)w(y, t),
\]

\( w(y, 0) = w^0(y) := \sqrt{\varrho(y)}u^0(y), \quad y \in \mathbb{R}, \ t > 0. \tag{2.8} \)

The coefficients \( c \) and \( d \) in (2.8) are given by

\[
c(y) := \frac{1}{\varrho(y)} \quad \text{and} \quad d(y) := \frac{1}{2}\left(\frac{1}{\varrho}\right)'(y). \tag{2.9} \]

The \( L^2(\mathbb{R}) \)-norm of \( w \) is time conservative. For all \( \epsilon > 0 \), any initial datum \( u^0 \) in (1.1), for \( w \) being the solution of (2.8) with \( u^0 = \sqrt{\varrho}u^0 \) and using property (2.7), we can express the time conservative total energy \( \mathcal{E}_\rho(u(\cdot, t)) \) in (1.2) of the solution \( u \) of (1.1) in terms of the Wigner transform at scale \( \epsilon \) of \( w \) as follows:

\[
\mathcal{E}_\rho(u(\cdot, t)) = ||w(\cdot, t)||^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} W^\epsilon[w](x, t, \xi) \, dx \, d\xi \tag{2.10}.
\]

Using the arguments in [22, 27] or [33], the following result concerning the transport Eq. (2.8) can be proved (for the sake of completeness, we will give a sketch of this proof in “Appendix”):

**Theorem 2.1** For any coefficient \( c \in C^{0,1}(\mathbb{R}) \) in (2.8) and any initial data \( w^0 = w^{\epsilon,0} \) in (2.8) bounded in \( L^2(\mathbb{R}) \) as \( \epsilon \to 0 \), there exists a positive Radon measure \( \mathcal{W}(y, t, \xi) \) defined on \( \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \) such that, eventually after extracting subsequences, we get

\[
\mathcal{W}^\epsilon[w^\epsilon](y, t, \xi) \rightharpoonup \mathcal{W}(y, t, \xi) \quad \text{weakly star in} \quad \mathcal{S}'(\mathbb{R}_y \times \mathbb{R}_\xi), \tag{2.11}
\]

uniformly on each compact set of the time interval \( \mathbb{R}_+ \). Moreover, the measure \( \mathcal{W} \) satisfies the following Liouville equation:

\[
\partial_t \mathcal{W}(y, t, \xi) = -P_{c,\xi}\mathcal{W}(y, t, \xi), \quad \mathcal{W}(y, 0, \xi) = \mathcal{W}^0(y, \xi) := \lim_{\epsilon \to 0} \mathcal{W}^\epsilon[w^{\epsilon,0}](y, \xi). \tag{2.12}
\]

For all \( C^{0,1} \)-functions \( c = c(x) \) and \( \omega = \omega(\xi) \), the operator \( P_{c,\omega} \) is defined as follows:

\[
P_{c,\omega} := c(x)\omega'(\xi)\partial_x - c'(x)\omega(\xi)\partial_\xi. \tag{2.13}
\]
2.4 Existing Results on Wigner Measures for Continuous Wave Equations

For $u$ being the solution of (1.4), set $(w(y, t), \tilde{w}(y, t)) := (\sqrt{\rho(y)}\partial_y u(y, t), \sqrt{\sigma(y)}\partial_y u(y, t))$. Then, $(w, \tilde{w})$ (whose $(L^2(\mathbb{R}))^2$-norm is conserved in time) satisfies the system of first-order PDEs:

\[
\begin{cases}
\left( \begin{array}{c}
\partial_t w(y, t) \\
\partial_y \tilde{w}(y, t)
\end{array} \right) = \mathcal{A} \left( \begin{array}{c}
w(y, t) \\
\tilde{w}(y, t)
\end{array} \right), \quad \forall y \in \mathbb{R}, \ t > 0, \\
(w(y, 0), \tilde{w}(y, 0)) = (w^0(y), \tilde{w}^0(y)) := (\sqrt{\rho(y)}u^1(y), \sqrt{\sigma(y)}(u^0)'(y))
\end{cases}
\]

(2.14)

The coefficients $c$, $d$, and $e$ are given by

\[
c(y) := \sqrt{\sigma(y)} / \rho(y), \quad d(y) := \frac{\sigma'(y)}{2\sqrt{\sigma(y)}\rho(y)} \quad \text{and} \quad e(y) := -\frac{\sqrt{\sigma(y)}\rho'(y)}{2\rho(y)\sqrt{\rho(y)}}
\]

(2.15)

so that

\[
d(y) + e(y) = c'(y) \quad \text{and} \quad d(y) - e(y) = \frac{(\sqrt{\rho\sigma})'(y)}{\rho(y)}.
\]

(2.16)

Let $J_d$ be the $d$-dimensional exchange matrix, i.e.,

\[
J_d := \begin{pmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
1 & \cdots & 0 & 0
\end{pmatrix}.
\]

Since $\mathcal{A}^* = -\mathcal{A} + (d + e - c')J_2$, the operator $\mathcal{A}$ in (2.14) is anti-self-adjoint if and only if $d + e = c'$. This is precisely our case.

Let us introduce the following notation $w^\pm(y, t) = (w(y, t) \pm \tilde{w}(y, t))/\sqrt{2}$, where $(w(y, t), \tilde{w}(y, t))$ is the solution of (2.14). Then, the pair $(w^+(y, t), w^-(y, t))$ solves the following coupled system of PDEs:

\[
\begin{cases}
\left( \begin{array}{c}
w^+_t(y, t) \\
w^-_t(y, t)
\end{array} \right) = \tilde{\mathcal{A}} \left( \begin{array}{c}
w^+(y, t) \\
w^-(y, t)
\end{array} \right), \quad \text{with} \quad \tilde{\mathcal{A}} := \begin{pmatrix}
c(y)\partial_y + \frac{1}{2}c'(y) & -\frac{1}{2}\tilde{c}(y) \\
\frac{1}{2}\tilde{c}(y) & -c(y)\partial_y - \frac{1}{2}c'(y)
\end{pmatrix}, \\
w^\pm(y, 0) = w^{0, \pm}(y) := \frac{w^0(y)\pm w^0(y)}{\sqrt{2}}.
\end{cases}
\]

(2.17)

Here, $\tilde{c}(y) := d(y) - e(y)$ ($d$, $e$ as in (2.15)). The operator $\tilde{\mathcal{A}}_{11}$ is precisely the one involved in the scalar transport Eq. (2.8) (with a different $c := \sqrt{\sigma/\rho}$), while $\tilde{\mathcal{A}}_{22} = -\tilde{\mathcal{A}}_{11}$. System (2.17) is similar to the one verified by $v_{n, \pm}$ in [21], (4.28), $\partial_t v_{n, \pm} = \pm i|\nabla|v_{h, \pm}$ for $t$ constant coefficient multidimensional wave equation ($|\nabla|$ being the pseudo-differential operator generated by the symbol $|\xi|$).
Set
\[ \mathcal{W}^{\epsilon, \pm} := \mathcal{W}^{\epsilon}[w^{\pm}] \text{ and } \tilde{\mathcal{W}}^{\epsilon, \pm} := \mathcal{W}^{\epsilon}[w^{+}, w^{-}] \pm \mathcal{W}^{\epsilon}[w^{-}, w^{+}], \] (2.18)
where \( \mathcal{W}^{\epsilon} \) is the Wigner transform at scale \( \epsilon \) in (2.6) and \( w^{\pm} \) is the solution of (2.17).

Note that, for all \( \epsilon > 0 \), the time conservative energy \( E_{\rho, \sigma}(u_{t}(\cdot, t), u_{t}(\cdot, t)) \) in (1.5) of the solution of the wave Eq. (1.4) admits the following equivalent representations in terms of the solution \((w, \tilde{w})\) of (2.14), the solution \((w^{+}, w^{-})\) of (2.17) and the Wigner transform of \((w^{+}, w^{-})\):
\[
E_{\rho, \sigma}(u_{t}(\cdot, t), u_{t}(\cdot, t)) = \frac{1}{2} \int_{\mathbb{R}} (|w(y, t)|^2 + |\tilde{w}(y, t)|^2) dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} (|w^{+}(y, t)|^2 + |w^{-}(y, t)|^2) dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \text{tr} \mathcal{W}^{\epsilon}[w^{+}, w^{-}](y, t, \xi) dy d\xi
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (\mathcal{W}^{\epsilon,+}(y, t, \xi) + \mathcal{W}^{\epsilon,-}(y, t, \xi)) dy d\xi. \quad (2.19)
\]

The following result explains the behavior of \( \mathcal{W}^{\epsilon} := \text{tr} \mathcal{W}^{\epsilon}[w^{+}, w^{-}] \) as \( \epsilon \to 0 \):

**Theorem 2.2** For any coefficient \( c \in C^{0,1}(\mathbb{R}) \) and any initial data \((w^{\epsilon,0,+}, w^{\epsilon,0,-})\) in (2.17) bounded in \((L^{2}(\mathbb{R}))^{2}\) as \( \epsilon \to 0 \), there exists a positive Radon measure \( \mathcal{W}(y, t, \xi) \) defined on \( \mathbb{R} \times \mathbb{R}_{+} \times \mathbb{R} \) such that, eventually after extracting subsequences, (2.11) holds uniformly on each compact set of the time interval \( \mathbb{R}_{+} \).

Moreover, \( \mathcal{W} \) can be split into two positive Radon measures as \( \mathcal{W} = \mathcal{W}^{+} + \mathcal{W}^{-} \), where \( \mathcal{W}^{\pm} := \lim_{\epsilon \to 0} \mathcal{W}^{\epsilon, \pm} \) is the Wigner measures of \( w^{\epsilon, \pm} \) solving (2.17) with initial data \( w^{\epsilon,0,\pm} \). Each Wigner measure \( \mathcal{W}^{\pm} \) satisfies the Liouville equation
\[
\partial_{t}\mathcal{W}^{\pm}(y, t, \xi) = \pm P_{c, \xi} \mathcal{W}^{\pm}(y, t, \xi),
\]
\[
\mathcal{W}^{\pm}(y, 0, \xi) = \lim_{\epsilon \to 0} \mathcal{W}^{\epsilon, \pm}(y, 0, \xi), \quad y, \xi \in \mathbb{R}, \ t > 0, \quad (2.20)
\]
where the operator \( P_{c, \xi} \) is as in (2.13) with \( \omega(\xi) = \xi \).

### 2.5 Concentrated and Oscillatory Initial Data

The most singular initial data for the transport and wave Eqs. (1.1) and (1.4) are the highly concentrated and oscillatory ones, i.e.,
\[
u_{0}^{0}(y) = u_{0}^{\epsilon,0}(y) := \epsilon^{\alpha} f \left( \frac{y - y_{0}}{\epsilon^{\beta}} \right) \exp \left( \frac{iy_{0} \xi_{0}}{\epsilon} \right), \quad 2\alpha + \beta = 0, \ 0 < \beta < 1, \quad (2.21)
\]
with \( f \in L^2(\mathbb{R}) \) for the transport equation, and

\[
\begin{align*}
\quad u^0(y) &= u^{\varepsilon,0}(y) := \varepsilon^\alpha f \left( \frac{y-y_0}{\varepsilon^\beta} \right) \exp \left( \frac{iy\xi_0}{\varepsilon} \right), \\
\quad u^1 &= u^{\varepsilon,1} := cu^{\varepsilon,0}_y, \quad 2(\alpha - 1) + \beta = 0, \ 0 < \beta < 1, \tag{2.22}
\end{align*}
\]

with \( f \in H^1(\mathbb{R}), (y_0, \xi_0) \in \mathbb{R}^2, \xi_0 \neq 0 \) and \( c \) as in (2.15) for the wave equation (see [36] for \( \beta = 1/2 \)).

The role of the scaling factors \( \varepsilon^\alpha \) in (2.22) (and similarly in (2.21)) is to make the total energy \( \mathcal{E}_{\rho,\sigma}(u^{\varepsilon,0}, u^{\varepsilon,1}) \) to be bounded as \( \varepsilon \to 0 \). These initial data are concentrated at scale \( \varepsilon^{-\beta} \) through the factor \( f((y-y_0)e^{-\beta}) \) around the point \( y_0 \) in space and oscillate at wavelength \( \varepsilon \) in the direction \( \xi_0 \). The first condition \( 2(\alpha - 1) + \beta = 0 \) on the exponents \( \alpha \) and \( \beta \) is to guarantee that the total energy of the initial data is uniformly bounded as \( \varepsilon \to 0 \). Indeed,

\[
\mathcal{E}_{\rho,\sigma}(u^{\varepsilon,0}, u^{\varepsilon,1}) = ||\sqrt{\sigma} u^{\varepsilon,0}_y||^2_{L^2} \sim \varepsilon^{2\alpha - \beta} ||f'||^2_{L^2} + \varepsilon^{2(\alpha - 1) + \beta} \xi_0^2 ||f||^2_{L^2}.
\]

Moreover, for this total energy to be of order \( O(1) \), there are two options: i) \( 2\alpha - \beta = 0 \) and \( 2(\alpha - 1) + \beta \geq 0 \) or ii) \( 2\alpha - \beta \geq 0 \) and \( 2(\alpha - 1) + \beta = 0 \). In both cases, \( \alpha \geq 1/2 \), while \( \beta \geq 1 \) and \( \beta \leq 1 \) for the first/second inequalities system. The fact that \( \beta > 0 \) is needed to obtain concentrated initial data. We can exclude the case \( \beta \geq 1 \) since we want the semiclassical measure \( \mathcal{W}^+(y, 0, \xi) \) of \( u^{\varepsilon,0, +} = (\sqrt{\sigma} u^{\varepsilon,0}_y + \sqrt{\rho} u^{\varepsilon,1})/\sqrt{2} = \sqrt{2\sigma} u^{\varepsilon,0}_y \) to be \( \delta_{y_0}(y) \otimes \delta_{\xi_0}(\xi) \) (modulo a multiplicative constant). More precisely, for the choice (2.22) of the initial data \( (u^0, u^1) = (u^{\varepsilon,0}, u^{\varepsilon,1}) \) in (1.4), we can prove that

\[
\mathcal{W}^+(y, 0, \xi) = 2\sigma (y_0) \xi_0^2 ||f||^2_{L^2} \delta_{y_0}(y) \otimes \delta_{\xi_0}(\xi). \tag{2.23}
\]

Consequently, at future times \( t > 0 \), the Wigner measure \( \mathcal{W}^+ \), which is the solution of the transport Eq. (2.20) in phase space takes the explicit form \( \mathcal{W}^+(y, t, \xi) = 2\sigma (y_0) \xi_0^2 ||f||^2_{L^2} \delta_{y_0}^{-}(y) \otimes \delta_{\xi_0}^{-}(\xi) \) and propagates along the characteristics \( (y^{-}(t), \xi^{-}(t)) \) being the solution of the ODE system (2.4) corresponding to the initial data \( (y_0, \xi_0) \). For the same initial data (2.22) in (1.4), \( \mathcal{W}^-(y, 0, \xi) \) in (2.20) vanishes. Note that if \( f \) in (2.21) or (2.22) is compact supported, then the support of order \( \varepsilon^\beta \) of the envelope \( f((y-y_0)/\varepsilon^\beta) \) is asymptotically larger as \( \varepsilon \to 0 \) than the wavelength of order \( \varepsilon \) of the data \( u^{\varepsilon,0} \) when \( \beta < 1 \).

2.6 Interpretation of Theorems 2.1 and 2.2

The corresponding solution of (1.1) is driven by characteristics, so that if the initial datum \( u^0 = u^{\varepsilon,0} \) is concentrated at some point, the solution \( u^\varepsilon(\cdot, t) \) is necessarily concentrated at any further time \( t > 0 \) for all \( \varepsilon > 0 \). However, the analysis of the concentration of energy for the solutions of the second-order wave equation is more subtle. More precisely, the energy density \( d[u^\varepsilon, u^\varepsilon_\sharp] \) corresponding to initial data \( (u^{\varepsilon,0}, u^{\varepsilon,1}) \)
as in (2.22) propagates along the space component \(y^-(t)\) of the solution \((y^-(t), \xi^-(t))\) of (2.4) starting at \((y_0, \xi_0)\), in the sense that

\[
\lim_{\epsilon \to 0} \int_{D_r(t)} d[u^\epsilon, u^\epsilon_t](y, t) \, dy = 0,
\]

\[
d[u, u_t](y, t) := \frac{1}{2} \left( q(y)|u_t(y, t)|^2 + \sigma(y)|u_y(y, t)|^2 \right)
\]

for all \(r > 0\), with \(D_r(t) := \mathbb{R} \setminus B(y^-(t), r)\) and \(B(x, r) := (x - r, x + r)\). Indeed, using the properties (2.7) of the Wigner transform, we obtain (with \(D_{R, r}(t) := B(0, R) \setminus B(y^-(t), r)\), \(D_R := \mathbb{R} \setminus B(0, R)\)):

\[
\int_{D_r(t)} d[u^\epsilon, u^\epsilon_t](y, t) \, dy = A^\epsilon_R + \frac{1}{2} \int_{D_{R, r}(t)} \left( |w^\epsilon,+(y, t)|^2 + |w^\epsilon,-(y, t)|^2 \right) \, dy
\]

\[
= A^\epsilon_R + \frac{1}{2} \int_{D_{R, r}(t)} \left( \mathcal{W}^{\epsilon, +}(y, t, \xi) + \mathcal{W}^{\epsilon, -}(y, t, \xi) \right) \, d\xi \, dy
\]

\[
= A^\epsilon_R + B^\epsilon_{R, r} + C^\epsilon_{R, r}.
\]

Set \(\chi_A^\delta\) a regularization of size \(\delta\) of the characteristic function of \(A\), \(\chi_A\). The regularization parameter \(\delta\) is chosen such that \(\chi_{D_{R, r}(t)}^\delta(y^-) = 0\). In the above identity, \(A^\epsilon_R\), \(B^\epsilon_{R, r}\) and \(C^\epsilon_{R, r}\) are given by

\[
A^\epsilon_R := \frac{1}{2} \int_{D_R} \left( |w^\epsilon,+(y, t)|^2 + |w^\epsilon,-(y, t)|^2 \right) \, dy,
\]

\[
B^\epsilon_{R, r} := \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{D_{R, r}(t)}^\delta(y) \chi_{B(0, R)}^\delta(\xi) \left( \mathcal{W}^{\epsilon, +}(y, t, \xi) + \mathcal{W}^{\epsilon, -}(y, t, \xi) \right) \, d\xi \, dy,
\]

\[
C^\epsilon_{R, r} := \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{D_{R, r}(t)}^\delta(y) \left( 1 - \chi_{B(0, R)}^\delta(\xi) \right) \left( \mathcal{W}^{\epsilon, +}(y, t, \xi) + \mathcal{W}^{\epsilon, -}(y, t, \xi) \right) \, d\xi \, dy.
\]

For any \(R > |y^-|\), using (2.23), the fact that \(W^- \equiv 0\) and that \(\chi_{D_{R, r}(t)}^\delta(y^-) = 0\), we obtain that \(B^\epsilon_{R, r} \to 0\) as \(\epsilon \to 0\). Since \(w^\epsilon,\pm\) belongs to \(L^2(\mathbb{R})\) uniformly as \(\epsilon \to 0\), then the sequences \(w^\epsilon,\pm\) are compact at infinity in the sense that

\[
\lim_{\epsilon \to 0} \sup_{|y| > R} \int |w^\epsilon,\pm(y, t)|^2 \, dy \to 0 \text{ as } R \to \infty,
\]

so that for each \(\epsilon\), there exists \(R\) large enough such that \(A^\epsilon_R\) is as small as one wants.
3 Main Results and Bibliographical Comments

**Notation 1** By lowercase letters accompanied by the superscript $h$ (e.g., $f^h$), we denote an infinite column vector associating with each node $x_j$ of the uniform grid $G^h$ an unique value $f_j$, while $f^h$ associates with each $x_j$ a $d$-dimensional vector of values $f_j = (f_{j,1}, \ldots, f_{j,d})$.

### 3.1 Discrete Pseudo-differential Operators and Wigner Transforms

Let us start this section by introducing the notions of discrete pseudo-differential operator and Wigner transform that will be systematically used along this section and in Sect. 4.

For a $d \times d$ - matrix-valued function $\Theta^h(x, \xi)$, let us introduce the discrete pseudo-differential operator $\Theta^h(x, h\partial_x)$ by (cf. [39])

$$\Theta^h(x, h\partial_x)f_j := \frac{1}{2\pi} \int_{\Pi^h} \Theta(x_j, h\xi)^\Gamma f(\xi) \exp(i\xi x_j) \, d\xi. \quad (3.1)$$

Here, $fh := (f_j)_{j \in \mathbb{Z}}$, with $f_j = (f_{j,1}, \ldots, f_{j,d})$, $\Pi^h := [-\pi/h, \pi/h]$ and $\Gamma = (\Gamma^h,k)_{1 \leq k \leq d}$ being the column vector containing the semi-discrete Fourier transforms (SDFT) $\Gamma^h,k$ at scale $h$ of the components of $f^h,k = (f^k_j)_{j \in \mathbb{Z}}$. Recall that the SDFT for a sequence $fh := (f_j)_{j \in \mathbb{Z}} \in \ell^2$ is given by (cf. [47]):

$$\Gamma^h,k f(\xi) := h \sum_{j \in \mathbb{Z}} f_j \exp(-i\xi x_j) \quad \forall \xi \in \Pi^h. \quad (3.2)$$

When $\Theta(x, \xi) = A(x)B(\xi)$ is a matrix in separated variables, we denote by $\Theta^h_{A,B}$ the corresponding pseudo-differential operator.

We define the discrete Wigner transform matrix at scale $\epsilon$ to be (see, e.g., [34] or [35]):

$$\mathcal{W}_\epsilon[f^h,1,f^h,2](\frac{X_m}{2}, \xi) := \frac{h}{2\pi \epsilon} \sum_{n=m} f_{m,n}^1 \otimes \left( f_{m,n}^2 \right)^* \exp\left( i \frac{x_n \xi}{\epsilon} \right), \quad (3.3)$$

where $f^h,1, f^h,2$ are two discrete functions associating with each grid point $x_j$ the $d$-dimensional vectors $f^1_j$ and $f^2_j$ and $\xi \in [-\epsilon \pi/h, \epsilon \pi/h]$. Here, and in the sequel, $\equiv$ designs the equivalence modulo 2. In this paper, we restrict the analysis to the cases $d = 1$ and $d = 2$ (corresponding to the discrete $1-d$ transport/wave equation). When $d = 1$, the tensor product on the right-hand side of (3.3) is simply the scalar multiplication. To emphasize the fact that, for $d = 1$, $\mathcal{W}_\epsilon[f^h,1,f^h,2]$ is a scalar quantity, we will replace the bold symbol in $\mathcal{W}_\epsilon[f^h]$ by $\mathcal{W}_\epsilon^h$ and denote by $\mathcal{W}_\epsilon^h[f^h,1,f^h,2]$ the scalar discrete Wigner transform. We also set $\mathcal{W}_\epsilon[f^h] := \mathcal{W}_\epsilon^h[f^h]$. 

\[ \mathcal{W}_\epsilon^h[f^h] := \mathcal{W}_\epsilon^h[f^h,1,f^h,2] \]
The analysis in this paper is restricted to the discrete solutions corresponding to the highest frequencies for which \( \epsilon = h, \xi \in [-\pi, \pi] \) and \( \exp(ix_n \xi / \epsilon) = \exp(in\xi) \). Observe that when \( m \) is even, \( \mathcal{W}^h[f^{h,1}, f^{h,2}] (x_m / 2, \cdot) \) is a \( \pi \)-periodic function in \( \xi \), while when \( m \) is odd, \( \mathcal{W}^h[f^{h,1}, f^{h,2}] (x_m / 2, \cdot) \) is \( 2\pi \)-periodic in \( \xi \). Thus, for any value of \( m \in \mathbb{Z} \), \( \mathcal{W}^h[f^{h,1}, f^{h,2}] (x_m / 2, \cdot) \) is \( 2\pi \)-periodic. This justifies the fact that, for the discrete case, \([-\pi, \pi]\) is the suitable choice for the domain of the phase variable \( \xi \).

**Notation 2** Let us introduce the Hilbert space \( l^2 \) of square summable discrete functions with inner product given by \( (f^{h,1}, f^{h,2}) := h \sum_{j \in \mathbb{Z}} f_{j}^{h,1} f_{j}^{h,2} \) and \( \| \cdot \|_{l^2} \) be the corresponding norm. The space \( l^1 \) contains the discrete functions \( f^h \) such that \( \partial_h^+ f^h \in l^2 \) and is endowed with the inner product \( (f^{h,1}, f^{h,2}) := (\partial_h^+ f^{h,1}, \partial_h^+ f^{h,2})_{l^2} \) and the norm \( \| \cdot \|_{l^1} \). We also employ the notation \( x^h := (x_j)_{j \in \mathbb{Z}} \) and \( f(x^h) := (f(x_j))_{j \in \mathbb{Z}} \), for any continuous function \( f \).

### 3.2 The Discrete Transport Equation

The finite difference numerical scheme (1.6) approximating the transport Eq. (1.1) converges in the classical sense of numerical analysis. More precisely, the following convergence result (whose proof is provided in “Appendix”) holds:

**Proposition 3.1** Assume that \( \varrho \in C^1(\mathbb{R}) \) in (1.1) is such that there exists \( \varrho^+, \varrho^- \) so that \( 0 < \varrho^- \leq \varrho(y) \leq \varrho^+ \) and \( |\varrho'(y)| \leq \varrho^+ \) for all \( y \in \mathbb{R} \) and that the initial datum \( u^0 \) belongs to \( C_{d}^2(\mathbb{R}) \). We also consider quasi-uniform grids, i.e., non-uniform grids \( G^h \) given by \( g \in C^1(\mathbb{R}) \) such that there exists \( g_{d}^\pm \) so that \( 0 < g_{d}^- \leq |g'(x)| \leq g_{d}^+ \) for all \( x \in \mathbb{R} \). Under these hypotheses, the numerical approximation scheme (1.6) with initial datum \( u^{h,0} := (u^0(g_j))_{j \in \mathbb{Z}} \) converges with order \( O(h) \) to the solution of the transport Eq. (1.1) in the \( l^2 \)-norm, i.e., there exists a constant \( C(t, g, \varrho, u^0) \) independent of \( h \) such that

\[
\| \mathbf{u}^h(t) - u(g(x^h), t) \|_{l^2} \leq hC(t, g, \varrho, u^0). \tag{3.4}
\]

The function \( v(x, t) := u(g(x), t) \), where \( u(y, t) \) is the solution of (1.1), solves the transport equation

\[
g'(x)\varrho(g(x))v_t(x, t) + v_x(x, t) = 0, \quad v(x, 0) = u(g(x), 0) = u^0(g(x)), \quad x \in \mathbb{R}, \ t > 0. \tag{3.5}
\]

Thus, by reconsidering the values \( u_j(t) \) associated initially with the points of the non-uniform grid as being values \( v_j(t) = u_j(t) \) at points of the uniform grid, we obtain that \( v_j(t) \) is the solution of the following finite difference scheme for the transport Eq. (3.5) on the uniform grid \( G^h \):

\[
\partial_h^+ g_{j} \varrho(g_{j}) \partial_t v_{j}(t) + \partial_h^+ v_{j}(t) = 0, \quad v_{j}(0) = v_{j}^0 = u_{j}^0, \ j \in \mathbb{Z}. \tag{3.6}
\]
Indeed, from Proposition 3.1, we see that $\psi^h(t) \sim v(x^h, t)$, where $v(x, t)$ and $\psi^h(t)$ are the solutions of (3.5) and (3.6). This idea of reorganizing numerical schemes on non-uniform meshes in order to obtain approximations of PDEs of the same type with different variable coefficients on uniform meshes is not new, and it has been used, for example, in [44] to construct preconditioning strategies for systems arising from discretizations of elliptic PDEs on non-uniform meshes based on known algorithms for approximating PDEs with variable coefficients on uniform meshes.

The null bi-characteristic lines associated with the principal symbol (1.10) of the discrete transport Eq. (3.6) are the solutions of the following system

$$
\begin{align*}
X'(s) &= \partial_{\xi} \varphi = - \cos(\Xi(s)), & \xi'(s) &= \partial_{\tau} \varphi = - g'(X(s)) \varrho(g(X(s))), \\
\Xi'(s) &= - \partial_{x} \varphi = \tau (g'(. \varrho(g(.)))'(X(s))), & \tau'(s) &= - \partial_{t} \varphi = 0,
\end{align*}
$$

subjected to the initial data $(X(0), \tau(0), \Xi(0), \varrho(0)) = (x_0, 0, \xi_0, \tau_0)$ such that $\varphi(x_0, 0, \xi_0, \tau_0) = 0$. Then, $(\tau(s) = \tau_0$ and $\varphi(X(s), \tau(s), \Xi(s), \varrho(s))) = 0$ are conserved in $s$. Set $(x(t), \xi(t)) := (X(s), \Xi(s))$ to be the solutions of (3.7) as functions of $t$. Then, $(x(t), \xi(t))$ is the solution of the following Hamiltonian system:

$$
\begin{align*}
x'(t) &= c_g(x(t)) \cos(\xi(t)), & \xi'(t) &= - c'_g(x(t)) \sin(\xi(t)), \\
x(0) &= x_0, & \xi(0) &= \xi_0, & c_g(x) := \frac{1}{g'(x)}.
\end{align*}
$$

(3.8)

The characteristic rays on the non-uniform grid $G^h_g$ are the curves $(y(t) := g(x(t)), \xi(t))$, with $(x(t), \xi(t))$ being the solution of (3.8). Therefore, $(y(t), \xi(t))$ is the solution of the non-Hamiltonian system below, with $c$ as in (2.9) and $c_g$ as in (3.8), but for which the quantity $\sin(\xi(t))/c_g(g^{-1}(y(t)))$ is time conservative:

$$
\begin{align*}
y'(t) &= c(y(t)) \cos(\xi(t)), & \xi'(t) &= - c'_g(g^{-1}(y(t))) \sin(\xi(t)), \\
y(0) &= y_0 := g(x_0). & \xi(0) &= \xi_0.
\end{align*}
$$

(3.9)

As in the continuous case, we will not apply the discrete Wigner transform directly on the solution $v_j$ of (3.6), but on $w_j := v_j(j) / \sqrt{\delta^h g_j \rho(g_j)}$ whose $\ell^2$-norm is conserved in time.

**Lemma 3.2** For the solution $v^h(t)$ of (3.6), set $w_j(t) := v_j(t) / \sqrt{\delta^h g_j \rho(g_j)}$. The discrete function $w^h(t)$ conserves its $\ell^2$-norm and is the solution of the system

$$
\begin{align*}
\partial_t w_j(t) &= - \gamma_j \partial^h w_j(t) - \frac{1}{2} \delta_j (w_{j+1}(t) + w_{j-1}(t)), \\
w_j(0) &= \frac{1}{\alpha_j} v_j^0 := \frac{1}{\alpha_j} u_j^0, \quad j \in \mathbb{Z}.
\end{align*}
$$

(3.10)

Here, $\gamma_j := \gamma^h(x_j)$ and $\delta_j := \delta^h(x_j)$, where

$$
\gamma^h(x) := \frac{\alpha^h(x)(\alpha^h(x + h) + \alpha^h(x - h))}{2}.
$$
\[ \delta^h(x) := \alpha^h(x) \partial^h \alpha^h(x) \text{ and } \alpha^h(x) := \frac{1}{\sqrt{\partial^h g(x) g(g(x))}}. \quad (3.11) \]

Remark that the following approximations hold:

\[ \gamma^h(x) \sim c_g(x), \quad \delta^h(x) \sim \frac{1}{2} c'_g(x) \quad \text{and} \quad \alpha^h(x) \sim \sqrt{c_g(x)} \quad \text{as } h \to 0, \]

with \( c_g(x) := \frac{1}{g(g(x)) g'(x)} \).

\[ (3.12) \]

**Proof of Lemma 3.2** By replacing \( v_j(t) = \alpha_j w_j(t) \) in (3.6) (which can be written as \( \partial_t v_j(t) / \alpha_j^2 + \partial^h v_j(t) = 0 \)), we obtain

\[ \partial_t w_j(t) + \frac{\alpha_{j+1} \alpha_j w_{j+1}(t) - \alpha_j \alpha_{j-1} w_{j-1}(t)}{2h} = 0, \quad (3.13) \]

but (3.13) can be reorganized as (3.10) using the obvious identity \( ab - cd = (a + c)(b - d)/2 + (a - c)(b + d)/2 \) in the particular case \( a = \alpha_{j+1} \alpha_j, b = w_{j+1}(t), c = \alpha_j \alpha_{j-1} \) and \( d = w_{j-1}(t) \).

System (3.10) is just a finite difference discrete analogue of (2.8) on the uniform mesh \( \mathcal{G}^h \) in which \( c(y) \) is replaced by \( c_g(x) \). From now on, unless stated, we will use only the discrete transport Eq. (3.10) since it is the one conserving the \( \ell^2 \)-norm of the numerical solution. As we will see, the Wigner measure \( W := \lim_{h \to 0} W^h[W^h] \) of the solution \( w^h(t) \) propagates along the bi-characteristic curves \( (x(t), \xi(t)) \) described by (3.8). Having the Eq. (3.10) defined on the uniform grid \( \mathcal{G}^h \) has the advantage that, using the results in [35], we know how to pass to the limit in the Wigner transform of the initial data \( w_{h,0} \) in (2.21) (thus, \( \alpha = -\beta/2 \) and any \( \beta \in (0,1) \)) on the uniform grid \( \mathcal{G}^h \), then the corresponding Wigner measure is \( W(x,0,\xi) = \delta_{y_0}(x) \otimes \delta_{\xi_0}(\xi) \).

Let \( \lambda_{c,\omega}(x, \xi) = c(x) \omega(\xi) \) be a scalar function in separated variables. Using the notation for pseudo-differential operators in Sect. 3.1, the discrete transport Eq. (3.10) can be written as

\[ \partial_t w^h(t) = -\frac{i}{h} \lambda_{g,h,\omega}(x, h \partial_x) w^h(t) - \lambda_{g,h,\omega}(x, h \partial_x) w^h(t), \quad \text{with } \omega(\xi) = \sin(\xi). \quad (3.14) \]

Our first result (to be proved in Sect. 4) describes the asymptotic behavior as \( h \to 0 \) of the discrete Wigner transform \( W^h[W^h] \), where \( w^h(t) \) is the solution of the transport Eq. (3.10).

**Theorem 3.3** For any coefficient \( g \) in (1.1), any non-uniform grid \( \mathcal{G}_g^h \) obtained by a transformation \( g \) so that \( c_g \) in (3.12) belongs to \( \mathcal{C}^{0,1}(\mathbb{R}) \) and for any initial data \( w_{h,0} \) in (3.10) bounded in \( \ell^2 \) as \( h \to 0 \), there exists a positive Radon measure \( W = W(x, t, \xi) \) defined on \( \mathbb{R} \times \mathbb{R}^+ \times [-\pi, \pi] \) such that, eventually after extracting a subsequence, we get
\[ \mathcal{W}^h[w^h](x, t, \xi) \rightarrow \mathcal{W}(x, t, \xi) \text{ weakly star in } \mathcal{S}'(\mathbb{R}_+) \times \mathcal{D}'([-\pi, \pi]), \quad (3.15) \]

uniformly on each compact set of the time interval \( \mathbb{R}_+ \). Moreover, the measure \( \mathcal{W} \) satisfies the Liouville equation (we follow the terminology in [33])

\[ \partial_t \mathcal{W}(x, t, \xi) = -P_{c, \omega} \mathcal{W}(x, t, \xi), \quad \mathcal{W}(x, 0, \xi) = \mathcal{W}^0(x, \xi) : = \lim_{h \to 0} \mathcal{W}^h[w^{h,0}](x, \xi) \quad (3.16) \]

with \( \omega(\xi) = \sin(\xi) \) and the operator \( P_{c, \omega} \) defined as in (2.13).

Remark 1 Theorem 3.3 states that the Wigner measure corresponding to the discrete transport Eq. (3.10) propagates according to a transport equation in both phase-space variables \( (x, \xi) \) whose solution propagates along the characteristic rays described by the Hamiltonian system (3.8).

Remark 2 A particular class of bounded initial data in \( \ell^2 \) as \( h \to 0 \) is given by \( w^{h,0} : = (u^{h,0}(x_j))_{j \in \mathbb{Z}} \), with \( u^{h,0} \) as in (2.21).

Remark 3 The dynamics of the solution to the discrete problem (3.10) is determined by the symbol \( \lambda_{p^{h,0}}(x, \xi) \sim \lambda_{c, \omega}(x, \xi) = c(x)\omega(\xi) \) (\( \omega(\xi) = \sin(\xi) \)) generating the leading pseudo-differential operator in (3.14) (i.e., the term of order \( h^{-1} \) in the right-hand side of (3.14)).

3.3 The Discrete Wave Equation

The following convergence result for the solution of the discrete wave Eq. (1.7) holds (the corresponding proof is provided in “Appendix”):

Proposition 3.4 Assume \( \rho \in C^1(\mathbb{R}) \) and \( \sigma \in C^2(\mathbb{R}) \) in (1.4) such that there exists \( \rho^-, \rho^+, \rho_d^+, \rho_d^-, \sigma^-, \sigma^+, \sigma_{dd}^-, \sigma_{dd}^+ \) so that \( 0 < \rho^- \leq \rho(y) \leq \rho^+, \ |\rho'(y)| \leq \rho_d^+ \), \( 0 < \sigma^- \leq \sigma(y) \leq \sigma^+, \ |\sigma''(y)| \leq \sigma_{dd}^+, \) for all \( y \in \mathbb{R} \), and compactly supported initial data \( (u^0, u^1) \in C^3 \times C^2(\mathbb{R}) \). We also consider non-uniform grids \( G^h \) given by \( g \in C^2(\mathbb{R}) \) such that there exists \( g_d^+, g_d^-, g_{dd}^+ > 0 \) so that \( 0 < g_d^- \leq |g'(x)| \leq g_d^+ \) and \( |g''(x)| \leq g_{dd}^+ \) for all \( x \in \mathbb{R} \). Under these hypotheses, the numerical approximation (1.6) with initial data \( (u^{h,0}, u^{h,1}) : = (u^0(g_j), u^1(g_j))_{j \in \mathbb{Z}} \) is a convergent scheme of order \( O(h) \) for the wave Eq. (1.4) in the \( h^{1/2} \times \ell^2 \)-norm, i.e., there exists a constant \( C(t, g, \rho, \sigma, u^0, u^1) \) independent of \( h \) such that

\[ ||(u^h(t) - u(g(x^h), t), \partial_{x}u^h(t) - \partial_{x}u(g(x^h), t))||_{h^{1/2} \times \ell^2} \leq hC(t, g, \rho, \sigma, u^0, u^1). \quad (3.17) \]

The function \( v(x, t) : = u(g(x), t) \) \( (u(y, t) \) being the solution of (1.4)) solves the wave equation
Thus, by reconsidering the values \( u_j(t) \) associated initially with the points of the non-uniform grid as being values \( v_j(t) = u_j(t) \) at points of the uniform grid, we obtain that \( v_j(t) \) is the solution of the finite difference approximation for the wave Eq. (3.18) on the uniform grid \( \mathcal{G}^h \)

\[
\frac{\partial^2 h g_j \rho(g_j)}{\partial t^2} v_j(t) - \frac{\rho(g_j) \partial^2 v_j(t)}{\partial t^2} = 0. \tag{3.19}
\]

Indeed, from Proposition 3.4, we see that \( u^h(t) = v^h(t) \sim v(x^h, t) = u(g(x), t) \), where \( (v(x, t), v^h(t)) \) are the solutions of ((3.18),(3.19)).

The null bi-characteristic lines associated with the principal symbol (1.11) of the numerical approximation (3.19) of the wave equation are the solutions of the following system:

\[
\begin{cases}
X'(s) = \partial_t \varphi = 2 \sin(\Xi(s)) \frac{\sigma(g(X(s)))}{g(X(s))}, \quad \tau'(s) = \partial_t \varphi = -2g'(X(s))\rho(g(X(s)))\tau, \\
\Xi'(s) = -\partial_s \varphi = \tau^2 (g(X(s))\rho(g(X(s)))' - 4 \sin^2 (\frac{\Xi(s)}{2}) \frac{\sigma(g(X(s)))}{g'(X(s))}) (X(s)), \quad \tau'(s) = -\partial_s \varphi = 0,
\end{cases} \tag{3.20}
\]

subjected to the initial data \((X(0), t(0), \Xi(0), \tau(0)) = (x_0, 0, \xi_0, \tau_0) = 0\). Then, \(\tau(s) = \tau_0\) and \(\varphi(X(s), t(s), \Xi(s), \tau(s)) = 0\) are conserved in \(s\). Set \((x^\pm(t), \xi^\pm(t)) := (X(s), \Xi(s))\) to be the solution of (3.20) corresponding to one of the two possible roots \(\tau_0^\pm\) of \(\varphi(X(s), t(s), \Xi(s), \tau_0) = 0\) given by \(\tau_0^\pm = \pm 2 \sin(\Xi(s)/2) \sigma_g(X(s))\), with \(c_g(x) := \sqrt{\sigma(g(x))/\rho(g(x))/g'(x)}\). Then, \((x^\pm(t), \xi^\pm(t))\) is the solution of the following Hamiltonian system:

\[
(x^\pm)'(t) = \mp c_g(x^\pm(t)) \cos \left(\frac{\xi^\pm(t)}{2}\right), \quad (\xi^\pm)'(t) = \pm 2 c_g'(x^\pm(t)) \sin \left(\frac{\xi^\pm(t)}{2}\right),
\]

\[
x(0) = x_0, \quad \xi(0) = \xi_0. \tag{3.21}
\]

Remark 4 The Hamiltonian system (3.21) can be also obtained by considering in (3.20) the principal symbol \(\varphi(x, t, \xi, \tau)\) below instead of (1.11)

\[
\varphi(x, t, \xi, \tau) := -\rho(g(x))\tau^2 + 4 \sin^2 \left(\frac{\xi}{2}\right) \frac{\sigma(g(x))}{g'(x)^2}. \tag{3.22}
\]

This last symbol can be seen intuitively by taking plane wave solutions of the form \(u_j(t) = \exp(i \tau t / h + i \xi j)\) in the numerical scheme (1.7) on the non-uniform grid \(\mathcal{G}_g^h\).

Let us consider the following continuous functions:
\[ \begin{align*}
\alpha^h(x) & := \frac{1}{\sqrt{\frac{\partial^h g(x) \rho(g(x))}}}, \quad \beta^h(x) := \sqrt{\frac{\sigma(g(x) + h/2)}{\partial^h g(x)}}, \\
\gamma^h(x) & := \alpha^h(x) \beta^h(x) \text{ and } \delta^{h, \pm}(x) := \beta^h(x) \partial^{h, \pm} \alpha^h(x) \pm \alpha^h(x) \partial^{h, -} \beta^h(x). \quad (3.22)
\end{align*} \]

Associated with them, we consider the discrete functions \( \alpha_j := \alpha^h(x_j), \beta_j := \beta^h(x_j), \gamma^h(x_j) \text{ and } \delta^h_j := \delta^{h, \pm}(x_j), \) for all \( j \in \mathbb{Z}. \) Remark that, as \( h \to 0, \) the following approximations hold:

\[ \begin{align*}
\alpha^h(x) & \sim \frac{1}{\sqrt{\rho(g(x))g'(x)}}, \quad \beta^h(x) \sim \sqrt{\frac{\sigma(g(x))}{g'(x)}}, \\
\gamma^h(x) & \sim c^h(x) := \frac{1}{g'(x)} \sqrt{\frac{\sigma(g(x))}{\rho(g(x))}}, \\
\gamma^h(x) & \sim c^h(x) \text{ and } \delta^h_j \sim \frac{\sigma(g(x))}{g'(x)} \left( \frac{1}{\sqrt{\rho(g(x))}} \right)'(x).
\end{align*} \] (3.23)

In the semi-discrete wave Eq. (3.19), set \( w_j(t) := \frac{v_j(t)}{\alpha_j} \text{ and } \tilde{w}_j(t) := \beta_j \partial^{h, +} v_j(t). \) Due to the conservation of energy, the norm \( \|(w^h(t), \tilde{w}^h(t))\|_{(L^2)^2} \) is conserved in time. Furthermore, the two functions \( w^h(t) \) and \( \tilde{w}^h(t) \) are the solutions of the coupled system of discrete transport equations

\[ \begin{align*}
\partial_t w_j(t) & = \gamma_j \partial^{h, -} \tilde{w}_j(t) + \alpha_j (\partial^{h, -} \beta_j) \tilde{w}_{j-1}(t), \\
\partial_t \tilde{w}_j(t) & = \gamma_j \partial^{h, +} w_j(t) + \beta_j (\partial^{h, +} \alpha_j) w_{j+1}(t), \quad (3.24)
\end{align*} \]

with initial data \( w_j(0) = w_j^0 := \frac{u_j}{\alpha_j} \text{ and } \tilde{w}_j(0) = \tilde{w}_j^0 := \beta_j \partial^{h, +} u_j^0, \) for all \( j \in \mathbb{Z}. \)

Let us observe that (3.24) can be also written in terms of the two pseudo-differential operators \( \Theta^h_1(x, h\partial_x) \) and \( \Theta^h_0(x, h\partial_x) \) (here, the subscripts 0 and 1 indicate the smallest power of \( \xi \) appearing in the Taylor expansions around \( \xi = 0 \) of \( \Theta_1 \) and \( \Theta_0 \) for a fixed \( x \)) generated by the matrices

\[ \Theta_1(x, \xi) := \begin{pmatrix}
0 & \gamma^h(x)(1 - \exp(-i\xi)) \\
\gamma^h(x)(\exp(i\xi) - 1) & 0
\end{pmatrix} \]

and

\[ \Theta_0(x, \xi) := \begin{pmatrix}
0 & \alpha^h(x) \partial^{h, -} \beta^h(x) \exp(-i\xi) \\
\beta^h(x) \partial^{h, +} \alpha^h(x) \exp(i\xi) & 0
\end{pmatrix} \]

as follows

\[ \begin{pmatrix}
\partial_t w^h(t) \\
\partial_t \tilde{w}^h(t)
\end{pmatrix} = \frac{1}{h} \Theta^h_1(x, h\partial_x) \begin{pmatrix}
w^h(t) \\
\tilde{w}^h(t)
\end{pmatrix} + \Theta^h_0(x, h\partial_x) \begin{pmatrix}
w^h(t) \\
\tilde{w}^h(t)
\end{pmatrix}. \] (3.25)
The matrix $Θ_1(x, ξ)$ admits the spectral decomposition $Θ_1(x, ξ) = iΔ(ξ)Λ(x, ξ)Δ^*(ξ)$ ($Λ^*$ being the conjugate transpose of the matrix $Λ$), where
\[
Δ(x, ξ) := \begin{pmatrix} λ^{+}_{γ^h,ω}(x, ξ) & 0 \\ 0 & λ^{-}_{γ^h,ω}(x, ξ) \end{pmatrix},
\]
\[
Δ(ξ) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\exp(iξ/2) \\ -\exp(iξ/2) & 1 \end{pmatrix},
\]
with $λ^{±}_{γ^h,ω}(x, ξ) = ±γ^h(x)ω(ξ)$, $γ^h$ as in (3.22) and $ω(ξ) = 2\sin(ξ/2)$. Note that the eigenvalues $λ^{±}_{γ^h,ω}$ of $Θ_1$ have the same structure in separated variable of the symbol $λ_{γ^h,ω}(x, ξ) = γ^h(x)ω(ξ)$ generating the leading pseudo-differential operator in the transport Eq. (3.14). The function $γ^h(x)$ approximates $c_g(x)$ for both models, where the functions $c_g$ are given by (3.12) and (3.23) for the discrete transport/wave equations and depend on the variable coefficients appearing in the corresponding continuous model and on the grid mapping $g$. Another difference between $λ_{γ^h,ω}(x, ξ)$ and $λ^{±}_{γ^h,ω}(x, ξ)$ in (3.14) and (3.25) is the Fourier symbol $ω$, so that $ω(ξ) = \sin(ξ)$ and $ω(ξ) = 2\sin(ξ/2)$ in the first/second case. These trigonometric functions $ω$ are simply the classical dispersion relations for the finite difference semi-discretizations of the constant coefficients transport/wave equations on uniform meshes.

Remark also that the matrix of eigenvectors $Δ(ξ)$ in (3.26) does not depend on the space variable $x$. Let us introduce the projectors for the numerical scheme (3.24) to be
\[
Δ^+ := Δ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Δ^* \quad \text{and} \quad Δ^- := Δ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Δ^*,
\]
i.e. $Δ^\pm = \frac{1}{2} \begin{pmatrix} 1 & ±\exp(-iξ/2) \\ ±\exp(iξ/2) & 1 \end{pmatrix}$, (3.27)

where $Δ$ is the matrix containing the eigenvectors of $Θ_1$ in (3.26). Let us denote the solution of (3.25) by $w^h(t) = (w^h(t), \tilde{w}^h(t))$ and by $W^h[w^h](x_m/2, t, ξ)$ the Wigner transform matrix of $w^h$. Set
\[
W^h,±\left(\frac{x_m}{2}, t, ξ\right) := Δ^±(ξ)W^h[w^h]\left(\frac{x_m}{2}, t, ξ\right)Δ^±(ξ)
\]
and $W^h,±\left(\frac{x_m}{2}, t, ξ\right) := \text{tr}(W^h,±\left(\frac{x_m}{2}, t, ξ\right))$. (3.28)

Here, $\text{tr}(A)$ is the trace of the matrix $A$. Our second result (also to be proved in Sect. 4) states that the Wigner measure of $w^h(t)$ can be decomposed into two parts following two Wigner measures as $h → 0$, each of them satisfying a transport equation in the space-phase variables $(x, ξ)$. Firstly, set $W^h(x_m/2, t, ξ) := \text{tr}(W^h[w^h](x_m/2, t, ξ))$.

**Theorem 3.5** For each coefficient $ρ$ and $σ$ in (1.4), each non-uniform grid $G^h_g$ obtained by transformations $g$ so that $c_g$ in (3.23) belongs to $C^{1,1}(\mathbb{R})$ and for any initial data $(w^{h,0}, \tilde{w}^{h,0})$ in (3.24) bounded in $ℓ^2$ as $h → 0$, there exists a positive Radon measure $\mathcal{W} = \mathcal{W}(x, t, ξ)$ defined on $\mathbb{R} × \mathbb{R}_+ × [−π, π]$ such that, eventually after extracting...
subsequences, we get (3.15) uniformly on each compact set of the time interval \( \mathbb{R}_+ \).

Moreover, the measure \( \mathcal{W} \) can be decomposed as \( \mathcal{W} = \mathcal{W}^+ + \mathcal{W}^- \), where \( \mathcal{W}^\pm = \lim_{h \to 0} \mathcal{W}^{h, \pm} \) (as in (3.28)) satisfies the Liouville equation

\[
\partial_t \mathcal{W}^\pm(x, t, \xi) = \pm P_{c_g, \omega}(x, t, \xi) \mathcal{W}^\pm(x, t, \xi),
\]

\[
\mathcal{W}^\pm(x, 0, \xi) = \mathcal{W}^{h, \pm}(x, 0, \xi).
\]

Here, the operator \( P_{c_g, \omega}(x, t, \xi) \) is defined by (2.13) with \( c = c_g \) given by (3.23) and \( \omega(\xi) = 2 \sin(\xi/2) \).

Remark 5  The propagation properties of the discrete wave Eq. (3.24) are determined by the two eigenvalues \( \lambda_{\gamma_h, \omega}^\pm(x, \xi) \sim \lambda_{c_g, \omega}^\pm(x, \xi) \) of the principal symbol \( \Theta_1(x, \xi) \) in (3.25).

Remark 6  The positivity of the Radon measures \( \mathcal{W} \) in Theorems 2.1, 3.3, 2.2, and 3.5 follows, on one hand, by the results in [35] relating a discrete function with a continuous one by using different reconstruction procedures (splines, Shannon, or wavelets) and, on the other hand, by the classical results on the positivity of the Wigner measures at the continuous level (see, e.g., Theorem 1.1 in [35]). The trace of the Wigner transform matrix \( \mathcal{W}^r[f] \) of a vector-valued function \( f \) is not a positive distribution, while the limiting Wigner measure \( \mathcal{W} \) is positive. The proof of the positivity part in Theorem 1.1 in [35] is based on the fact that the limits as \( \epsilon \to 0 \) of the so-called Husimi (wave packet) transform and of the Wigner transform coincide, but the Husimi transform is a positive distribution and, consequently, its limit has to be positive. See also [22] and [33].

The following consequence of Theorem 3.5 (whose proof relies on the same arguments presented in Sect. 2.6 and based on the discrete Wigner transform and its properties) holds:

**Corollary 3.6** The energy of the solution \( \mathbf{w}^h(t) = (\mathbf{w}^h(t), \tilde{\mathbf{w}}^h(t)) \) of (3.24) can be arbitrarily concentrated along one of the rays \( x^ \pm(t) \) in (3.21), in the sense that for any \( r > 0 \) and for the initial data in (3.24)

\[
\mathbf{w}^{h, 0} = \left( \pm \alpha_j^{-1} c_g(x_j) \partial_{x_j}^h + u^{h, 0}(x_j) \right)_{j \in \mathbb{Z}} \quad \text{and} \quad \tilde{\mathbf{w}}^{h, 0} = \left( \beta_j \partial_{x_j}^h + u^{h, 0}(x_j) \right)_{j \in \mathbb{Z}},
\]

we get

\[
\lim_{h \to 0} h \sum_{|x_j - x^ \pm(t)| > r} (|w_j(t)|^2 + |\tilde{w}_j(t)|^2) = 0.
\]

Here, \( x^ \pm(t) \) is the solution of (3.21) with data \( (x_0, \xi_0), \alpha_j, \beta_j \) as in (3.22), \( u^{\epsilon, 0} \) is as in (2.22) with \( y_0 \) replaced by \( x_0 := g^{-1}(y_0) \) and \( f \) is such that (with \( \omega(\xi) = 2 \sin(\xi/2) \))

\[
\sigma(y_0)|\omega(\xi_0)|^2 ||f||_{L^2(\mathbb{R})}^2 / g'(g^{-1}(y_0)) = 1.
\]
Of course, this Corollary 3.6 and the fact that there exists a constant \( g_d^- > 0 \) such that \(|g'(x)| \geq g_d^-\) for all \( x \in \mathbb{R} \) imply that the energy of the original discrete wave Eq. (1.7) on the non-uniform grid \( G^h_g \) corresponding to the initial data \( u^{h,0} = (uh,0_j(g_j))_{j \in \mathbb{Z}} \) and \( u^{h,1} = (\pm cg(x_j)\partial h,±u^0_j)_{j \in \mathbb{Z}} \) may be concentrated arbitrarily close to the ray \( y^\pm(t) := g(x^\pm(t)) \).

3.4 Group Velocity

We define the group velocity associated with the numerical schemes (1.6) and (1.7) for the transport or the wave equations to be \((x'(t) \text{ (i.e., } c_g(x(t)) \cos(\xi(t)) \text{)})\), where \( c_g \) is as in (3.12) and \((x(t), \xi(t))\) is the solution of (3.8) or \((x^\pm')'(t) \text{ (i.e., } c_g(x^\pm(t)) \cos(\xi^\pm(t)/2) \text{)}\), where \( c_g \) is as in (3.23) and \((x^\pm(t), \xi^\pm(t))\) is the solution of (3.21) (cf. [7], pp. 65, or [49], Chapter 15).

This definition of the group velocity is consistent with the well known one for the homogeneous discrete medium (i.e., constant coefficients and on a uniform mesh) (see [16,47] or [51]). In that case, \( c_g \equiv 1 \), so that the solution of the second equations in (3.8) and (3.21) is \( \xi^\pm(t) = \pm \xi_0 \). Consequently, the first equations in (3.8) and (3.21) become \((x'(t) = \mp \cos(\xi_0)) \text{ and } (x'(t) = \mp \cos(\xi_0/2), \text{ so that the group velocity does not depend on time and the characteristics } x^\pm(t) \text{ are simply straight lines.})\)

It is well known that, for homogeneous media, a vanishing group velocity (at \( \xi_0 = \pi/2 \) or at \( \xi_0 = \pi \) for the transport/wave equation) yields spurious solutions that do not propagate in time. These fictitious solutions make the uniform (with respect to the mesh size) observability property to fail. In that case, it is necessary to apply filtering techniques on the initial data (e.g., Fourier truncation, bi-grid algorithms) to avoid the high-frequency pathological effects (cf. [16,51]).

However, for heterogeneous media (variable coefficients and/or non-uniform meshes), the group velocity depends on the time \( t \) through the two components \((x(t), \xi(t)) \text{ or } (x^\pm(t), \xi^\pm(t)) \text{ solving (3.8) or (3.21).}\)

3.5 Related Bibliography

In [5] and [46], the problem of computing spectral distributions for locally Toeplitz matrix sequences has been considered and the results have been applied to design efficient preconditioning strategies for linear systems arising in the numerical approximation of elliptic PDEs in [44].

Locally, Toeplitz matrices appear naturally, indeed, in the discretization process of PDEs with variable coefficients on non-uniform meshes \( G^h_g \), \( g : [0,1] \to [0,1], \quad g \in C^1(0,1) \). For simplicity, consider the \( 1-d \) Laplace problem

\[-(\sigma(y)u')'(y) = f(y), \quad y \in (0,1), \quad u(0) = u(1),\]

for which a typical discretization is the three-point finite difference scheme below \((h := 1/(N+1), N \in \mathbb{N})\)

\[
A^h_g u^h = f^h \text{ or, more precisely, } -\frac{1}{\partial h g_j} \partial h,\sigma^{j+1/2} \partial h,±u_j = f_j, \]

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When \( \sigma \) is a continuous function, any sub-block of fixed size of the matrix \( A^h \) of the system tends to a Toeplitz matrix as \( N \to \infty \). The spectral distribution \( \mu \) of the sequence of stiffness matrices of this scheme is the measure defined as

\[
\langle \mu, F \rangle = \lim_{N \to \infty} N^{-1} \sum_{j=1}^{N} F(\lambda_j),
\]

where \( (\lambda_j)_{1 \leq j \leq N} \) are the eigenvalues of \( A^h \) and \( F \) is any \( C_c(\mathbb{R}) \)-function. In [5] (variable coefficients and non-uniform meshes) and [46] (variable coefficients and uniform meshes), it is proved that the spectral distribution satisfies the identity

\[
\langle \mu, F \rangle = \int_{0}^{1} \int_{-\pi}^{\pi} F(\lambda(x, \xi)) \frac{d\xi}{2\pi},
\]

where \( \lambda(x, \xi) := \sigma(g(x)) \omega_2(\xi)/|g'(x)|^2 \) and \( \omega_2(\xi) = 4 \sin^2(\xi/2) \) is the Fourier symbol of the three-point scheme for the Laplace equation with constant coefficients on the uniform mesh \( G^h \). This is in agreement with our results for the wave Eq. (1.4), for which the characteristics depend on the symbol \( \lambda(x, \xi) \) : \( c(g(x)) \omega(\xi)/|g'(x)| \), with \( c := \sqrt{\sigma/\rho} \).

In [14], the observability problem for mixed finite element semi-discretizations of the \( 1-d \) wave equation has been considered. More precisely, for quasi-uniform meshes (for which the ratio between the maximum cell size and the minimum one remains bounded when refining the mesh), it is proved that the approximation under consideration inherits its well-known good features on uniform grids, for which observability holds uniformly with respect to the mesh size parameter. However, the results are based on a precise description of the spectrum of the numerical scheme on non-uniform meshes and they cannot be generalized to other numerical approximations.

In [15], it is shown via spectral estimates that for all convergent numerical approximations (thus including the ones on non-uniform meshes), after truncating some high-frequency modes, one can always prove uniform observability estimates for the remaining low-frequency ones. Our analysis in this paper shows the necessity of such filtering mechanisms for finite difference numerical schemes on non-uniform meshes for the \( 1-d \) wave equation.

In [35], Macià analyzes to which extent the oscillation and the concentration effects of a sequence of functions are preserved or eliminated by sampling and reconstruction operators on uniform meshes. The main tool of that analysis is based, as in this paper, on Wigner measures. The oscillation/concentration effects of samplings of continuous functions analyzed in that paper are important ingredients when dealing with the Wigner measure of the initial data in the discrete problems (3.10) and (3.24). Thus, our analysis on the evolution problems under consideration combined with the one in [35] on the initial data give the complete picture on the behavior of the Wigner measures leaded by the Cauchy problems (3.16) and (3.29).

### 4 Proof of the Main Results

#### 4.1 Proof of Theorem 3.3

We follow the same steps in the proof of Theorem 2.1 in Sect. 2. For the simplicity of notation, in this proof, we write \( \mathcal{W}^h \) instead of \( \mathcal{W}^h[\mathbf{w}^h] \).

**Step 1** Using the boundedness as \( h \to 0 \) of the \( \ell^2 \)-norm of the initial data \( \mathbf{w}^{h,0} \) in (3.10) and the time conservation of the \( \ell^2 \)-norm of \( \mathbf{w}^h(\cdot, t) \), we obtain that \( \mathcal{W}^h \) is...
bounded in \( S'([\mathbb{R}_x] \times \mathcal{D}'([-\pi, \pi])) \) for all \( t \geq 0 \). By using the Eqs. (4.4) and (4.5) and similar arguments to the ones in Step I in the proof of Theorem 2.1 (see “Appendix”), we also obtain the boundedness of \( \partial_t \mathcal{W}^h \) in \( S'([\mathbb{R}_x] \times \mathcal{D}'([-\pi, \pi])) \) for all \( t > 0 \) and then the equicontinuity of the family \( \{\mathcal{W}^h\}_h \) in the time variable. Consequently, there exists a positive Radon measure \( \mathcal{W} \) such that

\[
\mathcal{W}^h(x, t, \xi) \rightharpoonup \mathcal{W}(x, t, \xi)
\]

weakly star in \( S'([\mathbb{R}_x] \times \mathcal{D}'([-\pi, \pi])) \) as \( h \to 0 \), \( \forall t \geq 0 \),

(4.1)

**Step II** Set \( \hat{\mathcal{W}}^h(x_m/2, t, n) \) to be the \( n \)-th Fourier coefficient of \( \mathcal{W}^h(x_m/2, t, \xi) \) as function of \( \xi \). Following (3.3), it is easy to see that

\[
\hat{\mathcal{W}}^h\left(\frac{x_m}{2}, t, n\right) = w_{m+n}(t)\hat{w}_{m+n}(t).
\]

(4.2)

Define \( \partial^h_x \) and \( \mathcal{M}^h_x \) to be the centered derivative/average operators in the variable \( x \). More precisely,

\[
\partial^h_x f(x_m/2, \xi) := \frac{f(x_{m+1}/2, \xi) - f(x_{m-1}/2, \xi)}{h}
\]

and

\[
\mathcal{M}^h_x f(x_m/2, \xi) := \frac{f(x_{m+1}/2, \xi) + f(x_{m-1}/2, \xi)}{2},
\]

where \( \xi \) stores the other possible variables defining the function \( f(x, \xi) \). By taking time derivative in (4.2) and using the numerical scheme (3.10), we get the following equation verified by \( \hat{\mathcal{W}}^h(x_m/2, t, n) \):

\[
\partial_t \hat{\mathcal{W}}^h\left(\frac{x_m}{2}, t, n\right) = -\left(\frac{1}{2}K^h_{\gamma^h}\left(\frac{x_m}{2}, n\right)\left(\partial^h_x \hat{\mathcal{W}}^h\left(\frac{x_m}{2}, t, n+1\right) + \partial^h_x \hat{\mathcal{W}}^h\left(\frac{x_m}{2}, t, n-1\right)\right) + \frac{1}{2}K^h_{\gamma^h}\left(\frac{x_m}{2}, n\right)\left(\mathcal{M}^h_x \hat{\mathcal{W}}^h\left(\frac{x_m}{2}, t, n+1\right) - \mathcal{M}^h_x \hat{\mathcal{W}}^h\left(\frac{x_m}{2}, t, n-1\right)\right) + \frac{1}{2}K^h_{\delta^h}\left(\frac{x_m}{2}, n\right)\left(\partial^h_x \mathcal{W}^h\left(\frac{x_m}{2}, t, n+1\right) + \partial^h_x \mathcal{W}^h\left(\frac{x_m}{2}, t, n-1\right)\right) + \frac{1}{4}K^h_{\delta^h}\left(\frac{x_m}{2}, n\right)\left(\partial^h_x \mathcal{W}^h\left(\frac{x_m}{2}, t, n+1\right) - \partial^h_x \mathcal{W}^h\left(\frac{x_m}{2}, t, n-1\right)\right)\right),
\]

(4.3)

where \( K^h_{\epsilon^h, \pm} \) are as in (7.2) (with \( \epsilon = h \)) and \( \gamma^h, \delta^h \) are as in (3.11).

**Step III** Set

\[
\langle \hat{f}_1, \hat{f}_2 \rangle_h := \frac{1}{2\pi} \frac{h}{2} \sum_{m=\pi} \int_{\mathbb{Z}_2^n} \hat{f}_1\left(\frac{x_m}{2}, \xi\right) \hat{f}_2\left(\frac{x_m}{2}, \xi\right) d\xi.
\]

and

\[
\langle f_1, f_2 \rangle_{h, \mathcal{D}', \mathcal{D}} := \frac{h}{2} \sum_{m=\pi} \int_{\mathbb{Z}_2^n} f_1\left(\frac{x_m}{2}, \xi\right) f_2\left(\frac{x_m}{2}, \xi\right) d\xi.
\]
Consider $a \in S_{\delta}(\mathbb{R}) \times D_{\delta}([-\pi, \pi])$. Let us multiply (4.3) by $h^{-1}\delta(x_m/2, n)$ and add in $m, n \in \mathbb{Z}$ so that $n \equiv m \mod 2$. By Parseval identity in $\xi$, we obtain

$$
\langle \partial_t \mathcal{W}^h(\cdot, t, \cdot), a \rangle_{h, D', D} = \langle \partial_t \tilde{\mathcal{W}}^h(\cdot, t, \cdot), \tilde{a} \rangle_h = -\left( I_1^h(t) + I_2^h(t) + I_3^h(t) + I_4^h(t) \right),
$$

where

$$
I_1^h(t) = \left( \partial_x \tilde{\mathcal{W}}^h(\cdot, t, \cdot) + \partial_x \tilde{\mathcal{W}}^h(\cdot, t, -1), \frac{1}{2} \mathcal{K}_{\gamma h}^{+, \frac{\pi}{a}} \right)_h,
$$

$$
I_2^h(t) = \left( \mathcal{M}_x \tilde{\mathcal{W}}^h(\cdot, t, \cdot) - \mathcal{M}_x \tilde{\mathcal{W}}^h(\cdot, t, -1), \frac{1}{2} \mathcal{K}_{\gamma h}^{+, \frac{\pi}{a}} \right)_h,
$$

$$
I_3^h(t) = \left( \mathcal{M}_x \tilde{\mathcal{W}}^h(\cdot, t, \cdot) + \mathcal{M}_x \tilde{\mathcal{W}}^h(\cdot, t, -1), \mathcal{K}_{\delta h}^{+, \frac{\pi}{a}} \right)_h,
$$

$$
I_4^h(t) = \left( \partial_x \tilde{\mathcal{W}}^h(\cdot, t, \cdot) - \partial_x \tilde{\mathcal{W}}^h(\cdot, t, -1), \frac{h^2}{4} \mathcal{K}_{\delta h}^{+, \frac{\pi}{a}} \right)_h.
$$

By passing all the discrete derivatives and averages from $\tilde{\mathcal{W}}^h$ to the accompanying factors and applying Parseval identity, we get

$$
I_j^h(t) := \langle \mathcal{W}^h(\cdot, t, \cdot), (a^* c \tilde{g})^h_j \rangle_{h, D', D},
$$

where

$$
(a^* c \tilde{g})_1^h \left( \frac{x_m}{2}, \xi \right) := -\cos(\xi) \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \partial_x^h \left( \mathcal{K}_{\gamma h}^{+, \frac{\pi}{a}} \right) \left( \frac{x_m}{2}, n \right) \exp(i \xi n),
$$

$$
(a^* c \tilde{g})_2^h \left( \frac{x_m}{2}, \xi \right) := i \sin(\xi) \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \mathcal{M}_x^h \left( \mathcal{K}_{\gamma h}^{+, \frac{\pi}{a}} \right) \left( \frac{x_m}{2}, n \right) \exp(i \xi n),
$$

$$
(a^* c \tilde{g})_3^h \left( \frac{x_m}{2}, \xi \right) := 2 \cos(\xi) \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \mathcal{M}_x^h \left( \mathcal{K}_{\delta h}^{+, \frac{\pi}{a}} \right) \left( \frac{x_m}{2}, n \right) \exp(i \xi n)
$$

and

$$
(a^* c \tilde{g})_4^h \left( \frac{x_m}{2}, \xi \right) := -i \sin(\xi) \frac{h^2}{2\pi} \sum_{n \in \mathbb{Z}} \partial_x^h \left( \mathcal{K}_{\delta h}^{+, \frac{\pi}{a}} \right) \left( \frac{x_m}{2}, n \right) \exp(i \xi n).
$$

By the Taylor expansion (7.5), we get $\mathcal{K}_{c}^{+, \frac{\pi}{a}}(x_m/2, n) \sim c(x_m/2)$ and $\mathcal{K}_{c}^{-, \frac{\pi}{a}}(x_m/2, n) \sim nc'(x_m/2)$ as $h \to 0$. Moreover, by taking into account (3.12), we obtain

$$
(a^* c \tilde{g})_1^h \left( \frac{x_m}{2}, \xi \right) \sim -\cos(\xi) \left( c'_g \left( \frac{x_m}{2} \right) \bar{a} \left( \frac{x_m}{2}, \xi \right) + c_g \left( \frac{x_m}{2} \right) \partial_x \bar{a} \left( \frac{x_m}{2}, \xi \right) \right),
$$

$$
(a^* c \tilde{g})_4^h \left( \frac{x_m}{2}, \xi \right) \sim 0,
$$

(4.6)
and
\[
(a^*_N c_g h) \left( \frac{x_m}{2}, \xi \right) \sim \sin(\xi) c_g' \left( \frac{x_m}{2} \right) \frac{\partial}{\partial \xi} \tilde{a} \left( \frac{x_m}{2}, \xi \right),
\]
\[
(a^*_N c_g h) \left( \frac{x_m}{2}, \xi \right) \sim \cos(\xi) c_g' \left( \frac{x_m}{2} \right) \tilde{a} \left( \frac{x_m}{2}, \xi \right).
\]

Let us denote by \( \langle \cdot, \cdot \rangle \) the duality product between \( S'(\mathbb{R}) \times D([-\pi, \pi]) \) and \( S(\mathbb{R}) \times D([-\pi, \pi]) \). From (4.6) and (4.7), we obtain that \( I_j^h(t) \to I_j(t) \) as \( h \to 0 \) for each \( I_j^h(t) \) in (4.5), \( 1 \leq j \leq 4 \), with
\[
I_1(t) := \langle \mathcal{W}(\cdot, t, \cdot), -\cos(\xi) \partial_x (c_g(x) a) \rangle,
I_2(t) := \langle \mathcal{W}(\cdot, t, \cdot), \sin(\xi) c_g'(x) a \xi \rangle,
I_3(t) := \langle \mathcal{W}(\cdot, t, \cdot), \cos(\xi) c_g'(x) a \rangle \text{ and } I_4(t) = 0.
\]

We conclude the proof of Theorem 3.3 by passing the derivatives of \( a \) to the accompanying factors in each \( I_j(t), 1 \leq j \leq 4 \).

\( \square \)

4.2 Proof of Theorem 3.5

We use similar arguments to the ones in the alternative proof of Theorem 2.2. **Step I. Equation (4.11) of the Wigner transform matrix** \( \mathcal{W}^h[w^h] \) of the solution \( w^h(t) \) of (3.25). Set \( \Theta := \Theta_1 + h\Theta_0 \), where \( \Theta_0, \Theta_1 \) are the ones in (3.25). For simplicity, during this proof, we use \( \mathcal{W}^h \) to denote the Wigner transform matrix \( \mathcal{W}^h[w^h] \). Using the definition (3.3) and the Eq. (3.25), we obtain that the Fourier coefficients of \( \mathcal{W}^h \) with respect to \( \xi \) verify the equation
\[
\partial_t \mathcal{W}^h \left( \frac{x_m}{2}, t, n \right) = \frac{1}{h} \Theta^h(x, h \partial_x) w_{m-n}(t) \otimes w_{m+n}^*(t)
+ w_{m+n}(t) \otimes \left( \frac{1}{h} \Theta^h(x, h \partial_x) w_{m-n}(t) \right)^*.
\]

We multiply equation (4.8) by \( \frac{x_m}{2m/n/4\pi} \) and add in \( m, n \in \mathbb{Z} \) such that \( n \equiv m \), where \( a \in S(\mathbb{R}) \times D([-\pi, \pi]) \). After writing explicitly how the pseudo-differential operator \( \Theta^h(x, h \partial_x) \) acts, we get
\[
\langle \partial_t \mathcal{W}^h(\cdot, t, \cdot), \frac{x_m}{2} \rangle_h = A^h + B^h,
\]

where
\[
A^h := \frac{1}{(2\pi)^2} \frac{h^2}{2} \sum_{m \in \mathbb{Z}} \sum_{n \equiv m} \sum_{p \in \mathbb{Z}} \int \frac{1}{h} \Theta \left( \frac{x_m-n}{2}, h\xi \right) w_p(t)
\]
\[
\otimes w_{m+n}^*(t) \frac{x_m}{2} \tilde{a} \left( \frac{x_m}{2}, n \right) \exp \left( i \xi \left( x_{m-n} - x_p \right) \right) d\xi.
\]
and
\[
B^h := \frac{1}{(2\pi)^2} \frac{h^2}{2} \sum_{m \in \mathbb{Z}} \sum_{n=m}^{m+p \in \mathbb{Z}} \int_{\mathbb{T}^h} \frac{1}{h} w_{m,n}(t) \otimes w^*_p(t) \Theta^* \left( \frac{x_{m+n}}{2}, h, \xi \right) \tilde{a}(x_n, n) \exp \left( -i \xi \left( \frac{x_{m+n}}{2} - x_p \right) \right) \, d\xi.
\]

Here, $\mathbb{T}^h := [-\pi/h, \pi/h]$. In both $A^h$ and $B^h$, we do the change in variable $\eta := \xi h$. Moreover, in $A^h$ and $B^h$, we change $m$ by $m' := p + (m+n)/2$ and $m' := p + (m-n)/2$, respectively. In that case, we observe that $m \equiv n$ is equivalent to $m', n \in \mathbb{Z}$. In $B^h$, we also change $p$ by $p' := m' - p$. After renaming $m'$, $p'$ as $m$, $p$ and $\eta$ as $\xi$, we get
\[
A^h := \frac{1}{(2\pi)^2} \frac{h^2}{2} \sum_{m,n,p \in \mathbb{Z} \setminus \pi} \int_{\mathbb{T}^h} \frac{1}{h} \Theta(x_{m-n-p}, \xi) w_p(t) \otimes w^*_{m-p}(t) \tilde{a}(x_{m-n-p}, n) \exp \left( i \xi (m - 2p - n) \right) \, d\xi
\]
and
\[
B^h := \frac{1}{(2\pi)^2} \frac{h^2}{2} \sum_{m,n,p \in \mathbb{Z} \setminus \pi} \int_{\mathbb{T}^h} \frac{1}{h} w_p(t) \otimes w^*_{m-p}(t) \Theta^* \left( x_{p+n}, \xi \right) \tilde{a}(x_{p+n}, n) \exp \left( i \xi (m - 2p - n) \right) \, d\xi.
\]

As in the continuous case, we set $y := x_{m}/2$, $y_1 := x_{m/2-n-p}$ and $y_2 := x_{m/2-p-n/2}$ and consider the Taylor expansions (7.16) of $\Theta(y \pm y_1, \xi)$ and $\tilde{a}(y \pm y_2, n)$ at $x = y$. We also decompose $A^h$ and $B^h$ as $A^h := \widehat{A}^h + \mathcal{R}_A^h$ and $B^h := \widehat{B}^h + \mathcal{R}_B^h$, where $\widehat{A}^h$ and $\widehat{B}^h$ are the sums of the same type as $A^h$ and $B^h$, retaining from the terms $\Theta(y \pm y_1, \xi)\tilde{a}(y \pm y_2, n)$ in $A^h$ and $B^h$ only $\Theta(y, \xi)\tilde{a}(y, n) \pm y_1 \partial_x \Theta(y, \xi)\tilde{a}(y, n)$.

In what follows, we use the equivalent expression below the discrete Wigner transform (3.3)
\[
\mathcal{W}^\epsilon [f_{h,1}, f_{h,2}](x_{m}/2, \xi) := \frac{h}{2\pi \epsilon} \sum_{n \in \mathbb{Z}} f_{n}^1 \otimes (f_{m-n}^2)^* \exp \left( \frac{i x_{m-2n} \xi}{\epsilon} \right).
\]

As in the continuous case, we also take into account the fact that the term $x_{m/2-p}$ appearing in both $y_1$ and $y_2$ yields $h \partial_\xi \mathcal{W}^h/2i$ when add in $p$, while $-x_n \partial_x \tilde{a}(y, n)$ yields $h \partial_\xi \partial_x a(y, \xi) / i$ when add in $n$ (with $\alpha = 0$ or $\alpha = 1$). Thus, $\widehat{A}^h := \widehat{A}^h_1 + \widehat{A}^h_2 + \hat{A}^h_3$, where
\[
\widehat{A}^h_1 := \frac{1}{h} \Theta \mathcal{W}^h(\cdot, \cdot, \cdot)_{h, D^t, D^t}, \quad \widehat{A}^h_2 := -\frac{1}{2i} \partial_\xi \Theta \mathcal{W}^h(\cdot, \cdot, \cdot)_{h, D^t, D^t}, \quad \hat{A}^h_3 := \Theta \mathcal{W}^h(\cdot, \cdot, \cdot)_{h, \overline{D}^t, D^t}.
\]
and

\[ \tilde{A}^h := -\frac{1}{2i} \left\{ \partial_x \Theta \partial_\xi \mathcal{W}^h(\cdot, t, \cdot) + 2\partial^2_{x\xi} \Theta \mathcal{W}^h(\cdot, t, \cdot), a \right\}_{h, D', D} \]

and a similar expression for \( \tilde{B}^h \) in which the signs of the second and of the third term in \( \tilde{A}^h \) are changed, \( \Theta \) and its derivatives are replaced by their conjugate transpose matrices and interchanged with the corresponding derivatives of the discrete Wigner transform matrix \( \mathcal{W}^h \). Thus, similarly to the continuous case, by taking into account that \( \Theta = \Theta_1 + h\Theta_0 \) and that \( \Theta_1^* = -\Theta_1 \), we obtain the following equation for \( \mathcal{W}^h \):

\[
\partial_t \mathcal{W}^h = \frac{\Theta_1^* \mathcal{W}^h - \mathcal{W}^h \Theta_1}{h} + (\Theta_0 \mathcal{W}^h + \mathcal{W}^h \Theta_0^*) - \frac{1}{2i} \mathfrak{D} (\partial_\xi \Theta_1 \mathcal{W}^h + \mathcal{W}^h \partial_\xi \Theta_1) - \frac{1}{i} (\partial^2_{x\xi} \Theta_1 \mathcal{W}^h + \mathcal{W}^h \partial^2_{x\xi} \Theta_1) + h\mathcal{R}^h \\
=:\sum_{j=1}^5 \tilde{\Theta}_j^h + h\mathcal{R}^h, \tag{4.11}
\]

where the operator \( \mathfrak{D} \) (standing for derivative) is given by \( \langle \mathfrak{D} f, a \rangle_{h, D', D} = \langle f, \partial_x a \rangle_{h, D', D} \) and \( \mathcal{R}^h \) is bounded in \( S'(\mathbb{R}) \times D'(\mathbb{R}) \) as \( h \to 0 \).

**Equation of the projections of the Wigner measure matrix on the Fourier modes of \( \Theta_1 \).** As in the continuous case, we denote by \( \mathcal{W}(x, t, \xi) \) to be the weak limit of the matrix \( \mathcal{W}^h(x, t, \xi) \) and by \( \mathcal{W}^\pm \) the weak limit of \( \mathcal{W}^{h, \pm} := \Delta^\pm \mathcal{W}^h \Delta^\pm \) in (3.28), with \( \Delta^\pm(\xi) \) being the projectors in (3.27). Passing to the limit in the term of order \( h^{-1} \) in the right-hand side of (4.11), we obtain

\[
\tilde{\Theta}_1(x, \xi) \mathcal{W}(x, t, \xi) = \mathcal{W}(x, t, \xi) \tilde{\Theta}_1(x, \xi), \quad \forall x \in \mathbb{R}, \xi \in [-\pi, \pi]. \tag{4.12}
\]

where \( \tilde{\Theta}_1(x, \xi) \) is the limit as \( h \to 0 \) of \( \Theta_1(x, \xi) \) in (3.25) in which \( \gamma^h(x) \) is replaced by \( c_g(x) \) in (3.23).

Let us observe that the projectors \( \Delta^\pm(\xi) \) in (3.27) verify the same properties (7.13) as in the continuous case. Both matrices \( \Theta_1 \) and \( \tilde{\Theta}_1 \) admit the decomposition (7.12) with \( \omega(\xi) = 2 \sin(\xi/2) \), while \( c = \gamma^h \) in the first case and \( c = c_g \) in the second one.

By multiplying by \( \Delta^\pm \) to the left and to the right in (4.11), the left-hand side becomes \( \partial_t \mathcal{W}^{h, \pm} \), while the term of order \( h^{-1} \) on the right-hand side, \( \Delta^\pm \tilde{\Theta}^h \Delta^\pm \), disappears due to identity (7.22), in which \( \omega(\xi) = 2 \sin(\xi/2) \) and \( c = c_g \). It is easy to check that

\[
\Delta^\pm(\xi) \Theta_0(x, \xi) \Delta^\pm(\xi) = \pm \frac{1}{2} (\alpha^h(x) \partial^h;^+ - \beta^h(x) + \beta^h(x) \partial^h;^+ + \alpha^h(x)) \cos(\xi/2) \Delta^\pm(\xi) \\\n+ \pm \frac{1}{4} (\alpha^h(x) \partial^h;^+ - \beta^h(x) - \beta^h(x) \partial^h;^+ + \alpha^h(x))(2i \sin(\xi/2)) \Delta^\pm(\xi). \tag{4.13}
\]
This is the analogue of (7.24) in the continuous case. Taking into account (7.23), (7.26), and the approximations (3.23), we get

$$
\Delta^\pm(\xi) \Sigma^h_2 \Delta^\pm(\xi) \sim \pm c'_h(x) \cos(\xi/2) W^\pm(x, t, \xi),
$$

(4.14)
as $h \to 0$, for all $x \in \mathbb{R}$, $t > 0$ and $\xi \in [-\pi, \pi]$.

Since the projectors $\Delta^\pm(\xi)$ do not depend on $x$, the operator $\mathcal{D}$ in the third term on the right-hand side of (4.11) commutes with the projectors. Taking into account (7.13), we obtain

$$
\begin{align*}
\frac{1}{2i} \Delta^\pm \partial_\xi \Theta_1 W^h \Delta^\pm &= \frac{1}{2} \partial_\xi \lambda^\pm_{\gamma^h,\omega} W^{h,\pm} + \frac{1}{2} \lambda^\pm_{\gamma^h,\omega} \Delta^\pm \partial_\xi \Delta^\pm W^h \Delta^\pm + \frac{1}{2} \\
&\times \lambda^\mp_{\gamma^h,\omega} \Delta^\pm \partial_\xi \Delta^\mp W^h \Delta^\pm.
\end{align*}
$$

(4.15)

By taking derivative in $\xi$ in the first identity in (7.13) and multiplying the result to both sides by $\Delta^\pm$, we get

$$
\Delta^\pm \partial_\xi \Delta^\pm \Delta^\pm = 0_2 \text{ and } \Delta^\pm \partial_\xi \Delta^\mp \Delta^\pm = 0_2.
$$

(4.16)

Using (7.13) and the definition of $W^{h,\pm} := \Delta^\pm W^h \Delta^\pm$, the following identity holds

$$
\Delta^\pm \partial_\xi \Delta^\pm W^h \Delta^\pm = \Delta^\pm \partial_\xi \Delta^\pm \Delta^\pm W^{h,\pm} + \Delta^\pm \partial_\xi \Delta^\mp \Delta^\mp W^h \Delta^\pm.
$$

(4.17)

From (4.16), the first term on the right-hand side of (4.17) vanishes, while from (7.26), we get that the last one tends to zero as $h \to 0$ for $\xi \neq 0$. The last term in the right-hand side of (4.15) tends to zero by similar arguments. Thus,

$$
\langle \Delta^\pm \Sigma^h_3 \Delta^\pm, a \rangle_{h, \mathcal{D}'} \mathcal{D} \to -\langle \partial_\xi \lambda^\pm_{\gamma^h,\omega} W^{\pm}, \partial_x a \rangle,
$$

(4.18)
as $h \to 0$ for any smooth test function $a$. By passing the derivative of $a$ to the accompanying factors, we get

$$
-\langle \partial_\xi \lambda^\pm_{\gamma^h,\omega} W^{\pm}, \partial_x a \rangle = \langle \partial_x^2 \lambda^\pm_{\gamma^h,\omega} W^{\pm} + \partial_\xi \lambda^\pm_{\gamma^h,\omega} \partial_x W^{\pm}, a \rangle.
$$

Due to the fact that the projectors $\Delta^\pm$ do not depend on the space variable $x$, but depend on $\xi$, we get the following two identities (we also use (7.12) with $c = \gamma^h$ and $\omega(\xi) = 2 \sin(\xi/2)$):

$$
\Delta^\pm \Sigma^h_4 \Delta^\pm = -\partial_x \lambda^\pm_{\gamma^h,\omega} \Delta^\pm \partial_\xi W^h \Delta^\pm
$$

(4.19)

and

$$
\Delta^\pm \Sigma^h_5 \Delta^\pm = -2 \partial_x^2 \lambda^\pm_{\gamma^h,\omega} W^{h,\pm} - \partial_\xi \lambda^\pm_{\gamma^h,\omega} \left( \Delta^\pm \partial_\xi \Delta^\pm W^h \Delta^\pm + \Delta^\pm W^h \partial_\xi \Delta^\pm \right)
$$

(4.20)
Using the third identity in (7.13), we obtain the following two identities:

\[
\partial_\xi \Delta^{\pm} \mathcal{W}^h \Delta^{\pm} = \partial_\xi \Delta^{\pm} \mathcal{W}^h, \quad \Delta^{\pm} \mathcal{W}^h \partial_\xi \Delta^{\pm} = \mathcal{W}^h \Delta^{\pm} + \Delta^{\pm} \mathcal{W}^h \partial_\xi \Delta^{\pm}.
\]  

(4.21)

From (4.17) combined with (7.26) and (4.16), we obtain that the last two terms in the right-hand side of (4.20) tend to 0 as \( h \to 0 \).

From (7.26), the second term in the right-hand side of each identity in (4.21) tends to zero as \( h \to 0 \).

Thus, by passing to the limit as \( h \to 0 \) in (4.11), we obtain that each matrix \( \mathcal{W}^{\pm} \) verifies the problem (with \( \lambda_{c,\omega}^{\pm} := \pm c(x)\omega(\xi), \omega(\xi) := 2 \sin(\xi/2) \) and \( c_g \) as in (3.23))

\[
\partial_t \mathcal{W}^{\pm} = \{\lambda^{\pm} c_g, \mathcal{W}^{\pm}\} + \partial_\xi \lambda^{\pm} c_g, \mathcal{W}^{\pm} + \mathcal{W}^{\pm} \partial_\xi \Delta^{\pm}. \]

(4.22)

Here, \( \{p, q\} := \partial_\xi p \partial_q q - \partial_\xi q \partial_p p \) is the Poisson bracket in (7.18). Observe that, compared with (7.29), there is an additional term in the right-hand side of (4.22), due to the fact that, in the discrete case, the projectors \( \Delta^{\pm} \) depend on the phase variable \( \xi \).

In order to conclude (3.29) in Theorem 3.5, we should prove that

\[
\text{tr} \left( \partial_\xi \Delta^{\pm} \mathcal{W}^{\pm} + \mathcal{W}^{\pm} \partial_\xi \Delta^{\pm} \right) = 0. \]

(4.23)

Writing explicitly both identities (7.26) (corresponding to the + and the – sign) in terms of the components of the matrix \( \mathcal{W} \), we obtain that the components of \( \mathcal{W} \) satisfy the following two identities

\[
W_{11} = W_{22} \quad \text{and} \quad W_{12} \exp(i\xi/2) = W_{21} \exp(-i\xi/2). \]

(4.24)

Intuitively, this means that, asymptotically, the kinetic and the potential energies of (1.7) propagate identically. Using these two identities, we can show that \( \mathcal{W}^{\pm} = (W_{11} \pm W_{12} \exp(i\xi/2))\Delta^{\pm} \). Thus, by deriving in \( \xi \) the first identity in (7.13), we obtain

\[
\partial_\xi \Delta^{\pm} \mathcal{W}^{\pm} + \mathcal{W}^{\pm} \partial_\xi \Delta^{\pm} = (W_{11} \pm W_{12} \exp(i\xi/2)) \left( \partial_\xi \Delta^{\pm} \Delta^{\pm} + \Delta^{\pm} \partial_\xi \Delta^{\pm} \right) \]

(4.25)

\[
= (W_{11} \pm W_{12} \exp(i\xi/2)) \partial_\xi \Delta^{\pm}.
\]

Then, (4.23) follows by observing that, since \( \Delta^{\pm} \) has constant elements on the main diagonal, \( \partial_\xi \Delta^{\pm} \) has trivial elements on the main diagonal. This concludes the proof of Theorem 3.5.

\[\square\]

5 Numerical Experiments and Their Interpretation

This section is devoted to present and interpret in terms of our theoretical results in Sect. 3 several numerical simulations for the following two equations:
Fig. 4  Representation of the two non-uniform grids $g_h,1$ and $g_h,2$. By cross markers we indicate the grid points and by solid line the application $g$ generating the non-uniform grid.

- the transport one (1.1) with constant coefficient $\varrho = 1$ on the finite space interval $[-1, 1]$ with periodic boundary conditions $u(-1, t) = u(1, t)$ on the time interval $[0, T]$ with $T = 5$. As numerical method, we use the finite difference approximation (1.6) with $u_0(t) = u_N(t)$ and $u_1(t) = u_{N+1}(t)$.
- the wave one (1.4) with constant coefficients $\rho = \sigma = 1$ on the finite space interval $[-1, 1]$ with homogeneous Dirichlet boundary conditions on the time interval $[0, T]$ with $T = 5$. As numerical method, we use the finite difference approximation (1.7) with $u_0(t) = u_{N+1}(t) = 0$.

Here, $h := 1/(N + 1)$ with $N = 249$. Set $x^h$ to be the uniform mesh of size $h$ of the interval $[-1, 1]$. Let us define the following two non-uniform meshes (see Fig. 4)

$$g_{h,1} := \tan(\pi x^h/4)$$ and $$g_{h,2} := 2 \sin(\pi x^h/6).$$ (5.1)

Both transformations $g$ map the interval $[-1, 1]$ into itself. The two meshes $g_{h,1}$ and $g_{h,2}$ are symmetric with respect to zero. The first one is finer in the middle and coarser at the endpoints, while for the second one the situation is opposite. Both grids satisfy the regularity assumptions of Theorems 3.3 and 3.5.

The time discretization is done by means of the centered scheme $(f^{n+1} - f^{n-1})/2\delta t$ for the transport equation and by the leap-frog scheme $(f^{n+1} - 2f^n + f^{n-1})/\delta t^2$ for the wave equation. Here, $f^n$ is an approximation of the function $f(t)$ at the time $t = n\delta t$. Since these times schemes are explicit, we use the Courant–Friedrichs–Lewy (CFL) condition $\delta t = \mu h$, with $\mu = 0.1$.

The initial data in (1.6) and (1.7) are constructed out of the following Gaussian profile with $\gamma = h^{-0.9}$:

$$G_\gamma(y) = \exp(-\gamma(g^{-1}(y) - g^{-1}(y_0))^2/2) \exp(i\xi_0 g^{-1}(y)/h).$$ (5.2)

Here, $g^{-1}$ denotes the inverse of function $g$. For the transport equation (1.6), we consider the initial data $u^{h,0} := G_\gamma(g^h)$. In Figs. 5 (for $g^h = g_{h,1}$) and 6 (for $g^h = g_{h,2}$).
Fig. 5 Numerical solution of the transport equation (1.1) (with constant coefficient $\varrho = 1$) and the corresponding generalized ray, with $g(x) = \tan(\pi x/4)$. a Numerical solution for $\xi_0 = 3\pi/4$, $y_0 = 1/2$, c numerical solution for $\xi_0 = 3\pi/4$, $y_0 = 0$, e numerical solution for $\xi_0 = \pi/2$, $y_0 = 1/2$, g numerical solution for $\xi_0 = \pi/2$, $y_0 = 0$, b, d, f, h the corresponding rays of Geometric Optics
Fig. 6 Numerical solution of the transport equation (1.1) (with constant coefficient $\varrho = 1$) and the corresponding generalized ray, with $g(x) = 2 \sin(\pi x/6)$. a Numerical solution for $\xi_0 = 3\pi/4$, $y_0 = 1/2$, c numerical solution for $\xi_0 = 3\pi/4$, $y_0 = 0$, e numerical solution for $\xi_0 = \pi/2$, $y_0 = 1/2$, g numerical solution for $\xi_0 = \pi/2$, $y_0 = 0$. b, d, f, h the corresponding rays of Geometric Optics.
\(g^h = g^{h,2}\), we represent these solutions in the column on the left for the particular cases (a) \((\xi_0, y_0) = (3\pi/4, 1/2)\), (c) \((\xi_0, y_0) = (3\pi/4, 0)\), (e) \((\xi_0, y_0) = (\pi/2, 1/2)\), and (g) \((\xi_0, y_0) = (\pi/2, 0)\). In each subfigure on the right, we represent the corresponding **generalized ray of Geometric Optics**. For the transport equation with **periodic boundary conditions**, the generalized rays are an union of curves based on the solution of the system (3.8) obtained as follows. Firstly, since in this section we deal with the transport equation (1.6), we plot \(g(x_1(t))\) for \(t \in [0, t_1]\), where \((x_1(t), \xi_1(t))\) is the solution of (3.8) with initial data \(x_1(0) = g^{-1}(y_0)\) and \(\xi_1(0) = \xi_0(y_0, \xi_0\) being the ones in (5.2)) and \(t_1\) is the first time when \(g(x_1(t))\) reaches one of the endpoints of the computational domain \([-1, 1]\), \(y^*\). Then, for \(j \geq 2\), we plot \(g(x_j(t))\) for \(t \in [t_{j-1}, t_j]\), where \((x_j(t), \xi_j(t))\) is the solution of (3.8) with initial data \(x_j(t_{j-1}) = -y^*\) (which is the opposite endpoint of \(y^*\)) and \(\xi_j(t_{j-1}) = \xi_j(t_{j-1})\) and \(t_j\) is the first time when \(g(x_j(t))\) reaches the endpoint \(y^*\). This iterative process finishes when \(t = T\). The resolution of system (3.8) is done numerically by using the command `ode45` of the MATLAB environment.

Let us denote by \((\lambda^k)_{-N \leq k \leq N}\) the square roots of the **eigenvalues** \(\Lambda^k\) in the **spectral problem** below associated with the numerical approximation (1.7) for the wave equation (with constant coefficients \(\sigma = \rho = 1\) on the interval \((-1, 1)\) with **homogeneous Dirichlet boundary conditions**) and by \(\phi^{h,k} = (\phi^k_j)_{-N \leq j \leq N}\) the corresponding **eigenvector**:

\[
-\frac{1}{h} \left( \frac{\partial^h_+ \varphi^k_j}{\partial \xi^h_+ g_j} - \frac{\partial^h_- \varphi^k_j}{\partial \xi^h_- g_j} \right) = \lambda^k \partial^h \varphi^k_j, \quad -N \leq j \leq N, \quad \varphi^{k,-(N+1)} = \varphi^{N+1} = 0.
\]  

(5.3)

The spectral decomposition of the solution of (1.7) is

\[
u^h(t) = \sum_{\pm} \sum_{-N \leq k \leq N} \left( \frac{1}{2} \left( \hat{u}^{k,0} \pm \frac{\hat{u}^{k,1}}{i\lambda^k} \right) \exp(\pm it\lambda^k) \right) \phi^{h,k},
\]  

(5.4)

where \(\hat{u}^{k,0}\) and \(\hat{u}^{k,1}\) are the **Fourier coefficients** of the initial data \(u^{h,0}\) and \(u^{h,1}\) in (1.7) defined as \(\hat{u}^{k,i} := (u^{h,i}, \phi^{h,k})_{\ell^2}\), \(i = 0, 1\). Set \(\hat{G}^k_\gamma := (G_\gamma(u^h), \phi^{h,k})_{\ell^2} (-N \leq k \leq N)\) to be the **Fourier coefficients** of the vector \(G_\gamma(u^h)\).

We take the following initial data in (1.7)

\[
u^{h,0} := \sum_{k=-N}^{N} \hat{G}^k_\gamma \phi^{h,k} \quad \text{and} \quad u^{h,1} \quad \text{such that its Fourier coefficients are} \quad \hat{u}^{k,1} := i\lambda^k \hat{u}^{k,0}.
\]  

(5.5)

Under the restriction (5.5) on the initial velocity \(u^{h,1}\), the solution of (1.7) becomes \(u^h(t) := \sum_{k=-N}^{N} \hat{G}^k_\gamma \exp(it\lambda^k) \phi^{h,k}\).

In Figs. 7 (for \(g^h = g^{h,1}\)) and 8 (for \(g^h = g^{h,2}\), we represent these solutions in the column on the left for the particular cases (a) \((\xi_0, y_0) = (\pi, 1/2)\), (c) \((\xi_0, y_0) = (\pi/2, 0)\), (g) \((\xi_0, y_0) = (\pi/2, 0)\).
Fig. 7  Numerical solution of the wave equation (1.4) (with constant coefficient $\rho = \sigma = 1$) and the corresponding generalized ray, with $g(x) = \tan(\pi x/4)$. a Numerical solution for $\xi_0 = \pi, \gamma_0 = 1/2$, c numerical solution for $\xi_0 = \pi, \gamma_0 = 0$, e numerical solution for $\xi_0 = \pi/2, \gamma_0 = 1/2$, g numerical solution for $\xi_0 = \pi/2, \gamma_0 = 0$, b, d, f, h the corresponding rays of Geometric Optics.
In this section, the functions $ag$ and $\xi$ for the constant coefficients (corresponding to the + sign) with initial data $x_1^+(0) = g^{-1}(y_0)$ and $\xi_1^+(0) = \xi_0$ ($y_0, \xi_0$ being the ones in (5.2)) and $t_1$ is the first time when $g(x_1^+(t))$ reaches one of the endpoints of the computational domain $[-1, 1]$. $y^*$. We chose the system (3.21) with the + sign at step $S_1$ since in (5.5) we take the Fourier coefficients of the initial velocity $u^{h,1}$ of the form $\hat{u}^{k,1} = s(1)i\lambda^k\hat{u}^{k,0}$, with $s(1) = +$ being precisely the sign we chose at the first step. In general, we set $s(j) = \pm$ for even/odd $j$. At step $S_j$, with $j \geq 2$, we plot $g(x_j^{s(j)}(t))$ for $t \in [t_{j-1}, t_j]$, where $(x_j^{s(j)}(t), \xi_j^{s(j)}(t))$ is the solution of (3.8) with $s(j)$ sign and initial data $x_j^{s(j)}(t_{j-1}) = s(j-1)y^*$ and $\xi_j^{s(j)}(t_{j-1}) = \xi_j^{s(j-1)}(t_{j-1})$ and $t_j$ is the first time when $g(x_j^{s(j)}(t))$ reaches the endpoint $s(j)y^*$. This process ends when $t = T$.

On the right column of Figs. 5, 6, 7, 8, we represented $g(x^\pm(t))$, where $x^\pm(t)$ is the space component of the solution of the coupled systems (3.8) or (3.21). Remark that $(y^\pm(t) = g(x^\pm(t)), \xi^\pm(t))$ satisfies the system

$$
(y^\pm)'(t) = \mp ag(y^\pm(t))\omega(\xi^\pm(t)), \quad (\xi^\pm)'(t) = \pm bg(y^\pm(t))\omega(\xi^\pm(t)),
$$

$$
y^\pm(0) = y_0, \quad \xi^\pm(0) = \xi_0.
$$

(5.6)

Here, $ag(y) := (g'c_g)(g^{-1}(y))$ and $bg(y) := c_g(g^{-1}(y))$, where $c_g$ is as in (3.8) or as in (3.21). For the discrete transport, the symbol $\omega$ is given by $\omega(\xi) := \sin(\xi)$, while for discrete waves $\omega(\xi) = 2\sin(\xi/2)$. When we deal with numerical schemes for the constant coefficients transport and wave equations on non-uniform meshes as in this section, the functions $ag$ and $bg$ in (5.6) take the simpler form

$$
a_g(y) := 1 \quad \text{and} \quad b_g(y) := \left(\frac{1}{g}\right)'(g^{-1}(y)).
$$

(5.7)

Set $f^\pm(y, \xi) := (\mp\omega'(\xi), \pm bg(y)\omega(\xi))$ (with $bg$ as in (5.7)) and observe that (5.6) can be written as

$$
((y^\pm)'(t), (\xi^\pm)'(t)) = f^\pm(y^\pm(t), \xi^\pm(t)), \quad y^\pm(0) = y_0, \quad \xi^\pm(0) = \xi_0.
$$

(5.8)

5.1 High-Frequency Pathologies

Let us point out the following three pathologies of the solutions of the discrete transport and wave equations and of the rays of Geometric Optics that we observe in Figs. 5, 6, 7, 8:
Fig. 8  Numerical solution of the wave equation (1.4) (with constant coefficient $\rho = \sigma = 1$) and the corresponding generalized ray, with $g(x) = 2\sin(\pi x/6)$. a Numerical solution for $\xi_0 = \pi$, $y_0 = 1/2$, c numerical solution for $\xi_0 = \pi$, $y_0 = 0$, e numerical solution for $\xi_0 = \pi/2$, $y_0 = 1/2$, g numerical solution for $\xi_0 = \pi/2$, $y_0 = 0$, b, d, f, h the corresponding rays of Geometric Optics.
• **Non-propagating Waves.** In Figs. 5, 6g, h and 7, 8c, d, we observe waves/rays that do not propagate. As we will see from the phase portrait analysis, they correspond to equilibrium (fixed) points \((y^♯, ξ^♯)\) (the green ones) on the corresponding phase diagrams in Figs. 9 and 10. However, there is a big difference concerning the dispersion along the ray between Figs. 5g and 6g and between Figs. 7c and 8c. When the wave is very dispersive, the equilibrium point is a saddle point, whereas when the wave is not dispersive, the fixed point is a center. These non-propagating solutions are well known to hold for numerical schemes on uniform meshes at wave numbers \(ξ^♯\) where the group velocity \(ω'(ξ^♯)\) vanishes.

• **Trapped rays in the interior of the domain or at one endpoint.** It is well known that, for the continuous case or for numerical waves on uniform meshes concentrated on frequencies where the group velocity is not trivial, all the generalized rays are straight lines reflecting at both endpoints. Instead, when the mesh is non-uniform, the corresponding rays are in general strongly curved (see Figs. 5a, c, e, 6a, e, g) or mildly curved (see Figs. 6a, c, 8e, g). The completely new pathology of the waves on non-uniform meshes is the presence of i) waves oscillating in the interior of the computational domain without reflections on the boundary (see Figs. 5a, c, e, 7a, e, g) and ii) waves that oscillate in the interior of the domain and reflect only at one of the endpoints (see Fig. 8a). The trajectories of these trapped rays always remain in the red area of the phase portraits in Figs. 9 and 10. More precisely, the situation (i) corresponds to periodic orbits in the phase diagram which are completely included in the region between the two dotted black vertical asymptotes indicating the computational domain \([-1, 1]\), while (ii) is related to the red lateral area limited by separatrices and the dotted black vertical asymptotes in Figs. 9 and 10b. Recall that a saddle point \(O\) is characterized by the fact that the space around it is divided into four sectors by two curves (the separatrices) passing by \(O\). Moreover, in each of the four sectors, the trajectories are arcs of hyperbolas of center \(O\) (cf. [26]).

• **Generalized rays starting in the non-physical sense.** The solutions for the continuous transport equation (1.1) with constant coefficient \(ρ = 1\) are of the form \(u(y, t) = u^0(y - t)\), so that they propagate to the right. The solutions of the wave equation (1.4) with constant coefficients \(ρ = σ = 1\) and initial data \(u^1(y) = (u^0)'(y)\) (which is the analogue of the second condition in (5.5)) are of the form \(u(y, t) = u^0(y + t)\), so that they propagate to the left. These directions in the continuous case coincide with the arrow sense in the phase portraits in Figs. 9 and 10 at frequency \(ξ = 0\). However, as it could be observed in Figs. 5a, c, e, 6a, c and 8a, the high-frequency discrete waves could start in the non-physical sense. But in Fig. 9 for \(ξ ∈ [π/2, 3π/2]\) and in Fig. 10 for \(ξ ∈ [π, 2π]\), we see that the arrow orientation changes. This is not a new phenomenon for the discrete transport equation since when the grid is uniform, the group velocity \(ω'(ξ) = \cos(ξ)\) changes the sign for \(ξ ∈ [π/2, 3π/3]\) and this yields waves propagating to the left. But this pathology of having waves moving in the non-physical sense is completely new for the finite difference approximation of the wave equation on uniform meshes where the group velocity \(ω'(ξ) = \cos(ξ/2)\) does not change the sign for \(ξ ∈ [-π, π]\).
5.2 Phase Diagram Analysis

The following energy of the solutions \((y^{\pm}(t), \xi^{\pm}(t))\) of (5.8) is conserved along trajectories:

\[
E_{g, \omega}(t) := \frac{1}{g'}(g^{-1}(y^{\pm}(t)))\omega(\xi^{\pm}(t)). \tag{5.9}
\]

The equilibrium (fixed) points \((y^{\sharp}, \xi^{\sharp})\) of the phase portrait of the solution \((y^{\pm}(t), \xi^{\pm}(t))\) satisfy \(f^{\pm}(y^{\sharp}, \xi^{\sharp}) = (0, 0)\), so that \((y^{\sharp}, \xi^{\sharp})\) is the solution of

\[
\omega'(\xi^{\sharp}) = 0 \text{ and } b_g(y^{\sharp}) = 0, \tag{5.10}
\]

with \(b_g\) as in (5.7). From the condition that the second component \(f^h(y^{\sharp}, \xi^{\sharp})\) vanishes, we should normally obtain an alternative equation of \(b_g(y^{\sharp}) = 0\), which is \(\omega(\xi^{\sharp}) = 0\). But since the symbol \(\omega(\xi)\) is a trigonometric one, it is in general difficult to guarantee simultaneously \(\omega(\xi^{\sharp}) = \omega'(\xi^{\sharp}) = 0\).

The first equation in (5.10) yields precisely the wave numbers \(\xi^{\sharp}\) where the group velocity \(\omega'(\xi)\) for the same numerical scheme on the uniform mesh vanishes, so that,

\[\text{Fig. 9 Phase portrait of the system (5.8) for the numerical transport equation (i.e., } \omega(\xi) = \sin(\xi)\text{) and the grid transformations } g(x) = \tan(\pi x / 4) \text{ (left) and } g(x) = 2 \sin(\pi x / 6) \text{ (right). We put } y(t), \xi(t) \text{ on the horizontal/vertical direction.}\]

\[\text{Fig. 10 Phase portrait of the system (5.8) for the numerical wave equation (i.e., } \omega(\xi) = 2 \sin(\xi / 2)\text{) and the grid transformations } g(x) = \tan(\pi x / 4) \text{ (left) and } g(x) = 2 \sin(\pi x / 6) \text{ (right). We put } y^+(t), \xi^+(t) \text{ on the horizontal/vertical direction.}\]
for the finite difference approximation of the transport equation, we get $\xi^\sharp \in \{(2k + 1)\pi/2, k \in \mathbb{Z}\}$, while for the numerical wave equation $\xi^\sharp \in \{(2k + 1)\pi, k \in \mathbb{Z}\}$. Thus, within the frequency range $\xi \in [0, 2\pi]$ for the phase portraits in Figs. 9 and 10, we obtain $\xi^\sharp \in \{\pi/2, 3\pi/2\}$ for the transport equation and $\xi^\sharp = \pi$ for the wave equation.

For the tangential mesh $g(x) = \tan(\pi x/4)$, we get $b_g(g(x)) = -\sin(\pi x/2)$, which vanishes at any $x^\sharp \in 2\mathbb{Z}$. The points $y^\sharp$ in (5.10) satisfy $y^\sharp = g(x^\sharp)$. For $x^\sharp = -2, 0, 2$ and $g(x) = \tan(\pi x/4)$, we get $y^\sharp = -\infty, 0, \infty$. For the sinusoidal mesh $g(x) = 2 \sin(\pi x/6)$, we get $b_g(g(x)) = \sin(\pi x/6)/(2 \cos^2(\pi x/6))$, which vanishes for any $x^\sharp \in 6\mathbb{Z}$, so that $y^\sharp = 0$, for each $x^\sharp \in 6\mathbb{Z}$. Consequently, on both tangential and sinusoidal meshes, there are two fixed points for the transport equation, $(y^\sharp, \xi^\sharp) = (0, \pi/2)$ and $(y^\sharp, \xi^\sharp) = (0, 3\pi/2)$, while for the wave equation, there is an unique equilibrium point, $(y^\sharp, \xi^\sharp) = (0, \pi)$.

However, in the sinusoidal case, $b_g(g(x))$ blows up for $x^\flat \in 6\mathbb{Z} + 3$, yielding $y^\flat = g(x^\flat) = 2(-1)^k$ for $x^\flat = 3(2k + 1), k \in \mathbb{Z}$. Thus, $b_g$ is $C^{0,1}(-2+\delta, 2-\delta)$ for each $\delta > 0$. On the phase portraits in Fig. 9b, the points $(y, \xi) \in (-2, \pi), (2, 0), (2, 2\pi)$ and $(y, \xi) \in (2, \pi), (-2, 0), (2, 2\pi)$ seem to be attracting stable and repulsive unstable nodes, respectively. The same happens with $(y, \xi) \in \{(-2, 0), (2, 2\pi)\}$ and $(y, \xi) \in \{(2, 0), (-2, 2\pi)\}$ in Fig. 10b, but we cannot find anyone of these points as solutions of the corresponding $f^\pm(y, \xi) = 0$ (with $f^\pm$ as in (5.8)). Moreover, in [45], p. 160, we see that a conservative system cannot have any attracting point, but in [45], the author assumes that $f$ is Lipschitz with respect to both components ($y, \xi$). However, a function is not Lipschitz at its blow-up points, so that there is a lack of regularity for $b_g$ in the case of the sinusoidal mesh. Nevertheless, $b_g$ has the required regularity $C^{0,1}$ on the computational domain $[-1, 1]$.

In order to see the nature of the stable points of system, let us firstly observe that the Jacobian matrix of $f^\pm$ is as follows:

$$Jf^\pm(y, \xi) := \begin{pmatrix} 0 & \mp \omega'(\xi) \\ \pm b_g'(y)\omega(\xi) & \pm b_g(y)\omega'(\xi) \end{pmatrix}.$$ 

However, due to (5.10), the component $(2, 2)$ of the matrix $Jf^\pm$ at any fixed point $(y^\sharp, \xi^\sharp)$ vanishes, so that the two eigenvalues of $Jf^\pm(y^\sharp, \xi^\sharp)$ are solutions of the quadratic equation:

$$\lambda^2 + b_g'(y^\sharp)\omega(\xi^\sharp)\omega'(\xi^\sharp) = 0. \quad (5.11)$$

For both numerical transport and wave equations and for any $\xi^\sharp$ satisfying (5.10), we get $\omega(\xi^\sharp)\omega'(\xi^\sharp) = -1$, so that the eigenvalues $\lambda$ in (5.11) do not depend on the equation type (i.e., transport or waves). For both grid applications $g$ considered in this section, $y^\sharp = 0$. Thus, for the tangential grid, we obtain a negative $b_g'(0)(= -2)$, while for the sinusoidal one, we get a positive $b_g'(0)(= 1/4)$.

In the tangential case, the two eigenvalues in (5.11) are purely imaginary, $\lambda^\pm = \pm \sqrt{2}i$, so that the equilibrium point is not a hyperbolic one in the sense of Chapter 7 in [43] and the Hartman–Grobman Theorem (cf. [43]) cannot be applied directly. However, due to the conservation of the energy $E_{g, \omega}(t)$ in (5.9) along the trajectories,
the qualitative behavior of the linearized system at the fixed points still coincides with
the one of the nonlinear one (5.8) (cf. section 6.5 in [45]). Thus, for both transport and
wave equations on the tangential mesh, all the equilibrium points \((y^\#, \xi^\#)\) in (5.10)
are centers (and the trajectories around them are periodic orbits).

In the sinusoidal case, the two eigenvalues in (5.11) are real and of opposite signs,
\(\lambda_{\pm} = \pm 1/2\), so that the equilibrium point is a hyperbolic one and the Hartman–
Grobman Theorem (cf. [43]) works. Thus, for both transport and wave equations on
the sinusoidal mesh, all the fixed points \((y^\#, \xi^\#)\) are of saddle type.

6 Comments and Open Problems

In this article, we have developed a microlocal approach for the analysis of the propa-
gation properties of solutions of 1D heterogeneous wave equations under non-uniform
finite difference discretizations. We have seen that a careful study of the phase por-
trait of the Hamiltonian system giving the characteristic rays indicates with precision
important qualitative properties of the rays (their possibility to reach the boundary in
finite time, or to have a stationary trajectory, or to be reflected inside the domain when
the mesh becomes too coarse to resolve the corresponding wave number, etc.). How-
ever, compared to our understanding on wave propagation in homogeneous media,
there are plenty of phenomena to be understood of which we list the following ones:

1. **Irregular meshes.** Our analysis is limited to the case of smooth non-uniform
   meshes that can be obtained by diffeomorphic transformations of an uniform one.
   Very likely, in the case of more irregular meshes, obtained as deformations of the
   uniforms one through singular maps, other new phenomena and pathologies will
   appear. In the case of 1D waves in continuous heterogeneous media, the lack of
   \(C^1\)-regularity of the coefficients allows exhibiting unexpected concentration phe-
   nomena of the high-frequency waves that contradict all the well-known propagation
   and dispersion properties of waves in homogeneous media (see [9]). Systematic
   analysis of this extra possible pathologies related to the irregularity of the numeri-
   cal meshes is to be developed. We refer the interested reader to the book by Cohen
   [11] where a careful analysis of transmission–reflection phenomena is carried on
   in the case of two uniform grids with different meshes sizes.

2. **Other numerical schemes.** In this paper, we have considered the case of finite
difference approximations. It would be interesting to develop the symbolic calculus
under consideration for other numerical schemes such as finite elements, mixed or
 discontinuous Galerkin finite elements. This will be done in a forthcoming paper.
In particular, for the mixed finite elements in [14], we expect the corresponding principal symbol to be
\(p(x, t, \xi, \tau) = g(x)\phi(g(x))\tau^2 - 4 \tan^2(\xi/2)\sigma(g(x))/g'(x)\)
for which the corresponding Hamiltonian system is
\(x'(t) = cg(x(t))/\cos^2(\xi(t)/2)\)
and \(\xi'(t) = -2c_g(x(t))\tan(\xi(t)/2)\), where \(c_g(x) := \sqrt{\sigma}\phi(g(x))/g'(x)\). The
good observability features of the mixed finite element scheme proved in [14] could
be explained by the absence of fixed points for the above Hamiltonian system since
\(x'(t)\) cannot vanish.

3. **Multidimensional waves.** The extension of the analysis in this paper to the mul-
tidimensional case is a challenging problem. Our techniques can be employed to
deal with non-uniform meshes obtained as diffeomorphic transformations of an uniform grid, for instance, in the context of finite differences, but, of course, in the finite element setting, it is common to use and deal with meshes that are not topologically equivalent to an uniform one. Adapting our analysis to that framework requires significant further developments.

4. Filtering mechanisms on non-uniform meshes. Our analysis in this paper shows the necessity of using filtering mechanisms on non-uniform regular meshes whose corresponding Hamiltonian systems have fixed points. In [37], we proved the efficiency of the numerical viscosity method in the context of the boundary stabilization for the variable coefficients wave equation approximated by three-point finite difference schemes on regular non-uniform meshes. However, to the best of our knowledge, nothing is known concerning the extension to the non-uniform mesh case of other well known filtering techniques on uniform media like the Fourier truncation and the bi-grid techniques [16].

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7 Appendix: Proof of Some Technical Results

7.1 Proof of Theorem 2.1

We deduce only equation (2.12) for the limit measure \( W(y, t, \xi) \).

**Step I. Weak convergence of the Wigner transforms** \( W(\epsilon)^{w(\epsilon)} \). In order to simplify the notation, in this proof, we skip the argument \( w(\epsilon) \) accompanying the Wigner transform \( W(\epsilon)^{w(\epsilon)}(y, t, \xi) \) to write \( W(\epsilon)(y, t, \xi) \). Using the boundedness of \( (w(\epsilon, 0))_{\epsilon} \) in \( L^2(\mathbb{R}) \) as \( \epsilon \to 0 \) and the conservation in time of the \( L^2 \)-norm of \( w(\epsilon)(\cdot, t) \), it can be proved that the corresponding Wigner transform \( W(\epsilon)(y, t, \xi) \) is bounded in \( S'(\mathbb{R} \times \mathbb{R}_\xi) \) for all \( t \geq 0 \). Also, from (7.2), we have

\[
|\langle \partial_t \hat{W}^{\epsilon}, \hat{a} \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)}| \leq ||\hat{W}^{\epsilon}||_{L^\infty(\mathbb{R}^2)} ||(K_{c^{\epsilon}}^{z^\pm} \hat{a})_y + (K_{c^{\epsilon}}^{z^\pm} \hat{a})_z - 2c_d^{\epsilon} \hat{a}||_{L^1(\mathbb{R}^2)} \\
\leq \max \{ ||K_{c^{\epsilon}}^{z^\pm}||_{L^\infty}, ||K_{c^{\epsilon}}^{z^\pm}||_{L^\infty}, ||K_{c^{\epsilon}}^{z^\pm} / z||_{L^\infty} \}(||\hat{a}||_{L^1} + ||\partial_y \hat{a}||_{L^1} + ||\partial_z \hat{a}||_{L^1}),
\]

so that, by the boundedness of \( ||\hat{W}^{\epsilon}||_{L^\infty(\mathbb{R}^2)} \) and the fact that \( ||K_{c^{\epsilon}}^{z^\pm}||_{L^\infty} \leq ||c'||_{L^\infty}, ||K_{c^{\epsilon}}^{z^\pm}||_{L^\infty} \leq ||c'||_{L^\infty} \) and \( ||K_{c^{\epsilon}}^{z^\pm} / z||_{L^\infty} \leq ||c'||_{C^0, 1} \), we obtain that \( \partial_t W^{\epsilon} \) is bounded in \( S'(\mathbb{R}^2) \) and then the equicontinuity of the family \( W^{\epsilon} \) in the time variable follows (for a similar argument for a system of transport equations associated with a constant coefficients pseudo-differential operator, see [22], Proof of Lemma 2.2, pp. 336).

Modulo extracting subsequences, we have

\[
W(\epsilon)(y, t, \xi) \rightharpoonup W(y, t, \xi) \text{ weakly star in } S'(\mathbb{R}_y \times \mathbb{R}_\xi) \text{ as } \epsilon \to 0,
\]

for any \( t \geq 0 \), where \( W \) is a positive Radon measure.
Step II. Write the equation satisfied by the Wigner transform $\mathcal{W}^\varepsilon$. Using the Eq. (2.8) for $w^\varepsilon(y, t)$, it is easy to see that the Fourier transform of $\mathcal{W}^\varepsilon$ in $\xi$, $\mathcal{W}^\varepsilon(y, t, z)$, satisfies the following equation:

$$\partial_t \mathcal{W}^\varepsilon(y, t, z) = -\left[ \mathcal{K}^\varepsilon_{c^+}(y, z) \partial_y \mathcal{W}^\varepsilon(y, t, z) + \mathcal{K}^\varepsilon_{c^-}(y, z) \partial_z \mathcal{W}^\varepsilon(y, t, z) + 2 \mathcal{K}^\varepsilon_d(y, z) \mathcal{W}^\varepsilon(y, t, z) \right],$$

(7.2)

where

$$\mathcal{K}^\varepsilon_{c^+}(y, z) := \frac{1}{2} \left( c(y + \frac{\varepsilon z}{2}) + c(y - \frac{\varepsilon z}{2}) \right)$$

and

$$\mathcal{K}^\varepsilon_{c^-}(y, z) := \frac{1}{2} \left( c(y + \frac{\varepsilon z}{2}) - c(y - \frac{\varepsilon z}{2}) \right).$$

Step III. Let us multiply $\mathcal{W}^\varepsilon_i(y, t, \xi)$ by $a \in S(\mathbb{R}_y \times \mathbb{R}_\xi)$ in the duality product $\langle \cdot, \cdot \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)}$, use Parseval identity in $\xi$ and (7.2) to obtain

$$\langle \partial_t \mathcal{W}^\varepsilon, a \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)} = \frac{1}{2\pi} \langle \partial_t \mathcal{W}^\varepsilon, \mathcal{F}\mathcal{W}^\varepsilon \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)} = -\mathcal{I}_1^\varepsilon(t) + \mathcal{I}_2^\varepsilon(t) + \mathcal{I}_3^\varepsilon(t)$$

$$= -\left( \frac{1}{2\pi} \langle \mathcal{K}^\varepsilon_{c^+} \partial_y \mathcal{W}^\varepsilon, \mathcal{F}\mathcal{W}^\varepsilon \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)} + \frac{1}{2\pi} \langle \mathcal{K}^\varepsilon_{c^-} \partial_z \mathcal{W}^\varepsilon, \mathcal{F}\mathcal{W}^\varepsilon \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)} + \frac{1}{2\pi} \langle 2 \mathcal{K}^\varepsilon_d \mathcal{W}^\varepsilon, \mathcal{F}\mathcal{W}^\varepsilon \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)} \right).$$

(7.3)

The terms $\mathcal{I}_1^\varepsilon(t)$ and $\mathcal{I}_2^\varepsilon(t)$ are generated by the principal operator in (2.8), $c(y) \partial_y$, and $\mathcal{I}_3^\varepsilon(t)$ is the contribution of the potential $d(y) w^\varepsilon$. Let us pass to the limit in each term $\mathcal{I}_j^\varepsilon(t)$, $1 \leq j \leq 3$. By passing the derivatives in $y$ and $z$ from $\mathcal{W}^\varepsilon$ to the other factors, taking into account that $\partial_y \mathcal{K}^\varepsilon_{c^+} = \mathcal{K}^\varepsilon_{c^+}$ and $\partial_z \mathcal{K}^\varepsilon_{c^-} = \mathcal{K}^\varepsilon_{c^-}$ and applying once more the Parseval identity in $z$, we have

$$\mathcal{I}_j^\varepsilon(t) = \langle \mathcal{W}^\varepsilon, (a \mathcal{W}^\varepsilon)_j \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)}, \quad 1 \leq j \leq 3,$$

(7.4)

where

$$(a \mathcal{W}^\varepsilon)_1^\varepsilon(y, \xi) = -\frac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{K}^\varepsilon_{c^+}(y, z) \mathcal{F}\mathcal{W}^\varepsilon(y, z) + \mathcal{K}^\varepsilon_{c^-}(y, z) \partial_y \mathcal{F}\mathcal{W}^\varepsilon(y, z)) \exp(i \xi z) \, dz,$$

$$(a \mathcal{W}^\varepsilon)_2^\varepsilon(y, \xi) = -\frac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{K}^\varepsilon_{c^+}(y, z) \mathcal{F}\mathcal{W}^\varepsilon(y, z) + \mathcal{K}^\varepsilon_{c^-}(y, z) \partial_z \mathcal{F}\mathcal{W}^\varepsilon(y, z)) \exp(i \xi z) \, dz$$

and

$$(a \mathcal{W}^\varepsilon)_3^\varepsilon(y, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}} 2 \mathcal{K}^\varepsilon_d(y, z) \mathcal{F}\mathcal{W}^\varepsilon(y, z) \exp(i \xi z) \, dz.$$
Let us remark that, by Taylor expansions of \( \tilde{c}(y \pm \epsilon z/2) \) around \( y \), for all regular functions \( \tilde{c} \), we get

\[
K_c^{e,+}(y, z) := \frac{1}{2} \left( \tilde{c} \left( y + \frac{\epsilon z}{2} \right) + \tilde{c} \left( y - \frac{\epsilon z}{2} \right) \right) \sim \tilde{c}(y)
\]

and

\[
K_c^{e,-}(y, z) := \frac{1}{\epsilon} \left( \tilde{c} \left( y + \frac{\epsilon z}{2} \right) - \tilde{c} \left( y - \frac{\epsilon z}{2} \right) \right) \sim \tilde{c}'(y)\epsilon
\]

as \( \epsilon \to 0 \) (\( C^1(\mathbb{R}) \) and \( C^2(\mathbb{R}) \) in the first/second case), so that \((a\hat{\pi} c)^f_j(y, \xi) \to (a\hat{\pi} c)_f(y, \xi)\) in \( S(\mathbb{R}^2) \), where

\[
(a\hat{\pi} c)_1(y, \xi) := -c'(y)\overline{\partial}(y, \xi) - c(y)\partial_y \overline{\partial}(y, \xi),
\]

\[
(a\hat{\pi} c)_2(y, \xi) := -c'(y)\overline{\partial}(y, \xi) + c'(y)\partial_\xi (\xi \overline{\partial}(y, \xi)).
\]

and

\[
(a\hat{\pi} c)_3(y, \xi) := 2d(y)\overline{\partial}(y, \xi).
\]

Passing the derivatives of \( a \) with respect to \( y \) or \( \xi \) on the right-hand side of (7.6) to \( \mathcal{W} \) in (7.4) and taking into account (2.9), we obtain

\[
\mathcal{I}_1^1(t) \to \langle c(y)\partial_y \mathcal{W}, a \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)}, \mathcal{I}_2^c(t) \to \langle -c'(y)\mathcal{W} - c'(y)\xi \partial_\xi \mathcal{W}, a \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)} \text{ and } \mathcal{I}_3^c(t) \to \langle c'(y)\mathcal{W}, a \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)} \quad \text{as} \quad \epsilon \to 0.
\]

This concludes the proof of (7.6).

\[ \square \]

7.2 Proof of Theorem 2.2

The first proof follows the same steps as the one of Theorem 2.1. **Step I.** We only note from (7.19) that the equicontinuity does not hold for the Wigner transform matrix \( \mathcal{W}' \) (due to the term of order 1/\( \epsilon \) on the right-hand side of (7.19)), but only for the projections \( \mathcal{W}^{\epsilon, \pm} \) on the two eigenmodes. The other arguments in Step I follow the ones in Theorem 2.1.

**Step II.** Let us remark that the Fourier transforms of \( \mathcal{W}^{\epsilon, \pm}(y, t, \xi) \) in \( \xi, \hat{\mathcal{W}}^{\epsilon, \pm}(y, t, z) \), satisfy the system

\[
\partial_t \hat{\mathcal{W}}^{\epsilon, \pm} = \pm K_c^{\epsilon,+}(y, z)\partial_y \hat{\mathcal{W}}^{\epsilon, \pm} \pm K_c^{\epsilon,-}(y, z)\partial_z \hat{\mathcal{W}}^{\epsilon, \pm} \pm K^{\epsilon,+}_c(y, z)\hat{\mathcal{W}}^{\epsilon, \mp} + \frac{1}{2} K^{\epsilon,-}_c(y, z)\hat{\mathcal{W}}^{\epsilon, +}.
\]

Here, \( K^{\epsilon, \pm}_c \) are as in (7.2). Observe that the first line in (7.7) is precisely of the same type as (7.2) in which the arguments in **Step III** in the proof of Theorem 2.1 suffice to pass to the limit. The only remaining thing is to pass to the limit in the second line of (7.7) which does not depend on \( \hat{\mathcal{W}}^{\epsilon, \pm} \), but on the Fourier transforms of \( \hat{\mathcal{W}}^{\epsilon, \pm} \) in (2.18) satisfying the system

\[
\partial_t \hat{\mathcal{W}}^{\epsilon, \pm} = -\frac{2}{\epsilon} K^{\epsilon,+}_c(y, z)\partial_y \hat{\mathcal{W}}^{\epsilon, \mp} - \frac{\epsilon}{2} K^{\epsilon,-}_c(y, z)\partial_y \hat{\mathcal{W}}^{\epsilon, \mp} - \frac{\epsilon}{2} K^{\epsilon,-}_c(y, z)\hat{\mathcal{W}}^{\epsilon, +} + \hat{A}^{\epsilon, \pm},
\]

\[ \square \]
\[ \tilde{A}^{\epsilon, +} := K_c^{\epsilon, +}(y, z)(\tilde{\mathcal{W}}^{\epsilon, +} - \tilde{\mathcal{W}}^{\epsilon, -}) \text{ and } \tilde{A}^{\epsilon, -} := \frac{\epsilon}{2} K_c^{\epsilon, -}(y, z)(\tilde{\mathcal{W}}^{\epsilon, +} + \tilde{\mathcal{W}}^{\epsilon, -}). \]

Remark that both \( \tilde{\mathcal{W}}^{\epsilon, \pm} \) are bounded as \( \epsilon \to 0 \) in \( \mathcal{S}'(\mathbb{R}^2) \), so that, modulo extracting subsequences, they converge weakly star to \( \tilde{\mathcal{W}}^{\pm} \) in \( \mathcal{S}'(\mathbb{R}^2) \). The second term of the second line in (7.7) is of order \( \epsilon \), so that it converge to zero as \( \epsilon \to 0 \). The first term is of order one, and following the same arguments in Step III of the proof of Theorem 2.1, we can prove that for all \( a \in \mathcal{S}(\mathbb{R}^2) \), the following convergence holds as \( \epsilon \to 0 \):

\[
\frac{1}{2} \frac{1}{2\pi} \left( K_c^{\epsilon, +} \tilde{\mathcal{W}}^{\epsilon, +}, a \right)_{\mathcal{S}'(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2)} \to \frac{1}{2} \left( \tilde{c}(y) \tilde{\mathcal{W}}^{+}, a \right)_{\mathcal{S}'(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2)}. 
\]

In order to get (2.20), we have to prove that, for all \( \xi \neq 0 \), \( \tilde{\mathcal{W}}^{+}(y, t, \xi) = 0 \). This is obtained by passing to the limit in (7.8). Let us observe that the first term in the right-hand side of (7.8) is of order \( \epsilon^{-1} \), the following two are of order \( \epsilon \), while the fourth one and the time derivative on the left-hand side are of order one. By multiplying (7.8) by \( \frac{\epsilon}{2\pi}(a \in \mathcal{S}(\mathbb{R}^2)) \) in the duality product \( (\cdot, \cdot)_{\mathcal{S}'(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2)} \) and passing to the limit as \( \epsilon \to 0 \), the duality products corresponding to all terms in (7.8) converge to zero, excepting for the one corresponding to the first term on the right-hand side. Thus,

\[
\lim_{\epsilon \to 0} \frac{1}{2\pi} \left( K_c^{\epsilon, +} \tilde{\mathcal{W}}^{\epsilon, \pm}, \tilde{\mathcal{W}}^{\pm} \right)_{\mathcal{S}'(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2)} = (c(y)(-i\xi)\tilde{\mathcal{W}}^{\pm}, a)_{\mathcal{S}'(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2)} = 0. 
\]

Since \( c \neq 0 \), we obtain that, for all \( \xi \neq 0 \), \( \tilde{\mathcal{W}}^{\pm} = 0 \). This concludes (2.20).

Set \( w(y, t) := (w(y, t), \tilde{w}(y, t)) \) to be the column vector containing the two unknowns in (2.14) and \( \Theta(y, \xi) := \Theta_1(y, \xi) + \epsilon \Theta_0(y, \xi) \), where

\[
\Theta_1(y, \xi) := \begin{pmatrix} 0 & c(y)i\xi \\ c(y)i\xi & 0 \end{pmatrix} \text{ and } \Theta_0(y, \xi) := \begin{pmatrix} 0 & d(y) \\ e(y) & 0 \end{pmatrix}. 
\]

Thus, using the notation in Subsection 2.2, system (2.14) can be written in its pseudo-differential form as

\[
\partial_t w(y, t) = \frac{1}{\epsilon} \Theta(y, \epsilon \partial_y) w(y, t) = \left( \frac{1}{\epsilon} \Theta_1(y, \epsilon \partial_y) + \Theta_0(y, \epsilon \partial_y) \right) w(y, t). 
\]

The matrix \( \Theta_1(y, \xi) \) admits the Fourier decomposition \( \Theta_1(y, \xi) = i \Delta \Lambda(y, \xi) \Delta^* \), where

\[
\Lambda(y, \xi) := \begin{pmatrix} \lambda_{c, \xi}^+(y, \xi) & 0 \\ 0 & \lambda_{c, \xi}^-(y, \xi) \end{pmatrix}, \quad \Delta := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. 
\]
and \( \lambda_{c,\xi}^\pm (y, \xi) := \pm c(y)\xi \). Remark that the eigenvector matrix \( \Delta \) in (7.11) does not depend on anyone of the variables \( y \) and \( \xi \). Let us define the projectors \( \Delta^\pm \) to be

\[
\Delta^+ := \Delta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Delta^* = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \Delta^- = \Delta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Delta^* = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]

Set \( \omega(\xi) := \xi \). Remark that the matrix \( \Theta_1(y, \xi) \) associated with the principal operator in (7.10) can be decomposed as

\[
\Theta_1 = i\lambda_{c,\xi}^+ \Delta^+ + i\lambda_{c,\xi}^- \Delta^-.
\]

Moreover, the following identities hold (\( 0_d, I_d \) being the \( d \)-dimensional \( \text{null and identity matrix} \)):

\[
\Delta^\pm \Delta^\pm = \Delta^\pm, \quad \Delta^\pm \Delta^\mp = 0_2, \quad \Delta^+ + \Delta^- = I_2.
\]

In what follows, we give a second proof of Theorem 2.2 which follows the one of Theorem 6.1 in [22], is more technical than the first proof we gave, but provides additional information with respect to that one. It highlights the interpretation of the results in Theorem 2.2 in terms of the spectral decomposition of \( \Theta_1(y, \xi) \).

Consider initial data \( w^{\epsilon,0} = (w^{\epsilon,0}, \tilde{w}^{\epsilon,0}) \) in (2.14) depending on a small parameter \( \epsilon \) and denote \( w^{\epsilon}(y, t) \) the corresponding solution. For simplicity, set \( W^{\epsilon} := W^{\epsilon}[w^{\epsilon}] \) to be the Wigner transform matrix of \( w^{\epsilon} \).

**Step I. Equation (7.19) of the Wigner transform matrix \( W^{\epsilon} \).** Using the definition (2.6) of the Wigner transform and equation (7.10), we obtain the following expression of the Fourier transform in \( \xi \) of \( W^{\epsilon}(y, t, \xi) \), \( \hat{W}^{\epsilon}(y, t, z) \):

\[
\partial_t \hat{W}^{\epsilon}(y, t, z) = \frac{1}{\epsilon} \Theta(y, \epsilon \partial_y) w^{\epsilon} \left( y - \frac{\epsilon z}{2}, t \right) \otimes \left( w^{\epsilon} \left( y + \frac{\epsilon z}{2}, t \right) \right)^* + \frac{1}{\epsilon} \left( \Theta(y, \epsilon \partial_y) w^{\epsilon} \left( y + \frac{\epsilon z}{2}, t \right) \right)^*.
\]

Let us consider \( a \in S(\mathbb{R}_y \times \mathbb{R}_\xi) \) and multiply (7.14) by \( \overline{a}(y, z)/2\pi \). After writing explicitly how the pseudo-differential operator \( \Theta(y, \epsilon \partial_y) \) acts in (7.14), we obtain

\[
\frac{1}{2\pi} (\partial_t \hat{W}^{\epsilon}, \overline{a})_{S'((\mathbb{R}^2_\xi), S(\mathbb{R}^2_y))} = A^{\epsilon} + B^{\epsilon},
\]

where

\[
A^{\epsilon} := \frac{1}{(2\pi)^{3}} \int_{\mathbb{R}^4} \frac{1}{\epsilon} \Theta(y - \frac{\epsilon z}{2}, \epsilon \eta) w^{\epsilon}(x, t) \otimes \left( w^{\epsilon} \left( y + \frac{\epsilon z}{2}, t \right) \right)^* \times \exp(i\eta \left( y - \frac{\epsilon z}{2} - x \right)) \overline{a}(y, z) \, dx \, dy \, dz \, d\eta.
\]
and

\[ B^\epsilon := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} w^\epsilon(y - \frac{\epsilon z}{2}, t) \otimes (w^\epsilon(x, t))^* \left( \frac{1}{\epsilon} \Theta\left(y + \frac{\epsilon z}{2}, \epsilon \eta\right) \right)^* \]

\[ \times \exp\left(-i \eta \left(y + \frac{\epsilon z}{2} - x\right)\right) \tilde{a}(y, z) \, dx \, dy \, dz \, d\eta. \]

In \( A^\epsilon \) and \( B^\epsilon \), consider the change in variable \( \zeta := \epsilon \eta \) (thus, \( d\eta := \epsilon^{-1} d\zeta \)). We also do the change in variable \( y \to y' \) so that \( y + \epsilon z/2 = 2y' - x \) (in \( A^\epsilon \)) and \( y - \epsilon z/2 = 2y' - x \) (in \( B^\epsilon \)) (thus, \( dy = 2 dy' \)). Moreover, in \( B^\epsilon \), we change \( x \to x' = 2y' - x \). After all these changes, \( A^\epsilon \) and \( B^\epsilon \) become

\[ A^\epsilon = \frac{1}{(2\pi)^2} \frac{2}{\epsilon} \int_{\mathbb{R}^4} \frac{1}{\epsilon} \Theta(2y - x - \epsilon z, \zeta) w^\epsilon(x, t) \otimes (w^\epsilon(2y - x, t))^* \]

\[ \times \exp\left(\frac{2i \zeta}{\epsilon} \left(y - x - \frac{\epsilon z}{2}\right)\right) \tilde{a}\left(2y - x - \frac{\epsilon z}{2}, z\right) \, dx \, dy \, dz \, d\zeta. \]

and

\[ B^\epsilon = \frac{1}{(2\pi)^2} \frac{2}{\epsilon} \int_{\mathbb{R}^4} w^\epsilon(x, t) \otimes (w^\epsilon(2y - x, t))^* \left( \frac{1}{\epsilon} \Theta(x + \epsilon z, \zeta) \right)^* \]

\[ \times \exp\left(\frac{2i \zeta}{\epsilon} \left(y - x - \frac{\epsilon z}{2}\right)\right) \tilde{a}\left(x + \frac{\epsilon z}{2}, z\right) \, dx \, dy \, dz \, d\zeta. \]

Remark that \( 2y - x - \epsilon z = y + y_1, x + \epsilon z = y - y_1, 2y - x - \epsilon z/2 = y + y_2, \) and \( x + \epsilon z/2 = y - y_2, \) with \( y_1 = y - x - \epsilon z \) and \( y_2 = y - x - \epsilon z/2 \). The following Taylor expansions of \( \Theta(y \pm y_1, \zeta) \) and \( \tilde{a}(y \pm y_2, z) \) about \( y \) hold

\[ \Theta(y \pm y_1, \zeta) = \Theta(y, \zeta) \pm y_1 \partial_y \Theta(y, \zeta) + \frac{y_1^2}{2} \mathcal{R}^\pm_\Theta \]

and

\[ \tilde{a}(y \pm y_2, z) = \tilde{a}(y, z) \pm y_2 \partial_y \tilde{a}(y, z) + \frac{y_2^2}{2} \mathcal{R}^\pm_\tilde{a}, \]

(7.16)

where \( \mathcal{R}^\pm_\Theta = \mathcal{R}^\pm_\Theta(y, y_1, \zeta) \) and \( \mathcal{R}^\pm_\tilde{a} = \mathcal{R}^\pm_\tilde{a}(y, y_2, z) \) are the corresponding Taylor remainders.

We write \( A^\epsilon \) and \( B^\epsilon \) as \( A^\epsilon := \hat{A}^\epsilon + \mathcal{R}^\epsilon_A \) and \( B^\epsilon := \hat{B}^\epsilon + \mathcal{R}^\epsilon_B \), where \( \hat{A}^\epsilon \) and \( \hat{B}^\epsilon \) are integrals of the same form as \( A^\epsilon \) and \( B^\epsilon \) retaining from the factors \( \Theta(y \pm y_1, \xi) \tilde{a}(y \pm y_2, z) \) appearing in \( A^\epsilon \) and \( B^\epsilon \) only the terms \( \Theta(y, \zeta) \tilde{a}(y, z) \pm y_2 \partial_y \tilde{a}(y, z) \). We follow by using the following equivalent form of the Wigner transform matrix (2.6) which can be obtained by the change in variable \( z \to y = x - \epsilon z/2 \) in (2.6):

\[ \mathcal{W}^\epsilon[f^1, f^2](x, \xi) := \frac{1}{2\pi \epsilon} \int_{\mathbb{R}} f^1(y) \otimes f^2(y) \exp\left(\frac{2i \xi (x - y)}{\epsilon}\right) \, dy. \]

(7.17)
We also take into account that the term $y - x$ entering in both $y_1$ and $y_2$ yields $\epsilon \partial_\xi W^e(y, t, \xi)/\epsilon i$ when integrate in $x$ in both $\hat{A}^e$ and $\hat{B}^e$, while the term $-\epsilon z \partial_\xi \hat{a}(y, z)$ yields $\epsilon \partial_\xi \partial_\eta^\beta a(y, \xi)/\epsilon i$ when integrate in $z$ (with $\alpha = 0$ or $\alpha = 1$). Thus,

$$\hat{A}^e := \frac{1}{\epsilon} \left\langle \Theta W^e(\cdot, t, \cdot), a \right\rangle_{S', S} + \frac{1}{2i} \left\langle \Theta \partial_\xi W^e(\cdot, t, \cdot), \partial_\eta^\beta a \right\rangle_{S', S}$$

$$+ \frac{1}{2i} \left\langle \Theta W^e(\cdot, t, \cdot), \partial_\eta^2 a \right\rangle_{S', S} + \frac{1}{2i} \left\langle \partial_\xi \Theta \partial_\xi W^e(\cdot, t, \cdot), a \right\rangle_{S', S}$$

and a similar expression for $\hat{B}^e$, in which the only changes are that the terms in the integrand from the second to the fifth ones change the sign, the matrix $\Theta(y, \xi)$ and its derivatives are replaced by the corresponding conjugate transpose matrices and interchanged with the corresponding derivatives of the Wigner transform matrix $W^e(y, t, \xi)$. Here, $(f, a)_{S', S} := \int_{\mathbb{R}^2} f(y, \xi) a(y, \xi) \, dy \, d\xi$. Passing all the derivatives of $a$ with respect to both $y$ and $\xi$ to the accompanying factors, we obtain

$$\hat{A}^e = \left. \frac{1}{\epsilon} \left\langle \Theta W^e(\cdot, t, \cdot), \partial_\eta^\beta \Theta W^e(\cdot, t, \cdot), a \right\rangle_{S', S} \right\rvert_{\xi, t} (7.18)$$

and a similar expression for $\hat{B}^e$, in which the sign of the third term is changed and the above-mentioned transformations concerning the conjugate transposition of $\Theta(y, \xi)$ and the interchange with $W^e(y, t, \xi)$ are done. Here, $\{p, q\} := \partial_\xi p \partial_\eta q - \partial_\eta p \partial_\xi q$ is the so-called Poisson bracket (cf. [22]).

Let us show that the terms $\mathcal{R}^e_A$ and $\mathcal{R}^e_B$ are small with respect to $\epsilon$. We explain how this can be proved for $\mathcal{R}^e_A$, for $\mathcal{R}^e_B$ the arguments being similar. Remark that $\mathcal{R}^e_A$ is an integral of the same type as $A^e$, in which the factor $\Theta(y + y_1, \xi) \partial_\xi a(y + y_2, z)$ is replaced by $y_2^a \Theta(y, \xi) \mathcal{R}_a^+ + y_1 y_2 \partial_\xi \Theta(y, \xi) \partial_\xi \hat{a}(y, z) + y_1 y_2 \partial_\xi \Theta(y, \xi) \mathcal{R}_a^+ + y_1^a \mathcal{R}_\Theta^+ \partial_\alpha \hat{a}(y, z) + y_2^a \mathcal{R}_\Theta^+ \mathcal{R}_a^+ \partial_\beta a(y, z)$. Each term of this integrand contains powers of the form $y_1^a y_2^\beta$, with $\alpha + \beta \geq 2$. In order to make clear the ideas, we analyze only the integral $\mathcal{I}^e$ involving the term $y_1 y_2 \partial_\xi \Theta(y, \xi) \partial_\xi \hat{a}(y, z)$. For the other ones, one can do a similar analysis, taking into account the fact that $\mathcal{R}_\Theta^+ \sim \partial_\xi \hat{a}(y, \xi)$ and $\mathcal{R}_a^+ \sim \partial_\xi \hat{a}(y, z)$. Observe that $y_1 y_2 = (y - x)^2 - 3 \epsilon \partial_\xi \hat{a}(y, z) + \epsilon^2 \partial_\xi \hat{a}(y, z)^2$. As before, we take into account the fact that $(y - x)^2$ yields $(\epsilon/2i)^2 \partial_\xi \hat{a}(y, \xi)$ when integrate in $x$, while any factor $(\epsilon^2) \partial_\xi \hat{a}(y, z)$ is converted into $-\epsilon/2 \partial_\xi \hat{a}(y, \xi)$ when integrate in $z$. In this way,

$$\mathcal{I}^e = -\frac{\epsilon}{4} \left\langle \partial_\xi \Theta \partial_\xi \hat{a}(y, \xi), a \right\rangle_{S', S} - \frac{\epsilon}{2} \left\langle \partial_\xi \Theta \partial_\xi \hat{a}(y, \xi), \partial_\xi \hat{a}(y, \xi) \right\rangle_{S', S}$$
Passing all the derivatives of $W^\xi$ with respect to $y$ or $\xi$ to the accompanying factors, we obtain

$$I^\xi = -\frac{\epsilon}{4}\left\{ \Theta_1 I_{y, \xi}^a(y, \zeta) + \Theta_0 I_{y, \xi}^a(\zeta) + \Theta_0 I_{y, \xi}^a(y, \zeta) \right\},$$

so that, taking into account that $W^\xi$ is bounded in $(S'(\mathbb{R}^y \times \mathbb{R}^\xi))^4$ and that $\Theta = \Theta_1 + \epsilon \Theta_0$, with $\Theta_0$ and $\Theta_1$ regular enough so that $\Theta_1 a$ and $\Theta_0 a$ are in $(L^\infty(\mathbb{R}^2))^4$, for all $i = 0, 1$ and $a \in S(\mathbb{R}^2)$, we conclude that $I^\xi = O(\epsilon)$.

By applying the Parseval identity in the left-hand side of (7.15), taking into account the above considerations on $A^\epsilon$ and $B^\epsilon$ (see (7.18)), the fact that $\Theta = \Theta_1 + \epsilon \Theta_0$ and that $\Theta_1 = -\Theta_1$, we conclude that the matrix $W^\xi(y, t, \xi)$ verifies the equation

$$\partial_t W^\xi = \frac{\Theta_1 W^\xi - W^\xi \Theta_1}{\epsilon} + (\Theta_0 W^\xi + W^\xi \Theta_0) + \frac{1}{2i}((\{\Theta_1, W^\xi\} - \{W^\xi, \Theta_1\}))
$$

$$- \frac{1}{2i}(\partial_{y, \xi}^2 \Theta_1 W^\xi + W^\xi \partial_{y, \xi}^2 \Theta_1) + \epsilon R^\epsilon,$n

where $R^\epsilon$ is bounded in $(S'(\mathbb{R}^2))^4$. Observe that this is a similar equation to (6.12) in [22], excepting the term $-(\partial_{y, \xi}^2 \Theta_1 W^\xi + W^\xi \partial_{y, \xi}^2 \Theta_1)/2i$. This comes from the fact that in that case, Eq. (7.10) involves the Weyl operator $\Theta^w(y, \epsilon \partial_y)$ instead of $\Theta(y, \epsilon \partial_y)$ (for the definition of the Weyl operator, see formula (1.4) in [22]). Of course, $W^\xi [(\Theta(y, \epsilon \partial_y) - \Theta^w(y, \epsilon \partial_y))u^\epsilon, v^\epsilon] \to 0$ as $\epsilon \to 0$ in $S'(\mathbb{R}^2)$, for all vector functions $u^\epsilon, v^\epsilon$ bounded in $(L^2(\mathbb{R}))^2$ as $\epsilon \to 0$. Due to the fact that the operator $\Theta(y, \epsilon \partial_y)$ in (7.10) is multiplied by a factor $\epsilon^{-1}$, the additional term in (7.19) with respect to (6.12) in [22] comes from the fact that $W^\xi [\epsilon^{-1}(\Theta(y, \epsilon \partial_y) - \Theta^w(y, \epsilon \partial_y))u^\epsilon, v^\epsilon]$ is not at all trivial as $\epsilon \to 0$.

**Step II. Equation (7.29) of the projections of the Wigner measure matrix on the Fourier modes of $\Theta_1(y, \xi)$**. Let us denote by $W(y, t, \xi)$ the weak limit of the Wigner transform matrix $W^\xi(y, t, \xi)$. Remark the following relationship between the limit $W$ in Theorem (2.2) which is a scalar quantity and $\mathcal{W}$:

$$W = \lim_{\epsilon \to 0} (W^\epsilon [w^\epsilon] + W^\epsilon [\tilde{w}^\epsilon]) = \lim_{\epsilon \to 0} \text{tr}(W^\epsilon) = \text{tr}(W).$$

Since the first term on the right-hand side of (7.19) is of order $\epsilon^{-1}$, we get directly from (7.19)

$$\Theta_1(y, \xi) W(y, t, \xi) = W(y, t, \xi) \Theta_1(y, \xi), \quad \forall y, \xi \in \mathbb{R}. \quad (7.21)$$

This is the analogous of (7.9) in the first proof of Theorem 2.2. Remark that, for $\xi = 0$, $\Theta_1(y, \xi)$ is the null matrix, so that (7.21) becomes an obvious identity from which one cannot get information about $W$.

Set $W^\pm(y, t, \xi) := \Delta^\xi W^\xi(y, t, \xi) \Delta^\xi$, where $W^\xi$ is the Wigner transform matrix in (7.19) and $W^\pm := \lim_{\epsilon \to 0} W^{\epsilon \pm}$. Let us multiply (7.19) to the left
and to the right by $\Delta^\pm$. In this way, the left-hand side of (7.19) is transformed into $\partial_1 \mathcal{W}^e, \pm(y, t, \xi)$. Remark that, by using the decomposition (7.12) of the matrix $\Theta_1(y, \xi)$, the first and the second identities in (7.13), we obtain

$$
\Delta^\pm \Theta_1(y, \xi) \mathcal{W}^e(y, t, \xi) \Delta^\pm = \Delta^\pm \mathcal{W}^e(y, t, \xi) \Theta_1(y, \xi) \Delta^\pm
$$

$$
= i\lambda_{c,\omega}^\pm (y, \xi) \mathcal{W}^{e, \pm}(y, t, \xi),
$$

(7.22)

with $\omega(\xi) = \xi$. Thus, under this process of multiplication of (7.19) by $\Delta^\pm$, the term of order $\epsilon^{-1}$ on the right-hand side of (7.19) vanishes. Now, let us remark that, using the last and the first identities in (7.13), we obtain

$$
\Delta^\pm \Theta_0(y, \xi) \mathcal{W}^e(y, t, \xi) \Delta^\pm = \Delta^\pm \Theta_0(y, \xi) \Delta^\pm \mathcal{W}^{e, \pm}(y, t, \xi)
$$

$$
+ \Delta^\pm \Theta_0(y, \xi) \Delta^\mp \mathcal{W}^e(y, t, \xi) \Delta^\pm.
$$

(7.23)

We also get the similar identity $\Delta^\pm \mathcal{W}^e \Theta_0^* \Delta^\pm = \mathcal{W}^{e, \pm} \Delta^\pm \Theta_0^* \Delta^\pm + \Delta^\pm \mathcal{W}^e \Delta^\mp \Theta_0^* \Delta^\pm$. An easy computation yields

$$
\Delta^\pm \Theta_0(y, \xi) \Delta^\pm = \Delta^\pm \Theta_0^*(y, \xi) \Delta^\pm = \pm \frac{1}{2} (d(y) + e(y)) \Delta^\pm,
$$

(7.24)

where the functions $d$ and $e$ have been introduced in (2.16). Thus, the second term in the right-hand side of (7.19) becomes

$$
\Delta^\pm (\Theta_0 \mathcal{W}^e + \mathcal{W}^e \Theta_0^*) \Delta^\pm = \pm (d + e) \mathcal{W}^{e, \pm} + \Delta^\pm \Theta_0^* \Delta^\mp \mathcal{W}^e \Delta^\pm
$$

$$
+ \Delta^\pm \mathcal{W}^e \Delta^\mp \Theta_0^* \Delta^\pm.
$$

(7.25)

In what follows, we show that the last two terms in the right-hand side of (7.25) converge to $\theta_2$ as $\epsilon \to 0$ for $\xi \neq 0$. To this aim, we multiply (7.21) by $\Delta^\mp$ to the left and by $\Delta^\pm$ to the right. Taking into account (7.12) and the first two identities in (7.13), we obtain $(\lambda_{c,\xi} \pm (y, \xi) - \lambda_{c,\xi}^\mp (y, \xi)) \Delta^\mp \mathcal{W}(y, t, \xi) \Delta^\pm = \theta_2$, and, by taking into account the fact that, for $\xi \neq 0, \lambda_{c,\xi} \pm (y, \xi) = \lambda_{c,\xi}^\mp (y, \xi)$, we finally get

$$
\Delta^\mp \mathcal{W} \Delta^\pm = \theta_2,
$$

(7.26)

where $\theta_2$ is the $2 - d$ column vector with null components. Due to (7.12), to the fact that the projectors $\Delta^\pm$ do not depend on anyone of the two variables $y$ and $\xi$ and to (7.13), we obtain the following two identities concerning the third and the fourth terms in (7.19):

$$
\frac{1}{2t} \Delta^\pm ([\Theta_1, \mathcal{W}^e] - [\mathcal{W}^e, \Theta_1]) \Delta^\pm = \{\lambda_{c,\xi}^\pm, \mathcal{W}^{e, \pm}\}
$$

(7.27)

and

$$
\frac{1}{2t} \Delta^\pm (\partial^2_{y^2} \Theta_1 \mathcal{W}^e + \mathcal{W}^e \partial^2_{y^2} \Theta_1) \Delta^\pm = \partial^2_{y^2} \lambda_{c,\xi}^\pm \mathcal{W}^{e, \pm}. \quad (7.28)
$$
Recollecting (7.22) and (7.25–7.28), we obtain that the matrix $W^\pm = \lim_{\epsilon \to 0} W^{\epsilon, \pm}$ verifies the problem $\partial_t W^\pm = \pm (d(y) + e(y)) W^\pm + \{\lambda^{\pm}_{c, \omega}, W^\pm\} - \partial^2_{\xi}\lambda^{\pm}_{c, \omega} W^\pm$, with $\omega(\xi) = \xi$. Using (2.16) and the fact that $\lambda^+_{c, \omega} = -\lambda^-_{c, \omega} = c(y)\xi$, we see that the first and the third terms on the right-hand side of this equation cancel, so that the equation for $W^\pm$ simplifies to

$$\partial_t W^\pm = \{\lambda^{\pm}_{c, \omega}, W^\pm\}. \quad (7.29)$$

Remark that by applying the trace operator to (7.29), we obtain that $W^\pm = \text{tr}(W^\pm)$ verifies equation (2.20). By passing to the limit as $\epsilon \to 0$ in the identity $W^{\epsilon, \pm} = \Delta^\pm W^\epsilon \Delta^\pm$, we obtain that $W^\pm = \Delta^\pm W \Delta^\pm$. We remark the following identities

$$W = \text{tr}(W) = \text{tr}(W^+) + \text{tr}(W^-) = W^+ + W^-,$$

where $W = \lim_{\epsilon \to 0} (W^\epsilon[w^\epsilon] + W^\epsilon[\tilde{w}^\epsilon])$ is the Wigner measure introduced in the statement of Theorem 2.2 and $W^\pm = \lim_{\epsilon \to 0} W^\epsilon[(w^\epsilon \pm \tilde{w}^\epsilon)/\sqrt{2}]$. This concludes the proof of Theorem 2.2.

### 7.3 Proof of Proposition 3.1

**Remark 7** Using the change in variable $\tilde{u}(\tilde{y}, t) = u(y, t)$, with $\tilde{y} = H(y)$ and $H'(y) = \varrho(y)$, the transport equation with variable coefficients (1.1) becomes the transport equation with constant coefficients below

$$\partial_t \tilde{u}(\tilde{y}, t) + \partial_y \tilde{u}(\tilde{y}, t) = 0, \quad \tilde{y} \in \mathbb{R}, \quad t > 0, \quad \tilde{u}(\tilde{y}, 0) = \tilde{u}^0(\tilde{y}) := u^0(H^{-1}(\tilde{y})), \quad (7.31)$$

for which the solution is $\tilde{u}(\tilde{y}, t) = \tilde{u}^0(\tilde{y} - t) = u^0(H^{-1}(\tilde{y} - t))$. Then, the solution of (1.1) is given by $u(y, t) = u^0(H^{-1}(H(y) - t))$.

We use Lax–Richtmyer Theorem (cf. [29]), guaranteeing that a numerical scheme is convergent if and only if both its consistency and stability properties hold. The stability of (1.6) means the existence of an uniform constant $C > 0$ as $h \to 0$ ($C := g_d \varrho^-$ in our case) such that (here, $f^{h, 1} \circ f^{h, 2} := (f^1_j f^2_j)_{j \in \mathbb{Z}}$)

$$(\partial^h g(x^h) \circ \varrho(g(x^h))) \circ \partial_x f^h(t) - \partial^h f^h(t), f^h(t))_{\ell^2_h} \geq C \partial_t (||f^h(t)||^2_{\ell^2_h}), \quad \forall f^h(t) \in \ell^2_h, \forall t > 0. \quad (7.32)$$

The consistency means to consider a solution $u(\cdot, t)$ of (1.1) belonging to $C^1(\mathbb{R}_+)$ for all $t > 0$ and to plug it in the numerical scheme (1.6) to obtain (by Taylor expansions):

$$\varrho(g_j) \partial_t u(g_j, t) + \frac{u(g_{j+1}, t) - u(g_{j-1}, t)}{g_{j+1} - g_{j-1}} = r_j(t), \quad (7.33)$$
where
\[ r_j(t) := \frac{1}{2} \left( \frac{g_{j+1} - g_j}{g_{j+1} - g_j} \right)^2 \partial^2_y u(\theta_{j+1/2}, t) - \frac{1}{2} \left( \frac{g_j - g_{j-1}}{g_{j+1} - g_j} \right)^2 \partial^2_y u(\theta_{j-1/2}, t) \]

and \( \theta_{j+1/2} \in (g_j, g_{j+1}) \), for all \( j \in \mathbb{Z} \) and all \( t > 0 \).

Using the hypothesis on \( g \), we obtain the following estimate on the residual \( r^h(t) := (r_j(t))_{j \in \mathbb{Z}} \):
\[ |\partial^h g_j r_j(t)| \leq \frac{h}{4} \left( \frac{g_{j+1} - g_j}{g_{j+1} - g_j} \right)^2 \left( |\partial_y^2 u(\theta_{j+1/2}, t)| + |\partial_y^2 u(\theta_{j-1/2}, t)| \right), \quad \forall j \in \mathbb{Z}, \forall t > 0. \] (7.34)

Let us remark that the error \( e^h(t) := u(g(x^h), t) - u^h(t) \) satisfies the problem
\[ \varrho(g(x_j)) \partial_t e_j(t) + \frac{e_{j+1}(t) - e_{j-1}(t)}{g_j - g_{j-1}} = r_j(t), \quad e_j(0) = 0, \quad \forall j \in \mathbb{Z}, t > 0. \] (7.35)

By multiplying (7.35) by \( h \partial^h g_j e_j(t) \), adding in \( j \in \mathbb{Z} \) and using the stability estimate (7.32), we obtain
\[ \varrho^{-} \partial_t (||e^h(t)||^2_{\ell^2_h}) \leq ||e^h(t)||_{\ell^2_h} ||\partial^h g^h \circ r^h(t)||_{\ell^2}, \] (7.36)

form where, using the hypothesis on \( g \), we have
\[ \partial_t (||e^h(t)||_{\ell^2}) \leq \frac{1}{2 \varrho^{-}} ||\partial^h g^h \circ r^h(t)||_{\ell^2}, \] (7.37)

or, taking into account the fact that \( e^h(0) = 0 \),
\[ ||e^h(t)||_{\ell^2} \leq \frac{1}{2 \varrho^{-}} \int_0^t ||\partial^h g^h \circ r^h(s)||_{\ell^2} ds \leq \frac{t}{2 \varrho^{-}} \sup_{s \in [0, t]} ||\partial^h g^h \circ r^h(s)||_{\ell^2}. \] (7.38)

From Remark 7, we see that
\[ u_{yy}(\theta_{j+1/2}, t) = (u^0)''(H^{-1}(H(\theta_{j+1/2}) + t)) \left( \frac{\varrho(\theta_{j+1/2})}{\varrho(H(\theta_{j+1/2}) + t)} \right)^2 \]
\[ + (u^0)'(H^{-1}(H(\theta_{j+1/2}) + t)) \]
\[ \times \frac{\varrho'(\theta_{j+1/2})\varrho(H(\theta_{j+1/2}) + t) - (\varrho(\theta_{j+1/2}))^2 \varrho'(H(\theta_{j+1/2}) + t)}{(\varrho(H(\theta_{j+1/2}) + t))^2}, \]
so that, using the hypothesis on the coefficient $\varrho$, we obtain

$$
|u_{yy}(\theta_{j\pm 1/2}, t)| \leq |(u^0)'(H^{-1}(H(\theta_{j\pm 1/2} + t)))(\frac{\varrho^+}{Q^-})^2$

$$
\quad + |(u^0)'(H^{-1}(H(\theta_{j\pm 1/2} + t))|\frac{\varrho^+ (1 + \varrho^+)}{(Q^-)^2}. \quad (7.39)
$$

Let us denote by $\theta^h_{\pm 1/2} := (\theta_{j\pm 1/2})_{j \in \mathbb{Z}}$. From (7.34) and (7.39), we obtain that

$$
||\partial^h s^h \tan \tan (s)||_{\ell^2} \leq \frac{h}{4}(g_d^+)^2(||\partial^2 u(\theta^h_{1/2}, s)||_{\ell^2} + ||\partial^2 u(\theta^h_{-1/2}, s)||_{\ell^2})
$$

$$
\quad \leq \frac{h}{4}(g_d^+)^2 \left[\left(\frac{\varrho^+}{Q^-}\right)^2||u^0)''(H^{-1}(H(\theta^h_{1/2} + s))||_{\ell^2} + ||u^0)'(H^{-1}
$$

$$
\quad \quad \times (H(\theta^h_{1/2} + s))||_{\ell^2}
$$

$$
\quad + \frac{\varrho^+ e_{d}^+(1 + \varrho^+)}{(Q^-)^2}(||(u^0)'(H^{-1}(H(\theta^h_{1/2} + s)))||_{\ell^2} + ||(u^0)'(H^{-1}(H(\theta^h_{-1/2} + s)))||_{\ell^2})\right].
$$

For a function $f \in C_c(\mathbb{R})$, we have

$$
||f(H^{-1}(H(\theta^h_{1/2} + s))||_{\ell^2} \leq ||f||_{L^\infty(\mathbb{R})}\left(h^2\{j \text{ s.t. } H^{-1}(H(\theta_{j\pm 1/2} + s) \in \text{ supp } f)\}^{1/2}
$$

$$
\quad \leq ||f||_{L^\infty(\mathbb{R})}(\mathcal{L}(g^{-1}(H^{-1}(H(\text{ supp } f - s)))))^{1/2},
$$

where $\mathcal{L}(A)$ is the Lebesgue measure of the set $A$.

The conclusion (3.4) follows by taking

$$
C(t, g, \varrho, u^0) := \frac{1}{4g_d^+} \left[\left(\frac{\varrho^+}{Q^-}\right)^2||u^0)''||_{L^\infty(\mathbb{R})}\left(\sup_{s \in [0, t]}\mathcal{L}(g^{-1}(H^{-1}(H(\text{ supp } u^0)'' - s)))\right)^{1/2}
$$

$$
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \frac{\varrho^+ e_{d}^+(1 + \varrho^+)}{(Q^-)^2}||u^0)'||_{L^\infty(\mathbb{R})}\left(\sup_{s \in [0, t]}\mathcal{L}(g^{-1}(H^{-1}(H(\text{ supp } u^0)' - s)))\right)^{1/2}\right]. \quad (7.40)
$$

7.4 Proof of Proposition 3.4

As in the proof of Proposition 3.1, the error $\epsilon^h(t) := u(g_j, t) - u_j(t)$ solves the equation
\[ \rho(g_j) \partial_t^2 \epsilon_j - \frac{\sigma(g(x_{j+1/2})) \epsilon_{j+1}(t) - \epsilon_j(t)}{g_{j+1} - g_j} - \frac{\sigma(g(x_{j-1/2})) \epsilon_j(t) - \epsilon_{j-1}(t)}{g_j - g_{j-1}} = r_j(t), \]

\[ \epsilon_j(0) = \epsilon_{j,t}(0) = 0, \]

(7.41)

for all \( j \in \mathbb{Z} \) and \( t > 0 \). Here, the residual \( r_j(t) \) below is the error obtained in the consistency analysis when the solution \( u(g_j, t) \) of the continuous equation (1.4) is introduced in the discrete equation (1.6):

\[ r_j(t) := r_j^1(t) + \cdots + r_j^7(t), \]

where

\[ r_j^1(t) := \frac{\sigma(g_j) \sum \pm (g_j \pm 1 - g_j)^2 \partial_\tau^3 u(\theta_{j+1/2}, t)}{3(g_{j+1} - g_{j-1})}, \]

\[ r_j^2(t) := -\sigma'(g_j) \partial_\tau u(g_j, t) \frac{g_{j+1} - 2g_{j+1/2} + 2g_{j-1/2} - g_j}{g_{j+1} - g_{j-1}}, \]

\[ r_j^3(t) := \frac{\sigma'(g_j) \partial_\tau^2 u(g_j, t) \sum \pm (g_j \pm 1/2 - g_j)(g_j \pm 1 - g_j)}{g_{j+1} - g_{j-1}}, \]

\[ r_j^4(t) := \frac{\sigma'(g_j) \partial_\tau^2 u(g_j, t) \sum \pm (g_j \pm 1/2 - g_j)(g_j \pm 1 - g_j)^2 \sigma'(g_j) \partial_\tau^3 u(\theta_{j+1/2}, t)}{3(g_{j+1} - g_{j-1})}, \]

\[ r_j^5(t) := \frac{\sum \pm (g_j \pm 1/2 - g_j)(g_j \pm 1 - g_j)^2 \sigma''(\theta_{j+1/4}) \partial_\tau u(g_j, t)}{g_{j+1} - g_{j-1}}, \]

\[ r_j^6(t) := \frac{\sum \pm (g_j \pm 1/2 - g_j)(g_j \pm 1 - g_j)^2 \sigma''(\theta_{j+1/4}) \partial_\tau^2 u(g_j, t)}{2(g_{j+1} - g_{j-1})}, \]

\[ r_j^7(t) := \frac{\sum \pm (g_j \pm 1/2 - g_j)(g_j \pm 1 - g_j)^2 \sigma''(\theta_{j+1/4}) \partial_\tau^3 u(\theta_{j+1/2}, t)}{6(g_{j+1} - g_{j-1})}, \]

with \( g_{j-1/2} < \theta_{j-1/4} < g_j < \theta_{j+1/4} < g_{j+1/2} \) and \( g_{j-1} < \theta_{j-1/2} < g_j < \theta_{j+1/2} < g_{j+1} \).

Let us remark that, using the hypothesis on \( g \) and \( \sigma \), we obtain

\[ |\partial_t g r_j^1(t)| \leq \frac{\sigma^+ (g_d^+)^2}{6} \int_0^t \sum \pm |\partial_\tau^3 u(\theta_{j+1/2}, t)|, \quad |\partial_t g r_j^2(t)| \leq \frac{\sigma^+ (g_d^+)^2}{4} \int_0^t |\partial_\tau u(g_j, t)|, \]

\[ |\partial_t g r_j^3(t)| \leq \frac{\sigma^+ (g_d^+)^2}{2} \int_0^t |\partial_\tau^2 u(g_j, t)|, \quad |\partial_t g r_j^4(t)| \leq \frac{(g_d^+)^3 \sigma^+}{12} \int_0^t \sum \pm |\partial_\tau^3 u(\theta_{j+1/2}, t)|, \]

\[ |\partial_t g r_j^5(t)| \leq \frac{\sigma^+ (g_d^+)^2}{4} \int_0^t |\partial_\tau u(g_j, t)|, \quad |\partial_t g r_j^6(t)| \leq \frac{\sigma^+ (g_d^+)^3}{8} \int_0^t |\partial_\tau^2 u(g_j, t)|, \]

\[ \square \]
and

\[ |\partial^h g_j r_j^7(t)| \leq \frac{\sigma_{d\alpha}(g_{d\alpha})^4}{48} h^3 \sum_{\pm} |\partial^3 y(\theta_{j\pm}, t)|. \]

By multiplying (7.41) by \( h^\alpha \partial^h g_j \epsilon_j^7(t) \), we obtain

\[
2 \min \left\{ \sqrt{g_d^\alpha \rho^\alpha}, \sqrt{\frac{\sigma^\alpha}{g_{d\alpha}}} \right\} \|||\epsilon^h(t), \partial_t \epsilon^h(t)||_{\dot{h}\times \ell^2_h}^2 \leq 2 (2 \epsilon^h_{\rho, \sigma, g}(\epsilon^h(t), \partial_t \epsilon^h(t)))^{1/2}
\]

\[
\leq \int_0^t \left( \sum_{j \in \mathbb{Z}} h \frac{\partial^h g_j}{\rho(g_j)} |r_j(s)|^2 \right)^{1/2} ds
\]

\[
\leq \frac{t}{\sqrt{\rho^h g_d^\alpha}} \sup_{s \in [0, t]} \left( ||\partial^h g^h \odot r^h(s)||_{\ell^\infty} (h^\alpha |j \text{ s.t. } r_j(s) \neq 0|)^{1/2} \right). \tag{7.42}
\]

Remark that

\[
||\partial^h g^h \odot r^h(s)||_{\ell^\infty} \leq (C_1 h + C_2 h^2 + C_3 h^3)||u(\cdot, s)||_{W^3, \infty(\mathbb{R})}. \tag{7.43}
\]

Let us introduce the function

\[
F(y) = \int_0^y \sqrt{\frac{\rho(z)}{\sigma(z)}} dz. \tag{7.44}
\]

For all \( x \in \mathbb{R} \) and \( s \in \mathbb{R}_+ \), set \( x^{s, \pm} := F^{-1}(F(x) \pm s) \). Let us remark that for the continuous wave equation (1.4), the space component \( y_{\pm}(t) \) of the two families of characteristics in (2.4) can be found independently on the phase variable \( \xi_{\pm}(t) \) and is precisely \( y_{\pm}(t) = y^{t, \pm} = F^{-1}(F(y) \pm t) \), with \( F \) as in (7.44).

The following result shows that if the initial data \((u^0, u^1)\) in (1.4) is compactly supported in \((a, b)\), then \( u(\cdot, s) \) is supported in \((a^{s, -}, b^{s, +})\).

**Lemma 7.1** Let \([a, b] = (supp u^0 \cup supp u^1) \) and \((y_0, y_1) \subset \mathbb{R} \setminus [a, b] \) be an interval in the complementary of the support of the initial data \((u^0, u^1)\) in (1.4). Then, for all \( s < T_F := (F(y_1) - F(y_0))/2 \),

\[
\int_{y_0^{s, -}}^{y_1^{s, -}} (\rho(y)|\partial_t u(y, s)|^2 + \sigma(y)|\partial_y u(y, s)|^2) dy = 0. \tag{7.45}
\]

By choosing \((y_0^a, y_0^a) \subset \mathbb{R} \setminus [a, b] \) such that \((y_0^{a, s, +}, y_1^{a, s, -})_a \) is a partition of \( \mathbb{R} \setminus [a^{s, -}, b^{s, +}] \), we get
\[
\int_{\mathbb{R} \setminus [a^{s-}, b^{s+}]} \left( \rho(y) |\partial_t u(y, s)|^2 + \sigma(y) |\partial_y u(y, s)|^2 \right) \, dy = 0,
\]

so that, using the decay properties of the solution \(u(\cdot, s)\) of (1.4) at infinity, we obtain that \(u(\cdot, s) \equiv 0\) in \(\mathbb{R} \setminus [a^{s-}, b^{s+}]\) for all \(s \geq 0\), meaning that \(\text{supp} u(\cdot, s) \subseteq [a^{s-}, b^{s+}]\). Thus,

\[
h^s \{ j \text{ s.t. } r_j(s) \neq 0 \} \leq h^s \{ j \text{ s.t. } g_j \in [a^{s-}, b^{s+}] \} = L\left( g^{-1}([a^{s-}, b^{s+}]) \right).
\]

(7.46)

It is well known that if the initial data \((u^0, u^1)\) in (1.4) belong to \(\dot{H}^k \times \dot{H}^{k-1}(\mathbb{R})\) and are compactly supported, then the solution \(u(\cdot, t)\) belongs to \(\dot{H}^k(\mathbb{R})\) and, as we saw, it is compactly supported. Then, \(u(\cdot, s) \in H^k(\mathbb{R})\). In order to guarantee that \(u(\cdot, s) \in W^{3,\infty}(\mathbb{R})\), we have to impose \(H^k(\mathbb{R}) \subseteq W^{3,\infty}(\mathbb{R})\), which, by the Sobolev embedding, holds if \(k \geq 3 + 1/2\).

We conclude the proof of (3.17) by plugging (7.43) and (7.46) into (7.42). \(\square\)

**Proof of Lemma 7.1** Let us consider the curved trapezoid \(T = T(y_0, y_1, F, s) := \{ y \in [y_{0}^{s+}, y_{1}^{s-}], t \in [0, s] \} \) (see Fig. 11), multiply Eq. (1.4) by \(\partial_t u(y, t)\) and integrate in \(T\).

By integrations by parts, we obtain only integrals on the boundary of the trapezoid. More precisely, by integrating \(\rho(y) \partial_t^2 u(y, t) \partial_t u(y, t)\), we obtain

\[
\int_T \rho(y) u_{tt}(y, t) u_t(y, t) \, dy \, dt = \frac{1}{2} \int_0^s \int_{y_0^{s+}} \rho(y) |u_t(y, t)|^2 \, dy \, dt
\]
\[
\begin{align*}
\left[\int_{y_0}^{y_0} F(y) - F(y_0) + \int_{y_0}^{y_1} s - \int_{y_1}^{y_1} F(y_1) - F(y)\right] \rho(y)|u_t(y, t)|^2 dt dy &= \\
\frac{1}{2} \int_{y_0}^{y_0} \rho(y)|\partial_t u(y, F(y) - F(y_0))|^2 dy + \frac{1}{2} \int_{y_0}^{y_1} \rho(y)|\partial_y u(y, s)|^2 dy \\
+ \frac{1}{2} \int_{y_1}^{y_1} \rho(y)|\partial_t u(y, F(y_1) - F(y))|^2 dy - \frac{1}{2} \int_{y_0}^{y_1} \rho(y)|\partial_t u(y, 0)|^2 dy.
\end{align*}
\] (7.47)

By integrating \(\partial_y (\sigma(y) \partial_y u)(y, t)\partial_t u(y, t)\), we obtain

\[
\int_{\mathcal{T}} \partial_y (\sigma(y) \partial_y u)(y, t)\partial_t u(y, t) dy dt = \int_{y_0}^{y_0} \int_{y_0}^{y_1} \partial_y (\sigma(y) \partial_y u)(y, t)\partial_t u(y, t) dy dt
\]

\[
= \mathcal{I}_3 - \mathcal{I}_5 - \int_{\mathcal{T}} \sigma(y) \partial_y u(y, t)\partial^2_{yy} u(y, t) dy dt,
\] (7.48)

where

\[
\mathcal{I}_3 := \int_{0}^{s} \sigma(y_1^{t, -}) \partial_y u(y_1^{t, -}, t)\partial_t u(y_1^{t, -}, t) dt
\]

and \(\mathcal{I}_5\)

\[
:= \int_{0}^{s} \sigma(y_0^{t, +}) \partial_y u(y_0^{t, +}, t)\partial_t u(y_0^{t, +}, t) dt.
\]

Since \(\sigma(y) \partial_y u(y, t)\partial^2_{yy} u(y, t) = \sigma(y) \partial_t (|\partial_y u(y, t)|^2) / 2\), the last term in the right-hand side of (7.48) can be treated similarly to (7.47). Finally, we obtain

\[
0 \leq \frac{1}{2} \int_{y_0}^{y_1} \rho(y)|\partial_t u(y, t)|^2 + \sigma(y)|\partial_y u(y, t)|^2 dy = \mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3 - \mathcal{I}_4 + \mathcal{I}_5,
\] (7.49)
\[ I_1 := \frac{1}{2} \int_{y_0}^{y_1} (\rho(y)u_1^2(y) + \sigma(y)(u_0')^2(y)) \, dy, \]

\[ I_2 := \frac{1}{2} \int_{y_0}^{y_0^{t,+}} (\rho(y)|\partial_t u(y, F(y) - F(y_0))|^2 + \sigma(y)|\partial_y u(y, F(y) - F(y_0))|^2) \, dy, \]

and

\[ I_4 := \frac{1}{2} \int_{y_1^{t,-}}^{y_1} (\rho(y)|\partial_t u(y, F(y_1) - F(y))|^2 + \sigma(y)|\partial_y u(y, F(y_1) - F(y))|^2) \, dy. \]

Due to the fact that \( u^0 = u^1 \equiv 0 \) in \((y_0, y_1)\), we have

\[ I_1 = 0. \quad (7.50) \]

By the changes in variable \( t \to y = y_0^{t,+} \) in \( I_3 \) and \( t \to y = y_1^{t,-} \) in \( I_5 \) and due to the fact that \((7.44)\) implies that \( F'(y) = \sqrt{\rho(y)/\sigma(y)} \), we obtain

\[ -I_2 - I_3 = -\frac{1}{2} \int_{y_0}^{y_0^{t,+}} \left| \sqrt{\rho(y)} \partial_t u(y, F(y) - F(y_0)) \right|^2 \, dy \leq 0 \quad (7.51) \]

and

\[ -I_4 + I_5 = -\frac{1}{2} \int_{y_1^{t,-}}^{y_1} \left| \sqrt{\rho(y)} \partial_t u(y, F(y_1) - F(y)) \right|^2 \, dy \leq 0. \quad (7.52) \]

We conclude \((7.45)\) by replacing \((7.50), (7.51), \) and \((7.52)\) in the right-hand side of \((7.49)\).

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