ON INTERSECTIONS OF CONJUGACY CLASSES AND
BRUHAT CELLS

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Abstract. For a connected complex semi-simple Lie group $G$ and a fixed pair $(B, B^-)$ of opposite Borel subgroups of $G$, we determine when the intersection of a conjugacy class $C$ in $G$ and a double coset $BwB^-$ is non-empty, where $w$ is in the Weyl group $W$ of $G$. The question comes from Poisson geometry, and our answer is in terms of the Bruhat order on $W$ and an involution $m_C \in W$ associated to $C$. We study properties of the elements $m_C$. For $G = SL(n+1, \mathbb{C})$, we describe $m_C$ explicitly for every conjugacy class $C$, and for the case when $w \in W$ is an involution, we also give an explicit answer to when $C \cap (BwB)$ is non-empty.

1. Introduction

1.1. The set up and the results. Let $G$ be a connected complex semi-simple Lie group, and let $B$ and $B^-$ be a pair of opposite Borel subgroups of $G$. Then $H = B \cap B^-$ is a Cartan subgroup of $G$. Let $W = N_G(H)/H$ be the Weyl group, where $N_G(H)$ is the normalizer of $H$ in $G$. One then has the well-known Bruhat decompositions

$$G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} BwB^- \quad \text{(disjoint unions)}.$$ 

Subsets of $G$ of the form $BwB$ or $Bw'B^-$, where $w, w' \in W$, will be called Bruhat cells in $G$. The Bruhat order on $W$ is the partial order on $W$ defined by

$$w_1 \leq w_2 \iff Bw_1B \subset Bw_2B, \quad w_1, w_2 \in W.$$ 

Given a conjugacy class $C$ of $G$, let

$$W_C = \{w \in W : C \cap (BwB) \neq \emptyset\},$$

$$W^-_C = \{w \in W : C \cap (BwB^-) \neq \emptyset\}.$$ 

The sets $W_C$ have been studied by several authors (see, for example, [8, 9] by Ellers and Gordeev and [4] by G. Carnovale) and are not easy to determine even for the case of $G = SL(n, \mathbb{C})$ (see [9]). On the other hand, let $m_C$ be the unique element in $W$ such that $C \cap (Bm_CB)$ is dense in $C$. It is easy to show (see Lemma 2.4) that $m_C$ is a unique maximal element in $W_C$ with respect to the Bruhat order on $W$. 

Our first result, Theorem 2.5, states that, for every conjugacy class $C$ in $G$,
\[ W^-_C = \{ w \in W : w \leq m_C \}. \]
Thus the set $W^-_C$ is completely determined by the element $m_C$ and the Bruhat order on $W$.

Theorem 2.5 is motivated by Poisson geometry. It is shown in [10] that the connected complex semi-simple Lie group $G$ carries a holomorphic Poisson structure $\pi_0$, invariant under conjugation by elements in $H$, such that the non-empty intersections $C \cap (BwB^-)$ are exactly the $H$-orbits of symplectic leaves of $\pi_0$, where $C$ is a conjugacy class in $G$ and $w \in W$. To describe precisely the symplectic leaves of $\pi_0$, one thus first needs to know when an intersection $C \cap (BwB^-)$ is non-empty. By [18, Theorem 1.4], the non-empty intersections $C \cap (BwB^-)$ are always smooth and irreducible. The geometry of such intersections and applications to Poisson geometry will be carried out elsewhere.

The elements $m_C$ play an important role in the study of spherical conjugacy classes, i.e., conjugacy classes in $G$ on which the $B$-action by conjugation has a dense orbit. In connection with their proof of the de Concini-Kac-Procesi conjecture on representations of the quantized universal enveloping algebra $U_\epsilon(g)$ at roots of unity over spherical conjugacy classes, N. Cantarini, G. Carnovale, and M. Costantini proved [2, Theorem 25] that a conjugacy class $C$ in $G$ is spherical if and only if $\dim C = l(m_C) + \text{rank}(1 - m_C)$, where $l$ is the length function on $W$, and $\text{rank}(1 - m_C)$ is the rank of the operator $1 - m_C$ in the geometric representation of $W$. It is also shown by M. Costantini [5], again for a spherical conjugacy class $C$, that the decomposition of the coordinate ring of $C$ as a $G$-module (for $G$ simply connected) is almost entirely determined by the element $m_C$ (see [5, Theorem 3.22] for the precise statement). When $G$ is simple, a complete list of the $m_C$’s, for $C$ spherical, is given by G. Carnovale in [3, Corollary 4.2].

In this paper, we study some properties of $m_C$ for every conjugacy class $C$ of $G$. After examining some properties of $W^-_C$, we show, in Corollary 2.11, that for each conjugacy class $C$ in $G$, $m_C \in W$ is one and the only one maximal length element in its conjugacy class in $W$. In particular, $m_C$ is an involution. When $C$ is spherical, the fact that $m_C$ is an involution is also proved in [2, Remark 4] and [3, Theorem 2.7]. For $m \in W$, denote by $O_m$ the conjugacy class of $m$ in $W$. Let
\[
M = \{ m \in W : m \text{ is the unique maximal length element in } O_m \}.
\]
Then $m_C \in M$ for every conjugacy class $C$ in $G$. It is thus desirable to study the set $M$.

When $G$ is simple, using arguments from [3], it is not hard to give a complete list of elements in $M$. It turns out that when $G$ is simple, the list of elements in
\( \mathcal{M} \) coincides with the list in [3, Corollary 4.2]. See §3 and in particular Theorem 3.10. Consequently, when \( G \) is simple, one has

\[
\mathcal{M} = \{ m_C \in W : C \text{ is a conjugacy class in } G \} = \{ m_C \in W : C \text{ is a spherical conjugacy class in } G \}.
\]

If \( G = G_1 \times G_2 \times \cdots \times G_k \) is semi-simple with simple factors \( G_j \) and Weyl groups \( W_j \) for \( 1 \leq j \leq k \), then

\[
\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_k,
\]

where for \( 1 \leq j \leq k \), \( \mathcal{M}_j \subset W_j \) is defined as in (1.3). Hence (1.4) also holds for \( G \) semi-simple. We have thus completely described the set \( \mathcal{M} \) for any connected semi-simple complex Lie group \( G \).

We consider the case of \( G = SL(n+1, \mathbb{C}) \) in §4. For any conjugacy class \( C \) in \( SL(n+1, \mathbb{C}) \) and any involution \( w \in W \equiv S_{n+1} \), we show in Theorem 4.2 that

\[
C \cap (BwB) \neq \emptyset \iff l_2(w) \leq r(C),
\]

where \( l_2(w) \) is the number of distinct 2-cycles in the cycle decomposition of \( w \), and

\[
r(C) = \min \{ \text{rank}(g - cI) : c \in C \}
\]

for any \( g \in C \). Theorem 4.2 is proved in §4.3 using a special case of a criterion by Ellers-Gordeev [9]. Since the proof of the Ellers-Gordeev criterion in [9] involves rather complicated combinatorics, we also give a direct proof of Theorem 4.2 in §4.4. Our direct proof also shows how to explicitly find an element in \( C \cap BwB \) when \( l_2(w) \leq l(C) \).

Combining Theorem 4.2 and a result of G. Carnovale [3, Theorem 2.7], one has, for a spherical conjugacy class \( C \) in \( SL(n+1, \mathbb{C}) \),

\[
W_C = \{ w \in S_{n+1} : w^2 = 1, \ l_2(w) \leq r(C) \}.
\]

As another consequence of Theorem 4.2, we show in Corollary 4.4 that for any conjugacy class \( C \) in \( SL(n+1, \mathbb{C}) \), if \( W_C \) contains an involution \( w \in S_{n+1} \), then \( W_C \) contains the whole conjugacy class of \( w \) in \( S_{n+1} \).

Finally, let \( m_0 = 1 \), and for an integer \( 1 \leq l \leq \left\lfloor \frac{n+1}{2} \right\rfloor \), let \( m_l \in S_{n+1} \) be the involution with the cycle decomposition

\[
m_l = (1, n+1)(2, n) \cdots (l, n+2-l).
\]

Corollary 4.8 says that for any conjugacy class \( C \) in \( SL(n+1, \mathbb{C}) \),

\[
m_C = \begin{cases} 
  w_0 & \text{if } r(C) \geq \left\lfloor \frac{n+1}{2} \right\rfloor, \\
  m_{r(C)} & \text{if } r(C) < \left\lfloor \frac{n+1}{2} \right\rfloor.
\end{cases}
\]

The explicit description of \( m_C \) for an arbitrary conjugacy class in other classical groups will be given in [6].
In the study of the symplectic leaves of certain Poisson structures on $G$ as well as on the de Concini-Procesi compactification of $G$ when $G$ is of adjoint type, one needs to consider intersections $C_\delta \cap (BwB^{-})$, where $\delta$ is an automorphism of $G$ preserving both $H$ and $B$ and $C_\delta$ is a $\delta$-twisted conjugacy class in $G$. See [14]. For such a conjugacy class $C_\delta$ in $G$, we have the element $m_{C_\delta} \in W$ which is the unique maximal length element in its $\delta$-twisted conjugacy class in $W$. See §2.3.

1.2. **Notation.** Let $\Delta$ be the set of all roots of $G$ with respect to $H$, let $\Delta^+ \subset \Delta$ be the set of positive roots determined by $B$, and let $\Gamma$ be the set of simple roots in $\Delta^+$. We also write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha \in \Delta^+$ (resp. $\alpha \in -\Delta^+$). Define $\delta_0 : \Delta \rightarrow \Delta : \delta_0(\alpha) = -w_0(\alpha), \quad \alpha \in \Delta.$ Then $\delta_0$ permutes $\Delta^+$ and $\Gamma$, and it induces an automorphism, still denoted by $\delta_0$, on $W$:

$$\delta_0 : W \rightarrow W : \delta_0(w) = w_0ww_0, \quad w \in W.$$ For $\alpha \in \Gamma$, let $s_\alpha \in W$ be the reflection determined by $\alpha$. For a subset $J$ of $\Gamma$, let $W_J$ be the subgroup of $W$ generated by $\{s_\alpha : \alpha \in J\}$, and let $w_0,J$ be the maximal length element in $W_J$. Let $W^J \subset W$ be the set of minimal length representatives of $W/W_J$. Set $w_0 = W_{0,\Gamma}$, so $w_0$ is the maximal length element in $W$. The length function on $W$ is denoted by $l$.

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2. **The sets $W_C$ and $W_C^-$ and the elements $m_C$**

2.1. **$W_C^-$ in terms of $m_C$.** We keep the notation as in §1.1. In particular, for each conjugacy class $C$ in $G$, we have the subsets $W_C$ and $W_C^-$ of $W$ as in (1.1) and (1.2).

**Lemma 2.1.** One has $W_C \subset W_C^-$ for every conjugacy class $C$ in $G$.

**Proof.** Let $w \in W$. If $C \cap (BwB) \neq \emptyset$, then $C \cap (Bw) \neq \emptyset$, so $C \cap (BwB^-) \neq \emptyset$.

Q.E.D.

**Lemma 2.2.** For any $w \in W$,

$$BwB^-B = \bigsqcup_{w' \leq w} Bw'B.$$
Proof. Clearly $BwB^{-1}B$ is the union of some $(B, B)$-double cosets. Let $w' \in W$. Then

$$Bw'B \subset BwB^{-1}B \iff (Bw') \cap (BwB^{-1}) \neq \emptyset \iff (Bw'B) \cap (BwB^{-1}) \neq \emptyset,$$

which, by [7], is equivalent to $w \leq w'$.

Q.E.D.

Lemma 2.3. Let $C$ be a conjugacy class in $G$ and let $w \in W$. Then $w \in W_{C^-}$ if and only if $w \leq w'$ for some $w' \in W_C$.

Proof. Since $C$ is conjugation invariant,

$$C \cap (BwB^{-1}) \neq \emptyset \iff C \cap (BwB^{-1}) \neq \emptyset,$$

which, by Lemma 2.2, is equivalent to $w \leq w'$ for some $w' \in W_C$.

Q.E.D.

For a subset $X$ of $G$, let $\overline{X}$ be the Zariski closure of $X$ in $G$. The following Lemma 2.4 can also be found in [2, §1].

Lemma 2.4. Let $C$ be a conjugacy class in $G$. Then

1) there is a unique $m_C \in W$ such that $C \cap (Bm_CB)$ is dense in $C$;

2) $w \leq m_C$ for every $w \in W_C$.

Proof. The decomposition $C = \bigsqcup_{w \in W_C} C \cap (BwB)$ gives

$$C = \bigsqcup_{w \in W_C} C \cap (BwB).$$

As $C$ is irreducible, there exists a unique $m_C \in W_C$ such that $\overline{C} = \overline{C \cap (Bm_CB)}$. If $w \in W_C$, then

$$\emptyset \neq C \cap (BwB) \subset \overline{C \cap (Bm_CB)} \subset Bm_CB,$$

so $w \leq m_C$.

Q.E.D.

Theorem 2.5. For every conjugacy class $C$ in $G$, $W_C^- = \{w \in W : w \leq m_C\}$.

Proof. Let $w \in W$. If $w \leq m_C$, then $w \in W_C^-$ by Lemma 2.3. Conversely, if $w \in W_C^-$, then again by Lemma 2.3, $w \leq w'$ for some $w' \in W_C$. Since $w' \leq m_C$ by Lemma 2.4, one has $w \leq m_C$.

Q.E.D.

Lemma 2.6. If $C$ and $C'$ are two conjugacy classes in $G$ such that $C' \subset \overline{C}$, then $m_{C'} \leq m_C$.
Proof. By definition, 
\[ \emptyset \neq C' \cap (Bm_cB) \subset \overline{C} = \overline{C \cap (Bm_cB)} \subset Bm_cB. \]
Thus \( m_{c'} \leq m_c \).
Q.E.D.

2.2. Some properties of \( W_c \) and \( m_c \). We recall some definitions and results from [8, 11, 12].

Definition 2.7. 1) [8, Definition 3.1] Let \( w, w' \in W \). An ascent from \( w \) to \( w' \) is a sequence \( \{\alpha_j\}_{1 \leq j \leq k} \) in \( \Gamma \) such that 
\[ w' = s_{\alpha_k} \cdots s_{\alpha_2} s_{\alpha_1} w \]
and \( l(s_{\alpha_j} \cdots s_{\alpha_2} s_{\alpha_1} w \cdot s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_j}) \geq l(s_{\alpha_{j-1}} \cdots s_{\alpha_2} s_{\alpha_1} w \cdot s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{j-1}}) \) for every \( 1 \leq j \leq k \). Write \( w' \hookrightarrow w \) if there is an ascent from \( w \) to \( w' \) or if \( w' = w \).

2) [12, §2.9] For \( w, w', x \in W \), write \( w \nsim w' \) if \( l(w) = l(w') \), \( w' = xw^{-1} \), and either \( l(w') = l(xw) + l(x) \) or \( l(w') = l(x) + l(wx^{-1}) \). Write \( w \sim w' \) if there exist sequences of \( \{x_j\}_{1 \leq j \leq k} \) and \( \{w_j\}_{1 \leq j \leq k} \) in \( W \) such that 
\[ w \nsim_1 w_1 \nsim_2 \cdots \nsim_k \sim w_k = w'. \]

3) Let \( O \) be a conjugacy class in \( W \). An element \( w \in O \) is called a maximal length element in \( O \) if \( l(w_1) \leq l(w) \) for all \( w_1 \in O \).

Proposition 2.8. [12, §2.9] Let \( O \) be any conjugacy class in \( W \).

1) For any \( w \in O \), there exists a maximal length element \( w' \in O \) such that \( w' \hookrightarrow w \).

2) If \( w' \) and \( w'' \) are two maximal length elements in \( O \), then \( w' \sim w'' \).

Proposition 2.9. Let \( C \) be a conjugacy class in \( G \), and let \( w, w' \in W \).

1) If \( w' \hookrightarrow w \) and \( w \in W_c \), then \( w' \in W_c \).

2) If \( w \sim w' \) and \( w \in W_c \), then \( w' \in W_c \).

Proof. 1) is just [8, Proposition 3.4]. To see 2), assume that \( w \nsim w' \) for some \( x \in W \), so \( w' = xw^{-1} \), and either \( l(w') = l(xw) + l(x) \) or \( l(w') = l(x) + l(wx^{-1}) \). Assume first that \( l(w') = l(xw) + l(x) \). Then
\[ C \cap (Bw'B) = C \cap (BxwBx^{-1}B) \supset C \cap (xwBx^{-1}) \neq \emptyset. \]
Thus \( C \cap (Bw'B) \neq \emptyset \) and \( w' \in W_c \). The case of \( l(w') = l(x) + l(wx^{-1}) \) is proved similarly.
Q.E.D.
Remark 2.10. We refer to [8, 9] for a more detailed study of the set \( W_C \) and in particular for the case of \( G = SL(n, \mathbb{C}) \). On the other hand, it is proved in [4] by G. Carnovale that a conjugacy class \( C \) in \( G \) is spherical if and only if \( W_C \) consists only of involutions. See also Corollary 4.6 in §4.2.

For \( w \in W \), let \( \mathcal{O}_w \) be the conjugacy class of \( w \) in \( W \).

Corollary 2.11. For any conjugacy class \( C \) in \( G \), \( m_C \) is the unique maximal length element in \( \mathcal{O}_{m_C} \).

Proof. By Proposition 2.8, there exists a maximal length element \( w' \in \mathcal{O}_{m_C} \) such that \( w' \rightarrow m_C \). By Proposition 2.9, \( w' \in W_C \), so \( w' \leq m_C \) by Lemma 2.4. Since \( l(w') \geq l(m_C) \), one has \( w' = m_C \). Thus \( m_C \) is a maximal length element in \( \mathcal{O}_{m_C} \). If \( w_1 \) is any maximal length element in \( \mathcal{O}_{m_C} \), then \( w_1 \sim m_C \) by Proposition 2.8, so \( w_1 \in W_C \) by Proposition 2.9, and thus \( w_1 \leq m_C \) by Lemma 2.4. Since \( l(w_1) = l(m_C) \), one has \( w_1 = m_C \). Thus \( m_C \) is the only maximal length element in \( \mathcal{O}_{m_C} \).

Q.E.D.

Consider now the bijection

\[
\phi : \ W \rightarrow W : \ w \mapsto w_0 w, \quad w \in W.
\]

Then under \( \phi \), the conjugation action of \( W \) on itself becomes the following \( \delta_0 \)-twisted conjugation action of \( W \) on itself:

\[
u \cdot w = \delta_0(u)w_0^{-1}, \quad u, w \in W.
\]

For \( w \in W \), let \( \mathcal{O}_{w_0}^{\delta_0} \) be the \( \delta_0 \)-twisted conjugacy class of \( w \), and say an element \( w' \in \mathcal{O}_{w_0}^{\delta_0} \) has minimal length if \( l(w') \leq l(w_1) \) for all \( w_1 \in \mathcal{O}_{w_0}^{\delta_0} \). Using the fact that \( l(w_0 u) = l(w_0) - l(u) \) for any \( u \in W \), it is easy to see that for any \( w \in W \), \( \phi \) maps maximal length elements in \( \mathcal{O}_w \) to minimal length elements in \( \mathcal{O}_{w_0 w}^{\delta_0} \).

Corollary 2.12. For any conjugacy class \( C \) in \( G \), \( w_0 m_C \) is the unique minimal length element in \( \mathcal{O}_{w_0 m_C}^{\delta_0} \).

Remark 2.13. Let \( \tilde{G} \) be the connected and simply connected cover of \( G \), let \( \pi : \tilde{G} \rightarrow G \) be the covering map, and let \( Z = \pi^{-1}(e) \), where \( e \) is the identity element of \( G \). Let \( \tilde{A} = \pi^{-1}(A) \), where \( A \in \{ \tilde{H}, \tilde{B}, B^- \} \). Identify the Weyl group for \( \tilde{G} \) with \( W \). For any conjugacy class \( C \) in \( G \), \( \pi^{-1}(C) \) is a union of conjugacy classes in \( \tilde{G} \). Since \( Z \subset \tilde{H} = \tilde{B} \cap \tilde{B}^- \), it is easy to see that for any conjugacy classes \( \tilde{C} \) in \( \pi^{-1}(C) \), \( \tilde{W}_C = W_C \) and \( \tilde{W}_C^- = W_C^- \), and in particular, \( m_C = m_{\tilde{C}} \). Thus the subset \( \{ m_C : C \text{ a conjugacy class in } G \} \) of \( W \) depends only on the isogeneous class of \( G \).
2.3. $\delta$-twisted conjugacy classes. Let $\delta$ be any automorphism of $G$ such that $\delta(B) = B$ and $\delta(H) = H$. Then $G$ acts on itself by $\delta$-twisted conjugation given by
\[ g \cdot \delta h = \delta(g)hg^{-1}, \quad g, h \in G. \]

A $\delta$-twisted conjugacy class in $G$ is defined to be a $G$-orbit of the $\delta$-twisted conjugation. Given a $\delta$-twisted conjugacy class $C_\delta$ of $G$, let
\[ W_{C_\delta} = \{ w \in W : C_\delta \cap (BwB) \neq \emptyset \}, \]
\[ W_{C_\delta}^- = \{ w \in W : C_\delta \cap (BwB^-) \neq \emptyset \}. \]

Then all the arguments in §2.1 carry through when $C$ is replaced by $C_\delta$. In particular, let $m_{C_\delta}$ be the unique element in $W$ such that $C_\delta \cap (Bm_{C_\delta}B)$ is dense in $C_\delta$. Then $m_{C_\delta} \in W_{C_\delta}$ and
\[ W_{C_\delta}^- = \{ w \in W : w \leq m_{C_\delta} \}. \]

Recall that $\Gamma$ is the set of simple roots determined by $(B, H)$. Since $\delta(H) = H$ and $\delta(B) = B$, $\delta$ acts on $\Gamma$ and thus also on $W$. For any automorphism $\sigma$ of $\Gamma$, define the $\sigma$-twisted conjugation of $W$ on itself by
\[ u \cdot \sigma v = \sigma(u)vu^{-1}, \quad u, v \in W, \]
and for $w \in W$, denote by $O_w^\sigma$ the $\sigma$-twisted conjugacy class of $w$ in $W$. Minimal length elements in $\sigma$-twisted conjugacy classes in $W$ have been studied by X. He in [13]. The map $\phi$ in (2.1) induces a bijection between $\delta$-twisted conjugacy classes and $\delta_0\delta$-twisted conjugacy in $W$. In particular, for any $w \in W$, $\phi$ maps maximal length elements in $O_w^\delta$ to minimal length elements in $O_{m_{C_\delta}}^{\delta_0\delta}$. Using the map $\phi$, one can translate the notions in [13, Section 3] and [13, Theorem 3.2] on minimal length elements in $\delta_0\delta$-twisted conjugacy classes to the analog of Proposition 2.8 on maximal length elements in $\delta$-twisted conjugacy classes. It is also straightforward to generalize Proposition 2.9 to the case of $\delta$-twisted conjugacy classes in $G$. We thus have the following conclusion.

**Proposition 2.14.** For any $\delta$-twisted conjugacy class $C_\delta$ in $G$, $m_{C_\delta}$ is the unique maximal length element in its $\delta$-twisted conjugacy class in $W$.

3. Conjugacy classes of $W$ with unique maximal length elements

3.1. The set $\mathcal{M}$. Introduce
\[ \mathcal{M} = \{ m \in W : m \text{ is the unique maximal length element in } O_m \}. \]

By Corollary 2.11, $m_C \in \mathcal{M}$ for every conjugacy class $C$ in $G$. It is thus desirable to have a precise description of elements in $\mathcal{M}$. Clearly $\mathcal{M}$ is in one-to-one correspondence with conjugacy classes in $W$ that have unique maximal length elements.
It is easy to see that if \( G = G_1 \times G_2 \times \cdots \times G_k \) is semi-simple with simple factors \( G_j \) and Weyl groups \( W_j \) for \( 1 \leq j \leq k \), then
\[
\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_k,
\]
where for \( 1 \leq j \leq k \), \( \mathcal{M}_j \subset W_j \) is defined as in (3.1). Therefore we only need to determine \( \mathcal{M} \) for \( G \) simple. This will be done in §3.3.

**Remark 3.1.** By [13, Corollary 4.5], in any \( \delta_0 \)-twisted conjugacy class in \( W \), a minimal element in the Bruhat order is also a minimal length element. Thus, for \( m \in W \), \( m \in \mathcal{M} \) if and only if \( m \) is the unique maximal element in \( O_m \).

**Lemma 3.2.** If \( m \in \mathcal{M} \), then \( m^2 = 1 \).

*Proof.* By [11, Corollary 3.2.14], \( m^{-1} \in O_m \). Since \( l(m) = l(m^{-1}) \), one has \( m = m^{-1} \).

Q.E.D.

### 3.2. The correspondence between \( \mathcal{M}' \) and \( J' \)

Introduce
\[
\mathcal{M}' = \{ m \in W : m^2 = 1 \text{ and } m \text{ is a maximal length element in } O_m \}.
\]

By Lemma 3.2, \( \mathcal{M} \subset \mathcal{M}' \). We first determine \( \mathcal{M}' \).

It is well-known that elements in \( \mathcal{M}' \) correspond to special subsets of the set \( \Gamma \) of simple roots. Indeed, minimal or maximal length elements in conjugacy classes of involutions in \( W \) have been studied (see, for example, [11, 13, 16, 17] and especially [11, Remark 3.2.13] for minimal length elements, [16, Theorem 1.1] for maximal length elements, and [13, Lemma 3.6] for minimal length elements in twisted conjugacy classes). We summarize the results on \( \mathcal{M}' \) in the following Proposition 3.6, and we give a proof of Proposition 3.6 for completeness.

**Lemma 3.3.** Let \( m \in W \) be an involution. If \( \alpha \in \Gamma \) is such that \( l(s_\alpha ms_\alpha) = l(m) \), then \( s_\alpha ms_\alpha = m \).

*Proof.* This is [11, Exercise 3.18]. If \( m(\alpha) > 0 \), then \( ms_\alpha > m \), and \( l(s_\alpha ms_\alpha) = l(m) \) implies that \( s_\alpha ms_\alpha < ms_\alpha \). Thus \( s_\alpha m(\alpha) = 0 \), so \( m(\alpha) = \alpha \). Similarly, if \( m(\alpha) < 0 \), then \( m(\alpha) = -\alpha \). In either case, \( s_\alpha ms_\alpha = m \).

Q.E.D.

**Lemma 3.4.** If \( m \in \mathcal{M}' \), then \( m = w_0w_{0,J} \), where \( J = \{ \alpha \in \Gamma : m(\alpha) = \alpha \} \), and \( J \) is \( \delta_0 \)-invariant.

*Proof.* Let \( m \in \mathcal{M}' \), and let \( x = w_0m \). Then \( x \) is a unique minimal length element in its \( \delta_0 \)-twisted conjugacy class \( O^\delta_{x_0} \) in \( W \). Let \( x = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k} \) be a reduced
A subset \( J \) of \( \Gamma \) is said to have Property (1) if \( J \) is \( \delta_0 \)-invariant and \(-w_0(\alpha) = -w_{0,J}(\alpha)\) for all \( \alpha \in J \).

Let \( J' \) be the collection of all subsets \( J \) of \( \Gamma \) that have Property (1). For \( J \in J' \), let \( m_J = w_0 w_{0,J} \). For \( m \in M' \), let

\[
J_m = \{ \alpha \in \Gamma : m(\alpha) = \alpha \} \subset \Gamma.
\]

It follows from Lemma 3.4 that \( J_m \in J' \) for every \( m \in M' \).

**Proposition 3.6.** 1) The map \( \psi : M' \to J' : m \mapsto J_m \) is bijective with inverse given by \( J \mapsto m_J \) for \( J \in J' \).

2) For \( J, K \in J' \), \( m_J \) and \( m_K \) are in the same conjugacy class in \( W \) if and only if there exists \( w \in W \) with \( \delta_0(w) = w \) such that \( w(J) = K \).

**Proof.** 1) Since \( m = w_0 w_{0,m} \) for every \( m \in M' \), \( \psi \) is injective. To show that \( \psi \) is surjective, let \( J \in J' \) and we will prove that \( m_J \in M' \). Since \( J \) is \( \delta_0 \)-invariant, \( m_J \) is an involution. Property (1) implies that \( s_\alpha m_J s_\alpha = m_J \) for every \( \alpha \in J \), so \( w m_J w^{-1} = m_J \) for every \( w \in W_J \). Thus, if \( u = w m_J w^{-1} \) is an element in \( O_{m_J} \), we can assume that \( w \in W_J \) (see notation in §1.2). Then

\[
l(u) \leq l(w) + l(m_J w^{-1}) = l(w) + l(w_0) - l(w_{0,J} w^{-1})
\]

\[
= l(w) + l(w_0) - l(w^{-1})
\]

\[
= l(m_J).
\]
We now turn to the set 

\[ \mathcal{M}' \]

If so, the same conjugacy class in 

\[ \mathcal{J} \]

Then it follows from \( m_K w = w m_j \) that for every \( \alpha \in J \),

\[ m_K w(\alpha) = w m_j(\alpha) = w(\alpha) > 0. \]

Thus \( w(\alpha) \in [K]^+ \), where \([K]^+\) denotes the set positive roots that are in the linear span of \( K \). Denote similarly by \([J]^+\) the set of positive roots in the linear span of \( J \). Then \( w([J]^+) \subset [K]^+ \). Since both \( m_j \) and \( m_K \) are maximal length elements in the same conjugacy class in \( W \), \( l(m_j) = l(m_K) \). Since

\[ l(m_j) = l(w_0) - ||J^+|| \quad \text{and} \quad l(m_K) = l(w_0) - ||[K]^+||, \]

one has \( ||J^+|| = ||[K]^+|| \). Here for a set \( A \), \( |A| \) denotes the cardinality of \( A \). Thus \( w([J]^+) = [K]^+ \). It follows that \( w(J) = K \). Now \( m_K = w m_j w^{-1} \) implies that

\[ w_0 K = \delta_0(w) w_0 J w^{-1}, \]

so \( \delta_0(w) = w_0 K w_0 J = w \).

Conversely, if \( J, K \in \mathcal{J}' \) are such that \( w(J) = K \) for some \( w \in W \) with \( \delta_0(w) = w \), then \( w_0 K = w w_0 J w^{-1} = \delta_0(w) w_0 J w^{-1}, \) so \( m_K = w m_j w^{-1} \).

Q.E.D.

3.3. The correspondence between \( \mathcal{M} \) and \( \mathcal{J} \). We now turn to the set \( \mathcal{M} \). Let \( \langle , \rangle \) be the bilinear form on \( \Gamma \) induced from the Killing form of the Lie algebra of \( G \). For a subset \( J \) of \( \Gamma \), an \( \alpha \in J \) is said to be isolated if \( \langle \alpha, \alpha' \rangle = 0 \) for every \( \alpha' \in J \setminus \{\alpha\} \). The following Definition 3.7 is inspired by [3, Lemma 4.1].

**Definition 3.7.** A subset \( J \) of \( \Gamma \) is said to have Property (2) if for every isolated \( \alpha \in J \), there is no \( \beta \in \Gamma \setminus \{\alpha\} \) with the following properties:

a) \( \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle \) and \( \langle \beta, \alpha \rangle \neq 0 \);

b) \( \langle \beta, \alpha' \rangle = 0 \) for all \( \alpha' \in J \setminus \{\alpha\} \);

c) \( -w_0(\beta) = \beta \).

**Lemma 3.8.** If \( m \in \mathcal{M} \), then \( J_m \) has Properties (1) and (2).

**Proof.** Let \( m \in \mathcal{M} \). By Lemma 3.4, \( J_m \) has Property (1). Suppose that \( \alpha \in J_m \) is an isolated point and that there exists \( \beta \in \Gamma \setminus \{\alpha\} \) with properties a), b) and c) in Definition 3.7. Let \( J'_m = J_m \setminus \{\alpha\} \). Since \( \alpha \in J_m \) is isolated, on has \( w_0(\alpha) = -\alpha \), so,

\[ m = w_0 s_{\alpha} w_0 J'_m = s_{\alpha} w_0 J'_m, \]
and by b) and c), \( m(\beta) = s_\alpha w_0 w_0' (\beta) = s_\alpha w_0(\beta) = -s_\alpha(\beta) < 0 \), and thus

\[
s_\beta ms_\beta = ms_m(\beta)s_\beta = ms_\beta s_\alpha s_\beta.
\]

By a), \( s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha \), so \( s_\beta ms_\beta = ms_\beta s_\alpha \), and thus

\[
s_\alpha s_\beta ms_\beta s_\alpha = s_\alpha ms_\beta.
\]

Since \( l(s_\alpha ms_\beta) \geq l(s_\alpha m) - 1 = l(m) \), and since \( m \) is the unique maximal length element in \( O_m \), \( s_\alpha ms_\beta = m \). It follows from \( ms_\alpha m = s_\alpha \) that \( s_\alpha s_\beta = 1 \) which is a contradiction.

Q.E.D.

Let \( J \) be the collection of all subsets \( J \) of \( \Gamma \) with Properties (1) and (2). A \( J \in J \) is said to be non-trivial if \( \Gamma \) is neither empty nor the whole of \( \Gamma \).

Identify \( \Gamma \) with the Dynkin diagram of \( G \) and a subset \( J \) of \( \Gamma \) as a sub-diagram of the Dynkin diagram. The following description of \( J \) for \( G \) simple is obtained in [3, Corollary 4.2]. We include the list here for the convenience of the reader and for completeness.

**Lemma 3.9.** Assume that \( G \) is simple and that the rank \( n \) of \( G \) is at least 2. The following is a complete list of non-trivial \( J \in J \) with points in \( J \) painted black:

1. \( A_n: \) \( J_l = \{ \alpha_i : l + 1 \leq i \leq n - l \} \) for \( 1 \leq l \leq \left[ \frac{n+1}{2} \right] - 1 \):

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_l-1 & \alpha_l+1 & \alpha_{l+2} & \alpha_{n-l-1} & \alpha_{n-l} & \alpha_{n-l+1} & \alpha_n \\
\end{array}
\]

2. \( B_n: \) \( J_{1,l} = \{ \alpha_i : l \leq i \leq n \} \) for \( 2 \leq l \leq n \):

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_l-1 & \alpha_l & \alpha_{l+1} & \alpha_{n-1} & \alpha_n \\
\end{array}
\]

\( J_{2,l} = \{ \alpha_1, \alpha_3, \ldots, \alpha_{2l-1} \} \cup \{ \alpha_i : 2l + 1 \leq i \leq n \} \), for \( 1 \leq l \leq \frac{n}{2} - 1 \):

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_{2l-1} & \alpha_{2l} & \alpha_{2l+1} & \alpha_{2l+2} & \alpha_{n-1} & \alpha_n \\
\end{array}
\]

If \( n = 2m \), \( J_3 = \{ \alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m-1} \} \):

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_{2m-3} & \alpha_{2m-2} & \alpha_{2m-1} & \alpha_{2m} \\
\end{array}
\]

If \( n = 2m + 1 \), \( J_4 = \{ \alpha_1, \alpha_3, \ldots, \alpha_{2m-1}, \alpha_{2m+1} \} \):

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_{2m-1} & \alpha_{2m} & \alpha_{2m+1} \\
\end{array}
\]
(3) $C_n$: $J_{1,l} = \{\alpha_i : l \leq i \leq n\}$ for $2 \leq l \leq n$:

$J_{2,l} = \{\alpha_1, \alpha_3, \ldots, \alpha_{2l-1}\} \cup \{\alpha_i : 2l + 1 \leq i \leq n\}$ for $1 \leq l \leq \frac{n}{2} - 1$:

If $n = 2m$, $J_3 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m-1}\}$:

If $n = 2m + 1$, $J_4 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-1}, \alpha_{2m+1}\}$:

(4) $D_{2m}$: $J_{1,l} = \{\alpha_i : 2l - 1 \leq i \leq 2m\}$ for $2 \leq l \leq m$:

$J_{2,l} = \{\alpha_1, \alpha_3, \ldots, \alpha_{2l-1}\} \cup \{\alpha_i : 2l + 1 \leq i \leq 2m\}$ for $1 \leq l \leq m - 1$:

$J_3 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m-1}\}$:

$J_4 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m}\}$:

(5) $D_{2m+1}$: $J_{1,l} = \{\alpha_i : 2l - 1 \leq i \leq 2m + 1\}$ for $2 \leq l \leq m$:

$J_{2,l} = \{\alpha_1, \alpha_3, \ldots, \alpha_{2l-1}\} \cup \{\alpha_i : 2l + 1 \leq i \leq 2m + 1\}$ for $1 \leq l \leq m - 1$:
\[ J_3 = \{ \alpha_1, \alpha_3, \ldots, \alpha_{2m-1} \} : \]

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_{2l-1} & \alpha_{2l} & \alpha_{2l+1} & \alpha_{2l+2} & \alpha_{2m-1} & \alpha_{2m} \\
\end{array}
\]

\[ J_3 = \{ \alpha_1, \alpha_3, \ldots, \alpha_{2m-1} \} : \]

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_{2m-3} & \alpha_{2m-2} & \alpha_{2m-1} & \alpha_{2m} \\
\end{array}
\]

\( J_3 : \)

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\end{array}
\]

\( J_3 : \)

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array}
\]

\( J_3 : \)

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array}
\]

\( J_3 : \)

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array}
\]

\( J_3 : \)

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array}
\]

\( J_3 : \)

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\end{array}
\]
Theorem 3.10. When \( G \) is simple, the map \( \psi : \mathcal{M} \to \mathcal{J} : m \mapsto J_m \) is a bijection.

Proof. It is clear that \( \psi \) is injective. To show that \( \psi \) is surjective, let \( J \in \mathcal{J} \). We need to show that \( m_J \in \mathcal{M} \), i.e., \( m_J \) is the unique maximal length element in its conjugacy class in \( \mathcal{O}_{m_J} \). Let \( m \) be any maximal length element in \( \mathcal{O}_{m_J} \). By Proposition 3.6, \( m = m_K \), where \( K \in \mathcal{J}' \) and there exists \( w \in W^J \) such that \( w(J) = K \) and \( \delta_0(w) = w \).

By examining the list of all \( J \)'s in \( \mathcal{J} \) in Lemma 3.9, every \( J \in \mathcal{J} \), when regarded as a Dynkin diagram, uniquely embeds in \( \Gamma \) with Property (1) except in the cases of \( J_{1,m}, J_3, J_4 \) for \( D_2m \) and \( J_4 \) for \( E_7 \). In these cases, one can use results in [16] to check directly that \( m_J \in \mathcal{M} \).

Q.E.D.

Remark 3.11. By [3, Remark 4.3], for every \( J \) in the list in Lemma 3.9, \( m_J = m_C \) for some spherical conjugacy class in \( G \), so in particular, \( m_J \in \mathcal{M} \). This gives another (indirect) proof of the surjectivity of the map \( \psi \) in Theorem 3.10.
4. The case of $G = SL(n + 1, \mathbb{C})$

In this section, for an arbitrary conjugacy class $C$ in $SL(n + 1, \mathbb{C})$, we give an explicit condition for $C \cap (BwB) \neq \emptyset$ when $w \in W \cong S_{n+1}$ is an involution. In particular, we describe $m_C \in S_{n+1}$ explicitly for every $C$.

4.1. Notation. As is standard, take the Borel subgroup $B$ (resp. $B^-$) to consist of all upper-triangular (resp. lower triangular) matrices in $SL(n + 1, \mathbb{C})$, so that $H = B \cap B^-$ consists of all diagonal matrices in $SL(n + 1, \mathbb{C})$. For an integer $p \geq 0$, denote by $I_p$ the identity matrix of size $p$ and by $\lfloor p/2 \rfloor$ the largest integer that is less than or equal to $p/2$.

Identify the Weyl group $W$ of $SL(n + 1, \mathbb{C})$ with the group $S_{n+1}$ of permutations on the set of integers between $1$ and $n + 1$. For $1 \leq i < j \leq n + 1$, let $(i, j)$ be the $2$-cycle in $S_{n+1}$ exchanging $i$ and $j$ and leaving every other $k \in [1, n + 1]$ fixed. If $w \in S_{n+1}$ is an involution, denote by $l_2(w)$ the number of $2$-cycles in the cycle decomposition of $w$.

Every conjugacy class $C$ in $SL(n + 1, \mathbb{C})$ contains some $g$ of (upper-triangular) Jordan form. We define the eigenvalues for $C$ to be the eigenvalues of such a $g \in C$ and similarly define the number and sizes of the Jordan blocks of $C$ corresponding to an eigenvalue. For $g \in GL(n + 1, \mathbb{C})$, define

$$d(g) = \max \{ \dim \ker (g - cI_{n+1}) : c \in \mathbb{C} \}$$

$$r(g) = n + 1 - d(g) = \min \{ \text{rank}(g - cI_{n+1}) : c \in \mathbb{C} \}$$

$$l(g) = \min \left\{ \frac{n + 1}{2}, r(g) \right\}.$$

For a conjugacy class $C$ in $SL(n + 1, \mathbb{C})$, define

$$d(C) = d(g), \quad r(C) = r(g) \quad \text{and} \quad l(C) = l(g), \quad \text{for any } g \in C.$$

Two elements in $SL(n + 1, \mathbb{C})$ are in the same conjugacy class in $SL(n + 1, \mathbb{C})$ if and only if they are in the same conjugacy class in $GL(n + 1, \mathbb{C})$. This fact will be used throughout the rest of this section.

4.2. The main theorem and its consequences.

**Lemma 4.1.** Let $C$ be a conjugacy class in $SL(n + 1, \mathbb{C})$ and let $w \in S_{n+1}$ be an involution. If $C \cap (BwB) \neq \emptyset$, then $l_2(w) \leq l(C)$.

**Proof.** Assume that $C \cap (BwB) \neq \emptyset$. Let $g \in C \cap (BwB)$, and write $g = b_1 \hat{w} b_2$, where $b_1, b_2 \in B$ and $\hat{w}$ is any representative of $w$ in the normalizer of $H$ in $G$. Then for any non-zero $c \in \mathbb{C}$,

$$\text{rank}(g - cI_{n+1}) = \text{rank}(b_1 \hat{w} b_2 - cI_{n+1}) = \text{rank}(\hat{w} - cb_1^{-1}b_2^{-1}).$$
Let \( w = (i_1, j_1) \cdots (i_{l_2(w)}, j_{l_2(w)}) \) be the decomposition of \( w \) into distinct 2-cycles, where \( i_1 < \cdots < i_{l_2(w)} \) and \( i_k < j_k \) for every \( 1 \leq k \leq l_2(w) \). It is easy to see that for any \( b \in B \), the columns of the matrix \( \dot{w} - b \) corresponding to \( i_1, \ldots, i_{l_2(w)} \) are linearly independent, so \( \text{rank}(\dot{w} - b) \geq l_2(w) \). Thus \( \text{rank}(g - cI_{n+1}) \geq l_2(w) \) for every non-zero \( c \in C \). Hence \( r(C) = r(g) \geq l_2(w) \). Since \( l_2(w) \leq \left\lfloor \frac{n+1}{2} \right\rfloor \), one has \( l_2(w) \leq l(C) \).

Q.E.D.

**Theorem 4.2.** Let \( C \) be a conjugacy class in \( SL(n + 1, \mathbb{C}) \) and let \( w \in S_{n+1} \) be an involution. Then \( C \cap (BwB) \neq \emptyset \) if and only if \( l_2(w) \leq l(C) \).

A proof of Theorem 4.2 using a result of Ellers-Gordeev [9] is given in §4.3, and a direct proof of Theorem 4.2 is given in §4.4. We now give some corollaries of Theorem 4.2.

**Corollary 4.3.** Let \( C \) and \( C' \) be two conjugacy classes in \( SL(n + 1, \mathbb{C}) \) such that \( C' \) is contained in the closure of \( C \). Let \( w \in S_{n+1} \) be an involution. If \( w \in W_{C'} \), then \( w \in W_C \).

**Proof.** It follows from the definition that \( r(C') \leq r(C) \), so \( l(C') \leq l(C) \). Corollary 4.3 now follows directly from Theorem 4.2.

Q.E.D.

Recall that for \( w \in S_{n+1} \), \( O_w \) denotes the conjugacy class of \( w \) in \( S_{n+1} \).

**Corollary 4.4.** Let \( w \in S_{n+1} \) be an involution and let \( C \) be a conjugacy class in \( SL(n + 1, \mathbb{C}) \). If \( w \in W_C \), then \( O_w \subset W_C \).

**Proof.** Since \( l_2(w') = l_2(w) \) for every \( w' \in O_w \), Corollary 4.4 follows directly from Theorem 4.2.

Q.E.D.

We now consider spherical conjugacy classes in \( SL(n + 1, \mathbb{C}) \).

**Lemma 4.5.** [1, 2] A spherical conjugacy class in \( SL(n + 1, \mathbb{C}) \) is either unipotent or semi-simple.

1) A unipotent conjugacy class in \( SL(n + 1, \mathbb{C}) \) is spherical if and only if all of its Jordan blocks are of size at most 2.

2) A semi-simple conjugacy class \( C \) in \( SL(n + 1, \mathbb{C}) \) is spherical if and only if it has exactly two distinct eigenvalues.
For a spherical conjugacy class $C$ in $SL(n+1, \mathbb{C})$, $r(C)$ is precisely the number of size 2 blocks in the Jordan form of $C$, and for a semi-simple spherical conjugacy class, $r(C)$ is equal to the smaller multiplicity of the two eigenvalues. In particular, $l(C) = r(C)$ for every spherical conjugacy class in $SL(n+1, \mathbb{C})$.

**Corollary 4.6.** For a spherical conjugacy class $C$ in $SL(n+1, \mathbb{C})$,

$$W_C = \{ w \in S_{n+1} : w^2 = 1 \text{ and } l_2(w) \leq r(C) \}.$$ 

**Proof.** Let $C$ be a spherical conjugacy class in $SL(n+1, \mathbb{C})$. By [3, Theorem 2.7], if $w \in W_C$, then $w$ is an involution, and by Theorem 4.2, $l_2(w) \leq r(C)$. Conversely, if $w \in S_{n+1}$ is an involution with $l_2(w) \leq r(C)$, then $w \in W_C$ by Theorem 4.2.

Q.E.D.

**Remark 4.7.** Fix $\xi \in \mathbb{C}$ such that $\xi^{n+1} = -1$. For an integer $0 \leq r \leq \left[ \frac{n+1}{2} \right]$, let

$$h_r = \begin{cases} \text{diag}(I_{n+1-r}, -I_r) & \text{if } r \text{ is even}, \\ \text{diag}(\xi I_{n+1-r}, -\xi I_r) & \text{if } r \text{ is odd}, \end{cases}$$

and let $C_{h_r}$ be the conjugacy class of $h_r$ in $SL(n+1, \mathbb{C})$. Every semi-simple spherical conjugacy class in $SL(n+1, \mathbb{C})$ is $SL(n+1, \mathbb{C})$-equivariantly isomorphic to $C_{h_r}$ for some $0 \leq r \leq \left[ \frac{n+1}{2} \right]$, which is also $SL(n+1, \mathbb{C})$-equivariantly isomorphic to the symmetric space

$$X = SL(n+1, \mathbb{C})/S(GL(n+1-r, \mathbb{C}) \times GL(r, \mathbb{C})).$$

Let $V$ be the set of $B$-orbits on $X$ and let $\phi : V \to I$ be the map defined in [19, Section 1.6] by Richardson and Springer, where $I$ is the set of all involutions in $S_{n+1}$. It is easy to see from the definitions that $W_C$ for $C = C_{h_r}$ is the same as $\text{Im}(\phi)$, the image of $\phi$. The fact that $\text{Im}(\phi)$ consists of all $w \in I$ with $l_2(w) \leq r$ is well-known (see, for example, [20]).

We now determine the element $m_C$ for every conjugacy class $C$ in $SL(n+1, \mathbb{C})$.

List the simple roots as $\Gamma = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$ in the standard way. Recall that $w_0$ is the longest element in $S_{n+1}$ and that for a subset $J$ of $\Gamma$, $w_{0,J}$ is the longest element in the subgroup of $S_{n+1}$ generated by simple roots in $J$. For an integer $0 \leq l \leq \left[ \frac{n+1}{2} \right]$, let

$$J_l = \begin{cases} \{ \alpha_{l+1}, \ldots, \alpha_{n-1} \}, & \text{if } 0 \leq l \leq \left[ \frac{n+1}{2} \right] - 1, \\ \emptyset, & \text{if } l = \left[ \frac{n+1}{2} \right], \end{cases}$$

and let $m_l = w_0 w_0, J_l$. Thus, $m_0 = 1$, and

$$m_l = (1, n + 1)(2, n) \cdots (l, n + 2 - l), \quad \text{if } 1 \leq l \leq \left[ \frac{n+1}{2} \right].$$
In particular, \( m_l = w_0 \) for \( l = \lfloor \frac{n+1}{2} \rfloor \). Note that for \( 0 \leq l_1, l_2 \leq \lfloor \frac{n+1}{2} \rfloor \),
\[
m_{l_1} \leq m_{l_2} \iff l_1 \leq l_2.
\]

**Corollary 4.8.** For any conjugacy class \( C \) in \( SL(n+1, \mathbb{C}) \), \( m_C = m_{l(C)} \), i.e.,
\[
m_C = \begin{cases} 
    w_0 & \text{if } r(C) \geq \left\lfloor \frac{n+1}{2} \right\rfloor, \\
    m_{r(C)} & \text{if } r(C) < \left\lfloor \frac{n+1}{2} \right\rfloor.
\end{cases}
\]

**Proof.** Let \( C \) be any conjugacy class in \( SL(n+1, \mathbb{C}) \). By Corollary 2.11, Lemma 3.8 and Lemma 3.9, \( m_C = m_l \) for some \( 0 \leq l \leq \left\lfloor \frac{n+1}{2} \right\rfloor \). Since \( C \cap (Bm_lB) \neq \emptyset \), \( l \leq l(C) \) by Theorem 4.2. Since \( C \cap (Bm_{l(C)}B) \neq \emptyset \) by Theorem 4.2, one also has \( l(C) \leq l \). Thus \( l = l(C) \).

Q.E.D.

**4.3. A proof of Theorem 4.2 using the Ellers-Gordeev criterion.**

**Notation 4.9.** First recall (see for example [9, Page 705]) that for an integer \( p > 0 \), a partition of \( p \) is a non-increasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_s) \) of positive integers such that \( \lambda_1 + \cdots + \lambda_s = p \), and \( s \) is called the length of \( \lambda \). The shape of a partition \( \lambda = (\lambda_1, \ldots, \lambda_s) \) of \( p \) consists of \( s \) rows of empty boxes left-aligned with \( \lambda_j \) boxes on the \( j \)-th row for each \( 1 \leq j \leq s \). The partition \( \lambda^* \) of \( p \) whose shape is obtained from switching the rows and columns of the shape of \( \lambda \) is called the dual of \( \lambda \). Let \( \lambda = (\lambda_1, \ldots, \lambda_s) \) and \( \mu = (\mu_1, \ldots, \mu_t) \) be two partitions of \( p \). Define \( \lambda \leq \mu \) if \( \sum_{j=1}^k \lambda_j \leq \sum_{j=1}^k \mu_j \) for every \( 1 \leq k \leq t \). One has (see [15, Section I.1.11]) \( \lambda \leq \mu \) if and only if \( \mu^* \leq \lambda^* \), where \( \mu^* \) and \( \lambda^* \) are the partitions of \( p \) that are dual to \( \mu \) and \( \lambda \) respectively.

For integers \( p > 0 \) and \( 0 \leq l \leq \lfloor p/2 \rfloor \), let \( \lambda(l, p) = (2, \ldots, p+1, \ldots, 1) \) be the partition of \( p \) with 2 appearing exactly \( l \) times.

**Lemma 4.10.** Let \( p > 0 \) be an integer and let \( 0 \leq l \leq \lfloor p/2 \rfloor \). Then for any partition \( \mu = (\mu_1, \ldots, \mu_s) \) of \( p \), \( \lambda(l, p) \leq \mu \) if and only if \( p - l \geq s \).

**Proof.** Let \( \lambda(l, p)^* \) and \( \mu^* \) be the partitions of \( p \) that are dual to \( \lambda(l, p) \) and \( \mu \) respectively. Then \( \lambda(l, p) \leq \mu \) if and only if \( \lambda(l, p)^* \geq \mu^* \), and the latter is equivalent to \( p - l \geq s \).

Q.E.D.

We now use [9, Theorem 3.20] to prove Theorem 4.2.

Let \( C \) be a conjugacy class in \( SL(n+1, \mathbb{C}) \) and assume that \( w \in S_{n+1} \) is an involution with \( l_2(w) \leq l(C) \), or, equivalently, \( l_2(w) \leq r(C) \). We need to show that \( C \cap (BwB) \neq \emptyset \). By [11, Theorem 3.2.9(a)], there exist \( w' \) which is a minimal length element in the conjugacy class of \( w \) in \( W \) and an ascent from \( w' \) to \( w \). Thus,
in the notation of [9], there is a tree \( \Gamma(w) \) with \( w' \in T(\Gamma(w)) \). By [9, Theorem 3.20], it is enough to show that \( \lambda(w') \leq \tilde{\nu}^* \), where \( \lambda(w') = \lambda(l_2(w), n + 1) \) is the partition \((2, \ldots, 2, 1, \ldots, 1)\) of \( n + 1 \) with 2 appearing \( l_2(w') \) times, and \( \tilde{\nu}^* \) is the partition of \( n + 1 \) associated to \( C \) as described at the beginning of [9, Section 3.4]. One checks from the definitions that the partition \( \tilde{\nu}^* \) has length \( d(C) \). By Lemma 4.10, \( \lambda(w') \leq \tilde{\nu}^* \) if and only if \( n + 1 - l_2(w) \geq d(C) \) which is equivalent to \( l_2(w) \leq r(C) \). This proves Theorem 4.2.

4.4. A direct proof of Theorem 4.2. The proof of Theorem 4.2 in §4.3 uses only a special case of of the Ellers-Gordeev criterion in [9, Theorem 3.20], and the proof of [9, Theorem 3.20] for the general case involves rather complicated combinatorics. We thus think that it is worthwhile to give a direct proof Theorem 4.2. Our direct proof also has the merit that it shows how to explicitly find an element in \( C \cap BwB \) when \( l_2(w) \leq l(C) \). We will use two lemmas from [9], namely [9, Lemma 3.3] and [9, Lemma 3.24] whose proofs as given in [9] are elementary.

For \( g = (g_{i,j}) \in SL(n + 1, \mathbb{C}) \) and \( 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor \), let \( g^{(i)} \) be the \( 2 \times 2 \) matrix

\[
g^{(i)} = \begin{pmatrix}
g_{2i-1,2i-1} & g_{2i-1,2i} \\
g_{2i,2i-1} & g_{2i,2i}
\end{pmatrix}.
\]

Recall that a square matrix is said to be regular if its characteristic polynomial is the same as its minimal polynomial. An upper-triangular matrix \( A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in GL(2, \mathbb{C}) \) is regular if and only if either \( x \neq z \), or \( x = z \) and \( y \neq 0 \), and in this case \( A \) is conjugate to some \( A_1 = \begin{pmatrix} x_1 & y_1 \\ u & z_1 \end{pmatrix} \) with \( u \neq 0 \).

**Proposition 4.11.** Let \( C \) be any conjugacy class \( C \) in \( SL(n + 1, \mathbb{C}) \) with \( l(C) > 0 \). Then there exists \( g \in C \cap B \) such that \( g^{(i)} \) is regular for every \( 1 \leq i \leq l(C) \).

Assuming Proposition 4.11, we now prove Theorem 4.2. Let \( C \) be a conjugacy class in \( SL(n + 1, \mathbb{C}) \) and \( w \in S_{n+1} \) an involution such that \( l_2(w) \leq l(C) \). We will show that \( C \cap (wB) \neq \emptyset \).

If \( l(C) = 0 \), then \( C \) consists of only one central element in \( SL(n + 1, \mathbb{C}) \), and \( C \) only intersects with \( B \) and Theorem 4.2 holds in this case. Thus we will assume that \( l(C) > 0 \). Since \( C \cap B \neq \emptyset \), we will also assume that \( l_2(w) > 0 \).

Let \( g \in C \cap B \) be as in Proposition 4.11 and let \( 1 \leq i \leq l_2(w) \). Since \( g^{(i)} \in GL(2, \mathbb{C}) \) is regular, there exists \( A_1 \in GL(2, \mathbb{C}) \) such that \( A_1g^{(i)}A_1^{-1} = \begin{pmatrix} x_i & y_i \\ u_i & z_i \end{pmatrix} \) with \( u_i \neq 0 \). Let \( A = \text{diag}(A_1, \ldots, A_{l_2(w)}, I_{n+1-2l_2(w)}) \) be the block diagonal matrix in \( GL(n + 1, \mathbb{C}) \). Then \( AgA^{-1} \in C \cap BuB \), where

\[
u = (1, 2)(3, 4) \cdots (2l_2(w) - 1, 2l_2(w)).
Since there is an ascent from some minimal length element in $O_w$ to $w$, we can assume that $w$ has minimal length in $O_w$, so $w$ is the following product of disjoint 2-cycles:

$$w = (i_1, i_1 + 1) \cdots (i_{l_2(w)}, i_{l_2(w)} + 1)$$

where $1 \leq i_1 < \cdots < i_{l_2(w)} \leq n$. We will now use [9, Lemma 3.3]. Let

$$X = \{\alpha_1, \alpha_3, \ldots, \alpha_{2l_2(w)}\}, \quad Y = \{\alpha_{i_1}, \ldots, \alpha_{i_{l_2(w)}}\},$$

and let $\omega$ be any element in $S_n$ such that $\omega(2s - 1) = i_s$ and $\omega(2s) = i_s + 1$ for $1 \leq s \leq l_2(w)$. Then $\omega^{-1}w\omega = u$, and $Y = \omega(X)$. Applying [9, Lemma 3.3] to the above $X, Y, \omega$ and $g_x = AgA^{-1}$, one sees, in the notation of [9, Lemma 3.3], that there exists $g_y \in C \cap BwB$. Thus $C \cap BwB \neq \emptyset$, and Theorem 4.2 is proved.

It remains to prove Proposition 4.11.

**Proof of Proposition 4.11 when $C$ has only one eigenvalue.** We will use induction on $n$. It is easy to see that Proposition 4.11 holds for $n = 1$ or $n = 2$.

Assume now that $n \geq 3$ and that Proposition 4.11 holds for conjugacy classes $C$ in $SL(p, \mathbb{C})$ for any $p < n + 1$ and any $C$ with only one eigenvalue. Assume that $C$ is a conjugacy class in $SL(n + 1, \mathbb{C})$ with one eigenvalue $c$. Since we are assuming that $l(C) > 0$, there exists a Jordan block of $C$ of size at least 2. Since Proposition 4.11 clearly holds when $C$ is regular, we also assume that $C$ has more than one Jordan block.

**Case 1.** There is a Jordan block of $C$ of size 1. In this case, choose $g \in C$ of the form $g = \begin{pmatrix} g' & 0 \\ 0 & c \end{pmatrix}$, where $g' \in GL(n, \mathbb{C})$ is of Jordan form with $c$ as the only eigenvalue. Then $d(g') = d(g) - 1$ and $r(g') = n - d(g') = r(g)$. Suppose that $r(g) \leq \left[ \frac{n+1}{2} \right] - 1$. Since $\left[ \frac{n+1}{2} \right] - 1 = \left[ \frac{n-1}{2} \right] \leq \left[ \frac{n}{2} \right]$, one has $l(g) = l(g')$, so by induction, Proposition 4.11 holds for $C$. Suppose that $r(g) \geq \left[ \frac{n+1}{2} \right]$. Then $l(g) = \left[ \frac{n+1}{2} \right]$ and $l(g') = \left[ \frac{n}{2} \right]$. If $n + 1$ is odd, then $\left[ \frac{n+1}{2} \right] = \left[ \frac{n}{2} \right]$, so $l(g) = l(g')$ and by induction, Proposition 4.11 holds for $C$. If $n + 1$ is even, then $l(g') = l(g) - 1$ and one could not use induction. However, since we are assuming that $C$ has a Jordan block of size at least 2, there is an element in $C$ of the form

$$\begin{pmatrix} J_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & J_k & 0 \\ 0 & \cdots & 0 & c \end{pmatrix}$$  \hspace{1cm} (4.1)$$

where $J_1, \cdots, J_k$ are of Jordan form and $J_k$ has size at least 2. By [9, Lemma 3.24], the matrix in (4.1) is conjugate to

$$g_1 = \begin{pmatrix} g'_1 & 0 & v \\ 0 & c & 1 \\ 0 & 0 & c \end{pmatrix},$$
where \( g_1' \in GL(n-1, \mathbb{C}) \) is of Jordan form, and \( v \) is the column vector in \( \mathbb{C}^{n-1} \) that has 1 for the last coordinate and 0 for all the other coordinates. Now \( d(g_1') = d(g) - 1 \), so \( r(g_1') = n - 1 - d(g_1') = r(g) - 1 \). Recall that we are assuming that \( n + 1 \) is even and \( r(g) \geq \left\lceil \frac{n+1}{2} \right\rceil \). Let \( n + 1 = 2m \). Since \( r(g_1') = r(g) - 1 \geq m - 1 = \left\lfloor \frac{n+1}{2} \right\rfloor \), \( l(g_1') = m - 1 = l(g) - 1 \). Applying induction to \( g_1' \), one sees that Proposition 4.11 holds for \( C \).

**Case 2.** All the Jordan blocks of \( C \) have sizes at least 2 and at least one of them has size 2. In this case, choose \( g \in C \) of the form

\[
g = \begin{pmatrix}
c & 1 & 0 \\
0 & c & 0 \\
0 & 0 & g'
\end{pmatrix},
\]

where \( g' \in GL(n-1, \mathbb{C}) \) is of Jordan form. Then \( d(g') = d(g) - 1 \), so \( r(g') = n - 1 - d(g') = r(g) - 1 \). Since all the Jordan blocks have sizes at least 2, one has \( 2d(g) \leq n + 1 \), so \( r(g) \geq \left\lceil \frac{n+1}{2} \right\rceil \), and \( r(g') \geq \left\lceil \frac{n+1}{2} \right\rceil - 1 = \left\lfloor \frac{n-1}{2} \right\rfloor \). Thus \( l(g') = \left\lceil \frac{n-1}{2} \right\rceil = l(g) - 1 \). Applying the induction assumption to \( g' \), one sees that Proposition 4.11 holds for \( C \).

**Case 3.** All the Jordan blocks of \( C \) have sizes at least 3. Then we can find \( g \in C \) of the form

\[
g = \begin{pmatrix}
c & 1 & 0 \\
0 & c & v \\
0 & 0 & g'
\end{pmatrix}
\]

where \( v \) is the row vector in \( \mathbb{C}^{n-1} \) which has 1 for the first coordinate and 0 for all the other coordinates, and \( g' \in GL(n-1, \mathbb{C}) \) is of Jordan form with \( d(g') = d(g) \), and thus \( r(g') = r(g) - 2 \). By assumption \( n + 1 \geq 3d(g) \geq 6 \), so \( r(g) \geq \frac{2(n+1)}{3} \) and \( r(g') = r(g) - 2 \geq \left\lfloor \frac{n-1}{2} \right\rfloor \). Thus \( l(g) = \left\lceil \frac{n+1}{2} \right\rceil \) and \( l(g') = \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \). Applying the induction assumption to \( g' \), one sees that Proposition 4.11 holds for \( C \).

This finishes the proof of Proposition 4.11 in the case when \( C \) has only one eigenvalue.

**Proof of Proposition 4.11 when \( C \) has more that one eigenvalue.** We again use induction on \( n \). Proposition 4.11 clearly holds for \( n = 0 \) or \( n = 1 \). Assume that Proposition 4.11 holds for \( GL(p, \mathbb{C}) \) for any \( p < n + 1 \) and any conjugacy class in \( GL(p, \mathbb{C}) \) with more than one eigenvalue. Let \( C \) be a conjugacy class in \( SL(n+1, \mathbb{C}) \) with distinct eigenvalues \( c_1, c_2, \ldots, c_k \), where \( k \geq 2 \), and for \( 1 \leq j \leq k \), let \( d_j \) be the number of Jordan blocks of \( C \) with eigenvalue \( c_j \). We will assume that \( d_1 \geq \cdots \geq d_k \). Then \( r(C) = n + 1 - d_1 \).
**Case 1.** \( r(C) > \left\lceil \frac{n+1}{2} \right\rceil \). Let \( g \in C \) be of the form

\[
g = \begin{pmatrix}
c_1 & 0 & v_1 \\
0 & c_2 & v_2 \\
0 & 0 & g'
\end{pmatrix}
\]

where \( v_1 \) and \( v_2 \) are row vectors of size \( n-1 \) and \( g' \in GL(n-1, \mathbb{C}) \). Then

\[
\text{rank}(g' - c_j I_{n-1}) \geq \text{rank}(g - c_j I_{n+1}) - 2, \quad 1 \leq j \leq k.
\]

Thus \( r(g') \geq r(g) - 2 \geq \left\lceil \frac{n-1}{2} \right\rceil \) and \( l(g') = l(g) - 1 \). If \( g' \) has only one eigenvalue, we have proved that Proposition 4.11 holds for the conjugacy class of \( g' \) and thus also holds for \( C \). If \( g' \) has more than one eigenvalue, one applies the induction assumption to \( g' \) to see that Proposition 4.11 holds for \( C \).

**Case 2.** \( r(C) \leq \left\lceil \frac{n+1}{2} \right\rceil \). Then \( d_1 \geq \frac{n+1}{2} \). If all the Jordan blocks of \( C \) with eigenvalue \( c_1 \) have sizes at least 2, then \( n + 1 \geq 2d_1 + 1 \), and \( d_1 \leq \frac{n}{2} \), which is a contradiction. Thus \( C \) has at least one Jordan block of size 1. Pick \( g \in C \) of the form

\[
(4.2) \quad g = \begin{pmatrix}
c_1 & 0 & 0 \\
0 & c_2 & v \\
0 & 0 & g'
\end{pmatrix}
\]

where \( v \) is a row vector of size \( n-1 \) and \( g' \in GL(n-1, \mathbb{C}) \). Then

\[
\text{rank}(g' - c_1 I_{n-1}) = \text{rank}(g - c_1 I_{n+1}) - 1 = n - d_1,
\]

\[
\text{rank}(g' - c_2 I_{n-1}) = \text{rank}(g - c_2 I_{n+1}) - 1 = n - d_2 \geq n - d_1, \quad \text{or}
\]

\[
\text{rank}(g' - c_2 I_{n-1}) = \text{rank}(g - c_2 I_{n+1}) - 2 = n - d_2 - 1.
\]

Moreover, if \( k \geq 3 \), then for every \( 3 \leq j \leq k \),

\[
\text{rank}(g' - c_j I_{n-1}) = \text{rank}(g - c_j I_{n+1}) - 2 = n - d_j - 1.
\]

If \( k \geq 3 \), then \( n - d_j - 1 \geq n - d_1 \) for every \( 3 \leq j \leq k \). Indeed, if \( n - d_j - 1 < n - d_1 \) for some \( j \geq 3 \), then \( n - d_2 - 1 \leq n - d_j - 1 < n - d_1 \), so \( d_1 = d_2 \). Since \( d_1 \geq \frac{n+1}{2} \) and \( d_1 + d_2 + d_j \leq n + 1 \), one has a contradiction. Thus

\[
r(g') = \min\{\text{rank}(g' - c_1 I_{n-1}), \text{rank}(g' - c_2 I_{n-1})\}.
\]

Consequently, \( r(g') = n - d_1 = r(g) - 1 \) unless \( \text{rank}(g' - c_2 I_{n-1}) = n - d_2 - 1 \) and \( n - d_2 - 1 < n - d_1 \). But in the latter case, \( d_1 = d_2 = \frac{n+1}{2} \) so \( n + 1 \) must be even and \( g \) is semi-simple. In particular, \( \text{rank}(g' - c_2 I_{n-1}) = n - d_2 \) which is a contradiction. Thus one always has \( r(g') = r(g) - 1 \) and \( l(g') = l(g) - 1 \). Induction on \( g' \) again yields Proposition 4.11.

This finishes the proof of Proposition 4.11.
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