The balanced quaternary sequence of odd period with low autocorrelation

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Abstract. In this paper, we construct some balanced quaternary sequences of odd period \( p \) with low autocorrelation from two types of Legendre sequences and its cyclic shift or complement and inverse Gray mapping, where \( p \) is odd prime. Firstly, we choose a Legendre sequence pair of period \( p \), which include \( p \equiv 1 (\text{mod} \ 4) \) and \( p \equiv 3 (\text{mod} \ 4) \), then we get a new Legendre sequence pair of period \( p \) by cyclic shift or complement, finally, we can get a quaternary sequence of period \( p \) by utilizing inverse Gray mapping into the new Legendre sequence pair.

1. Introduction

Quaternary sequence with low autocorrelation play an important role in practical communication systems, the employed sequences are required to have autocorrelation values as low as possible to reduce the interference and noise for acquiring the desired information from the received signals, therefore, it is worthwhile to construct such quaternary sequence with low autocorrelation.

Let \( a = (a(0), a(1), \ldots, a(N - 1)) \) and \( b = (b(0), b(1), \ldots, b(N - 1)) \) be two quaternary sequences of period \( N \) over \( \mathbb{Z}_4 \), the periodic correlation function between \( a \) and \( b \) at the shift \( 0 \leq \tau < N \) is defined by

\[
R_{a,b}(\tau) = \sum_{i=0}^{N-1} a(i)^{\tau}b(i+\tau)
\]  

(1)

Where \( \omega = \sqrt{-1} \). \( R_{a,b}(\tau) \) is called autocorrelation of \( a \) and \( b \) if \( a = b \) or cross-correlation of \( a \) and \( b \) otherwise. The maximum out-of-phase autocorrelation magnitude of \( a \) is defined as

\[
R_{\max}(a) = \max\{|R_{a}(\tau)|: 0 < \tau < N\}
\]

(2)

For quaternary sequence \( a \) of odd period \( N \), \( R_{\max}(a) \geq 1 \), the known quaternary with \( R_{\max}(a) = 1 \) was proposed in[5], the next smallest values for the maximum out-of-phase autocorrelation magnitude of quaternary sequence of odd period are \( \sqrt{3} \) or 3, which can refer to[12]. In[12], Yang et al. proposed quaternary sequences with odd period by inverse Gray mapping and binary sequences defined by cyclotomy of order 4. For quaternary sequence of even period \( N \), \( R_{\max}(a) = 2 \), which can refer to[1,2,10,6,8,4]. In[1,2], Jang et al. constructed new quaternary sequences of even period from binary sequences with ideal autocorrelation and Legendre sequences. In[10], Tang et al. proposed quaternary sequences by binary interleaved sequences and inverse Gray mapping. In[6], Shen et al. constructed quaternary sequences by using CRT and classical cyclotomy of order 4. In[8], Su et al. proposed a construction of quaternary sequences of even length with optimal autocorrelation magnitude by interleaving the twin-prime sequences pair, GMW sequences pair and binary sequences defined by cyclotomic classes of order 4 and inverse Gray mapping. In[4], Luo et al. constructed quaternary sequences by interleaving two inequivalent Tang-Lindner sequences[11].
2. Preliminaries

In this section, we will review some definitions and well-known lemmas, and adopt the following notations thought this paper without special explanation.

- Let \( q = (q(0), q(1), \ldots, q(N-1)) \) be a quaternary sequence of period \( N \) over \( Z_4 \), define \( \text{Car}_k(q) = \{0 \leq i < N; q(i) = k\} \) then \( q \) is said to be balanced if \( \max_{\text{Car}_k(q)} - \min_{\text{Car}_k(q)} \leq 1 \).

- The cyclic shift of \( q \) is defined by \( L^\tau(q) = (q(\tau), q(\tau+1), \ldots, q(\tau+N-1), q(0), \ldots, q(\tau-1)), 0 \leq \tau < N \).

- \( \frac{1}{2} \) is the multiplicative inverse of 2 (mod \( p \)), \( -\frac{1}{2} \pmod{\frac{p-1}{2}} \) is the additive inverse of \( \frac{1}{2} \pmod{\frac{p-1}{2}} \).

2.1. Gray mapping and its inverse

The well-known Gray mapping \( \varphi : Z_4 \rightarrow Z_2 \times Z_2 \) is defined as \( \varphi(0) = (0,0), \varphi(1) = (0,1), \varphi(2) = (1,1), \varphi(3) = (1,0) \).

Sing the inverse Gray mapping \( \varphi^{-1} : Z_2 \times Z_2 \rightarrow Z_4 \), i.e.

\[
\varphi^{-1}(0,0) = 0, \varphi^{-1}(0,1) = 1, \varphi^{-1}(1,1) = 2, \varphi^{-1}(1,0) = 3.
\]

Any quaternary sequence \( q = (q(0), q(1), \ldots, q(N-1)) \) can be obtained from two binary sequences \( c = (c(0), c(1), \ldots, c(N-1)) \) and \( d = (d(0), d(1), \ldots, d(N-1)) \) of the same period \( N \) as follows:

\[
q(i) = \varphi^{-1}(a(i), b(i)), 0 \leq i < N
\]

Here the binary sequence \( c \) and \( d \) are called the component sequences of \( q \).

Transforming the sequence \( q \) into its complex valued version, i.e.

\[
\omega^q(i) = \frac{1}{2}(1 + \omega)(-1)^{c(i)} + \frac{1}{2}(1 - \omega)(-1)^{d(i)}, 0 \leq i < N
\]

Where \( \omega = \sqrt{-1} \). Krone and Sarwate observed the following relation between their correlations\[3\].

Lemma1.\[3\]. The autocorrelation function of \( q \) is given by

\[
R_q(\tau) = \frac{1}{2}[R_a(\tau) + R_b(\tau)] + \frac{\omega}{2}[R_{a,b}(\tau) - R_{b,a}(\tau)]
\]

2.2. Two types of Legendre sequence

Let \( p \) be an odd prime, the Legendre sequence \( l = (l(0), l(1), \ldots, l(p-1)) \) of period \( p \) is defined as

\[
l(i) = \begin{cases} 
0 \text{ or } 1, & \text{if } i = 0 \\
1, & \text{if } i \in \text{QR}_p \\
0, & \text{if } i \in \text{NQR}_p 
\end{cases}
\]

Where \( \text{QR}_p \) is the set of quadratic residues modulo \( p \), and \( \text{NQR}_p \) is the set of quadratic non-residues modulo \( p \). Actually, the Legendre sequences are based on classical cyclotomy of order two with respect to \( Z_p \).

Definition1.\[7\]. Let \( p = 2f + 1 \) be an odd prime number, and let \( \alpha \) be a primitive root of \( Z_p \). Define

\[
D_i = \{\alpha^{i+2}(mod \ p)|j = 0,1, \ldots, f - 1 \text{ for } i = 0,1 \}
\]

Then these \( D_i \) are called cyclotomic classes of order two with respect to \( Z_p \).
For fixed $i$ and $j$, we define the cyclotomic number $(i, j)$ to be the number of solutions $(x, y)$ of the equation

$$x + 1 = y, (x, y) \in D_i \times D_j$$

Or, equivalently,

$$(i, j) = |(D_i + 1) \cap D_j|$$

From above, we can know $QR_p = D_0$ and $NQR_p = D_1$.

**Lemma 2.** The cyclotomic numbers of order two are given by

(a) $f$ even: $(0,0) = \frac{f-2}{2}; (0,1) = (1,0) = (1,1) = \frac{f}{2}$

(b) $f$ odd: $(0,1) = \frac{f+1}{2}; (0,0) = (1,0) = (1,1) = \frac{f-1}{2}$

Specially, $l$ is called the first Legendre sequence if $l(0) = 1$ or the second Legendre sequence if $l(0) = 0$. For simplicity, let $l$ and $l'$ be the first Legendre sequence and the second Legendre sequence respectively. There are the correlation properties of the Legendre sequence in the follows.

**Property 1.** Given $0 \leq \tau < p$, if $p \equiv 1 \pmod{4}$, then

$$R_l(\tau) = \begin{cases} p, & \tau = 0 \\ 1, & \tau \in QR_p \\ -3, & \tau \in NQR_p \end{cases}$$

And,

$$R_{l'}(\tau) = \begin{cases} p, & \tau = 0 \\ -3, & \tau \in QR_p \\ 1, & \tau \in NQR_p \end{cases}$$

If $p \equiv 3 \pmod{4}$, then

$$R_l(\tau) = R_{l'}(\tau) = \begin{cases} p, & \tau = 0 \\ -1, & \text{otherwise} \end{cases}$$

**Property 2.** Given $0 \leq \tau < p$, if $p \equiv 1 \pmod{4}$, then

$$R_{l,l'}(\tau) = R_{l',l}(\tau) = \begin{cases} p-2, & \tau = 0 \\ -1, & \text{otherwise} \end{cases}$$

If $p \equiv 3 \pmod{4}$, then

$$R_{l,l'}(\tau) = \begin{cases} p-2, & \tau = 0 \\ 1, & \tau \in QR_p \\ -3, & \tau \in NQR_p \end{cases}$$

And,

$$R_{l',l}(\tau) = \begin{cases} p-2, & \tau = 0 \\ -3, & \tau \in QR_p \\ 1, & \tau \in NQR_p \end{cases}$$

3. The construction of balanced quaternary sequence

In this section, we present the construction of quaternary sequences with low correlation from Legendre sequences and inverse Gray mapping.

**Construction 1:**

(i) Let $p$ be an odd prime, $l$ and $l'$ are the first and second Legendre sequence of period $p$, $b = (b(0), b(1)) \in \{0,1\}$, $\lambda = \frac{p+1}{2}$;

(ii) Define the component sequences of period $p$,

if $p \equiv 1 \pmod{4}$, then

$$a = l + b(0), b = L^3(l') + b(1)$$

Or

$$a = l' + b(0), b = L^3(l) + b(1);$$

If $p \equiv 3 \pmod{4}$, then...
Applying the inverse Gray mapping \( \varphi^{-1} \) to \( a \) and \( b \), i.e.

\[
q(i) = \varphi^{-1}(a(i), b(i)) , 0 \leq i < p.
\]

**Theorem 1.** The autocorrelation function of \( q \) generated by *Construction I* is given by

When \( p \equiv 1 \pmod{4} \),

\[
R_q(\tau) = \frac{1}{2} \{ R_l(\tau) + R_I(\tau) \} + (-1)^{b(0)+b(1)} \frac{\omega}{2} \{ R_{l\nu}(\tau + \lambda) - R_{l\nu}(\tau - \lambda) \}
\]

If \( a = l + b(0), b = L^2(l') + b(1) \), or

\[
R_q(\tau) = \frac{1}{2} \{ R_l(\tau) + R_I(\tau) \} + (-1)^{b(0)+b(1)} \frac{\omega}{2} \{ R_{l\nu}(\tau + \lambda) - R_{l\nu}(\tau - \lambda) \}
\]

If \( a = l' + b(0), b = L^2(l') + b(1) \),

\[
R_q(\tau) = \frac{1}{2} \{ R_l(\tau) + R_I(\tau) \} + (-1)^{b(0)+b(1)} \frac{\omega}{2} \{ R_{l\nu}(\tau + \lambda) - R_{l\nu}(\tau - \lambda) \}
\]

When \( p \equiv 3 \pmod{4} \),

\[
R_q(\tau) = \frac{1}{2} \{ R_l(\tau) + R_I(\tau) \} + (-1)^{b(0)+b(1)} \frac{\omega}{2} \{ R_{l\nu}(\tau + \lambda) - R_{l\nu}(\tau - \lambda) \}
\]

**Proof:** From (6), when \( p \equiv 1 \pmod{4} \), if \( a = l + b(0), b = L^2(l') + b(1) \), we have

\[
R_q(\tau) = \sum_{i=0}^{p-1} \omega^{q(i)-q(i+\tau)}
\]

\[
= \sum_{i=0}^{p-1} \omega^{q(i)} \omega^{-q(i+\tau)}
\]

\[
= \sum_{i=0}^{p-1} \frac{1}{2} (1 + \omega)(-1)^{l(i)+b(0)} + \frac{1}{2} (1 - \omega)(-1)^{l(i)+b(1)}
\]

\[
\cdot \left[ \frac{1}{2} (1 + \omega)(-1)^{l(i)+l(i+\tau)} + \frac{1}{2} (1 - \omega)(-1)^{l(i)+b(0)} \right]
\]

\[
= (-1)^{b(0)+b(1)} \frac{\omega}{2} \sum_{i=0}^{p-1} (-1)^{l(i)+l(i+\tau)} - (-1)^{b(0)+b(1)} \frac{\omega}{2} \sum_{i=0}^{p-1} (-1)^{l(i)+l(i+\tau) + l(i+\lambda)}
\]

\[
+ \frac{1}{2} \sum_{i=0}^{p-1} (-1)^{l(i)+l(i+\tau)} + \frac{1}{2} \sum_{i=0}^{p-1} (-1)^{l(i)+l(i+\lambda)}
\]

\[
= \frac{1}{2} \{ R_l(\tau) + R_I(\tau) \} + (-1)^{b(0)+b(1)} \frac{\omega}{2} \{ R_{l\nu}(\tau + \lambda) - R_{l\nu}(\tau - \lambda) \}
\]

Similarly, if \( a = l' + b(0), b = L^2(l') + b(1) \), we have

\[
R_q(\tau) = \frac{1}{2} \{ R_l(\tau) + R_I(\tau) \} + (-1)^{b(0)+b(1)} \frac{\omega}{2} \{ R_{l\nu}(\tau + \lambda) - R_{l\nu}(\tau - \lambda) \}
\]

When \( p \equiv 3 \pmod{4} \), \( a = l + b(0), b = L^2(l) + b(1) \), we have

\[
R_q(\tau) = \sum_{i=0}^{p-1} \omega^{q(i)-q(i+\tau)}
\]

\[
= \sum_{i=0}^{p-1} \frac{1}{2} (1 + \omega)(-1)^{l(i)+b(0)} + \frac{1}{2} (1 - \omega)(-1)^{l(i)+l(i+\lambda)+b(1)}
\]
\[
\frac{1}{2}(1 + \omega)(-1)^{l(i+\lambda+\tau)+b(1)} + \frac{1}{2}(1 - \omega)(-1)^{l(i+\tau)+b(0)}
\]

\[
= (-1)^{b(0)+b(1)} \frac{\omega}{2} \sum_{i=0}^{p-1} (-1)^{l(i)+l(i+\tau+\lambda)} - (-1)^{b(0)+b(1)} \frac{\omega}{2} \sum_{i=0}^{p-1} (-1)^{l(i+\tau)+l(i+\lambda)}
\]

\[
+ \frac{1}{2} \sum_{i=0}^{p-1} (-1)^{l(i)+l(i+\tau)} + \frac{1}{2} \sum_{i=0}^{p-1} (-1)^{l(i+\tau)+l(i+\lambda+\tau)}
\]

\[
= R_i(\tau) + (-1)^{b(0)+b(1)} \frac{\omega}{2} [R_i(\tau + \lambda) - R_i(\tau - \lambda)]
\]

Similarly, If \( a = l' + b(0) \), \( b = L^a(l') + b(1) \), we have

\[
R_q(\tau) = R_i(\tau') + (-1)^{b(0)+b(1)} \frac{\omega}{2} [R_i(\tau + \lambda) - R_i(\tau - \lambda)]
\]

**Theorem 2.** Let \( p \) be an odd prime, the quaternary sequence \( q \) of period \( p \) generated by Construction 1 has low autocorrelation function, i.e.

If \( p \equiv 1 \pmod{4} \),

\[
R_q(\tau) = \begin{cases}
-1 + (-1)^{b(0)+b(1)} \frac{p-1}{2} \omega, & \tau = -\lambda \\
-1 + (-1)^{b(0)+b(1)} \frac{1-p}{2} \omega, & \tau = \lambda \\
-1, & \text{otherwise}
\end{cases}
\]

where the component sequences \( a = l + b(0), b = L^a(l') + b(1) \) or \( a = l' + b(0), b = L^a(l) + b(1) \).

If \( p \equiv 3 \pmod{4} \),

\[
R_q(\tau) = \begin{cases}
-1 + (-1)^{b(0)+b(1)} \frac{p+1}{2} \omega, & \tau = -\lambda \\
-1 + (-1)^{b(0)+b(1)} \frac{-1-p}{2} \omega, & \tau = \lambda \\
-1, & \text{otherwise}
\end{cases}
\]

where the component sequences \( a = l + b(0), b = L^a(l) + b(1) \) or \( a = l' + b(0), b = L^a(l') + b(1) \).

**Proof:** From Theorem 1, Property 1 and 2, we can draw the conclusion, so we omit here.

**Theorem 3.** The quaternary sequences \( q \) with low autocorrelation generated by Construction 1 are balanced.

**Proof:** (i) \( p \equiv 1 \pmod{4} \), we denote \( q_1 = \varphi^{-1}(l + b(0), L^a(l') + b(1)) \), here we only discuss \( b = (0,0) \), i.e. \( q_1 = \varphi^{-1}(l, L^a(l')) \), the other cases are similar to it, we omit here. Then

\[
Car_0(q_1) = |NQR_p \cap (NQR_p \cup \{0\} + \frac{p-1}{2})|
\]

\[
= |2NQR_p \cap (2NQR_p \cup \{0\} + p - 1)|
\]

\[
= \begin{cases}
|QR_p \cap (QR_p \cup \{0\} - 1)| & \text{if } 2 \in NQR_p \\
|NQR_p \cap (NQR_p \cup \{0\} - 1)| & \text{if } 2 \in QR_p
\end{cases}
\]

\[
= \begin{cases}
1 + |QR_p \cap (QR_p + 1)| = \frac{p-1}{4}, & 2 \in NQR_p \\
|NQR_p \cap (NQR_p + 1)| = \frac{p-1}{4}, & 2 \in QR_p
\end{cases}
\]

\[
Car_1(q_1) = |NQR_p \cap (QR_p + \frac{p-1}{2})|
\]

\[
= |2NQR_p \cap (2QR_p + p - 1)|
\]

\[
= \begin{cases}
|QR_p \cap (NQR_p - 1)| & \text{if } 2 \in NQR_p \\
|NQR_p \cap (QR_p - 1)| & \text{if } 2 \in QR_p
\end{cases}
\]
\[\begin{align*}
&\left\{\begin{array}{ll}
|QR_p \cap (NQR_p + 1)| = \frac{p - 1}{4}, & 2 \in NQR_p \\
|NQR_p \cap (QR_p + 1)| = \frac{p - 1}{4}, & 2 \in QR_p
\end{array}\right.
\]

\[\text{Car}_2(q_1) = |QR_p \cup \{0\} \cap (QR_p + \frac{p - 1}{2})| = |2QR_p \cup \{0\} \cap (2QR_p + p - 1)| = |NQR_p \cup \{0\} \cap (NQR_p - 1)| = |QR_p \cup \{0\} \cap (QR_p - 1)| = \left\{\begin{array}{ll}
|NQR_p \cap (QR_p + 1)| = \frac{p - 1}{4}, & 2 \in NQR_p \\
1 + |QR_p \cap (QR_p + 1)| = \frac{p - 1}{4}, & 2 \in QR_p
\end{array}\right.
\]

\[\text{Car}_3(q_1) = |QR_p \cup \{0\} \cap (NQR_p \cup \{0\} + \frac{p - 1}{2})| = |2QR_p \cup \{0\} \cap (2NQR_p \cup \{0\} + p - 1)| = |NQR_p \cup \{0\} \cap (QR_p \cup \{0\} - 1)| = |QR_p \cup \{0\} \cap (NQR_p \cup \{0\} - 1)| = \left\{\begin{array}{ll}
1 + |NQR_p \cap (QR_p + 1)| = \frac{p + 3}{4}, & 2 \in NQR_p \\
1 + |QR_p \cap (NQR_p + 1)| = \frac{p + 3}{4}, & 2 \in QR_p
\end{array}\right.
\]

(ii) \(p \equiv 3 \pmod{4}\), we denote \(q_2 = \varphi^{-1}(l + b(0), L^*_l(1) + b(1))\), here we only discuss \(b = (0,0)\), i.e. \(q_2 = \varphi^{-1}(l, L^*_l(l))\), the other cases are similar to it, we omit here. Then

\[\text{Car}_0(q_2) = |NQR_p \cap (NQR_p + \frac{p - 1}{2})| = |2NQR_p \cap (2NQR_p + p - 1)| = |QR_p \cap (QR_p - 1)| = |NQR_p \cap (NQR_p - 1)| = \left\{\begin{array}{ll}
|NQR_p \cap (NQR_p + 1)| = \frac{p - 3}{4}, & 2 \in NQR_p \\
|QR_p \cap (QR_p + 1)| = \frac{p - 3}{4}, & 2 \in QR_p
\end{array}\right.
\]

\[\text{Car}_1(q_2) = |NQR_p \cap (QR_p \cup \{0\} + \frac{p - 1}{2})| = |2NQR_p \cap (2QR_p \cup \{0\} + p - 1)| = |QR_p \cap (QR_p \cup \{0\} - 1)| = |NQR_p \cap (QR_p \cup \{0\} - 1)| = |NQR_p \cap (QR_p \cup \{0\} + 1)| = |QR_p \cap (NQR_p \cup \{0\} + 1)| = \left\{\begin{array}{ll}
|NQR_p \cap (QR_p + 1)| = \frac{p + 1}{4}, & 2 \in NQR_p \\
1 + |QR_p \cap (NQR_p + 1)| = \frac{p + 1}{4}, & 2 \in QR_p
\end{array}\right.
\]

\[\text{Car}_2(q_2) = |QR_p \cup \{0\} \cap (QR_p \cup \{0\} + \frac{p - 1}{2})| = |2QR_p \cup \{0\} \cap (2QR_p \cup \{0\} + p - 1)|
\[
\begin{align*}
\sum_{Q} = & \left\{ |NQR_{p} \cup \{0\} \cap (NQR_{p} \cup \{0\} - 1)| \right. \\
& - \left. |QR_{p} \cup \{0\} \cap (QR_{p} \cup \{0\} - 1)| \right\} \\
& = \left\{ |QR_{p} \cup \{0\} \cap (QR_{p} \cup \{0\} + 1)| \right. \\
& \left. - |NQR_{p} \cup \{0\} \cap (NQR_{p} \cup \{0\} + 1)| \right\} \\
& = \left\{ 1 + |QR_{p} \cap (QR_{p} + 1)| = \frac{p + 1}{4}, \quad 2 \in NQR_{p} \right. \\
& \left. \right\} \\
& = \left\{ 1 + |NQR_{p} \cap (NQR_{p} + 1)| = \frac{p + 1}{4}, \quad 2 \in QR_{p} \right\} \\
\end{align*}
\]

\[Car_{3}(q_{2}) = |QR_{p} \cup \{0\} \cap (NQR_{p} + \frac{p-1}{2})| = |2QR_{p} \cup \{0\} \cap (2NQR_{p} + p - 1)| = |NQR_{p} \cup \{0\} \cap (QR_{p} - 1)| = |QR_{p} \cup \{0\} \cap (NQR_{p} - 1)| = |NQR_{p} \cup \{0\} \cap (QR_{p} + 1)| = \left\{ 1 + |QR_{p} \cap (QR_{p} + 1)| = \frac{p + 1}{4}, \quad 2 \in NQR_{p} \right. \]
\[\left. \right\} = \left\{ |NQR_{p} \cap (QR_{p} + 1)| = \frac{p + 1}{4}, \quad 2 \in QR_{p} \right\} \]

From above, if \( p \equiv 1 \pmod{4} \), we have \( \max_{Car_{3}(q_{2})} - \min_{Car_{3}(q_{2})} = \frac{p+3}{4} - \frac{p-1}{4} = 1 \leq 1 \); if \( p \equiv 3 \pmod{4} \), we have \( \max_{Car_{3}(q_{2})} - \min_{Car_{3}(q_{2})} = \frac{p+1}{4} - \frac{p-3}{4} = 1 \leq 1 \), so the quaternary sequences we constructed are balance.

4. Conclusion

In this paper, using two types of Legendre sequences and inverse mapping, we get some balanced quaternary sequences of odd period \( p \) with low autocorrelation. If \( p \equiv 1 \pmod{4} \), then \( R_{\max}(q) = (1 + \frac{(p-1)^{2}}{4})^{\frac{1}{2}} \) occurs twice; if \( p \equiv 3 \pmod{4} \), then \( R_{\max}(q) = (1 + \frac{(p+1)^{2}}{4})^{\frac{1}{2}} \) occurs twice, however, \( R_{q}(r) = -1 \) occurs \( p - 2 \) times for this two cases.

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