The Jumping Phenomenon of the Dimensions of Cohomology Groups of Tangent Sheaf

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Abstract

Let $X$ be a compact complex manifold, consider a small deformation $\phi : X \rightarrow B$ of $X$, the dimensions of the cohomology groups of tangent sheaf $H^q(X_t, T_{X_t})$ may vary under this deformation. This paper will study such phenomena by studying the obstructions to deform a class in $H^q(X, T_X)$ with the parameter $t$ and get the formula for the obstructions.

1 Introduction

Let $X$ be a compact complex manifold and $\phi : \mathcal{X} \rightarrow B$ be a family of complex manifolds such that $\phi^{-1}(0) = X$. Let $X_t = \phi^{-1}(t)$ denote the fibre of $\phi$ above the point $t \in B$. We denote by $\mathcal{O}_X$, $\Omega^p_X$ and $\mathcal{T}_X$ the sheaves of germs of $X$ of holomorphic functions, holomorphic $p$-forms and holomorphic tangent vector fields respectively. It is well known that the dimensions of cohomology groups of tangent sheaf may vary under small deformations, for example, the deformation of Hopf surfaces. In this paper, we will study such phenomena from the viewpoint of obstruction theory. More precisely, for a certain small deformation $\mathcal{X}$ of $X$ parametrized by a basis $B$ and a certain class $[\alpha]$ of the cohomology group $H^q(X, T_X)$, we will try to find out the obstruction to extending it to an element of the relative cohomology group $H^q(\mathcal{X}, T_{X/B})$. We will call those elements which have non trivial obstruction the obstructed elements.
In §2 we will summarize the results of Grauert’s Direct Image Theorems and we will try to explain why we need to consider the obstructed elements. Actually, we will see that these elements will play an important role when we study the jumping phenomenon of. Because we will see that the existence of the obstructed elements is a necessary and sufficient condition for the variation of the dimensions of cohomology groups.

**Theorem 3.4** Let \( \pi : X \to B \) be a deformation of \( \pi^{-1}(0) = X \), where \( X \) is a compact complex manifold. Let \( \pi_n : X_n \to B_n \) be the \( n \)th order deformation of \( X \). For arbitrary \( [\alpha] \) belongs to \( H^q(X, T_X) \), suppose we can extend \([\alpha]\) to order \( n - 1 \) in \( H^q(X_{n-1}, T_{X_n-1/B_{n-1}}) \). Denote such element by \([\alpha_{n-1}]\). The obstruction of the extension of \([\alpha]\) to \( n \)th order is given by:

\[
o_{n,n-1}(\alpha_{n-1}) = [\kappa_n, \alpha_{n-1}]_{rel,n-1},
\]

where \( \kappa_n \) is the \( n \)th order Kodaira-Spencer class and \([\cdot, \cdot]_{rel,n-1}\) is the Lie bracket induced from the relative tangent sheaf of the \( n - 1 \)th order deformation.

In §4 we will use this formula to study carefully the example given by Iku Nakamura, i.e. the small deformation of the Iwasama manifold and discuss some phenomenons.

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# 2 Grauert’s Direct Image Theorems and Deformation theory

In this section, let us first review some general results of deformation theory. Let \( X \) be a compact complex manifold. The manifold \( X \) has an underlying differential structure, but given this fixed underlying structure there may be many different complex structures on \( X \). In particular, there might be a range of complex structures on \( X \) varying in an analytic manner. This is the object that we will study.

**Definition 2.0** A deformation of \( X \) consists of a smooth proper morphism \( \phi : \mathcal{X} \to B \), where \( \mathcal{X} \) and \( B \) are connected complex spaces, and an isomorphism \( X \cong \phi^{-1}(0) \), where \( 0 \in B \) is a distinguished point. We call \( \mathcal{X} \to B \) a family of complex manifolds.
Although $B$ is not necessarily a manifold, and can be singular, reducible, or non-reduced, (e.g. $B = \text{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$), since the problem we are going to research is the phenomenon of the jumping of the dimensions of cohomology groups of tangent sheaf, we may assume that $X$ and $B$ are complex manifolds.

In order to study the jumping of the dimensions of cohomology groups of tangent sheaf, we need the following important theorem (one of the Grauert’s Direct Image Theorems).

**Theorem 2.1** Let $X$, $Y$ be complex spaces, $\pi : X \to Y$ a proper holomorphic map. Suppose that $Y$ is Stein, and let $\mathcal{F}$ be a coherent analytic sheaf on $X$. Let $Y_0$ be a relatively compact open set in $Y$. Then, there is an integer $N > 0$ such that the following hold.

I. There exists a complex

\[ \mathcal{E} : \ldots \to \mathcal{E}^{-1} \to \mathcal{E}^0 \to \ldots \to \mathcal{E}^N \to 0 \]

of finitely generated locally free $\mathcal{O}_{Y_0}$-modules on $Y_0$ such that for any Stein open set $W \subset Y_0$, we have

\[ H^q(\Gamma(W, \mathcal{E}')) \simeq \Gamma(W, R^q\pi_*(\mathcal{F})) \simeq H^q(\pi^{-1}(W), \mathcal{F}) \quad \forall q \in \mathbb{Z}. \]

II. (Base Change Theorem). Assume, in addition, that $\mathcal{F}$ is $\pi$-flat [i.e. $\forall x \in X$, the stalk $\mathcal{F}_x$ is flat over as a module over $\mathcal{O}_{Y, \pi(x)}$]. Then, there exists a complex

\[ \mathcal{E} : 0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \ldots \to \mathcal{E}^N \to 0 \]

of finitely generated locally free $\mathcal{O}_{Y_0}$-sheaves $\mathcal{E}^p$ with the following property:

Let $S$ be a Stein space and $f : S \to Y$ a holomorphic map. Let $X' = X \times_Y S$ and $f' : X' \to X$ and $\pi' : X' \to S$ be the two projections. Then, if $T$ is an open Stein subset of $f^{-1}(Y_0)$, we have, for all $q \in \mathbb{Z}$

\[ H^q(\Gamma(T, f^*(\mathcal{E}))) \simeq \Gamma(T, R^q\pi'_*(\mathcal{F}')) \simeq H^q(\pi'^{-1}(T), \mathcal{F}') \]

where $\mathcal{F}' = (f')^*(\mathcal{F})$.

Let $X, Y$ be complex spaces, $\pi : X \to Y$ a proper map. Let $\mathcal{F}$ be a $\pi$-flat coherent sheaf on $X$. For $y \in Y$, denote by $\mathcal{M}_y$ the $\mathcal{O}_Y$-sheaf of germs of holomorphic functions ”vanishing at $y”$: the stalk of $\mathcal{M}_y$ at $y$ is the maximal ideal of $\mathcal{O}_{Y,y}$; that at $t \neq y”$ is $\mathcal{O}_{Y,t}$. We set $\mathcal{F}(y) = \text{analytic restriction of } \mathcal{F} \text{ to } \pi^{-1}(y) = \mathcal{F} \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathcal{M}_y)$. Since we just need to study the local properties, we may assume, in view of Theorem 2.1, part II, that there is a
with the base change property in Theorem 2.1, part II. In particular, if $y \in Y$, we have

$$H^q(\pi^{-1}(y), \mathcal{F}(y)) \simeq H^q(\mathcal{E} \otimes (\mathcal{O}_Y / \mathcal{M}_y)).$$

Apply what we discussed above to our case $\phi : X \to B$, we get the following. There is a complex of vector bundles on the basis $B$, whose cohomology groups at the point identifies to the cohomology groups of the fiber $X_b$ with values in the considered vector bundle on $X$, restricted to $X_b$. Therefore, for arbitrary $p$, there exists a complex of vector bundles $(\mathcal{E}^\cdot, d^\cdot)$, such that for arbitrary $t \in B$, $H^q(X_t, \mathcal{T}_{X_t}) = H^q(E_t) = Ker(d^q)/Im(d^{q-1})$.

Via a local trivialisation of the bundle $E^i$, the differential of the complex $E^\cdot$ are represented by matrices with holomorphic coefficients, and follows from the lower semicontinuity of the rank of a matrix with variable coefficients, it is easy to check that the function $\text{dim}_C Ker(d^q)$ and $-\text{dim}_C Im(d^q)$ are upper semicontinuous on $B$. Therefore the function $\text{dim}_C H^q(E_t)$ is also upper semicontinuous. It seems that either the increasing of $\text{dim}_C Im(d^{q-1})$ or the decreasing of $\text{dim}_C Ker(d^q)$ will cause the jumping of $\text{dim}_C H^q(E_t)$, however, because of the following exact sequence:

$$0 \to Ker(d^q)_t \to E^q_t \to Im(d^q)_t \to 0 \quad \forall t,$$

which means the variation of $-\text{dim}_C Im(d^q)$ is exactly the variation of $\text{dim}_C Ker(d^q)$, we just need to consider the variation of $\text{dim}_C Ker(d^q)$ for all $q$.

In order to study the variation of $\text{dim}_C Ker(d^q)$, we need to consider the following problem. Let $\alpha$ be an element of $Ker(d^q)$ at $t = 0$, we try to find out the obstruction to extending it to an element which belongs to $Ker(d^q)$ in a neighborhood of 0. Such kind of extending can be studied order by order. Let $\mathcal{E}_0^q$ be the stalk of the associated sheaf of $E^q$ at 0. Let $m_0$ be the maximal idea of $\mathcal{O}_{B,0}$. For arbitrary positive interval $n$, since $d^q$ can be represented by matrices with holomorphic coefficients, it is not difficult to check $d^q(\mathcal{E}_0^q \otimes \mathcal{O}_{B,0} m_0^n) \subset \mathcal{E}_0^{q+1} \otimes \mathcal{O}_{B,0} m_0^n$. Therefore the complex of the vector bundles $(\mathcal{E}, d^\cdot)$ induces the following complex:

$$0 \to \mathcal{E}_0^0 \otimes \mathcal{O}_{B,0} \mathcal{O}_{B,0} / m_0^n \to \mathcal{E}_1^1 \otimes \mathcal{O}_{B,0} \mathcal{O}_{B,0} / m_0^n \to \ldots \to \mathcal{E}_N^N \otimes \mathcal{O}_{B,0} \mathcal{O}_{B,0} / m_0^n \to 0.$$
Definition 2.2 Those elements of $H^i(E_0)$ which can not be extended are called the first class obstructed elements.

Next, we will show the obstructions of the extending we mentioned above. For simplicity, my may assume that $\dim_CB = 1$, suppose $\alpha$ can be extended to an element $\alpha_{n-1}$ such that $j_0^{n-1}(d^i(\alpha_{n-1}))(t) = 0$, then $\alpha_{n-1}$ can be considered as the $n-1$ order extension of $\alpha$. Here $j_0^{n-1}(d^i(\alpha_{n-1}))(t)$ is the $n-1$ jet of $d^i(\alpha_{n-1})$ at $0$.

Define a map $\varphi^q_n : H^q(\mathcal{E}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n) \to H^{q+1}(E_0)$ by

$$[\alpha_{n-1}] \mapsto [j_0^n(d^q(\alpha_{n-1}))(t)/t^n].$$

Remark $\varphi^q_n$ is well defined [7].

There is natural a map $\rho^q_i : H^q(E_0) \to H^q(\mathcal{E}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^{i+1})$ given by

$$[\sigma] \mapsto [t^i\sigma], \forall [\sigma] \in H^q(E_0).$$

Denote the map $\rho^q_i \circ \varphi^q_n : H^q(\mathcal{E}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n) \to H^{q+1}(\mathcal{E}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^{i+1}), \forall i \leq n$ by $\varphi^q_{n,i}$.

Proposition 2.3[7] Let $\alpha_{n-1}$ be an $n-1$ order extension of $\alpha$, for arbitrary $i$, $0 < i \leq n$, $\alpha_{n-1}$ can be extended to $\alpha_n$ which is the $n$th order extension of $\alpha$ such that $j_0^{i-1}(\alpha_n - \alpha_{n-1})(t) = 0$ if and only if $\varphi^q_{n,n-1}([\alpha_{n-1}]) = 0$.

In the following, we will show that the obstructions $\varphi^q_{n}([\alpha_{n-1}])$ also play an important role when we consider about the jumping of $\dim C\text{Im}(d^q)$. Note that $\dim C\text{Im}(d^q)$ jumps if and only if there exist a section $\beta$ of $\dim C\text{Ker}(d^{q+1})$, such that $\beta_0$ is not exact while $\beta_t$ is exact for $t \neq 0$.

Definition 2.4 Those nontrivial elements of $H^i(E_0)$ that can always be extended to a section which is only exact at $t \neq 0$ are called the second class obstructed elements.

Note that if $\alpha$ is exact at $t = 0$, it can be extended to an element which is exact at every point. So the definition above does not depend on the element of a fixed equivalent class.

Proposition 2.5[7] Let $[\beta]$ be an nontrivial element of $H^{q+1}(E_0)$. Then $[\beta]$ is a second class obstructed element if and only if there exist $n \geq 0$ and $\alpha_{n-1}$ in $H^q(\mathcal{E}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n)$ such that $\varphi^q_n([\alpha_{n-1}]) = [\beta]$.

Proposition 2.6[7] Let $\alpha_{n-1}$ be an element of $H^q(\mathcal{E}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n)$ such that $\varphi^q_n([\alpha_{n-1}]) \neq 0$. Then there exists $n' \leq n$ and $\alpha'$ be an element of $H^q(\mathcal{E}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^{n'})$, such that $\rho^q_{n'} \circ \varphi^q_n([\alpha_{n-1}]) = \varphi^q_{n',n-1}([\alpha']) \neq 0$.

This proposition tells us that although $\varphi^q_n([\alpha_{n-1}]) \neq 0$ does not mean that $\varphi^q_{n,n-1}([\alpha_{n-1}]) \neq 0$, we can always find $\alpha'$ such that $\varphi^q_n([\alpha_{n-1}])$ comes from
obstructions like $o_{n,n-1}^q([\alpha])$. Therefore we can get the following corollary immediately from Proposition 2.5 and Proposition 2.6.

**Corollary 2.7** Let $[\beta]$ be a nontrivial element of $H^{q+1}(E_0)$. Then $[\beta]$ is a second class obstructed element if and only if there exist $n \geq 0$ and $\alpha_{n-1}$ in $H^q(E_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n)$ such that $o_{n,n-1}^q([\alpha_{n-1}]) = \rho_{n-1}^{q+1}([\beta])$.

Let us come back to our problem, suppose $\alpha$ can be extended to an element $\alpha_{n-1}$ such that $j^{n-1}_0(d^0(\alpha_{n-1}))(t) = 0$, since what we care is whether $\alpha$ can be extended to an element which belongs to $\ker(d^i)$ in a neighborhood of 0. So, if we have an $n$th order extension $\alpha_n$ of $\alpha$, it is not necessary that $j^{n-1}_0(\alpha_n - \alpha_{n-1})(t) = 0, \forall i, 1 < i < n$. What we need is just $j^0_0(\alpha_n - \alpha_{n-1})(t) = 0$ which means $\alpha_n$ is an extension of $\alpha$. So the “real” obstructions come from $o_{n,n-1}^q([\alpha_{n-1}])$. Since these obstructions is so important when we consider the problem of variation of the dimensions of cohomology groups of tangent sheaf, we will try to find out an explicit calculation for such obstructions in next section.

### 3 The Formula for the Obstructions

We are going to prove in this section an explicit formula (Theorem 3.4) for the abstract obstructions described above. Let $\pi : X \to B$ be a deformation of $\pi^{-1}(0) = X$, where $X$ is a compact complex manifold. For every integer $n \geq 0$, denote by $B_n = \text{Spec} \mathcal{O}_{B,0}/m_0^{n+1}$ the $n$th order infinitesimal neighborhood of the closed point $0 \in B$ of the base $B$. Let $X_n \subset X$ be the complex space over $B_n$. Let $\pi_n : X_n \to B_n$ be the $n$th order deformation of $X$. In order to study the jumping phenomenon of the cohomology groups of tangent sheaf, for arbitrary $[\alpha]$ belongs to $H^q(X, \mathcal{T}_X)$, suppose we can extend $[\alpha]$ to order $n - 1$ in $H^q(X_{n-1}, \mathcal{T}_{X_{n-1}}/B_{n-1})$. Denote such element by $[\alpha_{n-1}]$. In the following, we try to find out the obstruction of the extension of $[\alpha_{n-1}]$ to $n$th order. Denote $\pi^*(m_0)$ by $\mathcal{M}_0$. Consider the exact sequence

$$0 \to \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes_{\mathcal{O}_X} \mathcal{T}_{X_0/B_0} \to \mathcal{T}_{X_n/B_n} \to \mathcal{T}_{X_{n-1}/B_{n-1}} \to 0$$

which induces a long exact sequence

$$0 \to H^0(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_0/B_0}) \to H^0(X_n, \mathcal{T}_{X_n/B_n}) \to H^0(X_{n-1}, \mathcal{T}_{X_{n-1}/B_{n-1}}) \to H^1(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_0/B_0}) \to ....$$

The obstruction for $[\alpha_{n-1}]$ comes from the non trivial image of the connecting homomorphism $\delta^* : H^q(X_{n-1}, \mathcal{T}_{X_{n-1}/B_{n-1}}) \to H^{q+1}(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_0/B_0})$. We will calculate it by Čech calculation.
Cover $X$ by open sets $U_i$ such that, for arbitrary $i$, $U_i$ is small enough. More precisely, $U_i$ is Stein and we have the following exact sequences,

$$0 \rightarrow \mathcal{M}_0^{n+1} \otimes \mathcal{O}_X(U_i) \rightarrow \mathcal{O}_X(U_i) \rightarrow \mathcal{O}_{X_{n+1}}(U_i) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{M}_0^{n+1} \otimes \mathcal{O}_{X/B}(U_i) \rightarrow \mathcal{O}_{X/B}(U_i) \rightarrow \mathcal{O}_{X_n/B_n}(U_i) \rightarrow 0.$$

Denote the set of alternating $q$-cochains $\beta$ with values in $\mathcal{F}$ by $\mathcal{C}_q(U, \mathcal{F})$, i.e., to each $q+1$-tuple, $i_0 < i_1 \ldots < i_q$, $\beta$ assigns a section $\beta(i_0, i_1, \ldots, i_q)$ of $\mathcal{F}$ over $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_q}$.

Denote $r_{X_n}$ the restriction to the complex space $X_n$. Note that for arbitrary $U_i$, we have $[\Gamma(U_i, T_X \otimes \mathcal{O}_X \mathcal{M}_0^{n+1}), \Gamma(U_i, T_X)] \subset \Gamma(U_i, T_X \otimes \mathcal{O}_X \mathcal{M}_0^{n}), \forall n \in \mathbb{N}$. Therefore we can define the following Lie bracket of order $n$,

$$[\cdot, \cdot]_n : \Gamma(U_i, T_{X_{n+1}}) \times \Gamma(U_i, T_{X_n}) \rightarrow \Gamma(U_i, T_{X_n})$$

$$[\gamma, \beta]_n = r_{X_n} \circ [\tilde{\gamma}, \tilde{\beta}],$$

where $\tilde{\gamma}, \tilde{\beta}$ are sections of $\Gamma(U_i, T_X)$ such that their quotient images are $\gamma, \beta$.

On the other hand, we also have, $[\Gamma(U_i, T_{X/B} \otimes \mathcal{O}_X \mathcal{M}_0^{n+1}), \Gamma(U_i, T_{X/B})] \subset \Gamma(U_i, T_{X/B} \otimes \mathcal{O}_X \mathcal{M}_0^{n}), \forall n \in \mathbb{N}$. Therefore we can define the following Lie bracket of order $n$ for the relative tangent sheaves,

$$[\cdot, \cdot]_{rel,n} : \Gamma(U_i, T_{X_n/B_n}) \times \Gamma(U_i, T_{X_n/B_n}) \rightarrow \Gamma(U_i, T_{X_n/B_n})$$

$$[\gamma, \beta]_{rel,n} = r_{X_n} \circ [\tilde{\gamma}, \tilde{\beta}],$$

where $\tilde{\gamma}, \tilde{\beta}$ are sections of $\Gamma(U_i, T_{X/B})$ such that their quotient images are $\gamma, \beta$.

The relation between these two operators we defined above is the following.

**Lemma 3.0** Assume that $\gamma, \beta$ are sections of $T_{X_{n+1}}$ over $U_i$ such that, their restrictions to $X_{n-1}$ belong to $\Gamma(U_i, T_{X_{n-1/B_n-1}})$, then:

$$[\gamma, \beta]_n = [r_{X_{n-1}}(\gamma), r_{X_{n-1}}(\beta)]_{rel,n-1}.$$
and

\[ [r_{X_{n-1}}(\gamma), r_{X_{n-1}}(\beta)]_{rel,n-1} = r_{X_{n-1}} \circ [\gamma', \beta']. \]

Note that \([\Gamma(U_i, T_X \otimes_{O_X} M_{\gamma}^0), \Gamma(U_i, T_X \otimes_{O_X} M_{\beta}^0)] \subset \Gamma(U_i, T_X \otimes_{O_X} M_{\gamma}^0)\) and \([\Gamma(U_i, T_X \otimes_{O_X} M_{\beta}^0), \Gamma(U_i, T_X \otimes_{O_X} M_{\gamma}^0)] \subset \Gamma(U_i, T_X \otimes_{O_X} M_{\beta}^0)\). By the assumption, we have,

\[ r_{X_{n-1}} \circ [\gamma, \beta] - r_{X_{n-1}} \circ [\gamma', \beta'] = r_{X_{n-1}} \circ [\gamma - \gamma', \beta - \beta'] + r_{X_{n-1}} \circ [\gamma', \beta - \beta'] + r_{X_{n-1}} \circ [\gamma - \gamma', \beta - \beta'] = 0. \]

\[ \square \]

Define \([\cdot, \cdot]_n : \mathcal{C}^q(U, T_{X_{n+1}|X_n}) \times \mathcal{C}^k(U, T_{X_{n+1}|X_n}) \to \mathcal{C}^{q+k}(U, T_{X_n|X_{n-1}})\) by

\[ [\gamma, \beta]_n(i_0, i_1, ..., i_{q+k}) = [\gamma(i_0, i_1, ..., i_q), \beta(i_q, i_{q+1}, ..., i_{q+k})]_n, \]

where \(i_0 < i_1 < ... < i_{q+k}\).

We have the following lemma to describe the relation between the differential operator \(\delta\) and the operator we defined above.

**Lemma 3.1** For arbitrary \(\gamma\) (resp. \(\beta\)) belongs to \(\mathcal{C}^q(U, T_{X_{n+1}|X_n})\) (resp. \(\mathcal{C}^k(U, T_{X_{n+1}|X_n})\)) we have:

\[ \delta[\gamma, \beta]_n = [\delta(\gamma), \beta]_n + (-1)^q[\gamma, \delta(\beta)]_n. \]

**Proof.** By definition, we have:

\[
\delta[\gamma, \beta]_n(i_0, ..., i_{q+k+1}) = \sum_{j=0}^{q+k+1} (-1)^j [\gamma, \beta]_n(i_0, ..., \hat{i}_j, ..., i_{q+j+1})
\]

\[
= \sum_{j=0}^{q} (-1)^j [\gamma(i_0, ..., \hat{i}_j, ..., i_{q+1}), \beta(i_{q+1}, ..., i_{q+k+1})]_n
\]

\[
+ \sum_{j=q+1}^{q+k+1} (-1)^j [\gamma(i_0, ..., i_q), \beta(i_q, ..., \hat{i}_j, ..., i_{q+k+1})]_n,
\]

while

\[
[\delta(\gamma), \beta]_n(i_0, ..., i_{q+k+1}) = \left[ \sum_{j=0}^{q} (-1)^j \gamma(i_0, ..., \hat{i}_j, ..., i_{q+1}), \beta(i_{q+1}, ..., i_{q+k+1}) \right]_n
\]

\[
+ (-1)^q[\gamma(i_0, ..., i_q), \beta(i_q, ..., i_{q+k+1})]_n.
\]

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\[
[\gamma, \delta(\beta)]_n(i_0, \ldots, i_{q+k+1}) = [\gamma(i_0, \ldots, i_q), \sum_{j=0}^{k+1}(-1)^j \beta(i_{q+1}, \ldots, \hat{i}_{q+j}, \ldots, i_{q+k+1})]_n
\]

\[
= +[\gamma(i_0, \ldots, i_q), \beta(i_{q+1}, \ldots, i_{q+k+1})]_n
+ [\gamma(i_0, \ldots, i_q), \sum_{j=1}^{k+1}(-1)^j \beta(i_{q+1}, \ldots, \hat{i}_{q+j}, \ldots, i_{q+k+1})]_n.
\]

Now we are ready to calculate the formula for the obstructions. Let \( \tilde{\alpha} \) be an element of \( C^q(U, \mathcal{T}_{X_0/B_0}) \) such that its quotient image in \( C^q(U, \mathcal{T}_{X_{n-1}/B_{n-1}}) \) is \( \alpha_{n-1} \). Then \( \delta^*(\alpha_{n-1}) = [\delta(\tilde{\alpha})] \) which is an element of \( H^{q+1}(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_0/B_0}) \cong m_0/m_0^{n+1} \otimes H^{q+1}(X, \mathcal{T}_{X_0/B_0}). \)

Consider the following exact sequence. The connecting homomorphism of the associated long exact sequence gives the Kodaira-Spencer class of order \( n \) [4.1.3.2],

\[
0 \to \mathcal{T}_{X_{n-1}/B_{n-1}} \to \mathcal{T}_{X_n/X_{n-1}} \to \pi^*_n(\mathcal{T}_{B_n/B_{n-1}}) \to 0.
\]

In order to give the obstructions an explicit calculation, we need to consider the following map \( \rho : H^q(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_0/B_0}) \to H^q(X_{n-1}, \pi_{n-1}^*(\mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_{n-1}/B_{n-1}})) \), which is defined by

\[
\rho[\sigma] : \Gamma(X_{n-1}, \pi_{n-1}^*(\mathcal{T}_{B_n/B_{n-1}})) \to H^q(X_{n-1}, \mathcal{T}_{X_{n-1}/B_{n-1}})
\]

\[
[\tilde{\tau}] \mapsto [r_{X_{n-1}} \circ [\tilde{\tau}, \sigma]_{n+1}],
\]

where \( \tilde{\tau} \) is an element of \( C^0(U, \mathcal{T}_{X_{n+1}/X_n}) \) such that its quotient image in \( C^0(U, \pi_{n-1}^*(\mathcal{T}_{B_n/B_{n-1}})) \) is \( \tau \).

**Lemma 3.2** The map: \( \rho : H^q(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_0/B_0}) \to H^q(X_{n-1}, \pi_{n-1}^*(\mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_{n-1}/B_{n-1}})) \) is well defined.

**Proof.** At first, we need to show that if \( \sigma \) is closed, then \( r_{X_{n-1}} \circ [\tilde{\tau}, \sigma]_{n+1} \) is closed. In fact,

\[
\delta \circ r_{X_{n-1}} \circ [\tilde{\tau}, \sigma]_{n+1} = r_{X_{n-1}} \circ \delta[\tilde{\tau}, \sigma]_{n+1}
= r_{X_{n-1}} \circ [\delta(\tilde{\tau}),\sigma]_{n+1} + r_{X_{n-1}} \circ [\tilde{\tau}, \delta(\sigma)]_{n+1}
= r_{X_{n-1}} \circ [\delta(\tilde{\tau}),\sigma]_{n+1}.
\]

Note that \( r_{X_{n-1}} \circ \delta(\tilde{\tau}) \) is an element of \( C^0(U, \mathcal{T}_{X_{n+1}/B_{n-1}}) \) and \( \sigma \) is an element of \( C^q(U, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \mathcal{T}_{X_0/B_0}) \). So we have, \( r_{X_{n-1}} \circ [\delta(\tilde{\tau}),\sigma]_{n+1} = [\delta(\tilde{\tau}),\sigma]_n = \)

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[r_{X_{n-1} \circ \delta}(\bar{\tau}), r_{X_{n-1}}(\sigma)]_{rel,n-1} = 0.

Next we need to show that if \( \sigma \) is belongs to \( C^q(\mathcal{U}, \mathcal{M}_0^q/\mathcal{M}_0^{q+1} \otimes \Omega^p_{X_0/B_0}) \), then \( r_{X_{n-1}} \circ [\bar{\tau}, \delta(\sigma)] \) is exact. In fact, as the calculation above:

\[
r_{X_{n-1}} \circ [\bar{\tau}, (\delta\sigma)]_{n+1} = r_{X_{n-1}} \circ \delta [\bar{\tau}, \sigma]_{n+1} - r_{X_{n-1}} \circ [\delta(\bar{\tau}), \sigma]_{n+1} = \delta r_{X_{n-1}}[\bar{\tau}, \sigma]_{n+1}.
\]

In general, the map \( \rho \) is not injective. However, as we mentioned at the end of the previous section. The “real” obstructions are \( \theta_{n,n-1}^q([\alpha_{n-1}]) \), but not \( \rho^q([\alpha_{n-1}]) \). So we don’t need \( \rho \) to be injective. In the following, we will explain that \( \rho([\delta(\bar{\alpha})]) \) is exactly the “real” obstructions we need. In fact, \( H^q(X_{n-1}, \pi_{n-1}^*(\Omega_B|B_{n-1}) \otimes \mathcal{T}_{X_{n-1}/B_{n-1}}) = (\Omega_B|B_{n-1}) \otimes \mathcal{O}_{B_{n-1}} \).

Next we need to show that if \( \tau \) is belongs to \( \mathcal{M}_0 \), then \( [\bar{\tau}, \delta(\sigma)] \) is the \( n \)-order Kodaira-Spencer class. Let \( \bar{\tau} \) be an element of \( \mathcal{M}_0 \). For arbitrary \( \sigma \), let \( \bar{\alpha} \in H^q(X_{n-1}, \mathcal{T}_{X_{n-1}/B_{n-1}}) \). Then by a simple calculation, it is not difficult to check that \( \bar{\alpha} \) is the \( n \)-order Kodaira-Spencer class.

From the discussion above, we get the main theorem of this paper.

**Theorem 3.4** Let \( \pi : \mathcal{X} \to B \) be a deformation of \( \pi^{-1}(0) = X \), where \( X \) is a compact complex manifold. Let \( \pi_n : X_n \to B_n \) be the \( n \)-th order deformation of \( X \). For arbitrary \( [\alpha] \) belongs to \( H^q(X, \mathcal{T}_X) \), suppose we can extend \( [\alpha] \) to order \( n - 1 \) in \( H^q(X_{n-1}, \mathcal{T}_{X_{n-1}/B_{n-1}}) \). Denote such element by \( [\alpha_{n-1}] \). The obstruction of the extension of \( [\alpha] \) to \( n \)-th order is given by:

\[
o_{n,n-1}(\alpha_{n-1}) = [\kappa_n, \alpha_{n-1}]_{rel,n-1},
\]

where \( \kappa_n \) is the \( n \)-th order Kodaira-Spencer class and \( [\cdot, \cdot]_{rel,n-1} \) is the Lie bracket induced from the relative tangent sheaf of the \( n - 1 \)-th order deformation.
4 An Example

In this section, we will use the formula in previous section to study the jumping of the dimension of \( H^q(X, T_X) \) of small deformations of Iwasawa manifold. It was Kodaira who first calculated small deformations of Iwasawa manifold [2]. In the first part of this section, let us recall his result.

Set

\[
G = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\} \simeq \mathbb{C}^3
\]

\[
\Gamma = \left\{ \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} : \omega_i \in \mathbb{Z} + \mathbb{Z}\sqrt{-1} \right\}.
\]

The multiplication is defined by

\[
\begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2 + \omega_2 & z_3 + \omega_2 z_1 + \omega_3 \\ 0 & 1 & z_1 + \omega_1 \\ 0 & 0 & 1 \end{pmatrix}
\]

\( X = G/\Gamma \) is called Iwasawa manifold. We may consider \( X = \mathbb{C}^3/\Gamma \). \( g \in \Gamma \) operates on \( \mathbb{C}^3 \) as follows:

\[
z_1' = z_1 + \omega_1, \quad z_2' = z_2 + \omega_2, \quad z_3' = z_3 + \omega_1 z_2 + \omega_3
\]

where \( g = (\omega_1, \omega_2, \omega_3) \) and \( z' = z \cdot g \). There exist holomorphic 1-forms \( \varphi_1, \varphi_2, \varphi_3 \) which are linearly independent at every point on \( X \) and are given by

\[
\varphi_1 = dz_1, \quad \varphi_2 = dz_2, \quad \varphi_3 = dz_3 - z_1 dz_2,
\]

so that

\[
d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2.
\]

On the other hand we have holomorphic vector fields \( \theta_1, \theta_2, \theta_3 \) on \( X \) given by

\[
\theta_1 = \frac{\partial}{\partial z_1}, \quad \theta_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, \quad \theta_3 = \frac{\partial}{\partial z_3},
\]
It is easily seen that

\[ [\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \quad [\theta_1, \theta_3] = [\theta_2, \theta_3] = 0. \]

in view of Theorem 3 in [2], \( H^1(X, \mathcal{O}) \) is spanned by \( \varphi_1, \varphi_2 \). Since \( \Theta \) is isomorphic to \( \mathcal{O}^3 \), \( H^1(X, T_X) \) is spanned by \( \theta_i \varphi_i, i = 1, 2, 3, \lambda = 1, 2. \)

The small deformation of \( X \) is given by

\[ \psi(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \varphi_i t - (t_{11} t_{22} - t_{21} t_{12}) \theta_3 \varphi_3 t^2. \]

We summarize the numerical characters of deformations. The deformations are divided into the following three classes:

i) \( t_{11} = t_{12} = t_{21} = t_{22} = 0 \), \( X_t \) is a parallelisable manifold.

ii) \( t_{11} t_{22} - t_{21} t_{12} = 0 \) and \( (t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0) \), \( X_t \) is not parallelisable.

iii) \( t_{11} t_{22} - t_{21} t_{12} \neq 0 \), \( X_t \) is not parallelisable.

|     | \( h^0(T_X) \) | \( h^1(T_X) \) | \( h^2(T_X) \) | \( h^3(T_X) \) |
|-----|----------------|----------------|----------------|----------------|
| i)  | 3              | 6              | 6              | 3              |
| ii) | 2              | 5              | 5              | 2              |
| iii | 1              | 4              | 5              | 2              |

where \( h^q(T_X) = \dim \mathbb{C} H^q(X, T_X) \).

Now let us explain the jumping phenomenon of \( h^q(T_X), q = 0, 1, 2, 3 \) by using the obstruction formula. From Corollary 4.3 in [6], it follows that the cohomology groups of tangent sheaf are:

\[
\begin{align*}
H^0(X, T_X) &= \text{Span}\{[\theta_1], [\theta_2], [\theta_3]\}, \\
H^1(X, T_X) &= \text{Span}\{[\theta_1 \varphi_1], [\theta_2 \varphi_2]\}, i = 1, 2, 3, j = 1, 2, 3, \\
H^2(X, T_X) &= \text{Span}\{[\theta_1 \varphi_2 \wedge \varphi_3], [\theta_2 \varphi_3 \wedge \varphi_1]\}, i = 1, 2, 3, j = 1, 2, 3, \\
H^3(X, T_X) &= \text{Span}\{[\theta_1 \varphi_1 \wedge \varphi_2 \wedge \varphi_3]\}, i = 1, 2, 3.
\end{align*}
\]

For example, let us first consider \( h^0(T_X) \), in the ii) class of deformation. The Kodaira-Spencer class of this deformation is \( \psi_1(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \varphi_i \), with \( t_{11} t_{22} - t_{21} t_{12} = 0 \). It is easy to check that \( o_1(\theta_3) = [\psi_1(t), \theta_3] = 0, o_1(t_{11} \theta_1 + t_{21} \theta_2) = [t_{11} \theta_1 \varphi_1 + t_{12} \theta_1 \varphi_2 + t_{21} \theta_2 \varphi_1 + t_{22} \theta_2 \varphi_2, t_{11} \theta_1 + t_{21} \theta_2] = t_{21} t_{11} \theta_3 \varphi_1 + t_{21} t_{12} \theta_3 \varphi_2 - t_{11} t_{21} \theta_3 \varphi_1 - t_{11} t_{22} \theta_3 \varphi_2 = 0, \) and \( o_1(\theta_2) = t_{11} t_{22} \theta_1 + t_{12} \theta_2 \varphi_2, o_1(\theta_1) = -t_{21} \theta_3 \varphi_1 + t_{22} \theta_3 \varphi_2. \) Therefore, we have shown that for an element of the subspace \( \text{Span}\{[\theta_3], [t_{11} \theta_1 + t_{21} \theta_2]\} \), the first order obstruction
is trivial, while, since \((t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)\), at least one of the obstruction \(o_1(\theta_2)\), \(o_1(\theta_1)\) is non trivial which partly explain why the dimension \(h^0(T_X)\) jumps from 3 to 2. For another example, let us consider \(h^2(T_X)\), in the iii) class of deformation. It is easy to check that for an element of the subspace (the dimension of such a subspace is 5) \(\text{Span}\{[\theta_3 \varphi_2 \wedge \varphi_3], [\theta_3 \varphi_3 \wedge \varphi_1], [t_{22}\theta_1 \varphi_2 \wedge \varphi_3 - t_{21}\theta_1 \varphi_3 \wedge \varphi_1], [t_{12}\theta_2 \varphi_2 \wedge \varphi_3 - t_{11}\theta_2 \varphi_3 \wedge \varphi_1], [t_{11}\theta_1 \varphi_2 \wedge \varphi_3 + t_{21}\theta_2 \varphi_2 \wedge \varphi_3]\}\), the first order obstruction is trivial, while at least one of the obstruction \(o_1(\theta_1 \varphi_2 \wedge \varphi_3), o_1(\theta_1 \varphi_3 \wedge \varphi_1), o_1(\theta_2 \varphi_2 \wedge \varphi_3), o_1(\theta_2 \varphi_3 \wedge \varphi_1)\) is non trivial.

**Remark** It is easy to see that, in the ii) (resp. iii)) class of deformation, the first order obstruction for any element in \(H^1(X, T_X)\) is trivial. The reason of number \(h^1(T_X)\)’s jumping from 6 to 5 (resp. 4) comes from the existence of the second class obstructed elements \(o_1(\theta_2)\) or \(o_1(\theta_1)\) (resp. \(o_1(\theta_2)\) and \(o_1(\theta_1)\)). After simple calculation, it is not difficult to get the following equation of \(X_t, t \neq 0\).

\[
\begin{cases}
\overline{\partial} \theta_1 = t o_1(\theta_1), \\
\overline{\partial} \theta_2 = t o_1(\theta_2), \\
\overline{\partial} \theta_3 = 0,
\end{cases}
\]

which can be considered an example of proposition 2.5.

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