Linear-time calculation of the expected sum of edge lengths in planar linearizations of trees

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Dependency graphs have proven to be a very successful model to represent the syntactic structure of sentences of human languages. In these graphs, widely accepted to be trees, vertices are words and arcs connect syntactically-dependent words. The tendency of these dependencies to be short has been demonstrated using random baselines for the sum of the lengths of the edges or its variants. A ubiquitous baseline is the expected sum in projective orderings (wherein edges do not cross and the root word of the sentence is not covered by any edge). It was shown that said expected value can be computed in \(O(n)\) time. In this article we focus on planar orderings (where the root word can be covered) and present two main results. First, we show the relationship between the expected sum in planar arrangements and the expected sum in projective arrangements. Second, we also derive a \(O(n)\)-time algorithm to calculate the expected value of the sum of edge lengths. These two results stem from another contribution of the present article, namely a characterization of planarity that, given a sentence, yields either the number of planar permutations or an efficient algorithm to generate uniformly random planar permutations of the words. Our research paves the way for replicating past research on dependency distance minimization using random planar linearizations as random baseline.

1. Introduction

A successful representation of the structure of a sentence in natural language is a (labeled) graph indicating the syntactic relationships between words together with the encoding of the words’ order. In such a graph, the edge labels indicate the type of syntactic relationship between the words. Such combination of graph and linear ordering, as in Figure 1, is known as syntactic dependency structure (Nivre 2006). When the graph is (1) well-formed, namely, the graph is weakly connected, (2) is acyclic, that is, there are no cycles in the graph, (3) is single-headed, and (4) there is only one root word in the graph, then it is called a syntactic dependency tree (Nivre 2006). There exist formal constraints that are often imposed on dependency structures. One such constraint is projectivity: a dependency structure is projective if, for every vertex \(v\), all vertices reachable from \(v\) in the underlying graph form a continuous substring within the sentence (Kuhlmann and Nivre 2006) and the root word of the sentence is never covered (as in Figure 1(a)).

Another formal constraint is planarity, a generalization of projectivity where the root is allowed to be covered by one or more of the edges (as in Figure 1(b)). Figure 1(c) shows a sentence that is neither projective nor planar.

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In this article, we study statistical properties of syntactic dependency structures under the planarity constraint. Such structures are represented in this article as a pair consisting of a (free or rooted) tree and a linear arrangement of its vertices. Free trees are denoted as $T = (V, E)$, and rooted trees as $T^r = (V, E; r)$, where $V$ is the set of vertices, $E$ the set of edges, and $r \in V$ denotes the root vertex. Unless stated otherwise $n = |V|$, that is, $n$ denotes the number of nodes which equals the number of words in the sentence. A linear arrangement $\pi$ (also called embedding) of a tree is a function that maps every vertex $v \in V$ of a tree to a position $\pi(v)$. The positions of vertices are pairwise distinct and are considered to be in $[1, n]$, that is, $\pi(u) \in [1, n] = \{1, \cdots, n\}$.

Projectivity, as well as planarity, can be alternatively defined on linear arrangements using the concept of edge crossing. We say that any two (undirected) edges $\{s, t\}, \{u, v\}$ such that, without loss of generality, $\pi(s) < \pi(t)$, $\pi(u) < \pi(v)$ and $\pi(s) < \pi(u)$, cross in the linear ordering defined by $\pi$ if $\pi(s) < \pi(u) < \pi(t) < \pi(v)$. We denote the total number of edge crossings in an arrangement $\pi$ as $C_\pi(T)$. Then, an arrangement $\pi$ of a rooted tree $T^r$ is planar if $C_\pi(T^r) = 0$ and is projective if (a) it is planar and (b) the root of the tree is not covered, that is, there is no edge $(s, t)$ such that $\pi(s) < \pi(r) < \pi(t)$ or $\pi(t) < \pi(r) < \pi(s)$.

Planarity (Kuhlmann and Nivre 2006; Sleator and Temperley 1993) is a relaxation of projectivity where the root can be covered, that is, an arrangement $\pi$ of a free tree $T$ is planar if, more simply, $C_\pi(T) = 0$. Planar arrangements are also known in the literature as one-page book embeddings (Bernhart and Kainen 1979).

In this article, the main object of study is the expectation of the sum of edge lengths (or syntactic dependency distances) in planar arrangements of free trees. The length of an edge connecting two words, also known as dependency distance, is usually defined as

$$\delta_{uv}(\pi) = |\pi(u) - \pi(v)| \quad (1)$$

In words, the length of an edge connecting two syntactically-related words $u$ and $v$, or the syntactic dependency distance between them, is the number of intervening words inbetween $u$ and $v$ in the sentence plus 1 (Figure 1). More precisely, we define the total

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1 Notice that this notion of crossing does not depend on edge orientation.

2 Another popular definition is $\delta_{uv}(\pi) = |\pi(u) - \pi(v)| - 1$ (Liu, Xu, and Liang 2017).
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Edge lengths in random planar linearizations

Figure 2: Examples of sentences with their syntactic dependency structures; arc labels indicate dependency distance. The rectangles denote the root word in each sentence. Examples adapted from (Morrill 2000).

sum of edge lengths in \( \pi \) as

\[
D_\pi(T) = \sum_{u \in E} \delta_{uv}(\pi)
\]

Close attention has been paid to this metric in modern linguistic research since its causal relationship with cognitive cost was, to the best of our knowledge, first put forward by Hudson (1995). The main causal argument is that the longer the dependency, the greater the memory burden arising from decay of activation and interference (Hudson 1995; Liu, Xu, and Liang 2017). A number of studies have exposed the general tendency in languages to reduce \( D \), the total sum of edge lengths, a reflection of a potentially universal cognitive force known as the Dependency Distance Minimization principle (DDm) (Ferrer-i-Cancho 2004; Liu 2008; Futrell, Mahowald, and Gibson 2015; Liu, Xu, and Liang 2017; Ferrer-i-Cancho et al. 2022). As an example of such cognitive cost, consider the sentences in Figures 2(a) and 2(b): it is not surprising that the latter sentence is preferred over the former due to smaller total sum of edge lengths (Morrill 2000), the former’s being \( D = 18 \) and the latter’s being \( D = 12 \).

Statistical evidence of the DDm principle has been provided showing that dependency distances are smaller than expected by chance in syntactic dependency treebanks (Ferrer-i-Cancho 2004; Liu 2008; Park and Levy 2009; Gildea and Temperley 2010; Futrell, Mahowald, and Gibson 2015; Liu, Xu, and Liang 2017; Ferrer-i-Cancho et al. 2022; Kramer 2021). Typically, the random baseline is defined as a random shuffling of the words of a sentence. To the best of our knowledge, the first known instance of such an approach was done by Ferrer-i-Cancho (2004) who established the DDm principle by comparing the average real \( D(T) \) of sentences against its expected value in a uniformly random permutation of their words. More formally, Ferrer-i-Cancho (2004) calculated the expected value of \( D(T) \) when the words of the sentence are shuffled uniformly at random (u.a.r.), being all \( n! \) permutations equally likely, that is

\[
E[D(T)] = \frac{n^2 - 1}{3}
\]

In spite of the simplicity of the previous random baseline, the majority of researchers have used as random baseline \( E_{pr}[D(T^r)] \), the expected sum of edge lengths constrained to projective arrangements (Temperley 2008; Park and Levy 2009; Gildea and Temperley 2010; Futrell, Mahowald, and Gibson 2015; Kramer 2021). However, \( E_{pr}[D(T^r)] \) has been computed approximately via random sampling of projective arrangements. For these reasons, a formula to calculate the exact value of \( E_{pr}[D(T^r)] \) in
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linear time, has been derived (Alemany-Puig and Ferrer-i-Cancho 2022)

\[
E_{pr}[D(T^r)] = \frac{1}{6} \sum_{v \in V} s_r(u)(2d^+_r(u) + 1) - \frac{1}{6}
\]  

(4)

where \( s_r(u) \) denotes the size (in vertices) of the subtree of \( T^r \) rooted at \( u \), and \( d^+_r(u) \) is the out-degree of \( u \) in \( T^r \). In spite of its extensive use, the projective random baseline has some limitations. First, the percentage of non-projective sentences in languages ranges between 18.2 and 26.4 (Gómez-Rodríguez 2016) and raises the question if the projective baseline should be used for sentences that are not projective as it is customary in research on dependency distance minimization. In addition, projectivity per se implies a reduction in dependency distances, which raises the question if that rather strong constraint may mask the effect of the dependency distance minimization principle under investigation (Gómez-Rodríguez, Christiansen, and Ferrer-i-Cancho 2022). Here we aim to make a step forward by considering planarity, a generalization of projectivity, so as to increase the coverage of real sentences and reduce the bias towards dependency minimization in the random baseline. The percentage of non-planar sentences in languages ranges between 14.3 and 20.0 (Gómez-Rodríguez and Ferrer-i-Cancho 2017).

This article is part of a research program on the statistical properties of \( D(T) \) under constraints on the possible linear arrangements (Ferrer-i-Cancho 2019; Alemany-Puig, Esteban, and Ferrer-i-Cancho 2022; Alemany-Puig and Ferrer-i-Cancho 2022). The remainder of the article is structured as follows. We introduce notation used throughout the article in Section 2. In Section 3, we first present a characterization of planar arrangements so as to identify the underlying structure of such arrangements that the we apply to count their number for a given free, and later on in Section 4, to generate them u.a.r. by means of a novel \( O(n) \)-time algorithm. In Section 5, we use said characterization to prove the main result of the article, namely that expectation of \( D(T) \) in planar arrangements can be calculated from the expectation of projective arrangements, as the following theorem indicates.

Theorem 1

Given a free tree \( T = (V,E) \),

\[
E_{pl}[D(T)] = \frac{1}{n} \sum_{u \in V} E_{pr}^*[D(T^u) | u] \]

(5)

\[
= \frac{(n-1)(n-2)}{6n} + \frac{1}{n} \sum_{u \in V} E_{pr}[D(T^u)]
\]

(6)

where \( E_{pr}^*[D(T^u) | u] \) is the expected value of \( D(T^u) \) in uniformly random projective arrangements \( \pi \) of \( T^u \) such that \( \pi(u) = 1 \) and \( E_{pr}[D(T^u)] \) (Equation 4) is the expected value of \( D(T^u) \) in uniformly random projective arrangements of \( T^u \), the free tree \( T \) rooted at \( u \).

In Section 5, we also apply Theorem 1 to derive a \( O(n) \)-time algorithm to calculate \( E_{pl}[D(T)] \). Since Alemany-Puig and Ferrer-i-Cancho (2022) showed that \( E_{pr}[D(T^r)] \) can be evaluated in \( O(n) \) time, Equation 6 naturally leads to a \( O(n^2) \)-time algorithm if it is evaluated ‘as is’. However, we devise a \( O(n) \)-time algorithm to calculate \( E_{pl}[D(T)] \). Table 1 summarizes the results obtained in previous articles and those presented in this article. Both methods to generate planar arrangements and the \( O(n) \)-time calculation
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|                     | Projective (T^r)                      | Planar (T)                  | Unconstrained (T)          |
|---------------------|---------------------------------------|-----------------------------|---------------------------|
|                     | N_{pr}(T^r)                           | N_{pl}(T)                   | N(T)                      |
|                     | E_{pr}[\delta_{uv}]                  | E_{pl}[\delta_{uv}]        | E[\delta_{uv}]            |
|                     | \prod_{u \in V}(d^+_r(u) + 1)!       | n \prod_{u \in V} d(u)!    | n!                        |
|                     | 2s_r(u) + s_r(v) + 1)/6               | 1 + 1/n \sum_{s \in V \setminus \{u,v\}} E_{pl}[\delta_{uv} | s] | (n+1)/3                    |
|                     | (-1 + \sum_{v \in V} s_r(v)(2d^+_r(v) + 1))/6 | (n-1)(n-2)/6n + 1/n \sum_{u \in V} E_{pr}[D(T^u)] | (n^2 - 1)/3 |

Table 1: Summary of the main mathematical results presented in this article for the planar case along with previous results for the unconstrained and projective case from previous research. N_{pr}(T^r) is the number of distinct projective linear arrangements of a rooted tree. N_{pl}(T) and N(T) are, respectively, the number of distinct unconstrained and planar linear arrangements of a free tree. E_{pr}[\delta_{uv}], E_{pl}[\delta_{uv}] and E[\delta_{uv}] are the expected length of an edge in random linear arrangement for the projective, planar and unconstrained case, respectively. E_{pr}[\delta_{uv} | s] is E_{pr}[\delta_{uv}] conditioned to having vertex s as root of the tree.

of E_{pl}[D(T)] are freely available in the Linear Arrangement Library (Alemany-Puig, Esteban, and Ferrer-i-Cancho 2021).

2. Definitions and notation

In this article, we denote free trees as T = (V, E), and rooted trees as T^r = (V, E; r). We use u, v, w, z to denote vertices, r always denotes a root vertex, and i, j, k, p, q to denote integers. The edges of a free tree are undirected, and denoted as \{u, v\} = uv; those of rooted trees are directed and oriented away from r towards the leaves.

Let \Gamma(u) = \Gamma_T(u) denote the set of neighbors of u \in V in the free tree T, and let \Gamma^+_r(u) = \Gamma^+_T(u) denote the out neighbors (also, children) of u \in V in T^r. Notice that, \Gamma^+_r(u) \subseteq \Gamma(u) with equality if, and only if u = r. Let d^+_r(u) = d^+_T(u) = |\Gamma^+_r(u)| denote the (out) degree of vertex u of a rooted tree T^r, and let d(u) = d_T(u) = |\Gamma(u)| denote the degree of u in a free tree T. Notice that d^+_r(u) = d(u) - 1 when u \neq r and d^+_r(r) = d(r).

Furthermore, we denote the subtree rooted at v with respect to root u as T^r_v and its size as s_u(v) = s_{T^r_v}(v) = |V(T^r_v)| (Figure 3). Notice that T^r_v = T^r. We call this directional size (Hochberg and Stallmann 2003; Alemany-Puig, Esteban, and Ferrer-i-Cancho 2022). Note that s_u(u) + s_u(v) = n for any uv \in E.

As in previous research, we decompose an edge (r, u) in a projective arrangement \pi into two parts: its anchor and its coanchor, as in Figure 4 (Shiloach 1979; Chung 1984; Alemany-Puig and Ferrer-i-Cancho 2022). The length of an edge connecting r with u is
Figure 3: a) A free tree $T$, where $d(u) = 4$, and $d(v) = 5$; in this tree, $s_u(v) = 5$ and $s_v(u) = 4$. b) The free tree $T$ rooted at $u$, denoted as $T^u$, where $d^+_{T^u}(u) = d^+_{T^u}(v) = d(u) = 4$, and where $4 = d^+_{T^u}(v) = d^+_{T^u}(v) < d(v) = 5$. Figure borrowed from (Hochberg and Stallmann 2003; Alemany-Puig, Esteban, and Ferrer-i-Cancho 2022).

Figure 4: Illustration of an edge’s anchor $\alpha_{ru}(\pi)$ and coanchor $\beta_{ru}(\pi)$. In this figure, $u, v, w \in \Gamma^+_r(r)$. Figure adapted from (Alemany-Puig and Ferrer-i-Cancho 2022).

$$\delta_{ru}(\pi) = |\pi(r) - \pi(u)| = \alpha_{ru}(\pi) + \beta_{ru}(\pi),$$ where $\alpha_{ru}(\pi)$ is the length of the anchor and $\beta_{ru}(\pi)$ is the length of the coanchor.

Formally,

$$\alpha_{ru}(\pi) = |\pi(u) - \pi(z)| + 1 \quad (7)$$

$$\beta_{ru}(\pi) = |\pi(z) - \pi(r)| - 1 \quad (8)$$

where $z \in T^+_u$ is the vertex of $T^+_u$ closest to $r$ in $\pi$. The same notation with $\pi$ omitted, $\alpha_{ru}$ and $\beta_{ru}$ denote random variables. Furthermore, it will be useful to use the operator $\ast$ to denote sets of arrangements where the root is fixed at the first position of a linear ordering.

Finally, in this article we consider that two arrangements $\pi$ and $\pi'$ of the same tree $T$ are different if there is one vertex $u$ for which $\pi(u) \neq \pi'(u)$.

3. Counting planar arrangements

It is well known that the number of unconstrained arrangements of an $n$-vertex tree is $n!$. Building on the fact that projective arrangements span over contiguous intervals (Kuhlmann and Nivre 2006), Alemany-Puig and Ferrer-i-Cancho (2022) studied the expected value of the random variable $D(T^r)$ in such arrangements by defining, as usual, a set of segments $\Phi_u$ associated to each vertex $u$. For a vertex $u$, $\Phi_u$ is constructed from vertex $u$’s segment and the segments of its children, $\Gamma^+_u(u) = \{u_1, \ldots, u_k\}$, (Figure 5) each constructed with its corresponding permutation $\Phi_{u_1}, \ldots, \Phi_{u_k}$. Decomposing every vertex and its segments from the root to the leaves linearizes $T^r$ into a projective...
arrangement (Figure 5). This characterization led to a straightforward derivation of the total amount of projective arrangements of a rooted \( T^r \) (Table 1)

\[
N_{pr}(T^r) = \prod_{u \in V} (d_u^+(u) + 1)!
\]  

Using the structure of segments summarized above, we present a characterization of planar arrangements of free trees which helps to devise a method to generate planar arrangements u.a.r. (Section 4.2) and to prove Theorem 1 (Section 5).

**Proposition 1**

The number of planar arrangements of an \( n \)-vertex free tree \( T = (V, E) \), with \( V = \{u_1, \cdots, u_n\} \) is

\[
N_{pl}(T) = nN_{pr}^{*}(T^{u_1}) = \cdots = nN_{pr}^{*}(T^{u_n}) = n \prod_{u \in V} d(u)!
\]  

where \( N_{pr}^{*}(T^r) = |P_{pr}^{*}(T^r)| \) and \( P_{pr}^{*}(T^r) \) is the set of projective arrangements of a rooted tree \( T^r \) such that \( \pi(r) = 1 \).

**Proof.** Given a free tree \( T \), and any two distinct vertices \( u, v \), it holds that \( P_{pr}^{*}(T^u) \cap P_{pr}^{*}(T^v) = \emptyset \) because the vertex in the first position is never the same (recall that we consider arrangements to be of labeled trees). It is easy to see that when a vertex \( u \) is fixed to the leftmost position, the planar arrangements in \( P_{pr}^{*}(T^u) \) are obtained by arranging the subtrees \( T^v, v \in \Gamma(u) \), projectively to the right of \( u \) in the linear arrangement. This lets us partition \( P_{pl}(T) \) so that

\[
N_{pl}(T) = \sum_{u \in V} N_{pr}^{*}(T^u)
\]
It is easy to see that
\[ N_{\text{pr}}^*(T_u) = d(u)! \prod_{v \in \Gamma(u)} N_{\text{pr}}(T_u^v) = \prod_{v \in V} d(v)! \] (12)

We used Equation 9 in the second equality. Notice that, \( N_{\text{pr}}^*(T_{u1}) = \cdots = N_{\text{pr}}^*(T_{un}) \). Therefore, Equation 10 follows immediately.

Perhaps not so surprisingly, there are more planar arrangements of a free tree \( T \) than projective arrangements of any ‘rooting’ \( T_r \) of \( T \), formally \( N_{\text{pl}}(T) \geq N_{\text{pr}}(T_r) \). We can see this by noticing that, when given a ‘rooting’ of \( T \) at \( r \in V \),
\[ \frac{N_{\text{pl}}(T)}{N_{\text{pr}}(T_r)} = \frac{nd(r)! \prod_{u \in V \setminus \{r\}} d(u)!}{(d(r) + 1)! \prod_{u \in V \setminus \{r\}} d(u)!} = \frac{n}{d(r) + 1} \geq 1 \] (13)
with equality when \( T \) is a star tree and \( r \) is its vertex of highest degree.

4. Generating arrangements uniformly at random

Arrangements can be generated freely, imposing no constraint on the possible orderings, hence being all the \( n! \) possible orderings equally likely or imposing some constraint on the possible orderings. As for the former, the method is straightforward: it is well known that \( n \) elements can be permuted u.a.r. in \( O(n) \) time (Cormen et al. 2001). As for the latter, one of the most widely used constraints has been projecticity (Gildea and Temperley 2007; Liu 2008; Futrell, Mahowald, and Gibson 2015). First, we present a \( O(n) \)-time procedure to generate projective arrangements u.a.r. and review methods used in past research (Section 4.1). Then we present a novel \( O(n) \) procedure to generate planar arrangements u.a.r. (Section 4.2) which in turn involves the generation of random projective arrangements of a subtree.

4.1 Generating projective arrangements

The method we will present in detail here was outlined first by Futrell, Mahowald, and Gibson (2015). Here we borrow from recent theoretical research (Alemany-Puig and Ferrer-i-Cancho 2022) to derive a detailed algorithm to generate projective arrangements and prove its correctness.

In order to generate projective arrangements u.a.r., simply make random permutations of a vertex \( u \) and its children \( \Gamma_r^+(u) \), that is, choose one of the possible \((d_r^+(u) + 1)!\) permutations u.a.r. Algorithm 4.1 formalizes this brief description. The proof that Algorithm 4.1 produces projective arrangements of a rooted tree \( T_r \) u.a.r. is simple. The first call takes the root and its dependents and produces a uniformly random permutation with probability \( 1/(d_r^+(r) + 1)! \). Subsequent recursive calls (in Algorithm 4.2) produce the corresponding permutations each with its respective uniform probability, hence the probability of producing a particular permutation is the product of individual probabilities. Using Equation 9, we easily obtain that the probability of producing a certain projective arrangement is
\[ \prod_{u \in V} \frac{1}{(d_r^+(u) + 1)!} = \frac{1}{N_{\text{pr}}(T_r)} \] (14)
Algorithm 4.1: Generating projective arrangements u.a.r.

```plaintext
1 Function RANDOM_PROJECTIVE_ARRANGEMENT(Tr) is
   Input: Tr a rooted tree.
   Output: A projective arrangement π of Tr chosen u.a.r.
   π ← empty n-vertex arrangement
   // Algorithm 4.2
   RANDOM_PROJECTIVE_ARRANGEMENT_SUBTREE(Tr, r, 1, π)
   return π
```

Algorithm 4.2: Generating projective arrangements u.a.r. of a subtree.

```plaintext
1 Function RANDOM_PROJECTIVE_ARRANGEMENT_SUBTREE(Tr, u, p, π) is
   Input: Tr a rooted tree, u any vertex of Tr, p the starting position to arrange
   the vertices of T_u, π partially-constructed without T_u.
   Output: π partially-constructed with T_u.
   Φ_u ← a random permutation of Γ⁺_r(u) ∪ {u}
   for v ∈ Φ_u do
      if v = u then
         π(v) ← a
         p ← p + 1
      else
         RANDOM_PROJECTIVE_ARRANGEMENT_SUBTREE(Tr, v, p, π)
         p ← p + s_r(v)
   return π
```

4.1.1 Generation of projective arrangements in past research. Algorithm 4.1 is equivalent to the “fully random” method used by Futrell, Mahowald, and Gibson (2015) as witnessed by the implementation of their code available on Github, in particular in file cliqs/mindep.py (function _randlin_projective). Notice that Futrell, Mahowald, and Gibson (2015) outline (though vaguely) that a projective arrangement is generated randomly by “Starting at the root node of a dependency tree, collecting the head word and its dependents and order them randomly”.

Futrell, Mahowald, and Gibson (2015) present their method to generate random projective arrangements as though it was the same as that by Gildea and Temperley (2007, 2010), who introduced a method to generate random linearizations of a tree that consists of “simply by choosing a random branching direction for each dependent of each head, and – in the case of multiple dependents on the same side – randomly ordering them in relation to the head” (Gildea and Temperley 2010). However, Futrell, Mahowald, and Gibson (2015) do not actually implement Gildea & Temperley’s method as witnessed by their code. Critically, Gildea & Temperley’s method does not produce uniformly random linearizations as we show with a counterexample.

4 That is, as explained by Temperley and Gildea (2018), “choose a random assignment of each dependent to either the left or the right of its head.”
Consider a star tree rooted at its hub. Let $X$ be a random variable for the position of the root in a random projective linear arrangement ($1 \leq X \leq n$). We have $\Pr(X = x) = 1/n$ for all $x \in [1, n]$, therefore $X$ follows a uniform distribution and hence $E[X] = (n+1)/2$ and $\text{Var}[X] = (n^2 - 1)/12$ (Mitzenmacher and Upfal 2017). Let $X'$ be a random variable for the position of the root according to Gildea & Temperley’s method.

It is easy to see that $X' - 1$ follows a binomial distribution with parameters $n - 1$ and $1/2$. Namely, $\Pr(X' - 1 = x) = \binom{n-1}{x}/2^{n-1}$. We have that $E[X'] = 1 + E[X' - 1] = (n+1)/2 = E[X]$, but $\text{Var}[X'] = \text{Var}[X' - 1] = (n-1)/4$. Therefore, the variance in a truly uniformly random projective linear arrangement is $\Theta(n^2)$ while Gildea & Temperley’s method results in $\Theta(n)$, a much smaller dispersion. As $n \to \infty$, $X' - 1$ converges to a Gaussian distribution.

Gildea & Temperley’s method was introduced as a random baseline for the distance between syntactically-related words in languages and has been used with that purpose (Gildea and Temperley 2007, 2010; Temperley and Gildea 2018). Interestingly, the minimum baseline, namely, the minimum sum of dependency distances, results from placing the root at the center (Shiloach 1979; Chung 1984). The example above shows that Gildea & Temperley’s baseline tends to put the root at the center of the linear arrangement with higher probability than the truly uniform baseline. That behavior casts doubts on the power of that random baseline to investigate dependency distance minimization in languages since it tends to place the root at the center of the sentence, as expected from an optimal placement under projectivity (Gildea and Temperley 2007; Alemany-Puig, Esteban, and Ferrer-i-Cancho 2021) and does it with much lower dispersion around the center than in truly uniformly random linearizations.

4.2 Generating planar arrangements

Proposition 1 sheds light on a method to generate planar arrangements u.a.r. for a free tree $T$. The method we propose is detailed in Algorithm 4.3; notice that its cost is $O(n)$. Firstly, choose a vertex, say $u \in V$, u.a.r., and place it at one of the arrangement’s ends, say, the leftmost position; this vertex acts as a root for $T$. Secondly, choose u.a.r. one of the $d(u)!$ permutations of the segments of the subtrees $T^u_v$ u.a.r. Lastly, recursively choose u.a.r. a projective linearization of every subtree $T^u_v$ for $v \in \Gamma(u)$ (Algorithm 4.2). These steps generate a planar arrangement u.a.r. since the probability of producing a certain planar arrangement following these steps is, then,

$$\frac{1}{n} \cdot \frac{1}{d(u)!} \cdot \prod_{v \in \Gamma(u)} \frac{1}{N_{\text{pr}}(T^u_v)} = \frac{1}{n} \cdot \frac{1}{d(u)!} \cdot \prod_{v \in V \setminus \{u\}} \frac{1}{d(v)!} = \frac{1}{N_{\text{pl}}(T)}$$

(15)

The equalities follow from Proposition 1.

5. Expected sum of edge lengths

In this section we derive an arithmetic expression for $E_{\text{pl}}[D(T)]$ in Section 5.1, and an algorithm to evaluate it in $O(n)$ time in Section 5.2.

5.1 An arithmetic expression for $E_{\text{pl}}[D(T)]$

We now prove Theorem 1. To this aim, we define $E_{\text{pl}}[\alpha_{uv} | r] = E_{\text{pr}}[\alpha_{uv} | \pi(r) = 1]$ as the expected value of $\alpha_{uv}$ conditioned to the projective arrangements $\pi$ of $T^r$ such that
Algorithm 4.3: Generating planar arrangements u.a.r.

```plaintext
Function RANDOM_PLANAR_ARRANGEMENT(T) is
    Input: T a free tree.
    Output: A planar arrangement π of T chosen u.a.r.
    1
    2 π ← empty n-vertex arrangement
    3 u ← a vertex of T chosen u.a.r.
    4 π(u) ← 1
    5 Φ_u ← a random permutation of Γ(u)
    6 p ← 2
    7 for v ∈ Φ_u do
        // Algorithm 4.2
        8 R O M A N D _P R O J E C T I V E _A R R A N G E M E N T _ S U B T R E E(T_u, v, p, π)
        9 p ← p + s_u(v)
    10 return π
```

π(r) = 1; we define $E^*_{pr} [\beta_{uv} \mid r]$ likewise. The root is specified as a parameter of the expected value because we want to be able to use various roots.

**Proof of Theorem 1.** We first prove Equation 5. By the Law of Total Expectation,

$$E_{pl} [D(T)] = \sum_{u \in V} E_{pl} [D(T) \mid \pi(u) = 1] \mathbb{P}_{pl} (\pi(u) = 1)$$

(16)

It is easy to see that $\mathbb{P}_{pl}(\pi(u) = 1) = 1/n$ is the proportion of planar arrangements of T in which $\pi(u) = 1$ (Proposition 1). By noting that

$$E_{pl} [D(T) \mid \pi(u) = 1] = E_{pr} [D(T^u) \mid \pi(u) = 1] = E^*_{pr} [D(T^u) \mid u]$$

(17)

Equation 5 follows.

Now we write $E^*_{pr} [D(T^u) \mid u]$ as a function of $E_{pr} [D(T^u)]$. By decomposing all edges $uv$ of $T^u$ into anchor and coanchor, we get

$$E^*_{pr} [D(T^u) \mid u] = \sum_{v \in \Gamma_T(u)} \left( E^*_{pr} [\alpha_{uv} + \beta_{uv} \mid u] + E_{pr} [D(T^v)] \right)$$

(18)

By linearity of expectation and the fact that $E^*_{pr} [\alpha_{uv} \mid u] = E_{pr} [\alpha_{uv}]$,

$$E^*_{pr} [D(T^u) \mid u] = \sum_{v \in \Gamma_T(u)} \left( E_{pr} [\alpha_{uv}] + E^*_{pr} [\beta_{uv} \mid u] + E_{pr} [D(T^v)] \right)$$

(19)

The next step is to find the value of $E^*_{pr} [\beta_{uv} \mid u]$. The derivation has been moved to the Appendix since it is merely an adaptation of the proof of (Alemany-Puig and Ferrer-i-Cancho 2022, Lemma 1); it gives $E^*_{pr} [\beta_{uv} \mid u] = 3E_{pr} [\beta_{uv}] / 2$. Thus,

$$E^*_{pr} [D(T^u) \mid u] = \sum_{v \in \Gamma_T(u)} \left( E_{pr} [\alpha_{uv}] + \frac{3}{2} E_{pr} [\beta_{uv}] + E_{pr} [D(T^v)] \right)$$

(20)
\[ \sum_{v \in \Gamma_T(u)} \left( E_{pr}[\delta_{uv}] + E_{pr}[D(T^w)] + \frac{1}{2} E_{pr}[\beta_{uv}] \right) \]

(21)

\[ = E_{pr}[D(T^w)] + \frac{1}{2} \sum_{v \in \Gamma_T(u)} E_{pr}[\beta_{uv}] \]

(22)

In the third equality we have used the identity in (Alemany-Puig and Ferrer-i-Cancho 2022, Equation 24). Plugging Equation 22 into Equation 5 we get

\[ E_{pl}[D(T)] = \frac{1}{2n} \sum_{u \in V} \sum_{v \in \Gamma_T(u)} E_{pr}[\beta_{uv}] + \frac{1}{n} \sum_{u \in V} E_{pr}[D(T^u)] \]

(23)

We can use (Alemany-Puig and Ferrer-i-Cancho 2022)

\[ E_{pr}[\beta_{uv}] = \frac{s_r(c_r(u,v)) - s_r(f_r(u,v)) - 1}{3} = \frac{n - s_v(u) - 1}{3} \]

(24)

to further simplify Equation 23 and obtain

\[ \frac{1}{2n} \sum_{u \in V} \sum_{v \in \Gamma_T(u)} E_{pr}[\beta_{uv}] = \frac{(n - 1)(n - 2)}{6n} \]

(25)

Hence Equation 6.

For the sake of comprehensiveness, we also provide an arithmetic expression for the expected length of an edge \( uv \) of a free tree in uniformly random planar arrangements. To this aim, we further define \( E_{pl}^*[\delta_{uv} \mid \tau] = E_{pl}[\delta_{uv} \mid \pi(\tau) = 1] \) to be the expected value of the length of edge \( uv \in E(T) \) when the vertex \( \tau \in V(T) \) is fixed to the leftmost position in planar arrangements of \( T \). Similarly, given a rooting of \( T \) at \( \tau \), let \( E_{pr}^*[\delta_{uv} \mid \tau] = E_{pr}[\delta_{uv} \mid \pi(\tau) = 1] \) to be the expected value of the length of edge \( uv \in E(T') \) when the root vertex \( \tau \in V(T') \) is fixed to the leftmost position in projective arrangements of \( T' \).

Lemma 1

Given a free tree \( T = (V, E) \), for any \( uv \in E \) it holds that

\[ E_{pl}[\delta_{uv}] = 1 + \frac{1}{n} \sum_{r \in V \setminus \{u,v\}} E_{pr}[\delta_{uv} \mid \tau] \]

(26)

where (Alemany-Puig and Ferrer-i-Cancho 2022)

\[ E_{pr}[\delta_{uv} \mid \tau] = \frac{2s_r(c_r(u,v)) + s_r(f_r(u,v)) + 1}{6} \]

(27)

\( c_r(u,v) \) is the vertex of edge \( uv \) closest to vertex \( \tau \) and \( f_r(u,v) \) is the vertex of edge \( uv \) farthest from \( \tau \).
Proof. Following the characterization of planar arrangements described in Section 3, we have that \( \mathbb{P}_{\text{pl}} (\pi(r) = 1) = 1/n \). Then applying the Law of Total Expectation

\[
\mathbb{E}_{\text{pl}} [\delta_{uv}] = \sum_{r \in V} \mathbb{E}_{\text{pl}} [\delta_{uv} | \pi(r) = 1] \mathbb{P}_{\text{pl}} (\pi(r) = 1) = \frac{1}{n} \sum_{r \in V} \mathbb{E}^{*}_{\text{pl}} [\delta_{uv} | r] \tag{28}
\]

Now we calculate \( \mathbb{E}^{*}_{\text{pl}} [\delta_{uv} | r] \) by cases. When \( r \not\in \{u, v\} \),

\[
\mathbb{E}^{*}_{\text{pl}} [\delta_{uv} | r] = \mathbb{E}^{*}_{\text{pr}} [\delta_{uv} | r] = \mathbb{E}^{*}_{\text{pr}} [\alpha_{uv} + \beta_{uv} | r] = \mathbb{E}^{*}_{\text{pr}} [\alpha_{uv} | r] + \mathbb{E}^{*}_{\text{pr}} [\beta_{uv} | r] \tag{29}
\]

By denoting \( \tau \) the only vertex in \( \{u, v\} \setminus \{r\} \), then

\[
\mathbb{E}^{*}_{\text{pr}} [\alpha_{uv} | r] = \mathbb{E}^{*}_{\text{pr}} [\alpha_{uv} | r] = \frac{s_r(\tau) + 1}{2} \tag{30}
\]

Equation 31 relies on the fact that in a rooted tree \( T^r \), the expected length of the anchor of an edge incident to the root, say \( r w \in E(T^r) \), is given by \( \mathbb{E}^{*}_{\text{pr}} [\alpha_{r w} | r] = (s_r(w) + 1)/2 \) (Alemany-Puig and Ferrer-i-Cancho 2022). An arithmetic expression for \( \mathbb{E}^{*}_{\text{pr}} [\beta_{uv} | r] \) can be found by modifying the proof of Alemany-Puig and Ferrer-i-Cancho (2022, Lemma 1). Then we get (see Appendix),

\[
\mathbb{E}^{*}_{\text{pr}} [\beta_{uv} | r] = \frac{3}{2} \mathbb{E}^{*}_{\text{pr}} [\beta_{uv} | r] = \frac{n - s_r(\tau) - 1}{2} \tag{31}
\]

Therefore, adding Equations 31 and 32 we obtain

\[
\mathbb{E}^{*}_{\text{pl}} [\delta_{uv} | r] = \mathbb{E}^{*}_{\text{pr}} [\delta_{uv} | r] = \frac{s_r(\tau) + 1}{2} + \frac{n - s_r(\tau) - 1}{2} = \frac{n}{2} \tag{32}
\]

Equation 26 follows immediately after inserting Equations 33 and 29 in Equation 28. □

5.2 A linear-time algorithm to compute \( \mathbb{E}_{\text{pl}} [D(T)] \)

Here we consider algorithms of increasing efficiency. First, as \( \mathbb{E}_{\text{pr}} [D(T^w)] \) can be calculated in \( O(n) \)-time for any \( n \)-vertex rooted tree \( T^w \) (Alemany-Puig and Ferrer-i-Cancho 2022, Theorem 1), the evaluation ‘as is’ of Equation 6 leads to an \( O(n^2) \)-time algorithm.

Second, we could calculate the value \( \mathbb{E}_{\text{pr}} [D(T^u)] \) for all \( u \in V \) in \( O(n) \)-time and \( O(n) \)-space with the following procedure: (1) Precompute \( s_u(v) \) in \( O(n) \)-time (Alemany-Puig, Esteban, and Ferrer-i-Cancho 2022); (2) Choose an arbitrary vertex \( w \); (3) Calculate \( \mathbb{E}_{\text{pr}} [D(T^w)] \) in \( O(n) \)-time (Alemany-Puig and Ferrer-i-Cancho 2022); and finally (4) Perform a Breadth First Search (BFS) traversal of \( T \) starting at \( w \). In this traversal, when going from vertex \( u \) to vertex \( v \), the value of \( \mathbb{E}_{\text{pr}} [D(T^v)] \) is calculated applying the precomputed value of \( \mathbb{E}_{\text{pr}} [D(T^u)] \), and finally (4) Perform a Breadth First Search (BFS) traversal of \( T \) starting at \( w \). In this traversal, when going from vertex \( u \) to vertex \( v \), the value of \( \mathbb{E}_{\text{pr}} [D(T^v)] \) is calculated applying the precomputed value of \( \mathbb{E}_{\text{pr}} [D(T^u)] \) to \( \mathbb{E}_{\text{pr}} [D(T^v)] = \mathbb{E}_{\text{pr}} [D(T^u)] + \Delta \) with

\[
\Delta = \mathbb{E}_{\text{pr}} [D(T^w)] - \mathbb{E}_{\text{pr}} [D(T^v)] \tag{33}
\]
\[
= \frac{1}{6} [s_u(v)(2d(v) - 1) + 2n(d(u) - d(v)) - s_v(u)(2d(u) - 1)]
\] (35)

which follows from straightforward manipulations of Equation 4.

Third, we propose an alternative that is also \(O(n)\)-time yet simpler and faster in practice, based on Proposition 2.

**Proposition 2**

Given a free tree \(T = (V, E)\),

\[
\mathbb{E}_{pr}[D(T)] = \frac{(n - 1)(3n^2 + 2n - 2)}{6n} - \frac{1}{6n} \sum_{v \in V} \left(2d(v) - 1\right) \sum_{u \in \Gamma(v)} s_v(u)^2
\] (36)

**Proof.** Here we simplify the summation in Equation 6, which becomes (Alemany-Puig and Ferrer-i-Cancho 2022)

\[
\frac{1}{n} \sum_{u \in V} \mathbb{E}_{pr}[D(T^u)] = \frac{1}{6n} (f(T) - n)
\] (37)

with

\[
f(T) = \sum_{u \in V} \sum_{v \in V} s_u(v)(d_u^+(v) + 1)
\] (38)

Now we simplify \(f(T)\) by first removing the term \(d_u^+(v)\) and replacing it by \(d(v)\) so that we can swap the order of the summations afterwards, that is,

\[
f(T) = \sum_{u \in V} \left(s_u(u)(2d_u^+(u) + 1) + \sum_{v \in V \setminus \{u\}} s_u(v)(2d_u^+(v) + 1)\right)
\] (39)

\[
= \sum_{u \in V} n(2d(u) + 1) + \sum_{u \in V} \sum_{v \in V \setminus \{u\}} s_u(v)(2d(v) - 1)
\] (40)

\[
= n(5n - 4) - \sum_{u \in V} s_u(u)(2d(u) - 1) + 2 \sum_{u \in V} \sum_{v \in V} s_u(v)d(v) - \sum_{u \in V} \sum_{v \in V} s_u(v)
\] (41)

\[
= 2n^2 + g(T) - h(T)
\] (42)

with

\[
g(T) = 2 \sum_{u \in V} \sum_{v \in V} s_u(v)d(v)
\] (43)

\[
h(T) = \sum_{u \in V} \sum_{v \in V} s_u(v)
\] (44)
In the preceding derivation, the second equality holds due to $d_u^+(v) = d(v) - 1$ for $v \neq u$; the third and fourth steps, we apply the Handshaking lemma. These lead to

$$\frac{1}{n} \sum_{u \in V} \mathbb{E}_{pi} [D(T^u)] = \frac{1}{6n} \left[ n(2n - 1) + g(T) - h(T) \right]$$

(45)

It remains to simplify Equations 43 and 44. We start by changing the order of the summations in Equation 43,

$$g(T) = 2 \sum_{v \in V} \sum_{u \in V} s_u(v)d(v) = 2 \sum_{v \in V} d(v) \sum_{u \in V} s_u(v)$$

and continue simplifying the inner summation: for a fixed $v \in V$, we have that

$$\sum_{u \in V} s_u(v) = n + \sum_{u \in V\setminus\{v\}} s_u(v)$$

$$= n + \sum_{u \in \Gamma(v)} s_u(v)s_v(u)$$

$$= n + \sum_{u \in \Gamma(v)} (n - s_v(u))s_v(u)$$

$$= n^2 - \sum_{u \in \Gamma(v)} s_v(u)^2$$

(50)

Then

$$g(T) = 4n^2(n - 1) - 2 \sum_{v \in V} d(v) \sum_{u \in \Gamma(v)} s_v(u)^2$$

(51)

We use the result in Equation 50 to simplify Equation 44,

$$h(T) = \sum_{v \in V} \sum_{u \in V} s_u(v) = n^3 - \sum_{v \in V} \sum_{u \in \Gamma(v)} s_v(u)^2$$

(52)

By combining Equations 51 and 52 into Equation 45 and, after some effort, we obtain

$$\mathbb{E}_{pl} [D(T)] = \left( \frac{(n - 1)(n - 2)}{6n} \right) + \frac{1}{6n} \left( n(n - 1)(3n + 1) - \sum_{v \in V} (2d(v) - 1) \sum_{u \in \Gamma(v)} s_v(u)^2 \right)$$

(53)

which finally leads directly to Equation 36.

$$\square$$

**Lemma 2**

For any given free tree $T$, Algorithm 5.1 calculates $\mathbb{E}_{pl} [D(T)]$ in $O(n)$-time and $O(n)$-space.

**Proof.** The pseudocode to calculate $\mathbb{E}_{pl} [D(T)]$ based on Proposition 2 is given in Algorithm 5.1. This algorithm first calculates $s_u(v)$ for all edges $uv \in E$, for the given tree $T$ in $O(n)$ time using the pseudocode by Alemany-Puig, Esteban, and Ferrer-i-Cancho (2022,
Algorithm 2.1). Then it uses these values to calculate the sums of $s_v(u)^2$ for every vertex $v \in V$. Such sums are then used to evaluate Equation 36 hence obtaining $E_{pl}[D(T)]$ in $O(n)$ time.

**Algorithm 5.1: Calculation of $E_{pl}[D(T)]$. Cost $O(n)$-time, $O(n)$-space.**

1. **Function** `COMPUTE_EXPECTED_PLANAR(T)` is
2. **Input:** $T$ free tree.
3. **Output:** $E_{pl}[D(T)]$.
4. $S \leftarrow \text{COMPUTE_S_FT}(T)$ // (Alemany-Puig, Esteban, and Ferrer-i-Cancho 2022, Algorithm 2.1)
5. $L \leftarrow \{0\}^n$ // a vector of $n$ zeroes.
6. **for** $((u, v), s_u(v)) \in S$ **do**
7. \(L[u] \leftarrow L[u] + s_u(v)^2\)
8. **return** $((n-1)(3n^2 + 2n - 2) - \sum_{u \in V}(d(u) - 1)L[u])/6n$

### 5.3 A simple application

Let $E_{\geq 1}[D(T)]$ be the expected value of the sum of edge lengths conditioned to arrangements $\pi$ such that $C_\pi(T) \geq 1$. An immediate consequence of Lemma 2 is $E_{\geq 1}[D(T)]$ that can be computed easily as the following corollary states.

**Corollary 1**

For any free tree $T$, $E_{\geq 1}[D(T)]$ can be computed in $O(n)$-time and $O(n)$-space via

$$E_{\geq 1}[D(T)] = \frac{E[D(T)] - E_{pl}[D(T)]P(C(T) = 0)}{P(C(T) \geq 1)}$$

(54)

with $P(C(T) \leq 0) = N_{pl}(T)/n!$ and $P(C(T) \geq 1) = (n! - N_{pl}(T))/n!$.

**Proof.** Due to the Law of Total Expectation,

$$E[D(T)] = E_{pl}[D(T)]P(C(T) = 0) + E_{\geq 1}[D(T)]P(C(T) \geq 1)$$

(55)

and hence Equation 54. $N_{pl}(T)$ can be computed in $O(n)$-time with Equation 9 and $E_{pl}[D(T)]$ can be computed in $O(n)$-time and $O(n)$-space (Lemma 2). Hence all the components in the r.h.s. of Equation 54 can be computed in $O(n)$-time and $O(n)$-space.

### 6. Conclusions and future work

In Section 3, we have characterized planar arrangements of a given free tree $T$ using the concept of segment (Alemany-Puig and Ferrer-i-Cancho 2022). Using said characterization, we have shown that the number of planar arrangements of a free tree depends on its degree sequence (Proposition 1), in a similar way projective arrangements of a rooted tree do (Alemany-Puig and Ferrer-i-Cancho 2022). Moreover, we have given a procedure to generate u.a.r. planar arrangements of a given free tree in Section 4 (Algorithm 4.3) which can be easily adapted to generate such arrangements exhaustively. Interestingly, our algorithm to generate planar arrangements is based on the generation of projective...
arrangements of a rooted subtree. For the sake of completeness, we have detailed a procedure to generate u.a.r. projective arrangements of a given rooted tree (Algorithm 4.1). The identification of the underlying structure of planar arrangements have led us to derive an arithmetic expression, in Section 5, for $E_{pl}[D(T)]$ (Theorem 1) from which we devised a $O(n)$-time algorithm to calculate such value (Proposition 1, Algorithm 5.1).

In past research on syntactic dependency distance minimization, $E_{pr}[D(T)]$ has been the most widely used random baseline (Gildea and Temperley 2007; Liu 2008; Futrell, Mahowald, and Gibson 2015). However, projectivity has a lower coverage than planarity in real sentences (Gómez-Rodríguez and Ferrer-i-Cancho 2017). Thanks to the research in this article, we have paved the way for replicating that research replacing $E_{pr}[D(T)]$ by $E_{pl}[D(T)]$.

Planarity is a relaxation of projectivity but future work should address the problem of the expected value of $D(T)$ in classes of formal constraints with even more coverage (Gómez-Rodríguez 2016). A promising step is the investigation of $E_{\leq k}[D(T)]$, the expected value of $D(T)$ conditioned to arrangements $\pi$ such that $C_{\pi}(T) \leq k$. Notice that $E_{\leq 0}[D(T)] = E_{pl}[D(T)]$. In real languages, the average number of crossings ranges between 0.40 and 0.62 (Ferrer-i-Cancho, Gómez-Rodríguez, and Esteban 2018), suggesting that $E_{\leq k}[D(T)]$ with $k = 1$ or a small $k$ would suffice.

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Appendix A: Derivation of $E_{pr}^* [\beta_{uv} | u]$

Here we derive the expected length of the coanchor of a (directed) edge $uv \in E(T^u)$ in uniformly random projective arrangements of $T^u$ conditioned to $\pi(u) = 1$. Following Alemany-Puig and Ferrer-i-Cancho (2022), we decompose $\beta_{uv}$ as the sum of the lengths of the segments inbetween $u$ and $v$. Here we use $k_{uv}$ to denote the number of segments inbetween $u$ and $v$, and $\varphi_{uv}^{(i)}$ to denote the size of the $i$th segment, yielding (Alemany-Puig and Ferrer-i-Cancho 2022),

$$\beta_{uv} = \sum_{i=1}^{k_{uv}} \varphi_{uv}^{(i)} \quad (A.1)$$

By the Law of Total Expectation, we have that

$$E_{pr}^*[\beta_{uv} | u] = \sum_{k=1}^{d(u)-1} E_{pr}^*[\beta_{uv} | u, k_{uv} = k] P_{pr}^*(k_{uv} = k | u) \quad (A.2)$$

where $E_{pr}^*[\beta_{uv} | k_{uv} = k | u]$ is the expectation of $\beta_{uv}$ given that $u$ and $v$ are separated by $k$ segments, and $P_{pr}^*(k_{uv} = k | u)$ is the probability that $u$ and $v$ are separated by $k$ intermediate segments, both in uniformly random projective arrangements $\pi$ conditioned to
\( \pi(u) = 1 \), both conditioned to the root of the tree being vertex \( u \). On the one hand,

\[
E^*_{pr} [\beta_{uv} \mid u, k_{uv} = k] = \mathbb{E}^*_{pr} \left[ \sum_{i=1}^{k} \varphi(i) \mid u \right] = \frac{n - s_u(v) - 1}{d(u) - 1} \cdot k \tag{A.3}
\]

Notice that this is the same result as that obtained in (Alemany-Puig and Ferrer-i-Cancho 2022). Lastly, the proportion of arrangements in which the segment of \( v \) is at position \( k_{uv} + 1 \) equals \( (d(u) - 1)! \), therefore,

\[
P^*_{pr} \left( k_{uv} = k \mid u \right) = \frac{(d(u) - 1)! \prod_{v \in \Gamma_T(u)} N_{pr}(T^u)}{d(u)! \prod_{v \in \Gamma_T(u)} N_{pr}(T^u)} = \frac{1}{d(u)} \tag{A.4}
\]

Recalling that (Alemany-Puig and Ferrer-i-Cancho 2022)

\[
E_{pr} [\beta_{uv}] = \frac{n - s_u(v) - 1}{3} \tag{A.5}
\]

and plugging the results in Equations A.3 and A.4 into Equation A.2 we get

\[
E^*_{pr} [\beta_{uv} \mid u] = \frac{n - s_u(v) - 1}{d(u) - 1} \cdot \frac{1}{d(u)} \sum_{k=1}^{d(u)-1} k = \frac{n - s_u(v) - 1}{2} = \frac{3}{2} E_{pr} [\beta_{uv}] \tag{A.6}
\]

References

Alemany-Puig, Lluís, Juan Luis Esteban, and Ramon Ferrer-i-Cancho. 2021. The Linear Arrangement Library. A new tool for research on syntactic dependency structures. In Proceedings of the Second Workshop on Quantitative Syntax (Quasy, SyntaxFest 2021), pages 1–16, Association for Computational Linguistics, Sofia, Bulgaria.

Alemany-Puig, Lluís, Juan Luis Esteban, and Ramon Ferrer-i-Cancho. 2022. Minimum projective linearizations of trees in linear time. Information Processing Letters, 174:106204.

Alemany-Puig, Lluís and Ramon Ferrer-i-Cancho. 2022. Linear-time calculation of the expected sum of edge lengths in projective linearizations of trees. Computational Linguistics, pages 1–25.

Bernhart, Frank and Paul C. Kainen. 1979. The Book Thickness of a Graph. Journal of Combinatorial Theory, Series B, 27(3):320–331.

Chung, Fan R. K. 1984. On optimal linear arrangements of trees. Computers & Mathematics with Applications, 10(1):43–60.

Cormen, Thomas H., Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2001. Introduction to Algorithms, 2nd edition. The MIT Press, Cambridge, MA, USA.

Ferrer-i-Cancho, R., C. Gómez-Rodríguez, and J. L. Esteban. 2018. Are crossing dependencies really scarce? Physica A, 493:311–329.

Ferrer-i-Cancho, Ramon. 2004. Euclidean distance between syntactically linked words. Physical Review E, 70(5):5.

Ferrer-i-Cancho, Ramon. 2019. The sum of edge lengths in random linear arrangements. Journal of Statistical Mechanics, 2019(5):053401.

Ferrer-i-Cancho, Ramon, Carlos Gómez-Rodríguez, Juan Luis Esteban, and Lluís Alemany-Puig. 2022. Optimality of syntactic dependency distances. Phys. Rev. E, 105:014308.

Futrell, Richard, Kyle Mahowald, and Edward Gibson. 2015. Large-scale evidence of dependency length minimization in 37 languages. Proceedings of the National Academy of Sciences, 112(33):10336–10341.

Gildea, Daniel and David Temperley. 2007. Optimizing Grammars for Minimum Dependency Length. In Proceedings of the 45th Annual Meeting of the Association of Computational Linguistics, pages 184–191, Association for Computational Linguistics, Prague, Czech Republic.
Alemany-Puí & Ferrer-i-Cancho

Gildea, David and David Temperley. 2010. Do Grammars Minimize Dependency Length? Cognitive Science, 34(2):286–310.

Gómez-Rodríguez, Carlos. 2016. Restricted Non-Projectivity: Coverage vs. Efficiency. Computational Linguistics, 42(4):809–817.

Gómez-Rodríguez, Carlos, Morten H. Christiansen, and Ramon Ferrer-i-Cancho. 2022. Memory limitations are hidden in grammar. Glottometrics, 52:in press.

Gómez-Rodríguez, Carlos and Ramon Ferrer-i-Cancho. 2017. Scarcity of crossing dependencies: A direct outcome of a specific constraint? Phys. Rev. E, 96:062304.

Groß, Thomas and Timothy Osborne. 2009. Toward a practical dependency grammar theory of discontinuities. SKY Journal of Linguistics, 22:43–90.

Hochberg, Robert A. and Matthias F. Stallmann. 2003. Optimal one-page tree embeddings in linear time. Information Processing Letters, 87(2):59–66.

Hudson, Richard. 1995. Measuring syntactic difficulty. Unpublished paper.

Kramer, Alex. 2021. Dependency Lengths in Speech and Writing: A Cross-Linguistic Comparison via YouDePP, a Pipeline for Scraping and Parsing YouTube Captions. In Proceedings of the Society for Computation in Linguistics, volume 4, pages 359–365.

Kuhlmann, Marco and Joakim Nivre. 2006. Mildly Non-Projective Dependency Structures. In Proceedings of the COLING/ACL 2006 Main Conference Poster Sessions, COLING-ACL '06, pages 507–514.

Liu, Haitao. 2008. Dependency Distance as a Metric of Language Comprehension Difficulty. Journal of Cognitive Science, 9(2):159–191.

Liu, Haitao, Chunshan Xu, and Junying Liang. 2017. Dependency distance: A new perspective on syntactic patterns in natural languages. Physics of Life Reviews, 21:171–193.

Mitzenmacher, Michael and Eli Upfal. 2017. Probability and Computing. Randomization and Probabilistic Techniques in Algorithms and Data Analysis. Cambridge University Press.

Morril, Glyn. 2000. Incremental processing and acceptability. Computational Linguistics, 25(3):319–338.

Nivre, Joakim. 2006. Constraints on non-projective dependency parsing. In EACL 2006 - 11th Conference of the European Chapter of the Association for Computational Linguistics, Proceedings of the Conference, pages 73–80.