On a Family of Strictly Non-Volterra Quadratic Stochastic Operators

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Abstract. In this paper we consider a class of strictly non-Volterra quadratic stochastic operators defined on the two-dimensional simplex. We show that such operators have a unique fixed point and the set of limit points is either a single point or an infinite set.

1. Introduction
Quadratic stochastic operators (QSOs) were first introduced by Bernstein in [2]. Such operators frequently arise in many models of mathematical genetics, namely, in the theory of heredity (see [5]-[8],[11, 12],[15]-[18],[21, 22]). Consider a biological population, that is, a community of organisms closed with respect to reproduction. Assume that each individual in this population belongs precisely to one of the species (genotype) 1, . . . , m. The scale of species is such that the species of the parents i and j, unambiguously, determine the probability of every species k for the first generation of direct descendants. Denote this probability, called the heredity coefficient, by pij,k. It is then obvious that pij,k ≥ 0 for all i, j, k and that

$$\sum_{k=1}^{m} p_{ij,k} = 1, \quad i, j, k = 1, \ldots, m.$$  

The state of the population can be described by the tuple (x1, x2, . . . , xm) of species probabilities, that is, xk is the fraction of the species k in the total population. In the case of panmictia (random interbreeding) the parent pairs i and j arise for a fixed state x = (x1, x2, . . . , xm) with probability xi xj. Hence, the total probability of the species k in the first generation of direct descendants is defined by

$$x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \quad k = 1, \ldots, m.$$  

The association x 7→ x’ defines an evolutionary quadratic operator. Thus evolution of a population can be studied as a dynamical system of a quadratic stochastic operator [15].

The paper is organized as follows. In Section 2 we recall definitions and well known results from the theory of Volterra and non-Volterra QSOs. In Section 3 we consider a new class of non-Volterra QSOs which are generated by convex combination of two strictly non-Volterra QSOs. Then we show that each QSO from this class has a unique fixed point. Moreover, we prove that the set of limit points is either a single point or an infinite set.
2. Preliminaries

Let \( S_{m-1} = \{ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \text{for any } i, \ x_i \geq 0, \text{ and } \sum_{i=1}^{m} x_i = 1 \} \)

be the \((m-1)\)-dimensional simplex. A map \( V \) of \( S_{m-1} \) into itself is called a quadratic stochastic operator (QSO) if

\[
(Vx)_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j
\]

for any \( x \in S_{m-1} \) and for all \( k = 1, \ldots, m \), where

\[ p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k} \quad \text{for all } i, j, k; \quad \sum_{k=1}^{m} p_{ij,k} = 1. \]  \hspace{1cm} (2.2)

By \( \{ x^{(n)} \in S_{m-1} : n = 0, 1, 2, \ldots \} \) we denote the the trajectory of \( V \) starting from the initial point \( x \in S_{m-1} \), where \( x^{(n+1)} = V(x^{(n)}) \) for all \( n = 0, 1, 2, \ldots \), with \( x^{(0)} = x \). Recall that a point \( x \in S_{m-1} \) is called a fixed point of \( V \) if \( V(x) = x \).

**Definition 2.1.** A QSO \( V \) is called regular if for any initial point \( x \in S_{m-1} \), the limit \( \lim_{n \to \infty} V^n(x) \) exists.

Note that the limit point of the trajectory \( \{ x^{(n)} \} \) is a fixed point of \( V \). Thus, the fixed points of a QSO describe limit or long run behavior of the trajectories for any initial point. The limit behavior of trajectories and fixed points play an important role in many applied problems (see [7, 8, 14, 15, 16]). The biological treatment of the regularity of a QSO is rather clear: in the long run the distribution of species in the next generation coincides with the distribution of species in the previous one, i.e. it is stable.

In statistical mechanics, an ergodic hypothesis proposes a connection between dynamics and statistics. In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, more generally, that time averages may be replaced by space averages. For nonlinear dynamical systems (2.1) Ulam [21] suggested an analogue of a measure-theoretic ergodicity, the following ergodic hypothesis:

**Definition 2.2.** A QSO \( V \) is said to be ergodic if the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(x)
\]

exists for any \( x \in S_{m-1} \).

On the basis of numerical calculations Ulam, in 1960 [21], conjectured that the ergodic theorem holds for any QSO. In 1977, Zakharevich [22] proved that this conjecture is false in general. Later, in [5], a sufficient condition of non-ergodicity for QSOs defined on \( S^2 \) was established. The biological treatment of non-ergodicity of a QSO is the following: in the long run the behavior of the distributions of species is chaotic, i.e. it is unpredictable. Note that a regular QSO is ergodic, but in general from ergodicity does not follow regularity.

**Definition 2.3.** [13] Let \((X, d)\) be a metric space. A continuous map \( V : X \to X \) is called Li-Yorke chaotic if there exists an uncountable subset \( S \subset X \) such that for every pair \((x, y) \in S \times S\) of distinct points, we have that

\[
\lim_{n \to \infty} \inf d(V^n(x), V^n(y)) = 0
\]
\lim_{n \to \infty} \sup d(V^n(x), V^n(y)) > 0.

Note that the QSO which was studied by Zakharevich [22] exhibits the Li-Yorke chaos (see [20]).

Let \( \pi \) be a permutation of the set \( \{ 1, 2, \ldots, m \} \). Then we can define a one-to-one transformation \( T_\pi : S^{m-1} \to S^{m-1} \) as

\[ T_\pi(x_1, x_2, \ldots, x_m) = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)}). \tag{2.3} \]

It is evident that the transformation \( T_\pi^{-1}VT_\pi \) is a QSO and the limit behavior of trajectories of the QSO \( T_\pi^{-1}VT_\pi \) coincides with limit behavior of trajectories of the QSO \( V \) (see [11]).

The following notations will be used throughout this paper. We let \( \partial S^{m-1} \) denote the boundary of \( S^{m-1} \), \( \partial S^{m-1} = \{ x \in S^{m-1} : x_i = 0 \) for at least one \( i \in \{ 1, 2, \ldots, m \} \} \); the interior of \( S^{m-1} \) be the set \( \int S^{m-1} = \{ x \in S^{m-1} : x_1 x_2 \cdots x_m > 0 \} \).

Let \( \omega_V(x^0) \) be the set of limit points of the trajectory \( \{ V^k(x^0) \in S^{m-1} : k = 0, 1, 2, \ldots \} \). Using Lyapunov functions, one can handle on \( \omega \)-limit points. Recall the definition of a Lyapunov function.

**Definition 2.4.** A continuous function \( \varphi : \int S^{m-1} \to R \) is called a Lyapunov function for a QSO \( V \) if \( \varphi(V(x)) \geq \varphi(x) \) for all \( x \) (or \( \varphi(V(x)) \leq \varphi(x) \) for all \( x \)).

A Lyapunov function is very helpful to describe an upper estimate of \( \omega(x^0) \). However there is no general recipe on how to find such Lyapunov functions. In [1, 5, 7] the authors presented examples of Lyapunov functions. A QSO given by (2.1) is called Volterra operator, if \( p_{ij,k} = 0 \), whenever \( k \notin \{ i, j \} \) for every \( k, i, j \in \{ 1, \ldots, m \} \). Many properties of Lyapunov functions of Volterra operators have been investigated in [7]-[11],[17, 19]. In [23] it was defined an opposite class to Volterra operators. Such kind of operators are called strictly non-Volterra ones.

Now recall the definition of strictly non-Volterra operators.

**Definition 2.5.** [23] A quadratic stochastic operator (2.1),(2.2) is said a strictly non-Volterra operator if

\[ p_{ij,k} = 0, \quad \text{for } k \in \{ i, j \}, \quad i, j, k = 1, \ldots, m. \tag{2.4} \]

It is easy to see that for \( m = 2 \) conditions (2.2) and (2.4) cannot be satisfied simultaneously, hence strictly non-Volterra operators exist only when \( m \geq 3 \).

For \( m = 3 \) an arbitrary strictly non-Volterra operator has the following representation:

\[ V : \begin{cases} 
  x_1' = ax_1^2 + cx_2^2 + 2x_2x_3, \\
  x_2' = ax_1^2 + dx_3^2 + 2x_2x_3, \\
  x_3' = bx_1^2 + \beta x_2^2 + 2x_1x_2,
\end{cases} \tag{2.5} \]

where

\[ a, b, c, d, \alpha, \beta \geq 0, \quad \text{and} \quad a + b = c + d = \alpha + \beta = 1. \tag{2.6} \]

**Theorem 2.6** ([23]). For any parameters (2.6) the QSO (2.5) has a unique fixed point \( x^* \in S^2 \).

Let \( D_x V(x^*) = (\partial V_i/\partial x_j)(x^*) \) be a Jacobian of \( V \) at a point \( x^* \).

**Definition 2.7.** [3] A fixed point \( x^* \) is called hyperbolic if its Jacobian \( D_x V(x^*) \) has no eigenvalues on the unit circle.

**Definition 2.8.** [3] A hyperbolic fixed point \( x^* \) is called:

i) attracting if all the eigenvalues of the Jacobian \( D_x V(x^*) \) are less than 1 in absolute value;

ii) repelling if all the eigenvalues of the Jacobian \( D_x V(x^*) \) are greater than 1 in absolute value;

iii) a saddle otherwise.
3. Main result

In this section, we consider the following two strictly non-Volterra QSOs on the two-dimensional simplex

\[
V_1 : \begin{cases} 
  x'_1 = x_2^2 + 2x_2x_3, \\
  x'_2 = x_1^2 + 2x_1x_3, \\
  x'_3 = x_3^2 + 2x_1x_2,
\end{cases} \quad (3.1) \\
V_2 : \begin{cases} 
  x'_1 = x_2^2 + 2x_2x_3, \\
  x'_2 = x_1^2 + 2x_1x_3, \\
  x'_3 = x_3^2 + 2x_1x_2.
\end{cases} \quad (3.2)
\]

**Remark 3.1.** It is known that the operators (3.1), (3.2) have unique fixed point \( C(1/3, 1/3, 1/3) \) which is repelling and for any initial point \( x^{(0)} \in S^2 \) one has \( \omega_{V_i}(x^{(0)}) \subset \partial S^2 \), \( i = 1, 2 \) and \( \omega_{V_i}(x^{(0)}) \), \( i = 1, 2 \) is infinite set, moreover they are non-ergodic transformations (see [23]).

Note that in [9] a convex combination of a regular and non-ergodic QSOs was studied. It is shown that the properties of this operator depend on the combination parameter. Also some examples of invariant curves and the set of limit points of the trajectories of the convex combination operator are given. Now it seems natural to study dynamical systems of a QSO which is a convex combination of two non-ergodic QSOs.

In [10] the asymptotical behavior of the general form of homeomorphisms defined on the two-dimensional simplex was studied.

Let us consider the convex combination of given QSOs, that is convex linear combination of

\[
V_\theta = \theta V_1 + (1 - \theta) V_2.
\]

It is clear that \( V_\theta \) has the form

\[
V_\theta : \begin{cases} 
  x'_1 = \theta x_3^2 + (1 - \theta)x_2^2 + 2x_2x_3, \\
  x'_2 = \theta x_2^2 + (1 - \theta)x_3^2 + 2x_1x_3, \\
  x'_3 = \theta x_2^2 + (1 - \theta)x_3^2 + 2x_1x_2.
\end{cases} \quad (3.3)
\]

It is easy to see that the QSO (3.3) is strictly non-Volterra QSO and it coincides with operator (2.5) when \( a = c = \beta = \theta \). One can verify that \( C \) is the fixed point of (3.3) and by Theorem 2.6 it is the unique fixed point.

Note that the operator (3.3) doesn’t coincides with QSO which studied in [10].

**Theorem 3.2.** If \( \theta = \frac{1}{2} \) then \( C \) is non-hyperbolic; if \( \theta \neq \frac{1}{2} \) then \( C \) is repelling;

**Proof.** Using \( x_3 = 1 - x_1 - x_2 \) one can rewrite the QSO (3.3) as follows

\[
\begin{align*}
  x'_1 &= \theta x_1^2 - x_2^2 - 2(1 - \theta)x_1x_2 - 2\theta x_4 + 2(1 - \theta)x_2 + \theta, \\
  x'_2 &= -x_1^2 + (1 - \theta)x_2^2 - 2\theta x_1x_2 + 2\theta x_1 - 2(1 - \theta)x_2 + 1 - \theta.
\end{align*} \quad (3.4)
\]

where \((x_1, x_2) \in \{(x_1, x_2) : x_1, x_2 \geq 0, x_1 + x_2 \in [0, 1]\} \) and \( x_1, x_2 \) are the first two coordinates of a point lying in the simplex \( S^2 \). One can check that

\[-1 \pm \frac{\sqrt{3}}{3}|1-2\theta|i\]

are eigenvalues of Jacobian \( D_\theta V(x) \) of the the operator (3.4) at the point \( C \) and from it immediately implies the statements in the theorem. \( \square \)

Consider the operator \( W_\tau : S^2 \to S^2 \) defined as follows

\[
W_\tau : \begin{cases} 
  x'_1 = (1 - \tau)x_1^2 + \tau x_3^2 + 2x_1x_2, \\
  x'_2 = (1 - \tau)x_2^2 + \tau x_3^2 + 2x_2x_3, \\
  x'_3 = (1 - \tau)x_3^2 + \tau x_1^2 + 2x_1x_3.
\end{cases} \quad (3.5)
\]
Theorem 3.3. [4] The following statements hold true.

i) If $\tau = 1/2$ then $\varphi(x) = |x_1 - x_2||x_1 - x_3||x_2 - x_3|$ is a Lyapunov function for the operator (3.5). Moreover for any $x^{(0)} \in S^2$ one has that $\omega_{W_{\tau}}(x^{(0)}) = \{C\}$.

ii) If $\tau \neq 1/2$ then for any $x^{(0)} \in S^2 \setminus \{C\}$ one has that $\omega_{W_{\tau}}(x^{(0)}) \subset \text{int } S^2$ is an infinite compact subset.

Remark 3.4. On the basis of numerical calculations shown that the operator (3.5) is a nonergodic transformation and it exhibits the Li-Yorke chaos[4].

Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$

and the corresponding map $T_{\pi}$ defined by (2.3).

Proposition 3.5. One has that

i) $V_{\theta}(x) = W_{\theta}(T_{\pi}(x))$;

ii) $W_{\theta}^n(T_{\pi}(x)) = T_{\pi}(W_{\theta}^n(x))$ for all $n \geq 1$;

iii) For any $k = 0, 1, \ldots$ the following hold true

$$V_{\theta}^n(x) = \begin{cases} W_{\theta}^n(x), & \text{if } n = 3k, \\
T_{\pi}(W_{\theta}^n(x)), & \text{if } n = 3k + 1, \\
T_{\pi}^2(W_{\theta}^n(x)), & \text{if } n = 3k + 2. \end{cases}$$

Proof. i) Using (2.3) one has

$$V_{\theta}(x) = \begin{cases} x'_1 = (1 - \theta)x_2^2 + \theta x_3^2 + 2x_2x_3, \\
x'_2 = (1 - \theta)x_3^2 + \theta x_1^2 + 2x_1x_3, \\
x'_3 = (1 - \theta)x_1^2 + \theta x_2^2 + 2x_1x_2, \\
x'_4 = (1 - \theta)x_{\pi}^2(1) + \theta x_{\pi}^2(2) + 2x_{\pi}(1)x_{\pi}(2), \\
x'_5 = (1 - \theta)x_{\pi}^2(2) + \theta x_{\pi}^2(3) + 2x_{\pi}(2)x_{\pi}(3), \\
x'_6 = (1 - \theta)x_{\pi}^2(3) + \theta x_{\pi}^2(1) + 2x_{\pi}(1)x_{\pi}(2), \end{cases}$$

where $T_{\pi}(W_{\theta}^k(x)) = T_{\pi}(W_{\theta}(W_{\theta}^k(x))) = T_{\pi}(W_{\theta}(W_{\theta}(W_{\theta}(\ldots(W_{\theta}(x))))))$.

iii) Using $\pi^3 = id$, where $id$ is identity permutation and i), ii) we obtain $V_{\theta}^n(x) = T_{\pi}(W_{\theta}(W_{\theta}(\ldots(W_{\theta}(x)))))) = T_{\pi}^n(W_{\theta}^n(x)) = T_{\pi}^n(W_{\theta}(W_{\theta}(W_{\theta}(\ldots(W_{\theta}(x))))))$ which immediately implies the statement in the iii).

Since the QSO (3.3) is a permutation of the operator (3.5) using the results of Proposition 3.5 and Theorem 3.3 and by means of same methods and techniques which were used in [23], we shall prove the following Theorem.

Theorem 3.6. For the operator (3.3) the following statements hold true.

i) If $\theta = 1/2$ then $\varphi(x) = |x_1 - x_2||x_1 - x_3||x_2 - x_3|$ is a Lyapunov function. Besides for any $x^{(0)} \in S^2$ the trajectory converges to the center of simplex;

ii) If $\theta \neq 1/2$ then for any $x^{(0)} \in S^2 \setminus \{C\}$ one has that $\omega(x^{(0)}) \subset \text{int } S^2$ is an infinite compact subset.
Proof. i) Let $\theta = 1/2$ then since $\varphi(x) = |x_1 - x_2||x_1 - x_3||x_2 - x_3|$ is a Lyapunov function for the operator $W_\theta$ one has $\varphi(W_{1/2}(x)) \leq \varphi(x)$. We have $\varphi(V_{1/2}(x)) = \varphi(W_{1/2}(T_\pi(x))) \leq \varphi(T_\pi(x))$. It is easy to see that $\varphi(T_\pi(x)) = \varphi(x)$. So, $\varphi(x)$ is a Lyapunov function for QSO $V_{1/2}$.

Let $x^{(0)} \in S^2$ be an initial point. Using Proposition 3.5 and Theorem 3.3 one has

$$\lim_{n \to \infty} V_{1/2}^n(x^{(0)}) = \lim_{n \to \infty} W_{1/2}^n(T_\pi(x^{(0)})) = T_\pi\left(\lim_{n \to \infty} W_{1/2}^n(x^{(0)})\right) = C,$$

that is the trajectory converges to the center of the simplex. The operator $V_{1/2}$ has the regularity property.

ii) Let $\theta \neq 1/2$ and $x^{(0)} \in S^2 \backslash \{C\}$ be any initial point. Let $y \in \omega_{W_\theta}(x^{(0)})$ be any point. Then there exists a sequence $\{n_k\}, k = 1, 2, \ldots$ such that $\lim_{k \to \infty} W_{n_k}^\theta(x^{(0)}) = y$. It is evident that at least one of the following subsequences is infinite:

$$(n_k)_{k=1}^\infty \cap \{3t\}_{t=1}^\infty, \quad (n_k)_{k=1}^\infty \cap \{3t+1\}_{t=1}^\infty, \quad (n_k)_{k=1}^\infty \cap \{3t+2\}_{t=1}^\infty.$$

We denote this subsequence by $\{n_{k_s}\}_{s=1}^\infty$ then from Proposition 3.5 it follows

$$\lim_{s \to \infty} V_{\theta}^{n_{k_s}}(x^{(0)}) = y,$$

that is $y \in \omega_{W_\theta}(x^{(0)})$.

Consequently, for each $x^{(0)} \in S^2$ and each $y \in \omega_{W_\theta}(x^{(0)})$ we obtain that $|\omega_{W_\theta}(x^{(0)}) \cap \{y, T_\pi(y), T_{2\pi}(y)\}| \geq 1$, where $| \cdot |$ denotes the number of elements of a set. Hence cardinality of the set $\omega_{W_\theta}(x^{(0)})$ at least equal to cardinality of the set $\omega_{W_\theta}(x^{(0)})$. This by Theorem 3.3 ii) completes the proof.

Similarly to from Remark 3.4 we have

**Remark 3.7.** The operator (3.3) is a nonergodic transformation and it exhibits the Li-Yorke chaos.

**Remark 3.8.** The operator (3.3) at $\theta = 1/2$ is regular. The Lipschitz constant of an operator $V$ is

$$l(V) = \sup_{x \neq y} \frac{|V(x) - V(y)|}{|x - y|},$$

where $| \cdot |$ is some norm in $\mathbb{R}^m$. It is evident that if $l(V) < 1$ then the operator $V$ will be strict contraction and consequently one has that $V$ has a unique fixed point and the all trajectories of $V$ converges to this fixed point exponentially rapidly. By Theorem 8.2.1 of [15] we have

$$l(V) = \max_{i_1, i_2, j} \sum_{k=1}^m |p_{i_1, j, k} - p_{i_2, j, k}|. \text{ It is easy to check that } l(V) = 2 \text{ for the operator (2.5) with (2.6). Thus our operator (2.5) is not contraction.}$$

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