Higher colimits, derived functors and homology

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Abstract. We develop a theory of higher colimits over categories of free presentations. We show that different homology functors such as Hochschild and cyclic homology of algebras over a field of characteristic zero, simplicial derived functors, and group homology can be obtained as higher colimits of simply defined functors. Connes’ exact sequence linking Hochschild and cyclic homology was obtained using this approach as a corollary of a simple short exact sequence. As an application of the developed theory, we show that the third reduced $K$-functor can be defined as the colimit of the second reduced $K$-functor applied to the fibre square of a free presentation of an algebra. We also prove a Hopf-type formula for odd-dimensional cyclic homology of an algebra over a field of characteristic zero.

Bibliography: 17 titles.

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§1. Introduction

For an ‘algebraic object’ $A$ (for example, a group, an abelian group, an associative algebra, ...), denote by $\text{Pres}(A)$ the category of presentations of $A$ as a quotient of a free (or projective) object $F \to A$. In [6], the authors developed the theory of derived limits over categories of free presentations and showed that many functors can be obtained as (derived) limits of certain simply defined functors $\text{Pres}(A) \to (\text{modules over a commutative ring})$.

For a functor $\mathcal{F}$ from $\text{Pres}(A)$ to the category of modules over a fixed commutative ring and any $c \in \text{Pres}(A)$, there is a natural injection $\lim \mathcal{F} \hookrightarrow \mathcal{F}(c)$ and $\lim \mathcal{F}$ is the largest submodule of $\mathcal{F}(c)$ that does not depend on $c$. In other words, $\lim \mathcal{F}$ is the largest constant sub-functor of $\mathcal{F}$. This approach gives a method to construct various functors. For example, for a group $G$ and the category of free presentations $F \to G$, there is a natural description of the first derived functor (in the sense of Dold-Puppe) of the symmetric cube $L_1 S^3$ applied to the abelianization of $G$ as

$$L_1 S^3(G_{\text{ab}}) = \lim \frac{[F, F, F]}{[R, R, F][F, F, F]},$$

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where $R = \text{Ker}(F \to G)$ (here we use the left-normalized notation for commutator subgroups); see [12]. One can write any expression that functorially depends on a pair $(F, R)$ and apply the (derived) limit over $\text{Pres}(G)$. Here $R$ and $F$ are used as bricks in the constructor. Group homology, Hochschild and cyclic homology of algebras, and certain derived functors can be obtained in this way (see [6], [12] and [13]).

This paper concerns (higher, derived) colimits over the categories of presentations. In the above notation, for any $c \in \text{Pres}(A)$ we get a natural surjection $F(c) \twoheadrightarrow \text{colim } F$, and $\text{colim } F$ is the largest quotient of $F(c)$ that does not depend on $c$. In other words, $\text{colim } F$ is the largest constant quotient-functor of $F$.

A simple illustration of the above statement about the largest independent quotient is the following. For an algebra $A$ and the category of nonunital free presentations $F \to A$, one can obtain the first cyclic homology of $A$ as follows

$$\text{colim } R \cap [F, F] = HC_1(A),$$

where $R = \text{Ker}(F \to G)$. The required largest quotient independent of the presentation is given by the Hopf formula:

$$R \cap [F, F] \twoheadrightarrow \frac{R \cap [F, F]}{[R, F]} = HC_1(A).$$

At first glance, the theory of derived colimits looks like a mirror-dual analogue of the theory of derived limits developed in the papers of the authors. In a sense, this is true, however, there is no way to transfer the principal methods from limits to colimits because the category $\text{Pres}(A)$ is not self-dual. For example, for any functor $F$ from the category of presentations and $c \in \text{Pres}(A)$, $\lim F$ is the equalizer of two maps $F(c) \rightrightarrows F(c \sqcup c)$, where $\sqcup$ is the coproduct in $\text{Pres}(A)$ (see §2 in [13]). In the case of colimit, we cannot present $\text{colim } F$ as a coequalizer of two such maps because $c \times c$ does not exist in $\text{Pres}(A)$. We have to coequalize all the maps $F(c') \to F(c)$ for $c' \to c$ from $\text{Pres}(A)$. As a corollary, we do not have a simple criterion of triviality of colimits, as we have in the case of limits (the property called \textit{monoadditivity} in [6]). Unlike the case of limits, additive functors (with respect to coproducts in the category of presentations) can give nontrivial higher colimits in all degrees. For example, for a group $G$ and the category of free presentations $R \to F \to G$, $\lim_n F_{ab} = 0$ for all $n \geq 0$ (see [6]), however,

$$\text{colim}_n F_{ab} = H_{n+1}(G)$$

for $n \geq 0$ (Theorem 2).

In general, colimits can be viewed as a generalization of derived functors of nonadditive functors in the following sense. Let $C$ be a category with enough projectives and let $c \in C$. Denote by $\text{Pres}(c)$ the category of effective epimorphisms $p \to c$, where $p$ is a projective object. Assume that $\varepsilon_\bullet : p_\bullet \to c$ is a simplicial projective resolution of $c$ (see Definition 2) and $\varepsilon_\bullet^P \in \text{Pres}(c)^{\Delta^{\text{op}}}$ is the corresponding simplicial presentation. Then for any ‘good enough’ functor $\Phi : \text{Pres}(c) \to \text{Mod}(A)$ to the category of modules over some ring $\Lambda$ there is an isomorphism

$$\text{colim}_n \Phi = \pi_n(\Phi(\varepsilon_\bullet^P))$$
(see Theorem 1 for details). In particular, colimits give a way to define derived functors of nonadditive functors not using simplicial resolutions (see Corollary 1). For example, given a group \( G \) and a functor \( \Phi : \text{(Groups)} \to \text{(Abelian groups)} \), the derived colimits \( \operatorname{colim}_n \Phi(F) \) over the category of free presentations \( R \rightarrowtail F \rightarrowtail G \) coincide with simplicial derived functors of \( \Phi \):

\[
\operatorname{colim}_n \Phi(F) = L_n \Phi(G)
\]

(see Proposition 14). Another simple example of the use of Theorem 1 is the following formula for Andrè-Quillen homology of a commutative \( k \)-algebra \( A \) with coefficients in an \( A \)-module \( M \):

\[
D_n(A/k, M) = \operatorname{colim}_n \Omega^\text{comm}(F) \otimes_F M
\]  

(1.1)

(Proposition 13). Here the derived colimit is taken over the category of surjective homomorphisms \( F \rightarrow A \), where \( F \) is the (commutative) polynomial \( k \)-algebra, and \( \Omega^\text{comm}(F) \) is the module of Kähler differentials of \( F \).

The main results of this paper are the following.

1. A generalization of the description of the group homology mentioned above for the case of nontrivial coefficients. Let \( G \) be a group and \( M \) be a \( \mathbb{Z}[G] \)-module. For \( n \geq 0 \),

\[
H_{n+1}(G, M) = \operatorname{colim}_n H_1(F, M)
\]

(Theorem 2), where the colimits are taken over the category of free presentations \( F \rightarrowtail G \).

2. For the category of unital associative algebras over any field there is the following description of the Hochschild homology with coefficients in an \( A \)-bimodule \( M \) (Theorem 3):

\[
H_{n+1}(A, M) = \operatorname{colim}_n H_1(F, M) = \operatorname{colim}_n \Omega(F) \otimes_{F^e} M,
\]

\[
\text{HH}_{n+1}(A) = \operatorname{colim}_n \Omega(F)_{\sharp}
\]

for \( n \geq 1 \), where \( \Omega(F) := \text{Ker}(F \otimes F \rightarrow F) \) is the bimodule of noncommutative differential forms of \( F \), \( F^e = F^{\text{op}} \otimes F \) and

\[
M_{\sharp} := H_0(F, M) = \frac{M}{[M,F]}.
\]

(Compare the formula \( H_{n+1}(A, M) = \operatorname{colim}_n \Omega(F) \otimes_{F^e} M \) with (1.1).)

3. For the category of unital associative algebras over a field of characteristic zero, there is the following description of the Hochschild and reduced cyclic homology (see Proposition 20):

\[
\overline{HC}_{n+1}(A) = \operatorname{colim}_n [F,F], \quad n \geq 1,
\]

\[
\overline{HC}_{n+3}(A) = \operatorname{colim}_n [R,R], \quad n \geq 1,
\]

\[
\text{HH}_{n+2}(A) = \operatorname{colim}_n \frac{[F,F]}{[R,R]}, \quad n \geq 2.
\]
The natural short exact sequence

\[ 0 \rightarrow [R, R] \rightarrow [F, F] \rightarrow [F, F] \rightarrow [R, R] \rightarrow 0 \]

gives rise to the long exact sequence of (derived) colimits

\[ \cdots \rightarrow HC_6(A) \rightarrow HC_4(A) \rightarrow HH_5(A) \rightarrow HC_5(A) \]
\[ \rightarrow HC_3(A) \rightarrow HH_4(A) \rightarrow HC_4(A) \rightarrow HC_2(A). \]

Moreover, there is an isomorphism

\[ HC_n(A) = \text{colim}_n F_i \]

for \( n \geq 1 \) (Theorem 4) and the short exact sequence

\[ 0 \rightarrow F_i \rightarrow \Omega(F) \rightarrow [F, F] \rightarrow 0 \]

(Lemma 5) induces the Connes-like long exact sequence as above (Proposition 19).

4. Let \( k \) be a noetherian regular commutative ring and \( A \) be a unital \( k \)-algebra. Then

\[ \text{colim} \tilde{K}_2(F \times_A F) = \tilde{K}_3(A), \]

where the colimit is taken over the category of free presentations \( F \rightarrow A \) of unital \( k \)-algebras (Proposition 21,1). Here \( \tilde{K}_n \) is the reduced \( K \)-functor (\( n \)th homotopy group of the homotopy fibre of the map of spectra \( K(k) \rightarrow \tilde{K}(A) \)) and \( F \times_A F \) is the fibred square of the epimorphism \( F \rightarrow A \). Analogously, for a nonunital ring \( A \),

\[ \text{colim} K_2(F \times_A F) = K_3(A), \]

where the colimit is taken over the category of free presentations of nonunital rings (Proposition 21,2).

We also prove a Hopf-type formula for the odd-dimensional cyclic homology of algebras over a field \( k \) of characteristic zero (Theorem 7): for a nonunital free presentation of an algebra \( A, F \rightarrow A \), and \( n > 0 \), there is a natural isomorphism

\[ HC_{2n+1}(A) = \frac{R^{n+1} \cap [F, F]}{[R, R^n]}. \]

Note that this isomorphism is strict, without (co)limits, so that the right-hand side is independent of the presentation.

The paper is organized as follows. In §2, we consider general facts about colimits over categories. In particular, we show that the colimit of a functor over a strongly connected category is the largest constant quotient (Proposition 10). In §3 we show that the simplicial derived functors can be presented as higher colimits. Section 4 is about groups: we show that the group homology with coefficients in any module can be defined as higher colimits of the first homology of the free group over the category of free presentations (Theorem 2).

In §5 we give a description of the Hochschild and cyclic homology of unital algebras over fields of characteristic zero as higher colimits. As mentioned above,
the Connes exact sequence which connects Hochschild and cyclic homology can be obtained as a sequence of higher colimits. In the case of groups any functor that depends only on $R$ has trivial higher colimits (Proposition 15). In order to prove this we essentially use that a subgroup of a free group is free. However, this is not true for algebras. Moreover, the formula $\overline{HC}_{n+3}(A) = \text{colim}_n [R, R]$ holds (Proposition 18). It can be interpreted informally as the reason why the cyclic homology exists (and is different from the Hochschild homology): because a subalgebra of a free algebra is not necessarily free. We also prove a Hopf-type formula for the odd-dimensional cyclic homology of algebras over a field $k$ of characteristic zero (Theorem 5): for a free presentation of an algebra $A$, $F \twoheadrightarrow A$, and $n \geq 0$, there is a natural isomorphism

$$\overline{HC}_{2n+1}(A) = \frac{R^{n+1} \cap ([F, F] + k \cdot 1)}{[R, R^n]}.$$  

In §6, we give an analogue of the colimit formula and Hopf-type formulae for the cyclic homology of nonunital algebras. In §7, we consider $K_2$- and $K_3$-functors and give a description of the reduced $K_3$-functor as a colimit of reduced $K_2$-functors applied to the fibre squares.

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§2. General facts about derived colimits over categories

Let $\mathcal{C}$ be a small category. Recall that its nerve is a simplicial set $N\mathcal{C}$ such that

$$(N\mathcal{C})_0 = \text{ob}(\mathcal{C}) \quad \text{and} \quad (N\mathcal{C})_1 = \text{mor}(\mathcal{C}),$$

and the maps $d_0, d_1 : (N\mathcal{C})_1 \to (N\mathcal{C})_0$ and $s_0 : (N\mathcal{C})_0 \to (N\mathcal{C})_1$ are defined by

$d_0(\alpha) = \text{dom}(\alpha), \quad d_1(\alpha) = \text{codom}(\alpha) \quad \text{and} \quad s_0(c) = 1_c.$

For higher dimensions $(N\mathcal{C})_n$ is defined as the set of all sequences of $n$ composable morphisms

$$\bullet \xleftarrow{\alpha_1} \bullet \xleftarrow{\alpha_2} \ldots \xleftarrow{\alpha_n} \bullet.$$  

Faces are

$$d_i(\alpha_1, \ldots, \alpha_n) = \begin{cases} 
(\alpha_2, \ldots, \alpha_n), & i = 0, \\
(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n), & 1 \leq i \leq n - 1, \\
(\alpha_1, \ldots, \alpha_{n-1}), & i = n,
\end{cases}$$

and degeneracies are

$s_i(\alpha_1, \ldots, \alpha_n) = (\alpha_1, \ldots, \alpha_i, 1_{c_i}, \alpha_{i+1}, \ldots, \alpha_n),$

where $c_i = \text{dom}(\alpha_i)$ for $i \geq 1$ and $c_0 = \text{codom} (\alpha_1)$.

Fix an abelian category $\mathcal{A}$ with exact small direct sums (see [2], Appendix II) and enough projectives. Since $\mathcal{A}$ has all small direct sums, it also has all colimits. Consider the category $\mathcal{F}\mathcal{C}$ of functors $\mathcal{C} \to \mathcal{A}$. The following lemma seems to be well known but we cannot find a good reference.
**Lemma 1.** The category of functors $\mathcal{A}^{\mathcal{C}}$ is abelian, it has enough projective objects, and projective objects are direct summands of direct sums of functors of type $P^{(\mathcal{C}(c,:),)}$, where $P$ is a projective object of $\mathcal{A}$ and $P^{(X)} = \bigoplus_{X} P$ for a set $X$.

**Proof.** It is obvious that $\mathcal{A}^{\mathcal{C}}$ is abelian. Note that for any $c \in \mathcal{C}$ there is an adjunction $\mathcal{A}^{\mathcal{C}} \cong \mathcal{A}$ of functors $\mathcal{M} \mapsto \mathcal{M}(c)$ and $A \mapsto A^{(\mathcal{C}(c,:))}$. In other words, there is a Yoneda-like isomorphism with the same proof as that of the Yoneda lemma

$$\mathcal{A}^{\mathcal{C}}(A^{(\mathcal{C}(c,:))}, \mathcal{M}) = \mathcal{A}(A, \mathcal{M}(c)).$$

This isomorphism implies that the functor $\mathcal{A}^{\mathcal{C}}(P^{(\mathcal{C}(c,:))}, \cdot)$ is exact for any projective $P \in \mathcal{A}$, hence $P^{(\mathcal{C}(c,:))}$ is projective in the category of functors. For any functor $\mathcal{M}$ and any object $c \in \mathcal{C}$ we fix an epimorphism $P_{c} \rightarrow \mathcal{M}(c)$ from a projective object $P_{c}$ and consider the direct sum of adjoint morphisms $\bigoplus_{c \in \mathcal{C}} P_{c}^{(\mathcal{C}(c,:))} \rightarrow \mathcal{M}$. It is easy to see that it is an epimorphism. Thus $\mathcal{A}^{\mathcal{C}}$ has enough projectives. Moreover, if $\mathcal{M}$ is projective, the epimorphism splits and we obtain that $\mathcal{M}$ is a direct summand of a direct sum of functors $P_{c}^{(\mathcal{C}(c,:))}$.

The lemma is proved.

The functor of colimit

$$\text{colim} : \mathcal{A}^{\mathcal{C}} \rightarrow \mathcal{A}$$

is left adjoint to the diagonal functor. Hence it is right exact, and we denote its $n$th left derived functor by $\text{colim}_{n}$.

Let $\mathcal{M} \in \mathcal{A}^{\mathcal{C}}$. Consider the simplicial object $C_{\bullet}(\mathcal{C}, \mathcal{M})$ in $\mathcal{A}$ (see [2], Appendix II, 3.2) such that

$$C_{0}(\mathcal{C}, \mathcal{M}) = \bigoplus_{c \in (\mathcal{N}\mathcal{C})_{0}} \mathcal{M}(c),$$

$$C_{n}(\mathcal{C}, \mathcal{M}) = \bigoplus_{(\alpha_{1}, \ldots, \alpha_{n}) \in (\mathcal{N}\mathcal{C})_{n}} \mathcal{M}(\text{dom}(\alpha_{n})).$$

The boundary map $d_{1} : C_{n}(\mathcal{C}, \mathcal{M}) \rightarrow C_{n-1}(\mathcal{C}, \mathcal{M})$ is defined so that the restriction to the summand $\mathcal{M}(\text{dom}(\alpha_{n}))$ with index $(\alpha_{1}, \ldots, \alpha_{n})$ is just the embedding of the summand with index $d_{1}((\alpha_{1}, \ldots, \alpha_{n}))$ for $i \leq n - 1$ and the restriction of $d_{n}$ to the same summand is the map

$$\mathcal{M}(\alpha_{n}) : \mathcal{M}(\text{dom}(\alpha_{n})) \rightarrow \mathcal{M}(\text{codom}(\alpha_{n}))$$

composed with the embedding of the summand with index $d_{n}((\alpha_{1}, \ldots, \alpha_{n}))$. Degeneracy maps $s_{i}$ are defined so that the restriction to the summand with index $\alpha$ is an embedding in the summand with index $s_{i}(\alpha)$. As usual, we define

$$\delta_{n} = \sum (-1)^{i}d_{i} : C_{n}(\mathcal{C}, \mathcal{M}) \rightarrow C_{n-1}(\mathcal{C}, \mathcal{M})$$

and treat $C_{\bullet}(\mathcal{C}, \mathcal{M})$ as a complex. This complex computes the derived colimits.

**Proposition 1** (see [2], Appendix II, Proposition 3.3). Let $\mathcal{C}$ be a small category and $\mathcal{M} : \mathcal{C} \rightarrow \mathcal{A}$ be a functor. Then there is a natural isomorphism

$$\text{colim}_{n} \mathcal{M} \cong H_{n}(C_{\bullet}(\mathcal{C}, \mathcal{M})).$$
This isomorphism is natural in the following sense. Any functor Φ: \mathcal{C} → \mathcal{D}

between small categories defines a morphism of complexes

\[ C_\bullet(\varphi): C_\bullet(\mathcal{C}, \mathcal{M}) → C_\bullet(\mathcal{D}, \mathcal{M}) \]

that sends the summand with the index \((\alpha_1, \ldots, \alpha_n)\) to the summand with the index \((\Phi(\alpha_1), \ldots, \Phi(\alpha_n))\). This morphism of complexes induces a morphism on homology, whose composition with the isomorphisms coincides with the natural map

\[ \text{colim}_n \mathcal{M} \Phi → \text{colim}_n \mathcal{M}. \]

A simplicial set \(X\) is called contractible if \(|X|\) is contractible. This is also equivalent to the fact that \(X → \ast\) is a weak equivalence.

A category \(\mathcal{C}\) is said to be contractible if its nerve is contractible. Note that if there exists a sequence of functors connecting the identity functor and a constant functor \(\text{Id}_\mathcal{C} = \Phi_1, \ldots, \Phi_{2n} = \text{const}_0: \mathcal{C} → \mathcal{C}\) together with natural transformations

\[ \text{Id}_\mathcal{C} = \Phi_0 → \Phi_1 ← \Phi_2 → \cdots ← \Phi_{2n} = \text{const}_0, \]

then \(\mathcal{C}\) is contractible because these natural transformations induce homotopies between the corresponding maps on the nerve (see [15], Proposition 2).

**Proposition 2.** Assume that there exists an object \(c_0 \in \mathcal{C}\) such that for any object \(c \in \mathcal{C}\) the coproduct \(c ⊔ c_0\) exists. Then \(\mathcal{C}\) is contractible.

**Proof.** Consider the functor \(\Phi: \mathcal{C} → \mathcal{C}\) given by the formula \(\Phi(\cdot) = \cdot ⊔ c_0\). The maps \(c → c ⊔ c_0 ← c_0\) give natural transformations from the identity functor \(\text{Id}_\mathcal{C} → \Phi ← \text{const}_0\). It follows that \(\mathcal{C}\) is contractible.

**Proposition 3.** Let \(\mathcal{C}\) be a contractible category and \(A ∈ \mathcal{A}\) be an object. Consider a constant functor \(A^{\text{const}}: \mathcal{C} → \mathcal{A}\). Then

\[ \text{colim}_0 A^{\text{const}} = A \quad \text{and} \quad \text{colim}_n A^{\text{const}} = 0 \]

for \(n ≥ 1\).

**Proof.** Denote the category of free abelian groups by \(\text{fAb}\). Then there is a well-defined functor of tensor product \(⊗: \text{fAb} × \mathcal{A} → \mathcal{A}\) such that \(\mathbb{Z}^{(I)} ⊗ A = A^{(I)}\). Note that

\[ C_\bullet(\mathcal{C}, A^{\text{const}}) = C_\bullet(\mathcal{N}\mathcal{C}) ⊗ A, \]

where \(C_\bullet(\mathcal{N}\mathcal{C})\) is the standard chain complex of the simplicial set \(\mathcal{N}\mathcal{C}\). Since \(\mathcal{N}\mathcal{C}\) is contractible, the map \(C_\bullet(\mathcal{N}\mathcal{C}) → \mathbb{Z}[0]\) is a quasi-isomorphism. Using that the complex consists of free abelian groups, we obtain that the map \(C_\bullet(\mathcal{N}\mathcal{C}) ⇒ \mathbb{Z}[0]\) is a homotopy equivalence. Then \(C_\bullet(\mathcal{N}\mathcal{C}) ⊗ A → A[0]\) is a homotopy equivalence.

The proposition is proved.

Throughout the paper we use the homological notation for a complex \(M_\bullet:\)

\[ \cdots ← M_{n-1} ← M_n ← M_{n+1} ← \cdots. \]
Proposition 4 (spectral sequence of derived colimits I). Let \( C \) be a contractible category and \( \mathcal{M}_* \) be a complex of functors \( C \to \mathcal{A} \) such that \( \mathcal{M}_n = 0 \) for \( n \ll 0 \). Assume that \( H_n(\mathcal{M}_*) \) are isomorphic to constant functors. Then there exists a spectral sequence of homological type \( E \) in \( \mathcal{A} \) such that

\[
E_{n,m}^1 = \colim_m \mathcal{M}_n \implies H_{n+m}(\mathcal{M}_*).
\]

Proof. Consider two hyper-homology spectral sequences \( I^E \) and \( II^E \) for the functor \( \colim: \mathcal{A}^C \to \mathcal{A} \) and the complex \( \mathcal{M}_* \),

\[
I_{n,m}^2 = \colim_n H_m(\mathcal{M}_*) \implies \colim_{n+m}(\mathcal{M}_*),
\]

and

\[
II_{n,m}^1 = \colim_m \mathcal{M}_n \implies \colim_{n+m}(\mathcal{M}_*).
\]

Since \( H_m(\mathcal{M}_*) \) is constant, Proposition 3 implies that \( \colim_n H_m(\mathcal{M}_*) = 0 \) for \( n \geq 0 \), \( \colim_0 H_m(\mathcal{M}_*) = H_m(\mathcal{M}_*) \), hence \( \colim_m(\mathcal{M}_*) = H_m(\mathcal{M}_*) \). Then the second spectral sequence is the spectral sequence that we need.

The proposition is proved.

Proposition 5 (spectral sequence of derived colimits II). Let \( \mathcal{M}_* \) be a complex of functors \( C \to \mathcal{A} \) such that \( \mathcal{M}_n = 0 \) for small enough \( n \). Assume that \( \colim_m \mathcal{M}_n = 0 \) for \( m \neq 0 \) and any \( n \). Set \( \mathcal{M}_n^0 = \colim_0 \mathcal{M}_n \). Then there exists a converging spectral sequence of homological type such that

\[
E_{n,m}^2 = \colim_n H_m(\mathcal{M}_*) \implies H_{n+m}(\mathcal{M}_n^0),
\]

Proof. Consider two hyper-homology spectral sequences \( I^E \) and \( II^E \) for the functor \( \colim: \mathcal{A}^C \to \mathcal{A} \) and the complex \( \mathcal{M}_* \),

\[
I_{n,m}^2 = \colim_n H_m(\mathcal{M}_*) \implies \colim_{n+m}(\mathcal{M}_*)
\]

and

\[
II_{n,m}^1 = \colim_m \mathcal{M}_n \implies \colim_{n+m}(\mathcal{M}_*).
\]

Since \( \colim_m \mathcal{M}_n = 0 \) for \( m \neq 0 \), it follows that \( \colim_m(\mathcal{M}_*) = H_m(\mathcal{M}_n^0) \). Then the first spectral sequence is the spectral sequence that we need.

The proposition is proved.

For a functor \( \Phi: \mathcal{C} \to \mathcal{D} \) and an object \( d \in \mathcal{D} \) we denote by \( d \downarrow \Phi \) the comma-category. Its objects are couples \((c, \alpha: d \to \Phi(c))\), where \( c \in \mathcal{C} \) and \( \alpha \in \mathcal{D}(d, \Phi(c)) \). Its morphisms \( f: (c, \alpha) \to (c', \alpha') \) are morphisms \( f \in \mathcal{C}(c, c') \) such that \( \Phi(f)\alpha = \alpha' \).

Proposition 6 (cf. Quillen’s Theorem A; see [15]). Let \( \mathcal{C} \) and \( \mathcal{D} \) be small categories, \( \Phi: \mathcal{C} \to \mathcal{D} \) be a functor and let \( \mathcal{M}: \mathcal{D} \to \mathcal{A} \). Assume that the category \( d \downarrow \Phi \) is contractible for any \( d \in \mathcal{D} \). Then

\[
\colim_n \mathcal{M}\Phi \cong \colim_n \mathcal{M}
\]

for any \( n \).
Proof. Any contractible category is nonempty and connected. Since the category \( d \downarrow \Phi \) is not empty and connected, the functor \( \Phi \) is final (see [10], Ch. IX, §3), hence \( \text{colim} \mathcal{M} \Phi = \text{colim} \mathcal{M} \) for any \( \mathcal{M} \in \mathcal{A}^\Phi \).

The functor of composition \( - \circ \Phi: \mathcal{A}^\mathcal{D} \to \mathcal{A}^\mathcal{C} \) is exact. We prove that it sends projective objects to \( \text{colim}\)-acyclic objects. Projective objects of \( \mathcal{A}^\mathcal{D} \) are direct summands of direct sums of functors of the form \( P^\mathcal{D}(d, \cdot) \), where \( P \) is a projective object of \( \mathcal{A} \). Hence, we only need to show that the objects of \( \mathcal{C} \) of the form \( P^\mathcal{D}(d, \Phi(\cdot)) \) are \( \text{colim}\)-acyclic. Note that an \( n \)-simplex of \( N(d \downarrow \Phi) \) is the same as an \( n \)-simplex \( (c_0 \leftarrow \cdots \leftarrow c_n) \) of \( NC \) together with a morphism \( d \to \Phi(c_n) \). Therefore, we obtain an isomorphism

\[
\bigoplus_{(c_0 \leftarrow \cdots \leftarrow c_n) \in (N\mathcal{C})_n} P^\mathcal{D}(d, \Phi(\cdot)) = \bigoplus_{(\alpha_0 \leftarrow \cdots \leftarrow \alpha_n) \in (N(d \downarrow \Phi))_n} P.
\]

It is easy to check that this isomorphism is compatible with the face maps, hence

\[
C_\bullet(\mathcal{C}, P^\mathcal{D}(d, \Phi(\cdot))) = C_\bullet(d \downarrow \Phi, P).
\]

It follows that

\[
\text{colim}_* P^\mathcal{D}(d, \Phi(\cdot)) = \text{colim}_* P^\text{const}.
\]

Since \( d \downarrow \Phi \) is contractible, Proposition 3 implies that \( P^\mathcal{D}(d, \Phi(\cdot)) \) is \( \text{colim}\)-acyclic.

Then the spectral sequence of composition together with the fact that \( - \circ \Phi: \mathcal{A}^\mathcal{D} \to \mathcal{A}^\mathcal{C} \) is exact, implies the isomorphism \( \text{colim}_i \mathcal{M} \cong \text{colim}_i \mathcal{M} \Phi \).

The proposition is proved.

Proposition 7. Let \( \mathcal{C} \) be a full subcategory of a small category \( \mathcal{D} \) such that any object of \( \mathcal{D} \) is a retract of an object of \( \mathcal{C} \). Then for any functor \( \mathcal{M}: \mathcal{D} \to \mathcal{A} \) there is an isomorphism

\[
\text{colim}_n \mathcal{M}|_\mathcal{C} \cong \text{colim}_n \mathcal{M}
\]

for any \( n \).

Proof. Proposition 6 implies that we only need to prove that for any fixed object \( d_0 \in \mathcal{D} \) the category \( d_0 \downarrow \mathcal{C} \) is contractible. Since \( d_0 \) is a retract of an object of \( \mathcal{C} \), we can fix an object \( c_0 \in \mathcal{C} \) together with morphisms \( r_0: c_0 \to d_0 \) and \( s_0: d_0 \to c_0 \) such that \( r_0 s_0 = \text{id} \). Then for any object \( \alpha: d_0 \to c \) of \( d_0 \downarrow \mathcal{C} \) we obtain a commutative diagram

\[
\begin{array}{ccc}
\text{Id}_{d_0} & \xrightarrow{\alpha} & c \\
\downarrow{s_0} & \xrightarrow{\alpha r_0} & \\
c_0 & \xrightarrow{\alpha} & c
\end{array}
\]

which is natural by \( \alpha \). Therefore, we have constructed a natural transformation \( (d_0)^\text{const} \to \text{Id}_{d_0 \downarrow \mathcal{C}} \), and hence \( d_0 \downarrow \mathcal{C} \) is contractible.

The proposition is proved.
**Proposition 8.** Let $\mathcal{C}$ be a small category with pairwise coproducts, $\Phi: \mathcal{C} \times \mathcal{C} \to \mathcal{A}$ be a functor and $\Delta: \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the diagonal. Then

$$\text{colim}_n \Phi \Delta \cong \text{colim}_n \Phi$$

for any $n$.

**Proof.** The object $(c_1, c_2) \to (c_1 \sqcup c_2, c_1 \sqcup c_2)$ is the initial object of the category $(c_1, c_2) \downarrow \Delta$. Then the category $(c_1, c_2) \downarrow \Delta$ is nonempty and contractible. Proposition 6 implies the assertion.

The proposition is proved.

**Proposition 9** (Küneth theorem for derived colimits). Let $\mathcal{C}$ be a small category with pairwise coproducts, $R$ be a principal ideal domain, and $\Phi, \Psi: \mathcal{C} \to \text{Mod}(R)$ be functors. Assume that $\Psi(c)$ is flat over $R$ for any $c$. Denote by $\Phi \otimes_R \Psi$ the functor $\mathcal{C} \to \text{Mod}(R)$ that sends $c$ to $\Phi(c) \otimes_R \Psi(c)$. Then there is a short exact sequence of $R$-modules

$$\bigoplus_{i+j=n} (\text{colim}_i \Phi) \otimes_R (\text{colim}_j \Psi) \to \text{colim}_n (\Phi \otimes_R \Psi) \to \bigoplus_{i+j=n-1} \text{Tor}^R_1(\text{colim}_i \Phi, \text{colim}_j \Psi).$$

**Proof.** If $A_\bullet$ is a simplicial $R$-module, we denote by $CA_\bullet$ the corresponding complex with the differential $\delta_n = \sum (-1)^i d_i$ and set $H_n(A_\bullet) = H_n(CA_\bullet)$. Recall that if $A_\bullet$ and $B_\bullet$ are two simplicial $R$-modules and $A_\bullet \otimes^\text{simpl}_R B_\bullet$ is their tensor product which is defined level-wise $(A_\bullet \otimes^\text{simpl}_R B_\bullet)_n = A_n \otimes_R B_n$, then there is a natural isomorphism

$$H_*(A_\bullet \otimes^\text{simpl}_R B_\bullet) \cong H_*(CA_\bullet \otimes_R CB_\bullet),$$

where $CA_\bullet \otimes_R CB_\bullet$ is the total tensor product (see [17], §8.5.3). The Küneth theorem for complexes (see [17], §3.6.3) implies that if $B_n$ is $R$-flat for any $n$, then there is a short exact sequence

$$\bigoplus_{i+j=n} H_i(A_\bullet) \otimes_R H_j(B_\bullet) \to H_n(A_\bullet \otimes^\text{simpl}_R B_\bullet) \to \bigoplus_{i+j=n-1} \text{Tor}^R_1(H_i(A_\bullet), H_j(B_\bullet)).$$

Then we only need to note that

$$C_\bullet(\mathcal{C}, \Phi) \otimes^\text{simpl}_R C_\bullet(\mathcal{C}, \Psi) = C_\bullet(\mathcal{C} \times \mathcal{C}, \Phi \otimes_R \Psi),$$

where $\Phi \otimes_R \Psi$ is the functor $\mathcal{C} \times \mathcal{C} \to \text{Mod}(R)$ that sends $(c_1, c_2)$ to $\Phi(c_1) \otimes_R \Psi(c_2)$, and combine this with the identity $\Phi \otimes_R \Psi = (\Phi \otimes_R \Psi)\Delta$ and Proposition 8.

The proposition is proved.

**Definition 1** (strong connectedness). A category $\mathcal{C}$ is called strongly connected if $\mathcal{C}(c, c') \neq \emptyset$ (and $\mathcal{C}(c', c) \neq \emptyset$) for any two objects $c, c' \in \mathcal{C}$.
Proposition 10. If $\mathcal{C}$ is a strongly connected small category, then for any $\mathcal{M}: \mathcal{C} \rightarrow \mathcal{A}$ the map

$$i_{c_0}: \mathcal{M}(c_0) \rightarrow \text{colim} \mathcal{M}$$

is an epimorphism for any $c_0 \in \mathcal{C}$. Moreover, $\text{colim} \mathcal{M}$ is the largest constant quotient of $\mathcal{M}$. To be more precise, if we fix $c_0$, then for any two morphisms $\alpha, \beta: c \rightarrow c_0$ we have $i_{c_0}.\mathcal{M}(\alpha) = i_{c_0}.\mathcal{M}(\beta)$ and $i_{c_0}$ is initial among those. (Roughly speaking, $\text{colim} \mathcal{M}$ is the coequalizer of all morphisms $\mathcal{M}(\alpha): \mathcal{M}(c) \rightarrow \mathcal{M}(c_0)$.)

Proof. Consider a subcategory $\mathcal{C}_0$ of $\mathcal{C}$ with the same objects but with morphisms only to $c_0$, that is, $\text{ob}(\mathcal{C}_0) = \text{ob}(\mathcal{C})$, while $\mathcal{C}_0(c, c') = \emptyset$ if $c' \neq c_0$ and $\mathcal{C}_0(c, c_0) = \mathcal{C}(c, c_0)$. Denote by $\mathcal{M}_0: \mathcal{C}_0 \rightarrow \mathcal{A}$ the restriction of $\mathcal{M}$. Then the assertion can be reformulated as follows:

$$\text{colim} \mathcal{M} = \text{colim} \mathcal{M}_0.$$  

In order to prove this isomorphism, it is enough to prove that the embedding $\mathcal{C}_0 \rightarrow \mathcal{C}$ is a final functor (see [10], Ch. IX, §3). Fix an object $c_1 \in \mathcal{C}$ and prove that the category $c_1 \downarrow \mathcal{C}_0$ is not empty and connected. It is not empty because it contains the object $\text{id}_{c_1}: c_1 \rightarrow c_1$. It is connected because for any object $\alpha: c_1 \rightarrow c$ of it we can choose $\beta \in \mathcal{C}(c, c_0)$ and construct two morphisms in $c_1 \downarrow \mathcal{C}_0$ that connect $\alpha$ with the fixed object $\text{id}_{c_1}$.

The proposition is proved.

§ 3. Simplicial resolutions and derived colimits over the category of presentations

Let $\mathcal{C}$ be a small category with finite limits and colimits. Recall that a morphism $\alpha: c \rightarrow c'$ is called an effective epimorphism if it is a coequalizer of two arrows from the corresponding pullback of $\alpha$ with itself:

$$\alpha = \text{coeq}(c \times_{c'} c \rightrightarrows c).$$

An object $p$ of $\mathcal{C}$ is called projective if the hom-functor $\mathcal{C}(p, \cdot)$ sends effective epimorphisms to surjections.

A presentation of an object $c \in \mathcal{C}$ is an effective epimorphism $p \rightarrow c$ from a projective object. The category $\mathcal{C}$ is said to have enough projectives if any object has a presentation. Under this assumption a morphism $\alpha: c \rightarrow c'$ is an effective epimorphism if and only if $\mathcal{C}(p, \alpha): \mathcal{C}(p, c) \rightarrow \mathcal{C}(p, c')$ is surjective for any projective $p$ (see [14], Ch. II, §4, Proposition 2).

Assumptions. Below in this section we assume that $\mathcal{C}$ is a small category with finite limits and colimits and with enough projectives. Moreover, we denote by $\mathcal{A}$ an abelian category with exact small direct sums and enough projectives. The main examples for us are $\mathcal{A} = \text{Mod}(\Lambda)$ and $\mathcal{A} = \text{Mod}(\Lambda)^{\text{op}}$ for some ring $\Lambda$ (direct sums and projectives in $\text{Mod}(\Lambda)^{\text{op}}$ are direct products and injectives in $\text{Mod}(\Lambda)$).
Remark 1 (about set-theoretical assumptions). Here we just assume that $C$ is small. In what follows we will need to use Tarski-Grothendieck set theory. We will fix two universes $U$ and $U'$, $U \subseteq U'$, and consider the category $C$ of $U$-small algebraic objects of some kind (for example, the category of $U$-small groups). Then $C$ will be $U'$-small, and we need to consider an abelian category $\mathcal{A}$ with exact $U'$-small direct sums (for example, the category of $U'$-small modules over some ring). Thus our notion of smallness will be variable.

For an object $c \in C$ we consider three categories:

1) $C \downarrow c$, the category of objects over $c$;
2) its full subcategory $\text{Proj} \downarrow c$ of projective objects over $c$;
3) the most important one, the category of presentations $\text{Pres}(c)$.

The category $\text{Pres}(c)$ is the full subcategory of $\text{Proj} \downarrow c$ whose objects are presentations $p \to c$. One of the advantages of the category $\text{Pres}(c)$ is that it is strongly connected.

Lemma 2. The category $\text{Pres}(c)$ is strongly connected and has pairwise coproducts. In particular, it is contractible by Proposition 2.

Proof. Let $\sigma: p \to c$ and $\tau: q \to c$ be two objects of $\text{Pres}(c)$. Since $\tau$ is an effective epimorphism, the map $\mathcal{C}(p, \tau): \mathcal{C}(p, q) \to \mathcal{C}(p, c)$ is surjective. A preimage $\alpha \in \mathcal{C}(p, q)$ of $\sigma$ defines a morphism from $\sigma$ to $\tau$ in the category $\text{Pres}(c)$. Then it is strongly connected. Moreover, note that $p \amalg q$ is projective as well, and the map $(\sigma, \tau): p \amalg q \to c$ is the coproduct in this category.

The lemma is proved.

Definition 2 (simplicial projective resolution). We define a simplicial projective resolution of an object $c$ as a simplicial object $p_\bullet \in \mathcal{C}^{\Delta^{op}}$ whose components $p_n$ are projective, together with a morphism to the constant simplicial object $\varepsilon: p_\bullet \to c^{\text{const}}$ such that:

1) for any projective object $p'$ the morphism
$$\mathcal{C}(p', \varepsilon_\bullet): \mathcal{C}(p', p_\bullet) \to \mathcal{C}(p', c^{\text{const}})$$

is a trivial Kan fibration of simplicial sets;
2) $\varepsilon_0: p_0 \to c$ is a presentation.

Note that if $\varepsilon_\bullet: p_\bullet \to c^{\text{const}}$ is a simplicial projective resolution, then $\varepsilon_n = \varepsilon_0d^m_0$, $d^m_0: p_n \to p_0$ is a split epimorphism, hence $\varepsilon_n: p_n \to c$ is a presentation as well. Therefore, the simplicial projective resolution defines a simplicial object in the category of presentations that we denote by

$$\varepsilon^\mathcal{P}_\bullet \in \text{Pres}(c)^{\Delta^{op}}.$$

As usual, we treat a simplicial object $a_\bullet$ of an abelian category $\mathcal{A}$ as a complex with the differential $\delta_n = \sum (-1)^i d_i$, and denote

$$\pi_n(a_\bullet) = H_n(a_\bullet).$$

For a morphism $\alpha: c \to c'$ we denote by

$$\tilde{\alpha}: \text{Proj} \downarrow c \to \text{Proj} \downarrow c'$$

the corresponding functor of composition with $\alpha$. 
Proposition 11. Let \( \varepsilon_\bullet: p_\bullet \to c^{\text{const}} \) be a simplicial projective resolution of an object \( c \), \( \varepsilon_\bullet^p \) be the corresponding simplicial presentation, and \( \mathcal{M}: \text{Proj} \downarrow c \to \mathcal{A} \) be a functor. Then for any \( n \) there is an isomorphism

\[
\colim_{\text{Proj} \downarrow c} \mathcal{M} = \pi_n(\mathcal{M}(\varepsilon_\bullet^p)).
\]

Proof. The idea of the proof is the following. We construct a double complex \( D_{\bullet\bullet} \) whose vertical homology satisfies

\[
H_{\bullet,m}^{\text{vert}}(D_{\bullet\bullet}) = \begin{cases} 
\mathcal{M}(\varepsilon_\bullet^p), & m = 0, \\
0, & m \neq 0,
\end{cases}
\]

and the horizontal homology satisfies

\[
H_{n,\bullet}^{\text{hor}}(D_{\bullet\bullet}) = \begin{cases} 
C_\bullet(\text{Proj} \downarrow c, \mathcal{M}), & n = 0, \\
0, & n \neq 0.
\end{cases}
\]

In this case the two spectral sequences of the double complex \( D_{\bullet\bullet} \) imply that

\[
\colim_* \mathcal{M} = H_*(C_\bullet(\text{Proj} \downarrow c, \mathcal{M})) = H_*(\mathcal{M}(\varepsilon_\bullet^p)).
\]

Then we only need to construct a double complex \( D_{\bullet\bullet} \) satisfying (3.1) and (3.2). Set

\[
D_{nm} = C_m(\text{Proj} \downarrow p_n, \mathcal{M}\tilde{e}_n).
\]

The vertical differentials come from \( C_\bullet(\text{Proj} \downarrow p_n, \mathcal{M}\tilde{e}_n) \). Hence \( D_{n\bullet} = C_\bullet(\text{Proj} \downarrow p_n, \mathcal{M}\tilde{e}_n) \). The horizontal differentials \( \delta_{n,m}^{\text{hor}}: D_{n,m} \to D_{n-1,m} \) are defined by

\[
\delta_{n,m}^{\text{hor}} = \sum (-1)^i C_m(\tilde{d}_i): C_m(\text{Proj} \downarrow p_n, \mathcal{M}\tilde{e}_n) \to C_m(\text{Proj} \downarrow p_{n-1}, \mathcal{M}\tilde{e}_{n-1}).
\]

We prove (3.1). Since the category \( \text{Proj} \downarrow p_n \) has the terminal object \( \text{id}: p_n \to p_n \), we obtain \( \colim \mathcal{N} = \mathcal{N}(\text{id}) \) for any functor \( \mathcal{N}: \text{Proj} \downarrow p_n \to \mathcal{A} \). Therefore, \( \colim \mathcal{A}^{\text{Proj} \downarrow p_n} \to \mathcal{A} \) is exact and \( \colim_n \mathcal{N} = 0 \) for \( n \geq 1 \). It follows that

\[
H_m(C_\bullet(\text{Proj} \downarrow p_n, \mathcal{M}\tilde{e}_n)) = \begin{cases} 
\mathcal{M}\tilde{e}_n(\text{id}) = \mathcal{M}(\varepsilon_n), & m = 0, \\
0, & m \neq 0.
\end{cases}
\]

Therefore (3.1) is satisfied.

We prove (3.2). Note that if \( c' \) is an object of \( \mathcal{C} \), then an \( n \)-simplex of \( \mathcal{N}(\text{Proj} \downarrow c') \) is a sequence of morphisms of projective objects

\[
p^{(0)} \xleftarrow{\alpha_1} p^{(1)} \xleftarrow{\alpha_2} \ldots \xleftarrow{\alpha_m} p^{(m)}
\]

together with morphisms \( p^{(i)} \to c' \) such that the diagram is commutative. In order to fix the simplex it is enough to remember the sequence \( p^{(0)} \leftarrow p^{(1)} \leftarrow \ldots \leftarrow p^{(m)} \).
and the morphism $p^{(0)} \rightarrow c'$ because all other morphisms are compositions of these. So we have:

$$C_m(\text{Proj} \downarrow c', \mathcal{N}) = \bigoplus_{(p^{(0)} \rightarrow p^{(1)} \rightarrow \cdots \rightarrow p^{(m)})} \mathcal{N}(\varphi \alpha_1 \cdots \alpha_m).$$

Assume that we have a fixed morphism $\varepsilon': c' \rightarrow c$ and $\mathcal{N} = \mathcal{M}\varepsilon'$. Then $\mathcal{N}(\varphi \alpha_1 \cdots \alpha_m) = \mathcal{M}(\varepsilon' \varphi \alpha_1 \cdots \alpha_m)$, so it depends only on the composition $\psi := \varepsilon' \varphi: p^{(0)} \rightarrow c$. Therefore,

$$C_m(\text{Proj} \downarrow c', \mathcal{M}\varepsilon') = \bigoplus_{(p^{(0)} \rightarrow \cdots \rightarrow p^{(m)})} \mathcal{M}(\psi \alpha_1 \cdots \alpha_m),$$

where the last sum runs over the hom-set $(\text{Proj} \downarrow c)(\psi, \varepsilon')$. Since the summand $\mathcal{M}(\psi \alpha_1 \cdots \alpha_m)$ does not really depend on $\varphi$, we can rewrite this in the following form

$$C_m(\text{Proj} \downarrow c', \mathcal{M}\varepsilon') = \bigoplus_{(p^{(0)} \rightarrow \cdots \rightarrow p^{(m)})} \mathcal{M}(\psi \alpha_1 \cdots \alpha_m)((\text{Proj} \downarrow c)(\psi, \varepsilon')).$$

This isomorphism is natural by $\varepsilon'$. Therefore, in order to prove (3.2) it is sufficient to prove that

$$H_n(A((\text{Proj} \downarrow c)(\psi, \varepsilon))) = \begin{cases} A((\text{Proj} \downarrow c)(\psi, \text{id}_c)) = A, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

where $A$ is an object of $\mathcal{A}$ and $\psi: p^{(0)} \rightarrow c$ is a morphism from a projective object $p^{(0)}$.

According to the definition given in [2], Appendix II, § 4, we see that the left-hand homology is the homology of the simplicial set $(\text{Proj} \downarrow c)(\psi, \varepsilon)$ with coefficients in the object $A$. So we need to prove that

$$H_n((\text{Proj} \downarrow c)(\psi, \varepsilon), A) = \begin{cases} A, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Lemma 4.3 in [2], Appendix II, § 4, implies that it enough to prove that the simplicial set $(\text{Proj} \downarrow c)(\psi, \varepsilon)$ is contractible.

We prove that $(\text{Proj} \downarrow c)(\psi, \varepsilon)$ is contractible. Note that $(\text{Proj} \downarrow c)(\psi, \varepsilon_n)$ is the preimage of $\psi \in \mathcal{C}(p^{(0)}, c)$ under the map $\mathcal{C}(p^{(0)}, p_n) \rightarrow \mathcal{C}(p^{(0)}, c)$. It follows that $(\text{Proj} \downarrow c)(\psi, \varepsilon)$ is the fibre of the trivial Kan fibration $\mathcal{C}(p^{(0)}, p_\bullet) \rightarrow \mathcal{C}(p^{(0)}, c_{\text{const}})$. Hence $(\text{Proj} \downarrow c)(\psi, \varepsilon)$ is fibrant and acyclic. Thus it is contractible.

The proposition is proved.

Remark 2. The proof of Proposition 11 does not work for the category $\text{Pres}(c)$ because in this case $\varphi$ runs over the set of effective epimorphisms $\varphi: \psi \rightarrow \varepsilon'$ but not over the whole hom-set.

**Proposition 12.** Let $c \in \mathcal{C}$ be an object, $\mathcal{M}: \text{Proj} \downarrow c \rightarrow \mathcal{A}$ be a functor and $\mathcal{M}': \text{Pres}(c) \rightarrow \mathcal{A}$ be a restriction of it. Then there are isomorphisms

$$\text{colim}_n \mathcal{M}' \cong \text{colim}_n \mathcal{M}.$$
Proof. We only need to check that the embedding \( \text{Pres}(c) \hookrightarrow \text{Proj} \downarrow c \) satisfies the assumption of Proposition 6. So we need to prove that for any fixed morphism \( \varphi_0: p_0 \to c \) from a projective object \( p_0 \), the category \( \varphi_0 \downarrow \text{Pres}(c) \) is contractible. An object of the category \( \varphi_0 \downarrow \text{Pres}(c) \) is a triple \( C = (p, \pi, \alpha) \), where \( p \) is projective, \( \pi: p \to c \) is a presentation and \( \alpha: p_0 \to p \) is a morphism such that the diagram

\[
\begin{array}{ccc}
C: & p_0 & \alpha \to p \\
& \varphi_0 \downarrow & \pi \\
& c
\end{array}
\]

is commutative. Fix a presentation \( \varepsilon: p_1 \to c \) and consider the object \( C_0 = (p_0 \sqcup p_1, (\varphi_0, \varepsilon), i_{p_0}) \) of the category, where \( i_{p_0}: p_0 \to p_0 \sqcup p_1 \) is the standard embedding and \( (\varphi_0, \varepsilon): p_0 \sqcup p_1 \to c \) is the morphism whose restriction to \( p_0 \) is \( \varphi_0 \) and the restriction to \( p_1 \) is the standard projection \( \varepsilon: p_1 \to c \):

\[
C_0: \quad p_0 \overset{i_{p_0}}{\longrightarrow} p_0 \sqcup p_1
\]

From Proposition 2 we obtain that it is sufficient to prove that the coproduct \( C \sqcup C_0 \) exists for any object \( C = (p, \pi, \alpha) \). We construct it as follows. We consider the object \( C' = (p \sqcup p_1, (\pi, \varepsilon), i_p \circ \alpha) \)

\[
C': \quad p_0 \overset{i_p \circ \alpha}{\longrightarrow} p \sqcup p_1
\]

and prove that \( C' = C \sqcup C_0 \). Note that the big square

\[
\begin{array}{ccc}
p_0 & \overset{\varphi_0}{\longrightarrow} & p_0 \sqcup p_1 \\
& \varepsilon \downarrow & \varphi_0 \\
p \overset{\pi}{\longrightarrow} & c \end{array}
\]

is a pushout in the category \( \mathcal{C} \). Moreover, one can check that this is a pushout in the category \( \text{Proj} \downarrow c \). Generally, if \( \mathcal{D} \) is a category and \( d \in \mathcal{D} \), then the coproduct in \( d \downarrow \mathcal{D} \) is a pushout over \( d \). Therefore, the square is a coproduct in \( \varphi_0 \downarrow (\text{Proj} \downarrow c) \). Since it lies in \( \varphi_0 \downarrow \text{Pres}(c) \), it is a coproduct in \( \varphi_0 \downarrow \text{Pres}(c) \) as well.

**Theorem 1** (cf. [9], Proposition 5.5.8.15). Let \( \varepsilon_\bullet: p_\bullet \to c^\text{const} \) be a simplicial projective resolution, \( \varepsilon^P_\bullet \) be the corresponding simplicial presentation, \( \mathcal{M}: \text{Proj} \downarrow c \to \mathcal{A} \) be a functor, and \( \mathcal{M}' : \text{Pres}(c) \to \mathcal{A} \) be its restriction to the category of presentations. Then for any \( n \) there is an isomorphism

\[
\colim_n \mathcal{M}' \cong \pi_n(\mathcal{M}(\varepsilon^P_\bullet)).
\]
The proof follows from Proposition 11 and Proposition 12.

**Corollary 1.** Let $\Phi: \mathcal{C} \to \mathcal{A}$ be a functor, and $\varepsilon_\bullet: p_\bullet \to c^{\text{const}}$ be a simplicial projective resolution. Then for any $n \geq 0$ there is an isomorphism

$$\text{colim}_n \Phi' \cong \pi_n(\Phi(p_\bullet)),$$

where $\Phi': \text{Pres}(c) \to \mathcal{A}$ is the functor such that $\Phi'(p \to c) = \Phi(p)$.

**Remark 3.** Corollary 1 implies that the homotopy groups $\pi_n(\Phi(p_\bullet))$ in our general setting do not depend on the choice of the resolution $p_\bullet$. Thus $\text{colim}_n \Phi(p)$ can be considered as a version of a derived functor for a nonadditive functor.

**Remark 4.** Assume that in the category $\mathcal{C}$ there is a class of ‘free’ objects $\mathcal{F}$ which consists of projective objects, and any projective object is a retract of an object in $\mathcal{F}$. Denote by $\text{Pres}^\mathcal{F}(c)$ the full subcategory of $\text{Pres}(c)$,

$$\text{Pres}^\mathcal{F}(c) \subseteq \text{Pres}(c),$$

consisting of presentations $f \to c$, where $f \in \mathcal{F}$. For any functor $\mathcal{M}: \text{Pres}(c) \to \mathcal{A}$ we denote its restriction by $\mathcal{M}^\mathcal{F}: \text{Pres}^\mathcal{F}(c) \to \mathcal{A}$. Then Proposition 7 implies that there is an isomorphism

$$\text{colim}_* \mathcal{M} = \text{colim}_* \mathcal{M}^\mathcal{F}.$$ 

**Lemma 3.** Let $\mathcal{C}$ be a category together with a couple of adjoint functors

$$F: \text{Sets} \Rightarrow \mathcal{C}; U \Rightarrow \mathcal{C}$$

such that $U$ is a composition of some functor to the category of groups and the forgetful functor from groups to sets:

$$U: \mathcal{C} \xrightarrow{U'} \text{Gr} \to \text{Sets}.$$

Moreover, we assume that a morphism $\alpha$ of $\mathcal{C}$ is an effective epimorphism if and only if $U\alpha$ is a surjection.

Then

- $\mathcal{C}$ has enough projectives;
- an object is projective if and only if it is a retract of $FX$ for some set $X$;
- a simplicial object together with a morphism $\varepsilon_\bullet: p_\bullet \to c^{\text{const}}$ is a simplicial projective resolution if and only if $p_n$ is projective for any $n$ and $U'\varepsilon: U'p_\bullet \to U'c^{\text{const}}$ is a weak equivalence in $\text{Gr}$.

**Proof.** First note that, if we combine the fact that $U$ sends effective epimorphisms to surjections together with the isomorphism $\mathcal{C}(FX, \cdot) \cong \text{Sets}(X, U(\cdot))$, we obtain that $FX$ is projective for any set $X$. A retract of a projective object is projective. Hence a retract of $FX$ is projective. On the other hand, since the composition $Uc \to UFUc \to Uc$ is the identity, we obtain that $UFUc \to Uc$ is a surjection, hence $FUc \to c$ is an effective epimorphism. It follows that $\mathcal{C}$ has enough projectives. Moreover, a projective object $p$ is a retract of $FUp$. Hence an object is projective if and only if it is a retract of $FX$ for some $X$.

A retract of a Kan fibration is a Kan fibration. Thus $\varepsilon_\bullet: p_\bullet \to c$ is a simplicial projective resolution if and only if $\mathcal{C}(FX, p_\bullet) \to \mathcal{C}(FX, c)$ is a trivial Kan fibration.
and \( \mathcal{C}(FX, p_0) \to \mathcal{C}(FX, c) \) is surjective for any set \( X \). Using the adjunction we obtain that it is equivalent to the fact that \((U'p_\bullet)^X \to (U'c)^X\) is a trivial Kan fibration and \((U'p_0)^X \to (U'c)^X\) is an epimorphism. Since an epimorphism of simplicial groups is always a Kan fibration, this is equivalent to the fact that \((U'p_\bullet)^X \to (U'c)^X\) is a trivial Kan fibration and \((U'p_0)^X \to (U'c)^X\) is an epimorphism. Since an epimorphism of simplicial groups is always a Kan fibration, this is equivalent to the fact that \((U'p_\bullet)^X \to (U'c)^X\) is a trivial Kan fibration and \((U'p_0)^X \to (U'c)^X\) is an epimorphism. Since \(\pi_0(U'p_\bullet)^X \to (U'c)^X\) is a quotient of \(U'p_0^X \to (U'c)^X\) and \(\pi_0(U'c)^X = c\), we obtain that \((U'p_0)^X \to (U'c)^X\) is an epimorphism if \((U'p_\bullet)^X \to (U'c)^X\) is a weak equivalence. The assertion follows.

The lemma is proved.

**Proposition 13.** Let \( k \) be a commutative ring, \( A \) be a commutative \( k \)-algebra, and \( M \) be an \( A \)-module. Consider the category \( \text{Pres}^{\text{poly}}(A) \) of surjective homomorphisms \( F \mapsto A \), where \( F \) is a polynomial \( k \)-algebra. Then the André-Quillen homology can be presented as the following derived colimit over the category:

\[
D_n(A/k, M) = \text{colim}_n \Omega^{\text{comm}}(F) \otimes_F M,
\]

where \( \Omega^{\text{comm}}(F) \) is the module of Kähler differentials of \( F \).

**Proof.** Effective epimorphisms of the category of commutative \( k \)-algebras are surjective homomorphisms. Projective objects are retracts of polynomial algebras. Remark 4 shows that colimits over \( \text{Pres}(A) \) and \( \text{Pres}^{\text{poly}}(A) \) are the same. By definition

\[
D_n(A/k, M) = \pi_n(\Omega^{\text{comm}}(F) \otimes_F M)
\]

(see [17], Definition 8.8.2). Then the assertion follows from Theorem 1.

The proposition is proved.

### §4. Groups

In this section we fix a group \( G \), and all colimits are considered over the category of presentations \( \text{Pres}(G) \). A presentation here is a surjective homomorphism from a free group \( F \to G \). Its kernel is denoted by

\[
R = \text{Ker}(F \to G).
\]

We treat \( F \) and \( R \) as functors \( \text{Pres}(G) \to \text{Gr} \).

Here we fix two universes \( U \) and \( U' \), \( U \in U' \), and assume that \( G \) is \( U \)-small; a presentation \( F \to G \) is \( U \)-small; we denote by \( \text{Gr} \) the category of \( U \)-small groups, but denote by \( \text{Ab} \) the category of \( U' \)-small abelian groups.

**Proposition 14.** Let \( \Phi : \text{Gr} \to \text{Ab} \) be a functor and \( L_n \Phi \) be the simplicial derived functors of \( \Phi \). Then

\[
L_n \Phi(G) \cong \text{colim}_n \Phi(F),
\]

where \( \Phi(F) : \text{Pres}(G) \to \text{Ab} \) is the functor that sends \( (F \to G) \) to \( \Phi(F) \).

The proof follows from Corollary 1.

**Proposition 15.** Let \( \Phi : \text{Gr} \to \text{Ab} \) be a functor. Then

\[
\text{colim}_0 \Phi(R) = \Phi(1) \quad \text{and} \quad \text{colim}_n \Phi(R) = 0
\]

for \( n \geq 1 \), where \( \Phi(R) : \text{Pres}(G) \to \text{Ab} \) is the functor that sends \( (F \to G) \) to \( \Phi(R) = \Phi(\text{Ker}(F \to G)) \).
Proof. The kernel $R_\bullet = \text{Ker}(F_\bullet \to G^{\text{const}})$ is a free resolution of the trivial group. Then Theorem 1 implies that $\text{colim}_n \Phi(R) = L_n \Phi(1)$.

The proposition is proved.

**Theorem 2.** Let $M$ be a $G$-module. Then

$$H_{n+1}(G, M) = \text{colim}_n H_1(F, M)$$

for any $n \geq 0$. In particular,

$$H_{n+1}(G) = \text{colim}_n F_{ab}.$$

Proof. For a group $H$ and a $\mathbb{Z}[H]$-module $N$ we denote by $C_\bullet(H, N)$ the standard complex computing homology $H_n(H, N)$ such that $C_n(H, N) = \mathbb{Z}[H]^{\otimes n} \otimes N$.

Consider the complex of functors $P_\bullet = C_\bullet(F, I(F) \otimes M)$ from Proj $\downarrow G$ to Ab, where $P_n$ is the functor that sends $F \to G$ to $C_n(F, I(F) \otimes M)$ and $I(F)$ is the augmentation ideal of $\mathbb{Z}[F]$. Note that $H_0(P_\bullet) = H_0(F, I(F) \otimes M) = H_1(F, M)$.

We prove that $P_\bullet$ is a colim-acyclic resolution of the functor $H_1(F, M)$ in the category of functors Proj $\downarrow G \to \text{Ab}$. First note that for $n \geq 1$ we have

$$H_n(P_\bullet) = H_n(F, I(F) \otimes M) = H_{n+1}(F, M) = 0,$$

because $\mathbb{Z}[F]$ is hereditary (see [17], Corollary 6.2.7). Then we only need to prove that $P_n$ is colim-acyclic. Take a free simplicial resolution $F_\bullet \sim G$. Then $\pi_n(\mathbb{Z}[F_\bullet]) = H_n(F_\bullet)$, where $F_\bullet$ is considered as a simplicial set. Since $F_\bullet \to G^{\text{const}}$ is a weak equivalence of simplicial sets, this map induces an isomorphism on the level of homology of the simplicial sets. Then $\pi_n(\mathbb{Z}[F_\bullet]) = 0$ for $n \neq 0$ and $\pi_0(\mathbb{Z}[F_\bullet]) = \mathbb{Z}[G]$. Theorem 1 implies that

$$\text{colim}_0 \mathbb{Z}[F] = \mathbb{Z}[G] \quad \text{and} \quad \text{colim}_n \mathbb{Z}[F] = 0$$

for $n \neq 0$. Then the long exact sequence of colimits applied to the short exact sequence $I(F) \to \mathbb{Z}[F] \to \mathbb{Z}$ implies that

$$\text{colim}_0 I(F) = I(G) \quad \text{and} \quad \text{colim}_n I(F) = 0$$

for $n \neq 0$. Combining this with the Künneth theorem for colimits (Proposition 9) we obtain

$$\text{colim}_0 C_m(F, I(F) \otimes M) = C_m(G, I(G) \otimes M), \quad \text{colim}_n C_m(F, I(F) \otimes M) = 0 \quad (4.1)$$

for $n \neq 0$. In particular, $P_n$ is acyclic and $P_\bullet$ is a colim-acyclic resolution of $H_1(F, M)$.

Since $P_\bullet$ is a colim-acyclic resolution, we can use it to compute $\text{colim}_n H_1(F, M)$. Equation (4.1) implies that $\text{colim}_0 P_\bullet = C_\bullet(G, I(G) \otimes M)$. Then the assertion follows from the formula

$$H_n(C_\bullet(G, I(G) \otimes M)) = H_n(G, I(G) \otimes M) = H_{n+1}(G, M) \quad \text{for} \ n \geq 0.$$

The theorem is proved.
Remark 5. If we take a short exact sequence \( M_1 \hookrightarrow M_2 \twoheadrightarrow M_3 \) of \( G \)-modules and a presentation \( F \rightarrow G \), then we obtain an exact sequence

\[
0 \rightarrow H_1(F, M_1) \rightarrow H_1(F, M_2) \rightarrow H_1(F, M_3) \rightarrow (M_1)_G \rightarrow (M_2)_G \rightarrow (M_3)_G \rightarrow 0.
\]

If we apply the spectral sequence of colimits (Proposition 4) to this exact sequence and use that

\[
colim_n H_1(F, M_i) = H_{n+1}(G, M_i), \quad \colim_n (M_i)_G = 0 \quad \text{for } n \geq 1,
\]

and

\[
colim_0 (M_i)_G = (M_i)_G,
\]

then we obtain the long exact sequence

\[
\cdots \rightarrow H_n(G, M_1) \rightarrow H_n(G, M_2) \rightarrow H_n(G, M_3) \rightarrow H_{n-1}(G, M_1) \rightarrow \cdots.
\]

§ 5. Hochschild and cyclic homology of unital algebras

In this section we assume that all algebras are unital and associative. We assume that \( k \) is a field, denote the category of unital algebras over \( k \) by \( \text{Alg}^u \) and set \( \otimes = \otimes_k \). Effective epimorphisms in this category are surjective homomorphisms, and projective objects are retracts of free algebras. A free algebra \( F \) is isomorphic to the tensor algebra \( F \cong T(V) = \bigoplus_{n \geq 0} V^\otimes n \). Then the objects of the category of presentations \( \text{Pres}(A) \) are surjective homomorphisms \( F \rightarrow A \) from retracts of free algebras. In this section we consider only colimits of functors of the type \( \text{Pres}(A) \rightarrow \text{Vect} \).

Here we fix two universes \( U \) and \( U' \), \( U \in U' \), and assume that \( A \) is \( U \)-small and a presentation \( F \rightarrow A \) is \( U \)-small; we denote by \( \text{Alg}^u \) the category of \( U \)-small unital algebras, but denote by \( \text{Vect} \) the category of \( U' \)-small vector spaces.

Consider the subcategory

\[
\text{Pres}_\text{free}(A) \subseteq \text{Pres}(A)
\]

consisting of presentations \( F \rightarrow A \), where \( F \) is free. Remark 4 implies the isomorphism

\[
\colim_* \mathcal{M} \cong \colim_* \mathcal{M}
\]

for any functor \( \mathcal{M} : \text{Pres}(A) \rightarrow \text{Mod}(k) \). So we can limit ourselves to considering only presentations \( F \rightarrow A \), where \( F \) is a free algebra if the functor \( \mathcal{M} \) can be defined for all presentations. For a presentation \( F \rightarrow A \) we set \( R = \text{Ker}(F \rightarrow A) \).

For an algebra \( A \) and an \( A \)-bimodule \( M \) we set

\[
M_\sharp := \frac{M}{[M, A]} = HH_0(A, M).
\]

For an algebra \( A \) we set

\[
A^e = A^{op} \otimes A.
\]

If \( M \) is an \( A \)-bimodule, we can consider it both
(1) as a left $A^e$-module via $(a \otimes b) \ast m = bma$;
(2) as a right $A^e$-module via $m \ast (a \otimes b) = amb$.

Note that with this definition we have isomorphisms

$$M \otimes_{A^e} N \cong N \otimes_{A^e} M \cong (M \otimes_A N)_{\sharp} \cong (N \otimes_A M)_\sharp$$

for any two $A$-bimodules $M$ and $N$.

5.1. $\mathcal{O}$-modules and $\mathcal{O}$-bimodules. We denote by $\text{Mod}^r$ the category of couples $(M, B)$, where $B \in \text{Alg}^u$ and $M$ is a right $B$-module. Morphisms in this category are couples $(f, \varphi): (M, B) \to (M', B')$, where $\varphi: B \to B'$ is a homomorphism of algebras and $f: M \to M'$ is a homomorphism of $B$-modules, where $M'$ is considered as an $A$-module via $\varphi$. There is an obvious forgetful functor

$$\text{Mod}^r \to \text{Alg}^u, \quad (M, B) \mapsto B.$$

Consider the forgetful functor

$$\mathcal{O}: \text{Pres}(A) \to \text{Alg}^u, \quad \mathcal{O}(F \to A) = F.$$

A right $\mathcal{O}$-module is a functor $\mathcal{M}: \text{Pres}(A) \to \text{Mod}^r$ such that the diagram

$$\begin{array}{ccc}
\text{Pres}(A) & \xrightarrow{\mathcal{M}} & \text{Mod}^r \\
\downarrow{\mathcal{O}} & & \downarrow{\text{Mod}^r} \\
\text{Alg}^u & & \\
\end{array}$$

is commutative. An $\mathcal{O}$-homomorphism is a natural transformation $\mathcal{M} \to \mathcal{M}'$, whose second component consists of identity homomorphisms $\text{id}_F: F \to F$. Thus we obtain the category of right $\mathcal{O}$-modules $\text{Mod}^r(\mathcal{O})$. Similarly one can define the category of left $\mathcal{O}$-modules $\text{Mod}^l(\mathcal{O})$ and the category of $\mathcal{O}$-bimodules $\text{Bimod}(\mathcal{O})$.

By abuse of notation, for an $\mathcal{O}$-module $M$ we will identify $M(F \to A)$ with the underlying $F$-module $\mathcal{M}(\text{Pres}(A) \to \text{Mod}^r)$ such that the diagram

$$\begin{array}{ccc}
\text{Pres}(A) & \xrightarrow{\mathcal{M}} & \text{Mod}^r \\
\downarrow{\mathcal{O}} & & \downarrow{\text{Mod}^r} \\
\text{Alg}^u & & \\
\end{array}$$

is commutative. An $\mathcal{O}$-homomorphism is a natural transformation $\mathcal{M} \to \mathcal{M}'$, whose second component consists of identity homomorphisms $\text{id}_F: F \to F$. Thus we obtain the category of right $\mathcal{O}$-modules $\text{Mod}^r(\mathcal{O})$. Similarly one can define the category of left $\mathcal{O}$-modules $\text{Mod}^l(\mathcal{O})$ and the category of $\mathcal{O}$-bimodules $\text{Bimod}(\mathcal{O})$.

We claim that $\text{mod}^r : \text{mod}^r : \text{Alg}^u \to \text{Mod}^l$ is a vector space together with the structure of an $F$-module which is ‘natural by presentation’. For an $\mathcal{O}$-module $\mathcal{M}$ we set

$$\text{colim}_n \mathcal{M} := \text{colim}_n (\text{Pres}(A) \xrightarrow{\mathcal{M}} \text{Mod}^r \to \text{Vect}),$$

where $\text{Mod}^r \to \text{Vect}$ is the forgetful functor $(M, B) \mapsto M$.

We claim that $\text{mod}^r : \text{mod}^r : \text{Alg}^u \to \text{Mod}^l$ has the natural structure of an $A$-module. Indeed, Theorem 1 implies that $\text{colim}_0 F = A$, and the Künneth formula for colimits (Proposition 9) implies that $\text{colim}_0 F \otimes \mathcal{M}(F \to A) = A \otimes \mathcal{M}_0$. Therefore, the natural transformation $F \otimes \mathcal{M}(F \to A) \to \mathcal{M}(F \to A)$ induces a map $A \otimes \mathcal{M}_0 \to \mathcal{M}_0$. It is easy to check that this map defines the structure of a module on $\mathcal{M}_0$. Therefore, we obtain a well-defined functor

$$\text{Colim}_0: \text{Mod}^r(\mathcal{O}) \to \text{Mod}^r(A),$$

and similarly

$$\text{Colim}_0: \text{Mod}^l(\mathcal{O}) \to \text{Mod}^l(A) \quad \text{and} \quad \text{Colim}_0: \text{Bimod}(\mathcal{O}) \to \text{Bimod}(A).$$
Proposition 16. Let $\mathcal{M}$ and $\mathcal{N}$ be colim-acyclic $\mathcal{O}$-bimodules. Set $\mathcal{M}_0 = \text{Colim}_0 \mathcal{M}$ and $\mathcal{N}_0 = \text{Colim}_0 \mathcal{N}$. Then there is a spectral sequence of homological type

$$E^2_{n,m} = \text{colim}_n \text{Tor}^{\mathcal{O}}_m(\mathcal{M}, \mathcal{N}) \implies \text{Tor}^{\mathcal{O}}_{n+m}(\mathcal{M}_0, \mathcal{N}_0).$$

Proof. For an algebra $B$ we denote by $\text{Bar}_\bullet(B)$ the standard bar resolution of the bimodule $B$, where $\text{Bar}_n(B) = B^\otimes n+2$ and

$$d(b_0 \otimes \cdots \otimes b_{n+1}) = \sum (-1)^i b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_{n+1}.$$

For a right $B$-module $L$ and a left $B$-module $L'$ we set

$$C_{\mathcal{O}}^B(L, L') = L \otimes_B \text{Bar}_\bullet(B) \otimes_B L'.$$

It is easy to check that $\text{H}_n(C_{\mathcal{O}}^B(L, L')) = \text{Tor}_n^B(L, L')$ and $C_{\mathcal{O}}^B(L, L') = L \otimes B^\otimes n \otimes L'$ and the differential is given by the same formula as in $\text{Bar}_\bullet(B)$.

We consider the following complex of functors from $\text{Pres}(A)$ to $\text{Vect}$:

$$C_\bullet = C_{\mathcal{O}}^{F^e}(\mathcal{M}, \mathcal{N}).$$

Note that $C_n = \mathcal{M} \otimes F^{\otimes 2n} \otimes \mathcal{N}$ and $H_n(C_\bullet) = \text{Tor}_{F^e}^n(\mathcal{M}, \mathcal{N})$. The K"unneth formula for colimits together with the isomorphism $\text{colim}_n \mathcal{M} = \text{colim}_n \mathcal{N} = 0$ for $n \neq 0$ implies that

$$\text{colim}_n C_\bullet = 0 \quad \text{and} \quad \text{colim}_0 C_\bullet = C_{\mathcal{O}}^{A^e}(\mathcal{M}_0, \mathcal{N}_0)$$

for $n \neq 0$. Then the assertion follows from Proposition 5.

Corollary 2. Let $\mathcal{M}$ be a colim-acyclic $\mathcal{O}$-bimodule. Set $\mathcal{M}_0 = \text{Colim}_0 \mathcal{M}$. Then there is a long exact sequence that is natural with respect to $\mathcal{M}$

$$\text{colim}_{n-1} H_1(F, \mathcal{M}) \longrightarrow H_n(A, \mathcal{M}_0) \longrightarrow \text{colim}_n \mathcal{M}_0^e$$

$$\text{colim}_{n-2} H_1(F, \mathcal{M}) \longrightarrow H_{n-1}(A, \mathcal{M}_0) \longrightarrow \text{colim}_{n-1} \mathcal{M}_0^e$$

Proof. This follows from Proposition 16 if we take one of the $\mathcal{O}$-bimodules equal to $F$ and use that $H_n(F, \cdot) = 0$ for $n \geq 2$ and $H_0(F, \cdot) = (\cdot)^e_2$ (see [17], Proposition 9.1.6).

Corollary 3. Let $\mathcal{M}$ be a right $\mathcal{O}$-module and $\mathcal{N}$ be a left $\mathcal{O}$-module. Assume that $\mathcal{M}$ and $\mathcal{N}$ are colim-acyclic. Set $\mathcal{M}_0 = \text{Colim}_0 \mathcal{M}$ and $\mathcal{N}_0 = \text{Colim}_0 \mathcal{N}$. Then
there is a long exact sequence
\[ \text{colim}_{n-1} H_1(F, \mathcal{N} \otimes \mathcal{M}) \rightarrow H_n(A, \mathcal{N}_0 \otimes \mathcal{M}_0) \rightarrow \text{colim}_n \mathcal{M} \otimes_F \mathcal{N} \]
\[ \text{colim}_{n-2} H_1(F, \mathcal{N} \otimes \mathcal{M}) \rightarrow H_{n-1}(A, \mathcal{N}_0 \otimes \mathcal{M}_0) \rightarrow \text{colim}_{n-1} \mathcal{M} \otimes_F \mathcal{N} \]

**Proof.** This follows from Corollary 2 together with the isomorphism \((\mathcal{N} \otimes \mathcal{M})_\bullet = \mathcal{M} \otimes_F \mathcal{N}\) and the Künneth formula for colimits.

**Lemma 4.** For any \( n \) and \( 1 \leq m \leq l \) we have
\[ \text{colim}_n R^m = 0 \quad \text{and} \quad \text{colim}_n \frac{R^m}{R^l} = 0. \]

**Proof.** The second isomorphism follows from the first and the short exact sequence \( R^l \rightarrow R^m \rightarrow R^m/R^l \).

We prove the first equality. The proof is by induction. For \( m = 1 \) it follows from the fact that the map \( F \rightarrow A \) induces an isomorphism on the level of colimits \( \text{colim}_* F = \text{colim}_* A \) (which follows from Theorem 1). We make the step of induction. First note that for any two ideals \( a \) and \( b \) of a ring \( \Lambda \) there is a short exact sequence
\[ \text{Tor}_1^\Lambda (\Lambda/a, \Lambda/b) \rightarrow a \otimes_\Lambda b \rightarrow ab \]
(see [1], Chapter VI, Exercise 19). It follows that \( R \otimes_F R^m = R^{m+1} \). Moreover, \( R \) and \( R^m \) are projective as \( F \)-modules (right and left) because \( F \) is hereditary (see [17], Proposition 9.1.6). It follows that \( R^m \otimes R \) is a projective bimodule. Then the assertion follows from Corollary 3.

The lemma is proved.

**5.2. Hochschild homology.** Consider the kernel of the multiplication map
\[ \Omega(A) = \text{Ker}(A \otimes A \rightarrow A). \]
Since the beginning \( A^{\otimes 3} \rightarrow A^{\otimes 2} \rightarrow A \) of the bar resolution is exact, the map \( A^{\otimes 3} \rightarrow \Omega(A) \) given by \( a \otimes b \otimes c \rightarrow ab \otimes c - a \otimes bc \) is an epimorphism. Hence any element of \( \Omega(A) \) can be presented as a sum of elements of the form \( ab \otimes c - a \otimes bc \). If we identify \( A \otimes A = A^e \), then \( \Omega(A) \) is a right ideal of \( A^e \) but it is not necessarily a left ideal. The bimodule \( \Omega(A) \) is known as the bimodule of noncommutative differential forms and the map
\[ d: A \rightarrow \Omega(A), \quad d(a) = a \otimes 1 - 1 \otimes a, \]
is the universal derivation. Here we always consider \( A \otimes A \) as a bimodule with the following structure: \( a'(a \otimes b)b' = a'a \otimes bb' \). If we consider the long exact sequence of \( \text{Tor}_1^A(M, \cdot) \) applied to the short exact sequence \( \Omega(A) \rightarrow A^e \rightarrow A \) of right \( A^e \)-modules, we obtain the short exact sequence
\[ 0 \rightarrow H_1(A, M) \rightarrow \Omega(A) \otimes_{A^e} M \xrightarrow{\alpha_M} M \rightarrow M_2 \rightarrow 0. \]
Detailed understanding of the map $\alpha_M$ will be important. If we denote the embedding by $i_A : \Omega(A) \rightarrow A \otimes M$, then $\alpha_M : \Omega(A) \otimes_{A^e} M \rightarrow M$ can be written as

$$\alpha_M = i_A \otimes 1_M,$$

where $i_A \otimes 1_M$ is the composition of the map $i_A \otimes 1_M : \Omega(A) \otimes_{A^e} M \rightarrow A^e \otimes_{A^e} M$ and the isomorphism $A^e \otimes_{A^e} M \cong M$ given by $a \otimes b \otimes m \mapsto bma$. Therefore,

$$\alpha_M((ab \otimes c - a \otimes bc) \otimes m) = cmab - bcma = [cma, b].$$

In particular, we have the exact sequence

$$0 \rightarrow HH_1(A) \rightarrow \Omega(A) \xrightarrow{\alpha} A \rightarrow A_2 \rightarrow 0$$

where $\alpha(ab \otimes c - a \otimes bc + [\Omega(A), A]) = [ca, b]$.

**Theorem 3.** For any $A$-bimodule $M$ there is an isomorphism

$$H_{n+1}(A, M) \cong \text{colim}_n H_1(F, M)$$

for $n \geq 0$, and an isomorphism

$$H_{n+1}(A, M) \cong \text{colim}_n \Omega(F) \otimes_{F^e} M$$

for $n \geq 1$, and an isomorphism

$$HH_{n+1}(A) \cong \text{colim}_n \Omega(F) \cong \text{colim}_n \Omega(F) \otimes_{F^e} A$$

for $n \geq 1$. Moreover, the morphisms

$$H_1(F, A) \mapsto \Omega(F) \otimes_{F^e} A \mapsto \Omega(F)$$

induce isomorphisms

$$\text{colim}_n H_1(F, A) \cong \text{colim}_n \Omega(F) \otimes_{F^e} A \cong \text{colim}_n \Omega(F)$$

for $n \geq 1$, which are compatible with the above isomorphisms, and induce the following on the level of zero colimits:

$$\text{colim}_0 H_1(F, A) \cong HH_1(A) \mapsto \Omega(A) \cong \text{colim}_0 \Omega(F) \otimes_{F^e} A = \text{colim}_0 \Omega(F).$$

**Proof.** The first isomorphism follows from Corollary 2 if we take the constant $\mathcal{O}$-bimodule $\mathcal{M} = M$ and use that $M_2 = M/[M, A]$ is a constant functor and that higher colimits of a constant functor are trivial.

The second isomorphism follows from the first isomorphism and the short exact sequence with the constant last term

$$H_1(F, M) \mapsto \Omega(F) \otimes_{F^e} M \mapsto [M, F].$$

The $\mathcal{O}$-bimodules $F$ and $F \otimes F$ are colim-acyclic. Hence $\Omega(F)$ is a colim-acyclic $\mathcal{O}$-bimodule as well. The isomorphism

$$HH_{n+1}(A) \cong HH_n(A, \Omega(A)) \cong \text{colim}_n \Omega(F)$$
for \( n \geq 1 \) follows from Corollary 2 if we take the \( \mathcal{O} \)-bimodule given by \( \mathcal{M}(F \to A) = \Omega(F) \) and use that \( H_1(F, \Omega(F)) = 0 \).

Since \( \Omega(F) \) is a free \( F^e \)-module (see [16], Proposition 5.8, or [8], Remark 3.1.3), we obtain \( \text{Tor}_n^{F^e}(\Omega(F), R) = 0 \) for \( n \neq 0 \). Then Proposition 16 implies that

\[
\text{colim}_n \Omega(F) \otimes_{F^e} R = \text{Tor}_n^{A^e}(\Omega(A), 0) = 0.
\]

The short exact sequence \( R \to F \to A \) gives the following short exact sequence

\[
0 \to \Omega(F) \otimes_{F^e} R \to \Omega(F) \to \Omega(F) \otimes_{F^e} A \to 0.
\]

Combining this short exact sequence with the isomorphism \( \text{colim}_n \Omega(F) \otimes_{F^e} R \to 0 \), we obtain that the epimorphism \( \Omega(F) \to \Omega(F) \otimes_{F^e} A \) induces an isomorphism \( \text{colim}_n \Omega(F) \otimes_{F^e} A \).

Finally, the short exact sequence with constant last term

\[
0 \to H_1(F, A) \to \Omega(F) \otimes_{F^e} A \to [A, A] \to 0
\]

implies that the map \( H_1(F, A) \to \Omega(F) \otimes_{F^e} A \) induces an isomorphism \( \text{colim}_n H_1(F, A) = \text{colim}_n \Omega(F) \otimes_{F^e} A \) for \( n \neq 0 \) and the short exact sequence

\[
\text{colim}_n H_1(F, A) \to \Omega(A) \to [A, A].
\]

The theorem is proved.

**Proposition 17.** There is a natural isomorphism

\[
H_1(F, A^e) \cong \frac{R}{R^2}
\]

and a natural short exact sequence

\[
0 \to \frac{R^2 + [R, F]}{R^2} \to H_1(F, \Omega(A)) \to HH_2(A) \to 0
\]

such that the diagram

\[
\begin{array}{ccc}
\frac{R^2 + [R, F]}{R^2} & \to & H_1(F, \Omega(A)) \\
\downarrow & & \downarrow \\
R & \cong & H_1(F, A \otimes A)
\end{array}
\]

is commutative.

**Proof.** There is an exact sequence of \( A \)-bimodules

\[
0 \to \frac{R}{R^2} \to A \otimes_F \Omega(F) \otimes_F A \to A \otimes A
\]
(6), Proposition 7.2), where \( \delta(r + R^2) = 1 \otimes d(r) \otimes 1 \) and \( f \) is an \( A \)-bimodule homomorphism such that \( f(1 \otimes (x \otimes 1 - 1 \otimes x) \otimes 1) = \pi \otimes 1 - 1 \otimes \pi, \) where \( x \in F \) and \( \pi \) is its image in \( A \). We call this homomorphism the Magnus embedding for algebras.

Consider the isomorphism of \( A \)-bimodules

\[
A \otimes_F \Omega(F) \otimes_F A \cong \Omega(F) \otimes_{F^e} A^e
\]
given by \( a \otimes m \otimes b \mapsto m \otimes (a \otimes b) \). Here we assume that the structure of a left \( A^e \)-module on \( A^e \) is given by multiplication in \( A^e = A^{op} \otimes A \) (this structure is different from \( A \otimes A \) and uses the 'inner side': \( (a \otimes b) \ast (a' \otimes b') = a'a \otimes bb' \)). And the structure of a right \( A^e \)-module is given by multiplication as well but it is more standard because it uses 'exterior' multiplication and \( A^e = A \otimes A \) as right \( A^e \)-modules. Then we can rewrite this exact sequence as follows:

\[
0 \rightarrow \frac{R}{R^2} \xrightarrow{\delta'} \Omega(F) \otimes_{F^e} A^e \xrightarrow{f'} A^e, \tag{5.2}
\]

where \( \delta'(r + R^2) = d(r) \otimes 1 \otimes 1 \) and \( f' \) is a right \( A^e \)-module homomorphism such that \( f'(m \otimes 1 \otimes 1) = \overline{m} \), where \( m \in \Omega(F) \) and \( \overline{m} \) is its image in \( \Omega(A) \). Note that

\[
f' = i_F \tilde{\otimes} 1_{A^e} = \alpha_{A^e},
\]

where \( i_F \tilde{\otimes} 1_{A^e} \) is the composition of \( i_F \otimes 1_{A^e} : A^e \otimes \Omega(F) \rightarrow A^e \otimes_{F^e} F^e \) together with the isomorphism \( F^e \otimes_{F^e} A^e \cong A^e \). Indeed, both of them are \( A \)-bimodule homomorphisms that send \( m \otimes 1 \otimes 1 \) to \( \overline{m} \), where \( m \in \Omega(F) \) and \( \overline{m} \) is its image in \( \Omega(A) \). Therefore,

\[
\frac{R}{R^2} = \text{Ker}(f') = \text{Ker}(\alpha_{A^e}) = H_1(F, A^e).
\]

The image of \( f' = \alpha_{A^e} \) is equal to \( \Omega(A) \). Then we have the following exact sequence of right \( A^e \)-modules

\[
0 \rightarrow \frac{R}{R^2} \xrightarrow{\delta'} \Omega(F) \otimes_{F^e} A^e \xrightarrow{f'} \Omega(A) \rightarrow 0. \tag{5.3}
\]

Consider the map \( \text{tw} : A \otimes A \rightarrow A \otimes A \) given by \( \text{tw}(a \otimes b) = b \otimes a \) and set

\[
\Omega'(A) = \text{tw}(\Omega(A)).
\]

Then \( \Omega'(A) \) is a left ideal of \( A^e \). If we consider \( \Omega'(A) \) as an \( A \)-bimodule, then \( \Omega'(A) \cong \Omega(A) \). We tensor the short exact sequence (5.3) by \( \Omega'(A) \) and use that there is a monomorphism \( i'_A : \Omega'(A) \rightarrow A^e \) of left \( A^e \)-modules:

\[
\begin{array}{c}
\frac{R}{R^2} \otimes_{A^e} \Omega'(A) \xrightarrow{\delta \otimes 1} \Omega(F) \otimes_{F^e} \Omega'(A) \xrightarrow{\text{pr} \otimes 1} \Omega(A) \otimes_{A^e} \Omega'(A) \\
\downarrow \text{pr} \otimes i'_A \quad \downarrow \text{pr} \otimes i'_A \\
\frac{R}{R^2} \rightarrow \Omega(F) \otimes_{F^e} A^e \rightarrow \Omega(A)
\end{array}
\]
Here we use the isomorphism $M \otimes_{A^e} A^e = M$ given by $m \otimes (a \otimes b) \mapsto amb$. Then there is a short exact sequence

$$0 \to \text{Ker}(\text{pr} \otimes 1_{\Omega(A)}) \to \text{Ker}(\text{pr} \otimes i'_A) \to \text{Ker}(1_{\Omega(A)} \otimes i'_A) \to 0.$$  

We prove that this short exact sequence is the sequence that we need.

Since $\Omega(F)$ is a free bimodule, we get that the map

$$\Omega(F) \otimes_{F^e} \Omega(A) \to \Omega(F) \otimes_{F^e} A^e$$

is a monomorphism. It follows that the kernel $\text{pr} \otimes 1$ is isomorphic to the image of $R/R^2 \otimes_{A^e} \Omega'(A) \to R/R^2$, which is equal to $[R/R^2, A] = (R^2 + [R, F])/R^2$:

$$\text{Ker}(\text{pr} \otimes 1_{\Omega(A)}) = \frac{R^2 + [R, F]}{R^2}.$$  

Note that $\text{pr} \otimes i'_A = i_F \otimes 1_{\Omega'(A)}$. Indeed, both of them are induced by the multiplication in $A^e$. Therefore,

$$\text{Ker}(\text{pr} \otimes i'_A) = H_1(F, \Omega'(A)) = H_1(F, \Omega(A)).$$

Finally, for the same reason we see that

$$\text{Ker}(1_{\Omega(A)} \otimes i'_A) = \text{Ker}(i_A \otimes 1_{\Omega'(A)}) = H_1(A, \Omega'(A)) = HH_2(A).$$

The proposition is proved.

**Corollary 4.** For $n \geq 1$ there is an isomorphism

$$HH_{n+2}(A) \cong \text{colim}_n R^2 + [R, F],$$

and $\text{colim}_0 R^2 + [R, F] = 0$.

**Proof.** Theorem 3 implies that

$$\text{colim}_n H_1(F, \Omega(A)) = H_{n+1}(A, \Omega(A)) = HH_n(A)$$

for $n \geq 0$. The short exact sequence with constant functor at the end $(R^2 + [R, F])/R^2 \to H_1(F, \Omega(A)) \to HH_2(A)$ (Proposition 17) implies that $\text{colim}_n(R^2 + [R, F])/R^2 = HH_{n+2}(A)$ for $n \geq 1$ and $\text{colim}_0(R^2 + [F, F])/R^2 = 0$. Then, using that $\text{colim}_n R^2 = 0$ (Lemma 4) and the short exact sequence $R^2 \to (R^2 + [F, F]) \to (R^2 + [F, F])/R^2$ we obtain

$$\text{colim}_n (R^2 + [R, F]) = \text{colim}_n \frac{R^2 + [R, F]}{R^2}.$$  

The corollary is proved.
5.3. Reduced cyclic homology over a field of characteristic zero. In this subsection we will always assume that \( \text{char}(k) = 0 \).

For a unital algebra \( B \) we denote by \( \overline{B} \) the quotient \( \overline{B} = B / k \cdot 1 \). Moreover, for a free algebra \( F \) we set

\[
\overline{F}_1 = \frac{F}{k \cdot 1 + [F, F]}. 
\]

It is easy to see that there is a short exact sequence

\[
0 \to k \to F_1 \to \overline{F}_1 \to 0. 
\]

As usual, we consider \( \overline{F}_1 \) as the functor from the category of presentations that sends \( F \to A \) to \( \overline{F}_1 \).

We denote by \( \overline{HC}_n(A) \) the reduced cyclic homology of \( A \) (see [8], §2.2.13). Note that

\[
\overline{F}_1 = \overline{HC}_0(F). 
\]

**Theorem 4** (cf. [5]). Assume that \( \text{char}(k) = 0 \). Then for any \( n \geq 0 \) there is an isomorphism

\[
\text{colim}_n \overline{F}_1 = \overline{HC}_n(A). 
\]

**Proof.** We follow the notation of Loday (see [8], §2.1.9) and denote by \( \overline{B}(A) \) the double complex that computes the cyclic homology of the algebra \( A \), where

\[
\overline{B}(A)_{n,m} = A \otimes \overline{A}^{(m-n)}
\]

for \( m \geq n \) and \( \overline{B}(A)_{n,m} = 0 \) for \( m < n \). Further, following [8], §2.1.13, we denote by \( \overline{B}(B)_{\text{red}} \) the double complex that computes the reduced cyclic homology \( \overline{B}(A)_{\text{red}} = \overline{B}(A) / \overline{B}(k) \). It is easy to see that

\[
(\overline{B}(A)_{\text{red}})_{n,m} = \overline{A}^{(m-n+1)}
\]

for \( m \geq n \) and \( \overline{B}(A)_{n,m} = 0 \) otherwise. Consider the complex \( P_* \) of functors \( \text{Pres}(A) \to \text{Vect} \) given by

\[
P_*(F \to A) = \text{Tot} \overline{B}(F)_{\text{red}}.
\]

We prove that \( P_* \) is a colim-acyclic resolution of the functor \( \overline{F}_1 \). By Theorem 3.1.6 in [8] we have \( HC_n(F) = HC_n(k) \) for \( n \geq 1 \). Combining this with the fact that \( F \) is an augmented algebra we obtain \( HC_n(F) = 0 \) for \( n \geq 1 \). Therefore,

\[
H_n(P_*) = HC_n(F) = 0, \quad H_0(P_*) = HC_0(F) = F_1
\]

for \( n \geq 1 \). Now we need to prove that \( P_n \) is colim-acyclic. Theorem 1 implies that

\[
\text{colim}_n F = 0 \quad \text{and} \quad \text{colim}_0 F = A
\]

for \( n \geq 1 \). Using the short exact sequence \( k \to F \to \overline{F} \), we obtain

\[
\text{colim}_n F = 0 \quad \text{and} \quad \text{colim}_0 F = \overline{A}.
\]

Then the Künneth theorem for colimits (Proposition 9) implies that

\[
\text{colim}_n P_* = 0 \quad \text{and} \quad \text{colim}_0 P_* = \text{Tot} \overline{B}(A)_{\text{red}}
\]

for \( n \geq 1 \). It follows that \( P_* \) is a colim-acyclic resolution. Then the assertion follows from the equality \( \text{colim}_0 P_* = \text{Tot} \overline{B}(A)_{\text{red}} \).
Lemma 5. Let char($k$) = 0 and $F$ be a free $k$-algebra. Then there is a short exact sequence

$$0 \to F_{\natural} \xrightarrow{\tilde{d}} \Omega(F)_{\natural} \xrightarrow{\alpha} [F, F] \to 0,$$

where $\tilde{d}$ is induced by the universal derivation $d: F \to \Omega(F)$, $d(a) = a \otimes 1 - 1 \otimes a$ and $\alpha(ab \otimes c - a \otimes bc + [\Omega(F), F]) = [ca, b]$. In particular, there is an isomorphism $HH_1(F) = F_{\natural}$.

Proof. We denote the cyclic group generated by $t$ by $C_n = \langle t \mid t^n = 1 \rangle$. It is well known that

$$\cdots \xrightarrow{\frac{t-1}{N}} \mathbb{Z}[C_n] \xrightarrow{\frac{t}{N}} \mathbb{Z}[C_n] \xrightarrow{\frac{t-1}{N}} \mathbb{Z}[C_n] \to 0$$

is a projective resolution of the trivial module over $C_n$, where $N = 1 + t + \cdots + t^{n-1}$. Then for any $\mathbb{Z}[C_n]$-module $M$ the complex

$$\cdots \xrightarrow{\frac{t-1}{N}} M \xrightarrow{\frac{n}{N}} M \xrightarrow{\frac{t-1}{N}} M \to 0$$

computes $H_*(C_n, M)$.

If $M$ is a $k[C_n]$-module, we know that $H_n(C_n, M) = 0$ for $n \neq 0$ and $H_0(C_n, M) = M_{C_n}$ because char($k$) = 0. It follows that there is a four-term exact sequence

$$0 \to M_{C_n} \xrightarrow{\tilde{N}} M \xrightarrow{t-1} M \to M_{C_n} \to 0,$$  \hspace{1cm} (5.4)

where $\tilde{N}$ is induced by $N$.

Since $d: F \to \Omega(F)$ is a derivation, we have $d([a, b]) = [d(a), b] + [a, d(b)]$. It follows that $d([F, F]) \subseteq [\Omega(F), F]$. Moreover, $d(1) = 0$. Then $\tilde{d}: F_{\natural} \to \Omega(F)_{\natural}$ is well defined. In order to prove the statement, it is enough to prove that the sequence

$$0 \to F_{\natural} \xrightarrow{\tilde{d}} \Omega(F)_{\natural} \xrightarrow{\tilde{\alpha}} F \to F_{\natural} \to 0$$

is exact, where $\tilde{\alpha}$ is the composition of $\alpha$ with the embedding $[F, F] \subseteq F$. We know already that the sequence $\Omega(F)_{\natural} \to F \to F_{\natural} \to 0$ is exact. So we need to prove that Ker($\tilde{\alpha}$) = Im($\tilde{d}$) and that $\tilde{d}$ is a monomorphism.

Let $F = T(V)$ be the tensor algebra on a vector space $V$. Then it is well known that there is an isomorphism of bimodules.

$$F \otimes V \otimes F \cong \Omega(F),$$  \hspace{1cm} (5.5)

$$\alpha(a \otimes v \otimes b) = av \otimes b - a \otimes vb$$

(see [8], §3.1.3, [17], Proposition 9.1.6, or [16], Proposition 5.8, for example). Since $F \otimes V \otimes F$ is a free bimodule, and there are isomorphisms $(\cdot)_{\natural} \cong (\cdot) \otimes F^e$ and $(F \otimes V \otimes F)_{\natural} F = V \otimes F^e \otimes F^e F = V \otimes F$, we have an isomorphism

$$(F \otimes V \otimes F)_{\natural} = V \otimes F,$$  \hspace{1cm} (5.6)

$$\alpha(a \otimes v \otimes b + [F \otimes V \otimes F, F] \longmapsto v \otimes ba).$$

The composition of the isomorphism $V \otimes F \cong \Omega(F)_{\natural}$ and the map $\tilde{\alpha}$ is the map

$$\tilde{\alpha}' : V \otimes T(V) \to T(V),$$

$$\tilde{\alpha}'(v_0 \otimes v_1 \cdots v_n) = [v_1 \cdots v_n, v_0] = v_1 \cdots v_n v_0 - v_0 v_1 \cdots v_n.$$
Note that $\tilde{\alpha}' = \bigoplus \tilde{\alpha}'_n$, where

$$\tilde{\alpha}'_n : V \otimes V^{\otimes n-1} \to V^{\otimes n}, \quad \tilde{\alpha}'_n = \cdot(t-1),$$

where $t$ is given by the obvious action of $C_n$ on $V^{\otimes n}$. Since the sequence $\Omega(F)_\sharp \xrightarrow{\tilde{\alpha}} F \to F_\sharp \to 0$ is exact, we obtain $F_\sharp = k \oplus (\bigoplus_{n \geq 1}(V^{\otimes n})_{C_n})$. Therefore, $\overline{F}_\sharp = \bigoplus_{n \geq 1}(V^{\otimes n})_{C_n}$. The exact sequence (5.4) implies the exact sequence

$$0 \to (V^{\otimes n})_{C_n} \xrightarrow{\overline{N}} V^{\otimes n} \xrightarrow{t-1} V^{\otimes n} \to (V^{\otimes n})_{C_n} \to 0.$$

Then we only need to prove that $d' : F \to \Omega(F)$ composed with the isomorphisms $F_\sharp = \bigoplus_{n \geq 1}(V^{\otimes n})_{C_n}$ and $\Omega(F)_\sharp = \bigoplus_{n \geq 1} V \otimes V^{\otimes n-1}$ is given by the homomorphism $\overline{N}$ on each direct summand.

Indeed, the composition of the universal derivation $d : F \to \Omega(F)$ with the isomorphism $\Omega(F) = T(V) \otimes V \otimes T(V)$ is given by

$$d' : T(V) \to T(V) \otimes V \otimes T(V),$$

$$d'(v_1 \cdots v_n) = \sum v_1 \cdots v_{i-1} \otimes v_i \otimes v_{i+1} \cdots v_n.$$

If we pass to the quotients $T(V)_\sharp$ and $(T(V) \otimes V \otimes T(V))_\sharp$ and compose $d'$ with the isomorphism (5.6), we obtain the map

$$\tilde{d'} : T(V)_\sharp \to V \otimes T(V),$$

given by

$$\tilde{d'}(v_1 \cdots v_n + [T(V), T(V)]) = \sum v_i \otimes v_{i+1} \cdots v_n v_1 \cdots v_{i-1}$$

which is equal to $\overline{N}$.

Remark 6. For $\text{char}(k) \neq 0$ the isomorphism $HH_1(F) \cong \overline{F}_\sharp$ fails. This can be shown using Connes’ exact sequence for $F$ and the computation of $HC_*(F)$ (see [8], Theorem 3.1.6).

Theorem 5 (Hopf’s formula for $HC_{2n+1}$). Let $\text{char}(k) = 0$. Then for any $n \geq 0$ there is a natural isomorphism

$$HC_{2n+1}(A) \cong \frac{R^{n+1} \cap ([F, F] + k \cdot 1)}{[R, R^n]}.$$

Moreover, the functors $([F, F] + k \cdot 1)/[R, F]$ and $(R^{n+2} \cap [R, R^n])/[R, R^{n+1}]$ are constant for $n \geq 0$.

Proof. Quillen proved (see [16], Theorem 5.11) that there is an exact sequence

$$0 \to HC_{2n+1}(A) \to \frac{R^{n+1}}{[R, R^n]} \xrightarrow{\delta} \Omega(F)_\sharp,$$
where $\delta$ is induced by the universal derivation $d: F \to \Omega(F)$. Lemma 5 implies that $\delta$ factors through the monomorphism $\mathcal{F}_2 \xrightarrow{\tilde{d}} \Omega(F)_2$. Therefore, $\mathcal{HC}_{2n+1}(A)$ is the kernel of the map

$$\frac{R^{n+1}}{[R, R^{n+1}]} \to \frac{F}{[F, F] + k \cdot 1}.$$ 

Then

$$\mathcal{HC}_{2n+1}(A) = \frac{R^{n+1} \cap ([F, F] + k \cdot 1)}{[R, R^n]}.$$ 

If we take $n = 0$, we obtain a short exact sequence

$$0 \to \mathcal{HC}_1(A) \to \frac{[F, F] + k \cdot 1}{[R, F]} \to [A, A] + k \cdot 1 \to 0.$$ 

Since $\mathcal{HC}_1(A)$ and $[A, A] + k \cdot 1$ are constant, the middle term is constant as well. The functor $(R^{n+2} \cap [R, R^n])/[R, R^{n+1}]$ is the kernel of the morphism of constant functors $\mathcal{HC}_{2n+3}(A) \to \mathcal{HC}_{2n+1}(A)$, hence it is constant.

The theorem is proved.

**Proposition 18.** Let $\text{char}(k) = 0$. Then there are isomorphisms

$$\mathcal{HC}_{n+1}(A) = \text{colim}_n[F, F] = \text{colim}_n[R, F],$$

$$\text{colim}_0[R, F] = 0$$

for $n \geq 1$, and the embedding $[R, F] \subseteq [F, F]$ induces isomorphisms $\text{colim}_n[F, F] = \text{colim}_n[R, F]$ for $n \geq 1$. Moreover,

$$\mathcal{HC}_{n+2}(A) = \text{colim}_n H_1(F, R)$$

for $n \geq 0$, and

$$\mathcal{HC}_{n+3}(A) = \text{colim}_n [R, R]$$

for $n \geq 1$.

**Proof.** The short exact sequence $[F, F] + k \cdot 1 \to F \to \mathcal{F}_2$ implies that $\text{colim}_n([F, F] + k \cdot 1) = \text{colim}_{n+1} \mathcal{F}_2$ for $n \geq 1$ because $\text{colim}_n F = 0$. Theorem 4 implies that $\text{colim}_n([F, F] + k \cdot 1) = HC_{n+1}(A)$ for $n \geq 1$. The short exact sequence $[F, F] \to [F, F] + k \cdot 1 \to k$ implies that $\text{colim}_n[F, F] = HC_{n+1}(A)$ for $n \geq 1$. Since the functor $[F, F]/[R, F]$ is constant (Theorem 5), the short exact sequence $[R, F] \to [F, F] \to [F, F]/[R, F]$ implies that the embedding $[R, F] \hookrightarrow [F, F]$ induces isomorphisms $\text{colim}_n[R, F] = \text{colim}_n[F, F]$ for $n \geq 1$. Then

$$\text{colim}_n[R, F] = \mathcal{HC}_{n+1}(A)$$

for $n \geq 1$.

Since $\Omega(F)$ is a free $F$-bimodule, we have $\text{Tor}_m^{F_e} (\Omega(F), R) = 0$ for $m \neq 0$. Then Proposition 16 implies that $\text{colim}_n \Omega(F) \otimes_{F_e} R = 0$ for any $n \in \mathbb{Z}$. Hence the short exact sequence $H_1(F, R) \to \Omega(F) \otimes_{F_e} R \to [R, F]$ implies that

$$\text{colim}_n H_1(F, R) = \text{colim}_{n+1}[R, F]$$
for any $n \in \mathbb{Z}$. It follows that

$$\colim_n H_1(F, R) = \overline{HC}_{n+2}(A)$$

for $n \geq 0$ and $\colim_0 [R, F] = 0$.

Lemma 4 implies that $\colim_n R^2 = 0$. Then the short exact sequence $[R, R] \to R^2 \to R^2/[R, R]$ implies that

$$\colim_n [R, R] = \colim_{n+1} \frac{R^2}{[R, R]}$$

for all $n$. Quillen proved (see [16], Theorem 5.11) that there is an exact sequence

$$0 \to \overline{HC}_3(A) \to \frac{R^2}{[R, R]} \to H_1(F, R) \to \overline{HC}_2(A) \to 0.$$ 

Applying the spectral sequence of colimits (Proposition 4) to this small complex, and using that this spectral sequence converges to zero, we obtain

$$\colim_{n+1} \frac{R^2}{[R, R]} = \colim_{n+1} H_1(F, R)$$

for $n \geq 1$. Therefore, for $n \geq 1$ we get

$$\colim_n [R, R] = \overline{HC}_{n+3}(A).$$

The proposition is proved.

5.4. Connes’ exact sequence via derived colimits. In this subsection we give two ways of obtaining Connes’ exact sequence from the developed theory. Here we assume that $\text{char}(k) = 0$.

Proposition 19. Let $\text{char}(k) = 0$. Then the long exact sequence of derived colimits of the short exact sequence

$$0 \to F^\natural \to \Omega(F) \to [F, F] \to 0$$

(from Lemma 5) gives a long exact sequence

$$\cdots \to \overline{HC}_4 \to \overline{HC}_2 \to HH_3 \to \overline{HC}_3 \to \overline{HC}_1 \to HH_2 \to \overline{HC}_2,$$

where $\overline{HC}_n = \overline{HC}_n(A)$ and $HH_n = HH_n(A)$.

The proof follows from Theorem 4, Theorem 3 and Proposition 18.

Lemma 6. Let $\text{char}(k) = 0$. Then for $n \geq 2$ there is an isomorphism

$$\colim_n \frac{[R, F]}{[R, R]} \cong HH_{n+2}(A).$$
Proof. Consider the short exact sequence

\[ 0 \rightarrow \frac{R^2 \cap [R, F]}{[R, R]} \rightarrow \frac{[R, F]}{[R, R]} \rightarrow \frac{[R, F]}{R^2 \cap [R, F]} \rightarrow 0. \]

The first term is constant by Theorem 5. The last term is equal to \((R^2 + [R, F])/R^2\) by the third isomorphism theorem. Using that \(\text{colim}_n R^2 = 0\) we obtain

\[ \text{colim}_n \frac{[R, F]}{[R, R]} = \text{colim}_n R^2 + [R, F] \]

for \(n \geq 2\). Then the assertion follows from Corollary 4.

Proposition 20. Let \(\text{char}(k) = 0\). Then the long exact sequence of derived colimits of the short exact sequence

\[ 0 \rightarrow [R, R] \rightarrow [F, F] \rightarrow \frac{[F, F]}{[R, R]} \rightarrow 0 \]

gives a long exact sequence

\[ \cdots \rightarrow HC_6 \rightarrow HC_4 \rightarrow HH_5 \rightarrow HC_5 \rightarrow HC_3 \rightarrow HH_4 \rightarrow HC_4 \rightarrow HC_2. \]

Proof. This follows from Proposition 18 and Lemma 6.

§ 6. Cyclic homology of nonunital algebras

In this section we consider the category of nonunital algebras \(\text{Alg}_n\) and present nonunital versions of some results about cyclic homology in §5. We assume that \(\text{char}(k) = 0\). Effective epimorphisms of this category are surjective homomorphisms. Projective objects of this category are retracts of free nonunital algebras, where free nonunital algebras can be described as reduced tensor algebras \(F = T(V) = \bigoplus_{n \geq 1} V^{{\otimes}n}\). As in §5, Proposition 7 allows us to consider only presentations \(F \rightarrow A\), where \(F\) is a free algebra. We set \(F_\circ = HC_0(F) = F/[F, F]\) and \(R = \text{Ker}(F \rightarrow A)\).

Theorem 6 (cf. [5]). Assume that \(\text{char}(k) = 0\). Then for any \(n \geq 0\) there is an isomorphism

\[ \text{colim}_n F_\circ \cong HC_n(A). \]

The proof is similar to that of Theorem 4.

Theorem 7 (Hopf’s formula for \(HC_{2n+1}\)). Let \(\text{char}(k) = 0\). Then for any \(n \geq 0\) there is a natural isomorphism

\[ HC_{2n+1}(A) \cong \frac{R^{n+1} \cap [F, F]}{[R, R^n]}. \]

Proof. We denote by \(A_+\) the algebra with added formal unit \(A_+ = A \oplus k\). It is well known that \(HC_*(A) = HC_*(A_+)\). Theorem 5 implies that

\[ HC_{2n+1}(A) = \frac{R^{n+1} \cap ([F_+, F_+] + k \cdot 1)}{[R, R^n]}. \]

Then the assertion follows from the equations \([F_+, F_+] = [F, F]\) and \(F \cap ([F, F] + k \cdot 1) = [F, F]\).

The theorem is proved.
§ 7. $K$-functors

Recall that for any unital ring $A$ there is a notion of its Steinberg group $St(A)$. Some of its properties are listed here:

- $St(A)$ is the quotient of a free group on generators $e_{i,j}(x)$ for integers $i \neq j$ and $x \in A$ modulo the relations
  
  \[ e_{i,j}(x)e_{i,j}(y) = e_{i,j}(x+y), \]
  
  \[ [e_{i,j}(x), e_{j,k}(y)] = e_{i,k}(xy) \quad \text{if } i \neq k, \]
  
  \[ [e_{i,j}(x), e_{i',j'}(y)] = 1 \quad \text{if } i \neq j' \text{ and } j \neq i'. \]

- $H_1(St(A)) = H_2(St(A)) = 0$.
- $H_3(St(A)) = K_3(A)$ (see [3]).
- There is an exact sequence of groups
  
  \[ 1 \to K_2(A) \to St(A) \to E(A) \to 1 \]
  
  and, moreover, it is the universal central extension of
  
  \[ E(A) := [GL(A), GL(A)]. \]

For a ring homomorphism $A \xrightarrow{f} B$ we denote by $K_n(A, B)$ the $n$th relative $K$-theory group, that is, the $n$th homotopy group of the homotopy fibre of the induced map of spectra $K(A) \to K(B)$.

Now let $k$ be a noetherian regular commutative ring and $A$ be a $k$-algebra. We denote by $\widetilde{K}_n(A)$ the $n$th reduced $K$-theory group, $K_{n-1}(k, A)$. By definition there is an exact sequence

\[ 0 \to K_n(A) \to \widetilde{K}_n(A) \to \text{Ker}(K_{n-1}(k) \to K_{n-1}(A)) \to 0. \]

If $A$ admits an augmentation, $\widetilde{K}_n(A)$ is just the quotient $K_n(A)/K_n(k)$.

For any functor $\mathcal{F} \xrightarrow{\mathcal{F}} \mathcal{A}$ we denote by the same letter its extension to the category of nonunital rings $\text{Rngs}$ given by the formula

\[ \mathcal{F}(R) = \frac{\mathcal{F}(R \times \mathbb{Z})}{\mathcal{F}(\mathbb{Z})} = \text{Ker}(\mathcal{F}(R \times \mathbb{Z}) \to \mathcal{F}(\mathbb{Z})). \]

We will need the following lemma which is well known to specialists in the field. The essential ingredients for the lemma were given in Keune’s paper [7]. Note that in [11] the statement of the lemma is used as a definition of the relative $K_2$.

**Lemma 7.** Let $F \to A$ be a surjective ring homomorphism. Set $D = F \times_A F$. Then the group $K_2(F, A)$ is isomorphic to the quotient

\[ \frac{K_2(D)}{K_2(F) + \Gamma}, \]

where $K_2(F)$ is considered as a subgroup of $K_2(D)$ via the diagonal split embedding $F \to D$ and $\Gamma$ is the subgroup of $St(D)$ generated by the commutators $[e_{1,2}(x,0), e_{2,1}(0,y)]$ for all $x, y \in \text{Ker}(F \to A)$. 

**Proof.** For any surjective ring homomorphism $F \to A$ there is a group $\text{St}(F, A)$ and a map

$$\text{St}(F, A) \to \text{Ker}(E(F) \to E(A))$$

such that $K_2(F, A)$ is the kernel of this map (see [7], §5). By Theorem 12 in [7] the group is computed to be

$$\text{St}(F, A) = \frac{H}{\Gamma H},$$

where $H$ is the kernel of the projection $\text{St}(D) \to \text{St}(F)$ onto the first component. Note that

$$E(F \times_A F) = E(F) \times_{E(A)} E(F),$$

so every generator of $\Gamma$ is sent to a trivial element via the map $H \to E(D)$. This shows that the map

$$\text{St}(F, A) \to \text{Ker}(E(D) \to E(F)) \to \text{Ker}(E(F) \to E(A))$$

is well defined. Moreover, this tells us that $\Gamma$ is actually a subgroup of $K_2(D)$ and that $\text{St}(F, A) = H/\Gamma$ (since $K_2$ is central by Proposition 13 in [7]).

Now the natural map $\text{Ker}(K_2(D) \to K_2(F)) \to H$ induces a map on quotients

$$\frac{\text{Ker}(K_2(D) \to K_2(F))}{\Gamma} \to \frac{H}{\Gamma}.$$ 

This map is injective since the map $K_2(D) \to \text{St}(D)$ is. Moreover, an easy diagram-chasing argument shows that it surjects onto the kernel of the map from $H/\Gamma$ to $E(F \times_A F)$.

Indeed, we have the commutative diagram

$$
\begin{array}{cccc}
\text{Ker}(K_2(D) & \to K_2(F)) & \to & H \\
\uparrow & & & \uparrow \\
\text{Ker}(K_2(D) & \to K_2(F)) & \to & H \\
\uparrow & & & \uparrow \\
\downarrow & & & \downarrow \\
K_2(D) & \to & \text{St}(D) & \to E(D)
\end{array}
$$

whose lower row is exact.

Let $x \in H/\Gamma$ be an element mapped to a trivial elementary matrix. Denote its lift to $H$ by $\bar{x}$. By exactness of the lower row the image of $\bar{x}$ in $\text{St}(D)$ belongs to $K_2(D)$. It is also mapped to a trivial element of $\text{St}(F)$, and therefore it gives rise to an element $y$ of $\text{Ker}(K_2(D) \to K_2(F))$. It is mapped back to $\bar{x}$ by $f$ since the map $H \to \text{St}(D)$ is an inclusion. Now, the image of $y$ under the upper left vertical map is an element in the preimage of $x$. Hence we obtain an isomorphism

$$\frac{\text{Ker}(K_2(D) \to K_2(F))}{\Gamma} \cong K_2(F, A).$$
Lastly, the diagonal map $F \to D$ gives a splitting of the map $K_2(D) \to K_2(F)$ and an isomorphism

$$
\frac{\text{Ker}(K_2(D) \to K_2(F))}{\Gamma} \cong \frac{K_2(D)/K_2(F)}{\Gamma} = \frac{K_2(D)}{K_2(F) + \Gamma}.
$$

**Lemma 8.** Let $A$ be a ring without unit or a unital $k$-algebra. Consider $\Gamma$, the subgroup of $\text{St}(F \times_A F)$ generated by the commutators $[e_{1,2}(x,0), e_{2,1}(0,y)]$ for all $x,y \in \text{Ker}(F \to A)$, as a functor on the category of presentations of $A$. Then $\text{colim} \Gamma = 1$, where the colimit is taken over the category of presentations in the category of unital $k$-algebras (the category $\text{Rngs}$, respectively).

**Proof.** By Proposition 10 it suffices to show that for a fixed presentation $F \xrightarrow{p} A$ the coequalizer of all the maps $\Gamma(F) \to \Gamma(F)$ induced by maps of presentations $F \to F$ is trivial.

Choose a set $S$ such that $F = k\langle S \rangle$ ($F = F\langle S \rangle$, respectively). Adding a new variable if necessary we can assume that there is an element $s \in S$ such that $p(s) = 0$. For an element $a \in \text{Ker}(F \to A)$ consider the map of sets $S \to F$ that is identical on $S - \{s_0\}$ and sends $s_0$ to $a$. By the universal property of a free algebra (a free ring, respectively) the map extends uniquely to a map of rings $F \xrightarrow{f_r} F$. The equality $p \circ f_r(s) = p(s)$ holds for any $s \in S$ by design, hence $p \circ f_r = p$ by the universal property of a free algebra (a free ring, respectively).

The map $(f_r)_*$ sends $[e_{1,2}(s_0,0), e_{2,1}(0,b)]$ to $[e_{1,2}(a,0), e_{2,1}(0,b)]$ while the map $(f_0)_*$ sends the same element to $[e_{1,2}(0,0), e_{2,1}(0,b)] = 1$ for any $b \in k\langle S - \{s_0\} \rangle$ ($b \in F\langle S - \{s_0\} \rangle$, respectively). Moreover, for an arbitrary $b$, $(f_0)_*$ sends $[e_{1,2}(a,0), e_{2,1}(0,b)]$ to $[e_{1,2}(\tilde{a},0), e_{2,1}(0,\tilde{b})]$, where $\tilde{b} \in k\langle S - \{s_0\} \rangle$ ($\tilde{b} \in F\langle S - \{s_0\} \rangle$, respectively). Therefore, all generators of $\Gamma(F)$ become trivial in the coequalizer and the coequalizer itself is trivial.

The lemma is proved.

**Proposition 21.** 1. Let $A$ be a unital $k$-algebra. Then

$$
\text{colim} \tilde{K}_2(F \times_A F) = \tilde{K}_3(A),
$$

where the colimit is taken over the category of presentations in the category of unital $k$-algebras.

2. Let $A$ be a nonunital ring. Then

$$
\text{colim} K_2(F \times_A F) = K_3(A),
$$

where the colimit is taken over the category of presentations in the category $\text{Rngs}$.

**Proof.** Let $A$ be a unital $k$-algebra and let $F \to A$ be a surjective ring homomorphism where $F$ is a free unital $k$-algebra. By Lemma 7 we have an exact sequence

$$
\Gamma \to \frac{K_2(F \times_A F)}{K_2(F)} \to K_2(F,A) \to 0,
$$

where $\Gamma$ is the subgroup of $\text{St}(F \times_A F)$ generated by commutators $[e_{1,2}(x,0), e_{2,1}(0,y)]$ for all $x,y \in \text{Ker}(F \to A)$. 

By Corollary 3.9 in [4], the natural map $K_2(k) \to K_2(F)$ is an isomorphism, so by the 5-Lemma $\tilde{K}_3(A) = K_2(k, A) \to K_2(F, A)$ is also an isomorphism. Hence the exact sequence above is isomorphic to the exact sequence

$$\Gamma \to \tilde{K}_2(F \times_A F) \to \tilde{K}_3(A) \to 0.$$ 

Now Lemma 8 and right-exactness of the colimit functor imply the first part of the statement. The same argument applied to $F_+ \to A_+$, where $F \to A$ is a surjection from a free ring to a nonunital ring $A$, yields the second part of the statement.

The proposition is proved.

Bibliography

[1] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, NJ 1956, xv+390 pp.
[2] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergeb. Math. Grenzgeb., vol. 35, Springer-Verlag, New York 1967, x+168 pp.
[3] S.M. Gersten, “$K_3$ of a ring is $H_3$ of the Steinberg group”, *Proc. Amer. Math. Soc.* 37:2 (1973), 366–368.
[4] S.M. Gersten, “$K$-theory of free rings”, *Comm. Algebra* 1 (1974), 39–64.
[5] G. Donadze, N. Inassaridze and M. Ladra, “Cyclic homology via derived functors”, *Homology Homotopy Appl.* 12:2 (2010), 321–334.
[6] S.O. Ivanov and R. Mikhailov, “A higher limit approach to homology theories”, *J. Pure Appl. Algebra* 219:6 (2015), 1915–1939.
[7] F. Keune, “The relativization of $K_2$”, *J. Algebra* 54:1 (1978), 159–177.
[8] J.-L. Loday, *Cyclic homology*, Grundlehren Math. Wiss., vol. 301, Springer-Verlag, Berlin 1992, xviii+454 pp.
[9] J. Lurie, *Higher topos theory*, Ann. of Math. Stud., vol. 170, Princeton Univ. Press, Princeton, NJ 2009, xviii+925 pp.
[10] S. Mac Lane, *Categories for the working mathematician*, Grad. Texts in Math., vol. 5, Springer-Verlag, New York–Berlin 1971, ix+262 pp.
[11] B.A. Magurn, *An algebraic introduction to $K$-theory*, Encyclopedia Math. Appl., vol. 87, Cambridge Univ. Press, Cambridge 2002, xiv+676 pp.
[12] R. Mikhailov and I.B.S. Passi, “Generalized dimension subgroups and derived functors”, *J. Pure Appl. Algebra* 220:6 (2016), 2143–2163.
[13] R. Mikhailov and I.B.S. Passi, “Dimension quotients, Fox subgroups and limits of functors”, *Forum Math.* 31:2 (2019), 385–401; arXiv:1703.08304.
[14] D.G. Quillen, *Homotopical algebra*, Lecture Notes in Math., vol. 43, Springer–Verlag, Berlin–New York 1967, iv+156 pp.
[15] D. Quillen, “Higher algebraic $K$-theory. I”, *Algebraic $K$-theory*, vol. I: *Higher $K$-theories* (Battelle Memorial Inst., Seattle, WA 1972), Springer Lect. Notes Math., vol. 341, Springer, Berlin 1973, pp. 85–147.
[16] D. Quillen, “Cyclic cohomology and algebra extensions”, *K-Theory* 3:3 (1989), 205–246.
[17] C. A. Weibel, *An introduction to homological algebra*, Cambridge Stud. Adv. Math., vol. 38, Cambridge Univ. Press, Cambridge 1994, xiv+450 pp.

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