The geometry of supersymmetric coset models and superconformal algebras

G. Papadopoulos

Department of Physics,
Queen Mary and Westfield College,
Mile End Road,
London E1 4NS, UK.

ABSTRACT

An on-shell formulation of \((p, q), 2 \leq p \leq 4, 0 \leq q \leq 4\), supersymmetric coset models with target space the group \(G\) and gauge group a subgroup \(H\) of \(G\) is given. It is shown that there is a correspondence between the number of supersymmetries of a coset model and the geometry of the coset space \(G/H\). The algebras of currents of supersymmetric coset models are superconformal algebras. In particular, the algebras of currents of \((2,2)\) and \((4,0)\) supersymmetric coset models are related to the \(N=2\) Kazama-Suzuki and \(N=4\) Van Proeyen superconformal algebras correspondingly.

* Address from October 1st: Department of Mathematics, King’s College London, Strand, London WC2R 2LS.
1. Introduction

It has been known for many years that there is an interplay between the geometry of the target manifolds of supersymmetric sigma models and their number of supersymmetries [1]. In two dimensions, sigma models may have left- and/or right- handed supersymmetries and the geometry of their target manifolds has been extensively studied in the literature [2-6].

More recently, the authors of Ref [7] studied the two-dimensional supersymmetric gauged sigma models and found that there is a similar interplay between their number of supersymmetries and the geometry of their target manifolds. The same authors introduced off-shell multiplets and actions for all \((p,q)\), \(2 \leq p \leq 4\), \(0 \leq q \leq 2\), supersymmetric gauged sigma models by generalising a method employed in Refs [5,6] for the construction of off-shell multiplets and actions for the (ungauged) two-dimensional supersymmetric sigma models.

An interesting class of (supersymmetric) gauged sigma models arises from gauging the (supersymmetric) Wess-Zumino-Witten (WZW) models [8]. These models have been studied by many authors [9,10], they are (super)conformal and their algebras of currents realise (super)conformal algebras that are constructed from a method devised by Goddard, Kent and Olive known as the GKO or coset construction [11]. In the following, we will refer to gauged WZW models as coset models.

In Ref [12] Kazama and Suzuki (KS) constructed an N=2 superconformal algebra from an N=1 superconformal algebra using the GKO construction. The KS algebra may have a left and a right sector. A realisation of the left or right sector of the KS algebra was given in Ref [13] in terms of the currents of an off-shell (2,0) or (0,2) supersymmetric coset model. The expectation is that the KS superconformal algebra with both left and right sectors is realised as the algebra of currents of the (2,2) supersymmetric coset model. Several attempts were made to construct this model (see for example Ref [14]) including the off-shell formulation of Ref [7]. However the off-shell closure of the supersymmetry algebra imposes strong condi-
tions on the geometry of the sigma model target manifold. In the case of (2,2) supersymmetric coset model, these conditions restrict the geometry of the group manifold in such a way that the algebra of currents of these models is not the most general realisation of the KS algebra.

A. Van Proeyen (VP) in Ref [15] constructed an non-linear left-handed N=4 superconformal algebra using the GKO construction and the algebraic properties of tensors on Lie algebras that are associated with symmetric quaternionic Kähler manifolds (Wolf spaces). It is expected that this algebra can be realised as the algebra of the currents of the (4,0) supersymmetric coset model. However as in the case of the (2,2) supersymmetric coset model, the algebra of currents of the off-shell (4,0) model constructed in Ref [7] does not provide the most general realisation of the VP superconformal algebra.

In this paper, we will construct an on-shell formulation of all \((p, q)\), \(2 \leq p \leq 4\), \(0 \leq q \leq 4\) supersymmetric coset models with target manifold the group \(G\) and gauge group a subgroup \(H\) of \(G\). We will achieve this by starting with the (1,1) or (1,0) supersymmetric coset models and then by constructing the additional supersymmetry transformations necessary for the description of \((p, q)\) supersymmetric coset models. Then we will derive the conditions for the action of (1,1) or (1,0) coset model to be invariant under the action of \((p, q)\) supersymmetry transformations and give the conditions for the on-shell closure of the algebra of \((p, q)\) supersymmetry transformations. We will find that these conditions have a geometric interpretation. Indeed they will be understood in terms of the geometry of the coset space \(G/H\). In particular we will prove that \((2, q)\), \(0 \leq q \leq 2\), models exist provided that \(G/H\) is a Hermitian manifold with a holomorphic tangent bundle and \((4, q)\), \(0 \leq q \leq 4\) models exist provided that \(G/H\) is a quaternionic Kähler manifold with respect to its canonical connection. The algebras of currents of supersymmetric coset models are superconformal algebras. Finally, the algebraic closure properties of the algebra of supersymmetry transformations, the current content and the geometry of the coset spaces \(G/H\) of the (2,2) and (4,0) supersymmetric coset models indicate that the algebras of currents of these models
are (classical) realisations of the N=2 KS and N=4 VP superconformal algebras correspondingly.

This paper has been organised as follows: In section two, we will present the (1,1) and (1,0) supersymmetric gauged models and set up our notation. In section three, we will give the (2,2) supersymmetry transformations and calculate their commutator. In section four, we will derive the conditions necessary for the on-shell closure of the algebra of the supersymmetry transformations of a (2,2) supersymmetric coset model and relate them to the geometry of its target manifold. In addition, we will examine briefly the (2,0) and (2,1) supersymmetric coset models. In section five, we will construct the (4,0) supersymmetric coset model and discuss the geometry of the associated coset space $G/H$. Finally in section six, we will describe the rest of the $(p,q)$ models and give our conclusions.

2. The (1,1) and (1,0) supersymmetric coset models

The action of the (1,1) supersymmetric gauged sigma model with a Wess-Zumino term [16] is

$$L = g_{ij} \nabla_+ X^i \nabla_- X^j + b_{ij} \, D_+ X^i \, D_- X^j - A_+^a \, u_{ia} \, D_- X^i - A_-^a \, u_{ia} \, D_+ X^i + c_{ab} \, A_+^a \, A_-^b \quad (2.1)$$

where $b_{ij} = -b_{ji}$, $X$ is a section of a bundle with base space an (1,1) superspace with co-ordinates $(y^+, y^-, \theta^-, \theta^+)$ and fiber a manifold $M$, $i, j, k = 1, \cdots, \dim M$, and $\{D_+, D_-, \partial_+, \partial_-\}$ are the flat superspace derivatives ($D^2_+ = i\partial_+, D^2_- = i\partial_-)$. The target space $M$ admits a group action of a gauge group $H$ which generates the vector fields $\xi_a$, $a = 1, \cdots, \dim \text{Lie} H$, where $\text{Lie} H$ is the Lie algebra of the group $H$. The metric $g$ and the three form $B = 3db$ of $M$ are invariant tensors under the group action of $H$ on $M$. $\xi_a$ are Killing vector fields,

$$\xi^i_a B_{ijk} = 2 \, \partial_{[j} u_{k]} a \quad (2.2)$$
\[ c_{ab} = \xi^i_a u_{ib}. \]  

(2.3)

\( \{A_+, A_-, A_4, A_-\} \) are the components of a connection \( A \) and \( \{\nabla_+, \nabla_-, \nabla_4, \nabla_-\} \) are the corresponding covariant derivatives that satisfy the (super)algebra

\[
\begin{align*}
[\nabla_+, \nabla_-] &= F_{+-}, \\
[\nabla_-, \nabla_4] &= F_{-4}, \\
[\nabla_+, \nabla_4] &= F_{+4}, \\
[\nabla_+, \nabla_-] &= 2i\nabla_+, \\
[\nabla_-, \nabla_-] &= 2i\nabla_-.
\end{align*}
\]

(2.4)

The rest of the supercommutators vanish and \( F \) is the curvature of the connection \( A \). The action (2.1) is gauge invariant provided that \( L_{ab} u_b = f_{abc} u_c \) and \( c_{ab} = c_{[ab]} \) where \( f_{abc} \) are the structure constants of the Lie algebra \( \text{Lie} H \) and \( L_a \) is the Lie derivative of the vector field \( \xi_a \). Finally, \( \nabla_+ X^i = D_+ X^i + A^a_+ \xi^i_a \) and \( \nabla_- X^i = D_- X^i + A^a_- \xi^i_a \).

In the case of coset models, the target manifold \( M \) is a group \( G \) and \( H \) is a subgroup of \( G \). The group action of \( H \) on \( G \) is \( k \to hkh^{-1} \) where \( k \in G \) and \( h \in H \). This is the adjoint action of \( H \) into \( G \). The metric \( g \) is chosen to be a bi-invariant positive definite metric on \( G \), i.e. \( g_{ij} = L^A_i L^B_j \delta_{AB} = R^A_i R^B_j \delta_{AB} \), \( A, B = 1, \cdots, \dim \text{Lie} G \). \( k^{-1} dk = L^A t_A \) and \( dkk^{-1} = R^A t_A \), \( k \in G \), are the left and right frames correspondingly where \( t_A \) is an orthonormal basis of the Lie algebra \( \text{Lie} G \) of \( G \). \( B \) is a bi-invariant three form on \( G \) \( (B_{ijk} = -L^A_i L^B_j L^C_k f_{ABC} = -R^A_i R^B_j R^C_k f_{ABC}) \). The Killing vector fields are given by \( \xi_a = L_a - R_a \) and \( u_a = L_a + R_a \). \( \text{Lie} H \) is a subspace of \( \text{Lie} G \) and we have decomposed \( \text{Lie} G \) into \( \text{Lie} H \) and its orthogonal complement, i.e. \( \text{Lie} G = \text{Lie} H \oplus \mathcal{P} \). Under this decomposition, \( \text{Lie} H \) is reductive, i.e. \( [\text{Lie} H, \mathcal{P}] \subset \mathcal{P} \). Moreover due to a relative normalisation of the Wess-Zumino and the Kinetic terms of the theory, the connections

\[
\Gamma^{(\pm)i}_{jk} = \Gamma^i_{jk}(g) \pm \frac{1}{2} B^i_{jk}
\]

(2.5)

are flat where \( \Gamma(g) \) is the Levi-Civita connection of the bi-invariant metric \( g \).
The equations of motion of an (1,1) supersymmetric coset model are the following:

\[ \nabla^{(+)} \nabla_{+} X^i - L^i_a F^a_{+-} = 0, \quad (2.6) \]

\[ L_{ia} \nabla_{+} X^i = 0 \quad (2.7) \]

and

\[ R_{ia} \nabla_{-} X^i = 0, \quad (2.8) \]

where \( \nabla^{(\pm)} \) are the covariant derivatives of the \( \Gamma^{(\pm)} \) connections. A consequence of eqns. (2.6) and (2.7) is that the curvature \( F \) of the connection \( A \) is zero on-shell. This is a key property of the (1,1) ((1,0)) supersymmetric coset models that allows us to construct the additional supersymmetry transformations in order to describe the \( (p, q) \), \( 2 \leq p \leq 4 \), \( 0 \leq q \leq 4 \) models from the (1,1) (or (1,0)) ones and prove the on-shell closure of the associated supersymmetry algebras.

We conclude this section with a brief description of the (1,0) supersymmetric coset model. Let \( \xi_a \) be the vector fields generated by the group action of a group \( H \) on the target manifold \( M \) of an (1,0) supersymmetric sigma model with Wess-Zumino term \( b \) \((b_{ij} = -b_{ji})\). The action of the gauged sigma model with gauge group \( H \) is

\[ L = g_{ij} \nabla_{+} X^i \nabla_{-} X^j + b_{ij} \ D_{+} X^i \partial_{-} X^j - A^a_i u_{ia} \partial_{-} X^i + A^a_{ia} \ D_{+} X^i - c_{ab} A^a_{-} A^b_{+}, \quad (2.9) \]

where \( X \) is a section of bundle with base space the (1,0) superspace with coordinates \( (y^+, y^-, \theta^+) \) and fibre the target manifold \( M \), \( \{A_{+}, A_{-}, A_{\pm}\} \) are the components of the gauge connection \( A \) and \( \{\nabla_{+}, \nabla_{-}, \nabla_{\pm}\} \) are the corresponding
covariant derivatives that satisfy the (super)algebra
\[
[\nabla_+, \nabla_+] = 2i \nabla_+, \quad [\nabla_+, \nabla_-] = F_{+-}, \quad [\nabla_-, \nabla_4] = F_{-4}.
\] (2.10)

\(F\) is the curvature of the connection \(A\). \(D_+\) is the flat superspace derivative, \(D_+^2 = i\partial_4\). The rest of the notation is the same as in the (1,1) model and the action (2.9) is gauge invariant provided that \(L^a u_b = f_{ab}^c u_c\) and \(c_{ab} = c_{[ab]}\).

To describe the \((1,0)\) coset models, we take as a target manifold \(M\) to be a group \(G\). The gauge group \(H\) is a subgroup of \(G\) that acts on \(G\) with the adjoint action as in the (1,1) coset model. Similarly we choose the metric \(g\) and the Wess-Zumino term \(B = 3db\) of the model. The equations of motion of the \((1,0)\) coset model are
\[
\nabla_+ ^2 X^i - L^i_a F_{+-} = 0, \quad (2.11)
\]
\[
L^i_a \nabla_+ X^i = 0 \quad (2.12)
\]
and
\[
R_{ia} \nabla_- X^i = 0. \quad (2.13)
\]
A consequence of these equations of motion is that \(F = 0\) (on-shell).

3. The \((2,2)\) coset model

To describe the \((2,2)\) supersymmetric coset model, we begin with the \((1,1)\) supersymmetric coset model and then introduce the additional supersymmetry transformations necessary for the construction of this model. The \((2,2)\) supersymmetric coset model has two left-handed and two right-handed supersymmetry transformations. In the following, we concentrate on the left-handed symmetries of the action (2.1). At the end of this section, we will return to the right-handed symmetries and give the commutator of the left-handed transformations with the right-handed ones. It is straightforward to calculate the commutator of the right-handed transformations from the commutator of the left-handed ones.
The action (2.1) is manifestly (1,1) supersymmetric. The first left-handed supersymmetry and translation transformations are

\[
\begin{align*}
\delta_L X^i &= \eta \nabla_+ X^i - \frac{i}{2} D_+ \eta \nabla_+ X^i \\
\delta_L A^a_- &= -i \eta \nabla_+ F^a_+ + \frac{i}{2} D_+ \eta F^a_+ \\
\delta_L A^a_+ &= 0
\end{align*}
\]  

(3.1)

where \( \eta = \eta(x^+, \theta^+) \) are the parameters of the transformations.

The second left-handed supersymmetry transformation can be written as follows:

\[
\begin{align*}
\delta_L X^i &= a_- I^i_j \nabla_+ X^j, \\
\delta_L A^a_- &= a_- D_{ab} F^b_+; \\
\delta_L A^a_+ &= 0
\end{align*}
\]  

(3.2) and (3.3)

\( I \) is an invariant and \( D \) is an equivariant tensor of \( G \) under the adjoint action of the group \( H \) in \( G \) and \( a_- = a_- (x^+, \theta^+) \) is the parameter of the transformation. The transformations (3.2) and (3.3) are a special case of the most general higher spin transformations given in Ref [17]. The invariance property of \( I \) will be suitably modified in section five for the construction of (4,0) supersymmetric coset models. The transformations (3.2) and (3.3) are symmetries of the action (2.1) provided that

\[
\nabla^{(+)}_k I^i_j = 0,
\]  

(3.4)

\[
I_{ij} = I_{[ij]}
\]  

(3.5)

and

\[
L^k_a I_{ki} = - D^b_a L_{bi}.
\]  

(3.6)

To find the condition (3.4), we have used the fact that \( I \) is an invariant tensor under the adjoint action of \( H \) on \( G \). The commutator of two left-handed supersymmetry
transformations on the field $X$ is

$$[\delta_L, \delta'_L]X^i = \delta^{(1)}X^i + \delta^{(2)}X^i + \delta^{(3)}X^i$$  \hspace{1cm} (3.7)$$

where

$$\delta^{(1)}X^i = a'_- a_- N(I)_{jk}^i \nabla_+ X^j \nabla_+ X^k,$$  \hspace{1cm} (3.8)$$

$$\delta^{(2)}X^i = - D_+(a'_- a_-) I^i_k I^k_j \nabla_+ X^j$$  \hspace{1cm} (3.9)$$

and

$$\delta^{(3)}X^i = - 2ia'_- a_- I^i_k I^k_j \nabla_\mp X^j$$  \hspace{1cm} (3.10)$$

where $N(I)$ is the Nijenhuis tensor [19] of the tensor $I$ and is given by

$$N(I)^i_{jk} = 2(I^l_{[j} \partial_{[k]} I^i_{l]} - \partial_{[j} I^l_{k]} I^i_{l]}).$$  \hspace{1cm} (3.11)$$

The commutator (3.7) of two left-handed supersymmetry transformations (eqn. (3.2)) is similar to the commutator of two left-handed supersymmetry transformations of an ungauged N=1 supersymmetric sigma model. Indeed, the latter can be recovered from (3.7), if we replace in the eqns. (3.8)-(3.10) the covariant superspace derivatives $\nabla_+$ with partial superspace ones.

The commutator of two left-handed supersymmetry transformations on the field $A$ (eqn. (3.3)) is always proportional to the curvature $F_{+-}$ of the connection $A$ and its covariant derivatives. This can be verified by a short explicit calculation. Since one of the on-shell conditions of the (1,1) supersymmetric coset models is $F_{+-} = 0$, the transformation of eqn. (3.3) closes on-shell.
The right-handed supersymmetry transformations are

\[ \delta_R X^i = a_+ J^i_j \nabla_- X^j, \quad (3.12) \]

and

\[ \delta_R A^a_+ = a_+ E^a_{\ b} F^b_{+ \ -}; \]
\[ \delta_R A^a_- = 0, \quad (3.13) \]

where \( a_+ = a_+(x^-, \theta^-) \) is a parameter of the transformation. \( J \) is an invariant tensor and \( E \) is an equivariant tensor in the manifold \( G \) under the action of the group \( H \) as in the case of the left-handed transformations. The conditions for the invariance of the action (2.1) under the transformations (3.12) and (3.13) are given by eqns. (3.5) and (3.6), if we set \( J \) and \( E \) in the place of \( I \) and \( D \) correspondingly, and \( J \) is covariantly constant with respect to \( \nabla^{(-)} \) connection, i.e.

\[ \nabla_k^{(-)} J^i_j = 0. \quad (3.14) \]

The commutator of two right-handed supersymmetry transformations on the field \( X \) is similar as the commutator of two left-handed transformations. The commutator of two right-handed supersymmetry transformations on the field \( A \) closes on-shell. Finally the commutator of a left-handed with a right-handed supersymmetry transformation on the field \( X \) is

\[ [\delta_L, \delta_R]X^i = \delta^{(1)} X^i + \delta^{(2)} X^i + \delta^{(3)} X^i \quad (3.15) \]

where

\[ \delta^{(1)} X^i = a_- a_+ M(I, J)^i_{jk} \nabla_+ X^j \nabla_- X^k, \quad (3.16) \]

\[ \delta^{(2)} X^i = -a_- a_+ (J^i_k I^k_j - I^i_k J^k_j) \nabla_- \nabla_+ X^j \]
\[ + a_- a_+ I^i_k J^k_j F^a_{+ \ - \ al} \quad (3.17) \]
and
\[
\delta^{(3)}X^i = a_- a_+ \left( D^a_b J^j_j + E^a_b I^j_j \right) F^b_{+-} s^j_{a}.
\] (3.18)

The tensor $\mathcal{M}$ can be expressed in terms of $I$ and $J$ as follows:
\[
\mathcal{M}^i_{jk}(I, J) = \nabla_l J^l_j I^l_k + J^l_k \nabla_k I^l_j - \nabla_l I^l_j J^l_k - I^l_k \nabla_j J^l_k,
\] (3.19)

where $\nabla$ is the covariant derivative of the Levi-Civita connection of the metric $g$ of the group $G$. Finally, the commutator of a left-handed with a right-handed transformation on the field $A$ closes on-shell.

4. The geometry of the (2,2) coset model

In the previous section, we calculated the commutator of the (2,2) supersymmetry transformations necessary for the description of the (2,2) supersymmetric coset model. In this section, we will derive the conditions for the existence of (2,2) models and show that these conditions have a geometric interpretation in terms of the geometry of the coset space $G/H$.

First we begin with the commutator of left-handed with right-handed transformations (eqn. (3.15)). The terms in this commutator proportional to the curvature tensor $F$ are zero on-shell since the vanishing of $F$ is one of the on-shell conditions. The rest of the commutator vanishes on-shell as well. Indeed, the tensor $\mathcal{M}$ can be expressed in terms of $I$, $J$ and the Wess-Zumino term $B$ of the coset model using the fact that $I$ ($J$) is covariantly constant with respect to $\nabla^{(+)}$ ($\nabla^{(-)}$) covariant derivative. Then combining $\delta^{(1)}$ (eqn. (3.16)) and $\delta^{(2)}$ (eqn. (3.17)) in the commutator (3.15), we get
\[
[\delta_L, \delta_R]X^i = -a_- a_+ \left( J^i_k I^k_j - I^i_k J^k_j \right) \nabla^{(+)}_+ X^j \nabla^{(-)}_+ X^j
\] (4.1)

up to terms proportional to the curvature $F$. However $\nabla^{(+)}_+ X^j$ vanishes on-shell (eqn. (2.6)). Consequently the commutator (4.1) closes on-shell. In conclusion the
commutator of extended left-handed with right-handed supersymmetry transformations closes on-shell without any restrictions on the geometry of the target manifold $G$ of the (1,1) supersymmetric coset model.

Next we turn to study the conditions for the invariance of the action under both left- and right-handed supersymmetry transformations. The eqns. (3.4) and (3.14) can be solved as in the case of (ungauged) WZW models with extended supersymmetries. Indeed, following Refs [18] the most general solution of these equations is

$$I^i_j = L^i_A L^B_j I^{AB}_B, \quad J^i_j = R^i_A J^{AB}_B R^B_j$$

where $(I^{AB})$ and $(J^{AB})$ are constant matrices. Then equation (3.5) can be solved provided that $(I^{AB})$ and $(J^{AB})$ are antisymmetric matrices. Before we examine condition (3.6), we recall that both $I$ and $J$ are invariant tensors under the adjoint action of the group $H$ on $G$. This invariance property imposes the conditions

$$f_{aC}^A I^C_B - f_{aB}^C I^A_C = 0$$
$$f_{aC}^A J^C_B - f_{aB}^C J^A_C = 0$$

The solutions of (4.3) that impose the weakest restriction on the geometry of the target manifold $G$ and the gauge group $H$ of the (1,1) supersymmetric coset model are

$$(I^{AB})|_P = (I^n_m), \quad (J^{AB})|_P = (J^n_m)$$

and all the other components are zero, where the Lie algebra indices $A, B$ of $\text{Lie}G$ are split according to the vector space decomposition $\text{Lie}G = \text{Lie}H \oplus P$, i.e. $a = 1, \cdots, \dim \text{Lie}H$ and $n, m = 1, \cdots, \dim P$ ($\dim P = \dim \text{Lie}G - \dim \text{Lie}H$). The equation (4.3) then becomes

$$f_{ap}^m I^n_p - f_{an}^p I^n_m = 0$$
$$f_{ap}^m J^n_p - f_{an}^p J^n_m = 0$$

If the tensors $I$ and $J$ are chosen as in eqn. (4.4), we can solve equation (3.6) by
setting \( D = E = 0 \). This concludes the study of the conditions for the invariance of the action (2.1) under (2,2) supersymmetry transformations.

To study the conditions that a commutator (eqn. (3.7)) of two left-handed supersymmetry transformations closes on-shell, we write the Nijenhuis tensor \( \mathcal{N}(I) \) of \( I \) using eqn. (4.2) as

\[
N^i_{jk} = L^i_A N^A_{BC} L^B_j L^C_k.
\] (4.6)

The components of the Nijenhuis tensor \( \mathcal{N}(I) \) that contribute to the commutator (3.7) are \( N^A_{mn} \). The rest are proportional to the equations of motion (2.7) and they do not affect the on-shell closure of the algebra. Finally we can show using eqn. (4.4) that the commutator (3.7) closes on-shell to (1,1) supersymmetry transformations (eqn. (3.1)) provided that

\[
\mathcal{N}(I)^p_{mn} = 0, \quad \Gamma^m_p I^p_m = -\delta^m_n
\] (4.7)

and

\[
f^a_{lp} I^l_m I^p_n - f^a_{mn} = 0,
\] (4.8)

where \( l, m, n, p = 1, \cdots, \dim P \) and \( a = 1, \cdots, \dim \text{Lie} H \). The same analysis can be done for the right-handed supersymmetry transformations.

To find the geometric interpretation of the conditions (4.2), (4.4), (4.5), (4.7) and (4.8) in terms of the geometry of the coset space \( G/H \), we introduce a local section \( s \) of the principal bundle \( H \to G \to G/H \); \( H \) acts on \( G \) from the right, i.e. \( k \to kh \) where \( k \in G \) and \( h \in H \). The \( \text{Lie} G \)-algebra valued one-form \( s^{-1}ds \) on \( G/H \) can be decomposed as follows:

\[
s^{-1}ds = e^m t_m + \Omega^a t_a.
\] (4.9)

e is a frame and \( \Omega \) is the canonical connection of the coset space \( G/H \). The curvature \( F \) of the canonical connection is \( F^a_{\mu \nu} = -f^a_{mn} e^m_\mu e^n_\nu \) and its torsion \( T \)
is $T^\kappa_{\mu\nu} = -\frac{1}{2} e^\kappa f_{\mu \nu} e^m_p e^n_p$ where $\mu, \nu, \kappa = 1, \ldots, \dim G/H$. Using the frame $e$ of the coset space, we define the metric $g_{\mu\nu} = e^m_\mu e^n_\nu \delta_{mn}$ in $G/H$ and the tensor $I^\mu_{\nu} = e^n_\nu I^m_m e^\mu_m$ where $I^m_m$ is given in eqn. (4.4). The geometric interpretation of eqn. (4.5) is that the tensor $I^\mu_{\nu}$ is covariantly constant with respect to the canonical connection of the coset space $G/H$. This is equivalent to the fact that $I^\mu_{\nu}$ is a $G$-invariant tensor on the coset. From eqn. (4.7) we deduce that the tensor $I^\mu_{\nu}$ is an integrable complex structure on the coset space $G/H$ and since $(I_{mn})$ is antisymmetric matrix the metric $g_{\mu\nu}$ is an (1,1) tensor with respect to this complex structure, i.e. $G/H$ is a Hermitian manifold. Finally from eqn. (4.8) we find that the curvature $F$ of the canonical connection $\Omega$ is (1,1) $\text{Lie} H$-valued two-form with respect to the complex structure $I^\mu_{\nu}$. The latter implies that the tangent bundle of $G/H$ is a holomorphic vector bundle. Conversely, given a coset space $G/H$ which is a complex manifold with respect to a $G$-invariant complex structure such that the G-invariant metric in $G/H$ and the curvature of its canonical connection are (1,1) tensors, we can construct a (2,2) supersymmetric coset model with gauge group $H$.

An interesting class of (2,2) supersymmetric coset models arises whenever the target manifold $G$ is a semisimple Lie group and $G/H$ is a symmetric space. In this case the torsion $T$ of the canonical connection is zero and this connection becomes the Levi-Civita connection of the G-invariant metric on $G/H$ and the first condition of eqn. (4.5) implies eqn. (4.8). For symmetric coset spaces, $\mathcal{N}(I)^p_{mn}$ is identically zero and $I$ is covariantly constant with respect to the Levi-Civita connection of the invariant metric of the symmetric space, i.e. $G/H$ becomes a Kähler manifold.

Given a complex structure $I$ on $G/H$ necessary for the existence of the second left-handed supersymmetry transformation, it is straightforward to construct a complex structure $J$ necessary for the construction of a second right-handed supersymmetry transformation. Indeed we may set $J^i_j = R^i_A I^A_B R^B_j$. Because of this the (2,1) supersymmetric coset model is in fact invariant under (2,2) supersymmetry transformations.
It is straightforward to use the results of the previous section to treat the (2,0) supersymmetric coset models\(^\ast\). To do this we start from the (1,0) supersymmetric coset model and then introduce the additional left-handed supersymmetry transformations as in eqns. (3.2) and (3.3). The conditions for invariance of the action (2.9) under the transformations (3.2) and (3.3) are given by eqns. (3.4)-(3.6). The commutator of two left-handed transformations is given in eqn. (3.7). Finally the closure properties of two left-handed supersymmetry transformations are the same as the closure properties of two left-handed supersymmetry transformations of the (2,2) coset model. Consequently the geometry of the coset space \(G/H\) of the (2,0) supersymmetric coset model is the same as the geometry of the corresponding coset space of the (2,2) model.

5. The geometry of the (4,0) coset model

The discussion of sections three and four can be extended to describe the (4,0) supersymmetric coset model as well. This can be achieved by introducing three complex structures on the coset space \(G/H\) one for each additional supersymmetry. One of the conditions for the closure of the supersymmetry algebra is that the three complex structures that generate the left-handed supersymmetries satisfy the algebra of imaginary unit quaternions. In addition each complex structure must obey the condition (4.5). Thus all three complex structures are covariantly constant with respect to the canonical connection of the coset space \(G/H\). The latter is a very strong condition on the geometry of the coset manifolds. In particular, it excludes all the cases where \(G\) is semisimple and \(G/H\) is a symmetric space and consequently these models are not suitable for the realisation of the VP superconformal algebra. In the following we will describe a (4,0) model that has geometric properties similar to those of the VP superconformal algebra and at the conclusions we will comment how this method may be used to study the rest of the (4,\(q\)) models.

\(^\ast\) Our description of the (2,0) supersymmetric coset model is different from the one presented by the authors of Ref [13]. One of the differences is in the choice of the tensor \(I\).
To describe the (4,0) supersymmetric coset models, we begin from the off-shell formulation of (1,0) supersymmetric coset models of section two and introduce the additional left-handed supersymmetry transformations

\[
\delta L^i X_i = a^r I_r^i \nabla_+ X_j \\
\delta L^a A^a_-- = a^r D^a_+ b F^b_--; \\
\delta L^a A^a_+ = 0; \quad r = 1, 2, 3,
\]

where \(a^r\) are the parameters of the transformations and \(I_r\) are *equivariant* tensor fields on \(G\), i.e. \(\mathcal{L}_a I_r = -\omega_{ars} I_s\), \(r, s = 1, 2, 3\). \(\omega\) is a representation of \(\text{Lie}H\) on the space spanned by \(I_r\) and \(\mathcal{L}_a\) is the Lie derivative of of the vector field \(\xi_a\) on \(G\). The transformations (5.1) are symmetries of the (1,0) supersymmetric coset model provided that the eqns. (3.4), (3.5) and (3.6) are satisfied for every pair of \((I_r, D_r)\) and the parameters \(a^r_--\) are *covariantly* constant with respect to \(\nabla_-=\) covariant derivative, i.e.

\[
\nabla_- a^r_- = \partial_- a^r_- + A^b_- \omega_{bs}^- a^s_- = 0. \tag{5.2}
\]

From the conditions for the invariance of the action, each tensor \(I_r\) may be written as

\[
(I_r)^i_j = L_A^i (I_r)^A_B L_B^j \tag{5.3}
\]

where \(((I_r)^A_B)\) are constant antisymmetric matrices \((I_{rAB} = -I_{rBA})\). As in the case of (2,2) supersymmetric coset model in section three, we decompose \(\text{Lie}G\) into \(\text{Lie}H \oplus P\) and choose \(((I_r)^A_B)|_P = ((I_r)^n_m)\) with the rest of the components to vanish. Then we may set \(D_r = 0\) and all the conditions for the invariance of the action under the transformations (5.1) are satisfied. Finally, the equivariance property of \(I_r\) written in terms of \(((I_r)^n_m)\) is

\[
f_{ap}^m (I_r)_p^p - f_{an}^p (I_r)_n^m = -\omega_{ars} (I_s)_m^n. \tag{5.4}
\]

The commutator of any two left-handed transformations (eqn. (5.1)) closes on-shell to left-translations \(\delta X^i = \epsilon^A L_A^i\) and supersymmetry transformations (3.1),...
provided that the following conditions are satisfied:

\[ N(I_r, I_t)^p_{mn} = 0 \]
\[ (I_r)^n_p (I_t)^p_m = -\delta_{rt} + \epsilon r^a (I_s)^n_m \] \hspace{1cm} (5.5)

and

\[ f_{lp}^a (I_r)^l_m (I_r)^p_n - f_{mn}^a = k^a_r s (I_s)_{mn}, \] \hspace{1cm} (5.6)

where \( N(I_r, I_t)^p_{mn} \) are the components of the Nijenhuis tensor

\[ N(I_r, I_t)^i_{jk} = L^i_A N(I_r, I_t)^A_{BC} L^B_j L^C_k \]
\[ = [(I_r)^h_j \partial h (I_t)^i_k] - \partial h j (I_r)^h_k (I_t)^i_h + (I_r \rightarrow I_t)] \] \hspace{1cm} (5.7)

restricted on \( P \) and \( k^a_r = (k^a_r s), a = 1, \cdots, \dim \text{Lie}H, \) are constant \( 3 \times 3 \) matrices (indices in brackets ( ) are not summed over).

The algebra of left-handed supersymmetry transformations closes non-linearly to left-translations \( \delta X^i = \epsilon^A L^i_A \) where the parameter \( \epsilon \) of this symmetry includes the currents \( J_r = I_{rij} \nabla^i X^j \nabla^j X^j \) of the supersymmetry transformations (5.1), i.e. the algebra of supersymmetry transformations and left translations of the (4,0) supersymmetric coset model is a (super) W-algebra [20]. The non-linear closure of the algebra of the supersymmetry transformations is due to eqn. (5.6) and in particular to the fact that some of the \( k^a \) matrices are different from zero.

The conditions given in eqns. (5.5) and (5.6) are not all independent. In particular, if the components (5.5) of two of the six \( N(I_r, I_s) \) tensors vanish then the same components of the remaining four tensors vanish as well. Similarly, if two of the conditions (5.6) are satisfied, they imply the third.

To give a geometric interpretation for the conditions (5.3)-(5.6), we define the rank three \( \text{SO}(3) \)-vector bundle \( E \) over the coset space \( G/H \). This vector bundle is the bundle of the complex structures of \( G/H \) and admits three local sections that obey the algebra of imaginary unit quaternions. \( E \) is not always a trivial vector
bundle over $G/H$ and it does not admit global everywhere no-vanishing sections. We introduce the frame $e$ and the canonical connection $\Omega$ of the coset space as in the previous section. From eqn. (5.5), we can construct a local basis of the complex structures given by $I^\nu_r \mu = I^m_r n e^n \mu e^\nu_m$. The metric $g$ of $G/H$ is (1,1) with respect to all three complex structures. The latter follows because the matrices $(I_{rmn})$, $r = 1, 2, 3$, are antisymmetric. From eqn. (5.4), we find that the canonical connection $\Omega$ of $G/H$ preserves the bundle $E$, i.e. the covariant derivative of a section of $E$ with respect to the canonical connection is again a section of $E$. The equation (5.6) imposes restrictions on the components of the curvature tensor of the canonical connection. The manifold $G/H$ has a quaternionic structure. In fact it admits a quaternionic Kähler structure with respect to its canonical connection $\Omega$. However this quaternionic Kähler structure is not the conventional one [21, 22] since it is not with respect to a Levi-Civita connection.

An interesting class of (4,0) supersymmetric coset models arises whenever the target manifold $G$ is a semisimple group and $G/H$ is a symmetric space. In this case the eqn. (5.4) is not independent of eqn. (5.6) and the Nijenhuis tensors $N(I_r, I_t)$ are identically zero. The canonical connection of $G/H$ is the Levi-Civita connection of the invariant metric of $G/H$ and the manifold $G/H$ is a quaternionic Kähler manifold (Wolf space). The symmetric compact quaternionic Kähler manifolds were classified in Ref [23].

6. Concluding Remarks

To study the conditions for the existence of a $(p, q)$, $p \geq 3, q \geq 1$, supersymmetric coset model, we start with an off-shell formulation of the (1,1) supersymmetric coset model as in the construction of (2,2) one. A model with (3,q) supersymmetry is in fact invariant under (4,q) supersymmetry transformations. This is because given the two complex structures necessary to construct the three left-handed supersymmetry transformations, we can construct a third one by multiplying the two complex structures together. The third complex structure generates a fourth
left-handed supersymmetry transformation. Thus it is enough to examine the $(4,q)$
supersymmetric coset models. For this, we introduce additional $q-1$ right-handed
supersymmetry transformations

$$
\delta_R X^i = a^s_+ J^i_{ sj} \nabla^- X^j, \quad \delta_R A^a_+ = a^s_+ E^a_{sb} F^b_{+-},
$$

(6.1)
to the four left-handed ones, where $a^s_+$ are the parameters of the transformations.
The geometry of $(4,1)$ supersymmetric coset models is the same as the geometry
of the $(4,0)$ ones. There are two more models. These are the $(4,2)$ and the $(4,4)$
supersymmetric coset models. For both models, the conditions for the invariance
of the $(1,1)$ action under the additional left-handed supersymmetry transforma-
tions are the same as those found in the previous section for the invariance of
the $(1,0)$ action under the left-handed $(4,0)$ supersymmetry transformations. In
addition, the conditions for the closure of the algebra of additional left-handed
supersymmetry transformations are given in eqns. (5.5) and (5.6). In the case
of $(4,2)$ model, the tensor $J = J_r, \ r = 1$, that generates the additional right-
handed supersymmetry transformation is invariant under the adjoint action of the
group $H$ on $G$, and the action (2.1) is invariant under the $(4,2)$ supersymmetry
transformations provided the tensor $J$ satisfies the condition of eqn. (3.14) and is
antisymmetric. The algebra of right-handed supersymmetry transformations closes
on-shell subject to conditions satisfied by the tensor $J$ in section three where the
closure properties of the right-handed supersymmetry transformations of the $(2,2)$
model were examined. The commutator of left- with right-handed supersymmetry
transformations vanishes on-shell. Finally in the case of $(4,4)$ model, the tensors
$J_r, \ r = 1, 2, 3$, are equivariant in a way similar to that of the tensors $I_r, \ r = 1, 2, 3$.
The $(1,1)$ action is invariant under the additional right-handed supersymmetry
transformations provided that each tensor $J_r$ satisfies the condition of eqn. (3.14)
and is antisymmetric. The algebra of right-handed supersymmetry transforma-
tions closes under similar conditions on $J_r, \ r = 1, 2, 3$ as those described in section
five for the tensors $I_r$. The commutator of the left-handed with the right-handed
supersymmetry transformations closes on-shell.
From the conditions of existence of \((4,q)\), \(1 \leq q \leq 4\), supersymmetric coset models, it is clear that the models with \((4,1)\) and \((4,2)\) supersymmetry are in fact invariant under \((4,4)\) supersymmetry transformations. This is because we can construct the tensors \(J_r, r = 1, 2, 3\), necessary for the existence of four right-handed supersymmetries from the tensors \(I_r, r = 1, 2, 3\) that appear in the left-handed ones. Indeed we can achieve this by setting \(J_{rj}^i = R^i_A I^A_r R^B_j\) where \(R\) is the right frame of \(G\). Therefore the “independent” supersymmetric coset models are those with \((p,0)\) and \((p,p)\), \(p = 1, 2, 4\) supersymmetry.

In conclusion, each \((p,q)\) supersymmetric coset model has a left-handed and a right-handed Kac-Moody current, a left-handed and a right-handed energy momentum tensor that generates the translations and first supersymmetry transformations, and \(\mathcal{J}_{r+} = I_{rij} \nabla_i X^j \nabla_i X^j, 1 \leq r \leq p - 1\) and \(\mathcal{J}_{s-} = J_{sij} \nabla_i X^j \nabla_i X^j, 1 \leq s \leq q - 1\) conserved currents that generate the additional supersymmetry transformations. In the case of the \((4,0)\) supersymmetric coset model, the currents \(\mathcal{J}_{r+}\) are covariantly conserved. Similarly the currents \(\mathcal{J}_{r+}\) and \(\mathcal{J}_{s-}\) of the \((4,4)\) supersymmetric coset model are covariantly conserved as well. The \((p,q)\) supersymmetric coset models exist provided that geometry of the coset space \(G/H\) is restricted appropriately. In particular, the \((2,q)\), \(0 \leq q \leq 2\) models exist provided that \(G/H\) is a Hermitian manifold with a holomorphic tangent bundle and the \((4,q)\), \(0 \leq q \leq 4\) models exist provided that \(G/H\) is a quaternionic Kähler manifold with respect to the canonical connection of the coset space. The algebras of currents of the \((p,q)\) supersymmetric coset models are superconformal algebras. Finally, the algebraic closure properties of the algebra of supersymmetry transformations, the current content and the geometry of the coset spaces \(G/H\) of the \((2,2)\) and \((4,0)\) supersymmetric coset models indicate that the algebras of currents of these models are closely related to the N=2 Kazama-Suzuki and N=4 Van Proeyen superconformal algebras correspondingly.

Acknowledgements: I would like to thank C.M. Hull for advice and for taking part in the initial stages of this project, and B. Spence for useful discussions. This work was supported by the SERC.
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