Almost Optimal Sublinear Time Algorithm for Semidefinite Programming

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Abstract

We present an algorithm for approximating semidefinite programs with running time that is sublinear in the number of entries in the semidefinite instance. We also present lower bounds that show our algorithm to have a nearly optimal running time.

1 Introduction

We consider the following problem known as semidefinite programming

Find $X \succeq 0$  
subject to $A_i \cdot X \geq b_i$  
i = 1, ..., m

where $\forall i \in [m]$, $A_i \in \mathbb{R}^{n \times n}$ is w.l.o.g. symmetric and $b_i \in \mathbb{R}$.

Definition 1.1 ($\epsilon$-approximated solution). Given an instance of SDP of the form (1), a matrix $X \in \mathbb{R}^{n \times n}$ will be called an $\epsilon$-approximated solution if $X$ satisfies:

1. $A_i \cdot X \geq b_i - \epsilon$  
2. $X \succeq -\epsilon I$

The main result of this paper is stated in the following theorem.

\[ 1 \text{This work is a continuation and improvement of the sublinear SDP algorithm in [1].} \]
**Theorem 1.2.** There exists an algorithm that given $\epsilon > 0$ and an instance of the form (1) such that $\forall i \in [m], \|A_i\|_F \leq 1, |b_i| \leq 1$ and there exists a feasible solution $X^*$ such that $\|X^*\|_F \leq 1$, returns an $\epsilon$-approximated solution with probability at least $1/2$.

The running time of the algorithm is $O\left(\frac{m \log m}{\epsilon^2} + \frac{n^2 \log m \log n}{\epsilon^2}\right)$.

Our upper bound is completed by the following lower bound that states that the running time of our algorithm is nearly optimal.

**Theorem 1.3.** Given an instance of the form (1) such that $\forall i \in [m], \|A_i\|_F \leq 1, |b_i| \leq 1$, any algorithm that with probability at least $1/2$ does the following: either finds a matrix $X$ such that $X$ is an $\epsilon$-approximated solution and $\|X\|_F \leq 1$, or declares that no such matrix could be found, has running time at least $\Omega(\epsilon^{-2}(m + n^2))$.

## 2 Preliminaries

Denote the following sets:

\[
\mathbb{B}_F = \{ X \in \mathbb{R}^{n \times n} | \|X\|_F \leq 1 \}
\]

\[
\Delta_{m+1} = \{ p \in \mathbb{R}^m | \forall i \in [m] p_i \geq 0, \sum_{i=1}^m p_i \leq 1 \}
\]

\[
S_+ = \{ X \in \mathbb{R}^{n \times n} | X \succeq 0, \text{Tr}(X) \leq 1 \}
\]

We consider the following concave-convex problem

\[
\max_{X \in \mathbb{B}_F} \min_{p \in \Delta_{m+1}, Z \in S_+} \sum_{i=1}^m p_i(A_i \cdot X - b_i) + Z \cdot X \tag{2}
\]

The following claim establishes that in order to approximate (1) it suffices to approximate (2).

**Claim 2.1.** Given a feasible SDP instance of the form (1) let $X \in \mathbb{B}_F$ be such that

\[
\min_{p \in \Delta_{m+1}, Z \in S_+} \sum_{i=1}^m p_i(A_i \cdot X - b_i) + Z \cdot X \geq -\epsilon
\]

Then $X$ is an $\epsilon$-approximated solution.
Proof. Define $\text{Val}(X) = \min_{p \in \Delta_{m+1}, Z \in \mathbb{S}_+} \sum_{i=1}^{m} p_i (A_i \bullet X - b_i) + Z \bullet X$. For all $i \in [m]$ it holds by setting the dual variables to $p_i = 1$, $p_j = 0 \ \forall i \neq j$ and $Z = 0_{n \times n}$ that

$$A_i \bullet X - b_i \geq \text{Val}(X) \geq -\epsilon$$

Also, for any vector $v \in \mathbb{R}^n$ such that $\|v\|_2 \leq 1$ we set the dual variables to $p_i = 0 \ \forall i$ and $Z = vv^\top$ and thus is holds that

$$v^\top X v \geq \text{Val}(X) \geq -\epsilon$$

which implies that $X \geq -\epsilon I$. \hfill \Box

3 The Algorithm

In this section we present our algorithm that approximates the max-min objective in (2) up to a desired additive factor of $\epsilon$. Our algorithm can be viewed as a primal-dual algorithm that works in iterations, on each iteration performing a primal improvement step and a dual one. For this task we make use of online convex optimization algorithms which are known to be useful for solving concave-convex problems.

Consider the function $L : \mathbb{B}_F \times \Delta_{m+1} \times \mathbb{S}_+ \rightarrow \mathbb{R}$ given by

$$L(X, p, Z) = \sum_{i=1}^{m} p_i (A_i \bullet X - b_i) + Z \bullet X$$

The primal variable $X$ is updated by an online stochastic gradient ascent algorithm which updates $X$ by

$$X_{t+1} \leftarrow X_t + \eta \tilde{\nabla} t$$

where $\tilde{\nabla} t$ is an unbiased estimator for the derivative of $L(X, p, Z)$ with respect to the variable $X$, that is $\mathbb{E}[\tilde{\nabla} t | p, Z] = \sum_{i=1}^{m} p_i A_i + Z$. The parameter $\eta$ is the step size. Note that after such an update the point $X_{t+1}$ may be outside of the set $\mathbb{B}_F$ and we need to project it back to the feasible set which requires only to normalize the frobenius norm. Since we assume that the matrices $A_i$ are symmetric, then the primal variable $X$ is also always a symmetric matrix.

The dual variable $p$ which imposes weights over the constraints is updated by a variant of the well known multiplicative weights (MW) algorithm which performs the following updates:

$$w_{t+1} \leftarrow w_t e^{-\eta (A_i \bullet X - b_i)}, \quad p_{t+1} \leftarrow \frac{w_{t+1}}{\|w_t\|_1}$$
where \( w \) is the vector of weights prior to the normalization to have \( l_1 \) norm equals 1. This update increases the weight of constraints the are not satisfied well by the current primal solution \( X_t \).

The MW algorithm produces vectors \( p_t \) which lie in the simplex, that is \( \sum_{i=1}^{m} p_t(i) = 1 \). In our case we are interested that the sum of entries in \( p_t \) may be less then 1. We enable this by artificially adding an additional constraint to the sdp instance in the form \( 0_{n \times n} \cdot X \geq 0 \). And run the MW algorithm with dimension \( m + 1 \). By the MW update rule, the size of the entry \( p_{m+1} \) is fixed on all iteration and its entire purpose is to allow the sum of the first \( m \) entries to be less than 1. The added constraint is of course always satisfied and thus it does not affect the optimization.

An additional issue with the MW updates is that it requires to compute on each iteration the products \( A_i \cdot X_t \) for all \( i \in [m] \) which takes linear time in the number of entries in the sdp instance. We overcome this issue by only sampling these products instead of using exact computation. Given the matrix \( X \) we estimate the product \( A_i \cdot X \) by

\[
\tilde{v}_i \leftarrow \frac{A_i(j, l)\|X\|_F^2}{X(j, l)} \text{ with probability } \frac{X(j, l)^2}{\|X\|_F^2}.
\]

It holds that \( \mathbb{E} [\tilde{v}_i | X] = A_i \cdot X \).

On the down side the estimates \( v_i \) are unbounded which is important to get high probability concentration guarantees. We overcome this difficulty by clipping these estimates by taking \( v_i \leftarrow \max\{\min\{\tilde{v}_i, \eta^{-1}\}, -\eta^{-1}\} \). Note that \( v_i \) is no longer an unbiased estimator of \( A_i \cdot X \), however the resulting bias is of the order of \( \epsilon \) and thus does not hurt our analysis. Since the values \( v_i \) may still be large we use the variance of these variables to get better concentration guarantees. It holds that

\[
\mathbb{E} [v_i^2 | X] \leq \mathbb{E} [\tilde{v}_i^2 | X] = \|A_i\|_F^2 \|X\|_F^2
\]

Finally the dual variable \( Z \), unlike the variables \( X, p \) which are updated incrementally, is always locally-optimized by choosing

\[
Z \leftarrow \min_{M \in \mathbb{S}_+} M \cdot X
\]

Here we note that in case \( X \) is not PSD then without loss of generality \( Z \) is always a rank one matrix \( zz^\top \) such that \( z \) is an eigenvector of \( X \) corresponding to the most negative eigenvalue of \( X \). In case \( X \) is PSD then \( Z = 0_{n \times n} \). In any case \( \|Z\|_F \leq 1 \). \( Z \) could be approximated quite fast using an eigenvalue algorithm such as the Lanczos method. It will suffice to find a matrix \( Z \) such that the product \( Z \cdot X \) is \( O(\epsilon) \) far from the true minimum.

Finally the algorithm returns the average of all primal iterates.
Algorithm 1 SublinearSDP

1: Input: $\epsilon > 0$, $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}$ for $i \in [m]$.
2: Let $T \leftarrow 20^2\sqrt{40}\epsilon^{-2} \log m$, $\eta \leftarrow \sqrt{\frac{40\log m}{T}}$, $\epsilon' \leftarrow \epsilon/4$.
3: Let $Y_1 \leftarrow 0_{n \times n}$, $w_1 \leftarrow 1_m$.
4: Let $A_{m+1} = 0_{n \times n}$, $b_{m+1} = 0$.
5: for $t = 1$ to $T$ do
6: $X_t \leftarrow Y_t/\max\{1, \|Y_t\|_F\}$.
7: $p_t \leftarrow \frac{w_t}{\|w_t\|_1+1}$.
8: $Z_t \leftarrow Z \in \mathbb{R}^{n \times n}$ s.t. $Z \cdot X_t \leq \min_{Z \in S_+} Z \cdot X_t + \epsilon'$.
9: $i_t \leftarrow i \in [m]$ w.p. $p_t(i)$ and $i_t \leftarrow m + 1$ w.p. $1 - \sum_{i=1}^m p_t(i)$.
10: $Y_{t+1} \leftarrow Y_t + \sqrt{\frac{1}{2T}(A_{i_t} + Z_t)}$.
11: Choose $(j_t, l_t) \in [n] \times [n]$ by $(j_t, l_t) \leftarrow (j, l)$ w.p. $X_t(j, l)^2/\|X_t\|_F^2$.
12: for $i \in [m]$ do
13: $\tilde{v}_t \leftarrow A_i(j_t, l_t)\|X_t\|^2/\|X_t(j_t, l_t) - b_i\|$.
14: $v_t(i) \leftarrow \text{clip}(\tilde{v}_t(i), 1/\eta)$.
15: $w_{t+1}(i) \leftarrow w_t(i)(1 - \eta v_t(i) + \eta^2 v_t(i)^2)$
16: end for
17: end for
18: return $\bar{X} = \frac{1}{T} \sum_t X_t$

4 Analysis

The following lemma gives a bound on the regret of the MW algorithm (line 15), suitable for the case in which the losses are random variables with bounded variance. For a proof see [2] Lemma 2.3.

Lemma 4.1. The MW algorithm satisfies

$$\sum_{t \in [T]} p_t q_t \leq \min_{i \in [m]} \sum_{t \in [T]} \max\{q_t(i), -\frac{1}{\eta}\} + \frac{\log m}{\eta} + \eta \sum_{t \in [T]} p_t q_t^2$$

The following lemma gives concentration bounds on our random variables from their expectations. The proof is given in the appendix.

Lemma 4.2. For $1/4 \geq \eta \geq \sqrt{\frac{40\log m}{T}}$, with probability at least $1 - O(1/m)$, it holds that

(i) $\max_{i \in [m]} \left| \sum_{t \in [T]} (A_i \cdot X_t - b_i) - v_t(i) \right| \leq 3\eta T$

(ii) $\left| \sum_{t \in [T]} (A_{i_t} \cdot X_t - b_{i_t}) - \sum_{t \in [T]} p_t^T v_t \right| \leq 4\eta T$
The following Lemma gives a regret bound on the online gradient ascent algorithm used in our algorithm (line 10). For a proof see [8].

**Lemma 4.3.** Consider matrices \(M_1, \ldots, M_T \in \mathbb{R}^{n \times n}\) such that for all \(i \in [m]\)
\[
\|M_i\|_F \leq \rho.
\]
Let \(X_0 = 0_{n \times n}\) and for all \(t \geq 1\) let \(Y_{t+1} = X_t + \frac{\sqrt{2}}{\rho \sqrt{T}} M_t\) and
\[
X_{t+1} = \min_{X \in B_F} \|Y_{t+1} - X\|_F.
\]
Then
\[
\max_{X \in B_F} \sum_{t \in [T]} M_t \cdot X - \sum_{t \in [T]} M_t \cdot X_t \leq 2 \rho \sqrt{2T}
\]

We are now ready to prove our main theorem, theorem 1.2.

**Proof.** By applying lemma 4.3 with parameters \(M_t = A_{it} + Z_t\) and \(\rho = 2\) we get
\[
\max_{X \in B_F} \sum_{t \in [T]} (A_{it} + Z_t) \cdot X - \sum_{t \in [T]} (A_{it} + Z_t) \cdot X_t \leq 4 \sqrt{2T}
\]
Adding and subtracting \(\sum_{t=1}^T b_i\) gives
\[
\max_{X \in B_F} \sum_{t \in [T]} (A_{it} \cdot X - b_{it} + Z_t \cdot X) - \sum_{t \in [T]} (A_{it} \cdot X_t - b_{it} + Z_t \cdot X_t) \leq 4 \sqrt{2T}
\]
Since we assume that there exists a feasible solution \(X^* \in B_F\) we have that
\[
\sum_{t \in [T]} (A_{it} \cdot X_t - b_{it} + Z_t \cdot X_t) \geq -4 \sqrt{2T}
\]

Turning to the MW part of the algorithm, by lemma 4.1, and using the clipping of \(v_t(i)\) we have
\[
\sum_{t \in [T]} p_t v_t \leq \min_{p \in \Delta_{m+1}} \sum_{t \in [T]} p^\top v_t + (\log m) / \eta + \eta \sum_{t \in [T]} p_t v_t^2
\]
By lemma 4.2(i), with high probability and for any \(i \in [m]\),
\[
\sum_{t \in [T]} v_t(i) \leq \sum_{t \in [T]} A_i \cdot X_t - b_i + 3 \eta T
\]
Thus with high probability it holds that

$$\sum_{t \in [T]} p_t^T v_t \leq \min_{p \in \Delta_{m+1}} \sum_{t \in [T]} \sum_{i=1}^m p_i (A_i \cdot X_t - b_i) + (\log m)/\eta + \eta \sum_{t \in [T]} p_t^T v_t^2 + 3\eta T$$

Applying lemma 4.2 (ii) we get that with high probability

$$\sum_{t=1}^T (A_{i_t} \cdot X_t - b_{i_t}) \leq \min_{p \in \Delta_{m+1}} \sum_{t \in [T]} \sum_{i=1}^m p_i (A_i \cdot X_t - b_i) + (\log m)/\eta + \eta \sum_{t \in [T]} p_t^T v_t^2 + 7\eta T$$

Adding $\sum_{t=1}^T Z_t \cdot X_t$ to both sides of the inequality and using (3) yields

$$\min_{p \in \Delta_{m+1}} \sum_{t \in [T]} \left( \sum_{i=1}^m p_i (A_i \cdot X_t - b_i) + Z_t \cdot X_t \right) \geq -4\sqrt{2T} - (\log m)/\eta - \eta \sum_{t \in [T]} p_t^T v_t^2 - 7\eta T \quad (4)$$

It holds that

$$\sum_{t=1}^T Z_t \cdot X_t \leq \sum_{t=1}^T \min_{Z \in S_+} (Z \cdot X_t + \epsilon') \leq \min_{Z \in S_+} \sum_{t=1}^T (Z \cdot X_t + \epsilon')$$

Plugging the last inequality into (4) gives

$$\min_{p \in \Delta_{m+1}, Z \in S_+} \sum_{t \in [T]} \left( \sum_{i=1}^m p_i (A_i \cdot X_t - b_i) + Z \cdot X_t \right) \geq -4\sqrt{2T} - (\log m)/\eta - \eta \sum_{t \in [T]} p_t^T v_t^2 - 7\eta T - \epsilon' T \quad (5)$$

By a simple Markov inequality argument it holds that w.p. at least 3/4,

$$\sum_{t \in [T]} p_t^T v_t^2 \leq 4T$$

Plugging this bound into (5) and dividing through by $T$ gives with probability at least 1/2.
Theorem 1.3. Under the conditions stated in Theorem 1.3, any successful algorithm must read \( \Omega \left( \frac{m}{\epsilon^2} \right) \) entries from the input.

Proof. Assume that \( n \geq \frac{1}{\epsilon} \). Consider the following random instance. With probability \( 1/2 \) each of the constraint matrices \( A_i \) has a single randomly chosen entry \((i, j) \in \left[ \frac{1}{2^i} \right] \times \left[ \frac{1}{2^j} \right] \) that equals \( \sqrt{1 - \epsilon^2 \left( \frac{1}{4^k} - 1 \right)} \) and all other entries take random values from the interval \([0, \zeta]\) (the goal of these values is to prevent a sparse representation of the input). With the remaining probability of \( 1/2 \), all constraint matrices except one are exactly as before except for a single constraint matrix (chosen at random uniformly) that has all of its entries chosen at random from \([0, \zeta]\). In both cases for each constraint matrix \( A_i, i \in [m] \) it holds that \( \|A_i\|_F \leq 1 \). In the second case it clearly holds that for all \( X \in \mathbb{B}_F \), the theorem follows from plugging the values of \( T, \eta \) and \( \epsilon' \).
\[
\min_{i \in [m]} A_i \cdot X \leq \sqrt{\frac{1}{4\epsilon^2}} \cdot \zeta^2 = \frac{\zeta}{2\epsilon}
\]

In the first case we can construct a solution matrix \(X^*\) has follows: for each \((i, j) \in \left[\frac{1}{2}\right] \times \left[\frac{1}{2}\right], X^*(i, j) = 2\epsilon\) and 0 elsewhere. Clearly \(X^*\) is positive semi definite (since it is a symmetric rank-one matrix) and \(\|X\|_F = 1\). For each \(i \in [m]\) it holds that

\[A_i \cdot X^* \geq 2\epsilon \cdot \sqrt{1 - \zeta^2 \left(\frac{1}{4\epsilon^2} - 1\right)}\]

By choosing \(\zeta = \epsilon^2\) and in both cases \(b_i = 1.6\epsilon \forall i \in [m]\) we have that in the first case

\[
\min_{i \in [m]} A_i \cdot X - b_i \geq 2\epsilon \sqrt{1 - \epsilon^4 \left(\frac{1}{4\epsilon^2} - 1\right)} - 1.6\epsilon > \left(\sqrt{3} - 1.6\right) \epsilon > 0.1\epsilon
\]

In the second case, for all \(X \in \mathbb{B}_F\) it holds that,

\[
\min_{i \in [m]} A_i \cdot X - b_i \leq \frac{\epsilon}{2} - 1.6\epsilon = -1.1\epsilon
\]

Thus the first instance is feasible while the second one does not admit an \(\epsilon\)-approximated solution and the two instances differ by a single randomly chosen entry.

**Lemma 5.2.** Under the conditions stated in Theorem 1.3, any successful algorithm must read \(\Omega \left(\frac{n^2}{\epsilon^2}\right)\) entries from the input.

**Proof.** The proof follows the lines of the previous proof. Assume that \(m \geq \frac{1}{10\epsilon^2}\cdot \epsilon \geq \frac{1}{\sqrt{n}}\) and that \(n\) is even. Let \(p, q \in \mathbb{N}^n\) be two random permutations over the integers \(1..n/2\) and finally set \(q_i = \frac{n}{2} + q_i\). Consider the following random instance composed of \(\frac{1}{10\epsilon^2}\) constraint matrices \(A_i, i \in \left[\frac{1}{10\epsilon^2}\right]\). With probability 1/2 for each \(A_i\) we set the entry \(A_i(p_i, q_i)\) to equal \(\sqrt{1 - \zeta^2 (n^2 - 1)}\) and all other entries in \(A_i\) are sampled uniformly from \([0, \zeta]\). With the other 1/2 probability, all matrices are as before with the difference that we randomly pick a matrix \(A_j, j \in [m]\) and set \(A_j(p_j, q_j)\) to a value sampled uniformly from \([0, \zeta]\). In both cases it holds that
\[ \| A_i \|_F \leq 1 \text{ for all } i \in [m]. \]

In the second case it holds for all \( X \in \mathbb{B}_F \) that,

\[ \min_{i \in [m]} A_i \cdot X \leq n\zeta \]

In the first case we construct a solution \( X^* \) as follows. For every \( i \in [m] \) we define a matrix \( X_i^* \) such that \( X_i^*(p_i, q_i) = X_i^*(q_i, p_i) = X_i^*(p_i, p_i) = X_i^*(q_i, q_i) = 2\epsilon \) and \( X_i^* \) is zero elsewhere. Finally we take \( X^* = \sum_{i=1}^{m} X_i^* \).

Notice that \( X^* \) is the sum of symmetric rank-one matrices and thus it is positive semidefinite.

Since \( p, q \) are both permutations over disjoint sets we have that for every \( i, j \in [n] \times [n] \) it holds that \( |X^*(i, j)| \leq 2\epsilon \) and thus \( \| X^* \|_F^2 \leq \frac{1}{4\epsilon^2} \cdot 4 \cdot 4\epsilon^2 = 1. \)

By construction it holds for every \( i \in [m] \) that

\[ A_i \cdot X^* \geq 2\epsilon \sqrt{1 - \zeta^2 (n^2 - 1)} \]

By choosing \( \zeta = \frac{\epsilon}{2m} \) and in both cases \( b_i = 1.6\epsilon \forall i \in [m] \) we have that in the first case

\[ \min_{i \in [m]} A_i \cdot X^* - b_i \geq 2\epsilon \sqrt{1 - \frac{\epsilon^2}{4n^2} (n^2 - 1)} - 1.6\epsilon \]

\[ > (\sqrt{3} - 1.6) \epsilon > 0.1\epsilon \]

In the second case, for all \( X \in \mathbb{B}_F \) it holds that,

\[ \min_{i \in [m]} A_i \cdot X - b_i \leq \frac{\epsilon}{2} - 1.6\epsilon \leq -1.1\epsilon \]

Thus as before, the first instance is feasible while the second one does not have an \( \epsilon \) additive approximated solution and the two instances differ by a single entry.

Notice however that unlike the previous lemma, in this case because of the nature of our random construction, after reading \( k \) matrices it is suffices for an algorithm searching for the distinguishing entry, to only search \( \left( \frac{n^2}{2} - k \right)^2 \) entries in the next matrix. Nevertheless, by plugging the values of \( m \) and the lower bound on \( \epsilon \) we get that \( \left( \frac{n^2}{2} - m \right)^2 \geq \frac{n^2}{4} - \frac{n}{16}\epsilon \geq \frac{3n^2}{16} \) and thus any algorithm must still read an order of \( n^2 \) entries from each matrix.

\( \square \)
A Martingale and concentration lemmas

We first prove a lemma on the expectation of clipped random variables.

Lemma A.1. Let $X$ be a random variable, let $\bar{X} = \text{clip}(X, C) = \min\{C, \max\{-C, X\}\}$ and assume that $|E[X]| \leq C/2$ for some $C > 0$. Then

$$|E[X] - E[\bar{X}]| \leq \frac{2}{C} \text{Var}[X].$$

Proof. As a first step, note that for $x > C$ we have $x - E[X] \geq C/2$, so that

$$C(x - C) \leq 2(x - E[X])(x - C) \leq 2(x - E[X])^2.$$

Hence, we obtain

$$E[X] - E[\bar{X}] = \int_{x < -C} (x + C)d\mu_X + \int_{x > C} (x - C)d\mu_X$$

$$\leq \int_{x > C} (x - C)d\mu_X$$

$$\leq \frac{2}{C} \int_{x > C} (x - E[X])^2d\mu_X$$

$$\leq \frac{2}{C} \text{Var}[X].$$

Similarly one can prove that $E[X] - E[\bar{X}] \geq -2\text{Var}[X]/C$, and the result follows.

The following lemmas are used to prove lemma A.2.

In the following we assume only that $v_t(i) = \text{clip}(\tilde{v}_t(i), 1/\eta)$ is the clipping of a random variable $\tilde{v}_t(i)$, the conditional variance of $\tilde{v}_t(i)$ is at most one ($\text{Var}[\tilde{v}_t(i) | X_t] \leq 1$) and we use the notation $\mu_t(i) = E[\tilde{v}_t(i) | X_t] = A_t \cdot X_t^\top - b_t$. We also assume that the expectations of $\tilde{v}_t(i)$ are bounded in absolute value by a constant $|\mu_t(i)| = |A_t \cdot X_t - b_t| \leq C$, such that $2 \leq 2C \leq 1/\eta$.

Both lemmas are based on an application of Freedman’s inequality which is a Bernstein-like concentration inequality for martingales which we now state:

Lemma A.2 (Freedman’s inequality). Let $\xi_1, ..., \xi_T$ be a martingale difference sequence with respect to a certain filtration $\{S_t\}$, that is $E[\xi_t | S_t] = 0$ for every $t$. Assume also that for every $t$ it holds that $|\xi_t| \leq V$ and $E[\xi_t^2 | S_t] \leq s$. Then

$$P \left( \left| \sum_{t=1}^T \xi_t \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{\epsilon^2 / 2}{Ts + V\epsilon / 3} \right).$$
Lemma A.3. For $\frac{1}{2C} \geq \eta \geq \sqrt{\frac{4 \log (2m^2)}{T}}$ it holds with probability at least $1 - \frac{1}{m}$ that

$$\max_{i \in [m]} \left| \sum_{t=1}^{T} v_t(i) - \mu_t(i) \right| \leq 3\eta T$$

Proof. Given $i \in [m]$, consider the martingale difference sequence $\xi^i_t = v_t(i) - \mathbb{E}[v_t(i)]$ with respect to the filtration $S_t = (X_t)$. It holds that for all $t$, $|\xi^i_t| \leq \frac{2}{\eta}$ and $\mathbb{E}[|\xi^i_t|^2 | S_t] \leq 1$. Applying Freedman’s inequality we get

$$P\left( \left| \sum_{t=1}^{T} \xi^i_t \right| \geq \eta T \right) \leq 2 \exp \left( -\frac{\eta^2T^2/2}{T + (2/\eta)\eta T/3} \right) \leq 2 \exp \left( -\eta^2T/4 \right)$$

Using lemma A.1 the fact that $v_t(i)$ is the clipping of $\tilde{v}_t(i)$ and the triangle inequality we have,

$$P\left( \left| \sum_{t=1}^{T} v_t(i) - \mu_t(i) \right| \geq 3\eta T \right) \leq 2 \exp \left( -\eta^2T/4 \right)$$

Thus for $\eta \geq \sqrt{\frac{4 \log (2m^2)}{T}}$ we have that with probability at least $1 - \frac{1}{m}$,

$$\left| \sum_{t=1}^{T} v_t(i) - \mu_t(i) \right| \leq 3\eta T$$

The lemma follows from taking the union bound over all $i \in [m]$. \qed

Lemma A.4. For $\frac{1}{2C} \geq \eta \geq \sqrt{\frac{4 \log (2m^2)}{T}}$ it holds with probability at least $1 - \frac{1}{m}$ that

$$\left| \sum_{t \in [T]} p_t^\top v_t - \sum_{t \in [T]} p_t^\top \mu_t \right| \leq 3\eta T.$$ 

Proof. This Lemma is proven in essentially the same manner as Lemma A.3 and proven below for completeness.

Consider the martingale difference sequence $\xi_t = p_t^\top v_t - \mathbb{E}[p_t^\top v_t]$ with respect to the filtration $S_t = (X_t, p_t)$. It holds for all $t$ that $|\xi^i_t| \leq \frac{2}{\eta}$. Also by convexity it holds that $\mathbb{E}[\xi^i_t^2 | S_t] = $
\[ \mathbb{E}[(p_t^\top v_t)^2 \mid S_t] \leq \sum_{i=1}^m p_t(i) \mathbb{E}[v_t(i)^2 \mid S_t] \leq 1. \]

Applying Freedman’s inequality we have,

\[
P \left( \left| \sum_{t=1}^T \xi_t \right| \geq \eta T \right) \leq 2 \exp(-\eta^2 T/4)
\]

Using lemma [A.1] the fact that \( v_t(i) \) is the clipping of \( \tilde{v}_t(i) \) and the triangle inequality we have,

\[
P \left( \left| \sum_{t=1}^T p_t^\top v_t - p_t^\top \mu_t \right| \geq 3 \eta T \right) \leq 2 \exp(-\eta^2 T/4)
\]

Thus for \( \eta \geq \sqrt{\frac{4 \log (2m^2)}{T}} \) the lemma follows.

\[ \square \]

Lemma A.5. For \( \frac{1}{2C} \geq \eta \geq \sqrt{\frac{10C \log (2m)}{T}} \), with probability at least \( 1 - 1/m \),

\[ \left| \sum_{t \in [T]} \mu_t(i_t) - \sum_{t \in [T]} p_t^\top \mu_t \right| \leq \eta T. \]

**Proof.** Consider the martingale difference \( \xi_t = \mu_t(i_t) - p_t^\top \mu_t \), where now \( \mu_t \) is a constant vector and \( i_t \) is the random variable, and consider the filtration given by \( S_t = (X_t, p_t) \).

The expectation of \( \mu_t(i_t) \), conditioning on \( S_t \) with respect to the random choice of the index \( i_t \), is \( p_t^\top \mu_t \). Hence \( \mathbb{E}_t[\xi_t \mid S_t] = 0 \).

It holds that \( |\xi_t| \leq |\mu_t(i_t)| + |p_t^\top \mu_t| \leq 2C \). Also \( \mathbb{E}[\xi_t^2] = \mathbb{E}[(\mu_t(i_t) - p_t^\top \mu_t)^2] \leq 2\mathbb{E}[\mu_t(i_t)^2] + 2(p_t^\top \mu_t)^2 \leq 4C^2 \).

Applying Freedman’s inequality gives,

\[
P \left( \sum_{t=1}^T \xi_t \geq \eta T \right) \leq 2 \exp \left( -\frac{\eta^2 T^2 / 4C^2}{2C\eta T / 3} \right) \leq 2 \exp \left( -\eta^2 T / (10C^2) \right)
\]

where for the last inequality we use \( C \geq 1 \) and \( \eta \leq \frac{1}{C} \).

Thus for \( \eta \geq \sqrt{\frac{10C \log (2m)}{T}} \) the lemma follows. \( \square \)

Setting \( C = 2 \) and \( \eta = \sqrt{\frac{40 \log m}{T}} \) lemma A.3 yields part (i) of lemma 4.2 and combining combining lemmas A.4 and A.5 via the triangle inequality yields part (ii) of lemma 4.2.
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