UNIFORM STABILIZATION OF A WAVE EQUATION WITH PARTIAL DIRICHLET DELAYED CONTROL

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Abstract. In this paper, we consider the uniform stabilization of some high-dimensional wave equations with partial Dirichlet delayed control. Herein we design a parameterization feedback controller to stabilize the system. This is a new approach of controller design which overcomes the difficulty in stability analysis of the closed-loop system. The detailed procedure is as follows: At first we rewrite the system with partial Dirichlet delayed control into an equivalence cascaded system of a transport equation and a wave equation, and then we construct an exponentially stable target system; Further, we give the form of the parameterization feedback controller. To stabilize the system under consideration, we choose some appropriate kernel functions and define a bounded inverse linear transformation such that the closed-loop system is equivalent to the target system. Finally, we obtain the stability of closed-loop system by the stability of target system.

1. Introduction. Let \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be a bounded and connected domain (open set) with smooth boundary \( \partial \Omega \) (e.g., \( C^2 \)). Suppose that the boundary has a portion \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \) with condition that \( \Gamma_0, \Gamma_1 \) are disjoint parts of the boundary relatively open in \( \partial \Omega \), and \( int(\Gamma_0) \neq \emptyset \). Let us consider a wave equation on \( \Omega \) with partial Dirichlet delayed control, whose dynamical behaviour is governed by the partial differential equation

\[
\begin{cases}
  w_t(x,t) = \Delta w(x,t), & x \in \Omega, \quad t > 0, \\
  w(\xi,t) = 0, & \xi \in \Gamma_1, \quad t > 0, \\
  w(\xi,t) = u(\xi,t - \tau), & \xi \in \Gamma_0, \quad t > 0, \\
  w(x,0) = w_0(x), & x \in \Omega, \\
  w_t(x,0) = w_1(x), & x \in \Omega, \\
  u(\xi,s - \tau) = h(\xi,s), & \xi \in \Gamma_0, \quad s \in (0,\tau),
\end{cases}
\] (1)

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where \( u(\xi, t - \tau) \) is the delayed control with delay time \( \tau > 0 \) and \( h(\xi, s) \) is the memory of controller. \((w_0(x), w_1(x))\) is the initial state of system (1).

Let

\[
z(\xi, s, t) = u(\xi, t + s - \tau), \quad \xi \in \Gamma_0, \ s \in [0, \tau].
\]

Clearly, \( z(\xi, s, t) \) is a solution to the transport equation

\[
\begin{cases}
  z_t(\xi, s, t) = z_s(\xi, s, t), & \xi \in \Gamma_0, \ s \in (0, \tau), \\
  z(\xi, \tau, t) = u(\xi, t), & \xi \in \Gamma_0, \ t > 0, \\
  z(\xi, s, 0) = z_0(\xi, s) = h(\xi, s), & \xi \in \Gamma_0, \ s \in [0, \tau].
\end{cases}
\]

Thus, the system (1) is equivalent to a cascaded system with boundary control

\[
\begin{cases}
  z_t(\xi, s, t) = z_s(\xi, s, t), & \xi \in \Gamma_0, \ s \in (0, \tau), \ t > 0, \\
  z(\xi, \tau, t) = u(\xi, t), & \xi \in \Gamma_0, \ t > 0, \\
  w_{tt}(x, t) = \Delta w(x, t), & x \in \Omega, \ t > 0, \\
  w(\xi, t) = z(\xi, 0, t), & \xi \in \Gamma_0, \ t > 0, \\
  w(x, 0) = w_0(x), \ w_t(x, 0) = w_1(x), & x \in \Omega, \\
  z(\xi, s, 0) = z_0(\xi, s), & \xi \in \Gamma_0, \ s \in [0, \tau].
\end{cases}
\]  

Thus in the absence of time delay, the system (1) takes the following form

\[
\begin{cases}
  w_{tt}(x, t) = \Delta w(x, t), & x \in \Omega, \ t > 0, \\
  w(\xi, t) = u(\xi, t), & \xi \in \Gamma_0, \ t > 0, \\
  w(x, 0) = w_0(x), \ w_t(x, 0) = w_1(x), & x \in \Omega.
\end{cases}
\]

This is a classical problem of the partial Dirichlet boundary control with control space \( U = L^2(\Gamma_0) \).

1.1. Literature and motivation. In the last decades, controllability, observability and stabilization of systems described by PDEs have been studied extensively since this kind of systems can place the actuators and sensors on a part of the boundary of the spatial region. Due to the challenge in mathematical analysis, the boundary control problem of systems has been a hot topic in mathematical control field since 1970. For instance, Smyshlyaev et al. [30] considered the boundary stabilization of a 1-D wave equation with in-domain antidamping based on the backstepping method (similar method also appears in [3]). Guo et al. [11] solved the error feedback regulator problem for 1-D wave equation by using the adaptive control approach. In particular, the control problem of high-dimensional wave equations has been a difficult but hot topic in control field, here we refer to [4, 5, 13, 16] and the references therein. If the control acts on the Neumann boundary, the colocated feedback control law leads to a dissipative closed-loop system. Note that when the control acts on the Dirichlet boundary, the wave equation has a solution only in the space \( L^2(\Omega) \times H^{-1}(\Omega) \) for each \( L^2 \) control \( u(\xi, t) \). In this case the expression of feedback operator becomes a main difficulty. To study the Dirichlet control, Lasiecka et al. in [14] and McMillan in [19] considered the feedback control of finite region, and obtained some results on stability and instability. Slightly later, Lasiecka and Triggiani [15] gave the expression of feedback control for the wave equation with Dirichlet control and proved the uniform stabilization, that is the case of \( \Gamma_1 = \emptyset \). If \( \Gamma_1 \neq \emptyset \) and the control acts only on the part \( \Gamma_0 \), the controllability and stabilization of wave equation become more complicated, certain restrictive condition on \( \Gamma_0 \) is required [2, 31]. Yao in [35] firstly gave the controllability condition on \( \Gamma_0 \) for
the wave equation with variable coefficients. This condition was also useful for the controllability and stabilization of quasilinear wave equation in [36,38], and for the Cauchy problem with localized damping near infinity in [37].

According to earlier works e.g., see [2, 31] and [35], for a wave equation with constant coefficients, \( \Gamma_1 \) and \( \Gamma_0 \) must satisfy the condition that there is a smooth vector field \( h \) on \( \mathbb{R}^n \) such that

\[
\Gamma_1 = \{ \xi \in \partial \Omega \mid n_\xi \cdot h(\xi) < 0 \}, \quad \Gamma_0 = \{ \xi \in \partial \Omega \mid n_\xi \cdot h(\xi) > 0 \},
\]

where \( n_\xi \) is the outward unit normal at position \( \xi \). Let \( \mathcal{L} \) denote \(-\Delta\) with zero Dirichlet boundary condition and \( \Upsilon : L^2(\partial \Omega) \to L^2(\Omega) \) be the Dirichlet map on \( \Omega \). According to [15], the control operator can be expressed as \( Bu = \Delta \Upsilon(u) \) and the collocated observation is \( y(t) = -\partial \mathcal{L}^{-1} w(\xi,t) \).

For a wave equation with constant coefficients, partial boundary Dirichlet control, Ammari in [1] extended the stabilization result of [15] from \( \Gamma_1 = \emptyset \) to the case that \( \Gamma_1 \neq \emptyset \) but satisfying (4). The well-posedness and regularity of (3) were proved in [9]. For a wave equation with variable coefficients, partial boundary Dirichlet control and collocated observation, Guo and Zhang in [10, Theorem 1.1] proved that the system is well-posed in the sense of Salamon and regular in the sense of Weiss via the Riemannian geometry method.

We observed that the papers mentioned above mainly considered the control systems without time delay. When the system has a time delay, Nicaise and Pignotti in [25] studied the stability of a wave equation with localized Kelvin-Voigt damping and mixed boundary condition with time delay. And they gave the exponential stability results by using a frequency domain approach. Other results can also be found in [21, 23]. If there is a small delay in input, as shown by Datko [6–8], the collocated feedback control law may not stabilize the system. Xu et al. in [33] considered the stabilization problem of 1-d wave equation with difference-type control, i.e., \( \alpha u(t) + \beta u(t-\tau) \), and proved under the collocated feedback control law that the closed-loop system is exponentially stable as \( \alpha > \beta > 0 \) and instable as \( 0 < \alpha < \beta \). This is called the \( \frac{1}{2} \)-stability rule. Later many researchers verified the correctness of \( \frac{1}{2} \)-stability rule for different models, for example, see [20, 22, 24, 27] and the references therein. Ning and Yan in [26] discussed a wave equation with variable coefficients and a port Neumann boundary with the difference-type control. And they verified the uniform stabilization of system under the \( \frac{1}{2} \)-rule assumption. Essentially speaking, these papers mentioned above mainly studied the stability of systems under the collocated feedback control law, which do not include a new feedback controller design. To overcome the restriction of parameters \( \alpha \) and \( \beta \), Shang and Xu et al. started dynamic feedback controller design for the system with difference-type control. The similar results for other systems can also be seen in [28, 32, 34]. Slight later, Shang and Liu et al. [17, 18, 29] extended such dynamic feedback controller design from the difference-type control to more complicated cases that include multi-delay-time and the integral term. Since the dynamic feedback controller design is based on full state of the system, Han et al. [12] studied output-based stabilization for an Euler-Bernoulli beam with time-delay in boundary input. Such an investigation shows that the state observer and dynamic feedback controller can settle the delay control problem.

However, by using the state observer and dynamic feedback controller, we get a more complicated closed-loop system. The stability analysis of closed-loop system becomes a challenge work in mathematics, although the closed-loop system is stable.
from the view point of controller design. This is because the multiplier technique and Riesz basis approach used in earlier works cannot be applied to the resulted system. Up to now the dynamic feedback controller design only apply to 1-d models and the equations with constant coefficients. How to overcome the difficulty in dimensional problem and variable coefficients is a main issue in the stabilization of system with delay in control.

The main goal of this paper is to find a feedback control \( u(\xi, t) \) such that the closed-loop system corresponding to (2) is exponentially stable.

1.2. The idea of research. The same as earlier works we assume that the system (1) or (2) has the collocated observation

\[
y(t) = -\frac{\partial L^{-1}w(\xi, t)}{\partial n_\xi}
\]

and \( \Gamma_0, \Gamma_1 \) satisfy the condition (4). Then we can design a state observer for (2) as follows

\[
\begin{aligned}
\bar{z}(\xi, s, t) &= \bar{z}_0(\xi, s, t), & \xi \in \Gamma_0, & s \in [0, \tau], t > 0, \\
\bar{z}(\xi, \tau, t) &= u(\xi, t), & \xi \in \Gamma_0, & t > 0, \\
\bar{w}_0(t, x) &= \Delta \bar{w}(x, t), & x \in \Omega, & t > 0, \\
\bar{w}(\xi, t) &= 0, & \xi \in \Gamma_1, & t > 0, \\
\bar{w}(\xi, 0) &= \bar{w}_0(x), & \bar{w}(x, 0) &= \bar{w}_1(x), & x \in \Omega, \\
\bar{z}(x, s, 0) &= \bar{z}_0(x, s), & x \in \Gamma_0, & s \in [0, \tau]
\end{aligned}
\]

with \( k > 0 \).

To show that (5) is a state observer for system (2), we set

\[
v(\xi, s, t) = \bar{z}(\xi, s, t) - \bar{z}(\xi, s, t), \quad W(x, t) = w(x, t) - \bar{w}(x, t)
\]

and consider the error system

\[
\begin{aligned}
v_1(\xi, s, t) &= v_1(\xi, s, t), & \xi \in \Gamma_0, & s \in [0, \tau], t > 0, \\
v_1(\xi, \tau, t) &= 0, & \xi \in \Gamma_0, & t > 0, \\
W_{uu}(x, t) &= \Delta W(x, t), & x \in \Omega, & t > 0, \\
W(\xi, t) &= 0, & \xi \in \Gamma_1, & t > 0, \\
W(\xi, 0) &= v(\xi, 0, t) + k \frac{\partial L^{-1}w_1(\xi, t)}{\partial n_\xi}, & \xi \in \Gamma_0, & t > 0, \\
W(x, 0) &= W_0(x), & W_1(x, 0) &= W_1(x), & x \in \Omega, \\
v(x, s, 0) &= v_0(x, s), & x \in \Gamma_0, & s \in [0, \tau].
\end{aligned}
\]

Note that the \( v \)-part of (6) has a solution \( v(\xi, s, t) = v_0(\xi, t + s) \), so \( v(\xi, 0, t) = 0 \) as \( t > \tau \). In this case, the \( W \)-part satisfies the equation

\[
\begin{aligned}
W_{uu}(x, t) &= \Delta W(x, t), & x \in \Omega, & t > \tau, \\
W(\xi, t) &= 0, & \xi \in \Gamma_1, & t > \tau, \\
W(\xi, 0) &= \frac{\partial L^{-1}w_1(\xi, t)}{\partial n_\xi}, & \xi \in \Gamma_0, & t > \tau.
\end{aligned}
\]

Under the condition (4), the result of [1] asserts that (5) is a state observer for system (2). Therefore, without loss of generality we assume that the full state of system (2) is known.

Now we are in a position to find a feedback controller \( u(x, t) \) such the closed-loop system associated with (2) is exponentially stable. For this purpose we take the
state space as \( \mathcal{H} = L^2(\Gamma_0 \times [0, \tau]) \times L^2(\Omega) \times H^{-1}(\Omega) \). For the system (2), we take the state feedback control \( u(\xi, t) \) of the form

\[
u(\xi, t) = \mathbb{K}(z, w, w_\xi) + \gamma(\xi, t) z(\nu, r, t) \, dr \, dv
\]

where \( q(\xi, \tau - r), \gamma(\xi, \tau, y) \) and \( \eta(\xi, \tau, y) \) are unknown functions that will be determined later. Then the closed-loop system corresponding to (2) is exponentially stable in the sense of norm on space \( \mathcal{H} \).

\[ u(\xi, t) = \mathbb{K}(z, w, w_\xi) = \int_0^\tau \int_{\Gamma_0} q(\xi, \tau - r, \nu) z(\nu, r, t) dr \, dv \]

The control given by (7) is called parameterization feedback controller.

The main purpose of this paper is to provide a selecting approach of parameterization controller, i.e., choosing the functions \( q, \gamma \) and \( \eta \), under which the solution of system (8) is exponentially stable in the sense of norm on space \( \mathcal{H} \).

1.3. Statement of main result. In this paper, we shall prove that if \( q(\xi, s, \nu) = \frac{\partial \mathcal{L}^{-1} \eta(\xi, s, \nu)}{\partial \nu} \), \( \eta(\xi, s, y) = \gamma_s(\xi, s, y) \) and the function \( \gamma(\xi, s, y) \) satisfies the following partial differential equation

\[
\begin{cases}
\gamma_{ss}(\xi, s, y) = \Delta_y \gamma(\xi, s, y), & \xi \in \Gamma_0, \ s \in (0, \tau), \ y \in \Omega, \\
\gamma(\xi, s, \nu) = 0, & \xi \in \Gamma_0, \ s \in (0, \tau), \ \nu \in \partial \Omega, \\
\gamma(\xi, 0, y) = 0, & \xi \in \Gamma_0, \ y \in \Omega, \\
\gamma(\xi, 0, y) = k G_{D,1}(y, \xi), & \xi \in \Gamma_0, \ s \in [0, \tau],
\end{cases}
\]

where \( G_{D,1}(y, \xi) \) is given in Section 2. Then system (8) is equivalent to the following system

\[
\begin{cases}
v_t(\xi, s, t) = v_s(\xi, s, t), & \xi \in \Gamma_0, \ s \in [0, \tau], \ t > 0, \\
v(\xi, t) = 0, & \xi \in \Gamma_0, \ t > 0, \\
w_{tt}(x, t) = \Delta w(x, t), & x \in \Omega, \ t > 0, \\
w(\xi, t) = 0, & \xi \in \Gamma_1, \ t > 0, \\
w(x, 0) = w_0(x), \ w_1(x, 0) = w_1(x), & x \in \Omega, \\
v(\xi, s, 0) = v_0(\xi, s), & \xi \in \Gamma_0, \ s \in [0, \tau].
\end{cases}
\]

Our contribution of this paper is to give the equivalence transformation between (8) and (10), and to prove the boundedness of the transformation.

2. Preliminaries, space setting up and operators. In this section we shall give some basic operators and their integral representation that will be used later.

To formulate the system into an abstract evolution equation, we define an operator called the Laplace operator on \( L^2(\Omega) \) by

\[
\begin{cases}
\mathcal{L} f(x) = -\Delta f(x), & x \in \Omega, \\
f(\xi) = 0, & \xi \in \partial \Omega.
\end{cases}
\]

Clearly, \( D(\mathcal{L}) = H^2(\Omega) \cap H^1_0(\Omega) \), where \( H^1(\Omega) = \{ f \in H^1(\Omega) \mid f_{|\partial \Omega} = 0 \} \).
The following lemma gives the property of operator $L$.

**Lemma 2.1.** Let $L$ be defined as (11). Then the following statements are true:

1) $L$ is a self-adjoint and positive definite operator in $L^2(\Omega)$;
2) $L^{-1}$ is compact, and hence $\sigma(L) = \sigma_p(L) = \{\mu_n, n \geq 1\}$ consists of all isolated eigenvalues of $L$ and

$$0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_n \leq \mu_{n+1} \leq \cdots, \lim_{n \to \infty} \mu_n = +\infty;$$

3) Let $\varphi_n(x)$ be the eigenfunction corresponding to $\mu_n$ with $||\varphi_n||_{L^2} = 1$. Then $\{\varphi_n, n \geq 1\}$ forms an orthonormal basis for $L^2(\Omega)$.

Let us consider the boundary value problem

$$\begin{cases} -\Delta w(y) = 0, & y \in \Omega, \\
 w(\xi) = f(\xi), & \xi \in \partial \Omega,
\end{cases} \quad (12)$$

where $f \in L^2(\partial \Omega)$.

The following lemma gives the solvability of (12).

**Lemma 2.2.** For each $f \in L^2(\partial \Omega)$, (12) has a unique solution $w(y)$ in $L^2(\Omega)$. Moreover,

$$w(y) = \Upsilon(f) = \int_{\partial \Omega} G_D(y, \xi) f(\xi) d\xi, \quad y \in \Omega, \quad (13)$$

where $G_D(y, \xi)$ is the Green’s function defined as

$$G_D(y, \xi) = -\sum_{n=1}^{\infty} \frac{\varphi_n(y)}{\mu_n} \frac{\partial \varphi_n(\xi)}{\partial n_\xi}. \quad (14)$$

In particular, the map $\Upsilon$ is a bounded linear operator from $L^2(\partial \Omega)$ to $L^2(\Omega)$.

**Proof.** For any $\varphi_n$, it holds that

$$- (\Delta w, \varphi_n)_{H^{-2}, H^2} = - \int_{\partial \Omega} \frac{\partial w(\xi)}{\partial n_\xi} \varphi_n(\xi) d\xi + \int_{\partial \Omega} \frac{\partial \varphi_n(\xi)}{\partial n_\xi} w(\xi) d\xi - \int_{\Omega} w(y) \Delta \varphi_n(y) dy = \int_{\partial \Omega} \frac{\partial \varphi_n(\xi)}{\partial n_\xi} w(\xi) d\xi + \mu_n \int_{\Omega} w(y) \varphi_n(y) dy,$$

so we have

$$\int_{\Omega} w(y) \varphi_n(y) dy = -\frac{1}{\mu_n} \int_{\partial \Omega} \frac{\partial \varphi_n(\xi)}{\partial n_\xi} f(\xi) d\xi.$$

By the orthonormal basis property of the family $\{\varphi_n(x), n \geq 1\}$, the solution of (12) is given by

$$w(y) = \sum_{n=1}^{\infty} (w, \varphi_n)_{L^2} \varphi_n(y) = -\sum_{n=1}^{\infty} \frac{\varphi_n(x)}{\mu_n} \int_{\partial \Omega} \frac{\partial \varphi_n(\xi)}{\partial n_\xi} f(\xi) d\xi.$$

The desired result follows.
2.1. **Intermedia spaces.** From Lemma 2.1, we see that $\mathcal{L} : D(L) \to L^2(\Omega)$ is surjective. For $s \in [0,1]$, we can define the space $H^{2s}(\Omega)$ and fractional order operators $\mathcal{L}^s$ respectively by

$$H^{2s}(\Omega) = \left\{ f \in H^1(\Omega) \mid \sum_{n=1}^{\infty} \mu_n^{2s}|(f, \varphi_n)_{L^2}|^2 < \infty \right\}, \quad s \in \left[\frac{1}{2},1\right]$$

and

$$H^{2s}(\Omega) = \left\{ f \in L^2(\Omega) \mid \sum_{n=1}^{\infty} \mu_n^{2s}|(f, \varphi_n)_{L^2}|^2 < \infty \right\}, \quad s \in (0, \frac{1}{2}),$$

$$\mathcal{L}^s f(x) = \sum_{n=1}^{\infty} \mu_n^s(f, \varphi_n)_{L^2}\varphi_n(x), \quad D(\mathcal{L}^s) = H^{2s}(\Omega).$$

Note that $H^{2s}(\Omega)$ equipped with norm

$$||f||_{H^{2s}} = \left( \int_{\Omega} |\mathcal{L}^s f(x)|^2 dx \right)^{\frac{1}{2}} = \left( \sum_{n=1}^{\infty} \mu_n^{2s}|(f, \varphi_n)_{L^2}|^2 \right)^{\frac{1}{2}}$$

are Hilbert spaces, and $\mathcal{L}^s : H^{2s}(\Omega) \to L^2(\Omega)$ are the bounded linear operators.

So we can define the negative fractional order spaces by

$$H^{-2s}(\Omega) = \left\{ f = \sum_{n=1}^{\infty} f_n \varphi_n(x) \mid f_n \in \mathbb{C}, \sum_{n=1}^{\infty} \frac{|f_n|^2}{\mu_n^{2s}} < \infty \right\}$$

equipped with the norm

$$||f||_{H^{-2s}} = \left( \sum_{n=1}^{\infty} \mu_n^{-2s}|f_n|^2 \right)^{\frac{1}{2}}.$$

**Lemma 2.3.** Let $H^{2s}(\Omega)$ and $H^{-2s}(\Omega)$ be defined as before for $s \in (0,1)$. Then the following statements are true:

1) $H^{-2s}(\Omega)$ are Hilbert spaces;

2) $\mathcal{L}^s$ can be extended the bounded linear operators from $L^2(\Omega)$ to $H^{-2s}(\Omega)$, i.e.,

$$\mathcal{L}^s f = \sum_{n=1}^{\infty} \mu_n^s(f, \varphi_n)_{L^2}\varphi_n(x) \in H^{-2s}(\Omega), \forall f \in L^2(\Omega);$$

3) For $f \in H^{-2s}(\Omega)$, it holds that $f_n = (f, \varphi_n)_{H^{-2s},H^2}$ and

$$\mathcal{L}^{-s} f = \sum_{n=1}^{\infty} \left( \frac{f, \varphi_n}_{H^{-2s},H^2} \right) \varphi_n(x) \in L^2(\Omega).$$

Hence $\mathcal{L}^s L^2(\Omega) = H^{-2s}(\Omega)$ and $||\mathcal{L}^{-s} f||_{L^2} = ||f||_{H^{-2s}}$.

4) For any $f, g \in H^{-2s}(\Omega)$, it holds that

$$\int_{\Omega} \mathcal{L}^{-s} f(x) \mathcal{L}^{-s} g(x) dx = \int_{\Omega} \mathcal{L}^{-s} f(x) g(x) dx.$$

According to Lemma 2.3, we have the following result.

**Corollary 1.** Let $\mathcal{L}$ and $\mathcal{L}^\frac{1}{2}$ be defined as before. Then the following statements are true:

1) $\mathcal{L}$ is a bounded linear operator from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$, whose extension is a bounded linear operator from $L^2(\Omega)$ to $H^{-2}(\Omega)$. Moreover,

$$||\mathcal{L} f||_{H^{-1}} = ||f||_{H^1}, \quad f \in H^1_0(\Omega), \quad ||\mathcal{L} g||_{H^{-2}} = ||g||_{L^2}, \quad g \in L^2(\Omega).$$
2) $L^{\frac{1}{2}}$ is a bounded linear operator from $H^{1}_{0}(\Omega)$ to $L^{2}(\Omega)$, its extension is a bounded linear operator from $L^{2}(\Omega)$ to $H^{-1}(\Omega)$. In particular, $L^{\frac{1}{2}}H^{1}_{0}(\Omega) = L^{2}(\Omega)$ and $L^{\frac{1}{2}}L^{2}(\Omega) = H^{-1}(\Omega)$. Moreover, for any $f, g \in H^{-1}(\Omega)$, it has

$$\int_{\Omega} L^{-\frac{1}{2}} f(x) L^{-\frac{1}{2}} g(x) dx = \int_{\Omega} L^{-1} f(x) g(x) dx.$$

2.2. Integral representation of some operators.

**Lemma 2.4.** Let $\mathcal{L}$ and $\Upsilon$ be defined as before, and $\{\varphi_{n}(x), n \geq 1\}$ be the eigenfunctions of $\mathcal{L}$. We define functions

$$G_{D,1}(x, \xi) = -\sum_{n=1}^{\infty} \varphi_{n}(x) \frac{\partial \varphi_{n}(\xi)}{\partial n_{\xi}}, \quad (15)$$

and

$$G_{D,\frac{1}{2}}(x, \xi) = -\sum_{n=1}^{\infty} \frac{\varphi_{n}(x)}{\sqrt{\mu_{n}}} \frac{\partial \varphi_{n}(\xi)}{\partial n_{\xi}}. \quad (16)$$

Then we have

$$\mathcal{L} \Upsilon(f) = \int_{\partial \Omega} G_{D,1}(x, \xi) f(\xi) d\xi \in H^{-2}(\Omega)$$

and

$$\mathcal{L}^{\frac{1}{2}} \Upsilon(f) = \int_{\partial \Omega} G_{D,\frac{1}{2}}(x, \xi) f(\xi) d\xi \in H^{-1}(\Omega).$$

**Proof.** According to Lemma 2.2, for $f \in L^{2}(\partial \Omega)$, we have $\Upsilon(f) \in L^{2}(\Omega)$ which implies

$$\sum_{n=1}^{\infty} \frac{1}{\mu_{n}^{2}} \left| \int_{\partial \Omega} \frac{\partial \varphi_{n}(\xi)}{\partial n_{\xi}} f(\xi) d\xi \right|^{2} < \infty.$$  

Thus $-\Delta \Upsilon(f) = \mathcal{L} \Upsilon(f) \in H^{-2}(\Omega)$ and $\mathcal{L}^{\frac{1}{2}} \Upsilon(f) \in H^{-1}(\Omega)$.

Note that 

$$(\Upsilon(f), \varphi_{n})_{L^{2}} = -\frac{1}{\mu_{n}} \int_{\partial \Omega} \frac{\partial \varphi_{n}(\xi)}{\partial n_{\xi}} f(\xi) d\xi.$$  

By the definition of negative fractional order space, we have

$$\mathcal{L} \Upsilon(f) = \sum_{n=1}^{\infty} \mu_{n} (\Upsilon(f), \varphi_{n})_{L^{2}} \varphi_{n}(x) = -\sum_{n=1}^{\infty} \varphi_{n}(x) \int_{\partial \Omega} \frac{\partial \varphi_{n}(\xi)}{\partial n_{\xi}} f(\xi) d\xi \in H^{-2}(\Omega)$$

and

$$\mathcal{L}^{\frac{1}{2}} \Upsilon(f) = \sum_{n=1}^{\infty} \mu_{n}^{\frac{1}{2}} (\Upsilon(f), \varphi_{n})_{L^{2}} \varphi_{n}(x) = -\sum_{n=1}^{\infty} \varphi_{n}(x) \sqrt{\mu_{n}} \int_{\partial \Omega} \frac{\partial \varphi_{n}(\xi)}{\partial n_{\xi}} f(\xi) d\xi \in H^{-1}(\Omega).$$

From the definition of (15) and (16), it holds that

$$\mathcal{L} \Upsilon(f) = \int_{\partial \Omega} G_{D,1}(x, \xi) f(\xi) d\xi \in H^{-2}(\Omega)$$

and

$$\mathcal{L}^{\frac{1}{2}} \Upsilon(f) = \int_{\partial \Omega} G_{D,\frac{1}{2}}(x, \xi) f(\xi) d\xi \in H^{-1}(\Omega).$$

The desired result follows. \qed
Lemma 2.5. Let $L$ and $\Upsilon$ be defined as before, and $G_{D,1}(x, \xi)$ be defined as (15). Then for any $g \in H^2(\Omega) \cap H^1_0(\Omega)$, we have
\[
\frac{\partial g(\xi)}{\partial n_{\xi}} = -\Upsilon^* Lg = (g, G_{D,1}(\cdot, \xi)) = \int_{\Omega} G_{D,1}(x, \xi)g(x)dx,
\] (17)
where $n_{\xi}$ is the outward unit normal at position $\xi \in \partial \Omega$.
For any $g \in H^1_0(\Omega)$, we have
\[
\frac{\partial L^{-\frac{1}{2}} g(\xi)}{\partial n_{\xi}} = (g, G_{D,\frac{1}{2}}(\cdot, \xi)) = \int_{\Omega} G_{D,\frac{1}{2}}(x, \xi)g(x)dx.
\] (18)

Proof. For $g \in D(L)$, $\Delta g \in L^2(\Omega)$ and
\[
(\Delta g, \varphi_n)_{L^2} = \int_{\Omega} g(x)\Delta \varphi_n(x)dx = -\mu_n(g, \varphi_n)_{L^2},
\]
here we have used $g(\xi) = 0$ on $\partial \Omega$.
For any $f \in L^2(\partial \Omega)$, $(\Upsilon(f))_{L^2(\Omega)}$, it holds that
\[
(\Delta g, \Upsilon(f))_{L^2} = \int_{\partial \Omega} \frac{\partial g(\xi)}{\partial n_{\xi}} \Upsilon(f)d\xi = \int_{\partial \Omega} \frac{\partial g(\xi)}{\partial n_{\xi}} f(\xi)d\xi = (\Upsilon^*(\Delta g), f)_{L^2(\partial \Omega)}.
\]
Therefore, we obtain
\[
\frac{\partial g(\xi)}{\partial n_{\xi}} = -\Upsilon^* Lg.
\]
On the other hand,
\[
\Upsilon(f) = \int_{\partial \Omega} G_D(y, \xi)f(\xi)d\xi = -\sum_{n=1}^{\infty} \frac{\varphi_n(y)}{\mu_n} \int_{\partial \Omega} \frac{\partial \varphi_n(\xi)}{\partial n_{\xi}} f(\xi)d\xi,
\]
\[
\Delta g(y) = -\sum_{n=1}^{\infty} \mu_n(g, \varphi_n)_{L^2} \varphi_n(y) \in L^2(\Omega).
\]
A direct calculation gives
\[
(\Delta g, \Upsilon(f))_{L^2} = \sum_{n=1}^{\infty} \mu_n(g, \varphi_n)_{L^2} \frac{1}{\mu_n} \int_{\partial \Omega} \frac{\partial \varphi_n(\xi)}{\partial n_{\xi}} f(\xi)d\xi
\]
\[
= \sum_{n=1}^{\infty} (g, \varphi_n)_{L^2} \int_{\partial \Omega} \frac{\partial \varphi_n(\xi)}{\partial n_{\xi}} f(\xi)d\xi = \int_{\partial \Omega} \sum_{n=1}^{\infty} (g, \varphi_n)_{L^2} \frac{\partial \varphi_n(\xi)}{\partial n_{\xi}} f(\xi)d\xi
\]
\[
= -\left( \frac{\partial g}{\partial n_{\xi}}, f \right)_{L^2(\partial \Omega)}.
\]
This yields
\[
\frac{\partial g(\xi)}{\partial n_{\xi}} = -\sum_{n=1}^{\infty} (g, \varphi_n)_{L^2} \frac{\partial \varphi_n(\xi)}{\partial n_{\xi}} = -\int_{\Omega} g(y) \sum_{n=1}^{\infty} \varphi_n(y) \frac{\partial \varphi_n(\xi)}{\partial n_{\xi}} dy
\]
\[
= \int_{\Omega} g(y)G_{D,1}(y, \xi)dy = (g, G_{D,1}(\cdot, \xi)) = (g, G_{D,1}(\cdot, \xi)).
\]
Therefore, if $g \in H^2(\Omega) \cap H^1_0(\Omega)$, we have
\[
\frac{\partial g(\xi)}{\partial n_{\xi}} = -\Upsilon^* Lg = (g, G_{D,1}(\cdot, \xi)) = \int_{\Omega} g(y)G_{D,1}(y, \xi)dy.
\]
If \( g \in H^1_0(\Omega) \), then \( \mathcal{L}^{-\frac{1}{2}} g \in D(\mathcal{L}) \). Using equality (17), we have
\[
\frac{\partial \mathcal{L}^{-\frac{1}{2}} g(\xi)}{\partial n_{\xi}} = (\mathcal{L}^{-\frac{1}{2}} g, G_{D,1}(\cdot, \xi)) = \int_{\Omega} \mathcal{L}^{-\frac{1}{2}} g(y) G_{D,1}(y, \xi) dy = \int_{\Omega} g(y) \mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi) dy = \int_{\Omega} g(y) \mathcal{L}^{-\frac{1}{2}} g(y, \xi) dy.
\]
This ends the proof. \( \square \)

Now let \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be a bounded and connected domain (open set) with smooth boundary \( \partial \Omega \) (e.g., \( C^2 \)), where \( \partial \Omega \) has a portion \( \partial \Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1} \) with condition that \( \Gamma_0, \Gamma_1 \) are disjoint parts of the boundary relatively open in \( \partial \Omega \), and \( \text{int}(\Gamma_0) \neq \varnothing \). We consider the space \( H^2_{\Gamma_1}(\Omega) \) defined by
\[
H^2_{\Gamma_1}(\Omega) = \{ f \in H^2(\Omega) \mid f|_{\Gamma_1} = 0 \}.
\]
\( \gamma_{\Gamma_0} \) is the Dirichlet boundary map defined as
\[
\gamma_{\Gamma_0}(f) = f|_{\Gamma_0}, \quad \forall f \in H^2_{\Gamma_1}(\Omega).
\]
Now we define an operator \( \Upsilon_{\Gamma_0} : L^2(\Gamma_0) \rightarrow L^2(\Omega) \) by
\[
\Upsilon_{\Gamma_0}(f)(y) = \int_{\Gamma_0} G_D(y, \xi) f(\xi) d\xi.
\]
Clearly, \( \Upsilon_{\Gamma_0} \) is a restriction of \( \Upsilon \) on \( \Gamma_0 \).

For any \( f \in H^2_{\Gamma_1}(\Omega) \), it has a decomposition
\[
f = f_0 + \Upsilon \gamma_{\Gamma_0}(f), \quad f_0 \in H^2_0(\Omega) \cap H^1_0(\Omega).
\]
Hence,
\[
-\Delta f = -\Delta f_0 - \Delta \Upsilon \gamma_{\Gamma_0}(f) = \mathcal{L} f_0,
\]
where we have used \( \Upsilon \gamma_{\Gamma_0}(f) \in \mathcal{N}(\Delta) \), the null space of operator \( \Delta \).

Therefore,
\[
\mathcal{L}^{-1}(-\Delta) f = f_0 = f - \Upsilon \gamma_{\Gamma_0}(f).
\]
Moreover, for any \( g \in L^2(\Omega) \), it holds that
\[
(-\Delta) \mathcal{L}^{-1} g = g.
\]

3. Stabilization of (8). In this section we shall choose functions \( q, \gamma \) and \( \eta \) such that (8) is exponentially stable. For simplicity, we choose the error system (6) (that is also the system (10)) as a reference system, which called the target system, i.e.,
\[
\begin{align*}
\nu(t, \xi, s, t) = \nu_s(\xi, s, t), & \quad \xi \in \Gamma_0, \ s \in (0, \tau), \ t > 0, \\
v(\xi, \tau, t) &= 0, \quad \xi \in \Gamma_0, \ t > 0, \\
w_t(x, t) = \Delta w(x, t), & \quad x \in \Omega, \ t > 0, \\
w(\xi, t) &= 0, \quad \xi \in \Gamma_1, \ t > 0, \\
w(\xi, t) = k \frac{\partial \mathcal{L}^{-1} w(\xi, t)}{\partial n_{\xi}} + v(\xi, 0, t), & \quad \xi \in \Gamma_0, \ t > 0, \\
w(x, 0) = w_0(x), \ w_1(x, 0) = w_1(x), & \quad x \in \Omega, \\
v(\xi, s, 0) = \nu_0(\xi, s), & \quad \xi \in \Gamma_0, \ s \in [0, \tau],
\end{align*}
\]
where \( \Gamma_0 \) and \( \Gamma_1 \) satisfy the condition (4).

To study the system (19) and (8), we take the state space as
\[
\mathcal{H} = L^2(\Gamma_0 \times [0, \tau]) \times L^2(\Omega) \times H^{-1}(\Omega)
\]
Then the transformation

$$||z||_{L^2} + ||f||_{L^2} + ||L^{-\frac{1}{2}}g||_{L^2} \right)^{\frac{1}{2}}, \forall (z, f, g) \in \mathcal{H}. $$

It is easy to know that $\mathcal{H}$ is a Hilbert space. In the sequel we always assume that $\mathcal{H}$ is a real space.

**Theorem 3.1.** The solution of target system (19) decays exponentially in the sense of norm of $\mathcal{H}$.

### 3.1. Transformation from (8) to (19).

We define an operator $T$ on $\mathcal{H}$ by

$$
\begin{pmatrix}
v(\xi, s) \\
f(x) \\
g(x)
\end{pmatrix}
= T
\begin{pmatrix}
z \\
f \\
g
\end{pmatrix}
= \begin{pmatrix}
I - \hat{q} & -\hat{\gamma} & -\hat{\eta} \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
v(\xi, s) \\
f(x) \\
g(x)
\end{pmatrix}, \forall (z, f, g) \in \mathcal{H},
$$

(20)

where

$$
\hat{q}(z) = \int_0^s \int_{\Gamma_0} q(\xi, s - r, \nu) z(\nu, r) dr d\nu,
$$

$$
\hat{\gamma}(f) = \int_{\Omega} \gamma(\xi, s, y) f(y) dy, \hat{\eta}(g) = \int_{\Omega} L^{-\frac{1}{2}} \eta(\xi, s, y) L^{-\frac{1}{2}} g(y) dy.
$$

The following theorem gives a choice of kernel functions under which the system (8) is stable.

**Theorem 3.2.** Suppose that the kernel functions $q(\xi, s, \nu), \gamma(\xi, s, y)$ and $\eta(\xi, s, y)$ satisfy the following partial differential equation

$$
\begin{align*}
\gamma_s(\xi, s, y) + \eta(\xi, s, y) &= 0, \quad \xi \in \Gamma_0, s \in [0, \tau], y \in \Omega, \\
L^2 \gamma(\xi, s, y) &= \eta(\xi, s, y), \quad \xi \in \Gamma_0, s \in [0, \tau], y \in \Omega, \\
\gamma(\xi, s, \nu) &= 0, \quad q(\xi, s, \nu) = -\frac{\partial \gamma^{-1}(\xi, s, \nu)}{\partial \nu}, \quad \xi \in \Gamma_0, s \in [0, \tau], \nu \in \Gamma_0, \\
\text{Boundary condition respect to } y: &
\end{align*}
$$

(21)

$$
\begin{align*}
\gamma(\xi, 0, y) &= k G_{D, 1}(y, \xi), \\
\eta(\xi, 0, y) &= \xi \in \Gamma_0, y \in \overline{\Omega},
\end{align*}
$$

Then the transformation $T$ defined by (20) maps the solution of (8) to a solution of (19).

**Proof.** Let $(z(\xi, s, t), w(x, t), w_t(x, t)) \in \mathcal{H}$ be a classical solution to (8). And let

$$(v(x, s, t), w(x, t), w_t(x, t))^T = T(z, w, w_t)^T.$$

Then from the definition of (20) we know that

$$
v(\xi, s, t) = z(\xi, s, t) - \int_0^s \int_{\Gamma_0} q(\xi, s - r, \nu) z(\nu, r, t) dr d\nu - \int_{\Omega} \gamma(\xi, s, y) w(y, t) dy
$$

$$
- \int_{\Omega} L^{-\frac{1}{2}} \eta(\xi, s, y) L^{-\frac{1}{2}} w_t(y, t) dy,
$$

(22)

where $\xi \in \Gamma_0$ is regarded as a parameter.

In what follows, we shall verify $(v(\xi, s, t), w(x, t), w_t(x, t))$ satisfies the equation (19). We begin with checking the function $v(\xi, s, t)$.
Differentiating (22) with respect to $s$ and $t$ respectively, we have

$$v_s(\xi, s, t) = z_s(\xi, s, t) - \int_0^s q(\xi, 0, \nu)z(\nu, s, t) d\nu - \int_0^s \int_{\Gamma_0} q_s(\xi, s - r, \nu)z(\nu, r, t) drd\nu$$

$$- \int_\Omega \gamma_s(\xi, s, y)w(y, t)dy - \int_\Omega \mathcal{L}^{-\frac{1}{2}} \eta_s(\xi, s, y)\mathcal{L}^{-\frac{1}{2}} w_t(y, t)dy$$

(23)

and

$$v_t(\xi, s, t) = z_t(\xi, s, t) - \int_0^s \int_{\Gamma_0} q(\xi, s - r, \nu)z(\nu, r, t) drd\nu$$

$$- \int_\Omega \gamma(\xi, s, y)w_t(y, t)dy - \int_\Omega \eta(\xi, s, y)\mathcal{L}^{-1}\Delta y w(y, t)dy$$

(using the differential equations in (8))

$$= z_s(\xi, s, t) - \int_0^s \int_{\Gamma_0} q(\xi, s - r, \nu)z(\nu, r, t) drd\nu$$

$$- \int_\Omega \gamma(\xi, s, y)w_t(y, t)dy - \int_\Omega \eta(\xi, s, y)\mathcal{L}^{-1}\Delta y w(y, t)dy$$

(using equality $\mathcal{L}^{-1}(-\Delta) f = f - \mathcal{L}^{-1} f$)

$$= z_s(\xi, s, t) - \int_0^s \int_{\Gamma_0} q(\xi, s - r, \nu)z(\nu, r, t) drd\nu$$

$$- \int_\Omega \eta(\xi, s, y)[w(y, t) - \mathcal{L}^{-1} \eta(\xi, s, y)]dy$$

(using equality (23) and $\gamma(\xi, s, \nu) = 0, \nu \in \Gamma_0$)

$$= v_s(\xi, s, t) + \int_\Omega \gamma_s(\xi, s, y)w(y, t)dy + \int_\Omega \mathcal{L}^{-\frac{1}{2}} \eta_s(\xi, s, y)\mathcal{L}^{-\frac{1}{2}} w_t(y, t)dy$$

$$+ \int_{\Gamma_0} q(\xi, s, \nu)z(\nu, 0, t) d\nu - \int_\Omega \mathcal{L}^{\frac{1}{2}} \gamma(\xi, s, y)\mathcal{L}^{-\frac{1}{2}} w_t(y, t)dy$$

$$+ \int_0^t \frac{\partial \mathcal{L}^{-1}(\eta(\xi, s, \nu))}{\partial n_\nu} w(\nu, t) d\nu + \int_\Omega \eta(\xi, s, y)w(y, t)dy$$

(using the boundary condition of $w$ in (8))

$$= v_s(\xi, s, t) + \int_\Omega [\gamma_s(\xi, s, y) + \eta(\xi, s, y)]w(y, t)dy$$

$$+ \int_\Omega \mathcal{L}^{-\frac{1}{2}} [\eta_s(\xi, s, y) - \mathcal{L}\eta(\xi, s, y)] \mathcal{L}^{-\frac{1}{2}} w_t(y, t)dy$$

$$+ \int_{\Gamma_0} [q(\xi, s, \nu) + \frac{\partial \mathcal{L}^{-1}(\eta(\xi, s, \nu))}{\partial n_\nu}] w(\nu, t) d\nu$$

(using the differential equations in (21))

$$= v_s(\xi, s, t).$$

Moreover,

$$v(\xi, \tau, t) = z(\xi, \tau, t) - \int_0^\tau \int_{\Gamma_0} q(\xi, \tau - r, \nu)z(\nu, r, t) drd\nu - \int_\Omega \gamma(\xi, \tau, y)w(y, t)dy$$
So \( v(x, s, t) \) satisfies the differential equation and boundary condition in (19).

Next we verify that \( w(x, t) \) also satisfies the differential equation and boundary conditions in (19). In fact, we only need to check the boundary condition of \( w(x, t) \) on \( \Gamma_0 \). Since
\[
v(\xi, 0, t) = z(\xi, 0, t) - \int_\Omega \gamma(\xi, 0, y)w(y, t)dy - \int_\Omega L^{-\frac{1}{2}}\eta(\xi, 0, y)L^{-\frac{1}{2}}w_t(y, t)dy, \quad \xi \in \Gamma_0,
\]
we have
\[
\begin{align*}
&\quad w(\xi, t) - k \frac{\partial L^{-1}w_t(\xi, t)}{\partial n_\xi} - v(\xi, 0, t) \\
&= w(\xi, t) - k \frac{\partial L^{-1}w_t(\xi, t)}{\partial n_\xi} - z(\xi, 0, t) + \int_\Omega \gamma(\xi, 0, y)w(y, t)dy \\
&\quad + \int_\Omega L^{-\frac{1}{2}}\eta(\xi, 0, y)L^{-\frac{1}{2}}w_t(y, t)dy \\
&\quad \text{(using the equality (17))} \\
&= w(\xi, t) - z(\xi, 0, t) + \int_\Omega \gamma(\xi, 0, y)w(y, t)dy \\
&\quad + \int_\Omega \eta(\xi, 0, y)L^{-1}w_t(y, t)dy \\
&\quad \text{(using the initial value condition in (21))} \\
&= w(\xi, t) - z(\xi, 0, t) = 0.
\end{align*}
\]
Then \( w(x, t) \) satisfies the differential equation and boundary condition in (19).

In addition,
\[
v(\xi, s, 0) = z(\xi, s, 0) - \int_0^s \int_{\Gamma_0} q(\xi, s - r, \nu)z(\nu, r, 0)d\nu dr - \int_\Omega \gamma(\xi, s, y)w(y, 0)dy \\
- \int_\Omega L^{-\frac{1}{2}}\eta(\xi, s, y)L^{-\frac{1}{2}}w_t(y, 0)dy, \quad \xi \in \Gamma_0,
\]
that is
\[
v_0(\xi, s) = z_0(\xi, s) - \int_0^s \int_{\Gamma_0} q(\xi, s - r, \nu)z_0(\nu, r)d\nu dr - \int_\Omega \gamma(\xi, s, y)w_0(y)dy \\
- \int_\Omega L^{-\frac{1}{2}}\eta(\xi, s, y)L^{-\frac{1}{2}}w_1(y)dy, \quad \xi \in \Gamma_0.
\]
The proof is complete. \( \square \)
3.2. The inverse transformation from (19) to (2). We consider the transformation of the following form

\[
\begin{pmatrix}
  z(\xi, s) \\
  f(x) \\
  g(x)
\end{pmatrix} = \mathcal{S}\begin{pmatrix}
  v \\
  f \\
  g
\end{pmatrix} = \begin{pmatrix}
  I - \hat{q} & \hat{\gamma} & -\hat{\eta} \\
  0 & I & 0 \\
  0 & 0 & I
\end{pmatrix}\begin{pmatrix}
  v(\xi, s) \\
  f(x) \\
  g(x)
\end{pmatrix}, \quad \forall (v, f, g) \in \mathcal{H},
\]

where

\[
\hat{q}(v) = \int_0^s \int_\Gamma_0 q(\xi, s - r, \nu)v(\nu, r)drd\nu,
\]

\[
\hat{\gamma}(f) = \int_\Omega \hat{\gamma}(\xi, s, y)f(y)dy,
\]

\[
\hat{\eta}(g) = \int_\Omega \mathcal{L}^{-\frac{1}{2}}\hat{\eta}(\xi, s, y)\mathcal{L}^{-\frac{1}{2}}g(y)dy.
\]

By the transformation (24), it holds that

\[
z(\xi, s) = v(\xi, s) - \int_0^s \int_\Gamma_0 q(\xi, s - r, \nu)v(\nu, r)drd\nu - \int_\Omega \hat{\gamma}(\xi, s, y)f(y)dy
\]

\[
- \int_\Omega \mathcal{L}^{-\frac{1}{2}}\hat{\eta}(\xi, s, y)\mathcal{L}^{-\frac{1}{2}}g(y)dy, \quad \xi \in \Gamma_0.
\]

(25)

**Theorem 3.3.** Suppose that the functions \(q(\xi, s - r, \nu), \gamma(\xi, s, y)\) and \(\eta(\xi, s, y)\) satisfy the following partial differential equation

\[
\begin{cases}
\hat{\gamma}_s(\xi, s, y) + \hat{\eta}(\xi, s, y) = 0, & \xi \in \Gamma_0, \quad y \in \Omega, \\
\hat{\eta}_s(\xi, s, y) - \mathcal{L}\hat{\gamma}(\xi, s, y) = 0, & \xi \in \Gamma_0, \quad y \in \Omega, \\
\gamma(\xi, s, \nu) = 0, & \xi \in \Gamma_0, \quad \nu \in \Gamma_1, \\
\gamma(\xi, s, \nu) = -k\frac{\partial\mathcal{L}^{-1}\hat{\eta}(\xi, s, \nu)}{\partial\nu}, & \xi \in \Gamma_0, \quad \nu \in \Gamma_0, \\
\hat{\gamma}(\xi, 0, y) = 0, & \xi \in \Gamma_0, \quad y \in \Omega, \\
\hat{\eta}(\xi, 0, y) = -kG_{D,1}(y, \xi), & \xi \in \Gamma_0, \quad y \in \Omega, \\
q(\xi, s, \nu) + \frac{\partial\mathcal{L}^{-1}\hat{\eta}(\xi, s, \nu)}{\partial\nu} = 0, & \xi \in \Gamma_0, \nu \in \Gamma_0, \\
\hat{\gamma}(\xi, s, y) = \mathcal{L}^{-1}\hat{\eta}(\xi, s, y), & \xi \in \Gamma_0, \quad s \in [0, \tau], \quad y \in \Omega.
\end{cases}
\]

Then \(\mathcal{S}\) maps the solution of (19) to a solution of (2) with control

\[
u(\xi, t) = -\int_0^\tau \int_{\Gamma_0} \hat{q}(\xi, \tau - r, \nu)v(\nu, r)drd\nu - \int_\Omega \hat{\gamma}(\xi, \tau, y)f(y)dy
\]

\[
- \int_\Omega \mathcal{L}^{-\frac{1}{2}}\hat{\eta}(\xi, \tau, y)\mathcal{L}^{-\frac{1}{2}}g(y)dy, \quad \xi \in \Gamma_0.
\]

(27)

**Proof.** Let \((v(\xi, s, t), w(y, t), w_1(x, t))\) be a solution to (19) and \(\mathcal{S}\) be defined as (24). Set \((z, w, w_1)^T = \mathcal{S}(v, w, w_1)^T\). By (25), we have

\[
z(\xi, s, t) = v(\xi, s, t) - \int_0^s \int_{\Gamma_0} q(\xi, s - r, \nu)v(\nu, r)drd\nu - \int_\Omega \hat{\gamma}(\xi, s, y)w(y, t)dy
\]

\[
- \int_\Omega \mathcal{L}^{-\frac{1}{2}}\hat{\eta}(\xi, s, y)\mathcal{L}^{-\frac{1}{2}}w_1(y, t)dy, \quad \xi \in \Gamma_0.
\]

(28)

Thus,

\[
z(\xi, \tau, t) = v(\xi, \tau, t) - \int_0^\tau \int_{\Gamma_0} q(\xi, \tau - r, \nu)v(\nu, r)drd\nu - \int_\Omega \hat{\gamma}(\xi, \tau, y)w(y, t)dy
\]
\[- \int_{\Omega} L^{-\frac{1}{2}} \tilde{\eta}(\xi, \tau, y) L^{-\frac{1}{2}} w_t(y, t) dy \]

(Thanks to (27) and the boundary condition of \( v \))

\[ = u(\xi, t). \]

In what follows, we shall check the differential equations satisfied by \( z \) and \( w \). We begin with checking \( w \). Note that \( w_{tt}(x, t) = \Delta w(x, t) \) holds in (19) and (2), we only need to check the boundary conditions. Since

\[
z(\xi, 0, t) = v(\xi, 0, t) - \int_{\Omega} \tilde{\gamma}(\xi, 0, y) w(y, t) dy - \int_{\Omega} L^{-\frac{1}{2}} \tilde{\eta}(\xi, 0, y) L^{-\frac{1}{2}} w_t(y, t) dy,
\]

or equivalently,

\[
v(\xi, 0, t) = z(\xi, 0, t) + \int_{\Omega} \tilde{\gamma}(\xi, 0, y) w(y, t) dy + \int_{\Omega} L^{-\frac{1}{2}} \tilde{\eta}(\xi, 0, y) L^{-\frac{1}{2}} w_t(y, t) dy,
\]

then the boundary condition of \( w \) in (19) becomes

\[
0 = w(\xi, t) - k \frac{\partial L^{-1} w_t(\xi, t)}{\partial n_{\xi}} - v(\xi, 0, t)
\]

\[= w(\xi, t) - k \frac{\partial L^{-1} w_t(\xi, t)}{\partial n_{\xi}} - z(\xi, 0, t) - \int_{\Gamma_0} \gamma(\xi, 0, y) w(y, t) dy
\]

\[= z(\xi, 0, t) - \int_{\Omega} \tilde{\gamma}(\xi, 0, y) w(y, t) dy
\]

\[\text{(using the equality (17))}
\]

\[= w(\xi, t) - z(\xi, 0, t) - \int_{\Omega} \tilde{\gamma}(\xi, 0, y) w(y, t) dy
\]

\[-k \int_{\Gamma_0} \frac{\partial L^{-1} w_t(\nu, t)}{\partial n_{\nu}} G_D(\nu, \xi) d\nu - \int_{\Omega} \tilde{\eta}(\xi, 0, y) L^{-1} w_t(y, t) dy
\]

\[= w(\xi, t) - z(\xi, 0, t) - \int_{\Omega} \tilde{\gamma}(\xi, 0, y) w(y, t) dy
\]

\[-k \int_{\Omega} G_D,1(x, \xi) L^{-1} w_t(x, t) dx - \int_{\Omega} \tilde{\eta}(\xi, 0, y) L^{-1} w_t(y, t) dy
\]

\text{(using the initial value condition in (26))}

\[= w(\xi, t) - z(\xi, 0, t).
\]

So we have \( w(\xi, t) = z(\xi, 0, t) \).

Next we check \( z \) satisfies the differential equation in (2). Differentiating \( z(\xi, s, t) \) with respect to \( s \) and \( t \) respectively, we have

\[
z_s(\xi, s, t) = v_s(\xi, s, t) - \int_{\Omega} \gamma_s(\xi, 0, y) v(y, t) dy - \int_{\Omega} \tilde{\gamma}_s(\xi, s - r, \nu) v_s(\nu, r, t) dr d\nu
\]

\[\quad - \int_{\Gamma_0} \gamma_s(\xi, s, y) w(y, t) dy - \int_{\Omega} L^{-\frac{1}{2}} \tilde{\gamma}_s(\xi, s, y) L^{-\frac{1}{2}} w_t(y, t) dy,
\]

and

\[
z_t(\xi, s, t)
\]

\[= v_t(\xi, s, t) - \int_{\Omega} \gamma_t(\xi, s, y) w(y, t) dy - \int_{\Omega} \tilde{\gamma}_t(\xi, s, y) w_t(y, t) dy
\]
(using the differential equation in (19))

\[-\int_{\Omega} L^{-\frac{1}{2}} \bar{q}(\xi, s, y) L^{-\frac{1}{2}} w_t(y, t) dy\]

\[= v_s(\xi, s, t) - \int_{\Gamma_0} \int_{\Gamma_0} \bar{q}(\xi, s - r, \nu) v_r(\nu, r, t) dr d\nu - \int_{\Omega} \bar{\gamma}(\xi, s, y) w_t(y, t) dy\]

\[-\int_{\Omega} \bar{q}(\xi, s, y) L^{-1} \Delta_y w(y, t) dy\]

\[= v_s(\xi, s, t) - \int_{\Gamma_0} \bar{q}(x, 0, \nu) v(\nu, s, t) d\nu + \int_{\Gamma_0} \bar{q}(\xi, s, \nu) v(\nu, 0, t) d\nu\]

\[-\int_{\Omega} \int_{\Gamma_0} \bar{q}(\xi, s - r, \nu) v_r(\nu, r, t) dr d\nu - \int_{\Omega} \bar{\gamma}(\xi, s, y) w_t(y, t) dy\]

\[+\int_{\Omega} \bar{q}(\xi, s, y) [w(y, t) - \Upsilon \gamma_{\Gamma_0}(w)(y)] dy\]

\[= \int_{\Omega} \bar{\gamma}(\xi, s, y) w(y, t) dy\]

\[+\int_{\Gamma_0} \bar{q}(\xi, s, \nu) v(\nu, 0, t) d\nu - \int_{\Omega} \bar{\gamma}(\xi, s, y) w_t(y, t) dy\]

\[= \int_{\Omega} [L^{-1} \bar{q}(\xi, s, y) - \bar{\gamma}(\xi, s, y)] w_t(y, t) dy\]

\[+\int_{\Gamma_0} \bar{q}(\xi, s, \nu) v(\nu, 0, t) d\nu - \int_{\Omega} \Upsilon \gamma_{\Gamma_0}(\xi, s, \nu) w(\nu, t) d\nu\]

(using the boundary condition in (19))

\[= \int_{\Omega} [\bar{\gamma}(\xi, s, y) + \bar{\eta}(\xi, s, y)] w(y, t) dy\]

\[+\int_{\Omega} \left[ L^{-1} \bar{q}(\xi, s, y) - \bar{\gamma}(\xi, s, y) \right] w_t(y, t) dy + \int_{\Gamma_0} \bar{q}(\xi, s, \nu) \left[ z(\nu, 0, t) \right] dy\]

\[+\int_{\Gamma_0} \bar{\gamma}(\nu, 0, y) w(y, t) dy + \int_{\Omega} L^{-\frac{1}{2}} \bar{\eta}(\nu, 0, y) L^{-\frac{1}{2}} w_t(y, t) dy \]

\[+\int_{\Gamma_0} \frac{\partial L^{-1} \bar{\eta}(\xi, s, \nu)}{\partial \nu} z(\nu, 0, t) d\nu\]

(using the previous conditions)

\[= \int_{\Omega} [\bar{\gamma}(\xi, s, y) + \bar{\eta}(\xi, s, y)] w(y, t) dy\]

\[+\int_{\Omega} \left[ L^{-1} \bar{q}(\xi, s, y) - \bar{\gamma}(\xi, s, y) + \int_{\Gamma_0} \bar{q}(\xi, s, \nu) L^{-1} \bar{\eta}(\nu, 0, y) d\nu \right] w_t(y, t) dy\]

\[+\int_{\Gamma_0} \left[ \bar{q}(\xi, s, \nu) + \frac{\partial L^{-1} \bar{\eta}(\xi, s, \nu)}{\partial \nu} \right] z(\nu, 0, t) d\nu\]

(assuming previous conditions: \( \bar{\eta}(\xi, 0, y) = -k G_{D,1}(y, \xi) \))
\[ z_s(\xi, \eta, t) + \int_\Omega [\tilde{\gamma}_s(\xi, \eta, y) + \tilde{\eta}(\xi, \eta, y)] w(y, t) dy \\
+ \int_\Gamma \left( L^{-1} \tilde{\eta}(\xi, \eta, y) - \tilde{\gamma}(\xi, \eta, y) - k \int_{\Gamma_0} \tilde{q}(\xi, \eta, \nu) L_{\gamma}^{-1} G_{D,1}(y, \nu) d\nu \right) w(y, t) dy \\
+ \int_{\Gamma_0} \left[ \tilde{q}(\xi, \eta, \nu) + \frac{\partial L^{-1} \tilde{\eta}(\xi, \eta, \nu)}{\partial n_\nu} \right] z(\nu, 0, t) d\nu. \]

We take \( \tilde{\gamma}, \tilde{\eta} \) and \( \tilde{q} \) satisfy
\[
\begin{align*}
\left\{ \begin{array}{ll}
\tilde{\gamma}_s(\xi, \eta, y) + \tilde{\eta}(\xi, \eta, y) = 0, & \xi \in \Gamma_0, \ y \in \Omega, \\
L_y^{-1} \tilde{\eta}_s(\xi, \eta, y) - \tilde{\gamma}(\xi, \eta, y) = k L_y^{-1} \int_{\Gamma_0} \tilde{q}(\xi, \eta, \nu) G_{D,1}(y, \nu) d\nu, & \xi \in \Gamma_0, \ y \in \Omega, \\
\tilde{q}(\xi, \eta, \nu) + \frac{\partial L^{-1} \tilde{\eta}(\xi, \eta, \nu)}{\partial n_\nu} = 0, & \xi \in \Gamma_0, \ \nu \in \Gamma_0,
\end{array} \right.
\end{align*}
\]
then \( z_s(\xi, t) = z_s(\xi, s, t) \). Note that
\[
\int_{\Gamma_0} \tilde{q}(\xi, \eta, \nu) L_y^{-1} G_{D,1}(y, \nu) d\nu = - \int_{\Gamma_0} \frac{\partial L^{-1} \tilde{\eta}(\xi, \eta, \nu)}{\partial n_\nu} L_y^{-1} G_{D,1}(y, \nu) d\nu
= - Y_{\Gamma_0} \left( \frac{\partial L^{-1} \tilde{\eta}(\xi, \eta, \nu)}{\partial n_\nu} \right).
\]
So
\[
\begin{align*}
\left\{ \begin{array}{ll}
\gamma(\xi, \eta, y) = 0, & \xi \in \Gamma_0, \ y \in \Omega, \\
\gamma(\eta, \xi, y) = L \gamma(\xi, \eta, y) = 0, & \xi \in \Gamma_0, \ y \in \Omega, \\
\gamma(\xi, \eta, \nu) = 0, & \xi \in \Gamma_0, \ \nu \in \Gamma_1, \\
\tilde{q}(\xi, \eta, \nu) + \frac{\partial L^{-1} \tilde{\eta}(\xi, \eta, \nu)}{\partial n_\nu} = 0, & \xi \in \Gamma_0, \ \nu \in \Gamma_0.
\end{array} \right.
\end{align*}
\]

The proof of Theorem 3.3 is complete. \( \square \)

### 3.3. Solvability of (21) and (26)

In this subsection, we shall study the solvability of kernel equations (21) and (26).

Assume that \( \mathbb{H} = L^2(\Omega) \times H^{-1}(\Omega) \) and \( U = L^2(\Gamma_0) \). We recall the following wave equation with partial Dirichlet control and collocated observation:
\[
\begin{align*}
\begin{cases}
\Delta w(x, t) + \Delta w_t(x, t) & = 0, \ x \in \Omega, \ t > 0, \\
w(x, t) & = 0, \ x \in \Gamma_1, \ t > 0, \\
w(x, t) & = u(x, t), \ x \in \Gamma_0, \ t > 0, \\
y(\xi, t) & = - \frac{\partial L^{-1} w(\xi, t)}{\partial n_\xi}, \ \xi \in \Gamma_0, \ t > 0.
\end{cases}
\end{align*}
\]
(29)

The following result comes from [1] and [9], which will be used later.

**Theorem 3.4.** Let \( T > 0 \), \( (w_0, w_1) \in \mathbb{H} \), and \( u \in L^2(0, T; U) \). Then there exists a unique solution \( (w, w_t) \in C([0, T]; \mathbb{H}) \) to (29) satisfying \( w(\cdot, 0) = w_0 \) and \( w_t(\cdot, 0) = w_1 \). Moreover, there exists a constant \( C > 0 \), independent of \( (w_0, w_1, u) \) such that
\[
\| (w(\cdot, T), w_t(\cdot, T) \|_{\mathbb{H}}^2 + \| y \|_{L^2(0, T; U)}^2 \leq C \left( \| (w_0, w_1) \|_{\mathbb{H}}^2 + \| u \|_{L^2(0, T; U)}^2 \right). \tag{30}
\]

Theorem 3.4 implies that the system (29) is \( L^2 \) well-posed with state space \( \mathbb{H} \) and control space \( U \). From Proposition 2.2 of [1], we know that there exists a \( C^* > 0 \), independent of \( u \), such that
\[
\| y \|_{L^2(0, T; U)}^2 \leq C^* \| u \|_{L^2(0, T; U)}^2 \quad \text{when} \quad (w_0, w_1) = 0.
\]
For the sake of convenience, we rewrite kernel equations (21) and (26) as follows:

\[
\begin{align*}
\gamma_{ss}(\xi, s, y) &= \Delta_y \gamma(\xi, s, y), \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ y \in \Omega, \\
\gamma(\xi, s, \nu) &= 0, \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ \nu \in \partial \Omega, \\
\gamma(\xi, 0, y) &= 0, \quad \xi \in \Gamma_0, \ y \in \Omega, \\
\gamma_s(\xi, 0, y) &= -kG_{D,1}(y, \xi), \quad \xi \in \Gamma_0, \ y \in \Omega
\end{align*}
\]

with

\[
\begin{align*}
\eta(\xi, s, y) &= -\gamma_s(\xi, s, y), \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ y \in \Omega, \\
q(\xi, s, \nu) &= \frac{\partial \gamma_s(\xi, s, \nu)}{\partial \nu}, \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ \nu \in \Gamma_0
\end{align*}
\]

and

\[
\begin{align*}
\tilde{\gamma}_{ss}(\xi, s, y) &= \Delta_y \tilde{\gamma}(\xi, s, y), \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ y \in \Omega, \\
\tilde{\gamma}(\xi, s, \nu) &= 0, \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ \nu \in \Gamma_1, \\
\tilde{\gamma}(\xi, s, 0) &= 0, \\
\tilde{\gamma}_s(\xi, 0, y) &= kG_{D,1}(y, \xi), \quad \xi \in \Gamma_0, \ y \in \Omega.
\end{align*}
\]

with

\[
\begin{align*}
\tilde{\eta}(\xi, s, y) &= -\tilde{\gamma}_s(\xi, s, y), \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ y \in \Omega, \\
\tilde{q}(\xi, s, \nu) &= \frac{\partial \tilde{\gamma}_s(\xi, s, \nu)}{\partial \nu}, \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ \nu \in \Gamma_0
\end{align*}
\]

Assume that (31a) and (32a) have the collocated observation \(y(s) = -\frac{\partial \gamma_s(\xi, s, \nu)}{\partial \nu}\) and \(\tilde{y}(s) = -\frac{\partial \tilde{\gamma}_s(\xi, s, \nu)}{\partial \nu}\) respectively. For equations (31a) and (32a), Theorem 3.4 shows that the following result is true.

**Theorem 3.5.** The kernel equations (31a) and (32a) are well-posed in space \(L^2(\Gamma_0 \times \Omega) \times H^{-1}(\Gamma_0 \times \Omega)\), and have unique solutions \(\gamma\) and \(\tilde{\gamma}\) respectively. Moreover, for any \(\xi \in \Gamma_0\), there exist constants \(C_0, C_1 > 0\) such that

\[
\int_\Omega |\gamma(\xi, s, y)|^2 dy + \int_\Omega |L^{-\frac{1}{2}} \gamma_s(\xi, s, y)|^2 dy + \int_0^s \int_{\Gamma_0} \left| \frac{\partial L^{-1} \gamma_s(\xi, r, \nu)}{\partial \nu} \right|^2 dr d\nu \\
\leq C_0 \int_\Omega |L^{-\frac{1}{2}} kG_{D,1}(y, \xi)|^2 dy
\]

and

\[
\int_\Omega |\tilde{\gamma}(\xi, s, y)|^2 dy + \int_\Omega |L^{-\frac{1}{2}} \tilde{\gamma}_s(\xi, s, y)|^2 dy + \int_0^s \int_{\Gamma_0} \left| \frac{\partial L^{-1} \tilde{\gamma}_s(\xi, r, \nu)}{\partial \nu} \right|^2 dr d\nu \\
\leq C_1 \left[ \int_\Omega |L^{-\frac{1}{2}} kG_{D,1}(y, \xi)|^2 dy + \int_0^s \int_{\Gamma_0} \left| \frac{\partial L^{-1} \tilde{\gamma}_s(\xi, r, \nu)}{\partial \nu} \right|^2 dr d\nu \right].
\]

If we divide equation (32a) into two parts

\[
\begin{align*}
\tilde{\gamma}_{ss}^{(1)}(\xi, s, y) &= \Delta_y \tilde{\gamma}^{(1)}(\xi, s, y), \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ y \in \Omega, \\
\tilde{\gamma}^{(1)}(\xi, s, \nu) &= 0, \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ \nu \in \partial \Omega, \\
\tilde{\gamma}^{(1)}(\xi, s, 0) &= 0, \quad \xi \in \Gamma_0, \ y \in \Omega, \\
\tilde{\gamma}_s^{(1)}(\xi, 0, y) &= kG_{D,1}(y, \xi), \quad \xi \in \Gamma_0, \ y \in \Omega
\end{align*}
\]

and

\[
\begin{align*}
\tilde{\gamma}_{ss}^{(2)}(\xi, s, y) &= \Delta_y \tilde{\gamma}^{(2)}(\xi, s, y), \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ y \in \Omega, \\
\tilde{\gamma}^{(2)}(\xi, s, \nu) &= 0, \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ \nu \in \Gamma_1, \\
\tilde{\gamma}^{(2)}(\xi, s, 0) &= 0, \quad \xi \in \Gamma_0, \ s \in [0, \tau], \ \nu \in \Gamma_0, \\
\tilde{\gamma}_s^{(2)}(\xi, 0, y) &= 0, \quad \xi \in \Gamma_0, \ y \in \Omega, \\
\tilde{\gamma}_s^{(2)}(\xi, 0, y) &= 0, \quad \xi \in \Gamma_0, \ y \in \Omega
\end{align*}
\]
Remark 1. Theorem 3.4 shows that equations (35) and (36) are $L^2$ well-posed with the collocated observation \( \tilde{y}^{(1)}(s) = -\frac{\partial L^{-1}\gamma^{(1)}(\xi,s,\nu)}{\partial n_\nu} \) and \( \tilde{y}^{(2)}(s) = -\frac{\partial L^{-1}\gamma^{(2)}(\xi,s,\nu)}{\partial n_\nu} \) respectively. And there exist \( C_2, C_3 > 0 \) such that \( \tilde{\gamma}^{(1)} \) and \( \tilde{\gamma}^{(2)} \) satisfy

\[
\int_\Omega |\tilde{\gamma}^{(1)}(\xi, s, y)|^2 dy + \int_0^s \int_{\Gamma_0} \left| \partial L^{-1}\gamma^{(1)}_s(\xi, r, \nu) \right|^2 dr d\nu \\
\leq C_2 \int_0^s \int \left| \partial L^{-\frac{1}{2}}kD,1(y, \xi) \right|^2 dy,
\]

(37)

\[
\int_\Omega |\tilde{\gamma}^{(2)}(\xi, s, y)|^2 dy + \int_0^s \int_{\Gamma_0} \left| \partial L^{-1}\gamma^{(2)}_s(\xi, r, \nu) \right|^2 dr d\nu \\
\leq C_3 \int_0^s \int_{\Gamma_0} \left| k \partial L^{-\frac{1}{2}}\gamma_s(\xi, r, \nu) \right|^2 dr d\nu,
\]

(38)

where \( \xi \in \Gamma_0 \).

3.4. The boundedness of transformations (20) and (24). In this subsection, we shall prove the operators \( T \) and \( S \) are bounded linear operators.

The following theorems give the desired results.

**Theorem 3.6.** Let operator \( T \) be defined as (20) and \( \gamma(\xi, s, y) \) be a solution to (31a). Then there exists \( M_1 > 0 \) such that the following asserts are true:

1. For any \( f(y) \in L^2(\Omega) \),

\[
\int_0^\tau \int_{\Gamma_0} \left| \int_\Omega \gamma(\xi, s, y)f(y)dy \right|^2 d\xi ds \leq M_1 \int_\Omega |f(y)|^2 dy;
\]

2. For any \( g(y) \in H^{-1}(\Omega) \),

\[
\int_0^\tau \int_{\Gamma_0} \left| \int_\Omega L^{-\frac{1}{2}}\eta(\xi, s, y) L^{-\frac{1}{2}}g(\nu)dy \right|^2 d\xi ds \leq M_1 \int_\Omega |L^{-\frac{1}{2}}g(\nu)|^2 dy;
\]

3. For any \( z(\nu, r) \in L^2(\Omega \times [0, \tau]) \),

\[
\int_0^\tau \int_{\Gamma_0} \left| \int_\Omega q(\xi, s - r, \nu)z(\nu, r)drd\nu \right|^2 d\xi ds \leq M_1 \int_0^\tau \int_{\Gamma_0} |z(\nu, r)|^2 dr d\nu.
\]

**Proof.** For any \( f(y) \in L^2(\Omega) \), combining with (33), we have

\[
\int_0^\tau \int_{\Gamma_0} \left| \int_\Omega \gamma(\xi, s, y)f(y)dy \right|^2 d\xi ds \\
\leq \int_0^\tau \int_{\Gamma_0} \left[ \int_\Omega |\gamma(\xi, s, y)| \cdot |f(y)|dy \right]^2 d\xi ds \\
\leq \int_0^\tau \int_{\Gamma_0} \left[ \int_\Omega |\gamma(\xi, s, y)|^2 dy \right] d\xi ds \cdot \int_\Omega |f(y)|^2 dy \\
\leq C_0 \int_0^\tau \int_{\Gamma_0} \left[ \int_\Omega \left| -kL^{-\frac{1}{2}}D,1(y, \xi) \right|^2 dy \right] d\xi ds \cdot \int_\Omega |f(y)|^2 dy
\]

\[
= C_0 k^2 \tau \int_{\Gamma_0} \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi)|^2 dy \, d\xi \cdot \int_{\Gamma_0} |f(y)|^2 dy.
\]

Since \( G_{D,1}(y, \xi) \in H^{-1}(\Omega) \), we obtain
\[
\int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi)|^2 dy = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \left| \frac{\partial \varphi_n(\xi)}{\partial n_\xi} \right|^2 < \infty.
\]

Thus there exists \( M > 0 \) such that
\[
\int_0^T \int_{\Gamma_0} \int_{\Omega} \gamma(\xi, s, y) f(y) dy \, d\xi ds \leq C_0 k^2 \tau \int_0^T \left( \sum_{n=1}^{\infty} \frac{1}{\mu_n} \left| \frac{\partial \varphi_n(\xi)}{\partial n_\xi} \right|^2 \right) d\xi \cdot \int_{\Omega} |f(y)|^2 dy \\
\leq C_0 k^2 \tau M \int_{\Omega} |f(y)|^2 dy = M_1 \int_{\Omega} |f(y)|^2 dy.
\]

The first assertion holds.

Similarly, for any \( g(y) \in H^{-1}(\Omega) \), we have
\[
\int_0^T \int_{\Gamma_0} \int_{\Omega} \mathcal{L}^{-\frac{1}{2}} \eta(\xi, s, y) \mathcal{L}^{-\frac{1}{2}} g(y) dy \, d\xi ds \\
= \int_0^T \int_{\Gamma_0} \int_{\Omega} \mathcal{L}^{-\frac{1}{2}} (-\gamma_s(\xi, s, y)) \mathcal{L}^{-\frac{1}{2}} g(y) dy \, d\xi ds \\
\leq \int_0^T \int_{\Gamma_0} \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} \gamma_s(\xi, s, y)|^2 dy \, d\xi ds \cdot \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} g(y)|^2 dy \\
\leq C_0 k^2 \int_0^T \int_{\Gamma_0} \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi)|^2 dy \, d\xi ds \cdot \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} g(y)|^2 dy \\
\leq M_1 \cdot \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} g(y)|^2 dy.
\]

On the other hand, for \( z(\nu, r) \in L^2(\Gamma_0 \times [0, \tau]) \), it holds that
\[
\int_0^T \int_{\Gamma_0} \int_0^s \int_0\int_{\Gamma_0} q(\xi, s - r, \nu) z(\nu, r) dr d\nu \, d\xi ds \\
= \int_0^T \int_{\Gamma_0} \int_0^s \int_0\int_{\Gamma_0} \frac{\partial \gamma_s(\xi, s - r, \nu)}{\partial n_\nu} z(\nu, r) dr d\nu \, d\xi ds \\
\leq \int_0^T \int_{\Gamma_0} \int_0^s \int_0\int_{\Gamma_0} \left| \frac{\partial \gamma_s(\xi, s - r, \nu)}{\partial n_\nu} \right|^2 dr d\nu \, d\xi ds \cdot \int_0^T \int_{\Gamma_0} |z(\nu, r)|^2 dr d\nu \\
\leq C_0 k^2 \tau \int_{\Gamma_0} \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi)|^2 dy \, d\xi ds \cdot \int_0^T \int_{\Gamma_0} |z(\nu, r)|^2 dr d\nu \\
\leq M_1 \cdot \int_0^T \int_{\Gamma_0} |z(\nu, r)|^2 dr d\nu.
\]

All assertions are proved.

\[ \square \]

**Theorem 3.7.** Let operator \( \mathcal{S} \) be defined as (24) and \( \tilde{\gamma}(\xi, s, y) = \tilde{\gamma}^{(1)}(\xi, s, y) + \tilde{\gamma}^{(2)}(\xi, s, y) \) be the solution to (32a), where \( \tilde{\gamma}^{(1)}(\xi, s, y) \) and \( \tilde{\gamma}^{(2)}(\xi, s, y) \) satisfy the equations (35) and (36) respectively. Then there exist \( M_3, M_4 > 0 \) such that the following asserts are true:
(1) For any $f(y) \in L^2(\Omega)$,
\[ \int_0^T \int_{\Gamma_0} \left| \int_{\Omega} \tilde{\gamma}(\xi, s, y)f(y)dy \right|^2 \, d\xi ds \leq M_3 \int_{\Omega} |f(y)|^2 \, dy; \]

(2) For any $g(y) \in H^{-1}(\Omega)$,
\[ \int_0^T \int_{\Gamma_0} \left| \int_{\Omega} \mathcal{L}^{-\frac{1}{2}} \tilde{\eta}(\xi, s, y) \mathcal{L}^{-\frac{1}{2}} g(y)dy \right|^2 \, d\xi ds \leq M_3 \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} g(y)|^2 \, dy; \]

(3) For any $v(\nu, r) \in L^2(\Gamma_0 \times [0, \tau])$,
\[ \int_0^T \int_{\Gamma_0} \left| \int_{\Gamma_0} \tilde{\gamma}(\xi, s, r, \nu) \int_{\Gamma_0} v(\nu, r) \, d\nu \right|^2 \, d\xi ds \leq M_k \int_0^T \int_{\Gamma_0} |v(\nu, r)|^2 \, d\nu. \]

Proof. According to $\tilde{\gamma}(\xi, s, y) = \tilde{\gamma}^{(1)}(\xi, s, y) + \tilde{\gamma}^{(2)}(\xi, s, y)$ and (38), we have
\[ \int_0^T \int_{\Gamma_0} \left| \frac{\partial \mathcal{L}^{-\frac{1}{2}} \tilde{\gamma}(\xi, s, \nu)}{\partial n_\nu} \right|^2 \, dsd\nu \leq 4 \int_0^T \int_{\Gamma_0} \left| \frac{\partial \mathcal{L}^{-\frac{1}{2}} \tilde{\gamma}^{(1)}(\xi, s, \nu)}{\partial n_\nu} \right|^2 \, dsd\nu + 4 \int_0^T \int_{\Gamma_0} \left| \frac{\partial \mathcal{L}^{-\frac{1}{2}} \tilde{\gamma}^{(2)}(\xi, s, \nu)}{\partial n_\nu} \right|^2 \, dsd\nu + 4C_3k^2 \int_0^T \int_{\Gamma_0} \left| \frac{\partial \mathcal{L}^{-\frac{1}{2}} \tilde{\gamma}(\xi, s, \nu)}{\partial n_\nu} \right|^2 \, dsd\nu. \]

Suppose that $1 - 4C_3k^2 > 0$, then there exists $M_2 = \frac{4}{1 - 4C_3k^2} > 0$ such that
\[ \int_0^T \int_{\Gamma_0} \left| \frac{\partial \mathcal{L}^{-\frac{1}{2}} \tilde{\gamma}(\xi, s, \nu)}{\partial n_\nu} \right|^2 \, dsd\nu \leq M_2 \int_0^T \int_{\Gamma_0} \left| \frac{\partial \mathcal{L}^{-\frac{1}{2}} \tilde{\gamma}^{(1)}(\xi, s, \nu)}{\partial n_\nu} \right|^2 \, dsd\nu. \] (39)

(39) combined with (34) and (37) implies that for any $f(y) \in L^2(\Omega)$,
\[ \int_0^T \int_{\Gamma_0} \left| \int_{\Omega} \tilde{\gamma}(\xi, s, y)f(y)dy \right|^2 \, d\xi ds \]
\[ \leq \int_0^T \int_{\Gamma_0} \left[ \int_{\Omega} |\tilde{\gamma}(\xi, s, y)|^2 dy \right] \, d\xi ds \cdot \int_{\Omega} |f(y)|^2 \, dy \]
\[ \leq C_1 \int_0^T \int_{\Gamma_0} \left[ \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} kG_{D,1}(y, \xi)|^2 dy \right] \, d\xi ds \cdot \int_{\Omega} |f(y)|^2 \, dy \]
\[ \leq k^2 C_1 \int_0^T \int_{\Gamma_0} \left[ \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi)|^2 dy \right] \, d\xi ds \cdot \int_{\Omega} |f(y)|^2 \, dy \]
\[ + M_2 \int_0^T \int_{\Gamma_0} \left[ \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} \tilde{\gamma}^{(1)}(\xi, s, \nu)|^2 dy \right] \, d\xi ds \cdot \int_{\Omega} |f(y)|^2 \, dy \]
\[ \leq k^2 C_1 \int_0^T \int_{\Gamma_0} \left[ \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi)|^2 dy \right] \, d\xi ds \cdot \int_{\Omega} |f(y)|^2 \, dy \]
\[ + M_2 C_2 \int_0^T \left[ \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi)|^2 dsd\nu \right] \, d\xi ds \cdot \int_{\Omega} |f(y)|^2 \, dy \]
\[ = (1 + M_2 C_2 k^2) \int_0^T \int_{\Gamma_0} \left[ \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi)|^2 dy \right] d\xi ds \cdot \int_{\Omega} |f(y)|^2 \, dy \]
Proof. The transformation (24) implies that
\[ \| (z(\xi, s), f(x), g(x)) \|_{\mathcal{H}} \leq \| S \| \cdot \| (v(\xi, s), f(x), g(x)) \|_{\mathcal{H}}. \]

The result follows directly from Theorem 3.1.

\[ \leq (1 + M_2C_2k^2)k^2C_1\tau M \int_{\Omega} |f(y)|^2dy = M_3 \int_{\Omega} |f(y)|^2dy. \]

Moreover, for any \( g(y) \in H^{-1}(\Omega) \), a direct calculation gives
\[ \int_0^T \int_{\Gamma_0} \left| \int_{\Omega} \mathcal{L}^{-\frac{1}{2}} \overline{\eta}(\xi, s, y) \mathcal{L}^{-\frac{1}{2}} g(y)dy \right|^2 d\xi ds \]
\[ = \int_0^T \int_{\Gamma_0} \left| \int_{\Omega} \mathcal{L}^{-\frac{1}{2}} (-\overline{\gamma}_s(\xi, s, y)) \mathcal{L}^{-\frac{1}{2}} g(y)dy \right|^2 d\xi ds \]
\[ \leq \int_0^T \int_{\Gamma_0} \left[ \left( \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} \overline{\gamma}_s(\xi, s, y)|^2 dy \right) \right] d\xi ds \cdot \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} g(y)|^2 dy \]
\[ \leq C_1 \int_0^T \int_{\Gamma_0} \left[ \left( \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} kG_{D,1}(y, \xi)|^2 dy \right) \right] d\xi ds \cdot \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} g(y)|^2 dy \]
\[ \leq M_3 \int_{\Omega} |\mathcal{L}^{-\frac{1}{2}} g(y)|^2 dy. \]

For any \( v(\nu, r) \in L^2(\Gamma_0 \times [0, \tau]) \), using (39) and (37), we can get easily that
\[ \int_0^T \int_{\Gamma_0} \left| \int_{\Gamma_0} \overline{q}(\xi, s-r, \nu) v(\nu, r) \right|^2 d\xi ds \]
\[ = \int_0^T \int_{\Gamma_0} \left| \int_{\Gamma_0} \frac{\partial \mathcal{L}^{-\frac{1}{2}} \overline{\gamma}_s(\xi, s-r, \nu)}{\partial n_\nu} v(\nu, r) drdv \right|^2 d\xi ds \]
\[ \leq \int_0^T \int_{\Gamma_0} \left[ \left( \int_{\Gamma_0} \left| \frac{\partial \mathcal{L}^{-\frac{1}{2}} \overline{\gamma}_s(\xi, s-r, \nu)}{\partial n_\nu} \right|^2 drdv \right) \right] d\xi ds \cdot \int_0^T \int_{\Gamma_0} |v(\nu, r)|^2 drdv \]
\[ \leq \left( M_2 \int_{\Gamma_0} \left[ \left( \int_{\Omega} \left| \frac{\partial \mathcal{L}^{-\frac{1}{2}} \overline{\gamma}_s(\xi, s-r, \nu)}{\partial n_\nu} \right|^2 dy \right) \right] d\xi ds \cdot \int_0^T \int_{\Gamma_0} |v(\nu, r)|^2 drdv \]
\[ \leq M_2C_2 \int_{\Gamma_0} \left[ \left( \int_{\Omega} \left| k \mathcal{L}^{-\frac{1}{2}} G_{D,1}(y, \xi) \right|^2 dy \right) \right] d\xi ds \cdot \int_0^T \int_{\Gamma_0} |v(\nu, r)|^2 drdv \]
\[ \leq M_2C_2k^2\tau M \int_{\Gamma_0} |v(\nu, r)|^2 drdv = M_4 \int_{\Gamma_0} |v(\nu, r)|^2 drdv. \]

The proof is complete.

Summarizing discussions above, we have the following results.

Corollary 2. Let \( \mathcal{T} \) and \( \mathcal{S} \) be defined by (20) and (24) respectively, then \( \mathcal{S} = \mathcal{T}^{-1} \).

Theorem 3.8. Let \( u(x, t) \) be defined as (7) where the kernel functions satisfy equation (21). Then the closed-loop system (8) is exponentially stable in the sense of \( \mathcal{H} \).

Proof. The transformation (24) implies that
\[ \| (z(\xi, s), f(x), g(x)) \|_{\mathcal{H}} \leq \| \mathcal{S} \| \cdot \| (v(\xi, s), f(x), g(x)) \|_{\mathcal{H}}. \]
4. Conclusion. In this paper, we consider the uniform stabilization of a wave equation with partial Dirichlet delayed control. We present a kind of new feedback controller that is called parameterization state feedback controller to stabilize the system. Different from the existing dynamic feedback controller, this approach of controller design overcomes the difficulty in stability analysis of the closed-loop system in dimensional problem. By choosing a suitable target system and giving the form of parameterization state feedback controller, we can focus on the selection of kernel functions and then define a bounded inverse linear transformation that establishes the feedback equivalence between the system under consideration and the target system of exponential stability. In the whole process of controller design, the important issue is the selection of kernel functions. By the detailed discussion, we find that the selection of kernel functions depends only on the original system and target system. From the present paper we see that the kernel function $\gamma(\xi, s, y)$ is a solution to the original system without control, satisfying certain initial value conditions, and the kernel function $\tilde{\gamma}(\xi, s, y)$ is a solution to the $w$-part of target system as $t > \tau$, satisfying other certain initial value conditions. Further, thanks to the boundedness and invertibility of transformation, we conclude that the closed-loop system we considered is equivalent to the target system. Hence we assert that the closed-loop system is exponentially stable.

On the other hand, we shall note that in order to prove the boundedness and invertibility of the transformation, we do not need an explicit expression of the kernel function, we only need the $L^2$ well-posedness of the partial differential equation satisfied by the kernel function with partial Dirichlet control and collocated observation. This gives a method for selecting the target system.

Although such a new approach of feedback controller design proposed in this paper is carried out for a high-dimensional wave equation with partial Dirichlet delayed control, it fits more general case theoretically. We hope it can be applied more complicated control form, for example, difference-type control or including a distributed term. In the future, we shall use this approach to study more complicated high-dimensional models.

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