Provably Efficient Safe Exploration via Primal-Dual Policy Optimization

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Abstract

We study the Safe Reinforcement Learning (SRL) problem using the Constrained Markov Decision Process (CMDP) formulation in which an agent aims to maximize the expected total reward subject to a safety constraint on the expected total value of a criterion function (e.g., utility). We focus on an episodic setting with the function approximation where the reward and criterion functions and the Markov transition kernels all have a linear structure but do not impose any additional assumptions on the sampling model. Designing SRL algorithms with provable computational and statistical efficiency is particularly challenging under this setting because of the need to incorporate both the safety constraint and the function approximation into the fundamental exploitation/exploration tradeoff. To this end, we present an Optimistic Primal-Dual Proximal Policy Optimization (OPDOP) algorithm where the value function is estimated by combining the least-squares policy evaluation and an additional bonus term for safe exploration. We prove that the proposed algorithm achieves an $\tilde{O}(d^{1.5} H^{3.5} \sqrt{T})$ regret and an $\tilde{O}(d^{1.5} H^{3.5} \sqrt{T})$ constraint violation, where $d$ is the dimension of the feature mapping, $H$ is the horizon of each episode, and $T$ is the total number of steps. We establish these bounds under the following two settings: (i) Both the reward and criterion functions can change adversarially but are revealed entirely after each episode. (ii) The reward/criterion functions are fixed but the feedback after each episode is bandit. Our bounds depend on the capacity of the state space only through the dimension of the feature mapping and thus our results hold even when the number of states goes to infinity. To the best of our knowledge, we provide the first provably efficient policy optimization algorithm for CMDPs with safe exploration.

1 Introduction

Reinforcement Learning (RL) studies how an agent learns to maximize its expected total reward by interacting with an unknown environment over time [40]. Safe RL (SRL) involves extra restrictions or specifications arising from the concept of safety in real-world problems [22, 4, 19]. Examples

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include the collision-avoidance in self-driving cars [21], the switching cost limitations in medical applications [6], and the legal and business restrictions in financial management [1]. One standard environment model for SRL is the Constrained Markov Decision Process (CMDP) [3] that extends the classical MDP by adding an extra safety-related criterion, e.g., utility or negative cost, and translates the safety requirement into a constraint on the expected total criterion [2]. The presence of safety constraints makes the classical exploration-exploitation trade-off [14] more challenging.

Most SRL algorithms for CMDPs are policy-based whose goal is to find a single policy that maximizes the time average or discounted reward from a given batches of data. Various constrained optimization methods are used to incorporate the constraint in the policy search/optimization, e.g., Constrained Policy Gradient (CPG) [43], Lagrangian-based Actor-Critic (AC) [9, 8, 15, 41, 30], Primal-Dual Policy Optimization (PDPO) [34], Constrained Policy Optimization (CPO) [2], Reward Constrained Policy Optimization (RCPO) [41], and Interior-Point Policy Optimization (IPPO) [32]. These SRL algorithms either do not have theoretical guarantee or can only be shown to converge asymptotically in the batch offline setting where an inner loop is required to solve a policy or value optimization problem for each outer iteration.

In this work, we look at a more challenging problem of provably efficient RL, which focuses on finding a sequence of policies in response to streaming reward functions and transition samples so that the time average reward approaches that of the best fixed policy in hindsight with theoretical guarantees. The notion of safe policy exploration, on the other hand, refers to the scenario where an agent explores an unknown environment while avoiding the bad policy that possibly violates some safety constraint [4]. If the safety criterion is known \emph{a priori}, heuristics algorithms have been proposed, e.g., Gaussian process prior [42, 7, 45] and Lyapunov-based approach [16, 17]. On the other hand, recent policy-based SRL algorithms for CMDPs, e.g., CPO [2] and PDPO [34], seek a single safe policy via the constrained policy optimization whose sample efficiency guarantees over streaming or time-varying data are largely unknown. Therefore, it is less studied how to design SRL algorithms for CMDPs which are also provably efficient. Formally, in this paper, we seek to answer the following theoretical question:

**Is it possible to design a provably sample efficient online policy optimization algorithm for CMDP?**

More specifically, we propose the first provably efficient SRL algorithm for the constrained Markov decision process with an unknown transition model in the linear episodic setting – the Optimistic Primal-Dual Proximal Policy Optimization (OPDOP) algorithm – where the value function is estimated by combining the least-squares policy evaluation and an additional bonus term for safe exploration. Theoretically, we prove that the proposed algorithm achieves an $\tilde{O}(d^{1.5}H^{3.5}\sqrt{T})$ regret and the same $\tilde{O}(d^{1.5}H^{3.5}\sqrt{T})$ constraint violation, where $d$ is the dimension of the feature mapping, $H$ is the horizon of each episode, and $T$ is the total number of steps. We establish these bounds under the following two settings: (i) Both the reward and criterion functions can change adversarially but are revealed entirely after each episode. (ii) The reward/criterion functions are fixed but the feedback after each episode is bandit. Our bounds depend on the capacity of the state space only through the dimension of the feature mapping and thus our results hold even when the number of states goes to infinity. To the best of our knowledge, our result is the first provably efficient policy optimization algorithm for CMDPs with safe exploration.
1.1 Related Work

**Provably Efficient RL:** The exploration-exploitation trade-off [14] is crucial for RL algorithms to have good sample efficiency. Our work is related to a line of efficient RL algorithms based on linear function approximation where the exploration is achieved by adding an Upper Confidence Bound (UCB) bonus [49, 50, 24]. In our current result, we exploit the linear structure of the transition model and the value functions to develop an efficient SRL algorithm with safe exploration. A closely-related recent work is Proximal Policy Optimization (PPO) [38]. As is shown in [31, 13], PPO converges to the optimal policy sublinearly and an optimistic variant of PPO is sample efficient with UCB exploration in the linear setting. However, such results only hold for unconstrained RL problems. We make the first attempt to study an optimistic variant of PPO for CMDPs with UCB exploration and prove the sample efficiency in terms of the regret and the constraint violation.

**Constrained MDP (CMDP):** The CMDP is a well-established model in the SRL research where the safety is represented as a constraint on some expected cumulative criterion. For the large CMDP with unknown model, there is a line of work that relates to the policy optimization with constraints, e.g., CPG [43], CPO [2], RCPO [41], and IPPO [32]. However, their theoretical guarantees still need further research. Compared with this line of work, we study a primal-dual type PPO for CMDPs, featured with UCB exploration, and provide theoretical guarantees on both the regret and the constraint violation. Our methodology is also related to several Lagrangian-based methods, e.g., [8, 41, 30, 34]. However, we particularly focus on the theoretical guarantees in the linear episodic setting with an unknown transition model. Moreover, our reward/criterion functions can be full-information feedback, but adversarial, or bandit feedback.

**Safe Exploration:** Safe exploration, referring to the scenario where constraints on policies are not known a priori, remains an open challenge for SRL. It appears in many real-world problems, e.g., the therapy test [36] and the space exploration [45], where the agent has to learn the safety criterion by exploring an unknown environment. Some recent works, e.g. [47, 52], consider regret minimization in online CMDPs where the transition model is known and proves sublinear regret bounds and constraint violations. In a concurrent work [35], the authors consider a finite state-action stochastic shortest path problem with constraints and unknown transitions. With a model-based transition estimation method, the work shows $O(\sqrt{T})$ regret and constraint violations. On the contrary, we consider a more challenging linear MDP scenario where state space is allowed to be infinite and its dimension can be much larger than the implicit dimension after feature map. Our method is policy based and model-free with a regret bound depending on the implicit dimension as oppose to the true dimension of the state space.

1.2 Notation

Let $[N]$ denote the set of integers $\{1, 2, \ldots, N\}$ for $N \in \mathbb{N}$. We denote by $\|\cdot\|_2$ the $\ell_2$-norm of a vector or the spectral norm of a matrix. We use $\Delta(A)$ to represent the set of probability distributions on a set $A$. The KL-divergence is $D(p_1 \| p_2) = \sum_{a \in A} p_1(a) \log(p_1(a)/p_2(a))$ where $p_1, p_2 \in \Delta(A)$. We denote a sequence of real functions $f_1(\cdot), \ldots, f_H(\cdot)$ by $\{f_h(\cdot)\}_{h=1}^H$ or $\{f_h(\cdot)\}_H$ if the index is clear from the context. For any set $S$ and integer $H \in \mathbb{N}$, a collection of probability distributions is denoted by $\Delta(A \| S, H) = \{\{\pi_h(\cdot \| \cdot)\}_{h=1}^H \mid \pi_h(\cdot \| x) \in \Delta(A) \text{ for any } x \in S \text{ and } h \in [H]\}$. For a real function $f(\cdot, \cdot) : S \times A \to \mathbb{R}$, we denote the inner product with $\pi(\cdot \| x) \in \Delta(A)$ as $\langle f(x, \cdot), \pi(\cdot \| x) \rangle_A = \sum_{a \in A} \langle f(x, a), \pi(a \| x) \rangle$ given $x \in S$ and we denote it as $\langle f, \pi \rangle$ if the
dependence on $x$ is clear from the context.

2 Problem Setup

We consider an episodic Markov decision process (MDP), $\text{MDP}(S, A, H, \mathbb{P}, r)$, where $S$ is a state space, $A$ is an action space, $H$ is a fixed length of each episode, $\mathbb{P} = \{\mathbb{P}_h\}_{h=1}^H$ is a collection of transition probability measures, and $r = \{r_h\}_{h=1}^H$ is a collection of reward functions in the $k$th episode. We assume that $S$ is a measurable space with possibly infinite number of elements. Moreover, for each step $h \in [H]$, $\mathbb{P}_h(\cdot | x, a)$ is a transition kernel over next state if action $a$ is taken for state $x$ and $r_h: S \times A \rightarrow [0, 1]$ is a reward function. The constrained MDP, denoted by $\text{CMDP}(S, A, H, \mathbb{P}, r, g)$, also contains criterion functions $g = \{g_h\}_{h=1}^H$ where $g_h: S \times A \rightarrow [0, 1]$ is the criterion function in the $k$th episode, e.g., utility. We assume that both reward and criterion functions are deterministic; generalization to a random setup is straightforward.

A policy of an agent is a collection of probability distributions $\pi^k \in \Delta(A | S, H)$ where $\pi^k_h(\cdot | x)$: $S \rightarrow A$ is the action that the agent takes at state $x$ and step $h$ in the $k$th episode. For simplicity, we set the initial state $x_1$ to be the same in each episode. The agent interacts with the MDP in the $k$th episode as follows. At the beginning, the agent determines a policy $\pi^k$. Then, at each step $h \in [H]$, the agent observes the state $x^k_h \in S$, determines an action $a^k_h$ following the policy $\pi^k_h(\cdot | x^k_h)$, and receives a reward $r^k_h$ together with a criterion $g^k_h$. Meanwhile, the MDP evolves into next state $x^k_{h+1}$ drawing from the probability measure $\mathbb{P}_h(\cdot | x^k_h, a^k_h)$. The episode terminates at state $x^k_{H+1}$; when this happens, no control action is taken and both reward and criterion functions are equal to zero.

Given a policy $\pi \in \Delta(A | S, H)$, the value function $V^\pi_{r, h}: S \rightarrow \mathbb{R}$ associated with the reward function $r^k$ at each step $h$ is the expected value of total rewards received under policy $\pi$,

$$V^\pi_{r, h}(x) = \mathbb{E}_\pi \left[ \sum_{i=h}^H r^k_i(x_i, a_i) \mid x_h = x \right],$$

where the expectation $\mathbb{E}_\pi$ is taken over the random state-action pairs $\{(x_h, a_h)\}_{i=h}^H$. Here, the action $a_h$ follows the policy $\pi(\cdot | x_h)$ at the state $x_h$ and the next state $x_{h+1}$ follows the transition dynamics $\mathbb{P}_h(\cdot | x_h, a_h)$. Thus, the action-value function $Q^\pi_{r, h}(x, a): S \times A \rightarrow \mathbb{R}$ associated with the reward function $r^k$ is the expected value of total rewards when the agent starts from an arbitrary state-action pair at step $h$ and follows policy $\pi$,

$$Q^\pi_{r, h}(x, a) = \mathbb{E}_\pi \left[ \sum_{i=h}^H r^k_i(x_i, a_i) \mid x_h = x, a_h = a \right].$$

In addition, we introduce the value function $V^\pi_{g, h}: S \rightarrow \mathbb{R}$ associated with the criterion function $g^k$,

$$V^\pi_{g, h}(x) = \mathbb{E}_\pi[\sum_{i=h}^H g^k_i(x_i, a_i) \mid x_h = x].$$

Similarly, we define the action-value function $Q^\pi_{g, h}(x, a): S \times A \rightarrow \mathbb{R}$ associated with the criterion function $g^k$. 

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We consider the constrained problem for the episodic CMDP,

\[
\begin{align*}
\text{maximize} & \quad \sum_{k=1}^{K} V_{r,1}^{\pi,k}(x_1) \\
\text{subject to} & \quad \frac{1}{K} \sum_{k=1}^{K} V_{g,1}^{\pi,k}(x_1) \geq b, \\
\end{align*}
\]

(1)

where the objective is the sum of \( K \) expected total rewards and the safety constraint is enforced on the average of \( K \) expected total criterions. For notational simplicity, we only consider one constraint in (1); extension to multiple constraints is straightforward. We are interested in two types of the reward/criterion information provided by environment:

(i) **Full-information Feedback.** The entire reward/criterion functions \( r_h^k(\cdot, \cdot) \) and \( g_h^k(\cdot, \cdot) \) are revealed at the end of each step \( h \). The superscript \( k \) implies that these functions can vary adversarially over different episodes. This depicts a typical scenario where reward/criterion functions are not \textit{a priori} known \cite{39,42}. Additional information for adversarially changing reward/criterion functions can be found in the work \cite{45}.

(ii) **Bandit Feedback.** The agent only observes the values of reward/criterion functions, \( r_h^k(x_h^k, a_h^k) \) and \( g_h^k(x_h^k, a_h^k) \), at visited state-action pair \( (x_h^k, a_h^k) \). We assume that reward/criterion functions are fixed over episodes. Thus, we may simplify notation as \( r_h(x_h^k, a_h^k) \) and \( g_h(x_h^k, a_h^k) \). This scenario includes examples in the work \cite{36,20,34} where only partial observation is available.

The episodic instance of a classical CMDP \cite{3} is a special case of (1) in the bandit setting. In this case, the objective in (1) becomes \( KV_{r,1}^{\pi}(x_1) \) and the constraint reduces to \( V_{g,1}^{\pi}(x_1) \geq b \).

We make the following two assumptions about problem (1).

**Assumption 1** (Slater condition). \textit{There exists \( \epsilon > 0 \) and \( \bar{\pi} \in \Delta(\mathcal{A}|\mathcal{S}, H) \) such that \( V_{g,1}^{\pi}(x_1) \geq b + \epsilon \) for all \( k \in [K] \).}

**Assumption 2** (Common subset). \textit{There exists a nonempty set of \( \pi \in \Delta(\mathcal{A}|\mathcal{S}, H) \) such that \( V_{g,1}^{\pi}(x_1) \geq b \) for all \( k \in [K] \). We call this set a common subset.}

The Slater condition is mild in practice, e.g., the minimal utility should be achievable by the greedy policy in every episode so that all constraints become loose. We note that the Slater condition implies the existence of a common subset. Such conditions also find use in the work \cite{33}.

To simplify notation, let us introduce \( \mathbb{P}_h V_{\ell,h+1}^{\pi}(x,a) := \mathbb{E}_{x' \sim \mathbb{P}_h(\cdot|x,a)} V_{\ell,h+1}^{\pi}(x') \) for \( \ell = r \) or \( g \). For the reward/criterion value functions, the Bellman equations associated with a policy \( \pi \) become

\[
\begin{align*}
Q_{\ell,h}^{\pi,k}(x,a) = (\ell_h + \mathbb{P}_h V_{\ell,h+1}^{\pi,k}(x,a)) \\
V_{\ell,h}^{\pi,k}(x) = \langle Q_{\ell,h}^{\pi,k}(x, \cdot), \pi_h(\cdot|x) \rangle_{\mathcal{A}}.
\end{align*}
\]

(2)

which hold for all \( (x,a) \in \mathcal{S} \times \mathcal{A} \). Here the symbol \( \ell_h \) denotes the function \( r_h^k \) or \( g_h^k \) and the subscript \( \ell \) is an index.
2.1 Learning Performance

In constrained problem (1), the learning goal of the agent is to maximize the expected total reward while meeting safety performance specifications.

In the full-information setting, we take any fixed policy from the common subset in Assumption 2, denoted as $\pi^\star$. We measure the learning performance by the regret of the agent associated with the reward value functions, and constraint violation associated with the criterion value functions. We introduce the regret as a difference between the total reward value of the best policy $\pi^\star$ and that of the agent’s policy $\pi_k$ over $K$ episodes,

$$\text{Regret}(K) = \sum_{k=1}^{K} \left( V_{r,1}^{\pi^\star,k}(x_1) - V_{r,1}^{\pi_k,k}(x_1) \right).$$  \hspace{1cm} (3)

Similar to (3), we introduce the constraint violation as the difference between $Kb$ and the total criterion value of the agent’s policy $\pi_k$ over $K$ episodes,

$$\text{Violation}(K) = Kb - \sum_{k=1}^{K} V_{g,1}^{\pi_k,k}(x_1).$$  \hspace{1cm} (4)

In the bandit setting, we define learning performance similarly. We note that the reward/criterion functions do not vary over episodes in this setting. Although we use notation of (3) and (4), they only depend on policy $\pi_k$ in each episode given $x_1$. Thus, we may simplify notation, e.g., use $V_{r,1}^{\pi_k}(x_1)$ for $V_{r,1}^{\pi_k,k}(x_1)$. Accordingly, inequalities in Assumption 1 and problem (1) hold by removing $k \in [K]$. This further clarifies why (1) covers the episodic CMDP.

In this paper, we design algorithms, taking full-information or bandit feedback of the reward/criterion functions, with both regret and constraint violation being sublinear in the total number of steps $T = HK$. Put differently, the algorithm should ensure that given $\epsilon > 0$, if $T = \tilde{O}(1/\epsilon^2)$, then with high probability we have

$$\frac{1}{K} \sum_{k=1}^{K} V_{r,1}^{\pi^\star,k}(x_1) - \frac{1}{K} \sum_{k=1}^{K} V_{r,1}^{\pi_k,k}(x_1) \leq \epsilon$$

and $b - \frac{1}{K} \sum_{k=1}^{K} V_{g,1}^{\pi_k,k}(x_1) \leq \epsilon$.

This shows an $\epsilon$-approximation of the problem (1) with the safety satisfaction. We show how to achieve this goal in Section 3.

2.2 Linear Function Approximation

We focus on a particular Markov decision process where the transition kernels, the reward functions, and the criterion functions are linear in feature maps.

Assumption 3. The CMDP($\mathcal{S}, \mathcal{A}, H, \mathbb{P}, r, g$) is a linear MDP with a feature map $\phi: \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^d$, if for any $(h, k) \in [H] \times [K]$, there exist $d$ unknown (signed) measures $\mu_h = (\mu_{h1}^1, \ldots, \mu_{h1}^d)$ over $\mathcal{S}$
and two unknown vectors $\theta^k_{r,h}, \theta^k_{g,h} \in \mathbb{R}^d$ such that for any $(x,a) \in S \times A$,

\[
\mathbb{P}_h (\cdot | x, a) = \langle \phi (x, a), \mu_h (\cdot) \rangle,
\]

\[
r^k_h(x, a) = \langle \phi (x, a), \theta^k_{r,h} \rangle \quad \text{and} \quad \theta^k_{g,h}(x, a) = \langle \phi (x, a), \theta^k_{g,h} \rangle.
\]

Moreover, we assume $\|\phi(x,a)\|_2 \leq 1$ for all $(x,a) \in S \times A$, $\max(||\theta^k_{r,h}||_2, ||\theta^k_{g,h}||_2) \leq \sqrt{d}$ and \(\sum_{i=1}^{d} \|\mu_i^h\|_1^2 \leq d\) for all $(h,k) \in [H] \times [K]$ where $\|\mu_i^h\|_1 = \int_S |\mu_i^h| dx$.

Assumption 3 extends the linear MDP [23, 24] to CMDPs. Linear MDP examples, e.g., the finite MDP or the MDP with simplex feature space [24], can be trivially extended to linear CMDPs by adding linear constraints. Additional linear MDP examples can be found in the work [49, 50]. For utility of linear structure, see discussions in the work [18, 44, 27]. The design of provably efficient RL algorithms for the general transition dynamics remains an open problem.

For linear MDPs, the action-value functions $Q^\pi_{r,h}$ and $Q^\pi_{g,h}$ are linear in the feature map for any policy [24]. In what follows, we focus on linear parameterizations of $Q^\pi_{r,h}$ and $V^\pi_{r,h}$ where $\ell = r$ or $g$.

3 Main Results

3.1 Algorithm

In Algorithm 1, we present a variant of proximal policy optimization – an Optimistic Primal-Dual Proximal Policy OPtimization (OPDOP) algorithm. We effectuate the optimism through the Upper-Confidence Bounds (UCB) and address the constraints using Lagrange multipliers. In each episode, our algorithm consists of three main stages. The first stage (lines 2-10) executes the policy improvement to update policy $\pi^k$ based on previous $\pi^{k-1}$. The second stage (line 11) updates the dual variable $Y^k$ based on the constraint violation induced by previous policy $\pi^k$. The third stage (line 12) corresponds to the policy evaluation via the least-squares policy evaluation with an additional UCB bonus term for exploration.

Policy Improvement. For the $k$th episode, we are supposed to optimize a simple Lagrangian $L^{\pi-1}(\pi, Y) = V^{\pi,k-1}_{\ell,1}(x_1) - Y(b - V^{\pi,k-1}_{g,1}(x_1))$ where $Y \geq 0$ is the Lagrange multiplier. However, it is not computable since the model is unknown. It is also online infeasible if we use an inner loop of policy optimization or value iteration [29, 34]. Instead, we treat the unknown reward and criterion value functions via the performance difference lemma and approximate value functions $V^{\pi,k-1}_{\ell,1}(x_1)$, $\ell = r$ or $g$, at the previously known policy $\pi^{k-1}$ as follows [37, 38],

\[
V^{\pi,k-1}_{\ell,1}(x_1) = V^{\pi,k-1,k-1}_{\ell,1}(x_1) + E^{\pi,k-1}_{\ell,1} \left[ \sum_{h=1}^{H} \langle Q^{\pi,k-1,k-1}_{\ell,h}(x_h, \cdot), (\pi_h - \pi^{k-1}_h)(\cdot|x_h) \rangle | x_1 \right].
\]

Thus, we define the approximation of $V^{\pi,k-1}_{\ell,1}(x_1)$ as

\[
L^{\pi-1}_{\ell}(\pi) = V^{\pi,k-1}_{\ell,1}(x_1) + \sum_{h=1}^{H} \langle Q^{k-1}_{\ell,h}(x_h, \cdot), (\pi_h - \pi^{k-1}_h)(\cdot|x_h) \rangle,
\]

for all $(x,a) \in S \times A$.


where $Q_{k-1}^{g,h}$ and $V_{k-1}^{g}$ are estimated from the policy evaluation. It allows us to approximate the Lagrangian $L_{k-1}^{k}$ and update the policy $\pi^{k}$ via online mirror descent,

$$\max_{\pi \in \Delta(A,S,H)} \psi L_{r}^{k-1}(\pi) - Y^{k-1}(b - L_{g}^{k-1}(\pi)) - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_{h}(\cdot | x_{h}) | \bar{\pi}_{h}^{k-1}(\cdot | x_{h})),$$

where $\bar{\pi}_{h}^{k-1}(\cdot | x_{h}) = (1 - \theta) \pi_{h}^{k-1}(\cdot | x_{h}) + \theta |\mathcal{A}|$ is a mixed policy of the previous one and the uniform distributed with $\theta \in (0, 1]$. The constant $\psi, \alpha > 0$ are trade-off parameters, $Y^{k-1} \geq 0$ is from the dual update, $D(\pi | \bar{\pi}^{k-1})$ is the KL divergence between $\pi$ and $\bar{\pi}^{k-1}$ such that $\pi$ is absolutely continuous in $\bar{\pi}^{k-1}$. The policy mixing step ensures the such absolute continuity and implies uniformly bounded KL divergence (see Lemma 19 in Appendix E). Ignoring other $\pi$-irrelevant terms, we update $\pi^{k}$ in terms of previous policy $\pi^{k-1}$ by

$$\arg \max_{\pi \in \Delta(A,S,H)} \sum_{h=1}^{H} \langle (\psi Q_{r}^{k-1} + Y^{k-1} Q_{g}^{k-1})(x_{h}, \cdot), \pi_{h}(\cdot | x_{h}) \rangle - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_{h}(\cdot | x_{h}) | \bar{\pi}_{h}^{k-1}(\cdot | x_{h})).$$

Since the above update is separable over states $\{x_{h}\}_{h=1}^{H}$, we can update the policy $\pi^{k}$ as line 6 in Algorithm 1. It has the following closed solution for any step $h \in [H]$,

$$\pi_{h}^{k}(\cdot | x_{h}) \propto \bar{\pi}_{h}^{k-1}(\cdot | x_{h}) e^{\alpha (\psi Q_{r}^{k-1} + Y^{k-1} Q_{g}^{k-1})(x_{h}, \cdot)}. \quad (5)$$

If we set $Y^{k-1} = 0$ and $\theta = 0$, the above update reduces to one step in an optimistic variant of PPO [13]. The idea of KL-divergence regularization in policy optimization has been widely used in many unconstrained scenarios, e.g., NPG [25], TRPO [37], PPO [38]. This seems to the first application of such technique in the constrained scenario.

**Dual Update.** Once we obtain a new policy $\pi^{k}$, we estimate $V_{g,1}^{k-1}(x_{1})$ by the linear approximation $L_{g}^{k-1}(\pi^{k})$, and infer the constraint violation for the dual update. We update the Lagrangian multiplier $Y \geq 0$ by moving $Y^{k}$ to the direction of minimizing the Lagrangian function $L_{k-1}^{k}(\pi, Y)$ over $Y$,

$$Y^{k} \leftarrow \max \left( Y^{k-1} + (b - L_{g}^{k-1}(\pi^{k})), 0 \right).$$

Again, we estimate $Q_{g,h}^{k-1,k-1}$ using $Q_{g,h}^{k-1}$ from the policy evaluation. Combining this with the above update yields line 11 in Algorithm 1. The dual update works as a trade-off between the reward maximization and the constraint violation reduction. If the current policy $\pi^{k}$ satisfies the approximated constraint, i.e., $b - L_{g}^{k-1}(\pi^{k}) \leq 0$, we may put less weight on the action-value function associated with the criterion and maximize the reward; otherwise, we may sacrifice the reward to satisfy the constraint. The dual update has been commonly used for dealing with constraints in CMDP, e.g., Lagrangian-based AC [15, 30]. Other use in online constrained optimization can be found in the work [51, 48].

**Policy Evaluation.** The last stage of the $k$th episode takes the Least-Squares Temporal Difference (LSTD) [11, 10, 28, 26] to evaluate the policy $\pi^{k}$ based on previous $k - 1$ historical trajectories. In Algorithm 1, depending on the revealed information on the reward/criterion in line 8, we choose one of the following two cases for the policy evaluation in line 12.
Algorithm 1 Optimistic Primal-Dual Proximal Policy OPtimization (OPDOP)

1: **Initialization:** Let $\{Q^0_{r,h}, Q^0_{g,h}\}_{h=1}^H$ be zero functions, $\{\pi^0_h\}_{h \in [H]}$ be uniform distributions on $\mathcal{A}$, $V^0_{g,1}$ be $b$, $Y^0$ be 0, and $\alpha, \beta, \psi, \lambda > 0, \theta \in (0, 1]$.

2: for episode $k = 1, \ldots, K$ do

3: Set the initial state $x_1^k = x_1$.

4: for step $h = 1, 2, \ldots, H$ do

5: Mix the policy $\tilde{\pi}^{k-1}_h(\cdot | x_h) = (1 - \theta) \pi^{k-1}_h(\cdot | x_h) + \theta \frac{1}{|\mathcal{A}|} 1$.

6: Update the policy $\pi^k_h(\cdot | x_h)$ by

$$\pi^k_h(\cdot | x_h) \leftarrow \arg\max_{\pi \in \Delta(\mathcal{A})} \langle (\psi Q^{k-1}_{r,h} + Y^{k-1}_{g,h}(x_h, \cdot), \pi(\cdot | x_h) \rangle - \frac{1}{\alpha} D(\pi_h(\cdot | x_h) | \tilde{\pi}^{k-1}_h(\cdot | x_h))$$.

7: Take an action $a^k_h \sim \pi^k_h(\cdot | \cdot)$

8: Receive reward/criterion

$$\begin{cases} r^k_h(\cdot, \cdot), g^k_h(\cdot, \cdot) & \text{Full-information;} \\ r_h(x^k_h, a^k_h), g_h(x^k_h, a^k_h) & \text{Bandit.} \end{cases}$$

9: Observe the next state $x_{h+1}^k$.

10: end for

11: Update the dual variable $Y^k$ by

$$Y^k \leftarrow \max \left( Y^{k-1} + b - \sum_{h=1}^H \langle Q^{k-1}_{g,h}(x_h, \cdot), (\pi^k_h - \pi^{k-1}_h)(\cdot | x_h) \rangle - V^{k-1}_{g,1}(x_1), 0 \right) .$$

12: Estimate the reward/criterion value functions $\{Q^k_{r,h}(\cdot, \cdot), Q^k_{g,h}(\cdot, \cdot)\}_{h=1}^H$ via one subroutine,

$$\begin{cases} \text{LSTD-Full} \left( \{x^k_h, a^k_h, r^k_h(\cdot, \cdot), g^k_h(\cdot, \cdot)\}_{h=1}^H \right); \\ \text{LSTD-Bandit} \left( \{x^k_h, a^k_h, r_h(x^k_h, a^k_h), g_h(x^k_h, a^k_h)\}_{h=1}^H \right). \end{cases}$$

13: end for
In the full-information setting, for each step $h \in [H]$, instead of $\mathbb{P}_h V^{r,k}_{r,h+1}$ in the Bellman equations (2), we estimate $\mathbb{P}_h V^{r,k}_{r,h+1}$ by $\phi^T w^{k}_{r,h}$ where $w^{k}_{r,h}$ is updated by the minimizer of the regularized least-squares problem over $w$,

$$
\sum_{\tau=1}^{k-1} (V^{k}_{r,h+1}(x^{\tau}_{h+1}) - \phi(x^{\tau}_{h}, a^{\tau}_{h})^T w)^2 + \lambda \|w\|^2,
$$

where $V^{k}_{r,h+1}(x^{\tau}_{h+1}) = (Q^{k}_{r,h+1}(x^{\tau}_{h+1}, \cdot), \pi^{k}_{h+1}(\cdot|x^{\tau}_{h+1}))$ for $h \in [H - 1]$ and $V^{k}_{H+1} = 0$, and $\lambda > 0$ is the regularization parameter. We display the least-squares solution in Algorithm 2. We update the estimated action-value function iteratively with reward functions $r^{k}_{h}$ directly as line 8 where $\phi^T w^{k}_{r,h}$ is an estimate of $\mathbb{P}_h V^{k}_{r,h+1}$; we add an UCB bonus $\Gamma^{k}_{h}(\cdot, \cdot): \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^+$ so that $\phi^T w^{k}_{r,h} + \Gamma^{k}_{h}$ becomes an upper confidence bound for exploration. We take $\Gamma^{k}_{h} = \beta(\phi^T (\Lambda^{k}_{h})^{-1} \phi)^{1/2}$ and leave the parameter $\beta > 0$ to be tuned later. Moreover, the bounded reward $r^{k}_{h} \in [0, 1]$ shows that $\mathbb{P}_h V^{k}_{r,h+1} \in [0, H - h]$. Similarly, we estimate $Q^{k}_{g,h}$ and $V^{k}_{g,h}$.

### Algorithm 2 Least-Squares Temporal Difference with Full-Information Feedback (LSTD-Full)

1. **Input:** $\{x^{k}_{h}, a^{k}_{h}, r^{k}_{h}(\cdot, \cdot), g^{k}_{h}(\cdot, \cdot)\}_{H=1}^{H}$
2. **Initialization:** Set $\{V^{k}_{r,h+1}, V^{k}_{g,h+1}\}$ be zero functions
3. **for** step $h = H, H-1, \cdots, 1$ **do**
   4. $\Lambda^{k}_{h} \leftarrow \sum_{\tau=1}^{k-1} \phi(x^{\tau}_{h}, a^{\tau}_{h})\phi(x^{\tau}_{h}, a^{\tau}_{h})^T + \lambda I.$
   5. $\Gamma^{k}_{h}(\cdot, \cdot) \leftarrow \beta(\phi(\cdot, \cdot)^T (\Lambda^{k}_{h})^{-1} \phi(\cdot, \cdot))^{1/2}$.
   6. $w^{k}_{r,h} \leftarrow (\Lambda^{k}_{h})^{-1} \sum_{\tau=1}^{k-1} \phi(x^{\tau}_{h}, a^{\tau}_{h})V^{k}_{r,h+1}(x^{\tau}_{h+1}).$
   7. $w^{k}_{g,h} \leftarrow (\Lambda^{k}_{h})^{-1} \sum_{\tau=1}^{k-1} \phi(x^{\tau}_{h}, a^{\tau}_{h})V^{k}_{g,h+1}(x^{\tau}_{h+1}).$
   8. $Q^{k}_{r,h}(\cdot, \cdot) \leftarrow q^{k}_{h}(\cdot, \cdot) + \min (\phi(\cdot, \cdot)^T w^{k}_{r,h} + \Gamma^{k}_{h}(\cdot, \cdot), H - h)^+.$
   9. $Q^{k}_{g,h}(\cdot, \cdot) \leftarrow q^{k}_{h}(\cdot, \cdot) + \min (\phi(\cdot, \cdot)^T w^{k}_{g,h} + \Gamma^{k}_{h}(\cdot, \cdot), H - h)^+.$
10. $V^{k}_{r,h}(\cdot) \leftarrow \langle Q^{k}_{r,h}(\cdot, \cdot), \pi^{k}_{h}(\cdot, \cdot) \rangle_A.$
11. $V^{k}_{g,h}(\cdot) \leftarrow \langle Q^{k}_{g,h}(\cdot, \cdot), \pi^{k}_{h}(\cdot, \cdot) \rangle_A.$
12. **end for**
13. **Return:** $\{Q^{k}_{r,h}(\cdot, \cdot), Q^{k}_{g,h}(\cdot, \cdot)\}_{H=1}^{H}$

In the bandit setting, for each step $h \in [H]$, we estimate $Q^{w,k}_{r,h}$ by $\phi^T w^{k}_{r,h}$ where $w^{k}_{r,h}$ is updated by the minimizer of another regularized least-squares problem over $w$,

$$
\sum_{\tau=1}^{k-1} (r^{k}_{h}(x^{\tau}_{h}, a^{\tau}_{h}) + V^{k}_{r,h+1}(x^{\tau}_{h+1}) - \phi(x^{\tau}_{h}, a^{\tau}_{h})^T w)^2 + \lambda \|w\|^2.
$$

The least-squares solution gives Algorithm 3. We update the estimated action-value function directly as line 8 where $\phi^T w^{k}_{r,h}$ gives an estimate of $Q^{k}_{r,h} \in [0, H]$ and we add an UCB bonus $\Gamma^{k}_{h}(\cdot, \cdot): \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^+$ so that $\phi^T w^{k}_{r,h} + \Gamma^{k}_{h}$ becomes an upper confidence bound. We take $\Gamma^{k}_{h} = \beta(\phi^T (\Lambda^{k}_{h})^{-1} \phi)^{1/2}$ and leave the parameter $\beta > 0$ to be tuned later. Moreover, the bounded reward $r^{k}_{h} \in [0, 1]$ shows that $Q^{k}_{r,h} \in [0, H]$. Similarly, we estimate $Q^{k}_{g,h}$ and $V^{k}_{g,h}$.
We remark the computational efficiency of Algorithm 1. For the time complexity, since line 6 has a closed form solution (5) and line 11 is a scalar update, they need $O(|\mathcal{A}|T)$ time. Another dominating calculation is from line 6 and line 7 in Algorithm 2 or Algorithm 3. If we use the Sherman-Morrison formula for computing $(\Lambda^k_h)^{-1}$, in total it takes $O(d^2T)$ time. Thus, the time complexity is $O(d^2|\mathcal{A}|T)$. For the space complexity, in the full-information setting, we store $\pi^k$, $\tilde{\pi}^k$, $Y^k$, $\Lambda_h^k$, and $w_h^k$, $w_{g,h}^k$, and it takes $O(d^2H + |\mathcal{A}|H)$ space. In the bandit setting, we also need to store $r_h(x_h^k, a_h^k)$ and $g_h(x_h^k, a_h^k)$ and it takes $O(d^2H + |\mathcal{A}|T)$ space.

**Algorithm 3 Least-Squares Temporal Difference with Bandit Feedback (LSTD-Bandit)**

1. **Input:** $\{x_h^k, a_h^k, r_h(x_h^k, a_h^k), g_h(x_h^k, a_h^k)\}_{h=1}^H$
2. **Initialization:** Set $\{V_{r,h+1}^k, V_{g,h+1}^k\}_{h=1}$ be zero functions
3. **for** step $h = H, H - 1, \cdots, 1$ **do**
   4. $\Lambda^k_h \leftarrow \sum_{\tau=1} \phi(x_h^\tau, a_h^\tau)\phi(x_h^\tau, a_h^\tau)^T + \lambda I$.
   5. $\Gamma^k_h(\cdot, \cdot) \leftarrow \beta(\phi(\cdot, \cdot)\big(\Lambda^k_h)^{-1}\phi(\cdot, \cdot)\big)\cdot 1/2$.
   6. $w_{r,h}^k \leftarrow (\Lambda^k_h)^{-1}\sum_{\tau=1} \phi(x_h^\tau, a_h^\tau)(r_h(x_h^\tau, a_h^\tau) + V_{r,h+1}^k(x_h^\tau))$.
   7. $w_{g,h}^k \leftarrow (\Lambda^k_h)^{-1}\sum_{\tau=1} \phi(x_h^\tau, a_h^\tau)(g_h(x_h^\tau, a_h^\tau) + V_{g,h+1}^k(x_h^\tau))$.
   8. $Q_{r,h}^k(\cdot, \cdot) \leftarrow \min_{\cdot} (\cdot, \cdot)^T w_{r,h}^k + \Gamma^k_h(\cdot, \cdot, H)^+$.
   9. $Q_{g,h}^k(\cdot, \cdot) \leftarrow \min_{\cdot} (\cdot, \cdot)^T w_{g,h}^k + \Gamma^k_h(\cdot, \cdot, H)^+$.
   10. $V_{r,h}^k(\cdot) \leftarrow \langle Q_{r,h}^k(\cdot, \cdot), \pi_h^k(\cdot | \cdot) \rangle_A$.
   11. $V_{g,h}^k(\cdot) \leftarrow \langle Q_{g,h}^k(\cdot, \cdot), \pi_h^k(\cdot | \cdot) \rangle_A$.
12. **end for**
13. **Return:** $\{Q_{r,h}^k(\cdot, \cdot), Q_{g,h}^k(\cdot, \cdot)\}_{h=1}^H$

### 3.2 Regret and Constraint Violation

We now show that the regret and the constraint violation for Algorithm 1 are sublinear in $T := KH$ that is the total number of steps taken by the algorithm where $K$ is the total number of episodes and $H$ is the episode horizon. We recall that $|\mathcal{A}|$ is the cardinality of action space $\mathcal{A}$ and $d$ is the dimension of the feature map.

#### 3.2.1 Full-information Setting

**Theorem 1** (Regret Bound and Constraint Violation). Let Assumptions 1, 2, and 3 hold. Fix $p \in (0, 1)$. In Algorithm 1 with the full-information setting, we set $\alpha = \sqrt{\log |\mathcal{A}|/(H^3K)}$, $\beta = C_1dH\sqrt{\log (dT/p)}$, $\psi = \sqrt{K}$, $\theta = 1/(K\log |\mathcal{A}|)$, and $\lambda = 1$ where $C_1$ is an absolute constant.
Suppose \( \log |A| = O \left( d^{1.5} \log^2 \left( dT/p \right) \right) \). Then, the regret (3) and the constraint violation (4) satisfy

\[
\text{Regret}(K) \leq C d^{1.5} H^{3.5} \sqrt{T} \log \left( \frac{dT}{p} \right),
\]
\[
\text{Violation}(K) \leq C' d^{1.5} H^{3.5} \sqrt{T} \log^2 \left( \frac{dT}{p} \right),
\]

with probability \( 1 - p \) where \( C, C' \) are absolute constants.

**Proof.** In Section 4, we provide a proof sketch that supports complete proof in Appendix B.

The above result establishes that OPDOP in the full-information setting enjoys an \( \tilde{O}(d^{1.5} H^{3.5} \sqrt{T}) \) regret and an \( \tilde{O}(d^{1.5} H^{3.5} \sqrt{T}) \) constraint violation if we set algorithm parameters \( \{\alpha, \beta, \psi, \theta, \lambda\} \) properly. Our results have the optimal dependence on the total number of steps \( T \) up to some logarithmic factors. The \( d^{1.5} \) dependence occurs due to the uniform concentration for controlling the fluctuations in the least-squares policy evaluation. This matches the existing bounds in the unconstrained linear MDP setting [24, 13, 46]. Our bounds differ from them only on the dependence on \( H \), which, essentially, is introduced by the uniform bound on the constraint violation. This seems to be unique in our framework.

It is noticed that although it is a full-information setting, we still allow both the reward and the criterion to vary adversarially in each episode. In this sense, OPDOP is a robust SRL algorithm with efficient safe exploration.

### 3.2.2 Bandit Setting

**Theorem 2** (Regret Bound and Constraint Violation). Let Assumptions 1, 2, and 3 hold. Fix \( p \in (0, 1) \). In Algorithm 1 with the bandit setting, we set \( \alpha = \sqrt{\log |A|/(H^3 K)} \), \( \beta = C_1 dH \sqrt{\log (dT/p)} \), \( \psi = \sqrt{K} \), \( \theta = 1/(K \log |A|) \), and \( \lambda = 1 \) where \( C_1 \) is an absolute constant. Suppose \( \log |A| = O \left( d^{1.5} \log^2 \left( dT/p \right) \right) \). Then, the regret (3) and the constraint violation (4) satisfy

\[
\text{Regret}(K) \leq C'' d^{1.5} H^{3.5} \sqrt{T} \log \left( \frac{dT}{p} \right),
\]
\[
\text{Violation}(K) \leq C''' d^{1.5} H^{3.5} \sqrt{T} \log^2 \left( \frac{dT}{p} \right),
\]

with probability \( 1 - p \) where \( C'', C''' \) are absolute constants.

**Proof.** See a complete proof in Appendix C.

Theorem 2 proves that OPDOP in the bandit setting achieves similar bounds as in the full-information setting. Although it is in the bandit setting, Algorithm 3 can still utilize the historical data to estimate the reward/criterion value functions and OPDOP achieves the same statistical efficiency.
4 Proof Sketch

We sketch the proof for Theorem 1. The proof of Theorem 2 is similar.

4.1 Regret Analysis

The analysis is based on the regret decomposition of (3),

\[ \text{Regret}(K) = (\text{R.I}) + (\text{R.II}), \]

where \( (\text{R.I}) = \sum_{k=1}^{K} (V_{r,1}^{\pi^*,k}(x_1) - V_{r,1}^{k}(x_1)) \) and \( (\text{R.II}) = \sum_{k=1}^{K} (V_{r,1}^{k}(x_1) - V_{r,1}^{\pi^*,k}(x_1)) \). The inserted value \( V_{r,1}^{k}(x_1) \) is estimated from the policy evaluation; the policy \( \pi^* \) in hindsight is from the common subset in Assumption 2.

To bound \( \text{Regret}(K) \), we analyze \( (\text{R.I}) \) and \( (\text{R.II}) \) separately. For this purpose, we define the model prediction error associated with the reward \( r \) as \( \iota_{r,h}^k := r^k + \mathbb{P}_h V_{r,h+1}^k - Q_{r,h}^k \), which depicts the error using \( V_{r,h+1}^k \) for \( V_{r,h}^k \) in the Bellman equations (2). Similarly, we define \( \iota_{g,h}^k \) for the criterion.

First, we connect the update of line 6 in Algorithm 1 with \( \text{Regret}(K) \) via the performance difference lemma (see Lemma 3.2 in the work [13]) which enables the following expansion of \( V_{g,1}^{\pi^*,k}(x_1) - V_{g,1}^{k}(x_1) \),

\[ \sum_{h=1}^{H} \left( \mathbb{E}_{\pi^*} \left[ (Q_{g,h}^k(x_h, \cdot), (\pi^*_h - \pi_h^k)(\cdot | x_h)) \right] | x_1 \right) + \mathbb{E}_{\pi^*}[\iota_{g,h}^k(x_h, a_h) | x_1] \).

Similarly, we expand \( V_{r,1}^{\pi^*,k}(x_1) - V_{r,1}^{k}(x_1) \). We explicitly add the UCB bonus \( \Gamma_{r,h}^k = \beta(\phi^T (\Lambda_h^k)^{-1} \phi)^{1/2} \) into the estimation of \( Q_{r,h}^k \) and \( Q_{g,h}^k \) in Algorithm 2. This is known as the mechanism of ‘Optimistic in the Face of Uncertainty’ [5, 12]. As we see in Lemma 13 in Appendix E, for a fixed \( p \in (0, 1) \), if we set \( \lambda = 1 \) and \( \beta = C_1 dH \sqrt{\log(dT/p)} \) where \( C_1 \) is an absolute constant, then for all \( (k, h) \in [K] \times [H] \) and \( (x, a) \in S \times A \), with probability \( 1 - p/3 \),

\[ Q_{r,h}^k(x, a) \geq \iota_{r,h}^k(x, a) + (\mathbb{P}_h V_{r,h+1}^k) + \mathbb{P}_h V_{r,h+1}^k(x, a) \]

or, equivalently, \( \iota_{r,h}^k(x, a) \leq 0 \) where \( \ell = r \) or \( g \).

This allows us to utilize the previous expansion of \( V_{g,1}^{\pi^*,k}(x_1) - V_{g,1}^{k}(x_1) \) to show that \( V_{g,1}^{\pi^*,k}(x_1) \) is bounded by \( V_{g,1}^{k}(x_1) + \mathbb{E}_{\pi^*}[(Q_{g,h}^k(x_h, \cdot), (\pi^*_h - \pi_h^k)(\cdot | x_h)) | x_1] \). With the above observations, we establish the first bound on \( (\text{R.I}) \) using the parameters in Theorem 1 (see Lemma 1 in Appendix B.1),

\[ (\text{R.I}) \leq C_2 H^{3.5} \sqrt{T \log |A|} + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*}[\iota_{r,h}^k(x_h, a_h) | x_1] \]

with probability \( 1 - p/3 \) where \( C_2 \) is an absolute constant.

Second, as shown in Appendix D.2, we expand \( (\text{R.II}) \) into

\[ (\text{R.II}) = - \sum_{k=1}^{K} \sum_{h=1}^{H} \iota_{r,h}^k(x_h^k, a_h^k) + M_{r,H_2}^k, \]
where $M_{r,H}^{K}$ defines a particular martingale. Now, if we substitute the above bounds on (R.I) and (R.II) into (6), we still need to bound two quantities: $\sum_{k=1}^{K} \sum_{h=1}^{H} (E_{\pi^{k}}[r_{h}(x_{h}, a_{h}) | x_{1}] - r_{h}^{k}(x_{h}, a_{h}^{k}))$ and $M_{r,H}^{K}$. In Lemma 2 in Appendix B.1, we apply the other bound on $r_{h}^{k}$ in Lemma 13 in Appendix E that $-2\gamma^{k}_{h} \leq r_{h}^{k}$ and the elliptical potential lemma (see Lemma 16) to bound the first quantity by $2C_{1}\sqrt{2d^{3}H^{3}T \log (K + 1) \log (dT/p)}$ with probability $1 - p/3$. For the second quantity, we apply the Azuma-Hoeffding inequality in Lemma 3 to bound it by $4\sqrt{HT^{2}T \log (6/p)}$ with probability $1 - p/3$. With the help of these bounds, we obtain the regret bound in Theorem 1.

### 4.2 Constraint Violation Analysis

We first decompose the constraint violation (4) into,

\[
\text{Violation}(K) = \sum_{k=1}^{K} (b - V_{g,1}^{k}(x_{1})) + (\text{V.II}),
\]

where (V.II) = $\sum_{k=1}^{K} (V_{g,1}^{k}(x_{1}) - V_{g,1}^{\pi^{k},k}(x_{1}))$ and $V_{g,1}^{k}(x_{1})$ is estimated from the policy evaluation.

First, we seek to bound $\sum_{k=1}^{K} (b - V_{g,1}^{k}(x_{1}))$. We analyze the random process $\{Y_{k}\}_{k=1}^{K}$ in Lemma 4 and Lemma 5 (see them in Appendix B.2) via the tool of drift analysis [51]. As before, we expand $V_{g,1}^{\pi^{k},k}(x_{1})$ using the performance difference lemma (see Lemma 3.2 in the work [13]) first, and then apply the UCB result in Lemma 13 in Appendix E show that $V_{g,1}^{\pi^{k},k}(x_{1})$ is bounded by $V_{g,1}^{k}(x_{1}) + \sum_{h=1}^{H} E_{\pi}([O_{g,1}^{k}(x_{h}, \cdot), (\pi_{g,1} - \pi^{k}_{g,1})(\cdot | x_{h})] | x_{1})$. With this observation, in Lemma 4 and Lemma 5, we utilize the drift bound of random process from Lemma 17 in Appendix E to show that $E_{t}[Y_{k}] \leq C_{4}H^{3.5}\sqrt{T \log (|A|)}$ with probability $1 - p/3$ where $C_{4}$ is an absolute constant. Next, we relate this bound with $\sum_{k=1}^{K} (b - V_{g,1}^{k}(x_{1}))$ in Lemma 6 in Appendix B.2 and conclude that,

\[
\sum_{k=1}^{K} (b - V_{g,1}^{k}(x_{1})) \leq C_{5}H^{3.5}\sqrt{T} (\log |A| + \sqrt{\log |A| \log H}).
\]

Second, we similarly expand (V.II) as we did for (R.II). Then we apply the UCB result in Lemma 13 and the Azuma-Hoeffding inequality. We finally combine these bounds to obtain the constraint violation bound in Theorem 1.

### 5 Concluding Remarks

In this paper, we have developed a provably efficient safe reinforcement learning algorithm in the linear MDP setting. The algorithm extends the proximal policy optimization to the constrained MDP and incorporates the UCB exploration. We prove that the proposed algorithm obtains an $\tilde{O}(\sqrt{T})$ regret and an $\tilde{O}(\sqrt{T})$ constraint violation under mild regularity conditions where $T$ is the total number of steps taken by the algorithm. Moreover, our algorithm works in settings where reward/criterion functions are given by full-information or bandit feedback. To the best of our knowledge, our algorithm is the first provably efficient policy optimization algorithm for CMDP with safe exploration. We hope that our work provides a step towards a principled way to design efficient safe reinforcement learning algorithms.
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A Preliminaries

Our analysis begins with the regret decomposition,

$$\text{Regret}(K) = \sum_{k=1}^{K} \left( V_{r,1}^{\pi_k}(x_1) - V_{r,1}^{\pi_k}(x_1) \right) + \sum_{k=1}^{K} \left( V_{r,1}^{\pi_k}(x_1) - V_{r,1}^{\pi_k}(x_1) \right), \quad (7)$$

where we add and subtract the value $V_{r,1}^{\pi}(x_1)$ estimated from the policy evaluation; the policy $\pi^*$ in hindsight is the best policy from the common subset in Assumption 2. To bound the total regret (7), we would like to analyze (R.I) and (R.II) separately.

First, we define the model prediction error for the reward as $\epsilon_{r,h} = r_h + \mathbb{P}_h V_{r,h+1}^k - Q_{r,h}^k$ for all $(k, h) \in [K] \times [H]$, which describes the prediction error in the Bellman equations (2) using $V_{r,h+1}^k$ instead of $V_{r,h+1}^{\pi_k,k}$. With this notation, we expand (R.I) into

$$\text{(R.I)} = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \langle Q_{r,h}^k(x_h, \cdot), \pi^*_h(\cdot \mid x_h) - \pi_h^k(\cdot \mid x_h) \rangle \mid x_1 \right] + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \epsilon_{r,h}^k(x_h, a_h) \mid x_1 \right], \quad (8)$$

where the first double sum is linear in terms of the policy difference and the second one describes the total model prediction error. The above expansion is based on the known performance difference lemma (see Lemma 3.2 [13]) and we provide a proof in Section D.1 for reference. Meanwhile, if we define the model prediction error for the criterion as $\epsilon_{g,h}^k = g_h + \mathbb{P}_h V_{g,h+1}^k - Q_{g,h}^k$, then, similarly, we can expand $\sum_{k=1}^{K} \left( V_{g,1}^{\pi_k,k}(x_1) - V_{g,1}^k(x_1) \right)$ into

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \langle Q_{g,h}^k(x_h, \cdot), \pi^*_h(\cdot \mid x_h) - \pi_h^k(\cdot \mid x_h) \rangle \mid x_1 \right] + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \epsilon_{g,h}^k(x_h, a_h) \mid x_1 \right]. \quad (9)$$

Such derivation also applies to the auxiliary regret, $\overline{\text{Regret}}(K) := \sum_{k=1}^{K} \left( V_{g,1}^{\pi,k}(x_1) - V_{g,1}^{\pi,k}(x_1) \right) = \text{(V.I)} + \text{(V.II)}$, where (V.I) = $\sum_{k=1}^{K} \left( V_{g,1}^{\pi,k}(x_1) - V_{g,1}^{k}(x_1) \right)$ and (V.II) = $\sum_{k=1}^{K} \left( V_{g,1}^{k}(x_1) - V_{g,1}^{\pi,k,k}(x_1) \right)$ where the policy $\pi$ in hindsight satisfies the Slater condition in Assumption 1. Meanwhile, we have such kind of expansion of (V.I) as follows,

$$(V.I) = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi} \left[ \langle Q_{g,h}^k(x_h, \cdot), \pi^*_h(\cdot \mid x_h) - \pi_h^k(\cdot \mid x_h) \rangle \mid x_1 \right] + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi} \left[ \epsilon_{g,h}^k(x_h, a_h) \mid x_1 \right], \quad (10)$$

where $\epsilon_{g,h}^k = g_h + \mathbb{P}_h V_{g,h+1}^k - Q_{g,h}^k$ is the model prediction error for the criterion $g$. We omit the verification of (10) since it follows the proof of (8) in Section D.1.

The constraint violation analysis is based on the following decomposition,

$$\text{Violation}(K) = \sum_{k=1}^{K} \left( b - V_{g,1}^{k}(x_1) \right) + \sum_{k=1}^{K} \left( V_{g,1}^{k}(x_1) - V_{g,1}^{\pi,k,k}(x_1) \right),$$

where $b$ is the constraint violation for the common subset in Assumption 2.
which the inserted value $V^k_{g,1}(x_1)$ is estimated from the policy evaluation.

It is noticed that the above decompositions and expansions are valid in both full-information and bandit settings. In what follows we take them as the first step and delve into the analysis of the regret and the constraint violation in two settings, individually.

**B Proof of Theorem 1**

In the full-information setting, the reward/criterion $r$ and $g$ can vary adversarially over episodes, but entirely known at the end of each episode. For notational simplicity, we introduce the underlying probability structure first. For any $(k,h) \in [K] \times [H]$, we define $\mathcal{F}^k_{h,1}$ as a $\sigma$-algebra generated by state-action sequences, reward and criterion functions,

$$
\{ (x^T_i, a^T_i) \}_{(\tau,i) \in [k-1] \times [H]} \cup \{ r^T, g^T \}_{\tau \in [k]} \cup \{ (x^k_i, a^k_i) \}_{i \in [h]}.
$$

Similarly, we define $\mathcal{F}^k_{h,2}$ as an $\sigma$-algebra generated by

$$
\{ (x^T_i, a^T_i) \}_{(\tau,i) \in [k-1] \times [H]} \cup \{ r^T, g^T \}_{\tau \in [k]} \cup \{ (x^k_i, a^k_i) \}_{i \in [h]} \cup \{ x^k_{h+1} \}.
$$

Here, $x^k_{H+1}$ is a null state for any $k \in [K]$. A filtration is a sequence of $\sigma$-algebras $\{ \mathcal{F}^k_{h,m} \}_{(k,h,m) \in [K] \times [H] \times [2]}$ in terms of time index $t(k,h,m) := 2(k-1)H + 2(h-1) + m$. It holds that $\mathcal{F}^k_{h,m} \subset \mathcal{F}^k_{h',m'}$ for any $t \leq t'$. The estimated reward/criterion value functions, $V^k_{r,h}, V^k_{g,h}$, and the associated $Q$-functions, $Q^k_{r,h}, Q^k_{g,h}$, are $\mathcal{F}^k_{1,1}$-measurable since they are obtained from previous $k-1$ historical trajectories. Meanwhile, the reward/criterion $r^k$ and $g^k$ are also $\mathcal{F}^k_{1,1}$-measurable and they can be adversarially chosen by the environment before the $k$th episode starts. With these notations, we can expand (R.II) in the regret (7) into

$$
(R.II) = - \sum_{k=1}^{K} \sum_{h=1}^{H} \ell^k_{r,h}(x^k_h, a^k_h) + M^K_{r,H,2},
$$

where $\{ M^k_{r,h,m} \}_{(k,h,m) \in [K] \times [H] \times [2]}$ is a martingale adapted to the filtration $\{ \mathcal{F}^k_{h,m} \}_{(k,h,m) \in [K] \times [H] \times [2]}$ in terms of time index $t$. Similarly, we have it for (V.II),

$$
(V.II) = - \sum_{k=1}^{K} \sum_{h=1}^{H} \ell^k_{g,h}(x^k_h, a^k_h) + M^k_{g,H,2},
$$

where $\{ M^k_{g,h,m} \}_{(k,h,m) \in [K] \times [H] \times [2]}$ is a martingale adapted to the filtration $\{ \mathcal{F}^k_{h,m} \}_{(k,h,m) \in [K] \times [H] \times [2]}$ in terms of time index $t$. We verify (11) in Section D.2 which can be found in Lemma 4.2 [13] and (12) is similar.

We recall the UCB bonus $\Gamma^k_h := \beta(\phi^T (\Lambda^k_h)^{-1} \phi)^{1/2}$ in the action-value function estimation of Algorithm 2. According to the UCB result in Lemma 13, for a fixed $p \in (0,1)$, if we set $\lambda = 1$ and $\beta = C_1 dH \sqrt{\log(dT/p)}$ where $C_1$ is an absolute constant, then for all $(k,h) \in [K] \times [H]$ and $(x,a) \in \mathcal{S} \times \mathcal{A}$, we have

$$
-2\Gamma^k_h(x,a) \leq \ell^k_{\ell,h}(x,a) \leq 0
$$

(13)
with probability $1 - p/3$ where the symbol $\ell = r$ or $g$.

Next, we divide the proof into two parts for the regret bound and the constraint violation, respectively, in Section B.1 and Section B.2. We recall that $T := KH$ is the total number of steps taken by algorithm, $K$ is the total number of episodes, $H$ is the episode horizon, $|A|$ is the cardinality of $A$, and $d$ is the dimension of the feature map $\phi$.

### B.1 Regret Bound

First, we analyze the linear term of (R.I) in (8) via the drift analysis and show the following bound on (R.I).

**Lemma 1.** Let Assumption 2 and Assumption 3 hold. Fix $p \in (0,1)$. In Algorithm 1 with the full-information setting, if we set $\alpha = \sqrt{\log |A|/(H^3 K)}$, $\psi = \sqrt{K}$, and $\theta = 1/(K \log |A|)$ then it holds with probability $1 - p/3$ that

$$
\text{(R.I)} \leq C_2 H^{3.5} \sqrt{T \log |A|} + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \psi_{r,h}(x_h, a_h) | x_1 \right],
$$

where $C_2$ is an absolute constant and $T = HK$.

**Proof.** We begin with the drift of the dual update, $\Delta(Y^k) = (|Y^k|^2 - |Y^{k-1}|^2)/2$. Expanding the quadratic term $|Y^k|^2$ using line 11 in Algorithm 1 shows that

$$
|Y^k|^2 - |Y^{k-1}|^2
$$

$$
= \max^2 \left( Y^{k-1} + \left( b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}(x_h, \cdot), \pi_h^k(\cdot | x_h) - \pi_h^{k-1}(\cdot | x_h) \rangle - V_{g,1}^{k-1}(x_1) \right), 0 \right) - \left( Y^{k-1} \right)^2
$$

$$
\leq 2Y^{k-1} \left( b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}(x_h, \cdot), \pi_h^k(\cdot | x_h) - \pi_h^{k-1}(\cdot | x_h) \rangle - V_{g,1}^{k-1}(x_1) \right)
$$

$$
+ \left( b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}(x_h, \cdot), \pi_h^k(\cdot | x_h) - \pi_h^{k-1}(\cdot | x_h) \rangle - V_{g,1}^{k-1}(x_1) \right)^2.
$$

To simplify notation, we may write $\langle Q_{g,h}^{k-1}(x_h, \cdot), \pi_h^k(\cdot | x_h) - \pi_h^{k-1}(\cdot | x_h) \rangle$ as $\langle Q_{g,h}^{k-1}, \pi_h^k - \pi_h^{k-1} \rangle$ if the inner product is clear from the context, and denote $D(\pi_h(\cdot | x_h), \pi_h^{k-1}(\cdot | x_h))$ as $D(\pi_h | \pi_h^{k-1})$.

Denote $B := b + H + H^2$. Clearly, $B^2 \geq \left( b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi_h^k - \pi_h^{k-1} \rangle - V_{g,1}^{k-1}(x_1) \right)^2$. Therefore, we have

$$
\sum_{h=1}^{H} \psi(Q_{r,h}^{k-1}, \pi_h^k - \pi_h^{k-1}) - \Delta(Y^k) - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^k(\cdot | x_h) | \pi_h^{k-1}(\cdot | x_h))
$$

$$
\geq \sum_{h=1}^{H} \psi(Q_{r,h}^{k-1}, \pi_h^k - \pi_h^{k-1}) - \frac{1}{2} B^2 - Y^{k-1} \left( b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi_h^k - \pi_h^{k-1} \rangle - V_{g,1}^{k-1}(x_1) \right) - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^k(\cdot | x_h) | \pi_h^{k-1}(\cdot | x_h)).
$$
We recall that line 6 in Algorithm 1 gives the solution to

\[
\max_{\pi \in \Delta(A|S,H)} \sum_{h=1}^{H} \langle \psi Q_{r,h}^{k-1} + Y^{k-1} Q_{g,h}^{k-1}, \pi_h \rangle - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h (\cdot \mid x_h) \mid \pi_h^{k-1} (\cdot \mid x_h)). \tag{16}
\]

The above problem (16) is in form of the subproblem in Lemma 18 and we directly apply the pushback property with \( x^* = \pi_h^k, y = \pi_h^{k-1} \) and \( z = \pi_h^\star \).

\[
\sum_{h=1}^{H} \langle \psi Q_{r,h}^{k-1} + Y^{k-1} Q_{g,h}^{k-1}, \pi_h \rangle - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h (\cdot \mid x_h) \mid \pi_h^{k-1} (\cdot \mid x_h))
\]

\[
\geq \sum_{h=1}^{H} \langle \psi Q_{r,h}^{k-1} + Y^{k-1} Q_{g,h}^{k-1}, \pi_h^\star \rangle - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^\star (\cdot \mid x_h) \mid \pi_h^{k-1} (\cdot \mid x_h)) + \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^\star (\cdot \mid x_h) \mid \pi_h^k (\cdot \mid x_h)).
\]

Applying the above inequality to the right-hand side of (15) yields

\[
\sum_{h=1}^{H} \psi(\langle Q_{r,h}^{k-1}, \pi_h^\star - \pi_h^{k-1} \rangle) - \Delta(Y^k) - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^\star (\cdot \mid x_h) \mid \pi_h^{k-1} (\cdot \mid x_h))
\]

\[
\geq \sum_{h=1}^{H} \psi(\langle Q_{r,h}^{k-1}, \pi_h^\star - \pi_h^{k-1} \rangle) - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^\star (\cdot \mid x_h) \mid \pi_h^{k-1} (\cdot \mid x_h)) + \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^\star (\cdot \mid x_h) \mid \pi_h^k (\cdot \mid x_h)).
\]

Notice that \(|b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi_h^\star - \pi_h^{k-1} \rangle - V_{g,1}^{k-1}(x_1)| \leq B\). Applying the UCB result (13) to the regret decomposition for \( V_{g,1}^{k-1}(x_1) - V_{g,1}^{k-1}(x_1) \) in (9) shows that it holds with probability \( 1 - p/3 \) that,

\[
V_{g,1}^{\pi^\star, k-1}(x_1) - V_{g,1}^{k-1}(x_1) \leq \sum_{h=1}^{H} E_{\pi^\star} \left[ \langle Q_{g,h}^{k-1}, \pi_h^\star - \pi_h^{k-1} \rangle \mid x_1 \right],
\]

which further implies that

\[
b - E_{\pi^\star} \left[ \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi_h^\star - \pi_h^{k-1} \rangle \mid x_1 \right] - V_{g,1}^{k-1}(x_1) \leq b - V_{g,1}^{\pi^\star, k-1}(x_1) \leq 0,
\]

where \( \pi^\star \) is the best policy satisfying Assumption 2. After taking the expectation \( E_{\pi^\star} \), on both sides of (17), we utilize the above inequality to simplify the right-hand side,

\[
\sum_{h=1}^{H} E_{\pi^\star} \left[ \psi(\langle Q_{r,h}^{k-1}, \pi_h^\star - \pi_h^{k-1} \rangle) \mid x_1 \right] - E_{\pi^\star} [\Delta(Y^k)] - \frac{1}{\alpha} \sum_{h=1}^{H} E_{\pi^\star} \left[ D(\pi_h^\star \mid \pi_h^{k-1}) \right]
\]

\[
\geq \sum_{h=1}^{H} E_{\pi^\star} \left[ \psi(\langle Q_{r,h}^{k-1}, \pi_h^\star - \pi_h^{k-1} \rangle) \mid x_1 \right] - \frac{1}{\alpha} \sum_{h=1}^{H} E_{\pi^\star} \left[ D(\pi_h^\star \mid \pi_h^{k-1}) \right] + \frac{1}{\alpha} \sum_{h=1}^{H} E_{\pi^\star} \left[ D(\pi_h^\star \mid \pi_h^k) \right].
\]

(18)
According to the Hölder’s inequality and the Pinsker’s inequality, we first have

\[
\begin{align*}
&\sum_{h=1}^{H} \psi \left( Q_{r,h}^{k-1}, \pi_h^k - \pi_h^{k-1} \right) - \frac{1}{\alpha} \sum_{h=1}^{H} D \left( \pi_h^k \mid \tilde{\pi}_h^{k-1} \right) \\
= &\sum_{h=1}^{H} \psi \left( Q_{r,h}^{k-1}, \pi_h^k - \tilde{\pi}_h^{k-1} \right) - \frac{1}{\alpha} \sum_{h=1}^{H} D \left( \pi_h^k \mid \tilde{\pi}_h^{k-1} \right) + \sum_{h=1}^{H} \psi \left( Q_{r,h}^{k-1}, \tilde{\pi}_h^{k-1} - \pi_h^{k-1} \right) \\
\leq &\sum_{h=1}^{H} \left( \psi \left\| Q_{r,h}^{k-1} \right\|_\infty \left\| \pi_h^k - \tilde{\pi}_h^{k-1} \right\|_1 - \frac{1}{2\alpha} \left\| \pi_h^k - \tilde{\pi}_h^{k-1} \right\|_1^2 \right) + \sum_{h=1}^{H} \psi \left\| Q_{r,h}^{k-1} \right\|_\infty \left\| \tilde{\pi}_h^{k-1} - \pi_h^{k-1} \right\|_1.
\end{align*}
\]

Then, using the square completion,

\[
\psi \left\| Q_{r,h}^{k-1} \right\|_\infty \left\| \pi_h^k - \tilde{\pi}_h^{k-1} \right\|_1 - \frac{1}{2\alpha} \left\| \pi_h^k - \tilde{\pi}_h^{k-1} \right\|_1^2 = -\frac{1}{2\alpha} \left( \alpha \psi \left\| Q_{r,h}^{k-1} \right\|_\infty - \left\| \pi_h^k - \tilde{\pi}_h^{k-1} \right\|_1 \right)^2 + \frac{\alpha \psi^2}{2} \left\| Q_{r,h}^{k-1} \right\|_\infty^2
\]

and \( \left\| \tilde{\pi}_h^{k-1} - \pi_h^{k-1} \right\|_1 \leq \theta \), we drop off the first quadratic term in the right-hand side of the above equality to obtain,

\[
\sum_{h=1}^{H} \psi \left( Q_{r,h}^{k-1}, \pi_h^k - \pi_h^{k-1} \right) - \frac{1}{\alpha} \sum_{h=1}^{H} D \left( \pi_h^k \mid \pi_h^{k-1} \right) \leq \frac{\alpha \psi^2}{2} \sum_{h=1}^{H} \left\| Q_{r,h}^{k-1} \right\|_\infty^2 + \theta \psi \sum_{h=1}^{H} \left\| Q_{r,h}^{k-1} \right\|_\infty \leq \frac{\alpha \psi^2}{2} \left( H + 2 \right)^3 + \theta \psi \left( H + 1 \right)^2,
\]

where the last inequality is due to \( \left\| Q_{r,h}^{k-1} \right\|_\infty \leq H - h + 1 \), an immediate result from line 7 in Algorithm 2, and \( \sum_{h=1}^{H} \left( H - h + 1 \right)^2 \leq (H + 2)^2 / 3 \), \( \sum_{h=1}^{H} (H - h + 1) \leq (H + 1)^2 / 2 \). Taking the expectation \( \mathbb{E}_{\pi_*} \) on both sides of the above inequality and substituting it into the left-hand side of (18) show that

\[
\frac{\theta \psi (H + 1)^2}{2} + \frac{\alpha \psi^2 (H + 2)^3}{6} - \mathbb{E}_{\pi_*} [\Delta(Y^k)] + \frac{1}{2} B^2
\]

\[
\geq \sum_{h=1}^{H} \mathbb{E}_{\pi_*} \left[ \psi \left( Q_{r,h}^{k-1}, \pi_h^* - \pi_h^{k-1} \right) \mid x_1 \right] - \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi_*} \left[ D \left( \pi_h^* \mid \tilde{\pi}_h^{k-1} \right) \right] + \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi_*} \left[ D \left( \pi_h^* \mid \pi_h^{k-1} \right) \right].
\]

Then, using the fact that \( D \left( \pi_h^* \mid \tilde{\pi}_h^{k-1} \right) - D \left( \pi_h^* \mid \pi_h^{k-1} \right) \leq \theta \log |A| \) from Lemma 19, we have

\[
\frac{\theta H \log |A|}{\alpha} + \frac{\theta \psi (H + 1)^2}{2} + \frac{\alpha \psi^2 (H + 2)^3}{6} - \mathbb{E}_{\pi_*} [\Delta(Y^k)] + \frac{1}{2} B^2
\]

\[
\geq \sum_{h=1}^{H} \mathbb{E}_{\pi_*} \left[ \psi \left( Q_{r,h}^{k-1}, \pi_h^* - \pi_h^{k-1} \right) \mid x_1 \right] - \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi_*} \left[ D \left( \pi_h^* \mid \pi_h^{k-1} \right) \right] + \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi_*} \left[ D \left( \pi_h^* \mid \pi_h^{k-1} \right) \right].
\]

Notice that we set \( Q_{r,h}^0 = 0 \) in Algorithm 1. We now take a telescoping sum of both sides of (21)
from $k = 1$ to $k = K + 1$ and shift the index $k$ by one,
\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \psi(Q_{r,h}^k, \pi_h^* - \pi_h^k) \mid x_1 \right] \leq \frac{(K+1)\theta H \log |A|}{\alpha} + \frac{(K+1)\theta \psi(H+1)^2}{2} \\
+ \frac{\alpha(K+1)\psi(H+2)^3}{6} + \frac{(K+1)B^2}{2} + \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \mid \pi_h^0) \right],
\]
where we dropoff $\alpha^{-1} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} [D(\pi_h^* \mid \pi_h^{K+1})]$ and $|YK+1|^2/2$ without changing the direction of the inequality. Since $\pi_h^*$ is uniform over $A$, we know that
\[
D(\pi_h^* \mid \pi_h^0) = \sum_{a \in A} \pi_h^*(a \mid x_h) \log (|A| \pi_h^*(a \mid x_h)) \leq \log |A|,
\]
where we dropoff the entropy term $\sum_{a \in A} \pi_h^*(a \mid x_h) \log (\pi_h^*(a \mid x_h))$ that is nonpositive. We substitute the above inequality into the right-hand side of (22) and utilize the regret decomposition for $V_{r,1}^{\pi^*,k}(x_1) - V_{r,1}^k(x_1)$ in (8) again to show,
\[
\sum_{k=1}^{K} (V_{r,1}^{\pi^*,k}(x_1) - V_{r,1}^k(x_1)) \leq \frac{(K+1)\theta H \log |A|}{\alpha\psi} + \frac{(K+1)\theta(H+1)^2}{2} \\
+ \frac{\alpha(K+1)\psi(H+2)^3}{6} + \frac{(K+1)B^2}{2} + \frac{H \log |A|}{\alpha\psi} \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \nu(x_h, a_h) \mid x_1 \right].
\]
If we set $\alpha = \sqrt{\log |A|/(H^3K)}$, $\psi = \sqrt{K}$, and $\theta = 1/(K \log |A|)$, then the sum of first five terms in the right-hand side of the above inequality has the order of $H^{3.5} \sqrt{T \log |A|}$. Therefore, we prove (14).

Now, we collect the results (11) and (14) for the regret (7) to obtain the following tractable form,
\[
\text{Regret}(K) = C_2H^{3.5} \sqrt{T \log |A|} + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \nu(x_h, a_h) \mid x_1 \right] - \nu(x_h, a_h^k) + M_{r,H}^K.
\]
We recall the UCB result (13) and have the following lemma.

**Lemma 2.** Let Assumption 3 hold. Fix $p \in (0, 1)$. In Algorithm 1 with the full-information setting, if we set $\lambda = 1$ and $\beta = C_1dH \sqrt{\log (dT/p)}$, then with probability $1 - p/3$ it holds that
\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \left( \mathbb{E}_{\pi^*} \left[ \nu(x_h, a_h) \mid x_1 \right] - \nu(x_h, a_h^k) \right) \leq 2C_1 \sqrt{2d^3H^3T \log (K+1) \log \left( \frac{dT}{p} \right)},
\]
where $C_1$ is an absolute constant and $T = HK$.  

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The proof is based on the UCB result \((13)\). Since \(|\iota_{r,h}^k(x_h^k, a_h^k)| \leq 2\Gamma_h^k(x_h^k, a_h^k)|_{\mathbb{P}} \leq 2\Gamma_h^k(x_h^k, a_h^k)|_{\mathbb{P}} \), with probability \(1 - p/3\) it holds that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \left( \mathbb{E}_{x^1} \left[ \iota_{r,h}^k(x_h^k, a_h^k) \mid x_1 \right] - \iota_{r,h}^k(x_h^k, a_h^k) \right) \leq 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k(x_h^k, a_h^k),
\]

where \(\Gamma_h^k = \beta(\phi^T (\Lambda_h^k)^{-1} \phi)^{1/2}\) is the UCB bonus term. Application of the Cauchy-Schwarz inequality shows that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k(x_h^k, a_h^k) \leq \beta \sum_{h=1}^{H} \left( \sum_{k=1}^{K} \phi(x_h^k, a_h^k)^T (\Lambda_h^k)^{-1} \phi(x_h^k, a_h^k) \right)^{1/2}.
\]

Using Lemma 16, for any \(h \in [H]\) it holds that

\[
\sum_{k=1}^{K} \phi(x_h^k, a_h^k)^T (\Lambda_h^k)^{-1} \phi(x_h^k, a_h^k) \leq 2 \log \left( \frac{\det (\Lambda_h^k)}{\det (\Lambda_h^1)} \right).
\]

Due to \(\|\phi\| \leq 1\) in Assumption 3 and \(\Lambda_h^1 = \lambda I\) in Algorithm 1, it is clear that for any \(h \in [H]\),

\[
\Lambda_h^{K+1} = \sum_{k=1}^{K} \phi(x_h^k, a_h^k) \phi(x_h^k, a_h^k)^T + \lambda I \preceq (K + \lambda)I.
\]

Therefore, we have

\[
\log \left( \frac{\det (\Lambda_h^{K+1})}{\det (\Lambda_h^1)} \right) \leq \log \left( \frac{\det ((K + \lambda)I)}{\det (\lambda I)} \right) \leq d \log \left( \frac{K + \lambda}{\lambda} \right).
\]

Applying the above inequality to \((24)\) yields

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k(x_h^k, a_h^k) \leq \beta \sum_{h=1}^{H} \left( \sum_{k=1}^{K} \phi(x_h^k, a_h^k)^T (\Lambda_h^k)^{-1} \phi(x_h^k, a_h^k) \right)^{1/2} \leq \beta H \sqrt{2dK \log \left( \frac{K + \lambda}{\lambda} \right)}.
\]

Finally, we obtain \((23)\) by setting \(\lambda = 1\) and \(\beta = C_1 dH \sqrt{\log (dT/p)}\).

**Lemma 3.** Fix \(p \in (0, 1)\). In Algorithm 1 with the full-information setting, it holds with probability \(1 - p/3\) that

\[
|M_{r,H,2}^K| \leq 4 \sqrt{H^2 T \log \left( \frac{6}{p} \right)},
\]

where \(T = HK\).
Proof. In the verification of (11) (see Section D.2), we introduce the following martingale,

$$M^K_{r,h,2} = \sum_{k=1}^{K} \sum_{h=1}^{H} (D^k_{r,h,1} + D^k_{r,h,2}),$$

where

$$D^k_{r,h,1} = (\mathcal{I}_h(Q^r_{r,h} - Q^{\pi^k}_{r,h}) (x^k_{r,h}) - (Q^r_{r,h} - Q^{\pi^k}_{r,h}) (x^k_{r,h}, a^k_{r,h}),$$

$$D^k_{r,h,2} = (\mathcal{P}_h V^r_{r,h+1} - \mathcal{P}_h V^{\pi^k}_{r,h+1}) (x^k_{r,h}, a^k_{r,h}) - (V^r_{r,h+1} - V^{\pi^k}_{r,h+1}) (x^k_{r,h+1}),$$

where $\mathcal{I}_h^k f(x) := \langle f(x, \cdot), \pi^k_{r,h}(\cdot|x) \rangle$.

Due to the truncation in line 7 of Algorithm 2, we know that $Q^r_{r,h}, Q^{\pi^k}_{r,h}, V^r_{r,h+1}, V^{\pi^k}_{r,h+1} \in [0, H]$.

This shows that $|D^k_{r,h,1}|, |D^k_{r,h,2}| \leq 2H$ for all $(k, h) \in [K] \times [H]$. Application of the Azuma-Hoeffding inequality yields,

$$P(|M^K_{r,K,H,2}| \geq t) \leq 2 \exp \left( \frac{-t^2}{16H^2T} \right).$$

For $p \in (0, 1)$, if we set $t = 4H \sqrt{T \log (6/p)}$, then the inequality (26) holds with probability at least $1 - p/3$. \qed

Now, we are ready to show the desired regret bound. Substituting (14) into the right-hand side of (7) first and using (11), we have

$$\text{Regret}(K) = C_2 H^{3.5} \sqrt{T \log |A|} + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{x, a} \left[ \ell^k_{r,h}(x_h, a_h) | x_1 \right] + \sum_{k=1}^{K} \left( V^k_{r,1}(x_1) - V^{\pi^k}_{r,1}(x_1) \right)

= C_2 H^{3.5} \sqrt{T \log |A|} + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{x, a} \left[ \ell^k_{r,h}(x_h, a_h) | x_1 \right] - \ell^k_{r,h}(x_h, a_h) + M^K_{r,H,2}.$$

Applying (23) and (26) on the right-hand side of the above equality, we have

$$\text{Regret}(K) \leq C_2 H^{3.5} \sqrt{T \log |A|} + 2C_1 \sqrt{2d^3 H^3 T \log (K + 1) \log \left( \frac{dT}{p} \right)} + 4\sqrt{H^2 T \log \left( \frac{6}{p} \right)}$$

with probability $1 - p$ where $C_1, C_2$ are absolute constants. If $\log |A| = O \left( d^{1.5} \log^2 (dT/p) \right)$, then with probability $1 - p$ it holds that

$$\text{Regret}(K) \leq C d^{1.5} H^{3.5} \sqrt{T \log \left( \frac{dT}{p} \right)},$$

where $C$ is an absolute constant.

### B.2 Constraint Violation

To establish a bound on the constraint violation, we begin with the drift analysis for the random process $\{Y^k\}_{k=1}^K$. 26
Lemma 4. Let Assumptions 1 and 3 hold. Fix $k_0 \in \mathbb{N}$. In Algorithm 1 with the full-information setting, for the $k_0$ step drift, it holds with probability $1 - p/3$ that

$$
k_0C_3 + 2k_0^2B^2 - 2k_0\varepsilon \mathbb{E}[Y^{k-1}] + \frac{2H}{\alpha} \log \left( \frac{|A|}{\theta} \right) \geq \mathbb{E}\left[ |Y^{k+k_0-1}|^2 - |Y^{k-1}|^2 \right],
$$

where $C_3 := 2\psi H^2 + \theta \psi (H + 1)^2 + 2\alpha \psi^2 (H + 2)^3 / 3 + 2H \theta \log |A| / \alpha + B^2$ and $B := b + H + H^2$.

Proof. We begin with the subproblem (16) (see line 6 in Algorithm 1). Applying the pushback property in Lemma 18 with $x^* = \pi_h^k$, $y = \tilde{\pi}_h^{k-1}$ and $z = \pi_h$ yields

$$
\sum_{h=1}^{H} \langle \psi Q_{r,h}^{k-1} + Y^{k-1}Q_{g,h}^{k-1}, \pi_h^{k-1} \rangle - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^k(\cdot | x_h) | \tilde{\pi}_h^{k-1}(\cdot | x_h))
\geq \sum_{h=1}^{H} \langle \psi Q_{r,h}^{k-1} + Y^{k-1}Q_{g,h}^{k-1}, \pi_h \rangle - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \tilde{\pi}_h^{k-1}(\cdot | x_h)) + \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \pi_h^k(\cdot | x_h)),
$$

where $\pi_h$ satisfies the Slater condition in Assumption 1: $V_{g,1}^{\pi_h}(x_1) = b + \epsilon$ for all $k \in [K]$ and $\epsilon > 0$. Recall two immediate results (15) and (19) from the proof of Lemma 1. Substituting the above inequality into the right-hand side of (15) yields

$$
\sum_{h=1}^{H} \psi \langle Q_{r,h}^{k-1}, \pi_h - \pi_h^{k-1} \rangle - \Delta(Y^k) - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \tilde{\pi}_h^{k-1}(\cdot | x_h))
\geq \sum_{h=1}^{H} \psi \langle Q_{r,h}^{k-1}, \pi_h - \pi_h^{k-1} \rangle - \frac{1}{2} B^2 - Y^{k-1} \left( b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi_h - \pi_h^{k-1} \rangle - V_{g,1}^{k-1}(x_1) \right)
- \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \tilde{\pi}_h^{k-1}(\cdot | x_h)) + \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \pi_h^k(\cdot | x_h)),
$$

where $B := b + H + H^2 \geq b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi_h - \pi_h^{k-1} \rangle - V_{g,1}^{k-1}(x_1)$.

Applying the result (19) to the left-hand side of (28) yields

$$
\frac{\theta \psi (H + 1)^2}{2} + \frac{\alpha \psi^2 (H + 2)^3}{6} - \Delta(Y^k) + \frac{1}{2} B^2
\geq \sum_{h=1}^{H} \psi \langle Q_{r,h}^{k-1}, \pi_h - \pi_h^{k-1} \rangle - Y^{k-1} \left( b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi_h - \pi_h^{k-1} \rangle - V_{g,1}^{k-1}(x_1) \right)
- \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \tilde{\pi}_h^{k-1}(\cdot | x_h)) + \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \pi_h^k(\cdot | x_h)).
$$

Due to the fact $\langle Q_{r,h}^{k-1}, \pi_h - \pi_h^{k-1} \rangle \leq H$, we further simplify the above inequality as

$$
\psi H^2 + \frac{\theta \psi (H + 1)^2}{2} + \frac{\alpha \psi^2 (H + 2)^3}{6} + \frac{1}{2} B^2 + Y^{k-1} \left( b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi_h - \pi_h^{k-1} \rangle - V_{g,1}^{k-1}(x_1) \right)
+ \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \tilde{\pi}_h^{k-1}(\cdot | x_h)) - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h(\cdot | x_h) | \pi_h^k(\cdot | x_h))
\geq \Delta(Y^k) := \frac{1}{2} (|Y^k|^2 - |Y^{k-1}|^2).
$$
Let $C_\alpha := \psi H^2 + \theta (H + 1)^2/2 + \alpha \psi^2 (H + 2)^3/6 + B^2/2$. Fix $k_0 \in \mathbb{N}$, summing both sides of the above inequality from $k$ to $k + k_0 - 1$ yields

$$k_0 C_\alpha + \sum_{\tau = k}^{k + k_0 - 1} \sum_{\tau = k}^{k + k_0 - 1} Y^{\tau - 1} \left( b - \sum_{h = 1}^{H} \langle Q_{g,h}^{\tau - 1}, \bar{\pi}_h - \pi_h^{\tau - 1} \rangle - V_{g,1}^{\tau - 1}(x_1) \right)$$

$$+ \frac{1}{\alpha} \sum_{h = 1}^{H} \sum_{\tau = k}^{k + k_0 - 1} \left( D \left( \bar{\pi}_h(\cdot \mid x_h) \mid \pi_h^{\tau - 1} \right) - D \left( \bar{\pi}_h(\cdot \mid x_h) \mid \pi_h^{\tau - 1} \right) \right) \geq \frac{1}{2} \left( |Y^{k + k_0 - 1}|^2 - |Y^{k - 1}|^2 \right).$$

Using line 11 in Algorithm 1, it is easy to verify that

$$\sum_{\tau = k}^{k + k_0 - 1} \left( Y^{\tau - 1} - Y^{k - 1} \right) \left( b - \sum_{h = 1}^{H} \langle Q_{g,h}^{\tau - 1}, \bar{\pi}_h - \pi_h^{\tau - 1} \rangle - V_{g,1}^{\tau - 1}(x_1) \right)$$

$$\leq \sum_{\tau = k}^{k + k_0 - 1} \sum_{\tau = k}^{k + k_0 - 1} b - \sum_{h = 1}^{H} \langle Q_{g,h}^{\tau - 1}, \bar{\pi}_h - \pi_h^{\tau - 1} \rangle - V_{g,1}^{\tau - 1}(x_1) \right) \geq \frac{1}{2} \left( |Y^{k + k_0 - 1}|^2 - |Y^{k - 1}|^2 \right).$$

The above inequality further simplifies (29) as

$$k_0 C_\alpha + k_0^2 B^2 + Y^{k - 1} \sum_{\tau = k}^{k + k_0 - 1} \left( b - \sum_{h = 1}^{H} \langle Q_{g,h}^{\tau - 1}, \bar{\pi}_h - \pi_h^{\tau - 1} \rangle - V_{g,1}^{\tau - 1}(x_1) \right)$$

$$+ \frac{1}{\alpha} \sum_{h = 1}^{H} \sum_{\tau = k}^{k + k_0 - 1} \left( D \left( \bar{\pi}_h(\cdot \mid x_h) \mid \pi_h^{\tau - 1} \right) - D \left( \bar{\pi}_h(\cdot \mid x_h) \mid \pi_h^{\tau - 1} \right) \right) \geq \frac{1}{2} \left( |Y^{k + k_0 - 1}|^2 - |Y^{k - 1}|^2 \right).$$

Applying the UCB result (13) to the expansion in (10) shows that it holds with probability $1 - p/3$ that,

$$b - \mathbb{E}_{\pi} \left[ \sum_{h = 1}^{H} \langle Q_{g,h}^{\tau - 1}, \bar{\pi}_h - \pi_h^{\tau - 1} \rangle \mid x_1 \right] - V_{g,1}^{\tau - 1}(x_1) \leq b - V_{g,1}^{\pi,\tau - 1}(x_1) \leq - \epsilon,$$

where $\bar{\pi}_h$ satisfies the Slater condition in Assumption 1. Taking the expectation $\mathbb{E}_{\pi}$ on both sides of (30) and substituting the above inequality into the right-hand side of (30) yield

$$k_0 C_\alpha + k_0^2 B^2 - k_0 \epsilon \mathbb{E}_{\pi} \left[ Y^{k - 1} \right]$$

$$+ \frac{1}{\alpha} \sum_{h = 1}^{H} \sum_{\tau = k}^{k + k_0 - 1} \mathbb{E}_{\pi} \left[ D \left( \bar{\pi}_h(\cdot \mid x_h) \mid \pi_h^{\tau - 1} \right) \right] - D \left( \bar{\pi}_h(\cdot \mid x_h) \mid \pi_h^{\tau - 1} \right) \right] \geq \frac{1}{2} \mathbb{E}_{\pi} \left[ |Y^{k + k_0 - 1}|^2 - |Y^{k - 1}|^2 \right].$$
Notice that
\[
\sum_{\tau = k}^{k+k_0-1} \left( D\left( \bar{\pi}_h(\cdot \mid x_h) \mid \bar{\pi}_h^{\tau-1}(\cdot \mid x_h) \right) - D\left( \bar{\pi}_h(\cdot \mid x_h) \mid \bar{\pi}_h^\tau(\cdot \mid x_h) \right) \right)
= \left( D\left( \bar{\pi}_h \mid \bar{\pi}_h^{k-1} \right) - D\left( \bar{\pi}_h \mid \bar{\pi}_h^{k+k_0-1} \right) \right) + \sum_{\tau = k+1}^{k+k_0-1} \left( D\left( \bar{\pi}_h \mid \bar{\pi}_h^{\tau-1} \right) - D\left( \bar{\pi}_h \mid \bar{\pi}_h^\tau \right) \right),
\]
where \( D\left( \bar{\pi}_h \mid \bar{\pi}_h^{k-1} \right) \leq \log(\|A\|/\theta) \) and \( D\left( \bar{\pi}_h \mid \bar{\pi}_h^{\tau-1} \right) - D\left( \bar{\pi}_h \mid \bar{\pi}_h^\tau \right) \leq \theta \log |A| \) are from Lemma 19. This observation allows us to further simplify (31) as
\[
k_0C_\alpha + k_0^2B^2 - k_0\epsilon E_\#[Y^{k-1}] + \frac{k_0H\theta \log |A|}{\alpha} + \frac{H}{\alpha} \log \left( \frac{|A|}{\theta} \right) \geq \frac{1}{2} E_\# \left[ |Y^{k+k_0-1}|^2 - |Y^{k-1}|^2 \right],
\tag{32}
\]
which shows (27) by noting the definition of \( C_3 \).

Proof. We apply Lemma 17 to the stochastic process \( \{Y^k\}_{k=1}^{K} \) which is adapted to the filtration \( \{\mathcal{F}^k_{0,1}\}_{k=1}^{K} \). Clearly, \( Y^1 = 0 \) and \( \mathcal{F}^1_{0,1} \) is the \( \sigma \)-algebra to begin with. Following line 11 in Algorithm 1, it is easy to verify that
\[
|Y^{k+1} - Y^k| = |Y^{k+1} - Y^k| \leq B := \delta_{\max},
\]
where \( B := b + H + H^2 \). This also implies that \( E_\# \left[ |Y^{k+k_0} - Y^k| \right] \leq t_0\delta_{\max} \). On the other hand, if we choose
\[
Y^k \geq \frac{k_0C_3 + 2k_0^2B^2 + \frac{2H}{\alpha} \log \left( \frac{|A|}{\theta} \right)}{k_0\epsilon} := s,
\]
then (27) becomes \(-k_0\epsilon Y^k \geq \left| Y^{k+k_0} - Y^k \right|^2 - |Y^k|^2 \) and thus,
\[
E_\# \left[ |Y^{k+k_0}|^2 \mid \mathcal{F}^k_{0,1} \right] \leq |Y^k|^2 - k_0\epsilon Y^k \leq \left( Y^k - \frac{k_0\epsilon}{2} \right)^2.
\]
Taking square root of both sides of the above inequality and using the Jensen’s inequality yield,
\[
E_\# \left[ Y^{k+k_0} \mid \mathcal{F}^k_{0,1} \right] \leq \sqrt{E_\# \left[ |Y^{k+k_0}|^2 \mid \mathcal{F}^k_{0,1} \right]} \leq Y^k - \frac{k_0\epsilon}{2},
\]
where we can set \( \zeta = \epsilon/2 \leq \delta_{\max} \). By Lemma 17, we have
\[
E_\# \left[ Y^k \right] \leq \frac{k_0C_3 + 2k_0^2B^2 + \frac{2H}{\alpha} \log \left( \frac{|A|}{\theta} \right)}{k_0\epsilon} + \frac{8B^2}{\epsilon} \log \left( \frac{32B^2}{\epsilon^2} \right) + \frac{C_3}{\epsilon} + \frac{2k_0B^2}{\epsilon} + \frac{2H}{k_0\alpha\epsilon} + \frac{8B^2}{\epsilon} \log \left( \frac{32B^2}{\epsilon^2} \right),
\]

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where \( C_3 := 2\psi H^2 + \theta\psi(H + 1)^2 + 2\alpha\psi^2(H + 2)^3/3 + 2H\theta \log |\mathcal{A}|/\alpha + B^2. \) Notice that \( \theta = 1/(K \log |\mathcal{A}|), \alpha = \sqrt{\log |\mathcal{A}|/(H^3 K)}, \) and \( \psi = \sqrt{K}. \) If we choose \( k_0 = \sqrt{K}, \) then (33) holds. \( \square \)

Next, we relate the bound on the Lagrangian multiplier in Lemma 5 to the constraint violation.

**Lemma 6.** Let Assumptions 1 and 3 hold. Let \( \theta = 1/(K \log |\mathcal{A}|), \alpha = \sqrt{\log |\mathcal{A}|/(H^3 K)}, \) and \( \psi = \sqrt{K} \) in Algorithm 1 with the full-information setting. Then,

\[
\sum_{k=1}^{K} V_{g,1}(x_1) \geq Kb - \mathbb{E}_\pi [Y^{K+1}] - \frac{H}{\alpha} \log |\mathcal{A}| + \frac{\sqrt{\log |\mathcal{A}|}}{H}(\sqrt{K} + \mathbb{E}_\pi [Y^k]) \quad (34)
\]

**Proof.** We begin with line 11 of Algorithm 1 and apply the Cauchy-Schwarz inequality,

\[
Y^k \geq Y^{k-1} + (b - V_{g,1}^{k-1}(x_1)) - \sum_{h=1}^{H} \|Q_{g,h}^{k-1}\|_\infty \|\pi_h^k - \pi_h^{k-1}\|_1.
\]

Taking a telescoping sum of both sides of the above inequality from \( k = 2 \) to \( k = K + 1 \) and using \( Q_{g,h}^0 = 0, V_{g,1}^0 = b, \) and \( Y^0 = Y^1 = 0 \) yields

\[
Y^{K+1} \geq Y^1 + \sum_{k=2}^{K+1} (b - V_{g,1}^{k-1}(x_1)) - \sum_{k=2}^{K+1} \sum_{h=1}^{H} \|Q_{g,h}^{k-1}\|_\infty \|\pi_h^k - \pi_h^{k-1}\|_1.
\]

Then we use \( \|Q_{g,h}^{k-1}\|_\infty \leq H \) and shift \( k \) by one to obtain

\[
\sum_{k=1}^{K} V_{g,1}(x_1) \geq Kb - Y^{K+1} - \frac{H}{\alpha} \log |\mathcal{A}| + \frac{\sqrt{\log |\mathcal{A}|}}{H}(\sqrt{K} + \mathbb{E}_\pi [Y^k]) \quad (35)
\]

To bound \( \|\pi_h^{k+1} - \pi_h^k\|_1, \) we recall the subproblem (16) (see line 6 in Algorithm 1) again. For each \( h \in H, \) using the pushback property with \( x^* = \pi_h^k, y = \pi_h^{k-1} \) and \( z = \pi_h^{k-1}, \) we obtain

\[
\langle \psi Q_{r,h}^{k-1} + Y^{k-1} Q_{g,h}^{k-1}, \pi_h^k \rangle - \frac{1}{\alpha} D(\pi_h^k (\cdot | x_h) \mid \pi_h^{k-1} (\cdot | x_h))
\]

\[
\geq \langle \psi Q_{r,h}^{k-1} + Y^{k-1} Q_{g,h}^{k-1}, \pi_h^{k-1} (\cdot | x_h) \rangle + \frac{1}{\alpha} D(\pi_h^{k-1} (\cdot | x_h) \mid \pi_h^k (\cdot | x_h))
\]

or, equivalently,

\[
D(\pi_h^{k-1} \mid \pi_h^k) + D(\pi_h^k \mid \pi_h^{k-1}) \leq \alpha \langle \psi Q_{r,h}^{k-1} + Y^{k-1} Q_{g,h}^{k-1}, \pi_h^k - \pi_h^{k-1} \rangle.
\]

For the above inequality, we apply the Pinsker’s inequality to the left-hand side and the Cauchy-Schwarz inequality to the right-hand side to show

\[
\|\pi_h^{k-1} - \pi_h^k\|_1^2 \leq \alpha \langle \psi Q_{r,h}^{k-1} + Y^{k-1} Q_{g,h}^{k-1}, \pi_h^k - \pi_h^{k-1} \rangle
\]

\[
\leq \alpha H(\psi + Y^{k-1}) \|\pi_h^k - \pi_h^{k-1}\|_1,
\]

30
where we also use $Q_{g,h}^{-1}, Q_{g,h}^{-1} \leq H$ in the second inequality. Solving the above quadratic inequality in terms of $\|\pi_h^k - \bar{\pi}_h^{k-1}\|_1$ and summing it up over $h \in [H]$ from both sides yield

$$\sum_{h=1}^{H} \|\pi_h^k - \bar{\pi}_h^{k-1}\|_1 \leq \alpha H^2 (\psi + Y^{k-1}).$$

Notice that $\|\pi_h^k - \bar{\pi}_h^{k-1}\|_1 \geq \|\pi_h^k - \bar{\pi}_h^{k-1}\|_1 - \theta$. Finally, we substitute the above inequality into the right-hand side of (35) with expectation $E_\pi$ to obtain

$$\sum_{k=1}^{K} V_{g,1}^k(x_1) \geq Kb - E_\pi [Y^{K+1}] - H \sum_{k=1}^{K} (\theta + \alpha H^2 (\psi + E_\pi [Y^k])).$$

We conclude (34) by noting $\alpha = \sqrt{\log |A|/(H^3 K)}$, $\psi = \sqrt{K}$, and $\theta = 1/(K \log |A|)$.

Now, we are ready to show the desired constraint violation bound. Substituting (33) into the right-hand side of (34) yields

$$\sum_{k=1}^{K} V_{g,1}^k(x_1) \geq Kb - C_4 H^{3.5} \sqrt{T \log (|A|)} - \frac{H}{\log |A|} - (\sqrt{K} + C_4 H^{3.5} \sqrt{T \log (|A|)} ) \sqrt{\log |A|}.$$ 

Here we denote by $C_5 H^{3.5} \sqrt{T (\log |A| + \sqrt{\log |A|} \log H)}$ the sum of negative terms on the right-hand side of the above inequality where $C_5$ is an absolute constant. We recall the decomposition (12) for $\sum_{k=1}^{K} (V_{g,1}^k(x_1) - V_{g,1}^{\pi^{k-1}}(x_1))$. Thus, we have

$$\text{Violation}(K) = Kb - \sum_{k=1}^{K} V_{g,1}^k(x_1) + \sum_{k=1}^{K} (V_{g,1}^k(x_1) - V_{g,1}^{\pi^{k-1}}(x_1))$$

$$\leq C_5 H^{3.5} \sqrt{T (\log |A| + \sqrt{\log |A|} \log H)} - \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k(x_h^k, a_h^k) + M_{g,H,2}^K$$

$$\leq C_5 H^{3.5} \sqrt{T (\log |A| + \sqrt{\log |A|} \log H)} + 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k(x_h^k, a_h^k) + |M_{g,H,2}^K|.$$ (36)

Finally, we recall two immediate results of Lemma 2 and Lemma 3. Fix $p \in (0, 1)$, the proof of Lemma 2 also shows that with probability $1 - p/3$,

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k(x_h^k, a_h^k) \leq C_1 \sqrt{2d^3 H^3 T \log (K + 1) \log \left(\frac{dT}{p}\right)},$$ (37)

and Lemma 3 shows that with probability $1 - p/3$,

$$|M_{g,H,2}^K| \leq 4 \sqrt{H^2 T \log \left(\frac{6}{p}\right)}.$$
Applying the above two probability bounds to (36), with probability \(1 - p\) it holds that

\[
\text{Violation}(K) \leq C_5 H^{3.5} \sqrt{T \left( \log |A| + \sqrt{\log |A| \log H} \right)} + 2C_1 \sqrt{2d^3 H^3 T \log (K + 1) \log \left( \frac{dT}{p} \right)} + 4 \sqrt{H^2 T \log \left( \frac{6}{p} \right)}.
\]

Notice that \(\log |A| = O \left(d^{1.5} \log^2 (dT/p)\right)\). We combine the last three negative terms to conclude the following probability bound,

\[
\text{Violation}(K) \leq C' d^{1.5} H^{3.5} \sqrt{T \log \left( \frac{dT}{p} \right)}.
\]

where \(C'\) is an absolute constant.

### C Proof of Theorem 2

In the bandit setting, the reward/criterion \(r\) and \(g\) do not change over episodes, but not entirely known at the end of each episode. We recall the model prediction error \(\iota_{\ell,h} := \ell_h + \mathbb{P}_h V_{\ell,h+1}^k - Q_{\ell,h}^k\) where \(\ell = r\) or \(g\), and the UCB bonus \(\Gamma_{h}^k := \beta (\phi^T (A_{h}^k)^{-1} \phi)^{1/2}\) in the action-value function estimation of Algorithm 3. According to the UCB result in Lemma 14, if we set \(\lambda = 1\) and \(\beta = C_1 dH \sqrt{\log (dT/p)}\) where \(C_1\) is an absolute constant, then for all \((k,h) \in [K] \times [H]\) and \((x,a) \in \mathcal{S} \times \mathcal{A}\), we have

\[
|\iota_{\ell,h}^k (x,a)| \leq \Gamma_{h}^k (x,a) \text{ and } Q_{\ell,h}^k (x,a) \geq Q_{\ell,h}^k (x,a),
\]

with probability \(1 - p/3\) where the symbol \(\ell = r\) or \(g\).

Let \(\delta_{\ell,h}^k := V_{\ell,h}^k (x_h^k) - V_{\ell,h}^{\pi_{k,k}} (x_h^k)\) and \(\zeta_{\ell,h+1}^k := \mathbb{E} \left[ \delta_{\ell,h+1}^k | x_h^k, a_h^k \right] - \delta_{\ell,h}^k\) where \(\ell = r\) or \(g\). The above inequalities further imply that

\[
(R.II) \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \zeta_{r,h}^k + 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_{h}^k,
\]

\[
(V.II) \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \zeta_{g,h}^k + 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_{h}^k.
\]

We verify (39) in Section D.3 following Lemma B.6 in [24] and (40) is similar.

Similarly, we have two parts for the regret bound and the constraint violation, respectively, in Section C.1 and Section C.2. We recall that \(T := KH\) is the total number of steps taken by algorithm, \(K\) is the total number of episodes, \(H\) is the episode horizon, \(|A|\) is the cardinality of \(\mathcal{A}\), and \(d\) is the dimension of the feature map \(\phi\).
C.1 Regret Bound

Similar to Lemma 1, we analyze the linear term of (R.I) in (8) to obtain the following bound.

**Lemma 7.** Let Assumption 3 hold. Fix \( p \in (0, 1) \). In Algorithm 1 with the bandit setting, if we set \( \alpha = \sqrt{\log |A|/(H^3 K)} \), \( \psi = \sqrt{K} \), and \( \theta = 1/(K \log |A|) \) then it holds that

\[
\text{(R.I)} \leq C_0 H^{3.5} \sqrt{T \log |A|} + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \nu_{r,h}(x_h, a_h) \mid x_1 \right], \tag{41}
\]

where \( C_0 \) is an absolute constant and \( T = HK \).

**Proof.** The proof follows the procedure in proving Lemma 1. Recall \( \Delta(Y^k) = (|Y^k|^2 - |Y^{k-1}|^2)/2 \), \( B := b + H + H^2 \), and the inequality (17). It is easy to verify that

\[
b - \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi^*_h - \pi_h^{k-1} \rangle - V_{g,1}^{k-1}(x_1)
\]

\[
= b - \sum_{h=1}^{H} \langle Q_{g,h}^{\pi,\pi_{k-1}}, \pi^*_h - \pi_h^{k-1} \rangle - V_{g,1}^{\pi,\pi_{k-1}}(x_1)
\]

\[
- \sum_{h=1}^{H} \langle Q_{g,h} - Q_{g,h}^{\pi,\pi_{k-1}}, \pi^*_h \rangle
\]

\[
+ \sum_{h=1}^{H} \langle Q_{g,h}^{\pi,\pi_{k-1}}, \pi^*_h - \pi_h^{k-1} \rangle - V_{g,1}^{\pi,\pi_{k-1}}(x_1) - V_{g,1}^{k-1}(x_1)
\]

\[
\leq b - \sum_{h=1}^{H} \langle Q_{g,h}^{\pi,\pi_{k-1}}, \pi^*_h - \pi_h^{k-1} \rangle - V_{g,1}^{\pi,\pi_{k-1}}(x_1),
\]

with the probability \( 1 - p/3 \), where we use the UCB result (38) and the definition of value functions to obtain the inequality. If we take the expectation \( \mathbb{E}_{\pi^*} \) on both sides of (42) and apply the performance difference lemma (see Lemma 3.2 \([13]\)), then

\[
b - \mathbb{E}_{\pi^*} \left[ \sum_{h=1}^{H} \langle Q_{g,h}^{k-1}, \pi^*_h - \pi_h^{k-1} \rangle \mid x_1 \right] - V_{g,1}^{k-1}(x_1) \leq 0,
\]

where \( \pi^* \) is the best policy satisfying Assumption 2. Taking the expectation \( \mathbb{E}_{\pi^*} \) on both sides of (17) and using the above inequality to simplify the right-hand side of (17) show

\[
\sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \psi \langle Q_{r,h}^{k-1}, \pi_h^* - \pi_h^{k-1} \rangle \mid x_1 \right] - \mathbb{E}_{\pi^*}[\Delta(Y^k)] - \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \mid \pi_h^{k-1}) \right]
\]

\[
\geq \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \psi \langle Q_{r,h}^{k-1}, \pi_h^* - \pi_h^{k-1} \rangle \mid x_1 \right] - \frac{1}{2} B^2
\]

\[
- \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \mid \pi_h^{k-1}) \right] + \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \mid \pi_h^k) \right]. \tag{43}
\]
Since line 7 in Algorithm 3 shows \( \|Q_{r,h}^{k-1}\|_\infty \leq H \), the previous inequality (19) becomes
\[
\sum_{h=1}^{H} \psi \langle Q_{r,h}^{k-1}, \pi_h^{k-1} - \tilde{\pi}_h^{k-1} \rangle - \frac{1}{\alpha} \sum_{h=1}^{H} D(\pi_h^k \| \tilde{\pi}_h^{k-1}) \leq \frac{\alpha \psi^2}{2} \sum_{h=1}^{H} \|Q_{r,h}^{k-1}\|_\infty^2 + \theta \psi \sum_{h=1}^{H} \|Q_{r,h}^{k-1}\|_\infty \leq \frac{\alpha \psi^2 H^3}{2} + \theta \psi H^2.
\] (44)

Taking the expectation \( \mathbb{E}_{\pi^*} \) on both sides of the above inequality and substituting it into the left-hand side of (43) show that
\[
\theta \psi H^2 + \frac{\alpha \psi^2 H^3}{2} - \mathbb{E}_{\pi^*} [\Delta(Y^k)] + \frac{1}{2} B^2 \geq \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \psi \langle Q_{r,h}^{k-1}, \pi_h^* - \tilde{\pi}_h^{k-1} \rangle \ | \ x_1 \right] - \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \| \tilde{\pi}_h^{k-1}) \right] + \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \| \pi_h^k) \right].
\]
Then, using the fact that \( D(\pi_h^* \| \tilde{\pi}_h^{k-1}) - D(\pi_h^* \| \pi_h^{k-1}) \leq \theta \log |A| \) from Lemma 19, we have
\[
\frac{\theta H \log |A|}{\alpha} + \theta \psi H^2 + \frac{\alpha \psi^2 H^3}{2} - \mathbb{E}_{\pi^*} [\Delta(Y^k)] + \frac{1}{2} B^2 \geq \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \psi \langle Q_{r,h}^{k-1}, \pi_h^* - \tilde{\pi}_h^{k-1} \rangle \ | \ x_1 \right] - \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \| \pi_h^{k-1}) \right] + \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \| \pi_h^k) \right].
\] (45)

We now take a telescoping sum of both sides of (45) from \( k = 1 \) to \( k = K + 1 \) and shift the index \( k \) by one,
\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \psi \langle Q_{r,h}^k, \pi_h^* - \pi_h^k \rangle \ | \ x_1 \right] \leq \frac{(K+1)\theta H \log |A|}{\alpha} + (K+1) \theta \psi H^2
\] + \frac{\alpha (K+1) \psi^2 H^3}{2} + \frac{(K+1) B^2}{2} + \frac{1}{\alpha} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ D(\pi_h^* \| \pi_h^0) \right],
\] (46)

where we dropoff \( \alpha^{-1} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} [D(\pi_h^* \| \pi_h^{K+1})] \) and \( |Y^{K+1}|^2/2 \) without changing the direction of the inequality. Since \( \pi_h^0 \) is uniform over \( A \), we know that
\[
D(\pi_h^* \| \pi_h^0) = \sum_{a \in A} \pi_h^0(a \mid x_h) \log (|A| \pi_h^0(a \mid x_h)) \leq \log |A|,
\]
where we dropoff the entropy term \( \sum_{a \in A} \pi_h^0(a \mid x_h) \log (\pi_h^0(a \mid x_h)) \) that is nonpositive. We substitute the above inequality into the right-hand side of (46) and utilize the regret decomposition for \( V_{r,1}^{\pi^*,k}(x_1) - V_{r,1}^k(x_1) \) in (8) again,
\[
\sum_{k=1}^{K} (V_{r,1}^{\pi^*,k}(x_1) - V_{r,1}^k(x_1)) \leq \frac{(K+1)\theta H \log |A|}{\alpha \psi} + (K+1) \theta H^2
\]
+ \frac{\alpha (K+1) \psi H^3}{2} + \frac{(K+1) B^2}{2 \psi} + \frac{H \log |A|}{\alpha \psi}
\] + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ e_{r,h}^k(x_h, a_h) \ | \ x_1 \right].
\]

34
If we set $\alpha = \sqrt{\log |A|/(H^3K)}$, $\psi = \sqrt{K}$, and $\theta = 1/(K \log |A|)$, then the sum of first five terms in the right-hand side of the above inequality has the order of $H^{3.5} \sqrt{T \log |A|}$. Therefore, we conclude (14).

Lemma 8. Let Assumption 3 hold. Fix $p \in (0, 1)$. If we set $\beta = C_1 dH \sqrt{\log (dT/p)}$ in Algorithm 1 with the bandit setting, then with probability $1 - p/3$ it holds that

$$
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \ell_{r,h}(x_h, a_h) \mid x_1 \right] \leq C_1 \sqrt{2d^3H^3T \log (K + 1) \log \left( \frac{dT}{p} \right)},
$$

where $C_1$ is an absolute constant and $T = HK$.

Proof. The proof is based on the UCB result (38). Since $|\ell_{r,h}(x_h, a_h)| \leq \Gamma_h^k(x_h, a_h)$, with probability $1 - p/3$ it holds that

$$
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \ell_{r,h}(x_h, a_h) \mid x_1 \right] \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k(x_h, a_h),
$$

where $\Gamma_h^k(\cdot, \cdot) = \beta(\phi(\cdot, \cdot)^T (\Lambda_h^k)^{-1} \phi(\cdot, \cdot))^{1/2}$ is the bonus. Then, we apply the result (25) to the right-hand side of the above inequality and set $\beta = C_1 dH \sqrt{\log (dT/p)}$ and $\lambda = 1$ to obtain (47).

Lemma 9. Fix $p \in (0, 1)$. In Algorithm 1 with the bandit setting, it holds with probability $1 - p/3$ that

$$
\sum_{k=1}^{K} \sum_{h=1}^{H} \xi_{r,h}^k \leq \sqrt{2H^2T \log \left( \frac{6}{p} \right)},
$$

where $T = HK$.

Proof. We notice that $\{\xi_{r,h}^k\}$ forms a martingale that satisfies $|\xi_{r,h}^k| \leq H$ for all $(k, h) \in [K] \times [H]$. Therefore, application of the Azuma-Hoeffding inequality yields,

$$
P \left( \sum_{k=1}^{K} \sum_{h=1}^{H} \xi_{r,h}^k \geq t \right) \leq 2 \exp \left( \frac{-t^2}{2H^2T} \right).$$

For $p \in (0, 1)$, if we set $t = \sqrt{2H^2T \log (6/p)}$, then the inequality (48) holds with probability at least $1 - p/3$.

Now, we are ready to show the desired regret bound. Substituting (41) into the right-hand side of the regret (7) first and then using (39), we have

$$
\text{Regret}(K) = C_0 H^{3.5} \sqrt{T \log |A|} + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \ell_{r,h}(x_h, a_h) \mid x_1 \right] + \sum_{k=1}^{K} \left( V_{r,1}^k(x_1) - V_{r,1}^{\pi^*k}(x_1) \right) + \sum_{k=1}^{K} \sum_{h=1}^{H} V_{r,h}^k + 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k.
$$
Applying (47), (48) and (37) to the right-hand side of the above equality, we have

$$\text{Regret}(K) \leq C_6 H^{3.5} \sqrt{T \log |A|} + 3C_1 \sqrt{2d^3 H^3 T \log (K + 1) \log \left(\frac{dT}{p}\right)} + \sqrt{2H^2 T \log \left(\frac{6}{p}\right)},$$

with probability $1 - p$ where $C_1, C_6$ are absolute constants. If $\log |A| = O \left(d^{1.5} \log^2 (dT/p)\right)$, then with probability $1 - p$ it holds that

$$\text{Regret}(K) \leq C'' d^{1.5} H^{3.5} \sqrt{T \log \left(\frac{dT}{p}\right)},$$

where $C''$ is an absolute constant.

### C.2 Constraint Violation

Similar to Lemma 10, we have the following lemma on the random process $\{Y^k\}_{k=1}^K$.

**Lemma 10.** Let Assumptions 1 and 3 hold. Fix $k_0 \in \mathbb{N}$. In Algorithm 1 with the bandit setting, for the $k_0$ step drift, it holds with probability $1 - p/3$ that

$$k_0 C_7 + 2k_0^2 B^2 - 2k_0 \epsilon \mathbb{E}_\pi [Y^{k-1}] + \frac{2H}{\alpha} \log \left(\frac{|A|}{\theta}\right) \geq \mathbb{E}_\pi \left[|Y^{k+k_0-1}|^2 - |Y^{k-1}|^2\right],$$

where $C_7 := 2\psi H^2 + 2\theta \psi H^2 + \alpha \psi^2 H^3 + 2H \theta \log |A|/\alpha + B^2$ and $B := b + H + H^2$.

**Proof.** The proof follows the procedure proving Lemma 4. Recall the inequality (28). Applying the result (44) to the left-hand side of (28) yields

$$\theta \psi H^2 + \frac{\alpha \psi^2 H^3}{2} - \Delta(Y^k) + \frac{1}{2} B^2$$

$$\geq \sum_{h=1}^H \psi \langle \tilde{Q}_{g,h}^{k-1}, \tilde{\pi}_h - \pi_h^{k-1} \rangle - Y^{k-1} \left(b - \sum_{h=1}^H \langle \tilde{Q}_{g,h}^{k-1}, \tilde{\pi}_h - \pi_h^{k-1} \rangle - V_{g,h}^{k-1}(x_1) \right)$$

$$- \frac{1}{\alpha} \sum_{h=1}^H D(\tilde{\pi}_h(\cdot \mid x_h) \mid \tilde{\pi}_h^{k-1}(\cdot \mid x_h)) + \frac{1}{\alpha} \sum_{h=1}^H D(\tilde{\pi}_h(\cdot \mid x_h) \mid \pi_h^k(\cdot \mid x_h)).$$

The rest follows Lemma 4 with a different $C_\alpha = \psi H^2 + \theta \psi H^2 + \alpha \psi^2 H^3/2 + B^2/2$. Similar to (42), using $\tilde{\pi}_h$ instead of $\pi_h^*$, we can show that with probability $1 - p/3$,

$$b - \mathbb{E}_\pi \left[\sum_{h=1}^H \langle Q_{g,h}^{\tau-1}, \tilde{\pi}_h - \pi_h^{\tau-1} \rangle \mid x_1\right] - V_{g,1}^{\tau-1}(x_1) \leq b - V_{g,1}^{\pi,k-1}(x_1) \leq -\epsilon,$$

where $\tilde{\pi}_h$ satisfies the Slater condition in Assumption 1.

Next, we restate Lemma 5 and Lemma 6 for the bandit setting.
**Lemma 11.** Let Assumptions 1 and 3 hold. Let $\theta = 1/(K \log |\mathcal{A}|)$, $\alpha = \sqrt{\log |\mathcal{A}|/(H^3 K)}$, and $\psi = \sqrt{K}$ in Algorithm 1 with the bandit setting. Then, for any $k \in [K]$, it holds with probability $1 - p/3$ that

$$E_{\pi}[Y^k] \leq C_8 H^{3.5} \sqrt{T \log (H|\mathcal{A}|)}$$

(50)

where $C_8$ is an absolute constant.

**Proof.** The proof is similar as the proof of Lemma 5. \hfill \Box

**Lemma 12.** Let Assumptions 1 and 3 hold. Let $\theta = 1/(K \log |\mathcal{A}|)$, $\alpha = \sqrt{\log |\mathcal{A}|/(H^3 K)}$, and $\psi = \sqrt{K}$ in Algorithm 1 with the bandit setting. Then,

$$\sum_{k=1}^{K} V_{g,1}^k(x_1) \geq Kb - E_{\pi}[Y^{K+1}] - \frac{H}{K} \sum_{k=1}^{K} \left( \frac{1}{\log |\mathcal{A}|} + \frac{\sqrt{\log |\mathcal{A}|}}{H} (\sqrt{K} + E_{\pi}[Y^k]) \right).$$

(51)

**Proof.** The proof is similar as the proof Lemma 6. \hfill \Box

Substituting (50) into the right-hand side of (51) yields

$$\sum_{k=1}^{K} V_{g,1}^k(x_1) \geq Kb - C_8 H^{3.5} \sqrt{T \log (H|\mathcal{A}|)} - \frac{H}{H \log |\mathcal{A}|} - H^2 (\sqrt{K} + C_8 H^{3.5} \sqrt{T \log (H|\mathcal{A}|)}) \sqrt{\log |\mathcal{A}|}.$$

Here we denote by $C_9 H^{3.5} \sqrt{T} (\log |\mathcal{A}| + \sqrt{\log |\mathcal{A}| \log H})$ the sum of negative terms on the right-hand side of the above inequality where $C_9$ is an absolute constant. We recall the regret decompositions (40) for $\sum_{k=1}^{K} (V_{g,1}^k(x_1) - V_{\pi,1}^k(x_1))$. Thus, we have

$$\text{Violation}(K) = Kb - \sum_{k=1}^{K} V_{g,1}^k(x_1) + \sum_{k=1}^{K} (V_{g,1}^k(x_1) - V_{\pi,1}^k(x_1))$$

$$\leq C_9 H^{3.5} \sqrt{T} (\log |\mathcal{A}| + \sqrt{\log |\mathcal{A}| \log H}) + \sum_{k=1}^{K} \sum_{h=1}^{H} \zeta_{g,h}^k + 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k$$

$$\leq C_9 H^{3.5} \sqrt{T} (\log |\mathcal{A}| + \sqrt{\log |\mathcal{A}| \log H}) + \sqrt{2H^2 T \log \left( \frac{6}{p} \right)} + 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k.$$  

(52)

Finally, we recall an immediate result of Lemma 2. Fix $p \in (0,1)$, the proof of Lemma 2 also shows that with probability $1 - p/3$,

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma_h^k(x_h^k, a_h^k) \leq C_1 \sqrt{2d^3 H^3 T \log (K+1) \log \left( \frac{dT}{p} \right)}.$$

Applying the above probability bound to (52), with probability $1 - p$ it holds that

$$\text{Violation}(K) \leq C_9 H^{3.5} \sqrt{T} (\log |\mathcal{A}| + \sqrt{\log |\mathcal{A}| \log H}) + 2C_1 \sqrt{2d^3 H^3 T \log (K+1) \log \left( \frac{dT}{p} \right)}$$

$$+ \sqrt{2H^2 T \log \left( \frac{5}{p} \right)}.$$
Notice that \( \log |\mathcal{A}| = O \left( d^{1.5} \log^2 (dT/p) \right) \). We combine the last three negative terms to conclude the following probability bound,

\[
\text{Violation}(K) \leq C'' d^{1.5} H^{3.5} \sqrt{T} \log^2 \left( \frac{dT}{p} \right).
\]

where \( C'' \) is an absolute constant.

## D Other Verifications

In this section, we collect some verifications for the reader's convenience.

### D.1 Verify (8) and (10)

For any \((k, h) \in [K] \times [H]\), we recall the definitions of \( V_{r, h}^{\pi^{*, k}} \) in the Bellman equations (2) and \( V_{r, h}^{k} \) from line 17 in Algorithm 1,

\[
V_{r, h}^{\pi^{*, k}}(x) = \langle Q_{h}^{\pi^{*, k}}(x, \cdot), \pi_{h}^{*}(\cdot | x) \rangle \quad \text{and} \quad V_{r, h}^{k}(x) = \langle Q_{h}^{k}(x, \cdot), \pi_{h}^{k}(\cdot | x) \rangle.
\]

We can expand the difference \( V_{r, h}^{\pi^{*, k}}(x) - V_{r, h}^{k}(x) \) into

\[
V_{r, h}^{\pi^{*, k}}(x) - V_{r, h}^{k}(x) = \langle Q_{h}^{\pi^{*, k}}(x, \cdot), \pi_{h}^{*}(\cdot | x) \rangle - \langle Q_{h}^{k}(x, \cdot), \pi_{h}^{k}(\cdot | x) \rangle
= \langle Q_{h}^{\pi^{*, k}}(x, \cdot) - Q_{h}^{k}(x, \cdot), \pi_{h}^{*}(\cdot | x) \rangle + \langle Q_{h}^{k}(x, \cdot), \pi_{h}^{*}(\cdot | x) - \pi_{h}^{k}(\cdot | x) \rangle
= \langle Q_{h}^{\pi^{*, k}}(x, \cdot) - Q_{h}^{k}(x, \cdot), \pi_{h}^{*}(\cdot | x) \rangle + \xi_{h}^{k}(x),
\]

where \( \xi_{h}^{k}(x) := \langle Q_{h}^{k}(x, \cdot), \pi_{h}^{*}(\cdot | x) - \pi_{h}^{k}(\cdot | x) \rangle \).

Recall the equality in the Bellman equations (2) and the model prediction error,

\[
Q_{r, h}^{\pi^{*, k}} = r_{h} + \mathbb{P}_{h} V_{r, h+1}^{\pi^{*, k}} \quad \text{and} \quad \iota_{r, h}^{k} = r_{h} + \mathbb{P}_{h} V_{r, h+1}^{k} - Q_{r, h}^{k}.
\]

As a result of the above two, it is easy to see that

\[
Q_{r, h}^{\pi^{*, k}} - Q_{r, h}^{k} = \mathbb{P}_{h} (V_{r, h+1}^{\pi^{*, k}} - V_{r, h+1}^{k}) + \iota_{r, h}^{k}.
\]

Substituting the above difference into the right-hand side of (53) yields,

\[
V_{r, h}^{\pi^{*, k}}(x) - V_{r, h}^{k}(x) = \langle \mathbb{P}_{h} (V_{r, h+1}^{\pi^{*, k}} - V_{r, h+1}^{k}) (x, \cdot), \pi_{h}^{*}(\cdot | x) \rangle + \langle \iota_{r, h}^{k}(x), \pi_{h}^{*}(\cdot | x) \rangle + \xi_{h}^{k}(x),
\]

which establishes a recursive formula over \( h \). Thus, we expand \( V_{r, 1}^{\pi^{*, k}}(x_{1}) - V_{r, 1}^{k}(x_{1}) \) recursively with \( x = x_{1} \) as

\[
V_{r, 1}^{\pi^{*, k}}(x_{1}) - V_{r, 1}^{k}(x_{1}) = \langle \mathbb{P}_{1} (V_{r, 2}^{\pi^{*, k}} - V_{r, 2}^{k}) (x_{1}, \cdot), \pi_{1}^{*}(\cdot | x_{1}) \rangle + \langle \iota_{r, 1}^{k}(x_{1}), \pi_{1}^{*}(\cdot | x_{1}) \rangle + \xi_{1}^{k}(x_{1})
= \langle \mathbb{P}_{1} \mathbb{P}_{2} (V_{r, 3}^{\pi^{*, k}} - V_{r, 3}^{k}) (x_{2}, \cdot), \pi_{2}^{*}(\cdot | x_{2}) \rangle (x_{1}, \cdot), \pi_{1}^{*}(\cdot | x_{1}) \rangle
+ \langle \mathbb{P}_{1} (\iota_{r, 2}^{k}(x_{2}, \cdot), \pi_{2}^{*}(\cdot | x_{2}) \rangle (x_{1}, \cdot), \pi_{1}^{*}(\cdot | x_{1}) \rangle + \langle \iota_{r, 1}^{k}(x_{1}, \cdot), \pi_{1}^{*}(\cdot | x_{1}) \rangle
+ \langle \mathbb{P}_{1} \iota_{2}^{k}(x_{1}, \cdot), \pi_{1}^{*}(\cdot | x_{1}) \rangle + \xi_{1}^{k}(x_{1}).
\]
For notational simplicity, for any \((k, h) \in [K] \times [H]\), we define an operator \(\mathcal{I}_h\) for function \(f : S \times A \rightarrow \mathbb{R}\),

\[
(\mathcal{I}_h f)(x) = \langle f(x, \cdot), \pi^*_h(\cdot | x) \rangle.
\]

With this notation, repeating the above recursion (54) over \(h \in [H]\) yields

\[
V^\pi_{r,1}(x_1) - V^k_{r,1}(x_1) = \mathcal{I}_1 \mathcal{P}_1 \mathcal{I}_2 \mathcal{P}_2 \cdots \mathcal{I}_{H-1} \mathcal{P}_h \mathcal{I}_{H+1} \mathcal{P}_{H+1} V^\pi_{r,H+1} - V^k_{r,H+1} = 0,
\]

Finally, notice that \(V^\pi_{r,H+1} = V^k_{r,H+1} = 0\), we use the definitions of \(\mathcal{P}_h\) and \(\mathcal{I}_h\) to conclude (8). Similarly, we can also use the above argument to verify (10) with \(\bar{\pi}\), instead of \(\pi^*\).

**D.2 Verify (11) and (12)**

We recall the definition of \(V^\pi_{r,h}\) and define an operator \(\mathcal{I}^k_h\) for function \(f : S \times A \rightarrow \mathbb{R}\),

\[
V^\pi_{r,h}(x) = \langle Q^\pi_{r,h}(x, \cdot), \pi^k_h(\cdot | x) \rangle \quad \text{and} \quad \left(\mathcal{I}^k_h f\right)(x) = \langle f(x, \cdot), \pi^k_h(\cdot | x) \rangle.
\]

We expand the model prediction error \(\iota^k_{r,h}\) into,

\[
\iota^k_{r,h}(x^k_h, a^k_h) = r^k_{r,h}(x^k_h, a^k_h) + (\mathcal{P}_h V^k_{r,h+1})(x^k_h, a^k_h) - Q^k_{r,h}(x^k_h, a^k_h)
\]

\[
= \left( r^k_{r,h}(x^k_h, a^k_h) + (\mathcal{P}_h V^k_{r,h+1})(x^k_h, a^k_h) - Q^\pi_{r,h}(x^k_h, a^k_h) \right) + \left( Q^\pi_{r,h}(x^k_h, a^k_h) - Q^k_{r,h}(x^k_h, a^k_h) \right),
\]

where we use the Bellman equation \(Q^\pi_{r,h}(x^k_h, a^k_h) = r^k_{r,h}(x^k_h, a^k_h) + (\mathcal{P}_h V^\pi_{r,h+1})(x^k_h, a^k_h)\) in the last equality. With the above formula, we expand the difference \(V^k_{r,1}(x_1) - V^\pi_{r,1}(x_1)\) into

\[
V^k_{r,h}(x^k_h) - V^\pi_{r,h}(x^k_h) = \left( \mathcal{I}^k_h (Q^k_{r,h} - Q^\pi_{r,h}) \right)(x^k_h) - \iota^k_{r,h}(x^k_h, a^k_h)
\]

\[
\quad + \left( \mathcal{P}_h V^k_{r,h+1} - \mathcal{P}_h V^\pi_{r,h+1} \right)(x^k_h, a^k_h) + \left( Q^\pi_{r,h} - Q^k_{r,h} \right)(x^k_h, a^k_h).
\]

Let

\[
D^k_{r,h,1} := \left( \mathcal{I}^k_h (Q^k_{r,h} - Q^\pi_{r,h}) \right)(x^k_h) - \left( Q^k_{r,h} - Q^\pi_{r,h} \right)(x^k_h, a^k_h),
\]

\[
D^k_{r,h,2} := \left( \mathcal{P}_h V^k_{r,h+1} - \mathcal{P}_h V^\pi_{r,h+1} \right)(x^k_h, a^k_h) - \left( V^k_{r,h+1} - V^\pi_{r,h+1} \right)(x^k_{h+1}).
\]

Therefore, we have the following recursive formula over \(h\),

\[
V^k_{r,h}(x^k_h) - V^\pi_{r,h}(x^k_h) = D^k_{r,h,1} + D^k_{r,h,2} + \left( V^k_{r,h+1} - V^\pi_{r,h+1} \right)(x^k_{h+1}) - \iota^k_{r,h}(x^k_h, a^k_h).
\]
Notice that $V_{r,H+1}^{\pi,k} = V_{r,H+1}^k = 0$. Summing the above equality over $h \in [H]$ yields

$$V_{r,1}(x_1) - V_{r,1}^{\pi,k}(x_1) = \sum_{h=1}^H \left( D_{r,h,1}^k + D_{r,h,2}^k \right) - \sum_{h=1}^H \ell_{r,h}(x_h, a_h^k). \tag{55}$$

Following the definitions of $\mathcal{F}_{h,1}^k$ and $\mathcal{F}_{h,2}^k$, we know $D_{r,h,1}^k \in \mathcal{F}_{h,1}^k$ and $D_{r,h,2}^k \in \mathcal{F}_{h,2}^k$. Thus, for any $(k, h) \in [K] \times [H]$,

$$\mathbb{E} \left[ D_{r,h,1}^k | \mathcal{F}_{h-1,1}^k \right] = 0 \quad \text{and} \quad \mathbb{E} \left[ D_{r,h,2}^k | \mathcal{F}_{h,1}^k \right] = 0.$$

Notice that $t(k, 0, 2) = t(k-1, H, 2) = 2H(k-1)$. Clearly, $\mathcal{F}_{0,2}^k = \mathcal{F}_{H,2}^{k-1}$ for any $k \geq 2$. Let $\mathcal{F}_{1}^1$ be empty. We define a martingale sequence,

$$M_{r,h,m}^k = \sum_{\tau=1}^{k-1} \sum_{i=1}^H (D_{r,i,1}^\tau + D_{r,i,2}^\tau) + \sum_{i=1}^{h-1} \left( D_{r,i,1}^k + D_{r,i,2}^k \right) + \sum_{\ell=1}^m D_{r,h,\ell} \tag{55}$$

where $t(k, h, m) := 2(k-1)H + 2(h-1) + m$ is the time index. Clearly, this martingale is adapted to the filtration $\{\mathcal{F}_{h,m}^k\}_{(k,h,m) \in [K] \times [H] \times [2]}$, and particularly,

$$\sum_{k=1}^K \sum_{h=1}^H (D_{r,h,1}^k + D_{r,h,2}^k) = M_{r,H,2}^K.$$

Finally, we combine the above martingale with (55) to obtain (11). Similarly, we can show (12).

### D.3 Verify (39) and (40)

According to Lemma 14, for any $(k, h) \in [K] \times [H]$ and $(s, a) \in \mathcal{S} \times \mathcal{A},$

$$Q_{\ell,h}^k(x, a) - Q_{\ell,h}^{\pi,k}(x, a) \leq \mathbb{P}_h(V_{\ell,h+1}^k - V_{\ell,h+1}^{\pi,k})(x, a) + 2\Gamma_h^k(x, a),$$

and utilize $\delta_{\ell,h}^k = V_{\ell,h}^k(x_h^k) - V_{\ell,h}^{\pi,k}(x_h^k)$ and $\zeta_{\ell,h+1}^k = \mathbb{E}[\delta_{\ell,h+1}^k | x_h^k, a_h^k] - \delta_{\ell,h}^k$ where $\ell = r$ or $g$ to obtain that

$$\delta_{\ell,h}^k \leq \delta_{\ell,h+1}^k + \zeta_{\ell,h+1}^k + 2\Gamma_h^k.$$

Finally, applying the above inequality recursively and summing over $k \in [K]$ prove (39) and (40).

### E Supporting Lemmas

In this section, we collect some known results that are used in our proof.

First, we recall the UCB bonus $\Gamma_h^k := \beta(\phi^T (\Lambda_h^k)^{-1} \phi)^{1/2}$ in the action-value function estimation of Algorithm 2 or Algorithm 3 and the model prediction errors,

$$\ell_{t,h}^k := \ell_h^k + \mathbb{P}_h V_{t,h+1}^k - Q_{t,h}^k,$$

where we abuse the symbol $\ell$ a bit and it represents index or function for $r$ or $g.
Lemma 13 (Upper Confidence Bound, Full-information Setting). Let Assumption 3 hold. Fix $p \in (0,1)$. In Algorithm 2, we set $\lambda = 1$ and $\beta = C_1dH\sqrt{\log(dT/p)}$ where $C_1$ is an absolute constant. Then, for all $(k,h) \in [K] \times [H]$ and $(x,a) \in \mathcal{S} \times \mathcal{A}$, we have

$$-2\Gamma^k_h(x,a) \leq \ell^k_{t,h}(x,a) \leq 0,$$

with probability $1 - p/3$ where the symbol $\ell = r$ or $g$.

Proof. See the proof of Lemma 4.3 in [13].

We recall $\ell^k_{t,h} := \mathbb{P}_h(V^k_{t,h+1} - V^\pi_{t,h+1}) + \ell^k_{t,h}(Q^\pi_{t,h} - Q^k_{t,h})$ and have the following lemma.

Lemma 14 (Upper Confidence Bound, Bandit Setting). Let Assumption 3 hold. Fix $p \in (0,1)$. In Algorithm 3, we set $\lambda = 1$ and $\beta = C_1dH\sqrt{\log(dT/p)}$ where $C_1$ is an absolute constant. Then, for all $(k,h) \in [K] \times [H]$ and $(x,a) \in \mathcal{S} \times \mathcal{A}$, we have

$$|\ell^k_{t,h}(x,a)| \leq \Gamma^k_h(x,a) \text{ and } Q^\pi_{t,h}(x,a) \leq Q^k_{t,h}(x,a),$$

with probability $1 - p/3$ where the symbol $\ell = r$ or $g$.

Proof. The first inequality follows Lemma B.4 in [24]. If we use $Q^\pi_{t,h}$ instead of $Q^r_t$, the second inequality follows Lemma B.5 in [24] where $\ell = r$ or $g$.

Lemma 13 and Lemma 14 are based on the following concentration lemma that essentially shows the model prediction error in the least-squares policy evaluation is well-bounded.

Lemma 15 (Concentration of Self-normalized Process). Let $\lambda = 1$ and $\beta = C_1dH\sqrt{\log(dT/p)}$ in Algorithm 2 or Algorithm 3 where $C_1$ is an absolute constant. Fix $p \in (0,1)$. Then, for any $(k,h) \in [K] \times [H]$ it holds for $\ell = r$ or $g$ that

$$\left\| \sum_{t=1}^{k-1} \phi(x^T_h,a^T_h)((V^k_{t,h+1}(x^T_h+1) - (\mathbb{P}_hV^k_{t,h+1})(x^T_h,a^T_h)) \right\|_{(\Lambda^k_h)^{-1}} \leq CdH \sqrt{\chi}$$

with probability at least $1 - p/3$ where $\chi = \log(3(C_1 + 1)dT/p)$ and $C > 0$ is an absolute constant.

Proof. See the proof of Lemma B.3 in [24].

Lemma 16 (Elliptical Potential Lemma). Let $\{\phi_t\}_{t=1}^\infty$ be a sequence of functions in $\mathbb{R}^d$ and $A_0 \in \mathbb{R}^{d \times d}$ be a positive definite matrix. Let $A_t = A_0 + \sum_{s=1}^{t-1} \phi_s \phi_s^T$. Assume $\|\phi_t\|_2 \leq 1$ and $\lambda_{\min}(A_0) \geq 1$. Then for any $t \geq 1$ it holds that

$$\log \left( \frac{\det(A_{t+1})}{\det(A_1)} \right) \leq \sum_{s=1}^{t} \phi_s^T A_{s-1}^{-1} \phi_s \leq 2 \log \left( \frac{\det(A_{t+1})}{\det(A_1)} \right).$$

Proof. See the proof of Lemma D.2 in [24] or [13].
Lemma 17 (Drift Analysis of Random Process). Let \( \{ X_t \}_{t=1}^{\infty} \) be a discrete time stochastic process adapted to a filtration \( \{ F_t \}_{t=0}^{\infty} \) with \( X_1 = 0 \) and \( F_1 = \{ \emptyset, \Omega \} \). Suppose there exist \( t_0 \in \mathbb{N}, s \in \mathbb{R}, \) and \( \delta_{\text{max}} \in \mathbb{R}^+ \) such that for any \( t \in \mathbb{N}, \)

\[
|X_{t+1} - X_t| \leq \delta_{\text{max}},
\]

\[
\mathbb{E}[X_{t+t_0} - X_t | F_t] \leq \begin{cases} t_0 \delta_{\text{max}} & \text{when } X_t < s; \\ -t_0 \zeta & \text{when } X_t \geq s, \end{cases}
\]

where \( 0 < \zeta \leq \delta_{\text{max}} \). Then,

\[
\mathbb{E}[X_t] \leq s + t_0 \frac{4\delta_{\text{max}}^2}{\zeta} \log \left( \frac{8\delta_{\text{max}}^2}{\zeta^2} \right) \text{ for all } t \in \mathbb{N}.
\]

Proof. See the proof of Lemma 5 in [51].

Lemma 18 (Pushback Property of KL-divergence). Let \( f : \Delta \to \mathbb{R} \) be a concave function where \( \Delta \) is a probability simplex in \( \mathbb{R}^d \). Let \( \Delta^o \) be the interior of \( \Delta \). Let \( x^* = \text{argmax}_{x \in \Delta} f(x) - \alpha^{-1}D(x, y) \) for a fixed \( y \in \Delta^o \) and \( \alpha > 0 \). Then, for any \( z \in \Delta, \)

\[
f(x^*) - \frac{1}{\alpha}D(x^*, y) \geq f(z) - \frac{1}{\alpha}D(z, y) + \frac{1}{\alpha}D(z, x^*).
\]

Proof. See the proof of Lemma 14 in [48].

Lemma 19 (Bounded KL-divergence Difference). Let \( \pi_1, \pi_2 \) be two probability distributions in \( \Delta(\mathcal{A}) \). Let \( \tilde{\pi}_2 = (1 - \theta)\pi_2 + \theta/|\mathcal{A}| \) where \( \theta \in (0, 1] \). Then,

\[
D(\pi_1 | \tilde{\pi}_2) - D(\pi_1 | \pi_2) \leq \theta \log |\mathcal{A}|.
\]

Moreover, we have an uniform bound, \( D(\pi_1 | \tilde{\pi}_2) \leq \log(|\mathcal{A}|/\theta). \)

Proof. See the proof of Lemma 31 in [48].