The Generalized Fractional Benjamin-Bona-Mahony Equation: Analytical and Numerical Results

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Abstract
The generalized fractional Benjamin-Bona-Mahony (gfBBM) equation models the propagation of small amplitude long unidirectional waves in a nonlocally and nonlinearly elastic medium. The equation involves two fractional terms unlike the well-known fBBM equation. In this paper, we prove local existence and uniqueness of the solutions for the Cauchy problem. The sufficient conditions for the existence of solitary wave solutions are obtained. The Petviashvili method is proposed for the generation of the solitary wave solutions and their evolution in time is investigated numerically by Fourier spectral method. The efficiency of the numerical methods is tested and the relation between nonlinearity and fractional dispersion is observed by various numerical experiments.

Keywords: Generalized Fractional Benjamin-Bona-Mahony equation, Conserved Quantities, Local Existence, Solitary Waves, Petviashvili Method

1. Introduction
This paper is concerned with the generalized fractional Benjamin-Bona-Mahony (gfBBM) equation
\begin{equation}
    u_t + u_x + \frac{1}{2}(u^{p+1})_x + \frac{3}{4}D^\alpha u_x + \frac{5}{4}D^\alpha u_t = 0,
\end{equation}

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which models the propagation of small amplitude long unidirectional waves in a nonlocally and nonlinearly elastic medium. The equation is derived in [1, 2] as a special case of the generalized fractional Camassa-Holm equation by using an asymptotic expansion technique. Here $p$ is a positive integer and the operator $D^\alpha = (-\Delta)^{\alpha/2}$ denotes the Riesz potential of order $-\alpha$, for any $\alpha \in \mathbb{R}$. The operator can be defined via Fourier transform by

$$\hat{D}^\alpha q(\xi) = |\xi|^\alpha \hat{q}(\xi),$$

where $\hat{q}$ is the Fourier transform of a function $q$.

The effects of the relation between the nonlinearity and the dispersion on the dynamics of solutions has been the focus of many studies. The problem mostly handled by fixing the dispersion and increasing the nonlinearity. Studies for the more physically significant case with lower dispersion has become popular only in the recent years. The well-known Korteweg-de Vries (KdV) and Benjamin-Bona-Mohany (BBM) equations are investigated by using corresponding fractional forms such as

$$u_t + u_x + w^p u_x + D^\alpha u_x = 0$$

fractional KdV (fKdV) equation and

$$u_t + u_x + w^p u_x + D^\alpha u_t = 0$$

fractional BBM (fBBM) equation. These equations have been intensively studied in the past years for $\alpha \geq 1$ in terms of global well-posedness, stability and blow-up etc. We refer [3, 4, 5] for a more detailed discussion and review. The research on the more delicate case $\alpha \in (0, 1)$ has been increased in the last few years. For $p = 1$, Linares et. al. [6] proved the local well-posedness for the Cauchy problem and they have also investigated the solitary wave solutions in terms of existence and stability in [7]. The stability and linear instability results for a general nonlinearity are obtained by Pava [7]. In [8] the blow-up and the global existence problems are handled and solitary wave solutions for the fKdV are constructed numerically. Duran used efficient numerical methods to investigate the solitary wave solution of the fKdV equation in [9].

The gfBBM equation contains the fractional terms of both the fKdV and the fBBM equations, but unlike them, the gfBBM equation models a physical phenomena. The effects of these terms on the solutions when occurring
together, such as well-posedness of the Cauchy problem, existence of solitary waves and the nature of solutions in time is therefore a curious problem. The aim of the current study is to investigate the dynamics of the gfBBM equation with a general power type nonlinear term.

The paper is organized as follows: In Section 2, we prove the local existence and uniqueness of the Cauchy problem for the gfBBM equation together with the initial condition

$$u(x, 0) = \phi(x)$$

(1.2)

by the help of deriving a suitable energy estimate. In Section 3, we derive the conserved quantities for the gfBBM equation. The existence-nonexistence results for solitary wave solutions are given in Section 4. We use Pohozaev type identities to show the non-existence of solitary wave solutions and the results of [10] are applied for the existence of positive solitary waves for certain values of $\alpha$, $p$ and the wave speed $c$. Section 5 is devoted to numerical investigation of the solutions; we construct the solitary wave solutions numerically by using Petviashvili method and propose a Fourier pseudo-spectral method for the time evolution of the solutions. We perform numerical experiments for several values of $\alpha$ and $p$ to investigate the effects of dispersion and nonlinearity.

Throughout this study, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\| \cdot \|_{L^p}$ for $1 \leq p \leq \infty$. $H^s(\mathbb{R})$ is the Sobolev space with the norm

$$\| u \|_{H^s} = \left( \int_\mathbb{R} (1 + |k|^2)^s |\hat{u}(k)|^2 dk \right)^{1/2}$$

for $s \in \mathbb{R}$ and $C$ denotes the generic constant. Here, the Fourier transform and its inverse for the given function $u \in L^2(\mathbb{R})$ defined as

$$u(x) = \frac{1}{2\pi} \int_\mathbb{R} \hat{u}(k)e^{ikx} dk, \quad \hat{u}(k) = \int_\mathbb{R} u(x)e^{-ikx} dx.$$  

(1.3)

$J^s = (I-\Delta)^{\frac{s}{2}}$ denotes the Bessel potentials of order $-s$ with $\|J^s u\|_{L^2} = \|u\|_{H^s}$.

2. Local Existence and Uniqueness

In this section, we obtain the local well-posedness of the gfBBM equation. The main result of the Section is as follows:
Theorem 2.1. Let $0 < \alpha < 1$. For the initial data in $H^{s + \frac{\alpha}{2}}(\mathbb{R})$ with $s \geq \frac{3}{2}(1 - \alpha)$, there is some $T > 0$ such that the initial value problem (1.1) - (1.2) has a unique solution in $C([0, T], H^{s + \frac{\alpha}{2}}(\mathbb{R}))$.

Proof: Applying the operator $J^s$ to the eq. (1.1), multiplying both sides of the equation by $J^s u$ and then integrating on the whole line, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (|J^s u|^2 + \frac{5}{4}|J^{s+\frac{\alpha}{2}} u|^2)dx = -\frac{p + 1}{2} \int_{\mathbb{R}} J^s u J^s (u^p u_x)dx. \quad (2.2)$$

Using the fractional Leibniz rule \cite{11}, the term $J^s (u^p u_x)$ is written as

$$J^s (u^p u_x) = u^p J^s u_x + u_x J^s u^p + R,$$

where $R$ is the remainder. Then, the eq. (2.2) becomes

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (|J^s u|^2 + \frac{5}{4}|J^{s+\frac{\alpha}{2}} u|^2)dx = -\frac{p + 1}{2} \left( \int_{\mathbb{R}} u^p J^s u_x J^s udx + \int_{\mathbb{R}} u_x J^s u^p J^s udx + \int_{\mathbb{R}} R J^s udx \right). \quad (2.3)$$

By integration by parts and the Hölder’s inequality, the first term of RHS is estimated as

$$\int_{\mathbb{R}} |(u^p)_x (J^s u)^2|dx \leq C \|(u^p)_x\|_{L^{q_1}} \|J^s u\|_{L^{q_2}}^2 \leq C \|(u^p)_x\|_{H^{s+\frac{\alpha}{2}-1}} \|J^s u\|_{H^{\frac{\alpha}{2}}}^2 \leq C \|u\|_{H^{s+\frac{\alpha}{2}}}^p \|J^s u\|_{H^{\frac{\alpha}{2}}}^2 \leq C \|u\|_{H^{s+\frac{\alpha}{2}}}^{p+2}. \quad (2.4)$$

Here, we have used the following Sobolev imbeddings

$$H^{s+\frac{\alpha}{2}-1} \hookrightarrow L^{q_1}, \text{ for } q_1 \leq \frac{2}{3 - 2s - \alpha},$$
$$H^{\frac{\alpha}{2}} \hookrightarrow L^{q_2}, \text{ for } q_2 \leq \frac{2}{1 - \alpha}.$$
where $\frac{1}{q_1} + \frac{2}{q_2} \leq 1$ implies $s \geq \frac{3}{2}(1 - \alpha)$. Note that $H^r(\mathbb{R})$ with $r > \frac{1}{2}$ is an algebra. Therefore $u \in H^{s+\frac{\alpha}{2}}(\mathbb{R})$ ensures $u^p \in H^{s+\frac{\alpha}{2}}(\mathbb{R})$, as the above condition $s \geq \frac{3}{2}(1 - \alpha)$ guarantees that $H^{s+\frac{\alpha}{2}}(\mathbb{R})$ is an algebra. The estimation of the second term in RHS of the eq. (2.3) is given as

$$\int_{\mathbb{R}} |u_x J^s u J^s u^p| dx \leq \|u_x\|_{L^{q_3}} \|J^s u\|_{L^{q_4}} \|J^s u^p\|_{L^{q_5}} \leq C \|u_x\|_{H^{s+\frac{\alpha}{2}-1}} \|J^s u\|_{H^{\frac{\alpha}{2}}} \|J^s u^p\|_{H^{\frac{\alpha}{2}}} \leq C \|u\|_{H^{s+\frac{\alpha}{2}}}^{p+2},$$

(2.5)

where $\frac{1}{q_3} + \frac{1}{q_4} + \frac{1}{q_5} \leq 1$. Here the Sobolev imbeddings

$$H^{s+\frac{\alpha}{2}-1} \hookrightarrow L^{q_3}, \text{ for } q_3 \leq \frac{2}{3 - 2s - \alpha};$$

$$H^{\frac{\alpha}{2}} \hookrightarrow L^{q_4}, \text{ for } q_4 \leq \frac{2}{1 - \alpha};$$

$$H^{\frac{\alpha}{2}} \hookrightarrow L^{q_5}, \text{ for } q_5 \leq \frac{2}{1 - \alpha};$$

provide $s \geq \frac{3}{2}(1 - \alpha)$.

The estimation of last term of the eq. (2.3),

$$\int_{\mathbb{R}} |R J^s u| dx \leq \|R\|_{L^{2/(1+\alpha)}} \|J^s u\|_{L^{2/(1-\alpha)}} \leq C \|R\|_{L^{2/(1+\alpha)}} \|u\|_{H^{s+\frac{\alpha}{2}}},$$

follows directly from Hölder’s inequality and the Sobolev imbedding $H^{\frac{\alpha}{2}} \hookrightarrow L^{2/(1-\alpha)}$. Following [6, 11] and using

$$H^{\frac{\alpha}{2}+\epsilon}(\mathbb{R}) \hookrightarrow L^{4/(1+\alpha)}, \quad \frac{4}{1+\alpha} \leq \frac{2}{1 - 2(s + \frac{\alpha}{2} + \epsilon)};$$

$$H^{s+\frac{\alpha}{2}-1-\epsilon}(\mathbb{R}) \hookrightarrow L^{4/(1+\alpha)}, \quad \frac{4}{1+\alpha} \leq \frac{2}{1 - 2(s + \frac{\alpha}{2} - 1 - \epsilon)}$$

one gets

$$\|R\|_{L^{2/(1+\alpha)}} \leq C \|J^{s-\epsilon} u\|_{L^{4/(1+\alpha)}} \|J^s u_x\|_{L^{4/(1+\alpha)}} \leq C \|u\|_{H^{s+\frac{\alpha}{2}}} \|u\|_{H^{s+\frac{\alpha}{2}}},$$

and finally

$$\int_{\mathbb{R}} |R J^s u| dx \leq C \|u\|_{H^{s+\frac{\alpha}{2}}}^3.$$
Choosing a suitable $\epsilon$, last restriction provides $s \geq s_2(1 - \alpha)$ as above. Combining the eqs. (2.4), (2.5) and (2.6), we have

$$\frac{d}{dt} \| J^{s+\frac{\alpha}{2}} u(.,t) \|_{L^2}^2 \leq C \| J^{s+\frac{\alpha}{2}} u(.,t) \|_{L^2}^{p+2}.$$ 

By a standard compactness method as in [6], we get the existence of a solution $u \in L^\infty(0, T, H^{s+\frac{\alpha}{2}}(\mathbb{R}))$ where $T = T(\|u_0\|_{H^{s+\frac{\alpha}{2}}}) > 0$.

We now prove the uniqueness of the corresponding solution. Let $u_1$ and $u_2 \in C([0, T], H^{s+\frac{\alpha}{2}}(\mathbb{R}))$ be two solutions of the eq. (1.1) then $v = u_1 - u_2$ satisfies the following equation

$$v_t + v_x + \frac{3}{4}D^\alpha v_x + \frac{5}{4}D^\alpha v_t + \frac{1}{2}(vg(u_1, u_2))_x = 0, \quad (2.7)$$

where $g(u_1, u_2) = u_1^p + u_1^{p-1}u_2 + \ldots + u_1u_2^{p-1} + u_2^p$. Multiplying both sides of the equation by $v$, integrating on the whole line we have

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}} (|v|^2 + \frac{5}{4}|D^\frac{\alpha}{2} v|^2) dx \right) + \frac{1}{2} \int_{\mathbb{R}} vv_x g(u_1, u_2) + \frac{1}{2} \int_{\mathbb{R}} v^2 g_x(u_1, u_2) = 0.$$ 

Using integration by parts, we get

$$\frac{d}{dt} \left( \int_{\mathbb{R}} (|v|^2 + \frac{5}{4}|D^\frac{\alpha}{2} v|^2) dx \right) \leq \frac{1}{2} \int_{\mathbb{R}} |v|^2 |g_x(u_1, u_2)| dx. \quad (2.8)$$

By the help of Hölder inequality and Sobolev imbedding theorem, the RHS of the eq. (2.8) can be estimated as

$$\frac{d}{dt} \| v \|_{H^{\frac{\alpha}{2}}}^2 \leq \frac{1}{2} \| v \|_{L^{q_6}}^2 \| g_x(u_1, u_2) \|_{L^{q_7}} dx \leq \frac{1}{2} \| v \|_{H^{\frac{\alpha}{2}}}^2 \| g(u_1, u_2) \|_{H^{s+\frac{\alpha}{2}}} dx,$$

where $\frac{2}{q_6} + \frac{1}{q_7} \leq 1$ and

$$H^{\frac{\alpha}{2}} \hookrightarrow L^{q_6}, \quad \text{for} \quad q_6 \leq \frac{2}{1 - \alpha},$$

$$H^{s+\frac{\alpha}{2}-1} \hookrightarrow L^{q_7}, \quad \text{for} \quad q_7 \leq \frac{2}{3 - 2s - \alpha},$$

for all $t \in [0, T]$. Using Gronwall’s lemma, we get

$$\| v \|_{H^{\frac{\alpha}{2}}}^2 = \| u_1 - u_2 \|_{H^{\frac{\alpha}{2}}}^2 \leq \exp(CK)\| u_1(x, 0) - u_2(x, 0) \|_{H^{\frac{\alpha}{2}}}^2,$$
where
\[ K = \int_0^T \|g(u_1, u_2)\|_{H^{s+\frac{1}{2}}} dt. \]

Finally we conclude uniqueness result of Theorem (2.1) by choosing \( u_1(x, 0) = u_2(x, 0) \).

3. Conserved Quantities

In this section, we derive the conserved quantities of the gfBBM eq. (1.1) for the smooth enough solutions which tend to 0 as \( x \to \mp \infty \). The equation is rewritten in the conservative like form

\[ u_t + \partial_x (I + \frac{5}{4} D^\alpha)^{-1} [u + \frac{1}{2} u^{p+1} + \frac{3}{4} D^\alpha u] = 0. \] (3.9)

Therefore, the first conserved integral is

\[ I_0 = \int \limits_{\mathbb{R}} u(x, t) dx. \] (3.10)

Multiplying the eq. (1.1) by \( u \) and integrating on the whole line, we have

\[
\int \frac{u^2}{2} dx + \int \frac{u^2}{2} dx + \frac{1}{2} \int R \frac{p+1}{p+2} (u^{p+2})_x dx \\
+ \frac{3}{4} \int R uD^\alpha u_x dx + \frac{5}{4} \int R uD^\alpha u_t dx = 0.
\] (3.11)

Thanks to the Plancherel theorem, one can write

\[
\int \frac{u^2}{2} dx = \int |\xi|^{\alpha} \hat{u}_t(\xi, t) \hat{u}(\xi, t) d\xi, \\
= \frac{1}{2} \frac{d}{dt} \int |D^\frac{\alpha}{2} u|^2 dx.
\] (3.12)

Rewriting the eq. (3.11) in the form

\[
\frac{1}{2} \frac{d}{dt} \int (u^2 + \frac{5}{4} |D^\alpha u|^2) dx + \frac{1}{2} \int (u^2 + \frac{p+1}{p+2} u^{p+2} + \frac{3}{4} |D^\alpha u|^2)_x dx = 0
\]
allows us to write a second conserved integral

\[ I_1 = \int_{\mathbb{R}} \left(u^2 + \frac{5}{4} |D^\frac{\alpha}{2} u|^2\right) dx \]  

(3.13)

for any \( \alpha \). Therefore, the Cauchy problem for the eq. (1.1) admits a unique global weak solution in \( L^\infty(\mathbb{R}, H^\frac{\alpha}{2}(\mathbb{R})) \).

The equivalent form for the eq. (3.9) gives us the Hamiltonian formulation

\[ \partial_t u + J_\alpha \nabla u H(u) = 0. \]

Here, skew-adjoint operator \( J_\alpha \) is

\[ J_\alpha = \partial_x (I + \frac{5}{4} D^\alpha)^{-1}, \]

and the Hamiltonian is

\[ H(u) = \frac{1}{2} \int_{\mathbb{R}} \left(u^2 + \frac{u^{p+2}}{p+2} + \frac{3}{4} |D^\frac{\alpha}{2} u|^2\right) dx. \]  

(3.14)

The Sobolev Imbedding \( H^{\alpha/2} \hookrightarrow L^{p+2} \) yields that \( H(u) \) is well-defined for \( \alpha \geq \frac{p}{p+2} \).

4. Solitary Wave Solutions

To find the localized solitary wave solutions of the eq. (1.1), we use the ansatz \( u(x,t) = Q_c(\xi), \ \xi = x - ct \) with \( \lim_{|\xi| \to \infty} Q_c(\xi) = 0 \) which leads to the ordinary differential equation

\[ -c Q'_c + Q'_c + \frac{1}{2} (Q_c^{p+1})' + \frac{3}{4} D^\alpha Q'_c - \frac{5}{4} D^\alpha Q_c' = 0. \]

Here ‘ denotes the derivative with respect to \( \xi \). Integrating the above equation, we have

\[ \left(\frac{5}{4}c - \frac{3}{4}\right) D^\alpha Q_c + (c - 1)Q_c - \frac{1}{2} (Q_c)^{p+1} = 0. \]  

(4.15)

The following theorem shows the non-existence of the nontrivial solutions of the eq. (4.15) for some values of \( \alpha, p \) and \( c \).
Theorem 4.1. Assume that one of the following cases

i. \( c \in (\frac{3}{5}, 1) \) and \( \alpha \geq \frac{p}{p+2} \),

ii. \( c \not\in (\frac{3}{5}, 1) \) and \( \alpha \leq \frac{p}{p+2} \),

iii. \( c = \frac{3}{5} \) or \( c = 1 \),

is satisfied. Then, the eq. (4.15) does not admit any nontrivial solution \( Q_c \in H^\frac{\alpha}{2}(\mathbb{R}) \cap L^{p+2}(\mathbb{R}) \).

Proof: Let \( Q_c \) be any nontrivial solution of the eq. (4.15) in the class \( H^\frac{\alpha}{2}(\mathbb{R}) \cap L^{p+2}(\mathbb{R}) \). Multiplying the eq. (4.15) by \( Q_c \) and integrating on \( \mathbb{R} \), we get

\[
\left( \frac{5}{4} c - \frac{3}{4} \right) \int_\mathbb{R} Q_c D^\alpha Q_c dx + (c - 1) \int_\mathbb{R} Q_c^2 dx = \frac{1}{2} \int_\mathbb{R} Q_c^{p+2} dx. \tag{4.16}
\]

By using the Plancherel’s formula

\[
\int_\mathbb{R} Q_c D^\alpha Q_c dx = \int_\mathbb{R} |\xi|^\alpha \hat{Q}_c(\xi) \overline{\hat{Q}_c(\xi)} d\xi = \int_\mathbb{R} |D^\frac{\alpha}{2} Q_c|^2 dx,
\]

(4.16) becomes

\[
\left( \frac{5}{4} c - \frac{3}{4} \right) \int_\mathbb{R} |D^\frac{\alpha}{2} Q_c|^2 dx + (c - 1) \int_\mathbb{R} Q_c^2 dx = \frac{1}{2} \int_\mathbb{R} Q_c^{p+2} dx. \tag{4.17}
\]

On the other hand, multiplying the eq. (4.15) by \( xQ'_c \) and integrating over \( \mathbb{R} \), we have

\[
\left( \frac{5}{4} c - \frac{3}{4} \right) \int_\mathbb{R} xQ'_c D^\alpha Q_c dx + (c - 1) \int_\mathbb{R} xQ'_c Q_c dx - \frac{1}{2} \int_\mathbb{R} xQ'_c Q_c^{p+1} dx = 0. \tag{4.18}
\]

Later, the equality

\[
\int_\mathbb{R} xQ'_c D^\alpha Q_c dx = \frac{\alpha - 1}{2} \int_\mathbb{R} |D^\frac{\alpha}{2} Q_c|^2 dx, \tag{4.19}
\]

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in \[1\] and several integration by parts turns the eq. (4.18) into the Pohozaev type identity
\[
\left(\frac{5}{4}c - \frac{3}{4}\right)\frac{1-\alpha}{2} \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 dx + \frac{c-1}{2} \int_{\mathbb{R}} Q_c^2 dx - \frac{1}{2(p+2)} \int_{\mathbb{R}} Q_c^{p+2} dx = 0.
\]  
(4.20)

Finally, by gathering the eqs. (4.17) and (4.20), we obtain
\[
\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 dx = \frac{4p(c-1)}{(5c-3)[\alpha(p+2)-p]} \int_{\mathbb{R}} Q_c^2 dx
\]
and the proof of the theorem follows directly from the positivity of the left hand integral.

Combining the results of Theorem 4.1 and the condition \(\alpha \geq \frac{p}{p+2}\) which ensures that the Hamiltonian (3.14) is well-posed for \(0 < \alpha < 1\) we conclude that in order to have a non-trivial solution we must have \(\alpha \geq \frac{p}{p+2}\) and \(c < 3/5\) or \(c > 1\).

On the other hand if we assume that the solution \(Q_c\) is positive and \(c < 3/5\) then the eq. (4.16) gives a contradiction while the RHS of the equation is positive and the LHS is negative. Therefore we are able to say that the eq. (4.15) has no positive solitary wave solutions unless \(c > 1\).

In order to show the existence and uniqueness of the solitary wave solutions, we recall the results of Frank and Lenzmann [10].

**Definition 4.2.** (Definition 2.1 of [10], Definition 1.1 of [5]) Let \(Q \in H^{\alpha/2}(\mathbb{R})\) be an even and positive solution of the equation
\[
D^{\alpha} Q + Q - Q^{p+1} = 0.
\]  
(4.21)

If
\[
J^{\alpha,p}(Q) = \inf \{ J^{\alpha,p}(u) | u \in H^{\alpha/2}(\mathbb{R}) \backslash \{0\} \}
\]  
(4.22)
then \(Q \in H^{\alpha/2}(\mathbb{R})\) is a ground state solution of the eq. (4.21) where \(J^{\alpha,p}\) is the Weinstein functional defined by
\[
J^{\alpha,p}(u) = \left( \int_{\mathbb{R}} |u|^{p+2} dx \right)^{-1} \left( \int_{\mathbb{R}} |D^{\alpha/2} u|^2 dx \right)^{p/2\alpha} \left( \int_{\mathbb{R}} |u|^2 dx \right)^{p(\alpha-1)/2\alpha+1}.
\]

The scaling
\[
Q_c(\xi) = (2(c-1))^{1/p} Q \left( \frac{4(c-1)}{5c-3} \right)^{1/\alpha} \xi
\]

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converts the eq. (4.15) into the eq. (4.21). In Proposition 1 and Theorem 4 of [10], Frank and Lenzmann prove the existence and uniqueness of the positive ground state solutions of the eq. (4.21) when $0 < \alpha < 2$ and $0 < p < p_{\text{max}}$ holds, where the critical exponent is defined as

\[
p_{\text{max}}(\alpha) = \begin{cases} 
  \frac{2\alpha}{1-\alpha}, & \text{for } 0 < \alpha < 1 \\
  \infty, & \text{for } 1 \leq \alpha < 2.
\end{cases}
\] (4.23)

Therefore, the eq. (4.15) has a unique positive ground state solution \(Q_c \in H^{\alpha/2}(\mathbb{R})\) for \(c > 1\) and \(\alpha > \frac{p}{p+2}\). That is why in the following section, we choose the parameters \(c, \alpha\) and \(p\) satisfying these conditions to obtain positive solitary wave solutions, numerically.

5. Numerical results for gfBBM Equation

In this section, we discuss the numerical solutions of the gfBBM equation. We first construct the solitary wave solutions by using the Petviashvili’s iteration method and then we investigate time evolution of the constructed solutions by a Fourier pseudo-spectral method.

5.1. Numerical Generation of Solitary Waves

Since we do not know the analytical solitary wave solutions of the eq. (1.1) for any \(\alpha \in (0, 1)\), we use the Petviashvili’s iteration method [12, 13, 14, 15] to construct the solitary wave solution numerically. The solitary wave solution of the gfBBM equation satisfies the eq. (4.15). Applying the Fourier transform to the eq. (4.15) yields

\[
\hat{Q}_c(k)[\frac{5c}{4} - \frac{3}{4}|k|^\alpha + c - 1] = \frac{1}{2} \hat{Q}_c^{p+1}(k).
\]

We propose standard iterative algorithm in the form

\[
\hat{Q}_{n+1}(k) = \frac{1}{2[\frac{5c}{4} - \frac{3}{4}|k|^\alpha + c - 1]} \hat{Q}_n^{p+1}(k),
\] (5.24)

where \(Q\) is used instead of \(Q_c\) for simplicity. The condition \(c > 1\) guarantees the non-resonance condition \((\frac{5c}{4} - \frac{3}{4}|k|^\alpha + c - 1) \neq 0\) for any \(k \in \mathbb{R}\). The main idea in the Petviashvili method is to add a stabilizing factor into the
fixed-point iteration in order to prevent the iterated solution to converge to zero solution or diverge. Then, the Petviashvili method for the gfBBM eq. is given by

\[
\hat{Q}_{n+1}(k) = \frac{M^n}{2[\left\{\left(\frac{5c}{4} - \frac{3}{4}\right)k^\alpha + c - 1\right\}]} \hat{Q}^{p+1}(k) \tag{5.25}
\]

with stabilizing factor

\[
M^n = \frac{\int \left[\left(\frac{5c}{4} - \frac{3}{4}\right)k^\alpha + c - 1\right]\hat{Q}_n(k)^2 dk}{\int \frac{1}{2}\hat{Q}^{p+1}(k)\hat{Q}_n(k) dk},
\]

for some parameter \(\nu\). Pelinovsky et. al. in \[12\] showed that the fastest convergence occurs when \(\nu = (p + 1)/p\). Therefore to reduce the CPU time, we use \(\nu = (p + 1)/p\) for the rest of the paper. The Fourier pseudo-spectral method is employed to implement the scheme \(5.25\). The MATLAB functions “fft” and “ifft” compute the discrete Fourier transform and its inverse for any function \(f(x)\) by using efficient Fast Fourier Transform. We note that the Petviashvili iteration method can also be used for approximating the periodic waves \[15\].

A solitary wave solution \(u(x, t) = Q_c(x - ct)\) of the equation

\[
u_t + uu_x - D^\alpha u_x = 0 \tag{5.26}
\]

satisfies the ODE

\[D^\alpha Q_c + cQ_c - \frac{1}{2}Q_c^2 = 0. \tag{5.27}\]

The eq. \(5.27\) has the solution

\[
Q_c(x, t) = \frac{4c}{1 + c^2(x - ct)^2} \tag{5.28}
\]

for \(\alpha = 1\) in \[8\]. By the convenient scaling, the exact solitary wave solution of gfBBM eq. can be written as

\[
Q_{\text{exact}}(x, t) = \frac{4(c - 1)}{1 + \left(\frac{4(c - 1)}{5c - 3}\right)^2(x - ct)^2} \tag{5.29}
\]

for \(\alpha = 1\) and \(p = 1\).
Figure 1: Difference between the exact and the numerical solutions for $\alpha = 1$, $p = 1$ and the variation of the $\text{Error}(n)$, $|1 - M_n|$ and $\text{RES}$ with the number of iterations in semi-log scale.

In the first numerical experiment, we test our scheme by comparing the numerical result with the exact solution. The space interval and number of grid points are chosen as $x \in [-2048, 2048]$ and $N = 2^{18}$, respectively. The overall iterative process is controlled by the error,

$$\text{Error}(n) = \|Q_n - Q_{n-1}\|, \quad n = 0, 1, \ldots$$

between two consecutive iterations defined with the number of iterations, the stabilization factor error

$$|1 - M_n|, \quad n = 0, 1, \ldots$$

and the residual error

$$\text{RES}(n) = \|SQ_n\|_\infty, \quad n = 0, 1, \ldots$$
where
\[
S Q = \left( \frac{3}{4} - \frac{5c}{4} \right) D^\alpha Q - (1 - c)Q - \frac{1}{2}Q^{p+1}.
\]
\[ (5.30) \]

In Figure 1, we present the difference between the obtained numerical and exact solitary wave solution and the variation of three different errors with the number of iterations in semi-log scale. As it is seen from the Figure 1, our proposed numerical scheme captures the solution remarkably well.

In Figure 2, we illustrate the solitary wave profiles generated by Petviashvili’s iteration method for various values of $\alpha$ and for $c = 1.1$, $p = 1$ in Figure 2. We observe that the solution becomes more and more peaked with decreasing values of $\alpha$.

Since we do not know the exact solitary wave solution for different values of $\alpha$, we cannot compare the numerical solution with the exact solution. Therefore, the iteration, stabilization factor and the residual errors are depicted in Figure 3, respectively, for $\alpha = 0.4$, $\alpha = 0.6$ and $\alpha = 0.8$. These results show that the solitary wave solution generated by Petviashvili’s method converges rapidly to the accurate solution. In order to understand the effects of the nonlinearity, we present the solitary wave profiles generated by Petviashvili’s iteration method for various nonlinearities with fixed $\alpha = 0.8$ in Figure 4. This numerical result agrees well with the fact that the wave steepens with increasing nonlinear effects.
Figure 3: The variation of the iteration, stabilization factor and the residual errors with the number of iterations in the semi-log scale ($c = 1.1$, $p = 1$).

Figure 4: Solitary wave profiles for various nonlinearities ($\alpha = 0.8$, $c = 1.1$).

In Figure 5, we illustrate the variation of the amplitude with the speed parameter for various values of $p$ and the fixed $\alpha = 0.8$ and various values of $\alpha$ and fixed nonlinearity $p = 1$. 

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Figure 5: Speed and amplitude relation for various values of $p$ and the fixed $\alpha = 0.8$ (right panel) and for various values of $\alpha$ and fixed nonlinearity $p = 1$ (left panel).

5.2. The Fourier-pseudospectral method for gfBBM Equation

In this subsection, we propose a Fourier pseudo-spectral method for the time evolution of numerically generated solitary waves. Since the fractional derivative in the gfBBM equation is defined by a Fourier multiplier, the Fourier spectral method will be the most appropriate method for investigating the evolution of the solution in time. We assume that $u(x,t)$ has periodic boundary condition $u(-L,t) = u(L,t)$ on the truncated domain $(x,t) \in [-L,L] \times [0,T]$.

In order to use the MATLAB functions “fft” and “ifft”, the spatial period is normalized from finite interval $x \in [-L,L]$ to $X \in [0,2\pi]$ using the transformation $X = \pi (x + L)/L$. In this case, the eq. (1.1) becomes

$$
\left[ I + \frac{5}{4}\left(\frac{\pi}{L}\right)^{\alpha} D^{\alpha}\right] u_t = -\frac{\pi}{L} u_X - \frac{\pi}{2L} (u^{p+1})_X - \frac{3}{4}\left(\frac{\pi}{L}\right)^{\alpha+1} D^{\alpha} u_X. \tag{5.31}
$$

The interval $[0,2\pi]$ is divided into $N$ equal subintervals with grid spacing $\Delta X = 2\pi/N$, where the integer $N$ is even. The spatial grid points are given by $X_j = 2\pi j/N$, $j = 0, 1, 2, ..., N$. The time interval $[0,T]$ is divided into $M$ equal subintervals with time step $\Delta t$. The temporal grid points are shown by $t_m = \frac{mT}{M}$, $m = 0, ..., M$. The discrete Fourier transform of the sequence
\{U_j\}, i.e.
\[
\tilde{U}_k = \mathcal{F}_k[U_j] = \frac{1}{N} \sum_{j=0}^{N-1} U_j e^{-i k X_j}, \quad -N/2 \leq k \leq N/2 - 1
\]  
(5.32)
gives the corresponding Fourier coefficients. Likewise, \{U_j\} can be recovered from the Fourier coefficients by the inversion formula for the discrete Fourier transform \(5.32\), as follows:
\[
U_j = \mathcal{F}_j^{-1}[\tilde{U}_k] = \sum_{\xi = -N/2}^{N/2 - 1} \tilde{U}_k e^{i k X_j}, \quad j = 0, 1, 2, ..., N - 1.
\]  
(5.33)

Here \(\mathcal{F}\) denotes the discrete Fourier transform and \(\mathcal{F}^{-1}\) its inverse. Applying the discrete Fourier transform to the eq. \(5.31\), we get the ordinary differential equation given by
\[
(\tilde{U}_k)_t = -\frac{\pi}{L} i k \tilde{U}_k - \frac{\pi}{2L} i k (\overline{U_{p+1}})_k - \frac{3}{4} \left(\frac{\pi}{L}\right)^{\alpha+1} i k |k|^\alpha \tilde{U}_k 
\]  
\[1 + \frac{5}{4} \left(\frac{\pi}{L}\right)^{\alpha} |k|^\alpha \].
(5.34)

We then use the fourth order Runge-Kutta method to solve the resulting ODE in time. Finally, we find the approximate solution by using the inverse discrete Fourier transform.

We first show that the numerical solution captures the exact solution \(5.29\) for \(\alpha = 1\) well enough. We use the initial data \(5.29\) with \(c = 1.1, \alpha = 1, p = 1\). The problem is solved in the space interval \(-2048 \leq x \leq 2048\) up to \(T = 20\). We set the number of grid points as \(N = 2^{18}, M = 4000\). Figure 6 illustrates variations of the conserved quantities \(I_0\) and \(I_1\) with time and shows that they are preserved by the numerical scheme.

Figure 7 shows the wave profile calculated by the Fourier pseudo-spectral method with time step \(\Delta t = 0.005\) at \(t = 10\) and \(t = 20\). To observe the wave profile more clear, we focus on the space interval \(-150 \leq x \leq 150\). We also show the change in the conserved quantities \(I_0\) and \(I_1\) in Figure 7. It is seen from the figure, the conserved quantities remain constant in time and this behavior provides a valuable check on the numerical results.
Figure 6: Variations of the conserved quantities $I_0$ and $I_1$ with time

Figure 7: The change in the conserved quantities $I_0$ and $I_1$ with time (upper panels) and time evolution of the wave profile ($c = 1.1$, $p = 1$, $\alpha = 0.6$) at several times (lower panel)
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