The patch like model of galaxies formation: the virial paradox, core–cusp and missing satellite problems

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(Dated: September 10, 2020)

The patch like model of the hierarchical galaxy formation in the ΛCDM cosmological model with small damping scale is considered. In this model galaxies and clusters of galaxies are identified with rare high density peaks, what suppresses the action of random factors in the vicinity of peaks and makes the process of halos formation more rapid and regular. High concentration of irregular subhalos surrounding the central peaks and their subsequent merging just after formation allows to consider this medium as a mixture of collisionless dispersed dark matter (DM) particles and collisional subhalos. Merging of these subhalos with the central dominating halo is accompanied by tidal destruction of the central cusp, what progressively shallows the density profile and promotes formation of super massive central black holes. The simulations 1–3 provide some quantitative characteristics of these processes.

In the framework of this model we can reproduce the observed correlation of mass and density of virialized galaxies and clusters of galaxies known as the virial paradox 4, 5. These correlations are closely linked with the composition of DM and the shape of the power spectrum of density perturbations what allows to restrict them using already available observations. In particular, these correlations put constraints on the HDM and WDM models and allow to test models of cosmological inflation. We confirm that the missing satellite problem is directly linked with the virial paradox and reheating of the Universe which increases temperature and entropy of the baryons, prevents formation of first stars and divides halos into two populations: the first one includes galaxies formed before reheating which are mainly concentrated in the vicinity of the massive ones while the second population – numerous dark halos formed after reheating — accumulates majority of DM but does not contain stars. Their spatial distribution is more homogeneous.

INTRODUCTION

During the last decade much progress has been achieved in observations of the cosmic microwave radiation (WMAP and Planck missions, 6, 7, see also 8) and simulations of the Large Scale Structure of the Universe, DM halos, galaxies and clusters of galaxies. Many recent reviews 9–18 present various aspects of these processes.

Now the main attention is concentrated on evolution of the baryonic matter and formation of observed luminous galaxies 13, 14, 25. However a few fundamental problems of galaxy formation remain unsolved and are now actively discussed. First of all these are the core–cusp and missing satellites problems. These problems arise when one compares the observed galaxies and the present day simulated DM halos. In the last few years unexpected discovery of the ultra diffuse galaxies 20, galaxies with a deficit of DM component 27, 28 and a progress in understanding of the Ly-α forest 4, 24, 31 complicates the problem of DM – baryons interconnection.

Present day high resolution simulations provide several representative samples of DM halos ranging from dwarf galaxies and up to rich clusters of galaxies. In majority of simulations 1, 3, 32 the density profile near the center of halos is cusp–like:

\[ \rho(r) \propto \rho_0/r^\alpha, \quad 1.5 \geq \alpha \sim 1. \]  (1)

This profile reproduces reasonably well the observed one in clusters of galaxies. However in less massive galaxies the density profile is more shallow 33, 35 and can be expressed by the relation 1 with the power index \( \alpha \simeq 0.4 \leq 1 \). This is the core-cusp problem. After 20 years of study discrepancies between theoretical and simulated models of CDM universe and observations of dwarf galaxies are still pronounced.

Now many models attempt to explain this problem. It has been suggested that the core-cusp conflict could be
resolved by fluctuations of the inner gravitational potential. The most popular explanation of these fluctuations relies on sudden removal of gas from centers of cuspy DM halos caused by energy injected by supernovae [32, 37]. Progressive disruption of the DM cusp owing to its tidal interaction with the gaseous clouds, stars and protostars is discussed in [38–41]. In [42] erosion of the cusp is related to accretion of a suitable spherical DM shells.

Of course, these factors lead to some cusp erosion, but a correct estimate of their efficiency is lacking. Recent analysis [14, 43] indicates insufficient variety of mass profiles to explain the observed diversity of dwarf galaxy rotation curves. In turn recent models [42] require a very special kind of DM accretion and conformity between masses of the cusp and shells.

Another possibility is to use a more complex dark matter model. It remains an open possibility that these tensions may point to exotic particle physics. Such models – the scalar field dark matter, Bose – Einstein condensate, or ultralight axion DM [44, 45] are identical to the CDM model at cosmological scales but differ at galactic scales. All these problems are also reviewed in [18] with more attention put on exotic particle physics.

However the simplest and the most promising models of the cusp disruption were discussed in [1, 3, 16, 17] where it was shown that the cusp becomes shallower in the earlier formed low mass DM halos owing to merging of subhalos. This means that the core–cusp problem is mostly a result of insufficient resolution of present day simulations and it disappears in CDM models with a sufficiently small dissipative scale.

In the standard ΛCDM model gravitationally bound DM structures build up hierarchically by a combination of accretion of the diffused surrounding matter and continuous absorption of smaller surrounding halos [17–22]. During the period of mildly nonlinear matter evolution the formation of structure elements is driven by the random velocity field, and at all redshifts it leads to significant matter concentration in filaments and sheets [48–55]. These elements of the structure represent the intermediate asymptotic of the matter condensation and are observed as the Large and Super Large Scale Structure [53]. Later on some fraction of this matter is accumulated into compact halos. Formation of DM halos is nicely described by the Press–Schechter model (PS) [54–56].

In this paper we reconsider the process of DM halos formation in the framework of the ΛCDM model with the power spectrum of density perturbations with a small scale damping [60]. In this model galaxies are associated with the very rare random highest peaks of density perturbations surrounded by many smaller peaks in the immediate vicinity of the central one. These special features lead to a very rapid regular growth of mass of the main halo and progressive tidal disruption of both the absorbed subhalos and the central cusp.

This is the patch like process of formation of massive DM halos when at high redshifts the active creation of new halos is concentrated only in the vicinity of the central peak. Later on at $z \leq 10$ many DM halos are formed in all of space. This model preserves the main features of the usually discussed models of galaxy formation and the large scale matter distribution but the internal structure of early formed halos is more shallow.

Owing to limited resolution, the early period of halos formation is poorly reproduced by present day simulations [1, 3] and for qualitative estimates we have to rely on the Press–Schechter approach [3, 57–59]. Such analysis demonstrates rapid acceleration of growth of halos at high redshifts $z \geq 10$ and very important role of merging of earlier formed subhalos. Traces of these processes are seen in some present day simulations and are discussed in [12, 17, 61].

As was shown in [5] for objects with virial masses $10^5 M_\odot \leq M_{vir} \leq 10^{14} M_\odot$ the virial density is a regular function of the mass. For galaxies this correlation is traced up to redshift $z \sim 4$ [4], but it fades for both the observed and simulated low mass DM halos [1, 3]. This property of DM halos – the virial paradox – is also reproduced by the considered model and allows to restrict the shape of the small scale power spectrum. It can be useful for discussion of the cosmological inflation and puts restrictions on the WDM models with light DM particles.

Thus considered here patch like model provides satisfactory description of the observed Universe and attenuates differences between properties of the observed and simulated matter distributions. It leads to more shallow internal structure of halos, introduces differences between galaxies and later formed dark halos and thereby explains both the virial paradox and the missing satellite problem. Simple limited versions of this model have been simulated in [11, 14, 17, 62, 64]. Further progress can be achieved with special more refined and representative simulations [55].

This paper is organized as follows: the basic properties of the PS and Zel’dovich approaches are discussed in Sec. 2 & 3, some aspects of the process of halos formation in the patch like model are discussed in Sec. 4. Conclusions can be found in Sec. 5. Statistical characteristics of the Zel’dovich approach are presented in the Appendix.

2. BASIC PARAMETERS OF THE ΛCDM MODEL

The standard ΛCDM model assumes the isotropic matter expansion with the Hubble constant $H_0$, adiabatic density perturbations with the Harrison – Zel’dovich primordial power spectrum $P(k)$, the dimensionless densities of dark energy, $\Omega_M$, dark matter, $\Omega_{DM}$ and baryonic matter, $\Omega_b$. The density of nonrelativistic matter, DM, and baryons together, is determined as $\Omega_m = \Omega_{DM} + \Omega_b$. Observations of Planck [7] allowed to measure these pa-
parameters with high precision

\[ H^2 = H_0^2[(1 + z)^3\Omega_m + \Omega_\Lambda], \quad H_0 \simeq 67.8 \text{km/s/Mpc} , \]
\[ \Omega_\Lambda \simeq 0.72, \quad \Omega_{DM} \simeq 0.24, \quad \Omega_b \simeq 0.04, \quad \Omega_m = 0.28 . \quad (2) \]

Here \( z \) is redshift and the density of nonrelativistic matter

\[ \langle \rho_m \rangle = 33(1 + z)^3\Theta_m M_\odot/kpc^3, \quad \Theta_m = \Omega_m/0.28 . \quad (3) \]

For this model the growth of perturbations in the linear theory can be approximately described as

\[ D(z \geq 1) \approx \frac{1.3}{1 + z}, \quad \beta = \frac{1 + z \frac{dD(z)}{D(z)} dz}{D(z)} \approx 1. \quad (4) \]

This simple fit is reasonably accurate for more interesting case \( z \geq 1 \). More refined expression normalized by the condition \( D(0) = 1 \) can be found in [59].

2.2 Characteristics of the random density and velocity fields

In this paper we consider the power spectrum with the Harrison – Zel’dovich asymptotic, \( P(k) \propto k \), at \( k \to 0 \), and CDM-like transfer function, \( T^2(k) \), introduced in [65]

\[ P(k) = \frac{A^2}{4\pi^3} k^2 T^2(k_0) D_w(kl_D) , \quad (5) \]

\[ l_0 = \frac{Mpc}{\Omega_m h^2} \simeq 7.14 \frac{\Theta_m Mpc}{\Theta_m} , \quad M_0 = \frac{4\pi}{3} (\rho_m)^0 5 \cdot 10^{13} M_\odot . \]

Here \( k \) is the comoving wave number and \( A \) is the dimensionless amplitude of perturbations. The damping function \( D_w \) and the damping scale, \( l_D \), describe damping of perturbations owing to the random motions of DM particles. According to [60] (see also [63]) the damping mass \( M_D \) can be taken as

\[ M_D \simeq \frac{4\pi}{3} (\rho_m)^0 l_D \simeq 10^{-6} M_\odot . \]

For the power spectrum [58], the dispersion of the density perturbations is divergent and it is measured in units \( \sigma_8 \) which is the relative density perturbation, \( \delta \rho/\langle \rho \rangle \), in a sphere of radius \( R_8 = 8h^{-1} Mpc = 1.6l_0 \),

\[ \sigma_8^2 = \int_0^\infty d^3k P(k) W^2(R_8) \approx \frac{A^2}{236} = 0.64, \quad (6) \]

\[ A \approx 12, \quad W(x) = 3(\sin x - x \cos x)/x^3 . \]

Here, \( W(Rk) \) is the Fourier transform of the real space top-hat filter corresponding to a sphere of radius \( R \) and mass \( \mu = M/M_0 \).

In this model the amplitudes of displacement, \( \sigma_s \), and velocity, \( \sigma_u \), are

\[ \sigma_s^2 = \int_0^\infty d^3k P(k)/k^2 \approx (1.8l_0)^2, \]
\[ \sigma_u \simeq 13 Mpc, \quad \sigma_u = H_0 \sigma_s \simeq 900 \text{km/s} . \quad (7) \]

3. THE PRESS–SCHETZCHEK AND ZEL'DOVICH MODELS OF STRUCTURE FORMATION

The most popular description of evolution of perturbations is the linear theory, discussed in many publications. Unfortunately nonlinear studies of matter condensation can be described analytically only for special cases. In spite of the limited applicability of these models they allow to describe and illustrate the action of some factors that are important for the structure formation and evolution.

3.1 Extended Press – Schechter model

Evolution of spherical compact high density objects – DM halos, galaxies and clusters of galaxies – can be approximated by the Press–Schechter (PS) model [56–58]. This model considers the successive spherical halo formation around random density peaks. It assumes the Gaussian distribution function for the masses accumulated in a spherical volume of radius \( R \) and dispersion

\[ \sigma_m^2(\mu) = \int_0^\infty d^3k P(k) W^2(kR), \quad \mu = \frac{4\pi (\rho_m) R^3}{3 M_0} . \quad (8) \]

Formation of halos is determined by the condition

\[ D(z) A_{rnd} \sigma_m(M) = 1.686, \quad 1 + z \simeq 0.77 A_{rnd} \sigma_m(M) . \quad (9) \]

For the spectrum [58] with the transfer function from [65] and \( M_D \simeq 10^{-6} M_\odot \) the important function \( \sigma_m(\mu) \) can be fitted by the expression

\[ \sigma_m(\mu) \approx \frac{3\mu^{-0.06}}{1 + 1.82\mu^{0.24}} . \quad (10) \]

For the WDM model with cutoff at \( M_{mn} = 10^4 M_\odot \) the corresponding fit is

\[ \sigma_m(\mu) \approx \frac{11}{1 + 10\mu^{0.2}} . \quad (11) \]

Comparison of expressions [110] and [111] demonstrates the impact of small scale part of the power spectrum.

The redshift evolution of the fraction of compressed matter, \( f_m(z, M) \), is given by [58, 59]

\[ \frac{df_m(z, M)}{dM} = 0.37 \frac{d\mu}{dM} \exp(-y^2)[1 + 0.81/y^{0.6}] , \quad (12) \]

\[ f_m(z, M_{mn}) = 0.18\Gamma(0.5, y_{mn}) + 0.144\Gamma(0.2, y_{mn}) . \]

\[ y(z, M) \simeq 1/D(z)/\sigma_m(\mu), \quad y_{mn}(z, M_{mn}) = y(z, M_{mn}) . \]
Here $M_{\text{min}} \leq M \leq \infty$, $\Gamma(\beta, x)$ is the incomplete gamma function. For $y_{\text{min}} \simeq 0.25(1 + z)\mu_{\text{min}}^{0.06} \ll 1$ we get

$$f_m \simeq 1 - 0.27y_{\text{min}} - 0.65y_{\text{min}}^{0.4} + ..$$  \hspace{1cm} (13)

For small $M_{\text{min}}$ this fit correctly describes the function $f_m$ for $1 + z \leq 10$. For the mean mass of halo we have

$$\langle M(z, M_{\text{min}}) \rangle = \frac{1}{f_m(z, M_{\text{min}})} \int_{M_{\text{min}}}^{\infty} M df_m.$$  \hspace{1cm} (14)

### 3.2 The Zel’ dovich model

The first analytic theory of structure formation was provided by the Zeldovich approximation [48, 50]. This theory correctly describes the early anisotropic stage of matter condensation and formation of elements of the Large Scale Structure (LSS) of the Universe – network of filaments and walls – Superclusters (Zel’dovich pancakes) [49, 52, 54, 68]. At all redshifts these elements are formed in the course of mildly nonlinear self similar process of matter condensation.

In the Zel’dovich approximation the Eulerian, $r_i$, and the Lagrangian, $q_i$, coordinates of particles (fluid elements) and their velocities are related by

$$r_i = (1 + z)^{-1}[q_i - D(z)S_i(q)],$$  \hspace{1cm} (15)

$$v_i = dr/dt = H(z)(1 + z)^{-1}[q_i - D(z)\beta(z)S_i(q)],$$

$$\beta(z) = 1 - d \ln D(z)/d \ln (1 + z).$$

Here the Lagrangian coordinates of a particle, $q_i$, are its unperturbed coordinates in the real space, $r_i(z = 0) = q_i$, $v_i$ is the velocity of particle, and the random vector $S_i(q)$ characterizes the shift of particle from the unperturbed position. The function $D(z)$ is given by [4].

The statistical aspects of this theory had been discussed in [49, 51, 53, 67, 68] and are briefly presented in the Appendix. The random displacement of a particle $S_i(q)$ is described by the Gaussian distribution function with the correlation functions

$$\Psi_{ij}(q) = \frac{(S_i(q_1)S_j(q_2))}{\sigma_s^2} = \int_0^{\infty} dk \frac{k_i k_j}{k^2} \frac{P(k)W(kq)}{\sigma_s^2},$$  \hspace{1cm} (16)

$$\Psi_{ij}(q) = \frac{1}{3} \delta_{ij}G_1(q) + \frac{q_i q_j}{3}G_2(q), \hspace{0.5cm} \Psi_{ij}(0) = \frac{1}{3} \delta_{ij},$$

$$G_1(q) = \frac{4\pi}{\sigma_s^2} \int_0^{\infty} dk W(qk)P(k), \hspace{0.5cm} G_2(q) = \frac{dG_1(q)}{dq}.$$

### 3.3 Angular momentum of DM halos

The Zel’dovich theory predicts an asymmetrical collapse what decreases the matter compression and generates the angular momentum for both the separate particles and for halos as a whole. This problem was discussed in [53, 68, 73].

The transverse components of the velocity are characterized by the functions

$$u_i(q) = H(z)e_i k q_j S_j(q).$$  \hspace{1cm} (18)

FIG. 1: For spectrum \([5]\) the functions $\sigma_m$, with $\mu \geq 10^{-11}$, and $\mu \geq 10^{-9}$, and function $\sqrt{-G_2}$ are plotted vs. dimensionless mass $\mu$ by solid, dashed and long dashed lines.
\[ H^{-2}(u_1(q)u_1(p)) = (q_2p_2 + q_3p_3)G_1 + (q_2p_3 - q_3p_2)^2G_2, \]
\[ H^{-2}(u_1(q)u_2(p)) = -q_2p_1G_1 + (q_1p_3 - q_3p_1)(q_3p_2 - q_2p_3)G_2, \]
\[ G_1 = G_1(p - q), \quad G_2 = G_2(p - q), \]
and similar relations for other indexes. Thus we get for the angular momentum of one particle
\[ p = q, \quad u^2(q) \propto 2q^2 \propto \mu^{2/3}. \] (19)

For a halo the angular momentum is determined by an integral over the halo and for a symmetrical volume we get
\[ J_1^2 = H^2 \int dq dp (q_2p_3 - q_3p_2)^2 G, \] (20)

4. PATCH LIKE GALAXY FORMATION IN THE ΛCDM MODEL

In the standard ΛCDM model a gravitationally bound DM structure is built up hierarchically by a combination of accretion of the diffuse surrounding matter and sequential merging of subhalos [17, 19, 22, 61]. In the ΛCDM model with a small damping scale galaxies can be identified with the very rare high density peaks while between them the low amplitude perturbations evolve in the linear regime. This difference results in a patch-like galaxy formation which is operating up to small redshifts.

The massive halos accumulate many less massive subhalos, which could contain in turn smaller subhalos. High concentration of subhalos near the highest peaks accelerates this merging as compared with accretion of the dispersed DM particles. In turn tidal interactions of the merged loose subhalos with the central cusp of the major halos leads to destruction of subhalos, flattening of the cusp [1, 13, 17, 61], and facilitates formation of super massive black holes. Efficiency of these random processes depends upon the peak amplitude: it is high at high redshifts and decreases with time. On the basis of the present-day simulations the process of halos formation, the important role of subhalos, their tidal disruption and heating etc. are discussed in [12, 17].

Unfortunately, these simulations cannot describe evolution of a box \( L \geq 10 Mpc \) with mass resolution \( M_{\text{min}} \leq M_{\odot} \) and the noticeable patch like halos formation at redshifts \( z \geq 30 - 50 \). Thus to illustrate evolution of DM halos we have to use the Press-Schechter formalism and compare its predictions with several numerical simulations [1, 3].

4.1 Characteristics of relaxed halos

In spite of active discussions the adequate description of the violent relaxation of DM halos is not yet available. Detailed dynamical analysis of this process was performed in [76, 77] for slightly perturbed spherical clouds. The density profile [1] with

\[ \alpha \simeq 1.8 - 2, \]

was found in both publications. However this result has a very limited applicability as the spherical collapse is very rare [51]. At high redshifts the simulations [1, 3, 17] prefer the density profile with \( \alpha \simeq 1.5 \).

Efficient method of identification of the distinct virialized elliptical halos has not been proposed yet. The popular phenomenological description of the virial density

\[ \rho_{\text{vir}} \simeq 18\pi^2 \langle \rho(z) \rangle \simeq 6.6 \cdot 10^3 (1 + z)^3 M_\odot/kpc^3, \] (21)

provides rough estimate for spherical systems, but it overestimates \( \rho_{\text{vir}} \) for the most abundant elliptical systems.

The estimate (21) is based on two assumptions. Firstly, according to [78, 79], collapse of a spherical dust cloud at rest of radius \( R_0 \) and density \( \rho_0 \) with conservation of mass, \( M \), and energy, \( E \), results in a virialized state

\[ E = -\frac{3}{5} G M^2 \rho_0 = \frac{1}{2} U = -\frac{3}{5} G M_{\text{vir}}^2 \rho_{\text{vir}}. \]

Here \( U \) is the potential energy of the halo and, therefore,

\[ R_{\text{vir}} = R_0/2, \quad \rho_{\text{vir}} = 8 \rho_0. \] (22)

Secondly, at the moment of collapse the average density of a spherical halo exceeds the mean density [80]

\[ \rho_0 = 4.5 \pi^2 \langle \rho(z) \rangle. \]

what in combination with (22) results in expression (21).

This estimate remains correct for the Tolmen model of evolution of a spherical dust cloud [51], but for deviations from spherical symmetry, such as ellipsoidal deformations [82, 83], the energy \( E \) decreases with compression of DM.

The kinetic energy of rotation and turbulent motions also decrease the expected density (21). Thus the expression (21) can be considered only as an approximate phenomenological estimate of the complex process of relaxation of collisionless DM. Owing to its approximate character the coefficient 500 is often used in expression (21) instead of the theoretical coefficient \( 18\pi^2 \).

Moreover the expression (21) leads to an unexpected inference that the virial density depends only upon redshift and, therefore, it is the same for all halos at a given redshift. The wide variety of observed and simulated relaxed halos at \( z \leq 1 \) indicates that the actual situation is
more complex, and it is necessary to restrict such universality. For this purpose it is convenient to introduce the redshift of halo formation, \( z_{24} \) and to use \( z_{21} \) with \( z_{cr} \). Such a modified version of \( z_{24} \) agrees with the model of Lacey and Cole [80].

Next problem is the correct determination of the shape and boundary of galaxies and clusters [51, 52, 88] and determination of halos density in observations and simulations. Thus, very detailed analysis of halo evolution [59] is performed without consideration of possible anisotropy of matter distribution. This factor is specially important for earlier halos, shapes of which are often close to flattened ellipsoid (Zel’dovich pancakes) [51]. Dependence of the halo parameters upon its internal structure [89–91] and/or complex environment [92] can be discussed separately.

### 4.2 DM halos as counterparts of galaxies

Some information about the process of galaxy formation can be obtained using the standard technique developed for description of evolution of random density and velocity fields [2, 13, 53, 65, 67, 68]. In order to identify density peaks with galaxies and clusters of halos we can compare their mean number densities \( \langle n_{cls} \rangle \) and \( \langle n_{pk} \rangle \).

In the SDSS for the observed clusters of galaxies with \( M_{cls} \geq 10^{13} M_\odot \), the mean number density \( \langle n_{cls} \rangle \) and the mean cluster separation, \( \langle d_{cls} \rangle \), are estimated as [92]

\[
\langle n_{cls} \rangle \sim 10^{-5} Mpc^{-3}, \quad \langle d_{cls} \rangle \sim 45 Mpc \sim 6 l_0 .
\]  

(23)

In turn for the power spectrum [5] the mean number density of high peaks is determined by the scale \( l_0 \). This means that the clusters are associated with only a small fraction of the high density peaks.

As was shown in [63] the cumulative number density of high peaks of a random scalar field can be roughly estimated as

\[
\langle n_{pk} \rangle \approx \frac{10^{-2}}{l_0^3} \exp \left( - \frac{A_{25}^2}{2} \right) \approx n_{cls},
\]  

(24)

where \( A_{25} \) is the peak amplitude. This means that clusters with parameters [23] are identified with the peaks with the amplitude

\[
A_{25} \sim \sqrt{2 \ln \langle n_{pk} / \langle n_{cls} \rangle \rangle} \approx 1.4 .
\]  

(25)

For comparison, assuming cosmological origin of Super Massive Black Holes, we can estimate the corresponding peak amplitude for \( M_{BH} \sim 10^9 M_\odot \) as [94]

\[
n_{BH} \sim 3 \cdot 10^{-9} Mpc^{-3}, \quad A_{25} \sim 4.2 .
\]  

(26)

For galaxies the random amplitude \( A_{25} \) can be estimated from [9] and the assumption that the reionization of the Universe at redshifts \( 1 + z \sim 10 \) is caused by halos with mass \( M \sim 10^6 M_\odot \), \( \mu \sim 2 \cdot 10^{-5} \). These parameters correspond to \( \sigma_m(\mu) \sim 0.7 \)–0.8 and thus \( A_{25} \sim 1 \)–1.2. More accurate estimate of the peak’s amplitude associated with galaxies – \( A_{25} \sim 1 \)–2 will be given in the next subsection.

Of course, this picture predicts formation of many low mass halos in the vicinity of the main halo in a wide range of scales. This successive formation of many low mass DM subhalos continues up to small redshifts. In turn, large scale perturbations regulate the spatial distribution of low mass halos, provide their higher concentration in the immediate vicinity of the central peak and regulate the further transformation of the system of subhalos into the distinct massive objects – galaxies, filaments or sheet–like superclusters.
and the maximal value is achieved for $\mu_{\text{max}} \approx 0.03$. This value exceeds the observed one by a factor $\sim 10$ and the shape of this function differs from (27).

Differences between the observed function $G_{\rho}^{\text{obs}}(\mu)$ and $G_{\rho}^{\text{th}}(\mu)$ can be reduced either by increasing the observed virial mass of galaxies (by a factor $\sim 10$) or by increase of the amplitude of small scale parts of the power spectrum $\mathcal{P}$. Its large scale shape and amplitude are fixed by observations of the relic microwave radiation [6, 7] (see also [8]), but we can change the small scale shape of the power spectrum by introduction of a correction function

$$
\psi_{\text{cor}}(q) = 1 + \frac{q^2}{1 + a_c q^2}, \quad q = k l_0, \quad a_c \simeq 0.2. \quad (30)
$$

This function increases the amplitude of the small scale power spectrum by the factor of $1/a_c$ and retains its shape. For so corrected power spectrum we have

$$
\sigma_{\text{m}}(\mu) \simeq \frac{5\mu^{-0.066}}{1 + 3\mu^{0.3}}, \quad (31)
$$

and for the reduced virial density

$$
G_{\rho}^{\text{th}}(\mu) \simeq \frac{5 A_{nd}^{0.3}}{(1 + 3\mu^{0.3})^2}. \quad (32)
$$

Comparison (27) and (32) shows that for the amplitude $A_{nd} \sim 2$ these functions become identical.

This approach is more sensitive for the small scale power spectrum but its reliability is not very high. Indeed, it is based on a limited statistic, observations are performed with limited precision, the theoretical base is the phenomenological PS model. Non the less these results confirm that for galaxies and clusters of galaxies the correlation of the virial density and the virial mass is a natural result for the corrected power spectrum (5,30).

One would expect that this approach will be also useful for models of inflation [97, 98], and, in particular, it favors more complex models of inflation. It also allows to restrict parameters of WDM models and properties of hypothetical exotic DM particles [18, 44, 45].

Similar links between the halos masses and sizes are found for 160 systems of metal lines observed in absorption spectra of quasars [4], for 30 massive clusters in simulations [97] and for three clusters of galaxies at $z \approx 0.4$ [57]. However for simulated low mass halos the reduced virial density is much smaller than (27). Thus for both halos associated with the $10^3$ Ly-$\alpha$ absorbers [4] and $\sim 10^6$ low mass simulated halos [5] we get that

$$
\langle G_{\rho}^{\text{sim}} \rangle \leq 2 \cdot 10^{-2}.
$$

Simulations demonstrate [5] that for redshifts $z \lesssim 10$ the reduced density $G_{\rho}$ of low mass halos is a many–valued function of the mass. This means that either these halos are not virialized, or the virial density is not described by the function (27) and, as it was discussed in Sec. 4.1, we need to use a more complex expression corrected, for instance, by inclusion $z_{\text{cr}}$ – the redshift of halo formation. After such correction the many–valued character of the simulated virial density gets reasonable explanation: halos with the same $\mu$ are formed with the same $A_{nd} D(z_{\text{cr}})$ [9, 37, 38] but different $z_{\text{cr}}$, $\rho_{\text{vir}}$ and $G_{\rho}$. These expectations can be easy tested in simulations. For observed galaxies and clusters of galaxies such ambiguity is not so evident owing to the limited number of observed halos.

This unexpected difference of observed and simulated halos can be formulated as the virial paradox. It shows that there are at least two populations of DM halos with different evolution and different properties. One of them is formed at high redshifts ($z \gtrsim 10$) and is observed as galaxies, and the other is formed at small redshifts, do not contain stars and is observed, in particular, as Ly-$\alpha$ forest and circum galactic halos [4]. This effect is closely linked with the missing satellite problem.

The discussed models of halos formation allow to describe some properties of the virialized halos, but they do not consider the internal structure of halos. Thus for comparison with the popular description of the core we can introduce the virial power index

$$
\gamma_{\text{vir}} = -\frac{M_{\text{vir}}}{\rho_{\text{vir}} d M_{\text{vir}}}. \quad (33)
$$

As is seen from (27, 29, 32) it can be expected that

$$
\gamma_{\text{vir}} \simeq 0.18. \quad (34)
$$
However for the low mass simulated halos it is  
\[ \gamma_{\text{sim}} \simeq 0.08. \]

### 4.4 Evolution of massive halos

In the PS model formation of halos is determined by the condition (35) and usually the model is applied for description of evolution of the mass function (12) and the mean mass of halos (14).

For \( M_{mn} = 10M_\odot \) we get for the mean mass

\[ \langle M(z) \rangle = \frac{5.5 \cdot 10^8 M_\odot x_{10}^{11.5}}{1 + 1.5x_{10}^{0.5} + 1.8x_{10}^{0.5}}, \quad x_{10} = \frac{10}{1 + z}. \]  

This function is plotted in Fig. 3. At small redshifts \( 1 \leq z \leq 10 \) the mean mass of halos is described as

\[ \langle M \rangle = 1.7 \cdot 10^{14} M_\odot \left( \frac{1 \pm 0.2}{(1 + z)^{4.7}} \right). \]  

The mass of a separate halo with the random peak amplitude \( A_{\text{rand}} \) can be found with Eqs. (36) and (37)

\[ \mu \simeq \frac{1.6 \cdot 10^{-4} x_{17.3}^2}{1 + 12x_{17.3}^2 + x_{11.8}^2}, \quad x = \frac{7A_{\text{rand}}}{1 + z}. \]

For \( z \gg 1, x \ll 1 \) this expression is quite similar to (35) and for \( x \geq 1 \) it is similar to (36). As is seen from (37) for \( x \ll 1 \) we have

\[ \mu \simeq \left[ 4A_{\text{rand}} \frac{1 + z}{1 + z} \right]^{17}. \]

In these expressions the crucial role of the redshift \( z \approx 10 \) is clearly seen as the strong change of the rate of matter concentration in the massive halos. In turn, this change of the evolutionary rate is the direct result of the shape of the power spectrum (35). These results are illustrated in Fig. 3.

The PS model with a small minimal mass shows that already during the early period of evolution DM halos can accumulate significant mass fraction and it rapidly increases with time. Thus for the minimal mass \( M_{mn} = 10^{-2}M_\odot \) at \( z \approx 50 \) the matter fraction \( f_m \sim 0.1 \) is already concentrated in halos with \( \langle M \rangle \sim 0.2M_\odot \), and at \( z \approx 25 \) the matter fraction \( f_m \sim 0.3 \) is accumulated in halos with \( \langle M \rangle \sim 10^3 M_\odot \). At \( z \approx 10 \) the matter concentration \( f_m \sim 0.5 \) is concentrated in halos with \( \langle M \rangle \sim 10^8 M_\odot \), but majority of DM is concentrated in less massive halos. At the same time halos with masses \( M \geq 10^6 M_\odot \), \( \langle M \rangle \sim 10^3 M_\odot \) accumulate only \( f_m \sim 0.1 \) but in such halos the first stars are formed, and they are responsible for reionization and reheating.

There are six natural consequences of the patch like model of galaxy formation and identification of galaxies with the highest density peaks:

1. At high redshifts \( z \geq 10 \) rapid \( M \propto (1 + z)^{-17} \) regular growth of the central halo is accompanied by creation of many subhalos in the immediate vicinity of the central peak and their rapid merging.

2. During this period density of the central halo and merged subhalos are similar to each other and their shapes are far from spherical. These peculiarities allow to consider this medium as a mixture of collisionless dispersed DM particles and collisional irregular DM subhalos.

3. Rapid infall of surrounding subhalos into the central halo and their tidal disruption just after merging partly transforms the dissipationless evolution of the dispersed DM into the dissipational one of subhalos. Impact of low entropy baryons amplifies this effect.

4. Accretion of the diffuse DM particles leads to formation of a cusp-like density profile (see, e.g., [32], [33], but successive merging of surrounding DM subhalos rapidly increases the central halo and gradually makes its cusp more and more shallow [1 3 10 17].

5. Accretion of the collisional fraction – DM subhalos – increases concentration of DM in the central core of halos and promotes formation of massive and supermassive central black holes.

These comments clarify qualitative differences in evolution of DM halos at high and small redshifts. Weak
traces of these processes are seen in the present day simulations and are discussed in \cite{12,17}. More accurate quantitative estimates could be obtained from suitable simulations such as \cite{3,17,51}.

For the later period \((z \leq 5)\) of evolution growth of halos mass slows down \cite{50} and there is a large gap between formation of massive clusters of galaxies at \(z \approx 1\) and high density galaxies at \(z \geq 2\). In contrast with the high redshift evolution, collisions and stripping of galaxies are rare, and in the observed clusters both the cusp like density profile and distinct high density galaxies could survive.

4.5 Evolution of the central cusp

At present the structure and evolution of the core of DM halos remains unclear. Unfortunately it can not be described analytically and in simulations it is investigated only at low redshift (see \cite{98,100}). At \(z \leq 10\) modern high resolution simulations provide a set of representative DM halos ranging from dwarf galaxies and up to rich clusters of galaxies. Their profile is found to be cusp–like and close to the two parametric Navarro-Frenk-White (NFW) profile \cite{32}

\[
\rho_{\text{NFW}}(r) \sim \frac{\rho_0}{x^2(1 + x)^2}, \quad x = r/r_0, \quad \alpha = 1. \tag{39}
\]

This profile reproduces reasonably well that observed for majority of clusters of galaxies. However in less massive galaxies the observed density profile is more shallow (Table \ref{tab:1}) and can be expressed by the relation \cite{11} with the power index \(\alpha \leq 1\). Strong influence of random factors for galaxies manifests itself as larger scatter of the power index \(\alpha\). A review of observations is presented in \cite{33}.

Now four models of flattening of the central cusp are discussed:

1. The most popular explanation is the cusp destruction owing to sudden removal of gas from the center of a cuspy DM halo caused by energy injection by explosions of supernovae \cite{32}.

2. In \cite{38,41} the authors consider progressive destruction of the DM cusp owing to its tidal interaction with gaseous clouds, first stars and protostars.

3. In \cite{42} the cusp erosion is related to accretion of a suitable DM spherical shell.

4. Perhaps, further progress can be achieved by considering exotic particle physics \cite{18,44,45}.

Of course, these factors influence the density profile but they are of secondary importance with moderate efficiency which depends upon thermal evolution of baryons, star formation or high energy physic. Recent discussion of this problem \cite{16,43} confirms the critical role of the threshold of stars formation and insufficient variety of mass profiles. Authors of \cite{16,43} believe that their simulations cannot explain the observed diversity of galaxy rotation curves.

However, these inferences relate to simulations with the cutoff of power spectrum which cannot represent the earlier stage of halos formation responsible for the structure of the central region. It can be expected that matter compression has gone through pancake–like anisotropic stage and the final density profile is formed by a complex relaxation processes.

In \cite{51} these problems have been analyzed in detail using simulations at redshifts \(0 \leq z \leq 3\) and the Minimal Spanning Tree technique. Main results of this investigation can be formulated as follow:

1. the shape of a halo with mass \(M\) is elliptical with half axes \(a_1\) and velocity dispersion \(w_i\)

\[
a_1 : a_2 : a_3 \sim (6 : 2 : 1)\sqrt{M/\langle M \rangle}, \quad w_i \propto \sqrt{M/\langle M \rangle}. \tag{40}
\]

2. in accordance with the tidal torque theory \cite{69,71} the angular momentum of clouds can be approximated by \(|\mathbf{J}| \approx 0.17|w_i|R_{\text{vir}}\) with the exponential PDF.

3. the turbulent motions can be approximated by the angular momentu \(|\mathbf{j}| \approx 0.8|w_i|R_{\text{vir}}\) with the Gaussian PDF.

These results indicate limited application of the spherical approach and a high influence of velocities for evolution of central regions of DM halos. Some applications of the tidal torque theory are discussed in \cite{72,73}.

High efficiency of tidal interactions for flattening of the cusp was demonstrated in \cite{38,41} where the cusp disruption is explained by absorption and tidal destruction of a suitable set of stars and protostars. High efficiency of merging of surrounding DM subhalos and tidal heating of the central cores are confirmed by simulations \cite{1–3,17,61,107}. These papers illustrate the crucial role of initial stages of halos formation for correct reproduction of structure of halos and, in particular, for successive transformations of the central cusp to a core. They demonstrate that

1. For the low mass DM subhalos the central cusp is steeper than for the NFW model.

| Table 1: Parameters of observed density profile |
|-----------------------------------------------|
| \(N_{\text{obj}}\) \(\alpha_{mn} \leq \alpha \leq \alpha_{max} (\alpha)\) |
| 20 clusters \(0.5 \leq \alpha \leq 1.5 1.02 \pm 0.08 \) [101] |
| 26 galaxies \(0 \leq \alpha \leq 1.2 0.2 \pm 0.2 \) [102] |
| 15 galaxies \(0 \leq \alpha \leq 1.2 0.6 \pm 0.35 \) [103] |
| 7 galaxies \(0 \leq \alpha \leq 1.2 0.29 \pm 0.07 \) [104] |
| 26 galaxies \(0 \leq \alpha \leq 1.2 0.32 \pm 0.24 \) [105] |
| 7 galaxies \(0.5 \leq \alpha \leq 0.73 0.67 \pm 0.10 \) [106] |


2. The central cusp becomes shallower owing to major merging processes as the halos mass increases.

3. The simulated mass dependence of the power index can be roughly fitted as

$$\alpha \simeq 0.123 \log \left( \frac{M}{10^9 M_\odot} \right).$$

This expression describes both the small $\alpha$ at $M \sim (10^6 - 10^9) M_\odot$ and $\alpha \sim 1$ for clusters with $M \sim 10^{14} M_\odot$. This trend is consistent with results of (34). The small simulated box and redshift interval limits obtained there promising results and they should be repeated in larger boxes and extended to smaller redshifts.

### 4.6 The missing satellites problem

The missing satellites problem is formulated as a strong discrepancy between the number of observed satellites of the Milky Way ($\sim 30 - 40$ at distances $D \leq 1 Mpc$) and the number of simulated DM subhalos around massive galaxies (108). It can be reformulated as high difference between the matter fraction concentrated in luminous galaxies and in the DM halos ($\sim 70\%)$. Estimates of (108) show that only moderate fraction of baryons,

$$\Omega_{fum} \simeq 0.20(1 \pm 0.2) \Omega_b,$$

is concentrated in luminous objects (stars, galaxies, clusters of galaxies).

The main difference between galaxies and DM halos is the presence or absence of stars, what reduces the discussion to the problem of formation of stars – or even first stars. The virial paradox discussed in Sec. 4.3 shows that here we have to deal with two different populations of halos and it is closely linked with the shape of the primordial power spectrum of density perturbations. Observations of the ultra diffuse galaxies (20) suggest that there is continuous transition between these populations.

The matter fraction (42) related to galaxies is comparable with that accumulated by high density massive halos before reheating of the Universe when the temperature of the low density baryons was rapidly increasing from $\sim 1 K$ up to $\sim 10^4 K$. (Next problem is the topology of the reionization bubble network (110)). The fraction of high density baryons kept low entropy and the ability to form first stars. This means that we can consider the stars as the trademark of such halos. The multitude of low mass DM halos formed later do not contain neither the low entropy baryons nor stars.

Conversion of DM halos into the observed luminous galaxies is a very complex multistep process (14, 111–114) which can be described only phenomenologically. These complexities prevent discrimination between simulated galaxies and invisible DM halos (115). Besides, papers (20, 23) show some unexpected features of dwarf galaxies what emphasizes again a complex character of their evolution. The PS formalism reproduces the observed estimate (42) and confirms that the missing satellites problem is deeply linked with the virial paradox and the identification of galaxies and simulated DM halos.

As is illustrated in (116) metal production in dwarf galaxies is irregular and randomized. Present day simulations can reproduce these processes only phenomenologically with special assumptions. This means that the missing satellites problem requires more adequate simulations with restoration of the patch like formation of galactic counterparts and the reheating process.

### 5. CONCLUSIONS

In this paper we consider evolution of the $\Lambda$CDM cosmological model with a small damping scale. In this model the patch like character of halos formation leads to creation of two different populations of objects. The first population includes high density halos formed before reheating in immediate vicinity of high density peaks identified with galaxies. Such halos contain stars and low entropy baryons and are observed as galaxies. Low mass halos of the second population are formed after reheating and they do not contain stars and low entropy baryons. Some of them are observed as the Ly--$\alpha$ forest (4) and circumgalactic matter (30).

Evolution of the second population is investigated in many simulations. In contrast, evolution of the first population is presented only in a few simulations (1–3, 46, 47). For its description we have to use abilities of analytically extended Press–Schechter and Zel’dovich approaches. Such models allow to reveal the main specific features of halos evolution, to clarify differences in properties of these populations and to explain the virial paradox, the core – cusp and the missing satellite problems. As discussed in Sec. 4, properties of these two families of halos are determined by the shape of the initial power spectrum of density perturbations.

Thus the patch like model demonstrates that:

1. The rapid formation of many subhalos at $z \geq 10$ in the immediate vicinity of the rare high density peaks and their rapid merging just after formation lead to very rapid growth of mass of the central halo (38) and Fig. 4.

2. Tidal interaction of irregular merged subhalos with the central cusp makes it more and more shallow, what allows to explain the core – cusp problem and accelerates formation of the central black holes. These inferences are consistent with simulations (1–3, 46, 47, 62).
3. The patch like model demonstrates strong differences between characteristics of populations of galaxies created before reheating and numerous population of low mass dark DM halos created later. The last population does not contain low entropy baryons and stars. This division explains the missing satellite problem.

4. The path like model explains the virial paradox – observed correlations of the density and mass of first population of halos – galaxies and clusters of galaxies [4, 5] and links it with the cosmological power spectrum of density perturbations.

5. This link allows to estimate the shape of the small scale power spectrum (Fig. 2) and to place new constrains on the parameters of DM particles, WDM models and models of cosmological inflation. This approach deserves further refined investigations in both observations and simulations.

Traces of these processes are revealed in present day simulations and are discussed in [12, 17]. However these simulations cannot adequately reproduce both the early and later periods of structure evolution as well as the reheating of the intergalactic matter. This means that for our analysis we have to use theoretical models. Results obtained in this way cannot be considered as an actual proof of declared inferences but they reveal new important factors, actions of which were underestimated in previous discussions, and point out promising ways for further progress.

Acknowledgments

The work is supported by the new scientific group of LPI.

APPENDIX: STATISTICAL CHARACTERISTICS OF THE ZEL’DOVICH THEORY.

The Zel’dovich approximation [48, 51] correctly describes the early anisotropic period of matter condensation and formation of elements of the Large Scale Structure of the Universe – network of filaments and walls– superclusters (Zel’dovich ‘pancakes’) [49, 52, 54, 68]. At all redshifts these elements are formed in the course of mildly nonlinear self similar matter compression described by Eq. [13, 16].

The original Zel’dovich model describes the matter condensation with the deformation tensor. However, for the CDM models with a small scale cutoff of the power spectrum this approach has to be reformulated in terms of displacements of particles what allows to prevent singularities at kl0 ≫ 1 and to concentrate more attention on observed and simulated scales. This requires modification of the statistical description of these processes. These problems have been discussed in [49, 51, 53, 67, 68] and are shortly presented here.

As it follows from [16] for the differences of displacements ΔSi we get

\[ \Phi_{ij}(q_1, q_2) = \langle \Delta S_i(q_1) \Delta S_j(q_2) \rangle = 2\Psi_{ij}(0) - 2\Psi_{ij}(q_{12}) , \]

\[ \Delta S_i(q) = S_i(q) - S_i(-q) , \quad q_{12} = |q_1 - q_2| , \]

\[ G_{11} = \Phi_{11}(q_1, q_2) = \frac{2}{3} \left[ 1 - G_1(2q_1) - 4q_1^2 G_2(2q_1) \right] , \quad (43) \]

\[ G_{12} = \Phi_{12}(q_1, q_2) = -\frac{8}{3} q_1^2 G_2(\sqrt{2}q_1) , \quad |q_1| = |q_2| . \]

These relations and eq. [17] allow us to estimate the coefficient correlations of the orthogonal displacements for the spectra discussed in Secs. 3 & 4 as

\[ r_{12} = r_{13} = r_{23} = G_{12}(q_1, q_2)/G_{11}(q_1, q_1) \approx 2/3 . \quad (44) \]

Structure characteristics of uncorrelated distribution function of displacements

According to Eq. [13, 16] the Zel’dovich approach describes the matter condensation in compact clouds, filaments and walls. In particular it allows to estimate the evolution of matter fractions associated with – walls Ww, filaments Wf, clouds Wc and voids Wv. For illustration we can ignore correlations between orthogonal displacements ΔSi ΔSk, i ≠ k and to assume that the distribution function for these displacements is Gaussian

\[ dW = \Phi(\xi_1, \xi_2, \xi_3)d^3\xi = 0.75 \exp(-Q)d^3\xi , \quad (45) \]

\[ Q = (\xi_1^2 + \xi_2^2 + \xi_3^2)/2 , \quad -\infty \leq \xi_3 \leq \xi_2 \leq \xi_1 \leq \infty . \]

\[ \Phi_{ij}(q_1, q_2) = \langle \Delta S_i(q_1) \Delta S_j(q_2) \rangle = 2\Psi_{ij}(0) - 2\Psi_{ij}(q_{12}) , \]

\[ \Delta S_i(q) = S_i(q) - S_i(-q) , \quad q_{12} = |q_1 - q_2| , \]

\[ G_{11} = \Phi_{11}(q_1, q_2) = \frac{2}{3} \left[ 1 - G_1(2q_1) - 4q_1^2 G_2(2q_1) \right] , \quad (43) \]

\[ G_{12} = \Phi_{12}(q_1, q_2) = -\frac{8}{3} q_1^2 G_2(\sqrt{2}q_1) , \quad |q_1| = |q_2| . \]

These relations and eq. [17] allow us to estimate the coefficient correlations of the orthogonal displacements for the spectra discussed in Secs. 3 & 4 as

\[ r_{12} = r_{13} = r_{23} = G_{12}(q_1, q_2)/G_{11}(q_1, q_1) \approx 2/3 . \quad (44) \]
and \( \xi = D(z)\Delta S_i/\sigma_i \).

According to Eq. (33) the matter fractions accumulated by structure elements are determined by a common threshold \( \zeta \geq 0 \) and we get for voids,

\[
\zeta \geq \xi_1 \geq \xi_2 \geq \xi_3 \geq -\infty \quad W_v = 0.125(1 + e(\zeta))^{3},
\]

where \( e(\zeta) = erf(\zeta) \) and for the distinct clouds, formed directly from a weakly perturbed matter

\[
\infty \geq \xi_1 \geq \xi_2 \geq \xi_3 \geq \zeta, \quad W_{cl} = 0.125(1 - e(\zeta))^{3}.
\]

The matter fraction accumulated by filaments, \( W_f \), and walls, \( W_w \), are determined by similar conditions

\[
\infty \geq \xi_1 \geq \xi_2 \geq \xi_3 \geq -\zeta \geq -\infty \quad W_f = 3W_{cl}\frac{1 + e(\zeta)}{1 - e(\zeta)}
\]

\[
\infty \geq \xi_1 \geq \zeta \geq \xi_2 \geq \xi_3 \geq -\infty \quad W_w = 3W_v\frac{1 - e(\zeta)}{1 + e(\zeta)}.
\]

Evidently, \( W_w + W_f + W_{cl} + W_v = 1 \). Thus, for \( \zeta = 0 \) we get

\[
W_w = W_f = 3/8, \quad W_{cl} = W_v = 1/8,
\]

but for \( \zeta \geq 0 \) the symmetry is destroyed (Table III) and voids accumulate dominant matter fraction.

However as was shown in the first Zel’dovich paper [48] the size of high density multistream regions associated with walls exceeds the size determined by the condition \( \zeta = 1 \) by a factor \( \sqrt{3} \). This means that the condition \( \zeta \simeq 0.5 \) more correctly describes the compressed matter fractions.

**Structure characteristics of the correlated distribution functions of displacements**

The impact of correlations of the orthogonal displacements leads to more cumbersome estimates and can be suitably analyzed numerically using 10^7 random realizations. The correlation coefficient \( \kappa \) depends upon the power spectrum and for the power spectrum [3] \( \kappa \sim 2/3 \). For three amplitudes \( \zeta \) the matter fractions accumulated by structure elements are presented in Table III.

As is seen from this Table the matter fraction directly accumulated by clouds, \( W_{cl} \), is minimal, but for the amplitude \( \zeta = 0.5 \) the matter fraction accumulated by high density LSS elements, \( 1 - W_v \), increases up to 77% what is comparable with simulated results. As was shown in [117] and [118] after formation of high density filaments and walls they are rapidly disrupted into distinct relaxed clouds. This inference is also confirmed by simulations.

In the general case the PDF of the random displacements is Gaussian with correlation coefficients \( r_{ij} \), \( i, j = 1, 2, 3, \ -\infty \leq \xi_i \leq \xi_2 \leq \xi_1 \leq \infty \)

\[
Q = \frac{\xi_1^2 + \xi_2^2 + \xi_3^2}{2} + \kappa_{12}\xi_1\xi_2 + \kappa_{13}\xi_1\xi_3 + \kappa_{23}\xi_2\xi_3 , \quad (46)
\]

![FIG. 4: The probability distribution functions for three displacements, \( \Delta S_1 \geq \Delta S_2 \geq \Delta S_3 \), are plotted vs. \( x = \Delta S_i/\sigma_i \), \( \langle x_i \rangle = 1.07, 0, -1.07 \). \( \sigma_i = 0.76, 0.64, 0.76 \).](attachment:image.png)

\[
\xi_i = \varepsilon_{jk}\Delta S_i, \quad \kappa_{ij} = \frac{r_{ij} - r_{ik}r_{jk}}{\sqrt{(1 - r_{ik}^2)(1 - r_{jk}^2)}}, \quad i \neq j \neq k,
\]

\[
\varepsilon_{jk} = (1 - r_{jk}^2)/D, \quad D = 1 - r_{12}^2 - r_{23}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23}.
\]

As it follows from [119] for the spectra discussed in Sec. 3 & 4 we get

\[
r_{12} \simeq 2/3, \quad D \simeq 7/27, \quad \omega_{22}^2 = 15/7, \quad \kappa_{ij} = 2/5.
\]

These PDFs of the displacements are plotted in Fig. IV and some numerical estimates are given in Table III.

To convert the expression (36) to the orthogonal form we use transformation

\[
Q = \frac{1}{2}(\eta_1^2 + \eta_2^2 + \eta_3^2), \quad (47)
\]

\[
\xi_3 = \omega_{33}\eta_3, \quad \xi_2 = \omega_{22}\eta_2 - \omega_{23}\eta_3, \quad \xi_1 = \eta_1 - \kappa_{12}\xi_2 - \kappa_{13}\xi_3,
\]

\[
\omega_{33} = \sqrt{1 - \kappa_{12}^2/\kappa_{12}^2} \simeq 1.045, \quad \omega_{22} = \frac{1}{\sqrt{1 - \kappa_{12}^2}} \simeq 1.033,
\]

\[
\omega_{23} = \frac{\kappa_{23} - \kappa_{12}\kappa_{13}}{\sqrt{(1 - \kappa_{12}^2)DD}} \simeq 0.21,
\]

\[
DD = 1 - \kappa_{12}^2 - \kappa_{13}^2 - \kappa_{23}^2 - 2\kappa_{12}\kappa_{13}\kappa_{23} \simeq 27/32.
\]

[1] Ishiyama, T., 2014, ApJ, 788, 27
[2] Angulo, R., Hahn, O., Ludlow, A., ey al., 2017, MNRAS, 471, 4667
