How Many Samples is a Good Initial Point Worth?

Gavin Zhang  
Department of Electrical and Computer Engineering  
University of Illinois Urbana Champaign  
Illinois, IL61820  
jialun2@illinois.edu

Richard Y. Zhang  
Department of Electrical and Computer Engineering  
University of Illinois Urbana Champaign  
Illinois, IL61820  
ryz@illinois.edu

Abstract

Given a sufficiently large amount of labeled data, the non-convex low-rank matrix recovery problem contains no spurious local minima, so a local optimization algorithm is guaranteed to converge to a global minimum starting from any initial guess. However, the actual amount of data needed by this theoretical guarantee is very pessimistic, as it must prevent spurious local minima from existing anywhere, including at adversarial locations. In contrast, prior work based on good initial guesses have more realistic data requirements, because they allow spurious local minima to exist outside of a neighborhood of the solution. In this paper, we quantify the relationship between the quality of the initial guess and the corresponding reduction in data requirements. Using the restricted isometry constant as a surrogate for sample complexity, we compute a sharp “threshold” number of samples needed to prevent each specific point on the optimization landscape from becoming a spurious local minima. Optimizing the threshold over regions of the landscape, we see that, for initial points not too close to the ground truth, a linear improvement in the quality of the initial guess amounts to a constant factor improvement in the sample complexity.

1 Introduction

A perennial challenge in non-convex optimization is the possible existence of bad or spurious critical points and local minima, which can cause a local optimization algorithm like gradient descent to slow down or get stuck. Several recent lines of work showed that the effects of non-convexity can be tamed through a large amount of diverse and high quality training data [17][11][9][8][18][12]. Concretely, these authors showed, for classes of problems based on random sampling, that spurious critical points and local minima become progressively less likely to exist with the addition of each new sample. After a sufficiently large number of samples, all spurious local minima are eliminated, so any local optimization algorithm is guaranteed to converge to the globally optimal solution starting from an arbitrary, possibly random initial guess.

This notion of a global guarantee—one that is valid starting from any initial point—is considerably stronger than what is needed for empirical success to be observed [8]. For example, the existence of a spurious local minimum may not pose an issue if gradient descent does not converge towards it. However, a theoretical guarantee is no longer possible, as starting the algorithm from the spurious local minimum would result in failure [22]. As a consequence, these global guarantees tend to

Preprint. Under review.
be pessimistic, because the number of samples must be sufficiently large to eliminate spurious local minima everywhere, even at adversarial locations. By contrast, the weaker notion of a local guarantee [11, 10, 15, 19, 5, 7, 20, 13]—one that is valid only for a specified set of initial points—is naturally less conservative, as it allows spurious local minima to exist outside of the specified set.

In this paper, we provide a unifying view between the notions of the global and local guarantees by quantifying the relationship between the sample complexity and the quality of the initial point. We restrict our attention to the matrix sensing problem, which seeks to recover a rank-$r$ positive semidefinite matrix $M^* = ZZ^T \in \mathbb{R}^{n \times n}$ with $Z \in \mathbb{R}^{n \times r}$ from $m$ sub-Gaussian linear measurements of the form

$$b = A(ZZ^T) = [(A_1, M^*) \cdots (A_m, M^*)]^T$$

by solving the following non-convex optimization problem:

$$\min_{X \in \mathbb{R}^{n \times r}} f_A(X) \equiv \|A(XX^T - ZZ^T)\|^2 = \sum_{i=1}^{m} \left(\langle A_i, XX^T - ZZ^T \rangle - b_i\right)^2. \quad (2)$$

We characterize a sharp "threshold" on the number of samples $m$ needed to prevent each specific point on the optimization landscape from becoming a spurious local minima. While the threshold is difficult to solve, we derive a lower-bound in closed-form based on spurious critical points, and show that it constitutes a sharp lower-bound on the original threshold of interest. The lower-bound reveals a simple geometric relationship: a point $X$ is more likely to be a local minimum if the incidence angle $\phi$ satisfying $\langle XX^T, ZZ^T \rangle = \|XX^T\|^2\|ZZ^T\|\cos\phi$ is closer to orthogonal. Optimizing the closed-form lower-bound over regions of the landscape, we show for initial points not too close to the ground truth, that a constant factor improvement of the initial point amounts to a constant factor reduction in the number of samples needed to guarantee recovery.

## 2 Our Approach: Threshold RIP Constant

Previous work that studied the global optimization landscape of problem (2) typically relied on the restricted isometry property (RIP) of $A$. It is now well-known that if the measurement operator $A$ satisfies the restricted isometry property with a sufficiently small constant $\delta < 1/5$ then problem (2) contains no spurious local minima; see Bhojanapalli et al. [2].

**Definition 1 ($\delta$-RIP).** Let $A : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^m$ be a linear measurement operator. We say that $A$ satisfies the $\delta$-restricted isometry property (or simply $\delta$-RIP) if it satisfies the following inequality

$$\|1 - \delta\|M\|_F^2 \leq \|A(M)\|^2 \leq (1 + \delta)\|M\|_F^2, \quad \forall M \in \mathcal{M}_{2r},$$

where $\mathcal{M}_{2r} = \{X \in \mathbb{R}^{n \times n} : \text{rank}(X) = 2r\}$ denotes the set of rank-$2r$ matrices. The RIP constant of $A$ is the smallest value of $\delta$ such that the inequality above holds.

Let $\delta \in [0, 1)$ denote the RIP constant of $A$. It is helpful to view $\delta$ as a surrogate for the number of measurements $m \geq 0$, with a large value of $\delta$ corresponding a smaller value of $m$ and vice versa. For a wide range of sub-Gaussian measurement ensembles, if $m \geq C_0nr/\delta^2$ where $C_0$ is an absolute constant, then $A$ satisfies $\delta$-RIP with high probability [2, 16].

Take $X \in \mathbb{R}^{n \times r}$ to be a spurious point such that $XX^T \neq ZZ^T$. Our approach in this paper is to define a threshold number of measurements that would be needed to prevent $X$ from becoming a local minimum for problem (1). Viewing the RIP constant $\delta$ as a surrogate for the number of measurements $m$, we follow a construction of Zhang et al. [23], and instead define a threshold $\delta_{\text{soc}}(X)$ on the RIP constant $\delta$ that would prevent $X$ from becoming a local minimum for problem (1). Such a construction must necessarily take into account all choices of $A$ satisfying $\delta$-RIP, including those that adversarially target $X$, bending the optimization landscape into forming a region of convergence around the point. On the other hand, such adversarial choices of $A$ must necessarily be defeated for a sufficiently small threshold on $\delta$, as we already know that spurious local minima cannot exist for $\delta < 1/5$. The statement below makes this idea precise, and also extends it to a set of spurious points.

**Definition 2 (Threshold for second-order condition).** Fix $Z \in \mathbb{R}^{n \times r}$. For $X \in \mathbb{R}^{n \times r}$, if $XX^T = ZZ^T$, then define $\delta_{\text{soc}}(X) = 1$. Otherwise, if $XX^T \neq ZZ^T$, then define

$$\delta_{\text{soc}}(X) \equiv \min_{A} \left\{ \delta : \nabla f_A(X) = 0, \quad \nabla^2 f_A(X) \succeq 0, \quad A \text{satisfies } \delta\text{-RIP} \right\} \quad (3)$$

where the minimum is taken over all linear measurements $A : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^m$. For $\mathcal{W} \subseteq \mathbb{R}^{n \times r}$, define

$$\delta_{\text{soc}}(\mathcal{W}) = \inf_{X \in \mathcal{W}} \delta_{\text{soc}}(X).$$
If \( \delta < \delta_{\text{soc}}(X) \), then \( X \) cannot be a spurious local minimum by construction, or it would contradict the definition of \( \delta_{\text{soc}}(X) \) as the minimum value. By the same logic, if \( \delta < \delta_{\text{soc}}(W) \), then no choice of \( X \in W \) can be a spurious local minimum. In particular, it follows that \( \delta_{\text{soc}}(\mathbb{R}^{n \times r}) \) is the usual global RIP threshold: if \( A \) satisfies \( \delta \)-RIP with \( \delta < \delta_{\text{soc}}(\mathbb{R}^{n \times r}) \), then \( f_A(X) \) is guaranteed to admit no spurious local minima. Starting a local optimization algorithm from any initial point guarantees exact recovery of an \( X \) satisfying \( XX^T = ZZ^T \).

Now, suppose we are given an initial point \( X_0 \). It is natural to measure the quality of \( X_0 \) by its relative error, as in \( \varepsilon = \|XX^T - ZZ^T\|_F / \|ZZ^T\|_F \). If we define an \( \varepsilon \)-neighborhood of all points with the same relative error

\[
B_\varepsilon = \{ X \in \mathbb{R}^{n \times r}, \|XX^T - ZZ^T\|_F \leq \varepsilon \|ZZ^T\|_F \} \tag{4}
\]

then it follows that \( \delta_{\text{soc}}(B_\varepsilon) \) is an analogous local RIP threshold: if \( A \) satisfies \( \delta \)-RIP with \( \delta < \delta_{\text{soc}}(B_\varepsilon) \), then \( f_A(X) \) is guaranteed to admit no spurious local minima over all \( X \in B_\varepsilon \). Starting a local optimization algorithm from the initial point \( X_0 \) guarantees either exact recovery of an \( X \) satisfying \( XX^T = ZZ^T \), or termination at a strictly worse point \( X \) with \( \|XX^T - ZZ^T\|_F > \|X_0X_0^T - ZZ^T\|_F \). Imposing further restrictions on the algorithm prevents the latter scenario from occurring (local strong convexity with gradient descent \( [19] \), strict decrements in the levels set \( [10, 23] \), see also \( [8] \)), and so exact recovery is guaranteed.

The numerical difference between the global threshold \( \delta_{\text{soc}}(\mathbb{R}^{n \times r}) \) and the local threshold \( \delta_{\text{soc}}(B_\varepsilon) \) is precisely the number of samples that an \( \varepsilon \)-quality initial point \( X_0 \) is worth, up to some conversion factor. But two major difficulties remain in this line of reasoning. First, evaluating \( \delta_{\text{soc}}(X) \) for some \( X \in \mathbb{R}^{n \times r} \) requires solving a minimization problem over the set of \( \delta \)-RIP operators. Second, evaluating \( \delta_{\text{soc}}(B_\varepsilon) \) in turn requires minimizing \( \delta_{\text{soc}}(X) \) over all choices of \( X \) within an \( \varepsilon \)-neighborhood. Regarding the first point, Zhang et al. \( [23] \) showed that \( \delta_{\text{soc}}(X) \) is the optimal value to a convex optimization problem, and can therefore be evaluated to arbitrary precision using a numerical algorithm. In the rank-1 case, they solved this convex optimization in closed-form, and used it to optimize over all \( X \in B_\varepsilon \). Their closed-form solution spanned 9 journal pages, and evoked a number of properties specific to the rank-1 case (for example, \( xy^T + yx^T = 0 \) implies \( x = 0 \) and \( y = 0 \), but \( XY^T + YX^T = 0 \) may hold for \( X \neq 0 \) and \( Y \neq 0 \)). The authors noted that a similar closed-form solution for the general rank-\( r \) case appeared exceedingly difficult. While overall proof technique is sharp and descriptive, its applicability appears to be entirely limited to the rank-1 case.

## 3 Main results

In this paper, we bypass the difficulty of deriving a closed-form solution for \( \delta_{\text{soc}}(X) \) altogether by adopting a sharp lower-bound. This is based on two key insights. First, a spurious local minimum must also be a spurious critical point, so the analogous threshold over critical points would give an obvious lower-bound \( \delta_{\text{loc}}(X) \leq \delta_{\text{soc}}(X) \).

**Definition 3** (Threshold for first-order condition). Fix \( Z \in \mathbb{R}^{n \times r} \). For \( X \in \mathbb{R}^{n \times r} \), if \( XX^T = ZZ^T \), then define \( \delta_{\text{loc}}(X) = 1 \). Otherwise, if \( XX^T \neq ZZ^T \), then define

\[
\delta_{\text{loc}}(X) \equiv \min_A \{ \delta : \nabla f_A(X) = 0, \quad A \text{ satisfies } \delta \text{-RIP} \}, \tag{5}
\]

where the minimum is taken over all linear measurements \( A : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^m \). For \( \mathcal{W} \subseteq \mathbb{R}^{n \times r} \), define \( \delta_{\text{loc}}(\mathcal{W}) = \inf_{X \in \mathcal{W}} \delta_{\text{loc}}(X) \).

Whereas the main obstacle in Zhang et al. \( [23] \) is the considerable difficulty in deriving a closed-form solution for \( \delta_{\text{soc}}(X) \), we show in this paper that it is relatively straightforward to solve \( \delta_{\text{loc}}(X) \) in closed-form, to result in a simple, geometry solution.

**Theorem 4.** Fix \( Z \in \mathbb{R}^{n \times r} \). Given \( A \) satisfying \( \delta \)-RIP and \( X \in \mathbb{R}^{n \times r} \) such that \( XX^T \neq ZZ^T \), we have \( \delta_{\text{loc}}(X) = \cos \theta \), where

\[
\sin \theta = \|Z^T(1 - XX^+)Z\|_F / \|XX^T - ZZ^T\|_F. \tag{6}
\]

and \( X^+ \) denotes the pseudo-inverse of \( X \). It follows that if \( \delta < \cos \theta \), then \( X \) is not a spurious critical point of \( f_A(X) \). If \( \delta \geq \cos \theta \), then there exists some \( A^* \) satisfying \( \cos \theta \)-RIP such that \( \nabla f_A(X) = 0 \).
Theorem 5. Let \( \delta \). The proof of this theorem is in Section 6.2. Theorem 5 says that the number of samples needed to eliminate spurious critical points decreases dramatically as \( \varepsilon \) becomes small. Given that \( m \geq C_0 \sqrt{r} / \delta^2 \) sub-Gaussian measurements are needed to satisfy \( \delta \)-RIP, we can translate Theorem 5 into the following sample complexity bound.

\[ \delta < \max \left\{ \sqrt{1 - C \varepsilon^2}, \delta^* \right\} \]

where \( \delta^* = 1/2 \) if \( r = 1 \) and \( \delta^* = 1/5 \) if \( r > 1 \), then \( f_A(X) \) has no spurious critical point within an \( \varepsilon \)-neighborhood of the solution:

\[ \nabla f_A(X) = 0, \quad \| X X^T - ZZ^T \|_F \leq \varepsilon \| ZZ^T \|_F \iff X X^T = ZZ^T. \]

The proof of this theorem is in Section 6.2. Theorem 5 says that the number of samples needed to eliminate spurious critical points decreases dramatically as \( \varepsilon \) becomes small. Given that \( m \geq C_0 \sqrt{r} / \delta^2 \) sub-Gaussian measurements are needed to satisfy \( \delta \)-RIP, we can translate Theorem 5 into the following sample complexity bound.

\[ \delta < \max \left\{ \sqrt{1 - C \varepsilon^2}, \delta^* \right\} \]

where \( \delta^* = 1/2 \) if \( r = 1 \) and \( \delta^* = 1/5 \) if \( r > 1 \).
Corollary 6 (Informal). Let \( A : \mathbb{R}^{n \times n} \to \mathbb{R}^m \) be a sub-Gaussian measurement ensemble. If
\[
m \geq \min \left\{ \frac{C_0 nr}{(1 - C_0^2)} + 25C_0 nr \right\}
\]
then with high probability there are no spurious local minima within \( B_\varepsilon \).

The proof of Corollary 6 follows immediately from Theorem 5 and the 25\( C_0 nr \) global sample complexity figure of Bhojanapalli et al. \([2]\). For large values of \( \varepsilon \), we see that the number relationship between the quality of the initial point and the number of samples saved is essentially linear. For example, improving \( \varepsilon \) from 1 to 0.6 would reduce the number of measurements \( m \) by a factor of 3. On the other hand, for small values of \( \varepsilon \), we have \( m \geq \frac{C_0 nr}{(1 - C_0^2)} + 25C_0 nr + CC_0 nr \varepsilon^2 + O(\varepsilon^4) \), so the reduction in the number of samples is marginal.

4 Related Work

Local Guarantees. The earliest work on exact guarantees for non-convex optimization focused on generating a good initial guess within a local region of attraction. For instance, in \([21, 24]\), the authors showed that when \( A \) satisfies 6\( \delta \)-RIP with a constant \( \delta \leq 1/10 \), and there exists a initial point sufficiently close to the ground truth, gradient descent starting from this initial point has a linear convergence rate. The typical strategy to find such the initial point is spectral initialization [11, 10, 21, 19, 8, 14, 6]: using the singular value decomposition on a surrogate matrix to find low-rank factors that are close to the ground truth.

In this paper, we focus on the trade-off between the quality of an initial point and the number of samples needed to prevent the existence of spurious local minima, while sidestepping the question of how it is found. We note, however, that the number of samples needed to find an \( \varepsilon \)-good initial guess (e.g. via spectral initialization) forms an interesting secondary trade-off. It remains a future work to study the interactions between these two points.

Global Guarantees. Recent work focused on establishing a global guarantee that is independent of the initial guess [17, 1, 9, 3, 15, 12]. For our purposes, Bhojanapalli et al. \([2]\) showed that RIP with \( \delta_{2r} < 1/5 \) eliminates all spurious local minima, while Zhang et al. \([23]\) refined this to \( \delta_{2r} < 1/2 \) for the rank-1 case, and showed that this is both and necessary and sufficient. This paper is inspired by proof techniques in the latter paper; a part of our contributions is modifying the technique in order to accommodate for matrices of arbitrary rank.

5 Proof of Main Results

5.1 Notation and Definitions

We use \( \| \cdot \| \) for the vector 2-norm and use \( \| \cdot \|_F \) to denote the Frobenius norm of a matrix. For two square matrices \( A \) and \( B \), \( A \succeq B \) means \( B - A \) is positive semidefinite. The trace of a square matrix \( A \) is denoted by \( \text{tr}(A) \). The vectorization \( \text{vec}(A) \) is the length-\( mn \) vector obtained by stacking the columns of \( A \). Let \( A : \mathbb{R}^{n \times n} \to \mathbb{R}^m \) be a linear measurement operator, and let \( Z \in \mathbb{R}^{r \times r} \) be a fixed ground truth matrix. We define \( A = [\text{vec}(A_1), \ldots, \text{vec}(A_m)] \) as the matrix representation of \( A \), and note that \( \text{vec}[A(X)] = A \text{vec}(X) \). We define the error vector \( e \) and its Jacobian \( X \) to satisfy
\[
e = \text{vec}(XX^T - ZZ^T) \quad (10a)
\]
\[
X \text{vec}(Y) = \text{vec}(YY^T + YX^T) \quad \text{for all } Y \in \mathbb{R}^{n \times r}. \quad (10b)
\]

5.2 Proof Sketch of Theorem 4

A complete proof of Theorem 4 relies on a few technical lemmas, so we defer the complete proof to the Appendix. The key insight is that \( \delta_{\text{loc}}(X) \) is the solution to a convex optimization problem, which we can solve in closed-form. At first sight, evaluating \( \delta_{\text{loc}}(X) \) seems very difficult as it involves solving an optimization problem over the set of \( \delta \)-RIP operators, as defined in equation 5. However, we can modify the arguments of Zhang et al. \([23]\) to show that \( \delta_{\text{loc}}(X) \) can be reformulated as a convex optimization problem of the form
\[
\eta(X) = \max_{\eta, H} \left\{ \eta : X^T \text{He} = 0, \quad \eta I \preceq H \leq I \right\}. \quad (11)
\]
In fact, this reformulation of $\delta_{\text{loc}}(X)$ is exact, as stated in the theorem below. The proof of this theorem is mostly similar to the proof of Theorem 8 in [23], so we defer it to Appendix A.

**Theorem 7** (Exact convex reformulation). Let $\eta^*(X)$ be the optimal value of problem (11). Then

$$\delta_{\text{loc}}(X) = 1 - \eta^*(X).$$

Moreover, every minimizer $H$ for problem (11) is related to the minimizer $A^*$ in equation (9) via $H = (A^*)^T A^*$.

We will show that problem (11) actually has a simple closed-form solution. First, we write its Lagrangian dual as

$$\begin{align*}
\text{minimize}_{y, U_1, U_2} & \quad \text{tr}(U_2) \\
\text{subject to} & \quad (Xy)e^T + e(Xy)^T = U_1 - U_2 \\
& \quad \text{tr}(U_1) = 1, \quad U_1, U_2 \geq 0.
\end{align*}$$

Notice that strong duality holds because Slater’s condition is trivially satisfied by the dual: $y = 0$ and $U_1 = U_2 = 2I/n(n + 1)$ is a strictly feasible point. It turns out that the dual problem can be rewritten as an optimization problem over the eigenvalues of the matrix $(Xy)e^T + e(Xy)^T$. The proof of this in Appendix A.

More precisely, let $\lambda_i(y)$ denote the eigenvalues of the rank-2 matrix $(Xy)e^T + e(Xy)^T$. Then the dual problem can be rewritten as

$$\min_{y} \sum_{i=1}^{n} (-\lambda_i(y))_+ \quad \text{where} \quad (\alpha)_+ = \begin{cases} \alpha & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}$$

It is easy to verify that the only two non-zero eigenvalues of $(Xy)e^T + e(Xy)^T$ are

$$\|Xy\| \|e\| \cos^2 \theta_y \pm 1, \quad \text{where} \quad \cos \theta_y = \frac{e^T Xy}{\|e\| \|Xy\|}.$$ 

It follows that

$$\eta(X) = \min_{y} 1 - \cos \theta_y$$

and therefore

$$\delta_{\text{loc}}(X) = \max_{y} \cos \theta_y = \max_{y} \frac{e^T Xy}{\|e\| \|Xy\|}.$$ 

Let $y^*$ be the optimizer of the optimization problem above, then $\theta_{y^*}$ is simply the incidence angle between the error vector $e$. Thus we have $y^* = \arg \min_y \|e - Xy\|$. Using Lemma 7 in the Appendix, we show that solving for $y^*$ yields a closed-form expression for $\theta_{y^*}$ in the form

$$\sin \theta_{y^*} = \frac{\|Z^T (I - XX^*) Z\|_F}{\|XX^T - ZZ^T\|_F}.$$ 

Hence we have $\delta_{\text{loc}}(X) = \cos \theta$, with $\theta$ given by the equation above.

### 5.3 Proof Sketch of Theorem 5

The proof of Theorem 5 requires some technical computation so we only provide a sketch of the proof here. The complete version can be found in Appendix B.

The main idea behind the proof is that $\delta_{\text{loc}}(X)$, which is equal to $\cos \theta$, can be lower bounded in the $\varepsilon$-neighborhood $B_\varepsilon$. This allows us to obtain a lower bound for $\delta_{\text{loc}}(B_\varepsilon)$. Roughly speaking, $\cos \theta$ depends on the geometric relationship between $X$ and $Z$. In $B_\varepsilon$, this relationship is restricted, and as a result $\cos \theta$ cannot be arbitrarily small.

To obtain a lower bound on $\cos \theta$, it suffices to derive an upper bound for $\sin \theta$. According to Theorem 4 we have

$$\sin \theta = \frac{\|Z^T (I - XX^*) Z\|_F}{\|XX^T - ZZ^T\|_F}.$$ 


It turns out that $\sin \theta$ mainly depends on two geometric properties of $X$ and $Z$: the ratios between $\|XX^T\|_F$ and $\|ZZ^T\|_F$ and the angle between $XX^T$ and $ZZ^T$. To simplify notations, we define the parameters
\[
\rho^2 = \frac{\|XX^T\|_F}{\|ZZ^T\|_F}, \quad \cos^2 \phi = \frac{(XX^T, ZZ^T)}{\|XX^T\|_F \|ZZ^T\|_F}.
\] (14)

In Section 6, we provide some geometric intuition for why these two parameters help us to obtain a bound on $\sin \theta$. Now to upper bound $\sin \theta$, we need to upper bound the numerator. Some computations yields
\[
\|ZZ^T(I - XX^+)Z\|_F^2 \leq \|ZZ^T\|_F^2 (1 + \cos^2 \phi) \sin^2 \phi.
\] (15)

We also rewrite the denominator as
\[
\|XX^T - ZZ^T\|_F^2 = \|ZZ^T\|_F^2 [(\rho^2 - 1)^2 + 2 \rho^2 \sin^2 \phi].
\]

Thus
\[
\sin \theta \leq \frac{(1 + \cos^2 \phi) \sin^2 \phi}{\sqrt{(\rho^2 - 1)^2 + 2 \rho^2 \sin^2 \phi}} \leq \frac{(1 + \cos^2 \phi) \sin \phi}{\sqrt{\rho^4 + 1}}.
\] (16)

Suppose that $X \in B_{\varepsilon}$, then $\|XX^T - ZZ^T\|_F^2 \leq \varepsilon^2 \|ZZ^T\|_F^2$. This implies that $(\rho^2 - 1)^2 + 2 \rho^2 \sin^2 \phi \leq \varepsilon^2$. Therefore, we need to upper bound the right-hand side of Equation 16 subject to this constraint. Some more algebraic manipulation yields $\sin \theta \leq \tau(\varepsilon)$, where
\[
\tau(\varepsilon) = \frac{\varepsilon (2 - \varepsilon^2)}{(1 - \varepsilon)^2 + 1}, \text{ if } \varepsilon^2 \leq 2/3,
\]
and
\[
\tau(\varepsilon) = \frac{4 \sqrt{2 / 3}}{3 (1 - \varepsilon)^2 + 3}, \text{ otherwise}.
\]

Finally, we get
\[
\cos \theta = \sqrt{1 - \sin^2 \theta} > \sqrt{1 - \tau(\varepsilon)^2}.
\]

It is easy to see that $\tau(\varepsilon)$ is dominated by a linear function on the interval $[0, 1]$. In fact, we have $\tau(\varepsilon) \leq C \varepsilon$ where $C \approx 1.5$. If follows that $\delta_{\text{loc}}(B_{\varepsilon}) \geq \sqrt{1 - C \varepsilon^2}$. This completes the proof.

6 Numerical Results

In this section we numerically verify that $\delta_{\text{loc}}(X)$ and $\delta_{\text{soc}}(X)$ both becomes very close to 1 when $\varepsilon$ is small, as predicted by Theorem 5. We also show that $\delta_{\text{loc}}(X)$ becomes a tight lower bound for $\delta_{\text{soc}}(X)$ when $\varepsilon$ is small, thus providing numerical evidence that the bound in Theorem 6 is tight. Moreover, we give some geometric intuition for motivation behind the proof of Theorem 5, i.e., the value of $\delta_{\text{loc}}(X)$ mainly depends on two parameters defined in equation 14.

Our main results hold for any rank, but for ease of visualization we focus on the rank-1 case. Since $X$ and $Z$ are now just vectors, it is easy to see that the parameters in equation 14 reduces to
\[
\rho = \frac{\|X\|}{\|Z\|}, \quad \cos \phi = \frac{XX^T}{\|X\| \|Z\|}.
\]

For rank-1, these two parameters completely determines the values of both $\delta_{\text{loc}}(X)$ and $\delta_{\text{soc}}(X)$. See Section 8.1 for a proof of this fact. This allows us to plot the level curves of $\delta_{\text{loc}}(X)$ and $\delta_{\text{soc}}(X)$ over the parameter space $\rho$ and $\phi$ in Figure 2. This is shown by the blue curves. The red curves show the level sets of the function $\|XX^T - ZZ^T\|_F^2/\|ZZ^T\|_F$. The horizontal axis is the value of $\rho \cos \phi$ and the vertical axis is the value of $\rho \sin \phi$.

We can immediately see that when $\sin \phi$ is small, both $\delta_{\text{soc}}(X)$ and $\delta_{\text{loc}}(X)$ are very close to 1. Geometrically, this means that in regions in the optimization landscape where $X$ is more aligned with $Z$, the values of both threshold functions tend to be high, thus a relatively small number of samples suffices to prevent $X$ from becoming a spurious critical point. However, when $X$ and $Z$ becomes closer to being orthogonal, $\delta_{\text{loc}}(X)$ becomes arbitrarily small, and $\delta_{\text{soc}}(X)$ also becomes
smaller, albeit to a lesser extent. Thus, preventing $X$ from becoming a spurious critical point (or spurious local minima) in these regions require many more samples. This intuition also permeates to the high-rank case, even though visualization becomes difficult. For instance, we can see from equation $15$ that when $\sin \phi$ becomes small, so $\sin \theta$ is also small and $\delta_{\text{loc}}(X)$ becomes large. Similar to the rank-1 case, this means that in regions where $XX^T$ and $ZZ^T$ are more aligned, the sample complexity required to eliminate spurious critical points is small.

In addition, if we look at the level sets of the level sets of $\|XX^T - ZZ^T\|_F / \|ZZ^T\|_F$, we see that in regions close to the ground truth, both $\delta_{\text{soc}}(X)$ and $\delta_{\text{loc}}(X)$ are very close to $1$. This is in perfect agreement with our results in Theorem 5, where we should that a small $\epsilon$ results in a large $\delta_{\text{loc}}(X)$. Moreover, the shapes of the level curves of $\delta_{\text{soc}}$ and $\delta_{\text{loc}}$ that flow through the regions near the ground truth are almost identical. This provides numerical evidence that $\delta_{\text{loc}}(X)$ is in fact a sharp lower bound for $\delta_{\text{soc}}(X)$ when $\epsilon$ is small.

Figure 2: (a) the level sets of $\delta_{\text{loc}}$ and $\|XX^T - ZZ^T\|_F / \|ZZ^T\|_F$ (b) the level sets of $\delta_{\text{soc}}$ and $\|XX^T - ZZ^T\|_F / \|ZZ^T\|_F$

7 Conclusions

The problem of matrix sensing have been shown to contain no spurious local minima when there are sufficiently large amount of samples. On the other hand, previous results have also indicated that a good initial point can guarantee convergence to global minimum. In this paper we study the sharp threshold on the number of samples required to prevent each specific point on the optimization landscape from becoming a spurious local minima. We overcome the difficulty of computing a closed-form formula for this threshold by deriving sharp lower bounds based on spurious critical points instead. We show that this lower bound has a simple formula which admits a nice geometric meaning. As a result, we are able to characterize the precise relationship between the quality of an initial point and the sample complexity required to eliminate spurious local minima. For initial points not too close to the ground truth, a constant factor improvement of the initial point is worth a constant factor decrease in the sample complexity. Our work provides a new perspective on the value of a good initial point in low-rank matrix factorization. We hope that our results is another step towards understanding the importance of a good initialization in more complicated problems such as deep learning.

Broader Impact

Many modern applications in engineering and computer science, and in machine learning in particular often have to deal with non-convex optimization. However, many aspects of non-convex optimization are still not well understood. Our paper provides more insight into the optimization landscape of a
particular problem: low-rank matrix factorization. In addition, the methods we develop can be used to understand the many other non-convex problems. This is a step towards a more thorough analysis of current algorithms for non-convex optimization and also a step towards developing better and more efficient algorithms with theoretical guarantees.

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Appendix A

In this section we fill out the missing details in the proof sketch of Section 5.2, completing the proof of Theorem 4.

**Theorem 7** (Exact convex reformulation). Let \( \eta^*(X) \) be the optimal value of problem 11. Then
\[
\delta_{\text{soc}}(X) = 1 - \eta^*(X).
\]

Moreover, every minimizer \( \mathbf{H} \) for problem 11 is related to the minimizer \( \mathcal{A}^* \) in equation 5 via
\[
\mathbf{H} = \mathcal{A}^T \mathcal{A}^*.
\]

**Proof.** The proof of this theorem relies Theorem 8 of [23], where the authors showed a similar result for \( \delta_{\text{soc}}(X) \) instead. Repeating their arguments step by step while ignoring the second-order constraint \( \nabla^2 f(X) \geq 0 \), we have \( \delta_{\text{soc}}(X) = \text{LMI}(X) \), where \( \text{LMI}(X) \) is defined in [23] as the solution to the following optimization problem:
\[
\text{LMI}(X) \equiv \min_{\mathbf{H} = \mathcal{A}^T \mathcal{A}} \delta
\]
subject to \( f(X) = \frac{1}{2} \| \mathcal{A} \left( XX^T - ZZ^T \right) \|^2 \)
\[
\nabla f(X) = 0
\]
\[
(1 - \delta)I \preceq \mathcal{A}^T \mathcal{A} \preceq (1 + \delta)I.
\]

Now simply note that \( \eta^*(X) \), the optimal value of problem (11), satisfies
\[
\eta^*(X) = 1 - \text{LMI}(X).
\]

It follows that \( \delta_{\text{soc}}(X) = 1 - \eta^*(X) \).

In Section 5.2, we stated that the optimization problem (13) can be rewritten as an optimization problem over the eigenvalues of a rank-2 matrix. This is given in the proposition below.

**Proposition 1.** The optimization problem
\[
\begin{align*}
\min_{y, U_1, U_2} & \quad \text{tr}(U_2) \\
\text{subject to} & \quad (Xy)e^T + e(Xy)^T = U_1 - U_2 \\
& \quad \text{tr}(U_1) = 1, \quad U_1, U_2 \succeq 0.
\end{align*}
\]

can be rewritten as
\[
\min_y \sum_{i=1}^n \lambda_i(y) \mathbf{e}_i
\]
where \( \alpha \equiv \left\{ \begin{array}{c}
\alpha < 0 \\
0 \\
\alpha \geq 0
\end{array} \right. ,
\]

where \( \lambda_i(y) \) denotes the eigenvalues of the rank-2 matrix \( (Xy)e^T + e(Xy)^T \).

Before we prove Proposition 1, we state a technical lemma that makes the mechanics of this optimization problem clearer.

**Lemma 1.** Given a positive semidefinite matrix \( M \) we split the matrix \( M \) into a positive and negative part satisfying
\[
M = M_+ - M_- \quad \text{where} \quad M_+, M_- \succeq 0, \quad M_+ M_- = 0.
\]

Then the following problem has solution
\[
\text{tr} (M_-) / \text{tr} (M_+) = \min_{\alpha \in \mathbb{R}} \{ \text{tr}(V) : \text{tr}(U) = 1, \alpha M = U - V \}.
\]

**Proof.** (Lemma 1) Let \( p^* \) be the optimal value. Then we have
\[
p^* = \max_{\beta} \min_{\alpha \in \mathbb{R}} \{ \text{tr}(V) + \beta \cdot [\text{tr}(U) - 1] : \alpha M = U - V \}
\]
\[
= \max_{\beta \geq 0} \min_{\alpha \in \mathbb{R}} \{ -\beta + \min_{U, V \succeq 0} \{ \text{tr}(V) + \beta \cdot \text{tr}(U) : \alpha M = U - V \} \}
\]
\[
= \max_{\beta \geq 0} \{ -\beta : \text{tr}(M_-) + \beta \cdot \text{tr}(M_+) \}
\]
\[
= \text{tr} (M_-) / \text{tr} (M_+).
\]
Note that in the first line we converted an equality constraint into a Lagrangian. In the second line we optimize over $U,V$ with $\beta \geq 0$, noting that $\beta < 0$ will cause $\text{tr}(U)$ to go to negative infinity.

Now we are ready to prove Proposition 1.

**Proof.** (Proposition 1) First, suppose keep the length of $y$ fixed, and optimize problem (1) over $U_1$ and $U_2$. Applying Lemma 1 to this problem allows us to write the problem as a minimization over the ratio of the trace of the negative and positive component of $(X \hat{y})e^T + e(X \hat{y})^T$. Finally, optimizing over the length of $y$ gives the desired result.

Finally, the final steps in Section 5.2 relied on the fact that the incidence angle $\theta$ between $e$ and $X$ can be solved in closed form. We give a proof of this fact below.

**Lemma 7.** Let $X,Z$ be $n \times r$ matrices of any rank, and define $e$ and $X$ as in equations (9a) and (9b). Then, the incidence angle $\theta$ between $e$ and $X$, defined as in

$$
\cos \theta = \max_y \left\{ \frac{e^T X y}{\|e\| \|X y\|} \right\},
$$

has closed-form expression

$$
\sin \theta = \frac{\|Z^T (I - XX^+) Z\|_F}{\|XX^T - ZZ^T\|_F}
$$

where $X^+$ denotes the Moore–Penrose pseudoinverse of $X$.

**Proof.** Define $y^* = \arg \min_y \|e - X y\|$ and decompose $e = X y^* + w$. The optimality condition for $y^*$ reads $X^T (e - X y^*) = X^T w = 0$, so we have

$$
\|e\| \cos \theta = \|e\| \max_y \left\{ \frac{e^T X y}{\|e\| \|X y\|} \right\} = \max \left\{ \frac{(y^*)^T X^T X y}{\|X y\|} \right\} = \|X y^*\|,
$$

and therefore $\|e\| \sin \theta = \|w\| = \min_y \|e - X y\|$. Now, define $Q = \text{orth}(X) \in \mathbb{R}^{n \times q}$ where $q = \text{rank}(X) \leq r$, and define $P \in \mathbb{R}^{n \times (n-q)}$ as the orthogonal complement of $Q$. Decompose $X = Q \tilde{X}$, and $Z = Q \tilde{Z}_1 + P \tilde{Z}_2$, and note that

$$
\|w\| = \min_y \|e - X y\|
= \min_y \|((X X^T - ZZ^T) - (XY^T + YX^T))\|_F
= \min_{\tilde{Y}_1, \tilde{Y}_2} \|\begin{bmatrix} \tilde{X} \tilde{X}^T - \tilde{Z}_1 \tilde{Z}_1^T & -\tilde{Z}_1 \tilde{Z}_2^T \\ -\tilde{Z}_2 \tilde{Z}_1^T & -\tilde{Z}_2 \tilde{Z}_2^T \end{bmatrix} - \begin{bmatrix} \tilde{X} \tilde{Y}_1^T + \tilde{Y}_1 \tilde{X}^T & \tilde{X} \tilde{Y}_2^T \\ \tilde{Y}_2 \tilde{X}^T & 0 \end{bmatrix}\|_F
= \|\tilde{Z}_2 \tilde{Z}_2^T\|_F
$$

The third line uses the fact that the $q \times r$ matrix $\tilde{X}$ has full row rank, so that $\tilde{X} \tilde{X}^T \succ 0$ and $\tilde{X} \tilde{X}^+ = I_q$. Finally, note that $Q \tilde{Z}_1 = XX^+ Z$ and $P \tilde{Z}_2 = (I - XX^+) Z$ and that

$$
\|\tilde{Z}_2 \tilde{Z}_2^T\|_F^2 = \|P \tilde{Z}_2 \tilde{Z}_2^T P^T\|_F^2
= \|(I - XX^+) ZZ^T (I - XX^+)\|_F^2
= \text{tr}[(I - XX^+) ZZ^T (I - XX^+)(I - XX^+)]
= \text{tr}[Z^T (I - XX^+) ZZ^T (I - XX^+) Z]
= \|Z^T (I - XX^+) Z\|_F^2.
$$

Substituting the definition of $e$ completes the proof. □

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Appendix B

In this section we provide a complete proof of Theorem 5, which includes all the intermediate calculations that was skipped in Section 5.3.

Proof. Define
\[ \rho = \frac{\|XX^T\|_F}{\|ZZ^T\|_F}, \quad \cos^2 \phi = \frac{\langle XX^T, ZZ^T \rangle}{\|XX^T\|_F \|ZZ^T\|_F}. \]
and note that
\[ \|ZZ^T(I - XX^+)Z\|_F^2 = \min_Y \|ZZ^T - XY^T - YX^T\|_F^2 \]
\[ \leq \min \|ZZ^T - \alpha XX^T\|_F^2 \]
\[ = \|ZZ^T\|_F^2 + \min \{\|\alpha XX^T\|_F^2 - 2\alpha \langle ZZ^T, XX^T \rangle\} \]
\[ = \|ZZ^T\|_F^2 - \frac{\langle ZZ^T, XX^T \rangle^2}{\|XX^T\|_F^2} \]
\[ = \|ZZ^T\|_F^2 (1 - \cos^4 \phi) \]
\[ = \|ZZ^T\|_F^2 (1 + \cos^2 \phi) \sin^2 \phi \]
and
\[ \|XX^T - ZZ^T\|_F^2 \]
\[ = \|XX^T\|_F^2 - 2\langle XX^T, ZZ^T \rangle + \|ZZ^T\|_F^2 \]
\[ = \|ZZ^T\|_F^2 (\rho^2 - 2\rho \cos^2 \phi + 1) \]
\[ = \|ZZ^T\|_F^2 [\rho^2 - 2\rho \sin^2 \phi] \]
and therefore
\[ \sin \theta \leq \frac{(1 + \cos^2 \phi) \sin^2 \phi}{\sqrt{(\rho - 1)^2 + 2\rho \sin^2 \phi}} \leq \frac{(1 + \cos^2 \phi) \sin \phi}{\sqrt{\rho^2 + 1}}. \]
Since \( \|XX^T - ZZ^T\|_F^2 \leq \varepsilon^2 \|ZZ^T\|_F^2 \), we have
\[ (\rho - 1)^2 + 2\rho \sin^2 \phi \leq \varepsilon^2. \]
It follows that
\[ (\rho^2 + 1) \sin^2 \phi \leq (\rho - 1)^2 + 2\rho \sin^2 \phi \leq \varepsilon^2, \]
and
\[ (\rho - 1)^2 \leq \varepsilon^2. \]
Thus \( \sin^2 \phi \leq \varepsilon^2 \) and \( 1 - \varepsilon \leq \rho \leq 1 + \varepsilon. \) Set \( x = \sin^2 \phi \), then
\[ (1 + \cos^2 \phi) \sin \phi = \sqrt{x(2 - x)}. \]
If \( \varepsilon^2 > 3/2 \), the maximum of the equation above is achieved at \( \sin^2 \phi = 2/3. \) Otherwise, the maximum is achieved at \( \sin^2 \phi = \varepsilon^2. \) Therefore,
\[ \sin \theta \leq \frac{(1 + \cos^2 \phi) \sin \phi}{\sqrt{\rho^2 + 1}} \leq \tau(\varepsilon). \]
where
\[ \tau(\varepsilon) = \begin{cases} \frac{\varepsilon(2 - \varepsilon^2)}{1 - \varepsilon^2 + \varepsilon^4}, & \text{if } \varepsilon^2 \leq 2/3 \\ \frac{4\sqrt{2/3}}{3(1-\varepsilon)^2+3}, & \text{otherwise}. \end{cases} \]
It follows that
\[ \cos \theta = \sqrt{1 - \sin^2 \theta} > \sqrt{1 - \tau(\varepsilon)^2}. \]
It is easy to see that \( \tau(\varepsilon) \) is dominated by a linear function on the interval \([0, 1]\). In fact, we have
\( \tau(\varepsilon) \leq C\varepsilon \) where \( C \approx 1.5. \) It follows that \( \delta_{\text{loc}}(B) \geq \sqrt{1 - C\varepsilon^2}. \) \( \square \)