Universality of Holographic Phase Transitions and Holographic Quantum Liquids

PAOLO BENINCASA

Center for Particle Theory & Department of Mathematical Sciences
Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
paolo.benincasa@durham.ac.uk

Abstract

We explore the phase structure for defect theories in full generality using the gauge/gravity correspondence. On the gravity side, the systems are constructed by introducing $M$ (probe) $D(p + 4 - 2k)$-branes in a background generated by $N$ $Dp$-branes to obtain a codimension-$k$ intersection. The dual gauge theory is a $U(N)$ Supersymmetric Yang-Mills theory on a $(1 + p - k)$-dimensional defect with both adjoint and fundamental degrees of freedom. We focus on the phase structure in the chemical potential versus temperature ($\mu, T$) plane. We observe the existence of two universality classes for holographic gauge theories, which are identified by the order of the phase transition in the interior of the ($\mu, T$)-plane. Specifically, all the sensible systems with no defect show a third order phase transition. Gauge theories on a defect with $(p - 1)$-spatial directions are instead characterised by a second order phase transition. One can therefore state that the order of the phase transition in the interior of the ($\mu, T$)-plane is intimately related to the codimensionality of the defect. We also discuss the massless hypermultiplet at low temperature, where a thermodynamical instability seems to appear for $p < 3$. Finally, we comment on such an instability.

October 2009
1 Introduction

Gauge/gravity correspondence [1–4] provides a powerful tool to investigate the dynamics of strongly coupled gauge theories. The original formulation [1] conjectures the equivalence between supergravity on the “near-horizon” geometry generated by a stack of $N$ coincident D3-branes ($AdS_5 \times S^5$) and the gauge theory (four-dimensional $\mathcal{N} = 4$ SU($N$) Supersymmetric Yang-Mills) living on the boundary of $AdS_5$, which describes the brane modes decoupled from the bulk. It can be straightforwardly extended to any asymptotically $AdS \times M$ geometry, $M$ being a compact manifold. This conjectured equivalence is made precise by identifying the string partition function with the generating function for the gauge theory correlators, with the boundary value of the bulk modes acting as source of the correspondent gauge theory operator [3].

It can be extended to the case of arbitrary $D_p$-branes ($p \neq 3$), for which the world-volume gauge theory is again equivalent to the supergravity on the near-horizon background generated by the $D_p$-branes [5]. The dual gauge theory is a $(p + 1)$-dimensional $U(N)$ supersymmetric Yang-Mills theory. Contrarily to the case of the D3-branes, it has a dimensionful coupling constant and the effective coupling depends on the energy scale: the gauge theory is no longer conformal. More specifically, there exists a frame [6] in which the near-horizon geometry induced by the $D_p$-branes is conformally $AdS_{p+2} \times S^{8-p}$ [7–9]. In this frame, the existence of a generalized conformal symmetry [10] becomes manifest and the radial direction (transverse to the boundary) acquires the meaning of energy scale of the dual gauge theory [9, 11], as in the original $AdS/CFT$-correspondence. Moreover, the holographic RG flow turns out to be trivial and the theory flows just because of the dimensionality of the coupling constant. In the case of the D4-branes, the theory flows to a 6-dimensional fixed point at strong coupling: the world-volume theory of D4-branes flows to the world-volume theory of M5-branes.

Gauge/gravity correspondence can be further generalized by inserting extra degrees of freedom in the theory. More precisely, one can add a finite number of branes and consider the probe approximation, so that the backreaction on the background geometry can be neglected. Inserting probe branes introduces a fundamental hypermultiplet in the gauge theory, partially or completely breaking the original supersymmetries [12].

Here we are mainly interested in the phase structure of the BPS brane intersections at finite temperature and finite chemical potential. The phase diagram temperature versus chemical potential has been studied in details especially in relation to the D3/D7 system, where the probe D7-branes are parallel to the background D3-branes [13–20]. A similar analysis has been carried out for the D4/D6 system with the direction of the background D4-branes which is not parallel to the probe D6-branes compactified to a circle so that the
system has effectively codimension 0 [21]. Considering a system at finite temperature and finite (“baryonic”) chemical potential means considering a black brane background solution and a non-trivial (“electric”) profile for the world-volume gauge field on the probe branes: the temperature of the system is given by the Hawking temperature of the black brane background and the chemical potential is provided by the boundary value of the time component of the world-volume gauge potential.

Along the temperature axis, the system undergoes a first order phase transition (Figure 1), which can be seen as a meson dissociation transition [14,17,22]. From the branes perspective,

![Minkowski Embedding](image1)

![Critical Embedding](image2)

![Black Hole Embedding](image3)

(a) Minkowski-Embedding  
(b) Critical-Embedding  
(c) Black-Hole-Embedding

Figure 1: First order phase transition. At low enough temperature \((T < T_d)\) the probe branes lie outside the black hole (Minkowski embedding). Increasing the temperature, the black hole increases in size and the probe branes starts to bend towards it more and more until they touch the black hole at \(T = T_d\) (Critical embedding). For \(T > T_d\) part of the brane falls inside the black hole (Black hole embedding). The transition from Minkowski embedding to black-hole one is of first order.

at sufficiently small temperatures, the probe branes lie outside the black-hole background (Minkowski embedding - Figure 1(a)). Increasing the temperature, the size of the black hole
increases and, consequently, the probe branes start to feel stronger and stronger the black hole attraction, bending more and more towards the black hole until they touch the black hole horizon at one point (critical embedding) at temperature $T = T_d$ (Figure 1(b)). Continuing increasing the temperature part, of the probe branes fall into the black hole horizon (black hole embedding - Figure 1(c)).

Along the chemical potential axis, the system instead undergoes a second order phase transition [23]. In the interior of the $(\mu, T)$-plane there has been identified a transition curve $\mu = m(T)$: below this curve the system is in a Minkowski phase in which the “quark” density is zero, while above it the system is in a black-hole phase with non-vanishing “quark” density. This transition curve was numerically found to be first order [16]. However, recently an analytic computation by Faulkner and Liu [20] showed that this transition is actually of third order. In this picture, in the region $\mu < m(T)$ of the phase diagram (Minkowski embedding), the DBI-action of the D7-branes provides the dominant contribution to description of the brane embedding. For $\mu > m(T)$ string worldsheet instantons contribute as well [24], creating an instability: the instantons condense and create a neck between the probe D7-branes and the black hole so that at $\mu = m(T)$ the Minkowski embedding goes over to a black-hole embedding (Figure 2).

![Minkowski Embedding](image1)

![Instanton Condensation](image2)

(a) Minkowski-Embedding  (b) Instanton-Condensation

Figure 2: Worldsheat instanton effects. In the region $\mu > m$ the instanton corrections to the DBI-action becomes relevant given their dependence on $\sim e^{-|n|\beta \left(m(T) - \frac{\mu}{m}\right)}$. Their contribution becomes dominant so that they condense following a neck until the black hole, sending the system to the black hole phase.

A natural question to ask is how universal are these phase transitions and which type of phase transitions these systems allow to investigate. The intersecting brane constructions provide an arena to construct model which may points towards a deeper theoretical understanding of QCD-like [25–29] and condensed-matter-like [30–37] features.
In this paper, we investigate the phase structure in the plane chemical potential versus temperature for BPS intersecting brane systems, which are constructed by introducing a stack of $M$ probe $D(p + 4 - 2k)$-branes in a background generated by $N$ $D_p$-branes ($M \ll N$). The parameter $k$ ($k = 0, 1, 2$) indicates the codimensionality of the intersection, i.e., the number of the spatial directions of the background branes along which the probe branes do not extend. The dual gauge theory is therefore a $(p + 1)$-dimensional $U(N)$ Supersymmetric Yang-Mills theory with a $(p + 1 - k)$-dimensional defect on which the fundamental degrees of freedom propagate.

We start with considering the massive fundamental hypermultiplet, which means describing the embedding of the probe branes through a coordinate of the transverse space. In order to introduce the chemical potential, we turn on an electric ansatz for the world-volume gauge field.

At zero temperature one can identify a brane/anti-brane phase and a “black-hole” crossing phase. In the first case, the probe brane can extend from the boundary to a minimum distance at which the branes turn and go back hitting the boundary again, forming therefore a brane/anti-brane configuration. In the second case, the probe branes can extend down to the location of the background branes. The latter phase is thermodynamically favoured, the grand-potential being negative. The phase transition between these two configurations is of second order as in [23]. The presence of a second order phase transition along the chemical potential axis is therefore a universal property of the $D_p/D(p + 4 - 2k)$ system. The previous analysis applies to systems with $k = 0, 1$. The case $k = 2$ is very different and subtle: turning on the embedding mode in the transverse space does no longer correspond to a mass deformation, but rather it provides a vacuum-expectation-value for the dual operator. However, suitably defining the “physical” chemical potential as well as the “physical” mass, one can show that also gauge theories dual to these brane constructions fall in this universality class.

At finite temperature, the systems show a transition line in the chemical potential versus temperature plane. We analyse such a phase transition analytically following the approach proposed in [20]. The equations of motion for the embedding function and the gauge potential are too involved to be analytically solved. The idea in [20] is to perturbatively solve them considering the “quark”-density (suitably rescaled to be a dimensionless parameter) as parameter expansion. Such a perturbative analysis allows to obtain an analytical expression for the chemical potential which in turn allows to analytically study the order of the phase transition. In the case of systems with no defect, the chemical potential acquires the following expression up to first order in the quark density $c_f$.

$$\mu = m(T) + s_1(T) c_f - s_2(T) c_f \log c_f + O(c_f^2),$$  \hspace{1cm} (1.1)
\textit{i.e.} the structure found in [20] for the D3/D7 system generalises to more general D$_{p}$/D($p + 4$) systems.

In the case of codimension-1 defect systems, there is no logarithmic term at first order. This is actually crucial. The presence of the logarithmic term in (1.1) implies that the phase transition is of third order. In the case no logarithmic term is present at first order in $c_f$, the system show a second order phase transition. It is thus possible to state that the order of the transition line in the chemical potential versus temperature plane is strictly tied to the codimensionality of the system under examination: systems with a codimension-0 defect (\textit{i.e.} no defect) show a third order phase transition, while systems with a codimension-1 defect are characterised by a second order phase transition. In the latter case, the whole transition line is of second order. For $k = 0$, the function $s_2$ has a point at $T = T_c < T_d$ where it vanishes: this point represents a tricritical point at which the phase transition becomes of second order.

We also consider the case of massless hypermultiplets in the fundamental representation. As mentioned before, such description is provided by fixing the probe branes to wrap the maximal $S^{3-k} \subset S^{8-p}$ with the probe branes wrapping in a $(p + 2 - k)$-dimensional subspace of the $(p + 2)$-dimensional non-compact manifold.

For defect theories, this embedding of the probe branes is described by the scalar mode $x^p \equiv z(\rho)$, with $x^p$ being the direction of the background branes along which the probe branes do not extend. When such mode has a non trivial profile, the supersymmetries are broken since it provides a vacuum-expectation-value to its dual operator. The supersymmetries are instead preserved if the scalar mode $z(\rho)$ has a trivial profile (\textit{i.e.} no scalar mode is turned on). For codimension-0 ($k = 0$) systems, the supersymmetric embeddings are the only possible. In [38] massless $\mathcal{N} = 2$ hypermultiplets in $\mathcal{N} = 4$ Supersymmetric Yang Mills were studied via probe $D(7-2k)$-branes wrapping an $AdS_5 \times S^{3-k}$ space. In this framework, the specific heat and the zero-sound mode for holographic quantum fluids have been discussed, \textit{i.e.} translationally invariant systems at low temperature and finite chemical potential. The zero sound mode turns out to persist to all the value of the hypermultiplet mass for such systems [39] and was extensively discussed for D4/D8/$\bar{D}8$ systems [40]. In [41] the analysis of [38] was extended to D$p$/D$q$ systems, where both massless and massive deformations were analysed. In real world, there are two main classes of quantum liquids in $(1 + d)$-dimensions, with $d > 1$: the Bose and Fermi fluids. In the first case, the low energy elementary excitations are given by superfluids phonons with linear dispersion relation and specific heat $c_v \sim T^d$. Fermi liquids have both bosonic and fermionic quasi-particle excitations, with the latter dominating at low temperature and fixing the specific heat to scale directly proportionally to the temperature: $c_v \sim T$ ($\forall d > 1$). This Fermi-liquid type of behaviour has been observed
in holographic gravity duals constructed with background D4-branes [40, 41]. The \((1 + 1)\)-dimensional case is indeed peculiar and can never be described through Bose or Fermi liquids. First, in \((1 + 1)\)-dimensions just collective motion is possible. The reason is straightforward to understand. The excitations have one-direction only in which they can propagate and therefore they would necessarily scatter with other excitations put them in motion. Such systems are thought to be described by the Tomonaga-Luttinger liquids [42–47]. They do not have quasi-particle peaks and are characterised by two elementary excitations: plasmons, \(i.e.\) sound waves as response of charge density to external perturbations with speed dependent on the interactions, and spin density waves which propagates independently of the plasmons. A first attempt to extensively discuss a holographic realisation of \((1 + 1)\)-dimensional quantum liquids has been made using a D3/D3-system [37]. At low temperature, the specific heat in the dual field theory scales quadratically in the temperature, while for Tomonaga-Luttinger liquid the specific heat scales linearly if the system is not interacting and scales as \(\sim T^{\alpha(K)}\) for interacting systems (the power \(\alpha(K)\) depends on the Luttinger parameter \(K\) and acquires different forms according to the repulsive or attractive nature of the interaction).

In this paper we focus again on the massless hypermultiplet. As in [41], we notice that at low temperature both the entropy density and the specific heat scale with the temperature as \(T^{\frac{3-2p}{3}}\), which implies that the systems with \(p < 3\) have divergent entropy density and divergent specific heat as the temperature approaches zero. Furthermore, the specific heat appears to be negative. For \(p = 3\) the system is characterised by a non-zero entropy density at zero temperature and the specific heat scaling with the temperature depends on the codimensionality of the defect: \(c_v \sim T^{2(3-k)}\). Finally, for \(p = 4\), the entropy density and the specific heat scales linearly in the temperature and, therefore, such a system is characterised by a zero entropy density and a zero specific heat at zero temperature. While the thermodynamical behaviour for \(p \geq 3\) is physically intuitive, it is not the same for \(p < 3\). First, a scaling \(T^{-\gamma}\) \((\gamma > 0)\) for the entropy density and specific heat seems to violate the third law of thermodynamics, which requires the entropy density of a system to reach a minimum\(^1\) at zero temperature (and, consequently, to have zero specific heat). We study the thermodynamical stability of these systems. Since we work in the grand-canonical ensemble, we need to check the positive-definiteness of the Hessian matrix of (minus) the grand-potential. The Hessian

\(^1\)Strictly speaking the third law of thermodynamics (Nernst theorem) states that at zero temperature the entropy vanishes - provided that this limit is taken by keeping the other thermodynamical quantities fixed. However, there exist systems which Nernst theorem does not apply to or, more precisely, where its statement needs to be modified. For example, the previous formulation does not apply to amorphous solids, which are not in equilibrium. They have a finite non-zero entropy at zero temperature (holographic quantum liquids constructed with background D3-branes have such a characteristic [38]). Therefore, a more general formulation of the third law of thermodynamics is that the entropy reach a minimum in the limit of zero temperature, which is the one we stated in the text.
matrix turns out to be positive definite for $p \geq 3$ and negative definite for $p < 3$. This is a signature of a thermodynamical instability at low temperature for $p < 3$. The natural question to ask is what is the interpretation of such an instability and how it matches with the existence of perfectly well behaved backreacted solutions at zero temperature involving D-branes with $p < 3$, such as the Cherkis-Hashimoto solution for D2/D6 systems [48]. We will argue that most likely this is a signature of the breaking down of the probe approximation. As we just mentioned, the Hessian matrix of the grand-potential turns out to be negative definite with just one of the eigenvalues being negative. One might think to try to stabilise the system by turning on also a magnetic component for the world-volume gauge field strength. This however will just produce an overall factor dependent on the magnetic charge and, as a consequence, it does not stabilise the system. The only tunable parameter is the number $M$ of probe branes. Increasing it, one obtains a family of potential curves and it is possible to keep increasing the number of branes until the Hessian changes sign. We will elaborate more extensively on this point in the Conclusion section.

The paper is organized as follows. In Section 2 we introduce the codimension-$k$ defect theories by discussing their brane realisation. The codimension-$k$ defect is created by introducing probe D($p + 4 - 2k$)-brane in the background generated by the D$p$-branes in a suitable way. In Section 3 we consider the probe brane embedding configuration which introduces massive fundamental degrees of freedom in the $(p + 1 - k)$-dimensional defect. We explore the $(\mu, T)$-plane for such systems. We show that along the chemical potential axis ($T = 0$) there is a second order phase transition for any sensible system of codimension $k = 0, 1$. We briefly discuss the case $k = 2$. We also investigate the existence of a phase transition in the interior of the $(\mu, T)$-plane, showing that the order of the phase transition crucially depends on the codimensionality of the system. In Section 4 we consider the probe brane embedding configuration which introduces massless degrees of freedom. We discuss the behaviour of the low temperature density entropy and specific heat. We analyse the stability of the system by studying the positive-definiteness properties of the Hessian matrix and we show that the systems with $p < 3$ are thermodynamically unstable at low temperature. Finally, Section 5 contains conclusion and a summary of the results.

## 2 Codimension-$k$ Defect Theories

Consider the background generated by a stack of $N$ black D$p$-branes in the near-horizon limit

\[
\begin{align*}
    ds^2_{10} &= g_{MN} dx^M dx^N = \\
    &= \left( \frac{r}{r_p} \right)^{\frac{7-p}{2}} \left[ -h_p(r) dt^2 + d\vec{x}^2 \right] + \left( \frac{r}{r_p} \right)^{\frac{-p}{2}} \left[ \frac{dr^2}{h_p(r)} + r^2 d\Omega_{8-p}^2 \right],
\end{align*}
\]

(2.1)
where the constant \( r_p \) and the function \( h_p(r) \) are respectively given by

\[
r_p^{7-p} \equiv (2\sqrt{\pi})^{5-p} \Gamma \left( \frac{7-p}{2} \right) g_s N (\alpha')^{\frac{7-p}{2}} \equiv d_p g_s N (\alpha')^{\frac{7-p}{2}},
\]

\[
h_p(r) = 1 - \frac{r_p^{7-p}}{r^{7-p}},
\]

with \( r_h \) parametrising the position of the black brane horizon The dilaton and the background \((p + 1)\)-form are respectively

\[
e^\phi = g_s \left( \frac{r_h}{r_p} \right)^{\frac{(7-p)(p-3)}{4}}, \quad C_{\alpha\ldots p} = g_s^{-1} \left( \frac{r}{r_p} \right)^{7-p}.
\]

The coupling constant \( g_{YM} \) of the dual gauge theory is dimensionful (for \( p \neq 3 \)) and defined by

\[
g_{YM}^2 \equiv g_s (2\pi)^{p-2} \left( \alpha' \right)^{\frac{p-3}{2}}.
\]

The temperature of the background is given by the Hawking temperature

\[
T = \frac{\kappa}{2\pi} = \frac{7-p}{4\pi r_p} \left( \frac{r_h}{r_p} \right)^{\frac{5-p}{2}}.
\]

Let us redefine the radial coordinate according to the following differential relation

\[
\frac{d\sigma}{\sigma} = \frac{dr}{r \sqrt{h_p(r)}},
\]

so that the background metric \((2.1)\) takes the form

\[
ds_{10} = \left( \frac{\sigma}{r_p} \right)^{\frac{7-p}{2}} h_+ \left[ - \frac{h_p^2}{h_+^2} dt^2 + d\mathbf{x}^2 \right] + \left( \frac{r}{\sigma} \right)^{\frac{7-p}{2}} h_+^{\frac{p-3}{2}} \left[ d\sigma^2 + \sigma^2 d\Omega_{8-p}^2 \right],
\]

where the new radial coordinate \( \sigma \) has been rescaled

\[
\sigma \rightarrow \frac{\sigma}{\sigma_h}, \quad \sigma_h \equiv \frac{r_h}{2^{\frac{p}{2}}}, \quad \sigma \in [\sigma_h, +\infty[\]

and the functions \( h_+(\sigma) \) is defined as

\[
h_+(\sigma) \equiv 1 + \frac{\sigma_h^{7-p}}{\sigma^{7-p}}.
\]

The position of the black-brane horizon is now parametrized by \( \sigma_h \) and the background temperature can be rewritten as

\[
T = \frac{9-p}{2^{7-p} \pi r_p \left( \frac{\sigma_h}{r_p} \right)^{\frac{5-p}{2}}}
\]
The dilaton and the background \((p + 1)\)-form respectively become

\[
e^{\phi} = g_s \, h^{\rho=3}_{1/2} \left( \frac{\sigma}{r_p} \right)^{\frac{(7-p)(p-3)}{4}}, \quad C_{0...p} = g_s^{-1} h^{\rho=2}_{1/2} \left( \frac{\sigma}{r_p} \right)^{7-p}.
\]  

(2.11)

For later convenience it is also useful to express the background metric (2.1) in a frame in which it is manifestly conformal to an \(AdS_{p+2} \times S^{8-p}\) black-hole space for \(p \neq 5\) (the so-called “dual” frame) \([9, 11]\). This can be easily seen by redefining the radial coordinate according to

\[
u^2 = \frac{r^{5-p}}{r_p^{5-p}}, \quad u_p = \frac{5-p}{2}
\]

(2.12)

and rewriting the line element (2.1) as

\[
ds^2_{10} = \left( N e^{\phi} \right)^{2/(7-p)} ds^2_{10},
\]

(2.13)

so that the line element \(ds^2_{10}\) describes an \(AdS_{p+2} \times S^{8-p}\) black-hole geometry

\[
ds^2_{10} = g_s \, \frac{r}{u_p} \left( \frac{r}{u_p} \right)^2 e^{\frac{2}{7-p} \phi} \left\{ u^2 \left[ -h_p(u) \, dt^2 + d\vec{x}^2 \right] + \left[ h_p(u) \right]^{-1} \frac{du^2}{u^2} + u_p^2 \, d\Omega^2_{8-p} \right\}
\]

(2.14)

with

\[
h_p(u) = 1 - \left( \frac{u}{u_p} \right)^{\frac{5-p}{2}}, \quad u_h = \frac{u_p}{r_p^2} \frac{r_h}{r_p^2}.
\]

(2.15)

Parametrising the position of the event horizon by \(u_h\), the background temperature can be conveniently written as

\[
T = \frac{7-p}{4\pi} \frac{u_h}{u_p} = \frac{7-p}{2\pi(5-p)} u_h.
\]

(2.16)

It is useful to redefine the radial coordinate through the following differential relation

\[
[h_p(u)]^{-\frac{1}{2}} \frac{du}{u} = \frac{d\rho}{\rho},
\]

(2.17)

so that the boundary and the black-hole horizon are now located at \(\rho = 0\) and \(\rho = 1\) respectively and the function \(h_p(u)\) can be conveniently written as

\[
h_p(u) = \frac{h^2(\rho)}{h^2_+(\rho)}, \quad h_+(\rho) = 1 + \rho^{\frac{7-p}{5-p}}.
\]

(2.18)

Let us rescale the radial coordinate \(\rho\) as follows

\[
\rho \rightarrow \rho, \quad \rho_h = \frac{u_h}{2^{7-p}},
\]

(2.19)

where \(\rho_h\) parametrises the position of the black hole horizon. In these coordinates, the background metric, the dilaton and the background \((p+1)\)-form becomes

\[
ds^2_{10} = \left( \frac{r}{u_p} \right)^{\frac{2}{3-p}} \frac{h^{\rho=3}_{\rho=3}}{h^{\rho=3}_{\rho=3}} \left\{ \frac{2^{5-p}}{r_p^{5-p}} (\rho)^{5-p} \left[ \frac{h^2_+ (\rho)}{h^2_+ (\rho)} dt^2 + d\vec{x}^2 \right] + \frac{dp^2}{\rho^2} + u_p^2 d\Omega^2_{8-p} \right\}
\]

(2.20)

\[
e^{\phi} = g_s \, \left( \frac{r}{u_p} \right)^{\frac{(p-3)(7-p)}{2(5-p)}} \frac{h^{\rho=3}_{\rho=3}}{h^{\rho=3}_{\rho=3}} \left( \frac{r}{u_p} \right)^{\frac{(p-3)(7-p)}{2(5-p)}} \left[ \frac{h^2_+ (\rho)}{h^2_+ (\rho)} dt^2 + d\vec{x}^2 \right] + \frac{dp^2}{\rho^2} + u_p^2 d\Omega^2_{8-p} \right\}
\]

\[
C_{0...p} = g_s^{-1} \left( \frac{r}{u_p} \right)^{\frac{2}{3-p}} \frac{h^2_+ (\rho)}{h^2_+ (\rho)} \rho^{\frac{7-p}{5-p}}.
\]
From now on we will work with the coordinates (2.7), unless otherwise specified.

In the geometry (2.7) let us introduce $M$ parallel probe $D(p+4-2k)$-branes ($M \ll N$) according to the following intersection configuration

|   | 0 | 1 | ... | p-3 | p-2 | p-1 | p | p+1 | p+2 | p+3 | p+4 | ... | 8 | 9 |
|---|---|---|-----|-----|-----|-----|---|-----|-----|-----|-----|-----|---|---|
| $Dp$ | X | X | ... | X | X | X | X | X | X | X | X | X | X | X | ... |
| $k = 0$ | X | X | ... | X | X | X | X | X | X | X | X | X | X | X | ... |
| $k = 1$ | X | X | ... | X | X | X | X | X | X | X | X | X | X | X | ... |
| $k = 2$ | X | X | ... | X | X | X | X | X | X | X | X | X | X | X | ... |

intersecting the background $Dp$-branes along $p-k$ directions $\{x_1^{p-k}\}_{i=1}$ and thus forming a defect of codimension $k$ ($k = 0, 1, 2$). The probe branes wrap an internal $(3-k)$-sphere $S^{3-k} \subset S^{8-p}$. The transverse space can be parametrized as

$$
\begin{align*}
\text{ds}_T^2 &= d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\Omega_{3-k}^2 + \cos^2 \theta d\Omega_{4-p+k}^2) \\
\text{ds}_T^2 &= d\varrho^2 + dy^2 + \varrho^2 d\Omega_{3-k}^2 + y^2 d\Omega_{4-p+k}^2,
\end{align*}
$$

(2.21)

where the two parametrisation in (2.21) are related by $y = \sigma \cos \theta$, $\varrho = \sigma \sin \theta$. The above configuration for the brane intersections ensures that the systems are BPS at zero temperature and therefore no stability issues arise for the ground state.

The presence of probe branes introduces hypermultiplets in the fundamental representation propagating on a $(1 + (p-k))$-dimensional defect. Given the presence of a codimension-$k$ defect, the embedding of the $D(p+4-2k)$ branes in the $Dp$-brane background can in principle be described through two functions $x^p \equiv z(\rho)$ (for $k \neq 0$) and either $\theta \equiv \theta(\rho)$ or $y \equiv y(\rho)$ dependently on the parametrisation (2.21) chosen. Let us comment on these two classes of embeddings.

Fixing the position of the probe branes in the $(1 + (p-k))$-dimensional non-compact submanifold at $z = 0$, the probe branes embedding can be controlled by a scalar mode which can be turned on by requiring that a coordinate in the transverse space has a non trivial profile. As mentioned earlier, the embeddings can be parametrized either through $\theta(\rho)$ or $y(\rho)$, dependently on the parametrisation of the transverse space (2.21). In this class of embeddings, the space-time distance between the background branes and the probe ones in the transverse space appears as a parameter which is related to the mass of the fundamental hypermultiplet. Thus, analysing this class of embeddings is equivalent to consider massive hypermultiplets propagating in a $(1 + (p-k))$-dimensional defect.

This physical interpretation holds for $k = 0, 1$. In the case of codimension-2 intersections, the gauge theory on both branes stays dynamical and can be viewed as two gauge-theories coupled through bifundamental hypermultiplets living on a $(1 + (p-2))$-dimensional defect (the D3/D3 system was discussed in [49], while the Higgs branch for Dp/Dp systems was
analysed in [50]). Furthermore, the coefficient of the normalizable mode determines the vev of the operator dual to the embedding function, implying that the space-time separation between the background Dp-branes and the probe Dp-branes is no longer a parameter but rather provides a vev for a dynamical field.

The second class of embeddings describes the embedding of the \((p + 2 - k)\)-dimensional submanifold \(M_{p+2-k}\) in the non-compact \((p + 2)\)-dimensional manifold \(M_{p+2}\) keeping the position of probe branes in the transverse space fixed to wrap the maximal sphere \(S^{3-k}\). This corresponds to set the mass of the fundamental hypermultiplet to zero. It can be parametrized through the coordinate \(x^p \equiv z(\rho)\), with \(\theta = 0\). The embedding mode \(z\) is related to the vev of its dual operator \(O_z\). If it is has a non-trivial profile, the operator \(O_z\) acquires a non-zero vev breaking the supersymmetries. The supersymmetries are instead restored if the embedding mode is constant (\(z = 0\)). In the case the probe branes fill the whole non-compact manifold \(M_{p+2}\), obviously it is not possible to turn on such a mode and the description of the massless excitations is necessarily supersymmetric.

For the time being, let us keep both of the two embedding functions and consider an electric component for the world-volume gauge field strength \(F_{AB}\)

\[
F_2 = -f' (\rho) \, dt \wedge d\rho.
\]

This means that the gauge field on the boundary field theory couples to a \(U(1) \subset U(M)\) current. With such an ansatz, the probe branes are described through the DBI-action which acquires the form

\[
S_{D(p + 4 - 2k)} = -M \, T_{D(p + 4 - 2k)} \int d^{p+5-2k} \xi \, e^{-\phi} \sqrt{-\det \{g_{AB} + F_{AB}\}} =
\]

\[
= -M \, T_{D(p + 2)} \, N_{k} \int d^{p+2-k} \xi \, h_{+}^{\frac{p+1-2k}{2-\frac{p}{2}}} \, \frac{g^{-3-k}}{2} \times
\]

\[
\times \left[ 1 + (y')^2 + \left( \frac{\sigma}{r_p} \right)^{7-p} \, \frac{g^{\frac{2-p}{2}}}{h_+^{\frac{2-p}{2}}} \left( z' \right)^2 - \frac{\frac{2-p}{2}}{h_+^2} \left( f' \right)^2 \right]^{\frac{1}{2}}.
\]

The second line of (2.23) has been written by using the solution for the dilaton (2.11) and the usual factor \((2\pi\alpha')\) in front of the gauge field strength has been absorbed in \(F_2\). The induced metric on the \(D(p + 4 - 2k)\)-brane world-volume metric is

\[
ds_{p+5-2k}^2 = \left( \frac{\sigma}{r_p} \right)^{7-p} \, h_+ \left[ \frac{h_+^2}{2} \, dt^2 + d\hat{x}^2 \right] + \left( \frac{r_p}{\sigma} \right)^{7-p} \, h_+^{\frac{p-3}{2}} \times
\]

\[
\times \left[ 1 + (y')^2 + \left( \frac{\sigma}{r_p} \right)^{7-p} \, \frac{g^{\frac{2-p}{2}}}{h_+^{\frac{2-p}{2}}} \left( z' \right)^2 \right] \, d\hat{y}^2 + \left( \frac{r_p}{\sigma} \right)^{7-p} \, h_+^{\frac{p-3}{2}} \, g^2 d\Omega_{3-k}^2.
\]

where \(\hat{x}\) indicates the (spatial) coordinates \(\{x^i\}_{i=1}^{p-1}\) on the defect, the constant \(N_k\) is \(N_k = g_s^{-1} \text{Vol} \{S^{3-k}\}\), the prime \(\prime\) indicates the first derivative with respect to the radial coordinate
and \( \sigma^2 = g^2 + y^2 \). The action (2.23) depends on the embedding function \( z(\rho) \) and the gauge field \( f(\rho) \) through their first derivatives only. There is therefore one first integral of motion related to each of them

\[
\begin{align*}
c_f &= g^{3-k} h_+ \frac{11-p-2k}{7-p} \left( \frac{\sigma}{\tau_p} \right)^{7-p} \frac{-f'}{\sqrt{1 + (y')^2 + (\sigma r_p)^{7-p} (z')^2 - \frac{\tau_p}{h^c} (f')^2}} \\
c_z &= g^{3-k} h_+ \frac{11-p-2k}{7-p} \left( \frac{\sigma}{\tau_p} \right)^{7-p} \frac{z'}{\sqrt{1 + (y')^2 + (\sigma r_p)^{7-p} (z')^2 - \frac{\tau_p}{h^c} (f')^2}}
\end{align*}
\]

(2.25)

Notice that, in the case of a black hole embedding phase for the class of embeddings \( z \), the regularity condition at the horizon, fixes the first integral of motion \( c_z \) to be zero and therefore the embedding function \( z \) must have a trivial profile. In order to have a non-trivial profile for the embedding mode, one would need to introduce a magnetic component for the world-volume 2-form so that the action for the probe branes has also a Wess-Zumino term. The Wess-Zumino action would induce an extra term in the equation of motion (2.25), whose value at the horizon fixes \( c_z \). We will not discuss the case of the presence of a magnetic component for the world-volume gauge field. This means that the regularity condition at the horizon forces the system to be supersymmetric.

The first integral of motion \( c_f \) is related to the charge density \( n \), which is defined as the canonical momentum conjugate to \( f(\rho) \) evaluated at the boundary:

\[
n = \lim_{\rho \to \infty} \frac{\partial L}{\partial f'} = MT_{D(p+2)} N_k (2\pi \alpha') c_f,
\]

(2.26)

where the factor \( 2\pi \alpha' \) has been restored. For later convenience, let us rescale the coordinates, the gauge field \( f \) and \( c_f \) so that they are dimensionless:

\[
\{ t, \vec{x}, \varrho, y; f \} \rightarrow L_* \{ t, \vec{x}, \varrho, y; f \}, \quad c_f \rightarrow L_3^{3-k} c_f.
\]

(2.27)

With such a rescaling, the position of the black brane horizon gets parametrized by the dimensionless quantity \( \hat{\sigma}_h = \sigma_h / L_* \).

We will discuss the two classes of embeddings separately, \textit{i.e.} we will discuss both the massive and massless degrees of freedom.

### 3 Massive hypermultiplet and \((\mu, T)\) phase diagram

Let us now fix the position of the probe \( D(p+4-2k) \)-branes in the non-compact \((1+(p-k))\)-dimensional submanifold \((z=0)\) and let us consider their embedding in the transverse
space, which has been parametrized through the angular coordinate \( \theta \ (\rho) \). It is actually more convenient to parametrize differently the embedding, using a function \( y (\varrho) \) of a redefined radial coordinate \( \varrho \) according to (2.7) and (2.21). Let us write explicitly the induced metric on the world-volume of the probe branes

\[
\begin{align*}
  ds^2_{p+5-2k} &= \left( \frac{\sigma}{r_p} \right)^{\frac{7-p}{2}} h^+ \left[ -\frac{h^2}{h^2_p} + d\xi^2 \right] + \left( \frac{r_p}{\sigma} \right)^{\frac{7-p}{2}} h^{-\frac{p-3}{p}} \left[ (1 + (y')^2) \, d\varrho^2 + \varrho^2 d\Omega^2_{3-k} \right], \\
  (3.1)
\end{align*}
\]

with \( \{ x^i \}_{i=1}^{p-k}, \sigma^2 = \varrho^2 + y^2 \) and the dilaton given by

\[
  e^\phi = g \varrho^{\frac{p-3}{2}} \left( \frac{\sigma}{r_p} \right)^{\frac{(7-p)(p-3)}{4}}, \quad h^+ = 1 \mp \left( \frac{\sigma h}{\sigma} \right)^{7-p}, \quad (3.2)
\]

where the coordinates are dimensionless. For this class of embeddings, the action of the probe branes is given just by the DBI-action

\[
  S_{D(p+4-2k)} = -MT_{D(p+4-2k)} N_k \int d^{p-k+2} \xi \, h^+ \varrho^{\frac{p-1-2k}{2}} \left[ 1 + (y')^2 - \frac{h^+}{h^2} \left( f' (\varrho) \right)^2 \right]^{\frac{1}{2}}.
  \]

(3.3)

As noticed earlier, in the most general case, the action (3.3) depends on the electric gauge potential \( f (\varrho) \) just through its first derivative, so that there exists a first integral of motion \( c_f \) related to it. In the limit of zero temperature (i.e. in any point of the chemical potential axis in the phase diagram \((\mu, T)\)), the action also depends on the embedding function \( y \) through its first derivative only, so that there is a further integral of motion \( c_y \), beside \( c_f \). In the most general case, the equations of motion are

\[
  f' (\varrho) = c_f^{\frac{h}{h^+}} \frac{\sqrt{1 + (y')^2}}{\sqrt{c_f^2 + \varrho^{3-k} h^+^{1-\frac{k}{p}}}},
\]

\[
  0 = \frac{y''}{1 + (y')^2} + \frac{3 - k}{\varrho} y' + 2 \left( \frac{\sigma h}{\sigma} \right)^{7-p} \varrho y' - y \left[ \frac{3 + k - p + (4 - k) \left( \frac{\sigma h}{\sigma} \right)^{7-p}}{3} - \right] - \frac{c_f^2}{c_f^2 + \varrho^{2(3-k)} h^+^{1-\frac{k}{p}}} \left[ \frac{3 - k}{\varrho} y' - 2 (3 - k) \left( \frac{\sigma h}{\sigma} \right)^{7-p} \varrho y' - y \right],
\]

(3.4)

In the following subsections, we explore the whole phase diagram for this class of systems, including also the simplest cases.

### 3.1 The chemical potential axis

As mentioned earlier, in the limit of zero temperature there is a first integral of motion \( c_y \) related to the embedding function \( y \), beside \( c_f \). The equations of motion (3.4) have the
corresponds to a black-hole crossing phase, in which the chemical potential and the embedding function can be explicitly written as

\[ m > \mu \quad \text{(the region \( m < \mu \))}

They can be easily integrated out. Notice that we need to distinguish three cases, according to the sign of \( c_f^2 - c_y^2 \). In this case this difference is negative, the probe branes can extend from the boundary to a minimum distance \( \varrho_{\text{min}} = (c_y^2 - c_f^2)^{1/(3-k)} \), where the branes turns back and hit the boundary again. The system is therefore a D\((p + 4 - 2k)/D(p + 4 - 2k)\) in a Dp-brane background. In this phase, the chemical potential is

\[
\mu = \int_{\varrho_{\text{min}}}^{\infty} d\varrho \frac{c_f}{\sqrt{\varrho^{2(3-k)} + c_f^2 - c_y^2}} = \frac{1}{2(3-k)} \frac{c_f}{(c_y^2 - c_f^2)^{2/(3-k)}} B \left( \frac{2-k}{2(3-k)}, \frac{1}{2} \right),
\]

while the embedding function can be explicitly written as

\[
y(\varrho) = \int_{\varrho_{\text{min}}}^{\infty} d\varrho \frac{c_y}{\sqrt{\varrho^{2(3-k)} + c_f^2 - c_y^2}} = \frac{1}{4} \frac{c_y}{(c_y^2 - c_f^2)^{2/(3-k)}} \left[ B \left( \frac{2-k}{2(3-k)}, \frac{1}{2} \right) - B \left( \frac{\varrho_{\text{min}}}{\varrho}, \frac{2-k}{2(3-k)}, \frac{1}{2} \right) \right].
\]

In (3.6) and (3.7), we set \( f(\varrho_{\text{min}}) = 0 = y(\varrho_{\text{min}}) \). Notice that the expressions (3.6) and (3.7) for the chemical potential and the embedding functions are valid for \( k = 0, 1 \). For the codimension-2 systems, the integrals (3.6) and (3.7) are divergent as \( \varrho \to \infty \). We will come back to this issue later. For the time being, we focus on the case \( k = 0, 1 \).

Taking the limit \( \varrho \to \infty \) of (3.7) (in this limit the incomplete Beta-function vanishes), one obtains the “quark” mass \( m \) in terms of the first integral of motions \( c_f \) and \( c_y \). It is easy to invert this expression, together with the result for the chemical potential (3.6), to obtain the first integral of motions in terms of \( m \) and \( \mu \):

\[
c_f = \left[ \frac{2(3-k)}{B \left( \frac{2-k}{2(3-k)}, \frac{1}{2} \right)} \right]^{3-k} \mu \left( m^2 - \mu^2 \right)^{\frac{2-k}{2}} \quad \text{and} \quad c_y = \left[ \frac{2(3-k)}{B \left( \frac{2-k}{2(3-k)}, \frac{1}{2} \right)} \right]^{3-k} m \left( m^2 - \mu^2 \right)^{\frac{2-k}{2}}.
\]

The condition \( c_f^2 - c_y^2 < 0 \) implies necessarily that \( m > \mu \) for \( k \) even. For \( k = 1 \), it is straightforward to realize that the expressions (3.8) are meaningful just for in the region \( m > \mu \) (the region \( m < -\mu \) is not physical).

In the case \( c_f^2 - c_y^2 > 0 \), the probe branes can extend to \( \varrho = 0 \). Such a configuration corresponds to a black-hole crossing phase, in which the chemical potential and the embedding
function are expressed by

\[
\mu = \int_0^\infty d\rho \frac{c_f}{\sqrt{\rho^{2(3-k)} + c_f^2 - c_y^2}} = \frac{1}{2(3-k)} \frac{c_f}{(c_f^2 - c_y^2)^{\frac{2-k}{2(3-k)}}} B \left( \frac{2-k}{2(3-k)}, \frac{1}{2} \right),
\]

\[
y(\rho) = \frac{1}{2(3-k)} \frac{c_y}{(c_f^2 - c_y^2)^{\frac{2-k}{2(3-k)}}} \left[ B \left( \frac{2-k}{2(3-k)}, \frac{1}{2} \right) - B \left( \frac{c_f^2 - c_y^2}{\rho^{2(3-k)}}, \frac{2-k}{2(3-k)}, \frac{1}{2} \right) \right].
\]

(3.9)

At the boundary, the embedding function provides the “quark” mass, which is provided by the first term in the second expression in (3.9) (the incomplete Beta-function vanishes). The first integrals of motion \(c_f\) and \(c_y\) can be easily expressed in terms of \(m\) and \(\mu\):

\[
c_f = \left[ \frac{2(3-k)}{B \left( \frac{2-k}{2(3-k)}, \frac{1}{2(3-k)} \right)} \right]^{\frac{3-k}{2(3-k)}} \mu \left( \mu^2 - m^2 \right)^{\frac{2-k}{2(3-k)}}, \quad c_y = \left[ \frac{2(3-k)}{B \left( \frac{2-k}{2(3-k)}, \frac{1}{2(3-k)} \right)} \right]^{\frac{3-k}{2(3-k)}} m \left( \mu^2 - m^2 \right)^{\frac{2-k}{2(3-k)}}.
\]

(3.10)

Similarly to the previous case, it is straightforward to see that equations (3.10) hold in the phase diagram region \(\mu > m\).

Let us now turn to the thermodynamics of our systems in these two phases, by considering in particular the grand-potential \(\Omega\). It is given, up to a sign, by the renormalized on-shell action\(^2\)

\[
\Omega = -S_{D(p+4-2k)} \bigg|_{\text{ren}} = -\lim_{\Lambda \to \infty} \left[ S_{D(p+4-2k)} \bigg|_{\text{on-shell}} + S_{D(p+4-2k)} \bigg|_{\text{ct}} \right]
\]

(3.11)

where

\[
S_{D(p+4-2k)} \bigg|_{\text{on-shell}} = -MTD_{D(p+4-2k)} \tilde{N}_k \int_{\tilde{\rho}}^\Lambda d\rho \frac{\rho^{2(3-k)}}{\sqrt{\rho^{2(3-k)} + c_f^2 - c_y^2}}
\]

(3.12)

\[
S_{D(p+4-2k)} \bigg|_{\text{ct}} = -MTD_{D(p+4-2k)} \tilde{N}_k \left( -\frac{\Lambda^{4-k}}{4-k} \right),
\]

(the overall factor \(\tilde{N}_k\) is a redefinition of \(N_k\) including \(\int d^{p+k-1}\xi\), and \(\tilde{\rho}\) is 0 for \(c_f^2 - c_y^2 > 0\) or \(\rho_{\text{min}}\) if \(c_f^2 - c_y^2 < 0\)).

Integrating the expressions (3.12), the grand-potential in the two phases is given by

\[
\Omega = \begin{cases} 
\left( \frac{c_f^2 - c_y^2}{2(4-k)} B \left( \frac{2-k}{2(3-k)(4-k)} \right) \right)^{(4-k)} \left( \frac{2-k}{2} \right) & \quad \mu < m \\
\left( \frac{c_f^2 - c_y^2}{2(3-k)(4-k)} B \left( \frac{2-k}{2(3-k)} \right) \right)^{(4-k)} \left( \frac{1}{2} \right) & \quad \mu > m
\end{cases}
\]

(3.13)

\(\text{The holographic renormalization of probe D-branes in non-conformal background was extensively discussed in [51].}\)

16
where \( a_1 \) and \( a_2 \) are two positive constants.

Some comments are now in order. Notice that the phase ("black-hole" crossing phase) for which \( c_f^2 - c_y^2 > 0 \) is thermodynamically favoured with respect to the phase ("brane/anti-brane"-phase) for which this difference is negative, given that in the first case the grand-potential is negative while in the second case it is positive. Let us analyze the derivatives of the grand-potential \( \Omega \):

\[
\frac{\partial \Omega}{\partial \mu} \bigg|_{\mu \to m} = \begin{cases} 
-a_1 (4 - k) \mu \left( \mu^2 - m^2 \right)^{\frac{2-k}{2}} & \mu \to m \\
-a_2 (4 - k) \mu \left( m^2 - \mu^2 \right)^{\frac{2-k}{2}} & \mu \to m \\
-a_1 (4 - k) \frac{(3-k)\mu^2 - m^2}{(\mu^2 - m^2)^{\frac{3}{2}}} & \mu \to m \\
-a_2 (4 - k) \frac{(3-k)\mu^2 - m^2}{(\mu^2 - m^2)^{\frac{3}{2}}} & \mu \to m \\
\end{cases}
\]

(3.14)

The point \( \mu = m \) represents a second order phase transition. This is a universal feature for gauge theories whose holographic bulk dual is constructed in terms of flavoured branes. In this section we showed how this holds for gauge theories with no defect \((k = 0)\) and on a defect with \((p - 1)\) spatial directions \((k = 1)\). The presence of this second order phase transition is however common to gauge theories on a defect with \((p - 2)\) spatial directions as well, which is dual to the other possible BPS brane construction \((k = 2)\). In this last case one needs to take into account some subtleties which we will discuss in the next subsection.

### 3.1.1 Codimension-2 systems

As pointed out, the analysis in the previous subsection holds for codimension-0 and codimension-1 systems, \(i.e.\) \(D_p/D(p+4)\) and \(D_p/D(p+2)\) intersections respectively. The \(D_p/D_p\) systems are quite different from the higher dimensional defect theories. The first difference lies in the divergence structure of the on-shell action. The case we are now analysing is the only one in which the embedding mode saturates the Breitenhoner-Freedman bound which implies the presence of a logarithmic divergence \([51, 52]\). Furthermore, the role of the normalizable and non-normalizable modes are exchanged, with the coefficient of the normalizable mode determining the vacuum-expectation-value of the operator dual to the embedding function, so that the brane separation appears as a vev rather than as a parameter. A similar discussion

---

\(^3\)For the discussion of the D3/D3 system see section 6 of [52]. The generalisation to any \(D_p/D_p\) system \((p < 5)\) was discussed in [51]
applies for the gauge potential $f(\rho)$. For both $y(\rho)$ and $f(\rho)$, the asymptotic expansion near the boundary $\rho \to \infty$ shows a logarithmic term

$$
y(\rho) \overset{\rho \to \infty}{=} m + c_y \log \rho, \quad f(\rho) \overset{\rho \to \infty}{=} \mu + c_f \log \rho.
$$

The divergences of the action (3.12) are cured by the counter-terms found in [51]. In principle, the integrals of the equations of motion (3.5), which define $y(\rho)$ and $f(\rho)$, remain divergent as near the boundary. One can define $\mu$ and $m$ respectively as

$$
\mu = -\frac{\partial \Omega}{\partial c_f}, \quad m = \frac{\partial \Omega}{\partial c_y}. \quad (3.16)
$$

Through (3.16), one can rewrite the grand-potential in the form (3.13) (with $k = 2$). Therefore, codimension-2 systems show a second order phase transition on the chemical potential axis at $\mu = m$ as well.

Thus one can conclude that the existence of a second order phase transition on the chemical potential axis is a universal feature of gauge theories with a gravitational bulk dual constructed via $D_p/D(p + 4 - 2k)$ systems.

Before turning on the temperature, let us discuss the behaviour of the chemical potential and of the embedding function in the black-hole crossing phase in the limit of small density $c_f$. This will turn out to be useful for the finite temperature case, where the analytic analysis is performed in such a limit.

### 3.1.2 Small density expansion

Consider the expression for the embedding function (3.7) taking the limit $\varrho \to \infty$, so that it provides an expression of the “quark”-mass in terms of the two first integral of motion $c_f$ and $c_y$. Such an expression can be inverted in order to express $c_y$ in terms of the density $c_f$.

In the limit $c_f \to 0$, one obtains:

$$
c^2_f - c^2_y = \kappa^2 \frac{3-k}{2-k} c_f \frac{2-k}{2-k} \left[1 - \frac{3-k}{2-k} \kappa^2 \frac{3-k}{2-k} c_f \frac{2-k}{2-k} + O \left( \frac{1}{c_f} \right) \right], \quad (3.17)
$$

where $\kappa$ is constant which contains the “quark” mass as well as other numerical coefficients.

This suggests that a perturbative expansion around $c_f$ becomes subtle in the region of the radial axis for which $\varrho^{2(3-k)}$ is of the same order of the leading term in (3.17), i.e.

$$
\varrho \sim c_f^{\frac{1}{3-k}}. \quad (3.18)
$$

Therefore, this region of the radial axis can be conveniently studied by redefining the radial coordinate as $\tau = c_f^{\frac{1}{3-k}} \varrho$. Keeping this in mind, it is possible to integrate the equation
for the gauge field (3.5) to obtain the chemical potential in the small density limit at zero temperature

\[ \mu = m_o \left[ 1 + \frac{\kappa}{2} c_f^{\frac{2}{2-k}} + O \left( c_f^{-\frac{4}{2-k}} \right) \right]. \]

(3.19)

Given equation (3.19), one can expect that the expansion parameter for the chemical potential (as well as the embedding function) is \( c_f^{\frac{2}{2-k}} \) rather than other powers of the “quark”-density \( c_f \).

3.2 Chemical potential-temperature plane

In this section we consider the systems at finite temperature and with chemical potential. For convenience we rewrite the equations of motions here

\[ f'(\varrho) = c_f \frac{h_{-}}{\sqrt{c_f^{2} + g^{2(3-k)}}} \frac{\sqrt{1 + (y')^2}}{h_{+}} \]

\[ 0 = \frac{y''}{1 + (y')^2} + \frac{3 - k}{\varrho} y' + 2 \left( \frac{\sigma}{\sigma} \right)^{7-p} \frac{g y' - y}{\sigma^2 h_{+}} \left[ 3 + k - p + (4 - k) \left( \frac{\sigma}{\sigma} \right)^{7-p} \right] \]

\[ - \frac{c_f^2}{c_f^2 + g^{2(3-k)}} \left[ \frac{3 - k}{\varrho} y' - 2 (3 - k) \left( \frac{\sigma}{\sigma} \right)^{7-p} \frac{g y' - y}{\sigma^2 h_{+}} \right] \]

(3.20)

In the black-hole crossing phase, the regularity condition at the horizon imposes the following boundary conditions for the embedding function

\[ y'(\varrho_h) = \frac{y_{h}}{\varrho_{h}} \quad \text{with} \quad \sigma^2_{h} = \varrho_{h}^2 + y_{h}^2, \]

(3.21)

which prevents the last two terms in (3.20) to blow-up as the horizon is approached. From the first equation in (3.20), the chemical potential can be written as

\[ \mu = c_f \int_{\varrho_{h}}^{\varrho} d\varrho \frac{h_{-}}{h_{+}} \frac{\sqrt{1 + (y')^2}}{\sqrt{c_f^{2} + g^{2(3-k)}}} \]

(3.22)

In order to study the transition line in the \((\mu, T)\)-plane analytically, we closely follow the small density approach of [20].

3.2.1 Small density expansion of the embedding function

As noted in [20], the small density expansion for the embedding function and the chemical potential is subtle. This is due to the fact that the last term in the embedding function
equation (3.20) for \( c_f \to 0 \) can be considered as an actual perturbation of the equation at zero condensate as long as \( \rho^{2(3-k)} h^{4-\frac{1-k}{p}} \) does not become of order \( \mathcal{O}(c_f^2) \). For such values of the radial coordinates, this term is no longer small. Let us divide the radial direction in two regions:

1. \( \rho \in ]\rho_{\Lambda}, \infty[ \),
   where \( \rho_{\Lambda} \) is a cut-off distance until which a uniform perturbative expansion can be considered;

2. \( \tau \in ]\tau_h, \Lambda[ \),
   where \( \tau \) is defined as \( \rho = c_f^{\frac{1}{2-k}} \tau \), and \( \tau_h \) and \( \Lambda \) respectively as \( \rho_{\Lambda} = c_f^{\frac{1}{2-k}} \tau_h \) and \( \rho_{\Lambda} = c_f \Lambda \).

The idea is to found a (perturbative) solution in both regions 1 and 2 and then matching them in the limit \( c_f \to 0, \rho_{\Lambda} \to 0 \) and \( \Lambda \to \infty \), keeping \( \tau \) finite. Notice that the above splitting of the radial coordinate axis does not hold for \( k = 2 \).

Let us start with finding the solution for the embedding function equation (3.20) in the region 1, where, for small \( c_f \), the solution can be expanded as follows

\[
y(\rho) = \sum_{i=0}^{\infty} c_i^f y_i(\rho) . \tag{3.23}
\]

The zero-th order term \( y_0(\rho) \) satisfied the equation (3.20) at \( c_f = 0 \) with boundary condition \( y_0(0) = 1 \). In a small \( \rho \) expansion, the 0-th order (in \( c_f \)) solution is given by

\[
y_0(\rho) = 1 + y_0^{(2)} \rho^2 + y_0^{(4)} \rho^4 + \mathcal{O}(\rho^6) , \tag{3.24}
\]

where the coefficients \( y_0^{(2)}, y_0^{(4)} \) and \( y_0^{(6)} \) are given by

\[
y_0^{(2)} = \frac{(3 + k - p) + (4 - k) \sigma_{h}^{7-p}}{(4 - k) \left(1 - \sigma_{h}^{2(7-p)}\right)} \sigma_{h}^{7-p} ,
\]

\[
y_0^{(4)} = \frac{\sigma_{h}^{7-p}}{4(6 - k)(4 - k)^3 \left(1 - \sigma_{h}^{2(7-p)}\right)} \sum_{i=0}^{5} b_i^{(p, k)} \sigma_{h}^{i(7-p)} , \tag{3.25}
\]

\[
y_0^{(6)} = -\frac{\sigma_{h}^{7-p}}{24(8 - k)(6 - k)(4 - k)^5 \left(1 - \sigma_{h}^{2(7-p)}\right)} \sum_{i=0}^{9} c_i^{(p, k)} \sigma_{h}^{i(7-p)}
\]

where the coefficients \( b_i^{(p, k)} \) and \( c_i^{(p, k)} \) are constants dependent on the spatial dimensions of the background branes \( p \) and the codimensionality of the defect \( k \) and are explicitly given
in Appendix A. At the next order in \(c_f\), the solution in a neighbourhood of \(\varrho \sim 0\) can be generally written as

\[
y_1(\varrho) = \frac{a_{-(2-k)}}{\varrho^{2-k}} + a_0 + b_0 \log \varrho + a_{(2-k)}\varrho^{2-k} + \mathcal{O}(\varrho^{2-k}).
\] (3.26)

The coefficient of the leading order \(a_{-(2-k)}\) and the zero order coefficient \(a_0\) are constant of integration which will be fixed by matching the solutions of region 1 and 2. For \(k = 0\) only the coefficients related to even powers of \(\varrho\) are non-vanishing, while for \(k = 1\) both even and odd powers are admitted. A crucial difference between the cases \(k = 0\) and \(k = 1\) is the presence of logarithmic terms in the \(k = 0\) expansion, which do not instead appear for \(k = 1\). The coefficient of the logarithmic term \(b_0\) for \(k = 0\) is fixed in terms of the integration constant \(a_{-2}\)

\[
b_0 = \left(\sigma_h^7 - p\right)^4 \left(6 - p\right)(5 - p) - \left(p^2 + 60 - 29\right)\sigma_h^{7-p} + 4(7 - p)(5 - p)\sigma_h^{2(7-p)} + 4\sigma_h^{3(7-p)}
\]

\[
2 \left(1 - \sigma_h^{2(7-p)}\right)^2 a_{-2}.
\] (3.27)

In the expansion for \(k = 1\), the coefficient \(a_1\) is determined in terms of the integration constant \(a_{-1}\):

\[
a_1 = \frac{(7 - p)\sigma_h^{7-p}}{2 \left(1 - \sigma_h^{2(7-p)}\right)^2} \left[ (11 - 2p) + 6\sigma_h^{7-p} + (11 - 2p)\sigma_h^{2(7-p)} \right] a_{-1}.
\] (3.28)

Let us now consider the region 2, by recasting the equation (3.20) in terms of the independent variable \(\tau\). For the solution one can consider the following perturbative expansion

\[
y(\tau) = \sum_{i=0}^{\infty} c_i^{(2)} \dot{z}_i(\tau).
\] (3.29)

Also, the position of the horizon \(\tau_h\) admits a perturbative expansion for small \(c_f\)

\[
\tau_h = \sum_{i=0}^{\infty} c_i^{(2)} \tau_i.
\] (3.30)

The 0-th order equation is

\[
0 = \frac{\dot{z}_0}{z_0^2} + (3 - k) \left[ 1 + \left(\frac{\sigma_h}{z_0}\right)^{7-p} \right] \frac{4}{\left(7-p\right)} \tau^{5-2k} \dot{z}_0 +
\]

\[
+ 2 \left(\frac{\dot{z}_0}{z_0}\right)^{7-p} \frac{\tau \dot{z}_0 - z_0}{z_0^2} \left[ 1 - \left(\frac{\sigma_h}{z_0}\right)^{2(7-p)} \right] \left[ 6 - p + \left(\frac{\sigma_h}{z_0}\right)^{7-p} \right],
\] (3.31)
where the dot $\dot{}$ indicates the derivative with respect to $\tau$. The conditions at the horizon and at the boundary are

\[
\text{horizon: } z_0(\tau_0) = \sigma_h, \quad z_0' = \frac{\sigma_h}{\tau_0}, \tag{3.32}
\]

\[
\text{boundary: } z_0(\tau_0) = 1
\]

In order to match the solutions in region 1 and 2, it is suitable to look in the region 2 for a solution at large $\tau$

\[
z_0(\tau) = 1 + \sum_{i=1}^{\infty} \frac{\zeta_{(2-k)i}}{\tau(2-k) i}, \quad \tag{3.33}
\]

where the very first coefficient $\zeta_{(2-k)}$ is

\[
\zeta_{(2-k)} = -\frac{1}{(2-k) \left( 1 + \hat{\sigma}_h^{7-p} \right)^2}, \quad \tag{3.34}
\]

At first order, the perturbative solution shows a singular term, which is of order $O(\tau)$:

\[
z_1(\tau) = v_{2-k} \tau^{2-k} + v_0 + w_0 \log \tau + \frac{v_{-(2-k)}}{\tau^{2-k}} + O\left( \frac{1}{\tau^{2(2-k)}} \right), \quad \tag{3.35}
\]

where $v_0$ is an integration constant and the coefficients $v_{2-k}$ and $w_0$ are fixed in terms of the zero-th order solution to be:

\[
v_{2-k} = \begin{cases} \left. \frac{(3+k-p)+(4-k)\hat{\sigma}_h^{7-p}}{(4-k)(1-\hat{\sigma}_h^{2(7-p)})} \hat{\sigma}_h^{7-p} \right|_{k=0} & \text{for } k = 0 \\ 0 & \text{for } k = 1. \end{cases} \quad \tag{3.36}
\]

We are now in condition to compare the solutions in the two different regions of the radial coordinate. First, it is convenient to rewrite the expansion in the region 2 just in terms of the radial coordinate $\varrho$ as well as the density $c_f$. We will rewrite the expansions both in region 1 and 2 so that the comparison between the two results becomes straightforward

\[
y(\varrho)|_{\text{reg 1}} = 1 + y_0^{(2)} \varrho^2 + O\left( \varrho^4 \right) + c_f \left[ \frac{a_{-(2-k)}}{\varrho^{2-k}} + a_0 + b_0 \log \varrho + \frac{a_{(2-k)}}{\varrho^{2-k}} + O\left( \varrho^{2(2-k)} \right) \right],
\]

\[
y(\varrho)|_{\text{reg 2}} = 1 + v_{(2-k)} \varrho^{2-k} + \eta_{2(2-k)} \varrho^{2(2-k)} + O\left( \varrho^{3(2-k)} \right) +
\]

\[
+ c_f \left[ \frac{\zeta_{2-k}}{\varrho^{2-k}} + v_0 + w_0 \log \varrho - \frac{w_0}{2-k} \log c_f + O\left( \varrho^{2-k} \right) \right],
\]

where the coefficient $\eta_{2(2-k)}$ comes from the equations at order $c_f^2$, which we have not written down explicitly. The matching at zero-th order in $c_f$ fixes some of the integration constants and provides some non-trivial consistency check. Notice that this matching provides a non-trivial check on the expansion:

\[
v_{2-k}|_{k=0} = y_0^{(2)}
\]

\[
v_{2-k}|_{k=1} = 0, \quad \tag{3.38}
\]

22
where the coefficients in (3.38) are provided in (3.25) and (3.36) and identically satisfy the equalities in (3.38). The matching at first order in $c_f$ fixes some integration constants:

$$a_{-(2-k)} = \zeta_{2-k}, \quad \nu_0 = \nu_0, \quad \nu_0 = a_0 + \frac{\nu_0}{2-k} \log c_f,$$

(3.39)

where the second equality becomes the identity $0 = 0$ for $k = 1$. The constant of integration $a_0$ is determined by the boundary condition $y_0(\varrho)|_{\varrho \to \infty} \to 0$.

This small density analysis of the embedding function showed that the appearance of logarithmic terms is common to all the systems with no defect while it disappears when a codimension-1 defect is introduced.

### 3.2.2 Small density expansion of chemical potential

We can now use the previous analysis to explicitly compute the chemical potential in the small density limit. The chemical potential is given by the integral in (3.22) with extreme of integration $[\varrho_\Lambda, \infty]$ and $[\varrho_h, \varrho_\Lambda]$ respectively in region 1 and region 2.

In region 1, the leading term of the chemical potential in a $c_f$ perturbative expansion is of order one

$$\left. \mu \right|_{\text{reg } 1} = c_f \int_{\varrho_\Lambda}^{\infty} d\varrho \left( \varrho^2 + y_0^2 \right)^{2-k} \frac{\left( \varrho^2 + (y_0^2)^{7-p} - \sigma_h^{7-p} \right)^{\frac{11-p-2k}{7-p}}}{\varrho^{3-k}} \sqrt{1 + (y_0^2)^2} + \mathcal{O} \left( \epsilon_f^2 \right).$$

(3.40)

Using the perturbative solution (3.24) for $y_0$, it is easy to compute the first orders of the chemical potential in a neighbourhood of $\varrho \sim 0$

$$\left. \mu^{(1)} \right|_{\text{reg } 1} = \frac{m_{-(2-k)}}{\varrho_\Lambda^{2-k}} + m_0 \log \varrho_\Lambda + m_1 \varrho_\Lambda + \mathcal{K}(\varrho_\Lambda),$$

(3.41)

where

$$m_{-(2-k)} = \frac{1 - \sigma_h^{7-p}}{(2-k) \left( 1 + \sigma_h^{7-p} \right)^{\frac{11-p-2k}{7-p}}},$$

$$m_0 = \delta_{k,0} m_k, \quad m_1 = \delta_{k,1} m_k,$$

$$m_k = \frac{1}{(4-k)^2 \left( 1 - \sigma_h^{2(7-p)} \right) \left( 1 + \sigma_h^{(7-p)} \right)^{\frac{2(7-p)}{7-p}}} \sum_{i=0}^{4} c_{i}^{(p, k)} \sigma_h^{i(7-p)},$$

(3.42)

and the last term $\mathcal{K}(\varrho_\Lambda)$ in (3.41) is finite and defined by

$$\mathcal{K}(\varrho_\Lambda) = \int_{\varrho_\Lambda}^{\infty} d\varrho \left. \frac{f'(\varrho)}{c_f} \right|_{y = y_0} - \frac{m_{-(2-k)}}{\varrho_\Lambda^{2-k}} - m_0 \log \varrho_\Lambda - m_1 \varrho_\Lambda,$$

(3.43)
where the zero-th order term, which is fixed by the solution \((3.33)\) for \(z_0\), can be conveniently written as

\[
\mu^{(0)}|_{\text{reg } 2} = m - \int_{\tau_0}^{\Lambda} d\tau \frac{\hat{z}_0}{z_0} \frac{z_0^{7-p} - \hat{\sigma}_h^{7-p}}{\left(z_0^{7-p} + \hat{\sigma}_h^{7-p}\right)^{\frac{1}{7-p}}}.
\]

(3.45)

Using the perturbative expansion \((3.35)\) as \(\tau \to \infty\), the zero-th order chemical potential can be obtained as an expansion in \(\Lambda^{-1}\)

\[
\mu^{(0)}|_{\text{reg } 2} = m - \frac{1 - \hat{\sigma}_h^{7-p}}{(2 - k) \left(1 + \hat{\sigma}_h^{7-p}\right)^{\frac{11 - p - 2k}{7-p}}} \frac{1}{\Lambda^{2-k}} + \mathcal{O}\left(\frac{1}{\Lambda^{2(2-k)}}\right).
\]

(3.46)

At first order in \(c_f\), the chemical potential receives contributions from the zero-th order solution \(z_0\) in \((3.33)\) as well as the first order one \(z_1\) in \((3.35)\). In full generality, it can be written as

\[
\mu^{(1)}|_{\text{reg } 2} = \int_{\tau_0}^{\Lambda} d\tau \left\{ \partial_\tau \left[ \frac{z_0^{7-p} - \hat{\sigma}_h^{7-p}}{z_0^{7-p} + \hat{\sigma}_h^{7-p}} \right] + \delta_{k,0} \left[ \frac{z_0^{7-p} - \hat{\sigma}_h^{7-p}}{2 z_0^2 \left(z_0^{7-p} + \hat{\sigma}_h^{7-p}\right)^{\frac{1}{7-p}}} \right] + \right.

\left. \tau^2 \left[ 2 z_0^{10} \hat{\sigma}_h^{7-p} \left(\hat{\sigma}_h^{7-p} + (6 - p) z_0^{7-p}\right) - \tau^4 \left(\frac{2(7-p)}{z_0^{7-p}} - \frac{2(7-p)}{\hat{\sigma}_h^{7-p}} \left(z_0^{7-p} + \hat{\sigma}_h^{7-p}\right)^{\frac{12}{7-p}} \right) \hat{z}_0 \right] \right\}
\]

(3.47)

The first term in \((3.47)\) can be easily integrated. Notice that in the case of codimension-1 systems, it is the only term contributing to \(\mu^{(1)}|_{\text{reg } 2}\). Using the expansions \((3.33)\) and \((3.35)\), as well as the relations \((3.38)\) and \((3.39)\), in the case of \(k = 1\) it acquires the form

\[
\mu^{(1)}|_{\text{reg } 2}^k = \frac{1 - \hat{\sigma}_h^{7-p}}{(1 + \hat{\sigma}_h^{7-p})^{\frac{1}{7-p}}} v_0.
\]

(3.48)

Therefore one can write down the chemical potential in the small density expansion as follows

\[
\mu = m + \tilde{s}(T) c_f + \mathcal{O}\left(c_f^2\right),
\]

(3.49)

where \(\tilde{s}(T)\) is defined as

\[
\tilde{s}(T) = \lim_{\varrho_\lambda \to 0} K(\varrho_\lambda) + \mu^{(1)}|_{\text{reg } 2},
\]

(3.50)
which should vanish in the zero temperature limit. In principle one would need to fix the constant \( v_0 \) by imposing the boundary condition \( y_1|_{\rho \to \infty} = 0 \). However, for our purposes, *i.e.* to check the order of the phase transition, it will not be necessary. The function \( \tilde{s}(T) \) in (3.49) may be both non-zero and zero. In the second case one would need to go to next order in \( c_f \), in which case the small density expansion of the chemical potential \( \mu \) can *at most* acquire the form:

\[
\mu = m + s_1(T) c_f^2 - s_2(T) c_f^3 \log c_f
\]

(3.51)

However, in any case the order of the phase transition will be the same. We will comment on this in the next subsection. For the time being, let us focus now on the structure of the small density expansion of the chemical potential for system with no defect.

In the case of \( k = 0 \) we have a non-trivial contribution to \( \mu^{(1)}|_{\text{reg 2}} = 0 \) from both the terms in (3.47). For convenience let us write (3.47) as

\[
\mu^{(1)}|_{\text{reg 2}} = \mu_a^{(1)}|_{\text{reg 2}} + \mu_b^{(1)}|_{\text{reg 2}},
\]

(3.52)

where \( \mu_a^{(1)}|_{\text{reg 2}} \) indicates the total term in (3.47). The first term in (3.52) is easy to obtain and can be written as

\[
\mu_a^{(1)}|_{\text{reg 2}} = \frac{3 - p + 4\tilde{\sigma}_h^{-p} - \hat{\sigma}_h^{-p} \Lambda^2}{4} + \frac{1 - \tilde{\sigma}_h^{7-p}}{1 + \tilde{\sigma}_h^{7-p}} v_0 - (6 - p)(3 - p) + (27 - 5p)\tilde{\sigma}_h^{-p} + 4\hat{\sigma}_h^{2(7-p)} \hat{\sigma}_h^{2(7-p)} + 4 \left( 1 - \tilde{\sigma}_h^{2(7-p)} \right) \left( 1 + \tilde{\sigma}_h^{7-p} \right) \frac{25}{7-p} \log \Lambda,
\]

(3.53)

where, similarly to (3.48), we would not need to fix the coefficient \( v_0 \) explicitly. The second term in (3.52) instead acquires the form

\[
\mu_b^{(1)}|_{\text{reg 2}} = \frac{3 - p + 4\tilde{\sigma}_h^{7-p}}{4} - \frac{2(p^2 - 9p + 12) + (53 + 14p - 3p^2)\tilde{\sigma}_h^{-p} - 2(12 + 5p - p^2)\hat{\sigma}_h^{2(7-p)} - 4\hat{\sigma}_h^{3(7-p)}}{4 \left( 1 + \tilde{\sigma}_h^{7-p} \right) \frac{25}{7-p} \left( 1 - \tilde{\sigma}_h^{7-p} \right) \left( 1 + \tilde{\sigma}_h^{7-p} \right)} \hat{\sigma}_h^{7-p} \log \Lambda + \mathcal{W}
\]

(3.54)
where, similarly to $K$, $W$, is defined as

$$W = \lim_{\Lambda \to \infty} \mu_b^{(1)} \mid_{\text{reg} 2} + \frac{3 - p + 4\hat{\sigma}_h^{7-p}}{4 \left(1 + \hat{\sigma}_h^{7-p}\right)^{2\frac{p-3}{p}}} \hat{\sigma}_h^{7-p} \Lambda^2 + C \left(\hat{\sigma}_h^{7-p}\right) \log \Lambda,$$  

(3.55)

where $C \left(\hat{\sigma}_h^{7-p}\right)$ indicates the term of (3.54) in curl bracket. Notice that the first (divergent) term in (3.54) exactly cancels the quadratic divergence (as $\Lambda \to \infty$) in (3.53). Summing the contributions (3.41), (3.46), (3.53) and (3.54), as well as using the third relation (3.39), the chemical potential in the small density expansion can be expressed as

$$\mu = m + s_1(T) c_f - s_2(T) c_f \log c_f,$$  

(3.56)

where $s_1(T)$ and $s_2(T)$ are given respectively by

$$s_1(T) = \mathcal{K} + \frac{1 - \hat{\sigma}_h^{7-p}}{\left(1 + \hat{\sigma}_h^{7-p}\right)^{\frac{5-p}{2-p}}} a_0 \left(6 - p\right) \left(3 - p\right) + \frac{27 - 5p}{2} \hat{\sigma}_h^{7-p} + 4\hat{\sigma}_h^{2(7-p)} \right) \frac{2^{\frac{5-p}{2}}}{\left(1 + \hat{\sigma}_h^{7-p}\right)^{2^{\frac{5-p}{2}}}} \hat{\sigma}_h^{7-p} \Lambda^2 + C \left(\hat{\sigma}_h^{7-p}\right) \log \Lambda,$$  

(3.57)

where $x$ and $y$ are the coefficients of log $\Lambda$ in (3.53) and (3.54) respectively, and $b_0$ is given in (3.27). The expressions (3.49) and (3.56) can be written as

$$\mu = m + s_1(T) c_f^\frac{2}{7-p} - s_2(T) c_f^\frac{2}{7-p} \log c_f,$$  

(3.58)

In the next subsection, we will discuss in detail the order the phase transition in the interior of the plane $(\mu, T)$ using the results of this subsection.

### 3.3 Phase transitions in the $(\mu, T)$-plane

In the previous section we were able to obtain a very general analytic expression for the chemical potential in the small density limit. Let us emphasise again an important point. The charge density $n$, which is given in (2.26), is proportional to the density $c_f$. The Minkowski and black-hole embeddings are characterised by zero and non-zero $n$ respectively. Therefore, working in the small density limit with black-hole embedding boundary conditions means focusing on a neighbourhood of the transition curve in the $(\mu, T)$-plane, so that the limit $c_f \to 0$ send us on the transition curve.

Let us now look in more detail at this phase transition. Since we are in the grand-canonical ensemble, the right thermodynamical potential to analyse is the grand potential $\Omega$
and its derivatives. In the Minkowski embedding phase \((i.e. \mu < m)\), all the derivatives of the grand-potential \(\Omega\) turn out to be zero:

\[
\frac{\partial \Omega}{\partial \mu} \bigg|_{\mu < m} \sim n \bigg|_{\mu < m} = 0, \quad \frac{\partial^2 \Omega}{\partial \mu^2} \bigg|_{\mu < m} = 0, \quad \forall s \geq 1. \quad (3.59)
\]

In the black-hole phase, \((i.e. \mu > m)\), we can use the expression \((3.58)\) for the chemical potential \((if \tilde{s}(T) = 0)\), then take the limit \(c_f \to 0\) to go on the transition curve \(\mu = m\) and match the result with \((3.59)\):

\[
\frac{\partial^2 \Omega}{\partial \mu^2} \bigg|_{\mu > m} \sim \frac{\tilde{s}_1(T)}{2-k} - \frac{\tilde{s}_2(T)}{2-k} \log c_f \quad \text{as } c_f \to 0 \to \infty, \quad \forall k = 0, 1. \quad (3.60)
\]

Already from \((3.60)\) one can infer that, while for \(k = 0\) there is no discontinuity in the second derivative of the chemical potential, for \(k = 1\) it diverges. For codimension-1 systems, the phase transition in the \((\mu, T)\)-plane is of second order.

Let us now consider the third-derivative of the grand-potential \(\Omega\) for \(k = 0\):

\[
\frac{\partial^3 \Omega}{\partial \mu^3} \bigg|_{\mu > m, k = 0} \sim -\frac{\tilde{s}_2(T)}{c_f (\tilde{s}_1(T) - \tilde{s}_2(T) - \tilde{s}_2(T) \log c_f)^2} \quad \text{as } c_f \to 0 \to \infty. \quad (3.61)
\]

As a consequence of such a divergence, the phase transition for codimension-0 systems is of third order.

In the case \(\tilde{s}(T) \neq 0\) for codimension-1 defects, then the second derivative of \(\Omega\) with respect the chemical potential \(\mu\) acquires the form

\[
\frac{\partial^2 \Omega}{\partial \mu^2} \bigg|_{\mu > m} \sim \frac{1}{\tilde{s}_1(T)} + O(c_f) \quad \text{as } c_f \to 0 \to \frac{1}{\tilde{s}_1(T)}, \quad k = 1. \quad (3.62)
\]

Even with the form \((3.49)\), the second derivative shows a discontinuity.

One comment is now in order. The results \((3.60), (3.61)\) and \((3.62)\) imply that the order of the phase transition in the \((\mu, T)\)-plane is tied to the codimensionality of the defect theory. We therefore identified two universality classes of theories determined by their codimensionality, or equivalently, by the order of the phase transition in the \((\mu, T)\)-plane.

The existence of a zero in the function \(\tilde{s}_2(T)\) would implicate the presence for \(k = 0\) systems of a critical point \((\mu_c, T_c)\) at which the phase transition becomes of second order.

### 4 Massless hypermultiplet and quantum fluids from non-conformal branes backgrounds

In order to consider massless degrees of freedom on a \((1+p-k)\)-defect, we need to fix the probe branes to wrap the maximal sphere \(S^{3-k}\). The probe branes therefore wrap \(\mathcal{M}_{p+2-k} \times S^{3-k}\).
Figure 3: Phase diagram ($\mu, T$). The axes are defined as $\mu/m$ and $T/T_d$. The phase diagram shows a first order phase transition at $T = T_d$ along the temperature axis (systems at zero baryonic chemical potential), while at zero temperature and finite density all the $D_p/D(p + 4 - 2k)$ systems are characterised by a second order phase transition at $\mu = m_0$. The curve $\mu = m(T)$ actually represents a transition curve in the interior of the ($\mu, T$) plane. The region $\mu < m(T)$ corresponds to Minkowski-like embeddings with $n = 0$, while the region $\mu > m(T)$ corresponds to a black hole phase, with the transition between the two phases is of second order for theories with a codimension-1 defect and of third order for theories with no defect.
We will work directly in the black hole embedding: since the probe branes wrap the black hole geometry $M'_{p+2-k}$, the induced metric contains a black hole as well. This class of embedding can therefore be described through the linear coordinate $x^p \equiv z(r)$ for defect theories. If the embedding mode $z(r)$ has a non-trivial profile, its dual operator $O_z$ acquires a non-zero vacuum-expectation-value breaking the supersymmetries. If the embedding mode $z(r)$ is a constant (namely $z = 0$), the probe branes have a fixed position in the $(p+2-k)$-dimensional non-compact manifold, the operator $O_z$ has a zero vev and the supersymmetries are not broken. This is the only possible configurations for theories with no defect (i.e. $Dp/D(p+4)$ systems). As pointed out in Section 2, regularity condition at the horizon forces the embedding function to have a trivial profile. We therefore need to consider the supersymmetric case. A similar analysis was carried out in [41]. We will begin with following [38, 41] by computing thermodynamical quantities such as the entropy density and the specific heat.

Let us turn on a non-trivial profile for the world-volume gauge field using the ansatz (2.22) so that the dual gauge theory has a chemical potential. The DBI action and the equation of motion for the gauge potential $f(r)$ are

$$S_{D(p+4-2k)} = -MT_{D(p+4-2k)} \int dt d^{p-k}x dr d^{3-k} \varphi e^{-\phi} \sqrt{-\det \{g_{AB} + F_{AB}\}} =$$

$$= -MT_{D(p+4-2k)} g_s^{-1} \text{Vol} \{S^3 \} \int dt d^{p-k}x dr r^{3-k} \sqrt{1 - [f'(r)]^2} \quad (4.1)$$

The contribution to the grand-potential from the fundamental degrees of freedom is given, up to a sign, by the renormalized on-shell action

$$\Omega_{\text{fun}} = -\lim_{\Lambda \to \infty} [S^\Lambda|_{\text{on-shell}} + S^\Lambda|_{\text{ct}}] \quad (4.2)$$

(the holographic renormalization of probe branes in non-conformal backgrounds was extensively studied in [51]). We can directly compute $\Omega_{\text{fun}}$ in the low-temperature limit (i.e. $r_h \to 0$), which is the regime we are interested in

$$\begin{align*}
\Omega_{\text{fun}} &= \Omega_{\text{fun}}^{(T=0)} - MT_{D(p+4-2k)}N_{p-k} \int_0^{r_h} dr \frac{r^{2(3-k)}}{\sqrt{r^{2(3-k)} + c_f^2}} + MT_{D(p+4-2k)}N_{p-k} \frac{r_h^{4-k}}{2(4-k)} = \\
&\equiv 0 \quad \Omega_{\text{fun}}^{(T=0)} - \frac{MT_{D(p+4-2k)}N_{p-k} r_h^{2(3-k)+1}}{2(3-k)+1} \frac{2(3-k)+1}{c_f} \left[ 1 + O \left( \frac{r_h^{2(3-k)}}{c_f} \right) \right] + \\
&+ MT_{D(p+4-2k)}N_{p-k} \frac{r_h^{4-k}}{2(4-k)},
\end{align*}$$

Here we indicate $z$ as function of $r$ since we will consider the background line element (2.1).
where \( \Omega_{\text{fun}}^{(T = 0)} \) is the grand-potential at zero temperature, which is explicitly provided in Section 3.1 (setting \( c_p = 0 \)) and we rewrite here for future convenience:

\[
\Omega_{\text{fun}}^{(T = 0)} = -\frac{M T D(p + 4 - 2k) N_{\bar{p} - k}}{(4 - k) a^{3 - k}} (\mu^{(T = 0)})^{4 - k}.
\]  

(4.4)

Furthermore, the constant \( \hat{N}_{\bar{p} - k} \) is a redefinition of \( N_k \) by including the volume of the defect:

\[ \hat{N}_{\bar{p} - k} \equiv N_k V_{\bar{p} - k} = N_k \int dp^{-k} x. \]

Notice that full grand-potential for the system is given by the contribution (4.3) from the fundamental degrees and the one from the adjoint ones:

\[ \Omega_{\text{adj}} = -(5 - p) N^2 \varpi \lambda^{\frac{p - 3}{5 - p}} T^{2 \frac{5 - p}{5 - p}}. \]

The latter contribution, as well as the last term in (4.3) are independent of the baryon density. Therefore, in order to study the features of the “quantum liquids” we need to focus just on the first two terms of (4.3) which will be indicated as \( \hat{\Omega}_{\text{fun}} \).

Similarly, we can compute the chemical potential in the low-temperature limit by integrating the equation of motion (4.1) for the gauge potential \( f(r) \):

\[
\mu = \mu_0 - \int_0^{r_h} \frac{c_f}{\sqrt{r^2(3 - k) + c_f^2}} = T \mu_0 - r_h \left[ 1 - \frac{1}{2[2(3 - k) + 1]} \frac{r_h^{2(3 - k)}}{c_f^2} + O \left( \frac{r_h^{2(3 - k)}}{c_f^2} \right) \right],
\]

(4.5)

with \( \mu_0 = a^{1 - \frac{k}{2}} \) being the chemical potential at zero temperature. Notice that the parameter expansion in (4.3) and (4.5) is the dimensionless ratio \( r_h / c_f \) or, equivalently in terms of the temperature and the chemical potential at zero temperature, \( T^{\frac{2 - p}{5 - p}} / \mu_0 \).

The entropy density can be now computed as a function of \( T \) and \( c_f \):

\[
s(T, c_f) = -\frac{1}{V_{\bar{p} - k}} \frac{\partial \hat{\Omega}_{\text{fun}}}{\partial T} = -\frac{1}{V_{\bar{p} - k}} \left[ \frac{\partial \hat{\Omega}_{\text{fun}}}{\partial T} \right]_{c_f} - \frac{\partial \hat{\Omega}_{\text{fun}}}{\partial c_f} \left[ \frac{\partial \mu}{\partial T} \right]_{c_f} =
\]

\[
= MT D(p + 4 - 2k) N_{\bar{p} - k} \frac{2}{5 - p} \left( \frac{4\pi}{7 - p} \right)^{\frac{2 - p}{5 - p}} c_f T^{\frac{2 - p}{5 - p}} \times
\]

\[
\times \left[ 1 + \left( \frac{4\pi}{7 - p} \right)^{\frac{4 - k}{2}} \frac{T^{\frac{4 - k}{2} - p}}{2 c_f^2} - \left( \frac{4\pi}{7 - p} \right)^{\frac{8 - 3k}{2}} \frac{T^{\frac{8 - 3k}{2} - p}}{8 c_f^2} + O \left( \frac{T^{12 - 3k - p}}{c_f^6} \right) \right].
\]

(4.6)

The leading term in the entropy density in \( s(T, c_f) \) scales with the temperature as \( s \sim T^{\frac{2 - p}{5 - p}} \), which is independent of the codimensionality of the defect. Notice that actually this term is

\[
\varpi = \left[ \frac{2^{2p - 5p} \pi^{13 - 3p}}{(7 - p)^{3(7 - p)}} \Gamma \left( \frac{7 - p}{2} \right) \right]^{\frac{2}{5 - p}}
\]

\[ \text{30} \]
of order one in the conformal case \( p = 3 \), while for \( p < 3 \) it decreases with the temperature as \( s \sim T^{-\frac{3-p}{5-p}} \) and for \( p = 4 \) it increases linearly as \( s \sim T \). This would imply that only in the conformal case, the entropy at zero temperature is non-zero. Not only. In the limit \( T \to 0 \) the entropy density seems to blow up for \( p < 3 \). We will comment later on this.

From the entropy density (4.6), it is straightforward to compute the specific heat \( c_V \) at constant volume and density (in the limit of low temperature).

\[
c_V = T \left. \frac{\partial s}{\partial T} \right|_{c_f} =
\]

\[
= M T_D(p+4-2k) N_k \left( \frac{2}{5-p} \left( \frac{4\pi}{7-p} \right)^{\frac{2}{5-p}} \frac{T^{\frac{p-3}{5-p}}}{c_f} \left[ \frac{p-3}{5-p} + \frac{9+p-4k}{2(5-p)} \left( \frac{4\pi}{7-p} \right)^{\frac{4\pi-1}{5-p}} T^{\frac{12-4k}{5-p}} \right] \right.
\]

\[
- \frac{21 + p - 8k}{8(5-p)} \left( \frac{4\pi}{7-p} \right)^{\frac{4\pi-1}{5-p}} \frac{T^{\frac{12-4k}{5-p}}}{c_f^4} + O\left( \frac{T^{12-4k}}{c_f^6} \right) \right].
\]  

(4.7)

For \( p = 3 \) the first term in (4.7) vanishes so that the leading term for the specific heat \( c_V \) scales with the temperature as \( T^{2(3-k)} \) (see [38]). For \( p = 4 \), \( c_V \sim T \) independently of the dimension of the defect in which the fundamental massless degrees of freedom propagates. This is actually the same behaviour of Fermi liquids. It is interesting to notice that the leading term for the specific heat \( c_V \) becomes negative for \( p < 3 \). This would be indeed a signature of a thermodynamical instability in the canonical ensemble. However, our analysis is carried out in the grand canonical ensemble where the thermodynamical stability of the system is guaranteed if and only if the Hessian of (minus) the grand potential is positive definite

\[
\mathcal{H} \left( \hat{\Omega}_{\text{fun}} \right) = \frac{\partial^2 \left( -\hat{\Omega}_{\text{fun}} \right)}{\partial \mathcal{X}_i \partial \mathcal{X}_j} = \begin{pmatrix} \Omega_{TT} & \Omega_{T\mu} \\ \Omega_{T\mu} & \Omega_{\mu\mu} \end{pmatrix}, \quad \mathcal{X} = \{T, \mu\}.
\]  

(4.8)
where the elements \( \{ \Omega_{TT}, \Omega_{T\mu}, \Omega_{\mu\nu} \} \) of the Hessian (4.8) are explicitly given by

\[
\begin{align*}
\Omega_{TT} &\equiv \left. \frac{\partial^2 (-\Omega)}{\partial T^2} \right|_\mu = \frac{2}{(5-p)} \mathcal{N} \left( \frac{4\pi}{7-p} \right) \frac{2}{7-p} c_f T^{-2 \frac{4}{3} + \frac{4}{5} - p} \times \\
&\times \left[ \frac{p - 3}{5-p} + \frac{2}{5-p} \left( \frac{4\pi}{7-p} \right) \frac{4}{5-p} \right] \tau \frac{T^{\frac{2}{5} - p}}{\mu_0} + \frac{9 + 4k}{2(5-p)} \left( \frac{4\pi}{7-p} \right) \frac{8}{5-p} c_f T^{-2 \frac{4}{3} + \frac{8}{5} - p} + \\
&\quad + \mathcal{O} \left( \frac{T^{2 \frac{4}{3} - \frac{2}{5} - p}}{\mu_0 c_f^2} \right) \right]
\end{align*}
\]

\[
\begin{align*}
\Omega_{T\mu} &\equiv \left. \frac{\partial^2 (-\Omega)}{\partial T \partial \mu} \right|_T = \mathcal{N} \left( 3 - k \right) c_f \frac{c_f}{\mu_0} \times \\
&\times \left[ 1 - \frac{3 - k}{2(3 - k) + 1} \left( \frac{4\pi}{7-p} \right) \frac{2 \frac{4}{3} K}{2} \mu_0 c_f^2 + \frac{3}{8} \left( \frac{4\pi}{7-p} \right) \frac{5 - k}{2} \frac{T^{\frac{2}{5} - p}}{c_f^2} + \mathcal{O} \left( \frac{T^{2 \frac{4}{3} - \frac{2}{5} - p}}{\mu_0 c_f^2} \right) \right] + \frac{1}{T} \frac{\Omega_{TT}}{\mu_0 c_f^2} + \ldots (4.9)
\end{align*}
\]

\[
\begin{align*}
\Omega_{\mu\nu} &\equiv \left. \frac{\partial^2 (-\Omega)}{\partial \mu^2} \right|_T = \mathcal{N} (3 - k) \frac{c_f}{\mu_0} \times \\
&\times \left[ 1 + \frac{3 - k}{2(3 - k) + 1} \left( \frac{4\pi}{7-p} \right) \frac{2 \frac{4}{3} K}{2} \mu_0 c_f^2 - \frac{3}{2(3 - k) + 1} \left( \frac{4\pi}{7-p} \right) \frac{5 - k}{2} \frac{T^{\frac{2}{5} - p}}{c_f^2} + \mathcal{O} \left( \frac{T^{2 \frac{4}{3} - \frac{2}{5} - p}}{\mu_0 c_f^2} \right) \right] \ldots,
\end{align*}
\]

where \( \mathcal{N} = MT_{D(p + 4 - 2k)N_{p - k}} \). In order to have the stability of the system guaranteed, the Hessian \( \mathcal{H} \left( \Omega_{\text{fin}} \right) \) in (4.8) needs to be (semi)-definite positive, which means that the eigenvalues need to be positive (or at most zero). This is also equivalent to requiring the non-negativity of all the principal minors of \( \mathcal{H} \left( \Omega_{\text{fin}} \right) \). Being a \( 2 \times 2 \) matrix, there are just two principal minors: \( \Omega_{TT} \) and the determinant \(|\mathcal{H}|\) of the Hessian. From the explicit expression in (4.9), the smallest minor \( \Omega_{TT} \) becomes negative for \( p < 3 \). As far as the determinant \(|\mathcal{H}|\) is concerned, it has the following form:

\[
|\mathcal{H}| = 2 \frac{3 - k}{5 - p} \mathcal{N}^2 \left( \frac{4\pi}{7-p} \right) \frac{2}{7-p} c_f^2 \frac{T^{-2 \frac{4}{3} + \frac{4}{5} - p}}{\mu_0} \left[ \frac{p - 3}{5-p} + \frac{9 + 4k}{2(5-p)} \right] \ldots (4.10)
\]

As for the minor \( \Omega_{TT} \), the determinant (4.10) has the leading term in the low temperature expansion which becomes negative for \( p < 3 \). Therefore, the Hessian turns out to be negative definite for \( p < 3 \). In the conformal case \( p = 3 \), the first term in (4.10) vanishes and the new dominant contribution is always positive. Finally, for \( p = 4 \) all the terms in (4.10) are positive. The Hessian is therefore positive definite for \( p \geq 3 \). The positive-definiteness of the Hessian for the cases \( p = 3, 4 \) insures stability. Moreover, the entropy density at
zero temperature is either finite ($p = 3$) or zero ($p = 4$), in agreement with the third law of thermodynamics. The specific heat turns out to behave as $\sim T^{2(3-k)}$ ($p = 3$) or $\sim T$ ($p = 4$).

A comment is now in order. The statement of the negative-definiteness of the Hessian is a local statement. This means that, in principle, the systems could tend to a stable configuration. Moreover, the brane configurations analysed in this section show a very peculiar behaviour for entropy density (4.6) and specific heat (4.7) at low temperature for $p < 3$. For convenience, let us rewrite it below:

$$s(T, c_f) = \tilde{N} \frac{2}{5 - p} \left( \frac{4\pi}{7 - p} \right)^{\frac{2}{5 - p}} c_f T^{-\frac{3 - p}{5 - p}} + \ldots, \quad c_V = \tilde{N} 2(5 - p)^2 \left( \frac{4\pi}{7 - p} \right)^{\frac{2}{5 - p}} T^{-\frac{3 - p}{5 - p}} c_f + \ldots$$

(4.11)

The behaviour (4.11) implies that both entropy density and specific heat increase as the temperature approaches to zero. This seems to violate the third law of thermodynamics according to which the density entropy reaches a minimum value as the temperature approaches to zero. As a consequence, the specific heat should vanish in the same limit. This, together with the negative-definiteness of the Hessian (which, at the end of the day, is a consequence of the behaviour (4.11)), implies that the configuration analysed here are not thermodynamically stable for $p < 3$. However, we need to recall that zero temperature backreacted solutions do exist for $p < 3$ which are well-behaved, like the D2/D6 solution found in [48]. It is therefore natural to ask how the appearance of this low-temperature instability connects with the existence of well-behaved backreacted solutions. One explanation to this question can be provided by the fact that these systems can become stable once one goes beyond the probe approximation, i.e. it is the the backreaction which stabilises them. The only tunable parameter available is the number $M$ of probe branes. One can thus think to increase the number of D$(p + 4 - 2k)$-branes until the Hessian becomes positive definite and thus the systems stabilise. This would imply a modification of the potential curve and therefore the effect of increasing $M$ can’t be seen from our computation, given that the potential curves of the probe case and of the backreacted one are different. In other words, tuning $M$ one gets a family of potentials until stability is reached at the backreaction, when the probe approximation breaks down.

One might argue that we are just computing thermodynamical quantities which refer just to the fluid (i.e. to the fundamental degrees of freedom), and that our stability analysis does not take into account all the degrees of freedom. This can be done by starting from the full grand potential, which contains both a contribution from the adjoint degrees of freedom and from the fundamental ones (including the density independent term in (4.3)). The contributions from $\Omega_{\text{adj}}$ scales with the temperature as $T^{2 \frac{-p}{5 - p}}$, while the last term in (4.3) as $T^{2 \frac{k}{5 - p}}$. It easy to see that such a contribution will not affect the leading behaviour at low temperature of any of the relevant thermodynamical quantities, such as the density
entropy, the specific heat and the Hessian of the grand potential. Thus the instability we are observing is not indeed due to not keeping into account the whole grand potential.

In principle, looking at the eigenvalues of the Hessian, one might think that the system can be driven to a high temperature stable point. In this case, the contribution from the adjoint degrees of freedom becomes more and more important as the temperature increases. Simultaneously the probe branes get heat up and can acquire enough stress-energy to eventually backreact. The probe approximation may break down anyway. It seems reasonable to think that the backreaction can stabilise the system, and the negative-definite Hessian, together with an apparent violation of the third law of thermodynamics, is a signature of the breaking down of the probe approximation.

We would like to stress that a deeper analysis of such an instability is indeed needed, since our arguments do not provide a robust proof of the nature of such an instability as the breaking-down of the probe approximation.

5 Conclusion

In this paper, we investigate the phase structure of $Dp/D(p + 4 - 2k)$-systems, where the $D(p + 4 - 2k)$ branes are considered in the probe approximation. These systems are BPS and introduce a $(p+1-k)$-dimensional defect in the $(p+1)$-dimensional $U(N)$ gauge theory. We consider both the probe brane configurations which introduce a massive fundamental hypermultiplet and the ones which introduce massless excitations in the $(p+1-k)$-dimensional defect.

The probe $D(p + 4 - 2k)$-branes wrap a $M_{p+2-k} \times S^{3-k}$ subspace of the background geometry $M_{p+2} \times S^{8-p}$ generated by a stack of Dp-branes. The embedding of the probe branes can be controlled by a scalar mode, which is provided by one of the angular coordinates in the transverse space. The separation between the stack of probe branes and the background ones is a parameter in the theory related to the quark mass. In this setting we consider the system to be at finite temperature (i.e. non-extremal black hole background) and at finite chemical potential, by turning on a non-trivial profile for the gauge field on the probe branes world-volume. On the chemical potential axis ($T = 0$), the system undergoes a second order phase transition for $k = 0, 1$. In the region $\mu < m_0$ the probe branes cannot reach the location of the probe branes, but they have an extreme point where they turn back and hit the boundary again: the probe branes are in a $D(p + 4 - 2k)/\bar{D}(p + 4 - 2k)$ configuration. For $\mu > m_0$ the probe brane can reach the location of the background branes, in a black-hole crossing phase. The second phase is thermodynamically favoured since the grand-potential is negative, while in the brane/anti-brane phase it is positive. The case $k = 2$ is a bit special given its different physical interpretation: turning on a non-trivial embedding mode.
in the transverse space does no longer correspond to a mass-deformation, i.e. the separation between the probe branes and the background one can no longer be interpreted as a mass, but it provides a non-zero vacuum-expectation-value for its dual operator.

In the interior of the \((\mu, T)\)-plane, it is possible to identify a transition-curve \(\mu = m(T)\) between Minkowski embeddings and black-hole embeddings. The order of the phase transition across this curve is strongly tied to the codimensionality \(k\) of the defect. In the case of \(k = 0\), the result of [20] can be extended to any sensible \(Dp/D(p+4)\)-systems: the transition across the curve \(\mu = m(T)\) is a third order phase transitions. Third-order phase transitions are not very common in nature. However, there are some meaningful examples to keep into consideration. One example is indeed provided by the Gross-Witten model [53] in which a third-order phase transition in the large-\(N\) lattice gauge theory in two-dimensions was observed and then extended to four-dimensions. Indeed, the physics of the phase transition in the Gross-Witten model is very different from the one of the class of theories discussed in this paper. It is a weak-to-strong coupling phase transition which occurs at fixed ‘t Hooft coupling \((\lambda = \lambda_c)\) and at large-\(N\). Here instead, the system is studied at both large-\(N\) and infinite ‘t Hooft coupling and the transition occurs in the strongly-coupled regime (for \(p \geq 0\)). The gauge theories dual to these brane construction define a new class of theories with a third order phase transition.

For such systems, the transition line \(\mu = m(T)\) represents a third order phase transition until it reaches the temperature \(T = T_c\) at which the phase transition becomes of second order.

In the case \(k = 1\), the phase transition along \(\mu = m(T)\) is of second order. This is a direct consequence of the fact that, for such systems, the small “quark”-density expansion of the chemical potential does not show any logarithmic behaviour at first order in “quark”-density.

Theories with a codimension-1 defect can be reduced to effective codimension-0 theories by compactifying the direction of the background on which the probe branes do not extend. An example of such systems was studied in [17, 21, 26]. One can wonder if and how the reduction of theories with a codimension-1 defect to effective theories with no defect affects the order of the phase transition along the curve \(\mu = m(T)\). Most likely, it is the case because of the introduction of a new scale provided by the compactification radius. However, we did not check this explicitly and we leave it for future work. Indeed, if the compactification has the effect of changing the order of the phase transition along \(\mu = m(T)\), it would strengthen our statement of the existence of two universality classes of theories with baryonic chemical potential, which are identified by the codimensionality of the defect, i.e. the order of the phase transition.

The massless degrees of freedom on the \((p+1-k)\)-dimensional defect are studied by fixing
the probe branes to wrap the maximal \((3 - k)\)-sphere in the transverse space. Fixing also
the position of the probe-branes in the non-compact directions provides a supersymmetric
description of the system. In principle one can turn on a non-trivial profile for \(x^p = z(\rho)\)
(indeed this description is valid for \(k \neq 0\)). In this case, the operator \(O\), would acquire a
non-zero vev and the supersymmetries get broken. We consider the supersymmetric case and
focus on the low temperature properties of such finite density systems. Following [38, 41], we
compute the density entropy and the specific heat in such a regime. Both those quantities
turn out to scale with the temperature as
\[
s \sim T^{\frac{p-3}{5-p}}, \quad c_V \sim (p - 3)T^{\frac{p-3}{5-p}}.
\]
(5.1)

It is interesting to notice that the powers in (5.1) become negative for \(p < 3\). Indeed, this
type of behaviour seems to be at least counter-intuitive. For the third law of thermodynamics
one would expect the specific heat to vanish as the temperature goes to zero and the density
entropy to reach zero as well or, anyway, a finite value. Instead, for the class of systems
identified by \(p < 3\) both these quantities appear to blow up in the zero temperature limit.
In order to investigate this issue, we explicitly analyse the stability of these systems. A
necessary and sufficient condition for the (local) stability of these systems is that the Hessian
matrix is positive definite. In the case of interest \((p < 3)\), we show that the Hessian is
actually negative definite, which identifies an instability at low temperature explaining the
observed behaviour of entropy and specific heat. However, it arises new questions. First,
what is the nature of such an instability? How can these systems be stabilised? How does
this result connects with the existence of a zero-temperature backreacted solution such as the
D2/D6 of [48]? We pointed out that the only tunable parameter in the theory is the number
\(M\) of probe branes. So one can think to increase it until the potential changes in such a
way to make the Hessian positive definite, at which point most likely the \(M \ D(p + 4 - 2k)\)-
branes \((p < 3)\) will start backreacting stabilising the system and the probe approximation
breaks down. Another possibilities is that the analysis of the eigenvalues of the Hessian may
suggest that the system can be lead to a stable point at high temperature. In this case, the
contribution from the adjoint degrees of freedom as well as from the density independent
term from the fundamental ones will start to be more and more relevant. Simultaneously,
the probe branes get heat up to the point that they can start to backreact. This would lead
again to the breaking-down of the probe approximation. It is therefore reasonable to think
that the backreaction is needed to stabilise the system, and consider it as best candidate for
explaining the appearance of such an instability at low temperature. Once again, we want to
reiterate the idea that a more detailed analysis about this stability issue is needed, but our
observation stresses the necessity of an extensive discussion about the validity of the probe
approximation as well as the need of a deeper understanding of backreacted solutions and
the related physics.

Acknowledgment

It is a pleasure to thank Michael Haack and Suresh Nampuri for hospitality at LMU as well as the organisers of the Workshop on the Fluid Gravity Correspondence held in Munich and 5th Aegean Summer School in Milos for the stimulating environment. I am very thankful to Suresh Nampuri for discussions and correspondence. I would like to thank Johanna Erdmenger, Michael Haack and the string theory groups at LMU and MPI Munich for the possibility to present preliminary results of this work in their local seminar, as well as Giuseppe Policastro and the LPTENS for the possibility to present them in the “Rencontres Théoriciennes”. It is also a pleasure to thank for stimulating discussions and interest Marcello Dalmonte, Jan de Boer, Pau Figueras, Veronika Hubeny, David Mateos, Mukund Rangamani, Andrea Scaramucci. This work is supported by STFC Grant.

A Coefficients for the small density expansions

In this section we explicitly write down the numerical coefficients in the small density expansion of section 3.2, where we emphasised as in [20] that such an expansion is subtle and it is necessary to split the radial axis in two regions. In region 1, at zero order in $c_f$ the embedding function is conveniently expanded in a neighbourhood of $\tilde{\rho} \sim 0$ and such expansion turns out to involve even powers only of the radial coordinate (3.24). The coefficients of this expansion have been written in (3.25) in terms of other coefficients $b^{(p,k)}_{i}$ and $c^{(p,k)}_{i}$ which depend on the spatial dimensions $p$ on the background branes and on the codimensionality $k$ of the defect. Here we list the coefficients $b^{(p,k)}_{i}$ and $c^{(p,k)}_{i}$:

- Coefficients $b^{(p,k)}_{i}$ in $y_{0}^{(4)}$

\begin{align*}
    b^{(p,k)}_{0} &= -(4 - k)^2 (p^2 - (12 + k)p + 9(k + 3)), \\
    b^{(p,k)}_{1} &= 2(4 - k)^2 (p^3 - 2(k + 8)p^2 + (2k^2 + 18k + 85)p - 2(9k^2 - 2k + 109)), \\
    b^{(p,k)}_{2} &= -2(4 - k) (4p^3 - 3(k^3 - 12k^2 + 52k - 52)p^2 + \\
    &\qquad + (3k^4 - 2k^3 - 252k^2 + 1512 - 2068)p - (3 + k)(25k^3 - 296k^2 + 1224k - 1564)), \\
    b^{(p,k)}_{3} &= 2(4 - k) ((4 - k)p^3 + 2(k^2 + k - 14)p^2 - (2k^3 - 2k^2 + 49k - 124)p + \\
    &\qquad + 4(3k^3 - 19k^2 + 95k - 137)), \\
    b^{(p,k)}_{4} &= -(4 - k)^2 ((4 - k)p^2 + (k^2 - 4k + 24)p - 3(12 - k)(3 + k)), \\
    b^{(p,k)}_{5} &= 4(6 - k)(4 - k)^3.
\end{align*}

(A.1)
\[ \begin{align*}
\text{Coefficients } c^{(p,k)}_i \text{ in } y_6^{(6)} \\
c^{(p,k)}_0 &= (6 - k)(4 - k)^5(11 - p)(9 - p)(3 + k - p), \\
c^{(p,k)}_1 &= 2(4 - k)^6(9 - p) \left[ (16 - 3k)p^3 + (6k^2 + 17k - 264)p^2 + \\
&\quad - (k^3 + 92k^2 - 347k - 960)p + (15k^3 + 242k^2 - 1753k + 24) \right], \\
c^{(p,k)}_2 &= 4(4 - k)^3(3 + k - p) \left[ 2(5 - k)p^4 + 4(k^2 + 29k - 256)p^3 + \\
&\quad + (4k^3 - 156k^2 + 33k + 1258)p^2 - (74k^3 - 1884k^2 + 7565k - 3970)p - \\
&\quad - (4 - k)(340k^2 - 5848k + 9573) \right], \\
c^{(p,k)}_3 &= -2(4 - k)^2 \left[ 88p^5 + (11k^3 - 136k^2 + 208k - 2616)p^4 - \\
&\quad - 2(11k^4 + 18k^3 - 1564k^2 + 3552k - 17176)p^3 - \\
&\quad - (3k^5 - 714k^4 + 6150k^3 + 4896k^2 - 37152k + 209968)p^2 + \\
&\quad + 2(2k^5 - 3147k^4 + 37726k^3 - 107556k^2 + 1458884k + 175420)p + \\
&\quad + (203k^5 + 14898k^4 - 226797k^3 + 941720k^2 - 1807856k + 755560) \right], \\
c^{(p,k)}_4 &= 2(3 + k - p) \left[ 8(2k^4 - 35k^3 + 228k^2 - 648k + 696)p^4 - \\
&\quad - 8(4k^5 - 30k^4 - 299k^3 + 3940k^2 - 142898k + 17504)p^3 + \\
&\quad + (35k^6 - 198k^5 - 1664k^4 - 3224k^3 + 197280k^2 - 924160k + 1288320)p^2 - \\
&\quad - 4(165k^6 - 3024k^5 + 21040k^4 - 89722k^3 + 372024k^2 - 1148640k + 1484736)p + \\
&\quad + (3069k^6 - 68754k^5 + 605952k^4 - 2847248k^3 + 8357440k^2 - 15749056k + 14671808) \right], \\
c^{(p,k)}_5 &= -2(4 - k) \left[ 8(4 - k)p^5 - 55k^4 - 964k^3 + 5936k^2 - 17880k + 25120)p^4 + \\
&\quad + 2(55k^5 - 424k^4 - 3884k^3 + 48176k^2 - 173528k + 237536)p^3 - \\
&\quad - (73k^6 + 422k^5 - 15982k^4 + 42672k^3 + 415744k^2 - 2346864k + 3647040)p^2 + \\
&\quad + 2(560k^6 - 7005k^5 + 13600k^4 + 40732k^3 + 483088k^2 - 3681404k + 6445680)p - \\
&\quad - (4187k^6 - 80082k^5 + 551079k^4 - 1930700k^3 + 5556816k^2 - 15048120k + 20930976) \right], \\
c^{(p,k)}_6 &= 4(4 - k)^2(3 + k - p) \left[ 2(5 - k)(4 - k)p^4 + (4 - k)(4k^2 + 11k - 306)p^3 + \\
&\quad + (32k^4 - 496k^3 + 2921k^2 - 10622k + 20264)p^2 - \\
&\quad - (398k^4 - 7196k^3 + 4654k^2 - 149796k + 190264)p + \\
&\quad + (1100k^4 - 22516k^3 + 162269k^2 - 504752k + 629168) \right], \\
c^{(p,k)}_7 &= 2(4 - k)^3 \left[ 3(4 - k)(16 - 3k)p^4 + 2(4 - k)(9k^2 + 16k - 516)p^3 + \\
&\quad + (27k^4 - 206k^3 + 450k^2 - 8200k + 33792)p^2 - \\
&\quad - 2(136k^4 - 1896k^3 + 9704k^2 - 32996k + 79184)p + \\
&\quad + (405k^4 - 8014k^3 + 56877k^2 - 182228k + 322176) \right], \\
\end{align*} \]
\[c_s^{(p, k)} = (4 - k)^2(3 + k - p) \left[9(6 - k)(4 - k)p^2 - 4(4 - k)(154 - 3k)p + (131k^2 - 782k + 2952)\right],
\]
\[c_0^{(p, k)} = 8(4 - k)^5(11k^2 - 24k - 32).\]

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* 2 (1998) 231–252, arXiv:hep-th/9711200.

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett.* B428 (1998) 105–114, arXiv:hep-th/9802109.

[3] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* 2 (1998) 253–291, arXiv:hep-th/9802150.

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” *Phys. Rept.* 323 (2000) 183–386, arXiv:hep-th/9905111.

[5] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” *Phys. Rev.* D58 (1998) 046004, arXiv:hep-th/9802042.

[6] M. J. Duff, G. W. Gibbons, and P. K. Townsend, “Macroscopic superstrings as interpolating solitons,” *Phys. Lett.* B332 (1994) 321–328, arXiv:hep-th/9405124.

[7] H. J. Boonstra, B. Peeters, and K. Skenderis, “Duality and asymptotic geometries,” *Phys. Lett.* B411 (1997) 59–67, arXiv:hep-th/9706192.

[8] H. J. Boonstra, B. Peeters, and K. Skenderis, “Brane intersections, anti-de Sitter spacetimes and dual superconformal theories,” *Nucl. Phys.* B533 (1998) 127–162, arXiv:hep-th/9803231.

[9] H. J. Boonstra, K. Skenderis, and P. K. Townsend, “The domain wall/QFT correspondence,” *JHEP* 01 (1999) 003, arXiv:hep-th/9807137.

[10] A. Jevicki, Y. Kazama, and T. Yoneya, “Generalized conformal symmetry in D-brane matrix models,” *Phys. Rev.* D59 (1999) 066001, arXiv:hep-th/9810146.

[11] K. Skenderis, “Field theory limit of branes and gauged supergravities,” *Fortsch. Phys.* 48 (2000) 205–208, arXiv:hep-th/9903003.
[12] A. Karch and E. Katz, “Adding flavor to AdS/CFT,” JHEP 06 (2002) 043, arXiv:hep-th/0205236.

[13] J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik, and I. Kirsch, “Chiral symmetry breaking and pions in non-supersymmetric gauge / gravity duals,” Phys. Rev. D69 (2004) 066007, arXiv:hep-th/0306018.

[14] D. Mateos, R. C. Myers, and R. M. Thomson, “Holographic phase transitions with fundamental matter,” Phys. Rev. Lett. 97 (2006) 091601, arXiv:hep-th/0605046.

[15] T. Albash, V. G. Filev, C. V. Johnson, and A. Kundu, “A topology-changing phase transition and the dynamics of flavour,” Phys. Rev. D77 (2008) 066004, arXiv:hep-th/0605088.

[16] S. Kobayashi, D. Mateos, S. Matsuura, R. C. Myers, and R. M. Thomson, “Holographic phase transitions at finite baryon density,” JHEP 02 (2007) 016, arXiv:hep-th/0611099.

[17] D. Mateos, R. C. Myers, and R. M. Thomson, “Thermodynamics of the brane,” JHEP 05 (2007) 067, arXiv:hep-th/0701132.

[18] K. Ghoroku, M. Ishihara, and A. Nakamura, “D3/D7 holographic Gauge theory and Chemical potential,” Phys. Rev. D76 (2007) 124006, arXiv:0708.3706 [hep-th].

[19] D. Mateos, S. Matsuura, R. C. Myers, and R. M. Thomson, “Holographic phase transitions at finite chemical potential,” JHEP 11 (2007) 085, arXiv:0709.1225 [hep-th].

[20] T. Faulkner and H. Liu, “Condensed matter physics of a strongly coupled gauge theory with quarks: some novel features of the phase diagram,” arXiv:0812.4278 [hep-th].

[21] S. Matsuura, “On holographic phase transitions at finite chemical potential,” JHEP 11 (2007) 098, arXiv:0711.0407 [hep-th].

[22] C. Hoyos-Badajoz, K. Landsteiner, and S. Montero, “Holographic Meson Melting,” JHEP 04 (2007) 031, arXiv:hep-th/0612169.

[23] A. Karch and A. O’Bannon, “Holographic Thermodynamics at Finite Baryon Density: Some Exact Results,” JHEP 11 (2007) 074, arXiv:0709.0570 [hep-th].

[24] T. Faulkner and H. Liu, “Meson widths from string worldsheet instantons,” Phys. Lett. B673 (2009) 161–165, arXiv:0807.0063 [hep-th].
[25] X.-J. Wang and S. Hu, “Intersecting branes and adding flavors to the Maldacena-Nunez background,” JHEP 09 (2003) 017, arXiv:hep-th/0307218.

[26] M. Kruczenski, D. Mateos, R. C. Myers, and D. J. Winters, “Towards a holographic dual of large-N(c) QCD,” JHEP 05 (2004) 041, arXiv:hep-th/0311270.

[27] T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” Prog. Theor. Phys. 113 (2005) 843–882, arXiv:hep-th/0412141.

[28] T. Sakai and S. Sugimoto, “More on a holographic dual of QCD,” Prog. Theor. Phys. 114 (2005) 1083–1118, arXiv:hep-th/0507073.

[29] B. A. Burrington and J. Sonnenschein, “Holographic Dual of QCD from Black D5 Branes,” arXiv:0903.0628 [hep-th].

[30] J. L. Davis, P. Kraus, and A. Shah, “Gravity Dual of a Quantum Hall Plateau Transition,” JHEP 11 (2008) 020, arXiv:0809.1876 [hep-th].

[31] M. Ammon, J. Erdmenger, M. Kaminski, and P. Kerner, “Superconductivity from gauge/gravity duality with flavor,” arXiv:0810.2316 [hep-th].

[32] R. C. Myers and M. C. Wapler, “Transport Properties of Holographic Defects,” JHEP 12 (2008) 115, arXiv:0811.0480 [hep-th].

[33] M. Fujita, W. Li, S. Ryu, and T. Takayanagi, “Fractional Quantum Hall Effect via Holography: Chern-Simons, Edge States, and Hierarchy,” JHEP 06 (2009) 066, arXiv:0901.0924 [hep-th].

[34] M. Ammon, J. Erdmenger, M. Kaminski, and P. Kerner, “Flavor Superconductivity from Gauge/Gravity Duality,” arXiv:0903.1864 [hep-th].

[35] K. Peeters, J. Powell, and M. Zamaklar, “Exploring colourful holographic superconductors,” arXiv:0907.1508 [hep-th].

[36] M. C. Wapler, “Holographic Experiments on Defects,” arXiv:0909.1698 [hep-th].

[37] L.-Y. Hung and A. Sinha, “Holographic quantum liquids in 1+1 dimensions,” arXiv:0909.3526 [hep-th].

[38] A. Karch, D. T. Son, and A. O. Starinets, “Zero Sound from Holography,” arXiv:0806.3796 [hep-th].

[39] M. Kulaxizi and A. Parnachev, “Comments on Fermi Liquid from Holography,” Phys. Rev. D78 (2008) 086004, arXiv:0808.3953 [hep-th].
[40] M. Kulaxizi and A. Parnachev, “Holographic Responses of Fermion Matter,” *Nucl. Phys. B815* (2009) 125–141, arXiv:0811.2262 [hep-th].

[41] A. Karch, M. Kulaxizi, and A. Parnachev, “Notes on Properties of Holographic Matter,” arXiv:0908.3493 [hep-th].

[42] S.-i. Tomonaga, “Remarks on Bloch’s method of sound waves applied to many-fermion problems,” *Progress of Theoretical Physics* 5 (1950) no. 4, 544–569. http://ptp.ipap.jp/link?PTP/5/544/.

[43] J. Luttinger, “An exactly soluble model of a many-fermion system,” *J. Math. Phys.* 4 (1963) 1154.

[44] D. C. Mattis and E. H. Lieb, “Exact Solution of a Many-Fermion System and Its Associated Boson Field,” *J. Math. Phys.* 6 (1965) 304–312.

[45] F. D. M. Haldane, “Luttinger Liquid Theory’ of One-Dimensional Quantum Liquid: I. Properties of the Luttinger Model and Their Extension to the General 1D Interacting Spinless Fermi Gas,” *J. Phys. C* 14 (1981) 2585–2609.

[46] H. J. Schulz, G. Cuniberti, and P. Pieri, “Fermi Liquids and Luttinger Liquids,” *G. Morandi et al. eds, Springer* (2000), arXiv:cond-mat/9807366.

[47] T. Giamarchi, “Quantum physics in one-dimensions,” *Oxford University Press* (2003).

[48] S. A. Cherkis and A. Hashimoto, “Supergravity solution of intersecting branes and AdS/CFT with flavor,” *JHEP* 11 (2002) 036, arXiv:hep-th/0210105.

[49] N. R. Constable, J. Erdmenger, Z. Guralnik, and I. Kirsch, “Intersecting D3-branes and holography,” *Phys. Rev.* D68 (2003) 106007, arXiv:hep-th/0211222.

[50] D. Arean, A. V. Ramallo, and D. Rodriguez-Gomez, “Holographic flavor on the Higgs branch,” *JHEP* 05 (2007) 044, arXiv:hep-th/0703094.

[51] P. Benincasa, “A Note on Holographic Renormalization of Probe D-Branes,” arXiv:0903.4356 [hep-th].

[52] A. Karch, A. O’Bannon, and K. Skenderis, “Holographic renormalization of probe D-branes in AdS/CFT,” *JHEP* 04 (2006) 015, arXiv:hep-th/0512125.

[53] D. J. Gross and E. Witten, “Possible Third Order Phase Transition in the Large N Lattice Gauge Theory,” *Phys. Rev.* D21 (1980) 446–453.