Near Optimal Compressed Sensing of a Class of Sparse Low-Rank Matrices via Sparse Power Factorization
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Abstract

Compressed sensing of simultaneously sparse and low-rank matrices enables recovery of sparse signals from a few linear measurements of their bilinear form. One important question is how many measurements are needed for a stable reconstruction in the presence of measurement noise. Unlike conventional compressed sensing for sparse vectors, where convex relaxation via the $\ell_1$-norm achieves near optimal performance, for compressed sensing of sparse low-rank matrices, it has been shown recently \cite{2} that convex programmings using the nuclear norm and the mixed norm are highly suboptimal even in the noise-free scenario.

We propose an alternating minimization algorithm called sparse power factorization (SPF) for compressed sensing of sparse rank-one matrices. For a class of signals whose sparse representation coefficients are fast-decaying, SPF achieves stable recovery of the rank-1 matrix formed by their outer product and requires number of measurements within a logarithmic factor of the information-theoretic fundamental limit. For the recovery of general sparse low-rank matrices, we propose subspace-concatenated SPF (SCSPF), which has analogous near optimal performance guarantees to SPF in the rank-1 case. Numerical results show that SPF and SCSPF empirically outperform convex programmings using the best known combinations of mixed norm and nuclear norm.

I. INTRODUCTION

A. Problem statement

Let $X \in \mathbb{C}^{n_1 \times n_2}$ be an unknown rank-$r$ matrix whose singular value decomposition is written as $X = UAV^*$, where $\Lambda \in \mathbb{R}^{r \times r}$ is a strictly positive diagonal matrix, $U \in \mathbb{C}^{n_1 \times r}$, and $V \in \mathbb{C}^{n_2 \times r}$ satisfy $U^*U = V^*V = I_r$. We further assume that $X$ is sparse in the following senses: i) either $U$ or $V$ is
row-$s$-sparse, that is, has at most $s$ nonzero rows;\footnote{Without loss of generality, we assume that $U$ is row-$s$-sparse.} or ii) $U$ and $V$ are row-$s_1$-sparse and row-$s_2$-sparse, respectively.

Suppose that the measurement vector $b \in \mathbb{C}^m$ of $X$ is obtained using a known linear operator $A : \mathbb{C}^{n_1 \times n_2} \to \mathbb{C}^m$ as

$$b = A(X) + z,$$

(1)

where $z \in \mathbb{C}^m$ denotes additive noise.

We study the problem of stable reconstruction of the unknown \textit{simultaneously sparse and low-rank} $X$ from the noisy linear measurements $b$. Our goal is to find a good estimate $\hat{X}$ of $X$ from a minimal number of measurements using a computationally efficient algorithm, which satisfies the following stability criterion:

$$\frac{\|\hat{X} - X\|_F}{\|X\|_F} \leq C \cdot \frac{\|z\|_2}{\|A(X)\|_2}$$

(2)

for all $z \in \mathbb{C}^m$ and some absolute (dimension-independent) constant $C$. The condition in (2) implies that the normalized reconstruction error is at most a constant factor of the noise-to-signal ratio in the measurements, which, in the absence of noise, automatically implies the perfect reconstruction of $X$.

\textbf{B. Motivating applications: sparse bilinear inverse problems}

Bilinear inverse problems arise ubiquitously in a variety of areas. For example, \textit{blind deconvolution} (cf. \cite{3} and references therein) factors two input signals given their convolution, which is a bilinear function. In general, bilinear inverse problems involve various ambiguities and do not admit a unique solution. For example, any bilinear inverse problem suffers from scaling ambiguity and the best result one can get is to identify the solution up to a scalar factor. Besides the fundamental ambiguities that cannot be overcome by any method, it is still challenging to identify the solution up to an appropriate equivalence class. To overcome this difficulty, various sparsity models were introduced and the resulting \textit{sparse} bilinear inverse problem has been shown empirically to admit good solutions in various real-world applications.

In blind deconvolution, signals of interest admit sparse representations \cite{4} and these sparse signal models have been exploited in denoising, compression, compressed sensing, etc. The impulse responses of convolution systems in applications have sparse representations too. For example, high definition television (HDTV) channels, hilly terrain delay profiles, and underwater acoustic or reverberant room...
channels, all have sparse channel coefficients (see [5] and references therein). These sparsity models were employed to solve blind deconvolution problems, e.g., in the context of blind echo cancellation [5], [6].

In these applications, sparsity models narrow down the solution set, which, in a bilinear inverse problem, is a product of subspaces, by replacing the subspaces by unions of low-dimensional subspaces. This makes robust reconstruction possible even when subsampling is present, which is often desired in applications such as calibration-free parallel imaging.

Recently, it has been proposed to reformulate bilinear inverse problems as the recovery of a low-rank matrix from its linear measurements, through the so-called “lifting” procedure [3], [7]. Ahmed et al. [3] first introduced this idea to solve the blind deconvolution problem as a matrix-valued linear inverse problem, and further showed that nuclear-norm minimization is nearly optimal for certain random linear operators. On the other hand, Choudhary et al. [7] showed negative results in the setup where both the input and linear operator are deterministic. In a nutshell, in the lifted formulation, one obtains a solution to a bilinear system \( f_i(x, y) = b_i, \ i = 1, \ldots, m \) from a low-rank solution to a linear system \( \mathcal{A}(X) = b \) in the matrix-valued unknown \( X \). In the lifted formulation of blind deconvolution, the unknown matrix \( X \) is rank-one. On the other hand, in MIMO channel identification [8], the measurements are given as superpositions of convolutions and therefore the solution to the lifted formulation is low-rank, where the rank is determined as the number of the input channels.

In this paper, we consider the lifted linear inverse problem, although we adopt a nonconvex approach. Then, the scaling ambiguity is absorbed into the factorization of the matrix-valued solution to the lifted formulation. In the lifted formulation of the sparse bilinear inverse problem, the unknown \( X \) has a sparsity model corresponding to those imposed on the unknowns of the original bilinear problem.

The product of two compatible matrices is also bilinear in the individual matrices; hence, matrix factorization is another bilinear inverse problem (cf. [9]). Sparsity models also arise in certain matrix factorization problems. For example, dictionary learning aims to find a good sparse representation for a given data set, which can be formulated as a matrix factorization problem with a sparsity prior [9]. Learning a sparsifying dictionary or transform from compressive measurements [10]–[12] has a similar flavor, but is a more difficult problem, since fewer equations are available to determine the unknowns. Compressive blind source separation [13] is yet another matrix factorization problem that exploits a sparsity prior. These applications can be naturally formulated as the recovery of a sparse and low-rank
matrix from its linear measurements.

C. Related work

The recovery of a sparse and low-rank matrix from its minimal incoherent linear measurements is a special case of compressed sensing of general low-rank matrices. Compressed sensing of low-rank matrices without sparsity constraints has been well studied as an extension of compressed sensing of sparse vectors. Recht et al. [14] presented the analogy between the two problems and showed that the minimum nuclear norm solution to the linear system given by the measurements is guaranteed to recover the unknown low-rank matrix under the rank-restricted isometry property. Greedy recovery algorithms for compressed sensing of sparse vectors and their performance guarantees under the restricted isometry property (RIP) have been also extended to analogous algorithms (e.g., ADMiRA [15], SVP [16]) with corresponding guarantees for compressed sensing of low-rank matrices under the rank-restricted isometry property. An alternating minimization algorithm called power factorization (PF) has been proposed as a computationally efficient heuristic [17] for the recovery of general low-rank matrices, and its performance guarantee in terms of the rank-restricted isometry property was presented recently [18]. In particular, for a certain class of sensing systems, it has shown [14], [19], [20] that $O(nr)$ or slightly more, by a logarithmic factor, measurements suffice for stable recovery of an $n \times n$ matrix of rank-$r$.

When the unknown matrix is not only low-rank but also sparse, the number of compressed sensing measurements required for its recovery is further reduced. Suppose that the unknown $n \times n$ matrix is of rank $r$ and has up to $s$ nonzero rows and up to $s$ nonzero columns, and its noise-free measurements are obtained as the inner products with i.i.d. Gaussian matrices. Oymak et al. [2] showed that by solving a combinatorial optimization problem, exact recovery is guaranteed with $O(\max\{rs, s\log(en/s)\})$ measurements. However, they also showed the following negative result: Combining convex surrogates for multiple nonconvex priors does not improve the recovery performance compared to the case of using just one of the priors via a convex surrogate. More precisely, exact recovery using combinations of the nuclear norm and the $\ell_1,2$ norm requires $\Omega(\min\{rn, sn\})$ measurements. This is significantly worse than the sample complexity that can be obtained by solving a combinatorial optimization problem.

One might attempt to modify ADMiRA or SVP to exploit the low-rank and sparsity priors simultaneously. Unfortunately, the key procedure in these algorithms is to project a given matrix onto a set of low-rank and sparse matrices, which is another challenging open problem, in the sense that there is no
algorithm with a performance guarantee for this problem.

A closely-related statistical problem involving both sparsity and low-rankness is sparse principal component analysis (SPCA) [21]–[26], which deals with estimating the principal subspaces of a covariance matrix when the singular vectors are sparse. For instance, in the rank-one case, one observes independent samples from $\mathcal{N}(0, I_p + \lambda vv^*)$ and estimates the leading singular vector $v$ which is known to be sparse a priori. The minimax estimation error of SPCA has been characterized within constant factors in [23]–[26]. However, it has been shown [27] that attaining the minimax rate of SPCA can be reduced to planted clique, an open problem that is believed to be hard. Another related problem is submatrix detection or biclustering [28], [29], where $X = \lambda vv^* + Z$ is observed with $Z$ i.i.d. Gaussian and $u, v$ sparse binary vectors. The goal is to decide whether or not $\lambda = 0$. It has been shown that attaining the optimal rate for this problem is computationally hard in a similar sense [30]. The main distinction between these problems and our sparse inverse problem (1) is that in the former the low-rank and sparse signals (either the covariance or the mean matrix) are observed directly from noisy samples, whereas in the latter the signal is only observed indirectly, through linear measurements that mix the components. Therefore, our sparse inverse problem (1) is harder than the other two problems. However, the relative difficulty arising from indirect access to the measurement through noisy linear measurements can be overcome if the linear operator satisfies the restrict isometry property. All the aforementioned problems are difficult in general. In our problem, under an additional “peakiness” assumption, we manage to solve the problem with a provably near optimal performance guarantee.

D. Main contributions

As discussed earlier, in the existing theory for compressed sensing of simultaneously sparse and low-rank matrices, the best known performance guarantee on the sample complexity for polynomial-time algorithms is significantly worse than the theoretical optimum.\footnote{When we posted the first draft of this work to arXiv [31], there were no algorithms that achieve a near optimal sample complexity for recovering simultaneously sparse and low-rank matrices. Subsequently, it has been shown that for a special measurement scheme with nested structures, a two-step approach achieves near optimal sample complexity [32], [33]. However, designing such nested structured sensing mechanisms are impossible in many applications such as blind deconvolution. Very recently, we have learned that the linear operator arising in blind deconvolution satisfies the RIP [34]. Therefore, the performance guarantee for SPF in this paper applies not only to compressed sensing with i.i.d. Gaussian measurements but also to blind deconvolution.} Toward closing this gap, in this paper, we propose a set of recovery algorithms that provide near optimal performance guarantees at low computational cost. Motivated by the lifted formulation of various sparse bilinear inverse problems, we
first focus on the rank-one case and will later extend both the algorithms and the analysis to the rank-$r$ case.

We propose an alternating minimization algorithm called *sparse power factorization* (SPF), which reconstructs the unknown sparse rank-one matrix from its linear measurements. SPF is obtained by modifying the updates of estimates of $u$ and $v$ in PF (see Section II-A for a summary) to exploit their sparsity priors. In principle, any algorithm for recovery of sparse vectors from linear measurements can be employed for these steps. In this paper, we focus on a specific procedure called hard thresholding pursuit (HTP) [35]. For recovering sparse vectors and under RIP assumptions, HTP provides performance guarantees for both estimation error and convergence rate, which can be further generalized in the presence of noisy measurements. Exploiting these guarantees for HTP, we show that the iterative updates in SPF converge linearly under RIP assumption.

Like most alternating minimization methods, the empirical performance of PF and SPF depends crucially on the initial values. Furthermore, to obtain a provable guarantee, it is important to design the initialization procedure carefully. Let $A^*$ denote the adjoint operator of $A$. Jain et al. [18] showed that PF initialized by the leading right singular vector of the proxy matrix $A^*(b)$ [36] provides stable recovery of a rank-$r$ matrix under the rank-2$r$ RIP. In particular, if the unknown $n_1 \times n_2$ matrix is rank-one, then their guarantee holds with $O(\max\{n_1, n_2\})$ i.i.d. Gaussian measurements.

When the unknown $n_1 \times n_2$ matrix is row-$s$-sparse, the initialization needs to be modified accordingly. We propose to initialize the SPF algorithm by the leading right singular vector of a submatrix of $A^*(b)$ whose rows are restricted to an estimated row-support of the left singular vector $u$, SPF initialized with a good approximation on either the left or the right singular vector provides near-optimal performance guarantee whenever the linear operator satisfies RIP for rank-2 and row-3$s$-sparse matrices, which holds with $O((s + n_2) \log(en_1/s))$ i.i.d. Gaussian measurements. In particular, when the entries of $u$ are fast-decaying, which is often satisfied by signal models in practice, we show that a simple thresholding algorithm provides such a good initialization.

In the case when the unknown $n \times n$ matrix is doubly-$s$-sparse (both row and column sparse), similarly to the previous case, SPF initialized with a good approximation on either $u$ or $v$ has a performance guarantee under the rank-2 and $(3s, 3s)$-sparse RIP, which holds with $O(s \log(en/s))$ i.i.d. Gaussian measurements, and significantly improves on the guarantee for PF. In particular, under extra decay conditions on the nonzero entries of $u$ and $v$, we show that a simple thresholding algorithm produces a desired good
initialization for SPF.

Next, for the sparse and rank-$r$ matrices with $r > 1$, we extend the SPF algorithm and its performance guarantees accordingly. The generalization is non-trivial in the sense that it is unclear whether the straightforward rank-$r$ extension of the SPF algorithm can be guaranteed to recover the unknown rank-$r$ and doubly-$s$-sparse matrix from $O(rs \log n)$ measurements. More specifically, the number of measurements for a performance guarantee depends on a power of the rank $r$ rather than linearly on $r$. To fix this, we considered a variation of SPF called subspace-concatenated SPF (SCSPF) which provably achieves a sample complexity that scales linearly in the rank $r$. Again, similarly to the rank-one cases, the success of cheap initialization requires extra technical conditions that are analogous to the fast decay properties.

Even in the absence of sparsity where simple initialization works, our results improve the state of the art for recovering low-rank matrices using alternating minimization. Specifically, for rank-$r$ matrices with conditioning number at most $\kappa$, we show that SCSPF succeeds with $m = O(\kappa^2 rn)$ measurements, which significantly improves on the previous result of $m = O(\kappa^4 r^3 n)$ [18].

How close is the performance of the SPF algorithm to optimality? We show that stable recovery of sparse rank-one matrices in the sense of (2) requires at least $(s_1 + s_2 - 3r/2)r$ measurements, where $s_1$ and $s_2$ are the row-sparsity levels of the left and right factor $U$ and $V$, respectively. Note that this lower bound coincides with the number of degrees of freedom in the singular vectors. While the parameter-counting argument is heuristic, our converse is obtained via information-theoretic arguments, which provide necessary conditions for stable recovery by any reconstruction method from any measurement mechanism – linear or not. It follows that our performance guarantees for SPF are near-optimal in the sense that SPF achieves robust reconstruction with a number of measurements that is within at most a logarithmic factor of the fundamental limit. Similar near-optimal guarantees for the structured (rather than just random Gaussian) measurements that arise in practical applications are presented in a companion paper [37].

In addition to its near-optimal theoretic guarantees, SPF also outperforms competing convex approaches in the following practical aspects:

- SPF requires vastly less memory, since it solves the bilinear formulation with an explicit rank-$r$ factorization of the unknown with $r(n_1 + n_2)$ variables. In contrast, the linear formulation solves an optimization problem with $n_1 n_2$ variables.
• SPF has lower computational cost. The SPF algorithm converges superlinearly fast and each of the sparse recovery steps (inner iteration using HTP) converges in $O(s)$ iterations. Furthermore, each iteration is fast because it only updates $r n_1$ or $r n_2$ variables instead of $n_1 n_2$ variables. Also note that these guarantees are derived for the initialization method that only involves simple thresholding on the row and column norms of the $n_1 \times n_2$ matrix $A^*(b)$ and the truncated singular value decomposition of a reduced $s_1 \times s_2$ matrix up to the first $r$ factors.

• As demonstrated in Section VI, the empirical performance of SPF is significantly better than that of convex approaches. In fact, extensive numerical experiments suggest that the performance guarantee of SPF/SCSPF continues to hold even in the absence of the technical assumptions (e.g., sufficiently high SNR and fast decaying magnitudes). Therefore, we suspect that these technical conditions are just artifacts in the proofs.

E. Organization

The sparse power factorization algorithms are described in detail in Section II, followed by their performance guarantees in Section III. The extension of both algorithms and performance guarantees to the general rank-$r$ case is presented in Section IV. An information-theoretic lower bound on the number of measurements for stable recovery of sparse rank-one matrices is given in Section V. After reporting on the empirical performance of sparse power factorization algorithms in Section VI, we conclude the paper in Section VII. Proofs of the main results are given in Section VIII, with proofs of several technical lemmas deferred to the appendix.

F. Notations

Let $\mathbb{N} = \{1, 2, \cdots \}$ denote the set of natural numbers and $[n] \triangleq \{1, \ldots, n\}$ for $n \in \mathbb{N}$. For a complex vector $x \in \mathbb{C}^n$, its $k$th element is denoted by $[x]_k$ and the element-wise complex conjugate of $x$ is denoted by $\overline{x}$. The identity operator on $\mathbb{C}^{n_1 \times n_2}$ is denoted as “id”. The Frobenius norm, the spectral norm, and the Hermitian transpose of $X \in \mathbb{C}^{n_1 \times n_2}$ are denoted by $\|X\|_F$, $\|X\|_*$, and $X^*$, respectively. The matrix inner product is defined by $\langle A, B \rangle = \text{trace}(A^* B)$. For a linear operator $A$ between two vector spaces, the range space is denoted by $\mathcal{R}(A)$ and the adjoint operator of $A$ is denoted by $A^*$ such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $x$ and $y$.

For a subspace $S$ of $\mathbb{C}^n$, let $P_S \in \mathbb{C}^{n \times n}$ denote the orthogonal projection onto $S$. The coordinate
projection $\Pi_J \in \mathbb{C}^{n \times n}$ is defined by

$$[\Pi_J x]_k = \begin{cases} [x]_k & \text{if } k \in J \\ 0 & \text{else} \end{cases}$$

for $J \subset [n]$. Then, $\Pi_J^\perp \in \mathbb{C}^{n \times n}$ is defined as $I_n - \Pi_J$ where $I_n$ is the $n \times n$ identity matrix.

II. Sparse Power Factorization Algorithms

In this section, we present alternating minimization algorithms for compressed sensing of sparse rank-one matrices. To describe these algorithms, we first introduce linear operators that describe the restrictions of the linear operator $A : \mathbb{C}^{n_1 \times n_2} \to \mathbb{C}^m$ acting on rank-one matrix $xy^* \in \mathbb{C}^{n_1 \times n_2}$ when either $x$ or $y$ are fixed.

In linear sensing schemes each measurement amounts to a matrix inner product. Indeed, there exist matrices $(M_\ell)_{\ell=1}^m \subset \mathbb{C}^{n_1 \times n_2}$ that describe the action of $A$ on $Z \in \mathbb{C}^{n_1 \times n_2}$ and that of its adjoint $A^*$ on $z = [z_1, \ldots, z_m]^\top \in \mathbb{C}^m$ by

$$A(Z) = [(M_1, Z), \ldots, (M_m, Z)]^\top$$

and

$$A^*(z) = \sum_{\ell=1}^m z_\ell M_\ell,$$

respectively. Using $(M_\ell)_{\ell=1}^m$, we define linear operators $F : \mathbb{C}^{n_2} \to \mathbb{C}^{m \times n_1}$ and $G : \mathbb{C}^{n_1} \to \mathbb{C}^{m \times n_2}$ by

$$F(y) \triangleq \begin{bmatrix} y^* M_1^* \\ y^* M_2^* \\ \vdots \\ y^* M_m^* \end{bmatrix} \quad \text{and} \quad G(x) \triangleq \begin{bmatrix} x^* M_1 \\ x^* M_2 \\ \vdots \\ x^* M_m \end{bmatrix},$$

respectively, for $y \in \mathbb{C}^{n_2}$ and $x \in \mathbb{C}^{n_1}$. Then, since $A(xy^*)$ is sesqui-linear in $(x, y)$, $F$ and $G$ satisfy

$$A(xy^*) = [F(y)]x = [G(x)]y.$$

A. Review of power factorization

Power factorization (PF) [17] is an alternating minimization algorithm that estimates a rank-$r$ matrix $X \in \mathbb{C}^{n_1 \times n_2}$ from its linear measurements $b = A(X) + z$. In this section, we specialize PF to the
rank-one case. Let \( t \geq 0 \) denote the iteration index. With a certain initialization \( v_0 \), PF iteratively updates estimates \( X_t = u_t v_t^* \) by alternating between the following procedures:

- For fixed \( v_{t-1} \), update \( u_t \) by
  \[
  u_t = \arg\min_{\tilde{u}} \| b - A(\tilde{u}v_{t-1}^*) \|_2^2. 
  \]  
  (7)

- For fixed \( u_t \), update \( v_t \) by
  \[
  v_t = \arg\min_{\tilde{v}} \| b - A(u_t \tilde{v}^*) \|_2^2. 
  \]  
  (8)

Using \( F \) and \( G \) defined in (6), the update rules in (7) and (8) can be rewritten respectively as

\[
 u_t = \arg\min_{\tilde{u}} \| b - [F(v_{t-1})] \tilde{u} \|_2^2 
\]  
(9)

and

\[
 v_t = \arg\min_{\tilde{v}} \| b - [G(u_t)] \tilde{v} \|_2^2. 
\]  
(10)

**B. Sparse power factorization (SPF)**

We propose an alternating minimization algorithm, called *sparse power factorization* (SPF), which recovers a row-sparse rank-one matrix \( X = \lambda uv^* \in \mathbb{C}^{n_1 \times n_2} \) with \( s_1 \)-sparse left singular vector \( u \) and \( s_2 \)-sparse right singular vector \( v \). SPF is obtained by modifying the updates of \( u_t \) and \( v_t \) in PF as follows.

Note that the measurement vector \( b \) of \( X \) can be expressed as

\[
 b = A(\lambda uv^*) + z = [F(v)](\lambda u) + z. 
\]  

For fixed \( v \), alternatively, \( b \) can be understood as the measurement vector of the \( s \)-sparse vector \( \lambda u \) using the sensing matrix \( F(v) \). When \( v_{t-1} \), normalized in the \( \ell_2 \) norm, corresponds to an estimate of the right singular vector \( v \), the matrix \( F(v_{t-1}) \) can be interpreted as an estimate of the unknown sensing matrix \( F(v) \). In the PF algorithm, the update of \( u_t \) in (9) corresponds to the least squares solution to the linear system consisting of the perturbed sensing matrix \( F(v_{t-1}) \) and the measurement vector \( b \). In contrast, SPF exploits the sparsity of \( u \) (when \( s_1 < n_1 \)) and updates the left factor \( u_t \) by an \( s_1 \)-sparse estimate of \( \lambda u \) from \( b \) using the perturbed sensing matrix \( F(v_{t-1}) \). Existing sparse recovery algorithms such as CoSaMP [36], subspace pursuit [38], and hard thresholding pursuit (HTP) [35] provide good estimates of \( \lambda u \) at low computational cost. Under certain conditions on the original and perturbed sensing matrices,
these algorithms are guaranteed to have small estimation error. In this paper, we focus on a particular instance of SPF that updates \( u_t \) using HTP, which is summarized in Alg. 1. (For completeness, the HTP algorithm is detailed in Alg. 2.) However, the results in this paper readily extend to instances of SPF employing other sparse recovery algorithms with a similar performance guarantee to that of HTP. The step size \( \gamma > 0 \) in HTP depends on the scaling of the sensing matrix \( F(v_t) \); hence, to fix the step size as \( \gamma = 1 \), we normalize \( v_{t-1} \) in the \( \ell_2 \) norm before the HTP step. Likewise, in the presence of column sparsity \( (s_2 < n_2) \), the update of \( v_t \) from \( u_t \) is modified to exploit the sparsity prior on \( v \) by using HTP.

**Algorithm 1:** \( \hat{X} = \text{SPF}_\text{HTP}(A, b, n_1, n_2, s_1, s_2, v_0) \)

1. while stop condition not satisfied do
2. \( t \leftarrow t + 1; \)
3. \( v_{t-1} \leftarrow \frac{v_{t-1}}{\|v_{t-1}\|_2}; \)
4. if \( s_1 < n_1 \) then
5. \( u_t \leftarrow \text{HTP}(F(v_{t-1}), b, s_1); \)
6. else
7. \( u_t \leftarrow \text{argmin}_x \|b - [F(v_{t-1})]x\|_2^2; \)
8. end
9. \( u_t \leftarrow \frac{u_t}{\|u_t\|_2}; \)
10. if \( s_2 < n_2 \) then
11. \( v_t \leftarrow \text{HTP}(G(u_t), b, s_2); \)
12. else
13. \( v_t \leftarrow \text{argmin}_y \|b - [G(u_t)]y\|_2^2; \)
14. end
15. end
16. return \( \hat{X} \leftarrow u_t v_t^*; \)

**Algorithm 2:** \( \hat{x} = \text{HTP}(\Phi, b, s) \)

1. while stop condition not satisfied do
2. \( t \leftarrow t + 1; \)
3. \( J \leftarrow \text{supp}(H_s[ x_{t-1} + \gamma \Phi^*(b - \Phi x_{t-1})]); \)
4. \( x_t \leftarrow \text{argmin}_x \{\|b - \Phi x\|_2 : \text{supp}(x) \subset J\}; \)
5. end
6. return \( \hat{x} \leftarrow x_t; \)

The performance of alternating minimization algorithms usually depends critically on the initialization. A typical heuristic (cf. [17]) is to select the best solution \( \hat{X} \) that minimizes \( \|b - A(\hat{X})\|_2^2 \) among solutions

\(^3\)This step size leads to a guarantee using the sparsity-restricted isometry property of \( F(v_{t-1}) \) [35].
obtained by multiple random initializations. However, no theoretical guarantee has been shown for this heuristic. Instead, Jain et al. [18] proposed to set \( v_0 \) to the leading right singular vector of the proxy matrix \( \mathcal{A}^*(b) \in \mathbb{C}^{n_1 \times n_2} \), and provided a performance guarantee of PF with this initialization under the rank-restricted isometry property of \( \mathcal{A} \). However, when applied to the sparse rank-one matrix recovery problem, this procedure does not exploit the sparsity of the eigenvectors, leading to highly suboptimal performance in the sparse regime.

Algorithm 3: \( v_0^{th} = \text{thres}_\text{init} (\mathcal{A}, b, n_1, n_2, s_1, s_2) \)

1. \( M \leftarrow \mathcal{A}^*(b) \)
2. for \( k = 1, \ldots, n_1 \) do
3. \( \zeta_k \leftarrow \ell_2 \) norm of the \( s_2 \)-sparse approx. of the \( k \)th row of \( M \)
4. end
5. \( \hat{J}_1 \leftarrow \text{indices of the } s_1 \text{ entries of } \zeta \text{ with the largest magnitude} \)
6. \( \hat{J}_2 \leftarrow \text{indices of the } s_2 \text{ columns of } \Pi_{\hat{J}_1} M \text{ with the largest } \ell_2 \text{ norm} \)
7. \( v_0^{th} \leftarrow \text{the first dominant right singular vector of } \Pi_{\hat{J}_1} M \Pi_{\hat{J}_2} \)
8. return \( v_0^{th} \)

To achieve near optimal recovery of sparse rank-one matrices, we propose a simple initialization method that exploits the sparsity structure, which is summarized in Algorithm 3. Although the initialization \( v_0^{Th} \) is practical thanks to its low computational cost, the success of Algorithm 3 requires an extra condition on the unknown singular vectors. It is of interest to design a more sophisticated initialization with a better performance at an increased computational cost. For example, similarly to the initialization for PF by Jain et al. [18], if the best sparse and rank-one approximation of the matrix \( \mathcal{A}^*(b) \) is available, then one can compute a good initialization as follows. Compute estimates \( \hat{J}_1 \) on the support \( J_1 \) of \( u \) and \( \hat{J}_2 \) on the support \( J_2 \) of \( v \) by solving

\[
(\hat{J}_1, \hat{J}_2) \triangleq \arg\max_{|\hat{J}_1|=s_1, |\hat{J}_2|=s_2} ||\Pi_{\hat{J}_1}[\mathcal{A}^*(b)]\Pi_{\hat{J}_2}||.
\]

The leading right singular of \( \Pi_{\hat{J}_1}[\mathcal{A}^*(b)]\Pi_{\hat{J}_2} \), denoted by \( v_0^{opt} \), is used as the initialization for SPF. We refer to this procedure as optimal initialization. Solving (11) involves searching over all possible support sets, which can be computationally demanding in high-dimensional settings. Iterative algorithms developed for sparse principal component analysis (e.g., [39]) might be employed to get a good approximate solution to (11). In this paper, we will focus only on the simple thresholding initialization \( v_0^{th} \) by Algorithm 3 and the optimal initialization \( v_0^{opt} \). Performance guarantees of the SPF algorithms equipped with these
initialization schemes are presented in the next section.

III. RANK-1 RECOVERY GUARANTEES

In this section we provide upper bounds on the number of linear measurements that guarantee the stable recovery of sparse and rank-one matrices by SPF with high probability. We consider Gaussian sensing schemes with the linear operator $A: \mathbb{C}^{n_1 \times n_2} \rightarrow \mathbb{C}^m$ given by

$$A(Z) = [(M_1, Z), \ldots, (M_m, Z)]^T,$$

where $M_\ell \in \mathbb{C}^{n_1 \times n_2}$ has i.i.d. $\mathcal{CN}(0, 1/m)$ entries. We call such an $A$ an i.i.d. Gaussian measurement operator.

Recall the two initialization schemes defined in Section II-B. Our main results are stated in the following theorem.

**Theorem III.1.** Let $A: \mathbb{C}^{n_1 \times n_2} \rightarrow \mathbb{C}^m$ be an i.i.d. Gaussian measurement operator. There exist absolute constants $c_1$, $c_2$, $c_3$, $c_4$, and $C$ such that for all $s_1 \in [n_1], s_2 \in [n_2]$, the following statement holds with probability at least $1 - \exp(-c_2 m)$. If $m \geq c_1 (s_1 + s_2) \log(\max\{en_1/s_1, en_2/s_2\})$, then when initialized by $v^\text{Th}_0$, SPF outputs $\hat{X}$ that satisfies

$$\frac{\|\hat{X} - X\|_F}{\|X\|_F} \leq C \frac{\|z\|_2}{\|A(X)\|_2} \quad (12)$$

for all $(s_1, s_2)$-sparse and rank-one $X = \lambda uv^*$ with $\|u\|_2 = \|v\|_2 = 1$ and $\min(\|u\|_\infty, \|v\|_\infty) \geq c_4$, and for all $z$ with $\|z\|_2 \leq c_3 \|A(X)\|_2$. In the special cases of $s_2 = n_2$ (row sparsity), the “peakiness” condition $\|u\|_\infty \|v\|_\infty \geq c_4$ is replaced by $\|u\|_\infty \geq c_4$.

The probability in Theorem III.1 is with respect to the selection of an i.i.d. Gaussian measurement operator, and the guarantee applies uniformly to the set of all matrices following the underlying model. In particular, this result achieves (to within a logarithmic factor) the fundamental limit on the number $m$ of measurements, in comparison to the corresponding necessary condition in Section V.

Theorem III.1 claims that SPF initialized by $v^\text{Th}_0$ provides stable reconstruction when the singular vectors $u$ and $v$ of the unknown matrix $X$ are heavily peaked in the sense that both $\|u\|_\infty$ and $\|v\|_\infty$ are larger than an absolute constant. Intuitively, in the presence of a few dominant components in $u$ and $v$, the simple thresholding heuristic in Algorithm 3 can capture the location of these peaks although it might not identify the entire support sets. This peakiness property is satisfied by certain classes of “fast-decaying”
signals. Let \( u^{(k)} \) denote the \( k \)th largest magnitudes of \( u \). For example, if \( u^{(k)} \leq ck^{-\alpha} \) for \( \alpha > \frac{1}{2} \) or \( u^{(k)} \leq c\beta^k \) for \( \beta \in (0, 1) \), then \( \|u\|_\infty \) is larger than an absolute constant. These fast-decaying-magnitudes models on the sparse vector \( u \) are often relevant to practical applications. For example, the magnitudes of the wavelet coefficients of piecewise smooth signals decay geometrically across the scales of the wavelet tree [4].

**Proposition III.2.** In the setup of Theorem III.1, SPF initialized by \( v_0^{\text{th}} \) provides the same recovery guarantee from \( m = O(s_1 s_2 \log(\max\{en_1/s_1, en_2/s_2\})) \) measurements without requiring the peakiness condition.

**Remark III.3.** The performance guarantee in Proposition III.2 is only as good as those for other recovery algorithms with provable guarantees, which ignore the rank-one prior in the matrix structure and only exploit the sparsity prior (e.g., basis pursuit). We included Proposition III.2 to demonstrate that SPF initialized by \( v_0^{\text{th}} \) is as good as existing guaranteed algorithms even when the peakiness condition is not satisfied. Furthermore, SPF is still preferable to other methods that do not exploit the rank-one prior because it solves the un-lifted formulation and has much lower computational cost.

As we show in Section III-B, given a good initialization, the convergence of the subsequent iterations is shown without the heavily-peakedness condition. The following proposition demonstrates that the initialization \( v_0^{\text{opt}} \) from the exact solution to (11) enables the performance guarantee for SPF without the heavily-peakedness condition. Recall that the computation of \( v_0^{\text{opt}} \) involves exhaustive search over all support sets of cardinality \( s_1 \) and \( s_2 \). In fact, with this enumeration, by applying guaranteed algorithms for low-rank matrix recovery for each choice of the support, one can get the same sample complexity result as in Proposition III.4 easily. Nonetheless, the success of SPF initialized by \( v_0^{\text{opt}} \) opens up the possibility of finding better initialization schemes using a practical approximate algorithm to solve (11).

**Proposition III.4.** In the setup of Theorem III.1, SPF initialized by \( v_0^{\text{opt}} \) provides the same recovery guarantee from \( m = O((s_1 + s_2) \log(\max\{en_1/s_1, en_2/s_2\})) \) measurements without requiring the peakiness condition.

The rest of this section is devoted to proving Theorem III.1, Proposition III.2, and Proposition III.4. The outline of the proof is the following:

1) Theorem III.9 gives a deterministic guarantee for SPF under the condition that the linear operator
satisfies certain RIP conditions and the initial value is reasonably close to the true singular vector.

2) Theorem III.7 shows that the Gaussian measurement operator satisfies the desired RIP if the number of measurements is lower bounded accordingly.

3) The sufficiency of the initialization methods $v_{0}^{\text{opt}}$ and $v_{0}^{\text{th}}$, defined in Section II-B, to satisfy the conditions in Theorem III.9, is established under respective conditions.

A. Restricted isometry properties

A sufficient condition for stable recovery of SPF is that the linear operator $A$ satisfies certain restricted isometry property (RIP) conditions. The original version of RIP [40], denoted by $s$-sparse RIP in this paper, refers to a linear operator being a near isometry when restricted to the set of $s$-sparse vectors. This notion has been extended to a similar near isometry property restricted to the set of rank-$r$ matrices [14]. Here, the relevant RIP condition to the analysis of SPF is the near isometry on the set of rank-$r$ matrices with at most $s_{1}$ nonzero rows and at most $s_{2}$ nonzero columns.

**Definition III.5** (Rank-$r$ and $(s_{1}, s_{2})$-sparse RIP). A linear operator $A : \mathbb{C}^{n_{1} \times n_{2}} \to \mathbb{C}^{m}$ satisfies the rank-$r$ and doubly $(s_{1}, s_{2})$-sparse RIP with isometry constant $\delta$ if

$$(1 - \delta)\|Z\|_{F}^{2} \leq \|A(Z)\|_{2}^{2} \leq (1 + \delta)\|Z\|_{F}^{2}$$

for all $Z \in \mathbb{C}^{n_{1} \times n_{2}}$ such that $\text{rank}(Z) \leq r$, $\|Z\|_{0,2} \leq s_{1}$, and $\|Z^{*}\|_{0,2} \leq s_{2}$.

**Remark III.6.** The special case of the rank-$r$ and $(s_{1}, s_{2})$-sparse RIP with $s_{2} = n_{2}$ (resp. $s_{1} = n_{1}$) is called the rank-$r$ and row-$s_{1}$-sparse RIP (resp. the rank-$r$ and column-$s_{2}$-sparse RIP).

The following result gives a sufficient condition for the Gaussian measurement operator to satisfy the RIP condition defined in Definition III.5.

**Theorem III.7.** Let $A : \mathbb{C}^{n_{1} \times n_{2}} \to \mathbb{C}^{m}$ be an i.i.d. Gaussian measurement operator. If

$$m \geq c_{1}r(s_{1} + s_{2}) \log \left( \max \left\{ \frac{en_{1}}{s_{1}}, \frac{en_{2}}{s_{2}} \right\} \right),$$

then $A$ satisfies the rank-$r$ and $(s_{1}, s_{2})$-sparse RIP with isometry constant $\delta$ with probability at least $1 - \exp(-c_{2}\delta^{2}m)$, where $c_{1}, c_{2}$ are absolute constants.

The proof of Theorem III.7 is rather straightforward using standard mathematical tools in the literature [14], [19], [41]; hence, we only provide a sketch. It follows from the standard volume argument and the
exponential concentration of the i.i.d. Gaussian measurement operator [41]. The only difference from the derivation of the standard $s$-sparse RIP [41] is to use the $\epsilon$-net for all unit-norm rank-$r$ matrices, the cardinality of which is bounded according to the following lemma.

**Lemma III.8** (Size of $\epsilon$-net of rank-$r$ matrices [19]). Let $\mathcal{S} = \{X \in \mathbb{C}^{n_1 \times n_2} : \text{rank}(X) \leq r, \|X\|_F = 1\}$. There exists a subset $\mathcal{S}_\epsilon$ of $\mathcal{S}$ such that

$$\sup_{X \in \mathcal{S}} \inf_{\hat{X} \in \mathcal{S}_\epsilon} \|X - \hat{X}\|_F \leq \epsilon$$

and

$$|\mathcal{S}_\epsilon| \leq \left(\frac{9}{\epsilon}\right)^{r(n_1 + n_2 + 1)}.$$

**B. RIP-based recovery guarantees for SPF**

Performance guarantees for recovery by SPF are derived using the rank-2 and $(3s_1, 3s_2)$-sparse RIP of $\mathcal{A}$. The next theorem shows that given a good initialization, SPF provides stable recovery.

**Theorem III.9** (RIP-guarantee for SPF with good initialization). Suppose the followings:

1) $X = \lambda uv^*$ satisfies $\|u\|_0 \leq s_1$ and $\|v\|_0 \leq s_2$.

2) $\mathcal{A}$ satisfies the rank-2 and $(3s_1, 3s_2)$-sparse RIP with isometry constant $\delta = 0.08$.

3) $b = \mathcal{A}(X) + z$ where $z$ and $\mathcal{A}(X)$ satisfy

$$\frac{\|z\|_2}{\|\mathcal{A}(X)\|_2} \leq \nu$$

with $\nu = 0.08$.

4) The initialization $v_0$ of SPF satisfies

$$\|P_{\mathcal{R}(v)} - P_{\mathcal{R}(v_0)}\| < 0.85.$$  \hspace{1cm} (14)

Then, the output $(X_t)_{t \in \mathbb{N}}$ of SPF satisfies

$$\limsup_{t \to \infty} \frac{\|X_t - X\|_F}{\|X\|_F} \leq 8.3 \frac{\|z\|_2}{\|\mathcal{A}(X)\|_2}.$$  \hspace{1cm} (15)

Moreover, the convergence in (15) is superlinear, i.e., for any $\epsilon > 0$, there exists $t_0 = O(\log(1/\epsilon))$ that satisfies

$$\frac{\|X_{t_0} - X\|_F}{\|X\|_F} \leq 8.3 \frac{\|z\|_2}{\|\mathcal{A}(X)\|_2} + \epsilon.$$  \hspace{1cm} (16)
Proof: See Section VIII-A.

Theorem III.9 implies that under the rank-2 and \((3s_1, 3s_2)\)-sparse RIP assumption on \(A\), With a good initialization \(v_0\), which is close to the unknown \(v\) in the principal angle, SPF converges superlinearly to a robust reconstruction of \(X\) in the sense of (2). In particular, in the noiseless case \((z = 0)\), SPF recovers \(X\) perfectly. Note that the performance guarantee in Theorem III.9 is obtained under the conservative assumption that the noise variance is below a certain constant threshold. However, empirically, SPF still provides stable recovery of \(X\) even when the additive noise is stronger than the threshold in Theorem III.9 (See Section VI).

Next, we address the question of finding a good initialization. The performance guarantee for SPF in Theorem III.9 holds subject to the condition that the initialization satisfies (14). We study the performance of the two initialization methods proposed in Section II and present the corresponding performance guarantees below.

**Theorem III.10 (RIP-guarantees: doubly sparse case).** Suppose that \(X = \lambda uv^* \) satisfies \(\|u\|_0 \leq s_1\) and \(\|v\|_0 \leq s_2\).

1) Suppose that \(A\) satisfies the rank-2 and \((3s_1, 3s_2)\)-sparse RIP with isometry constant \(\delta = 0.04\), and that the SNR condition in (13) holds with \(\nu = 0.04\). Then, SPF initialized by \(v_0^{opt}\) provides a performance guarantee as in Theorem III.9.

2) Suppose that either one of the following conditions is satisfied:

   a) \(A\) satisfies the rank-2 and \((3s_1, 3s_2)\)-sparse RIP with isometry constant \(\delta = 0.04\), the SNR condition in (13) holds with \(\nu = 0.04\), \(\|u\|_\infty \geq 0.78\|u\|_2\), and \(\|v\|_\infty \geq 0.78\|v\|_2\).

   b) \(A\) satisfies the RIP with isometry constant \(\delta = 0.02\), when restricted to the set of matrices with up to \(9s_1s_2\) nonzero entries, and the SNR condition in (13) holds with \(\nu = 0.02\).

Then, SPF initialized by \(v_0^{Th}\) provides a performance guarantee as in Theorem III.9.

Proof: See Section VIII-B.

**Remark III.11.** It is noteworthy that the two different performance guarantees in Part 2 of Theorem III.10 are achieved by a single algorithm. In fact, when \(A\) satisfies the \((3s_1, 3s_2)\)-sparse RIP, nuclear norm minimization achieves a performance guarantee, which applies to all \((s_1, s_2)\)-sparse matrices (not necessarily of rank-1). Part 2-(b) of Theorem III.10 asserts that SPF with the thresholding initialization provides a comparable performance guarantee in this scenario (only by the \((3s_1, 3s_2)\)-sparse RIP without any further
condition on $u$ and $v$). The performance guarantee in Part 2-(b) of Theorem III.10 only recovers existing results. However, by Part 2-(a) of Theorem III.10, when $u$ and $v$ have large entries, unlike the nuclear norm minimization that discards the rank-1 constraint, the same algorithm (SPF with the thresholding initialization) achieves a better performance guarantee by a weaker RIP.

It is straightforward to check that the performance guarantees in Theorem III.10 apply to the row-sparse (resp. the column sparse case) by letting $s_2 = n_2$ (resp. $s_1 = n_1$). However, in the row-sparse case (resp. the column-sparse case), the near optimal performance guarantee for SPF initialized by $v_0^{th}$ requires only the additional condition on $\|u\|_\infty$ (resp. $\|v\|_\infty$), as stated in the following result.

**Theorem III.12** (RIP-guarantee: row-sparse case). Suppose the followings:

1) $X = \lambda uv^*$ satisfies $\|u\|_0 \leq s_1$ and $\|u\|_\infty \geq 0.4 \|u\|_2$.

2) $A$ satisfies the rank-2 and row-$3s_1$-sparse RIP with isometry constant $\delta = 0.08$.

3) The SNR condition in (13) holds with $\nu = 0.04$.

Then, SPF initialized by $v_0^{th}$ provides a performance guarantee as in Theorem III.9.

**Proof:** See Section VIII-C.

**Remark III.13.** The constants in Theorems III.10 and III.12 are not optimized, but rather were chosen conservatively to simplify the proofs and the statement of the results.

We conclude this section with the proofs of Theorem III.1, Proposition III.2, and Proposition III.4.

**Proof:** In view of Theorem III.7, the RIP conditions in Theorem III.10 are satisfied by corresponding conditions on the number of i.i.d. Gaussian measurements as follows. First, the performance guarantee for SPF initialized by $v_0^{opt}$ is given by the rank-2 and $(3s_1, 3s_2)$-sparse RIP; hence, it holds with high probability for $m = O((s_1 + s_2) \log(\max\{en_1/s_1, en_2/s_2\})$ i.i.d. Gaussian measurements. Thus, Proposition III.4 follows from Part 1 of Theorem III.10. Next, the RIP conditions in Part 2-(a) of Theorem III.10 and in Theorem III.12 are similarly expressed as conditions on $m$. This proves Theorem III.1. Finally, noting that the RIP condition in Part 2-(b) of Theorem III.10 holds for an i.i.d. Gaussian measurement operator with $m = O(s_1s_2 \log(\max\{en_1/s_1, en_2/s_2\})$ proves Proposition III.2.

**IV. Extension to the Rank-$r$ Case**

In this section, we extend the results in Section III from the rank-1 case to a more general rank-$r$ case.
A. Algorithms for the rank-$r$ case

First, we generalize the definition of $F(\cdot)$ and $G(\cdot)$ in (6) to the rank-$r$ case. Recall that there exist matrices $M_1, M_2, \ldots, M_m \in \mathbb{C}^{n_1 \times n_2}$ such that

$$A(X) = [(M_1, X), \ldots, (M_m, X)]^\top.$$  

For $V \in \mathbb{C}^{n_2 \times r}$, a linear operator $F(V) : \mathbb{C}^{n_1 \times r} \to \mathbb{C}^m$ parameterized by $V$ is defined by

$$[F(V)](U) := [(M_1 V, U), \ldots, (M_m V, U)]^\top, \quad \forall U \in \mathbb{C}^{n_1 \times r}. \quad (17)$$

For $U \in \mathbb{C}^{n_1 \times r}$, a linear operator $G(U) : \mathbb{C}^{n_2 \times r} \to \mathbb{C}^m$ parameterized by $U$ is defined by

$$[G(U)](V) := [(M_1^* U, V), \ldots, (M_m^* U, V)]^\top, \quad \forall V \in \mathbb{C}^{n_2 \times r}. \quad (18)$$

Then,

$$A(UV^*) = [F(V)](U) = [G(U)](V).$$

When $r = 1$, the above definitions reduce to the corresponding part in Section II.

With $F$ and $G$ defined respectively in (17) and (18), similarly to the PF algorithm [17], SPF in Algorithm 1 extends naturally to the rank-$r$ case, which is summarized in Algorithm 4. We also extend the thresholding initialization in Section II to the rank-$r$ case for both the initial estimates $U_0$ and $V_0$. Theses algorithms are summarized in Algorithms 6 and 7, respectively.

As demonstrated in Section VI-C, empirically, the natural extension of SPF (Algorithm 4) outperformed the convex method. However, in our attempt to extend Theorem III.1 to the rank-$r$ case, instead of the linear dependence, the sample complexity for performance guarantees had higher order dependence on the rank $r$. This is suboptimal in order compared to the matching lower bound. To overcome this limitation, we modify the natural rank-$r$ sparse power factorization (Algorithm 4) into the subspace-concatenated sparse power factorization (SC-SPF), summarized in Algorithm 8. The most important difference between SC-SPF and SPF is that in every iteration of SC-SPF, the initial estimate is used in concatenation with the estimate from the previous iteration. As we show in the next section, with this subspace concatenation, the sample complexity for recovering a low-rank and sparse matrix scales linearly in the rank, which is optimal.
Algorithm 4: $\hat{X} = \text{rSPF}_\text{HTP}(A, b, n_1, n_2, r, s_1, s_2, U_0, V_0)$

1. while stop condition not satisfied do
2.   $t \leftarrow t + 1$
3.   $V_{t-1} \leftarrow \text{orth}(V_{t-1})$ \quad // ortho-basis for $\mathcal{R}(V_{t-1})$
4.   if $s_1 < n_1$ then
5.     $U_t \leftarrow \text{B-HTP}(F(V_{t-1}), b, s_1)$
6.   else
7.     $U_t \leftarrow \text{argmin}_{U'} \|b - [F(V_{t-1})](U')\|_2^2$
8.   end
9.   $U_t \leftarrow \text{orth}(U_t)$
10. if $s_2 < n_2$ then
11.     $V_t \leftarrow \text{B-HTP}(G(U_t), b, s_2)$
12.   else
13.     $V_t \leftarrow \text{argmin}_{V'} \|b - [G(U_t)](V')\|_2^2$
14.   end
15. end
16. return $\hat{X} \leftarrow U_t V_t^*$

Algorithm 5: $\hat{x} = \text{B-HTP}(\Phi, b, s)$

1. while stop condition not satisfied do
2.   $t \leftarrow t + 1$;
3.   $\hat{X} \leftarrow X_{t-1} + \gamma \Phi^*(b - \Phi(X_{t-1}))$;
4.   $J \leftarrow \text{indices of the } s \text{ rows of } \hat{X} \text{ with the largest } \ell_2 \text{ norm}$;
5.   $X_t \leftarrow \text{argmin}_{X'} \{\|b - \Phi(X')\|_2 : \Pi_J X' = X'\}$;
6. end
7. return $\hat{X} \leftarrow X_t$;

Algorithm 6: $V_{0}^\text{Th} = \text{INIT_SC_SPF_V}(A, b, n_1, n_2, r, s_1, s_2)$

1. $M \leftarrow A^*(b)$;
2. for $k = 1, \ldots, n_1$ do
3.   $\zeta_k \leftarrow \ell_2 \text{ norm of the } s_2\text{-sparse approx. of the } k\text{th row of } M$;
4. end
5. $\hat{J}_1 \leftarrow \text{indices of the } s_1 \text{ entries of } \zeta \text{ with the largest magnitude}$;
6. $\hat{J}_2 \leftarrow \text{indices of the } s_2 \text{ columns of } \Pi_{\hat{J}_1} M \text{ with the largest } \ell_2 \text{ norm}$;
7. $V_{0}^\text{Th} \leftarrow r \text{ leading right singular vectors of } \Pi_{\hat{J}_1} M \Pi_{\hat{J}_2}$;
Algorithm 7: $U_{0}^{Th} = \text{INIT\_SC\_SPF\_U}(A, b, n_{1}, n_{2}, r, s_{1}, s_{2})$

1 $M \leftarrow A^{*}(b)$;
2 for $k = 1, \ldots, n_{2}$ do
3     $\zeta_{k} \leftarrow \ell_{2}$ norm of the $s_{1}$-sparse approx. of the $k$th column of $M$;
4 end
5 $\hat{J}_{2} \leftarrow$ indices of the $s_{2}$ entries of $\zeta$ with the largest magnitude;
6 $\hat{J}_{1} \leftarrow$ indices of the $s_{1}$ rows of $M\Pi_{\hat{J}_{2}}$ with the largest $\ell_{2}$ norm;
7 $V_{0}^{Th} \leftarrow r$ leading left singular vectors of $\Pi_{\hat{J}_{1}}M\Pi_{\hat{J}_{2}}$;

Algorithm 8: $\hat{X} = \text{SC\_SPF\_HTP}(A, b, n_{1}, n_{2}, r, s_{1}, s_{2}, U_{0}, V_{0})$

1 while stop condition not satisfied do
2     $t \leftarrow t + 1$;
3     $\tilde{V} \leftarrow \text{orth}([V_{t-1}, V_{0}])$;
4     if $s_{1} < n_{1}$ then
5         $\tilde{U} \leftarrow \text{B-HTP}(F(\tilde{V}), b, s_{1})$;
6     else
7         $\tilde{U} \leftarrow \arg\min_{U'} \|b - [F(\tilde{V})](U')\|_{2}^{2}$;
8     end
9     $U_{t} \leftarrow$ (the best rank-$r$ approximation of $\tilde{U}$);
10    $\tilde{U} \leftarrow \text{orth}([U_{t}, U_{0}])$;
11    if $s_{2} < n_{2}$ then
12        $\tilde{V} \leftarrow \text{B-HTP}(G(\tilde{U}), \bar{b}, s_{2})$;
13    else
14        $\tilde{V} \leftarrow \arg\min_{V'} \|\bar{b} - [G(\tilde{U})](V')\|_{2}^{2}$;
15    end
16    $V_{t} \leftarrow$ (the best rank-$r$ approximation of $\tilde{V}$);
17 end
18 return $\hat{X} \leftarrow U_{t}V_{t}^{*}$;

B. Performance guarantees

Similar to the performance guarantee in Section III, we derive a sufficient condition for sparse recovery by SC-SPF from i.i.d. Gaussian measurements.

Theorem IV.1. Let $A : \mathbb{C}^{n_{1} \times n_{2}} \rightarrow \mathbb{C}^{m}$ be an i.i.d. Gaussian measurement operator. There exist absolute constants $c_{1}, \ldots, c_{5}$, and $C$ such that the following statement holds. Let $X \in \mathbb{C}^{n_{1} \times n_{2}}$ be a fixed matrix of rank-$r$, where $X = U\Lambda V^{*}$ denotes the singular value decomposition of $X$. Suppose that the following conditions hold:

1) The condition number of $X$ is no greater than $\kappa \leq c_{1}$. 


2) \( U \in \mathbb{C}^{n_1 \times r} \) and \( V \in \mathbb{C}^{n_2 \times r} \) are row-s\(_1\)-sparse and row-s\(_2\)-sparse, respectively.

3) \( \sigma_r(\Pi_{J_1} U) \geq c_2 \) and \( \sigma_r(\Pi_{J_2} V) \geq c_2 \), where \( J_1 \) and \( J_2 \) are respectively defined by

\[
\tilde{J}_1 \triangleq \arg\max_{\gamma \subset [n_1]: |\gamma| = r} \Pi_\gamma U \quad \text{and} \quad \tilde{J}_2 \triangleq \arg\max_{\gamma \subset [n_2]: |\gamma| = r} \Pi_\gamma V.
\]

4) \( b = \mathcal{A}(X) + z \) where \( \|z\|_2 \leq c_3(\|X\|/\|X\|_F)\|\mathcal{A}(X)\|_2 \).

5) \( m \geq c_4 \kappa^2 r (s_1 + s_2) \log(\max\{en_1/s_1, en_2/s_2\}) \).

Then initialized by \((U_0^{\text{Th}}, V_0^{\text{Th}})\), SC-SPF outputs \( \hat{X} \) that satisfies

\[
\frac{\|\hat{X} - X\|_F}{\|X\|_F} \leq C\kappa^2 \left( \frac{\|z\|_2}{\|\mathcal{A}(X)\|_2} \right).
\]

with probability at least \( 1 - \exp(-c_5 m) \).

**Remark IV.2.** Assumption 3 in Theorem IV.1 generalizes the peakiness assumption, \( \|u\|_\infty \geq c_2 \) and \( \|v\|_\infty \geq c_2 \), of Theorem III.1. In the rank-1 case, \( \tilde{J}_1 \) (resp. \( \tilde{J}_2 \)) in (19) reduces to the index of the largest entry of \( u \) (resp. \( v \)) in magnitude. Therefore, Assumption 3 in Theorem IV.1 reduces to the corresponding peakiness assumption in Theorem III.1.

**Remark IV.3.** The dependence of the sample complexity on the condition number \( \kappa \) is due to the estimation of subspaces. We can always apply the algorithm with parameter \( r' < r \) to decrease \( \kappa \) and the reconstruction error will depends on the best rank-\( r' \) approximation of \( X \) (which is now absorbed into the measurement error \( z \)).

When the unknown rank-\( r \) matrix \( X \) is not sparse (\( s_1 = n_1 \) and \( s_2 = n_2 \)), there is no need to estimate the support sets and the initialization \((U_0^{\text{Th}}, V_0^{\text{Th}})\) by Algorithms 6 and 7 is trivially obtained as the singular vectors of the rank-\( r \) approximation of \( \mathcal{A}^*(b) \). Furthermore, the iterative updates in Algorithm 8 are done by solving least squares problems. In this scenario, the guarantee in Theorem IV.1 holds only with Assumption 4 (sufficiently high SNR) as shown in the next corollary.

**Corollary IV.4** (Non-sparse case). Let \( \mathcal{A} : \mathbb{C}^{n_1 \times n_2} \to \mathbb{C}^m \) be an i.i.d. Gaussian measurement operator. There exist absolute constants \( c_3, c_4, c_5, \) and \( C \) such that the following statement holds. Let \( X \in \mathbb{C}^{n_1 \times n_2} \) be a fixed matrix of rank-\( r \), where \( X = U \Lambda V^* \) denotes the singular value decomposition of \( X \). Suppose that Assumption 4 in Theorem IV.1 holds. If \( m \geq c_2 \kappa^2 r (n_1 + n_2) \), then initialized by \((U_0^{\text{Th}}, V_0^{\text{Th}})\), SC-SPF outputs \( \hat{X} \) that satisfies (20) with probability at least \( 1 - \exp(-c_5 m) \).
Corollary IV.4 implies that the recovery of an $n$-by-$n$ matrix of rank $r$ is guaranteed from $m = O(\kappa^2rn)$ measurements\(^4\), which significantly improves on the previous result $m = O(\kappa^4r^3n)$ [18] also using alternating minimization and i.i.d. Gaussian measurements.

In the remainder of this section we provide a proof of Theorem IV.1. To this end, we first derive a deterministic RIP-based guarantee for SC-SPF assuming good initialization, then a RIP-based guarantee using the initialization in Algorithms 6 and 7, and finally the sample complexity for the relevant RIP using a Gaussian measurement operator.

**Theorem IV.5 (RIP-guarantee for SC-SPF with good initialization).** Suppose the followings:

1) $X = U\Lambda V^*$ denotes the singular value decomposition of a rank-$r$ matrix $X \in \mathbb{C}^{n_1 \times n_2}$, where $U \in \mathbb{C}^{n_1 \times r}$ and $V \in \mathbb{C}^{n_2 \times r}$ are row-$s_1$-sparse and row-$s_2$-sparse, respectively.

2) The condition number of $X$ is no greater than $\kappa$.

3) $A$ satisfies the rank-$2r$ and $(3s_1, 3s_2)$-sparse RIP with isometry constant $\delta = \frac{0.04}{\kappa}$.

4) $b = A(X) + z$ where $z$ and $A(X)$ satisfy

$$\frac{\|X\|_F}{\|X\|} \cdot \frac{\|z\|_2}{\|A(X)\|_2} \leq \nu$$

with $\nu = \frac{0.04}{\kappa}$.

5) The initialization $(U_0, V_0)$ of SC-SPF satisfies

$$\max(\|P_{\mathcal{R}(U)} P_{\mathcal{R}(U_0)}\|, \|P_{\mathcal{R}(V)} P_{\mathcal{R}(V_0)}\|) < 0.95.$$  \hspace{1cm} (22)

Then the output $(X_t)_{t \in \mathbb{N}}$ of SC-SPF satisfies

$$\limsup_{t \to \infty} \frac{\|X_t - X\|_F}{\|X\|_F} \leq (55\kappa^2 + 3\kappa + 3) \frac{\|z\|_2}{\|A(X)\|_2}.$$  \hspace{1cm} (23)

Moreover, the convergence in (23) is linear, i.e., for any $\epsilon > 0$, there exists $t_0 = O(\log \frac{1}{\epsilon})$ that satisfies

$$\frac{\|X_t - X\|_F}{\|X\|_F} \leq (55\kappa^2 + 3\kappa + 3) \frac{\|z\|_2}{\|A(X)\|_2} + \epsilon.$$  \hspace{1cm} (24)

Theorem IV.5 implies that starting from good initial estimates $U_0$ and $V_0$, SC-SPF provide stable recovery of the unknown matrix $X$ under the rank-$2r$ and doubly $(3s_1, 3s_2)$-sparse RIP of $A$. In particular, when the unknown matrix is only of rank $r$ without sparsity ($s_1 = n_1$ and $s_2 = n_2$), one can obtain

\[^4\]The logarithmic term comes from the unknown support and disappears in the non-sparse case.
good initial estimates satisfying (22) by a single step of the singular value projection [18]. In this case, Theorem IV.5, combined with Theorem III.7 and Lemma IV.8, shows that stable recovery of an unknown \( n \times n \) matrix of rank-\( r \) from \( O(\kappa^2rn) \) i.i.d. Gaussian measurements is guaranteed. With the sparsity model, we provide a performance guarantee of SC-SPF with initial estimates using Algorithms 6 and 7 in the following theorem.

**Theorem IV.6 (RIP-guarantee).** Suppose the followings:

1) \( X = UAV^* \) is the singular value decomposition of a rank-\( r \) matrix \( X \in \mathbb{C}^{n_1 \times n_2} \), where \( U \in \mathbb{C}^{n_1 \times r} \) and \( V \in \mathbb{C}^{n_2 \times r} \) are row-\( s_1 \)-sparse and row-\( s_2 \)-sparse, respectively.

2) \( \sigma_r(\Pi_{\tilde{J}_1}U) \geq 0.9 \) and \( \sigma_r(\Pi_{\tilde{J}_2}V) \geq 0.9 \), where \( \tilde{J}_1 \) and \( \tilde{J}_2 \) are defined in (19).

3) The condition number of \( X \) is no greater than \( \kappa \leq 4 \).

4) \( A \) satisfies the rank-\( 2r \) and \( (3s_1,3s_2) \)-sparse RIP with isometry constant \( \delta = \frac{0.04}{\kappa} \).

5) \( A \) and \( X \) satisfy

\[
\sup_{|J_1| \leq s_1} \sup_{|J_2| \leq s_2} \| \Pi_{J_1}[(A^*A - \text{id})(X)]\Pi_{J_2} \| \leq \delta \|X\|
\]

for \( \delta = \frac{0.04}{\kappa} \).

6) \( b = A(X) + z \) where the SNR condition in (21) holds with \( \nu = \frac{0.04}{\kappa} \).

Then Algorithm 8 initialized by \( (U_{0}^T,V_{0}^T) \) provides a performance guarantee as in Theorem IV.5.

In particular, when the unknown rank-\( r \) matrix \( X \) is not sparse (\( s_1 = n_1 \) and \( s_2 = n_2 \)), the same guarantee holds without Assumptions 2 and 3.

**Remark IV.7.** In the rank-1 case, \( \sigma_r(\Pi_{\tilde{J}_1}U) \) and \( \sigma_r(\Pi_{\tilde{J}_2}V) \) reduce to \( \|u\|_\infty \) and \( \|v\|_\infty \), respectively.

The next lemma provides an RIP-like property of an i.i.d. Gaussian operator. Theorem IV.1 is then obtained by combining Theorems IV.6, III.7, and Lemma IV.8.

**Lemma IV.8.** Let \( X \in \mathbb{C}^{n_1 \times n_2} \) be an arbitrarily fixed rank-\( r \) matrix. Let \( A \) be defined in (4) where \( M_1, \ldots, M_m \) are mutually independent random matrices whose entries are i.i.d. following \( \mathcal{CN}(0,1/m) \).

Then, with probability \( 1 - \epsilon \),

\[
\sup_{|J_1| \leq s_1} \sup_{|J_2| \leq s_2} \| \Pi_{J_1}[(A^*A - \text{id})(X)]\Pi_{J_2} \| \leq \delta r^{-1/2} \|X\|_F \leq \delta \|X\|
\]

provided

\[
m \geq Cr\delta^{-2} \max\left( (s_1 + s_2) \log(\max\{en_1/s_1, en_2/s_2\}), \log(\epsilon^{-1}) \right)
\]
for an absolute constant $C > 0$.

**Proof of Lemma IV.8:** Let

$$V_Z \triangleq \frac{1}{\sqrt{m}} (I_m \otimes \text{vec}(Z)^\top),$$

where $\otimes$ denotes the Kronecker product and

$$\xi \triangleq [\text{vec}(M_1)^\top, \ldots, \text{vec}(M_m)^\top]^\top.$$

Note that $\xi \in \mathbb{C}^{mn_1 n_2}$ is a Gaussian vector with $E \xi \xi^* = I_{mn_1 n_2}$. Then, the action of $\mathcal{A}$ on $Z \in \mathbb{C}^{n_1 \times n_2}$ is expressed as

$$\mathcal{A}(Z) = V_Z \xi.$$

Therefore, it follows that

$$\sup_{|J_1| \leq s_1} \sup_{|J_2| \leq s_2} \| \Pi_{J_1} [\mathcal{A}^* \mathcal{A} - \text{id}](X) \Pi_{J_2} \| = \sup_{V_Z \in \Xi} \| \langle V_X \xi, V_Z \xi \rangle - E\langle V_X \xi, V_Z \xi \rangle \|,$$

where

$$\Xi \triangleq \{ V_Z : Z = uv^*, \|u\|_2 = 1, \|u\|_0 \leq s_1, \|v\|_2 = 1, \|v\|_0 \leq s_2 \}.$$

Then, the radius of $\Xi$ in the spectral norm is

$$d_{2\rightarrow2}(\Xi) = \frac{1}{\sqrt{m}}$$

and the radius of $\Xi$ in the Frobenius norm is

$$d_F(\Xi) = 1.$$

Furthermore, Talagrand’s $\gamma_2$ functional [42] of $\Xi$ with respect to the spectral norm is upper bounded by

$$\gamma_2(\Xi, \| \cdot \|) \lesssim \sqrt{\frac{s_1 \log(en_1/s_1)}{m}} + \sqrt{\frac{s_2 \log(en_2/s_2)}{m}}.$$

Then, the conclusion follows from [43, Theorem 2.3].

**V. Fundamental Limits: Necessary Conditions for Robust Reconstruction**

In this section we give necessary conditions for robust reconstruction of sparse rank-one matrices by considering a Bayesian version of (1). Denote the $r$-dimensional complex Stiefel manifold $V(\mathbb{C}^n, r) \triangleq$
\{V \in \mathbb{C}^{n \times r} : V^*V = I_r\}$. Following the information-theoretic approach in [44], we prove a non-asymptotic lower bound that holds for any non-linear measurement mechanisms (encoders) and reconstruction algorithms (decoders). The setup is illustrated in Fig. 1, where

- $X = UV^* \in \mathbb{C}^{n_1 \times n_2}$ is a rank-$r$ random matrix, where $U \in \mathbb{C}^{n_1 \times r}$ and $V \in \mathbb{C}^{n_2 \times r}$ are independent random matrices. Furthermore, $U$ is row-sparse with row support $S = \text{supp}(U)$ chosen uniformly at random from all subsets of $[n_1]$ of cardinality $s_1$, and the non-zero rows $U_S$ are uniformly distributed on $\mathcal{V}(\mathbb{C}^{s_1}, r)$. Similarly, $V$ is $s_2$-sparse with uniformly chosen support and the non-zero component is uniform on $\mathcal{V}(\mathbb{C}^{s_2}, r)$.
- $Z \sim \mathcal{CN}(0, \sigma^2 I_m)$ denotes additive complex Gaussian noise, whose real and imaginary parts are independently distributed according to $\mathcal{N}(0, \frac{\sigma^2}{2} I_m)$.
- The encoder $f : \mathbb{C}^{n_1 \times n_2} \rightarrow \mathbb{C}^m$ satisfies the average power constraint
  \[ \mathbb{E}\|f(X)\|^2 \leq m \] (25)
- The decoder $g : \mathbb{C}^m \times \mathbb{C}^{n_1 \times n_2} \rightarrow \mathbb{C}^m$ outputs a rank-$r$ matrix, namely, $g(f(X) + Z) = \hat{U}\hat{V}^*$ with $\hat{U} \in \mathcal{V}(\mathbb{C}^{n_1}, r)$ and $\hat{V} \in \mathcal{V}(\mathbb{C}^{n_2}, r)$.

![Fig. 1: Bayesian setup for the compressed sensing problem (1), allowing possibly non-linear measurement mechanisms.](image)

**Theorem V.1.** Let $f$ satisfy the average power constraint (25). Let $g$ achieve the reconstruction error

\[ \mathbb{E}\|\hat{X} - X\|_F^2 \leq rD \] (26)

where $D > 0$. Then for any $s_1, s_2, r \in \mathbb{N}$ with $r \leq \min\{s_1, s_2\}$,

\[ m \geq \left( (s_1 + s_2)r - \frac{3r^2}{2} \right) \frac{\log \frac{m}{c}}{\log \left( 1 + \frac{1}{\sigma^2} \right)}. \] (27)

where $c$ is a universal constant.

**Proof:** Section VIII-F.
As a consequence of Theorem V.1, we conclude that the minimum number of measurements (sample complexity) for robust reconstruction of doubly sparse rank-\(r\) matrices, i.e., achieving finite noise sensitivity with \(D \leq C\sigma^2\) for all \(\sigma^2 > 0\) and some absolute constant \(C\), must satisfy

\[
m \geq (s_1 + s_2 - \frac{3r}{2})r.
\]  

(28)

which holds even if the recovery algorithm is allowed to depend on the noise level. This follows from (27) by sending \(\sigma \to 0\). When \(r = 1\), since \(m\) is an integer, we conclude that stable recovery of doubly-sparse rank-one matrices requires at least

\[
m \geq s_1 + s_2 - 1.
\]  

(29)

number of measurements.

In view of the lower bounds (28)–(29), we conclude that the number of measurements in Theorems III.1 and IV.1 are optimal within logarithmic factors. To see this, we argue that the additional spikiness assumption on the matrix imposed in Theorems III.1 and IV.1 does not change the sample complexity of the problem. To adapt Theorem V.1 to this scenario, one can simply consider \(X = UV^*\), where \(U = \frac{1}{\sqrt{2}} [I_r \tilde{U}]\) and \(V = \frac{1}{\sqrt{2}} [I_r \tilde{V}]\), and \(\tilde{U}\) and \(\tilde{V}\) are distributed according to Theorem V.1 with ambient dimension \(n_1\) (resp. \(n_2\)) replaced by \(n_2 - r\) (resp. \(n_1 - r\)). Then \(U\) and \(V\) satisfy the spikiness assumptions in Theorems III.1 and IV.1. Note that since \(r \leq \min\{s_1, s_2\}\) by definition, the lower bound (28) is always at least \(\frac{1}{4}r(s_1 + s_2)\). Therefore as long as the rank is not too high, namely, \(r \leq \min\{n_1 - s_1, n_2 - s_2\}\), the information-theoretic lower bounds (28)–(29) continue to hold and we conclude that SPF algorithm achieves the fundamental limits for stably recovering sparse low-rank matrices within logarithmic factors.

Remark V.2. Compared to [44, Theorem 10] for compressed sensing of sparse vectors proved under the high-dimensional scaling, the lower bound (28) is non-asymptotic. Moreover, even if we relax the stability requirement in (26) to \(\|\hat{X} - X\|_F^2 \leq \sigma^{2\alpha}\) for some \(\alpha \in (0, 1)\), the number of measurements still satisfies the lower bound \(m \geq \alpha(s_1 + s_2 - 2r)r\).

Remark V.3. The lower bound (28) can be heuristically understood by counting the number of (real) degrees of freedom in \(X = UV^*\), which turns out to be \(2(s_1 + s_2)r - r^3\). Note that the Stiefel manifold \(\mathbb{V}(\mathbb{C}^n, r)\) has topological (real) dimension\(^5\) \(\sum_{i=1}^{r} (2n - 2i + 1) = 2nr - r^2\). Since the non-zero parts

\[\text{5} \]This follows from choosing the first column \(v_1\) of \(V\) from the complex unit sphere which has \(2n_1 - 1\) real variables, then the second column \(v_2 \perp v_1\) which gives two equations (real and imaginary parts) and leaves \(2n_1 - 3\) free variables, etc.
of $U$ and $V$ belongs to $V(C^{s_1}, r)$ and $V(C^{s_2}, r)$ respectively, the total number of the (real) degrees of freedom in $U$ and $V$ is $2(s_1 + s_2)r - 2r^2$. However, since $UV^* = UR(VR)^*$, we need to quotient out the orthogonal group $O(r)$ in $\mathbb{C}^r$, which has dimension $\sum_{i=1}^{r}(2r - 2i + 1) = r^2$. Therefore the total degrees of freedom in $X$ is $\dim V(C^{s_1}, r) + \dim V(C^{s_2}, r) - \dim O(r) = 2(s_1 + s_2)r - 3r^2$. Hence, intuitively, we expect at least half of this number of complex linear measurements for stable recovery. Theorem V.1 gives a rigorous information-theoretic justification of this heuristic. See Remark VIII.19 for more detailed discussion on counting degrees of freedom.

**Remark V.4** (Bayesian v.s. minimax lower bound). The lower bound in Theorem V.1 is obtained under a Bayesian setup where the left and right singular vectors have uniformly drawn support and non-zeros. On the other hand, the upper bounds in Theorems III.1 and IV.1 are obtained under an adversarial setting where both the unknown matrix $X$ and the noise $z$ are deterministic. It is unclear whether the extra logarithmic factor for the number of Gaussian measurements is necessary in a minimax setting, where, for instance, the noise is additive Gaussian and the unknown rank-one matrix $X$ is adversarial.

The proof of Theorem V.1 relies on a rate-distortion lower bound for random subspaces, given in Theorem VIII.17 in Section VIII-F, which might be of independent interest.

VI. **NUMERICAL RESULTS**

In this section, we compare the empirical performance of SPF to those of PF and other popular recovery algorithms based on convex optimization [2]. The simulation setup is as follows: Let $\mathcal{A}$ be an i.i.d. Gaussian measurement operator. The unknown sparse rank-one matrix is generated with support uniformly drawn at random and the nonzero elements of the singular vectors are uniform on the complex sphere. The recovery performance of different procedures was compared in a Monte-Carlo study, averaging over 100 instances of signal matrices and measurement operators drawn at random.

A. **Row sparsity**

We compare the recovery performance of a row-sparse rank-one matrix by SPF in both noiseless and noisy cases against PF as well as the following convex-optimization approaches: basis pursuit with the row-sparsity prior (BP_RS), with the low-rank prior (BP_LR), and with both priors combined (BP_RSLR),
Fig. 2: Phase transition of the empirical success rates in the recovery of a rank-1 and row-$s$-sparse matrix of size $n_1 \times n_2$ ($x$-axis: $s/n_1$, $y$-axis: $m/n_1 n_2$, $n_1 = 128$). The empirical success rate is on a grayscale (black for zero and white for one). The yellow line corresponds to the fundamental limit $m \geq s + n_2 - 2$.

i.e.,

\[
\text{BP_RS} : \quad \hat{X} = \arg\min_{\tilde{X}} \left\{ \| \tilde{X} \|_{1,2} : A(\tilde{X}) = b \right\}
\]
\[
\text{BP_LR} : \quad \hat{X} = \arg\min_{\tilde{X}} \left\{ \| \tilde{X} \|_* : A(\tilde{X}) = b \right\}
\]
\[
\text{BP_RSLR} : \quad \hat{X} = \arg\min_{\tilde{X}} \left\{ \max \left( \frac{\| \tilde{X} \|_{1,2}}{\| X \|_{1,2}}, \frac{\| \tilde{X} \|_*}{\| X \|_*} \right) : A(\tilde{X}) = b \right\}
\]

where $\| \cdot \|_*$ denotes the nuclear norm (sum of singular values), and $\| \cdot \|_{1,2}$ denotes the mixed norm (sum of row norms). BP_RSLR corresponds to the optimal convex approach among all methods that minimize combinations of the nuclear norm and mixed norm [2]. The weights used in BP_RSLR are functions of the unknown matrix $X$ and therefore BP_RSLR is considered as an oracle method.

a) Noiseless case: The empirical phase transitions of success rates of various algorithms are given in Fig. 2, where success corresponds to achieving a reconstruction signal-to-noise-ratio (SNR) higher than 50 dB. SPF improves the performance of PF significantly by exploiting the sparsity structure and outperforms all three convex approach. Moreover, empirically the phase transition boundary of SPF is close to the fundamental limit $m \geq s_1 + n_2 - 2$ given in Section V. A detailed comparison of different phase transition boundaries is given below:
• Both PF and BP_LR exploit the rank-one constraint requiring

\[ m \geq C \max(n_1, n_2) \]  

number of measurements where \( C \) denotes some generic absolute constant. The difference is that PF iterates between left and right singular vectors and BP_LR promotes low-rankness by minimizing the nuclear norm. As the number of columns \( n_2 \) in the unknown matrix increases, the number of singular values \( \min(n_1, n_2) \) increases; hence, the rank-one constraint becomes more informative in the sense that it further narrows down the solution set. Therefore both PF and BP_LR benefit from a larger \( n_2 \). However, since neither procedure takes into account the row sparsity, their performance does not improve with smaller relative sparsity \( s/n_1 \). This results in a flat phase transition boundary. On the other hand, SPF achieves a better performance by exploiting the sparsity in the left singular vectors.

• Similarly, BP_RS, which minimizes the mixed norm to promote row sparsity, provides perfect reconstruction with number of measurements

\[ m \geq C s n_2 \log(en_1/s) \]  

which decreases as the relative sparsity \( s/n_1 \) decreases, but its performance is indifferent to the relative low-rankness \( 1/\min(n_1, n_2) \).

• BP_RSLR, which mixes BP_RS and BP_LR optimally, performs only as well as, but no better than, the best of BP_LR and BP_RS, achieving a success region as the union of the two. It requires number of measurements to exceed the smallest of the right-hand sides of (30) and (31), which can far exceed the fundamental limit \( m \geq s_1 + n_2 \). The intrinsic suboptimality of BP_RSLR stems
from the following observation: Although equivalent under the rank-one constraint, row-sparsity of
the matrix is excessive relaxation for the sparsity of the left singular vector, and simply promoting
low-rankness seems to be not enough. By thresholding the singular vectors in the power iteration,
SPF achieves near-optimal performance.

Figure 3 compares the phase transition of empirical success rates for SPF and BP_RSLR when the
number of columns $n_2$ and the sparsity level $s$ vary while the number of measurements $m$ is fixed to
be 128. SPF outperforms BP_RSLR by succeeding roughly under the condition $m \geq 4(s + n_2)$. This
empirical observation is aligned with the theoretical analysis in this paper.

b) Noisy case: Figure 4 visualizes the performance of SPF in the presence of noise. While the
theoretical analysis of SPF in this paper is restricted to the conservative case where SNR exceeds an
absolute constant level, empirically the reconstruction error of SPF is robust with respect to increased
noise level. In particular, with $m \geq 3(s + n_2)$, the noise amplification in the reconstruction remains less
than 1 for all SNR greater than 6 dB.

![Reconstruction SNR](image1)
![Noise amplification](image2)

Fig. 4: Relative error and noise amplification constant of SPF for varying noise strength $\nu$ (x-axis: $\nu$, y-axis: $m/(s + n_2)$, $n_1 = 256, s_1 = 32, n_2 = 64$). Reconstruction SNR = $\min \{50, 20 \log_{10} \|\hat{X} - X\|_F/\|X\|_F\}$. Noise amplification = $\min \{3, \log_{10} \|\hat{X} - X\|_F/\|\nu \|X\|_F\}$. 

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B. Row and column sparsity

When the unknown rank-one matrix $X$ is both column and row-sparse, we compare the recovery performance by SPF to the recovery using the following convex optimization approaches:

\[
\text{BP_DS}: \quad \hat{X} = \arg \min_Z \left\{ \max \left( \|Z\|_{1,2}, \|Z^*\|_{1,2}, \|X\|_{1,2}, \|X^*\|_{1,2} \right) : A(Z) = b \right\}
\]

\[
\text{BP_DSLR}: \quad \hat{X} = \arg \min_Z \left\{ \max \left( \|Z\|_{1,2}, \|Z^*\|_{1,2}, \|X\|_{1,2}, \|X^*\|_{1,2}, \|Z\|_{*,*}, \|X\|_{*,*} \right) : A(Z) = b \right\}.
\]

Both BP_DS and BP_DSLR use the oracle optimal weights, which are functions of the unknown matrix $X$.

![Figure 5](image)

**Fig. 5:** Phase transition for the empirical success rate in the recovery of a rank-1 and doubly $(s, s)$-sparse matrix of size $n \times n$ ($x$-axis: $s/n$, $y$-axis: $m/n$, $n = 64$). The yellow line corresponds to the fundamental limit $m = 2s$.

Figure 5 plots the empirical success rates of SPF and competing convex approaches. The unknown rank-one matrix of size $n \times n$ is assumed doubly $(s, s)$-sparse. In this setup, BP_DSLR performs as well as BP_DS in the very sparse case ($s = 1$). Otherwise, the rank-one prior dominates the double sparsity prior and the performance of BP_DSLR coincides with that of BP_LLR. On the contrary, SPF significantly improves on PF by exploiting the double sparsity prior. Fig. 6a (resp. Fig. 6b) shows that, when $m = 1.5n$ (resp. $m = 2n$), SPF empirically succeed roughly under the condition $m \geq 6s$, which is aligned with the performance guarantee of SPF in Section III and the lower bound $m \geq 2s - 2$ in Section V. As shown in Fig. 6c, even with more measurements, BP_DSLR completely failed in this setup except for the corner case of either $s_1 = 1$ or $s_2 = 1$.

C. Sparse and low rank matrices

In the scenario where the rank of the unknown sparse matrix is low but larger than 1, we compare the recovery performance of SCSPF to the natural rank-$r$ extension of SPF without subspace concatenation (Algorithm 4) and to BP_DSLR. Figure 7 compares the three algorithms when the condition number of...
Fig. 6: Phase transition for the empirical success rate in the recovery of a rank-1 and doubly $(s_1, s_2)$-sparse matrix of size $64 \times 64$ using SPF when the number of measurements $m$ is fixed ($x$-axis: $s_1$, $y$-axis: $s_2$).

The unknown matrix is ideally fixed as 1. SCSPF and SPF outperform BP_DSLR similarly to the previous sections. The empirical phase transition occurs at a sample complexity proportional to the rank of the unknown matrix, which is aligned with the presented theory for SCSPF. Although the sample complexity for the performance guarantee for SCSPF in Theorem IV.1 is proportional to the square of the condition number $\kappa$, as shown in Figure 8, when $\kappa$ is small, 5 in this figure, the empirical performance of the three algorithms looked similar to the ideal case in Figure 7. In both figures, in fact, the natural rank-$r$ SPF provided a better empirical performance compared to SCSPF. We suspect that the subspace concatenation in SCSPF was necessary because of artifacts in our proof techniques. It might be possible that a more careful analysis of the natural rank-$r$ SPF provide a performance guarantee at the sample complexity depending linearly on the rank.

Fig. 7: Phase transition for the empirical success rate in the recovery of a rank-$r$ and doubly $(s, s)$-sparse matrix of size $n \times n$ ($x$-axis: $r$, $y$-axis: $m/n$, $n = 64$, $s = 16$, $\kappa = 1$).
VII. CONCLUSION

We proposed an alternating minimization algorithm called sparse power factorization (SPF) that reconstructs sparse rank-one matrices from their linear measurements. We showed that under variants of the restricted isometry property corresponding to the underlying matrix models, SPF provides provable performance guarantees. Information-theoretic determination of the minimal number of measurements is another contribution of this paper. In particular, when the measurements are given using an i.i.d. Gaussian measurement operator, SPF with the initialization \(v_0^{\text{opt}}\) provides a near-optimal performance guarantee that holds with a number of measurements, which exceed the fundamental lower bound only by a logarithmic factor. On the other hand, when the unknown matrix has sparse singular vectors with fast-decaying magnitudes, the performance guarantee for SPF with the computationally efficient initialization \(v_0^{\text{Th}}\) holds with the same number of measurements. Similar performance guarantees in the context of blind deconvolution are presented in a companion paper [37]. Furthermore, through numerical experiments, we showed that the empirical performance of SPF dominates that of competing convex approaches, which is consistent with the theoretical analysis.

We conclude the paper by discussing the computational aspect of our sensing problem. As mentioned in Section III, the performance of SPF hinges on the initialization. The near optimal number of measurements \(O((s_1 + s_2) \log \max(en_1/s_1, en_2/s_2))\) in Theorem III.1 with no additional assumption is obtained using the initial value \(v_0^{\text{opt}}\) defined in Section II-B, which can be expensive to compute. In view of the failure of convex programming via nuclear/mixed norms [2], as well as the recently established hardness of sparse PCA [27] and biclustering [30], it is an open problem to determine whether achieving stable recovery using \(O(s_1 + s_2)\) or \(O((s_1 + s_2) \log \max(en_1/s_1, en_2/s_2))\) measurements is computationally intractable.
VIII. Proofs

A. Proof of Theorem III.9

For \( t \geq 0 \), denote the angle between \( u \) and \( u_t \) (resp. between \( v \) and \( v_t \)) by \( \theta_t \) (resp. \( \phi_t \)):

\[
\theta_t \triangleq \cos^{-1} \left( \frac{|u^* u_t|}{\|u\|_2 \|u_t\|_2} \right) \quad \text{and} \quad \phi_t \triangleq \cos^{-1} \left( \frac{|v^* v_t|}{\|v\|_2 \|v_t\|_2} \right),
\]

which are in \([0, \pi/2]\).

We derive recursive relations between the sequences \((\theta_t)_{t \in \mathbb{N}}\) and \((\phi_t)_{t \in \mathbb{N}}\). This roadmap is similar to that in a previous work [18] deriving a performance guarantee in terms of the rank-\(2r\) RIP for the recovery of a rank-\(r\) matrix using PF.

The updates of \( u_t \) and/or \( v_t \) using HTP, in the presence of the corresponding sparsity prior, are new components in SPF compared to PF, and require a different analysis (Lemma VIII.3). Although the least squares update of \( u_t \) or \( v_t \), in the absence of the corresponding sparsity prior, is common to both SPF and PF, we provide a new analysis of this step (Lemma VIII.5) that improves on the previous analysis [18] by sharpening the inequalities involved.

First, we analyze the update of \( u_t \) using HTP. To this end, we present a modified guarantee for HTP in the following lemma.

**Lemma VIII.1** (RIP-guarantee for HTP). Let \( b = \Phi x + z \in \mathbb{C}^m \) denote the noisy measurement vector of an \( s \)-sparse \( x \in \mathbb{C}^n \) obtained using a sensing matrix \( \Phi \in \mathbb{C}^{m \times n} \). Let \((x_k)_{k \in \mathbb{N}}\) be the iterates of HTP using a perturbed sensing matrix \( \hat{\Phi} \). Suppose that \( \hat{\Phi} \) satisfies the \( 3s \)-sparse RIP with isometry constant \( \delta < 0.5 \), and that there exists \( \vartheta \in [0, 1) \) such that

\[
\| \Pi_{\hat{J}} \hat{\Phi}^* (\Phi - \hat{\Phi}) \Pi_{\hat{J}} \| \leq \vartheta \delta, \quad \forall \hat{J} \subset [n], \ |\hat{J}| \leq 3s.
\]

Then, there exist \( L_{\delta}^{\text{HTP}} \in \mathbb{N} \) and \( K_{\delta}^{\text{HTP}} > 0 \) such that

\[
\| x_k - x \|_2 \leq C_{\delta}^{\text{HTP}} (\vartheta \delta \|x\|_2 + \sqrt{1 + \delta} \|z\|_2)
\]

for all \( k \geq \lceil L_{\delta}^{\text{HTP}} + K_{\delta}^{\text{HTP}} s \rceil \), where the constants \( L_{\delta}^{\text{HTP}}, K_{\delta}^{\text{HTP}}, C_{\delta}^{\text{HTP}} \) depend only on \( \delta \).

**Remark VIII.2.** Given \( \delta \), the constants \( L_{\delta}^{\text{HTP}}, K_{\delta}^{\text{HTP}}, \) and \( C_{\delta}^{\text{HTP}} \) are computed explicitly. For example, if \( \delta = 0.08 \), then \( L_{\delta}^{\text{HTP}} = 3, K_{\delta}^{\text{HTP}} = 3.17, \) and \( C_{\delta}^{\text{HTP}} = 2.86 \).

Unlike the original performance guarantee for HTP [35], in Lemma VIII.1, the recovery of the unknown
via HTP uses a perturbed sensing matrix \( \hat{\Phi} \). Furthermore, the error bound in Lemma VIII.1 applies to the estimate after \( O(s) \) iterations rather than to the limit to which HTP converges to. This non-asymptotic bound is useful in the sense that the resulting final guarantee for SPF in Theorem III.9 applies to the case where the number of inner iterations for the HTP steps is bounded. As a result, compared to [35], the bound increases slightly by a constant factor 1.01, which appears in the definition of \( C_{\delta}^{\text{HTP}} \) in the proof of Lemma VIII.1.

**Proof of Lemma VIII.1:** Let \( E \triangleq \Phi - \hat{\Phi} \). Then

\[
b = \hat{\Phi}x + Ex + z,
\]

which can be viewed as the measurement vector \( \hat{\Phi}x \) corrupted by the additive noise \( Ex + z \). Then, we can apply the conventional analysis of HTP [35] to \( \hat{\Phi} \). However, due to the near bi-orthogonality between \( \hat{\Phi}_J \) and \( E_J \) in (33), the noise term \( Ex \) and \( z \) propagate with different amplification coefficients. The resulting guarantee is summarized as follows: Under the assumptions in Lemma VIII.1, we have

\[
\|x_k - x\|_2 \leq \rho \|x_{k-1} - x\|_2 + \tau (\vartheta \delta \|x\|_2 + \sqrt{1 + \delta} \|z\|_2), \quad \forall k \in \mathbb{N} \tag{34}
\]

and

\[
\|x_k - x\|_2 \leq \rho' \|\Pi_{J_k}^\perp x\|_2 + \tau' (\vartheta \delta \|x\|_2 + \sqrt{1 + \delta} \|z\|_2), \quad \forall k \in \mathbb{N}, \tag{35}
\]

where \( J_k \) denotes the support of \( x_k \) and the constants are given by

\[
\rho = \sqrt{\frac{2 \delta^2}{1 - \delta^2}}, \quad \tau = \sqrt{\frac{2}{1 - \delta^2}} + \frac{1}{1 - \delta}, \quad \rho' = \frac{1}{\sqrt{1 - \delta^2}}, \quad \tau' = \frac{1}{1 - \delta}.
\]

(The modification of the conventional analysis of HTP to this version is straightforward. Hence, we omit the detail for deriving this step.)

Applying (34) and (35) to [45, Lemma 2.11] yields that if

\[
k > L + \frac{s \ln \left( 1 + 2 \left( \rho' + \frac{\tau' - \rho}{2} \right) \right)}{\ln \left( \frac{2}{1 + \rho} \right)}
\]

for \( L \in \mathbb{N} \), then

\[
\|x_k - x\|_2 \leq \frac{\tau [1 + \rho^L (2 \rho' - \rho)]}{1 - \rho} \cdot (\vartheta \delta \|x\|_2 + \sqrt{1 + \delta} \|z\|_2).
\]

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Set the parameter $L$ to

$$L = \left\lceil \frac{\ln(100(2\rho' - \rho))}{\ln(\rho^{-1})} \right\rceil.$$ 

Then, $(*)$ is bounded from above by $\frac{1.01\tau}{1-\rho}$; hence, if

$$k > \left\lceil \frac{\ln(100(2\rho' - \rho))}{\ln(\rho^{-1})} \right\rceil + s \cdot \frac{\ln \left[ 1 + 2 \left( \rho' + \frac{\tau'}{\tau} \frac{1-\rho}{2} \right) \right]}{\ln \left( \frac{2}{1+\rho} \right)} K_{\Delta}^{\text{HTP}},$$

then

$$\|x_k - x\|_2 \leq \frac{1.01\tau}{1-\rho} \cdot \left( (\theta s)\|z\|_2 + \sqrt{1 + \delta}\|z\|_2 \right).$$

Next, using Lemma VIII.1, we analyze the update of $u_t$ using HTP.

**Lemma VIII.3 (Sparse update of $u_t$).** Suppose the hypotheses of Theorem III.9 hold. Let $u_t$ be obtained as the $[L_{\delta}^{\text{HTP}} + K_{\delta}^{\text{HTP}} s]$th iterate of HTP applied to the sensing matrix $F(v_{t-1})$ and the measurement vector $b$, where $L_{\delta}^{\text{HTP}}$ and $K_{\delta}^{\text{HTP}}$ are constants determined by $\delta$ in the proof of Lemma VIII.1. Then,

$$\sin \theta_t \leq C_{\delta}^{\text{HTP}} \left[ \delta \tan \phi_{t-1} + (1 + \delta) \sec \phi_{t-1} \frac{\|z\|_2}{\|A(X)\|_2} \right].$$

where $\theta_t$ and $\phi_{t-1}$ are defined in (32).

**Proof:** Recall that $v_{t-1}$ was normalized in the $\ell_2$ norm before the update of $u_t$. Decomposing $v$ as

$$v = P_{\mathcal{R}(v_{t-1})}v + \underbrace{P_{\mathcal{R}(v_{t-1})}^* v}_{\zeta} = (v_{t-1}^* v) v_{t-1} + \zeta,$$ (37)

we have

$$\|\zeta\|_2^2 = 1 - |v_{t-1}^* v|^2 = \sin^2 \phi_{t-1}$$

and

$$\|\zeta\|_0 \leq \|v\|_0 + \|v_{t-1}\|_0 \leq 2s_2.$$  

By (37), we have

$$(v_{t-1}^* v) F(v_{t-1}) = F(v) - F(\zeta).$$ (38)
By (38), we can rewrite the measurement vector \( b \) as

\[
b = \mathcal{A}(X) + z \\
= \mathcal{A}(\lambda w^*) + z \\
= [F(v)](\lambda u) + z \\
= [(v_{t-1}^* v) F(v_{t-1}) + F(\zeta)](\lambda u) + z \\
= \left[ F(v_{t-1}) + (v_{t-1}^* v)^{-1} F(\zeta) \right] ((v_{t-1}^* v) \lambda u) + z.
\]

Since \( \|v_{t-1}\|_2 = 1 \) and \( \|v_{t-1}\|_0 \leq s_2 \), the rank-2 and \((3s_1, 3s_2)\)-sparse RIP of \( \mathcal{A} \) implies by Lemma B.1 that \( F(v_{t-1}) \) satisfies the \( 3s_1 \)-sparse RIP with isometry constant \( \delta \). Similarly, since \( \langle v_{t-1}, \zeta \rangle = 0 \) and \( \|v_{t-1}\|_0 + \|\zeta\|_0 \leq 3s_2 \), the rank-2 and \((3s_1, 3s_2)\)-sparse RIP of \( \mathcal{A} \) implies by Lemma B.1 that

\[
\|\Pi_{\bar{J}}[F(v_{t-1})]^*[F(\zeta)]\Pi_{\bar{J}}\| \leq \delta \|\zeta\|_2, \quad \forall \bar{J} \subset [n_1], |\bar{J}| \leq 3s_1.
\]

Therefore, by Lemma VIII.1, we have

\[
\|u_t - (v_{t-1}^* v) \lambda u\|_2 \leq C_{\delta}^{\text{HTP}} \delta \|\zeta\|_2 \|\lambda u\|_2 + C_{\delta}^{\text{HTP}} \sqrt{1 + \delta} \|z\|_2 \\
\leq \lambda C_{\delta}^{\text{HTP}} \left( \delta \sin \phi_{t-1} + \sqrt{1 + \delta} \frac{\|z\|_2}{\lambda} \right) \\
= \lambda C_{\delta}^{\text{HTP}} \left( \delta \sin \phi_{t-1} + \sqrt{1 + \delta} \frac{\|z\|_2}{\|X\|_F} \right)
\]

and \( \sin \theta_t \) is rewritten as

\[
\sin \theta_t = \|P_{\mathcal{R}(u_t)}^* P_{\mathcal{R}(u)}\| \\
= \|P_{\mathcal{R}(u_t)}^* u\|_2 \\
= \|P_{\mathcal{R}(u_t)}^* (v_{t-1}^* v) \lambda u\|_2 \\
= \|P_{\mathcal{R}(u_t)}^* (v_{t-1}^* v) \lambda u\|_2 \\
= \frac{\lambda \cos \phi_{t-1}}{(v_{t-1}^* v) \lambda u - u_t} \\
\leq \frac{\alpha}{\cos \phi_{t-1}},
\]
where (c) follows since
\[ \|P_{R(u_t)} (v_{t-1}^* v) \lambda u\|_2 = \min_{\tilde{u}} \{ \|(v_{t-1}^* v) \lambda \tilde{u} - \tilde{u}\|_2 : \tilde{u} \in R(u_t) \} \].

It remains to apply the following upper bound on \( \|z\|_2 / \|X\|_F \)
\[ \frac{\|z\|_2}{\|A(X)\|_2} \geq \frac{1}{\sqrt{1 + \delta}} \frac{\|z\|_2}{\|X\|_F} \] (40)
implied by the rank-2 and row-3s-sparse RIP of \( A \) with isometry constant \( \delta \).

In the doubly sparse case \( (s_1 < n_1 \) and \( s_2 < n_2 \)), by the symmetry in the updates of \( u_t \) and \( v_t \), we get the following corollary as a direct implication of Lemma VIII.3.

**Corollary VIII.4 (Sparse update of \( v_t \)).** Suppose the hypotheses of Theorem III.9 hold. Let \( v_t \) be obtained as the \( \lceil L_{\delta}^{\text{HTP}} + K_{\delta}^{\text{HTP}} \rceil \)th iterate of HTP applied to the sensing matrix \( G(u_t) \) and the measurement vector \( \overline{b} \), where \( L_{\delta}^{\text{HTP}} \) and \( K_{\delta}^{\text{HTP}} \) are constants determined by \( \delta \) in the proof of Lemma VIII.1. Then,
\[ \sin \phi_t \leq C_{\delta}^{\text{HTP}} \left[ \delta \tan \theta_t + (1 + \delta) \sec \theta_t \frac{\|z\|_2}{\|A(X)\|_2} \right] \] (41)
where \( \theta_t \) and \( \phi_t \) are defined in (32).

In the row-sparse case \( (s_2 = n_2) \), the right factor \( v_t \) is updated using \( u_t \) by solving the least squares problem in (10). The angle between the resulting \( v_t \) and \( v \) is analyzed in the following lemma.

**Lemma VIII.5 (LS update of \( v_t \)).** Suppose the hypotheses of Theorem III.9 hold. Then,
\[ \sin \phi_t \leq \frac{1}{1 - \delta} \left[ \delta \tan \theta_t + (1 + \delta) \sec \theta_t \frac{\|z\|_2}{\|A(X)\|_2} \right] \] (42)

**Proof:** Recall that \( u_t \) is normalized in the \( \ell_2 \) norm before the update of \( v_t \) and that \( v_t \) is updated as
\[ v_t = [G(u_t)]^\dagger \overline{b} \]
where \( [G(u_t)]^\dagger \) denotes the pseudo-inverse of \( [G(u_t)] \).

Decomposing \( u \) as
\[ u = P_{R(u_t)} u + P_{R(u_t)^\perp} u = (u_t^* u) u_t + \xi \]
\[ \xi \]

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yields
\[ \|\xi\|^2 = 1 - |u^*_t u|^2 = \sin^2 \theta. \]

By the anti-linearity of \( G \), we have
\[ (u^*_t u)G(u_t) = G(u) - G(\xi). \]

We can rewrite the complex conjugate of the measurement vector as follows:
\[
\bar{b} = \overline{A(X)} + \bar{z} \\
= \overline{A(\lambda uv^*)} + \bar{z} \\
= [G(u)](\lambda v) + \bar{z} \\
= [(u^*_t u)G(u_t) + G(\xi)](\lambda v) + \bar{z} \\
= [G(u_t)](\overline{(u^*_t u)\lambda v}) + [G(\xi)](\lambda v) + \bar{z}.
\]

Since \( \|u_t\|_2 = 1 \) and \( \|u_t\|_0 \leq s_1 \), the rank-2 and \((3s_1, 3s_2)\)-sparse RIP of \( A \) implies by Lemma B.1 that
\[ \|[(G(u_t)]^*[G(u_t)]^{-1}\| \leq \frac{1}{1 - \delta} \] (43)

and
\[ \|[G(u_t)]^*z\|_2 \leq \sqrt{1 + \delta}\|z\|_2, \quad \forall z \in \mathbb{C}^n. \] (44)

Similarly, since \( \langle u_t, \xi \rangle = 0 \) and \( \|u_t\|_0 + \|\xi\|_0 \leq 3s_1 \), the rank-2 and \((3s_1, 3s_2)\)-sparse RIP of \( A \) implies by Lemma B.1 that
\[ \|G(u_t)^*G(\xi)\| \leq \delta\|\xi\|_2. \] (45)

Note that \( v_t \) is written as
\[
v_t = [G(u_t)]^T[(G(u_t)](\overline{(u^*_t u)\lambda v}) + [G(\xi)](\lambda v) + \bar{z}) \\
= (u^*_t u)\lambda v + [G(u_t)]^T[G(\xi)](\lambda v) + [G(u_t)]^T\bar{z}.
\]
By (43), (44), and (45), it follows that

\[
\|v_t - (u^*_t u)\lambda v\|_2 \leq \|[G(u_t)]^\dagger[G(\xi)](\lambda v)\|_2 + \|[G(u_t)]^\dagger z\|_2
\]

\[
\leq \frac{\delta}{1 - \delta} \lambda \|\xi\|_2 + \frac{\sqrt{1 + \delta}}{1 - \delta} \|z\|_2
\]

\[
= \lambda \left( \frac{\delta}{1 - \delta} \sin \theta_t + \frac{\sqrt{1 + \delta}}{1 - \delta} \cdot \frac{\|z\|_2}{\lambda} \right)
\]

\[
= \lambda \cdot \frac{1}{1 - \delta} \left( \delta \sin \theta_t + \sqrt{1 + \delta} \cdot \frac{\|X\|_F}{\lambda} \right)
\]

\[
\leq \lambda \cdot \frac{1}{1 - \delta} \left( \delta \sin \theta_t + (1 + \delta) \frac{\|z\|_2}{\alpha \|A(X)\|_2} \right)
\]

where the last step follows by (40).

Then, \( \sin \phi_t \) is bounded from above by

\[
\sin \phi_t = \|[P_{R(v)}] : P_{R(v)}\|_2
\]

\[
= \|[P_{R(v)}] : v\|_2
\]

\[
= \|[P_{R(v)}] : (u^*_t u)\lambda v\|_2
\]

\[
= \|[u^*_t u] \lambda \| \lambda \|v\|_2
\]

\[
\leq \|(u^*_t u)\lambda v - v_t\|_2 \lambda \cos \theta_t
\]

\[
\leq \frac{\alpha}{\cos \theta_t}
\]

where the first inequality holds using a similar argument to (a) in the proof of Lemma VIII.3.

Remark VIII.6. In fact, since \( C^\text{HTP}_\delta \geq (1 - \delta)^{-1} \), the upper bound on \( \phi_t \) by Lemma VIII.5 is tighter than that by Corollary VIII.4. In other words, Corollary VIII.4 also applies to the row-sparse case. By symmetry, we also conclude that Lemma VIII.3 also applies to the column-sparse case (\( s_1 = n_1 \)).

Next, Lemma VIII.3 and Corollary VIII.4 provide recursive relations that alternate between the two sequences \((\phi_t)_{t \in \mathbb{N}}\) and \((\theta_t)_{t \in \mathbb{N}}\). From these results, we deduce a recursive relation on \((\theta_t)_{t \in \mathbb{N}}\) that leads to a convergence of SPF in the following lemma, the proof of which is deferred to later.

Lemma VIII.7. Suppose the hypotheses of Theorem III.9 hold. Define

\[
\Omega \triangleq \{ \omega \in [0, \pi/2) : \omega \geq \sin^{-1}(C^\text{HTP}_\delta \delta \tan \omega + (1 + \delta) \nu \sec \omega) \}
\]

\[
\]
and let $\omega_{\text{sup}}$ is the supremum of $\Omega$. Suppose

$$\|P_{\mathcal{R}(v^1)} P_{\mathcal{R}(v_0)}\| < \sin \omega_{\text{sup}}. \quad (46)$$

In fact, for $\delta = 0.08$ and $\nu = 0.08$ as in Theorem III.9, then (46) coincides with (14). Then,

$$\limsup_{t \to \infty} \sin \theta_t \leq C \frac{\|z\|_2}{\|A(X)\|_2},$$

where the constant $C$ depends only on $\delta$ and $\nu$. Furthermore, $\max(0, \sin \theta_t - C\|z\|_2/\|A(X)\|_2)$ converges to 0 superlinearly.

The next lemma provides an upper bound on the normalized estimation error $\|X - u_t v_t^*\|_F/\|X\|_F$ in terms of $\sin \theta_t$.

**Lemma VIII.8.** Suppose the hypotheses of Theorem III.9 hold. Then,

$$\frac{\|X - u_t v_t^*\|_F}{\|X\|_F} \leq \sqrt{1 + 2(\delta C^{\text{HTP}})^2} \sin \theta_t + \sqrt{2(1 + \delta)C^{\text{HTP}}} \frac{\|z\|_2}{\|A(X)\|_2}. \quad (47)$$

**Remark VIII.9.** For fixed $\delta$ and $\nu$, the noise amplification in Lemma VIII.7 is explicitly bounded. For example, if $\delta \leq 0.08$ and $\nu \leq 0.08$, then

$$\limsup_{t \to \infty} \sin \theta_t \leq 3.75 \frac{\|z\|_2}{\|A(X)\|_2}$$

By Lemma VIII.8, this implies

$$\limsup_{t \to \infty} \frac{\|X - X_t\|_F}{\|X\|_F} \leq 8.3 \frac{\|z\|_2}{\|A(X)\|_2}.$$

Combining Lemmas VIII.7 and VIII.8 yields the proof of Theorem III.9. We conclude this section with the proofs of Lemmas VIII.7 and VIII.8.

**Proof of Lemma VIII.7:** Define $f : [0, \pi/2) \to [0, \pi/2)$ by

$$f(\omega) \triangleq \sin^{-1} \left(C^{\text{HTP}} \left[ \delta \tan \omega + (1 + \delta) \sec \omega \frac{\|z\|_2}{\|A(X)\|_2} \right] \right)$$

and let $\Omega_f \triangleq \{ \omega : \omega \geq f(\omega) \}$. Let $\omega_{f,\text{inf}}$ and $\omega_{f,\text{sup}}$ denote the infimum and supremum of $\Omega_f$, respectively. Since $\sin \omega$ is monotone increasing and concave on $[0, \pi/2]$, it follows that $\sin^{-1} \nu$ is monotone increasing and convex on $[0, 1]$. Furthermore, both $\tan \omega$ and $\sec \omega$ are monotone increasing and convex on $[0, \pi/2]$. Therefore, $f$ is monotone increasing and convex on $[0, \pi/2)$. Then it follows
that
\[
\omega \geq f(\omega), \quad \forall \omega \in [\omega_{f,\text{inf}}, \omega_{f,\text{sup}}],
\]
\[
\omega < f(\omega), \quad \forall \omega \in [0, \omega_{f,\text{inf}}) \cup (\omega_{f,\text{sup}}, \pi/2).
\] (48)

First, we show that
\[
\phi_0 < \omega_{f,\text{sup}}.
\] (49)

Since \(\|z\|_2/\|A(X)\|_2 \leq \nu\), \(\Omega\) is a subset of \(\Omega_f\). Then it follows that \(\omega_{\text{sup}} \leq \omega_{f,\text{sup}}\). The inequality in (49) holds since we assumed that \(\phi_0 < \omega_{\text{sup}}\).

In the first iteration, by Lemma VIII.3, \(\theta_1\) is upper-bounded by
\[
\theta_1 \leq f(\phi_0).
\] (50)

Then, by (48) and (50), either of the following cases holds: if \(\phi_0 > \omega_{f,\text{inf}}\), then \(\theta_1 < \phi_0\); if \(\phi_0 \leq \omega_{f,\text{inf}}\), then \(\theta_1 \leq \omega_{f,\text{inf}}\).

Next, in the second iteration, by Corollary VIII.4, \(\phi_1\) is upper-bounded by
\[
\phi_1 \leq f(\theta_1).
\] (51)

Similarly to the previous case, since \(\theta_1 < \omega_{\text{sup}}\), (51) implies that \(\phi_1\) satisfies either \(\phi_1 < \theta_1\) or \(\phi_1 \leq \omega_{f,\text{inf}}\).

By induction, both \((\phi_t)_{t \in \mathbb{N}}\) and \((\theta_t)_{t \in \mathbb{N}}\) converge to the set \([0, \omega_{\text{inf}}]\). The convergence is superlinear because of the convexity of \(f\).

Finally, it remains to compute an upper bound on \(\sin \omega_{f,\text{inf}}\). Define \(\hat{f} : [0, \pi/2) \rightarrow [0, \pi/2)\) by
\[
\hat{f}(\omega) \triangleq \sin^{-1}\left(\frac{C_{\text{HTP}}}{\cos \omega_{\text{sup}}} \left[\delta \sin \omega + (1 + \delta) \frac{\|z\|_2}{\|A(X)\|_2}\right]\right)
\]
and let \(\hat{\Omega}_f \triangleq \{\omega \in [0, \omega_{\text{sup}}] : \omega \geq \hat{f}(\omega)\}\). Then \(\hat{\Omega}_f\) is a subset of \(\Omega_f\) and it follows that \(\omega_{f,\text{inf}}\) is no greater than the infimum of \(\hat{\Omega}_f\). Thus we have
\[
\sin \omega_{f,\text{inf}} \leq \left(\frac{1 + \delta}{\cos \omega_{\text{sup}} - \delta C_{\text{HTP}} \|A(X)\|_2}\right) \frac{\|z\|_2}{C},
\]
where \(C\) depends only on \(\delta\) and \(\nu\).

**Proof of Lemma VIII.8:** Without loss of generality, we may assume that \(\|u_t\|_2 = 1\). Note that the
estimation error is decomposed as

\[\lambda vv^* - u^*_t v^*_t = \lambda P_{R(u_t)} vv^* - u^*_t v^*_t + \lambda P_{R(u_t)^\perp} vv^* = u_t [(u^*_tu_t)\lambda - v_t] + \lambda P_{R(u_t)^\perp} vv^*.\]

As shown in the proof of Lemma VIII.3, we have

\[\|v_t - (u^*_tu_t)\lambda - (u^*_tu_t)\lambda - v_t\|_2 \leq \lambda C_{\delta}^\text{HTP} \left( \frac{\|P_{R(u_t)^\perp} v\|_2}{\|\mathcal{A}(X)\|_2}\right).\]

Therefore,

\[\|\lambda vv^* - u_t v_t^*\|_F^2 = \|u_t [(u^*_tu_t)\lambda - v_t]\|_F^2 + \lambda^2 \|P_{R(u_t)^\perp} vv^*\|_F^2 \]

\[\leq \lambda^2 \left[ 1 + 2(\delta C_{\delta}^\text{HTP})^2 \right] \|P_{R(u_t)^\perp} P_{R(u)}\|_2^2 + 2 \left[ \lambda (1 + \delta) C_{\delta}^\text{HTP} \frac{\|\tilde{z}\|_2}{\|\mathcal{A}(X)\|_2}\right]^2,\]

which implies (47).

\[\|\lambda vv^* - u_t v_t^*\|_F^2 = \|u_t [(u^*_tu_t)\lambda - v_t]\|_F^2 + \lambda^2 \|P_{R(u_t)^\perp} vv^*\|_F^2 \]

\[\leq \lambda^2 \left[ 1 + 2(\delta C_{\delta}^\text{HTP})^2 \right] \|P_{R(u_t)^\perp} P_{R(u)}\|_2^2 + 2 \left[ \lambda (1 + \delta) C_{\delta}^\text{HTP} \frac{\|\tilde{z}\|_2}{\|\mathcal{A}(X)\|_2}\right]^2,\]

B. Proof of Theorem III.10

The lemmas in this section assumes that \(\mathcal{A}\) satisfies the rank-2 and \((3s_1,3s_2)\)-sparse RIP with isometry constant \(\delta\), unless otherwise stated. Through the following results, we show that the problem of finding a good initialization, which satisfies (14), reduces to finding good support estimates of \(u\) and \(v\). Once the satisfaction of (14) is shown, the performance guarantee is automatically implied by Theorem III.9.

First, we present the following lemma that computes an upper bound on the angle between \(v\) and \(v_0\).

Lemma VIII.10. Let \(\tilde{J}_1 \subset [n_1]\) and \(\tilde{J}_2 \subset [n_2]\) satisfy \(|\tilde{J}_1| \leq s_1\) and \(|\tilde{J}_1| \leq s_2\), respectively. Let \(v_0\) be obtained as the leading right singular vector of \(\Pi_{\tilde{J}_1} [A^*(b)]\Pi_{\tilde{J}_2}\). Then,

\[\|P_{R(v)} P_{R(v)}\|_2 \leq \frac{\|\Pi_{\tilde{J}_1} u\|_2 \|\Pi_{\tilde{J}_2} v\|_2 + \delta + (1 + \delta)\nu}{\|\Pi_{\tilde{J}_1} u\|_2 - \delta - (1 + \delta)\nu}\].

\[(52)\]

Proof: See Appendix C.

By Lemma VIII.10, if the right-hand-side of (52) is smaller than \(\sin \omega_{\text{sup}}\), then the inequality in (14) holds. Rearranging this, we get a sufficient condition for (14) given by

\[\delta + (1 + \delta)\nu < \frac{\|\Pi_{\tilde{J}_1} u\|_2 \left( \sin \omega_{\text{sup}} - \|\Pi_{\tilde{J}_2} v\|_2 \right)}{1 + \sin \omega_{\text{sup}}}.

\[(53)\]

Finding estimates \(\tilde{J}_1\) and \(\tilde{J}_2\) of the supports of \(u\) and \(v\) that satisfy (53) is much easier than finding
the supports of $u$ and $v$ exactly. We show that the two support estimation schemes in the initialization of SPF, proposed in Section II, are guaranteed to satisfy (53) under the RIP assumption.

1) Proof of Part 1: Recall that $v_{0}^{\text{opt}}$ was the leading right singular vector of $\Pi_{\hat{J}_{1}}[A^*(b)]\Pi_{\hat{J}_{2}}$, where $\hat{J}_{1}$ and $\hat{J}_{2}$ are given in (11). In this case, the product of $\|\Pi_{\hat{J}_{1}}u\|_{2}$ and $\|\Pi_{\hat{J}_{2}}v\|_{2}$ is lower-bounded by the following lemma.

**Lemma VIII.11.** Let $\hat{J}_{1} \subset [n_{1}]$ and $\hat{J}_{2} \subset [n_{2}]$ be given by (11). Then,

$$\|\Pi_{\hat{J}_{1}}u\|_{2} \cdot \|\Pi_{\hat{J}_{2}}v\|_{2} \geq \sin \left[ \sin^{-1} \left( \frac{1 - \delta - 2(1 + \delta)\nu}{\sqrt{\delta^2 + (1 + \delta)^2}} \right) - \sin^{-1} \left( \frac{\delta}{\sqrt{\delta^2 + (1 + \delta)^2}} \right) \right].$$

**Proof:** See Appendix D.

Let

$$\varphi(\delta, \nu) \triangleq \sin^{-1} \left( \frac{1 - \delta - 2(1 + \delta)\nu}{\sqrt{\delta^2 + (1 + \delta)^2}} \right) - \sin^{-1} \left( \frac{\delta}{\sqrt{\delta^2 + (1 + \delta)^2}} \right).$$

Since $\|\Pi_{\hat{J}_{1}}u\|_{2} \leq 1$ and $\|\Pi_{\hat{J}_{2}}v\|_{2} \leq 1$, by Lemma VIII.11, we have

$$\min(\|\Pi_{\hat{J}_{1}}u\|_{2}, \|\Pi_{\hat{J}_{2}}v\|_{2}) \geq \sin \varphi(\delta, \nu),$$

which also implies

$$\|\Pi_{\hat{J}_{2}}v\|_{2} \leq \cos \varphi(\delta, \nu).$$

Therefore, a sufficient condition for $v_{0}^{\text{opt}}$ to satisfy (14) is given by

$$\delta + (1 + \delta)\nu < \frac{\sin \varphi(\delta, \nu) (\sin \omega_{\text{sup}} - \cos \varphi(\delta, \nu))}{1 + \sin \omega_{\text{sup}}}. \tag{55}$$

Owing to the monotonicity of components in (55), a set of $(\delta, \nu)$ where (55) holds is determined. For example, it holds if $\delta \leq 0.04$ and $\nu \leq 0.04$.

2) Proof of Part 2-(a): Toward a performance guarantee using a computationally efficient algorithm, we analyze the performance of the initialization with $v_{0}^{\text{Th}}$.

The assumptions on $\delta$, $\nu$, $\|u\|_{\infty}$, and $\|v\|_{\infty}$ imply that

$$\delta + \nu + \delta \nu \leq \frac{\|u\|_{\infty} \|v\|_{\infty} + \sin \omega_{\text{sup}} - \sqrt{2}}{3 + \sin \omega_{\text{sup}}}, \tag{56}$$

where $\sin \omega_{\text{sup}} \geq 0.97$. 

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Let $j_0 \in [n_1]$ denote the index of the largest entry of $u$ in magnitude, i.e.,

$$j_0 \triangleq \arg\max_{j \in [n_1]} |u_j|.$$ 

Then, $|u_{j_0}| = \|\Pi_{\{j_0\}} u\|_2 = \|u\|_\infty$.

First, we show that $j_0 \in \hat{J}_1$ using the following lemma.

**Lemma VIII.12.** Let $\hat{J}_1 \subset [n_1]$ be the support estimate for $v_0^{\text{Th}}$. If there exists $\tilde{J}_1 \subset \text{supp}(u)$ such that

$$2\delta + 2(1 + \delta)\nu < \min_{j \in \tilde{J}_1} |u_j|,$$  

(57)

then $\tilde{J}_1 \subset \hat{J}_1$.

**Proof:** See Appendix E. \(\square\)

We apply Lemma VIII.12 with $\tilde{J}_1 = \{j_0\}$. Since (56) implies (57), we have shown $j_0 \in \hat{J}_1$.

Next, we derive a lower bound on $\|\Pi_{\hat{J}_1} u\|_2 \|\Pi_{\hat{J}_2} v\|_2$. Define

$$k_0 \triangleq \arg\max_{k \in [n_2]} |v_k|,$$

$$k_1 \triangleq \arg\max_{k \in [n_2]} \|\Pi_{\hat{J}_1} [A^*(b)]\Pi_k\|_F,$$

$$k_2 \triangleq \arg\max_{k \in [n_2]} \|\Pi_{\{j_0\}} [A^*(b)]\Pi_k\|_F.$$

Then, by the selection of $\hat{J}_2$ in computing $v_0^{\text{Th}}$, we have $k_1 \in \hat{J}_2$. On the other hand, by definition of $k_1$ and $k_2$, it follow that

$$\|\Pi_{\hat{J}_1} [A^*(b)]\Pi_{\{k_1\}}\|_F \geq \|\Pi_{\hat{J}_1} [A^*(b)]\Pi_{\{k_2\}}\|_F \
\geq \|\Pi_{\{j_0\}} [A^*(b)]\Pi_{\{k_2\}}\|_F$$  

(58)

where the second inequality holds since $j_0 \in \hat{J}_1$.  

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The left-hand-side of (58) is further upper-bounded by
\[
(a) \leq \|\Pi_{\tilde{J}_1}X\Pi_{\{k_1\}}\|_F + \|\Pi_{\tilde{J}_1}(A^*A - \text{id})(X)\Pi_{\{k_1\}}\|_F + \|\Pi_{\tilde{J}_1}[A^*(z)]\Pi_{\{k_1\}}\|_F \\
\leq \lambda\|\Pi_{\tilde{J}_1}u\|_2\|\Pi_{\{k_1\}}v\|_2 + \delta\lambda + \sqrt{1 + \delta}\|z\|_2 \\
\leq \lambda\|\Pi_{\tilde{J}_1}u\|_2\|\Pi_{\tilde{J}_2}v\|_2 + \delta\lambda + \sqrt{1 + \delta}\|z\|_2,
\]
where the second inequality follows by Lemmas B.2 and B.3, and the last step holds since $k_1 \in \tilde{J}_2$.

The right-hand-side of (58) is lower-bounded by
\[
(b) \geq \|\Pi_{\{j_0\}}X\Pi_{\{k_0\}}\|_F - \|\Pi_{\{j_0\}}(A^*A - \text{id})(X)\Pi_{\{k_0\}}\|_F - \|\Pi_{\{j_0\}}[A^*(z)]\Pi_{\{k_0\}}\|_F \\
\geq \lambda\|u\|_\infty\|v\|_\infty - \delta\lambda - \sqrt{1 + \delta}\|z\|_2.
\]
Applying (59) and (60) to (58) with (40) yields
\[
\|\Pi_{\tilde{J}_1}u\|_2\|\Pi_{\tilde{J}_2}v\|_2 \geq \|u\|_\infty\|v\|_\infty - 2(\delta + \nu + \delta\nu).
\]

Then, by applying (61) to (53) with some rearrangement, we get a sufficient condition for (53) given by
\[
\|\Pi_{\tilde{J}_1}u\|_2^2 < \left[\sin\sup\|\Pi_{\tilde{J}_1}u\|_2 - (1 + \sin\sup)(\delta + \nu + \delta\nu)\right]^2 \\
+ \left[\|u\|_\infty\|v\|_\infty - 2(\delta + \nu + \delta\nu)\right]^2.
\]
By convexity of the scalar quadratic function, (62) is implied by
\[
\sqrt{2}\|\Pi_{\tilde{J}_1}u\|_2 < \sin\sup\|\Pi_{\tilde{J}_1}u\|_2 + \|u\|_\infty\|v\|_\infty - (3 + \sin\sup)(\delta + \nu + \delta\nu).
\]
Here, we used the fact that $\sin\sup\|\Pi_{\tilde{J}_1}u\|_2 - (1 + \sin\sup)(\delta + \nu + \delta\nu) > 0$, which is implied by (56).

Note that (63) is equivalently rewritten as
\[
\delta + \nu + \delta\nu < \frac{\|u\|_\infty\|v\|_\infty - (\sqrt{2} - \sin\sup)\|\Pi_{\tilde{J}_1}u\|_2}{3 + \sin\sup},
\]
which is implied by (56) since $\sin\sup < \sqrt{2}$.

3) Proof of Part 2-(b): Unlike the previous parts of Theorem III.10, Part 2-(b) assumes a stronger RIP of $A$ that applies to all matrices with up to $9s_1s_2$ nonzero entries.

Let $\mathcal{S}$ denote the set of $n_1$-by-$n_2$ matrices such that there are at most $s_1$ nonzero rows and each row has at most $s_2$ nonzero elements. Let $\mathcal{P}_S$ denote the orthogonal projection onto $\mathcal{S}$. Then, $\tilde{J}_1$ coincides with the row-support of $\mathcal{P}_S[A^*(b)]$.
The RIP assumption admits an upper bound on \( \| \mathcal{P}_S[A^*(b)] - X \|_F \), which is direct implication of the analogous result in compressed sensing of a sparse vector. We use the version by Foucart [46, Theorem 3] given by

\[
\| \mathcal{P}_S[A^*(b)] - X \|_F \leq 2\delta \| X \|_F + 2\sqrt{1+\delta} \| z \|_2. \tag{64}
\]

Define a two-dimensional coordinate projection \( \tilde{\Pi}_J : \mathbb{C}^{n_1 \times n_2} \rightarrow \mathbb{C}^{n_1 \times n_2} \) associated with an index set \( J \subset [n_1] \times [n_2] \) by

\[
[\tilde{\Pi}_J(Z)]_{i,j} = \begin{cases} 
[Z]_{i,j} & (i,j) \in J, \\
0 & \text{else},
\end{cases}
\]

for all \( Z \in \mathbb{C}^{n_1 \times n_2} \). Then, it follows that \( \tilde{\Pi}_{\tilde{J}_1 \times \tilde{J}_2} (Z) = \Pi_{\tilde{J}_1} Z \Pi_{\tilde{J}_2} \). Furthermore, there exists a subset \( \tilde{J} \subset [n_1] \times [n_2] \) such that

\[
\mathcal{P}_S[A^*(b)] = \tilde{\Pi}_{\tilde{J}}[A^*(b)]. \tag{65}
\]

Let \( \tilde{M} \in \mathbb{C}^{n_1 \times n_2} \) denote the best rank-1 approximation of \( \Pi_{\tilde{J}_1} [A^*(b)] \Pi_{\tilde{J}_2} \) in the Frobenius norm. Then, we have

\[
\| \Pi_{\tilde{J}_1} [A^*(b)] \Pi_{\tilde{J}_2} - \tilde{M} \|_F \leq \| \Pi_{\tilde{J}_1} [A^*(b)] \Pi_{\tilde{J}_2} - X \|_F.
\]

Therefore,

\[
\| \tilde{M} - X \|_F \leq \| \tilde{M} - \Pi_{\tilde{J}_1} [A^*(b)] \Pi_{\tilde{J}_2} \|_F + \| \Pi_{\tilde{J}_1} [A^*(b)] \Pi_{\tilde{J}_2} - X \|_F \\
\leq 2\| \Pi_{\tilde{J}_1} [A^*(b)] \Pi_{\tilde{J}_2} - X \|_F \\
\leq 2\| \Pi_{\tilde{J}_1} [A^*(b)] \Pi_{\tilde{J}_2} - \tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2) \cup \tilde{J}} [A^*(b)] \|_F + 2\| \tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2) \cup \tilde{J}} [A^*(b)] - X \|_F \\
\leq 4\| \tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2) \cup \tilde{J}} [A^*(b)] - X \|_F,
\]

where the last step follows since

\[
\Pi_{\tilde{J}_1}[A^*(b)] \Pi_{\tilde{J}_2} = \text{argmin}_Z \{ \| \tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2) \cup \tilde{J}} [A^*(b)] - Z \|_F : \text{Z is (s_1,s_2)-sparse} \}.
\]

The last term in (66) is further upper-bounded by

\[
\| \tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2) \cup \tilde{J}} [A^*(b)] - X \|_F \leq \| \tilde{\Pi}_{\tilde{J}} [A^*(b)] - X \|_F \leq \| \tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2) \setminus \tilde{J}} [A^*(b)] \|_F. \tag{67}
\]

Since \( \tilde{\Pi}_{\tilde{J}}[A^*(b)] = \mathcal{P}_S[A^*(b)] \), the first term in the right-hand-side of (67) is upper-bounded by (64).
Next, by the triangle inequality, the second term in the right-hand-side of (67) is upper-bounded by

\[
\|\tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2)\setminus \tilde{J}} [A^\ast (b)]\|_F
\]

\[
= \|\tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2)\setminus \tilde{J}} [\tilde{X} + (A^\ast A - \text{id})(X) + A^\ast (z)]\|_F
\]

\[
\leq \|\tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2)\setminus \tilde{J}} (X)\|_F + \|\tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2)\setminus \tilde{J}} (A^\ast A - \text{id})(X)\|_F + \|\tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2)\setminus \tilde{J}} (A^\ast (z))\|_F
\]

\[
\leq \|\tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2)\setminus \tilde{J}} (X)\|_F + \delta \|X\|_F + \sqrt{1 + \delta} \|z\|_2,
\]

where the last step follows by the RIP of $A$ with $|(J_1 \times J_2) \setminus \tilde{J}| \leq s_1 s_2$.

Let $J_1$ and $J_2$ denote the support of $u$ and $v$, respectively. In other words, $J_1$ and $J_2$ are the row-support and column-support of $X$, respectively. Then,

\[
\tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2)\setminus \tilde{J}} (X) = \tilde{\Pi}_{[(\tilde{J}_1 \cap J_1) \times (\tilde{J}_2 \cap J_2)]\setminus \tilde{J}} (X).
\]

Therefore, (a) is upper-bounded by

\[
\|\tilde{\Pi}_{(\tilde{J}_1 \times \tilde{J}_2)\setminus \tilde{J}} (X)\|_F
\]

\[
= \|\tilde{\Pi}_{\tilde{J}} (X) - X\|_F
\]

\[
= \|\tilde{\Pi}_{\tilde{J}} [A^\ast (b) - (A^\ast A - \text{id})(X) - A^\ast (z)] - X\|_F
\]

\[
\leq \|\tilde{\Pi}_{\tilde{J}} [A^\ast (b)] - X\|_F + \|\tilde{\Pi}_{\tilde{J}} [(A^\ast A - \text{id})(X)]\|_F + \|\tilde{\Pi}_{\tilde{J}} [A^\ast (z)]\|_F
\]

\[
\leq 3 \delta \|X\|_F + 3 \sqrt{1 + \delta} \|z\|_2,
\]

where the last step follows by (65), (64), and the RIP of $A$ with $|\tilde{J}| \leq s_1 s_2$.

By applying these upper bounds back to (66), we get

\[
\|\hat{M} - X\|_F \leq 24 \delta \|X\|_F + 24 \sqrt{1 + \delta} \|z\|_2.
\]

Let $v^\text{Th}_0$ be the leading right singular vector of $M$. Then, by the non-Hermitian sin $\theta$ theorem [47] (Lemma C.1),

\[
\|P_{R(v)} \cdot P_{R(v^\text{Th})}\| \leq \frac{\|M - X\|_F}{\|X\|_F} \leq 24 \delta + 24 (1 + \delta) \nu,
\]

where the second step follows from (68) and (40).
Therefore, a sufficient for (14) is given by

$$24\delta + 24(1 + \delta)\nu < \sin \omega_{\sup},$$

which holds, for example, if $\delta \leq 0.02$ and $\nu \leq 0.02$.

C. Proof of Theorem III.12

In the row-sparse case ($s_2 = n_2$), the support estimate for $v$ is trivially given as $\widehat{J}_2 = [n_2]$. The assumptions of Theorem III.12 imply

$$\delta + \nu + \delta \nu < \min\left(\frac{\sin \omega_{\sup} \|u\|_\infty, \|u\|_\infty}{1 + \sin \omega_{\sup}}, \frac{\|u\|_\infty}{2}\right)$$

(69)

Let $j_0 \in [n_1]$ denote the index of the largest entry of $u$ in magnitude, i.e.,

$$j_0 \triangleq \arg\max_{j \in [n_1]} |u_j|.$$  

Then, $|u_{j_0}| = \|\Pi_{\{j_0\}}u\|_2 = \|u\|_\infty$.

Since (69) implies

$$2\delta + 2(1 + \delta)\nu < \|u\|_\infty,$$

(70)

by Lemma VIII.12 with $\widetilde{J}_1 = \{j_0\}$, it follows that $j_0 \in \widetilde{J}_1$; hence,

$$\|\Pi_{\widetilde{J}_1} u\|_2 \geq \|u\|_\infty.$$  

(71)

On the other hands, since $\Pi_{\widetilde{J}_2} v = 0$, by Lemma VIII.10, we get a sufficient condition for (14) given by

$$\delta + (1 + \delta)\nu < \frac{\sin \omega_{\sup} \|\Pi_{\widetilde{J}_1} u\|_2}{1 + \sin \omega_{\sup}}.$$  

(72)

Finally, note that (69) also implies (72) since (71) holds.

D. Proof of Theorem IV.5

The proof will use the following lemma.

Lemma VIII.13 (Iterative update with subspace concatenation). Suppose the hypotheses of Theorem IV.5
hold. Then, for all \( t \geq 1 \),
\[
\frac{\|P_{\mathcal{R}(U_t)}^*P_{\mathcal{R}(U)}\|_F}{\sqrt{r}} \leq \frac{2kC^{\text{HTP}}_\delta}{\sqrt{1 - \|P_{\mathcal{R}(U_0)}^*P_{\mathcal{R}(U)}\|^2}} \left[ \frac{\delta \|P_{\mathcal{R}(V_{t-1})}^*P_{\mathcal{R}(V)}\|_F}{\sqrt{r}} + \frac{(1 + \delta)\|z\|_2}{\|A(X)\|_2} \right]
\]
and
\[
\frac{\|P_{\mathcal{R}(V_t)}^*P_{\mathcal{R}(V)}\|_F}{\sqrt{r}} \leq \frac{2kC^{\text{HTP}}_\delta}{\sqrt{1 - \|P_{\mathcal{R}(U_0)}^*P_{\mathcal{R}(U)}\|^2}} \left[ \frac{\delta \|P_{\mathcal{R}(U_{t-1})}^*P_{\mathcal{R}(U)}\|_F}{\sqrt{r}} + \frac{(1 + \delta)\|z\|_2}{\|A(X)\|_2} \right] .
\]

**Proof of Lemma VIII.13:** We provide the proof only for the first part. The proof for the second part follows by symmetry.

Let \( \tilde{V} \in \mathbb{C}^{n_2 \times r} \) denote an orthonormal basis for the subspace spanned by \( V_{t-1} \), where \( V_{t-1} \) is the estimate of \( V \) obtained in the previous iteration. Let \( V_0 \in \mathbb{C}^{n_2 \times r} \) denote the initial estimate of \( V \). Note that \( V_0 \) and \( V \) were respectively obtained as singular vectors of a certain matrix. Then we have \( \tilde{V}^*\tilde{V} = V_0^*V_0 = I_r \).

The algorithm uses the subspace spanned by \( \tilde{V} \) and does not depend on a specific choice of an orthonormal basis. Our proof will use \( \tilde{V} \) constructed as follow. Let \( Q \) be a matrix representing an orthonormal basis of \( P_{\mathcal{R}(\tilde{V})^*\mathcal{R}(V_0)} \). Let \( \tilde{V} = [\tilde{V}, Q] \). Then, \( \tilde{V} \) satisfies
\[
\mathcal{R}(\tilde{V}) = \mathcal{R}(\tilde{V}) + \mathcal{R}(V_0)
\]
and
\[
\tilde{V}e_k = \tilde{V}e_k, \quad \forall k = 1, \ldots, r.
\]

Let \( \tilde{r} \) denote the rank of \( \tilde{V} \). Then, \( r \leq \tilde{r} \leq 2r \).

Since \( \tilde{V}^*\tilde{V} \) is an identity matrix, the projection operator \( P_{\mathcal{R}(\tilde{V})} \) is expressed as
\[
P_{\mathcal{R}(\tilde{V})} = \tilde{V}(\tilde{V}^*\tilde{V})^{-1}\tilde{V}^* = \tilde{V}\tilde{V}^* .
\]
The measurement vector \( b \) is then written as

\[
b = A(U\Lambda V^*) + z
\]

\[
= A[U\Lambda V^*(P_{R(\tilde{v})} + P_{\perp R(\tilde{v})})] + z
\]

\[
= A(U\Lambda V^*P_{R(\tilde{v})}) + A(U\Lambda V^*P_{\perp R(\tilde{v})}) + z
\]

\[
= A(U\Lambda V^*\tilde{V}^*) + A(U\Lambda V^*P_{\perp R(\tilde{v})}) + z
\]

\[
= [\mathcal{F}(\tilde{V})](UAV^*\tilde{V}) + [\mathcal{F}(P_{\perp R(\tilde{v})}V)](U\Lambda) + z.
\]

In the scenario \( s_1 < n_1, \tilde{U} \) is obtained by B-HTP with the sensing linear operator \( \mathcal{F}(\tilde{V}) : \mathbb{C}^{n_1 \times \tilde{r}} \rightarrow \mathbb{C}^m \). The performance guarantee for HTP in Lemma VIII.1 extends to the case of B-HTP. The derivation is done in a straightforward way that involves replacing a few symbols. By this RIP-guarantee for B-HTP, the error in the estimate \( \tilde{U} \) of \( U\Lambda V^* \tilde{V} \) is upper-bounded by

\[
\| \tilde{U} - U\Lambda V^* \tilde{V} \|_F \leq C_{\delta}^{\text{HTP}} |\langle W, [\mathcal{F}(\tilde{V})]^* [A(U\Lambda V^*P_{\perp R(\tilde{v})}) + z] \rangle | \tag{73}
\]

for some \( W \in \mathbb{C}^{n_1 \times \tilde{r}} \) such that \( \| W \|_F = 1 \) and \( W \) is row \( s_1 \)-sparse. By applying the triangle inequality to the right-hand-side of (73), \( \| \tilde{U} - U\Lambda V^* \tilde{V} \|_F \) is further upper-bounded by

\[
\| \tilde{U} - U\Lambda V^* \tilde{V} \|_F \leq C_{\delta}^{\text{HTP}} |\langle W, [\mathcal{F}(\tilde{V})]^* A(U\Lambda V^*P_{\perp R(\tilde{v})}) \rangle | + C_{\delta}^{\text{HTP}} |\langle W, [\mathcal{F}(\tilde{V})]^* z \rangle |. \tag{74}
\]

Then the right-hand-side of (74) is upper-bounded as follows. Note that \( \langle W, [\mathcal{F}(\tilde{V})]^* A(U\Lambda V^*P_{\perp R(\tilde{v})}) \rangle \) is rewritten as

\[
\langle W, [\mathcal{F}(\tilde{V})]^* A(U\Lambda V^*P_{\perp R(\tilde{v})}) \rangle = \langle [\mathcal{F}(\tilde{V})]W, A(U\Lambda V^*P_{\perp R(\tilde{v})}) \rangle
\]

\[
= \langle A(W\tilde{V}^*), A(U\Lambda V^*P_{\perp R(\tilde{v})}) \rangle
\]

\[
= \langle W\tilde{V}^*, A^* A(U\Lambda V^*P_{\perp R(\tilde{v})}) \rangle
\]

\[
= \langle W\tilde{V}^*, (A^* A - \text{id})(U\Lambda V^*P_{\perp R(\tilde{v})}) \rangle,
\]

where the last step follows since \( \langle W\tilde{V}^*, U\Lambda V^*P_{\perp R(\tilde{v})} \rangle = 0 \). Thus, by the rank-2\( r \) and doubly \((3s_1, 3s_2)\)-
sparse RIP of $\mathcal{A}$, the first term in the right-hand-side of (74) is upper-bounded by

$$C^\text{HTP}_\delta |\langle W, [\mathcal{F}(\tilde{V})]^* \mathcal{A}(U\Lambda V^* P_{\mathcal{R}(V)}^\perp) \rangle| \leq C^\text{HTP}_\delta \|W\tilde{V}^*\|_F \|U\Lambda V^* P_{\mathcal{R}(V)}^\perp\|_F$$

$$\leq C^\text{HTP}_\delta \|A\|_F \|P_{\mathcal{R}(V)}^\perp P_{\mathcal{R}(V)}\|_F$$

$$\leq C^\text{HTP}_\delta \|A\|_F \|P_{\mathcal{R}(V)}^\perp P_{\mathcal{R}(V)}\|_F,$$

where the last step holds since $\mathcal{R}(\tilde{V}) \subset \mathcal{R}(V)$. On the other hand, since $\langle W, [\mathcal{F}(\tilde{V})]^* z \rangle$ is rewritten as

$$\langle W, [\mathcal{F}(\tilde{V})]^* z \rangle = \langle [\mathcal{F}(\tilde{V})] W, z \rangle = \langle \mathcal{A}(W\tilde{V}^*), z \rangle,$$

by the RIP of $\mathcal{A}$ together with the fact that $W$ is row-$s_1$-sparse and $\tilde{V}$ is row-$2s_2$-sparse, the second term in the right-hand-side of (74) is upper-bounded by

$$C^\text{HTP}_\delta |\langle W, [\mathcal{F}(\tilde{V})]^* z \rangle| \leq C^\text{HTP}_\delta \|\mathcal{A}(W\tilde{V}^*)\|_2 \|z\|_2 \leq C^\text{HTP}_\delta \sqrt{1 + \delta} \|W\tilde{V}^*\|_F \|z\|_2 \leq C^\text{HTP}_\delta \sqrt{1 + \delta} \|z\|_2.$$

Therefore, (73) implies

$$\|\tilde{U} - U\Lambda V^* \tilde{V}\|_F \leq C^\text{HTP}_\delta \left( \|A\|_F \|P_{\mathcal{R}(V)}^\perp P_{\mathcal{R}(V)}\|_F + \sqrt{1 + \delta} \|z\|_2 \right). \quad (75)$$

On the other scenario $s_1 = n_1$, matrix $\tilde{U}$ is obtained by

$$\tilde{U} = [\mathcal{F}(\tilde{V})]^b$$

$$= [\mathcal{F}(\tilde{V})]^\dagger \left([\mathcal{F}(\tilde{V})](U\Lambda V^*) + [\mathcal{F}(P_{\mathcal{R}(V)}^\perp V)](U\Lambda) + z \right)$$

$$= U\Lambda V^* \tilde{V} + [\mathcal{F}(\tilde{V})]^\dagger [\mathcal{F}(P_{\mathcal{R}(V)}^\perp V)](U\Lambda) + [\mathcal{F}(\tilde{V})]^\dagger z$$

Then, by Lemma B.4, we have

$$\|\tilde{U} - U\Lambda V^* \tilde{V}\|_F \leq \frac{1}{1 - \delta} \left( \|A\|_F \|P_{\mathcal{R}(V)}^\perp P_{\mathcal{R}(V)}\|_F + \sqrt{1 + \delta} \|z\|_2 \right). \quad (76)$$

Since $C^\text{HTP}_\delta \geq \frac{1}{1 - \delta}$, (76) implies (75). Therefore, we will use (75) regardless of whether $s_1 < n_1$ or not.

Next, we derive a lower bound on the left-hand-side of (75). Let $\tilde{U} \in \mathbb{C}^{n_1 \times r}$ be a matrix with the first $r$ left singular vectors of $\tilde{U}$. Then, $\tilde{U}\tilde{U}^* \tilde{U}$ is the best rank-$r$ approximation of $\tilde{U}$ and by the optimality we have

$$\|\tilde{U}\tilde{U}^* \tilde{U} - U\Lambda V^* \tilde{V}\|_F \leq \|\tilde{U}\tilde{U}^* \tilde{U} - \tilde{U}\|_F + \|\tilde{U} - U\Lambda V^* \tilde{V}\|_F \leq 2\|\tilde{U} - U\Lambda V^* \tilde{V}\|_F. \quad (77)$$
On the other hand, since $\tilde{U}\tilde{U}^*\tilde{U}$ is spanned by $\tilde{U}$, we have

$$
\|\tilde{U}\tilde{U}^*\tilde{U} - U\Lambda V^*\tilde{V}\|_F \geq \|P_{R(\tilde{U})}^* U\Lambda V^*\tilde{V}\|_F
$$
$$
= \|P_{R(\tilde{U})}^* U\Lambda V^*\tilde{V}\|_F
$$
$$
= \|P_{R(\tilde{U})}^* U\Lambda V^*P_{R(V)}\|_F
$$
$$
\geq \|P_{R(\tilde{U})}^* U\Lambda V^*P_{R(V_0)}\|_F
$$

\[ (78) \]

where the second inequality holds since $R(V_0) \subset R(\tilde{V})$.

By combining (75), (77) and (78), we get

$$
\|P_{R(\tilde{U})}^* P_{R(U)}\|_F \leq 2C_\delta^{\text{HTP}} \left( \frac{\kappa\delta\|P_{R(V)}^* P_{R(V)}\|_F}{\sqrt{1 - \|P_{R(V_0)} P_{R(V)}\|^2}} + \frac{\sqrt{1 + \delta}\|z\|_2}{\sigma_r(\Lambda)\sqrt{1 - \|P_{R(V_0)} P_{R(V)}\|^2}} \right).
$$

\[ (79) \]

Finally, note that

$$
\sqrt{\tau}\sigma_r(\Lambda) = \frac{\sqrt{\tau}\|A\|}{\kappa} \geq \frac{\|X\|_F}{\kappa}.
$$

\[ (80) \]

Applying (80) to (79) completes the proof.

Define

$$
\rho \triangleq \tilde{C}_\delta \cdot \delta \quad \text{and} \quad \tau \triangleq \tilde{C}_\delta \cdot (1 + \delta),
$$

where

$$
\tilde{C}_\delta \triangleq \max \left\{ \frac{2\kappa C_\delta^{\text{HTP}}}{\sqrt{1 - \|P_{R(V_0)} P_{R(V)}\|^2}}, \frac{2\kappa C_\delta^{\text{HTP}}}{\sqrt{1 - \|P_{R(V_0)} P_{R(V)}\|^2}} \right\}.
$$

Then, by Lemma VIII.13, we have

$$
\|P_{R(U_0)} P_{R(U)}\|_F \leq \frac{\rho}{\sqrt{\tau}} \|P_{R(V_0)} P_{R(V)}\|_F + \tau \frac{\|z\|_2}{\|A(X)\|_2}
$$

\[ (81) \]

and

$$
\|P_{R(V_0)} P_{R(V)}\|_F \leq \frac{\rho}{\sqrt{\tau}} \|P_{R(U_0)} P_{R(U)}\|_F + \tau \frac{\|z\|_2}{\|A(X)\|_2}
$$

\[ (82) \]
for all \( t \geq 1 \). Furthermore, by the choice of \( \delta = \frac{0.01}{\kappa} \) and (22), we have \( 0 < \rho < 1 \). Therefore, from the alternating recursion formula in (81) and (82), the left-hand-side of (81) converges linearly as

\[
\limsup_{t \to \infty} \frac{\|P_{\mathcal{R}(U_t)} + P_{\mathcal{R}(U_t)}\|_F}{\sqrt{r}} \leq \frac{\tau}{1 - \rho} \frac{\|z\|_2}{\|A(X)\|_2}. \tag{83}
\]

It remains to bound the error term \( \|UAV^* - U_t V_t^*\|_F \) in terms of \( \|P_{\mathcal{R}(U_t)} + P_{\mathcal{R}(U_t)}\|_F \). Without loss of generality, we may assume that \( U_t^* U_t = I_r \). (Indeed, the update of \( V_t \) depends on \( U_t \) only in terms of the subspace spanned by it.) Note that the estimation error is decomposed as

\[
UAV^* - U_t V_t^* = P_{\mathcal{R}(U_t)} UAV^* - U_t V_t^* + P_{\mathcal{R}(U_t)} UAV^*
= U_t [(U_t^* U)AV^* - V_t^*] + P_{\mathcal{R}(U_t)} UAV^*.
\]

Furthermore, it follows from the RIP-guarantee of B-HTP that \( V_t \) satisfies

\[
\|V_t^* - U_t^* UAV^*\|_F \leq 2C_{\delta}^{\text{HTP}} \left[ \delta \|A\| \|P_{\mathcal{R}(U_t)} U\|_F \sqrt{1 + \delta \|z\|_2} \right] \leq 2C_{\delta}^{\text{HTP}} \left[ \delta \|A\| \|P_{\mathcal{R}(U_t)} U\|_F + \|X\|_F (1 + \delta \nu) \right].
\]

Therefore,

\[
\|UAV^* - U_t V_t^*\|^2_F = \|P_{\mathcal{R}(U_t)} (UAV^* - U_t V_t^*)\|^2_F + \|P_{\mathcal{R}(U_t)} (UAV^* - U_t V_t^*)\|^2_F
= \|U_t [(U_t^* U)AV^* - V_t^*]\|^2_F + \|P_{\mathcal{R}(U_t)} UAV^*\|^2_F
\leq \|U_t^* UAV^* - V_t^*\|^2_F + \|A\|^2 \|P_{\mathcal{R}(U_t)} U\|^2_F
\leq \|A\|^2 \left[ 1 + 2(2\delta C_{\delta}^{\text{HTP}})^2 \right] \|P_{\mathcal{R}(U_t)} P_{\mathcal{R}(U_t)}\|^2_F + 2 \left[ \|X\|_F (1 + \delta) C_{\delta}^{\text{HTP}} \nu \right]^2,
\]

which implies

\[
\frac{\|X - U_t V_t^*\|_F}{\|X\|_F} \leq \kappa \sqrt{1 + 8(\delta C_{\delta}^{\text{HTP}})^2} \frac{\|P_{\mathcal{R}(U_t)} P_{\mathcal{R}(U_t)}\|_F}{\sqrt{r}} + \sqrt{2(1 + \delta) C_{\delta}^{\text{HTP}}} \frac{\|z\|_2}{\|A(X)\|_2}. \tag{84}
\]

Finally, by applying (83) to (84), we obtain

\[
\limsup_{t \to \infty} \frac{\|X - U_t V_t^*\|_F}{\|X\|_F} \leq \frac{\tau \kappa}{1 - \rho} \sqrt{1 + 8(\delta C_{\delta}^{\text{HTP}})^2} + \sqrt{2(1 + \delta) C_{\delta}^{\text{HTP}}} \frac{\|z\|_2}{\|A(X)\|_2},
\]

where the factor \( C' \) is no greater than \( 55\kappa^2 + 3\kappa + 3 \) under the hypotheses of Theorem IV.5. This completes the proof.
E. Proof of Theorem IV.6

It suffices to show that the condition in (22) is satisfied by the initial estimates \( U_0 = U_0^\text{Th} \) and \( V_0 = V_0^\text{Th} \), where \( U_0 \) and \( V_0 \) are generated respectively by Algorithms 7 and 6. Then, the result follows from Theorem IV.5. Specifically, we need to show that \( V_0^\text{Th} \) satisfies

\[
\| P_{R(V)} P_{R(V_0^\text{Th})}^\perp \| < 0.95, \tag{85}
\]

and \( U_0^\text{Th} \) satisfies

\[
\| P_{R(U)} P_{R(U_0^\text{Th})}^\perp \| < 0.95. \tag{86}
\]

We present the proof only for (85). The proof for (86) follows by symmetry.

The first step of our proof is to show that the support-estimate \( \widehat{J}_1 \) (resp. \( \widehat{J}_2 \)) in Algorithm 6 includes \( \tilde{J}_1 \) (resp. \( \tilde{J}_2 \)), which are defined in Theorem IV.1. The following lemmas provide a sufficient condition for the containment.

**Lemma VIII.14.** Suppose the hypotheses of Theorem IV.6 hold. Let \( J_1 \subset [n_1] \) denote the set of the indices of the nonzero rows of \( U \). Let \( \tilde{J}_1 \subset J_1 \) be defined in (19). If

\[
2\kappa (\delta + (1 + \delta)\nu) < \sigma_r(U^* \Pi \tilde{J}_1),
\]

then \( \tilde{J}_1 \subset \widehat{J}_1 \).

**Proof of Lemma VIII.14:** See Appendix F.

**Lemma VIII.15.** Suppose the hypotheses of Theorem IV.6 hold. Let \( J_2 \subset [n_2] \) denote the set of the indices of the nonzero rows of \( V \). Let \( \tilde{J}_2 \) be a subset of \( J_2 \). If

\[
2\kappa (\delta + (1 + \delta)\nu) < \sigma_r(U^* \Pi \tilde{J}_1) \sigma_r(V^* \Pi \tilde{J}_2),
\]

then \( \tilde{J}_2 \subset \widehat{J}_2 \).

**Proof of Lemma VIII.15:** See Appendix G.

It is straightforward to verify that (87) and (88) are satisfied under the hypotheses of Theorem IV.6.

Next, from the results by Lemmas VIII.14 and VIII.15, we derive an upper bound on \( \| P_{R(V_0)} P_{R(V)}^\perp \| \) using the \( \sin \theta \) theorem, which is summarized in the following lemma.

**Lemma VIII.16.** Suppose the hypotheses of Theorem IV.6 hold. Let \( \tilde{J}_1 \subset [n_1] \) and \( \tilde{J}_2 \subset [n_2] \) satisfy...
\( |\hat{J}_1| \leq s_1 \) and \( |\hat{J}_2| \leq s_2 \), respectively. Let \( V_0 \) be a matrix whose columns are the \( r \) leading singular vectors of \( \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \). Then,

\[
\|P_{R(V_0)} - P_{R(V)}\| \leq \frac{\delta + \sqrt{1 - \sigma_r^2(V^* \Pi_{\hat{J}_2})} + (1 + \delta)\nu}{\sigma_r(\Pi_{\hat{J}_1} U)/\kappa - \delta - (1 + \delta)\nu}.
\]  

(89)

**Proof of Lemma VIII.16:** See Appendix H.

We verify that (85) is obtained from the upper bound in Lemma VIII.16 under the hypotheses of Theorem IV.6.

Finally, in the case when \( s_1 = n_1 \) and \( s_2 = n_2 \), we have \( \hat{J}_1 = [n_1] \) and \( \hat{J}_2 = [n_2] \). These support estimates are trivially obtained without requiring that \( \sigma_r(V^* \Pi_{\hat{J}_2}) \) and \( \sigma_r(\Pi_{\hat{J}_1} U) \) exceed a constant. Furthermore, it follows that \( \sigma_r(V^* \Pi_{\hat{J}_2}) = 1 \) and \( \sigma_r(\Pi_{\hat{J}_1} U) = 1 \). Thus (89) reduces to

\[
\|P_{R(V_0)} - P_{R(V)}\| \leq \frac{\delta + (1 + \delta)\nu}{1/\kappa - \delta - (1 + \delta)\nu}.
\]  

(90)

Recall that it was assumed that \( \delta = \frac{0.04}{\kappa} \) and \( \nu = \frac{0.04}{\kappa} \) in Theorem IV.6. Then the upper bound in the right-hand-side of (90) is less than 0.95 for any \( \kappa \geq 1 \). Therefore, the condition \( \kappa \leq 4 \) is not necessary in this case. This completes the proof.

**F. Proof of Theorem V.1**

The proof of the rate-distortion lower bound for rank-\( r \) matrix in Theorem V.1 relies on two auxiliary results, which we present first:

(a) Theorem VIII.17: a rate-distortion lower bound for a \( r \)-dimensional uniform random subspace in \( \mathbb{C}^n \), where the distortion measure is the squared distance between subspaces, more precisely, the Frobenius norm between the respective projection matrices;

(b) Theorem VIII.18: a rate-distortion lower bound for a complex orthogonal matrix uniformly distributed with respect to the quadratic distortion.

Both results are tight asymptotically (see Remark VIII.19); however, the main point here is a non-asymptotic bound.

**Theorem VIII.17.** Let \( V \) be uniformly distributed on the \( r \)-dimensional complex Stiefel manifold \( V(\mathbb{C}^n, r) \).
There exists a universal constant \(c > 0\) such that for any \(n \geq 2, r \in [n]\) and \(D > 0\),

\[
\inf_{P_{\hat{V}|VV^*}} \{I(VV^*; \hat{V}\hat{V}^*) : E\|V V^* - \hat{V}\hat{V}^*\|_F^2 \leq rD\} \geq (n - r)r \log \frac{c}{D},
\]

where the infimum is taken over all probability transition kernels such that \(\hat{V} \in V(\mathbb{C}^n, r)\).

Theorem VIII.18. Let \(T\) be uniformly distributed on the orthogonal group \(O(r)\) in \(\mathbb{C}^r\). There exists a universal constant \(c' > 0\) such that for any \(r \in \mathbb{N}\) and \(D > 0\),

\[
\inf_{P_{\hat{T}|T}} \{I(T; \hat{T}) : E\|T - \hat{T}\|_F^2 \leq rD\} \geq \frac{r^2}{2} \log \frac{c'}{D},
\]

where the infimum is taken over all probability transition kernels from \(O(r)\) to itself.

Remark VIII.19. The lower bound in Theorem VIII.17 is in fact sharp within constant factors and, in addition, asymptotically sharp in the low-distortion regime, and the rate-distortion function on the left-hand side of (91) is in fact \((n - r)r \log \frac{c}{D})(1 + o(1))\) as \(D \to 0\), since a matching upper bound can be obtained by quantization and using the covering number bound of the Grassmannian manifold [48].

The pre-log factor \((n - r)r\) in (91) deserves a careful explanation. We first recall that the low-distortion asymptotics of the rate-distortion function obtained in [49] for mean-square distortion:

\[
R_X(D) = \frac{d(X)}{2} \log \frac{1}{D}(1 + o(1)),
\]

(93)

where \(d(X)\) is the information dimension of the random vector \(X\) [50], which, in the absolute continuous case, coincides with the (real) topological dimension of the support of \(X\). As mentioned in Remark V.3, the number of free (real) variables in \(V\) is \(2nr - r^2\). Furthermore, the loss function \(\|VV^* - \hat{V}\hat{V}^*\|\) corresponds to the subspace distance which is rotationally invariant. This further reduces the number of free variables by the degrees of freedom of the orthogonal group \(O(r)\) in \(\mathbb{C}^r\), which is \(\sum_{i=1}^r (2r - 2i + 1) = r^2\). Therefore under the subspace distance distortion metric, the effective number of degrees of freedom in \(V\) is \(2r(n - r)\), and the corresponding rate-distortion behavior (91) is consistent with the information dimension characterization (93).

Similarly, Theorem VIII.18 is tight when \(D \to 0\), which is met by the covering number bound of the orthogonal group [48].

To prove Theorem VIII.17 first we need an auxiliary result from linear algebra.
Lemma VIII.20. Let $O(r)$ be the set of $r \times r$ complex orthogonal matrices. Then for any $V, \hat{V} \in V(\mathbb{C}^n, r)$, there exists $A \in O(r)$, such that

$$
\|V - \hat{V} A\|_F \leq \|VV^* - \hat{V} \hat{V}^*\|_F.
$$

(94)

Proof:

$$
\|VV^* - \hat{V} \hat{V}^*\|_F^2 = 2 \sum_{i=1}^{r} \sin^2 \theta_i = 2r - 2 \sum_{i=1}^{r} \cos^2 \theta_i
$$

$$
= 2r - 2 \sum_{i=1}^{r} (\text{Re}\langle u_i, \hat{u}_i \rangle)^2
$$

where $\theta_1, \ldots, \theta_r$ are the principal angles between subspaces $\text{span}(V), \text{span}(\hat{V})$, and $U = [u_1, \ldots, u_r], \hat{U} = [\hat{u}_1, \ldots, \hat{u}_r]$, where $u_i, \hat{u}_i$ are the corresponding principal vectors. Note that $U = VR, \hat{U} = \hat{V} \hat{R}$, for some $R, \hat{R} \in O(r)$. Then

$$
\|U - \hat{U}\|_F^2 = 2r - 2 \text{Re}\langle U, \hat{U} \rangle = 2r - 2 \sum_{i=1}^{r} \text{Re}\langle u_i, \hat{u}_i \rangle = 2r - 2 \sum_{i=1}^{r} \cos \theta_i.
$$

(95)

Since $0 \leq \cos \theta_i \leq 1$, $\|VV^* - \hat{V} \hat{V}^*\|_F^2 \geq \|U - \hat{U}\|_F^2$. Moreover,

$$
\|U - \hat{U}\|_F^2 = \|VR - \hat{V} \hat{R}\|_F^2 = \|V - \hat{V} \hat{R} R^*\|_F^2,
$$

(96)

thus by choosing $A = \hat{R} R^*$, $\|V - \hat{V} A\|_F \leq \|VV^* - \hat{V} \hat{V}^*\|_F$. $\square$

Proof of Theorem VIII.17: When $r = n$, the lower bound is automatically true (and tight since the subspace has no randomness). Henceforth we assume that $r \leq n - 1$. Also, it is sufficient to consider $D < 1$. Under the mean-square distortion, the rate-distortion function for $CN(0, 1)$ is given by

$$
R(D) = \log^+ \frac{1}{D}, \quad D > 0,
$$

(97)

where $\log^+ \triangleq \max\{\log, 0\}$. Let $W$ be an $n \times r$ random matrix with i.i.d. entries drawn from $CN(0, 1)$. As a consequence of (97), for any $a > 0$,

$$
\inf_{P_{W|\hat{W}}} \{I(W; \hat{W}) : \mathbb{E}\|\hat{W} - W\|_F^2 \leq n r a\} \geq n r \log^+ \frac{1}{a}.
$$

(98)

Let $W = V R$ be the QR decomposition of $W$, where $V \in V(\mathbb{C}^n, r)$ and $R$ is a $r \times r$ upper triangular matrix with real-valued diagonals and complex-valued off-diagonals. Since the law of $W$ is left rotationally
invariant, \( V \) and \( R \) are independent. Moreover, \( V \) is Haar distributed; the entries of \( R \) are independent, where the off-diagonals \( \{ R_{ij} : i < j \} \) are standard complex Gaussian and \( 2R^2_{ii} \) are independent \( \chi^2_{2(n-i+1)} \) for \( i \in [r] \) (see, e.g., [51, Lemma 2.1]).

The main idea of obtaining the lower bound (91) is to combine a given compressor of the column subspace of \( W \) together with that of the matrix \( R \) to yield a lossy compressor of the Gaussian matrix \( W \), and the overall performance must obey the rate-distortion function of \( W \). To implement this program, fix \( 0 < D < 1 \) and fix any probability transition kernel \( P_{\tilde{V}\cdot|V\cdot} \) such that \( \tilde{V} \in V(\mathbb{C}^n, r) \) and \( \mathbb{E}\|VV^* - \tilde{V}\tilde{V}^*\|^2_F \leq Dr \). By Lemma VIII.20, there exists \( A = A(V, \tilde{V}) \in O(r) \), such that \( \|V - \tilde{V}A\|_F \leq \|VV^* - \tilde{V}\tilde{V}^*\|_F \). Next we use the metric entropy bound for the orthogonal group to produce a quantized version of \( A \). By [48] (see also [52]), there exists a universal constant \( c_0 > 1 \), such that the covering number of \( O(r) \) with respect to \( \|\cdot\|_F \) is at most \( (\frac{\sqrt{n}}{\epsilon})^{r^2/2} \) for any \( r \in \mathbb{N} \) and any \( \epsilon \in (0, \sqrt{r}) \), where the \( r^2 \) is the (real) topological dimension of \( O(r) \) and \( \sqrt{r} \) is the diameter. Therefore for \( \epsilon = \sqrt{D} \), there exists \( T_1, \ldots, T_m \in O(r) \) with \( m \leq (\frac{\sqrt{n}}{\epsilon})^{r^2/2} \), that constitute an \( \epsilon \)-covering of \( O(r) \), namely, for any \( S \in O(r) \), there exists \( i = i(S) \in [m] \) such that \( \|S - T_i(S)\|_F \leq \epsilon \). Let \( A_m = T_i(A) \) denote the closest \( T_i \) to \( A \). Then \( \|A_m - A\|_F \leq \sqrt{D} \). Set \( \tilde{V} = \tilde{V}A_m \). Then

\[
\|V - \tilde{V}\|_F \leq \|V - \tilde{V}A\|_F + \|\tilde{V}(A - A_m)\|_F \leq \|V - \tilde{V}A\|_F + \sqrt{D}
\]

and hence

\[
\mathbb{E}\|V - \tilde{V}\|^2_F \leq 2\mathbb{E}\|V - \tilde{V}A\|^2_F + 2\epsilon D \leq 2\mathbb{E}\|VV^* - \tilde{V}\tilde{V}^*\|^2_F + 2\epsilon D \leq 4\epsilon D.
\] (99)

Next we use \( P_{\tilde{V}|V} \) to design a lossy compressor for the standard Gaussian matrix \( W \). Fix a transition kernel \( P_{\tilde{R}|R} \) to be specified later and set \( \tilde{W} = \tilde{V}\tilde{R} \). The dependence diagram for all random variables is as follows:

\[
\begin{array}{cccc}
R & \rightarrow & \tilde{R} & \rightarrow & \tilde{W} \\
V & \rightarrow & \tilde{V} & \rightarrow & \tilde{V} \\
\downarrow & & \downarrow & & \downarrow \\
A & \rightarrow & A_m & \rightarrow & A_m
\end{array}
\]
which implies that

\[
I(W; \bar{W}) = I(V, R; \bar{V}, \bar{R}) = I(R; \bar{R}) + I(V; \bar{V}) \leq I(R; \bar{R}) + I(V; \bar{V}, A_m)
\]

\[
\leq I(R; \bar{R}) + I(V; \bar{V}) + H(A_m)
\]

\[
\leq I(R; \bar{R}) + I(V; \bar{V}) + \frac{r^2}{2} \log \frac{c_0}{D},
\]

(100)

where the last step follows from the fact that \( A_m \) is a random variable which takes no more than \( m \) values, hence \( H(A_m) \leq \log m \leq \frac{r^2}{2} \log \frac{c_0}{D} \).

Note that for each \( i \in [r] \), \( \mathbb{E} R_{ii}^2 = 2m \) and \( \mathbb{E} R_{ii} = \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m)} \), where \( m = n - i + 1 \geq 2 \) by assumption that \( r < n \). Since \( \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m)^{m-1}} \geq 1 \) for any \( m \geq 2 \), we have \( \text{var} R_{ii} \leq 1 \). By [53, Theorem 4.3.3], the rate-distortion function of \( R_{ii} \) is majorized by that of the standard normal distribution, i.e.,

\[
\min_{P_{R_{ii}}} I(R_{ii}; \tilde{R}_{ii}) \leq \frac{1}{2} \log \frac{1}{D}, \quad D > 0.
\]

For the off-diagonal \( R_{ij} \) which is independent standard complex normal, by (97), there exists \( P_{\tilde{R}_{ij}} \) such that \( \mathbb{E} |\tilde{R}_{ij} - R_{ij}|^2 \leq D \) and \( I(R_{ij}; \tilde{R}_{ij}) = \log^+ \frac{1}{D} \). Let \( P_{\tilde{R}} = \prod_{i \leq j} P_{\tilde{R}_{ij}} \). Then

\[
\mathbb{E} \| \tilde{R} - R \|^2_F \leq \frac{r(r+1)D}{2},
\]

(101)

and

\[
I(R; \tilde{R}) = \sum_{1 \leq i \leq j \leq r} I(R_{ij}; \tilde{R}_{ij}) \leq \frac{r(r-1)}{2} + \frac{r}{2} \log^+ \frac{1}{D} = \frac{r^2}{2} \log^+ \frac{1}{D}.
\]

(102)

To bound the overall distortion for \( W \), note that

\[
\| W - \bar{W} \|^2_F \leq 2 \| (V - \bar{V}) R \|^2_F + 2 \| V(R - \bar{R}) \|^2_F
\]

\[
\leq 2 \| V - \bar{V} \|^2_F \| R \|^2 + 2 \| R - \tilde{R} \|^2_F.
\]

(103)

(104)

Note that \( \| R \| = \| W \| \) is the largest singular value of an \( n \times r \) standard complex Gaussian matrix. It is well-known that\(^6\) \( \mathbb{E} \| R \|^2 \leq c_1 n \) for some absolute constant \( c_1 \). Since \( R \) is independent of \( \{V, \bar{V}\} \), in view of (99) and (101) we have

\[
\mathbb{E} \| W - \bar{W} \|^2_F \leq 2 \mathbb{E} \| V - \bar{V} \|^2_F \mathbb{E} \| R \|^2 + 2 \mathbb{E} \| R - \tilde{R} \|^2_F \leq 8c_1 nrD + r(r+1)D \leq c_2 nrD
\]

(105)

\(^6\)This follows from Gordon’s inequality and Davidson-Szarek theorem cf. e.g., [54, Theorems 2.6 and 2.11]
Similarly, define the other hand, where we used Cauchy-Schwarz inequality and for that
where the second inequality follows analogously. Note that to introduce the rotation matrix
\( \sqrt{I} \) then similar to (105), we have
\[ \|W - \hat{W}\|_F^2 \leq 2\mathbb{E}\|R - \hat{R}\|_F^2 + 2\mathbb{E}\|T - \hat{T}\|_F^2 \mathbb{E}\|R\|_F^2 \leq C r^2 D \]
for some absolute constant \( C \). Similar to (100), we have \( I(W; \hat{W}) \leq I(T; \hat{T}) + I(R; \hat{R}) \). Then the desired lower bound (92) on \( I(T; \hat{T}) \) follows from (102) and (98) with \( c' = 1/C \).

Now we are ready to prove the main lower bound for robust reconstruction of rank-\( r \) matrices:

**Proof of Theorem VIII.18:** The proof follows the same course of proving Theorem VIII.18 (with \( n = r \)) and is even simpler, since the loss function is the usual Frobenius norm and hence there is no need to introduce the rotation matrix \( A \). Recall that \( W \) is a \( r \times r \) complex Gaussian matrix and \( W = TR \) be its QR decomposition. Given any feasible \( P_{T|T} \) in (92), let \( \hat{W} = \hat{T}\hat{R} \), where \( \hat{R} \) fulfills (102) and (101). Then similar to (103)-(105), we have
\[ \mathbb{E}\|W - \hat{W}\|_F^2 \leq 2\mathbb{E}\|R - \hat{R}\|_F^2 + 2\mathbb{E}\|T - \hat{T}\|_F^2 \mathbb{E}\|R\|_F^2 \leq C r^2 D \]
for some absolute constant \( C \). This follows from (102) and (98) with \( c' = 1/C \).

Now we are ready to prove the main lower bound for robust reconstruction of rank-\( r \) matrices:

**Proof of Theorem VI.1:** As in Fig. 1, let \( \hat{X} = \hat{U}\hat{V}^* = g(f(UV^*) + Z) \) be the reconstruction, where \( U, \hat{U} \in \mathbb{C}^{n_1, r} \) and \( V, \hat{V} \in \mathbb{C}^{n_2, r} \). Note that
\[ \|UV^* - \hat{U}\hat{V}^*\|_F^2 = 2r^2 - 2\text{Re}(\hat{U}^*U, \hat{V}^*V) \geq 2r^2 - 2\|\hat{U}^*U\|_F \|\hat{V}^*V\|_F \geq 2r^2 - 2\|\hat{U}^*U\|_F , \]
(106)
where we used Cauchy-Schwarz inequality and \( \|V^*V\|_F^2 = \langle \hat{V}^*V, \hat{V}V^* \rangle \leq \|V^*V\|_F \|V^*V\|_F = r \). On the other hand, \( \|UU^* - \hat{U}\hat{U}^*\|_F^2 = 2r^2 - 2\|\hat{U}^*U\|_F^2 \). Substituting into (106) and using the fact that \( \sqrt{1 - \frac{1}{2}} \leq 1 - x/2 \) for all \( 0 \leq x \leq 1 \), we obtain
\[ \|UU^* - \hat{U}\hat{U}^*\|_F^2 \leq 2\|UU^* - \hat{U}\hat{V}^*\|_F^2 \]
By the assumption that \( \mathbb{E}\|UU^* - \hat{U}\hat{V}^*\|_F^2 \leq rD \), we have
\[ \mathbb{E}\|UU^* - \hat{U}\hat{U}^*\|_F^2 \leq 2rD, \quad \mathbb{E}\|VV^* - \hat{V}\hat{V}^*\|_F^2 \leq 2rD, \]
(107)
where the second inequality follows analogously. Note that \( UU^* \) is in one-to-one correspondence to the column span of \( X \). Furthermore, given \( UU^* \), let \( \hat{U} \in \mathbb{C}^{n_1, r} \) be an arbitrary basis of the column span, so that \( \hat{U}\hat{U}^* = UU^* \). In other words, \( \hat{U} \) is a deterministic function of \( UU^* \). Similarly, define \( \hat{V} \) as a function of \( VV^* \).

\(^7\)To be definitive, one can obtain \( \hat{U} = [\hat{u}_1, \ldots, \hat{u}_r] \) recursively as follows: let \( \hat{u}_1 \) be the normalized first column of \( UU^* \), and for \( i = 2, \ldots, r \) then let \( \hat{u}_i \) be the normalized first nonzero column of \( P_{E_i^+}UU^*P_{E_i^+} \), where \( E_i = \text{span}(u_1, \ldots, u_{i-1}) \).
In view of the Markov chain \( X \to Y \to \hat{Y} \to \hat{X} \), we obtain the following inequalities in parallel to the joint-source-channel-coding converse in Shannon theory:

\[
m \log \left( 1 + 1/\sigma^2 \right)
\geq I(Y; Y + Z) \tag{108}
\geq I(X; \hat{X}) \tag{109}
= I(X, UU^*, VV^*; \hat{X}) = I(UU^*, VV^*; \hat{X}) + I(X; \hat{X}|UU^*, VV^*) \tag{110}
= I(UU^*, VV^*; \hat{X}) + I(T; \hat{T}|UU^*, VV^*) \tag{111}
\geq I(UU^*; \hat{X}) + I(VV^*; \hat{X}) + I(T; \hat{T}) \tag{112}
\geq I(UU^*; \hat{X}) + I(VV^*; \hat{X}) + I(T; \hat{T}) \tag{113}
\geq \inf_{E: \|UU^* - \hat{X}\|_F \leq 2rD} I(UU^*; \hat{X}) + \inf_{E: \|VV^* - \hat{X}\|_F \leq 2rD} I(VV^*; \hat{X})
+ \inf_{E: \|T - \hat{T}\|_F \leq rD} I(T; \hat{T}) \tag{114}
\geq \left( (s_1 - r) + (s_2 - r)r \right) \log \frac{c}{2D} + \frac{r^2}{2} \log \frac{c'}{D}. \tag{115}
\]

where
\begin{itemize}
  \item (108): by the complex Gaussian channel capacity formula with the average power constraint \( \mathbb{E}\|Y\|_2^2 \leq m \), which is guaranteed by (25);
  \item (109): by the data processing inequality for mutual information;
  \item (110): by the fact the pair \((UU^*, VV^*)\) is a deterministic function of \(X = UV^*\);
  \item (111): we defined \(T \triangleq \hat{X} V\), \(T \triangleq \hat{X} V\);
  \item (112): by the mutual independence of \(T, UU^*\) and \(VV^*\) and the property of mutual information \(I(A, B; C) \geq I(A; C) + I(B; C)\) whenever \(A\) and \(B\) are independent;
  \item (113): by the fact the pair \((\hat{X} V^*\hat{X})\) is a deterministic function of \(\hat{X}\);
  \item (114): by (107) and the fact that \(\|T - \hat{T}\|_F = \|\hat{X} V - \hat{X}\|_F \leq \|X - \hat{X}\|_F\), where the infima are over \(P_{UU^*, \text{supp}(U)}\), \(P_{VV^*, \text{supp}(V)}\), and \(P_{\hat{T}T}\) respectively;
  \item (115): conditioned on \(\text{supp}(U)\) (resp. \(\text{supp}(V)\)), the non-zeros of \(U\) (resp. \(V\)) is uniform on the Stifled manifold of dimension \(r\) in \(\mathbb{C}^{s_1}\) (resp. \(\mathbb{C}^{s_2}\)). Furthermore, \(T\) is uniform over \(O(r)\). Applying Theorem VIII.17 and Theorem VIII.18 yields the desired lower bound.
\end{itemize}
ACKNOWLEDGEMENTS

K. Lee and Y. Bresler are supported in part by the National Science Foundation under Grants CCF 10-18789, CCF 10-18660, and IIS 14-47879. K. Lee would like to thank Marius Junge and Angelia Nedić for discussions related to this paper. The authors thank J.A. Geppert and F. Krahmer for their comments on an earlier version [31].

APPENDIX A

NOTATION

We will use the following notation in the appendix: For $A_1 \in \mathbb{C}^{n_1 \times n_1}$ and $A_2 \in \mathbb{C}^{n_2 \times n_2}$, $A_1 \otimes A_2 : \mathbb{C}^{n_1 \times n_2} \to \mathbb{C}^{n_1 \times n_2}$ denotes the linear operator defined by $(A_1 \otimes A_2)X = A_1XA_2^*$ for all $X \in \mathbb{C}^{n_1 \times n_2}$.

APPENDIX B

RIP LEMMAS

In this section, we present a set of lemmas as consequences of the rank-2 and doubly $(3s_1,3s_2)$-sparse RIP of $A$ with isometry constant $\delta$.

Lemma B.1. Let $F : \mathbb{C}^{n_2} \to \mathbb{C}^{m \times n_1}$ and $G : \mathbb{C}^{n_1} \to \mathbb{C}^{m \times n_2}$ be defined from $A$ by (6). Then, $F$ and $G$ satisfy
\[
\|\Pi_{\tilde{J}_1}([F(y])^*[F(\zeta)] - \langle \zeta, y \rangle I_{n_1})\Pi_{\tilde{J}_1}\| \leq \delta \|y\|_2 \|\zeta\|_2, \quad \forall \tilde{J}_1 \subset [n_1], \quad |\tilde{J}_1| \leq 3s_1 \quad (116)
\]
for all $y, \zeta \in \mathbb{C}^{n_2}$ with $\|y\|_0 + \|\zeta\|_0 \leq 3s_2$, and
\[
\|\Pi_{\tilde{J}_2}([G(x])^*[G(\xi)] - \langle \xi, x \rangle I_{n_2})\Pi_{\tilde{J}_2}\| \leq \delta \|x\|_2 \|\xi\|_2, \quad \forall \tilde{J}_2 \subset [n_2], \quad |\tilde{J}_2| \leq 3s_2 \quad (117)
\]
for all $x, \xi \in \mathbb{C}^n$ with $\|x\|_0 + \|\xi\|_0 \leq 3s_1$, respectively.

Proof: Since (116) is homogeneous in $y$ and $\zeta$, without loss of generality, we may assume that $\|y\|_2 = \|\zeta\|_2 = 1$. 
Let \( u, \tilde{u} \in \mathbb{C}^n \) satisfy \( \|u\|_2 = \|\tilde{u}\|_2 = 1 \). Let \( v, \tilde{v} \in \mathbb{C}^d \) satisfy \( \|v\|_2 = \|\tilde{v}\|_2 = 1 \). Then,

\[
\langle uv^*, (\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(y)})(A^*A)(\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(\zeta)})(\tilde{u}\tilde{v}^*) \rangle = \langle uv^*, (\Pi_{\tilde{J}_1} \otimes yy^*)(A^*A)(\Pi_{\tilde{J}_1} \otimes \zeta\zeta^*)(\tilde{u}\tilde{v}^*) \rangle
\]

\[
= \langle (\Pi_{\tilde{J}_1} \otimes yy^*)uv^*, (A^*A)(\Pi_{\tilde{J}_1} \otimes \zeta\zeta^*)(\tilde{u}\tilde{v}^*) \rangle
\]

\[
= \langle A(\Pi_{\tilde{J}_1} \otimes yy^*)(uv^*), A(\Pi_{\tilde{J}_1} \otimes \zeta\zeta^*)(\tilde{u}\tilde{v}^*) \rangle
\]

\[
= \langle A(\Pi_{\tilde{J}_1} uv^*yy^*), A(\Pi_{\tilde{J}_1} \tilde{u}\tilde{v}^*\zeta\zeta^*) \rangle
\]

\[
= (y^*v)(\tilde{v}^*\zeta)\langle [F(y)]\Pi_{\tilde{J}_1}u, [F(\zeta)]\Pi_{\tilde{J}_1}\tilde{u} \rangle
\]

\[
= (y^*v)(\tilde{v}^*\zeta)\langle u, \Pi_{\tilde{J}_1}[F(y)]^*[F(\zeta)]\Pi_{\tilde{J}_1}\tilde{u} \rangle.
\]

Similarly,

\[
\langle uv^*, (\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(y)})(\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(\zeta)})(\tilde{u}\tilde{v}^*) \rangle = (y^*v)(\tilde{v}^*\zeta)\langle \zeta, y \rangle \langle u, \Pi_{\tilde{J}_1}\tilde{u} \rangle.
\]

Therefore,

\[
\langle uv^*, (\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(y)})(A^*A - \text{id})(\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(\zeta)})(\tilde{u}\tilde{v}^*) \rangle = (y^*v)(\tilde{v}^*\zeta)\langle u, \Pi_{\tilde{J}_1}([F(y)]^*[F(\zeta)] - \langle \zeta, y \rangle I_n)\Pi_{\tilde{J}_1}\tilde{u} \rangle.
\]

In fact, the operator norm of \((\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(y)})(A^*A - \text{id})(\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(\zeta)})\) is achieved by maximizing the left hand side of (118) over \( u \) and \( \tilde{u} \) supported on \( \tilde{J}_1 \) for \( v = y \) and \( \tilde{v} = \zeta \). Therefore,

\[
\|\Pi_{\tilde{J}_1}([F(y)]^*[F(\zeta)] - \langle \zeta, y \rangle I_n)\Pi_{\tilde{J}_1}\|
\]

\[
= \|((\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(y)})(A^*A - \text{id})(\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(\zeta)})\|
\]

\[
= \|((\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(y)+\mathcal{R}(\zeta)})(A^*A - \text{id})(\Pi_{\tilde{J}_1} \otimes P_{\mathcal{R}(y)+\mathcal{R}(\zeta)})\|
\]

\[
\leq \delta
\]

where the last step follows since \((A^*A - \text{id})\) is restricted to a set of rank-2 and doubly \((3s_1, 3s_2)\)-sparse matrices.
The inequality in (116) is derived similarly using the following identity

\[
\langle uv^*, (P_{R(u)} \otimes \Pi_{\tilde{J}_2})(A^* A - \text{id})(P_{R(v)} \otimes \Pi_{\tilde{J}_2})\tilde{v}^* \rangle = (u^* x)(\xi^* \tilde{v}) \langle \tilde{v}, \Pi_{\tilde{J}_2}([G(\xi)]^* G(x)) - \langle x, \xi \rangle I_{n_2})\Pi_{\tilde{J}_2} v \rangle.
\]

(119)

\[\text{Lemma B.2.}\] Let \( \tilde{J}_1 \subset [n_1] \) and \( \tilde{J}_2 \subset [n_2] \) satisfy \(|\tilde{J}_1| \leq 2s_1 \) and \(|\tilde{J}_2| \leq 2s_2\), respectively. Suppose that \( X \in \mathbb{C}^{n_1 \times n_2} \) is a doubly \((s_1, s_2)\)-sparse rank-one matrix. Then,

\[\|\Pi_{\tilde{J}_1}[(A^* A - \text{id})(X)]\Pi_{\tilde{J}_2}\| \leq \delta \|X\|_F.\]

(120)

\[\text{Proof:}\] Suppose that \( \xi \in \mathbb{S}^{n_1-1} \) and \( \zeta \in \mathbb{S}^{n_2-1} \) are supported on \( \tilde{J}_1 \) and \( \tilde{J}_2 \), respectively. Since (120) is homogeneous, without loss of generality, we may assume that \( \|X\|_F = 1 \). Let \( X = \lambda uv^* \) denote the singular value decomposition of \( X \). Then, \( X = P_{R(u)} XP_{R(v)} = (P_{R(u)} \otimes P_{R(v)})(X) \); hence,

\[\|\langle \xi \zeta^*, \Pi_{\tilde{J}_1}[(A^* A - \text{id})(X)]\Pi_{\tilde{J}_2}\rangle \| = \|\langle P_{R(\xi)} \xi \zeta^* P_{R(\zeta)}(A^* A - \text{id})(X) \rangle \|
\]

\[\leq \|P_{R(\xi)} \otimes P_{R(\zeta)}(A^* A - \text{id})(P_{R(u)} \otimes P_{R(v)})(X)\|\]

\[\leq \|P_{R(\xi)} \otimes P_{R(\zeta)}(A^* A - \text{id})(P_{R(\xi)} \otimes P_{R(v)} + P_{R(\zeta)} \otimes P_{R(u)})(X)\|\]

\[\leq \delta.\]

Maximizing this inequality over \( \xi \) and \( \zeta \) yields the desired claim.

\[\text{Lemma B.3.}\] Let \( \tilde{J}_1 \subset [n_1] \) and \( \tilde{J}_2 \subset [n_2] \) satisfy \(|\tilde{J}_1| \leq 3s_1 \) and \(|\tilde{J}_2| \leq 3s_2\), respectively. Then,

\[\|\Pi_{\tilde{J}_1}[(A^* A)z]\Pi_{\tilde{J}_2}\| \leq \sqrt{1 + \delta}\|z\|_2, \quad \forall z \in \mathbb{C}^{m}.\]

\[\text{Proof:}\] Let \( \xi \in \mathbb{S}^{n_1-1} \) and \( \zeta \in \mathbb{S}^{n_2-1} \).

\[\|\langle \xi \zeta^*, \Pi_{\tilde{J}_1}[(A^* A)z]\Pi_{\tilde{J}_2}\rangle \| = \|\langle A(\Pi_{\tilde{J}_1} \xi \zeta^* \Pi_{\tilde{J}_2}), z \rangle \|
\]

\[\leq \|A(\Pi_{\tilde{J}_1} \xi \zeta^* \Pi_{\tilde{J}_2})\|_2 \|z\|_2\]

\[\leq \sqrt{1 + \delta}\|\Pi_{\tilde{J}_1} \xi \zeta^* \Pi_{\tilde{J}_2}\|_F \|z\|_2\]

\[\leq \sqrt{1 + \delta}\|z\|_2.\]
Maximizing this inequality over $\xi$ and $\zeta$ yields the desired claim. □

The following lemma is a consequence of the rank-$2r$ and doubly $(s_1, s_2)$-sparse RIP of $A$ with isometry constant $\delta$.

**Lemma B.4.** Suppose that $A : \mathbb{C}^{n_1 \times n_2} \rightarrow \mathbb{C}^m$ satisfies the rank-$2r$ and doubly $(s_1, s_2)$-sparse RIP with isometry constant $\delta$. Let $F$ and $G$ be defined from $A$, respectively, by (17) and (18). Let $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) denote the Hilbert space of $n_1$-by-$r$ (resp. $n_2$-by-$r$) matrices with the Frobenius norm. Fix $r$ be an arbitrary positive integer satisfying $r \leq \min(n_1, n_2)$. Then, the followings holds:

1) For all $V \in \mathbb{C}^{n_2 \times r}$ such that $V^*V = I_r$ and $V$ is row $s_2$-sparse, the linear operator $[F(V)]^*[F(V)]$ on $\mathcal{H}_1$ satisfies

$$\max_{J_1 \subset [n_1]; |J_1| \leq s_1} \| (\Pi_{J_1} \otimes I_r)([F(V)]^*[F(V)] - \text{id}_{\mathbb{C}^{n_1 \times r}})(\Pi_{J_1} \otimes I_r) \|_{\mathcal{H}_1 \rightarrow \mathcal{H}_1} \leq \delta.$$ 

2) For all $V, \tilde{V} \in \mathbb{C}^{n_2 \times r}$ such that $\tilde{V}^*V = 0$ and $[V, \tilde{V}]$ is row $s_2$-sparse, the linear operator $[F(\tilde{V})]^*[F(V)]$ on $\mathcal{H}_1$ satisfies

$$\max_{J_1 \subset [n_1]; |J_1| \leq s_1} \| (\Pi_{J_1} \otimes I_r)[F(\tilde{V})]^*[F(V)](\Pi_{J_1} \otimes I_r) \|_{\mathcal{H}_1 \rightarrow \mathcal{H}_1} \leq \delta \|V\|\|\tilde{V}\|.$$ 

3) For all $U \in \mathbb{C}^{n_1 \times r}$ such that $U^*U = I_r$ and $U$ is row $s_1$-sparse, the linear operator $[G(U)]^*[G(U)]$ on $\mathcal{H}_2$ satisfies

$$\max_{J_2 \subset [n_2]; |J_2| \leq s_2} \| (\Pi_{J_2} \otimes I_r)([G(U)]^*[G(U)] - \text{id}_{\mathbb{C}^{n_2 \times r}})(\Pi_{J_2} \otimes I_r) \|_{\mathcal{H}_2 \rightarrow \mathcal{H}_2} \leq \delta.$$ 

4) For all $U, \tilde{U} \in \mathbb{C}^{n_1 \times r}$ such that $\tilde{U}^*U = 0$ and $[U, \tilde{U}]$ is row $s_1$-sparse, the linear operator $[G(\tilde{U})]^*[G(U)]$ on $\mathcal{H}_2$ satisfies

$$\max_{J_2 \subset [n_2]; |J_2| \leq s_2} \| (\Pi_{J_2} \otimes I_r)[G(\tilde{U})]^*[G(U)](\Pi_{J_2} \otimes I_r) \|_{\mathcal{H}_2 \rightarrow \mathcal{H}_2} \leq \delta \|U\|\|\tilde{U}\|.$$

*Proof of Lemma B.4:* We only prove the first two results since the third and fourth results are derived by symmetry.

Fix an arbitrary $V \in \mathbb{C}^{n_2 \times r}$ so that $V^*V = I_r$ and $V$ is row $s_2$-sparse. Fix an arbitrary $J_1 \subset [n_1]$ so
that \(|J_1| \leq s_1\). Fix arbitrary \(U, \tilde{U} \in \mathbb{C}^{n_2 \times r}\). Then, we have

\[
\langle \tilde{U}, (\Pi_{J_1} \otimes I_r)([\mathcal{F}(V)]^* [\mathcal{F}(V)] - \text{id}_{\mathbb{C}^{n_1 \times r}})(\Pi_{J_1} \otimes I_r)U \rangle \\
= \langle (\Pi_{J_1} \otimes I_r)\tilde{U}, ([\mathcal{F}(V)]^* [\mathcal{F}(V)] - \text{id}_{\mathbb{C}^{n_1 \times r}})(\Pi_{J_1} \otimes I_r)U \rangle \\
= \langle \Pi_{J_1} \tilde{U}, ([\mathcal{F}(V)]^* [\mathcal{F}(V)] - \text{id}_{\mathbb{C}^{n_1 \times r}})(\Pi_{J_1} U) \rangle \\
= \langle \Pi_{J_1} \tilde{U}, [\mathcal{F}(V)]((\Pi_{J_1} U)) - (\Pi_{J_1} U, \Pi_{J_1} U) \rangle - (\Pi_{J_1} \tilde{U}, \Pi_{J_1} UV^* V) \\
= \langle A(\Pi_{J_1} \tilde{U} V^*), A(\Pi_{J_1} UV^*)) - (\Pi_{J_1} \tilde{U} V^*, \Pi_{J_1} UV^*) \rangle \\
= \langle \Pi_{J_1} \tilde{U} V^*, (A^* A - \text{id})(\Pi_{J_1} UV^*) \rangle,
\]

where the fourth step holds since \(V^* V = I_r\) and the fifth step holds by the definition of \(\mathcal{F}\). Therefore, since \([\Pi_{J_1} U, \Pi_{J_1} \tilde{U}] = \Pi_{J_1} [U, \tilde{U}]\) is row \(s_1\)-sparse and \(V\) is row \(s_2\)-sparse, by the rank-2\(r\) and doubly \((s_1, s_2)\)-sparse RIP of \(A\), it follows that

\[
|\langle \tilde{U}, (\Pi_{J_1} \otimes I_r)([\mathcal{F}(V)]^* [\mathcal{F}(V)] - \text{id}_{\mathbb{C}^{n_1 \times r}})(\Pi_{J_1} \otimes I_r)U \rangle | \\
\leq \delta \|\Pi_{J_1} UV^*\|_F \|\Pi_{J_1} \tilde{U} V^*\|_F \\
= \delta \|\Pi_{J_1} U\|_F \|\Pi_{J_1} \tilde{U}\|_F \\
\leq \delta \|U\|_F \|\tilde{U}\|_F.
\]

By maximizing over \(U\) and \(\tilde{U}\) within the unit ball in \(\mathcal{H}_1\), we have

\[
\|(\Pi_{J_1} \otimes I_r)([\mathcal{F}(V)]^* [\mathcal{F}(V)] - \text{id}_{\mathbb{C}^{n_1 \times r}})(\Pi_{J_1} \otimes I_r)\|_{\mathcal{H}_1 \rightarrow \mathcal{H}_1} \leq \delta.
\]

By maximizing over \(J_1\), we get the first result.

The second result is proved in a similar way. Fix arbitrary \(V, \tilde{V} \in \mathbb{C}^{n_2 \times r}\) so that \(\langle \tilde{V}, V \rangle = 0\) and \([V, \tilde{V}]\) is row \(s_2\)-sparse. Fix an arbitrary \(J_1 \subset [n_1]\) so that \(|J_1| \leq s_1\). Fix arbitrary \(U, \tilde{U} \in \mathbb{C}^{n_1 \times r}\). Then,
similar to the previous case, we have

\[
\langle \tilde{U}, (\Pi_{J_1} \otimes I_r)[\mathcal{F}(\tilde{V})]^*[\mathcal{F}(V)](\Pi_{J_1} \otimes I_r)U \rangle \\
= \langle \Pi_{J_1} \tilde{U} \tilde{V}^*, A^* A (\Pi_{J_1} UV^*) \rangle, \\
= \langle \Pi_{J_1} \tilde{U} \tilde{V}^*, (A^* A - \text{id}) (\Pi_{J_1} UV^*) \rangle,
\]

where the last step holds since \( \langle \Pi_{J_1} \tilde{U} \tilde{V}^*, \Pi_{J_1} UV^* \rangle = 0 \), which follows from \( \tilde{V}^* V = 0 \). Therefore, since \([\Pi_{J_1} U, \Pi_{J_1} \tilde{U}] = \Pi_{J_1} [U, \tilde{U}] \) is row \( s_1 \)-sparse and \([V, \tilde{V}] \) is row \( s_2 \)-sparse, by the rank-2r and doubly \((s_1, s_2)\)-sparse RIP of \( A \), it follows that

\[
||\langle \tilde{U}, (\Pi_{J_1} \otimes I_r)[\mathcal{F}(\tilde{V})]^*[\mathcal{F}(V)](\Pi_{J_1} \otimes I_r)U \rangle|| \\
\leq \delta \|\Pi_{J_1} UV^*\|_F \|\Pi_{J_1} \tilde{U} \tilde{V}^*\|_F \\
= \delta \|\Pi_{J_1} U\|_F \|V\| \|\Pi_{J_1} \tilde{U}\|_F \|\tilde{V}\| \\
\leq \delta \|U\|_F \|\tilde{U}\|_F \|V\| \|\tilde{V}\|.
\]

By maximizing over \( U \) and \( \tilde{U} \) within the unit ball in \( \mathcal{H}_1 \), we have

\[
\|\langle \Pi_{J_1} \otimes I_r)[\mathcal{F}(\tilde{V})]^*[\mathcal{F}(V)](\Pi_{J_1} \otimes I_r)\|_{\mathcal{H}_1 \to \mathcal{H}_1} \leq \delta \|V\| \|\tilde{V}\|.
\]

By maximizing over \( J_1 \), we get the second result.

\[\blacksquare\]

**APPENDIX C**

**PROOF OF LEMMA VIII.10**

To prove Lemma VIII.10, we use the non-Hermitian \( \sin \theta \) theorem [47, pp. 102–103].

**Lemma C.1** (Non-Hermitian \( \sin \theta \) theorem [47]). Let \( M \in \mathbb{C}^{n \times d} \) be a rank-\( r \) matrix. Let \( \hat{M} \in \mathbb{C}^{n \times d} \) be the best rank-\( r \) approximation of \( M + \Delta \) in the Frobenius norm. Then,

\[
\max \left( \sin \theta(\mathcal{R}(M), \mathcal{R}(\hat{M})), \sin \theta(\mathcal{R}(M^*), \mathcal{R}(\hat{M}^*)) \right) \leq \frac{\max(\|P_{\mathcal{R}(M)} \Delta\|, \|\Delta P_{\mathcal{R}(M^*)}\|)}{\sigma_r(M) - \|M + \Delta - \hat{M}\|}.
\]

**Proof of Lemma VIII.10:** Let \( M \triangleq \Pi_{\tilde{J}_1} X \) where \( X = \lambda uv^* \) with \( s_1 \)-sparse \( u \in \mathbb{S}^{n_1 - 1} \) and \( s_2 \)-sparse \( v \in \mathbb{S}^{n_2 - 1} \). Let \( \hat{M} \) denote the best rank-one approximation of \( \Pi_{\tilde{J}_1} [A^*(b)] \Pi_{\tilde{J}_2} \) in the spectral norm. Let
\[ \Delta \triangleq \Pi_{j_1} [A^* (b)] \Pi_{j_2} - M. \] Then,

\[
\| M + \Delta - \hat{M} \| = \| \Pi_{j_1} [A^* (b)] \Pi_{j_2} - \hat{M} \|
\leq \| \Pi_{j_1} [A^* (b)] \Pi_{j_2} - \Pi_{j_1} X \Pi_{j_2} \|
\leq \| \Pi_{j_1} [(A^* A - \text{id})(X)] \Pi_{j_2} + \Pi_{j_1} [A^* (z)] \Pi_{j_2} \|
\leq \| \Pi_{j_1} [(A^* A - \text{id})(X)] \Pi_{j_2} \| + \| \Pi_{j_1} [A^* (z)] \Pi_{j_2} \|
\leq \delta \| X \|_F + \sqrt{1 + \delta} \| z \|_2
\]

where the second inequality follows from Lemma B.2 and Lemma B.3.

Similarly, the difference \( \Delta \) is bounded in the spectral norm by

\[
\| \Delta \| = \| \Pi_{j_1} [A^* (b)] \Pi_{j_2} - \Pi_{j_1} X \Pi_{j_2} - \Pi_{j_1} X \Pi_{\perp j_2} \|
= \| \Pi_{j_1} [(A^* A - \text{id})(X)] \Pi_{j_2} - \Pi_{j_1} X \Pi_{\perp j_2} + \Pi_{j_1} [A^* (z)] \Pi_{j_2} \|
\leq \| \Pi_{j_1} [(A^* A - \text{id})(X)] \Pi_{j_2} \| + \| \Pi_{j_1} X \Pi_{\perp j_2} \| + \| \Pi_{j_1} [A^* (z)] \Pi_{j_2} \|
\leq \delta \| X \|_F + \| X \|_F \| \Pi_{j_1} u \|_2 \| \Pi_{\perp j_2} v \|_2 + \sqrt{1 + \delta} \| z \|_2
\]

where the last step follows from Lemma B.2 and Lemma B.3.

Note that \( \| M \| \) is rewritten as

\[
\| M \| = \| \Pi_{j_1} X \| = \| \Pi_{j_1} (\lambda u v^*) \| = \lambda \| \Pi_{j_1} u \|_2 = \| X \|_F \| \Pi_{j_1} u \|_2.
\]

Since \( v_0 \) is the right singular vector of \( \hat{M} \) and \( \mathcal{R}(M^*) = \mathcal{R}(X^*) \), by Lemma C.1, we have

\[
\sin(\mathcal{R}(X^*), \mathcal{R}(v_0)) \leq \frac{\| \Delta \|}{\| M \| - \| M + \Delta - \hat{M} \|}
\leq \frac{\| X \|_F \| \Pi_{j_1} u \|_2 \| \Pi_{\perp j_2} v \|_2 + \delta \| X \|_F + \sqrt{1 + \delta} \| z \|_2}{\| X \|_F \| \Pi_{j_1} u \|_2 - \delta \| X \|_F - \sqrt{1 + \delta} \| z \|_2}.
\]

Applying (40) to this result, we get the assertion in Lemma VIII.10.
APPENDIX D

PROOF OF LEMMA VIII.11

Let $J_1$ and $J_2$ denote the support of $u$ and $v$, respectively. By the definition of $(\tilde{J}_1, \tilde{J}_2)$ in (11), we have

$$\|\Pi_{\tilde{J}_1} [A^*(b)]\Pi_{\tilde{J}_2} \| \geq \|\Pi_{J_1} [A^*(b)]\Pi_{J_2} \|. \quad (121)$$

First, the right-hand-side of (121) is bounded from below by

$$\|\Pi_{J_1} [A^*(b)]\Pi_{J_2} \| = \|\Pi_{J_1} [X + (A^*A - \text{id})(X) + A^*(z)]\Pi_{J_2} \|$$
$$\geq \|X\| - \|\Pi_{J_1} [(A^*A - \text{id})(X)]\Pi_{J_2} \| - \|\Pi_{J_1} [A^*(z)]\Pi_{J_2} \|$$
$$\geq (1 - \delta)\lambda - \sqrt{1 + \delta}\|z\|_2$$

where the first inequality follows from Lemma B.2 and Lemma B.3.

Next, the left-hand side of (121) is bounded from above by

$$\|\Pi_{\tilde{J}_1} [A^*(b)]\Pi_{\tilde{J}_2} \| \leq \|\Pi_{\tilde{J}_1} [(A^*A)(\Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2})]\Pi_{\tilde{J}_2} \|$$

\(\text{(a)}\)
$$+ \|\Pi_{\tilde{J}_1} [(A^*A)(X - \Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2})]\Pi_{\tilde{J}_2} \| + \|\Pi_{\tilde{J}_1} [A^*(z)]\Pi_{\tilde{J}_2} \|. \quad (b)$$

where (a) and (b) are further bounded using Lemma B.2 by

$$\|\Pi_{\tilde{J}_1} [(A^*A)(\Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2})]\Pi_{\tilde{J}_2} \|$$
$$= \|((\Pi_{\tilde{J}_1} \otimes \Pi_{\tilde{J}_2})(A^*A)(\Pi_{\tilde{J}_1} \otimes \Pi_{\tilde{J}_2})(X))\|$$
$$\leq (1 + \delta)\|\Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2}\|_F$$

and

$$\|\Pi_{\tilde{J}_1} [(A^*A)(X - \Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2})]\Pi_{\tilde{J}_2} \|$$
$$= \|((\Pi_{\tilde{J}_1} \otimes \Pi_{\tilde{J}_2})(A^*A)(X - \Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2}))\|$$
$$= \|((\Pi_{\tilde{J}_1} \otimes \Pi_{\tilde{J}_2})(A^*A - \text{id})(X - \Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2}))\|$$
$$\leq \delta\|X - \Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2}\|_F$$
$$= \delta(\|X\|_F^2 - \|\Pi_{\tilde{J}_1} X \Pi_{\tilde{J}_2}\|_F^2)^{1/2}$$
respectively, and the noise term (c) is bounded using Lemma B.3 by

\[ \| \Pi_{\hat{J}_1} [A^*(z)] \Pi_{\hat{J}_2} \| \leq \sqrt{1 + \delta} \| z \|_2. \]

Combining all, the left-hand side of (121) is bounded from above by

\[ \| \Pi_{\hat{J}_1} [A^*(b)] \Pi_{\hat{J}_2} \| \leq (1 + \delta) \| \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \|_F + \delta (\| X \|_F^2 - \| \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \|_F^2)^{1/2} + \sqrt{1 + \delta} \| z \|_2. \quad (123) \]

Then, (121) is implies

\[ (1 + \delta) \| \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \|_F + \delta (\| X \|_F^2 - \| \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \|_F^2)^{1/2} \geq (1 - \delta) \lambda - 2 \sqrt{1 + \delta} \| z \|_2. \quad (124) \]

Applying (40) to (124), we get another necessary condition for (121) given by

\[
\begin{align*}
&\frac{1 + \delta}{\sqrt{\delta^2 + (1 + \delta)^2}} \frac{\| \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \|_F}{\| X \|_F} \\
&\quad + \frac{\delta}{\sqrt{\delta^2 + (1 + \delta)^2}} \left( \frac{\| X \|_F^2 - \| \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \|_F^2}{\| X \|_F} \right)^{1/2} \\
&\geq \frac{(1 - \delta) - 2(1 + \delta) \nu}{\sqrt{\delta^2 + (1 + \delta)^2}}.
\end{align*}
\]

Define \( \alpha, \beta \in [0, \pi/2] \) by

\[ \alpha \triangleq \sin^{-1} \left( \frac{\| \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \|_F}{\| X \|_F} \right) \quad \text{and} \quad \beta \triangleq \sin^{-1} \left( \frac{\delta}{\sqrt{\delta^2 + (1 + \delta)^2}} \right). \]

Then, using a trigonometric identity, we get

\[ \sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin(\alpha + \beta) \geq \frac{(1 - \delta) - 2(1 + \delta) \nu}{\sqrt{\delta^2 + (1 + \delta)^2}}, \]

which is rewritten as

\[ \alpha \geq \sin^{-1} \left( \frac{(1 - \delta) - 2(1 + \delta) \nu}{\sqrt{\delta^2 + (1 + \delta)^2}} \right) - \sin^{-1} \left( \frac{\delta}{\sqrt{\delta^2 + (1 + \delta)^2}} \right). \]

Therefore, (121) implies

\[ \frac{\| \Pi_{\hat{J}_1} X \Pi_{\hat{J}_2} \|_F}{\| X \|_F} \geq \sin \left[ \sin^{-1} \left( \frac{(1 - \delta) - 2(1 + \delta) \nu}{\sqrt{\delta^2 + (1 + \delta)^2}} \right) - \sin^{-1} \left( \frac{\delta}{\sqrt{\delta^2 + (1 + \delta)^2}} \right) \right]. \]
APPENDIX E

PROOF OF LEMMA VIII.12

Let $J_1$ and $J_2$ denote the support of $u$ and of $v$, respectively. For $\tilde{J}_1 \subset \tilde{J}_1$, it suffices to show

$$\min_{j \in \tilde{J}_1} \| e^*_j [A^*(b)] \|_{s_2} > \max_{j \in [n_1] \setminus J_1} \| e^*_j [A^*(b)] \|_{s_2}$$

(125)

where $e_j \in \mathbb{C}^{n_1}$ denotes the $j$th column of the $n_1 \times n_1$ identity matrix. Let

$$u_{\min} \triangleq \min_{j \in J} |u_j|.$$

Then, for $j \in \tilde{J}_1$, we have

$$\| e^*_j [A^*(b)] \|_{s_2} \geq \| e^*_j [A^*(b)] \Pi_{J_2} \|_2$$

$$= \| e^*_j [(A^* A - \text{id})(X) + X + A^*(z)] \Pi_{J_2} \|_2$$

$$\geq \| e^*_j X \|_2 - \| e^*_j [(A^* A - \text{id})(X)] \Pi_{J_2} \|_2 - \| e^*_j [A^*(z)] \Pi_{J_2} \|_2$$

$$\geq \lambda |u_j| - \delta \lambda - \sqrt{1 + \delta} \| z \|_2$$

$$\geq \lambda [u_{\min} - \delta - (1 + \delta) \nu]$$

where the third inequality follows from Lemma B.2 and Lemma B.3.

Next, for $j \in [n_1] \setminus J_1$, there exists $J' \subset [n_2]$ with $|J'| \leq s_2$ such that

$$\| e^*_j [A^*(b)] \|_{s_2} = \| e^*_j [A^*(b)] \Pi_{J'} \|_2$$

$$= \| e^*_j [(A^* A - \text{id})(X) + X + A^*(z)] \Pi_{J'} \|_2$$

$$\leq \lambda [\delta + (1 + \delta) \nu]$$

where the last step follows by Lemma B.2, Lemma B.3, and $e_j^* X = 0$.

Therefore, a sufficient condition for (125) is given by

$$u_{\min} > 2\delta + 2(1 + \delta) \nu.$$
APPENDIX F
PROOF OF LEMMA VIII.14

A sufficient condition for \( \tilde{J}_1 \subset \tilde{J}_1 \) is given by

\[
\min_{j \in \tilde{J}_1} \left\| e_j^* [A^*(b)] \right\|_{s_2} > \min_{j \in [n_1] \setminus J_1} \left\| e_j^* [A^*(b)] \right\|_{s_2}.
\] (126)

Therefore, it suffices to show (126) holds.

We first derive a lower bound on the left-hand-side of (126). Let \( J_2 \subset [n_2] \) denote the set of the indices of the nonzero rows of \( V \). For any \( j \in \tilde{J}_1 \), we have

\[
\left\| e_j^* [A^*(b)] \right\|_{s_2} \geq \left\| e_j^* [A^*(b)] \Pi_{J_2} \right\|_2
\]

\[
= \left\| e_j^* [(A^*A - \text{id})(X) + X + A^*(z)] \Pi_{J_2} \right\|_2
\]

\[
\geq \left\| e_j^* X \Pi_{J_2} \right\|_2 - \left\| e_j^* [(A^*A - \text{id})(X)] \Pi_{J_2} \right\|_2 - \left\| e_j^* [A^*(z)] \Pi_{J_2} \right\|_2
\]

\[
\geq \left\| e_j^* X \Pi_{J_2} \right\|_2 - \delta \|X\| - \sqrt{1 + \delta} \|z\|_2
\]

\[
= \|e_j^* X\|_2 - \delta \|X\| - \sqrt{1 + \delta} \|z\|_2,
\]

where the fourth step follows from Lemmas IV.8, B.2, and B.3.

Next, we drive an upper bound on the right-hand-side of (126). For any \( j \in [n_1] \setminus J_1 \), there exists \( J' \subset [n_2] \) with \( |J'| \leq s_2 \) such that

\[
\left\| e_j^* [A^*(b)] \right\|_{s_2} = \left\| e_j^* [A^*(b)] \Pi_{J'} \right\|_2
\]

\[
= \left\| e_j^* [(A^*A - \text{id})(X) + X + A^*(z)] \Pi_{J'} \right\|_2
\]

\[
\leq \left\| e_j^* X \Pi_{J'} \right\|_2 + \left\| e_j^* [(A^*A - \text{id})(X)] \Pi_{J'} \right\|_2 + \left\| e_j^* [A^*(z)] \Pi_{J'} \right\|_2
\]

\[
\leq \delta \|X\| + \sqrt{1 + \delta} \|z\|_2,
\]

where the last step follows from Lemmas IV.8, B.2, B.3, and \( e_j^* X = 0 \).

By these bounds and (126), we get a sufficient condition for (126) given by

\[
2 \left( \delta \|X\| + \sqrt{1 + \delta} \|z\|_2 \right) < \min_{j \in \tilde{J}_1} \|e_j^* X\|_2.
\] (127)
The right-hand-side of (127) is lower-bounded by
\[ \min_{j \in \tilde{J}_1} \|e^*_j X\|_2 \geq \sigma_r(X) \min_{j \in \tilde{J}_1} \|e^*_j U\|_2 \geq \sigma_r(X) \sigma_r(U^* \Pi_{\tilde{J}_1}). \] (128)

By (127) and (128) with \( \|X\| \leq \kappa \sigma_r(X) \), we show that (127) is implied by
\[ 2\kappa \left[ \delta + \sqrt{1 + \delta} \frac{\|z\|_2}{\|X\|} \right] < \sigma_r(U^* \Pi_{\tilde{J}_1}). \] (129)

Furthermore, by the rank-2r and doubly \((3s_1, 3s_2)\)-RIP of \(A\),
\[ \frac{\|z\|_2}{\|X\|} \leq \frac{\|X\|_F}{\|X\|_F} \cdot \frac{\|z\|_2}{\|X\|_F} \leq \frac{\|X\|_F}{\|X\|} \cdot \frac{\sqrt{1 + \delta} \|z\|_2}{\|A(X)\|_2} \leq \sqrt{1 + \delta}, \] (130)

where the last step follows from (21). By (130) and (129), we show that (129) implies (87). This completes the proof.

APPENDIX G

PROOF OF LEMMA VIII.15

A sufficient condition for \( \tilde{J}_2 \subset \hat{J}_2 \) is given by
\[ \min_{j \in \tilde{J}_2} \|\Pi_{\tilde{J}_1} [A^*(b)]e_j\|_2 > \min_{j \in [n_2] \setminus J_2} \|\Pi_{\tilde{J}_1} [A^*(b)]e_j\|_2. \] (131)

Therefore, it suffices to show (131) holds.

We first derive a lower bound on the left-hand-side of (131). For any \( j \in \tilde{J}_2 \), we have
\[ \|\Pi_{\tilde{J}_1} [A^*(b)]e_j\|_2 = \|\Pi_{\tilde{J}_1} [(A^*A - \text{id})(X) + X + A^*(z)]e_j\|_2 \]
\[ \geq \|\Pi_{\hat{J}_1} X e_j\|_2 - \|\Pi_{\tilde{J}_1} [(A^*A - \text{id})(X)]e_j\|_2 - \|\Pi_{\tilde{J}_1} [A^*(z)]e_j\|_2 \]
\[ \geq \|\Pi_{\hat{J}_1} X e_j\|_2 - \delta \|X\| - \sqrt{1 + \delta} \|z\|_2, \]

where the last step follows from Lemmas IV.8, B.2, and B.3.

Next, we drive an upper bound on the right-hand-side of (131). For any \( j \in [n_2] \setminus J_2 \), there exists
$J' \subset [n_1]$ with $|J'| \leq s_1$ such that
\[
\|\Pi_{\tilde{J}_1}[A^*(b)]e_j\|_2 = \|\Pi_{\tilde{J}_1}[(A^*A - \text{id})(X) + X + A^*(z)]e_j\|_2 \\
\leq \|\Pi_{\tilde{J}_1}Xe_j\|_2 + \|\Pi_{\tilde{J}_1}[(A^*A - \text{id})(X)]e_j\|_2 + \|\Pi_{\tilde{J}_1}[A^*(z)]e_j\|_2 \\
\leq \delta \|X\| + \sqrt{1 + \delta} \|z\|_2,
\]
where the last step follows from Lemmas IV.8, B.2, B.3, and $Xe_j = 0$.

By these bounds and (131), we get a sufficient condition for (131) given by
\[
2 \left(\delta \|X\| + \sqrt{1 + \delta} \|z\|_2\right) \leq \min_{j \in \tilde{J}_2} \|\Pi_{\tilde{J}_1}Xe_j\|_2.
\]

The right-hand-side of (132) is lower-bounded by
\[
\min_{j \in \tilde{J}_2} \|\Pi_{\tilde{J}_1}Xe_j\|_2 \geq \sigma_r(X) \sigma_r(U^*\Pi_{\tilde{J}_1}) \min_{j \in \tilde{J}_2} \|e_j^*V\|_2 \geq \sigma_r(X) \sigma_r(U^*\Pi_{\tilde{J}_1}) \sigma_r(V^*\Pi_{\tilde{J}_2}).
\]

Similarly to the proof of Lemma VIII.15, we show that (87) implies (131). This completes the proof.

APPENDIX H
PROOF OF LEMMA VIII.16

To prove Lemma VIII.16, we use the non-Hermitian sin $\theta$ theorem [47, pp. 102–103].

Let $X = UA^*$ denote the singular value decomposition of $X$. Then, $U$ is row $s_1$-sparse and $V$ is row $s_2$-sparse. Let $M = \Pi_{\tilde{J}_1}X$. Let $\widehat{M}$ be the best rank-$r$ approximation of $\Pi_{\tilde{J}_1}[A^*(b)]\Pi_{\tilde{J}_2}$ in the spectral norm. Let $\Delta \triangleq \Pi_{\tilde{J}_1}[A^*(b)]\Pi_{\tilde{J}_2} - M$. Then,
\[
\|M + \Delta - \widehat{M}\| = \|\Pi_{\tilde{J}_1}[A^*(b)]\Pi_{\tilde{J}_2} - \widehat{M}\| \\
\leq \|\Pi_{\tilde{J}_1}[A^*(b)]\Pi_{\tilde{J}_2} - \Pi_{\tilde{J}_1}X\Pi_{\tilde{J}_2}\| \\
= \|\Pi_{\tilde{J}_1}[(A^*A - \text{id})(X)\Pi_{\tilde{J}_2} + \Pi_{\tilde{J}_1}[A^*(z)]\Pi_{\tilde{J}_2}\| \\
\leq \|\Pi_{\tilde{J}_1}[(A^*A - \text{id})(X)]\Pi_{\tilde{J}_2}\| + \|\Pi_{\tilde{J}_1}[A^*(z)]\Pi_{\tilde{J}_2}\| \\
\leq \delta \|X\| + \sqrt{1 + \delta} \|z\|_2,
\]
where the last step follows from Lemmas IV.8 and B.3.
Similarly, the difference $\Delta$ is upper-bounded in the spectral norm by

$$\|\Delta\| = \|\Pi_{\tilde{J}}[A^*(b)]\Pi_{\tilde{J}_{\tilde{s}}} - \Pi_{\tilde{J}}X\Pi_{\tilde{J}_{\tilde{s}}} - \Pi_{\tilde{J}}X\Pi_{\tilde{J}_{\tilde{s}}}^{\perp}\|$$

$$= \|\Pi_{\tilde{J}}[(A^*A - \text{id})(X)]\Pi_{\tilde{J}_{\tilde{s}}} - \Pi_{\tilde{J}}X\Pi_{\tilde{J}_{\tilde{s}}}^{\perp} + \Pi_{\tilde{J}}[A^*(z)]\Pi_{\tilde{J}_{\tilde{s}}}\|$$

$$\leq \|\Pi_{\tilde{J}}[(A^*A - \text{id})(X)]\Pi_{\tilde{J}_{\tilde{s}}}\| + \|\Pi_{\tilde{J}}X\Pi_{\tilde{J}_{\tilde{s}}}^{\perp}\| + \|\Pi_{\tilde{J}}[A^*(z)]\Pi_{\tilde{J}_{\tilde{s}}}\|$$

$$\leq \delta \|X\| + \|\Pi_{\tilde{J}}X\Pi_{\tilde{J}_{\tilde{s}}}^{\perp}\| + \sqrt{1 + \delta^2} \|z\|_2,$$

where the last step follows from Lemma IV.8 and Lemma B.3.

The minimum singular value of $M$ is lower-bounded by

$$\sigma_r(M) \geq \sigma_r(\Lambda) \sigma_r(\Pi_{\tilde{J}}U).$$

Since $V_0$ consists of the $r$ singular vectors of $\tilde{M}$ and $\mathcal{R}(M^*) = \mathcal{R}(X^*)$, by Lemma C.1, we have

$$\sin(\mathcal{R}(X^*), \mathcal{R}(V_0)) \leq \frac{\|\Delta\|}{\sigma_r(M) - \|M + \Delta - \tilde{M}\|}$$

$$\leq \frac{\delta \|X\| + \|\Pi_{\tilde{J}}X\Pi_{\tilde{J}_{\tilde{s}}}^{\perp}\| + \sqrt{1 + \delta^2} \|z\|_2}{\sigma_r(\Lambda) \sigma_r(\Pi_{\tilde{J}}U) - \delta \|X\| - \sqrt{1 + \delta^2} \|z\|_2}$$

$$\leq \frac{\delta \|X\| + \|\Pi_{\tilde{J}}U\| \|X\| \|\Pi_{\tilde{J}_{\tilde{s}}} V\| + \|X\|(1 + \delta) \nu}{\sigma_r(\Lambda) \sigma_r(\Pi_{\tilde{J}}U) - \delta \|X\| - \|X\|(1 + \delta) \nu}$$

$$= \frac{\delta + \|\Pi_{\tilde{J}}U\| \|\Pi_{\tilde{J}_{\tilde{s}}} V\| + (1 + \delta) \nu}{\sigma_r(\Pi_{\tilde{J}}U)/\kappa - \delta - (1 + \delta) \nu},$$

where the third inequality follows from (21) and the rank-$2r$ and doubly $(3s_1, 3s_2)$-sparse RIP of $A$.

Finally, we derive an upper bound on $\|\Pi_{\tilde{J}_{\tilde{s}}} V\|$. Since $J_2 \subset \tilde{J}_2$, we have

$$\|\Pi_{\tilde{J}_{\tilde{s}}} V\| \leq \|\Pi_{\tilde{J}_{\tilde{s}}} V\|. \quad (135)$$

Thus we will derive an upper bound on $\|\Pi_{\tilde{J}_{\tilde{s}}} V\|$. By the relations between the principal angles between the two $r$-dimensional subspaces $\mathcal{R}(V)$ and $\mathcal{R}(\Pi_{\tilde{J}})$, we have

$$\sigma_r^2(\mathcal{P}_{\mathcal{R}(V)} \mathcal{P}_{\mathcal{R}(\Pi_{\tilde{J}_{\tilde{s}}})}) + \|\mathcal{P}_{\mathcal{R}(V)} \mathcal{P}_{\mathcal{R}(\Pi_{\tilde{J}_{\tilde{s}}})}\|^2 = 1. \quad (136)$$

Since the projections are expressed as

$$\mathcal{P}_{\mathcal{R}(V)} = VV^* \quad \text{and} \quad \mathcal{P}_{\mathcal{R}(\Pi_{\tilde{J}_{\tilde{s}}})} = \Pi_{\tilde{J}_{\tilde{s}}},$$

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(136) is equivalently rewritten to

\[ \sigma^2_r(VV^*\Pi_{\tilde{J}_2}) + \|VV^*\Pi_{\tilde{J}_2}\|^2 = 1. \]

Then it follows that

\[ \|\Pi_{\tilde{J}_2} V\| = \|VV^*\Pi_{\tilde{J}_2}\| = \sqrt{1 - \sigma^2_r(VV^*\Pi_{\tilde{J}_2})} = \sqrt{1 - \sigma^2_r(V^*\Pi_{\tilde{J}_2})}. \] (137)

Applying (135) and (137) to (134) completes the proof.

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