The algebra of invariants for the adjoint action of the unitriangular group

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Abstract. In this paper the algebra of invariants for the adjoint action of the unitriangular group in the nilradical of a parabolic subalgebra is studied. We prove that the algebra of invariants is finitely generated.

§1. Introduction

Let $G$ be the general linear group $GL(n, K)$ over an algebraically closed field $K$ of characteristic zero. Let $B$ ($N$, respectively) be its Borel (maximal unipotent, respectively) subgroup, which consists of upper triangular matrices with nonzero (unit, respectively) elements on the diagonal. We fix a parabolic subgroup $P \supset B$. Let $p$, $b$ and $n$ be the Lie subalgebras in $\mathfrak{gl}(n, K)$ corresponding to $P$, $B$ and $N$, respectively. We represent $p = r \oplus m$ as the direct sum of the nilradical $m$ and a block diagonal subalgebra $r$ with sizes of blocks $(n_1, \ldots, n_s)$. The subalgebra $m$ is invariant relative to the adjoint action of the group $P$:

for any $g \in P$ we have $x \in m \mapsto \text{Ad}_g x = gxg^{-1}$.

Therefore $m$ is invariant relative to the adjoint action of the subgroups $B$ and $N$. We extend this action to the representation in the algebra $K[m]$ and in the field $K(m)$:

for any $g \in P$ we have $f(x) \in K[m] \mapsto f(\text{Ad}_{g^{-1}}x)$.

The complete description of the field of invariants $K(m)^N$ for any parabolic subalgebra is a result of [S1]. In this paper a notion of an extended base is introduced. Elements of the extended base correspond to a set of algebraically independent $N$-invariants. These invariants generate the field of invariants.

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Further in the paper [S2] the structure of the algebra of invariants $K[m]^N$ is considered. If the sizes of diagonal blocks are $(2, k, 2)$, $k > 2$, or $(1, 2, 2, 1)$, then the invariants constructed on the extended base do not generate the algebra of invariants and the algebra of invariants is not free. Besides, the additional invariants in both cases are constructed, which together with the main list of the invariants constructed on the extended base generate the algebra of invariants. Also, the relations between these invariants are provided.

The aim of this paper is to prove that the algebra of invariants $K[m]^N$ is finitely generated. We show this as follows. Let $P = L \ltimes U$, where $L$ is the Levi subgroup and $U$ is the unipotent radical. Then $N = U_L \ltimes U$, where $U_L$ is the maximal unipotent subgroup of $L$. One has

$$K[m]^N = K[K[m]^U]^{U_L}.$$ 

In this paper we show that the algebra of invariants $K[m]^U$ is a finitely generated, free algebra and we present its generating invariants. Then by Khadzhiev’s theorem (see Theorem [5]), we get our main result:

**Theorem 1.1.** The algebra of invariants $K[m]^N$ is finitely generated.

§2. Main statements and definitions

We begin with definitions. Let $b = n \oplus \mathfrak{h}$ be a triangular decomposition. Let $\Delta$ be the root system relative to $\mathfrak{h}$ and let $\Delta^+$ be the set of positive roots. Let $\{\varepsilon_i\}_{i=1}^n$ be the standard basis of $\mathbb{C}^n$. Every positive root $\gamma$ in $\mathfrak{gl}(n, K)$ can be represented as $\gamma = \varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq n$ (see [GG]). We identify a root $\gamma$ with the pair $(i, j)$ and the set of the positive roots $\Delta^+$ with the set of pairs $(i, j)$, $i < j$. The system of positive roots $\Delta^+_r$ of the reductive subalgebra $\mathfrak{r}$ is a subsystem in $\Delta^+$.

Let $\{E_{i,j} : i < j\}$ be the standard basis in $n$. Let $E_\gamma$ denote the basis element $E_{i,j}$, where $\gamma = (i, j)$.

Let $M$ be a subset of $\Delta^+$ corresponding to $m$ that is

$$m = \bigoplus_{\gamma \in M} E_\gamma.$$ 

We identify the algebra $K[m]$ with the polynomial algebra in variables $x_{i,j}$, $(i, j) \in M$.

We define a relation in $\Delta^+$ such that $\gamma' > \gamma$ whenever $\gamma' - \gamma \in \Delta^+$. Note that the relation $>$ is not an order relation.
The roots \( \gamma \) and \( \gamma' \) are called *comparable*, if either \( \gamma' > \gamma \) or \( \gamma > \gamma' \).

We will introduce a subset \( S \) in the set of positive roots such that every root from this subset corresponds to some \( N \)-invariant.

**Definition 2.1.** A subset \( S \) in \( M \) is called a *base* if the elements in \( S \) are not pairwise comparable and for any \( \gamma \in M \setminus S \) there exists \( \xi \in S \) such that \( \gamma > \xi \).

Let us show that the base exists. We need the following

**Definition 2.2.** Let \( A \) be a subset in \( M \). We say that \( \gamma \) is a *minimal element* in \( A \) if there is no \( \xi \in A \) such that \( \xi < \gamma \).

For a given parabolic subgroup we will construct a diagram in the form of a square array. The cell of the diagram corresponding to a root of \( S \) is labeled by the symbol \( \otimes \). Symbols \( \times \) will be explained below.

**Example 2.3.** Diagram 1 represents the parabolic subalgebra with sizes of its diagonal blocks \((2, 1, 3, 2)\). In this case minimal elements in \( M \) are \((2, 3)\), \((3, 4)\) and \((6, 7)\).

![Diagram 1](image)

We construct the base \( S \) by the following algorithm.

**STEP 1.** Put \( M_0 = M \) and \( i = 1 \). Let \( S_1 \) be the set of minimal elements in \( M_0 \).

**STEP 2.** Put \( M_i = M_{i-1} \setminus \{ S_i \cup \{ \gamma \in M_{i-1} : \exists \xi \in S_i, \xi < \gamma \} \} \). Let \( S_i \) be the set of minimal elements of \( M_{i-1} \). Increase \( i \) by 1 and repeat Step 2 until \( M_i \) is empty.

Denote \( S = S_1 \cup S_2 \cup \ldots \). The base \( S \) is unique.

We have \( S_1 = \{(2, 3), (3, 4), (6, 7)\} \) and \( S_2 = \{(1, 5), (5, 8)\} \) in Example 2.3.

Let \((r_1, r_2, \ldots, r_s)\) be the sizes of the diagonal blocks in \( \mathfrak{r} \). Put

\[
R_k = \sum_{i=1}^{k} r_i.
\]
Let us present \( N \)-invariant corresponding to a root of the base. Consider the formal matrix \( X \) of variables
\[
\begin{pmatrix}
0 & X_{1,2} & X_{1,3} & \cdots & X_{1,s} \\
0 & 0 & X_{2,3} & \cdots & X_{2,s} \\
0 & 0 & 0 & \cdots & X_{s-1,s} \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]
where the size of \( X_{i,j} \) is \( r_i \times r_j \).

The matrix \( X \) can be represented as a block matrix

\[
X = \begin{pmatrix}
0 & X_{1,2} & X_{1,3} & \cdots & X_{1,s} \\
0 & 0 & X_{2,3} & \cdots & X_{2,s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & X_{s-1,s} \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

where \( X_{i,j} \) is \( r_i \times r_j \).

\[
X_{i,j} = \begin{pmatrix}
X_{R_i-1+1,R_j-1+1} & X_{R_i-1+1,R_j-1+2} & \cdots & X_{R_i+1,R_j} \\
X_{R_i-1+1,R_j-1+2} & X_{R_i-1+2,R_j-1+2} & \cdots & X_{R_i+1+2,R_j} \\
\vdots & \vdots & \ddots & \vdots \\
X_{R_i,R_j-1+1} & X_{R_i,R_j-1+2} & \cdots & X_{R_i,R_j}
\end{pmatrix}.
\] (1)

**Lemma 2.4.** The roots corresponding to the antidiagonal elements in \( X_{i,i+1} \) (from the lower left element towards right upper direction) are in the base.

Thus the roots of the base in the blocks \( X_{i,i+1} \) are as follows.

![Roots in \( X_{i,i+1} \)](image)

**Proof.** By definition 2.2 for any \( i \) the root \((R_i, R_i + 1)\) is minimal. Therefore \( M \setminus M_1 \) contains roots corresponding to all cells in the row \( R_i \) and the column \( R_i + 1 \). Hence \((R_i - 1, R_i + 2) \in S_2\) if \( r_i, r_{i+1} > 1 \) and all roots of \( M \) in the rows \( R_i \), \( R_i - 1 \) and in the columns \( R_i + 1 \), \( R_i + 2 \) belong to \( M \setminus M_2 \). Hence \((R_i - 2, R_i + 3) \in S_3\) if \( r_i, r_{i+1} > 2 \) etc. \( \square \)

There are roots in \( S \) such that these roots do not correspond to elements of the secondary diagonal in \( X_{i,i+1} \), for example \((1, 5)\) in Example 2.3.

For any root \( \gamma = (a, b) \in M \) let \( S_{\gamma} = \{(i, j) \in S : i > a, j < b\} \). Let \( S_{\gamma} = \{(i_1, j_1), \ldots, (i_k, j_k)\} \). Note that if \( \gamma \) is minimal in \( M \), then \( S_{\gamma} = \emptyset \).
Denote by $M_\gamma$ a minor $X_I^J$ of the matrix $X$ with ordered systems of rows $I$ and columns $J$, where

$$I = \text{ord}\{a, i_1, \ldots, i_k\}, \quad J = \text{ord}\{j_1, \ldots, j_k, b\}.$$ 

**Example 2.5.** Let us continue Example 2.3. For the root $(1, 6)$ we have $S_{(1,6)} = \{(2, 3), (3, 4)\}$, $I = \{1, 2, 3\}$, $J = \{3, 4, 6\}$, and

$$M_{(1,6)} = \begin{vmatrix} x_{13} & x_{14} & x_{16} \\ x_{23} & x_{24} & x_{26} \\ 0 & x_{34} & x_{36} \end{vmatrix}.$$ 

All minors $M_\xi$ for $\xi \in S$ are following

$$M_{(2,3)} = x_{23}, \quad M_{(3,4)} = x_{34}, \quad M_{(6,7)} = x_{67},$$

$$M_{(5,8)} = \begin{vmatrix} x_{57} & x_{58} \\ x_{67} & x_{68} \end{vmatrix}, \quad M_{(1,5)} = \begin{vmatrix} x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & x_{25} \\ 0 & x_{34} & x_{35} \end{vmatrix}.$$ 

**Lemma 2.6.** For any $\xi \in S$ the minor $M_\xi$ is $N$-invariant.

**Notation 2.7.** The group $N$ is generated by the one-parameter subgroups $g_{i,j}(t) = I + tE_{i,j}$, where $1 \leq i < j \leq n$ and $I$ is the identity matrix. The adjoint action of any $g_{i,j}(t)$ makes the following transformations of a matrix:

1) the $j$th row multiplied by $t$ is added to the $i$th row,

2) the $i$th column multiplied by $-t$ is added to the $j$th column, i.e. for a variable $x_{a,b}$ we have

$$\text{Ad}_{g_{i,j}(t)}x_{a,b} = \begin{cases} x_{a,b} + tx_{j,b} & \text{if } a = i; \\ x_{a,b} - tx_{a,i} & \text{if } b = j; \\ x_{a,b} & \text{otherwise.} \end{cases}$$

**Proof.** By the notation it is sufficient to prove that for any $\xi = (k, m) \in S$ the minor $M_\xi$ is invariant under the adjoint action of $g_{i,j}(t)$ for any $i < j$. If $i < k$, then the $i$th row does not belong to the minor $M_\xi$ and the adding of the $j$th row to the $i$th row leaves $M_\xi$ unchanged. Let $M_\xi = X_I^J$ for some collections of rows $I$ and columns $J$. If $i \geq k$, then since the numbers in $I$ are consecutive, the number of any nonzero row $j$ at the intersection with columns $J$ belongs to $I$. Then the adding of the $j$th row to the $i$th row leaves
$M_\xi$ unchanged again. Using the similar reasoning for columns, we get that $M_\xi$ is $N$-invariant. □

The set $\{M_\xi, \xi \in S\}$ does not generate all the $N$-invariants. There is the other series of $N$-invariants. To present it we need

**Definition 2.8.** An ordered set of positive roots

$$\{\varepsilon_{i_1} - \varepsilon_{j_1}, \varepsilon_{i_2} - \varepsilon_{j_2}, \ldots, \varepsilon_{i_s} - \varepsilon_{j_s}\}$$

is called a *chain* if $j_1 = i_2, j_2 = i_3, \ldots, j_{s-1} = i_s$.

**Definition 2.9.** We say that two roots $\xi, \xi' \in S$ form an *admissible pair* $q = (\xi, \xi')$ if there exists $\alpha_q$ in the set $\Delta^+_r$ corresponding to the reductive part $r$ such that the ordered set of roots $\{\xi, \alpha_q, \xi'\}$ is a chain. In other words, roots $\xi = \varepsilon_i - \varepsilon_j$ and $\xi' = \varepsilon_k - \varepsilon_l$ are an admissible pair if $\alpha_q = \varepsilon_j - \varepsilon_k \in \Delta^+_r$. Note that the root $\alpha_q$ is uniquely determined by $q$.

**Example 2.10.** In the case of Diagram 1 we have three admissible pairs $q_1 = (\xi_1, \xi_3), q_2 = (\xi_2, \xi_3), q_3 = (\xi_1, \xi_4)$, where $\xi_1 = (2,3), \xi_2 = (1,5), \xi_3 = (6,7)$, and $\xi_4 = (5,8)$.

Let the set $Q := Q(p)$ consist of admissible pairs. For every admissible pair $q = (\xi, \xi')$ we construct a positive root $\varphi_q = \alpha_q + \xi'$, where $\{\xi, \alpha_q, \xi'\}$ is a chain. Consider the subset $\Phi = \{\varphi_q : q \in Q\}$ in the set of positive roots. The cell of the diagram corresponding to a root of $\Phi$ is labeled by $\times$.

**Example 2.11.** The roots of $\Phi$ for the admissible pairs in Example 2.10 are $\varphi_{q_1} = (4,7), \varphi_{q_2} = (5,7), \varphi_{q_3} = (4,8)$.

Now we are ready to present the $N$-invariant corresponding to a root $\varphi \in \Phi$.

Let admissible pair $q = (\xi, \xi')$ correspond to $\varphi_q \in \Phi$. We construct the polynomial

$$L_{\varphi_q} = \sum_{\substack{\alpha_1, \alpha_2 \in \Delta^+_r \cup \{0\} \\ \alpha_1 + \alpha_2 = \alpha_q}} M_{\xi + \alpha_1} M_{\alpha_2 + \xi'}.$$

(2)

**Example 2.12.** Continuing the previous example, we have

$$L_{(4,7)} = x_{3,4}x_{4,7} + x_{3,5}x_{5,7} + x_{3,6}x_{6,7},$$

$$L_{(4,8)} = x_{3,4} \begin{vmatrix} x_{4,7} & x_{4,8} \\ x_{6,7} & x_{6,8} \end{vmatrix} + x_{3,5} \begin{vmatrix} x_{5,7} & x_{5,8} \\ x_{6,7} & x_{6,8} \end{vmatrix},$$

$$L_{(5,7)} = \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & x_{3,4} & x_{3,5} \end{vmatrix} \begin{vmatrix} x_{5,7} + \varepsilon_i & x_{1,3} & x_{1,4} \\ x_{2,3} & x_{2,4} & x_{2,6} \\ 0 & x_{3,4} & x_{3,6} \end{vmatrix} x_{6,7}. \]
Lemma 2.13. The polynomial $L_{\varphi}$ is $N$-invariant for any $\varphi = \varphi_q \in \Phi$, $q = (\xi, \xi')$.

Proof. By Notation 2.7 it is sufficient to check the action of $g_{i,j}(t)$. Let $\xi = (a, b)$, $\xi' = (a', b')$. Using the definition of admissible pair, we have $a < b < a' < b'$, $\alpha_q = (b, a') \in \Delta^+_t$, and $\varphi = (b, b')$. If $i < b$ or $j > a'$, then using the same arguments as in the proof of the invariance of $M_{\xi}$ for $\xi \in S$, we have that the minors of the right part of (2) are $g_{i,j}(t)$-invariant.

Let $b \leq i < j \leq a'$. Denote $\gamma_1 = (b, i)$, $\gamma_2 = (j, a')$, $\beta = (i, j)$; then $\alpha_q = \gamma_1 + \beta + \gamma_2$ and $\gamma_1 + \gamma_2 + \beta + \gamma_2 \in \Delta^+_t \cup \{0\}$. We have

$$\begin{cases} T_{g_{i,j}(t)} M_{\xi+\gamma_1+\beta} = M_{\xi+\gamma_1+\beta} + t M_{\xi+\gamma_1}, \\ T_{g_{i,j}(t)} M_{\beta+\gamma_2+\epsilon'} = T_{\beta+\gamma_2+\epsilon'} - t M_{\gamma_2+\epsilon'}. \end{cases}$$

(3)

The other minors of (2) are invariant under the action of $g_{i,j}(t)$. Combining (2) and (3), we get

$$(T_{g_{i,j}(t)} L_{\varphi}) - L_{\varphi} = M_{\xi+\gamma_1} (M_{\beta+\gamma_2+\epsilon'} - t M_{\gamma_2+\epsilon'}) + (M_{\xi+\gamma_1+\beta} + t M_{\xi+\gamma_1}) M_{\gamma_2+\epsilon'} - M_{\xi+\gamma_1} M_{\beta+\gamma_2+\epsilon'} - M_{\xi+\gamma_1+\beta} M_{\gamma_2+\epsilon'} = 0. \quad \square$$

Thus we proved the first part of

Theorem 2.14. For an arbitrary parabolic subalgebra, the system of polynomials

$$\{M_{\xi}, \xi \in S, L_{\varphi}, \varphi \in \Phi, \}$$

is contained in $K[m]_N$ and is algebraically independent over $K$.

To show the algebraic independence, consider the restriction homomorphism $f \mapsto f|_{\mathcal{Y}}$, where

$$\mathcal{Y} = \left\{ \sum_{\xi \in S \cup \Phi} c_{\xi} E_{\xi} : c_{\xi} \neq 0 \forall \xi \in S \cup \Phi \right\},$$

from $K[m]$ to the polynomial algebra $K[\mathcal{Y}]$ of $x_{\xi}, \xi \in S$, and of $x_{\varphi}, \varphi \in \Phi$. Direct calculations show that the system of the images

$$\{M_{\xi}|_{\mathcal{Y}}, \xi \in S, L_{\varphi}|_{\mathcal{Y}}, \varphi \in \Phi\}$$

is algebraically independent over $K$. Therefore, the system (4) is algebraically independent over $K$ (see details in [PS]).

Definition 2.15. The set $S \cup \Phi$ is called an extended base.

Definition 2.16. The matrices of $\mathcal{Y}$ are called canonical.
By [SI] one has the following theorems.

**Theorem 2.17.** There exists a nonempty Zariski-open subset $W \subset m$ such that the $N$-orbit of any $x \in W$ intersects $Y$ at a unique point.

**Theorem 2.18.** The field of invariants $K(m)^N$ is the field of rational functions of $M_\xi$, $\xi \in S$, and $L_\varphi$, $\varphi \in \Phi$.

§3. Invariants of the unipotent subgroup in the Levi decomposition of $P$

Let us consider the decomposition of a parabolic group $P$ into the semi-direct product of the Levi subgroup $L$ and the unipotent radical $U$. Let $U_L$ be the maximal unipotent subgroup in the Levi group $L$. One has $N = U_L \rtimes U$. The aim is to describe the algebra of invariants $K[m]^U$.

As above, we will introduce some subset $T \subset \Delta^+$ and construct a corresponding invariant $N_\xi \in K[m]^U$ for every root $\xi \in T$.

**Definition 3.1.** A root $\xi \in \Delta^+$ belongs to a broad base $T \subset \Delta^+$ if one of the following conditions holds:

1) the root $\xi$ belongs to $S$;

2) there exists a root $\gamma \in S$ such that $\xi > \gamma$ and the variables $x_\xi$ and $x_\gamma$ are located in the same block $X_{i,j}$.

**Example 3.2.** The diagram presents roots of the broad base $T$ for the diagonal blocks $(2, 1, 3, 2)$. The cells of the diagram corresponding to roots of $S$ (resp. $T \setminus S$) are labeled by the symbol $\otimes$ (resp. $\boxtimes$).

Let $M' = \{\xi \in M : E_\xi \in m^2\}$. In other words, if an element corresponding to a root $\xi \in M$ does not belong to blocks $X_{k,k+1}$ for any $k$, then $\xi \in M'$. 

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We have
\[
\sum_{\xi \in \mathcal{M}'} x_\xi E_\xi = \begin{pmatrix}
0 & 0 & X_{1,3} & \ldots & X_{1,s} \\
0 & 0 & 0 & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix},
\]

\[
\sum_{\xi \in \mathcal{M}\setminus\mathcal{M}'} x_\xi E_\xi = \begin{pmatrix}
0 & X_{1,2} & 0 & \ldots & 0 \\
0 & 0 & X_{1,3} & \ldots & 0 \\
0 & 0 & 0 & \ldots & X_{s-1,s} \\
0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix},
\]

where \(X_{i,j}\) is the block \((1)\).

If \(\xi \in \mathcal{M} \setminus \mathcal{M}'\), then \(x_\xi\) is in some block \(X_{k,k+1}\). We have \(\xi \in S\) or using Lemma 2.4 there is \(\gamma \in S\) such that \(\xi\) is to the right or above \(\gamma\). In both cases \(\xi \in T\). Therefore \(\mathcal{M} \setminus \mathcal{M}' \subset T\). Example 3.2 shows that \(\mathcal{M} \setminus \mathcal{M}' \neq T\) in general case.

For \(\xi = (i, j) \in T\) let \(N_\xi \in \mathcal{K}[\mathbf{m}]\) be defined as follows
\[
N_\xi = \begin{cases}
x_{i,j} & \text{if } \xi \in \mathcal{M} \setminus \mathcal{M}'; \\
M_\xi & \text{if } \xi \in \mathcal{M}'.
\end{cases}
\]

Example 3.3. Let us write all \(U\)-invariants \(N_\xi\) in the case \((2,1,3,2)\) for \(\xi \in T \cap \mathcal{M}'\).

\[
N_{(1,5)} = \begin{vmatrix}
x_{1,3} & x_{1,4} & x_{1,5} \\
x_{2,3} & x_{2,4} & x_{2,5} \\
0 & x_{3,4} & x_{3,5}
\end{vmatrix},
N_{(1,6)} = \begin{vmatrix}
x_{1,3} & x_{1,4} & x_{1,6} \\
x_{2,3} & x_{2,4} & x_{2,6} \\
0 & x_{3,4} & x_{3,6}
\end{vmatrix}.
\]

Lemma 3.4. The minor \(N_\xi\) is invariant under the adjoint action of the unipotent group \(U\) for any \(\xi \in T\).

Proof. The group \(U\) is generated by the one-parameter subgroups (see Notation 2.7)
\[
g_{i,j}(t) = I + tE_{i,j}, \text{ where } (i, j) \in \mathcal{M}.
\]

There are two cases of a root \(\xi \in T\). The first case is \(\xi \in \mathcal{M} \setminus \mathcal{M}'\) and the second one is \(\xi \in \mathcal{M}' \cap T\).

1. Suppose \(\xi = (a, b) \in T\) belongs to the set \(\mathcal{M} \setminus \mathcal{M}'\); then \(N_\xi = x_{a,b}\) and there is some \(k\) such that the variable \(x_{a,b}\) is in the block \(X_{k,k+1}\). Using the Notion 2.7 for \(t \neq 0\) we have \(\text{Ad}_{g_{i,j}^{-1}(t)}x_{a,b} \neq x_{a,b}\) if \(a = i\) and \(x_{i,j}\) is in \(X_{k,k+1}\) or \(b = j\) and \(x_{a,i}\) is in \(X_{k,k+1}\). In both cases the root \((i, j)\) belongs to \(\Delta^+_T\). Therefore \((i, j) \notin \mathcal{M}\) and \(g_{i,j}(t) \notin U\). Hence \(x_{a,b}\) is an \(U\)-invariant.
2. If the root \( \xi = (a, b) \in T \) does not belong to \( M \setminus M' \), then by definition of \( T \), there exists a root \( \gamma \in S \) such that \( \gamma = (i, b) \), \( i > a \), or \( \gamma = (a, j) \), \( j < b \), and \( x_\gamma \) and \( x_\gamma \) are in the same block \( X_{l,m} \), \( l < m + 1 \). Suppose \( \gamma = (i, b), i > a \). The case \( \gamma = (a, j) \) is similar. Let \( M_\eta = X_\gamma^I \) be a minor of order \( k \) of the formal matrix with rows \( I = \{ i, i + 1, \ldots, i + k - 1 \} \) and columns \( J = \{ b - k + 1, b - k + 2, \ldots, b \} \), then \( N_\xi = X_\gamma^I, \) where \( I' = \{ a, i + 1, \ldots, i + k - 1 \} \). Note that all rows of \( N_\xi \) except the row \( a \) and all columns are consecutive. Since a minor is not changed by addition to a row (resp. column) any other its row (resp. column), the adjoint action of \( g_\alpha(t) \) can change \( X_\gamma^I \) if \( u = a \) and \( v \leq i \). Let \( \text{Ad}_{g_\alpha^{-1}(t)} N_\xi \neq N_\xi \) for \( t \neq 0 \). Since \( x_\xi \) and \( x_\gamma \) are in the same block \( X_{l,m} \) and \( u = a \) and \( v \leq i \), then \( x_{(u,b)} \) and \( x_{(v,b)} \) are in the same block \( X_{l,m} \). Hence \( (u, v) \in \Delta_\xi^u \) and \( g_\alpha(t) \notin U \). So we have that \( N_\xi \) is an \( U \)-invariant. □

**Definition 3.5.** The remoteness of a root \( \gamma \in M \) is called the maximum number \( s \) of roots \( \gamma_i \in M \) such that \( \gamma = \gamma_1 > \gamma_2 > \ldots > \gamma_s \).

**Example 3.6.** The remoteness of the root \((1, 6)\) in Example 2.3 equals 5, we have

\[(1, 6) > (1, 5) > (2, 5) > (3, 5) > (3, 4)\].

**Lemma 3.7.** The system of polynomials \( \{ N_\xi, \xi \in T \} \) is algebraically independent over \( K \).

**Proof.** Assume the converse, namely that the system \( \{ N_\xi, \xi \in T \} \) is algebraically dependent. Hence there is a polynomial \( f \) such that for some \( \xi_1, \ldots, \xi_k \) we have

\[f(N_{\xi_1}, N_{\xi_2}, \ldots, N_{\xi_k}) = 0.\]

Suppose that the degree of the polynomial \( f \) is minimal. Let \( \xi_1 \) be a root with the maximal remoteness. If \( \xi \in T \) has a \( k \)th remoteness, then all roots \( \gamma \neq \xi \) for variables \( x_\gamma \) in the polynomial \( N_\xi \) have a remoteness smaller than \( \xi \). The variable \( x_\xi \) is in the first row and the last column of the minor \( N_\xi \). Let us expand \( N_\xi \) according to the first row. We have \( N_\xi = ax_\xi + b \) for some polynomials \( a \) and \( b \) and all variables in \( a \) and \( b \) correspond to the roots with less remoteness than the remoteness of \( \xi \). Then the variable \( x_{\xi_1} \) is included into the single minor \( N_{\xi_1} \).

We have

\[0 = f(N_{\xi_1}, \ldots, N_{\xi_k}) =
= f_m(N_{\xi_2}, \ldots, N_{\xi_k})N_{\xi_1}^m + f_{m-1}(N_{\xi_2}, \ldots, N_{\xi_k})N_{\xi_1}^{m-1} + \ldots + f_0(N_{\xi_2}, \ldots, N_{\xi_k}).\]
Since $N_{\xi} = ax_1 + b$ and $a \neq 0$, we conclude that the coefficient of the highest power for the variable $x_1$ is $f_m(N_{\xi_2}, \ldots, N_{\xi_k})a^m$. Therefore
\[ f_m(N_{\xi_2}, \ldots, N_{\xi_k}) = 0. \]
This contradicts the minimality of $f$ and completes the proof. \(\square\)

§4. The algebra of $U$-invariants

Let $Z = \left\{ \sum_{\xi \in T} c_\xi E_\xi : c_\xi \in K \right\}$.

**Proposition 4.1.** There exists a nonempty Zariski-open subset $V \subset m$ such that for any $x \in V$ the $U$-orbit of the element $x$ intersects $Z$ at a unique point.

**Proof.** By Theorem 2.17 there exists a nonempty Zariski-open subset $W$ such that for any $x \in m$ there exists $g \in N$ satisfying $\text{Ad}_g x \in Y$. Fix any $x \in W$, there is an element $g \in N$ corresponding to $x$. Since $N = U_L \ltimes U$, $g \in N$ can be represented as the product $g = g_1g_2$, where $g_1 \in U_L$ and $g_2 \in U$. Then $g_1^{-1}g \in U$. Let us show that we can take $V = W$ and $\text{Ad}_{g_1^{-1}}z \in Z$.

Since $Y \subset Z$ and one-parameter subgroups $g_{i,j}(t) = I + tE_{i,j}$, where $(i, j) \in \Delta^+_r$, generate the group $U_L$, it is enough to show that for any $g_{i,j}(t) \in U_L$ we have $\text{Ad}_{g_{i,j}(t)}Z \subset Z$. Suppose $g_{i,j}(t) \in U_L$; then $(i, j) \in \Delta^+_r$. This means that there exists $k$ such that $R_{k-1} < i < j \leq R_k$. If some element is changed after the action of the one-parameter subgroup $g_{i,j}(t)$, then this element is $(i, a)$ or $(b, j)$ for some $a > i$ and $b < j$. In the first case the $j$th row is added to the $i$th row
\[ \text{Ad}_{g_{i,j}(t)}x_{i,a} = x_{i,a} + tx_{j,a}. \]
We have that the variables $x_{(i, a)}$ and $x_{(j, a)}$ are in the same block $X_{k,l}$. In the second case the $i$th column is added to the $j$th column
\[ \text{Ad}_{g_{i,j}(t)}x_{b,j} = x_{b,j} - tx_{b,i}. \]
Similarly, the variables $x_{(b, j)}$ and $x_{(b, i)}$ are in the same block $X_{m,k}$. By the definition of $T$, in the case $(i, a)$ we have that if the root $(j, a) \in T$, then $(i, a) \in T$. This means that the $g_{i,j}$-action does not change the set $Z = \sum_{\xi \in T} c_\xi E_\xi$. Similarly, if $(b, i) \in T$, then $(b, j) \in T$.

By Lemmas 3.4 and 3.7 any $z \in Z$ such that $N_{\xi,z} \neq 0$ for any $\xi \in T$ is a representative of some $U$-orbit. \(\square\)
Let $S$ be the set of denominators generated by invariants $N_\xi$, $\xi \in T$. Denote by $K[m]_S^U$ localization of the algebra $K[m]^U$ on $S$. Let

$$\pi : K[m]^U \to K[Z]$$

be the restriction homomorphism, $f \in K[m]^U \twoheadrightarrow f|_Z$, where the algebra $K[Z]$ is a polynomial algebra of variables $x_\xi$, $\xi \in T$. Extend $\pi$ to the mapping $\tilde{\pi} : K[m]_S^U \to K[c_{\xi_1}^{\pm 1}, c_{\xi_2}^{\pm 1}, \ldots, c_{\xi_s}^{\pm 1}]$, where $\xi_1, \xi_2, \ldots, \xi_s$ are all roots in $T$.

**Proposition 4.2.** Let $\{\xi_1, \xi_2, \ldots, \xi_s\}$ be a collection of roots of the broad base $T$. The mapping $\tilde{\pi} : K[m]_S^U \to K[c_{\xi_1}^{\pm 1}, c_{\xi_2}^{\pm 1}, \ldots, c_{\xi_s}^{\pm 1}]$ is an isomorphism and $K[m]_S^U = K[N_{\xi_1}^{\pm 1}, N_{\xi_2}^{\pm 1}, \ldots, N_{\xi_s}^{\pm 1}]$.

**Proof.** Let us show that $\tilde{\pi}$ is a monomorphism. Indeed, if $f \in K[m]_S^U$ satisfies $\tilde{\pi}(f) = 0$, then $f|_Z = 0$. By Proposition 4.1 Ad$_UZ$ is dense in $m$, therefore $f(m) = 0$. So $f \equiv 0$ and $\pi$ is a monomorphism.

To prove that $\tilde{\pi}$ is an epimorphism, we will show that for any $\xi \in T$ the element $c_\xi$ has a preimage in $K[N_{\xi_1}^{\pm 1}, N_{\xi_2}^{\pm 1}, \ldots, N_{\xi_s}^{\pm 1}]$. The proof is by induction on the remoteness of $\xi$. Since for any root $\xi \in M \setminus M'$ the polynomial $N_\xi = x_\xi$ is an $U$-invariant, then $\tilde{\pi}(N_\xi) = c_\xi$ and the base of induction is evident. Suppose for a root $\xi$ with remoteness less than $k$ we have that $c_\xi$ has a preimage in $K[N_{\xi_1}^{\pm 1}, N_{\xi_2}^{\pm 1}, \ldots, N_{\xi_s}^{\pm 1}]$. Let us show the statement for $k$. Consider a relation $\prec$ on $T$, defined by $\varphi_1 \prec \varphi_2$ whenever $i_1 > i_2$ and $j_1 < j_2$, where $\varphi_1 = (i_1, j_1)$ and $\varphi_2 = (i_2, j_2)$. Let $\xi \in T$ have a $k$th remoteness, then

$$N_\xi = x_\xi \prod_{\varphi \prec \xi} N_\varphi + b,$$

where the product is taken on all roots $\varphi \prec \xi$ such that $\varphi \in S$ and $\varphi$ is maximal in the sense of the relation $\prec$. For Example 3.2 we have

$$\prod_{\varphi \prec (1,6)} N_\varphi = N_{(2,3)} N_{(3,4)}.$$

Note that all variables $c_\gamma$ in the polynomial $\tilde{\pi}(b)$ correspond to the roots $\gamma$ with less remoteness than the remoteness of $\xi$. Therefore by the induction assumption, for all these roots $\gamma$ we have that $c_\gamma$ has a preimage in the localization $K[N_{\xi_1}^{\pm 1}, N_{\xi_2}^{\pm 1}, \ldots, N_{\xi_s}^{\pm 1}]$. Hence there is a function $\phi(y_1, \ldots, y_s) \in K[y_1^{\pm 1}, y_2^{\pm 1}, \ldots, y_s^{\pm 1}]$ such that $\tilde{\pi}(b) = \tilde{\pi}(\phi(N_{\xi_1}, \ldots, N_{\xi_s}))$. Then

$$\tilde{\pi}^{-1}(c_\xi) = \frac{N_\xi - \phi(N_{\xi_1}, \ldots, N_{\xi_s})}{\prod_{\varphi \prec \xi} N_\varphi} \in K[N_{\xi_1}^{\pm 1}, N_{\xi_2}^{\pm 1}, \ldots, N_{\xi_s}^{\pm 1}]. \quad \Box$$
Theorem 4.3. The algebra of invariants $K[m]^U$ is a polynomial algebra of $N_\xi, \xi \in T$.

Proof. Let us show that for $L \in K[m]^U$ one has

$$L \in K[N_{\xi_1}, N_{\xi_2}, \ldots, N_{\xi_s}]$$

where $\{\xi_1, \xi_2, \ldots, \xi_s\}$ is a collection of roots of the broad base $T$. By Proposition 4.2 there exists a polynomial $f$ and integers $l_1, l_2, \ldots, l_k$ such that

$$L = f(N_{\xi_1}, N_{\xi_2}, \ldots, N_{\xi_k}) \prod_{i=1}^k N_{\xi_i}^{l_i}$$

(5)

By the induction on the number of $N_\xi$ in the denominator it is sufficient to prove that if $LN_\xi \in K[N_{\xi_1}, \ldots, N_{\xi_s}]$ for some $\xi \in T$ and for some $L \in K[m]$, then $L \in K[N_{\xi_1}, \ldots, N_{\xi_s}]$.

We fix a root $\xi$. Suppose $\xi = (i, j)$ and consider the case $\xi \in M'$. If some root $\gamma$ in the broad base $T$ has the form $(i-1, b)$ for some $b > j$, then denote $\mu_\gamma = (a, b)$ for some $a > i$ such that $\mu_\gamma \notin T$. If $\gamma = (a, j+1)$ for some $a < i-1$, then denote $\mu_\gamma = (a, b)$ for some $b < j$ such that $\mu_\gamma \notin T$. For the other roots $\gamma \in T$ and in the case $\xi \notin M'$ we have $\mu_\gamma = \gamma$.

The existence of this root $\mu_\gamma$ in the case $\mu_\gamma \neq \gamma$ is explained as follows. Since $\xi \in M'$, then $x_\xi$ is the block $X_{k,m}$ for some $k, m$ and $m > k + 1$. Evidently, the roots $(R_k, R_k+1)$ and $(R_{m-1}, R_{m-1}+1)$ are minimal in $M$ and belong to $S$. By definition of $T$, we have $(R_k, u) \notin T$ and $(v, R_{m-1}+1) \notin T$ for $u \geq j$ and $v \leq i$. These roots can be chosen for $\mu_\gamma$.

Example 4.4. Let us take the root $\xi = (2, 7)$. The symbol \(\bullet\) marks this root on the diagram. The roots $\mu_\gamma = \gamma$ in $T$ are pointed out by the symbol $\boxtimes$. The single root $\mu_\gamma \neq \gamma$ is marked by $\odot$.

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*Diagram 3*
Consider a set of matrices
\[ A = \left\{ \sum_{\gamma \in T} c_{\mu, \gamma} E_{\mu, \gamma}, \text{ where } c_{\mu, \gamma} \text{ such that } N_{\gamma}|_A \neq 0 \text{ for } \gamma \neq \xi \text{ and } N_{\xi}|_A = 0 \right\}. \]

Consider a subset \( \mathcal{X} = \{(N_{\xi_1}|_A, \ldots, N_{\xi_s}|_A)\} \) of the vector space \( K^s \). Evidently, \( \mathcal{X} \subset \text{Ann } N_{\xi} \). Let us show that the system of polynomials
\[ \{N_{\gamma}|_A, \gamma \neq \xi\} \]
is algebraically independent. The proof is by induction on the number of roots. Since we have \( N_{\gamma}|_A = x_\gamma \) for any \( \gamma \in M \setminus M' \) and \( N_{\gamma}|_A = N_{\gamma}|_Z \) for any \( \gamma < \xi \), the set \( B = \{\gamma \in T : \gamma < \xi\} \cup (M \setminus M') \) is the base of induction. Suppose that for a subset \( B \subset T \) such that for any root \( \gamma \in B \) with the maximal remoteness and for any \( \eta \in T \) we have \( \mu_\eta < \mu_\gamma \), then \( \eta \in B \). Suppose that the polynomials \( N_{\gamma}|_A, \gamma \in B \), are algebraically independent.

Denote \( I_{\mathcal{X}} = \{\varphi \in K[y_{\xi_1}, \ldots, y_{\xi_s}] : \varphi(\mathcal{X}) = 0\} \) and \( I = \langle y_\xi \rangle \). Now let us prove that \( I_{\mathcal{X}} = I \). Obviously, \( I_{\mathcal{X}} \supset I \), hence \( \mathcal{X} \subset \text{Ann } I \). Since the dimension of \( \text{Ann } I \) is the degree of transcendence of the algebra \( K[y_{\xi_1}, \ldots, y_{\xi_s}]/I \) over the main field \( K \), we have
\[ \dim \text{Ann } I = \degtr_K K[y_{\xi_1}, \ldots, y_{\xi_s}]/I = s - 1, \]
\[ \dim \mathcal{X} = s - 1. \]

Therefore, \( \text{Ann } I = \mathcal{X} \). Suppose \( g \in I_{\mathcal{X}} \), then there exists \( m \in \mathbb{N} \) such that \( g^m \in I \) by the Hilbert’s Nullstellensatz. Since \( I \) is a prime ideal, we obtain \( g \in I \). This means \( I_{\mathcal{X}} = I = \langle y_\xi \rangle \). To conclude the proof, it remains to note that there exists a polynomial \( p = p(y_{\xi_1}, \ldots, y_{\xi_s}) \) such that
\[ LN_{\xi} = N_{\xi} p(N_{\xi_1}, \ldots, N_{\xi_s}). \]

Finally, we have \( L \in K[N_{\xi_1}, \ldots, N_{\xi_s}] \). \( \square \)

By [Kh] one has

**Theorem (Khadzhiev) 4.5.** Let \( H \) be a connected reductive group and \( U \) its maximal reductive subgroup. Then for any finitely generated algebraic \( H \)-algebra \( A \) the algebra \( A^U \) is finitely generated.
Corollary 4.6. The algebra of invariants $K[m]^N$ is finitely generated.

Proof. By Theorem 4.3, the algebra of invariants $A = K[m]^U$ is finitely generated. Therefore the algebra of invariants

$$A^U_L = K[K[m]^U]^U_L = K[m]^N$$

under the adjoint action of the reductive group $U_L$, where $U_L$ is the Levi subgroup of the parabolic group $P$, is finitely generated too by the Khadzhiev’s theorem. □

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