Amenability of semigroups and the Ore condition for semigroup rings

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Abstract
It is known that if a cancellative monoid $M$ is left amenable then the monoid ring $K[M]$ satisfies the Ore condition, that is, there exist nontrivial common right multiples for the elements of this ring. Donnelly (Semigroup Forum 81:389–392, 2010) shows that a partial converse to this statement is true. Namely, if the monoid $\mathbb{Z}^+[M]$ of all elements of $\mathbb{Z}[M]$ with positive coefficients has nonzero common right multiples, then $M$ is left amenable. He asks whether the converse is true for this particular statement. We show that the converse is false even for the case of groups. If $M$ is a free metabelian group, then $M$ is amenable but the Ore condition fails for $\mathbb{Z}^+[M]$. Besides, we study the case of the monoid $M$ of positive elements of R. Thompson’s group $F$. The amenability problem for $F$ is a famous open question. It is equivalent to left amenability of the monoid $M$. We show that for this case the monoid $\mathbb{Z}^+[M]$ does not satisfy the Ore condition. That is, even if $F$ is amenable, this cannot be shown using the above sufficient condition.

Keywords Left amenable semigroup · Ore condition · Semigroup ring · R. Thompson’s group $F$

1 Preliminaries

Let us recall one of the definitions of left amenable semigroups.

Let $S$ be a semigroup. Suppose that there exists a mapping $\mu: \mathcal{P}(S) \to [0, 1]$ from the power set of $S$ into the unit interval satisfying the following conditions.

1. $\mu$ is additive, that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subseteq S$;
Another equivalent definition can be done in terms of left invariant means on the set of bounded real-valued functions on $S$ (cf. [7]).

A convenient criterion for amenability of groups was proved by Følner in [9]. It can be generalized to the case of cancellative semigroups [6].

Let $S$ be a cancellative semigroup. It is left amenable iff there exists $\delta > 0$ such that for any nonempty finite set $A$ in $S$ there exists a nonempty finite set $E$ such that $|aE \cap E| > \delta |E|$ for any $a \in A$. One more equivalent statement says that $S$ is left amenable whenever for any nonempty finite subset $A$ in $S$ there exists an "almost invariant" finite set $E$. This means that for any $\varepsilon > 0$ the set $E$ can be chosen in such a way that $|aE \Delta E| < \varepsilon |E|$ for all $a \in A$.

The definition of right amenable semigroups is similar. For the case of groups, both properties are equivalent.

A semigroup $S$ has common right multiples whenever for any $a, b \in S$ there exist $u, v \in S$ such that $au = bv$. If $S$ has a zero element (say, $S$ is a monoid ring or so), then we say that $S$ satisfies the Ore condition whenever for any $a, b \in S$ there exist $u, v \in S$ such that $au = bv$ excluding the case $u = v = 0$. In this case we can also say that $S$ has nonzero (or nontrivial) common right multiples.

The following well-known result belongs to Tamari [14]. If $M$ is left amenable cancellative monoid, then for any field $K$, the monoid ring $K[M]$ satisfies the Ore condition.

Let $\mathbb{Z}^+[M]$ denote the set of linear combinations of elements of $M$ with positive integer coefficients. The monoid ring $\mathbb{Z}[M]$ here is assumed to be cancellative. In [7] Donnelly proves that the following partial converse to the above statement holds.

If $\mathbb{Z}^+[M]$ has nonzero common right multiples, then $M$ is left amenable.

It is asked in [7] whether the converse to this particular statement is true. Also he asks whether the Ore condition for $K[M]$, where $K$ is a field, implies that $\mathbb{Z}^+[M]$ satisfies the Ore condition. We answer both questions in the negative in the next section (even for the case of groups).

We are also going to discuss the above conditions for the case of $R$. Thompson’s group $F$ and its positive monoid $M$. Let us recall some definitions.

Let $M$ be a monoid given by the following monoid presentation:

$\langle x_0, x_1, x_2, \ldots | x_j x_i = x_i x_{j+1} \ (0 \leq i < j) \rangle$.  \hspace{1cm} (1)$

It is well known that $M$ is cancellative and has common right multiples. By Ore’s theorem, it is embeddable into a group given by the same presentation. This group usually denoted by $F$ was found by Richard J. Thompson in the 60s. We refer to the survey [5] for details. (See also [2–4].) This is the group of right quotients of $M$ so that $F = MM^{-1}$. We refer to $M$ as the positive monoid of $F$.

The group $F$ has presentation with 2 generators and 2 defining relations. Brin and Squier proved in [2] that there are no free subgroups of rank $\geq 2$ in $F$. The famous open problem whether $F$ is amenable, usually attributed to Geoghegan, exists for more
than 40 years (see [10]). It is equivalent to the left amenability of the monoid $M$ for which is known it is not right amenable [12].

It follows from the Tamari result that amenability of $F$ implies that the group ring $K[F]$ over any field satisfies Ore condition. Recently Kielak [1] showed that the converse is also true. From elementary reasons it follows that the Ore condition for the group ring $K[F]$ is equivalent to the Ore condition for monoid ring $K[M]$. So this is an open question, and we are going to clarify whether $\mathbb{Z}^+[M]$ satisfies Ore condition, where $M$ is the positive monoid of $F$. In this case amenability of $F$ would follow from Donnelly’s result. However, we will show in the next section that the Ore condition does not hold for $\mathbb{Z}^+[M]$.

2 Main results

Lemma 1 Let $M$ be a monoid embeddable into a group $G$. Suppose that the monoid $\mathbb{Z}^+[M]$ has nonzero common right multiples. Then for any $a, b \in M$ there exists a relation of the form $a^{\pm 1}b^{\pm 1} \cdots a^{\pm 1}b^{\pm 1} = 1$ that holds in $G$.

Proof Consider the equation $(1 + a)u = (1 + b)v$ in $\mathbb{Z}^+[M]$. It has a nontrivial solution. Therefore, $u$ and $v$ can be presented as sums of elements of $M$. That is, there exist $g_1, \ldots, g_m \in M$ and $h_1, \ldots, h_n \in M$ such that $(1 + a)(g_1 + \cdots + g_m) = (1 + b)(h_1 + \cdots + h_n)$. Notice that elements in the lists may have repetitions.

Since $g_1 + \cdots + g_m + ag_1 + \cdots + ag_m = h_1 + \cdots + h_n + bh_1 + \cdots + bh_n$, the multisets $\{g_1, \ldots, g_m, ag_1, \ldots, ag_m\}$ and $\{h_1, \ldots, h_n, bh_1, \ldots, bh_n\}$ coincide. In particular, $m = n$.

Let us consider a directed graph with the vertex set $V$ indexed by the elements of the above multiset. For any $1 \leq i \leq n$ let us add a directed edge from $ag_i$ to $g_i$ labelled by $a$. Also add a directed edge labelled by $b$ from $bh_i$ to $h_i$ for any $i$. We assume that all our graphs are in the sense of Serre, that is, each directed edge $e$ has a formal inverse $e^{-1}$. Its label is an inverse letter.

We get a directed labelled graph $\Gamma$. Every vertex $v$ has exactly one outcoming edge labelled by $a^{\pm 1}$ and exactly one outcoming edge labelled by $b^{\pm 1}$. Therefore, the edges of $\Gamma$ form a union of disjoint cycles. The label of each of these cycles forms a relation that holds in the group $G$. □

Lemma 2 Let $G$ be a free metabelian group with basis $\{a, b\}$. Then $G$ has no relations between $a$ and $b$ of the form $a^{\pm 1}b^{\pm 1} \cdots a^{\pm 1}b^{\pm 1} = 1$.

Proof Let $R$ be a normal subgroup in the free group $F_m$ with $m$ generators. To any word $w$ in these generators one can assign a unique path $p = p(w)$ in the Cayley graph of $F_m/R$ starting at the identity. The word $w$ belongs to the derived subgroup $R'$ of $R$ if and only if for any edge $e$ of the Cayley graph, the number of occurrences of $e$ in $p = p(w)$ equals the number of occurrences of $e^{-1}$ in $p$. This solves the word problem in $F_m/R'$ provided it is solvable for $F_m/R$. The proof can be found in [8]; see also [13, Lemma 3].

Let us apply this for the case $R = F_m'$. Suppose that the word $w = a^{\pm 1}b^{\pm 1} \cdots a^{\pm 1}b^{\pm 1}$ represents the identity in the free metabelian group, that is, in
$F_m/R'$. Then the path $p = p(w)$ in the Cayley graph of free Abelian group $F_m/R$ satisfies the property from the previous paragraph.

Let $e$ be an edge from this path. Without loss of generality, assume it has label $a$. The path $p = p(w)$ has an occurrence of $e^{-1}$. Up to a cyclic shift, let $p = ege^{-1}s$ for some paths $q, s$. We see that $q$ is a loop, and the label of it has the form $b^\pm 1 \cdots a^\pm 1 b^\pm 1$. However, the length of this word is odd so it cannot represent an identity in the free Abelian group.

Now we can show that the converse to Donnelly’s implication from [7] does not hold.

**Theorem 1** There exists a left amenable cancellative monoid $M$ (actually, an amenable group) such that the monoid $\mathbb{Z}^+[M]$ does not satisfy the Ore condition.

**Proof** Let $M$ be the free metabelian groups on 2 generators. It is well known that all soluble groups are amenable [11]. Therefore, $M$ is a left amenable cancellative monoid. If $\mathbb{Z}^+[M]$ satisfies the Ore condition then by Lemma 1 $M$ has a relation of the form $a^\pm 1 b^\pm 1 \cdots a^\pm 1 b^\pm 1 = 1$ between its free generators. This contradicts Lemma 2.

Our example answers one more question from [7]. If we take the free metabelian group as the monoid $M$, then for any field $K$, the group ring $K[M]$ satisfies the Ore condition by the result of Tamari. However, this does not imply that $\mathbb{Z}^+[M]$ has nonzero common right multiples.

Now we want to clarify the situation for the case of the Ore condition for $\mathbb{Z}^+[M]$, where $M$ is the positive monoid of R. Thompson’s group $F = MM^{-1}$.

**Lemma 3** Let $x_0, x_1, \ldots, x_m, \ldots$ be the standard generating set for R. Thompson’s group $F$. Then any word of the form $w = x_{i_1}^\pm 1 x_{j_1}^\pm 1 \cdots x_{i_k}^\pm 1 x_{j_k}^\pm 1 (k \geq 1)$ does not represent the identity element of $F$ provided all $i_1, \ldots, i_k$ are even and all $j_1, \ldots, j_k$ are odd.

**Proof** Let us call such relations alternating, where odd end even subscripts of the generators alternate. We proceed by induction on the length of the word $w$. Let $\alpha$ be the minimal subscript on letters in $w$. It is well known that $x_i \mapsto x_{i+\alpha} (i \geq 0)$ induces a monomorphism from $F$ into itself. Therefore one can subtract $\alpha$ from all indices getting a relation $w'$ in $F$. Clearly, it is also alternating. Since the algebraic sum of exponents on $x_0$ in a relation must be zero, we obtain that $w'$ has a cyclic subword of the form $\ldots x_0^{-1} u x_0 \ldots$, where no $x_0^{\pm 1}$ occurs in $v$.

Applying defining relations of the form $x_i^{-1} x_j x_i = x_{j+1} (i < j)$ in $F$, we see that conjugation of $v = v(x_1, x_2, \ldots)$ by $x_0$ increases all indices of letters in $v$ by 1. The first and the last subscripts of letters in $v$ were odd; after conjugation they will be even. The subscript on a letter before $x_0^{-1}$ stays odd; the same for the letter after $x_0$. So the resulting word has the same structure where odd and even subscripts alternate. Its length decreases so the inductive assumption can be applied. This completes the proof.  

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Theorem 2 Let $M$ be positive monoid of $R$. Thompsón’s group $F$. Then $\mathbb{Z}^+[M]$ does not satisfy the Ore condition.

Proof Lemma 3 implies that $F$ has no relations of the form $a^{\pm 1}b^{\pm 1}\cdots a^{\pm 1}b^{\pm 1} = 1$, where $a = x_0$, $b = x_1$. Therefore, by Lemma 1, the equation $(1 + x_0)u = (1 + x_1)v$ does not have nonzero solutions in $\mathbb{Z}^+[M]$.

Notice that there are many solutions of the equation $(1 \pm x_0)u = (1 \pm x_1)v$ in the monoid ring $\mathbb{Z}[M]$. We can give a precise description of all their solutions. Details will appear elsewhere.

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