Modeling of charged anisotropic compact stars in general relativity

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\textbf{Abstract.} A charged compact star model has been determined for anisotropic fluid distribution. We have solved the Einstein-Maxwell field equations to construct the charged compact star model by using the radial pressure, the metric function $e^\lambda$ and the electric charge function. The generic charged anisotropic solution is verified by exploring different physical conditions like causality condition, mass-radius relation and stability of the solution (via the adiabatic index, TOV equations and the Herrera cracking concept). It is observed that the present charged anisotropic compact star model is compatible with the star PSR 1937+21. Moreover, we also presented the EOS $\rho = f(p)$ for the present charged compact star model.

1 Introduction

The search for theoretical compact star modeling, one of the main focuses of general relativity, has been a work in progress for over five decades. Over the years the researchers in general relativity had been using the perfect fluid as a source of stellar structures to form models for compact stars [1–7]. Later on, it was observed that for stars with highly dense core realistic results were not found with the perfect fluid as their source. It was proposed by Ruderman [8] and Canuto [9] that the stars may have high density ranges, when the nuclear matter has anisotropic features. Consequently fluids with anisotropy in pressure were chosen as source to depict the stellar models in order to improve the results considerably [10–23]. On the other hand, the gravitational collapse is one of the most dramatic phenomena in the universe. When the pressure is not sufficient to balance the gravitational attraction inside the star, the star undergoes a sudden gravitational collapse and the physical characteristics of the star change dramatically. The presence of charged compact objects may avert the gravitational attraction which is counterbalanced by the repulsive Colombian force in addition to the pressure gradient. There have been several investigations of compact star models in the presence of an electric field in recent years on the Einstein-Maxwell system of equations [24–31]. The presence of charge affects the values of redshift, luminosity and maximum mass for stars where the gravitational fields are strong and are described by solutions of the coupled Einstein-Maxwell system of equations. Moreover the charged analogues of the said anisotropic models further improved the situation [32–34]. Lake [3] and Herrera et al. [35] have developed the algorithm for all spherical symmetric solutions corresponding to perfect and anisotropic fluid distribution, respectively. Later on, Maurya et al. [36] have generalized this algorithm for charged anisotropic fluid distributions. Also, we have given several conditions on the physical quantities for constructing the realistic models. Several researchers have obtained solutions of Einstein’s equations in different approaches which can be seen in recent refs. [37–46].

Recently Piyali et al. [47,48] have obtained the anisotropic solutions by taking the radial pressure and mass function’ but both these solutions are unstable at some points inside the star due to the fact that the anisotropic factor $\Delta$ is negative. However in the present paper a class of more generalized stellar models is developed in the same manner by considering physically suitable expressions of radial pressure, charged intensity and to some extent mass function (by choosing $e^\lambda$ suitably). The models so obtained are neatly analyzed subject to adequate physical conditions. We found that the present charged anisotropic solution is stable at every point inside the star. In addition to that we have also provided the EOS for the present compact star which has the form $\rho = f(p)$.

The outline of the paper is as follows: In sect. 2, the Einstein-Maxwell equations have been presented including the spherically symmetric metric. The new charged anisotropic solution is derived by means of $e^\lambda, q$ and $p^r$ in...
sect. 3. We have mentioned the equation of state (EOS), which is a most vital one, in sect. 4. Moreover, in sect. 5 the boundary conditions have been applied to determine the constants. The physical features of the models with stability conditions have been discussed in details in sect. 6. Finally, in sect. 7, we discuss the result with concluding remarks.

2 The Einstein-Maxwell field equations for spherical symmetric anisotropic fluid distribution

Let us consider the spherical symmetric metric in curvature coordinates as
\[
d s^2 = -e^{\lambda(r)}d t^2 + e^{\nu(r)}d r^2 + r^2(d \theta^2 + \sin^2 \theta d \phi^2). \tag{1}
\]

Then the Einstein-Maxwell equations for anisotropic fluid distribution is defined as (by taking \( G = c = 1 \))
\[
-8 \pi (T_j^i + E_j^i) = R_j^i - \frac{1}{2} R \delta_j^i, \tag{2}
\]
where the energy momentum tensor \((T_j^i)\) and the electromagnetic field tensor \((E_j^i)\) can be written as
\[
T_j^i = [(\rho + p_t)u^i u_j + (p_r - p_t)\mu^i \mu_j - p_t \delta_j^i], \tag{3}
\]
\[
E_j^i = \frac{1}{4\pi}[-F^{ik}F_{kj} + \frac{1}{4} \delta^{ij} F^{kl} F_{kl}], \tag{4}
\]
where \(u^i\) is the four-velocity, \(\mu^i = e^{\nu(r)/2}\delta^i_r\), and \(\mu^i\) is the unit spacelike vector in the direction of the radial vector, \(\mu^i = e^{\lambda(r)/2}\delta^i_r\), \(\rho\) is the energy density, \(p_r\) is the pressure in the direction of \(\mu^i\) (normal or radial pressure) and \(p_t\) is the pressure orthogonal to \(\mu^i\) (transverse or tangential pressure). Also, the anti-symmetric electromagnetic field tensor \(F_{ij}\) satisfies the Maxwell equations
\[
F_{ik,j} + F_{kj,i} + F_{ji,k} = 0, \tag{5}
\]
and
\[
\frac{\partial}{\partial x^i} (\sqrt{-g} F^{ik}) = -4 \pi \sqrt{-g} J^i, \tag{6}
\]
where \(J^i\) is the four-current vector defined by
\[
J^i = \frac{\sigma}{\sqrt{g_{44}}} \frac{dx^i}{dx^4} = \sigma u^i, \tag{7}
\]
where \(\sigma\) is the charged density.

For static matter distribution the only non-zero component of the four-current is \(J^t\). Because of the spherical symmetry, the four-current component is only a function of radial distance, \(r\). The only non-vanishing components of the electromagnetic field tensor are \(F^{41}\) and \(F^{14}\), related by \(F^{41} = -F^{14}\), which describe the radial component of the electric field. From eq. (6), one obtains the following expression for the electric field:
\[
F^{41} = e^{-(\nu + \lambda)/2} \left[ \frac{q(r)}{r^2} \right], \tag{8}
\]
where \(q(r)\) represents the total charge contained within the sphere of radius \(r\), which is defined by
\[
q(r) = 4 \pi \int_0^r \sigma r^2 e^{\lambda/2} dr = r^2 \sqrt{-F_{44}} = r^2 F^{41} e^{(\nu + \lambda)/2}. \tag{9}
\]

Equation (9) can be treated as the relativistic version of Gauss’s law, which gives
\[
\frac{\partial}{\partial r} (r^2 F^{41} e^{(\nu + \lambda)/2}) = -4 \pi r^2 e^{(\nu + \lambda)/2} J^4. \tag{10}
\]

In view of metric (1), The Einstein-Maxwell field equations (2), (5), and (7) give the following differential equations:
\[
-8 \pi T^1_1 = \frac{\nu'}{r} e^{-\lambda} - \frac{(1 - e^{-\lambda})}{r^2} = 8 \pi p_t - \frac{q^2}{r^4}, \tag{11}
\]
\[
-8 \pi T^2_2 = -8 \pi T^3_3 = \left( \frac{\nu''}{2} + \frac{\lambda'}{4} + \frac{\nu'}{4} - \frac{\lambda'}{2r} \right) e^{-\lambda} = 8 \pi p_r + \frac{q^2}{r^4}, \tag{12}
\]
\[
-8 \pi T^4_4 = \frac{\lambda'}{r} e^{-\lambda} + \frac{(1 - e^{-\lambda})}{r^2} = 8 \pi \rho + \frac{q^2}{r^4}. \tag{13}
\]
where the prime denotes differential with respect to \(r\) and \(E = \frac{1}{r^2} \int_0^r 4 \pi r^2 \sigma e^{\lambda/2} = \frac{q^2}{r^4} \). Moreover \(\sigma\) is the charge density.

3 New charged anisotropic solution for compact star

To solve the Einstein-Maxwell equations (11)–(13), we have considered the following metric function \(e^{\lambda(r)}\) and electric charge function \(q(r)\) as
\[
e^{\lambda} = \frac{(1 + 2br^2 + b^2r^4)}{(1 + ar^2 + a^2r^4)}, \tag{14}
\]
where \(a\) and \(b\) are parameters with unit of length\(^{-2}\). Recently Takisa et al. [49] have obtained the stellar model PSR J1614-2230 in the quadratic form with quadratic equation of state by taking the metric function \(e^\lambda = (1 + ar^2)/(1 + br^2)\). Moreover, the proposed metric function is also used by Newton et al. [50] in class one. Also the gravitational metric function is singularity free at the centre as \(e^{\lambda(0)} = 1\) and it is monotonically increasing with \(r\). The behavior is shown by fig. 1.

Now we have also considered the electric charged function to solve the system of equations which is given by
\[
\frac{q^2}{r^4} = \frac{a \nu^2}{(1 + 2br^2 + b^2r^4)}. \tag{15}
\]

For the physical validity of the solution, the electric charge must vanish at the centre and it should increase with \(r\). This is clear form fig. 2, the electric charge function is increasing away from the centre and it is vanishing at
Table 1. Values of different physical parameters of PSR 1937+21 for $a = 0.002$, $b = 0.0094$, $\alpha = 0.00002$, $\beta = 0.009$, $M = 2.4115M_\odot$, $R = 10.3142$ km, $P_t = 8\pi p_r$, $P_r = 8\pi p$, $D = 8\pi q$, $\Delta_i = 8\pi \Delta$.

| $r/R$ | $P_r$ | $P_t$ | $D$  | $q$   | $\Delta_1$ | $V_r$  | $V_t$  | $e^r$ | $e^A$ | $Z$   |
|-------|-------|-------|------|-------|------------|-------|-------|-------|-------|-------|
| 0.0   | 0.009 | 0.009 | 0.444| 0.000 | 0.0000     | 0.5065| 0.5025| 0.135 | 1.015 | 1.718|
| 0.1   | 0.0087| 0.0087| 0.434| 0.0049| 4.74 x 10^{-6} | 0.5071| 0.502  | 0.137 | 1.015 | 1.714|
| 0.2   | 0.008 | 0.008 | 0.405| 0.0377| 2.47 x 10^{-5} | 0.5090| 0.5004| 0.143 | 1.063 | 1.651|
| 0.3   | 0.0069| 0.0069| 0.363| 0.1216| 7.11 x 10^{-5} | 0.5115| 0.4982| 0.153 | 1.143 | 1.571|
| 0.4   | 0.0056| 0.0057| 0.314| 0.2707| 1.48 x 10^{-4} | 0.5141| 0.4959| 0.164 | 1.258 | 1.466|
| 0.5   | 0.0043| 0.0045| 0.265| 0.4907| 2.48 x 10^{-4} | 0.5159| 0.4939| 0.182 | 1.407 | 1.343|
| 0.6   | 0.0031| 0.0033| 0.222| 0.7794| 3.54 x 10^{-4} | 0.5161| 0.4924| 0.205 | 1.595 | 1.207|
| 0.7   | 0.0021| 0.0024| 0.181| 1.1296| 4.46 x 10^{-4} | 0.5137| 0.4913| 0.235 | 1.207 | 1.065|
| 0.8   | 0.0012| 0.0016| 0.148| 1.532 | 5.13 x 10^{-4} | 0.5078| 0.4902| 0.270 | 2.083 | 0.922|
| 0.9   | 0.00052| 0.0009| 0.121| 1.9764| 5.51 x 10^{-4} | 0.4979| 0.4882| 0.314 | 2.383 | 0.782|
| 1.0   | 0.000 | 0.0004| 0.097| 2.4535| 5.61 x 10^{-4} | 0.483 | 0.4841| 0.367 | 2.719 | 0.649|

Fig. 1. Variation of the metric function $e^\lambda$ with the fractional coordinate $r/R$ for PSR 1937+21. For this graph, the numerical values of the parameters are as follows: $a = 0.002$, $b = 0.0094$, $R = 10.3142$, $M = 2.4115M_\odot$.

The variation of the electric charge $q$ with the fractional coordinate $r/R$ for PSR 1937+21 is shown in Fig. 2. For this graph, the values of the parameters are: $b = 0.0094$, $\alpha = 0.00002$, $R = 10.3142$, $M = 2.4115M_\odot$.

The variation of density $\rho$ with the fractional coordinate $r/R$ for PSR 1937+21 is shown in Fig. 3. For this graph, the parameter values of the constants are as follows: $a = 0.002$, $b = 0.0094$, $\alpha = 0.00002$, $\beta = 0.009$, $R = 10.3142$, $M = 2.4115M_\odot$.

The density profile has been shown in fig. 3; we observed that the density is maximum at the centre and it is decreasing throughout the star.
we get, from the centre. Pressure is maximum at the centre and is decreasing away.

Fig. 4. Variation of mass function $m(r)$ with the radial vector $r$. For this graph, the parameter values of constants are as follows: $a = 0.002$, $b = 0.0094$, $\alpha = 0.00002$, $\beta = 0.009$.

By inserting the value of $\lambda(r)$ and $q(r)$ into eq. (17), we get

$$m(r) = \frac{r^3}{2} \left[ (-2a + 2b - a^2 r^2 + b^2 r^2 + \alpha r^2) \right] \frac{1}{(1 + br^2)^2}. \tag{18}$$

For any physical acceptable models the mass function must be increasing with its radial coordinate and it should be zero at the centre. As we can see from fig. 4 the mass function is monotonically increasing with $r/R$ and is zero at the centre.

Now our next aim is to determine the metric function $\nu$ for examining other physical properties of the compact star. For this purpose we will consider the expression for radial pressure which is positive finite inside the star and zero at the boundary of the star. Let us consider the expression of radial pressure $p_r$ as

$$p_r = \frac{\beta (1 - br^2)}{8\pi (1 + 2br^2 + b^2r^4)}. \tag{19}$$

Since $p_r$ is zero at $r = \frac{1}{\sqrt{\beta}}$, i.e. $p_r(\frac{1}{\sqrt{\beta}}) = 0$, then $R = \frac{1}{\sqrt{\beta}}$ will be the radius of the star. The behavior of radial pressure is shown in fig. 5, which shows that the radial pressure is maximum at the centre and is decreasing away from the centre.

From eq. (11) together with eqs. (14), (15) and (19), we get,

$$\nu' = \left[ \frac{(2b - 2a + \beta) + (b^2 - a^2 - \beta b - \alpha)r^2}{1 + ar^2} \right] \frac{r}{(1 + 2br^2 + b^2r^4)} \tag{20}$$

We integrate the above equation (20) with respect to "$r$", then we get the metric function $\nu$ of the form

$$\nu = \frac{(2b - 2a + \beta) + (b^2 - a^2 - \beta b - \alpha)r^2}{a^2(1 + ar^2)} \ln(1 + ar^2) \tag{21}$$

where $A$ is the constant of integration.

Let us discuss about the physical behavior of the metric function $\nu$. For any physically acceptable models, Lake [3] has proved in general that the metric function $e^\nu$ must increase with $r$ and it should be free from singularity at any point of the star. From the plot of fig. 6, $e^\nu$ is monotonically increasing with the increase of $r$ and it is free from singularity at every point.

Then the tangential pressure $p_t$ and the anisotropic factor, $\Delta = p_t - p_r$, are determined as

$$p_t = \frac{p_{t1} + r^2[p_{t2} + p_{t3} + p_{t4}]}{32\pi (1 + ar^2)^2(1 + br^2)^3} \tag{22}$$

$$\Delta = \frac{\beta(-1 + b^2r^4)(1 + ar^2)^2 + p_{t1} + r^2[p_{t2} + p_{t3} + p_{t4}]}{32\pi(1 + ar^2)^2(1 + br^2)^3} \tag{23}$$
compact objects. The behavior of the anisotropy allows to construct more realistic star, while the anisotropy factor is zero at the center and monotonically increasing away from the centre. This feature of the anisotropy allows to construct more realistic compact objects. The behavior of the anisotropy factor can be seen in fig. 8.

\[ p_{t1} = \beta^2 r^2 (-1 + br^2)^2 (1 + br^2) - 2 \beta [ -2 + a^2 r^4 + \alpha^4 + 2b^2 r^6 + b^4 r^8 + 2b^2 (1 + 4ar^2 + 2a^2 r^4) + b^2 r^4 - b^2 (a^2 + \alpha)^2]; \]

\[ p_{t2} = 16a^3 r^2 + 5b^4 r^4 + b^6 r^6 + a^4 r^4 (5 + br^2) + a (-12 + \alpha^2) + bar^2 (12 + \alpha^2) r^4. \]

\[ p_{t3} = -2b^2 (6 + \alpha^2) - 6b^2 (-2 + \alpha^2) - 8a^2 r^2 + 2a^2 r^2 + b (3 + \alpha^2)^3; \]

\[ p_{t4} = -2a^2 [-6 + 5b^2 r^4 + 3\alpha^4 + b^4 r^6 + br^2 (10 + \alpha^2)]. \]

The plot for \( p_t \) is shown in fig. 7, it is clear from the figure that \( p_t \) is monotonically decreasing throughout the star, while the anisotropy factor is zero at the centre and monotonically increasing away from the centre. This feature of the anisotropy allows to construct more realistic compact objects. The behavior of the anisotropy factor can be seen in fig. 8.

\[ \rho = \frac{[3b^2 f_1 - \alpha f_1 + b^3 f_1^2 + f_2] \rho_r}{\beta (1 - f_b) (1 + f_b)}, \]  
\[ \rho = \frac{4 (1 + f_a)^2 [3b^2 f_1 - \alpha f_1 + b^3 f_1^2 + f_2] \rho_r}{\beta^2 f_1 (-1 + f_b)^2 (1 + f_b) + f_3 + f_4 [f_4 + f_5 + f_6]}, \]

where

\[ f_1 = \frac{-(16 \pi p_e + \beta) + \sqrt{64 \pi p_e \beta + \beta^2}}{16 \pi b p_r}, \]

\[ f_2 = 2a (-3 + b f_1) - a^2 f_1 (5 + b f_1) + (6 - \alpha f_1^2). \]

\[ f_{t1} = \alpha f_1, \quad f_{t2} = b f_1, \quad f_{t3} = -2 \beta [-2 + f_a + \alpha f_1 + 2 f_3^2 - 2 f_3^2 + f_5 + f_6 + 2 (1 + f_a) + 2f_3^2] + f_2^2 - b^2 (a^2 + \alpha)^2 f_4 + 16 a^2 f_a + b^2 f_3^2 + b^2 f_3^2 + 2f_3^2 + f_5 + f_6 + 2 (1 + f_a) + 2f_3^2 + f_2^2 - b^2 (a^2 + \alpha)^2 f_4; \]

\[ f_{t4} = -2b^2 f_6 (-6 + \alpha f_1^2) - 6b^2 (-2 + \alpha f_1^2) - 8a [-b f_6 + 2a f_1 + b (3 + \alpha)^2]; \]

\[ f_6 = -2a^2 [-6 + 5f_2^2 + 3\alpha f_1^2 + f_3^2 + f_6 (10 + \alpha f_1^2)]. \]

The above relations (eqs. (24) and (25)) implies that the density is purely function of both radial and tangential pressure. So it may represent an equation of state (EOS) for the present charged compact star.

\[ \rho = \frac{1}{\sqrt{\rho}}. \]  
\[ e^{-\lambda (r)} = 1 - \frac{2M}{r} + \frac{Q^2}{R^2}, \]  
\[ e^{\nu (r)} = 1 - \frac{2M}{R} + \frac{Q^2}{R^2}, \]  
\[ q(R) = Q. \]

4 Equation of state for the present charged compact star

The equation of state is an important relation in pressure and density for any realistic matter and it is necessary to provide the equation of state (EOS) \( \rho = f(p) \) to model the present charged compact star.

From eqs. (16) and (19), we get

\[ \rho = \frac{[3b^2 f_1 - \alpha f_1 + b^3 f_1^2 + f_2] \rho_r}{\beta (1 - f_b) (1 + f_b)}, \]  
\[ \rho = \frac{4 (1 + f_a)^2 [3b^2 f_1 - \alpha f_1 + b^3 f_1^2 + f_2] \rho_r}{\beta^2 f_1 (-1 + f_b)^2 (1 + f_b) + f_3 + f_4 [f_4 + f_5 + f_6]}, \]

5 Boundary condition for determining the constants

To determine the arbitrary constants of the anisotropic charged fluid solution we must join the interior solution of metric (1) to the Reissner-Nordström metric at the boundary of the star \( r = R \). The Reissner-Nordström metric is given as

\[ ds^2 = -
\frac{(1 - 2M/r + Q^2/r^2)^{-1}}{r^2 + (1 - 2M/r + Q^2/r^2) dr^2}
- r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

The joining of metric conditions demands the continuity of \( e^\lambda, e^\nu \) and \( Q \) across the boundary \( r = R \), so that

\[ e^{-\lambda(R)} = 1 - \frac{2M}{R} + \frac{Q^2}{R^2}, \]  
\[ e^{\nu(R)} = 1 - \frac{2M}{R} + \frac{Q^2}{R^2}, \]  
\[ q(R) = Q. \]
Moreover, it is also required that the radial pressure, \( p_r \), should be zero at the boundary (continuity of the second fundamental form at the boundary), so that
\[
R = \sqrt{\frac{1}{b}}. \tag{30}
\]
Conditions (20) and (21) give the value of \( A \) as
\[
A = \frac{(1 + aR^2)^2}{(1 + bR^2)^2} e^{-\frac{2a+b+\sqrt{a^2+b^2}}{2a+b+\sqrt{a^2+b^2}}} \tag{31}
\]

### 6 Physical features of the charged anisotropic models

#### 6.1 Causality condition

For any physical acceptable anisotropic charged fluid models, the speed of sound must be less than the speed of light inside stars. Then the following condition must be satisfied: \( V_r = \sqrt{\frac{d\rho_r}{d\rho}} < 1 \) and \( V_t = \sqrt{\frac{d\rho_t}{d\rho}} < 1 \). From fig. 9, we have observed that the radial and transverse speed of sound are always less than the speed of light.

#### 6.2 Stability condition

##### 6.2.1 Adiabatic index

Heintzmann and Hillebrandt [51] have proposed that neutron star models are stable with anisotropic equation of state if \( \Gamma_i > 4/3 \). Also in Newton’s theory of gravitation, it is well demonstrated that there is no upper mass limits when the equation of state has an adiabatic index \( \Gamma_i > 4/3 \). The adiabatic indexes corresponding to radial and transverse pressure are defined as
\[
\Gamma_r = \left( \frac{\rho + p_r}{p_r} \right) \left( \frac{dp_r}{d\rho} \right), \tag{32}
\]
\[
\Gamma_t = \left( \frac{\rho + p_t}{p_t} \right) \left( \frac{dp_t}{d\rho} \right). \tag{33}
\]

As we can see from fig. 10 the radial and tangential adiabatic indexes have values higher than 4/3 everywhere inside the star. Then our charged fluid models are stable.

##### 6.2.2 Herrera Cracking concept

In the charged anisotropic compact star models, to examine the stability of the model we plot the radial \( V_r^2 = dp_r/d\rho \) and transverse \( V_t^2 = dp_t/d\rho \) sound speeds in fig. 11. It shows that \( V_r^2 \) and \( V_t^2 \) satisfy the inequalities \( 0 \leq V_r^2 \leq 1 \) and \( 0 \leq V_t^2 \leq 1 \) everywhere within the charged anisotropic stellar object [52,53].

Also, to discuss about the stability of an anisotropic charged compact star i.e., whether the anisotropic charged matter distribution is stable or not, we will use the concept of Herrera [52] cracking concept which states that the region is potentially stable where the radial velocity of sound is greater than the tangential velocity of sound.

6.2.3 Tolman-Oppenheimer-Volkoff (TOV) equation

Let us write the generalized Tolman-Oppenheimer-Volkoff (TOV) equation to examine the stability of the model under stable equilibrium configuration as
\[
-M_G(\rho + p_r) \frac{e^{\lambda - \nu}/2}{r^2} - \frac{dp_r}{dr} + \sigma q \frac{q}{4\pi r^2} \frac{d\nu}{dr} + \frac{2}{r}(p_t - p_r) = 0, \tag{34}
\]
where \( M_G \) is the gravitational mass within the radius \( r \) and \( \sigma \) is the charged density, which are defined as
\[
M_G(r) = \frac{1}{2} r^2 \nu e^{(\nu - \lambda)/2} \quad \text{and} \quad \sigma = \frac{e^{-\lambda/2}}{4\pi} \frac{dq}{dr}. \tag{35}
\]

On inserting the value of \( M_G \) and \( \sigma \) into the above equations, we get
\[
\frac{1}{2} \nu'(\rho + p_r) - \frac{dp_r}{dr} + \sigma q \frac{dq}{4\pi r^2} \frac{d\nu}{dr} + \frac{2}{r}(p_t - p_r) = 0. \tag{36}
\]

The above TOV equation describes the equilibrium condition for a charged anisotropic fluid subject to gravitational force \( (F_g) \), hydrostatic force \( (F_h) \), electric charge force \( (F_e) \) and anisotropic stress \( (F_a) \) so that
\[
F_g + F_h + F_e + F_a = 0, \tag{37}
\]
where the explicit form of the above forces can be defined as
\[
F_g = -\frac{1}{2} \nu'(\rho + p_r) = -\frac{F_{g1}}{32\pi(1 + ar^2)^2}\left[\frac{F_{g2}}{(1 + br^2)^3}\right], \tag{38}
\]
\[
F_h = -\frac{dp_r}{dr}, \tag{39}
\]
\[
F_e = \frac{q}{4\pi r^2} \frac{dq}{dr} = \frac{\alpha r^3}{4\pi} \frac{(3 + b^2 r^2)}{(1 + br^2)^3}, \tag{40}
\]
\[
F_a = \frac{2}{r}(p_t - p_r) \tag{41}
\]
where
\[
F_{g1} = r[2b - 2a + \alpha + (b^2 - a^2 - \beta b - a)r^2],
\]
\[
F_{g2} = \beta + 6b + 3b^2 r^2 - \alpha^2 - \beta b r^4 + b^3 r^4 - \beta a r^4 + 2a(-3 + br^2) - a^2 r^2(5 + br^2).
\]

In fig. 12 we have shown the variation of the different forces of the TOV equation. From this plot it is observed that the system is in static equilibrium under four different forces, e.g. gravitational, hydrostatic, electric and anisotropic forces. Moreover, the strong gravitational force is counterbalanced by the joint action of hydrostatic and electric forces and anisotropic force. Also as we can see that the effect of the anisotropic stress has a negligible effect to balance this mechanism.
Fig. 9. Variation of the radial velocity (left panel) and tangential velocity (right panel) with the fractional coordinate $r/R$ for PSR 1937+21. For this graph, the parameter values of the constants are as follows: $a = 0.002$, $b = 0.0094$, $\alpha = 0.00002$, $\beta = 0.009$, $R = 10.3142$, $M = 2.4115M_\odot$.

Fig. 10. Variation of the adiabatic index corresponding to radial pressure (left panel) and tangential pressure (right panel) with the fractional coordinate $r/R$ for PSR 1937+21. For this graph, the parameter values of the constants are as follows: $a = 0.002$, $b = 0.0094$, $\alpha = 0.00002$, $\beta = 0.009$, $R = 10.3142$, $M = 2.4115M_\odot$.

Fig. 11. Variation of the square of the radial velocity, $V_r^2$, and tangential velocity, $V_t^2$, with the fractional coordinate $r/R$ for PSR 1937+21. For this graph, the parameter values of the constants are as follows: $a = 0.002$, $b = 0.0094$, $\alpha = 0.00002$, $\beta = 0.009$, $R = 10.3142$, $M = 2.4115M_\odot$. 
Fig. 12. Variation of different forces with the fractional coordinate \( r/R \) for PSR 1937+21. For this graph, the parameter values of the constants are as follows: \( a = 0.002, b = 0.0094, \alpha = 0.00002, \beta = 0.009, R = 10.3142, M = 2.4115M_\odot \).

Fig. 13. Variation of the redshift \( Z \) with the fractional coordinate \( r/R \) for PSR 1937+21. For this graph, the parameter values of constants are as follows: \( a = 0.002, b = 0.0094, \alpha = 0.00002, \beta = 0.009, R = 10.3142, M = 2.4115M_\odot \).

6.3 Maximum limit of mass-radius ratio

The maximum limit of the mass-radius ratio is proposed by Buchdahl [5] for an isotropic compact star, which satisfies the inequality \( 2M/R \leq 8/9 \). This inequality implies that the mass of a static isotropic compact star cannot be arbitrarily large. Also Bohmer and Harko [54] provided the lower limit of the mass-radius ratio for the charged compact star model with \( Q < M \),

\[
\frac{Q^2}{2R} \left( \frac{18R^2 + Q^2}{12R^2 + Q^2} \right) \leq \frac{2M}{R}. \tag{42}
\]

In addition to that the upper bound of the mass of the charged sphere was generalized by Andréasson [55] and it was proved that

\[
\sqrt{M} \leq \frac{\sqrt{R}}{3} + \sqrt{\frac{R^2 + 3Q^2}{9R}}. \tag{43}
\]

Then the range of mass \( M \) can be given by the following inequality:

\[
\frac{Q^2}{2R} \left( \frac{18R^2 + Q^2}{12R^2 + Q^2} \right) \leq M \leq \frac{\sqrt{R}}{3} + \sqrt{\frac{R^2 + 3Q^2}{9R}}. \tag{44}
\]

6.4 Surface redshift

Let us write the effective mass for the charged compact star as

\[
M_{\text{eff}} = 4\pi \int_0^R \left( \rho + \frac{E^2}{8\pi} \right) r^2 dr = \frac{1}{2} R \left[ 1 - e^{-\lambda(R)} \right]. \tag{45}
\]

Then the compactness factor \( u \) for above mass is defined as

\[
u = \frac{M_{\text{eff}}}{R}. \tag{46}
\]

Now the surface redshift for the above compactness \( u \) can be written as

\[
Z_s(R) = \sqrt{\frac{1}{1 - 2u}} - 1 = e^{\lambda(R)/2} - 1. \tag{47}
\]

As we can see from eq. (47) the surface redshift function is dependent on the compactness factor \( u \). Since \( Z_s(R) \) is monotonically increasing with increase of \( u \), then the surface redshift function cannot have any arbitrary values as \( u \) is always less than \( 4/9 \) (Buchdahl limit). The variation of the redshift of the charged anisotropic solution is shown in fig. 13. We observed from this figure that the redshift is maximum at the centre and minimum at the boundary of the star.

7 Conclusion

In the present article we have obtained the charged anisotropic compact star model via radial pressure. The
Einstein–Maxwell field equations are solved by using the metric function $\Lambda$ [which is a generalized form of the earlier work by Maurya et al. [56]], the radial pressure and the electric charge function. The obtained generic function $\nu$ is physically valid due to its monotonic increasing nature away from the centre and $\nu(0) \neq 0$. The charged anisotropic solution having the following features:

i) The physical requirements of the mass function have been investigated. Recently Lake [3] and Maurya et al. [56] have proved in general that the mass function should be zero at the centre and must have increasing nature throughout away from the centre. In the present article, the investigated mass function is satisfying the above physical requirements (fig. 4).

ii) The energy density of the models is positive and decreasing away from the centre, which can be seen from fig. 3.

iii) The radial pressure at the centre is $p_r(0) = \beta/8\pi$, which is positive. Moreover it is zero at the boundary i.e. $p_r(R) = p_r(\frac{\beta}{8\pi}) = 0$, which gives the radius of the star, $R = \frac{\beta}{8\pi}$. From fig. 5, it is observed that $p_r$ is decreasing outward and zero on the boundary.

iv) The metric functions $e^\nu$ and $e^\lambda$ at the centre are:

$$e^{\lambda(0)} = 1$$
$$e^{\nu(0)} = A\sqrt{\frac{a^2 + b^2 - 2ab\beta + \beta(a + b) - a}{2a^2}}.$$  

As we can observe, both metric functions are positive and non-singular at the centre and also both have increasing nature (figs. 1 and 6).

v) The anisotropic factor $\Delta = p_t - p_r$ is always positive inside the star. This implies that $p_t > p_r$ and the force $F_a = \frac{2(p_t - p_r)}{r}$ is directed outward, which allows the construction of more realistic compact objects [17]. The behavior is shown in fig. 8.

vi) It is observed from fig. 9 that the speed of sound is less than the speed of light throughout inside the star. Moreover, the radial velocity has no monotonic decreasing nature, while the tangential velocity has monotonic decreasing nature inside the star.

vii) Stability conditions: we have investigated the stability of the charged compact star model via three different ways: Herrera cracking concept, adiabatic index and TOV equations. According to the Herrera cracking concept, the region is potentially stable where the radial speed of sound is greater than the tangential velocity and there is no change in sign of $V_r^2 - V_t^2$ everywhere inside the star, which can be observed from fig. 11 and fig. 14. Moreover the sufficient condition of stability for any models has an adiabatic index $\Gamma_r > 4/3$. It is observed from fig. 10, that $\Gamma_r$ and $\Gamma_t$ have values higher than 4/3 everywhere inside the star. Now let us examine the equilibrium condition for the present models, we observe from fig. 12 that the gravitational force $F_g$ is counterbalanced by the joint action of the hydrostatic force $F_s$ and the electric force $F_e$, while the anisotropic stress has a negligible effect to balance this system. This implies that our system is in equilibrium stage.

The surface redshift of the model depends upon the compactness factor $u$. If the value of compactness increases, then the corresponding surface redshift will also increase. This imply that the redshift of the model cannot be arbitrarily large. The surface redshift of the model is turns out to be $Z_s = 0.6491$, which is in good agreement with the present charged compact star model [54, 57, 58]. As a final comment, the author has presented a charged anisotropic solution which represents the known compact star with their observed mass and radius (table 2). We have also obtained the equation of state (EOS) for the present compact star model, which is the most important physical property to describe the structure of any realistic matter. As we can see from eqs. (24) and (25) the density is purely a function of pressure. Hence we conclude that this approach may help to describe the structure of compact stars. Thus our present charged anisotropic solution might have some astrophysical application in the future.

SKM acknowledges support and encouragement from the Authority of University of Nizwa, Nizwa, Sultanate of Oman for this research work.

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