Asymptotic stability of vacuum twisting type II metrics

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Abstract

We generalize the result of Lukács et al. on asymptotic stability of the Schwarzschild metric with respect to perturbations in the Robinson-Trautman class of metrics to the case of Petrov type II twisting metrics, under the condition of asymptotic flatness at future null infinity. The Bondi energy is used as the Lyapunov functional and we prove that the “final state” of such metrics is the Kerr metric.

Conventions and notation

The notation used here is similar to that of [1], with few exceptions. Partial derivatives are denoted by comma. Whenever asymptotically flat space-time is mentioned, it is understood as space-time that admits a piece of future null infinity \( R \times S^2 \). The standard metric of \( S^2 \) in stereographic coordinates is

\[
2P^{-2}d\xi d\bar{\xi}, \quad \text{where} \quad P = 1 + \frac{1}{2}|\xi|^2.
\]

Any other function \( P \) divided by \( P \) will be denoted by \( \hat{P} \).

1 Introduction

The purpose of this paper is to show the Lyapunov stability of metrics known as diverging, twisting Petrov type II metrics. They can be expressed by a null tetrad [1]

\[
\begin{align*}
g & = 2(\theta^0 \theta^1 - \theta^2 \theta^3), \\
\omega & = du + (Ld\xi + \text{c.c.}), \\
\theta^1 & = H\omega + dr + (Wd\xi + \text{c.c.}), \\
\theta^2 & = P^{-1}(r - i\Sigma)d\xi, \\
\theta^3 & = \bar{\theta}^2,
\end{align*}
\]

\[
\begin{align*}
\omega \wedge d\omega & = \frac{2\Sigma}{iP^2}du \wedge d\xi \wedge d\bar{\xi}, \\
H & = -r(\ln P)_u - \frac{mr + M\Sigma}{r^2 + \Sigma^2} + K/2, \\
K & = 2P^2 \text{Re}[\partial(\bar{\partial}\ln P - \bar{L},u)], \\
\partial & = \partial_{\xi} - L\partial_u, \\
W & = -(r + i\Sigma)L_u + i\partial\Sigma.
\end{align*}
\]

(1.1)

The shear-free repeated principal null direction \( \partial_r \) of the Weyl tensor is related to \( \omega \) via \( \omega = g(\partial_r) \). \( P, m \) and \( L \) are, respectively, two real and one complex functions independent of \( r \). Einstein equations in vacuum reduce to

\[
P^{-3}M = \text{Im} \partial^2\bar{\partial}^2V,
\]

(1.2a)

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\[ \partial(m + iM) = 3(m + iM)L_{uu}, \]  
(1.2b)

\[ (P^{-3}(m + iM) - \partial^2 \bar{\partial}^2 V)_{uu} = -P|I|^2, \]  
(1.2c)

where \( P = V_u \) and \( I = (\partial + (\partial \ln P - \bar{L}_u))(\partial \ln P - \bar{L}_u) \). Define \( G = -\partial \ln P + L_u \).

Equation (1.2c) can be interpreted as the Bondi mass-loss formula\([3]\): if \( g \) is asymptotically flat, \( |I|^2 \) is proportional to the Bondi news function, and the quantity differentiated with respect to \( u \) on the l.h.s. of (1.2c) is related to the modified Bondi mass aspect—the Bondi energy is given by \([3]\)

\[ E = \frac{1}{4\pi} \int_{S^2} \left( \frac{m}{P^3} - P^3 \text{Re} \partial^2 \bar{\partial}^2 V \right). \]  
(1.3)

Condition (1.2a) is compatible with (1.2c) because the r.h.s of (1.2c) is real.

Apart from the freedom of fixing the origin of \( r \), the allowed coordinate transformations are of the form

\[ u \mapsto u' = F(u, \xi, \bar{\xi}), \]  
(1.4a)

\[ \xi \mapsto \xi' = h(\xi), \quad h, \bar{h} = 0, \]  
(1.4b)

Under (1.4a), \( P \mapsto P' = (F_u)^{-1}P \) and \( L \mapsto L' = -\partial F \). In particular, by means of (1.4a) one can achieve \( P_u = 0 \). Then the remaining freedom is (1.4b) with \( F_{uu} = 0 \). The function \( G \) transforms as \( G \mapsto G' = h'^{-1}G - h''/(2h') \) under (1.4b) and is preserved by (1.4a).

Asymptotic stability of the Schwarzschild solution within the Robinson-Trautman (RT) class of metrics\([4]\) has been established in \([5]\). The Lyapunov functional used by Lukács et al. was \( \int S K^2 \), where \( S = \{u = \text{const}, r = \text{const}\} \) and \( K \) is its Gauss curvature\([4]\), and they had to apply the Laplace operator of \( S (2P^2 \partial_\xi \partial_{\bar{\xi}}) \) to the Robinson-Trautman equation to make it an evolution of \( K \) alone. This seems rather complicated as compared to noting that \( \int \hat{P}^{-3} \) also decreases with \( u \). The authors of \([5]\) most probably were unaware that the average \( \langle m\hat{P}^{-3} \rangle_{u=\text{const}} \) is the Bondi mass of an asymptotically flat RT space-time and that the RT equation is responsible for the energy loss, but still it is evident that for positive, smooth \( \hat{P} \) and \( m > 0 \) that average is positive and that its \( u \)-derivative is negative (being the average of the news function). Establishing this is attributed to D. Singleton in \([6]\).

In the work presented here, we have no natural “finite \( r \)” and “constant retarded time” family of two-dimensional closed surfaces. Our idea is to assume asymptotic flatness of \( g \) and use the Lyapunov functional which is identical with the Bondi energy (which we also assume to be positive), \( \mathcal{L}[g] = E \), where \( E \) is given by (1.3). Then the mass-loss equation (1.2c) shows that \( \frac{d}{du}\mathcal{L}[g] \leq 0 \) and \( \mathcal{L}[g] = \text{const} \iff I = 0 \). Therefore it is important

\(^2\)In the published version this sentence is typeset in a way that makes no sense
to know the properties of twisting metrics with vanishing news, and whether \( I = 0 \) is compatible with asymptotic flatness. We will show that the only possible “fixed point” solution that is asymptotically flat is the Kerr metric.

2 “Non-radiating” type II metrics

The condition \( I = 0 \) is written as

\[
0 = \bar{I} = (G - \partial) G .
\]

(2.1)

In the gauge \( P = 1, G = L_u \) and this is integrated with the help of the Cauchy-Riemann function, similarly as \((1.2b)\)\[7\]. Suppose we have such a function \( \chi \) that \( \partial \bar{\chi} = 0 \). Then \( L = \frac{\bar{\chi}}{\chi} \), and due to \( [\partial_u, \partial] = -L_u \partial_u \) we have \( G = \bar{\chi}_u \) and \( m + iM = (\bar{\chi}_u)^3 \), but such solution is defined up to multiplication by a Cauchy-Riemann function. Let us take, similarly as in \[1\], \( m + iM = 2G^3 A(u, \xi, \bar{\xi}) \). Imposing \((1.2b)\) we obtain \( \partial A = 0 \). But the field equation \((1.2c)\) yields \( (m + iM)_u = 0 \), so \( G^3 A \) is \( u \)-independent. Applying \( \partial \) to \( G^3 A \) we get

\[
\partial(G^3 A) = (G^3 A)_{\bar{\xi}} = 3AG^2 \partial G = 3G^4 A .
\]

(2.2)

Taking \( u \)-derivatives of both sides of \((2.2)\) gives

\[
0 = (G^3 A)_{\xi u} = 3[G(G^3 A)]_{,u} = 3G_{,u} G^3 A + 3G(G^3 A)_{,u} = 3G^3 A G_{,u} .
\]

(2.3)

The conclusion is that \( G_{,u} = 0 \) is not an extra assumption as stated in \[1\]. Also, \( A_{,u} = A_{,\xi} = 0 \) and we replace \( A \) by \( \bar{a}(\xi) \), an antiholomorphic function. The condition \((2.1)\) gives\[1\] \( G = -(\xi + \bar{g}(\xi))^{-1} \), where \( g \) is holomorphic (not to be misidentified with the metric).

Remark 2.1 The function \( G^{-1} \) is harmonic.

Now, following \[1\], we consider separately the following two subclasses of metrics with vanishing “news”, for which \( L_u \) is either transformable to zero or not.

2.1 \( I = 0, L_u \) transformable to 0

In this case \( \partial_u \) is a Killing vector of \( g \). The field equations give

\[
P = A\xi + (B\xi + \text{c.c.}) + C ,
\]

\[
m + iM = \bar{z}(\xi) ,
\]

\[
L = P^{-2} \left( \bar{l}(\xi) - \frac{1}{2} \int \frac{\bar{z}(\xi)}{(A\xi + B)^2} d\xi \right) .
\]

(2.4)

For \( AC - |B|^2 > 0 \), linear transformation of \( \xi \) gives \( P = P_S \). Note that \( m + iM \) cannot be regular unless \( z = \text{const} \) and \( l \) is linear, and the metric
cannot be asymptotically flat unless $M = 0$. The reason is that $\xi^{-1}$ terms in front of $m$ can be absorbed in $u$ by means of (1.4a) with $F_{uu} = 0$, under which $L \rightarrow L - F_{\xi}$.

Similar terms in front of $M$ would survive since $F$ is real. Therefore $g$ is a vacuum Kerr-Schild metric. According to [8], the only metric of this kind with singularities inside a spatially bounded region is the Kerr metric. (Another way to see this is to use the Weyl scalar $s$ [9] and deduce that $I = 0$ implies that $g$ is of Petrov type D, and the only vacuum solution of this type with a $S_2$ set of generators of future scri is the Kerr metric.)

2.2 $I = 0$, $L_u$ not transformable to 0

In this case for $P_s = 0$, say $P = P_s$, the field equations give

$$m + iM = 2P_s^3G^3a(\bar{\xi})$$,
$$L = (G + (\ln P)_{\xi})u + P^{-1}G\ell(\xi, \bar{\xi})$$,
$$\ell = -\int a(\xi)\bar{G}(\bar{\xi}, \xi)G(\xi, \bar{\xi})^{-1}d\xi + \varphi(\bar{\xi})$$,  \hspace{1cm} (2.5)$$

where $a$ and $\varphi$ are arbitrary holomorphic functions of $\xi$. To begin discussing regularity of the metric components, consider the blow-up points of $G$, i.e. the set $N = \{\xi \in \mathbb{C}: \xi + \bar{g}(\bar{\xi}) = 0\}$. From the harmonicity of $G^{-1}$ it follows that $N$ cannot be a two-dimensional subset of $R^2$, so (if nonempty), it must be either a curve or a set of points.

If $N \neq \emptyset$, we can solve its defining equation. Let $\xi = |\xi|e^{i\beta}$ and $g(\xi) = |g|e^{i\gamma}$, where $\beta$ and $\gamma$ are real phases. On $N$ we have $|g| = |\xi|$ and since $\xi = -\bar{g}$, their phases satisfy $\pi - (\beta + \gamma) = 0 \mod 2\pi$. This gives $g = -|\xi|e^{-i\beta}$, but $g$ is holomorphic, so the only possibility is $|\xi| = \text{const}$ and $\bar{g} = \text{const}\xi^{-1}$. Therefore $N$ is a circle or a point (respectively, when const $> 0$ or const $= 0$). In both cases we cannot cancel such set of zeros of $G^{-1}$ by specifying the function $\bar{a}$, because it is antiholomorphic.

The most interesting case is that of $N = \emptyset$. A family of $g$’s giving rise to regular $m + iM$ is given by

$$\overline{g(p,\alpha)} = |p|e^{i\alpha}\bar{\xi}^{-1}, \hspace{1cm} R \ni \alpha \neq 0 \mod \pi$$,  \hspace{1cm} (2.6)$$

Remark 2.2 $P_sG(p,\alpha)\bar{\xi}^{-1}$ is a regular function, so the regularity of $m + iM$ can be achieved by taking $\bar{a} = \text{const}\bar{\xi}^{-3}$.

Remark 2.3 Adding a linear function of $\bar{\xi}$ to $\overline{g(p,\alpha)}$ does not spoil asymptotics of $G$, but always introduces nontrivial zeros in its denominator.

This, together with harmonicity of $G^{-1}$ suggests that $G(p,\alpha)$ defined by (2.6) is the only possible function $G$ that can give finite $m + iM$. Note that for
\( \alpha \notin \pi Z \), the Newman-Unti-Taub (NUT) charge \( M \) cannot vanish: \( L_u \) is not transformable to zero, so \( m + iM \) cannot be made constant by means of (1.4a), therefore \( M \neq 0 \), and “scri” would have topology \( S_1 \times S_2 \) rather than \( R \times S_2 \).

A more direct argument showing singularity in \( L \) is based on finding \( \ell \sim |\xi|^2 + 2|p|^{-1}(\sin \alpha) \ln(|\xi|^2 + |p|^{-1} e^{i\alpha}) \) and noting that \( P_{SG}^{-1}G_{(p,\alpha)}\ell \) must be singular because of the term \( |\xi|^{-2} \), since \( P_{SG}G_{(p,\alpha)}\xi^{-1} \) is well defined.

Summary

**Proposition 1** Regularity conditions imposed on the two subclasses (2.4) and (2.5) of “news-free” metrics (1.1) rule out the second subclass and single out the Kerr solution out of the first subclass.

**Proposition 2** Suppose \( g \) given by (1.1) is asymptotically flat and equation (1.2c) admits a solution that can be continued to \( u = \infty \). Then the Kerr metric is asymptotically stable and \( g \) tends to Kerr as \( u \to \infty \).

Problem of the existence and regularity of solutions of (1.2c) as well as their approximations shall be treated elsewhere.

Note that if one wishes to treat these metrics as perturbations of the Schwarzschild metric, a stationary perturbation leading to the Kerr metric must be included. In other words: metrics of Petrov type II with twist cannot be viewed as nontrivially evolving corrections to Robinson-Trautman metrics.

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References

[1] Stephani H., Kramer D., MacCallum M. A. H., Hoenselaers C. and Herlt E., *Exact Solutions to Einstein’s Field Equations, Second Edition*, 2003 (Cambridge: Cambridge University Press)

[2] Robinson I., Robinson J. R., *Vacuum metrics without symmetry*, *Int. J. Theor. Phys.*, 7 (1969), 231.
[3] Natorf W., Tafel J. *Asymptotic flatness and algebraically special metrics*, Class. Quantum Grav. **21** (2004), 5397.

[4] Robinson I., Trautman A., *Some spherical gravitational waves in general relativity*, Proc. Roy. Soc. Lond. A **265** (1962), 463.

[5] Lukács B., Perjés Z., Porter J., Sebestyén Á., *Lyapunov Functional Approach to Radiative Metrics*, Gen. Rel. Grav. **16** (1984), 691.

[6] Chow E. W. M., Lun A. W.-C., *Apparent Horizons in Vacuum Robinson-Trautman Spacetimes*, J. Austral. Math. Soc. Ser. B **41** (1999), 217.

[7] Stephani H., *Algebraically special, diverging vacuum and pure radiation fields revisited*, Gen. Rel. Grav. **16** (1983), 173.

[8] Kerr R.P., Wilson W.B. *Singularities in the Kerr-Schild metrics*, Gen. Rel. Grav. **10** (1979), 273.

[9] G. J. Weir, Kerr R. P., *Diverging type-D metrics*, Proc. R. Soc. London, Ser. A **355** (1977), 31.