Nonleptonic $K \to 2\pi$ decay dynamics *

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Abstract

Using Chiral Perturbation Theory to properly account for the dynamics of nonleptonic $K \to 2\pi$ decays, we found the Standard Model prediction for the CP violating ratio $\text{Re}(\epsilon'/\epsilon) = (14 \pm 5) \times 10^{-4}$, where isospin breaking effects are included, in perfect agreement with the current experimental world average. Similar results have been reported by a recent release of improved lattice data.

Keywords: Kaon decays, CP violation, Standard Model, Chiral Perturbation Theory, Lattice QCD

1. Introduction

The large asymmetry between matter and antimatter in the observable universe requires the existence of additional CP sources beyond the Standard Model (SM). Accordingly, the CP violating ratio $\epsilon'/\epsilon$ provides an excellent road towards the discovery of New Physics (NP), since it is linearly proportional to the only existing CP source in the SM. In addition, its study offers a formidable test of flavor-changing neutral-currents (FCNCs) transitions and therefore the Glashow-Iliopoulos-Maiani (GIM) mechanism.

The current experimental world average of $\epsilon'/\epsilon$ from NA48 [1] and KTeV [2, 3],

$$\text{Re}(\epsilon'/\epsilon)_{\text{exp}} = (16.6 \pm 2.3) \cdot 10^{-4},$$

exhibits the existence of direct CP violation in decay transitions of $K \to 2\pi$, in addition, it is highly sensitive to new sources of CP violation due to its small size.

Since the establishment of its experimental measurement, the SM prediction of $\epsilon'/\epsilon$ has been subject of several years of discussions. The first next-to-leading order (NLO) calculations obtained SM values of order $\sim 10^{-4}$ [4–9], in clear conflict with Eq. (1). However, it was soon realised by the authors of Refs. [10–12], that an important ingredient had been missed in the previous predictions, the inclusion of final-state interactions (FSI) between the two emitted pions. These corrections were computed within the low-energy effective realization of quantum chromodynamics (QCD), that is Chiral Perturbation theory ($\chi$PT). Once they were included in the calculation, the SM prediction was found consistent with Eq. (1), albeit with large uncertainties of non-perturbative origin.

The development of advanced lattice QCD techniques as well as the increasing computer capabilities resulted in a successful calculation of the $\Delta I = 3/2$ $K^+ \to \pi^+\pi^0$ amplitude [13–15], and the first statistically-significant signal of the $\Delta I = 1/2$ enhancement [16], both carried out by RBC-UKQCD collaboration in agreement with previous analytical computations [17–26].

Based on these results, the RBC-UKQCD collaboration reported as well the prediction for the CP violating ratio, $\text{Re}(\epsilon'/\epsilon) = (1.4 \pm 6.9) \cdot 10^{-4}$ [15, 27], with a tension of $\sim 2\sigma$ with Eq. (1). This result revived again the old debate [28–31], bringing back the past approaches with low $\epsilon'/\epsilon$ predictions [32, 33]. However, the same
lattice study \[27\] reported a $I = 0$ phase shift about $\sim 3\,\sigma$ away from its dispersive prediction, exhibiting that the long-distance re-scattering of the final pions in $K \to 2\pi$ was not well-controlled in the simulation.

Although the importance of the pion dynamics in the determination of $e'/\epsilon$ was already pointed out in Refs. \[10\]-\[12\], the results required a thorough revision with an update of the different inputs. At the end of 2017, the updated prediction of $e'/\epsilon$ \[35\] was found again in good agreement with Eq. (1).

Last year, the authors of Ref. \[36\] performed the update of the last missing ingredient, isospin breaking corrections to $e'/\epsilon$. Including all the sources of uncertainty, the prediction for the CP violating ratio was found in perfect agreement with Eq. (1).

The results of Refs. \[36\] were presented to the high energy physics community in several international conferences \[37\]-\[39\], making necessary a revision of the lattice QCD prediction \[15\]-\[27\] to confirm the findings of Refs. \[35\]-\[36\].

After years of effort to develop the lattice understanding of the pion dynamics, the RBC-UKQCD outcomes \[40\] emerged in April 2020, which are outlined in the following \[15\]-\[40\]:

- Lattice strong phase shifts in accordance with dispersive predictions.
- $K \to 2\pi$ isospin amplitudes $A_{0,2}$ in good agreement with their experimental measurements.
- $e'/\epsilon$ prediction in agreement with Eq. (1),

$$\text{Re} (e'/\epsilon)_{\text{lattice}} = (22 \pm 8) \times 10^{-4}, \quad (2)$$

where isospin breaking effects are not included in the central value.

These lattice results close the long debate on the SM prediction of $e'/\epsilon$, confirming the observations made 20 years earlier in chPT \[10\]-\[12\] \[41\]-\[42\], and offering good evidence for their updates \[35\]-\[36\] \[43\].

In the following sections, we highlight the essential dynamical features of $K \to 2\pi$ decays and briefly summarize the updated $e'/\epsilon$ prediction from Ref. \[36\].

2. Dynamics of $K \to 2\pi$ decays

The physical kaon decay amplitudes can be expressed in terms of isospin components as

$$A(K^0 \to \pi^+\pi^-) = \mathcal{A}_{1/2} + \frac{1}{\sqrt{2}} (\mathcal{A}_{3/2} + \mathcal{A}_{5/2}) \quad (1)$$

$$A(K^+ \to \pi^+\pi^0) = \mathcal{A}_{1/2} - \frac{\sqrt{2}}{3} (\mathcal{A}_{3/2} + \mathcal{A}_{5/2}) \quad (3)$$

with $\mathcal{A}_{1/2} = A_0 e^{i\chi_0}, \mathcal{A}_{3/2} = \frac{1}{3} A_2 e^{i\chi_2} + \frac{1}{3} A_2^* e^{-i\chi_2}$, and $\mathcal{A}_{5/2} = \frac{1}{3} A_2 e^{i\chi_2} - \frac{1}{3} A_2^* e^{-i\chi_2}$. $A_0$, $A_2$ and $A_2^*$ are real and positive amplitudes if the CP-conserving limit is assumed. In the isospin limit, $\mathcal{A}_{5/2}$ is zero, resulting in only two isospin amplitudes $A_0$ and $A_2 = A_2^*$ corresponding to final states $(\pi\pi)_{I=0,2}$, in addition to its respective phase differences $\chi_0$ and $\chi_2 = \chi_2^*$ which represent the $S$-wave scattering phase shifts.

Information regarding amplitudes and phase differences can be extracted from the experimental $K \to 2\pi$ branching ratios \[44\]:

$$A_0 = (2.704 \pm 0.001) \times 10^{-7} \text{ GeV},$$

$$A_2 = (1.210 \pm 0.002) \times 10^{-8} \text{ GeV},$$

$$\chi_0 - \chi_2 = (47.5 \pm 0.9) \circ, \quad (4)$$

which allow us to learn two things about the dynamics of these decays:

1. Substantial enhancement in the isoscalar amplitude relative to the isotensor one,

$$\omega \equiv \text{Re} A_2/\text{Re} A_0 \approx 1/22. \quad (5)$$

2. Large difference between $\chi_0$ and $\chi_2$, which implies by Eq. (3) that half of the ratio $\mathcal{A}_{1/2}/\mathcal{A}_{3/2}$ is generated from its absorptive contribution:

$$\text{Abs} (\mathcal{A}_{1/2}/\mathcal{A}_{3/2}) \approx \text{Dis} (\mathcal{A}_{1/2}/\mathcal{A}_{3/2}). \quad (6)$$

The above statements are general, and represent excellent control tests for $K \to 2\pi$ amplitude predictions when a specific theoretical framework is adopted.

In the presence of CP violation, $\text{Im} A_{0,2}$ are nonzero and allows us to define the CP violating observable \[7\]

$$e' = -\frac{i}{\sqrt{2}} e^{i(\omega - x)} \omega \text{Im} A_0 \text{Re} A_0 \left[1 - \frac{1}{\omega} \text{Im} A_2\right]. \quad (7)$$

Eq. (7) shows that $e'/\epsilon$ is approximately real ($\chi_2 - \chi_0 - \pi/2 \approx 0$), and also that $e'$ is suppressed by the ratio $\omega$. In view of the latter, one could be worried about an additional suppression from a possible subtle numerical cancellation emerging from

$$x \equiv 1 - \frac{1}{\omega} \frac{\text{Im} A_2}{\text{Im} A_0}. \quad (8)$$

\[1\] Using the recent lattice results \[15\]-\[30\], the authors of Ref. \[43\] have also reported a prediction of $e'/\epsilon$ in agreement with Eq. (1).

\[2\] Eq. (7) is leading order (LO) in CP violation. Corrections of $O(\text{Im} A_{0,2}^2)$ are very suppressed.
The study of Eq. (8) requires the estimation of \( \text{Im} A_{0,2} \), which can be done in the limit of a large number of QCD colours where the T-product of two colour singlet currents factorizes

\[
\langle Q \rangle = \langle J \cdot J \rangle = \langle J \rangle \langle J \rangle B_1 ,
\]

and the current \( \langle J \rangle \) has a well-known representation at LO in the \( 1/N_C \) expansion. Here, \( B_1 \) parametrizes the deviation of the true hadronic matrix element \( \langle Q \rangle \) from its large-\( N_C \) approximation \( \langle J \rangle \langle J \rangle \).

Due to the chiral enhancement of \((V-A) \times (V+A)\) operators, as well as the size of their respective Wilson coefficients, \( \text{Im} A_{0,2} \) are mainly dominated by \( Q_{S,8} \), respectively. Considering only these two operators, one finds

\[
x \approx 1 + \frac{\omega^{-1}}{2 \sqrt{2}} \left( \frac{F_K}{F_\pi} - 1 \right)^{-1} \frac{\gamma_6(\mu)}{\gamma_6(\mu)} \frac{\bar{B}}{B} \approx 1 - \frac{1}{2} \frac{\bar{B}}{B} ,
\]

with \( \bar{B} \equiv \frac{\beta_0^{(2)}}{\beta_0^{(0)}} \). \( F_K \) and \( F_\pi \) are the kaon and pion decay constants, respectively. The CP-odd components of the Wilson coefficients at the short-distance scale \( \mu \) are denoted by \( \gamma_6(\mu) \). Since the anomalous dimensions of \( \gamma_6 \) and \( \gamma_8 \) are the same at \( N_C \to \infty \), the ratio \( \gamma_6(\mu)/\gamma_6(\mu) \) is independent of \( \mu \) in that limit, showing how the product of two colour-singlet quark currents factorizes, as previously seen in Eq. (9).

In the large-\( N_C \) limit, \( x \approx 1/2 \) and therefore \( O(1/N_C) \) corrections are numerically relevant in Eq. (10). Performing an expansion of \( B_6^{(1/2)} \) and \( B_8^{(3/2)} \) in powers of \( 1/N_C \), that is \( B_6 = 1 + \frac{1}{N_C} b_6 + O \left( \frac{1}{N_C^2} \right) \), where \( b_6 \) are \( O(1) \), Eq. (10) becomes

\[
x \approx 1 - \frac{1}{2} \frac{b_8 - b_6}{3} ,
\]

where \( N_C = 3 \) has been employed. Eq. (11) shows explicitly the possibility of a strong cancellation between the two isospin contributions, reaching maximal efficiency for \( b_8 - b_6 \approx \frac{1}{2} \), however this cancellation depends on the predicted values of \( b_6 \) and \( b_8 \) which are \( O(1) \) as already mentioned.

Although isospin breaking effects are suppressed in almost all the phenomenological observables, \( \varepsilon'/\varepsilon \) represents an exception. In the second term of Eq. (7), we observe that a tiny isospin violating correction in \( \text{Im} A_2 \) can break its own suppression through the large enhancement of \( \omega^{-1} \approx 22 \) consequence of the \( \Delta I = 1/2 \) rule. These effects can be naively accounted in Eq. (11), inserting an additional parameter \( \Omega_{\text{naive}} \).

\[
x \approx 1 - \frac{1}{2} \frac{b_8 - b_6}{3} + \Omega_{\text{naive}} ,
\]

As an example, we observe that LO isospin breaking corrections originated from \( \pi^0\eta \) mixing can be estimated as \( \omega_{\text{naive}} \sim O(\omega^{-1})^2 \sim O(10^{-1}) \), whereas \( \omega(2) \sim O(10^{-2}) \) is the tree-level mixing angle. Therefore, isospin corrections in \( x \) are relevant and even more if Eq. (11) results to have a strong cancellation.

For a proper control of the \( \varepsilon'/\varepsilon \) prediction, it is useful to rewrite Eq. (7) to first order in isospin breaking corrections \( A_{0,2}^0 \),

\[
\text{Re} \left( \frac{\varepsilon'}{\varepsilon} \right) = -\frac{\omega_{\text{eff}}}{\sqrt{2} |\varepsilon|} \left[ \frac{\text{Im} A_{0,2}^0}{\text{Re} A_{0,2}^0} (1 - \Omega_{\text{eff}}) - \frac{\text{Im} A_{0,2}^\text{emp}}{\text{Re} A_{0,2}^\text{emp}} \right] .
\]

The superscript \( (0) \) denotes the isospin limit, \( \text{Im} A_{0,2}^\text{emp} \) contains the \( I = 2 \) contribution from the electromagnetic penguin operator, and \( \omega_{\text{eff}} \equiv \text{Re} A_{2,0}^0/ \text{Re} A_{0,2}^0 \). The parameter \( \Omega_{\text{eff}} \) encodes isospin-breaking corrections.

To minimize the theoretical uncertainty in our prediction, the CP-conserving amplitudes \( \text{Re} A_{0,2}^0 \), and consequently \( \omega_{\text{eff}} \), are set to their experimental values, provided in Eq. (4). Only \( \text{Im} A_{0,2}^0 \), \( \text{Im} A_{0,2}^\text{emp} \), and \( \Omega_{\text{eff}} \) (naively related with \( b_6 \), \( b_8 \) and \( \Omega_{\text{naive}} \), respectively) require an analytical computation within a theoretical framework. The later has to account for the dynamics of \( K \to 2 \pi \) decays, raised at the beginning of this section, as well as good theoretical control of the different contributions, to disentangle the possibility of cancellation between different contributions as seen in Eq. (12). The theoretical framework adopted for our prediction is presented in the following section.

3. Theoretical framework

The physical origin of \( \varepsilon'/\varepsilon \) is at the electroweak scale where all the flavor-changing processes are defined in terms of quarks and gauge bosons. The gluonic corrections to the \( K \to 2 \pi \) amplitudes are amplified with large logarithms due to the existence of very different mass scales \( (M_{\pi} < M_{K} < M_{W}) \), which can be summed up using the Operator Product Expansion (OPE) and the Renormalization Group Equations (RGEs), all the way down to \( \mu < m_t \) scales. Finally, in the three-flavor theory, one gets the following effective Lagrangian \( L_{\text{eff}}^{\Delta I=1} \),

\[
L_{\text{eff}}^{\Delta I=1} = -\frac{G_F}{\sqrt{2}} V_{us} V_{ub}^{*} \sum_{i=1}^{10} C_i(\mu) Q_i(\mu) ,
\]

which is a sum of local operators weighted by \( C_i(\mu) \) short-distance coefficients which depend on the parameters of the heavy masses \( (\mu > M) \) and the Cabibbo-Kobayashi-Maskawa (CKM) matrix. At NLO, the Wilson coefficients \( C_i(\mu) \) are known \( [46,49] \). This includes both \( O(\alpha_s^2 \mu^0) \) and \( O(\alpha_s^3 \mu^0) \) corrections, where
\[ t \equiv \log(M_1/M_2) \] applies to any ratio logarithm with \( M_1, M_2 \geq \mu \). Some next-to-next-to-leading-order (NNLO) corrections are already established \([50, 51]\) and an attempt for the full short-distance calculation at the NNLO is underway \([52]\).

We can use symmetry considerations below the resonance region, where perturbation theory no longer works, to describe another effective field theory in terms of the QCD Goldstone bosons \((\pi, K, \eta)\). The pseudoscalar and axial-vector spectral functions are defined by Eq. (14), leading to \( \mathcal{L}^{\pi \pi = 1} = \sum_{n=2}^{\infty} a_n^{\pi \pi = 1} \), where the cut in \( n \) depends on how accurate we want to be in our predictions. At LO, only three terms are allowed by symmetries

\[ \mathcal{L}^{\pi \pi = 1} = G_8 \mathcal{L}_8 + G_{27} \mathcal{L}_{27} + G_{\text{ewk}} g_{\text{ewk}} \mathcal{L}_{\text{ewk}}. \quad (15) \]

The chiral low-energy constants (LECs), i.e. \( G_8, G_{27} \) and \( G_{\text{ewk}} g_{\text{ewk}} \), can not be determined by symmetry principles, however Eq. (15) allows us to perform a matching between Eqs. (14) and (15) in the large-\( N_C \) limit, which gives rise to chiral LECs in terms of Wilson coefficients, and so the short-distance dynamics. Since the large-\( N_C \) limit is only applied in the matching, missing 1/\( N_C \) corrections are not enhanced by large logarithms, i.e. the naive estimation of these contributions is \( b_{6,8}^{\text{matching}} \sim \log(\mu/M_\rho) \sim O(10^{-1}) \), one order of magnitude smaller than its natural size, and then numerically irrelevant to play a game in the cancellation of Eq. (12).

In contrast, the decay amplitudes get large logarithmic corrections from pion loops which are a direct consequence of the large phase shift difference between \( \chi_0 \) and \( \chi_2 \). The naive estimation of these contributions implies \( b_6^{\text{matching}} \sim \log(\mu/M_\rho) \sim O(1) \), and then they are crucial in Eq. (12). These corrections can be rigorously calculated using the usual \( \chi \) PT methods \([10, 12]\), and they turn out to be positive for \( A_0|q_0 \), while negative for \( A_2|q_0 \), or in other words \( b_6 < 0 \) and \( b_8 > 0 \). Therefore, the numerical cancellation between the \( I = 0 \) and \( I = 2 \) terms in Eq. (12) is completely destroyed by the chiral loop corrections, leading to a sizeable enhancement of the SM prediction for \( \epsilon' / \epsilon \), presented in the next section.

4. SM prediction of \( \epsilon' / \epsilon \)

Taking into account all computed corrections in Eq. (13), which we refer for details to \([35, 36]\), our SM prediction for \( \epsilon' / \epsilon \) is

\[
\text{Re}(\epsilon'/\epsilon) = \left( 13.8 \pm 1.3 \gamma_5 \pm 2.5 \text{LECs} \pm 3.5 \text{1/} N_C \right) \cdot 10^{-4}
\]

\[
= (14 \pm 5) \cdot 10^{-4}.
\]

where isospin breaking corrections have been included, and result in \( \Omega_{\text{eff}} = 0.11 \pm 0.09 \) \([53]\). The first error represents the uncertainty from NNLO corrections to the Wilson coefficients accounted through the choice of the scheme for \( \gamma_5 \). The second error comes from the input values of the strong LECs (mainly \( L_{5,7,8} \)). The last error parameterizes our ignorance about 1/\( N_C \)-suppressed contributions in the matching region which have been estimated conservatively through the variation of \( \mu \) and \( \gamma_5 \) in the intervals [0.9, 1.2] GeV and [0.6, 1] GeV, respectively.

Our SM prediction for \( \epsilon' / \epsilon \) is in perfect agreement with the measured value, which gives strong support to the validity of both effective field theory and dispersive techniques \([10, 12, 55, 56]\).

- NNLO computation of the Wilson coefficients is close to being finished \([52]\).
- The \( O(\epsilon^2 \rho^0) \) coupling \( G_8 g_{\text{ewk}} \) can be expressed as a dispersive integral over the hadronic vector and axial-vector spectral functions. The \( \tau \) decay data can be used to perform a direct determination of this LEC \([53, 54]\).
- Estimation of higher-order chiral logarithmic corrections could be feasible either through explicit two-loop calculations or with dispersive techniques \([10, 12, 55, 56]\).
- A matching calculation of the weak LECs at NLO in 1/\( N_C \) remains a very challenging task. A fresh view to previous attempts \([17, 26, 57, 51]\) could suggest new ways to face this unsolved problem.

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