Quantum Langlands duality and mirror symmetry

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1 Introduction

There are recent indications that geometric Langlands (Beilinson-Drinfeld) duality [2, 3] is related to S-duality in QFT and string theories (cf. [30]). In this paper we discuss a peculiar relation of this Beilinson-Drinfeld duality and Givental’s version of mirror symmetry. It should be noted that mirror symmetry is a T-duality from the physical viewpoint [29].

Consider two (infinite-dimensional) ind-varieties: the n-reduced Sato Grassmannian \( GR(n) \) and the periodic (or affine) flag variety \( FL(n) \). The Sato Grassmannian \( GR \) is the set of subspaces \( W \subset C((t)) \) commensurable with \( C[t^{-1}] \), i.e., such that the projection

\[
\gamma_W : W \rightarrow C((t))/tC[t]
\]

is Fredholm (with finite kernel and cokernel). Then

\[
GR(n) = \{ W \in GR \mid t^{-n}W \subset W \}
\]

(1.2)

and

\[
FL(n) = \{ W_0 \subset W_1 \subset \cdots \subset W_n \mid \forall i W_i \in GR(n), \text{virt.
 dim. } W_i = i - n \text{ and } t^{-n}W_n = W_0 \}
\]

(1.3)

where

\[
\text{virt.
 dim. } W = \text{ind } \gamma_W = \dim \text{Ker } \gamma_W - \dim \text{Coker } \gamma_W
\]

is the virtual dimension of \( W \).

We construct \( D \)-modules of Beilinson-Drinfeld type on \( GR(n) \) parametrized by local affine \( GL_n \)-opers. In the same manner, we construct \( D \)-modules of Givental type on \( FL(n) \) parametrized by local affine Miura \( GL_n \)-opers.

**Main theorem.** Givental (quantum) \( D \)-modules on \( FL(n) \) are transformed to Beilinson-Drinfeld (quantum) \( D \)-modules on \( GR(n) \) via the affine Miura transformation.
The outline of this paper is as follows. First of all, we recall some facts about the gravitational quantum cohomology and Givental’s construction of a quantum $D$-module $D(X)$ for a smooth projective variety $X$. We would like to construct a family of $D$-modules on infinite-dimensional varieties $\mathcal{G}^{(n)}$ and $\mathcal{F}^{(n)}$ parametrized by affine opers. It is possible to generalize Givental’s approach using Guest-Otofuji results [14]. However, in this paper we adopt a different strategy. In section 4 we recall Beilinson-Drinfeld construction of $D$-modules on the moduli stack $\mathbb{B}un_G(X)$ of principal $G$-bundles on a curve $X$. This construction uses a quantization of the Hitchin system. In the following two sections (5 and 6) we sketch a local affine version of the Beilinson-Drinfeld construction using the formalism of null vectors. In some sense, it is related to the quantization of Korteweg-de Vries hierarchies. A similar approach is used later to construct $D$-modules on $\mathcal{F}^{(n)}$ parametrized by local affine Miura $GL_n$-opers. In section 9 we deduce our main theorem from standard results about the Miura transformation. Finally, we indicate how one can follow very closely the Beilinson-Drinfeld construction using analogues of the Hitchin fibration.

2 Background from gravitational quantum cohomology

The surveys [3] and [23] may serve as a nice introduction. Physically, (gravitational) quantum cohomology is a cohomological realization of non-linear $\Sigma$-models of two-dimensional field theory (coupled to gravity). One considers interacting “strings” propagating inside a variety $X$, i.e., maps $\Sigma \to X$ where $\Sigma$ is a Riemann surface swept out by these strings. The main statistical objects of interest are the correlators (vacuum expectation values) of the primary fields, represented by cocycles $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{Q})$, and their gravitational descendants $\tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n (d_i \in \mathbb{N})$. Fix a homology class $\beta \in H_2(X, \mathbb{Z})$ and consider the moduli stack $\mathcal{M}_{g,n}(X,\beta)$ of stable maps $f: C \to X$ of algebraic curves of genus $g$ with $n$ marked points such that $f_*[C] = \beta$. Denote by $\mathcal{M}_{g,n}(X,\beta)_{\text{virt}}$ the virtual fundamental class (the orbifoldic Poincaré dual of $1 \in H^*(\mathcal{M}_{g,n}(X,\beta), \mathbb{Q})$). In addition, let

$$e_i : \mathcal{M}_{g,n}(X,\beta) \to X, \quad e_i(f : C \to X, p_1, \ldots, p_n) = f(p_i)$$

be the evaluation maps at $i$th marked point and $\mathcal{L}_i$, “the cotangent line at $p_i$”, i.e., a line bundle over $\mathcal{M}_{g,n}(X,\beta)$ whose fiber over $(f : C \to X, p_1, \ldots, p_n)$ is the cotangent space $T_{p_i}^* C$. Now the gravitational correlators are defined by

$$\langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle_{g,\beta} = \int_{\mathcal{M}_{g,n}(X,\beta)} I_{g,n,\beta}(\tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n)$$

where

$$I_{g,n,\beta}(\tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n) = (c_1(\mathcal{L}_1)^{d_1} \cup e_1^*(\gamma_1)) \cup \cdots \cup (c_1(\mathcal{L}_n)^{d_n} \cup e_n^*(\gamma_n))$$

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are the gravitational Gromov-Witten classes. If \( d_1 = \cdots = d_n = 0 \) then
\[
I_{g,n,\beta}(\tau_{d_1}, \ldots, \tau_{d_n}, \gamma) = I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)
\] (2.2)
are the usual Gromov-Witten classes and
\[
\langle \tau_{d_1}, \ldots, \tau_{d_n}, \gamma \rangle_{g,\beta} = \langle I_{g,n,\beta} \rangle(\gamma_1, \ldots, \gamma_n)
\] (2.3)
are the Gromov-Witten invariants.

3 Givental quantum \( \mathcal{D} \)-modules

First of all, we recall some basic facts about Givental quantum connection and associated quantum \( \mathcal{D} \)-modules. Let \( X \) be a smooth projective variety and \( T_0, \ldots, T_m \) a basis of \( H^*(X, \mathbb{C}) \). We consider \( H^*(X, \mathbb{C}) \) as a Frobenius supermanifold with respect to supercommutative variables \( t_0, \ldots, t_m \) associated to \( T_i \) (cf. [3, sect. 10.1.1]). The Givental quantum connection \( \nabla^h \) is defined on the trivial cohomology bundle \( H^*(X, \mathbb{C}[t_0, \ldots, t_m]) \) over \( H^*(X, \mathbb{C}) \) by
\[
\nabla^h_{\partial t_i} \left( \sum_j a_j T_j \right) = h \sum_j \frac{\partial a_j}{\partial t_i} T_j - \sum_j a_j T_j \ast T_i
\] (3.1)
where \( \ast \) is the big quantum product on \( H^*(X, \mathbb{C}[t_0, \ldots, t_m]) \).

Consider the genus \( g \) couplings
\[
\langle \langle \tau_{d_1}, \ldots, \tau_{d_n}, \gamma \rangle \rangle_g = \sum_{k=0}^{\infty} \sum_{\beta} \frac{1}{k!} \langle \langle \tau_{d_1}, \ldots, \tau_{d_n}, \gamma \rangle \rangle_{g,\beta} q^g
\] (3.2)
where \( \gamma = \sum_i t_i T_i \) [3, sect. 10.1.1]. The formal sections
\[
s_a = T_a + \sum_j \left\langle \left\langle \frac{T_a}{h-c} T_j \right\rangle_0 \right\rangle T^j,
\] (3.3)
where \( c = c_1(L_1) \), form a basis of \( \nabla^h \)-flat sections of \( H^*(X, \mathbb{C}[t_0, \ldots, t_m]) \). This means that
\[
h \frac{\partial s_a}{\partial t_i} = T_i \ast s_a, \quad a, i = 0, \ldots, m
\] (3.4)
[3, sect. 10.2.1].

**Definition 3.1** Denote
\[
\mathcal{D} = \mathbb{C} [h\partial / \partial t_i, \exp t_i, h].
\] (3.5)
The elements of \( \mathcal{D} \), called quantum differential operators, are polynomials in the quantities

\[
h \partial/\partial t_0, \ldots, h \partial/\partial t_r, e^{t_0}, \ldots, e^{t_r}, h.
\]

The \( \mathcal{D} \)-module

\[
\mathcal{D}(X) = \mathcal{D}/I, \quad I = \{ D \in \mathcal{D} \mid DS_a = 0, \ 1 \leq a \leq r \}
\]
generated by \( S_a = \langle s_a, 1 \rangle \) is called the Givental quantum \( \mathcal{D} \)-module of \( X \).

Givental and Kim showed that the quantum \( \mathcal{D} \)-module of the flag variety \( G/B \) is generated by non-constant first integrals of the quantum Toda chain for the Langlands dual \( G^\vee \) [12, 20]. Guest and Otofuji [14] generalized some of these results to the case of the affine flag variety \( \mathcal{F}_L^{(n)} \).

We use this cohomological Givental construction as a motivation and adopt a different strategy in the rest of the paper.

### 4 Hitchin systems, \( G \)-opers and Beilinson-Drinfeld duality

Let \( X \) be a smooth projective curve of genus \( g \geq 2 \) and \( \text{Bun}_G(X) \) the moduli space of stable principal \( G \)-bundles on \( X \). If \( G = \text{GL}_n \) we denote \( \text{Bun}_{X,n} \) the moduli space of stable vector bundles of rank \( n \). Hitchin [15] constructed a map

\[
h : T^* \text{Bun}_G(X) \to V
\]

from the cotangent bundle \( T^* \text{Bun}_G(X) \) to a certain vector space \( V \) of dimension

\[
\dim V = \dim \text{Bun}_G(X) = (g - 1) \dim G + \dim Z(G).
\]

If \( G = \text{GL}_n \) then

\[
( \text{Bun}_G(X) )_P \simeq H^0(X, \text{End} P \otimes K_X)
\]

where \( K_X \) denotes the canonical bundle of \( X \). In other words, \( T^* \text{Bun}_{X,n} \) consists of pairs \((P, \varphi)\) where \( P \in \text{Bun}_{X,n} \) and \( \varphi : P \to P \otimes K_X \) a twisted endomorphism. The Hitchin map is defined by

\[
h : T^* \text{Bun}_{X,n} \to V = \bigoplus_{i=1}^n H^0(X, K_X^i)
\]

\[
(P, \varphi) \mapsto \text{ch}(\varphi) = (\text{tr} \varphi, \text{tr}(\wedge^2 \varphi), \ldots, \text{tr}(\wedge^n \varphi))
\]

where \( \text{ch}(\varphi) \) is the \( n \)-tuple of the coefficients of the characteristic polynomial of \( \varphi \).
Hitchin has shown that for any basis \( \{ f_1, \ldots, f_r \} \) of \( V^* \) the functions \( h^* f_i \) and \( h^* f_j \) commute with respect to the standard Poisson bracket:

\[
\{ h^* f_i, h^* f_j \} = 0 \tag{4.6}
\]
on \( T^* \text{Bun}_G(X) \). Denote by \( B_G(X) \) the ring of polynomial functions on \( V \). The Hitchin map induces a morphism

\[
\pi_{cl} : B_G(X) \to \{ \text{functions on } T^* \text{Bun}_G(X) \} \tag{4.7}
\]

Suppose now, for the sake of simplicity, that \( G \) is semi-simple and simply connected. Beilinson and Drinfeld constructed a quantization (\( h \)-deformation) of \( \pi_{cl} \), that is, a graded \( \mathbb{C} \)-algebra \( A_G(X) \) such that \( \text{gr} A_G(X) = B_G(X) \) equipped with a morphism

\[
\pi : A_G(X) \longrightarrow H^0( \text{Bun}_G(X), \mathcal{D}_{1/2} ) = H^0( \text{Bun}_G(X), \mathcal{D}_{-h^\vee} ) \tag{4.8}
\]
compatible with \( \pi_{cl} \). Here \( h^\vee \) is the dual Coxeter number of \( G \) and \( \mathcal{D}_{1/2} = \mathcal{D}_{-h^\vee} \) is the sheaf of twisted differential operators (of critical level \( k = -h^\vee \)) acting on \( K_{\text{Bun}_G(X)}^{1/2} = \eta^{-h^\vee} \), where \( \eta^k, k \in \mathbb{Z} \), denotes a line bundle on \( \text{Bun}_G(X) \) obtained via the generalized Borel-Weil-Bott (Kumar-Mathieu) theorem.

The elements of \( A_G(X) \) are called \( G^\vee \)-opers. Denote by \( \mathcal{M}( \text{Bun}_G(X) ) \) (resp. \( \mathcal{M}( X \times \text{Bun}_G(X) ) \)) the category of \( \mathcal{D} \)-modules on \( \text{Bun}_G(X) \) (resp. \( X \times \text{Bun}_G(X) \)).

**Theorem 4.1 (Beilinson-Drinfeld)** There exists a family of \( \mathcal{D} \)-modules \( \{ \mathcal{D}_\mathcal{L} \} \) on \( \text{Bun}_G(X) \) parametrized by \( G^\vee \)-opers \( \{ \mathcal{L} \} \) which are eigensheaves of Hecke operators:

\[
T_\chi : \mathcal{M}( \text{Bun}_G(X) ) \longrightarrow \mathcal{M}( X \times \text{Bun}_G(X) ), \quad T_\chi \mathcal{D}_\mathcal{L} = V^X_\mathcal{L} \otimes \mathcal{D}_\mathcal{L} \tag{4.9}
\]
for any dominant coweight \( \chi \in P_+( G^\vee ) \) where \( V^X_\mathcal{L} \) is a \( G^\vee \)-local system on \( X \) associated to a couple \( ( \mathcal{L}, \chi ) \).

One can give an explicit description of \( \text{GL}_n \)-opers.

**Definition 4.1** A \( \text{GL}_n \)-oper over \( X \) is a local system \( ( \mathcal{E}, \nabla ) \) adapted to a complete flag

\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E} \tag{4.10}
\]
of vector bundles over \( X \) such that
(i) \( \text{gr}_i(\mathcal{E}) \) are invertible,
(ii) \( \nabla(\mathcal{E}_i) \subset \mathcal{E}_{i+1} \otimes K_X \),
(iii) \( \nabla \) induces an isomorphism

\[
\text{gr}_i(\mathcal{E}) \xrightarrow{\sim} \text{gr}_{i+1}(\mathcal{E}) \otimes K_X. \tag{4.11}
\]
In the local coordinates a $\text{GL}_n$-oper is of the form:

$$\partial_t + \begin{pmatrix}
* & * & \cdots & * \\
+ & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & * 
\end{pmatrix}$$

(4.12)

where elements denoted by * are arbitrary and elements denoted by + are non-zero. Locally, one can write a $\text{GL}_n$-oper in the form

$$\partial_t - \begin{pmatrix}
q_1 & q_2 & \cdots & q_n \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & \cdots 
\end{pmatrix}, \ q_i \in \mathbb{C}[t],$$

(4.13)

equivalent to a differential operator

$$L = \partial_t^n - q_1 \partial_t^{n-1} - \cdots - q_n.$$  

(4.14)

Thus, a local $\text{GL}_n$-oper is a differential operator of order $n$ whose principal symbol is equal to 1.

5 Affine $\text{GL}_n$-opers and KdV hierarchies

We suppose now that $X$ is a complete (possibly singular) curve and $P$ is a smooth closed point on $X$.

**Definition 5.1** A local affine $\text{GL}_n$-oper is a pair $(\hat{E}, \hat{\nabla})$ where $\hat{E}$ is a formal torsion-free sheaf of rank $n$ over $X \times \hat{D}_t = X \times \text{Spec} \mathbb{C}[t]$ (an infinitesimal deformation of a torsion-free sheaf $\mathcal{E}$ of rank $n$ over $X$) equipped with a parabolic structure along of $P \times \hat{D}_t$:

$$\hat{E} = \hat{E}_0 \subset \hat{E}_1 \subset \cdots \subset \hat{E}_n = \hat{E}(P)$$

(5.1)

Moreover, $\hat{\nabla}$ is a (micro)connexion on $\hat{E}$, that is, a map

$$\hat{\nabla} : \left( \varprojlim \hat{E}_i \right) \longrightarrow \left( \varprojlim \hat{E}_i \right)$$

(5.2)

satisfying the Leibniz rule

$$\hat{\nabla}(fs) = f\hat{\nabla}(s) + \frac{\partial f}{\partial t} \cdot s$$

(5.3)

for any function $f$ on $X \times \hat{D}_t$ and any section $s$, and such that

(i) $\text{gr}_i(\hat{E})$ are invertible and $\chi(\hat{E}_{i+1}) = \chi(\hat{E}_i) + 1,$
Proposition 5.1 (Drinfeld \cite{8}) There exists a natural bijection between local affine $GL_n$-opers and Krichever modules, i.e., commutative subrings $R \subset \mathbb{C}[t][\partial_t]$, of rank $n$.

Remark 5.1 In positive characteristic, this corresponds to a bijection between elliptic sheaves over a field and Drinfeld modules.

The famous Krichever map \cite{25} associates to any local affine $GL_n$-oper a point $W$ on the $n$-reduced Sato Grassmannian $G\mathcal{G}(n)$.

The $n$th Korteweg-de Vries (KdV) hierarchy describes isospectral deformations of a differential operator $L \in \mathbb{C}[t][\partial_t]$ of order $n$. It may be written in the Lax form:

$$\frac{\partial L}{\partial t_r} = [L^{r/n}, L], \quad r \in \mathbb{N},$$

(5.4)

where $L^{r/n}$ is the positive part of the microdifferential operator

$$L^{r/n} \in \mathbb{C}[t]\left(\partial_t^{-1}\right).$$

(5.5)

Indeed, by a lemma of Schur \cite{26, p. 140}, any differential operator $L \in \mathbb{C}[t][\partial_t]$ of order $n$ has an (essentially unique) root $L^{1/n} \in \mathbb{C}[t]\left(\partial_t^{-1}\right)$.

One can view the $n$th KdV hierarchy as a dynamical system on the space of local affine $GL_n$-opers.

6 Semi-infinite $q$-wedges, quantum $\tau$-functions and quantum $\mathcal{D}$-modules for KdV theory

Consider the infinite-dimensional Clifford algebra $\mathcal{C}\mathcal{L}$ generated by $\psi_n^\pm$, $n \in \mathbb{Z} + 1/2$, satisfying the following relations:

$$[\psi_m^+, \psi_n^+]_+ = 0, \quad [\psi_m^+, \psi_n^-]_+ = \delta_{m,-n}.$$  

(6.1)

This Clifford algebra admits a unique irreducible representation (called spin representation) in an infinite-dimensional vector space $V$ (resp. $V^*$) which is a left (resp. right) module admitting a non-zero vacuum vector $|0\rangle$ (resp. $\langle 0|$) such that

$$\psi_i^\pm |0\rangle = 0 \quad (\text{resp.} \quad \langle 0|\psi_{-i}^\pm = 0) \quad \text{for} \quad i > 0.$$  

(6.2)
Denote by \( \mathfrak{gl}_\infty \) the Lie algebra of infinite matrices indexed by \( \mathbb{Z}+1/2 \) with almost all entries equal to zero. It has a canonical basis consisting of matrices \( E_{ij} \), \( i, j \in \mathbb{Z}+1/2 \), with 1 as the \((i, j)\)-entry and 0 elsewhere. The map \( E_{ij} \mapsto \psi^+_j \psi^-_j \) defines a bijection between \( \mathfrak{gl}_\infty \) and quadratic elements \( \sum a_{ij} \psi^+_i \psi^-_j \) of \( \mathbb{C} \mathcal{L} \). Later \( \mathfrak{gl}_\infty \) will also be identified with normally ordered expressions of the type \( \sum a_{ij} \psi^+_i \psi^-_j \). 

Using the exponential map we obtain a representation \( R \) of the group \( \text{GL}_\infty \) on \( V \) and \( V^* \). For a positive integer \( m \) denote 

\[
| \pm m \rangle = (0)\psi^+_{1/2} \cdots \psi^+_{m-1/2} \in V^* \quad \text{and} \quad | \pm m \rangle = \psi^-_{-m+1/2} \cdots \psi^-_{-1/2}(0) \in V.
\]

This defines the charge decomposition \( V = \oplus V^{(m)} \) (cf. below).

This spin representation has the following semi-infinite wedge realization. Let \( \mathbb{C}^{\infty} \) be an infinite-dimensional complex vector space with a fixed basis \( \{v_i\}_{i \in \mathbb{Z}+1/2} \). Then the semi-infinite wedge (or fermionic Fock) space \( V = \wedge^{\infty/2} \mathbb{C}^{\infty} \) is generated by semi-infinite monomials of the form 

\[
v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots \quad \text{where} \quad i_1 > i_2 > i_3 > \cdots
\]

and \( i_k+1 = i_k - 1 \) for \( k \gg 0 \). (6.3) 

The generators of the infinite-dimensional Clifford algebra are represented as the wedging and contracting operators defined by:

\[
\psi^+_j(v_{i_1} \wedge v_{i_2} \wedge \cdots) = v_{-j} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots = (-1)^s v_{i_1} \wedge \cdots \wedge v_{i_s} \wedge v_{-j} \wedge v_{i_{s+1}} \cdots \quad \text{if} \quad i_s > -j > i_{s+1}
\]

and zero in all other cases. By definition, the charge of \( v_{i_1} \wedge v_{i_2} \wedge \cdots \) is equal to \( m \) if \( i_k + k = m + 1/2 \) for \( k \gg 0 \). This defines the charge decomposition as above. In particular,

\[
|m\rangle = v_{m-1/2} \wedge v_{m-3/2} \wedge v_{m-5/2} \cdots \in V^{(m)}.
\]

(6.4) 

The space \( V^{(m)} \) is an irreducible highest weight representation of \( \mathfrak{gl}_\infty \subset \mathbb{C} \mathcal{L} \) with the highest weight vector \(|m\rangle\).

The \( \text{GL}_\infty \)-orbit of the vacuum vector \(|0\rangle\) is closely related to the infinite-dimensional Sato Grassmannian. Indeed, let us identify \( \mathbb{C}^{\infty} \) with \( \mathbb{C}(t) \) via \( v_i \mapsto t^{i+1/2} \). Then the map

\[
\varphi : \text{GL}_\infty |0\rangle \rightarrow \mathcal{G} \mathcal{R}, \quad \tau = \bigwedge \{u_i \mapsto \sum_{\{i=-1/2, -3/2, \cdots\}} \mathcal{C} u_i\}
\]

is well-defined since \( u_{-i} = v_{-i} \) for \( i \) big enough. This map is surjective and we can also write

\[
\mathcal{G} \mathcal{R} = \text{GL}_\infty |0\rangle / \mathbb{C}^*.
\]

(6.5)
By definition, semi-infinite wedges lying in the \( \text{GL}_\infty \)-orbit of \(|0\rangle\) are called \( \tau \)-functions in the fermionic picture.

Consider the affine Hecke (or Iwahori-Hecke) algebra \( \hat{\mathcal{H}}_N(q) \) with the generators \( T_{i}^{\pm 1}, X_{i}^{\pm 1}, 1 \leq i \leq N - 1 \), and relations:

\[
\begin{align*}
T_i T_i^{-1} &= T_i^{-1} T_i = X_i X_i^{-1} = X_i^{-1} X_i = 1; \quad (6.6a)
T_i T_j &= T_j T_i \text{ if } |i - j| > 1, \quad X_i X_j = X_j X_i \forall i, j; \quad (6.6b)
T_i T_{i+1} T_i &= T_{i+1} T_i T_i, \quad (T_i + 1)(T_i - q) = 0; \quad (6.6c)
X_i T_i &= T_i X_i \text{ if } j \neq i, i + 1, \quad T_i X_i T_i = q X_i X_{i+1}. \quad (6.6d)
\end{align*}
\]

Its infinite counterpart \( \hat{\mathcal{H}}_\infty(q) \) is generated by a countable number of generators \( T_{i}^{\pm 1}, X_{i}^{\pm 1}, i \in \mathbb{N} \), satisfying the same relations.

Let \( V \) be a complex vector space of dimension \( n \) with a basis \( \{e_1, \ldots, e_n\} \) and let \( V(\hat{\mathcal{H}}_N(q^2)) \) be the \( \hat{\mathcal{H}}_N(q^2) \)-module on \( V^{(n)} \). There are well-known right actions of \( \hat{\mathcal{H}}_N(q^2) \) on \( V^{(n)} \) and of \( \hat{\mathcal{H}}_\infty(q^2) \) on the so-called thermodynamic limit \( V^{(\infty)} \) \cite{19,28}. The latter is well-defined since any \( T_i \) acts only in a pair of adjacent factors. The \( q \)-antisymmetrization procedure (loc. cit.) gives the \( q \)-wedge spaces

\[
\bigwedge_q V^{(n)} = V^{(n)} \bigoplus_{i=1}^{n-1} \text{Ker}(T_i + 1) \quad (6.7a)
\]

and

\[
\bigwedge_{q}^{\infty} V^{(\infty)} = V^{(\infty)} \bigoplus_{i=1}^{\infty} \text{Ker}(T_i + 1). \quad (6.7b)
\]

In the same manner, the \( q \)-deformed Fock space \( V_q^{(m)} \) of charge \( m \) is the quotient of the space \( U_q^{(m)} \) of pure tensors \( u_{m_1} \otimes u_{m_1} \otimes \cdots \) where \( m_k = m - k + 1/2 \) for \( k \gg 1 \) by \( \sum_{i=1}^{\infty} (T_i + 1) \).

**Definition 6.1** The elements of the \( q \)-antisymmetrized \( \text{GL}_\infty \)-orbit of the vacuum vector \(|0\rangle\) will be called quantum \( \tau \)-functions.

Using fermionic normal ordering

\[
: \psi_m \psi_n^* : = \begin{cases} 
\psi_m \psi_n^* & \text{if } m < 0 \text{ or if } n > 0, \\
-\psi_n^* \psi_m & \text{if } m > 0 \text{ or if } n < 0
\end{cases} \quad (6.8)
\]

one can define operators

\[
H_n = \sum_{i \in \mathbb{Z}+1/2} : \psi_i^+ \psi_{i+n}^- ; \quad (6.9)
\]
satisfying the commutation relations:

\[ [H_m, H_n] = m \delta_{m,-n}. \]  

(6.10)

It is easy to see that for any given element \( u \) of the Fock space, the expression \( H_n |u\rangle \) is a finite sum. Consider also the generating function

\[ H(t) = \sum_{n=1}^{\infty} t_n H_n \]  

(6.11)

called the Hamiltonian where \( t = (t_1, t_2, \ldots) \). In the bosonic picture, the \( \tau \)-function \( \tau_g \), associated to \( g = \sum a_{ij} \psi^+_i \psi^-_j \in \mathfrak{gl}_\infty \), is the correlation function

\[ \tau_g(t) = \langle 0 | e^{H(t)} g | 0 \rangle = \langle 0 | e^{H(t)} e^{\sum a_{ij} \psi^+_i \psi^-_j} | 0 \rangle \]  

(6.12)

(c.f. [6, 7, 16]). The wedging and contracting operators \( \psi^+_i \) and \( \psi^-_j \) correspond to the intertwiners:

\[ \Gamma^+_i (z) : V(z) \otimes \mathcal{V}(m) \longrightarrow \mathcal{V}(m+1) \]  

and \( \Gamma^-_j (z) : \mathcal{V}(m) \longrightarrow \mathcal{V}(m-1) \otimes V(z) \).  

(6.13)

Now let \( W \) be a point of \( \mathcal{G} \mathcal{R} \) associated to a local affine \( \mathfrak{gl}_n \)-oper \( (\hat{E}, \hat{\nabla}) \) via the Krichever map (c.f. sect. 5). Consider the associated \( \tau \)-function

\[ \tau_W(t) = \langle 0 | e^{H(t)} g_W | 0 \rangle = \langle 0 | e^{H(t)} e^{\sum a_{ij} \psi^+_i \psi^-_j} | 0 \rangle \]  

(6.14)

The wedging and contracting operators \( \psi^+_i \) and \( \psi^-_j \) correspond to the intertwiners of level-1 \( \hat{\mathfrak{sl}}_n \)-modules

\[ \Gamma^+_i (z) : V(z) \otimes V_\Lambda_m \longrightarrow V_\Lambda_{m+1} \]  

and \( \Gamma^-_j (z) : V_\Lambda_m \longrightarrow V_\Lambda_{m-1} \otimes V(z) \)  

(6.15)

where \( V_\Lambda_m \) are the \( \hat{\mathfrak{sl}}_n \)-modules with fundamental weights \( \Lambda_m \).

In general, consider a singular vector of a highest weight \( \hat{\mathfrak{sl}}_n \)-module \( V_\Lambda \). It is an eigenvector of the Cartan subalgebra of \( \hat{\mathfrak{sl}}_n \) annihilated by \( U_+ (\hat{\mathfrak{sl}}_n) \) different from the highest weight vector \( v_\Lambda \). It may be written in the form \( S v_\Lambda \) where \( S \) is a uniquely defined element of \( U_- (\hat{\mathfrak{sl}}_n) \). Then

\[ \langle 0 | (S v_\Lambda) g | 0 \rangle = D \langle 0 | v_\Lambda g | 0 \rangle = 0. \]  

(6.16)

for a certain differential operator \( D \) [9, sect. 5.3].

In our case, any \( \tau \)-function \( \tau_W \) defines a \( \mathcal{D} \)-module \( \mathcal{D}(\tau_W) \) on \( \mathcal{G} \mathcal{R} \). These \( \mathcal{D} \)-modules are closely related to the so-called deformed Knizhnik-Zamolodchikov equations for form factors (c.f. sect. 6]).

Remark 6.1 All \( \tau \)-functions satisfy bilinear Hirota equations (of KP hierarchy) related to the time evolution \( e^{H(t)} \). These equations characterize the Sato Grassmannian itself.
Some remarks can be made, however, in our infinite-dimensional setting. The ind-varieties $\mathcal{GR}$ and $\mathcal{GR}^{(n)}$ have two equivalent ind-structures given by pole orders and by Schubert varieties. Denote by $\mathcal{GR} = \lim_{\to} \mathcal{GR}_i$ one of these structures and let $\mathcal{D}(\mathcal{GR}) = \lim_{\to} \mathcal{D}((\mathcal{GR}_i))$ be the sheaf of differential operators on $\mathcal{GR}$. Then one considers right $D(\mathcal{GR}) = \lim_{\to} D(\mathcal{GR}_i)$-modules in the abelian category $\mathcal{M}(\mathcal{GR},\mathcal{O})$ of $\mathcal{O}$-modules (cf. [2, sect. 7.11] for a detailed exposition).

Quantum correlations functions constructed from intertwiners of $U_q(\widehat{\mathfrak{sl}}_n)$-modules should satisfy quantum differential equations in the sense of Frenkel-Reshetikhin [11]. We use the notation $D_q(\tau_W)$ for associated quantum $D$-modules on $\mathcal{GR}^{(n)}$ parametrized by affine local $GL_n$-opers.

7 Affine Miura opers and Miura transformation

We use [4, 7.2 and 8.3.5] as a reference for affine Miura opers. Let $G$ be a reductive group, $X$ a smooth complex curve and $P \in |X|$. Denote $L_G \subset LG$ the subgroup consisting of loops which extends to $X \setminus P$.

Let $B$ be a Borel subgroup of $G$ and $B^-$ a transverse Borel subgroup (such that $[B^-] \subset G/B$ lies in the open $B$-orbit). Choose a point $Q$ on $X$ distinct from $P$ and consider the subgroup $L_G^{-} \subset LG$ consisting of “negative” loops whose values at $Q$ lie in $B^{-}$.

**Definition 7.1** A local affine Miura $G$-oper is a quadruple $(\mathcal{U}, \nabla, \mathcal{U}^{-}, \mathcal{U}^{+})$ where $\mathcal{U}$ is an $L_G$-torsor on $\widehat{D}_t$ with a connection $\nabla$, a flat reduction $\mathcal{U}^{-}$ to $L_G^{-}$ and a reduction $\mathcal{U}^{+}$ to $L_G^{+}$ in “tautological relative position” with respect to $\nabla$ (loc. cit.).

**Remark 7.1** One can define local affine $G$-opers in the same vein replacing $L_G^{-}$ by $L_G^{+}$.

Local Miura $GL_n$-oper is a connection of the type

$$\partial_t - \begin{pmatrix} \chi_1 & 0 & 0 & \cdots & 0 \\ 1 & \chi_2 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \chi_n \end{pmatrix}, \chi_i \in \mathbb{C}[t], \quad (7.1)$$

on a vector bundle of rank $n$ over $\widehat{D}_t$. The Miura transformation is the transformation from this connection to the unique gauge equivalent connection of the form $[1.13]$. Geometrically, it may be described as follows [10, 3.8]. Let $(\mathcal{E}, \nabla)$ be a local $GL_n$-oper with another full flag of subbundles

$$\mathcal{E}_1' \subset \mathcal{E}_2' \cdots \subset \mathcal{E}_n' \quad (7.2)$$
preserved by $\nabla$. Such a flag defines a connection $\nabla'$ on $\text{gr}(E) = \oplus_{i=1}^{n} \text{gr}_i(E)$. Then the Miura transformation associates $(E, \nabla)$ to $(\text{gr}(E'), \nabla')$.

Now let $(\hat{E}, \hat{\nabla})$ be a local affine $\text{GL}_n$-oper and

$$\hat{E} = \hat{E}_0 \subset \hat{E}_1 \subset \hat{E}_2 \subset \cdots \subset \hat{E}_n = \hat{E}'(Q)$$

(7.3)

another parabolic structure preserved by $\hat{\nabla}$. As above, the affine Miura transformation associates $(\hat{E}, \hat{\nabla})$ to $(\text{gr}(E'), \nabla')$.

**Proposition 7.1** Let $(\text{gr}(E'), \nabla')$ be a local affine Miura $\text{GL}_n$-oper and $(\hat{E}, \hat{\nabla})$ the associated local affine $\text{GL}_n$-oper. Let $W$ be a point of the $n$-reduced Sato Grassmannian $\mathfrak{S}(n)$ which is the image of $(\hat{E}, \hat{\nabla})$ under the Krichever map. Then

$$(\text{gr}(E'), \nabla') \mapsto (W, \hat{E}_1 / \hat{E}_1 \subset \hat{E}_2 / \hat{E}_2 \subset \cdots \subset \hat{E}_n / \hat{E}_n)$$

(7.4)

defines a point of the affine flag variety $\mathcal{F}L(n)$.

**Proof.** Recall that there is a natural fibration $\mathcal{F}L(n) \to \mathfrak{S}(n)$ whose fiber is the usual flag variety $\text{GL}_n / B$. Thus, the pair

$$(W, \hat{E}_1 / \hat{E}_1 \subset \hat{E}_2 / \hat{E}_2 \subset \cdots \subset \hat{E}_n / \hat{E}_n)$$

(7.5)

defines, indeed, a point of $\mathcal{F}L(n)$ under this fibration.

## 8 $\mathcal{D}$-modules for affine Toda theories

Consider the generating functions

$$\psi^+(\mathfrak{z}) = \sum_{i \in \mathbb{Z}} \psi^+_i \mathfrak{z}^i \quad \text{and} \quad \psi^-(\mathfrak{z}) = \sum_{i \in \mathbb{Z}} \psi^-_i \mathfrak{z}^{-i}$$

(8.1)

of free fermions $\psi^+_i$ indexed by $i \in \mathbb{Z}$ (cf. sect. 3). Introduce “positive” and “negative” times $t = (t_1, t_2, \cdots)$ and $t' = (t'_1, t'_2, \cdots)$ resp., and denote $\xi(t, \mathfrak{z}) = \sum_{n \geq 0} t_n \mathfrak{z}^n$. The **two-dimensional Toda theory** is related to the following time evolution $g(t, t')$:

$$\psi^+(\mathfrak{z}) \mapsto \mathfrak{z}^n e^{\xi(t, \mathfrak{z}) - \xi(t', \mathfrak{z}^{-1})} \psi(\mathfrak{z})$$

(8.2)

$$\psi^-(\mathfrak{z}) \mapsto \mathfrak{z}^{-n} e^{-\xi(t, \mathfrak{z}) + \xi(t', \mathfrak{z}^{-1})} \psi(\mathfrak{z})$$

(8.3)

(with essential singularities at $\mathfrak{z} = 0, \infty$) of $g = \exp \left( \sum_i a_i \psi^+(p_i) \psi^-(q_i) \right)$ [B], §9. The $\tau$-functions are defined as before by

$$\tau_g(t, t') = \langle 0 | g(t, t') | 0 \rangle$$

(8.4)
The usual reduction to $A_n^{(1)}$ [16, §8,9] defines $\tau$-functions of affine Toda chains.

Let $(\text{gr}(\widehat{E}', \widehat{\nabla}'))$ be a local affine Miura $\text{GL}_n$-oper and denote $(W, \widehat{E}'_\bullet / \widehat{E}')$ the associated point of $FL^{(n)}$. For any $\tau$-function

$$\tau_W(t, t') = \langle 0 | g_W(t, t') | 0 \rangle$$

(8.5)
corresponding to $(W, \widehat{E}'_\bullet / \widehat{E}')$, we construct $D$-modules $D(\tau_W)$ and $D_q(\tau_W)$ as in sect. 6.

9 Proof of the main theorem

The result follows easily from the constructions above.

Let $(\text{gr}(\widehat{E}', \widehat{\nabla}'))$ be a local affine Miura $\text{GL}_n$-oper and $(W, \widehat{E}'_\bullet / \widehat{E}')$ the associated point of $FL^{(n)}$. Under the Miura transformation, we get a local affine $\text{GL}_n$-oper $(E, \nabla)$. It is easy to see that the Miura transform of $(W, \widehat{E}'_\bullet / \widehat{E}')$ coincides with the point $W \in \mathcal{X}^{(n)}$ obtained from $(E, \nabla)$ via the Krichever map.

Moreover, the $\tau$-function $\tau_W(t, t')$ of the affine Toda $\widehat{\text{sl}}_n$-chain becomes the $\tau$-function $\tau_W(t)$ of the $n$th KdV hierarchy. Clearly, this transformation preserves null vectors. Thus, the Givental type $D$-modules $D(\tau_W)$ and $D_q(\tau_W)$ on $FL^{(n)}$ are transformed to Beilinson-Drinfeld type $D$-modules on $\mathcal{X}^{(n)}$.

10 Alternative construction of $D$-modules

In this section we briefly indicate how one can construct $D$-modules on $\mathcal{X}^{(n)}$ and $FL^{(n)}$ using a local affine version of the Beilinson-Drinfeld approach.

In the case of the $n$th KdV hierarchy, an analogue of the Hitchin fibration is the fibration $S \to \mathcal{M}^{(n)}$ whose fibers are (Jacobians) of spectral curves. Here $\mathcal{M}^{(n)}$ denotes the moduli space of local affine $\text{GL}_n$-opers (so-called abelianized Grassmannian). Recall (cf. [1, 22]) that any local affine $\text{GL}_n$-oper $m \in \mathcal{M}$, i.e., any Krichever module, defines a spectral curve $X_m$. The fibration above is closely related to $T^*\mathcal{X}^{(n)}$ and one can construct Beilinson-Drinfeld type $D$-modules on $\mathcal{X}^{(n)}$ indexed by local affine $\text{GL}_n$-opers.

In the case of the affine Toda $\widehat{\text{sl}}_n$-chain, an analogue of the Hitchin fibration is the fibration $S \to \mathcal{O}^{(n)}$ whose fibers are (Jacobians) of spectral curves of this chain [24, sect. 1]. Here $\mathcal{O}^{(n)}$ denotes the moduli space of local affine Miura $\text{GL}_n$-opers. Using the Beilinson-Drinfeld procedure one can construct $D$-modules on $FL^{(n)}$ indexed by such opers.

Acknowledgements. I am grateful to Mark Spivakovsky for useful remarks.
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