**q-CATALAN BASES**
**AND THEIR DUAL COEFFICIENTS**

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**Abstract.** We define \(q\)-Catalan bases which are a generalization of the \(q\)-polynomials \(z^n(z, q)_n\). The determination of their dual bases involves some \(q\)-power series termed dual coefficients. We show how these dual coefficients occur in the solution of some equations with \(q\)-commuting coefficients and solve an abstract \(q\)-Segner recursion. We study the connection between this theory and Garsia’s (1981). The overall flavor of this work is to show how some properties of \(q\)-Catalan numbers are in fact instances of much more general results on dual coefficients.

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1. **Introduction.** The purpose of this paper is to revisit the old problem of determining the dual bases of some bases of power series, show some new connections with other problems and give a different perspective on some results pertaining to analytic combinatorics and the theory of \(q\)-series. Numerous papers deal with describing dual bases, but to broadly fix the ideas, while this work is not properly on combinatorics, it is inspired by results in that area, especially the works of Garsia (1981), Krattenthaler (1984) and Fürlinger and Hofbauer (1985) related to \(q\)-Catalan numbers.

In this paper we identify two new objects, a class of formal power series, which we call \(q\)-Catalan bases, and some \(q\)-polynomials or power series, which we call dual coefficients. While these dual coefficients are instrumental in determining the dual bases of \(q\)-Catalan bases, they are interesting quantities to study for their own sake, since we will see that they connect different problems, such as finding power series expansions to the solutions to some equations with \(q\)-commuting coefficients and solving some recursions or functional equations of \(q\)-type. Moreover, in some special cases, these dual coefficients have a combinatorial interpretation such as Carlitz’s (1972) \(q\)-Catalan numbers, and our result leads to a new characterization of these numbers in terms of power series solution of a quadratic equation with \(q\)-commuting coefficients.
The organization of this paper is as follows. Section 2 contains the main definitions, in particular that of \(q\)-Catalan basis and dual coefficients, and the main result of this section is a description of the dual bases of \(q\)-Catalan bases. In section 3 we examine further properties of the dual coefficients. Because the results of sections 2 and 3 are so intimately related, section 4 gathers their proofs. In section 5 we consider some particular \(q\)-Catalan bases; we show how some previous results by Garsia (1981) and Krattenthaler (1988) can be viewed in this setting, and how the Rogers-Ramanujan continuous fraction (see Flajolet and Sedgewick, 2009; example V.9) can be extended in this setting. Finally section 6 discusses further specialization related to the combinatorics of trees and lattice paths.

**Notation.** Following the custom in \(q\)-series (see e.g. Andrews, Askey, Roy, 2000), we write for any \(n\) positive

\[
(z,q)_n = \prod_{0 \leq j < n} (1 - zq^j),
\]

with \((z,q)_0 = 1\) and \((z,q)_\infty = \prod_{j \geq 0} (1 - zq^j).

When indexing quantities by pairs \((i,j)\), such as \(e_{(i,j)}\) we tend to drop the pair notation and write instead \(e_{i,j}\). We write \(\mathbb{C}[[z]]\) for the set of all formal power series with complex coefficients.

Throughout the paper, we will consider formal power series. Conditions under which those formal power series are convergent ones are usually easy to determine.

2. \(q\)-Catalan bases and their dual forms. Our main object of study is a special type of power series in two variables and some related bases of the space of power series.

**Definition 2.1.** A power series \(P(z,t)\) is a Catalan power series if there exists a power series \(\tilde{P}(z,t)\) such that

\[
P(z,t) = t - z\tilde{P}(z,t)t^2.
\]

We say that \(\tilde{P}\) is associated to \(P\).

While some nontrivial examples are developed in the fifth and sixth section, we will run a couple of trivial ones through this section in order to make things more concrete and explain the terminology.
Examples. a) $P(z,t) = t$ is a Catalan power series with $\tilde{P}(z,t) = 0$.
b) $P(z,t) = t - zt^2$ is a Catalan power series with $\tilde{P}(z,t) = 1$.

Viewing $P(z,t)$ as a power series in $t$ with coefficients in $\mathbb{C}[[z]]$, it is a Catalan power series if the coefficient of $t$ is the power series in $z$ which is constant and equal to 1. If $P(z,t)$ and $Q(z,t)$ are two Catalan power series, so is $\alpha P(z,t) + (1 - \alpha)Q(z,t)$ for any real number $\alpha$, and so are $P(z,t)Q(z,t)/t$ and $t^2/P(z,t)$.

Next, we define what will turn out to be bases for the space of power series.

Definition 2.2. A family of power series $(e_k(z,q))_{k \geq 0}$ is a $q$-Catalan basis if there exists a Catalan power series $P$ such that for any nonnegative integer $k$,

$$\frac{e_k(qz)}{e_k(z)} = \frac{P(z,q^k)}{P(z,1)}. \quad (2.1)$$

We then say that $P$ is associated to $e_k$, or also that $e_k$ is associated to $P$.

Examples. (continued) a) $e_k(z) = z^k$.
b) $e_k(z) = z^k(z,q)_k$.

Note that in both examples $e_k(z)$ is a polynomial in $z$, of order $k$. Our first lemma shows that if $(e_k)$ is a Catalan basis then $e_k$ is of order $k$, and, consequently, that $(e_k)$ is a basis for the space of power series; hence the terminology.

Lemma 2.3. If $(e_k)$ is a q-Catalan basis, then each $e_k$ has order $k$.

To understand better the relationship between Catalan power series and $q$-Catalan bases, we need the following definition.

Definition 2.4. A $q$-Catalan basis $(e_k)$ is normalized if $[z^k]e_k = 1$.

A $q$-Catalan basis is normalized if its term of lowest degree has coefficient 1.
Among other things, our next result asserts that there is a bijection between Catalan power series and normalized $q$-Catalan bases.

**Lemma 2.5.** (i) If $P$ is a Catalan power series, the unique normalized $q$-Catalan basis associated to $P$ is given by

$$e_k(z) = z^k \prod_{j \in \mathbb{N}} \frac{1 - q^j z \tilde{P}(q^j z,1)}{1 - q^{k+j} z \tilde{P}(q^j z,q^k)}.$$ 

(ii) A $q$-Catalan basis is associated with a unique Catalan power series.

Lemma 2.5 makes clear what the difference between $q$-Catalan bases and Garsia’s (1981) powers is: with the notation of Lemma 2.5, Garsia’s powers are obtained if $\tilde{P}(z,t)$ is a power series in the product $zt$. In general, $q$-powers of Hofbauer (1984), as extended by Krattenthaler (1984), are not $q$-Catalan bases. Krattenthaler’s (1988) results cover $q$-Catalan bases, but, not surprisingly, our less general assumptions entail for a more specific theory which emphasizes less on the linear algebraic part of the theory and more on its connection with other problems, and which is perhaps easier to apply in some cases. Our assumptions should be viewed as in between those of Garsia (1981) and Krattenthaler (1988), though closer to Garsia’s (1981).

Having a basis, a natural question is how to decompose functions in this basis. If $(e_n)$ is a basis, recall that its dual basis consists of the linear forms $[e_n]$ such that $[e_n](e_k) = \delta_{k,n}$. Thus the question of decomposing functions in a basis $(e_n)$ is really that of finding the dual basis.

Concerning the notation, note that when we consider the basis of the monomials $(z^k)$ then $[z^k]$ are the corresponding dual forms, and $[z^k]f(z)$ is the coefficient of $z^k$ in $f$. For power series of two variables $(z,t)$ say, then $[(z^i t^j)]f(z,t)$ is the coefficient of $z^i t^j$ in the power series $f(z,t)$.

In order for us to describe the dual bases of $q$-Catalan ones, we need to define some functions of two variables. For this purpose, let

$$e_1 = (1,0) \quad \text{and} \quad e_2 = (0,1)$$

be the canonical basis of $\mathbb{R}^2$, and let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbb{R}^2$. 
Definition 2.6. Let $P$ be a Catalan power series. Its predual basis is the array $(\tilde{e}_i(z,t))_{i \in \mathbb{N}^2}$ of power series of two variables

$$\tilde{e}_i(z,t) = z^{(i,e_1)} \prod_{0 \leq j < (i,e_2)} P(q^j z, t),$$

where it is agreed that a product over an empty index set is 1.

By extension, we also say that $(\tilde{e}_i)_{i \in \mathbb{N}^2}$ is a predual basis for any Catalan basis associated to $P$.

While the term predual is convenient and as we will see meaningful, it has no connection with the algebraic notion of preduall of a vector space.

Examples. (continued) a) $\tilde{e}_i(z,t) = z^{(i,e_1)} t^{(i,e_2)}$.

b) $\tilde{e}_i(z,t) = z^{(i,e_1)} t^{(i,e_2)} (zt, q)_{(i,e_2)} = z^{(i,e_1-e_2)} e_{(i,e_2)} (zt)$.

Given the convention that a product over an empty set is 1, we have $\tilde{e}_{(n,0)} = z^n$ for all nonnegative integers $n$.

Our next result asserts that a predual basis is indeed a basis.

Lemma 2.7. If $(e_k)_{k \in \mathbb{N}}$ is a Catalan basis, its predual basis $(\tilde{e}_i(z,t))_{i \in \mathbb{N}^2}$ is a basis of the vector space of power series in $(z,t)$.

As a consequence of Lemma 2.7, the following definition makes sense.

Definition 2.8. Given a Catalan power series $P$, or a Catalan basis $(e_k)_{k \in \mathbb{N}}$ associated to a Catalan power series $P$, its dual coefficients are the unique $(T_i)_{i \in \mathbb{N}^2}$ defined by

$$\sum_{i \in \mathbb{N}^2} T_i \tilde{e}_i(z,t) = t. \quad (2.2)$$

Note that since $(\tilde{e}_k)$ depends on $q$, the dual coefficients depend also on $q$, but not on $z$ or $t$. A recursive way of calculating the dual coefficients is given in Theorem 3.3 in the next section.

Examples. (continued) a) $\sum_{i \in \mathbb{N}^2} T_i z^{(i,e_1)} t^{(i,e_2)} = t$ forces $T_i = 0$ if $i \neq (0,1)$ and $T_{0,1} = 1$. 

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b) Multiplying both sides of (2.2) by $z$, we obtain

$$
\sum_{i \in \mathbb{N}^2} T_i z^{(i,e_1-e_2)+1} e_{(i,e_2)}(zt) = zt. \tag{2.3}
$$

Since $(e_n)$ is a basis and $e_0 = 1$, there exists a unique sequence $(C_n)$ such that

$$
\sum_{n \geq 0} C_n e_{n+1}(z) = z. \tag{2.4}
$$

Then, for (2.3) to hold, we need $T_i$ to vanish when $(i,e_1-e_2) + 1$ does not. Thus, $T_i = 0$ if $i \not\in \{(n,n+1) : n \in \mathbb{N}\}$ and for $n$ in $\mathbb{N}$, $T_{n,n+1} = C_n$. The sequence $(C_n)_{n \in \mathbb{N}}$ satisfying (2.4) is known as Carlitz’s (1972) $q$-Catalan numbers and does not have a known nice closed form. It has been studied by Andrews (1975), Fürlinger and Hofbauer (1985) among others.

While the specific values of the dual coefficients depend on the Catalan series $P$, some coefficients are universal, and, in particular, the following lemma contains the important value $T_{0,1} = 1$.

**Lemma 2.9.** For any dual coefficients $(T_i)_{i \in \mathbb{N}^2}$,

(i) $T_{n,0} = 0$ for any $n$ in $\mathbb{N}$;

(ii) $T_{n,1} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$

The dual coefficients allow us to express the dual basis of a Catalan basis $(e_k)$ associated to the same Catalan power series $P$. Viewing $P(z,t)$ as a power series in $t$, and given that for any nonnegative integer $n$ the ratio $(s^n - t^n)/(s-t)$ is a polynomial in $(s,t)$,

$$
\Delta P(z,s,t) = \frac{P(z,s) - P(z,t)}{s-t}
$$

is a power series in $(z,s,t)$; we extend this definition to $\partial P(z,s)/\partial s$ on tuples of the form $(z,s,s)$. In the power series $\Delta P(z,s,t)$ we agree to write the monomial in the form $z^i s^j t^k$, with the variables in this order. If $T$ is an operator then $\Delta P(z,T,t)$ makes sense.

**Examples.** (continued) a) $\Delta P(z,s,t) = 1$.

b) $\Delta P(z,s,t) = 1 - z(s+t)$. 

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Theorem 2.10. The dual forms of a Catalan basis \((e_k)_{k\in\mathbb{N}}\) with dual coefficients \((T_i)_{i\in\mathbb{N}^2}\) are given as follows. Let

\[
T(f)(z) = \sum_{i\in\mathbb{N}^2} T_i \hat{e}_i(z,1)f(q^{(i,e_2)}z).
\]

Then

\[
[e_n]f = q^n[z^0]\left(\frac{\Delta P(z,T,q^n)f(z)}{e_n(qz)P(z,1)}\right).
\]

Put differently, this result asserts that if one knows how to decompose the projection \((z,t)\mapsto t\) over the predual basis \((\hat{e}_i)_{i\in\mathbb{N}^2}\), viz. (2.2), then there is an explicit formula for the dual forms \([e_n]\).

Examples. (continued) a) We have \(Tf(z) = T_{0,1} \hat{e}_{0,1}(z,1)f(qz) = f(qz)\). Thus,

\[
[e_n]f = q^n[z^0] \frac{f(z)}{q^n z^n} = [z^n]f(z).
\]

b) We have

\[
Tf(z) = \sum_{n\in\mathbb{N}} C_n e_{n,n+1}(z,1)f(q^{n+1}z)
\]

\[
= \sum_{n\in\mathbb{N}} C_n z^{-1} e_{n+1}(z)f(q^{n+1}z).
\]

Since \(\Delta P(z,T,q^n) = 1 - z(T + q^n)\), we obtain

\[
[e_n]f = q^n[z^0] \frac{f(z) - zTf(z) - zq^n f(z)}{e_n(qz)P(z,1)}
\]

\[
= [z^0] \frac{f(z) - zTf(z) - zq^n f(z)}{z^n(z,q)_{n+1}}
\]

\[
= [z^n] \frac{f(z)}{(z,q)_n} - [z^{n-1}] \frac{Tf(z)}{(z,q)_{n+1}}.
\]

In general \(\Delta P(z,T,q^n)\) involves powers of \(T\). Thus, to use (2.6), we need to be able to calculate powers of \(T\). If an operator can be diagonalized, then its powers can be calculated very efficiently. In the present situation, we will see in section 4 that \(T\) is diagonal in the basis \((e_k)\) and more precisely that \(Te_k = q^k e_k\). Therefore, to use a diagonalization method to calculate \(T^n f\) requires one to expand \(f\).
on the basis \( e_n \), which is in effect to know the dual basis! A more effective solution is to use the next result. We define the real-valued maps \( B_n \) on \((\mathbb{R}^2)^n\) by

\[
B_n(u_1, \ldots, u_n) = \begin{cases} 
\sum_{1 \leq i < j \leq n} \langle u_i, e_2 \rangle \langle u_j, e_1 \rangle & \text{if } n \geq 2, \\
0 & \text{for } n = 0, 1.
\end{cases}
\] (2.7)

A simple consideration of \( B_3(u_1, u_2, u_3) \) shows that this map is not multilinear in general, and, comparing with \( B_3(u_2, u_1, u_3) \), is not symmetric in general.

**Proposition 2.11.** For any integer \( n \) positive,

\[
T^n f(z) = \sum_{i_1, \ldots, i_n \in \mathbb{N}^2} T_{i_1} \cdots T_{i_n} q^{B_n(i_1, \ldots, i_n)} \tilde{e}_{i_1 + \cdots + i_n}(z, 1) f(q^{i_1 + \cdots + i_n, e_2} z).
\]

3. Some equations with \( q \)-commuting coefficients, generating functions and recursions. The purpose of this section is to show that, underpinning Theorem 2.10, there is an interesting connection between some algebraic equations in some noncommuting variables and some generating functions. In essence, we show the equivalence between the problem of determining dual bases to Catalan bases, determining dual coefficients, solving some equations in \( q \)-commuting variables via a \( q \)-commuting analogue of Puiseux series, and solving some recursions which generalize the Segner recursion for Catalan numbers. In short, for a variety of different problems, solving one allows one to solve all the others.

We will not be interested in general noncommuting variables, but in \( q \)-commuting ones in the following sense.

**Definition 3.1.** Two variables \((A, M)\) \( q \)-commute if \( AM = qMA \).

Since we will consider power series of \( q \)-commuting variables, we agree on the following in order to avoid any ambiguity.

**Convention.** (i) When writing power series in a pair of \( q \)-commuting variable \((X, Y)\), we always write the monomials in the form \( X^i Y^j \).
(ii) If $i = (i_1, i_2)$ is in $\mathbb{N}^2$, we write $(X,Y)^i$ for $X^{i_1}Y^{i_2}$.

Our next result builds upon Newton’s method for finding Puiseux series expansion of roots of polynomials (see Walker, 1950) and its multivariable extension of McDonald (1995). It shows that solving some equations in $q$-commuting coefficients is equivalent to determining dual coefficients.

**Theorem 3.2.** Let $(M, A)$ be two $q$-commuting variables. Let $P$ be a Catalan power series and let $(T_i)_{i \in \mathbb{Z}^2}$ be its dual coefficients. Then

$$T = \sum_{i \in \mathbb{N}^2} T_i (M, A)^i$$

is the unique power series in $(M, A)$ solving the equation

$$A = P(M, T).$$

Stated differently, Theorem 3.2 asserts that if $\sum_{i \in \mathbb{N}^2} T_i (M, A)^i$ is a solution of $A = P(M, T)$, then the generating function of the coefficients $(T_i)_{i \in \mathbb{N}^2}$ with respect to the basis $(\tilde{e}_i(z,t))_{i \in \mathbb{N}^2}$ is easy to calculate, for it is exactly $t$ thanks to (2.2).

**Examples.** (continued) a) Since only $T_{0,1}$ does not vanish and is equal to 1, Theorem 3.2 asserts that $T_{0,1}(M,A)^{(0,1)} = A$ is the only solution of the equation $A = T$.

b) Assume that $q = 1$, and let us switch to lower case symbols to stress the commutativity. The equation $a = t - mt^2$ has solutions

$$t_+ = \frac{1 + \sqrt{1 - 4am}}{2m} \quad \text{and} \quad t_- = \frac{1 - \sqrt{1 - 4am}}{2m}.$$ 

Of those two solutions only $t_-$ is a power series in $(a, m)$ since $t_+$ is a Laurent series having a term $1/m$ of degree $-1$. Recall that the Catalan numbers

$$C_i = \frac{(2i)!}{i!(i + 1)!}$$

have the generating function

$$\hat{C}(z) = \sum_{i \in \mathbb{N}} C_i z^i = \frac{1 - \sqrt{1 - 4z}}{2z}.$$
Since $t_\_ = a\hat{C}(am)$, we have

$$[(a,m)]t_\_ = C_{(i,e_2)}\delta_{(i,e_1-e_2),1}.$$ 

When $q$ is not 1, then Carlitz’s $q$-Catalan numbers, $(C_n)$, are involved. Since by the discussion following (2.4) the only nonvanishing dual coefficients are $T_{n,n+1} = C_n$, $n \geq 0$, Theorem 3.2 asserts that

$$T = \sum_{n \in \mathbb{N}} C_n M^n A^{n+1}$$

solves $A = T - MT^2$. This provides a new characterization of Carlitz’s $q$-Catalan numbers in terms of a power series solution of a quadratic equation with some $q$-commuting coefficients.

To motivate what follows, recall that the Catalan numbers obey the Segner recursion

$$C_n = \sum_{0 \leq i \leq n-1} C_{n-1-i} C_i$$

(3.1)

with the initial condition $C_0 = 1$ (see Koshy, 2009). Thus, there is some form of recursion involving the coefficients of $t_\_$ in example b). The question arises as to whether such a recursion exists in the more general context of Theorem 3.2, or, equivalently, for the dual coefficients as defined in Definition 2.8. Recall that the maps $B_n$ are defined in (2.7).

**Theorem 3.3.** Let $P$ be a Catalan power series, and set $R(z,t) = \hat{P}(z,t)t$. Let $R_{i,j} = [z^i t^j]R(z,t)$. The dual coefficients $(T_i)_{i \in \mathbb{N}^2}$ associated to $P$ are the unique solution to the recursion

$$T_r = 1\{ r = (0,1) \} + \sum_{(i,j) \in \mathbb{N}^2} R_{i,j} \sum_{k_1,\ldots,k_{j+1} \in \mathbb{N}^2} T_{k_1} \cdots T_{k_{j+1}} q^{B_{j+1}(k_1,\ldots,k_{j+1})} 1\{ k_1 + \cdots + k_{j+1} = r - (i + 1,0) \}.$$  

(3.2)

If this is not clear that (3.2) is indeed a recursion, the proof of Theorem 3.3 makes it clear.

**Examples.** (continued) a) Since $P(z,t) = t$ we see that $R$ is the constant function 0. Thus, (3.2) gives $T_r = 1\{ r = (0,1) \}$. Hence, $T_{0,1} = 1$ and all the other $T_{i,j}$ vanish.
b) Since $R(z, t) = t$, we have

$$R_{i,j} = \begin{cases} 1 & \text{if } (i, j) = (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Recursion (3.2) with $r = (0, m)$ becomes

$$T_{0,m} = 1\{ m = 1 \} + \sum_{k_1, k_2 \in \mathbb{N}^2} T_{k_1} T_{k_2} q^{B_2(k_1, k_2)} \cdot 1\{ k_1 + k_2 = (-1, m) \}$$

$$= 1\{ m = 1 \}.$$

Assume now that we proved that $T_{i, j} = 0$ for any $i < n$ and any $j \neq i + 1$, which we just did for $n = 1$. Then, for $n > 1$, (3.2) yields

$$T_{n,m} = \sum_{k_1, k_2 \in \mathbb{N}^2} T_{k_1} T_{k_2} q^{B_2(k_1, k_2)} \cdot 1\{ k_1 + k_2 = (n - 1, m) \}.$$

Using our induction hypothesis

$$T_{n,m} = \sum_{i,j \in \mathbb{N}} T_{i,i+1} T_{j,j+1} q^{(i+1)j} \cdot 1\{ i + j = n - 1; i + j + 2 = m \}$$

$$= \sum_{i,j \in \mathbb{N}} T_{i,i+1} T_{j,j+1} q^{(i+1)j} \cdot 1\{ i + j = n - 1; n + 1 = m \}$$

Thus, $T_{n,m} = 0$ if $m \neq n + 1$ and

$$T_{n,n+1} = \sum_{i,j \in \mathbb{N}} T_{i,i+1} T_{j,j+1} q^{(i+1)j} \cdot 1\{ i + j = n - 1 \}.$$

Writing $C_n$ for $T_{n,n+1}$, we obtain

$$C_n = 1\{ n = 0 \} + \sum_{i,j \in \mathbb{N}} C_i C_j q^{j(i+1)} \cdot 1\{ i + j = n - 1 \},$$

that is $C_0 = 1$ and for $r \geq 1$,

$$C_r = \sum_{0 \leq i \leq r-1} C_i C_{r-1-i} q^{(r-1-i)(i+1)}. \quad (3.3)$$

As shown in Fürlinger and Hofbauer (1985; display preceding (2.2)), this is the recursion for the Carlitz $q$-Catalan numbers. When $q$ is 1, the $C_i$ are the usual Catalan numbers and (3.3) is the Segner
recursion (3.1). Therefore, recursion (3.2) may be viewed as an abstract form of the Segner one.

Note that once we know the result, it is a little easier to calculate the dual coefficients from Theorems 3.2. Indeed, the equation corresponding to that in Theorem 3.2 is \( A = T - MT^2 \). Knowing that only the \( T_{n,n+1} \) may not vanish, we write \( C_n \) for \( T_{n,n+1} \),

\[
T = \sum_{i \in \mathbb{N}} C_i M^i A^{i+1}.
\]

Substituting this form for \( T \) into the equation, we should have

\[
\sum_{i \in \mathbb{N}} C_i M^i A^{i+1} = A + \sum_{i,j \in \mathbb{N}} C_i C_j M^{i+1} A^{i+1} M^j A^{j+1}.
\]

Therefore,

\[
C_r = 1 \{ r = 0 \} + \sum_{i,j \in \mathbb{N}} C_i C_j q^{j(i+1)} 1 \{ i + j = r - 1 \},
\]

which is (3.1) again.

4. Proofs. This section contains the proofs of the results stated in sections 2 and 3.

4.1. Proof of Lemma 2.3. Let \( P \) be the power series associated to \( (e_k) \). Since \( P \) is a Catalan power series, \( P(0,t) = t \). Therefore, (2.1) implies \( [z^0](e_k(qz)/e_k(z)) = q^k \). Since \( e_k \) is a power series, this implies that it is of order \( k \).

4.2. Proof of Lemma 2.5. Let \( (e_k) \) be a normalized \( q \)-Catalan basis associated to \( P \). Since \( e_k \) is of order \( k \) and is normalized, there exists a sequence of power series \( (f_k) \) such that \( [z^0]f_k = 1 \) and \( e_k(z) = z^k f_k(z) \). Then

\[
q^k \frac{f_k(qz)}{f_k(z)} = \frac{e_k(qz)}{e_k(z)} = \frac{P(z,q^k)}{P(z,1)} = q^k \frac{1 - z \tilde{P}(z,q^k)q^k}{1 - z P(z,1)}.
\]

Thus,

\[
f_k(z) = \frac{1 - z \tilde{P}(z,1)}{1 - z P(z,q^k)q^k} f_k(qz) .
\]
By repeated substitution and using that \( f_k(0) = 1 \), we obtain
\[
f_k(z) = \prod_{j \geq 0} \frac{1 - q^j z \tilde{P}(q^j z, 1)}{1 - q^{j+k} z \tilde{P}(q^j z, q^k)}.
\]

(ii) Let \((e_k)\) be a \(q\)-Catalan basis associated to some Catalan power series \(P\).

If \(Q\) is another Catalan power series associated to \((e_k)\) then
\[
e_k(qz) e_k(z) = \tilde{P}(z, q^k) = Q(z, q^k) Q(z, 1).
\]

Setting \(c(z) = Q(z, 1)/P(z, 1)\), this implies \(Q(z, t) = c(z) P(z, t)\).

Since both \(P\) and \(Q\) are Catalan, their coefficient of \(t\) is 1. Therefore, \(c(z) = 1\) and \(Q = P\).

4.3. Proof of Lemma 2.7. (i) Since \(P\) is a Catalan power series,
\[
\prod_{0 \leq n < j} P(q^n z, t) = t^j \prod_{0 \leq n < j} (1 - q^n z t \tilde{P}(q^n z, t)).
\]

Therefore, writing \((z, t)^i\) for \(z^{(i,e_1)} t^{(i,e_2)}\), there exist power series \(f_i, i \in \mathbb{N}^2\), such that \(f_i(0,0) = 0\) and
\[
\tilde{e}_i(z, t) = (z, t)^i (1 + f_i(z, t)). \tag{4.3.1}
\]

Since \(\tilde{e}_0 = 1\) we see that \(f_0 = 0\). We then need to show that any power series
\[
g(z, t) = \sum_{i \in \mathbb{N}^2} g_i(z, t)^i \tag{4.3.2}
\]
can be represented in the basis \(\tilde{e}_i\) as \(\sum_{i \in \mathbb{N}^2} G_i \tilde{e}_i\).

We now claim that we can determine the \(G_i\) recursively by traveling through the indices \(i\) in \(\mathbb{N}^2\) in the order indicated in the following picture.

To make this explicit, considering the coefficient of \((z, t)^0\), we must have \(G_0 = g_0\).

For \(i, j \in \mathbb{N}^2\), we write \(i \prec j\) if \(\langle i, e_1 \rangle\) is at most \(\langle j, e_1 \rangle\), \(\langle i, e_2 \rangle\) is at most \(\langle j, e_2 \rangle\), and \(i\) is not \(j\); geometrically, that means that \(i\) is in the rectangle determined by the origin and the point \(j\), with the vertex \(j\) excluded.
Applying \([(z, t)^j]\) to both sides of (4.3.2),

\[
g_j = \sum_{i \in \mathbb{N}^2} G_i[(z, t)^j] \tilde{e}_i.
\]

But (4.3.1) implies that for \([(z, t)^j]\tilde{e}_i\) not to be 0, we must have \(i \prec j\) or \(i = j\). Therefore, since (4.3.1) also implies that \([(z, t)^j]\tilde{e}_j = 1\), we have

\[
g_j = \sum_{i \prec j} G_i[(z, t)^j] \tilde{e}_i + G_j.
\]

In other words, we can express \(G_j\) in terms of \(g_j\) and the \(G_i\) with \(i \prec j\). The result is then clear.

### 4.4. Proof of Lemma 2.9.

Given definitions 2.1, 2.6 and 2.8, we have

\[
\sum_{i \in \mathbb{N}^2} T_i z^{(i, e_1)} t^{(i, e_2)} \prod_{0 \leq j < (i, e_2)} (1 - zt \tilde{P}(q^j z, t)) = t.
\]  \hspace{1cm} (4.4.1)

We then view both sides of (4.4.1) as power series in \(t\) with coefficients in \(\mathbb{C}[[z]]\).

(i) The coefficient of \(t^0\) is obtained when \((i, e_2) = 0\) and is \(\sum_{n \in \mathbb{N}} T_{n,0} z^n\). Thus, given the right hand side of (4.4.1), we obtain \(T_{n,0} = 0\) for any nonnegative integer \(n\).

(ii) The coefficient of \(t\) in the left hand side of (4.4.1) is obtained when \((i, e_2) = 1\) and is

\[
\sum_{n \in \mathbb{N}} T_{n,1} z^n [t^0] (1 - zt \tilde{P}(z, t)) = \sum_{n \in \mathbb{N}} T_{n,1} z^n.
\]

Given (4.4.1), this series must be 1 and assertion (ii) of the lemma follows.

### 4.5. Some equations with \(q\)-commuting coefficients and recursions.

In order to prove Theorems 2.10, 3.2 and 3.3, we first prove a form of equivalence between Theorems 3.2 and 3.3.

#### Theorem 4.5.1.

Let \((A, M)\) be some \(q\)-commuting variables, and let \(P\) be a Catalan power series and set \(R(z, t) = \tilde{P}(z, t)t\). The equation \(A = P(M, S)\) has a unique power series \(S = \sum_{i \in \mathbb{N}^2} S_i(M, A)^i\) solution, which is defined by the recursion

\[
S_r = \mathbb{1}\{r = (0, 1)\} + \sum_{(i,j) \in \mathbb{N}^2} R_{i,j} \sum_{k_1, \ldots, k_{j+1} \in \mathbb{N}^2} S_{k_1} \cdots S_{k_{j+1}} q^{B_{j+1}(k_1, \ldots, k_{j+1})} \mathbb{1}\{k_1 + \cdots + k_{j+1} = r - (i + 1, 0)\}.
\]  \hspace{1cm} (4.5.1)
Proof. The proof consists in considering a generic power series $S$ in $(M, A)$, and show that the relation $A = P(M, S)$ allows us to calculate recursively the coefficients of the various powers of $(M, A)$ in $S$.

The maps $B_n$ defined in (2.7) come from the following lemma which shows that it allows us to keep track of the exponent of $q$ when multiplying monomials in two $q$-commuting variables.

Lemma 4.5.2. Let $(A, M)$ be some $q$-commuting variables. For any $k_1, \ldots, k_n$ in $\mathbb{N}^2$,

$$(M, A)^{k_1} \cdots (M, A)^{k_n} = q^{B_n(k_1, \ldots, k_n)} (M, A)^{k_1 + \cdots + k_n}.$$ 

Proof. The proof is by induction on $n$. The statement is obvious if $n$ is 1. Assume that this identity holds for $n$. Then

$$(M, A)^{k_1} \cdots (M, A)^{k_n+1} = q^{B_n(k_1, \ldots, k_n)} M^{(k_1 + \cdots + k_n, e_1)} A^{(k_1 + \cdots + k_n, e_2)} M^{(k_{n+1}, e_1)} A^{(k_{n+1}, e_2)}.$$

Since $(A, M)$ $q$-commute, this is

$$q^{B_n(k_1, \ldots, k_n) + \langle k_{n+1}, e_1 \rangle + \langle k_{n+1}, e_2 \rangle} (M, A)^{k_1 + \cdots + k_{n+1}}.$$

The exponent of $q$ in this formula is

$$B_n(k_1, \ldots, k_n) + \sum_{1 \leq i < n+1} \langle k_i, e_2 \rangle \langle k_{n+1}, e_1 \rangle,$$

which is indeed $B_{n+1}(k_1, \ldots, k_{n+1})$.

We can now prove Theorem 4.5.1. We express the equation $A = P(M, S)$ as

$$S = A + MR(M, S) S,$$

meaning

$$S = A + \sum_{(i,j) \in \mathbb{N}^2} R_{i,j} M^{i+1} S^{j+1}.$$ 

We substitute our tentative expansion for $S$, obtaining

$$\sum_{k \in \mathbb{N}^2} S_k (M, A)^k = A + \sum_{(i,j) \in \mathbb{N}^2} R_{i,j} \sum_{k_1, \ldots, k_{j+1} \in \mathbb{N}^2} S_{k_1} \cdots S_{k_{j+1}} M^{i+1}(M, A)^{k_1} \cdots (M, A)^{k_{j+1}}.$$
Using Lemma 4.5.2, we rewrite this equation as

$$\sum_{k \in \mathbb{N}^2} S_k(M, A)^k = A + \sum_{(i,j) \in \mathbb{N}^2} R_{i,j} \sum_{k_1, \ldots, k_{j+1} \in \mathbb{N}^2} S_{k_1} \ldots S_{k_{j+1}} q^{B_{j+1}(k_1, \ldots, k_{j+1})} (M, A)^{k_1 + \cdots + k_{j+1} + (i+1,0)}.$$  

Equating the coefficients of $(M, A)^r$ on both sides of the identity, we should have for any $r$ in $\mathbb{N}^2$,

$$S_r = 1 \{ r = (0, 1) \} + \sum_{(i,j) \in \mathbb{N}^2} R_{i,j} \sum_{k_1, \ldots, k_{j+1} \in \mathbb{N}^2} S_{k_1} \ldots S_{k_{j+1}} q^{B_{j+1}(k_1, \ldots, k_{j+1})} 1 \{ k_1 + \cdots + k_{j+1} = r - (i+1, 0) \}.$$  

(4.5.3)

Think of the $S_r$ as placed on the lattice $\mathbb{N}^2$ with $S_r$ sitting at site $r$. The above equality asserts that $S_r$ can be calculated from the various $S_{n,0}$ with $(k, e_1) \leq \langle r, e_1 \rangle - i - 1$, and so $\langle k, e_1 \rangle < \langle r, e_1 \rangle$. In other words, we can calculate these coefficients recursively once we know all the coefficients $S_{n,0}$, $n \geq 0$. To show that all those vanish, we consider $r$ of the form $(n, 0)$. Note that the second indicator function in (4.5.3) is 0 if $n - i - 1$ is negative. This is the case in particular if $n = 0$. Therefore, (4.5.3) implies $S_{0,0} = 0$.

Next, note that if $k_1 + \cdots + k_{j+1} = (n - i - 1, 0)$ and all the $k_i$ are in $\mathbb{N}^2$, then necessarily $\langle k_i, e_2 \rangle = 0$ for $i = 1, \ldots, j+1$; therefore, for $n$ positive, (4.5.3) yields

$$S_{n,0} = \sum_{0 \leq i \leq n-1} R_{i,j} \sum_{n_1, \ldots, n_{j+1} \in \mathbb{N}} S_{n_1,0} \ldots S_{n_{j+1},0} 1 \{ n_1 + \cdots + n_{j+1} = n - i - 1 \}.$$  

This allows us to calculate $S_{n,0}$ from $S_{n-1,0}, \ldots, S_{0,0}$, and, if $S_{n-1,0}, \ldots, S_{0,0}$ all vanish, so does $S_{n,0}$. Since we have seen that $S_{0,0}$ vanishes, all $S_{n,0}$ do as well, and this proves Theorem 4.5.1.

**4.6. A first expression for the dual forms.** The purpose of this subsection is to prove a form of Theorem 2.10 which is of interest for theoretical purposes, notably to prove Theorem 2.10, but does not give yet a computable form of the dual basis.

**Theorem 4.6.1.** Let $(e_k)_{k \in \mathbb{N}}$ be a Catalan basis with associated series $P$, and let $T$ be the linear operator on power series defined by $T e_k = q^k e_k$, $k \in \mathbb{N}$. Then

$$[e_n] f = q^n [z^0] \left( \frac{\Delta P(z, T, q^n) f(z)}{e_n(qz) P(z, 1)} \right).$$  

(4.6.1)
Note that while this theorem provides the dual basis, it is not explicit. Indeed, the operator $T$ is defined on the basis $(e_k)$, and in order to calculate $Tf$ for arbitrary power series $f$, we need a priori to decompose $f$ on the basis $(e_k)$, and, at this stage of the proof, we do not know how to do such a decomposition since we do not know the dual forms $[e_k]$. However, comparing (4.6.1) with (2.6), we see that Theorem 2.10 can be proved by showing that $T$, as defined in Theorem 4.6.1, can be represented as in (2.5).

**Proof.** Define the operator $\Delta_q$ by

$$\Delta_q f(z) = f(qz) - f(z).$$

This operator has the property that for any power series $f$,

$$[z^0]\Delta_q f(z) = 0.$$

Following Hofbauer (1984) and Krattenthaler (1984), we consider

$$\Delta_q \left( \frac{e_k}{e_n} \right)(z) = \frac{e_k(z)}{e_n(qz)} \left( \frac{e_k(qz)}{e_n(z)} - \frac{e_n(qz)}{e_n(z)} \right).$$

Therefore, since $(e_k)$ is a Catalan basis with associated series $P$,

$$\Delta_q \left( \frac{e_k}{e_n} \right)(z) = \frac{e_k(z)}{e_n(qz)} P(z, 1) (q^k - q^n) \Delta P(z, q^k, q^n).$$

This implies that if $k \neq n$ then

$$[z^0] \left( \frac{e_k(z)}{e_n(qz)} P(z, 1) \Delta P(z, q^k, q^n) \right) = 0,$$

while, if $k = n$,

$$[z^0] \left( \frac{e_k(z)}{e_n(qz)} P(z, 1) \Delta P(z, q^k, q^n) \right) = [z^0] \left( \frac{1}{P(z, q^k)} \frac{\partial}{\partial t} P(z, t)|_{t=q^n} \right) = \frac{1}{P(0, q^n)} \frac{\partial}{\partial t} P(0, t)|_{t=q^n}.$$

Since $P$ is a Catalan power series,

$$P(0, q^n) = q^n \quad \text{and} \quad \frac{\partial}{\partial t} P(0, t) = 1.$$
Therefore,
\[
q^n [z^0] \left( \frac{e_k(z)}{e_n(qz)P(z, 1)} \Delta P(z, q^k, q^n) \right) = \delta_{k, n}.
\]
Since \( \Delta P(z, s, t) \) is a power series in \((z, s, t)\),
\[
e_k(z) \Delta P(z, q^k, q^n) = \Delta P(z, T, q^n)e_k(z).
\]
This is the result.

4.7. Connecting Theorems 4.5.1 and 4.6.1. The purpose of this subsection is to show that the operator \( T \) involved in Theorem 4.6.1 solves a particular case of the equation involved in Theorem 4.5.1.

**Theorem 4.7.1.** Let \((e_k)\) be a Catalan basis associated to a power series \( P \), and define a linear operator \( T \) by \( Te_k = q^ke_k \). Consider the operators

\[ Af(z) = P(z, 1)f(qz) \quad \text{and} \quad Mf(z) = zf(z). \]

Then,
(i) \((A, M)\) \( q \)-commute;
(ii) \( A = P(M, T) \).

**Proof.** Since (i) is trivial, only (ii) needs to be proved. Since \((e_k)\) is a Catalan basis with associated power series \( P \),
\[
P(z, 1)e_k(qz) = P(z, q^k)e_k(z),
\]
that is
\[
Ae_k(z) = P(M, T)e_k(z).
\]
Since this equality holds on the basis, the result follows.

4.8. Proof of Theorem 3.3. To explain the spirit of the proof, consider an equation \( a = P(m, t) \) in real or complex variable. Assume that there is a power series solution
\[
t = \sum_{i \in \mathbb{N}^2} t_i(m, a)^i. \quad (4.8.1)
\]
There is a dual viewpoint, which is to consider the array \((t_i)_{i \in \mathbb{N}^2}\) as a given. It has a generating function \( \hat{t}(u, v) = \sum_{i \in \mathbb{N}^2} t_i(u, v)^i \). Then,
this power series satisfies the equation \( a = P(m, \hat{t}(m, a)) \). The proof of Theorem 3.3 is a \( q \)-analogue: starting with the definition of the dual coefficients through their generating function, our goal is to show that their generating function satisfies a functional equation which is similar to (4.8.1).

Throughout this subsection, we consider a Catalan power series with dual coefficients \((T_i)_{i \in \mathbb{N}^2}\), and we set \( R(z, t) = \tilde{P}(z, t) t \). Let \((\tilde{e}_i)_{i \in \mathbb{N}^2}\) be the corresponding predual basis. For any positive integer \( n \) we consider a map \( \Pi_n \), which maps \( n \) power series of two variables to a single power series in two variables, is \( n \)-linear, and is defined by

\[
\Pi_n(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_n})(z, t) = \tilde{e}_{i_1}(z, t) \tilde{e}_{i_2}(q^{i_1, e_2} z, t) \cdots \tilde{e}_{i_n}(q^{i_1 + i_2 + \cdots + i_{n-1}, e_2} z, t);
\]

this map is extended by \( n \)-linearity to power series, agreeing that for \( f(j)(z, t) = \sum_{i \in \mathbb{N}^2} f(j, i) \tilde{e}_i \), \( 1 \leq j \leq n \), we set

\[
\Pi_n(f(1), \ldots, f(n)) = \sum_{i_1, \ldots, i_n \in \mathbb{N}^2} f(1, i_1) \cdots f(n, i_n) \Pi_n(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_n}).
\]

The following proposition expresses some important properties of these \( n \)-linear maps. If \( i, j \) are in \( \mathbb{N}^2 \), we write \( \det(i, j) \) for the determinant of the matrix whose first column is the column vector \( i \) and second column is \( j \).

**Proposition 4.8.1.**

1. \( \Pi_n(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_n}) = q^{B_n(i_1, \ldots, i_n)} \tilde{e}_{i_1 + \cdots + i_n} \).
2. \( \Pi_n(f_1, \ldots, f_n) = \Pi_2(f_1, \Pi_{n-1}(f_2, \ldots, f_n)) = \Pi_2(\Pi_{n-1}(f_1, \ldots, f_{n-1}), f_n) \).
3. If \( f = \sum_{i \in \mathbb{N}^2} f_i \tilde{e}_i \) and \( g = \sum_{i \in \mathbb{N}^2} g_i \tilde{e}_i \), then

\[
\Pi_2(f, g) = \sum_{i, j \in \mathbb{N}^2} q^{B_2(i, j) + \det(i, j)} g_i f_j \tilde{e}_{i+j}.
\]

Note that assertion (iii) shows that \( \Pi_2(f, g) \) is not \( \Pi_2(g, f) \) because of the terms \( q^{\det(i, j)} \).
Proof. (i) We proceed by induction. For \( n = 2 \), the definition of \( \Pi_2 \) and the definition of the \( \tilde{e}_i \) we have

\[
\Pi_2(\tilde{e}_i, \tilde{e}_j)(z, t) = z^{(i, e_1)} \prod_{0 \leq n < \langle i, e_2 \rangle} P(q^n z, t) \times (zq^{(i, e_2)})^{(j, e_1)} \prod_{0 \leq n < \langle j, e_2 \rangle} P(q^n zq^{(i, e_2)}, t)
\]

The right hand side of this identity is

\[
q^{(i, e_2)}(j, e_1) z^{(i+j, e_1)} \prod_{0 \leq n < \langle i+j, e_2 \rangle} P(q^n z, t),
\]

which is indeed \( q^{B_2(i, j)} \tilde{e}_{i+j}(z, t) \). The induction is then immediate.

(ii) It follows by induction from the proof of (i).

(iii) Using (i), we have

\[
\Pi_2(f, g) = \sum_{i, j \in N^2} q^{B_2(i, j)} f_i g_j \tilde{e}_{i+j}.
\]

Writing \( i = (i_1, i_2) \) and \( j = (j_1, j_2) \), we have

\[
B_2(i, j) = (i, e_2)(j, e_1)
= B_2(j, i) + i_2j_1 - j_2i_1
= B_2(j, i) - \det(i, j),
\]

and the result follows.

Garsia’s (1981) tangled product is defined on power series through the bilinear mapping

\[
\Gamma_2(z^i, z^j) = z^i(q^i z)^j = q^{ij} z^{i+j}.
\]

The operator \( \Pi_2 \) defines a tangled product by

\[
\Pi_2(\tilde{e}_i, \tilde{e}_j) = q^{B_2(i, j)} \tilde{e}_{i+j},
\]

which is somewhat an analogue of Garsia’s, but at the level of the predual basis. However, while \( z^{i+j} \) relates to \( z^i \) and \( z^j \) by a simple product, \( \tilde{e}_{i+j} \) does not relate in a simple way to \( \tilde{e}_i \) and \( \tilde{e}_j \); this precludes us from defining an analogue of Garsia’s roofing operator.
in our setting which would trivialize tangled products into ordinary ones.

The following result asserts that $\Pi_n(t, \ldots, t)(z, t) = t^n$ but expresses this equality in a more readable way.

**Proposition 4.8.2.** The generating function

$$\hat{T}(z, t) = \sum_{i \in \mathbb{N}^2} T_i \tilde{e}_i(z, t),$$

which by Definition 2.8 is $t$, satisfies $\Pi_n(\hat{T}, \ldots, \hat{T}) = t^n$.

**Proof.** Given (2.2), we have for any $i$ in $\mathbb{N}^2$,

$$\sum_{j \in \mathbb{N}^2} T_j \tilde{e}_j(q^{(i,e_2)} z, t) = t.$$

This identity can be multiplied on both sides by $T_i \tilde{e}_i$ and then summed over $i$ to obtain

$$t^2 = \sum_{i_1 \in \mathbb{N}^2} \left( T_{i_1} \tilde{e}_{i_1}(z, t) \sum_{i_2 \in \mathbb{N}^2} T_{i_2} \tilde{e}_{i_2}(q^{(i_1,e_2)} z, t) \right).$$

More generally, using the same principle, $\hat{T}(z, t)^m$, that is, $t^m$, is

$$\sum_{i_1 \in \mathbb{N}^2} \left( T_{i_1} \tilde{e}_{i_1}(z, t) \sum_{i_2 \in \mathbb{N}^2} T_{i_2} \tilde{e}_{i_2}(q^{(i_1,e_2)} z, t) \sum_{i_3 \in \mathbb{N}^2} T_{i_3} \tilde{e}_{i_3}(q^{(i_1+i_2,e_2)} z, t) \right) \cdots$$

$$= \sum_{i_1, \ldots, i_m \in \mathbb{N}^2} T_{i_1} T_{i_2} \ldots T_{i_m} \Pi_m(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_m})(z, t) \quad (4.8.2)$$

$$= \Pi_m(\hat{T}, \ldots, \hat{T})(z, t).$$

We can now conclude the proof of Theorem 3.3. Identity (4.8.2) and Proposition 4.8.1.(i) yield

$$t^m = \sum_{i_1, \ldots, i_m} T_{i_1} \ldots T_{i_m} q^{B_m(i_1, \ldots, i_m)} \tilde{e}_{i_1 + \ldots + i_m}(z, t). \quad (4.8.3)$$
Therefore, by uniqueness of the decomposition in the basis \((\tilde{e}_i)\), for any \(r \in \mathbb{N}^2\),

\[
\sum_{i_1 + \cdots + i_m = r} T_{i_1} \cdots T_{i_m} q^{B_m(i_1,\ldots,i_m)} = [\tilde{e}_r] t^m.
\]

Referring to the sums involved in (3.2), we then have

\[
\sum_{(i,j) \in \mathbb{N}^2} R_{i,j} \sum_{k_1,\ldots,k_{j+1} \in \mathbb{N}^2} T_{k_1} \cdots T_{k_{j+1}} q^{B_{j+1}(k_1,\ldots,k_{j+1})} \mathbb{1}\{k_1 + \cdots + k_{j+1} = r - (i+1,0)\}
= \sum_{(i,j) \in \mathbb{N}^2} R_{i,j}[\tilde{e}_{r-(i+1,0)}] t^{j+1}.
\]

(4.8.4)

Since \(z^i \tilde{e}_j(z,t) = \tilde{e}_{j+(i,0)}(z,t)\), we have

\[
[\tilde{e}_{r-(i+1,0)}] f(z,t) = [\tilde{e}_r] z^{i+1} f(z,t).
\]

Therefore, (4.8.4) is

\[
\sum_{(i,j) \in \mathbb{N}^2} R_{i,j}[\tilde{e}_r] (z^{i+1} t^{j+1}) = [\tilde{e}_r] \left( \sum_{(i,j) \in \mathbb{N}^2} R_{i,j} z^{i+1} t^{j+1} \right)
= [\tilde{e}_r] (zR(z,t) t)
= [\tilde{e}_r] (-P(z,t) + t)
= -[\tilde{e}_r] P(z,t) + [\tilde{e}_r] t.
\]

Now, (2.2) implies \([\tilde{e}_r] t = T_r\), and since \(P(z,t) = \tilde{e}_{0,1}(z,t)\), we also have \([\tilde{e}_r] P(z,t) = \mathbb{1}\{r = (0,1)\}\). Thus, we obtain that

\[
\sum_{(i,j) \in \mathbb{N}^2} R_{i,j} \sum_{k_1,\ldots,k_{j+1} \in \mathbb{N}^2} T_{k_1} \cdots T_{k_{j+1}} q^{B_{j+1}(k_1,\ldots,k_{j+1})} \mathbb{1}\{k_1 + \cdots + k_{j+1} = r - (i+1,0)\}
= -\mathbb{1}\{r = (0,1)\} + T_r,
\]

which is recursion (3.2). Since the recursion has a unique solution, this proves Theorem 3.3.

4.9. Proof of Theorem 2.10. Theorem 4.6.1 gives us an expression for the dual basis in terms of the operator \(T\) defined by \(Te_k = q^k e_k\). Since (4.6.1) and (2.6) are the same expression,
it suffices to show that this operator $T$ acts on function as indicated by (2.5).

Theorem 4.7.1 asserts that $T$ satisfies the equation $A = P(M, T)$ for the specific operators $A$ and $M$ given in that theorem.

Theorem 4.5.1 asserts that a solution of the equation $A = P(M, T)$ is given by $\sum_{i \in \mathbb{N}^2} T_i(M, A)^i$ where the $T_i$ obey the recursion (3.2). Theorem 3.3 asserts that those $T_i$ are precisely the dual coefficients. Therefore, it only remains to prove that the particular solution $\sum_{i \in \mathbb{N}^2} T_i(M, A)^i$ of the equation $A = P(M, T)$ is the one we are looking for; or, put differently, that the specific $T$ we are interested in, which is defined a priori by $Te_k = q^k e_k$, coincides with the power series $\sum_{i \in \mathbb{N}^2} T_i(M, A)^i$, and acts on functions according to (2.5). Thus, we need to prove the following.

**Theorem 4.9.1.** Let $A$ and $M$ be as in Theorem 4.7.1, and let $(T_i)$ be defined by (2.2). Furthermore, let $T = \sum_{i \in \mathbb{N}^2} T_i(M, A)^i$. Then

(i) $Te_k = q^k e_k$ for any nonnegative integer $k$;
(ii) $Tf(z) = \sum_{i \in \mathbb{N}^2} T_i \tilde{e}_i(z, 1)f(q^{(i, e_2)}z)$.

**Proof.** (i) An induction shows that

$$A^i f(z) = \prod_{0 \leq j < i} P(q^{i-j}z, 1)f(q^i z).$$

Since $(e_k)$ is a Catalan basis with associated series $P$,

$$e_k(q^i z) = \frac{P(q^{i-1}z, q^k)}{P(q^{i-1}z, 1)}e_k(q^{i-1}z) \prod_{0 \leq j < i} P(q^j z, q^k) \frac{1}{\prod_{0 \leq j < i} P(q^j z, 1)} e_k(z).$$

Consequently,

$$A^i e_k(z) = \prod_{0 \leq j < i} P(q^j z, q^k)e_k(z),$$
and
\[ \sum_{i \in \mathbb{N}^2} T_i(M,A)^i e_k(z) = \sum_{i \in \mathbb{N}^2} T_i z^{(i,e_1)} \prod_{0 \leq j < (i,e_2)} P(q^j z, q^k) e_k(z) = \sum_{i \in \mathbb{N}^2} T_i \tilde{e}_i(z, q^k) e_k(z). \]

Since \( (T_i)_{i \in \mathbb{N}^2} \) are the dual coefficients, \( \sum_{i \in \mathbb{N}^2} T_i \tilde{e}_i(z, q^k) = q^k. \)

(ii) We have
\[ T f(z) = \sum_{i \in \mathbb{N}^2} T_i z^{(i,e_1)} A^{(i,e_2)} f(z) = \sum_{i \in \mathbb{N}^2} T_i z^{(i,e_1)} \prod_{0 \leq j < (i,e_2)} P(qz^j, 1) f(q^{(i,e_2)} z) = \sum_{i \in \mathbb{N}^2} T_i \tilde{e}_i(z, 1) f(q^{(i,e_2)} z). \quad (4.9.1) \]

This proves Theorem 4.9.1.

\[ \text{Remark. Note that writing} \ t \ \text{for the power series} \ t(z, t) = t, \ \text{that is the projection} \ (z, t) \mapsto t, \ (2.2) \ \text{asserts that} \ t = \sum_{i \in \mathbb{N}^2} T_i \tilde{e}_i. \ \text{Thus,} \ (4.9.1) \ \text{could also be rewritten as the simpler looking expression} \ T f(z) = \Pi_{\tilde{e}}(f)(z, 1). \]

4.10. Proof of Theorem 3.2. Consider the dual coefficients \( (T_i)_{i \in \mathbb{N}^2} \). Let \( (A, M) \) be, as in Theorem 3.2, arbitrary \( q \)-commuting variables. By Theorem 3.3, \( (T_i) \) is the unique solution of the recursion (3.2). Theorem 4.5.1 then gives that \( T = \sum_{i \in \mathbb{N}^2} T_i(M,A)^i \) is a power series solving \( A = P(M,T). \)

Conversely, Theorem 4.5.1 implies that the only power series solution in \( (M,A) \) of \( A = P(M,T) \) is given by \( \sum_{i \in \mathbb{N}^2} S_i(M,A)^i \) where \( S_i \) satisfies (4.5.1). But the recursion (4.5.1) is (3.2) (just substitute the letter \( T \) for the letter \( S \) in (4.5.1)). Therefore, by Theorem 3.3, \( (S_i)_{i \in \mathbb{N}^2} \) are the dual coefficients, and this proves Theorem 3.2.

4.11. Proof of Proposition 2.11. For \( n = 1 \), this is the definition of \( T \). We then proceed by induction, assuming that the statement is correct for \( n \). Then \( T^{n+1} f(z) \) is
\[ \sum_{i_2, \ldots, i_{n+1} \in \mathbb{N}^2} T_{i_2} \cdots T_{i_{n+1}} q^{B_{n}(i_2, \ldots, i_{n+1})} T(\tilde{e}_{i_2+\cdots+i_{n+1}}(z, 1) f(q^{(i_2+\cdots+i_{n+1}, e_2)} z)) . \quad (4.11.1) \]
But setting $i = i_2 + \cdots + i_{n+1}$ and using (2.5),

$$T(\tilde{e}_i(z,1)f(q^{(i,e_2)}z)) = \sum_{j \in \mathbb{N}^2} T_j \tilde{e}_j(z,1)\tilde{e}_i(q^{(j,e_2)}z,1)f(q^{(j+i,e_2)}z)$$

$$= \sum_{j \in \mathbb{N}^2} T_j \Pi_2(\tilde{e}_j,\tilde{e}_i)(z,1)f(q^{(j+i,e_2)}z). \quad (4.11.2)$$

Using the first assertion of Proposition 4.8.1 and substituting $i_1$ for $j$, we obtain that (4.11.2) is

$$\sum_{i_1 \in \mathbb{N}^2} T_{i_1} q^{B_2(i_1,i)}\tilde{e}_{i_1+i}(z,1)f(q^{(i_1+i,e_2)}z).$$

Substituting this expression in (4.11.1), the result follows from the identity

$$B_n(i_2,\ldots,i_{n+1}) + B_2(i_1,i_2 + \cdots + i_{n+1}) = B_{n+1}(i_1,\ldots,i_{n+1}). \quad \blacksquare$$

5. $p$-ary powers. In general the dual coefficients form a two dimensional array. However, in example b) of section 2 and 3, it was in fact one-dimensional, since only the $T_{n,n+1}$ could not vanish. More can be said on this situation and this is related to $P(z,t)$ having a specific form and $e_k$ being similar to Garsia’s (1981) powers, whose definition we will recall.

**Definition 5.1.** Let $p$ be an integer at least 2. Given a power series $\phi$ of order 1, its $p$-ary powers are the power series

$$\phi_{p,k,q}(z) = z^k \prod_{0 \leq j < k(p-1)} (1 - \phi(q^j z)), \quad k \in \mathbb{N}, \quad (5.1)$$

with $\phi_{p,0,q} = 1$.

Garsia (1981) considers powers defined as follows: start with a power series $\psi$ of order 1, without loss of generality satisfying $[z]\psi(z) = 1$; he considers the series

$$\psi_{[k,q]}(z) = \prod_{0 \leq j < k} \psi(q^j z). \quad (5.2)$$

Defining $\phi$ by $\psi(z) = z - z\phi(z)$, we see that $\psi_{[k,q]} = q^{(k)}\phi_{2,k,q}$. Thus, up to a normalization, Garsia’s powers coincide with 2-ary

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powers. Gessel (1980) deals with the more general $p$-ary powers (see his section 12) and his functional $\alpha(x_1,\ldots,x_n)$ for general noncommuting variables $x_1,\ldots,x_n$ is the analogue to our $B_n(i_1,\ldots,i_n)$ in his setting.

It is possible to interpret $p$-ary powers as Garsia’s powers in at least two different ways. Indeed, if $\phi$ is a power series of order 1, we can consider the series

$$\psi(z) = z \prod_{0 \leq j < p-1} \left(1 - \phi(q^j z)\right).$$

For this specific $\psi$,

$$\phi_{p,k,q}(z) = q^{-\binom{k}{2}}(p-1)^{-1} \psi_{[k,q/p-1]}(z).$$

However, this interpretation fails if we allow for $q$ to vary, for

$$q^{-\binom{k}{2}}(p-1)^{-1} \psi_{[k,r,p-1]}(z) = q^{-\binom{k}{2}}(p-1)^{-1} z^k \prod_{0 \leq i < k} \prod_{0 \leq j < p-1} \left(1 - \phi(q^i r^{\binom{j}{2}}(p-1) z)\right)$$

is not $\phi_{p,k,r}(z)$.

A more useful interpretation is obtained as follows. For any integer $p$ at least 2, define the linear operator $K_p$ by

$$K_p f(z) = f(z^{p-1}).$$

We then have

$$K_p \phi_{p,k,q}(z) = z^{k(p-1)} \prod_{0 \leq j < k(p-1)} \left(1 - \phi(q^j z^{p-1})\right) = (K_p \phi)_{2,k(p-1),q^{1/(p-1)}}(z).$$

Thus, up to a normalizing factor, $\phi_{p,k,q}(z^{p-1})$ is the Garsia power $(\text{Id}(1 - K_p \phi))_{[k(p-1),q^{1/(p-1)}]}$.

Despite these two interpretations of $p$-ary powers as Garsia powers, introducing the notion will make our results easier to prove, and we will see in the next section that the notion occurs naturally while the interpretation in terms of Garsia power is somewhat contrived.

Our next result shows that $p$-ary powers are $q$-Catalan bases associated to some Catalan power series that have a special form, and that the dual coefficients have a specific sparsity.
Proposition 5.2. The $p$-ary powers $\phi_{p,k,q}$ form the normalized $q$-Catalan basis associated with the Catalan power series $P(z,t) = t - t\phi(t^{p-1}z)$. The dual coefficients are determined by

$$T_i = 0 \text{ if } i \not\in \{ (n, n(p - 1) + 1) : n \in \mathbb{N} \},$$

and

$$\sum_{n \geq 0} q^{-n}T_{n,n(p-1)+1}\phi_{p,n,q}(qz) = \frac{1}{1 - \phi(z)}.$$  \hspace{1cm} (5.4)

Proof. Since $\phi$ is of order at least 1 and $p$ is at least 2, the power series $P$ given in the statement is a Catalan one. Since

$$\frac{\phi_{p,k,q}(qz)}{\phi_{p,k,q}(z)} = q^k \frac{1 - \phi(q^{k(p-1)}z)}{1 - \phi(z)} = \frac{P(z,q^k)}{P(z,1)},$$

the first assertion of the Proposition follows from Lemma 2.5.

The predual basis associated to $(\phi_{p,k,q})$ is then

$$\tilde{e}_i(z,t) = z^{(i,e_1)}t^{(i,e_2)} \prod_{0 \leq j < (i,e_2)} (1 - \phi(q^j t^{p-1}z)).$$

Therefore,

$$\tilde{e}_i(z^{p-1},t) = z^{(p-1)(i,e_1)}t^{(i,e_2)} \prod_{0 \leq j < (i,e_2)} (1 - \phi(q^j (tz)^{p-1})) .$$

By definition of the dual coefficients, $t = \sum_{i \in \mathbb{N}^2} T_i \tilde{e}_i(z,t)$; substituting $z^{p-1}$ for $z$ in this identity and then multiplying both sides by $z$,

$$zt = \sum_{i \in \mathbb{N}^2} T_i z \tilde{e}_i(z^{p-1},t)$$

$$= \sum_{i \in \mathbb{N}^2} T_i z^{(p-1)(i,e_1)+1-(i,e_2)}(zt)^{(i,e_2)} \prod_{0 \leq j < (i,e_2)} (1 - \phi(q^j (tz)^{p-1})) .$$  \hspace{1cm} (5.5)

Setting $\tau = zt$ we see from (5.5) that whenever $T_i$ does not vanish we must have

$$(p - 1)(i,e_1) + 1 - (i,e_2) = 0.$$
Thus, only $T_{n,n(p-1)+1}$, $n \in \mathbb{N}$, may not vanish. Consequently, (5.5) yields
\[
\tau = \sum_{n \in \mathbb{N}} T_{n,n(p-1)+1} \prod_{0 \leq j < n(p-1)+1} (1 - \phi(q^j t^{p-1})) ,
\]
that is, after simplifying both sides of this identity by $\tau$ and setting $z = \tau t^{p-1}$,
\[
1 = \sum_{n \in \mathbb{N}} T_{n,n(p-1)+1} z^n \prod_{0 \leq j < n(p-1)+1} (1 - \phi(q^j z))
= \left(1 - \phi(z)\right) \sum_{n \in \mathbb{N}} q^{-n} T_{n,n(p-1)+1} \phi_{p,n,q}(qz) .
\]

Starting with a power series $\phi(z)$ of order 1, we consider the $p$-ary powers $\phi_{p,k,q}(z)$; they induce a linear operator $U_{p,\phi,q}$ defined by
\[
U_{p,\phi,q} z^k = q^{\frac{k(p-1)}{2}} \phi_{p,k,q}(z) .
\]
Following Garsia (1981) and Krattenthaler (1988), extending their result from Garsia powers to $p$-ary powers, we will now construct the inverse of $U_{p,\phi,q}$.

Given Proposition 5.2 the dual coefficients for the basis $(\phi_{p,n,q})$ are defined by (5.4). We define
\[
\phi^{op}(z) = - \sum_{n \geq 1} T_{n,n(p-1)+1} q^{-\frac{(n(p-1)+1)}{2}} z^n .
\]
Note that the map $\phi \mapsto \phi^{op}$ depends on $q$, a dependence lost in the notation $\phi^{op}$. In order to avoid any ambiguity in the notation, we agree that the mapping $\phi \mapsto \phi^{op}$ has the precedence over $\phi \mapsto \phi_{p,k,q}$, so that, for instance, $\phi^{op}_{p,k,q}$ means $(\phi^{op})_{p,k,q}$.

**Theorem 5.3.** $U_{p,\phi,q}$ and $U_{p,\phi^{op},1/q}$ are inverse of each others.

**Proof.** We will first prove the result when $p$ is 2, which corresponds to the Garsia powers, and extend this result to arbitrary $p$ via (5.3).

**Case p = 2.** We assume for the time being that $p$ is 2. The heart of the proof is identity (4.8.3) and the following lemma.

**Lemma 5.4.** Let $(u_i)_{1 \leq i \leq m}$ be $m$ vectors in $\mathbb{N}^2$ with
\[
\langle u_i, e_2 \rangle = \langle u_i, e_1 \rangle + 1
\]

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for \( i = 1, \ldots, m \). Set \( v = u_1 + \cdots + u_m \). Then

\[
B_m(u_1, \ldots, u_m) = \binom{\langle v, e_2 \rangle}{2} - \sum_{1 \leq i \leq m} \binom{\langle u_i, e_2 \rangle}{2} - \sum_{1 \leq i \leq m} \langle u_i, e_2 \rangle (m - i).
\]

**Proof.** Write \( n_i \) for \( \langle u_i, e_2 \rangle \). Since by hypothesis \( \langle u_j, e_1 \rangle = n_j - 1 \),

\[
B_m(u_1, \ldots, u_m) = \sum_{1 \leq i < j \leq m} n_i (n_j - 1)
\]

\[
= \frac{1}{2} \left( \sum_{1 \leq i \leq m} n_i \right)^2 - \frac{1}{2} \sum_{1 \leq i \leq m} n_i^2 - \sum_{1 \leq i \leq m} n_i (m - i)
\]

\[
= \frac{1}{2} \langle v, e_2 \rangle^2 - \frac{1}{2} \sum_{1 \leq i \leq m} n_i^2 - \sum_{1 \leq i \leq m} n_i (m - i).
\]

Note that for any integer \( n \),

\[
\frac{n^2}{2} = \binom{n}{2} + \frac{n}{2}.
\]

Therefore, \( B_m(u_1, \ldots, u_m) \) is

\[
\binom{\langle v, e_2 \rangle}{2} + \langle v, e_2 \rangle - \sum_{1 \leq i \leq m} \binom{n_i}{2} - \frac{1}{2} \sum_{1 \leq i \leq m} n_i - \sum_{1 \leq i \leq m} n_i (m - i).
\]

This is the result since \( \langle v, e_2 \rangle = \sum_{1 \leq i \leq m} n_i \).

Throughout the proof we will use the \( q \)-Catalan basis \( (e_k) = (\phi_{2,k,q}) \), and switch freely between the notation \( e_k \) and \( \phi_{2,k,q} \) according to the context. In particular \( [e_k] \) is the dual form of \( \phi_{2,k,q} \). The predual basis associated to \( (e_k) \) is

\[
\tilde{e}_i(z,t) = z^{[i,e_1-e_2]} e_{(i,e_2)}(zt).
\]

Proposition 5.2 shows that the dual coefficients are determined by

\[
\sum_{n \geq 0} q^{-n} T_{n,n+1} \phi_{2,n,q}(qz) = \frac{1}{1 - \phi(z)}.
\]
Identity (4.8.3) is then
\[ t^m = \sum_{i_1, \ldots, i_m \in \mathbb{N}^2} T_{i_1} \cdots T_{i_m} q^{B_m(i_1, \ldots, i_m)} z^{(i_1 + \cdots + i_m, e_1 - e_2)} e_{(i_1 + \cdots + i_m, e_2)}(zt). \]

Multiplying both sides by \( z^m \) and substituting \( \zeta \) for \( zt \),
\[ \zeta^m = \sum_{i_1, \ldots, i_m \in \mathbb{N}^2} T_{i_1} \cdots T_{i_m} q^{B_m(i_1, \ldots, i_m)} z^{(i_1 + \cdots + i_m, e_1 - e_2) + m} e_{(i_1 + \cdots + i_m, e_2)}(\zeta). \] (5.6)

Given Proposition 5.2 with \( p = 2 \), only the \( T_i \) for which \( i = (n - 1, n) \), \( n \geq 1 \), may not vanish. Moreover, writing \( i_j \) as \( (n_j - 1, n_j) \), we see that
\[ (i_1 + \cdots + i_m, e_1 - e_2) + m = 0, \]
and Lemma 5.4 gives
\[ B_m(i_1, \ldots, i_m) = \left( n_1 + \cdots + n_m \right) - \sum_{1 \leq i \leq m} \binom{n_i}{2} - \sum_{1 \leq i \leq m} n_i (m - i). \]

Thus, (5.6) gives
\[ \zeta^m = \sum_{n_1, \ldots, n_m \geq 1} T_{n_1 - 1, n_1} \cdots T_{n_m - 1, n_m} q^{\binom{n_1 + \cdots + n_m}{2} - \sum_{1 \leq i \leq m} \binom{n_i}{2} - \sum_{1 \leq i \leq m} n_i (m - i)} e_{n_1 + \cdots + n_m}(\zeta). \]

Substituting \( z \) for \( \zeta \) and applying \( [e_k] \) to both sides,
\[ [e_k] z^m = \sum_{n_1, \ldots, n_m \geq 1} 1\{ n_1 + \cdots + n_m = k \} T_{n_1 - 1, n_1} \cdots T_{n_m - 1, n_m} q^{\binom{k}{2} - \sum_{1 \leq i \leq m} \binom{n_i}{2} - \sum_{1 \leq i \leq m} n_i (m - i)}. \]

The key of the proof is to multiply both sides of this identity by \( z^k \) and notice the nice way to factor the right hand side, obtaining
\[ z^k [e_k] z^m = q^{\binom{k}{2}} \sum_{n_1, \ldots, n_m \geq 1} 1\{ n_1 + \cdots + n_m = k \} \prod_{1 \leq i \leq m} T_{n_i - 1, n_i} q^{-\binom{n_i}{2}} \left( \frac{z}{q^{m-1}} \right)^{n_i}. \]
We then multiply both sides by \( q^{-\binom{m}{2}} \) and sum over \( k \) and use \( T_{0,1} = 1 \) to obtain, with \( \theta(z) = z(1 - \phi^{2})(z) \), that

\[
\sum_{k \geq 0} q^{-\binom{k}{2}} z^k [e_k] z^m = \prod_{1 \leq i \leq m} \left( \sum_{n \geq 1} T_{n-1,n} q^{-\binom{n}{2}} \left( z \left( \frac{z}{q^{m-i}} \right) \right)^n \right)
= \prod_{1 \leq i \leq m} \theta \left( \frac{z}{q^{m-i}} \right)
= U_{2,\phi^{2},1/q} z^m.
\] (5.7)

We apply \( U_{2,\phi,q} \) on both sides of the identity and use that \( ([e_k]) \) is the dual basis of \( (\phi_{2,k,q}) \) to get

\[
z^m = U_{2,\phi,q} U_{2,\phi^{2},1/q} z^m,
\]
and therefore \( U_{2,\phi,q} U_{2,\phi^{2},1/q} = \text{Id} \).

To prove that \( U_{2,\phi^{2},1/q} U_{2,\phi,q} = \text{Id} \), we can either use that since both \( U_{2,\phi,q} \) and \( U_{2,\phi^{2},1/q} \) map basis to basis they are invertible and therefore their right and left inverse coincide; or, we can note that (5.7) asserts that

\[
\sum_{k \geq 0} q^{-\binom{k}{2}} z^k [e_k] (z^\ell) = q^{-\binom{\ell}{2}} \phi^{2}_{2,\ell,1/q}(z).
\] (5.8)

We then have

\[
U_{2,\phi^{2},1/q} U_{2,\phi,q} z^m = U_{2,\phi^{2},1/q} q^{\binom{m}{2}} \sum_{\ell \in \mathbb{N}} [z^\ell](\phi_{2,m,q}) z^\ell
= q^{\binom{m}{2}} \sum_{\ell \in \mathbb{N}} [z^\ell](\phi_{2,m,q}) q^{-\binom{\ell}{2}} \phi^{2}_{2,\ell,1/q}(z).
\] (5.9)

Thus, using (5.8), the right hand side of (5.9) is

\[
q^{\binom{m}{2}} \sum_{k,\ell \in \mathbb{N}} [z^\ell](\phi_{2,m,q}) q^{-\binom{\ell}{2}} z^k [e_k] (z^\ell) = q^{\binom{m}{2}} \sum_{k \in \mathbb{N}} q^{-\binom{k}{2}} z^k [e_k] \phi_{2,m,q}.
\]

Since \( \phi_{2,m,q} = e_m \), we obtain that the right hand side of (5.9) is

\[
q^{\binom{m}{2}} \sum_{k \in \mathbb{N}} q^{-\binom{k}{2}} z^k \delta_{k,m} = z^m,
\]
and therefore, \( U_{2,\phi^{2},1/q} U_{2,\phi,q} z^m = z^m \). This proves Theorem 5.3 when \( p \) is 2.
Case \( p \geq 3 \). We assume now that \( p \) is at least 3. We set \( r = q^{1/(p-1)} \).

The identity (5.3) asserts that

\[
K_p U_{p,\phi,q}^k z^k = U_{2,K_p,\phi,r} K_p^* z^k .
\]

Given its left hand side, this equality shows that the image of \( U_{2,K_p,\phi,r} K_p \) is in the range of \( K_p \). Therefore,

\[
U_{p,\phi,q} = K_p^{-1} U_{2,K_p,\phi,r} K_p ,
\]

and, consequently, using the present theorem with \( p = 2 \),

\[
U_{p,\phi,q}^{-1} = K_p^{-1} U_{2,K_p,\phi,r} K_p = K_p^{-1} U_{2,(K_p \phi)^{\circ 2,1/r}} K_p . \tag{5.10}
\]

For this formula to be useful, we need to calculate \((K_p \phi)^{\circ 2}\). Thus, we consider the 2-ary powers \((K_p \phi)^{2}_{n,r}\), and, in order to avoid any confusion, we write \((S_i)_{i \in \mathbb{N}^2}\) for their dual coefficients. Applying Proposition 5.2, these dual coefficients are defined by

\[
\sum_{n \in \mathbb{N}} r^{-n} S_{n,n+1}(K_p \phi)^{2}_{n,r}(r z) = \frac{1}{1 - K_p \phi(z)} . \tag{5.11}
\]

Since \((K_p \phi)^{2}_{n,r}\) is in \( z^n \mathbb{C}[[z^{p-1}]] \) and \( 1/(1 - K_p \phi(z)) \) is in \( \mathbb{C}[[z^{p-1}]] \), (5.11) implies that the coefficients \( S_{n,n+1} \) must vanish if \( n \) is not a multiple of \( p - 1 \). Therefore, only \( S_{n(p-1),n(p-1)+1} \), \( n \in \mathbb{N} \), may not vanish. It follows that

\[
(K_p \phi)^{\circ 2}(z) = - \sum_{n \geq 1} S_{n(p-1),n(p-1)+1} r^{-n(p-1)/2} z^{n(p-1)} . \tag{5.12}
\]

Moreover, since only \( S_{n(p-1),n(p-1)+1} \), \( n \in \mathbb{N} \), may not vanish, (5.11) can then be rewritten as

\[
\sum_{n \in \mathbb{N}} S_{n(p-1),n(p-1)+1} z^{n(p-1)} \prod_{0 \leq j < n(p-1)} \left( 1 - \phi\left((zr^{j+1})^{p-1}\right) \right) = \frac{1}{1 - \phi(z^{p-1})} .
\]

Substituting \( z \) for \( z^{p-1} \), this means

\[
\sum_{n \in \mathbb{N}} q^{-n} S_{n(p-1),n(p-1)+1} \phi_{p,n,q}(q z) = \frac{1}{1 - \phi(z)} .
\]

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Consequently, given (5.4), $S_{n(p-1),n(p-1)+1} = T_{n,n(p-1)+1}$ and (5.12) becomes

$$(K_p \phi)^{\circ 2}(z) = - \sum_{n \geq 1} T_{n,n(p-1)+1} r^{-(n(p-1)+1)} z^{n(p-1)}.$$ 

Given how $\phi^{\circ p}$ is defined, this means $(K_p \phi)^{\circ 2}(z) = (K_p(\phi^{\circ p}))(z)$. Going back to (5.10), this implies

$$K_p^{-1} U_{2,(K_p \phi)^{\circ 2},1/r} K_p z^m$$

$$= K_p^{-1} r^{-(m(p-1))} (K_p(\phi^{\circ p}))_{2,m(p-1),1/r}(z)$$

$$= r^{-(m(p-1))} K_p^{-1} \left( z^{m(p-1)} \prod_{0 \leq j < m(p-1)} \left( 1 - \phi^{\circ p} \left( \frac{z^{p-1}}{r^{(p-1)/j}} \right) \right) \right)$$

$$= r^{-(m(p-1))} z^m \prod_{0 \leq j < m(p-1)} \left( 1 - \phi^{\circ p} \left( \frac{z}{q^j} \right) \right)$$

$$= U_{p,\phi^{\circ p},1/q} z^m.$$ 

This proves Theorem 5.3 by (5.10).

**Corollary 5.5.** For every $p$ at least 2, the map $(\phi, q) \mapsto (\phi^{\circ p}, 1/q)$ is an involution.

**Proof.** Theorem 5.3 implies that $U_{p,\phi,q}^{-1} = U_{p,\phi^{\circ p},1/q}$. Substituting $(\phi^{\circ p}, 1/q)$ for $(\phi, q)$, we obtain $U_{p,\phi^{\circ p},1/q}^{-1} = U_{p,(\phi^{\circ p})^{\circ p},q}$. Therefore, $U_{p,(\phi^{\circ p})^{\circ p},q} = U_{p,\phi,q}$, which forces $(\phi^{\circ p})^{\circ p} = \phi$. 

Starting with a power series $\phi(z)$ of order 1 and $[z] \phi(z) = 1$, the power series $P(z,t) = \phi(zt)/z$ is a Catalan one. For this specific type of Catalan power series, define $P^{\circ p}(z,t) = \phi^{\circ p}(zt)/z$. Corollary 5.5 implies that $(P,q) \mapsto (P^{\circ p}, 1/q)$ is an involution as well.

To make explicit the connection between Theorem 5.3 and the work of Garsia (1981) and Krattenthaler (1988), let $\phi$ be a power series of order 1 and set $\psi(z) = z(1 - \phi(z))$. Krattenthaler’s (1988) identity (6.11) asserts that the relation

$$\sum_{k \in \mathbb{N}} a_k z^k = \sum_{k \in \mathbb{N}} b_k \prod_{0 \leq j < k} \psi(q^j z),$$

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that is,
\[
\sum_{k \in \mathbb{N}} a_k z^k = U_{2, \phi, q} \sum_{k \in \mathbb{N}} b_k z^k,
\]
is equivalent to
\[
\sum_{k \in \mathbb{N}} b_k z^k = U_{2, \phi \circ_2 1/q} \sum_{k \in \mathbb{N}} a_k z^k,
\]
which is clear from Theorem 5.3.

Similarly, Krattenthaler's (1988) identity (6.10) states that if \( F_{k, \ell} \) are defined by
\[
z_\ell = \sum_{k \geq \ell} F_{k, \ell} \prod_{0 \leq j < k} \psi(q^j z) = U_{2, \phi, q} \sum_{k \geq \ell} F_{k, \ell} z^k,
\]
then, setting \( \Psi(z) = z(1 - \phi \circ_2(z)) \),
\[
\sum_{k \geq \ell} F_{k, \ell} z^k = U_{2, \phi, q}^{-1} z^\ell = U_{2, \phi \circ_2 1/q} \sum_{0 \leq j < \ell} \Psi(z/q^j).
\]

Theorems 3.2 and 3.3 can be viewed as different definitions of the dual coefficients, in which case Definition 2.8 asserts that a form of generating function of the dual coefficients, \( \sum_{n \in \mathbb{N}} T_{\ell} \bar{e}_i(z, t) \), has a particular value. However, this 'form' of generating function is not as useful as a proper generating function. While we do not know how to calculate the generating function of the dual coefficients in general, our next result shows that in the particular context of \( p \)-ary powers, the generating function of the suitably multiplied dual coefficients obeys a \( q \)-functional relation, and, as this will be clear in the next paragraph, this is related to Theorem 5.3 and its proof.

Given a power series \( \phi \) of order 1 and the corresponding \( p \)-ary powers \( \phi_{p, k, q} \), Proposition 5.2 asserts that only the \( T_{n, n(p-1)+1}, n \in \mathbb{N} \), may not vanish. The generating function of these dual coefficients is then \( \sum_{n \in \mathbb{N}} T_{n, n(p-1)+1} z^n \). Instead, it will be easier to consider the generating function of \( q^{-(p-1)(\ell)_2} T_{n, n(p-1)+1} \),
\[
T_{\phi, p}(z) = \sum_{n \in \mathbb{N}} q^{-(p-1)(\ell)_2} T_{n, n(p-1)+1} z^n.
\]
As an indication that this generating function is related to Theorem 5.3, note the identity

\[ T_{\phi,p}(z) = 1 + \sum_{n \geq 1} q^{-\left(\frac{n(p-1)+1}{2}\right)} q^{np/2} T_{n,n(p-1)+1} z^n \]

\[ = 1 - \phi^{q^p/2} z. \]

For the following result, it is of interest to use Garsia’s roofing operator, defined on power series \( f(z) = \sum_{n \in \mathbb{N}} f_n z^n \) by

\[ \hat{f}(z) = \sum_{n \in \mathbb{N}} q^{-\left(\frac{n^2}{2}\right)} f_n z^n, \]

as well as his reciprocal staring operator

\[ ^*f(z) = \prod_{n \in \mathbb{N}} f(z/q^n). \]

The roofing operator may be iterated, setting \( f^{\wedge p} = f^{\wedge (p-1)} \), or, more explicitly,

\[ f^{\wedge p}(z) = \sum_{n \in \mathbb{N}} q^{-\left(\frac{n^2}{2}\right)} f_n z^n. \]

**Theorem 5.6.** Let \( \phi(z) = \sum_{i \geq 1} \phi_i z^i \) be a power series of order 1. The generating function \( T_{\phi,p} \) obeys the functional relation

\[ T_{\phi,p}(z) = 1 + \sum_{i \geq 1} \phi_i z^i q^{-(i-1)/2} \prod_{0 \leq j \leq (p-1)i} T_{\phi,p}(q^{-j} z). \quad (5.13) \]

and satisfies the identity

\[ T_{\phi,p}(z) = \frac{\left(^* (1 - \phi) \right)^{(p-1)}(z/q)}{\left(^* (1 - \phi) \right)^{(p-1)}(z)}. \]

Before proving this result, some remarks and an example may help to understand its content.

**Remark.** Defining

\[ \Phi_q(z) = \prod_{n \in \mathbb{N}} (1 - \phi(q^n z)), \]
we see that \((1 - \phi)(z) = \Phi_{1/q}(z)\). Set

\[ g(z) = \sum_{n \in \mathbb{N}} q^{-(p-1)n} z^n \frac{\Phi_{1/q}(z)}{[z^n]}, \]

that is \(g(z) = \Phi_{1/q}^{(p-1)}\). The second identity in Theorem 5.6 gives that \(T_{\phi,p}(z) = g(z/q)/g(z)\).

**Example.** Continuing example b) of sections 2 and 3, and given Proposition 5.2, the Catalan power series \(P(z,t) = t - t^2z\) corresponds to \(\phi(z) = z\) and \(p = 2\). In this case, \([z]\phi(z) = 1\), that is, \(\phi_1 = 1\), and all the other \(\phi_i\) vanish. The assertion of Theorem 5.6 is then

\[ T_{\phi,2}(z) = 1 + zT_{\phi,2}(z/q)T_{\phi,2}(z). \]

This implies

\[ T_{\phi,2}(z) = \frac{1}{1 - zT_{\phi,2}(z/q)}, \]

and, by iterated substitution, we obtain the continued fraction,

\[ T_{\phi,2}(z) = \frac{1}{1 - \frac{z/q}{1 - \frac{z/q}{1 - \frac{z/q}{1 - \ldots}}}}, \tag{5.14} \]

as indicated in Garsia (1981).

With the notation of the remark preceding this example, we have here

\[ \Phi_{1/q}(z) = \prod_{n \in \mathbb{N}} (1 - z/q^n) = (z, 1/q)_\infty. \]

Using Euler’s identity (see Andrews, Askey and Roy, 1999, Corollary 10.2.2),

\[ [z^n]\Phi_{1/q}(z) = (-1)^n \frac{q^{-(2)}(z)}{(1/q, 1/q)_n} \]

Thus, since \(p = 2\),

\[ g(z) = \sum_{n \in \mathbb{N}} (-1)^n \frac{q^{-n(n-1)}}{(1/q, 1/q)_n} z^n. \]
Recall that, following Ismail (2009, formula (21.7.3)) the $q$-Airy function is defined by

$$Ai_q(z) = \sum_{n \geq 0} (-1)^n \frac{q^{n^2} z^n}{(q,q)_n}.$$

Thus $g(z) = Ai_{1/q}(qz)$. Theorem 5.6 and the remark following it imply that in this case

$$T_{\phi,2}(z) = \frac{Ai_{1/q}(z)}{Ai_{1/q}(z/q)}, \quad (5.15)$$

the identity between (5.14) and (5.15) being the Rogers-Ramanujan continuous fraction.

**Remark.** It is interesting to compare Theorem 5.6 with Gessel’s (1980) Theorem 12.2. When $q$ is 1, (5.13) is

$$T_{\phi,p}(z) = 1 + T_{\phi,p}(z)\phi(zT_{\phi,p}(z)^{p-1}). \quad (5.16)$$

The function $T_{\phi,p}$ can then be obtained by Lagrange inversion. Setting $y = z T_{\phi,p}(z^{p-1})$ and $g(y) = 1/(1 - \phi(y))$, substituting $z^{p-1}$ for $z$ in (5.16), we can rewrite (5.16) as $y = zg(y^{p-1})$. This is to compare with (12.1) in Gessel (1980), which, when $q$ is 1, would be $y = g(z^{p-1})$. The second equality in Theorem 5.6 can then be interpreted as a Lagrange type inversion formula for the functional equation (5.13).

To further compare Theorem 5.6 with Gessel’s (1980) Theorem 12.2, setting $r = 1/q$ and $m = p - 1$, and writing now $f$ for $T_{\phi,p}$ and recalling the notation in (5.2), (5.13) is

$$f(z) = 1 + f(z) \sum_{i \geq 1} \phi_i z^i r^{(p-1)(i)} f_{[mi,r]}(rz)$$

$$= 1 + f(z) \sum_{i \geq 1} r^{-mi/2} \phi_i r^{mi^2/2} z^i f_{[mi,r]}(rz). \quad (5.17)$$

Setting $g(z) = \phi(r^{-m/2}z)$ and $g_i = [z^i]g(z)$, (5.17) becomes

$$f(z) = 1 + f(z) \sum_{n \geq 1} g_n r^{mn^2/2} z^n f_{[mn,r]}(rz).$$

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This expression is to be compared with (12.7) in Gessel (1980) which is (with Gessel’s $g_0$ set to 1)

$$f(z) = 1 + \sum_{n \geq 1} g_n r^{mn^2/2} z^n f_{[mn,r]}(z).$$

In our setting, the analogue of Gessel’s (1980) identity (12.7) is the second assertion of Theorem 5.6, which provides, with $H = (\ast(1 - \phi))^{-(p-1)}$,

$$f_{[k,r]}(z) = H(zr^k)/H(z),$$

and the analogue of his (12.4) is that

$$(\ast(1 - \phi))^{-(p-1)}(z) (T_{\phi,p} k,1/q)(z) = (\ast(1 - \phi))^{-(p-1)}(z/q^{k-1}),$$

or, written differently, $H(z)f_{[k,r]}(z) = H(zr^k)$. This last identity implies $[z^n](H(z)f_{[k,r]}(z)) = r^{nk}[z^n]H(x)$, which is the analogue of Gessel’s (1980) identity (12.11) in our setting.

**Remark.** Identity (5.13) can be put in a fairly compact form using the operator $U_{p,\psi,q}$. Indeed, since $T_{0,1} = 1$ (see Lemma 2.9), the power series $\psi(z) = 1 - T_{\phi,p}(z)$ is of order 1. Considering the right hand side of (5.13), we have

$$q^{-(p-1)\langle\frac{i}{2}\rangle} \prod_{0 \leq j < (p-1)i} T_{\phi,p}(q^{-j}z) = q^{-(p-1)\langle\frac{i}{2}\rangle} \psi_{p,i,1/q}(z)$$

$$= q^{(p-2)i/2} U_{p,\psi,1/q}(z_i).$$

Therefore, since $T_{\phi,p} = 1 - \psi$, (5.13) can be rewritten as

$$-\psi(z) = \sum_{i \geq 1} \phi_i q^{(p-2)i/2} U_{p,\psi,1/q} z^i$$

$$= U_{p,\psi,1/q} \sum_{i \geq 1} \phi_i q^{(p-2)i/2} z^i$$

$$= U_{p,\psi,1/q} \phi(q^{(p-2)/2} z).$$

Writing now $g(z) = -\phi(q^{p/2} z)$, we see that $\psi$ is the function $f$ that solves

$$f(z) = U_{p,f,1/q} g(z/q).$$
Up to substituting \( q \) for \( 1/q \), this is exactly Gessel’s (1980) equation (12.1), which is \( f(z) = U_f,p,q g(qz) \).

**Proof.** The proof has two parts corresponding to the two assertions. **Part 1. Proof of the functional relation.** In order to keep the subscripting within reason, we will write \( t_n \) for \( T_{n,n(n-1)+1} \). As in Proposition 5.2, consider the Catalan power series \( P(z,t) = t - t \phi(t^{p-1}z) \), for which \( t_n \) are the dual coefficients. Those dual coefficients obey the recursion given by Theorem 3.3. To write this recursion, we have, with \( R(z,t) = \tilde{P}(z,t) t \),

\[
R(z,t) = \phi(t^{p-1}z)/z = \sum_{i \geq 1} \phi_i t^{(p-1)i} z^{i-1}.
\]

Thus, writing \( R_{i,j} \) for the coefficient of \( z^i t^j \) in \( R(z,t) \), we have

\[
R_{i-1,(p-1)i} = \phi_i, \quad i \geq 1,
\]

and \( R_{i,j} = 0 \) if \( j \neq (p-1)(i + 1) \) for nonnegative \( i \). Writing \( k_i \) for \( (n_i, n_i(p-1) + 1) \), recursion (3.2) takes the form

\[
t_n = 1 \{ n = 0 \} + \sum_{i \geq 1} \phi_i \sum_{n_1,\ldots,n_{(p-1)i+1} \in \mathbb{N}} t_{n_1} \cdots t_{n_{(p-1)i+1}} q^{B(p-1)i+1(k_1,\ldots,k_{(p-1)i+1})} 1 \{ n_1 + \cdots + n_{(p-1)i+1} = n - i \}. \tag{5.18}
\]

Arguing as in the proof of Lemma 5.4, \( B_m(k_1,\ldots,k_m) \) is

\[
\sum_{1 \leq i < j \leq m} ((p-1)n_i + 1)n_j = (p-1) \left( \sum_{1 \leq i \leq m} n_i \right) - (p-1) \sum_{1 \leq i \leq m} \left( \begin{array}{c} n_i \\ 2 \end{array} \right) + \sum_{1 \leq j \leq m} (j-1)n_j.
\]

Hence, (5.18) yields

\[
t_n = 1 \{ n = 0 \} + \sum_{i \geq 1} \phi_i \sum_{n_1,\ldots,n_{(p-1)i+1} \in \mathbb{N}} t_{n_1} \cdots t_{n_{(p-1)i+1}} q^{(p-1)(n_i - 1) - (p-1) \sum_{1 \leq j \leq (p-1)i+1} \left( \begin{array}{c} n_j \\ 2 \end{array} \right) - \sum_{1 \leq j \leq (p-1)i+1} (j-1)n_j} 1 \{ n_1 + \cdots + n_{(p-1)i+1} = n - i \}. \tag{5.19}
\]
Set \( \tilde{t}_n = q^{-(p-1)\binom{n}{2} / n} t_n \). We see that if \( n_1 + \cdots + n_{(p-1)i+1} = n - i \), then
\[
\begin{align*}
\left( \frac{n - i}{2} \right) - \left( \frac{n}{2} \right) &= -in + \frac{i(i+1)}{2} \\
&= -\left( \frac{i}{2} \right) - i(n_1 + \cdots + n_{(p-1)i+1}).
\end{align*}
\]

Multiplying both sides of (5.19) by \( z^n q^{-(p-1)\binom{n}{2}} \), a quantity which is 1 when \( n = 0 \), yields
\[

tilde{t}_n = 1 + \sum_{i \geq 1} \phi_i z^i q^{-(p-1)\binom{i}{2}} \sum_{n_1, \ldots, n_{(p-1)i+1}} \tilde{t}_{n_1} \cdots \tilde{t}_{n_{(p-1)i+1}} \\
\times q^{\sum_{1 \leq j \leq (p-1)i+1} \binom{j-1-(p-1)i}{j-1-(p-1)i} n_j} z^{n_1 + \cdots + n_{(p-1)i+1}} \\
\times 1 \{ n_1 + \cdots + n_{(p-1)i+1} = n - i \}
\]
\[
= 1 + \sum_{i \geq 1} \phi_i z^i q^{-(p-1)\binom{i}{2}} \sum_{n_1, \ldots, n_{(p-1)i+1}} \left( \prod_{1 \leq j \leq (p-1)i+1} \tilde{t}_{n_j} q^{(j-1-(p-1)i) n_j} z^{n_j} \right) 1 \{ n_1 + \cdots + n_{(p-1)i+1} = n - i \}.
\]

We sum over \( n \) to obtain
\[
\mathcal{T}_{\phi,p}(z) = 1 + \sum_{i \geq 1} \phi_i z^i q^{-(p-1)\binom{i}{2}} \prod_{1 \leq j \leq (p-1)i+1} \mathcal{T}_{\phi,p}(q^{j-1-(p-1)i} z),
\]
that is, after substituting \( j \) for \( (p-1)i - j + 1 \) in the product, the first assertion of Theorem 5.6.

**Part 2. Proof of the second assertion.** The proof is an abstraction of some calculations in Prellberg and Brak (1995) — but see also Gessel (1980, section 12) where similar ideas are used. Considering the functional relation, we see that it is nonlinear because of the product term \( \prod_{0 \leq j \leq (p-1)i} \mathcal{T}_{\phi,p}(q^{j-(p-1)i} z) \). As noticed in Prellberg and Brak (1995) in some special cases, writing \( \mathcal{T}_{\phi,p}(z) \) as \( g(z/q) / g(z) \) for some power series \( g \) to be determined simplifies this product considerably, and in fact linearizes the functional equation. Repeated substitutions in the relation \( g(z) = g(z/q) / \mathcal{T}_{\phi,q}(z) \) shows that \( g(z) = 1 / \prod_{j \geq 0} \mathcal{T}_{\phi,q}(z/q^j) \) is this function; but this product form is not useful beyond showing the existence and uniqueness of the power
series $g$. This change of power series leads to

$$
\prod_{0 \leq j \leq (p-1)i} T_{\phi,p}(q^{-j}z) = \prod_{0 \leq j \leq (p-1)i} \frac{g(q^{-j-1}z)}{g(q^{-j}z)} = \frac{g(q^{-(p-1)i}z)}{g(z)}.
$$

Thus, multiplying both sides of (5.13) by $g(z)$, we obtain

$$
g(z/q) = g(z) + \sum_{i \geq 1} \phi_i z^i q^{-(p-1)(i^2)} g(q^{-(p-1)i-1}z),
$$

an equation which is linear in $g$. We write $g(z) = \sum_{n \in \mathbb{N}} g_n z^n$ for the power series expansion of $g$, and set $g_i = 0$ if $i$ is negative. Considering the coefficient of $z^n$ in this equation, we obtain the identity

$$
q^{-n}g_n = g_n + \sum_{i \geq 1} \phi_i q^{-(p-1)(i^2)-(n-i)((p-1)i+1)} g_{n-i}.
$$

To understand where the Garsia roofing operator comes from in this calculation, note that given the complicated power of $q$ in right hand side of (5.21), it is quite natural to make a change of variable, setting $g_n = q^{-\alpha_n}a_n$, and determine the sequence $(\alpha_n)$ which allows us to rewrite (5.21) in the simplest manner. Making this change of variable, and multiplying both sides of (5.21) by $q^{\alpha_n}$ yields

$$
q^{-n}a_n = a_n + \sum_{i \geq 1} \phi_i q^{-(p-1)(i^2)-(n-i)((p-1)i+1)-\alpha_{n-i}+\alpha_n} a_{n-i}.
$$

The exponent of $q$ is

$$
-p - (n^2 - (n-i)^2) + \frac{p-1}{2} i - n + i - \alpha_{n-i} + \alpha_n.
$$

Therefore, choosing $\alpha_n = (p-1)n(n-1)/2$ changes this exponent into $i - n$, and yields the recursion

$$
q^{-n}a_n = a_n + \sum_{i \geq 1} \phi_i q^{i-n}a_{n-i},
$$

(5.22)
with \( a_i = 0 \) if \( i \) is negative. Consider the generating function of \( a_n \), that is, \( A(z) = \sum_{n \geq 0} a_n z^n \). Identity (5.22) asserts that

\[
[z^n]A(z/q) = [z^n]A(z) + \sum_{i \geq 1} [z^i] \phi(z) [z^{n-i}] A(z/q)
\]

Thus, \( A(z/q) = A(z) + \phi(z) A(z/q) \), and therefore, \( A(z) = (1 - \phi(z)) A(z/q) \). By repeated substitutions, \( A(z) = \prod_{i \geq 0} (1 - \phi(q^{-1} z)) \).

Given the remark following Theorem 5.6, we see that \( A \) is \( \Phi_{1/q} \) and that the function \( g \) in this proof coincides with that given in the remark. This proves the result.

As a consequence of Theorem 5.6, we can express the generating function \( T_{\phi,p} \) as a ratio of two constant terms. They involve the Jacobi style theta function

\[
\theta(u, q) = \sum_{n \in \mathbb{N}} q^{(n-1)/2} u^n.
\]

**Proposition 5.7.** With the notation of Theorem 5.6,

\[
T_{\phi,p}(z) = \frac{[u^0] \left( \theta \left( \frac{z}{u^{q^{-1}}}, \frac{1}{q^{p-1}} \right) \Phi_{1/q}(u) \right)}{[u^0] \left( \theta \left( \frac{z}{u^{q^{-1}}}, \frac{1}{q^{p-1}} \right) \Phi_{1/q}(u) \right)}.
\]

**Proof.** Note that

\[
[u^{-n}] \theta(z/u, 1/q^{p-1}) = q^{-(p-1)} z^n.
\]

Thus, the function \( g \) in the remark following Theorem 5.6 can be written as

\[
g(z) = \sum_{n \in \mathbb{N}} [u^{-n}] \theta(z/u, 1/q^{p-1}) [u^n] \Phi_{1/q}(u)
\]

\[
= [u^0] \left( \theta(z/u, 1/q^{p-1}) \Phi_{1/q}(u) \right).
\]
The result follows from Theorem 5.6.

6. *q*-analogue of the Fuss-Catalan numbers. A *p*-ary tree is a tree for which each parent has *p* children. The Fuss-Catalan number $C_{p,n}$ counts the numbers of *p*-ary trees with *np* + 1 nodes. From this definition, these numbers obey the recursion

$$C_{p,n} = \begin{cases} n = 0 \\ \sum_{r_1, \ldots, r_p \in \mathbb{N}} \mathbb{1}\{r_1 + \cdots + r_p = n - 1\} C_{p,r_1} \cdots C_{p,r_p} \end{cases}.$$  
(6.1)

Alternative definition in terms of lattice paths, dissections of polygons, or staircase tilings exist, as shown for instance in Hilton and Perderson’s (1991) and Heubach, Li and Mansour (2008) papers. They also arise in some probabilistic problems (Bajunaid, Cohen, Colonna, Singman, 2005; Liggett, 2000). From the recursion (6.1), the generating function

$$\hat{C}_p(z) = \sum_{n \in \mathbb{N}} C_{p,n} z^n$$

solves the equation

$$t^p z - t + 1 = 0.$$  
(6.2)

Lagrange inversion leads to the explicit formula

$$C_{p,n} = \frac{1}{(p-1)n + 1} \binom{pn}{n}.$$  

We will now define some *q*-Fuss-Catalan numbers $(C_{p,n})_{n \in \mathbb{N}}$ in the spirit of Carlitz’s (1972) *q*-Catalan numbers. It is unclear from the definition that we will use that these numbers have a combinatorial interpretation in terms of lattice paths; while one may suspect, by analogy, some connection with *(p − 1)*-Dyck paths, the results of the previous sections provide various interpretations for those numbers; and we will see that one of those interpretations allows us to indeed derive a combinatorial one in terms of *(p − 1)*-Dyck paths.

Given (6.2), consider the polynomial $P(z,t) = t - t^p z$. It is a Catalan power series of the form $t - t\phi(t^{p-1}z)$ with $\phi(z) = z$. The corresponding *p*-ary powers are

$$\phi_{p,n,q}(z) = z^n \prod_{0 \leq j < n(p-1)} (1 - \phi(q^j z))$$

$$= z^n (z,q)_{n(p-1)}.$$  

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Proposition 5.2 asserts that the corresponding dual coefficients 
\((T_i)_{i \in \mathbb{N}^2}\) can be identified with a sequence \((C_{p,n})_{n \in \mathbb{N}}\) as follows. Define \((C_{p,n})_{n \in \mathbb{N}}\) as the unique sequence such that

\[
\sum_{n \in \mathbb{N}} C_{p,n} z^n (z,q)_{n(p-1)+1} = 1.
\]

We then have

\[
T_i = \begin{cases} 
C_{p,n} & \text{if } i = (n, n(p-1) + 1) \text{ for some } n \in \mathbb{N}; \\
0 & \text{otherwise}.
\end{cases}
\]

Alternatively, using Theorem 5.6, the sequence \((C_{p,n})\) may be defined through the generating function

\[
T(z) = T_{\phi,p}(z) = \sum_{n \in \mathbb{N}} q^{-(p-1)\binom{z}{2}} C_{p,n} z^n
\]

by \(T(0) = 1\) and the functional relation

\[
T(z) = 1 + z \prod_{0 \leq j \leq p-1} T(q^{-j} z).
\]

Comparing this functional relation with display (12) in Bergeron (2012), we see that \(q^{-(p-1)\binom{z}{2}} C_{p,n}\) has a combinatorial interpretation for it counts the area of \((p-1)\)-Dyck paths. This section then yields new results concerning the area of these path, which are the analogue of known results on the usual Dyck paths and the analogue of results on \(q\)-Catalan numbers related to the usual Dyck paths.

Theorem 5.6 allows us to express this generating function with a new form of \(q\)-Airy function. Indeed, we saw in the example following Theorem 5.6 that with \(\phi(z) = z\) we have

\[
[z^n] \Phi_{1/q}(z) = (-1)^n \frac{q^{-\binom{z}{2}}}{(1/q, 1/q)_n}.
\]

We have, with the notation of the remark following Theorem 5.6,

\[
g(z) = \sum_{n \in \mathbb{N}} \frac{q^{-p\binom{z}{2}} (-1)^n z^n}{(1/q, 1/q)_n}.
\]
This function $g$ is a new $q$-analogue of the Airy function, call it $\text{Ai}_{p, 1/q}$. Then Theorem 5.6 asserts that

$$\mathcal{T}(z) = \frac{\text{Ai}_{p, 1/q}(z/q)}{\text{Ai}_{p, 1/q}(z)}.$$

Theorem 3.2 provides an alternative viewpoint. Let $(M, A)$ be two $q$-commuting variables and consider the equation $A = T - T^p M$ with unknown $T$. The unique power series in $(M, A)$ solving this equation is

$$T = \sum_{i \in \mathbb{N}^2} T_i (M, A)^i = \sum_{n \in \mathbb{N}} C_{p,n} M^n A^{n(p-1)+1}.$$

Finally, Theorem 3.3 provides yet another definition. Indeed, with the notation of that theorem, $R(z, t) = t^{p-1}$ and therefore $R_{i,j} = 1 \{ i = 0 ; j = p - 1 \}$. Recursion (3.2) asserts that

$$C_{p,n} = 1 \{ n = 0 \} + \sum_{k_1, \ldots, k_p \in \mathbb{N}} C_{p,k_1} \cdots C_{p,k_p} q^{(p-1) \sum_{1 \leq i < j \leq p} k_i k_j + \sum_{1 \leq j \leq p} (j-1) k_j} 1 \{ k_1 + \cdots + k_p = n - 1 \},$$

which is a $q$-analogue of (6.1) and a $p$-ary analogue of (3.3). Given this recursion, it appears that these $q$-Fuss-Catalan numbers are different than those introduced by Haiman (1994), Garsia and Haiman (1996) or those of type A and B defined by Stump (2008; section 2.4).

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