Sensivity of the Hermite rank

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Sensitivity of the Hermite rank

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Abstract

The Hermite rank appears in limit theorems involving long memory. We show that an Hermite rank higher than one is unstable when the data is slightly perturbed by transformations such as shift and scaling. We carry out a “near higher order rank analysis” to illustrate how the limit theorems are affected by a shift perturbation that is decreasing in size. We also consider the case where the deterministic shift is replaced by centering with respect to the sample mean. The paper is a companion of Bai and Taqqu [2] which discusses the instability of the Hermite rank in the statistical context.

1 Introduction

A stationary sequence \( \{X(n)\} \) with finite variance is said to have long memory or long-range dependence, if

\[
\text{Cov}[X(n), X(0)] \approx n^{2H-2}
\]

as \( n \to \infty \), where \( \approx \) means asymptotic equivalence up to a positive constant, the parameter \( H \in (1/2, 1) \) is called the Hurst index. See, e.g., the recent monographs Giraitis et al. [9], Beran et al. [3], Samorodnitsky [22] and Pipiras and Taqqu [19] for more information on the notion long memory.

The study of the asymptotic behavior of partial sums of long-memory sequences has been of great interest in probability and statistics. In particular, complete results have been obtained for the following class of models called Gaussian subordination. Let \( \{Y(n)\} \) be a stationary Gaussian process with long memory in the sense of \( \square \), which we suppose to have mean 0 and variance 1 without loss of generality. Now consider

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1In general, we can allow inserting a slowly varying factor (Bingham et al. [4]), e.g., a logarithmic function, in the asymptotic relation \( \square \), but we shall not do that for simplicity.
the transformed process
\[ X(n) = G(Y(n)), \]  
where \( G(\cdot) : \mathbb{R} \to \mathbb{R} \) is a function such that \( \mathbb{E}X(n)^2 = \mathbb{E}G(Y(n))^2 < \infty \). The goal is to establish limit theorems for the normalized sums of \( X(n) \), the transformed stationary process.

To develop the limit theorems, one has to use the notion of Hermite rank, which is an integer attached to the function \( G(\cdot) \). It is defined as follows. Let \( Z \) denote a standard Gaussian random variable and let \( \varphi(dx) = (2\pi)^{-1/2}e^{-x^2/2}dx \) denote its distribution. The Hermite rank is associated with the orthogonal decomposition of \( G(\cdot) \) in the space
\[ L^2(\gamma) = \{ G(\cdot) : \mathbb{E}G(Z)^2 < \infty \} \]
into Hermite polynomials, which are defined as
\[ H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad \text{and more generally,} \]
\[ H_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2} \quad \text{for} \quad m \geq 1. \]  
Then \( \sqrt{m}H_m(\cdot), m \geq 0 \) forms an orthonormal basis of \( L^2(\gamma) \) (see Pipiras and Taqqu [19], Proposition 5.1.3).

**Definition 1.1.** The Hermite rank \( k \) of \( G(\cdot) \in L^2(\gamma) \) is defined as
\[ k = \inf \left\{ m \geq 1 : \mathbb{E}G(Z)H_m(Z) = \int_{\mathbb{R}} G(x)H_m(x)\gamma(dx) \neq 0 \right\}, \]  
where \( H_m(\cdot) \) is the \( m \)-th order Hermite polynomial. Equivalently, \( k \) is the order of the first nonzero coefficient in the \( L^2(\gamma) \)-expansion:
\[ G(\cdot) - \mathbb{E}G(Z) = \sum_{m=k}^{\infty} c_m H_m(\cdot), \quad k \geq 1, \]  
that is, \( c_m = 0 \) for \( m < k \) and \( c_k \neq 0 \) where
\[ c_m = \frac{\mathbb{E}[G(Z)H_m(Z)]}{m!} \quad \text{for} \quad m \geq 0. \]  

The following celebrated results due to Dobrushin and Major [7], Taqqu [24] and Breuer and Major [5] have been established involving the Hermite rank.

**Theorem 1.2.** Suppose that \( G(\cdot) \in L^2(\gamma) \) has Hermite rank \( k \geq 1 \). Then the following conclusions hold.
- **Central limit case:** suppose that \( 1/2 < H < 1 - \frac{1}{2k} \) (this implies \( k \geq 2 \)). Then \( \{G(X(n))\} \) has short memory in the sense that
  \[ \sigma^2 := \sum_{n=-\infty}^{\infty} \text{Cov}[G(X(n)),G(X(0))] \]
  converges absolutely and
  \[ \frac{1}{N^{1/2}} \sum_{n=1}^{[Nt]} \left( G(X(n)) - \mathbb{E}G(X(n)) \right) \overset{f.d.d.}{\longrightarrow} \sigma B(t), \quad t \geq 0, \]  
where \( B(t) \) is a standard Brownian motion.
where \( \overset{f.d.d.}{\to} \) denotes convergence of the finite-dimensional distributions and \( B(t) \) is the standard Brownian motion.

- Non-central limit case: suppose that \( 1 - \frac{4}{\pi^2} < H < 1 \). \( \{G(X(n))\} \) has long memory with Hurst index:

\[
H_G = (H - 1)k + 1 \in \left(\frac{1}{2}, 1\right).
\]

Furthermore, as \( N \to \infty \), we have

\[
\frac{1}{N^{H_G}} \sum_{n=1}^{[Nt]} \left(G(X(n)) - E G(X(n))\right) \overset{f.d.d.}{\to} c Z_{H,k}(t), \tag{9}
\]

for some \( c \neq 0 \), and

\[
Z_{H,k}(t) = \int_{\mathbb{R}^k} \left[ \int_0^t \prod_{j=1}^{k} (s - x_j)^{H-3/2} ds \right] B(dx_1) \ldots B(dx_k), \tag{10}
\]

is the so-called \( k \)-th order Hermite process, where \( \int_{\mathbb{R}^k} B(dx_1) \ldots B(dx_k) \) denotes the \( k \)-tuple Wiener-Itô integral with respect to the standard Brownian motion \( B(\cdot) \) (Major \[15\]).

Bai and Taqqu \[2\] point out that despite the probabilistic interest of Theorem 1.2, its straightforward application to large-sample statistical theories can be problematic. This is due to a strong instability feature in the notion Hermite rank. This paper can be viewed as a technical companion to Bai and Taqqu \[2\] containing some mathematical characterization of such instability and related problems. In particular, the paper contains the following results:

1. The instability of the Hermite rank with respect to transformations including shift, scaling, etc.
2. A “near higher order rank” analysis of Theorem 1.2 perturbed by a diminishing shift.
3. Coincidence of Hermite rank and power rank (Ho and Hsing \[11\]) in the Gaussian case.

The third result is not closely related to the first two, but it is obtained as a direct byproduct of the analysis of the instability problems. The paper is organized as follows: the results described above are stated in Sections 2, 3, and 4. Section 5 contains some auxiliary results involving analytic function theory. The theorems are proved in Section 6.

Throughout the paper, \( a_N \ll b_N \) means as \( N \to \infty \), \( a_N/b_N \to 0 \), and \( a_N \approx b_N \) means \( a_N/b_N \to c \) for some positive constant \( c > 0 \).

## 2 How a transformation affects the Hermite rank

First we state the results regarding the instability of the Hermite rank under transformation. Their relevance is explained in Remark 2.4 which follows. The proofs are given in Section 6.

\[2\] In fact, we have weak convergence in the space \( D[0,1] \) with uniform metric.
Recall that $x$ is an accumulation point or limit point of a set $E \subset \mathbb{R}^p$, if every neighborhood of $x$ contains an infinite number of elements of $E$. The so-called derived set $E'$ consists of all the accumulation points of $E$. For example, if $E = \{0\} \cup [1, 2)$, then $E' = [1, 2]$.

**Theorem 2.1** (Instability with respect to shift). Suppose that the measurable function $G(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is not a.e. constant and $G(\cdot \pm M) \in L^2(\gamma)$ for some $M > 0$. Then $G(\cdot + x) \in L^2$ for all $x \in [-M, M]$, and the interval $(-M, M)$ contains no accumulation point of the set

$$E = \{ x \in (-M, M) : G(\cdot + x) \text{ has Hermite rank } \geq 2 \},$$

namely, $E' \cap (-M, M) = \emptyset$.

**Theorem 2.2** (Instability with respect to scaling). Suppose that the measurable function $G : \mathbb{R} \rightarrow \mathbb{R}$ is not a.e. symmetric and $G(\cdot \times M) \in L^2(\gamma)$ for some $M > 1$. Then $G(\cdot \times y) \in L^2$ for all $y \in (0, M]$, and the interval $(0, M)$ contains no accumulation point of the set

$$E = \{ y \in (0, M) : G(\cdot \times y) \text{ has Hermite rank } \geq 2 \},$$

namely, $E' \cap (0, M) = \emptyset$.

**Remark 2.3.** The requirement of $G$ being non-symmetric in Theorem 2.2 is essential because any non-zero symmetric function $f \in L^2(\gamma)$ has Hermite rank 2 since due to the symmetry of the measure $\gamma$

$$\int_{\mathbb{R}} f(z)H_1(z)\gamma(dz) = \int_{\mathbb{R}} f(z)z\gamma(dz) = 0,$$

while

$$\int_{\mathbb{R}} f(z)(z^2 - 1)\gamma(dz) = \int_{\mathbb{R}} f(z)H_2(z)\gamma(dz) = 2\int_{0}^{\infty} f(z)z^2\gamma(dz) > 0$$

due to symmetry of $\gamma$.

**Remark 2.4.** The preceding two theorems point to the instability of the Hermite rank. They imply, for example, that the particular values $x = 0$ in Theorem 2.1 or $y = 1$ in Theorem 2.2 cannot be an accumulation point of the set $E$ with Hermite rank greater or equal to 2. Hence some neighborhood of $x = 0$ or $y = 1$ contains at most one point ($x = 0$ or $y = 1$) with Hermite rank $k \geq 2$. This means that a slight level shift or scale change (when the original transformation is non-symmetric) will force the rank to change from perhaps being $\geq 2$ to being 1. In the case of shift, this is stated as Theorem 2.14 in Bai and Taqqu [2]. Note that the rank 2 of a non-constant symmetric function in Remark 2.3 is still unstable with respect to a level shift by Theorem 2.1.

In addition, one can consider the shift and scaling joint together, namely, deal with an affine transformation.

**Theorem 2.5** (Instability with respect to affine transformation). Suppose that the measurable $G(\cdot)$ is not constant a.e. and $G(\pm M_1 + \cdot \times M_2) \in L^2(\gamma)$. Then the set

$$E = \{(x, y) \in (-M_1, M_1) \times (0, M_2) : G(x + \cdot \times y) \text{ has Hermite rank } \geq 2 \}$$
has Hausdorff dimension (see Falconer [8], Section 2.2) not exceeding 1. So in particular, \( E \) has 0 two-dimensional Lebesgue measure.

To illustrate Theorem 2.5, consider the function \( G(z) = z^2 = 1 + H_2(z) \) which has Hermite rank 2. Suppose that it is perturbed by an affine transformation involving a shift \( x \) and a scale \( y \) and becomes
\[
(x + zy)^2 = x^2 + 2xyz + y^2 z^2 = x^2 + y^2 + (2xy)z + y^2(z^2 - 1) = x^2 + y^2 + (2xy)H_1(z) + y^2 H_2(z).
\]
After centering, the centered function in \( z \) becomes \( (2xy)H_1(z) + y^2 H_2(z) \), which has Hermite rank 1 if and only if \( x \neq 0 \) (it no longer has Hermite rank 2 as \( G(z) \) does). The set \( E \) in Theorem 2.5 corresponds to the one-dimensional line \( \{(x, y): x = 0\} \) which has Lebesgue measure zero.

Finally, we can formulate an abstract result about the instability with respect to general nonlinear transformations.

**Theorem 2.6.** Suppose that \( G \in L^2(\gamma) \) and let \( F_\theta \) be a family of transformations parameterized by \( \theta \in D \subset \mathbb{R}^p \), where \( D \) is an open region containing the origin \( 0 = (0, \ldots, 0) \) and let \( F_0 \) be the identity transformation. Suppose that \( G \) is perturbed and becomes \( G \circ F_\theta \).

Assume that \( G \circ F_\theta \in L^2(\phi) \) for all \( \theta \in D \), \( Z \) is a standard normal random variable, and let
\[
U(\theta) := \mathbb{E}[(G \circ F_\theta)(Z) \times H_1(Z)] = \mathbb{E}[(G \circ F_\theta)(Z) \times Z].
\]
be the first coefficient of the Hermite expansion of \( G \circ F_\theta \), and assume that the following properties hold:

(a) \( U(\theta) \) is real analytic in \( \theta \in D \);

(b) If \( U(\theta) = 0 \) holds for all \( \theta \in D \), then \( G \) is constant a.e..

Then either \( G \) is constant a.e., or if not, then the set
\[
E = \{ \theta \in D: G \circ F_\theta \text{ has Hermite rank } \geq 2 \} = \{ \theta \in D: U(\theta) = 0 \}
\]
has Hausdorff dimension not exceeding \( p - 1 \). In particular, \( E \) has zero \( p \)-dimensional Lebesgue measure.

**Remark 2.7.** Here is a more informative description of the set \( E \) in Theorem 2.5 and 2.6 above. First, \( E \) is called the zero set of \( U \) if it consists of all \( \theta \in D \) such that \( U(\theta) = 0 \). Second, the zero set \( E = \{ \theta \in D: U(\theta) = 0 \} \) of a (multivariate) real analytic function \( U \) is called a real analytic variety. According to Lojasiewicz’s Structure Theorem (Theorem 6.3.3 of Krantz and Parks [13]), \( E \subset \mathbb{R}^p \) can be expressed as a union of submanifolds of dimensions 0, 1, \ldots, \( p - 1 \). We state here the results in terms of the Hausdorff dimension for simplicity.

### 3 Being near a higher order Hermite rank

In the so-called near integration analysis of unit root (e.g., Phillips [18]), one studies the asymptotic behavior of an autoregressive model when the autoregression coefficient tends to 1 (unit root) as the sample size
increases. Such analysis sheds lights on the situation where the coefficient is close to but not exactly 1. Here
in a similar spirit, we carry out a “near higher order rank” analysis for Theorem 2.1 regarding the limit
theorem for the sum \( \sum_{n=1}^{N} G(Y(n) + x_N) \), where the shift \( x_N \to 0 \) as the number of summands \( N \to \infty \).
The result is given in the following theorem, where \( c, c' \)'s denote constants whose value can change from line
to line.

**Theorem 3.1.** Let \( \{Y(n)\} \) be a standardized stationary Gaussian process whose covariance satisfies [1] with
Hurst index \( H \in (1/2, 1) \). Suppose that the function \( G(\cdot) \) is not constant a.e., \( G(\cdot + M) \in L^2(\gamma) \) for some
\( M > 0 \), and has an Hermite rank \( k \geq 1 \). Set

\[
S_N(x_N) = \sum_{n=1}^{N} [G(Y(n) + x_N) - \mathbb{E}G(Y(n) + x_N)],
\]

where \( x_N \to 0 \).

- **Central limit case:** suppose that \( 1/2 < H < 1 - \frac{1}{2k} \) (if \( k = 1 \) this case does not exist). Then as \( N \to \infty \),
  \( N^{(1/2-H)/(k-1)} \to 0 \) and

  \[
  (a) \text{ if } x_N \ll N^{(1/2-H)/(k-1)}; \quad N^{-1/2}S_N(x_N) \xrightarrow{f.d.d.} cB(t);
  \]

  \[
  (b) \text{ if } x_N \approx N^{(1/2-H)/(k-1)}; \quad N^{-1/2}S_N(x_N) \xrightarrow{f.d.d.} c_1Z_{H,1}(t) + c_2B(t);
  \]

  \[
  (c) \text{ if } N^{(1/2-H)/(k-1)} \ll x_N \to 0; \quad N^{-H}x_N^{-k}S_N(x_N) \xrightarrow{f.d.d.} cZ_{H,1}(t);
  \]

where the fractional Brownian motion \( Z_{H,1}(t) \) and the Brownian motion \( B(t) \) are independent.

- **Non-central limit case:** suppose that \( 1 - \frac{1}{2k} < H < 1 \). Let \( H_G \) be as in [3]. Then as \( N \to \infty \), \( N^{H-1} \to 0 \), and

  \[
  (a) \text{ if } 0 < x_N \ll N^{H-1}; \quad N^{-H}S_N(x_N) \xrightarrow{f.d.d.} cZ_{H,k}(t);
  \]

  \[
  (b) \text{ if } x_N \approx N^{H-1}; \quad N^{-H}S_N(x_N) \xrightarrow{f.d.d.} \sum_{m=1}^{k} c_mZ_{H,m}(t);
  \]

  \[
  (c) \text{ if } N^{H-1} \ll x_N \to 0; \quad N^{-H}x_N^{-k}S_N(x_N) \xrightarrow{f.d.d.} cZ_{H,1}(t);
  \]

where the Hermite processes \( Z_{H,m}(t) \)'s are defined through the same Brownian motion in [3].

The terms ”central limit case” and ”non-central limit case” in the preceding theorem refer to the terminol-
ogy used in Theorem 1.2 and thus to the type of limits one obtains when there is no shift, that is when
\( x_N = 0 \).

**Remark 3.2.** Theorem 3.1 has some interesting implications. In the central limit case, the critical order
\( N^{(1/2-H)/(k-1)} \) depends negatively on \( H \). The larger \( H \), the smaller the critical order. Thus the larger \( H \)
(but below \( 1 - 1/(2k) \)) is, the more easily the effect of a higher-order rank in the limit theorem gets surpassed
by a shift perturbation. In the non-central limit case, however, the relation is reversed since the critical order
\( N^{H-1} \) depends positively on \( H \). Note that in the non-central limit case, the order \( N^{H-1} \) determining the
border of the regimes does not depend on the rank \( k \).
In Theorem 3.3 below, we provide a result on the non-centered sum \( \sum_{n=1}^{N} G(Y(n) + x_N) \), assuming that \( \mathbb{E}G(Y(n)) = 0 \). It is expected that if \( x_N \) tends to 0 too slowly, then a deterministic trend will appear in the limit.

**Theorem 3.3.** Under the assumptions of Theorem 3.1, if in addition \( \mathbb{E}G(Y(n)) = 0 \) and \( S_N \) is replaced by

\[
\tilde{S}_N(x_N) = \sum_{n=1}^{N} G(Y(n) + x_N),
\]

then the following conclusions hold:

- **Central limit case:** suppose that \( H < 1 - \frac{1}{2k} \). Then as \( N \to \infty \):
  
  \begin{align*}
  (a) \text{ if } x_N \ll N^{-1/(2k)}: & \quad N^{-1/2} \tilde{S}_{[N]}(x_N) \xrightarrow{f.d.d.} cB(t) \\
  (b) \text{ if } x_N \approx N^{1-1/(2k)}: & \quad N^{-1/2} \tilde{S}_{[N]}(x_N) \xrightarrow{f.d.d.} c_1 t + c_2 B(t) \\
  (c) \text{ if } N^{1-1/(2k)} \ll x_N \to 0: & \quad N^{-1} x_N^{-k} \tilde{S}_{[N]}(x_N) \xrightarrow{f.d.d.} ct;
  \end{align*}

  where \( Z_{H,k}(t) \) and \( B(t) \) are independent.

- **Non-central limit case:** suppose that \( H > 1 - \frac{1}{2k} \). Let \( H_G \) be as in (8). Then as \( N \to \infty \):
  
  \begin{align*}
  (a) \text{ if } 0 < x_N \ll N^{H-1}: & \quad N^{-H} \tilde{S}_{[N]}(x_N) \xrightarrow{f.d.d.} cZ_{H,k}(t) \\
  (b) \text{ if } x_N \approx N^{H-1}: & \quad N^{-H} \tilde{S}_{[N]}(x_N) \xrightarrow{f.d.d.} c_0 t + \sum_{m=1}^{k} c_m Z_{H,m}(t) \\
  (c) \text{ if } N^{H-1} \ll x_N \to 0: & \quad N^{-1} x_N^{-k} \tilde{S}_{[N]}(x_N) \xrightarrow{f.d.d.} ct;
  \end{align*}

  where the Hermite processes \( Z_{H,m}(t) \)’s are defined through the same Brownian motion in (10).

In the next theorem the argument \( x_N \) in Theorem 3.3 is replaced by a subtracted sample mean.

**Theorem 3.4.** Under the assumptions of Theorem 3.3 assume in addition that \( G(\cdot) \) is a polynomial. Let \( \tilde{Y}_N = N^{-1} \sum_{n=1}^{N} Y(n) \). Then the following conclusions hold:

- **Central limit case:** suppose that \( H < 1 - \frac{1}{2k} \). Then as \( N \to \infty \),
  
  \[
  N^{-1/2} \tilde{S}_{[N]}(-\tilde{Y}_N) \xrightarrow{f.d.d.} cB(t)
  \]

- **Non-central limit case:** suppose that \( H > 1 - \frac{1}{2k} \). Let \( H_G \) be as in (8). Then as \( N \to \infty \),
  
  \[
  N^{-H} \tilde{S}_{[N]}(-\tilde{Y}_N) \xrightarrow{f.d.d.} \sum_{m=1}^{k} c_m Z_{H,m}(t)
  \]

  where the Hermite processes \( Z_{H,m}(t) \)’s are defined through the same Brownian motion in (10).

**Remark 3.5.** As an example, one may take \( G(z) = z^2 \) in (12), which leads to the sample variance as considered in Hosking [12] and Dehling and Taqqu [6]. See also Section 3.1 of Bai and Taqqu [2]. Comparing Theorem 3.4 with Theorem 1.2 centering \{Y(n)\} by subtracting the sample mean may not affect the fluctuation order of the sum, but can change the limit distribution.

The proofs of Theorem 2.1, 2.2, 2.3 and Theorems 3.1, 3.3 and 3.4 can be found in Section 6 and are all based on the analysis of analytic functions.
4 On the coincidence of Hermite and power ranks

The Hermite rank has been defined in (4). We now define the power rank which is used in the approach of Ho and Hsing [11] for limit theorems for transformations of long-memory moving-average processes. Given a function $G(\cdot)$ and a random variable $Y$ satisfying $\mathbb{E}G(Y)^2 < \infty$, let
\[ G_\infty(y) = \mathbb{E}G(Y + y) \] (13)
given that the expectation exists and suppose that $G_\infty(\cdot)$ has derivatives of order sufficiently high. The power rank of $G(\cdot)$ with respect to $Y$ is defined as
\[ \inf\{m \geq 1 : G^{(m)}_\infty(0) \neq 0\}, \] (14)
where $G^{(m)}_\infty(y)$ denotes the $m$-th derivative of $G_\infty(y)$. The following fact was stated in Ho and Hsing [11] without a detailed proof. It was proved by Lévy-Leduc and Taqqu [14] in the case where $G(\cdot)$ is a polynomial.

**Proposition 4.1.** Suppose that $G(\cdot) \in L^2(\gamma)$. Then the power rank in (14) coincides with the Hermite rank if $Y$ is Gaussian.

**Proof.** Proposition 4.1 is a direct consequence of (17) below, namely,
\[ \sum_{m=0}^{\infty} \frac{G^{(m)}_\infty(0)}{m!} x^m = \sum_{m=0}^{\infty} \frac{\mathbb{E}[G(Z)H_m(Z)]}{m!} x^m, \]
whenever the integrability holds.

5 Auxiliary results

We prove here some auxiliary results involving the Weierstrass transform and analytic function theory.

Let $Z$ denote throughout a standard Gaussian random variable and let $\phi = d\gamma/dx = (2\pi)^{-1/2}e^{-x^2/2}$ denote its density. Define the function
\[ G_\infty(x) = \mathbb{E}G(Z + x) = \int_{\mathbb{R}} G(z + x)\phi(z)dz = \int_{\mathbb{R}} G(z)\phi(z - x)dz, \] (15)
whenever the integrability holds. The function $G$ may not be smooth, but the function $G_\infty$ is smooth due to the convolution with the smooth $\phi(z)$. $G_\infty$ is called the Weierstrass transform of $G$ (see Hirschman and Widder [10], Chapter VIII).

First we state some preliminary facts. Recall that a function $f : \mathbb{R}^p \to \mathbb{R}$ is real analytic over an open domain $D \subset \mathbb{R}^p$, if at every point $y = (y_1, \ldots, y_p) \in D$, there exists a neighborhood $B \subset D$ of $y$, such that
\[ f(x_1, \ldots, x_p) = \sum_{i_1, \ldots, i_p=1}^{\infty} a_{i_1,\ldots,i_p} (x_1 - y_1)^{i_1} \cdots (x_p - y_p)^{i_p} \]
for some coefficient $a(i_1, \ldots, i_k)$, where the series converges absolutely in $B$. It is well-known that $f$ is infinitely differentiable, and common elementary operations including composition, affine transform, multiplication preserve analyticity. See, e.g., Krantz and Parks [13], Chapter 1.
Lemma 5.1.

(a) If \( g(\cdot + a), g(\cdot + b) \in L^1(\phi) \), \( a < b \), then

(a1) \( g(\cdot + x) \in L^1(\phi) \) for any \( a < x < b \);

(a2) the Weierstrass transform \( \mathbb{E}g(Z + x) \) is a real analytic function in \( x \in (a,b) \).

(b) If \( g(\cdot + c) \in L^1(\phi) \), then \( g(\cdot + y) \in L^1(\phi) \) for any \( 0 < y < c \).

Proof. (a) The first statement (a1) follows from the fact that \( \phi(z - x) \leq A\phi(z - a) + B\phi(z - b) \) for some sufficiently large constants \( A, B > 0 \) by exploring the shape of \( \phi \). To obtain statement (a2), first as in item 2.2 of Hirschman and Widder [10], Chapter VIII, the Weierstrass transform \( (Wg)(x) := \mathbb{E}g(Z + x) = (g*\phi)(x) \) can be expressed by a bilateral Laplace transform \((\mathcal{L}g)(x) := \int_{\mathbb{R}} e^{-zx} g(z) dz\) as

\[
(Wg)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) e^{-(x-z)^2/2} dz = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(z)^2/2} |f(z)e^{-(z)^2/2}| dz = \frac{e^{-x^2/2}}{\sqrt{2\pi}} (\mathcal{L}g)(-x),
\]

where \( g(z) = f(z)e^{-z^2/2} \). Note that \( e^{-z^2/2} \) is analytic. Then the analyticity of \( Wg \) follows from the analyticity of the bilateral Laplace transform \( \mathcal{L}g \). See Widder [25] Chapter VI, p.140. The statement (b) follows from \( \phi(x/y) \leq C\phi(x/c) \) for some sufficiently large constant \( C > 0 \), because \( \int_{\mathbb{R}} |g(xc)|\phi(x/c) dx = c^{-1}\int_{\mathbb{R}} |g(x)|\phi(x/c) dx < \infty \)

Lemma 5.2. Suppose \( G \in L^2(\phi) \). Then \( \mathbb{E}|G(Z + x)| < \infty \), namely, \( G(x + \cdot) \in L^1(\phi) \), for all \( x \in \mathbb{R} \). Furthermore, \( G_\infty(x) \) in [12] admits the analytic expansion

\[
G_\infty(x) = \sum_{m=0}^{\infty} \frac{G_\infty^{(m)}(0)}{m!} x^m = \sum_{m=0}^{\infty} c_m x^m, \tag{17}
\]

where \( G_\infty^{(m)} \) is the \( m \)-th derivative of \( G_\infty \) and

\[
c_m = \frac{\mathbb{E}[G(Z)H_m(Z)]}{m!}
\]

are the coefficients of the Hermite expansion in [14].

If in addition, \( G(\cdot + M) \in L^2(\phi), M > 0 \), then

\[
G_\infty^{(m)}(x) = \mathbb{E}[G(Z + x)H_m(Z)], \quad |x| < M, \quad m = 0, 1, 2, \ldots \tag{18}
\]

Proof. Let \( H_m(x) \)'s be the Hermite polynomials defined in [13]. Then by the Cauchy-Schwartz inequality and Corollary 5.1.2 of Pipiras and Taqqu [19], one has for any \( x \in \mathbb{R} \) that

\[
\sum_{m=1}^{\infty} \frac{|x|^m}{m!} \int_{\mathbb{R}} |G(z)H_m(z)|\phi(z) dz \leq \sum_{m=1}^{\infty} \frac{|x|^m}{m!} \|G\|_{L^2(\phi)} \|H_m\|_{L^2(\phi)} = \|G\|_{L^2(\phi)} \sum_{m=1}^{\infty} \frac{|x|^m}{\sqrt{m!}} < \infty \tag{19}
\]

since \( \|H_m\|_{L^2(\phi)} = \sqrt{m!} \). By Proposition 1.4.2 of Nourdin and Peccati [17], one has

\[
\sum_{m=0}^{\infty} \frac{x^m}{m!} H_m(z) = \exp(-(z - x)^2/2 + z^2/2).
\]
So
\[
\sum_{m=0}^{\infty} \frac{x^m}{m!} \int_\mathbb{R} G(z)H_m(z)\phi(z)dz = \int_\mathbb{R} G(z)e^{-(z-x)^2/2}dz = \int_\mathbb{R} G(z)\phi(z-x)dz = G_\infty(x),
\]
where the change of the order between the sum and the integral in the first equality can be justified by (19) and Fubini’s Theorem. This shows that \( G_\infty \) has the desired expansion (17). Note that the same computation as above with \( G \) replaced by \(|G|\) shows that \( \mathbb{E}|G(Z+x)| = \int_\mathbb{R} |G(z)|\phi(z-x)dz < \infty \) by (19).

The formula (18) follows from
\[
G^{(m)}_\infty(x) = \frac{d^m}{dx^m} \int_\mathbb{R} G(z)\phi(z-x)dz = \int_\mathbb{R} G(z)\frac{d^m}{dx^m}\phi(z-x)dz = \int_\mathbb{R} G(z)H_m(z-x)\phi(z-x)dx
\]
by (3), where the differentiation under the integral can be justified by the Dominated Convergence Theorem and the Mean Value Theorem as in the proof of Lemma 5.3 below.

\section*{Lemma 5.3 (Uniqueness of Weierstrass transform).} If \( f \in L^1(\phi) \) and the convolution \( f * \phi = 0 \) a.e., then \( f = 0 \) a.e..

\begin{proof}
First, express as in (16) the Weierstrass transform by a bilateral Laplace transform. Then the conclusion follows from the uniqueness of the bilateral Laplace transform. See Widder [25] Chapter VI, Theorem 6b.
\end{proof}

\section*{Lemma 5.4.} Suppose that \( G(\cdot \times M) \in L^2(\phi) \) for some \( M \in (1, +\infty) \). Then \( \mathbb{E}|G(yZ)Z| < \infty \) and
\[
F(y) := \mathbb{E}G(yZ)Z
\]
is an analytic function in \( y \in (0, M) \).

\begin{proof}
Since \( \phi(t/y) \leq c\phi(t/M) \) for some constant \( c > 0 \),
\[
\mathbb{E}|G(yZ)Z| = \int_\mathbb{R} |G(yz)z|\phi(z)dz = \frac{1}{y^2} \int_\mathbb{R} |G(t)\phi(t/y)|dt \leq C \int_\mathbb{R} |G(Mz)z|\phi(z)dz,
\]
and hence by Cauchy-Schwartz,
\[
\mathbb{E}|G(yZ)Z| \leq C\|G(\cdot \times M)\|_{L^2(\phi)} < \infty
\]
for some constant \( C > 0 \) depending only on \( y \). Next applying the change of variable \( u = (yz)^2 \) on \( z \geq 0 \) and on \( z < 0 \), we have
\[
F(y) = \mathbb{E}G(yZ)Z = \int_\mathbb{R} zG(yz)e^{-z^2/2\pi}dz = \frac{1}{2y^2\sqrt{2\pi}} \int_0^\infty [G(u^{1/2}) - G(-u^{1/2})] \exp[-u(2y^2)^{-1}]]du . \quad (20)
\]
The integral in (20) is a Laplace transform evaluated at \((2y^2)^{-1}\). The Laplace transform is real analytic over the open half interval where the integrability holds (Widder [25] Chapter II, Theorem 5a). The function \( y^{-2} \) is also analytic in \( y \in (0, M) \). The conclusion then follows since elementary operations preserve analyticity.
\end{proof}
Lemma 5.5. Let \(D = (-M_1, M_1) \times (0, M_2), \ M_1 > 0, \ M_2 > 1\). Suppose that \(g\) is a measurable function such that \(g(\pm M_1 + \cdot \times M_2) \in L^2(\phi)\). Then the function

\[
f(x, y) := \int_\mathbb{R} g(u) \exp\left[-\frac{(u-x)^2}{2y^2}\right] du
\]

is real analytic over \((x, y) \in D\).

Proof. The assumption implies that \(g(x + \cdot \times y) \in L^2(\phi)\) for all \((x, y) \in D\) by Lemma 5.1. By Theorem 1 of Sičiak [23], one needs to show that

(1) \(f\) is \(C^\infty\) (infinitely smooth);

(2) \(f\) is univariately analytic along any direction at an arbitrary point \((x_0, y_0) \in D\).

For part (1), we need to check that the partial derivatives of all orders of \(f(x, y)\) exist and are continuous.

First, the derivatives with respect to \(x\) and \(y\) can be taken under the integral sign by applying the Dominated Convergence Theorem and the Mean Value Theorem with the help of the following facts which are justified afterwards:

1. For any integers \(i, j \geq 0\),

\[
\frac{\partial^{i+j}}{\partial x^i \partial y^j} \exp\left(-\frac{(u-x)^2}{2y^2}\right) = Q(u, x, y^{-1}) \exp\left(-\frac{(u-x)^2}{2y^2}\right)
\]

for some trivariate polynomial \(Q(x_1, x_2, x_2)\).

2. For any \(x_0 \in \mathbb{R}\) and small \(\delta > 0\), there exists a polynomial \(Q^*\) with non-negative coefficient such that

\[
\sup_{|x-x_0| \leq \delta} |Q(u, x, y^{-1})| \leq Q^*(|u|, |y|^{-1});
\]

A similar bound holds for \(\sup_{|y-y_0| \leq \delta} |Q(u, x, y^{-1})|\).

3. If \(\delta\) is chosen sufficiently small so that \(\exp\left(-\frac{4\delta^2}{y^2}\right) \geq 1/2\), then

\[
\sup_{|x-x_0| \leq \delta} \exp\left(-\frac{(u-x)^2}{2y^2}\right) \leq \exp\left(-\frac{(u-x_0 + \delta)^2}{2y^2}\right) + \exp\left(-\frac{(u-x_0 - \delta)^2}{2y^2}\right),
\]

and for any \(\delta > 0\),

\[
\sup_{|y-y_0| \leq \delta} \exp\left(-\frac{(u-x + \delta)^2}{2y^2}\right) = \exp\left(-\frac{(u-x + \delta)^2}{2(y_0 + \delta)^2}\right).
\]

4. For any integer \(n \geq 0\) and polynomial \(Q\),

\[
\int |g(u)Q(|u|) \exp\left(-\frac{(u-x)^2}{2y^2}\right) du < \infty
\]

for all \((x, y) \in D\).
Item 1 can be verified easily by induction and the elementary differentiation rules.

To see Item 2, write the polynomial as \( Q(u, x, v) = a_0(x) + a_1(x)u + a_2(x)v + \ldots \). So \(|Q(u, x, v)| \leq |a_0(x)| + |a_1(x)u| + |a_2(x)v| + \ldots \). Note that each coefficient \(|a_i(x)|\) is a continuous function. Over a compact interval \([x_0 - \delta, x_0 + \delta]\), each \(|a_i(x)|\) is bounded by a positive constant. Replacing \(|a_i(x)|\)'s by these constants yields the conclusion.

For Item 3, the maximum of the function \( \exp(-\frac{(u-x)^2}{2y^2}) \) in \( x \) over the interval \([x_0 - \delta, x_0 + \delta]\) can be either 1 if \( u \in [x_0 - \delta, x_0 + \delta] \), or \( \exp(-\frac{(u-x_0+\delta)^2}{2y^2}) \) otherwise. In the case where \( u \in [x_0 - \delta, x_0 + \delta] \), by the choice of \( \delta \) described in Item 3 and \(|u - x_0| \leq 2\delta\), we have
\[
\exp(-\frac{(u-x_0+\delta)^2}{2y^2}) + \exp(-\frac{(u-x_0-\delta)^2}{2y^2}) \geq 1/2 + 1/2 \geq 1;
\]
in the case where \( u \notin [x_0 - \delta, x_0 + \delta] \), e.g., if \( u < x_0 - \delta \), then one has \( \exp(-\frac{(u-x_0)^2}{2y^2}) \leq \exp(-\frac{(u-x_0+\delta)^2}{2y^2}) \) by monotonicity. The second inequality follows easily from monotonicity.

To obtain Item 4, by Cauchy-Schwartz and (21), we have
\[
\int |g(u)|Q(|u|)\exp(-\frac{(u-x)^2}{2y^2})du \leq y(2\pi)^{1/2}\|Eg(x+yZ)^2\|^{1/2}\|EQ(|x+yZ|^2)^2\|^{1/2} < \infty
\]

Continuity of \( f(x, y) \) can as well be verified using Dominated Convergence and the facts 2~4 above.

To check claim (2), set \( x(t) = x_0 + at, y = y_0 + bt \) where \((a, b) \in \mathbb{R}^2\) satisfying \( a^2 + b^2 = 1 \). Suppose that \((x(t), y(t)) \in D\) when \(|t| < \delta\), where \( \delta > 0 \) will be adjusted smaller if necessary later. We want to check the real analyticity of
\[
f(t) := f(x(t), y(t)) = \int_{\mathbb{R}} g(u) \exp\left[ -\frac{(u-x_0-at)^2}{2(y_0+bt)^2} \right] du. \tag{22}
\]
at \( t = 0 \). Note that the function
\[
h(u, z) := \exp(A(u, z)) := \exp\left[ -\frac{(u-x_0-a\bar{z})^2}{2(y_0+b\bar{z})^2} \right]
\]
is complex analytic in \( z \in B_\delta := \{z \in \mathbb{C} : |z| < \delta\} \) for each fixed \( u \in \mathbb{R} \) since \( y_0 + bz \neq 0 \). Next, by choosing \( \delta \) small enough, one can ensure that for some constant \( c, \mu > 0 \) sufficiently large and \( \sigma \in (0, M_2) \), such that
\[
\text{Re}[A(u, z)] = -\text{Re}\left[\frac{(u-x_0-a\bar{z})^2(y_0+b\bar{z})^2}{2|y_0+b\bar{z}|^4}\right] = -\frac{y_0^2 u^2 + \text{Re}[]=}{2|y_0+b\bar{z}|^4} = -\frac{u^2 + \text{Re}[]=}{2|y_0|^2 + b\bar{z}/y_0^{1/2}|^4}. \tag{23}
\]
Since \( y_0 \in (0, M_2) \) and \(|z| < \delta\) where \( \delta \) is sufficiently small, there exists \( \sigma \in (0, M_2) \), so that
\[
|y_0^{1/2} + b\bar{z}/y_0^{1/2}|^4 \geq \sigma^2
\]
for all \(|z| < \delta\). On the other hand, the bracket \([\ldots]\) in (23) is of the form
\[
c_1(z, \bar{z}, x_0)u + c_0(z, \bar{z}, x_0),
\]
where \( c_1 \) and \( c_0 \) are polynomials. For all \(|z| < \delta\), one has for some large \( \mu, c > 0 \) that
\[
\text{Re}[]= \leq |c_1(z, \bar{z}, x_0)|u |u| + |c_0(z, \bar{z}, x_0)| \leq 2\mu|u| - \mu^2 + 2\sigma^2c. \tag{25}
\]
Combining (23), (24) and (25), one has
\[ \text{Re}[A(u, z)] \leq -u^2 + 2\mu|u| - \frac{\mu^2}{2\sigma^2} + c \]

Hence for some constant \( C > 0 \),
\[ |h(u, z)| = \exp(\text{Re}[A(u, z)]) \leq C[\exp \left( -(u^2 + \mu^2)/(2\sigma^2) \right) + \exp \left( -(u^2 - \mu^2)/(2\sigma^2) \right)], \quad z \in B_\delta. \quad (26) \]

In \( \int_R |g(u)h(u, z)|du \), replace \( |h(u, z)| \) by the preceding bound (26), and note that \( g(\pm \mu + \cdot \times \sigma) \in L^1(\phi) \) for any \( \mu \in \mathbb{R} \) and \( \sigma \in (0, M) \) by Lemma 5.1 (b) and Lemma 5.2. Then one deduces that
\[ \sup_{z \in B_\delta} \int_R |g(u)h(u, z)|du < \infty. \]

This fact with Fubini’s Theorem justifies the following order change of integrals:
\[ \oint_\Delta f(z)dz = \oint_\Delta dz \int_R g(u)h(u, z)du = \int_R g(u)du \oint_\Delta h(u, z)dz = 0, \]
where \( \Delta \) is a closed triangle within \( B_\delta \), and the last equality is due to Cauchy’s theorem since \( h(u, z) \) is complex analytic in \( z \in B_\delta \). Then the complex analyticity of \( f(z) \) over \( B_\delta \) follows from Morera’s Theorem (Theorem 10.17 of Rudin [21]). Restricting \( B_\delta \) to \( \text{Im}(z) = 0 \) yields the real analyticity.

6 Proof of the theorems

We prove here Theorems 2.1, 2.2, 2.5 involving the instability of the Hermite rank and and Theorems 3.1, 3.3 and 3.4 involving ”near higher order rank”.

Proof of Theorem 2.1. Suppose that the function \( G \) is not a.e. constant. Define \( E = \{ x \in (-M, M) : G(\cdot + x) \text{ has Hermite rank } \geq 2 \} \). Note first that we have \( E = \{ x \in (-M, M) : G^{(1)}(x) = 0 \} \) in view of Lemma 5.2 applied to \( G(\cdot + x) \). Indeed, since the Hermite rank of \( G(\cdot + x) \) is greater than 1, we have \( G^{(1)}(x) = \mathbb{E}G(Z + x)H_1(Z) = 0 \). Now suppose by contradiction that \( E \) has an accumulation point in \( (-M, M) \). Since \( G_\infty \) is analytic by Lemma 5.2 so is the derivative \( G^{(1)}_\infty(x) \). By Rudin [20] Theorem 8.5, \( G^{(1)}_\infty \) is identically zero on \( (-M, M) \) and hence \( G_\infty \) is a constant on \( (-M, M) \). But by \( \text{[30]} \), \( G_\infty = G \ast \phi \). By Lemma 5.3 and linearity, \( G \) is a.e. a constant as well which contradicts the theorem assumption.

Proof of Theorem 2.2. Suppose that the function \( G \) is not a.e. symmetric. Now the set with Hermite rank \( \geq 2 \) is \( E = \{ y \in (0, M) : F(y) := \mathbb{E}G(yZ)Z = 0 \} \). The proof is similar to that of Theorem 2.1 above. Suppose by contradiction that \( E \) has an accumulation point in \( (0, M) \). Since \( F \) is analytic on \( (0, M) \) by Lemma 5.4 so \( F(y) \) is identically zero on \( (0, M) \). Then we apply the uniqueness of the Laplace transform (Widder [22] Chapter II, Theorem 6.3) to relation (20) to conclude that \( G(z) = G(-z) \) a.e., which contradicts the assumption.
Proof of Theorem 2.5. Define
\[ F(x, y) = E[ZG(x + yZ)]. \]
As before \( E = \{(x, y) \in D : F(x, y) = 0\} \). By a change of variable \( z = (u - x)/y \) we can write
\[
F(x, y) = \frac{1}{y^2 \sqrt{2\pi}} \int_{\mathbb{R}} (u - x) G(u) \exp \left[ -\frac{(u-x)^2}{2y^2} \right] du.
\]
(27)
Then \( F(x, y) \) is a bivariate real analytic function by Lemma 5.5 since \((\pm M_1 + M_2 u)G(\pm M_1 + M_2 u) \in L^2(\phi)\) by Cauchy-Schwartz. If \( F(x, y) \equiv 0 \), which implies \( F(x, 1) \equiv 0 \), then the Hermite rank of \( G(\cdot + x) \) is greater than 1 for all \( x \in (-M_1, M_1) \), which contradicts Theorem 2.1. So \( F(x, y) \) is not identically zero. The claimed properties of \( E \), the zero set of the analytic \( F \), then follow from Mityagin 10 (see also Krantz and Parks 13, Section 4.1).

\[ \square \]

Proof of Theorem 2.6. If \( G \) is not a constant a.e., by assumption (b), the analytic \( U(\theta) \) is not identically zero on \( D \). So taking into account assumption (a), the zero set \( E \) has the claimed properties by Mityagin 16.

\[ \square \]

Proof of Theorem 3.1. First by the Hermite expansion (5) we can write:
\[ G(Z + x_N) = \sum_{m=0}^{\infty} c_m(x_N) H_m(Z), \]
(28)
We set as in (15) that \( G_\infty(x_N) = \mathbb{E}G(Z + x_N) \), where \( Z \sim N(0, 1) \). In view of Lemma 5.2 we have
\[ c_m(x_N) = \frac{1}{m!} \mathbb{E}G(Z + x_N) H_m(Z) = \frac{1}{m!} G_\infty^{(m)}(x_N). \]
and
\[ F_\infty(x_N) := \mathbb{E}G(Z + x_N)^2 = \sum_{m=0}^{\infty} c_m(x_N)^2 \mathbb{E}H_m(Z)^2 = \sum_{m=0}^{\infty} c_m(x_N)^2 m!. \]
(29)
Since \( G \) has Hermite rank \( k \), in view of Lemma 5.2 we have \( G_\infty^{(1)}(0) = \ldots = G_\infty^{k-1}(0) = 0 \neq G_\infty^{k}(0) \). By the analytic expansion,
\[ G_\infty(x_N) = G_\infty^{(k)}(0) \frac{x_N^k}{k!} + O(x_N^{k+1}) \]
and
\[ G_\infty^{(m)}(x_N) = \frac{d^m G}{dx^m}(x_N) = \frac{1}{(k-m)!} g_k x_N^{k-m} + O(x_N^{k-m+1}), \quad m = 1, \ldots, k, \]
where
\[ g_k = G_\infty^{(k)}(0). \]
Then
\[
S_{Nt}(x_N) = \sum_{n=1}^{[Nt]} \left[ G(Y(n) + x_N) - \mathbb{E}G(Y(n) + x_N) \right] = A_N(t) + B_N(t) + C_N(t) + D_N(t) + H.O.T. = A_N(t) + B_N(t) + R_N(t) + H.O.T.
\]
(30)
We now focus on $B$ namely, indeed, when $m > k$

In view of (33), when $m > k$

Consider first the central limit case where $H_o(Y) = Y$; the term $B_N(t)$ involves the second up to the $(k-1)^{th}$ Hermite polynomial; the term $C_N(t)$ involves the $k^{th}$ Hermite polynomial; the term $D_N(t)$ involves all the Hermite polynomials of orders higher than $k$. $R_N(t)$ involves the terms $m \geq k$ in (28).

We assume for simplicity that $H \neq 1/(2m)$, $m = 1, \ldots, k$. Otherwise the proof undergoes a slight modification involving an logarithmic factor. We have to consider the behavior of $\sum_{n=1}^{[Nt]} H_m(Y(n))$ for all $m \geq 1$. Let

$$k_o = \sup \{ m \in \mathbb{Z}_+ : H > 1 - 1/(2m) \}.$$ (33)

In view of (33), when $m > k_o$ the central limit theorem in Theorem 1 holds and when $m < k_o$ it is the non-central limit theorem that holds. Alternatively, the central limit theorem holds when $H < 1 - 1/(2k)$ and the non-central limit theorem holds when $H > 1 - 1/(2k)$.

- Consider first the central limit case where $H < 1 - 1/(2k)$.

In this case, $A_N(t)$ is associated with the order $x_N^{k-1} N^H$ and $R_N(t)$ is associated with the order $N^{1/2}$. We now focus on $B_N(t)$ and claim that it is asymptotically negligible compared with either $A_N(t)$ or $R_N(t)$, namely,

$$x_N^{k-m} N^{H[m]} \ll \max(x_N^{k-1} N^H, N^{1/2}), \quad m = 2, \ldots, k-1.$$ (34)

Indeed, when $m > k_o$, this is obvious since $H[m] = 1/2$; Suppose now $m \leq k_o$. In this case, $H[m] = (H - 1)m + 1$. However, in cases (a) and (b), where $x_N \ll$ or $\approx N^{(1/2-H)/(k-1)}$, the factor $x_N^{k-m} N^{H[m]}$ has an exponent bounded by

$$\frac{(1/2 - H)(k - m)}{k - 1} + (H - 1)m + 1 = \frac{(m - 1)k}{k - 1} H + \frac{k + m - 2km}{2(k - 1)} + 1 < \frac{1}{2},$$ (35)

where the last inequality can be obtained by plugging in the inequality $H < 1 - 1/(2k)$ and some elementary computation. Hence in cases (a) and (b), the relation (34) is proved. In the case (c) where $x_N \gg$
\[ N^{(\frac{1}{2} - H)/(k-1)}, \text{it can be checked that } x_N^{k-m} N^{(H-1)m+1} \ll x_N^{k-1} N^H \text{ using the inequality } \frac{1/2-H}{k-1} > H - 1, \]
and the latter inequality follows from \( H < 1 - 1/(2k) \). We have thus proved (34) and hence \( B_N(t) \) is asymptotically negligible. We are left with \( A_N(t) \) and \( R_N(t) \).

The different asymptotic regimes in the central limit case in Theorem 3 come about when comparing the order \( x_N^{k-1} N^H \) of \( A_N(t) \) with the order \( N^{1/2} \) of \( R_N(t) = C_N(t) + D_N(t) \). But we also have to deal with the convergence of the processes. By Bai and Taqqu [1], we have the joint convergence
\[
\left( \frac{1}{N^H} \sum_{n=1}^{[Nt]} Y(n), \frac{1}{N^{1/2}} \sum_{n=1}^{[Nt]} \sum_{m=k}^{\infty} \frac{g_m}{m!} H_m(Y(n)) \right) \xrightarrow{f.d.d.} (c_1 Z_{H,1}(t), c_2 B(t))
\]
where the fractional Brownian motion \( Z_{H,1}(t) \) and the Brownian motion \( B(t) \) are independent. From (36) we set
\[ R'_N(t) = \sum_{n=1}^{[Nt]} \sum_{m=k}^{\infty} \frac{g_m}{m!} H_m(Y(n)), \]
which is \( R_N(t) \) with \( x_N = 0 \).

We cannot use (36) directly because we have \( R_N(t) \) instead of \( R'_N(t) \). We thus need to compare them first. Since \( Y(n) \) is standardized, we have \( |\gamma_Y(n)|^m \leq |\gamma_Y(n)|^k \) if \( m \geq k \), where \( \gamma_Y(n) = \text{Cov}[Y(n), Y(0)] \). Thus by a computation similar to those on p.299 of Pipiras and Taqqu [19], using the orthogonality of the Hermite polynomials,
\[
\mathbb{E}[R_N(t) - R'_N(t)]^2 = \sum_{m>K} m! (c_m(x_N) - c_m(0))^2 \sum_{|n|<[Nt]} (|[Nt] - |n|) \gamma_Y(n)^m
\]
\[
\leq (Nt) \left( \sum_{n=-\infty}^{\infty} |\gamma_Y(n)|^k \right) \sum_{m=k}^{\infty} m! (c_m(x_N) - c_m(0))^2. \tag{37}
\]
Note that for a \( K > k \),
\[
\sum_{m=K}^{\infty} m! (c_m(x_N) - c_m(0))^2 \leq \sum_{m=k}^{K} m! (c_m(x_N) - c_m(0))^2 + 2 \sum_{m>K} m! (c_m(x_N)^2 + c_m(0)^2). \tag{38}
\]
Recall \( F_\infty(x) = \sum_{m=0}^{\infty} m! c_m(x)^2 \) in (29), which is analytic in view of Lemma 5.1. Hence
\[
\sum_{m>K} m! c_m(x_N)^2 = F_\infty(x_N) - F_\infty(0) + F_\infty(0) - \sum_{m=0}^{K} m! c_m(x_N)^2.
\]
As \( N \to \infty \), by the continuity of \( F_\infty \) and \( c_m(x) = C^{(m)}_\infty(x)/m! \), we have
\[
\sum_{m>K} m! c_m(x_N)^2 \to \sum_{m>K} m! c_m(0)^2. \tag{39}
\]
Take the limit \( N \to \infty \) in (38) using (39) and the continuity of \( c_m \), the right-hand side of (38) is only left with the term \( 2 \sum_{m>K} m! c_m(0)^2 \), which tends to 0 as \( K \to \infty \). Combining this with (37), we have
\[
\lim_{N \to \infty} N^{-1} \mathbb{E}[R_N(t) - R'_N(t)]^2 = 0. \tag{40}
\]
It thus follows from (36) that

\[
\left( \frac{1}{N^{H}} \sum_{n=1}^{[Nt]} Y(n), \frac{1}{N^{1/2}} \sum_{n=1}^{[Nt]} R_N(t) \right) \overset{f.d.d.}{\longrightarrow} (c_1 Z_{H,1}(t), c_2 B(t)).
\]

In case (a), we have \(x_N^{k-1} N^H \ll N^{1/2}\) and thus as \(N \to \infty\), \(A_N(t)\) is negligible compared to \(R_N(t)\) and thus we get \(N^{-1/2} R_N(t) \overset{f.d.d.}{\longrightarrow} c_2 B(t)\). In case (c), \(x_N^{k-1} N^H \gg N^{1/2}\) and the limit is \(c_1 Z_{H,1}(t)\). In case (b), where \(x_N^{k-1} N^H \approx N^{1/2}\), both limits appear. Finally, note that \(N^{-H} x_N^{1-k} A_N(t)\) is proportional to the first component on the left-hand side of (36), while by (40), the term \(N^{-1/2} R_N(t)\) can be asymptotically replaced by \(N^{-1/2} R'_N(t)\), which is the second component on the left-hand side of (36). The rest of the proof in the central limit case can be carried out easily.

\[\bullet\] Now we consider the non-central limit case where \(H > 1 - \frac{1}{2k}\). We have to study, as before, the behavior of \(\sum_{n=1}^{[Nt]} H_m(Y(n))\) for all \(m \geq 1\).

We first consider the terms \(A_N(t), B_N(t)\) and \(C_N(t)\) which involve \(m \leq k\). In this case, we have the following relation: any \(n, m \in \mathbb{Z}_+\),

\[x_N^{k-m} N^{(H-1)m+1} \ll x_N^{k-n} N^{(H-1)n+1} \iff x_N \ll N^{H-1},\]

which holds with \(\ll\) replaced by \(\approx\) or \(\gg\) as well.

Hence in case (a), the term \(A_N(t)\) in (39) contributes; in case (b), the terms \(A_N(t), B_N(t)\) and \(C_N(t)\) all contribute; in case (c), the term \(C_N(t)\) contributes.

We shall show below that the term \(D_N(t)\), which involves \(m > k\), is negligible. Set \(\gamma_Y(n) = \text{Cov}(Y(n), Y(0)) \approx n^{2H-2}\). By a similar computation as in (37),

\[\mathbb{E}D_N(t)^2 = \sum_{m>k} m! c_m(x_N)^2 \sum_{\lceil n \rceil < \lceil Nt \rceil} ([Nt] - |n|) \gamma_Y(n)^m.\]  \hspace{1cm} (41)

For an arbitrarily small \(\epsilon > 0\), we have for \(m > k\) that

\[\left| \sum_{\lceil n \rceil < \lceil Nt \rceil} ([Nt] - |n|) \gamma_Y(n)^m \right| \leq c_1(Nt) \sum_{\lceil n \rceil < \lceil Nt \rceil} n^{(2H-2)(k+1)} \leq c_2(Nt)^{2H[k+1]},\]  \hspace{1cm} (42)

for some constants \(c_1, c_2 > 0\). where \(H[m]\) is as in (32). Since \(H[k+1] < H[k] \) when \(H > 1 - 1/(2k)\), in view of (41), (42) and (29),

\[\mathbb{E}D_N(t)^2 \leq c_2(Nt)^{2H[k+1]} F_\infty(x_N) \ll N^{2H[k]} \approx \mathbb{E}C_N(t)^2.\]

So the order of \(D_N(t)\) is always dominated by that of \(C_N(t)\). So we only need to focus on \(A_N(t), B_N(t)\) and \(C_N(t)\). Then the rest of the proof can be carried out using the following consequence of Bai and Taqqu [1]:

\[\left( \frac{1}{N^{H-1} m+1} \sum_{n=1}^{[Nt]} H_m(Y(n)), m = 1, \ldots, k \right) \overset{f.d.d.}{\longrightarrow} \left( c_m Z_{H,m}(t), m = 1, \ldots, k \right)\]  \hspace{1cm} (43)

where the Hermite processes \(Z_{H,m}(t)\)'s are defined through the same Brownian motion in (10).
Proof of Theorem 3.3. The proof is done as the proof of Theorem 3.1 based on (36) and (33), and we thus only provide an outline. By Taylor expansion of $G_\infty$ in (17), and since $\mathbb{E}G(Y(n)) = g_0 = G_\infty(0) = 0$ and the Hermite rank is $k$, we have

$$
\mathbb{E}G(Y(n) + x_N) = G_\infty(x_N) = \frac{g_k}{k!} x_N^k + O(x_N^{k+1}).
$$

Adding the leading term above to (30), we have

$$
\tilde{S}_{[N]}(x_N) = Z_N(t) + A_N(t) + B_N(t) + C_N(t) + D_N(t) + H.O.T. = Z_N(t) + A_N(t) + B_N(t) + R_N(t) + H.O.T.
$$

(44)

where

$$
Z_N(t) = \frac{g_k}{k!} (x_N[Nt]) + H.O.T.
$$

In the central limit case where $H < 1 - 1/(2k)$, the term $B_N(t)$ is negligible compared to $A_N(t)$ and $R_N(t)$ as in the proof of Theorem 3.1 above. We thus compare the orders $x_N^k N$ of $Z_N(t)$, $x_N^{k-1} N^H$ of $A_N(t)$ and $N^{1/2}$ of $R_N(t)$. Indeed, if $x_N < N^{-1/(2k)}$, then $x_N^k N < N^{1/2}$ and $x_N^{k-1} N^H < N^{1/2}$, so only $R_N(t)$ contributes; if $x_N \approx N^{-1/(2k)}$, then $x_N^k N \approx N^{1/2}$ and $x_N^{k-1} N^H \ll N^{1/2}$, so $Z_N(t)$ and $R_N(t)$ contribute; if $x_N \gg N^{-1/(2k)}$, then $x_N^k N \gg N^{1/2}$ and $x_N^{k-1} N^H$, so only $Z_N(t)$ contributes.

A similar analysis can be carried out in the non-central limit case. As in the proof of Theorem 3.1 the term $B_N(t)$ only contributes in the case $x_N \approx N^{H-1}$. The asymptotic regimes in the other cases are determined by comparing the orders $x_N^k N$ of $Z_N(t)$, $x_N^{k-1} N^H$ of $A_N(t)$ and $N^{(H-1)k+1}$ of $C_N(t)$.

Proof of Theorem 3.4. The proof is again similar to those of Theorem 3.1 and 3.3 and we thus only provide an outline. Since $G(\cdot)$ is a polynomial, in (41) the $\infty$ in the sum defining $D_N(t)$ is replaced by a finite integer. This is important since the arguments leading to (40) can no longer be applied.

The key is to note that $-\tilde{\gamma}_N$ is associated with the order $N^{H-1}$, which replaces $x_N$ in all the terms in (44). More precisely, in the central limit case where $H < 1 - 1/(2k)$, comparing the order $N^{k(H-1)}$ of $Z_N(t)$, the order $N^{(k-m)(H-1)+H[m]}$ of $A_N(t)$ and $B_N(t)$, $0 < m < k$, and the order $N^{1/2}$ of $R_N(t)$, where $H[m]$ is as in (32), one can find that $R_N(t)$ is the contributing term. The conclusion then follows from (36) and the Slutsky Lemma using the continuity of the coefficients $c_m(\cdot)$. In the non-central limit case where $H > 1 - 1/(2k)$, we compare the order $N^{k(H-1)}$ of $Z_N(t)$, the order $N^{k(H-1)+1}$ of $A_N(t), B_N(t), C_N(t)$, $0 < m \leq k$, and the order $N^{H[k+1]}$ of $D_N(t)$. In this case, the terms $A_N(t)$ and $B_N(t)$ contribute.

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