A differential bialgebra associated to a set theoretical solution of the Yang-Baxter equation

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Abstract

For a set theoretical solution of the Yang-Baxter equation \((X, \sigma)\), we define a d.g. bialgebra \(B = B(X, \sigma)\), containing the semigroup algebra \(A = k\{X\}/\langle xy = zt : \sigma(x, y) = (z, t) \rangle\), such that \(k \otimes_A B \otimes_A k\) and \(\text{Hom}_{A \otimes A}(B, k)\) are respectively the homology and cohomology complexes computing biquandle homology and cohomology defined in [CEGN, CJKS] and other generalizations of cohomology of rack-quandle case (for example defined in [CES2]). This algebraic structure allows us to show the existence of an associative product in the cohomology of biquandles, and a comparison map with Hochschild (co)homology of the algebra \(A\).

1 Introduction

A quandle is a set \(X\) together with a binary operation \(\ast : X \times X \to X\) satisfying certain conditions (see definition on example 1 below), it generalizes the operation of conjugation on a group, but also is an algebraic structure that behaves well with respect to Reidemeister moves, so it is very useful for defining knot/links invariants. Knot theorists have defined a cohomology theory for quandles (see [CJKS] and [CES1]) in such a way that 2-cocycles give rise to knot invariants by means of the so-called state-sum procedure. Biquandles are generalizations of quandles in the sense that quandles give rise to solutions of the Yang-Baxter equation by setting \(\sigma(x, y) := (y, x \ast y)\). For biquandles there is also a cohomology theory and state-sum procedure for producing knot/links invariants (see [CES2]).

In this work, for a set theoretical solution of the Yang-Baxter equation \((X, \sigma)\), we define a d.g. algebra \(B = B(X, \sigma)\), containing the semigroup algebra \(A = k\{X\}/\langle xy = zt : \sigma(x, y) = (z, t) \rangle\), such that \(k \otimes_A B \otimes_A k\) and \(\text{Hom}_{A \otimes A}(B, k)\) are respectively the standard homology and cohomology complexes attached to general set theoretical solutions of the Yang-Baxter equation. We prove that this d.g. algebra has a natural structure of d.g. bialgebra (Theorem 2). Also, depending on properties of the solution \((X, \sigma)\) (square free, …
quandle type, biquandle, involutive,...) this d.g. bialgebra $B$ has natural (d.g. bialgebra) quotients, giving rise to the standard sub-complexes computing quandle cohomology (as sub-complex of rack homology), biquandle cohomology, etc.

As a first consequence of our construction, we give a very simple and purely algebraic proof of the existence of a cup product in cohomology. This was known for rack cohomology (see [Cl]), the proof was based on topological methods, but it was unknown for biquandles or general solutions of the Yang-Baxter equation. A second consequence is the existence of a comparison map between Yang-Baxter (co)homology and Hochschild (co)homology of the semigroup algebra $A$. Looking carefully this comparison map we prove that it factors through a complex of ”size” $A \otimes B \otimes A$, where $B$ is the Nichols algebra associated to the solution $(X, -\sigma)$. This result leads to new questions, for instance when $(X, \sigma)$ is involutive (that is $\sigma^2 = \text{Id}$) and the characteristic is zero we show that this complex is acyclic (Proposition 24), we wander if this is true in any other characteristic, and for non necessarily involutive solutions.

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1.1 Basic definitions

A set theoretical solution of the Yang-Baxter equation (YBeq) is a pair $(X, \sigma)$ where $\sigma : X \times X \to X \times X$ is a bijection satisfying

$$(\text{Id} \times \sigma)(\sigma \times \text{Id})(\text{Id} \times \sigma) = (\sigma \times \text{Id})(\text{Id} \times \sigma)(\sigma \times \text{Id}) : X \times X \times X \to X \times X \times X$$

If $X = V$ is a $k$-vector space and $\sigma$ is a linear bijective map satisfying YBeq then it is called a braiding on $V$.

Example 1. A set $X$ with a binary operation $\triangleright X \times X \to X \times X$ is called a rack if

- $- \triangleright x : X \to X$ is a bijection $\forall x \in X$ and
- $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z) \forall x, y, z \in X$.

$x \triangleright y$ is usually denoted by $x^y$.

If $X$ also verifies that $x \triangleright x = x$ then $X$ is called a quandle.

An important example of rack is $X = G$ a group, $x \triangleright y = y^{-1}xy$.

If $(X, \triangleright)$ is a rack, then

$$\sigma(x, y) = (y, x \triangleright y)$$

is a set theoretical solution of the YBeq.

Let $M = M_X$ be the monoid freely generated in $X$ with relations

$$xy = zt$$

$\forall x, y, z, t$ such that $\sigma(x, y) = (z, t)$. Denote $G_X$ the group with the same generators and relations. For example, when $\sigma = \text{flip}$ then $M = N_0^{(X)}$ and $G_X = Z_0^{(X)}$. If $\sigma = \text{Id}$ then $M$ is the free (non abelian) monoid in $X$. If $\sigma$ comes from a rack $(X, \triangleright)$ then $M$ is the monoid with relation $xy = y(x \triangleright y)$ and $G_X$ is the group with relations $x \triangleright y = y^{-1}xy$. 

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2 A d.g. bialgebra associated to \((X, \sigma)\)

Let \(k\) be a commutative ring with 1. Fix \(X\) a set, and \(\sigma : X \times X \to X \times X\) a solution of the YBeq. Denote \(A_\sigma(X)\), or simply \(A\) if \(X\) and \(\sigma\) are understood, the quotient of the free \(k\) algebra on generators \(X\) modulo the ideal generated by elements of the form \(xy - zt\) whenever \(\sigma(x, y) = (z, t)\):

\[
A := k \langle X \rangle / \langle xy - zt : x, y \in X, (z, t) = \sigma(x, y) \rangle = k[M]
\]

It can be easily seen that \(A\) is a \(k\)-bialgebra declaring \(x\) to be grouplike for any \(x \in X\), since \(A\) agrees with the semigroup-algebra on \(M\) (the monoid freely generated by \(X\) with relations \(xy \sim zt\)). If one considers \(G_X\), the group freely generated by \(X\) with relations \(xy = zt\), then \(k[G_X]\) is the (non commutative) localization of \(A\), where one has inverted the elements of \(X\). An example of \(A\)-bimodule that will be used later, which is actually a \(k[G_X]\)-module, is \(k\) with \(A\)-action determined on generators by

\[
x\lambda y = \lambda, \forall x, y \in X, \lambda \in k
\]

We define \(B(X, \sigma)\) (also denoted by \(B\)) the algebra freely generated by three copies of \(X\), denoted \(x, e_x\) and \(x'\), with relations as follows: whenever \(\sigma(x, y) = (z, t)\) we have

- \(xy \sim zt\), \(xy' \sim z't\), \(x'y' \sim z't'\)
- \(xe_y \sim e_z t\), \(e_x y' \sim z'e_t\)

Since the relations are homogeneous, \(B\) is a graded algebra declaring

\[
|x| = |x'| = 0, \quad |e_x| = 1
\]

**Theorem 2.** The algebra \(B\) admits the structure of a differential graded bialgebra, with \(d\) the unique superderivation satisfying

\[
d(x) = d(x') = 0, \quad d(e_x) = x - x'
\]

and comultiplication determined by

\[
\Delta(x) = x \otimes x, \quad \Delta(x') = x' \otimes x', \quad \Delta(e_x) = x' \otimes e_x + e_x \otimes x
\]

By differential graded bialgebra we mean that the differential is both a derivation with respect to multiplication, and coderivation with respect to comultiplication.

**Proof.** In order to see that \(d\) is well-defined as super derivation, one must check that the relations are compatible with \(d\). The first relations are easier since

\[
d(xy - zt) = d(x)y + xd(y) - d(z)t - zd(t) = 0 + 0 - 0 - 0 = 0
\]

and similar for the others (this implies that \(d\) is \(A\)-linear and \(A'\)-linear). For the rest of the relations:

\[
d(xe_y - e_z t) = xe(e_y) - d(e_z)t = x(y - y') - (z - z')t
\]
If one wants to write it in a normal form (say, every \(x\) and the \(e\)’s in the middle), then one should use the relations in \(B\): this might be a very complicated formula, depending on the braiding. We give examples in some particular cases. Let’s denote \(\sigma(x, y) = (\sigma^1(x, y), \sigma^2(x, y))\).

\[
= xy - zt - (xy' - z't) = 0
\]

\[
d(e_x y' - e_z t) = (x - x')y' - z'(t - t') = xy' - z't - (x'y' - z't') = 0
\]

Remark 3. \(\Delta\) is coassociative.

For a particular element of the form \(b = e_{x_1} \ldots e_{x_n}\), the formula for \(d(b)\) can be computed as follows:

\[
d(e_{x_1} \ldots e_{x_n}) = \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \ldots e_{x_{i-1}} d(e_{x_i}) e_{x_{i+1}} \ldots e_{x_n}
\]

\[
= \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \ldots e_{x_{i-1}} (x_i - x'_i) e_{x_{i+1}} \ldots e_{x_n}
\]

\[
= \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \ldots e_{x_{i-1}} x_i e_{x_{i+1}} \ldots e_{x_n} - \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \ldots e_{x_{i-1}} x'_i e_{x_{i+1}} \ldots e_{x_n}
\]
Example 4. In low degrees we have

- \( d(e_x) = x - x' \)
- \( d(e_x e_y) = (e_z t - e_x y) - (x' e_y - z' e_t) \), where as usual \( \sigma(x, y) = (z, t) \).
- \( d(e_{x_1} e_{x_2} e_{x_3}) = A_I - A_{II} \) where
  \[ A_I = e_{\sigma'(x_1,x_2)} e_{\sigma'(\sigma^2(x_1,x_2),x_3)} \sigma^2(x_2,x_3) - e_{x_1} e_{\sigma'(x_2,x_3)} \sigma^2(x_2,x_3) + e_{x_1} e_{x_2} x_3 \]
  \[ A_{II} = x'_1 e_{x_2} e_{x_3} - \sigma'(x_1,x_2) e_{\sigma^2(x_2,x_3)} e_{x_3} + \sigma'(x_1, \sigma^2(x_2,x_3)) e_{\sigma^2(x_2,x_3)} e_{\sigma^2(x_2,x_3)} \]

In particular, if \( f : B \rightarrow k \) is an \( A-A' \) linear map, then

\[
\begin{align*}
  f(d(e_{x_1} e_{x_2} e_{x_3})) &= f(e_{\sigma'(x_1,x_2)} e_{\sigma'(\sigma^2(x_1,x_2),x_3)}) - f(e_{x_1} e_{\sigma'(x_2,x_3)}) + f(e_{x_1} e_{x_2}) \\
  &- f(e_{x_2} e_{x_3}) + f(e_{\sigma^2(x_2,x_3)} e_{x_3}) - f(e_{\sigma^2(x_1, \sigma^2(x_2,x_3))} e_{\sigma^2(x_2,x_3)})
\end{align*}
\]

Erasing the \( e \)'s we notice the relation with the cohomological complex given in [CES2], see Theorem 5 below.

If \( X \) is a rack and \( \sigma \) the braiding defined by \( \sigma(x, y) = (y, x \triangleleft y) = (x, x^y) \), then:

- \( d(e_x) = x - x' \)
- \( d(e_x e_y) = (e_y x^y - e_x y) - (x' e_y - y' e_x) \)
- \( d(e_x e_y e_z) = e_x e_y z - e_x e_z y^z + e_y e_z x^{y^z} - x' e_y e_z + y' e_x e_z - z' e_x e_y^z \).

- In general, expressions I and II are

\[
\begin{align*}
  I &= \sum_{i=1}^{n} (-1)^{i+1} x_{i_1} \ldots x_{i_{i-1}} e_{x_{i+1}} \ldots e_{x_n} x^{i+1 \ldots n} \\
  II &= \sum_{i=1}^{n} (-1)^{i+1} x'_1 x_{i_1} x_{i_2} \ldots e_{x_{i_{i-1}}} e_{x_{i+1}} \ldots e_{x_n}
\end{align*}
\]

then

\[
\partial f(x_1, \ldots, x_n) = f(d(e_{x_1} \ldots e_{x_n})) = \sum_{i=1}^{n} (-1)^{i+1} \left( f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) x^{i+1 \ldots n} - x'_1 f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \right)
\]

Let us consider \( k \otimes_{k[M]} B \otimes_{k[M]} k \) then \( d \) represents the canonical differential of rack homology and \( \partial f(e_{x_1} \ldots e_{x_n}) = f(d(e_{x_1} \ldots e_{x_n})) \) gives the traditional rack cohomology structure.

In particular, taking trivial coefficients:

\[
\partial f(x_1, \ldots, x_n) = f(d(e_{x_1} \ldots e_{x_n})) = \sum_{i=1}^{n} (-1)^{i+1} \left( f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - f(x_1 x_{i+1}, \ldots, x_{i_{i-1}}, x_{i+1}, \ldots, x_n) \right)
\]

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Theorem 5. Taking in \( k \) the trivial \( \mathcal{A}'-\mathcal{A} \)-bimodule, the complexes associated to set theoretical Yang-Baxter solutions defined in [CES2] can be recovered as

\[
(C_\bullet(X, \sigma), \partial) \simeq (k \otimes_{\mathcal{A}'} B_\bullet \otimes_A k, \partial = id_k \otimes_{\mathcal{A}'} d \otimes_A id_k)
\]

\[
(C^\bullet(X, \sigma), \partial^\bullet) \simeq (\text{Hom}_{\mathcal{A}'-\mathcal{A}}(B, k), \partial^\bullet = d^\bullet)
\]

In the proof of the theorem we will assume first Proposition [12] that says that one has a left \( A' \)-linear and right \( A \)-linear isomorphism:

\[
B \cong A' \otimes TE \otimes A
\]

where \( A' = TX'/\langle x'y' = z't' : \sigma(x, y) = (z, t) \rangle \) and \( A = TX/\langle xy = zt : \sigma(x, y) = (z, t) \rangle \).

We will prove Proposition [12] later.

Proof. In this setting every expression in \( x, x', e_x \), using the relations defining \( B \), can be written as \( x_{i_1} \cdots x_{i_n} e_{x_1} \cdots e_{x_k} x_j \cdots x_{j_l} \), tensorizing leaves the expression

\[
1 \otimes e_{x_1} \cdots e_{x_k} \otimes 1
\]

This shows that \( T = k \otimes_{k[M]} B \otimes_{k[M]} k \simeq T\{e_x\}_{x \in X} \), where \( \simeq \) means isomorphism of \( k \)-modules. This also induces isomorphisms of complexes

\[
(C_\bullet(X, \sigma), \partial) \simeq (k \otimes_{\mathcal{A}'} B_\bullet \otimes_A k, \partial = id_k \otimes_{\mathcal{A}'} d \otimes_A id_k)
\]

\[
(C^\bullet(X, \sigma), \partial^\bullet) \simeq (\text{Hom}_{\mathcal{A}'-\mathcal{A}}(B, k), d^\bullet)
\]

\( \square \)

Now we will prove Proposition [12]. Call \( Y = \langle x, x', e_x \rangle_{x \in X} \) the free monoid in \( X \) with unit 1, \( k \langle Y \rangle \) the \( k \)-algebra associated to \( Y \). Let \( S = \{r_1, r_2, r_3\} \) be the reduction system defined as follows: \( r_i : k \langle Y \rangle \to k \langle Y \rangle \) the families of \( k \)-module endomorphisms such that \( r_i \) fix all elements except

\[
r_1(xy) = z't, \quad r_2(xe_y) = e_x t \quad \text{and} \quad r_3(e_x y') = z'e_t.
\]

Note that \( S \) has more than 3 elements, each \( r_i \) is a family of reductions.

Definition 6. A reduction \( r_i \) acts trivially on an element \( a \) if \( w_i \) does not appear in \( a \), ie: \( Aw_iB \) appears with coefficient 0.

Following [B], \( a \in k \langle Y \rangle \) is called irreducible if \( Aw_iB \) does not appear for \( i \in \{1, 2, 3\} \). Call \( k_{\text{irr}}(Y) \) the \( k \) submodule of irreducible elements of \( k \langle Y \rangle \). A finite sequence of reductions is called final in \( a \) if \( r_{i_n} \circ \cdots \circ r_{i_1}(a) \in k_{\text{irr}}(Y) \). An element \( a \in k \langle Y \rangle \) is called reduction-finite if for every sequence of reductions \( r_{i_n} \circ \cdots \circ r_{i_1}(a) \) for sufficiently large \( n \). If \( a \) is reduction-finite, then any maximal sequence of reductions, such that each \( r_i \) acts nontrivially on \( r_{i_{n-1}} \circ \cdots \circ r_{i_1}(a) \), will be finite, and hence a final sequence. It follows that the reduction-finite elements form a \( k \)-submodule of \( k \langle Y \rangle \) \( a \in k \langle Y \rangle \) is called reduction-unique if is reduction finite and it’s image under every finite sequence of reductions is the same. This common value will be denoted \( r_s(a) \).
**Definition 7.** Given a monomial \( a \in k(Y) \) we define the disorder degree of \( a \), \( \text{disdeg}(a) = \sum_{i=1}^{n} r_{p_{i}} + \sum_{j=1}^{n'} (p_{j}) \), where \( r_{p_{i}} \) is the position of the \( i \)-th letter “\( x \)” counting from right to left, and \( l_{p_{j}} \) is the position of the \( i \)-th letter “\( x' \)” counting from left to right.

If \( a = \sum_{i=1}^{n} k_{i} a_{i} \) where \( a_{i} \) are monomials in letters of \( X, X', e_{X} \) and \( k_{i} \in K - \{0\} \),

\[
\text{disdeg}(a) := \sum_{i=1}^{n} \text{disdeg}(a_{i})
\]

**Example 8.**

\( \bullet \) \( \text{disdeg}(x_{1}e_{y}x_{2}z'_{1}x_{3}z'_{2}) = (2 + 4 + 6) + (4 + 6) = 22 \)

\( \bullet \) \( \text{disdeg}(xe_{y}z') = 3 + 3 = 6 \) and \( \text{disdeg}(x'e_{y}z) = 1 + 1 \)

\( \bullet \) \( \text{disdeg}(\prod_{i=1}^{n} x_{i}' \prod_{j=1}^{m} e_{y} \prod_{k=1}^{l} z_{i}) = \frac{n(n+1)}{2} + \frac{k(k+1)}{2} \)

The reduction \( r_{1} \) lowers disorder degree in two and reductions \( r_{2} \) and \( r_{3} \) lowers disorder degree in one.

**Remark 9.** \( k_{irr}(Y) = \{ \sum A' e_{B}C : A' \text{ word in } X', e_{B} \text{ word in } e_{x}, C \text{ word in } X \} \).

\( k_{irr} \simeq TX' \otimes TE \otimes TX \)

Take for example \( a = xe_{y}z' \), there are two possible sequences of final reductions: \( r_{3} \circ r_{1} \circ r_{2} \) or \( r_{2} \circ r_{1} \circ r_{3} \). The result will be \( a = A'e_{B}C \) and \( a = D'e_{E}F \) respectively, where

\[
A = \sigma^{(1)}(\sigma^{(1)}(x, y), \sigma^{(2)}(x, y, z))
\]

\[
B = \sigma^{(2)}(\sigma^{(1)}(x, y), \sigma^{(2)}(x, y, z))
\]

\[
C = \sigma^{(2)}(\sigma^{(2)}(x, y, z))
\]

\[
D = \sigma^{(1)}(x, \sigma^{(1)}(y, z))
\]

\[
E = \sigma^{(1)}(\sigma^{(2)}(x, \sigma^{(1)}(y, z), \sigma^{(2)}(y, z))
\]

\[
F = \sigma^{(2)}(\sigma^{(2)}(x, \sigma^{(1)}(y, z), \sigma^{(2)}(y, z))
\]

We have \( A = D, B = E \) and \( C = F \) as \( \sigma \) is a solution of \( YBeq \), hence \( r_{3} \circ r_{1} \circ r_{2}(xe_{y}z') = r_{2} \circ r_{1} \circ r_{3}(xe_{y}z') \).

A monomial \( a \) in \( k(Y) \) is said to have an overlap ambiguity of \( S \) if \( a = ABCDE \) such that \( w_{1} = BC \) and \( w_{3} = CD \). We shall say the overlap ambiguity is resolveable if there exist compositions of reductions, \( r, r' \) such that \( r(Ar_{1}(BC)DE) = r'(ABr_{1}(CD)E) \). Notice that it is enough to take \( r = r_{s} \) and \( r' = r_{s} \).

**Remark 10.** In our case, there is only one type of overlap ambiguity and is the one we solved previously.

**Proof.** There is no rule with \( x' \) on the left nor rule with \( x \) on the right, so there will be no overlap ambiguity including the family \( r_{1} \). There is only one type of ambiguity involving reductions \( r_{2} \) and \( r_{3} \).

Notice that \( r_{s} \) is a projector and \( I = \langle xy' - z't, xe_{y} - e_{z}t, e_{z}y' - z'e_{i} \rangle \) is trivially included in the kernel. We claim that it is actually equal:

**Proof.** As \( r_{s} \) is a projector, an element \( a \in ker \) must be \( a = b - r_{s}(b) \) where \( b \in k(Y) \). It is enough to prove it for monomials \( b \).

\( \bullet \) if \( a = 0 \) the result follows trivially.
if not, then take a monomial \( b \) where at least one of the products \( xy' \), \( xe_y \) or \( e_x y' \) appear. Let\u2019s suppose \( b \) has a factor \( xy' \) (the rest of the cases are analogous).

\[
b = Axy'B \quad \text{where} \ A \text{ or } B \text{ may be empty words. } \ r_1(b) = Ar_1(xy')B = Az'tB. \]

Now we can rewrite:

\[
b - r_s(b) = \underbrace{Axy'B - Az'tB + Az'tB - r_s(b)}_{\epsilon I} \quad \text{where} \ \text{as usual} \ \text{disdeg in} \ \text{two, we have}
\]

\[
\text{disdeg}(A\epsilon I B - r_s(b)) < \text{disdeg}(b - r_s(b)) \quad \text{then in a finite number of steps we get}
\]

\[
b = \sum_{k=1}^N i_k \quad \text{where } i_k \in I. \quad \text{It follows that } b \in I.
\]

\[\square\]

**Corollary 11.** \( r_s \) induces a \( k \)-linear isomorphism:

\[
k\langle Y \rangle / (xy' - z't, xe_y - e_z t, e_x y' - z'e_t) \to TX' \otimes TE \otimes TX
\]

Returning to our bialgebra, taking quotients we obtain the following proposition:

**Proposition 12.** \( B \simeq (TX'/(x'y' = z't')) \otimes TE \otimes (TX/(xy = zt)) \)

Notice that \( \prod_1 \ldots x_n = \prod [\beta_m \circ \cdots \circ \beta_1(x_1, \ldots, x_n)] \) where \( \beta_i = \sigma_i^{+1} \), analogously with \( \prod_1 \ldots x'_n \).

This ends the proof of Theorem 5

**Example 13.**

If the coefficients are trivial, \( f \in C^1(X, k) \) and we identify \( C^1(X, k) = kX \), then

\[
(\partial f)(x, y) = f(d(e_x e_y)) = -f(x) - f(y) + f(z) + f(t)
\]

where as usual \( \sigma(x, y) = (z, t) \) (If instead of considering Hom\(_{A' - A} \), we consider Hom\(_{A - A'} \), then \( \partial f)(x, y) = f(d(e_x e_y)) = f(x) + f(y) - f(z) - f(t) \) but with \( \sigma(z, t) = (x, y) \)).

Again with trivial coefficients, and \( \Phi \in C^2(X, k) \cong kX^2 \), then

\[
(\partial \Phi)(x, y, z) = \Phi(d(e_x e_y e_z)) = \Phi \left( I_{x e_y e_z} - I_{x' e'_y e_z} - I_{x z e_y e_z} + e_x y' e_z + e_x e_y e_z - e_x e_y e_z \right)
\]

If considering Hom\(_{A' - A} \) then, using the relations defining \( B \), the terms \( I, III, IV \) and \( VI \) changes leaving

\[
\partial \Phi(x, y, z) = \Phi(\sigma^1(x, y), \sigma^1(\sigma^2(x, y), z)) - \Phi(y, z) - \Phi(x, \sigma^1(y, z)) + \\
\Phi(\sigma^2(x, y), z) + \Phi(x, y) - \Phi(\sigma^2(x, \sigma^1(y, z)), \sigma^2(y, z))
\]

If \( M \) is a \( k[T] \)-module (notice that \( T \) need not to be invertible as in \([CESI]\)) then \( M \) can be viewed as an \( A' - A \)-bimodule via

\[
x' \cdot m = m, \quad m \cdot x = Tm
\]

The actions are compatible with the relations defining \( B \):
\[(m \cdot x) \cdot y = T^2 m, \quad (m \cdot z) \cdot t = T^2 m\]

and
\[x' \cdot (y' \cdot m) = m, \quad z' \cdot (t' \cdot m) = m\]

Using these coefficients we get twisted cohomology as in [CES1] but for general YB solutions. If one takes the special case of \((X, \sigma)\) being a rack, namely \(\sigma(x, y) = (y, x \triangle y)\), then the general formula gives
\[
\partial f(x_1, \ldots, x_n) = f(d(e_{x_1} \ldots e_{x_n})) = \sum_{i=1}^{n} (-1)^{i+1} \left( T f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, x_i, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \right)
\]

that agree with the differential of the twisted cohomology defined in [CES1].

Remark 14. If \(c(x \otimes y) = f(x, y)\sigma^4(x, y) \otimes \sigma^2(x, y)\), then \(c\) is a solution of YBeq if and only if \(f\) is a 2-cocycle.

\[
c_1 \circ c_2 \circ c_1 (x \otimes y \otimes z) = \underbrace{a \sigma^1(\sigma^4(x, y), \sigma^4(\sigma^2(x, y), z)) \otimes \sigma^2(\sigma^1(x, y), \sigma^4(\sigma^2(x, y), z))}^{I} \otimes \sigma^2(\sigma^1(x, y), \sigma^2(\sigma^4(x, y), z))
\]

where
\[
a = f(x, y) f(\sigma^2(x, y), z) f(\sigma^4(x, y), \sigma^4(\sigma^2(x, y), z))
\]

\[
c_2 \circ c_1 \circ c_2 (x \otimes y \otimes z) = \underbrace{b \sigma^4(x, \sigma^4(y, z)) \otimes \sigma^1(\sigma^2(x, \sigma^4(y, z)), \sigma^2(y, z))}^{II} \otimes \sigma^2(\sigma^1(x, \sigma^4(y, z)), \sigma^2(y, z))
\]

where
\[
b = f(y, z) f(x, \sigma^4(y, z)) f(\sigma^2(x, \sigma^4(y, z)), \sigma^2(y, z))
\]

Writing YBeq with this notation leaves:
\[
\sigma \text{ is a braid} \iff I = II \quad (1)
\]

Take \(f\) a two-cocycle, then
\[
0 = \partial f(x, y, z) = f(d(e_x e_y e_z)) = f(((x - x') e_y e_z - e_x (y - y') e_z + e_x e_y (z - z'))
\]

is equivalent to the following equality
\[
f(x e_y e_z) + f(e_x y' e_z) + f(e_x e_y z) = f(x' e_y e_z) + f(e_x y e_z) + f(e_x e_y z')
\]
using the relations defining $B$ we obtain
\[
f(\epsilon_{\sigma(x,y)}\epsilon_{\sigma(\sigma^2(x,y),z)}\sigma^2(\sigma^2(x,y)z)) + f(\sigma^1(x,y)\epsilon_{\sigma^2(x,y)}\epsilon_z) + f(\epsilon_x e_y z) = f(x' e_y e_z) + f(\epsilon_x e_{\sigma(y,z)}\sigma^2(y,z)) + f(\sigma^1(x,\sigma^1(y,z))\epsilon_{\sigma^2(y,z)})
\]

If $G$ is an abelian multiplicative group and $f : X \times X \to (G, \cdot)$ then the previous formula says
\[
f(\epsilon_{\sigma(x,y)}\epsilon_{\sigma(\sigma^2(x,y),z)}\sigma^2(\sigma^2(x,y)z)) f(\sigma^1(x,y)\epsilon_{\sigma^2(x,y)}\epsilon_z) f(\epsilon_x e_y z) = f(x' e_y e_z) f(\epsilon_x e_{\sigma(y,z)}\sigma^2(y,z)) f(\sigma^1(x,\sigma^1(y,z))\epsilon_{\sigma^2(y,z)})
\]
which is exactly the condition $a = b$.

Notice that if the action is trivial, then the equation above simplifies giving
\[
f(\epsilon_{\sigma(x,y)}\epsilon_{\sigma(\sigma^2(x,y),z)}\sigma^2(\sigma^2(x,y)z)) f(\epsilon_x e_y z) = f(e_y e_z) f(\epsilon_x e_{\sigma(y,z)}\sigma^2(y,z)) f(\epsilon_{\sigma^2(x,\sigma(y,z))}\sigma^2(y,z))
\]
which is precisely the formula on [CES2] for Yang-Baxter 2-cocycles (with $R_1$ and $R_2$ instead of $\sigma^1$ and $\sigma^2$).

3 1st application: multiplicative structure on cohomology

**Proposition 15.** $\Delta$ induces an associative product in $\text{Hom}_{A' - A}(B, k)$ (the graded Hom).

**Proof.** It is clear that $\Delta$ induces an associative product on $\text{Hom}_k(B, k)$ (the graded Hom), and $\text{Hom}_{A' - A}(B, k) \subset \text{Hom}_k(B, k)$ is a $k$-submodule. We will show that it is in fact a subalgebra.

Consider the $A' - A$ diagonal structure on $B \otimes B$ (i.e. $x'_1(b \otimes b') x_2 = x'_1 b x_2 \otimes x'_1 b' x_2$) and denote $B \otimes^D B$ the $k$-module $B \otimes B$ considered as $A' - A$-bimodule in this diagonal way. We claim that $\Delta : B \to B \otimes^D B$ is a morphism of $A' - A$-modules:
\[
\Delta(x'_1 y x_2) = x'_1 y x_2 \otimes x'_1 y x_2 = x'_1(y \otimes y) x_2
\]
same with $y'$, and with $e_x$:
\[
\Delta(x'_1 e_y x_2) = (x'_1 \otimes x'_1)(y' \otimes e_y + e_y \otimes y)(x_2 \otimes x_2) = x'_1 \Delta(e_y) x_2
\]

Dualizing $\Delta$ one gets:
\[
\Delta^* : \text{Hom}_{A' - A}(B \otimes^D B, k) \to \text{Hom}_{A' - A}(B, k)
\]
consider the natural map
\[
i : \text{Hom}_k(B, k) \otimes \text{Hom}_k(B, k) \to \text{Hom}_k(B \otimes B, k)
\]
\[
i(f \otimes g)(b_1 \otimes b_2) = f(b_1)g(b_2)
\]
and denote $i|_{\text{Hom}_{A' - A}(B,k) \otimes \text{Hom}_{A' - A}(B,k)}$ by
\[
i| = i|_{\text{Hom}_{A' - A}(B,k) \otimes \text{Hom}_{A' - A}(B,k)}
\]
Let us see that
\[ \text{Im}(\iota) \subset \text{Hom}_{A' - A}(B \otimes B, k) \subset \text{Hom}_k(B \otimes B, k) \]

If \( f, g : B \to k \) are two \( A' - A \)-module morphisms (recall \( k \) has trivial actions, i.e. \( x' \lambda = \lambda \) and \( \lambda x = x \)), then
\[
\iota(f \otimes g)(x' (b_1 \otimes b_2)) = f(x'b_1)g(x'b_2) = (x'f(b_1))(x'g(b_2)) \\
= f(b_1)g(b_2) = x'\iota(f \otimes g)(b_1 \otimes b_2) \\
\iota(f \otimes g)((b_1 \otimes b_2)x) = f(b_1)xg(b_2x) = (f(b_1)x)(g(b_2)x) \\
= (f(b_1)g(b_2))x = \iota(f \otimes g)(b_1 \otimes b_2)x
\]

So, it is possible to compose \( \iota \mid \) and \( \Delta \), and obtain in this way an associative multiplication in \( \text{Hom}_{A' - A}(B, k) \).

Now we will describe several natural quotients of \( B \), each of them give rise to a subcomplex of the cohomological complex of \( X \) with trivial coefficients that are not only subcomplexes but also subalgebras; in particular they are associative algebras.

### 3.1 Square free case

A solution \( (X, \sigma) \) of YBeq satisfying \( \sigma(x, x) = (x, x) \forall x \in X \) is called square free. For instance, if \( X \) is a rack, then this condition is equivalent to \( X \) being a quandle.

In the square free situation, namely when \( X \) is such that \( \sigma(x, x) = (x, x) \) for all \( x \), we add the condition \( e_x e_x \sim 0 \).

If \( (X, \sigma) \) is a square-free solution of the YBeq, let us denote \( sf \) the two sided ideal of \( B \) generated by \( \{e_x e_x\}_{x \in X} \).

**Proposition 16.** \( sf \) is a differential Hopf ideal. More precisely,
\[
d(e_x e_x) = 0 \quad \text{and} \quad \Delta(e_x e_x) = x' x' \otimes e_x e_x + e_x e_x \otimes xx.
\]

In particular \( B/sf \) is a differential graded bialgebra. We may identify \( \text{Hom}_{A' - A}(B/sf, k) \subset \text{Hom}_{A' - A}(B, k) \) as the elements \( f \) such that \( f(\ldots, x, x, \ldots) = 0 \). If \( X \) is a quandle, this construction leads to the quandle-complex. We have \( \text{Hom}_{A' - A}(B/sf, k) \subset \text{Hom}_{A' - A}(B, k) \) is not only a subcomplex, but also a subalgebra.

### 3.2 Biquandles

In [KR], a generalization of quandles is proposed (we recall it with different notation), a solution \( (X, \sigma) \) is called non-degenerated, or birack if in addition,

1. for any \( x, z \in X \) there exists a unique \( y \) such that \( \sigma^1(x, y) = z \), (if this is the case, \( \sigma^1 \) is called left invertible),
2. for any \( y, t \in X \) there exists a unique \( x \) such that \( \sigma^2(x, y) = t \), (if this is the case, \( \sigma^2 \) is called right invertible),
A birack is called *biquandle* if, given \( x_0 \in X \), there exists a unique \( y_0 \in X \) such that \( \sigma(x_0, y_0) = (x_0, y_0) \). In other words, if there exists a bijective map \( s : X \to X \) such that

\[
\{(x, y) : \sigma(x, y) = (x, y)\} = \{(x, s(x)) : x \in X\}
\]

**Remark 17.** Every quandle solution is a biquandle, moreover, given a rack \((X, \triangleleft)\), then \( \sigma(x, y) = (y, x \triangleleft y) \) is a biquandle if and only if \((X, \triangleleft)\) is a quandle.

If \((X, \sigma)\) is a biquandle, for all \( x \in X \) we add in \( B \) the relation \( e_x e_{s(x)} \sim 0 \). Let us denote \( BQ \) the two sided ideal of \( B \) generated by \( \{e_x e_{s(x)}\}_{x \in X} \).

**Proposition 18.** \( BQ \) is a differential Hopf ideal. More precisely, \( d(e_x e_{s(x)}) = 0 \) and \( \Delta(e_x e_{s(x)}) = x' s(x)' \otimes e_x e_{s(x)} + e_x e_{s(x)} \otimes x s(x) \).

In particular \( B/bQ \) is a differential graded bialgebra. We may identify \( \text{Hom}_{A'A'}(B/bQ, k) \cong \{ f \in \text{Hom}_{A'A'}(B, k) : f(\ldots, x, s(x), \ldots) = 0 \} \subset \text{Hom}_{A'A'}(B, k) \).

In [CES2], the condition \( f(\ldots, x_0, s(x_0), \ldots) = 0 \) is called the *type 1 condition*. A consequence of the above proposition is that \( \text{Hom}_{A'A'}(B/bQ, k) \subset \text{Hom}_{A'A'}(B, k) \) is not only a subcomplex, but also a subalgebra. Before proving this proposition we will review some other similar constructions.

### 3.3 Identity case

The two cases above may be generalized in the following way:

Consider \( S \subseteq X \times X \) a subset of elements verifying \( \sigma(x, y) = (x, y) \) for all \((x, y) \in S\). Define \( idS \) the two sided ideal of \( B \) given by \( idS = \langle e_x e_y / (x, y) \in S \rangle \).

**Proposition 19.** \( idS \) is a differential Hopf ideal. More precisely, \( d(e_x e_y) = 0 \) for all \((x, y) \in S\) and \( \Delta(e_x e_y) = x' y' \otimes e_x e_y + e_x e_y \otimes x y \).

In particular \( B/idS \) is a differential graded bialgebra. If one identifies \( \text{Hom}_{A'A'}(B/sf, k) \subset \text{Hom}_{A'A'}(B, k) \) as the elements \( f \) such that

\[
f(\ldots, x, y, \ldots) = 0 \quad \forall (x, y) \in S
\]

We have that \( \text{Hom}_{A'A'}(B/idS, k) \subset \text{Hom}_{A'A'}(B, k) \) is not only a subcomplex, but also a subalgebra.

### 3.4 Flip case

Consider the condition \( e_x e_y + e_y e_x \sim 0 \) for all pairs such that \( \sigma(x, y) = (y, x) \). For such a pair \((x, y)\) we have the equations \( xy = yx \), \( x'y' = y'x' \), \( x'y' = y'x' \) and \( x e_y = e_y x \). Note that there is no equation for \( e_x e_y \). The two sided ideal \( D = \langle e_x e_y + e_y e_x : \sigma(x, y) = (y, x) \rangle \) is a differential and Hopf ideal.

Moreover, the following generalization is still valid:
3.5 Involutive case

Assume \(\sigma(x, y)^2 = (x, y)\). This case is called involutive in \[ESS\]. Define \(\text{Invo}\) the two sided ideal of \(B\) given by \(\text{Invo} = \langle e_x e_y + e_z e_t : (x, y) \in X, \sigma(x, y) = (z, t) \rangle\).

**Proposition 20.** \(\text{Invo}\) is a differential Hopf ideal. More precisely, \(d(e_x e_y + e_z e_t) = 0\) for all \((x, y) \in X\) (with \((z, t) = \sigma(x, y)\)) and if \(\omega = e_x e_y + e_z e_t\) then \(\Delta(\omega) = x'y' \otimes \omega + \omega \otimes xy\).

In particular \(B/\text{Invo}\) is a differential graded bialgebra. If one identifies \(\text{Hom}_{A' A}(B/\text{Invo}, k) \subset \text{Hom}_{A' A}(B, k)\) then \(\text{Hom}_{A' A}(B/\text{Invo}, k) \subset \text{Hom}_{A' A}(B, k)\) is not only a subcomplex, but a subalgebra.

**Conjecture 21.** \(B/\text{Invo}\) is acyclic in positive degrees.

**Example 22.** If \(\sigma = \text{flip}\) and \(X = \{x_1, \ldots, x_n\}\) then \(A = k[x_1, \ldots, x_n] = SV\), the symmetric algebra on \(V = \bigoplus_{x \in X} kx\). In this case \((B/\text{Invo}, d) \cong (S(V) \otimes \Lambda V \otimes S(V), d)\) gives the Koszul resolution of \(S(V)\) as \(S(V)\)-bimodule.

**Example 23.** If \(\sigma = \text{Id}\), \(X = \{x_1, \ldots, x_n\}\) and \(V = \bigoplus_{x \in X} kx\), then \(A = TV\) the tensor algebra. If \(\frac{1}{2} \in k\), then \((B/\text{invo}, d) \cong TV \otimes (k \oplus V) \otimes TV\) gives the Koszul resolution of \(TV\) as \(TV\)-bimodule. Notice that we don’t really need \(\frac{1}{2} \in k\), one could replace \(\text{invo} = (e_x e_y + e_x e_y : (x, y) \in X \times X)\) by \(\text{idXX} = (e_x e_y : (x, y) \in X \times X)\).

The conjecture above, besides these examples, is supported by next result:

**Proposition 24.** If \(Q \subseteq k\), then \(B/\text{Invo}\) is acyclic in positive degrees.

**Proof.** In \(B/\text{Invo}\) it can be defined \(h\) as the unique (super)derivation such that:

\[
h(e_x) = 0; \quad h(x) = e_x, \quad h(x') = -e_x
\]

Let us see that \(h\) is well defined:

\[
h(xy - zt) = e_x y + x e_y - e_z t - z e_t = 0
\]
\[
h(xy' - z't) = e_x y' - x e_y + e_z t - z' e_t = 0
\]
\[
h(x'y' - z't') = -e_x y' - x' e_y + e_z t' + z' e_t = 0
\]
\[
h(x e_y - e_z t) = e_x e_y + e_z e_t = 0
\]

Notice that in particular next equation shows that \(h\) is not well-defined in \(B\).

\[
h(e_x y' - z' e_t) = e_x e_y + e_z e_t = 0
\]
\[
h(z t' - x' y) = e_z t' - z e_t + e_x y - x' e_y = 0
\]
\[
h(z e_t - e_x y) = e_z e_t + e_x e_y = 0
\]
\[
h(e_z t' - x' e_y) = e_z e_t + e_x e_y = 0
\]
\[
h(e_x e_y + e_z e_t) = 0
\]
Since (super) commutator of (super)derivations is again a derivation, we have that \([h, d] = hd + dh\) is also a derivation. Computations on generators:

\[
h(e^x) = 2e^x, \quad h(x) = x - x', \quad h(x') = x' - x
\]
or equivalently

\[
h(e^x) = 2e^x, \quad h(x + x') = 0, \quad h(x - x') = 2(x - x')
\]

One can also easily see that \(\mathcal{B}/\text{Invo}\) is generated by \(e^x, x_{\pm}\), where \(x_{\pm} = x \pm x'\), and that their relations are homogeneous. We see that \(hd + dh\) is nothing but the Euler derivation with respect to the grading defined by

\[
\deg e^x = 2, \quad \deg x = 0, \quad \deg x' = 2,
\]

We conclude automatically that the homology vanish for positive degrees of the \(e^x\)'s (and similarly for the \(x_{\pm}\)’s).

Next, we generalize Propositions 16, 18, 19 and 20. 

### 3.6 Braids of order \(N\)

Let \((x_0, y_0) \in X \times X\) such that \(\sigma^N(x_0, y_0) = (x_0, y_0)\) for some \(N \geq 1\). If \(N = 1\) we have the ”identity case” and all subcases, if \(N = 2\) we have the ”involutive case”. Denote \((x_i, y_i) := \sigma^i(x_0, y_0) 1 \leq i \leq N - 1\)

Notice that the following relations hold in \(B\):

\[
\begin{align*}
\ast x_{N-1}y_{N-1} & \sim x_0y_0, \quad x_{N-1}y'_{N-1} \sim x_0'y_0, \quad x'_{N-1}y'_{N-1} = x'_0y'_0 \\
\ast x_{N-1}e_{y_{N-1}} & \sim e_{x_0}y_0, \quad e_{x_{N-1}}y'_{N-1} \sim x'_0e_{y_0}
\end{align*}
\]

and for \(1 \leq i \leq N - 1:\)

\[
\begin{align*}
\ast x_{i-1}y_{i-1} & \sim x_iy_i, \quad x_{i-1}y'_{i-1} \sim x'_iy_i, \quad x'_{i-1}y'_{i-1} = x'_iy'_i \\
\ast x_{i-1}e_{y_{i-1}} & \sim e_{x_i}y_i, \quad e_{x_{i-1}}y'_{i-1} \sim x'_ie_{y_i}
\end{align*}
\]

Take \(\omega = \sum_{i=0}^{N-1} e_{x_i}e_{y_i}\), then we claim that

\[
d\omega = 0
\]

and

\[
\Delta \omega = x_0y_0 \otimes \omega + \omega \otimes x'_0y'_0
\]

For that, we compute

\[
d(\omega) = \sum_{i=0}^{N-1} \left( x_i - x'_i \right) e_{y_i} - e_{x_i} (y_i - y'_i) = \]

\[
\sum_{i=0}^{N-1} (x_i e_{y_i} - e_{x_i} y_i) - \sum_{i=0}^{N-1} (x'_i e_{y_i} - e_{x_i} y'_i) = 0
\]
For the comultiplication, we recall that
\[ \Delta(ab) = \Delta(a)\Delta(b) \]
where the product on the right hand side is defined using the Koszul sign rule:
\[ (a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{|a_2||b_1|}a_1b_1 \otimes a_2b_2 \]
So, in this case we have
\[ \Delta(\omega) = \sum_{i=0}^{N-1} \Delta(e_x e_{y_i}) = \sum_{i=0}^{N-1} (x'_i y'_i \otimes e_x e_{y_i} - x'_i e_{y_i} \otimes e_x y_i + e_x y_i \otimes x_i e_{y_i} + e_x e_{y_i} \otimes x_i y_i) \]
the middle terms cancel telescopically, giving
\[ = \sum_{i=0}^{N-1} (x'_i y'_i \otimes e_x e_{y_i} + e_x e_{y_i} \otimes x_i y_i) \]
and the relation \( x_i y_i \sim x_{i+1} y_{i+1} \) gives
\[ = x'_0 y'_0 \otimes \left( \sum_{i=0}^{N-1} e_x e_{y_i} \right) + \left( \sum_{i=0}^{n-1} e_x e_{y_i} \right) \otimes x_0 y_0 \]
\[ = x'_0 y'_0 \otimes \omega + \omega \otimes x_0 y_0 \]
Then the two-sided ideal of \( B \) generated by \( \omega \) is a Hopf ideal. If instead of a single \( \omega \) we have several \( \omega_1, \ldots, \omega_n \), we simply remark that the sum of differential Hopf ideals is also a differential Hopf ideal.

**Remark 25.** If \( X \), is finite then for every \( (x_0, y_0) \) there exists \( N > 0 \) such that \( \sigma^N(x_0, y_0) = (x_0, y_0) \).

**Remark 26.** Let us suppose \( (x_0, y_0) \in X \times X \) is such that \( \sigma^N(x_0, y_0) = (x_0, y_0) \) and \( u \in X \) an arbitrary element. Consider the element
\[ ((\text{Id} \times \sigma)(\sigma \times \text{Id})(u, x_0, y_0) = (\tilde{x}_0, \tilde{y}_0, u'') \]
graphically

![Diagram](image)

then \( \sigma^N(\tilde{x}_0, \tilde{y}_0) = (\tilde{x}_0, \tilde{y}_0) \).
Proof.

\[(\sigma^N \times \text{id})(\tilde{x}_0, \tilde{y}_0, u'') = (\sigma^N \times \text{id})(\text{id} \times \sigma)(\sigma \times \text{id})(u, x_0, y_0) = \]

\[(\sigma^{N-1} \times \text{id})(\sigma \times \text{id})(\text{id} \times \sigma)(\sigma \times \text{id})(u, x_0, y_0) = \]

using YBeq

\[(\sigma^{N-1} \times \text{id})(\sigma \times \text{id})(\text{id} \times \sigma)(\sigma \times \text{id})(u, x_0, y_0) = \]

repeating the procedure \(N - 1\) times leaves

\[(\sigma^N - 1 \times \text{id})(\sigma \times \text{id})(\text{id} \times \sigma)(\sigma \times \text{id})(u, x_0, y_0) = \]

\[\text{Corollary 27. For all } A\text{-bimodule } M, \text{ there exists natural maps} \]

\[\tilde{\text{id}}_* : H^*_{YB}(X, M) \rightarrow H^*_*(A, M) \]

\[\tilde{\text{id}}^* : H^*_{YB}(A, M) \rightarrow H^*_{YB}(X, M) \]

that are the identity in degree zero and 1.

Moreover, one can choose an explicit map with extra properties. For that we recall some definitions: there is a set theoretical section to the canonical projection from the Braid group to the symmetric group

\[\mathbb{B}_n \xrightarrow{\tilde{s}} \mathbb{S}_n \]

\[T_s := \sigma_{i_1} \cdots \sigma_{i_k} \leftarrow s = \tau_{i_1} \cdots \tau_{i_k} \]

where

- \(\tau \in \mathbb{S}_n\) are transpositions of neighboring elements \(i\) and \(i + 1\), so-called simple transpositions,
• $\sigma_i$ are the corresponding generators of $\mathbb{B}_n$,
• $\tau_{i_1} \ldots \tau_{i_k}$ is one of the shortest words representing $s$.

This inclusion factorizes through
\[ S_n \hookrightarrow \mathbb{B}_n^+ \hookrightarrow \mathbb{B}_n \]
It is a set inclusion not preserving the monoid structure.

**Definition 28.** The permutation sets
\[ \text{Sh}_{p_1, \ldots, p_k} := \{ s \in S_{p_1 + \ldots + p_k} / s(1) < \cdots < s(p_1), \ldots, s(p + 1) < \cdots < s(p + p_k) \} , \]
where $p = p_1 + \cdots + p_{k-1}$, are called shuffle sets.

**Remark 29.** It is well known that a braiding $\sigma$ gives an action of the positive braid monoid $B_n^+$ on $V^\otimes n$, i.e. a monoid morphism
\[ \rho : B_n^+ \rightarrow \text{End}_K(V^\otimes n) \]
defined on generators $\sigma_i$ of $B_n^+$ by
\[ \sigma_i \mapsto \text{Id}_V^{\otimes (i-1)} \otimes \sigma \otimes \text{Id}_V^{\otimes (n-i+1)} \]

Then there exists a natural extension of a braiding in $V$ to a braiding in $T(V)$.

\[ \sigma(v \otimes w) = (\sigma_k \ldots \sigma_1) \circ \cdots \circ (\sigma_{n+k-2} \ldots \sigma_{n-1}) \circ (\sigma_{n+k-1} \ldots \sigma_n)(vw) \in V^k \otimes V^n \]
for $v \in V^\otimes n$, $w \in V^k$ and $vw$ being the concatenation.

Graphically
\[ \cdots \otimes \cdots \]

**Definition 30.** The quantum shuffle multiplication on the tensor space $T(V)$ of a braided vector space $(V, \sigma)$ is the $k$-linear extension of the map
\[ \sqcup \sqcup = \sqcup_{p,q} : V^\otimes p \otimes V^\otimes q \rightarrow V^\otimes (p+q) \]
\[ \overline{v} \otimes \overline{w} \mapsto \overline{v} \sqcup \overline{w} := \sum_{s \in \text{Sh}_{p,q}} T^\sigma_s(\overline{vw}) \]

Notation: $T^\sigma_s$ stands for the lift $T_s \in \mathbb{B}_n^+$ acting on $V^\otimes n$ via the braiding $\sigma$. The algebra
\[ Sh_\sigma(V) := (TV, \sqcup_\sigma) \]
is called the quantum shuffle algebra on $(V, \sigma)$.

It is well-known that $\sqcup_\sigma$ is an associative product on $TV$ (see for example [Le] for details) that makes it a Hopf algebra with deconcatenation coproduct.
Definition 31. Let $V$ be a braided vector space, then the quantum symmetrizer map $\sqcup_{\sigma} : V^\otimes n \to V^\otimes n$ defined by

$$QS_{\sigma}(v_1 \otimes \cdots \otimes v_n) = \sum_{\tau \in S_n} T_{\sigma}^\tau (v_1 \otimes \cdots \otimes v_n)$$

where $T_{\sigma}^\tau$ is the lift $T_{\sigma}^\tau \in B_n^+$ of $\tau$, acting on $V^\otimes n$ via the braiding $\sigma$.

In terms of shuffle products the quantum symmetrizer can be computed as

$$\omega \sqcup_{\sigma} \eta := \sum_{\tau \in Sh_{p,q}} T_{\sigma}^\tau (\omega \otimes \eta)$$

The quantum symmetrizer map can also be defined as

$$QS_{\sigma}(v_1 \otimes \cdots \otimes v_n) = v_1 \sqcup_{\sigma} \cdots \sqcup_{\sigma} v_n$$

With this notation, next result reads as follows:

Theorem 32. The $A'\otimes A$-linear quantum symmetrizer map

$$A'V^\otimes n A \xrightarrow{\tilde{\text{Id}}} A \otimes A^\otimes n \otimes A$$

$$a'_1e_{x_1} \cdots e_{x_n}a_2 \mapsto a_1 \otimes (x_1 \sqcup_{\sigma} \cdots \sqcup_{\sigma} x_n) \otimes a_2$$

is a chain map lifting the identify. Moreover, $\tilde{\text{Id}} : B \to (A \otimes TA \otimes A, b')$ is a differential graded algebra map, where in $TA$ the product is $\sqcup_{-\sigma}$, and in $A \otimes TA \otimes A$ the multiplicative structure is not the usual tensor product algebra, but the braided one. In particular, this map factors through $A \otimes B \otimes A$, where $B$ is the Nichols algebra associated to the braiding $\sigma'(x \otimes y) = -z \otimes t$, where $x, y \in X$ and $\sigma(x, y) = (z, t)$.

Remark 33. The Nichols algebra $B$ is the quotient of $TV$ by the ideal generated by (skew)primitives that are not in $V$, so the result above explains the good behavior of the ideals $\text{invo}, \text{idS}$, or in general the ideal generated by elements of the form $\omega = \sum_{i=0}^{N-1} e_{x_i}e_{y_i}$ where $\sigma(x_i, y_i) = (x_{i+1}, y_{i+1})$ and $\sigma^N(x_0, y_0) = (x_0, y_0)$. It would be interesting to know the properties of $A \otimes B \otimes A$ as a differential object, since it appears to be a candidate of Koszul-type resolution for the semigroup algebra $A$ (or similarly the group algebra $k[G_X]$).

The rest of the paper is devoted to the proof of 32. Most of the Lemmas are ”folklore” but we include them for completeness. The interested reader can look at [Le2] and references therein.

Lemma 34. Let $\sigma$ be a braid in the braided (sub)category that contains two associative algebras $A$ and $C$, meaning there exists bijective functions

$$\sigma_A : A \otimes A \to A \otimes A, \quad \sigma_C : C \otimes C \to C \otimes C, \quad \sigma_{C,A} : C \otimes A \to A \otimes C$$

such that

$$\sigma_*(1, -) = (-, 1) \text{ and } \sigma_*(-, 1) = (1, -) \text{ for } * \in \{A, C; C, A\}$$

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\[ \sigma_{C,A} \circ (1 \otimes m_A) = (m_A \otimes 1)(1 \otimes \sigma_{C,A})(\sigma_{C,A} \otimes 1) \]

and

\[ \sigma_{C,A} \circ (m_C \otimes 1) = (1 \otimes m_C)(\sigma_{C,A} \otimes 1)(1 \otimes \sigma_{C,A}) \]

Diagrammatically

Assume that they satisfy the braid equation with any combination of \( \sigma_A, \sigma_C \) or \( \sigma_{A,C} \). Then, \( A \otimes_{\sigma} C = A \otimes C \) with product defined by

\[ (m_A \otimes m_C) \circ (\text{Id}_A \otimes \sigma_{C,A} \otimes \text{Id}_C) : (A \otimes C) \otimes (A \otimes C) \to A \otimes C \]

is an associative algebra. In diagram:

Proof. Take \( m \circ (1 \otimes m)((a_1 \otimes c_2) \otimes ((a_2 \otimes c_2) \otimes (a_3 \otimes c_3)) \) use \([*]\), associativity in \( A \), associativity in \( C \) then \([**]\) and the result follows.

Lemma 35. Let \( M \) be the monoid freely generated by \( X \) module the relation \( xy = zt \) where \( \sigma(x,y) = (z,t) \), then, \( \sigma : X \times X \to X \times X \) naturally extends to a braiding in \( M \) and verifies
Proof. It is enough to prove that the extension mentioned before is well defined in the quotient. Inductively, it will be enough to see that \( \sigma(a_{xy}b, c) = \sigma(a_{zt}b, c) \) and \( \sigma(c, a_{xy}b) = \sigma(c, a_{zt}b) \) where \( \sigma(x, y) = (z, t) \), and this follows immediately from the braid equation:

A diagram for the first equation is the following:

As \( \alpha \beta = \alpha^\ast \beta^\ast \) the result follows.

Lemma 36. \( m \circ \sigma = m \), diagrammatically:
Proof. Using successively that \( m \circ \sigma_i = m \), we have:

\[
m \circ \sigma(x_1 \ldots x_n, y_1 \ldots y_k) = m \left( (\sigma_k \ldots \sigma_1) \ldots (\sigma_{n+k-1} \ldots \sigma_n)(x_1 \ldots x_n y_1 \ldots y_k) \right)
\]

\[
= m \left( (\sigma_{k-1} \ldots \sigma_1) \ldots (\sigma_{n+k-1} \ldots \sigma_n)(x_1 \ldots x_n y_1 \ldots y_k) \right) = \ldots
\]

\[
= m(x_1 \ldots x_n, y_1 \ldots y_k)
\]

\[\square\]

Corollary 37. If one considers \( A = k[M] \), then the algebra \( A \) verifies all diagrams in previous lemmas.

Lemma 38. If \( T = (TA, \sqcup) \) there are bijective functions

\[
\sigma_{T,A} := \sigma|_{T \otimes A} : T \otimes A \to A \otimes T
\]

\[
\sigma_{A,T} := \sigma|_{A \otimes T} : A \otimes T \to T \otimes A
\]

that verifies the hypothesis of Lemma 34, and the same for \((TA, \sqcup_{-a})\).

Corollary 39. \( A \otimes (TA, \sqcup_{-a}) \otimes A \) is an algebra.

Proof. Use 34 twice and the result follows. \[\square\]

Corollary 40. Taking \( A = k[M] \), then the standard resolution of \( A \) as \( A \)-bimodule has a natural algebra structure defining the braided tensorial product as follows:

\[
A \otimes TA \otimes A = A \otimes_\sigma (T^c A, \sqcup_{-a}) \otimes_\sigma A
\]

Recall the differential of the standard resolution is defined as \( b' : A^{\otimes n+1} \to A^{\otimes n} \)

\[
b'(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n
\]

for all \( n \geq 2 \). If \( A \) is a commutative algebra then the Hochschild resolution is an algebra viewed as \( \oplus_{n \geq 2} A^{\otimes n} = A \otimes TA \otimes A \), with right and left \( A \)-bilinear extension of the shuffle product on \( TA \), and \( b' \) is a (super) derivation with respect to that product (see for instance Prop. 4.2.2 [Lo]). In the braided-commutative case we have the analogous result:

Lemma 41. \( b' \) is a derivation with respect to the product mentioned in Corollary 40.

Proof. Recall the commutative proof as in Prop. 4.2.2 [Lo]. Denote \( \ast \) the product

\[
(a_0 \otimes \ldots \otimes a_{p+1}) \ast (b_0 \otimes \ldots \otimes b_{q+1}) = a_0 b_0 \otimes ((a_1 \ldots \otimes a_p) \sqcup (b_1 \otimes \ldots \otimes b_q)) \otimes a_{p+1} b_{q+1}
\]

Since \( \oplus_{n \geq 2} A^{\otimes n} = A \otimes TA \otimes A \) is generated by \( A \otimes A \) and \( 1 \otimes TA \otimes 1 \), we check on generators. For \( a \otimes b \in A \otimes A \), \( b'(a \otimes b) = 0 \), in particular, it satisfies Leibnitz rule for elements in \( A \otimes A \). Also, \( b' \) is \( A \)-linear on the left, and right-linear on the right, so

\[
b'((a_0 \otimes a_{n+1}) \ast (1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)) = b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1})
\]

\[
= a_0 b'(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)a_{n+1} = (a_0 \otimes a_{n+1}) \ast b'(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)
\]
and 

\[ A \text{ is a differential graded algebra morphism, } T_V \]

**Corollary 42.** There exists a comparison morphism \( f : (B, d) \to (A \otimes TA \otimes A, b') \) which is a differential graded algebra morphism, \( f(d) = b'(f) \), simply defining it on \( e_x (x \in X) \) and verifying \( f(x' - x) = b'(f(e_x)) \).

**Proof.** Define \( f \) on \( e_x \), extend \( k \)-linearly to \( V \), multiplicatively to \( TV \), and \( A' \)-\( A \) linearly to \( A' \otimes TV \otimes A = B \). In order to see that \( f \) commutes with the differential, by \( A' \)-\( A \)-linearity it suffices to check on \( TV \), but since \( f \) is multiplicative on \( TV \) it is enough to check on \( V \), and by \( k \)-linearity we check on basis, that is, we only need \( f(de_x) = b'f(e_x) \). \( \square \)

**Corollary 43.** \( f|_{TX} \) is the quantum symmetrizer map, and therefore \( \text{Ker}(f) \cap TX \subset B \) defines the Nichol’s ideal associated to \( -\sigma \).

**Proof.**

\[
f(e_{x_1} \cdots e_{x_n}) = f(e_{x_1}) \cdots f(e_{x_n}) = (1 \otimes x_1 \otimes 1) \cdots (1 \otimes x_n \otimes 1) = 1 \otimes (x_1 \sqcup \cdots \sqcup x_n) \otimes 1
\]

\( \square \)

The previous corollary explains why \( \text{Ker}(\text{Id} - \sigma) \subset B_2 \) gives a Hopf ideal and also ends the proof of Theorem 32.

**Question 44.** \( \text{Im}(f) = A \otimes \mathcal{B} \otimes A \) is a resolution of \( A \) as a \( A \)-bimodule? namely, is \( (A \otimes \mathcal{B} \otimes A, d) \) acyclic?
This is the case for involutive solutions in characteristic zero, but also for \( \sigma = \text{flip} \) in any characteristic, and \( \sigma = \text{Id} \) (notice this Id-case gives the Koszul resolution for the tensor algebra). If the answer to that question is yes, and \( \mathcal{B} \) is finite dimensional then \( A \) have necessarily finite global dimension. Another interesting question is how to relate generators for the relations defining \( \mathcal{B} \) and cohomology classes for \( X \).

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