High-energy expansion of Coulomb corrections to the $e^+e^-$ photoproduction cross section

R.N. Lee, A. I. Milstein, and V.M. Strakhovenko

Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia

(Dated: August 8, 2018)

Abstract

First correction to the high-energy asymptotics of the total $e^+e^-$ photoproduction cross section in the electric field of a heavy atom is derived with the exact account of this field. The consideration is based on the use of the quasiclassical electron Green function in an external electric field. The next-to-leading correction to the cross section is discussed. The influence of screening on the Coulomb corrections is examined in the leading approximation. It turns out that the high-energy asymptotics of the corresponding correction is independent of the photon energy. In the region where both produced particles are relativistic, the corrections to the high-energy asymptotics of the electron (positron) spectrum are derived. Our results for the total cross section are in good agreement with experimental data for photon energies down to a few $MeV$. In addition, the corrections to the bremsstrahlung spectrum are obtained from the corresponding results for pair production.

PACS numbers: 32.80.-t, 12.20.Ds

Keywords: $e^+e^-$ photoproduction, bremsstrahlung, Coulomb corrections, screening

*Electronic address: R.N.Lee@inp.nsk.su
†Electronic address: A.I.Milstein@inp.nsk.su
‡Electronic address: V.M.Strakhovenko@inp.nsk.su
I. INTRODUCTION

Knowledge of the photoabsorption cross sections is very important in various applications, see, e.g., [1]. The relevant processes are the atomic photoeffect, nuclear photoabsorption, incoherent and coherent photon scattering and $e^+e^-$ pair production. In the coherent processes, by definition, there is no excitation or ionization of an atom. The high-accuracy estimation of the corresponding cross sections is required. They have different dependence on the photon energy $\omega$. At $\omega \gtrsim 10\text{MeV}$, the cross section of $e^+e^-$ pair production becomes dominant [2]. The coherent contribution $\sigma_{coh}$ to the pair production cross section is roughly $Z$ times larger than the incoherent one ($Z$ is the atomic number), thereby being the most important for heavy atoms. Just the coherent pair production is considered below.

The theoretical and experimental investigation of the coherent pair production has a long history, see [2]. In the Born approximation, the cross section $\sigma_B$ is known for arbitrary photon energy [3, 4]. The account of the effect of screening is straightforward in this approximation and can be easily performed if the atomic form factor is known [5]. For heavy atoms it is necessary to take into account the Coulomb corrections $\sigma_C$,

$$\sigma_{coh} = \sigma_B + \sigma_C. \quad (1)$$

These corrections are higher order terms of the perturbation theory with respect to the atomic field. The magnitude of $\sigma_C$ depends on $\omega$ and the parameter $Z\alpha$ ($\alpha = 1/137$ is the fine-structure constant). The formal expression for $\sigma_C$, exact in $Z\alpha$ and $\omega$, was derived by Øverbø et al. [6]. This expression has a very complicated form causing severe difficulties in computations. The difficulties grow as $\omega$ increases, so that numerical results in [6] were obtained only for $\omega < 5\text{MeV}$.

In the high-energy region $\omega \gg m$ ($m$ is the electron mass, $\hbar = c = 1$), the consideration is greatly simplified. As a result, a rather simple form was obtained in [7, 8] for the Coulomb corrections in the leading approximation with respect to $m/\omega$. However, the theoretical description of the Coulomb corrections at intermediate photon energies ($5 \div 100\text{MeV}$) has not been completed. At present, all estimates of $\sigma_C$ in this region are based on the ”bridging” expression derived by Øverbø [9]. This expression is actually an extrapolation of the results obtained for $\omega < 5\text{MeV}$. It is based on some assumptions on the form of the asymptotic expansion of $\sigma_C$ at high photon energy. It is commonly believed that the ”bridging” ex-
pression has an accuracy providing the maximum error in $\sigma_{coh}$ of the order of a few tens of percent.

Here we develop a description of $e^+e^-$-pair production at intermediate photon energies by deriving the next-to-leading term of the high-energy expansion of $\sigma_C$. First we consider a pure Coulomb field and represent $\sigma_C$ in the form

$$\sigma_C = \sigma_C^{(0)} + \sigma_C^{(1)} + \sigma_C^{(2)} + \ldots$$

The term $\sigma_C^{(n)}$ has the form $(m/\omega)^n S^{(n)}(\ln \omega/m)$, where $S^{(n)}(x)$ is some polynomial. The $\omega$-independent term $\sigma_C^{(0)}$ corresponds to the result of Davies et al. [8]. In the present paper we derive the term $\sigma_C^{(1)}$. It turns out that $S^{(1)}$ is $\omega$-independent in contrast to a second-degree polynomial suggested by Øverbø [9]. We present an ansatz for $\sigma_C^{(2)}$, which provides a good agreement with available experimental data for $\omega > 5\text{MeV}$.

The high-energy expansion of the Coulomb corrections to the spectrum has the form similar to (2). In the region $\varepsilon_+ \gg m$, we derive the term $d\sigma_C^{(1)}/dx$, where $\varepsilon_-$ and $\varepsilon_+$ are the electron and positron energy, respectively, $x = \varepsilon_-/\omega$. The term $d\sigma_C^{(1)}/dx$ may turn important, e.g., for description of the development of electromagnetic showers in a medium. The correction found is antisymmetric with respect to the permutation $\varepsilon_+ \leftrightarrow \varepsilon_-$ and does not contribute to the total cross section. In fact, $\sigma_C^{(1)}$ originates from two energy regions $\varepsilon_+ \sim m$ and $\varepsilon_- \sim m$, where the spectrum is not known. However, our result for $\sigma_C^{(1)}$ allows us to claim that the spectrum in these regions differs drastically from the result obtained by Davies et al. [8] for $\varepsilon_+ \gg m$, if the latter is formally applied at $\varepsilon_- \sim m$ or $\varepsilon_+ \sim m$.

The effect of screening on $\sigma_C$ at $\omega \gg m$ is considered quantitatively. In the leading approximation, we find the corresponding correction $\sigma_C^{(\text{scr})}$, which is $\omega$-independent similar to $\sigma_C^{(0)}$. So, for the atomic field, $\sigma_C^{(\text{scr})}$ should be added to the right-hand side of Eq. (2). The screening correction to the spectrum is also obtained.

In this paper we present the explicit calculations of the corrections, which have been given without derivation in our recent work [10] and used for the detailed comparison of theory with experimental data.
II. GENERAL DISCUSSION

The cross section of $e^+e^-$ pair production by a photon in an external field reads

$$d\sigma_{coh} = \frac{\alpha}{(2\pi)^3} \omega \int dp \, dq \, \delta(\omega - \varepsilon_+ - \varepsilon_-) |M|^2,$$

where $\varepsilon_+ = \varepsilon_p = \sqrt{p^2 + m^2}$, $\varepsilon_-$, and $p$, $q$ are the electron and positron momenta, respectively. The matrix element $M$ has the form

$$M = \int dr \, \bar{\psi}_p^{(+)}(r) \hat{e} \psi_q^{(-)}(r) \exp(ikr).$$

Here $\psi_p^{(+)}$ and $\psi_q^{(-)}$ are positive-energy and negative-energy solutions of the Dirac equation in the external field, $e_\mu$ is the photon polarization 4-vector, $k$ is the photon momentum, $\hat{e} = e_\mu \gamma^\mu$, $\gamma^\mu$ are the Dirac matrices. It is convenient to study various processes in external fields using the Green function $G(r_2, r_1 | \varepsilon)$ of the Dirac equation in this field. This Green function can be represented in the form

$$G(r_2, r_1 | \varepsilon) = \sum_n \frac{\psi_n^{(+)}(r_2) \bar{\psi}_n^{(+)}(r_1)}{\varepsilon - \varepsilon_n + i0} + \int \frac{dp}{(2\pi)^3} \left[ \frac{\psi_p^{(+)}(r_2) \bar{\psi}_p^{(+)}(r_1)}{\varepsilon - \varepsilon_p + i0} + \frac{\psi_p^{(-)}(r_2) \bar{\psi}_p^{(-)}(r_1)}{\varepsilon + \varepsilon_p - i0} \right],$$

where $\psi_n^{(+)}$ is the discrete-spectrum wave function, $\varepsilon_n$ is the corresponding binding energy. The regularization of denominators in (5) corresponds to the Feynman rule. From (5),

$$\int \Omega_q \, \psi_q^{(-)}(r_2) \bar{\psi}_q^{(-)}(r_1) = -i(2\pi)^2 \frac{q^{\varepsilon_q}}{q^{\varepsilon_q}} \delta G(r_2, r_1 | -\varepsilon_q),$$

$$\int \Omega_p \, \psi_p^{(+)}(r_1) \bar{\psi}_p^{(+)}(r_2) = i(2\pi)^2 \frac{p^{\varepsilon_p}}{p^{\varepsilon_p}} \delta G(r_1, r_2 | \varepsilon_p),$$

where $\Omega_p$ is the solid angle of $p$, and $\delta G = G - \tilde{G}$. The function $\tilde{G}$ is obtained from $G$ by the replacement $i0 \leftrightarrow -i0$.

Taking the integrals over $\Omega_p$ and $\Omega_q$ in (3), we obtain the electron spectrum, which is the cross section differential with respect to the electron energy $\varepsilon_-$. Using relations (5), we express this spectrum via the Green functions:

$$\frac{d\sigma_{coh}}{d\varepsilon_-} = \frac{\alpha}{\omega} \int dr_1 \, dr_2 \, e^{-ikr} \, \text{Sp} \{ \delta G(r_2, r_1 | \varepsilon_-) \hat{e} \delta G(r_1, r_2 | -\varepsilon_+) \hat{e} \},$$

where $r = r_2 - r_1$ and $\varepsilon_+ = \omega - \varepsilon_-$ is the positron energy. Since the spectrum is independent of the photon polarization, here and below we assume $e^* = e$ (linear polarization).
Due to the optical theorem, the process of pair production is related to the process of Delbrück scattering (coherent scattering of a photon in the electric field of an atom via virtual electron-positron pairs). At zero scattering angle, the amplitude $M_D$ of Delbrück scattering reads

$$M_D = 2i\alpha \int d\varepsilon \int d\mathbf{r}_1 d\mathbf{r}_2 e^{-i\mathbf{k}\mathbf{r}} Sp \{ G(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) \hat{\varepsilon} G(\mathbf{r}_1, \mathbf{r}_2|\varepsilon - \omega) \hat{\varepsilon} \}.$$ (8)

It is necessary to subtract, from the integrand in (8), the value of this integrand at zero external field ($Z\alpha = 0$). Below, such a subtraction is assumed to be made.

It follows from Eqs. (7), (8) and the analytical properties of the Green function that

$$\frac{1}{\omega} \text{Im} M_D = \sigma_{\text{coh}} + \sigma_{\text{bf}}.$$ (9)

Here

$$\sigma_{\text{bf}} = -\frac{2i\pi\alpha}{\omega} \int d\mathbf{r}_1 d\mathbf{r}_2 e^{-i\mathbf{k}\mathbf{r}} \sum_n Sp \{ \rho_n(\mathbf{r}_2, \mathbf{r}_1) \hat{\varepsilon} \delta G(\mathbf{r}_1, \mathbf{r}_2|\varepsilon_n - \omega) \hat{\varepsilon} \},$$

$$\rho_n(\mathbf{r}_2, \mathbf{r}_1) = \lim_{\varepsilon \to \varepsilon_n} (\varepsilon - \varepsilon_n) G(\mathbf{r}_2, \mathbf{r}_1|\varepsilon).$$ (10)

The quantity $\sigma_{\text{bf}}$ coincides with the total cross section of the so-called bound-free pair production when an electron is produced in a bound state. In fact, due to the Pauli principle, there is no bound-free pair production on neutral atoms. Nevertheless, the term $\sigma_{\text{bf}}$ should be kept in the r.h.s. of (9). In a Coulomb field, the total cross section $\sigma_{\text{bf}}$ was obtained in \[11\] for $\omega \gg m$. In this limit, $\sigma_{\text{bf}} \propto 1/m\omega$ and should be taken into account when using the relation (9) for the calculation of the corrections to $\sigma_{\text{coh}}$ from. The main contribution to $\sigma_{\text{bf}}$ comes from the low-lying bound states [11] when screening can be neglected. So, in [9] we can use $\sigma_{\text{bf}}$ obtained in [11].

It is convenient to represent $d\sigma_{\text{coh}}/d\varepsilon_-$ and $M_D$ in another form using the Green function $D(\mathbf{r}_2, \mathbf{r}_1|\varepsilon)$ of the squared Dirac equation,

$$G(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) = \left[ \gamma^0 (\varepsilon - V(\mathbf{r}_2)) - \gamma \mathbf{p}_2 + m \right] D(\mathbf{r}_2, \mathbf{r}_1|\varepsilon), \quad \mathbf{p}_2 = -i\nabla_2$$ (11)

According to [12], we can rewrite Eq. (7) in the form

$$\frac{d\sigma_{\text{coh}}}{d\varepsilon_-} = \frac{\alpha}{2\omega} \int d\mathbf{r}_1 d\mathbf{r}_2 e^{-i\mathbf{k}\mathbf{r}}$$

$$\times Sp\{[(2\mathbf{e}\mathbf{p}_2 - \hat{\varepsilon}\hat{\mathbf{k}})\delta D(\mathbf{r}_2, \mathbf{r}_1|\varepsilon_-)][(2\mathbf{e}\mathbf{p}_1 + \hat{\varepsilon}\hat{\mathbf{k}})\delta D(\mathbf{r}_1, \mathbf{r}_2|\varepsilon_+)]\},$$ (12)
and Eq. (8) as

\[ M_D = i\alpha \int d\varepsilon \int d\mathbf{r}_1 d\mathbf{r}_2 e^{-ikr} \]
\[ \times \text{Sp}\{[(2e\mathbf{p}_2 - \hat{e}\hat{k})D(\mathbf{r}_2, \mathbf{r}_1|\omega - \varepsilon)][(2e\mathbf{p}_1 + \hat{e}\hat{k})D(\mathbf{r}_1, \mathbf{r}_2| - \varepsilon)]\} \]
\[ + 2i\alpha \int d\varepsilon \int dr \text{Sp}D(\mathbf{r}, \mathbf{r}|\varepsilon). \] (13)

The last term in (13) is \( \omega \)-independent, and has no imaginary part. Therefore, it does not contribute to the relation (9).

III. GREEN FUNCTION

To obtain the spectrum (7), (12) and the Delbrück scattering amplitude (8), (13) it is necessary to know the explicit form of the Green function of the Dirac equation in the Coulomb potential \( V(r) = -Z\alpha/r \). An integral representation for \( G(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) \) has been obtained in (13). For \( |\varepsilon| > m \) it has the form

\[ G(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) = \frac{i}{4\pi r_2 r_1 \kappa} \int_0^{\infty} ds \exp[2iZ\alpha s \lambda + i\kappa(r_2 + r_1)\coth s] T \]
\[ T = [1 - (\gamma \cdot \mathbf{n}_2)(\gamma \cdot \mathbf{n}_1)][(\gamma^0 \varepsilon + m)\frac{y}{2} \partial_y S_B - iZ\alpha\gamma^0 \kappa \coth s S_B] \]
\[ + [1 + (\gamma \cdot \mathbf{n}_2)(\gamma \cdot \mathbf{n}_1)](\gamma^0 \varepsilon + m)S_A + imZ\alpha\gamma^0 \gamma \cdot (\mathbf{n}_2 + \mathbf{n}_1)S_B \]
\[ + \frac{i\kappa^2(r_2 - r_1)}{2\sinh^2 s} \gamma \cdot (\mathbf{n}_2 + \mathbf{n}_1)S_B - \kappa \coth s \gamma \cdot (\mathbf{n}_2 - \mathbf{n}_1)S_A. \] (14)

In this formula

\[ S_A = \sum_{l=1}^{\infty} e^{-i\pi\nu} J_{2\nu}(y) l[P_l'(x) + P_{l-1}'(x)], \quad S_B = \sum_{l=1}^{\infty} e^{-i\pi\nu} J_{2\nu}(y)[P_l'(x) - P_{l-1}'(x)], \]
\[ \nu = \sqrt{l^2 - (Z\alpha)^2}, \quad \kappa = \sqrt{\varepsilon^2 - m^2}, \quad \lambda = \varepsilon/\kappa, \]
\[ y = 2\kappa\sqrt{r_2/r_1}\sinh s, \quad x = \mathbf{n}_1 \cdot \mathbf{n}_2, \quad \mathbf{n}_{1,2} = \mathbf{r}_{1,2}/|\mathbf{r}_{1,2}|. \] (15)

\( J_{2\nu}(y) \) are Bessel functions and \( P_l(x) \) are Legendre polynomials, \( P_l'(x) = \partial_x P_l(x) \). The Green function \( D(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) \) can be obtained from (14) by keeping in \( T \) the terms \( \propto m \):

\[ D(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) = \frac{i}{4\pi r_2 r_1 \kappa} \int_0^{\infty} ds \exp[2iZ\alpha s \lambda + i\kappa(r_2 + r_1)\coth s] \]
\[ \times \{[1 - (\gamma \cdot \mathbf{n}_2)(\gamma \cdot \mathbf{n}_1)]\frac{y}{2} \partial_y S_B + [1 + (\gamma \cdot \mathbf{n}_2)(\gamma \cdot \mathbf{n}_1)]S_A \]
\[ + iZ\alpha\gamma^0 \gamma \cdot (\mathbf{n}_2 + \mathbf{n}_1)S_B \}. \] (16)
We are going to derive the high-energy asymptotic expansion of the spectrum in the region \( \varepsilon_\pm \gg m \). For the first two terms of such an expansion the main contribution to the integral in (7), (12) is given by the region \( r = |r_2 - r_1| \sim \omega/m^2 \), see [14, 15]. Let us introduce the variable \( \rho \) as the component of \( r_1 \) (or \( r_2 \)) perpendicular to \( r \):

\[
\rho = \frac{r \times [r_1 \times r_2]}{r^2} \tag{17}
\]

As shown in [15], the main contribution to the Coulomb corrections to the spectrum originates from the region \( \rho \sim 1/m \) and \( \theta, \psi \sim m/\omega \ll 1 \), where \( \theta \) is the angle between the vectors \( r_2 \) and \( -r_1 \), and \( \psi \) is the angle between the vectors \( r \) and \( k \). In this region we have \( \theta \approx r \rho / r_1 r_2 \). The argument of the Legendre polynomials in (15) is \( x = n_1 \cdot n_2 \approx -1 + \theta^2/2 \). Besides, the term \( \kappa (r_2 + r_1) \sim \omega^2/\omega^2 \gg 1 \) in the exponents in (14), (16) is large, and the integral is determined by large \( s \). Then \( \coth s \approx 1 + 2 \exp(-2s) \), and from (16) we have \( \exp(-2s) \sim 1/kr \sim m^2/\omega^2 \). The argument, \( y \), of the Bessel functions in \( S_{A,B} \) can be estimated as \( y \sim kr / \sinh s \gg 1 \).

A simple method of the calculation of \( S_{A,B} \) at \( y \gg 1, 1 + x \approx \theta^2/2 \ll 1, \) and \( y \theta \sim 1 \) has been formulated in the Appendix of [15]. It turns out that the leading term and the first correction are determined by values of \( l \sim y \sim \omega/m \) in sums \( S_{A,B} \). This fact is in agreement with the evident estimate \( l \sim \varepsilon \rho \sim \omega/m \gg 1 \). In the same way as in [15] we obtain for \( S_{A,B} \) with the first correction taken into account

\[
S_A = -\frac{y^2}{8} J_0(y \theta/2) \left[ 1 + i \frac{\pi(Z\alpha)^2}{y} \right], \quad S_B = -\frac{y}{2\theta} J_1(y \theta/2) \left[ 1 + i \frac{\pi(Z\alpha)^2}{y} \right] \tag{18}
\]

Let us pass in (16) from the integration over \( s \) to the integration over \( y \), see (15). We have

\[
\exp[2iZ\alpha \lambda s] \approx \left( \frac{4\kappa \sqrt{T_1 T_2}}{y} \right)^{2iZ\alpha \lambda}, \quad \coth s \approx 1 + \frac{y^2}{8\kappa^2 r_1 r_2}.
\]

Then we obtain

\[
D(r_2, r_1|\varepsilon) = \frac{ie^{i(k_1 + k_2)}}{16\pi \kappa r_2 r_1} \int_0^\infty y \, dy \left( \frac{4\kappa \sqrt{T_1 T_2}}{y} \right)^{2iZ\alpha \lambda} \exp \left[ \frac{i(r_2 + r_1)y^2}{8\kappa r_1 r_2} \right]
\times \left\{ 1 + i\pi(Z\alpha)^2/y \right\} J_0(y \theta/2) + 2iZ\alpha \frac{J_1(y \theta/2)}{y \theta} \alpha \cdot (n_2 + n_1)
\]

\[
+ \pi(Z\alpha)^2 \frac{J_1(y \theta/2)}{y^2 \theta} |n_2 \times n_1| \cdot \Sigma \right\}, \tag{19}
\]

where \( \alpha = \gamma^0 \gamma, \Sigma = (i/2)[\gamma \times \gamma] \). This expression is the quasiclassical Green function of the squared Dirac equation with the first correction taken into account. The leading term
in this expression, as well as the corresponding expression for \( G(r_2, r_1) \), has been derived in [14, 15]. It is convenient to rewrite (19) in another form using the relations

\[
\int_0^\infty q \, dq J_0(q\theta) g(q) = \int \frac{dq}{2\pi} e^{i q \cdot \theta} g(q), \quad \int_0^\infty q \, dq J_1(q\theta) g(q) = -i \int \frac{dq}{2\pi} \frac{(q \cdot \theta)}{q\theta} e^{i q \cdot \theta} g(q),
\]

where \( g(q) \) is an arbitrary function, \( q \) and \( \theta \) are two-dimensional vectors. In our case we direct the vector \( \theta \) along \( \rho \) so that \( \theta = r \rho / r_1 r_2 \). Using (20) we have

\[
D(r_2, r_1|\varepsilon) = \frac{i e^{i k(r_1 + r_2)}}{8\pi r_1 r_2} \int dq \left( \frac{2\kappa \sqrt{r_1 r_2}}{q} \right)^{2i Z\alpha} \exp \left[ \frac{i(r_2 + r_1)q^2}{2\kappa r_1 r_2} + i q \cdot \theta \right] \times \left\{ 1 + \frac{i \pi (Z\alpha)^2}{2q} \right\} \left[ 1 + Z\alpha \frac{q \cdot \theta}{q^2} - \frac{i \pi (Z\alpha)^2}{4q^3} (r/r) \cdot [q \times \Sigma] \right\}.
\]

The leading term of this formula has been obtained in [16, 17]. Let us integrate by parts the term containing \( \alpha \)-matrix and make the change of variable \( q \to \kappa(q - \rho) \). We obtain

\[
D(r_2, r_1|\varepsilon) = \frac{i e^{i k r}}{8\pi r_1 r_2} \int dq \exp \left[ i \frac{\kappa r q^2}{2r_1 r_2} \right] \left( \frac{2\sqrt{r_1 r_2}}{|q - \rho|} \right)^{2i Z\alpha} \times \left\{ 1 + \frac{r}{2r_1 r_2 \alpha \cdot q} \right\} \left( 1 + i \frac{\pi (Z\alpha)^2}{2\kappa |q - \rho|} \right) - \frac{\pi (Z\alpha)^2}{4\kappa^2} \frac{\gamma \cdot (q - \rho)}{|q - \rho|^3} \right\}. \tag{22}
\]

Note that in this formula and below we can set \( \lambda = \text{sign} \, \varepsilon \).

It is easy to check, that within our accuracy the contribution of the last term in braces vanishes after taking the trace in (12). Therefore, this term can be omitted in the problem under consideration. The remaining terms in (22) can be represented in the form

\[
D(r_2, r_1|\varepsilon) = \left[ 1 + \frac{\alpha \cdot (p_1 + p_2)}{2\varepsilon} \right] D^{(0)}(r_2, r_1|\varepsilon), \tag{23}
\]

\[
D^{(0)}(r_2, r_1|\varepsilon) = \frac{i e^{i k r}}{8\pi r_1 r_2} \int dq \exp \left[ i \frac{\kappa r q^2}{2r_1 r_2} \right] \left( \frac{2\sqrt{r_1 r_2}}{|q - \rho|} \right)^{2i Z\alpha} \left( 1 + i \frac{\pi (Z\alpha)^2}{2\kappa |q - \rho|} \right). \tag{24}
\]

The function \( D^{(0)}(r_2, r_1|\varepsilon) \) is nothing but the quasiclassical Green function of the Klein-Gordon equation in the Coulomb field. The function \( \delta D \) in (12) is defined as \( \delta D = D - \tilde{D} \), where \( \tilde{D} \) is obtained from (23) by the replacement \( D^{(0)} \to D^{(0)*} \).

IV. COULOMB CORRECTIONS TO THE SPECTRUM

In this section we consider the Coulomb corrections to the spectrum, \( d\sigma_C/dx \), for \( \varepsilon_\pm \gg m \) taking into account terms of the order \( m/\varepsilon_\pm \). According to [8], the higher order terms of the perturbation theory with respect to the external field (Coulomb corrections) are not
seriously modified by screening. However, this question has not been studied quantitatively so far. The influence of screening on Coulomb corrections is investigated in detail in Section VI. In the present Section we calculate $d\sigma_C/d\varepsilon_-$ in a pure Coulomb field.

Substituting (23) in (12) and taking the trace, we obtain

$$
\frac{d\sigma_C}{d\varepsilon_-} = \frac{4\alpha}{\omega} \Re \int d\mathbf{r}_1 d\mathbf{r}_2 e^{-ik \cdot r} \left\{ 4[\mathbf{e} \cdot \mathbf{p}_2 D^{(0)}_+][\mathbf{e} \cdot \mathbf{p}_1 D^{(0)}_-] - \frac{\omega^2}{\varepsilon_- \varepsilon_+}[\mathbf{e} \cdot (\mathbf{p}_1 + \mathbf{p}_2) D^{(0)}_-][\mathbf{e} \cdot (\mathbf{p}_1 + \mathbf{p}_2) D^{(0)}_+] \right\},
$$

$$
D^{(0)}_+ = D^{(0)}(\mathbf{r}_2, \mathbf{r}_1 | \varepsilon_-), \quad D^{(0)}_- = D^{(0)}(\mathbf{r}_1, \mathbf{r}_2 | - \varepsilon_+). \quad (25)
$$

The terms $\propto D^{(0)} D^{(0)*}$ are omitted in this formula since they do not contribute to the leading term and the correction we are interested in. Besides, we have integrated by parts the terms containing second derivatives of $D^{(0)}$. In this formula and below we assume the subtraction from the integrand the terms of the order $(Z\alpha)^0$ and $(Z\alpha)^2$. Then we use the relation

$$(\mathbf{e} \cdot \mathbf{p}_{1,2}) D^{(0)}(\mathbf{r}_2, \mathbf{r}_1 | \varepsilon) = \frac{i\kappa_2 e^{i k r}}{8\pi^2 r_1 r_2} \int d\mathbf{q} \exp \left[ i \frac{\kappa r q^2}{2 r_1 r_2} \left( \frac{2\sqrt{r_1 r_2}}{q - \rho} \right)^{2iZ\alpha} \right] \times \left( 1 + i \frac{\pi (Z\alpha)^2}{2\kappa} (\mathbf{r} \cdot \mathbf{q}) \right), \quad (26)
$$

and pass from the variables $\mathbf{r}_{1,2}$ to the variables

$$
\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \quad \rho = \mathbf{r} \times [\mathbf{r}_1 \times \mathbf{r}_2] \quad \frac{r^2}{r_2^2}, \quad z = -\frac{(\mathbf{r} \cdot \mathbf{r}_1)}{r_2^2}. \quad (27)
$$

In terms of these variables $d\mathbf{r}_1 d\mathbf{r}_2 = r d\mathbf{r} d\mathbf{\rho} dz$, and within our accuracy $r_1 = rz$, $r_2 = r(1 - z)$. We obtain from (25)

$$
\frac{d\sigma_C}{d\varepsilon_-} = \frac{\alpha\varepsilon_- \varepsilon_+}{16\pi^4 \omega} \Re \int d\mathbf{r} \int_0^1 \frac{dz}{z^2(1 - z)^2} \int d\mathbf{q}_- d\mathbf{q}_+ d\mathbf{\rho} \left( \frac{Q_+}{Q_-} \right)^{2iZ\alpha} \exp \left[ \frac{i\omega r}{2} \left( \psi^2 - \frac{m^2}{\varepsilon_- \varepsilon_+} \right) + i \frac{\varepsilon_- q_+^2 + \varepsilon_+ q_-^2}{2rz(1 - z)} \right] \left[ 1 + \frac{i\pi (Z\alpha)^2}{2} \left( \frac{1}{\varepsilon_- Q_-} + \frac{1}{\varepsilon_+ Q_+} \right) \right] \times \left\{ 4\varepsilon_- \varepsilon_+ \left[ \mathbf{e} \cdot \mathbf{r} + \frac{e \cdot q_+}{1 - z} \right] \left( -\mathbf{e} \cdot \mathbf{r} + \frac{e \cdot q_+}{z} \right) - \frac{\omega^2}{z^2(1 - z)^2} (\mathbf{e} \cdot \mathbf{q}_-)(\mathbf{e} \cdot \mathbf{q}_+) \right\}, \quad (28)
$$

where $Q_{\pm} = |\mathbf{q}_{\pm} - \mathbf{\rho}|$ and $\psi$ is the angle between vectors $\mathbf{r}$ and $\mathbf{k}$. Since $d\sigma_C/d\varepsilon_-$ is independent of the photon polarization, we can replace in (28) $e^i \mathbf{e}^j$ by $\frac{1}{2} \delta^i_1 \delta^j_2 = \frac{1}{2} (\delta^i_2 - k^i k^j / \omega^2)$.

The integral over $\mathbf{\rho}$ can be taken with the help of the relations (see Appendix)

$$
f(Z\alpha) = \frac{1}{2\pi (Z\alpha)^2 q_+^2} \int d\mathbf{\rho} \left[ \left( \frac{Q_+}{Q_-} \right)^{2iZ\alpha} - 1 + 2(Z\alpha)^2 \ln^2 Q_+ / Q_- \right] = \Re [\psi(1 + iZ\alpha) + C],
$$

$$
g(Z\alpha) = \frac{i}{4\pi q} \int \frac{d\mathbf{\rho}}{Q_+} \left[ \left( \frac{Q_+}{Q_-} \right)^{2iZ\alpha} - 1 \right] = Z\alpha \frac{\Gamma(1 - iZ\alpha) \Gamma(1/2 + iZ\alpha)}{\Gamma(1 + iZ\alpha) \Gamma(1/2 - iZ\alpha)}, \quad (29)
$$
where $\psi(t) = d\ln \Gamma(t)/dt$, $C = 0.577...$ is the Euler constant, $q = |q_- - q_+|$. We have

$$
\frac{d\sigma_C}{d\varepsilon_-} = -\frac{\alpha(Z\alpha)^2\varepsilon_-\varepsilon_+}{2\pi^2\omega} \text{Re} \int_0^\infty dr \int_0^\infty r^3 \psi d\psi \int_0^1 \frac{dz}{z^2(1-z)^2} \int dq_- dq_+ \\
\times \exp \left[ \frac{i\omega r}{2} \left( \psi^2 - \frac{m^2}{\varepsilon_-\varepsilon_+} \right) + i\frac{\varepsilon_-q_+^2 + \varepsilon_+q_-^2}{2rz(1-z)} \right] \left[ q^2 f(Z\alpha) + \pi q \left( \frac{g(Z\alpha)}{\varepsilon_+} - \frac{g^*(Z\alpha)}{\varepsilon_-} \right) \right] \\
\times \left\{ 4\varepsilon_-\varepsilon_+ \left( -r^2\psi^2 + \frac{q_-\cdot q_+}{z(1-z)} \right) - \frac{\omega^2}{z^2(1-z)^2} (q_-\cdot q_+) \right\},
$$

(30)

Passing to the variables $\tilde{q} = q_- + q_+, q = q_- - q_+$, we take all integrals in the following order: $d\psi, d\tilde{q}, dq, dr, dz$. The final result for the Coulomb corrections to the spectrum reads

$$
\frac{d\sigma^{(0)}_C}{dx} + \frac{d\sigma^{(1)}_C}{dx} = -4\sigma_0 \left[ 1 - \frac{4}{3} x(1-x) \right] f(Z\alpha) \\
- \frac{\pi^3(1-2x)m}{8x(1-x)\omega} \left( 1 - \frac{3}{2} x(1-x) \right) \text{Re} g(Z\alpha),
$$

(31)

In (31), the term $\propto f(Z\alpha)$ corresponds to the leading approximation $d\sigma^{(0)}_C/dx$ [8], the term $\propto \text{Re} g(Z\alpha)$ is the first correction $d\sigma^{(1)}_C/dx$. In contrast to the leading term, this correction is antisymmetric with respect to the permutation $\varepsilon_+ \leftrightarrow \varepsilon_-$ (or $x \leftrightarrow 1-x$) and, therefore, does not contribute to the total cross section. Besides, the correction is an odd function of $Z\alpha$ due to the charge-parity conservation and the antisymmetry mentioned above. The antisymmetric contribution enhances the production of electrons at $x < 1/2$ and suppresses it at $x > 1/2$. Evidently, the opposite situation occurs for positrons. Qualitatively, such a behavior of the spectrum takes place for any $\omega$ being the most pronounced at low photon energy [6]. At intermediate photon energies, the spectrum (31) essentially differs from that given by the leading approximation. We illustrate this statement in Fig. II where $\sigma_0^{-1}d\sigma_C/dx$ with correction (solid line) and without correction (dashed line) are plotted for $Z = 82$ and $\omega = 50\,\text{MeV}$.

Due to the antisymmetry of $d\sigma^{(1)}_C/dx$ at $\varepsilon_\pm \gg m$, the term $\sigma^{(1)}_C$ in the total cross section may originate only from the energy regions $\varepsilon_- \sim m$ and $\varepsilon_+ = \omega - \varepsilon_- \sim m$. The quasiclassical approximation can not be used directly in these regions, and another approach is needed to calculate the spectrum. We are going to do this elsewhere. However, for the total cross section, it is possible to overcome this difficulty by means of dispersion relations (see Section V).
As known [see, e.g., 18], the spectrum of bremsstrahlung can be obtained from the spectrum of pair production. This can be performed by means of the substitution $\varepsilon_+ \to -\varepsilon$, $\omega \to -\omega'$, and $dx \to ydy$, where $y = \omega' / \varepsilon$; $\omega'$ is the energy of an emitted photon, $\varepsilon$ is the initial electron energy. Using (31), we obtain for the Coulomb corrections to the bremsstrahlung spectrum

$$y \frac{d\sigma_C}{dy} = -4\sigma_0 \left[ \left( y^2 + \frac{4}{3}(1-y) \right) f(Z\alpha) \right. \right.$$

$$\left. - \frac{\pi^3(2-y)m}{8(1-y)\varepsilon} \left( y^2 + \frac{3}{2}(1-y) \right) \text{Re} \, g(Z\alpha) \right].$$

(32)

This formula describes bremsstrahlung from electrons. For the spectrum of photons emitted by positrons, it is necessary to change the sign of $Z\alpha$ in (32). Our result (32) coincides with that obtained in [19] if the obvious mistake in the latter is corrected by changing

$$\frac{1}{\gamma} \to \frac{1}{2} \left( \frac{m}{\varepsilon} + \frac{m}{\varepsilon - \omega'} \right) = \frac{(2-y)m}{2(1-y)\varepsilon}$$

in Eq.(22) of [19]. The correction (32) is the most important at $y$ close to unity, see Fig. 2 where $\sigma_0^{-1}y\sigma_C^C/dy$ with correction (solid line) and without correction (dashed line) are shown for $Z = 82$ and $\varepsilon = 50\,\text{MeV}$.

FIG. 1: The dependence of $\sigma_0^{-1}d\sigma_C/dx$ on $x$, see (31), for $Z = 82$, $\omega = 50\,\text{MeV}$. Dashed curve: leading approximation; solid curve: first correction is taken into account.
V. COULOMB CORRECTIONS TO THE TOTAL CROSS SECTION

In the leading approximation, the Coulomb corrections, \( \sigma^{(0)}_C \), to the total cross section of pair production for \( \omega \gg m \) were obtained in [8]. Using this result and dispersion relations, the corresponding term, \( M^{(0)}_{DC} \), in the forward Delbrück scattering amplitude \( M_D \) was obtained in [20]. These two quantities read:

\[
\sigma^{(0)}_C = -\frac{28}{9} \sigma_0 f(Z\alpha), \quad M^{(0)}_{DC} = -i\frac{28}{9} \omega \sigma_0 f(Z\alpha), \quad (33)
\]

where \( \sigma_0 \) and \( f(Z\alpha) \) are defined in (31) and (29), respectively.

In this Section we derive the correction \( \sigma^{(1)}_C \) by means of the relation (9). Starting from (13) and performing the same calculations as in the previous Section we obtain

\[
M^{(0)}_{DC} + M^{(1)}_{DC} = -4i\omega \sigma_0 \int_0^1 dx \left[ \left(1 - \frac{4}{3}x(1-x)\right) f(Z\alpha) - \frac{\pi^3 m}{8\omega} \left(1 - \frac{3}{2}x(1-x)\right) \left(\frac{g^*(Z\alpha)}{x} - \frac{g(Z\alpha)}{1-x}\right) \right]. \quad (34)
\]

Here the integration over \( x \) corresponds to the integration over \( \varepsilon/\omega \). After the integration the \( M^{(0)}_{DC} \) in (31) coincides with that in (33). The integral in \( M^{(1)}_{DC} \) is logarithmically divergent. Note that we have obtained the integrand in (33) under the conditions \( x \gg m/\omega \) and
Taking the integral from $\delta$ to $1 - \delta$, where $\delta \gtrsim m/\omega$, we find within logarithmic accuracy that $\text{Im} M^{(1)}_{\text{DC}}$ vanishes and

$$\text{Re} M^{(1)}_{\text{DC}} = \frac{\alpha(Z\alpha)^2 \pi^3 \text{Im} g(Z\alpha)}{m} \ln \frac{\omega}{m}. \quad (35)$$

The quantity $\text{Im} M^{(1)}_{\text{DC}}$ does not contain $\ln(\omega/m)$ and is determined by the regions of integration over $\varepsilon$, where $\varepsilon \sim m$ and $\omega - \varepsilon \sim m$, and, therefore, the quasiclassical approximation is invalid. Nevertheless, this quantity, which is related to $\sigma^{(1)}_C$ [2], can be obtained from the dispersion relation for $M_D$ [20]:

$$\text{Re} M_D(\omega) = \frac{2}{\pi} \omega^2 \text{P} \int_0^\infty \frac{\text{Im} M_D(\omega') \, d\omega'}{\omega' (\omega'^2 - \omega^2)}. \quad (36)$$

Using this relation, it can be easily checked that the high-energy asymptotics (35) unambiguously corresponds to the $\omega$-independent high-energy asymptotics

$$\text{Im} M^{(1)}_{\text{DC}} = -\frac{\alpha(Z\alpha)^2 \pi^4 \text{Im} g(Z\alpha)}{2m}, \quad (37)$$

Substituting (37) into (2) and using $\sigma_{bf}$ from [11] in the form

$$\sigma_{bf} = 4\pi\sigma_0 (Z\alpha)^3 f_1(Z\alpha) \frac{m}{\omega}, \quad (38)$$

we have for $\sigma^{(1)}_C$

$$\sigma^{(1)}_C = -\sigma_0 \left[ \frac{\pi^4}{2} \text{Im} g(Z\alpha) + 4\pi (Z\alpha)^3 f_1(Z\alpha) \right] \frac{m}{\omega}. \quad (39)$$

The function $f_1(Z\alpha)$ is plotted in Fig. 3.

The quantity $(\omega/m)\sigma^{(1)}_C/\sigma^{(0)}_C$ is shown in Fig. 4 (solid curve). It is seen that this ratio is numerically large for any $Z$. Therefore, the term $\sigma^{(1)}_C$ gives a significant contribution to $\sigma_C$ for intermediate photon energies. Dashed curve in Fig. 4 gives the same ratio when $\sigma_{bf}$ in (39) is omitted. It is seen that the relative contribution of the term $\propto f_1(Z\alpha)$ in (39) is numerically small.

**VI. SCREENING CORRECTIONS**

In two previous Sections the cross section of $e^+e^-$ pair production has been considered for a pure Coulomb field. The difference, $\delta V(r)$, between an atomic potential and a Coulomb potential of a nucleus leads to the modification of this cross section known as the effect of
screening. In the Born approximation, this effect was studied long ago [see, e.g., 5]. Let us consider now $\sigma^{(\text{scr})}_C$ characterizing the influence of screening on the Coulomb corrections. Recollect that the Coulomb corrections denote the higher-order terms of the perturbation theory with respect to the atomic field. So far it was only known that the correction $\sigma^{(\text{scr})}_C$
is not large\[8\]. Here we consider this issue quantitatively.

The quasiclassical Green function \(D^{(0)}(r_2, r_1|\varepsilon)\) for an arbitrary localized potential \(V(r)\) has been obtained in \[21\] with the first correction taken into account. The leading term has the form (see also \[12\])

\[
D^{(0)}(r_2, r_1|\varepsilon) = \frac{i\kappa e^{i\sigma r_1 r_2}}{8\pi^2r_1 r_2} \int dq \exp \left[ i\kappa r q^2 - i\lambda r \int_0^1 dx V (r_1 + x r - q) \right]. \tag{40}
\]

Substituting this formula into (25), we obtain (cf. (28))

\[
\begin{align*}
\frac{d\sigma_C}{d\varepsilon_-} &= -\frac{\alpha\varepsilon_-}{16\pi^4\omega} \Re \int \frac{dr}{r^5} \int_0^1 \frac{dz}{z^2(1-z)^2} \int \int dq_-dq_+ dp \\
& \times \exp \left\{ i\Phi + \frac{i\omega r}{2} \left( \psi^2 - \frac{\varepsilon_-q_+^2}{\varepsilon_-\varepsilon_+} + i\frac{\varepsilon_-q_+^2 + \varepsilon_+q_+^2}{2r(z(1-z)} \right) \right\} \\
& \times \left\{ 4\varepsilon_-\varepsilon_+ \left( e \cdot r + \frac{e \cdot q_+}{1-z} \right) \left( e \cdot r + \frac{e \cdot q_+}{z} \right) - \frac{\omega_2^2}{z^2(1-z)}(e \cdot q_-(e \cdot q_+) \right\},
\end{align*}
\]

\[
\Phi = r \int_0^1 dx [V(r_1 + x r - q_+) - V(r_1 + x r - q_-)]. \tag{41}
\]

The phase \(\Phi\) can be represented as

\[
\Phi = 2Z\alpha \ln(Q_+/Q_-) + \Phi^{(scr)}
\]

\[
= 2Z\alpha \ln(Q_+/Q_-) + r \int_0^1 dx [\delta V(r_1 + x r - q_+) - \delta V(r_1 + x r - q_-)]. \tag{42}
\]

As in the case of a pure Coulomb field, the main contribution to the Coulomb corrections comes from the region of integration \(q_\pm \sim \rho \sim 1/m\). The main contribution to the integral over \(x\) in (42) comes from the narrow region around the point \(x_0 = -r_1 \cdot r/r^2 = z, \delta x = \rho/r \ll 1\). Therefore, it is possible to perform the integration in (12) from \(-\infty\) to \(\infty\). Thus we can estimate \(\Phi^{(scr)}\) as \(\Phi^{(scr)} \sim \rho \delta V(\rho) \sim Z\alpha \delta V(\rho)/V(\rho) \ll 1\). In our calculation of \(\sigma_C^{(scr)}\), we retain the linear term of expansion in \(\Phi^{(scr)}\). By definition,

\[
\delta V(r) = \int \frac{d\Delta}{(2\pi)^3} e^{i\Delta r} F(\Delta) \frac{4\pi Z\alpha}{\Delta^2}, \tag{43}
\]

where \(F(Q)\) is the atomic electron form factor. Substituting this formula to (12) and taking the integral over \(x\) from \(-\infty\) to \(\infty\), we obtain for \(\Phi^{(scr)}\)

\[
\Phi^{(scr)} = \int \frac{d\Delta_\perp}{(2\pi)^2} \left( e^{i\Delta_\perp(\rho - q_+)} - e^{i\Delta_\perp(\rho - q_-)} \right) F(\Delta_\perp) \frac{4\pi Z\alpha}{\Delta_\perp^2}, \tag{44}
\]

where \(\Delta_\perp\) is two-dimensional vector lying in the plane perpendicular to \(r\). Then we use the identity, see Eqs. (22), (23) in \[17\]

\[
\int d\rho \left( \frac{\rho - q_+}{\rho - q_-} \right)^{2iZ\alpha} \exp [i\Delta_\perp \cdot (\rho - q_\pm)]
\]

15
\[ q = q_- - q_+ \] and \[ f_\pm = f \pm \Delta_\perp. \] Expanding (41) in \( \Phi^{(scr)} \) and using the identity (45), we take the integrals over variables \( q_\pm, r, \) and \( z \) and obtain

\[
\frac{d\sigma_C^{(scr)}}{dx} = \frac{8\alpha(Z\alpha)}{3\pi} \int \frac{d\Delta_\perp}{\Delta_\perp^4} F(\Delta_\perp) \int \frac{df}{2\pi} \left( \frac{f_+}{f_-} \right)^{2iZ\alpha} \left[ \frac{R(\xi_-, a)}{f_+^2} - \frac{R(\xi_+, a)}{f_-^2} \right],
\]

where

\[
R(\mu, a) = \frac{(\mu - 1)}{4\mu^2} \left\{ \frac{1}{2\sqrt{\mu}} \left[ 18 - 6\mu + a(\mu^2 + 2\mu - 3) \right] \ln \left[ \frac{\sqrt{\mu} + 1}{\sqrt{\mu} - 1} \right] 
- 18 - a(\mu - 3) \right\},
\]

\[
\xi_\pm = 1 + 16m^2/f_\pm^2, \quad a = 6x(1 - x).
\]

Using the trick introduced in Section IX in [17], we rewrite this formula in another form. Let us multiply the integrand in (46) by

\[
1 = \int_{-1}^{1} dy \delta \left( y - \frac{2f \cdot \Delta_\perp}{f^2 + \Delta_\perp^2} \right)
= \left( f^2 + \Delta_\perp^2 \right) \int_{-1}^{1} \frac{dy}{|y|} \delta((f - \Delta_\perp/y)^2 - \Delta_\perp^2(1/y^2 - 1)),
\]

change the order of integration over \( f \) and \( y \), and make the shift \( f \to f + \Delta_\perp/y \). After that the integration over \( f \) can be done easily. Then we make the substitution \( y = \tanh \tau \) and

![Graph](image-url)

**FIG. 5:** The ratio \( \sigma_C^{(scr)}/\sigma_C^{(0)} \) as a function of \( Z \)
obtain

\[
\frac{d\sigma_C^{(scr)}}{dx} = \frac{32}{3} \sigma_0 m^2 \int_0^\infty \frac{dQ}{Q^3} F(Q) \int_0^\infty \frac{d\tau}{\sinh \tau} \left[ \frac{\sin(2Z\alpha\tau)}{2Z\alpha} - \tau \right] \\
\times \int_0^{2\pi} \frac{d\varphi}{2\pi} \left[ e^{\tau} R(\mu_+, a) - e^{-\tau} R(\mu_-, a) \right],
\]

\[
\mu_\pm = 1 + \frac{8m^2 e^{\pm\tau} \sinh^2 \tau}{Q^2 (\cosh \tau + \cos \varphi)}.
\]

(48)

Integrating over \(x\), we have

\[
\sigma_C^{(scr)} = \frac{32}{3} \sigma_0 m^2 \int_0^\infty \frac{dQ}{Q^3} F(Q) \int_0^\infty \frac{d\tau}{\sinh \tau} \left[ \frac{\sin(2Z\alpha\tau)}{2Z\alpha} - \tau \right] \\
\times \int_0^{2\pi} \frac{d\varphi}{2\pi} \left[ e^{\tau} R(\mu_+, 1) - e^{-\tau} R(\mu_-, 1) \right].
\]

(49)

Similar to \(\sigma_C^{(0)}\), this correction is \(\omega\)-independent. Shown in Fig. 5 is the \(Z\)-dependence of the ratio \(\sigma_C^{(scr)}/\sigma_C^{(0)}\) calculated with the use of the form factors taken from [22]. As seen from Fig. 5, this ratio is approximately fitted by the linear function, \(\sigma_C^{(scr)} \approx -5.4 \cdot 10^{-4} \cdot Z\sigma_C^{(0)}\).

The corresponding correction to the bremsstrahlung spectrum is obtained from (48) by means of the same substitutions as in Section IV. So that the quantity \(y^{-1}d\sigma_C^{(scr)}/dy\) is given by the right-hand side of (48) if we set \(a = 6(y - 1)/y^2\).

**VII. ESTIMATION OF \(\sigma_C^{(2)}\) FROM EXPERIMENTAL DATA**

The most detailed and accurate experimental data have been obtained just in the region of intermediate photon energies. In this region, the first correction \(\sigma_C^{(1)}\), obtained above, becomes large, see Fig. 4 for \((\omega/m)\sigma_C^{(1)}/\sigma_C^{(0)}\), and the next term \(\sigma_C^{(2)}\) in the expansion \(\sigma_C\) may be significant. Using the arguments similar to those presented by Davies et al. [8] the following ansatz for \(\sigma_C^{(2)}\) has been suggested in our recent paper [10]

\[
\sigma_C^{(2)} = \sigma_0 \left[ b \ln(\omega/2m) + c \right] \left( \frac{m}{\omega} \right)^2,
\]

(50)

where \(b\) and \(c\) are some functions of \(Z\alpha\). It was shown in [10] that experimental data for \(\sigma_{coh}\) are well described by the formula

\[
\sigma_{coh} = \sigma_B + \sigma_C^{(0)} + \sigma_C^{(scr)} + \sigma_C^{(1)} + \sigma_C^{(2)},
\]

(51)

where \(\sigma_C^{(0)}\), \(\sigma_C^{(scr)}\), and \(\sigma_C^{(1)}\) are from Eqs. (33), (49), and (39), respectively; \(\sigma_C^{(2)}\) is given by (50) with \(b = -3.78(\omega/m)\sigma_0^{-1} \sigma_C^{(1)}\), \(c = 0\).
It is interesting to compare our predictions for the Coulomb corrections to the total cross section with the results of Øverbø [9]. Shown in Figs. 6 and 7 is the ratio $S = (\sigma_{\text{coh}} - \sigma_B)/\sigma^{(0)}_C$, which is the Coulomb corrections in units of $\sigma^{(0)}_C$, (33). Our results are represented by solid curves, those of Øverbø are shown as dashed curves. The values of $S$ extracted from

![Graph](image1)

**FIG. 6:** The $\omega$-dependence of $S = (\sigma_{\text{coh}} - \sigma_B)/\sigma^{(0)}_C$ for Bi. Solid curve: our result; dashed curve: the result of Øverbø [9]; experimental data from [23].

![Graph](image2)

**FIG. 7:** Same as Fig. 6 but for Pb; experimental data from [24, 25].
the experimental data are also shown. The results for $Bi$ are plotted in Fig. 6 with the experimental data taken from [23]. The results for $Pb$ are plotted in Fig. 7 with the experimental data taken from [24, 25]. It is seen that the difference between our results and those of Øverbø is small at relatively low energies and becomes noticeable as $\omega$ increases. According to our results, this difference tends to a constant $\sigma_C^{(scr)} / \sigma_C^{(0)}$ at $\omega \to \infty$. The experimental data are, on the whole, in a better agreement with our results than with those of Øverbø.

VIII. CONCLUSION

For the $e^+e^-$ photoproduction, we have calculated the leading correction (31) to the electron spectrum in the region $\varepsilon_\pm \gg m$. This contribution noticeably modifies the spectrum at intermediate photon energy. It turns out that the correction is antisymmetric with respect to the permutation $\varepsilon_+ \leftrightarrow \varepsilon_-$ and hence does not contribute to the total cross section. The leading correction to the total cross section, $\sigma_C^{(1)}$, originates from two regions, $\varepsilon_+ \sim m$ and $\varepsilon_- \sim m$. We have obtained $\sigma_C^{(1)}$ (39) using dispersion relations. In contrast to the form of the fit suggested by Øverbø [9], the quantity $\sigma_C^{(1)}$ does not contain any powers of $\ln(\omega/m)$.

We have also performed the quantitative investigation of the influence of screening on the Coulomb corrections (48), (49). It is important that $\sigma_C^{(scr)}$ does not vanish in the high-energy limit. We have suggested a form for the next-to-leading correction, $\sigma_C^{(2)}$, to the total cross section. Altogether, the corrections found allow one to represent well the available experimental data.

Starting with the results obtained for the $e^+e^-$ photoproduction spectrum, we have obtained the corresponding corrections to the bremsstrahlung spectrum as well.

Acknowledgments

We are indebted to J.H. Hubbell for his continuing interest to this work. This work was supported in part by RFBR Grants 01-02-16926 and 03-02-16510.
In this appendix we derive the formulas (29). In the integral for \( f(Z\alpha) \) let us make the change of variables \( \rho \to (\rho + q + q^-)/2 \):

\[
f(Z\alpha) = \frac{1}{8\pi(Z\alpha)^2q^2} \int d\rho \left[ \frac{2iZ\alpha}{\rho - q} \right]^{2iZ\alpha} - 1 + 2(Z\alpha)^2 \ln \frac{\rho + q}{\rho - q} \]  

(A1)

Let us multiply the integrand in (46) by

\[
1 \equiv \int_{-1}^{1} dy \delta \left( y - \frac{2\rho \cdot q}{\rho^2 + q^2} \right) = (\rho^2 + q^2) \int_{-1}^{1} dy \frac{1}{|y|} \delta ((\rho - q/y)^2 - q^2(1/y^2 - 1)),
\]

(A2)

change the order of integration over \( \rho \) and \( y \), and make the shift \( \rho \to \rho + q/y \). Then the integral over \( \rho \) becomes trivial and we have

\[
f(Z\alpha) = \frac{1}{4(Z\alpha)^2} \int_{-1}^{1} dy \left[ \left( \frac{1 + y}{1 - y} \right)^{iZ\alpha} - 1 + \frac{(Z\alpha)^2}{2} \ln \left( \frac{1 + y}{1 - y} \right) \right]
\]

(A3)

Then we make the substitution \( y = \tanh \tau \) and obtain

\[
f(Z\alpha) = \frac{1}{2(Z\alpha)^2} \int_{0}^{\infty} d\tau \cosh \frac{\tau}{\sinh^{3/2} \tau} \left[ \cos(2Z\alpha\tau) - 1 + 2(Z\alpha)^2 \tau^2 \right]
= \int_{0}^{\infty} d\tau e^{-\tau} \sinh \tau \left[ 1 - \cos(2Z\alpha\tau) \right] = \text{Re}[\psi(1 + iZ\alpha) + C].
\]

(A4)

In order to calculate the function \( g(Z\alpha) \) we make a shift \( \rho \to \rho + q_+ \) in (29) and use the exponential parameterization

\[
A^\nu = \frac{e^{i\pi\nu/2}}{\Gamma(-\nu)} \int_{0}^{\infty} ds \frac{ds}{s^{1+\nu}} \exp[iAs].
\]

(A5)

Then we have

\[
g(Z\alpha) = \frac{ie^{iZ\alpha/2}}{4\pi q \Gamma(iZ\alpha)} \int d\rho \rho^{-1+2iZ\alpha} \int_{0}^{\infty} ds s^{-1+iZ\alpha} \left\{ \exp[i\rho (\rho - q)] - \exp[i\rho^2] \right\}.
\]

(A6)

Taking the integral over the angles of two-dimensional vector \( \rho \) and making the substitutions \( \rho \to \rho/\sqrt{s} \), \( s \to s/q^2 \) we come to

\[
g(Z\alpha) = \frac{ie^{iZ\alpha/2}}{2\Gamma(iZ\alpha)} \int_{0}^{\infty} d\rho \rho^{2iZ\alpha} \exp[i\rho^2] \int_{0}^{\infty} ds s^{-3/2} \left[ e^{i\rho_0^2(2\rho\sqrt{s} - 1)} \right].
\]

(A7)
Taking the integrals over $s$ and then over $\rho$ we finally obtain

$$g(Z\alpha) = Z\alpha \frac{\Gamma(1-iZ\alpha)\Gamma(1/2+iZ\alpha)}{\Gamma(1+iZ\alpha)\Gamma(1/2-iZ\alpha)}.$$  \hspace{1cm} (A8)

[1] J. H. Hubbell, Rad. Phys. Chem. 59, 113 (2000).
[2] J. H. Hubbell, H. A. Gimm, and I. Øverbø, J. Phys. Chem. Rev. Data 9, 1023 (1980).
[3] H. A. Bethe and W. Heitler, Proc. R. Soc. London A146, 83 (1934).
[4] G. Racah, Nuovo Cim. 11, 461 (1934).
[5] R. Jost, J. M. Luttinger, and M. Slotnick, Phys. Rev. 80, 189 (1950).
[6] I. Øverbø, K. J. Mork, and H. A. Olsen, Phys. Rev. 175, 1978 (1968).
[7] H. A. Bethe and L. C. Maximon, Phys. Rev. 93, 768 (1954).
[8] H. Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. 93, 788 (1954).
[9] I. Øverbø, Phys. Lett. B71, 412 (1977).
[10] R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, hep-ph/0307388.
[11] A. I. Milstein and V. M. Strakhovenko, JETP 76, 775 (1993), [Zh. Eksp. Teor. Fiz. 103 (1993) 1584].
[12] R. N. Lee and A. I. Milstein, Phys. Lett. A198, 217 (1995).
[13] A. I. Milstein and V. M. Strakhovenko, Phys. Lett. A90, 447 (1982).
[14] A. I. Milstein and V. M. Strakhovenko, Phys. Lett. A95, 135 (1983).
[15] A. I. Milstein and V. M. Strakhovenko, JETP 58, 8 (1983), [Zh. Eksp. Teor. Fiz. 85 (1983) 14].
[16] R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, JETP 85, 1049 (1997), [Zh. Eksp. Teor. Fiz. 112 (1997) 1921].
[17] R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, PRA 57, 2325 (1998).
[18] V. B. Berestetski, E. M. Lifshits, and L. P. Pitayevsky, Quantum electrodynamics (Pergamon, Oxford, 1982).
[19] V. N. Baier and V. M. Katkov, Sov. Phys. Docl. 21, 150 (1976), [Docl. Akad. Nauk SSSR, 227 (1976) 325].
[20] F. Rohrlich, Phys. Rev. 108, 169 (1957).
[21] R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, JETP 90, 66 (2000), [Zh. Eksp. Teor. Fiz. 117 (2000) 75].

[22] J. H. Hubbell and I. Øverbø, J. Phys. Chem. Ref. Data 8, 69 (1979).

[23] N. K. Sherman, C. K. Ross, and K. H. Lokan, Phys. Rev. C 21, 2328 (1980).

[24] E. S. Rosenblum, E. F. Shrader, and R. M. Warner, Jr., Phys. Rev. 88, 612 (1952).

[25] H. A. Gimm and J. H. Hubbell, NBS technical note 968, National Bureau of Standards (1978).