THE MATSUMOTO-YOR PROPERTY AND ITS CONVERSE ON SYMMETRIC CONES

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Abstract. The Matsumoto-Yor (MY) property of the generalized inverse Gaussian and gamma distributions has many generalizations. As observed in [Letac - Wesołowski, Ann. Probab. 28 (2000), 1371–1383] the natural framework for multivariate MY property are symmetric cones, however they prove the result for the cone of symmetric positive definite real matrices only.

In this paper we prove the converse to the symmetric cone-variate MY property, which extends earlier results. The smoothness assumption for densities of respective variables is reduced to continuity only. It was possible due to the new solution of a related functional equation for real functions defined on symmetric cones, which seems to be of independent interest.

1. Introduction

Matsumoto and Yor [2001, 2003] have shown that, if \( X \) and \( Y \) are independent random variables, \( Y \) is gamma distributed with shape parameter \( p \) and a scale parameter \( a \) and \( X \) has a generalized inverse Gaussian distribution (GIG) with parameters \((-p,a,b)\), then the random variables \( U = (X + Y)^{-1} \) and \( V = X^{-1} - (X + Y)^{-1} \) are independent with respective distributions GIG with parameters \((-p,b,a)\) and gamma with parameters \( p \) and \( b \).

Matsumoto and Yor asked about the converse theorem based on the independence of \( U \) and \( V \). Assume that \( X \) and \( Y \) are non-degenerate non-negative independent random variables, such that \( U \) and \( V \) are independent. Does this imply that \( X \) and \( Y \) must follow GIG and gamma distributions, respectively?

A positive answer to this question (in the univariate case) was given by Letac and Wesołowski [2000], with the use of Laplace transforms. In that article both the Matsumoto-Yor property and its converse (with additional smoothness assumptions) were generalized to the cone \( \Omega_+ \) of symmetric positive definite \((r,r)\) real matrices.

For \( p > (r - 1)/2 \) and \( a, b \in \Omega_+ \), consider two independent random variables \( X \) and \( Y \) with following distributions

\[
\mu_{-p,a,b}(dx) = c_1 (\det x)^{-p-(r+1)/2} \exp \left(-\operatorname{tr}(a \cdot x) - \operatorname{tr}(b \cdot x^{-1})\right) I_{\Omega_+}(x) dx,
\]

\[
\gamma_{p,a}(dy) = c_2 (\det y)^{p-(r+1)/2} \exp(-\operatorname{tr}(a \cdot y))I_{\Omega_+}(y) dy.
\]

The distribution of \( X \) is the GIG with parameters \((-p,a,b)\) and the distribution of \( Y \) is the Wishart distribution with shape parameter \( p \) and scale parameter \( a \). Letac and Wesołowski have shown that, if \( X \) and \( Y \) are as above, then \( U \) and \( V \) (defined as above) are independent with respective distributions GIG with parameters \((-p,b,a)\) and Wishart with parameters \((p,b)\). As was observed by the authors, the natural framework for Matsumoto-Yor property are symmetric cones. Statement of a symmetric-cone version of Matsumoto-Yor property is given in Section 3.

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In the following paper we give the new proof of converse result of Matsumoto-Yor, when \( X \) and \( Y \) are valued in any irreducible symmetric cone. The smoothness assumption is reduced from \( C^2 \) densities in \cite{Letac2000} and differentiability in \cite{Wesolowski2002a} to continuity only. The new solution of related functional equation on symmetric cones (see Theorem 4.5) was found under the assumption of continuity of respective functions with the use of corresponding univariate result due to Wesolowski \cite{Wesolowski2002a}.

It is worth to mention related one dimensional results due to Chou and Huang \cite{Chou2004}, Koudou and Vallois \cite{Koudou2012} as well as results for random matrices in \cite{Letac2000} and Koudou \cite{Koudou2012}.

While solving the functional equation, we use the Hua’s identity, which allows to write \( V^{-1} \) in a more convenient form:

\[
V^{-1} = X + X \cdot Y^{-1} \cdot X.
\]

Hua’s identity already proved useful in problems related to GIG and Wishart distributions - see Bernadac \cite{Bernadac1995}, who used it to analyze random continued fractions on symmetric cones.

The paper is organized as follows. We start in the next section with basic definitions and theorems regarding analysis on symmetric cones. In Section \ref{section:matsumoto-yor} we define GIG and Wishart distributions and state the Matsumoto-Yor property on symmetric cones. A core of the proof of the converse to Matsumoto-Yor property is a solution of some functional equation for real functions in arguments from the cone, which seems to be of independent interest. Section \ref{section:functional-equation} is devoted to consideration of this functional equation. The statement and proof of the main result are given in Section \ref{section:main-result}.

### 2. Preliminaries

In this section we give a short introduction to the theory of symmetric cones. For further details we refer to Faraut and Korányi \cite{Faraut1994}.

A **Euclidean Jordan algebra** is a Euclidean space \( E \) (endowed with scalar product denoted \( \langle x, y \rangle \)) equipped with a bilinear mapping (product)

\[
E \times E \ni (x, y) \mapsto xy \in E
\]

and a neutral element \( e \) in \( E \) such that for all \( x, y, z \) in \( E \):

- \( xy = yx \),
- \( x(y^2) = x^2(y) \),
- \( xe = x \),
- \( (x, yz) = \langle xy, z \rangle \).

For \( x \in E \) let \( L(x) : E \to E \) be linear map defined by

\[
L(x)y = xy,
\]

and define

\[
P(x) = 2L^2(x) - L(x^2).
\]

The map \( P : E \to \text{End}(E) \) is called the *quadratic representation* of \( E \).

An element \( x \) is said to be **invertible** if there exists an element \( y \) in \( E \) such that \( L(x)y = e \). Then \( y \) is called the *inverse* of \( x \) and is denoted by \( y = x^{-1} \). Note that the inverse of \( x \) is unique. It can be shown that \( x \) is invertible if and only if \( P(x) \) is invertible and in this case \( (P(x))^{-1} = P(x^{-1}) \).

Euclidean Jordan algebra \( E \) is said to be **simple** if it is not a Cartesian product of two Euclidean Jordan algebras of positive dimensions. Up to linear isomorphism there are only five kinds of Euclidean simple Jordan algebras. Let \( K \) denote either the real numbers \( \mathbb{R} \), the complex ones \( \mathbb{C} \), quaternions \( \mathbb{H} \) or the octonions \( \mathbb{O} \), and write \( S_r(K) \) for the space of \( r \times r \) Hermitian matrices valued in \( K \), endowed with the Euclidean structure \( (x, y) = \text{Trace}(x \cdot y) \) and with the Jordan product

\[
xy = \frac{1}{2}(x \cdot y + y \cdot x),
\]
where \( \mathbf{x} \cdot \mathbf{y} \) denotes the ordinary product of matrices and \( \mathbf{y} \) is the conjugate of \( \mathbf{y} \). Then \( S_r(\mathbb{R}) \), \( r \geq 1 \), 
\( S_r(\mathbb{C}) \), \( r \geq 2 \), \( S_r(\mathbb{H}) \), \( r \geq 2 \), and the exceptional \( S_3(\mathbb{O}) \) are the first four kinds of Euclidean simple Jordan algebras. Note that in this case

\[
\mathbb{P}(\mathbf{y}) \mathbf{x} = \mathbf{y} \cdot \mathbf{x} \cdot \mathbf{y}.
\]

The fifth kind is the Euclidean space \( \mathbb{R}^{n+1}, n \geq 2 \), with Jordan product

\[
(x_0, x_1, \ldots, x_n) (y_0, y_1, \ldots, y_n) = \left( \sum_{i=0}^{n} x_i y_i, x_0 y_1 + y_0 x_1, \ldots, x_0 y_n + y_0 x_n \right).
\]

To each Euclidean simple Jordan algebra one can attach the set of Jordan squares

\[
\Omega = \{ \mathbf{x} \in \mathbb{E} : \text{there exists } \mathbf{y} \in \mathbb{E} \text{ such that } \mathbf{x} = \mathbf{y}^2 \}.
\]

The interior \( \Omega \) is a symmetric cone. Moreover \( \Omega \) is irreducible, i.e. it is not the Cartesian product of two convex cones. One can prove that an open convex cone is symmetric and irreducible if and only if it is the cone \( \Omega \) of some Euclidean simple Jordan algebra. Each simple Jordan algebra corresponds to a symmetric cone, hence there exist, up to linear isomorphism, also only five kinds of symmetric cones. The cone corresponding to the Euclidean Jordan algebra \( \mathbb{R}^{n+1} \) equipped with Jordan product \( \mathbb{P} \) is called the Lorentz cone.

We will now introduce a very useful decomposition in \( \mathbb{E} \), called spectral decomposition. An element \( \mathbf{c} \in \mathbb{E} \) is said to be a primitive idempotent if \( \mathbf{c} \mathbf{c} = \mathbf{c} \neq 0 \) and if \( \mathbf{c} \) is not a sum of two non-null idempotents. A complete system of primitive orthogonal idempotents is a set \( \{\mathbf{c}_1, \ldots, \mathbf{c}_r\} \) such that

\[
\sum_{i=1}^{r} \mathbf{c}_i = \mathbf{e} \quad \text{and} \quad \mathbf{c}_i \mathbf{c}_j = \delta_{ij} \mathbf{c}_i \quad \text{for } 1 \leq i \leq j \leq r.
\]

The size \( r \) of such system is a constant called the rank of \( \mathbb{E} \). Any element \( \mathbf{x} \) of a Euclidean simple Jordan algebra can be written as \( \mathbf{x} = \sum_{i=1}^{r} \lambda_i \mathbf{c}_i \) for some complete \( \{\mathbf{c}_1, \ldots, \mathbf{c}_r\} \) system of primitive orthogonal idempotents. The real numbers \( \lambda_i, i = 1, \ldots, r \) are the eigenvalues of \( \mathbf{x} \). One can then define trace and determinant of \( \mathbf{x} \) by, respectively, \( \text{tr } \mathbf{x} = \sum_{i=1}^{r} \lambda_i \) and \( \det \mathbf{x} = \prod_{i=1}^{r} \lambda_i \). An element \( \mathbf{x} \in \mathbb{E} \) belongs to \( \Omega \) if and only if all its eigenvalues are strictly positive.

Note that up to a constant, \( \text{tr } (\mathbf{x} \mathbf{y}) \) is the only scalar product of \( \mathbb{E} \). Henceforth we assume that \( \Omega \) is an irreducible cone and that corresponding Jordan algebra \( \mathbb{E} \) is equipped with canonical scalar product \( \langle \mathbf{x}, \mathbf{y} \rangle = \text{tr } (\mathbf{x} \mathbf{y}) \).

The rank \( r \) and \( \dim \Omega \) of irreducible symmetric cone are connected through relation

\[
\dim \Omega = r + \frac{dr(r-1)}{2},
\]

where \( d \) is an integer called the Peirce constant.

The important property of determinant is that

\[
\det (\mathbb{P}(\mathbf{x}) \mathbf{y}) = (\det \mathbf{x})^2 \det \mathbf{y}, \quad \langle \mathbf{x}, \mathbf{y} \rangle \in \Omega^2.
\]

It turns out that property \[2\] characterizes determinant - see Lemma 4.2. Moreover (see \cite{Faraut and Korányi, 1994}, Proposition II.4.2)

\[
\det (\mathbb{P}(\mathbf{x})) = (\det \mathbf{x})^{2 \dim \Omega / r},
\]

where \( \text{Det} \) denotes the determinant in the space of endomorphisms on \( \Omega \).

In the proof of our main theorem we will need following the identity (called Hua’s identity - see \cite{Faraut and Korányi, 1994}, Exercise 5c, p39)

\[
\mathbf{a}^{-1} - (\mathbf{a} + \mathbf{b})^{-1} = (\mathbf{a} + \mathbb{P}(\mathbf{a}) \mathbf{b}^{-1})^{-1}.
\]
when \( a \in \Omega, \ b \in \mathbb{E} \) are such that \( b, a + b \) and \( a + P(a)b^{-1} \) are invertible. Note that if \( a, b \in \Omega \), then \( a^{-1} - (a + b)^{-1} \in \Omega \).

3. Probability distributions on symmetric cones

The Wishart distribution \( \gamma_{p,a} \) in \( \bar{\Omega} \) is defined for any \( a \in \Omega \) and any \( p \) in the set
\[
\Lambda = \{0, d/2, d, \ldots, d(r-1)/2\} \cup \{d(r-1)/2, \infty\}
\]
by its Laplace transform
\[
\int_\Omega \exp(-\langle \sigma, y \rangle)\gamma_{p,a}(dy) = \left( \frac{\det a}{\det (a + \sigma)} \right)^p
\]
for any \( \sigma, a \in \Omega \). If \( p > \dim \Omega/r - 1 \), then \( \gamma_{p,a} \) is absolutely continuous with respect to Lebesgue measure and has density
\[
\gamma_{p,a}(dx) = \left( \frac{\det a}{\Gamma_\Omega(p)} \right)^p (\det x)^{p-\dim \Omega/r} e^{-\langle a, x \rangle} I_\Omega(x) \, dx, \quad x \in \Omega,
\]
where \( \Gamma_\Omega \) is the Gamma function of the symmetric cone \( \Omega \) (see [Faraut and Korányi 1994, p124]).

Absolutely continuous generalized inverse Gaussian distribution \( \mu_{p,a,b} \) on \( \Omega \) is defined for \( a, b \in \Omega \) and \( p \in \mathbb{R} \) by density
\[
\mu_{p,a,b}(dx) = \frac{1}{K_p(a,b)} (\det x)^{p-\dim \Omega/r} e^{-\langle a, x \rangle} e^{-\langle b, x^{-1} \rangle} I_\Omega(x) \, dx, \quad x \in \Omega,
\]
where \( K_p(a,b) \) is a normalizing constant.

In Letac and Wesolowski 2000 the analogue of following theorem for the cone of symmetric, positive definite real matrices \( \Omega_- \) (see their Theorem 3.1) has been proved. As it was observed by the authors, symmetric cones are the natural framework for considering Matsumoto-Yor property. We state following theorem without proof as it only mimics the argument for \( \Omega_- \). The original proof relies on the properties of Bessel-like functions \( (K_p(a,b)) \) introduced in Herz 1955, which maintain their properties in the symmetric cone setting.

**Theorem 3.1** Let \( p \) be in \( \Lambda \) and \( a \) and \( b \) in irreducible symmetric cone \( \Omega \). Let \( X \) and \( Y \) be independent random variables in \( \Omega \) and \( \bar{\Omega} \) with respective distributions \( \mu_{-p,a,b} \) and \( \gamma_{p,a} \). Then random variables
\[
U = (X + Y)^{-1} \quad \text{and} \quad V = X^{-1} - (X + Y)^{-1}
\]
are independent with respective distributions \( \mu_{-p,b,a} \) and \( \gamma_{p,b} \).

4. Functional equations on symmetric cones

At the beginning of the present section we state three results that will be useful in the proof of the main technical result - Theorem 4.3. The first one regards regular additive functions (see Kuczma 2000) on symmetric cone.

**Lemma 4.1** (Additive Cauchy functional equation) Let \( f : \Omega \to \mathbb{R} \) be continuous function satisfying
\[
f(x) + f(y) = f(x + y), \quad (x, y) \in \Omega^2.
\]
Then there exists \( f \in \mathbb{E} \) such that \( f(x) = \langle f, x \rangle \) for any \( x \in \Omega \).

Elementary proof in the symmetric cone setting of this theorem may be found in Kołodziejek 2013a.

The following lemma was recently proved in Kołodziejek 2013b.

**Lemma 4.2** (Logarithmic Pexider functional equation) Let \( f_1, f_2, f_3 : \Omega \to \mathbb{R} \) be continuous functions satisfying
\[
f_1(x) + f_2(y) = f_3 \left( P \left( x^{1/2} \right)_y \right), \quad (x, y) \in \Omega^2.
\]
Then there exists a constant $q \in \mathbb{R}$ and constants $\gamma_1, \gamma_2 \in \mathbb{R}$ such that for all $x \in \Omega$,
\[
\begin{align*}
f_1(x) &= q \log \det x + \gamma_1, \\
f_2(x) &= q \log \det x + \gamma_2, \\
f_3(x) &= q \log \det x + \gamma_1 + \gamma_2.
\end{align*}
\]

The main technical result will rely on the following univariate result due to Wesołowski [2002b].

**Theorem 4.3** Let $A$, $B$, $C$ and $D$ be locally integrable real functions defined on $(0, \infty)$ satisfying the equation
\[
(5) \quad g(x(x+y)) - g(y(x+y)) = \alpha(x) - \alpha(y), \quad (x, y) \in (0, \infty)^2.
\]
Then there exist real numbers $A$, $B$, $C$ and $D$ such that for any $x > 0$,
\[
g(x) = Ax + B \log x + C, \quad \alpha(x) = Ax^2 + B \log x + D.
\]

The following result then follows easily from Theorem 4.3

**Theorem 4.4** Let $A$, $B$, $C$ and $D$ be locally integrable real functions defined on $(0, \infty)$ satisfying the equation
\[
(6) \quad A(x) + B(y) = C((x+y)^{-1}) + D(x^{-1} - (x+y)^{-1}), \quad (x, y) \in (0, \infty)^2.
\]
Then there exist real numbers $p, f, g$ and $C_i$, $i = 1, \ldots, 4$, such that for any $x > 0$,
\[
\begin{align*}
A(x) &= -p \log x + fx + gx^{-1} + C_1, \\
B(x) &= p \log x + fx + C_2, \\
C(x) &= -p \log x + gx + fx^{-1} + C_3, \\
D(x) &= p \log x + gx + C_4,
\end{align*}
\]
and $C_1 + C_2 = C_3 + C_4$.

**Proof.** Denote $g_1(x) = A(x^{-1}) - B(x^{-1})$ and $\alpha_1(x) = D(x^2)$. Interchange the roles of $x$ and $y$ in (6) and subtract from the original equation. Then one gets
\[
g_1(x^{-1}) - g_1(y^{-1}) = \alpha_1 \left( \frac{y}{x(x+y)} \right) - \alpha_1 \left( \frac{x}{y(x+y)} \right).
\]
Inserting $x = (u(u+v))^{-1}$ and $y = (v(u+v))^{-1}$ we arrive at (5) with $g$ and $\alpha$ replaced, respectively, with $g_1$ and $\alpha_1$.

Substituting $x \mapsto (x+y)^{-1}$ and $y \mapsto x^{-1} - (x+y)^{-1}$ in (6) we obtain
\[
A((x+y)^{-1}) + B(x^{-1} - (x+y)^{-1}) = C(x) + D(y), \quad (x, y) \in (0, \infty)^2.
\]
Analogously as before, denoting $g_2(x) = C(x^{-1}) - D(x^{-1})$ and $\alpha_2(x) = B(x^2)$ and subtracting the same equation with $x$ and $y$ interchanged, we see that (5) holds true for $g_2$ and $\alpha_2$ also. Functions $g_i$ and $\alpha_i$, $i = 1, 2$, are locally integrable, because for $g_1$ we have
\[
\int_K |A(x^{-1}) - B(x^{-1})| \, dx = \int_{\partial(K)} |A(y) - B(y)|/y \, dy \leq c \int_{\partial(K)} |A(y) - B(y)| \, dy < \infty
\]
for all compact sets \( K \subset (0, \infty) \), where \( \phi(K) \) is the (compact) image of \( K \) under \( \phi(x) = x^{-1} \). Analogously for \( g_2, \alpha_1 \) and \( \alpha_2 \). Thus by Theorem 4.3 we obtain (notation borrowed from Theorem 4.3):

\[
B(x) = \alpha_2(\sqrt{x}) = A_2x + B_2/2 \log x + D_2,
\]

\[
D(x) = \alpha_1(\sqrt{x}) = A_1x + B_1/2 \log x + D_1,
\]

\[
A(x) = A(x) - B(x) + B(x) = g_1(x^{-1}) + \alpha_2(\sqrt{x}) = A_2x + A_1x^{-1} - (B_1 - B_2/2) \log x + C_1 + D_2,
\]

\[
C(x) = C(x) - D(x) + D(x) = g_2(x^{-1}) + \alpha_1(\sqrt{x}) = A_1x + A_2x^{-1} - (B_2 - B_1/2) \log x + C_2 + D_1,
\]

Inserting it back into (6) it can be quickly verified that \( B_1 = B_2 = B \). \( \square \)

We are now ready to state and solve the functional equation related to the Matsumoto-Yor property on symmetric cones.

**Theorem 4.5** Let \( a, b, c \) and \( d \) be continuous real functions defined on \( \Omega \) satisfying the equation

(7)

\[
a(x) + b(y) = c(x + y^{-1}) + d(x^{-1} - (x + y)^{-1}) , \quad (x, y) \in \Omega^2.
\]

Then there exist constants \( q \in \mathbb{R}, \ f, g \in \mathbb{E} \) and \( \gamma_i \in \mathbb{R}, \ i = 1, 2, 3 \), such that for any \( x \in \Omega \),

\[
a(x) = q \log \det x + f(x) + g(x^{-1}) + \gamma_1 + \gamma_3,
\]

\[
b(x) = -q \log \det x + f(x) + \gamma_2,
\]

\[
c(x) = q \log \det x + f(x) + g(x^{-1}) + \gamma_3,
\]

\[
d(x) = -q \log \det x + f(x) + \gamma_1 + \gamma_2.
\]

**Proof.** By inserting \((x, y) = (\alpha z, \beta z)\) for \( \alpha, \beta > 0 \) and \( z \in \Omega \) into (7), we arrive at the equation (6) with \( A(\alpha) := a(\alpha z), \ B(\alpha) := b(\alpha z), \ C(\alpha) := c(\alpha z^{-1}) \) and \( D(\alpha) := d(\alpha z^{-1}) \). Functions \( A, B, C \) and \( D \) are continuous, thus they are locally integrable. Therefore, by Theorem 4.3 for any \( z \in \Omega \) there exists constants \( p(z), f(z), g(z) \) and \( C_i(z), i = 1, \ldots, 4 \), such that

(8)

\[
a(\alpha z) = -p(z) \log \alpha + f(z) \alpha + g(z) \alpha^{-1} + C_1(z),
\]

\[
b(\alpha z) = p(z) \log \alpha + f(z) \alpha + C_2(z),
\]

\[
c(\alpha z^{-1}) = -p(z) \log \alpha + g(z) \alpha + f(z) \alpha^{-1} + C_3(z),
\]

\[
d(\alpha z^{-1}) = p(z) \log \alpha + g(z) \alpha + C_4(z),
\]

\[
C_1(z) + C_2(z) = C_3(z) + C_4(z),
\]

for any \( \alpha > 0 \) and \( z \in \Omega \). Functions \( z \mapsto p(z), \ z \mapsto f(z), \ z \mapsto g(z) \) and \( z \mapsto C_i(z), i = 1, \ldots, 4 \), are continuous, because \( a, b, c \) and \( d \) are continuous. Let \( \beta > 0 \). By the equality \( a(\alpha \beta z) = a(\alpha z) \), we obtain for any \( \alpha > 0 \),

\[
a(\alpha \beta z) = -p(z) \log \alpha \beta + f(z) \alpha \beta + g(z) \alpha^{-1} \beta^{-1} + C_1(z)
\]

\[
= -p(\beta z) \log \alpha + f(\beta z) \alpha + g(\beta z) \alpha^{-1} + C_1(\beta z),
\]

hence

(9)

\[
f(\beta z) = \beta f(z), \quad g(\beta z) = \beta^{-1} g(z),
\]

\[
p(\beta z) = p(z), \quad C_1(\beta z) = C_1(z) - p(z) \log \beta.
\]

Following the same procedure for functions \( b, c \) and \( d \), we have

(10)

\[
C_i(\beta z) = C_i(z) + p(z) \log \beta, \quad i = 2, 3,
\]

\[
C_4(\beta z) = C_4(z) - p(z) \log \beta.
\]
Using (8) for $\alpha = 1$ in (7), we get
\begin{equation}
(11) \quad f(x) + g(x) + C_1(x) + f(y) + C_2(y) = g(x + y) + f(x + y) + C_3(x + y) + g((x^{-1} - (x + y)^{-1} - 1)) + C_4((x^{-1} - (x + y)^{-1} - 1)).
\end{equation}
Consider the above equation for $(\alpha^{-1}x, \alpha^{-1}y) \in \Omega^2$, $\alpha > 0$. Then, by (11),
\begin{equation}
\alpha^{-1}f(x) + \alpha g(x) + C_1(\alpha^{-1}x) + \alpha^{-1}f(y) + C_2(\alpha^{-1}y) = \alpha g(x + y) + \alpha^{-1}f(x + y) + C_3(\alpha^{-1}(x + y)) + \alpha g((x^{-1} - (x + y)^{-1} - 1)) + C_4(\alpha^{-1}(x^{-1} - (x + y)^{-1} - 1)).
\end{equation}
Multiplying both sides of the above equation by $\alpha$ and passing to the limit as $\alpha \to 0$, by (10) we obtain
\begin{equation}
f(x) + f(y) - f(x + y) = \lim_{\alpha \to 0} \alpha \left\{ C_3(\alpha^{-1}(x + y)) + C_4(\alpha^{-1}(x^{-1} - (x + y)^{-1} - 1)) - C_1(\alpha^{-1}x) - C_2(\alpha^{-1}y) \right\}.
\end{equation}
By (9) and (10) the limit on the right hand side of the above equations equals 0. Thus, by Lemma (11) there exists $f \in \mathbb{E}$ such that $f(x) = \langle f, x \rangle$. Analogously, consider (11) for $(\alpha x, \alpha y) \in \Omega^2$, $\alpha > 0$, multiply both sides of equation by $\alpha$ and pass to the limit as $\alpha \to 0$. Then
\begin{equation}
g(x) - g(x + y) = \lim_{\alpha \to 0} \alpha \left\{ C_3(\alpha(x + y)) + C_4(\alpha(x^{-1} - (x + y)^{-1} - 1)) - C_1(\alpha x) - C_2(\alpha y) \right\} = 0.
\end{equation}
Define $\bar{g}(x) = g(x^{-1})$. Then,
\begin{equation}
\bar{g}(x^{-1}) = \bar{g}(x + y) - \bar{g}(x^{-1} - (x + y)^{-1}).
\end{equation}
Thus, $\bar{g}$ is additive, i.e. there exists $g \in \mathbb{E}$ such that $g(x) = \langle g, x^{-1} \rangle$.

Using above results for $f$ and $g$, (11) simplifies to
\begin{equation}
C_1(x) + C_2(y) = C_3(x + y) + C_4((x^{-1} - (x + y)^{-1} - 1)).
\end{equation}
Recall that by Hua’s identity (11), the argument of $C_4$ above may be written as
\begin{equation}
(x^{-1} - (x + y)^{-1} - 1) = x + \mathbb{P}(x)y^{-1}.
\end{equation}
Using this fact along with (11) in (12) for $y = \alpha z$, we obtain
\begin{equation}
C_1(x) + C_2(z) + p(z) \log \alpha
= C_1(x) + C_2(\alpha z) = C_3(x + \alpha z) + C_4(\alpha^{-1}(\alpha x + \mathbb{P}(x)z^{-1}))
= C_3(x + \alpha z) + C_4(\alpha x + \mathbb{P}(x)z^{-1} + p(\alpha x + \mathbb{P}(x)z^{-1}) \log \alpha).
\end{equation}
Passing to the limit as $\alpha \to 0$ (recall that $C_1$ are continuous on $\Omega$), we obtain
\begin{equation}
C_1(x) + C_2(z) = C_3(x + \alpha z) - C_3(x) - C_4(\mathbb{P}(x)z^{-1}) = \lim_{\alpha \to 0} \log \alpha \left\{ p(\alpha x + \mathbb{P}(x)z^{-1} - p(z)) \right\}
\end{equation}
for any $(x, z) \in \Omega^2$. A necessary condition for the limit on the right hand side to exist is
\begin{equation}
\lim_{\alpha \to 0} \left\{ p(\alpha x + \mathbb{P}(x)z^{-1}) - p(z) \right\} = 0.
\end{equation}
But $p$ is continuous and $\lim_{\alpha \to 0} p(\alpha x + \mathbb{P}(x)z^{-1}) = p(\mathbb{P}(x)z^{-1})$, hence $p(z) = p(\mathbb{P}(x)z^{-1})$. Thus, function $p$ is constant and the right hand side of (13) is equal to 0. Hence, substituting $z = y^{-1}$ and $x \mapsto x^{1/2}$ in (13), we get
\begin{equation}
C_1(x^{1/2}) - C_3(x^{1/2}) + C_2(y^{-1}) = C_4(\mathbb{P}(x^{1/2})y).
\end{equation}
Define $f_1(x) := C_1(x^{1/2}) - C_3(x^{1/2})$, $f_2(x) := C_2(x^{-1})$ and $f_3(x) := C_4(x)$ for $x \in \Omega$. Then
\[
f_1(x) + f_2(y) = f_3(P(x^{1/2})y), \quad (x, y) \in \Omega^2.
\]
By Lemma 12 there exist real constants $q, \gamma_1$ and $\gamma_2$ such that for any $x \in \Omega$,
\[
f_1(x) = q \log \det x + \gamma_1,
\]
\[
f_2(x) = q \log \det x + \gamma_2,
\]
\[
f_3(x) = q \log \det x + \gamma_1 + \gamma_2,
\]
that is,
\[
C_1(x) = C_3(x) + 2q \log \det x + \gamma_1,
\]
\[
C_2(x) = -q \log \det x + \gamma_2,
\]
\[
C_4(x) = q \log \det x + \gamma_1 + \gamma_2.
\]
Let us go back to (12) and use the above result. Then
\[
C_3(x) + 2q \log \det x - q \log \det y = C_3(x + y) + q \log \det (x + P(x)y^{-1}), \quad (x, y) \in \Omega^2.
\]
Since $\det(x + P(x)y^{-1}) = \det(x^2) \det(x^{-1} + y^{-1})$, we obtain
\[
C_3(x) - q \log \det y = C_3(x + y) + q \log \det (x^{-1} + y^{-1}).
\]
One can interchange $x$ and $y$ on the right hand side to obtain
\[
C_3(x) + q \log \det x = C_3(y) + q \log \det y = \text{const} := \gamma_3,
\]
that is, $C_3(x) = -q \log \det x + \gamma_3$, what completes the proof.

5. Converse to Matsumoto-Yor property on symmetric cones

In the following section we prove our main result, which is a converse to the Matsumoto-Yor property in the symmetric cone variate case. We lessen the smoothness conditions for densities from $C^2$ densities in [Letac and Wesolowski 2000] and differentiability in [Wesolowski 2002a] to continuity only.

**Theorem 5.1** Let $X$ and $Y$ be independent random variables in $\Omega$ with continuous and strictly positive densities. If the random variables $U = (X + Y)^{-1}$ and $V = X^{-1} - (X + Y)^{-1}$ are independent, then there exists $p > \dim \Omega/r - 1$, $a$ and $b$ in $\Omega$ such that $X$ and $Y$ follow respective distributions $\mu_{-p,a,b}$ and $\gamma_{p,a}$.

**Proof.** Define the map $\Psi: \Omega^2 \to \Omega^2$ by $\Psi(x, y) = ((x + y)^{-1}, x^{-1} - (x + y)^{-1}) = (u, v)$. Obviously $(U, V) = \Psi(X, Y)$. Function $\Psi$ is a bijection. In order to find the joint density of $(U, V)$ the essential computation is involved with finding the Jacobian $J$ of the map $\psi^{-1}$, that is, the determinant of the linear map
\[
\begin{pmatrix}
du \\
dv
\end{pmatrix} \mapsto \begin{pmatrix}
dx \\
dy
\end{pmatrix} = \begin{pmatrix}
dx/du & dx/dv \\
dy/du & dy/dv
\end{pmatrix} \begin{pmatrix}
du \\
dv
\end{pmatrix}.
\]
It is easy to see that $\Psi = \Psi^{-1}$, that is $(x, y) = ((u + v)^{-1}, u^{-1} - (u + v)^{-1})$. Note that, the derivative of the map $x \mapsto x^{-1}$ is $-P(x)^{-1}$. Thus
\[
J = \begin{vmatrix}
-P(u + v)^{-1} & -P(u + v)^{-1} \\
-P(u)^{-1} + P(u + v)^{-1} & P(u + v)^{-1}
\end{vmatrix} = \begin{vmatrix}
-P(u + v)^{-1} & -P(u + v)^{-1} \\
-P(u)^{-1} & 0
\end{vmatrix} = \det (P(u + v)^{-1}P(u)^{-1}).
\]
By (3) we get
\[
J = (\det u \det (u + v))^{-2\dim \Omega/r}.
\]
Since \((X, Y)\) and \((U, V)\) have independent components, the following identity holds almost everywhere with respect to Lebesgue measure:

\[
f_U(u)f_V(v) = (\det u \det(u+v))^{-2 \dim \Omega/r} f_X \left( (u+v)^{-1} \right) f_Y \left( u^{-1} - (u+v)^{-1} \right),
\]

where \(f_X\), \(f_Y\), \(f_U\) and \(f_V\) denote densities of \(X\), \(Y\), \(U\) and \(V\), respectively. Since the respective densities are assumed to be continuous, the above equation holds for every \((u, v) \in \Omega^2\). Taking logarithms of both sides of the above equation (it is permitted since \(f_X, f_Y > 0\) on \(\Omega\)) we get

\[
a(u) + b(v) = c ((u+v)^{-1}) + d (u^{-1} - (u+v)^{-1}),
\]

where

\[
a(x) = \log f_U(x) + \frac{2 \dim \Omega}{r} \log \det x,
\]

\[
c(x) = \log f_X(x) + \frac{2 \dim \Omega}{r} \log \det x,
\]

\[
b = \log f_Y, \quad d = \log f_Y.
\]

By Theorem 4.5 there exist constants \(q \in \mathbb{R}\), \(f, g \in \mathbb{E}\) and \(\gamma_i \in \mathbb{R}\), \(i = 1, 2, 3\), such that for any \(x \in \Omega\),

\[
c(x) = -q \log \det x + \langle g, x \rangle + \langle f, x^{-1} \rangle + \gamma_3,
\]

\[
d(x) = q \log \det x + \langle g, x \rangle + \gamma_1 + \gamma_2,
\]

that is,

\[
f_X(x) = e^{\gamma_3 (\det x)^{-q-2 \dim \Omega/r} e^{\langle g, x \rangle + \langle f, x^{-1} \rangle}},
\]

\[
f_Y(x) = e^{\gamma_1 + \gamma_2 (\det x)^{q} e^{\langle g, x \rangle}}.
\]

Since \(f_X\) and \(f_Y\) are densities, we have \(a = -g \in \Omega, b = -f \in \Omega\) and \(q = p - \dim \Omega/r > -1\). Thus, \(X \sim \mu_{-p,a,b}\) and \(Y \sim \gamma_{p,a}\). \(\square\)

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