An Inverse Problem for a Non-Homogeneous Time-Space Fractional Equation

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Abstract: We consider the inverse problem of finding the solution of a generalized time-space fractional equation and the source term knowing the spatial mean of the solution at any times \( t \in (0, T] \), as well as the initial and the boundary conditions. The existence and the continuity with respect to the data of the solution for the direct and the inverse problem are proven by Fourier’s method and the Schauder fixed-point theorem in an adequate convex bounded subset. In the published articles on this topic, the incorrect use of the estimates in the generalized Mittag–Leffler functions is commonly performed. This leads to false proofs of the Fourier series’ convergence to recover the equation satisfied by the solution, the initial data or the boundary conditions. In the present work, the correct framework to recover the decay of fractional Fourier coefficients is established; this allows one to recover correctly the initial data, the boundary conditions and the partial differential equations within the space-time domain.

Keywords: inverse problem; fractional derivative; time-space fractional equation; integral equations; biorthogonal system of functions; Fourier series

1. Introduction

Consider the following non-homogeneous fractional evolution boundary value problem:

\[
\begin{aligned}
\mathcal{D}_t^\gamma u(x, t) - \sum_{i=1}^{d} \mathcal{D}_{x_i}^{\alpha_i} u(x, t) &= p(x) f(t), \quad (x, t) \in \Omega \times (0, T), \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times [0, T], \\
u(x, 0) &= \varphi(x), \quad x \in \Omega,
\end{aligned}
\]

where \( \Omega = (0, 1)^d \), \( d \geq 1, x = (x_1, \ldots, x_d), \mathcal{D}_t^\gamma u(x, t) \) denotes the (time) fractional derivative in the sense of Caputo, with \( 0 < \gamma < 1 \), and \( \mathcal{D}_{x_i}^{\alpha_i} u(x, t) \); \( 1 \leq i \leq d \), is the (space) fractional derivative, in the sense of Riemann–Liouville, with \( 1 < \alpha_i < 2 \); and \( T > 0 \) denotes the length of the time interval.

Let us fix the framework: When the initial data \( \varphi \) and the source terms \( (p, f) \) are given, the direct problem consists of looking for the unknown function \( u \) in an adequate functional space. On the other hand, when the initial data \( \varphi \) and only the spatial part \( p \) are given, the inverse problem consists of looking for the unknown functions \( (u, f) \) via, for example, the supplementary data:

\[
\int_\Omega u(x, t) \, dx = \psi(t), \quad t \in [0, T],
\]
where \( \psi \) is a given function of \([0, T]\).

Notice that the fractional operator \( D_{\alpha}^{x} \) is unbounded, non-self-adjoint with compact resolvent admitting a countable complete basis of eigenfunctions, in \( L^{2}(\Omega, \mathbb{C}) \) [1]. This will allow us to look for classical solutions via the biorthogonal representation with respect to \( x \). Indeed, the uniform convergence of the resulting series in \( \Omega \times [0, T] \) (and of their derivatives) and the uniqueness of the corresponding limits, with respect to the norms \( \| \cdot \|_{2} \) and \( \| \cdot \|_{\infty} \), will lead to the existence of classical solutions of the direct problem. For the inverse problems, we will make use of the Schauder fixed point theorem in an adequate convex and bounded subset.

Before we present and prove our results, let us dwell on the existing literature. An analytical and numerical study of inverse problems for fractional diffusion equations has undergone intensive development over the present decade. Kirane and Malik [2] considered at some later date a one-dimensional fractional diffusion equation with a non-local, non-self-adjoint boundary condition. They developed a solution and source function using a bi-orthogonal method and obtained the uniqueness. In addition, they have prove an existence result when both the initial and the final conditions are smooth enough. In [3], the problem in [2] is generalized. Furati et al. used the same bi-orthogonal spatial system to obtain a separable formal solution and source function and to show their uniqueness. They then used the asymptotic expansion of the generalized Mittag–Leffler function to achieve a result of existence under certain smoothness criteria on the initial and final conditions. Aleroev et al. [1] consider a linear heat equation, with a non-local boundary condition, involving a fractional derivative in time. They establish a space independent source term and the distribution of temperature for a problem with integral type over determining condition. They proved the solution’s existence and uniqueness, and its continuous dependence on the data. Among other papers devoted to determining unknown source terms, we may cite [4–6].

As was mentioned in the abstract, an incorrect use of the estimates in the generalized Mittag–Leffler functions is commonly performed in the literature on this topic. This leads to false proofs of the Fourier series’ convergence to recover the equation satisfied by the solution, the initial data or the boundary conditions. Remark 1 explains this fairly widespread mistake. In the present work, the correct framework to recover the decay of fractional Fourier coefficients is established; this allows one to correctly recover the initial data, the boundary conditions and the partial differential equation satisfied within the space-time domain.

The manuscript is organized as follows: In Section 2, fundamental concepts of fractional calculus are recalled. In Section 3, we obtain results for the direct problem using Fourier’s method; we determine the existence, uniqueness and continuous dependence on the data. In Section 4, the inverse problem associated with the direct problem is considered, and we show the results of the existence, uniqueness and continuous dependence of the solution on the data.

2. Preliminaries

In this section, we recall basic definitions and notations from fractional calculus [7,8]. Let \( f : [a, b] \rightarrow \mathbb{R} \) be an integrable function. The left-sided Riemann–Liouville fractional integral of order \( \alpha > 0 \) is defined by:

\[
I_{a}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(t)}{(x-t)^{1-\alpha}} dt,
\]

where \( \Gamma(\alpha) \) is the Euler’s Gamma function.

For arbitrary values of \( \alpha > 0 \), the left sided Riemann–Liouville fractional derivative of order \( \alpha \) is defined by:

\[
D_{a}^{\alpha} f(x) = \frac{d^{n}}{dx^{n}} I_{a}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{a}^{x} (x-t)^{n-1-\alpha} f(t) dt, \; n = [\alpha] + 1,
\]
Then, the function defined by the series \( \alpha_1. \) For a given \( \alpha_r \) where
\[
\left\lceil \alpha_r \right\rceil \end{equation}

Let \( \alpha \rightarrow 0. \) Furthermore, from [7], we have:
\[
\int_0^\zeta t^{\beta-1}E_{\alpha,\beta}(\lambda t^\beta) \, dt = z^\beta E_{\alpha,\beta+1}(\lambda z^\beta), \quad (\alpha > 0, \beta > 0, \lambda \in \mathbb{R}).
\end{equation}

In the case \( \beta = 1, \) the function \( E_{\alpha,1} \) is the classical Mittag–Leffler function denoted by \( E_{\alpha}. \) From [9], we have the following property:
\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + \beta)}, \quad \text{for any } z \in \mathbb{C}. \quad (5)
\end{equation}

\[\text{Lemma 1.} \quad \text{Suppose the following.} \]
\[\text{Furthermore, from [7], we have:} \]
\[\text{Lemma 1.} \quad \text{Let } (f_i)_{i \in \mathbb{N}} \text{ be a sequence of functions defined on the interval } (a, b]. \text{ Suppose the following conditions are fulfilled:} \]
1. \( \text{For a given } \alpha > 0, \text{ the fractional derivative } D_+^\alpha f_i \text{ exists for all } i \in \mathbb{N} \text{ and } z \in (a, b]; \)
2. \( \text{Both series } \sum_{i=1}^\infty f_i(z) \text{ and } \sum_{i=1}^\infty D_+^\alpha f_i(z) \text{ are uniformly convergent on the interval } [a+\epsilon, b] \text{ for any } \epsilon > 0. \)

Then, the function defined by the series \( \sum_{i=1}^\infty f_i(z) \) is \( \alpha \)-differentiable on \((a, b]\) and satisfies:
\[
D_+^\alpha \left( \sum_{i=1}^\infty f_i(z) \right) = \sum_{i=1}^\infty D_+^\alpha f_i(z). \quad (9)
\end{equation}

\[\text{3. Direct Problem} \]

In this section, we will study the forward problem:
\[
\begin{cases}
D_t^\gamma u(x, t) - \sum_{i=1}^d D_x^\nu_i u(x, t) = p(x) \, f(t), \quad (x, t) \in \Omega \times (0, T), \\
u(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, T], \\
u(x, 0) = \varphi(x), \quad x \in \bar{\Omega},
\end{cases}
\end{equation}

\[\text{where } [\alpha] \text{ denotes the integer part of } \alpha. \text{ The left-sided Caputo fractional derivative of order } \alpha > 0 \text{ is defined by:} \]
\[
D_+^\alpha f(x) = I_{-a}^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x (x - t)^{n-1-\alpha} f(t) \, dt, \quad n = [\alpha] + 1.
\end{equation}

\[\text{where } f^{(n)} \text{ denotes the } n\text{-th classical derivative of } f. \]

The Mittag–Leffler function with two parameters \( \alpha \) and \( \beta \) is defined as:
\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + \beta)}, \quad \text{for any } z \in \mathbb{C}. \quad (5)
\end{equation}

\[\text{Theorem 1.} \quad \text{Let } \theta \in (0, 2), \nu > 0 \text{ and } \mu \text{ be such that } \frac{\theta \nu}{\pi} < \mu < \min(\pi, \theta \pi). \text{ Then, there is a constant } c = c(\theta, \nu, \mu) \geq 0 \text{ such that:} \]
\[
|E_{\theta,\nu}(z)| \leq \frac{c}{1 + |z|}, \quad \forall z \in \mathbb{C}, \quad \mu \leq |\arg(z)| \leq \pi \quad (7)
\end{equation}
\[
|E_{\theta,\nu}(z)| \leq c \left( \frac{1}{1 + |z|} + (1 + |z|)^{\frac{1}{\nu} + \frac{\pi}{\theta \pi}} R e(z^\beta) \right), \quad \forall z \in \mathbb{C}, \quad |\arg(z)| \leq \mu, \quad (8)
\end{equation}

\[\text{where } Re(z) \text{ denotes the real part of the complex number } z. \]
where \( p, f \) and \( \varphi \) are given functions such that \( p, \varphi \in \mathcal{C}([\Omega, \mathbb{R}) \) and \( f \in \mathcal{C}([0, T], \mathbb{R}) \).

We are interested in classical solutions to Problem (10), that is, solutions \( u(x, t) \) which are continuous on \( \Omega \times (0, T) \) and satisfy \( D_t^1 u(x, t) \in L^2(\Omega \times (0, T)) \) and \( D_{x_i}^n u(x, t) \in L^2(\Omega \times (0, T)) \), for any \( (x, t) \in \Omega \times (0, T) \).

**Example 1.** Consider the case \( d = 2 \) and \( \Omega = (0, \pi) \times (0, \pi) \), with \( 0 < \gamma < 1 \) and \( 1 < \alpha_1, \alpha_2 < 2 \). We fix the initial condition \( \varphi(x_1, x_2) \), the spatial and temporal source terms \( p(x_1, x_2) \) and \( f(t) \) for any \((x_1, x_2) \in \Omega \) and \( t > 0 \) as the following:

\[
\begin{align*}
\varphi(x_1, x_2) &= 0, \\
p(x_1, x_2) &= \sin(x_1) \sin(x_2) + x_1^{1-\alpha_1} E_{\frac{1}{2},2-\alpha_1}(-x_1^2) \sin(y) + \sin(x_2) x_2^{1-\alpha_2} E_{\frac{1}{2},2-\alpha_2}(-x_2^2), \\
f(t) &= -E_{\gamma,1}(-t^\gamma),
\end{align*}
\]

The exact solution is then \( u(x_1, x_2, t) = \sin(x_1) \sin(x_2) E_{\gamma,1}(-t^\gamma) \), since it is known that

\[
\begin{align*}
D_t^\gamma E_{\gamma,1}(-t^\gamma) &= -E_{\gamma,1}(-t^\gamma), \quad t > 0, \\
D_{x_i}^n \sin(x_i) &= x_i^{1-\alpha_i} E_{\frac{1}{2},2-\alpha_i}(-x_i^2), \quad i = 1, 2.
\end{align*}
\]

Since the problem (10) is linear, we can split its solution as follows:

\[
u(x, t) = v(x, t) + w(x, t),
\]

where \( v(x, t) \) is the solution of the homogeneous problem:

\[
\begin{align*}
D_t^\gamma v(x, t) - \sum_{i=1}^d D_{x_i}^n v(x, t) &= 0, \quad (x, t) \in \Omega \times (0, T), \\
v(x, t) &= 0, \quad (x, t) \in \partial\Omega \times [0, T], \\
v(x, 0) &= \varphi(x), \quad x \in \Omega,
\end{align*}
\]

and \( w(x, t) \) is the solution of the problem:

\[
\begin{align*}
D_t^\gamma w(x, t) - \sum_{i=1}^d D_{x_i}^n w(x, t) &= p(x) f(t), \quad (x, t) \in \Omega \times (0, T), \\
w(x, t) &= 0, \quad (x, t) \in \partial\Omega \times [0, T], \\
w(x, 0) &= 0, \quad x \in \Omega.
\end{align*}
\]

For every direction \( x_i, 1 \leq i \leq d \), the spectral problem associated with Problem (12) is:

\[
D_{x_i}^n X_i(x_i) = \lambda X_i(x_i), \quad 0 \leq x_i \leq 1, \quad \text{with } X_i(0) = X_i(1) = 0.
\]

The eigenfunctions of (14) are given by [9]:

\[
X_{n,i}^i(x_i) = x_i^{\alpha_n-1} E_{\alpha_n,\alpha_i}(\lambda_{n,i}^i x_i), \quad n \geq 1,
\]

where \( \lambda_{n,i}^i \) are the associated eigenvalues. Notice that these eigenvalues form a countable set that is denoted by \( \lambda_{n,i}^i \). Moreover, they satisfy the following:

\[
\begin{align*}
\text{(P1)} & \quad \text{There are only finitely many real eigenvalues and the rest appears as complex conjugate pairs;} \\
\text{(P2)} & \quad |\lambda_{n,i}^i| \sim (2 \pi n)^{\alpha_i} \text{ as } n \to +\infty; \\
\text{(P3)} & \quad \frac{\alpha_i \pi}{2} < \arg(\lambda_{n,i}^i) \leq \pi \text{ for } n \text{ sufficiently large and } \arg(\lambda_{n,i}^i) \sim \frac{\alpha_i \pi}{2} \text{ as } n \to +\infty.
\end{align*}
\]
Now, consider a real number $\mu$ such that:

$$\mu \in \left(\frac{\gamma \pi}{2}, \min\left(\frac{\alpha_i \pi}{2}, \gamma \pi\right)\right),$$

and this is possible since $\alpha_i > \gamma$ and consequently the interval $\left(\frac{\gamma \pi}{2}, \min\left(\frac{\alpha_i \pi}{2}, \gamma \pi\right)\right)$ is not empty. Therefore, the property (P3) implies that

$$\mu < \frac{\alpha_i \pi}{2} < \arg\left(\lambda_n^i\right) \leq \pi \text{ for } n \text{ large enough.}$$

Since $\left(\frac{\gamma \pi}{2}, \min\left(\frac{\alpha_i \pi}{2}, \gamma \pi\right)\right) \subset \left(\frac{\gamma \pi}{2}, \min\left(\gamma \pi, \pi\right)\right)$, then applying Theorem 1, we obtain the estimate:

$$\exists \ c \geq 0, \forall t \in [0, T], \ |E_{\gamma, 1}(\lambda_n^i t^\gamma)| \leq \frac{c}{1 + |\lambda_n^i| t^\gamma}. \quad (17)$$

**Remark 1.** In the published papers on this topic, the mistake commonly made is due to a misapplication of Theorem 1. Indeed, the following estimate

$$\exists \ c \geq 0, \forall n \geq 1, \forall z \in [0, 1], \ |E_{\alpha, \beta}(\lambda_n^i z^{\alpha_i})| \leq \frac{c}{1 + |\lambda_n^i| z^{\alpha_i}}. \quad (18)$$

cannot hold true for all the eigenvalues $\lambda_n^i, n \geq 1$, since $\lim_{n \to \infty} \arg\left(\lambda_n^i\right) = \frac{\alpha_i \pi}{2}$. Consequently, there is no real number $\mu \in \left(\frac{\alpha_i \pi}{2}, \min(\pi, \alpha_i \pi)\right)$ such that

$$\forall n \geq 1, \mu \leq \arg\left(\lambda_n^i\right) \leq \pi.$$

Therefore, the conditions for applying Theorem 1 are not satisfied to obtain (18).

Based on [11], for every $1 \leq i \leq d$, the family $(X^i_n)_{n \geq 1}$ is a basis of $L^2((0, 1), \mathbb{C})$, which is not orthogonal, because the operator $D^\alpha_{x_i}$ is not self-adjoint. Consequently, for each $n \geq 1$, we introduce the family:

$$\mathbb{X}_{x_i}^i(n) = (1 - x_i)^{a_i - 1}E_{\alpha_i, \alpha_i}(\lambda_n^i (1 - x_i)^{\alpha_i}). \quad (19)$$

corresponding to the eigenfunctions of the adjoint operator of $D^\alpha_{x_i}$, with respect to the inner product in $L^2((0, 1), \mathbb{C})$:

$$\langle f, g \rangle_{L^2((0,1),\mathbb{C})} = \int_0^1 f(x_i) \overline{g(x_i)} \ dx_i, \quad (20)$$

where $\overline{g(x_i)}$ denotes the conjugate of the complex number $g(x_i)$. In the sequel, the inner products in $L^2((0, 1), \mathbb{C})$ and $L^2([0, T], \mathbb{C})$ will be simply denoted by $\langle \cdot, \cdot \rangle_{L^2((0,1),\mathbb{C})}$ and $\langle \cdot, \cdot \rangle_{L^2([0,T],\mathbb{C})}$, respectively.

Notice that the adjoint operator of $D^\alpha_{x_i}$ is the right-sided Riemann–Liouville fractional derivative of order $\alpha_i > 0$ defined by:

$$x_i D^\alpha f(x) = \frac{1}{\Gamma(n - \alpha_i)} \frac{d^n}{dt^n} \int_{x_i}^1 (t - x_i)^{n-1-\alpha_i} f(t) \ dt, \ n = [\alpha_i] + 1. \quad (21)$$

Moreover, the family $(\mathbb{X}_{x_i}^i)_{n \geq 1}$ is biorthogonal to $(X^i_n)_{n \geq 1}$ in $L^2((0,1), \mathbb{C})$ and satisfies

$$\langle X^i_n, \mathbb{X}_{x_i}^i \rangle_{L^2((0,1),\mathbb{C})} = 0 \text{ for } n \neq m \text{ and } \langle X^i_n, \mathbb{X}_{x_i}^i \rangle_{L^2([0,T],\mathbb{C})} = C^i_n > 0. \quad (22)$$
On the other hand, for any integers \( n_1 \geq 1, \ldots, n_d \geq 1 \), the resulting problem in the variable \( t \) is:

\[
\mathcal{D}_t^\gamma T(t) = \left( \sum_{i=1}^{d} \lambda_i^n \right) T(t), \quad 0 < t < T, \quad \text{with } T(0) = \varphi_{n_1 \ldots n_d},
\]

whose solution is given by:

\[
T_{n_1 \ldots n_d}(t) = \varphi_{n_1 \ldots n_d} E_{\gamma,1} \left( \left( \sum_{i=1}^{d} \lambda_i^n \right) t^{\gamma} \right),
\]

where:

\[
\varphi_{n_1 \ldots n_d} = \frac{\langle \varphi, \bigotimes_{i=1}^{d} X_i^{n_i} \rangle_{L^2(\Omega)}}{\langle \bigotimes_{i=1}^{d} X_i^{n_i} \rangle_{L^2(\Omega)}},
\]

since \( \varphi \in L^2(\Omega) \). The solution of Problem (12) is then formally given by:

\[
v(x,t) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \varphi_{n_1 \ldots n_d} E_{\gamma,1} \left( \left( \sum_{i=1}^{d} \lambda_i^n \right) t^{\gamma} \right) \bigotimes_{i=1}^{d} X_i^{n_i}.
\]

In the same way, we can look for the classical solution of the problem (13) as:

\[
w(x,t) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} w_{n_1 \ldots n_d}(t) \bigotimes_{i=1}^{d} X_i^{n_i},
\]

where the coefficients \( w_{n_1 \ldots n_d}(t) \) are to be found. To this end, if \( p \in L^2(\Omega) \), then:

\[
p(x) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} p_{n_1 \ldots n_d} \bigotimes_{i=1}^{d} X_i^{n_i},
\]

with

\[
p_{n_1 \ldots n_d} = \frac{\langle p, \bigotimes_{i=1}^{d} X_i^{n_i} \rangle_{L^2(\Omega)}}{\langle \bigotimes_{i=1}^{d} X_i^{n_i} \rangle_{L^2(\Omega)}}.
\]

Substituting the solution \( w \) given by (25) in Equation (13), we find formally:

\[
\sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \left( \mathcal{D}_t^\gamma - \left( \sum_{i=1}^{d} \lambda_i^n \right) \right) w_{n_1 \ldots n_d}(t) \bigotimes_{i=1}^{d} X_i^{n_i}(x_i) = f(t) \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} p_{n_1 \ldots n_d} \bigotimes_{i=1}^{d} X_i^{n_i}(x_i).
\]

Therefore, for each \( (n_1, \ldots, n_d) \), the coefficient \( w_{n_1 \ldots n_d}(t) \) satisfies the equation:

\[
\mathcal{D}_t^\gamma w_{n_1 \ldots n_d}(t) - \left( \sum_{i=1}^{d} \lambda_i^n \right) w_{n_1 \ldots n_d}(t) = f(t) p_{n_1 \ldots n_d}.
\]

Equation (28) is supplemented with the initial condition:

\[
w_{n_1 \ldots n_d}(0) = \frac{\langle w(.,0), \bigotimes_{i=1}^{d} X_i^{n_i} \rangle_{L^2(\Omega)}}{\langle \bigotimes_{i=1}^{d} X_i^{n_i} \rangle_{L^2(\Omega)}} = 0.
\]
Following [12], the solution of (28) is:

\[
\omega_{n_1 \cdots n_d}(t) = p_{n_1 \cdots n_d} \int_0^t s^{\gamma-1} E_{\gamma,\gamma} \left( \left( \sum_{i=1}^d \lambda_{n_i}^i \right) s^\alpha \right) f(t-s) \, ds.
\] (30)

Therefore, the solution of problem (13) is written as:

\[
\omega(x,t) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} p_{n_1 \cdots n_d} \bigotimes_{i=1}^d X_{n_i}(x_i) \int_0^t s^{\gamma-1} E_{\gamma,\gamma} \left( \left( \sum_{i=1}^d \lambda_{n_i}^i \right) s^\alpha \right) f(t-s) \, ds.
\] (31)

In what follows, \( c \) is a constant that might change from line to line.

**Theorem 2.** Let \( p \in C(\overline{\Omega}, \mathbb{R}) \), \( f \in C([0,T], \mathbb{R}) \) and \( \varphi \in C(\overline{\Omega}, \mathbb{R}) \) such that \( D_t^\gamma \sum \cdots D_t^\gamma p \in C(\overline{\Omega}, \mathbb{R}), D_x^\alpha \cdots D_x^\alpha \varphi \in C(\overline{\Omega}, \mathbb{R}) \) and \( \varphi|_{\partial \Omega} = 0 \). Then, there exists a unique classical solution of Problem (10).

**Proof.** Existence of a solution:

To lighten the proof without loss of generality, we present the case \( d = 1 \) and explain below the general case \( d \geq 1 \). In this situation, we consider the problem:

\[
\begin{align*}
D_t^\gamma \omega(x,t) - D_x^\alpha \omega(x,t) &= p(x) f(t), \quad (x,t) \in (0,1) \times (0,T), \\
u(x,t) &= 0, \quad (x,t) \in \{0,1\} \times [0,T], \\
u(x,0) &= \varphi(x), \quad x \in [0,1],
\end{align*}
\]

with \( 0 < \gamma < 1 \) and \( 1 < \alpha < 2 \). We need to prove the convergence of series (24), (31) and the series associated with their fractional derivatives in space and time. We set \( \Phi := D_x^\alpha \varphi \in L^2(0,1) \). Then:

\[
\Phi = \sum_{n \geq 1} \Phi_n X_n
\] (32)

where

\[
\Phi_n = \frac{\left\langle \Phi, X_n \right\rangle_{L^2(0,1)}}{\left\langle X_n, X_n \right\rangle_{L^2(0,1)}}
\]

\[
= \frac{\left\langle D_x^\alpha \varphi, X_n \right\rangle_{L^2(0,1)}}{\left\langle X_n, X_n \right\rangle_{L^2(0,1)}}
\]

\[
= \frac{\left\langle \varphi, x D_x^\alpha X_n \right\rangle_{L^2(0,1)}}{\left\langle X_n, X_n \right\rangle_{L^2(0,1)}}
\]

\[
= \frac{\left\langle \varphi, X_n \right\rangle_{L^2(0,1)}}{\left\langle X_n, X_n \right\rangle_{L^2(0,1)}}
\]

\[
= \frac{\varphi_n}{\lambda_n} X_n(0)
\]

Then, there is a constant \( c \geq 0 \) such that, for every \( x \in [0,1], t \in [0,T] \) and \( n \) are sufficiently large such that:

\[
\left| \varphi_n X_n(x) E_{\gamma,1} (\lambda_n t^\gamma) \right| \leq \frac{\left\langle \Phi_n X_n(0) \right\rangle_{L^2(0,1)}}{\lambda_n} E_{\gamma,1} (\lambda_n t^\gamma) \leq \frac{c}{\lambda_n} \leq \frac{c}{n^2}.
\] (33)
Remark 2. Proceeding as above with $\phi$ convergence on the whole set $\sum_{n=1}^\infty$ of initial and the boundary conditions of (12) and (13), respectively.

P for the series (31) by setting $P = D_1^2 p \in L^2(0, 1)$. Hence, the functions $\nu$ and $w$ verify the initial and the boundary conditions of (12) and (13), respectively.

Finally, to prove the convergence of the series associated to the fractional derivatives of $\nu$ and $w$, it suffices to prove the convergence of series associated with: $D_1^\alpha v, D_1^\alpha w, D_1^\alpha u$ and $D_1^\alpha \nu$. To this end, let

$$\nu_n(x, t) = \phi_n X_n(x) E_{\nu,1}(\lambda_n t^\gamma),$$

and

$$\tilde{w}_n(x, t) = p_n X_n(x) \int_0^t s^{\gamma-1} E_{\nu,1}(\lambda_n s^\gamma) f(t - s) \, dt;$$

so $\nu(x, t) = \sum_{n=1}^\infty \nu_n(x, y, t)$ and $w(x, t) = \sum_{n=1}^\infty \tilde{w}_n(x, t)$.

From above, for any $n \geq 1$, $x \in (0, 1)$ and $t \in (0, T)$, one has:

$$|D_1^\alpha \nu_n(x, t)| = \left| \phi_n X_n(x) \lambda_n E_{\nu,1}(\lambda_n t^\gamma) \right| = \left| \frac{\phi_n}{\lambda_n} X_n(x) \right| \times |\lambda_n E_{\nu,1}(\lambda_n t^\gamma)|$$

$$\leq \frac{c}{n^\alpha} \times \frac{|\lambda_n|}{1 + |\lambda_n| t^{\gamma}} (c \text{ is independent of } x, t, \text{ and } n)$$

and

$$|D_1^\alpha w_n(x, t)| = |\lambda_n \phi_n X_n(x) E_{\nu,1}(\lambda_n t^\gamma)| = |D_1^\alpha \nu_n(x, t)| \leq \frac{c}{n^\alpha t^{\gamma}}. \quad (34)$$

Therefore, the series $\sum_{n=1}^\infty D_1^\alpha \nu_n(x, t)$ and $\sum_{n=1}^\infty D_1^\alpha w_n(x, t)$ converges normally on $[0, 1] \times [\epsilon, T]$, for every $\epsilon \in (0, T)$.

Applying the same ideas, we obtain the normal convergence of the series $\sum_{n=1}^\infty D_1^\alpha \tilde{w}_n(x, t)$ and $\sum_{n=1}^\infty D_1^\alpha \tilde{w}_n(x, t)$ on $[0, 1] \times [\epsilon, T]$, for every $\epsilon \in (0, T)$. Whence, we obtain the desired convergence on the whole set $[0, 1] \times (0, T]$ by Lemma 1, which achieves the claim.

– Uniqueness of the solution:

Let $u_1, u_2$ be two classical solutions of Problem (10) and set $u^* \equiv u_1 - u_2$. Then, $u^*$ satisfies the following problem:

$$\begin{cases} D_1^\alpha u^*(x, t) - D_2^\alpha u^*(x, t) = 0, & (x, t) \in (0, 1) \times (0, T) \\ u^*(0, t) = u^*(1, t) = 0, & t \in [0, T], \\ u^*(x, 0) = 0, & x \in [0, 1]. \end{cases}$$

Proceeding as above with $\phi \equiv 0$, we obtain that $\phi_n = 0$, for every $n \geq 1$, which implies that $u^* \equiv 0$. \hfill \square

Remark 2. The generalization of the previous proof to the case $d \geq 1$ proceeds as follows: We set $\Phi := D_{i_1}^{\alpha_1} \cdots D_{i_d}^{\alpha_d} \phi \in L^2(\Omega)$. Then,

$$\Phi(x) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \Phi_{n_1 \cdots n_d} \otimes X_{n_1}(x_1).$$
Direct computations give:

\[ \Phi_{n_1 \cdots n_d} = \prod_{i=1}^{d} \lambda_{n_i} \times \Phi_{n_1 \cdots n_d} \]

- The analogous of the estimate (33) becomes, for any \( x \in \Omega, \ t \in [0, T] \) and \( n_1, \cdots, n_d \) sufficiently large:

\[
\left| \varphi_{n_1 \cdots n_d} \bigotimes_{i=1}^{d} X_{n_i}^i (x_i) E_{\Gamma, 1} \left( \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) t^\Gamma \right) \right| \leq \frac{c}{\prod_{i=1}^{d} \lambda_{n_i}^i} \leq \frac{c}{\prod_{i=1}^{d} n_i^{n_i}}.
\]

- The analogous of the time derivative estimate (34) becomes, for any \( x \in \Omega, \ t \in (0, T) \) and \( n_1, \cdots, n_d \) sufficiently large:

\[
\left| D^{\gamma}_{t} v_{n_1 \cdots n_d} (x, t) \right| = \left| \varphi_{n_1 \cdots n_d} \bigotimes_{k=1}^{d} X_{n_k}^k (x_k) \left( \sum_{k=1}^{d} \lambda_{n_k}^k \right) E_{\Gamma, 1} \left( \left( \sum_{k=1}^{d} \lambda_{n_k}^k \right) t^\Gamma \right) \right| \\
\leq \frac{c}{\prod_{k=1}^{d} \lambda_{n_k}^k} \times \left| \sum_{k=1}^{d} \lambda_{n_k}^k \right| \leq \frac{c}{\prod_{k=1}^{d} n_k^{n_k}} \times \frac{1}{1 + t^\Gamma} \left| \sum_{k=1}^{d} \lambda_{n_k}^k \right| \\
\leq \frac{c}{t^\Gamma \prod_{k=1}^{d} n_k^{n_k}}.
\]

- The analogues of the space-derivative estimate (34) becomes, for any \( x \in \Omega, \ t \in (0, T) \) and \( n_1, \cdots, n_d \) sufficiently large:

\[
\left| \sum_{k=1}^{d} D^{\alpha}_{x_k} v_{n_1 \cdots n_d} (x, t) \right| = \left| \left( \sum_{k=1}^{d} \lambda_{n_k}^k \right) \varphi_{n_1 \cdots n_d} \bigotimes_{k=1}^{d} X_{n_k}^k (x_k) E_{\Gamma, 1} \left( \left( \sum_{k=1}^{d} \lambda_{n_k}^k \right) t^\Gamma \right) \right| \\
= \left| D^{\gamma}_{t} v_{n_1 \cdots n_d} (x, t) \right| \\
\leq \frac{c}{t^\Gamma \prod_{k=1}^{d} n_k^{n_k}}.
\]

These new estimates lead to the same conclusions about the Fourier series in the case \( d \geq 1 \) as in the case \( d = 1 \).

**Theorem 3.** Under the same conditions as in Theorem 2, the solution of the direct problem (10) depends continuously on the given data.

**Proof.** The arguments developed in Remark 2 allow, without loss of generality, to restrict ourselves to the case \( d = 1 \).
Let $u$ and $\tilde{u}$ be the solutions to the direct problem, corresponding to the data $\{p, f, \varphi\}$ and $\{\tilde{p}, \tilde{f}, \tilde{\varphi}\}$, respectively. Using (11), we find

$$|u(x, t) - \tilde{u}(x, t)| \leq |v(x, t) - \tilde{v}(x, t)| + |w(x, t) - \tilde{w}(x, t)|,$$

where $\{v, w\}$ and $\{\tilde{v}, \tilde{w}\}$ correspond to the data $\{p, f, \varphi\}$ and $\{\tilde{p}, \tilde{f}, \tilde{\varphi}\}$, respectively. The same arguments as applied to (24) before lead to

$$|v(x, t) - \tilde{v}(x, t)| \leq c \|\varphi - \tilde{\varphi}\|_{L^\infty(0, 1)},$$

where $c$ is a positive constant. On the other hand, a direct computation gives

$$|w_n - \tilde{w}_n| \leq \int_0^t (t - s)^\gamma E_{\gamma, \gamma}(\lambda_n (t - s)^\gamma)(f(s) p_n - \tilde{f}(s) \tilde{p}_n) ds.$$

It holds that

$$|f(s) p_n - \tilde{f}(s) \tilde{p}_n| = |f(s) p_n - f(s) \tilde{p}_n + f(s) \tilde{p}_n - \tilde{f}(s) \tilde{p}_n| \leq \|f\|_{L^\infty((0,T])} |p_n - \tilde{p}_n| + \|\tilde{p}\|_{L^\infty((0,T])} \|f - \tilde{f}\|_{L^\infty((0,T])}.$$

Therefore, there is a constant $c > 0$ such that

$$|w(x, t) - \tilde{w}(x, t)| \leq c (\|f\|_{L^\infty(0, 1)} \|p - \tilde{p}\|_{L^\infty(0, 1)} + \|\tilde{p}\|_{L^\infty(0, 1)} \|f - \tilde{f}\|_{L^\infty(0, T)}),$$

which achieves the proof. \(\square\)

4. The Inverse Problem

Let us consider the problem:

$$\begin{align*}
D_t^\gamma u(x, t) - \sum_{i=1}^d D_{x_i}^{\alpha_i} u(x, t) &= p(x) f(t), \quad (x, t) \in \Omega \times (0, T), \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times [0, T], \\
u(x, 0) &= \varphi(x), \quad x \in \Omega, \\
\int_\Omega u(x, t) dx &= \psi(t), \quad t \in [0, T].
\end{align*} \tag{35}$$

where the functions $p, \varphi$ and $\psi$ are given, and the functions $u$ and $f$ are unknown.

**Example 2.** As above, consider the case $d = 2$ and $\Omega = (0, \pi) \times (0, \pi)$, with $0 < \gamma < 1$ and $1 < \alpha_1, \alpha_2 < 2$. We fix the initial condition $\varphi(x_1, x_2)$, the spatial source term $p(x_1, x_2)$ for any $(x_1, x_2) \in \Omega$ and the “mean value” $\psi(t)$, $t \in [0, T]$ of the solution $u(x_1, x_2, t)$ on $\Omega$:

$$\begin{align*}
\varphi(x_1, x_2) &= 0, \\
p(x_1, x_2) &= \sin(x_1) \sin(x_2) + x_1^{1-\alpha_1} E_{1,2-\alpha_1} (-x_1^2) \sin(y) + \sin(x) x_2^{1-\alpha_2} E_{1,2-\alpha_2} (-x_2^2), \\
\psi(t) &= 4 E_{1,1} (-t^\gamma), \quad t \in [0, T].
\end{align*}$$

The unknown temporal source term is given by $f(t) = -E_{\gamma, 1} (-t^\gamma)$ and the exact solution $u(x_1, x_2, t) = \sin(x_1) \sin(x_2) E_{\gamma, 1} (-t^\gamma)$.

To solve Problem (35), we start by expanding $u(x, t)$ and $p(x)$, using systems (15) and obtain:

$$u(x, t) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} c_{n_1 \cdots n_d}(t) \bigotimes_{i=1}^d X^i_{n_i}(x_i), \tag{36}$$
with
\[ c_{n_1 \ldots n_d}(t) = \frac{\langle u(\cdot, t) \otimes_{i=1}^{d} \tilde{X}_{n_i} \rangle}{\langle \otimes_{i=1}^{d} X_{n_i} \otimes_{i=1}^{d} \tilde{X}_{n_i} \rangle}_{L^2(\Omega)}, \]  
(37)

\[ p(x) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} p_{n_1 \ldots n_d} \otimes X_{n_i}(x_i), \]  
(38)
and
\[ p_{n_1 \ldots n_d} = \frac{\langle p \otimes_{i=1}^{d} X_{n_i} \rangle}{\langle \otimes_{i=1}^{d} X_{n_i} \rangle}_{L^2(\Omega)}. \]  
(39)

By substitution in (35), we formally obtain:
\[ \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \left( \mathcal{D}^i - \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) \right) c_{n_1 \ldots n_d}(t) \otimes X_{n_i}(x_i) = f(t) \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} p_{n_1 \ldots n_d} \otimes X_{n_i}(x_i). \]  
(40)

It follows that
\[ \mathcal{D}^i c_{n_1 \ldots n_d}(t) = \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) c_{n_1 \ldots n_d}(t) = f(t) p_{n_1 \ldots n_d}, \quad 0 < t < T, \]  
(41)

with the initial condition \( c_{n_1 \ldots n_d}(0) = \varphi_{n_1 \ldots n_d} \) and \( \varphi_{n_1 \ldots n_d} \) is defined by (23). Using (36) and (6) we obtain:
\[ \psi(t) = \int_{\Omega} u(x, t) \, dx \]
\[ = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) \sum_{i=1}^{d} x_i^{\alpha_{n_i} - 1} E_{\alpha_{n_i}} \left( \sum_{k=1}^{d} \lambda_{n_k}^k \right) \right) dx_i \]
\[ = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \prod_{i=1}^{d} E_{\alpha_{n_i}+1} \left( \sum_{k=1}^{d} \lambda_{n_k}^k \right) \times c_{n_1 \ldots n_d}(t). \]

Let us set
\[ \Gamma_{n_1 \ldots n_d} = \prod_{i=1}^{d} E_{\alpha_{n_i}+1} \left( \sum_{k=1}^{d} \lambda_{n_k}^k \right). \]

Using (41), we formally find:
\[ \mathcal{D}^i \psi(t) - \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) \Gamma_{n_1 \ldots n_d} \times c_{n_1 \ldots n_d}(t) = f(t) \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \Gamma_{n_1 \ldots n_d} p_{n_1 \ldots n_d}, \]  
(42)

Therefore, we obtain:
\[ f(t) = \frac{\mathcal{D}^i \psi(t) - \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) \Gamma_{n_1 \ldots n_d} \times c_{n_1 \ldots n_d}(t)}{\sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \Gamma_{n_1 \ldots n_d} p_{n_1 \ldots n_d}}. \]  
(43)

According to [12], the solution of (41) is given by:
\[ c_{n_1 \ldots n_d}(t) = \varphi_{n_1 \ldots n_d}(t) E_{\gamma,1} \left( \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) t^\gamma \right) + p_{n_1 \ldots n_d} \int_0^t s^{\gamma-1} E_{\gamma,\gamma} \left( \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) s^\gamma \right) f(t-s) \, ds. \]  
(44)
Replacing $c_{n_1\ldots n_d}(t)$ given by (44) in (43), we then find the integral equation satisfied by the unknown function $f(t)$:

\[
f(t) = \frac{1}{a} \left( D^i_0 \psi(t) - \phi(t) - \int_0^t (t-s)^{i-1} f(s) \rho(t-s) ds \right)
\]

where

\[
a = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \Gamma_{n_1\ldots n_d} p_{n_1\ldots n_d}
\]

(46)

\[
\phi(t) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) \phi_{n_1\ldots n_d} E_{\gamma,1} \left( \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) t^\gamma \right) \Gamma_{n_1\ldots n_d}
\]

(47)

and

\[
\rho(t) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) p_{n_1\ldots n_d} E_{\gamma,1} \left( \left( \sum_{i=1}^{d} \lambda_{n_i}^i \right) t^\gamma \right) \Gamma_{n_1\ldots n_d}
\]

(48)

**Theorem 4.** Let

- $p \in C(\overline{\Omega}, \mathbb{R})$ such that $D_{x_1}^{d_1} \cdots D_{x_d}^{d_d} p \in C(\overline{\Omega}, \mathbb{R})$ and $\int_{\Omega} p(x) \, dx \neq 0$.

- Let $\psi \in C([0, T], \mathbb{R})$ such that $D^i \psi \in C([0, T], \mathbb{R})$.

- $\phi \in C(\overline{\Omega}, \mathbb{R})$ such that $D_{x_1}^{d_1} \cdots D_{x_d}^{d_d} \phi \in C(\overline{\Omega}, \mathbb{R})$ and $\phi|_{\partial \Omega} = 0$.

Then, there exists at least one solution to Problem (35).

Before proving Theorem 4, we introduce the functional framework for the fixed point integral Equation (45). First, applying the same arguments as before, we can show that the functions $\rho$ and $\phi$ defined by (48) and (47) are $\gamma$-differentiable on $[0, T[$. Moreover, the condition $\int_{\Omega} p(x) \, dx \neq 0$ implies that the real number $a \neq 0$.

At this stage, we will establish the existence of at least a solution to the inverse problem (35) and that this solution depends continuously on the data. First of all, we will use the Schauder fixed point theorem in Banach spaces with the Arzela–Ascoli compactness result. To this end, we start by defining the following operator:

\[
B : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})
\]

\[
f \rightarrow B(f) : t \mapsto \frac{1}{a} \left( D^i_0 \psi(t) - \phi(t) - \int_0^t (t-s)^{i-1} f(s) \rho(t-s) ds \right)
\]

(49)

To prove that the operator $B$ admits a fixed point, start by showing that $B$ maps a certain closed convex set into itself, in the space $C([0, T], \mathbb{R})$ equipped with the Bielecki norm. For every $\delta > 0$, we introduce the Bielecki norm:

\[
\| u \|_\delta = \sup_{t \in [0, T]} \left( e^{-\delta t} | u(t) | \right)
\]

(50)

The space $(C([0, T], \mathbb{R}), \| \cdot \|_\delta)$ is a Banach space, and the two norms $\| \cdot \|_\delta$ and $\| \cdot \|_\infty$ are equivalent.

**Lemma 2.** Under the above notations, there exists a positive constant $\delta_* > 0$ such that for any $\delta > \delta_*$, there is a radius $R > 0$ such that the closed convex ball

\[
K = \{ f \in C([0, T], \mathbb{R}) : \| f \|_\delta \leq R \}
\]

is stable by the operator $B$; that is, $B(K) \subset K$. 

Proof. For any $f \in C([0, T], \mathbb{R})$, $t \in [0, T]$ and $\delta > 0$, we have:

$$
e^{-\delta t} |B(f)(t)| = \frac{1}{|a|} \left| \int_0^t e^{-\delta t} \left( D_t^\gamma \psi(t) - \phi(t) - \int_0^t (t-s)^{\gamma-1} \rho(t-s) f(s) \, ds \right) \right|,$$

$$= \frac{1}{|a|} \left| \int_0^t e^{-\delta t} \left( D_t^\gamma \psi(t) - \phi(t) \right) - \int_0^t e^{-\delta(t-s)} (t-s)^{\gamma-1} \rho(t-s) e^{-\delta s} f(s) \, ds \right|,$$

$$\leq \frac{1}{|a|} \left( \|D_t^\gamma \psi\|_\infty + \|\phi\|_\infty + \|f\|_\infty \right) \int_0^t e^{-\delta s} s^{\gamma-1} |\rho(s)| \, ds.$$

At this stage, recall that for any $\gamma > 0$, the function $E_\delta : s \mapsto \frac{1}{|a|} e^{-\delta s} s^{\gamma-1} \rho(s)$ is integrable on $[0, T]$. Moreover, it holds that $\lim_{\delta \to +\infty} \|E_\delta\|_{L^1([0, T])} = 0$; consequently:

$$\exists \delta_* > 0, \forall \delta \geq \delta_*, \quad \frac{1}{|a|} \int_0^T e^{-\delta s} s^{\gamma-1} |\rho(s)| \, ds \leq \frac{1}{|a|} \int_0^T e^{-\delta s} s^{\gamma-1} \rho(s) \, ds < 1.$$

That is, there are two real constants $c_1(\delta) := \frac{1}{|a|} (\|D_t^\gamma \psi\|_\infty + \|\phi\|_\infty)$ and $c_2 := \frac{1}{|a|} \int_0^T e^{-\delta s} s^{\gamma-1} |\rho(s)| \, ds$ satisfying $c_1(\delta) \geq 0$ and $0 \leq c_2 < 1$ such that:

$$\forall \delta \geq \delta_*, \quad \|B(f)\|_\delta \leq c_1(\delta) + c_2 \|f\|.$$

Whence, for any $\delta \geq \delta_*$, there exists $R > 0$ such that the closed ball $K$ of radius $R$ in $(C([0, T], \mathbb{R}), \| \cdot \|_\delta)$ is stable by the operator $B$; this achieves the proof. \(\square\)

**Remark 3.** Using the Bielecki norm allows one to have no constraint on the maximum value that $T$ can take. On the other hand, if we use the classical infinite norm, then $T$ must be less than a finite quantity depending on the data of the problem.

**Lemma 3.** The family $(B(f))_{f \in K}$ is equicontinuous, that is:

$$\forall t \in [0, T], \forall \epsilon > 0, \exists \tau > 0 : \forall t' \in [0, T] \cap [t - \tau, t + \tau], \forall f \in K, \quad |B(f)(t) - B(f)(t')| \leq \epsilon.$$

**Proof.** It suffices to prove that the family $(B_1(f))_{f \in K}$ is equicontinuous, where

$$B_1 : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R}),$$

$$f \mapsto B_1(f) : t \mapsto \int_0^t (t-s)^{\gamma-1} \rho(t-s) f(s) \, ds. \quad (51)$$

We introduce the functions $W_1 : s \mapsto s^{\gamma-1} \rho(s)$ and $W_2 : s \mapsto e^{-\delta s} W_1(s)$, which are clearly integrable on $[0, T]$. Let $t$ and $t'$ in $[0, T]$. We can assume, without loss of generality, that $0 < t < t' \leq T$.

$$|B_1(f)(t') - B_1(f)(t)| = \left| \int_0^t \left( W_1(t') - W_1(t-s) \right) f(s) \, ds + e^{\delta t} \int_t^{t'} W_2(t-s) e^{-\delta s} f(s) \, ds \right|,$$

$$\leq e^{\delta t} \|f\|_\infty \left( \int_0^t |W_1(t') - W_1(t-s)| \, ds + \int_0^{t'-t} |W_2(s)| \, ds \right).$$

First, for every $s \in (0, t)$, we have

$$|W_1(t') - W_1(t-s)| \leq (t-s)^{\gamma-1} |\rho(t'-s)| + (t-s)^{\gamma-1} |\rho(t-s)|,$$

$$\leq 2 \|\rho\|_\infty (t-s)^{\gamma-1}.$$
Using the fact that the map \( s \mapsto (t-s)^{\gamma-1} \) is in \( L^1(0, t) \) and applying the dominated convergence theorem of Lebesgue, we deduce that

\[
\lim_{\epsilon \to 0} \int_0^t |W_1(t' - s) - W_1(t - s)| \, ds = 0.
\]

Fixing now an \( \epsilon > 0 \) and \( t \in (0, T) \), it follows that

\[
\exists r_1 > 0, \forall t' \in [0, T] \cap |t - t_1, t + r_1|, \int_0^{t'} |W_1(t' - s) - W_1(t - s)| \, ds \leq \frac{e^{-\delta T}}{4R} \| \rho \|_\infty \epsilon.
\]

On the other hand, using the fact that the map \( W_2 \) is in \( L^1(0, T) \) and applying the dominated convergence theorem of Lebesgue, we deduce that

\[
\lim_{\epsilon \to 0} \int_0^{t'} |W_2(s)| \, ds = 0.
\]

That is:

\[
\exists r_2 > 0, \forall t' \in [0, T] \cap |t - t_2, t + r_2|, \int_0^{t'} |W_2(s)| \, ds \leq \frac{e^{-\delta T}}{2R} \epsilon.
\]

Let us point out that radii \( r_1 \) and \( r_2 \) depend only on \( t \) and \( \epsilon \), but are independent of \( f \). Whence, for any \( t \in [0, T] \) and \( \epsilon > 0 \), there exists \( r := \min \{r_1, r_2\} > 0 \) such that

\[
\forall t' \in [0, T] \cap |t - r, t + r|, \forall f \in K, \ |B_1(f)(t) - B_1(f)(t')| \leq \epsilon,
\]

which achieves the proof. \( \Box \)

By Arzela–Ascoli theorem, \( B(K) \) is relatively compact. Now, we are able to use the Schauder fixed-point theorem [13].

**Theorem 5** (Schauder fixed-point theorem). Let \( (E, \| . \|) \) be a Banach space, and let \( K \subseteq E \) be convex and closed. Let \( L : K \to K \) be a continuous operator such that \( L(K) \) is relatively compact. Then, \( L \) has a fixed point in \( K \).

**Proof of Theorem 4.** We know from Lemmas 2 and 3 that \( B \) is a continuous operator and \( B(K) \) is relatively compact, with \( K \) convex and closed. By Theorem 5, the operator \( B \) admits a fixed point in \( K \), so the integral equation (45) has a solution in \([0, T]\).

Once the existence of the source term \( f \) is established, the existence of \( u \) follows in a similar way with the arguments developed above for the direct problem. \( \Box \)

**Theorem 6.** Under the conditions of Theorem 4, every solution of the inverse problem (35) depends continuously on the given data \( \varphi, p \) and \( \psi \).

**Proof.** Let \( \{u(x, t), f(t)\}, \{\tilde{u}(x, t), \tilde{f}(t)\} \) be two solution sets of the inverse problem, corresponding to the data \( \{p, \varphi, \psi\}, \{\tilde{p}, \tilde{\varphi}, \tilde{\psi}\} \), respectively. Using (45), we have:

\[
f(t) = \frac{D_0^\gamma \varphi(t)}{a} - \frac{\phi(t)}{a} - \int_0^t (t-s)^{\gamma-1} f(s) \frac{D_0^\gamma (t-s)}{a} \, ds,
\]

and

\[
\tilde{f}(t) = \frac{D_0^\gamma \tilde{\varphi}(t)}{a} - \frac{\tilde{\phi}(t)}{a} - \int_0^t (t-s)^{\gamma-1} \tilde{f}(s) \frac{D_0^\gamma (t-s)}{a} \, ds,
\]

where \( \{a, \phi, \rho\}, \{\tilde{a}, \tilde{\phi}, \tilde{\rho}\} \) correspond to the data \( \{p, \varphi, \psi\}, \{\tilde{p}, \tilde{\varphi}, \tilde{\psi}\} \), respectively.
Remark first that $\rho$, $a$ and $\phi$ depend continuously on $p$ and $\varphi$. Indeed, for the function $\rho$, using (39) and (48), we have the estimate:

$$\|\rho - \bar{\rho}\|_{L^\infty(0,T)} \leq c \|p - \bar{p}\|_{L^\infty(\Omega)}.$$  

(52)

For the real constant $a$, using (46) and (39), we have the estimate:

$$|a - \bar{a}| \leq c \|p - \bar{p}\|_{L^\infty(\Omega)}.$$  

(53)

For the function $\phi$, using (47) and (23), we have the estimate:

$$|\phi(t) - \bar{\phi}(t)| \leq c \|\varphi - \bar{\varphi}\|_{L^\infty(\Omega)}. $$  

(54)

Let $g(t)$ and $\bar{g}(t)$ be such that

$$g(t) = \int_0^t (t - s)^{\gamma - 1} f(s) \frac{\rho(t - s)}{a} \, ds, \quad \bar{g}(t) = \int_0^t (t - s)^{\gamma - 1} \tilde{f}(s) \frac{\bar{\rho}(t - s)}{a} \, ds.$$  

Then, for any $\delta > 0$:

$$e^{-\delta t} |f(t) - \tilde{f}(t)| \leq e^{-\delta t} \left( \left| \frac{D_t^\gamma \psi(t)}{\bar{a}} - \frac{D_t^\gamma \bar{\psi}(t)}{\bar{a}} \right| + \left| \frac{\phi(t)}{\bar{a}} - \frac{\bar{\phi}(t)}{\bar{a}} \right| + |g(t) - \bar{g}(t)| \right).$$  

(55)

Now,

$$e^{-\delta t} \left| \frac{D_t^\gamma \psi(t)}{\bar{a}} - \frac{D_t^\gamma \bar{\psi}(t)}{\bar{a}} \right| \leq \frac{\|D_t^\gamma \psi\|_{L^\infty(0,T)}}{\|\bar{a}\|} \|p - \bar{p}\|_{L^\infty(\Omega)} + \frac{1}{|\bar{a}|} \|D_t^\gamma \psi - D_t^\gamma \bar{\psi}\|_{L^\infty(0,T)}.$$  

and consequently,

$$e^{-\delta t} \left| \frac{\phi(t)}{\bar{a}} - \frac{\bar{\phi}(t)}{\bar{a}} \right| \leq c \frac{M_1}{|\bar{a}|} \|p - \bar{p}\|_{L^\infty(\Omega)} + \frac{1}{|\bar{a}|} \|\psi - \bar{\psi}\|_{L^\infty(\Omega)}.$$  

where $M_1$ is a positive constant such that $\|D_t^\gamma \psi\|_{L^\infty(0,T)} \leq M_1$. Similarly, we obtain:

$$e^{-\delta t} \left| \frac{\phi(t)}{\bar{a}} - \frac{\bar{\phi}(t)}{\bar{a}} \right| \leq c \|\phi\|_{L^\infty(0,T)} \frac{1}{|\bar{a}|} \|p - \bar{p}\|_{L^\infty(\Omega)} + \frac{1}{|\bar{a}|} \|\phi - \bar{\phi}\|_{L^\infty(\Omega)}.$$  

It follows that

$$e^{-\delta t} \left| \frac{\phi(t)}{\bar{a}} - \frac{\bar{\phi}(t)}{\bar{a}} \right| \leq c \frac{M_2}{|\bar{a}|} \|p - \bar{p}\|_{L^\infty(\Omega)} + \frac{c}{|\bar{a}|} \|\varphi - \bar{\varphi}\|_{L^\infty(\Omega)},$$  

where $M_2$ is a positive constant such that $\|\phi\|_{L^\infty(\Omega)} \leq M_2$. Finally, we obtain:

$$e^{-\delta t} |g(t) - \bar{g}(t)| = e^{-\delta t} \int_0^t (t - s)^{\gamma - 1} f(s) \frac{\rho(t - s)}{a} - \tilde{f}(s) \frac{\bar{\rho}(t - s)}{a} \, ds \leq e^{-\delta t} \int_0^t (t - s)^{\gamma - 1} f(s) \frac{\rho(t - s)}{a} - \tilde{f}(s) \frac{\rho(t - s)}{a} \, ds + e^{-\delta t} \int_0^t (t - s)^{\gamma - 1} \tilde{f}(s) \frac{\rho(t - s)}{a} - \tilde{f}(s) \frac{\bar{\rho}(t - s)}{a} \, ds \leq \|\tilde{f}\|_{L^\infty(0,T)} \int_0^t (t - s)^{\gamma - 1} \frac{\rho(t - s)}{a} \, ds = \frac{M_3}{|a|} \|\tilde{f}\|_{L^\infty(\Omega)},$$  

where $\delta$ is a positive constant such that $\|\phi\|_{L^\infty(\Omega)} \leq \bar{\phi}$. Finally, we obtain:
where $M_3$ is a positive constant such that $\|p\|_{L^\infty(0,T)} \leq M_3$ and
\[
\|I\| = \int_0^T (t-s)^{\gamma-1} e^{-\delta(t-s)} \, ds.
\] (56)

Similarly, we obtain:
\[
\left| \frac{\rho(t-s)}{a} - \frac{\check{\rho}(t-s)}{\bar{a}} \right| \leq \frac{cM_3}{|a\ddot{a}|} \|p - \check{p}\|_{L^\infty(\Omega)} + \frac{1}{|a|} \|\rho - \check{p}\|_{L^\infty(0,T)},
\]
\[
\leq \frac{c(M_3+1)}{|a\ddot{a}|} \|p - \check{p}\|_{L^\infty(\Omega)}.
\]

Then,
\[
e^{-\delta t} \int_0^T (t-s)^{\gamma-1} \left| \frac{\rho(t-s)}{a} - \frac{\check{\rho}(t-s)}{\bar{a}} \right| \, ds \leq \frac{c(M_3+1)}{|a\ddot{a}|} \frac{T^\gamma}{\gamma} \|p - \check{p}\|_{L^\infty(\Omega)}.
\]

Therefore, we obtain the estimate:
\[
|g(t) - \check{g}(t)| \leq \frac{T^\gamma}{\gamma} \frac{c(M_3+1)}{|a\ddot{a}|} \|p - \check{p}\|_{L^\infty(\Omega)} + \frac{M_3\|I\|}{|a|} \|f - \check{f}\|_{L^\infty(0,T)}.
\]

Combining the last estimate and (55), we deduce the estimate on the source term:
\[
\left(1 - \frac{M_3\|I\|}{|a|}\right) \|f - \check{f}\| \leq \tilde{c} \left( \|D^\gamma_\gamma \psi - D^\gamma_\gamma \check{\psi}\|_{L^\infty(0,T)} + \|\psi - \check{\psi}\|_{L^\infty(\Omega)} + \|p - \check{p}\|_{L^\infty(\Omega)} \right),
\] (57)

where $\tilde{c}$ is a positive constant depending on $a, \ddot{a}, T, M_1, M_2$ and $M_3$.

Since $\lim_{\delta \to +\infty} \|I\| = 0$, there is $\delta^* > 0$ such that
\[
1 - \frac{M_3\tilde{c}(\delta)}{|a|} > 0, \quad \text{for any} \quad \delta \geq \delta^*
\]
holds true. Whence, there is a positive constant $C > 0$, independent of $f$ and $\check{f}$ such that
\[
\|f - \check{f}\| \leq C \left( \|D^\gamma_\gamma \psi - D^\gamma_\gamma \check{\psi}\|_{L^\infty(0,T)} + \|\psi - \check{\psi}\|_{L^\infty(\Omega)} + \|p - \check{p}\|_{L^\infty(\Omega)} \right).
\]

That is, the solution $f$ depends continuously on the data.

The continuity of the part $u$ of the solution follows with classical arguments for linear problems with the norm: $\|u\|_{\infty,\delta} = \sup_{x \in \Omega} \sup_{t \in [0,T]} e^{-\delta I_u(x,t)}$. $\Box$

5. Conclusions

We studied the inverse problem of finding the solution of a generalized time-space fractional equation and the source term, knowing the spatial mean of the solution at any time. The existence and the continuity with respect to the data of the solution for the direct and the inverse problem are proved by Fourier’s method and the Schauder fixed-point theorem, in an adequate convex bounded subset.

In the literature concerning this topic, an incorrect use of the estimates in the generalized Mittag–Leffler functions is commonly performed. This leads to false proofs of the Fourier series’ convergence to recover the partial differential equation satisfied by the solution, the initial data or the boundary conditions. In the present manuscript, the correct framework to recover the decay of fractional Fourier coefficients is established; this allows one to correctly recover the initial data, the boundary conditions and the partial differential equation satisfied within the space-time domain.
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