ANALYSIS OF THE NEWSBOY PROBLEM SUBJECT TO PRICE DEPENDENT DEMAND AND MULTIPLE DISCOUNTS

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Abstract. Existing papers on the Newsboy Problem that deal with price dependent demand and multiple discounts often analyze those two problems separately. This paper considers a setting where price dependence and multiple discounts are observed simultaneously, as is the case of the apparel industry. Henceforth, we analyze the optimal order quantity, initial selling price and discount scheme in the News-Vendor Problem context. The term of discount scheme is often used to specify the number of discounts as well as the discount percentages. We present a solution procedure of the problem with general demand distributions and two types of price-dependent demand: additive case and multiplicative case. We provide interesting insights based on a numerical study. An approximation method is proposed which confirms our numerical results.

1. Introduction. The Newsboy Problem, also known as the single-period inventory problem or News-Vendor Problem (NVP), is a classical problem in inventory management aiming at finding the optimal order quantity which maximizes the expected profit under probabilistic demand. The optimal order quantity is deduced from the trade-off between two situations: if the order quantity is not enough, the newsboy loses some possible profit; on the other hand, if the order quantity is too large, overstock happens. The Newsboy Problem reflects many real situations: service industries [17], fashion and sporting industries [4], etc. and interest in Newsboy Problem continues unabated [18, 19].

Pricing and multiple discounting are common features observed in real life Newsboy Problems. In many situations, demand depends on product’s selling price since

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demand would increase when selling price decreases. This relationship enables retailers to adjust the selling price to influence demand. Furthermore, multiple discounts mean that the retailer uses a certain number of discounts to sell excess inventory, rather than performing only one discount. In realistic situations, multiple discounts are progressively used to sell excess inventory, that, in turn, impact demand. This is often encountered in the apparel industry where the initial selling price has an important influence on demand realized during the regular selling period and discounts get deeper as the season draws to the end. This end of season, for example, is called the discount period in France, which happens twice every year.

Our work is motivated by the fact that most retailers use several discounts to sell excess inventory. In this situation demand depends on the selling price and discounts are a certain percentage of the initial selling price. The term of discount scheme is often used to specify the number of discounts as well as the discount percentages. A special discount scheme where the discount prices are equally spaced, is called a linear discount scheme. In this work, given the unit purchasing cost, salvage value and the price-demand relationship, we are concentrating on the determination of the order quantity, the initial selling price and the discount scheme that would maximize the expected profit. Two special demand-price relations are considered: additive and multiplicative cases. In the additive case, the mean demand decreases linearly with the selling price, while in the multiplicative case, the mean demand decreases exponentially. These two relations are common expressions used to represent the price-dependent demand in practice [13]. [6] obtains the optimality condition of the order quantity for a NV considering multiple discounts. [8] extends to the case where multiple discounts are used and the demand is price-dependent. The concavity is proved for the NVP with uniformly distributed demand, the condition of optimal order quantity is obtained while the discount prices are linear and the demand-price relationship is considered to be additive.

This paper extends the work of [8] since: (1) we demonstrate the concavity for the NVP with multiple discounts and price-dependent demand under general demand distributions and obtain the optimality condition of the order quantity, i.e. the concavity is not limited to uniform distribution; (2) we provide a simple expression of the optimal expected profit; (3) we obtain optimality conditions of the order quantity for both additive and multiplicative demand case; (4) based on a numerical study, we show some new insights, e.g. on the optimal discount scheme; (5) under some conditions we write the expected profit function in a manner that enables to search the numerical optimal initial selling price. This approximation method confirms the insights observed in numerical studies.

The rest of the paper is organized as follows. Section 2 presents the related literature review. In section 3, we formulate the multiple discounts and price-dependent Newsboy Problem. In Section 4, we solve the order quantity and initial pricing decisions with the objective of maximizing the expected profit, for additive price-dependent demand. Numerical examples are then provided. In Section 5, we treat the case of multiplicative demand in the same way. Section 6 contains further discussions and some suggestions for future research.

2. Literature review. Interest in price-dependent and multiple discounts problem goes on in the last decades. One of the latest work is [3] who consider the
price-dependent and multiple discounts problem with multiple periods over a product’s life. [3] review works on price-dependent and multiple discounts problem, but they are not focused on the Newsboy Problem. Therefore, we review the earlier achievements in the area of NVP, which consists of two streams, i.e.: (1) the NVP with price-dependent demand and (2) the NVP with multiple discounts.

In the classic Newsboy Problem, the selling price is considered as exogenous, over which the retailer has no control. This is true in a perfectly competitive market where buyers are mere pricetakers. However, retailers may adjust the current selling price in order to increase or decrease demand. Therefore, several researchers have suggested extensions of Newsboy Problem in which demand is assumed to be price dependent. [11] assumes that price-dependent demand is affected additively by a random variable, which is independent of the selling price. [5] introduce the case of a multiplicative model in which the stochastic demand is affected multiplicatively by a random variable. [10] examine the pricing and ordering policies of a newsboy facing a random price-dependent demand under two different objectives, (1) the objective of maximizing the expected profit and (2) the objective of maximizing the probability of achieving a certain profit level. Analytical solutions are obtained for the additive price-demand relationship with normal distribution. They develop numerical procedures for another case of demand: the demand distribution is constructed using a combination of statistical data analysis and experts’ subjective estimates. [12] investigates the joint pricing and ordering decisions under general demand uncertainty, aiming to reveal the fundamental properties independent of demand pattern. Unimodality of the expected profit function that traces the best price trajectory over the order-up-to decision was proved under the assumptions that the mean demand is a monotone decreasing function of price. [2] investigate the price-dependent newsvendor model in a competitive environment. They show the conditions for the existence of the pure-strategy Nash equilibrium and its uniqueness. [15] introduces a price-dependent demand with stochastic selling price into the classical Newsboy Problem, analyses the expected average profit for a general distribution function of price and obtains an optimal order quantity. [1] studies the channel coordination with a return policy that lets the manufacturer share the risk of demand uncertainty. The manufacturer’s decision is to identify both the optimal wholesale price and the return policy, based on the retailer’s reaction on that offer. The retailer in turn optimizes the retail price and the order quantity to meet a price-dependent uncertain demand. [14] develops a distribution free approach to News-Vendor Problem with price-dependent demand for the situations in which the NV may be missing demand distribution information or historical demand data may not fit any standard probability distributions. Lower bounds on the expected profit are shown to be jointly concave in price and order quantity.

[6] solves a Newsboy Problem in which multiple discounts are used to sell excess inventory. In this model, retailers progressively increase the number of discounts until all excess inventories are sold out. The product is initially sold at a regular price $v_0$. After some time, if any inventories remain, the unit price is reduced to $v_1$, $v_0 > v_1$. Then, a second discount with a selling price $v_2 (v_1 > v_2)$ is made, etc. The amount demanded for each value of $v_i$ is assumed to be a multiple of the demand realized at the regular selling price and moreover, the coefficients are supposed to be given. [6] solves the problem under two objectives: (a) maximizing the expected profit and (b) maximizing the probability of achieving a target profit. [6] shows that the expected profit is concave and derived the sufficient optimality
parameter | price-demand relation | demand distribution | discount prices |
---|---|---|---|
[6] | fixed | general | known |
[8] | additive | uniform and normal | linear |
our paper | additive and multiplicative | general | all types |

Table 1. Comparison with the work of Khouja (1995,2000)

condition for the order quantity. A closed-form expression for the optimal order quantity is obtained for the objective of maximizing the probability of achieving a target profit. [7] develops an algorithm for identifying the optimal order quantity for the multi-discount Newsboy Problem when the supplier offers the newsboy an all-units quantity discount. [9] provide a solution algorithm to the multi-product multi-discount constrained Newsboy Problem. [8] extends the Newsboy Problem to the case where demand is additively price dependent and multiple discount prices are used to sell excess inventory. Given the initial price and linear discount scheme, he solved the condition of the order quantity which maximizes the expected profit prior to any demand being realized. [16] consider an inventory problem for gradually obsolescent products with price-dependent demand and multiple discounts. They assume that the increase of demand due to price change is linearly correlated with the difference between prior and present prices. However, the demand is supposed to be deterministic as a function of time, which is a limited assumption for the News-Vendor Problem context.

Our work focuses on the News-Vendor Problem and differs from previous works according to the different points summarized in Table 1. We generalize the Newsboy Problem with multiple discounts in three aspects: price-demand relation, demand distribution and discount scheme.

3. The problem under study. Figure 1 represents the sequence of events in a selling season. A season consists of n+1 sub-periods where each sub-period i (i=0,...,n) is characterized by a unit selling price and a stochastic demand which depends on the selling price offered to customers during the sub-period. At the beginning of the season, the newsboy buys from the supplier a quantity Q of products at unit price w. This quantity has to cover all demand during the selling season since it is assumed that the newsboy can not buy products during the season. In sub-period i=0, i.e. the regular period, the product unit selling price is v₀, the demand is x₀. In sub-period i=1, i.e. the first discount period, the product unit selling price is v₁, the total demand until the end of this period is x₁. The rest of periods can be deduced in the same way. As selling season goes on, the discounts get deeper and the NV captures some additional demand in each discount period, until the final discount period, i.e. sub-period i=n, where all remaining products are disposed of at a unit price s where s = vₙ. These discount prices are not given, but for a linear scheme, the discount prices are equally spaced between v₀ and s. Otherwise, we call it a non-linear scheme.

The objective of our problem is to find the order quantity Q that maximizes the expected profit.

Define the following notations:
THE NVP WITH MULTIPLE DISCOUNTS

Figure 1. sequence of events for a selling season

\[ x_0 \] Demand during the regular period with mean \( \mu_0 \) and standard deviation \( \sigma_0 \), a random variable,

\[ f \] Density function of \( x_0 \),

\[ F \] Cumulative distribution of \( x_0 \),

\[ x_i (i > 0) \] Demand accumulated till the sub-period \( i \), a random variable.

Given variables:

\[ s \] Dispose price per unit, \( s = v_n \),

\[ w \] Purchase price per unit.

Decision variables:

\[ v_0 \] Regular selling price (initial price) per unit,

\[ n \] The number of discounts during the season,

\[ v_i \] Unit selling price at the \( i \)-th discount period,

\[ v_0 > v_1 > \cdots > v_i > \cdots > v_n \],

\[ Q \] Order quantity.

The random profit function is a multivariate function of selling prices and the order quantity:

\[
\pi(Q) = \begin{cases} 
  v_0Q - wQ & \text{if } x_0 > Q \\
  v_0x_0 + (Q - x_0)v_1 - wQ & \text{if } x_0 \leq Q < x_1 \\
  v_0x_0 + (x_1 - x_0)v_1 + (Q - x_1)v_2 - wQ & \text{if } x_1 \leq Q < x_2 \\
  \vdots & \vdots \\
  v_0x_0 + (x_i - x_0)v_1 + \cdots + (x_{i-1} - x_{i-2})v_{i-1} + (Q - x_{i-1})v_i - wQ & \text{if } x_{i-1} \leq Q < x_i \\
  \vdots & \vdots \\
  v_0x_0 + (x_1 - x_0)v_1 + \cdots + (x_{n-1} - x_{n-2})v_{n-1} + (Q - x_{n-1})v_n - wQ & \text{if } x_{n-1} \leq Q 
\end{cases}
\]

(3.1)

The profit related to the interval: \( x_{i-1} < Q < x_i \), is the sum of the revenue of the regular period \( v_0x_0 \), the first \( i-1 \) periods \( (x_1 - x_0)v_1 + \cdots + (x_{i-1} - x_{i-2})v_{i-1} \), and the \( i \)-th period \( (Q - x_{i-1})v_i \), subtracted by the total purchase cost \( wQ \).

Let’s remark that the demand accumulated till the \( i \)-th discount period \( x_i \) is a random variable dependent on the demand of the first (regular) period \( x_0 \). The relationship is affected by the selling price associated with the discount period, thus it depends on how the price-demand relation is modeled. Hence in this paper two cases are considered, the additive price-dependent demand (cf. Section 4) and the multiplicative price-dependent demand (cf. Section 5).
4. Optimal pricing and ordering decisions, for additive price-dependent demand. In the case of additive price-dependent demand, the mean demand $\mu$ decreases linearly with the price $v$, i.e., $\mu = a - bv$, $a$ and $b$ are both positive constants obtained from historical data. [8] assumes that $v = W - Bx$, where $B$ is a positive constant known to the NV (it equals to $1/b$ in our model), and $W$ is a random variable with a known probability distribution. When the NV is ordering, he/she does not know the realization of $W$. The realization of $W$ becomes known at the end of the regular selling period. We refer readers for more details to [8], which has considered the additive price-dependent demand case, as explained in the literature review section. Then the relationship between two random variables, $x_i$ and $x_0$, can be written as:

$$x_i - x_0 = \mu_i - \mu_0$$ (4.1)

$\mu_i$ is defined as the mean demand (accumulated demand of each sub-period) corresponding the unit selling price $v_i$, $i=0,1,...,n$, and $\mu_n = \infty$.

4.1. Optimal expected profit and the optimal order quantity. If we replace $x_i$ in the profit function $\pi(Q)$ by $x_0$ (equation 3.1), we can derive the expected profit function $E(\pi(Q))$ (c.f. Appendix 1).

The expected profit can be developed to (c.f. Appendix 1):

$$E(\pi(Q)) = Q[-w + v_0 + \sum_{i=0}^{n-1} (v_{i+1} - v_i)F(Q + \mu_0 - \mu_i)] + \sum_{i=0}^{n-1} \int_0^{Q + u_0 - u_i} (v_i - v_{i+1})(x + u_i - u_0)f(x)dx$$ (4.2)

Lemma 1. The expected profit function $E(\pi(Q))$ is concave.

Proof. The proof is provided in Appendix 2. \hfill \Box

The condition of the optimal order quantity is given by:

$$\sum_{i=0}^{n-1} (v_i - v_{i+1})F[Q^* + \mu_0 - \mu_i] - v_0 + w = 0$$ (4.3)

When $n=1$, we get the optimality condition for the classical newsboy problem:

$$F(Q^*) = \frac{v_0 - w}{v_0 - s}$$ (4.4)

According to equation 4.3, the first term of equation 4.2 is zero for $Q^*$. So, we have:

$$E(\pi(Q^*)) = \sum_{i=0}^{n-1} \int_0^{Q^* + \mu_0 - \mu_i} (v_i - v_{i+1})(x + \mu_i - \mu_0)f(x)dx$$ (4.5)

Equation 4.5 gives the optimal expected profit. We note that when $n=1$, we get the profit associated with the classical newsboy problem:

$$E(\pi(Q^*)) = \int_0^{Q^*} (v_0 - s)xf(x)dx$$ (4.6)
4.2. Numerical analysis. We use normally distributed demand (e.g. [8]) in our examples. Other demand distributions will also work. The concavity enables us to search the optimal order quantity by using a Golden Section method (The golden section search is a technique for finding the extremum of a strictly unimodal function by successively narrowing the range of values inside which the extremum is known to exist). Given a discount scheme, we obtain the expected profit for an arbitrary value of the initial selling price by equation 4.5. Thus we can search the optimal initial price by numerical global optimization methods.

Consider a practical example: A supplier provides a new type of T-shirt at a price \( w = 3 \) Euros per piece. The amount of demand \( (x_0) \) has a normal distribution \( N(\mu_0, \sigma_0) \), the mean \( \mu_0 \) will decrease linearly with the price \( (v_0) \): \( \mu_0 = a - bv_0 \). T-shirts can be disposed of at the end of the selling season with a price \( s \). A manager finds that multiple discounts can improve the profit. The problem is: before the selling season, he needs to determine the order quantity, the initial selling price and discount scheme in order to maximize the profit.

4.2.1. Numerical example 1: Linear discount scheme. \( v_i = \alpha_i \cdot v_0 \), and in the linear discount case, \( \alpha_i = 1 - \frac{1}{\pi i}(i = 1, ..., 5) \). Consider \( \sigma_0 = 0 \) (deterministic distribution), 2, 4, 6, 8, \( n \) increases from 2 to 21, \( a = 80 \), \( b = 8 \), and \( s = 2 \). By setting \( v_0 = 8 \), we have \( \mu_0 = 16 \). Figure 2 shows \( E(\pi(Q^*)) \) as a function of \( n \).

![Figure 2. Expected profit \( E(\pi(Q^*)) \), as a function of the discount number, for normally distributed demand](image1)

The graph shows that the expected profit \( E(\pi(Q^*)) \) increases with the discount number \( n \) (with \( \sigma = 2 \), the expected profit is improved by about 100% with \( n = 5 \) compared with the classical case \( n = 1 \)), but the increase speed is decreasing and tends to be 0 when \( n \to \infty \). The reason for this result is that when \( n > 1 \), the newsboy has more opportunity to sell more products at unit price bigger than \( s \) and the opportunity tends to a limit when \( n \to \infty \). We find that the expected profit decreases with \( \sigma_0 \). This is reasonable because for the classical NVP with normally distributed demand, the expected profit decreases with the uncertainty too. We repeated computations similar to figure 2 for many different combinations of \( w(w \in [2, 4]) \), \( s(s \in [1, 3]) \), \( a(a \in [60, 100]) \), \( b(b \in [6, 12]) \), \( v_0(v_0 \in [6, 12]) \), and similar results are obtained.
In real life, the value of $n$ would be limited. So we consider $n = 5$ in our analysis. Then $v_0$ changes from 7 to 11. For each value of $v_0$, equation 4.5 gives the related expected profit value. Figure 3 is the computing graph of the expected profit.

![Figure 3. Expected profit $E(\pi(Q^*))$, as a function of the initial price](image)

The graph shows that for the different $\sigma_0 = 0, 2, 4, 6, 8$, the expected profit is concave, thus we can derive the optimal value of the initial price with a Golden Section method. However, the concavity of the expected profit in $(Q,v_0)$ was not demonstrated. Even in the case of the News-Vendor problem with price-dependent demand and only one discount [10], the concavity was not proved; Similar results are got repeating the computations with different combinations of $w, s, a, b, n$.

$\sigma_0$ reflects the degree of uncertainty in demand forecast and according to $\mu_0 = a - bv_0$, b’s magnitude reflects demand’s sensitivity to price. The value of $\sigma_0$ and $\mu_0$ determine the probability function $f(x)$. Using equation 4.5, table 2 gives the values of $v_0^*$, $Q^*$ and $E(\pi(Q^*, v_0^*))$ for various combinations of $b$, $\sigma_0$ and $n$. $v_0^*$, $Q^*$ and $E(\pi(Q^*, v_0^*))$ all increase with $n$; The effects of $b$ follow intuitive expectation too: a lower value of $b$ enables the firm to set a higher price, have a larger quantity of products, and realize a higher expected profit. When the uncertainty increases, $E(\pi(Q^*, v_0^*))$ decreases. This reflects the potential value of reducing demand uncertainty.

4.2.2. Numerical example 2: Non-linear discount scheme. Consider the numerical example $n = 5, \sigma = 4, w = 3, s = 2, a = 80, b = 8$. The discount prices were produced as: $\alpha_1v_0$, $\alpha_2v_0$, $\alpha_3v_0$, $\alpha_4v_0$, $s$. In the linear case, $\alpha_i = 1 - \frac{1 - 5}{5}i(i = 1, \ldots, 5)$. Then $\alpha_i$ is generated by adding a term to these proportions for non-linear cases. We produce a series of discount scheme produced with a certain logic in order to assess the sensitivity of the linear discount scheme.

$\alpha_i = 1 - \frac{1 - 5}{5}i + \text{coe}(5-i)i(i = 1, \ldots, 5)$. We change the coefficient $\text{coe}$ to control the perturbation of the linear discount scheme. We show here 7 series of discounts ($\text{coe}=-0.03, -0.02, -0.01, 0, 0.01, 0.02, 0.03$) (Figure 4), and compute the optimal expected profit (Table 3).

The first line in table 3 is the linear case. Others are non-linear. For $\text{coe} > 0$, when $\text{coe}$ is larger, the discount scheme curve is farther from the linear discount line and we found that the maximum expected profit decreases. The optimal initial price tends to decrease too. For $\text{coe} < 0$, when $\text{coe}$ decreases, we find the same
properties. And when the \( \text{coe} \) has the same absolute value, the positive one lead to bigger expected profit, e.g. the expected profit of scheme 5 is bigger than that of scheme 2. In our cases, the linear scheme gives almost the largest expected profit, but the expected profit of scheme 4 is a little bigger. The extreme case of non-linearity is the case where all the first four discounts are 100\% or the same to \( s \). This is the same to the case that only one discount \( s \) happens: the classical NVP.

To summarize, after the discount number is fixed, it is more profitable to cut down the price slowly at the beginning of the season and then at a faster magnitude at

\[

test & n & b & \sigma_0 & v_0^* & Q^* & E(\pi(Q^*, v_0^*)) \\
1 & 4 & 6 & 2 & 10.20 & 55.8 & 249.0 \\
2 & 4 & 6 & 4 & 10.18 & 55.9 & 246.9 \\
3 & 4 & 6 & 6 & 10.24 & 56.1 & 245.0 \\
4 & 4 & 6 & 8 & 10.23 & 56.9 & 243.4 \\
5 & 4 & 8 & 2 & 8.54 & 50.4 & 153.3 \\
6 & 4 & 8 & 4 & 8.58 & 49.8 & 151.6 \\
7 & 4 & 8 & 6 & 8.59 & 49.6 & 150.2 \\
8 & 4 & 8 & 8 & 8.57 & 50.0 & 148.6 \\
9 & 4 & 10 & 2 & 6.60 & 46.3 & 95.0 \\
10 & 4 & 10 & 4 & 6.64 & 44.5 & 94.3 \\
11 & 4 & 10 & 6 & 6.64 & 44.3 & 93.6 \\
12 & 4 & 10 & 8 & 6.61 & 44.6 & 92.2 \\
13 & 5 & 6 & 2 & 11.41 & 56.6 & 263.9 \\
14 & 5 & 6 & 4 & 11.51 & 56.4 & 262.0 \\
15 & 5 & 6 & 6 & 11.47 & 56.7 & 260.2 \\
16 & 5 & 6 & 8 & 11.54 & 57.4 & 258.2 \\
17 & 5 & 8 & 2 & 8.81 & 51.9 & 159.8 \\
18 & 5 & 8 & 4 & 8.71 & 50.9 & 158.6 \\
19 & 5 & 8 & 6 & 8.75 & 50.8 & 157.4 \\
20 & 5 & 8 & 8 & 8.81 & 51.2 & 155.8 \\
21 & 5 & 10 & 2 & 7.09 & 45.7 & 100.1 \\
22 & 5 & 10 & 4 & 7.06 & 45.0 & 99.8 \\
23 & 5 & 10 & 6 & 7.01 & 45.1 & 98.8 \\
24 & 5 & 10 & 8 & 7.09 & 45.3 & 97.6 \\
25 & 6 & 6 & 2 & 11.90 & 57.6 & 271.5 \\
26 & 6 & 6 & 4 & 11.90 & 57.2 & 270.0 \\
27 & 6 & 6 & 6 & 11.88 & 57.5 & 268.3 \\
28 & 6 & 6 & 8 & 12.0 & 58.2 & 266.3 \\
29 & 6 & 8 & 2 & 8.91 & 52.6 & 164.5 \\
30 & 6 & 8 & 4 & 8.91 & 51.5 & 163.7 \\
31 & 6 & 8 & 6 & 8.94 & 51.6 & 162.6 \\
32 & 6 & 8 & 8 & 8.91 & 52.1 & 161.0 \\
33 & 6 & 10 & 2 & 7.16 & 44.8 & 103.8 \\
34 & 6 & 10 & 4 & 7.18 & 45.7 & 103.3 \\
35 & 6 & 10 & 6 & 7.19 & 45.8 & 102.3 \\
36 & 6 & 10 & 8 & 7.18 & 46.1 & 100.0 \\

Table 2. The optimal order initial price, order quantity and expected profit for different combinations of \( n, b, \sigma_0 \) for normally distributed demand

4.3. Approximation of the optimal expected profit and condition for the optimal initial price. The above numerical examples show some interesting properties, e.g. the expected profit seems to be a parabola function of the initial price and the optimal initial prices for different demand uncertainties are close to each other, see figure 3. However, they are not obvious to be explained from equation 4.5. An approximation method is proposed in order to explain them and it provides a faster way for the NV to make decisions. We write equation 4.5 in another way, by two steps. In the first step we consider the deterministic demand case. In the second step, we introduce the impact of the uncertainty of demand. The equation of the expected profit can therefore be decomposed in 2 components:

\[ \mathbb{E}(\pi(Q^*)) = \mathbb{E}_\sigma + \mathbb{E}_v + \epsilon \]  

(4.7)

\( \mathbb{E}_\sigma \) is a part of expected profit depending on \( \sigma \) only, \( \mathbb{E}_v \) is a part of expected profit depending on the prices the NV use only and \( \epsilon \) is an error. Equation 4.7 allows us to get the optimality condition of \( v_0 \). In order to be clearer, principle results are presented in Table 4.

We have:

\[ E(\pi(Q^*)) = \sum_{i=0}^{n-1} \int_0^{Q^*+\mu_0-\mu_i} (v_i - v_{i+1})(x + \mu_i - \mu_0)f(x)dx; \]

and

\[ \sum_{i=0}^{n-1} (v_i - v_{i+1})F[Q^* + \mu_0 - \mu_i] = v_0 - w \]
For a NVP with uniformly distributed demand for example, if \( \forall j, u_j - u_{j-1} > \sigma_0/2 \).
There must be a \( i \) that if \( j > i \), then \( F(Q^* - \mu_j + \mu_0) = 0 \) and if \( j < i \), we have \( F(Q^* - \mu_j + \mu_0) = 1 \).
Thus, \( F(Q^* - \mu_i + \mu_0) = \frac{v_i - w}{v_{i+1} - v_i} \).

We have \( F(Q^* - \mu_i + \mu_0) \geq 0 \), as a result, \( \frac{v_i - w}{v_{i+1} - v_i} \geq 0 \), thus \( v_i \geq w \). In other words, the inventory with quantity \( Q^* \) is all sold with prices higher than the purchasing price \( w \). In fact, when the total discount number \( n \) is fixed, the latter discounts are unused, as a result, if the NV cut the price slower in the beginning (before the price is cutten to be lower than \( w \)), more discounts are really used. This explains why the best discount scheme cuts down the price slowly in the beginning of the selling season and faster in the ending, in our numerical examples. Though the demand distributions have higher uncertainties, it can be explained in the same way. The optimal expected profit can be written as: \( E(\pi(Q^*)) = (v_0 - v_1)\mu_0 + (v_1 - v_2)\mu_1 + \ldots + (v_{i-1} - v_i)(\mu_{i-1}) + \int_0^{Q^*+\mu_0-\mu_i}(v_i - v_{i+1})(x + \mu_i - \mu_0)f(x)dx \).
We have \( \int_0^{Q^*+\mu_0-\mu_i}(v_i - v_{i+1})(x + \mu_i - \mu_0)f(x)dx = (v_i - w)\mu_i - \frac{\sigma_0^2}{4}(v_i - v_{i+1})(1 - (2\frac{v_i - w}{v_{i+1} - v_i} - 1)^2) \). Then, \( E(\pi(Q^*)) = E_v + E_\sigma \), with
\[
E_\sigma = \frac{-\sigma_0^2}{4}(v_i - v_{i+1})(1 - (2\frac{v_i - w}{v_{i+1} - v_i} - 1)^2) = O(\sigma_0)
\]
\[
E_v = (v_0 - v_1)\mu_0 + (v_1 - v_2)\mu_1 + \ldots + (v_{i-1} - v_i)(\mu_{i-1}) + (v_i - w)\mu_i \quad (4.8)
\]

**Lemma 2.** For an additive price dependent demand with uniform distribution \( U[\mu_0 - \sigma_0, \mu_0 + \sigma_0] \), the optimal expected profit \( E(\pi(Q^*)) \) is the sum of \( E_v \), \( E_\sigma \) and an error \( \epsilon \); \( E_\sigma = O(\sigma_0) \), a function of the uncertainty \( \sigma_0 \); if \( \forall i, \sigma_0 \leq \frac{\mu_i - \mu_{i-1}}{2}, \epsilon = 0 \).

Similarly, for any demand distribution function who has an upper bound and a lower bound, the optimal expected profit can be developed to the sum of \( E_v \), \( E_\sigma \) and \( \epsilon \). The most used distributions, like normal distribution, Poisson distribution, can be approximated by bounded distributions. For example, we can use triangular distribution to approximate normal distribution. Here we give the expressions of \( E_\sigma \) for normal distribution and uniform distribution and the conditions that makes \( \epsilon = 0 \):

For uniform distribution, if \( \forall j, \sigma_0 \leq \frac{\mu_i - \mu_{i-1}}{2}, \epsilon = 0 \),
\[
E_\sigma = \frac{-\sigma_0^2}{4}(v_i - v_{i+1})(1 - (2\frac{v_i - w}{v_{i+1} - v_i} - 1)^2) \quad (4.9)
\]
For normal distribution, if \( \forall j, \sigma_0 \leq \frac{\mu_i - \mu_{i-1}}{4}, \epsilon = 0 \),
\[
E_\sigma \approx \frac{-\sigma_0^2}{4}(v_i - v_{i+1})f(F^{-1}(\frac{v_i - w}{v_{i+1} - v_i})) \quad (4.10)
\]
The “\( \approx \)” comes from the fact that it’s not a finite distribution.

| Distribution | \( U[\mu_0 - \sigma_0, \mu_0 + \sigma_0] \) | \( N(\mu_0, \sigma_0) \) |
|--------------|---------------------------------|-------------------|
| Condition for \( \epsilon = 0 \) | \( \forall j, \sigma_0 \leq \frac{\mu_i - \mu_{i-1}}{2} \) | \( \forall j, \sigma_0 \leq \frac{\mu_i - \mu_{i-1}}{4} \) |
| \( E(\pi(Q^*)) \) | \( E_\sigma + E_v \) | \( E_\sigma + E_v \) |
| \( E(\pi(Q^*)) \) for linear case | equation 4.11 | equation 4.11 |
| \( E_v \) | equation 4.9 | equation 4.8 |
| \( E_\sigma \) | equation 4.10 | equation 4.10 |

**Table 4.** Expected profit function for uniform and normal distributions

The relationships between \( \mu \) and \( \sigma \) and \( v \) and \( w \) are
\[ E = \int_0^{\pi} \frac{(v_i - v_{i+1})(x + \mu_i - \mu_0)f(x)dx}{\sigma_0^2} \approx \frac{-\sigma_0^2}{4}(v_i - v_{i+1})(1 - (2\frac{v_i - w}{v_{i+1} - v_i} - 1)^2) \]
A numerical example can well verify these results. Consider the following example: the demand is normally distributed \( N(\mu_0, \sigma_0) \) respectively uniformly distributed \( U[\mu_0 - \sigma_0, \mu_0 + \sigma_0] \), \( w = 3, a = 80, b = 8, s = 2, \mu_0 = a - b v_0 \) and \( v_0 = 8 \), then we have \( \mu_0 = 16 \). Setting \( \sigma_0 = 0 \) (deterministic demand), \( 2, 4, 6, 8 \), and \( n \) increases from 2, the value of the expected profit \( E(\pi(Q^*)) \) is calculated by equation 4.5. Figure 5 and 6 show the values of \( E(\pi(Q^*)) - E_\sigma \) and \( E_v \).

![Figure 5](image1.png)

**Figure 5.** The value of \( (E(\pi(Q^*)) - E_\sigma) \), as a function of discount number, with normal distribution.

![Figure 6](image2.png)

**Figure 6.** The value of \( (E(\pi(Q^*)) - E_\sigma) \), as a function of discount number, with uniform distribution.

For the case of uniform distribution, the graph shows that \( \epsilon = 0 \) for \( n < 11 \); for the normally distributed demand case, \( \epsilon = 0 \) for \( n < 7 \). For a News-Vendor problem, generally it is enough to have 7 discount periods. Repeat the computation with different combinations of \( w, s, a, b, v_0 \), similar results were got.

Therefore, it is practically feasible for the manager to approximate the expected profit by \( E_\sigma + E_v \), and numerically it’s faster. Taking the classical NVP with uniform distribution for example: \( E_v = (v_0 - v_1)\mu_0 + (v_0 - w)\mu_0 = (v_0 - w)\mu_0; E_\sigma = -\frac{\sigma_0}{4}(v_0 - s)(1 - (\frac{v_0 - w}{v_0 - s} - 1)^2); \) if \( \forall \sigma_0 \leq \frac{u_1 - u_0}{2}, \epsilon = 0 \), in fact this condition can be satisfied for all \( \sigma_0 \) because \( u_1 = \infty \). Developing equation 4.6, we get the same equation as \( E_v + E_\sigma \).
For a NV with additive demand, \( x_i = \mu_i + \epsilon_i \), and \( \mu_i = a - bv_i \). \( a \) and \( b \) are both positive constants, and \( \epsilon_i \) is a random variable with a probability density function and cumulative distribution function with a mean of zero. When discounts are linearly decreasing, \( v_i = v_0 - (v_0 - s)i/n \). In conditions that make \( \epsilon = 0 \), the optimal expect profit is developed to:

\[
E(\pi(Q^*)) = E_\sigma + \left( -\frac{b}{2} - \frac{b}{2n} \right) v_0^2 + a v_0 + \frac{b(w + s)}{2n} v_0 + \frac{b}{2} w^2 - aw - \frac{bws}{2n} + \frac{b}{2}(w - v_i)(v_{i+1} - w)
\]

(4.11)

Subtract \( E(\pi(Q^*)) \) by \( E_\sigma \) and the last term, it turns to be a parabola of \( v_0 \) and hyperbola of \( n \).

\[
E(\pi(Q^*)) = E_\sigma + \left( -\frac{b}{2} - \frac{b}{2n} \right) v_0^2 + a v_0 + \frac{b(w + s)}{2n} v_0 + \frac{b}{2} w^2 - aw - \frac{bws}{2n} + \frac{b}{2}(w - v_i)(v_{i+1} - w)
\]

(4.12)

Then

\[
E(\pi(Q^*)) = E_\sigma + \left( -\frac{b}{2} - \frac{b}{2n} \right) v_0^2 + a v_0 + \frac{b(w + s)}{2n} v_0 + \frac{b}{2} w^2 - aw - \frac{bws}{2n} + \frac{b}{2}(w - v_i)(v_{i+1} - w)
\]

(4.13)

In conditions that make \( \epsilon = 0 \), the expected profit function is the sum of \( E_\sigma \) with order \( \frac{1}{2} \), the last term with order \( n^2 \) (we have \( 0 \leq \frac{b}{2}(w - v_i)(v_{i+1} - w) \leq \frac{b}{2}(w - s)^2 ) \) and a function of \( v_0 \) and \( n \).

This function is a parabola of \( v_0 \) and hyperbola of \( n \). This explains the numerical results that the expected profit increases with \( n \) but has an upper limit.

We define \( v_p \) the optimal condition of the parabola, \( v_p = \frac{2mu + b(w + s)}{2b(n + 1)} \). Obviously, the \( v_p \) increases with \( n \).

5. Optimal pricing and ordering decisions, for multiplicative price-dependent demand. In the case of multiplicative price-dependent demand, \[5\] assume that: \( x = \mu(v)\epsilon, \epsilon \) is a random variable independent of price whose actual realization is not known when the NV is ordering; the realization of \( \epsilon \) becomes known at the end of the regular selling period. Then the relationship between two random variables, \( x_i \), can be written as:

\[
\frac{x_i}{x_0} = \frac{\mu_i}{\mu_0}
\]

(5.1)

\( \mu_i \) is defined as the mean demand (accumulated demand of each sub-period) corresponding the unit selling price \( v_i \), \( i=0,1,...,n \), and \( \mu_n = \infty \).

Let us provide some argument in support of the assumption of multiplicative price-dependent demand. The idea is as follows. The actual sale of the product during the season depends on whether or not customers like that particular product. In terms of modeling, this is represented through the random term that affects the sales. The higher the random terms (compared to the average value of one), the larger the actual sales. And conversely, the lower the random terms (compared to the average value of one), the lower the actual sales. If we assume that customers coming during the sales season will statistically have the same behavior as those coming during the regular season, it is therefore consistent to use the same random term to reflect whether the product under consideration is successful. Let us illustrate this further. Consider a sweater with two colors. Color 1 has been very
successful during the regular season and the actual sales were 40% higher than the expected value. On the other hand, customers did not like very much Color 2 and the actual sales were 30% lower than expected. It is reasonable to assume that the sales of Color 1 sweater during the sales season will be 40% higher than expected while the sales of Color 2 sweater during the sales season will be 30% lower than expected.

5.1. Optimal expected profit and the optimal order quantity. Replace $x_i$ in the profit function $\pi(Q)$ by $x_0$ (equation 3.1). The expected profit function is derived in the Appendix 3.

Lemma 3. The expected profit function $E(\pi(Q))$ is concave.

Proof. The proof is provided in the Appendix (c.f. Appendix 4).

The condition of optimal order quantity is given by:

$$\sum_{i=0}^{n-1} (v_i - v_{i+1}) F\left[Q \frac{\mu_0}{\mu_i}\right] - v_0 + w = 0 \quad (5.2)$$

When $n=1$, we get the optimality condition for the classical newsboy problem:

$$F(Q^*) = \frac{v_0 - w}{v_0 - s}$$

Similar to the additive demand case, the optimal expected profit is:

$$E(\pi(Q^*)) = \sum_{i=0}^{n-1} \int_0^{Q^*} \frac{\mu_0}{\mu_i} (v_i - v_{i+1}) \mu_i f(x) dx \quad (5.3)$$

Let’s note that when $n=1$, we get the profit for the classical newsboy problem:

$$E(\pi(Q^*)) = \int_0^{Q^*} (v_0 - s) xf(x) dx$$

5.2. Approximation of the expected profit function. The approximation method in the additive case inspires us to do the same thing in this multiplicative case in the same way. For this reason we propose the approximation method first and then after we will give the numerical examples for both section 5.1 and 5.2. In this way we can make comparisons between results in these two sections. The equation of the expected profit can therefore be decomposed in 3 components too:

$$E(\pi(Q^*)) = E_\sigma + E_v + \epsilon \quad (5.4)$$

$E_\sigma$ is a part of expected profit depending on $\sigma$ only, $E_v$ is a part of expected profit depending on the prices the NV use only and $\epsilon$ is an error. Equation 5.4 allows us to get the optimality condition of $v_0$. In order to be clearer, principle results are presented in Table 5.

$$E_v = (v_0 - v_1) \mu_0 + (v_1 - v_2) \mu_1 + \cdots + (v_{i-1} - v_i) (\mu_{i-1}) + (v_i - w) \mu_i \quad (5.5)$$

Lemma 4. For a uniform distribution $U[\mu_0 - \sigma_0, \mu_0 + \sigma_0]$, the optimal expected profit is the sum of $E_v$, $E_\sigma$ and $\epsilon$; $E_\sigma = O(\sigma_0)$; if $\forall i, \sigma_0 \leq \frac{\mu_i - \mu_{i-1}}{2}$, $\epsilon = 0$. 
Fix \( n \), for \( \pi^* \) and \( \nu_i \), \( \nu_i = s/v_i \) have always an \( i \) that \( v_i < w \). This simplifies equation 5.5.

\[
\text{Exponential case:} \quad \text{For normal distribution, if } \mu_0, \sigma_0 = 4000, \text{ and } \epsilon = 0, \text{ the optimal expected profit can be obtained by equation 5.3.}
\]

\[
\lim_{n \to \infty} \mathbb{E}(\pi(Q^*)) = av_0^{1-b} \frac{1 - (w/v_0)^{1-b}}{1-b}
\]  
(5.9)

Fix \( n \), for \( w = \nu_i \) or \( w = \nu_{i+1} \), \( \mathbb{E}(\pi(Q^*)) =
\]

\[
av_0^{1-b} \frac{1 - (\frac{w}{v_0})^{1/n}}{1 - (\frac{w}{v_0})^{1-b/n}} (1 - (\frac{w}{v_0})^{1-b})
\]  
(5.10)

No direct expression for optimal initial price is obtained. But equation 5.10 can help a manager to get an approximate value of it. Numerical examples will show more insights on it.

### 5.3. Numerical analysis

We use normal distributed demand \( \mathcal{N}(\mu_0, \sigma_0) \) in our examples. Let’s note that other distributions will also work well. Give \( s = 2, w = 3, a = 4000, b = 4 \). The optimal expected profit is obtained by equation 5.3.

#### 5.3.1. Numerical example 3: Discount prices are exponentially declining

We work on the multiplicative price-dependent demand in an exponential declining discount case. According to lemma 4, \( \mathbb{E}(\pi(Q^*)) - \mathbb{E}_v - \mathbb{E}_\sigma = \epsilon \), and \( \epsilon = 0 \) in the conditions obtained. According to equation 5.9, the expected profit should have a limit close to \( av_0^{1-b} \frac{1-(w/v_0)^{1-b}}{1-b} = 38.7 \). Set \( v_0 = 5, \sigma_0 = 0 \) (deterministic demand), 0.1\( \mu_0 \), and

| Distribution | \( \mathbb{E}[\mu_0 - \sigma_0, \mu_0 + \sigma_0] \) | \( \mathbb{N}(\mu_0, \sigma_0) \) |
|-------------|-----------------|-----------------|
| Condition that \( \epsilon = 0 \) | \( \forall j, \sigma_0 \leq \frac{\mu_j - \mu_{j+1}}{4} \) | \( \forall j, \sigma_0 \leq \frac{\mu_j - \mu_{j+1}}{4} \) |
| \( \mathbb{E}(\pi(Q^*)) \) | \( E_\sigma + E_v \) | \( E_\sigma + E_v \) |
| \( E_v \) | equation 5.8 | equation 5.8 |
| \( E_\sigma \) | equation 5.6 | equation 5.7 |

**Table 5.** Expected profit function for uniform and normal distributions

Similarly, for any demand distribution function who has an upper bound and a lower bound, the optimal expected profit can be developed to the sum of \( E_v, E_\sigma \) and \( \epsilon \). Here we give the expressions of \( E_\sigma \) for normal distribution and uniform distribution and the conditions that makes \( \epsilon = 0 \).

For uniform distribution, if \( \forall j, \sigma_0 \leq \frac{\mu_j - \mu_{j+1}}{4}, \epsilon = 0, E_\sigma = \]

\[
- \frac{\sigma_0 \mu_j}{4 \mu_0} (v_i - v_{i+1}) (1 - (w/v_i - v_{i+1}))
\]

(5.6)

For normal distribution, if \( \forall j, \sigma_0 \leq \frac{\mu_j - \mu_{j+1}}{4}, \epsilon = 0, E_\sigma \approx \]

\[
- \frac{\sigma_0^2 \mu_j}{4 \mu_0} (v_i - v_{i+1}) f(F^{-1}(\frac{v_i - w}{v_i - v_{i+1}}))
\]

(5.7)

We work on the multiplicative price-dependent demand in an exponential declining discount case. According to lemma 4, \( \mathbb{E}(\pi(Q^*)) - \mathbb{E}_v - \mathbb{E}_\sigma = \epsilon \), and \( \epsilon = 0 \) in the conditions obtained. According to equation 5.9, the expected profit should have a limit close to \( av_0^{1-b} \frac{1-(w/v_0)^{1-b}}{1-b} = 38.7 \). Set \( v_0 = 5, \sigma_0 = 0 \) (deterministic demand), 0.1\( \mu_0 \), and

\[
\lim_{n \to \infty} \mathbb{E}(\pi(Q^*)) = av_0^{1-b} \frac{1 - (w/v_0)^{1-b}}{1-b}
\]  
(5.9)

Fix \( n \), for \( w = \nu_i \) or \( w = \nu_{i+1} \), \( \mathbb{E}(\pi(Q^*)) =
\]

\[
av_0^{1-b} \frac{1 - (\frac{w}{v_0})^{1/n}}{1 - (\frac{w}{v_0})^{1-b/n}} (1 - (\frac{w}{v_0})^{1-b})
\]  
(5.10)
The expected profit $E(\pi(Q^*))$ is calculated by equation 5.3. Figure 7 shows the values of $E(\pi(Q^*)) - E_\sigma$ and $E_v$.

The graph shows that $E(\pi(Q^*)) - E_\sigma$ and $E_v$ increase with the number of discounts; the increase speed is decreasing and tends to be 0 when $n \to \infty$. These results are similar to the additive demand case. When $n < 7$, $\epsilon = 0$ for these values of $\sigma_0$; then $\epsilon$ will increase with $n$, but even at $n=20$, $\epsilon < 3.6\%E_v$. Repeat computations with different combinations of $s, w, a, b, v_0$, we get similar results. So it is practically feasible to calculate the expected profit by the sum of $E_v$ and $E_\sigma$. And numerically it’s much quicker.

5.3.2. Numerical example 4: The prices are not exponentially declining. We take in our analysis $n = 6$, $\sigma_0 = 0.1\mu_0$, and $v_0$ changes from 3 to 12. The discount prices were produced as: $\alpha_1 v_0$, $\alpha_2 v_0$, $\alpha_3 v_0$, $\alpha_4 v_0$, $\alpha_5 v_0$, $s$, $\alpha_i = (s v_0)^{i/n}(1 + coe(n-i)i)(i = 1, \ldots, n)$. We change $coe$ to control the perturbation of the exponential discount scheme. When $coe = 0$, it is the exponentially declining case. We show here 7 series of discounts (coe=-0.03, -0.02, -0.01, 0, 0.01, 0.02, 0.03), and compute the expected profit by equation 5.3 (Figure 9). Figure 9 shows also the approximate expected profit value for the exponential discount scheme (equation 5.10).

As Figure 9 shows, the approximate curve is concave, it has the optimal expected profit(29.7) at the initial price $v_0 = 6.3$, while the equation 5.3 gives two poles (scheme 0). The first maximum (30.2, which is also the global maximum) occurs at $v_0 = 6.1$. The difference between these two initial prices is 3.3%, and 2% between the optimal expected profits. We find that the two curves coincide at $v_0 = 6.7$: in this case, $v_4 = w = 3$, this is a special case when equation 5.10 equals to equation 5.8. When $v_0 < 5$, these two curves share the same values. But error of the approximate equation 5.10 turns bigger when initial price is bigger. This error comes from our assumption: $v_i = w$, while in fact, $v_i \leq w < v_{i-1}$. This assumption gives an error between $[0, v_{i-1} - v_i]$. In this case, $v_{i-1} - v_i = ((s v_0)^{i-1/n} - (s v_0)^{i/n})v_0$, it increases with $v_0$.

The expected profit can have several poles (e.g.scheme 6). Comparing the exponentially declining scheme to others, we get similar results to the additive case. The discount scheme 3 with $coe = 0.01$ gives the maximum value (31.0) of optimal expected profit, and it’s close to the exponentially declining case (30.2).

To conclude, the best discount scheme happens when the selling price is cutted down a little slower than the exponential case at the beginning of the selling season;
Figure 8. Discount percentages at $v_0 = 6$ for different schemes

Figure 9. Expected profit as function of initial price

the exponentially declining discount scheme brings an optimal expected profit which is very close to the best discount scheme; when the manager choose the exponentially declining discount scheme, an approximate function can be used to get the optimal initial price.

6. Conclusion. In this paper, we extend the classical newsboy problem to the case where the demand is price dependent and multiple discounts are used to sell excess inventory, which is disposed at the end of the selling season. We determine the optimal order quantity, the optimal initial selling price and the optimal discount scheme.

We develop a general profit formulation for a NVP with multiple discounts. We prove the concavity of the expected profit for both additive and multiplicative price-dependent demand cases under general demand distributions with no limit on the discount scheme (in other words, it works for any discount scheme with decreasing percentages). We develop then the optimality conditions of the order quantity for both cases. Furthermore, we provide a simple expression of the expected profit corresponding the optimal order quantity.

Numerical examples show that when the discount number is bigger, the NV should increase the order quantity. But when the demand sensitivity to price is higher, it is better for the NV to prepare less inventory before the selling season.
The expected profit increases with the discount number, but it has an upper bound. It is not reasonable to use too many discounts, because the increasing speed of the expected profit decreases and tends to be zero.

For additive and multiplicative demand, a common result is that it is not good to cut down prices fast in the beginning of the season. The optimal scheme in our examples cuts the price slowly in the beginning and faster in the end. For additive case, the expected profit is concave in the initial selling price, thus the optimal initial price can be obtained by a golden section method.

An approximation method is developed. We write the profit function as the sum of a function of price, a function of uncertainty and an error term. We derive conditions where this error is zero and the optimality conditions of the initial selling price. These expected profit equations show a much clearer insight into the impact of initial price and discount number on the expected profit and confirmed our numerical results. In additional case with linear discount scheme, the optimal initial price increases with discount number.

Future research can address several extensions of the above model. An extension dealing with the cost of discounting will make it possible to obtain the optimal discount number. In practice, we have a cost of discount (advertising cost, marking cost, etc.). The complexity of the problem will increase, so heuristics procedures may have to be used. The discount scheme issue is not completely solved in our paper, it will also be an interesting point for future research. Another extension can deal with the objective of maximizing the probability for achieving a target cost, etc.). The complexity of the problem will increase, so heuristics procedures may have to be used. The discount scheme issue is not completely solved in our paper, it will also be an interesting point for future research. Another extension can deal with the objective of maximizing the probability for achieving a target

In our paper, it will also be an interesting point for future research. Another extension can deal with the objective of maximizing the probability for achieving a target profit or suppose a second purchasing opportunity during the selling season. The topic of “multivariate distributions” is also interesting for future research, assuming that the demands during each discount period are independent random variables. However the complexity of the model as well as the formulation will be highly enlarged if multiple random demands are considered in the same time. To develop it further, restrictive assumptions need to be used. In addition, the right time to start a new discount period can also be interesting to study by taking into account of the revenue management. Starting the discount period too early leads to a loss for selling products at a low price, but if it is too late, the NV may lose customers. A possible way to solve this problem is to develop a time-dependent demand model and treat the selling price as an exogenous factor influencing demand, in this way the manager decides when to start a discount period according to the variation trend of the demand.

Appendices. Appendix 1: Expected profit of additive price-dependent demand:

\[
E(\pi(Q)) = \int_{Q}^{\inf} [v_0 Q - w Q] f(x) dx + \int_{\mu_0 - \mu_1 + Q}^{Q} [v_0 x + (Q - x)v_1 - w Q] f(x) dx + \\
+ \int_{\mu_0 - \mu_2 + Q}^{\mu_0 - \mu_1 + Q} [v_0 x + (\mu_1 - \mu_0)v_1 - (\mu_1 - \mu_0 + x)v_2 + (v_2 - w)Q] f(x) dx + \\
+ \int_{\mu_0 - \mu_1 + Q}^{\mu_0 - \mu_1 + Q} [v_0 x + (\mu_1 - \mu_0)v_1 + \cdots + (\mu_i - \mu_{i-2})v_{i-1} - (\mu_{i-1} - \mu_0 + x)v_i + (v_i - w)Q] f(x) dx + \\
+ \cdots + \int_{0}^{\mu_0 - \mu_{n-1} + Q} [v_0 x + (\mu_1 - \mu_0)v_1 + \cdots + (\mu_{n-1} - \mu_{n-2})v_{n-1} - (\mu_{n-1} - \mu_0 + x)v_n + (v_n - w)Q] f(x) dx \tag{6.1}
\]
Appendix 3: Expected profit of multiplicative price-dependent demand:

\[ E(\pi(Q)) = Q \int_{Q}^{\inf} (v_0 - w)f(x)dx + \sum_{i=1}^{n-1} \int_{\mu_{i-1} - \mu_i}^{\mu_{i-1} + Q} (v_i - w)f(x)dx + \sum_{i=1}^{n-1} \int_{\mu_{i-1} + Q}^{\mu_{i-1} + Q} (v_0x + (\mu_1 - \mu_0)v_1 + \cdots + (\mu_{i-1} - \mu_i)v_{i-1} + (\mu_{i-1} - \mu_i + \mu_0)xv_i) f(x)dx + \int_{0}^{\mu_{n-1} + Q} (v_0x + (\mu_1 - \mu_0)v_1 + \cdots + (\mu_{n-1} - \mu_n - 2)v_{n-1} - (\mu_{n-1} - \mu_n + x)v_{n}) f(x)dx. \]

\[ = Q[v_0 - w + \sum_{i=0}^{n-1} (v_i - v_{i+1})F(Q + \mu_0 - \mu_i)] + \sum_{i=1}^{n-1} (\int_{0}^{\mu_{i-1} + Q} (v_0x + (\mu_1 - \mu_0)v_1 + \cdots + (\mu_{i-1} - \mu_i)v_{i-1} + (\mu_{i-1} - \mu_i + \mu_0)xv_i) f(x)dx + \int_{0}^{\mu_{n-1} + Q} (v_0x + (\mu_1 - \mu_0)v_1 + \cdots + (\mu_{n-1} - \mu_n - 2)v_{n-1} - (\mu_{n-1} - \mu_n + x)v_{n}) f(x)dx. \]

Appendix 2: Proof of lemma 1:

Use Leibniz’s rule, we get the derivative of \(E(\pi(Q)):\)

\[ \frac{dE(\pi(Q))}{dQ} = -\sum_{i=0}^{n-1} (v_i - v_{i+1})F(Q + \mu_0 - \mu_i) + v_0 - w \quad (6.2) \]

The second derivative of \(E(\pi(Q))\) is:

\[ \frac{d^2E(\pi(Q))}{dQ^2} = -\sum_{i=0}^{n-1} (v_i - v_{i+1})f(Q + \mu_0 - \mu_i) \quad (6.3) \]

\(f(x) > 0, v_i - v_{i+1} > 0,\) so \(\frac{d^2E(\pi(Q))}{dQ^2} < 0,\) then \(E(\pi(Q))\) is concave.

Appendix 3: Expected profit of multiplicative price-dependent demand:

\[ E(\pi(Q)) = \int_{Q}^{\inf} (v_0Q - wQ)f(x)dx + \int_{\mu_{i-1}Q}^{\mu_0Q} (v_0x + (Q - x)v_1 - wQ)f(x)dx + \int_{\mu_{i-1}Q}^{\mu_0Q} (v_0x + \frac{x}{\mu_0}((\mu_1 - \mu_0)v_1 - \mu_1v_2) + (v_2 - w)Q)f(x)dx + \cdots + \int_{\mu_{i-1}Q}^{\mu_0Q} (v_0x + \frac{x}{\mu_0}((\mu_1 - \mu_0)v_1 + \cdots + (\mu_{i-1} - \mu_i)v_{i-1} - \frac{x_0v_1}{\mu_0}(\mu_{i-1}) + (v_1 - w)Q)f(x)dx + \cdots + \int_{0}^{\mu_{n-1}Q} (v_0x + \frac{x}{\mu_0}(\mu_1 - \mu_0)v_1 + \cdots +} \]
Use Leibniz’s rule, we get the derivative of $E(\pi(Q))$:

$$\frac{dE(\pi(Q))}{dQ} = -\sum_{i=0}^{n-1} (v_i - v_{i+1})F(Q_{\mu_0}/\mu_i) + v_0 - w$$  \hspace{1cm} (6.5)

Appendix 4: Proof of lemma 3:

The second derivative of $E(\pi(Q))$ is:

$$\frac{d^2E(\pi(Q))}{d^2Q} = -\sum_{i=0}^{n-1} (v_i - v_{i+1})f(Q_{\mu_0}/\mu_i)\frac{\mu_0}{\mu_i}$$  \hspace{1cm} (6.6)

$f(x) > 0$, $v_i - v_{i+1} > 0$, so $\frac{d^2E(\pi(Q))}{d^2Q} < 0$, then $E(\pi(Q))$ is concave.

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