Poisson-Gradient Dynamical Systems with Bounded Non-Linearity

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Abstract

We study the periodical solutions of a Poisson-gradient PDEs system with bounded non-linearity.

Section 1 introduces the basic spaces and functionals. Section 2 studies the weak differential of a function and establishes an inequality. Section 3 formulates some conditions under which the action functional is continuously differentiable. Section 4 analyzes the Poisson-gradient systems and some conditions that ensure periodical solutions.

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1 Introduction

We consider the point \( T = (T^1, ..., T^p) \) and the parallelepiped \( T_0 = [0, T^1] \times ... \times [0, T^p] \) in \( R^p \). We denote by \( W^{1,2}_T \) the Sobolev space of the functions
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\( u \in L^2 [T_0, R^n] \) which have the weak derivative \( \frac{\partial u}{\partial t} \in L^2 [T_0, R^n] \). The index \( T \) from the notation \( W^{1,2}_T \) comes from the fact that the weak derivatives are defined using the space \( C^\infty_T \) of all indefinitely differentiable multiple \( T \)-periodic functions from \( R^p \) into \( R^n \). We denote by \( H^1_T \) the Hilbert space \( W^{1,2}_T \). The norm used in \( H^1_T \) is the one induced by the scalar product

\[
\langle u, v \rangle = \int_{T_0} \left( \delta_{ij} u^i (t) v^j (t) + \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha} (t) \frac{\partial v^j}{\partial t^\beta} (t) \right) dt^1 \wedge \ldots \wedge dt^p.
\]

These are induced by the scalar product (Riemannian metric)

\[
G = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta^{\alpha\beta} \delta_{ij} \end{pmatrix}
\]

on \( R^{n+p} \) (multiphase space) and its associated Euclidean norm. We shall also use the scalar product \( (u, v) = \delta_{ij} u^i v^j \) and the norm \( |u| = \sqrt{\delta_{ij} u^i u^j} \) simultaneously, from the Euclidean space \( R^n \).

Let \( t = (t^1, \ldots, t^p) \) be a generic point in \( R^p \). Then the opposite faces of the parallelepiped \( T_0 \) can be described by the equations

\[
S^-_i : t^i = 0, S^+_i : t^i = T^i
\]

for each \( i = 1, \ldots, p \). We shall study the minimum of the action

\[
\varphi (u) = \int_{T_0} L \left( t, u (t), \frac{\partial u}{\partial t} \right) dt^1 \wedge \ldots \wedge dt^p,
\]

\[
L \left( t, u (t), \frac{\partial u}{\partial t} \right) = \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F (t, u (t))
\]

on the space \( H^1_T \), considering that the potential function \( F \) has the property of bounded non-linearity. We use the method of the minimizing sequences and the coercitivity condition \( \int_{T_0} F (t, u (t)) dt^1 \wedge \ldots \wedge dt^p \to \infty \) when \( |u| \to \infty \). The extremals of the action \( \varphi \) verifies the Euler-Lagrange equations with the boundary conditions

\[
u \big|_{S^-_i} = u \big|_{S^+_i}, \frac{\partial u}{\partial t} \big|_{S^-_i} = \frac{\partial u}{\partial t} \big|_{S^+_i}, i = 1, \ldots, p.
\]
Due to the particularity of the Lagrangian $L$, the Euler-Lagrange equations reduce to a PDEs system of the Poisson-gradient type

$$\Delta u(t) = \nabla F(t, u(t)).$$

The aim of this paper is to discuss the existence of solutions of this PDEs system with suitable boundary conditions. More precisely, we extend the theory in [2] from single-time to multi-time field theory, developing the ideas in the papers [6], [7], [9]. In this way we find positive answers for the existence of multi-periodical solutions of Euler-Lagrange equations that are Poisson-gradient PDEs with bounded non-linearity. The results can be applied to the multi-time geometric dynamics ([5], [8], [10]-[12]).

2 On the weak differential of a function

We consider $C^\infty_T$ the space of the indefinitely differentiable functions multiple periodical with the period $T = (T^1, ..., T^p)$, defined on $R^p$ taking values in $R^n$. We know that $C^\infty_T \subset W^{1,2}_T$. We establish some conditions satisfied by a function $u \in L^1[T_0, R^n]$ which has a weak differential.

**Theorem 1.** Let $u, v_\alpha \in L^1[T_0, R^n]$, $\alpha = 1, ..., p$, such that $v_\alpha dt^\alpha = (v_1^\alpha dt^\alpha, ..., v_p^\alpha dt^\alpha)$ is an integrable vector form. We consider $\hat{OT}$ an arbitrary curve from $T_0$, having the endings at $O = (0, ..., 0)$ and $T = (T^1, ..., T^p)$.

$$\int_{\hat{OT}} (u, df) = - \int_{\hat{OT}} (v_\alpha dt^\alpha, f),$$

(1)

for any $f \in C^\infty_T$, then $\int_{\hat{OT}} v_\alpha dt^\alpha = 0$ and it exists $c \in R^n$ such that $u(t) = \int_{\hat{Ot}} v_\alpha ds^\alpha + c$. Also $u(0) = u(T)$.

**Proof.** We choose $f = e^i = (0, ..., 0, 1, 0, ..., 0)$, with the value 1 on the position $i$. From the relation (1) we have $0 = - \int_{\hat{OT}} v_i^\alpha dt^\alpha$ and hence $\int_{\hat{Ot}} v_\alpha dt^\alpha = 0$.

We define $w \in C(T_0, R^n)$ by $w(t) = \int_{\hat{Ot}} v_\alpha ds^\alpha, t \in \hat{OT}$. By Fubini Theorem, the function $w$ satisfies the relation

$$\int_{\hat{OT}} (w(t), df) = \int_{\hat{OT}} \left( \int_{\hat{Ot}} v_\alpha ds^\alpha, df \right) = \int_{\hat{OT}} (\int_{\hat{sT}} (v_\alpha, df) ds^\alpha)$$
We consider now $u$ and we observe that (see the Fourier series theory)

$$c = \int_{\mathcal{O}^T} (v_\alpha, f (T) - f (s)) \, ds^\alpha = - \int_{\mathcal{O}^T} (v_\alpha, f (s)) \, ds^\alpha = \int_{\mathcal{O}^T} (u, df).$$

This means that

$$\int_{\mathcal{O}^T} (u - w, df) = 0. \quad (2)$$

We consider now $\gamma : [a, b] \to T_0, \gamma (\xi) = (t^1 (\xi), \ldots, t^n (\xi)), \gamma (a) = O, \gamma (b) = T$, a parameterization of the curve $\mathcal{O}T$. The equality (2) becomes

$$\int_a^b \left( u (t (\xi)) - w (t (\xi)), \left( \frac{\partial f_1}{\partial t^{\alpha}}, \ldots, \frac{\partial f_n}{\partial t^{\alpha}} \right) \right) d\xi = 0,$$

for any $f \in C^\infty_T$. We will particularize for the function sequences

$$f^{(k)}_j (t) = \begin{cases} \cos & \text{if } \sin \\ \left( \frac{2k\pi t^j}{T} \right) e^j, & k \in N \setminus \{0\}, \; 1 \leq j \leq n \end{cases}$$

and we observe that (see the Fourier series theory) $u (t) - w (t) = c$, $c \in R^a$ almost everywhere in $T_0$ (the constant is the only function orthogonal to the previous sequences). By replacing $w (t)$, we find that $u (t) = \int_{\mathcal{O}^T} v_\alpha ds^\alpha + c$

for any $t \in \mathcal{O}T$. The function $u$ satisfies $u (0) = c$ and $u (T) = \int_{\mathcal{O}^T} v_\alpha ds^\alpha + c = c$, so $u (0) = u (T)$. On the other side, the relation $u (t) - u (\tau) = \int_{\tau t} v_\alpha ds^\alpha$ implies that $u (t) = \int_{\tau t} v_\alpha ds^\alpha + u (\tau)$. The 1-form $v_\alpha dt^\alpha$ is called weak differential of the function $u$. By a Fourier series argument, the weak differential, if it exists, is unique. The weak differential of $u$ will be denoted by $du$. The existence of $du$ implies $u (0) = u (T)$.

**Theorem 2.** If $u = (u^1, \ldots, u^n) \in L^2 [T_0, R^n]$, \; $|u (t)|^2 = \delta_{ij} w^i (t) w^j (t)$, then

$$\left| \int_{T_0} u (t) \, dt^1 \wedge \ldots \wedge dt^n \right| \leq \left( n T^1 \ldots T^n \right)^{\frac{1}{2}} \left( \int_{T_0} |u (t)|^2 \, dt^1 \wedge \ldots \wedge dt^n \right)^{\frac{1}{2}}.$$

**Proof.** Successively we have the relations

$$\left| \int_{T_0} u (t) \, dt^1 \wedge \ldots \wedge dt^n \right| = \left| \int_{T_0} (u^1 (t), \ldots, u^n (t)) \, dt^1 \wedge \ldots \wedge dt^n \right|$$

$$= \left| \left( \int_{T_0} u^1 (t) \, dt^1 \wedge \ldots \wedge dt^n, \ldots, \int_{T_0} u^n (t) \, dt^1 \wedge \ldots \wedge dt^n \right) \right|$$
\[
\left( \int_{T_0} u^3(t) \, dt^1 \wedge ... \wedge dt^p \right)^2 + ... + \left( \int_{T_0} u^1(t) \, dt^1 \wedge ... \wedge dt^p \right)^2 \right)^{\frac{1}{2}} \\
\leq \left| \int_{T_0} u^1(t) \, dt^1 \wedge ... \wedge dt^p \right| + ... + \left| \int_{T_0} u^n(t) \, dt^1 \wedge ... \wedge dt^p \right|
\leq \int_{T_0} \left( |u^1(t)| + ... + |u^n(t)| \right) \, dt^1 \wedge ... \wedge dt^p
\]
\[
= \int_{T_0} \left( \left| u^1(t) \right|, ..., |u^n(t)| \right) \, (1, ..., 1) \, dt^1 \wedge ... \wedge dt^p.
\]

Using the Cauchy-Schwartz inequality, we obtain
\[
\left| \int_{T_0} u(t) \, dt^1 \wedge ... \wedge dt^p \right|
\leq \left( \int_{T_0} \left( |u^1(t)|^2 + ... + |u^n(t)|^2 \right) \, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \left( \int_{T_0} n \, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}}
\leq (nT^n ... T^p)^{\frac{1}{2}} \left( \int_{T_0} |u(t)|^2 \, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}}.
\]

## 3 Continuously differentiable action

The next theorem establishes some conditions in which the action
\[
\varphi : W^{1,2}_T \to R, \varphi (u) = \int_{T_0} L \left( t, \, u(t), \, \frac{\partial u}{\partial t}(t) \right) \, dt^1 \wedge ... \wedge dt^p
\]
is continuously differentiable. In this way we extend the particular case \( p = 1 \), studied in [3, Theorem 1.4].

**Theorem 3.** We consider \( L : T_0 \times \mathbb{R}^n \times \mathbb{R}^{np} \to R, (t, x, y) \to L(t, x, y) \), a measurable function in \( t \) for any \((x, y) \in \mathbb{R}^n \times \mathbb{R}^{np}\) and with the continuous partial derivatives in \( x \) and \( y \) for any \( t \in T_0 \). If here exist \( a \in C^1(R^+, R^+) \) with the derivative \( a' \) bounded from above, \( b \in C(T_0, \mathbb{R}^n) \) such that for any \( t \in T_0 \) and any \((x, y) \in \mathbb{R}^n \times \mathbb{R}^{np}\) to have

\[
|L(t, x, y)| \leq a \left( |x| + |y| \right) b(t),
|\nabla_x L(t, x, y)| \leq a \left( |x| \right) b(t),
|\nabla_y L(t, x, y)| \leq a \left( |y| \right) b(t),
\]
then, the functional \( \varphi \) has continuous partial derivatives in \( W^{1,2}_{T} \) and his gradient derives from the formula

\[
(\nabla \varphi (u), v) = \int_{T_0} \left[ \left( \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), v(t) \right) + \left( \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \right] dt^1 \wedge ... \wedge dt^p. \tag{4}
\]

**Proof.** It is enough to prove that \( \varphi \) has the derivative \( \varphi' (u) \in (W^{1,2}_{T})^* \) given by the relation (4) and the function \( \varphi' : W^{1,2}_{T} \to (W^{1,2}_{T})^* \), \( u \to \varphi' (u) \) is continuous. We consider \( u, v \in W^{1,2}_{T}, t \in T_0, \lambda \in [-1, 1] \). We build the functions

\[
F (\lambda, t) = L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right)
\]

and

\[
\Psi (\lambda) = \int_{T_0} F (\lambda, t) dt^1 \wedge ... \wedge dt^p.
\]

Because the derivative \( a' \) is bounded from above, exist \( M > 0 \) such that

\[
\frac{a (|u|) - a (0)}{|u|} = a' (c) \leq M. \text{ This means that } a (|u|) \leq M |u| + a (0).
\]

On the other side

\[
\frac{\partial F}{\partial \lambda} (\lambda, t) = \left( \nabla_x L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right), v(t) \right)
\]

\[
+ \left( \nabla_y L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \leq a \left( |u(t) + \lambda v(t)| \right) \]

\[
b (t) |v(t)| + a \left( \left| \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right| \right) b (t) \left| \frac{\partial v}{\partial t} (t) \right|
\]

\[
\leq b_0 \left( M \left| u(t) + v(t) \right| + a (0) \right) |v(t)| +
\]

\[
b_0 \left( M \left| \frac{\partial u}{\partial t} (t) \right| + \left| \frac{\partial v}{\partial t} (t) \right| + a (0) \right) \left| \frac{\partial v}{\partial t} (t) \right|
\]

where

\[
b_0 = \max_{t \in T_0} b (t).
\]
Then, we have $\left| \frac{\partial F}{\partial \lambda} (\lambda, t) \right| \leq d(t) \in L^1(T_0, R^+).$ Then Leibniz formula of differentiation under integral sign is applicable and

$$\frac{\partial \Psi}{\partial \lambda} (0) = \int_{T_0} \frac{\partial F}{\partial \lambda} (0, t) \, dt^1 \wedge \ldots \wedge dt^p = \int_{T_0} \left[ \left( \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), v(t) \right) \right.$$

$$+ \left. \left( \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \right] \, dt^1 \wedge \ldots \wedge dt^p.$$  

Moreover,

$$\left| \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| \leq b_0 (M |u(t)| + |a(0)|) \in L^1(T_0, R^+)$$

and

$$\left| \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| \leq b_0 \left( M \left| \frac{\partial u}{\partial t} (t) \right| + |a(0)| \right) \in L^2(T_0, R^+).$$

That is why

$$\int_{T_0} \left[ \left( \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), v(t) \right) \right.$$

$$+ \left. \left( \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \right] \, dt^1 \wedge \ldots \wedge dt^p$$

$$\leq \int_{T_0} \left| \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| |v(t)| \, dt^1 \wedge \ldots \wedge dt^p$$

$$+ \int_{T_0} \left| \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| \left| \frac{\partial v}{\partial t} (t) \right| \, dt^1 \wedge \ldots \wedge dt^p$$

$$\leq b_0 \int_{T_0} (M |u(t)| + |a(0)|) \, v(t) \, dt^1 \wedge \ldots \wedge dt^p$$

$$+ b_0 \int_{T_0} \left( M \left| \frac{\partial u}{\partial t} (t) \right| + |a(0)| \right) \left| \frac{\partial v}{\partial t} (t) \right| \, dt^1 \wedge \ldots \wedge dt^p.$$
By using the inequality Cauchy-Schwartz, we find

\[
\left| \frac{\partial \Psi}{\partial \lambda} (0) \right| \leq b_0 \left( \int_{T_0} (M |u(t)| + |a(0)|)^2 dt^1 \wedge ... \wedge dt^p \right)^\frac{1}{2}
\]

\[
\left( \int_{T_0} |v(t)|^2 dt^1 \wedge ... \wedge dt^p \right)^\frac{1}{2}
\]

\[+ b_0 \left( \int_{T_0} \left( M \left| \frac{\partial u}{\partial t} (t) \right| + |a(0)| \right)^2 dt^1 \wedge ... \wedge dt^p \right)^\frac{1}{2}
\]

\[
\left( \int_{T_0} \left| \frac{\partial v}{\partial t} (t) \right|^2 dt^1 \wedge ... \wedge dt^p \right)^\frac{1}{2}
\]

\[
\leq C_1 \left( \int_{T_0} |v(t)|^2 dt^1 \wedge ... \wedge dt^p \right)^\frac{1}{2} + C_2 \left( \int_{T_0} \left| \frac{\partial v}{\partial t} (t) \right|^2 dt^1 \wedge ... \wedge dt^p \right)^\frac{1}{2}
\]

\[
\leq \max \{C_1, C_2\} 2^{\frac{1}{2}} \left( \int_{T_0} \left( |v(t)|^2 + \left| \frac{\partial v}{\partial t} (t) \right|^2 \right) dt^1 \wedge ... \wedge dt^p \right)^\frac{1}{2} = C \|v\|.
\]

By consequence, the action \( \varphi \) has the derivative \( \varphi' \in (W^{1,2}_T)^* \) given by (4). The Krasnoselski theorem and the hypothesis (3) imply the fact that the application \( u \to \left( \nabla_x L \left( \cdot, u, \frac{\partial u}{\partial t} \right), \nabla_y L \left( \cdot, u, \frac{\partial u}{\partial t} \right) \right) \), from \( W^{1,2}_T \) to \( L^1 \times L^2 \), is continuous, so \( \varphi' \) is continuous from \( W^{1,2}_T \) to \( (W^{1,2}_T)^* \) and the proof is complete.

## 4 Poisson-gradient systems and their periodical solutions

### 4.1 Multi-time Euler-Lagrange equations

We consider the multi-time variable \( t = (t^1, ..., t^p) \in \mathbb{R}^p \), the functions \( x^i : \mathbb{R}^p \to \mathbb{R}, (t^1, ..., t^p) \to x^i (t^1, ..., t^p), i = 1, ..., n \), and the partial velocities \( x^i_{\alpha} = \)
\[
\frac{\partial x^i}{\partial t^\alpha}, \alpha = 1, ..., p. \text{ The Lagrangian}
\]

\[
L : \mathbb{R}^{p+n+np} \rightarrow \mathbb{R}, \left( t^\alpha, x^i, x^i_\alpha \right) \rightarrow L \left( t^\alpha, x^i, x^i_\alpha \right)
\]
determines the Euler-Lagrange equations

\[
\frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x^i_\alpha} = \frac{\partial L}{\partial x^i}, \ i = 1, ..., n, \ \alpha = 1, ... p
\]

\text{(PDEs system of second order in the n-dimensional space). We remark that in the left hand member we have summation after the index } \alpha \text{ (trace).}

4.2 An action that produces Poisson-gradient systems

Let \( \alpha = 1, ..., p, i = 1, ..., n, u^i : T_0 \rightarrow \mathbb{R}, t = (t^1, ..., t^p) \rightarrow u^i (t^1, ..., t^p), u : T_0 \rightarrow \mathbb{R}^n, u (t) = (u^1 (t), ..., u^n (t)) \)

We consider the Lagrangian

\[
L : T_0 \times \mathbb{R}^n \times \mathbb{R}^{np} \rightarrow \mathbb{R}, \left( t^\alpha, u^i, u^i_\alpha \right) \rightarrow L \left( t^\alpha, u^i, u^i_\alpha \right),
\]

\[
L \left( t^\alpha, u^i, u^i_\alpha \right) = \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t)).
\]

A function \( u \) (field) that realizes the minimum of the action

\[
\varphi (u) = \int_{T_0} L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) dt^1 \wedge ... \wedge dt^p,
\]

verifies a \textit{PDEs} system of Poisson-gradient type (Euler-Lagrange equations on \( H^1 \))

\[
\Delta u (t) = \nabla F \left( t, u (t) \right),
\]

together with the boundary conditions

\[
u \mid_{s_i^-} = u \mid_{s_i^+}, \ \frac{\partial u}{\partial t} \mid_{s_i^-} = \frac{\partial u}{\partial t} \mid_{s_i^+}, i = 1, ..., p.
\]
4.3 Periodical solutions of Poisson-gradient dynamical systems with bounded non-linearity

**Theorem 4.** Suppose the function $F : T_0 \times \mathbb{R}^n \to \mathbb{R}, (t, u) \to F(t, u)$ satisfies four properties:

1) $F(t, u)$ is measurable in $t$ for any $u \in \mathbb{R}^n$ and it is continuously differentiable in $u$ for any $t \in T_0$,

2) There exist the functions $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with the derivative $a'$ bounded from above and $b \in C(T_0, \mathbb{R}^+)$ such that for any $t \in T_0$ and any $u \in \mathbb{R}^n$ to have $|F(t, u)| \leq a(|u|) b(t)$ and $|\nabla_u F(t, u)| \leq a(|u|) b(t)$,

3) It exists $g \in C^1(T_0, \mathbb{R})$ such that for any $t \in T_0$ and any $u \in \mathbb{R}^n$, to have $|\nabla_u F(t, u)| \leq g(t)$,

4) The action $\varphi_1(u) = \int_{T_0} F(t, u(t)) \, dt^1 \wedge ... \wedge dt^p$ is weakly lower semi-continuous.

If $\int_{T_0} F(t, u) \, dt^1 \wedge ... \wedge dt^p \to \infty$ when $|u| \to \infty$, then the Dirichlet problem

$$\Delta u(t) = \nabla F(t, u(t)),$$

$$u \big|_{S_i^-} = u \big|_{S_i^+}, \quad \frac{\partial u}{\partial t} \big|_{S_i^-} = \frac{\partial u}{\partial t} \big|_{S_i^+}, \quad i = 1, ..., p,$$

has at least a solution which minimizes the action

$$\varphi(u) = \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t)) \right] \, dt^1 \wedge ... \wedge dt^p$$

in $H^1_T$.

**Proof.** We consider $u = \overline{u} + \tilde{u}$, where $\overline{u} = \frac{1}{T^1 \ldots T^p} \int_{T_0} u(t) \, dt^1 \wedge ... \wedge dt^p$. Then

$$\varphi(u) = \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t)) \right] \, dt^1 \wedge ... \wedge dt^p$$

$$= \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, \overline{u}) - F(t, \overline{u}) + F(t, u(t)) \right] \, dt^1 \wedge ... \wedge dt^p$$
\[= \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, \overline{u}(t)) \right] dt^1 \wedge ... \wedge dp + \int_{T_0} \int_0^1 (\nabla_u F(t, \overline{u} + s\tilde{u}(t)), \tilde{u}(t)) ds \wedge dt^1 \wedge ... \wedge dp.\]

According to property 3) from the hypothesis, we have the inequality
\[
(\nabla_u F(t, \overline{u} + s\tilde{u}(t)), \tilde{u}(t)) \leq |\nabla_u F(t, \overline{u} + s\tilde{u}(t))| |\tilde{u}(t)| \leq |g(t)||\tilde{u}(t)|
\]
from which we obtain the relation
\[
-|g(t)||\tilde{u}(t)| \leq (\nabla_u F(t, \overline{u} + s\tilde{u}(t)), \tilde{u}(t))
\]
for any \( t \in T_0 \). By using this inequality we obtain
\[
\varphi(u) = \int_{T_0} \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dp + \int_{T_0} F(t, \overline{u}) dt^1 \wedge ... \wedge dp
\]
\[
- \int_{T_0} |g(t)||\tilde{u}(t)| dt^1 \wedge ... \wedge dp
\]
\[
\geq \int_{T_0} \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dp + \int_{T_0} F(t, \overline{u}) dt^1 \wedge ... \wedge dp
\]
\[
- g_0 \int_{T_0} |\tilde{u}(t)| dt^1 \wedge ... \wedge dp,
\]
where \( g_0 = \max_{t \in T_0} |g(t)| \). According to the multi-time Wirtinger inequality [9], it exists \( C_1 > 0 \) such that
\[
\int_{T_0} |\tilde{u}(t)| dt^1 \wedge ... \wedge dp \leq C_1 \left( \int_{T_0} \left| \frac{\partial u}{\partial t}(t) \right|^2 dt^1 \wedge ... \wedge dp \right)^{\frac{1}{2}}.
\]
This means that
\[
\varphi(u) \geq \int_{T_0} \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dp + \int_{T_0} F(t, \overline{u}) dt^1 \wedge ... \wedge dp
\]
\[
- g_0 C_1 \left( \int_{T_0} \left| \frac{\partial u}{\partial t}(t) \right|^2 dt^1 \wedge ... \wedge dp \right)^{\frac{1}{2}}.
\]
Of course, if \( \|u\| \to \infty \), then, from the relation \( \|u\| \leq \|\pi\| + \|\tilde{u}\| \) it follows that \( \|\pi\| \to \infty \) or \( \|\tilde{u}\| \to \infty \). Because \( \pi \) is constant in \( R^n \), we have the equalities

\[
\|\pi\| = \|\pi\|_{W^{1,2}} = \left( \int_{T_0} \left( |\pi|^2 + \left| \frac{\partial \pi}{\partial t} \right| \right)^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}}
\]

\[
= \left( \int_{T_0} |\pi|^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} = |\pi| (T^1 \ldots T^p)^{\frac{1}{2}}.
\]

This means that if \( \|\pi\| \to \infty \), then \( |\pi| \to \infty \). Consequently using the hypothesis, we obtain

\[
\int_{T_0} F(t, \pi) dt^1 \wedge \ldots \wedge dt^p \to \infty.
\]

(5)

Also

\[
||\tilde{u}|| = \left( \int_{T_0} \left( |\tilde{u}(t)|^2 + \left| \frac{\partial \tilde{u}}{\partial t} (t) \right| \right)^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}}
\]

\[
= \left( \int_{T_0} \left( |\tilde{u}(t)|^2 + \left| \frac{\partial u}{\partial t} (t) \right| \right)^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}}.
\]

With the Wirtinger inequality we obtain

\[
||\tilde{u}|| \leq \left( \int_{T_0} \left( C \left| \frac{\partial \tilde{u}}{\partial t} (t) \right|^2 + \left| \frac{\partial u}{\partial t} (t) \right| \right)^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}}
\]

\[
= (C + 1) \left( \int_{T_0} \left| \frac{\partial \tilde{u}}{\partial t} (t) \right|^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}}.
\]

The condition \( ||\tilde{u}|| \to \infty \) implies

\[
\int_{T_0} \left| \frac{\partial u}{\partial t} (t) \right|^2 dt^1 \wedge \ldots \wedge dt^p \to \infty.
\]

(6)

From the hypothesis and (5) or (6) it follows that if \( ||u|| \to \infty \), then \( \varphi(u) \to \infty \). So \( \varphi \) is a coercitive application. This means that \( \varphi \) has a minimizing
bounded sequence \((u_k)\). The Hilbert space \(H^1_T\) is reflexive. By consequence, the sequence \((u_k)\) (or one of his subsequence) is weakly convergent in \(H^1_T\) with the limit \(u\). Because
\[
\varphi_2 (u) = \int_{T_0} \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha}(t) \frac{\partial v^j}{\partial t^\beta}(t) \, dt^1 \wedge ... \wedge dt^p
\]
is convex, it follows that \(\varphi_2\) is weakly lower semi-continuous, so that the action
\[
\varphi(u) = \varphi_1(u) + \varphi_2(u)
\]
is weakly lower semi-continuous and \(\varphi(u) \leq \lim \varphi(u_k)\). This means that \(u\) is minimum point of \(\varphi\).

We build the function
\[
\Phi : [-1, 1] \rightarrow \mathbb{R},
\]
\[
\Phi(\lambda) = \varphi(u + \lambda v)
\]
\[
= \int_{T_0} \left[ \frac{1}{2} \frac{\partial}{\partial t} (u(t) + \lambda v(t)) \right]^2 + F(t, u(t) + \lambda v(t)) \, dt^1 \wedge ... \wedge dt^p,
\]
where \(v \in C^\infty_T\). The point \(\lambda = 0\) is a critical point of \(\Phi\) if and only if the point \(u\) is a critical point of \(\varphi\). Consequently
\[
0 = \langle \varphi'(u), v \rangle = \int_{T_0} \left[ \delta^{\alpha\beta} \delta_{ij} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} + \delta_{ij} \nabla^i F(t, u(t)) v^j(t) \right] \, dt^1 \wedge ... \wedge dt^p,
\]
for all \(v \in H^1_T\) and hence for all \(v \in C^\infty_T\). According to the definition of the weak divergence, i.e.,
\[
\int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} \, dt^1 \wedge ... \wedge dt^p = - \int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial^2 u^i}{\partial t^\alpha \partial t^\beta} v^j \, dt^1 \wedge ... \wedge dt^p,
\]
the Jacobi matrix function \(\frac{\partial u}{\partial t}\) has weak divergence (the function \(u\) has a weak Laplacian) and
\[
\Delta u(t) = \nabla F(t, u(t))
\]
a.e. on \(T_0\). Also, the existence of weak derivatives \(\frac{\partial u}{\partial t}\) and \(\Delta u\) implies that
\[
u |_{S^-} = \frac{\partial u}{\partial t} \bigg|_{S^-}, \quad \frac{\partial u}{\partial t} \bigg|_{S^+} = \frac{\partial u}{\partial t} \bigg|_{S^+}.
\]
Remark. If the function $u$ is at least of class $C^2$, then the definition of the weak divergence of the Jacobian matrix $\frac{\partial u}{\partial t}$ (or of the weak Laplacian $\Delta u$) coincides with the classical definition. This fact is obvious if we have in mind the formula of integration by parts

$$\int_{T_0}^{\beta} \delta^\alpha \delta^\beta \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} dt^1 \wedge ... \wedge dt^p = \int_{T_0}^{\beta} \delta^\alpha \delta^\beta \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} dt^1 \wedge ... \wedge dt^p - \int_{T_0}^{\beta} \delta^\alpha \delta^\beta \frac{\partial^2 u^i}{\partial t^\alpha \partial t^\beta} v^j dt^1 \wedge ... \wedge dt^p.$$

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