Abstract

For moments of leptoproduction structure functions we show that all dependence on the renormalization and factorization scales disappears provided that all the ultraviolet logarithms involving the physical energy scale $Q$ are completely resummed. The approach is closely related to Grunberg's method of Effective Charges. A direct and simple method for extracting $\Lambda_{\overline{MS}}$ from experimental data is advocated.

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1 Introduction

The problem of renormalization scheme dependence in QCD perturbation theory remains an obstacle to making precise tests of the theory. In a recent paper \[1\] one of us pointed out that the renormalization scale dependence of dimensionless physical QCD observables, depending on a single energy scale $Q$, can be avoided provided that all ultraviolet logarithms which build the physical energy dependence on $Q$ are resummed. This was termed complete Renormalization Group (RG)-improvement in Ref.\[1\]. It was stressed that standard RG-improvement, as customarily applied with a $Q$-dependent scale $\mu = xQ$, omits an infinite subset of these logarithms. One should rather keep $\mu$ independent of $Q$, and then carefully resum to all-orders the RG-predictable ultraviolet logarithms. In this way all $\mu$-dependence cancels between the renormalized coupling and the logarithms of $\mu$ contained in the coefficients, and the correct physical $Q$-dependence is built. At next-to-leading order (NLO) the result is identical to the Effective Charge approach of Grunberg \[2, 3\]. We wish to extend this argument to processes involving factorization of operator matrix elements and coefficient functions, where a factorization scale $M$ arises in addition to the renormalization scale $\mu$. We shall use the prototypical factorization problem of moments of leptoproduction structure functions as a specific example. We shall identify the logarithms of $\mu$, $M$, and $Q$ which occur, and will show explicitly that on resumming all the ultraviolet logarithms the $\mu$ and $M$ dependence disappears. We shall organize the paper so that we review the treatment of Ref.\[1\] whilst showing how it generalizes for the moment problem. We begin in Section 2 by giving some basic definitions for the moments of structure functions. Section 3 considers the dependence of the perturbative coefficients on the parameters which label the renormalization procedure in both cases. Section 4 deals with the complete RG-improvement of the structure function moments and identifies and resums the physical ultraviolet logarithms. Finally, in Section 5 we discuss a more straightforward way of motivating this approach, and consider how to directly extract $\Lambda_{\overline{MS}}$ from data. We also give our Conclusions.
2 Structure Function Moments

In the prototypical factorization problem of deep inelastic leptoproduction the \( n \)th moment of a non-singlet structure function \( F(x) \),

\[
\mathcal{M}_n(Q) = \int_0^1 x^{n-2} F(x) \, dx ,
\]

(1)
can be factorized in the form

\[
\mathcal{M}_n(Q) = \langle \mathcal{O}_n(M) \rangle \mathcal{C}_n(Q, a(\mu), \mu, M) .
\]

(2)

Here \( M \) is an arbitrary factorization scale and \( a(\mu) \) is the RG-improved coupling \( \alpha_s(\mu) / \pi \) defined at a renormalization scale \( \mu \). The operator matrix element \( \langle \mathcal{O}_n(M) \rangle \) has an \( M \)-dependence given by its anomalous dimension,

\[
M \frac{\partial \langle \mathcal{O} \rangle}{\partial M} = \gamma_{\mathcal{O}}(a) = -d_1 a - d_2 a^2 - d_3 a^3 + \ldots
\]

(3)

For simplicity we shall from now on suppress the \( n \)-dependence of terms in equations, as we have done in Eq.(3). For a given moment \( d \) is independent of the factorization convention, whereas the higher \( d_i \), \( i \geq 1 \) depend on it. In Eq.(3) the coupling \( a \) is governed by the \( \beta \)-function equation

\[
M \frac{\partial a}{\partial M} = \beta(a) = -ba^2(1 + ca + c_2 a^2 + c_3 a^3 + \ldots) .
\]

(4)

Here \( b = (33 - 2N_f)/6 \), and \( c = (153 - 19N_f)/12b \), are the first two coefficients of the beta-function for SU(3) QCD with \( N_f \) active flavours of quark. They are universal, whereas the subsequent coefficients \( c_2, c_3, \ldots \) are scheme-dependent. Equation (3) can be integrated to

\[
\langle \mathcal{O}(M) \rangle = A \exp \left[ \int_0^a \frac{\gamma(1)(x)}{\beta(1)(x)} \, dx - \int_0^\infty \frac{\gamma(2)(x)}{\beta(2)(x)} \, dx \right] ,
\]

(5)

where \( \gamma(1) \) and \( \beta(2) \) denote these functions truncated at one and two terms, respectively. The factor \( A \) is scheme-independent and can be fitted to experimental data. The second integral in Eq.(5) is an infinite constant of
integration. In Eq.(2) \( C(Q, a(\mu), \mu, M) \) is the coefficient function and has the perturbation series

\[
C(Q, \tilde{a}, \mu, M) = 1 + r_1 \tilde{a} + r_2 \tilde{a}^2 + r_3 \tilde{a}^3 + \ldots .
\] (6)

We shall use \( \tilde{a} \) to stand for \( a(\mu) \) and \( a \) for \( a(M) \). After combining the integrals in Eq.(5) one obtains

\[
M = A \left( \frac{ca}{1 + ca} \right)^{d/b} \exp(\mathcal{I}(a)) \left( 1 + r_1 \tilde{a} + r_2 \tilde{a}^2 + r_3 \tilde{a}^3 + \ldots \right),
\] (7)

where \( \mathcal{I}(a) \) is the finite integral

\[
\mathcal{I}(a) = \int_0^a dx \frac{d_1 + (d_1c + d_2 - dc_2)x + (d_3 + cd_2 - c_3d)x^2 + \ldots}{b(1 + cx)(1 + cx + c_2x^2 + c_3x^3 + \ldots)} ,
\] (8)

which can be readily evaluated numerically. The coupling \( a(\tau) \) itself, where \( \tau \equiv b \ln(\mu/\tilde{\Lambda}) \), is obtained as the solution of the transcendental equation

\[
\frac{1}{a} + \ln \frac{ca}{1 + ca} = \tau - \int_0^a dx \left[ -\frac{1}{B(x)} + \frac{1}{x^2(1 + cx)} \right] ,
\] (9)

where \( B(x) \equiv x^2(1 + cx + c_2x^2 + c_3x^3 + \ldots) \).

### 3 RS and FS dependence of the coefficients

We first wish to parametrize the dependence of the \( r_n \) in the coefficient function on the renormalization scheme (RS) and factorization scheme (FS).

Recall first [6] that for the single scale case of a dimensionless observable \( \mathcal{R}(Q) \) with perturbation series

\[
\mathcal{R}(Q) = a + r_1 a^2 + r_2 a^3 + \ldots + r_n a^{n+1} + \ldots ,
\] (10)

the RS can be labelled by the non-universal coefficients of the beta-function \( c_2, c_3, \ldots \), and by \( \tau \), which can be traded as a parameter for \( r_1 \) since [2, 3, 6, 7]

\[
\tau - r_1 = \rho_0(Q) \equiv b \ln(Q/\Lambda_{\mathcal{R}}) ,
\] (11)

is an RS-invariant. Using the self-consistency of perturbation theory- that is that the difference between a N^nLO calculation (i.e up to and including
$r_n a^{n+1}$ performed with two different RS’s is $O(a^{n+2})$, one can derive expressions for the partial derivatives of the perturbative coefficients with respect to the scheme parameters. For $r_2$ for instance one has \[3\]

$$\frac{\partial r_2}{\partial r_1} = 2r_1 + c, \quad \frac{\partial r_2}{\partial c_2} = -1, \quad \frac{\partial r_2}{\partial c_3} = 0, \ldots \quad (12)$$

on integration one finds

$$r_2(r_1, c_2) = r_1^2 + cr_1 + X_2 - c_2$$

$$r_3(r_1, c_2, c_3) = r_1^3 + \frac{5}{2}cr_1^2 + (3X_2 - 2c_2)r_1 + X_3 - \frac{1}{2}c_3$$

$$\vdots \quad \vdots \quad \vdots \quad (13)$$

In general the structure is

$$r_n(r_1, c_2, \ldots, c_n) = \hat{r}_n(r_1, c_2, \ldots, c_{n-1}) + X_n - c_n/(n-1) \quad (14)$$

where $\hat{r}_n$ is RG-predictable from a complete $N^{n-1}$LO calculation (i.e. $r_2, r_3, \ldots, r_n$, and $c_2, c_3, \ldots, c_n$ have been computed in some RS), and the $X_n$ are $Q$-independent and RS-invariant constants of integration which are unknown unless a complete $N^n$LO calculation has been performed.

As we shall see the generalization to the moment problem is a dependence $r_n(\mu, M, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n)$ where the $c_i$ label the RS and the $d_i$ the FS. As before $M, \mu$ can be traded, in this case for $r_1(M)$ and $\tilde{r}_1 \equiv r_1(M = \mu)$. There will be analogous factorization and renormalization scheme (FRS) invariants, $X_n$, which represent the RG-unpredictable parts of $r_n$. Expressions for the dependence of the coefficients on FRS parameters have been derived before in Refs.\[4, 5, 8\], but there were some errors in Ref.\[4\], in particular the dependence of $r_2$ on $c_2$ was not recognized \[3\]. Partially differentiating Eq.(7) with respect to $\mu, M, c_2, c_3, d_1, d_2, d_3$, and demanding for consistency that it be $O(a^4)$, so that the coefficients of $a, a^2$ and $a^3$ vanish, one obtains analogous to Eqs.(12),

$$\mu \frac{\partial r_1}{\partial \mu} = 0, \quad \mu \frac{\partial r_2}{\partial \mu} = r_1 b, \quad \mu \frac{\partial r_3}{\partial \mu} = 2r_2 b + r_1 b c \quad ,$$

$$M \frac{\partial r_1}{\partial M} = d, \quad M \frac{\partial r_2}{\partial M} = d_1 + dr_1 - dL \quad ,$$
\[
M \frac{\partial r_3}{\partial M} = d_2 + d_1 r_1 + dr_2 - dr_1 L - 2d_1 L - dL^2,
\]
\[
\frac{\partial r_1}{\partial d_1} = -\frac{1}{b} \quad \frac{\partial r_2}{\partial d_1} = \frac{c}{2b} - \frac{L}{b} - \frac{r_1}{b},
\]
\[
\frac{\partial r_3}{\partial d_1} = \frac{cr_1}{2b} - \frac{c^2}{3b} + \frac{(c - r_1) L - r_2}{b} + \frac{c}{3b} - \frac{L^2}{b},
\]
\[
\frac{\partial r_1}{\partial d_2} = 0, \quad \frac{\partial r_2}{\partial d_2} = -\frac{1}{2b}, \quad \frac{\partial r_3}{\partial d_2} = \frac{c}{3b} - \frac{L}{b} - \frac{r_1}{2b},
\]
\[
\frac{\partial r_1}{\partial d_3} = 0, \quad \frac{\partial r_2}{\partial d_3} = 0, \quad \frac{\partial r_3}{\partial d_3} = -\frac{1}{3b},
\]
\[
\frac{\partial r_1}{\partial c_2} = 0, \quad \frac{\partial r_2}{\partial c_2} = \frac{3d}{2b}, \quad \frac{\partial r_3}{\partial c_2} = \frac{4d_1}{3b} + \frac{3dL}{b} + \frac{3dr_1}{2b} - \frac{r_1}{5} + \frac{c_2}{3b},
\]
\[
\frac{\partial r_1}{\partial c_3} = 0, \quad \frac{\partial r_2}{\partial c_3} = 0, \quad \frac{\partial r_3}{\partial c_3} = \frac{5d}{6b},
\]

(15)

Here we have defined for convenience \(L \equiv \ln(M/\mu)\). Consistently integrating the partial derivatives of \(r_1\) yields
\[
r_1 = \frac{d}{b} \tau_M - \frac{d_1}{b} - X_1(Q),
\]

(16)

where \(\tau_M \equiv \ln(M/\bar{\Lambda})\) and \(X_1(Q)\) is an FRS-invariant, analogous to \(\rho_0(Q)\) for the single scale problem defined in Eq.(11). Exactly analogous to \(\Lambda_R\), for the moment problem one can define an FRS-invariant \(\Lambda_M\) so that
\[
\frac{d}{b} \tau_M - \frac{d_1}{b} - r_1 = X_1(Q) \equiv d \ln \left( \frac{Q}{\Lambda_M} \right).
\]

(17)

Consistently integrating the remaining partial derivatives and using Eq.(16) to recast the \(M\) and \(\mu\) dependence in terms of \(r_1\) and \(\bar{r}_1\), one obtains the explicit dependence of \(r_2\) and \(r_3\) on the FRS parameters \(r_1, \bar{r}_1, d_1, d_2, d_3, c_2, c_3\),

\[
r_2 = \left(\frac{1}{2} - \frac{b}{2d}\right) r_1^2 + \frac{b}{d} \bar{r}_1 - \frac{cd_1}{2b} - \frac{d_2}{2b} - \frac{dc_2}{2b} + X_2
\]
\[
r_3 = \frac{b^2}{2d^2} - \frac{3b}{2d} + \frac{1}{2} \frac{r_1^3}{3} + \left(\frac{b^2}{2d^2} - \frac{b}{2d}\right) r_1^2 \bar{r}_1 + \left(\frac{bc}{d} + \frac{2bd_1}{d^2}\right) r_1 \bar{r}_1
\]
\[+ \left(-\frac{bc}{2d} - \frac{bd_1}{d^2} + \frac{d_1}{d}\right) r_1^2 + \left(-\frac{dc_2}{2b} + \frac{cd_1}{2b} + X_2 + \frac{d_1^2}{2db} + \frac{d_2}{2b} - \frac{d_2}{2b} - c_2\right) r_1\]
analogous to Eqs. (13) in the single scale case. Notice that we could equally use \( r_1 \) and \( L \) as parameters instead of \( \tilde{r}_1 \) and \( r_1 \), since \( L = \frac{(b/d)(r_1 - \tilde{r}_1)}{r_1 - \tilde{r}_1} \). As in the single scale case there are constants of integration \( X_n \) representing the RG-unpredictable part of \( r_n \). They are \( Q \)-independent and FRS-invariant.

In the single scale case parametrizing the RS-dependence using \( r_1, c_2, c_3, \ldots \) means that given a complete \( N^n \)LO calculation \( X_2, X_3, \ldots, X_n \) will be known. Using Eqs. (13) to sum to all-orders the RG-predictable terms, i.e. those not involving \( X_{n+1}, X_{n+2}, \ldots \), with coupling \( a(r_1, c_2, c_3, \ldots) \) is equivalent to \( N^n \)LO perturbation theory in the scheme with \( r_1 = c_2 = c_3 = \ldots = 0 \), and yields the sum

\[
\mathcal{R}^{(n)} = a_0 + X_2 a_0^2 + X_3 a_0^3 + \ldots + X_n a_0^n,
\]

where \( a_0 \equiv a(0,0,0,\ldots) \) is the coupling in this scheme. From Eqs. (9) and (11) it satisfies

\[
\frac{1}{a_0} + c \ln\left(\frac{c a_0}{1 + c a_0}\right) = b \ln\left(\frac{Q}{\Lambda_R}\right).
\]

In fact the solution of this transcendental equation can be written in closed form in terms of the Lambert \( W \)-function \([3, 10]\), defined implicitly by \( W(z) \exp(W(z)) = z \),

\[
a_0 = -\frac{1}{c[1 + W(z(Q))]},
\]

\[
z(Q) \equiv -\frac{1}{e} \left(\frac{Q}{\Lambda_R}\right)^{\frac{-b/c}{e}}.
\]

A similar expansion to Eq. (19), but motivated differently, has been suggested in Ref. [11].

In the moment problem by an exactly similar argument, with the chosen parametrization of FRS, given a complete \( N^n \)LO calculation (i.e. a calculation of \( r_1, r_2, \ldots, r_n \) and the \( d_1, d_2, \ldots, d_n \) and \( c_2, c_3, \ldots, c_n \) in some FRS) the
invariants \( X_2, X_3, \ldots, X_n \) will be known. Using Eqs.\( (18) \) to sum to all-orders the RG-predictable terms not involving \( X_{n+1}, X_{n+2}, \ldots, \), will be equivalent to working with an FRS in which all the FRS parameters are set to zero. \( \tilde{r}_1 = 0 \) means that \( \mu = M \). Setting \( r_1 = 0, d_1 = 0 \) in Eq.\( (17) \) yields \( \tau_M = \beta \ln(Q/\Lambda_M) \), so that \( a = \tilde{a} = a_0 \), given by Eq.\( (21) \) with \( \Lambda_R \) replaced by \( \Lambda_M \). Further, with \( c_i = d_i = 0 \) the integral \( \mathcal{I}(a) \) in Eq.\( (8) \) vanishes, so that finally the sum of all RG-predictable terms for the moment problem at N\(^a\)LO will be

\[
\mathcal{M} = A \left( \frac{c a_0}{1 + c a_0} \right)^{d/b} (1 + X_2 a_0^2 + X_3 a_0^3 + \ldots + X_n a_0^n), \tag{22}
\]

with an extremely similar structure to the single scale case in Eq.\( (19) \). Substituting for \( a_0 \) in terms of the Lambert \( W \)-function using Eq.\( (21) \) we then obtain

\[
\mathcal{M} = A \left[ -W(z(Q)) \right]^{b/d} (1 + X_2 a_0^2 + \ldots)
\]

\[
z(Q) \equiv -\frac{1}{e} \left( \frac{Q}{\Lambda_M} \right)^{-b/c}.
\tag{23}
\]

So that moments of structure functions have a \( Q \)-dependence naturally involving a power of the Lambert \( W \)-function.

As stressed in Ref.[1], the result of resumming all RG-predictable terms depends on the chosen parametrization of RS. By simply translating the parameters to a new set \( \tilde{r}_1 = r_1 - r_1, \tilde{c}_2 = c_2 - c_2, \ldots \), where the barred quantities are constants, one finds corresponding new constants of integration \( \tilde{X}_n \). The result of resumming all RG-predictable terms with this new parametrization then corresponds to standard fixed-order perturbation theory in the RS with \( r_1 = c_1, c_2 = c_2, \ldots \), or equivalently with \( \tilde{r}_1 = \tilde{c}_1 = \tilde{c}_2 = \ldots = 0 \). The key point is that \( r_1 \) has a special status since it contains the ultraviolet (UV) logarithms which build the physical \( Q \)-dependence of \( R(Q) \). Standard RG-improvement corresponds to shifting the parameter \( r_1 \), in which case the resulting constants of integration \( \tilde{X}_n \) contain physical UV logarithms which are not all resummed. Thus \( r_1 \) should be used as the parameter. An exactly similar statement holds for \( r_1 \) and \( \tilde{r}_1 \) in the moment problem. We shall identify the UV logarithms and show how their complete resummation builds the correct physical \( Q \)-dependence in the next section.
We shall refer to the expansions in Eqs. (19) and (22) as Complete RG-improved (CORGI) results. Whilst the parameters implicitly containing the UV logarithms do have a special status, the remaining dimensionless parameters $c_i$ and $d_i$ can be reparametrized as one pleases. As an example, in the Effective Charge approach of Grunberg \[2, 3\] one chooses $c_2, c_3, \ldots, c_n$ so that $\tilde{X}_2, \tilde{X}_3, \ldots, \tilde{X}_n$ are all zero at $N^n$LO, corresponding to $r_1 = r_2 = \ldots = r_n = 0$, and this is a priori equally reasonable. In the moment problem one can correspondingly choose the $c_i$ and $d_i$ so that at $N^n$LO the $\tilde{X}_i$ all vanish and $r_1 = r_2 = \ldots = r_n = 0$. If one further demands that the integral $I(a)$ in Eq. (8) vanishes order-by-order in $a$ a unique FRS is selected in which moments have the form

$\mathcal{M} = A \left( \frac{c \hat{R}}{1 + c \hat{R}} \right)^{d/b}$.

(24)

Where $\hat{R}$ is an effective charge which has a perturbation series of the form,

$\hat{R} = a + \hat{r}_1 a^2 + \hat{r}_2 + \ldots + \hat{r}_n a^{n+1} + \ldots$.

(25)

This is similar to Grunberg’s proposal \[3\] to associate an effective charge with $\mathcal{M}$ so that $\mathcal{M} = A(c \hat{R})^{d/b}$. The $\hat{r}_i$ are built from the $c_i, d_i, M$ and $\mu$, and are RS-dependent, but FS-independent. Effectively $\hat{R}$ can be RG-improved as in the single scale case. We have for instance

$\hat{r}_1 = b \ln(\mu/\tilde{\Lambda}) - b \ln(M/\tilde{\Lambda}) - \frac{b}{d} r_1 + d_1/d = \tau - X_1(Q)$,

(26)

where we have used Eq. (17). Comparing with Eq. (11) we see that treating $\hat{R}$ as a single scale problem we have $\rho_0(Q) = X_1(Q)$. This further implies that $\Lambda_{\hat{R}} = \Lambda_{\mathcal{M}}$ and so the corresponding CORGI couplings are identical. The CORGI expansion for $\hat{R}$ will be of the form

$\hat{R} = a_0 + \hat{X}_2 a_0^2 + \hat{X}_3 a_0^3 + \ldots$.

(27)

Inserting this result in Eq. (24) and re-expanding in $a_0$ will reproduce the CORGI expansion in Eq. (22).
4 Complete RG-improvement

In the single scale case using Eq.(11) one can write

\[ r_1 = b \left( \ln \frac{\mu}{\Lambda} - \ln \frac{Q}{\Lambda_R} \right). \] (28)

The first \( \mu \)-dependent logarithm depends on the RS, whereas the second \( Q \)-dependent UV logarithm will generate the physical \( Q \)-dependence and is RS-invariant. If one makes the simplification that \( c = 0 \) and sets \( c_2 = c_3 = \ldots = 0 \), then the coupling is given by

\[ a(\mu) = \frac{1}{b \ln \left( \frac{\mu}{\Lambda} \right)}. \] (29)

The sum to all-orders of the RG-predictable terms from Eqs.(13), given a NLO calculation of \( r_1 \), simplifies to a geometric progression,

\[ R = a + r_1 a + r_1^2 a^3 + \ldots + r_1^n a^{n+1} + \ldots. \] (30)

The idea of complete RG-improvement is that dimensionful renormalization scales, in this case \( \mu \), should be held strictly independent of the physical energy scale \( Q \) on which \( R(Q) \) depends. In this way the \( Q \)-dependence is built entirely by the “physical” UV logarithms \( b \ln(Q/\Lambda_R) \) contained in \( r_1 \), and the convention-dependent logarithms of \( \mu \) cancel between \( a(\mu) \) and \( r_1(\mu) \), when the all-orders sum in Eq.(30) is evaluated. The conventional fixed-order NLO truncation \( R = a(\mu) + r_1(\mu) a(\mu)^2 \), only makes sense if \( \mu = xQ \), but then the resulting \( Q \)-dependence involves the arbitrary parameter \( x \). In contrast using Eqs.(28),(29) and summing the geometric progression in Eq.(30) gives,

\[ R(Q) \approx a(\mu)/ \left[ 1 - \left( b \ln \frac{\mu}{\Lambda} - b \ln \frac{Q}{\Lambda_R} \right) a(\mu) \right] = 1/(b \ln(Q/\Lambda_R)), \] (31)

correctly reproducing the large-\( Q \) behaviour of \( R(Q) \),

\[ R(Q) \approx 1/b \ln(Q/\Lambda_R) + O(1/b \ln(Q/\Lambda_R)^3). \] (32)

In the moment problem the analogous UV logarithm is \( b \ln(Q/\Lambda_M) \) introduced in Eq.(17), and analogous to Eq.(28) we will have

\[ r_1 = d \left( \ln \frac{M}{\Lambda} - \ln \frac{Q}{\Lambda_M} \right) - \frac{d_1}{b}. \] (33)
Given a NLO calculation of \( r_1 \) we wish to see how the physical \( Q \)-dependence of \( \mathcal{M}(Q) \) arises on resumming to all-orders the UV logarithms contained in the RG-predictable terms from Eqs.(18). If we make similar approximations, so that \( c = 0 \) and the \( d_i \) and \( c_i \) are set to zero, then

\[
\mathcal{M} = A(c a(M))^{d/b}(1 + r_1 a(\mu) + r_2 a(\mu)^2 + \ldots).
\]  

(34)

We retain the overall factor of \( c^{d/b} \). The task is then to show that on resumming the RG-predictable terms in the coefficient function to all-orders the \( \ln(M/\Lambda) \) and \( \ln(\mu/\Lambda) \) contained in \( r_1 \) and \( \tilde{r}_1 \) cancel with those in the couplings \( a(M) \) and \( a(\mu) \) to yield the physical \( Q \)-dependence

\[
\mathcal{M}(Q) \approx A c^{d/b}(1/\ln(Q/\Lambda_M))^{d/b}(1 + O(1/\ln(Q/\Lambda_M))^2).
\]  

(35)

Again, the complete RG-improvement summing over all UV logarithms is forced on one if \( \mu \) and \( M \) are held independent of \( Q \).

The algebraic structure of the resummation of RG-predictable terms for the moment problem is considerably more complicated than the geometric progression of Eq.(30) encountered in the single scale case. With the simplifications \( c = 0, c_i = 0, d_i = 0 \) the first two RG-predictable coefficients from Eqs(18) are

\[
\begin{align*}
  r_2 &= \frac{1}{2} - \frac{b}{2d} r_1 + \frac{b}{d} r_1 \tilde{r}_1, \\
  r_3 &= \frac{b^2}{d^2} - \frac{3b}{2d} + \frac{1}{3} r_1^2 + \frac{-b^2}{d^2} + \frac{b}{d^2} r_1^2 \tilde{r}_1 + \frac{b^2}{d^2} r_1 r_1 \tilde{r}_1^2.
\end{align*}
\]

(36) \hspace{1cm} (37)

Suitably generalizing the partial derivatives in Eqs.(15) one can arrive at a general form for the RG-predictable terms. It is useful to arrange them in columns,

\[
\begin{pmatrix}
  r_1 & (\frac{b}{d} \tilde{r}_1)^0 r_1 \tilde{a} & 0 & 0 & \ldots \\
  r_2 & (\frac{b}{d} \tilde{r}_1)^1 r_1 \tilde{a}^2 & (1 - \frac{b}{d}) \frac{r_1^2}{2} \tilde{a}^2 & 0 & \ldots \\
  r_3 & (\frac{b}{d} \tilde{r}_1)^2 r_1 \tilde{a}^3 & 2(\frac{b}{d} \tilde{r}_1)(1 - \frac{b}{d}) \frac{r_1^2}{2} \tilde{a}^3 & (1 - \frac{b}{d})(\frac{1}{2} - \frac{b}{d}) \frac{r_1^3}{3} \tilde{a}^3 & \ldots \\
  r_4 & (\frac{b}{d} \tilde{r}_1)^3 r_1 \tilde{a}^4 & 3(\frac{b}{d} \tilde{r}_1)^2(1 - \frac{b}{d}) \frac{r_1^2}{2} \tilde{a}^4 & 3(\frac{b}{d} \tilde{r}_1)(1 - \frac{b}{d})(\frac{1}{2} - \frac{b}{d}) \frac{r_1^3}{3} \tilde{a}^4 & \ldots \\
  r_5 & (\frac{b}{d} \tilde{r}_1)^4 r_1 \tilde{a}^5 & 4(\frac{b}{d} \tilde{r}_1)^3(1 - \frac{b}{d}) \frac{r_1^2}{2} \tilde{a}^5 & 6(\frac{b}{d} \tilde{r}_1)^2(1 - \frac{b}{d})(\frac{1}{2} - \frac{b}{d}) \frac{r_1^3}{3} \tilde{a}^5 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(38)
The idea will be to resum each column separately. Denoting the sum of the $m^{th}$ column by $S_m$, one finds
\[ S_1 = r_1 \tilde{a} + \left( \frac{b}{d} \tilde{r}_1 \right) r_1 \tilde{a}^2 + \left( \frac{b}{d} \tilde{r}_1 \right)^2 r_1 \tilde{a}^3 + \left( \frac{b}{d} \tilde{r}_1 \right)^3 r_1 \tilde{a}^4 + \left( \frac{b}{d} \tilde{r}_1 \right)^4 r_1 \tilde{a}^5 + \ldots \]
\[ = r_1 \tilde{a} [1 + \left( \frac{b}{d} \tilde{r}_1 \right) \tilde{a} + \left( \frac{b}{d} \tilde{r}_1 \right)^2 \tilde{a}^2 + \left( \frac{b}{d} \tilde{r}_1 \right)^3 \tilde{a}^3 + \left( \frac{b}{d} \tilde{r}_1 \right)^4 \tilde{a}^4 + \ldots ] \]
\[ = r_1 \tilde{a} (1 - \frac{b}{d} \tilde{r}_1 \tilde{a})^{-1} \] \hfill (39)

Careful examination of the pattern of terms in Eq.(38) leads to the general result for $S_m$ for $m > 1$,
\[ S_m = (-1)^{m-1} \left( \frac{b}{d} - 1 \right) \left( \frac{b}{d} - \frac{1}{2} \right) \left( \frac{b}{d} - \frac{1}{3} \right) + \ldots + \left( \frac{b}{d} - \frac{1}{m-1} \right) \frac{S_1^{m-1}}{m} \] \hfill (40)

Finally the resummed RG-predictable terms in the coefficient function will follow from
\[ C = 1 + S_1 + S_2 + S_3 + \ldots + S_n + \ldots \]
Introducing for convenience $x = S_1 = r_1 \tilde{a} (1 - \frac{b}{d} \tilde{r}_1 \tilde{a})^{-1}$, we find
\[ C = 1 + x - \frac{b}{d} \left( \frac{b}{d} - 1 \right) \frac{x^2}{2} + \left( \frac{b}{d} - 1 \right) \left( \frac{b}{d} - \frac{1}{2} \right) \frac{x^3}{3} - \left( \frac{b}{d} - 1 \right) \left( \frac{b}{d} - \frac{1}{2} \right) \left( \frac{b}{d} - \frac{1}{3} \right) \frac{x^4}{4} + \ldots \]
\[ = 1 + \frac{d}{b} \left( \frac{bx}{d} \right) + \frac{1}{2!} \frac{d}{b} \left( \frac{d}{b} - 1 \right) \left( \frac{bx}{d} \right)^2 + \frac{1}{3!} \frac{d}{b} \left( \frac{d}{b} - 1 \right) \left( \frac{d}{b} - 2 \right) \left( \frac{bx}{d} \right)^3 + \ldots \]
\[ = (1 + \frac{b}{d} x)^{d/b} \] \hfill (41)

Substituting for $x$ yields
\[ C = \left[ 1 + \frac{b}{d} \tilde{r}_1 \tilde{a} (1 - \frac{b}{d} \tilde{r}_1 \tilde{a})^{-1} \right]^\frac{d}{b} = \left[ \frac{1 - \frac{b}{d} \tilde{r}_1 \tilde{a} + \frac{b}{d} \tilde{r}_1 \tilde{a}}{1 - \frac{b}{d} \tilde{r}_1 \tilde{a}} \right]^\frac{d}{b} \] \hfill (42)

We can write the numerator in Eq.(42) as
\[ (1 - \frac{b}{d} \tilde{r}_1 \tilde{a} + \frac{b}{d} r_1 \tilde{a}) = [1 + \tilde{a} b (\frac{r_1 - \tilde{r}_1}{d})] = (1 + \tilde{a} L) \] \hfill (43)

Where $L = b \ln(M/\mu) = b (r_1 - \tilde{r}_1)/d$. Since we are setting $c = c_2 = c_3 = \ldots = 0$ one has $(1 + \tilde{a} L)^{-1} = \tilde{a}/\tilde{a}$, substituting this into Eq.(42) gives
\[ C = \left[ (1 - \frac{b}{d} \tilde{r}_1 \tilde{a}) \tilde{a} \right]^\frac{d}{b} = \left[ \frac{1 - \frac{b}{d} \tilde{r}_1 \tilde{a}}{\tilde{a}} \right]^\frac{d}{b} \] \hfill (44)
Since $\tilde{a} = a(\mu) = 1/\tau$ we can rearrange Eq.(16) to obtain

$$\tilde{r}_1 = \frac{d}{b} \frac{1}{\tilde{a}} - d\ln \frac{Q}{\Lambda_M} ,$$

(45)

and substituting this result into Eq.(43) we find

$$\mathcal{C} = \left( \frac{1}{b\ln(Q/\Lambda_M)} \right)^{d/b} a^{-d/b} .$$

(46)

Combining this with the anomalous dimension part $(ca)^{d/b}$ we reproduce the physical $Q$-dependence of $\mathcal{M}(Q)$ in Eq.(35).

## 5 Discussion and Conclusions

An alternative and more straightforward way of understanding the CORGI proposal is as follows. Given a dimensionless observable $\mathcal{R}(Q)$, dependent on the single dimensionful scale $Q$, we clearly must have, on grounds of generalized dimensional analysis \[12\]

$$\mathcal{R}(Q) = \Phi \left( \frac{\Lambda}{Q} \right) ,$$

(47)

where $\Lambda$ is a dimensionful scale, connected with the universal dimensional transmutation parameter of the theory, whose definition will depend on the way in which ultraviolet divergences are removed, $\Lambda_{\overline{MS}}$ for instance. We can try to invert Eq.(47) to obtain

$$\frac{\Lambda}{Q} = \Phi^{-1}(\mathcal{R}(Q)) ,$$

(48)

where $\Phi^{-1}$ is the inverse function. This is indeed the basic motivation for Grunberg’s Effective Charge approach \[2, 3\]. We are assuming massless quarks here. The extension if one includes masses has been discussed in \[3, 13\]. The structure of $\Phi^{-1}$ is \[14, 15\]

$$\mathcal{F}(\mathcal{R}(Q))\mathcal{G}(\mathcal{R}(Q)) = \Lambda_{\overline{R}}/Q ,$$

(49)
where
\[ F(\mathcal{R}(Q)) \equiv e^{-1/b \mathcal{R}} (1 + 1/b \mathcal{R})^{c/b} \quad (50) \]
is a universal function of \( \mathcal{R} \). \( \Lambda_{\mathcal{R}} \) is connected with the universal parameter \( \Lambda_{\overline{MS}} \) by the relation
\[ \Lambda_{\mathcal{R}} = e^{r/b} \tilde{\Lambda}_{\overline{MS}} \quad (51) \]
which follows from Eq.(11), with \( r = r_1^{\overline{MS}}(\mu = Q) \) the NLO \( \overline{MS} \) coefficient. Note that \( r \) is \( Q \)-independent. The tilde over \( \Lambda \) reflects the convention assumed in integrating the beta-function equation to obtain Eq.(9) \[ \text{[3]}, \]
and \( \tilde{\Lambda}_{\overline{MS}} = (2c/b)^{-c/b} \Lambda_{\overline{MS}} \) in terms of the standard convention. The function \( G(\mathcal{R}(Q)) \) has the expansion
\[ G(\mathcal{R}(Q)) = 1 - \frac{X_2}{b} \mathcal{R}(Q) + O(\mathcal{R}^2) + \ldots \quad (52) \]
Here \( X_2 \) is the NNLO RS-invariant constant of integration which arises in Eqs.(13). Assembling all this we finally obtain the desired inverse relation between \( \mathcal{R} \) and \( \Lambda \), the universal dimensional transmutation parameter of the theory
\[ QF(\mathcal{R}(Q))G(\mathcal{R}(Q))e^{-r/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}} \quad (53) \]
Notice that all dependence on the subtraction scheme chosen resides in the single factor \( e^{-r/b} \), the remainder of the expression being independent of this choice. This corresponds to the observation of Celmaster and Gonshalves \[ \text{[4]}, \]
that \( \Lambda \)'s with different subtraction conventions can be exactly related given a one-loop (NLO) calculation. If only a NLO calculation has been performed \( G = 1 \) since \( X_2 \) will be unknown, so that the best one can do in reconstructing \( \Lambda_{\overline{MS}} \) is
\[ QF(\mathcal{R}(Q))e^{-r/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}} \quad (54) \]
This is precisely the result obtained on inverting the NLO CORGI result \( \mathcal{R} = a_0 \) given by Eq.(21).

The essential point is that the dimensional transmutation scale \( \Lambda \) is the fundamental object. In contrast the convention-dependent dimensionful scales \( \mu \) and \( M \) are ultimately irrelevant quantities which cancel out of physical predictions if one takes care to resum all of the ultraviolet logarithms that build the physical \( Q \)-dependence in association with \( \Lambda \). Our purpose
has been to indicate that the unphysical \( \mu \) and \( M \) dependence of conventional fixed-order perturbation theory reflects its failure to resum all of these RG-predictable terms. We have analyzed how Eq.(54) is built by explicitly resumming the convention-dependent logarithms together with the ultraviolet logarithms. Having done this, however, one can simply use Eq.(53) to test perturbative QCD. Given at least a NLO calculation for an observable \( \mathcal{R}(Q) \) one simply substitutes the data values into Eq.(53), where \( \mathcal{G}(\mathcal{R}(Q)) \) can include NNLO and higher corrections if known, and obtains \( \Lambda_{\overline{MS}} \). To the extent that remaining higher-order perturbative and possible power corrections are small, one should find consistent values of \( \Lambda_{\overline{MS}} \) for different observables. There is no need to mention \( \mu \) or \( M \) in this analysis, let alone to vary them over an \textit{ad hoc} range of values. For the moment problem the result corresponding to Eq.(53) is

\[
Q \mathcal{F} \left( \frac{M}{A} \right) \mathcal{G} \left( \frac{M}{A} \right) e^{-\hat{r}/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}},
\]

where \( \hat{r} \equiv \hat{r}_{1,\overline{MS}}(\mu = Q) \) is defined in Eq.(26). The modified functions \( \mathcal{F} \) and \( \mathcal{G} \) are most easily obtained by noting that \( \mathcal{R} \) in Eq.(24) is directly related to \( M/A \) and also satisfies Eq.(53). One finds

\[
\mathcal{F}(x) = \exp[bc(1 - x^{-b/d})(1 + bc(x^{-b/d} - 1))^{c/b}]
\]

\[
\mathcal{G}(x) = \left( 1 - \frac{X_2}{d} \frac{x^{b/d}}{c(1 - x^{b/d})} + \ldots \right).
\]

Where \( X_2 \) is the NNLO FRS-invariant which arises in Eqs.(18). The scheme-independent parameter \( A \) reflects a physical property of the operator \( O_n \) in Eq.(2). \( A_n \) and \( \Lambda_{\overline{MS}} \) should be fitted simultaneously to the data for \( M_n(Q) \) using Eq.(55).

We hope to report direct fits of data to \( \Lambda_{\overline{MS}} \) as outlined above, for both \( e^+e^- \) jet observables \cite{17} and structure functions and their moments \cite{18}, in future publications.

**Aknowledgements**

A.M. acknowledges the financial support of the Iranian government and also thanks S.J. Burby for useful discussions.
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