LARGE-TIME EXISTENCE FOR ONE-DIMENSIONAL GREEN-NAGHDI EQUATIONS WITH VORTICITY

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ABSTRACT. This essay is concerned with the one-dimensional Green-Naghdi equations in the presence of a non-zero vorticity according to the derivation in [5], and with the addition of a small surface tension. The Green-Naghdi system is first rewritten as an equivalent system by using an adequate change of unknowns. We show that solutions to this model may be obtained by a standard Picard iterative scheme. No loss of regularity is involved with respect to the initial data. Moreover solutions exist at the same level of regularity as for first order hyperbolic symmetric systems, i.e., with a regularity in Sobolev spaces of order $s > 3/2$.

1. Introduction. The water wave theory lies entirely inside hydrodynamics, a branch of fluid mechanics which deals with constant density fluids. Water wave problems have been widely used as an exploratory subject. Hydrodynamic processes identified with the motion of waves in shallow-water situations including dispersive (characterized by the shallowness parameter $\mu \ll 1$) and nonlinear phenomena (characterized by the so-called nonlinearity parameter $\varepsilon$, of order $O(1)$ here), as well as bathymetric effects (characterized by the depth parameter $\beta$, of order $O(\sqrt{\mu})$ here), are complex to study. Their understanding is essential for obtaining information that will help to tame natural phenomena such as tsunami waves, and what makes them more destructive than other ocean waves created by disturbances transferring energy into ocean water. The propagation of surface waves through an incompressible homogeneous inviscid fluid is described by the Euler equations combined with a nonlinear boundary condition at the free surface $\{z = \zeta(t, X), X \in \mathbb{R}^2\}$ and a no-slip condition at the bottom surface $\{z = -h_0 + b(X), X \in \mathbb{R}^2\}$, here described by a non-constant function $b$. The complexity of this problem drove physicists, oceanographers and mathematicians to derive simpler equations for specific physical regimes, say the shallow-water models or the deep-water models. The shallow-water models are widely used in oceanography and atmospheric science. In this framework, many approximate models have been derived. Some of them are the Boussinesq-type approximations that stimulated many wave motion issues in coastal engineering.

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and that may handle most of wave phenomena occurring in near-shore zones (such as refraction, diffraction, shoaling, frequency dispersion and nonlinear interaction). (See [2, 16, 18, 19, 29] for a justification of such approximations.) Nevertheless these models cannot predict either where and when a wave breaks or, particularly, the hydrodynamic features of a breaking wave. In 1953, Serre, and in 1976, Green and Naghdi introduced a higher order model. (See [2, 26, 27] for a justification.) This model has been widely used in coastal oceanography [3, 14, 15, 32] since it considers the dispersive impacts ignored by the shallow-water equations (or by the Saint-Venant approximation of order $O(\mu)$). The Green-Naghdi model is currently the most well-known model used for numerical simulations of waterfront streams, even in setups that incorporate vanishing depth (at the shoreline) and wave breaking. (See, for instance, [4, 7, 9, 13, 21, 24, 25, 30].) Regardless of their many favorable circumstances, the Green-Naghdi equations specially take into consideration neglected rotational effects, which are significant for wind-driven waves, waves riding upon a sheared current, waves near a ship, or tsunami waves approaching a shore. This is the case when vorticity is created: it is then a valuable instrument for showing how the ideal potential flow solutions may be changed to model real flows. (See [15, 23] for a derivation of these equations.)

The vorticity is defined as the curl of the velocity $U$,

$$\omega = \nabla_{X,z} \times U,$$

where the velocity $U = (V, w)^T$ at point $(X, z) = (x, y, z) \in \mathbb{R}^{2+1}$ is decomposed into its horizontal part $V = (u, v)$ and its vertical part $w$. When rotational effects are absent (i.e. $\omega = 0$), the Euler equations (in $\mathbb{R}^+ \times \mathbb{R}^{2+1}$) reduce to a Hamiltonian formulation, the Zakharov model [33], where all the functions are evaluated in terms of the free surface $\zeta(t, X), X \in \mathbb{R}^2$, and a specified function $\psi(t, X) = \phi(t, X, \zeta(t, X))$. On the other hand, in rotational setting this reduction strategy cannot easily be done. For this reason, several approaches have been proposed to take into account this difficulty. (See [5, 6, 8, 34].)

In this paper we will study the well-posedness of the one-dimensional Green-Naghdi equations in the presence of vorticity according to the model derived in [5]. We will make use of non-dimensional variables, and define the non-dimensional parameters

$$\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^3}{\lambda^2}, \quad \beta = \frac{b_0}{h_0}, \quad bo = \frac{\rho gh_0^2}{\sigma},$$

where $a$ is a typical amplitude of the waves, $h_0$ a reference depth of the liquid, $\lambda$ a typical wave-length of the waves, $b_0$ a typical amplitude of the variations of the bottom topography, and $\sigma$ the coefficient of interfacial surface tension. The letters $bo$ designate the so-called Bond number—which will be assumed bounded from below—, $\rho$ the density of the liquid, and $g$ the magnitude of gravity. Any function $f = f(t, X, z)$, defined at the point $(X, z)$ in the domain $\Omega_t$ of the flow, is decomposed into its vertically averaged part $\overline{f} = \overline{f}(t, X)$ and its zero mean value $f^* = f^*(t, X, z)$ as follows,

$$f(t, X, z) = \overline{f}(t, X) + f^*(t, X, z),$$

where

$$\overline{f}(t, X) = \frac{1}{h(t, X)} \int_{-1+\beta b(X)}^{\varepsilon \zeta(t, X)} f(t, X, z) \, dz,$$

and $h(t, X) := 1 + \varepsilon \zeta(t, X) - \beta b(X)$ is the non-dimensional total height of the fluid.
In the case of one-dimensional flows here considered, the space variable $X$ reduces to $X = (x, 0)$ with $x \in \mathbb{R}$, and the velocity $V = V(t, X)$ to its first horizontal component $V(t, X) = (u(t, x, z), 0)$. In particular the first component of the averaged velocity reads as follows,

$$\bar{u}(t, x) = {\frac{1}{h(t, x)}} \int_{-1+\beta b(x)}^{e \zeta(t, x)} u(t, x, z) \, dz.$$  

Moreover the vorticity reduces to its second component $\omega = (0, \omega, 0)$. The shear velocity $v_{sh}$ represents the contribution of the horizontal vorticity to the horizontal velocity, say

$$v_{sh}(t, x, z) = - \int_{z}^{e \zeta(t, x)} \omega(t, x, z) \, dz.$$  

For non-flat bottom surfaces (i.e. $\beta \neq 0$) of medium amplitude (i.e. $\beta = O(\sqrt{\mu})$) in the one-dimensional Green-Naghdi equations with vorticity and surface tension become, after dropping the $O(\mu^2)$ terms, a system written in terms of the free surface parametrization $\zeta$, the averaged velocity $\bar{u}$, and three other variables, $E$, $F$, and $v^\sharp$.

The variable $E$ is the second order self-interaction tensor of the zero-mean shear velocity, and $F$ is the third order self-interaction tensor of the same part of the velocity,

$$E = \int_{-1+\beta b}^{e \zeta} (v_{sh}^*)^2 \, dz, \quad F = \int_{-1+\beta b}^{e \zeta} (v_{sh}^*)^3 \, dz.$$  

The variable $v^\sharp$ is introduced to capture the corrections due to the non-hydrostatic effects of the vorticity and to the interaction of the shear velocity and of the dispersive vertical variations of the horizontal velocity,

$$v^\sharp = \frac{-24}{h^3} \int_{-1+\beta b}^{e \zeta} \int_{z}^{e \zeta} \int_{-1+\beta b}^{z_1} v_{sh}^* \, d\xi \, dz_1 \, dz.$$  

The Green-Naghdi equations then read as follows,

$$\begin{cases}  
\zeta_t + (h\bar{u})_x = 0, \\
(1+\mu T)(\bar{u}_t + \varepsilon \bar{u} \bar{u}_x) + (1 - \frac{\mu}{b_0} \partial^2_\omega) \zeta_x = \varepsilon \mu Q_1(\bar{u}) + \varepsilon \mu \frac{E_x}{h} + \varepsilon \mu^{3/2} C(v^\sharp, \bar{u}) = 0, \\
v^\sharp_t + \varepsilon \bar{u} v^\sharp_x + \varepsilon v^\sharp \bar{u}_x = 0, \\
E_t + \varepsilon \bar{u} E_x + 3\varepsilon \bar{u} E_{xx} + \varepsilon \sqrt{h} F_x = 0, \\
F_t + \varepsilon \bar{u} F_x + 4\varepsilon F_{xx} = 0. 
\end{cases}$$  

(1)

Recall that the unknown variable $h := 1 + \varepsilon \zeta - \beta b$ is the non-dimensional total height of the fluid. Here above, the following quantities have been introduced,

$$T u := - \frac{1}{3h} (h^3 u_x)_x + \frac{\beta}{2h} ((b_x h^2 u)_x - b_x h^2 u_x) + \beta^2 (b_x)^2 u,$$

$$Q_1(u) := \frac{2}{3h} (h^3 (u_x)^2)_x + \beta b_x h (u_x)^2 + \frac{\beta}{2h} (b_{xx} h^2 u^2)_x,$$

and

$$\forall v^\sharp, \forall u, \quad C(v^\sharp, u) := - \frac{1}{6h} \left(2h^3 v^\sharp u_{xx} + (h^3 v^\sharp)_x u_x \right)_x.$$  

(See [5] for a rigorous derivation and more explanations on the signification of all the variables.) As explained in [5], the velocity field in the two-dimensional fluid domain might be reconstructed from the one-dimensional equations (1).
Remark 1. In [5], the authors work with bo\(^{-1}\) = 0. However, it is clear that their results are still valid with a non-zero surface tension term, using the approximation

\[
-\mu \frac{\partial \xi}{bo} \left( k(\varepsilon \sqrt{\mu \zeta}) \right) - \mu \frac{\partial^2 \zeta}{bo} \leq \frac{\mu^2 \varepsilon^2}{bo} C(\mu \varepsilon^2, |\zeta|_{H^{s+2}}),
\]

where \(k(\zeta) = -\partial_{z} \left( \frac{1}{\sqrt{1+|z|^2}} \partial_{z} \zeta \right)\) denotes the mean curvature of the interface. (See [11] for this result and more explanation.) A small surface tension, expressed by bo\(^{-1}\) \leq bo\(_{\text{min}}\)\(^{-1}\), is needed here to use the previous approximation to exactly obtain the surface tension term present in system (1).

Because of the presence of a non-zero vorticity, this model takes into account the dispersive effects neglected in the case of the shallow-water equations; the presence of the terms due to non-zero vorticity makes the analysis more difficult. We will see that the construction of solutions with a standard Picard iterative scheme as in [1,11,17] cannot be achieved, at least as we proceed in our proof, without the addition of a small surface tension that smooths the way for controlling all the terms in the energy estimates. Our objective here is to demonstrate that it is also conceivable to utilize such an iterative scheme to study the well-posedness of the one-dimensional Green-Naghdi equations with vorticity and surface tension in the case of an uneven bottom geometry.

In the derivation of this model (as in [5]) and throughout this paper we consider \(\varepsilon = O(1), \beta = O(\sqrt{\mu})\), with \(\mu \ll 1\), and bo\(^{-1}\) \leq bo\(_{\text{min}}\)\(^{-1}\), with a given bo\(_{\text{min}}\) > 0. More precisely, we will assume parameters to be in the set,

\[
\mathcal{R}_{RG} := \{(\mu, \varepsilon, \beta, bo) : 0 < \mu < 1, 0 < \varepsilon \leq 1, \beta = O(\sqrt{\mu}), 0 < bo^{-1} \leq bo^{-1}_{\text{min}}\}.
\]

We also use the useful simplification \(\max(\beta, \varepsilon) = \varepsilon\), which can be regarded as a consequence of the regime being studied here.

**Notation.** We denote by \(C(\lambda_1, \lambda_2, ...)\) a constant depending on the parameters \(\lambda_1, \lambda_2, \ldots\) and whose dependence on the \(\lambda_j\)’s is always assumed to be nondecreasing. The notation \(a \lesssim b\) means that \(a \leq Cb\), for some nonnegative constant \(C\) whose exact expression is of no importance (in particular, \(C\) is independent of all the small parameters involved). As usual, \(L^2 = L^2(\mathbb{R})\) is the Hilbert space of all measurable functions \(f\) whose square is Lebesgue integrable, with the standard norm

\[
|f|_2 = \left( \int_{\mathbb{R}} f(x)^2 \, dx \right)^{1/2} < \infty.
\]

The inner product of two functions \(f\) and \(g\) in \(L^2(\mathbb{R})\) is denoted by

\[
(f,g) = \int_{\mathbb{R}} f(x)g(x) \, dx.
\]

The space \(L^\infty = L^\infty(\mathbb{R})\) consists of all measurable functions \(f\), which are bounded almost everywhere, and is endowed with the norm

\[
|f|_\infty = \text{ess sup}_{x \in \mathbb{R}} |f(x)| < \infty.
\]

For functions \(f = f(t,x)\) depending on the time and space variables, the partial derivative with respect to \(x\) is denoted by \(\partial_x f = f_x\) (and the partial derivative with respect to \(t\) is similarly so denoted). The space \(W^{1,\infty} = W^{1,\infty}(\mathbb{R}) = \{f \in L^\infty, f_x \in L^\infty\}\) is endowed with its canonical norm \(|\cdot|_{W^{1,\infty}}\). For any real number \(s, H^s =
\]
$H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions $f$ with the norm $|f|_{H^s} = |\Lambda^s f|_2 < \infty$, where $\Lambda$ is the pseudo-differential operator $\Lambda = (1 - \partial_x^2)^{1/2}$, and $H^\infty(\mathbb{R}) = \cap_{s \in \mathbb{R}} H^s$. For functions $f = f(t,x)$ and $g = g(t,x)$ defined on $[0,T] \times \mathbb{R}$ with $T > 0$, we denote the inner product in $L^2$, the $L^2$-norm, the $L^\infty$-norm, as well as the $H^s$-Sobolev norm, with respect to the spatial variable $x$, by $(f,g) = \langle f(t,\cdot),g(t,\cdot) \rangle$, $|f|_2 = |f(t,\cdot)|_2$, $|f|_\infty = |f(t,\cdot)|_\infty$, and $|f|_{H^s} = |f(t,\cdot)|_{H^s}$, respectively.

Let denote by $C^k(\mathbb{R})$ the space of $k$-times continuously derivable functions, and $C^\infty_0(\mathbb{R})$ the space of infinitely derivable functions, with compact support in $\mathbb{R}$. Let also denote by $C^\infty_b(\mathbb{R})$ the space of infinitely derivable functions that are bounded on $\mathbb{R}$, together with all their derivatives.

For any closed operator $T$ defined on a Banach space $X$ of functions, the commutator $[T,f]$ is defined by $[T,f]g = T(fg) - fT(g)$ with $f$, $g$ and $fg$ belonging to the domain of $T$.

**Remark 2.** Here and throughout the rest of this paper, and for sake of simplicity, we do not try to define the optimal regularity assumptions on the parametrization $b$ of the bottom surface. This could easily be done, but is of no interest for the present purpose. Consequently, we omit to write the dependence on $b$ of the different quantities that appear in the results and their proofs.

## 2. Well-posedness of the one-dimensional Green-Naghdi equations with vorticity and surface tension

In what follows, the letter $h$ will always denote the total non-dimensional height of the liquid, $h := 1 + \varepsilon \zeta - \beta b$. The Green-Naghdi equations (1) may be simplified by keeping only the terms up to order $O(\mu^2, \beta \mu^{3/2}, \beta^2 \mu)$. Moreover the system is rewritten after a change of unknowns inspired by the study presented in [24]. This change of unknowns is essential in the method being used here for proving the existence of solutions. In fact writing system (1) in the form (2) allows to obtain solutions with $\|u, v^x, E, F\|_\mathcal{E}$ of the same order of regularity, which implies the same result for $v^x$, $E$, and $F$ as well. This would not be the case if we directly study the original system (1).

After some straightforward computations and some rearrangements of terms, the system (1) may be written as the following system of equations, set in $\mathbb{R}_+ \times \mathbb{R}$,

\[
\begin{align*}
\begin{cases}
\zeta_t + (h u)_x &= 0, \\
(1 + \mu T)(u_t + \varepsilon (u + \sqrt{\mu} v^x)u_x) + \left(1 - \frac{\mu}{100} \partial_x^2\right)\zeta_x - \varepsilon \sqrt{\mu} v^x u_x &
+ \varepsilon \mu Q_1(u) + \varepsilon \mu^{3/2} c_2(v^x, u) + 3\varepsilon \mu h h_x E \left(\frac{E}{h^3}\right)_x + \varepsilon \mu h^2 \left(\frac{E}{h^3}\right)_x = 0, \\
\left(\frac{v^x}{h}\right)_t + \varepsilon u \left(\frac{v^x}{h}\right)_x &= 0, \\
\left(\frac{E}{h^3}\right)_t + \varepsilon u \left(\frac{E}{h^3}\right)_x &+ 4\varepsilon \sqrt{\mu} h x F \left(\frac{E}{h^3}\right)_x = 0, \\
\left(\frac{F}{h^3}\right)_t + \varepsilon u \left(\frac{F}{h^3}\right)_x &= 0,
\end{cases}
\end{align*}
\]

where the letter $u$ now designates the averaged velocity $\overline{v}$, and $c_2(v^x, u) := c(v^x, u) + \frac{1}{3} h (\partial_x^3 u^x u_{xx})_x = -\frac{1}{6h} \left(\partial_x^3 (v^x u_x)_x\right)_x$. 

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Define the non-zero depth condition as
\[ \exists h_1 > 0, \inf_{x \in \mathbb{R}} h(\cdot, x) \geq h_1, \]  
where as noted above, \( h(\cdot, x) := 1 + \varepsilon \zeta(\cdot, x) - \beta b(x) \) is the non-dimensional height of the liquid at \( x \in \mathbb{R} \). We also introduce the operators,
\[ T = T[\zeta] := h(1 + \mu T), \quad J_\nu := 1 - \nu \partial_x^2, \]
with \( \nu \in \mathbb{R}^*_+ \). Throughout this paper the lemmas given in Appendices A and B will be intensively used, often without referring to them.

2.1. The main result. The Green-Naghdi equations (2) are first written in a condensed form by use of the vector \( U = (\zeta, u, v^\#, E, F^3, F^4)^T \). The quantity containing \( Q_1(u) \) and \( C_2(v^#, u) \) in (2) is decomposed in the following way,
\[ \varepsilon \mu h \left( Q_1(u) + \mu^{1/2} C_2(v^#, u) \right) = Q_1[U]u_x + Q_2[U] \zeta_x + q(U), \]
with
\[ Q_1[U]f := \frac{2\varepsilon \mu}{3} \left( h^3 u_x f \right)_x + \left( \varepsilon \beta h^2 (b_x u)_x \right) f - \frac{\varepsilon \mu^{3/2}}{6} \left( h^3 v^#_x f \right)_x, \]
\[ Q_2[U]f := (\varepsilon \beta^2 \mu b_{xx} h u^2) f, \]
for \( f = f(x, t) \) regular enough, and
\[ q(U) := \frac{\varepsilon \beta \mu}{2} (b_{xx} - \beta (b^2_x)_x) h^2 u^2. \]
We observe that by Lemma A.1 the differential operator \( \mathfrak{T} \) is invertible. We multiply the second equation of (2) by \( h \), then apply \( \mathfrak{T}^{-1} \) to both sides of this equation, so that the system may be written as a single equation for the vector \( U = (\zeta, u, v^#, E, F^3, F^4)^T \),
\[ U_t + A[U]U_x + B(U) = 0. \]  
The operators \( A \) and \( B \) are defined by
\[
A[U] := \begin{pmatrix}
\varepsilon u & h & 0 & 0 & 0 \\
\varepsilon \mu T^{-1} \left( h(J_{\mu h} + 3\varepsilon^2 \mu h \frac{E^3}{F^3}) + Q_2[U] \right) & A(\cdot) & 0 & \varepsilon \mu T^{-1}(h^2 \cdot) & 0 \\
0 & 0 & \varepsilon u & 0 & 0 \\
4\varepsilon \sqrt{\mu} \frac{E^4}{F^4} & 0 & 0 & \varepsilon u & \varepsilon \sqrt{\mu} h \\
0 & 0 & 0 & 0 & \varepsilon u
\end{pmatrix},
\]
with \( A(f) := \varepsilon(u + \sqrt{\mu} v^\dagger)f + \Sigma^{-1}(-\varepsilon \sqrt{\mu} v^\dagger + Q_1[U])f \), and
\[
B(U) := \begin{pmatrix}
-\beta b_x u \\
\Sigma^{-1}(q(U) - 3\varepsilon \mu \beta b_x h^2 \frac{E}{h^2}) \\
0 \\
-\beta \varepsilon \sqrt{\mu} b_x \frac{E}{h^2} \\
0
\end{pmatrix}.
\]

The aim of this paper is to show the following result.

**Theorem 2.1** (Existence and uniqueness). Let \( \tau = (\mu, \varepsilon, \beta, b_0) \in \mathcal{R}_{RG}, b \in H^{s+3}(\mathbb{R}), s > 3/2 \). Let \( U_0 = (\zeta_0, u_0, \frac{v_0^\dagger}{h^2}, E_0, F_0)^T \) be given in \( X^s \), with \( h_0 := 1 + \varepsilon \zeta_0 - \beta b \) satisfying (3).

Then there exists a maximal time \( T_{\text{max}} > 0 \), which could be infinite, such that the Green-Naghdi equations (2) admit a unique solution \( U = (\zeta, u, \frac{v_\dagger}{h^2}, E, F)^T \) in the space \( C^0([0, T_{\text{max}}); X^s) \cap C^1([0, T_{\text{max}}); X^{s-1}) \), with initial condition \( U_0 \) at \( t = 0 \). In addition the nonvanishing depth condition (3) (with a different positive lower bound) is preserved on any compact interval in \([0, T_{\text{max}})\).

Moreover, there exist \( T', C_0 \) and \( \lambda \), all depending on \( (b_0^{-1}_{\min}, h_1^{-1}, |U_0|_{X^s}) \), independent of \( \tau = (\mu, \varepsilon, \beta, b_0) \in \mathcal{R}_{RG}, \) such that \( T'/\varepsilon \leq T_{\text{max}} \), and the solution satisfies the following energy estimate,
\[
\forall 0 \leq t \leq \frac{T'}{\varepsilon}, \quad |U(t, \cdot)|_{X^s} + |\partial_t U(t, \cdot)|_{X^{s-1}} \leq C_0 e^{\lambda t}.
\]

In particular, if \( T_{\text{max}} < \infty \), one has either
\[
|U(t, \cdot)|_{X^s} \rightarrow \infty \quad \text{as} \quad t \rightarrow T_{\text{max}},
\]
or
\[
\inf_{\mathbb{R}} h(t, \cdot) = \inf_{\mathbb{R}} (1 + \varepsilon \zeta(t, \cdot) - \beta b(\cdot)) \rightarrow 0 \quad \text{as} \quad t \rightarrow T_{\text{max}}.
\]

The Banach space \( X^s \) is defined in the next section.

**Remark 3.** In [10] the authors show existence and uniqueness of solutions for the two-dimensional Green-Naghdi system in the case of irrotational flows (i.e., with zero vorticity) without the presence of surface tension. The method used therein is different and implies to work with more regular initial data. (See also [12].) As noticed therein, the existence result is obtained with a velocity in the Sobolev space \( H^{s+1} \) with \( s \) integer, \( s > 4 \). We think that it might be possible to adapt their method to obtain a similar existence result for system (2) at the cost of having to assume more regular initial data. This matter would need more investigations.

### 2.2. Linear analysis

This subsection is devoted to the proof of energy estimates for the initial-value problem linearized around some reference state, say
\[
U = \left(\zeta, u, \frac{v^\dagger}{h^2}, \frac{E}{h^3}, \frac{F}{h^4}\right)^T.
\]
In what follows, an underlined quantity depending on $U$ means that $U$ is replaced by $U$ everywhere in this quantity: for example, $\hat{h} = 1 + \varepsilon \zeta - \beta b$, $\overline{h} = h(1 + \mu \overline{T})$, and $\frac{\mu}{\overline{T}} = \frac{\mu}{\overline{T}}$.

Consider the following linear initial-value problem,

$$\begin{aligned} U_t + A[U]U_x + B(U) &= 0, \\
U|_{t=0} &= U_0, \\
\end{aligned}$$

and define the energy spaces $X^s$ associated to this problem.

**Definition 1.** For all $s \geq 0$ and $T > 0$, $X^s$ denotes the vector space $H^{s+1}(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ endowed with the norm defined, for all $U = (\zeta, u, v^\sharp h, E^3, F^4) \in X^s$,

$$|U|^2_{X^s} = |\zeta|^2_{H^s} + \mu |u|^2_{H^s} + |u|^2_{H^s} + \mu |u|^2_{H^s} + \mu |v^\sharp h|^2_{H^s} + \mu \left| \frac{\mu}{\overline{T}} \right|^2_{H^s}.$$ 

The space $X^s_T$ is the space $C([0, T/\varepsilon]; X^s)$ endowed with its canonical norm.

First remark that a pseudo-symmetrizer for $A[U]$ is given by

$$S[U] := \begin{pmatrix} J_{\mu} & 0 & 0 & 0 \\
0 & \overline{T} & 0 & 0 \\
0 & 0 & J_{\mu} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & J_{\mu} \end{pmatrix}.$$ 

A natural energy for the initial-value problem (7) is then given by

$$E^s(U)^2 = (\Lambda^s U, S[U] \Lambda^s U).$$

The link between $E^s(U)$ and the $X^s$-norm is given in the following lemma, which is easily proven. (See also Lemma 3 of [17].)

**Proposition 2.2.** Let $b \in H^{s+3}(\mathbb{R})$, $s \geq 0$, and $\zeta \in W^{1,\infty}(\mathbb{R})$. Under the non-zero depth condition (3), the energy $E^s(U)$ is equivalent to the $|\cdot|_{X^s}$-norm, i.e. there exist some constants $C$ such that, for all $U \in X^s$,

$$E^s(U) \leq C(|h|_{\infty}, |\overline{h}|_{\infty}) |U|_{X^s},$$

and

$$|U|_{X^s} \leq C(1/h_1) E^s(U).$$

The constants $C$ are independent of $\tau = (\mu, \varepsilon, \beta, bo) \in \mathcal{R}_{RG}$: the first one depends only on $(bo_{\min}, h_1^{-1})$, while the second one depends only on $h_1^{-1}$.

We now prove the existence of solutions to Problem (7).
Proposition 2.3. Let \( b \in H^{s+3}(\mathbb{R}) \), \( s > 3/2 \). Let also \( \bar{U} = (\zeta, u, \frac{E'}{\lambda t}, \frac{F'}{\lambda t})^T \in X^s_T \) be such that \( \bar{U} \in X^{s-1}_T \) and \( h \) satisfies condition (3) on \( [0, \frac{T}{2}] \) for some \( T > 0 \).

Then for all \( U_0 \in X^s \), Problem (7) admits a unique solution \( U = (\zeta, u, \frac{E'}{\lambda t}, \frac{F'}{\lambda t})^T \) in \( X^{s-1}_T \), such that \( U \in X^{s-1}_T \) and \( h \) satisfies condition (3) on \( [0, \frac{T}{2}] \) (with a different positive lower bound).

Moreover, there exist some constants \( C \) and \( C_0 \) depending on the given state \( U \) such that for all \( t \in [0, \frac{T}{2}] \),

\[
E^s(U(t)) \leq e^{\lambda_T t} E^s(U(0)) + \varepsilon \int_0^t e^{\lambda_T (t-t')} C(E^s(U)(t'))dt',
\]

(9)

\[
|U_1|_{X^{s-1}} \leq C_0(E^s(U)) \left(e^{\lambda_T t} E^s(U(0)) + \varepsilon \left(1 + \int_0^t e^{\lambda_T (t-t')} C(E^s(U)(t'))dt'\right)\right),
\]

(10)

for some \( \lambda_T = \lambda_T(\sup_{t_0 \leq t \leq T/\varepsilon} E^s(U(t)), \sup_{0 \leq t \leq T/\varepsilon} |h_t(t)|_{\infty}) \). The constant \( \lambda_T \) is independent of \((\mu, \varepsilon, \beta, b_0) \in \mathcal{R}_{RG}\), but depends on \((b_{\min}^{-1}, h_1^{-1})\).

The proof of existence of solutions to the linear problem, as announced in Proposition 2.3, relies on the energy estimate (9), which is obtained through rather lengthy calculations because of the number of terms involved in the equations of the problem under study. So as to simplify the presentation, we first announce in the following lemma the result which is needed to show estimate (9).

Proposition 2.4. Under the hypotheses of Proposition 2.3, there exist some constants \( C \), depending on the fixed state \( U \), such that for all \( t \in [0, \frac{T}{2}] \),

\[
\partial_t (E^s(U)^2) \leq \varepsilon C(E^s(U), |h_t|_{\infty}) E^s(U)^2 + \varepsilon C(E^s(U)) E^s(U).
\]

The constants \( C \) are independent of \( t = (\mu, \varepsilon, \beta, b_0) \in \mathcal{R}_{RG}, \) but depend on the given parameters \((b_{\min}^{-1}, h_1^{-1})\).

We postpone the proof of this proposition to Section 3, directly going to the proof of Proposition 2.3.

Proof of Proposition 2.3.

Proof. We first derive estimate (9). Fix a \( \lambda \in \mathbb{R} \) and compute

\[
e^{\lambda T} \partial_t (e^{-\lambda t} E^s(U)^2) = -\varepsilon \lambda E^s(U)^2 + \partial_t (E^s(U)^2).
\]

Using Proposition 2.4 and using the fact that \( H^s(\mathbb{R}) \subset W^{1,\infty} \), we obtain

\[
e^{\lambda T} \partial_t (e^{-\lambda t} E^s(U)^2) \leq \varepsilon (C(E^s(U), |h_t|_{\infty}) - \lambda) E^s(U)^2 + \varepsilon C(E^s(U)) E^s(U).
\]

Taking \( \lambda = \lambda_T \) large enough—how large depends on \( \sup_{t \in [0, \frac{T}{2}]} C(E^s(U), |h_t|_{\infty}) \)—to have the first term of the right-hand side negative for all \( t \in [0, \frac{T}{2}] \), one deduces

\[
\forall t \in [0, \frac{T}{2}], \quad e^{\lambda T} \partial_t (e^{-\lambda t} E^s(U)^2) \leq \varepsilon C(E^s(U)) E^s(U).
\]

Integrating this differential inequality yields estimate (9).

To establish estimate (10) we use the system of equations (7). Indeed, one has

\[
|\partial_t U|_{X^{s-1}} = \left|\nabla \mathcal{A}(U) \partial_x U - \nabla \mathcal{B}(U)\right|_{X^{s-1}} \leq C(|U|_{X^s}) |U|_{X^s} + C(|U|_{X^s})
\]

\[
\leq C_0(E^s(U)) \left(e^{\lambda_T t} E^s(U(0)) + \varepsilon \left(1 + \int_0^t e^{\lambda_T (t-t')} C(E^s(U)(t'))dt'\right)\right),
\]

where we used Lemmas A.1 and B.1.
Existence and uniqueness of a solution to the initial value problem (7) follows, by standard techniques, from the a priori estimates (9) and (10). Condition (3) also holds on \([0, T/\varepsilon]\), with a different positive lower bound: the proof of this point is very similar to the one sketched in the proof of Theorem 2.1 here below. (See, for instance, [28,31] for more details, and [11,17] for a similar setting.)

2.3. Proof of Theorem 2.1. We are now in a position to prove Theorem 2.1.

Proof. Thanks to the energy estimates (9) and (10), we shall show the existence of a time, denoted

\[ T' = T'(E^s(U^0)) > 0, \] (11)

and a unique solution \( U \in C([0, T'/\varepsilon]; X^s(\mathbb{R})) \cap C^1([0, T'/\varepsilon]; X^{s-1}(\mathbb{R})) \) to (4), defined as the limit of the iterative scheme

\[
U^0 = U_0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \left\{ \begin{array}{l}
\partial_t U^{n+1}_t + A[U^n] \partial_x U^{n+1} + B[U^n] = 0, \\
U^{n+1}_{t=0} = U_0.
\end{array} \right. \tag{12}
\]

Existence of a maximal finite time of existence of solutions and alternative conditions (5) and (6) are standard for first order quasilinear system for which the existence time \( T' \) of solution verifies property (11). (See e.g. [1,31].)

For sake of completeness and clarity, we recall why the approximate solution \( U^n \) satisfies condition (3) uniformly in \( n \). Proposition 2.3 shows the existence of a time \( T_{n+1} > 0 \) and a unique solution \( U^{n+1}_n \in C^0([0, T_{n+1}/\varepsilon]; X^s) \cap C^1([0, T_{n+1}/\varepsilon]; X^{s-1}) \) to (12), provided

\[ U^n \in C^0([0, T_n/\varepsilon]; X^s) \cap C^1([0, T_n/\varepsilon]; X^{s-1}) \subset X^s_{T_n}, \]

and satisfies (3).

Existence and uniform control of the sequence \((U^n)_n\). The existence of \( T' > 0 \) such that the sequence \((U^n)_n \in X^s_{T'}\) is uniquely defined, having the regularity properties given in Proposition 2.3, and satisfying (3), uniformly in \( n \in \mathbb{N} \), is obtained by induction, as follows.

Proposition 2.3 yields, for all \( t \in [0, T_n/\varepsilon]\),

\[
\begin{cases}
E^s(U^{n+1})(t) \leq C_ne^{\lambda_n t}E^s(U_0), \\
|\partial_t U^{n+1}(t, \cdot)|_{X^{s-1}} \leq C_nE^s(U^{n+1}) \leq C_ne^{\lambda_n t}E^s(U_0),
\end{cases}
\tag{13}
\]

with the constants \( \lambda_n \) and \( C_n \) depending on \((\text{bo}_{\text{min}}, h_1^{-1}, \|U^n\|_{X^s_{T_n}})\), provided \( U^n \in X^s_{T_n} \) satisfies (3) on \([0, T_n/\varepsilon]\), with a positive lower bound \( h_{1,n} > 0 \).

Since \( U^n \) satisfies (12), one has

\[ \partial_t \zeta^{n+1} = -h^n u_x^{n+1} - \varepsilon \zeta_x^{n} u^{n+1} + \varepsilon b_2 u^{n+1}. \]

Using the continuous Sobolev embedding of \( H^{s-1} \) into \( L^\infty \) (here \( s - 1 > 1/2 \)), and the fact that \( h^n \) satisfies (3) on \([0, T_n/\varepsilon]\), with \( h_{1,n} > 0 \), we deduce that

\[ |\partial_t \zeta^{n+1}|_{L^\infty} \leq C(\text{bo}_{\text{min}}, h_1^{-1})\|U^n\|_{X^s_{T_n}}. \tag{14} \]

By use of the expression of \( h \) in terms of \( \zeta \), a simple integration in time obtains

\[ h^{n+1}(t, \cdot) = h^{n+1}(0, \cdot) + \varepsilon \int_0^t \partial_t \zeta^{n+1} \, dt, \]
Using equation (7) and integrating by parts, we obtain

$$|h^{n+1}(t, \cdot) - h^{n+1}(0, \cdot)|_{L^\infty} \leq \epsilon t b C(b_{\min}, h_{1,n}^{-1}) \|U^n\|_{X_{\alpha}^{\frac{1}{2}}}. $$

It is obtained immediately that $h^{n+1}(0, \cdot) = h^{0}(0, \cdot) \geq h_1 > 0$, for all $n$. Thus we may easily prove (by induction) that it is possible to chose $T' > 0$ such that, say, $h^{n+1} > h_{1}/2$ holds on $[0, T'/\epsilon]$, and the energy estimates (13) hold on $[0, T'/\epsilon]$, uniformly in $n$. More precisely, we may show by a straightforward induction that $h^n$ satisfies (3) with the lower bound $h_{1}/2 > 0$, and that the sequence $(U^n)_n$ satisfies, for $t \in [0, T'/\epsilon]$,

$$E^\ast(U^n)(t) \leq C_0 e^{\epsilon \lambda t} E^\ast(U_0), \quad \text{and} \quad |\partial_t U^n(t, \cdot)|_{X_{\alpha-1}} \leq C_0 e^{\epsilon \lambda t} E^\ast(U_0),$$

where the constants $\lambda$ and $C_0$ depend on $(b_{\min}, h_{1}^{-1}, [U_0]_{X^{\alpha}})$, uniformly in $n$.

Convergence of $(U^n)$ towards a solution of the nonlinear problem. The proof that the sequence $(U^n)$ converges towards a solution of the nonlinear problem is very classical and is identical to the one written in [11], page 42-43. We then refer the reader to this reference or to more classical ones such as [31] and [1].

3. Proof of Proposition 2.4. We first perform the calculation of $\partial_t \left( E^\ast(U^2) \right)$. From relation (8), it is deduced

$$\partial_t \left( E^\ast(U^2) \right) = 2 \left( \Lambda^\ast U_t, S[U] \Lambda^\ast U \right) + \left( \Lambda^\ast U_t, \left[ \partial_t S[U] \right] \Lambda^\ast U \right).$$

Using equation (7) and integrating by parts, we obtain

$$\frac{1}{2} e^{\epsilon \lambda t} \partial_t \left( e^{-\epsilon \lambda t} E^\ast(U^2) \right) = \frac{1}{2} \epsilon \lambda E^\ast(U^2) - \left( S[U] A[U] \left[ \Lambda^\ast U \right] X, \Lambda^\ast U \right) - \left( \Lambda^\ast U_t, S[U] \Lambda^\ast U \right) - \left( \Lambda^\ast B(U), S[U] \Lambda^\ast U \right) \quad (15)$$

We now show that the different terms of the right-hand side of (15) are bounded by either $\epsilon C(E^\ast(U), |h_k|_{\infty}) E^\ast(U^2)$ or $\epsilon C(E^\ast(U)) E^\ast(U)$. Moreover, all the constants $C$ involved in the following estimates are independent of $r = (\mu, \epsilon, \beta, b_0) \in R_{RG}$, and may only depend on $(b_{\min}, h_{1}^{-1})$.

These calculations will be the subjects of the following lemmas.

Lemma 3.1. There exists a constant $C$, depending on $U$, such that

$$\left| (S[U] A[U] \left[ \Lambda^\ast U \right] X, \Lambda^\ast U) \right| \leq \epsilon C(E^\ast(U), |h_k|_{\infty}) E^\ast(U^2).$$
Proof. Remarking that $S[U] A[U]$ is equal the to following matrix:

$$
\begin{pmatrix}
\varepsilon J_{\mu}(u) & J_{\mu}(h) & 0 & 0 & 0 \\
\frac{h}{\mu} \left( J_{\mu} + 3\varepsilon^2 \mu \frac{E}{h^3} \right) + Q_2[U] & \varepsilon \left( \mu \Lambda^\ast u_x \right) - \varepsilon \sqrt{\mu} h \partial_x^2 & 0 & \varepsilon \mu h^2 & 0 \\
0 & 0 & 0 & \varepsilon J_{\mu}(u) & 0 \\
A \varepsilon \sqrt{\mu} \frac{E}{h^3} & 0 & 0 & \varepsilon u & \varepsilon \sqrt{\mu} h \\
0 & 0 & 0 & 0 & \varepsilon J_{\mu}(u) \\
\end{pmatrix},
$$

we obtain

$$(S[U] A[U] \Lambda^s U_x, \Lambda^s U) = \varepsilon \left( J_{\mu}(u \Lambda^s \zeta_x), \Lambda^s \zeta \right) + \left( J_{\mu}(h \Lambda^s u_x), \Lambda^s \zeta \right) + \left( \frac{h}{\mu} J_{\mu}(\Lambda^s \zeta_x), \Lambda^s u \right) + 3\varepsilon^2 \mu \left( \frac{E}{h^3} \Lambda^s \zeta_x, \Lambda^s u \right) + Q_2[U] \Lambda^s \zeta_x, \Lambda^s u \right) + \left( \left( \varepsilon \partial_x^2 (u + \sqrt{\mu} \partial_x^2) - \varepsilon \sqrt{\mu} \partial_x^2 + Q_1[U] \right) \Lambda^s u_x, \Lambda^s u \right) + \varepsilon \left( \mu \Lambda^s \left( \frac{E}{h^3} \right)_x, \Lambda^s u \right) + \varepsilon \left( J_{\mu} \left( u \Lambda^s \left( \frac{E}{h^3} \right)_x \right), \Lambda^s \left( \frac{E}{h^3} \right) \right) + 4\varepsilon \sqrt{\mu} \left( \frac{E}{h^3} \Lambda^s \zeta_x, \Lambda^s \left( \frac{E}{h^3} \right) \right) + \varepsilon \left( u \Lambda^s \left( \frac{E}{h^3} \right)_x, \Lambda^s \left( \frac{E}{h^3} \right) \right) + \varepsilon \left( J_{\mu} \left( u \Lambda^s \left( \frac{E}{h^3} \right)_x \right), \Lambda^s \left( \frac{E}{h^3} \right) \right) := \sum_{i=1}^{12} A_i.$$

We now estimate each of the terms $A_i$, $i = 1 \cdots 12$, starting with the simplest terms to deal with. All these estimates are obtained by use of the Cauchy-Schwarz inequality (which we will not mention in general), in addition to other manipulations most of the times.

1. The $A_4 + A_5$ and $A_9 + A_{11}$ terms are straightforwardly estimated by using the Cauchy-Schwarz inequality,

$$|A_4 + A_5| \leq \varepsilon C \left( \sqrt{\mu} \partial_x, |h|_\infty, |u|_\infty, \left| \frac{E}{h^3} \right|_\infty \right) E^s(U)^2,$$

$$|A_9 + A_{11}| \leq \varepsilon C \left( \partial_x, |h|_\infty, \left| \frac{E}{h^3} \right|_\infty \right) E^s(U)^2.$$

2. The terms $A_7$ and $A_{10}$ are easily estimated by using a single integration by parts, so that we obtain

$$|A_7| \leq \varepsilon C \left( \left| \partial_x \right|_{W^{1,\infty}} \right) E^s(U)^2,$$
\[ |A_{10}| \leq \varepsilon C \left( |u|_{\infty} \right) E^s(U)^2. \]

3. The terms \( A_1 \) and \( A_8 + A_{12} \) are dealt with in the same way, so that we only do the work for \( A_1 \). Integrating several times by parts, we obtain
\[ A_1 = \varepsilon \left( J_{\mu} (u \Lambda^s \zeta_x), \Lambda^s \zeta \right) = -\frac{\varepsilon}{2} \left( \frac{\mu}{b_0} \right) \left( u_x \Lambda^s \zeta_x, \Lambda^s \zeta \right) + \frac{\varepsilon \mu}{2b_0} \left( u_x \Lambda^s \zeta_x, \Lambda^s \zeta_x \right), \]
so that we may conclude
\[ |A_1| \leq \varepsilon C \left( |u_x|_{\infty} \right) E^s(U)^2. \]

In the same way, we obtain the estimate
\[ |A_8 + A_{12}| \leq \varepsilon C \left( |u_x|_{\infty} \right) E^s(U)^2. \]

4. To estimate \( A_2 + A_3 \), we first remark that, after several integrations by parts,
\[ A_2 + A_3 = -\left( \frac{h_x \Lambda^s u}{2} \right) \Lambda^s \zeta_x - \mu \left( \frac{\mu}{b_0} \right) \left( u_x \Lambda^s u_x, \Lambda^s \zeta \right), \]
so that we obtain
\[ |A_2 + A_3| \leq \varepsilon C \left( |\zeta_x|_{H^{s-1}}, \left\| \frac{\mu}{b_0} \zeta_x \right\|_{H^{s-1}} \right) E^s(U)^2. \]
Note that, here above, \( h_x \) has been replaced by its value in terms of \( \zeta_x \) and \( \beta \leq \varepsilon \).

5. The last term to estimate is \( A_6 \), which is decomposed into two terms as follows,
\[ A_6 = \varepsilon \left( \left( \frac{3}{2} (u + \sqrt{\mu} v_x^T) - \sqrt{\mu} \right) \Lambda^s u_x, \Lambda^s u \right) + \left( Q_1 U \right) \Lambda^s u_x, \Lambda^s u \]
\[ = A_{61} + A_{62}. \]
Estimating both terms is straightforward: one term of \( A_{61} \) has to be integrated by parts, while it is the case for two terms of \( A_{62} \).

After one integration by parts, the term \( A_{61} \) reads as follows,
\[ A_{61} = \varepsilon \left( h \left( \frac{3}{2} (u + \sqrt{\mu} v_x^T) \right) \Lambda^s u_x, \Lambda^s u \right) - \varepsilon \sqrt{\mu} \left( u_x \Lambda^s u_x, \Lambda^s u \right) + \frac{\varepsilon \mu}{3} \left( h^3 \left( u_x \Lambda^s u_x, \Lambda^s u \right) \right) + \frac{\varepsilon \beta \mu}{2} \left( (b_x^2)_{x} (u + \sqrt{\mu} v_x^T) \Lambda^s u_x, \Lambda^s u \right) + \varepsilon \beta^2 \mu \left( (b_x^2) \Lambda^s u_x, \Lambda^s u \right), \]
so that we obtain
\[ |A_{61}| \leq C \left( |\zeta|_{W^{1,\infty}}, |u_x|_{\infty}, |v_x^T|_{\infty} \right) E^s(U)^2. \]
Integrating by parts two of the terms of \( A_{62} \), we calculate
\[ A_{62} = \left( Q_1 U \right) \Lambda^s u_x, \Lambda^s u \]
\[ = -\frac{2\varepsilon \mu}{3} \left( h^3 \Lambda^s u_x, \Lambda^s u_x \right) + \varepsilon \beta \mu \left( h^2 (b_x u)_x \Lambda^s u_x, \Lambda^s u \right) \]
\[ + \frac{\varepsilon \mu^{3/2}}{6} \left( (h^3 v_x^T)_{x} \Lambda^s u_x, \Lambda^s u_x \right). \]
therefore

\[ |A_{62}| \leq \varepsilon C(\xi_{W^{1,\infty}}, |u|_{W^{1,\infty}}, |v^\sharp|_{W^{1,\infty}})E^s(U)^2. \]

As a consequence, we obtain

\[ |A_6| \leq \varepsilon C(\xi_{W^{1,\infty}}, |u|_{W^{1,\infty}}, |v^\sharp|_{W^{1,\infty}})E^s(U)^2. \]

The estimates of each of the \( A_i \)'s show the result of this lemma.

We first show a result which will be needed in proving several estimates in Lemma 3.3.

\textbf{Lemma 3.2.} The function \( v^\sharp_x \) is estimated as follows,

\[ \|v^\sharp_x\|_H^s \leq C(E^s(U)). \]

\textit{Proof.} We calculate \( v^\sharp_x \) in terms of \( \frac{v^\sharp}{h} \) as follows:

\[ \|v^\sharp_x\|_H^s = \left| \left( \frac{v^\sharp}{h} \right)_x \right|_H^s \]

\[ = \left| \left( \frac{v^\sharp}{h} \right)_x + h_x \left( \frac{v^\sharp}{h} \right) \right|_H^s \]

\[ \leq \left| \left( \frac{v^\sharp}{h} \right)_x \right|_H^s (|h - 1|_H^s + 1) + |h_x|_H^s \left( \frac{v^\sharp}{h} \right)_H^s, \]

where

\[ |h_x|_H^s \leq \frac{\varepsilon H}{b_0} |\xi_x|_H^s + \beta |b_x|_H^s. \]

Therefore, we obtain the result of the lemma.

\textbf{Lemma 3.3.} The term \( \left( [\Lambda^s, A[U]] \partial_x U, S[U] \Lambda^s U \right) \) is estimated as follows:

\[ \left| \left( [\Lambda^s, A[U]] \partial_x U, S[U] \Lambda^s U \right) \right| \leq \varepsilon C(E^s(U), |h_x|_\infty)E^s(U)^2. \]
Proof. We calculate
\[
([\Lambda^s, A[U]] \partial_x U, S[U] \Lambda^s U) = \left( [\Lambda^s, \varepsilon \mu] \xi, J_\Lambda \Lambda^s \xi \right) + \left( [\Lambda^s, \mu] u_x, J_\Lambda \Lambda^s \xi \right)
+ \left( [\Lambda^s, \Xi^{-1} \left( h J_\Lambda + 3 \varepsilon^2 \mu \frac{E}{h^2} + Q_2[U] \right)] \xi, \Xi \Lambda^s u \right)
+ \left( [\Lambda^s, \varepsilon \mu] u_x, \Xi \Lambda^s u \right) + \left( [\Lambda^s, \varepsilon \sqrt{\mu} u] u_x, \Xi \Lambda^s u \right)
- \varepsilon \mu \left( [\Lambda^s, \Xi^{-1} (h^2)] u_x, \Xi \Lambda^s u \right)
+ ( [\Lambda^s, \Xi^{-1} Q_1[U]] u_x, \Xi \Lambda^s u )
+ \varepsilon \mu \left( [\Lambda^s, \Xi^{-1} (h^2)] \left( \frac{E}{h^3} \right)_x, \Xi \Lambda^s u \right)
+ \varepsilon \left( [\Lambda^s, \mu] \left( \frac{v^3}{h} \right)_x, J_\Lambda \Lambda^s \frac{v^3}{h} \right)
+ \varepsilon \sqrt{\mu} \left( [\Lambda^s, \frac{E}{h^3}] \xi, \Lambda^s \frac{E}{h^3} \right) + \varepsilon \left( [\Lambda^s, \mu] \left( \frac{E}{h^3} \right)_x, \Lambda^s \frac{E}{h^3} \right)
+ \varepsilon \sqrt{\mu} \left( [\Lambda^s, \frac{E}{h^3}] \xi, \frac{E}{h^3} \right)
+ \varepsilon \left( [\Lambda^s, \mu] \left( \frac{E}{h^3} \right)_x, \frac{E}{h^3} \right)
\]
\[= \sum_{i=1}^{13} B_i. \]

Here again, we may start by estimating the terms which are the easiest to deal with. Nonetheless, because the calculations are similar to the ones done for Lemma 3.1, we shall give the detailed calculation only for specific terms.

1. To estimate \( B_5 \), we calculate
\[
B_5 = \varepsilon \sqrt{\mu} \left( [\Lambda^s, v^3] u_x, \frac{h^3 \Lambda^s u_x}{h^3} \right) - \frac{\varepsilon \mu^{3/2}}{3} \left( [\Lambda^s, v^3] u_x, \frac{h^3 \Lambda^s u_x}{h^3} \right)
+ \frac{\varepsilon \beta^{3/2}}{2} \left( [\Lambda^s, v^3] u_x, (b_x h^2)_x \Lambda^s u \right)
\]
\[= B_{51} + B_{52} + B_{53}. \]

Each \( B_{5i} \), \( i = 1, 2, 3 \), is straightforwardly estimated in terms of \( E^s(U) \) and \( E^s(U) \) by using the Cauchy-Schwarz inequality and Lemma B.2. First, we obtain
\[
|B_{51}| \leq \varepsilon \sqrt{\mu} |h|_{\infty} |v^3|_{H^{-1}} |u_x|_{H^{-1}} |u|_{H^s}
\leq \varepsilon \sqrt{\mu} |h|_{\infty} (|h|_{-1} + 1) |v^3|_{H^s} |u|_{H^s}
\leq \varepsilon C(E^s(U)) E^s(U)^2.
\]

To estimate the next term, we calculate
\[
|B_{52}| = \frac{\varepsilon \mu^{3/2}}{3} \left( [\Lambda^s, v^3] u_x, \frac{h^3 \Lambda^s u_x}{h^3} \right) + \left( [\Lambda^s, v^3] u_x, \frac{h^3 \Lambda^s u_x}{h^3} \right)
\leq \frac{\varepsilon \mu^{3/2}}{3} |v^3|_{H^s} |u_x|_{H^{-1}} |\frac{h^3 \Lambda^s u_x}{h^3}|_{H^s} + \frac{\varepsilon \mu^{3/2}}{3} |v^3|_{H^s} |u_x|_{H^{-1}} |\frac{h^3 \Lambda^s u_x}{h^3}|_{H^s}.
\]
Using the result of Lemma 3.2, we obtain
\[ |B_{52}| \leq \varepsilon C(E^s(U)) E^s(U)^2. \]

We now calculate,
\[ |B_{53}| = \frac{\varepsilon \beta \mu^{3/2}}{2} \left| \left( [\Lambda^s, \varepsilon u_x] \zeta_x, \Lambda^s \zeta_{xx} \right) + \left( [\Lambda^s, \varepsilon u_x] \zeta_{xx}, \Lambda^s \zeta_x \right) \right|, \]
and the same derivation of the estimate as for \( B_{52} \) is also valid for \( B_{53} \). The term \( B_{54} \) also satisfies the same estimate, so that
\[ |B_5| \leq \varepsilon C(E^s(U)) E^s(U)^2. \]

2. To estimate \( B_1 + B_2 \), we calculate
\[ B_1 + B_2 = \left( [\Lambda^s, \varepsilon u] \zeta_x, \Lambda^s \zeta_{xx} \right) + \left( [\Lambda^s, \varepsilon u] \zeta_{xx}, \Lambda^s \zeta_x \right) - \frac{\varepsilon \mu}{b_0} \left( [\Lambda^s, \varepsilon u] \zeta_x, \Lambda^s \zeta_{xx} \right) \]
\[ - \frac{\mu}{b_0} \left( [\Lambda^s, \varepsilon u] \zeta_{xx}, \Lambda^s \zeta_x \right) \]
\[ := B_{121} + \cdots + B_{124}. \]

The term \( B_{121} \) is estimated as follows,
\[ |B_{121}| \leq \varepsilon |u_x|_{H^{-1}} |\zeta_x|_{H^{-1}} |\zeta|_{H^s} \]
\[ \leq \varepsilon |u|_{H^s} |\zeta_x|_{H^s}^2 \]
\[ \leq \varepsilon C(E^s(U)) E^s(U)^2, \]
and, similarly, for the term \( B_{122} \),
\[ |B_{122}| \leq \varepsilon C(E^s(U)) E^s(U)^2. \]

For the term \( B_{123} \), we may calculate
\[ B_{123} = \frac{\varepsilon \mu}{b_0} \left( [\Lambda^s, \varepsilon u] \zeta_x, \Lambda^s \zeta_{xx} \right) \]
\[ = -\frac{\varepsilon \mu}{b_0} \left( \partial_x [\Lambda^s, \varepsilon u] \zeta_x, \Lambda^s \zeta_{xx} \right) \]
\[ = -\frac{\varepsilon \mu}{b_0} \left( [\Lambda^s, \varepsilon u_x] \zeta_x, \Lambda^s \zeta_{xx} \right) - \frac{\varepsilon \mu}{b_0} \left( [\Lambda^s, \varepsilon u_{xx}] \zeta_x, \Lambda^s \zeta_{xx} \right) \]
\[ := B_{1231} + B_{1232}. \]

Both of these terms are estimated as follows,
\[ |B_{1231}| \leq \frac{\varepsilon \mu}{b_0} |u_x|_{H^{-1}} |\zeta_x|_{H^{-1}} |\zeta|_{H^s} \]
\[ \leq \frac{\varepsilon \mu}{b_0} \left( |u|_{H^s}^2 |\zeta_x|_{H^s}^2 + |\zeta|_{H^s}^2 \right) \]
\[ \leq \varepsilon C(E^s(U)) E^s(U)^2, \]
and
\[ |B_{1232}| \leq \frac{\varepsilon \mu}{b_0} |u_{xx}|_{H^{-1}} |\zeta_{xx}|_{H^{-1}} |\zeta_x|_{H^s} \]
\[ \leq \frac{\varepsilon \mu}{b_0} |u|_{H^s} |\zeta_x|_{H^s}^2 \]
\[ \leq \varepsilon C(E^s(U)) E^s(U)^2; \]
this shows the estimate
\[ |B_{123}| \leq \varepsilon C(E^s(U)) E^s(U)^2. \]
The term $B_{124}$ also satisfies the same estimate, so that
$$|B_3 + B_4| \leq \varepsilon C(E^s(\Omega))E^s(U)^2.$$ 

3. To estimate $B_4$, we calculate
$$B_4 = \varepsilon \left( [\Lambda^s, \underline{u}] u_x, \hat{h} \Lambda^s u \right) + \frac{\varepsilon \beta}{2} \mu \left( [\Lambda^s, \underline{u}] u_x, \partial_x (b_x \hat{h}^2 \Lambda^s u) \right)$$
$$- \frac{\varepsilon}{3} \left( [\Lambda^s, \underline{u}] u_x, \hat{h} \Lambda^s u \right) - \frac{\varepsilon \beta}{2} \mu \left( [\Lambda^s, \underline{u}] u_x, b_x \hat{h}^2 \Lambda^s u_x \right)$$
$$:= B_{41} + \ldots + B_{44}.$$  

The first term is estimated as follows,
$$|B_{41}| \leq \varepsilon |[\underline{u}] |_{H^{s-1}} |u_x|_{H^{s-1}} |\hat{h}|_{H^s} |u|_{H^s}$$
$$\leq \varepsilon |[\underline{u}] |_{H^{s-1}} |u_x|_{H^{s-1}} |\hat{h}|_{H^s} |u|_{H^s}$$
$$\leq \varepsilon C(E^s(\Omega))E^s(U)^2.$$ 

We may decompose the second term in the following way,
$$B_{42} = - \frac{\varepsilon \beta}{2} \mu \left( \partial_x [\Lambda^s, \underline{u}] u_x, b_x \hat{h}^2 \Lambda^s u \right)$$
$$= - \frac{\varepsilon \beta}{2} \mu \left( [\Lambda^s, \underline{u}] u_x, b_x \hat{h}^2 \Lambda^s u_x \right) - \frac{\varepsilon \beta}{2} \mu \left( \partial_x [\Lambda^s, \underline{u}] u_x, b_x \hat{h}^2 \Lambda^s u_x \right)$$
$$:= B_{421} + B_{422}.$$ 

Both terms are estimated as follows,
$$|B_{421}| \leq \frac{\varepsilon \beta}{2} \mu \left( b_x |\underline{u} x|_{H^{s-1}} |u_x|_{H^{s-1}} |\hat{h}^2|_{\infty} |u|_{H^s} \right)$$
$$\leq \frac{\varepsilon \beta}{2} \mu \left( b_x |\underline{u} x|_{H^{s-1}} |\hat{h}^2|_{\infty} |u|_{H^s} \right)$$
$$\leq \varepsilon C(E^s(\Omega))E^s(U)^2;$$

and
$$|B_{422}| \leq \frac{\varepsilon \beta}{2} \mu \left( b_x |\underline{u} x|_{H^{s-1}} |u_x|_{H^{s-1}} |\hat{h}^2|_{\infty} |u|_{H^s} \right)$$
$$\leq \frac{\varepsilon \beta}{2} \mu \left( b_x |\underline{u} x|_{H^{s-1}} |\hat{h}^2|_{\infty} |u|_{H^s} \right)$$
$$\leq \varepsilon C(E^s(\Omega))E^s(U)^2.$$ 

This shows that $B_{42}$ also satisfies the same estimate.

To estimate $B_{43}$, we calculate
$$B_{43} = \frac{\varepsilon}{3} \mu \left( \partial_x [\Lambda^s, \underline{u}] u_x, \hat{h}^3 \Lambda^s u_x \right)$$
$$= \frac{\varepsilon}{3} \mu \left( [\Lambda^s, \underline{u}] u_x, \hat{h}^3 \Lambda^s u_x \right) + \frac{\varepsilon}{3} \mu \left( \partial_x [\Lambda^s, \underline{u}] u_x, \hat{h}^3 \Lambda^s u_x \right)$$
$$:= B_{431} + B_{432}.$$ 

Both terms are estimated as follows,
$$|B_{431}| \leq \frac{\varepsilon}{3} \mu \left( [\underline{u}] |_{H^{s-1}} |u_x|_{H^{s-1}} |\hat{h}^3|_{\infty} |u_x|_{H^s} \right)$$
$$\leq \frac{\varepsilon}{3} \mu \left( [\underline{u}] |_{H^{s-1}} |\hat{h}^3|_{\infty} |u|_{H^s} \right)$$
$$\leq \varepsilon C(E^s(\Omega))E^s(U)^2.$$
and

\[ |B_{432}| \leq \frac{\varepsilon \mu}{3} |w|_{H^{r-1}} |u_{xx}|_{H^{r-1}} |h^3|_\infty |u_x|_{H^r} \]
\[ \leq \frac{\varepsilon}{3} |w|_{H^r} |h^3|_\infty |u_x|^2_{H^r} \]
\[ \leq \varepsilon C(E^*(U)) E^*(U)^2. \]

This shows that \(|B_{432}|\) is also bounded by \(C(E^*(U)) E^*(U)^2\). The same procedure applies for estimating \(|B_{444}|\), which is also bounded by the same quantity.

These ruminations show that the term \(B_4\) satisfies the same estimate,

\[ |B_4| \leq \varepsilon C(E^*(U)) E^*(U)^2. \]

4. To estimate \(B_3\), we first remark that

\[ \mathcal{T} \left[ \Lambda^*, \mathcal{T}^{-1} \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \right] \zeta_x \]
\[ = \mathcal{T} \left[ \Lambda^*, \mathcal{T}^{-1} \right] \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \zeta_x \]
\[ + \left[ \Lambda^*, \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \right] \zeta_x, \]

so that

\[ \mathcal{T} \left[ \Lambda^*, \mathcal{T}^{-1} \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \right] \zeta_x \]
\[ = \left[ \Lambda^*, \mathcal{T}^\dagger \right] \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \zeta_x \]
\[ + \left[ \Lambda^*, h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right] \zeta_x. \]

We now use the definition of \(\mathcal{T}\) and the fact that

\[ \left[ \Lambda^*, \partial_x(f \cdot g) \right] = \partial_x \left[ \Lambda^*, f \right] g, \]

and obtain

\[ B_3 = - \left( \left[ \Lambda^*, h \right] \mathcal{T}^{-1} \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \zeta_x, \Lambda^* u \right) \]
\[ - \frac{\mu}{3} \left( \left[ \Lambda^*, h^3 \right] \partial_x \mathcal{T}^{-1} \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \zeta_x, \Lambda^* u_x \right) \]
\[ + \frac{\beta \mu}{2} \left( \left[ \Lambda^*, h^2 b_2 \right] \mathcal{T}^{-1} \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \zeta_x, \Lambda^* u_x \right) \]
\[ + \frac{\beta \mu}{2} \left( \left[ \Lambda^*, h^2 b_2 \right] \partial_x \mathcal{T}^{-1} \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \zeta_x, \Lambda^* u \right) \]
\[ + \left( \left[ \Lambda^*, h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right] \zeta_x, \Lambda^* u \right) \]
\[ := B_{31} + \ldots + B_{35}. \]

The term \(B_{31}\) is estimated as follows,

\[ |B_{31}| \leq |h - 1|_{H^r} \mathcal{T}^{-1} \left( h J^\mu_\infty + 3 \varepsilon^2 \mu \frac{1}{h^2} E + Q_2[U] \right) \zeta_x |_{H^{r-1}} |u|_{H^r}, \]
where the term containing $\Sigma^{-1}$ needs some calculations to be estimated. Precisely,

$$
\left| \Sigma^{-1}\left( hJ_{\mu} + 3\varepsilon^{2}\mu \frac{1}{h^2}E + Q_2[U] \right) \zeta_x \right|_{H^{s-1}} \\
\leq |\Sigma^{-1}(h\zeta_x)|_{H^{s-1}} + \frac{\mu}{h_0} |\Sigma^{-1}(h\zeta_{xxx})|_{H^{s-1}} + 3\varepsilon^{2}\mu |\Sigma^{-1}\left( \frac{1}{h^2}E\zeta_x \right)|_{H^{s-1}} \\
+ |\Sigma^{-1}(Q_2[U]\zeta_x)|_{H^{s-1}} \\
\leq ||\Sigma^{-1}||_{H^{s-1}} \left( |(h - 1)\xi_x + \zeta_x|_{H^{s-1}} + \frac{\mu}{h_0} |\Sigma^{-1}(h\zeta_{xxx})|_{H^{s-1}} \right. \\
+ 3\varepsilon^{2}\mu ||\Sigma^{-1}||_{H^{s-1}} \left| (h - 1)E \frac{h}{h} + E \frac{h}{h_0} \zeta_x \right|_{H^{s-1}} + |Q_2[U]\zeta_x|_{H^{s-1}}.
$$

We calculate $h$ in terms of $\zeta$, so that

$$
h \zeta_{xxx} = \zeta_{xxx} + \varepsilon \zeta_{xxx} - \beta b \zeta_{xxx}.
$$

We then obtain

$$
\frac{\mu}{h_0} |\Sigma^{-1}(h\zeta_{xxx})|_{H^{s-1}} \\
\leq \frac{\mu}{h_0} |\Sigma^{-1}\partial_x \zeta_{xx}|_{H^{s-1}} + \varepsilon \frac{\mu}{h_0} |\Sigma^{-1}(\zeta_{xxx})|_{H^{s-1}} + \frac{\mu}{h_0} \beta |\Sigma^{-1}(b\zeta_{xxx})|_{H^{s-1}} \\
\leq \frac{\mu}{h_0} ||\Sigma^{-1}||_{H^{s-1}} \left| \partial_x \zeta_{xx} \right|_{H^{s-1}} + \varepsilon \frac{\mu}{h_0} |\Sigma^{-1}(\zeta_{xxx})|_{H^{s-1}} \\
+ \beta \frac{\mu}{h_0} |\Sigma^{-1}(b\zeta_{xxx})|_{H^{s-1}},
$$

where we decompose the quantity $\zeta_{xxx} = (\zeta_{xxx})_x - \zeta_x \zeta_{xx}$. Finally, we obtain the following estimate,

$$
\varepsilon \frac{\mu}{h_0} |\Sigma^{-1}(\zeta_{xxx})|_{H^{s-1}} \\
\leq \varepsilon \frac{\mu}{h_0} |\Sigma^{-1}(\zeta_{xxx})|_{H^{s-1}} + \varepsilon \frac{\mu}{h_0} |\Sigma^{-1}(\zeta_x \zeta_{xx})|_{H^{s-1}} \\
\leq \varepsilon \frac{\mu}{h_0} ||\Sigma^{-1}||_{H^{s-1}} \left| \partial_x \zeta_{xx} \right|_{H^{s-1}} + \varepsilon \frac{\mu}{h_0} |\Sigma^{-1}|_{H^{s-1}} \left| \zeta_x \zeta_{xx} \right|_{H^{s-1}} \\
\leq \varepsilon C(E^s(U)) E^s(U).
$$

Similarly, we have

$$
\frac{\mu}{h_0} \beta |\Sigma^{-1}(b\zeta_{xxx})|_{H^{s-1}} \leq \varepsilon C(E^s(U)) E^s(U),
$$

which implies

$$
\left| \Sigma^{-1}\left( hJ_{\mu} + 3\varepsilon^{2}\mu \frac{1}{h^2}E + Q_2[U] \right) \zeta_x \right|_{H^{s-1}} \leq \varepsilon C(E^s(U)) E^s(U).
$$

Therefore, $B_{31}$ satisfies the estimate

$$
|B_{31}| \leq \varepsilon C(E^s(U)) E^s(U)^2.
$$

In a similar way, we also obtain the same estimate for the term $B_{34}$.

We now go to estimating the term $B_{32}$ as follows,

$$
|B_{32}| \leq \frac{\mu}{3} |h^3 - 1|_{H^s} | \partial_x \Sigma^{-1}\left( hJ_{\mu} + 3\varepsilon^{2}\mu \frac{1}{h^2}E + Q_2[U] \right) \zeta_x \right|_{H^{s-1}} \left| u_x \right|_{H^s}.
$$

We may note that in the process of estimating $B_{32}$, the difficulty is due to the presence of the tension surface term, which is also essential in our method to provide more regularity on $\zeta$. 


For estimating the term $|\partial_x \tilde{\Sigma}^{-1} (\tilde{h} J \tilde{w}) \zeta_x|_{H^{r-1}}$, we define $g = \tilde{h}^{-1} (\tilde{h} J \tilde{w}) \zeta_x$, so that

$$
\tilde{g} = \frac{h}{\tilde{h}} \zeta_x - \frac{\mu}{b_0} \frac{1}{\tilde{h}^2} \zeta_{xx}.
$$

The second term on the right-hand side is developed as follows,

$$
- \frac{\mu}{b_0} \left( h \zeta_{xx} \right)_x = \frac{\mu}{3 \tilde{h}} \left( \frac{3}{b_0} \frac{1}{\tilde{h}^2} \zeta_x \right)_x - \frac{\mu}{3 \tilde{h}} \left( \frac{3}{b_0} \frac{1}{\tilde{h}^2} \right) \zeta_x,
$$

so that $\tilde{g}$ is now expressed in terms of $\tilde{g}(\zeta_x/\tilde{h}^2)$ as follows,

$$
\tilde{g} = \tilde{g} \left( \frac{3}{b_0} \frac{1}{\tilde{h}^2} \zeta_x \right) + R + R_b,
$$

with

$$
R = \frac{3 \mu}{b_0} \left( \frac{1}{2} b_{xx} \zeta_x + b_x \frac{h}{\tilde{h}} \zeta_x - \beta(b_x) \frac{1}{\tilde{h}} \zeta_x \right).
$$

We have hence obtained

$$
g = \frac{3}{b_0} \frac{1}{\tilde{h}^2} \zeta_x + \tilde{h}^{-1} (R - R_b),
$$

which implies

$$
\sqrt{\mu} g_x = \frac{3 \sqrt{\mu}}{b_0} \frac{1}{\tilde{h}^2} \zeta_{xx} - \frac{6 \sqrt{\mu}}{b_0} \frac{h}{\tilde{h}} \zeta_x + \sqrt{\mu} \partial_x \tilde{h}^{-1} (R - R_b),
$$

and, after estimating all the terms,

$$
\sqrt{\mu} |g_x|_{H^{r-1}} \leq C (E^s(U)^2) E^s(U),
$$

which shows the following estimate for $B_{32}$,

$$
|B_{32}| \leq \varepsilon C (E^s(U)) E^s(U)^2.
$$

We may also obtain the same estimate for $|B_{33}|$, by similar calculations, as well as for $B_{35}$ by using the relation

$$
[A^*, f \partial^2_x] g = \partial_x [A^*, f \partial_x] g - [A^*, \partial_x f] \partial_x g.
$$

We have then obtained the following estimate for $B_{33}$,

$$
|B_{33}| \leq \varepsilon C (E^s(U)) E^s(U)^2.
$$

To estimate the term $B_6$, we first remark that

$$
\tilde{\Sigma}[A^*, \tilde{\Sigma}^{-1} u_x] = \tilde{\Sigma}[A^*, \tilde{\Sigma}^{-1}] u_x + [A^*, u_x];
$$

with the use of the formula,

$$
[A^*, \tilde{\Sigma}^{-1}] = -\tilde{\Sigma}^{-1} [A^*, \tilde{\Sigma} \tilde{\Sigma}^{-1}],
$$

we deduce

$$
\tilde{\Sigma}[A^*, \tilde{\Sigma}^{-1} u_x] = -[A^*, \tilde{\Sigma} \tilde{\Sigma}^{-1} (u_x)] + [A^*, u_x] u_x.
$$
Using the definition of $\mathcal{S}$ and integrating by parts, we obtain

\[
B_6 = \varepsilon \mu \left( [\Lambda^s, h] \mathcal{S}^{-1}(v^x u_x), \Lambda^s u \right) - \frac{\varepsilon \mu^2}{3} \left( [\Lambda^s, h^3] \partial_x \mathcal{S}^{-1}(v^x u_x), \Lambda^s u \right)
+ \frac{\varepsilon \beta \mu^2}{2} \left( [\Lambda^s, b_x h^2] \mathcal{S}^{-1}(v^x u_x), \Lambda^s u \right) + \frac{\varepsilon \mu^2}{2} \left( [\Lambda^s, b_x h^2] \partial_x \mathcal{S}^{-1}(v^x u_x), \Lambda^s u \right)
- \varepsilon \mu \left( [\Lambda^s, v^x] u_x, \Lambda^s u \right),
\]

\[
:= B_{61} + ... + B_{65}.
\]

We present here below how to estimate the first two terms of $B_6$ in some details, the other terms being estimated in a very similar way. First, for $B_{61}$, we obtain

\[
|B_{61}| \leq \varepsilon \mu (|h|) x |H| s |\mathcal{S}^{-1}(v^x u_x)| H_{s-1} |u| H,
\]
\[
\leq \varepsilon \mu (|h - 1|) x |H| s |\mathcal{S}^{-1}(v^x u_x)| H_{s-1} |u| H
\]
\[
\leq \varepsilon \mu (|h|) x |H| s |\mathcal{S}^{-1}(v^x u_x)| H_{s-1} |u| H,
\]
\[
\leq \varepsilon \mu (|h - 1|) x |H| s |\mathcal{S}^{-1}(v^x u_x)| H_{s-1} |u| H
\]
\[
\leq \varepsilon \mu (|h|) x |H| s |\mathcal{S}^{-1}(v^x u_x)| H_{s-1} |u| H,
\]
\[
\leq \varepsilon C(E^s(\mathcal{U})) E^s(U)^2.
\]

For $B_{62}$, we write

\[
|B_{62}| \leq \varepsilon \mu^2 \frac{1}{3} |h^3 - 1| x |H| \partial_x \mathcal{S}^{-1}(v^x u_x) | H_{s-1} |u_x| H,
\]
\[
\leq \varepsilon \mu^2 \frac{1}{3} |h^3 - 1| x |H| \partial_x \mathcal{S}^{-1}(v^x u_x) | H_{s-1} |u_x| H,
\]
\[
\leq \varepsilon \mu^2 \frac{1}{3} |h^3 - 1| x |H| \partial_x \mathcal{S}^{-1}(v^x u_x) | H_{s-1} |u_x| H,
\]
\[
\leq \varepsilon \mu^2 \frac{1}{3} |h^3 - 1| x |H| \partial_x \mathcal{S}^{-1}(v^x u_x) | H_{s-1} |u_x| H,
\]
\[
\leq \varepsilon C(E^s(\mathcal{U})) E^s(U)^2.
\]

All these estimates show that $|B_6|$ is also bounded by $\varepsilon C(E^s(\mathcal{U})) E^s(U)^2$.

6. To estimate $B_7$, we first calculate

\[
\mathcal{S} \left[ [\Lambda^s, \mathcal{S}^{-1}(Q_1(\mathcal{U})) u_x, \mathcal{S}^{-1} \right] Q_1(\mathcal{U}) u_x + \left[ [\Lambda^s, Q_1(\mathcal{U}) \right] u_x
\]
\[
= - \left[ [\Lambda^s, \mathcal{S}^{-1}(Q_1(\mathcal{U})) u_x, \mathcal{S}^{-1} \right] u_x + \left[ [\Lambda^s, Q_1(\mathcal{U}) \right] u_x,
\]

which shows that $B_7$ might be decomposed into two terms, as $B_7 := B_{71} + B_{72}$. These terms are calculated here below. First, by using the definition of $\mathcal{S}$ we obtain

\[
B_{71} = - \left( [\Lambda^s, h] \mathcal{S}^{-1}(Q_1(\mathcal{U}) u_x), \Lambda^s u \right) - \frac{\mu}{3} \left( [\Lambda^s, h^3] \partial_x \mathcal{S}^{-1}(Q_1(\mathcal{U}) u_x), \Lambda^s u \right)
+ \frac{\beta \mu}{2} \left( [\Lambda^s, b_x h^2] \mathcal{S}^{-1}(Q_1(\mathcal{U}) u_x), \Lambda^s u \right) + \frac{\beta \mu}{2} \left( [\Lambda^s, b_x h^2] \partial_x \mathcal{S}^{-1}(Q_1(\mathcal{U}) u_x), \Lambda^s u \right)
\]
\[
:= B_{711} + \cdots + B_{714}.
\]

For $B_{711}$, we obtain at once

\[
|B_{711}| \leq |h - 1| x |H| \mathcal{S}^{-1}(Q_1(\mathcal{U}) u_x) | H_{s-1} |u| H,
\]
where we need to replace $Q_1[U]$ by its expression; we obtain

$$| \Sigma^{-1}(Q_1[U]u_x)|_{H^{s-1}}$$

$$\leq \frac{2}{3} \varepsilon \mu |\Sigma^{-1}(h^3 u_x u_x)_{1}|_{H^{s-1}} + \varepsilon \mu |\Sigma^{-1}(b_x h^3 u_x u_x)|_{H^{s-1}}$$

$$+ \varepsilon \mu |\Sigma^{-1}(b_x h^2 u_x)|_{H^{s-1}}$$

$$+ \frac{1}{6} \varepsilon |\Sigma^{1/2}(h^3 u_x u_x)|_{H^{s-1}} + \frac{1}{6} \varepsilon |\Sigma^{1/2}(h^3 u_x^2 u_x)|_{H^{s-1}}$$

$$\leq \frac{2}{3} \varepsilon (h^3 - 1)|_{H^{s-1}} + 1) |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$+ \varepsilon \mu |b_x|_{H^{s-1}}(h^2 - 1)|_{H^{s-1}} + 1) |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$+ \varepsilon \mu |b_x|_{H^{s-1}}(h^2 - 1)|_{H^{s-1}} + 1) |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$+ \frac{1}{6} \varepsilon |\Sigma^{1/2}(h^3 - 1)|_{H^{s-1}} |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$+ \frac{1}{6} \varepsilon |\Sigma^{1/2}(h^3 - 1)|_{H^{s-1}} |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$\leq \frac{2}{3} \varepsilon \mu \left[ (h^3 - 1)|_{H^{s-1}} + 1 \right] |u_x|_{H^{s-1}} |u_x|_{H^{s-1}} + \varepsilon \beta |b|_{H^{s-1}} \left[ (h^2 - 1)|_{H^{s-1}} + 1 \right] |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$+ \frac{1}{6} \varepsilon |\Sigma^{1/2}(h^3 - 1)|_{H^{s-1}} |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$\leq \varepsilon C(E(U))E^s(U),$$

which shows that $|B_{711}|$ is also bounded by $C(E^s(U))E^s(U)^2$. We have also used Lemma 3.2 here. The terms $B_{712}, B_{713}$ and $B_{714}$ are estimated in the same way, so that we obtain

$$|B_{71}| \leq \varepsilon C(E^s(U))E^s(U)^2.$$  

We now turn to estimating $B_{722}$. We calculate

$$B_{72} = (\Lambda^s, Q_1[U]u_x, \Lambda^s u)$$

$$= \frac{2}{3} \varepsilon \mu (\Lambda^s, h^3 u_x u_x, \Lambda^s u) + \varepsilon \beta |b|_{H^{s-1}} \left[ (h^2 - 1)|_{H^{s-1}} + 1 \right] |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$+ \frac{1}{6} \varepsilon |\Sigma^{1/2}(h^3 - 1)|_{H^{s-1}} |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$\leq B_{721} + ... + B_{725}.$$  

Here again, we give some details for estimating two terms, the other ones being estimated in a similar way. We start by the term $B_{721}$,

$$|B_{721}| \leq \frac{2}{3} \varepsilon (h^3 |_{H^{s-1}})|u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$\leq \frac{2}{3} \varepsilon |\Sigma^{1/2}(h^3 - 1)|_{H^{s-1}} |u_x|_{H^{s-1}} |u_x|_{H^{s-1}}$$

$$\leq \varepsilon C(E^s(U))E^s(U)^2.$$
Lemma 3.4. There exists a constant $C$ such that

$$| \langle \Lambda^* \mathbf{B}(\mathbf{U}), S(\mathbf{U}) \Lambda^* \mathbf{U} \rangle | \leq \varepsilon C(E^*(\mathbf{U})) E^*(\mathbf{U}).$$
Proof. We first calculate

\[
\begin{align*}
(\Lambda^* B(U), S(U) \Lambda^* U) &= -\beta \left( \Lambda^*(b_x u), J_{\mu H} \Lambda^* \zeta \right) - \left( [\Lambda^*, \overline{\mathcal{S}}]^{-1} q(U), \Lambda^* u \right) \\
&\quad + \left( \Lambda^* q(U), \Lambda^* u \right) + 3\varepsilon \mu \beta \left( [\Lambda^*, \overline{\mathcal{S}}]^{-1} \left( b_x \frac{E}{h^3} \right), \Lambda^* u \right) \\
&\quad - 3\varepsilon \mu \beta \left( \Lambda^* \left( b_x \frac{E}{h^3} \right), \Lambda^* u \right) - \varepsilon \beta \sqrt{\mu} \left( \left( \Lambda^* \left( b_x \frac{E}{h^3} \right), \Lambda^* \left( E \frac{E}{h^3} \right) \right) \right) \\
&:= \sum_{i=1}^{6} D_i.
\end{align*}
\]

1. The first term \( D_1 \) is calculated by using an integration by parts,

\[
D_1 = -\beta \left( \Lambda^*(b_x u), \Lambda^* \zeta \right) - \frac{\beta \mu}{b_0} \left( \Lambda^*(b_x u)_x, \Lambda^* \zeta_x \right) \\
\quad := D_{11} + D_{12}.
\]

The terms \( D_{11} \) and \( D_{12} \) are estimated as follows,

\[
|D_{11}| \leq \beta |b_x|_{H^1} |u|_{H^2} |\zeta|_{H^2} \\
\quad \leq \varepsilon C(E^*(U)) E^*(U),
\]

and

\[
|D_{12}| \leq \frac{\beta}{\sqrt{b_0}} \sqrt{\mu} |(b_x u)_x|_{H^1} \sqrt{\mu} |\zeta_x|_{H^1}.
\]

We remark that

\[
\sqrt{\mu} |\partial_x (b_x u)|_{H^1} \leq \sqrt{\mu} |b_{xx} u|_{H^1} + \sqrt{\mu} |b_x u_x|_{H^1} \\
\quad \leq |b_{xx}|_{H^1} |u|_{H^1} + \sqrt{\mu} |b_x|_{H^1} |u_x|_{H^1} \\
\quad \leq C(E^*(U)),
\]

which shows that \( D_{12} \) satisfies the estimate

\[
|D_{12}| \leq \varepsilon C(E^*(U), b_0) E^*(U).
\]

Hence, we have obtained

\[
|D_1| \leq \varepsilon C(E^*(U), b_0) E^*(U).
\]

2. Using the definitions of \( \overline{\mathcal{S}} \) and \( q(U) \), and Lemma A.2, we may estimate \( D_2 \) as follows,

\[
D_2 = -\frac{1}{2} \varepsilon \mu \beta \left( [\Lambda^*, h^3]^{-1} \left( b_{xxx} - \beta (b_x^2)_x \right) \frac{E}{h^2} u^2 \right), \Lambda^* u \) \\
\quad + \frac{1}{6} \varepsilon \mu^2 \beta \left( [\Lambda^*, h^4] \partial_x \overline{\mathcal{S}}^{-1} \left( b_{xxx} - \beta (b_x^2)_x \right) \frac{E}{h^2} u^2, \Lambda^* u_x \right) \\
\quad - \frac{1}{4} \varepsilon \beta^2 \mu^2 \left( [\Lambda^*, h^2 b_x] \overline{\mathcal{S}}^{-1} \left( b_{xxx} - \beta (b_x^2)_x \right) \frac{E}{h^2} u^2, \Lambda^* u_x \right) \\
\quad + \frac{1}{4} \varepsilon \beta^2 \mu^2 \left( [\Lambda^*, h^2 b_x] \partial_x \overline{\mathcal{S}}^{-1} \left( b_{xxx} - \beta (b_x^2)_x \right) \frac{E}{h^2} u^2, \Lambda^* u \right) \\
\quad := \sum_{i=1}^{4} D_{2i}.
\]
Lemma 3.5. There exists a constant $C$ such that

$$\|(\Lambda^s u, [\partial_t, \Sigma] \Lambda^s u)\| \leq C(E^*(U, |h|_{\infty})) E^*(U)^2.$$ 

Proof. Since $b$ is independent of $t$, we have

$$[\partial_t, b] \Lambda^s u = b_t \Lambda^s u,$$
$$[\partial_t, \partial_x (b^2 \partial_x^2)] \Lambda^s u = ((b^2)_{xx}) \Lambda^s u,$$
$$[\partial_t, \partial_x (b^2 b_x)] \Lambda^s u = b_x ((b^2)_{xx}) \Lambda^s u,$$
$$[\partial_t, b^2 b_x \partial_x] \Lambda^s u = b_x ((b^2)_t) \Lambda^s u.$$
Using the definition of $\Sigma$, we obtain the following,
\[
(\Lambda^s u, [\partial_t, \Sigma] \Lambda^s u) = (\Lambda^s u, \frac{\mu}{3} (\Lambda^s u_x, (\Lambda^s)^3_{,x}) + \frac{\varepsilon \mu}{2} (\Lambda^s u_x, b_2 (\Lambda^s)^2_{,x}) - \frac{\varepsilon \mu}{2} (\Lambda^s u_x, b_3 (\Lambda^s)^3_{,x})).
\]
Controlling these terms by $\varepsilon C(\varepsilon^{\delta}(U), \|u\|_{L^\infty}) E^\delta(U)^2$ follows directly from the Cauchy-Schwarz inequality and an integration by parts. This shows the result of this lemma.

\[
\square
\]

Appendix A. Some properties of the operator $\Sigma := h(1 + \mu T)$ and its inverse. In the following we give two lemmas concerning the operator $\Sigma := h(1 + \mu T)$, and its inverse.

**Lemma A.2.** Let $b \in C^\infty_0(\mathbb{R})$ and $\zeta \in W^{1,\infty}(\mathbb{R})$ be such that (3) is satisfied, where $h := 1 + \varepsilon \zeta - \beta b$. Then the self-adjoint operator

\[
\Sigma : H^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})
\]

is well defined, one-to-one and onto.

The following lemma gives some properties of the inverse operator $\Sigma^{-1}$.

**Lemma A.1.** Let $b \in C^\infty_0(\mathbb{R})$, $s_0 > 1/2$ and $\zeta \in H^{s_0+1}(\mathbb{R})$ be such that (3) is satisfied, where $h := 1 + \varepsilon \zeta - \beta b$. Then, the following assertions hold.

(i) $\forall 0 \leq s \leq s_0 + 1$, \[\|\Sigma^{-1} f\|_{H^s} + \sqrt{\mu} \|\partial_x \Sigma^{-1} f\|_{H^s} \leq C(\frac{1}{h_0}, |h - 1|_{H^{s_0+1}}) |f|_{H^s}, \text{ for } f \in H^s;\]

(ii) $\forall 0 \leq s \leq s_0 + 1$, \[\sqrt{\mu} \|\Sigma^{-1} \partial_x f\|_{H^s} \leq C(\frac{1}{h_0}, |h - 1|_{H^{s_0+1}}) |f|_{H^s}, \text{ for } f \in H^s;\]

(iii) If $s \geq s_0 + 1$ and $\zeta \in H^s(\mathbb{R})$, then

\[\|\Sigma^{-1} \|_{H^s(\mathbb{R}) \rightarrow H^s(\mathbb{R})} + \sqrt{\mu} \|\Sigma^{-1} \partial_x \|_{H^s(\mathbb{R}) \rightarrow H^s(\mathbb{R})} \leq c_s,\]

where $c_s$ is a constant depending on $\frac{1}{h_0}$, $|h - 1|_{H^{s_0+1}}$, and independent of $(\mu, \varepsilon) \in (0, 1)^2$.

Proofs of both lemmas are given in [17].

Appendix B. Product and commutator estimates in Sobolev spaces. We recall some product and commutator estimates in Sobolev spaces, which are intensively used throughout the present paper.

In the following lemmas, the notation $A_s = B_s + \langle C_s \rangle_{s > 2}$ is used to express that $A_s = B_s$ if $s \leq 2$, and $A_s = B_s + C_s$ if $s > 2$.

**Lemma B.1.** [Product estimates,]

Let $s \geq 0$. For all $f, g \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$,
\[
|f g|_{H^s} \leq |f|_{H^s} |g|_{H^s} + |f|_{H^s} |g|_{H^s}.
\]

Let $s > 1/2$. Then, thanks to the continuous embeddings into Sobolev spaces,
\[
|f g|_{H^s} \leq |f|_{H^s} |g|_{H^s}.
\]

More generally, for $s \geq 0$ and $s_0 > 1/2$, for all $f \in H^s(\mathbb{R}) \cap H^{s_0}(\mathbb{R}), g \in H^s(\mathbb{R})$,
\[
|f g|_{H^s} \leq |f|_{H^{s_0}} |g|_{H^s} + \langle |f|_{H^s} |g|_{H^{s_0}} \rangle_{s > s_0}.
\]
Let \( F \in C^\infty(\mathbb{R}) \) be such that \( F(0) = 0 \). If \( g \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) with \( s \geq 0 \), then \( F(g) \in H^s(\mathbb{R}) \) and
\[
\left| F(g) \right|_{H^s} \leq C \left( |g|_\infty, |F|_{C^\infty} \right) |g|_{H^s}.
\]

All these estimates are classical. (See [1, 20, 22].)

We now recall some commutator estimates, mainly due to Kato and Ponce [20], and improved by Lannes [22] (see Theorems 3 and 6 therein).

**Lemma B.2. [Commutator estimates.]**

Let \( s \geq 0 \). If \( f, g \in L^\infty(\mathbb{R}) \cap H^{s-1}(\mathbb{R}) \), then \( [\Lambda^s, f]g \) is in \( L^2 \) and
\[
\left| [\Lambda^s, f]g \right|_2 \lesssim |f_x|_{H^{s-1}} |g|_\infty + |f_x|_\infty |g|_{H^{s-1}}.
\]

Moreover, as a consequence of the continuous embeddings in Sobolev spaces, there holds, for \( s \geq s_0 + 1 \), \( s_0 > \frac{3}{2} \),
\[
\left| [\Lambda^s, f]g \right|_2 \lesssim |\partial_x f|_{H^{s-1}} |g|_{H^{s-1}}.
\]

More generally, assume \( s \geq 0 \), \( s_0 > 1/2 \), and \( \partial_x f, g \in H^{s_0}(\mathbb{R}) \cap H^{s-1}(\mathbb{R}) \), then
\[
\left| [\Lambda^s, f]g \right|_2 \lesssim |\partial_x f|_{H^{s_0}} |g|_{H^{s-1}} + \left( |\partial_x f|_{H^{s-1}} \right) |g|_{H^{s_0}}_{s > s_0 + 1}.
\]

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**REFERENCES**

[1] S. Alinhac and P. Gérard, *Opérateurs Pseudo-différentiels et Théorème de Nash-Moser*, Savoirs Actuels, InterEditions, Paris; Éditions du Centre national de la recherche scientifique, Meudon, 1991.

[2] B. Alvarez-Samaniego and D. Lannes, Large time existence for 3D water-waves and asymptotics, *Invent. Math.*, **171** (2008), 485–541.

[3] S. V. Basenkova, N. N. Morozov and O. P. Pogutse, Dispersive effects in two-dimensional hydrodynamics, *Dokl. Akad. Nauk*, **293** (1985), 818–822 (transl. *Sov. Phys. Dokl.*, **32** (1987), 262–264).

[4] P. Bonneton, F. Chazel, D. Lannes, F. Marche and M. Tissier, A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model, *J. Comput. Phys.*, **230** (2011), 1479–1498.

[5] A. Castro and D. Lannes, Fully nonlinear long-wave models in the presence of vorticity, *J. Fluid Mech.*, **759** (2014), 642–675.

[6] A. Castro and D. Lannes, Well-posedness and shallow-water stability for a new Hamiltonian formulation of the water waves equations with vorticity, *Indiana Univ. Math. J.*, **64** (2015), 1169–1270.

[7] Q. Chen, J. T. Kirby, R. A. Dalrymple, A. B. Kennedy and A. Chawla, Boussinesq modeling of wave transformation, breaking, and runup, Part II: Two horizontal dimensions, *J. Waterway Port Coastal Ocean Eng.*, **126** (2000), 48–56.

[8] Q. Chen, J. T. Kirby, R. A. Dalrymple, F. Shi and E. B. Thornton, Boussinesq modeling of longshore currents, *J. Geophys. Res.*, **108** (2003), 3362–3379.

[9] R. Cienfuegos, E. Barthélemy and P. Bonneton, A fourth-order compact finite volume scheme for fully nonlinear and weakly dispersive Boussinesq-type equations, Part I: Model development and analysis, *Int. J. Numer. Meth. Fluids*, **51** (2006), 1217–1253.

[10] V. Duchêne and S. Israwi, Well-posedness of the Green-Naghdi and Boussinesq-Peregrine systems, *Ann. Math. Blaise Pascal*, **25** (2018), 21–74.

[11] V. Duchêne, S. Israwi and R. Talhouk, A new fully justified asymptotic model for the propagation of internal waves in the Camassa-Holm regime, *SIAM J. Math. Anal.*, **47** (2015), 240–290.
[12] V. Duchêne, S. Israwi and R. Talhouk, A new class of two-layer Green-Naghdi systems with improved frequency dispersion, *Stud. Appl. Math.*, 137 (2016), 356–415.

[13] D. Dutykh, D. Clamond, P. Milewski and D. Mitsotakis, Finite volume and pseudo-spectral schemes for the fully nonlinear 1D Serre equations, *European J. Appl. Math.*, 24 (2013), 761–787.

[14] A. E. Green, N. Laws and P. M. Naghdi, On the theory of water waves, *Proc. Royal Soc. London Ser. A*, 338 (1974), 43–55.

[15] A. E. Green and P. M. Naghdi, A derivation of equations for wave propagation in water of variable depth, *J. Fluid Mech.*, 78 (1976), 237–246.

[16] T. Iguchi, A shallow water approximation for water waves, *J. Math. Kyoto Univ.*, 49 (2009), 13–55.

[17] S. Israwi, Large time existence for 1D Green-Naghdi equations, *Nonlinear Anal.*, 74 (2011), 81–93.

[18] S. Israwi and H. Kalisch, Approximate conservation laws in the KdV equation, *Phys. Lett. A*, 383 (2019), 854–858.

[19] T. Kano and T. Nishida, Sur les ondes de surface de l’eau avec une justification mathématique des équations des ondes en eau peu profonde, *J. Math. Kyoto Univ.*, 19 (1979), 335–370.

[20] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.*, 41 (1988), 891–907.

[21] M. Kazolea, A. I. Delis, I. K. Nikolos and C. E. Synolakis, An unstructured finite volume numerical scheme for extended 2D Boussinesq-type equations, *Coastal Eng.*, 69 (2012), 42–66.

[22] D. Lannes, Sharp estimates for pseudo-differential operators with symbols of limited smoothness and commutators, *J. Funct. Anal.*, 232 (2006), 495–539.

[23] D. Lannes and P. Bonneton, Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation, *Phys. Fluids*, 21 (2009), 016601.

[24] D. Lannes and F. Marche, A new class of fully nonlinear and weakly dispersive Green-Naghdi models for efficient 2D simulations, *J. Comput. Physics*, 282 (2015), 238–268.

[25] O. Le Métayer, S. Gavrilyuk and S. Hank, A numerical scheme for the Green-Naghdi model, *J. Comp. Phys.*, 229 (2010), 2034–2045.

[26] Y. A. Li, A shallow-water approximation to the full wave water problem, *Comm. Pure Appl. Math.*, 59 (2006), 1225–1285.

[27] N. Makarenko, The second long-wave approximation in the Cauchy-Poisson problem, *Dyn. Contin. Media*, 77 (1986), 56–72.

[28] G. Métivier, Para-differential Calculus and Applications to the Cauchy Problem for Nonlinear Systems, Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, Vol. 5, Scuola Norm. Sup. Pisa, 2008.

[29] L. V. Ovsjannikov, Cauchy problem in a scale of Banach spaces and its application to the shallow water theory justification, *In: Appl. Meth. Funct. Anal. Probl. Mech. (IUTAM/IMU-Symp., Marseille, 1975)*, Lect. Notes Math. 503, Springer, 1976, 426–437.

[30] M. Ricchiuto and A. G. Filippini, Upwind residual discretization of enhanced Boussinesq equations for wave propagation over complex bathymetries, *J. Comput. Physics*, 271 (2014), 306–341.

[31] M. E. Taylor, *Partial Differential Equations III*, Applied Mathematical Sciences, 117, Springer, 2011.

[32] G. Wei, J. T. Kirby, S. T. Grilli and R. Subramanya, A fully nonlinear Boussinesq model for surface waves, Part I. Highly nonlinear unsteady waves, *J. Fluid Mech.*, 294 (1995), 71–92.

[33] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Applied Mech. and Techn. Phys.*, 9 (1968), 190–194.

[34] Y. Zhang, A. B. Kennedy, N. Panda, C. Dawson and J. J. Westerink, Boussinesq-Green-Naghdi rotational water wave theory, *Coastal Eng.*, 73 (2013), 13–27.

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