Cosmic strings in hybrid metric-Palatini gravity

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CONTENTS

I. Introduction
II. Cosmic strings in hybrid gravity
A. Action and field equations
B. Metric of a cosmic string
C. Full field equations
D. Field equations with boost invariance
III. General solution of the field equations
IV. Specific cosmic string solutions
A. Constant scalar field potential: \( V = V_0 \)
   1. The particular case \( V = 0 \)
   2. The case \( V = V_0 \neq 0 \)
B. Power law potential: \( V(\phi) = V_0 \phi^{3/4} \)
C. Exponential potential: \( V(\phi) = V_0 e^{-\lambda \phi} \)
D. Higgs-type potential
V. Discussions and final remarks
Acknowledgments
References

We consider static and cylindrically symmetric interior string solutions in the scalar-tensor representation of the hybrid metric-Palatini modified theory of gravity. As a first step in our study, we obtain the gravitational field equations and further simplify the analysis by imposing Lorentz invariance along the \( t \) and \( z \) axes, which reduces the number of unknown metric tensor components to a single function. In this case, the general solution of the field equations can be obtained, for an arbitrary form of the scalar field potential, in an exact closed parametric form, with the scalar field \( \phi \) taken as a parameter. We consider in detail several exact solutions of the field equations, corresponding to a null and constant potential, and to a power-law potential of the form \( V(\phi) = V_0 \phi^{3/4} \), in which the behavior of the scalar field, metric tensor component and string tension can be described in a simple mathematical form. We also investigate the string models with exponential and Higgs type scalar field potentials by using numerical methods. In this way we obtain a large class of novel stable stringlike solutions in the context of hybrid metric-Palatini gravity, in which the basic parameters, such as the scalar field, metric tensor and string tension, depend essentially on the initial values of the scalar field, and of its derivative, along the string axis.

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I. INTRODUCTION

The formation of topological defects is a well-studied physical process in the context of condensed matter, namely, metal crystallization \([1]\), liquid crystals \([2, 3]\), superfluid helium-3 \([4]\) and helium-4 \([5]\), and superconductivity \([6]\). The formation of topological defects is a by-product of phase transitions and behind their formation lies a fundamental concept in physics, namely, spontaneous symmetry breaking (SSB). Although we can distinguish first and second order phase transitions, the essential features of such a concept can be illustrated by a simple Goldstone model \([1, 2]\). Here, the physical system of a Higgs field exhibits a non-degenerate vacuum expectation value at \( T > T_c \), but as the system is cooled, the minima of the potential becomes degenerate for \( T < T_c \) and the field will “roll” to the new minima, where \( T_c = \sqrt{6} \eta \) is the critical temperature in second order phase transitions, related to the energy breaking scale, \( \eta \). The new vacuum state now does not exhibit the same invariance as the previous minimum and, hence, the symmetry is spontaneously broken.

From the standpoint of cosmology, the formation of topological defects is related with the symmetries shown by the Standard Model of Particle Physics. In fact, many Grand Unification Theories (GUTs) postulate that the universe, as it cooled, underwent a series of phase transitions associated with SSB, meaning that at sufficiently high temperatures there was invariance under a more general group of symmetries. Each of these phase tran-
sitions may have left behind a network of topological defects \cite{10}. In fact, the Kibble and Zurek mechanism \cite{10,11} describes the non-equilibrium dynamics and the formation of topological defects in a system which is driven through a continuous phase transition at finite rate. When $T_c$ is reached, random fluctuations will dictate which of the minima state will be “chosen” by the field; regions of spacetime separated by a distance larger than the size of the particle horizon, will “choose” independent, but equivalent, states on the minima manifold. Indeed, the kind of defects we expect to be formed depend on the (non-trivial) topology of the minima manifold \cite{12}.

In the case of a discrete symmetry breaking, whenever the vacuum manifold is disconnected, a domain wall is formed, which is a surface that separates two patches with different vacuum expectation values (VEV). If the vacuum manifold contains unshrinkable surfaces, the field might develop non-trivial configurations corresponding to point-like defects, known as monopoles. In this work we limit ourselves to the investigation of what are considered to be the most viable types of topological defects, which may have already formed in the early Universe, namely, cosmic strings \cite{8,9}. These are line-like defects formed when the topology of the minima manifold is not simply-connected.

In field theoretical models strings can form once an axial symmetry is broken spontaneously. Strings can exist in the form of loops, or they can be infinitely long, spanning to the horizon. The equations of motion for two models of circular cosmic string loops with windings in a simply connected internal space were investigated numerically in \cite{13}. The Kosambi-Cartan-Chern theory was used to analyze the Jacobi stability of the string equations and determine bounds on the physical parameters that ensure dynamical stability of the windings. One may also consider more exotic defects, composed of combinations of strings. For a more comprehensive discussion of the set of possible topological defects we refer the reader to \cite{3,4}. The formation of a network of cosmic defects, and their symmetry breaking scale, is a key feature in many Grand Unified scenarios \cite{14}, and hence the search for the cosmological consequences of such defects is a key aspect for constraining different models.

Some defects tend to be inherently unstable \cite{8}, while domain walls and monopoles are either cosmologically catastrophic or severely constrained by current observations \cite{15}. On the other hand, the presence of cosmic strings can have important cosmological consequences, such as, for example, in the case of the Cosmic Microwave Background (CMB) anisotropies \cite{16}, for small scale structure formation \cite{17}, for the reionization history of the Universe \cite{18}, for Gamma Ray Bursts \cite{19,20}, for the gravitational lensing observations \cite{21}, and for the understanding of the formation of the super-massive black holes in the early universe \cite{22}, respectively.

Furthermore, the role of inflation on the survival of topological defects cannot be overstated, as defects formed too early would become diluted in the universe, which is useful in the case of domain walls or monopoles, but defects formed too late would become energetically dominant, changing drastically the standard cosmological model. However, it has been shown that strings are a by-product of several GUTs at the end of inflation \cite{14} and are stable topological defects, which make them good candidates for further analysis. Due to the existence of a magnetic flux inside the string \cite{22}, cosmic strings can either be infinite or form closed loops, which will oscillate and radiate energy via gravitational waves (GW), and thus decay. This radiation will cause a stochastic background in the GW spectra \cite{24}.

Even though most of the research on cosmic strings has been done in the framework of standard general relativity, the properties of cosmic strings have also been investigated in modified theories of gravity. String-like solutions have been found in $f(R)$ gravity \cite{25,27}. Spherically symmetric string solutions with constant Ricci curvature have been derived in \cite{23}, and it was shown that there is only one solution for $R = 0$. Families of vacuum solutions for which $R = const \neq 0$ were also found, representing $f(R)$ analogues of the Linet-Tian solution \cite{25,28}. In fact, the solution obtained in \cite{25} is a member of the general Tian family of solutions in general relativity, and therefore it can describe the exterior of a cosmic string. Kasner-type static, spherically symmetric interior string solutions in the framework of $f(R, L_m)$ gravity \cite{30,31} were considered in \cite{26}. Gravitational binding of general relativistic strings consisting of a Bose-Einstein condensate matter that is described, in the Newtonian limit, by the zero temperature time-dependent nonlinear Schrödinger equation (the Gross-Pitaevskii equation), with repulsive interparticle interactions were investigated in \cite{32}.

Cosmic strings have also been extensively explored in other extensions of general relativity, such as in scalar-tensor theories \cite{33–44}. An interesting aspect in these theories is the proof that the Vilenkin prescription in which an infinitely long straight static local gauge string satisfies the condition of the energy-momentum tensor $T^i_j = T^2_j \neq 0$ and all the other components $T^\mu_\nu = 0$ \cite{45} is inconsistent in Brans-Dicke theory of gravity \cite{33}. However, this inconsistency can be removed by including a cosmological constant \cite{14}, or by considering a more general scalar-tensor theory \cite{42}. This fact motivates investigating string type solutions in the scalar-tensor representation of the recently proposed hybrid metric-Palatini gravity \cite{46,47}, which is a modified theory of gravity that combines the metric and Palatini formalisms, already introduced in the study of standard general relativity, to construct a new gravitational Lagrangian.

In fact, theories of gravity with a gravitational action consisting of more general combinations of curvature invariants than the traditional Einstein-Hilbert term have been in recent years a source of intense scrutiny \cite{46,49–54}. Either by their ability to account for the late-time cosmic acceleration without dark energy \cite{55}, or by the
possibility of explaining the large scale dynamics of self-gravitating systems without the need for dark matter \cite{56, 60}. The hybrid metric-Palatini theory is one of these cases \cite{44, 48, 54, 61, 65}. From a theoretical point of view the main advantage of hybrid metric-Palatini gravity is that it is a viable gravity theory that includes elements of both Palatini and metric formalisms. A main success of the theory is the possibility of generating long-range forces that pass the classical local tests at the Solar System level of gravity. Another important advantage of the theory is that it admits an equivalent scalar-tensor representation, which greatly simplifies the analysis of the field equations, and the construction of their solutions.

In this work, we will analyze local gauge string solutions with a phenomenological energy momentum tensor, as prescribed by Vilenkin \cite{43}, in the context of the hybrid gravitational theory. The general solution of the field equations can be obtained in an exact parametric form for arbitrary scalar field potentials. Several solutions of the field equations, obtained for different functional forms of the scalar field potential are considered in detail. In particular we consider the cases of the null and constant potentials, as well as the power-law potential of the form \( V(\phi) = V_0 \phi^{1/4} \). For all these cases the solutions of the gravitational field equations can be represented in a simple mathematical form. Two important scalar field potentials are the exponential and the Higgs type scalar field potentials, which can be investigated only with the extensive use of numerical methods. As a result of our investigations we obtain several classes of novel stable stringlike solutions in hybrid metric-Palatini gravity. An interesting property of these solutions is that the behavior of all physical and geometrical quantities describing the strings (scalar field, metric tensor, and string tension), essentially depend on the initial values along the string axis of the scalar field, and of its derivative.

This work is organized in the following manner. In Sec. \ref{sec:II} we present the scalar-tensor representation of the hybrid metric-Palatini theory, by writing out the action and field equations for a general static, cylindrically symmetric metric. This is followed by Sec. \ref{sec:III} where we present the general solution of the field equations, for an arbitrary form of the scalar field potential, in an exact closed parametric form, with the scalar field \( \phi \) taken as a parameter. In Sec. \ref{sec:IV} we consider in detail several exact and numerical solutions of the field equations, by choosing several interesting choices for the potential. Finally, we summarize and discuss our results in Sec. \ref{sec:V}.

\section{Cosmic Strings in Hybrid Gravity}

\subsection{Action and field equations}

The action of hybrid metric-Palatini gravity is specified as \cite{47, 48}:

\[
S = \frac{1}{2 \kappa^2} \int d^4 x \sqrt{-g} \left[ R + f(R) \right] + S_m , \tag{1}
\]

where \( S_m \) is the matter action, \( \kappa^2 \equiv 8\pi G \), \( R \) is the Einstein-Hilbert term, \( R \equiv g^{\mu\nu} R_{\mu\nu} \) is the Palatini curvature, and \( R_{\mu\nu} \) is defined in terms of an independent connection \( \Gamma^\alpha_{\mu\nu} \) as

\[
R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\alpha\lambda} \Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\alpha\nu} . \tag{2}
\]

The action \((1)\) can be written in the scalar-tensor representation \cite{47}, by the following action

\[
S = \frac{1}{2 \kappa^2} \int d^4 x \sqrt{-g} \left[ (1 + \phi) R + \frac{3}{2 \phi} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + S_m . \tag{3}
\]

Performing the variation of the action with respect to the metric and the scalar field \( \phi \) yields the field equations

\[
(1 + \phi) G_{\mu\nu} = \kappa^2 T_{\mu\nu} + \nabla_\mu \nabla_\nu \phi - \nabla_\alpha \nabla^\alpha \phi g_{\mu\nu} - \frac{3}{2 \phi} \nabla_\mu \phi \nabla_\nu \phi + \frac{3}{4 \phi} \nabla_\lambda \phi \nabla^\lambda \phi g_{\mu\nu} - \frac{1}{2} V(\phi) g_{\mu\nu} , \tag{4}
\]

and

\[
- \nabla_\mu \nabla^\mu \phi + \frac{1}{2 \phi} \partial_\mu \phi \partial^\mu \phi + \frac{\phi}{3} [2 V - (1 + \phi) V_\phi] = \frac{\phi \eta^2}{3} T , \tag{5}
\]

where \( V_\phi \) denotes the derivative of \( V \) with respect to the scalar field. This equation of motion shows that, unlike in the Palatini case, the scalar field is dynamical and not affected by the microscopic instabilities found in Palatini models with infrared corrections \cite{47}.

\subsection{Metric of a cosmic string}

We now consider the specific case of a straight infinite cosmic string. In fact, a crucial parameter in cosmic string theory is the energy density, \( \mu \) (which is usually represented as a dimensionless quantity \( G \mu \)) closely related to the energy scale of the symmetry breaking, \( \eta \sim \sqrt{T} \): the tension of the string network, \( G \mu \) is significantly constrained (for a detailed discussion see \cite{14}), either by the CMB spectra (\( G \mu < 2.6 \times 10^{-7} \)) \cite{68}, gravitational lensing (\( G \mu < 10^{-9} \)) \cite{68}, 21-cm observations (\( G \mu < 10^{-10} \)) \cite{68} and, with the advent of LISA, it can be even more tightly constrained by the stochastic background GW spectra \cite{69}.

Throughout this paper, we will use Vilenkin’s prescription \cite{43}, given by

\[
T^t_t = T^z_z = -\sigma(r) , \tag{6}
\]

where \( \sigma \) is the string tension. We consider a general cylindrically symmetric static metric

\[
ds^2 = -e^{2(K-U)} dt^2 + e^{2(K-U)} dr^2 + e^{-2U} W^2 d\theta^2 + e^{2U} dz^2 , \tag{7}
\]

where \( t, r, \theta \) and \( z \) denote the time, radial, angular and axial cylindrical coordinates, respectively, and \( K, U \) and \( W \) are functions of \( r \) alone.
C. Full field equations

Taking into account the metric (7), then the field equation (1) provides the following non-zero components

\[(1 + \phi) \left( -U'^2 + K' \frac{W'}{W} - \frac{W''}{W} \right) = \phi'' - \frac{3}{4\phi} \phi'^2 - \left( K' - U' - \frac{W'}{W} \right) \phi' + \left( \kappa^2 \sigma + \frac{1}{2} V \right) e^{2(K-U)} ,\]  

(8)

\[(1 + \phi) \left( -U'^2 + K' \frac{W'}{W} \right) = -\frac{3}{4\phi} \phi'^2 - \left( K' - U' + \frac{W'}{W} \right) \phi' - \frac{1}{2} V e^{2(K-U)} ,\]  

(9)

\[(1 + \phi) \left( U'^2 + K'' - 2U' \frac{W'}{W} - 2U' \frac{W'}{W} + \frac{W''}{W} \right) = -\phi'' + \frac{3}{4\phi} \phi'^2 + \left( U' - \frac{W'}{W} \right) \phi' - \left( \kappa^2 \sigma + \frac{1}{2} V \right) e^{2(K-U)} \]  

(11)

Additionally, we can use Eq. (5) to determine the effective Klein-Gordon equation for the scalar field \( \phi \):

\[e^{-2(K-U)} \left( -\phi'' - \frac{W'}{W} \phi' + \frac{\phi'^2}{2\phi} \right) + \frac{\phi}{3} \left[ 2V - (\phi + 1)V_{\phi} \right] + \frac{2\phi\kappa^2\sigma}{3} = 0 .\]  

(12)

Since in this model the matter field couples minimally with curvature, it is possible to show that the energy conservation equation still holds, i.e.,

\[\nabla_\mu T^\mu = 0 \]  

(13)

which proves \( K' \sigma = 0 \), and apart from the trivial vacuum solution, \( \sigma = 0 \), this implies that \( K' = 0 \). Thus, we consider from now on that \( e^K = 1 \), so that Eqs. (8)-(11) simplify to the following relations

\[(1 + \phi) \left( -U'^2 - \frac{W''}{W} \right) = \phi'' - \frac{3}{4\phi} \phi'^2 + \left( U' + \frac{W'}{W} \right) \phi' + \left( \kappa^2 \sigma + \frac{1}{2} V \right) e^{-2U} ,\]  

(14)

\[(1 + \phi) U'^2 = \frac{3}{4\phi} \phi'^2 + \left( U' + \frac{W'}{W} \right) \phi' + \frac{1}{2} V e^{-2U} ,\]  

(15)

\[(1 + \phi) U'^2 = -\phi'' + \frac{3}{4\phi} \phi'^2 - U' \phi' - \frac{1}{2} V e^{-2U} ,\]  

(16)

\[(1 + \phi) \left( U'^2 - 2U'' - 2U' \frac{W'}{W} + \frac{W''}{W} \right) = -\phi'' + \frac{3}{4\phi} \phi'^2 + \left( U' - \frac{W'}{W} \right) \phi' - \left( \kappa^2 \sigma + \frac{1}{2} V \right) e^{-2U} ,\]  

(17)

respectively. Additionally, the effective Klein-Gordon equation for the scalar field \( \phi \) reduces to

\[e^{2U} \left( -\phi'' - \frac{W'}{W} \phi' + \frac{\phi'^2}{2\phi} \right) + \frac{\phi}{3} \left[ 2V - (\phi + 1)V_{\phi} \right] + \frac{2\phi\kappa^2\sigma}{3} = 0 .\]  

(18)

D. Field equations with boost invariance

Note that local gauge strings preserve boost invariance along the \( t \) and \( z \), so that this requires \( U = 0 \). Hence the metric of the cosmic string reduces to the form

\[ds^2 = -dt^2 + dr^2 + W^2 d\theta^2 + dz^2 .\]  

(19)

Applying this symmetry, the gravitational field equations simplify considerably,

\[(1 + \phi) \left( -\frac{W''}{W} \right) = \phi'' - \frac{3}{4\phi} \phi'^2 + \frac{W'}{W} \phi' + \kappa^2 \sigma + \frac{1}{2} V ,\]  

(20)

\[0 = \frac{3}{4\phi} \phi'^2 + \frac{W'}{W} \phi' + \frac{1}{2} V ,\]  

(21)

\[0 = -\phi'' + \frac{3}{4\phi} \phi'^2 - \frac{1}{2} V ,\]  

(22)

where Eqs. (20) and (23) become redundant. Combining Eqs. (21) and (22) yields the following relation for the potential \( V \):

\[V = -\phi'' - \frac{W'}{W} \phi' ,\]  

(24)

which substituting into the Klein-Gordon equation (18), the latter reduces to:

\[V (3 + 2\phi) - V_{\phi} \phi (\phi + 1) + 2\kappa^2 \sigma \phi + \frac{3\phi'^2}{2\phi} = 0 .\]  

(25)

Additionally, we can further deduce:

\[\kappa^2 \sigma = \frac{1}{W} \left( \frac{W}{W} \right) ',\]  

(26)

and

\[\frac{(1 + \phi)W''}{W} = -(V + \kappa^2 \sigma) .\]  

(27)

An important physical parameter characterizing the cosmic string properties is the mass per unit length of the string, which is defined as

\[m(r) = \int_0^{2\pi} d\theta \int_0^{R_s} \sigma(r)W(r)dr\]  

(28)

where \( R_s \) is the string radius.
III. GENERAL SOLUTION OF THE FIELD EQUATIONS

In the present section, we will consider the general solution of the field equations for a cosmic string in hybrid metric-Palatini gravity. It turns out that the system of gravitational equations describing a cosmic string can be solved analytically, with the solution represented in an exact (closed) form, with all the geometric and physical quantities expressed in a parametric form, with the scalar field \( \phi \) taken as a parameter. As an application of the obtained solution, in the next section, we will investigate the behavior of cosmic strings for several choices of the constant potential, including the cases of the exponential potential, and of the Higgs-type potential, respectively.

By taking into account Eq. (22), the field equations (20) and (23) reduce to the form

\[
(1 + \phi) \frac{W''}{W} = -\frac{W'}{W} \phi' - \kappa^2 \sigma. \tag{29}
\]

Equation (22) is independent of the metric tensor coefficient \( W \) and, from a mathematical point of view, it represents a second order nonlinear differential equation. In order to solve it we first rescale the radial coordinate \( r \) according to the transformation \( r = \beta \xi \). Hence Eq. (22) takes the form

\[
\frac{d^2 \phi}{d\xi^2} - \frac{3}{4 \phi} \left( \frac{d \phi}{d\xi} \right)^2 + \frac{1}{2} \beta^2 V(\phi) = 0. \tag{30}
\]

In order to solve Eq. (30) we introduce the transformations

\[
\frac{d\phi}{d\xi} = u, \quad \frac{d^2 \phi}{d\xi^2} = \frac{du}{d\xi} \frac{d\phi}{d\xi} = \frac{du}{d\phi} \frac{d\phi}{d\xi} = \frac{1}{2} \frac{d}{d\phi} u^2, \tag{31}
\]

and

\[
u^2 = v, \tag{32}
\]

respectively. Then Eq. (30) becomes a first order linear differential equation of the form

\[
\frac{dv}{d\phi} - \frac{3}{2 \phi} v + \beta^2 V(\phi) = 0, \tag{33}
\]

with the general solution given by

\[
v(\phi) = \phi^{3/2} \left[ C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi \right], \tag{34}
\]

where \( C \) is an arbitrary constant of integration. We immediately obtain

\[
u(\phi) = \phi^{3/4} \sqrt{\left[ C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi \right]}, \tag{35}
\]

and

\[
\xi + C_0 = \int \frac{\phi^{-3/4} d\phi}{\sqrt{\left[ C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi \right]}}, \tag{36}
\]

respectively, where \( C_0 \) is an arbitrary constant of integration.

Equation (21) can be successively transformed as

\[
\frac{1}{W} \frac{dW}{d\phi} \frac{d\phi}{d\xi} = -\frac{3}{4 \phi} \left( \frac{d \phi}{d\xi} \right)^2 - \frac{\beta^2}{2} V(\phi), \tag{37}
\]

and

\[
\frac{1}{W} \frac{dW}{d\phi} = -\frac{3}{4 \phi} - \frac{\beta^2}{2} \frac{\phi^{-3/2} V(\phi)}{\int \phi^{-3/2} V(\phi) d\phi} = -\frac{3}{4 \phi} + \frac{1}{2} \frac{d}{d\phi} \int \left[ C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi \right], \tag{38}
\]

yielding

\[
W(\phi) = W_0 \phi^{-3/4} \sqrt{C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi}, \tag{39}
\]

where \( W_0 \) is an arbitrary constant of integration.

As a last step we need to obtain the expression of \( \sigma \). Using Eq. (21), then Eq. (29) can be rewritten as

\[
(1 + \phi) \frac{1}{W} \frac{d^2 W}{d\xi^2} = \frac{3}{4 \phi} \left( \frac{d \phi}{d\xi} \right)^2 + \frac{1}{2} \beta^2 V(\phi) - \beta^2 k^2 \sigma. \tag{40}
\]

Taking into account the mathematical identities

\[
\frac{dW}{d\xi} = \frac{dW}{d\phi} \frac{d\phi}{d\xi} = \frac{dW}{d\phi} u, \tag{41}
\]

\[
\frac{d^2 W}{d\xi^2} = \frac{d^2 W}{d\phi^2} v + \frac{dW}{d\phi} \frac{dv}{d\phi}, \tag{42}
\]

respectively, Eq. (40) takes the form

\[
(1 + \phi) \left( \frac{1}{W} \frac{d^2 W}{d\phi^2} v + \frac{1}{2} \frac{dW}{d\phi} \frac{dv}{d\phi} \right) = \frac{3}{4 \phi} v + \frac{1}{2} \beta^2 V(\phi) - \beta^2 k^2 \sigma. \tag{43}
\]

Finally, after some simple calculations we obtain

\[
\kappa^2 \sigma(\phi) = \frac{1}{4 \phi} \left\{ 2(\phi + 1) V'(\phi) + 3 \sqrt{\phi} \int \frac{V'(\phi)}{\sqrt{\phi}} d\phi \right\} - 2(2 \phi + 3) V(\phi) - 3 \left( \frac{C}{\beta^2} \right) \sqrt{\phi}. \tag{44}
\]

Equation (36), (29) and (44) give the complete solution of the field equations describing the geometry of a cosmic string in hybrid metric-Palatini gravity. The solution is obtained in a parametric form, with \( \phi \) taken as a parameter. It also contains three arbitrary integration constants \( C_0, C, \) and \( W_0 \), respectively, which must be obtained from the initial or boundary conditions imposed on the cosmic string configuration.

As for the mass of the string, in the dimensionless variable \( \xi \) it can be obtained as

\[
m(\xi) = 2 \pi \beta \int_0^{\xi_s} \sigma(\xi) W(\xi) d\xi, \tag{45}
\]

where \( \xi_s = R_s/\beta \).
IV. SPECIFIC COSMIC STRING SOLUTIONS

In the present section, we consider specific applications of the general solution of the field equations for a cosmic string in hybrid metric-Palatini gravity, outlined in the previous section. We will investigate the behavior of cosmic strings for several choices of the scalar field potential, including the cases of the constant potential, of the exponential potential, and of the Higgs type potential, respectively.

A. Constant scalar field potential: $V = V_0$

As a first example of a cosmic string model in the hybrid metric-Palatini modified theory of gravity, we will assume that the potential of the scalar field is a constant, $V(\phi) = V_0 = \text{constant}$.

1. The particular case $V = 0$

In the particular case $V = 0$ the field equations describing the cosmic string configuration can be solved exactly. From Eq. (35), we immediately obtain

$$\phi(\xi) = C \left(\xi - 4C_0\right)^4. \tag{46}$$

The initial conditions $\phi(0) = \phi_0$ and $\phi'(0) = \phi'_0$ fix the constants $C_0$ and $C$ as

$$C_0 = -\frac{\phi_0}{\phi'_0}, \quad C = \frac{\phi'^4_0}{256\phi_0^4}. \tag{47}$$

For the metric tensor component $W$ we find

$$W^2(\xi) = \frac{w_0^2}{(\xi - 4C_0)^6}. \tag{48}$$

This metric tensor component does not satisfy the condition $W(0) = 0$. On the string axis the metric takes the finite value $W^2(0) = W_0^2 = w_0^2/4096C_0^4 = w_0^2\phi_0^6/4096\phi_0^6$, a condition that fixes the integration constant $w_0^2$ as $w_0^2 = 4096W_0\phi_0^6/\phi_0^6$. As for the energy density $\sigma$ of the string, it is given by

$$\kappa^2\sigma(\xi) = -\frac{12}{(\xi - 4C_0)^7}. \tag{49}$$

Both the metric and the energy density are singular at $\xi = 4C_0$. However, if $C_0 = -\phi_0/\phi'_0 < 0$, implying that both $\phi_0$ and $\phi'_0$ are positive, there is no infinite type singularity in the metric or energy density. The metric tensor and the energy density are monotonically increasing functions of the distance, while the scalar field is an increasing function of the radial coordinate, becoming infinite for $\xi \rightarrow \infty$.

As for the mass of the string, it is obtained as

$$m = 24\pi\beta W_0 \left[\frac{1}{1024C_0^4} - \frac{1}{4(\xi_s - 4C_0)^4}\right], \tag{50}$$

where $\xi_s$ is the string radius. If $\xi_s = 4C_0$, the total mass of the string is (negative) infinite. On the other hand, for $C_0 < 0$, the string extends to infinity, but its mass is finite, taking the value

$$m = \frac{3\pi\beta W_0}{128C_0^4} = \frac{3\pi\beta W_0\phi'_0^4}{128\phi_0^4}. \tag{51}$$

2. The case $V = V_0 \neq 0$

We will proceed now to the general case of a constant potential, $V = V_0 \neq 0$. Moreover, we will choose the scaling parameter of the radial coordinate $r$ so that $\beta^2V_0 = 1$, giving $\beta = 1/\sqrt{V_0}$, and $\xi = \sqrt{V_0}r$. Then the variation of the scalar field as a function of the radial coordinate is obtained from Eq. (35) as

$$\xi + C_0 = \int -\frac{\phi^{-3/4}d\phi}{\sqrt{C + 2\phi^{-1/2}}}. \tag{52}$$

giving

$$C (\xi + C_0) = 4\sqrt{\phi} \sqrt{C + \frac{2}{\sqrt{\phi}}}, \tag{53}$$

and

$$\phi(\xi) = \left[\frac{C^2(\xi + C_0)^2 - 32}{256C^2}\right]. \tag{54}$$

respectively. The integration constants $C_0$ and $C$ must be determined from the initial conditions $\phi(\xi_0) = \phi_0$ and $\phi'(\xi_0) = \phi'_0$, respectively, and they are given by

$$C_0 = \pm \frac{2\phi_0 - \phi'^2_0}{\phi'^3_0}, \quad C = \frac{4\phi_0\phi'_0}{\phi'^2_0 - 2\phi_0} - \xi_0. \tag{55}$$

The variation of the scalar field is represented in Fig. I. For large values of the radial coordinate $r = \xi/\sqrt{V_0}$ the scalar field is a monotonically decreasing function and, at large distances from the string, it reaches the value zero. The variation of $\phi$ is strongly dependent, from a quantitative point of view, on the initial conditions for the field on the string axis. For large values of $\phi'_0$ and near the axis of the string, the scalar field is an increasing function and, after reaching a maximal value at a finite $r$, $\phi$ begins to decrease tending towards zero for very large values of $r$.

For the metric tensor coefficient $W$ we obtain

$$W(\phi) = W_0\phi^{-3/4}\sqrt{C + 2\phi^{-1/2}}, \tag{56}$$

or

$$W(\xi) = \frac{64C^3W_0(\xi + C_0)}{\left[C^2(\xi + C_0)^2 - 32\right]^2}. \tag{57}$$
the Eqs. (64) of its derivative on the string axis. But if this relation is satisfied, as can be seen immediately from the second of the Eqs. (55), the constant $C$ is undefined, and diverges for $\xi = 0$. Therefore, the metric tensor is not defined on the string axis $r = 0$. The variation of the metric tensor coefficient $W^2(\xi)$ is represented, for $\phi(10^{-7}) = 1$, $W(10^{-7}) = 10^{-3}$, and for different values of $\phi'_0$, in Fig. 2.

![Graph](image)

**FIG. 1.** Variation of the scalar field of the cosmic string configuration in the presence of a constant potential for $\phi(0) = \phi_0 = 1$, and for different values of $\phi'_0$: $\phi'_0 = 0.012$ (solid curve), $\phi'_0 = 0.056$ (dotted curve), $\phi'_0 = 0.084$ (short dashed curve), $\phi'_0 = 0.126$ (dashed curve), and $\phi'_0 = 0.148$ (long dashed curve), respectively.

![Graph](image)

**FIG. 2.** Variation of the metric tensor component $W^2(\xi)$ of the cosmic string configuration in the presence of a constant potential for $\phi(10^{-7}) = 1$, $W(10^{-7}) = 10^{-3}$, and for different values of $\phi'_0$: $\phi'_0 = 0.010$ (solid curve), $\phi'_0 = 0.012$ (dotted curve), $\phi'_0 = 0.014$ (short dashed curve), $\phi'_0 = 0.016$ (dashed curve), and $\phi'_0 = 0.018$ (long dashed curve), respectively.

For $\xi = 0$ (on the string axis) we have $W(0) = 64 C^3 W_0 C_0 / (C^2 C_0^2 - 32)^2$. The condition $W(0) = 0$ would require to take $C_0 = 0$, which imposes the relation $2\phi_0 = \phi'_0$ between the initial values of the field and of its derivative on the string axis. But if this relation is satisfied, the energy density of the string is a monotonically decreasing function of the distance and, at least for the present choice of the initial conditions, it does not have any singularities.

For the total mass of the string we obtain the expression

$$m = \frac{64 \pi \beta \rho W_0 C W_0}{C^2 (C_0 + \xi)^2} \left\{ \frac{C^4 (C_0 + \xi)^2}{(C_0 + \xi)^2 - 32} \right\} \left\{ \frac{C^4 (C_0 + \xi)^2 + 64 (C_0 + 8) C^2 + 1024}{(C_0 + \xi)^2 - 32} \right\} + \frac{C^4 (C_0 + \xi)^2}{(C_0 + \xi)^2 - 32} \right\} (62)

The mass is divergent for $C_0^2 C^2 = 32$, and for $C^2 (C_0 + \xi)^2 \rightarrow 32$. If the string radius tends to infinity, the mass of the string is finite, and it is given by

$$m = \frac{64 \pi \beta \rho C W_0}{(C_0 + \xi)^2} \times \frac{(C_0 + \xi)^2 - 32}{C^2 (C_0 + \xi)^2} \times \left\{ \frac{C^4 (C_0 + \xi)^2 + 64 (C_0 + 8) C^2 + 1024}{(C_0 + \xi)^2 - 32} \right\} (63)

where we have assumed that $C_0^2 C^2 < 32$. The metric tensor component is divergent for $C^2 (C_0 + \xi)^2 - 32 = 0$, which gives for the value of the singular point $\xi_\infty$ the expression

$$\xi_\infty = (\phi'^2_0 - 2 \phi_0) \frac{4 \sqrt{2}}{4 \phi_0 \phi'_0 - \xi_0 (\phi'^2_0 - 2 \phi_0)} + \frac{1}{\phi'^{3/2}_0} . (58)$$

The position of the singular point is essentially determined by the initial values of the scalar field and of its derivative near the string axis. At the metric singularity the scalar field vanishes, as one can see immediately from Eq. (64). However, a different physical behavior is also possible, if near the origin the integration constants $C_0$ and $C$ satisfy the condition $C^2 C_0^2 \gg 32$, or, equivalently, $\phi'_0 \gg 2 \phi_0$. In this case the metric tensor can be approximated as

$$W^2(\xi) \approx \frac{4096 \rho W_0^2}{C^2 (C_0 + \xi)^2} . (59)$$

For $\xi \rightarrow \infty$, $W^2(\xi) \rightarrow 0$, and there are no infinity type singularities in the metric. However, a zero type singularity in the metric cannot be avoided even for this choice of the initial conditions.

As for $\sigma$, we easily find the expression

$$\kappa^2 \sigma(\phi) = V_0 \left\{ \frac{-3 C}{4 \sqrt{\phi}} \frac{\phi + 3}{\phi} \right\} . (60)$$

or

$$\kappa^2 \sigma(\xi) = \frac{V_0}{\left[ C^2 (\xi + C_0)^2 - 32 \right]^2} \left\{ C^2 \left\{ (\xi + C_0)^2 \times \left[ - (C^2 ((\xi + C_0)^2 + 12) - 64) \right] - 1024 \right\} \right\} . (61)$$

The variation of $\sigma(\xi)$ is represented, for $\phi(0) = 1$ and different values of $\phi'_0(0)$ in Fig. 3.
FIG. 3. Variation of the energy density $\kappa^2 \sigma(\xi)$ of the cosmic string configuration in the presence of a constant potential for $\phi(0) = 1$, and for different values of $\phi'_0$: $\phi'_0 = 1.272$ (solid curve), $\phi'_0 = 1.258$ (dotted curve), $\phi'_0 = 1.244$ (short dashed curve), $\phi'_0 = 1.230$ (dashed curve), and $\phi'_0 = 1.216$ (long dashed curve), respectively.

B. Power law potential: $V(\phi) = V_0 \phi^{3/4}$

The gravitational field equations describing a cosmic string in hybrid metric-Palatini gravity also admit another exact solution, corresponding to the power law type scalar field potential $V(\phi) = V_0 \phi^{3/4}$. We rescale the radial coordinate $r$ by imposing the condition $\beta^2 V_0 = 1$, which gives $r = \xi / \sqrt{V_0}$. With these choices from Eq. (63) we obtain explicitly the scalar field as a function of $\xi$, given by

$$\phi(\xi) = \frac{\left(\xi^2 \phi_0^{3/4} + 2 \xi \phi'_0 + 8 \phi_0\right)^4}{4096 \phi_0^3},$$

where we have used the usual initial conditions $\phi(0) = \phi_0$ and $\phi'(0) = \phi'_0$, respectively. For the metric tensor component $W$ we obtain

$$W(\xi) = \frac{W_0}{\left(\xi^2 \phi_0^{3/4} + 2 \xi \phi'_0 + 8 \phi_0\right)^4 \sqrt{2 \xi + 2 \phi'_0 / \phi_0^{3/4}}},$$

where $W_0$ is an arbitrary constant of integration. On the string axis, i.e., $\xi = 0$, we obtain that $W(0) = \pm W_0^2 / 524288 \phi_0^{3/4} \phi'_0$. Since the metric tensor component $W^2$ must be positive for all $\xi \geq 0$, it follows that the physical solution for the string configuration is the one with the positive sign. Hence in the case of the $V(\phi) = V_0 \phi^{3/4}$ potential, the solutions of the field equations describing a cosmic string in hybrid metric-Palatini gravity are

$$\phi(\xi) = \frac{\left(\xi^2 \phi_0^{3/4} + 2 \xi \phi'_0 + 8 \phi_0\right)^4}{4096 \phi_0^3},$$

$$W^2(\xi) = \frac{W_0^2}{\left(\xi^2 \phi_0^{3/4} + 2 \xi \phi'_0 + 8 \phi_0\right)^6 \left(2 \xi + 2 \phi'_0 / \phi_0^{3/4}\right)},$$

respectively, with $W_0^2 = 524288 W^2 (0) \phi^{21/4} \phi'_0$, a condition that implies $\phi_0 > 0$ and $\phi'_0 > 0$. For the string tension we obtain the expressions

$$\kappa^2 \sigma(\phi) = V_0 \frac{-6C - 5(\phi - 3) \sqrt{\phi}}{8 \sqrt{\phi}},$$

and

$$\kappa^2 \sigma(\xi) = \frac{V_0^3 \phi_0^{3/4}}{(\xi^2 \phi_0^{3/4} + 2 \xi \phi'_0 + 8 \phi_0)^2} \left\{ -48 C \phi_0^{3/4} - 5 \left(\xi^2 \phi_0^{3/4} + 2 \xi \phi'_0 + 8 \phi_0\right) \times \left[ \left(\xi^2 \phi_0^{3/4} + 2 \xi \phi'_0 + 8 \phi_0\right)^4 / 4096 \phi_0^3 \right] \right\},$$

respectively.

In this case, the scalar field is a monotonically increasing function of the radial distance from the string axis, and tends to infinity for $\xi \rightarrow \infty$. For the other hand the metric tensor component decreases monotonically from a finite value on the string axis to zero at infinity. For $\xi = 0$, the string tension takes the finite value $\sigma(0) = V_0 \left[ -48 C \phi_0^{3/4} - 40 (\phi_0 - 3) \phi_0 \right] / 64 \phi_0^{5/4}$, while $\lim_{\xi \rightarrow \infty} \sigma(\xi) = -\infty$, indicating that $\sigma$ is a monotonically decreasing function of the radial coordinate. In the first order of approximation we obtain for the mass of the string of radius $\xi_s$ the expression

$$m = \frac{\pi \beta W_0 \xi_s}{8192 \phi_0^{33/8} \phi'_0^{3/2}} \left[ 6C \xi_s \phi_0^{7/4} + 15 \xi_s \phi_0^{1/2} \left( C - 2 \sqrt{\phi_0} \right) - 4 \phi_0 \phi'_0 \left[ 6C + 5 (\phi_0 - 3) \sqrt{\phi_0} \right] + 5 \xi_s (\phi_0 - 3) \phi_0^2 \right].$$

In this approximation the mass is monotonically increasing with the string radius.

C. Exponential potential: $V(\phi) = V_0 e^{-\lambda \phi}$

As a second example of a string type configuration, we will consider the configuration generated by an exponential type potential, with $V(\phi) = V_0 e^{-\lambda \phi}$, where $V_0$ and $\lambda > 0$ are constants. The solutions of the gravitational field equations for different scalar field models with exponential potentials have been intensively investigated in the recent physical literature, including the cases of both homogeneous and inhomogeneous scalar fields [71,77]. In four-dimensional effective Kaluza-Klein or string-type theories an exponential potential is generated from the compactification of the higher dimensions [78]. Due to the curvature of the internal spaces or to the interaction with form fields on the internal spaces, the moduli fields may acquire exponential type potentials. Non-perturbative effects such as gaugino condensation can also lead to exponential type potentials for scalar fields [79].
In the case of the exponential potential, Eq. (39), giving the scalar field-radial coordinate dependence becomes

$$\xi + C_0 = \int \frac{e^{-\beta \phi}}{\sqrt{C + 2\sqrt{\pi} V_0 e^{\frac{1}{2}} \text{erf}(\sqrt{\lambda} \phi) + 2\beta^2 V_0 e^{-\lambda \phi} / \sqrt{\phi}}} \phi^{-3/4} d\phi$$

where $\text{erf}(x)$ is the error function, and cannot be represented in a closed form, therefore we will use a numerical approach to solve the field equations. We rescale first the scalar field so that $\phi = \Phi/\lambda$, and we choose the scaling parameter $\beta$ of the radial coordinate as $\beta = \sqrt{2/V_0 \lambda}$. Then Eq. (39), which gives the variation of the scalar field, takes the form

$$\frac{d^2 \Phi}{d\xi^2} - \frac{3}{4\Phi} \left( \frac{d\Phi}{d\xi} \right)^2 e^{-\Phi} = 0.$$  

The variation of the metric tensor component $W^2$ can be obtained from the equation

$$\frac{1}{W} \frac{dW}{d\xi} \frac{d\Phi}{d\xi} = -\frac{3}{4\Phi} \left( \frac{d\Phi}{d\xi} \right)^2 e^{-\Phi}.$$  

The behavior of the scalar field with exponential type potential is represented in Fig. 4. For the sake of comparison we have chosen the same initial values for the field $\Phi$ and for its derivative as in the case of the constant potential.

The variation of the metric tensor component $W^2(\xi)$ is represented in Fig. 5. For the adopted set of initial values the behavior of the scalar field and of the metric is very similar to the constant potential case. The scalar field is a monotonically decreasing function of the distance, and it reaches the value zero at a greater distance from the string axis than in the case of the constant potential. The behavior of the field is strongly dependent on the initial conditions. The metric tensor is a monotonically increasing function of $\xi$, and it is defined properly on the string axis. However, it becomes singular at a finite distance from the axis of the string, tending to infinity at a finite $\xi$. For distances in the range $\xi \in (0, 1)$, or $r \in (0, \sqrt{2/V_0 \lambda})$, the metric tensor is practically a constant, and its behavior is basically independent on the initial conditions of the scalar field. For the exponential potential the energy density of the string can be obtained generally as a function of the scalar field in the form

$$\beta^2 \sigma(\phi) = -\frac{3}{4\sqrt{\phi}} \left[ C + 2\sqrt{\pi} \beta^2 \sqrt{\lambda V_0 \text{erf}(\sqrt{\phi} \lambda)} \right]$$

$$-\frac{2\beta^2 V_0 e^{-\lambda \phi}}{4\phi} [\phi(\lambda \phi + \lambda + 2) + 6].$$

However, since the numerical solutions for $\phi(\xi)$ and $W(\xi)$ are known, it is more convenient to obtain $\sigma(\xi)$ from the equation

$$\beta^2 \kappa^2 \sigma(x) = -\frac{1 + \phi}{W} \left( \frac{d^2 W}{d\xi^2} + \frac{1}{1 + \phi} \frac{dW}{d\xi} \frac{d\phi}{d\xi} \right).$$

The variation of $\sigma$ as a function of the dimensionless radial coordinate distance is represented in Fig. 6. In order to obtain positive energy densities the initial values of $\Phi_0$ must be negative. There is a significant difference between the behavior of the energy density $\sigma$ as compared to the constant potential case. The energy density initially decreases for increasing values of the radial coordinate, but for $\xi > \xi_c$, the energy density begins to increase, and tends to infinity, and thus experiencing a singularity at large distances from the string axis.

In the first order of approximation, and after rescaling the variable $\phi$, the integrand in Eq. (69) can be approxi-
Within the framework of this approximation the metric tensor component $W$ is given by the differential equation

$$\frac{1}{W} \frac{dW}{d\xi} = -\frac{64e^{-\frac{\pi}{4}(C+2\sqrt\pi)^2\lambda(C_0+\xi)^4}}{(C+2\sqrt\pi)^2 \lambda (C_0 + \xi)^3} - \frac{3}{C_0 + \xi}$$ \hspace{1cm} (79)

with the general solution given by

$$W(\xi) = \frac{1}{(C_0+\xi)^{\nu}} \exp \left\{ 2 \left[ \frac{16e^{\frac{\pi}{4}(C+2\sqrt\pi)^2\lambda(C_0+\xi)^4}}{(C+2\sqrt\pi)^2(C+C+2\sqrt\pi)^2\lambda} \right. \right. \right.$$ 
$$\left. + \frac{\sqrt\pi}{(C+2\sqrt\pi)^{\nu}} \text{erf} \left( \frac{\pi}{2\sqrt\pi} (C+2\sqrt\pi) \sqrt\lambda (C_0 + \xi)^2 \right) \right\}. \hspace{1cm} (80)$$

Even in this approximation the full analysis of the behavior of the cosmic string configuration in hybrid metric-Palatini gravity in the presence of an exponential type potential can only be done using numerical methods.

### D. Higgs-type potential

Next we consider the case when the scalar field potential is of the Higgs-type, given by

$$V(\phi) = \frac{1}{2} \tilde{\mu}^2 \phi^2 + \frac{\nu}{4} \phi^4,$$ \hspace{1cm} (81)

where $\tilde{\mu}^2$ and $\nu$ are constants. In the following we will investigate only the case with $\tilde{\mu}^2 < 0$, that is, we will adopt the minus sign in the definition of the potential. By following the standard approach in elementary particle physics, we assume that the constant $\tilde{\mu}^2$ is related to the mass of the scalar field particle as $m_{\phi}^2 = 2\nu v^2 = 2\tilde{\mu}^2$, where $v^2 = \tilde{\mu}^2 / \xi$ gives the minimum value of the potential. The Higgs self-coupling constant $\nu$ can be obtained, in the case of strong interactions, from the determination of the mass of the Higgs boson in laboratory experiments, and its numerical value is of the order of $\nu \approx 1/8 [80]$. By rescaling the radial coordinate and the scalar field according to

$$r = \sqrt{2\tilde{\mu}} \xi, \quad \phi = \frac{\Phi}{(\nu\tilde{\mu})^{1/3}},$$ \hspace{1cm} (82)

then Eq. 34 provides the profile of the scalar field in the following form

$$\frac{d^2\Phi}{d\xi^2} - \frac{3}{4\Phi} \left( \frac{d\Phi}{d\xi} \right)^2 - \Phi^2 + \Phi^4 = 0.$$ \hspace{1cm} (83)

The general solution of this equation is given in a closed form by

$$\xi + C_0 = \int \frac{1}{\Phi^{3/4} \sqrt{C + \frac{2}{3\Phi} (7 - 3\Phi^2 \Phi^{3/2})}} d\Phi.$$ \hspace{1cm} (84)
However, this solution cannot be expressed in an analytical form in terms of known functions. In the first order approximation, we obtain

\[ \xi + C_0 \approx \frac{4\sqrt{\Phi}}{\sqrt{C}} - \frac{4\Phi^{7/4}}{21C^{3/2}} + O\left(\Phi^{9/4}\right), \]  

but this representation is not particularly useful from the point of view of concrete calculations.

The variation of the scalar field with Higgs potential supporting a string configuration in hybrid metric-Palatini gravity is represented in Fig. 7. There is a significant qualitative difference between this string model and the constant, simple power law or exponential potentials. Note that the scalar field for the Higgs-type potential shows a basically periodic structure, changing between successive maxima and minima. There are singularities in the field. Its behavior is strongly affected by the initial conditions on the string axis, and the field extends to infinity.

The variation of the metric tensor component \( W^2(\xi) \) in the presence of a Higgs potential is represented in Fig. 8. The same oscillatory pattern can also be observed in the case of the metric tensor component \( W^2 \). However, there is a difference in the phase of these quantities. When the field reaches its maximum at \( \xi \approx 1 \), the metric tends to zero, \( W^2(1) \approx 0 \). Then, while the scalar field decreases, the metric tensor increases, reaching its maximum at the minimum of the field, corresponding to \( \xi \approx 2 \). This pattern is repeated up to infinity.

The variation of the string tension with respect to the radial coordinate in the presence of the Higgs potential is depicted in Fig. 9. The variation of \( \sigma \) is in phase with that of the scalar field, and both quantities reach their maxima and minima at the same position. The string tension also has an oscillatory behavior, which is a general property of all physical and geometrical parameters of the string configurations supported by scalar fields with a Higgs-type potential.

V. DISCUSSIONS AND FINAL REMARKS

In the present paper we have investigated string type solutions in hybrid metric-Palatini gravity, which is an extension of general relativity that combines the metric and Palatini formalisms. From a theoretical point of view the main advantage of the hybrid theory is that it is a viable theory of gravity that includes elements of both formalisms. A main success of the theory is the possibility of generating long-range forces that pass the classical local tests at the Solar System level of grav-
ity. Another important advantage of the theory is that it admits an equivalent scalar-tensor representation, which greatly simplifies the analysis of the field equations, and the construction of their solutions. In this work, we have explored local gauge string solutions with a phenomenological energy momentum tensor, as prescribed in [44].

In fact, an important class of solutions of the gravitational field equations are represented by the string-like configurations, which are generally constructed by assuming a cylindrically symmetric metric of the form

$$ds^2 = -N^2(r)dt^2 + dr^2 + L^2(r)d\theta^2 + K^2(r)dz^2,$$  

(86)

where the functions $L$ and $N$ must satisfy the regularity conditions $L(0) = 0$, $L'(0) = 1$, $N(0) = 1$, and $N'(0) = 0$, respectively. The simplest model that gives rise to a string solution is based on the matter Lagrangian

$$L_m = |\partial_\mu \phi|^2 - \frac{\lambda}{4} (|\phi|^2 - \bar{\eta}^2)^2,$$  

(87)

where $\lambda$ is a dimensionless coupling constant, while $\bar{\eta}$ is the vacuum expectation value of the field $\Phi$. The model can also be extended to include gauge fields of the electromagnetic type [23]. In fact, the study of cosmic strings was pioneered in [81], and ever since it has become a popular subject of investigation. For a review of cosmic string and superstring properties see [82]. Cosmic strings have the interesting property that around a straight, local cosmic string the spacetime is flat. If the string is located around the $z$-axis, the spacetime metric is [15]

$$ds^2 = -dt^2 + dr^2 + (1 - 8\pi G\mu) r^2 d\theta^2 + dz^2,$$  

(88)

which has a flat geometry in terms of the modified azimuthal coordinate $\theta' = (1 - 4G\mu) \theta$, ranging from 0 to $2\pi - 8\pi G\mu$. More generally, the metric for a string oriented along the $z$ axis and having an infinite length can be written as [83]

$$ds^2 = -dt^2 + dr^2 + \left[1 - \frac{\mu(r)}{2\pi}\right]^2 r^2 d\theta^2 + dz^2.$$  

(89)

Another interesting string configuration is given by the Barriola and Vilenkin string [84], with metric

$$ds^2 = - \left(1 - 8\pi \bar{\eta}^2 - \frac{M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - 8\pi \bar{\eta}^2 - \frac{M}{r}\right)} + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right).$$  

(90)

This solution generalizes the matter Lagrangian [87] by considering a self-coupling scalar field triplet $\phi^a$, $a = 1, 2, 3$, so that the matter action is given by $L_m = (1/2)|\partial_\mu \phi^a|^2 \partial^a \phi^a - \lambda (\phi^a \phi^a - \bar{\eta}^2)^2$. To solve the gravitational field equations one uses the ansatz $\phi^a = \eta f(r) x^a/r$.

In the framework of the Brans-Dicke theory static cylindrically symmetric solutions of the field equations have been obtained in [44], for a gravitational action of the form

$$S = \int d^4x \sqrt{-g} \left\{ \phi (-R + 2\Lambda) + \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} + S_m[\Psi, g],$$  

(91)

where $\Lambda$ is the cosmological constant. For $\Lambda > 0$ the solution of the field equations is given by [44]

$$W(r) = A \sin^{(1/2)+(2)}(\beta r) \tan^{(-)}(\beta r),$$  

(92)

$$\phi(r) = A \sin^{(1/2)+(2)}(\beta r) \tan^{-1}(-\beta r) + \Lambda,$$  

(93)

where $\alpha = \sqrt{2\Lambda (2 + \bar{\omega})} > 0$, $\epsilon = \pm 1$, and $A$ is an integration constant. Similar solutions can be obtained in the case $\Lambda < 0$, with the trigonometric functions replaced by the hyperbolic ones, so that

$$W(r) = A \sinh^{(1/2)+(2)}(\beta r) \tanh^{(-)}(\beta r),$$  

(94)

$$\phi(r) = A \sinh^{(1/2)+(2)}(\beta r) \tanh^{-1}(\beta r) + \Lambda,$$  

(95)

where $\beta = \sqrt{2\Lambda (2 + \bar{\omega})}$. The solutions with $\epsilon = 1$ are smooth, regular, and free of any singularity on the string axis.

In our present investigation of the cosmic string type solutions in the context of the hybrid metric-Palatini gravitational theory we have adopted, after several simplifications, the metric [19], of the form similar to the metric [89], with $W^2(r) = [1 - \mu(r)/2\pi r^2] r^2$. By adopting the scalar-tensor representation, the gravitational field equations can be formulated in terms of the metric tensor $W$, the scalar field and its derivatives, and the string tension, respectively. The model also contains as an essential ingredient the scalar field potential $V$. Generally, for the case of an arbitrary potential, the general solution of the field equations can be obtained in a closed form. For at least three particular choices of the scalar field potential the solution of the field equations can be expressed in an exact (and simple) analytic form.

For some types of potentials the string solutions in the hybrid theory have some important distinctive features as compared to the other string models. First of all, the behavior of the solutions is strongly dependent on the initial conditions of the scalar field and of its derivative for $r = 0$. These initial conditions are rather arbitrary, since a large number of such field configurations can be constructed. Depending on the initial conditions for the field we obtain two distinct classes of solutions. The first class consists of the solutions that become singular for a finite value of the rescaled radial coordinate $\xi$. For example, in the string solution with $V = 0$, if $\phi_0 > 0$ and $\phi_0' < 0$, then $C_0 = \phi_0/\phi_0' > 0$, and the metric and the string tension are becoming singular (infinite) at $\xi_\infty =$
4\phi_0 \phi_0^\prime$. In this case the scalar field vanishes for \(\xi_\infty\). A completely different situation arises if both \(\phi_0\) and \(\phi_0^\prime\) are positive. In this case both the string tension and the metric tend to zero at infinity, with the scalar field becoming singular for \(\xi \to \infty\).

The exact solution of the field equations corresponding to the \(V = V_0 \phi^{3/4}\) corresponds to the second class. The requirement of the positivity of the metric tensor for \(r = 0\) imposes the condition that the metric tensor and the string tension are decreasing functions of the radial coordinate, since both \(\phi_0\) and \(\phi_0^\prime\) must be positive. As a consequence, it is the scalar field that diverges at infinity. However, in these cases one could also define a string radius by introducing an effective cut-off length \(\xi_\infty\) for the metric and scalar field, which would allow to construct finite string configurations, with finite values of the scalar field, string tension and metric tensor. But in this case the definition of \(\xi_\infty\) is either arbitrary, or based on some empirical considerations, such as the consistency with observational data.

In the case of the exponential and Higgs type scalar field potentials, one can use the condition of the vanishing of the string tension \(\sigma(r)\) to define a radius of the string. This can be done only numerically, and the numerical value of the string radius is strongly dependent on the initial conditions of the scalar field and its derivative at \(r = 0\). For example, in the case of the exponential potential, the string tension vanishes at \(\xi_\infty \approx 0.6\), which would allow to define a string radius \(R_s\) of the order of magnitude \(R_s \approx 0.6 \sqrt{2/V_0 \lambda \xi_\infty}\). Generally, the exponential potential \(\sigma\) reaches its zero/minimum value for finite values of the scalar field potential, and of the metric tensor, and hence the condition \(\sigma \approx 0\) may define the string radius for this type of scalar field potential.

The case of the Higgs potential is quite interesting. The string tension does not vanish for any value of the radial coordinate, and it reaches its minimum value for \(\xi \approx 3.5\). For this value of \(\xi\) the scalar field is at its minimum, while the metric tensor is singular, and tends to zero. Alternatively, one could define the string radius as corresponding to the first zero of the metric tensor \((\xi \approx 1)\), with the scalar field and the string tension reaching their first maxima. One can also introduce a cut-off radius for these potentials, with its value being determined from the confrontation of the theoretical predictions with observations.

An important geometrical quantity, the angular deficit \(\Delta \theta\) in the cylindrical symmetry, due to the presence of the string, is given by [84]

\[
\Delta \theta = 2\pi \left[1 - \lim_{r \to \infty} W'(r)\right],
\]

In the first order of approximation, and for strings with finite extension, we may replace \(W'(\infty)\) with \(W'(R_s)\) in Eq. (96), where \(R_s\) is the string radius, thus obtaining

\[
\Delta \theta \approx 2\pi \left[1 - W'(R_s)\right].
\]

In the variable \(\xi\) the angular deficit can be represented as

\[
\Delta \theta \approx 2\pi \left[1 - \frac{1}{\beta} W'\left(\xi_s\right)\right].
\]

Since the metric tensor \(W\) depends on the initial conditions of the scalar field on the string axis, the string geometries obtained in the present study allow a very large range of deficit angles, which significantly impact the topology of the spacetime near the string. For the solutions with \(\lim_{\xi \to \infty} W(\xi) \to 0\), generally also \(\lim_{\xi \to \infty} W'(\xi) \to 0\), like, for example, in the zero potential case with \(\phi_0\) and \(\phi_0^\prime\) positive. In this case we obtain \(\Delta \theta \approx 2\pi\). In the opposite limit of \(\lim_{\xi \to \infty} W'(\xi) \to \infty\), the deficit angle is formally infinite. We can still define a finite deficit angle by introducing a cut-off radius \(R_s\) that formally defines the radius of the string. For the exponential and Higgs potentials one can define an explicit string radius, which also allows the explicit estimation of the deficit angle.

Cosmic strings have a number of very intriguing properties. For example, as suggested by Witten [86], strings behave like superconducting wires. Hence they can interact with external cosmic electromagnetic fields, and as they move through cosmic magnetic fields they can develop electric currents. Therefore, short electromagnetic and highly beamed bursts can be emitted from some peculiar points (cusps), located on small string segments, where the velocity approaches the speed of light [87, 91]. Hence the cusp is a powerful source of electromagnetic radiation that may produce a jet of accelerated particles that may play an important role in many astrophysical phenomena, like the Gamma Ray Bursts prompt and afterglow emissions, respectively. It would be interesting to consider superconducting strings in the framework of modified theories of gravity, and in particular in hybrid metric-Palatini gravity. Such a study would offer some possibilities between discriminating standard cosmic strings from string-like structures that appear in modified theories of gravity.

Another important physical effect that could, at least in principle, discriminate between standard general relativistic cosmic strings, cosmic strings in modified gravity, and other filamentary matter distributions is gravitational lensing. According to the standard general relativistic string scenario, the curvature of the spacetime is not changed by a vacuum string. However, the topology of the spacetime is modified [92]. Hence photon beams are not bent by a cosmic string. But if two light rays travel on the different sides of the string, the presence of the specific conical structure of the spacetime geometry determines their later convergence at the same point of observation [93]. Hence, for a cosmic string located between a terrestrial observer and a distant cosmological source, the observer will detect two images of the light emitting source, separated by an angle \(\delta \theta = 8\mu_s \sin \alpha D_{LS}/D_{OS}\), where by \(\mu_s\) we have denoted the linear mass density of the string, \(\alpha\) represents the angle between the observer-source direction and the...
string, while $D_{\text{OS}}$ and $D_{\text{LS}}$ represent the distance between the observer and the source, and the lens and the source, respectively. Hence it follows that for the case of the standard general relativistic conical string, due to the string presence, the two images formed are identical to the original source, without any distortion or amplification. This effect is very different from the gravitational lensing by gas filaments, which show a very different image structure, formed from one or three elongated images. Hence the lensing properties of the string solutions in hybrid metric-Palatini gravity obtained in the present study could help in discriminating between these string solutions and the corresponding solutions obtained in standard string theory, or other modified gravity models. The systematic study of the lensing properties of hybrid metric-Palatini gravity strings, and their observational implications, will be considered in a future work.

The in-depth investigation of modified theories of gravity and of their astrophysical and cosmological implications is a major field of study in present day theoretical physics. Despite the fact that from the observational point of view cosmic strings are still elusive astrophysical and cosmological objects, the investigation of their theoretical properties may lead to a better understanding of the theoretical structure of modified gravity. In the present paper we have provided some basic theoretical tools that would enable the in-depth investigation of the properties of the cosmic strings in the hybrid metric-Palatini theory of gravity, and of their astrophysical and cosmological implications.

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[1] N. D. Mermin, Rev. Mod. Phys. 51, 591 (1979).
[2] J. W. O-T., Journ. of Molecular Struct., 29, 190 (1975).
[3] I. Chuang, R. Durrer, et al., Science 251, 1336 (1991).
[4] M. M. Salomaa and G. E. Volovik, Phys. Rev. Lett. 55, 1184 (1985).
[5] P. C. Hendry, N. S. Lawson, R. A. M. Lee, et al., Nature 368, 315 (1994).
[6] A. A. Abrikosov, Sov. Phys. JETP 5, 1174 (1957); [Zh. Eksp. Teor. Fiz. 32, 1442 (1957).
[7] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).
[8] A. Vilenkin and E. P. S. Shellard, Cosmic strings and other topological defects, Cambridge Monographs on Mathematical Physics, Cambridge, 2001.
[9] M. R. Anderson, The mathematical theory of cosmic strings: cosmic strings in the wire approximation, Institute of Physics Publishing, Bristol and Philadelphia, (2002).
[10] T. W. B. Kibble, J. Phys. A 9, 1387 (1976).
[11] W. H. Zurek, Nature 317, 505 (1985).
[12] P. Avelino et al., Symmetry 8, 70 (2016).
[13] M. J. Lake and T. Harko, The European Physical Journal C 76, 311 (2016).
[14] R. Jeannerot, J. Rocher, and M. Sakellariadou, Phys. Rev. D 68, 103514 (2003).
[15] Y. Nambu, H. Ishihara, N. Gouda, et al., Astrophysical Journal Letters 373, 35 (1991).
[16] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 571, A25 (2014).
[17] J.-H. P. Wu, P. P. Avelino, E. P. S. Shellard, International Journal of Modern Physics D 11, 61 (2002).
[18] K. D. Olum, and A. Vilenkin, 2006, Phys. Rev. D74, 063516 (2006).
[19] K. S. Cheng, Y.-W. Yu, and T. Harko, Phys. Rev. Lett. 104, 241102 (2010).
[20] K. S. Cheng, Y.-W. Yu, and T. Harko, Phys. Rev. Lett. 106, 259002 (2011).
[21] D. B. Thomas, C. R. Contaldi, and J. Magueijo, Physical Review Letters 103, 181301 (2009).
[22] M. J. Lake and T. Harko, Fortschrritte der Physik 65, 1600121 (2017).
[23] H. B. Nielsen and P. Olesen, Nucl. Phys. B 61, 45 (1973).
[24] B. P. Abbott, et al., Phys. Rev. D 97, 102002 (2018).
[25] A. Azadi, D. Momeni and M. Nouri-Zonoz, Phys. Lett. B 670, 210 (2008).
[26] D. Momeni, H. Gholizade, Int. J. Mod. Phys. D 18, 1719 (2009).
[27] M. Sharif and S. Arif, Mod. Phys. Lett. A. 27, 1250138 (2012).
[28] B. Linet, J. Math. Phys. 27, 1817 (1986).
[29] Q. Tian, Phys. Rev. D 33, 3549 (1986).
[30] O. Bertolami, C. G. Boehmer, T. Harko and F. S. N. Lobo, Phys. Rev. D 75, 104016 (2007).
[31] T. Harko and F. S. N. Lobo, Eur. Phys. J. C 70, 373 (2010).
[32] T. Harko and M. J. Lake, Eur. Phys. J. C 75, 60 (2015).
[33] T. Harko and M. J. Lake, Phys. Rev. D 91, 045012 (2015).
[34] C. Gundlach and M. E. Ortiz, Phys. Rev. D 42, 2521 (1990).
[35] A. Barros and C. Romero, J. Math. Phys. 36, 5800 (1995).
[36] M. E. X. Guimares, Class. Quantum Grav. 14, 435 (1997).
[37] B. Boisseau and B. Linet, Class. Quantum Grav. 14, 3063 (1997).
[38] F. Dahia and C. Romero, Phys. Rev. D 60, 104019 (1999).
[39] A. A. Sen, N. Banerjee, and A. Banerjee, Phys. Rev. D 56, 3706 (1997).
[40] A. Arai and C. Simeone, Gen. Relat. Gravit. 32, 2259 (2000).
[41] R. Gregory and C. Santos, Phys. Rev. D 55, 1194 (1997).
[42] A. A. Sen and N. Banerjee, Phys. Rev. D 57, 6558 (1998).
[43] A. A. Sen, Pramana 55, 369 (2000).
