Demonstration of Dynamic Topological Pumping Across Incommensurate Acoustic Meta-Crystals

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A Thouless topological pump [1] can be regarded as a dynamical version of the integer quantum Hall effect. In a finite configuration, a topological pump displays edge modes, which emerge dynamically from one bulk-band and dive into the opposite bulk-band, an effect that can be reproduced with both quantum and classical systems. In the classical setup, this phenomenon opens the possibility of controlled edge to edge energy transfer and many re-configurable metamaterials have been proposed for this purpose. However, driving them in adiabatic cycles requires synchronized bulk deformations of the meta-material and this proved to be extremely challenging. Here, we report the first un-assisted dynamic energy transfer across a meta-material via pumping of topological edge modes. The system is a topological aperiodic acoustic meta-crystal, with a phason degree of freedom that is experimentally accessible and easily adjustable. As a result, the phason can be fast and periodically driven in adiabatic cycles, without any outside intervention or assistance. Furthermore, when one edge of the meta-crystal is excited in a topological forbidden range of frequencies, a microphone placed at the other edge starts to pick up signal as soon as the dynamic pumping process is set in motion. In contrast, the microphone picks no signal when the forbidden range of frequencies is non-topological.

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More than 35 years ago, Thouless asked himself what happens with a filled sea of fermions when the underlying potential is slowly and periodically modulated in time [1]. He predicted that a precise non-fluctuating number of particles will be effectively transported from one side of the system to the other and that this number is determined by a topological invariant computed for a virtual system of one dimension higher than the original. The effect has been directly demonstrated recently, with both fermions and bosons [2][3]. It is now well established [4][5] that augmentation of a parameter space to a d-dimensional quantum or classical system can give access to topological effects that, in normal conditions, are observed in d + 1 or higher dimensions. One such example is the periodic 1-dimensional Rice-Mele model [6], where an adiabatic deformation of the parameters leads to a virtual 2-dimensional system whose energy bands support non-trivial Chern numbers [4]. As a result, the system displays chiral edge bands when driven in an adiabatic cycle. This was recently exploited in [7], where the Rice-Mele model was simulated with a classical magneto-mechanical system and dynamic pumping of mechanical energy from one edge of the system to the other was demonstrated for the first time. In this work, the couplings between different cells of the system were externally controlled and assisted, giving the platform a large flexibility at the expense of complexity.

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In this Letter, we report the first un-assisted dynamic energy pumping across a topological meta-material. Our main results are summarized in Fig. 1, where we present direct evidence of energy transfer from one end of a bulk structure to the other end, even though the frequency of the source falls in a forbidden wave-propagation range. This energy transfer happens in pumping conditions and when the source frequency is in a topological spectral gap of the meta-material. In contradistinction, when the frequency is adjusted in a non-topological spectral gap, there is no energy transfer even though same conditions of pumping are applied.

Our experimental platform consists of the aperiodic acoustic meta-crystal described in Fig. 1(a-c), with the aperiodicity being induced by the coupling of two incommensurate periodic arrays of acoustic resonators. The dimensions have been optimized to maximize the size of the topological gap. This type of patterned resonators was theoretically studied in [11], where it was found to support topological spectral gaps and topological edge modes. However, to our knowledge, this is the first time when coupled incommensurate chains are experimentally used to engineer chiral edge bands for topological pumping. Key to our experimental design was the replacement of any elaborate interconnections between acoustic resonators with a thin uniform spacer, extending from one end of the structure to the other. The resonators are attached to and coupled through this spacer. Note that this type of coupling does not allow fine-tuning but, as we mentioned, that is not necessary when using aperiodic principles, as long as the coupling is strong [10]. Furthermore, edges can be created by simply filling the spacer with solid material.

There are certain advantages in our design that made possible this first demonstration of dynamical pumping. More precisely: a) The phason can be driven in an adiabatic cycle by simply sliding the top array while holding the bottom one fixed; b) Since the bottom array is fixed, we can continuously pump energy at one edge by placing a source on the first bottom resonator; c) The left and right edges can be easily and independently adjusted to achieve the optimal dispersion of the edge modes.

The numerically simulated topological pumping process is reported in Fig. 2, where we also explain its mechanism. An animated version can be found in the Supplementary Material. Sure enough, the left and right chiral edge bands are present in the topological gap. Note their particular and optimal dispersion, which made the dynamical pumping possible. Indeed, it is very important that the right chiral edge band emerges from the top bulk-band shortly after the left chiral edge dived into the same band. The reason for this requirement is because the non-adiabatic effects cannot be prevented when the
FIG. 2. Principles and mechanism of our dynamical pumping. (a) The configuration of the system at the beginning of a pumping cycle. The top array is uniformly displaced to the right and, after a total displacement $d_2$, the system returns in its original configuration and completes a full pumping cycle. A speaker is inserted in resonator $S_1$ and is kept on at all times, while a microphone is inserted in the resonator $M_0$. (b) Simulation of the resonant spectrum as function of displacement. Chiral left and right edge bands are observed, which both connect two disjoint parts of the bulk spectrum. (c-h) Rendering of pumping mechanism: The left edge mode is loaded when the source frequency matches the mode frequency (c); The mode self-oscillates while its frequency is pushed up (d); The character of the mode changes from left-localized to delocalized (e); The character of the mode changes again from delocalized to right-localized (f); The mode self-oscillates as its frequency is pushed down (g); The cycle repeats itself as the top array is further pushed to the right (h). The microphone starts to pick up signal after the event (e). The simulations in panels (c-h) show the spatial profile of the resonant mode highlighted in the sub-panel below it. The shown microphone outputs are not from real measurements.

Pumping of energy is through the bulk states. As such, one has to optimize the pumping cycle such that there is a rapid change of the mode character from left-localized to extended and to right-localized, exactly as it can be seen in Fig. 2(c-h), where our pumping cycle was broken down into steps. In the standard picture of topological pumping, the mode self-oscillates after being loaded at the left edge, hence the pumping cycle must be performed fast enough to overcome the dissipation effects.

Given the particular engineering of our system, the pumping cycle can be performed extremely fast and repetitively, even without any external intervention. This enabled us to achieve the first un-assisted dynamical energy pumping via topological edge modes. Its dramatic manifestation is documented in Fig. 1(g), where a receiver placed opposite to an acoustic source is shown to pick up acoustic signal when the excitation frequency is in a topological resonant gap. In this experiment, 10 resonators were added beyond the edge to the left side of the top array, which resulted in the 10 pumping cycles visible in Fig. 1(g). The time period of the pumping cycle is approximately 0.12 seconds in Fig. 1(g). We have experimented with the time period of the cycle and found that the energy transfer is completely cut out when the period is about 1 second. This demonstrates that the pumping process is indeed essential for the energy transfer across the acoustic meta-crystal. Furthermore, when the source frequency is adjusted in a non-topological spectral gap, the receiver picks no signal whatsoever. We have experimented with different source frequencies inside the non-topological gap and we can confirm that the receiver does not pick any signal even when the frequency is very close to the bulk spectrum. This demonstrates that the chiral edge bands formed inside the topological spectral gap play an essential role for the energy transfer phenomena detected in our experiments.

The sound of the pumping reported in Fig. 1(g) can be played here, or from the audio files supplied in the Supporting Material. The sound of the pumping reported in Fig. 1(h) can be played here, or from the audio files supplied in the Supporting Material. As one can see, there is stark difference between the two pumpings reported in Fig. 1. Taking into account all the above facts, there can be no doubt that the energy
transfer across the meta-crystal was through a classic topological pumping process.

As we already mentioned, the resonator coupling through the spacer does not allow fine-tuning but that is not necessary when using aperiodic principles, as long as the coupling is strong \([10]\). To understand the mechanism of topological gap generation in our system, we show first in Fig. 1(d) the evolution of the simulated resonant spectrum with respect to the relative alignment of two identical arrays of resonators. As expected in any 1-dimensional periodic system, gaps appear in the resonant spectrum and, as the system switches between period-one and period-two, some of these spectral gaps close while other remain open. Regardless of that behavior, all these gaps are topologically trivial because the resonant bands seen in Fig. 1(d) result from dispersion-induced thickening of the discrete resonances of the individual resonators. However, when the lattice constant of the bottom array is varied and the system becomes aperiodic, these trivial bands are seen in Fig. 1(e) to become fragmented in sub-bands, exactly as it happens when a magnetic field is turned on a two dimensional electronic system \([18]\). In particular, the sub-bands carry non-zero Chern numbers \([11]\) and the presence of the chiral edge bands can be explained with the standard bulk-boundary correspondence arguments \([19, 20]\).

The simulated bulk spectrum is reproduce with high fidelity by the experimental measurements, as demonstrated in Fig. 3. In particular, well defined bulk-spectral gaps can be identified in the measured local density of states, which are well aligned with the theoretical predictions. The frequency 5.4 kHz used for topological pumping in Fig. 1(g) falls in the middle of one such gap. Furthermore, the signature of the non-zero Chern numbers, that is, the chiral edge bands, are also detected experimentally, as reported in Fig. 4. By comparing the panels (a) and (b), one can see that the experiment reproduces the simulations with very high fidelity.

Having demonstrated an un-assisted energy transfer via a topological pumping process, we have laid down a set of specific principles which could facilitate the engineering of the effect in many other contexts. The most important one is that fine-tuning is not necessary which, together with the many different ways of engineering phason spaces \([11]\), relaxes the design constraints giving scientists better chances with finding optimal and practical meta-structures. While for meta-materials this process is now more or less straightforward, it will be extremely interesting if these aperiodic principles can be successfully applied to mesoscopic systems and achieve electron pumping in conventional insulators.

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METHODS

The Phason Space. We recall from Fig. 5 how any finite piece of the incommensurate bilayer can be generated from a simple dynamical process over the non self-intersecting loop $\Omega$ shown in Fig. 5(a). When viewed from above, $\Omega$ appears as a figure eight as in Fig. 5(b), made out of two circles of perimeters $d_1$ and $d_2 < d_1$. We call this the projected $\Omega$. While walking on $\Omega$ in the positive direction indicated by the arrow in Fig. 5(c) one passes from the large circle to the small circle at point $J$, and then from the small circle back to the large circle at point $J'$. Consider now the group action $\tau : \mathbb{Z} \to \text{Homeo}(\Omega)$ such that $\tau_n$ moves a point $\omega$ by $nd_2$ along the projected $\Omega$ in the positive direction. On the real $\Omega$, this action also changes the vertical coordinate of the point. For $\omega \in \Omega$, let $d_{\omega/\omega} = \text{distance from } \omega \text{ to } J'$ as moving on the projected $\Omega$ in the positive/negative direction. The vertical coordinate of a point $\omega \in \Omega$ will be denoted by $Y(\omega)$. Then the algorithm for producing the incommensurate bilayer can be described as follows. We put our pencil on an arbitrary $\omega \in \Omega$ and, in the same time, we center a resonator at coordinates $(0, Y(\omega))$ of the physical space $\mathbb{R}^2$. Then we move $\omega$ in the positive direction by $d_2$ on the projected $\Omega$, and we center another resonator at horizontal and vertical coordinates $A_n(\omega) = \min\{d_2, d_{\omega/\omega}(\tau_1 \omega)\}$ and $\Delta Y(\omega) = Y(\tau_1 \omega) - Y(\omega)$, relative to the previous resonator. By repeating this step, we generate an iterative process $p_{n+1} = p_n + \Gamma(\tau_n \omega)$, with $\Gamma$ a continuous function from $\Omega$ to $\mathbb{R}^2$ defined by $\Gamma(\omega') = (\Delta_x(\omega'), \Delta_y(\omega'))$ for any $\omega' \in \Omega$. In Fig. 5(c), we show a pattern $\{p_n\}$ of 100 resonators generated with this algorithm and with the equation for $\Omega$ taken from Fig. 5. As one can see, it indeed reproduces a finite incommensurate bilayer.

When $\omega$ happens to land in the interval (1-2) between points (1) and (2) or the interval (3-4) between points (3) and (4) shown in Fig. 5(a), the algorithm will produce a point which is in between the bilayers. These are very rare events if the transitions between the top and bottom sections of $\Omega$ are made very sharp. Nevertheless, these transitions can be interpreted experimentally in the following way. In Fig. 5(d), we show the patterns generated with $\omega$ sitting at point (1) and at point (2). If we switch continuously between these two resonator configurations, the bottom resonator disappears beyond the edge while the top resonator gradually appears. Experimentally, we will associate these resonator configurations with an $\omega$ running continuously from location (1) to location (2) on $\Omega$. A similar correspondence can be worked out when $\omega$ runs in the interval (3-4). In the Supplementary Material, we provide all resonator configurations obtained by moving $\omega$ over the entire $\Omega$.

The algorithm can be seeded from any point $\omega$ of the loop $\Omega$ and this will result in different patterns of resonators. The difference between these patterns can be assessed as follows. First, note that if we start from $\tau_n \omega$ instead of $\omega$, we rigidly shift the pattern such that point $p_n$ sits at the origin. Equivalently, we can think that an observer has been moved from resonator $p_0$ to $p_n$. Now, as the observer moves from one resonator to another, he/she will see different patterns of resonators and attributes this phenomenon to the existence of a degree of freedom, which, by definition, is the phason itself. Certainly the phason takes the values $\tau_n \omega$ and, if $d_2$ and $d_1$ are incommensurate, then these values fill $\Omega$ densely as $n$ is varied in $\mathbb{Z}$. The conclusion is that $\Omega$ parametrizes all the patterns seen by the observer, hence the phason leaves on the set $\Omega$ described Fig. 5.

Origin of the Topological Gaps. Resolving the spectral and topological properties of aperiodic patterns of resonators starts with the computation of the algebra of observables $\mathcal{A}$. The $K$-theory of this algebra then supplies the unique topological labels for the spectral gaps $\mathcal{G}$. In Fig. 6, we report the first four elementary modes supported by the individual capped resonators. By examining their frequencies, it becomes clear that the collective modes reported in Figs. 4 and 5 practically involve only these four modes. As such, the dynamical matrix of the collective resonant modes takes a discrete form:

$$\hat{D}_\omega = \sum_{n,n'} w_{n,n'(\omega)} \otimes [n'](n)$$  \hspace{1cm} (1)

where $w_{n,n'} \in M_{4 \times 4}(\mathbb{C})$ is the matrix which couples the elementary modes supported by resonators $n$ and $n'$. The index $\omega$ reflects the dependence on the choice of the origin, where the edge will be placed, and consistency requires the following covariant relations:

$$w_{n'+a,n+a}(\omega) = w_{n',n}(\tau_a \omega), \quad \forall n,n',a \in \mathbb{Z}.$$  \hspace{1cm} (2)
Choosing $a = -n$ and replacing $\omega$ by $\tau_n \omega$, we find $\omega^{a,n}(\omega) = \omega^{-n} \omega^0(\tau_n \omega)$, hence we can drop one index and using $q = n' - n$, as well as the shift operator:

$$S(n) = |n + 1\rangle, \quad S^\dagger(n) = |n - 1\rangle, \quad SS^\dagger = S^\dagger S = I,$$

the generic dynamical matrix becomes:

$$D_\omega = \sum_q S^q \sum_n \omega_q(\tau_n \omega) \otimes |n\rangle\langle n|.$$

As one can see, the generic dynamical matrices belong to the algebra generated by the shift operator $S$ and by diagonal operators of the form $\sum_n f(\tau_n \omega) |n\rangle\langle n|$ with $f$ a continuous function over $\Omega$. Since any such function can be Fourier decomposed, we find that the algebra is in fact generated by just two operators, namely, $S$ and $V = \sum_n \exp(2\pi i \frac{\omega}{\Omega}) |n\rangle\langle n|$. Furthermore, a direct computation shows that:

$$VS = e^{2\pi i \phi} SV, \quad \phi = \frac{d_2}{d_1 + d_2}.$$

This is the commutation relation of the magnetic translations in 2-dimensions, provided we take the flux per unit cell to be $\phi$ in the quantum units of flux. Hence, the fragmentation of the bands seen in Fig. 2 is as in the Hofstadter butterfly [18]. In particular, the gaps can be labeled by Chern number.

**Spectral flow at the edges.** The bulk-boundary correspondence [19, 20] states that the bulk Chern number of a gap equals the number of chiral edge bands traversing that gap when the phason is pumped. Based on the equivalence between the $\Omega$ space and the resonator configurations explained above, we produced a physical pumping cycle which spends the entire space $\Omega$ of the phason (see Supplementary Material). This cycle is more complicated than the cycle shown in Fig. 2 and involves slidings of both the top and the bottom arrays of resonators. The emergence of chiral bands in a topological gap under this cycle is reported in Fig. 7. As one can see, the experiment reproduces the simulations with high fidelity. The dispersions seen in Fig. 7 are relatively complicated when the phason is driven over the entire loop $\Omega$. However, by examining the results in Fig. 7, we learned that chiral edge bands can be generated by driving the phason on a reduced cycle which involves only the circle of perimeter $d_2$. This is the cycle shown in Fig. 2 and the cycle that was used to achieve the dynamical pumping reported in Fig. 1.

**Numerical Simulations.** All numerical simulations were performed with COMSOL Multiphysics. This commercially available software was used to map the self-sustained acoustic wave-modes for different resonator configurations under hard boundary conditions.

**Fabrication.** All resonators were fabricated with an Anycubic Photon 3D printer which uses UV resin and has 47 um XY-resolution and 10 um Z-resolution. The dimensions of the resonators are supplied in Fig. 1. The resonators were mounted in laser-cut acrylic plates, such that their open ends were perfectly flushed. The spacer was also laser-cut from an acrylic plate and glued to the plate where the bottom resonators were mounted. Its dimensions are supplied in Fig. 1. The spacer was fitted with lateral guiding rails to hold in place the acrylic plate which supports the top resonators. For all laser-cutting, we used the Boss Laser-1630 Laser Engraver. The thickness of all the walls was large enough ($\geq$ 2mm) to justify rigid boundaries in our numerical simulations.

**Experimental Protocols.** The protocol for the acoustic bulk measurements reported in Fig. 3 was as follow. Sinusoidal signals of duration 1 s and amplitude of 0.5 V were produced with a Rigol DG 1022 function generator and applied on a speaker placed in a porthole opened in a resonator of the bottom row. A dbx RTA-M Reference Microphone with a Phantom Power was inserted in a porthole opened in a resonator of the top row and acquired the acoustic signals. To account for the frequency-dependent response of the components, several separate measurements were performed with the structure removed but speaker and microphone kept at the same positions. All our microphone readings are normalized by these reference measurements. The signals were read by a custom LabVIEW code via National Instruments USB-6122 data acquisition box and the data was stored on a computer for graphic renderings. To generate the data in Fig. 3, the microphone was placed on 14 different resonators, starting with the 5th and ending with the 18th top resonators, marked as M5 and M18 in Fig. 1(a). The speaker was placed on the resonator immediately below the microphone and, as a result, it was moved between the bottom resonators marked as S3 and S14 in Fig. 1(a). For each speaker-microphone configuration, the frequency was scanned from 3200 Hz to 6500 Hz in 25 Hz steps.

For the left edge acoustic measurements reported in Fig. 4, the same instrumentation was used but the speaker was inserted in the resonator marked as S1 in Fig. 3(a) and the microphone in the resonator marked as M1 in the same figure. For the right edge acoustic measurements, the speaker was inserted in the resonator marked as S2 in Fig. 3(a) and the microphone in the resonator marked as M2 in the same figure. For both left and right edges, the frequency was swept from 5.1 kHz to 5.8 kHz in steps of 25 Hz. The measurements were repeated with the relative position of the top row of resonators adjusted in steps of 1 mm.

For the dynamic pumping measurements reported in Figs. 1(g,h), ten additional resonators were added to left side of the top array, which was then pushed to right at an average speed that resulted in approximately one pumping cycle per 0.12 s. The speaker was inserted in the bottom resonator marked as S1 in Fig. 4(a) and was
kept on at all times while the microphone was inserted in the bottom resonator marked as M0 in the same figure. The frequency of the acoustic source was first set at 5400 Hz, which corresponds to a topological gap in Fig. 3, and then at 3200 Hz, which corresponds to a non-topological gap. A low-high pass filter of 2 kHz bandwidth centered on the driving frequency was applied to the recordings. Additional experimental data corresponding to different numbers of cycles and speeds are reported in the Supplementary Material.

FIG. 7. Topological edge spectrum under full phason cycle. (a-c) Same as Fig. 4 but this time with the phason driven over its full space $\Omega$.

[1] D. J. Thouless, Quantization of particle transport, Phys. Rev. B 27, 6083 (1983).
[2] S. Nakajima et al, Topological Thouless pumping of ultracold fermions, Nature Phys. 12, 296 (2016).
[3] M. Lohse, C. Schweizer, O. Zilberberg, M. Aidelsburger, I. Bloch, A Thouless quantum pump with ultracold bosonic atoms in an optical superlattice, Nature Phys. 12, 350 (2016).
[4] D. Xiao, Ming-Chie Chang, Qian Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. 82, 1959 (2010).
[5] E. Prodan, Virtual topological insulators with real quantized physics, Phys. Rev. B 91, 245104 (2015).
[6] M. J. Rice, E. J. Mele, Elementary excitations of a linearly conjugated diatomic polymer, Phys. Rev. Lett. 49, 1455 (1982).
[7] I. H. Grinberg, M. Lin, C. Harris, W. A. Benalcazar, C. W. Peterson, T. L. Hughes, G. Bahl, Robust temporal pumping in a magneto-mechanical topological insulator, arXiv:1905.02778v2 (2019).
[8] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, O. Zilberberg, Topological states and adiabatic pumping in quasicrystals, Phys. Rev. Lett. 109, 106402 (2012).
[9] Y. E. Kraus, O. Zilberberg, Quasiperiodicity and topology transcend dimensions, Nature Phys. 12, 624 (2016).
[10] D. J. Apigo, K. Qian, C. Prodan, E. Prodan, Topological Edge Modes by Smart Patterning, Phys. Rev. Materials 2, 124203 (2018).
[11] E. Prodan, Y. Shmalo, The K-Theoretic Bulk-Boundary Principle for Dynamically Patterned Resonators, Journal of Geometry and Physics 135, 135 (2019).
[12] D. J. Apigo, W. Cheng, K. F. Dobiszewski, E. Prodan, C. Prodan, Observation of Topological Edge Modes in a Quasi-Periodic Acoustic Waveguide, Phys. Rev. Lett. 122, 095501 (2019).
[13] X. Ni, K. Chen, M. Weiner, D. J. Apigo, C. Prodan, A. Ali, E. Prodan, A. B. Khanikaev, Observation of Hofstadter butterfly and topological edge states in reconfigurable quasi-periodic acoustic crystals, Commun. Physics 2, 55 (2019).
[14] M. I. N. Rosa, R. K. Pal, J. R. F. Arruda, M. Ruzzene, Edge states and topological pumping in spatially modulated elastic lattices, Phys. Rev. Lett. 123, 034301 (2019).
[15] Y. Xia, A. Erturk, M. Ruzzene, Topological edge states in quasiperiodically locally resonant metastructures, Phys. Rev. Applied 13, 014023 (2020).
[16] H. U. Voss, D. J. Ballon, Topological modes in radiofrequency resonator arrays, Phys. Lett. A 384, 126177 (2020).
[17] B. X. Wang, C. Y. Zhao, Topological quantum optical states in quasiperiodic cold atomic chains, arXiv:2005.05123 (2020).
[18] D. R. Hofstadter, Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields, Phys. Rev. B 14, 2239 (1976).
[19] J. Kellendonk, T. Richter, H. Schulz-Baldes, Edge current channels and Chern numbers in the integer quantum Hall effect, Rev. Math. Phys. 14, 87 (2002).
[20] E. Prodan, H. Schulz-Baldes, Bulk and Boundary Invariants for Complex Topological Insulators: From K-Theory to Physics, (Springer, Berlin, 2016).
[21] J. Bellissard, $K$-theory of $C^\ast$-algebras in solid state physics, in Lecture Notes in Physics, edited by T. Dorlas, M. Hugenholtz, and M. Winnink (Springer-Verlag, Berlin, 1986), Vol. 257, pp. 99156.
[22] J. Bellissard, Gap labeling theorems for Schrödinger operators, in From Number Theory to Physics, edited by M. Waldschmidt, P. Moussa, J.-M. Luck, and C. Itzykson (Springer, Berlin, 1995).