JÓNSSON’S THEOREM IN NON-MODULAR VARIETIES

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Abstract. A version of Jónsson’s Theorem, as previously generalized, holds in non-modular varieties.

Introduction

In this paper, we state and prove a further generalization of Jónsson’s Theorem which holds in non-modular varieties. Our starting point is [4, Theorem 10.1], a generalization of Jónsson’s Theorem due to Hagemann, Hermann, Freese, McKenzie, and Hrushovskii:

Theorem 1. Let $\mathcal{K}$ be a class of algebras, and suppose that $\text{HSP}(\mathcal{K})$ is congruence-modular. If $B \in \text{HSP}(\mathcal{K})$ is subdirectly irreducible and $\alpha$ is the centralizer of the monolith of $B$, then $B/\alpha \in \text{HSP}_u(\mathcal{K})$.

Here $\text{H}$, $\text{S}$, and $\text{P}$ are the standard operators on classes of algebras that close them under homomorphic images, subalgebras, and products respectively, and $\text{P}_u$ is the closure under ultraproducts. Also, recall that the monolith $\mu$ of a subdirectly irreducible algebra $B$ is the minimal nontrivial congruence of $B$, and the centralizer of $\mu$ is the largest congruence $\alpha$ such that $[\mu, \alpha] = \bot$, which exists because of the additivity of the commutator operation.

In §1, we introduce a notion of centrality which we call lax centrality, which reduces, as shown in §2, to the usual, commutator-theoretic notion in modular varieties. This definition was motivated by the requirements of the proof of the above theorem, and by its somewhat tenuous connection, discussed in §3, with the concept of the free intersection, which generalizes the modular commutator to non-modular varieties [2] [3]. In §4, we give the further generalization of Theorem 1 that is the goal of the paper. We make some closing remarks in §5.

Preliminaries

Category Theory. We follow [4] in terminology and notation.

Lattices. We use the symbols $\bot$ and $\top$ to denote the least and greatest elements of a lattice, assuming they exist.

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Universal Algebra. We assume the basic definitions of Universal Algebra, as found, for example, in [1]. However, in the definition of an algebra, we prefer to allow an algebra to be empty.

In §2, we will assume an acquaintance with Commutator Theory for congruence-modular varieties, as developed in [4]. Note that the notation used in [4] (and some other works on Commutator Theory) is different from ours. In particular, the least and greatest elements of a congruence lattice are denoted by 0 and 1 rather than \( \perp \) and \( \top \).

If \( A, B \) are algebras, \( f : A \to B \) is a homomorphism, and \( \alpha \in \text{Con} A \), we denote by \( \vec{f}\alpha \) the congruence of \( B \) generated by \( f(\alpha) \), the set of pairs \( (f(a), f(a')) \) such that \( a \alpha a' \).

**Theorem 2.** Let \( A, B \) be algebras and \( f : A \to B \) a homomorphism. Let \( \alpha \in \text{Con} A \). We have

1. If \( f \) is onto, then \( \vec{f}\alpha = f(\alpha \vee \ker f) \);
2. if \( \beta \in \text{Con} B \), then \( \vec{f}\alpha \leq \beta \iff \alpha \leq f^{-1}(\beta) \); and
3. if \( C \) is another algebra and \( g : B \to C \), then \( (g\vec{f})\alpha = \vec{g}(\vec{f}\alpha) \).

1. Lax Centrality

Let \( V \) be a variety of algebras of some type, and \( B \in V \). Let \( \mu \in \text{Con} B \). We say that \( \alpha \in \text{Con} B \) laxly centralizes \( \mu \) (with respect to the variety \( V \)) if there is an algebra \( C \in V \), an onto homomorphism \( \pi : C \to B \), and \( \beta, \gamma \in \text{Con} C \) such that \( \vec{\pi}\beta \geq \mu, \vec{\pi}\gamma \geq \alpha \), and \( \beta \wedge \gamma = \bot \).

Note that if \( C, \pi, \beta, \gamma \) are given such that \( \beta \wedge \gamma \leq \ker \pi \) and all of the other conditions of the definition are satisfied, then \( \alpha \) laxly centralizes \( \mu \), because we can replace \( C \) by \( C/(\beta \wedge \gamma) \).

Also, note that if \( \alpha \) laxly centralizes \( \mu, \alpha' \leq \alpha \), and \( \mu' \leq \mu \), then \( \alpha' \) laxly centralizes \( \mu' \).

For some notions of centrality, we know that, given \( \mu \in \text{Con} B \), there exists some maximal \( \alpha \) such that \( \alpha \) centralizes \( \mu \). However, we do not know this for lax centrality.

2. Lax Centrality and the Modular Commutator

The following theorem shows that if \( V \) is congruence-modular, the notion of lax centrality is identical to the notion of centrality given by commutator theory.

**Theorem 3.** Let \( V \) be a congruence-modular variety of algebras, and let \( B \in V \) and \( \mu, \alpha \in \text{Con} B \). Then \( \alpha \) laxly centralizes \( \mu \) iff \( [\mu, \alpha] = \bot \).

**Proof.** If \( \alpha \) laxly centralizes \( \mu \), let \( C, \pi, \beta, \gamma \) be given as in the definition of lax centrality. Let \( \ker \pi \) be denoted by \( \theta \). Since \( \beta \wedge \gamma = \bot \), we have \( [\beta, \gamma] = \bot \), which implies that \( [\vec{\pi}\beta, \vec{\pi}\gamma] = [\mu, \alpha] = \bot \).
If \([\mu, \alpha] = \perp\), we let \(C = B(\mu)\) be the subalgebra of \(B \times B\) of pairs related by \(\mu\), and form the pushout square

\[
\begin{array}{ccc}
C & \xleftarrow{\Delta} & B \\
\rho \downarrow & & \downarrow \text{nat } \alpha \\
P & \xleftarrow{\beta} & B/\alpha
\end{array}
\]

where \(\Delta\) is defined by \(\Delta(b) = \langle b, b \rangle\). Let \(\pi : C \to B\) be given by \(\pi : \langle b, c \rangle \mapsto c\), \(\beta = \ker \pi\mu\) where \(\pi_\mu : C \to B\) is given by \(\pi_\mu : \langle b, c \rangle \mapsto b\), and \(\gamma = \ker \rho\) where \(\rho\) is the homomorphism shown in the pushout square. Note that \(\rho\) is onto, because it is a pushout of the onto homomorphism \(\text{nat } \alpha\). In the notation of [4, Chapter IV], we have \(\gamma = \Delta_{\mu, \alpha}\).

Let \(\theta = \ker \pi\).

We have \(\overline{\pi} \beta = \mu\), and we have \(\overline{\pi} \rho = \alpha\) because that is the pushout of \(\rho\) along \(\pi\), and \(\pi \Delta = 1_B\).

To show that \(\beta \land \gamma = \perp\), we must use the fact that \([\mu, \alpha] = \perp\). Indeed, by [4, Theorem 4.9(iv)], we have

\[
\beta \land \gamma = \ker \pi_\mu \land \Delta_{\mu, \alpha}
\]

\[
= \{ \langle \langle x, y \rangle, \langle z, w \rangle \rangle | x = z \text{ and } y [\mu, \alpha] w \}
\]

\[
= \perp.
\]

3. Lax Centrality and the Free Intersection

Let \(B\) be an algebra in a variety \(V\), and \(\alpha, \beta \in \text{Con } B\). Let \(F\) be the relatively free algebra in \(V\) on the set of generators \(\{ x_b, y_b \}_{b \in B}\), i.e., on the disjoint union of two copies of the underlying set of \(B\). The free intersection of \(\alpha\) and \(\beta\) (with respect to \(V\)) is defined [2] [3] as \(\zeta(\overline{\alpha} \land \overline{\beta})\), where \(\overline{\alpha}\) is the congruence on \(F\) generated by the relation \(\{ \langle x_b, y_{b'} \rangle | b \alpha b' \}\) on the generators of \(F\), \(\overline{\beta}\) is the congruence generated by the relation \(\{ \langle y_b, y_{b'} \rangle | b \beta b' \}\), and \(\zeta\) is the onto homomorphism defined on generators by \(x_b \mapsto b\), \(y_b \mapsto b\). Note that \(\overline{\zeta} \overline{\alpha} = \alpha\) and \(\overline{\zeta} \overline{\beta} = \beta\).

We will denote the free intersection of \(\alpha\) and \(\beta\) by \([\alpha, \beta]\). This use of commutator notation is justified by the fact ([3, Theorem 5.14] or [2, Theorem 2.4]) that in modular varieties, the free intersection of two congruences is their commutator. Clearly, we have \([\alpha, \beta] = [\beta, \alpha]\), and the free intersection is monotonic in its arguments.

Let \(B\) belong to a variety \(V\) of algebras of some type, and let \(\mu, \alpha \in \text{Con } B\). Clearly, if \([\mu, \alpha] = \perp\), then \(\alpha\) laxly centralizes \(\mu\).

If the free intersection \([\mu, \alpha]\) is the minimum congruence of \(B\) of the form \(\overline{\pi}(\mu \land \overline{\alpha})\), where \(\pi\) is onto, \(\overline{\pi} \mu = \mu\), and \(\overline{\pi} \alpha = \alpha\), a proof of which is not known to us, that would imply the converse, i.e., that if \(\alpha\) laxly centralizes \(\mu\), then \([\mu, \alpha] = \perp\).
4. LAX CENTRALITY AND JÓNSSON’S THEOREM

We begin with a lemma:

**Lemma 4.** Let $\mathcal{K}$ be a class of algebras of some type, and let $B \in \text{HSP}(\mathcal{K})$ and $\alpha \in \text{Con} B$. If $\alpha$ laxly centralizes $\mu$, then there exist $C, \pi, \beta, \gamma, \nu$, as in the definition of lax centrality, such that in addition we have $C/\beta, C/\gamma \in \text{SP}(\mathcal{K})$.

*Proof.* Let $C_0, \pi_0, \beta_0,$ and $\gamma_0$ witness the fact that $\alpha$ laxly centralizes $\mu$. Since $\beta_0 \wedge \gamma_0 = \bot$, there is a natural one-one homomorphism $\iota_0 : C_0 \to C_0/\beta_0 \times C_0/\gamma_0$.

It is assumed in the definition of lax centrality that $C_0 \in \text{HSP}(\mathcal{K})$, and the same goes for $C_0/\beta_0$ and $C_0/\gamma_0$. Thus, there are $C_\beta, C_\gamma \in \text{SP}(\mathcal{K})$ and onto homomorphisms $\pi_\beta : C_\beta \to C_0/\beta_0, \pi_\gamma : C_\gamma \to C_0/\gamma_0$.

Let $C = \{ \langle c, b, g \rangle \mid c \in C_0, b \in C_\beta, g \in C_\gamma, c/\beta_0 = \pi_\beta(b), c/\gamma_0 = \pi_\gamma(g) \}$. Then the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\iota_1} & C_\beta \times C_\gamma \\
\pi_1 \downarrow & & \downarrow \pi_\beta \times \pi_\gamma \\
C_0 & \xrightarrow{\iota_0} & C_0/\beta_0 \times C_0/\gamma_0
\end{array}
$$

is a pullback square, where $\pi_1 \langle c, b, g \rangle = c$ and $\iota_1 \langle c, b, g \rangle = \langle b, g \rangle$. Clearly $\pi_1$ is onto, and since $\iota_0$ is one-one, $\iota_1$ is one-one as well.

Let $\beta$ be the kernel of the homomorphism $\nu_\beta : C \to C_\beta$ given by $\nu_\beta \langle c, b, g \rangle = b$, and $\gamma$ the kernel of the homomorphism $\nu_\gamma : C \to C_\gamma$ given by $\nu_\gamma \langle c, b, g \rangle = g$. Then $\beta \wedge \gamma = \bot$, because $\iota_1$ is one-one. Also, $\nu_\beta$ and $\nu_\gamma$ are easily seen to be onto. Thus, $C/\beta, C/\gamma \in \text{SP}(\mathcal{K})$.

Let $\pi = \pi_0 \pi_1$. To show that $C, \pi, \beta, \gamma$ witness the lax centrality of $\mu$ by $\alpha$, it suffices to prove that $\overline{\pi_1} \beta = \beta_0$ and $\overline{\pi_1} \gamma = \gamma_0$.

Suppose $c \beta_0 c'$. There is a $b \in C_\beta$ such that $\pi_\beta(b) = c/\beta_0 = c'/\beta_0$. Let $g, g' \in C_\gamma$ be such that $\pi_\gamma(g) = c/\gamma_0$ and $\pi_\gamma(g') = c'/\gamma_0$. Then $\langle c, b, g \rangle \beta \langle c', b, g' \rangle$. It follows that $\overline{\pi_1} \beta \geq \beta_0$.

On the other hand, $(\pi_1)^{-1}(\beta_0) = \{ \langle c, b, g \rangle \mid \beta \langle c', b, g' \rangle \in C^2 \mid c = c' \}$. If $\langle c, b, g \rangle \beta \langle c', b', g' \rangle$, then we have $b = b'$, which implies that $c \beta_0 c'$. Thus, we have $\beta \leq (\pi_1)^{-1}(\beta_0)$. It follows that $\overline{\pi_1} \beta \leq \beta_0$.

That $\overline{\pi_1} \gamma = \gamma_0$ follows by similar arguments.

Now for the main theorem:

**Theorem 5.** Let $\mathcal{K}$ be a class of algebras of some type, and let $B$ be a subdirectly irreducible algebra in $\text{HSP}(\mathcal{K})$, with monolith $\mu$. Let $\alpha \in \text{Con} B$ be maximal for the property that $\alpha$ laxly centralizes $\mu$. Then $B/\alpha \in \text{HSP}_u(\mathcal{K})$.

*Proof.* Let $C, \pi, \beta, \gamma$ be given as in the definition of lax centrality, and such that $C/\beta, C/\gamma \in \text{SP}(\mathcal{K})$, and denote ker $\pi$ by $\theta$.

Let $\{A_i\}_{i \in I}, \{A_i\}_{i \in \overline{I}}$ be tuples of algebras in $\mathcal{K}$, with $I$ and $\overline{I}$ disjoint, such that $C/\beta \hookrightarrow \Pi_I A_i$ and $C/\gamma \hookrightarrow \Pi_{\overline{I}} A_i$. Since $\beta \wedge \gamma = \bot$, we have embeddings $C \hookrightarrow C/\beta \times C/\gamma \hookrightarrow \Pi_{I \cup \overline{I}} A_i$. 

\[\square\]
Let $F$ be a filter on $I \cup \bar{I}$ maximal with respect to the property that $J \in F$ implies $\eta_J \leq \theta$, where $\eta_J$ is the kernel of the natural map from $C$ to $\Pi_{j \in J} A_j$. We have $I \notin F$, because $\beta \leq \theta$.

Let $U$ be an ultrafilter on $I \cup \bar{I}$ extending $F$ and containing $\bar{I}$. Such an ultrafilter exists, because $I \notin F$. We claim that if $J \in U$, then $\eta_J \leq \pi^{-1}(\alpha)$. For, if $J \notin F$, then $\eta_J \leq \theta$. On the other hand, if $J \in U - F$, we have by the maximality of $F$ that neither $J$ nor its complement $\bar{J}$ can be adjoined to $F$. This implies that there is a $K \in F$ such that $\eta_{J \cap K} \leq \theta$ and $\eta_{J \cap K} \leq \theta$. But $\eta_{J \cap K} \land \eta_{J \cap K} = \eta_K \leq \theta$. By the remark following the definition of lax centrality, $\bar{\pi}_{J \cap K}$ laxly centralizes $\bar{\pi}_{J \cap K}$. However, $J \cap K \subseteq J$, so $\eta_J \leq \eta_{J \cap K}$. Also, $\mu \leq \bar{\pi}_{J \cap K}$, because $\mu$ is a monolith. This holds for every $J \in U$, so it holds for $J \cap \bar{I}$ for any particular $J$. We have $\bar{\pi}_{J \cap \bar{I}} \leq \bar{\pi}_{I} = \bar{\pi}_\gamma \geq \alpha$ and laxly centralizing $\mu$, but $\alpha$ is maximal for that property. It follows that for any $J \in U$, $\bar{\pi}_{J \cap \bar{I}} = \alpha$, which verifies the claim.

Let $\eta_H$ be the restriction to $C$ of the ultrafilter congruence on $\Pi_{i \in I} A_i$. Since $\eta_J \leq \pi^{-1}(\alpha)$ for all $J \in U$, we have $\eta_H \leq \pi^{-1}(\alpha)$. Thus, $B/\alpha \in HSP_u(\mathcal{K})$. 

5. Remarks

In the main theorem, because we do not know whether the set of $\alpha$ laxly centralizing $\mu$ admits maximal elements, we had to assume that $\alpha$ is a maximal element of that set. Note that maximality of $\alpha$ is essential, even in the proof of the congruence-modular version of Jónssson’s Theorem. It is not the fact that $\alpha$ centralizes $\mu$ that implies that $B/\alpha \in HSP_u(\mathcal{K})$, but the fact that $\alpha$ is maximal for that property. The case where there is no such maximal $\alpha$, assuming this can indeed happen, is a likely point where further work may clarify the situation and perhaps allow a further elaboration of the theorem.

Lax centrality for the variety $V$ is a $V$-tuple of binary relations $\lambda_B$ on $\text{Con} B$, for $B \in V$. If $V$ is congruence-modular, then $\alpha \lambda_B \mu$ iff $[\mu, \alpha] = \bot$. We might ask, for general $V$, is there a suitable commutator, i.e., a $V$-tuple of binary relations $\kappa_B$ on $\text{Con} B$ for $B \in V$, such that $\alpha \lambda_B \mu$ iff $\mu \kappa_B \alpha = \bot$? There certainly is one, but it may not be very suitable:

$$\mu \kappa_B \alpha = \begin{cases} \bot, & \text{if } \alpha \text{ laxly centralizes } \mu; \\ \top, & \text{otherwise}. \end{cases}$$

In case $V$ is congruence-modular, we have $[\mu, \alpha] \leq [\bar{\pi}_\beta, \bar{\pi}_\gamma] = \bar{\pi}[\beta, \gamma] \leq \bar{\pi}(\beta \land \gamma)$, whenever $\pi : C \to B$ and $\beta, \gamma \in \text{Con} C$ are such that $\bar{\pi}_\beta \geq \mu$ and $\bar{\pi}_\gamma \geq \alpha$. On the other hand, if we take $C = B(\mu)$, $\pi$, $\beta$, and $\gamma$ as in the proof of Theorem 3, we have

$$\bar{\pi}(\beta \land \gamma) = \bar{\pi}\{ (x, y), (z, w) \mid x = z \text{ and } y [\mu, \alpha] w \}$$

$$= [\mu, \alpha].$$

Thus, we can define $[\mu, \alpha]$ to be the meet over all $(C, \pi, \beta, \gamma)$ (where $\bar{\pi} \geq \mu$ and $\bar{\pi}_\gamma \geq \alpha$) of $\bar{\pi}(\beta \land \gamma)$. Whether something similar can be done for arbitrary $V$ is an open question; it may be that the meet can be $\bot$ even when $\alpha$ does not laxly centralize $\mu$. 

The referee rightly pointed out that we did not give an effective method for computing an annihilator for $\mu$, and that such a method would be helpful in effectively applying Theorem 5.

More investigation will be required to fully understand lax centrality, its place in Commutator Theory, and its role in Jónsson’s Theorem.

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