New Inner and Outer Bounds for Gaussian Broadcast Channels with Heterogeneous Blocklength Constraints

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Abstract—We investigate novel inner and outer bounds on the rate region of a 2-user Gaussian broadcast channel with finite, heterogeneous blocklength constraints (HB-GBC). In particular, we introduce a new, modified Sato-type outer bound that can be applied in the finite blocklength regime and does not require the same marginal property. We then develop and analyze composite shell codes, which are suitable for the HB-GBC. Especially, to achieve a lower decoding latency for the user with a shorter blocklength constraint when successive interference cancellation is used, we derive the number of symbols needed to successfully early decode the other user’s message. We numerically compare our derived outer bound to the best known achievable rate regions. Numerical results show that the new early decoding performance in terms of latency reduction is significantly improved compared to the state of the art, and it performs very close to the asymptotic limit.

I. INTRODUCTION

As one of the application areas of 5G and beyond, Ultra-Reliable Low Latency Communication (URLLC) has attracted intensive attention. To achieve low latency, asymptotic assumptions on the codeword sizes are no longer valid, which motivates the finite blocklength analysis. Since the seminal work in [1], there have been several extensions to multi-user channels (e.g. [2]–[5]). Due to different demands on Quality of Service for the wide range of applications in modern communication systems, the latency requirements can differ among users. Therefore, in a realistic model, the two users may be required to decode after having received differing numbers of symbols $n_1$ and $n_2$, which leads to the model of the Gaussian broadcast channel with heterogeneous blocklength constraints (HB-GBC) [6]–[9]. The work [8] shows the possibility of performing superposition coding at the encoder and successive interference cancellation (SIC) at the decoder, even when the interfering codeword is not yet entirely received. This technique is called early decoding (ED). The sufficient conditions for a successful ED are: 1) the channel gain of the user with the stricter blocklength constraint is larger than that of the other user and 2) the shorter blocklength between the two codewords is still sufficiently large.

However, it is unclear how much the rates derived in [9] can be improved. Sato’s outer bound [10], a well-known asymptotic outer bound for broadcast channels, requires the same marginal property (SMP), which is invalid in the finite blocklength regime. Also, the authors of [8] and [9] only consider independent and identically distributed (i.i.d.) Gaussian codebooks, because shell codebooks bring unique challenges to the analysis in the heterogeneous blocklength scenario. Since shell codes are known to have a smaller dispersion than i.i.d. Gaussian codebooks, it is highly desirable to apply them in early decoding.

The main contributions of this paper are twofold. First, in Section III, we present a novel scheme of applying a Sato-type outer bound technique in the finite blocklength regime, where the SMP is invalid in general. Our scheme still allows the minimization of the outer bound with respect to the joint distributions having the same marginals. We then apply this idea to the HB-GBC. To the best of our knowledge, this leads to the first existing outer bound for this channel model. We combine the bounding technique with the information spectrum converse method [11] to derive the outer bounds. Second, we improve the achievable rate and the latency of the ED scheme proposed in [8] for the considered model in Section IV. For this purpose, we introduce a new class of shell codes, namely, composite shell codes, which is suitable for the heterogeneous blocklength scenario and leads to a second-order latency performance close to the asymptotic limit. The derived outer bounds are numerically compared to the best known achievable rate regions.

II. PRELIMINARIES AND SYSTEM MODEL

We consider a two-user Gaussian broadcast channel with heterogeneous blocklength constraints (HB-GBC) and quasi-static block flat-fading. The received signal at user $k$ at time $i \in \{1, ..., n_k\}$ is

$$Y_{k,i} = h_k^*X_i + Z_{k,i},$$

(1)

where $k = 1,2$ and $Z_{1,i} \sim \mathcal{N}(0,1)$ and $Z_{2,i} \sim \mathcal{N}(0,1)$ are i.i.d. and mutually independent additive white Gaussian noise random variables. The transmitter as well as the receivers have perfect knowledge of the channel gains $h_k$. Without loss of generality, in this paper we will always assume $n_1 \geq n_2$. Furthermore, we will consider the case that the shorter blocklength constraint belongs to the user with the larger channel gain, i.e., $h_2 \geq h_1$.

Definition 1. An $(n_1, n_2, M_1, M_2, \epsilon, F^{n_1})$-code for an HB-GBC $W$ consists of:

- two message sets $\mathcal{M}_k = \{1, ..., M_k\}$, $k = 1, 2$,
- one encoder $f : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow F^{n_1}$, where $F^{n_1} \subseteq \mathbb{R}^{n_1}$ is some pre-defined set of feasible codewords,
- two decoders $\phi_k : \mathbb{R}^{n_k} \rightarrow \mathcal{M}_k$, $k = 1, 2$,

such that the average system error probability satisfies

$$\epsilon(n_1, n_2) := \Pr[M_1 \neq M_1 | M_2 \neq M_2] \leq \epsilon.$$  

(2)

We denote by $C_W (n_1, n_2, F^{n_1}, \epsilon)$ the set of all second-order achievable message size pairs for an HB-GBC $W$, i.e., pairs of the form $\log(M_k) = n_k C_k + \sqrt{n_k V_k} + o(\sqrt{n_k})$, $k = 1, 2$, with some positive constants $C_k$ and $V_k$ for which an $(n_1, n_2, M_1, M_2, \epsilon, F^{n_1})$-code exists.
It is common to assume a power constraint $P$ on the codewords. In that case, the feasible set of channel inputs is
\[
F^{n_1} = F_{\text{max}}^{n_1}(P) := \{ x^{n_1} : \| x^{n_1} \|_2^2 \leq n_1 P \}.
\]

If superposition coding is used as an achievable scheme at the encoder, the codewords $x_1^{n_1}$ and $x_2^{n_2}$ are generated independently for user 1 and user 2 and then superimposed to get the channel input $x^{n_1} = x_1^{n_1} + x_2^{n_2}$, where $0^{n_1-n_2}$ is an $(n_1-n_2)$-dimensional vector consisting solely of zeros. In that case, we also call (3) sum power constraint (SPC). Alternatively, we can also impose a power constraint on the individual codewords $x_1^{n_1}$ and $x_2^{n_2}$. In that case, we call the individual power constraint (IPC), the feasible set of codewords is defined as
\[
F^{n_1} = F_{\text{max}}^{n_1}(P_1, P_2) := \{ x^{n_1} = x_1^{n_1} + x_2^{n_2}, 0^{n_1-n_2} : \| x_k^{n_k} \|_2^2 \leq n_k P_k, \; k = 1, 2 \}.
\]

In this paper, we will use individual error probabilities $\epsilon_1$ at user 1 and $\epsilon_{\text{SIC,1}}$ and $\epsilon_{\text{SIC,2}}$ for the different decoding steps of SIC at user 2. In order to conform to the constraint in (2), we have to choose these individual error probabilities properly s.t.
\[
\epsilon_1 + \epsilon_{\text{SIC,1}} + \epsilon_{\text{SIC,2}} \leq \epsilon.
\]

If user 2 can decode user 1’s message $m_1$ from the first $n_2$ received symbols $y_{2,1}, \ldots, y_{2,n_2}$, while fulfilling (2), we call that a successful early decoding [8].

Throughout this paper, we will use the symbol $p := n_2/n_1 \in (0, 1)$ for the blocklength ratio, $\hat{p} := 1 - p$, $C(x) = \frac{1}{2} \log(1 + x)$, $V(x) := \frac{\log^2 x (x+2)}{(x+1)^2}$, $V_C(x) := \log^2 x + 1$ and $Q^{-1}(\cdot)$ is the inverse $Q$-function. All logarithms are taken to the base 2.

### III. A MODIFIED SATO-TYPE OUTER BOUND

In the following, we use $W_{\text{SM}}(W)$ to denote the set of channels having the same conditional marginals $P_{Y_1|X}(y_1|x)$ and $P_{Y_2|X}(y_2|x)$ as a GBC $W = P_{Y_1,Y_2|X}(y_1, y_2|x)$.

#### A. The Same Marginal Property

Sato’s outer bound [10] relies on the SMP, i.e., the property that the capacity regions of Gaussian broadcast channels depend only on the marginal distributions of the channels [12, Lemma 5.1]. The SMP only holds as long as the decoding error probabilities at all the receivers are vanishing. In the finite blocklength regime, however, we deal with non-vanishing error probabilities. For the HB-GBC, the average system error probability in (2) decomposes into
\[
\epsilon^{(n_1, n_2)}(c_1, c_2, c_1^{(n_1, n_2)}, c_2^{(n_1, n_2)}),
\]
where $c_{k}^{(n_1)} := \Pr[M_{k} \neq M_{k}], \; k = 1, 2$, and $c_{1}^{(n_1, n_2)} := \Pr[M_{1} \neq M_{1} \text{ and } M_{2} \neq M_{2}]$. Since a channel $V \in W_{\text{SM}}(W)$ can have different $c_{1}^{(n_1, n_2)}$ from $W$ under otherwise fixed conditions, $C_{W}(n_1, n_2, F^{n_1}, \epsilon)$ and $C_{V}(n_1, n_2, F^{n_1}, \epsilon)$ can also differ. For example, consider a channel $W$ that has independent noise terms and another channel $V \in W_{\text{SM}}(W)$ that has highly correlated noise terms. Then we expect to have a larger $c_{1}^{(n_1, n_2)}$ than $W$ and as a consequence, a smaller $\epsilon^{(n_1, n_2)}$. As a result, we cannot use the SMP in the finite blocklength regime and as a consequence, neither the original Sato-type outer bound.

#### B. A Modified Sato-Type Outer Bound

Even though the SMP is not valid in the finite blocklength regime, $C_{V}(n_1, n_2, F^{n_1}, \epsilon)$ for a channel $V \in W_{\text{SM}}(W)$ can serve as an outer bound of that of $W$ if the error probability constraint for $V$ is relaxed from $\epsilon$ to $2\epsilon$. This is the statement of the following result, which can be used instead of the SMP for finite blocklength outer bounds.

**Lemma 1.** Let $W$ be an HB-GBC with blocklength constraints $n_1$ and $n_2$, $n_1 \geq n_2$, a set of feasible codewords $F^{n_1}$, and average system error probability constraint $\epsilon < 0.25$. Then for all $V \in W_{\text{SM}}(W)$, we have
\[
C_{W}(n_1, n_2, F^{n_1}, \epsilon) \subseteq C_{V}(n_1, n_2, F^{n_1}, 2\epsilon).
\]

Please refer to Appendix A for the proof. We now apply Lemma 1 to the HB-GBC with SPC to derive our main result as follows.

**Theorem 1** (Sato-type Outer Bound with Heterogeneous Blocklengths). For a two-user HB-GBC with feasible set $F_{\text{max}}^{n_1}(P)$ and channel gains $h_2 \geq h_1$, the message sizes of any coding scheme have to fulfill the following inequalities:
\[
\begin{align*}
\log M_k &\leq n_k C(h_k P_k - \sqrt{n_2 V(h_2 P_k)} Q^{-1}(\epsilon)) + 1/2 \log n_k + O(1), \quad k = 1, 2, \\
\log M_1 + \log M_2 &\leq n_1 C_s^*_s(h_1, h_2, p, P) - \frac{1}{2} \log n_1 + O(1),
\end{align*}
\]
where $P_k := \frac{n_k}{n_1}$, $k = 1, 2$, and
\[
C_s^*_s(h_1, h_2, p, P) := p C(h_2 P) + \bar{p} C(h_1 P) + \frac{1}{2} \log \frac{h_2}{1 + h_2 P} - \frac{h_1}{1 + h_1 P} P - \frac{2 h_2^2 P^2}{1 + h_1 P^2}.
\]

We can specialize Theorem 1 to the homogeneous blocklength case by increasing $n_2$ to $n_1$, i.e., inserting $p = 1$ into (9). Since increasing $n_2$ gives the cooperative receiver used in the Sato-type outer bound an additional advantage, the result is still an outer bound to the heterogeneous blocklength case.

**Corollary 1** (Sato-type outer bound with Homogeneous Blocklengths). For a two-user HB-GBC with feasible set $F_{\text{max}}^{n_1}(P)$ and channel gains $h_2 \geq h_1$, the message sizes of any coding scheme have to fulfill the inequalities (8) and
\[
\begin{align*}
\log M_k + \log M_2 &\leq n_1 C(h_2 P) - \sqrt{n_1 V(h_2 P)} Q^{-1}(2\epsilon) + 1/2 \log n_1 + O(1).
\end{align*}
\]

Note that an outer bound similar to Corollary 1 was suggested in [5], but the authors there do not discuss how they take the invalidity of the SMP into account, and therefore, it is unclear whether their bound really holds for finite blocklengths.
IV. IMPROVED ACHIEVABLE SCHEME

A. Composite Shell Codes

The authors in [8] use superposition coding with i.i.d. Gaussian codebooks. Since the dispersion of i.i.d. Gaussian codebooks is suboptimal, it is desirable to apply shell codebooks to the setup. The challenge is that, for early decoding, the codeword is not received completely, so the received codeword does not necessarily fulfill the IPC (4) with equality as required by the shell code definition. Therefore, we introduce a modified class of shell codebooks to solve this problem.

We define the set of composite shell codewords (CSC) as the set of all codewords composed of two sub-codewords that both fulfill (4) with equality, so the cost violation probability is equal to zero:

$$S^{(n_2,n_1)}(P) := \{ x^{n_1} : \|[x_1,\ldots,x_{n_2}]\|_2^2 = n_2P, \|[x_{n_2+1},\ldots,x_{n_1}]\|_2^2 = (n_1-n_2)P\}. \quad (14)$$

For every message $m_1$, we will choose a codeword according to the uniform distribution over $S^{(n_2,n_1)}(P)$, which can be obtained by uniformly and independently selecting the two sub-codewords from their respective power shells.

B. Improved Early Decoding

The following result is an improvement of [8, Theorem 1] and [9, Proposition 1] by using composite shell codes instead of an i.i.d. Gaussian codebook for user 1 under the IPC.

**Theorem 2.** Consider an HB-GBC with channel gains $h_1 \leq h_2$, blocklength constraints $n_1 \geq n_2$, feasible set $F_{\max}^{n_1,n_2}(P_1,P_2)$ and let $n_1$ be sufficiently large. Furthermore, let $\epsilon_1, \epsilon_{\text{SIC,1}}$ and $\epsilon_{\text{SIC,2}}$ be the target error probabilities at user 1 and at the two SIC steps at user 2, respectively, s.t. they fulfill (5). Then successful early decoding can be performed if the following inequality holds:

$$n_2 \geq \frac{\sqrt{V(g_2\bar{P}_1)}Q^{-1}(\epsilon_{\text{SIC,1}})}{2C(g_2\bar{P}_1)} \quad + \quad \sqrt{\frac{V(g_2\bar{P}_1)(Q^{-1}(\epsilon_{\text{SIC,1}}))^2}{4C(g_2\bar{P}_1)^2} + \frac{\log(M_1)}{\bar{C}(g_2\bar{P}_1)}}^2, \quad (15)$$

where $g_2 := \frac{h_2}{1+h_2\bar{P}_2}$ and $\bar{P}_2 := P_2 - \delta, \delta > 0$. In that case, all message size pairs fulfilling the following inequalities:

$$\log(M_1) \leq n_1\bar{C}_1 - \sqrt{n_1V_1Q^{-1}(\epsilon_1)} + O(1), \quad (16)$$

$$\log(M_2) \leq n_2C(h_2\bar{P}_2) - \sqrt{n_2V_{G}(h_2\bar{P}_2)Q^{-1}(\epsilon_{\text{SIC,2}})} + O(1) \quad (17)$$

are achievable, where $g_1 := \frac{h_1}{1+h_1\bar{P}_2}$ and

$$\bar{C}_1 := pC(g_1\bar{P}_1) + \bar{p}C(h_1\bar{P}_1), \quad (18)$$

$$\bar{V}_1 := pV(g_1\bar{P}_1) + \bar{p}V(h_1\bar{P}_1). \quad (19)$$

Please refer to Appendix C for the proof.

V. NUMERICAL RESULTS

A. Outer Bound

In this section, we compare the sum rate upper bounds from Theorem 1 and Corollary 1 to the achievable sum rates under the SPC that were derived in [9]. Therefore, in Fig. 1, we consider a HB-GBC with sum power constraint $P = 10$, $\epsilon = 2 \cdot 10^{-6}$, channel gains $h_1 = 1$ and $h_2 \in (1.5,10]$, fixed blocklength ratio $p = n_2/n_1 = 0.9$ and we vary $n_1 \in [128,2048]$. We assume $\epsilon_1 = 10^{-6}$, $\epsilon_{\text{SIC,1}} = \epsilon_{\text{SIC,2}} = 5 \cdot 10^{-7}$ for the achievable rate expressions from [9].

![Fig. 1. Comparison of the sum rate upper bounds from Theorem 1 (het) and Corollary 1 (hom) with the achievable sum rates based on [9] for two channels](image)

We can observe that for smaller $h_2$, the homogeneous upper bound and the heterogeneous upper bound are closer to each other and the latter always outperforms the former, which confirms the observation at the end of Section III. The gap between achievable rates and outer bounds is decreasing for larger $n_1$, but does not vanish, since Sato’s outer bound is not tight even in the asymptotic regime.

In Fig. 2, we compare the complete outer bounds from Theorem 1 and Corollary 1 to the achievable rate regions from the two achievable schemes early decoding and hybrid NOMA from [9]. We keep most of the system parameters the same as those in Fig. 1, except $h_2 = 50$, $n_1 = 1024$ and $n_2 = 840$. The maximal sum rate is achieved at $(0.22,3.48)$ because of the channel advantage of user 2. Again, the heterogeneous outer bound is tighter than the homogeneous outer bound. The gap between the achievable sum rates and the outer bound is smaller when they are close to the single-user rates. The sum rate upper bound is looser than the single-user rate upper bounds due to the assumption of a cooperative receiver for the sum rate upper bound.

B. Latency Reduction

We now turn to the HB-GBC with IPC and compare the number of necessary symbols $n_2$ for a successful ED from Theorem 2 (“Shell”) to that from [8] (“i.i.d.”) and the asymptotic result from [13], which is also considered in [8]. The considered system parameters are $\epsilon_1 = \epsilon_2 = 10^{-6}, h_1 = 1, P_1 = 8$ and $P_2 = 0.2$ and $n_1$ is varied. We numerically search for the best value of $\epsilon_{\text{SIC,1}} \in (0,\epsilon_2]$.

In Fig. 3, we observe that the improvement of Theorem 2 over [8] is significant. Our results are very close to the asymptotic limit for all considered blocklengths. From these results, we can conclude that ED is a promising technique, especially when CSCs are used.
We have applied this technique to the Gaussian broadcast and $\epsilon$ proofs, please refer to [14].

For a more detailed version of the achievability results based on a technique called early decoding channel with heterogeneous blocklength constraints using two different blocklengths for the i.i.d. Gaussian inputs [8], shell inputs for user 1

Fig. 2. Achievable rate regions using Early Decoding (ED) and Hybrid NOMA (HNOMA) as in [9] compared to our outer bounds under the SPC.

Fig. 3. Number of necessary symbols for a successful early decoding at different blocklengths for the i.i.d. Gaussian inputs [8], shell inputs for user 1 (Theorem 2) and the asymptotic analysis [13].

VI. CONCLUSION

In this paper, we have developed a novel approach of deriving Sato-type outer bounds in the finite blocklength regime. We have applied this technique to the Gaussian broadcast channel with heterogeneous blocklength constraints using two different bounding approaches. We have also improved previous achievability results based on a technique called early decoding by using a composite shell code for the user with the looser blocklength constraint. Numerical results show a significant improvement in terms of latency reduction over the previous early decoding result, and our results are now very close to the asymptotic limit.

APPENDIX

In the appendix, the main steps of the proofs for the results of this paper are outlined. For a more detailed version of the proofs, please refer to [14].

A. Proof of Lemma 1

According to (6), joint distributions with the same marginals affect the probability of a simultaneous decoding error $\tilde{e}_{1/2}^{(n_1,n_2)}$, but not the individual error probabilities $\tilde{e}_1^{(n_1)}$ and $\tilde{e}_2^{(n_2)}$. Therefore, if we consider a new channel $V \in \mathcal{W}_{SM}(W)$ with the same average system error probability $\epsilon$, $C_V(n_1,n_2, F^{n_1}, \epsilon)$ may be larger or smaller than $C_W(n_1,n_2, F^{n_1}, \epsilon)$. If, e.g., the joint error probability $\tilde{e}_{1/2}^{(n_1,n_2)}$ of the new channel is smaller than $\tilde{e}_{1/2}^{(n_1,n_2)}$, then by (6), we have to reduce $\tilde{e}_1^{(n_1)} + \tilde{e}_2^{(n_2)}$ in order to fulfill the same error probability constraint $\epsilon$. However, reducing the individual error probabilities makes the $C_V(n_1,n_2, F^{n_1}, \epsilon)$ smaller and therefore, we cannot simply use it as an outer bound on $C_W(n_1,n_2, F^{n_1}, \epsilon)$. As a result, to derive an outer bound on $C_W(n_1,n_2, F^{n_1}, \epsilon)$, we have to ensure that both the individual and total error probabilities are at least as large as those in the original channel $W$.

When transmitting through $W$, the individual error probabilities of the capacity-achieving code cannot be larger than $\epsilon$, since otherwise, the error probability constraint (2) would be violated. Therefore, we now fix the individual error probabilities of the new channel as $\tilde{e}_1^{(n_1)} = \tilde{e}_2^{(n_2)} = \epsilon$, and by (6), we get

$$\tilde{e}_{1/2}^{(n_1,n_2)} = \frac{\tilde{e}_1^{(n_1)} + \tilde{e}_2^{(n_2)} + \tilde{e}_{1/2}^{(n_1,n_2)} - \epsilon}{2} \geq \epsilon,$$

where $\tilde{e}_{1/2}^{(n_1,n_2)} \leq \min\{\tilde{e}_1^{(n_1)}, \tilde{e}_2^{(n_2)}\} = \epsilon$. Thus, we guarantee that the average system error probability of the new channel is at least as large as that of the original channel, which results in an outer bound on $C_W(n_1,n_2, F^{n_1}, \epsilon)$.

On the other side, since $\tilde{e}_{1/2}^{(n_1,n_2)} \geq 0$, we have $\tilde{e}_{1/2}^{(n_1,n_2)} \leq 2\epsilon$. Taking $\tilde{e}_{1/2}^{(n_1,n_2)} = 2\epsilon$ gives us a uniform upper bound of $\tilde{e}_{1/2}^{(n_1,n_2)}$ w.r.t. all possible $\tilde{e}_{1/2}^{(n_1,n_2)}$, so we can guarantee (7) to hold.

B. Proof of Theorem 1

The upper bounds on the individual rates directly follow from the single-user rate upper bounds [1, Theorem 65].

For the sum rate upper bound, we use the modified information spectrum approach presented in [15]. We consider a GBC where the two noise terms are correlated via a correlation $\rho \in [0,1]$. The modified information density for the cooperative receiver in this scenario is

$$\tilde{i}(x^{n_1}, Y_1^{n_1}, Y_2^{n_2}) = \sum_{i=1}^{n_1} \log \frac{dP_{Y_1,Y_2|X}(Y_1,i,Y_2,i|x_i)}{Q_{Y_1,Y_2}(Y_1,i,Y_2,i)}$$

$$+ \frac{n_1}{n_2+1} \log \frac{dP_{Y_1|X}(Y_1,i|x_i)}{Q_{Y_1}(Y_1,i)},$$

where $P_{Y_1,Y_2|X}$ is the channel distribution and $Q_{Y_1,Y_2}$ is the output distribution induced by i.i.d. Gaussian inputs. Its normalized expectation is

$$\mathbb{E} \left[ \frac{1}{n_1} \tilde{i}(x^{n_1}, Y_1^{n_1}, Y_2^{n_2}) \right] = C_{\rho,1} + \frac{C_{\rho,2}}{n_1} \cdot \sum_{i=n_2+1}^{n_1} x_i^2,$$

where

$$C_{\rho,1} := pC(h_\rho P) + \bar{p}C(h_1 P) + \frac{\bar{p} \log e}{2} \left( \frac{h_\rho}{1+h_\rho P} - \frac{h_1}{1+h_1 P} \right) P$$

and

$$C_{\rho,2} := \frac{\log e}{2} \left( \frac{h_1}{1+h_1 P} - \frac{h_\rho}{1+h_\rho P} \right).$$

Here, we have introduced the abbreviation $h_\rho := (h_1 + h_2 - 2\rho \sqrt{h_1 h_2})/(1-\rho^2)$. The variance of (21) is

$$\text{Var} \left[ \frac{1}{n_1} \tilde{i}(x^{n_1}, Y_1^{n_1}, Y_2^{n_2}) \right] = \mathbb{V}_{\rho,1} + \mathbb{V}_{\rho,2} \cdot \sum_{i=n_2+1}^{n_1} x_i^2,$$

where

$$\mathbb{V}_{\rho,1} = \frac{\bar{p}}{n_1} \cdot \frac{h_1 h_\rho}{(1+h_\rho P)^2} \cdot \left( \frac{h_1}{1+h_1 P} - \frac{h_\rho}{1+h_\rho P} \right)$$

and

$$\mathbb{V}_{\rho,2} = \frac{\bar{p}}{n_1} \cdot \frac{h_1 h_\rho}{(1+h_\rho P)^2} \cdot \left( \frac{h_1}{1+h_1 P} - \frac{h_\rho}{1+h_\rho P} \right).$$
where
\[
V_{p,1} := \frac{\log^2 e}{4} \left( p \cdot 2h_p^2 P^2 + 4h_p P \right) \left( 1 + h_p P \right)^2 + \hat{p} \left( 1 + h_{\hat{p}} P \right)^2 (26)
\]
and
\[
V_{p,2} := \frac{\log^2 e}{4} \left( \frac{4h_1}{(1 + h_1 P)^2} - \frac{4h_\rho}{(1 + h_\rho P)^2} \right). (27)
\]

We now employ the information spectrum converse [11], [15] and apply the Berry-Esseen Theorem to the error probability bound of the information spectrum converse, similar to, e.g., the achievability proofs in [9], as follows:
\[
P_{Y^n;x^n|z^n = y^n} \left( \frac{1}{n} \log^2 e \left( \frac{1}{n} \right) \right) \leq M \sqrt{n} - \frac{1}{n} I_{\rho} - \frac{1}{n} I_{\rho^*} - \frac{1}{n} \left( 1 + P \right) \left( 1 - P \right) \frac{1}{n} I_{\rho} \cdot \frac{1}{n} I_{\rho^*} \cdot \frac{1}{n} \left( 1 + P \right) \frac{1}{n} I_{\rho} \cdot \frac{1}{n} I_{\rho^*} (28)
\]
with a positive constant $B_1$. We observe that $C_{p,2}$ is negative and $V_{p,2}$ is positive if $P^2 \geq \frac{1}{n} I_{\rho}$. We can now upper bound $r_{m,\rho}(n_1)$ by using the following: $n_1 \cdot \frac{1}{n_2 + 1} x_i^2 \geq 0$ in the numerator and $0 \leq \sum_{i=n_2+1}^{n_1} x_i^2 \leq (n_1 - n_2) P$ in the denominator as follows:
\[
r_{m,\rho}(n_1) \leq \frac{C_{p,1} - \log(M/\sqrt{n_1})/n_1}{\sqrt{V_{p,1} + V_{p,2} \cdot (1 - P) \frac{1}{n_1} (P^2 \geq \frac{1}{n_1} I_{\rho})}}, (29)
\]
where $I(\cdot)$ is the indicator function. The remaining steps follow the standard steps for the information spectrum converse, but using $2\epsilon$ instead of $\epsilon$ as the error probability constraint according to Lemma 1.

By standard optimization methods, we find that $\rho = \sqrt{h_1/h_2}$ minimizes the first-order terms of the rate expression, leading to the sum rate upper bound of Theorem 1.

C. Proof of Theorem 2
First, we can derive that the uniform distribution on the composite power shell $S_{n_2,n_1}(P)$ can be described as follows:
\[
P_{Y^n,x^n|z^n = y^n} \left( \frac{1}{n} \log^2 e \left( \frac{1}{n} \right) \right) \leq \frac{M(n_1)}{\sqrt{n_1}} - \frac{1}{n_1} I_{\rho} - \frac{1}{n_1} I_{\rho^*} - \frac{1}{n_1} (1 + P)(1 - P) \frac{1}{n_1} I_{\rho} \cdot \frac{1}{n_1} I_{\rho^*} (30)
\]
where $S_n(r) := \frac{2n^{2n}}{r(2\pi)^n} e^{-n/2}$ and $\Gamma(\cdot)$ is the Gamma function. In the following, we derive an upper bound for a factor that is induced by the change of measure used in the proof. Let $P_{Y^n}(y^n)$ be the output distribution induced by $P_{n_2,n_1}(x^n)$ defined in (30) and $Q(Y^n)(y^n) = N(y^n, 0^n, (1 + P)(1 + P^2), \Gamma(\cdot))$. Then, for $n_1$ sufficiently large,
\[
\frac{dp_{Y^n}(y^n)}{dQ_{Y^n}(y^n)} \leq \tilde{K} := \frac{2}{n_1} (1 + P^2) (31)
\]
Equation (31) is proved by properly extending [4, Appendix B.A] to CSC and is sketched as follows. The Radon-Nikodym derivative for deriving CSC can be bounded by
\[
\frac{dp_{Y^n}(y^n)}{dQ_{Y^n}(y^n)} \leq K_{1}(P, t_1, t_{11}) \cdot e^{-\frac{1}{8} \left( f_1(P, t_1, t_{11}) - f_2(P, t) \right)}, (32)
\]
where $t_1 := \frac{\|y_1 - \cdots - y_{n_2}\|}{\sqrt{n_2}}$, $t_{11} := \frac{\|y_{n_2+1} - \cdots - y_{n_1}\|}{\sqrt{n_{1-n_2}}}$, $t := pt_1 + \frac{p}{t_{11}}$ and
\[
K_1(P, t_1, t_{11}) := \frac{729}{8} \frac{1}{4} \sqrt{1 + 4P t_1} \sqrt{1 + 4P t_{11}} (33)
\]
f_1(P, t_1, t_{11}) := (1 + P) + pt_1 + \frac{p}{t_{11}} - \frac{1}{2} \sqrt{1 + 4P t_{11}} (34)
f_2(P, t) := \ln \left( \frac{2(1 + P)}{1} \right) + \frac{t}{(1 + P)}. (35)

It can be shown that the optimal values of $t_1$ and $t_{11}$ are both $1 + P$. The rest of the proof follows the standard steps in the achievability analysis, where in the end, we drop the higher-order terms and solve for $n_2$ to obtain (15). In contrast to [8], we use the modified dependence testing (DT) bound [4, Theorem 3] for the single-user case with $K$ derived in (31) instead of the standard DT bound. The rate for user 1 follows by adapting the proof of [9, Proposition 1] to composite shell codes. The rate for user 2 is the standard single-user rate for i.i.d. Gaussian codebooks after a successful SIC. For more details, please refer to [14].

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