Periodic alternating 0, \( \pi \)-junction structures as realization of \( \varphi \)-Josephson junctions

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We consider the properties of a periodic structure consisting of small alternating 0- and \( \pi \)-Josephson junctions. We show that depending on the relation between the lengths of the individual junctions, this system can be either in the homogeneous or in the phase-modulated state. The modulated phase appears via a second order phase transition when the mismatch between the lengths of the individual junctions exceeds the critical value. The screening length diverges at the transition point. In the modulated state, the equilibrium phase difference in the structure can take any value from \(-\pi\) to \(\pi\) (\(\varphi\)-junction). The current-phase relation in this structure has very unusual shape with two maxima. As a consequence, the field dependence of the critical current in a small structure is very different from the standard Fraunhofer dependence. The Josephson vortex in a long structure carries partial magnetic flux, which is determined by the equilibrium phase.

The minimum energy of Josephson junction (JJ) usually corresponds to the zero phase difference of the superconducting order parameter. However, long time ago it has been predicted that the JJs with ferromagnetic interlayer (S-F-S junctions) may have ground state with phase difference equal to \(\pi\) ("\(\pi\)-junctions")\(^1\). Only recently this prediction has been experimentally verified\(^1\) and now the controllable fabrication of S-F-S \(\pi\)-junctions becomes possible. Another realization of \(\pi\)-junctions became possible due to the d-wave symmetry of the order parameter in high-temperature superconductors.\(^2\) Recent experiments on YBa\(_2\)Cu\(_3\)O\(_7\)-Nb ramp long JJs of zigzag geometry demonstrated that such junctions are composed of alternating facets of 0- and \(\pi\)-junctions.\(^3\) In a pioneering work Bul'evskii \textit{et al.} demonstrated that the spontaneous Josephson vortex (JV) carrying flux \(\Phi_0/2\) appears at the boundary between 0- and \(\pi\)-JJs. The structure of such semifluxon has been studied in more detail in Refs.\(^4\)\(^-\)\(^7\).

In the present work we study the properties of the periodic array of 0- and \(\pi\)-JJs. We show that depending on the ratio of lengths of 0- and \(\pi\)-JJs, such an array can be either in the homogeneous or modulated state. The second order phase transition between these states takes place when the length mismatch between 0- and \(\pi\)-JJs is small. At the transition point the screening length of the magnetic field diverges. In the modulated states the average phase difference \(\varphi_0\) can take any value between \(-\pi\) and \(\pi\). Further, we call such systems \(\varphi\)-junctions. Such structures were first predicted by Mints\(^8\) in the case of randomly alternating 0- and \(\pi\)-JJs. We study the properties of \(\varphi\)-junctions: effective Josephson current, Josephson length, and Fraunhofer-like oscillations of the critical current. Magnetic properties of long \(\varphi\)-junctions are determined by two types of Josephson vortices carrying partial fluxes \(\Phi_0\varphi_0/\pi\) and \(\Phi_0(\pi-\varphi_0)/\pi\) (see also\(^9\)\(^-\)\(^11\)). We find analytically the shapes of these unusual Josephson vortices. We also demonstrate that at the boundary between \(\varphi\)- and usual JJs a Josephson vortex appears carrying a partial flux \(\Phi_0\varphi_0/2\pi\).

Consider a periodic structure composed of alternating 0- and \(\pi\)-JJs of lengths \(d_0\) and \(d_\pi\) respectively. The energy per period of such a structure is given by

\[
F = \frac{\Phi_0j_c}{2\pi c} \int_{-d_0}^{d_0} \left( \frac{\lambda_j^2}{2} \left( \frac{d\varphi}{dx} \right)^2 + 1 - \cos \varphi \right) dx \\
+ \frac{\Phi_0j_c}{2\pi c} \int_{d_\pi}^{d_\pi} \left( \frac{\lambda_j^2}{2} \left( \frac{d\varphi}{dx} \right)^2 + 1 + \cos \varphi \right) dx,
\]

where \(j_c\) is the Josephson current density, \(\lambda_j\) is the Josephson length of the individual junctions, \(\lambda_j^2 = c\Phi_0/(8\pi^2t_jc)\), and \(t = t_0 + t_\pi = \lambda_j^2 t_\pi + \lambda_1 + \lambda_2\) is the effective junction thickness. For simplicity, we focus on the case when the critical current densities \(j_c\) are the same in 0- and \(\pi\)-JJs. Generalization for different critical currents is straightforward. Possible realization of such system is the zigzag junction between high-\(T_c\) and conventional superconductors as well as S-F-S junctions with periodically modulated thickness of the ferromagnetic interlayer. In the
case of the zigzag structure, $x$ is the coordinate along the zigzag boundary. The ground state phase distribution is determined by the equation

$$\frac{d^2\phi}{dx^2} + j(x) \sin \phi = 0, \quad -d_0 < x < d_\pi, \quad (2)$$

where $j(x) = -\lambda_j^2$ in 0-JJs and $j(x) = \lambda_j^2$ in \(\pi\)-JJs. At the boundaries, the phase and its derivative have to be continuous. Periodicity of \(\phi(x)\) implies that the solution is symmetric with respect to $x = -d_0/2$ and $d_\pi/2$, i.e., \(d\phi/dx|_{x=-d_0/2} = d\phi/dx|_{x=d_\pi/2} = 0\). Eq. (2) always has homogenous solutions \(\phi = 0\) and \(\phi = \pi\). However, these solutions give the ground state only in some range of the ratio $d_0/d_\pi$. In general, Eq. (2) also allows for the inhomogeneous solution. Consider the case $d_\pi < d_0$ and weak modulation around the \(\phi = 0\) state. In this case Eq. (2) can be linearized and its solution is given by

$$\phi = \begin{cases} A \cosh \frac{x+d_0/2}{\lambda_j}, & -d_0 < x < 0, \\ B \cos \frac{x-d_0/2}{\lambda_j}, & 0 < x < d_\pi. \end{cases} \quad (3)$$

Matching the logarithmic derivative $d \ln \phi/dx$ of these solutions at $x = 0$, we obtain the following condition

$$\tanh(d_0/2\lambda_j) = \tan(d_\pi/2\lambda_j), \quad (4)$$

for the onset of the modulated solution. The energy analysis shows that the uniform \(\phi = 0\) solution is favorable at \(\tanh(d_0/2\lambda_j) > \tan(d_\pi/2\lambda_j)\). In the opposite case, $d_\pi > d_0$, the uniform \(\phi = \pi\) solution is favorable at \(\tanh(d_\pi/2\lambda_j) > \tan(d_0/2\lambda_j)\). In the case of an infinite junction split into 0- and \(\pi\)-pieces, condition (1) was first derived in Ref. 10. These results are summarized in the phase diagram shown in Fig. 1. In the limit $d_0 \gg \lambda_j$ the condition for the modulated solution is given by $d_\pi > (\pi/2)\lambda_j$. In the case $d_0, d_\pi \ll \lambda_j$ the region of the modulated solution is given by

$$\frac{|d_0 - d_\pi|}{d_0} < \frac{d_\pi^2}{6\lambda_j^2} \ll 1, \quad (5)$$

i.e., the modulated solution exists only for a very small length mismatch. Since $\lambda_j$ is temperature dependent, the transition into the modulated state may occur with decreasing temperature.

In the case $d_0 = d_\pi = d$, the phase distribution is always modulated. Let us find this distribution. From symmetry $\phi(-x) = \pi - \phi(x)$, in particular, $\phi(0) = \pi/2$. The first integral of Eq. (2) for $x < 0$ is given by

$$\frac{\lambda_j^2}{2} \left( \frac{d\phi}{dx} \right)^2 + \cos \phi = \cos \phi_0 \quad (6)$$

with $\phi_0 = \phi(-d/2)$. The solution is given by

$$x(\phi) = -\frac{\lambda_j}{\sqrt{2}} \int_{\phi_0}^{\pi/2} \frac{d\phi'}{\sqrt{\cos \phi_0 - \cos \phi'}}, \quad (7)$$

where $\phi_0$ is determined by the condition $x(\phi_0) = -d/2$. Dependence $\phi_0(d/\lambda_j)$ and shapes of the phase variation $\phi(x)$ for different values of $d/\lambda_j$ are presented in Fig. 2. For small length $d \ll \lambda_j$, it is easy to find that the phase distribution for $-d < x < 0$ is $\phi(x) = \pi/2 + x(x + d)/\{8\lambda_j^2\}$, i.e., it varies weakly around $\pi/2$.

Consider a structure composed of short alternating 0 and $\pi$-JJs with a small length mismatch, $|d_0 - d_\pi| \ll d_0$. In this case the function $j(x)$ in Eq. (2) oscillates rapidly and we can present $\phi(x)$ as $\phi(x) = \varphi(x) + \xi(x)$, where the function $\varphi(x)$ varies slowly over the distance $d_0$ and $\xi(x)$ oscillates rapidly with $\xi(x) = 0$. In the limit $d_0, \pi \ll \lambda_j$, the oscillating term is small (as one can see from the right panel of Fig. 2) even at $d_0, d_\pi = \lambda_j$, (i) $|\xi(x)| \leq 0.13 \ll 1$ and (ii) the oscillating part is smaller than the average phase). In this case one can use a coarse-grained approximation (see, e.g., Ref. 17), meaning that we can average out the rapidly oscillating function $\xi(x)$ and derive the equation for the slowly varying function $\varphi(x)$ only:

$$\frac{d^2\varphi}{dx^2} = -\frac{d_0d_\pi}{24\lambda_j^2} \sin(2\varphi) + \frac{d_0 - d_\pi}{(d_0 + d_\pi)\lambda_j^2} \sin(\varphi), \quad (8)$$

Here the first term in the right hand side appears due to current modulations and the second term is proportional to the average Josephson current. A similar equation was derived in Ref. 16. The neglected terms in Eq. (8) are smaller by a factor of $(d_0/\lambda_j)^2$. Without the external magnetic field, when condition (5) is fulfilled, the equilibrium phase difference $\varphi_0$ is:

$$\cos \varphi_0 = f \equiv \frac{12\lambda_j^2(d_0 - d_\pi)}{d_0d_\pi(d_0 + d_\pi)}, \quad (9)$$

and for $|f| < 1$, the average phase difference lies in the region $0 < |\varphi_0| < \pi$, i.e., we have the realization of a JJ with arbitrary ground state phase difference (\(\varphi\)-junction). For $|f| > 1$ homogeneous 0- or $\pi$- phase is realized. An example of the dependence of the equilibrium phase $\varphi_0$ on length mismatch is shown in the inset of Fig. 11.

The energy in terms of the coarse-grained phase is

FIG. 2: Left panel: dependence of the minimum phase on the length of junction for $d_0 = d_\pi$. Right panel: shape of phase modulation at different junctions sizes.
This current-phase relation is quite peculiar (see left panel for the phase variation $\tilde{\phi}$ with values of the equilibrium phase $\phi$ is achieved at $\phi = \pm \phi_0$). Curves are marked by the value of $\phi_0$. Unit of current density $j$ in this plot is $d_0^2 j_c / (12 \lambda^2)$. Right panel: dependencies of critical current of small $0, \pi$-structure on the magnetic flux $\Phi$ through the structure, for different values of $\phi_0$. The current unit at the vertical axis is $j_j S d_0^2 / (12 \lambda^2)$, where $S$ is the cross section area of the structure.

Given by

$$F = \Phi_0 j_c \lambda^2 / 2 \pi c \int \left( \frac{1}{2} \left( \frac{d \phi}{dx} \right)^2 + (\cos \phi - f)^2 \right) dx,$$  \hspace{1cm} (10)

with $\lambda^2 = 12 \lambda^2_j / d_0^2 \gg \lambda^2_j$. The coarse-grained Josephson current density flowing through the array is given by

$$j(\phi) = \frac{d_0^2 j_c}{12 \lambda^2_j} \sin(\phi) (f - \cos \phi).$$ \hspace{1cm} (11)

This current-phase relation is quite peculiar (see left panel in Fig. 3). The current has two maxima and two minima, which are achieved at $\cos \phi = \phi = \phi_+ = (f + \sqrt{f^2 + 8}) / 4$. The absolute maximum

$$j_{\text{max}} = \frac{d_0^2 j_c}{48 \lambda^2_j} \sqrt{\frac{1}{2} - \frac{f}{8} (f - \sqrt{f^2 + 8})} \left(3f + \sqrt{f^2 + 8}\right)$$

is achieved at $\phi = \phi_-$. Consider screening of the weak magnetic field by the periodic structure. In the modulated phase linearizing Eq. 8 near $\phi_0, \phi = \phi_0 + \tilde{\phi}$, we obtain a linear equation for the phase variation $\tilde{\phi}$

$$\frac{d^2 \tilde{\phi}}{dx^2} = -\frac{\sin^2 \phi_0}{\lambda^2} \tilde{\phi},$$ \hspace{1cm} (12)

One can see that the effective screening length

$$\lambda_{\text{eff}} = \lambda^2 / \sin \phi_0 = \lambda^2 / \sqrt{1 - f^2}, \text{ for } |f| < 1$$ \hspace{1cm} (13)

diverges at the transition point. In the homogeneous phase, $|f| > 1$, similar derivation gives

$$\lambda_{\text{eff}} = \lambda^2 / \sqrt{|f| - 1}, \text{ for } |f| > 1.$$ \hspace{1cm} (14)

Because $\lambda^2(T)$ diverges at $T \to T_c$ faster than $f(T)$, $\lambda_{\text{eff}}(T)$ is nonmonotonic above the transition point. The minimum value of $\lambda_{\text{eff}}$, $\lambda_{\text{eff}} = 2d_0^2 / \sqrt{3}(d_0 - d_\text{c})$ is reached at $f = 2$.

Exactly at the transition point, $f = 1$, screening is nonlinear and the phase variation obeys equation

$$2 \lambda^2 \phi^2 d^2 \phi / dx^2 = \phi^3.$$ \hspace{1cm} (15)

The general solution of this equation is given by $\phi = \pm 2 \lambda^2 / (x + C)$. Using the phase-field relation $H = (\Phi_0 / 2 \pi t) d \phi / dx$, we derive from this solution that the magnetic field decays inside the structure as

$$H(x) = H_0 \sqrt{1 + x^2 / \left(\frac{\lambda^2 \Phi_0}{\pi t H_0}\right)}^{-2},$$ \hspace{1cm} (16)

where $H_0$ is the external magnetic field. It is interesting to note that the field tail at large $x$ does not depend on the applied field.

$$H(x) = \frac{\lambda^2 \Phi_0}{\pi t x^2}, \text{ for } x \gg \sqrt{\frac{\lambda^2 \Phi_0}{\pi t H_0}}.$$ \hspace{1cm} (17)

In addition to the usual $2\pi$-degeneracy, in the case $f < 1$, the ground state energy is also degenerate with respect to sign change of the equilibrium phase $\phi_0 \to -\phi_0$. This degeneracy leads to the appearance of two new kinds of solitons in which the phase sweeps either from $-\phi_0$ to $\phi_0$ or from $\phi_0$ to $2\pi - \phi_0$. Rewriting Eq. 8 in a more convenient form

$$\frac{d^2 \phi}{dx^2} = \frac{1}{\lambda^2} \sin(\phi) (\cos \phi_0 - \cos \phi),$$

we derive the shapes of the two solitons

$$\varphi_1(x) = 2 \arctan \left( \frac{\tanh (x \sin \phi_0)}{2 \lambda^2} \tan \left(\frac{\varphi_0}{2}\right) \right),$$ \hspace{1cm} (18a)

$$\varphi_2(x) = \pi + 2 \arctan \left( \frac{\tanh (x \sin \phi_0 / 2 \lambda^2)}{\tan \left(\frac{\varphi_0}{2}\right)} \right).$$ \hspace{1cm} (18b)

The soliton energies per unit length are given by

$$\varepsilon_1 = \varepsilon_{J0}(\lambda \lambda_J / 4 \lambda_{\text{c}}) (|\sin \phi_0| - \phi_0 \cos \phi_0),$$ \hspace{1cm} (19a)

$$\varepsilon_2 = \varepsilon_{J0}(\lambda \lambda_J / 4 \lambda_{\text{c}}) (|\sin \phi_0| + (\pi - \phi_0) \cos \phi_0),$$ \hspace{1cm} (19b)

where $\varepsilon_{J0} = 4 \Phi_0 j_c \lambda / (\pi c)$ is the energy of a single soliton in a uniform JJ. At the transition point, $f = 1$, the first soliton vanishes and the second one acquires the following shape, $\varphi_2(x) = \pi + 2 \arctan (x / \lambda_{\text{c}})$. Even though for $d_0 > d_\text{c}$ the first energy is smaller than the second one, both solitons play a role in the magnetic properties of the $\varphi$-junction. This is because without the second solitons the system can not contain two or more first solitons of the same sign due to topological constraints. In the external magnetic field $H$, the energy of the system with a small number of solitons of both kinds, $N_1$ and $N_2$, can be written as

$$E = \varepsilon_1 N_1 + \varepsilon_2 N_2 - \left( N_1 \frac{\varphi_0}{\pi} + N_2 \frac{\pi - \varphi_0}{\pi} \right) \frac{H \Phi_0}{4 \pi},$$ \hspace{1cm} (20)
and has to be supplemented by the topological constrain $|N_1 - N_2| \leq 1$. The penetration scenario is determined by the ratio of the soliton energy $\varepsilon_\alpha$ to its flux $\Phi_\alpha$. Simple analysis shows that for $d_0 > d_\alpha$ these ratios are arranged in the order $\varepsilon_1/\Phi_1 < (\varepsilon_1 + \varepsilon_2)/\Phi_0 < \varepsilon_3/\Phi_2$. These inequalities imply that a single first soliton appears at the field $H_1 = 4\pi^2 \varepsilon_1/\varepsilon_0 \Phi_0$. The soliton lattice, composed of alternating solitons of two types, penetrates into the structure at higher field $H_{c1} = 4\pi (\varepsilon_1 + \varepsilon_2)/\Phi_0$. In the case $d_0 = d_\alpha$, the fields $H_1$ and $H_{c1}$ coincide.

An interesting possibility to have an equilibrium $\varphi_0$ Josephson vortex is realized at the boundary between an alternating structure of 0- and $\pi$- JJ's and usual 0-JJ. We assume that the alternating structure occupies the region $x < 0$, while the 0-JJ with the Josephson penetration length $\Lambda$ occupies a positive semi-axis $x > 0$. In this case the phase difference must vary continuously from $\varphi_0$ for $x \to -\infty$ to zero for $x \to +\infty$. Due to the condition $\lambda_c > \Lambda \sim \lambda_J$, the value $\varphi(0) < 1$, which means that the boundary vortex is almost completely localized in region $x < 0$ and it is equivalent to half of the first soliton $\varphi_0$.

The metastable $\pi - 2\varphi_0$ boundary vortex is also possible and in the limit $\lambda_c \gg \Lambda$ its shape corresponds to the second soliton $\varphi_0$.

Consider a structure with the total length $L$ smaller that the screening length $\lambda_{\text{eff}}$ in an external magnetic field $H$, smaller than $\Phi_0/(d_0 + d_\alpha)$. In this case the coarse-grained approximation for the phase is justified and we can use the current-phase relation (11) where the coarse-grained phase depends on the coordinate as $\varphi(x) = h x + \beta$ with $\eta = 2\pi H/\Phi_0$ and $\beta$ is an arbitrary phase shift. The total current per unit thickness flowing through the structure is given by

$$I(\beta) = \frac{j_c \eta d_0^2}{12 \lambda^2} \int_{-L/2}^{L/2} dx \left( f \sin(hx + \beta) - \frac{\sin(2(hx + \beta))}{2} \right).$$

Calculating the integral, we reduce it to the form

$$I(\beta) = \frac{I_0}{\eta} \sin(\beta) \sin(\eta(f - \cos(\beta) \cos \eta)), \quad (21)$$

where $I_0 = j_c L d_0^2/(12 \lambda^2)$ and $\eta = hL/2 = \pi \Phi/\Phi_0$ and $\Phi = tLH$ is the total flux through the structure. The current reaches a maximum at

$$\cos(\beta) = \frac{\cos \varphi_0 - \sqrt{\cos^2 \varphi_0 + 8 \cos^2(\eta)}}{4 \cos(\eta)}. \quad (22)$$

Shapes of the field dependencies of the critical current at different values of $\varphi_0$ are plotted in the right panel of Fig. 3. These dependencies differ significantly from the Fraunhofer dependence in usual JJ's. They have large a component with half of the main period. At $\varphi_0 = \pi/2$ the dependence has the Fraunhofer shape but the period is two times smaller than in usual case. Note that the observed in the alternating 0-$\pi$ junction structure finite critical current at zero magnetic field is naturally obtained in the framework of our analysis. The theoretical analysis in Ref. 8, giving zero current at $H = 0$, is incomplete, because it does not take into account the current term $\propto \sin(2\varphi)$ coming from the rapidly oscillating phase.

In conclusion, the possibility to realize the transition into the $\varphi-$ junction state with decreasing the temperature from $T_c$, may be very helpful for experimental verification of the predicted effects. In particular, the observation of the striking non-monotonous variation of the screening length with temperature would provide an unambiguous proof of such a transition. Also, the studies of the fine structure of the critical current dependence vs magnetic field and the observation of a periodicity two times smaller than that expected for standard JJ, could be of considerable interest. Finally, note that the SQUID microscope may directly probe the specific shapes of the partial flux vortices. We would like to thank A. Lopatin and W. K. Kwok for critical reading of the manuscript. This work was supported by the U. S. DOE, Office of Science, under contract # W-31-109-ENG-38.

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