Selection principles and countable dimension

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Abstract

We consider player TWO of the game $G_1(A, B)$ when $A$ and $B$ are special classes of open covers of metrizable spaces. Our results give game-theoretic characterizations of the notions of a countable dimensional and of a strongly countable dimensional metric spaces.

The selection principle $S_1(A, B)$ states: There is for each sequence $(A_n : n \in \mathbb{N})$ of elements of $A$ a corresponding sequence $(b_n : n \in \mathbb{N})$ such that for each $n$ we have $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $B$. There are many examples of this selection principle in the literature. One of the earliest examples of it is known as the Rothberger property, $S_1(O, O)$. Here, $O$ is the collection of all open covers of a topological space.

The following game, $G_1(A, B)$, is naturally associated with $S_1(A, B)$: Players ONE and TWO play an inning per positive integer. In the $n$-th inning ONE first chooses an element $O_n$ of $A$; TWO responds by choosing an element $T_n \in O_n$. A play

$$ O_1, T_1, O_2, T_2, \ldots, O_n, T_n, \ldots $$

is won by TWO if $\{T_n : n \in \mathbb{N}\}$ is in $B$, else ONE wins.

TWO has a winning strategy in $G_1(A, B)$

$\downarrow$

ONE has no winning strategy in $G_1(A, B)$

$\downarrow$

$S_1(A, B)$.

There are many known examples of $A$ and $B$ where neither of these implications reverse.

Several classes of open covers of spaces have been defined by the following schema: For a space $X$, and a collection $T$ of subsets of $X$, an open cover $U$ of $X$ is said to be a $T$-cover if $X$ is not a member of $U$, but there is for each $T \in T$ a $U \in U$ with $T \subseteq U$. The symbol $O(T)$ denotes the collection of $T$-covers of $X$. In this paper we consider only $A$ which are of the form $O(T)$ and $B = O$. Several examples of open covers of the form $O(T)$ appear in the literature. To mention just a few: When $T$ is the family of one-element subsets of $X$, $O(T) = O$. When $T$ is the family of finite subsets of $X$, then members of $O(T)$ are called $\omega$-covers in $\mathbb{R}$. The symbol $\Omega$ denotes the family of $\omega$-covers of $\mathbb{R}$. 
X. When $T$ is the collection of compact subsets of $X$, then members of $O(T)$ are called $k$-covers in $[5]$. In $[5]$ the collection of $k$-covers is denoted $K$.

Though some of our results hold for more general spaces, in this paper “topological space” means separable metric space, and “dimension” means Lebesgue covering dimension. We consider only infinite-dimensional separable metric spaces. By classical results of Hurewicz and Tumarkin these are separable metric spaces which cannot be represented as the union of finitely many zerodimensional subspaces.

1 Properties of strategies of player TWO

Lemma 1 Let $F$ be a strategy of TWO in the game $G_1(O(T), B)$. Then there is for each finite sequence $(U_1, \cdots, U_n)$ of elements of $O(T)$, an element $C \in T$ such that for each open set $U \supseteq C$ there is an $U \in O(T)$ such that $U = F(U_1, \cdots, U_n, U)$.

Proof: For suppose on the contrary this is false. Fix a finite sequence $(U_1, \cdots, U_n)$ witnessing this, and choose for each set $C \subseteq X$ which is in $T$ an open set $U_C \supseteq C$ witnessing the failure of Claim 1. Then $U = \{U_C : C \subseteq X$ and $C \in T\}$ is a member of $O(T)$, and as $F(U_1, \cdots, U_n, U) = U_C$ for some $C \in T$, this contradicts the selection of $U_C$.

When $T$ has additional properties, Lemma 1 can be extended to reflect that. For example: The family $T$ is up-directed if there is for each $A$ and $B$ in $T$, a $C$ in $T$ with $A \cup B \subseteq C$.

Lemma 2 Let $T$ be an up-directed family. Let $F$ be a strategy of TWO in the game $G_1(O(T), B)$. Then there is for each $D \in T$ and each finite sequence $(U_1, \cdots, U_n)$ of elements of $O(T)$, an element $C \in T$ such that $D \subseteq C$ and for each open set $U \supseteq C$ there is an $U \in O(T)$ such that $U = F(U_1, \cdots, U_n, U)$.

Proof: For suppose on the contrary this is false. Fix a finite sequence $(U_1, \cdots, U_n)$ and a set $D \in T$ witnessing this, and choose for each set $C \subseteq X$ which is in $T$ and with $D \subseteq C$ an open set $U_C \supseteq C$ witnessing the failure of Claim 1. Then, as $T$ is up-directed, $U = \{U_C : D \subseteq C \subseteq X$ and $C \in T\}$ is a member of $O(T)$, and as $F(U_1, \cdots, U_n, U) = U_C$ for some $C \in T$, this contradicts the selection of $U_C$.

We shall say that $X$ is $T$-first countable if there is for each $D \in T$ a sequence $(U_n : n = 1, 2, \cdots)$ of open sets such that for all $n$, $T \subseteq U_{n+1} \subseteq U_n$, and for each open set $U \supseteq T$ there is an $n$ with $U_n \subseteq U$. Let $\langle T \rangle$ denote the subspaces which are unions of countably many elements of $T$.

Theorem 3 If $F$ is any strategy for TWO in $G_1(O(T), B)$ and if $X$ is $T$-first countable, then there is a set $S \in \langle T \rangle$ such that: For any closed set $C \subseteq X \setminus S$, there is an $F$-play $O_1, T_1, \cdots, O_n, T_n \cdots$ such that $\bigcup_{n=1}^{\infty} T_n \subseteq X \setminus C$.

More can be proved for up-directed $T$: 2
Theorem 4  Let $T$ be up-directed. If $F$ is any strategy for TWO in $G_1(\mathcal{O}(T), \mathcal{O})$ and if $X$ is $T$-first countable, then there is for each set $T \in \langle T \rangle$ a set $S \in \langle T \rangle$ such that: $T \subseteq S$ and for any closed set $C \subset X \setminus S$, there is an $F$-play $O_1, T_1, \cdots, O_n, T_n \cdots$ such that $T \subseteq \bigcup_{n=1}^{\infty} T_n \subseteq X \setminus C$.

Proof: Let $F$ be a strategy of TWO. Let $T$ be a given element of $\langle T \rangle$, and write $T = \bigcup_{n=1}^{\infty} T_n$, where each $T_n$ is an element of $T$.

Starting with $T_1$ and the empty sequence of elements of $\mathcal{O}(T)$, apply Lemma 2 to choose an element $S_0$ of $T$ such that $T_1 \subseteq S_0$, and for each open set $U \supseteq S_0$ there is an element $U \in \mathcal{O}(T)$ with $U = F(U)$. Since $X$ is $T$-first countable, choose for each $n$ an open set $U_n$ such that $U_n \supseteq U_{n+1}$, and for each open set $U$ with $S_0 \subseteq U$ there is an $n$ with $U_n \subseteq U$. Using Lemma 2 choose for each $n$ an element $U_n$ of $\mathcal{O}(T)$ such that $U_n = F(U_n)$.

Now consider $T_2$, and for each $n$ the one-term sequence $(U_n)$ of elements of $\mathcal{O}(T)$. Since $T$ is up-directed, choose an element $T$ of $T$ with $S_0 \cup T_2 \subseteq T$. Applying Lemma 2 to $T$ and $(U_n)$ choose an element $S(n) \in T$ such that for each open set $V \supseteq S(n)$ there is a $U \in \mathcal{O}(T)$ with $U = F(U, U)$. Since $X$ is $T$-first countable, choose for each $k$ an open set $U(n, k) \supseteq S(n)$ such that $U(n, k) \supseteq U(n, k+1) \supseteq S(n)$, and for each open set $V \supseteq S(n)$ there is a $k$ with $V \supseteq U(n, k)$. Then choose for each $n$ and $k$ an element $U(n, k)$ of $\mathcal{O}(T)$ such that $U(n, k) = F(U(n), U(n, k))$.

In general, fix $k$ and suppose we have chosen for each finite sequence $(n_1, \ldots, n_k)$ of positive integers, sets $S(n_1, \ldots, n_k) \in T$, open sets $U(n_1, \ldots, n_k)$ and elements $U(n_1, \ldots, n_k)$ of $\mathcal{O}(T)$, $n < \infty$, such that:

1. $T_1 \cup \cdots \cup T_k \subseteq S(n_1, \ldots, n_k)$;
2. $\{U(n_1, \ldots, n_k) : n < \infty\}$ witnesses the $T$-first countability of $X$ at $S(n_1, \ldots, n_k)$;
3. $U(n_1, \ldots, n_k) = F(U(n_1), \ldots, U(n_1, \ldots, n_k), U(n_1, \ldots, n_k), U(n_1, \ldots, n_k))$;

Now consider a fixed sequence of length $k$, say $(n_1, \ldots, n_k)$. Since $T$ is up-directed choose an element $T$ of $T$ such that $T_{k+1} \cup S(n_1, \ldots, n_k) \subseteq T$. For each $n$ apply Lemma 2 to $T$ and the finite sequence $(U(n_1), \ldots, U(n_1, \ldots, n_k))$: Choose a set $S(n_1, \ldots, n_k) \in T$ such that $T \subseteq S(n_1, \ldots, n_k)$ and for each open set $U \supseteq S(n_1, \ldots, n_k)$ there is a $U \in \mathcal{O}(T)$ such that $U = F(U(n_1), \ldots, U(n_1, \ldots, n_k), U(n_1, \ldots, n_k), U(n_1, \ldots, n_k))$. Since $X$ is $T$-first countable, choose for each $j$ an open set $U(n_1, \ldots, n_k, j)$ such that $U(n_1, \ldots, n_k, j+1) \subseteq U(n_1, \ldots, n_k, j)$, and for each open set $U \supseteq S(n_1, \ldots, n_k)$ there is a $j$ with $U \supseteq U(n_1, \ldots, n_k)$. Then choose for each $j$ an $U(n_1, \ldots, n_k, j) \in \mathcal{O}(T)$ such that $U(n_1, \ldots, n_k, j) = F(U(n_1), \ldots, U(n_1, \ldots, n_k), U(n_1, \ldots, n_k), U(n_1, \ldots, n_k, j))$.

This shows how to continue for all $k$ the recursive definition of the items $S(n_1, \ldots, n_k) \in T$, open sets $U(n_1, \ldots, n_k)$ and elements $U(n_1, \ldots, n_k)$ of $\mathcal{O}(T)$, $n < \infty$ as above.

Finally, put $S = \bigcup_{\tau \in \mathbb{N}} S_{\tau}$. It is clear that $S \in \langle T \rangle$, and that $T \subseteq S$. Consider a closed set $C \subseteq X \setminus S$. Since $C \cap S_0 = \emptyset$, choose an $n_1$ so that
Theorem 5 Let $X$ set $T$ cofinal set $F$ there is an $T$ such that $T \subseteq C$ with $G$ then TWO has a winning strategy in $T$-first countable. Call a subset $G$ of a topological space is a $\text{G}_\delta$-set if it is an intersection of countably many open sets.

When $T$ is a collection of compact sets in a metrizable space $X$ then $X$ is $T$-first countable. Call a subset $C$ of $T$ cofinal if there is for each $T \in T$ a $C \in C$ with $T \subseteq C$. As an examination of the proof of Theorem 4 reveals, we do not need full $T$-first countability of $X$, but only that $X$ is $C$-first countable for some cofinal set $C \subseteq T$. Thus, we in fact have:

**Theorem 5** Let $T$ be up-directed. If $F$ is any strategy for TWO in $G_1(\mathcal{O}(T), \mathcal{O})$ and if $X$ is $C$-first countable where $C \subseteq T$ is cofinal in $T$, then there is for each set $T \in \langle T \rangle$ a set $S \in \langle C \rangle$ such that: $T \subseteq S$ and for any closed set $C \subseteq X \setminus S$, there is an $F$-play

$$O_1, T_1, \cdots, O_n, T_n \cdots$$

such that $T \subseteq \bigcup_{n=1}^{\infty} T_n \subseteq X \setminus C$.

**2 When player TWO has a winning strategy**

Recall that a subset of a topological space is a $\text{G}_\delta$-set if it is an intersection of countably many open sets.

**Theorem 6** If the family $T$ has a cofinal subset consisting of $\text{G}_\delta$ subsets of $X$, then TWO has a winning strategy in $G_1(\mathcal{O}(T), \mathcal{O})$ if, and only if, the space is a union of countably many members of $T$.

**Proof:** $2 \Rightarrow 1$ is easy to prove. We prove $1 \Rightarrow 2$. Let $F$ be a winning strategy for TWO. Let $C \subseteq T$ be a cofinal set consisting of $\text{G}_\delta$-sets.

By Lemma 1 choose $C_0 \in T$ associated to the empty sequence. Since $C$ is cofinal in $T$, choose for $C_0$ a $\text{G}_\delta$ set $G_0$ in $C$ with $C_0 \subseteq G_0$. Choose open sets $(U_n : n \in \mathbb{N})$ such that for each $n$ we have $G_0 \subseteq U_{n+1} \subseteq U_n$, and $G_0 = \cap_{n \in \mathbb{N}} U_n$.

For each $n$ choose by Lemma 1 a cover $U_n \in \mathcal{O}(T)$ with $U_n = F(U_n)$. Choose for each $n$ a $C_n \in T$ associated to $(U_n)$ by Lemma 1. For each $n$ also choose a $\text{G}_\delta$-set $G_n \in C$ with $C_n \subseteq G_n$. For each $n_1$ choose a sequence $(U_{n_1n} : n \in \mathbb{N})$ of open sets such that $G_{n_1} = \cap_{n \in \mathbb{N}} U_{n_1n}$ and for each $n$, $U_{n_1n+1} \subseteq U_{n_1n}$. For each $n_1n_2$ choose by Lemma 1 a cover $U_{n_1n_2} \in \mathcal{O}(T)$ such that $U_{n_1n_2} = F(U_{n_1n_2})$. Choose by Lemma 1 a $C_{n_1n_2} \in T$ associated to $(U_{n_1n_2}, U_{n_1n_2})$, and then choose a $\text{G}_\delta$-set $G_{n_1n_2} \in C$ with $G_{n_1n_2} \subseteq G_{n_1n_2}$, and so on.

Thus we get for each finite sequence $(n_1n_2 \cdots n_k)$ of positive integers

1. a set $C_{n_1 \cdots n_k} \in T$,
2. a $\text{G}_\delta$-set $G_{n_1 \cdots n_k} \in T$ with $C_{n_1 \cdots n_k} \subseteq G_{n_1 \cdots n_k}$,
3. a sequence \((U_{n_1\cdots n_k} : n \in \mathbb{N})\) of open sets with \(G_{n_1\cdots n_k} = \cap_{n \in \mathbb{N}} U_{n_1\cdots n_k}\) and for each \(n\) \(U_{n_1\cdots n_k n+1} \subseteq U_{n_1\cdots n_k n}\), and

4. a \(U_{n_1\cdots n_k} \in \mathcal{O}(T)\) such that for all \(n\)

\[U_{n_1\cdots n_k n} = F(U_{n_1}, \ldots, U_{n_1\cdots n_k n}).\]

Now \(X\) is the union of the countably many sets \(G_{\tau} \in \mathcal{T}\) where \(\tau\) ranges over \(\omega^*\). For if not, choose \(x \in X\) which is not in any of these sets. Since \(x\) is not in \(G_{\emptyset}\), choose \(U_{n_1}\) with \(x \notin U_{n_1}\). Now \(x\) is not in \(G_{n_1}\), so choose \(U_{n_1 n_2}\) with \(x \notin U_{n_1 n_2}\), and so on. In this way we obtain the \(F\)-play

\[U_{n_1}, U_{n_1 n_2}, U_{n_1 n_2 n_3}, \ldots\]

lost by TWO, contradicting that \(F\) is a winning strategy for TWO.

Examples of up-directed families \(\mathcal{T}\) include:

- \([X]^{<\aleph_0}\), the collection of finite subsets of \(X\);
- \(\mathcal{K}\), the collection of compact subsets of \(X\);
- \(\text{KFD}\), the collection of compact, finite dimensional subsets of \(X\);
- \(\text{CFD}\), the collection of closed, finite dimensional subsets of \(X\);
- \(\text{FD}\), the collection of finite dimensional subsets of \(X\).

A subset of a topological space is said to be \textit{countable dimensional} if it is a union of countably many zero-dimensional subsets of the space. A subset of a space is \textit{strongly countable dimensional} if it is a union of countably many closed, finite dimensional subsets. Let \(X\) be a space which is not finite dimensional. Let \(\mathcal{O}_{\text{ctd}}\) denote \(\mathcal{O}(\text{CFD})\), the collection of CFD-covers of \(X\). And let \(\mathcal{O}_{\text{fd}}\) denote \(\mathcal{O}(\text{FD})\), the collection of FD-covers of \(X\).

**Corollary 7** For a metrizable space \(X\) the following are equivalent:

1. \(X\) is strongly countable dimensional.
2. TWO has a winning strategy in \(G_1(\mathcal{O}_{\text{ctd}}, \mathcal{O})\).

**Proof:** 1 \(\Rightarrow\) 2 is easy to prove. To see 2 \(\Rightarrow\) 1, observe that in a metric space each closed set is a \(G_\delta\)-set. Thus, \(T = \text{CFD}\) meets the requirements of Theorem 6.

For the next application we use the following classical theorem of Tumarkin:

**Theorem 8 (Tumarkin)** In a separable metric space each \(n\)-dimensional set is contained in an \(n\)-dimensional \(G_\delta\)-set.

**Corollary 9** For a separable metrizable space \(X\) the following are equivalent:

1. \(X\) is countable dimensional.
2. TWO has a winning strategy in $G_1(O_{fd}, O)$.

Proof: $1 \Rightarrow 2$ is easy to prove. We now prove $2 \Rightarrow 1$. By Tumarkin’s Theorem, $T = FD$ has a cofinal subset consisting of $G_δ$-sets. Thus the requirements of Theorem 6 are met. ♦

Recall that a topological space is perfect if every closed set is a $G_δ$-set.

Corollary 10 In a perfect space the following are equivalent:

1. TWO has a winning strategy in $G_1(K, O)$.

2. The space is $σ$-compact.

Proof: In a perfect space the collection of closed sets are $G_δ$-sets. Apply Theorem 6. ♦

And when $T$ is up-directed, Theorem 6 can be further extended to:

Theorem 11 If $T$ is up-directed and has a cofinal subset consisting of $G_δ$-subsets of $X$, the following are equivalent:

1. TWO has a winning strategy in $G_1(O(T), Γ)$.

2. TWO has a winning strategy in $G_1(O(T), Ω)$.

3. TWO has a winning strategy in $G_1(O(T), O)$.

Proof: We must show that $3 \Rightarrow 1$. Since $X$ is a union of countably many sets in $T$, and since $T$ is up-directed, we may represent $X$ as $\bigcup_{n=1}^{∞} X_n$ where for each $n$ we have $X_n \subset X_{n+1}$ and $X_n \in T$. Now, when ONE presents TWO with $O_n \in O(T)$ in inning $n$, then TWO chooses $T_n \in O_n$ with $X_n \subset T_n$. The sequence of $T_n$’s chosen by TWO in this way results in a $γ$-cover of $X$. ♦

3 Longer games and player TWO

Fix an ordinal $α$. Then the game $G_1^α(A, B)$ has $α$ innings and is played as follows. In inning $β$ ONE first chooses an $O_β \in A$, and then TWO responds with a $T_β \in O_β$. A play

$$0_0, T_0, ..., O_β, T_β, ..., β < α$$

is won by TWO if $\{T_β : β < α\}$ is in $B$; else, ONE wins.

In this notation the game $G_1(A, B)$ is $G_1^ω(A, B)$. For a space $X$ and a family $T$ of subsets of $X$ with $∪T = X$, define:

$$cov_X(T) = \min\{|S| : S \subseteq T \text{ and } X = ∪S\}.$$  

When $X = ∪T$, there is an ordinal $α \leq cov_X(T)$ such that TWO has a winning strategy in $G_1^ω(O(T), O)$. In general, there is an ordinal $α \leq |X|$ such that TWO has a winning strategy in $G_1^ω(O(T), O)$.

$$tp_{S_1(O(T), O)}(X) = \min\{α : \text{TWO has a winning strategy in } G_1^α(O(T), O)\}.$$
3.1 General properties

The proofs of the general facts in the following lemma are left to the reader.

**Lemma 12** 1. If $Y$ is a closed subset of $X$ then $\text{tp}_{S_1(\mathcal{O}(T),\mathcal{O})}(Y) \leq \text{tp}_{S_1(\mathcal{O}(T),\mathcal{O})}(X)$.

2. If $\alpha$ is a limit ordinal and if $\text{tp}_{S_1(\mathcal{O}(T),\mathcal{O})}(X_n) \leq \alpha$ for each $n$, then $\text{tp}_{S_1(\mathcal{O}(T),\mathcal{O})}(\bigcup_{n<\infty} X_n) \leq \alpha$.

We shall now give examples of ordinals $\alpha$ for which TWO has winning strategies in games of length $\alpha$. First we have the following general lemma.

**Lemma 13** Let $X$ be $T$-first countable. Assume that:

1. $T$ is up-directed;

2. $X \notin \langle T \rangle$;

3. $\alpha$ is the least ordinal such that there is an element $B$ of $\langle T \rangle$ such that for any closed set $C \subset X \setminus B$ with $C \notin T$, $\text{tp}_{S_1(\mathcal{O}(T),\mathcal{O})}(C) \leq \alpha$.

Then $\text{tp}_{S_1(\mathcal{O}(T),\mathcal{O})}(X) = \omega + \alpha$.

**Proof:** We must show that TWO has a winning strategy for $G_1^{\omega+\alpha}(\mathcal{O}(T),\mathcal{O})$, and that there is no $\beta < \omega + \alpha$ for which TWO has a winning strategy in $G_1^\beta(\mathcal{O}(T),\mathcal{O})$.

To see that TWO has a winning strategy in $G_1^{\omega+\alpha}(\mathcal{O}(T),\mathcal{O})$, fix a $B$ as in the hypothesis, and for each closed set $F$ disjoint from $B$, fix a winning strategy $\tau_F$ for TWO in the game $G_1^\alpha(\mathcal{O}(T),\mathcal{O})$ played on $F$. Now define a strategy $\sigma$ for TWO in $G_1^{\omega+\alpha}(\mathcal{O}(T),\mathcal{O})$ on $X$ as follows: During the first $\omega$ innings, TWO covers $B$. Let $T_1, T_2, \ldots$ be TWO’s moves during these $\omega$ innings, and put $C = X \setminus \bigcup_{n=1}^\omega T_n$. Then $C$ is a closed subset of $X$, disjoint from $B$. Now TWO follows the strategy $\tau_C$ in the remaining $\alpha$ innings, to also cover $C$.

To see that there is no $\beta < \omega + \alpha$ for which TWO has a winning strategy in $G_1^\beta(\mathcal{O}(T),\mathcal{O})$, argue as follows: Suppose on the contrary that $\beta < \omega + \alpha$ is such that TWO has a winning strategy $\sigma$ for $G_1^\beta(\mathcal{O}(T),\mathcal{O})$ on $X$. We will show that there is a set $S \in \langle T \rangle$ and an ordinal $\gamma < \alpha$ such that for each closed set $C$ disjoint from $S$, TWO has a winning strategy in $G_1^\gamma(\mathcal{O}(T),\mathcal{O})$ on $C$. This gives a contradiction to the minimality of $\alpha$ in hypothesis 3.

We consider cases: First, it is clear that $\alpha \leq \beta$, for otherwise TWO may merely follow the winning strategy on $X$ and relativize to any closed set $C$ to win on $C$ in $\beta < \alpha$ innings, a contradiction. Thus, $\omega + \alpha > \alpha$. Then we have $\alpha < \omega^2$, say $\alpha = \omega \cdot n + k$. Since then $\omega + \alpha = \omega \cdot (n+1) + k$, we have that $\beta$ with $\alpha < \beta < \omega + \alpha$ has the form $\beta = \omega \cdot n + \ell$ with $\ell \geq k$. The other possibility, $\beta = \omega \cdot (n+1) + j$ for some $j < k$, does not occur because it would give $\alpha + \omega > \beta = \omega \cdot n + (\omega + j) = (\omega \cdot n + k) + (\omega + j) = \alpha + \omega + j$.

Let $F$ be a winning strategy for TWO in $G_1^\beta(\mathcal{O}(T),\mathcal{O})$. By the second hypothesis and Theorem [5] we have $\beta > \omega$. By Theorem [4] fix an element $S \in \langle T \rangle$ such that $B \subset S$, and for any closed set $C \subset X \setminus S$, there is an...
versions the elements of the family $C$ and put $\alpha < \omega$ by Tumarkin’s Theorem, for some $V_3$: according to the winning strategy $F$ and put $C$ and $\gamma = \infty$ as strategy to play this game on $C$. Choose an hypothesis. Let ($\beta < \alpha$): $\alpha$ to the minimality of $F$-play ($\beta < \alpha$), and let ($\beta < \alpha$). The following is one of our main tools for these constructions:

Lemma 14 If $G$ is any $G_\delta$-subset of $\mathbb{R}^N$ with $\mathbb{R}_\infty \subset G$, then $G \setminus \mathbb{R}_\infty$ contains a compact nowhere dense subset $C$ which is homeomorphic to $[0, 1]^\mathbb{N}$.

We call $[0, 1]^\mathbb{N}$ the Hilbert cube. From now on assume the Continuum Hypothesis. Let ($F_\alpha : \alpha < \omega_1$) enumerate all the finite dimensional $G_\delta$-subsets of $\mathbb{R}^N$, and let ($C_\alpha : \alpha < \omega_1$) enumerate the $G_\delta$-subsets which contain $\mathbb{R}_\infty$. Recursively choose compact sets $D_\alpha \subset \mathbb{R}^N$, each homeomorphic to the Hilbert cube and nowhere dense, such that $D_0 \subset C_0 \setminus (\mathbb{R}_\infty \cup F_0)$, and for all $\alpha > 0$,

$$D_\alpha \subset (\cap_{\beta < \alpha} C_\beta) \setminus (\mathbb{R}_\infty \cup (\bigcup \{D_\beta : \beta < \alpha\}) \cup (\bigcup_{\beta < \alpha} F_\beta)).$$

Version 1: For each $\alpha$, choose a point $x_\alpha \in D_\alpha$ and put

$$B := \mathbb{R}_\infty \cup \{x_\alpha : \alpha < \omega_1\}.$$

Version 2: For each $\alpha$, choose a strongly countable dimensional set $S_\alpha \subset D_\alpha$ and put

$$B := \mathbb{R}_\infty \cup (\bigcup \{S_\alpha : \alpha < \omega_1\}).$$

Version 3: For each $\alpha$, choose a countable dimensional set $S_\alpha \subset D_\alpha$ and put

$$B := \mathbb{R}_\infty \cup (\bigcup \{S_\alpha : \alpha < \omega_1\}).$$

In all three versions, $B$ is not countable dimensional: Otherwise it would be, by Tumarkin’s Theorem, for some $\alpha < \omega_1$ a subset of $\bigcup_{\beta < \alpha} F_\beta$. Thus TWO has no winning strategy in the games $G_1(\mathcal{O}_\text{std}, \mathcal{O})$ and $G_1(\mathcal{O}_\text{std}, \mathcal{O})$. Also, in all three versions the elements of the family $\mathcal{C}$ of finite unions of the sets $S_\alpha$ are $G_\delta$-sets.
in X, and in fact X is C-first-countable. This is because the Dα’s are compact and disjoint, and ℝN is D-first countable, where D is the family of finite unions of the Dα’s, and this relativizes to X.

For Version 1 TWO has a winning strategy in Gω1(OSd, O) and in Gω1(OSd, C), and for Gωω(K, O). For Version 2 TWO has a winning strategy in Gωω(OSd, O), and for Version 3 TWO has a winning strategy in Gωω(OSd, C).

To see this, note that in the first ω innings, TWO covers ℝ∞. Let {Un : n ∈ ℕ} be TWO’s responses in these innings. Then G = ∪n=1 ∞ Un is an open set containing ℝ∞, and so there is an α < ω1 such that:

**Version 1:** B \ G ⊆ {xβ : β < α} is a closed, countable subset of X and thus closed, zero-dimensional. In inning ω + 1 TWO chooses from ONE’s cover an element containing the set B \ G.

**Version 2:** B \ G ⊆ ∪β<α Sβ. But ∪β<α Sα is strongly countable dimensional, and so TWO can cover this part of B in the remaining ω innings. By Lemma 13 TWO does not have a winning strategy in fewer then ω + ω innings.

**Version 3:** B \ G ⊆ ∪β<α Sβ. But ∪β<α Sα is strongly countable dimensional, and so TWO can cover this part of B in the remaining ω innings. By Lemma 13 TWO does not have a winning strategy in fewer then ω + ω innings.

With these examples established, we can now upgrade the construction as follows: Let α be a countable ordinal for which we have constructed an example of a subspace S of ℝN for which tpS1(OS(T), O)(S) = α. Then choose inside each Dβ a set Cβ for which tpS1(OS(T), O)(Cβ) = α. Then the resulting subset B constructed above has, by Lemma 12 tpS1(OS(T), O)(B) = ω + α. In this way we obtain examples for each of the lengths ω · n and ω · n + 1, for all finite n.

By taking topological sums and using part 2 of Lemma 12 we get examples for ω2.

## 4 Conclusion

One obvious question is whether there is, under the Continuum Hypothesis, for each limit ordinal α subsets Xα and Yα of ℝN such that tpS1(OSd, O)(Xα) = α, and tpS1(OSd, O)(Yα) = α + 1. And the same question can be asked for tpS1(OSd, C).

In [1] countable dimensionality of metrizable spaces were characterized in terms of the selective screenability game. A natural question is how S1(OSd, O) and S1(OSd, C) are related to selective screenability. It is clear that S1(OSd, O) ⇒ S1(OSd, C). The relationship among these two classes and selective screenability is further investigated in [2] where it is shown, for example, that S1(OSd, O) implies selective screenability, but the converse does not hold. Thus, these two classes are new classes of weakly infinite dimensional spaces.
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