TUKEY CLASSIFICATION OF SOME IDEALS ON $\omega$ AND THE LATTICES OF WEAKLY COMPACT SETS IN BANACH SPACES

A. AVILÉS, G. PLEBANEK, AND J. RODRÍGUEZ

ABSTRACT. We study the lattice structure of the family of weakly compact subsets of the unit ball $B_X$ of a separable Banach space $X$, equipped with the inclusion relation (this structure is denoted by $\mathcal{K}(B_X)$) and also with the parametrized family of “almost inclusion” relations $K \subseteq L + \varepsilon B_X$, where $\varepsilon > 0$ (this structure is denoted by $\mathcal{AK}(B_X)$). Tukey equivalence between partially ordered sets and a suitable extension to deal with $\mathcal{AK}(B_X)$ are used. Assuming the axiom of analytic determinacy, we prove that separable Banach spaces fall into four categories, namely: $\mathcal{K}(B_X)$ is equivalent either to a singleton, or to $\omega^{\omega}$, or to the family $K(\mathbb{Q})$ of compact subsets of the rational numbers, or to the family $[c]^{<\omega}$ of all finite subsets of the continuum. Also under the axiom of analytic determinacy, a similar classification of $\mathcal{AK}(B_X)$ is obtained. For separable Banach spaces not containing $\ell^1$, we prove in ZFC that $\mathcal{K}(B_X) \sim \mathcal{AK}(B_X)$ are equivalent to either $\{0\}$, $\omega^\omega$, $K(\mathbb{Q})$ or $[c]^{<\omega}$. The lattice structure of the family of all weakly null subsequences of an unconditional basis is also studied.

1. Introduction

The purpose of this paper is to establish a classification of separable Banach spaces according to how complicated the lattice of weakly compact subsets is. Let $\mathcal{K}(B_X)$ denote the family of all weakly compact subsets of the unit ball $B_X$ of a Banach space $X$, that we view as a partially ordered set endowed with inclusion. The way in which we measure the complexity of $\mathcal{K}(B_X)$ is through Tukey reduction. This has become a standard way to compare partially ordered sets, proven useful to isolate some essential features of the ordered structure \cite{Tukey}. Let us recall that two upwards-directed partially ordered sets are Tukey equivalent if and only if they are order isomorphic to cofinal subsets of some third upwards-directed partially ordered set. Our first main result is the following:

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Theorem A ($\Sigma^1_1D$). If $X$ is a separable Banach space, then $\mathcal{K}(B_X)$ is Tukey equivalent to one of the following partially ordered sets:

(i) either to a singleton,
(ii) or to $\omega^\omega$ (ordered pointwise),
(iii) or to the family $\mathcal{K}(\mathbb{Q})$ of compact subsets of the rational numbers (ordered by inclusion),
(iv) or to the family $[c]<\omega$ of all finite subsets of the continuum (ordered by inclusion).

The symbol ($\Sigma^1_1D$) in this and later results means that the statement holds under the axiom of analytic determinacy (which is consistent with ZFC if one believes in large cardinals). For the reader unfamiliar with determinacy axioms, one can think in practical terms that Theorem A holds for any reasonable Banach space, not arising from any set-theoretic oddity. The case (i) corresponds to reflexivity, so the result can be interpreted as saying that non-reflexive separable Banach spaces split into three categories, depending on three canonical patterns of disposition in the lattice of weakly compact sets. When $X^*$ (the dual of $X$) is separable (for the norm topology), Theorem A holds in ZFC without any determinacy axiom required, and (iv) never happens. This particular case is a corollary to a result of Fremlin [17], who established the Tukey classification of the lattices of compact subsets of coanalytic metric spaces. Our main contribution is therefore the case of non-separable dual. In the case of separable Banach spaces which not containing $\ell^1$, the classification of Theorem A corresponds to the following well-studied classes of spaces:

(i) reflexive spaces,
(ii) non-reflexive spaces with separable dual and the PCP (point of continuity property),
(iii) spaces with separable dual that fail the PCP
(iv) spaces with non-separable dual that do not contain copies of $\ell^1$,
see Theorem 6.1. When $\ell^1$ is present, the classification of Theorem A does not match, to the best of our knowledge, previously studied classes. Let us stress that analytic determinacy and case (iv) rise from the use of a Lusin gap dichotomy [35] for gaps which are more complex than analytic, which is in turn related to the validity of the open graph theorem for projective sets considered in [15]. The following purely combinatorial result is behind our approach to Theorem A:

Theorem B ($\Sigma^1_1D$). Let $\mathcal{I}$ be an analytic family of subsets of $\omega$. Then $\mathcal{I}^\perp$ (ordered by inclusion) is Tukey equivalent to either $\{0\}$, $\omega$, $\omega^\omega$, $\mathcal{K}(\mathbb{Q})$ or $[c]<\omega$.

The information provided by $\mathcal{K}(B_X)$ can be refined by taking into consideration not only the inclusion relation, but also the metric structure of $X$. With this purpose, we
introduce the object \( \mathcal{AK}(B_X) \) which consists again of the family of weakly compact subsets of \( B_X \), but now endowed with the family of binary relations \( K \subset L + \varepsilon B_X \), parametrized by \( \varepsilon > 0 \). We introduce a suitable notion of Tukey reduction that allows to compare such structures among them, and to compare them with ordinary partially ordered sets. For instance, the condition \( \mathcal{AK}(B_X) \sim \omega \) is equivalent to saying that \( X \) is non-reflexive and strongly weakly compactly generated in the sense of [33] (see Theorem 3.14). We obtain the following classification result:

**Theorem C (\( \Sigma_1^1 D \)).** If \( X \) is a separable Banach space, then:

(i) either \( \mathcal{AK}(B_X) \sim \{0\} \),
(ii) or \( \omega \preceq \mathcal{AK}(B_X) \preceq \omega^\omega \),
(iii) or \( \mathcal{AK}(B_X) \sim \mathcal{K}(\mathbb{Q}) \),
(iv) or \( \mathcal{AK}(B_X) \sim [c]^<\omega \).

We enumerate the cases in this way, because (i), (ii), (iii) and (iv) of Theorem C correspond exactly to cases (i), (ii), (iii) and (iv) of Theorem A. Theorem C provides finer information on the structure of the family of weakly compact sets, and its proof requires substantial extra effort with respect to Theorem A. Without assuming any determinacy axiom, we prove that for Banach spaces not containing \( \ell^1 \) the structures \( \mathcal{K}(B_X) \) and \( \mathcal{AK}(B_X) \) are equivalent (Theorem 6.1).

However, we do not know if there is a non-reflexive separable Banach space \( X \) (necessarily with non-separable dual) such that \( \mathcal{K}(B_X) \) is neither Tukey equivalent to \( \omega^\omega \) nor to \( \mathcal{K}(\mathbb{Q}) \) nor to \( [c]^<\omega \) (and the same question for \( \mathcal{AK}(B_X) \)). A possible example in the absence of analytic determinacy is provided in Theorem 7.12 by the construction of a peculiar unconditional basis from a coanalytic set of cardinality \( \aleph_1 \).

The structure of the paper is as follows. In Section 2 we review some basic facts about \( \mathcal{K}(B_X) \). In Section 3 we introduce the main object of our study, the structure \( \mathcal{AK}(B_X) \), and study its basic properties, including the connection with strongly weakly compactly generated spaces; some illustrating examples are given as well. Section 4 is devoted to the proofs of Theorems A and B while Section 5 deals with the proof of Theorem C. Once these general classification results are established, in the next two sections we try to identify which Banach spaces \( X \) satisfy that \( \mathcal{AK}(B_X) \) is equivalent to some of our five basic posets (\( \{0\}, \omega, \omega^\omega, \mathcal{K}(\mathbb{Q}) \) and \([c]^<\omega\)) in the following cases: spaces without copies of \( \ell^1 \) in Section 6 and spaces with unconditional basis in Section 7.

**Terminology.** Throughout this paper \( X \) is a (real) Banach space. The weak topology on \( X \) is denoted by \( w \) and the weak* topology on \( X^* \) is denoted by \( w^* \). The norm on \( X \) is denoted by \( \| \cdot \| \) or \( \| \cdot \|_X \) if needed explicitly. By a subspace of \( X \) we mean a closed linear subspace. Given a family \( \{X_i\}_{i \in I} \) of Banach spaces, the symbols \( \bigoplus_{i \in I} X_i \) and \( \bigoplus_{i \in I} X_i \) stand for their \( c_0 \)-sum and \( \ell^p \)-sum (\( 1 \leq p \leq \infty \)), respectively.
The set of all natural numbers (identified as the first infinite ordinal) is denoted by \(\omega = \{0, 1, \ldots\}\), while we write \(N = \{1, 2, \ldots\}\). Given a set \(S\), we denote by \([S]^{<\omega}\) the set of all finite subsets of \(S\), the cardinality of \(S\) is denoted by \(|S|\) and we write \(\mathcal{P}(S)\) for the power set of \(S\).

The Cantor set (the set of all infinite sequences of 0’s and 1’s) is denoted by \(2^{\omega}\) and we write \(2^{<\omega}\) for the dyadic tree, that is, the set of all finite sequences of 0’s and 1’s (the empty sequence is included here). Given \(t, s \in 2^{<\omega} \cup 2^\omega\), we write \(t \subseteq s\) if \(s\) extends \(t\). Given \(\sigma \in 2^\omega\) and \(m < \omega\), we write \(\sigma|_m\) to denote the unique element of \(2^{<\omega}\) such that \(\text{length}(\sigma|_m) = m\) and \(\sigma|_m \subseteq \sigma\). The concatenation of \(t, s \in 2^{<\omega}\) is denoted by \(t \bowtie s\). For every \(t = (t_0, \ldots, t_n) \in 2^{<\omega}\) we define \(t \bowtie \emptyset \in 2^\omega\) by \((t_0, \ldots, t_n, 0, 0, \ldots)\).

2. The lattice \(\mathcal{K}(B_X)\)

We first recall the concept of Tukey reduction between arbitrary binary relations. A suitable reference for the basic Tukey ordering theory is [18].

**Definition 2.1.** Let \((U, R)\) and \((V, S)\) be two sets equipped with binary relations (so \(R \subseteq U \times U\) and \(S \subseteq V \times V\)).

- A function \(f : U \to V\) is said to be a Tukey function if for every \(v_0 \in V\) there is \(u_0 \in U\) such that, for every \(u \in U\), the following implication holds:
  \[
  (f(u), v_0) \in S \implies (u, u_0) \in R.
  \]

- \((U, R)\) is said to be Tukey reducible to \((V, S)\) if there is a Tukey function \(f : U \to V\). In this case, we write \((U, R) \preceq (V, S)\) or simply \(U \preceq V\).

- \((U, R)\) and \((V, S)\) are said to be Tukey equivalent if both \((U, R) \preceq (V, S)\) and \((V, S) \preceq (U, R)\). In this case, we write \((U, R) \sim (V, S)\) or simply \(U \sim V\).

- We write \((U, R) \prec (V, S)\), or simply \(U \prec V\), whenever \((U, R) \preceq (V, S)\) but \((U, R)\) and \((V, S)\) are not Tukey equivalent.

Typically \(R\) and \(S\) are partial orders on the corresponding sets; note that in such a case a function \(f : U \to V\) is Tukey if and only if the preimage of every bounded subset of \(V\) is bounded above in \(U\).

If \(P\) is a partially ordered set (poset for short), then \(\text{cf}(P)\) denotes its cofinality (i.e. the least cardinality of a cofinal subset of \(P\)) and \(\text{add}_\omega(P)\) is the least cardinality of a set \(A \subseteq P\) which is not \(\sigma\)-bounded (i.e. \(A\) cannot be written as the union of countably many bounded above sets); we use the convention \(\text{add}_\omega(P) = \infty\) whenever \(P\) is \(\sigma\)-bounded. If \(Q\) is another partially ordered set and \(P \preceq Q\), then \(\text{add}_\omega(P) \geq \text{add}_\omega(Q)\) and \(\text{cf}(P) \leq \text{cf}(Q)\), see [18] Theorem 1J.

Given any topological space \(E\), we write \(\mathcal{K}(E)\) for the family of all compact subsets of \(E\). In the sequel we always assume that \(\mathcal{K}(E)\) is equipped with the relation of inclusion, i.e. we consider the partially ordered set \((\mathcal{K}(E), \subseteq)\).
It is time to present the five canonical partial orders that will show up once and
again along this paper:
- \{0\}, a singleton.
- \(\omega = \{0, 1, 2, \ldots\}\), endowed with its natural order.
- \(\omega^\omega\), the set of all sequences of natural numbers \(\{p_n\}_{n<\omega}\), endowed with the
pointwise order (i.e. \(\{p_n\}_{n<\omega} \leq \{q_n\}_{n<\omega}\) if and only if \(p_n \leq q_n\) for all \(n < \omega\)).
- \(\mathcal{K}(\mathbb{Q})\), endowed with the inclusion order.
- \([c]^{<\omega}\), endowed with the inclusion order.

The five posets are enumerated in increasing Tukey-complexity, that is
\[
\{0\} \prec \omega \prec \omega^\omega \prec \mathcal{K}(\mathbb{Q}) \prec [c]^{<\omega}
\]
(see [17]). The metric space \(\mathbb{Q}\) (the space of rational numbers) is homeomorphic to the
subset of \(2^\omega\) made up of all eventually zero sequences, and from now on we identify
both spaces. Note that \(\mathbb{Q}\) is Borel (hence coanalytic) in \(2^\omega\), but \(\mathbb{Q}\) is not a Polish
space. The coefficients \(\text{cf}(\cdot)\) and \(\text{add}_\omega(\cdot)\) of the partial orders above are as follows:
\[
\text{add}_\omega([c]^{<\omega}) = \omega_1, \quad \text{add}_\omega(\omega^\omega) = \text{add}_\omega(\mathcal{K}(\mathbb{Q})) = \mathfrak{b},
\]
\[
\text{cf}(\omega^\omega) = \text{cf}(\mathcal{K}(\mathbb{Q})) = \mathfrak{d}, \quad \text{cf}([c]^{<\omega}) = \mathfrak{c},
\]
see [11] and [17, Theorem 16(c)] for the case of \(\mathcal{K}(\mathbb{Q})\).

Our starting point is Fremlin’s classification of \(\mathcal{K}(E)\) when \(E\) is a separable metric
space which is coanalytic in some Polish space, see [17, Theorem 15].

**Theorem 2.2** (Fremlin). *Let \(E\) be a separable metric space which is coanalytic in
some Polish space. Then:*

(i) \(\mathcal{K}(E) \sim \{0\}\) if \(E\) is compact.
(ii) \(\mathcal{K}(E) \sim \omega\) if \(E\) is locally compact, not compact.
(iii) \(\mathcal{K}(E) \sim \omega^\omega\) if \(E\) is Polish, not locally compact.
(iv) \(\mathcal{K}(E) \sim \mathcal{K}(\mathbb{Q})\) if \(E\) is not Polish.

The Cartesian product of any family of partially ordered sets is endowed with
the coordinatewise order unless otherwise specified. We include a short proof of the
following known fact since we did not find a suitable reference for it.

**Lemma 2.3.** \(\mathcal{K}(\mathbb{Q})^\omega \sim \mathcal{K}(\mathbb{Q})\).

*Proof.* \(\mathbb{Q}^\omega\) is a separable metrizable space which is Borel (hence coanalytic) in \((2^\omega)^\omega\),
but \(\mathbb{Q}^\omega\) is not Polish. Hence Theorem [2.2] yields \(\mathcal{K}(\mathbb{Q}^\omega) \sim \mathcal{K}(\mathbb{Q})\). On the other hand,
we clearly have \(\mathcal{K}(\mathbb{Q}) \leq \mathcal{K}(\mathbb{Q})^\omega\). Finally, it is easy to check that the mapping
\[
f : \mathcal{K}(\mathbb{Q})^\omega \to \mathcal{K}(\mathbb{Q}^\omega), \quad f((K_n)) := \prod_{n<\omega} K_n,
\]
is Tukey, so \(\mathcal{K}(\mathbb{Q})^\omega \leq \mathcal{K}(\mathbb{Q}^\omega)\). It follows that \(\mathcal{K}(\mathbb{Q})^\omega \sim \mathcal{K}(\mathbb{Q})\).
\(\square\)
Remark 2.4. If $P$ is any upwards directed partially ordered set, then the mapping $f : P \to [P]^{<\omega}$ given by $f(p) := \{p\}$ is Tukey and so $P \succeq [P]^{<\omega}$.

Let us turn to the Banach space setting. We consider the Banach space $X$ equipped with its weak topology. Thus, for any $A \subseteq X$, the symbol $\mathcal{K}(A)$ denotes the family of all weakly compact subsets of $A$ ordered by inclusion.

Proposition 2.5 below explains that there is essentially no difference between considering $\mathcal{K}(X)$ and $\mathcal{K}(B_X)$. It also establishes that the lowest possible Tukey equivalence class for $\mathcal{K}(B_X)$, after $\{0\}$, is that of $\omega^\omega$. In particular, this excludes $\omega$. The proof of the Tukey reduction $\omega^\omega \leq \mathcal{K}(B_X)$ in (iii) is a simple adaptation of [17, Lemma 11].

Proposition 2.5.

(i) $\mathcal{K}(X) \sim \mathcal{K}(B_X) \times \omega$.

(ii) If $X$ is reflexive, then $\mathcal{K}(B_X) \sim \{0\}$ and $\mathcal{K}(X) \sim \omega$.

(iii) If $X$ is not reflexive, then $\omega^\omega \leq \mathcal{K}(B_X) \sim \mathcal{K}(X)$.

Proof. (i). Every $K \in \mathcal{K}(X)$ is bounded; let $n_K$ denote the least $n \in \mathbb{N}$ such that $K \subseteq nB_X$. The mapping $\tau : \mathcal{K}(X) \to \mathcal{K}(B_X) \times \omega$, where

$$\tau(K) := \left(\frac{1}{n_K}K, n_K\right),$$

is Tukey. Indeed, take any $(L, m) \in \mathcal{K}(B_X) \times \omega$ and define $\hat{L} := m \cdot \overline{\operatorname{acc}}(L) \in \mathcal{K}(X)$ (note that the closed absolutely convex hull $\overline{\operatorname{acc}}(L)$ of $L$ is weakly compact by the Krein-Smulian theorem). If $K \in \mathcal{K}(X)$ satisfies $\tau(K) \leq (L, m)$, then we have $n_K \leq m$ and $(1/n_K)K \subseteq L$, hence $K \subseteq n_KL \subseteq \hat{L}$.

On the other hand, to see that $\mathcal{K}(B_X) \times \omega \preceq \mathcal{K}(X)$ we can take the mapping $\tau' : \mathcal{K}(B_X) \times \omega \to \mathcal{K}(X)$ given by

$$\tau'(K, n) := K \cup \{nx_0\},$$

where $x_0$ is a fixed norm one vector in $X$. We claim that $\tau'$ is Tukey. Indeed, given any $L \in \mathcal{K}(X)$, we choose $n_0 < \omega$ large enough such that $\sup_{x \in L} \|x\| \leq n_0$ and we consider $(L \cap B_X, n_0) \in \mathcal{K}(B_X) \times \omega$. Clearly, if $(K, n) \in \mathcal{K}(B_X) \times \omega$ satisfies $\tau'(K, n) = K \cup \{nx_0\} \subseteq L$, then $(K, n) \leq (L \cap B_X, n_0)$.

(ii). If $X$ is reflexive, then $B_X$ is weakly compact and so $\mathcal{K}(B_X) \sim \{0\}$. By (i) we also have $\mathcal{K}(X) \sim \omega$.

(iii). If $X$ is not reflexive, then $B_X$ is not weakly compact, so there is a sequence $(x_n)$ in $B_X$ without weakly convergent subsequences. Define $f : \mathcal{K}(B_X) \times \omega \to \mathcal{K}(B_X)$ by $f(K, n) := K \cup \{x_0, \ldots, x_n\}$. Then $f$ is a Tukey map. Indeed, take any $L \in \mathcal{K}(B_X)$ and choose $n_0 < \omega$ such that $x_n \notin L$ for all $n \geq n_0$. Clearly, if $(K, n) \in \mathcal{K}(B_X) \times \omega$ satisfies $f(K, n) \subseteq L$, then $(K, n) \leq (L, n_0)$. Hence $\mathcal{K}(B_X) \times \omega \preceq \mathcal{K}(B_X)$. Bearing in mind (i), we conclude that $\mathcal{K}(X) \sim \mathcal{K}(B_X)$. 

For each \( n < \omega \) the ball \( \frac{1}{n+1} B_X \) is not weakly sequentially compact, so we may choose a sequence \((x_{ni})_i\) in \( \frac{1}{n+1} B_X \) without weakly convergent subsequences. We claim that for every \( \varphi \in \omega^\omega \) the set

\[
\tau(\varphi) := \{x_{ni} : i \leq \varphi(n)\} \cup \{0\}
\]

is norm compact. Indeed, let \((y_k)\) be a sequence in \( \tau(\varphi) \). If \((y_k)\) is not eventually 0, we can find a subsequence, not relabeled, of the form \( y_k = x_{n_ki_k} \) where \( i_k \leq \varphi(n_k) \).

Now there are two possibilities:

- There is a further subsequence \((y_{kj})\) such that \( n_{kj} < n_{kj+1} \) for every \( j < \omega \). Then \( \|y_{nkj}\| = \|x_{n_ki_k}\| \leq \frac{1}{n_{kj}+1} \to 0 \) as \( j \to \infty \).

- There exist a further subsequence \((y_{kj})\) and \( n < \omega \) such that \( n_{kj} = n \) for every \( j < \omega \). Since \( i_{kj} \leq \varphi(n) \) for every \( j < \omega \), the sequence \((y_{kj}) = (x_{n_ki_k})\) admits a constant subsequence.

This proves that \( \tau(\varphi) \) is norm compact.

We now check that the map \( \tau : \omega^\omega \to K(B_X) \) is Tukey. To this end, fix \( L \in K(B_X) \) and define \( \varphi_L \in \omega^\omega \) by \( \varphi_L(n) := \max\{i : x_{ni} \in L\} \). Take any \( \varphi \in \omega^\omega \) such that \( \tau(\varphi) \subseteq L \). Clearly, for every \( n < \omega \) we have \( \varphi(n) \leq \varphi_L(n) \), hence \( \varphi \leq \varphi_L \). \( \square \)

**Remark 2.6.** If \( Y \subseteq X \) is a subspace, then \( K(B_Y) \preceq K(B_X) \).

**Proof.** The mapping \( \tau : K(B_Y) \to K(B_X) \) given by \( \tau(K) := K \) is Tukey, because \( L \cap B_Y \in K(B_Y) \) for every \( L \in K(B_X) \). \( \square \)

The space \( X \) is said to have the Point of Continuity Property (PCP for short) if, for every weakly closed bounded set \( A \subseteq X \), the identity mapping on \( A \) has at least one point of weak-to-norm continuity. For instance, every Banach space with the Radon-Nikodým property has the PCP (see e.g. [12, Corollary 3.14]). In [12, Theorem A] it was proved that \((B_X, w)\) is a Polish space if and only if \( X \) has the PCP and \( X^* \) is separable. This equivalence and Theorem 2.2 yield the following result.

**Proposition 2.7.** Suppose \( X^* \) is separable. Then:

(i) \( K(B_X) \sim \{0\} \) if \( X \) is reflexive.

(ii) \( K(B_X) \sim \omega^\omega \) if \( X \) is not reflexive and has the PCP.

(iii) \( K(B_X) \sim K(Q) \) if \( X \) does not have the PCP.

**Proof.** (i) is clear and does not require the separability of \( X^* \). On the other hand, since \( X^* \) is separable, \((B_X, w)\) is separable metrizable and \((B_{X^{**}}, w^*)\) is a compact metrizable space (hence a Polish space). Let \( \{x_k : k < \omega\} \) be a norm dense subset of \( B_X \). Then

\[
B_X = \bigcap_{n<\omega} \left( x_k + \frac{1}{n+1} B_{X^{**}} \right) \cap B_{X^{**}},
\]
so $B_X$ is an $\mathcal{F}_{\sigma\delta}$ subset of $(B_X^*, w^*)$. In particular, $(B_X, w)$ is coanalytic in $(B_X^*, w^*)$.

By Theorem 2.2 we have $\mathcal{K}(B_X) \sim \omega^\omega$ if and only if $(B_X, w)$ is Polish but not locally compact, while $\mathcal{K}(B_X) \sim \mathcal{K}(\mathbb{Q})$ if and only if $(B_X, w)$ is not Polish. The possibility that $\mathcal{K}(B_X) \not\sim \omega$ is excluded by Proposition 2.5(iii). The conclusion now follows from the aforementioned [12, Theorem A].

In view of Proposition 2.7 we have $\mathcal{K}(c_0) \sim \mathcal{K}(\mathbb{Q})$ (as the space $c_0$ fails the PCP, see e.g. [12, Example 3.3]). Bearing in mind Remark 2.6, we get the following:

**Corollary 2.8.** If $X$ contains an isomorphic copy of $c_0$, then $\mathcal{K}(\mathbb{Q}) \preceq \mathcal{K}(B_X)$.

The converse is not valid in general: there exist Banach spaces with separable dual, not containing $c_0$, and failing the PCP (see [19, Section IV]).

The classification of Proposition 2.7 no longer holds without the separability assumption on $X^*$, as we show in Example 2.9 below, which turns out to be a particular case of a result in Section 3 (Corollary 3.17(iv)). The space $L^1[0, 1]$ fails the PCP (see e.g. [8, p. 94]).

**Example 2.9.** Let $X$ be a non-reflexive subspace of $L^1[0, 1]$. Then $\mathcal{K}(B_X) \sim \omega^\omega$.

**Proof.** The reduction $\omega^\omega \preceq \mathcal{K}(B_X)$ follows from Proposition 2.5(iii). We next prove the reduction $\mathcal{K}(B_X) \preceq \mathbb{N}^\mathbb{N}$ (where $\mathbb{N}^\mathbb{N}$ is equipped with the pointwise order).

We denote by $\lambda$ the Lebesgue measure on the Borel $\sigma$-algebra of $[0, 1]$. For a given $f \in L^1[0, 1]$ and $n \in \mathbb{N}$ we denote by $o(f, n)$ the least $k \in \mathbb{N}$ such that, for every Borel set $B \subseteq [0, 1]$, the following implication holds:

$$\text{if } \lambda(B) \leq \frac{1}{k} \text{ then } \left| \int_B f \, d\lambda \right| \leq \frac{1}{n}.$$ 

The classical Dunford-Pettis criterion (see e.g. [1, Theorem 5.2.9]) states that a bounded set $F \subseteq L^1[0, 1]$ is relatively weakly compact if and only if it is uniformly integrable, that is, $\{o(f, \cdot) : f \in F\}$ is bounded above in $\mathbb{N}^\mathbb{N}$. For every $K \in \mathcal{K}(B_X)$ we fix $\varphi_K \in \mathbb{N}^\mathbb{N}$ such that $o(f, \cdot) \leq \varphi_K$ for all $f \in K$. We claim that the mapping $\mathcal{K}(B_X) \to \mathbb{N}^\mathbb{N}$ given by $K \mapsto \varphi_K$ is Tukey. Indeed, fix $\psi \in \mathbb{N}^\mathbb{N}$ and define

$$K := \{f \in B_X : o(f, \cdot) \leq \psi\} = \bigcap_{n \in \mathbb{N}} \left\{f \in B_X : \left| \int_B f \, d\lambda \right| \leq \frac{1}{n} \text{ whenever } \lambda(B) \leq \frac{1}{\psi(n)} \right\}.$$ 

Since $K$ is weakly closed and $o(f, \cdot) \leq \psi$ for every $f \in K$, it follows that $K \in \mathcal{K}(B_X)$. Now, if $L \in \mathcal{K}(B_X)$ satisfies $\varphi_L \leq \psi$, then $L \subseteq K$. This shows that $\mathcal{K}(B_X) \preceq \mathbb{N}^\mathbb{N}$. □
3. The asymptotic structure $\mathcal{AK}(B_X)$

Our inspiration for introducing the notion of asymptotic structure (Definition 3.2 below) is the following class of Banach spaces.

**Definition 3.1.** A Banach space $X$ is called strongly weakly compactly generated (SWCG for short) if there exists a weakly compact set $K \subseteq X$ such that for every $\varepsilon > 0$ and every weakly compact set $L \subseteq X$ there is $n < \omega$ such that $L \subseteq nK + \varepsilon B_X$. In this case, we say that $K$ strongly generates $X$.

This is a well studied class of Banach spaces that includes reflexive spaces, separable spaces with the Schur property and the space $L^1(\mu)$ for any probability measure $\mu$, but excludes $c_0$ and $C[0,1]$ among others. For more information on SWCG spaces, we refer the reader to [20, Section 6.4] and [14, 24, 25, 29, 33].

Observe that the notion of SWCG space cannot be defined in terms of the partially ordered set $K(X)$, as it involves the relations of “almost inclusion” $L \subset K + \varepsilon B_X$ between weakly compact sets, not just the inclusion relation $L \subseteq K$. The following definitions are intended to develop a Tukey theory that takes into account these “almost inclusion” relations.

**Definition 3.2.** An asymptotic structure is a set $P$ endowed with a family of binary relations $\{\leq_t\}_{t>0}$ satisfying:

(i) $p \leq_t p$ for every $p \in P$ and every $t > 0$;
(ii) if $t < s$ and $p_1 \leq_t p_2$, then $p_1 \leq_s p_2$;
(iii) the binary relation $\bigcap_{t>0} \leq_t$ is a partial order on $P$.

Note that every poset $(P, \leq)$ can be viewed naturally as an asymptotic structure by declaring $\leq_t := \leq$ for all $t > 0$.

**Definition 3.3.** Let $P$ and $Q$ be asymptotic structures.

(i) A Tukey reduction $f : P \Rightarrow Q$ is a family of functions $\{f_\varepsilon : P \to Q\}_{\varepsilon>0}$ such that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$f_\varepsilon : (P, \leq_\varepsilon) \to (Q, \leq_\delta) \quad \text{is Tukey},$$

i.e. for every $q_0 \in Q$ there is $p_0 \in P$ such that $p \leq_\varepsilon p_0$ whenever $f_\varepsilon(p) \leq_\delta q_0$.

(ii) We say that $P$ is Tukey reducible to $Q$ (and we write $P \preceq Q$) if there is a Tukey reduction $P \Rightarrow Q$.

(iii) We say that $P$ and $Q$ are Tukey equivalent (and we write $P \simeq Q$) if both $P \preceq Q$ and $Q \preceq P$.

**Remark 3.4.** Tukey reduction between asymptotic structures is transitive.
Lemma 3.6. Let $P$ and $Q$ be asymptotic structures such that $P \preceq Q$. Then:

(i) $\text{cf}(P) \leq \text{cf}(Q)$;

(ii) $\text{add}_\omega(P) \geq \text{add}_\omega(Q)$.

Consequently, $\text{add}_\omega(P) = \text{add}_\omega(Q)$ and $\text{cf}(P) = \text{cf}(Q)$ whenever $P \sim Q$.

Proof. Let $\{f_\varepsilon : P \to Q\}_{\varepsilon > 0}$ be a family of functions giving a Tukey reduction $P \Rightarrow Q$.

(i) Let $D \subseteq Q$ be a cofinal set. For each $\varepsilon > 0$ and each $d \in D$, we fix

- $\delta_\varepsilon > 0$ such that $f_\varepsilon : (P, \leq_\varepsilon) \to (Q, \leq_{\delta_\varepsilon})$ is Tukey;
- $p_{d,\varepsilon} \in P$ such that $p \leq_\varepsilon p_{d,\varepsilon}$ whenever $f_\varepsilon(p) \leq_{\delta_\varepsilon} d$.

We claim that the set $C := \{p_{d,\varepsilon} : d \in D, \varepsilon \in Q^+\}$ is cofinal in $P$. Indeed, take $\varepsilon > 0$ and $p \in P$. Pick a rational $0 < \varepsilon' < \varepsilon$ and use the cofinality of $D$ to find $d \in D$ such that $f_{\varepsilon'}(p) \leq_{\delta_{\varepsilon'}} d$. Then $p \leq_{\varepsilon'} p_{d,\varepsilon'}$ and so $p \leq_\varepsilon p_{d,\varepsilon'}$ as well. This proves that $C$ is cofinal in $P$ and, bearing in mind that $|C| \leq \max\{|\aleph_0|, |D|\}$, we conclude that $\text{cf}(P) \leq \max\{|\aleph_0|, |D|\}$. As $D$ is an arbitrary cofinal subset of $Q$, we get $\text{cf}(P) \leq \text{cf}(Q)$.

(ii) Fix $\kappa < \text{add}_\omega(Q)$ and consider any set $A \subseteq P$ with $|A| \leq \kappa$. We shall check that $A$ is $\sigma$-bounded. Take any $\varepsilon > 0$ and choose $\delta > 0$ such that $f_\varepsilon : (P, \leq_\varepsilon) \to (Q, \leq_{\delta})$ is Tukey. Since $|f_\varepsilon(A)| \leq |A| \leq \kappa < \text{add}_\omega(Q)$, there is a sequence $(q_n)$ in $Q$ such that for every $p \in A$ we have $f_\varepsilon(p) \leq_\delta q_n$ for some $n < \omega$. Now, for each $n < \omega$ we take
Let us present an example of asymptotic structure in the Banach space setting.

**Definition 3.7.** Given $E \subseteq X$, we define an asymptotic structure $\mathcal{AK}(E)$ as follows:

(i) the underlying set is the family of all weakly compact subsets of $E$;
(ii) for every $t > 0$ the binary relation $\leq_t$ is defined by

$$K \leq_t L \iff K \subset L + tB_X.$$ 

In Proposition 3.11 we explore the general relations between $\mathcal{K}(B_X)$, $\mathcal{AK}(B_X)$ and $\mathcal{AK}(X)$. We first need some lemmata. The first one is sometimes called Grothendieck’s test of weak compactness (see e.g. [8, p. 227, Lemma 2]).

**Lemma 3.8.** A bounded set $C \subseteq X$ is relatively weakly compact if and only if for every $\varepsilon > 0$ there is $K \in \mathcal{K}(X)$ such that $C \subseteq K + \varepsilon B_X$.

**Lemma 3.9.** Let $C \subseteq X$ be a bounded set. The following statements are equivalent:

(i) $C$ is relatively weakly compact.
(ii) For every sequence $(x_n)$ in $C$ and every $\delta > 0$ there is $K \in \mathcal{K}(X)$ such that $x_n \in K + \delta B_X$ for infinitely many $n$’s.

**Proof.** The implication (i)⇒(ii) is obvious. Conversely, assume that (ii) holds. It suffices check that every sequence $(x_n)$ in $C$ admits a relatively weakly compact subsequence. By (ii) (taking $\delta = 1$) we can choose $N_0 \subseteq \omega$ infinite and $K_0 \in \mathcal{K}(X)$ such that $x_n \in K_0 + B_X$ for every $n \in N_0$. Condition (ii) applied to the subsequence $(x_n)_{n \in N_0}$ and $\delta = \frac{1}{2}$ ensures the existence of an infinite set $N_1 \subseteq N_0$ and $K_1 \in \mathcal{K}(X)$ such that $x_n \in K_1 + \frac{1}{2}B_X$ for every $n \in N_1$. Continuing in this manner, we can find a decreasing sequence $(N_m)$ of infinite subsets of $\omega$ and a sequence $(K_m)$ in $\mathcal{K}(X)$ such that

$$x_n \in K_m + \frac{1}{m+1}B_X \quad \text{for every } n \in N_m \text{ and } m < \omega.$$ 

Let $(n_m)$ be a strictly increasing sequence such that $n_m \in N_m$ for all $m < \omega$. We claim that $A := \{x_{n_m} : m < \omega\}$ is relatively weakly compact. Indeed, since $C$ is bounded, so is $A$. Fix $\varepsilon > 0$. Choose $m < \omega$ large enough such that $\frac{1}{m+1} \leq \varepsilon$. Given any $k \geq m$, we have $n_k \in N_k \subseteq N_m$ and (3.1) yields $x_{n_k} \in K_m + \frac{1}{m+1}B_X \subseteq K_m + \varepsilon B_X$. Hence

$$A \subseteq \left(\{x_{n_1}, \ldots, x_{n_m}\} \cup K_m\right) + \varepsilon B_X,$$ 

where $\{x_{n_1}, \ldots, x_{n_m}\} \cup K_m \in \mathcal{K}(X)$. An appeal to Lemma 3.8 ensures that $A$ is relatively weakly compact. □
Lemma 3.10. Let \( A \subseteq B_X, \ C \subseteq X \) and \( \varepsilon > 0 \). If \( A \subseteq C + \varepsilon B_X, \) then
\[
A \subseteq \left( \frac{1}{1 + \varepsilon} C \right) \cap B_X + \frac{2\varepsilon}{1 + \varepsilon} B_X.
\]

Proof. Note that
\[
(3.2) \quad A \subseteq C \cap (1 + \varepsilon)B_X + \varepsilon B_X.
\]
Indeed, given any \( x \in A \) we can write \( x = y + z \), where \( y \in C \) and \( z \in \varepsilon B_X \), and so \( \|y\| \leq \|x\| + \|z\| \leq 1 + \varepsilon \). Therefore
\[
A \subseteq C \cap (1 + \varepsilon)B_X + \varepsilon B_X \\
\subseteq \left( \frac{1}{1 + \varepsilon} C \right) \cap B_X + \frac{2\varepsilon}{1 + \varepsilon} B_X,
\]
as required. \( \square \)

Proposition 3.11.

(i) \( \mathcal{AK}(E) \leq \mathcal{K}(E) \) for every \( E \subseteq X \).

(ii) \( \mathcal{AK}(B_X) \leq \mathcal{AK}(X) \).

(iii) \( \mathcal{AK}(B_X) \times \omega \leq \mathcal{AK}(B_X) \) whenever \( X \) is non-reflexive.

(iv) If \( Y \) is a Banach space which is isomorphic to a complemented subspace of \( X \), then \( \mathcal{AK}(B_Y) \leq \mathcal{AK}(B_X) \).

(v) If \( X \) and \( Y \) are isomorphic Banach spaces, then \( \mathcal{AK}(B_X) \sim \mathcal{AK}(B_Y) \).

Proof. (i). A Tukey reduction \( f : \mathcal{AK}(E) \rhd \mathcal{K}(E) \) is defined by taking \( f_{\varepsilon}(K) := K \) for every \( \varepsilon > 0 \) and every weakly compact set \( K \subseteq E \).

(ii). We define \( f_{\varepsilon} : \mathcal{AK}(B_X) \rightarrow \mathcal{AK}(X) \) by \( f_{\varepsilon}(K) := K \) for every \( \varepsilon > 0 \). Let us check that \( \{f_{\varepsilon}\}_{\varepsilon > 0} \) is a Tukey reduction. Fix \( \varepsilon > 0 \), choose \( 0 < \varepsilon' < \min(\varepsilon, 2) \) and take \( \delta := \frac{\varepsilon'}{2 \epsilon'}. \) We shall prove that \( f_{\varepsilon} : (\mathcal{AK}(B_X), \leq_{\varepsilon}) \rightarrow (\mathcal{AK}(X), \leq_{\delta}) \) is Tukey. Indeed, given \( L_0 \in \mathcal{AK}(X) \), we take \( K_0 := (\frac{1}{1 + \delta} L_0) \cap B_X \in \mathcal{AK}(B_X) \). Then Lemma 3.10 implies that \( K \subseteq K_0 + \varepsilon B_X \) for every \( K \in \mathcal{AK}(B_X) \) satisfying \( K \subseteq L_0 + \delta B_X \).

(iii). Since \( X \) is not reflexive, \( B_X \) is not weakly compact and Lemma 3.9 ensures the existence of a sequence \( (x_n) \) in \( B_X \) and a constant \( \delta_0 \geq 0 \) such that for every \( K \in \mathcal{AK}(X) \) the set \( \{n < \omega : x_n \in K + \delta B_X\} \) is finite. For every \( \varepsilon > 0 \) we define
\[
\tau_{\varepsilon} : \mathcal{AK}(B_X) \times \omega \rightarrow \mathcal{AK}(B_X), \quad \tau_{\varepsilon}(K, n) := K \cup \{x_0, \ldots, x_n\}.
\]
We claim that the family \( \{\tau_{\varepsilon}\}_{\varepsilon > 0} \) defines a Tukey reduction \( \mathcal{AK}(B_X) \times \omega \rhd \mathcal{AK}(B_X) \). Indeed, fix \( \varepsilon > 0 \) and take \( \delta := \min(\delta_0, \varepsilon) \). Given \( L_0 \in \mathcal{AK}(B_X) \), there is \( n_0 < \omega \) such that \( x_n \not\in L_0 + \delta B_X \) for all \( n > n_0 \). Therefore, if \( (K, n) \in \mathcal{AK}(B_X) \times \omega \) satisfies \( \tau_{\varepsilon}(K, n) \subseteq L_0 + \delta B_X \), then \( K \subseteq L_0 + \varepsilon B_X \) and \( n \leq n_0 \).
(iv). Let $Z$ be a complemented subspace of $X$ which is isomorphic to $Y$. Let $T : Y \to Z$ be an isomorphism and let $P : X \to X$ be a projection onto $Z$. For every $\varepsilon > 0$ we define the map 

$$f_\varepsilon : \mathcal{A}K(B_Y) \to \mathcal{A}K(B_X), \quad f_\varepsilon(K) := \frac{1}{\|T\|}T(K).$$

Let us check that $\{f_\varepsilon\}_{\varepsilon > 0}$ defines a Tukey reduction $\mathcal{A}K(B_Y) \Rightarrow \mathcal{A}K(B_X)$. Fix $\varepsilon > 0$ and take $\delta > 0$ small enough such that

$$2\delta \|T\| \|T^{-1}\| \|P\| \leq \varepsilon.$$ 

Write $\varepsilon' := \delta \|T\| \|T^{-1}\| \|P\|$. Given $L_0 \in \mathcal{A}K(B_X)$, set

$$K_0 := \left(\frac{\|T\|}{1 + \varepsilon'}T^{-1}(P(L_0))\right) \cap B_Y \in \mathcal{A}K(B_Y).$$

Now if $K \in \mathcal{A}K(B_Y)$ satisfies $f_\varepsilon(K) \subseteq L_0 + \delta B_X$, then $K \subseteq \|T\| \|T^{-1}(P(L_0))\| + \varepsilon' B_Y$ and so Lemma 3.10 implies that

$$K \subseteq K_0 + \frac{2\varepsilon'}{1 + \varepsilon'} B_Y \leq K_0 + \varepsilon B_Y.$$

This finishes the proof of (iv). Finally, (v) follows at once from (iv). \qed

Theorem 7.5 in Section 7 will make clear that the assertion of Proposition 3.11(iv) is no longer true if $Y$ is just an uncomplemented subspace of $X$.

The following question remains open for us.

**Problem 3.12.** Is it true that $\mathcal{A}K(X) \sim \mathcal{A}K(B_X)$ for every non-reflexive $X$?

We presented SWCG Banach spaces as a motivation for introducing the asymptotic structure $\mathcal{A}K(B_X)$. To see how this class of spaces fits in the theory, we need the following elementary characterization (the equivalence (i)$\Leftrightarrow$(ii) was already pointed out in [33, Theorem 2.1]).

**Lemma 3.13.** The following statements are equivalent:

(i) $X$ is SWCG;

(ii) there exist countably many weakly compact sets $\{K_n : n < \omega\}$ in $X$ such that for every $\varepsilon > 0$ and every weakly compact set $L \subset X$ there is $n < \omega$ such that $L \subset K_n + \varepsilon B_X$;

(iii) there exist countably many weakly compact sets $\{S_n : n < \omega\}$ in $B_X$ such that for every $\varepsilon > 0$ and every weakly compact set $L \subset B_X$ there is $n < \omega$ such that $L \subset S_n + \varepsilon B_X$. 


Proof. (i)⇒(ii) is trivial.

(ii)⇒(iii). For each \( n < \omega \) and each \( p \in \mathbb{Q}^+ \), we consider the weakly compact set \( S_{n,p} := pK_n \cap B_X \). Fix \( \varepsilon > 0 \) and a weakly compact set \( L \subseteq B_X \). Choose \( \varepsilon' \in \mathbb{Q}^+ \) such that \( \frac{2\varepsilon'}{1 + \varepsilon'} \leq \varepsilon \). By the assumption, there is \( n < \omega \) such that \( L \subseteq K_n + \varepsilon' B_X \). An appeal to Lemma 3.10 yields

\[
L \subseteq \left( \frac{1}{1 + \varepsilon'} K_n \right) \cap B_X + \frac{2\varepsilon'}{1 + \varepsilon'} B_X \subseteq \left( \frac{1}{1 + \varepsilon'} K_n \right) \cap B_X + \varepsilon B_X.
\]

So, the family \( \{ S_{n,p} : n < \omega, p \in \mathbb{Q}^+ \} \) satisfies the required property.

(iii)⇒(ii). The family of weakly compact sets \( \{ pS_n : n < \omega, p \in \mathbb{Q}^+ \} \) fulfills the required property, as can be easily checked.

(iii)⇒(i). Define \( S'_n = \text{aco}(S_n) \) for every \( n < \omega \). Since each \( S'_n \) is weakly compact, the set

\[
K := \left\{ \sum_{n<\omega} \frac{1}{2^n} x_n : x_n \in S'_n \text{ for all } n < \omega \right\}
\]

is weakly compact as well. Indeed, note that the mapping

\[
\prod_{n<\omega} S'_n \to X, \quad (x_n) \mapsto \sum_{n<\omega} \frac{1}{2^n} x_n,
\]

is continuous when \( \prod_{n<\omega} S'_n \) is equipped with the product of the weak topology and \( X \) is equipped with the weak topology. Bearing in mind that \( S_n \subseteq 2^n K \) for every \( n < \omega \), it is easy to check that \( K \) strongly generates \( X \). □

**Theorem 3.14.**

(i) \( X \) is reflexive if and only if \( \mathcal{AK}(B_X) \sim \{0\} \).

(ii) \( X \) is SWCG and non-reflexive if and only if \( \mathcal{AK}(B_X) \sim \omega \).

Proof. (i). Since \( \mathcal{AK}(B_X) \preceq K(B_X) \) (Proposition 3.11(i)), we have \( \mathcal{AK}(B_X) \sim \{0\} \) when \( X \) is reflexive. Conversely, if \( X \) is non-reflexive, then Proposition 3.11(iii) yields \( \mathcal{AK}(B_X) \times \omega \preceq \mathcal{AK}(B_X) \) and so \( \omega \preceq \mathcal{AK}(B_X) \).

(ii). By the proof of (i), it only remains to prove that \( X \) is SWCG if and only if \( \mathcal{AK}(B_X) \preceq \omega \).

Suppose first that \( X \) is SWCG and let \( \{ S_n : n < \omega \} \) be a family of weakly compact subsets of \( B_X \) as in Lemma 3.13(iii). We can assume that \( S_n \subseteq S_{n+1} \) for every \( n < \omega \).

Given \( \varepsilon > 0 \), we define \( f_\varepsilon : \mathcal{AK}(B_X) \to \omega \) as follows: for every \( K \in \mathcal{AK}(B_X) \), we choose \( f_\varepsilon(K) < \omega \) satisfying \( K \subseteq S_{f_\varepsilon(K)} + \varepsilon B_X \). It is clear that \( \{ f_\varepsilon \}_{\varepsilon > 0} \) defines a Tukey reduction \( \mathcal{AK}(B_X) \Rightarrow \omega \). Hence \( \mathcal{AK}(B_X) \preceq \omega \).

Conversely, suppose there is a Tukey reduction \( f : \mathcal{AK}(B_X) \Rightarrow \omega \). For every \( \varepsilon \in \mathbb{Q}^+ \) and every \( n < \omega \), we take \( S_{\varepsilon,n} \in \mathcal{AK}(B_X) \) such that \( L \subseteq S_{\varepsilon,n} + \varepsilon B_X \) whenever
Theorem 3.16. If $\mathcal{AK}(B_X) \preceq P$ for a partially ordered set $P$, then $\mathcal{K}(B_X) \preceq P^\omega$.

Proof. Let $\{f_\varepsilon : \mathcal{AK}(B_X) \to P\}_{\varepsilon > 0}$ be a Tukey reduction. Define

$$F : \mathcal{K}(B_X) \to P^\omega, \quad F(L) := (f_1(L), f_2(L), f_3(L), \ldots).$$

We claim that $F$ is a Tukey map. Indeed, fix any $(p_n) \in P^\omega$. For every $n < \omega$ there is $K_n \in \mathcal{AK}(B_X)$ such that $L \subseteq K_n + \frac{1}{n+1}B_X$ whenever $L \in \mathcal{AK}(B_X)$ fulfills $f_{\frac{1}{n+1}}(L) \leq p_n$. Note that

$$K := \bigcap_{n<\omega} \left( K_n + \frac{1}{n+1}B_X \right)$$

is a weakly compact subset of $B_X$ (apply Lemma [3.8]). Clearly, if $L \in \mathcal{K}(B_X)$ satisfies $F(L) \leq (p_n)$, then $L \subseteq K$. It follows that $F$ is Tukey, as claimed. □

Statement (iv) of the following corollary generalizes Example [2.9].

Corollary 3.17.

(i) If $\mathcal{AK}(B_X) \sim P$ for a partially ordered set $P$, then $P \preceq \mathcal{K}(B_X) \preceq P^\omega$.

(ii) $\mathcal{K}(B_X) \sim \omega^\omega$ if and only if $\omega \preceq \mathcal{AK}(B_X) \preceq \omega^\omega$.

(iii) If $\mathcal{AK}(B_X) \sim \mathcal{K}(Q)$, then $\mathcal{K}(B_X) \sim \mathcal{K}(Q)$.

(iv) If $X$ is SWCG, then $\mathcal{K}(B_Y) \sim \omega^\omega$ for every non-reflexive subspace $Y \subseteq X$. 
Proof. (i) is a consequence of Theorem 3.16 and Proposition 3.11(i).

(ii). Suppose first that \( \mathcal{K}(B_X) \sim \omega^\omega \). On one hand \( \mathcal{AK}(B_X) \leq \mathcal{K}(B_X) \sim \omega^\omega \) (Proposition 3.11(i)). On the other hand, \( X \) is not reflexive, so by Proposition 3.11(iii) we get \( \omega \leq \mathcal{AK}(B_X) \). Conversely, if \( \omega \leq \mathcal{AK}(B_X) \), then Theorem 3.16 yields \( \mathcal{K}(B_X) \leq \omega^\omega \) (note that \( \omega^\omega \) and \( (\omega^\omega)^\omega \) are Tukey equivalent). Since \( X \) is not reflexive (because \( \mathcal{AK}(B_X) \not\sim \{0\} \), Proposition 2.7), and so it is reflexive if and only if it does not contain isomorphic copies of \( \ell^1 \) (thanks to Rosenthal’s \( \ell^1 \)-theorem, [13, Theorem 5.37]).

Finally, if \( Y \subseteq X \) is a non-reflexive subspace, then \( \omega^\omega \leq \mathcal{K}(B_Y) \) (by Proposition 2.5(iii)). Since \( X \) is SWCG, we have \( \mathcal{AK}(B_X) \leq \omega \) (by Theorem 3.14), and so Theorem 3.16 yields \( \mathcal{K}(B_X) \leq \omega^\omega \). Since \( \mathcal{K}(B_Y) \leq \mathcal{K}(B_X) \) (Remark 2.6), we get \( \mathcal{K}(B_Y) \sim \omega^\omega \). \( \square \)

The structure \( \mathcal{K}(B_X) \) cannot identify the class of SWCG spaces. Indeed, there exist non-reflexive Banach spaces with separable dual having the PCP (like the predual of the James tree space, see [12, Section 6]). Such a space \( X \) satisfies \( \mathcal{K}(B_X) \sim \omega^\omega \) (by Proposition 2.5, but it is not SWCG; note that any SWCG space is weakly sequentially complete [33, Theorem 2.5], and so it is reflexive if and only if it does not contain isomorphic copies of \( \ell^1 \) (thanks to Rosenthal’s \( \ell^1 \)-theorem, [13, Theorem 5.37]).

To illustrate the concept of Tukey reduction of asymptotic structures of weakly compact sets, we now exhibit some examples where \( \mathcal{AK}(B_X) \) can be identified. The first one says, in particular, that for \( X = c_0 \) we have \( \mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim \mathcal{K}(Q) \) (this is also a particular case of Theorem 6.1 in Section 6) and so

\[
\text{add}_\omega(\mathcal{AK}(C_{c_0})) = \text{add}_\omega(\mathcal{K}(Q)) = \mathfrak{b} \quad \text{and} \quad \text{cf}(\mathcal{AK}(C_{c_0})) = \text{cf}(\mathcal{K}(Q)) = \mathfrak{d}.
\]

**Example 3.18.** Let \((X_n)\) be a sequence of separable Banach spaces having the Schur property. Then \( X = (\bigoplus_{n<\omega} X_n)_{c_0} \) satisfies \( \mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim \mathcal{K}(Q) \).

*Proof.* For each \( x \in X \) we write \( |x| := (\|\pi_n(x)\|_{X_n}) \in \mathbb{R}^\omega \), where \( \pi_n : X \to X_n \) denotes the \( n \)-th coordinate projection. For each \( n < \omega \), we write \( \mathcal{K}_0(B_{X_n}) \) to denote the collection of all weakly compact subsets of \( B_{X_n} \) containing 0, which is partially ordered by the inclusion relation. Since \( X_n \) has the Schur property, its weakly compact subsets are norm compact. Recall that \( Q \) is identified with the subspace of \( 2^\omega \) made up of all eventually zero sequences. Using the previous identification, we write each \( a \in Q \) as a sequence \( a = (a(n)) \) of 0’s and 1’s which is eventually zero.

**Step 1.** For every \( A \in \mathcal{K}(Q) \) and every \( (L_n) \in \prod_{n<\omega} \mathcal{K}_0(B_{X_n}) \), the set

\[
K_{A,(L_n)} := \left\{ x \in \bigcap_{n<\omega} \pi_n^{-1}(L_n) : |x| \leq a \text{ for some } a \in A \right\} \subseteq B_X
\]

is weakly compact. Indeed, let \((x^k)\) be a sequence in \( K_{A,(L_n)} \). Since for each \( n < \omega \) the sequence \((\pi_n(x^k))\) is contained in the norm compact set \( L_n \), we can pass to a further
subsequence of \((x^k)\), not relabeled, such that for every \(n < \omega\) the sequence \((\pi_n(x^k))\) is norm convergent to some \(\varphi_n \in L_n\). For each \(k < \omega\) we fix \(a^k \in A\) such that \(|x^k| \leq a^k\). By passing to a further subsequence, we can assume that \(a^k \to a \in A\). Pick \(n_0 < \omega\) such that \(a(n) = 0\) for every \(n > n_0\) and define \(x \in X\) by declaring \(\pi_n(x) := \varphi_n\) for all \(n \leq n_0\) and \(\pi_n(x) := 0\) for all \(n > n_0\), so that \(x \in \bigcap_{n \leq \omega} \pi_n^{-1}(L_n)\). Observe that \(|x| \leq a\), because for every \(n < \omega\) we have

\[
\|\varphi_n\|_X = \lim_{k \to \infty} \|\pi_n(x^k)\|_X \leq \lim_{k \to \infty} a^k(n) = a(n).
\]

Hence \(x \in K_{A_\omega(L_n)}\). Note that \((3.4)\) also implies that \(\varphi_n = 0\) for all \(n > n_0\). Finally, observe that \((x^k)\) converges weakly to \(x\), because \((x^k)\) is bounded and for every \(n < \omega\) we have \(\pi_n(x^k) \to \varphi_n = \pi_n(x)\) in \(X_n\).

Step 2. Let \(\varepsilon > 0\) and define \(s_\varepsilon : X \to 2^\omega\) by

\[
s_\varepsilon(x)(n) := \begin{cases} 1 & \text{if } \|\pi_n(x)\|_X \geq \varepsilon, \\ 0 & \text{otherwise.} \end{cases}
\]

If \(K \in \mathcal{K}(B_X)\), then \(\tau_\varepsilon(K) := \{s_\varepsilon(x) : x \in K\} \subseteq \mathbb{Q}\), i.e. \(\tau_\varepsilon(K) \in \mathcal{K}(\mathbb{Q})\). Indeed, fix \(a \in \tau_\varepsilon(K)\) and let \((x^k)\) be a sequence in \(K\) such that \(s_\varepsilon(x^k) \to a\) in \(2^\omega\). We can additionally assume that \((x^k)\) converges weakly to some \(x \in K\), so that for every \(n < \omega\) we have \(\pi_n(x^k) \to \pi_n(x)\) in \(X_n\). Therefore, for each \(n < \omega\) we have

\[
\|\pi_n(x)\|_X \geq \varepsilon \text{ whenever } a(n) = \lim_{k \to \infty} s_\varepsilon(x^k)(n) = 1.
\]

Since the set \(\{n < \omega : \|\pi_n(x)\|_X \geq \varepsilon\}\) is finite, it follows that \(a \in \mathbb{Q}\).

Step 3. For every \(\varepsilon > 0\) we define the map

\[
F_\varepsilon : \mathcal{A}K(B_X) \to \mathcal{K}(\mathbb{Q}) \times \prod_{n < \omega} \mathcal{K}(B_{X_n}), \quad F_\varepsilon(K) := (\tau_\varepsilon(K), (\pi_n(K))_{n \in \omega}).
\]

Then \(\{F_\varepsilon\}_{\varepsilon > 0}\) defines a Tukey reduction \(\mathcal{A}K(B_X) \Rightarrow \mathcal{K}(\mathbb{Q}) \times \prod_{n < \omega} \mathcal{K}(B_{X_n})\). Indeed, fix \(A \in \mathcal{K}(\mathbb{Q})\) and \((L_n) \in \prod_{n < \omega} \mathcal{K}(B_{X_n})\). Write \(L'_n := L_n \cup \{0\} \in \mathcal{K}_0(B_{X_n})\) for all \(n < \omega\). Let \(K_{A_\omega(L'_n)} \in \mathcal{A}K(B_X)\) be as in Step 1. We next check that if \(K \in \mathcal{A}K(B_X)\) satisfies \(F_\varepsilon(K) \leq (A, (L'_n))\), then \(K \subseteq K_{A_\omega(L'_n)} + \varepsilon B_X\). To this end, take \(x \in K\). Then \(a := s_\varepsilon(x) \in \tau_\varepsilon(K) \subseteq A\). Define \(y \in B_X\) by declaring \(\pi_n(y) := \pi_n(x)\) whenever \(\|\pi_n(x)\|_X \geq \varepsilon\) and \(\pi_n(y) := 0\) otherwise. Clearly, we have \(|x - y|_X \leq \varepsilon\). On the other hand, since \(\pi_n(x) \in \pi_n(K) \subseteq L'_n\) and \(0 \in L'_n\) for every \(n < \omega\), we also have \(y \in \bigcap_{n < \omega} \pi_n^{-1}(L'_n)\); since \(a(n) = 1\) whenever \(\pi_n(x) \neq 0\), we conclude that \(y \in K_{A_\omega(L'_n)}\). This proves that \(K \subseteq K_{A_\omega(L'_n)} + \varepsilon B_X\).

Step 4. Since each \(X_n\) is SWCG, we have either \(\mathcal{K}(B_{X_n}) \sim \{0\}\) or \(\mathcal{K}(B_{X_n}) \sim \omega^\omega\) (apply Corollary 3.17(iv)), hence either \(\prod_{n < \omega} \mathcal{K}(B_{X_n}) \sim \{0\}\) or \(\prod_{n < \omega} \mathcal{K}(B_{X_n}) \sim \omega^\omega\).
and therefore
\[ \mathcal{AK}(B_X) \preceq \mathcal{K}(Q) \times \prod_{n<\omega} \mathcal{K}(B_{X_n}) \preceq \mathcal{K}(Q) \times \mathcal{K}(Q) \sim \mathcal{K}(Q). \]

**Step 5.** If \( K \subseteq B_X \) is weakly compact, then the set
\[ A_K := \{ a \in 2^\omega : a \leq 2|x| \text{ for some } x \in K \} \]
belongs to \( \mathcal{K}(Q) \). Indeed, since for every \( x \in X \) the set \( \{ n < \omega : \| \pi_n(x) \|_{X_n} \geq \frac{1}{2} \} \) is finite, \( A_K \subseteq Q \). Therefore, it suffices to check that \( A_K \) is closed in \( 2^\omega \). To this end, let \((a^k)\) be a sequence in \( A_K \) which converges to \( a \in 2^\omega \). For each \( k < \omega \) we pick \( x^k \in K \) such that \( a^k \leq 2|x^k| \). Since \( K \) is weakly compact, by passing to a subsequence we may assume that \((x^k)\) is weakly convergent to some \( x \in K \). There is \( n_0 < \omega \) such that for every \( n \geq n_0 \) we have \( \| \pi_n(x) \|_{X_n} < \frac{1}{2} \). For each \( n < \omega \) the sequence \((\pi_n(x^k))\) is norm convergent to \( \pi_n(x) \), hence
\[ a^k(n) \leq 2\| \pi_n(x^k) \|_{X_n} < 1 \text{ for } k \text{ large enough,} \]
and so \( a(n) = 0 \). It follows that \( a \in Q \).

**Step 6.** Fix a norm one vector \( x_n \in X_n \) for every \( n < \omega \). We define a mapping \( G : \mathcal{K}(Q) \rightarrow \mathcal{AK}(B_X) \) by \( G(A) := K_{A,(L_n)} \) where \( L_n := \{ 0, x_n \} \) for all \( n < \omega \). In order to check that \( G \) gives a Tukey reduction \( \mathcal{K}(Q) \rightarrow \mathcal{AK}(B_X) \), we only have to show that \( G : \mathcal{K}(Q) \rightarrow (\mathcal{AK}(B_X), \preceq_\delta) \) is Tukey for \( \delta = \frac{1}{2} \). Fix \( K \in \mathcal{AK}(B_X) \) and take \( A_K \in \mathcal{K}(Q) \) as in Step 4. We shall check that \( A \subseteq A_K \) for every \( A \in \mathcal{K}(Q) \) satisfying \( K_{A,(L_n)} \subseteq K + \frac{1}{2}B_X \). Indeed, fix \( a \in A \) and define \( x \in K_{A,(L_n)} \) by declaring \( \pi_n(x) := a(n)x_n \) for all \( n < \omega \). There is \( y \in K \) such that \( \| x - y \|_X \leq \frac{1}{2} \). Clearly, we have \( a \leq 2|y| \) and so \( a \in A_K \). This shows that
\[ \mathcal{K}(Q) \preceq \mathcal{AK}(B_X) \]
and the proof is finished. \( \square \)

The following result is an improvement of Corollary 2.8 within the class of Banach spaces having the **Separable Complementation Property** (SCP for short). Recall that \( X \) is said to have the SCP (see e.g. [13, Section 13.2]) if every separable subspace of \( X \) is contained in a complemented separable subspace of \( X \). For instance, any weakly compactly generated Banach space has the SCP.

**Corollary 3.19.** If \( X \) has the SCP and contains a copy \( c_0 \), then \( \mathcal{K}(Q) \preceq \mathcal{AK}(B_X) \).

**Proof.** This follows from Example 3.18 Proposition 3.11(iv) and a classical theorem due to Sobczyk, that \( c_0 \) is complemented in any separable superspace (see e.g. [11, Theorem 2.5.8]). \( \square \)

Our next example requires an auxiliary lemma.
Lemma 3.20. Let \( \{X_i\}_{i \in I} \) be a family of Banach spaces and let \( X = (\bigoplus_{i \in I} X_i)_{\ell^1} \). Then
\[
\mathcal{AK}(B_X) \preceq \prod_{i \in I} \mathcal{K}(B_{X_i}) \times [I]^{<\omega}.
\]

Proof. For each \( i \in I \), let \( \pi_i : X \to X_i \) be the \( i \)-th coordinate projection. Given any finite set \( J \subseteq I \) and weakly compact sets \( K_j \subseteq X_j \) for \( j \in J \), the set
\[
\bigoplus_{j \in J} K_j := \left( \bigcap_{j \in J} \pi_j^{-1}(K_j) \right) \cap \left( \bigcap_{i \in I \setminus J} \pi_i^{-1}(\{0\}) \right) \subseteq X
\]
is weakly compact. The following fact is folklore and can be deduced, for instance, by an argument similar to that of [9, p. 104, Theorem 4]; cf. [22, Lemma 7.2].

Fact. For every weakly compact set \( K \subseteq X \) and every \( \varepsilon > 0 \) there is a finite set \( J(K,\varepsilon) \subseteq I \) such that
\[
\sum_{i \in I \setminus J(K,\varepsilon)} \|\pi_i(x)\|_{X_i} \leq \varepsilon \quad \text{for all } x \in K
\]
and so
\[
(3.5) \quad K \subseteq \bigoplus_{j \in J(K,\varepsilon)} \pi_j(K) + \varepsilon B_X.
\]

We claim that the family \( \{f_\varepsilon\}_{\varepsilon > 0} \) defines a Tukey reduction \( \mathcal{AK}(B_X) \to P \times [I]^{<\omega} \). Indeed, fix \( \varepsilon > 0 \), take \( (L_i)_{i \in I} \in P \) and \( J \subseteq I \) finite. Consider the weakly compact subset of \( B_X \) defined by
\[
K_0 := \left( \frac{1}{1 + r(\varepsilon)} \bigoplus_{j \in J} L_j \right) \cap B_X.
\]

Let \( K \in \mathcal{AK}(B_X) \) such that \( f_\varepsilon(K) \leq ((L_i)_{i \in I}, J) \). Then \( J(K, r(\varepsilon)) \subseteq J \) and
\[
K \subseteq \bigoplus_{j \in J(K, r(\varepsilon))} \pi_j(K) + r(\varepsilon) B_X \subseteq \bigoplus_{j \in J} L_j + r(\varepsilon) B_X,
\]
which implies (via Lemma 3.10) that \( K \subseteq K_0 + \varepsilon B_X \). The proof is over.

Example 3.21. The space \( X = \ell^1(c_0) \) satisfies \( \mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim \mathcal{K}(Q) \).
Proof. We already know that $\mathcal{K}(B_{c_0}) \sim \mathcal{K}({\mathbb{Q}})$, hence $\mathcal{K}(B_{c_0})^\omega \sim \mathcal{K}({\mathbb{Q}})^\omega \sim \mathcal{K}({\mathbb{Q}})$ (see Lemma 2.3). Thus, Lemma 3.20 yields

$$\mathcal{A}K(B_X) \leq \mathcal{K}({\mathbb{Q}}) \times [\omega]^\omega \sim \mathcal{K}({\mathbb{Q}}) \times \omega \sim \mathcal{K}({\mathbb{Q}}).$$

On the other hand, since $X$ contains complemented subspaces isomorphic to $c_0$ and $\mathcal{A}K(B_{c_0}) \sim \mathcal{K}({\mathbb{Q}})$ (Example 3.18), we have $\mathcal{K}({\mathbb{Q}}) \leq \mathcal{A}K(B_X)$ (Proposition 3.11 iv). Thus $\mathcal{A}K(B_X) \sim \mathcal{K}(B_X)$ and so Corollary 3.17 (iii) applies to get $\mathcal{A}K(B_X) \sim \mathcal{K}(B_X)$.

□

The $\ell^1$-sum of countably many SWCG spaces is SWCG (see [33, Proposition 2.9]). Bearing in mind this fact, Theorem 3.14 and Corollary 3.17 (iv) yield:

**Example 3.22.** The space $X = \ell^1(\ell^2)$ satisfies $\mathcal{A}K(B_X) \sim \omega$ and $\mathcal{K}(B_X) \sim \omega^\omega$.

The case of $\ell^p$-sums, $1 < p < \infty$, is different. For instance, $\ell^p(\ell^1)$ is not isomorphic to a subspace of a SWCG space (see [23, Corollary 2.29]). In Example 7.4 we shall check that this space satisfies $\mathcal{A}K(B_X) \sim \mathcal{K}(B_X) \sim \omega^\omega$.

We finish this section with the following problem. We showed in Proposition 2.6 that $\omega^\omega$ was the lowest possible nontrivial value of the Tukey class of $\mathcal{K}(B_X)$. In the case of $\mathcal{A}K(B_X)$ we can get also $\omega$, but still we might ask if $\omega^\omega$ is the lowest possible value after $\omega$.

**Problem 3.23.** Is it true that for every Banach space $X$, either $\mathcal{A}K(B_X) \preceq \omega$ (i.e. $X$ is SWCG) or else $\omega^\omega \preceq \mathcal{A}K(B_X)$?

We next give a consistent non-separable counterexample using cardinal invariants, but the separable case remains open for us.

**Example 3.24.** The space $X = \ell^1(\omega_1)$ is not SWCG and so $\text{cf}(\mathcal{A}K(B_X)) \geq \omega_1$. By Lemma 3.20 we have $\mathcal{A}K(B_X) \preceq [\omega_1]^\omega$, hence $\text{cf}(\mathcal{A}K(B_X)) = \omega_1$. On the other hand, $\text{cf}(\omega^\omega) = \frak{d}$, so $\omega^\omega \not\preceq \mathcal{A}K(B_X)$ whenever $\omega_1 < \frak{d}$.

4. **Classification of $\mathcal{K}(B_X)$ under analytic determinacy**

This section is devoted to the proof of the following theorem, but the machinery developed here will be used in further sections.

**Theorem 4.1 ($\Sigma^1_1 D$).** If $X$ is separable, then $\mathcal{K}(B_X)$ is Tukey equivalent to either $\{0\}$, $\omega^\omega$, $\mathcal{K}({\mathbb{Q}})$ or $[c]^\omega$.

The first step is to state the consequences of the axiom of analytic determinacy (denoted by $\Sigma^1_1 D$) that we shall need: Theorems 4.2 and 4.7. We shall not state here the axiom and we refer the reader to [21] for basic knowledge on games, winning strategies and determinacy. (This is only needed to understand our proof of Theorem 4.2.) We say that a family $\mathcal{A}$ of subsets of $\omega$ is hereditary if for every $A \in \mathcal{A}$ and every $A' \subset A$ we have $A' \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are two families of subsets of $\omega$, we say that $\mathcal{A}$
and \( B \) are countably separated if there exists a countable family \( C \) of subsets of \( \omega \) such that for every \( a \in A \) and every \( b \in B \) with \( a \cap b = \emptyset \) there is \( c \in C \) such that \( a \subset c \) and \( b \subset \omega \setminus c \). By identifying each subset of \( \omega \) with its characteristic function, any family \( A \) of subsets of \( \omega \) can be viewed as a subset of \( 2^{<\omega} \), and in this sense we can say that \( A \) is analytic, coanalytic, etc.

Given \( i \in \{0,1\} \), an infinite set \( S \subseteq 2^{<\omega} \) is said to be an \( i \)-chain if it can be enumerated as \( S = \{ s^p : p < \omega \} \) where \( s^p \cap i \subseteq s^{p+1} \) for all \( p < \omega \). The set of all \( i \)-chains of \( 2^{<\omega} \) is denoted by \( C_i \).

**Theorem 4.2 (\( \Sigma^1_1D \)).** Let \( A_0 \) and \( A_1 \) be hereditary families of subsets of \( \omega \) such that \( A_0 \) is analytic and \( A_1 \) is coanalytic. Then:

(i) either \( A_0 \) and \( A_1 \) are countably separated,

(ii) or there exists an injective function \( u : 2^{<\omega} \to \omega \) such that \( u(S) \in A_i \) whenever \( S \subseteq 2^{<\omega} \) is an \( i \)-chain, \( i \in \{0,1\} \).

**Remark 4.3.** If both \( A_0 \) and \( A_1 \) are analytic, then the statement of Theorem 4.2 holds in ZFC, as shown by Todorcevic [35] in a slightly different language. His proof relies on the open graph theorem of Feng [15], who also proves that an open graph theorem holds for projective sets under projective determinacy. So it follows from combining the argument of both papers that Theorem 4.2 holds under projective determinacy when the families are projective. Here we present a direct proof for the case that we are going to use. Similar results for more than two families can be found in [4, 7].

We shall need the following elementary lemma.

**Lemma 4.4.** Let \( W \) be a set and let \( u' : 2^{<\omega} \to W \) be a function which is injective when restricted to each \( i \)-chain, \( i \in \{0,1\} \). Then there is a function \( v : 2^{<\omega} \to 2^{<\omega} \) such that:

(i) \( u' \circ v \) is injective;

(ii) \( v(S) \) is an \( i \)-chain whenever \( S \subseteq 2^{<\omega} \) is an \( i \)-chain, \( i \in \{0,1\} \).

**Proof.** The function \( v \) is constructed inductively following the lexicographical order \( \leq_{\text{lex}} \) of \( 2^{<\omega} \). Set \( v(\emptyset) := \emptyset \). Let \( s = (s_0, \ldots, s_p) \in 2^{<\omega} \) and suppose that \( v(t) \) is already defined for every \( t \leq_{\text{lex}} s \) with \( t \neq s \). Since \( u' \) is injective on every \( s_p \)-chain, there is \( n_s \in \mathbb{N} \) such that the element of \( 2^{<\omega} \) given by

\[
v(s) := v(s_0, \ldots, s_{p-1}) \bigcup (s_p, \ldots, s_p)
\]

satisfies that \( u'(v(s)) \neq u'(v(t)) \) for every \( t \leq_{\text{lex}} s \) with \( t \neq s \). By the very construction, \( u' \circ v \) is injective, and so is \( v \). Clearly, we also have

(4.1) \( v(t) \subseteq v(s) \) whenever \( t \subseteq s \).
Let $S = \{s^p : p < \omega\}$ be an $i$-chain, $i \in \{0, 1\}$, enumerated so that $s^p \cap i \subseteq s^{p+1}$ for all $p < \omega$. Then
\[
v(s^p) \cap i \subseteq v(s^p \cap i) \subseteq v(s^{p+1}) \quad \text{for every } p < \omega,
\]

hence $v(S)$ is an $i$-chain as well. \hfill \Box

**Proof of Theorem 4.2** Since $A_0$ is analytic, there is a continuous function $f : \omega^\omega \to 2^\omega$ such that $f(\omega^\omega) = A_0$. Consider an infinite game with two players where, at each step $k$, Player I plays $(m_k, \bar{m}_k) \in \omega \times \omega$, and Player II plays $i_k \in \{0, 1\}$. At the end of the game, Player I wins if and only if the following three conditions hold:

1. $m_k \neq m_{k'}$ for every $k \neq k'$,
2. $\{m_k : i_k = 0\} \subseteq f(\bar{m}_0, \bar{m}_1, \ldots)$,
3. $\{m_k : i_k = 1\} \in A_1$.

It is easy to check that the set of all infinite rounds $(m_0, \bar{m}_0, i_0, m_1, \bar{m}_1, i_1, \ldots)$ where Player I wins is coanalytic in the Polish space $(\omega \times \omega \times \{0, 1\})^\omega$. So, under the axiom of analytic determinacy, this game is determined and we have two cases.

**Case 1. Player I has a winning strategy.** We shall check that condition (ii) of the theorem holds. For each $s = (s_0, \ldots, s_p) \in 2^{<\omega}$, let $(m_{p+1}, \bar{m}_{p+1})$ be the move of Player I under his strategy after Player II has played $s_0, \ldots, s_p$ and Player I followed his strategy. Define the function $u' : 2^{<\omega} \to \omega$ by $u'(s) = m_{p+1}$. Now, let $S \subseteq 2^{<\omega}$ be any $i$-chain, $i \in \{0, 1\}$. Write $S = \{s^p : p < \omega\}$, where $s^p = (s_0, \ldots, s_p)$, the sequence $(n_p)$ is strictly increasing and $s_{n_p+1} = i$ for all $p < \omega$. Consider the infinite round $(m_0, \bar{m}_0, s_0, m_1, \bar{m}_1, s_1, \ldots)$ played according to the winning strategy of Player I. Then $u'(S) = \{m_{n_p+1} : p < \omega\} \subseteq \{m_k : s_k = i\}$. Since Player I is the winner of this round we conclude that $u'(S) \in A_i$ (bear in mind that $A_i$ is hereditary) and that the restriction $u'|_S$ is injective. An appeal to Lemma 4.4 ensures the existence of a function $u : 2^{<\omega} \to \omega$ as required in (ii).

**Case 2. Player II has a winning strategy.** We shall show that $A_0$ and $A_1$ are countably separated. For every $i \in \{0, 1\}$, every finite round of the game
\[
\xi = (m_0, \bar{m}_0, i_0, \ldots, m_k, \bar{m}_k, i_k)
\]
and every $\bar{m} \in \omega$, we define $c^1[\xi, \bar{m}]$ to be the set of all $m < \omega$ such that the strategy of Player II chooses $1 - i$ after
\[
(m_0, \bar{m}_0, i_0, \ldots, m_k, \bar{m}_k, i_k, m, \bar{m})
\]
is played. Notice that $c^0[\xi, \bar{m}] \cap c^1[\xi, \bar{m}] = \emptyset$. The countable family of subsets of $\omega$ that will witness the countable separation of $A_0$ and $A_1$ is:
\[
C := \{c \subseteq \omega : c \triangle c^0[\xi, \bar{m}] \text{ is finite for some } \bar{m} < \omega \text{ and some } \xi\}.
\]
Indeed, fix \( a_0 \in A_0 \) and \( a_1 \in A_1 \) such that \( a_0 \cap a_1 = \emptyset \). Take \( (\vec{n}_k) \in \omega^\omega \) such that \( a_0 = f((\vec{n}_k)) \). We say that a finite round of the game

\[
\xi = (m_0, \vec{m}_0, i_0, \ldots, m_k, \vec{m}_k, i_k)
\]

is acceptable if it is played according to the strategy of Player II and the following conditions hold:

(a) \( \vec{m}_j = \vec{n}_j \) for all \( j \);
(b) \( m_j \neq m_{j'} \) whenever \( j \neq j' \);
(c) \( \{m_j : i_j = 0\} \subset a_0 \);
(d) \( \{m_j : i_j = 1\} \subset a_1 \).

Let \( \Xi \) be the family of all acceptable \( \xi \)'s. Given \( \xi = (m_0, \vec{m}_0, i_0, \ldots, m_k, \vec{m}_k, i_k) \in \Xi \), we say that \( \xi \) is extensible if there exist \( m \in \omega \setminus \{m_0, \ldots, m_k\} \) and \( i \in \{0, 1\} \) such that

\[
(m_0, \vec{m}_0, i_0, \ldots, m_k, \vec{m}_k, i_k, m, \vec{m}_{k+1}, i) \in \Xi.
\]

Then there is some \( \xi \in \Xi \) which is not extensible, because otherwise there would exist an infinite round of the game, played according to the strategy of Player II, in which Player I wins. Fix a non-extensible \( \xi = (m_0, \vec{m}_0, i_0, \ldots, m_k, \vec{m}_k, i_k) \in \Xi \) and define

\[
c := (c^0[\xi, \vec{m}_{k+1}] \setminus \{m_0, \ldots, m_k\}) \cup \{m_j : a_0 : j \leq k\} \in C.
\]

We claim that \( a_0 \subseteq c \) and \( a_1 \cap c = \emptyset \). Indeed, this follows at once from the fact that \( c^0[\xi, \vec{m}_{k+1}] \cap c^1[\xi, \vec{m}_{k+1}] = \emptyset \) and the inclusion

\[
(a_i \subseteq c^i[\xi, \vec{m}_{k+1}] \cup \{m_0, \ldots, m_k\} \quad \text{for } i \in \{0, 1\}).
\]

To check \( (1.2) \), fix \( i \in \{0, 1\} \) and note that for every \( m \in a_i \setminus \{m_0, \ldots, m_k\} \) the non-extendability of \( \xi \) ensures that

\[
(m_0, \vec{m}_0, i_0, \ldots, m_k, \vec{m}_k, i_k, m, \vec{m}_{k+1}, i) \notin \Xi
\]

and so \( m \in c^i[\xi, \vec{m}_{k+1}] \). The proof of the theorem is over. \( \square \)

**Lemma 4.5.** Let \( D \) be a dense subset of a metric space \( E \) and let \( K_E(D) \) be the family of all subsets of \( D \) which are relatively compact in \( E \), ordered by inclusion. Then \( K_E(D) \sim K(E) \).

**Proof.** The function \( G : K_E(D) \to K(E) \) given by \( G(A) := \overline{A} \) is Tukey, because for each \( K \in K(E) \) the set \( K \cap D \in K_E(D) \) contains every \( A \in K_E(D) \) for which \( G(A) = \overline{A} \subset K \).

In order to check the Tukey reduction \( K(E) \preceq K_E(D) \), fix \( L \in K(E) \). Given any \( n < \omega \), let \( U_n^L \) be a finite cover of \( L \) by open balls of radius \( \frac{1}{n+1} \) such that \( L \cap B \neq \emptyset \) for
every $B \in \mathcal{U}_i^L$. Since $D$ is dense in $E$, we can select a finite set $F_n(L) \subseteq D \cap (\bigcup \mathcal{U}_i^L)$ such that $F_n(L) \cap B$ is a singleton for every $B \in \mathcal{U}_i^L$. Define

$$F(L) := \bigcup_{n < \omega} F_n(L) \subseteq D.$$  

Clearly, $L \subseteq \overline{F(L)}$. We claim that $F(L) \in \mathcal{K}_E(D)$. Indeed, let $(x_m)$ be a sequence in $F(L)$ with infinitely many distinct terms. Then we can find two strictly increasing subsequences $(m_k)$ and $(n_k)$ such that $x_{m_k} \in F_{n_k}(L)$ for all $k < \omega$. Pick a sequence $(y_k)$ in $L$ such that the distance between $y_k$ and $x_{m_k}$ is less than or equal to $\frac{2}{n_k+1}$ for all $k < \omega$. Since $L$ is compact, $(y_k)$ admits a convergent subsequence, and so does $(x_{m_k})$. This proves that $\overline{F(L)}$ is compact.

Finally, we check that $F : \mathcal{K}(E) \to \mathcal{K}_E(D)$ is Tukey. Indeed, if $R \in \mathcal{K}_E(D)$, then $R \subseteq \mathcal{K}(E)$ contains every $L \in \mathcal{K}(E)$ satisfying $F(L) \subseteq R$ (since $L \subseteq \overline{F(L)}$). \hfill \Box

The following lemma will also be needed in Section 5.

**Lemma 4.6.** Let $W$ be a set, $f : 2^{<\omega} \to W$ a function, $\mathcal{A} \subseteq \mathcal{P}(W)$ and “$\leq$” a binary relation on $\mathcal{P}(W)$ satisfying the following properties:

(i) $\mathcal{A}$ is closed under finite unions;

(ii) for every $A, B \subseteq W$ with $B \subseteq \bigcup \{C : C \leq A\}$ we have $B \leq A$;

(iii) $f(S) \in \mathcal{A}$ for every 1-chain $S \subseteq 2^{<\omega}$;

(iv) $f(S) \leq A$ for every 0-chain $S \subseteq 2^{<\omega}$ and every $A \in \mathcal{A}$.

Then $[c]^{\leq \omega} \leq (\mathcal{A}, \leq)$.

**Proof.** Let $G$ be the set of all $\sigma = (\sigma_k) \in 2^\omega$ having infinitely many 0’s and infinitely many 1’s. For such a $\sigma$, we consider the following 1-chains contained in $2^{<\omega}$:

$$S_1(\sigma) := \{ (\sigma_0, \ldots, \sigma_k) : \sigma_{k+1} = 1 \} \quad \text{and} \quad S_2(\sigma) := \{ (1 - \sigma_0, \ldots, 1 - \sigma_k) : \sigma_{k+1} = 0 \},$$

so that $F(\sigma) := f(S_1(\sigma)) \cup f(S_2(\sigma)) \in \mathcal{A}$ (by (i) and (iii)).

**Claim.** For every $A \in \mathcal{A}$ the set $\Omega_A := \{ \sigma \in G : F(\sigma) \leq A \}$ is finite. Our proof is by contradiction. Suppose $\Omega_A$ is infinite and let $(\sigma^n)$ be a sequence in $\Omega_A$ of pairwise distinct elements. Write $\sigma^n = (\sigma^n_k)$ for each $n < \omega$. By passing to a subsequence, we can assume that $(\sigma^n)$ converges to some $\sigma = (\sigma_k) \in 2^\omega$ such that $\sigma \neq \sigma^n$ for all $n < \omega$. Now, we can find recursively two strictly increasing subsequences $(n_j)$ and $(p_j)$ of $\omega$ such that, for every $j < \omega$, we have $\sigma^n_{n_j} = \sigma_k$ for all $k \leq p_j$ and $\sigma^n_{p_j+1} \neq \sigma_{p_j+1}$. Let $(j_m)$ be a strictly increasing subsequence of $\omega$ for which the sequence $(\sigma_{p_j+1})$ is constant. There are two cases:

- If $\sigma_{p_{jm}+1} = 0$ for every $m < \omega$, then the set $\{ (\sigma_0, \ldots, \sigma_{p_{jm}}) : m < \omega \} \subseteq 2^{<\omega}$ is a 0-chain, hence $B := \{ f(\sigma_0, \ldots, \sigma_{p_{jm}}) : m < \omega \} \not\subseteq A$ by property (iv). On
the other hand, we have
\[ f(\sigma_0, \ldots, \sigma_{pjm}) = f(\sigma^{\text{jm}}_0, \ldots, \sigma^{\text{jm}}_{pjm}) \subseteq f(S_1(\sigma^{\text{jm}}_m)) \subseteq F(\sigma^{\text{jm}}_m) \leq A \]
for every \( m < \omega \), hence property (ii) yields \( B \leq A \), which is a contradiction.

• If \( \sigma_{pjm+1} = 1 \) for every \( m < \omega \), the argument is similar. Indeed, the set \( \{(1 - \sigma_0, \ldots, 1 - \sigma_{pjm}) : m < \omega\} \subseteq 2^{<\omega} \) is a 0-chain and so (iv) yields
\[ B := \{f(1 - \sigma_0, \ldots, 1 - \sigma_{pjm}) : m < \omega\} \not\leq A. \]

On the other hand,
\[ f(1 - \sigma_0, \ldots, 1 - \sigma_{pjm}) = f(1 - \sigma^{\text{jm}}_0, \ldots, 1 - \sigma^{\text{jm}}_{pjm}) \subseteq f(S_2(\sigma^{\text{jm}}_m)) \subseteq F(\sigma^{\text{jm}}_m) \leq A \]
for every \( m < \omega \). It follows from (ii) that \( B \leq A \), again a contradiction.

This proves that \( \Omega_A \) is finite, as claimed.

Finally, the mapping
\[ \bar{F} : [G]^{<\omega} \rightarrow (A, \leq), \quad \bar{F}(B) := \bigcup_{\sigma \in B} F(\sigma), \]
is Tukey, because for every \( A \in \mathcal{A} \) the finite set \( \Omega_A \subseteq G \) contains any finite set \( B \subseteq G \) for which \( \bar{F}(B) \leq A \). Since \( G \) has cardinality \( c \), we get \( [c]^{<\omega} \leq (A, \leq) \).

Given a family \( \mathcal{I} \) of subsets of \( \omega \), its orthogonal is defined by
\[ \mathcal{I}^\perp = \{a \subseteq \omega : a \cap b \text{ is finite for every } b \in \mathcal{I}\}. \]

**Theorem 4.7 (\( \Sigma^1_1 \mathcal{D} \)).** Let \( \mathcal{I} \) be an analytic family of subsets of \( \omega \). Then \( \mathcal{I}^\perp \) (ordered by inclusion) is Tukey equivalent to either \( \{0\}, \omega, \omega^\omega, \mathcal{K}(\mathbb{Q}) \) or \( [c]^{<\omega} \).

**Proof.** Notice that the family \( \{a \subseteq \omega : a \cap b \text{ for some } b \in \mathcal{I}\} \) is also analytic, so we can assume that \( \mathcal{I} \) is hereditary. Since \( \mathcal{I}^\perp \) is coanalytic and hereditary, Theorem 12 can be applied to the families \( \mathcal{I} \) and \( \mathcal{I}^\perp \). Two cases arise.

**Case 1.** \( \mathcal{I} \) and \( \mathcal{I}^\perp \) are countably separated. By [1] Theorem 10 and Proposition 11], there exist a bijection \( f : \omega \rightarrow \mathbb{Q} \) and an analytic set \( F \subset 2^\omega \) such that \( \text{acc}(f(a)) \subseteq F \) (resp. \( \text{acc}(f(a)) \subseteq 2^\omega \setminus F \)) for every \( a \in \mathcal{I} \) (resp. \( a \in \mathcal{I}^\perp \)). Here we write \( \text{acc}(C) \) to denote the set of all accumulation points of a set \( C \subseteq 2^\omega \). Let \( K \) be the disjoint union \( \omega \cup 2^\omega \) equipped with the compact metrizable topology defined by:

- each \( n < \omega \) is isolated in \( K \);
- for each \( x \in 2^\omega \), the collection
\[ \{f^{-1}(O) \cup O : O \text{ is an open neighborhood of } x \text{ in } 2^\omega\} \]
is a basis of neighborhoods of \( x \) in \( K \).
In this way, the topology inherited by $2^\omega$ as a subspace of $K$ coincides with its usual topology. Define $E_0 := \omega \cup (2^\omega \setminus F) \subseteq K$, so that $E_0$ is coanalytic in $K$ and $\omega$ is dense in $E_0$. Let $g : \omega \to K$ be the identity mapping.

**CLAIM.** A set $a \subseteq \omega$ belongs to $\mathcal{I}^\perp$ if and only if $g(a) \in \mathcal{K}_{E_0}(\omega)$. Indeed, note first that the condition $g(a) \in \mathcal{K}_{E_0}(\omega)$ is equivalent to saying that $g(a) \subseteq E_0$ (because $K$ is compact metrizable). Suppose that $a \in \mathcal{I}^\perp$. Then acc$(f(a)) \subseteq 2^\omega \setminus F \subseteq E_0$. Take any $x \in g(a)$. If $x \in \omega$, then $x \in E_0$. If $x \notin \omega$, then $x \in g(a) \setminus g(a) \subseteq \text{acc}(g(a))$ and so $x \in \text{acc}(f(a)) \subseteq E_0$. This proves that $g(a) \subseteq E_0$ whenever $a \in \mathcal{I}^\perp$. Conversely, let $a \subseteq \omega$ be a set satisfying $g(a) \subseteq E_0$ and take any $b \in \mathcal{I}$. Then acc$(f(b)) \subseteq F$ and so

$$\text{acc}(f(a \cap b)) \subseteq \text{acc}(f(a)) \cap \text{acc}(f(b)) \subseteq g(a) \cap F \subseteq E_0 \cap F = \emptyset,$$

hence $a \cap b$ is finite. This shows that $a \in \mathcal{I}^\perp$ and finishes the proof of the claim.

Now, the mapping $\mathcal{I}^\perp \to \mathcal{K}_{E_0}(\omega)$ given by $a \mapsto g(a)$ is an isomorphism of partially ordered sets, hence $\mathcal{I}^\perp \sim \mathcal{K}_{E_0}(\omega)$. By Lemma 4.5 we have $\mathcal{K}_{E_0}(\omega) \sim \mathcal{K}(E_0)$ and so $\mathcal{I}^\perp \sim \mathcal{K}(E_0)$. Finally, an appeal to Fremlin’s Theorem 2.2 allows us to conclude that $\mathcal{I}^\perp$ is Tukey equivalent to either $\{0\}$, $\omega$, $\omega^\omega$ or $\mathcal{K}(\mathbb{Q})$.

**Case 2.** There is an injective function $u : 2^{<\omega} \to \omega$ such that $u(S) \in \mathcal{I}$ (resp. $u(S) \in \mathcal{I}^\perp$) whenever $S \subseteq 2^{<\omega}$ is a 0-chain (resp. 1-chain). In this case we have $\mathcal{I}^\perp \sim [c]^{<\omega}$. Indeed, since $\mathcal{I}^\perp$ is upwards directed and has cardinality $\leq c$, we have $\mathcal{I}^\perp \preceq [c]^{<\omega}$ (Remark 2.4). On the other hand, the Tukey reduction $[c]^{<\omega} \preceq \mathcal{I}^\perp$ follows from Lemma 4.6 applied to the function $f := u$ and the family $\mathcal{A} := \mathcal{I}^\perp$ equipped with the inclusion relation. The proof of the theorem is finished. 

The following result is similar to Lemma 4.5 now dealing with the weak topology of the Banach space $X$.

**Lemma 4.8.** Suppose $X$ is separable. Let $D \subseteq B_X$ be a norm dense set and let $\mathcal{RK}(D)$ denote the family of all subsets of $D$ which are relatively weakly compact, ordered by inclusion. Then $\mathcal{K}(B_X) \sim \mathcal{RK}(D)$.

**Proof.** It is clear that $F : \mathcal{RK}(D) \to \mathcal{K}(B_X)$, $F(A) := \overline{A}^\omega$, is a Tukey function. Let us check that $\mathcal{K}(B_X) \preceq \mathcal{RK}(D)$. Fix $L \in \mathcal{K}(B_X)$. Let $(x_n^L)$ be a norm dense sequence in $L$ such that every element is repeated infinitely many times. For each $n < \omega$, choose $y_n^L \in D$ such that $\|x_n^L - y_n^L\| \leq \frac{1}{n+1}$. Since $L$ is weakly compact and $y_n^L \in L + \frac{1}{n+1}B_X$ for every $n < \omega$, an appeal to Lemma 5.8 ensures that the set $\{y_n^L : n < \omega\}$ is relatively weakly compact. Define

$$G : \mathcal{K}(B_X) \to \mathcal{RK}(D), \quad G(L) := \{y_n^L : n < \omega\}.$$

To prove that $G$ is a Tukey function, fix $C_0 \in \mathcal{RK}(D)$ and define $K_0 := \overline{C_0}^\omega \in \mathcal{K}(B_X)$. Take any $L \in \mathcal{K}(B_X)$ satisfying $G(L) \subseteq C_0$. Given $x \in L$ and $\varepsilon > 0$, we can choose
Let $(x_n)$ be a sequence in $B_X$. Let $\Omega(x_n) \subseteq 2^\omega \times (B_X^*)^\omega$ be the set of all pairs $(a, (y_m^n))$ for which the iterated limits

$$\lim_{n \to \infty} \lim_{m \to \infty} y_m^*(x_n)$$ and $\lim_{m \to \infty} \lim_{n \to a} y_m^*(x_n)$$

exist and are distinct. Then $\Omega(x_n)$ is Borel when $2^\omega \times (B_X^*)^\omega$ is equipped with the product topology induced by the usual topology of $2^\omega$ and the $w^*$-topology of $X^*$.

Proposition 4.10. If $D \subseteq B_X$ is a countable set, then $\mathcal{RK}(D) \sim \mathcal{I}^\perp$ for some analytic family $\mathcal{I}$ of subsets of $\omega$.

Proof. If $D$ is finite, then $\mathcal{RK}(D) \sim \{0\} \sim P(\omega) = \mathcal{I}^\perp$ by taking $\mathcal{I} = \{\{\emptyset\}\}$. Suppose now that $D$ is infinite and enumerate $D = \{x_n : n < \omega\}$. Since $\mathcal{sp}(D)$ is separable, we can assume without loss of generality that $X$ is separable, so that $(B_X^*)^\omega$ is Polish when equipped with the product topology induced by the $w^*$-topology of $X^*$. By Lemma 4.3, the set

$$\mathcal{I} := \{a \in 2^\omega : (a, (y_m^*)) \in \Omega(x_n) \text{ for some } (y_m^n) \in (B_X^*)^\omega\}$$

is analytic. We shall prove that $\mathcal{I}^\perp \sim \mathcal{RK}(D)$. To this end, note that Grothendieck's double limit criterion (see e.g. [16, 1.6]) says that a set $C \subseteq D$ is relatively weakly compact if and only if for every subsequence $(x_{n_k})$ contained in $C$ and every sequence $(y_m^n)$ in $B_X^*$, the iterated limits

$$\lim_{k \to \infty} \lim_{m \to \infty} y_m^*(x_{n_k})$$ and $\lim_{m \to \infty} \lim_{k \to \infty} y_m^*(x_{n_k})$$

coincide whenever they exist. It is now easy to check the following statements:

- If $C \in \mathcal{RK}(D)$, then $F(C) := \{n \in \omega : x_n \in C\} \in \mathcal{I}^\perp$.
- If $a \in \mathcal{I}^\perp$, then $G(a) := \{x_n : n \in a\} \in \mathcal{RK}(D)$.

Clearly, the functions $F$ and $G$ are mutually inverse and give an isomorphism between the partially ordered sets $\mathcal{RK}(D)$ and $\mathcal{I}^\perp$. In particular, $\mathcal{RK}(D) \sim \mathcal{I}^\perp$. \qed

By combining Theorem 4.7 and Proposition 4.10, we get

Corollary 4.11 ($\Sigma_1^D$). If $D \subseteq B_X$ is a countable set, then $\mathcal{RK}(D)$ is Tukey equivalent to either $\{0\}$, $\omega$, $\omega^\omega$, $\mathcal{K}(\mathbb{Q})$ or $[\varepsilon]^\omega$.

We can now prove the main result of this section.

Proof of Theorem 4.4. By Lemma 4.8 and Corollary 4.11 (applied to a fixed countable norm dense set $D \subseteq B_X$), $\mathcal{K}(B_X)$ is Tukey equivalent to either $\{0\}$, $\omega^\omega$, $\mathcal{K}(\mathbb{Q})$ or $[\varepsilon]^\omega$ (remember that the case $\mathcal{K}(B_X) \sim \omega$ was excluded in Proposition 2.5). \qed
We finish this section by remarking that, even in the absence of analytic determinacy, the arguments in this section can still provide some information in ZFC.

**Theorem 4.12.** Let $I$ be an analytic family of subsets of $\omega$. Then either $I^\perp \sim [\omega]^<\omega$ or $I^\perp \leq K(Q) \times [\omega_1]^<\omega$. Therefore, the same holds for $K(B_X)$ when $X$ is separable.

**Proof.** We can distinguish two cases. In the first case, if $I$ is not countably separated from some analytic subfamily of $I^\perp$, then the argument of Case 2 of the proof of Theorem 4.7 can be applied (because Theorem 4.2 holds in ZFC when the families are both analytic, see the remarks before that theorem) and we obtain $I^\perp \sim [\omega]^<\omega$.

The second case is that $I$ is countably separated from every analytic subset of $I^\perp$. Since $I^\perp$ is coanalytic, we can write $I^\perp = \bigcup_{\alpha < \omega_1} J_{\alpha}$ where each $J_{\alpha}$ is analytic. The arguments of Case 1 of the proof of Theorem 4.7 can be applied to $I^\perp$ and $J_{\alpha}$ for each $\alpha < \omega_1$, obtaining that $J_{\alpha} \subset J'_{\alpha} \subset I^\perp$ where $J'_{\alpha} \sim K_{E_0}(\omega)$ for some coanalytic metric space $E_0$ in which $\omega$ is a dense subset. By Fremlin’s Theorem 2.2, for every $\alpha < \omega_1$ there is a Tukey reduction $\psi_{\alpha} : J'_{\alpha} \rightarrow K(Q)$. For every $a \in I^\perp$ we choose $\alpha(a) < \omega_1$ such that $a \in J_{\alpha(a)}$. Then the function

$$\psi : I^\perp \rightarrow K(Q) \times [\omega_1]^<\omega, \quad \psi(a) := (\psi_{\alpha(a)}(a), \{\alpha(a)\})$$

is Tukey and the proof is over. \qed

**Corollary 4.13.** Let $I$ be an analytic family of subsets of $\omega$. Then either $\text{cf}(I^\perp) = \omega$ or $\text{cf}(I^\perp) \leq \delta$. Therefore, the same holds for $K(B_X)$ when $X$ is separable.

5. **Classification of $\mathcal{A}K(B_X)$ under analytic determinacy**

This section is devoted to the proof of the following:

**Theorem 5.1 ($\Sigma^1_1$D).** If $X$ is separable, then one of the following holds:

(i) $\mathcal{A}K(B_X) \sim K(B_X) \sim \{0\}$,
(ii) $\omega \leq \mathcal{A}K(B_X) \leq \omega^\omega$ and $K(B_X) \sim \omega^\omega$,
(iii) $\mathcal{A}K(B_X) \sim K(B_X) \sim K(Q)$,
(iv) $\mathcal{A}K(B_X) \sim K(B_X) \sim [\omega]^<\omega$.

This will be obtained by gathering together Theorem 4.1 and Propositions 5.2 and 5.3 below. The general scheme is that we shall take the Tukey reductions that we constructed in the proof of Theorem 4.1 for $K(B_X)$ and we shall apply some refinements, mainly of Ramsey-theoretic nature, to obtain Tukey reductions for $\mathcal{A}K(B_X)$.

**Proposition 5.2 ($\Sigma^1_1$D).** If $X$ is separable, then $K(B_X) \sim [\omega]^<\omega$ if and only if $\mathcal{A}K(B_X) \sim [\omega]^<\omega$.

**Proposition 5.3 ($\Sigma^1_1$D).** If $X$ is separable, then $K(B_X) \sim K(Q)$ if and only if $\mathcal{A}K(B_X) \sim K(Q)$.
Proof of Theorem 5.1. By Theorem 4.1 we know that \( K(B_X) \) is Tukey equivalent to either \( \{0\} \), \( \omega^\omega \), \( K(Q) \) or \( [c]^{<\omega} \). Of course, we have \( K(B_X) \sim \{0\} \) if and only if \( AK(B_X) \sim \{0\} \) (if and only if \( X \) is reflexive). By Proposition 5.2 we have

\[
K(B_X) \sim [c]^{<\omega} \iff AK(B_X) \sim [c]^{<\omega},
\]

while Proposition 5.3 says that

\[
K(B_X) \sim K(Q) \iff AK(B_X) \sim K(Q).
\]

Finally, \( K(B_X) \sim \omega^\omega \) if and only if \( \omega \preceq AK(B_X) \preceq \omega^\omega \) (Corollary 3.17(ii)).

We next prove Propositions 5.2 and 5.3. We keep the notation of the previous section.

5.1. Case \( AK(B_X) \sim [c]^{<\omega} \). In this subsection we shall prove Proposition 5.2. To this end, the Ramsey principle needed is Lemma 5.5, which is a corollary of Milliken’s theorem [30] (cf. [5, Theorem 4]). Note that for any \( i \in \{0,1\} \) the set \( C_i \) of all \( i \)-chains of \( 2^{<\omega} \) is a closed subset of the compact metrizable space \( 2^{2^{<\omega}} \) (equipped with its usual product topology). To state Lemma 5.5 we need a definition.

**Definition 5.4.** A function \( u : 2^{<\omega} \to 2^{<\omega} \) is called a subtree if it is injective and for every \( t, s \in 2^{<\omega} \) and \( i \in \{0,1\} \) we have

\[
t \sim i \subseteq s \implies u(t) \sim i \subseteq u(s).
\]

Observe that any subtree maps \( i \)-chains to \( i \)-chains for \( i \in \{0,1\} \).

**Lemma 5.5.** Fix \( i \in \{0,1\} \). Let \( W \) be a finite set and \( c : C_i \to W \) an analytic measurable function (i.e. for every \( a \in W \) the set \( c^{-1}\{a\} \) belongs to the \( \sigma \)-algebra on \( C_i \) generated by the analytic sets). Then there exist a subtree \( u : 2^{<\omega} \to 2^{<\omega} \) and \( a \in W \) such that \( c(u(S)) = a \) for every \( S \in C_i \).

When using this principle, the elements of \( W \) are usually called colors and the function \( c \) is called a coloring of the \( i \)-chains. In this language, Lemma 5.5 states that if we color the \( i \)-chains with finitely many colors in a suitably measurable way, then we can pass to a subtree where all \( i \)-chains have the same color.

The following lemmas are needed to prove Proposition 5.2. Some of them will also be useful in other sections.

**Lemma 5.6.** Let \( P \subseteq 2^\omega \) be perfect. Then there is a subtree \( u : 2^{<\omega} \to 2^{<\omega} \) such that \( u(2^{<\omega}) \subseteq \{\sigma|_n : \sigma \in P, n < \omega\} \).

**Proof.** We define \( u \) inductively. Set \( u(\emptyset) := \emptyset \). Suppose \( u(t) \in T := \{\sigma|_n : \sigma \in P, n < \omega\} \)
has been constructed for \( t \in 2^{<\omega} \). We shall choose \( u(t \wedge 0), u(t \wedge 1) \in T \) such that
\[
u(t) \wedge i \subseteq u(t \wedge i) \quad \text{for } i \in \{0,1\}.
\]
To this end, write \( u(t) = \sigma|n \) for some \( \sigma \in P \) and \( n < \omega \). Since \( P \) is perfect, there is \( \sigma' \in P \setminus \{\sigma\} \) such that \( \sigma'|n = \sigma|n \). If \( m := \min\{k \geq n : \sigma'|k+1 \neq \sigma|k+1\} \), then we define \( u(t \wedge i) := \sigma|m \wedge i \) for \( i \in \{0,1\} \). It is clear that \( u \) satisfies the required properties. \( \square \)

**Lemma 5.7.** If \( X \) is separable, then the mapping \( p : X^{**} \to \mathbb{R} \) given by
\[
p(x^{**}) := d(x^{**}, X) = \inf_{x \in X} \|x^{**} - x\|
\]
is Borel\((X^{**}, w^*)\)-measurable.

**Proof.** Let \( X_0 \subseteq X \) be a countable dense set. Since \( B_{X^*} \) is \( w^* \)-separable, we can fix a countable \( w^* \)-dense set \( Z_0 \subseteq B_{X^*} \). Then for every \( x^{**} \in X^{**} \) we have
\[
p(x^{**}) = \inf_{x \in X_0} \|x^{**} - x\| = \inf_{x \in X_0} \sup_{x^* \in Z_0} x^*(x^{**} - x).
\]
Since for each \( x \in X_0 \) and \( x^* \in Z_0 \) the mapping \( x^{**} \mapsto x^*(x^{**} - x) \) is \( w^* \)-continuous, it follows that \( p \) is Borel\((X^{**}, w^*)\)-measurable. \( \square \)

**Definition 5.8.** Let \( \delta > 0 \). A sequence \((x_n)\) in \( X \) is called a \( \delta \)-controlled \( \ell^1 \)-sequence if it is bounded and
\[
\delta \sum_{i=0}^{n} |a_i| \leq \left\| \sum_{i=0}^{n} a_i x_i \right\|
\]
for every \( n < \omega \) and every \( a_0, \ldots, a_n \in \mathbb{R} \).

**Lemma 5.9.** Suppose \( X \) is separable. Let \( \{x_n : n < \omega\} \subseteq B_X \) be a countable set, \( f : 2^{<\omega} \to \omega \) an injective function and \( \delta > 0 \). Fix \( i \in \{0,1\} \). Each \( S \in \mathcal{C}_i \) is enumerated as \( S = \{S_n : n < \omega\} \) in such a way that \( S_n \wedge i \subseteq S_{n+1} \) for all \( n < \omega \). Then:

(i) The set \( W \) of all \( S \in \mathcal{C}_i \) such that \((x_{f(S_n)})\) is weakly Cauchy is coanalytic.

(ii) The set of all \( S \in \mathcal{C}_i \) such that \((x_{f(S_n)})\) is weakly Cauchy and
\[
d\left(w^* - \lim_{n \to \infty} x_{f(S_n)}, X\right) \geq \delta
\]
is coanalytic.

(iii) The set of all \( S \in \mathcal{C}_i \) such that \((x_{f(S_n)})\) is a \( \delta \)-controlled \( \ell^1 \)-sequence is Borel.

(iv) The set of all \( S \in \mathcal{C}_i \) such that \((x_{f(S_n)})\) is an \( \ell^1 \)-sequence is Borel.

**Proof.** Note first that, for each \( m < \omega \), the function
\[
e_m : \mathcal{C}_i \to X, \quad e_m(S) := x_{f(S_m)},
\]
is norm continuous. Indeed, let \((S^k)\) be a sequence in \(C_i\) converging to \(S \in C_i\). There is \(k_0 < \omega\) such that for every \(k \geq k_0\) and every \(s \in 2^{<\omega}\) with length\((s)\) \(\leq \) length\((s_m)\) we have
\[
 s \in S^k \iff s \in S,
\]
which implies that \(S^k_n = S_n\) for all \(n \leq m\). Hence \(e_m(S^k) = e_m(S)\) for every \(k \geq k_0\).

(i). By the continuity of the \(e_n\)'s, the set
\[
 E := \bigcup_{\varepsilon \in \mathbb{Q}^+} \bigcap_{n < \omega} \bigcap_{m,k \geq n} \left\{ (x^*, S) \in B_{X^*} \times C_i : \left| x^*(e_m(S)) - x^*(e_k(S)) \right| \geq \varepsilon \right\}
\]
is Borel in \(B_{X^*} \times C_i\), where \(B_{X^*}\) is equipped with the \(w^*\)-topology. Since \(X\) is separable, \((B_{X^*}, w^*)\) is metrizable and so it is a Polish space. If \(\pi : B_{X^*} \times C_i \to C_i\) stands for the second coordinate projection, then \(\pi(E)\) is analytic. Clearly, \(C_i \setminus \pi(E) = W\), the set of all \(S \in C_i\) for which \((x, S_n)\) is weakly Cauchy.

(ii). The function \(e : W \to X^{**}\) defined by
\[
 e(S) := w^* - \lim_{n \to \infty} e_n(S)
\]
is Borel\((X^{**}, w^*)\)-measurable (since each \(e_n\) is \(w^*\)-continuous) and so, by Lemma 5.7, the real-valued function defined on \(W\) by \(S \mapsto d(e(S), X)\) is Borel. Since \(W\) is coanalytic, the set \(\{ S \in W : d(e(S), X) \geq \delta \}\) is coanalytic as well.

(iii). The set of all \(S \in C_i\) such that \((e_n(S))\) is a \(\delta\)-controlled \(\ell^1\)-sequence can be written as
\[
 \bigcap_{I \in [\omega]^{<\omega}} \bigcap_{\phi \in \ell^I} \left\{ S \in C_i : \delta \left| \sum_{n \in I} \phi(n) \right| \leq \left\| \sum_{n \in I} \phi(n) e_n(S) \right\| \right\}.
\]
Since each \(e_n\) is norm continuous, the set above is Borel in \(C_i\). Finally, note that (iv) follows at once from (iii).

\[\square\]

**Lemma 5.10.** Let \((x_n)\) be a bounded sequence in \(X\) such that
\[
 \eta(x_n) := \inf \{ \| x^{**} - x \| : x^{**} \in \text{clust}_{X^{**}}(x_n), x \in X \} > 0
\]
where \(\text{clust}_{X^{**}}(x_n)\) stands for the set of all \(w^*\)-cluster points of \((x_n)\) in \(X^{**}\). Then for every \(0 < \delta < \eta(x_n)\) and every \(L \in K(X)\) the set \(\{ n < \omega : x_n \in L + \delta B_X \}\) is finite.

**Proof.** Fix any \(\delta > 0\). Suppose there is \(L \in K(X)\) such that \(x_n \in L + \delta B_X\) for infinitely many \(n\)'s. Then we can find a subsequence \((x_{n_k})\) and a weakly convergent sequence \((y_k)\) in \(L\) such that \(\| x_{n_k} - y_k \| \leq \delta\) for every \(k < \omega\). Let \(y \in L\) be the weak limit of \((y_k)\) and fix an arbitrary \(x^{**} \in \text{clust}_{X^{**}}(x_{n_k})\). Then
\[
 x^{**} - y \in \text{clust}_{X^{**}}(x_{n_k} - y_k) \subseteq \delta B_{X^{**}},
\]
hence \(\eta(x_n) \leq \delta\). This finishes the proof. \[\square\]

The following lemma was proved in [23, Lemma 5].
Lemma 5.11. Let $\delta > 0$. If $(x_n)$ is a $\delta$-controlled $\ell^1$-sequence in $X$, then $\eta(x_n) \geq \delta$.

Recall that $\mathcal{RK}(B_X)$ is the family of all relatively weakly compact subsets of $B_X$.

Lemma 5.12. Suppose there exist $\delta > 0$ and a function $f : 2^{<\omega} \to B_X$ such that:

(i) $f(S) \in \mathcal{RK}(B_X)$ for every $S \in C_0$;
(ii) $f(S) \nsubseteq L + \delta B_X$ for every $S \in C_1$ and every $L \in \mathcal{K}(B_X)$.

Then $[c]^{<\omega} \preceq \mathcal{AK}(B_X)$.

Proof. Consider the binary relation on $\mathcal{P}(B_X)$ defined by

$A \leq_\delta B :\iff A \subseteq B + \delta B_X$.

By (ii) we have $f(S) \nsubseteq_\delta A$ for every 1-chain $S \subseteq 2^{<\omega}$ and every $A \in \mathcal{RK}(B_X)$. Lemma 4.6 applied to the function $f$, the family $\mathcal{RK}(B_X)$ and the binary relation $\leq_\delta$ ensures that $[c]^{<\omega} \preceq (\mathcal{RK}(B_X), \leq_\delta)$. Since the function

$g : (\mathcal{RK}(B_X), \leq_\delta) \to (\mathcal{AK}(B_X), \leq_\delta), \quad g(A) := \overline{A}^w,$

is Tukey, we conclude that $[c]^{<\omega} \preceq \mathcal{AK}(B_X)$.

We can now prove the main result of this subsection.

Proof of Proposition 5.2. The “if part” (valid in ZFC) follows from Propositon 3.11(i) and the fact that $\mathcal{K}(B_X) \preceq [c]^{<\omega}$ whenever $X$ is separable (by Remark 2.4).

Conversely, suppose now that $\mathcal{K}(B_X) \sim [c]^{<\omega}$. Let $D = \{x_n : n < \omega\}$ be a countable dense subset of $B_X$ and let $\mathcal{I}$ be the set of all $A \subseteq \omega$ for which $\{x_n : n \in A\} \in \mathcal{RK}(D)$, i.e. $\{x_n : n \in A\}$ is relatively weakly compact. By the proofs of Theorems 4.1 and 4.7, there is an injective function $u : 2^{<\omega} \to \omega$ such that $u(S) \in \mathcal{I}$ (resp. $u(S) \in \mathcal{I}^\perp$) for every 0-chain (resp. 1-chain) $S \subseteq 2^{<\omega}$. Let $c : \mathcal{C}_1 \to \{0, 1, 2\}$ be the coloring defined by

$c(S) := \begin{cases} 
0 & \text{if } (x_{u(S_n)}) \text{ is weakly Cauchy}, \\
1 & \text{if } (x_{u(S_n)}) \text{ is an } \ell^1\text{-sequence}, \\
2 & \text{otherwise}.
\end{cases}$

(Here we follow the notation of Lemma 5.9) Since $c$ is analytic measurable (by Lemma 5.9), we can apply Lemma 5.5 to find a subtree $v : 2^{<\omega} \to 2^{<\omega}$ such that $c(v(\cdot))$ is constant on $\mathcal{C}_1$. Rosenthal’s $\ell^1$-theorem (see e.g. [13, Theorem 5.37]) states that every bounded sequence in a Banach space contains either a weakly Cauchy subsequence or an $\ell^1$-subsequence, so it is impossible that $c(v(S)) = 2$ for every $S \in \mathcal{C}_1$. Writing $w := u \circ v$, we are therefore reduced to consider two cases:

**Case 1:** $(x_{w(S_n)})$ is weakly Cauchy for every $S \in \mathcal{C}_1$. The $w^*$-limit $x^*_{S_n}$ of such a sequence belongs to $X^{**} \setminus X$, because otherwise $\{x_{w(S_n)} : n < \omega\} \in \mathcal{RK}(D)$, i.e.
Since we are assuming by \( \sigma \), consider
\[
\sigma^{(1)} := \{(\sigma_0, \ldots, \sigma_k) : \sigma_{k+1} = 1\} \in \mathcal{C}_1.
\]
Note that \( G \) is a \( \mathcal{G}_\delta \) subset of \( 2^\omega \), hence \( G \) is Polish. Write \( G = \bigcup_{n<\omega} A_n \), where
\[
A_n := \left\{ \sigma \in G : d(x_{\sigma^{(1)}}, X) > \frac{1}{n+1} \right\}.
\]
Each \( A_n \) is coanalytic, by Lemma \[5.9(ii)\] and the continuity of the map \( G \to \mathcal{C}_1 \) given by \( \sigma \mapsto \sigma^{(1)} \) (which can be proved easily). Fix \( m < \omega \) such that \( A_m \) is uncountable. Since we are assuming \( \Sigma^1_1 \mathbb{D} \), the uncountable coanalytic set \( A_m \) contains a set \( P \) homeomorphic to \( 2^\omega \) (see e.g. \[26, Theorem 32.2\]). By Lemma \[5.6\] there is a subtree \( \xi : 2^{<\omega} \to 2^{<\omega} \) such that \( \xi(2^{<\omega}) \subseteq T := \{\sigma_n : \sigma \in P, n < \omega\} \).

Claim: For every 1-chain \( S \subseteq 2^{<\omega} \) and every \( L \in \mathcal{K}(B_X) \) we have
\[
\{x_{w(\xi(S_n))} : n < \omega\} \not\subseteq L + \frac{1}{m+1} B_X.
\]
Indeed, \( \xi(S) \) is also a 1-chain, hence there exist \( \sigma \in G \) and a strictly increasing sequence \( (k_n) \) in \( \omega \) such that \( \xi(S_n) = \sigma|_{k_n} \) and \( \sigma|_{k_n} = 1 \) for all \( n < \omega \). Since \( \xi(2^{<\omega}) \subseteq T \), we have \( \sigma \in P \subseteq A_m \). Note that \( \sigma^{(1)} \supseteq \xi(S) \) and so \( x^{*\ast}_{\sigma^{(1)}} = x^{*\ast}_{\xi(S)} \), therefore
\[
d\left( w^* - \lim_{n \to \infty} x_{w(\xi(S_n))}, X \right) > \frac{1}{m+1}.
\]
An appeal to Lemma \[5.10\] finishes the proof of the claim.

Case 2: \( g(S) := (x_{w(S_n)}) \) is an \( \ell^1 \)-sequence for every \( S \in \mathcal{C}_1 \). We can write
\[
G = \bigcup_{n<\omega} B_n,
\]
where
\[
B_n := \left\{ \sigma \in G : g(\sigma^{(1)}) \text{ is } \frac{1}{n+1} \text{-controlled} \right\}.
\]
By Lemma \[5.9(iii)\] and the continuity of the mapping \( G \to \mathcal{C}_1 \) given by \( \sigma \mapsto \sigma^{(1)} \), each \( B_n \) is Borel, and so it contains a subset homeomorphic to \( 2^\omega \) whenever \( B_n \) is uncountable. As in Case 1, there exist \( m < \omega \) and a subtree \( \xi : 2^{<\omega} \to 2^{<\omega} \) such that, for every 1-chain \( S \subseteq 2^{<\omega} \), the sequence \( (x_{w(\xi(S_n))}) \) is a \( \frac{1}{m+1} \)-controlled \( \ell^1 \)-sequence, which implies that
\[
\{x_{w(\xi(S_n))} : n < \omega\} \not\subseteq L + \frac{1}{2(m+1)} B_X.
\]
for every \( L \in \mathcal{K}(B_X) \) (by Lemmas \[5.10\] and \[5.11\]).

In any of the two cases, we can apply Lemma \[5.12\] to the function \( f : 2^{<\omega} \to B_X \) given by \( f(t) := x_{w(\xi(t))} \) to conclude that \([c]^{<\omega} \leq \mathcal{AK}(B_X)\). \(\square\)
5.2. Case $\mathcal{AK}(B_X) \sim \mathcal{K}(\mathbb{Q})$. This subsection is devoted to sketching the proof of Proposition 5.3. To this end, we shall use a Ramsey theorem recently proved in [6]. Let us try to summarize the concepts and facts that we need, that can found in detail in [6] and in the more extended preprint [7]. Throughout this subsection $2^{<\omega}$ is equipped with the topology inherited from $2^\omega$ via the natural injection $s \mapsto s \upharpoonright 0$, so that $2^{<\omega}$ is homeomorphic to $\mathbb{Q}$. There are eight special types of infinite subsets of $2^{<\omega}$, denoted as $\mathfrak{T} = \{[0], [1], [01], [10], [011], [101], [011], [011]\}$.

Some properties of these types are the following:

(T1) For every set $A \subseteq 2^{<\omega}$ of type $\tau$ and every set $B \subseteq 2^{<\omega}$ of type $\tau' \neq \tau$, the intersection $A \cap B$ is finite.

(T2) Every infinite subset of $2^{<\omega}$ contains a further infinite subset which is of type $\tau$, for some $\tau \in \mathfrak{T}$.

(T3) Every set of type $\tau \in \{[0], [10]\}$ is relatively compact in $2^{<\omega}$, while every set of type $\tau \in \mathfrak{T} \setminus \{[0], [10]\}$ is closed and discrete in $2^{<\omega}$.

(T4) For every $\tau \in \mathfrak{T}$, the family $\mathcal{S}_\tau$ of all sets of type $\tau$ is a $G_\delta$ subset of the compact metrizable space $2^{2^{<\omega}}$, hence it can be viewed as a Polish space.

Associated to these types, we have the notion of nice embedding. The precise definition can be found in [6], but we do not give it here, let us just point out that a nice embedding is an injective function $u : 2^{<\omega} \to 2^{<\omega}$ that has the following properties:

(N1) For every $s, t \in 2^{<\omega}$, if length$(s) < \text{length}(t)$, then length$(u(s)) < \text{length}(u(t))$.

(N2) For every $s \in 2^{<\omega}$, we have:

- $u(s \upharpoonright 0) = u(s) \upharpoonright 0 \odot \cdots \odot 0$ for some number of 0’s;
- $u(s \upharpoonright 1) \supseteq u(s) \upharpoonright 1$.

(N3) For every $\tau \in \mathfrak{T}$, a set $A \subseteq 2^{<\omega}$ has type $\tau$ if and only if $u(A)$ has type $\tau$.

(N4) $u$ is a homeomorphism from $2^{<\omega}$ onto a closed subset of $2^{<\omega}$.

The Ramsey theorem from [6] that we mentioned is the following:

**Theorem 5.13.** Fix $\tau \in \mathfrak{T}$. Let $W$ be a finite set and $c : \mathcal{S}_\tau \to W$ an analytic measurable function (i.e. for every $a \in W$ the set $c^{-1}([a])$ belongs to the $\sigma$-algebra on $\mathcal{S}_\tau$ generated by the analytic sets). Then there is a nice embedding $u : 2^{<\omega} \to 2^{<\omega}$ such that $c$ is constant on $\{u(A) : A \in \mathcal{S}_\tau\}$.

In addition, we shall need some extension of this result for countable partitions, that holds only for some of the types; see [7] Lemma 2.8.1.

**Lemma 5.14.** Fix $\tau \in \mathfrak{T} \setminus \{[0], [10]\}$. Let $c : \mathcal{S}_\tau \to \omega$ be an analytic measurable function such that $c(A) = c(B)$ whenever $A \triangle B$ is finite. Then there is a nice embedding $u : 2^{<\omega} \to 2^{<\omega}$ such that $c$ is constant on $\{u(A) : A \in \mathcal{S}_\tau\}$. 

**Proof of Proposition 5.3.** Suppose first that $\mathcal{AK}(B_X) \sim \mathcal{K}(\mathbb{Q})$. Then

$$\mathcal{K}(\mathbb{Q}) \sim \mathcal{AK}(B_X) \leq \mathcal{K}(B_X)$$

(apply Proposition 3.11(i)). By Theorem 4.1 we get that either $\mathcal{K}(B_X) \sim \mathcal{K}(\mathbb{Q})$ or $\mathcal{K}(B_X) \sim [\mathbb{c}]^{<\omega}$. An appeal to Proposition 5.2 yields $\mathcal{K}(B_X) \sim \mathcal{K}(\mathbb{Q})$, as desired.

Suppose now that $\mathcal{K}(B_X) \sim \mathcal{K}(\mathbb{Q})$. We divide the proof that $\mathcal{AK}(B_X) \sim \mathcal{K}(\mathbb{Q})$ into several steps.

**Step 1.** There is an injective function $\nu : 2^{<\omega} \to B_X$ such that:

(*) a set $A \subseteq 2^{<\omega}$ is relatively compact in $2^{<\omega}$ if and only if $\nu(A)$ is relatively weakly compact in $X$.

To prove this, let $\{x_n : n < \omega\} \subseteq B_X$ be a countable dense set. Tracking the proof of Theorems 4.1 and 4.7, it follows that there is a metric space $(E_0, \rho)$ such that:

- $E_0$ is coanalytic (in some Polish space) but it is not Polish;
- $\omega$ is a dense subset of $E_0$;
- a set $C \subseteq \omega$ is relatively compact in $E_0$ if and only if $\{x_n : n \in C\}$ is relatively weakly compact.

By Hurewicz’s theorem (see e.g. [26, 21.18]), there is a closed set $F \subseteq E_0$ which is homeomorphic to $\mathbb{Q}$ (and so homeomorphic to $2^{<\omega}$). Enumerate $F = \{f_n : n < \omega\}$ and consider a set $G = \{g_n : n < \omega\} \subseteq \omega$ such that $\rho(f_n, g_n) \leq \frac{1}{n+1}$ for all $n < \omega$. Let $\varphi : 2^{<\omega} \to \omega$ be a bijection such that the induced mapping $t \mapsto f_{\varphi(t)}$ is a homeomorphism between $2^{<\omega}$ and $F$. It is clear that the function

$$\nu : 2^{<\omega} \to B_X, \quad \nu(t) := x_{g_{\varphi(t)}},$$

satisfies the required properties.

**Step 2.** For every infinite set $A \subseteq 2^{<\omega}$ we define $c_{\nu}(A) \in \{0, 1, 2\}$ by

$$c_{\nu}(A) := \begin{cases} 
0 & \text{if } \nu(A) \text{ is weakly Cauchy,} \\
1 & \text{if } \nu(A) \text{ is an } \ell^1\text{-sequence,} \\
2 & \text{otherwise,}
\end{cases}$$

where $\nu(A)$ is ordered as a sequence following the lexicographical order of $2^{<\omega}$. We can assume without loss of generality that $c_{\nu}$ is constant (equal to 0 or 1) on $S_\tau$ for every $\tau \in \mathfrak{T}$. To see this, enumerate $\mathfrak{T} = \{\tau_0, \ldots, \tau_\tau\}$. Since the restriction of $c_{\nu}$ to $S_{\tau_0}$ is analytic measurable, Theorem 5.13 ensures the existence of a nice embedding $u_0 : 2^{<\omega} \to 2^{<\omega}$ such that $c_{\nu}$ is constant on $\{u_0(A) : A \in S_{\tau_0}\}$. On the other hand, since $u_0$ is a homeomorphism onto a closed subset of $2^{<\omega}$ (by property (N4)), the composition $v_0 := \nu \circ u_0$ also has property (*) above. Clearly, $c_{v_0}$ is constant on $S_{\tau_0}$. By the same argument, now applied to the mapping $c_{v_{\tau_1}}$ and the type $\tau_1$, there is a nice embedding $u_1 : 2^{<\omega} \to 2^{<\omega}$ such that $c_{v_{\tau_1}}$ is constant on $\{u_1(A) : A \in S_{\tau_1}\}$. . .
Note that \( v_1 := v_0 \circ u_1 \) satisfies property (*) and that \( c_{v_1} \) is constant on \( \mathcal{S}_{\tau_0} \) and also on \( \mathcal{S}_{\tau_1} \). By continuing in this way, we find an injective function \( \tilde{v} : 2^{<\omega} \to B_X \) satisfying property (*) such that \( c_{\tilde{v}} \) is constant on \( \mathcal{S}_\tau \) for every \( \tau \in \mathfrak{T} \). Bearing in mind Rosenthal’s \( \ell^1 \)-theorem (see e.g. [13, Theorem 5.37]), the constant value cannot be 2.

**Step 3.** Let \( \tau \in \mathfrak{T} \setminus \{[0], [1^0] \} \) such that \( c_{\tilde{v}} \equiv 0 \) on \( \mathcal{S}_\tau \). Any \( A \in \mathcal{S}_\tau \) is infinite, closed and discrete in \( 2^{<\omega} \), hence it is not relatively compact and so \( v(A) \) is not relatively weakly compact. Since \( v(A) \) is weakly Cauchy, it converges to some \( x_A^{**} \in X^{**} \setminus X \). The function \( \hat{c} : \mathcal{S}_\tau \to \mathbb{N} \) given by

\[
\hat{c}(A) := \min \{ n \in \mathbb{N} : d(x_A^{**}, X) > \frac{1}{n} \}
\]

satisfies the requirements of Lemma [5.14] (use Lemma [5.7]), so there is a nice embedding \( u_\tau : 2^{<\omega} \to 2^{<\omega} \) such that \( \hat{c} \) is constant on \( \{ u_\tau(A) : A \in \mathcal{S}_\tau \} \), that is, there is \( \delta_\tau > 0 \) such that for every \( A \in \mathcal{S}_\tau \) we have \( d(x_{u_\tau(A)}^{**}, X) > \delta_\tau \), so \( v(u_\tau(A)) \not\subseteq L + \delta_\tau B_X \) for any weakly compact set \( L \subseteq X \).

**Step 4.** Let \( \tau \in \mathfrak{T} \setminus \{[0], [1^0] \} \) such that \( c_{\tilde{v}} \equiv 1 \) on \( \mathcal{S}_\tau \). Define a function

\[
\hat{c} : \mathcal{S}_\tau \to \mathbb{N}
\]

by declaring \( \hat{c}(A) \) to be the least \( n \in \mathbb{N} \) for which there is a finite set \( F \subseteq A \) such that \( v(A \setminus F) \) is a \( \frac{1}{n} \)-controlled \( \ell^1 \)-sequence. Since \( \hat{c} \) is analytic measurable, we can apply Lemma [5.14] to obtain a nice embedding \( u_\tau : 2^{<\omega} \to 2^{<\omega} \) such that \( \hat{c} \) is constant (say, equal to \( n_\tau \)) on \( \{ u_\tau(A) : A \in \mathcal{S}_\tau \} \). Writing \( \delta_\tau := \frac{1}{2n_\tau} \), an appeal to Lemma [5.11] ensures that for every \( A \in \mathcal{S}_\tau \) the set \( v(u_\tau(A)) \) is not contained in any set of the form \( L + \delta_B X \) with \( L \subseteq X \) weakly compact.

**Step 5.** We can assume that there is \( \delta > 0 \) such that for every \( \tau \in \mathfrak{T} \setminus \{[0], [1^0] \} \) and every \( A \in \mathcal{S}_\tau \), we have \( v(A) \not\subseteq L + \delta B_X \) for any weakly compact set \( L \subseteq X \). To check this, enumerate \( \mathfrak{T} \setminus \{[0], [1^0] \} = \{\tau_0, \ldots, \tau_5\} \). By the former steps applied to \( \tau_0 \), there exist a nice embedding \( u_0 : 2^{<\omega} \to 2^{<\omega} \) and \( \delta_0 > 0 \) such that, for every \( A \in \mathcal{S}_{\tau_0} \), we have \( v(u_0(A)) \not\subseteq L + \delta_0 B_X \) for any weakly compact set \( L \subseteq X \). Now, we can apply again Step 3 or 4 to the composition \( v \circ u_0 : 2^{<\omega} \to B_X \) and type \( \tau_1 \) to obtain a nice embedding \( u_1 : 2^{<\omega} \to 2^{<\omega} \) and \( \delta_1 > 0 \) such that, for every \( A \in \mathcal{S}_{\tau_1} \), we have \( v(u_0(u_1(A))) \not\subseteq L + \delta_1 B_X \) for any weakly compact set \( L \subseteq X \). By continuing in this manner, we obtain a function \( \tilde{v} : 2^{<\omega} \to B_X \) with the same properties as \( v \) and a constant \( \delta > 0 \) such that, for every \( \tau \in \mathfrak{T} \setminus \{[0], [1^0] \} \) and every \( A \in \mathcal{S}_\tau \), we have \( \tilde{v}(A) \not\subseteq L + \delta B_X \) for any weakly compact set \( L \subseteq X \).

**Step 6.** Finally, we shall check that \( \mathcal{K}(2^{<\omega}) \preceq \mathcal{A}\mathcal{K}(B_X) \). This will finish the proof since \( 2^{<\omega} \) is homeomorphic to \( \mathbb{Q} \). Define

\[
f : \mathcal{K}(2^{<\omega}) \to \mathcal{A}\mathcal{K}(B_X), \quad f(A) := \overline{v(A)}^p.
\]
We claim that for every $L \in \mathcal{AK}(B_X)$ there is $B \in \mathcal{K}(2^{<\omega})$ containing any $A \in \mathcal{K}(2^{<\omega})$ for which $f(A) \subseteq L + \delta B_X$. Indeed, it suffices to check that $B = v^{-1}(L + \delta B_X)$

is compact in $2^{<\omega}$, which is equivalent to saying (by the metrizability of $2^{<\omega}$) that every infinite subset of $v^{-1}(L + \delta B_X)$ contains a further infinite subset which is relatively compact in $2^{<\omega}$. Take an infinite set $C \subseteq v^{-1}(L + \delta B_X)$. By property (T2), there is an infinite set $D \subseteq C$ of some type $\tau$. From Step 5 and the fact that $v(D) \subseteq L + \delta B_X$ it follows that $\tau \in \{[0], [10]\}$. Property (T3) ensures that $D$ is relatively compact in $2^{<\omega}$ and the proof is over. \hfill \Box

6. Banach spaces not containing $\ell_1$

This section is entirely devoted to proving the following result, that holds in ZFC without need of additional axioms:

**Theorem 6.1.** If $X$ is separable and contains no copy of $\ell_1$, then $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X)$. More precisely:

(i) $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim \{0\}$ if $X$ is reflexive.

(ii) $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim \omega^\omega$ if $X$ is not reflexive, has separable dual and the PCP.

(iii) $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim \mathcal{K}(Q)$ if $X$ has separable dual and fails the PCP.

(iv) $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim [c]^{<\omega}$ if $X$ has non-separable dual.

We already know from Proposition 3.11(i) that $\mathcal{AK}(B_X) \preceq \mathcal{K}(B_X)$ holds for any Banach space $X$ and we also know from Proposition 2.7 the classification of $\mathcal{K}(B_X)$ when $X^*$ is separable. Since case (i) is trivial and $\mathcal{K}(B_X) \preceq [c]^{<\omega}$ whenever $X$ is separable (by Remark 2.4), in order to prove Theorem 6.1 it remains to check:

- $\omega^\omega \preceq \mathcal{AK}(B_X)$ if $X$ has separable dual and is not reflexive.
- $\mathcal{K}(Q) \preceq \mathcal{AK}(B_X)$ if $X$ has separable dual and fails the PCP.
- $[c]^{<\omega} \preceq \mathcal{AK}(B_X)$ if $X$ has non-separable dual and contains no copy of $\ell_1$.

We will prove these facts in Propositions 6.3, 6.4 and 6.12 below. Note that Theorem 6.1 implies, in particular, that if $X$ is separable and contains no copy of $\ell^1$, then $\text{cf}(\mathcal{AK}(B_X)) \in \{0, c\}$ and $\text{add}_\omega(\mathcal{AK}(B_X)) \in \{\omega_1, b\}$ unless $X$ is reflexive.

As we noted in Proposition 3.11(i), we have $\mathcal{AK}(E) \preceq \mathcal{K}(E)$ for every $E \subseteq X$. Lemma 6.2 below shows that under some assumption we have the converse reduction $\mathcal{K}(E) \preceq \mathcal{AK}(B_X)$. This lemma will be useful in the sequel.

**Lemma 6.2.** Let $E \subseteq B_X$. Suppose there is $\delta > 0$ such that for every $A \subseteq E$ we have

(i) either $A$ is contained in a weakly compact subset of $E$,

(ii) or $A \not\subseteq L + \delta B_X$ for any $L \in \mathcal{K}(X)$. 


Then $\mathcal{K}(E) \leq \mathcal{AK}(B_X)$. In particular, this Tukey reduction holds if there is $\delta > 0$ such that

$$\|x^{**} - x\| > \delta \quad \text{for every } x^{**} \in \overline{E^{w^*}} \setminus E \text{ and every } x \in X.$$  

(6.1)

**Proof.** We shall check that the function $f : \mathcal{K}(E) \to \mathcal{AK}(B_X)$ given by $f(K) := K$ is a Tukey reduction from $\mathcal{K}(E)$ to $(\mathcal{AK}(B_X), \leq_{\delta})$. Fix $L \in \mathcal{AK}(B_X)$ and define $L_0 := E \cap (L + \delta B_X)$. Then $L_0$ is contained in a weakly compact subset of $E$. Since $L_0$ is weakly closed in $E$, we conclude that $L_0$ is weakly compact. Obviously, for every $K \in \mathcal{K}(E)$ satisfying $K \subseteq L + \delta B_X$ we have $K \subseteq L_0$. This proves that $\mathcal{K}(E) \leq \mathcal{AK}(B_X)$. For the last statement, note that if $A \subseteq E \cap (L + \delta B_X)$ for some $L \in \mathcal{K}(X)$, then $\overline{A^{w^*}} \subseteq \overline{E^{w^*}} \cap (L + \delta B_{w^*})$; therefore, (6.1) implies that $\overline{A^{w^*}} \subseteq E$ and so $\overline{A^{w^*}} \in \mathcal{K}(E)$. The proof is over. 

\Box

**Proposition 6.3.** If $X^*$ is separable and $X$ is not reflexive, then $\omega^* \leq \mathcal{AK}(B_X)$.

**Proof.** Since $X^*$ is separable, $B_{X^{**}}$ is metrizable in the $w^*$-topology. Let $\rho$ be a metric on $B_{X^{**}}$ that metrizes the $w^*$-topology. Fix $0 < \theta < 1$. By the non-reflexivity of $X$ and Riesz’s Lemma (see e.g. [8, p. 2]), there is $x^{**} \in X^{**}$ with $\|x^{**}\| = 1$ such that $\|x^{**} - x\| > \theta$ for all $x \in X$. Since $B_X$ is $w^*$-dense in $B_{X^{**}}$, we can find a sequence $(x_n)$ in $B_X$ that $w^*$-converges to $x^{**}$. For each $n < \omega$, we define $y_n := \frac{1}{\theta}(x^{**} - x_n) \in B_{X^{**}}$ and we choose a sequence $(y_n^m)$ in $B_X$ such that

$$\rho(y_n^m, y_n) \leq \frac{1}{2^{n+m}} \quad \text{for every } m < \omega.$$  

(6.2)

We claim that the set $E := \{y_n^m : n, m < \omega\} \cup \{0\} \subseteq B_X$ satisfies the equality

$$\overline{E^{w^*}} = E \cup \{y_n : n < \omega\}.$$  

(6.3)

Indeed, let $(z_k)$ be a sequence in $E$ that $w^*$-converges to some $y \in \overline{E^{w^*}}$. If $z_k = 0$ only for finitely many $k$’s, then by passing to a subsequence we may assume that each $z_k$ is of the form $z_k = y_{n_k}^{m_k}$ for some $n_k, m_k < \omega$. Then there is a further subsequence of $(z_k)$, not relabeled, such that:

(a) either $n_k < n_{k+1}$ for all $k < \omega$,

(b) or there is $n < \omega$ such that $n_k = n$ for all $k < \omega$.

If (a) holds, then

$$\rho(y_{n_k}, y) \leq \rho(y_{n_k}, z_k) + \rho(z_k, y) \leq \frac{1}{2^{n_k+m_k}} + \rho(z_k, y) \leq \frac{1}{2^{n_k}} + \rho(z_k, y) \quad \text{for all } k < \omega,$$

hence $\rho(y_{n_k}, y) \to 0$ and so $y = 0 \in E$ (because $(y_n)$ is $w^*$-convergent to $0$). If (b) holds, then $z_k = y_{m_k}^{n_k}$ for all $k < \omega$ and so either $y \in E$ or $y = y_n$. This proves (6.3).

Note that $E$ satisfies the hypotheses of Lemma 6.2 by equality (6.3) and the fact that $\|y_n - x\| > \frac{\theta}{2}$ for every $n < \omega$ and every $x \in X$. Hence $\mathcal{K}(E) \leq \mathcal{AK}(B_X)$. 

We claim that the function $K$ is Tukey. Indeed, fix $y \in \mathcal{K}(E)$. For each $n < \omega$, we write

$$\rho(y^n_{s(n)}; y_n) \leq \frac{1}{2^{n+s(n)}} \leq \frac{1}{2^n} \quad \text{for all } n < \omega.$$ 

We claim that the function

$$F : \omega^\omega \to \mathcal{K}(E), \quad F(\varphi) := \{y^n_{\varphi(n)} : n < \omega\} \cup \{0\},$$

is Tukey. Indeed, fix $K \in \mathcal{K}(E)$. For each $n < \omega$, the set $\{m < \omega : y^n_m \in K\}$ is finite, because $y^n_m \to y_n \in X^{**} \setminus X$ in the $w^*$-topology as $m \to \infty$. Thus, there is $\varphi_K \in \omega^\omega$ such that $y^n_m \notin K$ for every $n < \omega$ and every $m > \varphi_K(n)$. Hence $\varphi \leq \varphi_K$ whenever $F(\varphi) \subseteq K$. This proves that $F$ is a Tukey function.

For the tree $\omega^{<\omega}$ of all finite sequences of natural numbers we use notations similar to those used for the dyadic tree $2^{<\omega}$. For instance, given $s = (s_0, \ldots, s_k) \in \omega^{<\omega}$ and $n < \omega$, we write $s \upharpoonright n = (s_0, \ldots, s_k, n)$.

**Proposition 6.4.** If $X^*$ is separable and $X$ fails the PCP, then $\mathcal{K}(\mathbb{Q}) \leq \mathcal{AK}(B_X)$.

**Proof.** We divide the proof into several steps.

**Step 1.** Since $X$ contains no copy of $\ell^1$ and fails the PCP, according to [27, Theorem 2.4] there exist $\varepsilon > 0$ and a set $\{x_s : s \in \omega^{<\omega}\} \subseteq X$ such that, for every $s \in \omega^{<\omega}$, we have:

$$\|x_s\| > \varepsilon, \quad \left\|\sum_{t \subseteq s} x_t\right\| \leq 1 \quad \text{and} \quad x_{s \upharpoonright n} \to 0 \text{ weakly as } n \to \infty.$$ 

For each $s \in \omega^{<\omega}$ we define $y_s := \sum_{t \subseteq s} x_t \in B_X$. Note that for every $s \in \omega^{<\omega}$ the sequence $(y_{s \upharpoonright n})$ is weakly convergent to $y_s$, while $\|y_{s \upharpoonright n} - y_s\| > \varepsilon$ for all $n < \omega$.

Since $X^*$ is separable, $B_X^{**}$ is metrizable in the $w^*$-topology. Let $\rho$ be a metric on $B_X^{**}$ that metrizes the $w^*$-topology. For every $s = (s_0, \ldots, s_k) \in \omega^{<\omega}$ we write

$$\Delta(s) := 2^{-(k+1+\sum_{i=0}^k s_i)}.$$

Clearly, we can construct recursively an injective function $u : \omega^{<\omega} \to \omega^{<\omega}$ such that, for each $s \in \omega^{<\omega}$, there is a strictly increasing $\varphi_s \in \omega^\omega$ such that

$$u(s \upharpoonright n) = u(s) \upharpoonright \varphi_s(n) \quad \text{and} \quad \rho(y_{u(s \upharpoonright n)}; y_{u(s)}) \leq \Delta(s \upharpoonright n) \quad \text{for every } n < \omega.$$ 

Therefore, replacing the family $\{y_s : s \in \omega^{<\omega}\}$ by $\{y_{u(s)} : s \in \omega^{<\omega}\}$ if necessary, we can assume without loss of generality that

$$\rho(y_{s \upharpoonright n}; y_s) \leq \Delta(s \upharpoonright n) \quad \text{for every } s \in \omega^{<\omega} \text{ and } n < \omega.$$ (6.4) 

Claim 1. For every $\sigma \in \omega^\omega$ the sequence $(y_{\sigma|n})$ is $w^*$-convergent to some $y_{\sigma}^{**} \in B_{X^{**}}$. Indeed, since $(B_{X^{**}}, \rho)$ is complete, it suffices to check that $(y_{\sigma|n})$ is $\rho$-Cauchy, and this clearly follows from the inequality

$$\rho(y_{\sigma|n+1}, y_{\sigma|n}) \leq \frac{\Delta(\sigma|n+1)}{2^{n+1}}$$

for all $n \in \mathbb{N}$.

Claim 2. The function $\omega^\omega \to B_{X^{**}}$ given by $\sigma \mapsto y_{\sigma}^{**}$ is continuous from the natural topology of $\omega^\omega$ to the $w^*$-topology of $B_{X^{**}}$. Indeed, given $s \in \omega^{<\omega}$ and $u = (u_0, \ldots, u_k) \in \omega^{<\omega}$, inequality (6.4) yields

$$\rho(y_{s\upharpoonright u}, y_s) \leq \sum_{i=0}^{k} \Delta(s \cap (u_0, \ldots, u_i)) \leq \sum_{i=0}^{k} \frac{\Delta(s)}{2^{i+1}} \leq \Delta(s).$$

It follows that $\rho(y_{\sigma}^{**}, y_{\sigma|n}) \leq \Delta(\sigma|n) \leq 2^{1-n}$ for every $\sigma \in \omega^\omega$ and every $n < \omega$. This implies that for every $\varepsilon > 0$ there is $n < \omega$ such that $\rho(y_{\sigma}^{**}, y_{\tau}^{**}) \leq \varepsilon$ whenever $\sigma, \tau \in \omega^\omega$ satisfy $\sigma|n = \tau|n$. Thus, the mapping $\sigma \mapsto y_{\sigma}^{**}$ is $w^*$-continuous.

Step 2. For each $n \in \mathbb{N}$ we define

$$A_n := \{ \sigma \in \omega^\omega : d(y_{\sigma}^{**}, X) \geq \frac{1}{n} \}$$

and we also set

$$A_0 := \{ \sigma \in \omega^\omega : y_{\sigma}^{**} \in X \} = \omega^\omega \setminus \bigcup_{n \in \mathbb{N}} A_n.$$

Note that $A_n$ is Borel for every $n < \omega$. Indeed, to check this for $n \geq 1$, it suffices to apply Lemma 5.7 and the $w^*$-continuity of the mapping $\sigma \mapsto y_{\sigma}^{**}$.

Now, we are going to apply some Ramsey-theoretic principles from [37, Section 7.2]. We fix a non-principal ultrafilter $\mathcal{U}$ on $\omega$. For coherence with the setting in which our reference [37] is written, let us consider the subtree $\omega^{<\omega} \subseteq \omega^{<\omega}$ consisting of all strictly increasing finite sequences in $\omega$ (which is naturally identified with the set $\mathbb{N}^{<\infty}$ of all finite subsets of $\mathbb{N}$). Let $\omega^{[\omega]} \subseteq \omega^\omega$ be the closed set of all strictly increasing infinite sequences in $\omega$ (which is naturally identified with the set $\mathbb{N}^{[\infty]}$ of all infinite subsets of $\mathbb{N}$). For every $m < \omega$ we define $A'_m := A_m \cap \omega^{[\omega]}$. Since $A'_m$ is Borel, we can apply [37, Theorem 7.42 and Lemma 7.36] to conclude that $A'_m$ is $\mathcal{U}$-Ramsey. Since $\omega^{[\omega]} = \bigcup_{m<\omega} A'_m$ and the family of $\mathcal{U}$-null sets is a proper $\sigma$-ideal of subsets of $\omega^{[\omega]}$ (see [37, Lemma 7.41]), there is $m < \omega$ such that $A'_m$ is not $\mathcal{U}$-null. It follows from [37, Definitions 7.37 and 7.39] that there is a $\mathcal{U}$-tree $\Upsilon \subseteq \omega^{<\omega}$ such that

$$[\Upsilon] := \{ \sigma \in \omega^{[\omega]} : \sigma|n \in \Upsilon \text{ for every } n < \omega \} \subseteq A'_m.$$

That $\Upsilon$ is a $\mathcal{U}$-tree means that it is a $\subseteq$-downwards closed subtree of $\omega^{<\omega}$ such that, for every $s \in \Upsilon$, the set $\{ n < \omega : s \upharpoonright n \in \Upsilon \}$ belongs to $\mathcal{U}$.

Step 3. Define $E := \{ y_s : s \in \Upsilon \} \subseteq B_X$. Note that $E$ equipped with the weak topology is a countable metrizable space without isolated points. Indeed, given $s \in \Upsilon$
and a weakly open set $V \subseteq X$ with $y_s \in V$, the fact that $(y_{s\sim n})$ is weakly convergent to $y_s$ ensures the existence of $n_0 < \omega$ such that $y_{s\sim n} \in V$ for all $n \geq n_0$. Since $\Upsilon$ is a $\mathcal{U}$-tree, the set $\{n \geq n_0 : s \sim n \in \Upsilon\}$ is infinite, so $E \cap V$ is infinite as well. It follows that $E$ is homeomorphic to $\mathbb{Q}$.

We claim that

(6.7)  \[ E^{w^*} = E \cup \{y_{\sigma}^{**} : \sigma \in [\Upsilon]\}. \]

Indeed, the inclusion “$\supseteq$” being obvious, let us prove “$\subseteq$”. Fix a sequence $(z_k)$ in $E$ that $w^*$-converges to some $y \in E^{w^*}$, and write $z_k = y_{s^k}$ for some $s^k \in \Upsilon$. It is not difficult to check that there is a subsequence of $(z_k)$, not relabeled, satisfying one of the following conditions:

- **Condition 1**: there exist $s \in \Upsilon$ and $\varphi \in \omega^{[\omega]}$ such that $s \sim \varphi(k) \subseteq s^k$ for every $k < \omega$. Then

\[
\rho(y_{s^k}, y_s) \leq \rho(y_{s^k}, y_{s\sim \varphi(k)}) + \rho(y_{s\sim \varphi(k)}, y_s) \leq \rho(y_{s\sim \varphi(k)}, y_s) \leq \frac{1}{2^{\varphi(k)}} + \rho(y_{s\sim \varphi(k)}, y_s) \quad \text{for every } k < \omega,
\]

hence $(y_{s^k})$ is weakly convergent to $y_s$ and so $y = y_s$.

- **Condition 2**: there exist $\sigma \in [\Upsilon]$ such that $\sigma|_k \subseteq s^k$ for every $k < \omega$. Then

\[
\rho(y_{s^k}, y_{\sigma}^{**}) \leq \rho(y_{s^k}, y_{\sigma|_k}) + \rho(y_{\sigma|_k}, y_{\sigma}^{**}) \leq \Delta(\sigma|_k) + \rho(y_{\sigma|_k}, y_{\sigma}^{**}) \leq \frac{1}{2^k} + \rho(y_{\sigma|_k}, y_{\sigma}^{**}) \quad \text{for every } k < \omega,
\]

therefore $(y_{s^k})$ is $w^*$-convergent to $y_{\sigma}^{**}$ and so $y = y_{\sigma}^{**}$.

This completes the proof of (6.7).

**Step 4.** If $m > 0$, then $E$ satisfies the requirements of Lemma (6.2) (by (6.6) and (6.7)), and we conclude that $\mathcal{K}(\mathbb{Q}) \sim \mathcal{K}(E) \preceq \mathcal{AK}(B_X)$. Therefore, in order to finish the proof it remains to check that $m > 0$. Our proof is by contradiction. If $m = 0$, then (6.6) and (6.7) yield $E^{w^*} \subseteq X$ and hence $E$ is relatively weakly compact. Therefore, $E$ is fragmented by the norm (see [31]), so there is a weakly open set $V \subseteq X$ such that $V \cap E \neq \emptyset$ and $\|y - y'\| \leq \varepsilon$ for every $y, y' \in V \cap E$. Arguing as in the proof that $E$ has no isolated points (Step 3), we find $s \in \Upsilon$ and $n < \omega$ such that $y_s, y_{s\sim n} \in V \cap E$, hence $\|y_s - y_{s\sim n}\| \leq \varepsilon$, a contradiction. The proof is over. \(\square\)

From now on we consider a topological space $J_3$ whose underlying set is $2^{<\omega} \cup 2^\omega$ and whose topology is defined by:

- all points from $2^{<\omega}$ are isolated;
- any $x \in 2^\omega$ has a neighborhood basis made of the sets $\{x\} \cup \{x|_k : k > n\}$, where $n < \omega$.
Let $K_3 := 2^{<\omega} \cup 2^\omega \cup \{\infty\}$ be its one-point compactification and consider its subspace $L_3 := 2^{<\omega} \cup \{\infty\}$. The following lemma provides a characterization of the compact subsets of these topological spaces. Recall that an antichain of $2^{<\omega}$ is a subset made up of pairwise incomparable elements.

**Lemma 6.5.**

(i) A set $C \subseteq J_3$ is relatively compact if and only if $C \cap 2^\omega$ is finite and $C$ contains no infinite antichain of $2^{<\omega}$.

(ii) An infinite set $C \subset L_3$ is compact if and only if $\infty \in C$ and $C \cap \{x|_n : n < \omega\}$ is finite for every $x \in 2^\omega$.

**Proof.** (i). Suppose first that $C$ is relatively compact. Since

$$\{\{t\} : t \in C \cap 2^{<\omega}\} \cup \\{\{x\} \cup \{x|_k : k < \omega\} : x \in C \cap 2^{<\omega}\}$$

is an open cover of $C$, it admits a finite subcover, that is, there exist finite sets $C_1 \subseteq C \cap 2^{<\omega}$ and $C_2 \subseteq C \cap 2^\omega$ such that

$$C \subseteq C_1 \cup C_2 \cup \{x|_k : x \in C_2, k < \omega\}.$$  

Clearly, this implies that $C \cap 2^{<\omega} = C_2$ and that $C$ contains no infinite antichain of $2^{<\omega}$.

In order to prove the converse, let $D \subseteq C$ be an infinite set. Since $C \cap 2^\omega$ is finite, the set $D \cap 2^{<\omega}$ is infinite and Ramsey’s theorem (see e.g. [21, Theorem 9.1]) ensures that $D \cap 2^{<\omega}$ contains either an infinite chain or an infinite antichain. The latter being impossible by assumption, we conclude that there exist $x \in 2^\omega$ and a strictly increasing sequence $(n_k)$ in $\omega$ such that $\{x|_{n_k} : k < \omega\} \subseteq D \cap 2^{<\omega}$. It is clear that $x$ is an accumulation point of $D$. This proves that $C$ is relatively compact.  

(ii). Let $C \subseteq L_3$ be a compact set. If $\infty \notin C$, then $C$ is finite because points of $L_3 \setminus \{\infty\}$ are isolated. Take any $x \in 2^\omega$. Since the sequence $(x|_n)$ converges to $x$ in $K_3$ and $C$ is a compact subset of $L_3$, the set $C \cap \{x|_n : n < \omega\}$ is finite.

Conversely, let $C \subseteq L_3$ be a set such that $\infty \in C$ and $C \cap \{x|_n : n < \omega\}$ is finite for every $x \in 2^\omega$. We shall prove that $K_3 \setminus C$ is open in $K_3$ (hence $C$ is compact). Clearly, if $t \in 2^{<\omega} \setminus C$, then its open neighborhood $\{t\}$ does not meet $C$. On the other hand, if $x \in 2^\omega \setminus C$, then there is $n < \omega$ such that the open neighborhood of $x$ given by $\{x\} \cup \{x|_k : k > n\}$ does not meet $C$. □

**Lemma 6.6.** $\mathcal{K}(L_3) \sim [\mathfrak{c}]^{<\omega}$.

**Proof.** Let $\mathcal{J}$ be the family of all sets $J \subseteq 2^{<\omega}$ such that $J \cap \{x|_n : n < \omega\}$ is finite for every $x \in 2^\omega$, ordered by inclusion.

Claim. $\mathcal{K}(L_3) \sim \mathcal{J}$. Indeed, by Lemma 6.5 we can define $f : \mathcal{K}(L_3) \to \mathcal{J}$ and $g : \mathcal{J} \to \mathcal{K}(L_3)$ by

$$f(C) := C \cap 2^{<\omega}, \quad g(J) := J \cup \{\infty\}.$$
It is clear that both \( f \) and \( g \) are Tukey functions, which proves the claim.

Therefore, it remains to show that \( \mathcal{J} \sim [\mathcal{C}]^{<\omega} \). The Tukey reduction \( \mathcal{J} \preceq [\mathcal{C}]^{<\omega} \) follows from Remark \([2,4]\). Given \( x = (x(n)) \in 2^\omega \), we define
\[
s_n^x := x|_n \sim (1 - x(n)) \in 2^{<\omega}\quad \text{for every } n < \omega
\]
and we write \( A_x := \{s_n^x : n < \omega\} \), so that \( A_x \in \mathcal{J} \) (since it is an antichain). Define
\[
f : [2^\omega]^{<\omega} \to \mathcal{J}, \quad f(F) := \bigcup_{x \in F} A_x.
\]
In order to check that \( f \) is a Tukey function it is enough to show that for every \( J \in \mathcal{J} \) the set \( H_J := \{x \in 2^\omega : A_x \subseteq J\} \) is finite. Our proof is by contradiction. If \( H_J \) is infinite, then by compactness there is a sequence \((x_j)\) in \( H_J \) converging to some \( x \in 2^\omega \) in the usual topology of \( 2^\omega \) such that \( x \neq x_j \) for all \( j < \omega \). Fix \( n < \omega \). Then there is \( j < \omega \) such that \( x_j|_n = x|_n \). Since \( x_j \neq x \), there is \( m \geq n \) such that \( x_j|_m = x|_m \) and \( x_j|_{m+1} \neq x|_{m+1} \), that is, \( x|_{m+1} = s_m^x \in A_{x_j} \subseteq J \). As \( n < \omega \) is arbitrary, we conclude that \( J \cap \{x|_n : n < \omega\} \) is infinite, a contradiction.

\[\square\]

**Definition 6.7.** A function \( \varphi : K_3 \to B_{X^{**}} \) is called a \( K_3 \)-embedding if

(i) it is \( u^* \)-continuous and one-to-one;
(ii) \( \varphi(\infty) = 0 \);
(iii) \( \varphi(2^{<\omega}) \subseteq X \);
(iv) \( \varphi(2^\omega) \subseteq X^{**} \setminus X \).

A \( K_3 \)-embedding \( \varphi \) is called regular if the following condition holds:

(v) there is \( \delta > 0 \) such that \( \|\varphi(\sigma) - x\| > \delta \) for every \( \sigma \in 2^\omega \) and every \( x \in X \).

**Lemma 6.8.** If \( X \) admits a regular \( K_3 \)-embedding, then \([\mathcal{C}]^{<\omega} \preceq AK(B_X)\).

**Proof.** Let \( \varphi : K_3 \to B_{X^{**}} \) be a regular \( K_3 \)-embedding and consider \( E := \varphi(L_3) \subseteq B_X \).

Since \( \varphi \) is a homeomorphism between \( K_3 \) and \( \varphi(K_3) \), we have \( \overline{E}^{u^*} = \varphi(K_3) \) and the restriction \( \varphi|_{L_3} \) is a homeomorphism between \( L_3 \) and \( E \). Let \( \delta > 0 \) be as in Definition \([6.7](v)\). Then \( \|x^{**} - x\| > \delta \) for every \( x^{**} \in \overline{E}^{u^*} \setminus E \) and every \( x \in X \). By Lemmas \([6.6]\) and \([6.2]\), we have \([\mathcal{C}]^{<\omega} \sim \mathcal{K}(L_3) \sim \mathcal{K}(E) \preceq AK(B_X) \). \[\square\]

Let us illustrate the notion of \( K_3 \)-embedding with an example.

**Example 6.9.** The space \( C[0, 1] \) satisfies \( AK(B_{C[0, 1]}) \sim \mathcal{K}(B_{C[0, 1]} \sim [\mathcal{C}]^{<\omega} \).

**Proof.** Since \( C[0, 1] \) and \( C(2^\omega) \) are isomorphic, it suffices to deal with \( X := C(2^\omega) \) (bear in mind Proposition \([3.11](v)\)). Define \( \varphi : K_3 \to B_{X^{**}} \) by declaring:
- \( \varphi(\infty) := 0 \);
- \( \varphi(u) := 1_{A_u} \) for every \( u \in 2^{<\omega} \), where \( 1_{A_u} \) is the characteristic function of the clopen \( A_u := \{\sigma \in 2^\omega : u \subseteq \sigma\} \);
• \( \varphi(\sigma)(\mu) := \mu(\{\sigma\}) \) for every \( \sigma \in 2^\omega \) and every \( \mu \in X^*; \) here \( X^* \) is identified with the space of all regular Borel measures on \( 2^\omega \) via Riesz’s theorem.

Clearly, \( \varphi \) is one-to-one and satisfies properties (ii), (iii) and (iv) of Definition 6.7. Writing \( \delta_\tau \in B_{X^*} \) to denote the point mass at \( \tau \in 2^\omega \), we have

\[
\|\varphi(\sigma) - f\| \geq \sup_{\tau \in 2^\omega} |\delta_\tau(\{\sigma\}) - f(\tau)| = \max\{ |1 - f(\sigma)|, \sup_{\tau \neq \sigma} |f(\tau)| \} \geq \frac{1}{2}
\]

for every \( \sigma \in 2^\omega \) and every \( f \in C(2^\omega) \). Therefore, \( \varphi \) fulfills condition (v) of Definition 6.7. In order to prove that \( \varphi \) is a regular \( K_3 \)-embedding, it remains to check that \( \varphi \) is \( w^* \)-continuous. Obviously, \( \varphi \) is continuous at each isolated point \( u \in 2^{<\omega} \), while the continuity at each \( \sigma \in 2^\omega \) follows from the fact that

\[
\varphi(\sigma)(\mu) = \mu(\{\sigma\}) = \lim_{n \to \infty} \mu(A_{\sigma|n}) = \lim_{n \to \infty} \varphi(\sigma|n)(\mu)
\]

for every \( \mu \in X^* \) (note that \( (A_{\sigma|n}) \) is a decreasing sequence of sets with intersection \( \{\sigma\} \)). To show that \( \varphi \) is continuous at \( \infty \), let \( W \subseteq X^{**} \) be any \( w^* \)-open neighborhood of \( 0 \). Then \( C := K_3 \setminus \varphi^{-1}(W) \) is relatively compact in \( J_3 \), by Lemma 6.5(i) and the following facts:

- \( C \cap 2^\omega \) is finite (bear in mind that \( w^* - \lim_{n \to \infty} \varphi(\sigma_n) = 0 \) for every sequence \( \{\sigma_n\} \) of distinct points of \( 2^\omega \);
- \( C \cap 2^{<\omega} \) contains no infinite antichain; indeed, if \( (u_n) \) is a sequence of incomparable elements of \( 2^{<\omega} \), then \( (1_{A_{u_n}}) \) is bounded and pointwise convergent to \( 0 \), hence weakly null (see e.g. [13, Corollary 3.138]).

It follows that \( \varphi^{-1}(W) \supseteq K_3 \setminus \overline{C}^J_3 \) is a neighborhood of \( \infty \) in \( K_3 \). This proves that \( \varphi \) is \( w^* \)-continuous at \( \infty \) and we conclude that \( \varphi \) is a regular \( K_3 \)-embedding.

By Lemma 6.8 we have \([c]^{<\omega} \leq AK(B_X) \). On the other hand, the reductions \( AK(B_X) \leq K(B_X) \leq [c]^{<\omega} \) follow from Proposition 3.11(i) and the separability of \( X \) (see Remark 2.1).

\[ \square \]

**Lemma 6.10.** If \( X \) is separable and admits a \( K_3 \)-embedding, then it also admits a regular \( K_3 \)-embedding. In particular, \( AK(B_X) \sim K(B_X) \sim [c]^{<\omega} \).

**Proof.** Let \( \varphi : K_3 \to B_{X^{**}} \) be a \( K_3 \)-embedding. For each \( n \in \mathbb{N} \), define

\[ A_n := \left\{ \sigma \in 2^\omega : d(\varphi(\sigma), X) \geq \frac{1}{n} \right\} \]

and observe that \( A_n \) is Borel (by Lemma 5.7 and the \( w^* \)-continuity of \( \varphi \)). Since \( 2^\omega = \bigcup_{n \in \mathbb{N}} A_n \), there is \( n \in \mathbb{N} \) such that \( A_n \) is uncountable. It follows that \( A_n \) contains a set \( P \) homeomorphic to \( 2^\omega \) (see e.g. [26, Theorem 13.6]). By Lemma 5.6 there is a subtree \( u : 2^{<\omega} \to 2^{<\omega} \) (as in Definition 5.4) such that

\[ u(2^{<\omega}) \subseteq T := \left\{ \sigma|n : \sigma \in P, n < \omega \right\}. \]
Claim 1. For every $\sigma \in 2^\omega$ there exists the limit $j(\sigma) := \lim_{n \to \infty} u(\sigma|_n)$ in $K_3$ and $j(\sigma) \in P$. Indeed, since $u(\sigma|_n) \subseteq u(\sigma|_{n+1})$ for all $n < \omega$, there exist $\tau \in 2^\omega$ and a strictly increasing sequence $(m_n)$ in $\omega$ such that $\tau|_{m_n} = u(\sigma|_n)$ for all $n < \omega$. Clearly, this implies that the sequence $(u(\sigma|_n))$ converges to $\tau$ in $K_3$. On the other hand, for every $n < \omega$ there is $\sigma_n \in P$ such that $\sigma_n|_{m_n} = u(\sigma|_n) = \tau|_{m_n}$. Therefore, the sequence $(\sigma_n)$ converges to $\tau$ in $2^\omega$ and, since $P$ is closed in $2^\omega$, we conclude that $\tau \in P$.

Define $\tilde{\varphi} : K_3 \to B_{X^{**}}$ as follows:

(i) $\tilde{\varphi}(\infty) := 0$;
(ii) $\tilde{\varphi}(t) := \varphi(u(t))$ for every $t \in 2^{<\omega}$;
(iii) $\tilde{\varphi}(\sigma) := \varphi(j(\sigma))$ for every $\sigma \in 2^\omega$.

Claim 2. $\tilde{\varphi}$ is $w^*$-continuous. Indeed, obviously $\tilde{\varphi}$ is continuous at each isolated point $t \in 2^{<\omega}$. On the other hand, for every $\sigma \in 2^\omega$ we have

$$\tilde{\varphi}(\sigma) = \varphi\left(\lim_{n \to \infty} u(\sigma|_n)\right) = \lim_{n \to \infty} \varphi(u(\sigma|_n)) = \lim_{n \to \infty} \tilde{\varphi}(\sigma|_n),$$

which implies the continuity of $\tilde{\varphi}$ at $\sigma$. Finally, in order to check the continuity at $\infty$, let $W \subseteq X^{**}$ be a $w^*$-open neighborhood of 0. Then $K := K_3 \setminus \varphi^{-1}(W) \subseteq J_3$ is compact and so $K \cap 2^\omega$ is finite and $K$ contains no infinite antichain of $2^{<\omega}$ (by Lemma 6.3(i)). Define

$$C := \{ t \in 2^{<\omega} : u(t) \in K \} \cup \{ \sigma \in 2^\omega : j(\sigma) \in K \} \subseteq J_3$$

and note that $C = K_3 \setminus \tilde{\varphi}^{-1}(W)$. Since $j : 2^\omega \to P$ is one-to-one, $C \cap 2^\omega$ is finite. Note that $C$ contains no infinite antichain of $2^{<\omega}$ (because if $t, s \in C$ are incomparable, then $u(t), u(s) \in K$ are incomparable as well). Another appeal to Lemma 6.3(i) ensures that $C$ is relatively compact in $J_3$. Then $K_3 \setminus C^3$ is an open neighborhood of $\infty$ in $K_3$ contained in $\tilde{\varphi}^{-1}(W)$. This finishes the proof of the claim.

Thus, $\tilde{\varphi}$ is a $K_3$-embedding. Since $j(\sigma) \in P \subseteq A_\omega$ for every $\sigma \in 2^\omega$, it follows that $\tilde{\varphi}$ is regular. From Lemma 6.8 we get $[c]^{<\omega} \preceq AK(B_X)$. On the other hand, the reductions $AK(B_X) \preceq K(B_X) \preceq [c]^{<\omega}$ follow from Proposition 3.1(i) and the separability of $X$.

Recall that the dual ball $B_{Y^*}$ of a Banach space $Y$ is is $w^*$-angelic if, for any set $A \subset B_{Y^*}$, every point in the $w^*$-closure of $A$ is the $w^*$-limit of a sequence contained in $A$. This property holds if $Y$ is WCG (cf. [13 Theorem 13.20]) and also if $Y$ is the dual of a separable Banach space not containing $\ell_1$, thanks to the Bourgain-Fremlin-Talagrand and Odell-Rosenthal theorems (cf. [13 Theorems 5.49 and 5.52]).

Lemma 6.11. Suppose that $B_{X^{**}}$ is $w^*$-angelic and that there is a biorthogonal system $\{(e_t, e_t^*) : t \in 2^{<\omega}\} \subseteq X \times X^*$ satisfying the following properties:

(i) $\|e_t\| \leq 1$ for every $t \in 2^{<\omega}$;
(ii) $\{e_t : t \in I\}^{**} = \{e_t : t \in I\} \cup \{0\}$ for every infinite antichain $I \subseteq 2^{<\omega}$;
(iii) for every $\sigma \in 2^\omega$ there exists $w^* - \lim_{n \to \infty} e_{\sigma|n} = e_{\sigma}^{**} \in X^{**} \setminus X$;
(iv) $e_{\sigma}^{**} \neq e_{\tau}^{**}$ whenever $\sigma \neq \tau$.

Then $E := \{e_{\sigma}^{**} : \sigma \in 2^\omega\}$ is $w^*$-discrete and $\overline{E}^{w^*} = E \cup \{0\}$.

Proof. We first note that $E \subseteq B_{X^{**}}$ and

\[(6.8) \quad \overline{E}^{w^*} \cap \{e_t : t \in 2^{<\omega}\} = \emptyset.
\]

Indeed, note that for each $t \in 2^{<\omega}$ the set $\{x^{**} \in X^{**} : x^{**}(e_t) > \frac{1}{2}\}$ is a $w^*$-open neighborhood of $e_t$ which is disjoint from $E$, by property (iii).

Claim 1. $\overline{E}^{w^*} \subseteq E \cup \{0\}$. To check this, note that condition (iii) ensures that $\overline{E}^{w^*} \subseteq \{e_t : t \in 2^{<\omega}\}^{w^*}$. Take any $x^{**} \in \overline{E}^{w^*}$. By the $w^*$-angelicity of $B_{X^{**}}$, there is a sequence $(t_n)$ in $2^{<\omega}$ such that $(e_{t_n})$ is $w^*$-convergent to $x^{**}$. Since $x^{**} \neq e_t$ for all $t \in 2^{<\omega}$ (by (6.8)), by passing to a further subsequence we can assume that $\text{length}(t_n) < \text{length}(t_{n+1})$ for all $n < \omega$. Ramsey’s theorem (see e.g. [21, Theorem 9.1]) ensures that the set $\{t_n : n < \omega\}$ contains either an infinite antichain or an infinite chain. In the first case, condition (ii) and (6.8) imply that $x^{**} = 0$. In the second case, there exist $\sigma \in 2^\omega$, a subsequence $(t_{n_k})$ and a strictly increasing sequence $(m_k)$ in $\omega$ such that $t_{n_k} = \sigma|_{m_k}$ for all $k < \omega$, hence (iii) yields $x^{**} = e_{\sigma}^{**}$. The claim is proved.

Claim 2. $E$ is $w^*$-discrete. Our proof is by contradiction. Suppose there is $\sigma_0 \in 2^\omega$ such that $e_{\sigma_0}^{**} \in \{e_{\sigma}^{**} : \sigma \in 2^\omega \setminus \{\sigma_0\}\}^{w^*}$. For each $\sigma \in 2^\omega \setminus \{\sigma_0\}$ we fix $n_\sigma < \omega$ such that $\sigma|_{n_\sigma} \neq \sigma_0|_{n_\sigma}$. Note that

\[e_{\sigma_0}^{**} \in \{e_{\sigma}^{**} : \sigma \in 2^\omega \setminus \{\sigma_0\}\}^{w^*} \ni \{e_{\sigma|n} : \sigma \in 2^\omega \setminus \{\sigma_0\}, n \geq n_\sigma\}^{w^*}.
\]

Arguing as in the proof of Claim 1 (bearing in mind that $e_{\sigma_0}^{**} \neq 0$), there exist $\sigma \in 2^\omega$, a sequence $(\tau_k)$ in $2^\omega \setminus \{\sigma_0\}$ and a strictly increasing sequence $(m_k)$ in $\omega$ with $m_k \geq n_{\tau_k}$ such that $w^* - \lim_{k \to \infty} e_{\tau_k|m_k} = e_{\sigma_0}^{**}$ and $\tau_k|m_k = \sigma|_{m_k}$ for all $k < \omega$. By (iii) we get $e_{\sigma_0}^{**} = e_{\sigma}^{**}$ and therefore (iv) yields $\sigma_0 = \sigma$. This contradicts that $\tau_k|m_{\tau_k} \neq \sigma_0|_{m_{\tau_k}}$ for all $k < \omega$ and the claim is proved.

It only remains to prove that $0 \in \overline{E}^{w^*}$. To this end, let $(\sigma_n)$ be a sequence of distinct elements of $2^\omega$. Since $(B_{X^{**}}, w^*)$ is sequentially compact (because it is an angelic compact), there exist $x^{**} \in B_{X^{**}}$ and a subsequence $(\sigma_{n_k})$ with $w^* - \lim_{k \to \infty} e_{\sigma_{n_k}}^{**} = x^{**}$. We can assume that $x^{**} \neq e_{\sigma_k}^{**}$ for all $k < \omega$. By Claim 1, we have $x^{**} \in E \cup \{0\}$. Since $E$ is $w^*$-discrete (Claim 2), we conclude that $x^{**} = 0$ and so $0 \in \overline{E}^{w^*}$. This finishes the proof of the lemma. \square

Proposition 6.12. If $X$ is separable, has non-separable dual and contains no copy of $\ell_1$, then $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim [e]^{<\omega}$. 
Proof. As we mentioned before Lemma 6.11, since $X$ is separable and contains no copy of $\ell^1$, the ball $B_{XX}$ is $w^*$-angelic. Theorem 3 in [10] (applied to the identity operator on $X$) ensures the existence of a biorthogonal system $\{(e_t, e_t^*) : t \in 2^{<\omega}\} \subseteq X \times X^*$ as in Lemma 6.11. Define a function $\varphi : K_3 \rightarrow B_{XX}$ by
\begin{itemize}
  \item $\varphi(\infty) := 0$;
  \item $\varphi(t) := e_t$ for every $t \in 2^{<\omega}$;
  \item $\varphi(\sigma) := e_\sigma^*$ for every $\sigma \in 2^\omega$.
\end{itemize}
Clearly, $\varphi$ is $w^*$-continuous at each point of $2^{<\omega} \cup 2^\omega$. To check that $\varphi$ is $w^*$-continuous at $\infty$, take any $w^*$-open set $W \subseteq X^{**}$ containing 0. Then:
\begin{enumerate}
  \item $(K_3 \setminus \varphi^{-1}(W)) \cap 2^\omega$ is finite. Indeed, it suffices to note that $0 \in \{e_{\sigma_n}^* : n < \omega\}^{w^*}$ for every sequence $(\sigma_n)$ of distinct elements of $2^\omega$ (see the end of the proof of Lemma 6.11).
  \item $(K_3 \setminus \varphi^{-1}(W)) \cap 2^{<\omega}$ contains no infinite antichain. Indeed, this follows at once from property (ii) in Lemma 6.11.
\end{enumerate}
Conditions (a) and (b) imply (use Lemma 6.5(i)) that $C := K_3 \setminus \varphi^{-1}(W)$ is relatively compact in $J_3$. Then $K_3 \setminus C^{J_3}$ is an open neighborhood of $\infty$ in $K_3$ contained in $\varphi^{-1}(W)$. This proves that $\varphi$ is $w^*$-continuous at $\infty$.

Therefore, $\varphi$ is a $K_3$-embedding. By Lemma 6.10, $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim [c]^{<\omega}$. \qed

The proof of Proposition 6.12 shows that $K_3$ embeds in a natural way into $(B_{XX}, w^*)$ whenever $X$ is separable, has non-separable dual and contains no copy of $\ell_1$. This statement is related to the fact that non-$G_δ$-points in Rosenthal compacta lie in the closure of a discrete set of size continuum [36].

7. Unconditional bases

Let us recall that an infinite countable set $B$ of non-zero vectors in the Banach space $X$ is called an unconditional basic sequence if there exists $C > 0$ such that for every finite sets $G \subset F \subset B$ and every $\alpha : F \rightarrow \mathbb{R}$ we have
$$\left\| \sum_{x \in G} \alpha(x) x \right\| \leq C \left\| \sum_{x \in F} \alpha(x) x \right\|.$$ 
If in addition $\text{span}(B) = X$, then $B$ is called an unconditional basis of $X$.

The interest of unconditional bases in this theory is that the partially ordered set $\mathcal{RK}(B)$ (made up of all subsets of $B$ which are relatively weakly compact, ordered by inclusion) is combinatorially easier to analyze than $\mathcal{K}(B_X)$ and $\mathcal{AK}(B_X)$, but yet it can provide information on these structures as Lemma 7.2 below shows.

We collect in the following lemma some well-known characterizations of relatively weakly compact subsets of unconditional basic sequences.
Lemma 7.1. Let \( \mathcal{B} \) be an unconditional basic sequence in \( X \). Then for any infinite set \( A \subset \mathcal{B} \) the following statements are equivalent:

(i) \( A \in \mathcal{RK}(\mathcal{B}) \);

(ii) \( A \) is weakly null, that is, \( A \) converges to 0 in the weak topology;

(iii) \( A \) is weakly Cauchy, that is, \( A \) converges in the weak\(^*\) topology of \( X^{**} \);

(iv) \( A \) is bounded and contains no subsequence equivalent to the basis of \( \ell^1 \).

The proof of the following lemma uses arguments by Johnson, Mercourakis and Stamati [29] (cf. [20, Theorems 7.40 and 7.41]), that we adapted to fit into our setting.

Lemma 7.2. If \( \mathcal{B} = \{ e_n : n < \omega \} \) is a semi-normalized unconditional basis of \( X \), then \( \mathcal{RK}(\mathcal{B}) \preceq \mathcal{AK}(B_X) \).

Proof. We begin by proving the following:

Claim. If \( K \subseteq X \) is weakly compact and \( \varepsilon > 0 \), then the set \( \Gamma(K, \varepsilon) := \{ e_n \in \mathcal{B} : \sup_{x \in K} |e^*_n(x)| \geq \varepsilon \} \in \mathcal{RK}(\mathcal{B}) \).

Our proof is by contradiction. If \( \Gamma(K, \varepsilon) \not\subseteq \mathcal{RK}(\mathcal{B}) \), then there is a strictly increasing sequence \( (n_j) \) in \( \omega \) such that \( e_{n_j} \in \Gamma(K, \varepsilon) \) for all \( j < \omega \) and \( (e_{n_j}) \) is equivalent to the basis of \( \ell^1 \) (apply Lemma 7.1). Let \( (x_j) \) be a sequence in \( K \) such that

\[
|e^*_n(x_j)| \geq \varepsilon \quad \text{for all } j < \omega.
\]

Since \( \mathcal{B} \) is an unconditional basis of \( X \), we can consider the (bounded and linear) operator

\[
P : X \to X, \quad P(x) := \sum_{j<\omega} e^*_n(x_j)e_{n_j},
\]

which is a projection onto \( Y := \text{span}\{e_{n_j} : j < \omega\} \). Since \( K \) is weakly compact and \( Y \) has the Schur property (because it is isomorphic to \( \ell^1 \)), the set \( P(K) \) is norm compact. Thus, by passing to a further subsequence, we may assume that there is \( x \in K \) such that \( ||P(x_j) - P(x)|| \to 0 \). Bearing in mind that \( \{e^*_n : n < \omega\} \) is bounded (because \( \mathcal{B} \) is an unconditional basis of \( X \) and \( \inf_{n<\omega} \|e_n\| > 0 \)), we conclude that

\[
|e^*_n(x_j) - e^*_n(x)| = |e^*_n(P(x_j)) - e^*_n(P(x))| \to 0.
\]

On the other hand, the convergence of the series \( \sum_{j<\omega} e^*_n(x_j)e_{n_j} \) and the fact that \( \inf_{j<\omega} \|e_{n_j}\| > 0 \) imply that \( e^*_n(x) \to 0 \), which combined with (7.2) yields \( e^*_n(x_j) \to 0 \). This contradicts (7.1) and the Claim is proved.

Define

\[
\delta := (2 \sup_{n<\omega} \|e^*_n\|)^{-1} > 0.
\]
Set $\rho := \sup_{n<\omega} \|e_n\|$. Finally, we prove that the function
$$f : \mathcal{RK}(\mathcal{B}) \to (\mathcal{AK}(\rho B_X), \leq_\delta), \quad f(A) := A^\omega,$$
is Tukey. Indeed, it suffices to prove if $K$ is a weakly compact subset of $\rho B_X$ and $A \in \mathcal{RK}(\mathcal{B})$ satisfies $A^\omega \subseteq K + \delta B_X$, then $A \subseteq \Gamma(K, \frac{1}{2})$. We argue by contradiction. Suppose there is $e_n \in A$ such that $|e_n^*(x)| < \frac{1}{2}$ for every $x \in K$. Write $e_n = x + y$, where $x \in K$ and $\|y\| \leq \delta$. Then
$$1 e_n^*(e_n) = e_n^*(x) + e_n^*(y) < \frac{1}{2} + \|e_n\| \delta \leq \frac{1}{2} + \frac{1}{2} = 1,$$
a contradiction which proves that $f$ is Tukey. Therefore, $\mathcal{RK}(\mathcal{B}) \leq \mathcal{AK}(\rho B_X)$. Since $\mathcal{AK}(\rho B_X) \sim \mathcal{AK}(B_X)$ (Proposition 3.11(v)), the proof is over. □

**Remark 7.3.** The equivalence $\mathcal{RK}(\mathcal{B}) \sim \mathcal{AK}(B_X)$ does not hold in general for an unconditional basis $\mathcal{B}$ of $X$. Indeed, the usual basis $\mathcal{B}$ of $c_0$ is weakly null and so it satisfies $\mathcal{RK}(\mathcal{B}) \not\sim \mathcal{K}(\mathcal{Q}) \sim \mathcal{AK}(B_{c_0})$ (Example 3.18).

As a first application of Lemma 7.2 we compute $\mathcal{AK}(B_X)$ and $\mathcal{K}(B_X)$ for the space $X = \ell^p(\ell^1)$, where $1 < p < \infty$.

**Example 7.4.** The space $X = \ell^p(\ell^1)$, $1 < p < \infty$, satisfies $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim \omega^\omega$.

**Proof.** It is known that a set $C \subseteq B_X$ is relatively weakly compact if and only if $\pi_n(C)$ is relatively norm compact in $\ell^1$ for every $n < \omega$, where $\pi_n : X \to \ell^1$ denotes the $n$-th coordinate projection. Therefore, if $\mathcal{B} = \{e_{nm} : n, m < \omega\}$ is the usual unconditional basis of $X$, then a set $C \subseteq \mathcal{B}$ belongs to $\mathcal{RK}(\mathcal{B})$ if and only if $C$ is contained in a set of the form
$$F(\varphi) := \{e_{nm} : n < \omega, m < \varphi(n)\}$$
for some $\varphi \in \omega^\omega$. Now, it is easy to check that the function $F : \omega^\omega \to \mathcal{RK}(\mathcal{B})$ is Tukey, hence $\omega^\omega \subseteq \mathcal{RK}(\mathcal{B})$. An appeal to Lemma 7.2 yields $\omega^\omega \leq \mathcal{AK}(B_X)$.

On the other hand, note that for every sequence $(L_n)$ of norm compact subsets of $B_{\ell^1}$, the set $\{x \in B_X : \pi_n(x) \in L_n \text{ for all } n < \omega\}$ is weakly compact. Therefore, the function
$$G : \mathcal{K}(B_X) \to (\mathcal{K}(B_{\ell^1}))^\omega, \quad G(L) := (\pi_n(L)), $$
is Tukey and so $\mathcal{K}(B_X) \not\leq (\mathcal{K}(B_{\ell^1}))^\omega \sim \omega^\omega$ (recall that $\mathcal{K}(B_{\ell^1}) \sim \omega^\omega$, by Example 2.9). Since $\mathcal{AK}(B_X) \leq \mathcal{K}(B_X)$ (Proposition 3.11(i)), we have $\mathcal{AK}(B_X) \sim \mathcal{K}(B_X) \sim \omega^\omega$. □

Recall that if $X$ is SWCG and $Y \subseteq X$ is a non-reflexive subspace, then $\mathcal{K}(B_Y) \sim \omega^\omega$ (Corollary 3.17(iv)) and $\mathcal{AK}(B_Y) \sim \omega$ under the additional assumption that $Y$ is complemented in $X$ (combine Theorem 3.14 and the fact that the SWCG property is inherited by complemented subspaces). Mercourakis and Stamati constructed in [20] Theorem 3.9(ii)) a subspace $Y \subseteq L^1[0, 1]$ which is not SWCG. The following theorem uses essentially the same construction.
Theorem 7.5. There is a subspace \( Y \) of \( L^1[0, 1] \) such that \( AK(B_Y) \sim \omega^\omega \).

To deal with the proof of Theorem 7.5, we need some lemmas.

Lemma 7.6. Let \( \varphi \in \mathbb{N}^\mathbb{N} \). Then there is \( \hat{\varphi} \in \mathbb{N}^\mathbb{N} \) such that:

(i) \( \varphi \leq \hat{\varphi} \);
(ii) \( \hat{\varphi} \) is strictly increasing;
(iii) \( n\hat{\varphi}(m) \geq (m + 1)\hat{\varphi}(n) \) for every \( m > n \);
(iv) \( \hat{\varphi}(n) \geq (n + 1)(n + 2) \) for every \( n \in \mathbb{N} \).

Proof. Define \( \hat{\varphi}(n) := 3^n \max\{2, \varphi(1), \ldots, \varphi(n)\} \) for all \( n \in \mathbb{N} \). We shall check that \( \hat{\varphi} \) satisfies the required properties. Clearly, \( \hat{\varphi} \) fulfills (i) and (ii). To check (iii), note that \( 3^j \geq j + 2 \) for all \( j \geq 1 \), hence if \( m > n \) then \( n(3^{m-n} - 1) \geq m - n + 1 \) and so \( n3^{m-n} \geq m + 1 \), therefore \( n\hat{\varphi}(m) \geq (m + 1)\hat{\varphi}(n) \). Finally, (iv) can be easily proved by induction with the help of (iii).

The straightforward proof of the following lemma is omitted. We denote by \( \lambda \) the Lebesgue measure on the Borel \( \sigma \)-algebra of \([0, 1]\).

Lemma 7.7. Let \( m, p \in \mathbb{N} \) with \( m \geq p > 2 \). Set \( c := \frac{1}{m} + \frac{1}{2} - \frac{1}{p} \in (\frac{1}{m}, \frac{1}{2}] \) and \( \alpha := \frac{1}{2(1-c)} \in (\frac{1}{2}, 1] \). Then the function \( f \in L^1[0, 1] \) defined by

\[
 f(t) := \begin{cases}
 \frac{m}{p} & \text{if } t \in [0, \frac{1}{m}] \\
 1 & \text{if } t \in [\frac{1}{m}, c] \\
 -\alpha & \text{if } t \in (c, 1]
\end{cases}
\]

satisfies \( \int_0^1 f \, d\lambda = 0 \) and \( \int_0^1 |f| \, d\lambda = 1 \).

We shall use the following notation as in the proof of Example 7.9. Given any probability space \((\Omega, \Sigma, \mu)\), any function \( f \in L^1(\mu) \) and \( n \in \mathbb{N} \), we denote by \( o(f, n) \) the least \( k \in \mathbb{N} \) such that, for every \( B \in \Sigma \), we have:

\[
\text{if } \mu(B) \leq \frac{1}{k} \text{ then } \left| \int_B f \, d\mu \right| \leq \frac{1}{n}.
\]

Lemma 7.8. Let \( \varphi \in \mathbb{N}^\mathbb{N} \) and \( n \in \mathbb{N} \), \( n > 2 \). Let \( \hat{\varphi} \in \mathbb{N}^\mathbb{N} \) be as in Lemma 7.6, and let \( f \in L^1[0, 1] \) be as in Lemma 7.7 by taking \( m = \hat{\varphi}(n) \) and \( p = n \). Then:

(i) \( o(f, k) \leq \hat{\varphi}(k) \) for every \( k \in \mathbb{N} \);
(ii) \( o(f, n) = \hat{\varphi}(n) \).

Proof. We divide the proof into several steps.

Step 1. \( o(f, k) \leq \hat{\varphi}(k) \) for every \( n \leq k \). Indeed, if \( B \subseteq [0, 1] \) is a Borel set with \( \lambda(B) \leq \frac{1}{\hat{\varphi}(k)} \), then

\[
\left| \int_B f \, d\lambda \right| \leq \frac{\hat{\varphi}(n)}{n} \cdot \lambda(B) \leq \frac{\hat{\varphi}(n)}{n} \cdot \frac{1}{\hat{\varphi}(k)}.
\]
If \( k = n \), then (7.3) yields \( |\int_B f \, d\lambda| \leq \frac{1}{n} \), while if \( k > n \) then property (iii) in Lemma 7.6 and (7.3) imply that

\[
|\int_B f \, d\lambda| \leq \frac{\tilde{\varphi}(n)}{n} \cdot \frac{1}{\varphi(k)} \leq \frac{\tilde{\varphi}(n)}{n} \cdot \frac{n}{(k+1)\tilde{\varphi}(n)} = \frac{1}{k+1} < \frac{1}{k}.
\]

This shows that \( o(f, k) \leq \tilde{\varphi}(k) \) whenever \( n \leq k \).

Step 2. \( o(f, n) = \tilde{\varphi}(n) \). Indeed, fix \( k < \tilde{\varphi}(n) \). Set \( c := \frac{1}{\tilde{\varphi}(n)} + \frac{1}{2} - \frac{1}{n} \in (\frac{1}{\tilde{\varphi}(n)}, \frac{1}{2}] \) and choose \( d \in \mathbb{R} \) such that

\[ \frac{1}{\tilde{\varphi}(n)} < d < \min\left\{ \frac{1}{k}, c \right\}. \]

By the definition of \( f \), we have

\[ \left| \int_0^d f \, d\lambda \right| = \frac{\tilde{\varphi}(n)}{n} \cdot \frac{1}{\varphi(k)} + \left( d - \frac{1}{\tilde{\varphi}(n)} \right) > \frac{1}{n}, \]

hence \( k > o(f, n) \). As \( k < \tilde{\varphi}(n) \) is arbitrary, we get \( o(f, n) = \tilde{\varphi}(n) \).

Step 3. \( o(f, k) \leq \tilde{\varphi}(k) \) for every \( k < n \). Indeed, let \( B \subseteq [0, 1] \) be a Borel set with \( \lambda(B) \leq \frac{1}{\tilde{\varphi}(k)} \). Define \( B_0 := B \cap [0, \frac{1}{\tilde{\varphi}(n)}] \). Then property (iv) in Lemma 7.6 yields

\[
|\int_B f \, d\lambda| \leq \left| \int_{B_0} f \, d\lambda \right| + \left| \int_{B \setminus B_0} f \, d\lambda \right| \leq \frac{1}{n} + \lambda(B) \leq \frac{1}{n} + \frac{\tilde{\varphi}(k)}{k+1} \leq \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} < \frac{1}{k}.
\]

It follows that \( o(f, k) \leq \tilde{\varphi}(k) \). The proof is over. \( \square \)

Proof of Theorem 7.5. Let \( \mathcal{R}_0 = \{ r_n : n \in \mathbb{N} \} \subseteq L^1([0, 1]) \) be the set made up of all functions as in Lemma 7.7. We shall now work in the space \( X := L^1([0, 1]^\mathbb{N}) \), which is isometric to \( L^1([0, 1]) \). For each \( n \in \mathbb{N} \), let \( f_n : [0, 1]^\mathbb{N} \to \mathbb{R} \) be given by \( f_n := r_n \circ \pi_n \), where \( \pi_n : [0, 1]^\mathbb{N} \to [0, 1] \) is the \( n \)-th coordinate projection. Then \( \mathcal{B} := \{ f_n : n \in \mathbb{N} \} \) is a (normalized) unconditional basic sequence in \( X \) (see the proof of [32], pp. 89-90 and the references therein). Let \( Y := \overline{\text{span}}(\mathcal{B}) \subseteq X \). By Lemma 7.2, Proposition 3.11(i) and Corollary 3.17(iv) we have

\[ \mathcal{R}K(\mathcal{B}) \leq \mathcal{AK}(\mathcal{B}_Y) \leq \mathcal{K}(\mathcal{B}_Y) \leq \mathbb{N}^\mathbb{N}. \]

So, in order to complete the proof we need to show that \( \mathbb{N}^\mathbb{N} \leq \mathcal{R}K(\mathcal{B}) \).

As we already pointed out in Example 2.3, a bounded set \( C \subseteq X \) is relatively weakly compact if and only if \( \{ o(f, \cdot) : f \in C \} \) is bounded above in \( \mathbb{N}^\mathbb{N} \). Therefore, we can consider the function

\[ F : \mathbb{N}^\mathbb{N} \to \mathcal{R}K(\mathcal{B}), \quad F(\varphi) := \{ f_n : o(f_n, \cdot) \leq \tilde{\varphi} \}, \]
where $\tilde{\varphi}$ is the function given by Lemma 7.6. We claim that $F$ is Tukey when $\mathbb{N}^\mathbb{N}$ is equipped with the binary relation $\leq_3$ defined by

$$\varphi \leq_3 \psi :\iff \varphi(n) \leq \psi(n) \quad \text{for every } n \geq 3.$$  

Indeed, fix $C_0 \in \mathcal{RK}(\mathcal{B})$ and take $\varphi_0 \in \mathbb{N}^\mathbb{N}$ such that $o(f_n, \cdot) \leq \varphi_0$ for all $f_n \in C_0$. We shall check that $\varphi \leq_3 \varphi_0$ whenever $\varphi \in \mathbb{N}^\mathbb{N}$ satisfies $F(\varphi) \subseteq C_0$. Take any $n \in \mathbb{N}$, $n \geq 3$. Let $r_j \in R_0$ be the function given by Lemma 7.7 by taking $m = \tilde{\varphi}(n)$ and $p = n$. By Lemma 7.8 we have $o(f_j, \cdot) = o(r_j, \cdot) \leq \tilde{\varphi}$ and $o(f_j, n) = o(r_j, n) = \tilde{\varphi}(n)$. Therefore, $f_j \in F(\varphi) \subseteq C_0$ and so $\varphi(n) \leq \tilde{\varphi}(n) \leq \varphi_0(n)$, which proves the claim.

Now, in order to finish the proof it suffices to check that $(\mathbb{N}^\mathbb{N}, \leq_3) \sim \mathbb{N}^\mathbb{N}$. On one hand, the identity mapping $(\mathbb{N}^\mathbb{N}, \leq_3) \to \mathbb{N}^\mathbb{N}$ is obviously a Tukey function. On the other hand, it is clear that the function $G : \mathbb{N}^\mathbb{N} \to (\mathbb{N}^\mathbb{N}, \leq_3)$ given by

$$G(\varphi)(n) := \max\{\varphi(1), \ldots, \varphi(n)\}$$

is Tukey as well. The proof is over. \hfill $\square$

We next explain a method to construct unconditional bases $\mathcal{B}$ for which $\mathcal{RK}(\mathcal{B})$ is Tukey equivalent to different posets. In this way, we shall get examples where $\mathcal{RK}(\mathcal{B})$ is equivalent to $\{0\}, \omega, \omega^\omega, \mathcal{K}(\mathbb{Q})$ and $[\mathbb{C}]^{<\omega}$. But more interesting, we shall also get a consistent example which is not equivalent to any of these (see Theorem 7.12 and the comments following it), thus showing that Theorem 4.7 and Corollary 4.11 do not hold in general in the absence of analytic determinacy. We believe that this example should also provide a consistent counterexample to the statements of Theorems 4.11 and 5.1 when analytic determinacy does not hold, but we were not able to prove it.

Recall that a family $\mathcal{A}$ of subsets of a given countable infinite set $D$ is called adequate if, for any $A \subseteq D$, we have $A \in \mathcal{A}$ if and only if $[A]^{<\omega} \subseteq \mathcal{A}$. In this case, we can define a norm on $c_{00}(D)$ (the linear space of all finitely supported real-valued functions on $D$) by the formula:

$$\|f\|_{\mathcal{A}} := \sup \left\{ \sum_{i \in T} |f(i)| : T \in \mathcal{A} \right\}, \quad f \in c_{00}(D).$$

In this way, the canonical Hamel basis of $c_{00}(D)$ becomes a (normalized) unconditional basis $\mathcal{B}_A = \{e_d : d \in D\}$ of the completion of $(c_{00}(D), \| \cdot \|_{\mathcal{A}})$, which we denote by $\mathcal{X}_A$. For more information on this space (sometimes denoted by $E_{0,1}(\mathcal{A})$), we refer the reader to [2]. Note that, as particular cases of this construction, we have:

- $\mathcal{X}_A = c_0$ if $D = \omega$ and $\mathcal{A} = \{\{n\} : n < \omega\} \cup \{\emptyset\}$;
- $\mathcal{X}_A = c_0(\ell^1)$ if $D = \omega \times \omega$ and $\mathcal{A}$ is the family made up of all sets of the form $
\{(n, m) : m \in F\}$, where $n < \omega$ and $F \subseteq \omega$.

**Lemma 7.9.** Let $\mathcal{A}$ be an adequate family of subsets of a countable infinite set $D$. Let $C \subseteq D$. The following statements are equivalent:

\begin{itemize}
\item $\mathcal{A}$ is Tukey equivalent to $\mathcal{A}_0$.
\item $C$ is Tukey equivalent to $\mathcal{A}_0$.
\end{itemize}
(i) \( C \in \mathcal{A}^\perp \) (i.e. \( C \cap A \) is finite for every \( A \in \mathcal{A} \));
(ii) \( \{e_d : d \in C\} \in \mathcal{RK}(\mathcal{B}_A) \).

Therefore, \( \mathcal{RK}(\mathcal{B}_A) \sim \mathcal{A}^\perp \) (ordered by inclusion).

Proof. The equivalence (i)\(\rightarrow\)(ii) is obvious if \( C \) is finite. Assume that \( C \) is infinite and enumerate \( C = \{d_n : n < \omega\} \). Bearing in mind Lemma 7.1 (ii) is equivalent to saying that the sequence \( (e_{d_n}) \) is weakly null. The set \( K_A := \{1_A : A \in \mathcal{A}\} \subseteq 2^D \) is closed, hence compact, when \( 2^D \) is equipped with its usual product topology. It is well-known (and easy to check) that \( X \) embeds isomorphically into \( C(K_A) \) via an operator \( T : \mathcal{X}_A \rightarrow C(K_A) \) such that \( T(e_d) \) is the \( d \)-th coordinate projection for all \( d \in D \). Since a bounded sequence in \( C(K_A) \) is weakly null if and only if it is pointwise null (see e.g. [13, Corollary 3.138]), we conclude that (ii) is equivalent to saying that, for each \( A \in \mathcal{A} \), we have \( T(e_{d_n}))(1_A) = 1_A(d_n) = 0 \) for \( n \) large enough, that is, \( C \cap A \) is finite. The proof is finished. \( \Box \)

Proposition 7.10. Let \( \mathcal{A} \) be the adequate family of all chains (including the finite ones) of \( D = 2^{<\omega} \). Then \( \mathcal{RK}(\mathcal{B}_A) \sim [c]^{<\omega} \).

Proof. Notice that \( \mathcal{A}^\perp \) is the family of all subsets of \( 2^{<\omega} \) which do not contain any infinite chain. We showed in the proof of Lemma 6.6 that this is Tukey equivalent to \( [c]^{<\omega} \). The conclusion now follows from Lemma 7.9. \( \Box \)

The following proposition, combined with Theorem 2.2, provides examples of unconditional bases \( \mathcal{B} \) for which \( \mathcal{RK}(\mathcal{B}) \) is Tukey equivalent to either \( \{0\}, \omega, \omega^\omega \) or \( \mathcal{K}(\mathbb{Q}) \).

If \( E \) is a coanalytic subset of some Polish space, then [3, Theorem 11] asserts that there is an adequate family \( \mathcal{A}_E \) of closed and discrete subsets of \( E \) with the following cofinality property: for every infinite closed and discrete set \( A \subseteq E \) there is an infinite set \( B \subseteq A \) such that \( B \in \mathcal{A}_E \).

Proposition 7.11. Let \( E \) be a coanalytic subset of some Polish space. Let \( D \subseteq E \) be a countable dense set and consider the adequate family of subsets of \( D \) defined by \( \mathcal{A}[E] := \{A \in \mathcal{A}_E : A \subseteq D\} \). Then \( \mathcal{RK}(\mathcal{B}_{\mathcal{A}[E]}) \sim \mathcal{K}(E) \).

Proof. By Lemma 7.9 we have \( \mathcal{RK}(\mathcal{B}_{\mathcal{A}[E]}) \sim \mathcal{A}[E]^\perp \). On the other hand, we claim that \( \mathcal{A}[E]^\perp \) coincides with the family \( \mathcal{K}_E(D) \) of all subsets of \( D \) which are relatively compact in \( E \). Indeed, a set \( C \subseteq E \) is relatively compact in the metric space \( E \) if and only if \( C \) contains no infinite closed and discrete set, which is equivalent to saying that \( C \) contains no infinite element of \( \mathcal{A}_E \). Therefore, a set \( C \subseteq D \) belongs to \( \mathcal{K}_E(D) \) if and only if \( C \in \mathcal{A}[E]^\perp \), as claimed. It follows that \( \mathcal{RK}(\mathcal{B}_{\mathcal{A}[E]}) \sim \mathcal{K}_E(D) \). An appeal to Lemma 4.3 finishes the proof. \( \Box \)

Theorem 7.12. If there exists a coanalytic set \( E \subseteq 2^\omega \) of cardinality \( \omega_1 \), then there is a normalized unconditional basis \( \mathcal{B} \) of a Banach space such that \( \mathcal{RK}(\mathcal{B}) \sim [\omega_1]^{<\omega} \).
Proof. We can suppose that $E$ is dense in $2^\omega$. Let us consider the set $K := 2^{<\omega} \cup 2^\omega$ equipped with the compact metrizable topology induced by the one-to-one mapping $f : K \to 2^\omega \times \mathbb{R}$ defined by

- $f(\sigma) := (\sigma, 0)$ for all $\sigma \in 2^\omega$;
- $f(t) := (t \& 0, (\text{length}(t) + 1)^{-1})$ for all $t \in 2^{<\omega}$.

Note that the topology inherited by $2^\omega \subseteq K$ is the usual one. Then $E' := 2^{<\omega} \cup E$ is a coanalytic subset of $K$. Let $\mathcal{A}_{E'}$ be an adequate family of closed and discrete subsets of $E'$ such that, for every infinite closed and discrete set $A \subseteq E'$, there is an infinite set $B \subseteq A$ such that $B \in \mathcal{A}_{E'}$ (apply [3, Theorem 11]). Of course, we can suppose that $\mathcal{A}_{E'}$ contains all singletons. For every $\sigma \in 2^\omega$ we denote $c(\sigma) := \{\sigma|_n : n < \omega\} \subseteq 2^{<\omega}$.

We define an adequate family of subsets of $2^{<\omega}$ by

$$\mathcal{A} := \{A \subseteq 2^{<\omega} : A \cap c(\sigma) \in \mathcal{A}_{E'} \text{ for all } \sigma \in 2^\omega\}.$$ 

We shall show that the unconditional basis $\mathcal{B}_{\mathcal{A}}$ is the one that we are looking for. By Lemma 7.9 we have $\mathcal{RK}(\mathcal{B}_{\mathcal{A}}) \sim \mathcal{A}^\perp$, so it is enough to prove that $\mathcal{A}^\perp \sim [E]^{<\omega}$.

We next check the following equality

$$(7.4) \quad \mathcal{A}^\perp = \{A \subseteq 2^{<\omega} : A \subseteq \bigcup_{\sigma \in F} c(\sigma) \text{ for some finite set } F \subseteq E\}.$$ 

For the inclusion “$\supseteq$” it suffices to prove that $c(\sigma) \in \mathcal{A}^\perp$ for every $\sigma \in E$. Given any $A \in \mathcal{A}$, we have $A \cap c(\sigma) \in \mathcal{A}_{E'}$ and so $A \cap c(\sigma)$ is closed in $E'$. Since the sequence $(\sigma|_n)$ converges to $\sigma \in E' \setminus A \cap c(\sigma)$ in the topology of $K$, the set $A \cap c(\sigma)$ is finite.

We divide the proof of the inclusion “$\subseteq$” in (7.4) into several steps. Fix $A \in \mathcal{A}^\perp$.

**Step 1.** Let $S$ be the set of all $\sigma \in 2^\omega$ for which $A \cap c(\sigma)$ is infinite. We claim that $S$ is finite. Our proof is by contradiction. If $S$ is infinite, then we can find a sequence $(\sigma^k)$ in $S$ converging to some $\sigma \in 2^\omega$ with $\sigma \neq \sigma^k$ for all $k < \omega$. Write

$$n_k := \min\{n < \omega : \sigma^k(n) \neq \sigma(n)\} \quad \text{for every } k < \omega.$$ 

By passing to a further subsequence, we can suppose that $n_k < n_{k+1}$ for all $k < \omega$. Now, we can pick $t^k \in A \cap c(\sigma^k)$ with length($t^k$) $> n_k$ (since $A \cap c(\sigma^k)$ is infinite) for every $k < \omega$. Notice that $B := \{t^k : k < \omega\} \subseteq A$ is an antichain, because for every $k < l < \omega$ we have

$$t^l(n_k) = \sigma^l(n_k) = \sigma(n_k) \quad \text{and} \quad t^k(n_k) = \sigma^k(n_k) \neq \sigma(n_k).$$

Therefore, $|B \cap c(\tau)| \leq 1$ for every $\tau \in 2^\omega$. Since all singletons of $2^{<\omega}$ belong to $\mathcal{A}_{E'}$, we have $B \in \mathcal{A}$, which contradicts that $A \in \mathcal{A}^\perp$, finishing the proof of Step 1.

**Step 2.** $S \subseteq E$. Indeed, suppose that there is $\sigma \in S \setminus E$. Since $A \cap c(\sigma)$ is infinite and the sequence $(\sigma|_n)$ converges to $\sigma \in K \setminus E'$, we have that $A \cap c(\sigma)$ is a closed and discrete subset of $E'$. Therefore, there is an infinite set $B \subseteq A \cap c(\sigma)$ such that
B \in \mathcal{A}_{E'}$. Bearing in mind that \( \mathcal{A} \) is hereditary, we conclude that \( B \in \mathcal{A} \), and this contradicts again that \( A \in \mathcal{A}^\perp \).

**Step 3.** The set \( B := A \setminus \bigcup_{\sigma \in S} c(\sigma) \) is finite. Again, our proof is by contradiction. If \( B \) is infinite, then Ramsey’s theorem (see e.g. [21, Theorem 9.1]) ensures that \( B \) contains either an infinite chain or an infinite antichain of \( 2^{<\omega} \). The first case is not possible (because \( B \cap c(\sigma) \) is finite for every \( \sigma \in 2^{\omega} \)), so there exists an infinite antichain \( B_0 \) contained in \( B \). Since \( |B_0 \cap c(\sigma)| \leq 1 \) for every \( \sigma \in 2^{\omega} \), we get \( B_0 \in \mathcal{A} \) (bear in mind that all singletons of \( 2^{<\omega} \) belong to \( \mathcal{A}_{E'} \)). This is a contradiction, because \( B_0 \) is infinite and \( B_0 \subseteq B \subseteq A \in \mathcal{A}^\perp \).

Finally, since \( E \) is dense in \( 2^{\omega} \), we have \( 2^{<\omega} \subseteq \bigcup_{\sigma \in E} c(\sigma) \). Let \( S_1 \subseteq E \) be a finite set such that \( B \subseteq \bigcup_{\sigma \in S_1} c(\sigma) \). Then \( S \cup S_1 \) is a finite subset of \( E \) such that \( A \subseteq \bigcup_{\sigma \in S\cup S_1} c(\sigma) \). This finishes the proof of (7.4).

Finally, note that equality (7.4) allows us to define a function \( f : [E]^{<\omega} \to \mathcal{A}^\perp \) by \( f(F) := \bigcup_{\sigma \in F} c(\sigma) \) and a function \( g : \mathcal{A}^\perp \to [E]^{<\omega} \) such that \( A \subseteq \bigcup_{\sigma \in g(A)} c(\sigma) \) for every \( A \in \mathcal{A}^\perp \). Clearly, both \( f \) and \( g \) are Tukey functions, so \( \mathcal{A}^\perp \sim [E]^{<\omega} \) and the proof is over. \( \square \)

There exists a model of set theory where the axioms \( MA_{\aleph_1} \) and Lusin’s hypothesis \( L \) (every subset of cardinality \( \omega_1 \) of a Polish space is coanalytic) both hold [28]. In such a model, the hypothesis of Theorem 7.12 holds, and moreover \( [\omega_1]^{<\omega} \) is not Tukey equivalent to any of \( \{ 0 \}, \omega, \omega^\omega, K(\mathbb{Q}) \) or \( [\omega]^{<\omega} \), because the cofinality of \( [\omega_1]^{<\omega} \) equals \( \aleph_1 \), while the other posets have cofinality either \( \omega_0 \), or \( d \) or \( c \) (see Section 2), but under \( MA_{\aleph_1} \) we have \( \aleph_1 < d \).

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**Departamento de Matemáticas, Facultad de Matemáticas, Universidad de Murcia, 30100 Espinardo, Murcia, Spain**

*E-mail address*: avileslo@um.es

**Instytut Matematyczny, Uniwersytet Wrocławska, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland**

*E-mail address*: grzes@math.uni.wroc.pl

**Departamento de Matemática Aplicada, Facultad de Informática, Universidad de Murcia, 30100 Espinardo, Murcia, Spain**

*E-mail address*: joserr@um.es