Cosmological Solutions of a Nonlocal Model with a Perfect Fluid

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Abstract. A nonlocal gravity model which does not assume the existence of a new dimensional parameter in the action and includes a function $f(\Box^{-1}R)$, with $\Box$ the d’Alembertian operator, is studied. By specifying an exponential form for the function $f$ and including a matter sector with a constant equation of state parameter, all available power-law solutions in the Jordan frame are obtained. New power-law solutions in the Einstein frame are also probed. Furthermore, the relationship between power-law solutions in both frames, established through conformal transformation, is substantially clarified. The correspondence between power-law solutions in these two frames is proven to be a very useful tool in order to obtain new solutions in the Einstein frame.

Keywords: nonlocal gravity, power-law solution, conformal transformation

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Contents

1 Introduction ................................................................. 2

2 Nonlocal gravitational action and the equations of motion ......... 4

3 Power-law solutions of the model with $f(\psi)$ an exponential function ......................................................... 5
   3.1 The model with $f(\psi)$ being an exponential function .... 5
   3.2 Solutions with $H = n/t$ .......................................... 6
   3.3 Proof of the absence of power-law solutions in the case $\psi_1 \neq 0$ .................................................. 9
   3.4 Special values of the power index $n$ ......................... 9
   3.5 Brief summary of the solutions in the Jordan frame ...... 10
   3.6 Local constraints .................................................. 10

4 Power-law solutions for the original nonlocal model ............ 12

5 Action and equation of motion in the Einstein frame ........... 14
   5.1 The Jordan and Einstein frames ................................. 14
   5.2 Conformal transformation ....................................... 15
   5.3 Equations of motion .............................................. 16
   5.4 The FLRW metric .................................................. 16

6 Power-law solutions in the Einstein frame for the model without matter ......................................................... 18

7 Power-law solutions in the Einstein frame for the model with matter .......................................................... 19
   7.1 The case $\Lambda = 0$ .............................................. 19
   7.2 The case $\Lambda \neq 0$ .............................................. 20
      7.2.1 $w_m = 1$ ...................................................... 20
      7.2.2 $w_m \neq 1$ ...................................................... 21

8 Relationship between power-law solutions in the Jordan and in the Einstein frames ..................................... 22
   8.1 Conformal transformation between power-law solutions .. 22
      8.1.1 General expression for the conformal factor $\Omega$ ...... 22
      8.1.2 Conformal factor corresponding to power-law solutions in the Jordan frame .................................. 23
   8.2 The case $C_1 = 0$ .............................................. 24
   8.3 The case $C_2 = 0$ .............................................. 25
   8.4 Radiation case .................................................... 26
   8.5 The case $\Lambda \neq 0$ .............................................. 27
   8.6 Brief summary of the solutions in the Einstein frame ...... 28

9 Conclusions ................................................................. 29
1 Introduction

The acceleration of the Universe expansion is presently supported by a large number of independent sets of observational data, of very different kind [1–7]. Modern cosmological surveys allow astronomers to obtain increasingly accurate joint constraints on the set of cosmological parameters (see, e.g., [8]). The usual assumption that General Relativity (GR) is the correct theory of gravity at all scales leads to the remarkable conclusion that about seventy per cent of the energy density of the Universe at present must be smoothly distributed under the form of a slowly varying cosmic fluid with negative pressure, called dark energy. Ordinarily, in order to specify a would-be component of the cosmic fluid use is made of its equation of state (EoS), namely a phenomenological relation between the pressure, $p$, and the energy density, $\rho$, corresponding to the considered component, e.g. $p = w \rho$, where $w$ is the EoS parameter. Contemporary experiments provide strong support to the statement that the dark energy EoS parameter is presently very close to $-1$. If this number were the exact value, this would lead us back to GR with a cosmological constant (and nothing else), but a small deviation from this value cannot be excluded by the most accurate astronomical data available. Moreover, the sign (positive or negative) or the tendency (e.g., the derivative) of this deviation cannot be clearly determined at present. This makes room for a number of theoretical models, derived from quite different fundamental theories, which can accommodate such situation.

Actually, a few types of models exist which are able to reproduce the observed late-time cosmic acceleration. The simplest, and most popular of them is $\Lambda$CDM, which fits a wide range of cosmological data [5]. In this model the dark energy component is just the cosmological constant which is added to the action corresponding to GR. Other models introduce a dynamical dark energy characterized by a varying EoS parameter. The standard way to obtain an evolving EoS parameter is the addition of scalar fields to the cosmological model. Actually, the evolution of the Universe is sufficiently well described by cosmological models with scalar fields, in particular, by quintom models, which involve two of them: a phantom scalar field and an ordinary scalar (see e.g. [9]). Quintom models are being very actively studied at present [10–14] (for reviews, see also [15]). We should note, however, that the origin of this fluid, which produces anti-gravitational effects, still remains a mystery. Other popular theories involve modifications of Einsteinian gravity, as for instance $F(R)$ gravity, with $F(R)$ an (in principle) arbitrary function of the scalar curvature $R$ (for recent reviews see, e.g., [16–22]).

Higher-derivative corrections to the Einstein–Hilbert action are being actively studied in the context of quantum gravity (as one of the first papers on this subject we can mention [23]; see also [24] and references therein). A nonlocal gravity theory obtained by taking into account quantum effects was proposed in [25]. Also, string/M theory is usually considered as a possible frame (expectedly, the ultimate one) for the discussion of all fundamental interactions, including gravity, consequently, the natural appearance of non-locality within string field theory provides a very strong motivation for studying nonlocal cosmological models. It should be emphasized in this context that most of the nonlocal cosmological models available explicitly include a function of the d’Alembertian operator, $\Box$, and either directly define a nonlocal modified gravity [26–40] or, alternatively, add a nonlocal scalar field, minimally coupled to gravity [41].

In the present paper, we consider a nonlocal gravity model which contains a function of the $\Box^{-1}$ operator but does not assume the existence of a new dimensional parameter in the action [25]. For this kind of nonlocal models, an explicit technique for choosing the
distortion function, \( f(\Box^{-1}) \), so as to fit an arbitrary expansion history, has been derived in [31], and the specific nonlocal model considered has a local scalar-tensor formulation [27]. The perturbation analysis of this model has been carried out and the Solar System test has been performed in [30]. De Sitter solutions and expanding universe solutions with \( a \sim t^n \) have been investigated in [27, 36–38]. In [29] the ensuing cosmology describing the four basic epochs was studied for nonlocal models involving, in particular, an exponential form for the function \( f(\eta) \). An explicit mechanism to screen the cosmological constant in nonlocal gravity was discussed in [35–37]. In the framework of the local formulation, a reconstruction procedure was proposed in [29] and it has been developed in [38–40].

The example most usually studied [27–29, 35–38] of a model of this kind is characterized by an exponential function \( f(\Box^{-1}R) = f_0 e^{\alpha(\Box^{-1}R)} \), where \( f_0 \) and \( \alpha \) are real parameters, a case that will be explicitly considered in this paper.

Conformal (Weyl) transformations are widely used in scalar-tensor theories of gravity, the theory of a scalar field coupled nonminimally to the Ricci curvature, \( R \), and in \( F(R) \) gravity theories [42–44] (see also [18, 20, 21]). The Hilbert–Einstein action and the modified gravity action can be related by the conformal transformation [18, 20, 21, 44–47], being the corresponding equations also connected by the same transformation. The very important issue concerning which of the conformal frames, Jordan or Einstein, is the true physical one, has been the subject of longstanding debate (see [43, 44], and references therein). In this respect, knowledge of the transitions between these frames is a very useful tool for the construction of new exact solutions. We prove this statement in Sect. 8.

The nonlocal model we will consider is usually studied in the Jordan frame, but recently its behavior in the Einstein frame, for the model without matter [36, 37], has been explored too. In this paper, we will focus our effort on the study of cosmological solutions of this model, both in the Jordan and in the Einstein frames, including the case with matter for the last one. We will consider gravity models with a cosmological constant \( \Lambda \) and including a perfect fluid, and study in detail their cosmological solutions with a power-law cosmic scalar factor: \( a \propto t^n \). The solutions thus obtained will be proven to generalize solutions found in [27, 36, 37].

In the Jordan frame we find a class of power-law solutions and prove, moreover, that other power-law solutions cannot be exact. We will analyze with care the correspondence existing between the solutions obtained in the different frames and will demonstrate explicitly how the explicit knowledge of power-law solutions in the Jordan frame can be used in order to get power-law solutions in the Einstein frame.

The paper is organized as follows. In Sect. 2, we start from the action for a general class of nonlocal gravity, without specifying the form of the function \( f(\psi) \), and derive the equations of motion for a spatially flat cosmology in the Jordan frame. The theory is specialized in Sect. 3 to a model characterized by the function \( f(\psi) = f_0 e^{\alpha \psi} \), and the corresponding power-law solutions are obtained. In Sect. 4, we consider the initial nonlocal model with a perfect fluid. Using the power-law solutions of Sect. 3, we then get the class of power-law solutions for this nonlocal model. In Sect. 5 we investigate a conformal transformation from the original (Jordan) to the Einstein frame, and derive the corresponding equations of motion (EOM). Vacuum power-law solutions in the Einstein frame are derived in Sect. 6. In Sect. 7, we include the matter sector and obtain the corresponding power-law solutions by directly solving the EOM. In Sect. 8 we use the correspondence between power-law solutions in the Jordan and in the Einstein frames in order to get brand new solutions in the Einstein frame. Finally, Sect. 9 is devoted to conclusions.
2 Nonlocal gravitational action and the equations of motion

We start by considering a class of nonlocal gravities, with action given by

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ R \left( 1 + f(\Box^{-1} R) \right) - 2\Lambda \right] + \mathcal{L}_m \right\}, \] (2.1)

where \( \kappa^2 = 8\pi G = 8\pi / M_{\text{Pl}}^2 \), the Planck mass being \( M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19} \text{ GeV} \), while \( f \) is a differentiable function, which characterizes the nature of nonlocality, \( \Box^{-1} \) being the inverse of the d'Alembertian operator, \( \Lambda \) the cosmological constant, and \( \mathcal{L}_m \) the matter Lagrangian.

For definiteness, we assume that matter is a perfect fluid. We use the signature \((-;+++)\), \( g \) being the determinant of the metric tensor, \( g^\mu_\nu \). Recall the covariant d'Alembertian for a scalar field, which reads

\[ \Box \equiv \nabla^\mu \nabla_\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right). \]

where \( \nabla^\mu \) is the covariant derivative.

For practical uses, introducing two scalar fields \( \psi = \Box^{-1} R \) and a Lagrange multiplier \( \xi \) (see [27]), action (2.1) can be recast as a local action, namely

\[ S_l = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ R \left( 1 + f(\psi) \right) + \xi \left( R - \Box \psi \right) - 2\Lambda \right] + \mathcal{L}_m \right\}. \] (2.2)

Therefore, the original action can actually be regarded as a local one (2.2) in the Jordan frame. By varying this action with respect to \( \xi \) and \( \psi \), one respectively gets the field equations

\[ \Box \psi = R, \quad \Box \xi = f,\psi(\psi) R, \] (2.3, 2.4)

where \( f,\psi(\psi) \equiv df/d\psi \). The corresponding Einstein equations are obtained by variation of the action (2.2) with respect to the metric tensor \( g^\mu_\nu \), as follows

\[ \frac{1}{2} g^\mu_\nu \left[ R \Psi + \partial_\mu \xi \partial_\nu \psi - 2(\Lambda + \Box \Psi) \right] - R^\mu_\nu \Psi - \frac{1}{2} \left( \partial_\mu \xi \partial_\nu \psi + \partial_\mu \psi \partial_\nu \xi \right) + \nabla_\mu \partial_\nu \Psi = -\kappa^2 T^{(m)}_{\mu\nu}, \] (2.5)

where \( \Psi \equiv 1 + f(\psi) + \xi \), and \( T^{(m)}_{\mu\nu} \) is the energy–momentum tensor of the matter sector, defined as

\[ T^{(m)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}. \] (2.6)

We note that the system of equations here considered does not include the function \( \psi \) itself, but instead \( f(\psi) \) and \( f,\psi(\psi) \), together with time derivatives of \( \psi \). Also, \( f(\psi) \) can only be determined up to a constant, since one may indeed add a constant to \( f(\psi) \) and subtract the same constant from \( \xi \) without changing the original equations at all.

In this paper, we assume a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe, with the space-time interval

\[ ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \] (2.7)
and consider the case where the scalar fields $\psi(t)$ and $\xi(t)$ are only functions of the cosmological time. Thus, the system of Eqs. (2.3)–(2.5) reduces to

$$3H^2\Psi = -\frac{1}{2}\dot{\xi}\dot{\psi} - 3H\dot{\Psi} + \Lambda + \kappa^2\rho_m,$$  \hspace{1cm} (2.8)

$$\left(2\dot{H} + 3H^2\right)\Psi = \frac{1}{2}\dot{\xi}\dot{\psi} - \ddot{\Psi} - 2H\dot{\Psi} + \Lambda - \kappa^2P_m,$$  \hspace{1cm} (2.9)

$$\ddot{\psi} = -3H\dot{\psi} - 6\left(\dot{H} + 2H^2\right),$$  \hspace{1cm} (2.10)

$$\ddot{\xi} = -3H\dot{\xi} - 6\left(\dot{H} + 2H^2\right)f_\psi(\psi),$$  \hspace{1cm} (2.11)

where a dot means differentiation with respect to time, $t$, in the Jordan frame: $\dot{A}(t) \equiv dA(t)/dt$, and $H = \dot{a}/a$ is the Hubble parameter. For a perfect matter fluid, we have $T_{(m)00} = \rho_m$ and $T_{(m)ij} = P_mg_{ij}$. The continuity equation is

$$\dot{\rho}_m = -3H(P_m + \rho_m).$$  \hspace{1cm} (2.12)

It is useful to add up (2.8) and (2.9), and get

$$\dddot{\Psi} + 5H\dot{\Psi} + \left(2\dot{H} + 6H^2\right)\Psi - 2\Lambda + \kappa^2(P_m - \rho_m) = 0.$$  \hspace{1cm} (2.13)

Note that Eq. (2.13) is a second-order linear differential equation for $\Psi$.

3 Power-law solutions of the model with $f(\psi)$ an exponential function

3.1 The model with $f(\psi)$ being an exponential function

Following [36–38], we consider the case where $f(\psi)$ is an exponential function, namely

$$f(\psi) = f_0 e^{\alpha\psi},$$  \hspace{1cm} (3.1)

with $f_0$ and $\alpha$ nonzero real parameters. The motivation for considering an exponential function $f(\psi)$ is not only because it is the simplest model with power-law and de Sitter solutions\(^1\), but also, because it is the better studied case among all possible functions $f(\psi)$ [27–29, 35–38] (de Sitter solutions for this model were discussed in [27, 36, 38], and expanding universe solutions with the Hubble parameter $H = n/t$, where $n$ is a nonzero constant, in [27, 37]). In the present paper we will investigate this last type of solutions in detail.

We consider matter with the EoS parameter $w_m \equiv P_m/\rho_m$ being a constant but not equal to $-1$. For power-law solutions $H = n/t$, Eq. (2.12) has the following general solution:

$$\rho_m(t) = \rho_0 t^{-3n(w_m+1)},$$  \hspace{1cm} (3.2)

where $\rho_0$ is an arbitrary constant.

\(^1\)In models with such solutions, the function $f(\psi)$ is either an exponential function or a sum of exponential functions [40].
3.2 Solutions with $H = n/t$

The goal of this section is to find the whole set of power-law solutions for the model, characterized by the function $f$ given in (3.1). In this subsection, we present some power-law solutions and, in the next one, we will show that no other power-law solutions exist.

Inserting $H = n/t$ into Eq. (2.10), the following solution $\psi(t)$ is obtained,

$$\psi(t) = \psi_1 t^{1-3n} - \frac{6n(2n-1)}{3n-1} \ln \left( \frac{t}{t_0} \right),$$  \hspace{1cm} (3.3)

where $\psi_1$ and $t_0$ are integration constants. We consider real solutions at $t > 0$, hence, $t_0 > 0$. Note that this solution is valid provided $n \neq 1/3$ and $n \neq 1/2$. Consequently, in this subsection the cases $n = 1/2$ and $n = 1/3$ will be excluded from our analysis. We also specify $\psi_1 = 0$, so that the function $f(\psi)$ takes the following form

$$f(\psi(t)) = f_0 \left( \frac{t}{t_0} \right)^m, \quad m = -6n(2n-1) \frac{n}{3n-1}. \hspace{1cm} (3.4)$$

We will show in the next subsection that there is no solution for $\psi_1 \neq 0$. The cases $n = 1/2$ and $n = 1/3$ will be considered in Sect. 3.4.

Inserting formulae (3.3) and (3.4) into (2.11) one obtains the following expression for $\xi(t)$

$$\xi(t) = \begin{cases} 
\xi_0 + \xi_1 \left( \frac{t}{t_0} \right)^{1-3n} + \frac{(3n-1)f_0}{3n+m-1} \left( \frac{t}{t_0} \right)^m, & \text{for } m \neq 1-3n, \\
\xi_2 - mf_0 \left( \frac{t}{t_0} \right)^m \ln \left( \frac{t}{t_1} \right), & \text{for } m = 1-3n, 
\end{cases} \hspace{1cm} (3.5)$$

where $\xi_0$, $\xi_1$, $\xi_2$, and $t_1$ are integration constants.

Furthermore, substituting the solutions described by formulae (3.2), (3.3), and (3.5), into Eqs. (2.8) and (2.9), constraints on these integration constants can be obtained

- For $m \neq 1 - 3n$, which equivalently implies the following constraint on the power index $n$

$$n \neq \frac{3(\alpha - 1) \pm \sqrt{3\alpha(3\alpha-2)}}{3(4\alpha-3)}, \hspace{1cm} (3.6)$$

in this case, we get constraints on the integration constants for $\Lambda = 0$ and $\Lambda \neq 0$ separately:

- For $\Lambda = 0$, by inserting the solutions (3.2), (3.3), and (3.5) into the system (2.8)–(2.11), the corresponding integration constants are fixed by

$$\begin{cases} 
\xi_0 = -1, \\
\rho_0 = \frac{6(3n-1+3\alpha-6n\alpha)f_0n^2}{(3n-1)\kappa^2} \frac{\kappa_n(2n-1)\alpha/(3n-1)}{t_0^3}, 
\end{cases} \hspace{1cm} (3.7)$$

with $t_0$ and $\xi_1$ to be determined by initial conditions, while the power index $n$ is constrained by

$$w_m + 1 - \frac{2}{3n} - \frac{2\alpha(2n-1)}{3n-1} = 0, \hspace{1cm} (3.8)$$
from which $n$ is expressed in terms of the parameters $w_m$ and $\alpha$:

$$n = n \pm = \frac{3w_m - 6\alpha + 9 \pm \sqrt{(3w_m - 6\alpha + 1)^2 + 8(1 - 3w_m)}}{6(3w_m - 4\alpha + 3)}. \quad (3.9)$$

This furthermore yields corresponding constraints on the parameters $\alpha$ and $w_m$ for a real number $n_\pm$:

$$(3w_m - 6\alpha + 1)^2 + 8(1 - 3w_m) \geq 0. \quad (3.10)$$

Interestingly enough, from Eq. (3.9) one finds that, for $\alpha \gg 1$, one of the power indices behaves as $n_\pm \rightarrow 1/2$, which implies that the Universe asymptotically evolves to a radiation-dominated phase, and this regardless of the details of the EoS for the matter sector. Thus, we manage to obtain an expanding universe without introducing a cosmological constant $\Lambda$. This is because the non-local term $\Box^{-1}R$ plays partially the role of a dark energy, though it is a decelerating expansion.

Furthermore, we pay attention to two special values of the EoS parameter:

(i) When $w_m = -1$ matter is nothing but just an effective cosmological constant. Therefore, this case corresponds to $\Lambda \neq 0$.

(ii) If the only matter is radiation, namely, $w_m = 1/3$, Eq. (3.9) leads to $n_- = 1/2$, which should be excluded. Therefore, the power index is $n = n_+$ for the radiation component.

For $\Lambda \neq 0$, the corresponding integration constants and parameters are constrained:

$$\begin{cases}
    m = 2, \\
    t_0^2 = \frac{6n(n + 1)f_0}{\Lambda}, \\
    \rho_0 = \frac{3(1 + \xi_0)n^2}{\kappa^2},
\end{cases} \quad (3.11)$$

the integration constants $\xi_0$ and $\xi_1$ in the solution (3.5) are to be fixed by the initial conditions, while the power-index $n$ is determined by $w_m$:

$$n = \frac{2}{3(1 + w_m)}. \quad (3.12)$$

Here we note that, by recalling the definition of $m$ in Eq. (3.4) and using Eq. (3.12), one finds that the parameter $\alpha$ is constrained by $m = 2$:

$$\alpha = \frac{3(1 - w_m^2)}{2(3w_m - 1)}. \quad (3.13)$$

Thus, unlike the situation for vacuum solutions with a nonzero cosmological constant in [37], here we find that with a matter sector the parameter of the model $\alpha$ is fixed by the EoS of matter in Eq. (3.13). In this sense, the model is spoiled since, in this case, it cannot yield a smooth evolution of the Universe for different stages with a given parameter $\alpha$.

From (3.13) it follows that there is no power-law solution for non-vanishing cosmological constant with matter whose EoS is $w_m = 1/3$ or $w_m = 1$, if $m \neq 1 - 3n$. 

-7-
The case $w_m = -1$ corresponds to the cosmological constant as the matter part. Hence, to obtain the solutions in this case we should put $\rho(t) = 0$, which corresponds to $\xi_0 = -1$ from Eq. (3.11). In this case, from the constraint $m = 2$, we also obtain two branches for the power-index:

$$n = n_\pm = \frac{3(\alpha - 1) \pm \sqrt{9\alpha^2 + 6\alpha + 9}}{12\alpha},$$

which coincide with Eq. (25) of Ref. [37], as expected. So the solutions obtained here contain the vacuum case where the universe asymptotically behaves like a radiation-dominated one.

- For $m = 1 - 3n$, the power-index $n$ is determined\(^2\) by the parameter $\alpha$:

  $$n = n_\pm = \frac{3(\alpha - 1) \pm \sqrt{3\alpha(3\alpha - 2)}}{3(4\alpha - 3)}.$$  

For $\Lambda = 0$, one finds power-law solutions with the following constraints on the integration constants

$$\begin{cases}
\xi_2 &= -1, \\
\rho_0 &= -\frac{3n(n-1)f_0}{\kappa^2} t_0^{2n-1},
\end{cases}$$

while, again, $n$ is determined by $w_m$

$$n = \frac{1}{3w_m},$$

which again implies that in this case the model cannot yield the different stages of the Universe evolution with a fixed $\alpha$.

For some special values of the parameter $\alpha$ additional solutions exist, namely:

- For $\alpha = 2/3$ and $\Lambda = 0$, there exists a solution with integral constants $\xi_2$, $t_0$, and $t_1$ to be specified by the initial conditions, while $n$, $w_m$, and $\rho_0$ are fixed by

$$n = 1, \quad w_m = -\frac{1}{3}, \quad \rho_0 = \frac{3(\xi_2 + 1)}{\kappa^2}.$$  

- When $\alpha = 6/5$ and $\Lambda \neq 0$ we obtain a solution with $t_1$ undetermined, while

$$n = -\frac{1}{3}, \quad t_0^2 = -\frac{4f_0}{3\Lambda}, \quad \rho_0 = 0, \quad \xi_2 = -1.$$  

Note that for $\alpha = 6/5$ we also have a solution (3.16) with $\Lambda = 0$ and $n = 5/9$.

Thus, we have obtained corresponding solutions for both nonzero and zero values of $\Lambda$. They generalize the ones previously found in the absence of matter [37]. All solutions for the case $m = 1 - 3n$ are new.

\(^2\)The equation is the following:

$$\alpha - \frac{(3n - 1)^2}{6n(2n - 1)} = 0.$$
3.3 Proof of the absence of power-law solutions in the case $\psi_1 \neq 0$

Recall at this point that our main goal is to find all solutions that correspond to $H = n/t$. In this subsection we consider the case $\psi_1 \neq 0$ with the hope to obtain new solutions or else rigorously prove that such solutions do not exist.

In the case $\psi_1 \neq 0$, Eq. (2.11) has no solution in terms of elementary functions, thus it is more convenient to solve Eq. (2.13), get $\Psi(t)$, and substitute $\xi(t) = \Psi(t) - f'(\psi(t))$ into Eq. (2.11), to then check if there exist values of the parameters for which the obtained $\xi(t)$ satisfies Eq. (2.11), or not. The type of solutions of Eq. (2.13) depends on the value of $n$, thus we consider different cases.

For $n \neq -1$ and $n \neq -1/3$, Eq. (2.13) has the following general solution

$$\Psi(t) = C_1 t^{-2n} + C_2 t^{1-3n} + \frac{\Lambda}{(n+1)(3n+1)} t^2 - \frac{\rho_0 \kappa^2 (w_m - 1)t^{2-3(1+w_m)n}}{(3nw_m - 1)(n + 3nw_m - 2)},$$  

(3.21)

where $C_1$ and $C_2$ are integral constants and $w_m$ is chosen so that $(3nw_m - 1)(n + 3nw_m - 2)$ be not equal to zero.

In the case $n \neq -1$ and $n \neq -1/3$, the solutions of Eqs. (2.10) and (2.13) are given by formulae (3.21) and (3.3), respectively. Substituting the $\xi(t) = \Psi(t) - f'(\psi(t))$ obtained into (2.11), we realize that this equation is not satisfied. In particular, the non-matching expression is proportional to

$$t^\gamma \exp \left( -\frac{\psi_1}{\beta} t^{1-3n} \right),$$

where $\beta$ and $\gamma$ are combinations of constants. The exponential term disappears only for $\psi_1 = 0$. Therefore, there is no solution for $\psi_1 \neq 0$ and $n \neq -1, n \neq -1/3$. Similar calculations show that, in the cases $n = -1$ and $n = -1/3$, solutions are absent, as well.

We thus have found all solutions which correspond to $H = n/t$. Note that, in contradistinction to the papers [27, 37], we here include an ideal perfect fluid in action (2.2) and do not impose any restrictions whatsoever on the parameters and integration constants.

3.4 Special values of the power index $n$

Let us consider the case $n = 1/2$, which corresponds to $R = 0$. Solving Eqs. (2.10) and (2.11), we get

$$\psi(t) = \psi_3 t^{-1/2} + \psi_4, \quad \xi(t) = \xi_3 t^{-1/2} + \xi_4,$$

(3.22)

where $\psi_3$, $\psi_4$, $\xi_3$, and $\xi_4$ are integral constants. Straightforward substitution of these functions and $H = 1/(2t)$ into Eqs. (2.8) and (2.9) yields a solution for $\Lambda = 0$ with the following conditions on the constants:

$$\psi_3 = 0, \quad \xi_4 = -1 - f_0 e^{\alpha \psi_4} + \frac{4}{3} \kappa^2 \rho_0, \quad w_m = \frac{1}{3},$$  

(3.23)

while $\rho_0$, $\psi_4$ and $\xi_3$ are to be determined by the initial conditions.

When $n = 1/3$, Eq. (2.10) has the solution:

$$\psi(t) = \frac{1}{3} \ln \left( \frac{t}{t_2} \right)^2 + \psi_5 \ln \left( \frac{t}{t_2} \right),$$  

(3.24)

where $\psi_5$ and $t_2$ are integration constants. The function $\xi(t)$, as a solution of Eq. (2.11), can be given in terms of quadratures only. At the same time, solving Eq. (2.13), we get $\Psi(t)$ in terms of elementary functions. Thus, if a solution exists, then the corresponding $\xi(t)$ should be an elementary function as well. We therefore arrive to a contradiction, what proves the absence of power-law solutions with $n = 1/3$. 


### Table 1. Solutions in the Jordan frame for \( m \neq 1 - 3n \)

| \( m = 1 - 3n \) | solutions | constraints |
|------------------|-----------|-------------|
| \( \Lambda = 0 \) | \[
\xi(t) = -1 + \xi_1 \left( \frac{t}{t_0} \right)^{1-3n} + \frac{(3n-1)f_0}{3n+m-1} \left( \frac{t}{t_0} \right)^m,
\]
| \[\rho_m(t) = \frac{6f_0n^2}{\kappa^2 t_0^2} \left[ 1 + \frac{3\alpha(1-2n)}{3n-1} \right] \left( \frac{t}{t_0} \right)^{3n(w_m+1)}\] | \[\text{Eq. (3.8)}\] \[\text{Eq. (3.9)}\] | \[\text{Eq. (3.10)}\] |
| \( \Lambda \neq 0 \) | \[
\xi(t) = \xi_0 + \xi_1 \left( \frac{t}{t_0} \right)^{1-3n} + \frac{(3n-1)f_0}{3n+1} \left( \frac{t}{t_0} \right)^2,
\]
| \[\rho_m(t) = \frac{3n^2(1+\xi_0)}{\kappa^2} t^{-3n(w_m+1)}\] | \[\text{Eq. (3.11)}\] \[\text{Eq. (3.12)}\] | \[\text{Eq. (3.13)}\] |

### Table 2. Solutions in the Jordan frame for \( m = 1 - 3n \)

| \( m = 1 - 3n \) | solutions | constraints |
|------------------|-----------|-------------|
| \( \Lambda = 0 \) | \[
\xi(t) = -1 + f_0(3n-1) \left( \frac{t}{t_0} \right)^{1-3n} \ln \left( \frac{t_1}{t} \right),
\]
| \[\rho_m(t) = \frac{3f_0n(1-n)}{\kappa^2 t_0^2} \left( \frac{t}{t_0} \right)^{-3n-1}\] | \[\text{Eq. (3.15)}\] \[\text{Eq. (3.17)}\] | \[\text{Eq. (3.18)}\] |
| \( \Lambda \neq 0 \) | \[
\xi(t) = -1 + \frac{3\Lambda t^2}{2} \ln \left( \frac{t}{t_1} \right),
\]
| \[\rho_m(t) = 0\] | \[\alpha = 6/5\] | \[\text{Eq. (3.20)}\] |

#### 3.5 Brief summary of the solutions in the Jordan frame

To make it easier for readers to look at the whole set of solutions, we list all those we have found in this section in Tables 1 and 2. We note that, in the case \( n \neq 1/2, 1/3 \), the solution for \( \psi(t) \) is uniquely given by Eq. (3.3) with \( \psi_1 = 0 \), i.e.

\[
\psi(t) = -\frac{6n(2n-1)}{3n-1} \ln \left( \frac{t}{t_0} \right),
\]  
(3.25)

and this expression is not repeated in the table. It should be noted that for \( n = 1/2 \) the solution is given by Eqs. (3.22) and (3.23), while no power-law solution exists for \( n = 1/3 \).

#### 3.6 Local constraints

Modified gravity theories are quite strictly constrained by local observations [18, 48]. Precise consideration [30] of the Newtonian limit of the theory, described by action (2.2), gives the
following restrictions on the post-Newtonian parameter $\gamma$:

$$|\gamma - 1| = \left| \frac{4f,\psi}{1 + f + \xi - 8f,\psi} \right| < 2.3 \times 10^{-5}. \quad (3.26)$$

In order to check whether the power-law solutions found in the previous sections can satisfy this constraint, we choose the solution where $m \neq 1 - 3n$ and $\Lambda = 0$, since this one may be most relevant to the deceleration expansion phase when matter fields dominate the expansion of the Universe. Hence, we take the corresponding solution

$$\psi(t) = -\frac{6n(2n-1)}{3n-1} \ln \left( \frac{t}{t_0} \right), \quad \xi(t) = \xi_1 \left( \frac{t}{t_0} \right)^{1-3n} - 1 + \frac{(3n-1)f_0}{3n+m-1} \left( \frac{t}{t_0} \right)^m \quad (3.27)$$

and obtain

$$\gamma - 1 = \frac{4f,\psi}{1 + f + \xi - 8f,\psi} = \frac{4f_0\alpha}{\xi_1 \left( \frac{t}{t_0} \right)^{1-3n-m} - f_0 \left( 8\alpha - 1 - \frac{3n-1}{3n+m-1} \right)} \quad (3.28)$$

Now, we discuss whether the constraint Eq. (3.26) can be satisfied or not, in each of the two different cases:

- $\xi_1 \neq 0$. The restrictions on the parameter $\gamma$ have been obtained by the consideration of the effects within the Solar System [48], so we can assume that $t$ is not small and that $t_0 \ll t$. Provided $1 - 3n - m > 0$, the constraint (3.26) can be easily fulfilled. For example, supposing that the power index $n > 1/2$, it translates into the following constraint on the parameter $\alpha$:

$$1 - 3n - m > 0 \implies \alpha > \frac{(3n-1)^2}{6n(2n-1)}. \quad (3.29)$$

- $\xi_1 = 0$. In this case Eq. (3.28) reduces to:

$$\gamma = 1 - \frac{4\alpha}{8\alpha - 1 - \left(1 + \frac{m}{3n-1}\right)^{-1}}, \quad (3.30)$$

where

$$\frac{m}{3n-1} = -\frac{6\alpha n(2n-1)}{(3n-1)^2}. \quad (3.31)$$

Without loss of generality, we assume $n \sim O(1)$. It is convenient to divide the discussion into three cases:

1) $|\alpha| \gg 1$. In this case, $|m/(3n-1)| \gg 1$, hence,

$$|\gamma - 1| \simeq \frac{1}{2}, \quad (3.32)$$

and the local constraint (3.26) cannot be satisfied.

2) $|\alpha| \ll 1$. In this case, $|m/(3n-1)| \ll 1$, hence,

$$|\gamma - 1| \simeq |\alpha| < 10^{-5}, \quad (3.33)$$
which implies that we need to tune $\alpha$ to a very small value.

3) $|\alpha| \sim O(1)$. In this case, we should recall that the power-index $n$ is related to $\alpha$ and the EoS parameter, $w_m$, by Eq. (3.9).

An especially interesting case is when the matter sector is composed of a non-relativistic matter fluid, i.e. $w_m = 0$. In this case, inserting $n_-$ into Eq. (3.30), one finds that we need to specify $\alpha$ to be of order $10^{-5}$ for the local constraint (3.26). If we take the $n_+$ branch, besides $|\alpha| \lesssim 10^{-5}$, there is another point $\alpha \approx 0.75$, but the allowed range around this value is about $10^{-5}$ for local constraints. Moreover, similar conclusions hold for radiation components, for which the EoS parameter $w_m = 1/3$.

Thus, depending on whether the integration constant $\xi_1$ is non-vanishing or not, we draw different conclusions concerning the constraint of the Post-Newtonian parameter $\gamma$: when $\xi_1 \neq 0$, the constraint can be easily satisfied for a wide range of choices of the parameter $\alpha$ in this model, but when $\xi_1 = 0$ one needs to tune the parameter $\alpha$ to at least $10^{-5}$ order, to satisfy the local constraint. Note that in previous papers [27, 37] the authors just set $\xi_1 = 0$ for simplicity. The analysis of the local constraint shows that solutions with nonzero $\xi_1$ allow to change the restrictions on the parameter $\alpha$, which are indeed necessary in order to make the model compatible with astronomical observations.

4 Power-law solutions for the original nonlocal model

In this section, we discuss the power-law solutions for the nonlocal model in the original form (2.1). When we vary the nonlocal action (2.1) with respect to the metric $g_{\mu\nu}$, under the spatially flat FLRW metric (2.7), the independent components of field equations can be expressed as follows [33, 34]:

$$
3H^2 + \Delta G_{00} = \kappa^2 \rho_m + \Lambda,
$$

$$
-2\dot{H} - 3H^2 + \frac{1}{3a^2} \delta^{ij} \Delta G_{ij} = \kappa^2 P_m - \Lambda,
$$

where $\Delta G_{00}$ and $\Delta G_{ij}$ denote the modifications coming from the nonlocal terms, namely

\[
\Delta G_{00} = \left(3H^2 + 3H\partial_t\right) \left\{ f(\Box^{-1}R) + \Box^{-1}\left[R \frac{df}{d(\Box^{-1}R)}\right]\right\} \\
+ \frac{1}{2} \partial_t \left(\Box^{-1}R\right) \partial_t \left(\Box^{-1}\left[R \frac{df}{d(\Box^{-1}R)}\right]\right),
\]

\[
\Delta G_{ij} = a^2 \delta_{ij} \left[ \frac{1}{2} \partial_t \left(\Box^{-1}R\right) \partial_t \left(\Box^{-1}\left[R \frac{df}{d(\Box^{-1}R)}\right]\right) \\
- \left[2\dot{H} + 3H^2 + 2H\partial_t + \partial_t^2\right] \left\{ f(\Box^{-1}R) + \Box^{-1}\left[R \frac{df}{d(\Box^{-1}R)}\right]\right\} \right].
\]

Hence, the identification of the scalar fields $\psi$ and $\xi$ with corresponding terms in the original action yields [31]

$$
\psi(t) = \Box^{-1}R,
$$

$$
\xi(t) = \Box^{-1}\left[R \frac{df}{d(\Box^{-1}R)}\right].
$$
Substituting these expressions into the system (4.1) we get (2.8) and (2.9). The fields $\psi(t)$ and $\xi(t)$, defined by (4.2) and (4.3) respectively, satisfy (2.3) and (2.4).

In the previous section we have obtained power-law solutions for the local formulation of the original nonlocal gravity, where two scalars, $\psi$ and $\xi$ have been introduced in the action (2.2). Thus, we come to conclusion that these solutions are solutions of the initial nonlocal model as well. This can also be checked immediately by direct substitution.

In the FLRW metric, the d’Alembert operator acting on a scalar $A(t)$ can be expressed as
\[
\Box A \equiv \frac{1}{\sqrt{-g}} \partial_{\rho} \left( \sqrt{-g} g^{\rho\sigma} \partial_{\sigma} \right) A = -\frac{1}{a^3(\dot{a})} \left( a^3 \frac{dA}{dt} \right),
\]
while its inverse operator reduces to a double integration [25, 31]:
\[
\Box^{-1} [A(t)] = -\int_{t_0}^{t} \frac{dt}{a^3(t)} \int_{\eta_0}^{\eta} d\eta a^3(\eta) A(\eta).
\]
where $t_0$ and $\eta_0$ are two initial boundaries for the integrals. For the power-law solution with $H = n/t$, we get $R = 6n(2n - 1)/t^2$ and the solutions can correspondingly be obtained by integration, as
\[
\psi(t) = -6n(2n-1) \int_{t_0}^{t} \frac{dt}{t^{3n}} \int_{\eta_0}^{\eta} \eta^{3n-2} d\eta = -\frac{6n(2n-1)}{3n-1} \ln \left( \frac{t}{t_0} \right) + \psi_1 t^{1-3n},
\]
where the integration constants, $t_0$ and $\psi_1$, are connected with $t_0$ and $\eta_0$. This solution coincides with (3.3). Setting $\psi_1 = 0$, which corresponds to $t_0 = t$ and $\eta_0 = 0$, we get
\[
\xi(t) = \frac{3n-1}{3n-1} m f_0 \int_{t_0}^{t} \frac{dt}{t^{3n}} \int_{t_1}^{t} \eta^{3n+m-2} d\eta,
\]
where $m$ is defined in Eq. (3.4). Therefore, we recover the solution (3.5). We thus conclude that the power-law solutions found in the previous sections are equivalent to those corresponding to the original form of the nonlocal theory.

It should be noted that the initial nonlocal model might be non-equivalent to its local formulation. Actually this non-equivalence does not arise from a difference in the equations, but from the initial (boundary) conditions. Let us make some further comments on this issue. By recasting the original form into the biscalar-tensor representation, one needs to invert the relationship $\psi = \Box^{-1} R$, in the form: $\Box \psi = R$. For a given background, the solution for the latter equation is unique up to a harmonic function $\chi$ which satisfies $\Box \chi = 0$, hence causing a legitimate problem, as reported in the first paper of Ref. [32].

Compared to the original form (2.1), the scalar-tensor presentation seems to have introduced a new degree of freedom $\chi$, as pointed out in [31, 49]. In fact, this can be seen more clearly if we write
\[
\psi \rightarrow \psi + \chi
\]

Note that in this paper the non-equivalence between biscalar-tensor representation of nonlocal gravity and its original form have been shown explicitly in the case of $f = \text{const}$ only, when the original model is local.
into (2.2), the corresponding term being
\[ \xi (\Box \psi - R) \longrightarrow \xi (\Box (\psi + \chi) - R), \tag{4.6} \]
and, after integration by parts, the change is
\[ g^{\mu \nu} \partial_\mu \xi \partial_\nu \psi \longrightarrow g^{\mu \nu} \partial_\mu \xi \partial_\nu (\psi + \chi). \tag{4.7} \]
Hence, it seems to have added an extra degree of freedom to the Lagrangian which is absent in the original form. However, if we impose an appropriate boundary condition, for example, \( \chi = 0 \) to recover the original form, then this would-be extra degree of freedom may be eliminated in this way. The issue on the choice of a correct boundary condition should be the only non-equivalence between the original form and its biscalare-tensor representation. Thus, for instance, in [31] the authors determine the inverse d’Alembert operator using the retarded Green function, in other words, they fix a solution of the equation \( \Box R = 0 \) putting \( \tilde{t}_0 = 0 \) and \( \eta_0 = 0 \).

Power-law solutions display a singularity at \( t = 0 \), so for such solutions it would be better to choose a positive value of \( \tilde{t}_0 = t_0 \). In Sect. 3 we obtain that for the model with nonzero \( \Lambda \) the value of \( t_0 \) is defined by \( \Lambda \), whereas at \( \Lambda = 0 \), \( t_0 \) is an arbitrary number, defined by the initial conditions, in particular by \( \rho_0 \).

A final comment is in order. As stated above, the biscalare-tensor representation introduces two scalars, \( \psi \) and \( \xi \), therefore, working in this way it seems that one will encounter a ghost-like behavior, as pointed out in Refs. [30, 35–37]. However, since the original nonlocal model does not introduce any new degree of freedom, the ghost-like behavior of the biscalare-tensor theory may not be physically relevant at all. Indeed, the associated terms can be cast as a boundary term of the nonlocal operators [49]. At the classical level, a necessary way to check whether the ghost-like behavior is physically relevant or not is by considering the equivalence of the solutions coming from the original nonlocal formulation and from its biscalare-tensor form, respectively.

5 Action and equation of motion in the Einstein frame

5.1 The Jordan and Einstein frames

Once a modified gravity theory is recast into its scalar-tensor presentation, it immediately follows that both the Jordan frame (where the matter sector minimally couples to gravity) and the Einstein one (where the Ricci is linear but matter couples to gravity non-minimally) are available (for a description see [42]). These two frames are related by conformal transformation
\[ g_{\mu \nu} = \Omega^2 g^{(E)}_{\mu \nu}, \tag{5.1} \]
where we denote the metric in the Jordan frame by \( g_{\mu \nu} \), while the one in the Einstein frame is labeled as \( g^{(E)}_{\mu \nu} \).

One soon realizes that the conformal transformation connecting both frames cannot be simply interpreted as a coordinate transformation of the theory, and this is the reason why there has been a long debate on which of these two frames is ‘the physical one’, regardless of the fact that the mathematical equivalence of the two frames is quite clear [43–45]. Recent researches have further clarified that, at least at the classical level, the two frames are physically equivalent, what means that all observational quantities (e.g., the redshift \( z \)) should yield the same value in both cases [47].
Based on the corresponding solutions in both frames, one can furthermore probe the equivalence of both frames in the framework of nonlocal gravity inspired models. It is of interest to check the precise behavior of the corresponding solutions in the Einstein frame to see if and, being the case, how much they differ from those obtained in the Jordan frame. Moreover, we will show that the transitions between these frames is a useful tool for the construction of power-law solutions in the Einstein one.

5.2 Conformal transformation

We are investigating the nonlocal model \((2.2)\) in the Einstein frame with a perfect fluid. It has been shown that those fluids have the same EoS in both frames \([20]\), which implies that the matter sector remains unaltered in both. Hence, it will be interesting to trace the behavior of the cosmological solutions in the Einstein frame, with the same matter fields.

On the other hand, it is known that a theory with higher derivatives in the action will often suffer from the ghost problem, i.e. a wrong sign in front of the kinetic terms, which will cause an instability problem thus making the theory physically irrelevant. Therefore, it is important to examine if the theory contains a ghost or not. To see this non-perturbatively, one needs first make a conformal transformation of the metric to bring the action into the form of the one in the Einstein frame \([35–37]\), namely the conformal frame in which the gravitational part of the action \((2.2)\) becomes purely Einsteinian. Note that the matter field is assumed to be minimally coupled to gravity in the Jordan frame, as given in action \((2.2)\).

Power-law solutions in the Einstein frame for the model given by the action \((2.2)\) without matter were considered in \([36, 37]\). In what follows we shall denote all quantities in the Einstein frame by adding a tag \((E)\) to the corresponding ones in the other frame, in order to avoid confusion.

Let us consider the conformal transformation \((5.1)\). Using \(g^{\mu\nu} = \Omega^{-2}g^{\mu\nu(E)}\), one obtains the relationship between the Ricci scalars in the two frames

\[
R = \Omega^{-2} \left[ R^{(E)} - 6 \left( \Box^{(E)} \ln \Omega + g^{\mu\nu(E)} \nabla_\mu^{(E)} \ln \Omega \nabla_\nu^{(E)} \ln \Omega \right) \right]
\]

and, inserting this into action \((2.2)\), one immediately identifies the conformal factor as \([35–37]\)

\[
\Omega^{-2} = \Psi \equiv 1 + f(\psi) + \xi,
\]

Then, by introducing a new field \(\phi\).

\[
\phi \equiv \ln \Omega = -\frac{1}{2} \ln (1 + f(\psi) + \xi) = -\frac{1}{2} \ln(\Psi),
\]

to remove the Lagrangian multiplier \(\xi\), we finally get the following action in the Einstein frame:

\[
S = \int d^4x \sqrt{-g^{(E)}} \left\{ \frac{1}{2\kappa^2} \left[ R^{(E)} - 6 \nabla^{(E)} \phi \nabla^{(E)} \phi - 2 \nabla^{(E)} \phi \nabla^{(E)} \psi \right. \right. \\
- \left. \left. e^{2\phi} f(\psi) \nabla^{(E)} \psi \nabla^{(E)} \psi - 2 e^{4\phi} \Lambda \right] + e^{4\phi} L_m(Q; e^{2\phi} g^{(E)}) \right\}.
\]

In what follows we will derive the corresponding equations of motion in the Einstein frame by varying this action. After solving them we will discuss specific cosmological behaviors.
5.3 Equations of motion

By varying action (5.5) with respect to the metric \( g^{\mu \nu(E)} \) one obtains the corresponding Einstein equations, as follows

\[
R^{(E)}_{\mu \nu} - \frac{1}{2} g^{(E)}_{\mu \nu} \left[ R^{(E)} - g^{\alpha \beta(E)} \left( 6 \partial_{\alpha} \phi \partial_{\beta} \phi + 2 \partial_{\alpha} \phi \partial_{\beta} \psi + e^{2 \phi} f_{, \psi} \partial_{\alpha} \psi \partial_{\beta} \psi \right) \right] - 2 e^{4 \phi} \Lambda - 6 \partial_{\mu} \phi \partial_{\nu} \phi - 2 \partial_{\mu} \phi \partial_{\nu} \psi - e^{2 \phi} f_{, \psi} \partial_{\mu} \psi \partial_{\nu} \psi = \kappa^2 T^{(E)}_{\mu \nu},
\]

(5.6)

where we recall that \( \phi \) is defined by (5.4), and the energy–momentum tensor in the Einstein frame as

\[
T^{(E)}_{\mu \nu} \equiv -\frac{2}{\sqrt{-g^{(E)}}} \frac{\delta \left( \Omega^4 \sqrt{-g^{(E)}} L_m \right)}{\delta g^{\mu \nu(E)}} = e^{2 \phi} T_{\mu \nu}.
\]

(5.7)

The field equations read

\[
\Box(E)(6 \phi + \psi) - e^{2 \phi} f_{, \psi} g^{\mu \nu(E)} \partial_{\mu} \psi \partial_{\nu} \psi - 4 e^{4 \phi} \Lambda + \kappa^2 e^{4 \phi} \left( 4 L_m + \frac{\partial L_m}{\partial \phi} \right) = 0,
\]

(5.8)

\[
2 \Box(E) \phi + 2 e^{2 \phi} f_{, \psi} \Box(E) \psi + g^{\mu \nu(E)} e^{2 \phi} \left( 4 f_{, \psi} \partial_{\mu} \phi \partial_{\nu} \psi + f_{, \psi} \partial_{\mu} \psi \partial_{\nu} \psi \right) = 0.
\]

(5.9)

At first sight, one may guess that the last term of Eq. (5.8) could be troublesome. However, this term can be substituted by a combination of the conformal factor and the trace part of the energy–momentum tensor in the Einstein frame. To achieve this, from Eqs. (5.1) and (5.7), we get

\[
T^{\nu(E)}_{\mu} = T^{(E)}_{\mu \alpha} g^{\alpha \nu(E)} = \Omega^2 T_{\mu \alpha} \Omega^2 g^{\alpha \nu} = \Omega^4 T^{(E)}_{\mu},
\]

(5.10)

\[
\frac{\partial L_m}{\partial \phi} = \frac{\partial L_m}{\partial g^{\mu \nu}} \frac{\partial g^{\mu \nu}}{\partial \phi} = \frac{\partial L_m}{\partial g^{\mu \nu}} \left[ \frac{\partial}{\partial g^{\mu \nu}} \left( \Omega^2 g^{\alpha \beta(E)} \right) \right] = -2 \Omega^{-2} g^{\mu \nu(E)} \frac{\partial L_m}{\partial g^{\mu \nu}} = -2 g^{\mu \nu} \frac{\partial L_m}{\partial g^{\mu \nu}},
\]

(5.11)

while from the definition of the energy–momentum tensor in the Jordan frame, we obtain

\[
T^{\mu(E)}_{\mu} = g^{\mu \nu} \left( g_{\mu \nu} L_m - 2 \frac{\partial L_m}{\partial g^{\mu \nu}} \right) = 4 L_m - 2 g^{\mu \nu} \frac{\partial L_m}{\partial g^{\mu \nu}}
\]

(5.12)

Inserting Eqs. (5.10) and (5.11) into (5.12), one recovers the last term of Eq. (5.8), under the form

\[
4 L_m + \frac{\partial L_m}{\partial \phi} = \Omega^{-4} T^{\mu(E)}_{\mu}.
\]

(5.13)

5.4 The FLRW metric

As is well known [20], conformally flat metrics are mapped into each other. The FLRW metric is conformally flat, so starting from the FLRW metric in the Jordan frame we obtain the corresponding FLRW metric in the Einstein one. This leads us to directly start from a FLRW metric with cosmic time in the Einstein frame

\[
ds^2 = -dt_E^2 + a_E^2(t_E) \delta_{ij} dx^i dx^j,
\]

(5.14)

where in \( dt_E \) and \( a_E(t_E) \) the index \( E \) denotes the corresponding quantities in the Einstein frame. We get

\[
dt_E = \Omega^{-1} dt, \quad a_E = \Omega^{-1} a, \quad H_E \equiv \frac{d \log a_E}{dt_E} = \Omega \left( H - \frac{\dot{\Omega}}{\Omega} \right)
\]

(5.15)
Under the conformal transformation (5.1), the energy–momentum tensor of a perfect fluid transforms as \([18, 20]\)

\[
T^\mu_\nu^{(E)} = \text{diag}(-\rho_E, P_E, P_E, P_E) = \Omega^4 \text{diag}(-\rho, P, P, P).
\]  

(5.16)

Using this equation together with Eq. (5.15), we obtain that the continuity (conservation) equation (2.12) is transformed into the following one in the Einstein frame

\[
\rho_E' + 3H_E (\rho_E + P_E) = \frac{\Omega'}{\Omega} (\rho_E - 3P_E),
\]  

(5.17)

where the prime denotes derivative with respect to the cosmological time in the Einstein frame, i.e. \(t' \equiv d/dt_E\).

From (5.16), we obtain the EoS parameter \(w_m = P_m/\rho_m = P_E/\rho_E\), therefore, the conservation law (5.17) can be rewritten as

\[
\rho_E' + \rho_E [3H_E(1 + w_m) + \phi'(3w_m - 1)] = 0.
\]  

(5.18)

It should be noted that if matter just reduces to radiation \((w_m = 1/3)\), then the conservation laws in the Jordan and Einstein frames coincide.

For the model with a perfect fluid, Eqs. (5.6)–(5.9) acquire the following form, in the FLRW metric,

\[
3H_E^2 - 3\phi'^2 - \phi' \psi' - \frac{e^{2\phi}}{2} f,_{\psi} \psi'^2 - e^{4\phi} \Lambda = \kappa^2 \rho_E,
\]  

(5.19)

\[
-6(H_E' + 2H_E^2) - 6\phi'^2 - 2\phi' \psi' - \frac{e^{2\phi}}{2} f,_{\psi} \psi'^2 + 4e^{4\phi} \Lambda = \kappa^2 \rho_E (3w_m - 1),
\]  

(5.20)

\[
6\psi'' + \psi' + 3H_E (6\phi' + \psi') = \frac{e^{2\phi}}{2} f,_{\psi} \psi'^2 + 4e^{4\phi} \Lambda = \kappa^2 \rho_E (3w_m - 1),
\]  

(5.21)

\[
2\phi'' + 6H_E \phi' + 2e^{2\phi} f,_{\phi} (\psi'' + 3H_E \psi') + 4e^{2\phi} f,_{\phi} \phi' \psi' = - \frac{e^{2\phi}}{2} f,_{\psi} \psi'^2,
\]  

(5.22)

where the first two are independent Einsteinian equations, while the other two are scalar field equations. Thus, the complete set of equations is given by (5.19)–(5.22). Eq. (5.18) follows from this system. Combining Eqs. (5.19) and (5.20), one obtains the remarkably useful result that the equation for a scalar \(\phi\) can be reduced to an algebraic equation, as follows,

\[
\Lambda e^{4\phi} = H_E' + 3H_E^2 + \frac{\kappa^2 \rho_E}{2} (w_m - 1).
\]  

(5.23)

For \(\Lambda \neq 0\), one can formally obtain from (5.23) the expression for the conformal factor \(\phi(t_E)\)

\[
\phi(t_E) = \frac{1}{4} \ln \left[ \frac{1}{\Lambda} \left( H_E' + 3H_E^2 + \frac{\kappa^2 \rho_E}{2} (w_m - 1) \right) \right].
\]  

(5.24)

It should also be noted that, by combining Eqs. (5.20) and (5.21), one can eliminate the \(f(\psi)\) and matter terms to obtain a second-order differential equation in terms of \(\psi(t_E)\) and \(\phi(t_E)\), namely

\[
6\psi'' + \psi' + 3H_E (6\phi' + \psi') + 6(H_E' + 2H_E^2) + 6\phi'^2 + 2\phi' \psi' = 0.
\]  

(5.25)

It is most convenient to derive the expression for \(\psi(t_E)\) from this equation once \(H_E\) and \(\phi(t_E)\) are known.
6 Power-law solutions in the Einstein frame for the model without matter

We here investigate power-law solutions (with $H_E = n_E/t_E$) in the Einstein frame. We consider the case of an exponential $f(\psi)$, given by (3.1). For the model with $\rho_E = 0$, the cases with and without matter will turn out to be essentially different.

If $\Lambda = 0$, from (5.23) we obtain the following power-law solution

$$H'_E = -3H_E^2, \quad \Rightarrow \quad H_E = \frac{1}{3(t_E - T_0)}, \quad (6.1)$$

meaning that all solutions correspond to the power-law behavior of the Hubble parameter with $n_E = 1/3$. Thus, the general solution of the system (5.19)–(5.22), which corresponds to arbitrary initial conditions for the scalar fields, yields just one cosmological evolution, specified by (6.1).

In the case where $\Lambda \neq 0$ and matter is absent, from (5.24) we obtain

$$\phi(t_E) = \frac{1}{4} \ln \left[ \frac{n_E(3n_E - 1)}{\Lambda t_E^2} \right]. \quad (6.2)$$

Note that there is no solution with $n_E = 1/3$ if $\Lambda \neq 0$. Eq. (5.25) is a linear differential equation for $\psi(t_E)$. Inserting the function $\phi(t_E)$ we just obtained into this equation, we get the following solution

$$\psi(t_E) = \tilde{\psi}_0 t_E^{2 - 3n_E} + m_E \ln \left( \frac{t_E}{\tilde{t}_0} \right), \quad (6.3)$$

where $m_E = 3(2n_E - 1)(4n_E - 3)/[2(2 - 3n_E)]$, while $\tilde{\psi}_0$ and $\tilde{t}_0$ are two integration constants. Inserting (6.2) and (6.3) into the system (5.19)–(5.22), one gets the following constraints on the integration constants

$$\tilde{\psi}_0 = 0, \quad \tilde{t}_0 = 3f_0(2n_E - 1)\sqrt{\frac{n_E}{\Lambda(3n_E - 1)}}, \quad (6.4)$$

$$\alpha m_E = 1 \iff \alpha + \frac{2(3n_E - 2)}{3(2n_E - 1)(4n_E - 3)} = 0, \quad (6.5)$$

from which the two branches corresponding to the index $n_E$ are expressed in terms of the parameter $\alpha$, as

$$\begin{cases} n_{E(1)} = \frac{15\alpha - 3 + \sqrt{3(3\alpha^2 + 2\alpha + 3)}}{24\alpha}, \\ n_{E(2)} = \frac{15\alpha - 3 - \sqrt{3(3\alpha^2 + 2\alpha + 3)}}{24\alpha}. \end{cases} \quad (6.6)$$

We note that this solution is the same as the one found in [37], where the vacuum solution with non-vanishing cosmological constant $\Lambda$ is constructed by conformal transformation from the corresponding one in the Jordan frame. Also we note that for any range of the parameter $\alpha$, the range for $n_{E(1)}$ is $n_{E(1)} \in (1/2, 3/4)$, while for $n_{E(2)}$ we have $n_{E(2)} \in (-\infty, 1/2) \cup (3/4, +\infty)$.

There are a few special cases of the parameter $n_E$ for which the above mentioned solution does not exist. In these cases the parameter $m_E$ is either equal to zero or it does not exist. Let us consider all of them in detail.
• If \( n_E = 1/2 \), then

\[
\psi(t_E) = c_1 + c_2 \sqrt{t_E}.
\]  

(6.7)

Substituting this solution into Eq. (5.19), we conclude that this equation is not satisfied whatever be the constants \( c_1 \) and \( c_2 \).

• If \( n_E = 2/3 \), then

\[
\psi(t_E) = \frac{1}{12} \ln \left( \frac{t_E}{T} \right)^2 + c_1 \ln \left( \frac{t_E}{T} \right).
\]  

(6.8)

Substituting this solution into Eq. (5.19), we conclude that this equation is not satisfied for any value of the constants \( c_1 \) and \( T \).

• If \( n_E = 3/4 \), then

\[
\psi(t_E) = c_1 + c_2 t_E^{-1/4}.
\]  

(6.9)

Substituting this solution into Eq. (5.19), with the condition \( c_2 = 0 \), from Eq. (5.22) we conclude that this equation is not satisfied for any value of \( c_1 \).

In summary, we have obtained power-law solutions in the Einstein frame for the model without matter, as follows,

\[
H(t_E) = \frac{n_E}{t_E},
\]  

(6.10)

\[
\phi(t_E) = \frac{1}{4} \ln \left[ \frac{n_E(3n_E - 1)}{\Lambda t_E^2} \right],
\]  

(6.11)

\[
\psi(t_E) = \frac{1}{\alpha} \ln \left( \frac{t_E}{4f_0(2n_E - 1)} \sqrt{\frac{\Lambda(3n_E - 1)}{n_E}} \right),
\]  

(6.12)

where the parameter \( n_E \) is connected with \( \alpha \) by (6.6). Note that \( n_E \neq 1/2, n_E \neq 2/3, n_E \neq 3/4 \), and \( n_E \neq 1/3 \). This is valid for \( \Lambda \neq 0 \). If \( \Lambda = 0 \), then all solutions of this system have power-law behavior with \( n_E = 1/3 \) and observe that the solution obtained has no arbitrary integration constant.

7 Power-law solutions in the Einstein frame for the model with matter

7.1 The case \( \Lambda = 0 \)

For \( \rho = n_E/t_E \), from Eq. (5.23) one gets that

\[
\rho(t_E) = \frac{2n_E(3n_E - 1)}{\kappa^2(1 - w_m)t_E^2}.
\]  

(7.1)

Inserting this into Eq. (5.18), the expression for \( \phi(t_E) \) follows

\[
\phi(t_E) = \frac{2 - 3n_E(1 + w_m)}{3w_m - 1} \ln \left( \frac{t_E}{\tilde{t}_1} \right),
\]  

(7.2)

where \( \tilde{t}_1 \) is an integration constant. Formula (7.2) is valid for \( n_E \neq 1/3, w_m \neq 1/3, \) and \( w_m \neq 1 \). Furthermore, using Eq. (5.25) we obtain the solution for \( \psi(t_E) \):

\[
\psi(t_E) = \tilde{\psi}_1 \left( \frac{t_E}{t_2} \right)^{1 + \frac{4 + 3n_E(w_m - 3)}{1 - 3w_m}} + \frac{12(2n_E - 1)\left[ 3(1 - w_m) + n_E(3w_m - 5) \right]}{5 - 3w_m + 3n_E(3w_m - 5)(3w_m - 1)} \ln \left( \frac{t_E}{t_2} \right).
\]  

(7.3)
where $\tilde{t}_2$ and $\tilde{\psi}_1$ are integration constants. Inserting Eqs. (7.1), (7.2), and (7.3) into the system (5.19)–(5.22), one obtains the following constraints on the integration constants and parameters

\begin{equation}
\tilde{\psi}_1 = 0, \quad n_{E\pm} = \frac{3 \left(3w_m^2 - 11\right) + 2\alpha (11 - 9w_m) \pm |1 - 3w_m| \sqrt{3 \left(3(2\alpha - w_m + 1)^2 - 16\alpha\right)}}{6 \left[3(w_m - 1)^2 - 4\alpha (5 - 3w_m) - 12\right]},
\end{equation}

\begin{equation}
\frac{\tilde{t}_2}{\tilde{t}_1} = \left[\frac{3n_m - 1)(3w_m - 1)}{6(2n_m - 1)(w_m - 1)f_0}\right]^{\frac{3w_m - 1}{2\left[2 - 3\kappa \psi(t_m)\left(1 + w_m\right)\right]}}.
\end{equation}

It follows that Eq. (7.5) gives rise to constraints on the parameters $\alpha$ and $w_m$, for $n_{E\pm}$ to be a real number:

\begin{equation}
3(2\alpha - w_m + 1)^2 - 16\alpha \geq 0.
\end{equation}

Interestingly enough, in the limit $\alpha \gg 1$, using Eq. (7.5), we obtain the following asymptotic behavior for the index $n_{E\pm}$:

\begin{equation}
n_{E\pm}(\alpha \gg 1) \approx \frac{11 - 9w_m \pm |1 - 3w_m|}{4(5 - 3w_m)}.
\end{equation}

One immediately realizes that, for $w_m = -1$, $n_{E+} \to 0.75$ asymptotically, what corresponds to an upper bound on the value for the index in the vacuum case, in the Einstein frame (see Eq. (38) of Ref. [37]). In fact, when $w_m = -1$, the solutions (7.1)–(7.6) reduce to the following form:

\begin{equation}
\rho_E(t_E) = \frac{n_{E}(3n_{E} - 1)}{\kappa^2 t_E^2}, \quad \phi(t_E) = -\frac{1}{2} \ln \left(\frac{t_E}{\tilde{t}_1}\right), \quad \psi(t_E) = \frac{1}{\alpha} \ln \left[\frac{3n_{E} - 1}{3f_0\tilde{t}_1(2n_{E} - 1)}\right],
\end{equation}

while the corresponding energy density in the Jordan frame is just the cosmological constant $\Lambda$. By using conformal transformation Eq. (5.16) and comparing it with the solution for $\rho_E(t_E)$ in (7.9), we can express the integration constant $\tilde{t}_1$ in terms of $\Lambda$, as

\begin{equation}
\tilde{t}_1 = \sqrt{\frac{n_{E}(3n_{E} - 1)}{\kappa^2 \Lambda}}.
\end{equation}

It is straightforward to see that Eq. (7.9) recovers the vacuum solutions, Eqs. (6.11)–(6.12).

Another interesting asymptotic behavior is the one for $w_m > 1/3$, $n_{E+} \to 1/2$, while for $w_m < 1/3$ we get $n_{E-} \to 1/2$, regardless of the value of the EoS parameter $w_m$.

We thus have derived solutions for all nonzero values of $n_E$, but for $n_E = 1/3$. Also, we assume that $w_m \neq 1/3$ and $w_m \neq 1$. Solutions in these special cases will be considered in the next section, by conformally transforming the power-law solutions obtained in the Jordan frame into the Einstein one.

### 7.2 The case $\Lambda \neq 0$

#### 7.2.1 $w_m = 1$

In this case, we cannot obtain a general solution for the system (5.19)–(5.22). Nevertheless, from Eq. (5.24) one finds that $w_m = 1$ is a special case where the function $\phi(t_E)$ has already been found. Thus, in the following we will consider the case $w_m = 1$. 

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**Note:** The quoted text appears to be a page from a scientific document discussing solutions to a system of equations related to cosmology or theoretical physics, involving indices and energy densities. The sections likely deal with asymptotic behaviors and constraints on parameters in these systems. The text also mentions conformal transformations and solutions for different cases, particularly focusing on $w_m = 1$. The notation used suggests advanced mathematical or physical concepts, typical of research-level discussions.
From Eq. (5.24) one obtains
\[
\phi(t_E) = \frac{1}{4} \ln \left[ \frac{(3n_E - 1)n_E}{\Lambda^2_E} \right],
\]  
(7.11)
and the condition \( n_E \neq 1/3 \). Substituting this function into Eq. (5.18), we get
\[
\rho_E(t_E) \propto \frac{1}{E^6}.
\]  
(7.12)
On the other hand, inserting the expression (7.11) into Eq. (5.25), \( \psi(t_E) \) can be solved as
\[
\psi(t_E) = -\frac{3(2n_E - 1)(4n_E - 3)}{2(3n_E - 2)} \ln \left( \frac{t_E}{T_1} \right) + C_1 t_E^{2-3n_E},
\]  
(7.13)
where \( C_1 \) and \( T_1 \) are arbitrary constants and \( n_E \neq 2/3, n_E \neq 1/2, n_E \neq 3/4 \).

However, substituting this function into (5.19), we discover that this equation cannot hold for any values of \( C_1 \) and \( T_1 \). Note that for \( n_E = 1/2, n_E = 2/3, \) and \( n_E = 3/4, \) Eq. (5.25) has different solutions\(^4\), but the system of equations (5.19)–(5.22) is not satisfied for these values of \( n_E \), either. Thus, we conclude that there is no consistent solution with \( w_m = 1 \).

7.2.2 \( w_m \neq 1 \)

If \( w_m \neq 1 \), then we can express \( \phi(t_E) \) via \( \rho_E(t_E) \) using Eq. (5.24) and, furthermore, obtain the following master equation for \( \rho_E(t_E) \) from Eq. (5.18):
\[
\frac{A_1(\rho_E) + A_2(\rho_E)}{4t_E \left[ \kappa^2 \rho_E t_E^2 (w_m - 1) + 2n_E (3n_E - 1) \right]} = 0,
\]  
(7.14)
where, to simplify the expression, we have defined \( A_1(\rho_E) \) and \( A_2(\rho_E) \) as the following two functions of \( \rho_E(t_E) \):
\[
A_1(\rho_E) \equiv 4n_E \rho_E \left[ 3\kappa^2 (w_m^2 - 1) \rho_E t_E^2 + (3n_E - 1) (6n_E(1 + w_m) + 1 - 3w_m) \right],
\]  
(7.15)
\[
A_2(\rho_E) \equiv \rho'_E t_E \left[ 3\kappa^2 (w_m^2 - 1) \rho_E t_E^2 + 8n_E (3n_E - 1) \right].
\]  
(7.16)
For generic values of the parameters \( w_m \) and \( n_E \) the general solution cannot be found in terms of elementary functions and it is only possible to cast this equation in transcendental form. At the same time, it is actually easy to solve it for some particular values of the constants, as the ones which follow.

- For \( n_E = 1/3 \), we obtain, from Eq. (5.24),
\[
\rho_E(t_E) = \frac{2 \Lambda e^{\phi(t_E)}}{(w_m - 1) \kappa^2},
\]  
(7.17)
while the master equation Eq. (7.14) has the general solution
\[
\rho_E(t_E) = \tilde{\rho}_0 t^{-4/3}_E.
\]  
(7.18)
A straightforward calculation shows that the system (5.19)–(5.22) has no solution for \( n_E = 1/3 \) if \( w_m \neq \pm 1 \). (Note that we always assume that \( w_m \neq -1 \), because matter with \( w_m = -1 \) coincides with the cosmological constant.)

\(^4\)These solutions coincide with (6.7)–(6.9).
\[ n_E = \frac{1}{2} \]

In this case, solving the system (5.19)–(5.22), it is found that there only exists the trivial solution \( \psi_E(t_E) = \text{const} \). Recalling the action (2.1) we are considering, such a constant solution just corresponds to a rescaling of the Ricci scalar \( R \), or equivalently, to a rescaling of both the Newtonian constant \( G \equiv 8\pi/\kappa^2 \) and the cosmological constant \( \Lambda \).

The correspondence of the special solutions considered above is that, in the Einstein frame, the case of a nonvanishing cosmological constant does not admit a power-law behaved matter sector other than \( \rho_E \propto t_E^{-2} \) for the scaling solution \( H_E \propto n_E/t_E \). To see this clearly, we consider the ansatz \( \rho_E(t_E) = \tilde{\rho}_1 t_E^p \). Then the master equation for the matter sector, Eq. (7.14), reduces to the following form

\[
\frac{4n_E(3n_E - 1)[1 + 2p - 3w_m + 6n_E(1 + w_m)] + 3\tilde{\rho}_1 \kappa^2 t_E^{2+p}(4n_E + p)(w_m^2 - 1)}{4t_E^2[\kappa^2 \tilde{\rho}_1 t_E^{p+2}(w_m - 1) + 2n_E(3n_E - 1)]} = 0. \tag{7.19}
\]

Let us first consider the case \( p \neq -2 \). Eq. (7.19) is then satisfied if and only if each of the two terms in the numerator are equal to zero. For the first term to vanish it is required that \( n_E = 1/3 \) or \( 1 + 2p - 3w_m + 6n_E(1 + w_m) = 0 \), while for the second, that \( p = -4n_E \) (recall that \( w_m = \pm 1 \) is excluded here), from where the only allowed possibility here is \( n_E = 1/3 \), \( p = -4/3 \). As discussed above, this does not satisfy the EOM (5.19)–(5.22).

Solutions with \( p = -2 \) correspond to the ones will be considered in Sect. 8.4.

8 Relationship between power-law solutions in the Jordan and in the Einstein frames

8.1 Conformal transformation between power-law solutions

8.1.1 General expression for the conformal factor \( \Omega \)

In Sects. 6 and 7 we have found solutions for the matter and massless sectors respectively. However, we recall that in Sect. 7.1, general solutions were found except for some singular values of \( n_E \) and \( w_m \), since it was not possible to separate variables in the system (5.19)–(5.22) to explicitly solve for the functions \( \psi \) and \( \phi \) in these special cases.

Meanwhile, in the vacuum case, by using conformal transformation [37] it has been found that some power-law solutions in the Jordan frame correspond to other power-law ones in the Einstein frame. Here we will generalize this correspondence for the model with a matter sector, and furthermore use the conformal transformation to obtain those special solutions in the Einstein frame which are very difficult to obtain by directly solving the system (5.19)–(5.22).

First, we formulate the differential equation for the conformal factor under which power-law solutions in the Jordan frame correspond to other power-law solutions in the Einstein frame. Using (5.15), we have

\[
H_E = \frac{n_E}{t_E} = \frac{n}{t} \Omega(t) - \dot{\Omega}(t), \tag{8.1}
\]

\(^5\text{We always assume that } n_E \neq 0.\)
where we recall that \( \dot{} \equiv d/dt \). This immediately gives a relationship between the cosmological times in the Jordan frame, \( t \), and in the Einstein one, \( t_E \), as follows
\[
t_E = \frac{n_E t}{n \Omega(t) - \dot{\Omega}(t)t}.
\] (8.2)

Taking the derivative of the equation above with respective to \( t \) and inserting \( dt_E/dt = \Omega^{-1} \), we obtain the following differential equation for the conformal factor \( \Omega(t) \):
\[
\ddot{\Omega} - \frac{1}{n_E \Omega} \dot{\Omega}^2 + \frac{n(2 - n_E)}{n_E t} \dot{\Omega} + \frac{n(n_E - n)}{n_E t^2} \dot{\Omega} = 0.
\] (8.3)

This equation has the following general solution:
\[
\Omega(t) = \begin{cases} 
\left( \frac{t}{T} \right)^n \left[ B_0 \left( \frac{t}{T} \right)^{1-n} + B_1 \right]^{n_{E^{-1}}}, & n \neq 1, \ n_E \neq 1, \\
\left( \frac{t}{T} \right) \left[ B_1 \ln \left( \frac{t}{T} \right) + 1 \right]^{n_{E^{-1}}}, & n = 1, \ n_E \neq 1, \\
B_1 t^n \exp \left[ \left( \frac{t}{T} \right)^{1-n} \right], & n \neq 1, \ n_E = 1, \\
\left( \frac{t}{T} \right)^{B_1}, & n = 1, \ n_E = 1,
\end{cases}
\] (8.4)

where \( B_0, B_1 \) and \( T \) are integration constants\(^6\).

In principle, as it was done in Ref. \([37]\), one can now obtain power-law solutions, by using Eq. (8.4), to conformally transform the corresponding solutions from the Jordan frame to the Einstein one. However, difficulties arise if one wants to directly obtain a general solution as (8.4).

### 8.1.2 Conformal factor corresponding to power-law solutions in the Jordan frame

As already stated above, although we have obtained the general solution (8.4), the complication of this expression renders the analysis of the solutions rather difficult. Another approach to construct the conformal factor \( \Omega \) is to insert solutions found in the Jordan frame into Eq. (5.3), to obtain the expression directly, case by case. Let us consider the case \( m \neq 1 - 3n \), i.e. use the corresponding solutions (3.3) and (3.5):
\[
\xi(t) = \xi_0 + \xi_1 \left( \frac{t}{t_0} \right)^{1-3n} + \frac{3n - 1}{3n + m - 1} \left( \frac{t}{t_0} \right)^m, \quad \psi(t) = \frac{m}{\alpha} \ln \left( \frac{t}{t_0} \right).
\] (8.5)

Thus, one obtains the expression for \( \Omega \) by employing Eq. (5.3)
\[
\Omega^{-2}(t) = C_1 t^m + C_2 t^{1-3n} + 1 + \xi_0 \quad \Longrightarrow \quad \Omega(t) = \frac{1}{\sqrt{C_1 t^m + C_2 t^{1-3n} + 1 + \xi_0}},
\] (8.6)

where, for simplicity of the expression, we have defined the two parameters \( C_1 \) and \( C_2 \) as follows
\[
C_1 \equiv \frac{6n + m - 2}{3n + m - 1} f_0^{1-m}, \quad C_2 \equiv \xi_1 t_0^{3n-1},
\] (8.7)

and again we recall that \( m \) is given by (3.4). Observing Eq. (8.6), we see we need the constraint on the integration constant \( \xi_0 = -1 \), since otherwise we can never find a correspondence of power-law solutions between the two frames. Recalling now Eqs. (3.7) and

\(^6\)Note that we use \( B_0 \) only to include the case \( B_0 = 0 \). A nonzero \( B_0 \) can always be put equal to one.
(3.11), one immediately realizes the implication of this constraint: there is no correspondence between power-law solutions in the case of non-vanishing cosmological constant $\Lambda$ and the matter sectors.

It is easy to see that there are two simple cases, namely $C_1 = 0$ and $C_2 = 0$ where, after conformal transformation, the power-law Hubble parameter in the Jordan frame $H$ yields a power-law function in the Einstein frame. We consider these cases separately in the following subsections. In Sects. 8.2–8.4 we will consider the case $\Lambda = 0$, whereas the case of nonzero $\Lambda$ will be dealt with in Sect. 8.5.

8.2 The case $C_1 = 0$

The case $C_1 = 0$ is special and corresponds to fixing the index $n$. Indeed, from $C_1 = 0$, Eq. (8.7) yields

$$n = n_0 = \frac{-6 + 3\alpha \pm \sqrt{3\alpha (3\alpha - 4)}}{6(2\alpha - 3)}. \quad (\text{8.8})$$

Comparing (3.9) with (8.8), one sees that $w_m = 1$. Inserting Eq. (8.6) into (5.15), we obtain

$$t = \left[ \frac{3(1 - n_0)}{2\sqrt{C_2}} t_E \right]^{\frac{2}{3(1 - n_0)}} \Omega^{-1} = \sqrt{C_2}^{\frac{1 - 3n_0}{2}} = \sqrt{C_2} \left[ \frac{3(1 - n_0)}{2\sqrt{C_2}} t_E \right]^{\frac{1 - 3n_0}{3(1 - n_0)}}, \quad (\text{8.9})$$

hence, using the relationship (5.15), we find

$$n_E = \frac{1}{3}, \quad \text{thus the Hubble parameter in the Einstein frame is}$$

$$H_E(t_E) = \frac{1}{3t_E}. \quad (\text{8.10})$$

Using the definition $\phi \equiv \ln \Omega$, we obtain the expression

$$\phi(t_E) = \frac{3n_0 - 1}{3(1 - n_0)} \ln \left( \frac{t_E}{t_3} \right), \quad \tilde{t}_3 \equiv \frac{2C_2^{\frac{3n_0 - 1}{2}}}{3(1 - n_0)}. \quad (\text{8.11})$$

Inserting (8.10) and (8.11) into Eq. (5.25), $\psi(t_E)$ is found to be

$$\psi(t_E) = \tilde{\psi}_4 \left( \frac{t_E}{t_3} \right)^{2 + \frac{4}{3(1 - n_0)}} + \frac{4n_0(2n_0 - 1)}{(n_0 - 1)(3n_0 - 1)} \ln \left( \frac{t_E}{t_*} \right), \quad (\text{8.12})$$

with two integration constants $\tilde{\psi}_4$ and $\tilde{t}_*$. On the other hand, from (8.5) and (8.9), we can rewrite the solution (3.3) in terms of the variables in the Einstein frame:\footnote{We recall that the integration constant $\psi_1 = 0$.}

$$\psi(t_E) = \frac{m}{\alpha} \ln \left( \frac{1}{t_0} \left[ \frac{3(1 - n_0)}{2\sqrt{C_2}} t_E \right]^{\frac{2}{3(1 - n_0)}} \right) = \frac{4n_0(2n_0 - 1)}{(n_0 - 1)(3n_0 - 1)} \ln \left( \frac{t_E}{\tilde{t}_4} \right), \quad (\text{8.13})$$

where we define the constant $\tilde{t}_4$ as follows:

$$\tilde{t}_4 = \frac{2\sqrt{C_2}}{3(1 - n_0)} t_0^{\frac{3}{2} \left( \frac{1 - n_0}{2} \right)}. \quad (\text{8.14})$$

Comparing Eqs. (8.12) and (8.13), one immediately finds that $\tilde{\psi}_4 = 0$ and $\tilde{t}_* = \tilde{t}_4$. Thus, we have obtained a new solution for $n_E = 1/3$ and $w_m = 1$ in the Einstein frame by using
conformal transformation, which would have been very difficult to obtain directly from the system (5.19)–(5.22).

To summarize, by using conformal transformation, we have found the following, new solution in the Einstein frame:

\[
\begin{align*}
H_E(t_E) &= \frac{1}{3t_E}, \\
\phi(t_E) &= \frac{3n_0 - 1}{3(1 - n_0)} \ln \left( \frac{t_E}{t_3} \right), \\
\psi(t_E) &= \frac{4n_0(2n_0 - 1)}{(n_0 - 1)(3n_0 - 1)} \ln \left( \frac{t_E}{t_4} \right), \\
\rho(t_E) &= \tilde{\rho}_3 \frac{4}{3} \left( \frac{t_E}{t_1} \right)^{n_0 - 1},
\end{align*}
\]  

(8.15)

with EoS \(w_m = 1\) and a constant \(\tilde{\rho}_3\) constrained by (5.19)–(5.22), as follows

\[
\tilde{\rho}_3 = \frac{8n_0 f_0}{3\kappa^2 (1 - n_0)^2} \left( \frac{\tilde{t}_3}{\tilde{t}_4} \right)^{\frac{2(3n_0 - 1)}{3(n_0 - 1)}},
\]  

(8.16)

where \(\tilde{t}_3\) and \(\tilde{t}_4\) are constants defined in Eqs. (8.11) and (8.14), respectively. Note that \(n_0\) is not a free parameter, because it is connected with \(\alpha\). Solutions have been found for arbitrary nonzero \(\alpha\), except for \(\alpha = 4/3\), which corresponds to \(n_0 = 1\).

### 8.3 The case \(C_2 = 0\)

In the case \(C_2 = 0\) (or, equivalently, \(\xi_1 = 0\)), similarly as in Sect. 8.2, the relationship between \(t\) and \(t_E\) can be obtained as

\[
t = \left( \frac{m + 2}{2\sqrt{C_1}} \right)^2 \frac{t_E}{t_1}.
\]  

(8.17)

Using this relation, the conformal factor can be expressed in terms of the variables in the Einstein frame, namely

\[
\Omega^{-2}(t_E) = C_1 \left( \frac{m + 2}{2\sqrt{C_1}} \right)^{\frac{2m}{m+2}} \left( \frac{t_E}{t_1} \right)^{\frac{2m}{m+2}}.
\]  

(8.18)

Inserting (8.18) into Eq. (5.15), we obtain the Hubble parameter in the Einstein frame

\[
H_E(t_E) = \frac{m + 2n}{(m + 2)t_E}.
\]  

(8.19)

Thus, by setting the integration constant \(\xi_1 = 0\), we obtain the correspondence between the power-law solutions in the Jordan and Einstein frames, and identify the index in the last with the corresponding one in the Jordan frame, as follows

\[
n_E = \frac{m + 2n}{m + 2} = \frac{3(2\alpha - 1)n + 1 - 3\alpha n}{6\alpha n^2 - 3n(1 + \alpha) + 1}.
\]  

(8.20)

The values of \(n_0\) equal to \(n_0 = 1/3\), which corresponds to \(\alpha = 0\), and \(n_0 = 1/2\), which does not correspond to any finite value of \(\alpha\), are also excluded (see Sect. 3).
We recall that the parameter $n$ is determined by $\alpha$ and $w_m$ in the constraint (3.9). For practical use, we rewrite this constraint as

$$\alpha = \frac{(3n - 2 + 3w_m, n)(3n - 1)}{6n(2n - 1)},$$

and substitute it into Eq. (8.20) to express $n$ in terms of $n_E$ and $w_m$:

$$n = \frac{2(2n_E - 1)}{3n_E(w_m + 1) - 1 - 3w_m}.$$  (8.22)

Inserting (8.22) into Eq. (8.18), we can also obtain the expression for the scalar field $\phi$

$$\phi(t_E) \equiv \ln \Omega = \frac{2 - 3n_E(1 + w_m)}{3w_m - 1} \ln \left(\frac{t_E}{\tilde{t}_5}\right),$$  (8.23)

where $\tilde{t}_5$ is an integral constant defined by

$$\tilde{t}_5 = \frac{2}{(m + 2)C_1^m},$$  (8.24)

hence connected with $t_0$ and $f_0$ by Eq. (8.7). We see that the expression (8.23) coincides with (7.2) and thus, generally speaking, we get only solutions previously obtained in Sect. 6.1. Nevertheless, for some values of the parameters we do not obtain the solutions of Sect. 6.1. Let us check the possibility to get these solutions using conformal transformation.

Recall that in Sect. 6.1 we did not find solutions for $w_m = 1/3$, $w_m = 1$, and $n_E = 1/3$. If $w_m = 1$, then from Eq. (5.23) it follows that $n_E = 1/3$ and vice versa. Thus, substituting $w_m = 1$ into (8.23), we get $\phi(t_E) = 0$. Straightforward substitution into Eqs. (5.19)–(5.22) shows that there is no solution in this case.

For $w_m = 1/3$ the formula (8.23) is not acceptable. We will consider this case in the next subsection.

### 8.4 Radiation case

Up to now we have not obtained any power-law solution in the Einstein frame for the case when $w_m = 1/3$, i.e. with a radiation sector. There, using the conservation law (5.18), we find

$$\rho(t_E) \propto t_E^{-3n_E},$$  (8.25)

while Eq. (5.23) with $\Lambda = 0$ yields

$$\rho_E(t_E) = \frac{3n_E(3n_E - 1)}{\kappa^2 t_E^2},$$  (8.26)

which implies that

$$n_E = \frac{1}{2}, \quad \rho_E(t_E) = \frac{3}{4\kappa^2 t_E^2}.$$  (8.27)

However, since $\phi(t_E)$ couples with $\psi(t_E)$, we cannot solve these equations directly from the EOM (5.19)–(5.22) in the Einstein frame. Recall now that, having obtained the solutions (3.3) and (3.5) in the Jordan frame we can, in principle, conformally transform both into their corresponding forms in the Einstein frame. To achieve this goal, we first note that,
using Eq. (8.20), we can find the power-index in the Jordan frame \( n \) from the corresponding one in the Einstein frame \( n_E = 1/2 \), as

\[
n = \frac{1}{3(1 - \alpha)}.
\] (8.28)

Inserting this into Eq. (8.18), we then get \( \phi(t_E) \) as

\[
\phi(t_E) = \frac{1 - 3\alpha}{6\alpha - 4} \ln \left( \frac{t_E}{\tilde{t}_2} \right),
\] (8.29)

where \( \tilde{t}_2 \) is an integration constant. And inserting this solution into Eq. (5.25), one obtains the solution for \( \psi(t_E) \):

\[
\psi(t_E) = \tilde{\psi}_2 \left( \frac{t_E}{t_3} \right)^\frac{3\alpha}{4 - 6\alpha} + \frac{3\alpha - 1}{\alpha (3\alpha - 2)} \ln \left( \frac{t_E}{t_3} \right),
\] (8.30)

with two integration constants, \( \tilde{\psi}_2 \) and \( \tilde{t}_3 \). Eqs. (5.19)–(5.22) introduce constraints on these integration constants, namely

\[
\tilde{\psi}_2 = 0,
\] (8.31)

\[
\tilde{t}_2 \tilde{t}_3 = \left[ \frac{1}{f_0} \left( 1 - \frac{3\alpha}{2} \right) \right]^{1 + \frac{1}{1 - 3\alpha}}.
\] (8.32)

Thus, Eqs. (8.27), (8.29)–(8.32) supplement the solutions in Sect. 7.1.

### 8.5 The case \( \Lambda \neq 0 \)

In the case \( \Lambda \neq 0 \) solutions in the Jordan frame are described by (3.3) and (3.5). To get the corresponding power-law solutions in the Einstein frame we need to select the case \( m \neq 1 - 3n \) and to put \( \xi_0 = -1 \). From (3.11), we get that the system does not include matter: \( \rho_0 = 0 \). It is easy to see that, for any nonzero \( n \), \( C_1 \neq 0 \) for \( m = 2 \), so we put \( \xi_1 = 0 \) and consider the case \( C_2 = 0 \). From (8.6), we obtain

\[
\Omega(t) = \frac{1}{\sqrt{C_1 t}}.
\] (8.33)

Therefore,

\[
t_E = \frac{\sqrt{C_1} t^2}{2}, \quad H_E = \frac{n + 1}{2t_E}, \quad n_E = \frac{n + 1}{2},
\] (8.34)

and

\[
\phi = -\ln(\sqrt{C_1 t}) = -\frac{1}{2} \ln \left( 2\sqrt{C_1} t_E \right) = \frac{1}{4} \ln \left( \frac{(3n_E - 1)n_E}{\Lambda t_E^2} \right),
\] (8.35)

where Eq. (8.7) is used in the last step. Thus, we reobtain the solution (6.2). Using (3.14), we reobtain the condition (6.6). Therefore, in the case of nonzero \( \Lambda \) we can use the power-law solutions of the Jordan frame to get the corresponding solutions in the Einstein frame, but this way is here not more effective than a straightforward search for power-law solutions of the system (5.19)–(5.22).

\(^9\)Note that, for \( n_E = 1/2 \) and \( w_m = 1/3 \), we cannot use (8.22) to define \( n \), because both numerator and denominator in this formula are equal to zero.
To render it easier for readers to examine the full set of solutions, in Tables 3, 4 and 5 we list all those corresponding to the Einstein frame. It should be noted that all solutions in these three tables correspond to the case $\Lambda = 0$. As discussed in Sects. 6 and 8.5, for the non-vanishing cosmological constant case in the Einstein frame only vacuum solution can be found, namely those of Eqs. (6.6), (6.11) and (6.12).
9 Conclusions

In General Relativity, power-law solutions of the type $H = n/t$ correspond to models with a perfect fluid whose EoS parameter $w_m \equiv P_m/\rho_m$ is related to the power-index by $w_m = -1 + 2/(3n)$. It is interesting to try to find similar power-law solutions in modified gravity theories, in order to check how much they deviate from those for GR. In this paper, we consider power-law solutions in a class of nonlocal gravity models stemming from the widely probed and very promising function $f(\Box^{-1}R) = f_0 e^{\alpha \Box^{-1}R}$ and which include a perfect fluid with constant EoS parameter $w_m$.

By recasting the original nonlocal action (2.1) into its local presentation (2.2), we have obtained power-law solutions for this model, with and without a cosmological constant and both in the Jordan and in the Einstein frames. We also show that power-law solutions, obtained in the Jordan frame satisfy the original nonlocal equations. In other words we get power-law solutions for the original nonlocal model (2.1) as well. In the Jordan frame we have reached the remarkable conclusion that all power-law solutions could be found (see Sect. 3), what is a most interesting outcome of this paper. In the Einstein frame, we have correspondingly obtained the power-law solutions either by directly solving the EOM, or by performing a conformal transformation of the solutions obtained in the Jordan frame. For this purpose, we have generalized the correspondence between power-law solutions in the Jordan and Einstein frames, as obtained in [37], in order to appropriately include the matter sector. By using this powerful, non-trivial tool, we have obtained the solutions when $w_m = 1/3$ and $w_m = 1$, in which cases it was very difficult to obtain the corresponding solutions by directly solving the system (5.19)–(5.22). Hence, we have shown explicitly how the construction of solutions by using conformal transformation between the two frames proceeds, thus proving that the method offers a valuable alternative in the search for new solutions.

In [40], it has been shown that not only models with exponential $f(\psi)$ can have power-law and de Sitter solutions. It would be interesting to consider power-law and de Sitter solutions in the models where $f(\psi)$ consists of a sum of exponentials. Another direct generalization of the present analysis is to include several perfect fluid components with different constant values of $w_m$.

In [36, 38], de Sitter solutions in nonlocal models were found. It will be interesting to check the possibilities for the Universe evolution as obtained from these models, from an inflationary de Sitter stage to the late power-law Universe and, furthermore, to check for deviations from the standard general relativity case, and its distinction from other modified gravity theories.

As is widely known, theories with higher derivatives often suffer from a ghost problem, namely a wrong sign in the kinetic term, resulting in a dangerous instability problem. A good aspect in making use of the conformal transformation technique between the two frames is to obtain the corresponding ghost-free conditions [35–37]. The biscalar-tensor representation introduces two extra scalars. As pointed out in [33, 35], they can lead to a ghost problem. We should note that the equivalence between the initial nonlocal theory and local formulation has not been established yet. The original nonlocal model has less degrees of freedom and, thus, the ghost-like behavior of the biscalar-tensor theory may not be a physical problem, since the associated terms can be cast as boundary terms of the nonlocal operators [49]. In other words, the would-be ghost mode might not be physically relevant since it would probably correspond to an inappropriate choice on the boundary condition. Anyway, it should be kept in mind that an appropriate choice of boundary condition is also necessary in the biscalar-
tensor presentation. We plan to consider this important question in more detail in further work on the original nonlocal model.

It will be most interesting, too, to test the solutions obtained in this paper to find the constraints on the parameters, hence to check for the possibility to obtain a realistic model which can be responsible for the current observed acceleration of the Universe expansion. This work is now in process. Also, an analysis of the stability of the solutions here encountered will be carried out [50].

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