A COUNTEREXAMPLE FOR THE OPTIMALITY OF KENDALL-CRANSTON COUPLING

KAZUMASA KUWADA
Department of Mathematics, Faculty of Science, Ochanomizu University, Tokyo 112-8610, Japan
email: kwada@math.ocha.ac.jp

KARL-THEODOR STURM
Institute for Applied Mathematics, University of Bonn, Wegelerstrasse 6, 53115 Bonn, Germany
email: sturm@uni-bonn.de

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Abstract
We construct a Riemannian manifold where the Kendall-Cranston coupling of two Brownian particles does not maximize the coupling probability.

1 Introduction

Given two stochastic processes $X_t$ and $Y_t$ on a state space $M$, a coupling $Z_t = (Z^{(1)}_t, Z^{(2)}_t)$ is a process on $M \times M$ so that $Z^{(1)}_t$ or $Z^{(2)}_t$ has the same distribution as $X_t$ or $Y_t$ respectively. Of particular interest in many applications is the distribution of the coupling time $T(Z) := \inf\{t > 0 ; Z^{(1)}_s = Z^{(2)}_s \text{ for all } s > t\}$. The goal is to make the coupling probability $\mathbb{P}[T(Z) \leq t]$ as large as possible by taking a suitable coupling. When $X$ and $Y$ are Brownian motions on a Riemannian manifold, Kendall [3] and Cranston [1] constructed a coupling by using the Riemannian geometry of the underlying space. Roughly speaking, under their coupling, infinitesimal motion $\Delta Y_t \in T_{Y_t}M$ at time $t$ is given as a sort of reflection of $\Delta X_t$ via the minimal geodesic joining $X_t$ and $Y_t$. Their coupling has the advantage of controlling the coupling probability by using geometric quantities such as the Ricci curvature. As a result, Kendall-Cranston coupling produces various estimates for heat kernels, harmonic maps, eigenvalues etc. under natural geometric assumptions.

On the other hand, there is the question of optimality. We say that a coupling $Z$ of $X$ and $Y$ is optimal at time $t$ if

$$\mathbb{P}[T(Z) \leq t] \geq \mathbb{P}[T(\tilde{Z}) \leq t]$$

holds for any other coupling $\tilde{Z}$. Though Kendall-Cranston coupling has a good feature as mentioned, in general there is no reason why it should be optimal.
The Kendall-Cranston coupling is optimal if the underlying space has a good symmetry. For example, in the case $M = \mathbb{R}^d$, the Kendall-Cranston coupling $(Z^{(1)}, Z^{(2)})$ is nothing but the mirror coupling. It means that $Z^{(2)}_t = \Psi(Z^{(1)}_t)$ up to the time they meet, where $\Psi$ is a reflection with respect to a hyperplane in $\mathbb{R}^d$ so that $\Psi(X_0) = Y_0$. It is well known that the mirror coupling is optimal. Indeed, it is the only coupling which is optimal and Markovian [2].

More generally, the same result holds if there is a sort of reflection structure like a map $\Psi$ on $\mathbb{R}$. We take three parameter $R > 0, \zeta > 0$ and $\delta > 0$ such that $\zeta < R/4$ and $\delta < \zeta/3$. Let $C = \mathbb{R} \times S^1$ be a cylinder with a flat metric such that the length of a circle $S^1$ equals $\zeta$. For simplicity of notation, we write $z = (r, \theta)$ for $z \in C$ where $r \in \mathbb{R}$ and $\theta \in (-\zeta/2, \zeta/2)$ such that the Riemannian metric is written as $dr^2 + d\theta^2$. If appropriate, any $\theta \in \mathbb{R}$ will be regarded mod $\zeta$ and considered as element of $(-\zeta/2, \zeta/2]$. We put

$$M_1 := ([-R, \infty) \times S^1) \setminus B_\zeta^C((0, \zeta/2)) \subset C$$

and write $\partial_{1,0} := \partial B_\zeta^C((0, \zeta/2))$ as well as $\partial_{1,2} := \{R\} \times S^1$ (see Fig.1). Let $C'$ be a copy of $C$. Then we put analogously

$$M_2 := ((-\infty, R] \times S^1) \setminus B_\delta^C((0, 0)) \subset C'$$

and write $\partial_{2,0} := \partial B_\delta^C((0, 0))$ as well as $\partial_{2,1} := \{R\} \times S^1$. Let $M_0 = S^1 \times [-1, 1]$ be another cylinder. We write $z \in M_0$ by $z = (\varphi, r)$ where $\varphi \in (0, 2\pi]$ and $r \in [-1, 1]$. Now we define a $C^\infty$-manifold $M$ (see Fig.2) by $M = M_0 \sqcup M_1 \sqcup M_2 / \sim$, where the identification $\sim$ means

$$\partial_{1,2} \ni (-R, \theta) \sim (R, \zeta/2 - \theta) \in \partial_{2,1} \quad \text{for } \theta \in (-\zeta/2, \zeta/2],$$

$$\partial_{1,0} \ni (\delta \cos \varphi, \zeta/2 - \delta \sin \varphi) \sim (\varphi, -1) \in M_0 \quad \text{for } \varphi \in (0, 2\pi],$$

$$\partial_{2,0} \ni (\delta \cos \varphi, \delta \sin \varphi) \sim (\varphi, 1) \in M_0 \quad \text{for } \varphi \in (0, 2\pi].$$

We endow $M$ with a $C^\infty$-metric $g$ such that $(M, g)$ becomes a complete Riemannian manifold and:

(i) $g|_{M_1}$ coincides with the metric on $M_1$ inherited from $C$,
(ii) $g|_{M_2}$ coincides with the metric on $M_2$ inherited from $C'$,

(iii) $g|_{M_0}$ is invariant under maps $(\theta, r) \mapsto (\theta, -r)$ and $(\theta, r) \mapsto (\theta + \varphi, r)$ on $M_0$,

(iv) $d((-1, 0), (1, 0)) = \zeta$ for $z_1 = (-1, 0), z_2 = (1, 0) \in M_0$

where $d$ is the distance function on $M$.

3 Comparison of coupling probabilities

Let $M$ be the manifold constructed above (with suitably chosen parameters $R$, $\zeta$ and $\delta$) and fix two points $x = (0, \zeta/6) \in M_1$ and $y = (0, \zeta/3) \in M_2$. In this paper, the construction of Kendall-Cranston coupling is due to von Renesse [5]. We will try to explain his idea briefly. His approach is based on the approximation by coupled geodesic random walks $\{\hat{\Xi}^k\}_{k \in \mathbb{N}}$ starting in $(x, y)$ whose sample paths are piecewise geodesic. Given their positions after $(n - 1)$-th step, one determines its next direction $\xi_n$ according to the uniform distribution on a small sphere in the tangent space and the other does it as the reflection of $\xi_n$ along a minimal geodesic joining their present positions. We obtain a Kendall-Cranston coupling $(X_t, Y_t)$ by taking the (subsequential) limit in distribution of them. We will construct another Brownian motion $(\hat{Y}_t)_{t \geq 0}$ on $M$ starting in $y$, again defined on the same probability space as we construct $(X_t, Y_t)$ such that

$$\mathbb{P}(X \text{ and } Y \text{ meet before time } 1) < \mathbb{P}(X \text{ and } \hat{Y} \text{ meet before time } 1).$$

In other words, if $Q$ denotes the distribution of $(X, Y)$ and $\hat{Q}$ denotes the distribution of $(X, \hat{Y})$ then

Proposition 3.1 $Q[T \leq 1] < \hat{Q}[T \leq 1]$.

Our construction of the process $\hat{Y}$ will be as follows. We define a map $\Phi : M_1 \to M_2$ by $\Phi((r, \theta)) = (-r, \zeta/2 - \theta)$ and then put
Lemma 3.2

(i) \( \hat{Y}_t = \Phi(X_t) \) for \( t \in [0, \tau_{\partial, 0} \wedge T] \);

(ii) \( X \) and \( \hat{Y} \) move independently for \( t \in [\tau_{\partial, 0}, T) \) in case \( \tau_{\partial, 0} < T \);

(iii) \( \hat{Y}_t = X_t \) for \( t \in [T, \infty) \).

Note that \( \tau_{\partial, 2} = T \) holds when \( \tau_{\partial, 2} \leq \tau_{\partial, 0} \) under \( \hat{Q} \).

Set \( H = S^1 \times \{0\} \subset M_0 \subset M \). For \( z_1, z_2 \in M \) and \( A \subset M \), minimal length of paths joining \( z_1 \) and \( z_2 \) which intersect \( A \) is denoted by \( d(z_1, z_2 ; A) \). We define a constant \( L_0 \) by

\[
L_0 := \inf \left\{ L \in (\delta, R] : d(z_1, z_2 ; H) \geq d(z_1, z_2 ; \partial, 2) \right\}
\]

for some \( z_1 = (L, \theta) \in M_1, z_2 = (L, \zeta/2 - \theta) \in M_2 \).

Lemma 3.2 \( R - \zeta < L_0 < R \).

Proof. First we show \( L_0 < R \). Let \( z_1 = (R, 0) \in M_1 \) and \( z_2 = (R, \zeta/2) \in M_2 \). Obviously there is a path of length \( 2R \) joining \( z_1 \) and \( z_2 \) across \( \partial, 2 \). Thus we have \( d(z_1, z_2 ; \partial, 2) \leq 2R \).

By symmetry of \( M \),

\[
d(z_1, z_2 ; H) = 2d(z_1, H) = 2 \left( d(z_1, \partial, 0) + \frac{\zeta}{2} \right) = 2 \left( \sqrt{R^2 + \zeta^2/4} - \delta \right) + \zeta > 2R,
\]

where the second equality follows from the third and fourth properties of \( g \) and the last inequality follows from the choice of \( \delta \). These estimates imply \( L_0 < R \).

Next, let \( z'_1 = (R - \zeta, \theta) \in M_1 \) and \( z'_2 = (R - \zeta, \zeta/2 - \theta) \in M_2 \). In the same way as observed above, we have

\[
d(z'_1, z'_2 ; H) = 2 \left( \sqrt{(R - \zeta)^2 + \theta^2} - \delta \right) + \zeta \leq 2R - 2\delta.
\]

Note that the length of a path joining \( z'_1 \) and \( z'_2 \) which intersects both of \( \partial, 2 \) and \( H \) is obviously greater than \( d(z'_1, z'_2 ; H) \). Thus, in estimating \( d(z'_1, z'_2 ; \partial, 2) \), it is sufficient to consider all paths joining \( z'_1 \) and \( z'_2 \) across \( \partial, 2 \) which do not intersect \( H \). Such a path must intersect both \( \{\delta\} \times S^1 \subset M_1 \) and \( \{-\delta\} \times S^1 \subset M_1 \) (see Fig.3). Thus we have

\[
d(z'_1, z'_2 ; \partial, 2) \geq d(z'_1, \delta) + d(\{-\delta\} \times S^1, \partial, 2) + d(\partial, 2, \delta)
\geq (R - \zeta - \delta) + (R - \delta) + \zeta
= 2R - 2\delta.
\]

Hence, the conclusion follows. \( \square \)

Set \( M'_1 := M_1 \cap [-L_0, L_0] \times S^1 \subset C \) and \( M'_2 := M_2 \cap [-L_0, L_0] \times S^1 \subset C \). We define a submanifold \( M' \subset M \) with boundary by \( M' = M_0 \cup M'_1 \cup M'_2 / \sim \) (see Fig.4). Let \( \Psi : M' \to M' \) be the reflection with respect to \( H \). For instance, for \( z = (r, \theta) \in M'_1, \Psi(z) = (r, \zeta/2 - \theta) \in M'_2 \).

Note that \( \Psi \) is an isometry, \( \Psi \circ \Psi = \text{id} \) and \( \{ z \in M' : \Psi(z) = z \} = H \).

Let \( X' \) be the given Brownian motion starting in \( x \) and now stopped at \( \partial M' \), i.e. \( X'_t = X_{t \wedge \tau_{\partial M'}} \).

Define a stopped Brownian motion starting in \( y \) by \( Y'_t = \Psi(X'_t) \) for \( t < \tau_H \) and by \( Y_t = X_t \) for \( t \geq \tau_H \) (that is, the two Brownian particles coalesce after \( \tau_H \)). Then we can prove the following lemma.
Lemma 3.3 The law of \((X_t\wedge\tau_{\partial_1}, Y_t\wedge\tau_{\partial_1})_{t\geq 0}\) coincides with that of \((X'_t, Y'_t)_{t\geq 0}\).

Proof. Note that the minimal geodesic in \(M\) joining \(z\) and \(\Psi(z)\) must intersect \(H\) for every \(z \in M'\) by virtue of the choice of \(L_0\). Thus, by the symmetry of \(M'\) with respect to \(H\), coupled geodesic random walks \(\hat{\Xi}_k\) are in \(E\) defined by
\[
E := \{(z^{(1)}, z^{(2)}) \in C([0, \infty) \to M \times M) ; z_1^{(2)} = \Psi(z_1^{(1)}) \text{ before } z_1^{(1)} \text{ exits from } M'\}
\]
(cf. Theorem 5.1 in \[4\]). Since \(E\) is closed in \(C([0, \infty) \to M \times M)\), \((X\cdot, Y\cdot) \in E\) holds \(\mathbb{P}\)-almost surely by taking a (subsequential) limit in distribution of \(\{\hat{\Xi}_k\}_{k \in \mathbb{N}}\). Thus the conclusion follows. \(\square\)

We now begin to show Proposition 3.1. First we give a lower estimate of \(\hat{Q}[T \leq 1]\). Let
\[
\gamma(a) := \{(x_1, x_2) \in \mathbb{R}^2 ; x_2 = a\}, a \in \mathbb{R},
\]
\[
A(\delta) := \bigcup_{n \in \mathbb{Z}} B_{\frac{\delta}{\sqrt{2}}}^{\mathbb{R}^2}(\left(\zeta(n + \frac{1}{2}), 0\right)).
\]
The remark after the definition of \(\hat{Q}\) implies
\[
\hat{Q}[T \leq 1] \geq \hat{Q}[T \leq 1, \tau_{0,2} < \tau_{0,0}] = \hat{Q}[\tau_{0,2} \leq 1 \wedge \tau_{0,0}].
\]
By lifting \(X_1\) to \(\mathbb{R}^2\), the universal cover of \(C\),
\[
\hat{Q}[\tau_{0,2} \leq 1 \wedge \tau_{0,0}] = \mathbb{P}^{\mathbb{R}^2}[\tau_{0,2} \leq 1 \wedge \tau_{A(\delta)}] \geq \mathbb{P}^{\mathbb{R}^2}[\tau_{0,2} \leq 1, \tau_{A(\delta)} > 1] \geq \mathbb{P}^{\mathbb{R}}[\tau_{R} \leq 1] - \mathbb{P}^{\mathbb{R}^2}[\tau_{A(\delta)} \leq 1].
\]
(3.1)

Here \(\mathbb{P}^{\mathbb{R}}\) and \(\mathbb{P}^{\mathbb{R}^2}\) denote the usual Wiener measure for Brownian motion (starting at the origin) on \(\mathbb{R}^2\) or \(\mathbb{R}\), resp. For simplicity, we write \(\tau_R\) instead of \(\tau_{1(R)}\).
Next we give an upper estimate of $Q[T \leq 1]$. Let $E := \{\tau_{\partial,0} < 1 \wedge \tau_{\partial,\partial'}\}$. Then

$$Q[E] = P[E] \leq P^{R^2} [\tau_{A(\delta)} < 1].$$

Note that, on $\{T \leq 1\} \cap E^c$, $X$ must hit $\partial \partial'$ before $T$. It means

$$Q[T \leq 1] \cap E^c = Q[\{\tau_{\partial,\partial'} < T \leq 1\} \cap E^c].$$

By Lemma 3.2, $Y_{\tau_{\partial,\partial'}} = \Psi(X_{\tau_{\partial,\partial'}})$ on $E^c$ under $Q$. In order to collide two Brownian motions starting at $X_{\tau_{\partial,\partial'}}$ and $\Psi(X_{\tau_{\partial,\partial'}})$, either of them must escape from the flat cylinder of length $2(L_0 - \delta)$ where its starting point has distance $L_0 - \delta$ from the boundary. This observation together with the strong Markov property yields

$$Q[\{\tau_{\partial,\partial'} < T \leq 1\} \cap E^c] = Q[Q(Y_{\tau_{\partial,\partial'}}) \wedge \tau_{\partial} < 1 \wedge \tau_{\partial,0}] \leq 2Q[P^{R^2} [\tau_{\partial} < 1 \wedge \tau_{\partial,0}] \wedge \tau_{\partial,\partial'} < 1 \wedge \tau_{\partial,0}].$$

By Lemma 5.2 and the definition of $\zeta$ and $\delta$, we have $L_0 - \delta \geq R - \zeta - \delta > 2R/3$. Thus

$$Q[P^{R^2} [\tau_{\partial} < 1 \wedge \tau_{\partial,0}] \wedge \tau_{\partial,\partial'} < 1 \wedge \tau_{\partial,0}] \leq 2 \exp \left(-\frac{(L_0 - \delta)^2}{2}\right) P[\tau_{\partial,\partial'} < 1 \wedge \tau_{\partial,0}] \leq \exp \left(-\frac{2R^2}{9}\right) P[\tau_{\partial,\partial'} < 1 \wedge \tau_{\partial,0}].$$

By lifting $X_t$ to $R^2$, we have

$$P[\tau_{\partial,\partial'} < 1 \wedge \tau_{\partial,0}] \leq P^{R^2} [\tau_{\partial} \wedge \tau_{\partial'} < 1 \wedge \tau_{\partial,0}] \leq 2P^{R^2} [\tau_{\partial} < 1] \leq 2P^{R^2} [\tau_{R - \zeta} < 1].$$

Here the last inequality follows from Lemma 5.2. Consequently, we obtain

$$Q[T \leq 1] \leq P^{R^2} [\tau_{A(\delta)} < 1] + 16 \exp \left(-\frac{2R^2}{9}\right) P^{R^2} [\tau_{R - \zeta} < 1].$$

(3.2)

Now take $R > 3\sqrt{2 \log 2}$. After that we choose $\zeta$ so small that $P^{R^2} [\tau_{R - \zeta} < 1] \approx P^{R^2} [\tau_R < 1]$. Finally we choose $\delta$ so small that $P^{R^2} [\tau_{A(\delta)} < 1] \approx 0$. Then Proposition 5.1 follows from 3.1 and 3.2.

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