Nonlocal Low-Rank Tensor Factor Analysis for Image Restoration

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Abstract

Low-rank signal modeling has been widely leveraged to capture non-local correlation in image processing applications. We propose a new method that employs low-rank tensor factor analysis for tensors generated by grouped image patches. The low-rank tensors are fed into the alternative direction multiplier method (ADMM) to further improve image reconstruction. The motivating application is compressive sensing (CS), and a deep convolutional architecture is adopted to approximate the expensive matrix inversion in CS applications. An iterative algorithm based on this low-rank tensor factorization strategy, called NLR-TFA, is presented in detail. Experimental results on noiseless and noisy CS measurements demonstrate the superiority of the proposed approach, especially at low CS sampling rates.

1. Introduction

Inspired by the nonlocal self-similarity of image patches [3], various image processing algorithms have been proposed to investigate the low-rank property of image patch groups [9, 11, 17, 31, 44]. In general, these methods first select a reference patch and then search for similar patches across the image to form a group. Following this, the patches in this group are vectorized and stacked to a matrix. Since these patches are similar, the constructed matrix has a low-rank property. Via performing this low-rank model on every (overlapping) patch in the image, state-of-the-art image restoration results have been achieved.

One common issue in the above algorithms is that the original two-dimensional (2D) patches are vectorized to construct the group matrix, which loses the spatial structure within the image patch. We propose a tensor based algorithm to retain this structure while still leveraging the advantages of low-rank patch models. Tensor factorization methods [19] offer a useful way to learn latent structure from complex multiway data, and have been used in image processing tasks [43]. These methods decompose the tensor data into a set of factor matrices (one for each mode or “way” of the tensor), that can be used as a latent feature representation for the objects in each of the tensor modes [33]. Recently tensor approaches have been applied in computer vision, such as image denoising [32], and video denoising [42]. We consider the general image restoration problem with specific applications to compressive sensing [12] using the tensor factor analysis (TFA) approach. In particular, instead of vectorizing the image patches, we impose low-rank TFA to the 3D image patch groups.

To be concrete, the image restoration problem aims to estimate the latent clean image $x$ given the observation $y$ (the corrupted/compressive measurement) and the measurement matrix $\Phi$, which can be formulated as

$$y = \Phi x + e,$$

where $e$ denotes the measurement noise and is usually modeled by $e \sim \mathcal{N}(0, \sigma^2 I)$. The problem has different variants

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for different $\Phi$: when $\Phi$ is the identity matrix, this is a de-
noising problem [16]; when $\Phi$ is a diagonal matrix whose
diagonal elements are either 1 or 0, keeping or removing

corresponding pixels, this becomes image inpainting [63];
and when $\Phi \in \mathbb{C}^{M \times N}$ and $M \ll N$, this is a compressive
sensing (CS) problem.

We focus on the CS problem [14, 28], and consider the
image reconstruction with a very limited number of compressive
measurements [50, 51, 52, 58]. A framework of
compressive measurements can be viewed as a process of linearly mapping an

high-dimensional signal from a small number of ran-
dom linear measurements. In this paper, the compressive sensing rate
is dramatically decreased after tensor decom-
position, i.e., from $I \times K \times J$ to $(I + K + J) \times R$.

Practically, most tensor decomposition problems are NP-
hard. However, in most real applications, as long as the
tensor does not have too many components, and the com-
ponents are not adversarially chosen, tensor decomposition

can be computed in polynomial time [34]. The tensor-

decomposition problem seeks to estimate $A, B, C$ and co-
efficients $\lambda$ from a tensor $T$. Adopting a least-squares fit-
ness criterion, the problem is

$$\min_{A,B,C,\lambda} \left\| T - \sum_{r=1}^{R} \lambda_r a_r \otimes b_r \otimes c_r \right\|_F$$

In this work, we employ Jenrich’s algorithm [21] to solve
this problem. In Algorithm 1, “+” denotes pseudo-inverse
of a matrix.

Algorithm 1 Jenrich’s Algorithm

Require: Tensor $T$.
1: Pick two random unit vectors $u$, $v$.
2: Compute $T_u = \sum_{i=1}^{k} u_i T[:, :, i]$;
3: Compute $T_v = \sum_{i=1}^{k} v_i T[:, :, i]$;
4: $a_v$’s are eigenvectors of $T_u (T_v\dagger)$, $b_r$’s are eigenvectors
   of $(T_u\dagger) (T_v\dagger)$;
5: Given $A$ and $B$, we can get $c_r$’s and $\lambda_r$’s by solving a
   linear system followed by normalization;
6: Output: Tensor factors $a_r$’s, $b_r$’s, $c_r$’s and coefficients
   $\lambda_r$’s

2.2. Compressive Sensing

Compressive sensing is a signal acquisition technique that enables sampling a signal at sub-Nyquist rates [12]. In
CS, a reconstruction algorithm is used to recover the original
high-dimensional signal from a small number of ran-
dom linear measurements. Taking compressive measure-
ments can be viewed as a process of linearly mapping an
$N$-dimensional signal vector $x$ to an $M$-dimensional mea-
surement vector $y$, $M \ll N$, using a measurement matrix
$\Phi \in \mathbb{C}^{M \times N}$, i.e., $y = \Phi x$. Since the matrix $\Phi$ is rank-
deficient, there exists more than one $x$ that yields the same
measurement $y$. In this paper, the compressive sensing rate
(CSR) is defined as $CSR = M/N$.

To recover $x$, one searches for the vector that possesses a
certain structure among all the vectors $x$ that satisfy $y \approx \Phi x$. In the case of a sparse $x$, a popular method is to solve the
optimization problem

$$x = \arg \min_x \| x \|_1, \text{ s.t. } \| y - \Phi x \| \leq \epsilon.$$
This problem is convex and known formally as basis pursuit denoising (BPDN). It has been shown in [6][13] that if \( x \) is sufficiently sparse and \( \Phi \) satisfies certain properties, then the \( s \)-sparse signal can be accurately recovered from \( m = O(s \log(n/s)) \) random linear measurements [5]. Equation (5) can be equivalently translated to the following unconstrained optimization problem

\[
x = \arg \min_\mathbf{x} \| \mathbf{y} - \Phi \mathbf{x} \|^2_2 + \lambda \| \mathbf{x} \|_1, \quad (6)
\]

where \( \lambda \) is a regularization parameter. Various methods can be used to solve the above minimization problem [49, 59, 57], and in this work we adopt the alternative direction multiplier method (ADMM) framework [2, 23]. Specifically, we consider the application of image CS [14, 51, 52], and beyond sparsity, we propose a new tensor factorization approach to exploit the high-order structure in the image patches, seeking high reconstruction performance at extremely low CSr, e.g., CSr<0.05. Refer to Fig. 3 for one example of reconstructed image using our proposed algorithm, compared with other leading algorithms at CSr= 0.02 (with image size 256 x 256).

### 3. Method

We propose a new model that recovers compressively sensed images using low-rank tensor factor analysis and ADMM. First, we generate a tensor from the estimated image based on patch grouping. Then we impose low rankness on the tensor after tensor decomposition. This new low-rank tensor is fed into a global objective function which is solved by ADMM. These two steps are iteratively performed until satisfying some criterion. The complete algorithm is shown in Algorithm 2.

#### 3.1. Patch-Based Low-Rank Tensor Factorization

Given the observation \( \mathbf{y} \), we first obtain the estimated image \( \hat{x} \) using some fast algorithm, e.g., the DCT or wavelet based algorithm [18, 59]. In denoising, \( \hat{x} \) can be simply set equal to \( \mathbf{y} \). Then the estimated image is divided into \( P \) overlapping patches \( \{x_1, ..., x_P\} \). The basic assumption underlying the proposed approach is that a sufficient number of similar patches can be found for any reference patch, a.k.a, the nonlocal self-similarity (NSS) prior [3]. For each reference patch \( x_p \), we perform a \( k \)-nearest-neighbor search for the nonlocal similar patches across the image to form a group \( \{x_{p,1}, x_{p,2}, ..., x_{p,k}\} \), where \( k \) is the number of similar patches (including the reference patch itself). Here, the Euclidean distance of pixel intensity is used as the metric to group patches. By concatenating the grouped patches on the third dimension in ascending order of Euclidean distances, we generate a tensor \( \mathcal{T}_p \) for reference patch \( x_p \)

\[
\mathcal{T}_p = [x_{p,1}, x_{p,2}, ..., x_{p,k}].
\]

As \( x_p \) has zeros distance to itself, it is always found as the leading patch in \( \mathcal{T}_p \), i.e., \( \mathcal{T}_p(:, :, 1) = x_p \). Eventually we have \( P \) tensors and each tensor corresponds to a reference patch. The coordinates of the grouped patches are also recorded for later image aggregation. Suppose the size of each patch is \( m \times n \), then the size of generated tensor is \( \mathcal{T}_p \in \mathbb{R}^{m \times n \times k} \). It has been shown that the grouped patches can be denoised by low-rank approximation [4]. In this work, the low-rankness is imposed on the tensor by taking the most significant tensor factors after tensor decomposition

\[
\mathcal{L}_p = \sum_{r \in S_{\ell}} \lambda_r \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r,
\]

where \( S_{\ell} \) selects the most significant \( \ell \) tensor factors of \( \mathcal{T}_p \). Therefore, \( \mathcal{L}_p \) has rank \( \ell \). The significance of tensor factors is evaluated by \( \lambda_r \)'s in (3).

Under the assumption that the grouped patches have similar structure, \( \mathcal{T}_p \) has a low-rank property, which ensures that \( \mathcal{T}_p \) can be represented by a relatively low-rank tensor \( \mathcal{L}_p \). This low-rank imposition shares the same spirit with the hard thresholding algorithm [1]. In Fig. 2, we impose low rankness on tensors with size \( 8 \times 8 \times 20 \) generated from a clean image using the above approach, and then aggregate the tensors back to images. As can be seen, reconstructed images with low-rank tensors can accurately approximate the original image. The images aggregated from rank-3 to
rank-8 tensors are highly similar (with PSNR merely increased by 0.55 dB), which indicates that in this case rank-3 tensors are adequate to represent the original image.

3.2. Image Recovery via ADMM

With the reconstructed low-rank tensors \( L_p \), the following optimization problem is proposed for CS recovery:

\[
\hat{x} = \arg \min_x \| y - \Phi x \|_2^2 + \eta \sum_p \| \bar{T}_p x - L_p \|_F^2, \tag{9}
\]

where \( \bar{T}_p x = \hat{T}_p \) denotes the tensor formed for each reference patch \( x_p \) and \( \eta \) is a regularization parameter. The closed-form solution for this quadratic optimization problem is

\[
x = (\Phi^H \Phi + \eta \sum_p \bar{T}_p^T \bar{T}_p)^{-1} (\Phi^H y + \eta \sum_p \bar{T}_p^T L_p), \tag{10}
\]

where \( H \) is the Hermitian transpose operation, \( \sum_p \bar{T}_p^T L_p \) denotes the results of averaging all of the similar patches for each reference patch, and each entry of \( \sum_p \bar{T}_p^T \bar{T}_p \) corresponds to an image pixel location whose value is the number of overlapping patches that cover this pixel location. In (10), the matrix to be inverted can be large, for which conjugate gradient descent (CG) is usually applied [8]. We adopt ADMM to solve this problem, introducing auxiliary variable \( z \). Applying ADMM to (9), we obtain the global objective function

\[
(\hat{x}, \hat{z}) = \arg \min_{x, z} \| y - \Phi x \|_2^2 + \beta \| x - z + \frac{\mu}{2\beta} \|_2^2
+ \eta \sum_p \| \bar{T}_p z - L_p \|_F^2, \tag{11}
\]

where \( \mu \) is the Lagrange multiplier, and \( \beta > 0 \) is the penalty parameter. Instead of minimizing \( x \) and \( z \) simultaneously, ADMM decomposes the problem into two subproblems that minimizes w.r.t \( x \) and \( z \), respectively. More specifically, the optimization problem in (11) consists of the following iterations:

\[
z^{t+1} = \arg \min_x \beta \| x^t - z + \frac{\mu^t}{2\beta} \|_2^2 + \eta \sum_p \| \bar{T}_p z - L_p \|_F^2, \tag{12}
\]

\[
x^{t+1} = \arg \min_x \| y - \Phi x \|_2^2 + \beta \| x - z^{t+1} + \frac{\mu^t}{2\beta} \|_2^2, \tag{13}
\]

\[
\mu^{t+1} = \mu^t + 2\beta (x^{t+1} - z^{t+1}). \tag{14}
\]

Both \( x \) and \( z \) admit closed-form solutions. For fixed \( x^t \) and \( \mu^t \),

\[
z^{t+1} = (\eta \sum_p \bar{T}_p^T \bar{T}_p + \beta I)^{-1} (\beta x^t + \frac{\mu^t}{2} + \eta \sum_p \bar{T}_p^T L_p), \tag{15}
\]

Then we use the updated \( z^{t+1} \) to update \( x \),

\[
x^{t+1} = U^{-1} (\Phi^H y + \beta z^{t+1} - \frac{\mu^t}{2}), \tag{16}
\]

where \( U^{-1} = (\Phi^H \Phi + \beta I)^{-1} \).

Algorithm 2 CS via Low-Rank TFA

Require: The observation measurement \( y \).

Initialization:
1. Set parameters \( \eta, \beta, \ell, m, n, k, I, \) and \( J \).
2. Pretrain \( C_\theta \) for inverting matrix \( \Phi^H \Phi + \beta I \) using (18);
3. Obtain an estimated image \( \hat{x} \) from observation \( y \) using a fast CS method;
4. Divide \( \hat{x} \) into a set of overlapping patches;
5. Initialize \( x^{(0)} = \hat{x} \);

Image Restoration:
6. for \( i = 0, 1, \ldots, I - 1 \) do
7. Divide \( x^{(i)} \) into a set of overlapping patches with size \( m \times n \);
8. Form a set of tensors with size \( m \times n \times k \) using patch block matching;
9. Decompose tensors using Jenrich’s Algorithm;
10. Impose low rankness on tensors via (8) and generate a set of rank-\( \ell \) tensors;

ADMM:
11. Initialize \( \mu^0 = 0, x^0 = x^{(i)} \);
12. for \( j = 0, 1, \ldots, J - 1 \) do
13. Update \( z^{j+1} \) via (15);
14. if \( \Phi \) is a partial Fourier transform matrix then
15. Update \( x^{j+1} \) via (21);
16. else
17. Update \( x^{j+1} \) via (19);
18. end if
19. Update \( \mu^{j+1} \) via (14);
20. end for
21. Update \( x^{(i+1)} = x^j \)
22. end for
23. Output: The reconstructed image \( \hat{x} = x^{(I)} \).

3.2.1 Pretrained Deep Convolutional Architectures

In (16), the term \( U^{-1} \) involves an expensive matrix inversion, which makes direct computation of \( x^{t+1} \) impractical. An inversion-free approach proposed by [41] addresses this problem by learning a convolutional neural network to approximate the matrix inversion. Note that training this neural network is data-independent since \( U^{-1} \) is only dependent on \( \Phi \). Applying the Sherman-Morrison-Woodbury formula, we reduce this matrix inversion to a smaller scale,

\[
U^{-1} = \beta^{-1} (I - \Phi^H \Phi V^{-1} \Phi), \tag{17}
\]
where \( V^{-1} = (\beta I + \Phi \Phi^H)^{-1} \) has the dimension \( m \times m \). To approximate \( V^{-1} \), a trainable deep convolutional neural network \( C_\theta \) parameterized by \( \theta \) is employed, i.e., \( C_\theta \approx V^{-1} \). \( C_\theta \) is learned by minimizing the sum of two reconstruction losses of two auto-encoders with shared weights

\[
\arg \min_\theta \mathbb{E}_x [||\epsilon - C_\theta V \epsilon||^2_2 + ||\epsilon - V C_\theta \epsilon||^2], \tag{18}
\]

where \( V = \beta I + \Phi \Phi^H \) can be computed directly, and \( \epsilon \) is sampled from publicly available image datasets [10]. Note that \( C_\theta \) is pretrained for different \( \Phi \) and \( \beta \). By plugging the learned \( C_\theta \) into (17), we obtain a reusable term \( U^{-1} = \beta^{-1}(I - \Phi^H C_\theta \Phi) \) as the replacement for the cumbersome inversion matrix. Hence \( x^{t+1} \) is updated by

\[
x^{t+1} = \beta^{-1}(I - \Phi^H C_\theta \Phi)(\Phi^H y + \beta z^{t+1} - \frac{\mu^t}{2}). \tag{19}
\]

### 3.2.2 Fourier space solution

Equation (16) can be solved by transforming the problem from the image space into the Fourier space when \( \Phi \) is a partial Fourier transform matrix [11]. For a down-sampling matrix \( D \) and a Fourier transform matrix \( F \), \( \Phi = DF \) is substituted into (16)

\[
x^{t+1} = (D^H D + \beta I)^{-1}(F^H D^H y + (\beta z^{t+1} - \frac{\mu^t}{2})), \tag{20}
\]

where the inverse matrix \((D^H D + \beta I)^{-1}\) is a diagonal matrix and thus can be computed easily. Equation (20) is equivalent to

\[
x^{t+1} = F^H \{(D^H D + \beta I)^{-1}(D^H y + F(\beta z^{t+1} - \frac{\mu^t}{2}))\}. \tag{21}
\]

Therefore, \( x^{t+1} \) can be obtained by applying inverse Fourier transform to terms in the brackets of the right hand side of (21).

### 4. Experiments

We conduct experiments on both noisy and noiseless CS measurements of 8 test images: Barbara, Boats, Cameraman, Foreman, House, Lena, Monarch, and Parrots. All images are resized to \( 256 \times 256 \). Since excellent results have been obtained when the CS rate is large, i.e., \( \text{CSr} > 0.1 \) [11, 29], we here focus on testing cases with limited number of measurements (\( \text{CSr} < 0.1 \)). The CS measurements are generated by pseudo-radial sampling of the test images in the Fourier domain. Unlike traditional random sampling schemes, pseudo-radial sampling produces streaking artifacts which are more difficult to remove. We also perform experiments using an \( M \times N \) random sensing matrix generated by a standard Gaussian distribution. The deep convolutional architecture \( C_\theta \) is pretrained by the sensing matrix then used in ADMM to solve the expensive matrix inversion. A DCT based CS recovery algorithm [52] is used for the initial image estimate. The main parameters of the proposed method are set as follows: patch size \( 4 \times 4 \), number of similar patches for each reference patch 50, and the imposed low rank is 20. The regularization parameter \( \eta \) and the ADMM parameter \( \beta \) are tuned separately for each sensing rate. In practice, we have found that ADMM converges fast after a few iterations and the performance gain is mainly from our low-rank TFA. Thus we set the number of outer-loop iterations to \( I = 50 \) and the number of inner loop iterations \( J = 2 \). Experimental results for noiseless CS measurements and noisy CS measurements are evaluated by peak signal-to-noise ratio (PSNR) and structural similarity (SSIM) [40].

![Figure 3: Reconstruction results from noiseless CS measurements for Foreman at CSr = 0.02. (a) Original image; (b) TV AL3 (23.35 dB); (c) NLR-CS (19.64 dB); (d) Ours (34.01 dB).](image-url)
low-rank regularization method along with ADMM to solve image CS problems. D-AMP employs a denoiser in an approximate message passing framework, achieving state-of-the-art performance on noisy measurements, especially at high CS rates. The source codes of these baseline methods are downloaded from the respective author's website and parameters for these algorithms are set to their default values.

### 4.2. Experiments with Noiseless CS Measurements

We first perform CS image restoration experiments from noiseless measurements using pseudo-radial sampling.
scheme at five different CS rates, i.e., CSr = {0.02, 0.04, 0.06, 0.08, 0.10}. Table 1 summarizes the results of our proposed algorithm compared with various CS inversion algorithms at different CSr values. Both BM3D-CS and the state-of-the-art methods NLR-CS and D-AMP suffer at extremely low CSr. As CSr increases, NLR-CS yields significant performance improvements, which leads one of its results at higher CSr (0.10) to surpass other methods. Our proposed method achieves the best performance in all cases when CSr < 0.1. On average, our proposed NLR-TFA algorithm outperforms all other competing methods at CSr = {0.02, 0.04, 0.06, 0.08, and 0.10}. The PSNR gains of our NLR-TFA over BM3D-CS, TVAL3, NLR-CS and D-AMP can be as much as 15.04 dB, 10.65 dB, 14.36 dB, and 14.58 dB, respectively. Furthermore, it can be observed that our average reconstruction PSNR only decreases by 2.33 dB as CSr decreases from 0.10 to 0.02, while this number for NLR-CS and D-AMP are 10.67 dB and 9.62 dB, respectively, indicating that the proposed method is very stable at low CSr, i.e., with limited number of measurements.

To evaluate the reconstruction visually, two examples of restored Foreman and Monarch images at CSr of 0.02 and 0.06 are shown in Figs. 3 and 4. It is evident that our method recovers the best visual quality among all competing methods. Large-scale sharp edges and small-scale fine structures are both reconstructed in two images. In particular, at extremely low CSr of 0.02, our NLR-TFA method (Fig. 3(d)) can effectively approximate the original image while some other methods, such as NLR-CS, can only reconstruct scratches (Fig. 3(c)).

Table 2: PSNR(dB) of reconstructions from measurements generated by random Gaussian sampling of 8 images at different CSr.

| Image    | Method | CSr          |
|----------|--------|--------------|
|          |        | 0.02 | 0.04 | 0.06 | 0.08 |
| Barbara  | D-AMP  | 15.79 | 17.56 | 19.20 | 21.96 |
|          | Ours   | 24.13 | 25.52 | 26.38 | 27.22 |
| Boats    | D-AMP  | 16.06 | 17.93 | 19.84 | 22.47 |
|          | Ours   | 26.04 | 27.44 | 28.59 | 29.51 |
| Cameraman| D-AMP  | 16.66 | 18.32 | 19.79 | 22.42 |
|          | Ours   | 23.59 | 24.66 | 25.41 | 26.15 |
| Foreman  | D-AMP  | 20.50 | 22.78 | 25.75 | 28.40 |
|          | Ours   | 27.72 | 29.23 | 30.66 | 31.47 |
| House    | D-AMP  | 18.61 | 20.85 | 24.42 | 26.97 |
|          | Ours   | 26.68 | 28.20 | 29.48 | 30.17 |
| Lena     | D-AMP  | 16.34 | 18.15 | 19.77 | 22.15 |
|          | Ours   | 24.63 | 26.04 | 26.99 | 27.78 |
| Monarch  | D-AMP  | 14.50 | 16.38 | 17.94 | 19.93 |
|          | Ours   | 21.57 | 23.01 | 24.19 | 24.89 |
| Parrots  | D-AMP  | 17.49 | 19.38 | 22.56 | 25.24 |
|          | Ours   | 25.92 | 27.19 | 27.96 | 28.54 |
| Average  | D-AMP  | 16.99 | 18.83 | 21.15 | 23.69 |
|          | Ours   | 25.04 | 26.41 | 27.46 | 28.22 |

We next generate CS measurements by sampling the test images with a random Gaussian sensing matrix. A deep convolutional neural network \(C_\theta\) is pretrained using the sensing matrix to solve the large-scale matrix inversion problem in ADMM. We compare our results with D-AMP which provides random Gaussian sensing implementations in their sourcing code. The reconstruction results at CSr of 0.02, 0.04, 0.06 and 0.08 are shown in Table 2. Our proposed method outperforms D-AMP on all test images and sensing rates. In addition, the runtime of the algorithm using this inversion-free approach is also much faster than using approaches with inner-loop updates, such as conjugate gradient (CG). This demonstrates the efficiency and accuracy of the deep convolutional architecture on approximating the inversion matrix.

**Patch Size Selection** We have conducted experiments using different patch sizes. As can be seen from Table 3 (in the pseudo-radial sampling scheme), smaller patch sizes lead to better reconstruction results. When we adopt patch size of 4 × 4 on the foreman image, the PSNR is significantly increased compared to results with larger patch sizes. Similar
Figure 5: Reconstruction results of Boats images from noisy CS measurements with noise standard deviation equal to 30 at CSr = 0.10. (a) Original image; (b) TVAL3 (24.02 dB); (c) NLR-CS (23.48 dB); (d) DAMP (24.71 dB); (e) Ours (30.97 dB).

Figure 6: Comparison of different CS methods with noisy measurements for Lena and Parrots images. (a) CSr= 0.08. (b) CSr=0.04.

Table 3: PSNR(dB) of reconstructions with variant patch sizes at different CSr for foreman image.

| Patch Size | CSr  | 0.02 | 0.04 | 0.06 | 0.08 | 0.10 |
|------------|------|------|------|------|------|------|
| 4 × 4      |      | 34.01| 34.61| 35.18| 35.33| 36.02|
| 6 × 6      |      | 31.16| 31.95| 32.72| 32.88| 33.59|
| 8 × 8      |      | 29.63| 30.41| 31.37| 31.68| 32.24|
| 10 × 10    |      | 28.57| 29.45| 30.44| 30.86| 31.33|
| 12 × 12    |      | 27.39| 28.61| 29.60| 30.11| 30.54|

conclusions are found on other images as well. In addition, the running time of our model is largely influenced by the tensor size. For tensor size of $4 \times 4 \times 50$ (image patch size $4 \times 4$ and 50 similar patches for each reference patch), the average running time at CSr=0.06 is about 10 minutes on a desktop with i7 CPU @3.4GHz and 24G RAM.

4.3. Experiments with Noisy CS Measurements

Similar experiments are conducted with noisy CS measurements to show the robustness of our algorithm to noise. The noisy CS measurements are obtained by adding random zero-mean Gaussian noise to the noiseless CS measurements. We test the algorithms at three levels of noise, with standard deviation $\sigma = \{10, 20, 30\}$. The PSNR comparison of all methods for Boats and Monarch images at sensing rates of 0.10 and 0.04, respectively, are shown in Fig. 6. Our proposed NLR-TFA outperforms other competing methods at all levels of noise. Furthermore, D-AMP shows great robustness to noise while BM3D-CS and NLR-CS are relatively sensitive to high-level noise. As the noise level increases, the reconstruction performance of our algorithm decreases slowly. This shows that the proposed method is robust to noise. In Fig. 5, we show the reconstructions of Boats images using various algorithms at CSr=0.10 and $\sigma = 30$. The proposed NLR-TFA clearly yields the best reconstruction and is robust to noise.

5. Conclusion

We have presented a low-rank tensor-factor-analysis-based approach to solve image-restoration problems. A tensor is generated by concatenating similar patches from the estimated image for each exemplar patch. Low-rank is imposed on the tensors to exploit non-local correlation and the high-order structure information via tensor factorization. ADMM is employed on low-rank tensors, where we use either a pretrained convolutional architecture or Fourier approaches to solve a matrix inversion. Experimental results demonstrate that the proposed NLR-TFA method outperforms state-of-the-art algorithms on CS image reconstruction at low CS sampling rates.

We have demonstrated the superiority of our model in image CS with three-way tensors, which can also be used in depth CS [25, 26, 53], polarization CS [39] and dynamic range CS [48]. Future work includes extending the model to four-way tensors for video CS [24, 27, 35, 36, 45, 46, 54, 55, 56, 60, 61], hyperspectral image CS [7, 62] and to five-way tensors for joint spectral-temporal CS [37, 38].
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