ON THE RENORMALIZED VOLUME OF TUBES OVER POLARIZED KÄHLER-EINSTEIN MANIFOLDS

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Abstract. A formula of the renormalized volume of tubes over polarized Kähler-Einstein manifolds is given in terms of the Einstein constant and the volume of the polarization.

1. Introduction

The renormalized volume of conformally compact Einstein manifolds was first introduced by Henningson-Skenderis [6] and Graham [3] in studies of AdS/CFT correspondence. Let $M$ be an $(m+1)$-dimensional compact manifold with boundary $\partial M$, and $x$ a defining function of the boundary. A Riemannian metric $g_+$ on the interior of $M$ is called a conformally compact Einstein metric if $x^2g_+$ extends to a Riemannian metric on $M$ and $\text{Ric}_{g_+} = -mg_+$ holds. The conformal class $C = [(x^2g_+)|_{\partial M}]$ on $\partial M$ is independent of the choice of $x$, called the conformal infinity of $g_+$. If we choose a representative $g$ of $C$, there exists a unique defining function $x$ near the boundary such that $|dx|_{g_+} = 1$ and $(x^2g_+)|_{\partial M} = g$. For such a defining function, consider the relatively compact domain $M^\epsilon = \{x < -\epsilon\}$ for $\epsilon > 0$. The volume $\text{Vol}_{g_+}(M^\epsilon)$ of $M^\epsilon$ has the following expansion, as $\epsilon \to +0$:

$$\text{Vol}_{g_+}(M^\epsilon) = \sum_{j=0}^{[m/2]-1} a_j \epsilon^{2j-m} + \begin{cases} V^o + O(1), & m \text{ odd}, \\ L \log \epsilon + V^e + O(1), & m \text{ even}. \end{cases}$$

The constant term $V^o$ or $V^e$ in this asymptotics is called the renormalized volume. Graham [3] has proved that $V^e$ and $L$ are independent of the choice of $x$. Moreover, Graham-Zworski [4] have shown that $L$ coincides with a constant multiple of the total $Q$-curvature of the boundary, a global conformal invariant; see also [2]. Fefferman-Graham [2] also have deduced a formula for $V^o$ in terms of the scattering matrix for $(M, g_+)$. A similar formula for $V^e$ has been proved by Yang-Chang-Qing [13].

Now we consider the renormalized volume of strictly pseudoconvex domains. Let $\Omega$ be a bounded strictly pseudoconvex domain in an $(n+1)$-dimensional complex manifold $X$. Assume that $\Omega$ has a complete Kähler-Einstein metric $\omega_+$ with Einstein constant $-(n+2)$. We also suppose that
$K_X$ has a Hermitian metric that is flat on the pseudoconvex side near the boundary. Then there exists a defining function $\rho$ such that $\omega_+\omega$ coincides with $-dd^c\log(-\rho)$ near the boundary, where $d^c=(\sqrt{-1}/2)(\overline{\partial} - \partial)$; see Lemma 2.2. For this $\rho$, the volume $\text{Vol}_{\omega_+}(\Omega^\epsilon)$ of the relatively compact domain $\Omega^\epsilon = \{\rho < -\epsilon\}$ has an expansion of the form

$$\text{Vol}_{\omega_+}(\Omega^\epsilon) = \sum_{j=0}^n b_j \epsilon^{j-n-1} + V + O(1),$$

as $\epsilon \to +0$. We call the constant term $V$ the renormalized volume of $(\Omega, \omega_+)$. Remark that Seshadri [10] has considered the renormalized volume for the choice of defining functions similar to the conformal case.

In this paper, we compute the renormalized volume of tubes over polarized Kähler-Einstein manifolds.

**Theorem 1.1.** Let $(Y, L)$ be a polarized $n$-dimensional Kähler manifold and $h_L$ a Hermitian metric of $L$ such that the curvature form $\omega = \sqrt{-1}\Theta h_L$ defines a Kähler-Einstein metric on $Y$ with Einstein constant $\beta$. If $\beta < 1$, there exists a complete Kähler-Einstein metric $\omega_+$ on $\Omega = \{v \in L^{-1} | h_L^{-1}(v,v) < 1\}$ with Einstein constant $-(n+2)$, where $h_{L^{-1}}$ is the dual metric of $h_L$, and the renormalized volume $V$ of $(\Omega, \omega_+)$ is given by

$$V = (2\pi)^{n+1} \left[ \frac{1}{n+1} \left( -\frac{\beta}{n+1} \right)^{n+1} - \left( \frac{1-\beta}{n+2} \right)^{n+1} \right] \text{Vol}(L).$$

Here $\text{Vol}(L) = \int_Y c_1(L)^n$ is the volume of $L$.

Note that the assumption $\beta < 1$ is a necessary and sufficient condition for the existence of a complete Kähler-Einstein metric on $\Omega$ with negative Einstein constant, which has been proved by van Coevering [12].

To prove Theorem 1.1 we apply the formula of the renormalized volume obtained by Hirachi-Marugame-Matsumoto [8]. From the assumption, there exists a lift $\tilde{c}_1(K_{\Omega}) \in H^2_c(\Omega; \mathbb{R})$ of the first Chern class $c_1(K_{\Omega}) \in H^2(\Omega; \mathbb{R})$.

**Theorem 1.2** ([8, Theorem 1.1]). The renormalized volume $V$ of $(\Omega, \omega_+)$ is given by

$$V = \frac{(-1)^{n+1}}{2(n!)^2(n+1)!} Q(\partial\Omega) + \left( \frac{2\pi}{n+2} \right)^{n+1} \int_{\Omega} \tilde{c}_1(K_{\Omega})^{n+1}.$$  

Here $Q(\partial\Omega)$ is the total Q-prime curvature of $\partial\Omega$; see [7, Proposition 5.5] for the definition.

In the setting of Theorem 1.1 the boundary $\partial\Omega$ is a Sasakian $\eta$-Einstein manifold, and the total Q-prime curvature for such CR manifolds has been computed by the author [11]. The result is

$$Q(\partial\Omega) = 2(n!)^2 \left( \frac{2\pi\beta}{n+1} \right)^{n+1} \text{Vol}(L).$$
Hence it is enough to compute the integral of $\tilde{c}_1(K_{\Omega})^{n+1}$. Here, we will give a little bit more general statement. We consider a general bounded strictly pseudoconvex domain $\Omega$, and make the following assumption:

**Assumption 1.3.** The maximal compact analytic set of $\Omega$ is a smooth complex hypersurface $D$, and there exists a Hermitian metric $h_{\Omega}$ of $K_{\Omega}$ such that the support of its curvature is contained in a tubular neighborhood of $D$.

Namely, we assume that the cohomology class $\tilde{c}_1(K_{\Omega})$ is localized along a smooth divisor. Then the quantity $\int_{\Omega} \tilde{c}_1(K_{\Omega})^{n+1}$ can be computed as follows.

**Theorem 1.4.** Under Assumption 1.3, the normal bundle $N_{D/\Omega}$ of $D$ satisfies $c_1(K_{D}) = \beta \cdot c_1(N_{D/\Omega})$ in $H^2(D;\mathbb{R})$ for some $\beta \in \mathbb{R}$, and the quantity $\int_{\Omega} \tilde{c}_1(K_{\Omega})^{n+1}$ is given by

$$\int_{\Omega} \tilde{c}_1(K_{\Omega})^{n+1} = (\beta - 1)^{n+1} \int_{D} c_1(N_{D/\Omega})^{n},$$

Additionally, assume that $\Omega$ has a Kähler-Einstein metric with negative Einstein constant. Then $N_{D/\Omega}$ is negative and $\beta < 1$ holds. In particular, $\int_{\Omega} \tilde{c}_1(K_{\Omega})^{n+1}$ is negative.

Note that Theorem 1.4 can be generalized to the case that the maximal compact analytic set consists of disjoint smooth complex hypersurfaces. We also remark that the constant $\beta$ is a rational number since $c_1(K_{D})$ and $c_1(N_{D/\Omega})$ are elements of $H^2(D;\mathbb{Z})$. In particular, $\int_{\Omega} \tilde{c}_1(K_{\Omega})^{n+1}$ is also a rational number.

This paper is organized as follows. In Section 2, we recall fundamental facts for strictly pseudoconvex domains. Section 3 provides the proof of Theorem 1.4. In Section 4, we prove Theorem 1.1.

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**2. Strictly pseudoconvex domains**

Let $\Omega$ be a bounded strictly pseudoconvex domain in an $(n+1)$-dimensional complex manifold $X$. If $X$ is the complex Euclidean space, $\Omega$ has a complete Kähler-Einstein metric with negative Einstein constant [11 Corollary 4.5]. For a general $X$, a necessary and sufficient condition for the existence of such a metric is obtained by van Coevering.
Theorem 2.1 ([12] Theorem 3.1 and Proposition 3.2). The domain $\Omega$ has a complete Kähler-Einstein metric $\omega_+$ with Einstein constant $-(n+2)$ if and only if $K_\Omega$ has a Hermitian metric with positive curvature. Moreover, such a metric is unique and invariant under biholomorphisms.

For $X = \mathbb{C}^{n+1}$, this metric has a global Kähler potential of the form $-\log(-\rho)$, where $\rho$ is a defining function of $\Omega$. This fact does not hold if, for example, $\Omega$ contains a compact analytic set with positive dimension. However, there exists such a Kähler potential near the boundary under an assumption for $K_\Omega$.

Lemma 2.2. Assume that $\Omega$ has a complete Kähler-Einstein metric $\omega_+$ with Einstein constant $-(n+2)$. There exists a defining function $\rho'$ of $\Omega$ such that $\omega_+ + dd^c \log(-\rho)$ has compact support in $\Omega$ if and only if $K_X$ has a Hermitian metric $h_X$ that is flat on the pseudoconvex side near the boundary.

Proof. Let $h'_X$ be a Hermitian metric of $K_X$. From the proof of [12, Theorem 3.1], there exists a defining function $\rho'$ of $\Omega$ such that $\omega_+ + dd^c \log(-\rho) = \sqrt{-1} \frac{1}{2} \pi \Theta_{h'_X}$. This expression proves the equivalence. $\square$

Remark that this condition is equivalent to the existence of a pseudo-Einstein contact form on the boundary [8, Proposition 2.6]. In particular, the compactly supported $(1,1)$-form $(\sqrt{-1}/2\pi)\Theta_{h_X}$ defines a cohomology class $\tilde{c}_1(K_\Omega)$ in $H^2_c(\Omega; \mathbb{R})$, which is a lift of the first Chern class $c_1(K_\Omega) \in H^2(\Omega; \mathbb{R})$. Note that the cohomology class $\tilde{c}_1(K_\Omega)^n+1 \in H^{2n+2}_c(\Omega; \mathbb{R})$ is independent of the choice of $h_X$. If $X = \mathbb{C}^{n+1}$, then $\tilde{c}_1(K_\Omega) = 0$ in $H^2_c(\Omega; \mathbb{R})$ since $X$ has a flat Kähler metric. However, as in the statement of Theorem 1.4, $\int_\Omega \tilde{c}_1(K_\Omega)^{n+1}$ may take a negative value.

Before the end of this section, we give a lemma for the normal bundle of the maximal compact analytic set.

Lemma 2.3. Let $D$ be a smooth complex hypersurface in $\Omega$ that is a connected component of the maximal compact analytic set of $\Omega$. Then the normal bundle $N_{D/\Omega}$ cannot be positive.

Proof. Suppose that $N_{D/\Omega}$ is positive. Then as in the proof of [11, Satz 1], there exist a small neighborhood $U$ of $D$ and a smooth strictly plurisubharmonic function $\psi$ on $U \setminus D$ such that $\psi^{-1}((c, \infty)) \cup D$ is an open neighborhood of $D$ for any $c \in \mathbb{R}$. On the other hand, there exists an analytic space $\Omega'$ and a proper surjective holomorphic map $\phi: \Omega \to \Omega'$ such that

1. $\phi$ maps $D$ to a single point $q \in \Omega'$;
2. $\phi: \Omega \setminus D \to \Omega' \setminus \{q\}$ is a biholomorphism;
3. $\phi_*(O_\Omega) = O_{\Omega'}$. 

Before the end of this section, we give a lemma for the normal bundle of the maximal compact analytic set.
Let $U' = \phi(U)$ be an open set in $\Omega'$. Then $\psi' = \psi \circ \phi^{-1}$ is a plurisubharmonic function on $U' \setminus \{q\}$ and $\psi'(q')$ goes to $\infty$ as $q' \to q$. However, from [5, Satz 4], $\psi'$ extends to a plurisubharmonic function on $U'$; this is a contradiction. □

3. Proof of Theorem 1.4

We first localize the integral $\int_{\Omega} \tilde{c}_1(K_\Omega)^{n+1}$ along $D$. From Assumption 1.3, there exists a Hermitian metric $h$ of the normal bundle $N_{D/\Omega}$ such that the domain
\[ V = \{ v \in N_{D/\Omega} \mid h(v, v) < 1 \} \]
is diffeomorphic (not necessarily biholomorphic) to a tubular neighborhood of $D$ containing the support of the curvature of $h_\Omega$; in the following we identify the zero section of $N_{D/\Omega}$ with $D$. Then
\[ \int_{\Omega} \tilde{c}_1(K_\Omega)^{n+1} = \int_{\Omega} \left( \frac{\sqrt{-1}}{2\pi} \Theta_{h_\Omega} \right)^{n+1} = \int_{V} \Pi^{n+1}, \]
where $\Pi$ is the 2-form on $V$ corresponding to $(\sqrt{-1}/2\pi)\Theta_{h_\Omega}$. Since $K_\Omega|D$ is isomorphic to $K_V|D$, the $(1,1)$-form $\omega = \Pi|_D$ is a representative of $c_1(K_\Omega|D) = c_1(K_V|D)$. If $\Omega$ has a Kähler-Einstein metric with negative Einstein constant, then the line bundle $K_\Omega|D \cong K_V|D$ is positive.

*Proof of Theorem 1.4.* Since $N_{D/\Omega}$ is a complex line bundle, it has a canonical $S^1$-action; its generator is denoted by $\xi$. Without loss of generality, we may assume that $\Pi$ is $S^1$-invariant. Then there exists an $S^1$-invariant 1-form $\mu$ on $N_{D/\Omega}$ such that
\[ \Pi - p^*\omega = -d\mu, \]
where $p: N_{D/\Omega} \to D$ is the projection. This is because the zero section is an $S^1$-equivariant deformation retract of $N_{D/\Omega}$. The boundary $S$ of $V$ is a principal $S^1$-bundle over $D$, and its Chern class is equal to $c_1(N_{D/\Omega})$. Near $S$, $p^*\omega = d\mu$ holds, and so
\[
d(\mu(\xi)) = \mathcal{L}_\xi \mu - i_\xi d\mu
= \mathcal{L}_\xi \mu - i_\xi (p^*\omega)
= 0.
\]
Hence $\mu(\xi)$ is a constant $\alpha \in \mathbb{R}$ near $S$. First assume that $\alpha = 0$. Then there exists a 1-form $\nu$ on $D$ such that $\mu|_S = p_S^*\nu$, where $p_S: S \to Y$ is the canonical projection, and $\omega = d\nu$ holds since $p_S^*\omega = p_S^*(d\nu)$. Thus $\Pi = d(p^*\nu - \mu)$ and
\[
\int_L \Pi^{n+1} = \int_V d(p^*\nu - \mu) \wedge \Pi^n
= \int_S (p^*\nu - \mu)|_S \wedge (\Pi|_S)^n
= 0.
\]
Note that \( c_1(K_V|_D) = 0 \) and
\[
c_1(K_D) = c_1(K_V|_D) + c_1(N_{D/\Omega}) = c_1(N_{D/\Omega})
\]
in \( H^2(Y; \mathbb{R}) \) since \( \omega = d\nu \in c_1(K_V|_D) \). Next, consider the case \( \alpha \neq 0 \). Then \( \alpha^{-1}\mu|_S \) is a connection 1-form for the principal \( S^1 \)-bundle \( p_S: S \to D \). In particular,
\[
c_1(N_{D/\Omega}) = \left[ -\frac{1}{2\pi} d(\alpha^{-1}\mu|_S) \right] = \left[ -\frac{1}{2\pi\alpha} \omega \right].
\]
Therefore,
\[
\int_V \Pi^{n+1} = \int_V (\Pi^{n+1} - (p^*\omega)^{n+1})
\]
\[
= \int_V (\Pi - p^*\omega) \wedge (\Pi^n + \cdots + (p^*\omega)^n)
\]
\[
= - \int_V d(\mu \wedge (\Pi^n + \cdots + (p^*\omega)^n))
\]
\[
= - \int_S \mu \wedge (p^*\omega)^n
\]
\[
= (-2\pi\alpha)^{n+1} \int_D c_1(N_{D/\Omega})^n.
\]
In the last equality, we use the integration along fibers. From \( \omega \in c_1(K_V|_D) \),
\[
c_1(K_V|_D) = (-2\pi\alpha)c_1(N_{D/\Omega}) \quad \text{and} \quad
\]
\[
c_1(K_D) = c_1(K_V|_D) + c_1(N_{D/\Omega}) = (1 - 2\pi\alpha)c_1(N_{D/\Omega}).
\]
This proves the first statement.

If \( K_V|_D \) is positive, then \( \alpha \) is non-zero since \( c_1(K_V|_D) \neq 0 \). Hence
\[
c_1(N_{D/\Omega}) = (-2\pi\alpha)^{-1}c_1(K_V|_D), \quad \text{and} \quad N_{D/\Omega}
\]
is either positive or negative. However, \( N_{D/\Omega} \) cannot be positive from Lemma 2.3 and so \( N_{D/\Omega} \) is negative. \( \square \)

4. PROOF OF THEOREM 1.1

To prove Theorem 1.1, we first show the following

**Lemma 4.1.** The domain \( \Omega \) in Theorem 1.1 satisfies Assumption 1.3.

**Proof.** Denote by \( X \) the total space of \( L^{-1} \). It is known that the maximal compact analytic set of \( \Omega \) is the zero section of \( L^{-1} \). Choose the whole \( \Omega \) as a tubular neighborhood of the zero section. It is enough to prove the existence of a Hermitian metric of \( K_X \) such that its curvature has compact support in \( \Omega \). Let \( h_Y \) be the Hermitian metric of \( K_Y \) induced from the Kähler metric \( \omega \). Then, \( \sqrt{-1}\Theta_{h_Y} = -\beta \cdot \omega \) holds. Since \( K_X \) is isomorphic to \( p^*K_Y \otimes p^*L \), where \( p: X \to Y \) is the projection, it has the Hermitian metric \( h_X ' \) induced from \( h_Y \) and \( h_L \), whose curvature \( \sqrt{-1}\Theta_{h_X} \) is equal to
\[
\sqrt{-1}\Theta_{h_X} = \sqrt{-1}(p^*\Theta_{h_Y} + p^*\Theta_{h_L}) = \sqrt{-1}(1 - \beta)p^*\Theta_{h_L}.
\]
Since $\sqrt{-1} p^* \Theta_{h_L}$ has a Kähler potential outside the zero section, we obtain a desired Hermitian metric of $K_X$ by modifying $h'_X$. □

**Proof of Theorem 1.1.** First, consider the existence of a Kähler-Einstein metric on $\Omega$. van Coevering [12, Corollary 5.6] has proved that $\Omega$ has a complete Kähler-Einstein metric with Einstein constant $-(n+2)$ if and only if $c_1(K_Y) + c_1(L) > 0$. Hence if $\beta < 1$, such a metric $\omega_+$ exists. Moreover, from the proof of Lemma 4.1, $K_X$ has a Hermitian metric whose curvature has compact support in $\Omega$. Thus the renormalized volume of $(\Omega, \omega_+)$ is well-defined. As discussed in Section 1, it is enough to compute the integral of $\tilde{c}_1(K_\Omega)^{n+1}$, and its formula is given by Theorem 1.4. □

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