Three Identical Fermions with Resonant $p$-wave Interactions in Two Dimensions

Chao Gao and Zhenhua Yu

Institute for Advanced Study, Tsinghua University, Beijing, 100084, China
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A new kind of “super-Efimov” states of binding energies scaling as $\ln |E_n| \sim - e^{3n\pi/4}$ were predicted by a field theory calculation for three identical fermions with resonant $p$-wave interactions in two dimensions [Phys. Rev. Lett. 110, 235301 (2013)]. However, the universality of these “super-Efimov” states has not been proved independently. In this Letter, we study the three fermion system through the hyperspherical formalism. Within the adiabatic approximation, we find that at $p$-wave resonances, the low energy physics of states of angular momentum $\ell = \pm 1$ crucially depends on the value of an emergent non-universal dimensionless parameter $Y$ determined by the details of the inter-particle potential. Only if $Y$ is exactly zero, the predicted “super-Efimov” states exist. If $Y > 0$, emerges a new scaling $\ln |E_n| \sim -(n\pi)^2/2Y$, while there are no shallow bound states if $Y < 0$. An analytic expression of $Y$ is derived. The values of $Y$ evaluated at various $p$-wave resonances for three model potentials are found to be different but all positive.

Introduction. The extreme experimental controllability and versatility of ultra-cold atomic gases provide an ideal platform either to test established theories or to simulate important theoretical models whose properties remain little known [1,3]. In the aspect of few-body physics, a landmark result is the Efimov bound states predicted theoretically long time ago for three-body systems with resonant $s$-wave interactions [4]. Such states have a peculiar scaling behavior: the ratio between the binding energies of successive bound states $E_{n+1}/E_n$ is a universal number, independent of the details of the inter-particle potentials [4-6]. Only recently, experimentalists succeeded in realizing resonant $s$-wave interactions in ultra-cold atomic gases by the technique of Feshbach resonance [7], and for the very first time revealed the Efimov states through detection of giant three-body recombination loss [8,9]. Enhanced atom-dimer inelastic collisions [10,11] and radio-frequency spectroscopy [12,13]. Further studies showed that even the three-body parameter which determines the absolute energy scale of the Efimov states has a universal feature for different atomic species [9,1,4,13].

The quest for universal physics at resonances of higher partial waves brought about a recent local quantum field theory calculation predicting that universal bound states exist for three identical fermions with resonant $p$-wave interactions in two dimensions [14]. These states have angular momentum $\ell = \pm 1$ and are called “super-Efimov” due to the fascinating scaling of the binding energies $\ln |E_{n+1}|/\ln |E_n| = e^{3\pi/4}$. However as pointed out by Ref. [21], a local quantum field theory calculation, when treating $p$-wave resonances, can lead to problematic results such as assigning negative probabilities to low energy states [22]. Therefore it is important to investigate the universality of the predicted “super-Efimov” states independently.

In this Letter, we use the hyperspherical formalism to study three identical fermions with resonant $p$-wave interactions in two dimensions. In the angular momentum $\ell = \pm 1$ channel, within the adiabatic approximation, we derive the lowest hyperspherical adiabatic potential $U_{\text{eff}}$ in the large hyperradius limit, which determines the low energy physics of the system. We find that $U_{\text{eff}}$ crucially depends on the value of an emergent non-universal dimensionless parameter $Y$: If $Y < 0$, $U_{\text{eff}}$ is repulsive and there are no shallow three-body bound states; only if $Y$ is exactly zero, the “super-Efimov” states appear; for $Y > 0$, the binding energies of bound states have a new scaling $\ln |E_n| \sim -(n\pi)^2/2Y$. We reveal that the emergence of $Y$ originates from the correlated motion of the three fermions on a hypersurface of fixed hyperradius; thus $Y$ can not be determined by a finite number of two-body scattering parameters. We derive an expression for $Y$ in terms of the inter-particle potential $V(r)$ and the zero energy $p$-wave radial wave-function $u_0(r)$. We calculate $Y$ for three different model potentials at various $p$-wave resonances; the value of $Y$ varies and is found to be all positive (see Table I). Finally we discuss the implication of our results regarding experimental observation of “super-Efimov” states.

Hyperspherical formalism. We consider three identical fermions with coordinates $r_1$, $r_2$ and $r_3$ interacting pairwise through a central potential $V(r)$ of finite range $r_0$ in two dimensions. The potential is fine tuned such that it is on a $p$-wave resonance. We introduce the Jacobi coordinates $x_i = r_j - r_k$ and $y_i = 2[r_j - (r_j + r_k)/2]/\sqrt{3}$ where $\{i,j,k\}$ takes the values of $\{1,2,3\}$ cyclically. The hyperradius is given by $\rho = \sqrt{x_i^2 + y_i^2}$ and the corresponding hyperspherical angles $\Omega_i = \{\alpha_i, \theta_{x_i}, \theta_{y_i}\}$ with $\alpha_i = \tan^{-1}(x_i/y_i)$. After separating out the center of mass part, we expand the wave-function of the system in terms of any set of $\Omega_i$ as

$$\Psi = \sum_n \rho^{-3/2} f_n(\rho) \Phi_n(\rho, \Omega_i).$$

(1)

The angular part $\Phi_n(\rho, \Omega_i)$ is required to satisfy the
eigenfunctions
\[ \hat{\Lambda}^2 + m\rho^2 \sum_{j=1}^{3} V(\rho \sin \alpha_j) \Phi_n(\rho, \Omega_i) = \lambda_n(\rho) \Phi_n(\rho, \Omega_i), \]
\[ (2) \]
with \( m \) the mass of each fermions. The total angular momentum operator is given by [23]
\[ \hat{\Lambda}^2 = -\frac{\partial^2}{\partial \alpha_i^2} - 2 \cot(2\alpha_i) \frac{\partial}{\partial \alpha_i} + \frac{E^2_{\alpha_i}}{\sin^2 \alpha_i} + \frac{E^2_{\beta_i}}{\cos^2 \alpha_i}. \]
\[ (3) \]
We take \( \hbar = 1 \) throughout. Consequently, the hyperradial part satisfies the coupled equations of eigen-energy \( E \) as [23]
\[ \left[ -\frac{d^2}{d\rho^2} + \frac{\lambda_n + 3/4}{\rho^2} - mE \right] f_n(\rho) = \sum_{n'} \left[ 2P_{nn'} \frac{d}{d\rho} + Q_{nn'} \right] f_{n'}(\rho). \]
\[ (4) \]
The couplings \( P_{nn'} = \langle \phi_n | \partial_\rho | \phi_{n'} \rangle \) and \( Q_{nn'} = \langle \phi_n | \partial_\rho^2 | \phi_{n'} \rangle \), with \( \langle \ldots \rangle \) standing for the integration over the hyperangles, are expected to be negligible in the large \( \rho \) limit and we assume them to be zero [23]. The eigenstates with \( E > 0 \) are governed by the lowest lying \( \lambda_0(\rho) \) in the region \( \epsilon \equiv r_0/\rho \ll 1 \) as Eq. (1) becomes decoupled.

We focus on the states with angular momentum \( |\ell| = |\ell_x| + |\ell_y| = 1 \) for which the “super-Efimov” scaling was found [20]. Given that \( \epsilon \) is so small, there are regions where \( \sin \alpha_i > \epsilon \) for any \( i \) and fermions feel no interaction. In such regions, from Eq. (2), we express the angular wave-function of \( \ell = 1 \) as
\[ \Phi_n = \sum_i \sin(\alpha_i) \left[ A_{1,0} P_{\nu_0}^{(1,0)}(-\cos 2\alpha_i) e^{-i\theta_{x_i}} \right. \]
\[ + A_{-1,2} \cos^2(\alpha_i) P_{\nu_0-1}^{(2,1)}(-\cos 2\alpha_i) e^{i(\theta_{x_i}-2\theta_{y_i})} \right], \]
\[ (5) \]
with \( P_{(a,b)} \) the Jacobi functions and \( 4(\nu_0 + 1)^2 = \lambda_n + 1 \). The first term in Eq. (5) corresponds to the channel with \( \ell_x = 1 \) and \( \ell_y = 0 \), and the second to the one with \( \ell_x = -1 \) and \( \ell_y = 2 \). The coefficients \( A_{1,0} \) and \( A_{-1,2} \) are to be determined. Note that at a p-wave resonance, channels of \( |\ell| \neq 1 \) would have negligible weight and have been dropped off in Eq. (5) [23].

On the other hand, we follow the procedure outlined in Ref. [23] solving the Faddeev equations corresponding to Eq. (2) in the region where only one pair of fermions can feel interaction, i.e., there is only one hyperangle, let us say \( \alpha_i \), small enough that \( \sin \alpha_i < \epsilon \). By connecting the solution in the region \( \sin \alpha_i < \epsilon \) and Eq. (5) at the point \( \alpha_i = \tilde{\alpha} = \sin^{-1}(\epsilon) \), we obtain the coupled eigenequations
\[ \frac{M_{\ell_x, \ell_y} Q_{\ell_z, \ell_y}}{M_{\ell_x, \ell_y} P_{\ell_z, \ell_y}} - \partial_\alpha Q_{\ell_z, \ell_y} \sin(\pi \nu_{\ell_x, \ell_y}) A_{\ell_x, \ell_y} = \cos(\pi \nu_{\ell_x, \ell_y}) A_{\ell_x, \ell_y} + 2 \sum_{(\ell_z', \ell_y')} R_{\ell_x, \ell_y}(\ell_z', \ell_y') A_{\ell_z', \ell_y'} \]
\[ (6) \]
where \( \{\ell_x, \ell_y\} \) and \( \{\ell_z', \ell_y'\} \) take \{1, 0\} or \{-1, 2\}. The notation \( P_{\ell_x, \ell_y} \) and \( Q_{\ell_x, \ell_y} \) stand for the regular and irregular Jacobi functions \( P_{\ell_x, \ell_y}(\cos 2\tilde{\alpha}) \) and \( Q_{\ell_x, \ell_y}(\ell_x, \ell_y) \) \( \cos 2\tilde{\alpha} \) respectively, and \( \nu_0 = \nu_{1,0} = \nu_{-1,2} + 4 = \sqrt{\lambda_0 + 1}/2 - 1 \). The rotation matrices \( R_{\ell_x, \ell_y}(\ell_z', \ell_y') \) are defined in Ref. [23] and found to be
\[ R_{(1,0),(1,0)} = -\frac{3(\nu_0 + 2) P_{\nu_{-1}}^{(1,2)}(1/2) + 4 P_{\nu_0}^{(0,1)}(1/2)}{8(\nu_0 + 1)} \]
\[ (7) \]
\[ R_{(1,0),(-1,2)} = \frac{3}{8} F_1 (1 - \nu_0, \nu_0 + 3; 3; 1/4) - \frac{1}{64} (\nu_0 - 1) \]
\[ \times (\nu_0 + 3) F_1 (2 - \nu_0, \nu_0 + 4; 4; 1/4) \]
\[ (8) \]
\[ R_{(-1,2),(1,0)} = -\frac{3}{8} (\nu_0 + 2) P_{\nu_{-1}}^{(1,2)}(1/2) \]
\[ (9) \]
\[ R_{(-1,2),(-1,2)} = -\frac{3(\nu_0 + 3) P_{\nu_{-2}}^{(3,2)}(1/2) + 4 P_{\nu_{-1}}^{(2,1)}(1/2)}{32\nu_0} \]
\[ (10) \]
The information of interactions is encoded in the quantities
\[ M_{\pm 1, \ell_y} = \partial_\alpha \ln u^{\ell_y} - \cot \tilde{\alpha} + |\ell_y| \tan \tilde{\alpha}, \]
\[ (11) \]
where the function \( u^{\ell_y} \) obeys
\[ [\Lambda^2 + m\rho^2 V(\rho \sin \alpha_i) - \lambda_n] u^{\ell_y}(\alpha_i) = 0, \]
\[ (12) \]
with \( L^2_{\alpha_i} \) and \( L^2_{\beta_i} \) in \( \Lambda^2 \) replaced by \( L^2_x \) and \( L^2_y \) respectively.

Adiabatic potential. To obtain the asymptotic behavior of the lowest lying \( \lambda_0 \) in the large \( \rho \) limit at p-wave resonances, we expand the coefficient of \( \sin(\pi \nu_{\ell_x, \ell_y}) A_{\ell_x, \ell_y} \) in Eq. (6) to the leading order of \( \epsilon \). Note that different from s-wave resonances in three dimensions, since \( Q_{\ell_x, \ell_y}(\ell_x, \ell_y) \sim 1/\pi(\nu + 1 + |\ell_y|) \epsilon^2 + O(\ln \nu, \epsilon^0) \), and \( M_{\pm 1, \ell_y} \sim -2/\epsilon + O(\epsilon) \) when on p-wave resonance, one must keep \( M_{\pm 1, \ell_y} \) to order \( O(\epsilon) \). Consequently the leading order of the coefficient is \( \ln \epsilon \) plus terms of \( O(\epsilon^0) \). We emphasize that it is crucial to retain these terms of \( O(\epsilon^0) \) which are functions of \( \lambda_0 \).

From Eq. (4) in which \( P_{nn'} \) and \( Q_{nn'} \) are neglected, a necessary condition for “super-Efimov” states is that when \( \rho \to \infty \), there is a solution \( \lambda_0(\rho) + 1 \to 0 \) (so \( \nu_0(\rho) + 1 \to 0 \) for Eq. (5). Actually we find that such a solution does exist. We further expand Eq. (6) to the
lowest order of $\nu_0 + 1$ and have
\[
\left[2(\nu_0 + 1) \ln \epsilon - \frac{Y}{2(\nu_0 + 1)}\right] A_{1,0} = \frac{4}{3} A_{1,2} \quad (13)
\]
\[-2(\nu_0 + 1) \ln \epsilon] A_{1,2} = \frac{4}{3} A_{1,0}, \quad (14)
\]
where $Y$ is a dimensionless parameter which depends on the
detail of the interaction potential $V(r)$ as we shall show later (see Eq. (25)). We substitute $\lambda_0$ obtained
from Eqs. (13) and (14) into Eq. (4) and find
\[
-\frac{d^2}{d\rho^2} - \frac{1}{4\rho^2} + U_{\text{eff}}(\rho) - mE \left[\frac{Y}{\ln(\rho/\rho_0)} + \frac{\rho^2}{\ln^2(\rho/\rho_0)}\right] f_0(\rho) = 0 \quad (15)
\]
with the adiabatic potential
\[
U_{\text{eff}}(\rho) = -\frac{1}{\rho^2} \left[\frac{Y}{\ln(\rho/\rho_0)} + \frac{\rho^2}{\ln^2(\rho/\rho_0)}\right], \quad (16)
\]
and $s_0^2 = 16/9$. We have checked that there is no other
$\lambda_n$ whose value can be smaller than $\lambda_0$ obtained above.
Depending on the value of $Y$, the adiabatic potential can be either repulsive or attractive for $\rho/\rho_0 \to \infty$.

Bound states. If $Y < 0$, $U_{\text{eff}}(\rho)$ is repulsive in the limit
$\rho \to \infty$; there are no arbitrarily shallow bound states. If $Y = 0$, $U_{\text{eff}} = -s_0^2/\rho^2 \ln^2(\rho/\rho_0)$. We evaluate the binding
ergies of shallow bound states by the WKB approximation.
Due to the singularity of $1/\ln^2(\rho/\rho_0)$ in $U_{\text{eff}}$, we need to transform the variables as $t = \ln(\rho/\rho_0)$ and
$f_0 = [\rho \ln(\rho/\rho_0)]^{1/2} h_0$ in Eq. (15) for $\rho > \rho_0$ and find
\[
\left(-\frac{d^2}{dt^2} - s_0^2 + \frac{1}{4}\right) h_0 = m r_0^2 E e^{2(\epsilon't + t)} h_0. \quad (17)
\]
The quantization condition for the $n$th state of binding energy $E_n$ is
\[
n\pi \approx \int_{t_0}^{t_T} dt \sqrt{s_0^2 - 1/4 - m r_0^2 |E_n| e^{2(\epsilon't + t)}}, \quad (18)
\]
where $t_0$ is a point above which $U_{\text{eff}}$ is applicable, and the turning point $t_T$ is given by $(s_0^2 - 1/4)e^{-2(\epsilon'T + t_T)} = m r_0^2 |E_n|$. As $n \to \infty$, $|E_n| \to 0$ and the leading contribution to the integral in Eq. (18) is $t_T \sqrt{s_0^2 - 1/4}$; we reproduce the “super-Efimov” scaling $\ln(m r_0^2 |E_n|) \sim -2 \exp(n\pi/\sqrt{16/9 - 1/4})$, but with a different scaling parameter from Ref. [20].

In the case $Y > 0$, in the limit $\rho \to \infty$, we keep the dominate term proportional to $Y$ in $U_{\text{eff}}$. We carry out the same variable transformations for the sake of the WKB approximation as above and find that Eq. (15) becomes
\[
\left(-\frac{d^2}{dt^2} - Ye^t\right) h_0 = m r_0^2 E e^{2(\epsilon't + t)} h_0. \quad (19)
\]
The corresponding new scaling is $\ln(m r_0^2 |E_n|) \sim -(n\pi^2)/2Y$.

We plot in Fig. (1) the binding energies $E_n$ calculated numerically from Eq. (15) with a hard wall boundary
imposed at $\rho/\rho_0 = 10$ for $Y = 0$ and $s_0^2 = 40$ and for $Y = 10$ and $s_0^2 = 0$. The results show good agreement
with the scalings derived from the WKB approximation.

Origin of non-universality. The non-universal parameter $Y$ enters into $\lambda_0$ through the $O(\epsilon^2)$ terms of $u^{\nu_5}(\alpha_i)$
in the calculation of Eq. (6). This parameter is peculiar to the three-body system and can not be determined by
a finite number of two-body scattering parameters parametrizing the $p$-wave phase shift $\delta_p$. To see the point,
let us calculate $u^{\nu_5}(\alpha_i)$ perturbatively to order $\epsilon^2$ in the region $\alpha_i \leq \tilde{\alpha}$. We introduce $\eta = x_i/\rho_0 = \sin\alpha_i/\sin\tilde{\alpha}$, and for $\eta \leq 1$, to order $\epsilon^2$, Eq. (12) becomes
\[
(H_0 + \epsilon^2 H_2 - \epsilon^2 \lambda_0) u^{\nu_5}(\eta) = 0 \quad (20)
\]
with $H_0 = -\partial^2_\eta - (1/\eta) \partial_\eta + \eta / \eta^2 + \tilde{V}(\eta)$, $H_2 = \partial^2_\eta + 3 \eta \partial_\eta + \eta^2 / \eta^2$ and $\tilde{V}(\eta) = m r_0^2 U(V(\rho_0 \eta))$. From Eq. (20), to the zero order of $\epsilon$, $u^{\nu_5}$ equals the zero energy two-body scattering wave-function $u_0$ which obeys $H_0 u_0 = 0$. One may analogize $\epsilon^2 \lambda_0$ to the two-body scattering energy. However, the term $\epsilon^2 H_2$ manifests the fundamental difference between the three-body system compared to the two-body one. The contribution of $\epsilon^2 H_2$ to the kinetic energy originates from the fact that Eq. (12) governs the motion of the three fermions on a hypersurface of fixed $\rho$. When $\alpha_i$ varies, so do both $x_i$ and $y_i$. While the kinetic energy in $H_0$ is associated with $x_i$, $\epsilon^2 H_2$ is the kinetic energy associated with $y_i$ under the constraint that $\rho$ is...
fixed. The relative weight between the two kinetic energies $\sim \epsilon^2$ can be understood from the uncertainty principle: Since Eq. (20) is valid in the region $x_1 \leq r_0$ and automatically $y_1 \sim \rho$, the relative weight is expected to be $\sim \frac{1}{\sqrt{\pi}} \frac{1}{2\pi} \sim \epsilon^2$. Thus the operator $H_2$ distinguishes $u^{\psi}$ from the low energy two-body scattering wave-function; on the hypersurface as two particles interact with each other, the system does know that there is a third one at a distance $\sim \rho$ moving correlatedly to maintain the fixed $\rho$.

In the cases of off resonance and $s$-wave resonances in three dimensions, in the calculation of Eq. (20), it is sufficient to keep the zero order of $u^{\psi}$ and the information of the third particle is lost, while in our present case, the second order contribution of $u^{\psi}$ does enter into $\lambda_0$. To see how $Y$ is related to the inter-particle interactions, we calculate the solution to Eq. (20) to order $\epsilon^2$ as $u^{\psi} = u_0 + \epsilon^2[\lambda_0u_2^{(1)} - u_2^{(2)}]$ with

$$u_2^{(1)}(\eta) = \int_0^\eta d\eta' K_0(\eta, \eta')u_0(\eta'), \quad (21)$$

$$u_2^{(2)}(\eta, \epsilon_0) = \int_0^\eta d\eta' K_0(\eta, \eta')h_2u_0(\eta'). \quad (22)$$

The integral kernel satisfies $H_2K_0(\eta, \eta') = \delta(\eta - \eta')$ and has the expression [25]

$$K_0(\eta, \eta') = \theta(\eta - \eta') \frac{v_0(\eta)v_0(\eta') - v_0(\eta)u_0(\eta')}{u_0(\eta')\partial_\eta v_0(\eta) - v_0(\eta')\partial_\eta u_0(\eta')}, \quad (23)$$

where $v_0$ is the irregular solution of $Hv_0 = 0$. Note that while $u_2^{(1)}$ would appear in the low energy expansion of the two-body scattering wave-functions, $u_2^{(2)}$ would not. After we substitute the solution of $u^{\psi}$ to order $\epsilon^2$ into Eq. (1), a lengthy but straightforward calculation yields

$$Y = \left[1 + \partial_\eta \left[ u_2^{(1)}(\eta) + u_2^{(2)}(\eta, 0) \right] \right]_{\eta=1} \int_0^{2\pi} d\eta' \frac{\tilde{V}(\eta)\tilde{V}(\eta')}{\eta|u_0(\eta)|^2}. \quad (24)$$

$$= -1 - \lim_{\eta \to \infty} \int_0^\infty d\eta' \tilde{V}(\eta)\tilde{V}(\eta') \frac{\eta|u_0(\eta)|^2}{\eta|u_0(\eta)|^2}. \quad (25)$$

The denominator in Eq. (25) simply cancels the normalization of $u_0$ appearing in the numerator since on $p$-wave resonance $u_0(\eta) \sim 1/\eta$ as $\eta \to \infty$. In Table I, we show Y calculated by Eq. (25) for the square well potential $V(r) = \tilde{g}\theta(r-b)/b^2$, the Gaussian potential $V(r) = e^{-r^2/b^2}/b^2$ and the “Hardcore+1/r^6” potential $V(r) = +\infty$ for $r < b$ and $V(r) = \tilde{g}b^4/r^6$ for $r > b$ at various $p$-wave resonances by tuning $\tilde{g}$ ($< 0$). We see that the values of Y are all positive and differ for different potentials and increase at resonances of deeper $\tilde{V}$ as naturally expected from Eq. (25).

**Final remarks.** It seems that $Y$ is generally positive. Consequently the scaling of the binding energies of three-body bound states would be $\ln(mr_0^2/E_n) \sim -(n\pi)^2/2Y$. The effects of these bound states on the three-body recombination loss are worth further investigation. Though at the moment we can not exclude the possibility that for some more complex model potentials, $Y$ can be fine tuned to zero. The emergence of $Y$ implies great challenges for observing “super-Efimov” states experimentally. In addition to the usual technique Feshbach resonance, one needs another method capable of tuning the inter-particle interactions independently such that both a $p$-wave resonance and $Y$ being zero can be achieved simultaneously.

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**Note added.** During completing this work, we became aware of the paper by A.G. Volosniev, D.V. Fedorov, A.S. Jensen, and N.T. Zinner, arXiv:1312.6535. Their conclusion on the “super-Efimov” states agrees with ours in the special case $Y = 0$.

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**TABLE I**: The parameter $Y$ calculated from Eq. (25) for different model potentials at from the first to the third $p$-wave resonance as the strength of the attractive potentials increases.

| Resonance | Square well | Gaussian | Hardcore+1/r^6 |
|-----------|-------------|----------|---------------|
| 1st       | 0.63        | 0.48     | 1.43          |
| 2nd       | 4.74        | 1.63     | 2.10          |
| 3rd       | 12.15       | 2.77     | 2.49          |

* Electronic address: huazhenyu2000@gmail.com

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