A Unified Approach Towards Monogamy of Quantum Correlations

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We prove that monogamy of a convex quantum correlation measure for an arbitrary multipartite pure quantum state leads to its monogamy for the mixed state in the same Hilbert space. In particular, we prove that the square of negativity is monogamous for n-qubit mixed state. Several derived monogamy relations are also discussed. We, primarily, aim to establish that a general quantum correlation obeys a set of monogamy relations.

I. INTRODUCTION

Quantum correlations [1, 2], of both entanglement and information-theoretic paradigm, is an indispensable resource in quantum information theory [3]. Entanglement, in particular, is the characteristic trait of quantum mechanics [4]. Seeing enormous applications of quantum correlations in quantum information theory huge attention is given to the study of quantum correlations. Monogamy [5, 6] of quantum correlations, a non-classical property and a consequence of the no-cloning theorem [7], restricts the arbitrary sharing of the quantum correlation among the multipartite quantum systems. It helps in classifying entanglement structure of quantum systems [5, 8] apart from its potential applications in generating secret quantum key distribution [9, 10] and distinguishing the Bell-like orthonormal bases [11]. Monogamy of quantum correlations is in vogue and is being studied extensively [12–22].

Consider that $Q$ is a bipartite entanglement measure. If for a multipartite quantum system described by a state $\rho_{AB_1B_2\cdots B_n}$, the following inequality

$$Q(\rho_{AB_1\cdots B_n}) \geq \sum_{j=1}^{n} Q(\rho_{AB_j}),$$

holds, then the state $\rho_{AB_1B_2\cdots B_n}$ is said to be monogamous under the quantum correlation measure $Q$. Otherwise, it is non-monogamous. Moreover, the deficit between the two sides is referred to as monogamy score [23], and is given by

$$\delta Q = Q(\rho_{A(B_1\cdots B_n)}) - \sum_{j=1}^{n} Q(\rho_{AB_j}).$$

Monogamy score can be interpreted as residual entanglement of the bi-partition $1 : rest$ of an $n$-party state that cannot be accounted for by the entanglement of two-qubit reduced density matrices separately.

Can there be more general and tighter monogamy relations than in Eq. (1)? Attempts have been made to address this question from different perspectives [20–22] recently. Indeed, a plethora of monogamy relations can be introduced. However, their validity remains to be examined thoroughly. In this paper we prove a few of monogamy relations analytically.

This paper is divided into four sections. In Sec. II, we prove monogamy of quantum correlations for mixed states. In Sec. III, we derive several monogamy relations from given monogamy inequality. Finally, we conclude in Sec. IV.

II. MONOGAMY OF QUANTUM CORRELATIONS FOR MIXED STATES

Is there any correspondence between the monogamy of a quantum correlation measure $Q$ for pure states and that for mixed states. In this section we prove that monogamy of $Q$ for multipartite pure states implies its monogamy for mixed states. This significantly simplifies the task of establishing the monogamy relations. Here it is assumed that $Q$ is convex in its arguments, that is, if $\rho = \sum_i p_i \rho^i$ then $Q(\rho) \leq \sum_i p_i Q(\rho^i)$.

Theorem 1: $Q^r$ is monogamous for mixed state if it is so for the pure state in given Hilbert space, where $Q$ is some bipartite entanglement measure and $r = 1, 2$.

Proof. Assume that $Q^r$, $(r = 1, 2)$, is monogamous for arbitrary multipartite pure state $|\psi\rangle_{AB} = |\psi\rangle_{AB_1B_2\cdots B_n}$ in some Hilbert space of dimension $d_A \otimes d_{B_1} \otimes d_{B_2} \cdots \otimes d_{B_n}$. That is,

$$Q^r(|\psi\rangle_{AB}) \geq \sum_{j=1}^{n} Q^r(|\rho^r_{AB_j}\rangle).$$

Let $\rho_{AB} = \sum_i p_i |\psi^i\rangle_{AB} \langle \psi^i | = \sum_i p_i \rho^i_{AB}$ be the optimal decomposition of $\rho_{AB}$ for $Q$, and $\rho^r_{AB_j} = \text{tr}_{A\text{rest}} \rho^r_{AB}$, $\rho_{AB_j} = \text{tr}_{A\text{rest}} \rho_{AB}$ be the reduced density matrices obtained after partial-tracing the sub-systems except $A$ and $B_j$ ($j = 1, 2, \cdots, n$).

Case 1: $r = 1$
We have
\[ Q(\rho_{AB}) = \sum_i p_i Q(|\psi^i\rangle_{AB}) \]
\[ \geq \sum_i p_i \sum_j Q(\rho^i_{AB,j}) \]
\[ = \sum_j \left( \sum_i p_i Q(\rho^i_{AB,j}) \right) \]
\[ \geq \sum_j Q\left( \sum_i p_i \rho^i_{AB,j} \right) \]
\[ = \sum_j Q(\rho_{AB,j}), \quad (4) \]

where the first inequality is due to monogamy of $Q$ for pure states and the second inequality is due to the convexity of $Q$.

In Ref. [13], it was shown numerically that entanglement measures become monogamous for pure states with increasing number of qubits. The above result then implies that multiqubit mixed states will also become monogamous.

**Case 2: $r = 2$**

Let us write
\[ Q(\rho_{AB}) = \sum_i p_i Q(|\psi^i\rangle_{AB}) = \sum_i Q_{ABi} \quad (5) \]
\[ Q'(\rho_{AB,j}) = \sum_i p_i Q(\rho^i_{AB,j}) = \sum_i Q_{AB,j,i} \geq Q(\rho_{AB,j}) \quad (6) \]

Then we have the following inequality
\[ Q^2(\rho_{AB}) - \sum_j Q^2(\rho_{AB,j}) \]
\[ = \left( \sum_i Q_{ABi} \right)^2 - \sum_j \left( \sum_i Q_{AB,j,i} \right)^2 \]
\[ = \sum_i \left( Q^2_{ABi} - \sum_j Q^2_{AB,j,i} \right) \]
\[ + 2 \sum_{i=1}^{n-1} \sum_{k=i+1}^n \left( Q_{ABi} Q_{ABk} - \sum_j Q_{AB,j,i} Q_{AB,j,k} \right) \geq 0, \quad (7) \]

because, in the second equation, the first term is non-negative due to monogamy of $Q^2$ for pure states and the second term is non-negative as shown below. We have, for arbitrary pure states $|\psi^i\rangle_{AB}$ and $|\psi^k\rangle_{AB}$,
\[ Q_{AB,i}^2 Q_{AB,k}^2 \geq \left( \sum_j Q_{AB,j,i}^2 \right) \left( \sum_j Q_{AB,j,k}^2 \right) \geq \sum_j Q_{AB,j,i} Q_{AB,j,k} \geq 0. \quad (8) \]

where the first inequality is due to monogamy of $Q^2$ for pure states while the second inequality follows from the Cauchy-Schwarz inequality, $\sum_i a_i b_i \leq \sqrt{\left( \sum_i a_i^2 \right) \left( \sum_i b_i^2 \right)}$. Hence,
\[ Q_{AB,i} Q_{AB,k} - \sum_j Q_{AB,j,i} Q_{AB,j,k} \geq 0. \quad (9) \]

Since $Q'(\rho_{AB,j}) \geq Q(\rho_{AB,j})$ (due to convexity of $Q$ as shown in Eq. (6)), we obtain the desired monogamy relation for mixed state,
\[ Q^2(\rho_{AB}) \geq \sum_j Q^2(\rho_{AB,j}). \quad (10) \]

**Corollary 1**: The squared negativity is monogamous for $n$-qubit mixed state.

**Proof**. Negativity is a convex function [25], and it has been proven that the square of negativity is monogamous for $n$-qubit pure states [26]. Hence the proof.

**Corollary 2**: The squared logarithmic-negativity is monogamous for $n$-qubit mixed state when $2N(\rho_{AB}) + 1 \geq \prod_j (2N(\rho_{AB,j}) + 1)$.

**Proof**. The logarithmic-negativity [27] is defined as $E_N(\rho) = \log(2N(\rho) + 1)$. Let $X = 2N(\rho_{AB}) + 1$ and $X_j = 2N(\rho_{AB,j}) + 1$. Since $0 \leq N(\rho) \leq 1$, by definition, $1 \leq X, X_j \leq 3$. Hence,
\[ (\log X)^2 - \sum_j (\log X_j)^2 \geq 0, \quad (11) \]

where the first inequality follows from $\sum_j (\log X_j)^2 \leq \left( \sum_j \log X_j \right)^2$ while the second inequality is due to the
given condition.
For the maximally entangled states \([28]\) whose reduced density matrices are maximally mixed, the squared logarithmic-negativity is monogamous.

### III. DERIVED MONOGAMY RELATIONS

In this section we derive several monogamy relations from given monogamy inequality and conditions.

**Theorem 2:** For non-zero \(Q(\rho_{ABi})\), \(Q^r(\rho_{AB}) \geq \sum_j Q^r(\rho_{ABj})\) implies logarithmic monogamy of \(Q\), i.e., \(\log(\rho(\rho_{AB})) \geq \sum_j \log(\rho(\rho_{ABj}))\).

**Proof.** \(Q^r(\rho_{AB}) \geq \sum_j Q^r(\rho_{ABj})\) implies \(Q^r(\rho_{AB}) \geq \prod_j Q^r(\rho_{ABj})\) which, in turn, implies \(Q(\rho_{AB}) \geq \prod_j Q(\rho_{ABj})\). Hence the proof.

**Theorem 3:** If \(Q^r(\rho_{AB}) \geq \sum_j Q^r(\rho_{ABj})\) then \(Q^\alpha(\rho_{AB}) \geq \sum_j Q^\alpha(\rho_{ABj})\) for \(\alpha \geq r \geq 1\).

**Proof.** We have the inequalities \((1 + x)^t \geq 1 + tx\) and \((\sum_i x_i^t) \geq \sum_i x_i^t\) where \(0 \leq x, x_i \leq 1\) and \(t \geq 1\). Now there exists \(1 \leq k \leq n\) such that \(\sum_{j \neq k} Q^r(\rho_{ABj}) \geq Q^r(\rho_{ABk})\). Now \(Q^r(\rho_{AB}) \geq \sum_j Q^r(\rho_{ABj})\) implies

\[
Q^\alpha(\rho_{AB}) \geq \left(\sum_j Q^r(\rho_{ABj})\right)^\frac{\alpha}{r} = \left(\sum_j Q^r(\rho_{ABj})\right)^\frac{\alpha}{r} \geq \left(1 + \frac{Q^r(\rho_{ABk})}{\sum_{j \neq k} Q^r(\rho_{ABj})}\right)^\frac{\alpha}{r} \geq \left(1 + \frac{Q^r(\rho_{ABk})}{\sum_{j \neq k} Q^r(\rho_{ABj})}\right) \geq \sum_j Q^\alpha(\rho_{ABj}) = \sum_j Q^\alpha(\rho_{ABj}),
\]

(12)

thus proving the theorem.

**Theorem 4:** If \(Q^r(\rho_{AB}) \leq \sum_j Q^r(\rho_{ABj})\) then \(Q^\alpha(\rho_{AB}) \leq \sum_j Q^\alpha(\rho_{ABj})\) for \(\alpha \leq r\).

**Proof.** The inequality \(Q^r(\rho_{AB}) \leq \sum_j Q^r(\rho_{ABj})\) implies \(Q^\alpha(\rho_{AB}) \leq \left(\sum_j Q^r(\rho_{ABj})\right)^\frac{\alpha}{r}\). Now above theorem can be proved by using the inequality \((1 + x)^t \leq 1 + tx^t\), for \(x > 0\) and \(t \leq 1\), repeatedly.

Sometimes an entanglement measure \(Q\) can be a function of another entanglement measure \(q\), say, \(Q = f(q^r)\). Depending on the nature of function \(f\) and monogamy of \(q\), the monogamy properties of \(Q\) can be derived.

**Theorem 5:** Given that \(q^r(\rho_{AB}) \geq \sum_j q^r(\rho_{ABj})\) and a monotonically increasing (decreasing) convex (concave) function \(f\) such that \(f^m(\sum_j q^r_j) \geq \sum_j f^m(q^r_j)\) then \(Q^m(\rho_{AB}) \geq (\leq) \sum_j Q^m(\rho_{ABj})\) where \(r\) and \(m\) are some positive numbers.

**Proof.** Let \(\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|\) be the optimal decomposition of \(\rho_{AB}\) for \(Q\). Then

\[
Q(\rho_{AB}) = \sum_i p_i Q(|\psi_i\rangle_{AB} \langle \psi_i|)
\]

\[
= \sum_i p_i f(q^r_{ABi})
\]

\[
\geq (\leq) f \left( \sum_i p_i q^r_{ABi} \right)
\]

\[
\geq (\leq) f \left( \sum_j f^m(q^r_{ABj}) \right)
\]

\[
= \left( \sum_j Q^m_{ABj} \right)^\frac{1}{m},
\]

(13)

where the first inequality is due to convexity (concavity) of \(f\), the second is due to monotonically increasing (decreasing) nature of \(f\) and \(\sum_i p_i x_i^t \geq (\sum_i p_i x_i)^t\), the third is due to monotonicity of \(f\) and because \(p_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|\) may not be the optimal decomposition of \(\rho_{AB}\) for \(q\) (that is, \(q(\rho_{AB}) \neq \sum_i p_i q(|\psi_i\rangle_{AB} \langle \psi_i|)\)), the fourth is due to monogamy of \(q\), and the fifth inequality follows from the constraint \(f^m(\sum_j q^r_j) \geq \sum_j f^m(q^r_j)\). Hence the theorem is proved.

Recently, it was shown in Ref. [19] that, for multiqubit systems, the \(r^\text{th}\)-power of concurrence [29] is monogamous for \(r \geq 2\) while non-monogamous for \(r \leq 0\), and the \(r^\text{th}\)-power of entanglement of formation (EoF) [30] is monogamous for \(r \geq \sqrt{2}\). Thus, by varying the exponent, monogamy relations can be preserved.
**Theorem 6:** For $|\psi\rangle_{ABC}$ of dimension $d_A \otimes d_B \otimes d_C$ with $d_C = \prod_{j=1}^m d_{C_j} = d_B^m$, $m \geq 1$, and $Q'(\langle \psi \rangle_{AXY}) \geq Q'(\rho_{AX}) + Q'(\rho_{AY})$ implies $Q^\alpha(\rho_{ABC}) \geq Q^\alpha(\rho_{AB}) + \sum_{k=1}^m Q^\alpha(\rho_{AC_k})$ when $\alpha \geq r$.

**Proof.** We have, for $r = 1$ and $2$,

\[ Q'(\langle \psi \rangle_{ABC}) \geq Q'(\rho_{AB}) + Q'(\rho_{AC}) \]
\[ \Rightarrow Q'(\rho_{ABC}) \geq Q'(\rho_{AB}) + Q'(\rho_{AC}) \]
\[ \Rightarrow Q^\alpha(\rho_{ABC}) \geq Q^\alpha(\rho_{AB}) + Q^\alpha(\rho_{AC}) \quad (14) \]

Also,

\[ Q^\alpha(\rho_{AC}) = \{ Q'(\rho_{AC_1(2\cdots m)}) \}^\alpha \]
\[ \geq \{ Q'(\rho_{AC_1}) + Q'(\rho_{A(2\cdots m)}) \}^\alpha \]
\[ \geq \sum_{j=1}^{k-1} Q^\alpha(\rho_{AC_j}) + Q^\alpha(\rho_{AC_{k-1}(2\cdots m)}) \quad (15) \]
\[ \geq \sum_{j=1}^m Q^\alpha(\rho_{AC_j}) \quad (16) \]

Thus, we obtain the following inequalities

\[ Q^\alpha(\rho_{ABC}) \geq Q^\alpha(\rho_{AB}) + \sum_{j=1}^{k-1} Q^\alpha(\rho_{AC_j}) + Q^\alpha(\rho_{AC_{k-1}(2\cdots m)}) \quad (17) \]
\[ \geq Q^\alpha(\rho_{AB}) + \sum_{k=1}^m Q^\alpha(\rho_{AC_k}) \quad (18) \]

It is remarked that above relations are also true for $r > 2$ when $Q'(\langle \psi \rangle_{AXY}) \geq Q'(\rho_{AX}) + Q'(\rho_{AY})$. Hence, we may define the hierarchical monogamy relations in following manner

\[ \delta Q^\alpha = Q^\alpha(\rho_{ABC}) - Q^\alpha(\rho_{AB}) - \sum_{j=1}^{k-1} Q^\alpha(\rho_{AC_j}) - Q^\alpha(\rho_{AC_{k-1}(2\cdots m)}) \quad (19) \]

**Theorem 7:** If $Q'(\rho_{AB}) \geq \sum_j Q'(\rho_{AB_j})$ then $Q^\alpha(\rho_{AB}) \geq \frac{1}{2^{n-1}} \sum_X Q^\alpha(\rho_{AX}) \geq \sum_j Q^\alpha(\rho_{AB_j})$ for $\alpha \geq r \geq 1$.

**Proof.** Let $B = \{B_1, B_2, \cdots, B_n\}$ be the set of sub-systems $B_j$’s, and $X = \{B_1, \cdots, B_n\}$ and $X^c = B - X$ be nonempty proper subsets of $B$. Thus $\rho_{AB} = \rho_{AXX^c}$. Applying monogamy inequality and Theorem 3, we get

\[ Q^\alpha(\rho_{AB}) \geq Q^\alpha(\rho_{AX}) + Q^\alpha(\rho_{AX^c}) \quad (20) \]

Since the set of all nonempty proper subsets of $B$ is same as the set of their complements, i.e., $\{X|X \subset B\} = \{X^c|X \subset B\}$, summing over all possible nonempty proper subsets $X$’s of $B$ leads to the following inequality,

\[ Q^\alpha(\rho_{AB}) \geq \frac{1}{2^{n-2}} \sum_X (Q^\alpha(\rho_{AX}) + Q^\alpha(\rho_{AX^c})) \]
\[ = \frac{1}{2^{n-1}} \sum_X Q^\alpha(\rho_{AX}). \quad (21) \]

We also have

\[ Q^\alpha(\rho_{AX}) + Q^\alpha(\rho_{AX^c}) \geq \sum_j Q^\alpha(\rho_{AB_j}). \quad (22) \]

Again summing over all possible nonempty proper subsets $X$’s of $B$, we obtain

\[ \frac{1}{2^{n-1}} \sum_X Q^\alpha(\rho_{AX}) \geq \sum_j Q^\alpha(\rho_{AB_j}). \quad (23) \]

Combining inequalities (21) and (23), we obtain the desired strong monogamy inequality for arbitrary multi-party quantum state $\rho_{AB_1B_2\cdots B_n}$.

Analogously, one can prove the following strong non-monogamy inequality using Theorem 4.

**Theorem 8:** If $Q'(\rho_{AB}) \leq \sum_j Q'(\rho_{AB_j})$ then $Q^\alpha(\rho_{AB}) \leq \frac{1}{2^{n-1}} \sum_X Q^\alpha(\rho_{AX}) \leq \sum_j Q^\alpha(\rho_{AB_j})$ for $\alpha \leq r$.

It was shown in Ref. [31] that entanglement of assistance [32] follows strong non-monogamy relation.

**IV. CONCLUSION**

We have proven that monogamy of a convex quantum correlation measure for multipartite pure states implies its monogamy for mixed states. This drastically simplifies the task of establishing the monogamy relations for mixed states. In particular, we have shown that the squared negativity is monogamous for multi-qubit mixed states. Further, we have derived various other monogamy relations under given conditions.

Our study partially answers the two conjectures in Ref. [18] in the sense that it now only remains to prove the monogamy of the squared entanglement of formation for pure states in arbitrary dimensions.

The monogamy relations discussed in this paper, by no means, exhaust all possible monogamy relations. Moreover, some monogamy relations may be specific to a particular quantum correlation measure. We believe that our study will be useful in quantum information theory.

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Appendix

Theorem: Theorem 1 is not, in general, true for \( r > 2 \).

Proof. Multinomial expansion is given by

\[
\left( \sum_i x_i \right)^r = \sum_{\{r_k\}} \prod_i x_i^{r_k} \tag{24}
\]

where \( \{r_k\} \) is the integer partition of \( r \) and the summation is over all permutations of such integer partitions of \( r \). Then, as in Theorem 1 (Case 2), we have

\[
\mathcal{Q}^r(\rho_{AB}) - \sum_j \mathcal{Q}^{r_j}(\rho_{AB_j}) = \left( \sum_i \mathcal{Q}_{ABi} - \sum_j \left( \sum_i \mathcal{Q}_{ABji} \right) \right)
\]

\[
= \sum_{\{r_k\}} \prod_i x_i^{r_i} \left( \prod_i \mathcal{Q}_{ABi}^{r_i} - \sum_j \prod_i \mathcal{Q}_{ABji}^{r_i} \right)
\]

\[
= \sum_i \left( \mathcal{Q}_{ABi} - \sum_j \mathcal{Q}_{ABji} \right)
\]

\[
+ \sum_{\{r_k\} \neq r} \prod_i x_i^{r_i} \left( \prod_i \mathcal{Q}_{ABi}^{r_i} - \sum_j \prod_i \mathcal{Q}_{ABji}^{r_i} \right). \tag{25}
\]

Although, in the third equality, the first term is non-negative due to the monogamy of \( \mathcal{Q}^r \) for pure states, we cannot say anything with certainty about the second term as we do not have Holder-type inequality for multivariables. However, the other way is always true, i.e., if \( \mathcal{Q}^r \) is monogamous for mixed states then it is certainly monogamous for pure states.