ON HOLOMORPHIC BANACH VECTOR BUNDLES OVER BANACH SPACES

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ABSTRACT. Let $X$ be a Banach space with a countable unconditional basis (e.g., $X = \ell_2$ Hilbert space), $\Omega \subset X$ pseudoconvex open, $E \to \Omega$ a locally trivial holomorphic vector bundle with a Banach space $Z$ for fiber type, $\mathcal{O}^E$ the sheaf of germs of holomorphic sections of $E \to \Omega$, and $Z_1$ the Banach space $Z_1 = \ell_p(Z) = \{z = (z_n) : z_n \in Z, \|z\| = (\sum_{n=1}^{\infty} \|z_n\|^p)^{1/p} < \infty\}$, $1 \leq p < \infty$. Then $E \oplus (\Omega \times Z_1)$ and $\Omega \times Z_1$ are holomorphically isomorphic, $\mathcal{O}^E$ is acyclic and $E$ is so to speak stably trivial over $\Omega$ in a generalized sense. We also show that if $E$ is continuously trivial over $\Omega$, then $E$ is holomorphically trivial over $\Omega$. In particular, if $Z = \ell_2$ or $\Omega$ is contractible, then $E$ is holomorphically trivial over $\Omega$. Some applications are also given.

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To my dear little daughter Sári Mangala on her third birthday.

1. INTRODUCTION.

In this paper we study holomorphic Banach vector bundles over pseudoconvex open subsets of Banach spaces. Under suitable conditions we show that any holomorphic Banach vector bundle can be exhibited as a direct summand of a trivial holomorphic Banach vector bundle. In this way we recover and substantially extend the following theorem of Lempert.

Theorem 1.1. (Lempert, [L3]) Let $X$ be a Banach space with a countable unconditional basis, $\Omega \subset X$ pseudoconvex open, $E \to \Omega$ a holomorphic Banach vector bundle, $\mathcal{O}^E \to \Omega$ the sheaf of germs of holomorphic sections of $E \to \Omega$. Then the sheaf cohomology groups $H^q(\Omega, \mathcal{O}^E)$ vanish for all $q \geq 1$.

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Following [L2] by Lempert we say that \textit{plurisubharmonic domination} holds in a complex Banach manifold \(\Omega\) if for every \(u: \Omega \to \mathbb{R}\) locally upper bounded there is a \(\psi: \Omega \to \mathbb{R}\) continuous and plurisubharmonic such that \(u(x) < \psi(x)\) for all \(x \in \Omega\).

**Theorem 1.2.** (Lempert, [L2]) If \(X, \Omega\) are as in Theorem 1.1, then plurisubharmonic domination holds in \(\Omega\).

We prove here the following Theorem 1.3.

**Theorem 1.3.** Let \(X\) be a Banach space with a Schauder basis, \(\Omega \subset X\) pseudoconvex open, \(E \to \Omega\) a holomorphic Banach vector bundle with a Banach space \(Z\) for fiber type. If plurisubharmonic domination holds in \(\Omega\), then we have the following.

(a) \(H^q(\Omega, \mathcal{O}^Z) = 0\) for \(q \geq 1\).

(b) Let \(Z_1 = \ell_p(Z), 1 \leq p < \infty\). Then \(E \oplus (\Omega \times Z_1)\) and \(\Omega \times Z_1\) are holomorphically isomorphic over \(\Omega\).

(c) \(H^q(\Omega, \mathcal{O}^E) = 0\) for \(q \geq 1\).

(d) If \(E\) is continuously trivial over \(\Omega\), then \(E\) is holomorphically trivial over \(\Omega\).

(e) If \(Z = \ell_2\), then \(E\) is holomorphically trivial over \(\Omega\).

(f) If \(\Omega\) is contractible, then \(E\) is holomorphically trivial over \(\Omega\).

The proof of Theorem 1.3 depends on an exhaustion procedure by Lempert, on a simple aspect of the Grauert–Oka principle in [P1], and on the contractibility of \(\text{GL}(\ell_2)\) by Kuiper in [K]. A few applications of Theorem 1.3 are given at the end of this paper. The methods and results here are further applied in [P4] on an analytic Koszul complex, and in [P5] on sheaf cohomology vanishing for a general class of sheaves that includes vector bundles and certain ideal sheaves. For background see [L1–L4, P1].

2. Exhaustion.

This section describes a way to exhaust a pseudoconvex open subset \(\Omega\) of a Banach space \(X\) that is convenient for proving vanishing results for sheaf cohomology over \(\Omega\). We follow here [L4, § 2].

We say that a function \(\alpha\), call their set \(\mathcal{A}'\), is an \textit{admissible radius function} on \(\Omega\) if \(\alpha: \Omega \to (0, 1)\) is continuous and \(\alpha(x) < \text{dist}(x, X \setminus \Omega)\) for \(x \in \Omega\). We say that a function \(\alpha\), call their set \(\mathcal{A}\), is an \textit{admissible Hartogs radius function} on \(\Omega\) if \(\alpha \in \mathcal{A}'\) and \(-\log \alpha\) is plurisubharmonic on \(\Omega\). Call \(\mathcal{A}\) \textit{cofinal} in \(\mathcal{A}'\) if for each \(\alpha \in \mathcal{A}'\) there is a \(\beta \in \mathcal{A}\) with \(\beta(x) < \alpha(x)\) for \(x \in \Omega\).

**Proposition 2.1.** Plurisubharmonic domination holds in \(\Omega\) if and only if \(\mathcal{A}\) is cofinal in \(\mathcal{A}'\).
Proof. Write $\alpha = e^{-u} \in A'$ and $\beta = e^{-\psi} \in A$. As plurisubharmonic domination holds on $\Omega$ for $u$ continuous if and only if for $u$ locally upper bounded, the proof of Proposition 3.1 is complete.

It will be often useful to look at coverings by balls $B_X(x, \alpha(x))$, $x \in \Omega$, $\alpha \in A'$, and shrink their radii to obtain a finer covering by balls $B_X(x, \beta(x))$, $x \in \Omega$, $\beta \in A$.

Let $e_n$, $n \geq 1$, be a Schauder basis in the Banach space $(X, \| \cdot \|)$. One can change the norm $\| \cdot \|$ to an equivalent norm so that $\| \sum_{i=m}^{n} x_i e_i \| \leq \| \sum_{i=M}^{n} x_i e_i \|$ for $0 \leq M \leq m \leq n \leq N \leq \infty$, $x_i \in \mathbb{C}$. Introduce the projections $\pi_N : X \to X$, $\pi_N \sum_{i=1}^{\infty} x_i e_i = \sum_{i=1}^{N} x_i e_i$, $x_i \in \mathbb{C}$, $\pi_0 = 0$, $\pi_\infty = 1$, $\varrho_N = 1 - \pi_N$, and define for $\alpha \in A$ and $N \geq 0$ integer the sets

\[
\begin{align*}
D_N(\alpha) &= \{ \xi \in \Omega \cap \pi_N X : (N + 1)\alpha(\xi) > 1 \}, \\
\Omega_N(\alpha) &= \{ x \in \pi_N^{-1} D_N(\alpha) : \varrho_N x \| < \alpha(\pi_N x) \}, \\
D^N(\alpha) &= \pi_{N+1} X \cap \Omega_N(\alpha), \\
\Omega^N(\alpha) &= \{ x \in \pi_{N+1}^{-1} D^N(\alpha) : \varrho_{N+1} x \| < \alpha(\pi_N x) \}, \\
\mathcal{B}(\alpha) &= \{ B_X(x, \alpha(x)) : x \in \Omega \}, \\
\mathcal{B}_N(\alpha) &= \{ B_X(x, \alpha(x)) : x \in \Omega_N(\alpha) \}.
\end{align*}
\]

(2.1)

These $\Omega_N(\alpha)$ are pseudoconvex open in $\Omega$, and they will serve to exhaust $\Omega$ as $N = 0, 1, 2, \ldots$ varies.

**Proposition 2.2.** (Lempert) Let $\alpha \in A$, and suppose that plurisubharmonic domination holds in $\Omega$.

(a) There is an $\alpha' \in A$, $\alpha' < \alpha$, with $\Omega_n(\alpha') \subset \Omega_N(\alpha)$ for all $N \geq n$. So any $x_0 \in \Omega$ has a neighborhood contained in all but finitely many $\Omega_N(\alpha)$.

(b) There are $\beta, \gamma \in A$, $\gamma < \beta < \alpha$, so that for all $N$ and $x \in \Omega_N(\gamma)$

\[
B_X(x, \gamma(x)) \subset \Omega_N(\beta) \cap \pi_N^{-1} B_X(\pi_N x, \beta(x)) \subset B_X(x, \alpha(x)).
\]

(2.2)

(c) If $8\alpha \in A$, $Y \subset X$ is a finite dimensional complex affine subspace, then $Y \cap \overline{\Omega_N(\alpha)}$ is plurisubharmonically convex in $Y \cap \Omega$.

(d) We have that $\Omega_N(\alpha) \subset \Omega^N(\alpha)$. If $4\alpha \in A$, then $\Omega^N(\alpha) \subset \Omega_N(2\alpha)$.

(e) There is a $\beta \in A$, $\beta < \alpha$, with $\Omega_N(\beta) \subset \Omega_N(\alpha) \cap \Omega_{N+1}(\alpha)$ for $N \geq 0$.

(f) There is an $\alpha' \in A$, $\alpha' < \alpha$, such that the covering $\mathcal{B}_N(\alpha)|\Omega_N(\alpha')$ has a finite basic refinement for all $N \geq 0$.

Proof. For (a) and (b) see [L4, Prop. 2.1], and [L3, Prop. 4.3], for (c) [L3, Prop. 4.3], for (d) [L3, Prop. 4.4], for (e) [L4, Prop. 2.3], and for (f) see [P4, Prop. 3.2(c)]. The proof of Proposition 2.2 is complete. (Remark for the record that (f) was not explicitly formulated by Lempert.)
The meaning of Proposition 2.2(b) is that certain refinement maps exist between certain open coverings, while (cd) are useful for Runge type approximation, and (ef) for exhaustion.

3. APPROXIMATION.

This section deals with some versions of Runge approximation.

**Proposition 3.1.** Let $X, Z$ be Banach spaces, $f \in \mathcal{O}(B_X(1), Z)$ bounded, $0 < \theta < 1$, $\epsilon > 0$. Then there is a polynomial $g \in \mathcal{O}(X, Z)$ with $\|f(x) - g(x)\| < \epsilon$ for $\|x\| < \theta$.

**Proof.** As well known we may take $g = \sum_{m=0}^{n} f_m$ for $n$ large enough, where $\sum_{m=0}^{\infty} f_m$ is the homogeneous expansion of $f$ about $x = 0$.

Let $Y, Z$ be Banach spaces, $D_1 \subset D_2 \subset D_3 \subset \subset \mathbb{C}^N$ pseudoconvex open, $R : D_3 \to (0, \infty)$ continuous with $-\log R$ plurisubharmonic on $D_3$. Let

\[(3.1) \quad \Omega(D_3, R) = \{(\zeta, y) \in D_3 \times Y : \|y\| < R(\zeta)\}.

**Proposition 3.2.** If $\overline{D_2}$ is holomorphically convex in $D_3$, $f \in \mathcal{O}(D_2, R, Z)$ is bounded and uniformly continuous, $0 < \theta < 1$, and $\epsilon > 0$, then there is a $g \in \mathcal{O}(D_3 \times Y, Z)$ that is bounded and uniformly continuous on any set of the form $K \times B_Y(r)$, where $0 < r < \infty$ and $K \subset \subset D_3$, and $\|f(\zeta, y) - g(\zeta, y)\| < \epsilon$ for $(\zeta, y) \in \Omega(D_1, \theta R)$.

**Proof.** See the proof of [L1, Thm. 6.1] and replace the reference there to Hypothesis($X, F$) by reference to Proposition 3.1 here.

Let $X$ be as in §2.

**Proposition 3.3.** Let $8 \alpha \in A$, and choose $\gamma \in A$ as in Proposition 2.2(b), and $\gamma' \in A$ as in Proposition 2.2(a). Let $f \in \mathcal{O}(\Omega_N(\alpha), Z)$ be bounded and uniformly continuous, and $\epsilon > 0$.

(a) There is a $g \in \mathcal{O}(\Omega_{N+1}(\alpha), Z)$ bounded and uniformly continuous with $\|f(x) - g(x)\| < \epsilon$ for $x \in \Omega_N(\gamma)$.

(b) There is a $g \in \mathcal{O}(\Omega, Z)$ that is bounded and uniformly continuous on $\Omega_{N+p}(\gamma')$ for all $p \geq 0$ with $\|f(x) - g(x)\| < \epsilon$ for $x \in \Omega_N(\gamma')$.

**Proof.** (a) Apply Proposition 2.2 and Proposition 3.2 as in the proof of [L3, Thm. 4.5].

(b) Letting $g_0 = f$, apply (a) repeatedly to find bounded and uniformly continuous functions $g_p \in \mathcal{O}(\Omega_{N+p}(\alpha), Z)$ for $p \geq 1$ with $\|g_p(x) - g_{p-1}(x)\| < \epsilon/2^p+1$ for $x \in \Omega_{N+p}(\gamma')$. Letting $g = g_0 + \sum_{p=1}^{\infty} (g_p - g_{p-1}) = \lim_{p \to \infty} g_p$ completes the proof of Proposition 3.3.

4. VANISHING IN THE MIDRANGE.
In this section we show that certain cocycles can be resolved.

Let $D \subset D' \subset \subset C^N$ be pseudoconvex open, $\mathcal{U}$ a covering of $D$ by open sets $U \subset D$. For $U \in \mathcal{U}$ let $U'$ be pseudoconvex open with $\overline{U} \subset U' \subset D'$ and assume that $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$ covers $D'$. Let $Y$ be a Banach space, $\pi : C^N \times Y \to C^N \times Y$ the projection $\pi(\xi, y) = (\xi, 0)$, $R : D' \to (0, \infty)$ be continuous and bounded away from zero and $-\log R$ plurisubharmonic, $0 < \theta < 1$, $\Omega' = \Omega(D', R)$, $\Omega = \Omega(D, \theta R)$ as in (3.1), $\mathcal{U}'(\Omega') = \{U'(\Omega') = \pi^{-1}(U') \cap \Omega' : U' \in \mathcal{U}'\}$, $\mathcal{U}(\Omega) = \{U(\Omega) = \pi^{-1}(U) \cap \Omega : U \in \mathcal{U}\}$.

We say that a cochain $f \in C^q(\mathcal{U}, \mathcal{O}Z)$, $q \geq 0$, is bounded and uniformly continuous if each of its components $f_\sigma$ is bounded and uniformly continuous on the body $|\sigma|$ of any $q$-simplex $\sigma$ of $\mathcal{U}$. This is most useful when the covering $\mathcal{U}$ is finite or has a finite refinement.

**Proposition 4.1.** For any bounded and uniformly continuous cocycle $f \in Z^q(\mathcal{U}'(\Omega'), \mathcal{O}Z)$, $q \geq 1$, there is a bounded and uniformly continuous cochain $g \in C^{q-1}(\mathcal{U}(\Omega), \mathcal{O}Z)$ with $f|\mathcal{U}(\Omega) = \delta g$.

**Proof.** This can be done using a smooth partition of unity on $D'$ and an integral operator solving a $\delta$-equation on a smooth strictly pseudoconvex complete Hartogs domain. See [P1, Prop. 5.1, Prop. 7.1].

### 5. INVERTIBLE MATRICES.

This section describes some properties of holomorphic functions with values in invertible matrices and invertible linear operators on a Banach space.

Let $Z$ be a Banach space, $\text{End}(Z) = \text{Hom}(Z, Z)$ the Banach algebra of all linear operators $A : Z \to Z$ endowed with the operator norm $\|A\|$, $G = \text{GL}(Z)$ the Banach Lie group of units of $\text{End}(Z)$, and $\hat{G} = \text{End}(Z)$ the Banach Lie algebra of $G$. Here $G$ is a pseudoconvex open subset of $\text{End}(Z)$, and we have a holomorphic map, called the exponential map, $\exp : \hat{G} \to G$ defined by $\exp(\xi) = \sum_{n=0}^{\infty} \xi^n/n!$, which is biholomorphic from a small ball $B_{\hat{G}}(\xi_0)$ of $\hat{G}$ to an open neighborhood of $1 \in G$, whose inverse is called logarithm and is denoted by log. In this case exp may equally be defined by $\exp(\xi) = \lim_{n \to -\infty} (1 + \frac{\xi}{n})^n$.

Let $(\Omega, d)$ be a metric space, and $f \in C(\Omega, G)$ a continuous function. If both $\|f(x)\|$ and $\|f(x)^{-1}\|$ are bounded for $x \in \Omega$, then we say that $f$ is bounded on $\Omega$, and write that $f \in C_b(\Omega, G)$. If for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|f(x)f(y)^{-1} - 1\| < \varepsilon$ for all $x, y \in \Omega$ with $d(x, y) < \delta$, then we say that $f$ is uniformly continuous on $\Omega$, and write that $f \in C_c(\Omega, G)$. If $f$ is both bounded and uniformly continuous on $\Omega$, then we write that $f \in C_{bu}(\Omega, G)$. If $\Omega$ is a complex Banach manifold with a fixed metric $d$ that induces the topology of $\Omega$, then we endow $\Omega \times [0, 1]$ with the metric
Let $\Omega$ be a domain in $\mathbb{C}$, let $G$ be a topological group, and let $f, g : \Omega \to G$ be holomorphic functions. We say that $f$ is a simple function on $\Omega$ if $f(\Omega)$ is a simple group.

A simple function $h : \Omega \to G$ is just a holomorphic function $f \in \mathcal{O}(\Omega, Z)$ that is bounded and uniformly continuous on $\Omega$ (null homotopy is automatic by taking $tf(x)$), and $h$ is just a homotopy $h \in C_{bu}(\Omega \times [0,1], G)$ such that $h(x, 1) = f(x)$ and $h(x, 0) = 1$ for all $x \in \Omega$. If $H$ is a topological group, then let $C_0([0,1], H)$ be the set of all continuous functions $f : [0,1] \to H$ with $f(0) = 1 \in H$. All that we state below in this section for $G = GL(Z)$ can easily and quite analogously be extended to the Banach Lie group $G = C_0([0,1], GL(Z))$, the path group of $GL(Z)$.

**Proposition 5.1.** (a) If $f, g \in \mathcal{O}_s(\Omega, G)$ are simple functions, then so are $fg \in \mathcal{O}_s(\Omega, G)$ and $f^{-1} \in \mathcal{O}_s(\Omega, G)$.

(b) If $f_n \in \mathcal{O}_s(\Omega, G)$ are simple functions on $\Omega$, and $f_n \to f$ uniformly on $\Omega$, i.e., $\lim_{n \to \infty} \sup_{x \in \Omega} \|f(x)f_n(x)^{-1} - 1\| = 0$, then their limit $f \in \mathcal{O}_s(\Omega, G)$ is also a simple function on $\Omega$.

(c) If $f \in \mathcal{O}(\Omega, G)$ is such that there is a homotopy $h \in C_{bu}(\Omega \times [0,1], G)$ with $h(x,t)$ holomorphic in $x \in \Omega$ for each $t \in [0,1]$, and $h(x,0) = 1$ and $h(x,1) = f(x)$ for $x \in \Omega$, then there are finitely many functions $g_1, \ldots, g_n \in \mathcal{O}_s(\Omega, \hat{G})$ such that $f(x) = \exp(g_1(x)) \ldots \exp(g_n(x))$ for all $x \in \Omega$.

(d) If $g \in \mathcal{O}_s^1(\Omega, \hat{G})$, then the solution $h \in \mathcal{O}_s^1(\Omega, G)$ of the parametric initial value problem $\frac{d}{dt}h(x,t) = g(x,t)h(x,t), h(x,0) = 1$ for the linear differential equation for the left logarithmic derivative is a simple $(h_1)$-function. Thus $f(x) = h(x,1)$ defines a simple function $f \in \mathcal{O}_s(\Omega, G)$.

**Proof.** As this proposition expresses some well known simple facts only, we just indicate the arguments.

(a) Boundedness and null homotopy are clear. Uniform continuity for $fg$ follows on writing $f(x)g(x)(f(y)g(y))^{-1} = [f(x)f(y)]^{-1}[g(x)g(y)^{-1}]f(y)$ from considering conjugation and the adjoint action of $G$ on $\hat{G}$, where $a^b = bab^{-1}$.

(b) The limit is clearly bounded and uniformly continuous. To see that it is also null homotopic note that $\|f(x)f_n(x)^{-1} - 1\| < 1/2$ for $x \in \Omega$ if $n$ is large enough. Fixing one such $n$ we can write $f(x)f_n(x)^{-1} = \exp(g(x))$, or $f(x) = \exp(g(x))f_n(x)$, from which $f$ is easily seen null homotopic.

(c) To write $f$ as a finite product of exponentials of simple functions we
consider the telescopic product
\[
f(x) = h(x,0)^{-1}h(x,1) = [h(x,0)^{-1}h(x,\frac{1}{n})][h(x,\frac{1}{n})^{-1}h(x,\frac{2}{n})] \ldots [h(x,\frac{n-1}{n})^{-1}h(x,1)],
\]
and take logarithm of \( h(x,\frac{i-1}{n})^{-1}h(x,\frac{i}{n}) \), \( i = 1, \ldots, n \), which is uniformly small if \( n \) is large enough.

(d) Our \( h \) is as claimed since we can write \( h(x,t) \) as a so-called product integral, which is a uniform limit as in (b) of products of finitely many simple functions. These products make a multiplicative analog of the ordinary Euler polygon method in linear ordinary differential equations. The proof of Proposition 5.1 is complete.

For the rest of this section see [P1, §§ 7-9], which has all that we need but in a slightly different form. We turn now to Runge approximation for simple functions. Resume the notation and assumptions of §3.

**Proposition 5.2.** If \( \overline{D_2} \) is holomorphically convex in \( D_3 \), \( f \in \mathcal{O}_s(\Omega(D_2,R),G) \) is a simple function, \( 0 < \theta < 1 \), and \( \varepsilon > 0 \), then there is a \( g \in \mathcal{O}(D_3 \times Y,G) \) that is homotopic through holomorphic maps \( D_3 \times Y \rightarrow G \) to 1 and is bounded and uniformly continuous on any set of the form \( K \times BY(r) \), where \( 0 < r < \infty \) and \( K \subset D_3 \), and \( \|f(\zeta,y)g(\zeta,y)^{-1} - 1\| < \varepsilon \) for \( (\zeta,y) \in \Omega(D_1,\theta R) \).

**Proof.** Define \( f' \in \mathcal{O}_s(\Omega(D_2,R),G) \) and \( f'' \in \mathcal{O}_s(D_2,G) \) by writing \( f(\zeta,y) = f'(\zeta,y)f''(\zeta) \), where \( f''(\zeta) = f(\zeta,0) \), so \( f'(\zeta,0) = 1 \). Looking at \( f'(\zeta,ty) \), \( t \in [0,1] \), Proposition 5.1(c) gives \( f'_i \in \mathcal{O}_s(\Omega(D_2,R),\hat{G}) \) with \( f'_i(\zeta,y) = \exp(f'_i(\zeta,y)) \ldots \exp(f'_n(\zeta,y)) \). After possibly shrinking \( D_2 \) arbitrarily slightly Grauert’s theorem [C, Théorème principal (ii)] yields \( f''_i \in \mathcal{O}_s(D_2,G) \) with \( f''(\zeta) = \exp(f''_1(\zeta)) \ldots \exp(f''_n(\zeta)) \). Let \( \eta > 0 \). Apply Proposition 3.2 to find functions \( g'_i \in \mathcal{O}(D_3 \times Y,\hat{G}) \) that approximate \( f'_i \) so that \( \|f'_i(\zeta,y) - g'_i(\zeta,y)\| < \eta \) for \( (\zeta,y) \in \Omega(D_1,\theta R) \). After possibly shrinking \( D_3 \) arbitrarily slightly Theorem A of Stein theory provides functions \( g''_j \in \mathcal{O}_s(D_3,\hat{G}) \) with \( \|f''_j(\zeta) - g''_j(\zeta)\| < \eta \) for \( \zeta \in D_1 \).

Letting \( g(\zeta,y) = \exp(g'_1(\zeta,y)) \ldots \exp(g'_m(\zeta,y)) \exp(g''_1(\zeta)) \ldots \exp(g''_n(\zeta)) \) will do if \( \eta > 0 \) is small enough. The proof of Proposition 5.2 is complete.

**Proposition 5.3.** Let \( 8\alpha \in A \), and choose \( \gamma \in A \) as in Proposition 2.2(b), and \( \gamma' \in A \) as in Proposition 2.2(a). Let \( f \in \mathcal{O}_s(\Omega_N(\alpha),G) \) be a simple function, and \( \varepsilon > 0 \).

(a) There is a simple function \( g \in \mathcal{O}_s(\Omega_{N+1}(\alpha),G) \) with \( \|f(x)g(x)^{-1} - 1\| < \varepsilon \) for \( x \in \Omega_{N}(\gamma) \).

(b) There is a \( g \in \mathcal{O}(\Omega,G) \) that is simple on \( \Omega_{N+p}(\gamma') \) for all \( p \geq 0 \) with \( \|f(x)g(x)^{-1} - 1\| < \varepsilon \) for \( x \in \Omega_{N}(\gamma') \).
Proof. (a) Apply Proposition 2.2 and Proposition 5.2 as in the proof of [L3, Thm. 4.5].

(b) Letting \( g_0 = f \), apply (a) repeatedly to find for an \( \eta > 0 \) simple functions \( g_p \in \mathcal{O}_s(\Omega_{N+p}(\alpha), G) \) for \( p \geq 1 \) with \( \| g_{p-1}(x)g_p(x)^{-1} - 1 \| < \eta/2^p+1 \) for \( x \in \Omega_{N+p}(\gamma) \). Setting \( g = \prod (g_p g_{p-1}) \cdots (g_1 g_0) \) completes the proof of Proposition 5.3 if \( \eta > 0 \) is small enough.

Now we look at cohomology vanishing for simple functions.

Let \( Y \) be a Banach space, \( D \subset D' \subset D'' \subset \mathbb{C}^N \) pseudoconvex open with \( \overline{D} \) holomorphically convex in \( D' \) and \( \overline{D'} \) holomorphically convex in \( D'' \), \( R : D'' \to (0, \infty) \) continuous with \( -\log R \) plurisubharmonic, \( 1/2 < \theta < 1 \), \( a < a' < b' < b \) reals, \( h \in \mathcal{O}(D') \), \( D'_{ab} = D' \cap \{ a < Re h < b \} \), \( D_1 = D \cap \{ Re h < b' \} \), \( D_2 = D \cap \{ a' < Re h \} \), \( \Omega^\kappa = \Omega(D^\kappa, \theta R) \) for \( \kappa = 1, 2 \) as in (3.1), \( \Omega^* = \Omega(D'_{ab}, R) \), \( \widehat{\omega} \subset \subset \mathbb{C}^{N+1} \) a \( C^\infty \)-smooth strictly smooth complete Hartogs domain with \( \{(\zeta, \lambda) \in D \times \mathbb{C} : |\lambda| < \theta R(\zeta)\} \subset \subset \widehat{\omega} \subset \subset \{(\zeta, \lambda) \in D' \times \mathbb{C} : |\lambda| < R(\zeta)\} \), \( \widehat{\omega}^1 = \{(\zeta, \lambda) \in \widehat{\omega} : Re h < b'\} \), \( \widehat{\omega}^2 = \{(\zeta, \lambda) \in \widehat{\omega} : a' < Re h\} \), \( \widehat{\Omega} = \{(\zeta, y) \in \mathbb{C}^N \times Y : (\zeta, \|y\|) \in \widehat{\omega}\} \), \( \widehat{\Omega}^\kappa = \{(\zeta, y) \in \mathbb{C}^N \times Y : (\zeta, \|y\|) \in \widehat{\omega}^\kappa\} \), \( \kappa = 1, 2 \). Let \( Z \) be a Banach space, \( G = GL(Z) \). Then \( \mathcal{O}_s(\widehat{\Omega}, Z) \) is a Banach space with the sup norm, and \( \mathcal{O}_s(\widehat{\Omega}, G) \) is a Banach Lie group with Banach Lie algebra \( \mathcal{O}_s(\widehat{\Omega}, G) \).

**Proposition 5.4.** There are bounded linear operators \( F_\kappa : \mathcal{O}_s(\widehat{\Omega}^1 \cup \widehat{\Omega}^2, Z) \to \mathcal{O}_s(\widehat{\Omega}^\kappa, Z) \), \( \kappa = 1, 2 \), with \( f = F_1(f) + F_2(f) \) for \( f \in \mathcal{O}_s(\widehat{\Omega}^1 \cap \widehat{\Omega}^2, Z) \).

**Proof.** See the proof of [P1, Prop. 7.1].

Note that part of Proposition 5.4 can be reformulated as saying that the Mayer–Vietoris sequence

\[
(5.1) \quad 0 \to \mathcal{O}_s(\widehat{\Omega}^1 \cup \widehat{\Omega}^2) \xrightarrow{r} \mathcal{O}_s(\widehat{\Omega}^1) \times \mathcal{O}_s(\widehat{\Omega}^2) \xrightarrow{a} \mathcal{O}_s(\widehat{\Omega}^1 \cap \widehat{\Omega}^2) \to 0
\]

is a split exact sequence of Banach spaces, where \( \mathcal{O}_s(\cdot) = \mathcal{O}_s(\cdot, Z) \), \( r(\Phi) = (\Phi|\widehat{\Omega}^1, -\Phi|\widehat{\Omega}^2) \), and \( a(f_1, f_2) = f_1 + f_2 \).

**Proposition 5.5.** There are an open neighborhood \( \mathcal{N} \) of \( 1 \) in \( \mathcal{O}_s(\widehat{\Omega}^1 \cap \widehat{\Omega}^2, G) \) and holomorphic maps \( F_\kappa : \mathcal{N} \to \mathcal{O}_s(\widehat{\Omega}^\kappa, G) \), \( \kappa = 1, 2 \), with \( F_\kappa(1) = 1 \), and \( f = F_1(f)F_2(f) \) for \( f \in \mathcal{N} \).

**Proof.** Apply Proposition 5.4 as in the proof of [P1, Prop. 7.2].

**Proposition 5.6.** There is an \( \epsilon_0 > 0 \) such that if \( f \in \mathcal{O}_s(\Omega^\ast, G) \) satisfies that \( \| f(x) - 1 \| < \epsilon_0 \) for \( x \in \Omega^\ast \), then there are \( f_\kappa \in \mathcal{O}_s(\Omega^\kappa, G) \), \( \kappa = 1, 2 \), with \( f = f_1f_2^{-1} \) on \( \Omega^1 \cap \Omega^2 \).

**Proof.** Apply Proposition 5.5.
Proposition 5.7. If \( \lambda \in \mathcal{O}_s(\Omega^*, G) \), \( g \in \mathcal{O}_s(\Omega^*, Z) \), then there are \( g_\kappa \in \mathcal{O}_s(\Omega^\kappa, Z) \) with \( g(x) = g_1(x) - \lambda(x)g_2(x) \) for \( x \in \Omega^1 \cap \Omega^2 \).

Proof. If \( \sup_{x \in \Omega^*} \| \lambda(x) - 1 \| < \varepsilon_0 \) for an \( \varepsilon_0 > 0 \) small enough, then Proposition 5.6 gives \( \lambda_\kappa \in \mathcal{O}_s(\Omega^\kappa, G) \) with \( \lambda = \lambda_1 \lambda_2^{-1} \). We seek \( g_\kappa \) in the form \( g_\kappa = \lambda_\kappa h_\kappa \), and let \( h = \lambda_1^{-1}g \). Proposition 4.1 gives \( h_\kappa \in \mathcal{O}_s(\Omega^\kappa, Z) \) with \( h = h_1 - h_2 \), completing the proof in the case \( \lambda \approx 1 \).

If \( \lambda \) is arbitrary, then we reduce to the case of \( \lambda \approx 1 \) treated above. Proposition 5.2 gives a \( \Lambda \in \mathcal{O}_s(D'' \times B_Y(2 \sup_D, R), G) \) such that \( \| \lambda(x)\Lambda(x)^{-1} - 1 \| < \varepsilon_0 \) for \( x \in \Omega^* \). We seek \( g_1, g_2 \) in the form \( g_1 = h_1, g_2 = \Lambda^{-1}h_2 \). As the Cousin problem \( g(x) = h_1(x) - \lambda(x)\Lambda(x)^{-1}h_2(x) \) can be solved by the already proved case \( \lambda \Lambda^{-1} \approx 1 \) above, the proof of Proposition 5.7 is complete.

Proposition 5.8. If \( f \in \mathcal{O}_s(\Omega^*, G) \), then there are \( f_\kappa \in \mathcal{O}_s(\Omega^\kappa, G), \kappa = 1, 2 \), with \( f = f_1f_2^{-1} \).

Proof. As in the proof of Proposition 5.2 we see that there is a homotopy \( h \in \mathcal{O}_{bu}(\Omega^* \times B_C(2), G) \) with with \( h(x, 1) = f(x) \) and \( h(x, 0) = 1 \) for \( x \in \Omega^* \). We seek to define \( f_\kappa(x) \) by \( f_\kappa(x) = h_\kappa(x, 1) \), where \( h_\kappa \in \mathcal{O}_{bu}(\Omega^\kappa \times B_C(2\theta), G) \) are to be chosen to satisfy

\[
(5.2) \quad h_1(x, t) = h(x, t)h_2(x, t).
\]

We will obtain \( h_\kappa \) from its logarithmic derivative \( \hat{h}_\kappa = h_\kappa t h_\kappa^{-1} \) via the parameteric initial value problem for ordinary differential equations

\[
(5.3) \quad \begin{cases}
\frac{d}{dt} h_\kappa(x, t) = \hat{h}_\kappa(x, t)h_\kappa(x, t) \\
h_\kappa(x, 0) = 1
\end{cases}
\]

Let \( \hat{h} = h_t h^{-1} \in \mathcal{O}_{bu}(\Omega^* \times B_C(2\theta), G) \) be the left logarithmic derivative of \( h \). The relation (5.2) follows by integration of ODEs from (5.3) together with the relation

\[
(5.4) \quad \hat{h}_1 = \hat{h} + \hat{h}_2 h^{-1},
\]

which is obtained from (5.2) by logarithmic differentiation. The equation (5.4) is a Cousin problem

\[
(5.5) \quad \dot{h} = \dot{h}_1 - \lambda \dot{h}_2,
\]

where \( \lambda = \text{Ad}(h) \in \mathcal{O}_{bu}(\Omega^* \times B_C(2), \text{GL}(G)) \). As (5.5) can be solved by Proposition 5.7 the proof of Proposition 5.8 is complete.

Proposition 5.9. If \( f \in \mathcal{O}_s^1(\Omega^*, G) \), then there are \( f_\kappa \in \mathcal{O}_s^1(\Omega^\kappa, G) \) solving \( f = f_1f_2^{-1} \) on \( (\Omega^1 \cap \Omega^2) \times [0, 1] \).
Proof. Grauert’s theorem [C, Théorème principal (iii)] furnishes $g_\kappa \in \mathcal{O}_s^1(D^\kappa, G)$ solving $f(\zeta, 0, t) = g_1(\zeta, t)g_2(\zeta, t)^{-1}$. Defining $f' \in \mathcal{O}_s^1(\Omega^*, G)$ by $f'(\zeta, y, t) = g_1(\zeta, t)^{-1}f(\zeta, y, t)g_2(\zeta, t)$ we see that $f'$ satisfies $f'(\zeta, 0, t) = 1$. Proposition 5.8 gives $f'_\kappa \in \mathcal{O}_s(\Omega^*, G)$ with $f'_\kappa(\zeta, 0) = 1$ solving $f'_\kappa(\zeta, y, 1) = f'_1(\zeta, y)f'_2(\zeta, y)^{-1}$. Define $f''(\zeta, y, t) = f_1(\zeta, ty)^{-1}f'(\zeta, y, t)f_2(\zeta, ty)$. Then $f''(\zeta, y, t) = 1$ for $t \in \{0, 1\}$. Let $f''_\kappa(\zeta, y, t) = f''(\zeta, \chi(\text{Re} h(\zeta))y, t)$, $f''_\kappa = 1$, where $\chi \in C^\infty(\mathbb{R}, [0, 1])$ is a smooth cutoff function that equals 1 on $[a', b']$, and 0 on $\mathbb{R} \setminus [a, b]$. Then setting $f_\kappa = g_\kappa f'_\kappa f''_\kappa \in \mathcal{O}_s^1(\Omega^*, G)$ completes the proof of Proposition 5.9.

6. VANISHING FOR A TRIVIAL BUNDLE.

In this section we complete the proof of Theorem 1.3(a). Resume the notation and hypotheses of Theorem 1.3.

Let $\mathcal{U}$ be an open covering of $\Omega$. We say that a cochain $f \in C^q(\mathcal{U}, \mathcal{O}^Z)$, $q \geq 1$, is simple and write $f \in C^q_s(\mathcal{U}, \mathcal{O}^Z)$ if its components $f_\sigma \in \mathcal{O}_s(\sigma, Z)$ are simple over the body $|\sigma|$ of any $q$-simplex $\sigma$ of $\mathcal{U}$. The coboundaries and finite sums of simple cochains are simple cochains. If $f \in C^q_s(\mathcal{U}, \mathcal{O}^Z)$, and $\delta f = 0$, then we call $f$ a simple cocycle, and write that $f \in Z^q_s(\mathcal{U}, \mathcal{O}^Z)$. Simple cocycles $Z^q_s(\mathcal{U}, \mathcal{O}^Z)$, $q \geq 1$, modulo simple coboundaries $\delta Z^{q-1}_s(\mathcal{U}, \mathcal{O}^Z)$ make up a group $H^q_s(\mathcal{U}, \mathcal{O}^Z)$. If an open covering $\mathcal{V}$ of $\Omega$ is any refinement of $\mathcal{U}$, and $f \in C^q_s(\mathcal{U}, \mathcal{O}^Z)$ is any simple cocycle of $\mathcal{U}$, then the image of $f$ under any refinement map from $\mathcal{V}$ to $\mathcal{U}$ is a simple cocycle of $\mathcal{V}$ in $C^q_s(\mathcal{V}, \mathcal{O}^Z)$. As $\mathcal{U}$ gets ever finer it is possible to take the direct limit of the $H^q_s(\mathcal{U}, \mathcal{O}^Z)$ to get a limit $H^q_s(\Omega, \mathcal{O}^Z)$. We claim that $H^q_s(\Omega, \mathcal{O}^Z) \cong H^q(\Omega, \mathcal{O}^Z)$, $q \geq 1$, are naturally isomorphic. To that end it is enough to show that any cocycle $f \in C^q(\mathcal{U}, \mathcal{O}^Z)$ can be represented by a simple cocycle $g \in C^q_s(\mathcal{U}, \mathcal{O}^Z)$. This can be seen as follows. Refine $\mathcal{U}$ so as to be locally finite ($\Omega$ is a metric space, hence it is paracompact). Then choose for each point $x \in \Omega$ a ball $V_x = B_X(x, \alpha(x)) \subset \Omega$ so small that if $\sigma$ is any of the finitely many $q$-simplices of $\mathcal{U}$ such that $|\sigma|$ intersects $V_x$, then $f_\sigma$ and its Fréchet differential $df_\sigma$ are bounded on $V_x$. Let $\mathcal{V} = \{V_x : x \in \Omega\}$, and $g$ the image of $f$ under any refinement map from $\mathcal{V}$ to $\mathcal{U}$. Then $\mathcal{V}$ and $g$ will do the job. The most useful case of simple cocycles $Z^q_s(\mathcal{U}, \mathcal{O}^Z)$ is that of $q = 1$, where approximation is necessary, and the extra regularity is welcome. If a covering $\mathcal{U}$ of $\Omega = \Omega(D, R)$ as in (3.1) has a finite basic refinement $\mathcal{U}(\Omega)$ as in §4, and $f \in Z^q_s(\mathcal{U}, \mathcal{O}^Z)$ is a simple 0-cocycle, then $f$ determines a unique simple function $f \in \mathcal{O}_s(\Omega, Z)$. This is due to the precompactness of $D$.

The above has a multiplicative version, too, for $G = GL(Z)$. Let $\mathcal{O}_1^G \to \Omega$ be the sheaf of germs of $(h_1)$-functions $f : \Omega \times [0, 1] \to G$, i.e., let $U \subset \Omega$ be open, and define $\mathcal{O}_1^G(U) = \mathcal{O}_1(U, G)$ as the set of all functions $f \in C(U \times [0, 1], G)$ with $f(x, 0) = 1$ and $f(x, 1)$ holomorphic for $x \in U$. We
say that a cochain \( f \in C^q(\mathcal{U}, \mathcal{O}^G_1) \), \( q = 0, 1 \), is **simple** and write that \( f \in C^q_s(\mathcal{U}, \mathcal{O}^G_1) \) if its components \( f_\sigma \in \mathcal{O}^1_s(\sigma, G) \) are simple \((h_1)\)-functions over the body \(|\sigma|\) of any \(q\)-simplex \(\sigma\) of \(\mathcal{U}\). The coboundaries \( f_{UV} f_{VW} f_{WU} \), \( f_{UV} f_{VW} f_{WU} \), and finite products and inverses of simple cochains are simple cochains. If \( f = (f_{UV}) \in C^1_s(\mathcal{U}, \mathcal{O}^G_1) \) and \( \delta f = (f_{UV} f_{VW} f_{WU}) = 1 \), then we call \( f \) a simple cocycle, and write that \( f \in Z^1_s(\mathcal{U}, \mathcal{O}^G_1) \). Between two simple cochains \( f = (f_{UV}), \ g = (g_{UV}) \in C^1_s(\mathcal{U}, \mathcal{O}^G_1) \) there is an equivalence relation \( f \sim g \) defined by making \( f \sim g \) if and only if there is a simple cochain \( c \in C^0_s(\mathcal{U}, \mathcal{O}^G_1) \) with \( f_{UV} = c_{UV}^{-1} g_{UV} c_V \). Simple cocycles \( Z^1_s(\mathcal{U}, \mathcal{O}^G_1) \) modulo this equivalence relation \( \sim \) make up a simple cohomology set \( H^1_s(\mathcal{U}, \mathcal{O}^G_1) \) with a distinguished element 1, which is just the class of the trivial cocycle 1. If an open covering \( \mathcal{W} \) of \( \Omega \) is any refinement of \( \mathcal{U} \), and \( f \in C^q_s(\mathcal{U}, \mathcal{O}^G_1) \) is any simple cochain, then the image of \( f \) under any refinement map \( \mathcal{W} \to \mathcal{U} \) is a simple cochain of \( \mathcal{W} \) in \( C^q_s(\mathcal{W}, \mathcal{O}^G_1) \), \( q = 0, 1 \). As \( \mathcal{U} \) gets ever finer it is possible to take the direct limit of the \( H^*_s(\mathcal{U}, \mathcal{O}^G_1) \) to get the simple cohomology group \( H^*_s(\Omega, \mathcal{O}^G_1) \) of \( \Omega \) with respect to the sheaf \( \mathcal{O}^G_1 \). If a covering \( \mathcal{U} \) of \( \Omega = \Omega(D, R) \) as in (3.1) has a finite basic refinement \( \mathcal{W}(\Omega) \) as in § 4, and \( f \in Z^0_s(\mathcal{U}, \mathcal{O}^G_1) \) is a simple 0-cocycle, then \( f \) determines a unique simple \((h_1)\)-function \( f \in \mathcal{O}^1_s(\Omega, G) \). This is due to the precompactness of \( D \).

**Proposition 6.1.** Let \( X \) be a Banach space with a Schauder basis, \( \Omega \subset X \) pseudoconvex open, and suppose that plurisubharmonic domination holds in \( \Omega \). Then for any \( \alpha \in \mathcal{A} \) there is a \( \gamma \in \mathcal{A} \) such that \( \gamma < \alpha \), and \( H^q(\mathcal{B}_N(\alpha), \mathcal{O}^Z) \mid \mathcal{B}_N(\gamma) = 0 \) for all \( N \geq 0 \) and \( q \geq 1 \).

**Proof.** We consider some open coverings and refinement maps of them. Let \( \alpha, \beta, \gamma \in \mathcal{A} \) be as in Proposition 2.2(b). Consider the open coverings \( \mathcal{B}_N(\alpha), \mathcal{U}_N = \{ U(x) = \Omega_N(\beta) \cap \pi_X^{-1} B_X(\pi_X x, \beta(x)) : x \in \Omega_N(\gamma) \} \), \( \mathcal{B}_N(\gamma), \) and their refinement maps \( \mathcal{U}_N \to \mathcal{B}_N(\alpha) \) given by \( U(x) \mapsto B_X(x, \alpha(x)) \), and \( \mathcal{B}_N(\gamma) \to \mathcal{U}_N \) given by \( B_X(x, \gamma(x)) \mapsto U(x) \). Due to the inequalities (2.2) the above are indeed refinement maps, and hence induce maps

\[
H^q_s(\mathcal{B}_N(\alpha), \mathcal{O}^Z) \to H^q_s(\mathcal{U}_N, \mathcal{O}^Z) \to H^q_s(\mathcal{B}_N(\gamma), \mathcal{O}^Z)
\]

in cohomology for \( N \geq 0 \) and \( q \geq 1 \). Since the first map has zero image by Proposition 4.1 the proof of Proposition 6.1 is complete.

**Proof of Theorem 1.3(a).** Let \( f \in H^q(\Omega, \mathcal{O}^Z) = H^q_s(\Omega, \mathcal{O}^Z) \) be a cohomology class that we would like to resolve. By plurisubharmonic domination in \( \Omega \) there is an \( \alpha \) such that \( 10\alpha \in \mathcal{A} \), and \( f \) can be represented by a simple cocycle \( f \in Z^q_s(\mathcal{B}(\alpha), \mathcal{O}^Z) \). On choosing a \( \gamma \in \mathcal{A} \) as in Proposition 6.1 we find \( g_N \in C^{q-1}_s(\mathcal{B}_N(\gamma), \mathcal{O}^Z), \ N \geq 0 \), with \( \delta g_N = f \mid \mathcal{B}_N(\gamma) \). We can extend the cochain \( g_N \) to a cochain \( g_N \in C^{q-1}_s(\mathcal{B}(\gamma), \mathcal{O}^Z) \) simply by defining \( g_N \) to be zero over simplices \( \bigcap_{i=1}^n B_X(x_i, \gamma(x_i)) \) if at least one vertex \( x_i \in \Omega_N(\gamma) \).
Proposition 2.2(e) gives a $\beta \in \mathcal{A}$, $\beta < \gamma$, with $\Omega_N(\beta) \subset \Omega_N(\gamma) \cap \Omega_{N+1}(\gamma)$ for $N \geq 0$. So $(g_{N+1} - g_N)|\mathfrak{B}_N(\beta) \in Z^0_{\beta}(\mathfrak{B}_N(\beta), \mathcal{O}^Z)$.

Suppose first that $q \geq 2$. Proposition 6.1 gives a $\beta' \in \mathcal{A}$ such that $\beta' < \beta$, and $H^{q-2}_{\beta}(\mathfrak{B}_N(\beta), \mathcal{O}^Z)|\mathfrak{B}_N(\beta') = 0$, so similarly extending a $(q - 2)$-cochain there is an $h_N \in C^q_s(\mathfrak{B}(\beta'), \mathcal{O}^Z)$ with $(g_{N+1} - g_N)|\mathfrak{B}_N(\beta') = \delta h_N|\mathfrak{B}_N(\beta')$. Letting $g_N = g_N|\mathfrak{B}(\beta') - \sum_{n=1}^{N-1} \delta h_n \in C^q_s(\mathfrak{B}(\beta'), \mathcal{O}^Z)$ Proposition 2.2(a) implies as $g_{N+1}'|\mathfrak{B}_N(\beta') = g_N|\mathfrak{B}_N(\beta')$ that $g_N'$ converges as $N \to \infty$ to a cochain $g \in C^q(\mathfrak{B}(\beta'), \mathcal{O}^Z)$ with $\delta g = f|\mathfrak{B}(\beta')$. Thus $f$ equals zero in $H^q(\Omega, \mathcal{O}^Z)$ for $q \geq 2$.

Let now $q = 1$. By Proposition 2.2(f) there is a $\beta' \in \mathcal{A}$, $\beta' < \beta$, such that the covering $\mathfrak{B}_N(\beta)|\Omega_N(\beta')$ has a finite refinement for all $N \geq 0$. As $(g_{N+1} - g_N)|\mathfrak{B}_N(\beta) = h_N \in Z^0_{\beta}(\mathfrak{B}_N(\beta), \mathcal{O}^Z)$ we see that over $\Omega_N(\beta')$ our $h_N|\Omega_N(\beta')$ matches up to simple function $h_N \in \mathcal{O}(\Omega_N(\beta'), Z)$. Proposition 3.3(b) gives a $\beta'' \in \mathcal{A}$, $\beta'' < \beta'$, and an $H_N \in \mathcal{O}(\Omega, Z)$ with $\|h_N(x) - H_N(x)\| < 1/2^N$ for all $N \geq 0$. Proposition 2.2(a) yields a $\beta''' \in \mathcal{A}$, $\beta''' < \beta''$ such that $\Omega_M(\beta''') \subset \Omega_N(\beta''')$ for all $N \geq M \geq 0$. Letting $g_N' = (g_N - \sum_{n=1}^{N-1} H_n)|\mathfrak{B}(\beta')$ defines a cochain $g'' \in C^0(\mathfrak{B}(\beta'''), \mathcal{O}^Z)$ that satisfies $g_1 + \sum_{n=1}^{\infty} (g_{n+1} - g_n - H_n) = \lim_{N\to \infty} (g_N - \sum_{n=1}^{N-1} H_n) = \lim_{N\to \infty} g_N' = g$, where the convergence is uniform on $\Omega_M(\beta''')$ for all $M \geq 0$. Thus the limit $g \in C^0(\mathfrak{B}(\beta'''), \mathcal{O}^Z)$ exists and satisfies $\delta g = \delta g_N|\mathfrak{B}(\beta''') = f|\mathfrak{B}(\beta''')$. The proof of Theorem 1.3(a) is complete.

Resume the context and notation of § 4.

**Proposition 6.2.** For any $f \in Z^1_s(\mathcal{U}(\mathcal{O}'), \mathcal{O}^G)$ with $f(\zeta, 0) = 1$ there is a $g \in C^0_s(\mathcal{U}(\Omega), \mathcal{O}^G)$ with $g(\zeta, 0) = 1$ such that $f_{U \cap V}(\zeta, y)|(U \cap V)(\Omega) = g_U(\zeta, y)g_V(\zeta, y)^{-1}$.

**Proof.** Letting $h \in \mathcal{O}(\mathcal{C}^N)$ be various linear functions $h(\zeta_1, \ldots, \zeta_N) = \zeta_i$ the usual induction process of Cousin and Cartan (see [H, § 7.2], or [Lt, Lemma 4.1]) relying on Proposition 5.8 completes the proof of Proposition 6.2.

**Proposition 6.3.** For any $f \in Z^1_s(\mathcal{U}(\mathcal{O}'), \mathcal{O}^G)$ with $f(\zeta, 0, t) = 1$ there is a $g \in C^0_s(\mathcal{U}(\Omega), \mathcal{O}^G)$ with $g(\zeta, 0, t) = 1$ such that $f_{U \cap V}(\zeta, y, t)|(U \cap V)(\Omega) = g_U(\zeta, y, t)g_V(\zeta, y, t)^{-1}$.

**Proof.** The proof is a similar induction as that of Proposition 6.2 relying on Proposition 5.9 this time.

**Proposition 6.4.** For any $f \in Z^1_s(\mathcal{U}(\mathcal{O}'), \mathcal{O}^G)$ there is a $g \in C^0_s(\mathcal{U}(\Omega), \mathcal{O}^G)$ with $f_{U \cap V}(\zeta, y, t)|(U \cap V)(\Omega) = g_U(\zeta, y, t)g_V(\zeta, y, t)^{-1}$.

**Proof.** Define $f' \in Z^1_s(\mathcal{U}', \mathcal{O}^G)$ and $f'' \in Z^1_s(\mathcal{U}'', \mathcal{O}^G)$ by writing
\[ f(\zeta, y, t) = f'(\zeta, t) f''(\zeta, y, t) \]

where \( f'(\zeta, t) = f(\zeta, 0, t) \), so \( f''(\zeta, 0, t) = 1 \). After possibly shrinking \( \mathcal{U}' \) arbitrarily slightly Grauert’s theorem [C, Théorème principal (iii)] gives a \( g' \in \mathcal{C}_s^1(\mathcal{U}', \mathcal{O}^G_\mathcal{U}) \) with \( f''(\zeta, y, t) = g''(\zeta, t) g'(\zeta, y, t) \). Define a new cocycle \( f'' \in \mathcal{Z}^1_s(\mathcal{U}'(\mathcal{O}'), \mathcal{O}^G_\mathcal{U}) \) by the formula \( f''(\zeta, y, t) = g''(\zeta, t) g'(\zeta, y, t) \). Then \( f''(\zeta, 0, t) = 1 \), and Proposition 6.3 provides a \( g'' \in \mathcal{C}_s^1(\mathcal{U}(\mathcal{O}), \mathcal{O}^G_\mathcal{U}) \) with \( g''(\zeta, 0, t) = 1 \), and \( f''(\zeta, y, t) = g''(\zeta, t) g'(\zeta, y, t) \). Letting \( g_v(\zeta, y, t) = g''(\zeta, t) g'(\zeta, y, t) \) completes the proof of Proposition 6.4.

We now turn to approximation. Resume the context and notation of \( \S 3 \) and Proposition 5.2.

**Proposition 6.5.** Let \( \overline{D_2} \) be holomorphically convex in \( D_3, 0 < \theta < 1, \varepsilon > 0, \) and \( f \in \mathcal{O}_s^1(\Omega(D_2, R), G) \).

(a) If \( f(\zeta, 0, t) = 1 \), then there is a \( g \in \mathcal{O}_s(\Omega(D_3 \times Y, G) \) with \( g(\zeta, 0, t) = 1 \) such that \( g(\zeta, y, t) \) is bounded and uniformly continuous on any set of the form \( K \times B_Y(r) \times [0, 1] \), where \( 0 < r < \infty \), and \( K \subset D_3, \) and \( \| f(\zeta, y, t) g(\zeta, y, t)^{-1} - 1 \| < \varepsilon \) for \( (\zeta, y) \in \Omega(D_1, \theta R) \), and \( t \in [0, 1] \).

(b) There is a \( g \in \mathcal{O}_s(\Omega(D_3 \times Y, G) \) such that \( g(\zeta, y, t) \) is bounded and uniformly continuous on any set of the form \( K \times B_Y(r) \times [0, 1] \), where \( 0 < r < \infty \), and \( K \subset D_3, \) and \( \| f(\zeta, y, t) g(\zeta, y, t)^{-1} - 1 \| < \varepsilon \) for \( (\zeta, y) \in \Omega(D_1, \theta R) \), and \( t \in [0, 1] \).

**Proof.** (a) Looking back at the proof of Proposition 5.2 we find that it gives a \( g' \in \mathcal{O}_s(\Omega(D_3 \times Y, G) \) with \( g'(\zeta, 0) = 1 \) and \( \| f(\zeta, y, 1) g'(\zeta, y)^{-1} - 1 \| < \varepsilon \) for \( (\zeta, y) \in \Omega(D_1, \theta R) \). Let \( \chi \in C^\infty(\mathbb{C}^N \setminus [0, 1]) \) be a smooth cutoff function with \( \chi = 1 \) on \( D_1, \) and \( \chi = 0 \) on \( \mathbb{C}^N \setminus D_2 \). Fix \( \theta' \) with \( 0 < \theta < \theta' < 1 \), and define \( \overline{f} \in C(D_3 \times Y \times [0, 1], G) \) by \( \overline{f}(\zeta, y, t) = f(\zeta, \chi(\zeta) \min(\| y \|, \theta' R(\zeta))) \). Let \( g(\zeta, y, t) = g'(\zeta, ty) \overline{f}(\zeta, ty, 1)^{-1} \). We check that this \( g \) will do. Indeed, \( g \) is defined and continuous on \( D_3 \times Y \times [0, 1], \) \( g(\zeta, y, 0) = 1, \) \( g(\zeta, 0, t) = 1, \) \( g(\zeta, y, 1) = g'(\zeta, y) \) is holomorphic on \( D_3 \times Y \), and for \( (\zeta, y) \in \Omega(D_1, \theta R) \), \( t \in [0, 1] \) we have that \( \| f(\zeta, y, t) g(\zeta, y, t)^{-1} - 1 \| < \varepsilon \) as \( f = \overline{f} \) on \( \Omega(D_1, \theta R) \times [0, 1], \) and \( (\zeta, ty) \in \Omega(D_1, \theta R) \).

(b) Let \( \eta > 0 \), and write \( f'(\zeta, t) = f(\zeta, 0, t), f(\zeta, y, t) = f'(\zeta, t) f''(\zeta, y, t), \) \( f' \in \mathcal{O}_s^1(\Omega(D_3, G)), f'' \in \mathcal{O}_s^1(\Omega(D_2, R), G). \) Grauert’s theorem [C, Théorème principal (ii)] gives a \( g' \in \mathcal{O}_s^1(\Omega(D_3, G)) \) with \( \| f'(\zeta, t) g'(\zeta, t)^{-1} - 1 \| < \eta \) for \( (\zeta, t) \in D_1 \times [0, 1]. \) Part (a) supplies a \( g'' \in \mathcal{O}_s^1(\Omega(D_3 \times Y, G) \) with \( g''(\zeta, 0, t) = 1 \) and \( \| f''(\zeta, y, t) g''(\zeta, y, t)^{-1} - 1 \| < \eta \) for \( (\zeta, y) \in \Omega(D_1, \theta R) \), and \( t \in [0, 1] \). Letting \( g = g' g'' \) completes the proof of Proposition 6.5 if \( \eta > 0 \) is small enough.

**Proposition 6.6.** Let \( 8\alpha \in \mathcal{A} \), and choose a \( \gamma \in \mathcal{A} \) as in Proposition 2.2(b), and a \( \gamma' \in \mathcal{A} \) as in Proposition 2.2(a). Let \( f \in \mathcal{O}_s^1(\Omega_N(\alpha), G) \) be a simple
(h$_1$)-function, and $\varepsilon > 0$.

(a) There is a simple (h$_1$)-function $g \in O^1_s(\Omega_{N+1} \langle \alpha \rangle, G)$ that satisfies $\|f(x, t)g(x, t)^{-1} - 1\| < \varepsilon$ for $x \in \Omega_N \langle \gamma \rangle$, $t \in [0, 1]$.

(b) There is a $g \in O_1(\Omega, G)$ that is a simple (h$_1$)-function on $\Omega_{N+p} \langle \gamma' \rangle$ for all $p \geq 0$ with $\|f(x, t)g(x, t)^{-1} - 1\| < \varepsilon$ for $x \in \Omega_N \langle \gamma' \rangle$, $t \in [0, 1]$.

Proof. Relying on Proposition 6.5(b) instead of Proposition 5.2 as in the proof of Proposition 6.3 completes the proof of Proposition 6.6.

**Proposition 6.7.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, and suppose that plurisubharmonic domination holds in $\Omega$. Then for any $\alpha \in A$ there is a $\gamma \in A$ such that $\gamma < \alpha$, and $H^1_s(\mathcal{B}_N(\alpha), O^G_1) | \mathcal{B}_N(\gamma) = 1$ for all $N \geq 0$.

Proof. We consider some open coverings and refinement maps of them. Let $\alpha, \beta, \gamma \in A$ be as in Proposition 2.2(b). Consider the open coverings $\mathcal{B}_N(\alpha), \mathcal{U}_N = \{U(x) = \Omega_N \langle \beta \rangle \cap \pi_N^{-1} B_X(\pi_N x, \beta(x)) : x \in \Omega_N \langle \gamma \rangle\}$, $\mathcal{B}_N(\gamma)$, and their refinement maps $\mathcal{U}_N \rightarrow \mathcal{B}_N(\alpha)$ given by $U(x) \mapsto B_X(x, \alpha(x))$, and $\mathcal{B}_N(\gamma) \rightarrow \mathcal{U}_N$ given by $B_X(x, \gamma(x)) \mapsto U(x)$. Due to the inequalities (2.2) the above are indeed refinement maps, and hence induce maps

$$H^1_s(\mathcal{B}_N(\alpha), O^G_1) \rightarrow H^1_s(\mathcal{U}_N, O^G_1) \rightarrow H^1_s(\mathcal{B}_N(\gamma), O^G_1)$$

in cohomology for $N \geq 0$. Since the image of the first map is 1 by Proposition 6.4 the proof of Proposition 6.7 is complete.

**Theorem 6.8.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, and suppose that plurisubharmonic domination holds in $\Omega$. Then $H^1_s(\Omega, O^G_1) = 1$.

Proof. Let $f \in H^1_s(\Omega, O^G_1)$ be a cohomology class that we would like to resolve. By plurisubharmonic domination in $\Omega$ there is an $\alpha$ such that $10\alpha \in A$, and $f$ can be represented by a simple cocycle $f \in Z^1_s(\mathcal{B}(\alpha), O^G_1)$. On choosing a $\gamma \in A$ as in Proposition 6.7 we find $g^N \in C^0_s(\mathcal{B}_N(\gamma), O^G_1)$, $N \geq 0$, with $f_{UV}|\mathcal{B}_N(\gamma) = g^N_U(g^N_V)^{-1}$. We can extend the cocycle $g^N$ to a cocycle $g^N \in C^0_s(\mathcal{B}(\gamma), O^G_1)$ simply by defining $g^N_U$ to be 1 over simplices $U = B_X(x, \gamma(x))$ if $x \notin \Omega_N \langle \gamma \rangle$. Proposition 2.2(e) gives a $\beta \in A$, $\beta < \gamma$, with $\Omega_N \langle \beta \rangle \subset \Omega_N \langle \gamma \rangle \cap \Omega_{N+1} \langle \gamma \rangle$ for $N \geq 0$. So $((g^N)^{-1}g^{N+1})|\mathcal{B}_N(\beta) \in Z^0_s(\mathcal{B}(\beta), O^G_1)$. By Proposition 2.2(f) there is a $\beta' \in A$, $\beta' < \beta$, such that the covering $\mathcal{B}_N(\beta)|\Omega_N \langle \beta' \rangle$ has a finite basic refinement for all $N \geq 0$. As $((g^N)^{-1}g^{N+1})|\mathcal{B}_N(\beta) = h_N \in Z^0_s(\mathcal{B}_N(\beta), O^G_1)$ we see that over $\Omega_N \langle \beta' \rangle$ our $h_N|((\mathcal{B}_N(\beta)|\Omega_N \langle \beta' \rangle)$ patches up to a simple (h$_1$)-function $h_N \in O^0_s(\Omega_N \langle \beta' \rangle, G)$. Let $a_1 = 1$. Repeated application of Proposition 6.6(b) gives a $\beta'' \in A$, $\beta'' < \beta'$, and a sequence $\tilde{g}^N \in C^0(\mathcal{B}(\beta''), O^G_1)$, $N \geq 1$, of the form $\tilde{g}_U^N = g_U^N a_N$, where $a_N \in O_1(\Omega, G)$ is such that $a_N|\Omega_{N+p} \langle \beta'' \rangle \in O^0_{N+p} \langle \beta'' \rangle$. 

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\(\Omega^1_N(\Omega_{pN}(\beta'''), G)\) is simple for all \(p \geq 0\), and \((\tilde{g}^n)^{-1} \tilde{g}^{n+1} = a^{-1}_N h_N a_{N+1}\) satisfies that \(\|a^{-1}_N h_N a_{N+1} - 1\| < 1/2^N\) on \(\Omega_N(\beta''')\). Proposition 2.2(a) yields a \(\beta''' \in A\), \(\beta''' < \beta''\) with \(\Omega_M(\beta''') \subset \Omega_N(\beta'')\) for all \(N \geq M \geq 0\). As \(\tilde{g}^N\) converges uniformly on \(\Omega_M(\beta''')\) for all \(M\) the limit \(g = \lim_{N \to \infty} \tilde{g}^N\) exists and satisfies that \(g \in C^0(\mathfrak{B}(\beta'''), \mathcal{O}^G_1), g|\mathfrak{B}(\beta''') \subset C^0(\mathfrak{B}(\beta'''), \mathcal{O}^G_1), \) and \(gvg^{-1}_V = f_{UV}|\mathfrak{B}(\beta'')\). Thus \(H_s^1(\Omega, \mathcal{O}^G_1) = 1\), and the proof of Theorem 6.8 is complete.

7. TOPOLOGICAL TRIVIALITY OF A VECTOR BUNDLE.

In this section we show that certain type of topological Banach vector bundles are topologically trivial.

Given a Banach space \(Z\), and a number \(1 \leq p < \infty\), let \(Z_1 = \ell_p(Z) = \{z = (z_n) : z_n \in Z, \|z\| = (\sum_{n=1}^{\infty} \|z_n\|^p)^{1/p} < \infty\}\). Note that \(Z_1, Z \oplus Z_1, Z_1 \oplus Z_1, \ell_p(Z_1)\) are isomorphic Banach spaces by permuting the coordinates. Denote by \(Z_1 \) the trivial Banach vector bundle \(\Omega \times Z_1\) over any base space \(\Omega\).

**Proposition 7.1.** Let \(X, Z\) be Banach spaces, \(X\) with a Schauder basis, \(\Omega \subset X\) open, \(E \to \Omega\) a topological Banach vector bundle with fiber type \(Z\), and \(F = E \oplus Z_1 \to \Omega\) the direct sum Banach vector bundle. Then \(F\) is topologically trivial: \(F\) is continuously isomorphic to \(\Omega \times Z_1\).

Proposition 7.1 is the main point of this section, and its proof will take us some steps.

Let \(\mathbb{N} = \{1, 2, 3, \ldots\}\), and fix bijections from \(\mathbb{N}\) to members of a partition of \(\mathbb{N}\) into two infinite subsets, say, odd or even numbers. Let \(I_2\) be the isomorphism \(I_2 \in \text{Hom}(Z_1, Z_1 \oplus Z_1)\) of Banach spaces induced by the above bijection, i.e., \(I_2((z_n)) = ((x_n), (y_n))\), where \(x_n = z_{2n-1}\) and \(y_n = z_{2n}\) for \(n \geq 1\). Fix a bijection \(N \to \mathbb{N} \times \mathbb{N}\), say, \(n \mapsto (i, j)\), where \(n = 2^i(2j - 1)\). Let \(I_\infty\) be the isomorphism \(I_\infty \in \text{Hom}(Z_1, \ell_p(Z_1))\) of Banach spaces induced by the above bijection, i.e., \(I_\infty((z_n)) = (i \mapsto (j \mapsto x_{ij}))\), where \(x_{ij} = z_n\), and \(n = 2^{i-1}(2j - 1)\). Then \(I_2, I_\infty\) are isometries and their operator norms are \(\|I_2\| = \|I_\infty\| = 1\).

Our vector bundle \(F\) has the stability property that \(F\) and \(F \oplus Z_1 = E \oplus Z_1 \oplus Z_1\) are isomorphic. This is due to the fact that \(Z_1\) and \(Z_1 \oplus Z_1\) are isomorphic. Let \(J \in C(\Omega, \text{Hom}(F, F \oplus Z_1))\) be an isomorphism of topological Banach vector bundles defined, e.g., by \(J(\xi, z_1) = (\xi, I_2(z_1))\), where \(\xi \in E_x, x \in \Omega\), and \(z_1 \in Z_1\).

Denote a finite or infinite block diagonal matrix \(A\) with diagonal blocks \(A_1, A_2, A_3, \ldots\) by \(A = \text{diag}(A_1, A_2, A_3, \ldots)\).

**Proposition 7.2.** Let \(U \subset \Omega\) be open, \(f \in C(U, \text{GL}(Z_1))\), and define \(f' \in\)
$C(U, GL(Z_1 \oplus Z_1))$ by $f'(x) = \text{diag}(f(x), 1)$. Then $f'$ is null homotopic, i.e., there is an $h \in C(U \times [0, 1], GL(Z_1 \oplus Z_1))$ with $h(x, 0) = f'(x)$, and $h(x, 1) = 1$ for all $x \in U$.

**Proof.** This is based on classical tricks with infinite matrices, see [K, Lemma 7]. As $Z_1 \oplus Z_1, Z_1 \oplus \ell_p(Z_1)$, and $\ell_p(Z_1)$ are isomorphic by $Z_1 \oplus Z_1 \ni (x, y) \mapsto (x, I_\infty(y)) \in Z_1 \oplus \ell_p(Z_1)$, and by $Z_1 \oplus \ell_p(Z_1) \ni (x, (y_n)) \mapsto (z_n) \in \ell_p(Z_1)$, where $z_1 = x$, and $z_n = y_{n-1}$ for $n \geq 2$, we see that $f'$ can be regarded as an element of $C(U, GL(\ell_p(Z_1)))$ defined by the infinite block diagonal matrix $f'(x) = \text{diag}(f(x), 1, 1, \ldots)$.

Our homotopy $h$ will be the concatenation $h = h_1 \vee h_2$ of two homotopies $h_i \in C(U \times [0, 1], GL(\ell_p(Z_1)))$, $i = 1, 2$, defined by $h(x, t) = h_1(x, 2t)$ for $x \in U$, $t \in [0, \frac{1}{2}]$, and $h(x, t) = h_2(x, 2t - 1)$ for $x \in U$, $t \in [\frac{1}{2}, 1]$, where we must have $h_1(x, 1) = h_2(x, 0)$ for all $x \in U$.

To define $h_1$ let $h_1(x, s)$ equal

$$
\text{diag}\left( f(x), \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} f(x) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \ldots \right),
$$

where the second diagonal block repeats along the diagonal, and $t = \frac{s}{2}$. Then we can easily compute that $h_1(x, 0) = \text{diag}(f(x), 1, 1, 1, \ldots)$, and $h_1(x, 1) = \text{diag}(f(x), f(x)^{-1}, f(x), f(x)^{-1}, \ldots)$.

We define $h_2$ by letting $h_2(x, s)$ equal

$$
\text{diag}\left( \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} f(x)^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} f(x) \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \ldots \right),
$$

where the first diagonal block repeats along the diagonal, and $t = \frac{s}{2}(s + 1)$. Then $h_2(x, 0) = h_1(x, 1)$, and $h_2(x, 1) = \text{diag}(1, 1, 1, 1, \ldots)$. The proof of Proposition 7.2 is complete.

Elementary algebraic topology tells us that null homotopic maps have an extension property.

**Proposition 7.3.** For any $F \subset U \subset \Omega$, with $F$ closed, $U$ open, and $f \in C(U, GL(Z_1))$ there is an $f'' \in C(\Omega, GL(Z_1 \oplus Z_1))$ with $f''(x) = f'(x)$ for $x \in F$, where $f'$ is as in Proposition 7.2.

**Proof.** Proposition 7.2 gives a homotopy $h \in C(U \times [0, 1], GL(Z_1 \oplus Z_1))$ with $h(x, 0) = f'(x)$ and $h(x, 1) = 1$ for all $x \in U$. Let $\chi \in C(\Omega, [0, 1])$ be a cutoff function that equals 1 on $F$ and 0 on $\Omega \setminus U$. Then letting $f''(x)$ equal $h(x, 1 - \chi(x))$ for $x \in U$ and 1 for $x \in \Omega \setminus U$ completes the proof of Proposition 7.3.
Proposition 7.4. Let \( U_1, U_2 \subset \Omega \) be open, and suppose that there are trivializations \( t_i \in C(U_i, \text{Hom}(F, Z_1)) \) of \( F|U_i, \ i = 1, 2 \). Then \( F \oplus Z_1 \) and \( F \) are trivial over \( U_1 \cup U_2 \).

Proof. Let \( U = U_1 \cap U_2 \), and define \( f \in C(U, \text{GL}(Z_1)) \) by \( f(x) = t_2(x)t_1(x)^{-1} \), and \( f' \in C(U, \text{GL}(Z_1 \oplus Z_1)) \) by \( f'(x) = \text{diag}(f(x), 1) \). Let \( V_i \subset \overline{V_i} \subset U_i \) be a shrinking of the covering \( \{U_1, U_2\} \) relative to \( U_1 \cup U_2 \). Proposition 7.3 gives an extension \( f'' \in C(\Omega, \text{GL}(Z_1 \oplus Z_1)) \) with \( f''(x) = f'(x) \) for \( x \in V_1 \cap V_2 \). Define the trivializations \( T_i \in C(V_i, \text{Hom}(F \oplus Z_1, Z_1 \oplus Z_1)) \) by \( T_1(x) = f''(x) \text{diag}(t_1(x), 1) \), and \( T_2(x) = \text{diag}(t_2(x), 1) \). As \( T_1(x) = T_2(x) \) for \( x \in V_1 \cap V_2 \), and \( V_1 \cup V_2 = U_1 \cup U_2 \), the \( T_1 \) and \( T_2 \) patch up to define a trivialization \( T \in C(U_1 \cup U_2, \text{Hom}(F \oplus Z_1, Z_1 \oplus Z_1)) \) by letting \( T(x) \) equal \( T_i(x) \) for \( x \in V_i, \ i = 1, 2 \). Hence we get that \( F \oplus Z_1 \) is trivial over \( U_1 \cup U_2 \). Defining \( S \in C(U_1 \cup U_2, \text{Hom}(F, Z_1)) \) by \( S(x) = I_2^{-1}(T(x), J(x)) \) trivializes \( F|U_1 \cup U_2 \) as well. The proof of Proposition 7.4 is complete.

Thus \( F \) is trivial over any finite union \( \bigcup U_i \) of open subsets \( U_i \) of \( \Omega \) over each of which \( F|U_i \) is trivial.

Proof of Proposition 7.1. Let \( \mathcal{A}' \) as in §2. There is an \( \alpha \in \mathcal{A}' \) such that \( F|B_X(x, \alpha(x)) \) is trivial for all \( x \in \Omega \). Proposition 2.2 (except its part (c)) remains true for any \( \alpha \in \mathcal{A}' \) as well. The proof is just like that of Proposition 2.2 for \( \mathcal{A} \) only it is easier, and plurisubharmonic domination in \( \Omega \) is not needed. Proposition 2.2(f) gives an \( \alpha' \in \mathcal{A}', \alpha' < \alpha \), such that \( \mathfrak{W}_N(\alpha)|\Omega_N(\alpha') \) has a finite refinement for all \( N \geq 0 \). Proposition 2.2(e) yields a \( \beta \in \mathcal{A}', \beta < \alpha' \), with \( \Omega_N(\beta) \subset \Omega_N(\alpha') \cap \Omega_{N+1}(\alpha') \) for all \( N \geq 0 \), and Proposition 2.2(a) a \( \gamma \in \mathcal{A}', \gamma < \beta \), with \( \Omega_n(\gamma) \subset \Omega_N(\beta) \) for \( N \geq n \).

We show by induction that there are for any \( N \geq 1 \) continuous trivializations \( T_N \in C(\Omega_N(\alpha'), \text{Hom}(F \oplus \ell_p(Z_1), \ell_p(Z_1))) \) of the form \( T_N(x) = \text{diag}(t_N(x), 1, 1, \ldots) \), where \( t_N \in C(\Omega_N(\alpha'), \text{Hom}(F \oplus Z_1^{\oplus N-1}, Z_1^{\oplus 2N-1})) \) are trivializations such that \( T_N = T_{N+1} \) over \( \Omega_M(\gamma) \) for all \( N \geq M \geq 0 \).

By the remark after the proof of Proposition 7.4 there is a trivialization \( t_1 \in C(\Omega_1(\alpha'), \text{Hom}(F, Z_1)) \). Define \( T_1 \) by \( T_1(x) = \text{diag}(t_1(x), 1, 1, \ldots, 1) \). Suppose that \( T_1, \ldots, T_N \) are already defined. By the remark after the proof of Proposition 7.4 there is a trivialization \( T_{N+1}' \in C(\Omega_{N+1}(\alpha'), \text{Hom}(F \oplus \ell_p(Z_1), \ell_p(Z_1))) \) of the form \( T_{N+1}'(x) = \text{diag}(t_{N+1}'(x), 1, 1, \ldots) \), where \( t_{N+1}' \in C(\Omega_{N+1}(\alpha'), \text{Hom}(F \oplus Z_1^{\oplus N}, Z_1^{\oplus 2N})) \) is a trivialization. We define \( f \in C(\Omega_N(\alpha') \cap \Omega_{N+1}(\alpha'), \text{GL}(Z_1^{\oplus 2N})) \) by \( f(x) = t_N(x)t_{N+1}'(x)^{-1} \), and \( f' \in C(\Omega_N(\alpha') \cap \Omega_{N+1}(\alpha'), \text{GL}(Z_1^{\oplus 2N} \oplus Z_1^{\oplus 2N})) \) by \( f'(x) = \text{diag}(f(x), 1) \). Then by Proposition 7.2 our \( f' \) is null homotopic, and thus by Proposition 7.3 there is an extension \( f'' \in C(\Omega, \text{GL}(Z_1^{\oplus 2N} \oplus Z_1^{\oplus 2N})) \) of \( f' \) with \( f''(x) = f'(x) \) for \( x \in \Omega_N(\beta) \).
Define $T_{N+1}$ by $T_{N+1}(x) = \text{diag}(f''(x), 1, 1, \ldots)T_N'$. As $T_{N+1}(x) = \text{diag}(f'(x), 1, 1, \ldots)\text{diag}(t_{N+1}', 1, 1, \ldots) = (t_N(x), 1, 1, \ldots) = T_N(x)$ for all $x \in \Omega_M(\gamma)$ for $N \geq M \geq 0$ we see that the sequence $T_N$ converges as $N \to \infty$ to a trivialization $T \in C(\Omega, \text{Hom}(F \oplus \ell_p(Z_1), \ell_p(Z_1)))$ in a quasi-stationary manner over $\Omega_M(\gamma)$ for all $M$. As $F \oplus \ell_p(Z_1) \cong F$, and $\ell_p(Z_1) \cong Z_1$ the proof of Proposition 7.1 is complete.

8. FROM CONTINUOUS TO LOCALLY LIPSCHITZ SECTIONS.

In connection with ‘telescopic products’ we shall need later that our sections (and homotopies) be locally more regular than just plain continuous. Smoothness (say, of class $C^1$) would be quite enough, but is hard to achieve if the ground Banach space lacks smooth partitions of unity. Instead, a notion of boundedness and uniform continuity with respect to a preferred trivialization would also do, but it is more convenient here to use locally Lipschitz sections (written $C^{0,1}_{\text{loc}}$), whose definition does not require a fixed preferred trivialization.

Let $X$ be a separable real Banach space, $\Omega \subset X$ open, $\Gamma \to \Omega$ a $C^3$-smooth real Banach Lie group bundle, $\hat{\Gamma} \to \Omega$ the associated $C^2$-smooth Banach Lie algebra bundle with a continuous norm $| \cdot |_x : \hat{\Gamma}_x \to [0, \infty)$, $P \to \Omega$ a $C^3$-smooth right $\Gamma$-principal bundle.

The main point of this section is the following Theorem 8.1.

**Theorem 8.1.** For any $f \in C(\Omega, P)$ there are a section $g \in C^{0,1}_{\text{loc}}(\Omega, P)$ and a homotopy $h \in C(\Omega \times [0, 1], P)$ such that $h(x, t) \in P_x$, $h(x, 0) = f(x)$, and $h(x, 1) = g(x)$ for all $(x, t) \in \Omega \times [0, 1]$.

Theorem 8.1 follows from Theorem 8.2 on approximating a continuous section $f$ by a locally Lipschitz one $g$.

**Theorem 8.2.** For any $f \in C(\Omega, P)$ and $\varepsilon \in C(\Omega, (0, \infty))$ there is a $g \in C^{0,1}_{\text{loc}}(\Omega, P)$ such that $| \log_x f(x)^{-1}g(x) |_x \leq \varepsilon(x)$ for all $x \in \Omega$.

The proof of Theorem 8.2 follows that of the Oka principle of Grauert, and even if much easier, it still takes a few propositions.

**Proposition 8.3.** Any open covering $\mathcal{U}$ of an open subset $\Omega$ of $X$ admits a Lipschitz partition of unity subordinate to $\mathcal{U}$.

Here a Lipschitz partition of unity is one whose members are Lipschitz functions $\chi$ on $\Omega$ the supports of which form a refinement of $\mathcal{U}$.

**Proof.** This is a standard fact; see [F].

We also introduce certain functions $\varepsilon_i, M_i \in C(\Omega, (0, \infty))$ which only depend on the geometry of the Banach Lie group bundle $\Gamma$, such as the radius
function of a normal neighborhood of $0$ in $\Gamma$.

**Proposition 8.4.** There are $\varepsilon_1, M_1 \in C(\Omega, (0, \infty))$ such that if $U, V \subset \Omega$ open, $h_{UV} \in C_{\text{loc}}^{0,1}(U \cap V) \times [0, 1], \hat{\Gamma}$, $\lambda \in C_{\text{loc}}^{0,1}(U \cap V) \times [0, 1], \text{End}(\hat{\Gamma})$, $\varepsilon \in C(\Omega, (0, \infty))$, $0 < \varepsilon(x) \leq \varepsilon_1(x)$ for $x \in \Omega$, $|h_{UV}(x,t)|x \leq \varepsilon(x)$, $\|\lambda(x,t) - 1\|_x \leq \varepsilon(x)$ for all $(x,t) \in (U \cap V) \times [0, 1]$, then there are $h_U \in C_{\text{loc}}^{0,1}(U \times [0, 1], \hat{\Gamma})$ and $h_V \in C_{\text{loc}}^{0,1}(V \times [0, 1], \hat{\Gamma})$ such that $h_{UV}(x,t) = h_U(x,t) - \lambda(x,t)h_U(x,t)$ for $(x,t) \in (U \cap V) \times [0, 1]$, $|h_U(x,t)|x \leq M_1(x)\varepsilon(x)$ for $(x,t) \in U \times [0, 1]$, and $|h_V(x,t)|x \leq M_1(x)\varepsilon(x)$ for $(x,t) \in V \times [0, 1]$.

Here $\|T\|_x$ is the operator norm of a linear operator $T : (\hat{\Gamma}_x, \|\cdot\|_x) \to (\hat{\Gamma}_x, \|\cdot\|_x)$.

**Proof.** Let $0 \leq \chi_U, \chi_V \leq 1$, $\chi_U + \chi_V = 1$, be a locally Lipschitz partition of unity over $U \cup V$ subordinate to the covering $\{U, V\}$ and as usual let $h_U(x,t) = -\chi_V(x)\lambda(x,t)^{-1}h_{UV}(x,t)$, and $h_V(x,t) = \chi_U(x)h_{UV}(x,t)$. Then $h_{UV} = h_V - \lambda h_U$, and the rest follows, completing the proof of Proposition 8.4.

**Proposition 8.5.** Let $\varepsilon_0 \in C(\Omega, (0, \infty))$ be the radius of a normal neighborhood of $0$ in $\hat{\Gamma}$. Then there are $\varepsilon_2, M_2 \in C(\Omega, (0, \infty))$ with $0 < \varepsilon_2(x) < \varepsilon_0(x)$ for $x \in \Omega$ such that if $x \in \Omega$, $0 < \varepsilon \leq \varepsilon_2(x)$, $a, b \in \Gamma_x$, $|\log_x a|_x, |\log_x b|_x \leq \varepsilon$, then $|\log_x ab|_x \leq M_2(\varepsilon_0(x))\varepsilon$.

**Proposition 8.6.** There are $\varepsilon_3, M_3 \in C(\Omega, (0, \infty))$ such that if $x \in \Omega$, $\xi, \eta \in \hat{\Gamma}_x$, $0 < \varepsilon \leq \varepsilon_3(x)$, $|\xi|_x + |\eta|_x \leq \varepsilon$, $\gamma \in [0, 1]$, then

\[
|\log_x[\exp_x(\xi) \exp_x(\gamma \log_x(\exp_x(-\xi) \exp_x(\eta)))]|_x \\
\leq |(1 - \gamma)\xi + \gamma\eta|_x + M_3(x)(|\xi|_x + |\eta|_x)^2 \\
\leq \varepsilon + M_3(\varepsilon_0(x))\varepsilon^2.
\]

**Proofs of Propositions 8.5 and 8.6.** These statements follow from the second order Taylor formula.

Let $U \subset \Omega$ be open. Given an $h \in C_{\text{loc}}^{0,1}(U \times [0, 1], \hat{\Gamma})$ look at the parametric initial value problem

\[
\frac{dH}{dt} = H \cdot h, \quad H(x,0) = 1,
\]

for an $H \in C_{\text{loc}}^{0,1}(U \times [0, 1], \Gamma)$, where the symbol $H(x,t) \cdot h(x,t)$ represents the left translate of the Lie algebra element $h(x,t) \in \hat{\Gamma}_x$ by the Lie group element $H(x,t) \in \Gamma$. The symbol $H(x,t) \cdot h(x,t)$ represents the left translate of the Lie algebra element $h(x,t) \in \hat{\Gamma}_x$ by the Lie group element $H(x,t) \in \Gamma_x$.

**Proposition 8.7.** There are functions $\varepsilon_4, M_4 \in C(\Omega, (0, \infty))$ such that if $\varepsilon \in C(\Omega, (0, \infty))$ with $0 < \varepsilon(x) \leq \varepsilon_4(x)$ for $x \in \Omega$, and $h \in C_{\text{loc}}^{0,1}(U \times [0, 1], \hat{\Gamma})$
with \(|h(x,t)|x \leq \varepsilon(x)| for \((x,t) \in U \times [0,1]\), then \(H \in C^{0,1}_{\text{loc}}(U \times [0,1], \Gamma)\) as in (8.2) satisfies that \(|\log_x H(x,t)|x \leq M_4(x)\varepsilon(x)| for \((x,t) \in U \times [0,1]\).

**Proof.** This follows from a standard a priori estimate for ordinary differential equations.

Given an \(\varepsilon \in C(\Omega, (0, \infty))\) define \(\varepsilon_5 \in C(\Omega, (0, \infty))\) by

\[
\varepsilon_5(x) = \frac{1}{4} \min\{\varepsilon_0(x), \varepsilon_1(x), \varepsilon_2(x), \varepsilon_4(x)\}/(1 + M_1(x)M_2(x)^2M_4(x)\}.
\]

**Proposition 8.8.** Let \(U, V \subset \Omega\) be open, \(f \in C(\Omega, P), \varepsilon \in C(\Omega, (0, \infty))\). If \(g_U \in C^{0,1}_{\text{loc}}(U, P), g_V \in C^{0,1}_{\text{loc}}(V, P)\) satisfy that \(|\log_x f(x)^{-1}g_U(x)|x \leq \varepsilon_5(x)| for \(x \in U\), and \(|\log_x f(x)^{-1}g_V(x)|x \leq \varepsilon_5(x)| for \(x \in V\), then there is a \(g \in C^{0,1}_{\text{loc}}(U \cup V, \Gamma)\) with \(|\log_x f(x)^{-1}g(x)|x \leq \varepsilon(x)| for \(x \in U \cup V\).

**Proof.** Define \(g_{UV} \in C^{0,1}_{\text{loc}}(U \cup V, \Gamma)\) by \(g_{UV}(x) = g_U(x)^{-1}g_V(x)\). By Proposition 8.5 we see that \(|\log_x g_{UV}(x)|x \leq M_2(x)\varepsilon_5(x)|. Define \(h_{UV} \in C^{0,1}_{\text{loc}}((U \cap V) \times [0,1], \Gamma)\) by \(h_{UV}(x,t) = t\log_x g_{UV}(x), H_{UV} \in C^{0,1}_{\text{loc}}((U \cap V) \times [0,1], \Gamma)\) by \(H_{UV}(x,t) = \exp_x h_{UV}(x,t)\), and \(\lambda \in C^{0,1}_{\text{loc}}((U \cap V) \times [0,1], \text{End}(\Gamma))\) by \(\lambda(x,t) = \text{Ad}_x(H_{UV}(x,t))\). Proposition 8.4 gives \(h_U \in C^{0,1}_{\text{loc}}(U \times [0,1], \Gamma)\) and \(h_V \in C^{0,1}_{\text{loc}}(V \times [0,1], \Gamma\) with \(|h_U(x,t)|x \leq M_1(x)M_2(x)\varepsilon_5(x)| for \(x, t \in U \times [0,1], |h_V(x,t)|x \leq M_1(x)M_2(x)\varepsilon_5(x)| for \(x, t \in V \times [0,1], and

\[
(8.3) \quad h_{UV}(x,t) = h_V(x,t) - \text{Ad}_x(H_{UV}(x,t))h_U(x,t)
\]

for \((x,t) \in (U \cap V) \times [0,1]\). Proposition 7.8 gives \(H_U \in C^{0,1}_{\text{loc}}(U \times [0,1], \Gamma)\) and \(H_V \in C^{0,1}_{\text{loc}}(V \times [0,1], \Gamma)\) that satisfy the analogs of (8.2), and the estimates \(|\log_x H_U(x,t)|x \leq M_1(x)M_2(x)M_4(x)\varepsilon_5(x)| for \((x,t) \in U \times [0,1]\), and \(|\log_x H_V(x,t)|x \leq M_1(x)M_2(x)M_4(x)\varepsilon_5(x)| for \((x,t) \in V \times [0,1]\). By (8.3) we see that \(H_{UV}(x,t) = H_U(x,t)^{-1}H_V(x,t)\) for \((x,t) \in (U \cap V) \times [0,1]\). In particular, for \(t = 1\) we get that \(g_U(x)^{-1}g_V(x) = H_U(x,1)^{-1}H_V(x,1)\). Define \(g \in C^{0,1}_{\text{loc}}((U \cup V) \times [0,1], \Gamma)\) by \(g(x) = g_U(x)H_U(x,1)^{-1} = g_V(x)H_V(x,1)^{-1}\). Then

\[
|\log_x f(x)^{-1}g(x)|x = |\log_x f(x)^{-1}g_U(x)H_U(x,1)^{-1}|x \\
\leq M_2(x)M_1(x)M_2(x)M_4(x)\varepsilon_5(x) < \varepsilon(x)
\]

if \(x \in U\), and similarly if \(x \in V\). The proof of Proposition 8.8 is complete.

Let \(f \in C(\Omega, P), U \subset \Omega\) open. For short we say that \(U\) is **good** for \(f\) if for any \(\varepsilon \in C(\Omega, (0, \infty))\) there is a \(g \in C^{0,1}_{\text{loc}}(U, P)\) with \(|\log_x f(x)^{-1}g(x)|x \leq \varepsilon(x)| for all \(x \in U\). In this language Proposition 8.8 says that the union of two (or finitely many) good open sets for \(f\) is a good open set for \(f\). If \(U\) is good for \(f\), and \(V \subset U\) is open, then clearly \(V\) is also good for \(f\).
**Proposition 8.9.** Any point \( x_0 \in \Omega \) has a good open neighborhood \( U \) for \( f \).

*Proof.* Let \( U \) be so small an open neighborhood of \( x_0 \) that \( \Gamma \) and \( P \) are trivial over \( U \), and let \( T : P|U \to G \) be a trivialization. Here the Banach Lie group \( G \) with Banach Lie algebra \( \dot{G} \) is the fiber type of \( \Gamma \), and define \( F : U \to G \) by \( F(x) = T(x)f(x) \). It is enough to show that \( x_0 \) has a good neighborhood \( U \) for \( F \). To this end by shrinking \( U \) towards \( x_0 \) we can find a constant \( c \in G \) such that \( F(x)c^{-1} \) is so nearly 1 that \( F(x) = c \exp \varphi(x) \) for \( x \in U \), where \( \varphi \in C(U, \dot{G}) \). Now we only need to show that \( U \) is a good neighborhood for \( \varphi \), which is clear by a Lipschitz partition of unity. The proof of Proposition 8.9 is complete.

Now we claim that there is an exhaustion of \( \Omega \) by open sets good for \( f \).

**Proposition 8.10.** There are good open sets \( \Omega_{n} \) for \( f \) with

\[
\Omega_1 \subset \overline{\Omega_1} \subset \Omega_2 \subset \overline{\Omega_2} \subset \ldots \subset \Omega_n \subset \overline{\Omega_n} \subset \ldots, \quad \text{and} \quad \bigcup_{n=1}^{\infty} \Omega_n = \Omega.
\]

*Proof.* Proposition 8.9 gives a covering \( \mathfrak{U} \) of \( \Omega \) by good open sets \( U \) for \( f \), and we may suppose that each \( U \subset \mathfrak{U} \) is bounded and is at a positive distance from \( X \setminus \Omega \) (in case \( \Omega \neq X \)). As \( \Omega \) is a Lindelöf space, being a separable metric space, we have a countable subcover \( V_n, n \geq 1 \), of \( \Omega \). Let \( U_n = \bigcup_{i=1}^{n} V_i \). Define \( \Omega_n, n \geq 1 \), by \( \Omega_n = \{ x \in \Omega : \text{dist}(x, X \setminus U_n) > 1/n \} \).

Then \( \Omega_n \) is an open subset of \( \Omega \) and is good for \( f \). To check that \( \overline{\Omega_n} \subset \Omega_{n+1} \), let \( x \in \overline{\Omega_n} \). Then \( \text{dist}(x, X \setminus U_n) \geq 1/n \), and since \( U_n \subset U_{n+1} \) we see that \( \text{dist}(x, X \setminus U_{n+1}) \geq \text{dist}(x, X \setminus U_n) \geq 1/n > 1/(n+1) \), so \( x \in \Omega_{n+1} \). To verify that \( \bigcup_{n=1}^{\infty} \Omega_n = \Omega \), let \( x \in \Omega \) be contained in \( U_m \) for an \( m \). There is an \( n \geq m \) for which the ball \( B_X(x_0, 2/n) \) is contained in the open set \( U_m \).

Then \( x \in \Omega_n \), since \( \text{dist}(x, X \setminus U_n) \geq \text{dist}(x, X \setminus U_m) \geq 2/n > 1/n \). The proof of Proposition 8.10 is complete.

**Proof of Theorem 8.2.** Let \( \Omega_n \) be as in Proposition 8.10, \( \varepsilon_5 \) as in Proposition 8.8, and take a convergent infinite product \( \prod_{n=1}^{\infty} a_n = 2 \) with terms \( a_n > 1 \), e.g., \( a_n = (1-(n+1)^{-2})^{-1} \), and define an \( \varepsilon_6 \in C(\Omega, (0, \infty)) \) such that \( \varepsilon_6(x) < \frac{1}{4} \min\{ \varepsilon_3(x), \varepsilon_5(x) \} \), and \( 1 + 4\varepsilon_6(x)M_3(x) < a_n \) for \( x \in \Omega_n \setminus \Omega_{n-1} \), \( n \geq 2 \).

We construct a sequence \( g_n \in C^{0,1}_{loc}(\Omega_n, P) \) for which \( g_{n+1} = g_n \) on \( \Omega_{n-1} \) for \( n \geq 2 \), and \( | \log_x f(x)^{-1}g_n(x) |_x \leq \varepsilon_6(x) \prod_{i=1}^{n-1} a_i \) for \( x \in \Omega_n \). If this can be done, then the limit \( g \) of \( g_n \) as \( n \to \infty \) will do.

Since \( \Omega_1 \) is good for \( f \) a \( g_1 \) can be chosen with \( | \log_x f(x)^{-1}g_1(x) |_x \leq \varepsilon_6(x) \) for \( x \in \Omega_1 \), and similarly for a \( g_2 \).
Suppose now that \( g_1, \ldots, g_n \) for \( n \geq 2 \) have already been defined, and define \( g_{n+1} \) as follows.

Choose an \( h \in C^{0,1}_{\text{loc}}(\Omega_{n+1}, P) \) with \( |\log_x f(x)^{-1}h(x)|_x \leq \varepsilon_6(x) \) on \( \Omega_{n+1} \), which is a good open set for \( f \), and a cutoff function \( \chi \in C^{0,1}_{\text{loc}}(\Omega, [0, 1]) \) with \( \chi = 1 \) on \( \Omega_{n-1} \), and \( \chi = 0 \) on an open neighborhood of \( \Omega \setminus \Omega_n \). Define \( g_{n+1} \) by \( g_{n+1}(x) = h(x) \exp_x(\chi(x) \log_x(h(x)^{-1}g_n(x))) \).

Then \( g_{n+1}(x) = g_n(x) \) for \( x \in \Omega_{n-1} \), and \( g_{n+1}(x) = h(x) \) for \( x \in \Omega_{n+1} \setminus \Omega_n \), so the required estimate \( |\log_x f(x)^{-1}g_{n+1}(x)|_x \leq \varepsilon_6(x) \prod_{i=1}^6 a_i \) holds there. If \( x \in \Omega_n \setminus \Omega_{n-1} \), then

\[
f(x)^{-1}g_{n+1}(x) = f(x)^{-1}h(x) \cdot \exp_x(\chi(x) \log_x((f(x)^{-1}h(x))^{-1} \cdot f(x)^{-1}g_n(x))),
\]

so by Proposition 8.6 we see that \( |\log_x f(x)^{-1}g_{n+1}(x)|_x \leq \varepsilon_6(x) \prod_{i=1}^6 a_i \) holds there, too. This completes the induction step, and with it the proof of Theorem 8.2.

**Proof of Theorem 8.1.** Theorem 8.2 gives for \( \varepsilon_0 \) as in Proposition 8.5 a \( g \in C^{0,1}_{\text{loc}}(\Omega, P) \) with \( |\log_x f(x)^{-1}g(x)|_x \leq \varepsilon_0(x)/2 \) for \( x \in \Omega \). Thus letting \( h(x, t) = f(x) \exp_x(t \log_x(f(x)^{-1}g(x))) \) completes the proof of Theorem 8.1.

The above proofs easily give approximation by smoother sections than locally Lipschitz if the ground Banach space admits smoother partitions of unity.

The general theme of Theorem 8.1 is to represent homotopy classes \([f]\) of continuous maps \( f : M \to N \) of Banach manifolds by smoother ones such as locally Lipschitz. Similarly the general theme of Theorem 8.2 is to approximate continuous maps \( f : M \to N \) of Banach manifolds by smoother ones such as locally Lipschitz. If \( N \) is finite dimensional or embeds in a Banach space \( Z \) as smooth neighborhood retract, then one can carry out this approximation simpler than suggested above for \( f \in C(\Omega, G) \) by first approximating \( f : M \to Z \) by a smoother map \( h : M \to Z \) then projecting back onto \( N \) as \( g = r \circ h \) with a smooth neighborhood retraction \( r : U \to N \). While this is much shorter when applicable, it is unclear if it applies to a map \( f : \Omega \to G \) into a general Banach Lie group as in Theorem 8.2.

**9. THE PROOF OF THEOREM 1.3(d).**

In this section we complete the proof of Theorem 1.3(d) that states that a continuously trivial holomorphic Banach vector bundle is holomorphically trivial in certain cases. Resume the context and notation of Theorem 1.3(d).

**Proof of Theorem 1.3(d).** Let \( G = \text{GL}(Z) \), and \( f \in Z^1(\mathcal{U}, \mathcal{O}^G) \) be a defining cocycle of our holomorphic Banach vector bundle \( E \to \Omega \). By plurisubharmonic domination in \( \Omega \) there is a Hartogs radius function \( \alpha \in \mathcal{A} \) such that
\( \mathfrak{B}(\alpha) \) is a refinement of \( \mathfrak{U} \), and the components \( f_{UV} \) of \( f \) are bounded and uniformly continuous on \( U \cap V \) for all \( U, V \in \mathfrak{B}(\alpha) \). As \( E \) is continuously trivial over \( \Omega \), there is a continuous cochain \( \varphi = (\varphi_U) \in C^0(\mathfrak{B}(\alpha), \mathcal{O}^G) \) with \( f_{UV}(x) = \varphi_U(x)\varphi_V(x)^{-1} \) for all \( x \in U \cap V, U, V \in \mathfrak{B}(\alpha) \), i.e., \( \varphi_U(x) = f_{UV}(x)\varphi_V(x) \). Thus our \( \varphi \) can be regarded as a continuous global section \( \varphi \) of a holomorphic principal \( G \)-bundle \( P \) over \( \Omega \) defined by \( f \). Theorem 8.1 shows that our continuous global section \( \varphi \) of \( P \) is continuously homotopic through global sections of \( P \) to a locally Lipschitz global section \( \psi \) of \( P \), i.e., there are \( ((x, t) \mapsto \varphi_U(x, t)) \in C(U \times [0, 1], G) \) with \( f_{UV}(x) = \varphi_U(x, t)\varphi_V(x, t)^{-1}, \varphi_U(x, 0) = \varphi_U(x) \), and \( \psi_U(x) = \varphi_U(x, 1) \) locally Lipschitz for \( x \in U, U \in \mathfrak{U} \). For a small enough \( \beta \in A, \beta < \alpha \), we can write \( \psi_U(x) = A_U \exp(\psi_U(x)) \) on \( U = B_X(x_0, \beta(x_0)) \), where \( A_U \in G \) is a constant, and \( \psi_U \in C^0(U, \mathfrak{G}) \) is bounded and uniformly continuous, in fact Lipschitz. Let \( \psi_U(x, t) = A_U \exp((1 - t)\psi_U(x)) \) for \( x \in U, t \in [0, 1], U \in \mathfrak{B}(\beta) \). Define \( f' \in Z^1(\mathfrak{B}(\beta), \mathcal{O}^G) \) by \( f'_{UV}(x, t) = \psi_U(x, t)^{-1}f_{UV}(x)\psi_U(x, t) \). This \( f'_{UV} \) is bounded and uniformly continuous on \( U \cap V \), \( f'_{UV}(x, 0) = 1 \), and \( f'_{UV}(x, 1) = A_U^{-1}f_{UV}(x)A_U \) is holomorphic for \( x \in U \cap V, U, V \in \mathfrak{B}(\beta) \). Theorem 6.8 gives a \( g' = (g'_{UV}) \in C^0(\mathfrak{B}(\beta), \mathcal{O}^G_1) \) that resolves \( f' \), i.e., \( f'_{UV}(x, t) = g'_{UV}(x, t)g'_{UV}(x, t)^{-1} \). Define \( g = (g_{UV}) \in C^0(\mathfrak{B}(\beta), \mathcal{O}^G) \) by \( g_{UV}(x) = A_U g'_{UV}(x, 1) \). Then \( g \) is holomorphic and satisfies that \( f_{UV}(x) = g_{UV}(x)g_{UV}(x)^{-1} \), i.e., our holomorphic Banach vector bundle \( E \) is indeed holomorphically trivial over \( \Omega \). The proof of Theorem 1.3(d) is complete.

10. THE PROOF OF THEOREM 1.3(b).

In this section we complete the proof of Theorem 1.3(b) that states that \( E \oplus Z_1 \) is holomorphically trivial in certain cases. Resume the context and notation of Theorem 1.3(b).

Proof of Theorem 1.3(b). Proposition 7.1 tells us that the holomorphic Banach vector bundle \( F = E \oplus Z_1 \to \Omega \) is continuously trivial. Theorem 1.3(d) shows then that \( F \) is in fact holomorphically trivial over \( \Omega \). As the fiber type of \( F \) is \( Z_1 \oplus Z_1 \cong Z_1 \), the proof of Theorem 1.3(b) is complete.

11. THE PROOF OF THEOREM 1.3(c).

In this section we complete the proof of Theorem 1.3(c) that states that \( \mathcal{O}^E \) is acyclic over \( \Omega \) in certain cases. Resume the context and notation of Theorem 1.3(c).

Proof of Theorem 1.3(c). As \( F = E \oplus Z_1 \to \Omega \) is holomorphically trivial by Theorem 1.3(b), we see by Theorem 1.3(a) that \( 0 = H^q(\Omega, \mathcal{O}^F) = H^q(\Omega, \mathcal{O}^E) \oplus H^q(\Omega, \mathcal{O}^{Z_1}) \) for \( q \geq 1 \). Thus \( H^q(\Omega, \mathcal{O}^E) = 0 \) for \( q \geq 1 \), and the proof of Theorem 1.3(c) is complete.

12. THE PROOF OF THEOREM 1.3(e).
In this section we complete the proof of Theorem 1.3(e) that says that holomorphic Hilbert vector bundles are holomorphically trivial in certain cases. Resume the context and notation of Theorem 1.3(e), and recall the following theorem of Kuiper.

**Theorem 12.1.** (Kuiper, [K]) The Banach Lie group $\text{GL}(\ell_2)$ is contractible, and thus any topological Hilbert vector bundle of fiber type $\ell_2$ over a paracompact Hausdorff space (e.g., a metric space) is continuously trivial.

Proof of Theorem 1.3(e). As $E$ is continuously trivial over $\Omega$ by Theorem 12.1 we can apply Theorem 1.3(d) and find that $E \to \Omega$ is holomorphically trivial over $\Omega$ as well. The proof of Theorem 1.3(e) is complete.

### 13. THE PROOF OF THEOREM 1.3(f).

In this section we finish the proof of Theorem 1.3(f) that says that a holomorphic Banach vector bundle $E$ over a contractible space $\Omega$ is holomorphically trivial in certain cases. Resume the context and notation of Theorem 1.3(f).

Proof of Theorem 1.3(f). As $\Omega$ is contractible, our bundle $E \to \Omega$ is continuously trivial by elementary algebraic topology. Theorem 1.3(d) then shows that $E \to \Omega$ is holomorphically trivial. The proof of Theorem 1.3(f) is complete.

### 14. APPLICATIONS.

This section points out some applications of Theorem 1.3.

The first application shows that Theorem 1.3 remains valid if the open set $\Omega$ is replaced by certain complex Banach manifolds $M$.

**Theorem 14.1.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $M$ a relatively closed complex Banach submanifold of $\Omega$ onto which there is a holomorphic retraction $r: \omega \to M$, where $\omega$ is pseudoconvex open with $M \subset \omega \subset \Omega$, and $r(x) = x$ if $x \in M$. If plurisubharmonic domination holds in $\omega$, then Theorem 1.3 holds with $\Omega$ replaced by $M$.

Proof. Consider the pullback bundle $r^*E \to \omega$, and apply to it Theorem 1.3, and then restrict back to $M$. The proof of Theorem 14.1 is complete.

Theorem 14.1 applies in many cases, e.g., it is shown in [P3] that if $M$ is a complete intersection in $\Omega$ in the sense that there is a holomorphic function $f \in \mathcal{O}(\Omega, Z)$ into a Banach space $Z$, $M = \{x \in \Omega : f(x) = 0\}$, and the Fréchet differential $df(x) \in \text{Hom}(X, Z)$ has split kernel for $x \in M$, then there is a holomorphic retraction $r: \omega \to M$ as in Theorem 14.1.
The total space of a direct summand of a trivial holomorphic Banach vector bundle is a complete intersection in the trivial bundle, hence is the following theorem that proves a vanishing result over the total space of a holomorphic Banach vector bundle in certain cases.

**Theorem 14.2.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $E' \to \Omega$ a holomorphic Banach vector bundle with a Banach space $Z'$ for fiber type, $E \to E'$ a holomorphic Banach vector bundle over the total space of the bundle $E' \to \Omega$, and $1 \leq p < \infty$. If $Z'_1 = \ell_p(Z')$ has a Schauder basis, and plurisubharmonic domination holds in $\Omega \times Z'_1$, then Theorem 1.3 holds with $\Omega$ replaced by $E'$.

**Proof.** Theorem 1.3(b) allows us to regard $E'$ as a direct summand in the trivial bundle $\Omega \times Z'_1$, and provides a projection $P \in \mathcal{O}(\Omega, \text{End}(Z'_1))$ with $P(x)^2 = P(x)$, and $\text{Im} P(x) = E'_x$, $x \in \Omega$. Thus $r : \Omega \times Z'_1 \to E' \subset \Omega \times Z'_1$ defined by $r(x, \xi) = (x, P(x)\xi)$ is a holomorphic retraction, and an application of Theorem 14.1 completes the proof of Theorem 14.2.

**Theorem 14.3.** If $X = \ell_2$ in Theorem 14.1 and $M$ is infinite dimensional, then the holomorphic tangent bundle $TM$ and cotangent bundle $T^*M$ of $M$ are holomorphically trivial over $M$, also so are the bundles $S^pM, \Lambda^pM$ of holomorphic symmetric or alternating $p$-forms on $M$ for $p \geq 1$.

**Proof.** As $TM$ and $T^*M$ are Hilbert bundles of fiber type $\ell_2$, they are holomorphically trivial by Theorem 14.1. As the bundles $S^pM, \Lambda^pM$ are associated bundles to the trivial bundle $TM$, they are themselves holomorphically trivial—they are defined by an action of the defining cocycle $f$ of $TM$ on the Banach spaces $S^p\ell_2, \Lambda^p\ell_2$, but as the defining cocycle $f$ is holomorphically trivial, so are the associated bundles $S^pM, \Lambda^pM$. The proof of Theorem 14.3 is complete.

So for example if $M$ is as in Theorem 14.3, then there are a holomorphic $(1,0)$-form $\alpha$ and a holomorphic $(1,0)$-vector field $\xi$ on $M$ that have no zeros on $M$. Can such an $\alpha$ be chosen exact on $M$, i.e., of the form $\alpha = df$, where $f \in \mathcal{O}(M)$? Can such a $\xi$ be chosen complete for real time on $M$?

We now look at analytic subsets $A$ of $\Omega$ and their neighborhood bases.

**Theorem 14.4.** With the notation and hypotheses of Theorem 1.3, any analytic subset $A$ of $\Omega$ that can be defined as a set by $A = \{x \in \Omega : s(x) = 0\}$, where $s \in \mathcal{O}(\Omega, E)$ is a holomorphic section of a holomorphic Banach vector bundle $E \to \Omega$ can in fact be defined as a set by $A = \{x \in \Omega : f(x) = 0\}$, where $f \in \mathcal{O}(\Omega, Z_1)$ is a holomorphic function with values in a Banach space $Z_1$.

**Proof.** Theorem 1.3(b) gives an injective map $I \in \mathcal{O}(\Omega, \text{Hom}(E, Z_1))$ of
holomorphic Banach vector bundles. Defining \( f \in \mathcal{O}(\Omega, Z_1) \) by \( f(x) = I(x)s(x) \) completes the proof of Theorem 14.4.

Theorem 14.4 applies in many cases. If \( A \subset \Omega \) is a possibly singular hypersurface, or an iterated hypersurface in the sense that \( A = M_n \subset M_{n-1} \subset M_1 \subset M_0 = \Omega \), where \( M_i \) is smooth complex hypersurface in \( M_{i-1}, i = 1, \ldots, n \), then \( A \) can be defined as a set by \( A = \{ x \in \Omega' : s(x) = 0 \} \) as in Theorem 14.4, where \( \Omega' \) is pseudoconvex open with \( A \subset \Omega' \subset \Omega \). See [P3], where it is also shown that if \( A \) can be defined as a set by \( A = \{ x \in \Omega : s(x) = 0 \} \) as in Theorem 14.4, then \( A \) has a neighborhood basis consisting of pseudoconvex open subsets of \( \Omega \).

In classical Stein theory the sheaves \( \mathcal{O}_Z, Z = \mathbb{C}^n, n \geq 1 \), are called free sheaves and serve as building blocks in acyclic resolutions of more general analytic sheaves, called coherent analytic sheaves. Let us call the sheaf \( \mathcal{O}_Z \) a model sheaf, where \( Z \) is any Banach space. These model sheaves can be used to define a class of sheaves that is analogous to the class of coherent analytic sheaves in finite dimensions, see [P5].

**Theorem 14.5.** With the notation and hypotheses of Theorem 1.3.

(a) There is a holomorphic operator-valued function \( A \in \mathcal{O}(\Omega, \text{End}(Z_1)) \) such that \( E \) is holomorphically isomorphic over \( \Omega \) to the cokernel of \( A \), i.e., there is a locally (and in fact globally) split short exact sequence \( 0 \rightarrow \Omega \times Z_1 \rightarrow \Omega \times Z_1 \rightarrow E \rightarrow 0 \) of holomorphic Banach vector bundles over \( \Omega \) with the second map being \( (x, \xi) \mapsto (x, A(x)\xi) \)

(b) The sheaf \( \mathcal{O}^E \) has a global resolution \( 0 \rightarrow \mathcal{O}^{Z_1} \rightarrow \mathcal{O}^{Z_1} \rightarrow \mathcal{O}^E \rightarrow 0 \) over \( \Omega \) by model sheaves that is exact on the level of germs and on the level of global section over any pseudoconvex open subset \( U \) of \( \Omega \), if plurisubharmonic domination holds in \( U \).

**Proof.** Part (a) follows easily from Theorem 1.3(b) while (b) follows from (a) noting that any locally split short exact sequence \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \) of holomorphic Banach vector bundles is in fact globally split since \( H^1(\Omega, \mathcal{O}^F) = 0 \) for \( F = \text{Hom}(E'', E') \) by Theorem 1.3(c). The proof of Theorem 14.5 is complete.

Vector bundles are also applicable to some questions on the algebra of holomorphic functions such as the weak Nullstellensatz or completion of holomorphic matrices to invertible holomorphic matrices.

**Theorem 14.6.** Let \( X \) be a Banach space with a Schauder basis, \( \Omega \subset X \) pseudoconvex open, and \( f = (f_n) \in \mathcal{O}(\Omega, \ell_2) \) with \( f(x) \neq 0 \) for \( x \in \Omega \). Then there is a \( g = (g_n) \in \mathcal{O}(\Omega, \ell_2) \) with \( \sum_{n=1}^{\infty} f_n(x)g_n(x) = 1 \) for \( x \in \Omega \).

**Proof.** Look at the short exact sequence \( 0 \rightarrow K \rightarrow \Omega \times \ell_2 \rightarrow \Omega \times \mathbb{C} \rightarrow 0 \) of
holomorphic Hilbert vector bundles over $\Omega$, where the second map is inclusion and the third is $(x, \xi) \mapsto F(x, \xi) = \sum_{n=1}^{\infty} f_n(x) \xi_n$. As $H^1(\Omega, \mathcal{O}^K) = 0$ by Theorem 1.3(c), our short exact sequence splits, i.e., there is a map $\xi = g(x)$ with $F(x, g(x)) = 1$ for $x \in \Omega$. The proof of Proposition 14.6 is complete.

Theorem 14.7. Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, and $f_1, \ldots, f_n \in \mathcal{O}(\Omega, \ell_2)$, $n \geq 1$, column vectors that are pointwise linearly independent in $\ell_2$. If plurisubharmonic domination holds in $\Omega$, then there are further columns $f_{n+1}, f_{n+2}, \ldots \in \mathcal{O}(\Omega, \ell_2)$ with the property that the matrix $A(x)$ whose $(i, j)$ component is the $i$'th component of the vector $f_j(x)$ defines an invertible operator $A \in \mathcal{O}(\Omega, GL(\ell_2))$ the first $n$ columns of which are the given $f_1, \ldots, f_n$.

Proof. Look at the short exact sequence $0 \to \Omega \times \mathbb{C}^n \to \Omega \times \ell_2 \to E \to 0$ of holomorphic Hilbert vector bundles, where the second map is $(x, \xi) \mapsto \sum_{i=1}^{n} f_i(x) \xi_i$. As $E$ is holomorphically isomorphic to $\Omega \times \ell_2$ by Theorem 1.3(d), and as the above short exact sequence holomorphically splits over $\Omega$ by Theorem 1.3(c), the proof of Theorem 14.7 is complete.

Below is a version of Theorem 14.7 for a finite matrix of determinant 1.

Theorem 14.8. Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, and $f_1, \ldots, f_n \in \mathcal{O}(\Omega, \mathbb{C}^N)$, $1 \leq n < N$, column vectors that are pointwise linearly independent in $\mathbb{C}^N$. If plurisubharmonic domination holds in $\Omega$, and there are further continuous columns $f_{n+1}, \ldots, f_N \in C(\Omega, \mathbb{C}^N)$ with the property that the matrix $A(x)$ whose $(i, j)$ component is the $i$'th component of the vector $f_j(x)$ has determinant 1 for $x \in \Omega$, and the first $n$ columns of which are the given $f_1, \ldots, f_n$, then there are holomorphic $f_{n+1}, \ldots, f_N \in \mathcal{O}(\Omega, \mathbb{C}^N)$ with the same property.

Proof. Look at the short exact sequence $0 \to \Omega \times \mathbb{C}^n \to \Omega \times \mathbb{C}^N \to E \to 0$ of holomorphic vector bundles, where the second map is $(x, \xi) \mapsto \sum_{i=1}^{n} f_i(x) \xi_i$. As $E$ is continuously trivial by the assumption about the existence of continuous augmentation $f_{n+1}, \ldots, f_N$, we see by Theorem 1.3(d) that $E$ is holomorphically trivial as well over $\Omega$, and as the above short exact sequence holomorphically splits over $\Omega$ by Theorem 1.3(c), the proof of Theorem 14.8 is complete. Note that if $\Omega$ is contractible, say, star-like or convex, then there is a continuous augmentation, and Theorem 14.8 applies.

The meaning of Theorem 14.8 is that under some conditions one (or several independent) unimodular row(s) of the ring $\mathcal{O}(\Omega)$ is (are) equivalent to the standard unimodular row $(1, 0, \ldots, 0)$ (or rows $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, \ldots, $(0, \ldots, 0, 1, 0, \ldots, 0)$) in the sense of algebra.

We now look at approximation and interpolation for holomorphic sections of a holomorphic Banach vector bundle in certain cases.
Theorem 14.9. With the notation and hypotheses of Theorem 1.3 the following version of Runge approximation holds for holomorphic sections of $E$. There are an embedding $I \in \mathcal{O}(\Omega, \text{Hom}(E, Z_1))$ of $E$ into the trivial bundle $\Omega \times Z_1$ as a direct summand $IE$, and a projection $P \in \mathcal{O}(\Omega, \text{End}(Z_1))$ onto $IE$. There is an $\alpha$ with $8\alpha \in A$ with $P$ bounded and uniformly continuous on $\Omega_N(\alpha)$ for all $N \geq 1$. Choose $\gamma \in A$ as in Proposition 2.2(b), and $\gamma' \in A$ as in Proposition 2.2(a). Let $\varepsilon > 0$, $N \geq 1$, and $f \in \mathcal{O}(\Omega_N(\alpha), E)$ such that $If \in \mathcal{O}(\Omega_N(\alpha), Z_1)$ is bounded and uniformly continuous. Then the following hold.

(a) There is a $g \in \mathcal{O}(\Omega_{N+1}(\alpha), E)$ with $Ig$ bounded and uniformly continuous such that $||I(x)(f(x) - g(x))|| < \varepsilon$ for $x \in \Omega_N(\gamma)$.

(b) There is a $g \in \mathcal{O}(\Omega, E)$ with $Ig$ bounded and uniformly continuous on $\Omega_{N+p}(\gamma')$ for all $p \geq 0$ such that $||I(x)(f(x) - g(x))|| < \varepsilon$ for $x \in \Omega_N(\gamma')$.

Proof. Theorem 1.3(b) provides $I$ and $P$, and for a given $P$ plurisubharmonic domination in $\Omega$ a required Hartogs radius function $\alpha$. An application of Proposition 3.3 completes the proof of Theorem 14.9.

Theorem 14.10. With the notation and hypotheses of Theorem 1.3. Let $x_n, n \geq 1$, be a sequence of points of $\Omega$, $\varepsilon_n > 0$, and suppose that the balls $B_n = B_X(x_n, 2\varepsilon_n)$ are disjoint from one another, and that the set $\{x_n : n \geq 1\}$ has no limit points in $\Omega$. Let $m_n \geq 0$ be integers for $n \geq 1$.

(a) Let $f_n \in \mathcal{O}(B_n, Z)$ for $n \geq 1$. Then there is an $f \in \mathcal{O}(\Omega, Z)$ with $||f(x) - f_n(x)|| = O(||x - x_n||^{m_n+1})$ as $x \to x_n$ for all $n \geq 1$. Let $J \to \Omega$ be the sheaf of germs of holomorphic functions $\Omega \to Z$ that vanish at the points $x_n$ to order $m_n$ for all $n \geq 1$. Then $H^q(\Omega, J) = 0$ for all $q \geq 1$.

(b) Suppose that $\varepsilon_n, n \geq 1$, are so small that $E|B_n$ has a trivialization $T_n \in \mathcal{O}(B_n, \text{Hom}(E, Z))$, and let $f_n \in \mathcal{O}(B_n, E)$ be local sections. Then there is an $f \in \mathcal{O}(\Omega, E)$ with $||T_n(x)(f(x) - f_n(x))|| = O(||x - x_n||^{m_n+1})$ as $x \to x_n$ for all $n \geq 1$. Let $J$ be the sheaf of germs of holomorphic sections $\Omega \to E$ that vanish at the points $x_n$ to order $m_n$ for all $n \geq 1$. Then $H^q(\Omega, J) = 0$ for all $q \geq 1$.

Proof. In both (a) and (b) the vanishing $H^q(\Omega, J) = 0$ for $q \geq 1$ follows from Theorem 1.3(ac) in a standard way if the said interpolations are possible.

To prove interpolation in (a) choose by plurisubharmonic domination in $\Omega$ an $\alpha$ with $10\alpha \in A$ so that the covering $\mathcal{B}(\alpha)$ by radius $\alpha$ balls is a refinement of the covering $\{\Omega \setminus \bigcup_{n=1}^{\infty} \overline{B_n}, B_n : n \geq 1\}$ of $\Omega$. Then in the exhausting sets $\Omega_N(\alpha)$ only finitely many points $x_n$ may fall for all $N \geq 1$. Complete the proof as in [P2, §8] relying on Proposition 3.2 here for Runge approximation there.
To prove interpolation in (b) choose $I$ and $P$ as in the proof of Theorem 14.9 and apply (a) to interpolate $I f_n$ by an $f'$, and project back to define $f$ by $f = P f'$. The proof of Theorem 14.10 is complete.

Note that in Theorems 14.4–14.10 the open set $\Omega$ can be replaced by a complex Banach manifold $M$ as in Theorem 14.1.

15. DISCUSSION.

In this section we make some remarks about the methods that we chose in this paper.

In the vanishing theorems of classical Stein theory such as Theorem B and the Grauert–Oka principle the main ingredient are holomorphically convex compact exhaustions, Runge-type approximation for holomorphic functions over holomorphically convex compact sets, the Oka coherence theorem, and some generalities on sheaf theory.

The Oka coherence theorem does not seem to have an analog in infinite dimensions, but since most of the interesting bundles such as the tangent bundle and its associated bundles are of infinite rank, the Oka coherence theorem, which is in essence a finiteness result, could not be profitable. It is an interesting question to find a class of analytic sheaves for which an analog of the classical Theorem B can be proved.

The most important properties of compact sets for the purposes of vanishing theorems seem to be that over them continuous and holomorphic functions are bounded and uniformly continuous. While boundedness would have sufficed in the proof of Theorem 1.3(a) for the additive case of $Z$, the uniform continuity was crucial for the classical trick of telescopic products in connection with $GL(Z)$ to go through. It is true that over sets $\Omega(D, R)$ as in (3.1) uniform continuity over an arbitrarily slightly smaller set of the same type can be deduced from boundedness by the Schwarz lemma for holomorphic functions, it was much more convenient to build it in the definitions. Especially so for Theorem 6.8 since continuous functions in infinite dimensions are not necessarily locally uniformly continuous. Another important property of compact sets is that they can be covered by finitely many open sets at an arbitrary scale. This, while cannot be fully matched in infinite dimensions with open sets of exhaustion, can be modelled by considering a fixed scale $\alpha \in \mathcal{A}$ as a Hartogs radius function at which the object at hand (say, a cocycle to be resolved) can be handled.

The Runge approximation was necessary only for bounded and uniformly continuous functions with values in a Banach space $Z$ or a Banach Lie group $G = GL(Z)$ for which it is straightforward.
The exhaustion was made possible by plurisubharmonic domination which in essence seems to be a useful holomorphic convexity property that allows one to make flexible exhaustions to match the size properties of the data. In the classical case one Stein exhaustion works for all cocycles of a coherent analytic sheaf, but in infinite dimensions such a one size fits all approach seems unavailable. The greatest virtue of plurisubharmonic domination is that it allows the construction of a fine scale based on the data at which to work.

It was László Lempert who introduced in [L1] (after some precursors; see [N]) the way to exhaust a pseudoconvex open set by so-called open sets of type (B) as in (3.1) here along with the fairly classical way of handling functions over one open set of type (B), in [L2] the idea of plurisubharmonic domination, and in [L3] how to use it to make an exhaustion tailored to the data.

The methods of this paper build on all of these ideas of Lempert along with some useful changes and twists. Our methods have the advantage (besides proving stronger results than in [L3]) that they make up a last on which a number of other vanishing theorems can be fashioned. Notice the analogous proofs of Theorem 1.3(a) and Theorem 6.8. Further applications include the proof of a fuller version of the Grauert–Oka principle in [P6], and amalgamation of syzygies in [P5].

Schauder basis for the ground Banach space $X$ is not fully necessary in Theorem 1.3, since the requirement that $X$ have a Schauder basis can be replaced by demanding that $X$ be a direct summand in a Banach space with a Schauder basis, i.e., $X$ have the bounded approximation property by a theorem of Pełczyński’s. Regarding the other hypothesis of Theorem 1.3 that in $\Omega$ plurisubharmonic domination hold, it is reasonable to hope (while not currently proved) that it may hold in any pseudoconvex open $\Omega$ subset of a Banach space $X$ with the bounded approximation property. Another advantage of plurisubharmonic domination over Runge approximation on balls seems to be that it can be formulated in Banach manifolds, it seems more directly relevant to the business of vanishing, and also it appears to inherit better to submanifolds.

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