Convergence to scale-invariant Poisson processes and applications in Dickman approximation*

Chinmoy Bhattacharjee† Ilya Molchanov‡

Abstract

We study weak convergence of a sequence of point processes to a scale-invariant simple point process. For a deterministic sequence \((z_n)_{n \in \mathbb{N}}\) of positive real numbers increasing to infinity as \(n \to \infty\) and a sequence \((X_k)_{k \in \mathbb{N}}\) of independent non-negative integer-valued random variables, we consider the sequence of point processes

\[ \nu_n = \sum_{k=1}^{\infty} X_k \delta_{z_k/z_n}, \quad n \in \mathbb{N}, \]

and prove that, under some general conditions, it converges vaguely in distribution to a scale-invariant Poisson process \(\eta_c\) on \((0, \infty)\) with the intensity measure having the density \(ct^{-1}, t \in (0, \infty)\). An important motivating example from probabilistic number theory relies on choosing \(X_k \sim \text{Geom}(1 - 1/\log p_k)\) and \(z_k = \log p_k, k \in \mathbb{N}\), where \((p_k)_{k \in \mathbb{N}}\) is an enumeration of the primes in increasing order. We derive a general result on convergence of the integrals \(\int_0^1 t \nu_n(dt)\) to the integral \(\int_0^1 t \eta_c(dt)\), the latter having a generalized Dickman distribution, thus providing a new way of proving Dickman convergence results.

We extend our results to the multivariate setting and provide sufficient conditions for vague convergence in distribution for a broad class of sequences of point processes obtained by mapping the points from \((0, \infty)\) to \(\mathbb{R}^d\) via multiplication by i.i.d. random vectors. In addition, we introduce a new class of multivariate Dickman distributions which naturally extends the univariate setting.

Keywords: Poisson processes; Vague convergence; scale invariance; random measures; Dickman distributions.

AMS MSC 2010: 60G55; 11K99; 60F05; 60G57.

Submitted to EJP on November 27, 2019, final version accepted on June 8, 2020.

Supersedes arXiv:1911.06229.

*Supported by the Swiss National Science Foundation Grant No. 200021_175584.
†Institut für Mathematische Statistik und Versicherungslehre, University of Bern, Switzerland.
E-mail: chinmoy.bhattacharjee@stat.unibe.ch, ilya.molchanov@stat.unibe.ch
1 Introduction

Consider a locally compact separable metric space $S$ with Borel $\sigma$-algebra $\mathcal{S}$. Let $\mathcal{M}(S)$ denote the space of all locally finite non-negative measures on $S$. This space is endowed with the vague topology generated by assuming continuity of the integration maps $\mu \mapsto \mu f = \int_S f(x)\mu(dx)$ for all $f$ from the family $\mathcal{C}_S$ of bounded non-negative continuous functions on $S$ with relatively compact support. A random measure $\xi$ is a random element in $\mathcal{M}(S)$, equivalently, $\xi A = \xi 1_A$ is a random variable for each relatively compact Borel set $A$. The associated notion of convergence in distribution of random measures is called vague convergence in distribution, denoted henceforth by $d_v$, see [11, 12]. When considering point processes, we restrict ourselves to the subclass $\mathcal{N}(S) \subset \mathcal{M}(S)$ of counting measures (that is, taking values in $\mathbb{N}_0$, the set of non-negative integers). A random measure $\xi$ is said to have a finite intensity if $E(\xi A) < \infty$ for every relatively compact Borel set $A$.

In this paper, we are particularly interested in vague convergence in distribution to scale-invariant Poisson processes. A random measure $\xi$ on $S$ is scale-invariant if its distribution is invariant with respect to a group of scaling transformations of $S$. Even though convergence to stationary Poisson processes has been extensively studied in the literature, studies regarding convergence to scale-invariant processes seem to be rare. Distributional properties of scale-invariant Poisson processes on the half-line $(0, \infty)$ are surveyed in [2]. While a simple transformation relates a scale-invariant Poisson process on $(0, \infty)$ to a stationary Poisson process on the line, such a transformation is not readily available in general Euclidean spaces.

Throughout the sequel, we take $S = \mathbb{R}^d \setminus \{0\}$, $d \in \mathbb{N}$, that is, the Euclidean space with the origin removed. On the half-line, for $c > 0$, we denote by $\eta_c$ the scale-invariant Poisson process on $(0, \infty)$ with intensity measure $ct^{-1}dt$, and we will simply write $\eta$ for $\eta_1$.

Scale-invariant processes naturally arise as limits of point processes when a scaling is applied to the support points of the point processes. For measures, this amounts to scaling of their arguments, namely, the scaling of $\nu \in \mathcal{M}(S)$ by $t > 0$ is defined as

$$ T_t\nu(A) = \nu(t^{-1}A), \quad A \in \mathcal{S}. \quad (1.1) $$

We call this operation intrinsic scaling. In Section 2, we show that random measures when intrinsically scaled, naturally yield scale-invariant measures as limits. As an application, we generalize a result in [10] proving that the intrinsically scaled process of jump sizes in a pure-jump subordinator converges vaguely in distribution to a scale-invariant Poisson process, and as a consequence, the sum of small jumps in the process converges to a Dickman distribution.

In this paper, our basic objects of interest are point processes on $(0, \infty)$ of the following type. Let $(z_k)_{k \in \mathbb{N}}$ be a sequence of positive deterministic numbers with $z_n \uparrow \infty$ as $n \to \infty$. For a sequence $(X_k)_{k \in \mathbb{N}}$ of independent random variables in $\mathbb{N}_0$, define the point process

$$ \nu = \sum_{k=1}^{\infty} X_k \delta_{z_k}, $$

where $\delta_x$ denotes the Dirac measure at $x$. Rescaling the support points of $\nu$ by $(z_n)_{n \in \mathbb{N}}$ yields the sequence of point processes

$$ \nu_n A = T_{z_n} \nu(A) = \nu(z_n^{-1}A), \quad A \in \mathcal{S}, \quad n \in \mathbb{N}. \quad (1.2) $$

In Section 3, we study the convergence of such processes; these results are extended to point processes in multidimensional Euclidean spaces in Section 4.
Our interest in the scale-invariant Poisson process $\eta_c$ also stems from its connection to the Dickman distributions. It is well known that the sum of points of $\eta_c$ lying in the interval $(0,1)$ is distributed as a generalized Dickman random variable denoted hereafter by $D_c$ for $c > 0$, with $D = D_1$ being a standard Dickman random variable. The generalized Dickman distribution with parameter $c > 0$ can be defined as the unique non-negative fixed point of the distributional transformation $W \mapsto W^*$ given by

$$W^* =_d Q^{1/c}(W + 1),$$

where $=_d$ denotes equality in distribution and $Q$ is a uniformly distributed random variable on $[0,1]$ independent of $W$. It was introduced in the work of Dickman [13] in the context of smooth numbers and since then has appeared, sometimes curiously, in various areas including probabilistic number theory [9, 23], minimal directed spanning trees [8, 21], quickselect sorting algorithm [15, 16] and log-combinatorial structures [4, 6].

Given the various application, not surprisingly, there have been many works studying weak convergence to Dickman distributions [16, 21, 23] and, more recently, Stein’s method has been used to provide non-asymptotic bounds for Dickman approximations [1, 9, 15]. In [22], Pinsky provided some general conditions under which certain randomly weighted Bernoulli sums converge to a generalized Dickman random variable. But, to the best of our knowledge, there has been no other attempt to characterize the domain of attraction of the Dickman distributions. Elaborating on [3], one aim of this work is to identify a broad class of random variables which asymptotically behave like a Dickman random variable. To do this, we make use of the fact that

$$D_c =_d \int_0^1 t\eta_c(dt) = \sum_{t \in \eta_c \cap (0,1)} t.$$

Hence, if a sequence of point processes converges vaguely in distribution to $\eta_c$, then, under certain natural additional conditions, sums of their points in the interval $(0,1)$ converge in distribution to the Dickman random variable $D_c$. Thus, our approach via scale-invariant Poisson processes yields a new tool to prove Dickman convergences and provides useful insights into why such convergences occur. We note here that a similar approach concerning limit theorems for point processes in relation to the behaviour of sums of their points has previously been discussed in [5]. Also, the simpler case of Poisson processes converging to $\eta_c$ on $(0, \infty)$ was considered in [10]. Scale-invariant Poisson processes also arise in limit theorems for records, see e.g. [7] and references therein.

In Section 5, we characterize scale-invariant Poisson processes in general dimension $d$, and show that any such process can be obtained by independently multiplying each point of a scale-invariant Poisson process on $(0,\infty)$ with independent and identically distributed unit vectors in $\mathbb{R}^d$. Such a characterization naturally leads to a multivariate generalization of the Dickman distribution. Analogous to the univariate case, these multivariate Dickman distributions are fixed points of a distributional transform

$$W^* =_d Q^{1/c}(W + U),$$

where $Q$ is a uniform random variable on $[0,1]$ and $U$ a unit random vector in $\mathbb{R}^d$, independent of everything else.

Some results concerning weak convergence of general point processes (not necessarily scale-invariant) are collected in the Appendix.

## 2 Intrinsic scaling of random measures

Let $\tilde{\mathcal{S}} \subset \mathcal{S}$ denote the family of relatively compact Borel sets in $S = \mathbb{R}^d \setminus \{0\}$ for some $d \in \mathbb{N}$. A subclass $\mathcal{U} \subset \tilde{\mathcal{S}}$ is called dissecting if every open set can be expressed as a
Convergence to scale-invariant Poisson processes

countable union of sets from $\mathcal{U}$ and every set in $\widehat{\mathcal{S}}$ can be covered by finitely many sets in $\mathcal{U}$. Recall that a subclass $\mathcal{I} \subset \widehat{\mathcal{S}}$ is a ring if it is closed under proper differences and under finite unions and intersections. In the special case of $(0, \infty)$, we will often take the dissecting ring $\mathcal{U}$ to be the family of finite unions of semi-open intervals $(a, b]$ with $0 < a < b < \infty$.

Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of point processes in $S$. It is well known that the vague convergence in distribution $\xi_n \xrightarrow{d} \xi$ for a simple $\xi$ follows from the one-dimensional weak convergences $\xi_n A \xrightarrow{d} \xi A$ for all $A$ from the dissecting ring

$$\mathcal{U} \subset \widehat{\mathcal{S}}_{\xi} = \{ B \in \widehat{\mathcal{S}} : \mathbb{E} \xi(\partial B) = 0 \},$$

where $\partial B$ denotes the boundary of $B$, see e.g. [19, Chapter 4]. A measure $\mu \in \mathcal{M}(S)$ is said to be scale-invariant if $T_c \mu = \mu$ for all $c > 0$, where $T_c$ is defined at (1.1). The next result shows that the limit of the sequence of random measures obtained by intrinsic scalings of a given random measure $\nu$ is necessarily scale-invariant under some mild conditions on the normalizing constants. For deterministic measures, similar results are known, see e.g. [20, Theorem 3.1]. We write $S^{d-1}$ for the $d$-dimensional unit sphere and $B_r$ for the closed ball of radius $r > 0$ around the origin.

**Lemma 2.1.** Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers increasing to infinity with $\lim_{n \to \infty} s_{n-1}/s_n = 1$, and let $\mu, \nu \in \mathcal{M}(S)$ be random measures with finite intensities such that $T_{s_n} \nu \xrightarrow{d} \mu$ as $n \to \infty$. Then $T_{t} \nu \xrightarrow{d} \mu$ as $t \to \infty$, and the limiting measure $\mu$ is scale-invariant.

**Proof.** Since $\mu$ has finite intensity, the family of sets

$$\mathcal{U} = \{ A \times [a, b] : \mathbb{E} \mu(\partial A \times (0, \infty)) = \mathbb{E} \mu(\partial(B_a) \cup \partial(B_b)) = 0, A \subseteq S^{d-1}, 0 < a < b < \infty \}$$

forms a dissecting semi-ring. Hence, the first claim will follow (see [17, Theorem 1.1]) by establishing that

$$(T_{t} \nu(A_i \times [a_i, b_i]))_{i \in [k]} \xrightarrow{d} (\mu(A_i \times [a_i, b_i]))_{i \in [k]} \quad \text{as} \quad n \to \infty \quad (2.1)$$

for all $k \in \mathbb{N}$ and $A_i \times [a_i, b_i] \in \mathcal{U}$, $i = 1, \ldots, k$.

To simplify the argument, assume that $k = 1$; for general $k \in \mathbb{N}$, one can argue similarly. For $t > 0$, let $n(t)$ be the integer such that $s_{n(t)} < t \leq s_{n(t)+1}$. Fix a Borel set $A \subseteq S^{d-1}$ and $0 < a < b < \infty$ with $A \times [a, b] \in \mathcal{U}$ and $\varepsilon \in (0, b-a)$. Since $\lim_{n \to \infty} s_{n-1}/s_n = 1$ and $n(t) \to \infty$ as $t \to \infty$,

$$\frac{a}{s_{n(t)+1}} > \frac{a - \varepsilon}{s_{n(t)}} \quad \text{and} \quad \frac{b}{s_{n(t)+1}} > \frac{b - \varepsilon}{s_{n(t)}}$$

for all sufficiently large $t$. Hence, for $t$ large enough, we have

$$T_{t} \nu(A \times [a, b]) \leq \nu(A \times [a/s_{n(t)+1}, b/s_{n(t)}]) \leq T_{s_{n(t)}} \nu(A \times [a - \varepsilon, b]).$$

A similar argument yields a lower bound, so that

$$T_{s_{n(t)}} \nu(A \times [a, b - \varepsilon]) \leq T_{t} \nu(A \times [a, b]) \leq T_{s_{n(t)}} \nu(A \times [a - \varepsilon, b])$$

for all sufficiently large $t$. Since $n(t) \to \infty$ as $t \to \infty$ and $T_{s_n} \nu \xrightarrow{d} \mu$ as $n \to \infty$, we obtain that

$$\limsup_{t \to \infty} \mathbb{P}\{ T_{t} \nu(A \times [a, b]) \leq x \} \leq \mathbb{P}\{ \mu(A \times [a, b - \varepsilon]) \leq x \}$$
and
\[
\liminf_{t \to \infty} P\{T_t \nu(A \times [a, b]) \leq x\} \geq P\{\mu(A \times [a - \varepsilon, b]) \leq x\}
\]
for \(x \geq 0\). Since \(E\|\partial(B_a) \cup \partial(B_b)\| = 0\),
\[
\lim_{\varepsilon \to 0} P\{\mu(A \times [a, b - \varepsilon]) \leq x\} = \lim_{\varepsilon \to 0} P\{\mu(A \times [a - \varepsilon, b]) \leq x\} = P\{\mu(A \times [a, b]) \leq x\},
\]
which, together with the two inequalities above yield (2.1), proving the first claim.

Finally, let \(v : S \to \mathbb{R}\) be a bounded continuous function with relatively compact support. For \(c > 0\), since \(T_t \nu \overset{d}{\to} \mu\) as \(t \to \infty\),
\[
\lim_{t \to \infty} T_t \nu(v) = \lim_{t \to \infty} \int_S v(x) T_t \nu(dx) = \lim_{t \to \infty} \int_S v(cx) T_t \nu(dx) = \int_S v(x) \mu(dx) = T_t \mu(v),
\]
which implies that
\[
T_t \nu \overset{d}{\to} \nu \quad \text{as} \quad t \to \infty.
\]
On the other hand, \(T_t \nu = T_{ct} \nu\) converges vaguely in distribution to \(\nu\) as \(t \to \infty\) by our assumption. Hence we obtain \(T_t \mu = \mu\), proving the scale invariance of \(\mu\). □

The following theorem proves Dickman convergence for the sums of small jump sizes in a pure-jump subordinator; we note here that the Dickman limit result is not new and has been proved in [10]. We prove a stronger result that the scaled point process of jump sizes converges to a scale-invariant Poisson process on \((0, \infty)\).

Let \(Y = (Y(t))_{t \geq 0}\) be a pure-jump subordinator with infinite Lévy measure \(\sigma\) and for \(\varepsilon > 0\), let \(Y_{\varepsilon}\) be the process obtained by removing the jumps of size larger than \(\varepsilon\) in the Lévy-Ito decomposition of \(Y\). For \(t > 0\), let \(\Pi_t\) denote the point process of jump sizes occurring in the time interval \([0, t]\). The scaled process \(T_{1/\varepsilon} \Pi_t\) consists of the points of \(\Pi_t\) scaled by \(\varepsilon\). Recall, \(D_c\) denotes a Dickman distributed random variable with parameter \(c > 0\).

**Theorem 2.2.** If \(\varepsilon^{-1} \int_0^\varepsilon x \sigma(dx) \to c > 0\) as \(\varepsilon \to 0\), then for any \(t > 0\),
\[
T_{1/\varepsilon} \Pi_t \overset{d}{\to} \eta_{ct} \quad \text{as} \quad \varepsilon \to 0.
\]
Moreover,
\[
\varepsilon^{-1} Y_{\varepsilon}(t) \overset{d}{\to} D_{ct} \quad \text{as} \quad \varepsilon \to 0.
\]

**Proof.** Arguing as in the proof of [10, Theorem 2.1], letting \(\psi\) and \(\psi_{\varepsilon}, \varepsilon > 0\) be the measures given by \(\psi(dx) = 1_{(0, 1)}(x) \sigma(dx)\) and \(\psi_{\varepsilon}(dx) = x \cdot T_{1/\varepsilon} \sigma(dx) = x \sigma(\varepsilon dx)\) respectively, for any \(p \in (0, 1)\), we have
\[
\psi_{\varepsilon}((0, p]) = \int_0^p x \sigma(dx) = \frac{1}{\varepsilon} \int_0^{\varepsilon p} z \sigma(dx) \to cp = \psi((0, p]) \quad \text{as} \quad \varepsilon \to 0.
\]
By Lemma A.2,
\[
T_{1/\varepsilon} \sigma((p, 1]) = \int_p^1 x^{-1} \psi_{\varepsilon}(dx) \to \int_p^1 x^{-1} \psi(dx) = c \log(1/p) \quad \text{as} \quad \varepsilon \to 0,
\]
which yields that the Poisson process on \((0, \infty)\) with intensity measure \(T_{1/\varepsilon} \sigma\) converges vaguely in distribution to \(\eta_{c}\) as \(\varepsilon \to 0\). Since \(Y\) is a Lévy process with Lévy measure \(\sigma\), the jump process \(\Pi_t\) is distributed as a Poisson process on \((0, \infty)\) with intensity measure \(t \sigma\); this proves the first claim.
Convergence to scale-invariant Poisson processes

Finally, note that \( \varepsilon^{-1} Y_\varepsilon(t) = \int_0^1 x \left( T_{1/\varepsilon} \Pi_1 \right)(dx) \). To prove the last claim, by Lemma A.3, it suffices to check that

\[
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} E \int_0^\delta x \left( T_{1/\varepsilon} \Pi_1 \right)(dx) = 0. \tag{2.2}
\]

Since \( \Pi_1 \) is a Poisson process with intensity measure \( t\sigma \), we have that \( T_{1/\varepsilon} \Pi_1 \) is distributed as a Poisson process on \((0, \infty)\) with intensity measure \( t T_{1/\varepsilon} \sigma \). Thus, using the Mecke equation in the first equality and that \( \varepsilon^{-1} \int_0^\delta x \sigma(dx) \to c \) as \( \varepsilon \to 0 \) in the third, we obtain

\[
\limsup_{\varepsilon \to 0} E \int_0^\delta x \left( T_{1/\varepsilon} \Pi_1 \right)(dx) = \limsup_{\varepsilon \to 0} t \int_0^\delta x T_{1/\varepsilon} \sigma(dx) = \limsup_{\varepsilon \to 0} t \varepsilon^{-1} \int_0^\varepsilon x \sigma(dx) = ct \delta
\]

which implies (2.2), concluding the proof. \( \Box \)

3 Convergence to scale-invariant Poisson processes

Now we move our attention to proving convergence to scale-invariant Poisson processes for sequences of general (not necessarily Poisson) point processes. The necessary and sufficient conditions for vague convergence in distribution of point processes to a simple point process given by Theorem A.1, when applied to \( \nu_n \), given by (1.2) with \( \eta_c \) being the limit, translate to the following simpler condition. For convenience, denote

\[
q_k^0 = P\{X_k = 0\} \quad \text{and} \quad q_k^1 = P\{X_k = 1\}, \quad k \geq 1.
\]

**Condition 3.1.** There exists \( c > 0 \) such that for all \( 0 < a < b < \infty \),

(i) \( \prod_{k:az_k < z_k \leq bz_k} q_k^0 \to (a/b)^c \) as \( n \to \infty \).

(ii) \( \liminf_{n \to \infty} \sum_{k:az_k < z_k \leq bz_k} q_k^1 / q_k^0 \geq c \log(b/a) \).

**Theorem 3.2.** A sequence of point processes \((\nu_n)_{n \in \mathbb{N}}\) given by (1.2) converges vaguely in distribution to \( \eta_c \) for some \( c > 0 \) as \( n \to \infty \) if and only if \( (q_k^1, q_k^0)_{k \in \mathbb{N}} \) and \((z_n)_{n \in \mathbb{N}}\) satisfy Condition 3.1.

**Proof.** Condition 3.1(i) for the dissecting ring composed of finite unions of semi-open intervals is equivalent to condition (i) in Theorem A.1. Condition (ii) in Theorem A.1 is equivalent to

\[
\liminf_{n \to \infty} \left[ 1 + \sum_{k:az_k < z_k \leq bz_k} q_k^1 / q_k^0 \right] \prod_{l:az_l < z_l \leq bz_l} q_l^0 \geq \left( \frac{a}{b} \right)^c \left( 1 + c \log \frac{b}{a} \right),
\]

which, given Condition 3.1(i), simplifies to Condition 3.1(ii), proving the result. \( \Box \)

The next result concerns vague convergence to scale-invariant Poisson processes for a large class of point processes \( \nu_n \) of the form (1.2) and, as a consequence, establishes weak convergence of sums of the points in \((0, 1)\) of \( \nu_n \) to a generalized Dickman distributed random variable \( D_c \). Note that such a convergence does not readily follow from the vague convergence since \( \eta_c \) has infinitely many points in any neighbourhood of zero.

**Theorem 3.3.** For a monotone sequence of positive numbers \((z_k)_{k \geq 0}\) increasing to infinity with \( \lim_{k \to \infty} z_k / z_{k-1} = 1 \), let \((X_k)_{k \in \mathbb{N}}\) be independent random variables in \( \mathbb{N}_0 \) with

\[
q_k^0 = (z_{k-1}/z_k)^c \quad \text{and} \quad q_k^1 = q_k^0 (1 - q_k^0)
\]

EJP 25 (2020), paper 79. http://www.imstat.org/ejp/
for some $c > 0$. Then the sequence $(\nu_n)_{n \in \mathbb{N}}$ defined at (1.2) converges vaguely in distribution to $\eta_c$ as $n \to \infty$. If, in addition, $EX_k = O(\eta_k^1)$, then

$$\frac{1}{z_n} \sum_{k=1}^{n} z_k X_k \xrightarrow{d} D_c \quad \text{as } n \to \infty. \quad (3.1)$$

**Proof.** Fix $0 < a < b < \infty$. Let $M = \inf\{k : a z_n < z_k \leq b z_n\}$ and $N = \sup\{k : a z_n < z_k \leq b z_n\}$. Letting $\delta_n = a z_n - M - 1$ and $\delta'_n = b z_n - N$, one has

$$\limsup_{n \to \infty} (\delta_n / z_n) = a / b.$$ 

Since $\lim_{k \to \infty} z_k / z_{k+1} = 1$ and $M \to \infty$ as $n \to \infty$,

$$\limsup_{n \to \infty} \frac{\delta_n}{z_n} \leq \lim_{n \to \infty} \frac{z_M - z_{M-1}}{z_M} \xrightarrow{z_n} 0,$$

and a similar argument shows that $\limsup_{n \to \infty} \delta'_n / z_n = 0$. Thus,

$$\prod_{k : a z_n < z_k \leq b z_n} q_k^0 = \prod_{k : a z_n < z_k \leq b z_n} \left( \frac{z_k - 1}{z_k} \right)^c = \left( \frac{z_{M-1}}{z_N} \right)^c \to \left( \frac{a}{b} \right)^c$$

as $n \to \infty$. Also,

$$\liminf_{n \to \infty} \sum_{k : a z_n < z_k \leq b z_n} \frac{q_k}{q_k} \geq \liminf_{n \to \infty} \left( \frac{z_n - 1}{z_n} \right)^c \liminf_{n \to \infty} \sum_{k : a z_n < z_k \leq b z_n} \frac{z_k^c - z_{k-1}^c}{z_k^c}$$

$$\geq \liminf_{n \to \infty} \frac{1}{\int_{c_{M-1}}^c} \frac{1}{t} \log \frac{z_N}{z_{M-1}} = c \log \frac{b}{a}.$$

Hence, Condition 3.1 is satisfied and the first claim follows by Theorem 3.2.

If $EX_k = O(q_k^1)$, then there exists $C > 0$ such that $EX_k \leq C q_k^1$ for all $k \in \mathbb{N}$. Denoting by $\lceil \cdot \rceil$ the ceiling function and using the simple inequality that $1 - (1 - x)^c \leq 2^c \cdot x$ for $x \in [0, 1]$ in the penultimate step, we have

$$E \int_0^\varepsilon t \nu_n(dt) = \frac{1}{z_n} \sum_{k : z_k \leq z_n \varepsilon} z_k X_k \leq C \frac{1}{z_n} \sum_{k : z_k \leq z_n \varepsilon} z_k q_k^1$$

$$\leq C \frac{1}{z_n} \sum_{k : z_k \leq z_n \varepsilon} z_k \left(1 - \left(1 - \frac{z_k - z_{k-1}}{z_k}\right)^c\right)$$

$$\leq C \frac{1}{z_n} \sum_{k : z_k \leq z_n \varepsilon} 2^c (z_k - z_{k-1}) \leq C 2^c \varepsilon.$$

Therefore,

$$\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} E \int_0^\varepsilon t \nu_n(dt) = 0. \quad (3.2)$$

Thus, invoking Lemma A.3 we obtain

$$\int_0^1 t \nu_n(dt) = \frac{1}{z_n} \sum_{k=1}^{n} z_k X_k \xrightarrow{d} D_c \quad \text{as } n \to \infty. \quad \square$$

**Remark 3.4.** Recall that $X$ is a geometric random variable with parameter $p \in (0, 1)$ if $P\{X = m\} = (1 - p)^m p$ for $m \geq 0$; we then write $X \sim \text{Geom}(p)$. For $(z_k)_{k \in \mathbb{N}}$ as in Theorem 3.3, clearly $X_k \sim \text{Geom}(q_k^0)$ satisfies the conditions therein. We can also take
Convergence to scale-invariant Poisson processes

the random variables $X_k \sim \text{Ber}(q_k^0)$ with $q_k^0$ as in Theorem 3.3, i.e. $X_k$ is a $\{0, 1\}$-valued random variable with $P\{X_k = 0\} = q_k^0$. In this case, a similar proof shows that

$$
\nu_n = \sum_{k=1}^{\infty} X_k \delta_{z_k/z_n} \xrightarrow{d} \eta_c \quad \text{as} \quad n \to \infty.
$$

Since $EX_k = q_k^1$, arguing like in Theorem 3.3, one can establish (3.1) in this case as well.

Remark 3.5. Even though under Condition 3.1 the sequence $\nu_n$ converges vaguely in distribution to a simple process, it is not necessarily true that the $X_k$’s are $\{0, 1\}$-valued almost surely for all sufficiently large $n$. Consider the sequence $\nu_n$ as in Theorem 3.3 with $c = 1$ and $z_k$ defined sequentially by letting $z_0 = z_1 = 1$ and $z_n/z_{n-1} = \sqrt{n}/(\sqrt{n} - 1)$ for $n \geq 2$. Since

$$
z_n = \frac{\sqrt{n}}{\sqrt{n} - 1} z_{n-1} \geq \frac{n}{n-1} z_{n-1} \geq \cdots \geq n z_1 = n,
$$

Theorem 3.3 yields that $\nu_n \xrightarrow{d} \eta$ as $n \to \infty$. Furthermore,

$$
\sum_{k=1}^{\infty} P\{X_k \geq 2\} = \sum_{k=1}^{\infty} (1 - q_k^0 - q_k^1) = \sum_{k=2}^{\infty} (1 - z_{k-1}/z_k)^2 \geq \sum_{k=2}^{\infty} k^{-1},
$$

which diverges. By the Borel-Cantelli lemma, $X_k$ is strictly greater than 1 for infinitely many $k$. However, after rescaling, the number of points with multiplicities more than 1 in any bounded interval $[a, b] \subset (0, \infty)$ converges to zero.

The processes in Theorem 3.3 do not necessarily satisfy (A.4), since only $q_k^0$ and $q_k^1$ are specified there and one can allocate the rest of the probability on a large number to make $EX_k$ sufficiently large so that (A.4) does not hold. Hence, an additional condition like $EX_k = O(q_k^1)$ is essential. Note that, for $X_k \sim \text{Geom}(q_k^0)$, we have $q_k^1 = q_k^0(1 - q_k^0)$ and

$$
EX_k = (1 - q_k^0)/q_k = (1/q_k)^2 q_k = O(q_k^1),
$$

since $q_k^0 \to 1$ as $k \to \infty$.

Next, we describe a sequence of point processes arising in probabilistic number theory which satisfies Condition 3.1, and hence, converges to the scale-invariant Poisson process $\eta$ by Theorem 3.2 and the sums of points in $(0, 1)$ converge to the standard Dickman distribution. For an enumeration $(p_k)_{k \in \mathbb{N}}$ of the prime numbers in increasing order, let $\Omega_n$ denote the set of positive integers having all its prime factors less than or equal to the $n^{th}$ prime $p_n$. Let $M_n$ be a random variable distributed according to the probability mass function $\Theta_n$ with $\Theta_n(m)$ being proportional to the inverse of $m$ for $m \in \Omega_n$. Then one can show that (see e.g. [23])

$$
\frac{\log M_n}{\log p_n} \xrightarrow{d} \frac{1}{\log p_n} \sum_{k=1}^{n} X_k \log p_k,
$$

(3.3)

where $X_1, \ldots, X_n$ are independent with $X_k \sim \text{Geom}(1 - 1/p_k)$ for $1 \leq k \leq n$. The distributional convergence of the right-hand side of (3.3) to the standard Dickman distribution was proved in [23] with optimal convergence rates provided in [9] using Stein’s method. We prove that this convergence is a consequence of the underlying sequence of point processes converging to $\eta$.

Theorem 3.6. Let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence of point processes defined at (1.2) with $z_k = \log p_k$ and $X_k \sim \text{Geom}(1 - 1/p_k)$ for $k \in \mathbb{N}$. Then $\nu_n \xrightarrow{d} \eta$ as $n \to \infty$ and

$$
\frac{1}{\log p_n} \sum_{k=1}^{n} X_k \log p_k \xrightarrow{D} D \quad \text{as} \quad n \to \infty.
$$

EJP 25 (2020), paper 79.  
http://www.imstat.org/ejp/
Convergence to scale-invariant Poisson processes

Proof. For the first part, by Theorem 3.2, we only need to check Condition 3.1. Since \( q_k^0 = (1 - 1/p_k) \), for \( 0 < a < b < \infty \), by Merten’s formula (see e.g. [25, Prop. 1.51]),

\[
\prod_{k : a x \leq k \leq b x} q_k^0 = \prod_{k : p_k^a < p_k \leq p_k^b} \frac{1 - 1/p_k}{p} \rightarrow \frac{a}{b} \quad \text{as} \quad n \rightarrow \infty.
\]

Hence, Condition 3.1(i) is satisfied. For Condition 3.1(ii), since \( q_k^1 = p_k^{-1}(1 - p_k^{-1}) \), Merten’s formula yields that

\[
\sum_{k : a x \leq k \leq b x} q_k^1 / q_k^0 = \sum_{k : p_k^a < p_k \leq p_k^b} \frac{1}{p_k} \log \frac{b}{a} \quad \text{as} \quad n \rightarrow \infty.
\]

Theorem 3.2 now yields the first part of the result.

For the second part, by Lemma A.3, it suffices to check (A.4). Since

\[
\sum_{p_k \leq n} p_k^{-1} \log p_k = \log n + O(1)
\]

(see [25, Prop. 1.51]), it follows that for \( \varepsilon > 0 \),

\[
\mathbb{E} \int_0^\varepsilon t \nu_n(dt) = \frac{1}{\log p_n} \mathbb{E} \sum_{k=1}^\infty X_k \log p_k 1_{(1 < p_k \leq p_n)} \leq \frac{2}{\log p_n} [\log p_n + O(1)],
\]

which converges to \( \varepsilon \) as \( n \rightarrow \infty \). Thus, \( (\nu_n)_{n \geq 1} \) satisfies (A.4), proving the result.

\[ \square \]

Remark 3.7. Let \( X_k \sim \text{Ber}(1/(1 + p_k)) \), where \( p_k \) is the \( k \)-th prime number and consider \( (\nu_n)_{n \in \mathbb{N}} \) defined in Theorem 3.6. One can argue as in the proof of Theorem 3.6 to show that \( \nu_n \overset{d}{\rightarrow} \eta \) as \( n \rightarrow \infty \) and

\[
\frac{1}{\log p_n} \sum_{k=1}^n X_k \log p_k \overset{d}{\rightarrow} D \quad \text{as} \quad n \rightarrow \infty.
\]

As mentioned above, if the \( X_k \)-s are distributed as geometric random variables given in Theorem 3.6, the induced distribution on \( M_n = \prod_{k=1}^n p_k^{X_k} \) is the reciprocal distribution on the set \( \Omega_n \) of positive integers with all prime factors less than or equal to \( p_n \). If \( X_k \sim \text{Ber}(1/(1 + p_k)) \), the induced distribution on \( M_n \) turns out to be the reciprocal distribution on the set of square-free positive integers with all its prime factors less than or equal to \( p_n \).

Next, we provide a few more examples that arise as special cases of the class of point processes considered in Theorem 3.2 and in Remark 3.4.

Example 3.8. Let \( X_k \sim \text{Ber}(1/k) \), \( k \geq 1 \), be independent and \( \nu_n = \sum_{k=1}^n X_k \delta_{k/n} \). In this case, one can easily check that Condition 3.1 and (A.4) are satisfied. Hence, \( \nu_n \overset{d}{\rightarrow} \eta \) and \( n^{-1} \sum_{k=1}^n k X_k \overset{d}{\rightarrow} D \) as \( n \rightarrow \infty \). This is a well-known example arising in the context of counting sums of ‘records’ in a random permutation. For a uniformly random permutation \( \sigma \) of \( \{1, \ldots, n\} \), let \( S_n \) be the sum of records, which are positions \( k \) such that \( \sigma(k) > \max_{i \in [k-1]} \sigma(i) \). One can check that \( S_n \) is indeed distributed as \( \sum_{k=1}^n k X_k \).

Example 3.9. Let \( \nu_n \) be as in (1.2) with \( z_k = \log k \) and independent \( X_k \sim \text{Geom}(1 - 1/(k \log k)) \), \( k \in \mathbb{N} \). In this case, it is straightforward to check that the conclusions of Theorem 3.3 hold. Heuristically, this is equivalent to Theorem 3.6, since, by the prime number theorem, one has that the \( k \)-th prime number \( p_k \) is asymptotically of the order \( k \log k \).

Example 3.10. Theorem 3.2 and Lemma A.3 apply if \( X_k \)-s are independent Poisson random variables with mean \( 1/p_k \) and \( \nu_n \) is given by (1.2) with \( z_k = \log p_k \), \( k \in \mathbb{N} \).

EJP 25 (2020), paper 79.

http://www.imstat.org/ejp/
4 Convergence of uplifted point processes

In this section, we consider convergence of certain general point processes to scale-invariant Poisson processes in dimension $d$. These point processes are obtained by first taking a point process on $(0, \infty)$ and transforming (uplifting) its points to $\mathbb{R}^d$ by multiplying them with random vectors taking values in $S = \mathbb{R}^d \setminus \{0\}$. We start with a point process $\xi = \sum_{k=1}^{\infty} X_k \delta_{Z_k}$ with finite intensity on the positive half-line. Let $V$ be a random vector in $S$ with i.i.d. copies $(V_k)_{k \in \mathbb{N}}$ which are independent of $\xi$. Define the uplifted process $\xi^V$ as

$$\xi^V = \sum_{k=1}^{\infty} X_k \delta_{V_k Z_k}.$$  \hspace{1cm} (4.1)

We need to impose some conditions on $\xi$ and $V$ to ensure that $\xi^V$ is locally finite on $S$. To this end, throughout this section, we assume for any uplifted process $\xi^V$ that $\xi$ and $V$ satisfy

$$\mathbb{E} \sum_{k=1}^{\infty} X_k I_{\{Z_k \|V_k\| \in [a,b]\}} < \infty \quad \text{for all} \quad 0 < a < b < \infty,$$ \hspace{1cm} (4.2)

where $\| \cdot \|$ denotes the Euclidean norm. Since $\xi$ has a finite intensity, this condition is always satisfied if $V$ is bounded away from 0 and $\infty$. In Lemma 5.1, we show that any scale-invariant Poisson process in $S$ has the same distribution as the uplifted process $\eta^U_c$ for some $c > 0$ and a unit random vector $U$ in $\mathbb{R}^d$. Thus, our uplifting scheme is a natural choice to recover all scale-invariant point processes in $S$.

It is well known that, if $\xi_n \Rightarrow \xi$ as $n \to \infty$, then (see e.g. [19, Theorem 4.11])

$$\mathbb{E} e^{-\xi_n f} \to \mathbb{E} e^{-\xi f} \quad \text{as} \quad n \to \infty.$$ \hspace{1cm} (4.3)

for any $f \in \hat{C}_S$. In order to handle uplifting transformations by a possibly unbounded random vector $V$, we need to consider test functions $f$ with unbounded support. The following result extends (4.3) to more general functions.

**Lemma 4.1.** Let $(\xi_n)_{n \in \mathbb{N}}$ and $\xi$ be point processes on a locally compact separable metric space $\Omega$ with $\xi$ having a finite intensity, such that $\xi_n \Rightarrow \xi$ as $n \to \infty$. Let $h$ be a non-negative continuous function on $\Omega$ such that for any $\varepsilon > 0$, there exists a relatively compact set $K_{\varepsilon}$ with

$$\limsup_{n \to \infty} \mathbb{E} \int_{K_{\varepsilon}} h(x) \xi_n (dx) \leq \varepsilon.$$ \hspace{1cm} (4.4)

Then

$$\mathbb{E} e^{-\xi_n h} \to \mathbb{E} e^{-\xi h} \quad \text{as} \quad n \to \infty.$$ \hspace{1cm} (4.5)

For a proof, see the Appendix. For $f \in \hat{C}_S$, define the function $h_f : \mathbb{N}_0 \times (0, \infty) \to \mathbb{R}$ as

$$h_f(x, y) = -\log \mathbb{E} e^{-xf(Vy)}.$$ \hspace{1cm} (4.6)

Note that by Jensen’s inequality, one has

$$h_f(x, y) \leq x \mathbb{E} f(Vy).$$ \hspace{1cm} (4.7)

Define the map $M : \mathcal{N}((0, \infty)) \to \mathcal{N}(\mathbb{N}_0 \times (0, \infty))$ at $\xi = \sum_{k=1}^{\infty} a_k \delta_{z_k}$ as

$$M(\xi) = \sum_{k=1}^{\infty} \delta_{(a_k, z_k)}.$$ \hspace{1cm} (4.8)

This map turns a counting measure with possibly multiple points into a simple counting measure in the product space $\mathbb{N}_0 \times (0, \infty)$. 

EJP 25 (2020), paper 79. http://www.imstat.org/ejp/
Convergence to scale-invariant Poisson processes

**Theorem 4.2.** Assume that a sequence of point processes \( \xi_n = \sum_{k=1}^{\infty} X_k \delta_{Z_k^n}, n \in \mathbb{N} \) converges vaguely to a simple point process \( \xi \) with finite intensity in \( \mathcal{N}((0, \infty)) \) as \( n \to \infty \). Moreover, let \( V \) be a random vector in \( S \) with i.i.d. copies \((V_k)_{k \in \mathbb{N}}\) such that for every \( f \in \hat{C}_S \) and \( \varepsilon > 0 \), there exists a compact set \( K_{f,\varepsilon} \subseteq \mathbb{N}_0 \times (0, \infty) \) such that
\[
\limsup_{n \to \infty} \mathbb{E} \sum_{(X_k, Z_k^n) \in K_{f,\varepsilon}} X_k f(V_k, Z_k^n) \leq \varepsilon. \tag{4.8}
\]
Then \( \xi_n^V \overset{d}{\longrightarrow} \xi^V \) as \( n \to \infty \).

**Proof.** Fix \( f \in \hat{C}_S \). Then
\[
\mathbb{E} e^{-\xi_n^f} = \mathbb{E} \prod_{k=1}^{\infty} \mathbb{E} \left[ \exp \{-X_k f(V_k, Z_k^n)\} \mid \xi_n\right] = \mathbb{E} \exp \left\{ - \sum_{k=1}^{\infty} h_f(X_k, Z_k^n) \right\} = \mathbb{E} e^{-\xi_n h_f},
\]
where \( \xi_n = M(\xi_n) \) and \( h_f \) is given by (4.5). Since \( \xi_n \overset{d}{\longrightarrow} \xi \) as \( n \to \infty \) with \( \xi \) being simple, Lemma A.4 and the continuous mapping theorem yield that \( \xi_n \overset{d}{\longrightarrow} \xi = M(\xi) \). Clearly, \( h_f \) is continuous as \( f \) is such. Also note that by (4.6) and (4.8), we have that \( h_f \) satisfies (4.4) with respect to the processes \((\xi_n)_{n \in \mathbb{N}}\). By Lemma 4.1,
\[
\mathbb{E} e^{-\xi_n^f} = \mathbb{E} e^{-\xi_n h_f} \to \mathbb{E} e^{-\xi h_f} \quad \text{as} \quad n \to \infty.
\]
Finally, noticing that
\[
\mathbb{E} e^{-\xi^V f} = \mathbb{E} [\mathbb{E}(e^{-\xi^V f} \mid \xi)] = \mathbb{E} e^{-\xi h_f},
\]
we obtain
\[
\mathbb{E} e^{-\xi_n^f} \to \mathbb{E} e^{-\xi^V f} \quad \text{as} \quad n \to \infty
\]
for all \( f \in \hat{C}_S \), which proves that \( \xi_n^V \overset{d}{\rightarrow} \xi^V \) as \( n \to \infty \).

The condition (4.8) in Theorem 4.2 that \((V_n)_{n \in \mathbb{N}}\) and \((\xi_n)_{n \in \mathbb{N}}\) are required to satisfy can be hard to check in general. In some special cases, one can find some easily verifiable conditions on \((\xi_n)_{n \in \mathbb{N}}\) and \( V \) so that (4.8) is satisfied. Throughout, \( \| \cdot \|_\infty \) denotes the supremum norm on \( \hat{C}_S \).

**Lemma 4.3.** Let \((\xi_n)_{n \in \mathbb{N}}\) be a sequence of simple point processes in \( (0, \infty) \). Let \( V \) be such that for some \( \alpha > 0 \),
\[
\limsup_{t \to \infty} t \mathbb{P} \{ \|V\| \geq t \} < \infty \quad \text{and} \quad \limsup_{t \to \infty} t^\alpha \mathbb{P} \{ \|V\| \leq 1/t \} < \infty. \tag{4.9}
\]
Moreover, assume that
\[
\lim_{r \to 0} \limsup_{n \to \infty} \mathbb{E} \int_0^r t \xi_n(dt) = 0 \quad \text{and} \quad \lim_{r \to 0} \limsup_{n \to \infty} \mathbb{E} \int_r^\infty t^{-\alpha} \xi_n(dt) = 0. \tag{4.10}
\]
Then the processes \((\xi_n)_{n \in \mathbb{N}}\) and i.i.d. copies \((V_n)_{n \in \mathbb{N}}\) of \( V \) satisfy (4.8).

**Proof.** Since \( \xi_n \) is simple, for \( f \in \hat{C}_S \), it suffices to check that \( h(y) = \mathbb{E} f(V y) \) satisfies
\[
\begin{align*}
\lim_{r \to 0} \limsup_{n \to \infty} \mathbb{E} \int_0^r h(y) \xi_n(dy) &= 0, \\
\lim_{r \to \infty} \limsup_{n \to \infty} \mathbb{E} \int_r^\infty h(y) \xi_n(dy) &= 0. \tag{4.11}
\end{align*}
\]
Convergence to scale-invariant Poisson processes

Since \( f \) is compactly supported, there exist \( 0 < a < b < \infty \) such that \( f(z) = 0 \) for \( |z| < a \) or \( |z| > b \). Thus, using (4.9) in the last step, we have

\[
\limsup_{y \to 0} \frac{h(y)}{y} = \limsup_{y \to 0} \frac{\mathbf{E} f(V)}{y} = \limsup_{y \to 0} \frac{\mathbf{E} [f(V)1_{\{|V| \geq a/y\}}]}{y} \\
\leq \|f\|_{\infty} \limsup_{y \to 0} y^{-1} \mathbf{P}\{\|V\| \geq a/y\} < \infty.
\]

Arguing similarly and using (4.9),

\[
\limsup_{y \to \infty} y^\alpha h(y) = \limsup_{y \to \infty} y^\alpha \mathbf{E} f(V) \\
= \limsup_{y \to \infty} y^\alpha \mathbf{E} [f(V)1_{\{|V| \leq b\}}] \\
\leq \|f\|_{\infty} \limsup_{y \to \infty} y^{-\alpha} \mathbf{P}\{|V| \leq b/y\} < \infty.
\]

Thus, \( \limsup_{y \to 0} h(y)/y < \infty \) and \( h(y) = O(y^{-\alpha}) \) as \( y \to \infty \). Together with (4.10), this implies that \( h \) satisfies (4.11).

\[\square\]

**Corollary 4.4.** Let \( (\xi_n)_{n \in \mathbb{N}} \) be a sequence of simple point processes converging vaguely in distribution to \( \eta \), as \( n \to \infty \). Assume that a random vector \( V \) in \( S \) and \( (\xi_n)_{n \in \mathbb{N}} \) satisfy (4.9) and (4.10), respectively. Then \( \xi_n^V \rightarrow \eta^V \) as \( n \to \infty \).

**Remark 4.5.** Fix \( \alpha > 0 \). For a sequence of point processes \( (\nu_n)_{n \in \mathbb{N}} \) as in Theorem 3.3 with \( \mathbf{E}X_k = O(q_k^b) \leq Cq_k^b \) for some \( C > 0 \), by (3.2) in the proof of Theorem 3.3, the first condition in (4.10) is satisfied. Letting \( N = \inf\{k : z_k > z_nr\} \) for \( r > 0 \) yields that

\[
\mathbf{E} \int_r^\infty t^{-\alpha} \nu_n(dt) = z_n^\alpha \sum_{k : z_k > z_nr} z_k^{-\alpha} \mathbf{E}X_k \leq Cz_n^\alpha (\sup_{k \leq N} q_k^b) \sum_{k = N}^\infty \frac{z_k^c - z_k^{c+1}}{z_k^{c+\alpha}} \\
\leq Cz_n^\alpha \left[ \frac{z_N^c - z_N^{c+1}}{z_N^{c+\alpha}} + \int_{(z_n)^2}^\infty \frac{1}{x(c+\alpha)/c} dx \right].
\]

Since the right-hand side converges to \( C(c/\alpha)^{-c/\alpha} \) as \( n \to \infty \),

\[
\lim_{r \to \infty} \limsup_{n \to \infty} \mathbf{E} \int_r^\infty t^{-\alpha} \nu_n(dt) = 0.
\]

Hence, these point processes satisfy (4.10).

**Example 4.6.** Consider the sequence of point processes \( (\nu_n)_{n \in \mathbb{N}} \) given by (1.2) with \( z_k = \log pk \) and \( X_k \sim \text{Geom}(1-1/pk) \). Since \( pk > k \log k \) (see e.g. [25]) and \( \log pk < 2 \log k \) for \( k \geq 6 \), (see e.g. [14, Lem. 1]), we have that \( N_n = \inf\{k : pk > p_n^*\} > n^{1/2} \) for \( n \) large enough. Hence,

\[
\sum_{k : pk > p_n^*} \frac{1}{pk(\log pk)^\alpha} \leq \sum_{k = N_n}^\infty \frac{1}{k \log k(\log k)^\alpha} \\
\leq \int_{N_n - 1}^\infty \frac{1}{x(\log x)^{1+\alpha}} dx = \frac{1}{\alpha} \frac{(\log (N_n - 1))^{-\alpha}}{\alpha} \leq \frac{2n(\log n)^{-\alpha}}{\alpha^{1+\alpha}}.
\]

Therefore,

\[
\limsup_{n \to \infty} \mathbf{E} \int_r^\infty t^{-\alpha} \nu_n(dt) = \limsup_{n \to \infty} (\log p_n)^\alpha \sum_{k : pk > p_n^*} \frac{2}{pk(\log pk)^\alpha} \\
\leq \limsup_{n \to \infty} (\log n)^{2+\alpha} \frac{2+\alpha}{\alpha^{2+2\alpha}} = \frac{2+2\alpha}{\alpha^{2+2\alpha}},
\]
which yields
\[
\lim_{r \to \infty} \limsup_{n \to \infty} E \int_r^\infty t^{-\alpha} \nu_n(dt) = 0.
\]

The other condition in (4.10) is easy to check using Merten’s formula. Hence, as \( \nu_n \xrightarrow{d} \eta \) as \( n \to \infty \) by Theorem 3.6, for \( V \) satisfying (4.9), Corollary 4.4 yields that \( \nu_n^{V} \xrightarrow{d} \eta^{V} \) as \( n \to \infty \).

We now return to our basic example of point processes given by (1.2). For a point process on \((0, \infty)\) with support points in a deterministic set, we can generalize the notion of uplifting. For \((\nu_n)_{n \in \mathbb{N}} \) given by (1.2), consider its uplifting by independent vectors \( V = (V_k)_{k \in \mathbb{N}} \) in \( S \) which are possibly non-identically distributed, allowing for possible dependence within the pairs \((V_k, X_k)\) for any \( k \in \mathbb{N} \). Assume that the conditional distribution of \( V_k \) given \( X_k \) is a function \( V(X_k) \) that does not depend on \( k \), i.e.,
\[
V(x) = d (V_k | X_k = x), \quad k \in \mathbb{N}.
\]

For instance, this is the case if the random vectors \((V_k)_{k \in \mathbb{N}}\) are i.i.d. and independent of the random variables \((X_k)_{k \in \mathbb{N}}\). We also assume that the random vectors \((V_k)_{k \in \mathbb{N}}\) are uniformly bounded away from 0 and \( \infty \) and define the \textit{uplifted} process \( \nu_n^{V} \) as
\[
\nu_n^{V} = \sum_{k=1}^{\infty} X_k \delta_{V_k z_k / z_n}.
\]

Finally, we assume that the random variables \((X_k)_{k \in \mathbb{N}}\) are \{0, 1\}-valued with high probability, i.e.,
\[
\prod_{k=1}^{\infty} (q^0_k + q^1_k) > 0.
\]

**Theorem 4.7.** For \((\nu_n)_{n \in \mathbb{N}} \) given by (1.2), assume that the \( X_k \)'s satisfy (14.3) and \( \nu_n \xrightarrow{d} \eta_c \) for some \( c > 0 \). Let \( V = (V_k)_{k \in \mathbb{N}} \) be a sequence of random vectors in \( S \) satisfying (14.12) with \( \epsilon \leq \|V(x)\| \leq r \) almost surely for all \( x \in \mathbb{N} \) for some \( 0 < \epsilon < r < \infty \). Then \( \nu_n^{V} \xrightarrow{d} \eta^{V(1)} \) as \( n \to \infty \), where \( \eta^{V(1)} \) is defined as in (4.1).

**Proof.** Let \( \tilde{X}_k = \mathbb{1}(X_k > 0) \). Let \((m(n))_{n \in \mathbb{N}} \) be such that \( m(n) \to \infty \) and \( z_{m(n)} = o(z_n) \) as \( n \to \infty \). Denote
\[
E_n = \{X_k = \tilde{X}_k \text{ for all } k \geq m(n)\}, \quad n \in \mathbb{N}.
\]

By Kolmogorov’s zero-one law and (14.3),
\[
\lim_{n \to \infty} P(E_n) = 1.
\]

Fix \( f \in \hat{C}_S \). Then, recalling that \( V(1) = d (V_k | X_k = 1) \), we have
\[
E \left[ e^{-\nu_n^{V} f | E_n} \right] = E \left[ \exp \left\{ -\sum_{k=1}^{m(n)-1} X_k f(V_k z_k / z_n) \right\} | E_n \right] = E \left[ \exp \left\{ -\sum_{k=1}^{m(n)-1} X_k f(V_k z_k / z_n) \right\} \right] \left[ \prod_{k=m(n), X_k=1}^{\infty} E \left[ e^{-f(V(1) z_k / z_n)} | E_n \right] \right].
\]

Since \( z_{m(n)} = o(z_n) \), the process \( \sum_{k=1}^{m(n)-1} X_k \delta_{V_k z_k / z_n} \) converges vaguely in distribution to the zero process in \( \mathcal{M}((0, \infty)) \) as \( n \to \infty \). Combined with our assumption that \( \epsilon \leq \|V(x)\| \leq r \) almost surely for all \( x \in \mathbb{N} \), this implies that the first factor on the
Convergence to scale-invariant Poisson processes

right-hand side of (4.14) converges to 1 as \( n \to \infty \). For the second factor in (4.14), we have

\[
\mathbb{E} \left[ \prod_{k=m(n), X_k=1}^{\infty} e^{-f(V(1)z_k/z_n)} \big| E_n \right] = \mathbb{E} \exp \left\{ - \sum_{k=m(n)}^{\infty} Y_k \tilde{h}(z_k/z_n) \right\},
\]

where \( Y_k \sim \text{Ber}\left( q_k^0/(q_k^0 + q_k^1) \right) \), \( k \geq m(n) \), has the same distribution as \( X_k \) conditional on \( E_n \), and

\[
\tilde{h}(t) = - \log \mathbb{E} e^{-f(V(1)t)}.
\]

Consider the point process \( \tilde{\nu}_n = \sum_{k=1}^{\infty} Y_k \delta_{z_k/z_n} \). Using (4.13) for the first equality, we have that for any \( 0 < a < b < \infty \),

\[
\lim_{n \to \infty} \prod_{k:a z_n < z_k \leq b z_n} \frac{q_k^0}{q_k^0 + q_k^1} = \lim_{n \to \infty} \prod_{k:a z_n < z_k \leq b z_n} q_k^0 = \left( \frac{a}{b} \right)^c,
\]

where in the last equality we have used our assumption that \( \nu_n \overset{d}{\to} \eta_c \) and Theorem 3.2. Hence, \( (\tilde{\nu}_n)_{n \in \mathbb{N}} \) satisfies Condition 3.1(i). That \( (\tilde{\nu}_n)_{n \in \mathbb{N}} \) satisfies Condition 3.1(ii) follows trivially by noticing that \( \nu_n \) satisfies Condition 3.1(ii). Thus, \( \tilde{\nu}_n \) converges vaguely in distribution to \( \eta_c \) as \( n \to \infty \) by Theorem 3.2. Again, we can ignore the first \( m(n) - 1 \) terms of the sum \( \tilde{\nu}_n \) as it converges to a zero process, whence

\[
\sum_{k=m(n)}^{\infty} Y_k \delta_{z_k/z_n} \overset{d}{\to} \eta_c \quad \text{as} \quad n \to \infty.
\]

By our assumption that \( V(1) \) is bounded away from 0 and \( \infty \) and that \( f \) is compactly supported, it follows that the function \( \tilde{h} \) has a relatively compact support in \((0, \infty)\). Clearly, \( \tilde{h} \) is continuous and bounded. Hence by (4.3),

\[
\mathbb{E} \exp \left\{ - \sum_{k=m(n)}^{\infty} Y_k \tilde{h}(z_k/z_n) \right\} \to \mathbb{E} e^{-\eta_c \tilde{h}} \quad \text{as} \quad n \to \infty.
\]

By (4.14),

\[
\mathbb{E} \left[ e^{-\nu_n^V} | E_n \right] \to \mathbb{E} e^{-\eta_c \tilde{h}} \quad \text{as} \quad n \to \infty.
\]

Finally, noticing that

\[
\mathbb{E} e^{-\eta_c \tilde{h}} = \mathbb{E} \left[ \mathbb{E} (e^{-\eta_c^{V(1)} f} | \eta_c) \right] = \mathbb{E} e^{-\eta_c^{V(1)} f},
\]

and that \( \mathbb{P}(E_n) \to 1 \) as \( n \to \infty \), we have

\[
\mathbb{E} e^{-\nu_n^V} = \mathbb{E} \left[ e^{-\nu_n^V f} | E_n \right] \mathbb{P}(E_n) + \mathbb{E} \left[ e^{-\nu_n^V f} | E_n^c \right] \mathbb{P}(E_n^c) \to \mathbb{E} e^{-\eta_c^{V(1)} f}
\]

as \( n \to \infty \) for any \( f \in \tilde{C}_S \), which yields that \( \nu_n^V \overset{d}{\to} \eta_c^{V(1)} \) as \( n \to \infty \).

**Example 4.8.** For \( z_k = \log p_k \) and \( X_k \sim \text{Geom}(1-1/p_k) \) or \( \text{Ber}(1/(p_k+1)) \), one can easily see that the conditions of Theorem 4.7 are satisfied by \( (\nu_n)_{n \in \mathbb{N}} \), and hence, for \( V \) as in Theorem 4.7, the conclusion of the result holds.

Note, if \( V_k \) is independent of \( X_k \) for all \( k \in \mathbb{N} \), then they are necessarily i.i.d. by (4.12). Now we consider an example when \( (X_k)_{k \in \mathbb{N}} \) and \( V \) are dependent.
Convergence to scale-invariant Poisson processes

**Example 4.9.** Let $d \geq 2$ and $m \in \mathbb{N}$ be positive integers. Let $X_k \sim \text{Geom}(1 - 1/p_k)$ be independent and $V_k = (mX_k)^{-1}(X_k^1, \ldots, X_k^d)1_{\{X_k > 0\}}$ for $k \in \mathbb{N}$, where $(X_k^1, \ldots, X_k^d)$ is multinomially distributed with the number of experiments $mX_k$ and the probabilities of outcomes $q_1, \ldots, q_d$ with $\sum_{i=1}^d q_i = 1$. Let

$$\nu_n = \sum_{k=1}^{\infty} X_k \delta \log p_k / \log p_n \quad \text{and} \quad \nu_n^V = \sum_{k=1}^{\infty} X_k \delta V_k \log p_k / \log p_n,$$

where $(p_k)_{k \in \mathbb{N}}$ is an enumeration of the primes. Clearly, the random variables $(X_k)_{k \in \mathbb{N}}$ satisfy (4.13). For each $k$, the random vector $V_k$ and hence $V_k(x)$ is almost surely bounded away from 0 and $\infty$ when $X_k = x > 0$. Since by Theorem 3.6 we have that $\nu_n$ converges vaguely in distribution to $\eta$ as $n \to \infty$, Theorem 4.7 yields that $\nu_n^V \overset{d}{\to} \eta^V(1)$ as $n \to \infty$, where $mV(1)$ is distributed as a multinomial random variable with $m$ experiments and probabilities of outcomes $q_1, \ldots, q_d$.

**5 Scale-invariant Poisson processes in higher dimensions and multivariate Dickman distributions**

In this section, we study and classify scale-invariant Poisson processes in higher dimensions and extend the generalized Dickman distributions in one dimension to its multivariate counterpart. For a simple point process $\xi$ in $(0, \infty)$ and a random vector $V$ taking values in $S = \mathbb{R}^d \setminus \{0\}$ bounded away from 0 and $\infty$ with i.i.d. copies $(V_k)_{k \in \mathbb{N}}$, recall that the uplifted point process $\xi^V$ is given by

$$\xi^V = \sum_{k=1}^{\infty} \delta V_k Z_k,$$

where $(Z_k)_{k \in \mathbb{N}}$ is an enumeration of the points in $\xi$.

**Lemma 5.1.** Any scale-invariant Poisson process in $S$ has the same distribution as $\eta^U_e$ for some $c > 0$ and unit random vector $U$ in $\mathbb{R}^d$. Moreover, for any random vector $V$ in $S$ with $\eta_e$ and $(V_k)_{k \in \mathbb{N}}$ satisfying (4.2), the uplifted point process $\eta^V_e$ has the same distribution as $\eta^U_e$ with $U = V/\|V\|$.

**Proof.** Let $\nu$ be a scale-invariant Poisson process in $S$. Hence $\nu(tB) = t \nu(B)$ for every Borel set $B \in \mathcal{G}$ and $t > 0$. Represent each point $x \in S$ as a pair $(u, r) \in S^{d-1} \times (0, \infty)$, where $u = x/\|x\|$ and $r = \|x\|$. For a measurable subset $A \subseteq S^{d-1}$ and $0 < a < b < \infty$, by scale invariance one has

$$\mathbb{E} \nu(A \times [a, b]) = \mathbb{E} \nu(A \times [a/b, 1]) = \mathbb{E} \nu(A \times [a/b, 1]).$$

(5.1)

For $p \in (0, 1)$ and $A \subseteq S^{d-1}$, define $\gamma_\nu(p, A) = \mathbb{E} \nu(A \times [p, 1])$ and $\gamma_\nu(1, A) = 0$. For every fixed $A \subseteq S^{d-1}$, notice that $\gamma_\nu$ satisfies

$$\gamma_\nu(p, A) + \gamma_\nu(q, A) = \gamma_\nu(pq, A), \quad p, q \in (0, 1).$$

By monotonicity, $\gamma_\nu(p, A) = -\gamma_\nu(A) \log p$ for $p \in (0, 1]$, where $\gamma_\nu$ is a locally finite measure on $S^{d-1}$ not depending on $p$. By (5.1),

$$\mathbb{E} \nu(A \times [a, b]) = \gamma_\nu(A) \log(b/a).$$
Convergence to scale-invariant Poisson processes

For a random vector $U$ in the unit sphere $S^{d-1}$ with distribution $\mu$, the uplifted process $\eta^U_c$ is also a Poisson process. Its intensity measure is given by

$$E \eta^U_c(A \times [a, b]) = \int_{a}^{b} ct^{-1} dt \mu(du) = c \mu(A) \log(b/a) \quad (5.2)$$

for all Borel $A \subseteq S^{d-1}$ and $0 < a < b < \infty$. It is immediately seen that $\eta^U_c$ is scale-invariant.

By comparing the two equations above, we obtain that $\eta$ has the same intensity measure as $\eta^U_c$ with $c = \gamma \nu(S^{d-1})$ and $U$ is distributed according to $\mu = \gamma \nu/c$. Thus $\nu = d \eta^U_c$ proving the first claim.

Next, for a random vector $V$ distributed on $S$ according to a probability measure $\psi$ with $\nu$ and $(V_k)_{k \in \mathbb{N}}$ satisfying (4.2), let $U = V/\|V\|$. Clearly, $\eta^V_c$ is also a Poisson process. For all $A \subseteq S^{d-1}$ and $0 < a < b < \infty$, using the substitution $z = \|v\|t$ in the second step, the intensity of $\eta^V_c$ can be expressed as

$$E \eta^V_c(A \times [a, b]) = \int_{v/\|v\| \in A, \|v\| \in [a, b]} ct^{-1} dt \psi(dv) = \int_{v/\|v\| \in A, \|v\| \in [a, b]} c \log(b/a) \psi(dv) = E \eta^V_c(A \times [a, b]),$$

where in the last step we have used (5.2). Hence, $\eta^V_c = d \eta^U_c$.

Recall that the generalized Dickman random variable $D_c$ with parameter $c > 0$ has the same distribution as the sum of points of $\eta_c$ in the interval $(0, 1)$. One can naturally generalize this definition to dimensions $d \geq 2$ by considering a scale-invariant Poisson process in $S$, which by Lemma 5.1 is of the form $\eta^U_c$ for some $c > 0$ and unit random vector $U$ in $\mathbb{R}^d$, and summing its points lying inside the unit ball $B_1$. The following definition makes this precise.

**Definition 5.2.** For a unit random vector $U$ in $\mathbb{R}^d$ and $c > 0$, the multivariate Dickman random variable $D^U_c$ with parameters $(c, U)$ is defined by

$$D^U_c = \int_{B_1} x \eta^U_c(dx) = \sum_{x \in \eta^U_c \cap B_1} x. \quad (5.3)$$

Note that the points of $\eta_c$ in the interval $(0, 1)$ are distributed as the collection $\{Q^U_1, (Q^U_1 Q^U_2), \ldots\}$, where $(Q_k)_{k \in \mathbb{N}}$ are independent copies of a random variable $Q$ which is uniformly distributed on $[0, 1]$. Thus, letting $(U_i)_{i \in \mathbb{N}}$ be i.i.d. copies of $U$, we can write

$$D^U_c = \sum_{x \in \eta^U_c} x \mathbb{1}_{\{\|x\| < 1\}} = \sum_{k=1}^{\infty} U_k \prod_{i=1}^{k} Q^U_i = d \prod_{i=1}^{k} Q^U_i = d \frac{1}{Q^U} (D^U_c + U^{*})$$

for $Q$ uniformly distributed on $[0, 1]$ and $U^{*} = d U$ independent of $D^U_c$. Thus, the random variable $D^U_c$ is the unique fixed point of the distributional transformation $W \mapsto W^{*}$ given by

$$W^{*} = d \frac{1}{Q^U} (W + U)$$

with $Q, U$ and $W$ mutually independent.

By Lemma 5.1, the sum of points from any scale-invariant Poisson process lying inside the unit ball is distributed as $D^U_c$ for some $c > 0$ and unit random vector $U$. In particular, for a general random vector $V$ in $S$, by Lemma 5.1, it is straightforward to see that the sum of points of $\eta^V_c$ inside the unit ball is distributed as $D^U_c$ with $U = V/\|V\|$. 

EJP 25 (2020), paper 79. 

http://www.imstat.org/ejp/
Convergence to scale-invariant Poisson processes

Also note that
\[ D_U^c = d \sum_{k=1}^{\infty} e^{-Z_k} U_k, \]
where \((U_k)_{k \geq 1}\) are i.i.d. copies of \(U\) and \((Z_k)_{k \in \mathbb{N}}\) is an enumeration of the points of a homogeneous Poisson process on the interval \((0, 1)\) with intensity \(c\). In particular, \(D_U^c\) is self-decomposable, see [24].

We finish this section with an example of weak convergence to a multivariate Dickman distribution as defined in (5.3). Consider the setting of Example 4.9 with \(d = 2\) and \(m = 1\). Let \(p_k, X_k\) and \(V_k\) be as in Example 4.9. For \(p \in (0, 1)\), let \(X_k^1 \sim \text{Bin}(X_k, p) 1_{\{X_k > 0\}}\) and \(X_k^2 = X_k - X_k^1\). Define
\[ W_n = \sum_{k=1}^{n} X_k V_k \log p_k / \log p_n = \frac{1}{\log p_n} \sum_{k=1}^{n} (X_k^1, X_k^2) \log p_k, \tag{5.4} \]
where \(V_k = X_k^{-1}(X_k^1, X_k^2) 1_{\{X_k > 0\}}\). Let \(D_U^1\) denote a Dickman random variable defined at (5.3), where \(U = (X, 1 - X)\) with \(X \sim \text{Ber}(p)\).

**Theorem 5.3.** Let \(W_n\) be given by (5.4). Then \(W_n \xrightarrow{d} D_U^1\) as \(n \to \infty\).

**Proof.** Define
\[ \tau_n = \sum_{k=\log n}^{\infty} Y_k \delta_{\log p_k / \log p_n}, \]
where the random variables \((Y_k)_{k \in \mathbb{N}}\) are independent with \(Y_k \sim \text{Ber}(1 + p_k)^{-1}\) for \(k \in \mathbb{N}\). Notice that \(\sum_{k=1}^{\log n - 1} Y_k \delta_{\log p_k / \log p_n}\) converges vaguely in distribution to the zero process on \((0, \infty)\) as \(n \to \infty\). By Remark 3.7, the process \(\sum_{k=1}^{\infty} Y_k \delta_{\log p_k / \log p_n}\) converges vaguely in distribution to \(\eta\) as \(n \to \infty\), hence, so does \((\tau_n)_{n \in \mathbb{N}}\). By Theorem 4.7, we obtain that \(\tau_n \xrightarrow{d} \eta\) as \(n \to \infty\).

Let \(E_n = \{X_k = \tilde{X}_k \text{ for all } k \geq \log n\}\), where \(\tilde{X}_k = 1_{\{X_k > 0\}}\). Notice that for each \(k\), the random variable \(Y_k\) has the same law as \(X_k\) conditional on the event \(X_k = \tilde{X}_k\). Hence, for each \(n\), conditional on \(E_n\), the point process \((X_k V_k \log p_k / \log p_n)_{\log n \leq k \leq n}\) has the same law as \(\tau_n^{\log n}\) restricted to the unit ball \(B_1\). Therefore, the conditional law of
\[ Z_n = \sum_{k=\log n}^{n} X_k V_k \log p_k / \log p_n \]
given \(E_n\) is the same as that of \(\int_{B_1} x \, d\tau_n^{\log n}\). Using [25, Prop. 1.51], notice that
\[ \lim_{\epsilon \to 0} \limsup_{n \to \infty} \epsilon \int_{B_{\epsilon}} x \tau_n^{\log n}(dx) \leq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{\log p_n} \sum_{k:p_k \leq p_n} \log p_k \leq 0. \]
Since \(\tau_n^{\log n} \xrightarrow{d} \eta\) as \(n \to \infty\), using [18, Theorem 4.28] and Lemma A.2, it is not hard to see that
\[ (Z_n | E_n) = d \int_{B_1} x \tau_n^{\log n}(dx) \xrightarrow{d} \int_{B_1} x \eta(dx) = d D_U^1 \text{ as } n \to \infty. \]
Since \(P(E_n) \to 1\) as \(n \to \infty\), this yields that \(Z_n \xrightarrow{d} D_U^1\) as \(n \to \infty\).

Finally, taking expectation and using [25, Prop. 1.51], it is straightforward to see that
\[ \sum_{k=1}^{\log n - 1} X_k V_k \log p_k / \log p_n \to 0 \text{ as } n \to \infty \]
in \(L^1\), hence, in probability as \(n \to \infty\). An application of Slutsky’s theorem yields the result. \(\square\)
A Results on vague convergence in distribution

Let $S$ be a locally compact separable metric space. The following result provides a necessary and sufficient condition for the vague convergence in distribution of a sequence of point processes to a simple point process. Recall that a semi-ring $I$ is a family of sets closed under finite intersections such that any proper difference of sets in $I$ is a finite, disjoint union of $I$-sets.

**Theorem A.1** (see [19, Theorem 4.15]). Let $(\xi_n)_{n \geq 1}$ be point processes on $S$, and fix a dissecting ring $\mathcal{U} \subset \mathcal{E}_{\mathcal{E}}$ and a semi-ring $I \subset \mathcal{U}$. Then $\xi_n \xrightarrow{d} \xi$ in $\mathcal{N}(S)$ as $n \to \infty$ for a simple point process $\xi$ if and only if

(i) $\lim_{n \to \infty} P\{\xi_n A = 0\} = P\{\xi A = 0\}$ for all $A \in \mathcal{U}$, and

(ii) $\limsup_{n \to \infty} P\{\xi_n B > 1\} \leq P\{\xi B > 1\}$ for all $B \in I$.

Recall that $\xi_n \xrightarrow{d} \xi$ is equivalent to (see e.g. [19, Theorem 4.11])

\[
\int f(x) \xi_n(dx) \xrightarrow{d} \int f(x) \xi(dx) \quad \text{as } n \to \infty \tag{A.1}
\]

for all $f \in \mathcal{C}_S$. By a standard argument, approximating an indicator function with a continuous function, it is straightforward to derive the following result.

**Lemma A.2.** Let $(\xi_n)_{n \geq 1}$, $\xi$ be random measures in $S$ such that $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$. For a relatively compact measurable set $K$, let $f : S \to \mathbb{R}$ be a non-negative function which is continuous when restricted to $K$ and $f(x) = 0$ for $x \notin K$. If $E\xi(\partial K) = 0$, then (A.1) holds.

Next we prove Lemma 4.1.

**Proof of Lemma 4.1.** Fix $\varepsilon > 0$ and $K_\varepsilon$ satisfying (4.4). Since $\xi$ has a finite intensity, without loss of generality, we can assume that $E\xi(\partial K_\varepsilon) = 0$. By Lemma A.2,

\[
\int_{K_\varepsilon} h(x) \xi_n(dx) \xrightarrow{d} \int_{K_\varepsilon} h(x) \xi(dx) \quad \text{as } n \to \infty.
\]

Hence,

\[
E\exp\left\{-\int_{K_\varepsilon} h(x) \xi_n(dx)\right\} \xrightarrow{} E\exp\left\{-\int_{K_\varepsilon} h(x) \xi(dx)\right\} \tag{A.2}
\]

as $n \to \infty$. Since $e^{EX} \leq E e^X$,

\[
\log E\exp\left\{-\int_{K_\varepsilon} h(x) \xi_n(dx)\right\} \geq -E\int_{K_\varepsilon} h(x) \xi_n(dx).
\]

Thus, by (4.4), we have that

\[
\lim_{\varepsilon \downarrow 0} \liminf_{n \to \infty} E\exp\left\{-\int_{K_\varepsilon} h(x) \xi_n(dx)\right\} = e^0 = 1. \tag{A.3}
\]

The same holds for the upper limit trivially. Combining (A.2) and (A.3) yields the desired result.

The following result which is a direct consequence of [18, Theorem 4.28] and Lemma A.2 provides conditions under which the vague convergence of a general sequence of point processes $(\nu_n)_{n \in \mathbb{N}}$ to $\eta_c$ implies the convergence of the sum of points in $(0, 1)$.
Convergence to scale-invariant Poisson processes

**Lemma A.3.** Let \((\nu_n)_{n \in \mathbb{N}}\) be a sequence of point processes in \((0, \infty)\) with \(\nu_n \overset{d}{\to} \eta_\varepsilon\) for \(c > 0\). If
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} E \int_0^c t \nu_n(dt) = 0,
\]
then
\[
\int_0^1 t \nu_n(dt) \overset{d}{\to} \int_0^1 t \eta_\varepsilon(dt) =_d D_c.
\]

**Lemma A.4** (Continuity of \(M\) restricted to simple counting measures). Let \((\xi_n)_{n \in \mathbb{N}}\) be a sequence of counting measures (deterministic) in \(\mathcal{N}(0, \infty)\) such that \(\xi_n\) converges vaguely to \(\xi\) as \(n \to \infty\) for a simple counting measure \(\xi\). If \(M\) is given by (4.7), then \(M(\xi_n)\) converges vaguely to \(M(\xi)\) as \(n \to \infty\).

**Proof.** Denote \(\tilde{\xi}_n = M(\xi_n)\) and \(\tilde{\xi} = M(\xi)\). Note that it suffices to show that, for all \(0 < a < b < \infty\) and \(k \in \mathbb{N}_0\),
\[
\tilde{\xi}_n([k, \infty) \times [a, b]) \to \tilde{\xi}([k, \infty) \times [a, b]) \quad \text{as} \quad n \to \infty.
\]
(A.5)

Since \(\xi\) is simple,
\[
\tilde{\xi}([k, \infty) \times [a, b]) = \begin{cases} \xi([a, b]) & \text{for } k = 0, 1, \\ 0 & \text{for } k > 1. \end{cases}
\]

Note that \(\tilde{\xi}_n([k, \infty) \times [a, b]) = \xi_n([a, b])\) for \(k = 0, 1\). Hence, (A.5) holds for \(k = 0, 1\) by our assumption that \(\xi_n \to \xi\) as \(n \to \infty\). Fix \(k > 1\). Let \(\xi([a, b]) = m\) for some \(m \geq 0\). If \(m = 0\), by our assumption we have \(\xi_n([a, b]) \to \xi([a, b]) = 0\) as \(n \to \infty\), which yields \(\xi_n([k, \infty) \times [a, b]) \to 0\) as \(n \to \infty\), showing (A.5). Next, assume that \(m \geq 1\). Since \(\xi\) is a locally finite counting measure, there are disjoint intervals \((I_i)_{1 \leq i \leq m}\) such that \(\xi(I_i) = 1\) for \(1 \leq i \leq m\) and \(\bigcup_{i=1}^m I_i = [a, b]_\infty\). By our assumption, \(\xi_n(I_i) \to \xi(I_i) = 1\) as \(n \to \infty\) for \(1 \leq i \leq m\). Since \(k > 1\), we have \(\xi_n([k, \infty) \times I_i) \to 0\) as \(n \to \infty\). Taking union over the \(m\) sets \([k, \infty) \times I_i, 1 \leq i \leq m\) proves (A.5), concluding the proof. \(\square\)

**References**

[1] Arras, B., Mijoule, G., Poly, G. and Swan, Y.: A new approach to the Stein–Tikhomirov method: with applications to the second Wiener chaos and Dickman convergence. *arXiv preprint arXiv:1605.06819*, (2016).

[2] Arratia, R.: On the central role of scale invariant Poisson processes on \((0, \infty)\). In *Microsurveys in Discrete Probability (Princeton, NJ, 1997)*, Amer. Math. Soc., Providence, RI, (1998), 21–41. MR-1630407

[3] Arratia, R.: Personal communications, (2017).

[4] Arratia, R., Barbour, A. D. and Tavaré, S.: *Logarithmic Combinatorial Structures: a Probabilistic Approach*. European Mathematical Society (EMS), Zürich, 2003. MR-2032426

[5] Arratia, R., Garibaldi, S. and Kilian, J.: Asymptotic distribution for the birthday problem with multiple coincidences, via an embedding of the collision process. *Random Structures Algorithms, 48*, (2016), 480–502. MR-3481270

[6] Barbour, A. D. and Nietlispach, B.: Approximation by the Dickman distribution and quasilogarithmic combinatorial structures. *Electron. J. Probab.*, 16, (2011), 880–902. MR-2793242

[7] Basrak, B., Planinic, H. and Soulier, P.: An invariance principle for sums and record times of regularly varying stationary sequences. *Probab. Theory Related Fields, 172*, (2018), 869–914. MR-3877549

[8] Bhatt, A. G. and Roy, R.: On a random directed spanning tree. *Adv. in Appl. Probab.*, 36, (2004), 19–42. MR-2035772

[9] Bhattacharjee, C. and Goldstein, L.: Dickman approximation in simulation, summations and perpetuities. *Bernoulli, 25*, (2019), 2758–2792. MR-4003654
Convergence to scale-invariant Poisson processes

[10] Covo, S.: On approximations of small jumps of subordinators with particular emphasis on a Dickman-type limit. *J. Appl. Probab.*, 46, (2009), 732–755. MR-2562319

[11] Daley, D. J. and Vere-Jones, D.: *An Introduction to the Theory of Point Processes. Vol. I: Elementary Theory and Methods*. Springer, New York, second edition, 2003. MR-1950431

[12] Daley, D. J. and Vere-Jones, D.: *An Introduction to the Theory of Point Processes. Vol. II: General Theory and Structure*. Springer, New York, second edition, 2008. MR-2371524

[13] Dickman, K.: On the frequency of numbers containing prime factors of a certain relative magnitude. *Arkiv för matematik, astronomi och fysik*, 22, (1930), 1–14.

[14] Dusart, P.: The 6th prime is greater than $k(\ln k + \ln \ln k - 1)$ for $k \geq 2$. *Math. Comp.*, 68(225), (1999), 411–415. MR-1620223

[15] Goldstein, L.: Non-asymptotic distributional bounds for the Dickman approximation of the running time of the Quickselect algorithm. *Electron. J. Probab.*, 23, (2018), 13 pp. MR-3862615

[16] Hwang, H. K. and Tsai, T. H.: Quickselect and the Dickman function. *Combin. Probab. Comput.*, 11, (2002), 353–371. MR-1918722

[17] Kallenberg, O.: Characterization and convergence of random measures and point processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, (27), (1973), 9–21. MR-0431374

[18] Kallenberg, O.: *Foundations of Modern Probability*. Springer, New York, second edition, 2002. MR-1876169

[19] Kallenberg, O.: *Random Measures, Theory and Applications*. Springer, Cham, 2017. MR-3642325

[20] Lindskog, F., Resnick, S. I. and Roy, J.: Regularly varying measures on metric spaces: hidden regular variation and hidden jumps. *Probab. Surv.*, 11, (2014), 270–314. MR-3271332

[21] Penrose, M. D. and Wade, A. R.: Random minimal directed spanning trees and Dickman-type distributions. *Adv. in Appl. Probab.*, 36, (2004), 691–714. MR-2079909

[22] Pinsky, R. G.: On the strange domain of attraction to generalized Dickman distribution for sums of independent random variables. *Electron. J. Probab.*, 23, (2018), 17 pp. MR-3751078

[23] Pinsky, R. G.: A natural probabilistic model on the integers and its relation to Dickman-type distributions and Buchstab’s function. In *Probability and Analysis in Interacting Physical Systems — in Honor of S.R.S. Varadhan*, Springer, (2019), 267–294. MR-3968515

[24] Sato, K. i.: Class L of multivariate distributions and its subclasses. *J. Multivariate Anal.*, 10, (1980), 207–232. MR-0575925

[25] Tao, T. and Vu, V.: *Additive Combinatorics*. Cambridge University Press, Cambridge, 2006. MR-2289012

**Acknowledgments.** We thank Matthias Schulte for a number of useful comments on the manuscript that greatly improved the paper. We would also like to thank Richard Arratia for pointing out the possible connection between convergence of sums and the convergence of underlying point processes.