Triangular invariants, three-point functions and particle stability on the de Sitter universe

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Abstract

We study a class of three-point functions on the de Sitter universe and on the asymptotic cone. A blending of geometrical ideas and analytic methods is used to compute some remarkable integrals, on the basis of a generalized star-triangle identity living on the cone and on the complex de Sitter manifold. We discuss an application of the general results to the study of the stability of scalar particles on the Sitter universe.

1 Prologue

The main result of this paper is the following formula:

\begin{equation}
\begin{aligned}
h_d(\kappa, \nu, \lambda) &\overset{\text{def}}{=} \int_{1}^{\infty} P_{-\frac{d}{2} + \frac{i\kappa}{2} + 1} (u) P_{-\frac{d}{2} + \frac{i\nu}{2} + 1} (u) P_{-\frac{d}{2} + \frac{i\lambda}{2} + 1} (u)(u^2 - 1)^{-\frac{d}{4}} \, du = \\
&= \frac{2^d}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d-1}{2}) \prod_{\varepsilon = \pm 1} \Gamma\left(\frac{d-1}{2} + i\varepsilon\kappa\right) \prod_{\varepsilon' = \pm 1} \Gamma\left(\frac{d-1}{2} + i\varepsilon'\nu\right) \prod_{\varepsilon'' = \pm 1} \Gamma\left(\frac{d-1}{2} + i\varepsilon''\lambda\right)}{
\prod_{\varepsilon, \varepsilon' = \pm 1} \Gamma\left(\frac{d-1}{4} + \frac{i\varepsilon\kappa + i\varepsilon'\nu + i\varepsilon''\lambda}{2}\right) \prod_{\varepsilon, \varepsilon' = \pm 1} \Gamma\left(\frac{d-1}{4} + i\varepsilon\nu + i\varepsilon'\nu\right) \prod_{\varepsilon, \varepsilon' = \pm 1} \Gamma\left(\frac{d-1}{4} + i\varepsilon\lambda + i\varepsilon'\lambda\right)}
\end{aligned}
\end{equation}

expressing the integral of a product of three Legendre functions of the first kind as a ratio of products of Euler Gamma functions. This beautifully symmetric formula is not listed in any of the handbooks on integrals of special functions available to us and appears to be new. The steps involved in the proof-computation also give rise to many interesting quantities having possibly geometrical interpretations that we have not yet fully explored in their mathematical and physical consequences.

The problem of computing the integral originates from the study of particle decays in a de Sitter universe. In that context, the real parameters $\kappa, \nu$ and $\lambda$ are related to the masses of the particles involved in the decay process and $d$ is the (complex) dimension of the de Sitter universe where the process takes place. However, the methods and the results we are going to present have presumably a wider interest and range of applications.
The study of particle decays in a de Sitter universe was initiated by O. Nachtmann \cite{4} in 1968. He showed, in a very special case, that while a Minkowskian particle can never decay into heavier products, a de Sitter particle can, although this effect is exponentially small in the de Sitter radius.

The subject has acquired a greater physical interest with the advent of inflationary cosmology. In particular, the idea that particle decays during the (quasi-)de Sitter phase may have important consequences on the physics of the early universe has been suggested recently \cite{5, 6} (see also the related works \cite{7, 8}). The mathematical and physical difficulties related to the lack of time-translation symmetry of the de Sitter universe, and more generally of non-static (cosmological) backgrounds, have been tackled in \cite{5, 6} by using the Schwinger-Keldysh formalism, which is suitable for studying certain aspects of quantum physics of systems out of equilibrium. However, the approach described in \cite{5, 6} necessitates the introduction of a practical notion of lifetime of an unstable particle which is completely different from the definition commonly used in quantum physics.

Actually the lack of a commutative symmetry group of spacetime translations renders the mathematics extremely complicated but does not prevent computing the inverse lifetime of an unstable de Sitter particle according to the usual definition, namely as the inclusive transition probability per unit time from an initial state to every possible final state: this computation, initiated by Nachtmann long ago, has been performed at first order in perturbation theory in \cite{2, 3}. After writing the relevant perturbative amplitude, the computation of the lifetime of the de Sitter unstable particle amounts to two essentially distinct and independent steps:

1. taking the adiabatic limit of the infrared regularized inclusive amplitude, i.e. removing the infrared cutoff coupling factor necessary to make the integral expressing the amplitude converge;

2. computing the so called “phase space” coefficient, a quantity which only depends on the masses of the particles involved in the decay process.

The first step has been largely discussed in \cite{2, 3} and the resulting mathematical structures elucidated there. The second step reduces to computing the integral at the RHS of (1).

The quantity \( h_{d}(\kappa, \nu, \nu) \), relative to a decay into two identical particles, has been computed in \cite{2, 3}. This special case already exhibits some concrete mathematical difficulties, and has been solved in a purely analytical way by the use of Mellin transform techniques and the evaluation of a Barnes-type integral \cite{2, 3}.

The above method fails however to provide a solution for the general case of the production of two non-identical particles, i.e. fails to give a solution to the general integral \cite{1}. For odd values of \( d \) the Legendre functions of the first kind reduce to trigonometric-type functions; in these cases a direct computation of \( h_{d}(\kappa, \nu, \lambda) \) is possible. To give an example, one can solve the three-dimensional problem \((d = 3)\) by an elementary computation. Indeed for \( d = 3 \) there holds the particularly simple expression of the Legendre function:

\[
P^{-\frac{1}{2} + \nu}(\cosh v) = \sqrt{\frac{2}{\pi \sinh v}} \frac{\sin \nu v}{\nu}.
\]  

A straightforward computation then gives:

\[
h_{3}(\kappa, \nu, \lambda) = \frac{1}{\sqrt{8\pi \kappa \nu \lambda}} \frac{\sinh(\pi \kappa) \sinh(\pi \lambda) \sinh(\pi \nu)}{\cosh \frac{\pi(\kappa - \lambda - \nu)}{2} \cosh \frac{\pi(\kappa + \lambda - \nu)}{2} \cosh \frac{\pi(\kappa - \lambda + \nu)}{2} \cosh \frac{\pi(\kappa + \lambda + \nu)}{2}}.
\]  

The general odd-dimensional case \( d = 2n + 1 \) can similarly be tackled by (increasingly cumbersome) elementary integration techniques. On the contrary, the computation of the even dimensional cases (including the physically relevant four-dimensional de Sitter universe) is very very far from obvious.
Some of the geometric ideas necessary to overcome the difficulties of the integral \( \text{(1)} \) were contained in an unpublished work by one of us \([\text{3} ] \). Combining those ideas with a the geometrical properties of the complex de Sitter manifold provides a way to solve the problem. The result is displayed in Eq. \( \text{(2)} \).

Beyond the study of the of de Sitter particle decays, there are other potential applications of the formula \( \text{(2)} \) and of the methods used to derive it which include the study of tensor product of representations of non-compact groups, many new integral relations involving products of hypergeometric functions, other applications to de Sitter and/or anti de Sitter QFT etc.

## 2 Legendre functions and de Sitter Klein-Gordon fields: a short review

The computation of the integral \( \text{(1)} \) requires several steps in which the geometrical features of the complex de Sitter manifold enter in a crucial way. The first important step consists in returning to the meaning of the Legendre functions of the first kind \( P(u) \) as \textit{two-point functions} of quantum fields on a complexified de Sitter spacetime \([\text{10} \text{]} \text{11} \text{12} \text{13}] \). The variable \( u \), appearing as integration variable in the r.h.s. of \( \text{(2)} \), is understood as a geometrical invariant \( u = u(x, x') \) relating two points \( x, x' \) of a (complex) de Sitter hyperboloid. This idea allows in particular a natural way for understanding many of the mathematical properties of the Legendre functions and gives also a simple procedure to build many of their integral representations. Here follows a short account of the construction.

Consider a \((d + 1)\)-dimensional Minkowski spacetime \( M_{d+1} \); an event \( x \) is parameterized by a set of inertial coordinates \( x^0, \ldots, x^d \); the scalar product of two events of \( M_{d+1} \) is the Lorentz-invariant product \( x \cdot x' = x^0 x'^0 - x^1 x'^1 - \ldots - x^d x'^d \). The \( d \)-dimensional de Sitter spacetime is represented as the one-sheeted hyperboloid

\[
X_d = \{ x \in M_{d+1} : x \cdot x = x^2 = -R^2 \}
\]

embedded in \( M_{d+1} \). The Lorentzian geometry of the de Sitter manifold is induced by the causal structure of the ambient spacetime:

\[
V^+ = \{ \xi \in M_{d+1} : \xi^2 = \xi \cdot \xi > 0, \; \xi^0 > 0 \}, \quad C^+ = \{ \xi \in M_{d+1} : \xi^2 = \xi \cdot \xi = 0, \; \xi^0 > 0 \};
\]

the future cone \( V^+ \) of the ambient spacetime induces the Lorentzian global causal ordering on the de Sitter universe: \( x \) is in the future of \( x' \) if and only if \( x - x' \in V^+ \). The forward light-cone \( C^+ \) of \( x \in \text{invariant trilinear in the vectors of the repres}

\[
\text{sentation of non-compact groups, many new integral relations involving products of hypergeometric functions, other applications to de Sitter and/or anti de Sitter QFT etc.}
\]

\[\int |I|^2 d\Omega_1 d\Omega_2 d\Omega_3 = C^2 \left( \frac{j_1}{m_1} \frac{j_2}{m_2} \frac{j_3}{m_3} \right)^2 = C^2\]

with the parametrization \( \xi = \cos \theta \, e^{i \phi} \) and \( \eta = \sin \theta \, e^{-i \phi} \). Since \( ||\psi_i \psi_j||^2 = \sin^2 \frac{\Delta m}{2} = \left( \frac{\Delta m}{2} \right)^2 \) one can recognize here the triangular invariant introduced in Section \([\text{5} ] \) and computed in Section \([\text{7} ] \). Here the calculation of the invariant \( J \) is done for non-integer values of the exponents (non-compact case). The method we will use generalize the partial integrations that were enough to solve the integer case in \([\text{9} ] \) by using the fractional calculs.
the ambient spacetime also plays the role of the space of momentum directions in de Sitter momentum space [12, 14]. The de Sitter invariance group is the Lorentz group of the ambient spacetime $SO(1, d)$.

A de Sitter generalized free field $\phi$ is fully characterized by its two-point vacuum expectation value $W(x, x')$ which is assumed to be be a local and de Sitter invariant distribution. Since there is no global de Sitter energy operator, a true spectral condition does not exist in the de Sitter spacetime; there is however a suitable replacement that can be formulated [12] by moving to the complex de Sitter manifold

$$X^{(c)}_d = \{ z \in M^{(c)}_{d+1} : z \cdot z = -R^2 \}$$

and requiring that $W(x, x')$ be the boundary value on the reals of a de Sitter invariant function $W_m(z, z')$ holomorphic in the tubular domain $T^- \times T^+$ with slow increase properties at infinity, where

$$T^\pm = \{(M_{d+1} \pm iV^+) \cap X^{(c)}_d \} = \{ z = x + iy \in X^{(c)}_d, \ y \cdot y > 0, \ \text{sign}(y^0) = \pm \}.$$

de Sitter invariance can then be used to show that $W(z, z')$ is actually maximally analytic, i.e. it is analytic in the domain

$$\Delta = \{(z, z') \in X^{(c)}_d \times X^{(c)}_d : \ (z - z')^2 \not\in \mathbb{R}^+ \}.$$

For a thermodynamical interpretation of the above analyticity property, see [12, 13, 15]. By introducing the de Sitter invariant variable

$$\zeta = \frac{z \cdot z'}{R^2}, \ (u \text{ when real and greater than one})$$

there holds a simple description of $W$ in terms of a function $w$ of the single variable $\zeta$, namely $w(\zeta) = W(z, z')$, holomorphic in image of the domain $\Delta$

$$\Pi = \{ \zeta \in \mathbb{C}, \ \zeta \not\in -1 - \mathbb{R}^+ \}.$$

For a Klein-Gordon field $\phi$ with mass $m \geq 0$ the two-point function must also be a bisolution of the Klein-Gordon equation

$$(\Box_x + m^2)W_m(x, x') = (\Box_{x'} + m^2)W_m(x, x') = 0$$

where $\Box$ is the Laplace-Beltrami operator relative to the de Sitter geometry. It is useful to introduce a dimensionless parameter $\nu$ related to the mass $m$ as follows:

$$m^2R^2 = \left( \frac{d-1}{2} \right)^2 + \nu^2.$$  

By abuse of language we will call $\nu$ a mass parameter even if it is dimensionless. Given a complex $\nu$ the corresponding two-point function $W_m(z, z') = W_\nu(z, z') = w_\nu(\zeta)$ is written in terms of Legendre functions of the first kind as follows:

$$W_\nu(z, z') = \frac{\Gamma \left( \frac{1}{2} + i\nu \right) \Gamma \left( \frac{d-1}{2} - i\nu \right)}{2(2\pi)^{\frac{d}{2}}R^{d-2}} (\zeta^2 - 1)^{-\frac{d-1}{2}} P_{-\frac{d-1}{2} + i\nu} \left( \zeta \right)$$

$z, z'$ are events belonging to $\Delta$; the normalization ensures that the canonical commutation relations hold with the correct coefficient.

The range $m \geq m_c = (d-1)/2R$ corresponds to the principal series of unitary irreducible representations of the de Sitter group ($\nu$ real) while $0 < m < m_c$ corresponds to the complementary series ($\nu$ imaginary). These restrictions ensure that the boundary value $W_m$ is positive definite and therefore a quantum theoretical interpretation is available. Note also the symmetry property

$$W_\nu(x, x') = W_{-\nu}(x, x') \quad \left( \text{implied by} \quad P_{-\frac{d-1}{2} + i\nu}(\zeta) = P_{\frac{d-1}{2} - i\nu}(\zeta) \right),$$

which holds for all $\nu$ and will play a role in one structural aspect of the derivation of formula $[11]$. 
2.1 Plane waves expansion of Legendre functions

There exists a Fourier-type representation of the two-point functions which is of fundamental importance to understand the above properties, and generally speaking to understand de Sitter QFT. It is constructed by using a natural basis of plane-wave solutions $\psi_\nu$ of the Klein-Gordon equation

\[(KG)_\nu \psi_\nu(z) = \left[ \Box_z + \left( \frac{d - 1}{2R} \right)^2 + \left( \frac{\nu}{R} \right)^2 \right] \psi_\nu(z) = 0, \quad (18)\]

which are parameterized by the choice of a lightlike vector $\xi \in C^+$ as follows:

\[\psi_\nu(z, \xi) = \left( \frac{z \cdot \xi}{R} \right)^{-\frac{d-1}{2}+i\nu}. \quad (19)\]

These waves are well-defined and analytic in each of the tubes $T^+$ and $T^-$. Then, for $z \in T^-$ and $z' \in T^+$ the following Fourier-type (i.e. momentum space) representation of the two-point function holds true:

\[W_\nu(z, z') = \Gamma\left( \frac{d-1}{2} + i\nu \right) \Gamma\left( \frac{d-1}{2} - i\nu \right) e^{-\pi \nu^2/d+1} \int_\gamma (z \cdot \xi)^{-\frac{d-1}{2}+i\nu} (\xi \cdot z')^{-\frac{d-1}{2}-i\nu} \alpha(\xi), \quad (20)\]

In standard coordinates the $(d-1)$-form $\alpha(\xi)$ is written

\[\alpha(\xi) = (\xi^0)^{-1} \sum_{j=1}^d (-1)^{j+1} \xi^j d\xi^1 \ldots \widehat{d\xi^j} \ldots d\xi^d. \quad (21)\]

In $(20)$, the $(d-1)$-form under the integration sign is closed and $\gamma$ denotes any $(d-1)$-cycle in the forward light-cone $C^+$ which is homologous to the following cycle $\gamma_0$: the support of $\gamma_0$ is represented as follows by the unit sphere, $S_{d-1}$ (equipped with its canonical orientation):

\[\text{supp} \gamma_0 = S_{d-1} = C^+ \cap \{ \xi : \xi^0 = 1 \} = \{ \xi \in C^+ : \xi^{12} + \ldots + \xi^{d2} = 1 \}. \quad (22)\]

With this choice $\alpha(\xi)$ coincides with the rotation invariant measure on $S_{d-1}$ normalized as follows:

\[\omega_d = \int_{\gamma_0} \alpha(\xi) = 2\pi^{\frac{d}{2}} \frac{d-1}{\Gamma\left( \frac{d}{2} \right)}. \quad (23)\]

2.2 Lobatchevski space and a remarkable representation of Legendre functions

A specially important parametrization of the Fourier-type representation is obtained by evaluating $(20)$ at the purely imaginary events $z = 0 - iy \in T^-$ and $z = 0 + iy' \in T^+$; $y$ and $y'$ can be visualized as points belonging to a Lobatchevski space, modeled as the upper sheet of a two-sheeted hyperboloid:

\[\mathbb{H}_d = \{ y \in M_{1,d} : y^2 = y \cdot y = R^2, \ y^0 > 0 \}. \quad (24)\]

We will make use of the following spherical parametrization of $\mathbb{H}_d$:

\[y(u, n) = R(u, n^1 \sqrt{u^2 - 1}, \ldots, n^d \sqrt{u^2 - 1}) \quad (25)\]

where $u \geq 1$ and $n \in S_{d-1}$; in these coordinates the Lorentz-invariant measure $dy$ is written

\[dy = R^d(u^2 - 1)^{\frac{d-2}{2}} dudu. \quad (26)\]
where \( dn \) denotes the rotation-invariant measure on the sphere \( S_{d-1} \) normalized as in Eq. (23). With the above specifications, Eq. (27) allow us to write:

\[
W_\nu(-iy, iy') = \frac{\Gamma \left( \frac{d-1}{2} + i\nu \right) \Gamma \left( \frac{d-1}{2} - i\nu \right)}{2^{d+1}\pi^d} \int_\gamma (y \cdot \xi)^{-\frac{d+1}{2}+i\nu} \left( \xi \cdot y' \right)^{-\frac{d+1}{2}-i\nu} d\mu_\gamma(\xi) = \\
= \frac{\Gamma \left( \frac{d-1}{2} + i\nu \right) \Gamma \left( \frac{d-1}{2} - i\nu \right)}{2(2\pi)^\frac{d}{2}} \left( (y \cdot y')^2 - 1 \right)^{-\frac{d+2}{4} - i\nu} P_{\frac{d-2}{4} + i\nu} (y \cdot y').
\]

(27)

Here and in the following we have set \( R = 1 \); by choosing in particular \( \gamma = \gamma_0 \) and \( y' = (1, 0, \ldots, 0) \) so that \( y \cdot y' = y \) \( m \) = \( u \geq 1 \), we then get the following integral representation:

\[
(\nu^2 - 1)^{-\frac{d+2}{4}} P_{\frac{d-2}{4} + i\nu} (u) = \frac{1}{(2\pi)^\frac{d}{2}} \int_{\gamma_0} (y \cdot \xi)^{-\frac{d+1}{2}+i\nu} \alpha(\xi).
\]

(28)

This formula will be of crucial importance for computing \( h_d(\kappa, \nu, \lambda) \), since it allows one to rewrite the integral in Eq. (2) as the following multiple integral over the manifold \( \mathbb{H}_d \times S_{d-1} \times S_{d-1} \times S_{d-1} \):

\[
h_d(\kappa, \nu, \lambda) = \frac{1}{(2\pi)^\frac{d}{2} \omega_{d-1}} \int_{\gamma_0} \int_{\gamma_0} \int_{\mathbb{H}_d} \int_{S_{d-1}} \frac{dw}{\omega_{d-1}} (y \cdot \xi_1)^{-\frac{d+1}{2} - i\nu} (y \cdot \xi_2)^{-\frac{d+1}{2} - i\nu} (y \cdot \xi_3)^{-\frac{d+1}{2} - i\nu} d\alpha(\xi_1) d\alpha(\xi_2) d\alpha(\xi_3)
\]

(29)

where we have used the measure (26) and the normalization (23).

### 2.3 Källén-Lehmann-type representation for general two-point functions

Consider again a general two-point function such that its reduced form \( w(\zeta) \) is analytic in the cutplane \( \Pi \) and uniformly bounded at infinity by a certain power \( |\zeta|^m_0 \). It has been shown in [10] that for \(-1 < m_0 < 0 \) there exists an integral representation of \( w(\zeta) \) the following form:

\[
w(\zeta) = \frac{1}{2(2\pi)^\frac{d}{2}} \int_{-\infty}^{\infty} \Gamma \left( m_0 + \frac{d-1}{2} + i\kappa \right) \Gamma \left( m_0 + d - 1 + i\kappa \right) \times
\]

\[
\times G(m_0 + i\kappa) \left( \zeta^2 - 1 \right)^{-\frac{d-2}{2}} P_{m_0 + \frac{d-2}{2} + i\kappa}(\zeta) d\kappa;
\]

(30)

the function \( G(m_0 + i\kappa) \) is the boundary value of a function \( G(s) \) holomorphic in the half-plane \( \text{Re } s > m_0 \). \( G \) is obtained as a Laplace-type transform of the discontinuity \( \Delta w(\zeta) \) of \( w(\zeta) \) across the cut \( ] -\infty, -1 [ \) (we will not need the explicit expression given in [10], Eqs. III 10 and III 11).

The results of [10] can be extended to the case \( m_0 = -\frac{d-1}{2} \), which is relevant for de Sitter quantum field theory, because, in that case, the Legendre functions involved in (30) are all the free-field two-point functions of the principal series. We omit the details of the proof of formula (30) under the assumption that \( |w(\zeta)| \) is bounded by \( |\zeta|^{-\frac{d-1}{2}} \). Inserting the value \( m_0 = -\frac{d-1}{2} \) in Eq. (30) and taking the symmetry condition (17) into account and puts, one obtains:

\[
w(\zeta) = \int_0^{\infty} \rho(\kappa) \left\{ \frac{\Gamma \left( \frac{d-1}{2} + i\kappa \right)}{2(2\pi)^\frac{d}{2}} \right\} \left( \zeta^2 - 1 \right)^{-\frac{d-2}{2}} P_{\frac{d-2}{4} + i\kappa}(\zeta) d\kappa^2 = \int_0^{\infty} \rho(\kappa) \ w_\kappa(\zeta) d\kappa^2.
\]

(31)

which is a genuine Källén-Lehmann-type representation of \( w(\zeta) \) with weight

\[
\rho(\kappa) = \frac{G \left( -\frac{d-1}{2} - i\kappa \right) - G \left( -\frac{d-1}{2} + i\kappa \right)}{2i}
\]

(32)
The computation the Källén-Lehmann weight \( \rho \) can also be tackled by invoking the generalized Mehler-Fock transformation theory [17], as we do here. Eq. (32) takes the following concrete form:
\[
\rho(\kappa) = 2 \left( \frac{2\pi}{d-2} \right)^{d-2} \sinh \pi \kappa \int_1^\infty w(\zeta) P_{\frac{d-2}{2}\mp \alpha}(\zeta) (\zeta^2 - 1)^{\frac{d-2}{2}} d\zeta. \tag{33}
\]

3 Decay of de Sitter unstable particles

The study of particle disintegration in the de Sitter universe has been initiated in a pioneering paper by Nachtmann. Consider three independent neutral Klein-Gordon scalar fields \( \phi_0, \phi_1, \phi_2 \) with real mass parameters \( \kappa, \nu, \lambda \) respectively (i.e. the fields in the principal series) and an interaction term of the form
\[
\int g(x) \mathcal{L}(x) dx, \quad \mathcal{L}(x) = : \phi_0(x) \phi_1(x) \phi_2(x) :\]
where \( g \) is a smooth spacetime dependent "switching-on factor" which, in the end, should be made to tend to the constant 1. Self-interactions \( \mathcal{L}(x) = : \phi(x)^3 : \) are a special case of this coupling. Let us consider the decay process
\[
0 \to 1 + 2 \tag{34}
\]
Let in particular \( \Psi_0 \) be a one-particle state of the form
\[
\Psi_0 = \int f(x) \phi_0(x) \Omega dx;
\]
the smooth test function \( f(x) \) contains the physical details about the preparation of the quantum state of the unstable particle whose disintegration we aim to study. The following general formula for the transition probability holds true [2]:
\[
\Gamma(0; 1, 2) = \frac{\pi^2}{\sinh \pi \kappa} \frac{\int g(x) |F(x)|^2 dx}{\int f(x) W_\kappa(x, y) f(y) dx dy} \rho_{\nu, \lambda}(\kappa). \tag{35}
\]
Here the convolution \( F(x) = \int W_\kappa(x, y) f(y) dy \) is the "positive-frequency" solution of the KG equation with mass \( \kappa \) associated with the test-function \( f \); the denominator is the squared norm of \( \Psi_0 \). Note that the first factor in this formula does not depend on the the decay particles but only on the wavefunction of the incoming unstable particle. The infrared problem is contained in this factor and has to be overcome when letting the remaining \( g(x) \) tend to 1 (adiabatic limit). We will not treat this problem here and refer to [3].

The second factor is the relevant Källén-Lehmann weight of the bubble diagram corresponding to two-point function of a composite field, obtained as the Wick product of the Klein-Gordon fields with mass parameters \( \nu \) and \( \lambda \):
\[
w(\zeta) = w_\nu(\zeta) w_\lambda(\zeta). \tag{36}
\]
This two-point function is well-defined and analytic in the cut-plane II. Moreover, for real values of \( \lambda \) and \( \nu \) (i.e. for fields belonging to the principal series) it is bounded in \( \Pi \) by \( |\zeta|^{-d-1} \) and therefore, a fortiori, by \( |\zeta|^{-\frac{d+1}{2}} \). In particular, the Laplace-type transform \( G_{\nu, \lambda}(s) \) of \( w_\nu(\zeta)w_\lambda(\zeta) \) is analytic in the half-plane \( \{ s \in \mathbb{C}; \text{Re } s > -(d - 1) \} \). Thus, there exists a Källén-Lehmann representation (31) of \( w_\nu(\zeta)w_\lambda(\zeta) \):
\[
w_\nu(\zeta) w_\lambda(\zeta) = \int_0^\infty \rho_{\nu, \lambda}(\kappa) w_\kappa(\zeta) d\kappa = \int_\mathbb{R} \kappa \rho_{\nu, \lambda}(\kappa) w_\kappa(\zeta) d\kappa. \tag{37}
\]
The weight \( \rho \), as given in Eq. (32) inherits analyticity properties from the aforementioned properties of \( G_{\nu,\lambda}(s) \) of (33) and is itself holomorphic in the strip \( \{s \in \mathbb{C}: -(d-1) < \Re s < 0\} \) and therefore it cannot vanish on any open interval of the line \( s = -\frac{d-1}{2} + ik \).

This immediately implies that, in de Sitter spacetime, there is nothing such as the "subadditivity condition" of the Minkowski case: that property would require that \( \rho_{\nu,\lambda}(\kappa) \) should vanish if \( \kappa < \nu + \lambda \) and this would forbid the decay of a particle of mass \( \kappa \) into a pair of particles of masses \( \nu \) and \( \lambda \) when \( \kappa < \nu + \lambda \). In contrast, in the de Sitter universe the disintegration of a particle of a given mass can give rise to two heavier particles if such a coupling enters in the interaction Lagrangian.

In the following sections we will explicitly compute the Källén-Lehmann weight by a mixture of geometrical insights and analytical techniques. By inserting the 16 of in the Mehler-Fock transform (33) of \( w_\nu w_\lambda \) it follows that \( \rho_{\nu,\lambda}(\kappa) \) is proportional to the integral (2); more precisely, one obtains:

\[
\rho_{\nu,\lambda}(\kappa) = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right) \Gamma\left(\frac{d-1}{2} - i\nu\right) \Gamma\left(\frac{d-1}{2} + i\lambda\right) \Gamma\left(\frac{d-1}{2} - i\lambda\right)}{2(2\pi)^{1+\frac{d}{2}} \sinh(\pi\kappa)} h_d(\kappa, \nu, \lambda), \tag{38}
\]

4 A special class of de Sitter three-point functions.

The general properties of de Sitter two-point functions can, to some extent, be generalized to a class \( \mathcal{C}_F \) of three-point functions \( W(x_1, x_2, x_3) \) on de Sitter spacetime, such that

1. \( W(x_1, x_2, x_3) \) is a distribution on \( X_d \times X_d \times X_d \) which is decomposable as a sum of two boundary values of holomorphic functions \( W_\varepsilon(z_1, z_2, z_3) \) from the respective tubular domains \( T^- \times T^+ \times T^+ \) where \( \varepsilon = + \) or \( - \).

2. Each function \( W_\varepsilon(z_1, z_2, z_3) \) is invariant under the complex de Sitter group \( SO_0(1, d)^{(c)} \) and therefore it coincides with a holomorphic function of the three complex invariants \( z_i \cdot z_j = \zeta_{ij}, \ i, j = 1, 2, 3 \), (since \( z_i^2 = -1 \)) in the image of the corresponding tubular domain.

3. Each distribution \( W_\varepsilon(x_1, x_2, x_3) \) admits a Fourier-type transform on the one-sheeted hyperboloid defined in terms of the plane waves (19) (see also (18) where the case \( d = 2 \) has been treated in detail).

We do not expect that the above properties hold for general interacting quantum field theories; in particular they do not apply to the general class introduced in (13). They can be however useful in a perturbative context.

The transform of \( W_\varepsilon(x_1, x_2, x_3) \) is a distribution on \( (\mathbb{C}^+) \times \mathbb{R}^3 \) defined by

\[
\tilde{W}_\varepsilon(\xi_1, \xi_2, \xi_3; \kappa, \nu, \lambda) = \int_{\mathbb{R}^3} (x_1 \cdot \xi_1)^{-\frac{d-1}{2} - i\nu} (x_2 \cdot \xi_2)^{-\frac{d-1}{2} - i\nu} (x_3 \cdot \xi_3)^{-\frac{d-1}{2} - i\lambda} \times
\]

\[
\times W_\varepsilon(x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3. \tag{39}
\]

Conversely, each holomorphic function \( W_\varepsilon \) is recovered in its respective domain \( T^- \times T^+ \times T^+ \) by an inversion formula, which includes an appropriate weight-function \( \sigma_d \) on \( \mathbb{R}^+ \):

\[
W_\varepsilon(z_1, z_2, z_3) = \int_{\mathbb{R}^+} \sigma_d(\kappa)\sigma_d(\nu)\sigma_d(\lambda) \, dk \, dv \, d\lambda \times
\]

\[
\times \int_{\mathbb{R}^3} (z_1 \cdot \xi_1)^{-\frac{d-1}{2} + i\kappa} (z_2 \cdot \xi_2)^{-\frac{d-1}{2} + i\nu} (z_3 \cdot \xi_3)^{-\frac{d-1}{2} + i\lambda} \tilde{W}_\varepsilon(\xi_1, \xi_2, \xi_3; \kappa, \nu, \lambda) \alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3). \tag{40}
\]
\( \hat{W}_c \) depends on \( (\xi_1, \xi_2, \xi_3) \) only through the (Lorentz) invariants \( \xi_1 \cdot \xi_2, \xi_2 \cdot \xi_3, \xi_1 \cdot \xi_1, \) (since \( \xi_2^2 = 0 \)).

The homogeneity properties of (39) w.r.t. the variables \( \xi_j \in C^+ \) imply that

\[
\sigma_d(\kappa)\sigma_d(\nu)\sigma_d(\lambda) \hat{W}_c(\xi_1, \xi_2, \xi_3; \kappa, \nu, \lambda) = \hat{\rho}_c(\kappa, \nu, \lambda) (\xi_1 \cdot \xi_2)^{a_3} (\xi_2 \cdot \xi_3)^{a_1} (\xi_3 \cdot \xi_3)^{a_2}. \tag{41}
\]

where we have introduced the parameters

\[
a_1 = -\frac{d-1}{4} - \frac{i}{2} \frac{\nu + \kappa - \lambda}{2}, \quad a_2 = -\frac{d-1}{4} - \frac{i}{2} \frac{\lambda + \kappa - \nu}{2}, \quad a_3 = -\frac{d-1}{4} - \frac{i}{2} \frac{\kappa + \nu - \lambda}{2}. \tag{42}
\]

The inversion formula (40) can therefore be rewritten as follows:

\[
\hat{W}_c(z_1, z_2, z_3) = \int_{\mathbb{R}^3} \hat{\rho}_c(\kappa, \nu, \lambda) w_{\kappa, \nu, \lambda}(z_1, z_2, z_3) \, d\kappa d\nu d\lambda \tag{43}
\]

where

\[
w_{\kappa, \nu, \lambda}(z_1, z_2, z_3) = \int_{\mathbb{R}^3} (z_1 \cdot \xi_1)^- \frac{dz_1 + i\kappa}{\sqrt{-d}} \cdot \frac{dz_2 + i\nu}{\sqrt{-d}} \cdot \frac{dz_3 - i\lambda}{\sqrt{-d}} 
\times (\xi_1 \cdot \xi_2)^{a_3} (\xi_2 \cdot \xi_3)^{a_1} (\xi_3 \cdot \xi_3)^{a_2} \alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3). \tag{44}
\]

The three-point function \( w_{\kappa, \nu, \lambda}(z_1, z_2, z_3) \) manifestly satisfies the triplet of Klein-Gordon equations:

\[
[(KG)_{\kappa}]_{z_1} w_{\kappa, \nu, \lambda}(z_1, z_2, z_3) = [(KG)_{\nu}]_{z_2} w_{\kappa, \nu, \lambda}(z_1, z_2, z_3) = [(KG)_{\lambda}]_{z_3} w_{\kappa, \nu, \lambda}(z_1, z_2, z_3) = 0 \tag{45}
\]

in the (non-connected) complex open set \( (z_1, z_2, z_3) \in T^\pm \times T^\pm \times T^\pm \) where it is holomorphically defined via Eq. (44). This set contains in particular the relevant tubular domains \( T^- \times T^\times \times T^+ \) in which the integral representation (43) is meaningful. Formula (43) has the shape of generalized Källén-Lehmann representation for all the three-point functions which belong to the class \( \mathcal{C}_F \) on the basis of three-point functions satisfying the Klein-Gordon system (15).

5 A star-triangle relation and a class of triangular invariants on the hypersphere

Before the computation of \( h_d(\kappa, \nu, \lambda) \) can be made possible we need to introduce two further ingredients: a generalized star-triangle relation and a class of triangular invariants on the hypersphere. They both come out from the study of the following integral on the Lobachevski manifold \( y \in \mathbb{H}_d \) (see Sec. 2.2):

\[
F_{a_1, a_2, a_3}(\kappa, \nu, \lambda) = \int_{\mathbb{H}_d} (y \cdot \xi_3)^{a_2+a_3+a_1} (y \cdot \xi_2)^{a_3} (y \cdot \xi_3)^{a_1+a_2} dy = c(a_1, a_2, a_3) (\xi_1 \cdot \xi_2)^{a_3} (\xi_2 \cdot \xi_3)^{a_1} (\xi_3 \cdot \xi_3)^{a_2} \tag{46}
\]

the second equality follows again from Lorentz (i.e. de Sitter) invariance and from the homogeneity properties of (16) with respect to the variables \( \xi_j \)'s; the constant \( c(a_1, a_2, a_3) \) remains to be determined.

This identity is a sort of “star-triangle relation” with one important difference w.r.t. what is usually called “star-triangle”: the center of the star is a point of \( \mathbb{H}_d \) while the legs belong to the asymptotic cone \( C^+ \) i.e. the center and the legs of the star do not belong to the same manifold. By integrating both sides over the spherical basis \( \gamma_0 \) of the cone we get

\[
f(a_1, a_2, a_3) = \int_{\gamma_0} \int_{\gamma_0} \int_{\gamma_0} F_{a_1, a_2, a_3}(\kappa, \nu, \lambda) \alpha(\xi_1) \alpha(\xi_2) \alpha(\xi_3)
= c(a_1, a_2, a_3) \int_{\gamma_0} \int_{\gamma_0} \int_{\gamma_0} (\xi_1 \cdot \xi_2)^{a_3} (\xi_2 \cdot \xi_3)^{a_1} (\xi_3 \cdot \xi_3)^{a_2} \alpha(\xi_1) \alpha(\xi_2) \alpha(\xi_3) \tag{47}
= c(a_1, a_2, a_3) \times \hat{J}(a_1, a_2, a_3) \tag{48}
\]
The integral at the r.h.s. is proportional to the value of the three-point function \( w_{\kappa,\nu,\lambda}(z_1, z_2, z_3) \) at the special complex event \( z_1 = z_2 = z_3 = iy_0 \in \mathcal{T}^+ \), \( y_0 = (1, 0, \ldots, 0) \):

\[
\begin{align*}
    w_{\kappa,\nu,\lambda}(iy_0, iy_0, iy_0) &= e^{\frac{\pi(\kappa + \nu + \lambda)}{2} + \pi \frac{3(\kappa - 1)\pi i}{4}} \int_{\mathcal{T}^+} (\xi_1 \cdot \xi_2)^{\kappa_1} (\xi_2 \cdot \xi_3)^{\kappa_2} (\xi_3 \cdot \xi_1)^{\kappa_3} \alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3).
\end{align*}
\]

(50)

From Eqs. (29), (32) and (40), it follows that computing \( h_d(\kappa, \nu, \lambda) \) is equivalent to integrating the star-triangle relation (48) w.r.t. the external legs:

\[
\begin{align*}
    h_d(\kappa, \nu, \lambda) &= \frac{1}{(2\pi)^\frac{d}{2} \omega_{d-1}} f(a_1, a_2, a_3) = \\
    &= \frac{1}{(2\pi)^\frac{d}{2} \omega_{d-1}} e^{\pi \frac{3(\nu + 1)\pi i}{4}} w_{\kappa,\nu,\lambda}(iy_0, iy_0, iy_0) \times c(a_1, a_2, a_3). 
\end{align*}
\]

(51)

Define

\[
\hat{J}(a_1, a_2, a_3) = \int_{\gamma_0} (\xi_1 \cdot \xi_2)^{a_1} (\xi_2 \cdot \xi_3)^{a_2} (\xi_3 \cdot \xi_1)^{a_3} \alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3),
\]

(52)

so that Eq. (48) is rewritten as follows:

\[
f(a_1, a_2, a_3) = c(a_1, a_2, a_3) \hat{J}(a_1, a_2, a_3).
\]

(53)

The integral (52) has a beautiful geometrical interpretation as a triangular invariant on the hypersphere \( S_{d-1} \). Consider indeed the squared distance \( \Delta n_{ik}^2 \) between two points \( n_i \) and \( n_k \) belonging to \( S_{d-1} \). The Lorentzian scalar product of two points \( \xi_i = (1, n_i) \) and \( \xi_k = (1, n_k) \) belonging to the spherical cycle \( \gamma_0 \) of the forward lightcone \( C^+ \) is proportional to the squared distance \( \Delta n_{ik}^2 \):

\[
\Delta n_{ik}^2 = (n_i - n_k)^2 = 2 - 2 n_i \cdot n_k = 2 \xi_i \cdot \xi_k.
\]

(54)

Given three points \( n_1, n_2 \) and \( n_3 \) of \( S_{d-1} \) and three complex numbers \( a_1, a_2 \) and \( a_3 \), we construct the rotation invariant quantity

\[
J = J(a_1, a_2, a_3) = \left\langle \left( \Delta n_{12}^2 \right)^{a_1} \left( \Delta n_{23}^2 \right)^{a_2} \left( \Delta n_{31}^2 \right)^{a_3} \right\rangle
\]

(55)

where \( \langle f \rangle \) denotes the average on \( S_{d-1} \); for three points

\[
\langle f \rangle = \frac{\int_{S_{d-1}} \, dn_1 \, dn_2 \, dn_3 \, f(n_1, n_2, n_3)}{\int_{S_{d-1}} \, dn_1 \, dn_2 \, dn_3}.
\]

(56)

It follows that

\[
J(a_1, a_2, a_3) = \frac{2^a}{\omega_d^d} \int (\xi_1 \cdot \xi_2)^{a_1} (\xi_2 \cdot \xi_3)^{a_2} (\xi_3 \cdot \xi_1)^{a_3} d\mu_{\gamma_0}(\xi_1) d\mu_{\gamma_0}(\xi_2) d\mu_{\gamma_0}(\xi_3) = \frac{2^a}{\omega_d^d} \hat{J}(a_1, a_2, a_3)
\]

(57)

and therefore, in view of (53):

\[
f(a_1, a_2, a_3) = 2^{-a} \omega_d^d c(a_1, a_2, a_3) J(a_1, a_2, a_3)
\]

(58)

In the following sections we will describe the details of the concrete evaluation of \( h_d(\kappa, \nu, \lambda) \). The method that we present below is based on the previous relations (48) and (51) applies to all values of the spacetime dimension \( d (d \geq 2) \).

There exists a remarkable symmetry relation between of \( c \) and \( J \) that follows from symmetry relation

\[
h_d(\kappa, \nu, \lambda) = h_d(-\kappa, -\nu, -\lambda)
\]

(59)
which in turn is a consequence for all real values of $\kappa, \nu, \lambda$, of the basic symmetry property (17) of the Legendre functions first-kind. If we introduce the corresponding transformation

$$a_j \to a'_j = -a_j - \frac{d-1}{2}; \quad j = 1, 2, 3,$$

and take Eqs. (52) and (51) into account, we obtain that for every triplet $(a_1, a_2, a_3)$ such that $\text{Re} a_i = \text{Re} a'_i = -\frac{d-1}{4}$, the following equality is valid

$$f(a_1, a_2, a_3) = f(a'_1, a'_2, a'_3)$$

or either (in view of Eq (55)):

$$2^{-d} a \frac{J(a_1, a_2, a_3)}{c(a'_1, a'_2, a'_3)} = 2^{-d} a' \frac{J(a'_1, a'_2, a'_3)}{c(a_1, a_2, a_3)}$$

which are both equivalent to (59).

This striking duality between two integrals over different manifolds is surprising at first sight. It will be made clear that the factorization (58) corresponds precisely to a splitting of the expression (2) of $h_d(\kappa, \nu, \lambda)$ into two parts which are symmetric under that parity transformation.

In the following section we will evaluate the functions $c$ and $\hat{J}$. The study of the integrals (46) and (52) will lead us to define and compute the functions $c$ and $\hat{J}$ in appropriate domains of the complex space $\mathbb{C}^3$ of the variables $(a_1, a_2, a_3)$. $h_d(\kappa, \nu, \lambda)$ is then obtained by taking the restriction of the holomorphic function $f = c \times \hat{J}$ to the linear real submanifold $L_d$ of $\mathbb{C}^3$ defined by the equations (42), namely $L_d = \{(a_1, a_2, a_3) \in \mathbb{C}^3; \text{Re} a_j = -\frac{d-1}{4}; \quad j = 1, 2, 3\}$.

### 6 Computing $c(a_1, a_2, a_3)$; more on the star-triangle relation

For computing the integral (46) and obtaining (17) with the complete expression of the function $c(a_1, a_2, a_3)$, we shall consider the following double Mellin transform:

$$\int_0^\infty \int_0^\infty (1 + v + z)^b v^{s-1} z^{t-1} dv dt = \frac{\Gamma(s)\Gamma(t)\Gamma(-b - s - t)}{\Gamma(-b)}; \quad (63)$$

this relation is valid for $\text{Re}(s) > 0, \text{Re}(t) > 0, \text{Re}[b + s + t] < 0$. Mellin’s inversion theorem then provides the following expansion:

$$(1 + v + z)^b = -\frac{1}{4\pi^2} \int_{\gamma-i\infty}^{\gamma+i\infty} ds \int_{\gamma'-i\infty}^{\gamma'+i\infty} dt \frac{\Gamma(s)\Gamma(t)\Gamma(-b - s - t)}{\Gamma(-b)}$$

where the integration paths lie in the strips allowed by the previous inequalities. The formula can be rendered symmetric by homogeneity:

$$(u + v + z)^b = -\frac{1}{4\pi^2} \int_{\gamma-i\infty}^{\gamma+i\infty} ds \int_{\gamma'-i\infty}^{\gamma'+i\infty} dt u^{b+s+t} v^{s} z^{-t} \frac{\Gamma(s)\Gamma(t)\Gamma(-b - s - t)}{\Gamma(-b)}.$$

Consider now the expression $(y \cdot \Xi)^{2a}$ where

$$\Xi = \xi_1 + \xi_2 + \xi_3, \quad a = a_1 + a_2 + a_3; \quad (66)$$
all the $\xi$’s are lightlike and therefore $\Xi$ is either timelike or lightlike (in the latter case $\xi_1, \xi_2$ and $\xi_3$ lie on the same generatrix of the cone):

$$\Xi^2 = 2 \xi_1 \cdot \xi_2 + 2 \xi_2 \cdot \xi_3 + 2 \xi_3 \cdot \xi_1 \geq 0.$$  \hspace{1cm} (67)

By application of (65) we get that

$$\begin{align*}
(y \cdot \Xi)^{2a} &= -\frac{1}{4\pi^2} \int_{\gamma-i\infty}^{\gamma+i\infty} ds \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{\Gamma(s)\Gamma(t)(-2a-s-t)}{\Gamma(-2a)} \left(y \cdot \xi_1\right)^{2a+s+t} \left(y \cdot \xi_2\right)^{-s} \left(y \cdot \xi_3\right)^{-t} \\hspace{1cm} (68)
\end{align*}
$$

with $\gamma > 0$, $\gamma' > 0$, $2 \Re(a) + \gamma + \gamma' < 0$.

Let us integrate the two members of (68) over the Lobatchevski manifold $\mathbb{H}_d$. The Lorentz invariance of the l.h.s. implies that the integral can be parametrized in terms of the hyperbolic angle $\lambda$ between $y$ and $\Xi$:

$$\int_{\mathbb{H}_d} (y \cdot \Xi)^{2a} dy = \frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{4}{2}\right)} \int_0^{\infty} (\cosh \alpha)^{2a} (\sinh \alpha)^{d-1} d\alpha = \frac{\pi^{\frac{3}{2}}}{\Gamma\left(\frac{4}{2}\right)} \left(\Xi^2\right)^{a};$$  \hspace{1cm} (69)

this result holds provided $\Re(a) < -(d-1)/2$. Thus

$$\begin{align*}
\frac{2^a\pi^{\frac{3}{2}}}{\Gamma\left(\frac{4}{2}\right)} \frac{\Gamma\left(\frac{1}{2} - a - \frac{\beta}{2}\right)}{\Gamma\left(\frac{1}{2} - a\right)} (\xi_1 \cdot \xi_2 + \xi_2 \cdot \xi_3 + \xi_3 \cdot \xi_1)^a &= \frac{1}{4\pi^2} \frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{4}{2}\right)} \frac{\Gamma\left(\frac{1}{2} - a - \frac{\beta}{2}\right)}{\Gamma\left(\frac{1}{2} - a\right)} \times \\hspace{1cm} (70)
\end{align*}
$$

In the second step we have applied once more Eq. (65), with $\delta > 0$, $\delta' > 0$, $\Re(a) + \delta + \delta' < 0$. Since Eq. (70) is valid for any choice of the vectors $\xi_1$, $\xi_2$ and $\xi_3$ we can multiply $\xi_2$ and $\xi_3$ by two complex numbers $\alpha$ and $\beta$ and obtain that

$$\begin{align*}
\int_{\gamma-i\infty}^{\gamma+i\infty} ds \alpha^{-s} \int_{\gamma'-i\infty}^{\gamma'+i\infty} dt \beta^{-t} \Gamma(s)\Gamma(t)(-2a-s-t) \int dy(y \cdot \xi_1)^{2a+s+t}(y \cdot \xi_2)^{-s}(y \cdot \xi_3)^{-t} = \\hspace{1cm} (71)
\end{align*}
$$

By changing the variables in the second integral as follows: $u = -a - s$, $v = a + s + t$. the r.h.s. becomes

$$\begin{align*}
\int_{-\Re(a)-\delta-i\infty}^{-\Re(a)-\delta+i\infty} ds \alpha^{-s} \int_{\delta+i\infty}^{\delta' + i\infty} dt \beta^{-t} \Gamma(-a-s)\Gamma(a+s+t) \times \\hspace{1cm} (72)
\end{align*}
$$

By Mellin’s inversion theorem we can now identify the integrands i.e.

$$\int dy(y \cdot \xi_1)^{2a+s+t}(y \cdot \xi_2)^{-s}(y \cdot \xi_3)^{-t} =$$
corresponds to setting $C$ where the integration is performed on the parabolic basis relation (79) with

Now we can follow the same steps as before and obtain the limiting conformal invariant star-triangle

Finally, by setting $-s = a_3 + a_1$ and $-t = a_1 + a_2$ the proof of the star-triangle relation (77) is completed with

$$c(a_1, a_2, a_3) = 2^{a_1 - 1} \pi^{\frac{a_1}{2}} \Gamma \left( \frac{3}{2} - a + \frac{d}{2} \right) \Gamma(-a - s) \Gamma(a + s + t) \Gamma(-a - t) \Gamma(s) \Gamma(t) \Gamma(-2a - s - t) \frac{\Gamma(-a_1) \Gamma(-a_2) \Gamma(-a_3)}{\Gamma(-a_2 - a_3) \Gamma(-a_3 - a_1) \Gamma(-a_1 - a_2)}$$

(74)

6.1 Conical limit

It is interesting to remark that, under certain conditions, a true star-triangle relation is obtained by integrating over a cycle of the lightcone. Consider indeed the three-point function

$$G_{a_1, a_2, a_3}(\xi_1, \xi_2, \xi_3) = \int \gamma_0 (\xi \cdot \xi_1)^{a_2 + a_3} (\xi \cdot \xi_2)^{a_3 + a_1} (\xi \cdot \xi_3)^{a_1 + a_2} \alpha(\xi),$$

(75)

where the integration is performed on the parabolic basis $C^+ \cap \{ \xi^0 + \xi^d = 1 \}$ of the future cone; this corresponds to setting $\lambda = 1$ in the parametrization

$$\xi(\lambda, \eta) = \begin{cases} \xi^0 = \frac{1}{2}(1 + \eta^2) \\ \xi^i = \lambda \eta^i, 0 < \lambda < \infty, \quad \eta \in \mathbb{R}^{d-2}, \\ \xi^{d-1} = \frac{1}{2}(1 - \eta^2) \end{cases}$$

(76)

and w.r.t. the Lebesgue measure $d\mu(\xi) = d\eta$. Concretely, since

$$2 \xi(\lambda, \eta) \cdot \xi'(1, \eta') = \lambda(\eta - \eta')^2$$

(77)

$$G_{a_1, a_2, a_3}(\xi_1, \xi_2, \xi_3) = \frac{\lambda_1^{a_2 + a_3} \lambda_2^{a_3 + a_1} \lambda_3^{a_1 + a_2}}{2^a} \int \gamma_0 ((\eta - \eta_1)^2)^{a_2 + a_3} [((\eta - \eta_2)^2)^{a_3 + a_1} ((\eta - \eta_3)^2)^{a_1 + a_2} d\eta.$$}

(78)

The result is of course invariant under Euclidean transformations in $(d-1)$ dimensions. There is however a special case to be considered: when $2a = (1-d)$ the integrand is a closed $(d-1)$-form on the future cone. By applying Stokes’ theorem one sees that the integral does not depend on the choice of the integration manifold and the result is fully $SO_0(1,d)$-invariant; this invariance can be now interpreted as Euclidean conformal invariance. As before, by exploiting the $SO_0(1,d)$-invariance of the integral and the homogeneity properties of the integrand one has that

$$\int_{M} (\xi \cdot \xi_1)^{a_2 + a_3} (\xi \cdot \xi_2)^{a_3 + a_1} (\xi \cdot \xi_3)^{a_1 + a_2} d\mu(\xi) = d(a_1, a_2, a_3)(\xi_1 \cdot \xi_2)^{a_3}(\xi_3 \cdot \xi_3)^{a_1}(\xi_3 \cdot \xi_1)^{a_2}.$$}

(79)

To determine the constant we need to compute the integral $\int (\xi \cdot y)^{1-d} \alpha(\xi)$ where again $y = \xi_1 + \xi_2 + \xi_3$. This is most easily done using the spherical basis of the cone $\gamma_0$ with respect to the measure $d\mu_{\gamma_0}(\xi)$ (see Eq. (23)). Calculating the previous integral in these coordinates is immediate:

$$\int \gamma_0 (\xi \cdot y)^{1-d} \alpha(\xi) = \omega_d (y \cdot y)^{-\frac{d+1}{2}}$$

(80)

Now we can follow the same steps as before and obtain the limiting conformal invariant star-triangle relation (79) with

$$d(a_1, a_2, a_3) = (2\pi)^{\frac{d+1}{2}} \Gamma(-a_1) \Gamma(-a_2) \Gamma(-a_3) \Gamma(-a_2 - a_3) \Gamma(-a_3 - a_1) \Gamma(-a_1 - a_2)$$

(81)
7 Computing $J(a_1, a_2, a_3)$: direct and indirect methods.

7.1 Probabilistic interpretation of $J$

To evaluate $J$ we make use of the standard parametrization of a point $n \in \mathbb{S}_{d-1}$ in terms of $(d-1)$ angles $\theta_i$, parametrization that we spell here for the reader’s convenience:

$$n = \begin{cases} 
\cos \theta_1 \\
\sin \theta_1 \cos \theta_2 \\
\vdots \\
\sin \theta_1 \ldots \sin \theta_{d-2} \cos \theta_{d-1} \\
\sin \theta_1 \ldots \sin \theta_{d-2} \sin \theta_{d-1}
\end{cases} \quad 0 < \theta_1 < \pi, \ldots, 0 < \theta_{d-2} < \pi \quad 0 < \theta_{d-1} < 2\pi \quad (82)
$$

so that

$$dn = \prod_{j=1}^{d-1} (\sin \theta_j)^{d-1-j} \, d\theta_j. \quad (83)$$

The rotation invariance of (85) implies that it is possible to perform the integrations by fixing one point, say $n_1 = (1,0,\ldots,0)$ and specializing the second point $n_2 = (\cos \phi_1, \sin \phi_1, 0, \ldots, 0)$ so that

$$n_1 \cdot n_2 = \cos \phi_1, \quad n_1 \cdot n_3 = \cos \theta_1, \quad n_2 \cdot n_3 = \cos \phi_1 \cos \theta_1 + \sin \phi_1 \sin \theta_1 \cos \phi_2. \quad (84)$$

It follows that

$$J = \frac{4^n \omega_{d-1} \omega_{d-2}}{\omega_d^2} \int_0^\pi (\sin \phi_1)^{d-2} d\phi_1 \int_0^\pi (\sin \theta_1)^{d-2} d\theta_1 \int_0^\pi (\sin \phi_2)^{d-3} d\phi_2 \times \left(1 - \cos \phi_1\right)^{a_3} \left(1 - \cos \theta_1\right)^{a_2} \left(1 - \cos \phi_1 \cos \theta_1 - \sin \phi_1 \sin \theta_1 \cos \phi_2\right)^{a_2}. \quad (85)$$

Before proceeding with the evaluation of $J$, let us change the integration variables at the r.h.s. of (85) and replace the angles used there with the distances

$$r_1 = \Delta n_{23}, \quad r_2 = \Delta n_{13}, \quad r_3 = \Delta n_{12}, \quad 0 \leq r_j \leq 1. \quad (86)$$

The Jacobian of the transformation is readily computed:

$$r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3 = (\sin \phi_1)^2 (\sin \theta_1)^2 \sin \phi_1 \, d\phi_1 \, d\theta_1 \, d\phi_2 \, d\theta_2 \, d\theta_3 \quad (87)$$

Since

$$(\sin \phi_1 \sin \theta_1 \sin \theta_2)^2 = 2(r_1^2 r_2^2 + r_2^2 r_3^2 + r_3^2 r_1^2) - (r_1^4 + r_2^4 + r_3^4) - r_1^2 r_2^2 r_3^2 \quad (88)$$

it follows that

$$J(a_1, a_2, a_3) = \int_D \rho(r_1, r_2, r_3) r_1^{2a_1} r_2^{2a_2} r_3^{2a_3} \, dr_1 \, dr_2 \, dr_3. \quad (89)$$

where

$$\rho(r_1, r_2, r_3) = \frac{4^n \omega_{d-1} \omega_{d-2}}{\omega_d^2} r_1 r_2 r_3 [2(r_1^2 r_2^2 + r_2^2 r_3^2 + r_3^2 r_1^2) - (r_1^4 + r_2^4 + r_3^4) - r_1^2 r_2^2 r_3^2]^{\frac{a_2}{2}}. \quad (90)$$

Below it will be proven that

$$\int \rho(r_1, r_2, r_3) \, dr_1 \, dr_2 \, dr_3 = 1 \quad (91)$$

$J(a_1, a_2, a_3)$ are therefore the moments of the probability density of three random points on $\mathbb{S}_{d-1}$ constituting a triangle whose sides have the sizes $r_1, r_2$ and $r_3$. 
### 7.2 Direct evaluation of $J$

By introducing a variable $z$ as follows:

$$z = \frac{1 - \cos \phi_1 \cos \theta_1}{\cos \phi_1 - \cos \theta_1}, \quad \sqrt{z^2 - 1} = \frac{\sin \phi_1 \sin \theta_1}{\cos \phi_1 - \cos \theta_1},$$

one can identify a well-known integral representation of a Legendre functions of the first kind entering at the r.h.s. (see e.g. [3], Eq. 3.7.6):

$$J = 4^a (2\pi)^{\frac{d-4}{2}} \int_0^\pi d\phi_1 \int_0^\pi d\cos \theta_1 \left(\frac{1 - \cos \phi_1}{2}\right)^{a_1} \left(\frac{1 - \cos \theta_1}{2}\right)^{a_1} \times \left(\frac{1 - \cos \theta_1}{2}\right)^{d-3} \right]^{d-3} \int_1^\infty \frac{dz}{z^{2-a_2}} P_{d-3-a_2}^{d-3} (z). \quad (92)$$

Evaluating this integral involves several nontrivial steps. In the first one we change to the variables $x = (1 - \cos \phi_1)/2$ and $y = (1 - \cos \theta_1)/2$ and restrict the domain of integration to the region $0 < y < x < 1$:

$$J_1 = 4^a \frac{\omega_{d-1}}{\omega_d} \int_0^1 dx \int_0^x dy \int_0^\infty dw \frac{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}; \quad (93)$$

in this Equation we have made use of the following remarkable integral representation of $P^\mu_\nu (z)$ that will be established in Appendix A:

$$\frac{z^2 - 1}{(z^2 - 1)^{d-1}} P_{\frac{d-3-a_2}{2}}^{d-3} (z) = 2 \frac{d-3-a_2}{2} \int_1^\infty \frac{dw}{z^2} \frac{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}. \quad (94)$$

In the second change of variables we replace $y$ by $\mu x$:

$$J_1 = 4^a \frac{\omega_{d-1}}{\omega_d} \int_0^1 dx \int_0^\infty dw \frac{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}; \quad (95)$$

$z$ can be expressed in terms of $x$ and $\mu$ as follows

$$z = z_\mu + (1 - z_\mu) x, \quad z_\mu = \frac{1 + \mu}{1 - \mu}. \quad (96)$$

Next, we use $z$ as integration variable in place of $x$ (at constant $\mu$) and we get that

$$\int_0^1 dx \frac{1}{z^{a_2+1}} \int_1^\infty \frac{dw}{z-w} \frac{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}} = \int_0^1 \frac{dz}{z^{a_2+1}} \frac{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}. \quad (97)$$

By setting $u = (z_\mu - w)/(z_\mu - 1)$ we obtain the final expression for $J_1$:

$$J_1 = 4^a \frac{\omega_{d-1}}{\omega_d} \int_0^1 dw \frac{(z_\mu - w)^{d-1}}{(z_\mu - 1)^{d-1}} \frac{(z_\mu - w)^{-a_2-1}(w^2-1)^{d-3+a_2}}{(z_\mu - w)^{-a_2-1}(w^2-1)^{d-3+a_2}} \times \int_0^1 \frac{dz}{z^{a_2+1}} \frac{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}{(z-w)^{-a_2-1}(w^2-1)^{d-3+a_2}}. \quad (98)$$
The integral $J_2$ over the domain $0 < x < y < 1$ is obtained from this expression by interchanging $a_1$ and $a_2$. Finally

$$J = J_1 + J_2 = \frac{2^{\frac{d-5}{2}} (2\pi)^{\frac{d-1}{2}} \omega_d^2}{\Gamma\left(\frac{d}{2} + 1\right)} \left( \frac{\Gamma(a + d - 1)}{\Gamma(a + 3 + d - 1)} \times \int_0^1 du \frac{u^{\frac{d-3}{2} + a_1}}{(1 - u)^{\frac{d-3}{2} + a_2}} \int_0^1 dv \frac{v^{\frac{d-3}{2} + a_1}}{(1 - v)^{\frac{d-3}{2} + a_2}} = \frac{2^{\frac{d-5}{2}} (2\pi)^{\frac{d-1}{2}} \omega_d^2}{\Gamma\left(\frac{d}{2} + 1\right)} \left( \frac{\Gamma(a + d - 1)}{\Gamma(a + 3 + d - 1)} \times \right. \left( \frac{\Gamma\left(\frac{d-1}{2} + a_1\right)}{\Gamma\left(\frac{d-1}{2} + a_2\right)} \right) \frac{\prod_{j=1}^3 \Gamma\left(\frac{d-1}{2} + a_j\right)}{\Gamma\left(d - 1 + a_j - a\right)} \right) \right)$$

In the end

$$h_d = \frac{1}{(2\pi)^{\frac{d+1}{2}}} 2^{-a} \omega_d^2 c(a_1, a_2, a_3) J(a_1, a_2, a_3)$$

and the formula (2) is proven for $d > 1$ integer. Estimates shown in Appendix (B) show that interpolation is permitted and the formula is valid for complex values of the dimension $d$. The analyticity in $\kappa$ of the expression at the r.h.s. also directly confirms our previous statement the analyticity of $\rho_{\nu, \lambda}(\kappa)$.

7.3 The $J/c$–duality and an indirect evaluation of $J(a_1, a_2, a_3)$

There is a possibility to avoid the computation based on the relation (52) and on the fact the integral (57) defines $J(a_1, a_2, a_3)$ as a holomorphic function of $(a_1, a_2, a_3)$ in the domain $\{(a_1, a_2, a_3) \in C^3; \Re(a_1 + a_2 + a_3) > -(d-1)\}$.

It then follows from the latter property and from the analyticity properties of the $\Gamma$–functions involved in the expression (74) of the function $c(a'_1, a'_2, a'_3)$ (with $a'_i = -a_i - \frac{d-1}{2}$) that the l.h.s. of (52) can be analytically continued as a holomorphic function $l(a_1, a_2, a_3)$ of $(a_1, a_2, a_3)$ in the tube $T_+ = \{(a_1, a_2, a_3) \in C^3; \Re a_i > -\frac{d-1}{2}; \ i = 1, 2, 3\}$, while the r.h.s. of (52) can be analytically continued as a holomorphic function $r(a_1, a_2, a_3)$ in the tube $T_- = \{(a_1, a_2, a_3) \in C^3; \Re a_i < -\frac{d-1}{2}; \ i = 1, 2, 3\}$. Note that the two tubes $T_+$ and $T_-$, which are symmetric of each other with respect to the submanifold $L_d$, have a nonempty intersection which is a tube containing $L_d$. Since both sides of (52) coincide on $L_d$, it follows that they both represent the same analytic function $E(a_1, a_2, a_3)$ in the complex domain $T_+ \cap T_-$, which therefore also admits an analytic continuation in $T_+ \cup T_-$. But in view of the tube theorem, the holomorphic envelope of $T_+ \cup T_-$ is the full space $C^3$, and thereby $E(a_1, a_2, a_3)$ is an entire function on $C^3$.

Moreover, one can show that the functions $l$ and $r$ are uniformly bounded by a constant $C$ in respective tubes $T'_+ \cup T'_-$, which are conical open neighborhoods of the respective "diagonal tubes" $\delta_+ = \{(a_1, a_2, a_3) \in C^3; \Re a_i = \alpha > -\frac{d-1}{2}; \ i = 1, 2, 3\}$ and $\delta_- = \{(a_1, a_2, a_3) \in C^3; \Re a_i = \alpha < -\frac{d-1}{2}; \ i = 1, 2, 3\}$. Since the convex hull of $T'_+ \cup T'_-$ is equal to $C^3$, it then follows that the entire function $E(a_1, a_2, a_3)$ is also uniformly bounded by $C$ in the whole space $C^3$ and is therefore a constant function. This entails that the l.h.s. (or r.h.s.) of (52) is a constant $\gamma_d$ (only depending on $d$) and that the integral (57) is directly obtained without additional computation by the formula

$$J(a_1, a_2, a_3) = \gamma_d 2^a c(a'_1, a'_2, a'_3) \quad \text{or} \quad J(a'_1, a'_2, a'_3) = \gamma_d 2^{a'} c(a_1, a_2, a_3)$$

(101)
Since $J(0, 0, 0) = 2^n$, the constant $\gamma_d$ is easily computed by choosing $a_j = 0$ in (101), which yields (in view of (74)):

$$\gamma_d = 2^{\frac{d-1}{2}} \pi^{-\frac{d-1}{2}} \frac{[\Gamma(d - 1)]^2}{[\Gamma(\frac{d}{2})]^3},$$

and therefore (in view of (58)):

$$h_d(\kappa, \nu, \lambda) = \frac{\omega_{d-1}^3}{(2\pi)^{\frac{d}{2}} \omega_d} \gamma_d c(a_1', a_2', a_3') \times c(a_1, a_2, a_3) = \frac{2^{d-\frac{3}{2}}}{\pi^d \Gamma(\frac{d}{2})} c(a_1', a_2', a_3') \times c(a_1, a_2, a_3)$$

(103)

7.4 A corollary

A formula by Hsu quoted in Szegő’s book ([19], page 390) gives a weighted integral of a product of three Gegenbauer polynomials, all having the same upper index $k = \frac{d-1}{2}$. That formula is a sort of extension to $SO(d)$ of a well known relation established for Legendre polynomials (upper index $k = 1$) and expressing the square of a Clebsh-Gordan coefficient of the rotation group $SO(3)$.

The Källén-Lehmann formula given in Eqs. (37) and (38) is an identity between two functions holomorphic in the variables $\zeta$ and $d$. It is possible to perform an analytic continuation of (37) to real values of the variable $\zeta$ belonging to the interval $]-1, 1[$ for every $d$.

By studying the behavior at infinity of the integrand one can show that the integral converges uniformly w.r.t. $\zeta$ in that interval (with the exclusion of the singular point $\zeta = -1$). Then, it is tempting to compute the r.h.s. as an infinite discrete series of residues (there is no difference in closing the contour to the right or to the left). But the so obtained series converges only for $k < 1$ i.e. $d < 3$. The residue of the pole in $i\kappa = k + n$ (with $k < 0$) of $w_\kappa$ is the Gegenbauer polynomial of degree $n$.

By specializing the l.h.s. of (37) to $i\nu = k + n_1$ and $i\lambda = k + n_2$ the series at the r.h.s. becomes a finite sum in the range $|n_1 - n_2| < n < |n_1 + n_2|$. The aforementioned formula by Hsu and Szegő is thus a corollary of our result (2) via the analyticity of the Källén-Lehmann formula (37).

8 Conclusions and outlook

The integral (10) that we have studied in this paper gives an exact evaluation of the “phase space” coefficient in the rate (35) of the decay process $1 \rightarrow 2$ in a de Sitter universe. The fact that $\rho_{\nu, \lambda}(k)$ is an holomorphic function in their arguments means that in the de Sitter universe all such decays are possible and there is no analogue of the mass subadditivity condition of flat spacetime. This phenomenon should not be ascribed to the thermal features of the de Sitter universe [15, 11, 12, 13].

Indeed the thermal features manifest themselves only when the field algebra is restricted to a wedge-shaped region bordered by bifurcate Killing horizons [20] while the amplitude (35) is integrated overall the de Sitter universe. Second, a similar computation performed in flat space thermal field theory does not exhibit the same phenomenon.

There several possible other investigation that are rendered possible by the results of the present paper. One is the exploration of the star-triangle relation that we have found and the associated Yang-Baxter equation. We will examine that question elsewhere.
A Appendix. Another integral representation for the associate Legendre functions of the first kind

In this appendix we show how to construct the integral representation of the Legendre function of the first kind that enters crucially in our computation of the triangular invariant $f(a_1,a_2,a_3)$.

The starting point is the usual expression for the associated Legendre functions as derivatives of Legendre polynomials commonly encountered in books of quantum mechanics:

$$P_l^m(x) = (x^2 - 1)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x) = \frac{1}{2^l l!}(x^2 - 1)^{\frac{m}{2}} \frac{d^{m+l}}{dx^{m+l}}(x^2 - 1)^l$$ \hfill (104)

valid for $x > 1$. The idea is to use the Riemann-Liouville fractional integration operator

$$(D^\nu f)(x) = \frac{1}{\Gamma(-\nu)} \int_1^x (x-y)^{-\nu-1} f(y)dy$$ \hfill (105)

to replace the standard derivative of integer order in (104). The conjectured interpolation formula for the Legendre function of first kind is then given by

$$P_l^m(x) \to \hat{P}_\lambda(z) = \frac{(x^2 - 1)^{\frac{m}{2}}}{2^l \Gamma(\lambda+1)\Gamma(-\mu - \lambda)} \int_1^x (z-y)^{-\mu - \lambda - 1}(y^2 - 1)^\lambda dy.$$ \hfill (106)

To directly evaluate the RHS of (106) we set $z - 1 = 2\xi$ with $-\pi < \text{Arg} \xi < \pi$ and $y - 1 = 2\lambda \xi$ with $0 \leq \lambda \leq 1$:

$$\hat{P}_\lambda(z) = \frac{(2\xi)^{-\mu} (z^2 - 1)^{\frac{m}{2}}}{\Gamma(\lambda+1)\Gamma(-\mu - \lambda)} \int_0^1 (1 - \lambda)^{-\mu - \lambda - 1} \lambda^\lambda (1 + \xi \lambda)^\lambda d\lambda$$

$$= \frac{(z - 1)^{-\frac{m}{2}} (z + 1)^{\frac{m}{2}}}{\pi} \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(-\mu + n + 1)\Gamma(\lambda - n + 1)\Gamma(n + 1)}$$

$$= \frac{(z - 1)^{-\frac{m}{2}} (z + 1)^{\frac{m}{2}}}{\Gamma(1 - \mu)} {}_2F_1 \left(-\lambda, \lambda + 1; 1 - \mu; -\frac{z}{2} \right) = \hat{P}_\lambda(z).$$ \hfill (107)

The last identification follows from \[1\] Eq. 3.2(14). The proof is valid in the domain of convergence of the hypergeometric series but the identification has of course a larger domain of applicability that is the cut-plane $\mathbb{C} \setminus \{-\infty, 1\}$ where the integral at the RHS of (106) converges. A second related way of evaluating (106) consists in using Mellin’s convolutions and writing

$$\int_1^x (x-y)^{-\mu - \lambda - 1}(y^2 - 1)^\lambda dy = x^{-\mu - \lambda - 1} \int_0^\infty f\left(\frac{x}{y}\right) g(y) \frac{dy}{y} = x^{-\mu - \lambda - 1} F * G(x)$$ \hfill (108)

where

$$f(x) = \left(1 - \frac{1}{x}\right)^{-\mu - \lambda - 1}, \quad g(x) = x(x^2 - 1)^\lambda.$$ \hfill (109)

The Mellin transforms $F(s)$ and $G(s)$ of these functions are readily computed and their product is the Mellin transform of the convolution (108)

$$F(s)G(s) = 2^{\lambda+\mu-1} \Gamma(-\mu - \lambda)\Gamma(\lambda + 1)$$

$$\frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{s}{2} - \frac{1}{2} - \lambda\right)}{\Gamma\left(-\frac{s}{2} + \frac{-\mu - \lambda + 1}{2}\right)\Gamma\left(-\frac{s}{2} + \frac{-\mu - \lambda + 1}{2}\right)}.$$ \hfill (110)
Inversion of the Mellin transform can be obtained by applying Slater’s theorem [21] or either directly by integrating à la Mellin-Barnes; after a few steps that we do not reproduce we get again that

\[ F \ast G(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s)G(s)x^{-s}ds = 2^\lambda \Gamma(\lambda + 1)\Gamma(-\mu - \lambda)(x^2 - 1)^{-\frac{\mu}{2}} P_\lambda^\mu(x) \]  

(111)

and the representation is proven to hold true.

There exist of course many other integral representations of Legendre functions. The advantage of this specific one is that it can be used easily for partial integrations and is very natural generalization of the classic expression (104). Strangely enough, we were not able to find this explicitly written in the form given in Eq. (106) in the literature available to us.

By using the integral representation (106) for real \( u > 1 \) we get the following bound:

\[ \left| P_\lambda^{\mu}(u) \right| \leq \frac{2^{\frac{d}{2}}(u + 1)^{-\frac{R_d+d}{2}}(u - 1)^{\frac{R_d-d}{2}}}{\Gamma\left(\frac{d}{2} - i\nu\right)\Gamma\left(\frac{d}{2} + i\nu\right)} \log \left( u + \sqrt{u^2 - 1} \right) \]  

(112)

so that

\[ \left| \Gamma^3\left(\frac{d-1}{2}\right) h_d(\kappa, \nu, \lambda) \right| \leq \frac{2^d C_d(\kappa, \nu, \lambda) \int_1^{\infty}(u + 1)^{-\Re d + 2}(u - 1)^{\Re d - 2} \left[ \log \left( u + \sqrt{u^2 - 1} \right) \right]^3 du}{\Gamma\left(\frac{d}{2} + i\kappa\right)\Gamma\left(\frac{d}{2} + i\nu\right)\Gamma\left(\frac{d}{2} + i\lambda\right)} \]  

(113)

where

\[ C_d(\kappa, \nu, \lambda) = \left| \frac{\Gamma^3\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2} - i\kappa\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)\Gamma\left(\frac{d-1}{2} - i\lambda\right)} \right|. \]  

(114)

The change of variable \( u = \cosh \phi \) allows to rewrite the integral at the RHS of Eq. (113) as follows:

\[ \int_0^{\infty} \left( \phi \coth \phi \right)^3 \left( \frac{\sinh \phi}{\cosh^2 \phi} \right)^{\Re d - 1} \phi^{\Re d} \, d\phi \simeq \frac{2 - \Re d}{\sqrt{\Re d}} \]  

(115)

The asymptotic behaviour at large \( d \) of the integral at the RHS has been estimated by the the steepest descent approximation; for large \( \Re d \) the maximum of the integrand is located near to \( \phi_c = 2 \log (1 + \sqrt{2}) \). The behaviour of \( C_d(\kappa, \nu, \lambda) \) can be obtained from Stirling’s formula (\[1\], p. 47) that tells us that it goes like a constant. It follows that

\[ \left| \Gamma^3\left(\frac{d-1}{2}\right) h_d(\kappa, \nu, \lambda) \right| \leq \text{const} \frac{2^{3\Re d/2}}{\sqrt{\Re d}} \]  

(116)

This asymptotic behavior will be promoted to a bound in the following section.

### B Appendix. Extension of the main formula to complex \( d \)

We have shown that the function \( h_d(\kappa, \nu, \lambda) \) defined in Eq. (11) coincides with

\[ g_d(\kappa, \nu, \lambda) = \frac{2^\frac{d}{2}(4\pi)^{\frac{d}{4}}\Gamma\left(\frac{d-1}{2}\right)^{\Re d - 1}}{\prod_{\epsilon = \pm 1} \Gamma\left(\frac{d-1}{2} + i\epsilon\kappa\right) \prod_{\epsilon' = \pm 1} \Gamma\left(\frac{d-1}{2} + i\epsilon'\nu\right) \prod_{\epsilon'' = \pm 1} \Gamma\left(\frac{d-1}{2} + i\epsilon''\lambda\right)} \]

where...
for all integer \( d \geq 2 \) and real \( \kappa, \nu, \) and \( \lambda \). In this appendix, we will show, by using Carlson’s Theorem \[22\], that these two functions coincide wherever they are both defined, i.e. when the integral in \[16\] converges. It is obviously sufficient to prove this for real values of \( \kappa, \nu, \) and \( \lambda \) satisfying \( |\kappa| < B, |\nu| < B, |\lambda| < B \) for some arbitrary \( B > 0 \).

**Theorem B.1 (Carlson)** Let \( f \) be a function holomorphic in the right half-plane \( \{ z \in \mathbb{C} : \Re z > 0 \} \), and satisfying

\[
|f(z)| \leq Ae^{k|z|} \quad \text{for all } z \text{ with } \Re z > 0,
\]

where \( A \geq 0 \) and \( 0 \leq k < \pi \). If \( f(z) = 0 \) for \( z = 1, 2, \ldots \), then \( f = 0 \).

Recall the formula \([1], 3.7(1) \text{ p. 155}\)

\[
P^\mu_\alpha(z) = \frac{2^{-\frac{\alpha}{2} - \frac{1}{2} - \mu/2}}{\Gamma(-\mu - \alpha)\Gamma(\alpha + 1)} \int_0^\infty (z + ch t)^{\mu - \alpha - 1}(sh t)^{2\alpha + 1} dt,
\]

valid for \( z \in \mathbb{C} \setminus (-\infty, 1] \) and \( \Re(-\mu) > \Re \alpha > -1 \). We apply this to the special case \( \Re \alpha = -1/2 \), which satisfies the above condition when \( -\Re \mu > 1/2 \). We shall suppose \(-\Re \mu \geq 0 \). In this case, for \( z > 1 \),

\[
|2^\alpha \Gamma(-\mu - \alpha)\Gamma(\alpha + 1)||P^\mu_\alpha(z)| \leq (z^2 - 1)^{-1/2}\Gamma(\Re \mu/2)\int_0^\infty (z + ch t)^{\Re \mu - 1/2} dt \\
\leq (z^2 - 1)^{-1/2}\int_0^\infty (z + 1)^{\Re \mu - 1/2 + \varepsilon}(sh t)^{-\varepsilon} dt \\
\leq (z^2 - 1)^{-1/2}\int_0^\infty 2\varepsilon e^{-t} dt \\
\leq 2\varepsilon z^{-1}(z^2 - 1)^{-1/2}\Gamma(\frac{3}{2}) (z + 1)^{\Re \mu - 1/2 + \varepsilon}
\]

This holds for all real \( \varepsilon \) such that \( 0 < \varepsilon < -\Re \mu + 1/2, \) in particular for \( 0 < \varepsilon < 1/2 \). Note that setting \( \varepsilon = (2 + \log z)^{-1} \) gives a more precise bound, but \([20]\) will suffice for our present purpose. Keeping \( \kappa, \nu, \) and \( \lambda \) real and setting \( \mu = -(d - 2)/2 \), with \( \Re d > 1 \), this gives

\[
|b_\mu(\kappa, \nu, \lambda)| \leq \frac{2^{\frac{d}{2} + 3\varepsilon} e^{-3}}{|\Gamma\left(\frac{d-1}{2} - i\kappa\right) \Gamma\left(\frac{d+1}{2} + i\kappa\right) \Gamma\left(\frac{1}{2} - i\nu\right) \Gamma\left(\frac{1}{2} + i\lambda\right)|^2} \\
\int_1^\infty (z^2 - 1)^{\Re \mu/2} (z + 1)^{-3\Re \mu/2 + 3\varepsilon} dz
\]

Bounding \((z^2 - 1)^2\) by \((z + 1)^2\), the last integral is bounded by

\[
\int_2^\infty t^{\Re \frac{d}{2} + 3\varepsilon} dt = \frac{2^{-\Re \frac{d}{2} + 3\varepsilon}}{\Re \frac{d}{2} - 3\varepsilon}
\]

provided \(3\varepsilon < \Re(d - 1)/2\). We choose e.g. \( \varepsilon \leq 1/12 \). Thus, for \( \Re d \geq 2 \),

\[
|h_\mu(\kappa, \nu, \lambda)| \leq \frac{2^{\frac{d}{2} + 6\varepsilon - \Re(d-1)/2} e^{-3}}{|\Gamma\left(\frac{d-1}{2} - i\kappa\right) \Gamma\left(\frac{d+1}{2} + i\kappa\right) \Gamma\left(\frac{1}{2} - i\nu\right) \Gamma\left(\frac{1}{2} + i\lambda\right)|^2}
\]

Recall Stirling’s formula \([1], \text{ p. 47}\):

\[
\Gamma(z) = (2\pi)^{\frac{1}{2}} e^{-z + (z - \frac{1}{2}) \log z} \left(1 + \frac{z^{-1}}{12} + \frac{z^{-2}}{288} + O(z^{-3})\right),
\]
valid for \( z \in \mathbb{C} \setminus \mathbb{R}^{+} \). In applying (124), we will ignore the last bracket, at the cost of introducing an unknown multiplicative constant and supposing \(|z| \geq a > 0\). Thus there is a constant \( K \) such that, for \( z = x + iy \), \( \text{Re} \, z > 0 \) and \(|z| \geq a\),

\[
|\Gamma(z)| \leq K|z|^{-1/2} \exp(-x + x \log |z| - \frac{\pi}{2}|y|), \quad |\Gamma(z)|^{-1} \leq K|z|^{1/2} \exp(x - x \log |z| + \frac{\pi}{2}|y|) .
\] (125)

If \(|\kappa| < B\), \( \Gamma(\frac{1}{2} + i\kappa) \) is bounded by a constant (depending on \( B \)). Assume \( \text{Re} \, d > 2 \) and \(|d - 1| > 2e + 2B\). Applying the second inequality in (125) with \( z = (d - 1)/2 - i\kappa \) we find that, for some \( K' > 0\),

\[
\frac{1}{|\Gamma \left( \frac{d-1}{2} - i\kappa \right) \Gamma \left( \frac{1}{2} + i\kappa \right) |} \leq K'|d - 1|^{1/2} \exp(\pi(d - 1)/4) .
\] (126)

Therefore, for \( \text{Re} \, d > 2 \), \(|d - 1| > 2e + 2B\), \(|\kappa| < B\), \(|\nu| < B\), and \(|\lambda| < B\), we find that, for some constant \( A_1 > 0 \) (depending on \( B \)),

\[
|h_d(\kappa, \nu, \lambda)| \leq A_1|d - 1|^{3/2}e^{3\pi|d - 1|/4} .
\] (127)

With the same assumptions on \( \kappa, \nu, \) and \( \lambda \), we now wish to apply (124) in formula (117), taking advantage of the cancellation of growth which occurs between numerator and denominator. Let \( z = x + iy = (d - 1)/2 \). The function \( g_d \) has the form

\[
g_d(\kappa, \nu, \lambda) = 2^\frac{d}{2}(4\pi)^{-\frac{d}{2}} \left[ \prod_{j=1}^{8} \Gamma \left( \frac{z}{2} + iu_j \right) \right] \left[ \prod_{k=0}^{6} \Gamma \left( z + iv_k \right)^{-1} \right] ,
\] (128)

where \( u_j, v_k \) are real, \(|u_j| < 2B\), \(|v_k| < B\), \((v_0 = 0)\). We suppose \( x > 1\), \(|z| > 4B + e\), \(|y| > 4B\). There is a constant \( A_2 > 0 \) such that

\[
|g_d(\kappa, \nu, \lambda)| \leq A_2|z|^{-1} \exp \left[ \sum_{j=1}^{8} \frac{x}{2}(\log |z| + \log |\frac{z}{2} + iu_j|/2|z|) - 1 - \frac{\pi}{2} \left( \frac{|y|}{2} - 2B \right) \right] + \left[ \sum_{k=0}^{6} x(\log |z| - \log |1 + iv_k|/|z|) + 1 + \frac{\pi}{2}(|y| + B) \right] .
\] (129)

With our assumptions, \( \log |\frac{z}{2} + iu_j|/2|z| < 0 \) and \( -\log |1 + iv_k|/|z| < 0 \), hence

\[
|g_d(\kappa, \nu, \lambda)| \leq A_2|z|^{-1} \exp \left[ -3x(\log |z| - 1) + \frac{3\pi|y|}{2} + 15\pi B/2 \right] ,
\] (130)

and finally, for some \( A_3 > 0\),

\[
|g_d(\kappa, \nu, \lambda)| \leq A_3|d - 1|^{-1} \exp \frac{3\pi|d - 1|}{4} .
\] (131)

Therefore the function \( d \mapsto h_d(\kappa, \nu, \lambda) - g_d(\kappa, \nu, \lambda) \) vanishes by Carlson’s theorem.

### C Appendix. Analytic continuation of the Källén-Lehmann weight

In this appendix, we use analytic continuation to obtain the Källén-Lehmann weight for a two-point function which is the product of the two-point functions of two free fields belonging to the
complementary series (a special case was done in [3]). We fix $R = 1$, and $d \geq 2$ is an integer. We
have obtained the representation
\[
\nu_{\nu}(\zeta)w_{\lambda}(\zeta) = \int_{R} \kappa \rho_{\nu,\lambda}(\kappa)w_{\nu}(\zeta) \, d\kappa ,
\]
with
\[
\kappa \rho_{\nu,\lambda}(\kappa) = \frac{1}{2^{\frac{d-1}{2}}} \pi^{\frac{d-2}{2}} R^{d-2} \Gamma \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-1}{2} + i\kappa \right) \Gamma \left( \frac{d-1}{2} - i\kappa \right) \times \prod_{\epsilon, \epsilon'} \Gamma \left( \frac{d-1}{4} + \epsilon \kappa + i\epsilon' \nu + i\epsilon'' \lambda \right) .
\]

Eq. (132) has been proved for real values of $\nu$ and $\lambda$. We know however that the l.h.s. of this
equation extends to a meromorphic function of $\nu$ and $\lambda$ with simple poles at $i\nu = \pm \left( \frac{d-1}{2} - n \right)$ and
$i\lambda = \pm \left( \frac{d-1}{2} - n \right)$, where $n \geq 0$ is an integer. As $\nu$ and $\lambda$ become complex, (132) will remain valid
as long as the r.h.s. can be continued. The integrand in the r.h.s. is a meromorphic function of $\kappa$,
$\nu$ and $\lambda$. We denote $\mu = \frac{d-1}{2}$. In the variable $\kappa$, the poles arise from the last product of Gamma
functions. When their arguments are close to $-n$, where $n$ is any integer $\geq 0$, these $\Gamma$ functions
behave as follows:
\[
\Gamma \left( \mu + \frac{i}{2}(\kappa + \epsilon' \nu + \epsilon'' \lambda) \right) \sim \frac{2(-1)^n}{n!i(\kappa - i(2\mu + 2n + i\epsilon' \nu + i\epsilon'' \lambda))} \quad (134)
\]
\[
\Gamma \left( \mu + \frac{i}{2}(-\kappa + \epsilon' \nu + \epsilon'' \lambda) \right) \sim \frac{-2(-1)^n}{n!i(\kappa + i(2\mu + 2n + i\epsilon' \nu + i\epsilon'' \lambda))} \quad (135)
\]

Thus the poles listed in (134) are the opposites of those in (135).

When $\nu$ and $\lambda$ are real, and more generally when $|\text{Im} \nu| + |\text{Im} \lambda| < 2\mu$, these poles do not touch the
real axis, so that (132) holds, by analytic continuation, for all such values.

For definiteness we first assume $0 < \beta = \text{Im} \lambda < \alpha = \text{Im} \nu$. We also temporarily assume that $\text{Re} \nu \neq 0$,
$\text{Re} \lambda \neq 0$, and $\text{Re} \nu \pm \text{Re} \lambda \neq 0$. The poles are at
\[
\epsilon \kappa = -\epsilon' \text{Re} \nu - \epsilon'' \text{Re} \lambda + i(2\mu + 2n - \epsilon' \alpha - \epsilon'' \beta),
\]

They can be grouped as follows:

1. **On the line** $-\text{Re} \nu - \text{Re} \lambda + iR$ :

   \[
   (\epsilon = 1, \, \epsilon' = \epsilon'' = 1) : \quad \kappa = -\text{Re} \nu - \text{Re} \lambda + i(2\mu + 2n - \alpha - \beta) ;
   \]

   \[
   (\epsilon = -1, \, \epsilon' = \epsilon'' = -1) : \quad \kappa = -\text{Re} \nu - \text{Re} \lambda + i(-2\mu - 2n - \alpha - \beta) ;
   \]

On this line, all poles move down as $\alpha + \beta$ increases, The mutual distances of these poles remain
constant.

2. **On the line** $\text{Re} \nu + \text{Re} \lambda + iR$ : poles opposite to the preceding.

3. **On the line** $-\text{Re} \nu + \text{Re} \lambda + iR$ :

   \[
   (\epsilon = 1, \, \epsilon' = -\epsilon'' = 1) : \quad \kappa = -\text{Re} \nu + \text{Re} \lambda + i(2\mu + 2n - \alpha + \beta) ;
   \]
\[(\epsilon = -1, \epsilon' = -\epsilon'' = -1) : \kappa = -\Re \nu + \Re \lambda + i(-2\mu - 2n - \alpha + \beta) ; \tag{140}\]

On this line, all poles move down as \(\alpha - \beta\) increases. The mutual distances of these poles remain constant.

4. **On the line** \(\Re \nu - \Re \lambda + iR\) : poles opposite to the preceding.

When \(\alpha + \beta\) increases and reaches \(2\mu\), the pole in (137) with \(n = 0\) reaches the real axis, as well as its opposite. Beyond this, eq. (132) acquires two discrete contributions provided by the residues of these poles; these can easily be seen to be equal. If \(\alpha + \beta\) increases further, poles in (137) with higher values of \(n\) cross the real axis, contributing new discrete terms to eq. (132). In the range \(2\mu + 2N < \alpha + \beta < 2\mu + 2N + 2\), (with integer \(N \geq 0\)), all the poles in (137) with \(0 \leq n \leq N\) have crossed (as well as their opposites).

Similarly if \(\alpha - \beta\) increases so that \(2\mu + 2N < \alpha - \beta < 2\mu + 2N + 2\), (with integer \(N \geq 0\)), all the poles in (139) with \(0 \leq n \leq N\) have crossed the real axis (as well as their opposites).

Therefore, if \(2\mu + 2N < \alpha + \beta < 2\mu + 2N + 2\) and \(2\mu + 2M < \alpha - \beta < 2\mu + 2M + 2\), where \(M\) and \(N\) are non-negative integers, and with our assumptions regarding \(\Re \nu\) and \(\Re \lambda\),

\[
w_\nu(z, z')w_\lambda(z, z') = \int_\mathbb{R} \kappa \rho_{\nu, \lambda}(\kappa) w_\kappa(z, z') d\kappa
\]

\[
+ \sum_{n=0}^{N} A_n(\nu, \lambda) w_{i(2\mu+2n+i\nu+i\lambda)} + \sum_{n=0}^{M} B_n(\nu, \lambda) w_{i(2\mu+2n+i\nu-i\lambda)} . \tag{141}\]

Here

\[
A_n(\nu, \lambda) = \frac{(-1)^n}{n!2^{\frac{d+1}{2}}R^{d-2} \Gamma\left(\frac{d-1}{2}\right) \Gamma(-2\mu - 2n - i\nu - i\lambda) \Gamma(2\mu + 2n + i\nu + i\lambda)} \times
\]

\[
\Gamma(-2n - i\nu - i\lambda) \Gamma(4\mu + 2n + i\nu + i\lambda) \times
\]

\[
\Gamma(-n - i\nu) \Gamma(-n - i\lambda) \Gamma(-n - i\nu - i\lambda) \times
\]

\[
\Gamma(2\mu + n + i\nu + i\lambda) \Gamma(2\mu + n + i\lambda) \Gamma(2\mu + n + i\nu) \Gamma(2\mu + n) , \tag{142}\]

and

\[
B_n(\nu, \lambda) = A_n(\nu, -\lambda). \tag{143}\]

In the expression for \(A_n(\nu, \lambda)\), we note that the poles in \(\Gamma(-n - i\nu - i\lambda)\) are cancelled by those of \(\Gamma(-2n - i\nu - i\lambda)\) in the denominator. The poles of \(\Gamma(2\mu + n + i\nu + i\lambda)\) are cancelled by those of \(\Gamma(2\mu + 2n + i\nu + i\lambda)\) in the denominator. Hence the possible poles of \(A_n(\nu, \lambda)\) come from the poles of:

\[
\Gamma(-n - i\nu) : \quad -n - i\nu = -m \quad \Leftrightarrow \quad \Re \nu = 0, \quad \alpha = n - m ; \tag{144}\]

\[
\Gamma(-n - i\lambda) : \quad -n - i\lambda = -m \quad \Leftrightarrow \quad \Re \lambda = 0, \quad \beta = n - m ;
\]

\[
\Gamma(2\mu + n + i\lambda) : \quad 2\mu + n + i\lambda = -m \quad \Leftrightarrow \quad \Re \lambda = 0, \quad \beta = 2\mu + n + m ;
\]

\[
\Gamma(2\mu + n + i\nu) : \quad 2\mu + n + i\nu = -m \quad \Leftrightarrow \quad \Re \nu = 0, \quad \alpha = 2\mu + n + m .
\]

Here \(m \geq 0\) is an integer. From here on we always assume \(0 \leq \beta < 2\mu\), so that only the first two lines in (144) remain possible. This assumption also prevents the occurrence of the terms containing \(B_n(\nu, \lambda)\) in (141). We now consider two special cases.
C.1 Case $\nu = i\alpha$, $\lambda = i\beta$, $0 < \beta < \alpha < 2\mu$

According to the above discussions, if $N$ is a non-negative integer such that $2\mu + 2N < \alpha + \beta < 2\mu + 2N + 2$,

$$w_{i\alpha}(z, z') w_{i\beta}(z, z') = \int_{\mathbb{R}} \kappa \rho_{i\alpha,i\beta}(\kappa) w_{\kappa}(z, z') d\kappa$$

$$+ \sum_{n=0}^{N} A_n(i\alpha, i\beta) w_{i(2\mu+2n-\alpha-\beta)}.$$  \hspace{1cm} (145)

provided neither $\alpha$ nor $\beta$ is an integer. If $\alpha + \beta < 2\mu$ the formula holds without the $A_n$ terms. It is easy to check that $\kappa \rho_{i\alpha,i\beta}(\kappa) \geq 0$. For $A_n(i\alpha, i\beta)$ we find

$$A_n(i\alpha, i\beta) = \frac{1}{n!2^n \pi^{d/2} R^{d-2} \Gamma\left(\frac{d-1}{2}\right) \Gamma(-2\mu - 2n + \alpha + \beta) \Gamma(-2n + \alpha + \beta) \Gamma(4\mu + 2n - \alpha - \beta) \times \Gamma(\alpha - n) \Gamma(\beta - n) \Gamma(\alpha + \beta - n) \Gamma(2\mu - \alpha + n) \Gamma(2\mu + \beta + n) \Gamma(2\mu + n) \times (-1)^n \frac{\Gamma(n + 2\mu - \alpha - \beta)}{\Gamma(2n + 2\mu - \alpha - \beta)}. \hspace{1cm} (146)$$

All factors in this expression except the last fraction are positive since the arguments of the Gamma functions are positive because of the conditions $2\mu > \alpha > \beta$ and $\alpha + \beta - 2\mu - 2n > 0$. The last fraction is of the form

$$\frac{(-1)^n \Gamma(n + x)}{\Gamma(2n + x)} = (-1)^n \prod_{q} (q + x)^{-1}. \hspace{1cm} (147)$$

The last product contains $n$ negative factors and the result is positive. The positive sesquilinear form defined on $SS(X_d)$ by the l.h.s. of (145) is thus, in this case, a positive superposition of positive sesquilinear forms, each corresponding to an irreducible unitary representation of the de Sitter group.

It follows that a particle with parameter $i\gamma$ (with $0 < \gamma < 2\mu$) can decay into two particles with parameters $i\alpha$ and $i\beta$ (with the preceding conditions satisfied) provided

$$\gamma = \alpha + \beta - 2\mu - 2n, \text{ i.e. } 2\mu - \gamma = (2\mu - \alpha) + (2\mu - \beta) + 2n,$$  \hspace{1cm} (148)

where $n$ is a non-negative integer.

C.2 Case $\nu = i\alpha$, $0 < \alpha < 2\mu$, $\text{Im} \lambda = 0$, $\text{Re} \lambda \neq 0$

Only a particle in the principal series can decay into two particles with such parameters. There is no other restriction on the mass of the decaying particle.

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