On Stein rational balls smoothly but not symplectically embedded in $\mathbb{CP}^2$

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Abstract
We extend recent work of Brendan Owens by constructing a doubly infinite family of Stein rational homology balls which can be smoothly but not symplectically embedded in $\mathbb{CP}^2$.

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1 | INTRODUCTION AND STATEMENT OF RESULTS

Let $p > q \geq 1$ be coprime integers and $B_{p,q}$ the rational homology ball smoothing of the quotient singularity $\frac{1}{p^2}(pq - 1, 1)$. Using results by Khodorovskiy [4], it is not hard to show [1, §2.1] that if the positive integers $p_1$, $p_2$ and $p_3$ form a Markov triple, that is $p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3$, then there are pairwise disjoint symplectic embeddings

$$B_{p_i,q_i} \subset \mathbb{CP}^2, \quad i = 1, 2, 3,$$

where $q_i = \pm 3 \frac{p_j}{p_k} \mod p_i$ with $\{i, j, k\} = \{1, 2, 3\}$. Note that the sign is irrelevant because $B_{p,q}$ is symplectomorphic to $B_{p,p-q}$ [1, Remark 2.8]. The existence of the simultaneous symplectic embeddings (1.1) comes from the fact that when $(p_1, p_2, p_3)$ is a Markov triple, there is a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{CP}^2$ of the weighted projective space $\mathbb{CP}(p_1^2, p_2^2, p_3^2)$. It is not possible to construct more than three disjoint symplectic embeddings using smoothings of singular surfaces. In fact, Hacking and Prokhorov [3] showed if $X$ is a projective surface with quotient singularities which has a $\mathbb{CP}^2$ smoothing, then $X$ is a $\mathbb{Q}$-Gorenstein deformation of such a weighted projective plane. Evans and Smith [1, Theorem 1.2] generalized this result to the symplectic category, showing that if $B_{p_i,q_i} \subset \mathbb{CP}^2, i = 1, \ldots, N$ is a collection of pairwise disjoint symplectic embeddings, then $N \leq 3$, the $p_i$ belong to Markov triples and the functions $q_i$ must satisfy certain constraints. In particular, if $B_{p,q} \subset \mathbb{CP}^2$ is a symplectic embedding, then $p$ must belong to a Markov triple and divide $q^2 + 9$. 

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Owens [9, Theorem 1] recently proved the existence of smooth embeddings

\[ B_{F_{2n+1}, F_{2n-1}} \subset \mathbb{C}P^2 \]

for each \( n \geq 1 \), where \( F_{2n-1} \) denotes the odd Fibonacci number, recursively defined by

\[ F_1 = 1, \quad F_3 = 2, \quad F_{2n+3} = 3F_{2n+1} - F_{2n-1}. \]

Moreover, he showed that the pair \((F_{2n+1}, F_{2n-1})\) satisfies the Evans–Smith constraints only if \( n = 1 \), and therefore that \( B_{F_{2n+1}, F_{2n-1}} \) does not embed symplectically in \( \mathbb{C}P^2 \) for \( n > 1 \).

In this paper, we extend Owens’ family of smooth embeddings to a two-parameter family of smooth embeddings \( B_{p,q} \subset \mathbb{C}P^2 \) such that \( B_{p,q} \) cannot be symplectically embedded in \( \mathbb{C}P^2 \).

Recall that to a string of integers \( s = (a_1, \ldots, a_n) \) is uniquely associated a smooth, oriented 4-dimensional plumbing \( P(s) = P(a_1, \ldots, a_n) \). When \( a_i \geq 2 \) for each \( i \), the Hirzebruch–Jung continued fraction

\[ [s] = [a_1, \ldots, a_n] = a_1 - \frac{1}{a_2 - \frac{1}{\ldots - \frac{1}{a_n}}} \]

is well defined, and the oriented boundary of \( P(s) \) is the lens space \( L(p, p - q) \), where \( \frac{p}{q} = [a_1, \ldots, a_n] \).

Given integers \( k \geq -1 \) and \( m \geq 1 \), define

\[ s_{k,m} := (2, (2^{[m-1]}, m + 2)^{[k+1]}, 2, 2, (2^{[m-1]}, m + 2)^{[k+1]}) \],

where \( x^{[n]} \) means \( x \) repeated \( n \) times if \( n > 0 \) and omitted when \( n = 0 \). We observe in Remark 2 below that the lens space \( L(s_{k,m}) = \partial P(s_{k,m}) \) is of the form \( L(p^2, pq - 1) \) for some \( p > q \geq 1 \). We denote by \( B(s_{k,m}) \) the corresponding rational homology ball \( B_{p,q} \).

When \( m = 1 \), \( s_{k,1} = (2, 3^{[k+1]}, 2, 2, 3^{[k+1]}) \) and using Riemenschneider’s point rule [10] one can check that if \( \frac{p}{q} = [s_{k,1}] \) then \( \frac{p}{p-q} = [3^{[k+1]}, 5, 3^{[k]}, 2] \). Moreover, the proof of [9, Theorem 1] shows that

\[ \frac{F_{2k+5}^2}{F_{2k+5}F_{2k+3} - 1} = [3^{[k+1]}, 5, 3^{[k]}, 2], \]

therefore \( B(s_{k,1}) = B_{F_{2k+5}, F_{2k+3}} \). Therefore, Owens’ family is precisely the one-parameter subfamily \{\( B(s_{k,1}) \)\}_{k \geq -1}. Note that the string \( s_{-1,m} \) reduces to \((2,2,2)\) for each \( m \). In this case, the ball \( B(s_{-1,m}) = B_{2,1} \) embeds symplectically in \( \mathbb{C}P^2 \) as the complement of a neighborhood of a smooth conic. The following is our main result.

**Theorem 1.** Let \( k \geq -1 \) and \( m \geq 1 \), \( m \) odd. Then:

1. \( B(s_{k,m}) \) smoothly embeds in \( \mathbb{C}P^2 \);
2. \( B(s_{k,m}) \) does not symplectically embed in \( \mathbb{C}P^2 \) if \( k \geq 0 \).
Remark 1. Theorem 1 is equivalent to [9, Theorem 1] when $m = 1$.

We prove Theorem 1(1) by showing that, for each $k \geq -1$ and $m \geq 1$, there is a smooth decomposition

$$\mathbb{C}\mathbb{P}^2 = S^1 \times D^3 \cup h_1 \cup h_2 \cup h_3 \cup S^1 \times D^3,$$

where $h_i$, for $i = 1, 2, 3$ is a 2-handle and $B(s_{k,m}) = S^1 \times D^3 \cup h_2$. Theorem 1(2) follows from [9, Theorem 1] if $m = 1$, while for $m > 1$ we show that $\partial B(s_{k,m})$ is of the form $L(p^2, pq - 1)$, where $p$ does not divide $q^2 + 9$. The conclusion follows by the results of [1].

In [9], Owens also proves another result (Theorem 2), which states that a disjoint union of two or more of the balls $B(s_{k,1})$ cannot be smoothly embedded in $\mathbb{C}\mathbb{P}^2$. This is viewed in [9] as mild support to a conjecture of Kollár [6], which would imply that at most three of the rational balls $B(s_{k,1})$ may embed smoothly and disjointly in $\mathbb{C}\mathbb{P}^2$. It is therefore natural to ask whether the analogue of [9, Theorem 2] holds for our extended family or rational balls:

**Question 1.** Can a disjoint union of two or more balls $B(s_{k,m})$ be smoothly embedded in $\mathbb{C}\mathbb{P}^2$?

We plan to address Question 1 in a future paper. This paper is organized as follows. In Section 2, we fix notation and collect some preliminary material. Section 3 contains the proof of Theorem 1.

## 2 | $SL_2(\mathbb{Z})$-framed chain links and $SL_2(\mathbb{Z})$-slam-dunks

Given a string of integers $s = (a_1, \ldots, a_n)$, let

$$K = K_1 \cup \ldots \cup K_n \subset S^3$$

be a chain link consisting of $n$ oriented, framed unknots, with framing coefficients specified by $s$. Performing Dehn surgery along each $K_i$ with coefficient $a_i$ gives rise, in the notation of Section 1, to the lens space $L(s) = \partial P(s)$. We shall need to keep track of detailed information about the gluing maps involved in the Dehn surgeries on the components of $K$. In order to do that, we are going to view the framed link $K$ as an $SL_2(\mathbb{Z})$-framed link in the sense of [5, Appendix], although we will use our own notation rather than the notation from [5].

Let $Y := S^3 \setminus N$ be the complement of a tubular neighborhood $N := N_1 \cup \ldots \cup N_n$ of $K_1 \cup \cdots \cup K_n$. We can express $L(s)$ as the result of gluing $n$ solid tori $V_1, \ldots, V_n$ to $Y$. The gluing maps $\varphi_i : \partial N_i \to \partial V_i$ are determined up to isotopy by $2 \times 2$ matrices if we specify, for each of the tori $\partial N_i$ and $\partial V_i$, two oriented curves that generate its first homology group — we identify such oriented curves with their homology classes and the maps $\varphi_i$ with the induced maps in homology. We can do this as follows.

- Orient $K_1, \ldots, K_n$ so that $\text{lk}(K_i, K_{i+1}) = -1$ for each $i$.
- In each $\partial N_i$, choose a canonical longitude $\lambda_i$ with the same orientation as $K_i$, and an oriented meridian $\mu_i$ that winds around $K_i$ according to the right-hand convention.
- Regarding each $V_i$ as the tubular neighborhood of an unknot in $S^3$, choose a canonical longitude $\ell_i$ and a meridian $m_i$ in $\partial V_i$ as above.
- For each $i$, choose the basis $(\lambda_i, \mu_i)$ for $H_1(\partial N_i)$ and the basis $(\ell_i, m_i)$ for $H_1(\partial V_i)$. 
Note that, with these assumptions, $Y$ and $V_1, \ldots, V_n$ have compatible orientations if and only if the matrices representing $\varphi_1, \ldots, \varphi_n$ with respect to the bases $(\lambda_i, \mu_i)$ and $(\ell_i, m_i)$ have determinant 1. With this in mind, and recalling that each $m_i$ must be sent by $\varphi_i^{-1}$ to $\lambda_i + a_i \mu_i$, we can choose $\varphi_i$ with matrix

$$A_{a_i}, \quad \text{where } A_m \text{ denotes the matrix } \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } m \in \mathbb{Z}. \quad (2.1)$$

After these choices, each component $K_i$ is decorated with the matrix $A_{a_i}$ rather than simply with the integer $a_i$, and $K$ becomes an $SL_2(\mathbb{Z})$-framed link. Moreover, a presentation $\{(K_i, A_{a_i})\}_{i=1}^n$ of $L$ can be modified via $SL_2(\mathbb{Z})$-slam-dunks (cf. [5, Lemma (A.2)]). We describe these modifications using our notation in the following proposition.

**Proposition 2.** Let $s = (a_1, \ldots, a_n)$ with $n > 1$ and $L = L(s)$. Then:

1. For $t = 1, \ldots, n$, the oriented meridian $\mu_t$ is isotopic to a curve lying in a regular neighborhood of $\partial V_1 \subset L$. Its homology class has coordinates, with respect to the basis $\ell_1, m_1$, given by the second column of $A_{a_1} \cdots A_{a_t}$.

2. For each $t = 2, \ldots, n$, the $SL_2(\mathbb{Z})$-framed link presentation $\{(K_i, A_{a_i})\}_{i=1}^n$ of $L$ can be modified into another presentation of $L$ given by $\{(K_i, B_i)\}_{i=1}^n$, where

$$B_t = A_{a_1} \cdots A_{a_t} \quad \text{and} \quad B_i = A_{a_i} \quad \text{for } i > t;$$

3. $L$ is orientation-preserving diffeomorphic to $L(p, p - q)$, where $\begin{pmatrix} p \\ q \end{pmatrix}$ is the first column of $A_{a_1} \cdots A_{a_n}$.

**Proof.** We first describe the case $t = 2$. Let $L'$ be the lens space arising from Dehn surgeries along all the components of the chain link except $K_1$, so that $L := L(s)$ is obtained from $L'$ by doing the remaining surgery along $K_1$. Since $K_1$ is a meridian of $K_2$, we can isotope it, as an oriented knot in $L'$, to $-\mu_2 = \varphi_2^{-1}(\ell_2)$ and then to the oriented core $K'_1$ of $V_2$. See Figure 1, where the oriented curve representing $-\mu_2$, mapped by $\varphi_2$ to $\ell_2$, and the oriented curve representing $\lambda_2 + a_2 \mu_2$, mapped by $\varphi_2$ to $m_2$, are shown. The isotopy from $K_1$ to $K'_1$ can be extended to an isotopy of tubular neighborhoods from $N_1$ to $N'_1 \subset V_2$, whose boundary is parallel to $\partial V_2$. Now $L$ is obtained by cutting $N'_1$ out of $V_2$ and pasting $V_1$ in its place, with the identification between $\partial N'_1$ and $\partial V_1$ given by a new gluing map $\varphi'_1$. Note that, since $V_2 \setminus N'_1$ is diffeomorphic to $T^2 \times [0, 1]$, we may unambiguously take $(\ell_2, m_2)$ as a basis for the domain of $\varphi'_1$ (regarded as a map between homology groups). With this assumption, $\varphi'_1$ is represented by the same matrix as $\varphi_1$. In fact, as already observed, $K'_1$ and $\ell_2$ are isotopic as oriented knots, and $K'_1$ admits $m_2$ as a right-hand-oriented meridian. Hence, $\ell_2$ and $m_2$ also play the role of the original $\lambda_1$ and $\mu_1$. Moreover, it makes sense to consider the composition

$$\varphi'_1 \varphi_2 : H_1(\partial N_2) \xrightarrow{\varphi_2} H_1(\partial V_2) = H_1(N'_1) \xrightarrow{\varphi'_1} H_1(\partial V_1),$$

which is represented by the matrix $A_{a_1} \cdot A_{a_2}$. This concludes the description of the $SL_2(\mathbb{Z})$-slam-dunk when $t = 2$.

We now describe the construction for $t > 2$ (assuming $n \geq 3$). We first apply an $SL_2(\mathbb{Z})$-slam-dunk to the first component, so that $K_1$ is removed from the chain link. Now $K_2$ is a meridian of
ON STEIN RATIONAL BALLS SMOOTHLY BUT NOT SYMPLECTICALLY EMBEDDED IN $\mathbb{CP}^2$

FIGURE 1 A neighborhood of $K_2$

$K_3$, so we can apply another $SL_2(\mathbb{Z})$-slam-dunk along $K_2$, and so on. In general, for each $1 \leq i < t$, we remove a tubular neighborhood $N'_i$ of the core of $V_{i+1}$, and identify its boundary with $\partial V_i$ via a gluing map $\varphi'_i$ represented by $A_{a_i}$ with respect to the bases $(\ell_{i+1}, m_{i+1})$ and $(\ell_i, m_i)$. By construction, for each $t = 2, \ldots, n$ the composition of gluing maps

$$\varphi'_1 \circ \cdots \circ \varphi'_{t-1} \circ \varphi_t : H_1(\partial N_t) \rightarrow H_1(\partial V_1)$$

identifies $\mu_t = -\varphi^{-1} t (\ell'_i)$ with a curve whose coordinates with respect to $\ell_1$ and $m_1$ are given by the second column of $A_{a_1} \cdots A_{a_t}$. Similarly, after gluing, the coordinates of $\lambda_t$ with respect to $\ell_1$ and $m_1$ are given by the first column of $A_{a_1} \cdots A_{a_t}$. This proves (1) and (2).

To prove (3), we choose $t = n$, so that the modified link has a single component. The result of gluing together all the ‘layers' $V_{i+1} \setminus N'_i$ for $i < n$ is diffeomorphic to $T^2 \times [0, 1]$ and the boundaries of the glued-up pieces are parallel tori. Moreover, the diffeomorphism with $T^2 \times [0, 1]$ can be chosen so that:

- $T^2 \times \{0\}$ and $T^2 \times \{1\}$ are identified with $\partial V_1$ and $\partial V_n$ respectively;
- the other parallel tori are identified with $T^2 \times \{h\}$ for $n - 2$ pairwise distinct values of $h \in (0, 1)$.

This shows that $L(s)$ results from gluing two solid tori to $T^2 \times [0, 1]$. Moreover, the boundaries of the meridian disks of the solid tori are $m_1 \subset T^2 \times \{0\}$ and $\varphi_n(\lambda_n) \subset T^2 \times \{1\}$. By construction, the curve $\varphi_n(\lambda_n)$ is isotopic to $p\ell_1 + q m_1$, where

$$A_{a_1} \cdots A_{a_n} = \begin{pmatrix} p & a \\ q & b \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Since $(p \ a \ b)^{-1} = (-q \ -p \ a)$, $L(s)$ is the result of a Dehn surgery with framing $\frac{p}{q}$ along an unknot, where $a(p - q) \equiv 1 \mod p$. Part (3) follows immediately from the fact that the lens spaces $L(p, q)$ and $L(p, q')$ are orientation-preserving diffeomorphic when $qq' \equiv 1 \mod p$. □
3 | PROOF OF THEOREM 1

The first part of Theorem 1 states that $B(s_{k,m})$ smoothly embeds in $\mathbb{CP}^2$ if $m$ is odd. We already observed in Section 1 that this is true if $k = -1$, therefore in the following we assume $k \geq 0$.

Consider the string $s_{k,m}$ of Section 1, with $k \geq 0$ and $m$ odd, and define:

- $s'_{k,m} := (2, (2m-1, m+2)_{k+1}, 1, 2, (2m-1, m+2)_{k+1})$;
- $s''_{k,m} := (2m-1, 1, m+2, (2m-1, m+2)_{k}, 2m, 1, m+2, (2m-1, m+2)_{k})$.

It is straightforward to check that the strings $s'_{k,m}$ and $s''_{k,m}$ are both obtained from $s_{k,m}$ by changing some terms from 2 to 1, and that they both 'blow-down' to (0) in the sense of [7, Definition 2.1], therefore $L(s'_{k,m}) = L(s''_{k,m}) = S^1 \times S^2$.

**Remark 2.** Applying [7, Lemma 2.4] to the string $s'_{k,m}$ immediately implies that $L(s_{k,m})$ is of the form $L(p^2, pq - 1)$ for some $p > q \geq 1$.

Denote by $\nu_2 \subset S^1 \times S^2$ the curve corresponding to the meridian of the (1)-framed unknot of the diagram associated with $s'_{k,m}$. In Section 2, the same meridian was denoted $\mu_{(k+1)m+2}$. Denote by $W$ the smooth 4-manifold with boundary obtained by viewing $S^1 \times S^2$ as the boundary of $S^1 \times D^3$ and attaching a 4-dimensional 2-handle along $\nu_2$ with framing $-1$. In view of [7, Theorem 1.1] and [8, Theorem 8.5.1], $W$ is orientation-preserving diffeomorphic to $B(s_{k,m})$.

We are going to prove Part (1) of Theorem 1 by showing that $\mathbb{CP}^2$ is obtained by attaching some 4-dimensional handles to $B(s_{k,m})$. First we attach two extra 2-handles along the meridians $\mu_m$ and $\mu_{(k+2)m+2}$, both with framing 1. Note that the indices $m$ and $(k+2)m + 2$ give the positions where $s_{k,m}$ and $s''_{k,m}$ are different. As before, we rename these two meridians as $\nu_3$ and $\nu_1$, respectively, so that we encounter $\nu_1$, $\nu_2$, and $\nu_3$ in this order as we move along the diagram from right to left. Denote by $X$ the smooth 4-manifold with boundary constructed so far. If we view $\nu_1$, $\nu_2$ and $\nu_3$ as part of a surgery presentation and blow them down we get a chain of unknots whose framing coefficients are exactly given by $s''_{k,m}$. This shows that the boundary of $X$ is $S^1 \times S^2$. We can now add a 3-handle and a 4-handle to $X$ and obtain a closed 4-manifold $\hat{X}$.

Our plan is to show that $\hat{X}$ is diffeomorphic to $\mathbb{CP}^2$. In order to do that, we view $\nu_1, \nu_2, \nu_3 \subset S^1 \times S^2$ as knots sitting inside a regular neighborhood $U$ of $\partial V_1 \subset S^1 \times S^2$ as in Part (1) of Proposition 2. The proof of Proposition 2 shows that $U$ can be identified with $T^2 \times [0, 1]$ in such a way that each $\nu_i$ is identified with a simple closed curve $T^2 \times \{h_i\}$, where $1 > h_1 > h_2 > h_3 > 0$. Moreover, the framing induced by $\partial N_i$ on $\nu_i$ coincides with the framing induced by $T^2 \times \{h_i\}$. We introduce the notation

$$
(\nu_1, \nu_2, \nu_3) = \left(\left(\begin{array}{c} p_1 \\ q_1 \end{array}\right)_{\delta_1}, \left(\begin{array}{c} p_2 \\ q_2 \end{array}\right)_{\delta_2}, \left(\begin{array}{c} p_3 \\ q_3 \end{array}\right)_{\delta_3}\right)
$$

(3.1)

to indicate that $\nu_i$ is $\delta_i$-framed (with $\delta_i = \pm 1$) with respect to the framing induced by $T^2 \times \{h_i\}$ and the coordinates of the homology class of $\nu_i$ with respect to the basis $\ell_1, m_1$ are $(p_i, q_i)$.

If we view $S^1 \times S^2$ as $L(s''_{k,m})$, applying Part (2) of Proposition 2 for $t = n$ gives the standard presentation of $S^1 \times S^2$ as $L((0))$, that is, as 0-surgery on an unknot. This way, $\partial V_1$ gets identified with the boundary of a neighborhood of such unknot, $m_1$ with a longitude and $\ell_1$ with a meridian.
Recall that, given a closed, oriented 3-manifold $M$ represented by a framed link with integer coefficients $\mathcal{L}$, there is a convenient way to represent handlebody decompositions of any 4-dimensional cobordism $X$ obtained by attaching 4-dimensional handles to $M \times [0,1]$ along $M \times \{1\}$. In fact, the attaching curves of the 2-handles can always be isotoped into the complement of the glued-in solid tori of $M \times \{1\}$, so that each 2-handle can be represented as an additional framed knot in $S^3 \setminus \mathcal{L}$. The union of all such framed knots with $\mathcal{L}$ is a relative Kirby diagram representing $X$. This representation requires a notation which distinguishes the role played by each component. If the framing coefficient of a knot $K$ is $n$, we are going to write it as $\langle n \rangle$ if $K$ is part of $\mathcal{L}$, and simply as $n$ if $K$ represents a 2-handle of $X$. Of course, we can also attach 3- and 4-handles as usual. There is a calculus for these handlebody presentations, usually called relative Kirby calculus. We refer the reader to [2, §5.5] for further details.

We are going to apply relative Kirby calculus to the handle decomposition of $\hat{X}$ we just described. It turns out that the effect of sliding the handle $h_{\nu_i}$ attached along $\nu_i$ over (an appropriate number of copies of) the handle $h_{\nu_{i+1}}$ attached over $\nu_{i+1}$, for $i = 1, 2$, was described in [11, Lemma 5.1]. In terms of our Notation (3.1), the action of such handle slides on the triples of coordinates is given by the following sliding map $F$, which can be applied to any two consecutive components of the triple as follows:

$$F \left( \left( \begin{array}{c} p \\ q \end{array} \right), \left( \begin{array}{c} p_0 \\ q_0 \end{array} \right), \delta \right) = \left( \left( \begin{array}{c} p_0 \\ q_0 \end{array} \right), \left( \begin{array}{c} p - \delta_0 \Delta_0 p_0 \\ q - \delta_0 \Delta_0 q_0 \end{array} \right), \delta \right),$$

where $\Delta_0 = p_0 q - q_0 p$. (3.2)

**Remark 3.** Recall that the curves $\nu_i$ are oriented, and therefore so are the handles $h_{\nu_i}$. The sliding map $F$ describes the change of coordinates of the homology classes of the attaching curves as a result of a handle addition of oriented 2-handles (cf. [2, §5.1]). On the other hand, the 4-manifold resulting from attaching a 2-handle does not depend on the choice of an orientation on the 2-handle, therefore a triple as in (3.1) can be modified by changing the signs of a pair $(p_i, q_i)$ (but not $\delta_i$) without changing the resulting 4-manifold up to diffeomorphisms.

Our strategy to prove that $\hat{X}$ is diffeomorphic to $\mathbb{C}P^2$ will be as follows. We will exhibit a sequence of slides such that the coordinates $p_i$ and $q_i$ gradually get smaller, until we end up with a familiar Kirby diagram for $\mathbb{C}P^2$.

We now show that, for any pair $(k, m)$ as above, the map $F$ transforms the starting triple $(\nu_1, \nu_2, \nu_3)$ into

$$\left( \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right).$$

In order to do that we need to determine the coordinates $(p_i, q_i)$ of (3.1) in terms of $k$ and $m$. These will be given by products of $2 \times 2$ matrices as in Proposition 2. Since the substring $(2^{m-1}, m + 2)$ occurs repeatedly in $s_{k,m}$, it will be useful to find a general formula for $A_2(A_{m-1}^m A_{m+2})^l$ (recall Notation 2.1). For this purpose, observe that an obvious induction gives

$$A_{m-1}^m = \left( \begin{array}{cc} m & 1 - m \\ m - 1 & 2 - m \end{array} \right).$$
Then, define

$$C := \begin{pmatrix} x + 1 & -1 \\ x & -1 \end{pmatrix} \in GL_2(\mathbb{Z}[x]).$$

It is easy to check that the matrix $A_m^{-1}A_{m+2}$ is obtained by evaluating the entries of $C^2$ at $m$.

Now for $l \in \mathbb{Z}$ let $M_l := A_2C^l$ and define the $\mathbb{Z}$-indexed sequences of polynomials $(P_l)$, $(Q_l)$, $(S_l)$ and $(T_l)$ by setting

$$\begin{pmatrix} P_l & -S_l \\ Q_l & -T_l \end{pmatrix} := M_l. \quad (3.3)$$

Since $C^2 = xC + I$ we have $M_{l+2} = xM_{l+1} + M_l$, therefore each sequence satisfies the recursive formula

$$f_{l+2} = x \cdot f_{l+1} + f_l. \quad (3.4)$$

Such sequences are completely determined by their values at two adjacent indices. Moreover:

- by setting $l = 0$ in (3.3), we immediately get $P_0 = 2$, $Q_0 = S_0 = 1$ and $T_0 = 0$;
- by setting $l = 1$ and computing $A_2C$, we get $P_1 = x + 2$, $Q_1 = x + 1$ and $S_1 = T_1 = 1$.

The following table shows a few terms of the four sequences:

| $l$ | $P_l$  | $Q_l$  | $S_l$  | $T_l$  |
|-----|--------|--------|--------|--------|
| $-1$ | $-x + 2$ | 1      | $1 - x$ | 1      |
| $0$  | 2      | 1      | 1      | 0      |
| $1$  | $x + 2$ | $x + 1$ | 1      | 1      |
| $2$  | $x^2 + 2x + 2$ | $x^2 + x + 1$ | $x + 1$ | $x$    |

The values in the table together with (3.4) imply that $S_l = Q_{l-1}$, therefore

$$M_l = A_2C^l = \begin{pmatrix} P_l & -Q_{l-1} \\ Q_l & -T_l \end{pmatrix} \quad \forall l \in \mathbb{Z}.$$ 

**Lemma 4.** The sequences $(P_l)$, $(Q_l)$ and $(T_l)$ satisfy the following identities:

1. $P_{l+1} - P_l = x \cdot Q_l$;
2. $Q_{l+1} - Q_l = x \cdot T_{l+1}$;
3. $Q_{l+1} + Q_l = P_{l+1}$;
4. $T_l + T_{l-1} = Q_l$;
5. $P_{l+1}Q_l - P_lQ_{l+1} = (-1)^{l+1}x$;
6. $Q_{2l}Q_{2l-1} - P_{2l}T_{2l} = 1$;
7. $P_{2l}T_{2l-1} - Q_{2l}^2 = 1$.

**Proof.** Both sides of (1)–(4) are the terms of two sequences of polynomials satisfying the recursive formula (3.4), therefore it is enough to verify the identities for two distinct values of $l$, say 0 and 1. (5) We first claim that $(P_{l+1}Q_l - P_lQ_{l+1})$ is a geometric progression with common ratio $-1$: by
ON STEIN RATIONAL BALLS SMOOTHLY BUT NOT SYMPLECTICALLY EMBEDDED IN $\mathbb{CP}^2$

(3.4), we have

$$P_{l+1}Q_l - P_lQ_{l+1} = (xP_l + P_{l-1})Q_l - P_l(xQ_l + Q_{l-1}) = -(P_lQ_{l-1} - P_{l-1}Q_l),$$

which proves the claim. Now it is enough to verify the identity for $l = 0$. (6) The left-hand side can be written as $\det(M_{2l}) = \det(A_2C^{2l})$, which is immediately seen to be $1$, since $\det(A_2) = \det(C^2) = 1$. Finally, (7) follows from (6) by substituting $Q_{2l} = P_{2l} - Q_{2l-1}$ and $T_{2l} = Q_{2l} - T_{2l-1}$, which is allowed by (3) and (4).

□

We can now compute the coordinates $(p_i, q_i)$ of $\nu_1, \nu_2$ and $\nu_3$:

- $\nu_3$ is given by the first column of $A_{m-1}^{m-1}$, which is $(\begin{smallmatrix} m \\ m-1 \end{smallmatrix})$;
- $\nu_2$ is given by the first column of $A_{m}C_{m}^{m+2} |_{x=m} = M_{2k+2} |_{x=m}$, which is $(P_{2k+1}(m) |_{x=m} A_{m-1}^{m-1})$;
- $\nu_1$ is given by the first column of $A_{m}C_{m}^{m+2} |_{x=m} A_{1}A_{m-1}^{m-1} = M_{2k+2} |_{x=m} A_{1}A_{m-1}^{m-1}$, which is

$$M_{2k+2} |_{x=m} A_{1} \begin{pmatrix} m \\ m-1 \end{pmatrix} = \begin{pmatrix} P_{2k+2}(m) & Q_{2k+2}(m) \\ Q_{2k+2}(m) & T_{2k+2}(m) \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} P_{2k+2}(m) \\ Q_{2k+2}(m) \end{pmatrix},$$

where the last equality holds by Identities (1) and (2) of Lemma 4.

Therefore, if for any pair $(l, m)$ of integers we define

$$\tau_{l,m} := \begin{pmatrix} P_{l+1}(m) \\ Q_{l+1}(m) \end{pmatrix}_{(-1)^l}, \begin{pmatrix} P_{l+2}(m) \\ Q_{l+2}(m) \end{pmatrix}_{(-1)^{l+1}}, \begin{pmatrix} m \\ m-1 \end{pmatrix}_{1},$$

the starting triple that arises from $B(s_{k,m})$ via the previous construction is $\tau_{2k,m}$.

Lemma 5. The following hold.

1. $\tau_{l,m}$ can be transformed into $\tau_{l+1,m}$ by applying the sliding map $F$ to the first two components.
2. Any pair of the form $(a+2 \ a+1)_{1,1}$, $(a \ a-1)_{1,1}$ can be transformed into $(a \ a-1)_{1,1}, (a-2 \ a-3)_{1,1}$ by applying $F$ and changing the signs in the second component.
3. $F^3(0 \ 1)_{1,1} = (1 \ 0)_{-1} = (1 \ 1)_{1}.$

Proof. We immediately see from (3.2) that $\tau_{l,m}$ is transformed into

$$\begin{pmatrix} P_{l+2}(m) \\ Q_{l+2}(m) \end{pmatrix}_{(-1)^{l+1}}, \begin{pmatrix} P_{l+1}(m) - \delta_0 \Delta_0 P_{l+2}(m) \\ Q_{l+1}(m) - \delta_0 \Delta_0 Q_{l+2}(m) \end{pmatrix}_{(-1)^{l+1}}, \begin{pmatrix} m \\ m-1 \end{pmatrix}_{1},$$

which clearly agrees with $\tau_{l+1,m}$ at the first and the third components and at the framing of the second one. Therefore, we are left with verifying that

$$P_{l+1}(m) - \delta_0 \Delta_0 P_{l+2}(m) = P_{l+3}(m) \quad \text{and} \quad Q_{l+1}(m) - \delta_0 \Delta_0 Q_{l+2}(m) = Q_{l+3}(m).$$

By (3.4), both these equalities follow from $\delta_0 \Delta_0 = -m$: this is true because

$$\delta_0 \Delta_0 = (-1)^{l+1}(P_{l+2}(m)Q_{l+1}(m) - P_{l+1}(m)Q_{l+2}(m)) = (-1)^{l+1}(-1)^{l}m = -m$$
where the second equality holds by Lemma 4(5). This proves (1). Finally, (2) and (3) follow from a straightforward computation; in particular, in order to prove (3), it is useful to observe that the quantity $\delta_0 \Delta_0$ stays unchanged at each step, since both $\delta_0$ and $\Delta_0$ change sign. □

Now, in order to prove Theorem 1(1), we must show that the triple $\tau_{2k,m}$ corresponds to a Kirby diagram for $\mathbb{CP}^2$. By Lemma 5(1), it is enough to prove this for

$$\tau_{-1,m} = \left( \begin{array}{c} 2 \\ 1 \end{array} \right)_{-1}, \left( \begin{array}{c} m + 2 \\ m + 1 \end{array} \right)_{1}, \left( \begin{array}{c} m \\ m - 1 \end{array} \right)_{-1}.$$

We can apply Lemma 5(2) several times to the last two components. Observe that all coordinates decrease by 2 at each step and recall that $m$ is odd. After $\frac{m-1}{2}$ applications of Lemma 5(2), we get

$$\left( \begin{array}{c} 2 \\ 1 \end{array} \right), \left( \begin{array}{c} 3 \\ 2 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_{-1},$$

and finally, applying Lemma 5(3) to the first two components,

$$\left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

The last step in the proof of Theorem 1(1) is the following:

**Lemma 6.** The triple $(0)_{1}, (0)_{-1}, (0)_{1}$ corresponds to a Kirby diagram for $\mathbb{CP}^2$.

**Proof.** We have three knots in $T^2 \times [0, 1] \subset S^1 \times S^2$, which can be glued to $V_1$ along $T^2 \times \{0\}$ to form a new solid torus, which we regard as the exterior of an unknot $\hat{K}$ in $S^3$ (as in the proof of Proposition 2). Consequently, we can regard $S^1 \times S^2$ as the result of a Dehn surgery along $\hat{K}$ with framing 0. Now, the attaching curves $\nu_1, \nu_2$ and $\nu_3$ of the 2–handles are contained in three nested tori, each of which bounds a regular neighborhood of $\hat{K}$. More precisely, $\nu_1$ is a parallel copy of $m_1$, hence a canonical longitude of $\hat{K}$, while $\nu_2$ and $\nu_3$ are two parallel copies of $\ell_1$, hence two unlinked meridians of both $\hat{K}$ and $\nu_1$. The left-most picture of Figure 2 illustrates the resulting handlebody decomposition. Performing the handle slide indicated by the horizontal arrow yields the second picture of Figure 2, canceling the obvious 1-2-handle pair yields the third picture, and canceling the 0-framed unknot with the 3-handle gives the well-known Kirby diagram for $\mathbb{CP}^2$. □
By Lemmas 5 and 6, the 4-manifold $\hat{X}$ is diffeomorphic to $\mathbb{CP}^2$. This proves the existence of the smooth embeddings, that is, Part (1) of Theorem 1.

Part (2) of Theorem 1 follows from [9, Theorem 1] if $m = 1$, so in the following we assume $m \geq 3$. By the results of Evans and Smith [1] recalled in Section 1, to show that $B(s_k,m)$ does not symplectically embed in $\mathbb{CP}^2$ it suffices to write the lens space $L(s_k,m) = \partial B(s_k,m)$ as $L(p^2, pq - 1)$ and show that $p$ does not divide $q^2 + 9$. By Proposition 2, we can find such $p$ and $q$ by computing the first column of $M_{2k+2}A_2M_{2k+2}|_{x=m}$: we have

$$M_{2k+2}A_2M_{2k+2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} P_{2k+2} & -Q_{2k+1} \\ Q_{2k+2} & -T_{2k+1} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{2k+2} \\ Q_{2k+2} \end{pmatrix}$$

$$= \begin{pmatrix} P_{2k+2} & -Q_{2k+1} \\ Q_{2k+2} & -T_{2k+1} \end{pmatrix} \begin{pmatrix} 2P_{2k+2} - Q_{2k+2} \\ P_{2k+2} \end{pmatrix} \begin{pmatrix} Q_{2k+2} & -T_{2k+2} \\ P_{2k+2} \end{pmatrix} \begin{pmatrix} P_{2k+2} + Q_{2k+1} \\ P_{2k+2} \end{pmatrix}$$

$$= \begin{pmatrix} P_{2k+2}(Q_{2k+2} - T_{2k+2} + Q_{2k+1}Q_{2k+2}) \\ P_{2k+2}Q_{2k+2} + 1 \end{pmatrix}.$$

The numbers above the equality symbols denote which identities from Lemma 4 have been used.

We can now obtain the first column of $M_{2k+2}A_2M_{2k+2}|_{x=m}$ by evaluating the above polynomials at $m$. We obtain $p = P_{2k+2}(m)$ and $q = P_{2k+2}(m) - Q_{2k+2}(m) \equiv Q_{2k+1}(m)$. By Lemma 4(7),

$$q^2 + 9 = Q_{2k+1}(m)^2 + 9 = P_{2k+2}(m)T_{2k+1}(m) + 8,$$

which is a multiple of $P_{2k+2}(m)$ if and only if $P_{2k+2}(m) \parallel 8$. However, we can easily observe that, for each $l \geq 1$, $P_l$ is a monic polynomial of degree $l$ with positive coefficients, so that $P_{2k+2}(m) \geq m^{2k+2} \geq m^2 \geq 9$, and in particular $P_{2k+2}(m) \parallel 8$. This concludes the proof of Theorem 1.

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