Appendix to “Welschinger invariant and enumeration of real rational curves”

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"Pilate answered: What I have written, I have written" (John 19: 22)

1 Prologue

Since the appearance of our note [9], we received several requests for a down-to-earth explanation of the invariance of the Welschinger numbers, an explanation which would avoid symplectic geometry technology required by a general symplectic setting chosen in the original Welschinger’s proof, see [13]. Such an explanation is certainly possible in the framework of the classical algebraic geometry, especially, if one restricts itself to the unnodal Del Pezzo surfaces. This class of surfaces includes all the surfaces which are subject of [9].

In the proof given in this Appendix we follow the strategy of Welschinger’s proof. However, several technical details are presented in another way because of our choice to work with equations instead of parametrizations. Such an approach can be useful for a study of higher genus cases in the algebraic geometry framework. For this reason, we give preferences to proofs which could be extended to higher genus even if they are a bit longer than proofs based on parametrizations.

An unnodal Del Pezzo surface is a complex surface whose anticanonical divisor is ample. In what follows we fix such an unnodal Del Pezzo surface Σ equipped with a real structure (i.e., an antiholomorphic involution) $c : \Sigma \to \Sigma$ and an ample real divisor $D \subset \Sigma$ such that $-DK_\Sigma - 2 > 0$.

As is known, the set $R_0 = R_0(D)$ of irreducible rational curves in the linear system $|D|$ is locally closed and its closure $R = R(D)$ is a projective variety. They are both of pure dimension $r = -DK_\Sigma - 1^*$ and are smooth at any $C \in |D|$.

*It seems that the only explicit (and not confused) formulation of the non-emptiness of $R_0(D)$ (for a nef $D$) with a complete proof is found in [7], though it may be deduced from the general Gromov-Witten theory; the count of the local dimensions is by nowadays a routine application of the deformation theory, which can be found almost everywhere.
representing a nodal rational curve. The Brusotti-Severi theorem (see [3] and [11]) implies the following statements: the tangent space to $R(D)$ at $C \in R^0(D)$ is the space $T_C R = \Lambda(\text{Sing } C)$ of curves in $|D|$ passing through $\text{Sing } C$, where $\text{Sing } C$ is the set of singular points of $C$; the conditions imposed by different singular points are independent, so $\text{codim} \Lambda(\text{Sing } C)$ is equal to the number $|\text{Sing } C|$ of points in $\text{Sing } C$; and any linear system of dimension $\text{codim} \Lambda(\text{Sing } C)$ intersecting $\Lambda(\text{Sing } C)$ at $C$ transversely, induces a joint versal deformation of all the singular points of $C$. Note also that according to the adjunction formula one has

$$D^2 = r + 2|\text{Sing } C| - 1. \quad (1)$$

Let us fix an integer $m$ such that $0 \leq 2m \leq r$ and introduce a real structure $c_{r,m}$ on $\Sigma^r$ which maps $(z_1, \ldots, z_r) \in \Sigma^r$ to $(z'_1, \ldots, z'_r) \in \Sigma^r$ with $z'_i = c(z_i)$ if $i > 2m$, and $(z'_{2j-1}, z'_{2j}) = (c(z_{2j}), c(z_{2j-1}))$ if $j \leq m$. With respect to this real structure, a point $w = (z_1, \ldots, z_r)$ is real, i.e., $c_{r,m}$-invariant, if and only if $z_i$ belongs to the real part $\mathbb{R}\Sigma$ of $\Sigma$ for $i > 2m$ and $z_{2j-1}, z_{2j}$ are conjugate for $j \leq m$. In what follows we work with an open dense subset $\Omega_{r,m}(\Sigma)$ of $\mathbb{R}\Sigma^r = \text{Fix } c_{r,m}$ constituted of $c_{r,m}$-invariant $r$-tuples $w = (z_1, \ldots, z_r)$ with pairwise distinct $z_i \in \Sigma$.

By abuse of language, we say that a curve $C$ in $\Sigma$ passes through $w \in \Sigma^r$ if $C$ contains all the components $z_i \in \Sigma$ of $w$. If $C$ is nonsingular at each of the components $z_i$ of $w$, we say that $C$ is nonsingular in $w$. Moreover, when it can not lead to a confusion we denote by the same symbol $w$ the set of its components.

In the spirit of [12, 13], given a generic $w \in \Omega_{r,m}(\Sigma)$, we count the number $N_{r,m}^{\text{even}}(w)$ (resp., $N_{r,m}^{\text{odd}}(w)$) of irreducible real rational curves in $|D|$ passing through $w$ and having even (resp., odd) number of solitary nodes. (Note that for a generic $w$ every rational curve in $|D|$ passing through $w$ is irreducible and nodal.) The Welschinger number $W_{r,m}(w)$ is defined by $W_{r,m}(w) = N_{r,m}^{\text{even}}(w) - N_{r,m}^{\text{odd}}(w)$.\footnote{This result was used by L. Brusotti for a construction of independent real variations of real nodes; note that Brusotti treated as well the case of reducible nodal curves; for far going contemporary generalizations see, for instance, [8], Theorem 6.1(ii).}

**Theorem 1** (J.-Y. Welschinger, see [12, 13]). The value $W_{r,m}(w)$ does not depend on the choice of a (generic) element $w$ in a given connected component of $\Omega_{r,m}(\Sigma)$. In particular, if $\mathbb{R}\Sigma$ is connected, and $w_1$ and $w_2$ are two generic elements of $\Omega_{r,m}(\Sigma)$, then $W_{r,m}(w_1) = W_{r,m}(w_2)$.

In a connected component of $\Omega_{r,m}(\Sigma)$, we can join any two generic elements by a sequence of paths such that

- either one of the real components of $w$, say $z_r$, travels in $\mathbb{R}\Sigma$ along a real segment of a smooth generic real algebraic curve avoiding $z_1, \ldots, z_{r-1}$ and crossing all $(-1)$-curves transversely, whereas the points $z_1, \ldots, z_{r-1}$ stay fixed,

- or a pair of imaginary conjugate components of $w$, say, $z_1, z_2$, travel along generic smooth conjugate arcs in the non-real part $\Sigma$, avoiding $z_3, \ldots, z_r$ and all $(-1)$-curves, whereas the components $z_3, \ldots, z_r$ stay fixed.

\footnote{In [9] we considered only the case $m = 0$ and used a slightly different notation.}
2 Moving a real point of the configuration

We will use the following observation.

Transfer of genericity. Let $Y$ be an irreducible complex variety equipped with a real structure. Assume that the real part $\mathbb{R}Y$ of $Y$ contains nonsingular points. Then any generic point of $\mathbb{R}Y$ is also a generic point of $Y$.

In particular, a generic point of $\Omega_{r,m}(\Sigma)$ is a generic point of $\Sigma^r$.

Assume that $2m < r$. Let $w' = (z_1, ..., z_{r-1}) \in \Omega_{r-1,m}(\Sigma)$ be generic, and let the point $z_r$ move along a segment $\sigma$ of a real part of some generic real smooth algebraic curve $S_0$ in $\Sigma$. In particular, we suppose that $\sigma$ does not contain any component of $w'$. Denote by $V(w') \subset |D|$ the set of irreducible rational nodal curves in $|D|$ which pass through $w'$ and are nonsingular at $w'$. Denote by $\tilde{V}(w')$ the closure of $V(w')$, by $\mathbb{R}\tilde{V}(w')$ the real part of $\tilde{V}(w')$, and by $\mathbb{R}V(w')$ the real part of $V(w')$.

Let $\Lambda(z_r) \subset |D|$ be the linear system of curves passing through $z_r$. We study bifurcations of the curves in the set $\Lambda(z_r) \cap \mathbb{R}\tilde{V}(w')$ along a move of $z_r$ through $\sigma$, and show that $W_{r,m}(w)$ does not change in these bifurcations.

Lemma 2 The set $V(w')$ (respectively, $\mathbb{R}V(w')$) is a smooth one-dimensional sub-variety of $|D|$ (respectively, $\mathbb{R}|D|$). Moreover, the intersection of $R$ with $\Lambda(w')$ is transversal along $\tilde{V}(w')$, and in its turn $V(w')$ at a point $C \in V(w')$ intersects transversely with a linear system $\Lambda(z_r)$, if $z_r \notin \text{Sing} C$.

Proof. All the claims follow from the fact that, for any $C \in V(w')$ and any $z_r \notin \text{Sing} C$, the points in $w' \cup \text{Sing} C$ impose independent conditions on the curves of $|D|$. The latter statement means that $\Lambda(w) \cap \Lambda(\text{Sing} C) = \{C\}$ and follows, for instance, from the Bézout theorem, since $C$ is irreducible and $r + 2|\text{Sing} C| = D^2 + 1 > D^2$ (see (1)).

In particular, Lemma 2 implies that $W_{r,m}(w)$ may change only when $\Lambda(z_r)$ crosses elements of $\mathbb{R}\tilde{V}(w') \setminus \mathbb{R}V(w')$ or becomes tangent to $\mathbb{R}V(w')$, and that such critical situations appear for at most finitely many $z_r$ in $\sigma$.

Lemma 3 The elements of $\tilde{V}(w') \setminus V(w')$ are

(i) irreducible rational curves whose collection of singular points consists of a cusp $A_2$ and $D(D + K_{\Sigma})/2$ nodes,

(ii) irreducible rational curves whose collection of singular points consists of a tacnode $A_3$ and $D(D + K_{\Sigma})/2 - 1$ nodes,

(iii) irreducible rational curves whose collection of singular points consists of a triple point $D_4$ and $D(D + K_{\Sigma})/2 - 2$ nodes,
(iv) reduced reducible curves, splitting into two rational nodal curves, intersecting transversely and only at their nonsingular points,

(v) irreducible rational nodal curves with $D(D + K_X)/2 + 1$ nodes such that exactly one of the nodes coincides with a point of $w'$.

Moreover, in the first four cases, the curves are nonsingular at $w'$.

Proof. According to the observation on the transfer of genericity, it is sufficient to prove the statement of Lemma for generic $w' \in \Sigma^{-1}$. Notice that the linear system $\Lambda(w')$ does not meet subvarieties of $|D|$ of dimension $< r - 1 = -DK_X - 2$. Indeed, choosing $z_1$ outside a curve $C \in X \subset |D|$, we obtain $\dim(\Lambda(z_1) \cap X) = \dim X - 1$ and then proceed by induction.

Thus, to show that the elements of $\tilde{V}(w') \setminus V(w')$ can not have more complicated singularities than those pointed in (i)–(iv) it is sufficient to carry out suitable dimension counts.

Assume that $C \in \tilde{V}(w')$ is reduced irreducible. Then it is rational. The Zariski tangent space to the germ at $C$ of the equisingular stratum in $|D|$ is the projectivization of $H^0(\Sigma, J_{Z^{es}(C)}(D))$, where $J_{Z^{es}(C)}$ is the ideal sheaf of the zero-dimensional scheme $Z^{es}(C)$ supported at $\text{Sing}(C)$ and defined at $z \in \text{Sing}(C)$ by the equisingular ideal (see [5, 15]). The scheme $Z^{es}(C)$ contains the conductor scheme $Z^{cond}(C)$ defined at any $z \in \text{Sing}(C)$ by the conductor ideal, and, furthermore,

$$\deg Z^{es}(C) - \deg Z^{cond}(C) = \sum_{p \in \text{Sing}(C)} (\tau'(p, C) - \delta(p, C)),$$

where $\tau'(p, C)$ is the codimension of the equisingular ideal, and $\delta(p, C)$ is the $\delta$-invariant. In our case $\deg Z^{cond}(C) = D(D + K_X)/2 + 1$, and hence (see, for example, [5]), $\deg Z^{es}(C) = D(D + K_X)/2 + 2$ only in cases (i)–(iv). For other collections of singularities, $\deg Z^{es}(C) \geq D(D + K_X)/2 + 3$. To finish our dimension count it remains to show that, in the latter case, $h^0(\Sigma, J_{Z^{es}(C)}(D)) \leq -DK_X - 2$. For this purpose, pick any zero-dimensional scheme $Z$ of degree $D(D + K_X)/2 + 3$ between $Z^{es}(C)$ and $Z^{cond}(C)$. Then

$$h^0(\Sigma, J_{Z^{es}(C)}(D)) \leq h^0(\Sigma, J_Z(D)) = \frac{D(D - K_X)}{2} + 1 - \deg Z + h^1(\Sigma, J_Z(D)) = -DK_X - 2 + h^1(\Sigma, J_Z(D)) = -DK_X - 2.$$

The relation

$$h^1(\Sigma, J_Z(D)) = 0,$$

which we use here, follows from $C \supset Z^{es}(C) \supset Z$ and from the exact sequence

$$0 = H^1(\Sigma, O_{\Sigma}) \to H^1(\Sigma, J_Z(D)) \to H^1(C, J_{Z/C} \otimes O_{\Sigma}(D)) = 0.$$

In turn, the latter vanishing comes from the Riemann-Roch theorem, since (see [11]) in view of $Z \supset Z^{cond}(C)$, the sheaf $J_{Z/C} \otimes O_{\Sigma}(D)$ on $C$ lifts up to a sheaf $O_{C^v}(D)$ of the normalization $C^v$, and

$$\deg O_{C^v}(D) = \deg J_{Z/C} \otimes O_{P^2}(d) = D^2 - \deg Z - \deg Z^{cond}(C)$$
Assume now that \( C \in \overline{V(w')} \setminus V(w') \) splits into irreducible components \( C_1, \ldots, C_k \), \( k \geq 2 \) (may be, coinciding). The classical theorem (see [10] for a modern proof) states that \( g(C_1) + \cdots + g(C_k) \leq g(C') = 0 \), where \( C' \) is a generic curve in \( V(w') \). Hence, all \( C_1, \ldots, C_k \) are rational. Since \(-C_iK_\Sigma - 1 \geq 0\) for any \( i = 1, \ldots, k \) (because the surface is unnodal), and since a rational curve \( C_i \) cannot contain more than \(-C_iK_\Sigma - 1\) generic points, the curve \( C \) does not contain more than \(-DK_\Sigma - k\) generic points. Thus, \( k = 2 \) and the rational curves \( C_1 \) and \( C_2 \) pass through \(-C_1K_\Sigma - 1\) and \(-C_2K_\Sigma - 1\) generic points of \( \Sigma \), respectively. Therefore, both the curves \( C_1 \) and \( C_2 \) are nodal and intersect each other transversely.

The last statement of Lemma also follows from a dimension count. Namely, we deduce from the preceding dimension count that the curves of the types described in the first four cases and having a singularity at a fixed generic point form a variety of dimension \( \leq (\overline{DK_\Sigma - 2}) - 2 \). Since the number of additional points to mark is equal to \(-DK_\Sigma - 3\), for generic marked points such curves do not appear.

According to a similar count, the curves with nodes at two marked points do not appear if the marked points are generic. \( \square \)

Let \( \mathcal{T} \) be the curve generated by the double points of \( C \in V(w') \), and \( \mathcal{T}' \) be the underlying reduced curve. To control the behavior of Welschinger number at critical moments we put special attention, in accordance with Lemmas 2 and 3, to the curves \( C \in \overline{V(w')} \setminus V(w') \) and \( \mathcal{T} \). Namely, we impose an extra condition on \( S_0 \) requiring that it crosses \( \mathcal{T}' \) and all \( C \in \overline{V(w')} \setminus V(w') \) transversely and at their generic nonsingular points.

**Lemma 4** The number \( W_{r,m}(w) \) does not change when \( z_r \) crosses \( \mathcal{T} \) along \( \sigma \).

**Proof.** Let \( z^0_r \) be such a crossing point. It does not belong to any \( C \in \overline{V(w')} \setminus V(w') \), as it follows from our choice of \( S_0 \). Pick a curve \( C^0 \in \Lambda(z^0_r) \cap V(w') \) with a node at \( z^0_r \). The equality \( [1] \) implies that the nearby curves \( C^i \in V(w') \) form a regular homotopy covering which covers twice a neighborhood of \( z^0_r \); each of the two branches of \( C^0 \) at \( z_r \) provides a leaf of this trivial 2-covering. The curve \( C^0 \) is real and the real parts \( \mathbb{R}C^i \) of the nearby \( C^i \in \mathbb{R}V(w') \) form a regular equivariant homotopy which covers twice a real neighborhood of \( z^0_r \) if \( z^0_r \) is not a solitary point of \( \mathbb{R}C^0 \), otherwise they cover exclusively \( \mathbb{R}\mathcal{T} \). Hence, the real solutions with \( z_r \) on one side of \( \mathbb{R}\mathcal{T} \) are real regular homotopy equivalent to the respective real solutions with \( z_r \) on the other side of \( \mathbb{R}\mathcal{T} \), and, therefore, \( W_{r,m}(w) \) does not change. \( \square \)

**Lemma 5** Let \( C \in \overline{V(w')} \) be as in the cases (i), (ii), or (iii) of Lemma \( \boxdot \) Then \( W_{r,m}(w) \) does not change when \( z_r \) crosses \( C \) along \( \sigma \).

**Proof.** Let us check, first, that the linear system \( \Lambda(w') \) induces a joint versal deformation of all the singular points of \( C \). Such a property of \( \Lambda(w') \) is equivalent to the transversality of intersection of \( \Lambda(w') \) and \( |J_{Z^{\sigma_n}(C)}(D)| \) in \( |D| \). As we have
seen in the proof of Lemma 3, \( \dim |D| = D(D-K_\Sigma)/2 \), \( \dim \Lambda(\mathbf{w}') = \dim |D| - r + 1 \), and

\[
\dim |\mathcal{J}_{Z_{es}(C)}(D)| = \dim |D| - \deg Z_{es} + h^1(\Sigma, \mathcal{J}_{Z_{es}(C)}(D)) \nonumber
\]

\[
= \frac{D(D-K_\Sigma)}{2} - \left( \frac{D(D+K_\Sigma)}{2} + 2 \right) + h^1(\Sigma, \mathcal{J}_{Z_{es}(C)}(D)) \equiv r - 1 .
\]

Hence it is sufficient to show that

\[
\dim \left( \Lambda(\mathbf{w}') \cap |\mathcal{J}_{Z_{es}(C)}(D)| \right) = 0
\]  \hspace{1cm} (3)

under a suitable choice of \( \mathbf{w}' \), and it remains to notice that choosing one-by-one generic points \( z_1, \ldots, z_{r-1} \) on \( C \) outside \( C_1 \in |\mathcal{J}_{Z_{es}(C)}(D)|, C_2 \in |\mathcal{J}_{Z_{es}(C)}(D)| \cap \Lambda(z_1), \ldots, \) we reduce the dimension of \( \Lambda(\mathbf{w}') \cap |\mathcal{J}_{Z_{es}(C)}(D)| \) up to zero.

On the other hand, as it follows from Lemma 3, the set \( \text{Sing} (C) \) contains a singular point \( z^* \) of type \( A_2, A_3 \) or \( D_4 \), and \( \text{Sing} (C) \setminus \{z^*\} \) consists of \( D(D+K_\Sigma)/2 + 2 - \mu(z^*, C) \) nodes. For any node \( q \) of \( C \), the germ \( N_q(C) \) at \( C \) of the set of curves in \( |D| \) having a node in a neighborhood of \( q \), is a smooth hypersurface in \( |D| \). Therefore, the above transversality implies that the \( \mu(z^*, C) \)-dimensional germ \( P = \Lambda(\mathbf{w}') \cap \bigcap_{q \in \text{Sing} (C) \setminus \{z^*\}} N_q(C) \) at \( C \) is smooth, and induces a versal deformation of the singular point \((C, z^*)\).

The germ of \( \bar{V}(\mathbf{w}') \) at \( C \) is contained in \( P \) and coincides with the \( \delta \)-constant stratum in the corresponding versal deformation. It is well-known that, for \( z^* \) of type \( A_2 \), this stratum has an ordinary cusp at \( C \) (it is described by the discriminant equation \( \lambda^3/27 + \mu^2/4 = 0 \) in the standard versal deformation \( x^3 + \lambda x + \mu \) of the cusp). For \( z^* \) of type \( A_3 \) or \( D_4 \), the germ of \( \bar{V}(\mathbf{w}') \) at \( C \) is smooth by [4], Proposition 4.17(2). Denote by \( T \) the (one-dimensional) tangent line to \( \bar{V}(\mathbf{w}') \) at \( C \). Let \( z^*_0 \) be an intersection point of \( \sigma \) with \( C \). The intersection of \( P \) and \( \Lambda(z^*_0) \) is transversal, and \( P \cap \Lambda(z^*_0) \) is a smooth line (resp., surface, three-fold) in the surface (resp., three-fold, four-fold) \( P \), if \( z^* \in C \) is of type \( A_2 \) (resp., \( A_3, D_4 \)). Furthermore, \( T_{C,P} \cap \Lambda(z^*_0) \) intersects transversally with the line \( T \) in the tangent space \( T_{C,P} \) to \( P \) at \( C \). Indeed, in view of the generic choice of \( z^*_0 \) on \( C \), the latter transversality simply means that \( T \) contains a curve different from \( C \). Then, in particular, any linear system \( \Lambda(z_r) \) with \( z_r \) belonging to the germ of \( \sigma \) at \( z^*_0 \), crosses \( \bar{V}(\mathbf{w}') \) transversely in a neighborhood of \( C \). Hence, varying \( z_r \) along the germ of \( \sigma \) at \( z^*_0 \), we observe for \( z_r \neq z^*_0 \), that

- if \( z^* \in C \) is of type \( A_2 \), then \( R(V(\mathbf{w}') \cap \Lambda(z_r) \cap P \) is empty or consists of two points, one corresponding to a curve with a solitary node in a neighborhood \( U^* \) of \( z^* \) in \( \Sigma \), the other corresponding to a curve with a non-solitary node in \( U^* \);
- if \( z^* \in C \) is a non-solitary tacnode \( A_3 \), then \( R(V(\mathbf{w}') \cap \Lambda(z_r) \cap P \) consists of one point, corresponding either to a curve with two non-solitary nodes in \( U^* \), or to a curve without nodes in \( U^* \);
- if \( z^* \in C \) is a solitary tacnode \( A_3 \), then \( R(V(\mathbf{w}') \cap \Lambda(z_r) \cap P \) consists of one point, corresponding either to a curve with two solitary nodes in \( U^* \), or to a curve without nodes in \( U^* \);
• if \( z^* \in C \) is of type \( D_4 \) with three real local branches, then \( \mathbb{R}V(w') \cap \Lambda(z_r) \cap P \) consists of one point, corresponding to a curve with three non-solitary nodes in \( U^* \);

• if \( z^* \in C \) is of type \( D_4 \) with one real local branch, then \( \mathbb{R}V(w') \cap \Lambda(z_r) \cap P \) consists of one point, corresponding to a curve with one solitary node in \( U^* \).

In all the above cases, the Welschinger number \( W_{r,m}(w) \) does not change when \( z_r \) crosses \( C \). \( \square \)

**Lemma 6** Let \( C \in \mathbb{R}V(w') \setminus \mathbb{R}V(w') \) be as in case (iv) of Lemma \( \square \). Then \( W_{r,m}(w) \) does not change when \( z_r \) crosses \( C \) along \( \sigma \).

**Proof.** According to Lemma \( \square \) the curve \( C \) splits into two irreducible rational nodal curves \( C_1 \) and \( C_2 \) which intersect each other transversely and only at their nonsingular points. These curves are either both real or complex conjugate to each other. Since by the Riemann-Roch theorem \( \dim \Lambda(w' \cup \text{Sing}(C)) = 0 \), the germ of \( \mathbb{R}V(w') \) at \( C \) is the union of \( s \) transverse smooth branches, where \( s \) is the number of real intersection points of \( C_1 \) and \( C_2 \), the tangent lines to these branches are \( T_q = \Lambda(w' \cup \text{Sing}(C) \setminus \{q\}) \), \( q \in C_1 \cap C_2 \cap \mathbb{R} \Sigma \), and when \( C \) moves along such a branch all the nodes are preserved, except \( q \) which undergoes a Morse transformation. By our convention on the choice of \( \sigma \), any intersection point \( z^0_r \) of \( \sigma \) and \( C \) lies outside some preselected generic curves in \( T_q \), \( q \in C_1 \cap C_2 \cap \mathbb{R} \Sigma \). Then, inside \( \Lambda(w') \), the linear system \( \Lambda(w', z^0_r) \) intersects transversely the branches of \( \mathbb{R}V(w') \) at \( C \). The number of solitary nodes (as well as the number of non-solitary nodes) is not changing under a Morse transformation outside the nodes; hence, the Welschinger number \( W_{r,m}(w) \) does not change in this situation. \( \square \)

**Lemma 7** Let \( C \in \mathbb{R}V(w') \setminus \mathbb{R}V(w') \) be as in the case (v) of Lemma \( \square \). Then \( W_{r,m}(w) \) does not change when \( z_r \) crosses \( C \) along \( \sigma \).

**Proof.** According to Lemma \( \square \) we can assume that \( 2m \leq r - 2 \), and \( C \) is an irreducible rational nodal curve with a node at \( z_{r-1} \in \mathbb{R} \Sigma \) and nonsingular at the other points of \( w' \). The germ of the smooth surface \( P = \Lambda(w') \cap \bigcap_{q \in \text{Sing}(C) \setminus \{z_{r-1}\}} N_q(C) \) (recall that \( N_q(C) \) is the germ at \( C \) of the set of curves in \( |D| \) having a node in a neighborhood of \( q \)) represents a hyperplane section of a versal deformation of the singularity \( z_{r-1} \in C \), namely the section determined by the condition to pass through \( z_{r-1} \). The germ of \( V(w') \) at \( C \) lies in \( P \) and consists of two smooth branches, \( B_1 \) and \( B_2 \), each representing an equisingular deformation. For each of the branches pick a curve representing a point different of \( C \) on the tangent to the branch. By the genericity assumption, we may suppose that \( \sigma \) crosses \( C \) at a point \( z^0_r \), which does not belong to the above curves. Then the linear system \( \Lambda(w', z^0_r) \) crosses transversely the tangents to \( B_1 \) and \( B_2 \).

If \( z_{r-1} \) is a solitary node of \( \mathbb{R}C \), then \( B_1 \) and \( B_2 \) are imaginary, and hence \( \Lambda(w) \cap \mathbb{R}V(w') \cap P = \emptyset \) as \( z_r \neq z^0_r \). If \( z_{r-1} \) is a non-solitary node of \( \mathbb{R}C \), then \( B_1 \) and \( B_2 \)
are real, and hence $\Lambda(w) \cap \mathbb{R}V(w') \cap P$ consists of two elements, and these elements correspond to real curves which are real equisingular deformation equivalent to $C$.

Thus, $W_{r,m}(w)$ does not change. \hfill \Box

In the case $m = 0$, Theorem 1 immediately follows from the results of this Section. For $m > 0$, it remains to prove the invariance of $W_{r,m}$ under moves of a pair of conjugated marked points.

### 3 Moving a pair of imaginary points of the configuration. End of the proof of Theorem 1

Assume that $m > 0$.

Let $w' = \{z_3, \ldots , z_r\}$ be a generic element in $\Omega_{r-2,m-1}(\Sigma)$, the point $z_1$ move along a generic smooth path $\sigma$ in $\Sigma \backslash \mathbb{R}\Sigma$, and $z_2 = \bar{z}_1$. Denote by $V(w')$ the set of irreducible rational nodal curves in $|D|$ which pass through $w'$ and are nonsingular at $w'$. Denote by $\bar{V}(w')$ the closure of $V(w')$, by $\mathbb{R}\bar{V}(w')$ the real part of $\bar{V}(w')$, and by $\mathbb{R}V(w')$ the real part of $V(w')$.

Let $\Lambda(z_1, z_2) \subset |D|$ be the linear system of curves passing through $z_1$ and $z_2$. We study bifurcations of the set $\Lambda(z_1, z_2) \cap \mathbb{R}\bar{V}(w')$ along the path $\sigma$.

**Lemma 8** The set $\mathbb{R}V(w')$ is a smooth two-dimensional subvariety of $\mathbb{R}|D|$. Moreover, the intersection of $R$ with $\Lambda(w')$ is transversal along $\mathbb{R}V(w')$, and in its turn $\mathbb{R}V(w')$ intersects transversely with the real part of any linear system $\Lambda(z_1, z_2)$, where $z_1 \in \sigma$, $z_2 = \bar{z}_1$.

**Proof.** The proof almost coincides with the proof of Lemma 2. The only new point is to show that, for generic $w'$ and $\sigma$, the curves $C \in \mathbb{R}V(w')$ are nonsingular at $z_1$ and $z_2$. The latter statement follows from the fact that $\sigma$ can be chosen in such a way that it avoids the singular points of curves $C \in \mathbb{R}V(w')$, which is a generic condition on $\sigma$. \hfill \Box

Lemma 8 implies that for all but finitely many positions of $z_1, z_2$ in a path, one has $\Lambda(z_1, z_2) \cap \bar{V}(w') \subset \Lambda(z_1, z_2) \subset V(w')$, and that $W_{r,m}(w)$ may change only when $\Lambda(z_1, z_2)$ crosses elements of $\mathbb{R}\bar{V}(w') \backslash \mathbb{R}V(w')$.

**Lemma 9** When $z_2 = \bar{z}_1$ and $z_1$ moves along $\sigma$, the linear system $\Lambda(z_1, z_2)$ crosses only one-dimensional strata of $\mathbb{R}\bar{V}(w') \backslash \mathbb{R}V(w')$, whose elements are as follows:

(i) irreducible rational curves with a collection of singular points, consisting of a cusp $A_2$ and $D(D + K_{\Sigma})/2$ nodes,

(ii) irreducible rational curves with a collection of singular points, consisting of a tacnode $A_3$ and $D(D + K_{\Sigma})/2 - 1$ nodes,
(iii) irreducible rational curves with a collection of singular points, consisting of a triple point \( D_4 \) and \( D(D + K) / 2 - 2 \) nodes,

(iv) reduced, reducible curves, splitting into two rational nodal curves, intersecting transversally and only at their nonsingular points,

(v) irreducible rational nodal curves with \( D(D + K) / 2 + 1 \) nodes such that exactly one of these nodes coincides with a real point of \( w' \).

Moreover, in the first four cases the curves are nonsingular at \( w' \).

**Proof.** By Lemma 3 the curves as in (i)–(v) form an open dense set in \( \tilde{V}(w') \setminus V(w') \). Denote it by \( Z \). The result follows as soon as we choose \( \sigma \) to avoid the singular points of the curves \( C \in Z \) as well as the points of the curves in the finite set \( (\tilde{V}(w') \setminus V(w')) \setminus Z \). \( \square \)

Now we specify a particular genericity condition and **require** that the path \( \sigma \) crosses the curves, mentioned in Lemma 9, only at their nonsingular points.

**Lemma 10** The value of \( W_{r,m}(w) \) does not change when \( z_1 \) and \( z_2 \) cross a curve of any of the types described in Lemma 9 along the paths \( \sigma \) and \( c(\sigma) \).

**Proof.** We use the transversality arguments as in the proofs of Lemmas 3, 6, and 8. The transversality is assured by the above requirement.

Assume, for instance, that \( \sigma \) crosses a curve \( C \) as described in case (i), having a cusp at \( z^* \in \mathbb{R} \Sigma \setminus \{z_1, ..., z_r\} \). Since \( z_1, ..., z_r \) are nonsingular points of \( C \) (the above requirement), the three-dimensional smooth germ of the variety \( P = \Lambda(w') \cap \bigcap_{q \in \text{Sing}(C) \setminus \{z^*\}} N_q(C) \) represents a versal deformation of the singular point \( z^* \) of \( C \), in which the discriminant is a surface with a cuspidal edge. The cuspidal edge divides the real part of the discriminant into a smooth surface corresponding to curves with a solitary node in a neighborhood of \( z^* \), and a smooth surface corresponding to curves with a non-solitary node in a neighborhood of \( z^* \). The (projective) tangent plane to the discriminant at \( C \) is naturally isomorphic to the linear system \( \Lambda(w' \cup \text{Sing}(C)) \), and, by the Riemann-Roch theorem, it intersects transversely the linear system \( \Lambda(z^0_1, z^0_2), z^0_1, z^0_2 \) standing for the travelling points \( z_1, z_2 \), corresponding to \( C \). Hence, varying \( z_1, z_2 \) along the germs of \( \sigma, c(\sigma) \) at \( z^0_1, z^0_2 \), respectively, we obtain a transversal intersection of the discriminant with \( \Lambda(z_1, z_2) \), which then consists of two imaginary curves, or of two real curves with the opposite contributions to \( W_{r,m}(w) \).

Similarly one proves the invariance of \( W_{r,m}(w) \) in other bifurcations. \( \square \)

Thus the proof of Theorem 1 is completed. \( \square \)
Let us look first at moves of a real marked point. Consider the (complex) surface \( V \) formed by all the curves \( C \in V(w') \) and the standard projection \( p : V \to \Sigma \). As it follows from the proofs in Section 2, the branch locus of \( p \) is formed by those curves \( C \) which have a cusp, and the branching index is two. Moreover, when \( z_r \) moves in \( \mathbb{R}\Sigma \) outside the ramification locus, the curve components \( C \) of \( p^{-1}(z_r) \), which correspond to the points of \( \Lambda(z_r) \cap V(w') \), are subject of a real regular homotopy with an exception of a finite number of real Morse transformations. Both the regular homotopies and the Morse transformations preserve the parity of the number of the solitary nodes (indeed, only one kind, up to reversing, of regular homotopies is changing this number: two solitary nodes can come together and then disappear together). At last, also up to reversing, when \( z_r \) crosses the ramification locus, two nodal curves come together, turn into a curve with a cusp and then disappear together. On one of them the cusp is generated by a solitary node, on the other one by a non-solitary node, and thus they give opposite contributions to the Welschinger number.

When we move a pair of complex conjugated points, we deal with the bifurcation locus of the correspondence \( C \in \mathbb{R}V(w') \cap \Lambda(z_1, z_2) \) between \( \mathbb{R}V(w') \) and \( \Sigma \times \Sigma \). As it follows from the proofs in Sections 2 and 3, the trace of the bifurcation locus on the real part \( z_2 = \bar{z}_1 \) of \( \Sigma \times \Sigma \) is a real algebraic variety of real dimension 3, and transversal crossings of the bifurcation locus provide the same bifurcations as in the case of a move of a real point (in fact, there is even one bifurcation less, since there are no more bifurcations as in Lemma 4). Hence, the Welschinger number is never changing.

Note that the bifurcations as in Lemma 4 are a source of non-invariance of the numbers \( W_{r,m} \) in the case of positive genus, see [9]. Note also that, in accordance with the non-genericity of the complex structure of nodal Del Pezzo surfaces, the numbers \( W_{r,m} \) are not invariant for nodal Del Pezzo surfaces; the simplest example is splitting out of a \((-2)\)-curve as in [4], Proposition 2.6(2b). That is why it is not clear how to use the Welschinger invariant for enumeration of curves on nodal Del Pezzo surfaces.

One may be interested to extend the results from \( \mathbb{R} \) to other real closed fields. To this end, consider the \( s \)-anticanonical models of real unnodal Del Pezzo surfaces of a given degree \( n \) (and with given \( s \) depending on \( n \)). They form a Zariski open subset \( \mathcal{H}^0_n(\mathbb{R}) \) in certain irreducible components of the corresponding Hilbert scheme \( \mathcal{H}_n(\mathbb{R}) \). Note that \( \mathcal{H}_n(\mathbb{R}) \) is defined over \( \mathbb{Q} \), and thus \( \mathcal{H}^0_n(\mathbb{R}) \) is defined over the field \( \mathbb{R}_{\text{alg}} \) of algebraic real numbers. Consider the family \( \mathcal{F}^0_n(\mathbb{R}) \) of configuration spaces \( \Omega_{r,m}(\Sigma) \) over \( \mathcal{H}^0_n(\mathbb{R}) \). The Welschinger number defines a function \( W \) on \( \mathcal{F}^0_n(\mathbb{R}) \). This function is semi-algebraic and defined over \( \mathbb{R}_{\text{alg}} \). Thus, \( W \) is defined for any real closed field \( \mathbb{K} \) and constitutes a semi-algebraic function on the corresponding family \( \mathcal{F}^0_n(\mathbb{K}) \).
Welschinger theorem for real closed fields. The function $W$ is constant on any semi-algebraically connected component of $F^0_n(K)$.

**Proof.** For $K = \mathbb{R}$ the statement follows from [13], Theorem 2.1 (for the definition of semi-algebraically connected components see [2]; in the case $K = \mathbb{R}$ semi-algebraically connected components coincide with the usual ones). According to the remarks on $W$ made above, the Tarski-Seidenberg principle (see, for example, [2]) applies and gives the statement required.

Note that the connected components of $F^0_n(\mathbb{R})$ are easy to describe in topological terms. Namely, $(\Sigma, w)$ and $(\Sigma', w')$, where $w \in \Omega_{r,m}(\Sigma)$ and $w' \in \Omega_{r,m}(\Sigma')$, are in the same component of $F^0_n(\mathbb{R})$ if $\mathbb{R}\Sigma$ and $\mathbb{R}\Sigma'$ are homeomorphic (see [6]).

3. When we were finishing these notes, we have become acquainted with a new preprint of Welschinger [14], where he proves the invariance of another characteristic, specific for rational curves in real algebraic convex 3-folds. The algebro-geometric approach he develops there (an approach which uses parametrizations) provides another strategy for a proof of Theorem 1 in the algebraic geometry framework.

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