On problem of optimal observers’ placement on plane

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Abstract. A plane problem for optimization of stationary sensors placement in a threat environment to counteract an evading object is considered. For this purpose, an auxiliary task of constructing locally optimal paths of the evading object for a given configuration of sensors is solved. Integral risk functional is minimized for the evading object. The minimum of this functional is required to be maximized in the task of sensors placement. The paper describes the methods of solving both problems and the results of numerical modeling for the case of 2 sensors: examples of local optimal path maps, and the best found configurations of sensors placement.

1. Introduction

Lately, in the class of control problems for mobile vehicles in a threat environment, non-traditional criteria are considered more often (see [1] and its bibliography). Instead of traditional minimizing of time or energy costs the criterium of increasing secrecy during the movement in a threat environment, taking into account the map of potential risks-threats [2], can be considered. The literature deals with the problem of moving vehicle evasion from observers represented by two types of locators: sensors and detectors. In this case, the location of the threat and its impact on the mobile object are assumed to be known, and the choice of its route and movement parameters are carried out to minimize the negative impact of the threat environment. For example, in papers [3-4], analytical solutions for optimal speed modes in problems with one sensor in plane statements were obtained. And in paper [1], solutions for optimal speed modes in problems with several conflicting objects are numerically simulated on the base of Dijkstra's algorithm.

In connection with the widespread use of unmanned aerial vehicles, in order to prevent unauthorized access to the protected area, the inverse problem is interesting: the optimization of sensors placement in the protected area to reduce the secrecy of the evading vehicle. Despite the relevance, in the works published on the problem of constructing a map of the several sensors optimal location, only the case when observers have a certain cone of view in which they fix a mobile object was considered.

In this paper, the plane problem of localization of stationary sensors with uniform circular detection fields for counteracting an evading object is solved. Also, the auxiliary problem of constructing of
locally optimal paths in a threat environment, represented by a fixed number of sensors, is considered. The signal-to-noise ratio at the inputs of the receiving systems of sensors is assumed to be small during the entire time the object moves along the route; therefore, an integral functional called risk is optimized [5].

2. The problem statement

The problem of placement of a certain fixed number $N$ of stationary sensors – points $L_i$ on the plane – is considered. A material point $M$ must move along this plane from fixed point $S$ to another point $F$ in a given time $T < \infty$. The control of the point $M$ is carried out by the magnitude and the direction of the velocity vector $v$ to minimize the functional

$$ I = \int_0^T \left( \sum_{i=1}^N q_i \frac{v^2}{r_i^2} \right) dt, $$

depending on the magnitude $v$ of the velocity vector of the point $M$, distances to each of the sensors $r_i$ and the weighting factors $q_i$ of each sensor $L_i$ influence.

The points $L_i$ can be placed inside some rectangle located between points $S$ and $F$. The location of the points $L_i$ should maximize the global minimum of the functional described above along all possible paths at fixed points $L_i$.

3. The solution method

To solve the placement problem, a rectangular grid is introduced, and the sensors are located at its nodes. For each possible sensors location, the auxiliary problem of finding the optimal paths is solved and the best value of the functional $I$, which the evading object can achieve, is memorized.

As a result, the value of the best functional from all paths found for a given configuration of the sensors was assigned to each grid node. All values were sorted by the value of the functional during the computation process. In the case of the presence of symmetry in the problem, computer calculations were carried out only for a part of the solution domain, then the solution continued to the rest of the domain by reflection [6].

4. The auxiliary problem

The auxiliary problem solution objective is to find locally optimal paths for a given sensor configuration in order to identify the functional of initial problem – value $J$ that is defined as the minimum of functional (1) that an evading vehicle can reach.

4.1. Auxiliary problem formalization

Let's introduce the Cartesian coordinate system on the plane as follows: the origin point $O$ coincides with the start point $S$, the $Oy$ axis passes through the start point $S$ and the finish point $F$ and is directed from $S$ to $F$. The $Ox$ axis is perpendicular to it. The unit section is chosen to make the ordinate of the end point $F$ equal to 1. In this coordinate system, the initial and final conditions at the time $t = 0$ and $t = T$ for the evading object are:

$$ x(0) = 0, y(0) = 0, x(T) = 0, y(T) = 1. $$

The movement of a material point $M$ with coordinates $(x(t), y(t))$ in the given coordinate system is described by a system of differential equations:

$$ \begin{aligned}
\dot{x} &= v \cos \varphi, \\
\dot{y} &= v \sin \varphi,
\end{aligned} $$

where $v$ is the magnitude of the point’s velocity vector, and $\varphi$ is the angle measured from the positive direction of the axis and specifying the direction of the velocity vector. The functions $v$ and $\varphi$ are control functions and are assumed to be bounded piecewise continuous functions:

$$ 0 \leq v(t) \leq v_{max} \leq \infty, $$

\[ 0 \leq \phi(t) \leq 2\pi \quad \forall t \in [0, T]. \]

For piecewise continuous bounded controls \( v, \phi \), the phase variables will be continuous piecewise smooth functions that satisfy the differential constraint equations on the continuity sections of their derivatives (3).

### 4.2. Auxiliary problem solution method

This problem is investigated using the L.S. Pontryagin’s maximum principle. Based on the analysis of the system of necessary optimality conditions [7], the solution to the original optimization problem (2)-(4) for minimization of functional (1) is reduced to the boundary value problem solving:

\[
\begin{aligned}
\dot{x} &= v \frac{p_x}{\|p\|}, \quad \dot{p}_x = -\sum_{i=1}^{N} q_i \frac{v^2 (x-a_i)}{(x-a_i)^2 + (y-b_i)^2}^2, \\
\dot{y} &= v \frac{p_y}{\|p\|}, \quad \dot{p}_y = -\sum_{i=1}^{N} q_i \frac{v^2 (y-a_i)}{(x-a_i)^2 + (y-b_i)^2}^2, \\
x(0) &= 0, \quad y(0) = 0, \quad x(T) = 0, \quad y(T) = 1,
\end{aligned}
\]

where \( v = \min \{ \frac{\|p\|}{\|p\|^2 + \sum_{i=1}^{N} q_i (x-a_i)^2 + (y-b_i)^2} \}, \quad v_{\max} \), (5)

\( (a_i, b_i) \) are the coordinates of the \( i \)-th sensor, \( p_x \) and \( p_y \) are conjugate variables, \( \|p\| = \sqrt{p_x^2 + p_y^2} \). To solve the boundary value problem (5), the shooting method [8] using the modified Newton method is applied with two parameters: the coordinates of the vector \( (p_x(0), p_y(0)) \), co-directional to the velocity vector at the initial moment of time. A series of Cauchy problems in Newton's method is solved numerically by the explicit Runge-Kutta method of the 8-th order DOPRI8, based on the calculation formulas of Dorman-Prince with automatic step selection due to the maximum permissible error boundary at a step by comparing the integration results on the current interval of methods 7 and precision 8 order. The authors have realized the corresponding programmatic code in the C language for constructing the Pontryagin extremals.

### 4.3. Auxiliary problem results

As a result of the auxiliary problem solving, a map of locally optimal paths is constructed for each given location of the sensors. The shooting method was converged dozens of times to each of the obtained extremals from different initial approximations. Numerically close trajectories were taken as one. And the extremals containing a full revolution made by the evading object around any sensor, were rejected, the value of their functional values usually turned out to be significantly higher than the other trajectories values. Figure 1 shows an example of the evading object found paths for the location of sensors at the common points (0.1, 0.1) and (-0.3, 0.4) with weights \( q_1 = q_2 = 1 \). The green trajectory is the best one found for the evading object, then blue, orange and cyan follow with the functional values ascending. These trajectories correspond to the next values of functional (1), respectively: 152.241267, 228.552752, 233.774312, 2048.684188.

In Figure 2a) it can be seen for the same location of the sensors that if we reduce the weight of the left sensor, then it is most advantageous for the evading object to move to the left of the segment connecting the start and finish points. The functional values on the best trajectories are 125.800797, 330.945701, 876.491069, 18487.568821.
Figure 2b) shows that if the weight of the right sensor is less, then it is more profitable to move to the right. The functional values on the best trajectories in this case are 126.961533, 162.541598, 306.479777, 3229.954910.

The worst of the extremals of Figures 1 - 2 turned out to be the trajectories on which the evading vehicle moves along the trajectories with the maximum total degree measure of revolutions around the detectors. The best turned out routes are trajectories without object movement between the sensors.

5. Results

The problem of sensors placement is solved numerically with the limitation of the maximum speed $v$. As a result, it is possible to visually simulate the developed maps of the optimal arrangement of the sensors. The best solutions are constructed, and parametric calculations are carried out. Results for $N = 2$ are shown as an example on a grid with steps 0.1, 0.1 by $x$ and $y$.

Figure 3. Dependence of the functional $J(x_1, y_1)$ on the location of one sensor, while the coordinates of the second sensor $(x_2, y_2)$ can take any values on the grid. Higher points correspond to the better sensors placement.

Figure 4. Evading object trajectories for the best sensors placement scheme found. Two sensors with coordinates $(0.1, 0.1)$ and $(0.9, 0.9)$, $q_1 = 1, q_2 = 1, T = 1.5, v_{max} = 2$. Trajectories of the same thickness have the same functional value.
Table 1. Found extremals

\( J \) is the smallest possible value of the functional \( J \) for the evading object with the fixed sensors configuration, \((x_1, y_1)\) and \((x_2, y_2)\) are sensors coordinates.

| № | \( J \)  | \( x_1 \) | \( y_1 \) | \( x_2 \) | \( y_2 \) | № | \( J \)  | \( x_1 \) | \( y_1 \) | \( x_2 \) | \( y_2 \) | № | \( J \)  | \( x_1 \) | \( y_1 \) | \( x_2 \) | \( y_2 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1132.9 | 0 | 0.1 | 0 | 0.9 | 4 | 666.4 | 0 | 0.1 | 0.1 | 0.1 | 7 | 666.4 | -0.1 | 0.1 | 0 | 0.1 |
| 2 | 747.4 | 0 | 0.9 | 0 | 0.9 | 5 | 666.4 | 0 | 0.9 | 0.1 | 0.1 | 8 | 666.4 | -0.1 | 0.9 | 0 | 0.9 |
| 3 | 747.4 | 0 | 0.1 | 0 | 0.1 | 6 | 666.4 | 0 | 0.9 | 0.1 | 0.9 | 9 | 666.4 | 0 | 0.1 | 0.1 | 0.9 |

Table 1 was calculated for parameters: \( q_1 = 1, q_2 = 1, T = 1.5, v_{\text{max}} = 2 \). It shows the 9 best values of the functional \( J \), not less than 660, and the corresponding locations of 2 sensors on a given grid. The first column is the number of the sensors placement scheme decreasing from the best one, second column contains \( J \) for a given location of sensors, then the coordinates \((x_1, y_1)\) of the first sensor and \((x_2, y_2)\) of the second sensor follow.

Table 1 shows the optimal solution in the case of two sensors placement optimization. One sensor should be as close as possible to the starting point, the other one, to the finish point. This solution has significantly better functional value in comparison with other solutions on the grid. The second solution is to place both sensors either near the start and near the finish. Figure 3 shows the best values of the functional \( J \) for all possible locations of one of the sensors, if the coordinates of the second sensor are fixed. Higher points in Figure 3 correspond to more favorable sensor locations for the original problem.

Figure 4 corresponds to the best sensors placement scheme found and evading object extremal paths for it. Two pairs of symmetrical trajectories with the same functional value can be observed.

Figure 5 corresponds to the next functional \( J \) decreasing value. The trajectories in Fig. 5a) correspond to record 2 from Table 1 with \( v_{\text{max}} = 2 \). If \( v_{\text{max}} \) decreases to 0.8 it can be seen that the "heart" from the figure 5a) is compressed at the sides and such trajectories are obtained in the Figure 5b). Each figure has a pair of symmetrical trajectories with the same functional values. Record 3 in Table 1 corresponds to an upside-down "heart".
6. Conclusion

The paper considers the problem of the sensors placement optimizing with uniformly spreading circular detection fields in a threat environment to counteract an evading vehicle. As the results for the case of two sensors, their best location is described, the comparison with the next worse solutions is given. Also the map of the possible locations of the second sensor for the fixed location of the first sensor is constructed.

The method based on the principle of maximum was developed to solve the auxiliary problem of planning a route in a threat environment on a plane. It allows numerically constructing maps of locally optimal paths for given sensor configurations and finding the best possible value of the functional. The corresponding software package in the C language has been implemented for constructing extremals.

As a development of the problem, it is possible to consider sensors with more complex field models, perform calculations on smaller grids using gradient methods, consider the original problem as a maximin problem, as well as a simplified model for constructing paths with discretization of the problem to speed up calculations.

This work was supported by the Youth Scientific School of the 38th laboratory of the ICS RAS.

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