THE PICARD GROUP OF THE LOOP SPACE OF THE RIEMANN SPHERE

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Abstract. The loop space $L\mathbb{P}_1$ of the Riemann sphere consisting of all $C^k$ or Sobolev $W^{k,p}$ maps $S^1 \to \mathbb{P}_1$ is an infinite dimensional complex manifold. We compute the Picard group $\text{Pic}(L\mathbb{P}_1)$ of holomorphic line bundles on $L\mathbb{P}_1$ as an infinite dimensional complex Lie group with Lie algebra the Dolbeault group $H^{0,1}(L\mathbb{P}_1)$. The group $G$ of Möbius transformations and its loop group $LG$ act on $L\mathbb{P}_1$. We prove that an element of $\text{Pic}(L\mathbb{P}_1)$ is $LG$-fixed if it is $G$-fixed, thus completely answer the question of Millson and Zombro about the $G$-equivariant projective embedding of $L\mathbb{P}_1$.

1. Introduction

Let $M$ be a finite dimensional complex manifold. We fix a smoothness class $C^k$, $k=1,2,\cdots,\infty$, or Sobolev $W^{k,p}$, $k=1,2,\cdots,1\leq p<\infty$, and consider the loop space $LM = L_kM$, or $L_{k,p}M$, of all maps $S^1 \to M$ with the given regularity. It is an infinite dimensional complex Banach/Fréchet manifold, see [4]. The goal of this paper is to study the Picard group $\text{Pic}(L\mathbb{P}_1)$ of holomorphic line bundles on the loop space $L\mathbb{P}_1$ of the Riemann sphere.

Let $G \simeq \text{PGL}(2,\mathbb{C})$ be the group of holomorphic automorphisms of $\mathbb{P}_1$. Its loop space $LG$ with pointwise group operation is again a complex Lie group and acts on $L\mathbb{P}_1$ holomorphically, thus also acts on $\text{Pic}(L\mathbb{P}_1)$ and the Dolbeault groups of $L\mathbb{P}_1$. In [7], Millson and Zombro raised the following question, which is a direct motivation to study holomorphic line bundles on $L\mathbb{P}_1$: does there exist a $G$-equivariant holomorphic embedding of $L\mathbb{P}_1$ into a projectivized Banach/Fréchet space? In [10], the infinite dimensional subgroup of $\text{Pic}(L\mathbb{P}_1)$ of $LG$-fixed elements was explicitly constructed; furthermore, it was proved that the space of holomorphic sections of any line bundle in this subgroup is finite dimensional. The Dolbeault group $H^{0,1}(L\mathbb{P}_1)$ was computed and its irreducible $G$-submodules were identified in [5]. Based on previous works, this paper offers a complete answer to the question of Millson and Zombro.

The following are the main results of this paper.

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Let $\text{Pic}^0(L\mathcal{P}_1) \subset \text{Pic}(L\mathcal{P}_1)$ be the subgroup of topologically trivial line bundles.

**Theorem 1.1.** There is an exact sequence of homomorphisms

\[
0 \to H^1(L\mathcal{P}_1, \mathbb{Z}) \to H^{0,1}(L\mathcal{P}_1) \xrightarrow{\mathcal{E}} \text{Pic}^0(L\mathcal{P}_1) \to 0.
\]

The map $\mathcal{E}$ in (1) is explicitly constructed, and it is equivariant with respect to the group of holomorphic automorphisms of $L\mathcal{P}_1$.

Theorem 1.1 is closely related to the Dolbeault isomorphism. Let $\mathcal{O}$ (resp. $\mathcal{O}^*$) be the sheaf of germs of holomorphic (resp. non-vanishing holomorphic) functions. The short exact sequence of sheaves

\[
0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^* \to 0
\]
on a complex manifold $N$ induces a long exact sequence of sheaf cohomology groups

\[
\cdots \to H^1(N, \mathbb{Z}) \to H^1(N, \mathcal{O}) \to H^1(N, \mathcal{O}^*) \to H^2(N, \mathbb{Z}) \to \cdots,
\]
where $H^1(N, \mathcal{O}^*) \approx \text{Pic}(N)$. Thus a map $H^{0,1}(N) \to \text{Pic}(N)$ similar to $\mathcal{E}$ in (1) would follow from the Dolbeault isomorphism $H^{0,1}(N) \simeq H^1(N, \mathcal{O})$. Such an isomorphism on the space $L^\infty \mathcal{P}_1$ of $C^\infty$ loops can be obtained from the exactness of the sequence of sheaves $\mathcal{E}_{0,0} \xrightarrow{\partial} \mathcal{E}_{0,1} \xrightarrow{\partial} \mathcal{E}_{0,2}$, where $\mathcal{E}_{p,q}$ is the sheaf of germs of $C^\infty$ forms of type $(p,q)$ (see [6, Appendix 3]), and from the existence of $C^\infty$ partitions of unity. However, the above isomorphism in general fails on an infinite dimensional complex Banach manifold or even an open subset of a complex Banach space, see [9], and it is not available on a general loop space. As a consequence of Theorem 1.1 we obtain the Dolbeault isomorphism $H^{0,1}(L\mathcal{P}_1) \simeq H^1(L\mathcal{P}_1, \mathcal{O})$.

Recall that $H^{0,1}(L\mathcal{P}_1)$, equipped with a natural topology, is a complex locally convex topological space, see [5, Section 0]. There is a unique complex Lie group structure on $\text{Pic}(L\mathcal{P}_1)$ such that the map $\mathcal{E}$ in (1) is holomorphic, actually it can be considered as $\exp(2\pi i \cdot)$, where $\exp$ is the exponential map of the Lie group.

If a group $\mathcal{G}$ acts on a set $V$, we write $V^\mathcal{G}$ for the $\mathcal{G}$-fixed subset.

**Theorem 1.2.** $\text{Pic}(L\mathcal{P}_1)^\mathcal{G} = \text{Pic}(L\mathcal{P}_1)^{L\mathcal{G}}$; $H^{0,1}(L\mathcal{P}_1)^\mathcal{G} = H^{0,1}(L\mathcal{P}_1)^{L\mathcal{G}}$.

By Theorem 1.2 and [10, Theorem 1.2], we obtain the following

**Corollary 1.3.** If $\Lambda \in \text{Pic}(L\mathcal{P}_1)^\mathcal{G}$, then $\dim H^0(L\mathcal{P}_1, \Lambda) < \infty$.

Therefore there does not exist a $\mathcal{G}$-equivariant holomorphic embedding of $L\mathcal{P}_1$ into a projectivized Banach/Fréchet space, otherwise the pull back of the hyperplane section bundle would be in $\text{Pic}(L\mathcal{P}_1)^\mathcal{G}$, and its space of holomorphic sections would be infinite dimensional, a contradiction.

This paper is organized as follows. In Section 2 we mainly recall some relevant facts about loop spaces and in particular, $L\mathcal{P}_1$. It was proved in [5] that $H^{0,1}(L\mathcal{P}_1)$ is isomorphic to a space $\mathcal{H}$ of normalized additive Čech...
1-cocycles with respect to a fixed open covering of $LP_1$. In Section \[3\] we first show that $\text{Pic}^0(LP_1)$ is isomorphic to a space $\mathcal{H}^*$ of normalized multiplicative Čech 1-cocycles with respect to the same open covering, then we study the relation between $\mathcal{H}$ and $\mathcal{H}^*$ and prove Theorem \[1\] followed by a couple of corollaries. At the end of this section, we briefly discuss the naturality and some special properties of the topology of $H^{0,1}(LP_1)$ and of $\text{Pic}(LP_1)$. Finally in Section \[4\] we study the structure of $\text{Pic}(LP_1)^G$ and of $H^{0,1}(LP_1)^G$ and prove Theorem \[1\]. Here a few key objects introduced in \[5\] to understand $H^{0,1}(LP_1)$ take very special forms and have alternative explanations when restricted to $H^{0,1}(LP_1)^G$.

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2. Preliminaries

If $\phi : M \to M'$ is a holomorphic map between finite dimensional complex manifolds, then $L\phi : LM \ni x \mapsto \phi \circ x \in LM'$ is holomorphic, and $L$ is functorial. Given $t \in S^1$, the evaluation map $E_t : LM \ni x \mapsto x(t) \in M$ is holomorphic, see \[1\]. The constant loops form a submanifold of $LM$, which can be identified with $M$. If $M$ is paracompact, so is $LM$, see \[2\] Proposition 42.3 for $L_\infty M$ and \[8\] Theorem 14.17 for all other cases.

Let $G \simeq PGL(2, \mathbb{C})$ be as in Section 1. If we apply the functor $L$ to the holomorphic action $G \times LP_1 \to LP_1$, then we obtain a holomorphic action $LG \times LP_1 \to LP_1$, which induces (right) $LG$-actions on $\text{Pic}(LP_1)$ and the Dolbeault groups of $LP_1$ by pull-backs. In particular, $G \subset LG$ acts on $LP_1$.

The group $H^{0,1}(LP_1)$ was computed in \[5\] as follows. Let $\mathcal{H}$ be the space of holomorphic functions $F : \mathbb{C} \times LC \to \mathbb{C}$ with properties

$$
\begin{align*}
F(\zeta/\lambda, \lambda^2y) &= O(\lambda^2) \quad \text{as } \mathbb{C} \ni \lambda \to 0, \\
F(\zeta, x + y) &= F(\zeta, x) + F(\zeta, y) \quad \text{if } \supp x \cap \supp y = \emptyset, \\
F(\zeta, y + \text{const}) &= F(\zeta, y).
\end{align*}
$$

With the compact-open topology $\mathcal{H}$ is a complex locally convex space. There is a holomorphic (right) $G$-action on $\mathcal{H}$: if $g \in G$, write $J_g(\zeta) = d(g\zeta)/d\zeta$, and set

$$
(Fg)(\zeta, y) = F(g\zeta, y/J_g(\zeta))J_g(\zeta)
$$

(which is defined for $g\zeta \neq \infty$ and can be extended to $\mathbb{C} \times LC$ by the first property of $F$). Then there is a $G$-isomorphism $H^{0,1}(LP_1) \simeq \mathbb{C} \otimes \mathcal{H}$ of locally convex spaces, where $G$ acts on the right by the second variable.

If $a, b, \cdots \in \mathbb{P}_1$, set $U_{ab\cdots} = \mathbb{P}_1 \setminus \{a, b, \cdots \}$. Thus

$$
L\Omega = \{LU_a : a \in \mathbb{P}_1\}
$$

is an open covering of $LP_1$. We consider $\mathbb{P}_1$ as $\mathbb{C} \cup \{\infty\}$ and identify $LU_\infty$ with $LC$, a Fréchet algebra. If $g \in G$, then $g(LU_{ab\cdots}) = LU_{g(a)g(b)\cdots}$.

Given $v : \mathbb{P}_1 \to \mathbb{C}$, finitely many $a, b, \cdots \in \mathbb{P}_1$ and a function $u : LU_{ab\cdots} \to \mathbb{C}$, we say that $u$ is cuspidal with respect to $v$ (as formulated in \[5\] Section
if

$$\lim_{\gamma \to \infty} g(x + \lambda) = v(\infty)$$

for any \(x \in L \mathcal{C}\) and \(v \in G\) that maps \(\infty\) to one of \(a, b, \cdots\). If \(u\) is holomorphic, then it follows from (5), where we set \(u|_{U_{ab}} = 0\) and \(u|_{U_{ab}} \equiv c\) for some constant \(c\) and \(u\) is cuspidal with respect to the constant \(c\). Let

$$\text{ind}_{ab}: LU_{ab} \ni x \mapsto \text{ind}_{ab}(x) \in \mathbb{Z}$$

(the winding number of \(x : S^1 \to U_{ab}\)). Then \(\text{ind}_{ab}\) is cuspidal with respect to the constant 0, and \((\text{ind}_{ab})_{a,b \in \mathbb{P}_1}\) is a \(G\)-fixed Čech 1-cocycle with respect to the covering \(L \mathcal{C}\) as in (4).

**Proposition 2.1.** The group \(H^1(L \mathbb{P}_1, \mathbb{Z}) \simeq \mathbb{Z}\) is generated by the Čech cohomology class of \((\text{ind}_{ab})\). Suppose \(e: \mathbb{P}_1 \to L \mathbb{P}_1\) is a holomorphic embedding and there exists \(t \in S^1\) such that \(E_t \circ e\), where \(E_t\) is the evaluation, is an isomorphism of \(L \mathbb{P}_1\). Then \(e^*: H^2(L \mathbb{P}_1, \mathbb{Z}) \to H^2(\mathbb{P}_1, \mathbb{Z})\) is an isomorphism.

**Proof.** Recall that \(H^1(L \mathbb{P}_1, \mathbb{Z}) \simeq \mathbb{Z}\) and \(H^2(L \mathbb{P}_1, \mathbb{Z}) \simeq \mathbb{Z}\) (e.g. see [1, Part II, Proposition 15.33] and [8, Theorem 13.14]). As every element of the open covering \(L \mathcal{C}\) is contractible, we have \(H^1(L \mathcal{C}, \mathbb{Z}) \simeq H^1(L \mathbb{P}_1, \mathbb{Z})\). Let \([\rho]\) be a generator of \(H^1(L \mathcal{C}, \mathbb{Z})\), where \(\rho\) is a 1-cocycle. We can write \((\text{ind}_{ab}) = m\rho + \rho'\), where \(m \in \mathbb{Z}\) and \(\rho'\) is exact. Since \(\text{ind}_{\rho_0} = 0\) takes value of any integer and \(\rho'_{\rho_0}\) is constant, we must have \(|m| = 1\). It is clear that the map \(e^*\) is surjective, and any surjective homomorphism \(\mathbb{Z} \to \mathbb{Z}\) is an isomorphism.

**Corollary 2.2.** Let \(e\) be as in Proposition 2.1. Then a bundle \(\Lambda \in \text{Pic}(L \mathbb{P}_1)\) is in \(\text{Pic}^0(L \mathbb{P}_1)\) if and only if \(\Lambda|_{e(\mathbb{P}_1)}\) (or in particular \(\Lambda|_{\mathbb{P}_1}\)) is trivial.

**Proof.** The conclusion immediately follows from Proposition 2.1 and the relation of the first Chern classes \(c_1(e^*\Lambda) = e^*c_1(\Lambda)\).

The proof of the following proposition is similar to that of [10] Proposition 2.3.

**Proposition 2.3.** Let \(M \xrightarrow{\pi} B\) be a holomorphic fiber bundle with fibers \(F_b\) finite dimensional connected compact complex manifolds, and \(E \to M\) a holomorphic line bundle. If \(\pi\) has a holomorphic section \(\sigma: B \to M\) such that \(E|_{\sigma(B)}\) and \(E|_{F_b}\), for all \(b \in B\), are holomorphically trivial, then \(E\) is holomorphically trivial.

**Proof.** Let \(u\) be a non-vanishing holomorphic section of \(E|_{\sigma(B)}\), and let \(s\) be the section of \(E \to M\) such that \(s|_{F_b}\) is the holomorphic section of \(E|_{F_b}\) satisfying \(s(\sigma(b)) = u(\sigma(b))\). It is clear that \(s\) is non-vanishing. Next we show that \(s\) is \(C^\infty\) and in turn holomorphic.

Let \(b_0 \in B\). By [3, Proposition 5.1] (where we choose \(p = 0\), \(q = 1\) and \(f = 0\)), there exist a neighborhood \(b_0 \in B_0 \subseteq B\) and a section \(v \in \text{Pic}(L \mathbb{P}_1)\) such that \(v|_{U_{ab}} = 0\) and \(\text{ind}_{ab} = m\rho + \rho'\), where \(m \in \mathbb{Z}\) and \(\rho'\) is exact. Since \(\text{ind}_{\rho_0} = 0\) takes value of any integer and \(\rho'_{\rho_0}\) is constant, we must have \(|m| = 1\). It is clear that the map \(e^*\) is surjective, and any surjective homomorphism \(\mathbb{Z} \to \mathbb{Z}\) is an isomorphism.
By choosing a sufficiently small $B_0$ we can assume that $\pi^{-1}(B_0) \to B_0$ is trivial and $v$ is non-vanishing. The function $s/v$ on $\pi^{-1}(B_0)$ is constant on each fiber $F_b$, and can also be considered as a function on $B_0$. Since $s/v(b) = u(\sigma(b))/v(\sigma(b))$, it follows that $s/v$ is $C^\infty$. Therefore $s$ is $C^\infty$ and holomorphic on each fiber $F_b$ as well as the cross section $\sigma(B)$. Apply Proposition 5.2 (ii) to the $(0,1)$ form $\bar{\partial}s$, we obtain that $\bar{\partial}s = 0$. \hfill $\square$

3. The map $\mathfrak{C} : H^{0,1}(L\mathbb{P}_1) \to \text{Pic}^0(L\mathbb{P}_1)$

In this section we construct the map $\mathfrak{C}$ in (1) and prove Theorem 1.1, then we compute $H^1(L\mathbb{P}_1, \mathcal{O})$ and $\text{Pic}(L\mathbb{P}_1)$ as corollaries. Finally we study certain special features of the topology of $H^{0,1}(L\mathbb{P}_1)$ and of $\text{Pic}(L\mathbb{P}_1)$.

Let $\mathfrak{H}$ (resp. $\mathfrak{H}^*$) be the linear space (resp. group) of holomorphic additive (resp. multiplicative) Čech 1-cocycles $(h_{ab} \in \mathcal{O}(LU_{ab}))_{a,b \in \mathbb{P}_1}$ (resp. $(h_{ab}^* \in \mathcal{O}^*(LU_{ab}))_{a,b \in \mathbb{P}_1}$) with respect to the covering $LU$ of $L\mathbb{P}_1$ such that every component $h_{ab}$ (resp. $h_{ab}^*$) is cuspidal with respect to $v \equiv 0$ (resp. $v \equiv 1$), see (5). The cocycle $(\text{ind}_{ab}) \in \mathfrak{H}$. The cuspidal property implies that $h_{ab}|_{U_{ab}} \equiv 0$ (resp. $h_{ab}^*|_{U_{ab}} \equiv 1$). The group $G$ acts on $\mathfrak{H}$ (resp. $\mathfrak{H}^*$) by pull-backs.

The space $\mathfrak{H}$ was defined in [5, Section 3] with the additional requirement that $h_{ab}(x)$ is holomorphic in $a$ and $b$, but we shall show in Corollary 3.3 that it is redundant.

It was proved in [5] that $H^{0,1}(L\mathbb{P}_1) \simeq \mathfrak{H}$ as $G$-modules. We shall construct a $G$-isomorphism $\text{Pic}^0(L\mathbb{P}_1) \simeq \mathfrak{H}^*$, and relate $H^{0,1}(L\mathbb{P}_1)$ and $\text{Pic}^0(L\mathbb{P}_1)$ by the exponential map $\mathfrak{H} \to \mathfrak{H}^*$.

We begin with a family of normalized non-vanishing holomorphic sections of a given bundle $\Lambda \in \text{Pic}^0(L\mathbb{P}_1)$ over $LU_a$, $a \in \mathbb{P}_1$, which depend on $a$ holomorphically.

**Proposition 3.1.** Let $\Lambda \in \text{Pic}^0(L\mathbb{P}_1)$, $v$ a holomorphic section of $\Lambda|_{\mathbb{P}_1}$ and $x_0 \in L\mathcal{C}^*$ (where $\mathcal{C}^* = \mathcal{C} \setminus \{0\}$). For any $a \in \mathbb{P}_1$ there is a unique $\sigma_a = \sigma_{a,v,x_0} \in H^0(LU_a, \Lambda)$ such that

$$\lim_{\zeta \to \lambda \to \infty} \sigma_a(g(x + \lambda x_0)) = v(a)$$

for all $x \in LU_\infty$ and $g \in G$ with $g(\infty) = a$. If $v$ is non-vanishing, so is $\sigma_a$; and $\sigma_a$ depends linearly on $v$. Furthermore, $\sigma(a,y) = \sigma_a(y)$ is holomorphic in $(a,y)$.

**Proof.** Let $\phi$ be a continuous complex linear functional on $LU_\infty$ such that $\phi(x_0) = 1$, let $Z = \ker(\phi)$ and consider the holomorphic map

$$P : \mathbb{P}_1 \times G \times Z \ni (\lambda, g, z) \mapsto g(z + \lambda x_0) \in L\mathbb{P}_1$$

and the pull-back bundle $P^*\Lambda$. Note that for any $g \in G$ and $z \in Z$, the map $P|_{\mathbb{P}_1 \times \{g\} \times \{z\}}$ is an embedding with image curve $C(g, z)$ passing through $g(\infty) \in \mathbb{P}_1 \subset L\mathbb{P}_1$. The topologically trivial line bundle $\Lambda|_{\mathbb{P}_1 \times \{g\} \times \{z\}}$ is also holomorphically trivial, so is $P^*\Lambda|_{\mathbb{P}_1 \times \{g\} \times \{z\}}$. Thus an equivalent statement
to (7) is that along each curve $C(g,z)$, where $g(\infty) = a$, $\sigma_a$ extends to be the unique holomorphic section of $\Lambda|_{C(g,z)}$ with value $v(a)$ at $a$.

The uniqueness of $\sigma_a$ follows from the fact that, for any fixed $g \in G$ with $g(\infty) = a$, the set $LU_a \cup \{a\}$ is covered by curves of the form $C(g,z)$, and $\sigma_a$ is the unique holomorphic section of $\Lambda$ on each curve satisfying (7). If $v(a) \neq 0$, then $\sigma_a$ is non-vanishing on each curve, and $\sigma_a$ depends on $v(a)$ linearly on each curve.

As to existence, since $P$ maps $\{\infty\} \times G \times Z$ to $\mathbb{P}_1 \subset L\mathbb{P}_1$, the bundle $P^*\Lambda$ is trivial on $\{\infty\} \times G \times Z$, which can be considered as a cross section of the trivial $\mathbb{P}_1$-bundle $\mathbb{P}_1 \times G \times Z$. It follows from Proposition 2.3 that $P^*\Lambda$ is trivial and we can uniquely extend $P^*v$ to a section $\tilde{\sigma} \in H^0(\mathbb{P}_1 \times G \times Z, P^*\Lambda)$. Note that, for any fixed $g \in G$ with $g(\infty) = a$, $P$ maps $\mathbb{C} \times \{g\} \times Z$ biholomorphically onto $LU_a$. Let $\tilde{P} : P^*\Lambda \rightarrow \Lambda$ be the bundle map associated with $P$. Then there exists $\sigma_a \in H^0(LU_a, \Lambda)$ such that

$$\tag{9} \sigma_a \circ P = \tilde{P} \circ \tilde{\sigma}$$

on $\mathbb{C} \times \{g\} \times Z$. It follows from the property of $\tilde{\sigma}$ that $\sigma_a$ satisfies (7).

Choose $g \in G$ with $g(\infty) = a$ such that $g$ depends holomorphically on $a$ (which can be done locally). If $y = P(\lambda,g,z) \in LU_a$, then we can write $\lambda$ and $z$ in terms of $y$ and $g$, and equation (9) becomes

$$\sigma_a(y) = \tilde{P} \left( \tilde{\sigma} \left( \phi \left( g^{-1}(y) \right), g, g^{-1}(y) - \phi \left( g^{-1}(y) \right) x_0 \right) \right),$$

i.e. $\sigma_a(y)$ is holomorphic in $(a,y)$. \hfill \Box

When $\Lambda$ is LG-fixed, two sections $\sigma_{a,v,x_0}$ and $\sigma_{a,v,x_1}$, $x_0, x_1 \in L\mathbb{C}^*$, as in Proposition 3.1 only differ by a multiplicative constant, see [10] Proposition 2.7. However, this is not true for a general $\Lambda \in \text{Pic}^0(L\mathbb{P}_1)$.

Choose $v$ to be non-vanishing and set $x_0 = 1$ in Proposition 3.1. Since $\Lambda$ determines $v$ up to a multiplicative constant, we can uniquely associate with $\Lambda$ the Čech 1-cocycle $(h_{ab}^*) = \sigma_b/\sigma_a)$. The property (7) of $\sigma_a$ and $\sigma_b$ implies that $(h_{ab}^*) \in \mathfrak{F}^*$.

**Proposition 3.2.** The $G$-morphism $\text{Pic}^0(L\mathbb{P}_1) \ni \Lambda \mapsto (h_{ab}^*) \in \mathfrak{F}^*$ is an isomorphism of groups.

**Proof.** It is clear that the kernel only contains the trivial bundle. Given $(h_{ab}^*) \in \mathfrak{F}^*$, we can construct a line bundle $\Lambda$ by taking the union of $LU_a \times \mathbb{C}$ over all $a \in \mathbb{P}_1$ and identifying $\{x\} \times \mathbb{C}$ in $LU_a \times \mathbb{C}$ and $LU_b \times \mathbb{C}$ via multiplication by $h_{ab}^*$. Let $\sigma_a$ be the section of $\Lambda$ on $LU_a$ corresponding to $LU_a \times \{1\}$. Then $h_{ab}^* = \sigma_b/\sigma_a$. Since $h_{ab}^*/LU_a \equiv 1$, there is a unique non-vanishing holomorphic section $v$ of $\Lambda|_{\mathbb{P}_1}$ such that $v|_{LU_a} = \sigma_a|_{LU_a}$. It follows from Corollary 2.2 that $\Lambda \in \text{Pic}^0(L\mathbb{P}_1)$. From the relation $\sigma_a = \sigma_b/h_{ab}^*$ and the cuspidal property of $h_{ab}^*$ we can obtain (7). \hfill \Box

Any $(h_{ab}) \in \mathfrak{F}$ (resp. $(h_{ab}^*) \in \mathfrak{F}^*$) can be considered as a function on

$$\Omega = \{(a,b,x) \in \mathbb{P}_1 \times \mathbb{P}_1 \times L\mathbb{P}_1 : a, b \not\in x(S^1)\}.$$
Lemma 3.5. The surjective, and its kernel consists of integer multiples of the cocycle $\omega$ exists for any $\omega$. Then $c(x,\tau)$ with respect to a constant $c$, such that $h^*_\omega = \sigma_b/\sigma_a$, and $\sigma_a(x)$ is holomorphic in $(a,x)$. The conclusion immediately follows for $(h^*_{ab})$. It is also true for $(h_{ab})$ because $(h_{ab})$ is a logarithm of $(\exp(h_{ab})) \in \mathcal{F}^*$.

As a subspace of $\mathcal{O}(\Omega)$ with the compact-open topology, $\mathcal{F}$ is a complete locally convex space. The isomorphism $H^{0,1}(L\mathbb{P}_1) \simeq \mathcal{F}$ is topological, see [5, Section 3].

Next we study the exponential map $\mathcal{F} \to \mathcal{F}^*$.

Proposition 3.4. Suppose that we are given finitely many $a, b, \cdots \in \mathbb{P}_1$, and a function $h^* \in \mathcal{O}^*(L U_{ab\cdots})$ cuspidal with respect to a constant $c^*$. If there exists $h \in \mathcal{O}(L U_{ab\cdots})$ such that $\exp(h) = h^*$, then $h$ is cuspidal with respect to a constant $c$, where $\exp(c) = c^*$.

Proof. The cuspidal property of $h^*$ implies that $h^*|_{U_{ab\cdots}} \equiv c^*$. So $h|_{U_{ab\cdots}} \equiv c$ for some constant $c$, where $\exp(c) = c^*$. Take any point from $\{a, b, \cdots\}$, say $a$, and fix $g \in G$ with $g(\infty) = a$. Since $h$ is a logarithm of $h^*$, the limit

$$\omega(x) \triangleq \lim_{\lambda \to \infty} h(g(x + \lambda))$$

exists for any $x \in L\mathcal{C}$, $\exp(\omega(x)) \equiv c^*$ and $\omega(0) = c$. Next we show that $\omega(x) \equiv c$.

Recall the map $P$ and the hyperplane $Z \subset L\mathcal{C}$ as in (8), where we set $x_0 = 1$. It follows from the definition of $\omega$ that, for any $x \in L\mathcal{C}$, there exists $z \in Z$ such that $\omega(x) = \omega(z)$. We only need to show that $\omega$ is continuous, thus constant on any line $\mathbb{C}z$ spanned by $z \in Z \setminus \{0\}$, so $\omega(z) = \omega(0) = c$.

Let $P_{g,z} : \mathbb{P}_1 \times \mathbb{C}z \to \mathbb{P}_1$ be the map obtained from $P$ with the fixed $g$ and $z$ above. Note that $P_{g,z}$ maps a neighborhood $U$ of the set $\{\infty\} \times \mathbb{C}z$ to $L U_{ab\cdots} \cup \{a\}$, and a point of $U$ is mapped to $a$ if and only if it is in $\{\infty\} \times \mathbb{C}z$. Define the function $\tau^*|_{U_{ab\cdots}}$ on $U$ as in the following: $\tau^*$ takes value $c^*$ on $\{\infty\} \times \mathbb{C}z$, and $\tau^* = P_{g,z}^* h^*$ otherwise. It follows from the cuspidal property of $h^*$ that $\tau^*$ is holomorphic with respect to each variable, thus holomorphic on $U$. Given $z_0 \in \mathbb{C}$, we choose a small neighborhood $V \subset U$ of $(\infty, z_0)$ such that $V$ is simply connected, and $V \setminus \{\infty\} \times \mathbb{C}z$ is connected. Let $\tau$ be the logarithm of $\tau^*$ on $V$ such that $\tau$ agrees with $P_{g,z}^* h$ on $V \setminus \{\infty\} \times \mathbb{C}z$. Then $\omega(z_0) = \tau(\infty, z_0)$ for $\zeta$ near $z_0$, and $\omega$ is continuous on $\mathbb{C}z$.

Lemma 3.5. The $G$-morphism $\mathcal{E} : \mathcal{F} \ni (h_{ab}) \mapsto (\exp(2\pi i h_{ab})) \in \mathcal{F}^*$ is surjective, and its kernel consists of integer multiples of the cocycle $(\text{ind}_{ab})$ as in (6).

Proof. Let $h^* \in \mathcal{F}^*$. Since the line bundle associated with $h^*$ is topologically trivial, there exist non-vanishing functions $\psi^*_a \in C(L U_a)$, $a \in \mathbb{P}_1$, such that $\psi^*_a/\psi^*_b = h^*_{ab}$. As $h^*_{ab}|_{U_{ab\cdots}} \equiv 1$, the functions $\psi^*_a|_{U_a}$, $a \in \mathbb{P}_1$, form a non-vanishing continuous function $v$ on $\mathbb{P}_1 \subset \mathbb{P}_1$. Fix a logarithm $u$ of $v$. The Picard group of the loop space of the Riemann sphere

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Since $LU_a$, $a \in \mathbb{P}_1$, are simply connected, there exist $\psi_a \in C(LU_a)$ such that $\exp(2\pi i u_a) = \psi_a^*$ and $2\pi iv_a|_{U_a} = u|_{U_a}$. Consider the Čech 1-cocycle $(h_{ab} = \psi_b - \psi_a)$. It is clear that $h_{ab}|_{U_ab} \equiv 0$ and $\exp(2\pi i h_{ab}) = h_{ab}^*$, so $h_{ab} \in O(LU_{ab})$. It follows from Proposition 3.4 that $h_{ab}$ is cuspidal with respect to the constant 0, therefore $(h_{ab}) \in \mathcal{H}.

The cocycle $(\text{ind} \ C)_{ab}$ is clearly in the kernel of $\mathcal{E}$. Suppose $h \in \mathcal{H}$ is in the kernel of $\mathcal{E}$, then it must be $\mathbb{Z}$-valued, thus generates a cohomology class in $H^1(LU, \mathbb{Z}) \simeq H^1(L\mathbb{P}_1, \mathbb{Z})$. It follows from Proposition 2.1 that $h = m(\text{ind} \ C)_{ab} + h'$, where $m \in \mathbb{Z}$ and $h'$ is an exact $\mathbb{Z}$-valued cochain of $L\mathbb{U}$, in particular, any component $h_{ab}'$ is constant. The cuspidal property of $h_{ab}'$ implies that $h_{ab}' \equiv 0$. □

Now $\mathcal{H}^*$, the quotient space of the complex locally convex space $\mathcal{H}$ modulo a discrete subgroup, is a complex Lie group with the quotient topology. We shall show in Proposition 3.11 that this topology of $\mathcal{H}^*$ is actually the compact-open topology.

Theorem 1.1 immediately follows from [3, Theorem 3.3] (that $H^{0,1}(L\mathbb{P}_1) \simeq \mathcal{H}$), Proposition 3.2, Lemma 3.5, and Proposition 2.1. The map $\mathcal{E}$ in Theorem 1.1 is the one in Lemma 3.5. If we endow $Pic^0(L\mathbb{P}_1)$ with the complex Lie group structure from $\mathcal{H}^*$, then the map $\mathcal{E}$ in Theorem 1.1 is just $\exp(2\pi i \cdot)$, where $\exp$ is the exponential map of the Lie group. There is a unique complex Lie group structure on $Pic(L\mathbb{P}_1)$ such that $Pic^0(L\mathbb{P}_1)$ is the component containing $1 \in Pic(L\mathbb{P}_1)$.

Next we draw a couple of corollaries of Lemma 3.5/Theorem 1.1.

Note that any 1-cocycle in $\mathcal{H}$ (resp. $\mathcal{H}^*$) generates a Čech cohomology class in $H^1(LU, \mathcal{O})$ (resp. $H^1(L\mathbb{U}, \mathcal{O}^*)$). From the definition of $\mathcal{H}$ (resp. $\mathcal{H}^*$) we obtain the following

**Proposition 3.6.** The map $\mathcal{H} \to H^1(L\mathbb{U}, \mathcal{O})$ (resp. $\mathcal{H}^* \to H^1(L\mathbb{U}, \mathcal{O}^*)$) is injective.

**Proof.** If $(h_{ab}) \in \mathcal{H}$ is exact, i.e. there exist $h_a \in O(LU_a)$, $a \in \mathbb{P}_1$, such that $h_{ab} = h_a - h_b$, it follows from the cuspidal property of $h_{ab}$ that

$$\lim_{\lambda \to \infty} h_a(g(x + \lambda)) = \lim_{\lambda \to \infty} (h_{ab} + h_b)(g(x + \lambda)) = h_b(a),$$

for any $x \in L\mathbb{C}$ and $g \in G$ with $g(\infty) = a$. So $h_a$ is constant along any curve of the form $\{g(x + \lambda) : \lambda \in \mathbb{P}_1\} \subset L\mathbb{P}_1$, and $h_a \equiv h_b(a)$. Also $h_b$ is constant, thus $h_{ab} \equiv 0$. The conclusion for $\mathcal{H}^*$ follows from similar arguments. □

Therefore we can consider $\mathcal{H}$ (resp. $\mathcal{H}^*$) as a subspace (resp. subgroup) of $H^1(L\mathbb{U}, \mathcal{O})$ (resp. $H^1(L\mathbb{U}, \mathcal{O}^*)$), which is, in turn, a subspace (resp. subgroup) of $H^1(L\mathbb{P}_1, \mathcal{O})$ (resp. $H^1(L\mathbb{P}_1, \mathcal{O}^*)$).

**Corollary 3.7.** The maps $\mathcal{H} \to H^1(L\mathbb{U}, \mathcal{O}) \to H^1(L\mathbb{P}_1, \mathcal{O})$ are isomorphisms of groups. In particular, $H^{0,1}(L\mathbb{P}_1) \simeq H^1(L\mathbb{P}_1, \mathcal{O})$.

**Proof.** Since $L\mathbb{P}_1$ is rationally connected, see the proof of Proposition 3.1. any holomorphic function on $L\mathbb{P}_1$ is constant. Therefore $H^0(L\mathbb{P}_1, \mathcal{O}) \simeq \mathbb{C}$
and $H^0(LP_1, O^*) \simeq \mathbb{C}^*$. The short exact sequence of sheaves $0 \to \mathbb{Z} \to O \xrightarrow{\exp(2\pi i) \cdot} O^* \to 0$ induces a long exact sequence of cohomology groups

$$0 \to \mathbb{Z} \to \mathbb{C} \to H^1(LP_1, \mathbb{Z}) \to H^1(LP_1, O) \xrightarrow{\exp(2\pi i) \cdot} H^1(LP_1, O^*) \to \cdots,$$

where the kernel of $\exp(2\pi i) \cdot$ is $H^1(LP_1, \mathbb{Z}) \subseteq \mathfrak{H} \subset H^1(LP_1, O)$, and the range of $\exp(2\pi i) \cdot$ is $\text{Pic}^0(LP_1) \subset H^1(LP_1, O^*)$. It follows from Lemma 3.5 and Propositions 2.1 and 3.2 that the map $\exp(2\pi i) \cdot$ and its restriction to $\mathfrak{K}$ have the same kernel and the same range, thus the map $\mathfrak{K} \to H^1(LP_1, O)$ is an isomorphism.

With the Dolbeault isomorphism in Corollary 3.7, the map $\mathfrak{K}$ in Theorem 1.1 can be considered as $\exp(2\pi i) \cdot$ in (10), thus it is equivariant with respect to the group of holomorphic automorphisms of $LP_1$.

Now $\text{Pic}^0(LP_1)$ can be computed as a quotient group by Theorem 1.1. To identify $\text{Pic}^0(LP_1)$ more precisely, we recall a few concepts which played key roles in [5], and which are also useful in the next section. Let $\mathfrak{F}$ be the function space as in [2]. In [5] Section 4], a map $\alpha : \mathfrak{H} \to \mathfrak{F}$ was introduced as follows. Given $\mathfrak{h} = (\mathfrak{h}_{ab}) \in \mathfrak{H}$, the cocycle relation implies that $d\zeta \mathfrak{h}_{a\zeta}(x)$ is independent of $a$. For $\zeta \in \mathbb{C}$ we can write

$$d\zeta \mathfrak{h}_{a\zeta}(x) = F \left( \zeta, \frac{1}{\zeta - x} \right) d\zeta, \quad x \in LU, \quad (11)$$

where $F \in O(\mathbb{C} \times LC)$. It turns out that $F \in \mathfrak{F}$, and we define $\alpha(\mathfrak{h}) = F$. The map $\alpha$ is a continuous $G$-morphism. It was proved in [5] Section 5] that the kernel of $\alpha$ is one-dimensional, spanned by the 1-cocycle $(\text{ind}_{ab})$ as in (6), and $\alpha$ has a right inverse $\beta$, which is continuous and $G$-equivariant.

Since the kernel of $\exp = \mathfrak{K}(\cdot/2\pi i) : \mathfrak{K} \to \mathfrak{K}^*$ is contained in the kernel of $\alpha : \mathfrak{K} \to \mathfrak{F}$, we have a well-defined continuous $G$-morphism $\alpha^* : \mathfrak{K}^* \to \mathfrak{F}$ such that

$$\alpha^* \circ \exp = \alpha. \quad (12)$$

**Proposition 3.8.** The map $\alpha^*$ has a continuous $G$-equivariant right inverse $\beta^*$, and its kernel consists of cocycles of the form

$$\left( \zeta^{\text{ind}_{ab}} \right), \quad \zeta \in \mathbb{C}^*. \quad (13)$$

Proof. The right inverse $\beta^*$ is the composition of the right inverse of $\alpha$ and the map $\exp : \mathfrak{K} \to \mathfrak{K}^*$. It is straightforward to verify that elements of $\ker(\alpha^*) = \exp(\ker(\alpha))$ take the form as in (13). \qed

Since $H^2(LP_1, \mathbb{Z}) \simeq \mathbb{Z}$, we obtain the following

**Corollary 3.9.** $\text{Pic}(LP_1) \simeq \mathbb{C}^* \times \mathfrak{F} \times \mathbb{Z}$.

Recall that $\mathfrak{H}$ (resp. $\mathfrak{H}^*$) is a subset of $O(\Omega)$ (resp. $O^*(\Omega)$). In the remainder of this section we are going to study the compact-open topology.
on $\mathcal{H}$ and on $\mathcal{H}^*$. If we consider the 1-cocycle $(\text{ind}_{ab})$ as a function $\text{ind} : \Omega \to \mathbb{Z}$, then for any $m \in \mathbb{Z}$, the subset $\text{ind}^{-1}(m) \neq \emptyset$ is a path connected component of $\Omega$.

Let $X \subset \mathbb{C} \times \mathbb{C}$ be a compact subset and let $X'$ be the inverse image of $X$ under the homeomorphism

$$\{(\zeta, x) \in \mathbb{C} \times \mathbb{C} : \lambda \in \mathbb{L}_{\zeta} \} \ni (\zeta, x) \mapsto (\zeta, 1/(\zeta - x)) \in \mathbb{C} \times \mathbb{C}$$

(see (11)). If $0 < r < \inf_{(\zeta, x) \in X', t \in S^1}|\zeta - x(t)|$ (where $|\zeta - x(t)| = +\infty$ when $x(t) = \infty \in \mathbb{P}_1$), then

$$(14) \quad K = K(X', r) = \{(\zeta, \zeta + z, x) : (\zeta, x) \in X', z \in \mathbb{C}, |z| \leq r\}$$

is a compact subset of $\{(a, b, x) \in \Omega : a, b \neq -\infty, \text{ind}_{ab}(x) = 0\}$.

**Proposition 3.10.** With notation above, we have

$$\sup_{X} |\alpha(h)| \leq r^{-1} \sup_{K} |h|$$

for any $h \in \mathcal{H}$; and if $0 < \delta < 1$, then there exists a constant $C_2 = C_2(\delta) > 0$ such that

$$\sup_{K} |h| \leq C_2 \sup_{K} |\exp h - 1|$$

for any $h \in \mathcal{H}$ with $\sup_{K} |\exp h - 1| \leq \delta$.

**Proof.** Taking the derivative of $h(\zeta, \zeta + z, x)$ with respect to $z$ at $z = 0$ for any $(\zeta, x) \in X'$, the inequality (15) follows from (11) and Cauchy’s estimate. Note that $h(\zeta, \zeta, x) = 0$ for all $(\zeta, x) \in X'$, i.e. on every path connected component of $K$ we can find a point where $h$ vanishes. Thus (16) follows from basic properties of the branch of $\ln w$ on $\{w \in \mathbb{C} : |w - 1| < \delta\}$ sending $w = 1$ to 0.

**Proposition 3.11.** Let $\omega_0 \in \Omega \setminus \text{ind}^{-1}(0)$ be fixed. Then the following topologies of $\mathcal{H}$ (resp. $\mathcal{H}^*$) are the same.

(a) The compact-open topology of $\mathcal{H}$ (resp. the quotient topology of $\mathcal{H}^*$ induced by the homomorphism $\mathcal{E} : \mathcal{H} \to \mathcal{H}^*$).

(b) The topology that has as a basis the sets $\{h \in \mathcal{H} : \sup_{K \cup \{\omega_0\}} |h - h_0| < \varepsilon\}$ (resp. $\{h^* \in \mathcal{H}^* : \sup_{K \cup \{\omega_0\}} |h^* - h_0^*| < \varepsilon\}$), where $K \subset \{(a, b, x) \in \Omega : a, b \neq -\infty, \text{ind}_{ab}(x) = 0\}$ is compact, $h_0 \in \mathcal{H}$ (resp. $h_0^* \in \mathcal{H}^*$), and $\varepsilon > 0$.

**Proof.** Let $\mathcal{T}_a$ and $\mathcal{T}_b$ denote the topologies in (a) and (b) respectively. It follows from Proposition 3.10 that the homomorphism $\alpha : (\mathcal{H}, \mathcal{T}_a) \to \mathcal{E}$ (resp. $\alpha^* : (\mathcal{H}^*, \mathcal{T}_b) \to \mathcal{E}$) between topological vector spaces (resp. topological groups) is continuous. With the relative topology its kernel is isomorphic to $\mathbb{C}$ (resp. $\mathbb{C}^*$), and its right inverse $\beta : \mathcal{E} \to (\mathcal{H}, \mathcal{T}_a)$ (resp. $\beta^* : \mathcal{E} \to (\mathcal{H}^*, \mathcal{T}_b)$) is also continuous (for $\mathcal{T}_a$ is finer than $\mathcal{T}_b$). So these homomorphisms induce isomorphisms $(\mathcal{H}, \mathcal{T}_a) \simeq \mathbb{C} \times \mathcal{E} \simeq (\mathcal{H}, \mathcal{T}_b)$ (resp. $(\mathcal{H}^*, \mathcal{T}_b) \simeq \mathbb{C}^* \times \mathcal{E} \simeq (\mathcal{H}^*, \mathcal{T}_b)$).
Since the compact-open topology of \( \mathfrak{H}^* \) is finer than the one in Proposition 3.11(b) and coarser than the other in (a), they are all the same.

4. The structure of \( G \)-fixed subgroups

In this section we study the structure of \( H^{0,1}(L\mathbb{P}_1)^G \) and of Pic\((L\mathbb{P}_1)^G \)

and in particular, prove Theorem [1.2]. We begin with explicit constructions of all elements of \( \mathfrak{H}^* \) and \( (\mathfrak{H}^*)^G \).

Let \( \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \) (resp. \( \text{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \)) be the linear space (resp. group) of all holomorphic homomorphisms from the loop group \( L\mathbb{C}^* \) to \( \mathbb{C} \) (resp. \( \mathbb{C}^* \)). With the compact-open topology \( \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \) is a complex locally convex space. If \( \psi \in \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \) and \( \varphi \in \text{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \), then \( \psi(z) = 0 \) and \( \varphi(z) = z^n \) for \( z \in \mathbb{C}^* \subseteq L\mathbb{C}^* \), where \( n \) is a fixed integer, and we call this \( n \) the order of \( \varphi \). We write \( \text{Hom}^0(L\mathbb{C}^*, \mathbb{C}^*) \) for the subgroup of zero order elements of \( \text{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \).

Let \((L\mathbb{C})'\) be the space of continuous linear functionals on \( L\mathbb{C} \) and let

\[ A = \{ \phi \in (L\mathbb{C})' : \phi|_{\mathbb{C}} \equiv 0 \}. \]

Associated with any \( \psi \in \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \) there is a commutative diagram

\[
\begin{array}{ccc}
\text{LC} & \xrightarrow{\phi} & \mathbb{C} \\
\exp \downarrow & & \exp \downarrow \\
\text{LC}^* & \xrightarrow{\varphi} & \mathbb{C}^*,
\end{array}
\]

(17)

where \( \varphi \in \text{Hom}^0(L\mathbb{C}^*, \mathbb{C}^*) \) is obtained by composition, and \( \phi \in A \) is the Lie algebra homomorphism of \( \psi \) and \( \varphi \). Let \( L^0\mathbb{C}^* \subseteq L\mathbb{C}^* \) be the subgroup of loops with winding number 0. Given \( \phi \in A \), the homomorphism \( L^0\mathbb{C}^* \ni x \mapsto \phi(\ln(x)) \in \mathbb{C} \) is independent of the choice of \( \ln(x) \) and therefore well defined, from which it follows that the space of \( \phi \) (resp. \( \varphi \)) generated as in (17) is all of \( A \) (resp. \( \text{Hom}^0(L\mathbb{C}^*, \mathbb{C}^*) \)). Thus \( \text{Hom}^0(L\mathbb{C}^*, \mathbb{C}^*) \), as the quotient space of \( \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \) modulo a discrete subgroup, is a complex Lie group.

Let \( \psi \in \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \) and \( g_{ab} \in G \subseteq LG \) such that \( g_{ab}(a) = \infty \) and \( g_{ab}(b) = 0 \). Consider \( \psi \) as a function on \( LU_{\infty 0} \) and define \( h_{ab}^\psi = g_{ab}^*\psi \in \mathcal{O}(LU_{ab}) \). The property \( \psi|_{\mathbb{C}^*} \equiv 0 \) implies that \( h_{ab}^\psi \) is independent of the choice of \( g_{ab} \). In particular,

\[
h_{ab}^\psi(x) = \begin{cases} 
\psi(x - b), & a = \infty; \\
-\psi(x - a), & b = \infty; \\
\psi ((x - b)(x - a)^{-1}), & \text{otherwise}.
\end{cases}
\]

(18)

Thus \( h^\psi = (h_{\infty 0}^\psi) \) is a Čech 1-cocycle of the covering \( L\mathbb{A} \). We claim that \( h_{ab}^\psi \) is cuspidal with respect to the constant 0. It follows from the definition that \( h^\psi \) is \( G \)-fixed, so we only need to show the cuspidal property of \( h_{\infty 0}^\psi = \psi \).
Indeed, \[ \lim_{\zeta \to \infty} \psi(x + \lambda) = \lim_{\lambda \to \infty} \psi(x/\lambda + 1) = 0. \]

Therefore \( h^\psi \in \mathfrak{g}_G^\circ \).

If we replace \( \psi \) by \( \varphi \in \text{Hom}^0(L\mathbb{C}^*, \mathbb{C}^*) \) in above, then the same procedure yields a multiplicative Čech 1-cocycle \( h^{*\varphi} = (g_{ab}^\varphi) \in (\mathfrak{g}_G^\circ)^G \), which is the same 1-cocycle as the one constructed in [10, (2.4)].

**Lemma 4.1.** Let \( \psi, \phi \) and \( \varphi \) be as in (17).

(a) If we consider elements of \( A \) as functions on \( \mathbb{C} \times L\mathbb{C} \) independent of the first variable, then \( A = \mathfrak{g}_G^\circ \) and \( \alpha(h^\psi) = \alpha^*(h^{*\varphi}) = \phi \).

(b) The map \( \psi \mapsto h^\psi \) (resp. \( \varphi \mapsto h^{*\varphi} \)) is an isomorphism \( \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \to \mathfrak{g}_G^\circ \) (resp. \( \text{Hom}^0(L\mathbb{C}^*, \mathbb{C}^*) \to (\mathfrak{g}_G^\circ)^G \)) of complex locally convex spaces (resp. complex Lie groups).

**Proof.** (a) It follows from the definition of \( \mathfrak{g} \) as in (2) and of the \( G \)-action on \( \mathfrak{g} \) as in (3) that \( A \subset \mathfrak{g}_G^G \). Let \( F \in \mathfrak{g}_G^G \). By considering the action of the Möbius transformations \( \zeta \mapsto \zeta + \lambda, \lambda \in \mathbb{C} \), and \( \zeta \mapsto \lambda \zeta, \lambda \in \mathbb{C}^* \), on \( F \), we can conclude that \( F \) is independent of the first variable and linear with respect to the second variable. The third property in the definition of \( \mathfrak{g} \) implies that \( F|_\mathbb{C} \equiv 0. \) Thus \( F \in A \) and \( \mathfrak{g}_G^G \subset A \).

Next we compute \( \alpha(h^\psi) \) by (11) and (18). Choose \( a = \infty \) in (11), and we only need to compute the derivative at \( \zeta = 0 \) (as \( \alpha(h^\psi) \in \mathfrak{g}_G^G \) is independent of \( \zeta \)). So

\[
\alpha(h^\psi)(-1/x) = \frac{d}{d\zeta} \bigg|_{\zeta=0} h^\psi_\infty(x) = \lim_{\zeta \to 0} \frac{\psi(x - \zeta) - \psi(x)}{\zeta}
= \lim_{\zeta \to 0} \frac{\psi(1 - \zeta/x)}{\zeta} = \phi(-1/x),
\]

where \( x \in LU_0 \), and the second limit above represents the derivative of \( \psi \) at \( 1 \in L\mathbb{C}^* \) in the direction of \(-1/x\). Thus \( \alpha(h^\psi) = \phi \). Note that \( \exp(h^\psi) = h^{*\varphi} \). It follows from the definition of \( \alpha^* \) that \( \alpha^*(h^{*\varphi}) = \phi \).

(b) As \( \psi = h^\psi_\infty \), the map \( \psi \mapsto h^\psi \) is clearly injective. Let \( h \in \mathfrak{g}_G^G \). By (a) there exists \( h^\psi \) such that \( \alpha(h^\psi) = \alpha(h) \in \mathfrak{g}_G^G \). Thus \( h - h^\psi = c(\text{ind}_{ab}) = h^{c \text{ind}_{\infty 0}} \), where \( c \in \mathbb{C} \) is a constant and \( \text{ind}_{\infty 0} \in \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \). So \( h = h^\psi + c \text{ind}_{\infty 0} \). Since \( h^\psi_{ab} = g_{ab}^\varphi \psi \) and \( \psi = h^\psi_\infty \), the isomorphism \( \psi \mapsto h^\psi \) is topological. The conclusion for the map \( \varphi \mapsto h^{*\varphi} \) follows from similar arguments.

It follows from the relation between \( \text{Hom}^0(L\mathbb{C}^*, \mathbb{C}^*) \) and \( (\mathfrak{g}_G^G)^G \) and Proposition 3.11 that the quotient topology of \( \text{Hom}^0(L\mathbb{C}^*, \mathbb{C}^*) \) inherited from \( \text{Hom}(L\mathbb{C}^*, \mathbb{C}) \) is the same as the compact-open topology.

**Proof of Theorem 1.2** Since each bundle \( \Lambda \in \text{Pic}(L\mathbb{P}_1) \) is a product of a \( LG \)-fixed bundle and a bundle in \( \text{Pic}^0(L\mathbb{P}_1) \), see [10, (2.5)], the conclusion \( \text{Pic}(L\mathbb{P}_1)^G = \text{Pic}(L\mathbb{P}_1)^{LG} \) follows from \( \text{Pic}^0(L\mathbb{P}_1)^G = \text{Pic}^0(L\mathbb{P}_1)^{LG} \). Let
\( \Lambda \in \text{Pic}^0(L\mathbb{P}_1)^G \). By Proposition 3.2 and Lemma 4.1(b) \( \Lambda \) is generated by a 1-cocycle of the form \( h^*\varphi, \varphi \in \text{Hom}^0(LC^*,C^*) \), the same one as in [10, (2.4)]. By [10, Propositions 2.2, 2.1] \( \Lambda \) is \( LG \)-fixed.

Let \( [f] \in H^{0,1}(L\mathbb{P}_1)^G \) and \( [f_0] \) a generator of \( \ker(E) \subset H^{0,1}(L\mathbb{P}_1)^{LG} \), see (1). Since \( E([f]) \) is \( LG \)-fixed, we have \( g[f] = [f] + m(g)[f_0] \), where \( g \in LG \), and \( m : LG \to \mathbb{Z} \) is a group homomorphism, constant on each component of \( LG \). Thus \( m \equiv 0 \) and \( [f] \in H^{0,1}(L\mathbb{P}_1)^{LG} \).

\[ \square \]

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