The rainbow connection number of enhanced power graph

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Abstract

Let $G$ be a finite group, the enhanced power graph of $G$, denoted by $\Gamma^e_G$, is the graph with vertex set $G$ and two vertices $x,y$ are edge connected in $\Gamma^e_G$ if there exist $z \in G$ such that $x,y \in \langle z \rangle$. Let $\zeta$ be a edge-coloring of $\Gamma^e_G$. In this article, we calculate the rainbow connection number of the enhanced power graph $\Gamma^e_G$.

Keywords: enhanced power graph; power graph; rainbow path; rainbow connection number.

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1 Introduction

Let $G$ be a finite group, the power graph of a finite group $G$ we denote the power graph by $\Gamma_G$, it is the graph whose vertex set are the elements of $G$ and two elements being adjacent if one is a power of the other. In [1] the authors found that the power graph is contained in the non-commuting graph and, they asked about how much the graphs are closer, and then, they defined the enhanced power graph of a finite group. We denoted to the enhanced power graph by $\Gamma^e_G$ whose vertex set is the group $G$ and two distinct vertices $x,y \in V(\Gamma^e_G)$ are adjacent if $x,y \in \langle z \rangle$ for some $z \in G$. Later, the enhanced power graph of a group was studied by Sudip Bera and A. K. Bhuniya [5].

In 2006, Chartrand, Johns, McKean and Zhang [8] introduced the concept of rainbow connection of graphs. This concept was motivated by communication of information between agencies of USA government after the September 11, 2001 terrorist attacks. The situation that helps to unravel this issue about communications has as graph-theoretic model the following. Let $\Gamma$ be a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We define a coloring $\zeta : E(\Gamma) \to \{1, \ldots, k\}$ with $k \in \mathbb{N}$. A path $P$ is a rainbow if any two edges of $P$ are colored distinct. If for each pair of vertices $u,v \in V(\Gamma)$, $\Gamma$ has a rainbow path from $u$ to $v$, then $\Gamma$ is

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rainbow-connected under the coloring $\zeta$, and $\zeta$ is called a rainbow $k$-coloring of $\Gamma$. The rainbow connection number of $\Gamma$, denoted by $rc(\Gamma)$ is the minimum $k$ for which there exists a rainbow $k$-coloring of $\Gamma$.

We will apply the idea of calculating the rainbow connected number of enhanced power graph through the graphs such that as was carried out by the authors from [10] about the power graph, with $InvMax_G$, the set of maximal involution of $G$, whose important theorems we can summarize in the following:

**Theorem 1.1.** Let $|InvMax_G| \neq \emptyset$ and $G$ be a finite group of order at least 3. Then

$$rc(\Gamma_G) = \begin{cases} 3, & \text{if } 1 \leq |InvMax_G| \leq 2; \\ |InvMax_G|, & \text{if } |InvMax_G| \geq 3. \end{cases}$$

If $|InvMax_G| = \emptyset$, let $G$ be a finite group

1. If $G$ is cyclic, then $rc(\Gamma_G) = \begin{cases} 1, & \text{if } |G| \text{ is a prime power}; \\ 2, & \text{otherwise} \end{cases}$
2. If $G$ es noncyclic, then $rc(\Gamma_G) = 2$ or 3.

In this paper we compute the rainbow connection number of $\Gamma^e_G$ and we characterize it in terms of independence cyclic set, whose particular case is maximal involution. This paper is organized as follows. In section 2 we put definitions and some properties about rainbow connection number and we describe a way for guarantee a coloring for enhanced power graphs. In section 3 we wrote the main theorems for determine $\Gamma^e_G$.

## 2 Definitions and properties

We start the section with a proposition from enhanced power graph definition.

**Proposition 2.1.** $rc(\Gamma^e_G) = 1$ if only if $\Gamma^e_G$ is complete if only if $G$ is cyclic.

**Definition 2.2.** Let $Max_G = \{x_1, \ldots, x_m\}$ be an essential cyclic set if

1. for all $g \in G$, $\langle g \rangle = \langle x_i \rangle$ for some $i$,
2. $\langle x_i \rangle \neq \langle x_j \rangle$ for $i \neq j$,
3. each $x_i$ is a maximal cyclic subgroup.

Therefore [2.1] can be rewritten as follows

**Proposition 2.3.** $|Max_G| = 1$ if only if $G$ is a cyclic group if only if $rc(\Gamma^e_G) = 1$

**Proposition 2.4.** If $|Max_G| = 2$, then $rc(\Gamma^e_G) = 2$

**Proof.** Since $\Gamma^e_G$ is not complete, we have $rc(\Gamma^e_G) > 2$, then we have
We can note that the only one path between \(x_1\) and \(x_2\), for all \(x_1, x_2 \in \langle x_1 \rangle \) and \(\langle x_2 \rangle \) is \((x_1, e, x_2)\), then the 2-coloring is given by \(\zeta : E(G) \rightarrow \{1, 2\}\) with \(f \mapsto i\), if \(f \in E_i\) is a rainbow 2-coloring of \(\Gamma^e_G\).

\[\begin{align*}
E_1 &= \{\{a, b\}| a, b \in \langle x_1 \rangle\} \\
E_2 &= \{\{a, b\}| a, b \in \langle x_2 \rangle\}
\end{align*}\]

We define the independence cyclic set of Max\(_G\), denoted by ics\(_G\), as 

\[\text{ics}(G) = \{x_i \in \text{Max}_G| \langle x_i \rangle \cap \langle x_j \rangle = e \text{ for } i \neq j\}\]

The independence cyclic number of Max\(_G\), denoted by icn\(_G\), is \(\text{icn}(G) = |\text{ics}(G)|\).

Remark 2.6. We note that 

\[\text{InMax}_G \subseteq \text{ics}(G) \subseteq \text{Max}_G\]

Proposition 2.7. If \(|\text{Max}_G| = 3\), then

\[\text{rc}(\Gamma^e_G) = \begin{cases} 2, & \text{if } \text{icn}(G) = 1 \\ 3, & \text{if } \text{icn}(G) = 3 \end{cases}\]

Proof. Let Max\(_G\) = \(\{x_1, x_2, x_3\}\) be an essential cyclic set.

Remark 2.8. We do not need to be concise with the path with both vertex in \(x_i\) for some \(i\), because with one color, we can coloring this path. The difficult is when both vertex are in different \(x_i\).

Case \(\text{cin}(G) = 1\) Without loss of generality we suppose \(\langle x_1 \rangle \cap \langle x_2 \rangle = e = \langle x_1 \rangle \cap \langle x_3 \rangle\) and \(\langle x_2 \rangle \cap \langle x_3 \rangle \neq e\). Since \(G\) is not cyclic group, then \(\text{rc}(\Gamma^e_G) \geq 2\). Let \(h \in \langle x_2 \rangle \cap \langle x_3 \rangle\) with \(h \neq e\), thus

\[\begin{align*}
E_1 &= \{\{a, b\}| a, b \in \langle x_1 \rangle\} \cup \{\{a, b\}| a, b \in \langle x_2 \rangle\text{ with } a, b \neq e\} \\
E_2 &= \{\{e, g\}| g \in \langle x_2 \rangle \cup \langle x_3 \rangle\} \cup \{\{a, b\}| a \in \langle x_3 \rangle \setminus \langle x_2 \rangle, \ b \in \langle x_2 \rangle \cap \langle x_3 \rangle, \ b \neq e\}
\end{align*}\]

In particular \(\{h, g\} \in E_2\) for all \(g \in \langle x_3 \rangle \setminus \langle x_2 \rangle\). Then, we will give a 2-coloring to \(\Gamma^e_G\):

\[
\zeta : E(G) \quad \begin{array}{c}
\rightarrow \{1, 2\} \\
\mapsto i & \text{if } i \in E_i
\end{array}
\]

(2.1)
Case $\text{cin}(G) = 3$ We suppose that $|\text{InMax}_G| = 0$, and without loss of generality $\langle x_i \rangle \cap \langle x_j \rangle = e$ for $1 \leq i < j \leq 3$. We will give a 3-coloring for $\Gamma^e_G$, with

\begin{align*}
E_1 &= \{ \{x_i, e\} | i = 1, 2, 3 \} \\
E_2 &= \{ \{e, x_{ij}\} | x_{ij} \in \bigcup_{i=1}^{3} \langle x_i \rangle \setminus \langle x_i \rangle \} \\
E_3 &= \{ \{a, b\} | a, b \in \langle x_i \rangle \text{ for } i = 1, 2, 3 \}
\end{align*}

With the coloring

$$
\zeta : E(G) \longrightarrow \{1, 2, 3\} \\
f \mapsto i \text{ if } i \in E_i
$$

(2.2)

Now, we suppose that $\text{InMax}_G = \text{Max}_G$, then with $E_i = \{ \{a, b\} | a, b \in \langle x_i \rangle \}$ be the edges set, and the coloring is given like \[2.2\]

We can not give a 2-coloring for $\Gamma^e_G$. We claim that there is a 2-coloring. Let $u \in \langle x_1 \rangle$, $v \in \langle x_2 \rangle$ and $w \in \langle x_3 \rangle$. Then, we have $\zeta(u, e) = 1$ and $\zeta(e, v) = 2$, thus $(u, e, v)$ is a desire rainbow path. Likewise $\zeta(u, e) = 1$ and $\zeta(e, w) = 2$, but for $(v, e, w)$ there is not a rainbow path. \qed

From \[2.7\] we can ask ourself about what happens whether no one of $\langle x_i \rangle$ can be intersected by another $\langle x_j \rangle$ with $i \neq j$ or, what happens if all $\langle x_i \rangle$ are intersected with some common elements. For this, we have the following prepositions.

The following preposition is just like [10, Proposition 2.4]

**Proposition 2.9.** Let $\text{Max}_G = \{x_1, ..., x_m\}$ be an essential cyclic set and $\text{InMax}_G = \text{Max}_G$. Then $\text{rc}(\Gamma^e_G) = m$.

**Proof.** For $\Gamma^e_G$ we will give a m-coloring. For each $i = 1, ..., m$ we have

$$E_i(G) = \{ \{a, b\} | a, b \in \langle x_i \rangle \}$$

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since for $u \in \langle x_i \rangle$ and $v \in \langle x_j \rangle$ with $i \neq j$ we have a only one path between them, which is $(u, e, v)$, and the coloring is given by

$$\zeta : E(G) \rightarrow \{1, \ldots, m\}$$

$$f \mapsto i \text{ if } f \in E_i$$

We can see the diagram in figure 1.

![Figure 1: InMax$_G = Max_G$](image)

**Proposition 2.10.** Let $Max_G = \{x_1, \ldots, x_m\}$ be an essential cyclic set with $m \geq 2$, and $h_{i,j} \in \langle x_i \rangle \cap \langle x_j \rangle$ for $1 \leq i < j \leq m$. If $h_{i,j} \neq h_{r,s}$, with $i \neq r$ or $j \neq s$, then $rc(\Gamma^e_G) = 2$.

**Proof.** By 2.8 we only give the coloring for $x_i$ and $x_j$ such that $i < j$. We fix

$$E_1(G) = \{\{a, h_{i,j}\} | a \in \langle x_i \rangle \setminus \langle x_j \rangle\} \cup \{\{a, b\} | a, b \in \langle x_i \rangle\}$$

$$E_2(G) = \{\{b, h_{i,j}\} | b \in \langle x_j \rangle \setminus \langle x_i \rangle\} \cup \{\{a, b\} | a, b \in \langle x_j \rangle\}$$

Then, we always have a path for $x_{is} \in \langle x_i \rangle$ to $x_{js} \in \langle x_j \rangle$ given by $(x_{is}, h_{i,j}, x_{js})$ with $i < j$, and the coloring is the same given in 2.1.

![Figure 2: Example for 2.10 for $m = 3$](image)

The next definition guarantees the existence of a coloring for $\Gamma^e_G$.

**Definition 2.11.** An *awning* is a collection $H_1, \ldots, H_{m-1}$ where the following occurs:

1. $H_i = A_i \cup B_i = \{h_{i,i+1}, \ldots, h_{i,m}\} \subset \langle X_i \rangle$ for $i = 1, \ldots, m - 1$
2. for all $i < j$, $h_{i,j} \in \langle x_i \rangle \cap \langle x_j \rangle$
3. for $i < j$ with $j = 2, ..., m - 1$, if $h_{j,s} = h_{i,r} \in H_j \cap H_i$ ($s \in \{j + 1, ..., m\}$, and $r \in \{i + 1, ..., m\}$), the following holds:

(a) $r = j$, $h_{i,r} \in A_i$, then $h_{j,s} \in B_j$
(b) $r = j$, $h_{i,r} \in B_i$, then $h_{j,s} \in A_j$
(c) $r = s > j$, $h_{i,r} \in A_i$, then $h_{j,s} \in A_j$
(d) $r = s > j$, $h_{i,r} \in B_i$, then $h_{j,s} \in B_j$

Remark 2.12. The case in 2.9 is a particular case where $G$ has not an awning. By definition of awning we want to say, if we have an awning, then we only need $H_i = \{e\}$ for only some $i$, and no more.

Corollary 2.13. If $G$ has an awning and $\vert Max_G \vert \geq 3$, then $icn(G) \leq 1$. In particular, $\vert InMax_G \vert \leq 1$.

Proof. Suppose that $icn(G) = 2$, then $H_{i_1} = H_{i_2} = \{e\}$. Hence $(x_{i_1}, e, x_{i_2})$ is a rainbow path, and $(x_{i_1}, e, x_{i_3})$ is another rainbow path, but in $(x_{i_2}, e, x_{i_3})$ we have not a rainbow path for $\Gamma e_G$. \hfill \Box

Corollary 2.14. If $\vert \cap H_i \vert \geq m - 1$ with $Max_G = m$, then $G$ has an awning.

Corollary 2.15. If $\vert Max_G \vert = 2$ then $icn(G) = 0$ or 2, and $G$ has an awning.

Corollary 2.16. If $G$ has an awning, then $icn(G) = 1$. In particular $\vert InMax_G \vert \leq 1$.

Remark 2.17. We note that the coloring whether we have to $InMax_G$ or $ics(G)$ does not change, both can be colored by only one color. The only one difference due to in $InMax_G$ there is only two elements in the subset of $G$ and, for a set taken of $ics(G)$ there are more than two elements but, the behaviour in coloring is exactly the same, because, in a set taken of $ics(G)$ all the elements are associated each them, then, one color is enough for coloring all set.

In the following properties we only consider the set $ics(G)$ unless otherwise indicated.

Proposition 2.18. If $G$ has an awning, then $rc(\Gamma _e^G) = 2$

Proof. We will give to $\Gamma _e^G$ a rainbow 2-coloring, for $1 \leq r < s \leq m$, let:

\[
\begin{align*}
E_{r,s}^1 &= \{ \{a, h_{r,s}\} \mid a \in \langle x_r \rangle \setminus \langle x_s \rangle; h_{r,s} \in A_r\} \\
E_{r,s}^2 &= \{ \{b, h_{r,s}\} \mid b \in \langle x_s \rangle \setminus \langle x_r \rangle; h_{r,s} \in B_r\} \\
E_{r,s}^1 &= \{ \{a, h_{r,s}\} \mid a \in \langle x_r \rangle \setminus \langle x_s \rangle; h_{r,s} \in A_r\} \\
E_{r,s}^2 &= \{ \{b, h_{r,s}\} \mid b \in \langle x_s \rangle \setminus \langle x_r \rangle; h_{r,s} \in B_r\}
\end{align*}
\]

Write $E_1 = \bigcup_{1 \leq r < s \leq m} E_{r,s}^1$ and $E_2 = \bigcup_{1 \leq r < s \leq m} E_{r,s}^2$ and we define a coloring

\[
\zeta : E(\Gamma _e^G) \longrightarrow \{1, 2\} \\
f \mapsto i, \text{ if } f \in E_i
\]
We go to check that, this is a 2-coloring for $\Gamma_G^e$. We will make a coloring for $j,s$-step. If this edges have been colored in a before step, i.e., if $h_{j,s} = h_{i,r}$ with $i < j$, thus we will have coloring problems with $r = j$ or $r = s$.

For $r = j$ ($r = s$), for (a)-(d) from 2.11 we can guarantee in before step we can conserve the coloring and that, not affect us with the 2-coloring that we gave.

**Proposition 2.19.** If $rc(\Gamma_G^e) = 2$, then for any order of $Max_G$, we have an awning.

**Proof.** We have $rc(\Gamma_G^e) = 2$ and suppose $E_1 \bigcup E_2 = E$ be the set of edges of $\Gamma_G^e$ and a 2-coloring give by 2.1 and let $Max_G = \{x_1, ..., x_m\}$ be an independence cyclic set of $\Gamma_G^e$, thus there is $h \in \langle x_i \rangle \cap \langle x_j \rangle$ such that $\{x_i, h\} \in E_1$ and $\{h, x_j\} \in E_2$ (or $\{x_i, h\} \in E_2$ and $\{h, x_j\} \in E_1$). We define $h_{i,j} := h$, moreover $H_i := \{h_{i,1}, ..., h_{i,m}\} =: A_i \bigcup B_i$ such that

$$A_i = \{h_{i,j}|\{x_i, h_{i,j}\} \in E_1\} \text{ and } B_i = \{h_{i,j}|\{x_i, h_{i,j}\} \in E_2\},$$

where (a) – (b) from 2.11 are met.

**Corollary 2.20.** If $G$ has an awning with any order on $Max_G$, then for every order, $G$ has an awning.

**Theorem 2.21.** $rc(\Gamma_G^e) = 2$ if only if $G$ has an awning and $G$ is not cyclic group.

**Proof.** By 2.18 and 2.19

By 2.6 we obtain a similar proposition like [10, Lemma 2.2].

**Lemma 2.22.** Let $Max_G = \{x_1, ..., x_m\}$ be an essential cyclic set. If $InMax_G \neq \emptyset$, then $|InMax_G| \leq rc(\Gamma_G^e)$.

**Proof.** As in the proof of [10, Lemma 2.2].

**Proposition 2.23.** Let $Max_G = \{x_1, ..., x_m\}$ be an essential cyclic set. If $ics(G) \geq 3$, then $3 \leq rc(\Gamma_G^e)$.

**Proof.** Suppose that $|InMax| = 0$ and $ics(G) = \{x_1, ..., x_k\}$ be an independence cyclic set with $k \geq 3$. We can not give a 2-coloring for the graph induced by $\langle x_1 \rangle \bigcup \cdots \bigcup \langle x_k \rangle$, but we will give a 3-coloring induced by the following edge sets

$$E_1 = \{\{x_i, e\}|i = 1, ..., m\}$$
$$E_2 = \{\{e, x_i\}|x_j \in \bigcup_{i=1}^{m}\langle x_i \rangle \setminus x_i\}$$
$$E_3 = \{\{a,b\}|a, b \in \langle x_i \rangle \text{ for each } i\}$$

with the rest edges just like 2.10 and 2.18. Thus the 3-coloring is given by 2.2.
If $|\text{InMax}_G| \geq 3$ then the edges set is

$E_1 = \{ \{x_i, e\} | i = l + 1, \ldots, m \}$

$E_2 = \{ \{e, x_i\} | x_i \in \bigcup_{i=l+1}^m \langle x_i \rangle \setminus x_i \}$

$E_3 = \{ \{a, b\} | a, b \in \langle x_i \rangle \text{ for each } i \}$

$E_i = \{ \{x_i, e\} | i = 1, \ldots, l \}$

and the coloring given by

$$\zeta : E(G) \longrightarrow \{1, \ldots, l\}$$

$$f \mapsto i \text{ if } f \in E_i$$

In particular we have the following

**Proposition 2.24.** Let $\text{Max}_G = \{x_1, \ldots, x_m\}$ be a essential cyclic group with $m \geq 4$ and $\text{icn}(G) \geq 2$, then $3 \leq \text{rc}(\Gamma_e^G)$.

**Remark 2.25.** We have $\text{rc}(\Gamma_e^G) \leq \text{rc}(\Gamma_G)$ because $E(\Gamma_G) \subseteq E(\Gamma_e^G)$.

## 3 Main theorems

In this section we prove our main theorems.

**Theorem 3.1.** Let $\text{Max}_G = \{x_1, \ldots, x_m\}$ be an essential cyclic set. If $\text{icn}(G) = 1$ then $\text{rc}(\Gamma_e^G) = 1$ if only if $m = 1$. In particular, if $|\text{InMax}_G| = 1$ then $\text{rc}(\Gamma_e^G) = 1$ if only if $G \cong \mathbb{Z}_2$.

**Proof.** By 2.1, 2.3 and 2.9
Theorem 3.2. Let $Max_G = \{x_1, \ldots, x_m\}$ be an essential cyclic set. If $icn(G) = 1$ then $rc(\Gamma_G^e) = 2$ if only if $G$ has an awning.

Proof. By 2.7, 2.16 and 2.18

Theorem 3.3. Let $Max_G = \{x_1, \ldots, x_m\}$ be a essential cyclic set with $m \geq 3$. If $icn(G) = 1$ then $rc(\Gamma_G^e) = 3$ if only if $G$ has not an awning.

Proof. By 2.23

Theorem 3.4. Let $Max_G = \{x_1, \ldots, x_m\}$ be a essential cyclic set with $m \geq 4$. If $icn(G) = 2$, then $rc(\Gamma_G^e) = 3$.

Proof. By 2.23 and 2.24

Theorem 3.5. Let $Max_G = \{x_1, \ldots, x_m\}$ be an essential cyclic set. If $icn(G) \geq 3$, then $rc(\Gamma_G^e) = |InMax_G|$.

Proof. By 2.23 and 2.9

Theorem 3.6. Let $Max_G = \{x_1, \ldots, x_m\}$ be an essential cyclic set with $icn(G) = 0$, then

\[
rc(\Gamma_G^e) = \begin{cases} 
1, & \text{if only if } G \text{ is a cyclic group} \\
2, & \text{if only if } G \text{ has an awning and } G \text{ is not cyclic.} \\
3, & \text{iff } G \text{ has not an awning.}
\end{cases}
\]

Proof. Case 1 By 2.3
Case 2 By 2.18
Case 3 By 2.12, 2.23

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