Homogenization of the heat equation with a vanishing volumetric heat capacity

Pernilla Johnsen, Tatiana Lobkova

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Abstract

This paper is devoted to the homogenization of the heat conduction equation, with a homogeneous Dirichlet boundary condition, having a periodically oscillating thermal conductivity and a vanishing volumetric heat capacity. A homogenization result is established by using the evolution settings of multiscale and very weak multiscale convergence. In particular, we investigate how the relation between the volumetric heat capacity and the microscopic structure effects the homogenized problem and its associated local problem. It turns out that the properties of the microscopic geometry of the problem give rise to certain special effects in the homogenization result.

1 Introduction

By means of periodic homogenization, we study the heat conduction equation with homogeneous Dirichlet boundary condition. Homogenization is a technique for mathematically investigating heterogeneous materials like e.g. composite materials and porous media. Thinking of the material contained in the domain as having periodically distributed heterogeneities where the period depends on a parameter \( \varepsilon \), we study the limit process as \( \varepsilon \) tends to zero.

We study the linear parabolic equation

\[
\varepsilon^q \partial_t u_\varepsilon(x,t) - \nabla \cdot \left( a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u_\varepsilon(x,t) \right) = f(x,t) \quad \text{in } \Omega_T,
\]

\[
u_\varepsilon(x,0) = u_0(x) \quad \text{in } \Omega,
\]

\[
u_\varepsilon(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T),
\]

where \( 0 < q < r \) are real numbers, \( f \in L^2(\Omega_T) \) and \( u_0 \in L^2(\Omega) \). We denote the domain by \( \Omega_T = \Omega \times (0,T) \), where \( \Omega \subset \mathbb{R}^N \) is open and bounded with smooth boundary \( \partial \Omega \) and \( (0,T) \subset \mathbb{R} \) is an open bounded interval. Here, the thermal conductivity is characterized by the function \( a \), which is periodic with respect to the unit cube \( Y \) in \( \mathbb{R}^N \) in its first variable and with respect to the unit interval \( S \) in \( \mathbb{R} \) in its second. The coefficient \( \varepsilon^q \) in front of the time derivative represents
the volumetric heat capacity. A more detailed description of the equation will be given in Section 3.

As \( \varepsilon \) tends to zero, we search for a weak limit \( u \) to the sequence of solutions \( \{u_\varepsilon\} \), where \( u \) is the solution to a so-called homogenized problem, which is characterized by a local problem. It turns out that equation (1) has two special features. The first one is that the homogenized problem is elliptic, for all \( 0 < q < r \), even though the original problem is parabolic. The second feature is that we have what we refer to as resonance, i.e. a parabolic local problem, for a different matching between the microscopic scales than the usual one. In [4] it was shown that parabolic equations usually have resonance if the temporal scale is the square of the spatial one. Several other studies of parabolic equations, both equations where the coefficient in front of the time derivative is identical to one and equations with oscillating coefficients, show resonance for the same type of matching, see e.g. [10], [15], [3], [7], [8], [19], [9] and [5]. As we will see in the homogenization result, equation (1) will have resonance if \( r = q + 2 \), i.e. the matching that gives a parabolic local problem is not when the temporal scale is equal to the spatial one.

In the homogenization procedure we use evolution settings of multiscale and very weak multiscale convergence. A gradient characterization and a very weak multiscale convergence compactness result for sequences bounded in \( W^{1,2}(0,T; H^1_0(\Omega), L^2(\Omega)) \), meaning that \( \{u_\varepsilon\} \) is bounded in \( L^2(0,T; H^1_0(\Omega)) \) and \( \{\partial_t u_\varepsilon\} \) is bounded in \( L^2(0,T; H^{-1}(\Omega)) \), can be found in e.g. [9]. Here, we use a different approach where the boundedness of the time derivative is replaced by a certain condition, see (2) in Theorem 5. This approach was, up to the authors’ knowledge, first used in [12] where compactness results for sequences defined on perforated domains were given. The corresponding compactness results for sequences defined on non-perforated domains, which we present in Theorem 5 and Theorem 8, are stated and proven in [11]. The present paper is a further development of the work in [11], where a homogenization result for equation (1) for the case when \( q = 1 \) and \( r = 3 \) was established.

**Notation 1** We use the notation \( Y_{n,m} = Y^n \times S^m \) with \( Y^n = Y_1 \times Y_2 \times \cdots \times Y_n \) and \( S^m = S_1 \times S_2 \times \cdots \times S_m \), where \( Y_1 = Y_2 = \cdots = Y_n = Y = (0,1)^N \) and \( S_1 = S_2 = \cdots = S_m = S = (0,1) \). Further, we let \( y^n = y_1, y_2, \ldots, y_n, dy^n = dy_1 dy_2 \cdots dy_n \), \( s^m = s_1, s_2, \ldots, s_m \) and \( ds^m = ds_1 ds_2 \cdots ds_m \). Moreover, we denote by \( W^{1,2}(0,T; H^1_0(\Omega), L^2(\Omega)) \) the space of all functions \( u \in L^2(0,T; H^1_0(\Omega)) \) such that \( \partial_t u \in L^2(0,T; H^{-1}(\Omega)) \). The subscript \( 2 \) is used on function spaces to denote periodicity of the functions involved over the domain in question. Lastly, for \( k = 1, \ldots, n \) and \( j = 1, \ldots, m \), the scale functions \( \varepsilon_k(\varepsilon) \) and \( \varepsilon_j'(\varepsilon) \) are strictly positive and tend to zero as \( \varepsilon \) does and we denote lists of spatial and temporal scales by \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) and \( \{\varepsilon'_1, \ldots, \varepsilon'_m\} \), respectively.
2 Preliminaries

Our main tools, evolution multiscale and very weak evolution multiscale convergence, are generalizations and modifications of the classical two-scale convergence. A sequence \( \{u_\varepsilon\} \) in \( L^2(\Omega) \) is said to two-scale converge to \( u_0 \in L^2(\Omega \times Y) \) if

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x) v \left( x, \frac{x}{\varepsilon} \right) dx = \int_{\Omega} \int_{Y} u_0(x,y) v(x,y) dy dx
\]

for all \( v \in L^2(\Omega; C^\#(Y)) \).

Two-scale convergence was introduced by Nguetseng in \[13\] and \[14\], where he uses the concept to homogenize a linear elliptic problem with one microscopic spatial scale. In \[1\], Allaire gives a compactness result for a different class of test functions and applies the concept to e.g. nonlinear elliptic problems and problems on perforated domains. The generalization of two-scale convergence to sequences with multiple microscopic scales in space was provided by Allaire and Briane in \[2\], where they give the definition and a compactness result for the concept. Following \[2\], we say that a sequence \( \{u_\varepsilon\} \) \((n+1)\)-scale converges to \( u_0 \in L^2(\Omega \times Y^n) \) if

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x) v \left( x, \frac{x}{\varepsilon_1}, \cdots, \frac{x}{\varepsilon_n} \right) dx = \int_{\Omega} \int_{Y^n} u_0(x,y^n) v(x,y^n) dy^n dx
\]

for all \( v \in L^2(\Omega; C^\#(Y^n)) \).

In \[16\] (see also the appendix of \[9\]), compactness results were given for an arbitrary number of scales in both space and time, extending the concept of multiscale convergence to an analogous evolution setting.

**Definition 2 (Evolution multiscale convergence)** A sequence \( \{u_\varepsilon\} \) in \( L^2(\Omega_T) \) is said to \((n+1,m+1)\)-scale converge to \( u_0 \in L^2(\Omega_T \times Y_{n,m}) \) if

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(x,t) v \left( x, t, \frac{x}{\varepsilon_1}, \cdots, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon_1}, \cdots, \frac{t}{\varepsilon_m} \right) dx dt = \int_{\Omega_T} \int_{Y_{n,m}} u_0(x,t,y^n,s^m) v(x,t,y^n,s^m) dy^n ds^m dx dt
\]

for all \( v \in L^2(\Omega_T; C^\#(Y_{n,m})) \). This is denoted by

\[
u_\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m).\]

Before we proceed with the compactness results we make some additional assumptions on the microscopic scales. The scales in a list are said to be separated if

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0
\]

and well-separated if there exists a positive integer \( \ell \) such that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_k} \left( \frac{\varepsilon_{k+1}}{\varepsilon_k} \right)^\ell = 0,
\]
where \( k = 1, \ldots, n - 1 \). When having two lists of microscopic scales, one spatial and one temporal, we have the following generalization of separatedness and well-separatedness, the concept of jointly (well-)separatedness. The definition was first given by Persson, see e.g. [18] where a more technically formulated version is given.

**Definition 3 (Jointly (well-)separated scales)** Let \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) and \( \{ \varepsilon'_1, \ldots, \varepsilon'_m \} \) be lists of (well-)separated scales. Collect all elements from both lists in one common list. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each such pair is removed and the list in order of magnitude of all the remaining elements is (well-)separated, the lists \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) and \( \{ \varepsilon'_1, \ldots, \varepsilon'_m \} \) are said to be jointly (well-)separated.

A compactness result for \((n + 1, m + 1)\)-scale convergence reads as follows.

**Theorem 4** Let \( \{ u_\varepsilon \} \) be a bounded sequence in \( L^2(\Omega_T) \) and suppose that the lists \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) and \( \{ \varepsilon'_1, \ldots, \varepsilon'_m \} \) are jointly separated. Then, up to a subsequence,

\[
u_\varepsilon (x, t) \overset{n+1,m+1}{\rightharpoonup} u_0 (x, t, y^n, s^m),
\]

where \( u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m}) \).

**Proof.** See Theorem A.1 in [9].

The following gradient characterization, which is adapted to our problem, will be important in the homogenization of (1).

**Theorem 5** Assume that \( \{ u_\varepsilon \} \) is bounded in \( L^2(0, T; H^1_0(\Omega)) \) and, for any \( v_1 \in D(\Omega), c_1 \in D(0, T), c_2 \in C^\infty_\#(S) \) and \( r > 0 \),

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon (x, t) v_1 (x) \partial_t \left( \varepsilon^r c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) dx dt = 0. 
\]

Then, for \( n = m = 1 \) with \( \varepsilon_1 = \varepsilon \) and \( \varepsilon'_1 = \varepsilon^r \), up to a subsequence,

\[
u_\varepsilon (x, t) \rightharpoonup u (x, t) \text{ in } L^2(0, T; H^1_0(\Omega))
\]

and

\[
\nabla u_\varepsilon (x, t) \overset{2,2}{\rightharpoonup} \nabla u (x, t) + \nabla_y u_1 (x, t, y, s),
\]

where \( u \in L^2(0, T; H^1_0(\Omega)) \) and \( u_1 \in L^2(\Omega_T \times S; H^1_2(Y)) \).

**Proof.** See Theorem 2.7 in [11].

Evolution multiscale convergence solely is not enough to handle certain sequences appearing in the homogenization of (1). Therefore, we introduce very weak evolution multiscale convergence. This type of convergence originates from [10], where it is used to obtain homogenization and corrector results for linear parabolic problems with one microscopic scale in space and time respectively. In [15], further progress in the context of \( \Sigma \)-convergence led to a closely related
result and a simplification for the applicability in the homogenization procedure. The present form of the concept was given for an arbitrary number of spatial scales in [6], where also the name “very weak multiscale convergence” was established. Following [17] and [9], we give the evolution version of very weak multiscale convergence including arbitrarily many spatial and temporal scales.

**Definition 6 (Very weak evolution multiscale convergence)** A sequence \( \{w_\varepsilon\} \) in \( L^1(\Omega_T) \) is said to (\( n+1, m+1 \))-scale converge very weakly to \( w_0 \in L^1(\Omega_T \times Y_{n,m}) \) if

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} w_\varepsilon(x,t) v_1 \left( \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_{n-1}} \right) v_2 \left( \frac{x}{\varepsilon_n}, c \left( \frac{t}{\varepsilon_1}, \ldots, \frac{t}{\varepsilon_m} \right) \right) dx dt \\
= \int_{\Omega_T} \int_{Y_{n,m}} w_0(x,t,y^n,s^m) v_1(x,y^{n-1}) v_2(y^n) c(t,s^m) dy^n ds^m dx dt
\]

for any \( v_1 \in D(\Omega; C_0^\infty(Y^{n-1})) \), \( v_2 \in C_0^\infty(Y_n)/\mathbb{R} \) and \( c \in D(0,T; C_0^\infty(S^m)) \), where

\[
\int_{Y_n} w_0(x,t,y^n,s^m) dy_n = 0. \tag{4}
\]

We write

\[
w_\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup}^{vw} w_0(x,t,y^n,s^m).
\]

**Remark 7** Due to (4) the limit is unique.

We now give a compactness result for very weak evolution multiscale convergence which will play a vital role, complementing Theorem 5 in the homogenization of [11]. Note that (5) in the theorem below is the same as (2) in Theorem 5.

**Theorem 8** Assume that \( \{u_\varepsilon\} \) is bounded in \( L^2(0,T; H_0^1(\Omega)) \) and, for any \( v_1 \in D(\Omega), c_1 \in D(0,T), c_2 \in C_0^\infty(S) \) and \( r > 0 \),

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(x,t) v_1(x) \partial_t \left( \varepsilon c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) dx dt = 0. \tag{5}
\]

Then, for \( n = m = 1 \) with \( \varepsilon_1 = \varepsilon \) and \( \varepsilon'_1 = \varepsilon^r \), up to a subsequence,

\[
\varepsilon^{-1} u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup}^{vw} u_1(x,t,y,s),
\]

where \( u_1 \in L^2(\Omega_T \times S; H_0^1(Y)/\mathbb{R}) \) is the same as in (3) in Theorem 5.

**Proof.** See Theorem 2.10 in [11].
3 Homogenization

Let us now investigate problem (1), i.e. establish a homogenization result for the equation

\[
\varepsilon^q \partial_t u_\varepsilon (x, t) - \nabla \cdot \left( a \left( \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^q} \right) \nabla u_\varepsilon (x, t) \right) = f(x, t) \text{ in } \Omega_T, \\
u_\varepsilon (x, 0) = u_0 (x) \text{ in } \Omega, \\
u_\varepsilon (x, t) = 0 \text{ on } \partial \Omega \times (0, T),
\]

where \(0 < q < r\), \(f \in L^2(\Omega_T)\) and \(u_0 \in L^2(\Omega)\). The coefficient \(a \in C^1(\mathcal{Y}_{1,1})^{N \times N}\) satisfies the coercivity condition

\[
a(y, s) \xi \cdot \xi \geq C_0 |\xi|^2
\]

for a.e. \((y, s) \in \mathcal{Y}_{1,1}\), for every \(\xi \in \mathbb{R}^N\) and for some \(C_0 > 0\). According to Section 23.7 in [20] the problem possesses a unique solution. The weak form of (6) is

\[
\int_{\Omega_T} -\varepsilon^q u_\varepsilon (x, t) v (x) \partial_t c (t) + a \left( \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^q} \right) \nabla u_\varepsilon (x, t) \cdot \nabla v (x) c(t) \, dx \, dt
\]

\[
= \int_{\Omega_T} f (x, t) v (x) c(t) \, dx \, dt,
\]

for all \(v \in H^1_0(\Omega)\) and \(c \in D(0, T)\).

We will now show that the solution to (6) is bounded in \(L^2(0, T; H^1_0(\Omega))\), i.e. it satisfies the a priori estimate

\[
\|u_\varepsilon\|_{L^2(0, T; H^1_0(\Omega))} \leq C,
\]

where \(C > 0\) is a constant independent of \(\varepsilon\). By Section 30.3 in [21], using \(u_\varepsilon \in W^{1,2}(0, T; H^1_0(\Omega), L^2(\Omega))\) as a test function, the operator form of (6) is

\[
\int_0^T \varepsilon^q (\partial_t u_\varepsilon, u_\varepsilon)_{H^{-1}(\Omega), H^1_0(\Omega)} \, dt + \int_{\Omega_T} a \left( \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^q} \right) \nabla u_\varepsilon (x, t) \cdot \nabla u_\varepsilon (x, t) \, dx \, dt
\]

\[
= \int_{\Omega_T} f (x, t) u_\varepsilon (x, t) \, dx \, dt.
\]

Multiplying by 2 and using the integration by parts formula (25) from Section 23.6 in [20] we obtain

\[
\int_\Omega \varepsilon^q \left((u_\varepsilon (x, T))^2 - (u_0 (x))^2\right) \, dx + 2 \int_{\Omega_T} a \left( \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^q} \right) \nabla u_\varepsilon (x, t) \cdot \nabla u_\varepsilon (x, t) \, dx \, dt
\]

\[
= 2 \int_{\Omega_T} f (x, t) u_\varepsilon (x, t) \, dx \, dt
\]
or, equivalently,
\[
\varepsilon^q \|u_\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 + 2 \int_{\Omega_T} a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u_\varepsilon(x, t) \cdot \nabla u_\varepsilon(x, t) \, dx dt
\]
\[
= \varepsilon^q \|u_0\|_{L^2(\Omega)}^2 + 2 \int_{\Omega_T} f(x, t) u_\varepsilon(x, t) \, dx dt.
\]

The coercivity condition (7) states that
\[
2 \int_{\Omega_T} a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u_\varepsilon(x, t) \cdot \nabla u_\varepsilon(x, t) \, dx dt
\]
\[
\geq 2C_0 \int_{\Omega_T} |\nabla u_\varepsilon|^2 \, dx dt = 2C_0 \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}^2 ,
\]
which gives us
\[
\varepsilon^q \|u_\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 + 2C_0 \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}^2
\]
\[
\leq \varepsilon^q \|u_0\|_{L^2(\Omega)}^2 + 2 \int_{\Omega_T} f(x, t) u_\varepsilon(x, t) \, dx dt.
\]

Using the Poincaré inequality
\[
\|u_\varepsilon\|_{L^2(\Omega)} \leq C_2 \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))} ,
\]
where \(C_2 > 0\) depends only on \(\Omega\), and the elementary inequality
\[
2xy \leq C_1 x^2 + C_1^{-1} y^2 ,
\]
with \(C_1 = C_0^{-1}C_2\), we have
\[
2 \int_{\Omega_T} f(x, t) u_\varepsilon(x, t) \, dx dt \leq C_0^{-1}C_2 \|f\|_{L^2(\Omega_T)}^2 + (C_0^{-1}C_2)^{-1} \|u_\varepsilon\|_{L^2(\Omega_T)}^2
\]
\[
\leq C_0^{-1}C_2 \|f\|_{L^2(\Omega_T)}^2 + C_0 C_2^{-1}C_2 \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}^2
\]
\[
= C_0^{-1}C_2 \|f\|_{L^2(\Omega_T)}^2 + C_0 \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}^2 .
\]

Now (10) becomes
\[
\varepsilon^q \|u_\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 + 2C_0 \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}^2
\]
\[
\leq \varepsilon^q \|u_0\|_{L^2(\Omega)}^2 + C_0^{-1}C_2 \|f\|_{L^2(\Omega_T)}^2 + C_0 \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}^2
\]
or, rewriting,
\[
\|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}^2 \leq \varepsilon^q C_0^{-1} \|u_0\|_{L^2(\Omega)}^2 + C_0^{-2}C_2 \|f\|_{L^2(\Omega_T)}^2 - \varepsilon^q C_0^{-1} \|u_\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 .
\]

Noting that
\[
\varepsilon^q C_0^{-1} \|u_\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 \geq 0
\]
we arrive at
\[ \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}^2 \leq \varepsilon^q C_0^{-1} \|u_0\|_{L^2(\Omega)}^2 + C_0^{-2} C_2 \|f\|_{L^2(\Omega_T)}^2, \]
which implies (9), i.e. we have shown that \( \{u_\varepsilon\} \) is bounded in \( L^2(0,T;H^1_0(\Omega)) \).

Before we are ready to give the homogenization result we prove that the assumption used in Theorems 5 and 8 is satisfied, i.e. that
\[ \|c\|_{L^2(\Omega)} \leq \varepsilon r \]
where \( 0 \leq r \) is bounded in \( L^2(\Omega) \) and by rearranging we obtain
\[ \lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(x,t) v_1(x) \partial_t \left( \varepsilon^r c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) \, dx \, dt = 0 \]  
for \( v_1 \in D(\Omega) \), \( c_1 \in D(0,T) \) and \( c_2 \in C^\infty(S) \) and \( r > 0 \). By using the weak form (8) with the test function
\[ v(x) c(t) = \varepsilon^{r-q} v_1(x) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right), \]
where \( 0 < q < r \), \( v_1 \in D(\Omega) \), \( c_1 \in D(0,T) \) and \( c_2 \in C^\infty(S) \), we get
\[
\begin{align*}
\int_{\Omega_T} & -\varepsilon^q u_\varepsilon(x,t) \varepsilon^{r-q} v_1(x) \partial_t \left( c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) \, dx \, dt \\
+ & \int_{\Omega_T} a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u_\varepsilon(x,t) \cdot \varepsilon^{r-q} \nabla v_1(x) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dx \, dt \\
= & \int_{\Omega_T} f(x,t) \varepsilon^{r-q} v_1(x) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dx \, dt
\end{align*}
\]
and by rearranging we obtain
\[
\begin{align*}
\int_{\Omega_T} u_\varepsilon(x,t) v_1(x) \partial_t \left( \varepsilon^r c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) \, dx \, dt \\
= & \int_{\Omega_T} \varepsilon^{r-q} a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u_\varepsilon(x,t) \cdot \nabla v_1(x) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dx \, dt \\
- & \int_{\Omega_T} \varepsilon^{r-q} f(x,t) v_1(x) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dx \, dt.
\end{align*}
\]
From (9) we know that \( \{u_\varepsilon\} \) is bounded in \( L^2(0,T;H^1_0(\Omega)) \) and therefore \( \{\nabla u_\varepsilon\} \) is bounded in \( L^2(\Omega_T)^N \) and we have
\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_{\Omega_T} & u_\varepsilon(x,t) v_1(x) \partial_t \left( \varepsilon^r c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) \, dx \, dt \\
= & \lim_{\varepsilon \to 0} \left( \int_{\Omega_T} \varepsilon^{r-q} a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u_\varepsilon(x,t) \cdot \nabla v_1(x) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dx \, dt \\
- & \int_{\Omega_T} \varepsilon^{r-q} f(x,t) v_1(x) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dx \, dt \right) = 0,
\end{align*}
\]}

meaning that (11) is fulfilled.

Finally, we give the homogenization result.
Theorem 9 Let \( \{u_\varepsilon\} \) be a sequence of solutions to (\text{10}) in \( W^{1,2}(0,T; H^1_0(\Omega), L^2(\Omega)) \). Then it holds that

\[
u_\varepsilon(x,t) \rightharpoonup u(x,t) \text{ in } L^2(0,T; H^1_0(\Omega)) \quad (12)
\]

and

\[
\nabla u_\varepsilon(x,t)^{2,2} \rightharpoonup \nabla u(x,t) + \nabla_y u_1(x,t,y,s), \quad (13)
\]

where \( u \in L^2(0,T; H^1_0(\Omega)) \) and \( u_1 \in L^2(\Omega_T \times S; H^1_1(Y)/\mathbb{R}) \). Here, \( u \) is the unique solution to the homogenized problem

\[
- \nabla \cdot (b \nabla u(x,t)) = f(x,t) \text{ in } \Omega_T, \quad (14)
\]

\[
u(x,t) = 0 \text{ on } \partial \Omega \times (0,T)
\]

with

\[
 b \nabla u(x,t) = \int_{Y \backslash Y_1} a(y,s) (\nabla u(x,t) + \nabla_y u_1(x,t,y,s)) \, dy \, ds. \quad (15)
\]

For \( q < r < q+2 \), \( u_1 \) is determined by the elliptic local problem

\[
- \nabla_y \cdot (a(y,s) (\nabla u(x,t) + \nabla_y u_1(x,t,y,s))) = 0 \quad (16)
\]

and for \( r = q+2 \) by the parabolic local problem

\[
\partial_s u_1(x,t,y,s) - \nabla_y \cdot (a(y,s) (\nabla u(x,t) + \nabla_y u_1(x,t,y,s))) = 0. \quad (17)
\]

For \( r > q+2 \), \( u_1 \) is determined by the elliptic local problem

\[
- \nabla_y \cdot \left( \left( \int_S a(y,s) \, ds \right) (\nabla u(x,t) + \nabla_y u_1(x,t,y)) \right) = 0 \quad (18)
\]

and since \( u_1 \) is independent of \( s \), the coefficient \( (15) \) can, in this case, be expressed as

\[
b \nabla u(x,t) = \int_Y \left( \int_S a(y,s) \, ds \right) (\nabla u(x,t) + \nabla_y u_1(x,t,y)) \, dy.
\]

Proof. Since (\text{10}) and (\text{11}) are satisfied the convergences (\text{12}) and (\text{13}) holds, according to Theorem 5. To obtain the homogenized problem we choose, in the weak form (\text{8}), the test function

\[
v(x) c(t) = v_1(x) c_1(t),
\]

where \( v_1 \in H^1_0(\Omega) \) and \( c_1 \in D(0,T) \), giving

\[
\int_{\Omega_T} -\varepsilon^q u_\varepsilon(x,t) v_1(x) \partial_t c_1(t) + a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^q} \right) \nabla u_\varepsilon(x,t) \cdot \nabla v_1(x) c_1(t) \, dx dt
\]

\[
= \int_{\Omega_T} f(x,t) v_1(x) c_1(t) \, dx dt.
\]
Letting $\varepsilon$ tend to zero we have
\[
\int_{\Omega_T} \int_{Y_{1,1}} a (y, s) (\nabla u (x, t) + \nabla_y u_1 (x, t, y, s)) \cdot \nabla v_1 (x) c_1 (t) \, dydsdxdt = \int_{\Omega_T} f (x, t) v_1 (x) c_1 (t) \, dxdt,
\]
and by the Variational lemma we arrive at
\[
\int_{\Omega} \int_{Y_{1,1}} a (y, s) (\nabla u (x, t) + \nabla_y u_1 (x, t, y, s)) \cdot \nabla v_1 (x) \, dydsdx = \int_{\Omega} f (x, t) v_1 (x) \, dx
\]
a.e. in $(0, T)$, which is the weak form of (14).

Now we continue by finding the local problem for each of the three cases.

Case 1: $0 < q < r < q + 2$. In (12) we choose the test function
\[
v (x) c (t) = \varepsilon v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right)
\]
where $v_1 \in D(\Omega)$, $v_2 \in C_\infty^2 (Y)/\mathbb{R}$, $c_1 \in D (0, T)$ and $c_2 \in C_\infty (S)$ and obtain, after differentiations
\[
\int_{\Omega_T} -\varepsilon^{q+1} u_\varepsilon (x, t) v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \partial_x c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt
\]
\[
- \int_{\Omega_T} \varepsilon^{q+1-r} u_\varepsilon (x, t) v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \partial_x c_1 (t) \partial_x c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt
\]
\[
+ \int_{\Omega_T} \varepsilon a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u_\varepsilon (x, t) \cdot \nabla v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt
\]
\[
+ \int_{\Omega_T} a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u_\varepsilon (x, t) \cdot \nabla_y v_2 \left( \frac{x}{\varepsilon} \right) \partial_x c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt
\]
\[
= \int_{\Omega_T} \varepsilon f (x, t) v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt.
\]
Letting $\varepsilon \to 0$, omitting terms that equal zero, we obtain
\[
\lim_{\varepsilon \to 0} \left( \int_{\Omega_T} -\varepsilon^{q+1-r} u_\varepsilon (x, t) v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \partial_x c_1 (t) \partial_x c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt \right) = 0.
\]
Using the fact that $r < q + 2$ and by observing that $\varepsilon^{q+1-r} = \varepsilon^{q+2-r} \cdot \varepsilon^{-1}$ the first term vanishes due to Theorem 8 and then Theorem 9 gives
\[
\int_{\Omega_T} \int_{Y_{1,1}} a (y, s) (\nabla u (x, t) + \nabla_y u_1 (x, t, y, s))
\times v_1 (x) \cdot \nabla_y v_2 (y) c_1 (t) c_2 (s) \, dydsdxdt = 0.
\]
The Variational lemma yields

$$\int_Y a(y, s)(\nabla u(x, t) + \nabla_y u_1(x, t, y, s)) \cdot \nabla_y v_2(y) \, dy = 0$$

a.e. in $\Omega_T \times S$ which is the weak form of (16).

Case 2: $r = q + 2$. Using the same test functions as in case 1 we arrive at (19). According to Theorems 5 and 8, since $r = q + 2$, we have

$$\int_{\Omega_T} \int_{Y_{1,1}} -u_1(x, t, y, s)v_1(x)v_2(y)c_1(t)c_2(s)dydsdxdt$$

$$+ \int_{\Omega_T} \int_{Y_{1,1}} a(y, s)(\nabla u(x, t) + \nabla_y u_1(x, t, y, s)) \cdot v_1(x) \cdot \nabla_y v_2(y)c_1(t)c_2(s)dyds = 0.$$

By applying the Variational lemma we get

$$\int_{Y_{1,1}} -u_1(x, t, y, s)v_2(y)\partial_sc_2(s)dyds$$

$$+ \int_{Y_{1,1}} a(y, s)(\nabla u(x, t) + \nabla_y u_1(x, t, y, s)) \cdot \nabla_y v_2(y)c_2(s)dyds = 0$$

a.e. in $\Omega_T$, which is the weak form of (17).

Case 3: $r > q + 2$. Before deriving the local problem for this case we establish the independence of $s$. By choosing the test function

$$v(x) = \varepsilon^{r-q-1}v_1(x)v_2\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right),$$

where $v_1 \in D(\Omega)$, $v_2 \in C^\infty_c(Y) \cap \mathbb{R}$, $c_1 \in D(0, T)$ and $c_2 \in C^\infty_c(S)$, the weak form, after differentiation, becomes

$$\int_{\Omega_T} -\varepsilon^{-1}u_1(x, t)v_1(x)v_2\left(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right)\partial_tc_1(t)c_2\left(t, \frac{t}{\varepsilon^r}\right) \, dxdt$$

$$- \int_{\Omega_T} \varepsilon^{-1}u_1(x, t)v_1(x)v_2\left(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right)c_1(t)\partial_sc_2\left(t, \frac{t}{\varepsilon^r}\right) \, dxdt$$

$$+ \int_{\Omega_T} \varepsilon^{-q-1}a\left(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right)\nabla u_1(x, t) \cdot \nabla v_1(x)v_2\left(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right) c_1(t)c_2\left(t, \frac{t}{\varepsilon^r}\right) \, dxdt$$

$$+ \int_{\Omega_T} \varepsilon^{-q-2}a\left(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right)\nabla u_1(x, t) \cdot \nabla v_2\left(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right) c_1(t)c_2\left(t, \frac{t}{\varepsilon^r}\right) \, dxdt$$

$$= \int_{\Omega_T} \varepsilon^{-q-1}f(x, t)v_1(x)v_2\left(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right) c_1(t)c_2\left(t, \frac{t}{\varepsilon^r}\right) \, dxdt.$$

Since $r > q + 2$ all terms but the second one vanish as $\varepsilon \to 0$. Due to Theorem 8 we have

$$\int_{\Omega_T} \int_{Y_{1,1}} -u_1(x, t, y, s)v_1(x)v_2(y)c_1(t)\partial_sc_2(s) \, dydsdxdt = 0.$$
and applying the Variational lemma we get
\[
\int _{S} -u _{1} (x, t, y, s) \partial _{s} c _{2} (s) \, ds = 0
\]
a.e. in \( \Omega _{T} \times Y \), which implies that \( u _{1} \) is independent of \( s \). Now, to find the local problem, we choose the test function
\[
v (x) c (t) = \varepsilon v _{1} (x) v _{2} \left( \frac{x}{\varepsilon} \right) c _{1} (t),
\]
where \( v _{1} \in D (\Omega) \), \( v _{2} \in C _{0} ^{\infty} (Y) / \mathbb{R} \) and \( c _{1} \in D (0, T) \). Carrying out differentiations, the weak form \( (8) \) becomes
\[
\int _{\Omega _{T}} -\varepsilon ^{q+1} u _{\varepsilon} (x, t) v _{1} (x) v _{2} \left( \frac{x}{\varepsilon} \right) \partial _{t} c _{1} (t) \, dx \, dt \\
+ \int _{\Omega _{T}} \varepsilon a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon ^{q}} \right) \nabla u _{\varepsilon} (x, t) \cdot \nabla v _{1} (x) v _{2} \left( \frac{x}{\varepsilon} \right) c _{1} (t) \, dx \, dt \\
+ \int _{\Omega _{T}} a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon ^{q}} \right) \nabla u _{\varepsilon} (x, t) v _{1} (x) \cdot \nabla _{y} v _{2} \left( \frac{x}{\varepsilon} \right) c _{1} (t) \, dx \, dt \\
= \int _{\Omega _{T}} \varepsilon f (x, t) v _{1} (x) v _{2} \left( \frac{x}{\varepsilon} \right) c _{1} (t) \, dx \, dt.
\]

Theorem \( 5 \) and the fact that \( u _{1} \) is independent of \( s \) gives
\[
\int _{\Omega _{T}} \int _{Y _{1,1}} a \left( y, s \right) \left( \nabla u (x, t) + \nabla _{y} u _{1} (x, t, y) \right) v _{1} (x) \cdot \nabla _{y} v _{2} (y) c _{1} (t) \, dy \, ds \, dx \, dt = 0
\]
and by the Variational lemma we have
\[
\int _{Y} \left( \int _{S} a \left( y, s \right) ds \right) \left( \nabla u (x, t) + \nabla _{y} u _{1} (x, t, y) \right) \cdot \nabla _{y} v _{2} (y) \, dy = 0
\]
a.e. in \( \Omega _{T} \), which is the weak form of \( (18) \). 

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