Non-extensive Statistical Mechanics and Black Hole Entropy From Quantum Geometry

Abhishek Majhi*
Instituto de Ciencias Nucleares
Universidad Nacional Autonoma de Mexico
A. Postal 70-543, Mexico D.F. 04510, Mexico

We apply non-extensive statistical mechanics, characterized by a free parameter $q$, to calculate black hole entropy from quantum geometry. For a given horizon area, the entropy of the black hole is given by the Bekenstein-Hawking area law for arbitrary real positive values of the Barbero-Immirzi parameter ($\gamma$). In the process, we find a specific correlation between $\gamma$ and $q$ that explains how the Chern-Simons gauge fields on the horizon is coupled to the bulk geometry exterior to the horizon. Precisely, it comes out to be such that the microstates of the horizon become more biased away from occurring with equal probability with the increasing strength of the coupling. This leads to a physical interpretation of the $q$-parameter in the context of quantum gravity. In passing, we deduce the non-additive entropy for a system of $N$ number of spins with arbitrary spin quantum numbers which can have applications in other fields related to quantum physics than black holes.

PACS numbers:

Failure of the application of extensive statistical mechanics in the context of gravitating systems due to the presence of long range interactions was pointed out by none other than J. W. Gibbs in his textbook back in 1902[1]. This issue has been repeatedly addressed in the literature and investigated upon in several instances e.g. [2–4](for a detailed history of literature in this context see Chapter 1 of [8]). One possible way to generalize the concept of entropy for systems with long range interactions and strongly coupled systems was proposed by Tsallis in 1988 which led to the following form of entropy(henceforth to be addressed as $q$-entropy)[5]¹:

$$S_q = \frac{(1 - \sum_{i=1}^{\Omega} p_i^q)}{(1 - q)}$$

(1)

where $\sum_{i=1}^{\Omega} p_i = 1$, $\Omega$ being the total number of microstates of the system under consideration and $p_i$ is the probability of the $i$-th microstate. The parameter $q$ is real. The Boltzmann constant that should be appeared as a multiplicative factor has been set to unity. The associated branch of statistical mechanics based on eq.(1) is known as non-extensive statistical mechanics[8]². In the limit $q \to 1$ one recovers the well known formula for the Shannon entropy

$$S_1 = -\sum_{i=1}^{\Omega} p_i \ln p_i$$

(2)

For $q \neq 1$, eq.(1) is non-additive. Physically, it can be thought of as an introduction of a bias in the microstates due to the presence of the long range interaction or strong coupling in the system, thus deviating from the postulate of equal a priori probability of the microstates[8].

Non-extensive statistical mechanics has been proved to be successfully applied in situations where extensive statistical mechanics fails to provide satisfactory answers[7]. Notably, in one instance it has been shown that non-extensive statistical mechanics reconciles the area law in quantum systems with classical thermodynamics[22]. As far as our knowledge is concerned, in spite of a plethora of examples of theoretical applications and experimental verifications of non-extensive statistical mechanics[7], it has not been seriously considered in case of black hole entropy calculation, except for one instance where the authors did not use any well defined microstates of black holes explicitly[6].

The works of Bekenstein[10] and Hawking[11] famously led to the result that the entropy of a black hole($S$) is given by $S = A/4\ell_p^2$ where $A$ is classical horizon area and $\ell_p$ is the Planck length. Although the physical arguments of Bekenstein in [10] began with eq.(2), the entropy was not really computed due to the lack of knowledge of the microstates of the black hole. Now, since black hole is certainly a prime example of an object with long range interaction, combined with the non-extensive nature of the black hole entropy, it is a natural question to ask whether we shall begin with eq.(1) instead of eq.(2) if we get to compute the entropy of a black hole from its microstates.

At present day, non-perturbative canonical quantum gravity provides us with an estimate of the microstates of black holes. Based on this estimate one can precisely compute the entropy of a black hole starting from eq.(2)[12, 13]. However, these calcu-

¹ Electronic address: abhishek.majhi@gmail.com
² The nomenclature ‘non-extensive’ is slightly misleading(see page 44 of [8]). However, we use it here as this branch of statistical mechanics is well known by this name.
lations only yield $S = (\lambda_0/\gamma)A/\ell_p^2$, where $\gamma$ is a free parameter of the theory, known as the Barbero-Immirzi parameter\[14-16] and $\lambda_0$ is a numerical constant resulting from the statistical mechanical exercise. Then, demanding that $S$ be given by $A/4\ell_p^2$, the value of $\gamma$ is fixed to the suitable value. This procedure of choosing $\gamma$ by hand just for the sake of obtaining the semi-classical area law is considered as a drawback of this approach of black hole entropy calculation\[17]. It is suggested that one should obtain the $S = A/4\ell_p^2$ for arbitrary $\gamma$. So, as it stands, we face a problem if we begin with eq.(2). Hence, we have a justified reason to begin with eq.(1). However, there lies a deeper physical motivation to choose this different path.

The theory of quantum geometry of black holes tells us that the microstates belong to the Hilbert space of an $SU(2)$ Chern-Simons(CS) theory on the horizon, which is coupled to the gravitational field of the exterior bulk\[12, 13]. For a black hole with given horizon area, $\gamma$ controls the strength with which the bulk fields couple to the CS theory on the horizon. Being purely gravitational, this is a system with long range interaction whose strength is dictated by $\gamma$ which is a free parameter of the theory. On the other hand, $q$ is a free parameter of non-extensive statistical mechanics signifying the degree of deviation of the microstates from being equally probable due to the presence of some long range interaction. Thus, one can immediately foresee some connection between $q$ and $\gamma$ on the physical ground that the bulk fields affect the horizon to create a bias in the microstates. But, how will those two parameters be related? The answer is: the coupling between the bulk and the horizon must be such that the $q$-entropy of the black hole be given by the BH area law. This will lead to the precise correlation between $q$ and $\gamma$, which may allow arbitrary real positive values of $\gamma$.

Technically, it is quite trivial to see this coming. If one begins with eq.(1) and calculates the entropy from quantum geometry, then one is expected to arrive at something like $S_q = f(\gamma, q, A)$, where $f$ is some function of $(\gamma, q, A)$. Consequently, demanding that $f(\gamma, q, A) = A/4$ one can have the precise relation between $\gamma$ and $q$ for every given $A$. As we shall see shortly that this is indeed the case and most importantly $\gamma$ can have arbitrary real positive values depending on $q$, for any given $A$. Remarkably, as we shall see later, the variation of $\gamma$ with $q$ will come out to be such that the bias in the microstates increases monotonically with the strength of the coupling of the bulk with the horizon i.e. as the bulk gets more strongly coupled to the horizon, the horizon microstates get more biased against being equally probable. This will provide us with a two-fold interdisciplinary gain. Firstly, this is one possible way to avoid the $\gamma$-tuning problem. Secondly, the parameter $q$ gains a physical interpretation in the context of quantum gravity. Considering all these, it is quite tempting to begin with eq.(1).

At first, we deduce the $q$-entropy for a system of $N$ number of spins with spin quantum numbers $(j_1, \cdots, j_N)$, which we shall apply to black holes. The $q$-entropy for a system of two statistically independent spin systems $j_1$ and $j_2$ is given by\[5\]

$$S_q(j_1, j_2) = S_q(j_1) + S_q(j_2) + (1 - q)S_q(j_1)S_q(j_2) \quad (3)$$

Now, one can add a third spin with value $j_3$ and compute out the formula for $S_q(j_1, j_2, j_3)$ using eq.(3) and consequently for more spins. It is straightforward to generalize this formula for a system of $N$ number of spins with values $(j_1, \cdots, j_N)$ by the method of mathematical induction to arrive at the following result:

$$S_q(j_1, \cdots, j_N) = \frac{\sum_{n=1}^{N} (1-q)^{n-1}}{n!} \sum_{i_1, \cdots, i_n} \prod_{i=1}^{n} S_q(j_{i_i}) \quad (4)$$

Although we are going to apply the formula for calculating black hole entropy, it can have potential applications in generic systems composed of arbitrary spin carrying particles. As far as our knowledge is concerned, eq.(4) has not appeared in the literature earlier. To verify with an already known formula one can check that if we have $j_1 = \cdots = j_N = s$ (say), then eq.(4) reduces to the following form:

$$S_q(s) := S_q(N \text{number of spin } s) = \frac{[(1 + 2s)(1-q)^N - 1]}{(1 - q)} \quad (5)$$

which was derived in \[9\].

Before applying the $q$-entropy to black holes, let us briefly discuss the essential structures of the quantum geometry of black holes\[12, 13\]. The quantum geometry of a cross-section of a black hole horizon is described by a two-sphere with marked points, usually called punctures, carrying ‘spin’ quantum numbers endowed by the edges of the spin network that span the bulk quantum geometry\[12\]. Quantum area of the black hole with spin quantum numbers $j_1, \cdots, j_N$ on $N$ punctures is given by $A_{qu} = 8\pi\gamma \sum_{i=1}^{N} \sqrt{j_i(j_i + 1)}$ in the units of Planck area which has been set to unity here. The Hilbert space associated with the black hole horizon is that of a quantized $SU(2)$ Chern-Simons(CS) theory, with level $k := A/4\pi\gamma$, coupled to the punctures\[12, 13, 18\]:

$$\mathcal{H}^k = \bigoplus_{p} \text{Inv} \left( \bigotimes_{i=1}^{N} \mathcal{H}_{j_i} \right) \quad (6)$$

\[3\] These ‘spin’ quantum numbers are not to be confused with particle spins. For an elaborate discussion see \[20\].
where \( \{P\} \equiv N; \frac{1}{2} \leq j_i \leq \frac{k}{2} \forall l \in [1, N] \) such that \( A_{pq} = A + \mathcal{O}(1) \) and ‘inv’ stands for the gauge invariance. Now, for a given set of \( N \) punctures with spins \( j_1, \cdots, j_N \) the number of microstates is given by[13]

\[
\Omega(j_1, \cdots, j_N) = \frac{2}{k+2} \sum_{a=1}^{k+1} \prod_{l=1}^{N} \sin \left[ \frac{(2j_l + a)\pi}{k+2} \right] (2j_l + 1) \tag{7}
\]

Since we shall consider black holes with \( A \gg \mathcal{O}(1) \), eq. (7) can be approximated in that limit by the method of saddle point approximation to yield[19]

\[
\Omega(j_1, \cdots, j_N) \simeq a^{-3/2} \prod_{l=1}^{N} (2j_l + 1) \tag{8}
\]

where the product structure results from the zeroth order term and the \( a \) is an overall effective correction from all the punctures that result from the second order term of the Taylor expansion[19]. For the sake of convenience we shall focus on the zeroth order term in this work. The correction term will be addressed in the context of non-extensive statistical mechanics of quantum geometry of black holes in a future study.

Hence, in the present scenario we shall look at a Hilbert space that looks like \( \mathcal{H}_{\text{eff}} \equiv \bigotimes_{j=1}^{N} \mathcal{H}_{j} \) and the microstate count is given by \( \Omega_{\text{eff}}(j_1, \cdots, j_N) = \prod_{l=1}^{N} (2j_l + 1) \). So, we practically have a system of \( N \) spins with quantum numbers \( (j_1, \cdots, j_N) \). Consequently, the \( q \)-entropy for the black hole with such a set of punctures is given by eq. (4). Then, to find the \( q \)-entropy for a given \( A \) of the black hole we need to take into account all possible such spin sets for which we will have \( A_{pq} = A + \mathcal{O}(1) \). Avoiding such a summation procedure and leaving it for future studies, as a first step of this exercise, we set \( j_i = 1/2 \forall l \in [1, N] \). Physically, this is a good approximation because we are interested in the scenario \( N \gg \mathcal{O}(1) \) which allows us to apply statistical mechanics to the system. Since \( 1/2 \leq j_i \leq k/2 \forall l \in [1, N] \) and we are looking at the quantum states with \( A_{pq} = A + \mathcal{O}(1) \) and \( A \gg \mathcal{O}(1) \) for large black holes, \( j_1 = \cdots = j_N = 1/2 \) satisfies all the conditions most strongly as it yields \( A_{pq} = 4\sqrt{3\pi}A \simeq A \). Thus, the \( q \)-entropy of a black hole can be obtained by putting \( s = 1/2 \) in eq. (5) and it leads to

\[
S_{q}^{(1/2)} = \frac{2^{(1-q)N} - 1}{(1-q)} \tag{9}
\]

Now, for \( j_1 = \cdots = j_N = 1/2 \), the area comes out to be \( A = 4\sqrt{3\pi}A \). Using this result in eq. (9) and demanding that \( S_{q}^{(1/2)} = A/4 \), one can solve for \( \gamma \) in terms of \( A \) and \( q \) to get

\[
\gamma = \frac{\ln 2}{\pi\sqrt{3}} \cdot \frac{4(1-q)}{\ln[1 + \frac{4}{3}(1-q)]} \tag{10}
\]

As a fiducial check, it is easy to see that in the limit \( q \to 1 \), we recover \( \gamma = \ln 2/\pi\sqrt{3} \) derived in [12] while calculating black hole entropy using eq. (2) for \( j_1 = \cdots = j_N = 1/2 \). However, the most important issue here is that, for a given value of \( A \), we can have arbitrary real positive values of \( \gamma \) depending on \( q \). A three dimensional plot of the surface given by eq. (10), viewed along the \( A \)-axis is shown in fig. (1). The cross-section of the surface for \( A = 10^6 \) yields the curve shown in fig. (2), shown as an example, which reveals explicitly the nature of \( \gamma-q \) correlation for this particular value of \( A \). Similarly, one can take other values of \( A(\gg \mathcal{O}(1)) \) and check the nature of \( \gamma-q \) curves to ensure that the overall nature of the curve remains the same.

![FIG. 1: In this plot we have shown the variation of \( \gamma \) with \( A \) and \( q \) using eq. (10). The view along \( A \)-axis is shown to reveal the variation of \( \gamma \) with \( q \) explicitly.](image)

Now, let us provide the physical interpretation of the correlation between \( \gamma \) and \( q \). From the statistical mechanical view point, as we depart from \( q = 1 \), the underlying microstates of the associated quantum system become biased i.e. all the microstates are not a priori equally probable[8]. To be more specific, \( q > 1 \) relatively enhances the frequent events i.e. increases the possibility of occurrence of the microstates whose probabilities are closer to unity and \( q < 1 \) relatively enhances the rare events i.e. increases the possibility of occurrence of the microstates whose probabilities are closer to zero.

On the other hand, from the quantum geometry view point \( k := A/4\pi\gamma \) is the coupling constant of the CS theory on the black hole horizon[23]. Mathematically, the field equations on the black hole horizon is given by

\[
F_{ab}^I = \frac{1}{k} \cdot \Sigma_{ab}^I = \frac{4\pi\gamma}{A} \cdot \Sigma_{ab}^I \tag{11}
\]
where $F^{I}_{ab}$ is the curvature of the CS gauge fields on the horizon and $\Sigma^{I}_{ab}$ stands for the pull-back of the soldering form constructed from the bulk tetrads. The superscript $I$ is the internal Lorentz index and the $a, b$ are the spacetime indices. It may be noted that the definition of $k$ may differ by some numerical constants in certain references depending on the redefinition of the fields. Eq.(11) is the equation of a CS theory coupled to an external source with the coupling strength being controlled by $k$. Hence, it is quite explicit that, for a black hole with a given area, $\gamma$ controls the strength with which the horizon field theory is sourced and affected by the bulk fields. Quantization of this CS theory coupled to the bulk source fields leads to the Hilbert space of the horizon that provides the estimate of the microstates given in eq.(7).

Now, reconciling the above facts associated with the two viewpoints, it is extremely tempting to make the conjecture that the interplay between $\gamma$ and $q$ signifies the role of the bulk fields in ‘forcing’ a bias into the microstates associated with the horizon. The Hilbert space of the quantum CS theory on the horizon provides the independent degrees of freedom. However, the fact that the horizon is not a completely independent entity is manifested through this bias created by the coupling with the bulk. For a black hole spacetime, the horizon is only a subsystem connected to the exterior through long range gravitational interaction. Hence, qualitatively, the correlation between $\gamma$ and $q$ is quite explicit. For instance, the interplay falls into place quite nicely.

To study this interplay between $\gamma$ and $q$ quantitatively, we study the graph $\gamma$ vs $q$ for $A = 10^6$ (such that $A \gg \mathcal{O}(1)$) in fig.(2). The plot shows that the value of $\gamma$ increases monotonically as $q$ decreases from unity. Since $k := A/4\pi \gamma$ and $k \rightarrow \infty$ is the weak coupling limit, it implies that the bulk affects the horizon and biases microstates more strongly as we depart more from $q = 1$. This is absolutely what is to be expected because of the following reason. Apart from handling systems with long range interactions, the other motive of the introduction of non-extensive statistical mechanics was to address strongly coupled systems. The degree of non-extensivity of the system is signified by the departure of $q$ from unity. In the present scenario, this is what is coming out from the quantitative study of the $\gamma$ vs $q$ interplay in the following way. As $q$ departs more from unity, the value of $\gamma$ increases which in turn lowers the value of $k$. This implies that the horizon gets more strongly coupled with the bulk and as an effect the microstates get more biased away from being equally probable. Thus we have a perfect matching of two free parameters in two branches of physics to give rise to a well known physical result in the form of Bekenstein-Hawking area law for black holes.

![Graph for $\gamma$ vs $q$](image)

**FIG. 2:** In this plot we have shown the variation of $\gamma$ with $q$ for $A = 10^6$ as one example. It shows that $\gamma$ can take arbitrary positive values depending on $q$ and it increases monotonically with decreasing $q$. This overall behaviour of the $\gamma$ vs $q$ remains the same for arbitrary values of $A \gg \mathcal{O}(1)$.

However, there is a section of the curve which extends in the range $q > 1$ where $\gamma < \ln 2/\pi \sqrt{3}$ as can be seen from fig.(3). In this regime, the above discussed physics fails, because with the departure of $q$ from unity $\gamma$ decreases, which in turn increases the value of $k$ implying that the coupling of the horizon with the bulk becomes weaker. But there is still a bias in the microstates due to $q \neq 1$, although in the opposite sense i.e. now the more probable microstates appear more frequently. Hence, this section of the curve for $q > 1$ represents some sort of anomaly which we are unable to address in the present discussion. We suspect that some explanation may come out if we study the underlying dynamics of the punctures in a similar line as in [21].

Further, an analysis of the eq.(10) shows that $\gamma \rightarrow 0$ as $q \rightarrow (1 + 4/A)$. This shows that we can have

![Graph for $A = 10^6$](image)

**FIG. 3:** In this plot we have shown a highly resolved version of fig.(2) to zoom into the region $q > 1$ to reveal its nature as $\gamma \rightarrow 0$. It also shows clearly that, at $q = 1$, $\gamma = \ln 2/\pi \sqrt{3} = 0.12738 \cdots$. 
values of $\gamma$ arbitrarily close to zero, which is not evident from the plot in fig.(3)(all the plots have been drawn in Mathematica v10). Therefore, we can conclude that we have obtained the Bekenstein-Hawking area law from non-perturbative canonical quantum gravity for arbitrary real positive values of $\gamma$. On the other hand, although in a restricted regime of validity($q \leq 1$), we have given a physical interpretation for the parameter $q$ of non-extensive statistical mechanics in the context of quantum gravity.

As a concluding remark it may be mentioned that although we have considered only spin 1/2 of the horizon punctures as a first step of application of non-extensive statistical mechanics to compute black hole entropy from quantum geometry, from previous evidences in literature concerning the application of extensive statistical mechanics in this context (see [24] and the references therein), we may expect that the inclusion of all spins will only slightly change the $\gamma$ vs $q$ interplay, keeping the overall physics same. Of course, we need to verify that in a quantitative manner, which we have left for future study.

Acknowledgments: The author wants to thank L. F. Camillo, F. Nettel and Y. Bonder for giving useful suggestions. This work is funded by DGAPA fellowship of UNAM.

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