Damage localization from vibration data using hierarchical a priori assumptions

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Abstract. An inverse method for damage identification is described. The method assumes that a forward model is available to predict hypothetical vibration data, for an arbitrarily varying profile of the local stiffness of the structure. In general, the inverse problem of identifying this stiffness profile from the observed vibration data is an ill-posed problem, therefore some form of regularization is required. In the case of this method, the regularization results from a Bayesian formulation of the a priori assumption that the stiffness change is smooth almost everywhere, but could have an unknown number of discontinuities at unknown locations.

1. Introduction
Damage is assumed to be associated with localized loss of stiffness in a structure. This loss of stiffness causes changes in modal parameters in the structure, namely the vibration frequencies and mode shapes. Reconstructing the changes in stiffness that resulted in the measured vibration characteristics is generally an ill-posed problem and cannot be solved without some sort of regularization.

In this paper we apply Bayesian inference to the problem and use a hierarchical model to express our prior knowledge about the damage. Using statistical methods allows us to model the measurement errors and thus to better retrieve meaningful information from the data. This framework is applied to a cantilever beam, with the prior assumption that the change in stiffness is piecewise smooth.

Sensitivity based methods that use statistical measure variables, which our method is an example of, have also been proposed in the past, for example in [1]. These methods use a parametrized model of the structure. The model parameters are changed so that a chosen statistical variable, that measures how well the model explains the measurements, is optimized.

A completely different way to look at the problem is proposed in [2, 3]. Flexibility-based methods are based on the fact, that the flexibility matrix can be quite accurately constructed by just using a few of the lowest modes. The flexibility matrix is calculated for both the undamaged and the damaged structure. Damage localization is then based on the principle that vectors which are in the null space of the change in flexibility, correspond to zero stress in the damaged areas.

A comprehensive study of various inverse methods used in damage identification is presented in [4]. An overview is given in [5].

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2. Forward model
We assume that the undamaged structure under free vibration can be modelled as

\[ M \ddot{x} + Kx = 0, \]  
(1)

where \( M \in \mathbb{R}^{n \times n} \) is the mass matrix and \( K \in \mathbb{R}^{n \times n} \) the stiffness matrix. The matrices \( M \) and \( K \) are assumed to be known for the structure. The vibration modes for such a structure are the solutions to the generalized eigenvalue problem

\[ Kx = \omega^2 Mx, \]  
(2)

where \( \omega \) is the angular frequency of the vibration mode and \( x \) is the mode shape. The measurement of modal parameters from actual structures is less trivial and is discussed in [6].

We assume that damage to the structure affects only its stiffness and not its mass. Hence, the damaged structure under free vibration can be modelled as

\[ M \ddot{x} + (K + \Delta K)x = 0. \]  
(3)

As the matrices \( M \) and \( K \) are commonly built using the finite element method, we are motivated to write the matrix \( K \) as

\[ K = \sum_{i=1}^{N} K^e_i, \]  
(4)

where \( K^e_i \) are the element stiffness matrices. The change in stiffness caused by the damage is then written as

\[ \Delta K = - \sum_{i=1}^{N} d_i K^e_i, \]  
(5)

where \( d_i \) is a “damage index” corresponding to the loss of stiffness in element \( i \). The \( d_i \) get values in the interval \([0, 1)\), where 0 means no damage and 1 would be complete loss of stiffness.

It is assumed that we can measure the frequencies for a few of the lowest modes, as well as some chosen components of the mode shapes for the same modes. The forward model is then a function \( f \), such that

\[ y = f(d), \]  
(6)

where \( d \) is a vector of the damage indices and \( y \) is a vector of the individual measurements in some order.

3. Hyperprior inverse method
In the Bayesian framework all variables are assumed to be random variables, which we only get realizations of. In this paper we refer to the random variables themselves with capital letters and to the realizations with lower case letters. Statistical inverse problems are discussed with more detail in [7].

It is assumed that measured data is produced by the forward model, but corrupted by random noise. Therefore we have the relation

\[ Y = f(D) + \epsilon, \]  
(7)

where \( \epsilon \) is the noise, which we assume to be zero mean Gaussian with covariance \( S^T S \). That is, \( \epsilon \) has the probability density function

\[ p_{\text{noise}}(\epsilon) = C \exp \left( -\frac{1}{2} \| S^{-1} \epsilon \|^2 \right), \]  
(8)
where $C$ is a normalization constant.

We can then write

$$\epsilon = Y - f(D).$$

(9)

For a given $D$, the conditional probability density function for $Y$ is

$$p_{Y|D}(y|d) = p_{\text{noise}}(y - f(d)) = C \exp \left( -\frac{1}{2} \left\| S^{-1}(y - f(d)) \right\|^2 \right).$$

(10)

By using Bayes’ formula, we get the conditional probability for $D$

$$p_{D|Y}(d|y) = \frac{p_{Y|D}(y|d)p_{D}(d)}{p_{Y}(y)}, \quad y = y_{\text{measured}}.$$  

(11)

From these we get the posterior probability density for a given measurement of $y$,

$$p_{\text{posterior}}(d|y) = \frac{p_{Y|D}(y|d)p_{\text{prior}}(d)}{p_{Y}(y)}, \quad y = y_{\text{measured}}.$$  

(12)

In our case, the prior knowledge itself depends on unknown variables, called hyperparameters, which are also estimated from the measured data. We assume that the hyperparameters are distributed according to a hyperprior distribution $p_{\text{hyper}}(\lambda)$. Thus we have the joint prior distribution

$$p_{\text{prior}}(d, \lambda) = p_{\text{prior}}(d|\lambda)p_{\text{hyper}}(\lambda).$$

(13)

Using this prior in Bayes’ formula we get the joint posterior density

$$p_{\text{posterior}}(d, \lambda|y) = \frac{p_{Y|D}(y|d)p_{\text{prior}}(d|\lambda)p_{\text{hyper}}(\lambda)}{p_{Y}(y)}, \quad y = y_{\text{measured}}.$$  

(14)

The posterior density is often considered the solution to an inverse problem. For single point estimates it is common to search for the mode of the posterior density, that is

$$(d, \lambda)_{\text{MAP}} = \arg \max_{y} p_{\text{posterior}}(d, \lambda|y), \quad y = y_{\text{measured}},$$

(15)

which is referred to as the maximum a posteriori estimate or MAP estimate.

4. Application to a one-dimensional beam model

An Euler-Bernoulli beam of length $L$ is discretized into $N$ elements, each of length $h = \frac{L}{N}$. In each element we assume that the stiffness is constant. This produces a vibration model in the form of (1).

Let $d$ be a vector of the damages. We know a priori that the stiffness along a cracked beam is piecewise smooth, by which we mean that the stiffness is smooth apart from an unknown number of jumps. The number and location of the jumps are to be estimated from the data.

We take

$$p_{\text{prior}}(d|\lambda) = C(\lambda) \exp \left( -\frac{1}{2} \sum_{j=1}^{N} \lambda_j \frac{1}{h^2} (d_j - d_{j-1})^2 \right) = C(\lambda) \exp \left( -\frac{1}{2} \| \Lambda^{1/2} Dd \|^2 \right)$$

(16)
as the prior distribution, where

$$D = \frac{1}{h} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{N \times N};$$

(17)

and

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N).$$

(18)

The factor $C(\lambda)$ is for normalization and it is known that

$$C(\lambda) = \tilde{C} \det(D^T \Lambda D)^{1/2} = \tilde{C} \det(D) \det(\Lambda)^{1/2}.$$  

(19)

If the hyperparameters $\lambda_i$ are all equal, this prior introduces a penalty for value changes in $d$, i.e. it favors smoothness of $d$. If, however, one $\lambda_i$ is smaller than the rest, $d$ can jump at that location. That is why we choose a hyperprior distribution that can allow for some extreme outliers. In this case, we have chosen

$$p_{\text{hyper}}(\lambda) = C p_+(\lambda) \exp\left(-\frac{\gamma}{2} \sum_{i=1}^{N} \lambda_i\right),$$

(20)

where $p_+(\lambda)$ is 1 when all of the $\lambda_i$ are non-negative and zero otherwise. The parameter $\gamma$ specifies the width of the distribution and is a tuning parameter.

Now we have

$$p_{\text{posterior}}(d, \lambda|y)$$

$$= C \exp\left(-\frac{1}{2} \|S^{-1}(y - f(d))\|^2\right) \det(\Lambda)^{1/2} \exp\left(-\frac{1}{2} \|\Lambda^{1/2} D d\|^2\right) \exp\left(-\frac{\gamma}{2} \sum_{i=1}^{N} \lambda_i\right)$$

$$= \prod_{i=1}^{N} \lambda_i^{1/2} C \exp\left(-\frac{1}{2} \|S^{-1}(y - f(d))\|^2 - \frac{1}{2} \|\Lambda^{1/2} D d\|^2 - \frac{\gamma}{2} \sum_{i=1}^{N} \lambda_i\right)$$

$$= \exp\left(-\frac{1}{2} \|S^{-1}(y - f(d))\|^2 - \frac{1}{2} \|\Lambda^{1/2} D d\|^2 - \frac{\gamma}{2} \sum_{i=1}^{N} \lambda_i + \frac{1}{2} \sum_{i=1}^{N} \log \lambda_i + \log C\right).$$

(21)

We are interested in the MAP estimate, which we get by maximizing the posterior density.

$$(d, \lambda)_{\text{MAP}} = \arg\max p_{\text{posterior}}(d, \lambda|y) = \arg\min -\log p_{\text{posterior}}(d, \lambda|y)$$

$$= \arg\min \left(\frac{1}{2} \|S^{-1}(y - f(d))\|^2 + \frac{1}{2} \|\Lambda^{1/2} D d\|^2 + \frac{\gamma}{2} \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i=1}^{N} \log \lambda_i - \log C\right)$$

$$= \arg\min \left(\frac{1}{2} \|S^{-1}(y - f(d))\|^2 + \frac{1}{2} \|\Lambda^{1/2} D d\|^2 + \frac{\gamma}{2} \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i=1}^{N} \log \lambda_i\right).$$

(22)

We split the optimization problem into two smaller problems that we iteratively solve. This method, proposed in [8], simplifies the optimization by avoiding the need to calculate Hessians.

1. Set $d^k = 0$, where $k = 1$.
2. Set $\lambda^k$ as the maximizer of $p(\lambda|d^k, y)$, which can be obtained analytically as

$$\lambda_i^k = \frac{1}{(D d^k)_i^2 + \gamma}.$$  

(23)
3. Set $d^{k+1}$ as the maximizer of $p(d | \lambda^k, y)$, which is obtained from the non-linear least squares problem

$$
d^{k+1} = \arg \min \left( \frac{1}{2} \| S^{-1}(y - f(d)) \|^2 + \frac{1}{2} \| \Lambda^{1/2} D d \|^2 \right).$$

(24)

4. Set $k = k + 1$ and repeat from 2. if not converged.

5. Results using simulated data

Figure 1. Reconstructions of simulated damage in a cantilever beam with a varying level of damage.

To avoid committing the inverse crime (which means using exactly the same model in simulating the data and as the forward model), the simulated measurements were taken from a finer finite element model than the one used in the forward model of the inverse method.

The forward model of the inverse method and the model used to simulate measurements both return frequencies and coarse shapes for 6 of the lowest modes. The coarse shapes are the mode shapes evaluated at 7 locations along the beam. This mimics measurement of the mode shapes by accelerometers or strain gauges. Gaussian noise is also added to the values when simulating measurements. The standard deviation of the noise is referred as the level of noise.

The simulated beam is 1.4 meters long, 60 millimeters wide and 5 millimeters thick (this is the direction of vibration) and is composed of steel. It is rigidly clamped on the left and is free on the right. The simulated accelerometers are located at 0.2, 0.4, 0.6, 0.8, 1.0, 1.2 and 1.4 meters from the left.

Figure 1 shows estimations for four cases. A different damage is induced to the same portion of the beam in each case. In each of the cases a constant damage index is set to the portion. The reconstructions have been calculated with $\gamma = 0.1$ and $N = 50$. Level of noise added to the frequencies is 0.01 Hz plus 0.1% of frequency magnitude and the level of noise added to each mode shape component is 1% of component magnitude.
Figure 2. Reconstructions of simulated damage in a cantilever beam with a varying level of noise.

The reconstruction responds to the damage in a highly nonlinear way. The low level of damage (damage indices at 0.05) is hardly reconstructed, while for the higher levels of damage the reconstruction get progressively better.

Figure 2 shows the effect of noise on the reconstruction. In each case a constant damage index of 0.2 is induced to the same portion of the beam. Only the level of simulated measurement noise is changed between the reconstructions. The reconstructions have been calculated with \( \gamma = 0.1 \) and \( N = 50 \). In the cases, the level of noise added to the frequencies is proportional to the magnitude of the frequencies and the level of noise added to the mode shape components is proportional to the component magnitudes. The proportionality constants are 0.1%, 0.5%, 1.0%, 1.5%, 2.0% and 3.0% for each case respectively.

As with all inverse problems, the reconstruction is found to be highly sensitive to the amount of noise. Higher levels of noise make the reconstruction progressively more difficult and ultimately impossible.

Figure 3 shows the effect of a small undamaged area between two damaged areas. Both of the damaged areas are 10% of the total length of the beam. The damage levels in the areas are kept constant between each case. Only the length of the undamaged gap is changed. Level of noise added to the frequencies was 0.01 Hz plus 0.1% of frequency magnitude. Level of noise
added to each mode shape component was 1% of component magnitude. The reconstructions have been calculated with $\gamma = 0.1$ and $N = 50$. The method starts to notice the gap even when it is fairly short. However, the gap length has to be quite long for the method to detect the damaged areas as completely separate. Nevertheless, the sensitivity is surprisingly high.

6. Conclusions and future work
We conclude that it is possible to solve the ill-posed problem of damage identification using the regularization method described, given enough meaningful data and that the damage is consistent with the prior information.

The method is highly sensitive to the amount of noise in the measurements, as are all inverse problems. This property can be explained by looking at the method as maximizing the likelihood that the difference in measurement and the estimate was produced by noise alone, given some prior information of the result. If the measurement noise is too high, the data might not be statistically meaningful enough, thus resulting in bad estimates.

Future research on the topic includes the measurement of damage in an actual beam. First an initial model is to be created for an undamaged beam. That model is then updated with the modal information obtained from measurements of the undamaged actual beam. Damage is then to be progressively inflicted on the beam and measurements are taken from several levels...
of damage. The described method is then to be used on damage identification for each of the
damage cases and the results compared to those of other methods.

Also, a method for selecting values for the tuning parameter $\gamma$ is required and is a topic for
further research.

### 7. Acknowledgements

This research is supported by “ISMO”, the Intelligent Structural Health Monitoring project of
the Multidisciplinary Institute for Digitalisation and Energy (MIDE) of the Helsinki University
of Technology. Additionally, S. Bossuyt is a researcher of the Research Foundation Flanders
(FWO - Vlaanderen).

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