Temporal instability of Walter’s B viscoelastic fluid film

Dharamendra¹ and Mukesh Kumar Awasthi

Department of Mathematics, Babasaheb Bhimrao Ambedkar University, Lucknow, 226025, India

¹Email: dharmendrakohli123@gmail.com

Abstract: The stability of an interface formed by Walter’s B viscoelastic liquid and viscous incompressible fluid is examined. These fluids lie in an annular region enclosed by two cylindrical shapes rigid boundaries. The surface tension effect is considered at the interface and therefore, the interface obeys capillary instability. The potential flow theory of viscoelastic-viscous fluids is utilized to compute the solution of modeled equations and the normal mode technique is enforced for the calculation of perturbation’s growth. A two-degree algebraic equation is established to obtain the stability/instability criterion. We found that the perturbations grow slower at the interface containing Walter’s B viscoelastic fluid than the corresponding Newtonian fluid.

Keywords: Walter’s B fluid; capillary instability; potential flow.

1. Introduction

When a liquid cylinder falls into another fluid, the capillary forces arise due to surface tension make the interface of the fluids unstable. This type of instability is called capillary instability and this instability can be seen in the liquid jet breakup, film boiling, etc.

The first study of instability of a liquid cylindrical column of the radius $R$ with the effect of capillary forces was done by Plateau [1]. He showed that the liquid cylinder is unstable for those perturbations who have wavelengths more than $2\pi R$. Rayleigh [2] also studied the stability of the liquid cylinder using potential flow theory but he ignores the effect of outside liquid. Rayleigh [3] added the viscous effect into his previous analysis but he again neglected the outside fluid effect. The effect of outside fluid on the instability of a liquid cylinder was included by Weber [4]. Tomotika [5] studied the stability of a viscous liquid cylinder into another viscous fluid. He achieved the dispersion relation for fully viscous fluids in the case of axisymmetric perturbations.

The Navier-Stokes equation reduces to Euler’s equation of motion when the flow is potential but viscosity is not zero. Joseph and Liao [6] presented a potential flow theory that includes viscosity
into the free surface problems. Funada and Joseph [7] studied instability of viscous liquid cylinder
surrounded by a viscous fluid utilizing potential flow theory. They found that the potential flow theory
gives very good agreement with fully viscous results.

The capillary instability of an Oldroyd B viscoelastic fluid surrounded by a viscous fluid was
investigated by Funada and Joseph [8]. They applied the potential flow theory and achieved a cubic
equation for the perturbation growth. Awasthi et al. [9] examined the breakup of the viscoelastic jet of
Oldroyd B fluid with the help of potential flow theory. They found that the growth of perturbations for
viscoelastic fluids is larger than the Newtonian fluids but smaller than the inviscid fluids.

The instability of the interface containing Walter’s B viscoelastic fluid was examined by
Kumar and Singh [10]. They formulated Rayleigh-Taylor instability and established a certain
condition for stability. The Kelvin-Helmholtz instability of Walter’s B fluid was carried out by El-
Sayed et al. [11]. They considered the flow region porous and the normal electric field was applied.
The effect of porous media, elasticity, suction/injection, in the flow of Walters’ B visco-elastic fluids
was studied by Barik et al. [12]. These studies are limited to inviscid fluids. The only study was done
by Moatimid and Zekry [13] for Walters’ B visco-elastic fluids includes viscosity into the analysis
along with viscoelasticity.

In this paper, we attempted to study the instability of Walter’s B viscoelastic fluid surrounded
by a viscous fluid. The potential flow theory is used and interfacial condition includes the viscosity of
both the fluids. The fluids are confined in an annular region bounded by rigid cylindrical boundaries.
The effect of gravity is neglected at the interface while the interface is experiencing a surface tension
effect. The well-known normal mode technique is applied and an algebraic equation for the growth of
perturbations is obtained. The instability is discussed through various plots.

2. Mathematical Statement of the problem

We consider a cylindrical layer of Walter’s B viscoelastic fluid of viscosity $\mu_i$, viscoelasticity
$\mu_i'$, and density $\rho_i$ surrounded by a viscous fluid of viscosity $\mu_o$ and density $\rho_o$. The surface tension
at the interface is $\sigma$. The viscous fluid is bounded by the outer cylinder of radius $r = r_o$ and the inner
cylinder $r = r_i$ bounds viscoelastic fluid. The Walter’s B viscoelastic fluid and viscous fluid are
separated by the cylindrical boundary $r = R$, initially.
If the velocity in the Walter’s B viscoelastic fluid phase is \( \mathbf{q}_i \), the governing equations for the viscoelastic fluid phase can be written as

\[
\nabla \cdot \mathbf{q}_i = 0 \\
\rho_i \left( \frac{\partial \mathbf{q}_i}{\partial t} + (\mathbf{q}_i \cdot \nabla) \mathbf{q}_i \right) = -\nabla p_i + \nabla \cdot \mathbf{\tau}_i
\]

(1)

The stress \( \mathbf{\tau}_i \) for Walter’s B viscoelastic is given by the expression

\[
\mathbf{\tau}_i = 2 \left( \mu_i - \mu_i' \frac{\partial}{\partial t} \right) \mathbf{D}_i
\]

(2)

where

\[
\mathbf{D}_i = \left( (\nabla \mathbf{q}_i) + (\nabla \mathbf{q}_i)^T \right)
\]

If the viscous fluid has velocity \( \mathbf{q}_o \), the governing equations can be written as

\[
\nabla \cdot \mathbf{q}_o = 0 \\
\rho_o \left( \frac{\partial \mathbf{q}_o}{\partial t} + (\mathbf{q}_o \cdot \nabla) \mathbf{q}_o \right) = -\nabla p_o + \mu_o \nabla^2 \mathbf{q}_o
\]

(3)

3. Stability Analysis

a. Basic state

Initially, the fluids are at rest in both the phases i.e. \( \mathbf{q}_o = (0,0,0) \) and \( \mathbf{q}_i = (0,0,0) \). Also, the interface is cylindrically flat in an equilibrium state and hence, \( p_o = \text{constant} = p_i \).

b. Perturbed state

A small perturbation is introduced to the basic state and therefore interface is located at \( r = R + \zeta(z,t) \). In the perturbed state, the velocities and pressures become \( \mathbf{q}_o = \mathbf{0} + \mathbf{q}_o' \), \( \mathbf{q}_i = \mathbf{0} + \mathbf{q}_i' \), and \( p_o + p_o' \), \( p_i + p_i' \), respectively. The linearized perturbations equations for both the phases can be written as;

\[
\nabla \cdot \mathbf{q}_i' = 0 \\
\rho_i \frac{\partial \mathbf{q}_i'}{\partial t} = -\nabla p_i' + \nabla \cdot \mathbf{\tau}_i'
\]

(4)

\[
\nabla \cdot \mathbf{q}_o' = 0 \\
\rho_o \frac{\partial \mathbf{q}_o'}{\partial t} = -\nabla p_o' + \mu_o \nabla^2 \mathbf{q}_o'
\]

(5)

The potential flow theory of viscous fluids is used here and therefore, \( \mathbf{q}_o' = \nabla \varphi_o \); \( \mathbf{q}_i' = \nabla \varphi_i \). Hence form continuity equations from (4) and (5), we have
\[ \nabla^2 \phi_o = 0; \quad \nabla^2 \phi_i = 0 \]  \hspace{1cm} (6)

c. Boundary and Interfacial Conditions

There is no flow in the perturbed state across the rigid cylindrical boundary and therefore,

\[ \frac{\partial \phi_o}{\partial r} = 0 \quad \text{at} \quad r = r_o \]
\[ \frac{\partial \phi_i}{\partial r} = 0 \quad \text{at} \quad z = r_i \]  \hspace{1cm} (7)

Also, the normal velocity of the interface is zero. Hence,

\[ \frac{\partial \phi_i}{\partial r} = \frac{\partial \zeta}{\partial t} \quad \text{at} \quad r = R \]  \hspace{1cm} (8)

4. Dispersion Relationship

The conventional normal mode technique is imposed to examine the stability of the interface. The interface distortion can be expressed as \[ \zeta(z, t) = B \exp[i k z - i \omega t] \] and other quantities \[ E(r, z, t) \] are depicted as \[ E(r, z, t) = f(r) \exp[i k z - i \omega t] \].

The potential functions \( \phi_o, \phi_i \) satisfying conditions (7) and (8) can be expressed as

\[ \phi_i = -\frac{i \omega}{k} A_i(kr) \zeta + c.c. \]  \hspace{1cm} (9)
\[ \phi_o = -\frac{i \omega}{k} A_o(kr) \zeta + c.c. \]  \hspace{1cm} (10)

Here, \( A_i(kr) = \frac{I_i(kr) K_i(kr) + I_i(kr) K_o(kr)}{I_i(kR) K_i(kr) - I_i(kr) K_i(kR)} \), \( A_o(kr) = \frac{I_o(kr) K_i(kr) + I_i(kr) K_o(kr)}{I_i(kR) K_i(kr) - I_i(kr) K_i(kR)} \).

The linearized stress balance equation at the interface containing viscosity can be expressed as

\[ \left[ \rho_o \left( \frac{\partial \phi_o}{\partial t} \right) + 2 \mu_o \frac{\partial^2 \phi_o}{\partial r^2} \right] - \left[ \rho_i \left( \frac{\partial \phi_i}{\partial t} \right) + 2 \left( \mu_i - \mu_i' \frac{\partial}{\partial t} \right) \frac{\partial^2 \phi_i}{\partial r^2} \right] = -\sigma \left( \frac{\partial^2 \zeta}{\partial z^2} + \frac{\zeta}{R^2} \right) \]  \hspace{1cm} (11)

The dispersion relationship can be obtained by using equations (9) and (10) into (11). This expression is given as

\[ A \omega^2 + B \omega + C = 0 \]  \hspace{1cm} (12)

\[ A = \rho_i A_i(kr) - \rho_o A_o(kr) - 2 \mu k^2 B_i(kr) \]
\[ B = 2 k^2 \left( \mu_i B_i(kr) - \mu_o B_o(kr) \right) \]
\[ C = -\sigma k \left( k^2 - \frac{1}{R^2} \right) \]
The capillary instability of two viscous fluids was considered by Funada and Joseph [7]. Their relation can be recovered by taking $\mu'$. Joseph et al. [8] have also discussed the Rayleigh-Taylor instability exists in viscoelastic flows but in their study, the viscoelastic fluid was Oldroyd B fluid.

As $\omega = \omega_{Re} + i\omega_1$, computing real and imaginary values of equation (12), we get

$$A(\omega^2_{Re} - \omega^2) - B\omega_1 + C = 0$$

(13)

$$A(2\omega_{Re}\omega_1) + B\omega_{Re} = 0 \Rightarrow \omega_{Re} = 0$$

(14)

Hence, equation (12) becomes

$$A\omega_1^2 + B\omega_1 - C = 0$$

(15)

For marginal stability analysis, $\omega_1 = 0$, therefore equation (15) changes to

$$C = 0 \Rightarrow \sigma k \left( k^2 - \frac{1}{R^2} \right) = 0$$

(16)

Equation (16) shows that viscosity and viscoelasticity do not affect the marginal stability criterion.

This result is inconsistent with the Funada and Joseph [7, 8].

The non-dimensional form of equation (14) can be written as

$$\left[ \rho A (\hat{k}\hat{r}) - A_s (\hat{k}\hat{r}) - 2\lambda\hat{k}^2 B_s (\hat{k}\hat{r}) \right] \hat{\omega}_1^2 + \left[ \frac{2\hat{k}^2}{Oh} (\mu B_s (\hat{k}\hat{r}) - B_s (\hat{k}\hat{r})) \right] \omega_1 + \hat{k} \left( \hat{k}^2 - \frac{1}{R^2} \right) = 0$$

(17)

5. Results and Discussions

In this section, we compute growth rates from the equation (17). The equation (17) has two roots and will plot the maximum of those two values. The effects of various flow parameters such as viscosity, viscoelasticity, etc. have been illustrated through plotted figures of growth rate curves.
In Figure 1, we have plotted the growth rate for interfaces containing Walter’s B viscoelastic fluid and viscous fluid interface. The outside fluid is taken as a viscous fluid which is the same for both cases. The third curves are plotted when both inside and outside fluids are inviscid fluid. The figure shows that the growth of Walter’s B viscoelastic fluid is the lowest among the three curves while the growth of inviscid fluids is highest. Funada and Joseph [7] found that the growth rate for Oldroyd B viscoelastic fluid lies above the viscous fluid. The growth for inviscid fluids always lies on the top. Hence, our result is different from Funada and Joseph [7]. Actually, in the case of Walter’s B viscoelastic fluid, the perturbed flow resists the viscoelasticity, and therefore, the system move towards stability.
Figure 2: Effect of viscoelasticity.

The variation of growth rate curves for the various values of viscoelasticity is shown in figure 2. It is clear from the constitutive equation (2) of Walter’s B viscoelastic fluid that if the viscoelasticity of the fluid increases, less energy is transferred to the interface. As viscoelasticity is present in equation (2) with a negative sign and therefore, on increasing viscoelasticity, energy dissipation will be lesser. As less energy is obtained by the interface so the perturbation’s amplitude will be low and perturbations will take more time to travel i.e. perturbation travels slower than usual. The same observation has been obtained from figure 2 that viscoelasticity has a stabilizing character.
Figure 3: Effect of Ohnesorge number.

The behavior of growth rate curves concerning the Ohnesorge number is shown in figure 3. As the Ohnesorge number increases, perturbations grow faster and the system gets destabilized. The surface tension is responsible for capillary forces and as surface tension increases, Ohnesorge number increases, and consequently, the perturbations grow fast. Hence, surface tension has a destabilizing nature. Also, the Ohnesorge number depends directly on the density of the outside fluid, and therefore, outside fluid density has destabilizing behavior. The Ohnesorge number increases, the outside fluid viscosity decreases, and therefore, outside fluid viscosity has stabilizing nature. As outside fluid viscosity increases, the flow resistance increases which makes the system stable.

Figure 4 shows the effect of the viscosity ratio on the growth of perturbations. The perturbation growth is slow on increasing the viscosity ratio of two fluids. Therefore, the viscosity of viscoelastic fluid has a stabilizing nature. If viscosity is high, more resistance to the flow will be high, and therefore, flow moves towards stability.
The effect of the density ratio of two fluids is studied in figure 5. The growth of perturbations decreases with the increment of density ratio and therefore, density ratio has a stabilizing effect. The density ratio depends directly on inside fluid density and inversely on outside fluid density. Hence, inside fluid density shows stabilizing character while outside fluid destabilizes the interface. As the inside fluid density increase, the inertia force increases which opposes the development of perturbations, and the system gets stabilized.
6. Conclusions

The linear temporal instability of the interface of Walter’s B viscoelastic fluid and viscous fluid is studied. The surface tension is present in the analysis while gravity is absent and hence, instability is modeled as capillary instability. The potential flow theory of viscous fluids is utilized to solve the mathematical equations. We achieve a second-order polynomial in the growth rate parameter and the imaginary part of the growth rate parameter is plotted to examine the effect of physical parameters like viscoelasticity, viscosity, etc. We found that viscosities of outside and inside fluids promote stability at the interface while the density of outside fluid has destabilizing nature. The viscoelasticity induces stability while surface tension has a reverse effect. The inside fluid density has a stabilizing character. We also observe that Walter’s B viscoelastic fluid is stable than the Newtonian fluid.

Acknowledgment

One of the Authors (Dharamendra) is thankful to the Council of Scientific and Industrial Research (CSIR), New Delhi for their financial support during this work.
References

[1] Plateau, Statique experimentale et theorique des liquide soumis aux seules forces moleculaire, vol. ii, 1873, p. 231.

[2] Rayleigh L., On the capillary phenomena of jets, Proc. Roy. Soc. London A 29 (1879) 71–79.

[3] Rayleigh L., On the instability of a cylinder of viscous liquid under capillary force, Phil. Mag. 34 (207) (1892) 145–154.

[4] Weber C., Zum Zerfall eines Flüssigkeitsstrahles. Ztschr. f. angew, Math. Mech. 11 (2) (1931) 136–154.

[5] Tomotika S., On the instability of a cylindrical thread of a viscous liquid surrounded by another viscous fluid, Proc. Roy. Soc. London A 150 (1935) 322–337.

[6] Joseph D.D., Liao T.Y., Potential flows of viscous and viscoelastic fluids, J. Fluid Mech. 265 (1994) 1–23.

[7] Funada T., Joseph D.D., Viscous potential flow analysis of capillary instability, Int. J. Multiphase Flow 28 (9) (2002) 1459–1478.

[8] Funada T., Joseph D.D., Viscoelastic potential flow analysis of capillary instability. J Non-Newton Fluid Mech 111(2003) 87–105.

[9] Awasthi M. K., Asthana R., and Agrawal G. S., Viscoelastic potential flow analysis of the stability of a cylindrical jet,” Sci. Iran., 21 (2014), 578–586.

[10] Kumar P., Singh G. J., On the stability of two stratified Walters B’ viscoelastic superposed fluids, Stud Geotech Mech XXXII (2010), 29–37.

[11] El-Sayed M. F., Eldabe N. T., Haroun M. H., Mostafa D. M., Nonlinear stability of viscoelastic fluids streaming through porous media under the influence of vertical electric fields producing surface charges. Int J Adv Appl Math Mech 2(2014), 110–125.

[12] Barik R. N., Dash G. C., Rath P. K., Steady laminar MHD of viscoelastic fluid through a porous pipe embedded in a porous medium. Alex Eng J 57 (2018), 973–982.

[13] Moatimid G. M., Zekry M. H., Nonlinear stability of electro-visco-elastic Walters’ B type in porous media, Microsystem Technologies 26 (2020), 2013–2027.