The chiral parametrization of gluons in $SU(3)$ QCD is proposed extending an approach developed earlier for $SU(2)$ case. A color chiral field is introduced, gluons are chirally rotated, and vector component of rotated gluons is defined on condition that no new color variables appeared with the chiral field. This condition associates such a vector component with $SU(3)/U(2)$ coset plus an $U(2)$ field. The topological action in $SU(3)$ QCD is derived. It is expressed in terms of axial vector component of rotated gluons. The vector field in $CP^2$ sector is studied in new variables of chiral parametrization.
1 Introduction

Confinement still remains one of the most important problems of QCD. The monopole condensation scenario for confinement \cite{1, 2, 3, 4} is considered as the most probable way. Therefore, much efforts were undertaken last decade, in order to find a proper parametrization of the gauge field in pure QCD (without quarks), which could contain necessary topological properties. The Faddeev soliton picture \cite{5} of QCD excitations was revived in the Faddeev-Niemi knot model \cite{6}, and two different approaches to the parametrization of the QCD gauge field were proposed \cite{7, 8}. One of them was related to the Cho decomposition of the QCD gauge field which included explicitly the magnetic connection \cite{9, 10}.

However, restriction to the pure QCD raises the question, whether such approximation is good. Due to the chiral anomaly \cite{11, 12}, the color gauge sector and the color chiral anomalous sector are parts of the total color space. An experience of the Schwinger model and related phenomenological discussion of 4d case shows \cite{13} that an impact of anomaly is important. The color chiral bosonization \cite{14} describes the anomalous sector in terms of an effective action, and it was found that such an action with a chromomagnetic vacuum background field admits stable soliton-skyrmion solutions \cite{15, 16}. If we neglect anomaly, i.e. if we neglect quark chiral phases then these solitons disappear. The problem of color degrees of quarks and gluons consists in their interrelation: are these degrees of freedom independent, or taking into account quark color chiral phases does not change the number of color degrees of freedom. The case of SU(2) color \cite{17} shows that the quark color field adds topological defects.

In this paper we consider $SU(3)$ color and extend to this case the chiral parametrization of the QCD gauge field developed for SU(2). Due to interdependence of chirally rotated gluons and induced axial vector field, the chiral parametrization for $SU(3)$ involves fields in $CP^2$ space and $U(2)$. We consider background field for gluons and quarks at one-loop level and calculate the Wess-Zumino-Witten topological action for the color chiral field expressed in terms of gluon variables.

In section 2 we introduce the left-right group, the chiral field and anomaly. In section 3 we explain special features of chiral parametrization in the case of $SU(3)$ and introduce $CP^2$, as underlying color space. In section 4 we present the topological action. Section 5 studies the $CP^2$ dynamics of the
vector component of the chiral parametrization. Section 6 discusses results.

2 Left-Right group $SU(3)_L \times SU(3)_R$, chiral field and the anomaly

In this section we introduce Left-Right group $SU(3)_L \times SU(3)_R$, chiral field and the anomaly. Consider massless fermions in external vector and axial vector fields $V_\mu, A_\mu$ and the Dirac operator

$$\mathcal{D}(V, A) = i\gamma^\mu (\partial_\mu + V_\mu + i\gamma_5 A_\mu)$$  \hspace{1cm} (1)

with antihermitean fields in algebra $su(3)$, so that $V_\mu = -V^a_\mu t_a, A_\mu = -A^a_\mu t_a$, where $t_a, a = 1, 2, \ldots 8$ are generators. The chiral transformation of fermions is given by

$$\psi'_L = \xi_L \psi_L, \psi'_R = \xi_R \psi_R, \psi = \psi_L + \psi_R$$  \hspace{1cm} (2)

where $\xi_L(x)$ and $\xi_R(x)$ are local chiral phase factors of left and right quarks $\psi_L$ and $\psi_R$, represented by unitary matrices in defining representations of left $SU(3)_L$ and right $SU(3)_R$ subgroups of the chiral group $G_{LR} = SU(3)_L \times SU(3)_R$. For $\psi_L = \frac{1}{2}(1 + \gamma_5) \psi, \psi_R = \frac{1}{2}(1 - \gamma_5) \psi$, generators $t_{La}$ and $t_{Ra}$ of left and right subgroups of $G_{LR}$ can be written as $t_{La} = \frac{1}{4}(1 + \gamma_5) \lambda_a, t_{Ra} = \frac{1}{4}(1 - \gamma_5) \lambda_a, [t_{La}, t_{Rb}] = 0$, where $\lambda_a, a = 1, 2, \ldots 8$ are the Gell-Mann matrices. Then quark left and right chiral phase factors $\xi_L, \xi_R$ arise from application of operators $\hat{\xi}_L = \text{exp}(-it_{La}\omega_{La}), \hat{\xi}_R = \text{exp}(-it_{Ra}\omega_{Ra})$ to left and right quarks $\psi_L$ and $\psi_R$. Vector gauge transformations $g(x)$ are associated with $t_a = t_{La} + t_{Ra} = \lambda_a/2$, i.e. $g(x)$ has properties of the product $\hat{\xi}_L(x)\hat{\xi}_R(x)$ of identical left and right rotations, $\omega_L = \omega_R = \alpha$. The generator of purely chiral transformations $g_5(x)$ is $t_{5a} = \gamma_5 \lambda_a/2 = t_{La} - t_{Ra}$; thus, $g_5(x)$ has properties of $\hat{\xi}_L(x)\hat{\xi}_R(x)$ for $\omega_L = \omega_R = \Theta$. Infinitesimally, the Dirac operator is transformed according to

$$\delta \mathcal{D} = [i\frac{1}{2}\alpha_\mu \lambda_a, \mathcal{D}] + \{i\frac{1}{2}\gamma_5 \Theta_a \lambda_a, \mathcal{D}\}$$  \hspace{1cm} (3)

Commutation relations for $t_a, t_{5a}$ are given by

$$[t_a, t_b] = if_{abc}t_c, [t_a, t_{5b}] = if_{abc}t_{5c}, [t_{5a}, t_{5b}] = if_{abc}t_c$$  \hspace{1cm} (4)
where $f_{abc}$ are antisymmetrical structure constants of $SU(3)$.

Instead of phases $\xi_L$ and $\xi_R$ one can work with the chiral field $U = \xi_R^+ \xi_L$, which describes rotation of only left quark leaving right quark in peace $\psi_L \rightarrow \psi'_L = U \psi_L$, $\psi_R \rightarrow \psi'_R = \psi_R$. The same result can be obtained by the chiral transformation $\psi_L \rightarrow \xi_L \psi_L$, $\psi_R \rightarrow \xi_R \psi_R$, followed by a vector gauge transformation with a gauge function $\xi_R^+$. The usual chiral gauge choice is $\xi_R = \xi_L^+$, then the chiral field is taken as squared left chiral phase:

$$U = \xi_L^2 = \exp i \Pi = \Pi_\alpha \lambda_\alpha$$  \hspace{1cm} (5)

where we used the flavor notation $\Pi$. To describe $U$ one can use the the Cartan basis with diagonal matrices $H_1, H_2$ and step up/down operators

$$E_{\pm 1} = \frac{1}{2\sqrt{2}} (\lambda_1 \pm i\lambda_2), E_{\pm 2} = \frac{1}{2\sqrt{2}} (\lambda_4 \pm i\lambda_5), E_{\pm 3} = \frac{1}{2\sqrt{2}} (\lambda_6 \pm i\lambda_7)$$

For applications related to SO(3) monopoles and $SU(3)/SO(3)$ coset, it is convenient to consider $SU(3)$ in the $SO(3)$ basis [18]. $SO(3)$ is the maximal subgroup of $SU(3)$. We use two hermitian combinations: $\hat{N}(x)$ and $\hat{N}^2(x)$, where $\hat{N}(x)$ is built on antisymmetric $\lambda$-matrices $\hat{N} = n_k O_k$, $O_k = (\lambda_7, -\lambda_5, \lambda_2)$, $n_k n_k = 1$, $N^3 = N$, while $\hat{N}^2$ contains only symmetric $\lambda$'s. Then general $SU(3)$-chiral field is

$$U(\alpha, \beta) = \exp i \Pi(\alpha, \beta) = \exp i \left( \hat{N} \alpha + \left( \frac{1}{2} \{ \hat{N}', \hat{N}'' \} - \frac{1}{3} tr N'N'' \right) \beta \right)$$ \hspace{1cm} (6)

where $\hat{N}'$, $\hat{N}''$ depend on $SO(3)$ unit vectors $n'_k$, $n''_l$. Three unit vectors plus $\alpha$, $\beta$, give altogether 8 parameters of $SU(3)$.

The chiral tranformation of fermions in the Dirac action is equivalent to the following change of the Dirac operator

$$\bar{\psi}' D(V, A) \psi' = \bar{\psi} D(V^U, A^U) \psi$$

$$V^U = \frac{1}{2} [U^+(\partial + V + A)U + (V - A)U], A^U = \frac{1}{2} [U^+(\partial + V + A)U - (V - A)]$$ \hspace{1cm} (7)

Consider now the fermionic path integral $Z_\psi[V, A] = \int d\mu \exp i \int dx \bar{\psi} / D(V, A) \psi$. Our starting point is such $Z'_\psi$, where fermion chiral phases are extracted from fermions by transformation $\psi \rightarrow \psi'$ in the Dirac action. A transformed fermionic path integral $Z'_\psi$ is equal to an original path integral
as a functional of transformed fields $Z'_\psi[V,A] = Z_\psi[V^U,A^U]$. Thus, we are going to calculate the path integral

$$Z'_\psi[V,A] = Z_\psi[V^U,A^U] = \int d\mu \exp i \int dx \bar{\psi} D(V^U,A^U) \psi,$$

where fields are chirally rotated: $V,A \rightarrow V^U,A^U$. The path integral $Z_\psi$ is invariant under vector gauge transformations of fermions, but undergoes changes under chiral transformations, because of non-invariance of the fermionic measure $d\bar{\psi}d\psi$: chiral transformations are anomalous. The chiral anomaly $A$ is defined by an infinitesimal change of $\ln Z_\psi$ due to an infinitesimal chiral transformation $\delta g_5 = i \theta_\alpha \tau_\alpha \equiv \Theta$.

We put $g_5(s) = \exp \gamma_5 \Theta s$ and write the anomaly $A(x,\Theta)$ at a chiral angle $\Theta$

$$A(x,\Theta) = \frac{1}{i} \frac{\delta \ln Z_\psi(\exp \Theta s)}{\delta s} \bigg|_{s=1}$$

In flavor bosonization the main question is, what is an effective action for the chiral field as a new independent variable. The usual way [12] to calculate such an effective chiral action $W_{\text{eff}}$ is to find the anomaly and integrate it over $s$ up to $g_5 = \exp \gamma_5 \Theta$

$$W_{\text{eff}} = -\int d^4x \int_0^1 ds A(x; s\Theta) \Theta(x) = \int dx L_{\text{eff}} - W_{WZW},$$

where the Wess-Zumino-Witten term $W_{WZW}$ describes topological properties of the chiral field $U$ and is represented by a five-dimensional integral with $x_5 = s$. In general, it cannot be expressed as a 4-dimensional integral over density $L_{WZW}$. The first term in $W_{\text{eff}}$ is of non-topological nature. It is the analogue of $W_{WZW}$ for color that we are interested in. However, in the case of color, the vector field $V_\mu$ is a dynamical one describing gluons, a dynamical axial vector field $A_\mu$ does not exist and the chiral field $U$ cannot be considered as an independent additional variable.

## 3 Chiral parametrization of the QCD vector field

In this section we apply the scheme sketched in the previous section to the case of the QCD vector field, which from now on is denoted by $V_\mu$. It is the field that enters the Dirac operator for quarks, and we should consider it as
a background field. In absence of a dynamical axial vector color field, $A^U$ denotes an axial component of the chirally rotated gauge field

$$V^U = \frac{1}{2}[U^+(\partial + V)U + V], A^U = \frac{1}{2}[U^+(\partial + V)U - V]$$

(11)

Thus, our initial setting for gluons plus quarks can be expressed by the following path integral

$$Z[V,U] = \int d\mu Q \exp i I_{eff} (V + Q) Z_{\psi} [V^U, A^U]$$

(12)

where $Q$ is a quantum field and $I_{eff}$ is the QCD effective action including the Faddeev-Popov ghosts and gauge fixing. In this paper, we are not going to consider an integration over $Q$, but we shall integrate over quarks in order to get the Wess-Zumino-Witten topological action in terms of the chiral field $U$. We work with an initial gauge field $V_\mu$ and two fields arising in its chiral transformation $U$, namely, a gauge field $V^U_\mu$ and an axial vector field $A^U_\mu$. However, by introducing the quark color chiral field $U$ we arrive at a system with too many degrees of freedom. A consistent parametrization of gluons and the chiral field within Left-Right color group becomes a special task. To eliminate superfluous variables, one should consider a relation between gluonic fields $V_\mu$ and chirally transformed field $V^U_\mu$ and find how different parts of these fields can be made from the same material, so that chiral field variables are either fixed, or fully incorporated into gauge field variables. This is a key point in our approach. When this task is accomplished, one can integrate chiral color anomaly and get a topological term, an analogue of the Wess-Zumino-Witten action.

Two simple expressions exist for $V^U \pm A^U$, namely

$$V^U_\mu + A^U_\mu = U^+(\partial_\mu + V_\mu)U, V^U_\mu - A^U_\mu = V_\mu$$

(13)

which mean that if the chiral field is considered as a regular gauge transformation, then $(V^U \pm A^U)_\mu$ combinations should have the same field strengths $(V^U \pm A^U)_\mu$. With topological $U$ these combinations can describe different situations.

In order to find a partial parametrization of gluonic field in terms of chiral parameters, let us at first define a combination of the gauge field $V^U_\mu$ and the chiral field $U$, which is invariant under chiral transformation:

$$(V^U_\mu)^U = V^U_\mu$$

(14)
A gauge field with this property we denote $V^\Omega_\mu$. It will contain $U$-variables. An axial vector field $A^\Omega_\mu$ calculated with $V^\Omega_\mu$ instead of $V_\mu$ is absent: $A^\Omega_\mu \equiv \frac{1}{2} [U^+(\partial_\mu + V^\Omega_\mu)U - V^\Omega_\mu] = 0$. It follows also that

$$U^2 = \exp i\zeta, \partial_\mu \zeta = 0$$

independently of the chiral color group. The invariance relation cannot be true in the whole space of chiral color, but it can be satisfied in a region with restricted number of variables. Such a region can be found by studying the $(V_\mu^U \to V_\mu)$ determinant.

Consider relation between $8\times8$ matrices of gluonic field $V_\mu$ and chirally rotated field $V_\mu^U (x)$ in adjoint representation

$$(V_\mu^U)_{ab} = \frac{1}{2} \left( 1 + R(U) \right)_{ab} V_{\mu b} + \frac{i}{g^2} (U \partial_\mu U^+)_{ab}, R_{ab} (U) = \frac{1}{2} \text{tr} \left( \lambda_a U \lambda_b U^+ \right)$$

where the chiral field $U = \xi_L^a = \exp i\Theta, \Theta = \lambda_a \Theta_a$, is defined in the chiral gauge $\xi_L = \xi_R^+$. Here $\lambda_a, a = 1, 2...8$, are the Gell-Mann matrices.

We write $U$ in flavor-like notation

$$U = \exp i\Pi, \Pi = \lambda_a \Pi_a$$

It is also convenient to write $\Pi$ in the SO(3) basis (see Section 2).

In order to calculate the determinant $\det \frac{1}{2} \left( 1 + R(U) \right)$, it is sufficient to consider the case $N' = N'' = N$, when

$$U \to U \left( \hat{N}, \alpha, \beta \right) = \exp \left( -\frac{2}{3} i\beta \right) \left[ 1 + i\hat{N} e^{i\beta} \sin \alpha + \hat{N}^2 \left( e^{i\beta} \cos \alpha - 1 \right) \right]$$

When eigenvalues of $\hat{N}$ are placed as diag(1, -1, 0), we see that $\alpha = \Theta_3, \beta = \Theta_8 \sqrt{3}$ and

$$\det \frac{1}{2} \left( 1 + R(U) \right) = \frac{1}{2} \left( 1 + \cos 2\alpha \right) \frac{1}{2} \left( 1 + \cos \left( \alpha + \beta \right) \right) \frac{1}{2} \left( 1 + \cos \left( \alpha - \beta \right) \right)$$

This result can be easily checked in (isospin $I_3$, hypercharge $Y$) basis, where the diagonal elements of $R(U) = \exp i \left( I_3 2\alpha + Y 2\beta \right)$, as of the adjoint representation of $U$, are nothing, but those of octet: pions ($I=1$, $Y=0$), K-mesons ($I=1/2$, $Y=1/2$), K-mesons ($I=-1/2$, $Y=-1/2$), $\sigma$-meson ($I=0, Y=0$).
This determinant is invariant under reflection \((\alpha, \beta) \rightarrow (-\alpha, -\beta)\). It disappears for values \((\alpha, \beta)\) equal to \((\pi/2, 0), (\pi/2, \pm \pi/2)\) and \((0, \pi)\) characterizing singularity surfaces in chiral color space (i.e. \(\gamma_5 \Theta\)). The first of these sets, \((\pi/2, 0)\), corresponds to \(SO(3)\) subgroup with one zero factor in \(U(N, \pi/2, 0) = \exp i \hat{N} \alpha = 1 + i \hat{N} - \hat{N}^2\).

For \(SU(3)\) we are interested in two coinciding zero factors of \(\det\) related to two simple roots of \(SU(3)\). Together with a singularity set related to a complex root, we can define three color chiral fields \(U(\hat{N}, \alpha, \beta)\) for \(SU(3)\).

For the pair \((\alpha, \beta) = (0, \pi)\) we get the chiral field 

\[
U(\hat{N}, 0, \pi) = (1 - 2\hat{N}^2) \exp(-i\frac{1}{3} \pi) = m_1 \exp(-i\frac{1}{3} \pi) \tag{20}
\]

For \((\alpha, \beta) = (\pi/2, \pi/2)\) we have

\[
U(\hat{N}, \pi/2, \pi/2) = (1 - N - N^2) \exp(-i\pi/3) = m_2 \exp(-i\pi/3) \tag{21}
\]

We denote \(U(\hat{N}, \pi/2, -\pi/2) = m_3 \exp(-i\pi/3)\). In all these cases \(m_k\) are normalized hermitian \(3 \times 3\) matrices in color space : \(m^2 = 1\). A product of two \(m\) 's is equal to the third \(m\) up to a constant phase. Their diagonal forms are \(m_1^0 = \text{diag}(-1, -1, 1), m_2^0 = \text{diag}(-1, 1, 1)\) and \(m_3^0 = \text{diag}(1, -1, 1)\). The same diagonal forms correspond to skyrmion-type embeddings of \(SU(2)\) in \(SU(3)\), when \(\Pi = \lambda_k \Pi_k + E_{33}, E_{33} = \text{diag}(0, 0, 1)\). In the general case, matrices \(m_k\) are given by

\[
m_k = S m_k^0 S^+ \tag{22}
\]

where \(S\) is a unitary \(SU(3)\) transformation common for all \(k = 1, 2, 3\). Then \(S\) is defined up to right multiplication by a diagonal unitary matrix, i.e \(S\) is in the coset \(SU(3)/U(1)^2\) and depends on six parameters associated with nondiagonal \(\lambda\)'s. When each \(m\) is considered separately, so that \(m_k = S(k)m_k^0 S^+(k)\), then \(S(k)\) is in the coset \(SU(3)/U(2)\), and different \(m_k\) do not commute. Unlike \(SU(2)\) unit matrix, the \(SU(3)\) matrices \(m_k\) are not traceless. Traceless diagonal matrices \(m_k^0 - 1/3, k = 1, 2, 3\), are in the following relation with roots \(r(k)\):

\[
m_k^0 - 1/3 = \pm D^{(k)} \tag{23}
\]

where \(D^{(k)}\) gives \(SU(3)\) magnetic monopole embedding along roots \(r(k)\).

Chirally invariant vector field \(V_\mu^\Omega\). Behavior of the \((V_\mu^U \rightarrow V_\mu)\) determinant \(\det \frac{1}{2} (1 + R)\) shows, how to restrict chiral color space in defining the
chiral field $U$, in order to construct the field $V^\Omega_\mu$. Gauge invariant chiral regions $\Omega$ are given by sets $(\alpha, \beta) = (0, \pi), (\pi/2, \pm \pi/2)$, where the chiral field $U = m_k$ is represented correspondingly by one of $3 \times 3$ unit matrices $m_k, k = 1, 2, 3$ up to a constant phase.

In the case of $SU(2)$ color, the chiral parametrization of gluons [17] defines the vector component $V^\Omega_\mu = \hat{n}C_\mu + \frac{1}{2}\hat{n}\partial_\mu\hat{n}$ in terms of unit vector $\hat{n} = \tau_k n_k$ belonging to the chiral field and which is covariantly constant: $D_\mu (V^\Omega_\mu) \hat{n} = 0$. The field $V^\Omega_\mu$ was introduced first in the decomposition of gluons in order to include topological structures into gluonic decomposition [9], [6] and exploit an idea of the abelian dominance. It was done without recourse to the chiral anomaly. Similar approaches to $SU(3)$ [23], [24] are based on the Cartan algebra $H_k, k = 1, 2, 3$, and lead to $SU(3)/U(1)^2$ submanifold $S$ with two independent vectors $n_k = SH_k S^{-1}$ and two independent $V^\Omega_\mu$-like fields.

The chiral parametrization changes the situation. Let us formulate first the result and then discuss the difference with the case, when the anomaly is neglected. The chiral parametrization of gluons in $SU(3)$ QCD contains only one unit $SU(3)$ matrix $m, m^2 = 1, \text{tr} m = 1$, defining a chiral field $U = m$, a direction in $SU(3)$ space. The chirally invariant vector field $V^\Omega_\mu$ is given by

$$V^\Omega_\mu = C_\mu (m - 1/3) + G_\mu + \frac{1}{2} m \partial_\mu m,$$

(24)

where $m$ denotes one of three unit matrices $m_k$ and $G_\mu$ is a $U(2)$ component of $V^\Omega_\mu$, so that $[m, G_\mu] = 0$. It can be considered as a result of chiral rotation $U = m$ applied to a pair $(V_\mu, A_\mu) = (C_\mu m + G_\mu, \frac{1}{2} m \partial_\mu m)$. The chirally rotated vector component reproduces itself,

$$(V^\Omega_\mu)^U = V^\Omega_\mu, V^\Omega_\mu = \frac{1}{2} \left( UV^\Omega_\mu U^+ + V^\Omega_\mu + U \partial_\mu U^+ \right), U = m$$

(25)

while $A^U_\mu$, as an axial complement of $V^\Omega_\mu$, disappears

$$A^U_\mu = \frac{1}{2} UD_\mu \left(V^\Omega \right) U^+ = 0$$

(26)

The matrix $m$ is covariantly constant: $D_\mu \left(V^\Omega \right) m = 0$.

In general, the chiral matrix $\Pi$ can be written as a sum over two independent roots, or two $m_k$’s, like $(am_1 + bm_2)$ instead of single $m$. However, the invariance condition can be satisfied only with $a = \pm b = \pi/2$, or $U = m_3$.  


Therefore, $U$ and $m \partial m$ term depend always only on one matrix $m$. It can be interpreted that the field $V_\mu^\Omega$ asymptotically might describe only one-monopole configurations.

Thus, the chiral anomaly imposes important restrictions on chiral field $U$ and subspace of $SU(3)$, which can be used for chiral parametrization of gluonic field $V_\mu$. Basic matrix $m$, which is a counterpart of traceless $SU(2)$ unit matrix $\hat{n}$, cannot be traceless, because it comes from an unitary matrix. Though we have choice of three possibilities $m_k$, the $SU(3)$ QCD uses only one. We shall take it as $m^0 = \text{diag}(1, 1, -1) = 2Y + 1/3$, where $Y$ denotes the color hypercharge. This corresponds to the chiral field $U = \exp iY \pi$. Then $Sm^0 S^{-1}$ is the orbit of $SU(3)$ through $Y$. The $SU(3)$ matrix $S$ is defined up to right multiplication $S \rightarrow Sg$, $g \in U(2)$, where $U(2)$ is built on $\lambda_k$, $k = 1, 2, 3, 8$. The $SU(3)$ matrix $S$ includes $\lambda_A$, $A = 4, 5, 6, 7$. Unit matrix $m$ commutes with $\lambda_k$ and anticommutes with $\lambda_A$. Matrix $\tilde{m}$ can be identified with $\lambda_3$.

Induced axial vector field $A_\mu^U$. Consider basic relation $V_\mu^U - A_\mu^U = V_\mu$. It follows from the structure of chirally rotated fields $V_\mu^U$ and $A_\mu^U$ that any field $B_\mu$ commuting with matrix $U = m$ contributes only to rotated vector field $V_\mu^U$, while a field $X_\mu$ anticommuting with $m$ contributes only to the induced axial vector field $A_\mu^U$, and this property does not depend on a particular color group. Thus, the latter set of $\lambda$’s can be used to build a contribution $X_\mu$ to field $A_\mu^U$

$$X_\mu = S \lambda_a x_{a\mu} S^+, \quad a = 4, 6, 6, 7,$$

with the property $\{X_\mu, m\} = 0$. Another example of matrices anticommuting with $m$ provide derivatives $\partial_\mu m$ and $m \partial_\mu m$, which we use to construct a contribution $Y_\mu$ to $A_\mu^U$

$$Y_\mu = \varphi \partial_\mu m + i \chi m \partial_\mu m$$

with $\varphi$ and $\chi$ are colorless functions. Such terms exist in the two-color decomposition [6]. The decomposition of gluons $V_\mu$ is given by

$$V_\mu = V_\mu^U - A_\mu^U = V_\mu^\Omega - \hat{A}_\mu, \quad \hat{A}_\mu = X_\mu + Y_\mu$$

It is easy to check that an axial field $\hat{A}_\mu$ arises in chiral rotation $U = m$ from $V_\mu = V_\mu^\Omega - \hat{A}_\mu$. If before chiral rotation $U = im$ the quark path integral depends on the gauge field only, $Z_\psi [V_\mu, 0] = Z_\psi [V_\mu^\Omega - \hat{A}_\mu, 0]$, then after this rotation we get $Z_\psi [V_\mu^\Omega, \hat{A}_\mu]$.
Thus, as it follows from expressions for the QCD vector field ('gluons') \( V_\mu \) there are two distinct sectors in the chiral decomposition of \( V_\mu = V_\mu^\Omega - \hat{A}_\mu \):

(a) The \( CP^2 \)-sector with the dynamical abelian field \( C_\mu \). The space \( CP^2 \) enters the scene with the chiral field \( U = m = S m_0 S^+ \), as the \( SU(3) \) orbit through \( m_0 \). The matrix \( m \) is the main element, which defines the vector field in the sector \((V_\mu^\Omega)_{CP^2} = m C_\mu + \frac{1}{2} m \partial_\mu m \), including the direction of an abelian field \( C_\mu \) in the \( SU(3) \) space. The axial component \( \hat{A}_\mu \) anticommutes with \( m \).

(b) An \( U(2) \) sector with the field \((V_\mu^\Omega)_{U(2)} = G_\mu \); the chiral field \( U = m \) commutes with \( G_\mu \).

\section{Topological action}

The effective action \( W_{\text{eff}} \)

\[ W_{\text{eff}} = W_{\text{WZW}} + W_{\text{an}} = - i \{ Z_\psi \left[ V_\mu^\Omega, \hat{A}_\mu \right] Z_\psi^{-1} \} \]  

(30)

describes the quark color chiral contribution to QCD dynamics and contains the topological term \( W_{\text{WZW}} \), which is an analogue of the chirally gauged Wess-Zumino-Witten action in flavor physics [20], [21], as well as a non-topological action \( W_{\text{an}} \). Non-topological term \( W_{\text{an}} \) has the same structure in \( SU(3) \) as in \( SU(2) \).

The gauged Wess-Zumino-Witten term \( W_{\text{WZW}} \) adapted for the general color case, (i.e. no dynamical axial vector, \( A_\mu = 0 \), and \( U \) belongs to \( SU(3) \)), is given in the Minkowski space by the five-dimensional integral (see also [22]):

\[ W_{\text{WZW}} = \frac{i}{96 \pi^2} \int d^5 x \varepsilon_{\mu \nu \sigma \lambda \rho} tr \left[ (j^-_\mu + j^+_{\mu}) V_{\nu \sigma} V_{\lambda \rho} + \right. \]

\[ \frac{1}{2} \left( j^-_\mu V_{\nu \sigma} U_s V_{\lambda \rho} U_s^{-1} + j^+_{\mu} V_{\nu \sigma} U_s^{-1} V_{\lambda \rho} U_s \right) - i V_{\mu \nu} \left( j^-_\sigma j^-_\lambda j^-_\rho + j^+_\sigma j^+_\lambda j^+_\rho \right) \]

\[ - \frac{2}{5} \varepsilon_{\mu \nu \sigma \lambda \rho} j^-_\mu j^-_\nu j^-_\sigma j^-_\lambda j^-_\rho \]  

(31)

where the following notations are used

\[ j^-_\mu = D_\mu U_s U_s^{-1}, j^+_{\mu} = U_s^{-1} D_\mu U_s, U_s = \exp s \Theta \]
\[ D_\mu U_s = \partial_\mu U_s + [V_\mu, U_s], x_5 = s \]

and the convention \( \mu, \nu, \ldots = 1, 2, 3, 4, 5; L_5 = R_5 = 0 \) implied. The fifth integration here is an \( s \) -integration over chiral anomaly. An additional integration arises from the relation \( \ln Z_\psi [V, 0] - \ln Z_\psi [V^U, A^U] = - \int ds \partial_s \ln Z_\psi [U(s)] \) with boundary points \( s = 0, U(0) = 1 \), and \( s = 1, U(1) = U \). For the case under consideration, when \( V^U_\mu = V_\mu^\Omega, A^U_\mu = \hat{A}_\mu, V_\mu = V_\mu^\Omega - \hat{A}_\mu \), we put \( U(s) = \exp\{ims\pi/2\} \) using relations \( D_\Omega^{\nu} = 0 \) and \( m^2 \hat{A}_\mu^\Omega + \hat{A}_\mu^\Omega m = 0 \). An integration over \( s \) is simple, so that in the case of color bosonization there exists a topological Lagrangian corresponding to \( W_{\text{WZW}} \)

\[
L_{\text{WZW}} = \frac{1}{32\pi} \varepsilon_{\nu\sigma\lambda\rho} \text{tr} m \left\{ \frac{1}{2} V^\Omega_\nu V^\Omega_\lambda - V^\Omega_\nu (\hat{A}_\lambda + [\hat{A}_\lambda, \hat{A}_\rho]) - [\hat{A}_\nu, \hat{A}_\sigma] \hat{A}_\lambda \right\}
\]

\[
\hat{A}_\nu = D^\Omega_\nu \hat{A}_\nu - D^\Omega_\sigma \hat{A}_\nu, D^\Omega_\nu = \partial_\nu + [V^\Omega_\nu, \hat{A}] \tag{32}
\]

where \( D^\Omega \) contains the field \( V^\Omega_\nu \), while \( V^\Omega_{\mu\nu} \) is the field strength of \( V_\mu^\Omega \). The Wess-Zumino-Witten part does not contain terms with four fields \( \hat{A} \).

### 5 Vector component \( V^\Omega_\mu \) in \( CP^2 \) sector

We are interested in variables appearing in \( CP^2 \) sector of the chiral decomposition of gluons \( V_\mu = V_\mu^\Omega - \hat{A}_\mu \). To this end, we consider the vector component \( V^\Omega_\mu \) only in \( CP^2 \)-sector, i.e. we neglect the \( U(2) \) field \( G_\mu \) and the axial part \( \hat{A}_\mu \). Then

\[
\Gamma_\mu = (V^\Omega_\mu)_{CP^2} = C_\mu \hat{m}' + \frac{1}{2} \hat{m} \partial_\mu \hat{m} \tag{33}
\]

where \( \hat{m}' = m - 1/3 \). The field strength \( \Gamma_{\mu\nu} \) is expressed in terms of basic unit \( SU(3) \) matrix \( \hat{m} \)

\[
\Gamma_{\mu\nu} = C_{\mu\nu} \hat{m}' + \frac{1}{4} [\partial_\mu \hat{m}, \partial_\nu \hat{m}] \tag{34}
\]

The matrix \( \hat{m} \) is normalized according to

\[
\hat{m}^2 = 1, \text{tr} \hat{m} = 1 \tag{35}
\]

and can be considered as a unitary transform of a constant diagonal matrix \( \hat{m}_0 = \text{diag} (1, 1, -1) \)

\[
\hat{m} = S \hat{m}_0 S^{-1}, SS^+ = 1 \tag{36}
\]
The unitary matrix $S$ is defined up to right $U(2)$ transformations $g : S \rightarrow Sg$, built on color analogues of isospin $I_C$ and hypercharge $Y_C$ generators.

$$g = \exp i (\lambda_k \beta_k + m_0 \phi) \equiv g_2 \exp im_0 \phi, gm_0 g^{-1} = m_0$$  \hspace{1cm} (37)

Then $S$ can be written as a function of four parameters associated with matrices $\lambda_4, \lambda_5, \lambda_6, \lambda_7$ and described by a hermitian matrix $\alpha$ and an $U(2)$-scalar $\omega$

$$S = \exp(i\alpha \omega), tr\alpha^2 = 2$$  \hspace{1cm} (38)

Explicitly

$$\hat{\alpha} = \hat{\xi} + \hat{\xi}^+$$
$$\hat{\xi} = \frac{1}{2} (\lambda_4 + i\lambda_5) \xi_1 + \frac{1}{2} (\lambda_6 + i\lambda_7) \xi_2$$  \hspace{1cm} (39)

Here isospinor components $\xi_1, \xi_2$ with quantum numbers of color $K$-fields $I_C = \pm \frac{1}{2}, Y_C = 1$ are normalized according to $\xi_1^+ \xi_1 + \xi_2^+ \xi_2 = 1$. Other components of color octet ($\pi$ and $\eta$) can be found in $\alpha^2$.

Under $m_0$-preserving transformation $g$ the matrix $\alpha$ and isospinor $\xi = \frac{1}{2} (1 + m_0) \alpha$ transform according to

$$\alpha' = g\alpha g^{-1}, \frac{1}{2} (1 + m_0) \alpha' = \xi' = g_2 \xi \exp im_0 \phi$$  \hspace{1cm} (40)

It is easy to verify that $\hat{\alpha} = \hat{\alpha}^3$, so that $\hat{\alpha}$-eigenvalues are $\pm 1, 0$. Thus, $S$ can be written as

$$S = 1 + i\hat{\alpha} \sin \omega + \hat{\alpha}^2 (\cos \omega - 1)$$  \hspace{1cm} (41)

Now, $P_\pm = \frac{1}{2} (\hat{\alpha}^2 \pm \hat{\alpha})$ and $P_0 = (1 - \hat{\alpha}^2)$ are projection operators on eigenvalues $\pm 1, 0$. We can build an isovector $n_k$ from isospinor $\xi$ with components $\xi_a = (\alpha)_{ka}, a = 1, 2$ and its conjugated isospinor $\xi^+$ with components $\xi^+_k = (\alpha)_{3a}$

$$n_k = \xi^+ \tau_k \xi$$  \hspace{1cm} (42)

while $(\alpha^2)_{ab} = \xi_a \xi^+_b$ and $(\alpha^2)_{33} = \xi^+ \xi = \xi^+_a \xi_a = 1$. An $SU(2)$ unit vector $\hat{n}$ is an useful collective variable entering the knot model \[5\]. Isospinors $\xi, \xi^+$, as members of octet, correspond to quark-antiquark variables, while isovector $\hat{n}$, as a bilinear combination of $\xi^+, \xi$, may represent four-quark degrees of freedom, though it belongs also to the same octet.

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A relation between $3 \times 3$ matrix $\alpha$, isospinor $\xi$, $2 \times 2$ unit matrix $\hat{n} = \tau_k n_k$ and $3 \times 3$ matrix $n$ is given by

$$\hat{\xi} = \frac{1}{2} (1 + m_0) \hat{\alpha}$$
$$\hat{\alpha}^2 = \frac{1}{2} (1 + n) = (\hat{n} + 1)/2 \quad (43)$$

It means also that in isospin subgroup a matrix $\hat{\alpha}^2$ acts as a projector on eigenvalue $\hat{n}' = +1$, while in $SU(3)$ it is a projector on eigenvalues $\alpha' = \pm 1$. The matrices $\lambda_4, \lambda_5, \lambda_6, \lambda_7$ anticommute with $m_0$, therefore

$$\hat{m} = S m_0 S^{-1} = U m_0 U = S^2 \quad (44)$$

Thus, the vector component $V^\alpha_\mu$ of the gluonic field can be expressed either in terms of $3 \times 3$ hermitian matrix $\alpha$ and an $SU(3)$ scalar $\omega$, or in terms of isospin doublets $\hat{\xi}$ and $\hat{\xi}^+$. We are interested in composite variables based on elementary $\hat{\xi}$ and $\hat{\xi}^+$. Then it is sufficient to consider the case of constant $\omega$. We have for $\omega = \pi/4$, when $(U)_{33} = 0$, the following relations

$$U \partial_\mu U^+ E_+ = -i (\eta_\mu + \eta_\mu^+) + b_\mu \left( 1 + \hat{\xi}^+ \right) + B_\mu U \partial_\mu U^+ E_- = b_\mu \quad (45)$$

where $b_\mu$ and $B_\mu$ are isoscalar and $2 \times 2$ matrix currents

$$b_\mu = (\xi^+ \partial_\mu \xi) = - (\partial_\mu \xi^+ \xi), B_\mu = \hat{\xi} \partial_\mu \hat{\xi}^+ - \partial_\mu \hat{\xi} \hat{\xi} \quad (46)$$

$\eta_\mu, \eta_\mu^+$ are gauge invariant isospinors

$$\eta_\mu = \partial_\mu \xi - b_\mu \xi, \eta_\mu^+ = \partial_\mu \xi^+ + b_\mu \xi^+ \quad (47)$$

and $E_{\pm}$ are projectors $E_{\pm} = \frac{1}{2} (1 \pm m_0)$. In the $SU(2)$ color case, the vector $b_\mu$ is called magnetic potential.

The field strength $\Gamma_{\mu\nu}$ in new variables $\hat{\xi}, \hat{\xi}^+$ is given by

$$\Gamma_{\mu\nu} = C_{\mu\nu} (m - 1/3) - \frac{1}{4} \left[ U \partial_\mu U^+, U \partial_\nu U^+ \right] = \Gamma^+_{\mu\nu} + \Gamma^-_{\mu\nu} + \Gamma^\xi_{\mu\nu}$$

$$\Gamma^+_{\mu\nu} = C_{\mu\nu} \left( 1 - \hat{\xi} \hat{\xi}^+ \right) - \frac{1}{4} \eta_{\mu\nu} \hat{\xi} \hat{\xi}^+ - \frac{i}{4} \left( \partial_\mu \left( \hat{\xi} \hat{\xi}^+ \right) b_\nu - \partial_\nu \left( \hat{\xi} \hat{\xi}^+ \right) b_\mu \right)$$

$$\Gamma^-_{\mu\nu} = C_{\mu\nu} \left( \hat{\xi} \hat{\xi}^+ \right) - \frac{1}{4} \eta_{\mu\nu} \hat{\xi} \hat{\xi}^+ - \frac{i}{4} \left( \partial_\mu \left( \hat{\xi} \hat{\xi}^+ \right) b_\nu - \partial_\nu \left( \hat{\xi} \hat{\xi}^+ \right) b_\mu \right)$$

$$\Gamma^\xi_{\mu\nu} = C_{\mu\nu} \left( \hat{\xi} \hat{\xi}^+ \right) - \frac{1}{4} \eta_{\mu\nu} \hat{\xi} \hat{\xi}^+ - \frac{i}{4} \left( \partial_\mu \left( \hat{\xi} \hat{\xi}^+ \right) b_\nu - \partial_\nu \left( \hat{\xi} \hat{\xi}^+ \right) b_\mu \right)$$
\[ \Gamma_{\mu\nu}^- = -\frac{1}{4} h_{\mu\nu} \Gamma^\xi_{\mu\nu} = -\frac{i}{4} \{ h_{\mu\nu} (\hat{\xi} - \hat{\xi}) - b_\mu \partial_\nu (\hat{\xi} - \hat{\xi}^+) + b_\nu \partial_\mu (\hat{\xi} - \hat{\xi}^+) \} \] (48)

It follows that the field strength \( \Gamma_{\mu\nu}^- \) describes two interacting systems. One of them is characterized by color isospin matrix variables \( \hat{\xi}, \hat{\xi}^+ \) with density \( \rho = \hat{\xi} \hat{\xi}^+ \), the second one is characterized by hypercharge like potential \( b_\mu \) and field strength \( h_{\mu\nu} \). The latter variable is directly related to an important quantity \( tr (\hat{m} [\partial_\mu \hat{m}, \partial_\nu \hat{m}] ) \), which for \( \omega = \text{const} \) reduces to

\[ tr (\hat{m} [\partial_\mu \hat{m}, \partial_\nu \hat{m}] ) = -h_{\mu\nu} \] (49)

An expression of \( h_{\mu\nu} \) in terms of the magnetic potential \( b_\nu \) involves a super-current \( \Sigma_{\mu\nu} \):

\[ h_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu - \Sigma_{\mu\nu} \] (50)

where \( \Sigma_{\mu\nu} = (\xi^+ [\partial_\mu, \partial_\nu] \xi) \). The quantity \( h_{\mu\nu} \) can be associated with magnetic field strength. In this picture of two interacting systems \( (\hat{\xi}, \hat{\xi}^+) \) and \( b_\mu \), kinetic terms in the Yang-Mills Lagrangian will be quadratic.

The WZW Lagrangian for \( \Gamma_\mu \) in new variables for the case \( \omega = \text{const} \) is given by

\[ L_{WZW} = \frac{1}{64 \pi} \varepsilon_{\nu\sigma\lambda\rho} tr \{ m \Gamma_{\nu\sigma} \Gamma_{\lambda\rho} \} = \] \[ = \frac{1}{64 \pi} \int d^4 x \varepsilon_{\nu\sigma\lambda\rho} \left\{ \frac{8}{3} C_{\mu\nu} C_{\lambda\sigma} - \frac{1}{2} C_{\mu\nu} h_{\lambda\sigma} + h_{\mu\nu} h_{\lambda\sigma} w (\omega) \right\} \] (51)

where \( w \) is a known function of \( \omega \).

Totally antisymmetric expression the action \( W_{WZW} \) is reproducing itself in partial integration, when

\[ \int d^4 x \partial_\mu \varepsilon_{\mu\nu\lambda\sigma} tr \{ \partial_\nu m \partial_\lambda m \partial_\sigma m \} = 0 \]

This leads to conservation of

\[ Q_m = \int d^3 x \varepsilon_{ijk} tr \{ \partial_i m \partial_j m \partial_k m \} \] (52)

In general, in a similar manner the action \( W_{WZW} \left( V_\mu^\Omega \right) \) tells what kind of charges can be introduced in \( V_\mu^\Omega \)-sector and what combination of charges is conserved.
6 Conclusions

It is shown that in the chiral parametrization of the QCD vector field for color $SU(3)$ an effective color space is defined by the hermitean color chiral field $U = m$ representing an orbit through hypercharge $SYS^{-1}$. The parametrization contains an abelian field directed along $m$ and an $U(2)$ field $G_\mu$ which commutes with $m$. The axial component $A_\mu$ anticommutes with $m$ and belongs to tangential bundle of $CP^2$. Thus, the chiral parametrization restricts the color space ascribed to gluons in absence of quark chiral color. This parametrization is quite different from those [23], [24],[25], where the chiral anomaly is neglected.

The $CP^2$ sector is studied. The chiral parametrization of the vector component $\Gamma_\mu$ involves an abelian field $C_\mu$, isospinors $\xi, \xi^+$ as building blocks for composite fields $B_\mu, b_\nu, h_{\mu\nu}$, as well as an isoscalar $\omega$. In the induced axial sector we have four components of axial vector field in the $X-$ sector and scalars $\varphi$ and $\chi$ in axial $Y-$ sector. The abelian fields $b_\nu, h_{\mu\nu}$ are of magnetic type (in language of $SU(2)$). The field strength in the vector sector $\Gamma_{\mu\nu}$ describes two interacting systems. One of them is characterized by color isospin matrix variables $\omega \xi, \omega \xi^+$, the second one is characterized by hypercharge like potential $b_\mu$ with field strength $h_{\mu\nu}$. Kinetic terms for these fields in the Yang-Mills Lagrangian are quadratic.

We hope to report applications of our parametrization of the QCD field in future papers.

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