NON-COMMUTATIVE POISSON STRUCTURE ON NON-COMMUTATIVE ALGEBRAIC TORUS ORBIFOLDS

SAFDAR QUDDUS

Abstract. We study the Gerstenhaber algebra structure on the algebraic non-commutative torus (also called quantum torus) orbifolds resulting by the action of finite subgroups of $SL_2(\mathbb{Z})$. We also examine the Poisson structure(s) and compute the Poisson cohomology.

0. Introduction

Inspired by this relationship between Poisson geometry and deformation theory, Jonathan Block and Ezra Getzler [BG] and Ping Xu [Xu] independently introduced a notion of a noncommutative Poisson structure and the non-commutative Poisson cohomology was studied by Xu explicitly for the smooth non-commutative 2-torus $A_\theta$. The paper of Halbout and Tang [HT] is the seminal paper for studying non-commutative Poisson structures on orbifolds or smash products. [HT] studied the Gestenhaber bracket and Poisson structure(s) on the $C^\infty(M) \rtimes G$, they showed that the Gerstenhaber bracket on the orbifold is a generalization of the classical Schouten-Nijenhuis bracket on manifold. This Gersten- haber bracket on the orbifold, called as the “twisted Schouten-Nijenhuis bracket” was used by them to understand the Poisson structures on the orbifold by solving $[\Pi,\Pi] = 0$ for $\Pi \in H^2(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G)$. This is how the non-commutative Poisson structure was studied for $A \rtimes G$ for $A = C^\infty(M)$. We can ask the question, what about such structures when $A$ is a non-commutative (/smooth)manifold.

The graded Lie bracket on Hochschild cohomology remains elusive in contrast to the cup product. The latter may be defined via any convenient projective resolution. But, the former is defined on the bar resolution, which is useful theoretically but not computationally, and one typically computes graded Lie brackets by translating to another more convenient resolution via explicit chain maps. Such chain maps are not always easy to find. Studying the Gerstenhaber bracket on the Hochschild cohomology also gives us an insight to the deformations of the algebra, Hochschild cohomology of the $A \rtimes G$, along with the Gerstenhaber bracket, is reflective of the deformation theory of modules, orbifold cohomology and Poisson cohomology. [PPTT] [LW1] [LW2] [BG].

Researchers like Witherspoon, Shepler, Negron and Zhou in various collaborations have studied the “twisted Schouten-Nijenhuis bracket” for several variant spaces and discrete group actions, including the polynomial ring $Sym(V)$ and the quantum polynomial ring $S_q(V)$ [SW1] [WZ] [NW]. They used appropriate Koszul resolution to study the Gerstenhaber bracket structure. In the paper [WZ] the discrete group action on the quantum symmetric polynomial algebra is considered for the special case when the action by a discrete group $G$ on it is diagonal($v_i \mapsto \chi_g v_i$ for some character $\chi: G \to \mathbb{C}^\times$). The Koszul resolution
for the quantum symmetric polynomial is due to Wambst [W1] which he obtained by generalizing the Connes’ Koszul resolution for non-commutative smooth 2-torus [C] to quantum symmetric algebra $S_q(V)$ associated to a vector space $V$. The algebraic non-commutative torus also known as the quantum torus, can be thought as a quantum symmetric algebra in 2-variables but unlike Witherspoon and Zhou we shall be considering non-diagonal discrete group action on it arising by the restriction of the action of $SL_2(\mathbb{Z})$ on the smooth non-commutative torus $A_\theta$.

In fact Gerstenhaber bracket is a generalization of the usual Schouten-brackets of multivector fields [T] and we saw in [HT] that the Schouten-bracket gets twisted over orbifolds of smooth manifolds. Here, in this example, we shall see that over the non-commutative orbifolds such a twist can in fact be similarly understood over each twisted component arising by the paracyclic or the spectral decomposition of the cohomology.

Non-commutative torus was studied extensively by Connes and Rieffel in the 1980’s and arose as a quantum deformation of the algebra of smooth functions on the classical 2-torus. It can also be seen as the irrational rotational algebra [R]. Several invariants of the same were studied, which conformed the smoothness of the “associated non-commutative space”. The existence of non-commutative Poisson structure [Xu] on the non-commutative torus establishes it as a preluding example to study noncommutative smooth spaces. What we study here is a dense sub-algebra of it. Let $\theta \in \mathbb{R}$, consider the group algebra $\mathbb{C} < U_1^{\pm 1}, U_2^{\pm 1} >$ of the free group on two generators $U_1$ and $U_2$, with coefficients in $\mathbb{C}$. We define

$$A_{\theta}^{alg} := \mathbb{C} < U_1^{\pm 1}, U_2^{\pm 1} > / < U_2 U_1 - \lambda U_1 U_2 >$$

for a fixed parameter $\lambda = e^{2\pi i \theta} \in \mathbb{C}$. Unless specified otherwise, we will assume in this paper that

$$\lambda^n \neq 1 \text{ for all } n \in \mathbb{Z} \quad (0.1)$$

The algebra $A_{\theta}^{alg}$ may be thought of as a ring of functions on a noncommutative algebraic torus. Geometrically, the algebras $A_{\theta}$, the non-commutative smooth 2-torus, arise as deformations of the ring $C^\infty(\mathbb{T})$ of smooth functions on the two-dimensional torus $\mathbb{T} = S^1 \times S^1$, and as such, these are fundamental examples of noncommutative differentiable manifolds in the sense of Connes [C]. On the other hand, algebraically, $A_{\theta}^{alg}$ is just a certain norm completion of $A_{\theta}$.

Xu studied the Poisson structure on non-commutative smooth torus $A_{\theta}[Xu]$. The unique 2-cocycle of the non-commutative smooth torus $A_{\theta}$ when $\theta$ satisfies the Diophantine condition gave the Poisson structure. The Poisson cohomology groups were also studied therein, the non-trivial Poisson 2 cocycles give rise to formal deformations of the algebra. This is how the formal deformations are studied and further one can ask about the index theory of such spaces.

We shall study the non-commutative Poisson structure on the orbifolds of $A_{\theta}^{alg}$ arising through the action of finite subgroups of $SL_2(\mathbb{Z})$. The action of $SL_2(\mathbb{Z})$ on $A_{\theta}^{alg}$ is described as follows. An element

$$g = \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix} \in SL_2(\mathbb{Z})$$

acts on the generators $U_1$ and $U_2$ as described below

$$\rho_1 U_1 = e^{(\pi i g_1, 2, 1)}g^0 U_1^{g_1, 1} U_2^{g_2, 1} \quad \text{and} \quad \rho_2 U_2 = e^{(\pi i g_1, 2, 2)}g^0 U_1^{g_1, 2} U_2^{g_2, 2}.$$ \hspace{1cm} \text{.}

We leave it to the readers to check that the above is an action of $SL_2(\mathbb{Z})$ on the algebra $A_{\theta}^{alg}$. We illustrate this action by considering the action of $\mathbb{Z}_4$ on the algebraic noncommutative torus, $A_{\theta}^{alg}$. We notice that the generator of $\mathbb{Z}_4$ in $SL_2(\mathbb{Z})$, $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ acts on $A_{\theta}^{alg}$ as follows

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} U_1 = U_2^{-1} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} U_2 = U_1.$$

1. Gestenhaber bracket and the Poisson Structure

For any algebra $A$ over a field $k$, Hochschild cohomology $HH^\bullet(A)$ is the space $Ext^\bullet_{A^{op}}(A, A)$, which has two compatible operations, cup product and bracket. Both operations are defined initially on the bar resolution, a natural $A \otimes A^{op}$-free resolution of $A$. It is a classical fact that the Hochschild cohomology $H^\bullet(A, A) := \bigoplus_{n \in \mathbb{Z}} H^n(A, A)$ of an associative algebra $A$ carries a Gerstenhaber algebra structure. A Gerstenhaber algebra is a graded associative algebra $(H^\bullet = \bigoplus_{n \in \mathbb{Z}} H^n, \cup)$ together with a degree $-1$ graded Lie bracket $[-,-]$ compatible with the product $\cup$ in the sense of the following Leibniz rule

$$\begin{equation}
[a \cup b, c] = [a, c] \cup b + (-1)^{|c|-1}|a|a \cup [b, c]
\end{equation} \hspace{1cm} (1.1)$$

For an associative algebra $(A, \mu)$, the Hochschild co-chain groups $C^\bullet(A, A)$ (cup) product $\cup : C^n(A, A) \otimes C^m(A, A) \to C^{n+m}(A, A)$ defined by

$$\begin{equation}
(f \cup g)(a_1, ..., a_{m+n}) = \mu(f(a_1, ..., a_m), g(a_{m+1}, ..., a_{m+n})). \hspace{1cm} (1.2)
\end{equation}$$

It turns out that the Hochschild coboundary operator $\delta$ is a graded derivation with respect to the cup product. Hence, it induces a cup product $\cup$ on the Hochschild cohomology $H^\bullet(A, A)$. Moreover, the co-chain groups $C^\bullet(A, A)$ carry a degree $-1$ graded Lie bracket compatible with the Hochschild co-boundary $\delta$ [G]. Therefore, it gives rise to a degree $-1$ graded Lie bracket on $H^\bullet(A, A)$. The cup product and the degree $-1$ graded Lie bracket on the Hochschild cohomology $H^\bullet(A, A)$ are compatible in the sense of (1.1) to make it into a Gerstenhaber algebra.

Using the Gerstenhaber bracket on $H^\bullet(C^\infty(M))$, one can define the Poisson structure on the manifold $M$, a Poisson structure on a smooth manifold $M$ is a Lie bracket $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad \text{for} \quad f, g, h \in C^\infty(M).$$

For $\Pi \in H^2(C^\infty(M))$ and $[\Pi, \Pi] = 0$, the Poisson bracket on $M$ can be defined as

$$\{f, g\} := [[\Pi, f], g].$$

The above characterization can be generalized to an associative algebra. Hence, the non-commutative Poisson structure of an associative algebra $A$ can be defined as follows:

**Definition 1.** Let $A$ be an associative algebra. A Poisson structure on $A$ is an element $\Pi \in H^2(A, A)$ such that $[\Pi, \Pi] = 0$. An algebra with a Poisson structure is called a Poisson algebra.
In this section we discuss about the Gerstenhaber brackets and Poisson structure on a general non-commutative orbifold. For a non-commutative manifold $\mathcal{A}$ and a discrete group $G$ acting on it, we can consider the orbifold $\mathcal{A} \rtimes G$. To understand the Gerstenhaber bracket and the possible Poisson structure on $\mathcal{A} \rtimes G$ we have to understand its Hochschild cohomology, $H^\bullet(\mathcal{A} \rtimes G, \mathcal{A} \rtimes G)$.

The Hochschild cohomology splits as $[GJ]$

$$H^\bullet(\mathcal{A} \rtimes G, \mathcal{A} \rtimes G) = \bigoplus_{\gamma \in G} H^\bullet(\mathcal{A}, \gamma \mathcal{A})^G.$$

Where $\gamma \mathcal{A}$ is set wise $\mathcal{A}$ with the left $\mathcal{A}$-module structure defined as $\alpha \cdot a = \gamma(\alpha)a$ where $\alpha \in \mathcal{A}$ and $a \in \gamma \mathcal{A}$ [Q1]. Witherspoon and Shepler [SW2] studied the same and constructed explicit isomorphism between the two cohomology groups described above. The general method is to decompose into the conjugacy classes of $G$ and using an appropriate projective resolution for $\mathcal{A}$. Using the resolution the Gerstenhaber bracket described over the bar complex is transferred via chain map to the convenient resolution and an explicit formulation of the Gerstenhaber bracket is then obtained. The article [SW2] had studied the Gerstenhaber bracket over the polynomial skew group algebra and provided necessary conditions to find possible Poisson structure for the polynomial skew group algebra.

We shall understand the Gerstenhaber bracket and then find the Poisson structure(s) and finally the associated Poisson cohomology for the non-commutative orbifolds arising from the action of the four finite subgroups of $SL_2(Z)$ on the algebraic non-commutative torus.

3. CONNES’ KOSZUL RESOLUTION

We briefly recall the Connes’ Koszul resolution in this section. It is a projective resolution for the non-commutative torus and the quantum torus. Wambst [W1] generalized it to the higher dimensional algebraic non-commutative torus and later computed its homology [W2]. Also Nest [N] gave a similar resolution for the higher dimensional smooth non-commutative torus, his resolution had different basis on the bar complex than Connes’.

$$\mathcal{A}_g^{alg} \xleftarrow{\epsilon} \mathcal{A}_g^{alg} e \otimes \mathbb{C} \xleftarrow{b_1} \mathcal{A}_g^{alg} e \otimes e_1 > \bigoplus (\mathcal{A}_g^{alg})^e \otimes e_2 > \bigoplus (\mathcal{A}_g^{alg})^e \otimes (e_1 \wedge e_2)$$

where, $(\mathcal{A}_g^{alg})^e = \mathcal{A}_g^{alg} \otimes (\mathcal{A}_g^{alg})^{op}$

$\epsilon(a \otimes b) = ab; b_1(1 \otimes e_j) = 1 \otimes U_j - U_j \otimes 1$ and

$b_2(1 \otimes (e_1 \wedge e_2)) = (U_2 \otimes 1 - \lambda \otimes U_2) \otimes e_1 - (\lambda U_1 \otimes 1 - 1 \otimes U_1) \otimes e_2$.

We shall use the above resolution to study the Hochschild cohomology $H^\bullet(\mathcal{A}_g^{alg})$, which is the cohomology of the following complex:

$$\gamma \mathcal{A}_g^{alg} \xrightarrow{\alpha_1} \gamma \mathcal{A}_g^{alg} \oplus \gamma \mathcal{A}_g^{alg} \xrightarrow{\alpha_2} \gamma \mathcal{A}_g^{alg} \to 0,$$

where the maps are as below:

$\gamma \alpha_1(\varphi) = (\gamma(U_1) \varphi - \varphi U_1, (\gamma(U_2) \varphi - \varphi U_2) ; \gamma \alpha_2(\varphi_1, \varphi_2) = (\gamma(U_2) \varphi_1 - \lambda \varphi_1 U_2 - \lambda(\gamma(U_1)) \varphi_2 + \varphi_2 U_1$.

We leave it upon the readers to verify these maps, which can be done easily using the Connes’ Koszul resolution.
4. Some Homology groups of $\mathcal{A}_\theta \rtimes \Gamma$

**Lemma 4.1.** $H^0(\mathcal{A}_\theta \rtimes \Gamma) \cong \begin{cases} 
\mathbb{C}^5 & \text{for } \Gamma = \mathbb{Z}_2 \\
\mathbb{C}^7 & \text{for } \Gamma = \mathbb{Z}_3 \\
\mathbb{C}^8 & \text{for } \Gamma = \mathbb{Z}_4 \\
\mathbb{C}^9 & \text{for } \Gamma = \mathbb{Z}_6. 
\end{cases}$

*Proof.* Using the resolution, we have, $H^0(\mathcal{A}_\theta) = \mathbb{C}$. It is generated by the $a_{0,0} \otimes 1 \otimes 1$ and since the chain maps at the zeroth cohomology are identity maps, the cocycle is $\Gamma$ invariant. Similarly, we see that $H^0(\gamma_\mathcal{A}_\theta)$ is the kernel of the map $\gamma_\alpha_1$. The result is a routine calculation and the invariance is also a straightforward computation. □

**Lemma 4.2.** For $\theta$ satisfying Diophantine condition, $H^2(\mathcal{A}_\theta)^{\Gamma} \cong \mathbb{C}$.

*Proof.* We know from [Xu, Lemma 4.1] that $H^2(\mathcal{A}_\theta) \cong \mathbb{C}$. To check its invariance under the action of $\Gamma$, we push the cocycle into the bar complex using the map $h_2$ [C] and after the action by the generator of $\Gamma$ we pull it back on to the Connes’ Koszul resolution and compare the equivalence classes. It can be easily checked that $H^2(\mathcal{A}_\theta)^{\Gamma} \cong \mathbb{C}$. We shall explicitly calculate for $\mathbb{Z}_3$, other cases too have similar computation. We abide by the notations of [C].

For $\varphi \in \omega \mathcal{A}_\theta$, let $\tilde{\varphi}$ be the corresponding element of $\text{Hom}_{\mathcal{A}_\theta^0}(\Omega_2, \omega \mathcal{A}_\theta)$. Then

$$\tilde{\varphi}(a \otimes b \otimes e_1 \wedge e_2)(x) = \varphi(\omega \cdot b) x a,$$

for all $a, b, x \in \mathcal{A}_\theta$. Let $\psi = k_2^* \tilde{\varphi} = \tilde{\varphi} \circ k_2$. We have

$$\psi(x, x_1, x_2) = \tilde{\varphi}(k_2(\mathbb{I} \otimes x_1 \otimes x_2))(x),$$

for all $x, x_1, x_2 \in \mathcal{A}_\theta$. The group $\omega$ acts on $\mathcal{A}_\theta$ in the bar complex as

$$\omega \cdot \chi(x, x_1, x_2) = \chi(\omega \cdot x, \omega \cdot x_1, \omega \cdot x_2).$$

Further we pull the map $\omega \psi := \omega \cdot \psi$ back on to the Connes complex via the map $h_2^*$. Let $w = h_2^*(\omega \psi)$ denote the pull-back of $\omega \psi$ on the Connes’ Koszul complex. We have

$$w(x) = \omega \psi(x, U_2, U_1) - \lambda \omega \psi(x, U_1, U_2) = \psi(\omega \cdot x, U_1 U_2^{-1}, U_2^{-1}) - \lambda \psi(\omega \cdot x, U_2^{-1}, U_1 U_2^{-1}) = \tilde{\varphi}(k_2(I \otimes U_2^{-1} \otimes U_2^{-1}))(\omega \cdot x) - \lambda \tilde{\varphi}(k_2(I \otimes U_1 U_2^{-1} \otimes U_1 U_2^{-1}))(\omega \cdot x).$$

Using the results of [C] and [Q1, Section 6], we have

$$k_2(I \otimes U_1 U_2^{-1} \otimes U_2^{-1}) - \lambda k_2(I \otimes U_2^{-1} \otimes U_1 U_2^{-1}) = (U_2^{-1} \otimes U_2^{-2}).$$

Hence,

$$\frac{1}{\sqrt{\lambda}}(\tilde{\varphi}(k_2(I \otimes U_1 U_2^{-1} \otimes U_2^{-1}))(\omega \cdot x) - \lambda \tilde{\varphi}(k_2(I \otimes U_2^{-1} \otimes U_1 U_2^{-1}))(\omega \cdot x)) = \frac{1}{\sqrt{\lambda}} \tilde{\varphi}((U_2^{-1} \otimes U_2^{-2}))(\omega \cdot x) = \sqrt{\lambda} \varphi(U_1 U_2^{-1} \cdot (\omega \cdot x) \cdot U_2^{-1}).$$

The cocycle $\varphi_{-1,-1} \in H^2(\mathcal{A}_\theta, \mathcal{A}_\theta)$ which on the Connes’ Koszul complex is supported at the $(-1, -1)$ coefficient is invariant under the action of $\mathbb{Z}_3$ because:
\[
\frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}(U_{2}^{-2}(\omega \cdot x)U_{2}^{-1}) = \frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}(U_{2}^{-2}(\omega \cdot (x_{-1,-1}U_{1}^{-1}U_{2}))U_{2}^{-1}) = \\
\frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}(U_{2}^{-2}x_{-1,-1}U_{2}(\frac{U_{1}U_{2}^{-1}}{\sqrt{\lambda}})U_{2}^{-1}) = \varphi_{-1,-1}(x_{-1,-1}U_{2}^{-2}U_{2}U_{1}^{-1}U_{2}^{-1}) = \\
\varphi_{-1,-1}(x_{-1,-1}U_{1}^{-1}U_{2}^{-1}) = x_{-1,-1}.
\]

Hence, \( H^{2}(A_{\theta})\mathbb{Z}_{3} \cong \mathbb{C} \).

\[\Box\]

5. Hochschild cohomology \( HH^{\bullet}(-) \) of \( A_{\theta}^{alg} \rtimes \Gamma \)

**Theorem 5.1.** Let \( \Gamma \) be any finite subgroup of \( SL_{2}(\mathbb{Z}) \) then we have the following:

\[
H^{0}(A_{\theta}^{alg} \rtimes \Gamma) \cong \begin{cases} 
\mathbb{C}^{5} & \text{for } \Gamma = \mathbb{Z}_{2} \\
\mathbb{C}^{7} & \text{for } \Gamma = \mathbb{Z}_{3} \\
\mathbb{C}^{8} & \text{for } \Gamma = \mathbb{Z}_{4} \\
\mathbb{C}^{9} & \text{for } \Gamma = \mathbb{Z}_{6}.
\end{cases}
\]

\( H^{\bullet}(A_{\theta}^{alg} \rtimes \Gamma) = 0 \) for \( \bullet > 2 \).

\( H^{1}(A_{\theta}^{alg} \rtimes \Gamma) \cong 0 \) and \( H^{2}(A_{\theta}^{alg} \rtimes \Gamma) \cong \mathbb{C} \).

**Proof of Theorem 5.1.** The Hochschild cohomology decomposes as follows:

\[
H^{\bullet}(A_{\theta}^{alg} \rtimes \Gamma) = \bigoplus_{\gamma \in \Gamma} H^{\bullet}(A_{\theta}^{alg} \rtimes \gamma A_{\theta}^{alg})\Gamma.
\]

It is clear from the Koszul resolution that the cohomology \( H^{\bullet}((A_{\theta}^{alg} \rtimes \Gamma)) \) for \( \bullet > 2 \) vanishes. The second cohomology \( H^{2}(A_{\theta}^{alg} \rtimes \Gamma) \) has the following decomposition:

\[
H^{2}(A_{\theta}^{alg} \rtimes \Gamma) \cong \bigoplus_{\gamma \in \Gamma} H^{2}(\gamma A_{\theta}^{alg})\Gamma.
\]

We have \( H^{2}(A_{\theta}^{alg})\Gamma = \mathbb{C} \), the proof is similar to the proof in the last section for the smooth orbifolds.

To calculate \( H^{2}(\gamma A_{\theta}^{alg})\Gamma \) we firstly study \( H^{2}(\gamma A_{\theta}^{alg}) = \gamma A_{\theta}^{alg} / im(\alpha_{2}) \) on the Connes’ Koszul complex. It is a straightforward computation that shows that for \( \gamma \in \Gamma \), on the Koszul complex, the group \( H^{2}(\gamma A_{\theta}^{alg}) \) is generated by equivalence classes of elements of the form

\[
a_{p,q} \otimes 1 \otimes e_{1} \wedge e_{2} \in \gamma A_{\theta}^{alg} \otimes (A_{\theta}^{alg})^{e} \otimes \Omega_{2},
\]

where \( p, q \in \{0, 1\} \). For example, for \( g = \sqrt{-1} \in \mathbb{Z}_{4} \); \( H^{2}(\gamma A_{\theta}^{alg}) \) is two dimensional and is generated by \( a_{0,0} \otimes 1 \otimes e_{1} \wedge e_{2} \) and \( a_{1,0} \otimes 1 \otimes e_{1} \wedge e_{2} \).

Similar to the method we adopted, we can check the \( \Gamma \) invariant cocycles of the \( g \)-twisted cohomology. It turns out that none of the cocycles are \( \Gamma \) invariant. Hence we have \( H^{2}(\gamma A_{\theta}^{alg})\Gamma \cong 0 \), for \( g \in \Gamma \) and \( \Gamma \) finite subgroups of \( SL_{2}(\mathbb{Z}) \).
The dimension of the zeroth cohomology can be concluded from the previous section. To determine the $H^1(A^\text{alg}_\theta \rtimes \Gamma)$ we firstly observe that $H^1(A^\text{alg}_\theta)$ is two dimensional and is generated by the equivalence class of elements supported at $\phi^1_{0,1}$ and $\phi^2_{0,-1}$. Using the maps $k_1$ and $n_1$ we check the invariance of these two cocycles in the similar way that we did for the 2-cocycle above. We can easily observe that $H^1(A^\text{alg}_\theta) = 0$ for all $\gamma \in \Gamma$. This statement is an straightforward corollary of the fact that $H^1(A^\text{alg}_\theta \rtimes \Gamma)$ is trivial.

6. Gerstenhaber Bracket for $A^\text{alg}_\theta \rtimes \Gamma$

We have described the above the method to compute the Gerstenhaber bracket by transporting the Gerstenhaber structure from the bar complex to convenient complex(here the $\gamma$-twisted Koszul complex) and then gain push forward to the bar complex for the explicit formula. It is to be noted that for $f \in H^n(A, \gamma A)$ and $g \in H^m(A, \mu A)$, for $\gamma, \mu \in \Gamma$, the Gerstenhaber bracket $[f, g]$ is a $n+m-1$ cocycle. With this in mind and the above theorem which computes the dimension of the Hochschild cohomology, we can easy say that for any $\Gamma < SL_2(\mathbb{Z})$, finite subgroup, the Gerstenhaber bracket on $H^\bullet(A^\text{alg}_\theta \rtimes \Gamma, A^\text{alg}_\theta)$ is trivial.

It is to be noted that $\Pi \in H^2(A^\text{alg}_\theta \rtimes \Gamma)$ for all $\gamma \in \Gamma$, that is the only 2-cocycle of $H^\bullet(A^\text{alg}_\theta \rtimes \Gamma)$ for all $\Gamma$ finite subgroups of $SL_2(\mathbb{Z})$ actually comes from the untwisted or the identity component of the decomposition $[Q3]$. Therefore we can finally conclude that:

**Lemma 6.1.** The Gerstenhaber bracket on the non-commutative torus orbifolds $A^\text{alg}_\theta \rtimes \Gamma$ for $\Gamma$ finite subgroups of $SL_2(\mathbb{Z})$ is 0. That is:

$$[-, -]: H^\bullet(A^\text{alg}_\theta \rtimes \Gamma) \to H^\bullet(A^\text{alg}_\theta \rtimes \Gamma).$$

is the zero map.

**Proof.** The only non-trivial 2-cocycle, $\Pi$, when paired with any of the 0cocycles should yield a 1-cocycle in $H^1(A^\text{alg}_\theta \rtimes \Gamma)$ (0). Enough to show that $[\Pi, \Pi] = 0$. This is clear as $\Pi \in H^2(A^\text{alg}_\theta, A^\text{alg}_\theta)$ explicitly, $\Pi = \delta_1 \wedge \delta_2$ where $\delta_1$ and $\delta_2$ are the two canonical derivations on $A^\text{alg}_\theta$ given by,

$$\delta_1(U^n_1 U^m_2) = 2\pi i n U^n_1 U^m_2$$

and

$$\delta_2(U^n_1 U^m_2) = 2\pi i m U^n_1 U^m_2$$

Since $\delta_1$ , $\delta_2$ commute on the smooth torus $[C]$, $[\Pi, \Pi] = 0$. Alternatively, $[\Pi, \Pi]$ is a 3-cocycle in $H^\bullet(A^\text{alg}_\theta \rtimes \Gamma)$ hence 0. □

Hence the only Poisson structure on $A^\text{alg}_\theta \rtimes \Gamma$ is induced from the only Poisson structure on $A\theta$.

7. Poisson Structure(s) and Cohomology of $A^\text{alg}_\theta \rtimes \Gamma$

The Poisson cohomology for a Poisson algebra is defined as follows. For an associative Poisson algebra $(A, \Pi)$, we define a co-chain complex $(H^\bullet(A, A), d_\Pi)$,

$$d_\Pi: H^\bullet(A, A) \to H^{\bullet+1}(A, A)$$
where 
\[ d_{\Pi}(U) = [\Pi, U], \text{ for all } U \in H^i(A, A). \]

From the condition that \([\Pi, \Pi] = 0\) and the graded Jacobi identity of the Gerstenhaber brackets, it follows that \(d_{\Pi}^2 = d_{\Pi} \circ d_{\Pi} = 0\). The cohomology of this complex \((H^*(A, A), d_{\Pi})\) is called Poisson cohomology of \((A, \Pi)\) and denoted by \(H^*_{\Pi}(A)\).

**THEOREM 7.1.** The \(H^0_{\Pi}(A_{\theta}^{alg} \rtimes \Gamma) \cong \begin{cases} \mathbb{C}^5 & \text{for } \Gamma = \mathbb{Z}_2 \\ \mathbb{C}^7 & \text{for } \Gamma = \mathbb{Z}_3 \\ \mathbb{C}^8 & \text{for } \Gamma = \mathbb{Z}_4 \\ \mathbb{C}^9 & \text{for } \Gamma = \mathbb{Z}_6. \end{cases} \)

and \(H^\bullet_{\Pi}(A_{\theta}^{alg} \rtimes \Gamma) = 0\) for \(\bullet > 2\), \(H^1_{\Pi}(A_{\theta}^{alg} \rtimes \Gamma) \cong 0\) and \(H^2_{\Pi}(A_{\theta}^{alg} \rtimes \Gamma) \cong \mathbb{C}.

**Proof.** The proof is straightforward consequence of Theorem 5.1 and Lemma 6.1. \(\Box\)

8. Conclusion

Although we had the machinery to compute the Gestenhaber brackets explicitly the bracket is trivial for the quantum orbifolds. The vanishing of the \(H^1(A_{\theta}^{alg} \rtimes \Gamma)\) for \(\Gamma\) is the key to isomorphism of the Hochschild cohomology and the Poisson cohomology. The lack of \(\gamma\)-twisted 2 cocycles\((\gamma \neq 1)\) in the Poisson cohomology \(H^2_{\Pi}(A_{\theta}^{alg} \rtimes \Gamma)\) indicates uninteresting deformation theory for orbifolds. It is well known that for a Poisson algebra \((A, \Pi)\) and a Poisson 2 cocycle \(\Pi_1 \in H^2_{\Pi}(A)\), the group \(H^3_{\Pi}(A)\) is the obstruction for existence of a deformation of \(\Pi\) with infinitesimal \(\Pi_1\). Since the Poisson cohomology \(H^2_{\Pi}(A_{\theta}^{alg} \rtimes \Gamma)\), is one dimensional, generated by the un-twisted cocycle. We do not have any non-trivial deformation of the orbifold. So to conclude, the formal deformations in the sense of Kontsevich\[K\] which was studied by Xu\[Xu\], Neumaier et al.\[NPPT\] amongst others does not exist in these non-commutative orbifolds except that which was inherited by the parent non-commutative manifold \(A_{\theta}^{alg}\).

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Safdar Quddus,
Department of Mathematics,
Indian Institute of Science, Bengaluru, India.
Email: safdarquddus@iisc.ac.in.