ON FOURTH ORDER RETARDED EQUATIONS WITH FUNCTIONAL BOUNDARY CONDITIONS: A UNIFIED APPROACH

ALESSANDRO CALAMAI AND GENNARO INFANTE

Abstract. By means of a recent Birkhoff-Kellogg type theorem, we discuss the solvability of a fairly general class of parameter-dependent fourth order retarded differential equations subject to functional boundary conditions. We seek solutions within a translate cone of nonnegative functions. We provide an example to illustrate our theoretical results.

Dedicated to Professor Jerome A. Goldstein in honor of his eightieth birthday

1. Introduction

In this paper we investigate the existence of positive solutions of the following class of fourth order parameter-dependent functional differential equations with functional boundary conditions (BCs):

\[ u^{(4)}(t) + \lambda F(t, u_t) = 0, \quad t \in [0, 1], \]  

with initial conditions

\[ u(t) = \psi(t), \quad t \in [-r, 0], \]  

and one of the following BCs

\[ u^{(j)}(1) = \lambda B[u], \]  

where \( j \) can be either 0 or 1, 2, 3.

Fourth order ODEs with nonlocal and/or nonlinear boundary terms have been studied in the past, also in view of applications to mechanical systems, see for example the recent papers [4, 7, 15, 21, 29] and the references therein. Functional differential equations, in the case of fourth and higher order, have also been considered in the past, we mention here the manuscripts [2, 16].

We show that, under suitable assumptions, the functional boundary value problem (FBVP) (1.1)–(1.2)–(1.3) admits a solution of the form \((\bar{\lambda}, \bar{u})\) with \(\bar{\lambda}\) positive and \(\bar{u}\) nontrivial and nonnegative. Under additional hypotheses on the nonlinearities we can prove that the FBVP (1.1)–(1.2)–(1.3) admits uncountably many solution pairs, this is illustrated in Example 4.1. In some sense we extend the existence result in [15], in which the author considered
the ODE case, namely
\[ u^{(4)}(t) + \lambda f(t, u(t), u'(t), u''(t), u'''(t)) = 0, \quad t \in [0, 1], \]
under cantilever-type BCs. We also mention the recent papers [18, 19, 27] for different and related existence results for fourth order ODEs with nonlocal boundary terms. We stress that the functional formulation \( B \) in (1.3) is fairly general and can be used to deal with the interesting cases of nonlinear and nonlocal BCs; we refer a reader interested in these topics to the reviews [3, 8, 20, 22, 24, 28] and the manuscripts [10, 17, 26]. Also note that, in our setting, the functional term \( B \) is allowed to depend on the datum \( \psi \), this is illustrated in (4.5).

Our approach is of topological nature and consists, roughly speaking, in rewriting the FBVP (1.1)–(1.2)–(1.3) into a perturbed Hammerstein integral equation of which we seek solutions in an appropriate subset of a Banach space. As in the recent papers [5, 6] we work within the context of affine cones. This setting seems to be quite natural for the case of delay equations subject to boundary conditions. We apply a kind of Birkhoff-Kellogg type theorem in affine cones, obtained recently by the authors in [6]. This result can be considered as a complement of the interesting topological results in affine cones proved by Djebali and Mebarki in [9].

We stress that, in the context of higher-order functional differential equations, the choice of the functional space is not automatic, see for example the Introduction of [2] for some remarks in this direction. If the fourth-order equation is associated to an initial-value problem, it seems to be convenient to work in the space \( C^3([-r, 0], \mathbb{R}) \) like in [2]. Here, on the other hand, when considering the BVP (1.1)–(1.2)–(1.3) we fix \( \psi \in C^2([-r, 0], \mathbb{R}) \) and look for solutions in the space \( C^2([-r, 0], \mathbb{R}) \).

Our starting point to build the solution operator will be the data \( \psi(0), \psi'(0), \psi''(0) \) and the Green’s functions associated to these data. This is a key difference with respect to the previous papers [5, 6], where we assumed homogeneous data for the function \( \psi \) and its derivatives at \( t_0 = 0 \). Another remarkable difference with [5, 6] is the following. Here we work in an affine cone which is not “centered” in the initial datum \( \psi \). Instead we consider a different translation obtained with a modified function \( \tilde{\psi} \) which also takes into account the BC (1.3): in fact, for each \( j \) we construct an affine cone with a different vertex. This is also a crucial methodological difference with respect to the paper [15], where no translation is needed due to the different boundary data.

We close the paper with an illustrating specific example, in the spirit of [6, 15], concerning the particular case of a parametrized fourth-order delay differential equation with three, possibly different, time-lags:
\[ u^{(4)}(t) + \lambda f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, 1]. \]
2. Setting of the problem

Let $I \subset \mathbb{R}$ be a compact real interval. By $C^2(I, \mathbb{R})$ we denote the Banach space of the twice continuously differentiable functions defined on $I$ with the norm

$$\|u\|_{I,2} := \max\{\|u\|_{I,\infty}, \|u'\|_{I,\infty}, \|u''\|_{I,\infty}\},$$

where $\|u\|_{I,\infty} := \sup_{t \in I} |u(t)|$.

In the paper we use the following notation, which is standard in retarded functional differential equations (cfr. [12]). Given a positive real number $r > 0$, a continuous function $u : J \to \mathbb{R}$, defined on a real interval $J$, and given any $t \in \mathbb{R}$ such that $[t-r, t] \subseteq J$, by $u_t : [-r, 0] \to \mathbb{R}$ we mean the function defined by $u_t(\theta) = u(t + \theta)$.

As pointed out in the Introduction, we study parametrized fourth order functional differential equations of type

$$u^{(4)}(t) + \lambda F(t, u_t) = 0, \ t \in [0,1],$$

with initial conditions

$$u(t) = \psi(t), \ t \in [-r,0],$$

(2.2)

together with one of the following functional (non necessarily local) BCs

$$u^{(j)}(1) = \lambda B[u],$$

(2.3)

where can be either 0 or 1, 2, 3.

In the BVP (2.1)–(2.2)–(2.3), $\lambda$ is a nonnegative parameter, $B$ is a suitable positive continuous functional, to be defined later, while the initial datum

$$\psi : [-r, 0] \to [0, +\infty)$$

is a given function twice continuously differentiable and such that $\psi'(0), \psi''(0)$ are nonnegative.

Regarding the operator $F : [0,1] \times C^2([-r,0], \mathbb{R}) \to [0, \infty)$, throughout the paper we will assume the following Carathéodory-type conditions (see also [12]):

- for each $\phi$, $t \mapsto F(t, \phi)$ is measurable;
- for a.e. $t$, $\phi \mapsto F(t, \phi)$ is continuous;
- for each $R > 0$, there exists $\varphi_R \in L^\infty[0,1]$ such that

$$F(t, \phi) \leq \varphi_R(t)$$

for all $\phi \in C^2([-r,0], \mathbb{R})$ with $\|\phi\|_{[-r,0],2} \leq R$, and a.e. $t \in [0,1]$.

2.1. A Birkhoff-Kellogg type theorem in affine cones. In this section we recall a Birkhoff–Kellogg type result in affine cones recently proved in [6]. The proof relies on classical fixed point index theory for compact maps. We refer a reader interested in the fixed point index to the review of Amann [1] and to the book by Guo and Lakshmikantham [11].
Let us first recall some useful notation. Let \((X, \| \cdot \|)\) be a real Banach space. A cone \(K\) of \(X\) is a closed set with \(K + K \subset K\), \(\mu K \subset K\) for all \(\mu \geq 0\) and \(K \cap (-K) = \{0\}\). For \(y \in X\), the translate of the cone \(K\) is defined as
\[
K_y := y + K = \{ y + x : x \in K \}.
\]
Given a bounded and open (in the relative topology) subset \(\Omega\) of \(K_y\), we denote by \(\overline{\Omega}\) and \(\partial \Omega\) the closure and the boundary of \(\Omega\) relative to \(K_y\). Given an open bounded subset \(D\) of \(X\) we denote \(D_{K_y} = D \cap K_y\), an open subset of \(K_y\).

We can now state the Birkhoff–Kellogg type result in affine cones.

**Theorem 2.1** ([6], Corollary 2.4). Let \((X, \| \cdot \|)\) be a real Banach space, \(K \subset X\) be a cone and \(D \subset X\) be an open bounded set with \(y \in D_{K_y}\) and \(\overline{D}_{K_y} \neq K_y\). Assume that \(F : \overline{D}_{K_y} \to K\) is a compact map and assume that
\[
\inf_{x \in \partial D_{K_y}} \| F(x) \| > 0.
\]
Then there exist \(x^* \in \partial D_{K_y}\) and \(\lambda^* \in (0, +\infty)\) such that \(x^* = y + \lambda^* F(x^*)\).

3. The associated Green’s functions and existence results

In order to illustrate our strategy, let us first focus on the following case, in which the nonlocal BC (1.3) involves the third derivative at \(t = 1\); namely:
\[
\begin{cases}
  u^{(4)}(t) + \lambda F(t, u_t) = 0, & t \in [0, 1], \\
  u(t) = \psi(t), & t \in [-r, 0], \\
  u''(1) = \lambda B[u],
\end{cases}
\tag{3.1}
\]
By means of a superposition principle, we associate to (3.1) a perturbed Hammerstein integral equation, in the spirit of [3] [13] [14]. Recall that \(\psi \in C^2([-r, 0], \mathbb{R})\), so the “functional” initial condition implies that \(u(0) = \psi(0), u'(0) = \psi'(0), u''(0) = \psi''(0)\).

First observe that, given a continuous function \(y\), the BVP
\[
\begin{cases}
  -u^{(4)}(t) = y(t), & t \in [0, 1], \\
  u(0) = u'(0) = u''(0) = u'''(1) = 0,
\end{cases}
\]
has the unique solution
\[
u(t) = \int_0^1 k(t, s)y(s)ds,
\]
where the Green’s function is, see for example [29],
\[
k(t, s) = \frac{1}{6} \begin{cases}
  t^3, & 0 \leq t \leq s \leq 1, \\
  s(3t^2 - 3ts + s^2), & 0 \leq s \leq t \leq 1,
\end{cases}
\]
In particular, for the kernel $k$ the following positivity property holds:

$$k(t, s) \geq 0 \text{ on } [0, 1] \times [0, 1],$$

A direct computation shows that the following problems have the corresponding solutions:

The BVP

$$\begin{cases}
  u^{(4)}(t) = 0, & t \in [0, 1], \\
  u(0) = 1, & u'(0) = u''(0) = u'''(1) = 0,
\end{cases}$$

is solved by the constant $\gamma_0(t) = 1$; the BVP

$$\begin{cases}
  u^{(4)}(t) = 0, & t \in [0, 1], \\
  u'(0) = 1, & u(0) = u''(0) = u'''(1) = 0,
\end{cases}$$

is solved by $\gamma_1(t) = t$; the BVP

$$\begin{cases}
  u^{(4)}(t) = 0, & t \in [0, 1], \\
  u''(0) = 1, & u(0) = u'(0) = u'''(1) = 0,
\end{cases}$$

is solved by $\gamma_2(t) = \frac{1}{2}t^2$; while for the condition at $t = 1$ we have

$$\begin{cases}
  u^{(4)}(t) = 0, & t \in [0, 1], \\
  u(0) = u'(0) = u''(0) = u'''(1) = 1,
\end{cases}$$

which is solved by $\gamma_3(t) = \frac{1}{6}t^3$.

Observe that $\gamma_i(t) \geq 0$ on $[0, 1]$, for $i = 0, \ldots, 3$. Moreover, one can check by direct computation that the maps

$$t \mapsto k(t, s)H(t) \quad t \mapsto \gamma_3(t)H(t), \quad t \in [-r, 1]$$

are of class $C^2([-r, 1], \mathbb{R})$, where

$$H(\tau) = \begin{cases} 
  1, & \tau \geq 0, \\
  0, & \tau < 0.
\end{cases}$$

Now, define

$$\hat{\psi}(t) := \begin{cases} 
  \psi(t), & t \leq 0, \\
  \gamma_0(t)\psi(0) + \gamma_1(t)\psi'(0) + \gamma_2(t)\psi''(0), & t > 0.
\end{cases} \tag{3.2}$$

and observe that, being $\psi$ of class $C^2$, the function $\hat{\psi}$ has the same regularity.

Due to the above setting, the FBVP (3.1) can be rewritten in the following form:

$$u(t) = \hat{\psi}(t) + \lambda \left( \int_{0}^{1} k(t, s)H(t)F(s, u_s) \, ds + H(t)\gamma_3(t)B[u] \right), \quad t \in [-r, 1] \tag{3.3}$$
By $K_0$ we denote the following cone of non-negative functions in the Banach space $C^2([-r, 1], \mathbb{R})$: 

$$K_0 = \{ u \in C^2([-r, 1], \mathbb{R}) : u(t) \geq 0 \ \forall t \in [-r, 1], \ \text{and} \ u(t) = u'(t) = u''(t) = 0 \ \forall t \in [-r, 0] \}.$$ 

Observe that that the function 

$$w(t) = \begin{cases} 0, & t \in [-r, 0], \\ t^3, & t \in [0, 1], \end{cases}$$

belongs to $K_0$, hence $K_0 \neq \{0\}$.

For a given $\Psi \in C^2([-r, 1], \mathbb{R})$, we let $K_\Psi$ be the following translate of the cone $K_0$ 

$$K_\Psi = \Psi + K_0 = \{ \Psi + u : u \in K_0 \}.$$ 

Definition 3.1. Given $\Psi \in C^2([-r, 1], \mathbb{R})$ and $\rho > 0$, we define the following subsets of $C^2([-r, 1], \mathbb{R})$: 

$$K_{0,\rho} := \{ u \in K_0 : \|u\|_{[0,1],2} < \rho \}, \ \ K_{\Psi,\rho} := \Psi + K_{0,\rho}.$$ 

As a consequence of the above Corollary 2.1, we get the following existence result.

**Theorem 3.2.** Let $\rho \in (0, +\infty)$ and assume the following conditions hold.

(a) There exist $\delta_\rho \in C([0,1], \mathbb{R}^+)$ such that 

$$F(t, \phi) \geq \delta_\rho(t), \ \text{for every} \ (t, \phi) \in [0, 1] \times C^2([-r, 0], \mathbb{R}) \ \text{with} \ \|\phi\|_{[-r,0],2} \leq \rho + \|\hat{\psi}\|_{[-r,1],2}.$$ 

(b) $B : K_{\hat{\psi},\rho} \to \mathbb{R}^+$ is continuous and bounded, in particular let $\eta_\rho \in [0, +\infty)$ be such that 

$$B[u] \geq \eta_\rho, \ \text{for every} \ u \in \partial K_{\hat{\psi},\rho}.$$ 

(c) The inequality 

$$\sup_{t \in [0,1]} \left\{ \gamma_3(t)\eta_\rho + \int_0^1 k(t,s)\delta_\rho(s) \ ds \right\} > 0 \quad (3.4)$$

holds.

Then there exist $\lambda_\rho \in (0, +\infty)$ and $u_\rho \in \partial K_{\hat{\psi},\rho}$ that satisfy the BVP (3.1). 

**Proof.** Let 

$$F(u(t) := \int_0^1 k(t,s)H(t)F(s, u_s) \ ds + H(t)\gamma_3(t)B[u].$$ 

Observe that the operator $F$ maps $K_{\hat{\psi},\rho}$ into $K_0$ and is compact. The compactness of the Hammerstein integral operator is a consequence of the regularity assumptions on the terms occurring in it combined with a careful use of the Arzelà-Ascoli theorem (see [25]), while the perturbation term $H(t)\gamma_3(t)B[\cdot]$ is a finite rank operator.

Take $u \in \partial K_{\hat{\psi},\rho}$. Then we have
\[\|F u\|_{[-r,1],2} \geq \|F u\|_{[-r,1],\infty} = \sup_{t \in [0,1]} \left| \int_0^1 k(t,s) F(s,u_s) \, ds + \gamma_3(t) B[u]\right| \geq \sup_{t \in [0,1]} \left\{ \gamma_3(t) \eta_\rho + \int_0^1 k(t,s) \hat{\lambda}_\rho(s) \, ds \right\}. \] (3.5)

Note that the RHS of (3.5) does not depend on the particular \(u\) chosen. Therefore we have

\[\inf_{u \in \partial K_{\tilde{\psi},\rho}} \|F u\|_{[-r,1],2} \geq \sup_{t \in [0,1]} \left\{ \gamma_3(t) \eta_\rho + \int_0^1 k(t,s) \hat{\lambda}_\rho(s) \, ds \right\} > 0,\]

and the result follows by Theorem 2.1.

3.1. A unified approach. Our purpose is now to show that, in the case of the BC (2.3) with \(j = 0, 1, 2\), we can follow the same procedure as above and obtain a result like Theorem 3.2. In other words, given \(j\), the corresponding FBVP can be rewritten in the form (3.3) for a suitable choice of \(\hat{\psi}, k, \gamma_3\) (recall that \(\gamma_i, i = 0, 1, 2\), are needed in (3.2), i.e. the definition of \(\hat{\psi}\)). Hereafter, with a slight abuse of notation, for simplicity we use the same notation for \(k\) and \(\gamma_i\) for all the FBVPs.

- The case \(j = 0\) in (2.3). Firstly we consider the case, in which the nonlocal BC (1.3) involves \(u(1)\).

\[
\begin{cases}
  u^{(4)}(t) + \lambda F(t, u_t) = 0, & t \in [0,1], \\
  u(t) = \psi(t), & t \in [-r,0], \\
  u(1) = \lambda B[u],
\end{cases}
\] (3.6)

We build up, as above, an integral equation equivalent to (3.6).

Given a continuous function \(y\), the BVP:

\[
\begin{cases}
  -u^{(4)}(t) = y(t), & t \in [0,1], \\
  u(0) = u'(0) = u''(0) = u(1) = 0,
\end{cases}
\]

has the unique solution

\[u(t) = \int_0^1 k(t,s)y(s) \, ds,\]

where the Green’s function is

\[k(t, s) = \begin{cases}
  \frac{1}{6} \left( t^3 (1 - s)^3, & 0 \leq t \leq s \leq 1, \\
  s(1 - t)(s^2 t^2 - 3st^2 + 3t^2 + s^2 t - 3st + s^2), & 0 \leq s \leq t \leq 1.
\end{cases}\]
In a similar way as above, direct computations show that the following problems have the corresponding solutions:

- \[
\begin{align*}
\begin{cases}
u^{(4)}(t) = 0, & t \in [0, 1], \\
u(0) = 1, & u'(0) = u''(0) = u(1) = 0,
\end{cases}
\end{align*}
\]
is solved by \( \gamma_0(t) = 1 - t^3 \);

- \[
\begin{align*}
\begin{cases}
u^{(4)}(t) = 0, & t \in [0, 1], \\
u'(0) = 1, & u(0) = u''(0) = u(1) = 0,
\end{cases}
\end{align*}
\]
is solved by \( \gamma_1(t) = t - t^3 \);

- \[
\begin{align*}
\begin{cases}
u''(0) = 1, & u(0) = u'(0) = u(1) = 0.
\end{cases}
\end{align*}
\]
is solved by \( \gamma_2(t) = \frac{1}{2}t^2(1 - t) \); while for the condition at \( t = 1 \) we have

- \[
\begin{align*}
\begin{cases}
u^{(4)}(t) = 0, & t \in [0, 1], \\
u(0) = u'(0) = u''(0) = 0, & u(1) = 1,
\end{cases}
\end{align*}
\]
which is solved by \( \gamma_3(t) = t^3 \). We can apply then apply the same procedure as in the previous case.

- **The case** \( j = 1 \) in \( \Box \). A similar approach can be followed for the case

- \[
\begin{align*}
\begin{cases}
u^{(4)}(t) + \lambda F(t, u_t) = 0, & t \in [0, 1], \\
u(t) = \psi(t), & t \in [-r, 0], \\
u'(1) = \lambda B[u].
\end{cases}
\end{align*}
\]

In fact, given a continuous function \( y \), the BVP:

- \[
\begin{align*}
\begin{cases}
-u^{(4)}(t) = y(t), & t \in [0, 1], \\
u(0) = u'(0) = u''(0) = u'(1) = 0,
\end{cases}
\end{align*}
\]
has the unique solution

- \[
u(t) = \int_0^1 k(t, s)y(s)ds,
\]
where the Green’s function is

- \[
k(t, s) = \frac{1}{6} \begin{cases} 
  t^3(1 - s)^2, & 0 \leq t \leq s \leq 1, \\
  s(st^3 - 2t^3 + 3t^2 - 3st + s^2), & 0 \leq s \leq t \leq 1.
\end{cases}
\]
Elementary computations show that the following problems have the corresponding solutions:

\[
\begin{cases}
  u^{(4)}(t) = 0, \; t \in [0, 1], \\
  u(0) = 1, \; u'(0) = u''(0) = u'(1) = 0,
\end{cases}
\]

is solved by \( \gamma_0(t) = 1 \);

\[
\begin{cases}
  u^{(4)}(t) = 0, \; t \in [0, 1], \\
  u'(0) = 1, \; u(0) = u''(0) = u'(1) = 0,
\end{cases}
\]

is solved by \( \gamma_1(t) = t - \frac{1}{3}t^3 \);

\[
\begin{cases}
  u^{(4)}(t) = 0, \; t \in [0, 1], \\
  u''(0) = 1, \; u(0) = u'(0) = u'(1) = 0,
\end{cases}
\]

is solved by \( \gamma_2(t) = t^2(\frac{1}{2} - \frac{1}{3}t) \); while for the condition at \( t = 1 \) we have

\[
\begin{cases}
  u^{(4)}(t) = 0, \; t \in [0, 1], \\
  u(0) = u'(0) = u''(0) = u'(1) = 1,
\end{cases}
\]

which is solved by \( \gamma_3(t) = \frac{1}{2}t^3 \).

- **The case** \( j = 2 \) in (2.3). For completeness we discuss also the case

\[
\begin{cases}
  u^{(4)}(t) + \lambda F(t, u_t) = 0, \; t \in [0, 1], \\
  u(t) = \psi(t), \quad t \in [-r, 0], \\
  u''(1) = \lambda B[u].
\end{cases}
\]

Notice that, given a continuous function \( y \), the BVP:

\[
\begin{cases}
  -u^{(4)}(t) = y(t), \; t \in [0, 1], \\
  u(0) = u'(0) = u''(0) = u''(1) = 0,
\end{cases}
\]

has the unique solution

\[ u(t) = \int_0^1 k(t, s)y(s)ds, \]

where the Green’s function is

\[ k(t, s) = \frac{1}{6} \begin{cases} 
  t^3(1 - s)^2, & 0 \leq t \leq s \leq 1, \\
  s(-t^3 + 3t^2 - 3st + s^2), & 0 \leq s \leq t \leq 1.
\end{cases} \]
Elementary computations in this case show that the following problems have the corresponding solutions:

\[
\begin{cases}
  u^{(4)}(t) = 0, & t \in [0, 1], \\
  u(0) = 1, & u'(0) = u''(0) = u''(1) = 0,
\end{cases}
\]

is solved by \( \gamma_0(t) = 1 \);

\[
\begin{cases}
  u^{(4)}(t) = 0, & t \in [0, 1], \\
  u'(0) = 1, & u(0) = u''(0) = u''(1) = 0,
\end{cases}
\]

is solved by \( \gamma_1(t) = t \);

\[
\begin{cases}
  u^{(4)}(t) = 0, & t \in [0, 1], \\
  u''(0) = 1, & u(0) = u'(0) = u''(1) = 0,
\end{cases}
\]

is solved by \( \gamma_2(t) = \frac{1}{2}t^2(1 - \frac{1}{3}t) \); while for the condition at \( t = 1 \) we have

\[
\begin{cases}
  u^{(4)}(t) = 0, & t \in [0, 1], \\
  u(0) = u'(0) = u''(0) = 0, & u''(1) = 1,
\end{cases}
\]

which is solved by \( \gamma_3(t) = \frac{1}{6}t^3 \).

Notice that in all the cases illustrated above the kernels \( k \) and the functions \( \gamma_i \) are non-negative.

4. Examples

We remark that our theory can be applied to delay differential equations (DDEs). Namely, let \( f : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, \infty) \) be a given Carathéodory map. Consider the parametrized fourth-order DDE with three time-lags

\[ u^{(4)}(t) + \lambda f(t, u(t), u'(t), u''(t), u(t - r_0), u'(t - r_1), u''(t - r_2)), \quad t \in [0, 1], \tag{4.1} \]

where the fixed (possibly different) delays \( r_i \) are positive, for \( i = 0, \ldots, 2 \). We can apply the techniques developed in this paper to the equation \( (4.1) \) with initial condition \( (2.2) \) along with one of the BCs \( (2.3) \).

To see this, observe that \( (4.1) \) is a special case of the functional equation \( (2.4) \), in which taking \( r := \max\{r_i, i = 0, \ldots, 2\} \), the operator \( F : [0, 1] \times C^2([-r, 0], \mathbb{R}) \to [0, \infty) \) is defined by

\[ F(t, \phi) = f(t, \phi(0), \phi'(0), \phi''(0), \phi(-r_0), \phi'(-r_1), \phi''(-r_2)). \]

Such an operator satisfies the above Carathéodory-type conditions if the map \( f \) satisfies analogous properties; namely,
for each \( R > 0 \), there exists \( \varphi^*_R \in L^\infty[0, 1] \) such that
\[
 f(t, u, v, w, \xi, \eta, \zeta) \leq \varphi^*_R(t) \quad \text{for all} \quad (u, v, w, \xi, \eta, \zeta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}
\]
with \( 0 \leq u, \xi \leq R, |v|, |w|, |\eta|, |\zeta| \leq R \), and a.e. \( t \in [0, 1] \).

In the following illustrating example (cfr. [6, 15]) we consider a specific DDE of type (4.1), with \( r = \max\{r_i, i = 0, \ldots, 2\} = 1/2 \). Observe in particular that the vertex \( \hat{\psi} \) of the affine cone, to which the solutions belong, is obtained by a \( C^2 \)-prolongation of the datum \( \psi \) after \( t_0 = 0 \).

**Example 4.1.** We consider the family of FBVPs
\[
 u^{(4)} + \lambda t e^{u(t)} + \left(u''(t - \frac{1}{4})\right)^2 \left(1 + \left(u'(t)\right)^2 + \left(u(t - \frac{1}{2})\right)^2 + \left(u''(t - \frac{1}{4})\right)^2\right), \quad t \in (0, 1),
\]
with the initial condition
\[
 u(t) = \psi(t), \quad t \in \left[-\frac{1}{2}, 0\right],
\]
with \( \psi(t) = H(-t) \cdot (1 - \cos t) \), and one of the four BCs (2.3). For example we choose \( j = 3 \), so that the functional BC is
\[
 u'''(1) = \lambda B[u],
\]
where we fix
\[
 B[u] = \frac{1}{1 + \left(u\left(\frac{1}{2}\right)\right)^2} + \int_{-\frac{1}{2}}^{1} t^3(u''(t))^2 \, dt.
\]
Thus the function \( \hat{\psi} \) is given by
\[
 \hat{\psi}(t) = \begin{cases} 
 1 - \cos t, & -\frac{1}{2} \leq t \leq 0, \\
 \frac{1}{2} t^2, & 0 < t \leq 1.
 \end{cases}
\]

Now choose \( \rho \in (0, +\infty) \). We may take
\[
 \eta_\rho(t) = \frac{1}{1 + \rho^2}, \quad \delta_\rho(t) = t.
\]
Therefore we have
\[
 \sup_{t \in [0,1]} \left\{ \frac{\frac{1}{2} t^2}{1 + \rho^2} + \int_0^{1} k(t, s) t \, ds \right\} \geq \frac{1}{6(1 + \rho^2)} > 0,
\]
which implies that (3.4) is satisfied for every \( \rho \in (0, +\infty) \).

Thus we can apply Theorem 3.2 obtaining uncountably many pairs of solutions and parameters \( (u_\rho, \lambda_\rho) \) for the FBVP (4.2)–(4.3)–(4.4).
Acknowledgements

The authors were partially supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). G. Infante is a member of the UMI Group TAA “Approximation Theory and Applications”.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM. Rev.*, **18** (1976), 620–709.

[2] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, On general properties of n-th order retarded functional differential equations, *Rend. Istit. Mat. Univ. Trieste*, **49** (2017), 73–93.

[3] A. Cabada, An overview of the lower and upper solutions method with nonlinear boundary value conditions, *Bound. Value Probl.* (2011), Art. ID 893753, 18 pp.

[4] A. Cabada and R. Jebari, Multiplicity results for fourth order problems related to the theory of deformations beams, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), no. 2, 489–505.

[5] A. Calamai and G. Infante, Nontrivial solutions of boundary value problems for second order functional differential equations, *Ann. Mat. Pura Appl.*, **195** (2016), 741–756.

[6] A. Calamai and G. Infante, An affine Birkhoff–Kellogg type result in cones with applications to functional differential equations, *Mathematical Methods in the Applied Sciences*, to appear. https://doi.org/10.1002/mma.8665

[7] F. Cianciaruso, G. Infante and P. Pietramala, Solutions of perturbed Hammerstein integral equations with applications, *Nonlinear Anal. Real World Appl.*, **33** (2017), 317–347.

[8] R. Conti, Recent trends in the theory of boundary value problems for ordinary differential equations, *Boll. Un. Mat. Ital.*, **22** (1967), 135–178.

[9] S. Djebali and K. Mebarki, Fixed point index on translates of cones and applications, *Nonlinear Stud.*, **21** (2014), 579–589.

[10] C. S. Goodrich, Pointwise conditions for perturbed Hammerstein integral equations with monotone nonlinear, nonlocal elements, *Banach J. Math. Anal.*, **14** (2020), 290–312.

[11] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[12] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.

[13] G. Infante, Positive solutions of differential equations with nonlinear boundary conditions, *Discrete Contin. Dyn. Syst., Suppl. Vol. 2003*, (2003), 432–438.

[14] G. Infante and J. R. L. Webb, Nonlinear nonlocal boundary value problems and perturbed Hammerstein integral equations, *textitProc. Edinb. Math. Soc.*, **49** (2006), 637–656.

[15] G. Infante, On the solvability of a parameter-dependent cantilever-type BVP, *Appl. Math. Lett.*, **132** (2022), 108090.

[16] T. Jankowski and R. Jankowski, Multiple Solutions of Boundary-Value Problems for Fourth-Order Differential Equations with Deviating Arguments, *J Optim. Theory Appl.*, **146**, 105–115 (2010).

[17] G. L. Karakostas and P. Ch. Tsamatos, Existence of multiple positive solutions for a nonlocal boundary value problem, *Topol. Methods Nonlinear Anal.*, **19** (2002), 109–121.
[18] A. Khanfer and L. Bougoffa, A cantilever beam problem with small deflections and perturbed boundary data, *J. Funct. Spaces*, **2021**, Article ID 9081623, 9 p. (2021).
[19] Y. Li, Existence of positive solutions for the cantilever beam equations with fully nonlinear terms, *Nonlinear Anal. Real World Appl.*, **27** (2016), 221–237.
[20] R. Ma, A survey on nonlocal boundary value problems, *Appl. Math. E-Notes*, **7** (2007), 257–279.
[21] Y. Ma, C. Yin and G. Zhang, Positive solutions of fourth-order problems with dependence on all derivatives in nonlinearity under Stieltjes integral boundary conditions, *Bound Value Probl.*, **2019**, 41 (2019).
[22] S. K. Ntouyas, Nonlocal initial and boundary value problems: a survey, *Handbook of differential equations: ordinary differential equations. Vol. II*, Elsevier B. V., Amsterdam, (2005), 461–557.
[23] M. Picone, Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **10** (1908), 1–95.
[24] A. Štikonas, A survey on stationary problems, Green’s functions and spectrum of Sturm-Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19** (2014), 301–334.
[25] J. R. L. Webb, Compactness of nonlinear integral operators with discontinuous and with singular kernels, *J. Math. Anal. Appl.*, **509** (2022), Paper No. 126000, 17 pp.
[26] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.*, **74** (2006), 673–693.
[27] M. Wei, Y. Li and G. Li, Lower and upper solutions method to the fully elastic cantilever beam equation with support, *Adv. Difference Equ.*, **2021**, 301 (2021).
[28] W. M. Whyburn, Differential equations with general boundary conditions, *Bull. Amer. Math. Soc.*, **48** (1942), 692–704.
[29] G. Zhang, Positive solutions to three classes of non-local fourth-order problems with derivative-dependent nonlinearities, *Electron. J. Qual. Theory Differ. Equ.*, **2022**, No. 11, 1–27.

**Alessandro Calamai**, Dipartimento di Ingegneria Civile, Edile e Architettura, Università Politecnica delle Marche Via Brecce Bianche I-60131 Ancona, Italy

Email address: calamai@dipmat.univpm.it

**Gennaro Infante**, Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy

Email address: gennaro.infante@unical.it