Conditional quantum state reduction in joint positive operator-valued measures

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We investigate conditional quantum state reduction associated with positive operator-valued measures describing joint measurements of incompatible observables. To this end we invoke consistency with very basic rules of conditional probabilities, as formulated in the standard Kolmogorovian-Bayesian formulation of statistics. We find an interesting relation with the nonclassical character of the statistics.

I. INTRODUCTION

Quantum state reduction associated with measurement is a rather peculiar but fundamental aspect of quantum mechanics without classical analog [1–6]. Actually we still lack a clear physical model supporting it [7, 8]. In many presentations of the quantum theory, it is introduced as a postulate in the form of an statistical, instantaneous, non-unitary, non-local, discontinuous transformation, very far from the unitary unobserved evolution. As discussed in Ref. [8], from a physical point of view the reduction seems rather unnatural and ad hoc, leaving room to the existence of a more intuitive explanation consistent with the everyday observed statistics.

An empiricist approach supports the rules of quantum state reduction in the measurement of Hermitian operators via practical models of system-apparatus couplings [2, 6, 9, 10]. In these models there is a neat separation between the Hilbert spaces of the system and the apparatus. These models explicitly leave aside the case of observables described by positive operator-valued measures, or POVMs [11], since in general there is no Hermitian operator nor an specific Hilbert space for the apparatus. There is the possibility of supplementing the POVMs via Neumark extensions [12], and then apply the standard procedure. But this goes against the very same idea of POVM and does not solve the questions addressed in this work, as we show in Appendix A.

So, there is room for the study of state reduction in POVMs invoking just consistency with very basic rules of conditional probabilities, as stated in the most standard Kolmogorovian-Bayesian formulation of statistics. In this work we address state reduction for POVMs describing the noisy joint observation of two observables. More specifically, we focus in the case that the POVM is actually the noisy joint measurement of two incompatible observables in a qubit system, which is a suitable arena to fundamental quantum issues such as complementarity and Bell-like tests [13, 14]. We find an interesting relation with the nonclassical character of the statistics as presented in Refs. [13, 14].

In Sec. II we provide the main settings and definitions required, including a brief recall of nonclassical statistics suited to this scenario. In Sec. III we address the determination of the reduced state associated with a conditional probability, presenting three main results that are further discussed in Secs. IV and V. Finally in the Appendix A we provide a physical realization of the POVM discussed in the main body of the paper showing that the state reduction in a Neumark scenario does not solve the question addressed here.

II. SETTINGS

Let us consider the joint measurement of two observables, which we call $X$ and $Y$, of a given physical system, $S$. The measurement should admit a joint probability distribution $p(x, y)$, that tells us the probability that the property, $X$ takes the value $x$ and $Y$ takes the value $y$. This implies that $X$ and $Y$ must be compatible, but we consider throughout this work that they actually provide the noisy joint observation of two incompatible observables, say $\sigma_X$ and $\sigma_Y$.

In a quantum scenario any $p(x, y)$ is linearly determined from the density-matrix of the system $\rho$ via a positive operator-valued measure $\Delta(x, y)$ as

$$p(x, y) = \text{tr}[\rho \Delta(x, y)], \quad (2.1)$$

where the statistical interpretation demands that

$$\Delta^1(x, y) = \Delta(x, y), \quad \Delta(x, y) \geq 0, \quad \sum_{x,y} \Delta(x, y) = I, \quad (2.2)$$

and $I$ is the identity. We assume a discrete character for $x,y$ just for the sake of simplicity, the extension to the continuum case can be carried straightforwardly and offers no new insights. The corresponding marginal POVMs for the observables $X$ and $Y$ are, respectively,

$$\Delta_X(x) = \sum_y \Delta(x, y), \quad \Delta_Y(y) = \sum_x \Delta(x, y), \quad (2.3)$$

so that the marginal probabilities are

$$p_W(w) = \text{tr}[\rho \Delta_W(w)], \quad (2.4)$$

where $W = X, Y$ and $w = x, y$. 

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A Kolmogorovian-Bayesian formulation of plain statistics leads us to define the conditional probability distribution \( p(x|y) \) of getting \( x \) provided that \( y \) was obtained as

\[
p(x, y) = p(x|y)p_Y(y). \tag{2.5}
\]

The main questions we address here is whether this conditional probability determines a legitimate and unique reduced state \( \rho_y \) associated with the result \( y \) of the \( Y \) measurement such that the conditional probability \( p(x|y) \) arises as the measurement of \( X \) on the system state \( \rho_y \), that is

\[
p(x|y) = \text{tr} [\rho_y \Delta_X(x)]. \tag{2.6}
\]

The case is straightforward when the POVM factorizes as the product of independent POVMs defined in different Hilbert spaces, namely

\[
\Delta(x, y) = \Delta_X(x) \otimes \Delta_Y(y), \tag{2.7}
\]

so that a simple ordering of the trace in Eq. (2.6) leads to

\[
\rho_y = \frac{\text{tr}_Y [\rho \Delta_Y(y)]}{\text{tr}_{X \otimes Y} [\rho \Delta_Y(y)]}, \tag{2.8}
\]

where for clarity the subscripts on the traces indicate the space where the trace is performed. This is a legitimate density matrix as it can be easily checked, including nonegativenss \( \rho_y \geq 0 \) since, for any vector \( |\psi\rangle \) in the \( X \) space

\[
\langle \psi | \rho_y | \psi \rangle \propto \text{tr}_{X \otimes Y} [\rho |\psi\rangle \langle \psi| \otimes \Delta_Y(y)] \geq 0, \tag{2.9}
\]

since \( \rho \geq 0, |\psi \rangle \langle \psi| \geq 0 \) and \( \Delta_Y(y) \geq 0 \).

Qubit system

The main objective of this work is to apply the above program to the most general POVM for two dichotomic observables \( X, Y \) in a qubit system

\[
\Delta(x, y) = \frac{1}{4} [\sigma_0 + S(x, y) \cdot \sigma], \tag{2.10}
\]

with \( x, y = \pm 1, \sigma \) are the Pauli matrices, \( \sigma_0 \) is the \( 2 \times 2 \) identity matrix, and

\[
S(x, y) = x \gamma_X S_X + y \gamma_Y S_Y + xy \gamma_{XY} S_{XY}, \tag{2.11}
\]

where \( S_X, S_Y, \) and \( S_{XY} \) are three-dimensional unit-modulus real vectors. The \( \gamma \) factors are real and positive expressing the accuracy of the observation as explained below. In any case all these elements are constrained so that \( |S(x, y)| \leq 1 \), so that conditions (2.2) are satisfied. For definiteness let us consider that \( S_X, S_Y \) and \( S_{XY} \) are mutually orthogonal

\[
S_X = (1, 0, 0), \quad S_Y = (0, 1, 0), \quad S_{XY} = (0, 0, 1), \tag{2.12}
\]

so that \( |S(x, y)| \leq 1 \) holds provided that

\[
\gamma^2_X + \gamma^2_Y + \gamma^2_{XY} \leq 1. \tag{2.13}
\]

In the Appendix A we provide a detailed derivation of this POVM in the particular but significant enough case of path-interference complementarity in a Young interferometer.

The joint POVM \( \Delta(x, y) \) leads to the marginals for \( X \) and \( Y \)

\[
\Delta_W(w) = \frac{1}{2} (1 + w \gamma_W \sigma_W), \tag{2.14}
\]

with \( W = X, Y, w = x, y \).

The system state can be expressed as

\[
\rho = \frac{1}{2} (\sigma_0 + s \cdot \sigma), \tag{2.15}
\]

where

\[
s = \text{tr} (\rho \sigma) = (s_X, s_Y, s_Z), \tag{2.16}
\]

is a three-dimensional real vector with \( |s| \leq 1 \). Then we have the following joint distribution

\[
p(x, y) = \frac{1}{4} (1 + x \gamma_X s_X + y \gamma_Y s_Y + xy \gamma_{XY} s_Z), \tag{2.17}
\]

with marginals

\[
p_X(x) = \frac{1}{2} (1 + x \gamma_X s_X), \quad p_Y(y) = \frac{1}{2} (1 + y \gamma_Y s_Y). \tag{2.18}
\]

Inversion and nonclassicality

The idea is that the POVM \( \Delta(x, y) \) in Eq. (2.10) provides a simultaneous noisy observation of two incompatible observables, say the ones represented by the operators \( \sigma_X \) and \( \sigma_Y \). So the observed marginals \( p_X(x) \) and \( p_Y(y) \) in Eq. (2.17) contain complete statistical information about \( \sigma_X \) and \( \sigma_Y \) in the state \( \rho \). This is that the true noiseless statistics of \( \sigma_W \) in the state \( \rho \), say \( P_W(w) \), for \( W = X, Y, w = x, y \), which are

\[
P_W(w) = \frac{1}{2} (1 + ws_W), \tag{2.19}
\]

can be obtained from the observed marginals \( p_W(w) \) via a simple state-independent inversion procedure

\[
P_W(w) = \sum_{w' = \pm 1} p_{W}(w, w') p_W(w'), \tag{2.20}
\]
with

\[ \mu_W(w, w') = \frac{1}{2} \left( 1 + \frac{ww'}{\gamma_W} \right), \quad (2.21) \]

which works equally well for all system states ρ.

Then we can apply the inversion procedure to the complete statistics \( p(x, y) \) to get a joint distribution \( P(x, y) \) that by construction provides the correct marginals for \( \sigma_X \) and \( \sigma_Y \), that is

\[ P(x, y) = \sum_{x', y'} \mu_X(x, x')\mu_Y(y, y')p(x', y'), \quad (2.22) \]

leading to

\[ P(x, y) = \frac{1}{4} \left( 1 + xs_X + ys_Y + xys_Z \frac{\gamma_{XY}}{\gamma_X \gamma_Y} \right). \quad (2.23) \]

As shown in Refs. 13-19, in a classical framework this inversion always produces a well behaved joint distribution, being always positive in particular. So we may refer to the case \( P(x, y) < 0 \) as evidence of nonclassical behavior.

**III. REDUCED STATE**

Let us then determine the reduced state \( \rho_y \) after the \( Y \) measurement. Combining Eqs. (2.15), (2.17) and (2.18) we readily get

\[ p(x|y) = \frac{1}{2} \left( 1 + x\gamma_X \frac{s_X + ys_Z \gamma_{XY}}{1 + y\gamma_Y s_Y} \right). \quad (3.1) \]

Expressing the reduced state as

\[ \rho_y = \frac{1}{2} (\sigma_0 + t \cdot \sigma), \quad (3.2) \]

and using Eq. (2.14) we get

\[ p(x|y) = \text{tr} [\rho_y \Delta_X(x)] = \frac{1}{2} \left( 1 + x\gamma_X t_X \right). \quad (3.3) \]

Then, equating (3.1) and (3.3) we obtain

\[ t_X = \frac{s_X + ys_Z \gamma_{XY}}{1 + y\gamma_Y s_Y}. \quad (3.4) \]

Let us derive three interesting conclusions from this result:

i.- Common quantum intuition suggests that the reduced state \( \rho_y \) should be determined by \( \Delta_Y(y) \), typically as \( \sqrt{\Delta_Y(y)\rho_y \sqrt{\Delta_Y(y)}} \). But in our case there seems to be no such a simple relation. In particular \( \Delta_Y(y) \) does not depend on \( \gamma_{XY} \) while \( \rho_y \) actually does.

ii.- The result (3.4) does no determine completely the reduced state \( \rho_y \), since \( t_Y \) and \( t_Z \) are left indeterminate. We may add the condition

\[ \text{tr} [\rho_y \Delta_Y(y)] = \text{tr} [\rho \Delta_Y(y)], \quad (3.5) \]

leading to \( t_Y = s_Y \), but this leaves still \( t_Z \) indeterminate.

iii.- There are situations in which there is no legitimate reduced state \( \rho_y \). This occurs when (3.4) is incompatible with the condition \( |t| \leq 1 \) required so that \( \rho_y \) is nonnegative \( \rho_y \geq 0 \). For example this is the case of \( s_X = s_Y = 0 \) provided that

\[ |t_X| = \left| y s_Z \frac{\gamma_{XY}}{\gamma_X} \right| > 1, \quad (3.6) \]

that holds provided that \( \gamma_{XY} \neq 0 \) and \( \gamma_X \) is small enough, a choice that is always possible.

**IV. NONEXISTENCE AND NONCLASSICALITY**

Let us show here that the nonexistence of \( \rho_y \) in Eq. (3.4) implies the nonclassical character of the recorded statistics \( p(x, y) \) as a necessary condition.

Let us begin with considering \( t_x > 0 \) so the nonexistence of \( \rho_y \) holds when \( t_x > 1 \). After Eq. (3.4) we have that \( t_x > 1 \) implies

\[ 1 - s_X + y\gamma_Y s_Y - y s_Z \frac{\gamma_{XY}}{\gamma_X} < 0. \quad (4.1) \]

Since always \( s_X \leq 1 \) then

\[ y\gamma_Y s_Y - y s_Z \frac{\gamma_{XY}}{\gamma_X} < 0, \quad (4.2) \]

and then, since \( 1 \geq \gamma_Y > 0 \)

\[ y s_Y - y s_Z \frac{\gamma_{XY}}{\gamma_X \gamma_Y} \leq y\gamma_Y s_Y - y s_Z \frac{\gamma_{XY}}{\gamma_X}, \quad (4.3) \]

which implies

\[ 1 - s_X + y s_Y - s_Z \frac{\gamma_{XY}}{\gamma_X \gamma_Y} < 0. \quad (4.4) \]

This readily means after Eq. (2.23) that the inverted joint distribution is pathological, i. e., \( t_x > 1 \) implies that \( P(x = -1, y) < 0 \). Likewise \( t_x < -1 \) implies that \( P(x = 1, y) < 0 \).

Inspired by this result, and looking for a more clear connection between nonexistence and nonclassicality, we may examine the reduction process after data inversion. This is replacing Eq. (2.23) by

\[ P(x, y) = P(x|y)P_Y(y), \quad (4.5) \]
so that after Eqs. (2.19) and (2.23) we get
\[
P(x|y) = \frac{1}{2} \left( 1 + x \frac{s_X + y s_Z \gamma_{XY}}{1 + y s_Y} \right). \tag{4.6}
\]
Asking for the \( \rho_y \) density matrix such that
\[
P(x|y) = \text{tr} [\rho_y \Omega_X (x)], \quad \Omega_X (x) = \frac{1}{2} (1 + x \sigma_X), \tag{4.7}
\]
where \( \Omega_X \) are the actual projectors on the eigenstates of \( \sigma_X \), and using the same notation in Eq. (3.2), we get
\[
t_X = \frac{s_X + y s_Z \gamma_{XY}}{1 + y s_Y}, \tag{4.8}
\]
so that in this case we readily get that \(|t_X| > 1\) holds if and if only if \(P(x = 1, y) < 0\) or \(P(x = -1, y) < 0\). This is to say that there is no legitimate \( \rho_y \) if and only if the joint statistics is nonclassical as discussed above.

\section{V. CONCLUSIONS}

We have examined quantum state reduction using just statistical Bayesian consistency applying it to POVMs representing noisy joint observations of incompatible observables. We have shown that there are situations where this Bayesian picture does not define a proper reduced state. We have found that this lack of existence has a clear and definite relation with the nonclassicality of the statistics defined by the very same POVM. This very interesting situation arises because in joint POVMs the observables involved lack their own separate Hilbert spaces, as the traditional models of quantum state reduction assume for the system and the apparatus as neatly separated entities.

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\appendix

\section*{Appendix A: Young interferometer realization}

Path-interference duality is a classic example of complementarity at work in a two-dimensional space, so it is a natural arena for a nontrivial joint observation of the kind we are addressing in this work. To this end we will follow the scheme in Ref. \[14\]. To fix ideas let us assume that we are dealing with a single-photon state of light.

Let us represent the state at the apertures of a Young interferometer by the two orthogonal kets \(|\pm\rangle\) that represent the presence of the photon at the corresponding aperture. They may be always regarded as the eigenvectors of the Pauli matrix \( \sigma_Y \), so let it be \( Y \propto \sigma_Y \) as the path observable.

Interference occurs by the coherent superposition of \(|\pm\rangle\), so we may represent interference by the observable \( X \propto \sigma_X \). We will consider that the observed interference \( X \) is directly measured on the system space \( S \) by projection on the eigenstates \(|x\rangle\) of \( \sigma_X \).

Path information, this is information about \( Y \), will be transferred from the system space to an auxiliary space \( A \), that will be the polarization of the photon at each aperture. The path information can be imprinted in the polarization state by a different phase plate placed on each aperture. When the apertures are illuminated by right-handed circularly polarized light, represented by the vector \(|\circ\rangle\) in the polarization space \( A \), the phase plates produce the following aperture-dependent polarization transformation
\[
|\pm\rangle \otimes |\circ\rangle \rightarrow |\pm\rangle \otimes |\pm \theta\rangle, \tag{A1}
\]
being
\[
|\pm \theta\rangle = \cos \frac{\theta}{2} |\circ\rangle \pm \sin \frac{\theta}{2} |\礭\rangle. \tag{A2}
\]
This corresponds to the action of the unitary operator
\[
U|\pm\rangle \otimes |\circ\rangle = |\pm\rangle \otimes |\pm \theta\rangle, \tag{A3}
\]
with
\[
U = e^{-i \frac{\phi}{2} \sigma_0 \otimes \Sigma_Y} = \cos \frac{\theta}{2} \sigma_0 \otimes \Sigma_0 - i \sin \frac{\theta}{2} \sigma_Y \otimes \Sigma_Y, \tag{A4}
\]
where for the sake of clarity we denote by capital \( \Sigma \) the Pauli matrices in the auxiliary polarization space \( A \), being \(|\circ\rangle\) and \(|\礭\rangle\) the eigenvectors of \( \Sigma_Z \), with eigenvalues +1 and −1 respectively.

Then, the path information is retrieved by measuring any combination of the observables represented by the Pauli matrices \( \Sigma_X \) and \( \Sigma_Y \) in the auxiliary polarization space, say
\[
\Sigma_\phi = \cos \varphi \Sigma_X + \sin \varphi \Sigma_Y. \tag{A5}
\]
This polarization measurement can be very easily achieved in practice with a linear polarizer, where \( \varphi \) represents the orientation of its axis. We denote as \(|y\rangle\) the eigenvectors of \( \Sigma_\phi \) with eigenvalue \( y = \pm 1 \), so the photon passing through the polarizer is represented by the vector \(|y = 1\rangle\) while the photon being stopped by the polarizer is represented by the vector \(|y = -1\rangle\).

Let us denote by \( \tilde{\Delta}(x, y) \) the actual projection-valued measure in the total space \( S \otimes A \)
\[
\tilde{\Delta}(x, y) = |x\rangle \langle x| \otimes |y\rangle \langle y|, \tag{A6}
\]
leading to the joint statistics
\[ p(x, y) = \text{tr}_{S \otimes A} \left[ U \rho \otimes \rho_A U^\dagger \Delta(x, y) \right] = \text{tr}_S \left[ \rho \Delta(x, y) \right], \]
where
\[ \rho_A = | \circ \rangle \langle \circ |, \]
and the subscripts \( A \) and \( S \) refers in each case to the auxiliary polarization space or the system space, and we use no subscript \( S \) in \( \rho \) and \( \Delta(x, y) \) to match the notation in the main body of the text. Then the actual POVM \( \Delta(x, y) \) just defined in the system space \( S \) is
\[ \Delta(x, y) = \text{tr}_A \left[ \rho_A U^\dagger \Delta(x, y) U \right], \]
which exactly of the form in Eqs. (2.10) and (2.11) with
\[ \gamma_X = \cos \theta, \quad \gamma_Y = \sin \theta \cos \varphi, \quad \gamma_{XY} = \sin \theta \sin \varphi, \]
that are actually points on the surface of a unit sphere,
\[ \gamma_X^2 + \gamma_Y^2 + \gamma_{XY}^2 = 1. \]

Let us investigate the reduced states \( \tilde{\rho}_y \) associated to the \( Y \) measurement, with a significant proviso: This is not what we were looking for in the main body of the paper, i.e. \( \rho_y \neq \tilde{\rho}_y \), mainly because
\[ \Delta_W(w) \neq | w \rangle \langle w |, \]
for \( W = X, Y, w = x, y \). To determine \( \tilde{\rho}_y \) we follow exactly the same steps mentioned in the case of a factorized POVM since \( X \) and \( Y \) have their own independent Hilbert spaces,
\[ \rho_y = \frac{\text{tr}_A \left[ U \rho \otimes \rho_A U^\dagger | y \rangle \langle y | \right]}{\text{tr}_{S \otimes A} \left[ U \rho \otimes \rho_A U^\dagger | y \rangle \langle y | \right]}, \]
leading to
\[ \tilde{\rho}_y = \frac{\Lambda_y \rho \Lambda_y^\dagger}{\text{tr}_S \left[ \Lambda_y^\dagger \Lambda_y \rho \right]}, \]
with
\[ \Lambda_y = \langle y | U | \circ \rangle = \frac{1}{\sqrt{2}} \left( ye^{i\varphi} \cos \frac{\theta}{2} \sigma_0 + \sin \frac{\theta}{2} \sigma_Y \right), \]
and
\[ \Lambda_y^\dagger \Lambda_y = \Delta_Y(y) = \frac{1}{2} (\sigma_0 + y \gamma_Y \sigma_Y). \]
Then it holds
\[ p(x|y) = \text{tr}_S \left[ \rho_y | x \rangle \langle x | \right], \]
which is different from the relation we were looking for
\[ p(x|y) = \text{tr}_S \left[ \rho_y \Delta_X(x) \right]. \]

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