Research Article

An Inverse Problem for Parabolic Partial Differential Equations with Nonlinear Conductivity Term

Ali Zakeri and Q. Jannati

Department of Mathematics, Faculty of Science, K. N. Toosi University of Technology, P.O. Box 16765-165, Tehran 19697 64499, Iran

Correspondence should be addressed to Ali Zakeri, azakeri@kntu.ac.ir

Received 28 December 2008; Revised 19 January 2009; Accepted 2 February 2009

We consider an inverse problem for partial differential equation with nonlinear conductivity term in one-dimensional space within a finite interval. In the considered problem, a temperature history is unknown in a boundary of domain. The homotopy perturbation technique is used. Moreover, we have presented a numerical example.

Copyright © 2009 A. Zakeri and Q. Jannati. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Inverse heat conduction problems (IHCPs) rely on temperature heat flux measurements for estimating unknown quantities in the analysis of physical problems in thermal engineering. As an example, inverse problems dealing with heat conduction have been generally associated with estimating an unknown boundary heat flux by using temperature measurements taken below the boundary surfaces. Therefore, while in the classical direct heat conduction problem the cause (boundary heat flux) is given and the effect (temperature field in the body) is determined, the inverse problem involves the estimation of the cause from the knowledge of the effect. An advantage of IHCP is that it enables a much closer collaboration between experimental and theoretical researchers in order to obtain the maximum of information regarding the physical problem under study.

Difficulties encountered in the solution of IHCPs should be recognized. IHCPs are mathematically classified as ill-posed in a general sense because their solutions may become unstable, as a result of the errors inherent to the measurements used in the analysis. Inverse problems were initially taken as not of physical interest due to their ill-posedness.

In recent years, some new methods for estimating surface heat flux in IHCPs developed in the theory and practice. Consequently, some methods for approximate solution of these problems have developed. To obtain stable results, special numerical techniques should be used. Examples include iterative gradient methods, optimization algorithms, regularization methods, function specification methods, space-marching method, conjugate gradient method, Levenberg-Marquardt method, iterative techniques, variational iteration, finite elements, finite volumes, boundary elements methods, and so on. In some works, the numerical methods have been used for IHCPs [1–14]. For instance, in [14] the variational iteration method is used to find the exact solution of a control parameter in parabolic equations. Unfortunately, most of these methods are useful in the linear IHCPs merely.

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician He [15–17]. The essential idea of this method is to introduce a homotopy parameter, say \( p \), which takes values from 0 to 1. When \( p = 0 \), the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As \( p \) is gradually increased to 1, the system goes through a sequence of deformations, the solution for each of which is close to that at the previous stage of deformation. Eventually, at \( p = 1 \), the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations [14–23]. The interested reader can see [21–23] for last development of HPM. This homotopy perturbation method will become a much more interesting method to solving nonlinear differential equations in science and engineering.
In this paper, we consider a one-dimensional nonlinear inverse heat conduction problem with nonlinear diffusivity term that temperature history is unknown in a boundary. Then, using finite difference method and discrete time variable, the partial differential equation converts to a system of nonlinear ordinary differential equations. Consequently, by applying the homotopy perturbation technique and estimate solution, the temperature distribution in domain at discrete times will be found. Finally, a numerical experiment is given.

2. Homotopy Perturbation Technique

We begin with the following definition which is presented in [24].

**Definition 1.** Let $X$ and $Y$ be topologic spaces. If $f$ and $g$ are continuous maps of the space $X$ into space $Y$, it is said that $f$ is homotopic to $g$, if there is a continuous map $F : X \times I = [0, 1] \rightarrow Y$ such that

$$
F(x, 0) = f(x), \quad F(x, 1) = g(x) \quad \text{for each} \ x \in X.
$$

The map $F$ is called a homotopy between $f$ and $g$.

To illustrate homotopy perturbation method, we consider the following nonlinear equation:

$$
A(u) - f(r) = 0, \quad (r \in \Omega),
$$

with the boundary conditions:

$$
B(u, \frac{\partial u}{\partial n}) = 0, \quad (r \in \Gamma),
$$

where $A$ is a general differential operator, $B$ is a boundary operator, $f(r)$ is a known analytic function, and $\Gamma$ is the boundary of domain $\Omega$. The operator $A$ can be generally divided in to two parts $F$ and $N$, where $F$ and $N$ are linear and nonlinear parts of $A$, respectively. However, (2) converts to the following form:

$$
L(u) + N(u) - f(r) = 0.
$$

In [15], the author constructs a homotopy $V : \Omega \times [0, 1] \rightarrow \mathbb{R}$ which satisfies

$$
H(v, p) = (1 - p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0,
$$

or

$$
H(v, p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0,
$$

where $r \in \mathbb{R}$ and $p \in [0, 1]$. In this situation, the parameter $p$ is called homotopy parameter and $v_0$ is an initial approximation of (2) which satisfies boundary conditions. When $p = 0$ or $p = 1$, we have

$$
H(v, 0) = L(v) - L(v_0) = 0,
H(v, 1) = A(v) - f(r) = 0.
$$

On the other hand, if $p \in (0, 1)$, then the homotopy $H(v, p)$ changes from $L(v) - L(v_0)$ to $A(v) - f(r)$.

Noticing that $0 \leq p \leq 1$ can be considered as a small parameter, applying the perturbation technique, we may assume that the solution of (5) or (6) can be expressed as a series in $p$, as follows:

$$
v = v_0 + p v_1 + p^2 v_2 + \cdots.
$$

When $p \rightarrow 1$, (5) or (6) corresponds to (4) and becomes the approximate solution of (4), that is, we have

$$
u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \cdots.
$$

Series (9) is convergent for most cases and the rate of convergence depends on $A(v)$ (for more details see [16, 17]).

3. Solution of Nonlinear Parabolic Problem by Homotopy Perturbation Technique

Let $\Phi(x, t)$ be given function in $\Omega \equiv [0, l] \times [0, T]$, $f(t)$, $g(t)$ and $s(x)$ known functions on $[0, T]$ and $[0, l]$, respectively.

Now, consider the following nonlinear differential equation:

$$
\frac{\partial u}{\partial t} - \left\{ (a(t)u + b(t))u_{xx} \right\} = \Phi(x, t),
$$

with initial condition:

$$
u(x, 0) = s(x), \quad x \in [0, l],
$$

and boundary conditions:

$$
u(0, t) = f(t), \quad t \in [0, T],
$$

$$
u(l, t) = g(t), \quad t \in [0, T],
$$

where $a, b, f, g$, and $s$ are known function such that $b(t)$ is far from zero in $[0, T]$.

Suppose that $u_0(x) = u(x, 0) = s(x)$, $n\Delta t = T$, $k = 1/\Delta t$, $t_j = j\Delta t$, $u_j(x) = u(x, t_j)$, and $\Phi_j(x) = \Phi(x, t_j)$ for $j \in J_n = \{1, 2, \ldots, n\}$. By using the backward finite difference scheme for term $u_t$ in (10) in the form

$$
u_j(x) = k(\nu_j(x) - \nu_{j-1}(x)), \quad j \in J_n,
$$

and substituting to (10), we find a system of second-order differential equations with respect to $x$. Then, we obtain

$$
-\frac{d}{dx} \left\{ (a(t_j)u_j(x) + b(t_j)) \frac{d}{dx} u_j(x) \right\} = \Phi_j(x),
$$

$$
1 \leq j \leq n,
$$

or

$$
\frac{d^2}{dx^2} u_j(x) - \left\{ \frac{k}{b(t_j)} (u_j(x) - u_{j-1}(x)) + a(t_j) \left( \frac{d}{dx} (u_j(x) \frac{d}{dx} u_j(x)) \right) \right\} = \Phi_j(x),
$$

$$
= \frac{-1}{b(t_j)} \Phi_j(x).
$$
Now, we can write (2) as follows:

\[ Au = L_xu - Nu = \Psi(x,t), \]  

(16)

where \( \Psi(x,t) = (-1/b(t_j)) \Phi_j(x) \), and \( L_x = d^2/dx^2 \) and \( Nu = (k/b(t_j))(u_j(x) - u_{j-1}(x)) + (a(t_j)/b(t_j))(d/dx)(u_j(x)/d/dx)u_j(x) \) are the linear and nonlinear parts of operator \( A \), respectively.

For simplicity, define \( u(x) = (u_1(x), u_2(x), \ldots, u_n(x))^T \) and \( D(u(x), v(x)) = (u_1(x)(d/dx)v_1(x), u_2(x)(d/dx)v_2(x), \ldots, u_n(x)(d/dx)v_n(x))^T \). Now, by putting

\[ m = \left( \begin{array}{c} -k/b(t_1) \\
\vdots \\
-k/b(t_n) \\
0 \\
0 \\
\end{array} \right) \]

\[ M_1 = \left( \begin{array}{ccc}
\frac{k}{b(t_1)} & 0 & 0 \\
-k/b(t_2) & \frac{k}{b(t_2)} & 0 \\
\vdots & \vdots & \ddots \\
0 & 0 & \cdots & \frac{k}{b(t_n)} \\
\end{array} \right), \]

\[ M_2 = \text{diag} \left( \frac{a(t_1)}{b(t_1)}, \frac{a(t_2)}{b(t_2)}, \ldots, \frac{a(t_n)}{b(t_n)} \right), \]

(17)

and \( D(u(x), u(x)) = (u_1(x)(d/dx)u_1(x), \ldots, u_n(x)(d/dx)u_n(x))^T \), where \( u_j(x) \) are the values of \( u \) at \( t = t_j \) for \( 1 \leq j \leq n \), and the notation \( \text{diag}(a_1, a_2, \ldots, a_n) \) refers to a diagonal matrix in the form

\[ \begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & a_n \\
\end{bmatrix}. \]

(18)

We can express (16) to the matrix form

\[ Nu = -M_2 \frac{d}{dx} (D(u, u)) + M_1 u + m. \]  

(19)

By homotopy perturbation method, we may choose a convex homotopy such that [18, 20]

\[ H(v(x), p) = v(x) - h(x) - p \int_0^x \int_0^x Nv(x)dx\,dx = 0, \]

(21)

\[ F(u(x)) = u(x) - h(x) = 0, \]

(22)

where

\[ h(x) = xg(t) + f(t) - \int_0^x \int_0^x \Psi(x)dx\,dx, \]

(23)

\[ v(x) = (v(t_1), \ldots, v(t_n))^T, \]  

and \( h(x) = (h(t_1), \ldots, h(t_n))^T \). By using (16), we can write

\[ v(x) = h(x) + p \int_0^x \int_0^x Nv(x)dx\,dx. \]

(24)

By combining (10) and (22), we obtain following results:

\[ v(x) = xg(t) + f(t) - \int_0^x \int_0^x \Phi_b(x)dx\,dx \]

\[ + p \int_0^x \int_0^x \left\{ M_1 (v(x) - u(x)) \right\} dx\,dx, \]

(25)

where \( u(x) = (s(x), u_1(x), \ldots, u_{n-1}(x))^T \), or

\[ v_0 = h(x) = xg(t) + f(t) - \int_0^x \int_0^x \Phi_b(x)dx\,dx, \]

\[ v_1 = \int_0^x \int_0^x \left\{ M_1 (v_0(x) - u(x)) - M_2 \frac{d}{dx} D(v_0, v_0) \right\} dx\,dx, \]

\[ v_2 = \int_0^x \int_0^x \left\{ \text{diag} \left( \frac{k}{b(t_1)}, \ldots, \frac{k}{b(t_n)} \right) v_1 \right. \]

\[ - M_2 ^{\frac{d}{dx}} D(v_0, v_1) + D(v_1, v_0) \right\} dx\,dx, \]

(26)

where the above relations are obtained of equating the terms with identical powers of \( p \) in (25). Obviously, if \( p \to 1 \), then the approximate solution is

\[ u(x) \approx v_0 + v_1 + v_2. \]  

(27)

In Section 5, we give a numerical sample. By using of homotopy perturbation technique, an approximate solution for nonlinear diffusion problem is obtained [15, 21].

4. Numerical Results

Let

\[ u_t = \frac{\partial}{\partial x} \left\{ \left( \frac{1}{6} e^{-t}u + (t + 5)e^{-t} \right) \frac{\partial u}{\partial x} \right\} \]

\[ = -\frac{7}{6} t + 9, \quad (x, t) \in [0,1] \times [0,1], \]

\[ u(x,0) = x^3, \quad 0 \leq x \leq 1, \]

\[ u(0,t) = t, \quad 0 \leq t \leq 1, \]

\[ u_u(0,t) = 0, \quad 0 \leq t \leq 1. \]  

(28)
### (a) Exact and approximate solution of $u_j(x)$ at $t_j = 0.25$. 

| $x$  | Exact solution | Approximate solution | Relative error |
|------|----------------|----------------------|----------------|
| 0.1  | 0.2628402542   | 0.2628399122        | $1.30 \times 10^{-6}$ |
| 0.2  | 0.3013610167   | 0.3013563938        | $1.53 \times 10^{-5}$ |
| 0.3  | 0.3655622875   | 0.3655396925        | $6.18 \times 10^{-5}$ |
| 0.4  | 0.4544406667   | 0.4553735995        | $1.54 \times 10^{-4}$ |
| 0.5  | 0.570063542    | 0.5708355086        | $2.29 \times 10^{-4}$ |
| 0.6  | 0.7122491501   | 0.7118965132        | $4.95 \times 10^{-4}$ |
| 0.7  | 0.8791724543   | 0.8783215163        | $7.40 \times 10^{-4}$ |
| 0.8  | 1.071776267    | 1.070669376         | $1.03 \times 10^{-3}$ |
| 0.9  | 1.290060588    | 1.288293071         | $1.37 \times 10^{-3}$ |
| 1    | 1.534025417    | 1.531339891         | $1.75 \times 10^{-3}$ |

### (b) Exact and approximate solution of $u_j(x)$ at $t_j = 0.5$. 

| $x$  | Exact solution | Approximate solution | Relative error |
|------|----------------|----------------------|----------------|
| 0.1  | 0.5164872127   | 0.5164865028        | $1.37 \times 10^{-6}$ |
| 0.2  | 0.5659488508   | 0.5659409374        | $1.39 \times 10^{-5}$ |
| 0.3  | 0.6483849144   | 0.6483481508        | $5.67 \times 10^{-5}$ |
| 0.4  | 0.7637954034   | 0.7636830954        | $1.47 \times 10^{-4}$ |
| 0.5  | 0.9121803178   | 0.9119111460        | $2.95 \times 10^{-4}$ |
| 0.6  | 1.093539658    | 1.092988531         | $5.03 \times 10^{-4}$ |
| 0.7  | 1.30873423     | 1.30862885          | $7.72 \times 10^{-4}$ |
| 0.8  | 1.555181613    | 1.553473905         | $1.09 \times 10^{-3}$ |
| 0.9  | 1.835464230    | 1.832754112         | $1.47 \times 10^{-3}$ |
| 1    | 2.148721271    | 2.144629711         | $1.90 \times 10^{-3}$ |

### (c) Exact and approximate solution of $u_j(x)$ at $t_j = 0.75$. 

| $x$  | Exact solution | Approximate solution | Relative error |
|------|----------------|----------------------|----------------|
| 0.1  | 0.7711700002   | 0.7711686828        | $1.70 \times 10^{-6}$ |
| 0.2  | 0.8346800007   | 0.8346667665        | $1.58 \times 10^{-5}$ |
| 0.3  | 0.9405300015   | 0.9404704769        | $6.32 \times 10^{-5}$ |
| 0.4  | 1.088720003    | 1.088540598         | $1.64 \times 10^{-4}$ |
| 0.5  | 1.279250004    | 1.278823053         | $3.33 \times 10^{-4}$ |
| 0.6  | 1.512120006    | 1.511249735         | $5.75 \times 10^{-4}$ |
| 0.7  | 1.787330008    | 1.785739520         | $8.89 \times 10^{-4}$ |
| 0.8  | 2.104880011    | 2.102199488         | $1.27 \times 10^{-3}$ |
| 0.9  | 2.464770014    | 2.460526294         | $1.72 \times 10^{-3}$ |
| 1    | 2.867000017    | 2.860607703         | $2.22 \times 10^{-3}$ |

### (d) Exact and approximate solution of $u_j(x)$ at $t_j = 1$. 

| $x$  | Exact solution | Approximate solution | Relative error |
|------|----------------|----------------------|----------------|
| 0.1  | 1.027182818    | 1.027180602         | $2.15 \times 10^{-6}$ |
| 0.2  | 1.108731273    | 1.108709841         | $1.93 \times 10^{-5}$ |
| 0.3  | 1.244645364    | 1.244550232         | $7.64 \times 10^{-5}$ |
| 0.4  | 1.434925092    | 1.434640074         | $1.98 \times 10^{-4}$ |
| 0.5  | 1.679570457    | 1.678894579         | $4.02 \times 10^{-4}$ |
| 0.6  | 1.978581458    | 1.977207417         | $6.94 \times 10^{-4}$ |
| 0.7  | 2.331958096    | 2.329452661         | $1.07 \times 10^{-3}$ |
| 0.8  | 2.739700370    | 2.735487036         | $1.53 \times 10^{-3}$ |
| 0.9  | 3.201808281    | 3.195152502         | $2.07 \times 10^{-3}$ |
| 1    | 3.718281828    | 3.708279035         | $2.69 \times 10^{-3}$ |
If we want to use our last notation, we have

$$\Phi(x, t) = -\frac{7}{3}t - 9,$$

$$a(t) = \frac{1}{6}e^{-t},$$

$$b(t) = (t + 5)e^{-t}. \quad (29)$$

Obviously, the above assumptions satisfy consideration of conditions. The exact solution is $$u(x, t) = x^2e^t + t.$$ In this sample, we obtain the solution in $$x = 0.2, 0.4, 0.8, 1$$ at $$t = 0.25, 0.5, 1.$$. Assume that $$\Delta t = 0.25$$, then we construct a homotopy as the same form as we describe in previous sections. Consequently, the solution will be constructed by

$$v_0(x) = h(x, t_j) = t_j - \frac{1}{(t_j + 5)}e^{-t_j} \left( -\frac{7}{3}t - 9 \right) \frac{x^2}{2},$$

$$v_1(x) = \int_0^x \int_0^x \int_0^x \int_0^x \frac{4e^t}{(t_j + 5)} v_0(x) - u(x, t_{j-1})$$

$$- \frac{1}{6} \frac{d}{(t_j + 5)} dx \left( v_0(x) \frac{d}{dx} v_0(x) \right) \right) dx,$$

$$v_2(x) = \int_0^x \int_0^x \int_0^x \int_0^x \frac{4e^t}{(t_j + 5)} v_1(x) - \frac{1}{6} \frac{d}{(t_j + 5)} dx$$

$$\times \left( v_0(x) \frac{d}{dx} v_1(x) + v_1(x) \frac{d}{dx} v_0(x) \right) \right) dx,$$

for $$j = 1, 2, 3, 4.$$

Exact solution, approximate solution, and relative error for the above problem are given in Table 1 at $$t = t_j = j\Delta t,$$ $$j = 1, 2, 3, 4.$$

5. Conclusion

In the obtained results of problem, we see that the approximate solutions for small increment $$\Delta t$$ have relative error at least of order $$O(E - 3).$$ This technique applied for some inverse problems and results of this approach are acceptable for small mesh size $$\Delta t.$$ In [25–27], the stability and convergency of HPM for heat transfer, KdV equation, and couple of systems of reaction-diffusion equations are discussed. The authors’ aim is to find the stability conditions of boundaries and initial data such that the solution is stable.

References

[1] J. V. Beck, B. Blackwell, and C. J. St. Clair Jr., *Inverse Heat Conduction: Ill-Posed Problems*, John Willey & Sons, New York, NY, USA, 1985.

[2] O. M. Alifonov, *Inverse Heat Transfer Problems*, Springer, Berlin, Germany, 1994.

[3] M. N. Özisik and H. R. B. Orlande, *Inverse Heat Transfer: Fundamentals and Applications*, Taylor & Francis, New York, NY, USA, 2000.

[4] H. M. Park and J. S. Chung, “A sequential method of solving inverse natural convection problems,” *Inverse Problems*, vol. 18, no. 3, pp. 529–546, 2002.

[5] J. R. Cannon, *The One-Dimensional Heat Equation*, Addison-Wesley, New York, NY, USA, 1984.

[6] A. Shidfar and A. Zakeri, "A numerical method for backward inverse heat conduction problem with two unknown functions," *International Journal of Engineering Science*, vol. 304, no. 16, pp. 71–74, 2008.

[7] A. Shidfar and A. Zakeri, “Asymptotic solution for an inverse parabolic problem,” *Mathematica Balkanica*, vol. 18, no. 3–4, pp. 475–483, 2004.

[8] A. Shidfar, A. Zakeri, and A. Neisi, “A two-dimensional inverse heat conduction problem for estimating heat source,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 10, pp. 1633–1641, 2005.

[9] T.-C. Chen, C.-C. Liu, H.-Y. Jang, and P.-C. Tuan, “Inverse estimation of heat flux and temperature in multi-layer gun barrel,” *International Journal of Heat and Mass Transfer*, vol. 50, no. 11–12, pp. 2060–2068, 2007.

[10] C.-H. Huang and S.-P. Wang, “A three-dimensional inverse heat conduction problem in estimating surface heat flux by conjugate gradient method,” *International Journal of Heat and Mass Transfer*, vol. 42, no. 18, pp. 3387–3403, 1999.

[11] J. Taler and W. Zima, “Solution of inverse heat conduction problems using control volume approach,” *International Journal of Heat and Mass Transfer*, vol. 42, no. 6, pp. 1123–1140, 1999.

[12] D. Lesnic, L. Elliott, and D. B. Ingham, “The solution of an inverse heat conduction problem subject to the specification of energies,” *International Journal of Heat and Mass Transfer*, vol. 41, no. 1, pp. 25–32, 1998.

[13] N. Al-Khalidy, “On the solution of parabolic and hyperbolic inverse heat conduction problems,” *International Journal of Heat and Mass Transfer*, vol. 41, no. 23, pp. 3731–3740, 1998.

[14] L. Jinbo and T. Jiang, “Variational iteration method for solving an inverse parabolic equation,” *Physics Letters A*, vol. 372, no. 20, pp. 3569–3572, 2008.

[15] J.-H. He, “Homotopy perturbation technique,” *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3–4, pp. 257–262, 1999.

[16] J.-H. He, “A coupling method of a homotopy technique and a perturbation technique for non-linear problems,” *International Journal of Non-Linear Mechanics*, vol. 35, no. 1, pp. 37–43, 2000.

[17] J.-H. He, “Homotopy perturbation method: a new non-linear analytical technique,” *Applied Mathematics and Computation*, vol. 135, no. 1, pp. 73–79, 2003.

[18] A. Zakeri, Q. Jannati, and A. Aminataei, “Application of He’s homotopy perturbation method for Cauchy problem in one-dimensional nonlinear equation of diffusion,” to appear in *International Journal of Engineering Science*.

[19] M. Ghasemi, M. Tavassoli Kajani, and A. Davari, “Numerical solution of two-dimensional nonlinear differential equation by homotopy perturbation method,” *Applied Mathematics and Computation*, vol. 189, no. 1, pp. 341–345, 2007.

[20] A. Yildirim, “Application of He’s homotopy perturbation method for solving the Cauchy reaction-diffusion problem,” *Computers & Mathematics with Applications*, vol. 57, no. 4, pp. 612–618, 2009.

[21] J.-H. He, “Some asymptotic methods for strongly nonlinear equations,” *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.

[22] J.-H. He, “Recent development of the homotopy perturbation method,” *Topological Methods in Nonlinear Analysis*, vol. 31, no. 2, pp. 205–209, 2008.
[23] J.-H. He, “An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering,” *International Journal of Modern Physics B*, vol. 22, no. 21, pp. 3487–3578, 2008.

[24] J. R. Munkres, *Topology*, Prentice-Hall, Upper Saddle River, NJ, USA, 2nd edition, 2000.

[25] D. D. Ganji, “The application of He’s homotopy perturbation method to nonlinear equations arising in heat transfer,” *Physics Letters A*, vol. 355, no. 4-5, pp. 337–341, July 2006.

[26] D. D. Ganji and M. Rafei, “Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation by homotopy perturbation method,” *Physics Letters A*, vol. 356, no. 2, pp. 131–137, 2006.

[27] D. D. Ganji and A. Sadighi, “Application of He’s homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations,” *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 4, pp. 411–418, 2006.