Gravitational Waves in Braneworld Scenarios with AdS Background

A. C. Amaro de Faria, and R. da Rocha
Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, 09210-170, Santo André, SP, Brazil

M. E. S. Alves
Instituto de Ciências Exatas, Universidade Federal de Itajubá, 37500-903 Itajubá, MG, Brazil

J. C. N. de Araújo
INPE - Instituto Nacional de Pesquisas Espaciais - Divisão de Astrofísica, Av. dos Astronautas 1758, São José dos Campos, 12227-010 SP, Brazil

In this paper, we investigate gravitational waves as metric perturbations around a general warped 5-dimensional background. We find an analytical solution in Randall-Sundrum braneworld model and analyze the implications of braneworld models in the gravitational waves propagation.

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I. INTRODUCTION

Braneworld scenarios wherein the observable universe is trapped on a brane, embedded in some higher-dimensional spacetime, can explain the hierarchy problem [1–8]. Braneworld models also provide alternatives to Kaluza-Klein compactification, where the topology has a compactification radius of the Planck length order. These possibilities come from developments in non-perturbative string theory, wherein the so-called D-branes are elicited and evinced as (D + 1)-dimensional manifolds in which the standard model of particles and fields can be consistently confined. A plausible reason for the weak appearance of the gravitational force, with respect to other forces, can be its dilution in a higher-dimensional bulk, where D-branes are embedded. D-branes are good candidates for braneworlds because among some outstanding features they possess gauge symmetries [10–12]. The gauge symmetry arises from open strings, which can collide to form a closed string — which simplest excitation modes correspond precisely to gravitons — that can leak into a higher-dimensional bulk. The possibility concerning the existence of extra dimensions may still ascertain physical aspects on string theory and D-branes. An alternative approach to the extra dimension compactification was pointed by Randall-Sundrum braneworld formalism [15, 16], where the electromagnetic, weak, and strong forces, together with all the matter in the universe as well are confined on a 3-brane, and only gravitons would be allowed to leave the surface and move into the AdS5 bulk.

At high energies, significant changes are introduced in gravitational dynamics, forcing general relativity to be emulated and overcome by a quantum gravity theory [14]. Randall-Sundrum braneworld models [15, 16] induce a volcano barrier shaped effective potential for gravitons around the brane [17]. The corresponding spectrum of gravitational perturbations has a massless bound state on the brane, and a continuum of bulk modes with suppressed couplings to brane fields, which introduces small corrections at short distances.

Although this alternative removes the hierarchy between the weak and the Planck scales, there is still a hierarchy between the weak and the compactification scales. However, the geometries arising from the Horava-Witten theory [18, 19] can explain the origin of this resulting hierarchy, as was treated by Randall-Sundrum [15, 16].

In this perspective, gravity can propagate in the higher-dimensional manifold without modifying Newton’s inverse square law [15, 16]. This is possible because a curved background could generate higher-dimensional modes of the graviton in the extra dimension.

In this context, the aim of the present work is to analyze gravitational waves as metric perturbations around a general 5-dimensional metric and also to reveal their prominent characteristics in braneworld scenarios, obtained in a background geometry. Such geometry associated would arise from a specific model characterized by a suitable action, which is equivalent to consider a general perturbation on the curved background. The gravitational waves propagation depends also on the extra dimension, in this formalism.

The paper is organized as follows: the background geometry on which the perturbations are expanded is briefly described in Section I. In Section III, in order to provide analytic solutions for the gravitational waves, we delve into the Randall-Sundrum formalism and find explicitly analytical solutions for the gravitational waves in this braneworld scenario, forthwith in full compliance to the standard usual 4-dimensional Minkowski solutions, which are the limit of the braneworld gravitational waves analytic solutions when the extra dimension tends to
II. THE BACKGROUND SPACETIME

Hereon \( \{ \epsilon_A \}, \mu = 0, 1, 2, 3 \) \( \{ \epsilon_A \}, A = 0, 1, 2, 3, 4 \) denotes a basis for the tangent space \( T_x M \) at a point \( x \) in a 3-brane \( M \) embedded in \( \mathbb{R}^5 \). Naturally the tangent space at \( x \) has an orthonormal basis \( \{ \theta^\mu \} \) \( \{ \theta^A \} \) such that \( \theta^\mu (\epsilon_A) = \delta^\mu_A \). If we choose a local coordinate chart, it is possible to represent \( \epsilon_A = \partial / \partial x^A \) and \( \theta^A = dx^A \).

Take \( n = n^A \epsilon_A \) a vector orthogonal to \( T_x M \) and let \( y \) be the Gaussian coordinate orthogonal to \( T_x M \), indicating how an observer upheavals out the brane into the bulk. In particular, \( n_A dx^A = dy \). A vector field \( v = x^A \epsilon_A \) in the bulk is split into components in the brane and orthogonal to the brane, respectively as \( v = x^\mu \epsilon_\mu + y \epsilon_A = (x^\mu, y) \). Since the bulk is endowed with a metric \((5) g = (5) g_{AB} dx^A dx^B \), the components of the metric in the brane and in the bulk, denoted respectively by \( g_{AB} \) and \( (5) g_{AB} \), are related by

\[
(5) g_{AB} = g_{AB} + n_AN_B.
\]  

The extrinsic curvature components can be defined via the Lie derivative as \( K_{AB} = \frac{1}{2} \mathcal{L}_\epsilon g_{AB} \). The extrinsic curvature of the brane localized at \( y = 0 \) describes the embedding of the brane in the bulk, and projects the bulk Riemann tensor on the brane as given by \( R_{ABCD} = (5) R_{EFGH} g_A^E g_B^F g_C^G g_D^H + 2 K_{[A}[C] K_{D]B]} \).

A background metric containing a warp factor which is a function of the extra dimension is now considered:

\[
(5) ds^2 = (5) g_{AB} dx^A dx^B = e^{2\zeta(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
\]  

The behavior of the so-called warped factor \( \zeta(y) \) can be derived from a general action \( \mathcal{L} \) with \( 2 \leq D \leq 4 \) as

\[
S_{gr} = S_{EH} + S_{GH} + S_1 + S_2 + \cdots
\]  

where the additional terms beyond \( S_2 \) make the action to be finite, since it presents logarithmic divergences \( \mathcal{L} \). In the most general case of a \( (D + 1) \)-dimensional spacetime, the first term is the usual Einstein-Hilbert action:

\[
S_{EH} = -\frac{1}{16\pi G_{D+1}} \int d^{D+1}x \sqrt{g_{D+1}} \left( R + \frac{D(D-1)}{\ell^2} \right),
\]  

where \( G_{D+1} \) denotes the \( (D + 1) \)-dimensional Newton constant \( 2\pi G_{D+1} = M_{D+1}^D G \), where \( M_{D+1} \) denotes the \( (D + 1) \)-dimensional Planck mass, \( G \) denotes the 4-dimensional gravitational constant, and \( D \) denotes the number of spatial dimensions. Also, \( R \) denotes the curvature scalar of the boundary, the metric \( g_{D+1} \) is the \( \text{AdS}_{D+1} \) metric and \( \ell \) denotes the \( \text{AdS}_{D+1} \) curvature radius.

The second term in the action \( (5) \) is given by

\[
S_{GH} = -\frac{1}{8\pi G_{D+1}} \int d^{D+1}x \sqrt{h} K,
\]  

where \( K \) is related to the boundary curvature and \( h \) denotes the determinant of the induced metric \( h \).

The first two counter terms in the action \( (5) \) are given by \( \gamma_1 \gamma_2 \gamma_3 \): \( S_1 = \frac{D-1}{8\pi G_{D+1} \ell} \int d^Dx \sqrt{h} \), \( S_2 = \frac{-\ell}{16\pi G_{D+1}(D-2)} \int d^Dx \sqrt{h} R \).

In the particular case of Randall-Sundrum braneworlds, gravity is localized in the brane by warped compactification, and what precludes gravity from leaking into the extra dimension \( y \) at low energies is a negative bulk cosmological constant, \( \Lambda_5 = -\frac{6\mu^2}{\ell^2} = -6\mu^2 \), where \( \mu \) denotes the corresponding energy scale. The curvature radius determines the magnitude of the Riemann tensor \( \gamma_2 \):

\[
(5) R_{ABCD} = -\frac{1}{\ell^2} \left( (5) g_{AC}(5) g_{BD} - (5) g_{AD}(5) g_{BC} \right).
\]  

The bulk cosmological constant \( \Lambda_5 \) imposes the gravitational field to be closer to the brane. In Gaussian normal coordinates \( X^A = (x^\mu, y) \) the \( \text{AdS}_5 \) metric takes the form

\[
(5) ds^2 = e^{-2\zeta(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
\]  

where \( \eta_{\mu\nu} \) denotes the Minkowski metric. The exponential warp factor reflects the confining role of the bulk cosmological constant. The \( \mathbb{Z}_2 \) symmetry about the brane at \( y = 0 \) is incorporated via the \( |y| \) term. In the bulk, this metric is a solution of the 5-dimensional Einstein equations, \( (5) G_{AB} = -\Lambda_5 (5) g_{AB} \). The brane is a flat Minkowski spacetime, with self-gravity in the form of brane tension.

The solution of Eq.\( (4) \) in the case of \( D = 4 \) presents the following warp factor \( \gamma_3 \) for the case of the Randall-Sundrum braneworld model\( ^1 \):

\[
\zeta(y) = -ky,
\]  

where \( k = \ell^{-1} \) is the inverse of the \( \text{AdS}_5 \) radius \( \ell \).

III. GRAVITATIONAL WAVES SOLUTION IN BRANEWORLD MODELS

Let us consider small perturbations around the background metric \( \eta_{\mu\nu} \mapsto h_{\mu\nu} + \eta_{\mu\nu} \), where \( h_{\mu\nu} \ll 1 \) in

\[\text{Reference}\]  

\[\text{footnote}^1\] For hybrid compactification, otherwise the warp factor is given by \( \zeta(y) = -|y| \).
Randall-Sundrum braneworld scenario

\[ ds^2 = e^{2\xi(y)}(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2, \]  
\[ (11) \]

In Gaussian coordinates the Lie derivative \( L_n \) equals \( \partial/\partial y \). Such approach was comprehensively used in different applications of braneworld formalism, as in [24, 29–33]. Calculating the perturbed vacuum field equations, the following equation for the 4-dimensional components is obtained:

\[ \partial_\nu \partial_\mu h_{\rho\nu} - \partial_\mu \partial_\rho h_{\nu\nu} + \Box h_{\mu\nu} - \eta_{\mu\nu} \zeta(2e^{2\xi}h_{\rho\rho}') - e^{2\xi}[2h''_{\mu\nu} + 4\zeta' h'_{\mu\nu} + 2(4\zeta'^2 + \zeta'')h_{\mu\nu}] = 0. \]  
\[ (12) \]

Using further gauge conditions \( \partial_\nu h_{\nu\nu} = 0 \), for each \( \nu = 0, 1, 2, 3 \), and \( h_{\nu\mu} = 0 \), the equations above simplify to:

\[ e^{2\xi}[2h''_{\mu\nu} + 4\zeta' h'_{\mu\nu} + 2(4\zeta'^2 + \zeta'')h_{\mu\nu}] - \Box h_{\mu\nu} = 0. \]  
\[ (13) \]

If the perturbation \( h_{\mu\nu} \) is written as the product

\[ h_{\mu\nu}(y, x^\rho) = \phi(y) \chi_{\mu\nu}(x^\rho), \]  
\[ (14) \]

and substituting Eq. (14) in Eq. (13), the usual 4-dimensional equations are obtained:

\[ \Box \chi_{\mu\nu} + n^2 \chi_{\mu\nu} = 0, \]  
\[ (15) \]

where \( n \) is a constant of separation. The solution of this equation is a linear superposition of plane waves:

\[ \chi_{\mu\nu} = \epsilon_{\mu\nu} \exp(ik_{\sigma}x^\sigma), \]  
\[ (16) \]

where the 4-wave vector satisfies \( k_{\mu}k^\mu = n^2 \). This condition upheals some informations with respect to the associated plane waves (16), for example the dispersion relation associated. In this context, an extra dimension dependence in the warp factor and consequently on the metric perturbation affects the plane waves propagation modes.

The function \( \phi(y) \) satisfies the equation

\[ \phi'' + 4\zeta' \phi' + 2(4\zeta'^2 + \zeta'' + n^2 e^{-2\zeta})\phi = 0, \]  
\[ (17) \]

that describes the dependence of the metric perturbation with respect to \( y \), once specified a particular warp factor \( \zeta(y) \). It is interesting to note that the 4-dimensional solution (16) is independent of the function \( \zeta(y) \).

Now, in order to find a solution for \( \phi(y) \) the function \( \zeta(y) \) is substituted by the solution (16) presented in [15, 16]. Equation (17) reads:

\[ \phi'' - 4k\phi' + 2(n^2 e^{2ky} + 4k^2)\phi = 0. \]  
\[ (18) \]

Using the suitable change of variables \( \xi = \frac{Z}{\sqrt{4k}}e^{ky} \), the solution of the equation above can be written as:

\[ \phi(\xi) = \xi^2 \left[ C_1 J_2(n\xi) + C_2 Y_2(n\xi) \right], \]  
\[ (19) \]

where the constants \( C_1 \) and \( C_2 \) can be suitably chosen in order to reproduce the standard general relativity at the brane. Indeed, as the 2nd order Bessel function \( Y_2(n\xi) \) diverges as \( n \to 0 \), in order to obtain physical solutions we impose the constant \( C_2 = 0 \).

Therefore, in the case where \( n = 0 \) we have \( k_\mu k^\mu = 0 \) and the solution for \( \phi \) is

\[ \phi(y) = e^{2ky} \left( A_1 e^{2iky} + A_2 e^{-2iky} \right), \]  
\[ (20) \]

where \( A_1 \) and \( A_2 \) are complex constants. Therefore, the well known case leading to general relativity is immediately obtained when \( h_{\mu\nu}(y = 0, x^\rho) \equiv \chi_{\mu\nu}(x^\rho) \) in Eq. (14). Indeed, when \( n = 0 \) Eq. (15) implies such limit case.

The functions \( J_2 \) and \( Y_2 \) are Bessel functions of the first and second kind of complex order \( 2i \), respectively. The real part of (19) represents the dependence of the gravitational waves amplitudes with the extra dimension \( y \). Its profile is related to the separation constant \( n \) and to the constant \( k \), which characterizes the Anti-de Sitter length scale. The smaller the \( \phi(y) \), the bigger is \( k \).

Furthermore, in spite of the behavior of (19) towards large values of \( y \), from the Randall-Sundrum model compactification topology \( S^1/Z_2 \), the increasing of the function \( \phi(y) \) would be limited by the extra dimension periodicity. Note that the increasing behavior of \( \phi(y) \) is also constrained by the ratio between the separation constant and the compactification radius of the extra dimension \( n/k \).

Following the analysis in [15, 16] and [18, 19] where the radius of compactification is of the order of Planck scale, we can consider the solution (19) in that regime. In this length scale the profile of solution in its asymptotic limit \( y \to 0 \) that presents a behavior given by expression in function of Bessel function of first kind \( J_\alpha(x) \sim \frac{\sin(x)}{x^{\alpha+1}}(\frac{2}{\pi x})^\alpha \) with \( \alpha \in 3 \). The same behavior happens when one analyze the Bessel function of second kind \( Y_\alpha \), that can be written in terms of \( J_\alpha \).

Standard classical 4-dimensional gravitational waves solution is obtained in the case when we consider \( y = 0 \), and we have in this case no extra dimension and the Minkowski metric. This can be seen substituting \( y = 0 \) in Eq. (17), where we obtain oscillatory solutions describing the gravitational waves.

IV. CONCLUDING REMARKS AND OUTLOOK

This work analyzed gravitational waves in scenarios where our universe is considered as a 3-brane embedded in five dimensions in accordance with Randall-Sundrum model. Essentially, we analyzed the metric perturbations around an AdS_5 metric, considering the Randall-Sundrum solution for the warp factor. The solution obtained is a linear combination of Bessel functions of first and second kind with complex order. Analyzing the real part of the solution, its behavior is characterized by the
The ratio between the separation constant and the constant $k$ which is of the order of the Planck length scale. The solution is consistent in the case of small scale length of extra dimension.

The prominent features of the analytical solutions in the Randall-Sundrum model: the extra dimension can be regulated by a separation constant different from zero, which implies also on the propagation of gravitational waves. At any rate the solution provides a dependence, in fact, of extra dimension on waves propagation. The propagation modes are affected by feature of topology of extra dimension and its particular and main features can be represented by real part of solution. The solution obtained here can be also consistent if one considers the $S^1/Z_2$ topology on the extra dimension, which make it periodic.

There is a question about the scenario in which are explored the gravitational waves. This is related mainly to the separation constant $n$. The whole spectrum of modes parameterized by it would imply in a dispersion relation characteristic.

These modes can, in fact, characterize the gravitational waves in scenarios with extra dimension. These modes could be interpreted like resonant modes in black hole perturbation theory context for example. Some additional aspects can be seen, e.g., in [37, 38]. There the equations governing gravity wave propagation support resonant solutions that are classified as quasi-normal modes. These resonant effects appear in many branches of physics and can, as was treated in this context, be used to characterize features of a given treatment metric perturbation.

Last but not least, recall that gravitational waves in general relativity present two polarization states, namely, the “+” and the “−”. In the present case, however, since the wave equation is different from that of the General Relativity, additional polarization states can well exist.

Therefore, the existence of additional states of polarizations could well give us some tips concerning the existence of extra dimensions. In a paper to appear elsewhere we will consider such an issue.

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