Hölder regularity of the solution to the complex Monge-Ampère equation with $L^p$ density

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Publication Details

Baracco, L., Khanh, T., Pinton, S. & Zampieri, G. (2016). Hölder regularity of the solution to the complex Monge-Ampère equation with $L^p$ density. *Calculus of Variations and Partial Differential Equations, 55* (4), 74-1-74-8.
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Abstract
On a smooth domain \( C \subset C^n \), we consider the Dirichlet problem for the complex Monge-Ampère equation \( ((dd^c u)^n = f dV, u|_{\partial C} = \phi) \). We state the Hölder regularity of the solution \( u \) when the boundary value \( \phi \) is Hölder continuous and the density \( f \) is only \( L^p \), \( p > 1 \). Note that in former literature (Guedj-Kolodziej-Zeriahi) the weakness of the assumption \( f \in L^p \) was balanced by taking \( \phi \in C^{1,1} \) (in addition to assuming strongly pseudoconvex).

Keywords
monge, complex, h, p, solution, l, density, equation, re, regularity, lder, amp

Disciplines
Engineering | Science and Technology Studies

Publication Details
Baracco, L., Khanh, T., Pinton, S. & Zampieri, G. (2016). Hölder regularity of the solution to the complex Monge-Ampère equation with L^p density. Calculus of Variations and Partial Differential Equations, 55 (4), 74-1-74-8.

This journal article is available at Research Online: http://ro.uow.edu.au/eispapers/5965
HÖLDER REGULARITY OF THE SOLUTION TO THE COMPLEX MONGE-AMPÈRE EQUATION WITH $L^p$ DENSITY

LUCA BARACCO, TRAN VU KHANH, STEFANO PINTON AND GIUSEPPE ZAMPIERI

Abstract. On a smooth domain $\Omega \subset \subset \mathbb{C}^n$, we consider the Dirichlet problem for the complex Monge-Ampère equation $((dd^c u)^n = f dV, u|_{b\Omega} = \varphi)$. We state the Hölder regularity of the solution $u$ when the boundary value $\varphi$ is Hölder continuous and the density $f$ is only $L^p$, $p > 1$. Note that in former literature (Guedj-Kolodziej-Zeriahi) the weakness of the assumption $f \in L^p$ was balanced by taking $\varphi \in C^{1,1}$ (in addition to assuming $\Omega$ strongly pseudoconvex).

MSC: 32U05, 32U40, 53C55

1. Introduction

For a bounded pseudoconvex domain $\Omega \subset \subset \mathbb{C}^n$, the Dirichlet problem for the Monge-Ampère equation consists in

\[
\begin{aligned}
(dd^c u)^n &= f dV \quad \text{in } \Omega, \\
u &= \varphi \quad \text{on } b\Omega.
\end{aligned}
\]  

(1.1)

In our discussion we take a density $0 \leq f \in L^p(\Omega)$, $1 < p \leq +\infty$, a boundary datum $\varphi$ in some Hölder class and look for a plurisubharmonic solution $u \in C^\beta(\bar{\Omega})$ for a certain $\beta$. Sometimes we use the notation $MA(\varphi, f)$ for the problem (1.1) and $u(\Omega, \varphi, f)$ for its solution. This problem has been extensively investigated in recent years under the assumption that $\Omega$ is strongly pseudoconvex. Bremermann [3], Walsh [15] and Bedford-Taylor [1] show that there is a solution $u \in C^0(\bar{\Omega})$ if $\varphi \in C^0(b\Omega)$ and $f \in C^0(\bar{\Omega})$. By the well known “comparison principle” (cf. Kolodziej [12]), the solution is unique; what matters is to prove the Hölder continuity of this $C^0$-solution. In this direction, in [1] is proved that $u \in C^{\frac{2}{\alpha}}(\bar{\Omega})$ if $\varphi \in C^\alpha(b\Omega)$, $f^{\frac{\alpha}{n}} \in C^{\frac{2}{\alpha}}(\bar{\Omega})$. A recent interest has been dedicated to the case when $\Omega$ is no longer strongly pseudoconvex but has a certain “finite type” $m$. Li proves in [14] that $u \in C^{\frac{\gamma}{\alpha}}(\bar{\Omega})$ if $\varphi \in C^\alpha(b\Omega)$ and $f^{\frac{\gamma}{n}} \in C^{\frac{\gamma}{\alpha}}(\bar{\Omega})$. Ha and Khanh in [8] get the same conclusion with a more geometric notion of finite type (cf. (1.2) below) and have also a generalization for the infinite type. Coming back to the case of $\Omega$ strongly pseudoconvex, Caffarelli, Kohn and Nirenberg prove in [4] that $u \in C^\infty(\bar{\Omega})$, for $\varphi \in C^\infty(b\Omega)$ and $f \in C^\infty(\bar{\Omega})$, in case $f > 0$ in $\Omega$. Lowering the smoothness of $f$ gives the problem additional difficulty. Guedj, Kolodziej and Zeriahi prove in [7] that if $f \in L^p(\Omega)$ with $p > 1$ and $\varphi \in C^{1,1}(b\Omega)$ then $u \in C^\gamma(\bar{\Omega})$ for any $\gamma < \gamma_p := \frac{2}{q_n+1}$ where $\frac{1}{p} + \frac{1}{q_n} = 1$. Recently, Charabati has obtained in [6] that $u \in C^{\frac{2}{\alpha}}(\bar{\Omega})$ for the same datum as in [7] on a bounded strongly hyperconvex Lipschitz domain i.e. on a domain for which there exists a Lipschitz plurisubharmonic defining function $\rho$ such that $(dd^c \rho)^n \geq c dV$. Our purpose is twofold: to lower the regularity of $\varphi$ and to allow a (geometric) finite type $m$ for $\Omega$ with some $m \geq 2$. What we get is that if $f \in L^p(\Omega)$ with $p > 1$ and

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The research of T. V. Khanh was supported by the Australian Research Council DE160100173.
\( \varphi \in C^\alpha(b\Omega) \) with \( 0 < \alpha \leq 2 \) then \( u \in C^\frac{\alpha}{m}(\bar{\Omega}) \) if \( \alpha < \gamma_p \) otherwise \( u \in C^\frac{\alpha}{m}(\bar{\Omega}) \) for any \( \gamma < \gamma_p \). To go into the detail of our geometric setting we consider a submanifold \( S \subset b\Omega \) of CR dimension 0. Let \( d_S \) be the distance to \( S \) and \( (L_{b\Omega}) \) be the Levi form of \( b\Omega \). We assume that \( b\Omega \) has finite type \( m \) along \( S \) in the sense that

\[
L_{b\Omega} \gtrsim d_S^{m-2}.
\]

(1.2)

To convert \( (1.2) \) into a suitable property for our use, we need two basic results. First, from Khanh and Zampieri [10], we know that \( (1.2) \) implies the potential-theoretic “\( t^{\frac{1}{m}} \)-property”.

By [9] and [8] this implies in turn that there is an exhaustion function \( \rho \) which defines \( \Omega \) by \( \rho < 0 \) such that

\[
i\bar{\partial}\rho \geq Id \text{ in } \Omega, \quad \rho \in C^2(m)(\bar{\Omega}).
\]

(1.3)

Remark 1.1. According to Catlin [5], if \( \Omega \) has finite D’Angelo type \( D \), then it has the “\( t^{\frac{1}{m}} \)-property” for \( \frac{1}{m} := D^{-\frac{n^2}{D_n^2}} \); again, this implies the existence of the exhaustion \( \rho \in C^{2D^{-\frac{n^2}{D_n^2}}}(\Omega) \) with \( i\bar{\partial}\rho \geq Id \text{ in } \Omega \).

It is (1.3) the property which rules many passages of this paper. Here is our result

Theorem 1.2. Let \( \Omega \subset\subset \mathbb{C}^n \) be a \( C^2 \)-smooth pseudoconvex domain of finite type \( m \) with \( m \geq 2 \) in the sense of (1.2) and let \( \varphi \in C^\alpha(b\Omega) \) with \( 0 < \alpha \leq 2 \) and \( f \in L^p(\Omega) \) with \( p > 1 \). Then the unique solution \( u \) to \( MA(\Omega, \varphi, f) \) is in \( C^{\min(\frac{\alpha}{m}, \gamma)}(\bar{\Omega}) \) with \( \gamma < \gamma_p \) where \( \gamma_p := \frac{2}{qn+1} \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

The proof follows in Section 3. Throughout the paper we use \( \lesssim \) and \( \gtrsim \) to denote an estimate up to a positive constant and \( \sim \) for the combination of \( \lesssim \) and \( \gtrsim \). Finally, the indices \( m, \alpha, p, \gamma \) and \( \gamma_p \) only take ranges as in Theorem 1.2.

Acknowledgments. The paper is largely inspired by S. Kolodziej on the stream of research initiated by Bedford and Taylor in 1976. The authors are deeply indebted to S. Kolodziej for his important advice during private communications. The authors are also grateful to the referee for his helpful comments.

2. Hölder regularity of a subsolution

We say that \( v \in C^0(\bar{\Omega}) \) is a subsolution to \( MA(\Omega, \varphi, f) \) if \( v \) is plurisubharmonic, \( v|_{\partial\Omega} = \varphi \) and \( (dd^c v)^n \geq f \) in \( \Omega \).

Proposition 2.1. Let \( \rho \) satisfy (1.3). Then there is a subsolution \( v \in C^0(\bar{\Omega}) \) to \( MA(\Omega, \varphi, f) \) for \( \varphi \in C^0(b\Omega) \) and \( f \in L^p(\Omega) \).

Proof. For a large ball \( \mathbb{B} \) containing \( \Omega \), we define

\[
\tilde{f}(z) := \begin{cases} f(z) & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{B} \setminus \Omega. \end{cases}
\]

We consider the solutions

\[
\begin{cases}
u_1 = u(\mathbb{B}, 0, \tilde{f}) \in C^0(\mathbb{B}) & \text{by Kolodziej on the ball } \mathbb{B} \text{ (strongly pseudoconvex) [13],} \\
u_2 = u(\Omega, (-u_1)|_{\partial\Omega}, 0) \in C^0(\bar{\Omega}) & \text{by Blocki [2].}
\end{cases}
\]
Taking summation \( u_1 + u_2 \) we have a subsolution to \( MA(\Omega, 0, f) \) in \( C^0(\Omega) \). Using the solution \( u(\Omega, \varphi, 0) \in C^0(\Omega) \) provided by [2] and putting

\[
v = u_1 + u_2 + u(\Omega, \varphi, 0),
\]

we get the desired subsolution.

\[\square\]

We change a little our setting and take \( \varphi \in C^\alpha(b\Omega) \) and \( f \in L^\infty(\Omega) \). If \( \zeta \) is a general point of \( b\Omega \) we set

\[
v_\zeta(z) := \begin{cases} 
\varphi(\zeta) - c[-\rho(z) + |z - \zeta|^2]^\frac{\alpha}{2} & \text{if } 0 < \alpha \leq 1, \\
\varphi(\zeta) - \sum_j 2\text{Re} \frac{\partial \varphi}{\partial z_j}(\zeta)(z_j - \zeta_j) - c[-\rho(z) + |z - \zeta|^2]^\frac{\alpha}{2} & \text{if } 1 < \alpha \leq 2.
\end{cases}
\]

(2.1)

If there is an exhaustion function \( \rho \in C_\infty^\infty(\bar{\Omega}) \) such that \( i\partial\bar{\partial}\rho \geq \text{Id} \) in \( \Omega \) then we can find \( c \), independent of \( \zeta \) and only depending on \( \|\varphi\|_{C^\alpha(\bar{\Omega})} \) and \( \|f\|_{L^\infty(\Omega)} \) such that (cf. [S [14]])

\[
\begin{cases} 
v_\zeta(z) \leq \varphi(z) & \text{if } z \in b\Omega, \\
v_\zeta(\zeta) = \varphi(\zeta), \\
(dd^c v_\zeta)^n \geq f & \text{in } \Omega, \\
v_\zeta \in C_\infty^\infty(\Omega).
\end{cases}
\]

(2.2)

Using the family \( \{v_\zeta\}_{\zeta \in b\Omega} \) it is readily seen (cf. [S [14]]) that for any plurisubharmonic \( C^0(\bar{\Omega}) \) solution to MA we have \( u(\Omega, \varphi, f) \in C_\infty^\infty(\bar{\Omega}) \) for \( \varphi \in C^\alpha(b\Omega) \) and \( f^\frac{1}{2} \in C_\infty^\infty(\bar{\Omega}) \); in particular, \( u(\Omega, \varphi, 0) \in C_\infty^\infty(\bar{\Omega}) \) for \( \varphi \in C^\alpha(b\Omega) \). We lower the smoothness of \( f \). We start from

**Proposition 2.2.** Let \( \rho \) satisfy [1.3]. Then there is a subsolution \( v \in C_\infty^\infty(\bar{\Omega}) \) to \( MA(\Omega, \varphi, f) \) for \( \varphi \in C^\alpha(b\Omega) \) and \( f \in L^\infty(\Omega) \).

*Proof.* We consider the solution \( u(\Omega, \varphi, 0) \in C_\infty^\infty(\bar{\Omega}) \) by [14] and [8] and define

\[
v = u(\Omega, \varphi, 0) + cp.
\]

For \( c \geq \|f\|_{L^\infty(\Omega)}^{-\frac{1}{2}} \), \( v \) is a subsolution.

\[\square\]

We now take \( f \in L^p(\Omega) \).

**Proposition 2.3.** Let \( \rho \) satisfy [1.3]. Then there is a subsolution \( v \in C_{\min(\frac{\alpha}{\min}, \frac{\alpha}{n})}(\bar{\Omega}) \) to \( MA(\Omega, \varphi, f) \) for \( \varphi \in C^\alpha(b\Omega) \) and \( f \in L^p(\Omega) \).

*Proof.* We define \( B \) and \( \tilde{f} \) as in the proof of Proposition 2.1. Since \( \tilde{f} \) is bounded near the boundary, we consider the solutions

\[
\begin{align*}
u_1 &= u(\mathbb{B}, 0, \tilde{f}) \in C^\alpha(\mathbb{B}) \quad \text{by [7]}, \\
u_2 &= u(\Omega, (-u_1)|_{\partial \mathbb{B}}, 0) \in C_\infty^\infty(\bar{\Omega}) \quad \text{by [14] and [8].}
\end{align*}
\]

Taking the solution \( u(\Omega, \varphi, 0) \in C_\infty^\infty(\bar{\Omega}) \) (cf. [8 [14]]) and taking summation \( v = u_1 + u_2 + u(\Omega, \varphi, 0) \) we have the conclusion.

\[\square\]
3. Hölder regularity of the solution - Proof of Thorem 1.2

We recall a crucial fact from [13]. For a general domain, not necessarily strongly pseudoconvex, the existence of \( u(\Omega, \varphi, 0) \in C^0(\Omega) \) (which turns out to be equivalent to the existence of a maximal function with boundary datum \( \varphi \)), in addition to the existence of a subsolution \( v \in C^0(\Omega) \) for \( \varphi \in C^0(b\Omega) \) and \( f \in L^p(\Omega) \), implies the existence of a solution \( u(\Omega, \varphi, f) \in L^\infty(\Omega) \). In particular,

**Theorem 3.1.** (Kolodziej [13]) Assume \( \Omega \) is defined by \( \rho < 0 \) for \( \rho \in C^0(\bar{\Omega}) \) such that \( i\partial\bar{\partial}\rho \geq \text{Id} \) in \( \Omega \). Then for any \( \varphi \in C^0(b\Omega) \), \( f \in L^p(\Omega) \) there is a (unique) plurisubharmonic solution \( u(\Omega, \varphi, f) \in L^\infty(\Omega) \).

**Proof.** By the property of \( \rho \), which implies b-regularity, there is a solution for continuous data, in particular for \( f = 0 \), that is \( u(\Omega, \varphi, 0) \) (cf. [2]); thus there is a maximal function for the given boundary data. Again by the property of \( \rho \), there is a subsolution for \( \varphi \in C^0(b\Omega) \), \( f \in L^p(\Omega) \) (Proposition 2.1 above). Then by [13] Thm. C p. 97 (3 lines after the statement) there is a solution in \( L^\infty(\Omega) \).

\[ \square \]

**Remark 3.2.** The solution \( u(\Omega, \varphi, f) \) for \( \varphi \in C^0(b\Omega) \), \( f \in L^p(\Omega) \) is in fact in \( C^0(\Omega) \) by Kolodziej [11]. Note that the paper makes the general assumption of pseudoconvexity of \( \Omega \) but this is needless for this specific conclusion. This is confirmed by private communication with the author.

We assume from now \( i\partial\bar{\partial}\rho \geq \text{Id} \) in \( \Omega \) for \( \rho \in C^2_{\bar{\Omega}}(\Omega) \). According to Proposition 2.3 above, when we take a smoother boundary datum \( \varphi \in C^0(b\Omega) \), there is a subsolution \( v \in C^{\min(\frac{n}{n}, \frac{n}{m})}(\Omega) \) for \( f \in L^p(\Omega) \). What follows is dedicated to show that, in this situation, the \( L^\infty \) plurisubharmonic solution \( u(\Omega, \varphi, f) \) is in fact in \( C^{\min(\frac{n}{n}, \frac{n}{m})}(\Omega) \).

Let \( w := u(\Omega, \varphi, 0) \in C^{\frac{n}{m}}(\bar{\Omega}) \) (cf. [8] [14]); comparison principle yields at once

\[ v \leq u(\varphi, f) \leq w. \quad (3.1) \]

By (3.1) and by the \( C^{\min(\frac{n}{n}, \frac{n}{m})} \) regularity of \( v \) and \( w \) we get

\[ |u(z) - u(\zeta)| \lesssim |z - \zeta|^\min(\frac{n}{n}, \frac{n}{m}), \quad z, \zeta \in b\Omega, \]

and therefore for \( \delta \) suitably small

\[ |u(z) - u(z')| \lesssim \delta^{\min(\frac{n}{n}, \frac{n}{m})}, \quad z, z' \in \Omega \setminus \Omega_\delta \text{ and } |z - z'| < \delta \quad (3.2) \]

where \( \Omega_\delta := \{ z \in \mathbb{C}^n : r(z) < -\delta \} \) and \( r \) is a \( C^2 \) defining function for \( \Omega \) with \( |\nabla r| = 1 \) in a neighborhood of \( b\Omega \). We have to prove that (3.2) also holds for \( z, z' \in \Omega_\delta \). We use the notation

\[ \begin{align*}
    u_\frac{\delta}{2} := & \sup_{|\zeta| < \frac{\delta}{2}} u(z + \zeta), & z \in \bar{\Omega}_\delta, \\
    \bar{u}_\frac{\delta}{2} := & \frac{1}{\sigma_{2n-1}(\frac{\delta}{2})^{2n-1}} \int_{b\mathbb{B}(z, \frac{\delta}{2})} u(\zeta)dS(\zeta), & z \in \bar{\Omega}_\delta,
\end{align*} \quad (3.3) \]

where \( \sigma_{2n-1}(\frac{\delta}{2})^{2n-1} = Vol(b\mathbb{B}(z, \frac{\delta}{2})) \). It is a classical consequence of Riesz Theorem that for a general plurisubharmonic function \( u \), not necessarily \( C^2 \), there is well defined \( \Delta u \) in the space of positive Borel measures. We use the notation \( \|\Delta u\|_\Omega \) for the total mass of \( \Delta u \) on \( \Omega \).
Theorem 3.3. Let $0 < \epsilon < 1$. We have

$$
\| \tilde{u}_\delta ^\frac{1}{2} - u \|_{L^1(\Omega_\delta)} \lesssim \delta^{1-\epsilon} \| (-r)^{1+\epsilon} \Delta u \|^{\frac{1}{2}}. 
$$

(3.4)

Proof. The proof is inspired by [7] Lemma 4.3; the novelty here consists in replacing $\delta^2$ by $\delta^{1-\epsilon}(-r)^{1+\epsilon}$. We start from

$$
\tilde{u}_\delta ^\frac{1}{2}(z) - u(z) \sim \frac{1}{\delta^{2n-1}} \int_{bB(0,\frac{1}{\delta})} (u(z + \xi) - u(z))dS(\xi)
$$

$$
\sim \frac{1}{\delta^{2n-2}} \int_{bB(0,\frac{1}{\delta})} dS(\xi) \int_0^1 \nabla u(z + s\xi) \cdot \frac{\xi}{\delta} ds
$$

$$
= \text{divergence thm.} \frac{1}{\delta^{2n-2}} \int_0^1 sds \int_{B(0,\frac{1}{\delta})} \Delta u(z + s\xi)
$$

$$
\sim \frac{1}{\delta^{2n-2}} \int_0^\frac{1}{\delta} t \frac{t^{-2n}}{\delta^{2n-2}} dt \int_{B(0,t)} \Delta u(z + \zeta).
$$

(3.5)

We denote by $\tau_\zeta$ the translation by $\zeta$ and observe that $\tau_\zeta \Omega_\delta \subset \Omega_{\delta^2} \subset \subset \Omega$ for any $\zeta \in B(0,t)$. Observing that the positive measure $\Delta u$ has finite mass on compact subsets of $\Omega$, in particular on $\Omega_{\delta^2}$, we get, for $t < \frac{1}{\delta}$

$$
\int_{\Omega_{\delta^2}} dV(z) \int_{B(0,t)} \Delta u(z + \zeta) \lesssim t^{2n} \int_{\Omega_{\delta^2}} \Delta u(z).
$$

(3.6)

We now perform integration $\int_{\Omega_{\delta^2}} dV(z)$ in both sides of (3.5), apply (3.6) and end up with

$$
\int_{\Omega_{\delta^2}} (\tilde{u}_\delta ^\frac{1}{2} - u)(z) dV(z) \lesssim \int_0^{\frac{1}{\delta}} t^{-2n+1}t^{2n} dt \int_{\Omega_{\delta^2}} \Delta u
$$

$$
\lesssim \int_0^{\frac{1}{\delta}} t\delta^{-(1+\epsilon)} dt \int_{\Omega_{\delta^2}} (-r)^{1+\epsilon} \Delta u
$$

$$
\sim \delta^{1-\epsilon} \| (-r)^{1+\epsilon} \Delta u \|^{\frac{1}{2}}.
$$

(3.7)

At this point, the problem is to prove the boundedness of $\| (-r)^{1+\epsilon} \Delta u \|^{\frac{1}{2}}$ uniformly in $\delta$. This holds (cf. Theorem 3.4 below) because of the presence of the factor $(-r)^{1+\epsilon}$. In absence of this factor, one should suppose from the beginning that $\Delta u$ has finite total mass on $\Omega$; in turn, this would be a consequence of the hypothesis $\varphi \in C^{1,1}$ (cf. [7]).

Theorem 3.4. We have

$$
\| (-r)^{1+\epsilon} \Delta u \|^{\frac{1}{2}} \lesssim \| (-r)^{-1+\epsilon} u \|_{L^1(\Omega)}.
$$

(3.8)

Proof. We take a system of smooth cut-off functions $\chi_\nu(|z|) \in C_\infty^\infty(B^{2n}(0,\frac{1}{\nu}))$, $\| \chi_\nu \|_{L^1} \equiv 1$, $\frac{1}{\nu} \to 0$, and regularize

$$
u := \int_{\Omega} u(\tau) \chi_\nu(|z - \tau|) dV(\tau).
$$
The $u_\nu$’s belong to $C^\infty(\Omega)$, converge to $u$ on $\Omega$, and satisfy
\[
\begin{align*}
\sup_{\Omega_1} |\nabla u_\nu| &= \sup_{\Omega_1} |\nabla (u * \chi_\nu)| \leq \nu\|u\|_{L^1(\Omega)} \\
\sup_{\Omega_1} u_\nu &\leq c \text{ independent of } \nu.
\end{align*}
\] (3.9)

Now that the $u_\nu$’s are regular, the $\Delta u_\nu$’s are well defined functions and hence we use the notation $\Delta u_\nu dV$ for the associated measures. We have
\[
\int_{\Omega_1} (-r)^{1+\epsilon} \Delta u_\nu dV(z) = \int_{\Omega_1} \text{div}((-r)^{1+\epsilon} \nabla u_\nu) dV(z) + (1 + \epsilon) \int_{\Omega_1} (-r)^{\epsilon} \nabla r \cdot \nabla u_\nu dV(z)
\]
\[
= \text{Stokes} \int_{\partial \Omega_1} (-r)^{1+\epsilon} \nabla r \cdot \nabla u_\nu ds^{2n-1}(z) + (1 + \epsilon) \int_{\Omega_1} (-r)^{\epsilon} \nabla r \cdot \nabla u_\nu dV(z)
\]
\[
= \int_{\partial \Omega_1} (-r)^{1+\epsilon} \nabla r \cdot \nabla u_\nu ds^{2n-1}(z) + (1 + \epsilon) \int_{\Omega_1} \text{div}((-r)^{\epsilon} (\nabla r u_\nu)) dV(z)
\]
\[
+ \epsilon(1 + \epsilon) \int_{\Omega_1} (-r)^{\epsilon-1} \nabla r \cdot \nabla ru_\nu dV(z) - (1 + \epsilon) \int_{\Omega_1} (-r)^{\epsilon} \Delta ru_\nu dV(z)
\]
\[
\leq O(\nu^{-\epsilon}) + (1 + \epsilon)O(\nu^{-\epsilon}) + \int_{\Omega_1} (-r)^{\epsilon-1} |u_\nu| dV(z) + \int_{\Omega_1} (-r)^{\epsilon} |u_\nu| dV(z)
\]
\[
\leq O(\nu^{-\epsilon}) + \|(-r)^{-1+\epsilon} u\|_{L^1(\Omega)}.
\] (3.10)

On the other hand, since $u$ is plurisubharmonic, then $\Delta u$ is a measure on $\Omega$ and $\Delta u_\nu dV \rightharpoonup \Delta u$. The conclusion follows from the following elementary Lemma

**Lemma 3.5.** Assume $\Delta u_\nu \geq 0$ and

\[
\begin{align*}
\int_{\Omega_1} (-r)^{1+\epsilon} \Delta u_\nu dV &\text{ are bounded} \\
\Delta u_\nu dV &\rightharpoonup \Delta u.
\end{align*}
\]

Then

\[
\int_{\Omega} (-r)^{1+\epsilon} \Delta u \text{ is bounded.}
\]

The proof is just a consequence of the dominated convergence theorem for the sequence $(-r)^{1+\epsilon} \psi_\nu \Delta u_\nu dV \rightharpoonup (-r)^{1+\epsilon} \Delta u$ where $\psi_\nu$ are the characteristic functions of the sets $\Omega_1$. With Lemma 3.5 in our hands, we get the conclusion of the proof of Theorem 3.4.
To end the proof of Theorem 1.2 we shall need the stability estimate (Theorem (1.1) in [7])

**Theorem 3.6.** Fix $0 \leq f \in L^p(\Omega), p > 1$. Let $U, W$ be two bounded plurisubharmonic functions in $\Omega$ such that $(dd^cU)^n = fdV$ in $\Omega$ and let $U \geq W$ on $\partial \Omega$. Fix $s \geq 1$ and $0 \leq \eta < \frac{1}{nq+s}, \frac{1}{p} + \frac{1}{q} = 1$. Then there exists a uniform constant $C = C(\eta, \|f\|_{L^p(\Omega)}) > 0$ such that

$$\sup_{\Omega}(W - U) \leq C\|\eta\|_{L^s(\Omega)}^n,$$

where $(W - U)_+ := \max(W - U, 0)$.

*End of Proof of Theorem 1.2.* Again, we follow the guidelines of [7]. Along with $\tilde{u}_\delta$ defined by (3.3) we introduce $\hat{u}_\delta := \frac{1}{\sigma_{2n(\delta)^m}} \int_{B(z,\delta)} u(\zeta)dV(\zeta), \quad z \in \Omega_\delta$. We recall that Lemma 4.2 of [7] states the equivalence between

$$\sup_{\Omega_\delta}(u_\delta^2 - u) \lesssim \delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)} \quad (3.11)$$

and

$$\sup_{\Omega_\delta}(\hat{u}_\delta^2 - u) \lesssim \delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)} \quad (3.12)$$

On the other hand, on account of the obvious inequalities

$$\hat{u}_\delta \leq \tilde{u}_\delta \leq u_\delta,$$

we see that whatever of (3.11) and (3.12) is equivalent to

$$\sup_{\Omega_\delta}(\hat{u}_\delta^2 - u) \lesssim \delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)} \quad (3.13)$$

We have thus to prove (3.13). To see it, we remark that

$$\|\tilde{u}_\delta^2 - u\|_{L^1(\Omega_\delta)} \lesssim \frac{\delta^{1-\epsilon}\|(-r)^{1+\epsilon} \Delta u\|_{\Omega_\delta^2}}{\delta^{1-\epsilon} \quad (3.14)}$$

By (3.2), we have for a suitable $c$

$$\tilde{u}_\delta^2 \leq u_\delta^2 \leq u + c\delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)} \quad \text{in a neighborhood of } b\Omega_\delta.$$

We are going to apply Theorem 3.6 for $\Omega_\delta$ with $U := u + c\delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)}, W := \tilde{u}_\delta$ and $s := 1$; thus we get

$$\sup_{\Omega_\delta}\left(\tilde{u}_\delta^2 - \left(u + c\delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)}\right)\right) \lesssim \|\tilde{u}_\delta^2 - \left(u + c\delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)}\right)\|_{L^1(\Omega_\delta)} \lesssim \|\tilde{u}_\delta^2 - u\|_{L^1(\Omega_\delta)} \lesssim \delta^{1-\epsilon}\|\eta\|_{L^s(\Omega_\delta)} \lesssim \delta^{1-\epsilon}\|\eta\|_{L^s(\Omega_\delta)} \quad (3.15)$$

for any $\eta < \frac{1}{2}\gamma_p = \frac{1}{np+1}$. It follows

$$\sup_{\Omega_\delta}\left(\tilde{u}_\delta^2 - u\right) \lesssim \delta^{1-\epsilon}\eta + \delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)} \lesssim \delta^{\frac{1}{2}} + \delta^{\min\left(\frac{\alpha_m}{m}, \frac{\gamma_m}{m}\right)} \quad (3.16)$$
and hence (3.13) is proved since \( m \geq 2 \). Here the last inequality follows by choosing \( \epsilon = \frac{\gamma - \gamma}{\gamma + \gamma} > 0 \) and \( \eta = \frac{1}{4}(\gamma_\rho + \gamma) < \frac{1}{2}\gamma_\rho \) since \( \gamma < \gamma_\rho \).

From (3.2) and (3.11) (which is equivalent to (3.13)), it is easy to prove that
\[
|u(z) - u(z')| \lesssim |z - z'|^{\min(\frac{\alpha m}{\gamma}, \frac{\gamma m}{\alpha})} \text{ for any } z, z' \in \Omega;
\]
thus the proof of Theorem 1.2 is complete. \( \square \)

References

[1] E. Bedford and B. A. Taylor, The Dirichlet Problem for a complex Monge-Ampère Equation, Inventiones Math. 37 (1976), 1–44.

[2] Z. Blocki, The complex Monge-Ampère operator in hyperconvex domains, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 23 (1996), 721–747.

[3] H. J. Bremermann, On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains, Characterization of Šilov boundaries, Trans. Amer. Math. Soc. 91 (1959), 246–276.

[4] L. Caffarelli, J. J. Kohn, L. Nirenberg, and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations II. Complex Monge-Ampère equations and uniformly elliptic equations, Comm. Pure Appl. Math. 38 (1985), 209–252.

[5] D. Catlin, Subelliptic estimates for the \( \overline{\partial} \)-Neumann problem on pseudoconvex domains, Ann. of Math. 126 (1987), 131–191.

[6] M. Charabati, Hölder regularity for solutions to complex Monge-Ampère equations, Ann. Polon. Math. 113 (2015), 109–127.

[7] V. Guedj, S. Kolodziej and A. Zeriahi, Hölder continuous solutions to Monge-Ampère equations, Bull. Lond. Math. Soc. 40(6) (2008), 1070–1080.

[8] L. K. Ha and T. V. Khanh, Boundary regularity of the solution to the Complex Monge-Ampère equation on pseudoconvex domains of infinite type, Math. Res. Lett. 22, (2015), 467–484.

[9] T. V. Khanh, Lower bounds on the Kobayashi metric near a point of infinite type, J. Geom. Anal. 26 (2016), 616–629.

[10] T. V. Khanh and G. Zampieri, Regularity of the \( \overline{\partial} \)-Neumann problem at point of infinite type, J. Funct. Anal. 259(11) (2010), 2760–2775.

[11] S. Kolodziej, Equicontinuity of families of plurisubharmonic functions with bounds on their Monge-Ampère masses, Math. Z. 240(4) (2002), 835–847.

[12] S. Kolodziej, The complex Monge-Ampère equation and pluripotential theory, Mem. Amer. Math. Soc. 178 (2005), no. 840, x+64 pp.

[13] S. Kolodziej, The complex Monge-Ampère equation, Acta Math. 180(1) (1998), 69–117.

[14] S. Y. Li, On the existence and regularity of Dirichlet problem for complex Monge-Ampère equations on weakly pseudoconvex domains, Calc. Var. 20 (2004), 119-132.

[15] J. B. Walsh, Continuity of envelopes of plurisubharmonic functions, J. Math. Mech. 18 (1968), 143–148.

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