A Connection on Manifolds with a Nilpotent Structure

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Abstract

All the connections, pure toward the nilpotent structure, are found. Examples of manifolds, for which the curvature tensor is pure or hybridous, are given. For a manifold of B-type a necessary and sufficient condition for purity of the curvature tensor is proved. It is verified that the conformal change of the metric of a B-manifold does not retain its purity.

Keywords: semi-Riemannian geometry, curvature tensor, geodesics, Einstein field equations, Schwarzschild solution.

We suppose that $B$ is a submanifold of $E$, where dim $B = n$, and dim $E = m + n$. We assume that

$$\delta : E \to B$$

is a submersion in $E$. If point $p$ belongs to the base $B$ then the set of points

$$\delta^{(-1)}(p) \subset E$$

are a layer over it. A priori we suppose a local triviality of $\delta$. By $TB$ we denote the tangential differentiation on the base $B$. If we designate by

$$(z^i, z^{n+i}, z^{2n+a}), \ i = 1, 2, \ldots, n; \ a = 1, 2, \ldots, m,$$

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then \( z^i \) are coordinates of a point from \( B \), \( z^{n+i} \) are coordinates of a point in a layer from \( TB(\pi : TB \to B) \), where \( \pi^{-1}(p) \) is a tangential layer over a point \( p \in B \), and \( z^{2n+a} \) are coordinates of a point from \( \sigma^{-1}(p) \). We can interpret the variables

\[
(z^i, z^{n+i}, z^{2n+a})
\]

as local coordinates of a point from a \( 2n + m \)-dimensional manifold \( M \) provided that these coordinates are replaced by the following rule [2]:

\[
z^i = \varphi^i(z^1, z^2, \ldots, z^n)
\]

\[
z^{n+i} = \sum_k \frac{\partial \varphi^i(z^1, z^2, \ldots, z^n)}{\partial z^k} z^{n+k}
\]

\[
z^{2n+a} = \Theta^a(z^1, z^2, \ldots, z^n; z^{n+1}, z^{n+2}, \ldots, z^{2n+m})
\]

(1)

It is proved in [2] that \( M \) tolerates an integratable nilpotent structure \( f \), called semitangential by the author of this paper, which commutates with the Jacobian of (1). With regard to a suitable local basis on \( M \), the matrix of \( f \) has the following form

\[
(f^\alpha_\beta) = \begin{pmatrix} E & \text{blank blocks} \end{pmatrix},
\]

\( E \) is a single matrix from row \( n \), and the blank blocks are zero blocks.

We assume that \( \nabla = g^\sigma_{\alpha\beta} \) is a torsionless connection on \( M \) with connection coefficients \( g^\sigma_{\alpha\beta} \). As regards \( \nabla \), we impose a purity condition of the coefficients toward \( f \) as well:

\[
g^\lambda_{\beta\sigma} f^\alpha_\beta = g^\alpha_{\alpha\lambda} f^\lambda_\beta = g^\alpha_{\lambda\sigma} f^\lambda_\beta, \quad \alpha, \beta, \sigma, \ldots = 1, 2, \ldots, m + 2n
\]

A solution to the above condition for purity is

\[
g^k_{is} = g^{n+k}_{i,n+s} = g^{n+k}_{n+i,s}, \quad g^{2n+a}_{\beta\sigma} = 0 \quad \text{and} \quad g^{2n+a}_{\beta\lambda} = 0
\]

(2)

In the special case \( m = 0 \), i.e. when \( M \) is the tangential differentiation of \( B \), the coefficients of the second series of solution (2) are

\[
g^{2n+a}_{\beta\sigma} = 0
\]

In this case, if \( \nabla = (g^h_{ik}) \) is a torsionless connection on the base, then we denote \( u^i = z^i \), \( y^i = z^{n+i} \) so therefore

\[
g^h_{ik} = g^h_{ik}; \quad g^{n+k}_{i,n+k} = g^{n+k}_{n+i,k} = g^{n+p}_{n+i,n+k} = 0
\]
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\[ g^{n+n}_{ik} = z^s \frac{\partial g^h_{ik}}{\partial u^s}, \]
\[ g^{n+h}_{i,n+k} = g^{n+h}_{n+i,k} = g^h_{ik}. \]

Thus, \( g^\alpha_{\beta\sigma} \) coincide with the coefficients of the complete lift of the connection \( \bar{\nabla} \).

**Lemma 1.** If a connection has pure coefficients with respect to one upper and one lower index, and \( f \) is covariantly constant, then \( f \) is an integratable structure.

**Proof.** From the condition

\[ \nabla_\sigma f^\alpha_\beta = \partial_\sigma f^\alpha_\beta - g^\alpha_\lambda f^\lambda_\beta + g^\nu_\sigma f^\alpha_\nu = 0 \]

follows that

\[ \nabla_\sigma f^\alpha_\beta = \partial_\sigma f^\alpha_\beta = 0, \]

which shows that \( f^\alpha_\beta = \text{const.} \)

**Corollary 1.** If \( f \) is an integratable structure and \( \nabla f = 0 \), then we have purity of the coefficients \( g^\alpha_{\beta\sigma} \) with respect to one upper and one lower index.

**Lemma 2.** If \( f \) is integratable and \( R^\sigma_{\alpha\beta\gamma} \) are the corresponding components of the curvature tensor for \( \nabla \), then the condition \( \nabla f = 0 \) is sufficient for \( R^\sigma_{\alpha\beta\gamma} \) to be pure with respect to one upper and one lower index.

**Proof.** From the Ricci identity

\[ \nabla_\alpha \nabla_\beta f^\gamma_\sigma - \nabla_\beta \nabla_\alpha f^\gamma_\sigma = R^\sigma_{\alpha\beta\gamma} f^\gamma_\sigma - R^\sigma_{\alpha\beta\gamma} f^\gamma_\lambda \]

the proof is obvious.

The purity of the Riemannian curvature tensor with respect to an upper and a lower index does not automatically mean purity with respect to two lower indices. An example of this are Riemannian manifolds, the functional tensor of which conforms in a suitable way with the semitangential structure \( f \).

**Example 1.** The Riemannian manifold \( \mathfrak{M}(g, f) \) with a metric tensor \( g = (g_{\alpha\beta}) \) and an assigned integratable nilpotent tensor field \( f \) of type (1,1) is called a manifold of \( B \)-type, if for every two vector fields \( x \) and \( y \) there follows:

\[ g(f x, y) = g(x, f y). \]
We note down that the last condition is equivalent to the purity of $g_{\alpha\beta}$, where $	ilde{g} = (g_{\alpha\lambda}f^\lambda_\beta)$ is a symmetrical tensor field, and under certain conditions $R^\sigma_{\alpha\beta\gamma}$ is pure with respect to all indices.

Detailed information about this type of manifolds and research work, connected with them, can be found in the studies of E. V. Pavlov.

**Example 2.** Riemannian manifolds $\mathcal{M}(g, f)$ with a metric tensor $g = g_{\alpha\beta}$ and an assigned integratable nilpotent tensor field $f$ of type (1,1) is called a manifold of Kähler type if for every two vector fields $x$ and $y$ there follows:

$$g(x, fy) = -g(fx, y)$$

The last equation is called a condition for the hybridity of $g$ toward $f$. In this example the tensor field $	ilde{g} = (g_{\alpha\lambda}f^\lambda_\beta = \tilde{g}_{\beta\alpha})$ is antisymmetric, $R^\sigma_{\alpha\beta\gamma}$ is not pure with respect to all indices.

Particular examples of this type of Kähler manifolds can be found in ([3], p. 137).

**Assertion 1.** If $\mathcal{M}(g, f)$ is a manifold from type $B$, then the components $R^\tau_{\sigma\alpha\beta}$ and $R_{\sigma\alpha\beta\gamma}$ are pure with respect to the two indices $(\beta, \gamma)$. If $\mathcal{M}(g, f)$ is of Kähler type, then $R_{\sigma\beta\alpha\gamma}$ and $R^\tau_{\sigma\beta\alpha}$ are hybrid with respect to $(\alpha, \gamma)$ and pure with respect to $(\alpha, \tau)$ respectively.

**Proof.** The condition in both cases follows from Lemma 2. We have

$$R^\lambda_{\alpha\beta\gamma} \cdot f^\sigma_\alpha = R^\sigma_{\alpha\beta\lambda} \cdot f^\lambda_\gamma$$

Hence follows that

$$R^\lambda_{\alpha\beta\gamma} F_{\lambda\sigma} = R^\sigma_{\alpha\beta\lambda} f^\lambda_\gamma,$$

where $F_{\lambda\sigma} = g_{\sigma\tau} f^\tau_\lambda$, and taking the antisymmetrization of $F_{\lambda\sigma}$ into consideration, we obtain

$$-R^\lambda_{\alpha\beta\gamma} f^\lambda_\sigma = R^\lambda_{\alpha\beta\gamma} f^\lambda_\gamma.$$

When $\mathcal{M}(g, f)$ is of type $B$, the proof is analogical. 

In connection with the $B$-type manifolds, two important theorems need to be mentioned, which are proved in [4] and [5] respectively.

**Theorem 1.** If the components $R_{\sigma\alpha\beta\lambda}$ of the curvature tensor of a $B$-type manifold are pure with respect to two indices, then they are pure with respect to all double indices.
Theorem 2. The components $R_{\sigma\alpha\beta\lambda}$ of the curvature tensor of a $B$-type manifold are pure then and only then when the partial derivatives of the metric tensor are pure.

Corollary from Assertion 1. Tensor $G$, defined by means of the equation

$$G(x, y, v, w) = g(x, v)g(y, w) - g(x, w)g(y, v)$$

is curvature-related for $\mathcal{M}(g, f)$. At that the tensor $\hat{G}$, for which

$$\hat{G}(x, y, v, w) = G(x, fy, v, fw) = -g(x, fw)g(fy, v)$$

is pure or hybrid depending on $\mathcal{M}(g, f)$. In case of

$$x \neq \ker f, \ G(x, f x, y, f y) = -[g(x, fy)]^2,$$

from where follows:

a) If $\mathcal{M}(g, f)$ is of type $B$, then $g(x, fx) \neq 0$. In this case, on the basis of Assertion 1 and Theorem 1, it is possible for $\mathcal{M}(g, f)$ to be flat on account of the fact that the holomorphic curvature in the direction of $\{x, fx\}$ is zero.

b) If $\mathcal{M}(g, f)$ is of Kähler type. Now $g(x, fx) = 0$. Therefore the holomorphic curvature in the direction of $\{x, fx\}$ is indefinite.

Theorem 3. If $\mathcal{M}(g, f)$ is a $B$-type manifold, then the Riemannian connection, originating from $g$, possesses pure connection coefficients with respect to all indices then and only then, when the partial derivatives

$$\partial_{\sigma}g_{\alpha\beta} = \frac{\partial}{\partial z^\sigma}g_{\alpha\beta}$$

are pure towards $f$.

Proof. We need to note down in advance that the symmetrical tensor field

$$\tilde{g} = (\tilde{g}_{\alpha\beta} = g_{\lambda\beta}f^\lambda_{\alpha})$$

is transferred parallely towards the Riemannian connection, originating from $g$. Because of $f_{\beta}^\lambda = const$ we have

$$\nabla_{\sigma}\tilde{g}_{\alpha\beta} = \partial_{\sigma}\tilde{g}_{\alpha\beta} - \tilde{g}_{\alpha\lambda}^\lambda\tilde{g}_{\lambda\beta} - \tilde{g}_{\alpha\beta}\tilde{g}_{\alpha\tau} = 0$$

Here we used the corollary from Lemma 1.
a) Let $\partial_\sigma g_{\alpha\beta}$ be pure toward $f$. In this case from the condition

$$2g^\sigma_{\alpha\beta} = g^\sigma_{\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta})$$

(3)

and the purity of $g^{\sigma\lambda}$, resulting from the equations

$$g^{\lambda\sigma}g_{\lambda\gamma} = \delta^\sigma_\gamma$$

$$\Rightarrow g^{\lambda\sigma}f^\nu_\sigma = f^\nu_\gamma \Rightarrow$$

$$\Rightarrow g^{\lambda\sigma}\delta^\beta_\lambda f^\nu_\sigma = f^\nu_\gamma g^{\sigma\beta} \Rightarrow$$

$$\Rightarrow g^{\beta\sigma}f^\nu_\sigma = g^{\sigma\beta}f^\nu_\gamma$$

(the purity of $g^{\alpha\beta}$),

there follows the purity of the connection coefficients with respect to all indices.

b) We assume that $g^{\sigma}_{\alpha\beta}$ are pure with respect to all indices. In this case from $\nabla_\sigma g_{\alpha\beta} = 0$ follow the conditions

$$\partial_\sigma \tilde{g}_{\alpha\beta} - g^\lambda_{\sigma\alpha} \tilde{g}_{\lambda\beta} - g^\nu_{\sigma\beta} \tilde{g}_{\alpha\nu} = 0$$

and

$$f^\tau_\sigma \partial_\tau g_{\alpha\beta} - g^\lambda_{\sigma\alpha} \tilde{g}_{\lambda\beta} - g^\nu_{\sigma\beta} \tilde{g}_{\alpha\nu} = 0 .$$

By means of their term-by-term subtraction we obtain

$$f^\tau_\sigma (\partial_\tau g_{\alpha\beta}) = \partial_\sigma (f^\lambda_\alpha g_{\lambda\beta}) .$$

Definition 1. If for $\mathcal{M}(g, f)$ the partial derivatives of the components of $g$ are pure, we say that $\mathcal{M}(g, f)$ is a $B$-manifold.

Theorem 4. If $\mathcal{M}(g, f)$ is a $B$-manifold and $\det(g + \tilde{g}) \neq 0$, then the metrics $g$ and $g + \tilde{g}$ originate one and same Riemannian connection.

Proof. Let us take into consideration that the matrices $(g_{\alpha\beta} + \tilde{g}_{\alpha\beta})$ and $(g^{\alpha\beta} - \tilde{g}^{\alpha\beta})$ are mutually inverse. Here we have denoted $\tilde{g}^{\alpha\beta} = g^{\sigma\lambda}f^\beta_\lambda$ and used the purity of $g_{\alpha\beta}$, which was proved in the previous theorem. If $\tilde{g}_{\alpha\beta}$ are the coefficients of the Riemannian connection, originating from $g + \tilde{g}$, we apply formula (3) but in reference to the metric $g + \tilde{g}$. In the course of the calculations we should keep in mind that the objects $\partial_\sigma g_{\alpha\beta}$ are pure. Thus there follows that $\tilde{g}^{\sigma}_{\alpha\beta} = g^{\sigma}_{\alpha\beta} . \qed
The theorem proved holds true in the most common case. In the special case where \( m = 0 \), i.e. when \( \mathcal{M}(g, f) \) is a tangential differentiation on base \( B \), a similar theorem is proved for the lifts of the base metric in \([7]\), p. 149. There \( g + \tilde{g} \) is designated as a metric \( I + II \). On the other hand, the change \( g \rightarrow g + \tilde{g} \) is the simplest CH-change. On condition that \( f^2 = I \) (\( I \) is the identical transformation), the CH-change is studied in detail in \([4]\). To the question whether it is the conformal change that needs to be investigated, or the generalization and the CH-change for \( \mathcal{M}(g, f) \) with \( f^2 = 0 \), we obtain an answer by means of

**Theorem 5.** If \( h(z^1, \ldots, z^{m+2n}) \) is a random manifold, and \( \mathcal{M}(g, f) \) is a \( B \)-manifold, then \( \mathcal{M}(hg, f) \) is a \( B \)-manifold then and only then, when \( h = \text{const} \).

**Proof.** If \( \mathcal{M}(hg, f) \) is a \( B \)-manifold, there follows that

\[
\partial_\lambda(hg_{\alpha\beta})f^\lambda_\sigma = \partial_\sigma(hg_{\lambda\beta}f^\lambda_\alpha) .
\]

Hence we obtain

\[
f^\lambda_\sigma h_{\lambda\delta}f^\delta_\alpha = h_\sigma f^\beta_\alpha , \quad (h_\lambda = \frac{\partial h}{\partial z^\lambda})
\]

This equation holds true for all values of the indices. For \( \beta = n + i \), \( \alpha = k \), \( f^k_{n+i} = \delta_k^i \) in particular, from the specified equation we have

\[
f^\lambda_\sigma h_{\lambda\delta}^i = h_\sigma \delta_k^i
\]

or \( h_\sigma = 0 \) for every \( \sigma \).

In the opposite case, where \( h = \text{const} \), it is obvious that \( \mathcal{M}(hg, f) \) is a \( B \)-manifold.

**Geodesic and holomorphic plane curves.** If we assume that on the manifold \( \mathcal{M}(f) \), which is provided with a pure connection with respect to \( f \), the curve \( c : z^\alpha = z^\alpha(+) \) is geodesic, after changing the parameter \( t : t = h(q) \), the equation of \( c \)

\[
\frac{\delta z^\alpha}{dt} = \frac{d^2 z^\alpha}{dt^2} + \gamma^\alpha_{\lambda\beta} \frac{dz^\lambda}{dt} \frac{dz^\beta}{dt} = 0
\]

\( (\dot{z}^\alpha = \frac{dz^\alpha}{dt} \) is the covariant differentiation toward the connection, which has the \( \gamma^\alpha_{\lambda\beta} \) coefficients) is equivalent to

\[
\frac{\delta}{dq} \left( \frac{dz^\alpha}{dq} \right) = \frac{dh}{dq} \frac{dz^\alpha}{dq} .
\]

It is possible to transfer the field \( v = (v^\alpha) \) parallely with respect to \( c \)
\[ \frac{\delta v^\alpha}{dq} = \frac{dv^\alpha}{dq} + \gamma^\alpha_{\beta \lambda} \frac{dv^\beta}{dt} = 0 \]

After changing \( v^\alpha \rightarrow \lambda(t)v^\alpha \) we have

\[ \frac{\delta}{dt}(\lambda(t)v^\lambda) = \lambda'(t)v^\alpha \]

In both of the examples given we will say that the directions \( \frac{dz^\alpha}{dt} \) and \( \lambda v^\alpha \) are transferred parallelly with respect to \( c \).

We say that the curve \( z^\alpha(t) \) from \( \mathcal{M}(f) \) is a holomorphic plane curve (a PH-curve) for \( \mathcal{M}(f) \), if \( z^\alpha(t) \) are the solutions to the differential equation

\[ \frac{\delta z^\alpha}{dt} = \frac{d^2 z^\alpha}{dt^2} + \gamma^\alpha_{\beta \lambda} \frac{dz^\lambda}{dt} \frac{dz^\beta}{dt} = a(t)\frac{dz^\lambda}{dt} + b(t)(f^\lambda_{\nu} \frac{dz^\nu}{dt}) \]$

Here \( a(t) \) and \( b(t) \) are functions. We will adopt the designation

\[ \frac{\widetilde{dz^\alpha}}{dt} = f^\lambda_{\nu} \frac{dz^\nu}{dt} \].

There exists a special case, in which the geodesic curves and the PH-curves coincide. That is when \( \mathcal{M}(f) \) is projective-Euclidean or PH-Euclidean. This peculiarity is illustrated best about the three-dimensional projective space \( B_3 \) (an extension of the corresponding Euclidean one). In \( B_3 \) the above curves coincide with the absolute straight line \( \omega \), which has a common point with each straight line from the absolutely congruent straight lines [8]. The absolute straight line \( \omega \) is the set of all points belonging to \( \ker f \).

**Assertion 2.** If \( z^\alpha(t) \) is geodesic for \( \mathcal{M}(f) \) and \( \dot{z}^\alpha \in \ker f \), then \( z^\alpha(t) \) is a PH-curve.

**Proof.** Let us recall that \( f \) is covariantly constant and the objects \( \gamma^\alpha_{\beta \lambda} \) are pure with regard to \( f \). In this case, from the differential equation of the geodesic curve there follows

\[ \frac{\delta}{dt} \dot{z}^\alpha = 0 \]

Let us consider the linear combination \( v^\alpha = h(t)\dot{z}^\alpha + l(t)\ddot{z}^\alpha \) for certain functions \( h(t) \) and \( l(t) \). For the vector field \( v = (v^\alpha) \) we have

\[ \frac{\delta v^\alpha}{dt} = [h(t)]' \dot{z}^\alpha + [l(t)]' \ddot{z}^\alpha \].
which shows that a random vector from the holomorphic vector space \( \{ \dot{z}^\alpha, \tilde{z}^\alpha \} \), at a parallel transfer with respect to \( z^\alpha(t) \), remains in \( \{ \dot{z}^\alpha, \tilde{z}^\alpha \} \), i.e. \( z^\alpha(t) \) is a PH-curve.

**Assertion 3.** If \( z^\alpha(t) \) is geodesic for \( \mathcal{M}(f) \) and \( \dot{z}^\alpha \in \ker f \), then \( z^\alpha(t) \) is a PH-curve and it belongs to the special plane (straight line) \( \omega \) in the biaxial space \( B_n(B_3) \).

**Proof.** The assertion is obvious because
\[
\frac{\delta}{dt} \dot{z}^\alpha = 0.
\]

**Assertion 4.** If \( z^\alpha(t) \) is a PH-curve and \( \dot{z}^\alpha \in \ker f \), then \( z^\alpha(t) \) is geodesic.

**Proof.** From the differential equation of PH-curves follows that
\[
\frac{\delta \dot{z}^\alpha}{dt} = a(t) \dot{z}^\alpha.
\]
The tangential direction is transferred parallely with respect to \( z^\alpha(t) \), therefore \( z^\alpha(t) \) is geodesic.

We note down that in this case all curves
\[
z^\alpha(t) = (c_1, \ldots, c_n, z^{n+1}(t), \ldots, z^{2n}(t), z^{2n+1}(t), \ldots, z^{2n+m}(t))
\]
possess one and the same tangential vector field. The special case, where \( n = 1, m = 0 \) in the two-dimensional affine plane \( 0xy \), is interesting. There \( z^\alpha(t) = (c, z(t)) \). The corresponding geodesic curve is a straight line, which is parallel to the axis \( 0y \).

Let us consider a particular case, where \( \mathcal{M}(f) \) is a tangential differentiation of the \( B \) manifold.

**Definition 2.** We say that the pair of functions \( p(u, v), q(u, v) \) is holomorphic if
\[
\frac{\partial p}{\partial u} = \frac{\partial q}{\partial v}
\]
and
\[
\frac{\partial p}{\partial v} = 0.
\]
On the basis of this definition of the two-dimensional surface

\[ S^H \subset \mathfrak{M}(f) : \{ z^i(u) ; z^{n+i}(u,v) = v \frac{dz^i}{du} \} , \]

we will say that it is holomorphic in \( \mathfrak{M}(f) \).

**Assertion 5.** As regards a holomorphic two-dimensional surface \( S^H \), with the change of the parameters \( \overline{u} = h(u), \overline{v} = t(u,v) \) in such a way that for the differentiable functions \( h(u) \) and \( t(u,v) \) there holds

\[ \overline{u} = h(u) \]

and

\[ \frac{dh}{du} = \frac{\partial t}{\partial v} , \]

the holomorphicity of \( S^H \) is preserved.

**Proof.** We substitute the values in the parametric equation of

\[ S^H : z^i = z^i(\overline{u}), z^{n+i} = \overline{v} \frac{dz^i}{d\overline{u}}(\overline{u},\overline{v}) \]

with their equals. In this case \( z^i \) depend only on the variables \( u \), and at that

\[ \frac{dz^i}{du} = \frac{dz^i}{d\overline{u}} \frac{dh}{du} ; \]

\[ \frac{\partial z^{n+i}}{\partial v} = \frac{\partial z^{n+1}}{\partial \overline{v}} \frac{dt}{dv} = \frac{\partial}{\partial \overline{v}}(\overline{v} \frac{dz^i}{du}) \frac{dt}{dv} = \frac{dz^i}{du} \frac{dh}{du} = \frac{dz^i}{du} , \quad \text{i.e.} \]

\[ \frac{\partial z^{n+i}}{\partial v} = \frac{dz^i}{du} . \]

**Theorem 6.** Every geodesic curve \( \beta(u) = (u = u, v = \text{const}) \) (\( u \)-line) from the two-dimensional holomorphic surface \( S^H \subset \mathfrak{M}(f) \) is a PH-curve for \( \mathfrak{M}(f) \).
Proof. Let $\gamma(v) = (u = \text{const}, v = v)$ be the $v$-lines for $S^H$, and

$$\dot{\beta} = (\dot{z}^i; v\ddot{z}^i; 0, \ldots, 0)$$

and

$$\dot{\gamma} = (0, \ldots; \dot{z}^i; 0, \ldots, 0)$$

are the corresponding velocities of $\beta(u)$ and $\gamma(u)$. Obviously

$$\dot{\gamma} = f \dot{\beta} .$$

Besides,

$$\frac{\delta \dot{\beta}_\alpha}{du} = \frac{d \dot{\beta}_\alpha}{du} + g^\lambda_{\alpha\nu} \dot{\beta}^\lambda \dot{\beta}^\nu = 0 .$$

Taking into consideration the purity of the objects $g^\alpha_{\lambda\nu}$ from the last two equations, we obtain

$$\frac{\delta \dot{\gamma}_\alpha}{du} = \frac{d \dot{\beta}_\alpha}{du} (f^\beta_\lambda \dot{\beta}^\lambda) + g^\alpha_{\nu\sigma} f^\nu_\lambda \dot{\beta}^\rho \dot{\beta}^\sigma =$$

$$= \frac{d \dot{\beta}_\alpha}{du} (f^\beta_\lambda \dot{\beta}^\lambda) + g^\lambda_{\rho\sigma} f^\nu_\lambda \dot{\beta}^\rho \dot{\beta}^\sigma =$$

$$= \left[ \frac{d \dot{\beta}_\lambda}{du} + g^\lambda_{\rho\sigma} \dot{\beta}^\rho \dot{\beta}^\sigma \right] f^\alpha_\lambda = 0 .$$

Therefore $\dot{\gamma}$ is transferred parallelly with respect to the $u$-lines $\beta$, i.e. $\beta$ is a PH-curve for $M(f)$. □

We note down that the theorem proved above is a confirmation of Assertion 2. Here the object of consideration is a holomorphic two-dimensional surface $S^H$ from $M(f)$, and we can always consider that on it there is set a geodesic vector field, which does not belong to $\ker f$. As it is recorded in [9], $S^H$ is interpreted as a real model of a geodesic curve on the manifold $\hat{M}$ over the algebra

$$\mathbb{R}(\varepsilon) = \{ a + \varepsilon b; \varepsilon^2 = 0; a, b \in \mathbb{R} \} .$$

There the surfaces $S^H$ are called analytical.

If $q = q_i$ is a form on $M(f)$, there can be found a lot of connections, which are pure with respect to $f$. The tensor of the affine deformation $T$, through which these connections are obtained, is

$$T^n_{ik} = \delta^n_i \tilde{q}_k + f^n_i q_k + \delta^n_k \tilde{q}_i + f^n_k q_i .$$
In the above equation $\tilde{q}_i := q_s f_i^s$.

Obviously $T$ is pure with respect to all indices. Therefore any connection
\[ \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + T^\alpha_{\beta\gamma} \]
is pure too.

**Theorem 7.** The connection $\nabla$ with coefficients $\Gamma^s_{\beta\sigma}$ possesses the following properties:

1. it has symmetrical and pure coefficients,
2. $\nabla f = 0$,
3. $\nabla$ and $\nabla$ have common PH-curves.

**Proof.**

1. The integrated tensor of $T$ is
\[ \tilde{T}^n_{is} = T^n_{in} f_s^i = f^n_i \tilde{q}_i + f^n_s \tilde{q}_i . \]
It hence follows that the connection with coefficients $\Gamma^\alpha_{\beta\gamma}$ is also pure.

2. Let us assume that $\nabla f = 0$. Then
\[ \nabla f^n_i = \tilde{\Gamma}^n_{sj} f^i_j - \tilde{\Gamma}^p_{sj} f^n_i = \Gamma^n_{sj} f^i_j + T^n_{sj} f^i_j - \Gamma^p_{sj} f^n_i - T^p_{sj} f^n_i = \nabla f + \tilde{T}^n_{si} - \tilde{T}^n_{si} = 0 . \]

3. We denote with $\tilde{\delta}$ the covariant differential as regards $\nabla$. Since
\[ \frac{\delta}{dt} \dot{x}^n = a(t) \dot{x}^n + b(t) \tilde{\dot{x}}^n , \]
there follows that
\[ \frac{\delta}{dt} \dot{x}^n = \dot{x} + [\Gamma^n_{ik} + \delta^n_i \tilde{p}_k + \delta^n_k \tilde{p}_i + f^n_i q_k + f^n_k q_i] \dot{x}^i \dot{x}^n = \]
\[ = \frac{\delta}{dt} \dot{x}^n + \dot{x}^n \tilde{p}(\dot{x}) + \dot{x}^n \tilde{p}(\dot{x}) + \tilde{x}^n q(\dot{x}) + \tilde{x}^n q(\dot{x}) . \]
Here we have designated $p_k := \tilde{q}_k$. Finally we obtain
\[ \frac{\delta}{dt} \dot{x}^n = (a + 2\tilde{p}(\dot{x})) \dot{x}^n + (b + 2q(\dot{x})) \tilde{x}^n . \]
References

[1] E. V. Pavlov, A. H. Hristov, Metric Manifolds With a Semi-Tangential Structure. (in Russian)

[2] V. V. Vishnevskiy. On the Geometric Model of a Semi-Tangential Structure. Higher School Bulletin, Mathematics, issue 3, 1983, pp. 73-75. (in Russian)

[3] V. V. Vishnevskiy, A. P. Shirokov, V. V. Shurigin. Spaces Over Algebras. Kazan University Press, 1988. (in Russian)

[4] Pavlov E. V., Manifolds With an Algebraic Structure and CH-Mapping. Research work qualifying for an academic degree. Plovdiv, 2003. (in Bulgarian)

[5] Vishnevskiy V. V. Certain Properties of Differential Geometric Structures That are Determined by Algebras., Scientific works of Plovdiv University, v. 10, b. 1, 1972 - mathematics. (in Russian)

[6] K. Yano. Differential Geometry on Complex and Almost Complex Spaces. The Macmillan Company, New York, 1965.

[7] Yano Kenturo, Shigera Ishihara, Tangent and Cotangent Bundles - New York, 1973.

[8] N.V. Talantova, A Biaxial Space of a Parabolic Type, Higher School Bulletin, Mathematics, 1959. (in Russian)

[9] V. S. Talapin, A-Planar Transformation of a Connection in Real Realization of Manifolds Over Algebras, Higher School Bulletin, Mathematics, 1980. (in Russian)

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