On higher order corrections to photon structure functions

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Abstract

The QCD corrections to photon structure functions are defined in a way consistent with the factorization scheme invariance. It is shown that the conventional DIS, factorization scheme does not respect this invariance and is thus deeply flawed. The origins of the divergent behavior of photonic coefficient function at large $x$ are analyzed and recipe to remove it is suggested.

1 Introduction

The recently completed evaluation of order $\alpha_s^3$ parton-parton [1, 2] and order $\alpha\alpha_s^2$ photon-parton splitting functions [3] has confirmed the earlier results [4] based on the evaluation of first six even moments. For both the quark distribution functions of the photon and the photon structure function, the results (for the latter shown in Fig. 1) exhibit very small difference between the next-to-leading (NLO) and next-to-next-to-leading order (NNLO) approximations defined in the standard way and evaluated in $\overline{\text{MS}}$ factorization scheme (FS).

This welcome feature stands in sharp contrast to the large difference, noted already in [5], between the standard LO and NLO approximations in the same $\overline{\text{MS}}$ FS. The standard formulation of the NLO approximation to photon structure function $F_2^\gamma(x, Q^2)$ suffers also from problems in the large $x$ region where it turns negative. In order to cure these two, related, problems, the so called DIS, FS has been proposed in [6].

The aim of this note is to show that, first, there is no such sharp difference between the LO and NLO approximations to $F_2^\gamma$ in the $\overline{\text{MS}}$ FS if these approximations are defined in a way consistent with the factorisation procedure. Second, I will argue that the way the DIS, FS is introduced violates the basic requirement of the FS invariance and, consequently, the related definition of parton distribution functions of the photon is deeply flawed. To see where the problem comes from, I will contrast the definition of DIS, FS with the analogous, but theoretically well-defined concept of DIS FS in the case of the proton structure function.

The main source of the difference between the standard treatment of the proton and photon structure functions can be traced back to the interpretation of the behaviour of parton distribution functions of the photon in perturbative QCD. In [7] I have discussed this point at length, but its conclusions have been largely ignored. I will therefore return to this point and present additional arguments showing that parton distribution functions of the photon behave like $\alpha$, rather than $\alpha/\alpha_s$ as claimed in [3–6] and most, though not all, other papers on photon structure functions. I will outline how the theoretically consistent LO, NLO and NNLO QCD approximations to $F_2^\gamma$ should be constructed from the results of [3,4].

Finally, I will discuss the origins of the problems in the large $x$ region and suggest how to remove them taking into account their pure QED nature.
2 Basic facts and notation

Let us start by briefly recalling the basic facts concerning the various ingredients of perturbative calculations involving (quasi)real photons in the initial state. In QCD the coupling of quarks and gluons is characterized by the renormalized colour coupling ("couplant" for short) \( \alpha_s(\mu) \), depending on the renormalization scale \( \mu \) and satisfying the equation

\[
\frac{d \alpha_s(\mu)}{d \ln \mu^2} \equiv \beta(\alpha_s(\mu)) = -\frac{\beta_0}{4\pi} \alpha_s^2(\mu) - \frac{\beta_1}{16\pi^2} \alpha_s^3(\mu) + \cdots,
\]

where, in QCD with \( n_f \) massless quark flavours, the first two coefficients, \( \beta_0 = 11 - 2n_f/3 \) and \( \beta_1 = 102 - 38n_f/3 \), are unique, while all the higher order ones are ambiguous. However, even for a given r.h.s. of (1) there is an infinite number of solutions, differing by the initial condition. This so called renormalization scheme (RS) ambiguity \(^1\) can be parameterized in a number of ways, for instance by the value of the renormalization scale, usually denoted \( \Lambda \), for which \( \alpha_s(\mu = \Lambda_{\text{RS}}) = \infty \). At the NLO the variation of both the renormalization scale \( \mu \) and the renormalization scheme \( \text{RS} \equiv \{\Lambda_{\text{RS}}\} \) is legitimate but redundant. It suffices to fix one of them and vary the other. In this paper we shall work in the standard \( \overline{\text{MS}} \) RS.

As nothing in my arguments depends in essential way on the numerical value of \( \beta_1 \), I will set in the following \( \beta_1 = 0 \). This assumption, simplifies many otherwise complicated formulae and lays bare the essential aspects of the problem. Note, however, that this assumption does not amount to working in the leading order of QCD, as all the relevant terms of higher order QCD approximations and their relations are kept.

Factorization scale dependence of PDF of the photon is determined by the system of inhomogeneous evolution equations

\[
\frac{d \Sigma(x, M)}{d \ln M^2} = k_\Sigma + P_{q\bar{q}} \otimes \Sigma + 2n_f P_{qG} \otimes G,
\]

\[
\frac{d G(x, M)}{d \ln M^2} = k_G + P_{Gq} \otimes \Sigma + P_{GG} \otimes G,
\]

\[
\frac{d q_{\text{NS}}(x, M)}{d \ln M^2} = k_{\text{NS}} + P_{\text{NS}} \otimes q_{\text{NS}},
\]

\(^1\)In higher orders this ambiguity includes also the arbitrariness of the coefficients \( \beta_i, i \geq 2 \).
The splitting functions choice of the factorization scheme distance, nonperturbative effects that go into the PDF and induce their dependence on the renormalization scale and described by the coefficient function \( s \), and those of the large ambiguities of the treatment of perturbatively calculable short distance physics, embodied in the coefficient functions as in [8–10] and the present paper. This alternative definition of \( \alpha \) way. For instance, in [11] the renormalization scale denotes the argument of the couplant \( C \) and the standard formula for \( C \):

\[
\Sigma(x, M) = \sum_{i=1}^{n_f} q_i^+(x, M) \equiv \sum_{i=1}^{n_f} [q_i(x, M) + \bar{q}_i(x, M)] , \tag{5}
\]

\[
q_{NS}(x, M) = \sum_{i=1}^{n_f} (e_i^2 - \langle e_i^2 \rangle) (q_i(x, M) + \bar{q}_i(x, M)) , \tag{6}
\]

\[
k_{NS} = \delta_{NS} k_q ; \quad \delta_{NS} = 6n_f (\langle e_i^4 \rangle - \langle e_i^2 \rangle^2) , \quad k_\Sigma = \delta_\Sigma k_q ; \quad \delta_\Sigma = 6n_f \langle e_i^2 \rangle . \tag{7}
\]

The splitting functions \( P_{ij} \) and \( k_i \) are given as power expansions in \( \alpha_s(M) \):

\[
k_i(x, M) = \frac{\alpha}{2\pi} \left[ k_i^{(0)}(x) + \frac{\alpha_s(M)}{2\pi} k_i^{(1)}(x) + \left( \frac{\alpha_s(M)}{2\pi} \right)^2 k_i^{(2)}(x) + \cdots \right] , \tag{8}
\]

\[
P_{ij}(x, M) = \frac{\alpha_s(M)}{2\pi} P_{ij}^{(0)}(x) + \left( \frac{\alpha_s(M)}{2\pi} \right)^2 P_{ij}^{(1)}(x) + \cdots , \tag{9}
\]

where \( i = q, G, NS \) and \( ij = qq, qG, Gq, GG, NS \). The leading order splitting functions \( k_q^{(0)}(x) \) and \( P_{ij}^{(0)}(x) \) are unique, while all higher order ones \( k_q^{(j)}, k_G^{(j)}, P_{kl}^{(j)}, j \geq 1 \) depend on the choice of the factorization scheme (FS). According to the factorization theorem, the photon structure function \( F_2^\gamma(x, Q^2) \) is given as the sum of the convolutions

\[
\frac{1}{x} F_2^\gamma(x, Q^2) = q_{NS}(M) \otimes C_q(Q/M) + \delta_{NS} C_\gamma + \langle e_i^2 \rangle \Sigma(M) \otimes C_q(Q/M) + \langle e_i^2 \rangle \delta_\Sigma C_\gamma + \langle e_i^2 \rangle G(M) \otimes C_G(Q/M) \tag{10}
\]

of PDF and coefficient functions \( C_q, C_G, C_\gamma \) admitting perturbative expansions

\[
C_q(x, Q/M) = \delta(1 - x) + \frac{\alpha_s(\mu)}{2\pi} C_q^{(1)}(x, Q/M) + \cdots , \tag{12}
\]

\[
C_G(x, Q/M) = \frac{\alpha_s(\mu)}{2\pi} C_G^{(1)}(x, Q/M) + \cdots , \tag{13}
\]

\[
C_\gamma(x, Q/M) = \frac{\alpha}{2\pi} \left[ C_\gamma^{(0)}(x, Q/M) + \frac{\alpha_s(\mu)}{2\pi} C_\gamma^{(1)}(x, Q/M) + \cdots \right] , \tag{14}
\]

where the standard formula for \( C_\gamma^{(0)} \) reads

\[
C_\gamma^{(0)}(x, Q/M) = (x^2 + (1 - x)^2) \ln \frac{Q^2(1 - x)}{M^2 x} + 8x(1 - x) - 1 . \tag{15}
\]

I am using the terms “renormalization” and “factorization” scales in standard way as discussed, for instance, in [8–10]. However, some authors use these concepts in a very different way. For instance, in [11] the renormalization scale denotes the argument of the couplant \( \alpha \) in the expansion (9) of the splitting functions, rather than in the expansions (12) of the coefficient functions as in [8–10] and the present paper. This alternative definition of the renormalization scale, though mathematically legitimate, lacks physical motivation as it fails to make the crucial difference, emphasized long time ago by Politzer [12], between the ambiguities of the treatment of perturbatively calculable short distance physics, embodied in the renormalization scale and described by the coefficient functions, and those of the large distance, nonperturbative effects that go into the PDF and induce their dependence on \( M \).
The renormalization scale $\mu$, used as argument of $\alpha_s(\mu)$ in (12–14), is in principle independent of the factorization scale $M$. Note that despite the presence of $\alpha_s(\mu)$ in (12–14), the coefficient functions $C_q, C_G$ and $C_\gamma$ are, if calculated to all orders in $\alpha_s$, independent of $\mu$ as well as of the RS. On the other hand, PDF and the coefficient functions $C_q, C_G$ and $C_\gamma$ do depend on both the factorization scale $M$ and factorization scheme, but in such a correlated manner that physical quantities, like $F_\gamma^2$, are independent of both $M$ and the FS, provided expansions (8–9) and (12–14) are taken to all orders. In practical calculations based on truncated forms of (8–9) and (12–14) this invariance is, however, lost and the choice of both $M$ and FS makes numerical difference even for physical quantities.

3 Parton distribution functions of the photon

The general solution of the evolution equations (2-4) can be written as the sum of a particular solution of the full inhomogeneous equation and the general solution of the corresponding homogeneous one, called hadronic. In rest of this note I will for technical reasons restrict the discussion to the nonsinglet quark distribution function of the photon and the corresponding nonsinglet part (10) of photon structure function. To simplify the formulae I will also drop the subscript “NS” and set $\delta_{NS} = 1$ everywhere.

As is well known, the subset of the solutions of the evolution equation (4) with the splitting functions including the first terms $k^{(0)}$ and $P^{(0)}$ only and vanishing at some scale $M_0$ results from the resummation of the contributions of the diagrams in Fig. (2). These, so called pointlike solutions, which start with the purely QED vertex $\gamma \rightarrow q\bar{q}$, define the standard “leading order” approximation and have, in momentum space, the form (again I will drop the specification “pointlike” throughout the rest of this paper)

$$q(n, M_0, M) = \frac{4\pi}{\alpha_s(M)} a(n) \left[ 1 - \left( \frac{\alpha_s(M)}{\alpha_s(M_0)} \right)^{1-2P^{(0)}(n)/\beta_0} \right],$$

(16)

where

$$a(n) \equiv \frac{\alpha}{2\pi \beta_0} \frac{k^{(0)}(n)}{1 - 2P^{(0)}(n)/\beta_0}. \quad (17)$$

As argued in [7] the fact that $\alpha_s(M)$ appears in the denominator of (16) does in no way mean that $q \propto \alpha/\alpha_s$. It is obvious that provided $M$ and $M_0$ are kept fixed when QCD is switched off by sending $\Lambda \rightarrow 0$ the expression (16) approaches

$$q(x, M, M_0) \rightarrow \frac{\alpha}{2\pi} k^{(0)}(x) \ln \frac{M^2}{M_0^2}, \quad (18)$$

corresponding to purely QED splitting $\gamma \rightarrow q\bar{q}$. Note that this limit holds even if we include in (4) the term proportional to $k^{(1)}$, which gives the lowest order QCD contribution to the inhomogeneous splitting function $k_q$. 

Figure 2: Diagrams defining the pointlike part of non-singlet quark distribution function of the photon in the leading logarithmic approximation.
To see what is wrong with the conventional way of arriving at the claim that \( q \propto \alpha/\alpha_s \), let us recast the evolution equation (4) into the equivalent form

\[
\frac{dq(n, Q)}{d\alpha_s} = -\frac{4\pi}{\beta_0} \left[ \frac{\alpha}{2\pi} \frac{k^{(0)}(n)}{\alpha_s^2} + \frac{\alpha}{2\pi} \frac{P^{(0)}(n)}{\alpha_s} q(n, Q) + \cdots \right] \tag{19}
\]

and take into account just the first term on its the right. Trivial integration then yields

\[
q(n, Q) = \frac{\alpha}{2\pi} \frac{4\pi}{\beta_0} \frac{k^{(0)}(n)}{\alpha_s(Q)} + A, \tag{20}
\]

where \( A \) denotes arbitrary integration constant specifying the boundary condition on the solution of (19). Choosing \( A = 0 \) might, but should not, mislead us to the usual claim that \( q \propto 1/\alpha_s \), because (19) is equivalent to (4), which in our approximation of keeping just the first term on its r.h.s. contains no trace of QCD, being of purely QED nature! Taking the difference of (20) for \( Q_1 \) and \( Q_2 \) and inserting the explicit expression for \( \alpha_s(M) \) we arrive at

\[
q(n, Q_1) - q(n, Q_2) = \frac{\alpha}{2\pi} \frac{4\pi}{\beta_0} \frac{k^{(0)}(n)}{\alpha_s(Q_1)} \left( \frac{1}{\alpha_s(Q_1^2)} - \frac{1}{\alpha_s(Q_2^2)} \right) = \frac{\alpha}{2\pi} \frac{k^{(0)}(n)}{\alpha_s(Q)} \frac{\ln Q_1^2}{Q_2^2}, \tag{21}
\]

the purely QED expression we can get directly from (4). We can further recast (21) into the form similar to (16)

\[
q(n, Q_1) - q(n, Q_2) = \frac{4\pi}{\alpha_s(Q_1)} \frac{\alpha}{2\pi} \frac{k^{(0)}(n)}{\alpha_s(Q)} d \left[ 1 - \left( \frac{\alpha_s(Q_1^2)}{\alpha_s(Q_2^2)} \right)^d \right] \tag{22}
\]

with \( d = 1 \). Including also the lowest order QCD term proportional to \( P^{(0)}(n) \) modifies the result slightly, but non-essentially by replacing \( d = 1 \) with \( d = 1 - 2P^{(0)}(n)/\beta_0 \). Note that since \( P^{(0)}_{qq}(0) = 0 \) the integral of quark distribution function is actually unchanged by QCD effects, but the standard claim would still be that it “behaves” as \( 1/\alpha_s \).

By the same reasoning we could “prove” that, for instance, also the vacuum polarization \( \Pi(Q^2) \) “behaves like \( 1/\alpha_s \)”! Indeed, taking the derivative of \( \Pi(Q^2) \) with respect to \( \ln Q \)

\[
D(Q^2) \equiv -\frac{d\Pi(Q^2)}{d\ln Q^2} = -\frac{d\Pi(Q^2)}{d\alpha_s(Q)} \frac{d\alpha_s(Q)}{d\ln Q^2} \tag{23}
\]

which defines the Adler function, and rewriting it in the way analogous to (19) we get

\[
\frac{d\Pi(Q)}{d\alpha_s(Q)} = -\frac{D(Q^2)}{d\alpha_s(Q)/d\ln Q^2} = \frac{4\pi}{\alpha_s(Q)} \frac{1 + \sum_{k=1}^{\infty} d_k \alpha_s^k(Q)}{\beta_0 \alpha_s^2 + \cdots}. \tag{24}
\]

Trivial integration then yields the advertised conclusion

\[
\Pi(Q^2) = -\frac{4\pi}{\beta_0 \alpha_s(Q)} + \cdots. \tag{25}
\]

Let me finally add another, quite different, argument demonstrating that PDF of the photon behave as \( O(\alpha) \). Consider the Mellin moments of the nonsinglet part of \( F^\gamma(x) \equiv F_{NS}^\gamma(x, Q)/x \)

\[
F^\gamma(n, Q) = q(n, M)C_q(n, Q/M) + C_\gamma(n, Q/M) \tag{26}
\]
for the general pointlike solution of (11), which like (16) vanishes at $M = M_0$. As $F_\gamma(n, Q)$ is independent of the factorization scale $M$, we can take any $M$ to evaluate it, for instance just $M_0$. However, for $M = M_0$ the first term in (26) vanishes and we get

$$F_\gamma(Q) = \frac{\alpha}{2\pi} \left[ C^{(0)}_\gamma(Q/M_0) + \frac{\alpha_s(\mu)}{2\pi} C^{(1)}_\gamma(Q/M_0) + \left( \frac{\alpha_s(\mu)}{2\pi} \right)^2 C^{(2)}_\gamma(Q/M_0, Q/\mu) + \cdots \right]$$

(27)

i.e. manifestly the expansion in powers of $\alpha_s(\mu)$ which starts with $O(\alpha)$ pure QED contribution ($\alpha/2\pi)C^{(0)}_\gamma$ and includes standard QCD corrections of orders $\alpha_s^k/k \geq 1$. Clearly this expansions vanishes when QCD is switched off and there is no trace of the supposed “$\alpha/\alpha_s$” behaviour. I will come back to this expression in Section 4.

4 Defining LO and NLO approximations for $F_\gamma(x, Q^2)$

Although semantics is a matter of convention, I think it is wise to define the terms “leading”, “next–to–leading” and higher orders in a way which guarantees that they have the same meaning in different processes. Recall that for the case of the familiar ratio

$$R_{e^+e^-}(Q) \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \left( 3 \sum_{i=1}^{n_f} e_i^2 \right) (1 + r(Q))$$

(28)

the prefactor $3 \sum_{i=1}^{n_f} e_i^2$, which comes from pure QED, is usually subtracted and only the QCD correction $r(Q)$ is considered for further analysis. For the quantity (28) the terms “leading” and “next–to–leading” thus apply only to genuine QCD effects as described by $r(Q)$, which starts as $\alpha_s/\pi$. Unfortunately, this practice is ignored in most analyses of $F_\gamma$, which count the QED term $k^{(0)}$ as the “leading order” [3–6].

In [7] I have proposed the definition of QCD approximations of $F_\gamma$ which follows closely the convention used in QCD analysis of quantities like (28). It starts with writing the (pointlike nonsinglet) quark distribution function $q(M)$ as the sum of the purely QED contribution

$$q_{\text{QED}}(M) \equiv \frac{\alpha}{2\pi} k^{(0)} \ln \frac{M^2}{M_0^2}$$

(29)

and the QCD correction satisfying the inhomogeneous evolution equation

$$\frac{dq_{\text{QCD}}(M)}{d\ln M^2} = \frac{\alpha_s}{2\pi} \left[ \frac{\alpha}{2\pi} k^{(1)} + P^{(0)} q_{\text{QED}}(M) \right] + \left( \frac{\alpha_s}{2\pi} \right)^2 \left[ \frac{\alpha}{2\pi} k^{(2)} + P^{(1)} q_{\text{QED}}(M) \right] + \cdots$$

$$+ \frac{\alpha_s}{2\pi} P^{(0)} q_{\text{QCD}}(M) + \left( \frac{\alpha_s}{2\pi} \right)^2 P^{(1)} q_{\text{QCD}}(M) + \cdots$$

(30)

The latter differs from that satisfied by the full quark distribution function not only by the absence of the term $(\alpha/2\pi)k^{(0)}$ but also by shifted appearance of higher order coefficients $k^{(i)}; i \geq 1$. For instance, the inhomogeneous splitting function $k^{(1)}$ enters (30) at the same order as homogeneous splitting function $P^{(0)}$ and thus these splitting functions will appear at the same order also in its solutions. Similarly, the simultaneous presence of $k^{(2)}$ and $P^{(1)}$ in the $O(\alpha_s^2)$ term of the inhomogeneous part of (30) implies that the NLO QCD analysis

\[ ^2\text{In the rest of this paper the dependence on the Mellin moment variable } n \text{ will be suppressed.} \]
of $F^\gamma$ requires the knowledge of $k^{(2)}$ etc. In Table 1 the terms included in the standard definition of the LO and NLO approximations are compared with those corresponding to my definition of these approximations. The difference between the two definitions is substantial.

|            | standard definition | my definition          |
|------------|---------------------|------------------------|
| QED        | does not introduce  | $k^{(0)}$, $C^{(0)}_\gamma$ |
| LO QCD     | $k^{(0)}$, $P^{(0)}$| $k^{(0)}$, $C^{(0)}_\gamma$, $k^{(1)}$, $C^{(1)}_\gamma$, $P^{(0)}$ |
| NLO QCD    | $k^{(0)}$, $P^{(0)}$, $k^{(1)}$, $C^{(0)}_\gamma$, $P^{(1)}$| $k^{(0)}$, $C^{(0)}_\gamma$, $k^{(1)}$, $C^{(1)}_\gamma$, $P^{(0)}$, $k^{(2)}$, $C^{(2)}_\gamma$, $P^{(1)}$ |

Table 1: Contributions included in the standard and correct definition of LO and NLO.

5 Factorization schemes and their choice - the proton

Let us first recall the definition of factorization schemes in the case of the (nonsinglet) proton structure function. Denoting $F^p \equiv F^p_{\text{NS}}/x$ we can write its moments as the product

$$ F^p(Q) = q(M) C_q(Q/M) $$

(31)

of the coefficient functions $C_q(Q/M)$, given in (12), and the quark distribution function $q(M)$ of the proton. If both the homogeneous splitting function (9) and coefficient function (12) are calculated to all orders in $\alpha_s$, the product on the r.h.s. of (31) is independent of both the factorization scale $M$ and the factorization scheme. It must thus hold

$$ \frac{dF(Q)}{d \ln M^2} = q(M) \left[ P(M) C_q(Q/M) + \dot{C}_q(Q/M) \right] = \frac{dF(Q)}{dC_q^{(j)}} = 0 $$

(32)

If, however, these expansions are truncated, the resulting finite order approximations for $F(Q)$ will depend on both the factorization scale and scheme. For a finite order approximation to be theoretically consistent, its dependence on all free parameters must be formally of higher order in $\alpha_s$ than those included in the approximation. From (32) we get, denoting $\dot{f}(M) \equiv df(M)/d \ln M^2$,

$$ P(M)C_q(Q/M) + \dot{C}_q(Q/M) = 0 $$

(33)

and expanding the splitting and coefficient functions to order $\alpha_s$ we arrive at the relation

$$ \dot{C}_q^{(1)}(Q/M) = -P^{(0)} \Rightarrow C_q^{(1)}(Q/M) = P^{(0)} \ln(Q^2/M^2) + C_q^{(1)}(1). $$

(34)

Taking into account the first two terms in (9) the solution of (4) reads

$$ q(M) = A(\alpha_s(M)) \frac{-2P^{(0)}}{\beta_0} \exp \left( \frac{-2P^{(1)} \alpha_s(M)}{\beta_0} \frac{\alpha_s(M)}{2\pi} \right), $$

(35)

where the $(n$-dependent) constant $A$ specifies the boundary condition $^3$. Inserting (35) into (31) yields

$$ F^p(Q) = A(\alpha_s(M)) \frac{-2P^{(0)}}{\beta_0} \exp \left( \frac{-2P^{(1)} \alpha_s(M)}{\beta_0} \frac{\alpha_s(M)}{2\pi} \right) \left[ 1 + \frac{\alpha_s(\mu)}{2\pi} C_q^{(1)}(Q/M) \right]. $$

(36)

\(^3\text{We can also use the above expression for } M = M_0 \text{ and by forming their ratio obtain instead of the usual relation between } q(M) \text{ and } q(M_0).
and expanding the exponential we get

\[ F^p(Q) = A(\alpha_s(M)) \frac{-2P^{(0)}}{\beta_0} \left[ 1 + \frac{\alpha_s(\mu)}{2\pi} \left( C_q^{(1)}(Q/M) - \frac{2P^{(1)}}{\beta_0} \right) \right]. \tag{37} \]

FS invariance of \[33\] implies that the non-universal functions \(C_q^{(1)}\) and \(P^{(1)}\) are related

\[ C_q^{(1)}(1) = \frac{2P^{(1)}}{\beta_0} + \kappa \quad \Leftrightarrow \quad P^{(1)} = \frac{\beta_0}{2} \left( C_q^{(1)}(1) - \kappa \right), \tag{38} \]

where the quantity \(\kappa = \kappa(n)\) is factorization scheme invariant. In other words either \(P^{(1)}\) or \(C_q^{(1)}\), but not both independently, can be chosen at will to specify the FS. For instance, \(F^p(Q)\) can be written as a function of \(C_q^{(1)}\) explicitly as

\[ F^p(Q) = A(\alpha_s(M)) \frac{-2P^{(0)}}{\beta_0} \exp \left[ -\frac{\alpha_s(M)}{2\pi} (C_q^{(1)} - \kappa) \right] \left( 1 + \frac{\alpha_s(\mu)}{2\pi} C_q^{(1)}(Q/M) \right). \tag{39} \]

The above expression is independent of \(C_q^{(1)}\) to the order considered but not exactly and thus its numerical value does depend on the choice of \(C_q^{(1)}\). Two points are worth noting.

First, the cancelation mechanism based on the relation \[38\] operates independently of the value of \(\alpha_s\) as well as for any fixed boundary condition specified by the constant \(A = A(n)\). These constants, being determined by the behaviour of \(q(M)\) at asymptotic values of \(M\), provide unambiguous way of specifying the initial condition on the solution of the evolution equations. The same boundary condition must therefore be used in all FS. If the boundary condition on \(q(M)\) is specified by the value \(q(M_0)\) at some initial \(M_0\), the situation is slightly more complicated. Imagine, we have \(q(M, M_0)\) given as the solution of \[41\] in a FS specified by \(P^{(1)}\), or equivalently \(C_q^{(1)}\), and with the boundary condition given by \(q(M_0)\) at some \(M_0\).

To get the boundary condition in a FS specified by \(\overline{\mathcal{P}}^{(1)}\), or equivalently \(\overline{C}_q^{(1)}\), which would yield the same proton structure function \(\overline{q}(1)\) we first use \[35\] to convert the information on \(q(M_0)\) into the knowledge of the constants \(A = A(n)\). Once we have them we can again use \[35\] to compute the appropriate boundary condition \(\overline{q}(M_0)\) in the new FS:

\[ \overline{q}(M_0) = q(M_0) \exp \left[ \frac{2 \left( \overline{P}^{(1)} - \overline{P} \right) \alpha_s(M_0)}{\beta_0} \right] = q(M_0) \exp \left[ \left( \overline{C}^{(1)} - \overline{C}_q^{(1)} \right) \frac{\alpha_s(M_0)}{2\pi} \right]. \tag{40} \]

This boundary condition is different from the one in the original FS at the same \(M_0\), and so would also be the solution \(q(M, M_0)\), but after inserting into \[31\] we get the same \(F^p(Q)\) since the change of \(q(M, M_0)\) is compensated by the accompanying change of \(C_q(Q/M)\). If we perform an analysis of experimental data on \(F^p(Q)\) by fitting the parameters specifying the initial \(q(M_0)\) (and \(\Lambda_{QCD}\)) we can thus work in any FS and choose in principle any initial \(M_0\) and get the same results \[^4\] for \(F^p(Q)\).

Second, the expression \[38\] connects quantities, \(C_q^{(1)}\) and \(P^{(1)}\), which come from evaluation of Feynman diagrams at different fixed orders: the former at \(\mathcal{O}(\alpha_s)\) order whereas the

[^4]: The independence of the chosen FS, as well as of the value of \(M_0\) does, however, requires sufficiently flexible form of the parametrization of \(q(M_0)\).
latter at $O(\alpha_s^2)$. This is due to the fact that the leading-logarithmic terms proportional to powers of $P^{(0)}$ are resummed to all orders thereby generating the first term on the r.h.s. of (39). Only after this resummation does $C_q^{(1)}$ contribute to $F^p$ at the same order as $P^{(1)}$.

In the standard \MSb FS both $C_q^{(1)}$ and $P^{(1)}$ are nonzero. From the infinity of other possible schemes only the so called “DIS” FS is regularly used. In this scheme one sets $Q = M$ and $C_q^{(1)} = 0$ in order to keep at the NLO the same relation between the proton structure function $F(Q)$ and the quark distribution function $q(Q)$ as at the LO. In some sense opposite to the DIS FS is the FS $P^{(1)} = 0$. In this FS the evolution equation for quark distribution function has the same form at the NLO as at the LO. As noted in [13] this FS corresponds to the point of local stability [14] of the expression (39) considered as function of $M, \mu$ and $C_q^{(1)}$.

6 Factorization schemes and their choice - the photon

For the hadronic part of the quark distribution function of the photon the factorization operates in exactly the same way as for the proton. Consequently $C_q^{(1)}$ can again be used to label different factorization schemes and also the relations (38) do hold.

For the pointlike part of the quark distribution function of the photon the situation is slightly more complicated as the expression for photon structure function $F^\gamma \equiv F_p^{\text{NS}}/x$ involves another coefficient function, namely the photonic $C^\gamma$

$$F^\gamma(Q) = q(M)C_q(Q/M) + C^\gamma(Q/M). \quad (41)$$

Consequently, the factorization scale invariance implies

$$\dot{F}^\gamma(Q) = \dot{q}(M)C_q(Q/M) + q(M)\dot{C}_q(Q/M) + \dot{C}^\gamma(Q/M) = 0$$

$$= \left[ P(M)C_q(Q/M) + \dot{C}_q(Q/M) \right] q(M) + k(M)C_q(Q/M) + \dot{C}^\gamma(Q/M) \quad (42)$$

The expression in the square bracket vanishes for the same reasons as for the hadronic structure function and we are thus left with

$$\dot{F}^\gamma(Q) = \frac{\alpha}{2\pi} \left\{ \left[ k^{(0)} + \frac{\alpha_s(M)}{2\pi} k^{(1)} + \cdots \right] \left[ 1 + \frac{\alpha_s(\mu)}{2\pi} C_q^{(1)}(Q/M) + \cdots \right] + \dot{C}^{\gamma(0)}(Q/M) + \frac{\alpha_s(\mu)}{2\pi} \dot{C}^{\gamma(1)}(Q/M) + \cdots \right\} = 0$$

$$= \frac{\alpha}{2\pi} \left\{ k^{(0)} + \dot{C}^{\gamma(0)}(Q/M) + \frac{\alpha_s}{2\pi} \left[ \dot{C}^{\gamma(1)}(Q/M) + k^{(1)} + k^{(0)}C_q^{(1)}(Q/M) \right] + \cdots \right\} = 0. \quad (43)$$

The argument of $\alpha_s$ in the expansion of the coefficient functions $C_q$ and $C^\gamma$, i.e. the renormalization scale $\mu$, is in general different from the factorization scale $M$. However, I did not write out the argument of $\alpha_s$ in the last line of (43) because the coefficient standing by it is independent of it. The Eq. (43) implies

$$\dot{C}^{\gamma(0)}(Q/M) = -k^{(0)}, \quad (44)$$

$$\dot{C}^{\gamma(1)}(Q/M) = -k^{(1)} - k^{(0)}C_q^{(1)}(Q/M), \quad (45)$$

This holds exactly for $\beta_1 = 0$, for realistic values of $\beta_1$ the saddle point is slightly away from $P^{(1)} = 0$. 
which upon integration and taking into account the relations (34) and (38) yields

\[ C^{(0)}_\gamma(Q/M) = k^{(0)} \ln(Q^2/M^2) + C^{(0)}_\gamma(1) \]  

\[ C^{(1)}_\gamma(Q/M) = \frac{k^{(0)} P^{(0)}}{2} \ln^2(Q^2/M^2) + (k^{(1)} + k^{(0)} C^{(1)}_q(1)) \ln(Q^2/M^2) + C^{(1)}_\gamma(1). \]  

(46)

(47)

Let us now consider the pointlike solution which results from taking into account beside the pure QED splitting function \( k^{(0)} \) the lowest order QCD splitting functions \( P^{(0)} \) and \( k^{(1)} \):

\[ q(M, M_0) = \frac{4\pi a}{\alpha_s(M)} \left[ 1 - \left( \frac{\alpha_s(M)}{\alpha_s(M_0)} \right)^{1-2P^{(0)}/\beta_0} \right] - \frac{\alpha}{2\pi} \left[ 1 - \left( \frac{\alpha_s(M)}{\alpha_s(M_0)} \right)^{-2P^{(0)}/\beta_0} \right] \frac{k^{(1)}}{P^{(0)}}, \]  

(48)

where \( a = a(n) \) is defined in (17). This solution satisfies \( q(M_0, M_0) = 0 \) and for small \( \alpha_s \) behaves as

\[ q(M, M_0) \approx \frac{\alpha}{2\pi} L_M \left[ k^{(0)} + \frac{\alpha_s(M)}{2\pi} \left( k^{(1)} + \frac{k^{(0)} P^{(0)}}{2} L_M \right) \right], \quad L_M \equiv \ln \frac{M^2}{M_0^2}. \]  

(49)

Multiplying (49) with \( C_q \) and adding the lowest two terms of \( C_\gamma \) we get

\[ \frac{2\pi}{\alpha} F^{(0)}(Q) \approx L_M k^{(0)} + C^{(0)}_\gamma(Q/M) \]

\[ + \frac{\alpha_s}{2\pi} \left[ C^{(1)}_\gamma(Q/M) + (k^{(1)} + k^{(0)} C^{(1)}_q(Q/M)) L_M + \frac{k^{(0)} P^{(0)}}{2} L_M^2 \right]. \]

Replacing \( C^{(1)}_\gamma \) with (17) and \( C^{(1)}_q \) with analogous expression (38), we find that all terms in the square bracket dependent on \( M \) cancel out \(^6\) and we are left with expression which is a function of \( Q/M_0 \) only. Moreover, as expected from (27), it is just \( C^{(1)}_\gamma(Q/M_0) \! \).  

Note that the only ambiguity in (27) comes from the freedom in the choice of the expansion parameter \( \alpha_s(\mu) \), i.e. the choice of the renormalization scale \( \mu \) and renormalization scheme. Because these ambiguities appear first at order \( \alpha_s^2 \), the lowest order QCD correction given by \( C^{(1)}_\gamma(Q/M_0) \) must be unique! Consequently, \( C^{(1)}_\gamma(1) \) as well as the combination

\[ k^{(1)} + k^{(0)} C^{(1)}_q(1) = \bar{\kappa} \]  

(51)

must also be unique, i.e. factorization scheme independent. The above relation implies that the first non-universal inhomogeneous splitting function \( k^{(1)} \) is, similarly as \( P^{(1)} \), a function of \( C^{(1)}_q(1) \) (or the other way around). So \( C^{(1)}_q(1) \) and its higher order partners \( C^{(i)}_q \) can be chosen to label the factorization scale ambiguity also for \( F^{(0)} \). Once these quantities (which are functions of \( n \), or equivalently, \( x \)) are chosen all other free parameters are determined by relation like those in (38) or (51).

7 What is wrong with the DIS\(_\gamma\) factorization scheme?

The freedom in the definition of quark distribution functions of the photon related to the non-universality of \( C^{(j)}_q, j \geq 1 \) is intimately connected with higher order QCD corrections. The purely QED contribution, given by the sum of first two terms in (50), is analogous

\(^6\)The analogous cancelation in the pure QED contribution to \( F^{(0)} \) is trivial.
to QED contribution \( 3 \sum_{i=1}^{n_f} e_i^2 \) in (28) and represents an input into QCD analysis. It is manifestly of order \( \alpha_s^0 = 1 \), but since in the standard approach the quark distribution function is - incorrectly as I have argued - assigned the order \( 1/\alpha_s \), the lowest order QCD contribution appears to be of lower order than \( C_q^{(0)} \). The latter is consequently assigned to the “next-to-leading” order [3, 5, 6] and treated in a similar way as the lowest order QCD coefficient function \( C_q^{(1)} \). This procedure has been motivated in part by the fact that \( C_q^{(0)}(x, M) \) as given in (15) turns negative at large \( x \) and even diverges to \( -\infty \) when \( x \to 1 \).

This has lead the authors of [5] to introduce “A factorization scheme avoiding the common perturbative instability problems encountered in the large-x region.” In this so called DIS\( _\gamma \) factorization scheme “… we can use the same boundary conditions for the pointlike LO and HO distributions

\[
q_{PL}(x, Q_0^2) = \bar{q}_{PL}(x, Q_0^2) = G_{PL}(x, Q_0^2) = 0 \tag{52}
\]

without violating the usual positivity requirements.” [5].

Their procedure amounts to of redefining the quark distribution function of the photon by absorbing the pure QED term \( C_q^{(0)} \) in \( q(M, M_0) \) according to (see eq. (5) of [6])

\[
\overline{q}(M, M_0) \equiv q(M, M_0) + \frac{\alpha}{2\pi} C_{\gamma}^{(0)}(1). \tag{53}
\]

As this redefinition involves the QED quantity \( C_{\gamma}^{(0)} \), and the notion of quark distribution function inside the photon is well-defined also in pure QED, the above procedure must make sense even there, without any QCD effects. Moreover, as the QCD contribution depends on the numerical value of \( \alpha_s \) it cannot cure any problem of the pure QED part.

However, in QED it is straightforward to see that in getting rid off the troubling \( C_q^{(0)} \) term the procedure proposed in [5] violates the requirement of factorization scheme invariance. Recall that in pure QED the contribution to \( F_\gamma \) coming from the box diagram regularized by explicit quark mass \( m_q \) reads [15]

\[
F_{\text{QED}}^\gamma(Q) = \frac{\alpha}{2\pi} C_{\gamma}^{(0)}(Q/m_q) = \frac{\alpha}{2\pi} \left[ k(0) \ln \frac{Q^2}{m_q^2} + C_{\gamma}^{(0)}(1) \right]. \tag{54}
\]

Introducing the arbitrary scale \( M \), we can split it into

\[
q_{\text{QED}}(M) \equiv \frac{\alpha}{2\pi} k(0) \ln \frac{M^2}{m_q^2} \tag{55}
\]

interpreted as quark distribution function, and \( C_{\gamma}^{(0)}(Q/M) \) given in [15]:

\[
F_{\text{QED}}^\gamma(Q) = q_{\text{QED}}(M) + \frac{\alpha}{2\pi} C_{\gamma}^{(0)}(Q/M). \tag{56}
\]

The sum (56) is manifestly \( M \)-independent. We can now redefine \( q_{\text{QED}}(M) \) by adding an arbitrary function \( f = f(n) \) according to

\[
q_{\text{QED},f}(M) \equiv q_{\text{QED}}(M) + \frac{\alpha}{2\pi} f. \tag{57}
\]

In order to keep the sum

\[
F_{\text{QED}}^\gamma(Q) = q_{\text{QED},f}(M) + \frac{\alpha}{2\pi} C_{\gamma,f}^{(0)}(Q/M) \tag{58}
\]
independent of $M$ and $f$ and equal to (54) induces the correlated change of $C_{\gamma}(0)$

$$C_{\gamma, f}(Q/M) \equiv C_{\gamma}(Q/M) - f.$$  \hspace{1cm} (59)

Note that the DIS$\gamma$ factorization scheme of [5] corresponds to $f = C_{\gamma}^{(0)}(1)$. We can write down the evolution equation for $q_{\text{QED}, f}(n, M)$ in the “$f$-factorization scheme”

$$\frac{d q_{\text{QED}, f}(M)}{d \ln M^2} = \frac{\alpha_s}{2\pi} k_f^{(0)},$$  \hspace{1cm} (60)

which is the same for all $q_{\text{QED}, f}(M)$. What we are, however, not allowed to do is to use the same boundary condition for all $q_{\text{QED}, f}(M)$, i.e. precisely what has been assumed in (52)! If we do that and impose the boundary condition $q_{\text{QED}, f}(M = m_q) = 0$, we get

$$F^\gamma(Q) = \frac{\alpha_s}{2\pi} \left( C_{\gamma}^{(0)}(Q/m_q) - f \right)$$  \hspace{1cm} (61)

which depends on the choice of $f$ and only for $f = 0$ coincides with the correct result (54). To get the latter we must impose on the solution of the evolution equation (60) in the $f$-factorization scheme the appropriate boundary condition: $q_{\text{QED}, f}(M = m_q) = (\alpha_s/2\pi)f$. The reason the QED expression (54) depends on the FS specified by the function $f = f(n)$ is clear: there is no all order resummation of the LL and NLL terms describing the multiple photon emission off the $qQ$ that would give rise to terms analogous to the first and second terms on the r.h.s. of (35).

In QCD the redefinition (57) (with $q_{\text{QED}}$ replaced with full quark distribution function $q(M, M_0)$) induces not only the shift (59) of $C_{\gamma}^{(0)}$ but also those of $k_f^{(1)}$ and $C_f^{(1)}$

$$k_f^{(1)} \equiv k_f^{(1)} - P^{(0)} f$$  \hspace{1cm} \hspace{1cm} (62)

$$C_{\gamma, f}(Q/M) \equiv C_{\gamma}^{(1)}(Q/M) - C_q^{(1)}(Q/M) f.$$  \hspace{1cm} (63)

As in pure QED the DIS$\gamma$ factorization scheme corresponds to $f = C_{\gamma}^{(0)}(1)$. In [5] and most other NLO QCD analysis the term proportional to $C_{\gamma}^{(1)}$ is assigned to the NNLO and thus the above second relation (63) is not written out explicitly.

The above substitutions (59-63) are legitimate but as in QED what we are not allowed to do is to impose on $q_f(M, M_0)$ the same boundary condition for all $f$. If we do that we straightforwardly find that the photon structure function can be written as

$$F(Q) = \frac{\alpha_s}{2\pi} \left[ (C_{\gamma}^{(0)}(Q/M_0) - f) + \frac{\alpha_s}{2\pi} \left( C_{\gamma}^{(1)}(Q/M_0) - C_q^{(1)}(Q/M) f \right) + \cdots \right]$$  \hspace{1cm} (64)

which coincides with (54) only for $f = 0$. Even if we sum (64) to all orders of $\alpha_s$ the result does depend on the chosen $f$, violating the fundamental requirement of factorization scheme independence of physical quantities!! As already emphasized, the QCD corrections cannot compensate the $f$-dependence of the first pure QED term. In fact even the all order sum of QCD terms in (64) is $f$-dependent.

8 Removing the problem in large $x$ region

Because the function $C_{\gamma}^{(0)}(x, Q/M)$ is of pure QED origin, one might expect it to cause problems also for leptonic structure function of the photon. This structure function, which
had been measured at LEP, is given by the same expression (54) as for quarks, except for the replacement of \( m_q \) with the lepton (muon or electron) mass \( m_l \):

\[
\frac{2\pi}{\alpha} F_{\text{lept}}^\gamma (x, Q) = \left( x^2 + (1 - x)^2 \right) \ln \left( \frac{Q^2(1 - x)}{m_l^2 x} \right) + 8x(1 - x) - 1
\]

\[
= \left( x^2 + (1 - x)^2 \right) \ln \left( \frac{W^2}{4m_l^2} \right) + \left( x^2 + (1 - x)^2 \right) \ln 4 + 8x(1 - x) - 1. \tag{65}
\]

The lepton mass regularizes mass singularities of (65), as does the quark mass in (54). Since in the kinematically accessible region \( W^2 \geq 4m_l^2 \), corresponding to \( x \leq 1/(1 + 4m_l^2/Q^2) \), both the function \( B(x) = (x^2 + (1 - x)^2) \ln 4 + 8x(1 - x) - 1 \) and the first term in (65) are positive, there is thus no problem with positivity of \( F_{\text{lept}}^\gamma \). The same holds for the QED contribution to quark distribution function with \( m_q \) as infrared regulator.

To identify the kinematical regions contributing to various parts of (54) or (65) we return to the \( x \)-space, where the limits on the virtuality \( \tau \) (see Fig. 2) of the quark included in the definition of the pointlike quark distribution function of the photon are given as [17]

\[
\tau_{\text{min}} = \frac{m_q^2}{1 - x} \leq \tau \leq \frac{Q^2}{x} = \tau_{\text{max}}. \tag{66}
\]

The potentially troubling term \( \ln(1 - x) \) in (15) comes from the lower limit on \( \tau \). So long as it appears in the combination \( \ln(Q^2(1 - x)/m_q^2 x) \) as it does in (54) it is harmless as the whole term \( \ln(Q^2(1 - x)/m_q^2 x) \) stays, as argued above, positive. The problem, does, however, arise when we replace \( m_q \) in this combination with an arbitrary initial \( M_0 \), which, moreover, is typically around 1 GeV and consider (54) for \( x \geq 1/(1 + 4M_0^2/Q^2) \), where \( \ln(Q^2(1 - x)/M_0^2 x) \leq 0 \). In this case we should drop the term proportional to \( \ln(1 - x) \) which comes from the region cut-off by the introduction of \( M_0 \). Moreover, we should also drop all other terms which come predominantly from the lower integration range in (65). As argued in [7, 17] this implies that \( C^{(0)}_\gamma \) as given in (15) should be replaced with

\[
C^{(0)}_\gamma (x, Q/M) = \left[ x^2 + (1 - x)^2 \right] \ln \frac{Q^2}{M^2} + \left[ x^2 + (1 - x)^2 \right] \ln \frac{1}{x} + 6x(1 - x) - 1, \tag{67}
\]

which implies decent behaviour of

\[
C^{(0)}_\gamma (x, 1) = \left[ x^2 + (1 - x)^2 \right] \ln \frac{1}{x} + 6x(1 - x) - 1 \tag{68}
\]

when \( x \to 1 \). Note that the expression (68) coincides with that used in [18].

### 9 Summary and Conclusions

The standard definition of LO and NLO approximations to photon structure function are shown to be inconsistent with the requirement of factorization scale and scheme invariance. The origin of this shortcoming is traced back to the usual but incorrect claim that quark distribution functions of the photon behave as \( \alpha/\alpha_s \).

Theoretically consistent definition of LO, NLO and NNLO approximations for the photon structure function are constructed and the potentially troubling behaviour of \( C^{(0)}_\gamma (x, Q/M) \)
at large $x$ is removed by discarding the terms coming from the region outside the validity of perturbation theory.

The so called DIS$_g$ factorization scheme invented in order to cure the mentioned problem with large-$x$ behaviour of $C_g^{(0)}(x, Q/M)$ is shown to be ill-defined as it is based on the procedure which violates the factorization scheme invariance of the photon structure function.

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References

[1] S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. B688 (2004), 101, hep-ph/0403192
[2] S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. B691 (2004), 129, hep-ph/0404111
[3] A. Vogt, talk at PHOTON2005 Symposium, Warsaw, September 2005, hep-ph/0511112
[4] S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. B621 (2002), 413, hep-ph/0110331
[5] M. Glück, E. Reya, A. Vogt, Phys. Rev. D45 (1992), 3986
[6] M. Glück, E. Reya, A. Vogt, Phys. Rev. D46 (1992), 1973
[7] J. Chýla, JHEP04(2000)007
[8] S. Catani, M. Dittmar, D. Soper, W.J. Stirling, S. Tapprogge (convenors), “1999 CERN Workshop on SM Physics (and more) at the LHC”, p. 8., hep-ph/0005025
[9] S. Catani, M. Dittmar, J. Huston, D. Soper, S. Tapprogge (conveners) The QCD and standard model working group: Summary Report, p. 16, hep-ph/0005114
[10] D. Soper, “Basics of QCD Perturbation Theory”, hep-ph/0011256 p. 31
[11] W.L. van Neerven, A. Vogt, Nucl. Phys. B 568 (2000), 263; B 588 (2000), 345
[12] H. D. Politzer, Nucl. Phys. B 194 (1982), 493
[13] J. Chýla, Z. Phys. C 43 (1989), 431
[14] P. M. Stevenson, Phys. Rev. D21 (1981), 2916
[15] E. Witten, Nucl. Phys. 120 (1977), 189
[16] W. Bardeen, A. Buras, Phys. Rev. D 20 (1979), 166
[17] J. Chýla, M. Taševský, Phys. Rev. D62 (2000), 114025
[18] G. Schuler, T. Sjöstrand, Z. Phys. C68 (1995), 607