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From symmetry breaking to Poisson point process in 2D Voronoi tessellations: the generic nature of hexagons. (English) Zbl 1214.82043
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Summary: We bridge the properties of the regular triangular, square, and hexagonal honeycomb Voronoi tessellations of the plane to the Poisson-Voronoi case thus analyzing in a common framework symmetry breaking processes and the approach to uniform random distributions of tessellation-generating points. We resort to ensemble simulations of tessellations generated by points whose regular positions are perturbed through a Gaussian noise, whose variance is given by the parameter $\alpha^2$ times the square of the inverse of the average density of points. We analyze the number of sides, the area, and the perimeter of the Voronoi cells. For all values $\alpha > 0$, hexagons constitute the most common class of cells, and 2-parameter gamma distributions provide an efficient description of the statistical properties of the analyzed geometrical characteristics. The introduction of noise destroys the triangular and square tessellations, which are structurally unstable, as their topological properties are discontinuous in $\alpha = 0$. On the contrary, the honeycomb hexagonal tessellation is topologically stable and, experimentally, all Voronoi cells are hexagonal for small but finite noise with $\alpha < 0.12$. For all tessellations and for small values of $\alpha$, we observe a linear dependence on $\alpha$ of the ensemble mean of the standard deviation of the area and perimeter of the cells. Already for a moderate amount of Gaussian noise ($\alpha > 0.5$), memory of the specific initial unp perturbed state is lost, because the statistical properties of the three perturbed regular tessellations are indistinguishable. When $\alpha > 2$, results converge to those of Poisson-Voronoi tessellations. The geometrical properties of $n$-sided cells change with $\alpha$ until the Poisson-Voronoi limit is reached for $\alpha > 2$; in this limit the Desch law for perimeters is shown to be not valid and a square root dependence on $n$ is established. This law allows for an easy link to the Lewis law for areas and agrees with exact asymptotic results. Finally, for $\alpha > 1$, the ensemble mean of the cells area and perimeter restricted to the hexagonal cells agree remarkably well with the full ensemble mean; this reinforces the idea that hexagons, beyond their ubiquitous numerical prominence, can be interpreted as typical polygons in 2D Voronoi tessellations.

MSC:
82B41 Random walks, random surfaces, lattice animals, etc. in equilibrium statistical mechanics
60D05 Geometric probability and stochastic geometry
60G55 Point processes (e.g., Poisson, Cox, Hawkes processes)

Keywords:
Voronoi tessellation; topological stability; random geometry; symmetry breaking; Poisson point process; Desch law; Lewis law; Gaussian noise

Software:
Qhull

Full Text: DOI arXiv Link

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