Asymptotics of the tacnode process: a transition between the gap probabilities from the tacnode to the Airy process

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Abstract
We study the gap probabilities of the single-time tacnode process. Through steepest descent analysis of a suitable Riemann–Hilbert problem, we show that under appropriate scaling regimes the gap probability of the tacnode process degenerates into a product of two independent gap probabilities of the Airy processes.

Keywords: point process, Fredholm determinant, Riemann–Hilbert problem, steepest descent method

Mathematics Subject Classification: 60J60, 35Q15, 34M60

1. Introduction

This paper deals with the analysis of a diffusion process obtained as a critical limit of Dyson’s non-intersecting Brownian motions [9].

Consider $n$ Brownian particles moving on the real line, conditioned never to intersect, with given starting and ending configurations. In particular, let us assume that all the particles start at two given fixed points and end at two other points (which may coincide with the starting points). For every time $t \in [0, 1]$ (1 being the end time where the particles collapse in the two final points), the positions of the Brownian paths form a determinantal process [20]. Moreover, as the number of particles tends to infinity, the paths fill a specific limit region which depends on the relative position of the starting and ending points.

There are three possible scenarios. The first is that there may be two independent connected components similar to ellipses (figure 1). The second is that there may be one connected...
Figure 1. Numerical simulation of 90 non-intersecting Brownian motions with two starting points ±α = 1 and two ending positions ±β = 1 in the case of large separation between the endpoints. For n → ∞ the positions of the Brownian motions fill a prescribed region in the time–space plane, which is bounded by the boldface lines in the figure. Here the horizontal axis denotes the time t and the vertical axis is the space.

component similar to two ‘merged’ ellipses (figure 2). It is well known that the microscopic behaviour of such an infinite particle system is regulated by the sine process in the bulk of the particle bundles [19], by the Airy process along the soft edges [16, 17, 21] and by the Pearcey process in the cusp singularity [22], when it occurs.

The third critical configuration can be seen as the limit of the large separation case when the two bundles are tangential to each other in one point, called a tacnode point (see figure 3). In a microscopic neighbourhood of this point the fluctuations of the particles are described by a new critical process called the tacnode process. In this limit setting, a parameter σ appears which controls the strength of the interaction between the left-most particles and the right-most ones (σ can be thought of as a pressure or temperature parameter).

The kernel of such a process in the single-time case was first introduced by Adler et al in [1] as the scaling limit of a model of random walks, and shortly after by Delvaux et al in [8], where the kernel was expressed in terms of a 4 × 4 matrix valued Riemann–Hilbert (RH) problem. In [18] Johansson formulated the multi-time (or extended) version of the process, remarking nevertheless on the fact that this extended version does not automatically reduce to the single-time version given in [8]. In this paper, for the first time, the kernel is expressed in terms of the resolvent and the Fredholm determinant of the Airy kernel.

In [2] the authors analysed the same process as arising from random tilings instead of self-avoiding Brownian paths and they proved the equivalency of all the above formulations. A similar result was obtained by Delvaux in [7], where a RH expression for the multi-time tacnode kernel is given. A more general formulation of this process has been studied in [10], where the limit shapes of the two groups of particles are allowed to be non-symmetric.

Physically, if we start from the tacnode configuration and push together the two ellipses, they will merge giving rise to the single connected component in figure 2, while if we pull the ellipses apart, we simply end up with two disjoint ellipses as in figure 1. It is thus natural to
Figure 2. Numerical simulation of 90 non-intersecting Brownian motions with two starting points $\pm \alpha = 0.5$ and two ending positions $\pm \beta = 0.5$ in case of small separation between the endpoints.

Figure 3. Numerical simulation of 90 non-intersecting Brownian motions with two starting points $\pm \alpha = 0.75$ and two ending positions $\pm \beta = 0.75$ in the case of critical separation between the endpoints.

expect that the local dynamic around the tacnode point will in either case degenerate into a Pearcey process or an Airy process, respectively.

The degeneration tacnode–Pearcey has been proven in [12] where the authors showed a uniform convergence of the tacnode kernel to the Pearcey kernel over compact sets in the limit as the two bundles are pushed to merge together. On the other hand, the method used
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in [12] cannot be extensively applied to the tacnode–Airy degeneration. The Airy process is structurally different from the Pearcey, since it has the feature of a ‘last particle’ (or largest eigenvalue in the random matrix setting), which is described by the well-known Tracy–Widom distribution [21]. The method above does not allow one to recover the emerging of the ‘last particle’ feature from the tacnode-to-Airy degeneration, which, on the other hand, is shown in the present paper.

The purpose of our paper is to study the asymptotic behaviour of the gap probability of the (single-time) tacnode process and its degeneration into the gap probability of the Airy process. There are two types of regimes in which this degeneration occurs: the limit as $\sigma \to +\infty$ (large separation), which physically corresponds to pulling apart the two sets of Brownian particles touching on the tacnode point, and the limit as $\tau \to \pm \infty$ (large time), which corresponds to moving away from the singular point along the boundary of the space–time region swept out by the non-intersecting paths.

An expression for the single-time tacnode kernel is the following (see [2, formula (19)])

$$K_{\text{tac}}(\tau; x, y) = K(\tau, -\tau) \text{Ai}(\sigma - x, \sigma - y) + \frac{3}{\sqrt{2}} \int_{\tilde{\sigma}}^{\infty} dw \int_{\tilde{\sigma}}^{\infty} \mathcal{A}_\tau^{-1}(w) \left( \frac{\text{Id} - K_{\text{Ai}}}{[\tilde{\sigma}, +\infty]} \right)^{-1} (z, w) \mathcal{A}_\tau^{-1}(z) (1.1)$$

with $\tilde{\sigma} := 2^{1/3} \sigma$ and

$$\mathbb{Ai}(t) := e^{x + \frac{1}{2} t^3} \text{Ai}(x) = \int_{\gamma_R} \frac{d\lambda}{2\pi i} e^{\frac{1}{3} t \lambda - \frac{1}{2} \lambda^3}$$

(1.2)

$$\mathbb{A}(\lambda) := \int_{\gamma_L} \frac{d\lambda}{2\pi i} e^{-\frac{1}{3} \lambda^3}$$

(1.3)

$$\mathcal{A}_\tau(z) := \mathbb{Ai}(t)(x + \sqrt{2} z) - \int_0^\infty dw \mathbb{Ai}(t)(-x + \sqrt{2} w) \text{Ai}(w + z) (1.4)$$

$$K_{\text{Ai}}(-\tau; -x, -y) := \int_0^\infty du \mathbb{Ai}(t)(-x + u) \mathbb{Ai}(u) (1.5)$$

$$K_{\text{Ai}}(z, w) := \int_0^\infty du \mathbb{Ai}(u) \mathbb{Ai}(w + u)$$

(1.6)

where the contour $\gamma_R$ is the contour extending to infinity in the $\lambda$-plane along the rays $e^{\pm \frac{1}{2} \pm \frac{1}{2} \lambda^3}$, oriented upwards and entirely contained in the right half plane ($\Re(\lambda) > 0$), and $\gamma_L := -\gamma_R$.

The quantity of interest, i.e. the gap probability of the process, is expressed in terms of the Fredholm determinant of an integral operator with kernel (1.1). Given a Borel set $\mathcal{I}$, then

$$P \left( \text{no particles in } \mathcal{I} \right) = \det \left( \text{Id} - \|K_{\text{tac}}\|_{\mathcal{I}} \right) (1.7)$$

The first difficulty in studying the tacnode process is the expression of its kernel, since it is highly transcendental and involves the resolvent of the Airy operator. It it thus necessary to reduce it to a more approachable form.

The first important step was [5, theorem 3.1] where it was proved that gap probabilities of the tacnode process can be defined as the ratio of two Fredholm determinants of explicit integral operators with kernels that only involve contour integrals, exponentials and Airy functions. This result, which will be recalled in section 3, will be our starting point in the investigation of the gap probabilities and their asymptotics. The second step will be to find an appropriate integral operator in the sense of Its–Izergin–Korepin–Slavnov (IIKS) [15] whose Fredholm determinant coincides with the quantity (1.7). In this way, it will be possible to
give a formulation of the gap probabilities of the tacnode in terms of a RH problem, naturally associated to an IIKS integral operator (see [13]). Finally, applying well known steepest descent methods to the above RH problem, we will be able to prove the conjectured degeneration into Airy processes.

The outline of the paper is as follows: in section 2 we state the main results of the paper, which will be proved in sections 3, 4 and 5. In particular, section 3 deals with some preliminary calculations which are necessary to set a RH problem on which we shall later perform some steepest descent analysis in the limit as $\sigma \to \infty$ (section 4) or $\tau \to \infty$ (section 5).

2. Results

The first results on asymptotic regime of the tacnode process were stated in [5]. We are recalling them here for the sake of completeness.

**Theorem 2.1.** Let $\mathcal{I} := \bigcup_{j=1}^{K} [a_{2j-1}, a_{2j}]$ be a collection of intervals, with $a_j = a(s_j) = -\sigma - \tau^2 + s_j$. Keeping the overlap $\sigma$ fixed, we have

$$\lim_{\tau \to \pm\infty} \det \left( \Id - \mathcal{K}^{\text{tac}}_{\mathcal{I}} \right) = \det \left( \Id - K_{\text{Ai}} \right) \quad (2.1)$$

with $J = \bigcup_{\ell=1}^{K} [s_{2\ell-1}, s_{2\ell}]$. Analogously, keeping $\tau$ fixed, we obtain

$$\lim_{\sigma \to \infty} \det \left( \Id - \mathcal{K}^{\text{tac}}_{\mathcal{I}} \right) = \det \left( \Id - K_{\text{Ai}} \right) \quad (2.2)$$

**Proof.** The convergence follows easily by directly studying the kernel of the extended tacnode process (see [2, formula (19)]), since the term involving the resolvent of the Airy kernel tends to zero, uniformly over compact sets of the spatial variables $x - \sigma - \tau^2$.

A more interesting situation is the one in which the tacnode process degenerates into a pair of Tracy–Widom distributions, in analogy with the Pearcey-to-Airy transition (see [4]). In this case, half of the space variables (endpoints of the gaps) move far away from the tacnode following the left branch of the boundary of the space–time region swept by the particles, and the other half goes in the opposite direction. Therefore, it is expected that the gap probability of the tacnode process for a ‘large gap’ factorize into two Fredholm determinants for semi-infinite gaps of the Airy process.

Numerically, these regimes are illustrated in the paper [5], where a conjecture of the above results has been proposed. However, all the results are rigorously proved here.

In the simple case with only one interval, we have the following theorems.

**Theorem 2.2 (Asymptotics as $\sigma \to +\infty$).** Let $\mathcal{K}^{\text{tac}}$ and $K_{\text{Ai}}$ be the kernels associated with the tacnode process and the Airy process, respectively. Let

$$a = a(t) = -\sigma - \tau^2 + t \quad b = b(s) = \sigma + \tau^2 - s \quad (2.3)$$

then as $\sigma \to +\infty$

$$\det \left( \Id - \mathcal{K}^{\text{tac}}_{[-\sigma - \tau^2, t, \sigma + \tau^2 - s]} \right) = \det \left( \Id - K_{\text{Ai}}_{[t, +\infty]} \right) \det \left( \Id - K_{\text{Ai}}_{[t, +\infty]} \right) \left( 1 + O(\sigma^{-1}) \right) \quad (2.4)$$

and the convergence is uniform over compact sets of the variables $s, t$ provided

$$-\infty < s, t < K_1(\sigma + \tau^2), \quad 0 < K_1 < 1. \quad (2.5)$$
Theorem 2.3 (Asymptotics as $\tau \to \pm \infty$). Let $K^{\text{tac}}$ be the tacnode process and $K_{\text{Ai}}$ the Airy process. Let

\[ a = a(t) = -\sigma - \tau^2 + t \quad b = b(s) = \sigma + \tau^2 - s \quad (2.6) \]

then as $\tau \to \pm \infty$

\[
\det \left( \mathbf{Id} - [K^{\text{tac}}]_{[-\sigma - \tau^2 + t, \sigma + \tau^2 - s]} \right) = \det \left( \mathbf{Id} - K_{\text{Ai}} \bigg|_{[s, +\infty]} \right) \det \left( \mathbf{Id} - K_{\text{Ai}} \bigg|_{[t, +\infty]} \right) \left( 1 + \mathcal{O}(\tau^{-1}) \right) \quad (2.7)
\]

and the convergence is uniform over compact sets of the variables $s, t$ provided

\[ -\infty < s, t < K_1(\sigma + \tau^2) \quad (2.8) \]

\[ t = 4\tau^2 - \delta, \quad 0 < \delta < \frac{7}{3}K_2\tau^2; \quad s = \tau^2 + 2\sigma - \delta, \quad 0 < \delta < K_3 \left( 2\sigma + \frac{2}{3}\tau^2 \right) \quad (2.9) \]

for some $0 < K_1, K_2, K_3 < 1$.

More generally, we consider the tacnode process restricted to a collection of intervals.

Theorem 2.4. Given

\[ I = \bigcup_{j=1}^{J} [a_{2j-1}, a_{2j}] \cup [a_{2J+1}, b_0] \cup \bigcup_{k=1}^{K} [b_{2k-1}, b_{2k}] \quad (2.10) \]

where

\[ a_{\ell} = a(s_{\ell}) = -\sigma - \tau^2 + t_{\ell} \quad b_{\ell} = b(t_{2K+1-\ell}) = \sigma + \tau^2 - s_{2K+1-\ell}, \quad (2.11) \]

then as $\sigma \to +\infty$

\[
\det \left( \mathbf{Id} - [K^{\text{tac}}]_{I} \right) = \det \left( \mathbf{Id} - K_{\text{Ai}} \bigg|_{J_1} \right) \det \left( \mathbf{Id} - K_{\text{Ai}} \bigg|_{J_2} \right) \left( 1 + \mathcal{O}(\sigma^{-1}) \right) \quad (2.12)
\]

or as $\tau \to \pm \infty$

\[
\det \left( \mathbf{Id} - [K^{\text{tac}}]_{I} \right) = \det \left( \mathbf{Id} - K_{\text{Ai}} \bigg|_{J_1} \right) \det \left( \mathbf{Id} - K_{\text{Ai}} \bigg|_{J_2} \right) \left( 1 + \mathcal{O}(\tau^{-1}) \right) \quad (2.13)
\]

where

\[ J_1 = \bigcup_{\ell=1}^{J} [t_{2\ell-1}, t_{2\ell}] \cup [t_{2J+1}, +\infty) \quad J_2 = \bigcup_{l=1}^{K} [s_{2l-1}, s_{2l}] \cup [s_{2K+1}, +\infty) \quad (2.14) \]

and the convergence is uniform over compact sets of the variables $s, t$ provided

\[ -\infty < s_{\ell}, t_{\ell} < K_1(\sigma + \tau^2) \quad (2.15) \]

\[ t_{\ell} = 4\tau^2 - \delta, \quad 0 < \delta < \frac{7}{3}K_2\tau^2; \quad s_{\ell} = \tau^2 + 2\sigma - \delta, \quad 0 < \delta < K_3 \left( 2\sigma + \frac{2}{3}\tau^2 \right) \quad (2.16) \]

for some $0 < K_1, K_2, K_3 < 1$.

The parametrization of the endpoints $a$ and $b$ in theorems 2.2 and 2.3 (and of $a_{\ell}$ and $b_{\ell}$ in theorem 2.4) has the following meaning. At the critical time $0 < t_{\text{tac}} < 1$, the two bulks tangentially touch at one point $P_{\text{tac}}$, the tacnode point. From the common tacnode point $a(t_{\text{tac}}) = b(t_{\text{tac}})$, two new endpoints $[a(t), b(t)]$ emerge and move away along the branches of the boundary.
The tacnode point process describes the statistics of the random walkers in a scaling neighbourhood of \( t = t_{\text{tac}} \) and \( a = b = P_{\text{tac}} \). The asymptotics as \( \tau \to \pm \infty \) given in theorem 2.3 is the regime where we look ‘away’ from the critical point (either in the future for \( \tau > 0 \) or in the past for \( \tau < 0 \)) and it is expected to reduce to two Airy point processes, which describe the edge-behaviour of the random walkers. Similarly, when we take the limit as \( \sigma \to +\infty \) (theorem 2.2) we are physically pushing away the two bulks from each other and the expected regime around the no longer critical time will again be a product of two Airy point processes.

The proof of these theorems relies essentially upon the construction of a RH problem deduced from a suitable IIKS integrable kernel \([15]\) and the Deift–Zhou steepest descent method \([6]\). In the next section we will show how to deduce such an integrable kernel from the tacnode kernel. We will start with considerations that apply to the more general case, but then we will specialize to the single interval case (theorems 2.2 and 2.3) in order to avoid unnecessary complications, which are purely notational and not conceptual.

3. The RH setting for the gap probabilities of the tacnode process

For the sake of clarity, we recall the definition of the tacnode kernel already given in the introduction (1.1)--(1.6). Referring to the formula given by Adler et al \([2]\), the single-time tacnode kernel reads (see \([2\), formula (19)])

\[
K_{\text{tac}}(\tau; x, y) = K_{\text{Ai}}^{(\tau,-\tau)}(\sigma - x, \sigma - y)
\]

\[
+ \sqrt{2} \int_{\tilde{\sigma}}^{\infty} dz \int_{\tilde{\sigma}}^{\infty} dw \ A_{\lambda - \sigma}^\tau(w) \left( I - K_{\text{Ai}} \right)_{\tilde{\sigma}}^{-1}(z, w) A_{\lambda - \sigma}^{-\tau}(w) \quad (3.1)
\]

where \( \tilde{\sigma} := 2^{1/2} \sigma \) and the functions appearing in the above definition are specified below:

\[
A_{\text{Ai}}^{(\tau)}(x) := e^{x + \tau z} A_{\text{Ai}}(x) = \int_{\gamma_R} \frac{d\lambda}{2\pi i} e^{\frac{\lambda}{2} \lambda^2 \tau - \lambda^2} \quad (3.2)
\]

\[
A_{\text{Ai}}(x) := \int_{\gamma_L} \frac{d\lambda}{2\pi i} e^{-\frac{\lambda}{2} \lambda^2} = - \int_{\gamma_R} \frac{d\lambda}{2\pi i} e^{\frac{\lambda}{2} \lambda^2} \quad (3.3)
\]

\[
A_{\lambda}^{\tau}(z) := A_{\text{Ai}}^{(\tau)}(x + \sqrt{2} z) = \int_{0}^{\infty} dw \ A_{\lambda}^{(\tau)}(-x + \sqrt{2} w) A_{\lambda}^{(-\tau)}(w + z) \quad (3.4)
\]

\[
K_{\text{Ai}}^{(\tau,-\tau)}(-x, -y) := \int_{0}^{\infty} du \ A_{\lambda}^{(\tau)}(-x + u) A_{\lambda}^{(-\tau)}(-y + u) \quad (3.5)
\]

\[
K_{\text{Ai}}(z, w) := \int_{0}^{\infty} du \ A_{\lambda}^{(z + u)}(w + u) \quad (3.6)
\]

The contour \( \gamma_R \) is a contour extending to infinity in the \( \lambda \)-plane along the rays \( e^{\pm i \frac{\pi}{4}} \), oriented upwards and entirely contained in the right half plane \( \Re(\lambda) > 0 \), and \( \gamma_L := -\gamma_R \).

First of all, since only the combination \( x - \sigma, y - \sigma \) appears, we shift the variables and perform a spatial rescaling of the form \( u = \sqrt{2} u' \). The resulting kernel is

\[
\tilde{K}(x, y) := \sqrt{2} K_{\text{tac}}(\sqrt{2} x, \sqrt{2} y)
\]

\[
= \sqrt{2} \int_{0}^{\infty} du \ A_{\lambda}^{(\tau)}(\sqrt{2} (u - x)) A_{\lambda}^{(-\tau)}(\sqrt{2} (u - y))
\]

\[+ \sqrt{2} \int_{\tilde{\sigma}}^{\infty} dz \int_{\tilde{\sigma}}^{\infty} dw \ A_{\lambda}^{(z)}(w) \left( I - K_{\text{Ai}} \right)_{\tilde{\sigma}}^{-1}(z, w) A_{\lambda}^{(-z)}(w) \quad (3.7)
\]
For the sake of brevity, we shall introduce the operators \( K_{\text{Ai}}, K_{\text{Ai}}^{(r,-r)} \), \( \mathfrak{A} \) (with abuse of notation) as the operators with the kernels,

\[
K_{\text{Ai}}^{(r,-r)} := K_{\text{Ai}}^{(r,-r)}(\sqrt{2}x, \sqrt{2}y) = \sqrt{2} \int_{0}^{\infty} \text{d}u \, A_{i}^{(r)}(\sqrt{2}(u-x)) \Lambda_{i}^{(r)}(\sqrt{2}(u-y))
\]

(3.8)

\[
K_{\text{Ai}} := K_{\text{Ai}}(x, y) \bigg|_{[\hat{\tau}, \infty)}
\]

(3.9)

\[
B_{r}(x, z) := 2^{\frac{1}{2}} A_{i}^{(r)}(\sqrt{2}(x + z)), \quad A(z, w) := A_{i}(z + w)
\]

(3.10)

\[
\mathfrak{A}_{r}(x, z) := A_{r}^{(3)} \tau (x, z) = B_{r}(x, z) - \int_{0}^{\infty} \text{d}w \, B_{r}(-x, w) \Lambda(w, z);
\]

(3.11)

moreover, we set \( \pi \) as the projector on the interval \([\hat{\tau}, \infty)\).

Given the above definitions, we can rewrite the tacnode kernel in the following way.

**Proposition 3.1.** The kernel \( \tilde{K} \) can be represented as

\[
\tilde{K}(x, y) = K_{\text{Ai}}^{(r,-r)}(x, y) + \int_{[\hat{\tau}, \infty)} \text{d}z \int_{[\hat{\tau}, \infty)} \text{d}w \, \mathfrak{A}_{r}(x, z) R(z, w) \mathfrak{A}_{r}(z, y)
\]

(3.12)

\[
R(z, w) := \left( \text{Id} - K_{\text{Ai}} \bigg|_{[\hat{\tau}, \infty)} \right)^{-1} (z, w).
\]

(3.13)

Alternatively,

\[
\tilde{K} = K_{\text{Ai}}^{(r,-r)} + \mathfrak{A}_{r} \pi \left( \text{Id} - K_{\text{Ai}} \bigg|_{[\hat{\tau}, \infty)} \right)^{-1} \pi \mathfrak{A}_{r}^{T}
\]

(3.14)

where we recall that \( \tilde{K} \) is the transform of the kernel \( K_{\text{Ai}}^{(r,-r)} \) under the change of variables \( u' = 2^{-r}(u - \tau) \).

Let \( \mathcal{I} = [a_1, a_2] \cup [a_3, a_4] \cdots \cup [a_{2K-1}, a_{2K}] \) and denote by \( \Pi \) the projector on \( \mathcal{I} \). We will denote with \( \hat{\Pi} \) the projection on the rescaled and translated collection of intervals \([\hat{a}_1, \hat{a}_2] \cup \cdots \cup [\hat{a}_{2K-1}, \hat{a}_{2K}]\), where \( \hat{a}_j := 2^{-r}(a_j - \tau) \). We are interested in studying the gap probability of the tacnode process restricted to this collection of intervals, namely

\[
\det(\text{Id} - \Pi K_{\text{Ai}}^{(r,-r)} \Pi) = \det \left( \text{Id} - 2^{\frac{1}{2}} \hat{\Pi} \left( K_{\text{Ai}}^{(r,-r)} + \mathfrak{A}_{r} \pi \left( \text{Id} - K_{\text{Ai}} \bigg|_{[\hat{\tau}, \infty)} \right)^{-1} \pi \mathfrak{A}_{r}^{T} \right) \hat{\Pi} \right).
\]

(3.15)

The following proposition is a restatement of theorem 3.1 from [5], adapted to the single-time case which we are examining.

**Proposition 3.2.** The gap probability of the tacnode process admits the following equivalent representation

\[
\det(\text{Id} - \Pi K_{\text{Ai}}^{(r,-r)} \Pi) = F_{2}(\hat{\Pi})^{-1} \det \left( \text{Id} - \hat{\Pi} \Pi \hat{\Pi} \right)
\]

\[
:= F_{2}(\hat{\Pi})^{-1} \det \left( \begin{bmatrix} \pi K_{\text{Ai}} \pi & -\sqrt{2} \pi \mathfrak{A}_{r}^{T} \Pi \hat{\Pi} \\ -\sqrt{2} \hat{\Pi} \mathfrak{A}_{r} \pi & \sqrt{2} \hat{\Pi} K_{\text{Ai}}^{(r,-r)} \Pi \hat{\Pi} \end{bmatrix} \right)
\]

(3.16)

where \( \hat{\Pi} \Pi \Pi \hat{\Pi} \) is an operator acting on the Hilbert space \( L^{2}(\hat{\tau}, \infty) \oplus L^{2}(\mathbb{R}) \), \( \hat{\Pi} := \pi \oplus \hat{\Pi} \)
and \( F_{2}(\hat{\Pi}) \) is the Tracy–Widom distribution

\[
F_{2}(\hat{\Pi}) := \det \left( \text{Id} - K_{\text{Ai}} \bigg|_{[\hat{\tau}, \infty)} \right).
\]

(3.17)
Remark 3.3. The projection $\pi$ in (3.16) is redundant since by definition the operator acts on the Hilbert space $L^2([\sigma, \infty))$, but we will keep it for convenience.

Therefore, the gap probabilities of the tacnode process are expressible as the ratio of two Fredholm determinants. We refer to [5, remark 3.1 and appendix] for a discussion about possible probabilistic interpretations of such a result.

Proof. The identity is based on the following operator identity (all being trace-class perturbations of the identity)

$$
\det \left( \Id - \left[ \frac{\pi K_{\tilde{A}_1} \pi}{\sqrt{2} \Pi_{\tilde{A}_1} \pi} \right] \right) = \det \left( \Id - \pi K_{\tilde{A}_1} \pi \right) \det \left( \frac{\Id}{\sqrt{2} \Pi_{\tilde{A}_1} \pi} \right)
$$

The kernels involved in the definitions (3.8)–(3.11) can be represented as the following contour integrals. The results are shown in the following two lemmas. Their proof is just a matter of straightforward calculations using Cauchy’s residue theorem.

Lemma 3.4. The kernels involved in the definitions (3.8)–(3.11) can be represented as the following contour integrals

$$
B^i(x, z) = 2^{-i} \int_{\gamma_R} \frac{d\lambda}{2\pi i} e^{\theta(x, z)} \quad A(z, w) = \int_{\gamma_R} \frac{d\lambda}{2\pi i} e^{\theta(z, w)}
$$

$$
A_r(x, z) = 2^{-i} \left[ -\int_{\gamma_L} \frac{d\mu}{2\pi i} \frac{e^{-\tilde{\theta}_r(z, \mu; x, z)}}{\mu - \lambda} \right]
$$

$$
K_{\tilde{A}_1}^{(\tau, -\tau)}(x, y) = \int_{\gamma_R} \frac{d\lambda}{2\pi i} \int_{\gamma_L} \frac{d\mu}{2\pi i} \frac{e^{\theta(z, \mu; x, y)}}{\sqrt{2(\mu - \lambda)}}
$$

$$
K_{\tilde{A}_1}(z, w) = \int_{\gamma_R} \frac{d\lambda}{2\pi i} \int_{\gamma_L} \frac{d\mu}{2\pi i} \frac{e^{\theta(z, \mu; x, \lambda)}}{\mu - \lambda}
$$

with $\tilde{\theta}_r(\lambda; x) := \frac{\lambda^2}{2} + \frac{x^2}{2} - x\lambda$ and $\theta(\lambda; x) := \frac{x^2}{2} - x\lambda$.

Moreover, if $\tilde{\Pi}$ is the projector on the collection of intervals $\bigcup_{j=1}^N [\tilde{a}_{2j-1}, \tilde{a}_{2j}]$ and $\pi$ is the projector on $(\tilde{\sigma}, \infty)$, a simple application of Cauchy’s theorem yields the following identities:

$$
\pi K_{\tilde{A}_1}(z, w) = \int_{\mathbb{R}} \frac{d\xi}{2\pi i} e^{\xi(z, \tilde{\sigma})} \int_{\mathbb{R}} \frac{d\zeta}{2\pi i} e^{\zeta(\tilde{\sigma} - w)} \int_{\gamma_R} \frac{d\lambda}{2\pi i} \int_{\gamma_L} \frac{d\mu}{2\pi i} e^{\theta(\lambda, \tilde{\sigma}, \tilde{\sigma} - \mu, \tilde{\sigma})}.
$$

Indeed, if $w > \tilde{\sigma}$ we can close the $\zeta$-integration with a big semicircle on the right-half plane, picking up the residue at $\lambda \in \gamma_R$; conversely, if $w < \tilde{\sigma}$ we close the $\zeta$-integration with a big semicircle in the left-half plane, which yields zero since there are no singularities within this contour of integration; the same argument applies for the variable $z$. 

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Similarly,
\[
\tilde{\Pi} A_i \pi(x, w) = \sum_{j} \frac{2^k}{\Pi_1^2} \int_{\mathbb{R}} \frac{d\xi}{2\pi} e^{\xi(i\tilde{a}_j - x)} \int_{\mathbb{R}} \frac{d\zeta}{2\pi} e^{\zeta(i\tilde{a}_j - y)} \left[ \int_{\gamma_\lambda} \frac{d\lambda}{2\pi i} \frac{e^{\tilde{\eta}_j(\lambda \tilde{a}_j + \tilde{\theta})}}{\xi - \lambda} \right] \] (3.24)

\[
\pi A_i \tilde{\Pi}(z, y) = \sum_{j} \frac{2^k}{\Pi_1^2} \int_{\mathbb{R}} \frac{d\xi}{2\pi} e^{\xi(i\tilde{a}_j - z)} \int_{\mathbb{R}} \frac{d\zeta}{2\pi} e^{\zeta(i\tilde{a}_j - y)} \left[ - \int_{\gamma_\lambda} \frac{d\mu}{2\pi i} \frac{e^{-\tilde{\eta}_j(\mu \tilde{a}_j + \tilde{\theta})}}{\xi - \mu} \right] \] (3.25)

\[
\tilde{\Pi} K_{\lambda_i}^{(\tau, -\tau)} \tilde{\Pi}(x, y) = \sum_{j,k} \frac{2^k}{\Pi_1^2} \int_{\mathbb{R}} \frac{d\xi}{2\pi} e^{\xi(i\tilde{a}_j - z)} \int_{\mathbb{R}} \frac{d\zeta}{2\pi} e^{\zeta(i\tilde{a}_j - y)} \int_{\gamma_\lambda} \frac{d\lambda}{2\pi i} \frac{e^{\tilde{\eta}_j(\lambda \tilde{a}_j + \tilde{\theta})}}{\xi - \lambda} \] (3.26)

We can now perform the Fourier transform of the above operators: we will use the convention that the sign of the transform in x, y or w, z is the opposite.

\[
f(x) = \int_{\mathbb{R}} e^{-i\xi x} \hat{f}(\xi) \frac{d\xi}{2\pi} \quad h(z) = \int_{\mathbb{R}} e^{iz\xi} \hat{h}(\xi) \frac{d\xi}{2\pi} \] (3.27)

**Lemma 3.5. The Fourier representation of the previous operators is the following**

\[
\mathcal{F}(\tilde{\Pi} A_i \pi)(\xi, \zeta) = \sum_{j} \frac{2^k}{\Pi_1^2} \frac{d\xi}{2\pi} e^{\xi(i\tilde{a}_j + \tilde{\theta})} \left[ \int_{\gamma_\lambda} \frac{d\lambda}{2\pi i} \frac{e^{\tilde{\eta}_j(\lambda \tilde{a}_j + \tilde{\theta})}}{\xi - \lambda} \right] \] (3.28)

\[
\mathcal{F}(\pi A_i \tilde{\Pi})(\xi, \zeta) = \sum_{j} \frac{2^k}{\Pi_1^2} \frac{d\xi}{2\pi} e^{\xi(i\tilde{a}_j - \tilde{\theta})} \left[ \int_{\gamma_\lambda} \frac{d\lambda}{2\pi i} \frac{e^{-\tilde{\eta}_j(\lambda \tilde{a}_j + \tilde{\theta})}}{\xi - \lambda} \right] \] (3.29)

\[
\mathcal{F}(\tilde{\Pi} K_{\lambda_i}^{(\tau, -\tau)} \tilde{\Pi})(\xi, \zeta) = \sum_{j,k} \frac{2^k}{\Pi_1^2} \frac{d\xi}{2\pi} e^{\xi(i\tilde{a}_j - \tilde{\theta})} \int_{\gamma_\lambda} \frac{d\lambda}{2\pi i} \int_{\gamma_\lambda} \frac{d\mu}{2\pi i} \frac{e^{\tilde{\eta}_j(\lambda \tilde{a}_j + \tilde{\theta})}}{\xi - \mu} \] (3.30)

\[
\mathcal{F}(\pi K_{\lambda_i} \pi)(\xi, \zeta) = \frac{1}{\Pi_1^2} \frac{d\xi}{2\pi} e^{\xi(i\tilde{\theta})} \int_{\gamma_\lambda} \frac{d\lambda}{2\pi i} \int_{\gamma_\lambda} \frac{d\mu}{2\pi i} \frac{e^{\theta(\lambda \tilde{\theta} - \tilde{\theta})(\lambda \tilde{\theta} - \tilde{\theta})}}{\xi - \mu} \] (3.31)

**All these kernels act on** $L^2(i\mathbb{R})$.

With the convention that $\rho, \xi, \xi \in \gamma_R$ and $\lambda \in \gamma_L$, $\mu \in \gamma_L$, we have the following result.
Lemma 3.6. The operators in lemma 3.5 can be represented as the composition of several operators:

\[ \mathcal{F}(\pi K\lambda_1\pi)(\xi, \zeta) = A(\xi, \mu)C(\mu, \lambda)B(\lambda, \zeta) \]  
(3.32)

\[ A(\xi, \mu) := \frac{e^{i(\mu - \xi)\sigma - \frac{\mu}{2}}}{2i\pi(\mu - \xi)}, \quad C(\mu, \lambda) := \frac{e^{i\pi(\lambda - \xi)}}{2i\pi(\lambda - \xi)} \]  
(3.33)

\[ F(\pi K\lambda_1\pi)(\xi, \zeta) = A_1(\xi, \mu)C_1(\mu, \lambda)B_1(\lambda, \zeta) \]  
(3.34)

\[ A_j(\xi, \mu) := \sum_j (-1)^j e^{i(\xi - \mu)\hat{\sigma}_j - 2i\pi\mu^2}, \quad B_k(\lambda, \zeta) := \sum_k (-1)^k e^{i(\lambda - \zeta)\hat{\sigma}_k + 2i\pi\lambda^2} \]  
(3.35)

Proposition 3.7. The following identity of determinants holds

\[ \det \left( \mathbb{I} - \begin{bmatrix} ACB & -Q_L & \tilde{H}_k + ACB_k \\ -H_j Q_R + A_j CB & \mathbb{I} \end{bmatrix} \right) = \det \left[ \begin{bmatrix} \mathbb{I} & B & B_k \\ \mathbb{I} & B & B_k \\ 0 & \tilde{H}_k & \mathbb{I} \end{bmatrix} \right] \]  
(3.45)

where by the \( \mathbb{I}_{X_j} \) we denote the identity operator on \( L^2(\mathbb{R}^j, \mathbb{C}) \) and the further subscript distinguishes orthogonal copies of the same space.
Proof. We start by noticing that all operators introduced in lemma 3.6 are Hilbert–Schmidt. Since a product of two such operators is a trace class operator, the first two determinants and the last one are ordinary Fredholm determinants; the third determinant should be understood as a Carleman regularized \( \det_2 \) determinant. However, since the operator whose determinant is computed is diagonal-free, the formal definition coincides with the usual Fredholm determinant. The first identity is seen by multiplying on the left by a proper lower triangular matrix, while the second one is given by multiplying the matrix

\[
\begin{pmatrix}
\text{Id}_{L_1} & 0 & 0 & 0 & 0 & \tilde{H}_k \\
0 & \text{Id}_{R_1} & 0 & 0 & 0 & Q_R \\
0 & 0 & \text{Id}_{L_2} & 0 & C & 0 \\
0 & 0 & 0 & \text{Id}_{R_2} & 0 & B \\
0 & 0 & 0 & 0 & \text{Id}_{R_3} & 0 \\
-\tilde{Q}_L & 0 & -A & 0 & 0 & \text{Id}_{R_3} \\
-\tilde{H}_j & -A_j & 0 & 0 & 0 & \text{Id}_{R_3}
\end{pmatrix}
\] (3.46)

on the left by

\[
\begin{pmatrix}
\text{Id}_{L_1} & 0 & 0 & 0 & 0 & 0 \\
0 & \text{Id}_{R_1} & 0 & 0 & 0 & 0 \\
0 & 0 & \text{Id}_{L_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \text{Id}_{R_2} & 0 & 0 \\
0 & 0 & 0 & 0 & \text{Id}_{R_3} & 0 \\
Q_L & 0 & A & 0 & 0 & \text{Id}_{R_3} \\
H_j & A_j & 0 & 0 & 0 & \text{Id}_{R_3}
\end{pmatrix}
\] (3.47)

where \( 0_j \) is a copy of the imaginary axis \( iR \). We now multiply the two matrices in reverse order, as we know that \( \det(\mathcal{N} \mathcal{M}) = \det(\mathcal{M} \mathcal{N}) \). In conclusion, we obtain the operator

\[
\begin{pmatrix}
\text{Id}_{L_1} & \tilde{H}_j & \tilde{H}_j A_j & 0 \\
Q_R Q_L & \text{Id}_{R_1} & Q_R A & 0 \\
0 & 0 & \text{Id}_{L_2} & C \\
B Q_L & B_k H_j & B A + B_k A_j & \text{Id}_{R_3}
\end{pmatrix}
\] (3.48)

where we have removed the trivial part involving the three copies of \( iR \). □

Collecting all the results found so far, we have

**Theorem 3.8.** The gap probability of the tacnode process at single time is

\[
\det(\text{Id} - \Pi \hat{\Pi}) = F_2(\hat{\sigma})^{-1} \cdot \det(\text{Id} - \mathcal{M})
\] (3.49)

where

\[
\mathcal{M} :=
\begin{pmatrix}
0_{L_1} & \tilde{H}_j & \tilde{H}_j A_j & 0 \\
Q_R Q_L & 0_{R_1} & Q_R A & 0 \\
0 & 0 & 0_{L_2} & C \\
-B Q_L & -B_k H_j & -(B A + B_k A_j) & 0_{R_3}
\end{pmatrix}
\] (3.50)

with

\[
Q_R Q_L(\lambda, \mu) = \frac{e^{\frac{\lambda^3}{2(\lambda - \mu)^2}}}{2\pi i(\lambda - \mu)} , \quad B Q_L(\lambda, \mu) = \frac{e^{\frac{\lambda^3}{2(\lambda - \mu)^2}}}{2\pi i(\lambda - \mu)}
\] (3.51)

\[
Q_R A(\lambda, \mu) = \frac{e^{\frac{\mu^3}{2(\lambda - \mu)^2}}}{2\pi i(\lambda - \mu)} , \quad C(\mu, \lambda) = \frac{e^{\frac{\mu^3}{2(\lambda - \mu)^2}}}{2\pi i(\lambda - \mu)}
\] (3.52)
\[
H_k H_j (\mu, \lambda) = \frac{\sum_{j=1}^{2K} (-1)^j h_j^{-1}(\mu) h_j(\lambda)}{2i\pi (\mu - \lambda)} 
\] (3.53)

\[
H_k A_j (\mu_1, \mu_2) = \frac{\sum_{j=1}^{2K} (-1)^j h_j^{-1}(\mu_1) g_j(\mu_2)}{2i\pi (\mu_1 - \mu_2)} 
\] (3.54)

\[
B_k H_j (\lambda_2, \lambda_1) = \frac{\sum_{j=1}^{2K} (-1)^j g_j^{-1}(\lambda_2) h_j(\lambda_1)}{2i\pi (\lambda_2 - \lambda_1)} 
\] (3.55)

\[
(B A + B_k A_j) (\lambda, \mu) = \frac{e^{\frac{i\pi}{2} \lambda - \frac{1}{2} (\mu - \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} + \sum_j (-1)^j g_j^{-1}(\lambda) g_j(\mu) \frac{2i\pi (\lambda - \mu)}{2i\pi (\lambda - \mu)} 
\] (3.56)

with

\[
h_j(\xi) := e^{\xi/12 + \frac{2i}{3\pi} (\xi - \tilde{\zeta}_j + \tilde{\sigma})}, \quad g_j(\xi) := e^{-\xi/12 + \frac{2i}{3\pi} (\xi - \tilde{\zeta}_j).} 
\] (3.57)

**Proof.** The first three kernels and the kernel \( B A \) follow from easy computations, by completing the contours to large semicircles.

\[
Q_R Q_L (\lambda, \mu) = \int_{\mathbb{R}} d\zeta \frac{e^{\frac{i\pi}{2} \zeta - \frac{1}{2} (\mu - \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} = \frac{e^{\frac{i\pi}{2} \mu - \frac{1}{2} (\mu - \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} 
\] (3.58)

\[
B Q_L (\lambda, \mu) = \int_{\mathbb{R}} d\zeta \frac{e^{\frac{i\pi}{2} \zeta - \frac{1}{2} (\mu + \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} = \frac{e^{\frac{i\pi}{2} \mu - \frac{1}{2} (\mu + \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} 
\] (3.59)

\[
Q_R A (\lambda, \mu) = \int_{\mathbb{R}} d\zeta \frac{e^{\frac{i\pi}{2} \zeta - \frac{1}{2} (\mu - \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} = \frac{e^{\frac{i\pi}{2} \mu - \frac{1}{2} (\mu - \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} 
\] (3.60)

\[
B A (\lambda, \mu) = \int_{\mathbb{R}} d\zeta \frac{e^{\frac{i\pi}{2} \zeta - \frac{1}{2} (\mu + \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} = \frac{e^{\frac{i\pi}{2} \mu - \frac{1}{2} (\mu + \lambda) \tilde{\sigma}}}{2i\pi (\lambda - \mu)} 
\] (3.61)

Next, we recall that the endpoints are ordered \( \tilde{a}_j < \tilde{a}_{j+1} \), so that we can pick up residues accordingly to the sign of \( \tilde{a}_j - \tilde{a}_k \) \( (j, k = 1, \ldots, 2K) \).

\[
\tilde{H}_k H_j (\mu, \lambda) = \sum_{j,k} (-1)^{j+k+1} \int_{\mathbb{R}} d\zeta \frac{e^{(\tilde{a}_j - \tilde{a}_k) \xi - \frac{1}{2} (\mu - \lambda) \tilde{\sigma}}}{2i\pi (\mu - \zeta)} \frac{e^{-\frac{1}{2} (\mu - \lambda) \tilde{\sigma} + \frac{1}{2i\pi} (\lambda^2 - \mu^2)}}{2i\pi (\mu - \lambda)} 
\]

\[
= \sum_{j<k} (-1)^{j+k} \frac{e^{(\mu - \lambda) \tilde{a}_j + (\mu - \lambda) \tilde{\sigma} + \frac{1}{2i\pi} (\lambda^2 - \mu^2)}}{2i\pi (\mu - \lambda)} + \sum_{j<k} (-1)^{j+k} \frac{e^{(\mu - \lambda) \tilde{a}_j + (\mu - \lambda) \tilde{\sigma} + \frac{1}{2i\pi} (\lambda^2 - \mu^2)}}{2i\pi (\mu - \lambda)} 
\]

\[
+ \sum_{j=1}^{2K} \frac{e^{(\mu - \lambda) \tilde{a}_j + (\mu - \lambda) \tilde{\sigma} + \frac{1}{2i\pi} (\lambda^2 - \mu^2)}}{2i\pi (\mu - \lambda)} 
\] (3.62)

Thanks to some cancellations, we are left with

\[
\tilde{H}_k H_j (\mu, \lambda) = \sum_{j=1}^{2K} (-1)^{j+1} \frac{e^{(\mu - \lambda) \tilde{a}_j + (\mu - \lambda) \tilde{\sigma} + \frac{1}{2i\pi} (\lambda^2 - \mu^2)}}{2i\pi (\mu - \lambda)} 
\] (3.63)
Similarly,

\[
B_k A_j(\lambda, \mu) := \int_{\mathbb{R}} \frac{d\zeta}{2i\pi} \sum_{k,j} \frac{(-1)^{k+j} e^{i\zeta(\lambda \zeta + \mu \zeta)}}{2i\pi (\lambda - \zeta)} \frac{\lambda^2 \mu^2}{(\zeta - \mu)}
\]

\[
= \sum_{j=1}^{2K} \frac{(-1)^j e^{i\zeta \mu^2}}{2i\pi (\lambda - \mu)} \frac{1}{\lambda^2} \lambda^2 (\lambda - \mu) \lambda_j.
\]

(3.64)

In the next computation, we set \(\lambda_1, \lambda_2 \in \gamma_R\):

\[
B_k H_j(\lambda_2, \lambda_1) = \int_{\mathbb{R}} \frac{d\zeta}{2i\pi} \sum_{k,j} \frac{(-1)^{k+j} e^{i\zeta(\lambda_2 \zeta + \mu \zeta)}}{2i\pi (\lambda_2 - \zeta)} \frac{\lambda_2^2 \mu^2}{(\zeta - \lambda_1)}
\]

\[
= \sum_{j \leq k} \left( -\frac{(-1)^j e^{i\zeta \mu^2}}{2i\pi (\lambda_2 - \lambda_1)} \right) \left( e^{(\lambda_2 - \lambda_1) \lambda_j} - e^{(\lambda_2 - \lambda_1) \mu_j} \right);
\]

(3.65)

the first term contributes only with the terms with even \(j\) (with a positive sign), the second only those with odd \(k\) with a negative sign so that

\[
B_k H_j(\lambda_2, \lambda_1) = \sum_{j=1}^{2K} \frac{(-1)^j e^{i\zeta \mu^2}}{2i\pi (\lambda_2 - \lambda_1)} \frac{1}{\lambda^2} \lambda^2 (\lambda - \mu) \lambda_j.
\]

(3.66)

Note that the kernel is regular at \(\lambda_1 = \lambda_2\) because the sum vanishes.

In a similar way

\[
\tilde{H}_k A_j(\mu_1, \mu_2) = \int_{\mathbb{R}} \frac{d\zeta}{2i\pi} \sum_{k,j} \frac{(-1)^{k+j+1} e^{(\mu_1 - \zeta) \lambda_2 + \mu_1 \zeta}}{2i\pi (\mu_1 - \zeta)} \frac{\lambda_2 \mu^2}{(\zeta - \mu_2)}
\]

\[
= \sum_{j \leq k} \left( -\frac{(-1)^j e^{i\zeta \mu^2}}{2i\pi (\mu_1 - \mu_2)} \right) \left( e^{(\mu_1 - \mu_2) \lambda_j} - e^{(\mu_1 - \mu_2) \mu_j} \right);
\]

(3.67)

\[
= \sum_{j=1}^{2K} \frac{(-1)^{j+1} e^{i\zeta \mu^2}}{2i\pi (\mu_1 - \mu_2)} \frac{1}{\lambda^2} \lambda^2 (\mu - \mu) \lambda_j.
\]

(3.68)

Now we recall that any operator acting on a Hilbert space of the type \(H = H_1 \oplus H_2 \oplus H_3 \oplus H_4\) can be decomposed as a \(4 \times 4\) matrix of operators with \((i, j)\)-entry given by an operator \(H_j \rightarrow H_i\). In conclusion, the kernel can be written as an integrable kernel in the sense of IJKS [15]:

\[
\mathcal{M}(\xi, \zeta) = \frac{f(\xi)^T \cdot g(\zeta)}{2\pi i (\lambda - \xi)}
\]

(3.71)

with

\[
f(\xi) = \frac{1}{2\pi i} \left[ \begin{array}{c}
-\text{e}^{-\xi/2} \chi_{x_2} \\
-\text{e}^{(\xi/2 - \xi)} \chi_{x_2} - \text{e}^{\xi/2} \chi_{x_1} \\
g_{-1}^{-1}(\xi) \chi_{x_2} - h_{-1}^{-1}(\xi) \chi_{x_1} \\
\vdots \\
-(-1)^{2K} g_{-2K}^{-1}(\xi) \chi_{x_2} + (\text{e}^{-\xi/2} h_{2K}^{-1}(\xi) \chi_{x_1})
\end{array} \right]
\]

(3.72)
\[ g(\zeta) = \begin{bmatrix} e^{-\frac{\zeta}{12}}X_{\xi 2} \\ e^{-\frac{\zeta}{12}}X_{\xi 1} + e^{-\frac{\zeta}{4\pi}}X_{\gamma 2} \\ \vdots \\ g_{\xi k}(\zeta)X_{\xi 2} + h_{\xi k}(\zeta)X_{\xi 1} \end{bmatrix} \tag{3.73} \]

and \( h_j, g_j \) as in (3.57).

Through Jacobi’s formula for variations of determinant, the study of the Fredholm determinant of the operator with kernel \( \mathcal{M} \) is directly linked to the invertibility of the operator \( \text{Id} - \mathcal{M} \). On the other hand, the theory of the integrable kernel in the sense of IIKS guarantees that the latter operator is invertible if and only if there exists a solution to the following RH problem. We refer to [13] for a detailed explanation.

**Proposition 3.9.** The Fredholm determinant \( \det(\text{Id} - \mathcal{M}) \) is linked through IIKS correspondence to the following \((2K + 2) \times (2K + 2)\) RH problem

\[
\Gamma_+(\lambda) = \Gamma_+(\lambda) - \frac{2\pi i}{\lambda} \mathcal{M} \to \infty, \quad \lambda \to \infty \tag{3.74}
\]

\[
J(\lambda) := I - 2\pi \mathcal{M}(\lambda) \mathcal{F}(\lambda)^T \tag{3.75}
\]

with

\[
\Theta_\gamma(\lambda) = \frac{\lambda^3}{3} - \bar{\sigma} \lambda, \quad \Theta_{\gamma,\tau}(\lambda) = \frac{\lambda^3}{6} - 2^{-\frac{5}{2}} \tau \lambda^2 - 2^{-\frac{1}{2}} (\alpha_i + \sigma) \lambda. \tag{3.76}
\]

**Proof.** It is simply a matter of straightforward calculations: starting from the formula \( J(\lambda) := I - 2\pi \mathcal{M}(\lambda) \mathcal{F}(\lambda)^T \) and writing explicitly the endpoints \( \bar{a}_i \) as functions of the original endpoints \( a_i \) (the change of variables is defined in proposition 3.1), we get the jump matrix as in (3.75), but with two distinct copies of \( \gamma_R \) and \( \gamma_L \), as specified in (3.72)–(3.73). On the other hand, it is easy to show that the jumps on, say, \( \gamma_R \) and \( \gamma_L \) commute, hence we can identify the two contours.

In particular, let us consider the simplest case where \( \mathcal{I} = [a, b] \) (\( K = 1 \), and the RH problem is \( 4 \times 4 \) with a jump matrix

\[
J(\lambda) = \begin{bmatrix} 1 & e^{\Theta_\gamma}X_{\xi 1} & e^{\Theta_{\gamma,\tau}}X_{\xi 2} & e^{\Theta_{\gamma,\tau}}X_{\gamma 2} \\ e^{\Theta_\gamma}X_{\xi 1} & 1 & e^{\Theta_{\gamma,\tau}}X_{\xi 2} & e^{\Theta_{\gamma,\tau}}X_{\gamma 2} \\ e^{\Theta_{\gamma,\tau}}X_{\xi 1} & e^{\Theta_{\gamma,\tau}}X_{\xi 2} & 1 & 0 \\ e^{\Theta_{\gamma,\tau}}X_{\gamma 2} & e^{\Theta_{\gamma,\tau}}X_{\xi 2} & 0 & 1 \end{bmatrix} \tag{3.77}
\]

where

\[
\Theta_\gamma(\lambda) = \frac{\lambda^3}{3} - \bar{\sigma} \lambda, \quad \Theta_{\gamma,\tau}(\lambda) = \frac{\lambda^3}{6} - 2^{-\frac{5}{2}} \tau \lambda^2 - 2^{-\frac{1}{2}} (\alpha_i + \sigma) \lambda. \tag{3.78}
\]

(we have renamed the contours \( R_1, R_2, R_3 \) and \( L_1, L_2, L_3 \)).
4. Proof of theorem 2.2

From now on, we are assuming $\tau > 0$. For $\tau \leq 0$ the calculations follow the same guidelines as below.

The phase functions $\Theta_{-\sigma, -b}(\lambda)$ and $\Theta_{-\tau, a}(\lambda)$ (appearing in the entries of the $2 \times 2$ off-diagonal blocks of the jump matrix (3.77)) have inflection points with zero derivative when the discriminant of the derivative vanishes, which occurs when

\[ a_{\text{crit}} + \sigma + \tau^2 = 0 \]  \hspace{1cm} (4.1)

\[ b_{\text{crit}} - \sigma - \tau^2 = 0 \]  \hspace{1cm} (4.2)

with critical values $\Theta_{-\sigma, -b}(\lambda) = 2^{1/3}\tau$ and $\Theta_{-\tau, a}(\lambda) = -2^{1/3}\tau$. The neighbourhood of the discriminant is parametrizable as follows

\[ a = a(t) = -\sigma - \tau^2 + t \]  \hspace{1cm} (4.3)

\[ b = b(s) = \sigma + \tau^2 - s. \]  \hspace{1cm} (4.4)

Thus, from (3.78) and substituting (4.3)–(4.4), we have the following expressions

\[ \Theta_{-\sigma, -b}(\lambda) = \frac{\xi^3}{3} - s\xi_s + \frac{\tau^3}{3} - s\tau, \quad \xi_s := \frac{\lambda - 2^{1/3}\tau}{2^{1/3}} \]  \hspace{1cm} (4.5)

\[ \Theta_{-\tau, a}(\lambda) = \frac{\xi^3}{3} - t\xi_t - \frac{\tau^3}{3} + t\tau, \quad \xi_t := \frac{\lambda + 2^{1/3}\tau}{2^{1/3}}. \]  \hspace{1cm} (4.6)

On the other hand, the phase $\Theta_{\tilde{\sigma}}$ in the entries (1, 2) and (2, 1) of (3.77) has a critical point at $\pm \sqrt{\sigma} = \pm \sqrt{2^{1/3}\sigma}$. 

**Figure 4.** The contour setting in the asymptotic limit as $\sigma \to +\infty$. 

We will now focus exclusively on the single-interval case and will apply a steepest descent method in order to prove the factorization of the gap probability of the tacnode process into two gap probabilities of the Airy process. The starting point is the $4 \times 4$ RH problem (3.77) with the contour configuration as in figure 4 or figure 6, depending on the scaling regime we are considering.
 Preliminary step.} We conjugate the matrix $\Gamma$ by the constant (with respect to $\lambda$) diagonal matrix

$$D := \text{diag}(1, 1, -K(t), -K(s))$$

(4.7)

where $K(u) := \frac{u}{\sqrt{2}} - u\tau$. As a result, the jump matrices (3.77) are also similarly conjugated and this has the effect of replacing the phases $\Theta_{\pm\tau, \mp a}$ and $\Theta_{\pm\tau, \mp b}$ by $\Theta_{\pm\tau, \mp a}/Theta1$ and $\Theta_{\pm\tau, \mp b}/Theta1$ respectively, so that their critical value is zero.

We denote by a hat the new matrix and respective jump

$$\hat{\Gamma} := e^{-D} \Gamma e^D, \quad \hat{J} := e^{-D} J e^D.$$  

(4.8)

Thus, the resulting jump $\hat{J}$ has the following form:

$$\begin{bmatrix}
1 & e^{-\Theta_{\pm \tau, \mp a}} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \text{ on } L_1,$$

$$\begin{bmatrix}
1 & 0 & e^{-\Theta_{\pm \tau, \mp a}} & 0 \\
0 & 1 & 0 & e^{-\Theta(\xi, s)} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \text{ on } L_2,$$

$$\begin{bmatrix}
1 & 0 & 0 & e^{-\Theta(\xi, s)} \\
0 & 1 & 0 & 0 \\
0 & e^{-\Theta_{\pm \tau, \mp a}} & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \text{ on } L_3,$$

on $L_1$, $L_2$, $L_3$, respectively, so that their critical value is zero.

We conjugate the matrix $P_L$ by $\hat{\Gamma}$ and respective jump $\hat{J}$:

$$P_{\sigma, R} := e^{-D} P_{\sigma, R} e^D.$$  

(4.9)

$$\begin{bmatrix}
1 & e^{\Theta_{\pm \tau, \mp a}} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \text{ on } R_1,$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \text{ on } R_2,$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \text{ on } R_3.$$  

(4.10)

where

$$\Theta(\xi_{\pm}, u) := \frac{\xi_{\pm}}{\sqrt{2}} - \xi_{\pm} u, \quad \xi_{\pm} := \frac{\lambda \pm \sqrt{2} \tau}{\sqrt{2}}$$

(4.12)

$$\Theta_{\pm \tau, \mp b}(u, s) := \frac{\lambda^2}{6} + \frac{\tau^2}{2} \lambda - \frac{2\sqrt{2}}{\sqrt{2}} \lambda s$$

(4.13)

$$\Theta_{\mp \tau, \pm a}(u, t) := \frac{\lambda^2}{6} - \frac{\tau^2}{2} \lambda - \frac{2\sqrt{2}}{\sqrt{2}} \lambda + \frac{t \lambda}{\sqrt{2}}$$

(4.14)

We choose the contours according to the following configuration (see figure 4):

- $L_2$ and $R_2$ are centred around the critical point $P_R := 2^+ \tau$
- $L_3$ and $R_3$ are centred around the critical point $P_L := -2^+ \tau$
- $L_1$ passes through the critical point $P_{\sigma, L} := -\sqrt{2}$ and $R_1$ passes through the critical point $P_{\sigma, R} := \sqrt{2}$; these points are thought of as very far from the origin, in the limit as $\sigma \gg 1$.

Remark 4.1. All the left jumps commute with themselves and similarly so do all the right jumps. Moreover, the jump matrices $L_2$ and $R_3$ commute.

The proof now proceeds along the following scheme (as $\sigma \to +\infty$):
1. the matrices $L_1$ and $R_1$ are exponentially close to the identity in every $L^p$ norm (lemma 4.2);
2. regarding the matrices $L_2$ and $R_3$, the entries of the form $\pm(\Theta_{\pm \tau, \mp a}/Theta1 - K(t))$ are exponentially small in every $L^p$ norm; the same behaviour will appear for the entries of the type $\pm(\Theta_{\pm \tau, \mp b} - K(s))$ in the matrices $L_3$ and $R_1$ (lemma 4.3);
3. for the remaining entries in the jumps $L_{2,3}$ and $R_{2,3}$ we will explicitly and exactly solve a (model) RH problem which will approximate the problem at hand.
4.1. Estimates on the phases

The proof of the first two points relies on the following lemmas.

Lemma 4.2. The jumps on the curves $L_1$ and $R_1$ are exponentially suppressed in any $L^p$ norm, $1 \leq p \leq \infty$, as $\sigma \to +\infty$.

Proof. A parametrization for the curves $L_1$ and $R_1$ is the following $\lambda = \pm \frac{2}{3} \sqrt{\sigma} + u \left( \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right)$. Therefore, we have (for both signs)

$$\Re \left[ \Theta_{b; R_1} \right] = -\frac{4}{3} \sigma^{3/2} - \frac{\sqrt{\sigma}}{2^{2/3}} u^2 - \frac{u^3}{3} \quad (4.15)$$

which implies

$$\left\| e^{\Theta_b} \right\|_{L^p(R_1)} = 2 \int_0^{\infty} e^{-\frac{4}{3} \sigma^{3/2}} \left( e^{\frac{2}{3} \sqrt{\sigma} + u \left( \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right)} \right) du \leq C e^{-\frac{2}{3} \sigma^{3/2}} \quad (4.16)$$

The same result holds for the contour $L_2$.

Lemma 4.3. Given $0 < K_1 < 1$ fixed and $s < K_1 (\sigma + \tau^2)$, then the function $e^{\Theta(-\tau, b) + K(s)}$ tends to zero exponentially fast in any $L^p(R_3)$ norm ($1 \leq p \leq \infty$) as $\sigma \to +\infty$:

$$\left\| e^{\Theta(-\tau, b) + K(s)} \right\|_{L^p(R_3)} \leq C e^{-2\tau (1 - K_1) \sigma} \quad (4.17)$$

Similarly, the function $e^{-\Theta(-\tau, b) - K(s)}$ is exponentially small in any $L^p(L_3)$ norm ($1 \leq p \leq \infty$). Moreover, the function $e^{\Theta_0 - K(s)}$ and $e^{\Theta_0 - K(s)}$ are exponentially small in any $L^p(L_2)$ and $L^p(R_2)$ norms, respectively ($1 \leq p \leq \infty$).

Proof. A parametrization of $R_3$ is $\lambda = \sqrt{2} \tau + u \left[ \frac{1}{2} \pm \frac{2}{\sqrt{3}} i \right]$, $u \geq 0$. This yields

$$\Re \left[ \Theta(-\tau, b) + K(s) \right] = -\frac{u^3}{6} - \frac{\delta u}{2^{5/3}} - 2\tau \sigma - 2\tau^3 + 2\tau \delta \quad (4.18)$$

where we set $s = 2\sigma + 2\tau^2 - \delta$, $0 < \delta < \sigma + \tau^2$, and this is valid for both branches of the curve.

Regarding the $L^p(R_3)$ norms, we have that $\left| e^{\Theta(-\tau, b) + K(s)} \right| = e^{\Re \left[ \Theta(-\tau, b) + K(s) \right]}$, therefore,

$$\left\| e^{\Theta(-\tau, b) + K(s)} \right\|_{L^p(R_3)} \leq 2Ce^{-2\tau (1 - K_1) \sigma} \left[ \int_0^{1} e^{-2\tau \sigma + \tau^3 - \frac{\delta u}{2^{5/3}}} du + \int_{1}^{\infty} e^{-2\tau \sigma - \delta u} du \right] \leq C e^{-2\tau (1 - K_1) \sigma} \quad (4.19)$$

$$\left\| e^{\Theta(-\tau, b) + K(s)} \right\|_{L^\infty(R_3)} = e^{-2\tau (\sigma + \tau^2 - \delta)} \leq C e^{-2\tau (1 - K_1) \sigma} \quad (4.20)$$

given that $s < K_1 (\sigma + \tau^2)$ with $0 < K_1 < 1$.

All the other cases are completely analogous. □
4.2. Global parametrix. The model problem

In this subsection we will use the Hastings–McLeod matrix (see [11], but in the normalization of [4]) as parametrix for the RH problem related to $\hat{\Gamma}$.

Let us consider the following model problem:

\[
\begin{align*}
\Omega_+(\lambda) &= \Omega_-(\lambda) J_R(\lambda) \quad \text{on } L_2 \cup R_2 \\
\Omega_+(\lambda) &= \Omega_-(\lambda) J_L(\lambda) \quad \text{on } L_3 \cup R_3 \\
\Omega(\lambda) &= I + O(\lambda^{-1}) \quad \text{at } \infty
\end{align*}
\]  

with jumps (see figure 5)

\[
J_R := \begin{bmatrix}
1 & 0 & 0 & e^{-\Theta_+} & 0 \\
0 & 0 & 1 & 0 & 0 \\
e^{-\Theta_+} & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

(4.22)

\[
J_L := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & e^{-\Theta_-} & 0 \\
e^{-\Theta_-} & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

(4.23)

and we recall $\xi_{\pm} := \frac{\lambda \pm \sqrt{2}}{\sqrt{2}}$ as defined in (4.12).

This model problem can be solved in exact form by considering two solutions of the Hastings–McLeod Painlevé II RH problem, namely

\[
\Phi_{HM}(s) \quad \text{and} \quad \tilde{\Phi}_{HM}(t) := \sigma_3 \sigma_2 \Phi_{HM}(t) \sigma_2 \sigma_3,
\]

(4.24)

where $\sigma_2$, $\sigma_3$ are Pauli matrices and $\Phi_{HM}(u)$ is the solution to a $2 \times 2$ RH problem with jump matrix

\[
\begin{bmatrix}
1 & e^{\Theta(\lambda, u)} \\
e^{-\Theta(\lambda, u)} & 1 \\
\end{bmatrix}, \quad \Theta(\lambda, u) = \frac{\lambda^3}{3} - u \lambda
\]

(4.25)

and behaviour at infinity normalized to the identity $2 \times 2$ matrix; as usual, $\gamma_R$ is a contour which extends to infinity along the rays arg$(\lambda) = \pm \frac{\pi}{3}$ and $\gamma_L = -\gamma_R$ (for more details see [4]).
The asymptotic behaviour of the functions (4.24) as $\xi \to \infty$ is
\[
\Phi(\xi_+, t) = I + \frac{1}{\xi_+} \begin{bmatrix} p(t) & q(t) \\ -q(t) & -p(t) \end{bmatrix} + O\left(\frac{1}{\xi_+^2}\right) \quad (4.26)
\]
and
\[
\hat{\Phi}(\xi_-, s) = \sigma_3 \sigma_2 \left[ I + \frac{1}{\xi_-} \begin{bmatrix} p(s) & q(s) \\ -q(s) & -p(s) \end{bmatrix} + O\left(\frac{1}{\xi_-^2}\right) \right] \sigma_2 \sigma_3
\]
\[
= I + \frac{1}{\xi_-} \begin{bmatrix} -p(s) & -q(s) \\ q(s) & p(s) \end{bmatrix} + O\left(\frac{1}{\xi_-^2}\right). \quad (4.27)
\]

The global parametrix, i.e. the exact solution of the model problem, is then easily verified to be given by
\[
\begin{bmatrix}
\Phi_{11}(\xi_-, s) & 0 & 0 & \Phi_{12}(\xi_-, s) \\
0 & \Phi_{11}(\xi_+, t) & \Phi_{12}(\xi_+, t) & 0 \\
0 & \Phi_{21}(\xi_+, t) & \Phi_{22}(\xi_+, t) & 0 \\
\Phi_{21}(\xi_-, s) & 0 & \Phi_{22}(\xi_-, s) & 0
\end{bmatrix}
\]
(4.28)

4.3. Approximation and error term for the matrix $\hat{\Gamma}$

The following relation holds
\[
\hat{\Gamma} = \mathcal{E} \cdot \Omega \quad (4.29)
\]
where $\mathcal{E}$ is the ‘error’ matrix. The goal is to show that the Riemann–Hilbert problem satisfied by the error matrix has a jump equal to a small perturbation of the identity matrix $I + O(\sigma^{-\infty})$, so that a standard small norm argument can be applied (see [14, section 5.1.3]).

Lemma 4.4. Given $s, t < K_1(\sigma + \tau^2)$ with $0 < K_1 < 1$, the error matrix $\mathcal{E} = \hat{\Gamma}(\lambda) \Omega^{-1}(\lambda)$ solves a RH problem with jumps on the contours as indicated in figure 4 and of the following orders
\[
\mathcal{E}_s(\lambda) = \mathcal{E}_-(\lambda) J_\mathcal{E}(\lambda) \quad \text{on } \Sigma
\]
\[
\mathcal{E}(\lambda) = I + O(\lambda^{-1}) \quad \text{as } \lambda \to \infty
\]
(4.30)

\[
J_\mathcal{E} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
O(\sigma^{-\infty}) \chi_{t_1} & 1 & 0 & 0 \\
O(\sigma^{-\infty}) \chi_{t_2} & 0 & 1 & 0 \\
-\sigma(\sigma^{-\infty}) \chi_{t_3} & 0 & 0 & 1
\end{bmatrix}
\]
(4.31)

and the $O$-symbols are valid in any $L^p$ norms ($1 \leq p \leq \infty$).

Proof. First of all, we notice that thanks to lemmas 4.2 and 4.3, all the extra phases that were not included in the model problem $\Omega$ behave like $O(\sigma^{-\infty})$ as $\sigma \to \infty$ in any $L^p$ norm. The jumps of the error problem are the remaining jumps appearing in the original $\hat{\Gamma}$-problem conjugated with the Hastings–McLeod solution $\Omega$, which is independent of $\sigma$:
\[
J_\mathcal{E} = \Omega^{-1} \hat{\Omega}
\]
\[
= \Omega^{-1} \begin{bmatrix}
e^{-\Theta_{s_1}} \chi_{t_1} & e^{-\Theta_{s_1} + K(t)} \chi_{t_2} & 0 \\
e^{-\Theta_{s_2}} \chi_{t_1} & 1 & 0 & 0 \\
e^{-\Theta_{s_3} - K(t)} \chi_{t_2} & 0 & 1 & 0 \\
e^{-\Theta_{s_3} - K(t)} \chi_{t_3} & 0 & 0 & 1
\end{bmatrix} \Omega
\]
\[
= \Omega^{-1} \left( I + O(\sigma^{-\infty}) \right) \Omega = I + O(\sigma^{-\infty}) \quad (4.32)
\]
since $\Omega$ and $\Omega^{-1}$ are uniformly bounded in $\sigma$. □
We recall that the small norm theorem says that uniformly on closed sets not containing the contours of the jumps
\[
\|E(\lambda) - I\| \leq C_{\text{dist}(\lambda, \Sigma)} \left( \|J_E - I\|_1 + \frac{\|J_E - I\|_2^2}{1 - \|J_E - I\|_\infty} \right)
\]  
(4.33)
where \( \Sigma \) is the collection of all contours. Thanks to lemma 4.4, we conclude
\[
\|E(\lambda) - I\| \leq C_{\text{dist}(\lambda, \Sigma)} e^{-K_4}
\]  
(4.34)
for some positive constants \( C \) and \( K \). The error matrix \( E \) is then found as the solution to the integral equation
\[
E(\lambda) = I + \int_{\Sigma} \frac{E(w) (J_E(\lambda) - I)}{2\pi i (w - \lambda)} d\lambda
\]  
(4.35)
and can be obtained by iterations
\[
E^{(0)}(\lambda) = I, \quad E^{(k+1)}(\lambda) = I + \int_{\Sigma} \frac{E^{(k)}(w) (J_E(\lambda) - I)}{2\pi i (w - \lambda)} d\lambda
\]  
(4.36)
and, thanks to lemma 4.4 we have
\[
E(\lambda) = I + \left( \frac{1}{\text{dist}(\lambda, \Sigma)} \right) O \left( e^{-K_4} \right). 
\]  
(4.37)

4.4. Conclusion of the proof of theorem 2.2

Using known results about Fredholm determinants of IIKS integrable kernels (see [3, section 5] and [4, section 2], in particular theorem 2.1) and adapting them to the case at hand we can state the following theorem.

**Theorem 4.5.** The Fredholm determinant \( \det(\text{Id} - \hat{\Pi} \hat{H} \hat{\Pi}) \) of (3.16) satisfies the following differential equations
\[
\partial_\rho \ln \det(\text{Id} - \hat{\Pi} \hat{H} \hat{\Pi}) = \omega(\partial_\rho) = \int_{\Sigma} \text{Tr} \left( \Gamma_{-1}^{-1}(\lambda) \Gamma^{-1}_-(\lambda) \partial_\rho \Xi(\lambda) \right) \frac{d\lambda}{2\pi i}.
\]  
(4.38)
More specifically,
\[
\partial_\rho \ln \det(\text{Id} - \hat{\Pi} \hat{H} \hat{\Pi}) = -\text{res}_{s=\infty} \text{Tr} \left( \Gamma^{-1}_{-1} \Gamma^{-1}_- \partial_\rho \right) = \frac{1}{\sqrt{2\lambda}} \Gamma_{1; (4.4)}
\]  
(4.39)
\[
\partial_t \ln \det(\text{Id} - \hat{\Pi} \hat{H} \hat{\Pi}) = -\text{res}_{s=\infty} \text{Tr} \left( \Gamma^{-1}_{-1} \Gamma^{-1}_- \partial_t \right) = -\frac{1}{\sqrt{2\lambda}} \Gamma_{1; (3,3)}
\]  
(4.40)
where \( \Gamma_1 := \lim_{\lambda \to \pm\infty} \lambda (\Gamma(\lambda) - I) \).

**Proof.** We notice that the original RHP for \( \Gamma \) (see (3.77)) is equivalent to a RH problem with constant jumps up to a conjugation with the matrix
\[
T = \text{diag} \left[ \frac{\kappa}{4}, -\Theta_{\Theta}, -\Theta_{\Theta} a, \frac{\kappa}{4}, -\Theta_{\Theta} b, \frac{\kappa}{4} \right]
\]  
(4.41)
\[
\kappa = \Theta_{\Theta} + \Theta_{\Theta} a + \Theta_{\Theta} b.
\]  
(4.42)
Thus, the matrix \( \Gamma := \Gamma_1 e^T \) solves a RHP with constant jumps and it is (sectionally) a solution to a polynomial ODE.
Applying the theorem [4, theorem 2.1], we have the equality (5.31). Moreover, using the Jimbo–Miwa–Ueno residue formula, we can explicitly calculate
\[ \partial_t \ln \det(\text{Id} - \hat{H} \hat{H}) = -\text{res}_{s, \to \infty} \text{Tr} (\Gamma^{-1} \Gamma' \partial_t T) \] (4.43)
\[ \partial_s \ln \det(\text{Id} - \hat{H} \hat{H}) = -\text{res}_{s, \to \infty} \text{Tr} (\Gamma^{-1} \Gamma' \partial_s T) . \] (4.44)

Taking into account the asymptotic behaviour at \( \infty \) of the matrix \( \Gamma \) we have
\[ \text{Tr} [\Gamma^{-1} \Gamma' \partial_t T] = \text{Tr} \left[ \left( -\frac{\Gamma_1}{\lambda^2} + O(\lambda^{-3}) \right) \left( \frac{\partial_t \kappa}{4} I - \partial_t \Theta_{t-a} E_{4,4} \right) \right] = -\frac{1}{\sqrt{2\lambda}} \Gamma_1; (4.4) \] (4.45)
\[ \text{Tr} [\Gamma^{-1} \Gamma' \partial_s T] = \text{Tr} \left[ \left( -\frac{\Gamma_1}{\lambda^2} + O(\lambda^{-3}) \right) \left( \frac{\partial_s \kappa}{4} I - \partial_s \Theta_{t-a} E_{3,3} \right) \right] = +\frac{1}{\sqrt{2\lambda}} \Gamma_1; (3.3) \] (4.46)
since \( \det \Gamma \equiv 1 \) which implies \( \text{Tr} \Gamma_1 = 0 \).

We now use the exact formula in theorem 4.5 to conclude the proof of theorem 2.2; recall that
\[ \Gamma(\lambda) = e^{\partial_\lambda \mathcal{E}(\lambda)} \Omega(\lambda) e^{-\partial_\lambda} \] (4.47)
and thanks to lemma 4.4 we have
\[ \Gamma_1 = e^{\partial_\lambda \mathcal{E}(\lambda)} e^{-\partial_\lambda} = \Omega_1 (I + O(\sigma^{-\infty})) \]
\[ \Gamma_1 = \sqrt{2} \begin{bmatrix} -p(s) & 0 & 0 & -q(s) \\ 0 & p(t) & q(t) & 0 \\ 0 & -q(t) & -p(t) & 0 \\ q(s) & 0 & p(t) & 0 \end{bmatrix} (I + O(\sigma^{-\infty})) \] (4.48)
which yields
\[ \Gamma_1; (4.4) = \Omega_1; (4.4) = \sqrt{2} p(s) + O(\sigma^{-\infty}) \] (4.49)
\[ \Gamma_1; (3.3) = \Omega_1; (3.3) = -\sqrt{2} p(t) + O(\sigma^{-\infty}). \] (4.50)
Recall that \( p(u) \) is the logarithmic derivative of the gap probability for the Airy process (i.e. the Tracy–Widom distribution); collecting all the previous results, we have
\[ \partial_{s,t} \ln \det \left( \text{Id} - \mathbb{H} \right) \bigg|_{(\sigma - \tau^2, \sigma + \tau^2 - 2)} \]
\[ = p(s) ds + p(t) dt + O(\sigma^{-\infty}) ds + O(\sigma^{-\infty}) dt + O(\sigma^{-\infty}) ds \ dt \] (4.51)
uniformly in \( s, t \) within the domain that guarantees the uniform validity of the estimates above as per lemma 4.4, namely, \( s, t < K_1(\sigma + \tau^2), 0 < K_1 < 1 \).

We now integrate from \((s_0, t_0)\) to \((s, t)\) with \( s_0 := a + \sigma + \tau^2, t_0 = -b + \sigma + \tau^2 \) and we get
\[ \ln \det \left( \text{Id} - \mathbb{H} \right) \bigg|_{(\sigma - \tau^2, \sigma + \tau^2 - 2)} \]
\[ = \ln \det \left( \text{Id} - K_{\mathcal{A}} \right) \bigg|_{(a, + \infty)} \] + \( \ln \det \left( \text{Id} - K_{\mathcal{A}} \right) \bigg|_{(t, + \infty)} \) + \( O(\sigma^{-1}) + C \) (4.52)
with \( C = \ln \det \left( \text{Id} - \mathbb{H} \right) \bigg|_{[a, b]} \).
In conclusion,
\[
\det \left( \Id - \left[ K^{(t,-)}_{[\sigma, \infty)} \right] \right) = e^C \det \left( \Id - K_{[\sigma, \infty)} \right) \det \left( \Id - K_{[\sigma, \infty)} \right) \left( 1 + O(\sigma^{-1}) \right)
\]
(4.53)

On the other hand, the Fredholm determinant of the Airy kernel appearing in the denominator tends to unity as \( \sigma \to \infty \), thus we only need to prove that the constant \( C \) is zero. Indeed this is the case.

**Lemma 4.6.** The constant of integration \( C \) in (4.52) is zero.

**Proof.** We recall the definition of the integral operator \( \hat{\Pi} \hat{\Pi} \Pi \) acting on \( \mathcal{H}_1 \oplus \mathcal{H}_2 = L^2([\bar{\sigma}, \infty)) \oplus L^2([\bar{\sigma}, \bar{\sigma}], \Pi) \), with kernel

\[
\hat{\Pi} \hat{\Pi} \Pi = \begin{bmatrix} \pi K_{\Lambda_1} & -\frac{\sqrt{2}\pi}{\sqrt{2\pi}} K^{(t,-)}_{\Lambda_1} \hat{\Pi} \\ -\frac{\sqrt{2}\pi}{\sqrt{2\pi}} K^{(t,-)}_{\Lambda_1} \hat{\Pi} & \frac{\sqrt{2}\pi}{\sqrt{2\pi}} K^{(t,-)}_{\Lambda_1} \hat{\Pi} \end{bmatrix}
\]
(4.54)

where \( \hat{\Pi} := \pi \oplus \Pi, \pi \) is the projector on \([\bar{\sigma}, \infty)\), \( \Pi \) is the projector on \([\bar{\sigma}, \bar{\sigma}]\) and

\[
K_{\Lambda_1}(x, y) := \int_0^\infty \Ai(x + u)\Ai(y + u) \, du
\]
(4.55)

\[
K^{(t,-)} \sigma_{-x, \sigma - y} := e^{\tau(y-x)} \int_0^\infty du \Ai(\sigma - x + \tau^2 + \sqrt{2}u)\Ai(\sigma - y + \tau^2 + \sqrt{2}u)
\]

(4.56)

\( \mathcal{A}_t(x, y) := \Ai^{(t)}(x - \sigma + \sqrt{2}y) - \int_0^\infty \Ai^{(t)}(\sigma - x + \sqrt{2}v)\Ai(v + y) \, dv 
\]

(4.57)

\[
\mathcal{A}^{(t,-)}_{-t}(x, y) := 2^{1/6} e^{-\tau(y - \sigma + \sqrt{2}x)} \Ai(y - \sigma + \sqrt{2}x + \tau^2) 
- 2^{1/6} \int_0^\infty dv e^{-\tau(y - \sigma + \sqrt{2}v)} \Ai(y - \sigma + \sqrt{2}v + \tau^2) \Ai(v + y)
\]

(4.58)

We would like to perform some uniform pointwise estimates on the entries of the kernel in order to prove that as \( \sigma \to +\infty \) the trace of the operator \( \hat{\Pi} \hat{\Pi} \Pi \) tends to zero. Indeed,

\[
|\pi K_{\Lambda_1}(u, v)| \leq C_1 e^{-\frac{\sqrt{2}}{3} u^{1/2} - \frac{\sqrt{2}}{3} v^{1/2}}
\]
(4.59)

\[
|\sqrt{2\pi} K^{(t,-)}_{\Lambda_1}(x, y) \hat{\Pi}| \leq C_2 e^{-\sigma^{1/2}}
\]
(4.60)

\[
|\sqrt{2\pi} \mathcal{A}_t(x, v) \tau| \leq C_3 e^{-\tau}(\sqrt{2}\sigma - \tau) \leq \frac{1}{2} (\sqrt{2}\sigma - \tau)^{1/2}
\]
(4.61)

\[
|\sqrt{2\pi} \mathcal{A}^{(t,-)}_{-t}(u, y) \hat{\Pi}| \leq C_4 e^{-\tau}(\sqrt{2}\sigma - \tau) \leq \frac{1}{2} (\sqrt{2}\sigma - \tau)^{1/2}
\]
(4.62)
for some positive constants $C_j$ ($j = 1, \ldots, 4$), where we have used the convention that $x, y$ are the variables running in $[a, b]$ and $u, v$ are the variables running in $[\tilde{a}, \tilde{b}]$. Such estimates follow from simple arguments on the asymptotic behaviour of the Airy function when its argument is very large.

Collecting all the estimates, we get

$$\pi K_{\lambda_0} \frac{\text{Ai} (\sqrt{2\pi} x)}{\sqrt{2\pi} x} \pi \leq C_\sigma \left[ \begin{array}{cc} f (u) f (v) & f (u) f (v) \\ f (v) & 1 \end{array} \right]$$

(4.63)

with $C_\sigma = \max \{C_j : j = 1, \ldots, 4\}$ and $f (z) = e^{-\sqrt{2\pi} (\sqrt{2\pi} x)}$. On the right hand side we have a new operator $L$ acting on the same Hilbert space $L^2([\tilde{a}, \tilde{b}]) \oplus L^2([\tilde{\sigma}, \infty))$ with trace

$$\text{Tr} L = \| f \|_{L^2([\tilde{\sigma}, \infty])}^2 + (\tilde{b} - \tilde{\alpha}) \leq C (b - a)$$

(4.64)

for some positive constant $C$, since $\| f \|_{L^2([\tilde{\sigma}, \infty])} \to 0$ as $\sigma \to +\infty$.

Concluding, keeping $[a, b]$ fixed,

$$| \ln \det (\text{Id} - \hat{H} \hat{H}^*) | = \sum_{n=1}^{\infty} \frac{\text{Tr} (\hat{H} \hat{H}^* \hat{H}^n)}{n} \leq \sum_{n=1}^{\infty} C_{\sigma} (b - a)^n \leq C_{\sigma} (b - a) \to 0$$

(4.65)

as $\sigma \to +\infty$. This implies that the constant of integration $C$ must be zero. \hfill \Box

\section{5. Proof of theorem 2.3}

We deal now with the case $\tau \to \pm \infty$, i.e. we are moving away from the tacnode point along the boundary curves of the domain so that there is one of the gaps that dvaricates as we proceed. From now on, we will only focus on the case $\tau \to +\infty$. The case $\tau \to -\infty$ is analogous.

The RH problem we are considering is the same as for the proof of theorem 2.2 (3.74)--(3.77). We conjugate the jumps with the constant diagonal matrix $D$ (see definition 4.7) and we have the same jump matrices as in (4.9)--(4.14).

The position of the curves is depicted in figure 6:

- $L_2$ and $R_2$ are centred around the critical point $P_R := 2^\frac{1}{2} \tau$
- $L_3$ and $R_3$ are centred around the critical point $P_L := -2^\frac{1}{2} \tau$
- $L_1$ passes through the critical point $P_{\sigma, L} := -\sqrt{\tilde{\sigma}}$ and $R_1$ passes through the critical point $P_{\sigma, R} := \sqrt{\tilde{\sigma}}$.

The points $P_{R/L} = \pm 2^\frac{1}{2} \tau$ are thought of as very far from the origin, in the limit as $\tau \gg 1$.

We need to perform certain ‘contour deformations’ and ‘jump splitting’ in the RHP (3.74)--(3.77). To explain these manipulations consider a general RHP with a jump on a certain contour $\gamma_0$ and with jump matrix $J(\lambda)$

$$\Gamma_+ (\lambda) = \Gamma_- (\lambda) J(\lambda), \quad \lambda \in \gamma_0.$$  

(5.1)

The ‘contour deformation’ procedure stands for the following: suppose $\gamma_1$ is another contour such that

- $\gamma_0 \cup \gamma_1^{-1}$ is the positively oriented boundary of a domain $D_{\gamma_0, \gamma_1}$, where $\gamma_1^{-1}$ stands for the contour traversed in the opposite orientation,
Figure 6. The contour setting in the asymptotic limit as $\tau \to +\infty$.

- $J(\lambda)$ and $J^{-1}(\lambda)$ are both analytic in $D_{\gamma_0,\gamma_1}$ and (in case the domain extends to infinity) $J(\lambda) \to I + O(\lambda^{-1})$ as $|\lambda| \to \infty$, $\lambda \in D_{\gamma_0,\gamma_1}$.

We define $\widetilde{\Gamma}(\lambda) = \Gamma(\lambda)$ for $\lambda \in \mathbb{C} \setminus D_{\gamma_0,\gamma_1}$ and $\widetilde{\Gamma}(\lambda) = \Gamma(\lambda)J(\lambda)^{-1}$ for $\lambda \in D_{\gamma_0,\gamma_1}$. This new matrix then has a jump on $\gamma_1$ with jump matrix $J(\lambda)$ ($\lambda \in \gamma_1$) and no jump (i.e. the identity jump matrix) on $\gamma_0$. While technically this is a new RH problem, we shall refer to it simply as the ‘deformation’ of the original one, without introducing a new symbol.

The ‘jump splitting’ procedure stands for a similar manipulation: suppose that the jump matrix relative to the contour $\gamma$ is factorizable into two (or more) matrices $J(\lambda) = J_0(\lambda)J_1(\lambda)$. Let $\widetilde{\gamma}, \widetilde{D}_{\gamma} \widetilde{\gamma} \gamma$ be exactly as in the description above. Then define $\widetilde{\Gamma}(\lambda) = \Gamma(\lambda)$ for $\lambda \in \mathbb{C} \setminus D_{\widetilde{\gamma}},\widetilde{\gamma}$ and $\widetilde{\Gamma}(\lambda) = \Gamma(\lambda)J_{0,1}(\lambda)^{-1}$ for $\lambda \in D_{\widetilde{\gamma},\widetilde{\gamma}}$. Then $\widetilde{\Gamma}$ has jumps

$$\widetilde{\Gamma}_+(\lambda) = \widetilde{\Gamma}_-(\lambda)J_0(\lambda), \quad \lambda \in \gamma_0, \quad \widetilde{\Gamma}_+(\lambda) = \widetilde{\Gamma}_-(\lambda)J_1(\lambda), \quad \lambda \in \gamma_1.$$

Also in this case, while this is technically a different RHP, we shall refer to it with the same symbol $\Gamma$. We will also refer to the inverse operation as ‘jump merging’.

With this terminology in mind, we deform $R_3$ on the left next to its critical point $-\sqrt{2}\tau$ which leads to a new jump matrix on $R_3$, due to conjugation with the curve $L_1$ (similarly for $L_2$)

$$J_3 := L_1R_3L_1^{-1} = R_1L_2R_1^{-1} =: J_2$$

$$= \begin{bmatrix}
1 & 0 & e^{-\Theta(\tau, a) + K(t)} & e^{-\Theta(\xi, t)} \\
0 & 1 & e^{\Theta(\xi, t)} & e^{-\Theta(-t, b) + K(t)} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

(5.2)

Again as before, the proof is based on estimating the phases in the jump matrices which are not critical and solving the RH problem by approximation with an exact solution to a model problem.

5.1. Estimates of the phases

First of all we notice that a similar version of lemma 4.2 does not apply here, since the phases on the contours $L_1$ and $R_1$ do not depend on $\tau$. On the other hand, we can partially restate lemma 4.3 applied to the case at hand when $\tau \to \infty$. 
Lemma 5.1. Given $0 < K_1 < 1$ fixed and $s < K_1(\sigma + \tau^2)$, then the function $e^{\Theta (-\tau, b) + K(t)}$ tends to zero exponentially fast in any $L^p(R_3)$ norm ($1 \leq p \leq \infty$) as $\tau \to +\infty$:
\[
\| e^{\Theta (-\tau, b) + K(t)} \|_{L^p(R_3)} \leq C e^{-2(1-K_1)\tau^3}.
\]
Similarly, the functions $e^{-\Theta (-\tau, b) - K(t)}$, $e^{\Theta - \tau - b + K(t)}$ and $e^{\Theta - \tau - b - K(t)}$ are exponentially small in any $L^p(L_3)$, $L^p(R_2)$ and $L^p(L_2)$ norms, respectively ($1 \leq p \leq \infty$).

Proof. Using the same parametrization as in lemma 4.3, we have
\[
\| e^{\Theta (-\tau, b) + K(t)} \|_{L^p(R_3)} \leq 2Ce^{-2p\tau(\sigma + \tau^2)} \left[ \int_0^1 e^{-2\tau^2}du + \int_1^\infty e^{-p\tau^2}du \right]
\]
\[
\leq Ce^{-2p(1-K_1)\tau^3} \quad \text{(5.4)}
\]
\[
\| e^{\Theta (-\tau, b) - K(t)} \|_{L^p(R_3)} \leq e^{-2\tau(\sigma + \tau^2 - \delta)} \leq Ce^{-2(1-K_1)\tau^3} \quad \text{(5.5)}
\]
where we set $s = 2\sigma + 2\tau^2 - \delta$, $0 < \delta < \sigma + \tau^2$. The proof for the other phases on the contours $L_3$, $R_2$ and $L_2$ is analogous.

Before estimating the entries of the jump matrices on $J_2$ and $J_3$, we factor the jumps in the following way. We split the jump $J_3$ into two jumps (and two curves): with abuse of notation we call the first one $J_2$ and we merge the second jump with the jump on $R_1$. Thus, the new jumps are the following (see figure 6):
\[
J_2 = \begin{bmatrix}
1 & 0 & e^{-\Theta (-\tau, b) + K(t)} & e^{-\Theta (-\tau, b) - K(t)} \\
0 & 1 & e^{\Theta (-\tau, b)} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\text{(5.6)}
\]
\[
\tilde{J}_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -e^{\Theta (-\tau, b) + K(t)} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\text{(5.7)}
\]
Analogously, we split the jump $J_1$ into two jumps: we again call the first one $J_3$ and we merge the second one with the jump on $L_1$. The new configuration of jump matrices is illustrated in figure 6.
\[
J_3 = \begin{bmatrix}
1 & 0 & 0 & e^{-\Theta (-\tau, b) - K(t)} \\
0 & 1 & e^{\Theta (-\tau, b) + K(t)} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\text{(5.8)}
\]
\[
\tilde{J}_3 = \begin{bmatrix}
1 & e^{-\Theta b} & 0 & e^{-\Theta (-\tau, b) - K(t)} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\text{(5.9)}
\]

Lemma 5.2. Let $\kappa := \frac{8}{3} - \frac{b}{p}$. Given $0 < K_2 < 1$ fixed and $t = 4\tau^2 - \delta$, $0 < \delta \leq K_2K\tau^2$, then the $(1, 3)$ and $(2, 3)$ entries of the jump matrix $J_2$ are exponentially suppressed as $\tau \to +\infty$ in $L^p$ norms with $p = 1, 2, +\infty$.

Given $0 < K_3 < 1$ fixed and $s = \tau^2 + 2\sigma - \delta$, $0 < \delta \leq K_3 \left(2\sigma + \frac{2}{3}\tau^2\right)$, the $(2, 4)$ entry of $J_2$ is exponentially suppressed in any $L^p$ norm ($1 \leq p \leq \infty$).

Similarly, the same results hold true for the $(1, 4)$ and $(2, 4)$ entries of $J_3$ and for the $(1, 3)$ entry of $\tilde{J}_3$. 

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We will now define a new 'model' RH problem which will eventually approximate the solution to our original problem \( \Gamma \).

We define the following RH problem:

\[
\begin{align*}
\Omega_+(\lambda) &= \Omega_-(\lambda) J_0(\lambda) \quad \text{on } L_1 \cup R_1 \\
\Omega_+(\lambda) &= \Omega_-(\lambda) J_R(\lambda) \quad \text{on } L_2 \cup R_2 \\
\Omega_+(\lambda) &= \Omega_-(\lambda) J_L(\lambda) \quad \text{on } L_3 \cup R_3 \\
\Omega(\lambda) &= I + O(\lambda^{-1}) \quad \text{as } \lambda \to \infty
\end{align*}
\]
with jumps

\[ J_{Ai} := \begin{bmatrix} 1 & e^{-\Theta_2} x_{i2} & 0 & 0 \\ e^{\Theta_2} x_{i1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \] (5.17)

\[ J_R := \begin{bmatrix} 1 & 0 & 0 & e^{-\Theta_1(\xi-s)} x_{22} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e^{\Theta_1(\xi-s)} x_{24} & 0 & 0 & 1 \end{bmatrix} \] (5.18)

\[ J_L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & e^{\Theta_1(\xi,t)} x_{32} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \] (5.19)

Let us denote by $/Psi_{a,b}$ the $4 \times 4$ solution to the Airy RHP related to the submatrix formed by the $a$-th row and column and by the $b$-th row and column. In particular, we call $/Psi_{1,2}$ the matrix solution to the Hastings–McLeod Airy RHP for the minor $(1, 2)$, related to the jump $J_{Ai}$, with asymptotic solution

\[ /Psi_{1,2}(\tilde{\sigma}) = I_{4 \times 4} + \frac{1}{\lambda} \begin{bmatrix} -p(\tilde{\sigma}) & -q(\tilde{\sigma}) & 0 & 0 \\ q(\tilde{\sigma}) & p(\tilde{\sigma}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + O\left(\frac{1}{\lambda^2}\right). \] (5.20)

We consider now the matrix $/Xi := /Omega /Psi_{1,2}^{-1}$. This matrix doesn’t have jumps on $L_1$ and $R_1$ by construction, but still has jumps on $L_2, R_2$ and $L_3, R_3$:

\[ \tilde{J}_L := /Psi_{1,2} J_L /Psi_{1,2}^{-1} \quad \text{and} \quad \tilde{J}_R := /Psi_{1,2} J_R /Psi_{1,2}^{-1}. \] (5.21)

On the other hand, as $\tau \to +\infty$ the critical points $\pm \sqrt{2}\tau$ as well as the curves $L_2, R_2, L_3, R_3$ go to infinity, while the matrix $/Psi_{1,2}$ is asymptotically equal to the identity matrix.

We are left with

\[ /Xi = /E_1 \cdot /Psi_{2,3}(t) \cdot /Psi_{1,4}(s) \] (5.22)

where $/Psi_{2,3}$ and $/Psi_{1,4}$ where defined in (4.24) and $/E_1$ is the error matrix.

Following the previous remark, it is easy to show that the error matrix $/E_1$ is a sufficiently small perturbation of the identity and therefore we can apply the small norm theorem and approximate the global parametrix $/Omega$ simply by the product of the matrices $/Psi_{a,b}$ ($(a, b) = (1, 2), (2, 3), (1, 4)$)

\[ /Omega = /Xi \cdot /Psi_{1,2}(\tilde{\sigma}) \sim /Psi_{2,3}(t) \cdot /Psi_{1,4}(s) \cdot /Psi_{1,2}(\tilde{\sigma}). \] (5.23)

### 5.3. Approximation and error term for the matrix $\hat{\Gamma}$

The relation between our original RH problem $\hat{\Gamma}$ and the global parametrix $/Omega$ is the following

\[ \hat{\Gamma} = /E_2 \cdot /Omega := /E_2 \cdot /Psi_{2,3}(t) \cdot /Psi_{1,4}(s) \cdot /Psi_{1,2}(\tilde{\sigma}) \] (5.24)

where $/E_2$ is again an error matrix, to which we will apply the small norm argument once again ([14, section 5.1.3]).
Lemma 5.4. In the estimates on s, t stated in lemmas 5.1 and 5.2, the error matrix $E = \hat{\Gamma}(\lambda)\Omega^{-1}(\lambda)$ solves a RH problem with jumps on the contours as indicated in figure 6 and of the following orders

$$\begin{align*}
E_+(\lambda) &= E_-(\lambda) J_E(\lambda) \quad \text{on} \quad \Sigma \\
E(\lambda) &= I + O(\lambda^{-1}) \quad \text{as} \quad \lambda \to \infty
\end{align*}$$

(5.25)

$$J_E = \begin{bmatrix}
1 & 0 & O(\tau^{-\infty})\chi_{s_1} + O(\tau^{-\infty})\chi_{h_1} & O(\tau^{-\infty})\chi_{h_1} \\
0 & 1 & O(\tau^{-\infty})\chi_{s_2} & O(\tau^{-\infty})\chi_{s_2} + O(\tau^{-\infty})\chi_{h_2} \\
O(\tau^{-\infty})\chi_{s_2} & 0 & 1 & 0 \\
0 & O(\tau^{-\infty})\chi_{s_1} & 0 & 1
\end{bmatrix}$$

(5.26)

where $\Sigma$ is the collection of all contours and the $O$-symbols are valid for $L^1$, $L^2$ and $L^\infty$ norms.

Proof. Due to lemmas 5.1 and 5.2, we know from the estimates above that all the extra phases that appear in the original RH problem for $\hat{\Gamma}$ are bounded by an exponential function of the form $C_1 e^{-C_2 \tau}$. The jumps of the error problem are the remaining jumps appearing in the RH problem conjugated with the global parametrix $\Omega$:

$$J_E = \Omega^{-1}(I + O(\tau^{-\infty}))\Omega = I + O(\tau^{-\infty}).$$

(5.27)

The last equality follows from the fact that the solution $\Omega$ depends on $\tau$ with a growth that is smaller than the bound $C_1 e^{-C_2 \tau}$ that we have for the phases. □

Thus, the small norm theorem can be applied

$$\|E(\lambda) - I\| \leq \frac{C}{\text{dist}(\lambda, \Sigma)} \left( \|J_E - I\|_1 + \frac{\|J_E - I\|_2}{1 - \|J_E - I\|_\infty} \right) \leq \frac{C}{\text{dist}(\lambda, \Sigma)} e^{-K\tau},$$

(5.28)

where $\Sigma$ is the collection of all contours for some positive constants $C$ and $K$. The error matrix $E$ is then found as the solution to an integral equation and, thanks to lemma 5.4, we have

$$E(\lambda) = I + \frac{1}{\text{dist}(\lambda, \Sigma)} O(\tau^{-\infty}).$$

(5.29)

We need the first coefficient $\hat{\Gamma}_1 = \hat{\Gamma}_1(s, t, \bar{s})$ of $\hat{\Gamma}(\lambda)$ at $\lambda = \infty$ and how it compares to the corresponding coefficient $\Omega_1$ of $\Omega(\lambda)$; the error analysis above shows that

$$\hat{\Gamma}_1 = \Omega_1 + O(\tau^{-\infty}).$$

(5.30)

5.4. Conclusion of the proof of theorem 2.3

Theorem 5.5. The Fredholm determinant $\det(\text{Id} - \hat{\Pi} \hat{\Pi})$ satisfies the following relations for any parameter $\rho$ on which the integral operator $\hat{\Pi} \hat{\Pi}$ may depend:

$$\partial_\rho \ln \det(\text{Id} - \hat{\Pi} \hat{\Pi}) = \omega(\partial_\rho) = \int_{\Sigma} \text{Tr} \left( \Gamma^{-1}_1(\lambda)\Gamma^{-1}_2(\lambda)\partial_\rho \Sigma(\lambda) \right) \frac{d\lambda}{2\pi i}.$$  

(5.31)

More specifically,

$$\partial_\rho \ln \det(\text{Id} - \hat{\Pi} \hat{\Pi}) = -\text{res}_{\rho=\infty} \text{Tr} \left( \Gamma^{-1}_1 \Gamma^{-1}_2 \partial_\rho T \right) = \frac{1}{\lambda} \Gamma_1; (2, 2)$$

(5.32)

$$\partial_{\bar{s}} \ln \det(\text{Id} - \hat{\Pi} \hat{\Pi}) = -\text{res}_{\rho=\infty} \text{Tr} \left( \Gamma^{-1}_1 \Gamma^{-1}_2 \bar{s} \partial_{\rho} T \right) = -\frac{1}{\sqrt{2}\lambda} \Gamma_1; (3, 3)$$

(5.33)

$$\partial_{\bar{s}} \ln \det(\text{Id} - \hat{\Pi} \hat{\Pi}) = -\text{res}_{\rho=\infty} \text{Tr} \left( \Gamma^{-1}_1 \Gamma^{-1}_2 \bar{s} \partial_{\rho} T \right) = \frac{1}{\sqrt{2}\lambda} \Gamma_1; (4, 4)$$

(5.34)

where $\Gamma_1 := \lim_{\lambda \to \infty} \lambda (\Gamma(\lambda) - I)$. 

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Proof. The first part of the Theorem is the same as theorem 4.5. Then, using the Jimbo–Miwa–Ueno residue formula, we have
\[ \partial_{s} \ln \det (\text{Id} - \Pi H \tilde{\Pi}) = -\text{res}_{\lambda \to \infty} \text{Tr} \left( \Gamma^{-1} \Gamma' \partial_{s} T \right) \]
(5.35)
with \( \rho = \bar{s}, s, t. \)

Taking into account the definition of the conjugation matrix \( T \) (see (4.41)) and the asymptotic behaviour of the matrix \( \Gamma \) at infinity, we get again
\[ \text{Tr} \left[ \Gamma^{-1} \Gamma' \partial_{s} T \right] = -\frac{\Gamma_{1; (2, 3)}}{\lambda}, \quad \text{Tr} \left[ \Gamma^{-1} \Gamma' \partial_{t} T \right] = -\frac{\Gamma_{1; (4, 4)}}{\sqrt{2} \lambda}, \quad \text{Tr} \left[ \Gamma^{-1} \Gamma' \partial_{t} T \right] = \frac{\Gamma_{1; (3, 3)}}{\sqrt{2} \lambda}. \]
(5.36)

On the other hand, thanks to lemma 5.4 and the small norm theorem, we can approximate the solution \( \Gamma \) with the global parametrix \( \Omega \) using (5.30) and we get
\[ d \ln \det \left( \text{Id} - \Pi K_{\text{vac}} \right)_{\mid s = t, \sigma = t \pm t, \tau = t - t} \]
\[ = p(s) ds + p(t) dt + p(\bar{s}) d\bar{s} + O \left( t^{-\infty} \right) ds + O \left( t^{-\infty} \right) dt + O \left( t^{-\infty} \right) d\bar{s} + O \left( t^{-\infty} \right) ds + O \left( t^{-\infty} \right) ds + O \left( t^{-\infty} \right) dt + O \left( t^{-\infty} \right) \]
\[ \quad + O \left( t^{-\infty} \right) d\bar{s}. \]
(5.37)
Integrating from a fixed point \( (s_0, t_0, \bar{s}_0) \) up to \( (s, t, \bar{s}), \)
\[ \det \left( \text{Id} - \Pi K_{\text{vac}} \right)_{\mid s = t, \sigma = t \pm t, \tau = t - t} \]
\[ = e^{C} \det \left( \text{Id} - K_{\lambda_{1}} \right)_{\mid s = t, \sigma = t \pm t, \tau = t - t} \det \left( \text{Id} - K_{\lambda_{2}} \right)_{\mid t = s, \sigma = t \pm t, \tau = t - t} \det \left( \text{Id} - K_{\lambda_{2}} \right)_{\mid \bar{s} = \infty} \left(1 + O(t^{-1}) \right) \]
\[ = e^{C} \det \left( \text{Id} - K_{\lambda_{1}} \right)_{\mid s = t, \sigma = t \pm t, \tau = t - t} \det \left( \text{Id} - K_{\lambda_{2}} \right)_{\mid t = s, \sigma = t \pm t, \tau = t - t} \left(1 + O(t^{-1}) \right) \]
(5.38)
with \( s, t \) within the domain that guarantees the uniform validity of the estimates above (see lemmas 5.1 and 5.2) and \( C = \ln \det (\text{Id} - \Pi K_{\text{vac}} \Pi_{\infty}) \).

Finally, we again need to show that the constant of integration \( C \) is equal to zero.

**Lemma 5.6.** The constant of integration \( C \) in the formula (5.38) is zero.

Proof. First of all we notice that (see lemma 3.2)
\[ \det \left( \text{Id} - \Pi K_{\text{vac}} \Pi \right) = \frac{\det \left( \text{Id} - \Pi \tilde{\Pi} \Pi \right)}{\det (\text{Id} - \pi K_{\lambda_{1}})} \]
\[ = \det \left( \begin{bmatrix} 0 & \Pi K_{\lambda_{1}} \Pi \end{bmatrix} \begin{bmatrix} \Pi K_{\lambda_{1}} \Pi \end{bmatrix} \right) \]
\[ = \det \left( \begin{bmatrix} \Pi K_{\lambda_{1}} \Pi \end{bmatrix} \begin{bmatrix} \Pi K_{\lambda_{1}} \Pi \end{bmatrix} \right) \]
(5.39)
where \( \tilde{\Pi} := \pi \oplus \Pi, \pi \) is the projector on \([\bar{s}, \infty]\) and \( \Pi \) is the projector on \([\tilde{a}, \tilde{b}]. \)
Along the same guidelines as the proof of Lemma 4.6, we will perform some uniform estimates on the entries of the kernel that will lead to the desired result.

We have

\[
\left| \sqrt{2} \Pi K_{\text{Ai}}^{(r,-\tau)}(u, v) \Pi \right| \leq \frac{C}{\tau} e^{-\frac{3}{4} \tau^2 - 2 \tau \sigma + 2 \tau v} \leq \frac{C_1}{\sqrt{\tau}} \tag{5.40}
\]

\[
\left| \sqrt{2} \Pi \mathcal{A}_{\tau}(x, v) \right| \leq \frac{C_2}{\sqrt{\tau}} \tag{5.41}
\]

\[
\left| 6 \sqrt{2} \Pi K_{\text{Ai}}^{\tau}(x, v) \right| \leq \frac{C_2}{\sqrt{\tau}} \tag{5.42}
\]

for some positive constants \( C_j \) (\( j = 1, 2, 3 \)), where the variables \( x, y \) run in \([\tilde{a}, \tilde{b}]\) and \( u, v \) run in \([\tilde{\sigma}, \infty)\). Such estimates follow again from simple arguments on the asymptotic behaviour of the Airy function when its argument is very large. Moreover, the resolvent of the Tracy–Widom distribution is uniformly bounded and independent of \( \tau \); here is the reason for the constant \( C_{\text{Ai}} \).

Collecting the above estimates, we have

\[
\begin{bmatrix}
0 \\
-\sqrt{2} \Pi (\text{Id} - \pi K_{\text{Ai}}^{\tau})^{-1} \Pi \mathcal{A}_{\tau} \end{bmatrix} \leq C \begin{bmatrix}
0 \\
f(u) \\
1 \\
1 
\end{bmatrix} \tag{5.43}
\]

with \( C := \max\{C_j, j = 1, 2, 3\} \) and \( f(u) = e^{-\tau (u - \sigma)} \). On the right hand side, we have a new operator \( \mathcal{M} \) acting on \( L^2([\tilde{\sigma}, \infty)) \oplus L^2([\tilde{a}, \tilde{b}]) \) with bounded trace

\[
\text{Tr} \mathcal{M} \leq \hat{C} \left( \| f \|_{L^2([\tilde{\sigma}, \infty))}^2 + (\tilde{b} - \tilde{a}) \right) \leq \hat{C} (b - a) \tag{5.44}
\]

for some positive constant \( \hat{C} \), since \( \| f \|_{L^2([\tilde{\sigma}, \infty))} \to 0 \) as \( \tau \to +\infty \).

Concluding, having \([a, b]\) fixed,

\[
| \ln \det(\text{Id} - \Pi K_{\text{Ai}}^{\tau} \Pi) | = \sum_{n=1}^{\infty} \frac{\text{Tr} \left( (\Pi K_{\text{Ai}}^{\tau} \Pi)^n \right)}{n} \leq \sum_{n=1}^{\infty} \frac{C^n \hat{C}^n (b - a)^n}{n} \leq \sum_{n=1}^{\infty} C_t^n \hat{C}^n (b - a)^n = \frac{C_t \hat{C} (b - a)}{1 - C_t \hat{C} (b - a)} \to 0 \tag{5.45}
\]

as \( \tau \to +\infty \).

Therefore, the constant of integration is equal to zero. \( \square \)

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