Is the SIC Outcome There When Nobody Looks?

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Informationally complete measurements are a dramatic discovery of quantum information science, and the symmetric IC measurements, known as SICs, are in many ways optimal among them. Close study of three of the “sporadic SICs” reveals an illuminating relation between different ways of quantifying the extent to which quantum theory deviates from classical expectations.

Any headline ending in a question mark can be answered with the single word No. — journalist’s adage (the Davis–Hinchliffe–Betteridge Law)

I. INTRODUCTION

What feature of quantum physics distinguishes it from classical mechanics? Schrödinger’s answer was “entanglement” [1]. Today, though, this response is rather passé. On the one hand, we have learned that entanglement is not unique to quantum mechanics, but occurs rather generically in nonclassical theories that lack superluminal signalling [2, 3]. On the other hand, we know that the mere occurrence of entanglement in a theory is, quantifiably, less exotic than the violation of a Bell inequality [4–7]. And if our answer is “quantum phenomena can violate a Bell inequality”, then a new question naturally arises. Is the specific numerical extent to which quantum theory violates a Bell inequality meaningful, and why? This article addresses that question by relating two measures of departure from classicality, one grounded in a Bell inequality and the other in recent progress on representing quantum theory in wholly probabilistic terms.

To accept quantum theory is to strive to maintain a certain peculiar consistency among expectations for mutually exclusive experiments. For example, consider the paradigmatic double-slit scenario. When only slit number 1 is open, the probability of a detector click as a function of detector position is given by some function, call it \( P_1(x) \). This function is quite mundane: It takes only nonnegative values and behaves in all ways like a classical probability. The same is true for the curves \( P_2(x) \) and \( P_{12}(x) \). The puzzle is that \( P_{12}(x) \neq P_1(x) + P_2(x) \). Closer investigation reveals that the mere fact of interference is not as enigmatic as it first appeared, and a Bell test can interrogate the exotic character of quantum physics more stringently than the double-slit experiment can. Remarkably, the same theme holds true in the more stringent inquiry: Given any specific experimental arrangement, the probabilities we compute with quantum theory appear quite ordinary. It is in the meshing of expectations for different interventions into the course of nature that the fundamental enigma of quantum physics manifests.

We will study this using examples in Hilbert spaces of decreasing dimension: first eight (three qubits), then four (two qubits) and finally three (a single qutrit). We begin with Mermin’s three-qubit Bell inequality [8–11]. From there, we will turn to the Hoggar SIC [12–16], an eight-dimensional structure that provides a common meeting ground for two ways of dis-
cussing the nonclassicality of quantum theory. On the one hand, it furnishes a SIC representation of eight-dimensional quantum state space \[17\], and so it exemplifies the nonclassical meshing of probability assignments described by Fuchs and Schack \[18–23\]. On the other hand, that same state space is what one requires for the GHZ \textit{gedankenexperiment} and for Mermin’s three-qubit Bell inequality. Considering the Hoggar SIC will be enough to answer the title question in the negative; we will then explore additional nuances by developing the theme using qubit and qutrit SICs. (The SICs for single qubits and qutrits, as well as the Hoggar SIC in dimension 8, stand out in some ways from the other known SICs and are collectively known as the \textit{sporadic SICs} \[15, 24, 25\].) Finally, we will conclude with some thoughts on the project of reconstructing quantum theory from physical principles.

II. MERMIN’S THREE-QUBIT BELL INEQUALITY

Let \(X, Y\) and \(Z\) denote the Pauli operators, and write \(XXX\) for \(X \otimes X \otimes X\) and so forth. Then we can write Mermin’s three-qubit Bell inequality \[9, 10\] in terms of a linear combination of expectation values:

\[
B(\rho) = \langle XXX \rangle - \langle XYY \rangle - \langle YXY \rangle - \langle YYX \rangle.
\]

One employs this inequality in the following manner. First, one argues that the hypothesis of local hidden variables implies

\[
-2 \leq B(\rho) \leq 2.
\]

A way to see why these bounds should be set at \(\pm 2\) is as follows. Suppose that each part of the tripartite system carries a pair of physical properties that respectively determine the outcomes of an \(X\) measurement and of a \(Y\) measurement performed on that part. As a whole, then, the system carries a set of properties

\[
\lambda = (\lambda_{1X}, \lambda_{1Y}, \lambda_{2X}, \lambda_{2Y}, \lambda_{3X}, \lambda_{3Y}),
\]

such that if we knew these values, we could say

\[
\langle XXX \rangle - \langle XYY \rangle - \langle YXY \rangle - \langle YYX \rangle = \lambda_{X1}\lambda_{X2}\lambda_{X3} - \lambda_{X1}\lambda_{Y2}\lambda_{Y3} - \lambda_{Y1}\lambda_{X2}\lambda_{Y3} - \lambda_{Y1}\lambda_{Y2}\lambda_{X3}.
\]

It is now a matter of arithmetic to verify that for each assignment of \(+1\) and \(-1\) to the six \(\lambda\)s, this quantity is either \(+2\) or \(-2\). The list of values denoted by \(\lambda\) is, in older jargon, a “dispersion-free state” \[26–29\]. Since the sum of expectation values given any dispersion-free state is \(\pm 2\), any probabilistic average over dispersion-free states will lie in the interval \([-2, 2]\).

Having established the bounds in (2), one then finds a state — for example, the GHZ state — for which those bounds are violated. This establishes that quantum probabilities cannot be accounted for by local hidden variables. The GHZ state \(\rho_{\text{GHZ}}\) is (by definition \[8\]) an eigenstate of the operator \(XXX\) with eigenvalue \(+1\), and it is also an eigenstate of \(XYY\), of \(YXY\) and of \(YYX\) with eigenvalue \(-1\). Therefore, \(B(\rho_{\text{GHZ}}) = 4\). This means that \(\rho_{\text{GHZ}}\) violates the inequality (2), and thus, the statistics encapsulated in \(\rho_{\text{GHZ}}\) defy local classical emulation.

In discussions of hidden-variable models, it usually does not particularly matter what other mathematical structure the set of all \(\lambda\)’s might have. The \(\lambda\)’s might, for all we end up
caring, be labeled by the elements of a group, or the morphisms of a groupoid, or the open sets of a topology; they could have any geometry, or none. (Indeed, it is fair to say that the nature of $\lambda$-space is “rarely subject to much critical scrutiny” [30].) What does matter is the hypothesis that each part of a system carries its part of $\lambda$ with it as an intrinsic physical property. In the example above, we hypothesized that each of the three qubits carried its own, intrinsic $\lambda_X$ and $\lambda_Y$. A preparation of the system naturally corresponds, then, to a probability distribution over the set of all possible $\lambda$’s, or in other words, to a point in the simplex whose vertices are labeled by the possible values of $\lambda$. We could choose to decorate these vertices with additional structure (say, making them into a group), but that extra mathematical ornamentation is secondary to the physical assumption which makes our state space into a simplex and, ultimately, powers the derivation of Bell inequalities [27].

III. PROBABILISTIC REPRESENTATION OF QUANTUM THEORY

A basic axiom of quantum theory is that to each physical system is associated a complex Hilbert space. In the domain of quantum information and computation, the Hilbert space is often taken to be finite-dimensional, with the dimension $d$ scaling with the available budget. We mathematically represent a measurement by a positive operator valued measure, or POVM, which is a set of positive semidefinite operators on the system’s Hilbert space that sum to the identity. Each operator in the set $\{E_i\}$ stands for an outcome of the measurement. When a physicist — let us call her Alice, per genre tradition — ascribes a quantum state to a system of interest, she writes a positive semidefinite operator of unit trace, i.e., a density matrix $\rho$. Alice’s probability for obtaining the outcome $E_i$ is

$$p(E_i) = \text{tr}(\rho E_i).$$

The POVM version of Gleason’s theorem establishes that any assignment of probabilities to experiment outcomes must take the form of Eq. (5) for some density matrix $\rho$, if the probability of an outcome is independent of the POVM in which it is embedded [31, 32].

If the elements of a POVM span the space of Hermitian operators, then we can write any density matrix $\rho$ as a linear combination of the POVM elements with real coefficients. This fact implies the possibility of informationally complete (IC) POVMs. Given a probability vector $p$ over the outcomes of an IC POVM, we can reconstruct the density matrix $\rho$, and so we can in principle do anything we would have done with $\rho$ using $p$ instead. An IC POVM must have at least $d^2$ elements to span the operator space. A minimal IC POVM, or MIC, has exactly $d^2$ elements. MICs can be constructed in any dimension $d$ [33, 34]; the question is how nice they can be made.

Let $\{|\pi_i\rangle\}$ be a set of $d^2$ unit vectors in a $d$-dimensional complex Hilbert space that enjoy the following property:

$$|\langle \pi_i | \pi_j \rangle|^2 = \frac{d \delta_{ij} + 1}{d + 1}.$$  

Such a set is called a SIC, where the $S$ stands for “symmetric” and the $IC$ for “informationally complete” as before [35–38]. The rank-1 projection operators

$$\Pi_i = |\pi_i\rangle \langle \pi_i|$$

form, after an appropriate scaling, a MIC:

$$H_i = \frac{1}{d} \Pi_i.$$  

Given a quantum state $\rho$, we have

$$p(H_i) = \frac{1}{d} \text{tr}(\rho \Pi_i),$$

and we can reconstruct $\rho$ from these probabilities by way of an appealingly simple formula:

$$\rho = \sum_i \left[ (d + 1)p(H_i) - \frac{1}{d} \right] \Pi_i.$$  \hfill (10)

Given any other POVM $\{D_j\}$, we can find its outcome probabilities by

$$q(D_j) = \text{tr}(\rho D_j) = \sum_i \left[ (d + 1)p(H_i) - \frac{1}{d} \right] p(D_j|H_i),$$

where the conditional probability on the right-hand side is

$$p(D_j|H_i) = \text{tr}(D_j \Pi_i).$$

Note that Eq. (11) has the form of the classical Law of Total Probability

$$p(D_j) = \sum_i p(H_i)p(D_j|H_i),$$

but with the probabilities $p(H_i)$ “deformed” by a rescaling and a shift. In prior work, the importance of Eq. (11) was recognized by designating it the "urgleichung" ("primal equation" in German, or perhaps Klingon).

Note that the bracketed quantity in the urgleichung, $(d + 1)p(H_i) - 1/d$, can go negative if $p(H_i)$ is sufficiently small. This deformation of the vector of $p(H_i)$ is technically what is sometimes known as a “quasi-probability” — a vector whose sum is normalized to unity, but whose elements are not confined to the unit interval [39–43]. Negative “quasi-probabilities” are an artifact of trying to squeeze something into the form of the Law of Total Probability that doesn’t actually fit. Generally, states that are close to orthogonal to one of the SIC vectors will pick up negativity in what we might call their quasi-probability representation. But it’s the $\{P(H_i)\}$ that are directly, operationally meaningful. There is an experiment that Alice could go into the lab and do, and $P(H_i)$ is how much she should bet on the $i^{th}$ outcome of it. Negativity of quasi-probability can become meaningful after one introduces a notion of “quasi-classical states”, or “states that are easy to emulate on a classical machine”. Once we bring in the ideas necessary to support a “resource theory”, then negativity can gain significance as a measure of how powerful a given resource is. But in the broader scheme of things, it is a secondary and somewhat incidental notion.\(^1\)

Any MIC will yield an expression akin to the urgleichung, but it can be proven that out of all MICs, the SICs furnish the expression that is as close as possible to the classical Law

\(^1\) Another reason to think of negativity as a secondary manifestation of nonclassicality is that we have considerable “gauge freedom” about where to put it. Adopting a vector notation for the urgleichung, we can express it as $Q(D) = P(D|H) \Phi P(H)$, where $\Phi$ is a linear combination of the identity and the all-ones matrix [23]. In this form, it is clear that we can multiply $\Phi$ to the right, turning the vector $P(H)$ into “quasi-probabilities”, or we could multiply $\Phi$ to the left, putting the negativity into the conditional probability matrix $P(D|H)$. We could even express $\Phi$ as $\Phi^{1/2}\Phi^{1/2}$ and split the negativity across both.
of Total Probability [23]. The intuition at work here is that, classically, an informationally complete measurement would be, e.g., one that reads off a system’s coordinates in phase space. Any other measurement would in principle be a coarse-graining of that information.

But in quantum theory, there is no underlying phase space, so we should not use a formula that depends upon the concept of one. By identifying this “minimum distance” between a probabilistic representation of quantum theory and classical probability, SICs provide a measure of exactly how nonclassical quantum physics is. This naturally raises the question of how this measure of nonclassicality relates to other such, of which the quintessential is the violation of a Bell inequality. We will answer this question in the next section.

IV. THE HOGGAR SIC

Consider the tensor product of three copies of qubit state space. We will take for our computational basis the tensor-product basis of Pauli $Z$ eigenstates.

Now, we construct the Hoggar SIC, which will provide a “Bureau of Standards” experiment — a reference measurement with respect to which we can represent quantum theory in wholly probabilistic terms [46]. This construction is an example of how all known SICs are generated: We begin with a fiducial vector and take its orbit under the action of a group [36]. A convenient fiducial for our present purpose is the vector given up to normalization by

$$|\pi_0^{(\text{Hoggar})}\rangle \propto (-1 + 2i, 1, 1, 1, 1, 1, 1, 1, 1)^T.$$  

(14)

We apply the three-qubit Pauli group to generate the Hoggar SIC [14, 15, 47]. This is a set of 64 equiangular unit vectors $\{|\pi_i\rangle\}$, which we can also represent in terms of the rank-1 projection operators $\Pi_i = |\pi_i\rangle\langle\pi_i|$. These define a representation of all three-qubit states as probability vectors:

$$p(H_i) = \frac{1}{d}\text{tr}(\rho\Pi_i).$$  

(15)

Given a probability distribution, we can construct the corresponding density matrix using Eq. (10).

Note what happens if we take the expectation value of an operator:

$$\langle A \rangle = \text{tr}(A\rho).$$  

(16)

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2 Classical measurements can, of course, disturb the system that they are applied to. But this is a largely uninteresting complication. In order to express Pauli’s “ideal of the detached observer” [22, 44, 45], reading off the system’s intrinsic properties without disruption is clearly the correct idealization. Comparing the quantum and the classical in a reasonable way requires the plainest expression of both.

3 When we count SICs in a dimension $d$, we do so up to unitary equivalence, since an overall unitary transformation of the entire set preserves the inner products between vectors. There are 240 distinct SICs of “Hoggar type”; i.e., by picking different fiducial vectors, one can construct 240 distinct sets of 64 lines apiece which all have the same symmetry and which are all orbits under the three-qubit Pauli group. All of these 240 sets are equivalent to one another under unitary or anti-unitary transformations [48], so for brevity, we can refer to the SIC constructed from the fiducial (14) as the Hoggar SIC.
Substituting in the expansion (10), we obtain
\begin{equation}
\langle A \rangle = \text{tr} \left[ A \sum_i \left( (d+1)p(H_i) - \frac{1}{d} \right) \Pi_i \right]
\end{equation}
\begin{equation}
= \sum_i \left( (d+1)p(H_i) - \frac{1}{d} \right) \text{tr}(A\Pi_i).
\end{equation}

Denote the expectation value of an operator \(A\) given the SIC state \(\Pi_i\) as
\begin{equation}
\langle A : i \rangle = \text{tr}(A\Pi_i).
\end{equation}

Then,
\begin{equation}
\langle A \rangle = (d+1) \sum_i p(H_i) \langle A : i \rangle - \frac{1}{d} \sum_i \langle A : i \rangle.
\end{equation}

We also know that
\begin{equation}
\sum_i \text{tr}(A\Pi_i) = \text{tr} \left[ A \sum_i \Pi_i \right] = d \text{tr} A.
\end{equation}

Each of the four operators \(XXX\), \(XYY\), \(YXY\) and \(YYX\) are themselves traceless. If we fix
\[\text{tr} A = 0,\]
then we obtain
\begin{equation}
\langle A \rangle = (d+1) \sum_i p(H_i) \langle A : i \rangle.
\end{equation}

This applies to each of the four operators, and also to linear combinations of them. It is more appropriate to use it for the individual operators, since those correspond to individual experiments, or to single trials in a multi-trial experiment.

If we followed classical intuition, we might say, “The expectation value for the random variable \(A\), if the system is in configuration \(\Pi_i\), is some number \(\langle A : i \rangle\). We don’t know what configuration the system is really in, so we have some probability spread over \(i\). To find the expectation value of \(A\), we just weight the \(\langle A : i \rangle\) according to those probabilities.” However, this does not give the correct answer. The classical result is off by a factor \((d+1)\).

We can calculate the \(\langle A : i \rangle\) for the Hoggar SIC. In fact, the peculiar symmetry of the Hoggar SIC makes the salient features of the computation rather easy to derive. The four operators in Mermin’s inequality are elements of the group \(\{D_k\}\) that generates the Hoggar SIC. Therefore, each of the four of them satisfies
\begin{equation}
|\langle \psi_i | D_k | \psi_i \rangle|^2 = \frac{1}{d+1} = \frac{1}{9}.
\end{equation}

Furthermore, each operator \(D_k\) is Hermitian, so its eigenvalues are real, as is its expectation value given any state. Consequently,
\begin{equation}
\langle D_k : i \rangle = \langle \psi_i | D_k | \psi_i \rangle = \pm \frac{1}{3}.
\end{equation}

This applies to each term in our linear combination of expectation values, Eq. (1). When we combine the expectation values for the four operators, the contributions might cancel each other, depending on the relative signs, but the absolute value of the sum total cannot
exceed $4/3$. This is safely within the interval that a local hidden variable explanation could account for. So, the Hoggar SIC states cannot be used as to violate the three-qubit Bell inequality. This will remain true for any SIC that is generated from a fiducial by applying the three-qubit Pauli group.

By doing the algebra explicitly, we find that the Hoggar SIC states do not even reach the bound of $4/3$ that we deduced. In fact,

$$|⟨XXX : i⟩ − ⟨XYY : i⟩ − ⟨XYX : i⟩ − ⟨YYX : i⟩| = \frac{2}{3} \forall i. \quad (26)$$

Furthermore, any probabilistic combination of the Hoggar SIC states will also be consistent with the LHV bound. That is, if we pick a state from the Hoggar SIC following the probability distribution $p(H_i)$, then the linear combination of the four expectation values will stay safely in the classical region. If we then average over $i$, then this will remain true, no matter what the distribution $p(H_i)$ is.

However! The GHZ state itself corresponds to some probability distribution $p_{GHZ}(H_i)$, because we can write any state in the Hoggar SIC representation. Let the index $O$ range over the four operators that we use to define the three-qubit Bell inequality:

$$O ∈ \{XXX, −XYY, −YXY, −YYX\}. \quad (27)$$

For any of our four operators $O$,

$$∑_{i} p_{GHZ}(H_i)⟨O : i⟩ = \frac{1}{9}, \quad (28)$$

meaning that the quantum expectation value is scaled up by the urgleichung’s factor $d + 1$:

$$⟨O⟩ = (d + 1) ∑_{i} p_{GHZ}(H_i)⟨O : i⟩ = 1. \quad (29)$$

Therefore,

$$∑_{O} ∑_{i} p_{GHZ}(H_i)⟨O : i⟩ = \frac{4}{9}. \quad (30)$$

This is within the classical interval $[-2, 2]$, but when we account for the extra factor in the urgleichung, we find

$$(d + 1) ∑_{O} ∑_{i} p_{GHZ}(H_i)⟨O : i⟩ = 4. \quad (31)$$

It is that factor of $(d + 1)$ that lifts us over the edge into nonclassical territory.

One way to interpret this result is as a bridge between interference experiments and Bell–Kochen–Specker phenomena. Interference phenomena are weakly nonclassical: That is, the bare fact of interference can occur in fundamentally classical theories [6, 49]. However, by adopting the proper mindset, we can strengthen the double-slit experiment into a genuine test for nonclassicality.

Interference between nonorthogonal alternatives — in other words, between alternative paths represented by nonorthogonal quantum states — can be a stronger test of nonclassicality than the double-slit experiment as it is normally described. This is because generalizing to nonorthogonal states allows the “which-way” information to be the outcome of an informationally complete measurement. (Heuristically speaking, this ties in with the idea that
pre- and post-selection effects with nonorthogonal states are more strongly nonclassical than they are when one considers only orthogonal states [49, 50].

Mermin wrote that the n-qubit GHZ state “combines two of the most peculiar features of the quantum theory” [9], interference of probabilities and the failure of local hidden-variable explanations. Using the Hoggar SIC, we have found a concise expression of this when \( n = 3 \). Correlations that violate the three-qubit Bell inequality encode a kind of interference that defies mimicking by classical randomness.

Mermin’s three-qubit Bell inequality is closely related to the GHZ thought-experiment, which is sometimes touted as an example where the distinction between quantum and classical is “all-or-nothing”. The hypothesis of local, intrinsic hidden variables implies one result with certainty, and quantum mechanics implies another, also with certainty. Stated carelessly, this can create the impression that probabilities are not involved. But a prediction made with probability 0 or 1 is still a probabilistic statement. Moreover, we can see the nontrivial probabilities churning just below the surface.

In the GHZ scenario, Alice measures the \( X \) observable on each of her three qubits and checks the parity of the answer. Writing \(|+\rangle\) and \(|-\rangle\) for the eigenstates of \( X \), and denoting the SIC representation of the state \(|++ +\rangle\) by \( p_{+++} \), she calculates that

\[
\sum_i p_{\text{GHZ}}(H_i)p_{+++}(H_i) = \frac{5}{288}.
\]  

(32)

The same result holds for the other states of the same parity, \( p_{+-+} \), \( p_{-+-} \) and \( p_{--+} \). Classical intuition would lead her to say that this number is the probability for obtaining each of the odd-parity outcomes, given a preparation described by \( p_{\text{GHZ}} \). In turn, the probability for getting any odd-parity outcome would be the sum of the probabilities for the four alternatives. But she knows to take the quantum correction, which is given by the urgleichung:

\[
P(\text{odd}) = d(d+1)\sum_i p_{\text{GHZ}}(H_i)[p_{+++}(H_i) + p_{+-+}(H_i) + p_{-+-}(H_i) + p_{--+}(H_i)]
- \sum_i [p_{++ +}(H_i) + p_{+--}(H_i) + p_{-- +}(H_i) + p_{---}(H_i)].
\]  

(33)

This evaluates to

\[
P(\text{odd}) = 72 \cdot 4 \cdot \frac{5}{288} - 4 = 1.
\]  

(34)

So, while ascribing the GHZ state does imply predictions with probability unity, that unity arises from the combination of many fractions.

V. QUBIT PAIRS AND TWINNED TETRAHEDRAL SICS

In this section, we change perspective slightly. Instead of applying one SIC measurement to the entirety of a tripartite system, we start with a smaller SIC and apply measurements based on it to each of two halves of a bipartite system. The end result will be a sharpened intuition for the nonclassicality of qubit pairs.

We have seen how attempting to interpret a SIC outcome as a specific, pre-existing physical property leads to a contradiction with the predictions of quantum theory. Any assumption which would incline us to interpret SIC outcomes in this way is, therefore, an assumption that would lead the unwitting physicist into error and would stand in the way of using quantum theory fruitfully. We can identify one such counterproductive idea — the EPR criterion of reality [51]:
If, without in any way disturbing a system one can [gather the information required to] predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

We now present a scenario in which the EPR criterion leads the unwitting physicist to conclude that SIC outcomes are pre-existing, specific “elements of physical reality.”

Alice arranges the following experiment. A device produces pairs of qubits to which Alice ascribes a maximally entangled state. Each qubit then travels to one of two widely separated instruments, which we can designate the left detector and the right detector. The detectors each have a control knob that can be turned to four different settings. Alice models the detectors using binary POVMs defined using the states comprising two SICs. The first SIC, which we can denote \( \{ \Pi_i^+ \} \), is a set of four projectors that together form the vertices of a tetrahedron inscribed in the Bloch sphere. The second SIC, \( \{ \Pi_i^- \} \), forms the tetrahedron whose vertices are antipodal to those of the first. Together, the two tetrahedra form a stellated octahedron. When the knob on a detector is set to position \( i \), it implements the POVM

\[
\{ \Pi_i^+, I - \Pi_i^+ \} = \{ \Pi_i^+, \Pi_i^- \}.
\] (35)

Consider first the case when Alice sets the two control knobs to the same position. She performs the measurement with one detector, say the one on the left. If she experiences the + outcome, she can predict with 100% certainty that she would experience the – outcome, if she were to walk over to the right-hand detector and test the other qubit. Likewise, if she experiences the – outcome on the left, she can predict with a probability of unity that she will experience + upon using the detector on the right. This holds true for all four values of the control setting \( i \).

Alice, deciding to entertain the EPR criterion, concludes that there exists within both particles emitted from the common source an “element of physical reality” that implies the outcome of each of the binary tests.

What happens when Alice chooses to set the two detectors differently? Now, if she performs test \( i \) on the left and obtains the + outcome, she updates her state for the right-hand particle to \( \Pi_i^- \). She then performs the test for some detector setting \( j \neq i \) on the right. Her probability of obtaining the – outcome on the right is

\[
\text{tr}(\Pi_i^- \Pi_j^-) = \frac{1}{d+1} = \frac{1}{3}.
\] (36)

Likewise, if Alice first experiences the – outcome on the left, she updates her state for the right-hand side to \( \Pi_i^+ \), and her probability for obtaining the + result on the right is

\[
\text{tr}(\Pi_i^+ \Pi_j^+) = \frac{1}{3}.
\] (37)

In summary, when Alice sets the detector controls to the same position, her probability of an anti-coincidence (+ on one device, – on the other) is unity. If she sets the detector controls differently, her probability of anti-coincidence is 1/3.

Can Alice account for these results in terms of hidden variables? Guided by the EPR criterion, she postulates that each particle carries an “instruction set” \([52]\) of the form \( \lambda_0 \lambda_1 \lambda_2 \lambda_3 \). Each \( \lambda_i \) is a pre-existing physical property of some kind, intrinsic to a particle, which can be thought of as taking values in the set \{+, –\}. The value of \( \lambda_i \) specifies the outcome of testing that particle with a detector configured to setting \( i \).
Alice hypothesizes that the source produces particle pairs with anticorrelated instruction sets:

\[
\begin{align*}
\{+ + --, -- ++, - + + +, + - - -, + + - -, - - + +\} \\
\end{align*}
\]

with

\[
\begin{align*}
\{- - ++, + + --, + - + -, - + --, - - + +, + + - -\}.
\end{align*}
\]

Alice finds that whichever instruction sets the particles carry, if she configures her two detectors identically, these instruction sets imply perfect anti-coincidence. If she instead sets her detector knobs to different positions, each choice of detector configurations will produce anti-coincidence with probability 1/3, provided that all six of these instruction-set pairs occur with equal probability.

How should Alice proceed from this point? She supposes, as a physicist naturally would, that whatever an instruction set is, a particle can carry one all by itself. The source in this experiment, Alice figures, happens to produce particles in pairs with perfectly anti-correlated instruction sets. To imagine that a particle only has an instruction set when it is produced as half of a pair strikes her as a touch pathological. A spinning top has angular momentum whether or not it is started into motion at the same time as another top, spun in the opposite direction.

Let \(T(i)\) be Alice’s probability for obtaining the + outcome when performing test \(i\) on an isolated system. Quantum theory tells us that we can craft another measurement corresponding to the four-outcome POVM

\[
\{\frac{1}{2} \Pi_0, \frac{1}{2} \Pi_1, \frac{1}{2} \Pi_2, \frac{1}{2} \Pi_3\}. \tag{39}
\]

Alice’s probability for obtaining outcome \(i\) in this experiment is

\[
p(H_i) = \frac{1}{d} \text{tr}(\rho \Pi_i) = \frac{1}{2} T(i). \tag{40}
\]

What is Alice’s interpretation of this four-outcome experiment in terms of her hidden-variable hypothesis? Suppose that she has \(T(0) = 1\). In quantum language, this means that her state for the system is the projector \(\Pi_0^+\). Referring back to the instruction sets listed in Eq. (38), Alice notes that three of them predict + for the binary test on the first element:

\[
\{+ + --, + - --, -- + +\}. \tag{41}
\]

Selecting a + at random from this list, Alice finds that she obtains a + in position 0 with probability 1/2, and in each of the other positions with probability 1/6. So, she can interpret \(p(H_i)\) as the probability that a + sign, chosen at random from all the + signs occurring in all possible instruction sets, falls in position \(i\).

The hypothesis of instruction sets implies that the outcome of a tetrahedral SIC measurement, Eq. (39), is a classical random variable. To adapt Einstein’s phrase, the SIC outcome is there even when nobody looks. Knowing that the SIC measurement is informationally complete, and seeing that its outcome probabilities are determined by the probability distribution over the six instruction sets, we conclude that the distribution over the instruction sets is all that is necessary to calculate the outcome statistics for any experiment.
There is another route to the instruction sets in Eq. (38), which begins with a set of desiderata that Spekkens provides for a noncontextual hidden-variable model [53]. The guiding philosophy of the Spekkens criteria is that two quantities which imply the same statistics should have the same representation in terms of probability distributions over the underlying hidden variables. If two preparations of a system yield the same statistics for all possible measurements, then those two preparations correspond to the same distribution over \( \lambda \). Likewise, if two measurements have the same statistics for all possible preparations, then those two measurements correspond to the same conditional probabilities of outcomes given \( \lambda \)'s. The key quantities are effects, that is, positive semidefinite operators that satisfy

\[
0 < E \leq I.
\]

(42)

Call the set of all effects \( \mathcal{E} \). By hypothesis, if Alice knows the ontic state \( \lambda \), she has a map from effects to probabilities:

\[
w : \mathcal{E} \rightarrow [0, 1].
\]

(43)

The function \( w \) will generally depend upon \( \lambda \). What properties will it satisfy? First, it obeys a sum rule. For any discrete set of effects \( \{E_i\} \subset \mathcal{E} \), if \( \sum_i E_i \) is also an effect, then

\[
w\left(\sum_i E_i\right) = \sum_i w(E_i).
\]

(44)

We will only need the particular special case of this in which the sum of the \( \{E_i\} \) is the identity operator, i.e., when the set of effects is a POVM. This is equivalent to saying that whatever the underlying ontic state of the system, when Alice applies a measurement, she is sure that something has to happen.

Furthermore, for any effect \( E \in \mathcal{E} \) and real number \( s \in [0, 1] \), if \( sE \in \mathcal{E} \), then

\[
w(sE) = sw(E).
\]

(45)

Again, we will only need a special case of this, specifically the case when \( s = 1/2 \). This is equivalent to saying that for any measurement, we can post-process the outcome by flipping a fair coin.

The identity effect is assigned unit probability:

\[
w(I) = 1.
\]

(46)

If Alice doesn’t care at all about what she does, then her probability of “whatever” happening is 1, regardless of the ontic state. When else can she have certainty? If and only if the effect in question is a projection [54]:

\[
w(E) \in \{0, 1\} \text{ if and only if } E^2 = E.
\]

(47)

What do these conditions imply for a qubit SIC? First, the SIC states form a POVM when scaled down by the dimension:

\[
\sum_i \frac{1}{2} \Pi_i = I.
\]

(48)

Therefore, it must be the case that

\[
\sum_i w\left(\frac{1}{2} \Pi_i\right) = 1.
\]

(49)
In turn, by the post-processing assumption,

\[ \sum_i w\left(\frac{1}{2}\Pi_i\right) = \frac{1}{2} \sum_i w(\Pi_i). \quad (50) \]

Each \( \Pi_i \) is a projector, so each \( w(\Pi_i) \) on the right-hand side must be either 0 or 1. Because the sum total must be normalized, exactly two terms are 0, while the other two both equal 1. Consequently, the instruction sets in Eq. (38) are the only configurations of hidden variables that are compatible with noncontextuality and with the structure of a qubit SIC measurement.

If we postulate that a tetrahedral SIC measurement \( \{\frac{1}{2}\Pi_i^+\} \) is possible, and we assert that the hidden-variable description of the qubit is noncontextual, then any quantum state for the qubit implies a probability distribution \( \varrho(\vec{\lambda}) \) over the six instruction sets in Eq. (38). In turn, such a probability distribution implies a \( p(H_i) \), specified by

\[ p(H_i) = \frac{1}{2} \sum \varrho(\vec{\lambda})\delta_{\lambda_i,+}. \quad (51) \]

This has a ready interpretation in terms of a two-step stochastic process. Effectively, we are picking an instruction set at random with probability \( \varrho(\vec{\lambda}) \), and then we are flipping a fair coin to select one of the two + signs in that instruction set.

By mapping points in the Bloch ball to density operators, and then solving for the corresponding hidden-variable distributions, we can map out the “classical region” of qubit state space. We define this region to be the subset of state space within which all elements of \( \varrho \) turn out nonnegative, meaning that the vector \( \varrho \) can be interpreted as an ordinary probability distribution, rather than a quasiprobability one. The eight states that comprise the vertices of the SICs \( \{\Pi_i^+\} \) and \( \{\Pi_i^-\} \) are classical, by this standard. The classical region of state space is the cube that is their convex hull.

Each of the six instruction sets in Eq. (38) is a “dispersion-free state” [26–29]. Using the SIC representation of qubit state space, we can see that they do not correspond to valid quantum states. Each instruction set implies a probability distribution \( p \) in which two elements equal \( 1/2 \) and the other two equal 0. Using Eq. (10), we can map these probability distributions to linear operators. The resulting operators will all be Hermitian, but they will not be positive semidefinite. Therefore, the dispersion-free states cannot be quantum states. Pictorially, they can be represented as the vertices of an octahedron outside the Bloch sphere: While the Bloch sphere has radius 1, the dispersion-free states all reside at a distance of \( \sqrt{3} \) from the origin.

We have seen that if we try to model the SIC states as essentially classical, then the eigenstates of the Pauli operators become maximally quantum, in that they lie as far as possible from the region of the Bloch ball for which a classical model exists. This is in a sense the dual of the statement that qubit SIC states are “magic states” when the Pauli eigenstates are treated as classical [55]. Consequently, we now have a certain intuition for the result of Andersson et al., who find that the maximal violation in an “elegant” two-qubit Bell inequality occurs when the measurements on one qubit are the Pauli eigenbases and the measurements on the other are the binary tests defined by pairs of antipodal SIC vectors [56].
VI. FAILURE OF HIDDEN VARIABLES FOR QUTRITS

The Spekkens criteria for hidden-variable models provide an alternative perspective on a Kochen–Specker proof that Bengtsson, Blanchfield and Cabello derive for qutrit systems [57]. Their proof relies upon a set of 21 vectors, 9 of which comprise a SIC and the other 12 of which form a particular set of orthonormal bases. These four bases have the nice property that they are all mutually unbiased with respect to one another. That is, the overlap of any vector from one basis with any vector from another basis is constant. In the Bengtsson et al. construction, the only properties that matter are the orthogonalities among the 21 vectors; by employing the fact that the set of 9 specifically form an informationally complete POVM, we can appreciate the result in a new way.

First, we note that if we have three vectors that form an orthonormal basis for \( \mathbb{C}^3 \), then the projectors onto those vectors add to the identity operator, meaning that by the Spekkens rules,

\[
 w(E_1) + w(E_2) + w(E_3) = 1. \tag{52}
\]

Furthermore, each term in the sum must be either 0 or 1, implying that whatever the underlying ontic state, exactly one vector in any orthonormal basis is assigned probability 1.

Now, we consider the cat’s cradle of vectors we encounter in dimension \( d = 3 \). First, there’s the Hesse SIC. Take \( \omega = e^{2\pi i/3} \), and construct the set of states \( \{ |\pi_j\rangle \} \) given by the columns of

\[
 \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & \omega & 0 & -\omega & \omega^2 & 0 & -\omega^2 \\ -1 & 1 & 0 & -\omega^2 & \omega^2 & 0 & -\omega & \omega & 0 \end{pmatrix}. \tag{53}
\]

We have a duality relation between the canonical mutually unbiased bases and the Hesse SIC. This relation is rather intricate: Each of the 9 SIC states is orthogonal to exactly 4 of the MUB states, and each of the MUB states is orthogonal to exactly 3 SIC states [19].

An easy way to remember these relationships is to consider the finite affine plane on nine points. Each of the points corresponds to a SIC vector, and each of the lines corresponds to a MUB vector, with point-line incidence implying orthogonality. The four bases are the four ways of carving up the plane into parallel lines (horizontals, verticals, diagonals and other diagonals).

Let the projectors onto the 9 SIC vectors be \( \Pi_1 \) through \( \Pi_9 \). We can uniquely identify each of the projectors onto the MUB vectors by the three SIC vectors to which they are orthogonal [19]. For example, \( M_{123} \) is orthogonal to \( \Pi_1, \Pi_2 \) and \( \Pi_3 \). The 12 MUB states are then

\[
 M_{123}, \; M_{456}, \; M_{789}; \\
 M_{147}, \; M_{258}, \; M_{369}; \\
 M_{159}, \; M_{267}, \; M_{348}; \\
 M_{168}, \; M_{249}, \; M_{357}; \tag{54}
\]

where each row corresponds to an orthonormal basis of \( \mathbb{C}^3 \).

Because \( \Pi_1 \) is orthogonal to \( M_{123} \), if the underlying ontic state \( \lambda \) implies \( w(M_{123}) = 1 \), then \( w(\Pi_1) = 0 \). The more of the \( \{ w(\Pi_i) \} \) that we can “zero out” in this way, the smaller their sum will be. By working through all the possibilities for assigning a \( w \) of unity to exactly one element of each basis, it is straightforward to show that whatever \( \lambda \) might be,

\[
 \sum_i w(\Pi_i) \leq 2. \tag{55}
\]
But from the post-processing rule,

$$w\left(\frac{1}{d}\Pi_i\right) = \frac{1}{d}w(\Pi_i),$$  \hspace{1cm} (56)

and from the sum rule,

$$\sum_i w\left(\frac{1}{d}\Pi_i\right) = w\left(\sum_i \frac{1}{d}\Pi_i\right) = w(I) = 1.$$  \hspace{1cm} (57)

Therefore,

$$\sum_i w(\Pi_i) = d\sum_i w\left(\frac{1}{d}\Pi_i\right) = 3.$$  \hspace{1cm} (58)

Our plans for a hidden-variable model have gone awry. The SIC states burst out of the confines that the orthonormal bases establish. The set of 9 and the set of 12 cannot coexist in the world of $\lambda$: If we take one set to have a classical representation, then the other cannot.

Bengtsson et al. derive a contradiction between the hypothesis of intrinsic hidden variables and the predictions of quantum theory by invoking the Born rule. This is equivalent to postulating a probability assignment for all POVMs, i.e., a frame function. If we know that the state space is the space of unit-trace positive semidefinite operators and that probabilities are calculated by the inner product between density matrices and effects, then we can say that

$$\sum_i \langle \Pi_i \rangle = \sum_i \text{tr}(\rho \Pi_i) = \text{tr}\rho = 3.$$  \hspace{1cm} (59)

By invoking Spekkens’ criteria for a hidden-variable model, we see that we do not have to postulate outcome statistics for all POVMs. Instead, we can derive the desired contradiction by considering only a discrete set of rays in the Hilbert space $\mathbb{C}^3$.

**VII. QUANTUM THEORY FROM NONCLASSICAL PROBABILITY MESHING**

Let us now return to the Hoggar SIC. We have seen how this configuration, and the representation of quantum state space that it furnishes, provides a link between interference phenomena and the failure of hidden-variable models. This mathematical construction — just sixty-four complex lines, making equal angles with one another — evidently cuts quite deeply into the quantum mysteries. Consider again our expression for the expectation value of an operator:

$$\langle A \rangle = (d + 1) \sum_i p(H_i)\langle A : i \rangle - \frac{1}{d} \sum_i \langle A : i \rangle.$$  \hspace{1cm} (60)

Seen in one way, this formula is a way to “do Feynman right”: Like the double-slit experiment, it captures the counterintuitive way quantum theory requires us to use expectations for one scenario to make deductions about another. However, it indicates a kind of interference that cannot be emulated by classical stochasticity. And, seen in another way, it opens the possibility of violating a Bell inequality.

This is a sufficiently appealing notion that one is naturally tempted to wonder how far it can go. If we take this idea as basic, if we make this way of relating expectations between counterfactual scenarios as a fundamental precept, what can we derive from it? The answer, potentially, is *quantum theory itself.*
During the twenty-first century, there has been increasing interest in the project of red-eriving or reconstructing the mathematical apparatus of quantum mechanics, by starting with a set of basic principles that, one hopes, are more meaningful or illuminating than the abstruse invocations with which one traditionally begins the subject [20]. These efforts begin with a set of axioms, typically expressed in operationalist terms as statements about what kinds of laboratory procedures are possible, and rederive quantum theory from that starting point [58–63]. Mathematically, these derivations are successful; however, they share the common feature that they make quantum theory as “benignly humdrum” as possible [20]. The remarkable and enigmatic phenomena seen within quantum physics are no closer to the surface than they were in the standard presentation of the formalism [46]. Indeed, the notions invoked in these axioms are often not that quantum at all. For example, the system of Chiribella, D’Ariano and Perinotti relies upon the purifiability of mixed states [64], which was originally discovered in quantum physics but actually arises naturally in the Spekkens toy model, a fundamentally classical theory [6, 65]. Moreover, it is not at first glance clear what these proposed sets of axioms have in common with each other, other than their conclusion.

In contrast, one research program aims to take the urgleichung as a basic postulate upon which quantum theory can be built [18, 20, 46]. The urgleichung embodies a rejection of the hypothesis of hidden variables, phrased in a way that does not depend upon the ordinary textbook formalism of Hilbert spaces and operators. The hope is that whatever deep lesson quantum physics has to teach us about the character of the natural world, we will see it most clearly by bringing the essential expression of it to the forefront, rather than deriving it as a consequence of “benignly humdrum” axioms. Postulating that the urgleichung is fundamental — that, try as one might to establish a standard reference measurement, nonclassical “deformation” of probabilities cannot be avoided — foregrounds the strangeness of quantum theory. What we know so far is that a reconstruction on these lines can be done, but at the price of invoking a couple additional presumptions that seem too specific to belong in the final answer. My own suspicion is that these additional requirements are stronger and more particular than is truly necessary. This is where drawing upon a variety of other reconstruction efforts may be helpful: They suggest that certain technical matters arising in the course of a reconstruction (e.g., the choice of a particular symmetry group) can be dispatched with relative ease.

In special relativity and in thermodynamics, one builds up the theory starting from postulates, the first of which has the character of a guarantee. (Inertial observers Alice and Bob can come to agree on the laws of physics; energy is conserved.) The second is a foil to the first, frustrating it and generating a degree of dramatic tension. (Alice and Bob cannot agree on a standard of rest, even by measuring the speed of light; entropy is nondecreasing.) Then comes a statement of unattainability, which is derived in one case (massive bodies cannot attain light speed) and assumed in the other (we cannot cool all the way to absolute zero). We might also draw an analogy between the clock postulate of special relativity, which lets us analyze accelerated motion using momentarily co-moving inertial frames, and the zeroth law of thermodynamics, which as it is applied in practice is a statement about momentary equilibrium between systems being a transitive condition. Might a similar story hold true for quantum mechanics as well? In the view of quantum theory we are developing here, the possibility of probability-1 predictions might be considered a guarantee (certainty is allowed). The urgleichung is then an axiom of frustration (certainty cannot be about hidden variables). Perhaps the rejection of hidden variables, carefully formulated in the
urgleichung, will one day be recognized as the Second Law of Quantum Mechanics.

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