On some 3-point functions in the $W_4$ CFT and related braiding matrix

P. Furlan* and V.B. Petkova**

*) Dipartimento di Fisica dell’Università di Trieste, Italy;
**) Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria

We construct a class of 3-point constants in the $sl(4)$ Toda conformal theory $W_4$, extending the examples in Fateev and Litvinov [1]. Their knowledge allows to determine the braiding/fusing matrix transforming 4-point conformal blocks of one fundamental, labelled by the 6-dimensional $sl(4)$ representation, and three partially degenerate vertex operators. It is a $3 \times 3$ submatrix of the generic $6 \times 6$ fusing matrix consistent with the fusion rules for the particular class of representations. We check a braiding relation which has wider applications to conformal models with $sl(4)$ symmetry. The 3-point constants in dual regions of central charge are compared in preparation for a BPS like relation in the $\hat{sl}(4)$ WZW model.

furlan@ts.infn.it, petkova@inrne.bas.bg
1. Introduction

The 2d conformal field theories (CFT) related to the \( sl(2) \) algebra, like the Virasoro, the WZW models with the affine \( sl(2) \) KM algebra and their supersymmetric extensions, are by now well established. This includes explicit expressions for basic data as the operator product expansion (OPE) coefficients (3-point functions) and the braiding/fusing matrices transforming conformal blocks. Much less is known about these structures in the CFT with higher rank symmetries, although a considerable progress in Toda CFT [2] was made by Fateev and Litvinov (FL) [1], [3]. Further advances in the field are important for the development of the higher rank 2d CFT as well as for potential applications in the string theory side of the AdS/CFT correspondence.

In the free field (Coulomb gas) approach the OPE constants are represented by complicated integrals which have to be computed explicitly before analytic continuation. The alternative derivation of functional relations arising from locality (crossing symmetry) of particular 4-point functions involving degenerate vertex operators requires the knowledge of fundamental braiding/fusing matrix elements, which in general are also part of the problem.

In [1], [3] Fateev and Litvinov developed a general method of recursively computing certain class of conformal integrals and gave explicit examples of 3-point constants. In the case of Toda \( W_3 \) theory they have as well computed the fundamental fusing matrix directly from the integral representations of the 4-point blocks; some partial results in the general \( W_n \) case were also obtained.

In this paper we are dealing with the \( sl(4) \) Toda conformal theory \( W_4 \). The 3-point functions known so far involve one vertex operator \( V_\beta \) with a degenerate charge \( \beta \) proportional to the fundamental weight \( \omega_1 \), or \( \omega_3 = \omega_1^* \), i.e., the highest weight of the 4-dimensional \( sl(4) \) representation. Our focus instead is on the symmetric representations \( \beta = \beta^* \) and, in particular, ”scalars” with \( \beta = -k\omega_2b \), where \( \omega_2 \) is the highest weight of the 6-dimensional fundamental \( sl(4) \) representation and \( k \) is arbitrary. The real parameter \( b \)

\[ 1 \] Apart from these traditional 2d methods a novel approach to the computation of the 3-point constants is provided by the (5d version of the) AGT-W relation [4], [5], see [6], where the main example of [1] has been recently reproduced, as well as references therein.

\[ 2 \] Here ”scalar” refers to the 4d context of conformal group representations, for which the components \( 2j_i = -\langle \beta, \alpha_i \rangle / b, i = 1, 3 \) label the \( SL(2, \mathbb{C}) \) spins, while \( \Delta = \langle \beta, \omega_2 \rangle / b \) corresponds to the 4d conformal dimension.
parametrises Toda central charge. In section 2 we compute a 3-point OPE constant for two scalar and one symmetric representations by deriving and solving a recurrence relation for the corresponding Coulomb gas integrals along the method of [7], [1] and then analytically continue it. A slightly more general 3-point constant is given in the Appendix.

In section 3 we use this data to derive the fusing matrix $F$ transforming the corresponding 4-point conformal blocks with one fundamental vertex $V_{-\omega_2 b}$. Here we follow a path somewhat opposite to the standard consideration in which one solves for the 3-point constants the system of equations implied by locality of the 4-point function. We shall not need the explicit integral realisation of this particular Toda 4-point function with three more partially degenerate representations of the type $\beta_a = -k_a \omega_2 b, a = 1, 2, 3$. In the intermediate channels appear also vertex operators with symmetric weights so that in the equations the more general constants of the type derived in section 3 are needed. The restriction to chiral vertex operators $V_{\beta_a}$ of such particular highest weights effectively restricts the braiding/fusing matrix to a $3 \times 3$ submatrix; its matrix elements are explicitly described.

Finally in this section we check a braiding identity, which is equivalent to a standard identity for the modular group on the sphere with 4 holes. This relation imposes restrictions solely on certain products of $F$ matrix elements and allows in principle for more general solutions for the individual $F$ matrix elements than the ones computed in the $W_4$ CFT. The semi-classical ”heavy charges” limit of the identity is a particular $sl(4)$ analog of the one exploited in the strong coupling $sl(2)$ sigma model constructions in [8], [9]. This suggests that the explicit expressions for the products of the fusing matrix elements extracted from Toda CFT (or their closely related WZW model counterparts) may eventually be used as a first step in higher rank generalisations of that work.

In the last section 4 we compare the 3-point constants in two regions of the central charge, analogs of the two Virasoro theory ingredients of Liouville gravity, $c > 25$ (Liouville) and $c < 1$ ("matter"). We discuss a BPS-like relation for the two sets of weights and show that the product of the two 3-point constants trivialises in the semi-classical ”light” charges” limit. Furthermore we discuss the possible implications for the related 3-point correlators in the corresponding WZW theories, the determination of which is still an open problem.
2. 3-point $W_4$ constants

We consider the $W_4$ CFT with central charge

$$c_T = 3(1 + 20Q^2) = 3(41 + 20(b^2 + \frac{1}{b^2})) > 243, \quad Q = \frac{1}{b} + b, \quad (2.1)$$

for real values of the parameter $b$. We shall skip the detailed presentation of the basics of Toda conformal theory and the free field (Coulomb gas) representation of the correlation functions: the reader is referred to [1], as well as to the original paper of Fateev and Lukyanov [2], formulated in the dual region of central charge with $b \to ib$ in (2.1).

The OPE constant of 2d scalar vertex operators is

$$c(\beta_1, \beta_2, 2\rho Q - \beta_3) = \lim_{x_3 \to \infty} (x_3^2)^{2\Delta(\beta_3)} \langle V_{2\rho Q - \beta_3}(x_3)V_{\beta_2}(1)V_{\beta_1}(0) \rangle_{\text{Coulomb}}$$

where the conformal dimension is given by the $sl(4)$ scalar product

$$\Delta(\beta) = \frac{1}{2}(\beta, 2\rho Q - \beta) \quad (2.2)$$

and $\rho = \sum_{i=1}^{3} \omega_i$ is the Weyl vector. The dimension (2.2), as well as the two other $W_4$ quantum numbers, are invariant with respect to an action of the Weyl reflection group

$$w \ast \beta = Q\rho + w(\beta - Q\rho). \quad (2.3)$$

The Coulomb gas representation of the OPE constant is defined for the screening charge conservation condition

$$\beta_{12}^3 + b \sum_i s_i \alpha_i := \beta_1 + \beta_2 - \beta_3 + b \sum_i s_i \alpha_i = 0 \Rightarrow bs_i = -(\beta_{12}^3, \omega_i). \quad (2.4)$$

The integers $s_i$ in front of the simple roots $\alpha_i$ in (2.4) count the number of screening charge vertex operators $V_{\beta_i}(z, \bar{z})$ of type $\beta_i = \alpha_i b, i = 1, 2, 3$, which are spinless fields of dimension $\Delta(\alpha_i b) = 1$. Formula (2.4) describes a generic fusion rule in which $\beta_3$ is obtained by a shift of, say, $\beta_1$ with the weight diagram $\Gamma_{-\beta_2/b} = \{-\beta_2/b - s_i \alpha_i\}$ of the representation of highest weight $-\beta_2/b$, times $(-b)$.

The OPE constant is given by a $\sum_i s_i$-- multiple 2d integral $I_{s_1,s_2,s_3}(\beta_1, \beta_2)$. We compute this integral in the particular case when $\beta_a = -k_a \omega_2 b, a = 1, 2$ and symmetric $\beta_3 = (-l(\omega_1 + \omega_3) - k_3 \omega_2)b = \beta^*$, so that in (2.4) $s_1 = s_3$. We shall skip the detailed computation since it follows straightforwardly the steps of the method explained in [1],
which is based on the use of a $sl(2)$ type duality formula [7] in order to derive recursion relations for $sl(n)$ Toda multiple 2d integrals. In our case after $s$ steps one gets an integral of type $I_{s_1-s,s_2-2s,s_3-s}(\beta_1+s\omega_2b,\beta_2+s\omega_2b)$ so that setting $s=s_3=s_1$ the integral is reduced to a Liouville type integral which is known. In particular for $\beta_2=-\omega_2b$ the resulting formula reproduces (in agreement with the general formula (1.51) of [1]) the structure constants $c_h:=c(\beta_1,-\omega_2b,2\rho Q-(\beta_1-hb))$ of the fusion of $V_{\beta_1}$ with the fundamental field $V_{-\omega_2b}$ corresponding to 3 of the 6 points of the weight diagram $\Gamma_{\omega_2}$, i.e.,

$$\beta_3 = \beta_1 - hb \text{ with } h = \omega_2; \ h = \omega_2 - \omega_2 = w_2(\omega_2); \ h = -\omega_2 = \omega_2 - 2\alpha_2 - \alpha_1 - \alpha_3 = w_{2312}(\omega_2).$$  

(2.5)

The next step is the standard analytic continuation of the constant, to be denoted $C(\beta_1,\beta_2,2\rho Q-\beta_3)$, in which one first gets rid of the integers $s_1$ and $s_2-2s_1$ exploiting the charge conservation condition (2.4), so that the Coulomb gas OPE constant is reproduced as a double residue. We shall write down the related formula for $\beta_3 = \beta_3^* \rightarrow 2\rho Q - \beta_3$, equivalently obtained by multiplication with the reflection amplitude $R(\beta_3)$ corresponding to the longest Weyl group element $w_{121321}$ [1]

$$R(\beta) = (b^{2(1-b^2)}\lambda_T)^{(2\rho Q - 2\beta, \omega)}/(2\rho Q - \beta)\prod_{\alpha > 0} \frac{Y_b((\rho Q,\beta,\alpha))}{Y_b((\beta,\rho Q,\alpha))},$$

(2.6)

namely,

$$C(\beta_1,\beta_2,\beta_3) = R(\beta_3)C(\beta_1,\beta_2,2\rho Q - \beta_3)$$

$$= (b^{2(1-b^2)}\lambda_T)^{(2\rho Q - 2\beta_3, \omega)}/(2\rho Q - \beta_3) \prod_{a=1,2} \frac{Y_b((\beta_a,\alpha_2))}{Y_b((\beta_{123} - 2\beta_a,\omega_2 - \omega_1))} \frac{Y_b((\beta_a - \rho Q,\alpha_24 + Q))}{Y_b((\beta_{123} - 2\beta_a,\omega_1) - Q)} \frac{Y_b((\beta_a,\alpha_24))}{Y_b((\beta_{123} - 2\beta_a,\omega_1) - Q)} \frac{Y_b((\beta_{123} - 2\beta_a,\omega_2 - \omega_1) - 2Q)}{Y_b((\beta_{123} - 2\beta_a,\omega_1) - 3Q)} \times$$

(2.7)

$3$ The formula derives from the observation that for the particular value $2b^2 = -1$ there are two free field representations for any $N$-point function in the Virasoro theory, s.t. in the second one all charges are replaced by their Weyl images, with the two numbers of screening charge operators summing up to $N-2$. The proportionality constant between the two integrals is given by the product of reflection amplitudes.

$4$ A more general OPE constant with $\beta_1 \neq \beta_1^*$ which yields a Coulomb gas correlator via three residua is given in the Appendix.
In (2.7) \( \lambda_T \) is proportional to the Toda cosmological constant, \( \lambda_T = \pi \mu T \gamma(b^2) \). In the products over positive roots in (2.7) the root \( \alpha_1 \) can be replaced with \( \alpha_3 \) and \( \alpha_{13} = \alpha_1 + \alpha_2 \) - with \( \alpha_{24} = \alpha_2 + \alpha_3 \) since \( \beta_3 = \beta_3^* \), \( \rho = \rho^* \). We have also used that the two weight \( \beta_1, \beta_2 \) have zero components \( (\beta_a, \alpha_i) = 0 \) for \( i = 1, 3 \) in order to write (2.7) in a form which makes it explicitly symmetric when the third weight \( \beta_3 \) is also chosen of this type, as we shall need it below: in that particular case the ratio in the second line of (2.7) produces a finite constant for \( (\beta_3, \alpha_1) \to 0 \), while \( (\beta, \omega_2 - \omega_1) = (\beta, \omega_1) \) in all scalar products of this type in the denominators in the last two lines.

Using (2.3) the terms in (2.7) depending on the three vertices, i.e., the eight \( \Upsilon_b \)-factors in the denominators of the last two lines, can be also written as points on an orbit of the Weyl group acting on the three weights (as discussed, e.g., in [6] for the FL example)

\[
\left( \Upsilon_b((\beta_{123} - 2\rho Q, \omega_1)) \Upsilon_b((w^{(3)}_{13} \star \beta_{123} - 2\rho Q, \omega_1)) \Upsilon_b((w^{(3)}_{121321} \star \beta_{123} - 2\rho Q, \omega_1)) \right)^{-1} \prod_{j=1}^{3} \Upsilon_b((w^{(j)}_{2132} \star \beta_{123} - 2\rho Q, \omega_1)) \prod_{i=1,2} \Upsilon_b((w^{(i)}_{2132} w^{(3)}_{13} \star \beta_{123} - 2\rho Q, \omega_1))
\]

(2.8)

where \( w^{(1)} w^{(3)} \star \beta_{123} = w' \star \beta_1 + \beta_2 + w \star \beta_3 \), etc..

3. Locality, fusing matrix, braiding identity

Consider the local 4-point function \( \langle V_f V_{\beta_1} V_{\beta_2} V_{\beta_3} \rangle \) of primary spinless operators \( V_\beta(z, \bar{z}) \). The function admits different equivalent diagonal decompositions in conformal blocks. They are related by braiding transformations, i.e., matrix realisation of the braiding group with generators \( e_i \), \( i = 1, 2, 3 \) on the plane (Riemann sphere) with 4 holes; \( e_i \) is exchanging the chiral vertex operators at the \( i \)-th and \( i + 1 \)-th points and the notation refers to the fixed ordered points, not to the labels of the concrete interchanged operators. In particular the generators \( e_2 \) (for the above order of the corresponding chiral vertex operators) is represented by non-trivial braiding matrix \( B \) proportional to the fusing matrix \( F \)

\[
B_{\beta_1 - h_s b, \beta_2 - h_t b} \left[ \begin{array}{c} \beta_1 \\ f \\ \beta_3 \end{array} \right] \left( \epsilon \right) = e^{i \pi \epsilon (\Delta(\beta_3) + \Delta(f) - \Delta(\beta_1 - h_s b) - \Delta(\beta_2 - h_t b))} F_{\beta_1 - h_t b, \beta_2 - h_t b} \left[ \begin{array}{c} \beta_1 \\ f \\ \beta_3 \end{array} \right],
\]

(3.1)

\[
B_{\gamma, \delta} \left[ \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \right] \left( \epsilon \right) B_{\delta, \gamma'} \left[ \begin{array}{c} \beta_2 \\ \beta_1 \\ \beta_3 \end{array} \right] (-\epsilon) = \delta_{\gamma \gamma'}
\]
while $e_1$ and $e_3$, which exchange the operators in the first two, respectively last two, fixed points, reduce due to triviality of $F$, to diagonal matrices. Locality (symmetry under exchange of two 2d fields) requires that the function is invariant under such transformations which results in equations involving fusing matrix elements and products of 3-point constants. In the case under consideration $f = -\omega_2 b$ is a degenerate field, so the equations take the form of a finite sum, e.g., for the exchange of $V_{\beta_1}$ and $V_{\beta_2}$ they read

$$
\sum_{h_s \in \Gamma_{\omega_2}} \frac{c_{h_s}(\beta_1)}{c_{h_t}(\beta_2)} \frac{C(\beta_1 - h_s b, \beta_2, \beta_3)}{C(\beta_1 - h_t b, \beta_2, \beta_3)} F_{\beta_1 - h_s b, \beta_2 - h_t b} F_{\beta_1 - h_s b, \beta_2 - h_u b} = \delta_{h_t, h_u}.
$$

(3.2)

Here $c_{h_s}(\beta_1)$ is a shorthand notation for the OPE constant given by a residue of $C(\beta_1, -\omega_2 b, 2\rho Q - \beta)$ at the values $\beta = (\beta_1 - h_s b)$, see the general formula (1.51) in [1]. In particular $c_{h=\omega_2} = 1$. In general $h_s$ stands for the weights of the weight diagram $\Gamma_{\omega_2}$ of the 6 dimensional representation, but for our restricted set of highest weights $\beta_a = -k_a \omega_2 b$, $a = 1, 2, 3$, the three of the OPE coefficients $c_{h_s}$ vanish so we are left with summation over 3 of the weights, as given in (2.5). A shorthand notation for the matrix $F_{h_s, h_t} = F_{\beta_1 - h_s b, \beta_2 - h_t b}$ in the last equality in (3.2) is used. As indicated in the r.h.s. the matrix formed by the ratio of constants times $F$ can be identified with the inverse matrix $F^{-1}$

$$
\frac{c_{h_s}(\beta_1)}{c_{h_t}(\beta_2)} \frac{C(\beta_1 - h_s b, \beta_2, \beta_3)}{C(\beta_2 - h_t b, \beta_1, \beta_3)} F_{\beta_1 - h_s b, \beta_2 - h_t b} = (F^{-1})_{h_t, h_s}.
$$

(3.3)

It is furthermore required that

$$
(F^{-1})_{h_t, h_s} = F_{h_t, h_s}(\beta_2, \beta_1),
$$

(3.4)

a consequence of the pentagon relation for $F$ (or of the normalization relation in (3.1)). In a shorthand notation we shall denote $F_{\beta_1 - k\omega_2 b, \beta_2 - t\omega_2 b}(\beta_1, \beta_2) = F_{s,t}(\beta_1, \beta_2) = F_{s,t}$, $s, t = \pm 1$, $F_{\beta_1 - h b, \beta_2 - t\omega_2 b} = F_{h,t}$, for $h := \omega_2 - \alpha_2$, etc., suppressing the dependence on the third argument $\beta_3$.

The ratios in (3.3) will be denoted

$$
U_{h, h'}(\beta_1, \beta_2) := \frac{c_h(\beta_1)}{c_{h'}(\beta_2)} \frac{C(\beta_1 - h b, \beta_2, \beta_3)}{C(\beta_2 - h' b, \beta_1, \beta_3)} = \frac{U_{h, +}(\beta_1, \beta_2) U_{h', +}(\beta_2, \beta_1)}{U_{+, +}(\beta_1, \beta_2) U_{h', +}(\beta_2, \beta_1)}
$$

(3.5)
and thus one needs to compute all $U_{h+}$. We give the explicit expression of the first of these ratios, computed from (2.7)

$$U_{+,+}(\beta_1, \beta_2) := \frac{C(\beta_1 - b\omega_2, \beta_2, \beta_3)}{C(\beta_1, \beta_2 - b\omega_2, \beta_3)} = \frac{\gamma(1 + b(\beta_2 - \rho Q, \alpha_2))\gamma(1 + b(\beta_2 - 2\rho Q, \alpha_2))}{\gamma(1 + b(\beta_1 - \rho Q, \alpha_2))\gamma(1 + b(\beta_1 - 2\rho Q, \alpha_2))} \times \frac{\gamma(b((\beta^2_{13}, \omega_1) - \frac{b}{2}))\gamma(b((\beta^2_{23}, \omega_1) - Q - \frac{b}{2}))}{\gamma(b((\beta^2_{23}, \omega_1) - \frac{b}{2}))\gamma(b((\beta^1_{23}, \omega_1) - Q - \frac{b}{2}))}.$$  \hspace{1cm} (3.6)

By analogy with the Liouville case this suggests the following ansatz for $F$:

$$F_{\beta_1 - \omega_2b, \beta_2 - \omega_2b} \left[ \begin{array}{c} \beta_1 \\ -\omega_2b \\ \beta_2 \\ \end{array} \right] = F_{++,+}(\beta_1, \beta_2) = \frac{\Gamma(b(\rho Q - \beta_2, \alpha_2))\Gamma(1 - b(\rho Q - \beta_1, \alpha_2))}{\Gamma(b(\beta^2_{13}, \omega_1) - \frac{b}{2})\Gamma(1 - b(\beta^2_{23}, \omega_1) - \frac{b}{2})} \times \frac{\Gamma(b(2\rho Q - \beta_2, \alpha_2))\Gamma(1 + b(\beta_1 - 2\rho Q, \alpha_2))}{\Gamma(b(\beta^2_{31}, \omega_1) - \frac{b}{2} - Q)\Gamma(1 - b(\beta^1_{23}, \omega_1) - \frac{b}{2} - Q)}.$$  \hspace{1cm} (3.7)

From (3.7) one computes $F_{++}(\beta_2, \beta_1)$ and confirms, using (3.3), that it indeed satisfies (3.4)

$$F_{++}(\beta_2, \beta_1) = \frac{C(\beta_1 - \omega_2b, \beta_2, \beta_3)}{C(\beta_1, \beta_2 - \omega_2b, \beta_3)} F_{++,+}(\beta_1, \beta_2) = (F^{-1})_{++}.$$  

Altogether we have

$$U_{+,+} F_{++}^2 = F_{++}(\beta_2, \beta_1) F_{++,+}(\beta_1, \beta_2) = \frac{\sin \pi((\beta^2_{13}, \omega_1) - \frac{b}{2})\sin \pi((\beta^2_{23}, \omega_1) - \frac{b}{2})b}{\sin \pi(b(\rho Q - \beta_1, \alpha_2))\sin \pi(b(\rho Q - \beta_2, \alpha_2))} \times \frac{\sin \pi((\beta^2_{31}, \omega_1) - \frac{b}{2} - Q)b}{\sin \pi((2\rho Q - \beta_1, \alpha_2)b)\sin \pi(2\rho Q - \beta_2, \alpha_2))} = \frac{ABA'B'}{P1[\beta_1]P1[\beta_2]P2[\beta_1]P2[\beta_2]}.$$  \hspace{1cm} (3.8)

Here $P[k][\beta] := \sin \pi b((\beta, \alpha_2) - kQ)$ and $A, B, A', B'$ denote the four sin's in the numerator correspondingly.

We proceed in this way to obtain $F_{h,+}$ for the other two shifts of $\beta_1 \to \beta_1 - hb$. Then with the help of simple trigonometric relations one checks and proves the first of the diagonal equations in (3.2), for $h_t = h_u = \omega_2$. Similarly one finds eight of the nine $F$ matrix elements checking the related equations. The expression for $U_{h,\tilde{h}}$, however, has a different structure, not suggesting straightforwardly an expression for $F_{h,\tilde{h}}$

$$U_{h,\tilde{h}}(\beta_1, \beta_2) = \frac{\gamma(3\rho Q - b(\beta_1, \alpha_2))}{\gamma(1 + b(\rho Q - \beta_1, \alpha_2))} \times \frac{\gamma((1 + b(3\rho Q, \alpha_2))}{\gamma(b(\beta_2 - \rho Q, \alpha_2))}.$$  \hspace{1cm} (3.9)

On the other hand writing the general expression of an inverse of a $3 \times 3$ matrix, $F^{-1}_{ij} = \frac{\delta_{ij}}{2 \det F} F_{mk} F_{nl}$ with $i, j, k, l, m, n = +, -, \tilde{h}$ we have, e.g.,

$$F_{\tilde{h}+,+}(\beta_2, \beta_1) = \frac{1}{\det F}(F_{-,+} F_{+,+} - F_{-,+} F_{+,+}).$$
etc. From this we can determine $\det F$:

$$
\det F(\beta_1, \beta_2) = - \prod_{\alpha = \alpha_2, \alpha_2 + \alpha_4, \alpha_1} \frac{(\beta_1 - \rho Q, \alpha)}{(\beta_2 - \rho Q, \alpha)}.
$$

(3.10)

Then, e.g., from

$$
F_{h, \bar{h}}(\beta_2, \beta_1) = \frac{1}{\det F}(F_{+,+}F_{-,+} - F_{+,+}F_{-,+})
$$

we can determine $F_{h, \bar{h}}$ and check the remaining identities in (3.2).

• Summarizing we get for the matrix elements of $F$ starting with (3.7)

$$
\begin{align*}
F_{-,+}(\beta_1, \beta_2) &= F_{++,}(w_{2132} \ast \beta_1, \beta_2) \\
F_{h,+}(\beta_1, \beta_2) &= F_{\beta_1 - b, \beta_2 - \omega_2 b} \left[ \begin{array}{c} \beta_1 \\ -\omega_2 b \\ \beta_3 \\ \beta_2 \end{array} \right] = \frac{\Gamma(1 - Qb)}{\Gamma(1 - 2Qb)} \times \frac{\prod \Gamma(1 + b(\beta_1 - 3\rho Q, \alpha_2)) \Gamma(1 - b(\beta_1 - \rho Q, \alpha_2)) \Gamma(b((3\rho Q - \beta_2, \alpha_2)) \Gamma(b((2\rho Q - \beta_2, \alpha_2))}{\prod \Gamma(b((\beta_{12}^3, \omega_1) - \frac{b}{2} - Q)) \Gamma(1 - b((\beta_{12}^3, \omega_1) - \frac{b}{2} - 2Q)) \Gamma(1 - b((\beta_{23}, \omega_1) - \frac{b}{2} - Q))}
F_{h,-}(\beta_1, \beta_2) &= F_{h,+}(w_{2132} \ast \beta_1, \beta_2) \\
F_{h,-}(\beta_1, \beta_2) &= F_{h,+}(w_{2132} \ast \beta_1, \beta_2) \\
F_{h,h}(\beta_1, \beta_2) &= \frac{\Gamma((\beta_2 - \rho Q, \alpha_2)b)\Gamma(1 + (\rho Q - \beta_1, \alpha_2)b)}{\Gamma(1 + (\beta_2 - 3\rho Q, \alpha_2)b)\Gamma((3\rho Q - \beta_1, \alpha_2)b)} \times \frac{(1 + 2\cos \pi Qb \sin \pi b((\beta_{12}^3, \omega_1) - \frac{b}{2} - Q) \sin \pi b((\beta_{12}^3, \omega_1) - \frac{b}{2} - 3Q)}{\sin \pi b((\beta_1, \alpha_2) - 3Q) \sin \pi b((\beta_2, \alpha_2) - 3Q)}.
\end{align*}
$$

(3.11)

The last matrix element can be written in various different ways.

Let us introduce some additional notation

$$
\begin{align*}
D &= D[\beta_1, \beta_2, \beta_3] := \sin \pi b((\beta_{123} - 2\rho Q, \omega_1) - \frac{b}{2}), \\
D' &= \sin \pi b((\beta_{123} - 2\rho Q, \omega_1) - \frac{b}{2} + Q) = D[\beta_1, \beta_2, w_{13} \ast \beta_3], \\
C &= \sin \pi b((\beta_{12}, \omega_1) - \frac{b}{2}) = D'[\beta_1, \beta_2, w_{2132} \ast \beta_3], \\
C' &= \sin \pi b((\beta_{12}, \omega_1) - \frac{b}{2} - Q) = D[\beta_1, \beta_2, w_{2132} \ast \beta_3],
\end{align*}
$$

(3.12)
and $A, A', B, B'$, explicitly described above, are also written in terms of Weyl group action

$$A = D'[\beta_1, w_{2132} \ast \beta_2, \beta_3], \quad A' = D[\beta_1, w_{2132} \ast \beta_2, \beta_3],$$

$$B = D'[w_{2132} \ast \beta_1, \beta_2, \beta_3], \quad B' = D[w_{2132} \ast \beta_1, \beta_2, \beta_3].$$

(3.13)

- Denoting $\tilde{F} = F(\beta_2, \beta_1)$, we have for the products of matrix elements in (3.2)

$$\tilde{F}_{++}F_{++} = \frac{AA'BB'}{P1[\beta_1]P2[\beta_1]P1[\beta_2]P2[\beta_2]}, \quad \tilde{F}_{+-}F_{--} = \frac{CC'DD'}{P3[\beta_1]P2[\beta_1]P1[\beta_2]P2[\beta_2]},$$

$$\tilde{F}_{-h}F_{h+} = \frac{2\cos \pi b^2 A'BCD'}{P1[\beta_1]P3[\beta_1]P1[\beta_2]P2[\beta_2]},$$

$$\tilde{F}_{-h}F_{h-} = \frac{2\cos \pi b^2 AB'C'D}{P1[\beta_1]P3[\beta_1]P3[\beta_2]P2[\beta_2]},$$

$$\tilde{F}_{h+h}F_{h-h} = \frac{(AA'BB' - CC'DD')^2}{P2[\beta_1]P2[\beta_2]P3[\beta_1]P3[\beta_2]P[\beta_1]P[\beta_2]} = \frac{(\cos \pi b^2 \cos \pi b((\beta_2, \alpha_2) - 2b) - \cos \pi b((\beta_1, \alpha_2) - 2b) \cos \pi b((\beta_2, \alpha_2) - 2b))^2}{P1[\beta_1]P1[\beta_2]P3[\beta_1]P3[\beta_2]}.$$ 

(3.14)

Compare with the Liouville case where $(\beta, \alpha) = 2\beta_L, (\beta, \omega) = \beta_L$ and

$$F_{s,t}^L = F_{\beta_1-s\omega b,\beta_2-t\omega b}^L \begin{pmatrix} \beta_1 & \beta_3 \\ -\omega b & \beta_2 \end{pmatrix}, \quad s, t = \pm 1$$

satisfying

$$F_{++}^L\tilde{F}_{++}^L = F_{++}^L \frac{1}{\det F} = F_{--}^L\tilde{F}_{--}^L = \frac{AB}{P1[\beta_1]P1[\beta_2]},$$

$$F_{+-}^L\tilde{F}_{--}^L = F_{+-}^L \frac{1}{\det F} = \tilde{F}_{--}^L = \frac{AB}{P1[\beta_1]P1[\beta_2]},$$

(3.15)

One can analogously compute the fusing matrix elements corresponding to the $sl(4)$ fundamental weights $\omega_1 = \omega_3^*$ using the 3-point constant computed by Fateev and Litvinov [1] in which two of the weights are arbitrary and the third is proportional to one of these

5 Analogous relations hold in the $sl(2)$ WZW case. The braiding matrices differ by $\beta$-independent phases (Q is replaced by $b$) and by normalisation so that effectively $\det F = 1 = \det B$, i.e., $F \in SL(2)$. 

9
fundamental weights. Partial data on the braiding matrices in that case is also provided (though in a different gauge) by the Boltzmann weights defining integrable $A_3^{(1)}$ lattice models [10] taking a proper limit of the spectral parameter.

- Finally we check a braiding relation relevant for the 4-point blocks under consideration, namely

$$\Omega_1\Omega_2\Omega_3 := (e_1^2)(e_2e_1^{-2})(e_3e_2^{-2}e_1^{-1}e_3^{-1}) = e^{-4\pi i \Delta(f)}$$

(3.16)

choosing the sign in (3.1) $\epsilon = 1$. In our case $\Delta(f) = \Delta(-\omega_2 b) = -\frac{5}{2} b^2 - 4$.

The meaning of the l.h.s of (3.16) is a composition of monodromies around the three vertex coordinates. On the sphere with 4 (ordered) points $e_1$ and $e_3$ are represented by diagonal braiding matrices; $e_2$ in $\Omega_2$ is represented by (3.1), while in $\Omega_3$ it is represented by the same braiding matrix with $\beta_3$ and $\beta_1$ exchanged. Using the defining relations of the braiding group

$$e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1}, \quad e_i e_j = e_j e_i \text{ for } j \neq i \pm 1$$

(3.17)

(3.16) is reduced to the first of the two additional relations on the generators \{ $e_i$ \} which characterise the modular group on the sphere with 4 holes [11]

$$e_1 e_2 e_3^2 e_2 e_1 = e^{-4\pi i \Delta(f)}, \quad (e_1 e_2 e_3)^4 = e^{-2\pi i (\Delta(f) + \sum_{a=1}^{3} \Delta(\beta_a))}.$$ 

(3.18)

Take the trace of (3.16)

$$\text{Tr}(\Omega_1 \Omega_2) = e^{-4\pi i \Delta(-\omega_2 b)} \text{Tr}(\Omega_3^{-1}).$$

(3.19)

The eigenvalues of the monodromy $\Omega$ are computed from the difference of Toda dimensions

$$p(\beta, h) = \Delta(\beta - h) - \Delta(\beta) - \Delta(-h\omega_2) = 2bQ + b(\beta - \rho Q, h)$$

(3.20)

where in general $h \in \Gamma_{\omega_2}$ is a weight in the 6-dim weight diagram $\Gamma_{\omega_2}$ of the fundamental representation $\omega_2 = (0, 1, 0)$, i.e., $h = \pm \omega_2, \pm(\omega_2 - \omega_1 - \omega_3), \pm(\omega_1 - \omega_3)$. Thus in general the trace of the diagonal monodromy

$$\text{Tr}(\Omega) = \text{Tr}(e_1^2) = \sum_{h \in \Gamma_{\omega_2}} e^{2\pi i p(\beta, h)} = e^{i\pi 4bQ} \chi_{\omega_2}(2\pi ib(\beta - \rho Q))$$

(3.21)
is proportional to the character \( \chi_{\omega_2}(\mu) \) of the fundamental representation \( \omega_2 = (0, 1, 0) \) evaluated at the "angle" \( \mu = 2\pi b(\beta - \rho Q) \). Denote by \( q(\beta) \) the normalized diagonal matrix 

\[
q_{h_s,h_t}(\beta) = \delta_{h_s,h_t} e^{2\pi i b(\beta - \rho Q,h_s)} = e^{-4\pi i Q \Omega}.
\]

In terms of \( F \) and its inverse \( \tilde{F} \) the relation (3.19) reads (collecting the three overall terms \( e^{4\pi i b^2} \) in the r.h.s.)

\[
\sum_{h_s,h_t} q_{h_s}(\beta_1) F_{h_s,h_t} \tilde{F}_{h_t,h_s} q_{h_t}(\beta_2) = e^{-4i\pi(3b^2 + \Delta(-\omega_2 b))} \text{Tr} q^{-1}(\beta_3). \tag{3.22}
\]

In our case of scalar (in 4d sense) weights \( \beta_a \) the relation involves a \( 3 \times 3 \) submatrix and accordingly \( q(\beta) \) reduces to the diagonal submatrix with matrix elements \( e^{2\pi i (\beta - \rho Q,h_s)} \), \( h_s = \pm \omega_2, \omega_2 - \alpha_2 \). Thus the sums in the l.h.s. of (3.22) run over these three weights, while the r.h.s. reduces to

\[
\text{r.h.s.'} = e^{-2i\pi b^2} (2 \cos 2\pi b(\beta - 2\rho Q, \alpha_2) + e^{2\pi i b^2}). \tag{3.23}
\]

Each of the products in (3.14) which appear in the l.h.s. of (3.22) in this case is a second order polynomial in \( 2 \cos \pi b(\beta - 2\rho Q, \alpha_2) \), as is the expression in (3.23), so the reduced relation is checked order by order.

Since in (3.22) \( F \) enters only through products \( F_{h_s,h_t} \tilde{F}_{h_t,h_s} \) it follows that (3.19) is a restriction on these products. Thus the products in (3.14) solving (3.19) with diagonal braiding determined from (3.20) may in principle admit more general solutions for the individual \( F \) matrix elements than the present Toda solution (3.7), (3.11). Indeed the \( sl(2) \) analog of this identity has been exploited in the recent papers [8], [9] on AdS\(_3 \times S^3\) sigma model correlators in the semi-classical strong 't Hooft coupling \( \lambda \) limit with large quantum numbers. In that case the eigenvalues of the monodromy matrix \( e^{\pm 2\pi i \eta(x)} \) depend on the spectral parameter \( x \) and the solution for the individual \( F = F(x) \) matrix elements depends nontrivially on the specific spectral curve. On the other hand the expression for the products \( F(x)\tilde{F}(x) \) functionally coincide with those in the WZW model (cf. (3.15) and footnote 5), taken in the semi-classical limit \( b^2 = 1/\sqrt{\lambda} \to 0 \) with three heavy charges \( \beta_a/b = \eta_a/b^2, \eta_a \) - finite, \( \eta_a = \eta(x) \). In this limit the phase in the r.h.s. of (3.16) reduces to 1.\(^6\) One may expect that the Toda theory data (3.14) can similarly be used as a starting point, although in this case the equation (3.19) is less restrictive by itself, compared with the \( sl(2) \) case where it uniquely determines the products \( F\tilde{F} \).

\(^6\) The assumption in [8],[9] concerning the identity (3.16) along with the hexagon identities (3.17) is equivalent to the statement that the 2d four-point solutions (normalized by the 3-point function) of the auxiliary linear system of equations transform linearly with respect to the \( B(x) \) matrix, so that the Wronskians of two related in these way solutions are expressed by the corresponding fusing matrix elements.
4. The 3-point functions in the compact ("matter") region. Comparisons.

By analogy with the Liouville gravity described by two dual Virasoro CFT with $c > 25$ (Liouville) and $c < 1$ ("matter") we shall extend here the results of section 2 to another region of central charge of the $W_4$ CFT, parametrised by the same real parameter $b$ as (2.1),

$$c_m = 3(1 - 20e_0^2) < 3, \ \ e_0 = \frac{1}{b} - b.$$  \hspace{1cm} (4.1)

The sum of central charges (2.1) and (4.1) is compensated by the contribution of the ghosts $c + c_m + c_{\text{ghost}} = 0$.

The conformal dimension of vertex operator $V_e$ is given by

$$\triangle_m(e) = \frac{1}{2} (e, e - 2\rho e_0),$$ \hspace{1cm} (4.2)

invariant under the action of the Weyl group

$$\hat{w}(e) := \rho e_0 + w(e - \rho e_0) = b(w \cdot \frac{e}{b} - \frac{1}{b^2} w \cdot 0)$$ \hspace{1cm} (4.3)

(i.e., the horizontal projection of the shifted action of the affine Weyl group elements $t_{-w,0}w$ on $(e/b + k\omega_0)$, times $b$, where $k + 4 = 1/b^2$). The minimal $W_4$ theory in the region (4.1) for rational $b^2$ has been discussed in [2]. Here the parameter $b$ is arbitrary and we shall consider vertex operators $V_e$ with symmetric charges $e = (r\omega_2 + s(\omega_1 + \omega_3))b = e^*$. Such $W_4$ representations are degenerate for nonnegative integers $r,s$. Once again we consider a 3-point function of vertex operators two of which have highest weights of type $e_a = r_a\omega_2b, \ a = 1, 2,$ and one with symmetric weight $e_3 = e_3^*$. The Coulomb gas computation is performed as before, or one can directly continue the Toda Coulomb gas OPE constants (being given by finite products of ratios of $\gamma$ functions) to $b^2 \to -b^2, Qb \to e_0b, \beta b \to eb$. This OPE constant can be expressed directly in terms of $\Upsilon_b$-functions with the result

$$C_m(e_1, e_2, e_3) = R_m(e_3) \frac{C_m(e_1, e_2, 2\rho e_0 - e_3)}{(b^2)^{Qb} b \lambda_m} \prod_{\alpha=1,2} \frac{\Upsilon_b((e_{123} - 2e_a, \omega_2 - \omega_1) + b)}{\Upsilon_b((e_a, \alpha_2) + b)} \frac{\Upsilon_b((e_{123} - 2e_0, \omega_1) + b)\Upsilon_b((e_{123} - 2e_0, \omega_1) - 3e_0 + b)}{\Upsilon_b((e_3, \alpha_2) + b)\Upsilon_b((e_3, \alpha_2) - e_0 + b)} \times \prod_{\alpha=1,2} \frac{\Upsilon_b((e_{123} - 2e_a, \omega_1) - e_0 + b)}{\Upsilon_b((e_a, \alpha_2) - e_0 + b)} \frac{\Upsilon_b((e_{123} - 2e_0, \omega_1) - e_0 + b)}{\Upsilon_b((e_{123} - 2e_0, \omega_1) - 3e_0 + b)}$$ \hspace{1cm} (4.4)
where $\lambda_m = \pi \mu_m \gamma(-b^2)$ with $\mu_m$ - the analog of the cosmological constant, multiplying the interaction term in the action. The reflection amplitude corresponding to the longest Weyl group element $w_{121321}$ is the analytic continuation of (2.6) (rewritten first as a finite ratio of $\gamma$-functions and then rewritten in terms of $\Upsilon_b$-functions)

$$R_m(e_3) = \left(b^2 Q^b \lambda_m\right)^{(2e_3 - 2\rho e_0, \rho)} \prod_{\alpha > 0} \frac{\Upsilon_b((e_3 - \rho e_0, \alpha) + b)}{\Upsilon_b((e_3 - \rho e_0, \alpha) + e_0 + b)}. \quad (4.5)$$

Analogously to (2.8) the eight three charge factors in (4.4) can be written as points on an orbit with respect to the shifted Weyl action (4.3). The $F$-matrix elements are obtained by the same analytic continuation of the Toda ones in (3.7), e.g.,

$$F_{e_1 + \omega_2, e_2 + \omega_2}^{e_1, e_2, e_3} = \frac{\Gamma(b(e_0 \rho - e_2, \alpha_2)) \Gamma(1 - b(\rho e_0 - e_1, \alpha_2))}{\Gamma(b((e_{23}, \omega_1) + b)) \Gamma(1 - b((e_{23}, \omega_1) + b))} \times \frac{\Gamma(b(2\rho e_0 - e_2, \alpha_2)) \Gamma(1 + b(e_1 - 2\rho e_0, \alpha_2))}{\Gamma(b((e_{31}, \omega_1) + b - e_0)) \Gamma(1 - b((e_{23}, \omega_1) + b - e_0))}. \quad (4.6)$$

etc. In what follows we shall restrict for simplicity to the case of three charges of type $e_a = (0, r_a b, 0)$.

- The $W_4$ CFT is described alternatively as the (principal) quantum Hamiltonian reduction of a $\hat{sl}(4)$ WZW model (or its dual). With the parametrisation in (2.1) and (4.1) in the noncompact and compact WZW analogs the corresponding Sugawara dimensions are given by

$$\Delta^S_u(\beta) = \frac{1}{2}(\beta, 2\rho b - \beta), \quad \Delta^S_m(e) = \frac{1}{2}(e, e + 2\rho b), \quad (4.7)$$

invariant (along with the higher Casimir eigenvalues) under the standard shifted action of the Weyl group on the $sl(4)$ weights $-\beta/b$ and $e/b$.

For any element $w$ of the Weyl group one has a BPS-like relation

$$\Delta^S_u(\beta) = \Delta^S_u(-b w \cdot \frac{e}{b}) = \Delta^S_u(b \rho - w(e + \rho b)) = -\Delta^S_m(e). \quad (4.8)$$

In particular there is only one nontrivial element of Weyl group, $w_{2132}$, s.t. its shifted action preserves the $sl(4)$ representations of type $(\lambda, \alpha_1) = 0 = (\lambda, \alpha_3)$, namely, $w_{2132} \cdot \lambda = -\lambda - 4\omega_2$, i.e., we have the relation

$$\beta = -b w_{2132} \cdot \frac{e}{b} = e + 4\omega_2 b \Rightarrow (\beta, \alpha_2)/b = (e, \alpha_2)/b + 4. \quad (4.9)$$
For the $W_4$ dimensions one has instead $\Delta(e + 4b\omega_2) + \Delta_m(e) = 8$. Therefore unlike the Virasoro case one cannot realise analogs of the tachyons simply by products of vertex operators from the two regions of the theory.\(^7\)

Nevertheless in view of the relation between the $\hat{sl}(4)$ WZW and the $W_4$ conformal theories we may expect that the 3-point constants in the two $W_4$ regions are closely related. Indeed, take all $e_a = (0, r_a, 0)b$. Then (4.4) is rewritten in terms of the inverse of the Toda constant in (2.7) with (4.9) imposed, i.e., $C(\beta_1, \beta_2, \beta_3) = C(e_1 + 4\omega_2 b, e_2 + 4\omega_2 b, e_3 + 4\omega_2 b)$

$$C_m(e_1, e_2, e_3) C(\beta_1, \beta_2, \beta_3) = \prod_{a=1}^{3} \phi((\beta_a, \alpha_2)) A(\beta_1, \beta_2, \beta_3) \bar{C}_m(e_1, e_2, e_3) \bar{C}(\beta_1, \beta_2, \beta_3)$$

$$= \lambda_T^{(2\rho Q - \beta_{123}, \rho)} \lambda_m^{(2\rho Q - 2\rho \alpha_0, \rho)} \prod_{a=1}^{3} \frac{(b^2)^3 \prod_{a=\alpha_2, \alpha_24, \alpha_14} \gamma((\beta_a - \rho b, \alpha)b)}{\gamma((\rho Q - \beta_a, \alpha_{24})b)} A(\beta_1, \beta_2, \beta_3),$$

where

$$A(\beta_1, \beta_2, \beta_3) = ((1 - b^4)^2((\beta_{123} - 2\rho Q, \omega_1)b)^2 \prod_{a=1}^{3}((\beta_{123} - 2\beta_a, \omega_1) - Q)b)^2)^{-1}.\quad (4.10)$$

The $\gamma$-factors in the second line of (4.10) (analogs of the leg factors) can always be removed by proper field normalisation. The intermediate notation $\bar{C}$ and $\bar{C}_m$ in the r.h.s. of the first equality refers to the constants obtained from the corresponding (2.7) and (4.4) by replacing $Q \to b$ and $e_o \to -b$, respectively, in the $\Upsilon_b$-functions. This is achieved by the use of the functional relations and produces finite products of $\gamma$-functions for each of the two constants, that are furthermore compensated in the product up to the factors $A(\beta_1, \beta_2, \beta_3)$ in (4.10) and $\prod_{a} \phi((\beta_a, \alpha_2))$, the explicit expression of which we skip.

The product $\bar{C}\bar{C}_m$ itself is trivial up to field renormalization: roughly the factor in the third (fourth) line in (2.7) cancels the one in the fourth (third) line in (4.4) respectively. In view of the relation of the WZW and Toda theory one may expect that the two modified constants $\bar{C}_m(e_1, e_2, e_3)$ and $\bar{C}(\beta_1, \beta_2, \beta_3)$ will describe the main structure (up to some field normalization) of the corresponding 3-point constants of the compact and noncompact WZW model.\(^8\) This conjecture remains to be checked. In any case the triviality (up to a an

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\(^7\) The same constant appears for the dual transformation $\Delta(-\frac{1}{b}w_{2132} \cdot (-be)) + \Delta_m(e) = 8$. Recall that in the Virasoro case the two relations yield $\Delta(ee + ab^2) + \Delta_m(e) = 1, \epsilon = \pm 1$ and describe the tachyons of the Liouville gravity [12],[13].

\(^8\) A heuristic argument in support: the integrands of the Coulomb gas representations of the WZW 3-point correlators (accounting for their isospin $SL(4)$ invariance) differ from the integrands of Toda ones by rational function of differences of coordinate and integration variables; this effectively modifies the weights $\beta$ in the powers by factors $k/b$. 

overall field renormalisation) of the product $\tilde{C}\tilde{C}_m$ whenever the relation (4.9) is imposed is a property expected for the correlators of BRST invariant states in the non-critical string model described by a $G/G$ topological CFT, see, e.g., [14].

In the semi-classical limit $b \to 0$ with "light" charges, i.e., $(\beta_a, \alpha_2)/b = \sigma_a$ are assumed finite, the factor in (4.10) which depends nontrivially on the three charges goes to a numerical constant, $A(\sigma_1 b, \sigma_2 b, \sigma_3 b) \to 1/9$. In other words in this limit the cancellation expected for the WZW counterparts of the $W_4$ constants holds true for these constants themselves.

We conclude with a remark about this "light-charge" limit of each of the constants $\tilde{C}_m(e_1, e_2, e_3)$ and $\tilde{C}(\beta_1, \beta_2, \beta_3)$ computed using the asymptotics of Barnes function

$$\lim_{b \to 0} \Upsilon_b(b)/\Upsilon_b(\sigma b) = \Gamma(\sigma)b^{\sigma-1}.$$ 

Take the limit of the subfactors in $\tilde{C}$ and $\tilde{C}_m$ corresponding to those given by the third line of (2.7) and (4.4) respectively: recall that $Q \to b$ and $e_0 \to -b$ and all weights are taken to be proportional to the second fundamental weight $\omega_2$. One recognizes in the resulting $\Gamma$-function ratios precisely the expressions of the 3-points constants of scalar 4d fields computed by integrating the boundary-bulk kernels over $AdS_5$ and $S^5$, respectively [15], [16]. In this comparison we identify the charges $(\beta_a, \alpha_2)/b$ - with the 4d scalar field conformal dimensions $\triangle_a$ and the weights $e_a/b$ (taking nonzero integer values) with the 4d isospins given by the SU(4) representation $(0, J_a, 0)$. The cancellation due to (4.9) implies $J_a = \triangle_a - 4$. On the other hand we can identify $(\beta_a, \alpha_2)/b = (\beta_a, \omega_2)/b$ with $\triangle_a + 4$. Then neither of the two factors in $\tilde{C}$ reproduces the $AdS_5$ result, but the trivialisation of the full $\tilde{C}\tilde{C}_m$ (and, in this limit, of the Toda constants product $CC_m$ itself) due to (4.9) holds true for $J_a = \triangle_a$, which is the actual 4d supersymmetric BPS condition for the given class of representations. Different identifications for the three weights are also possible (reminiscent of the mixed correlators discussed in [17]). The compensation mechanism is due to the integration over the full "bulk" group variables rather than the coset ones, cf. [18] for a direct classical computation of the Liouville correlator with light charges and [1] for generalisations to Toda theory.

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9 These are the $AdS_5 \times S^5$ free field ingredients of the 3-point function of "chiral primary operators" with $\triangle_a = J_a$ studied in [16]: the compensation trivializing the full constant is due to the additional factor in the coupling constant of the supergravity cubic interaction term.
5. Concluding remarks.

We have constructed 3-point functions in the $W_4$ Toda theory and have used them to derive novel data on a fundamental braiding/fusing matrix extending the rank 1 results. The solution described by a $3 \times 3$ matrix applies to a particular class of partially degenerate representations with highest weights proportional to the $sl(4)$ fundamental weight $\omega_2$. To the best of our knowledge these higher rank quantum ”6j-symbols” are new. The examples of OPE structure constants computed here are still quite simple and need to be extended which would allow to derive by the same method the full $6 \times 6$ fusing matrix. For that purpose the AGT-W approach [6] to the computation of Toda 3-point functions might be more constructive.

We have analysed a higher rank analog of the braiding relation which played a basic role in the construction of the semi-classical limit of the worldsheet $AdS_3 \times S^3$ 3-point functions [8],[9] and have identified it with a standard identity in the modular group on the plane with four holes. The explicit data (3.14) for the solutions of the braiding identity found in Toda CFT, in particular their ”heavy charge” limit, may thus find application to the quasiclassics of conformal sigma models described by compact and noncompact forms of $SL(4,\mathbb{C})$, generalising the $SL(2,\mathbb{C})$ results. Here again for a realistic application one needs first to extend the result beyond the particular class of representations.

More precisely, for this application one needs the extension of the Toda modular data to that of its WZW model counterpart; we hope to return to this problem. The computation of the corresponding $\hat{sl}(4)$ WZW 3-point functions is important also in view of the possible application to the $G/G$ models. As we have pointed out, there are indications that the affine $sl(4)$ WZW theories may alternatively describe the simplest BPS states in the ”light charge” classical limit by a different mechanism than the one provided by the supergravity approximation. The 2d CFT expected to describe the worldsheet realisation of the $\mathcal{N} = 4$ YM theory lacks the affine symmetry of the (super)conformal WZW models. Nevertheless further development of the latter may provide some inside on the structure of the former.

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Appendix A. More general 3-point constant

Case \((\beta_2, \alpha_i) = 0, i = 1, 3, (\beta_1, \alpha_1) = 0\). The OPE constant \(c(\beta_1, \beta_2, 2\rho Q - \beta_3)\) is computed under the condition \(s_3 \geq s_1\). After \(s_1\) steps the \(sl(4)\) type integral reduces to a \(sl(3)\) type \(I_{0, s_2 - 2s_1, s_3 - s_1}\) which furthermore is reduced to a \(sl(2)\) (Liouville) type \(I_{0, s_2 - s_1 - s_3, 0}\). For the analytic continuation (of the reflected third argument) one obtains

\[
C(\beta_1, \beta_2, \beta_3^*) = \frac{(b^{2\rho b} \lambda_T)^{(2\rho Q - \beta_1^2)} \gamma_3^3(b)}{\gamma_b((\beta_{12}^3, \omega_1)) \gamma_b((\beta_{12}^3, \omega_2 - \omega_1) - Q)} \prod_{\alpha > 0} \frac{\gamma_b((\rho Q - \beta_3^*, \alpha))}{\gamma_b((\beta_{123}, - 2\rho Q, \omega_2 - \omega_1) - Q) \gamma_b((\beta_{123}, - 2\rho Q, \omega_1))}
\]

\[
\gamma_b((\rho Q - \beta_1, \alpha_2)) \gamma_b((\rho Q - \beta_1, \alpha_2)) \gamma_b((\rho Q - \beta_2, \alpha_2)) \gamma_b((\rho Q - \beta_2, \alpha_13))
\]

\[
\gamma_b((\beta_{23}^1, \omega_1) - Q) \gamma_b((\beta_{23}^1, \omega_2 - \omega_1)) \gamma_b((\beta_{13}^2, \omega_2 - \omega_1)) \gamma_b((\beta_{13}^2, \omega_1) - Q)
\]

\[
\gamma_b((\rho Q - \beta_1, \alpha_3))
\]

\[
\gamma_b((\beta_{12}^3, \omega_3 - \omega_1) + (\beta_3 - \rho Q, \alpha_24)) \gamma_b((\beta_{123}, \omega_3 - \omega_1) + (\rho Q - \beta_3^*, \alpha_24))
\]

\[
1 \gamma_b((\beta_{123}, - 2\rho Q, \omega_2 - \omega_2)) \gamma_b((\beta_{12}^3, \omega_2)) \gamma_b((\beta_{12}^3, \omega_3 - \omega_1)) \gamma_b((\beta_{12}^3, \omega_3 - \omega_1))
\]

The triple residue of \(C(\beta_1, \beta_2, 2\rho Q - \beta_3)\) for \((\beta_1^2, \omega_i) = -s_i b, i = 1, 2, 3\) reproduces the OPE constant \(c(\beta_1, \beta_2, 2\rho Q - \beta_3)\). In particular the example \(\beta_2 = -\omega_2 b, s_3 = 1 = s_2, s_1 = 0\) reproduces the OPE formula [1] for the shift with \(\beta_3 = \beta_1 - (\omega_2 - \alpha_24)b = \beta_1 + (\omega_3 - \omega_1)b\).

Similarly one derives the analog of the constant (A.1) with \((\beta_1, \alpha_1) \neq 0\) which is to be used for the shift \(\beta_1 \rightarrow \beta_1 + (\omega_1 - \omega_3)b\). Both these more general formulae reproduce the formula (2.7) as a residue at \((\beta_{123}, \omega_1 - \omega_3) = 0\).
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