ON THE ASPHERICITY OF LOT-PRESENTATIONS OF GROUPS

S.V. IVANOV

Abstract. Let $U$ be an arbitrary word in letters $x_1^{\pm 1}, \ldots, x_m^{\pm 1}$ and $m \geq 2$. We prove that the group presentation $\langle x_1, \ldots, x_m \parallel Ux_iU^{-1} = x_{i+1}, i = 1, \ldots, m-1 \rangle$ is aspherical. The proof is based upon prior partial results of A. Klyachko and the author on the asphericity of such presentations.

Suppose that $U$ is a word in letters $x_1^{\pm 1}, \ldots, x_m^{\pm 1}$, $m \geq 2$, and a group $G$ is given by a presentation of the form

$$G = \langle x_1, \ldots, x_m \parallel Ux_iU^{-1} = x_{i+1}, i = 1, \ldots, m-1 \rangle. \quad (1)$$

It is proved by Klyachko and the author [6] that the following claims (C1)–(C2) hold. Note that for $m = 2$ these claims are immediate from classical Magnus’ and Lyndon’s results on one-relator groups, see [9], [8].

(C1) The presentation (1) of $G$ is aspherical if the conjugating word $U$ does not have the form $U_2U_1$, where $U_1$ is a word in letters $x_1^{\pm 1}, \ldots, x_{m-1}^{\pm 1}$ and $U_2$ is a word in letters $x_2^{\pm 1}, \ldots, x_m^{\pm 1}$.

(C2) The (images of) letters $x_1, \ldots, x_{m-1}$ freely generate a free subgroup of the group $G$ given by presentation (1) if and only if the conjugating word $U$ does not have the form $U_2U_1$, where $U_1$ is a word in letters $x_1^{\pm 1}, \ldots, x_{m-1}^{\pm 1}$ and $U_2$ is a word in letters $x_2^{\pm 1}, \ldots, x_m^{\pm 1}$.

The first claim (C1) is of interest because of still unsettled Whitehead asphericity conjecture [10] that states that every subcomplex of an aspherical 2-complex is aspherical (see [1], [2], [3], [4], [5]), or, equivalently, every subpresentation of an aspherical group presentation is aspherical. Observe that presentation (1) is a subpresentation of a balanced presentation of the trivial group (which is aspherical) obtained from (1) by adding a letter to the relator set. Claim (C1) is also of interest because the asphericity of presentation (1) is a special case of the separately conjectured asphericity of LOT-presentations of groups which, as was proved by Howie [2], is equivalent to the open problem on the asphericity of ribbon disk complements.

In this note, we will apply a "stabilization" trick to strengthen claim (C1) of [8] by lifting the restriction $U \neq U_2U_1$.

Theorem. The group presentation (1) is aspherical for an arbitrary word $U$.

Corollary. Let $k_1, \ldots, k_{m-1}$ be some integers, $U$ be a word in letters $x_1^{\pm 1}, \ldots, x_m^{\pm 1}$ and $m \geq 2$. Then a presentation of the form $G = \langle x_1, \ldots, x_m \parallel U^{k_1}x_iU^{-k_i} = x_{i+1}, i = 1, \ldots, m-1 \rangle$ is aspherical.

2000 Mathematics Subject Classification. Primary 20F05, 57M20.
Supported in part by NSF grants DMS 98-01500, DMS 00-99612.
Proof of Theorem. Note that for $m = 2$ we have a one-relator presentation and our Theorem follows from classical Lyndon’s results on one-relator groups, see [8]. Hence, arguing by induction on $m \geq 2$, in view of claim (C1), we can assume that $m > 2$ and the conjugating word $U$ has the form $U_2U_1$, where $U_1$ is a word in letters $x_1^{\pm 1}, \ldots, x_{m-1}^{\pm 1}$ and $U_2$ is a word in letters $x_2^{\pm 1}, \ldots, x_m^{\pm 1}$.

Let $S$ be a word in letters $x_1^{\pm 1}, \ldots, x_{m-1}^{\pm 1}$. By $S^\alpha$ denote the word obtained from $S$ by increasing the index of each letter of $S$ by 1. Suppose that $X, Y$ are some words in letters $x_1^{\pm 1}, \ldots, x_m^{\pm 1}$. We will write $X \overset{G}{=} Y$ if the natural images of words $X, Y$ in the group $G$ given by (1) are equal.

It easily follows from defining relations of the group $G$ that $UU_1U^{-1} \overset{G}{=} U_1^\alpha$ or $(U_2U_1)U_1(U_2U_1)^{-1} \overset{G}{=} U_1^\alpha$, whence

$$U_2U_1 \overset{G}{=} U_1^\alpha U_2.$$

(2)

Consider another group $H$ given by the following presentation

$$H = \langle x_1, \ldots, x_m \mid U_1^\alpha U_2x_i(U_1^\alpha U_2)^{-1} = x_{i+1}, i = 1, \ldots, m-1 \rangle.$$

(3)

Note that $U_1^\alpha U_2U_1(U_1^\alpha U_2)^{-1} \overset{H}{=} U_1^\alpha$. Hence,

$$U_2U_1 \overset{H}{=} U_1^\alpha U_2.$$

(4)

Consequently, it follows from equalities (2) and (4) that all defining relations of $H$ hold in $G$ and vice versa, that is, the groups $G$ and $H$ given by (1) and (3) are naturally isomorphic.

Denote $I_{m-1} = \{1, \ldots, m-1\}$ and consider a sequence of elementary Andrews-Curtis transformations (AC-moves) applied to presentation (1).

$$\langle x_1, \ldots, x_m \mid U_2U_1x_i(U_2U_1)^{-1} = x_{i+1}, i \in I_{m-1} \rangle \rightarrow \langle x_1, \ldots, x_m, z \mid z = 1, U_2U_1x_i(U_2U_1)^{-1} = x_{i+1}, i \in I_{m-1} \rangle \rightarrow \langle x_1, \ldots, x_m, z \mid zU_1^\alpha U_2(U_2U_1)^{-1} = 1, U_2U_1x_i(U_1^\alpha U_2)^{-1} = x_{i+1}, i \in I_{m-1} \rangle \rightarrow$$

$$K = \langle x_1, \ldots, x_m, z \mid zU_1^\alpha U_2(U_2U_1)^{-1} = 1, zU_1^\alpha U_2x_i(zU_1^\alpha U_2)^{-1} = x_{i+1}, i \in I_{m-1} \rangle.$$

Note that it follows from the last $m-1$ relations of the last presentation that in the group $K$ given by this presentation we have

$$zU_1^\alpha U_2U_1^{-1}(zU_1^\alpha U_2)^{-1} \overset{K}{=} (U_1^\alpha)^{-1}.$$

Therefore, we can continue the chain of elementary AC-moves, replacing the relator $R = zU_1^\alpha U_2(U_2U_1)^{-1}$ by

$$\langle zU_1^\alpha U_2U_1^{-1}(zU_1^\alpha U_2)^{-1}U_1^{-1} \rangle \rightarrow \langle R(U_1^\alpha)^{-1}z^{-1}U_1^{-1} \rangle \rightarrow \langle U_1^\alpha \rangle$$

or just by $z$.

$$\langle x_1, \ldots, x_m, z \mid zU_1^\alpha U_2(U_2U_1)^{-1} = 1, zU_1^\alpha U_2x_i(zU_1^\alpha U_2)^{-1} = x_{i+1}, i \in I_{m-1} \rangle \rightarrow \langle x_1, \ldots, x_m, z \mid z = 1, zU_1^\alpha U_2x_i(zU_1^\alpha U_2)^{-1} = x_{i+1}, i \in I_{m-1} \rangle \rightarrow$$

$$\langle x_1, \ldots, x_m, z \mid zU_1^\alpha U_2(U_1^\alpha U_2)^{-1} = x_{i+1}, i \in I_{m-1} \rangle \rightarrow \langle x_1, \ldots, x_m, z \mid z = 1, U_1^\alpha U_2x_i(U_1^\alpha U_2)^{-1} = x_{i+1}, i \in I_{m-1} \rangle \rightarrow$$

$$H = \langle x_1, \ldots, x_m \mid U_1^\alpha U_2x_i(U_1^\alpha U_2)^{-1} = x_{i+1}, i \in I_{m-1} \rangle.$$

Now we see that presentation (3) is Andrews-Curtis equivalent to presentation (1). Since elementary AC-moves preserve the asphericity of a presentation (see also [4], [3]), the asphericity of (1) would follow from the asphericity of (3). Since
the word $U_1^a U_2$ has no occurrences of $x_i^{+1}$, it follows that the asphericity of (3) is equivalent to the asphericity of presentation

$$\langle x_2, \ldots, x_m \mid U_1^a U_2 x_i (U_1^a U_2)^{-1} = x_{i+1}, \ i = 2, \ldots, m-1 \rangle.$$ 

Now it remains to refer to the induction hypothesis and Theorem is proved. $\square$

To prove Corollary, we note that introduction of new letters $x_{i,j}$ and splitting relations $U^k x_i U^{-k} = x_{i+1}$, where, say, $k_i > 0$, into several relations 

$$U x_i U^{-1} = x_{i,1}, \ldots, U x_i x_{i,1} U^{-1} = x_{i+1},$$

where $x_{i,1}, \ldots, x_{i,k_i-1}$ are new letters, result in a presentation whose asphericity is equivalent to the asphericity of the original presentation and for which (after obvious reindexing) all the numbers $k_i$ are equal to $\pm 1$. Applying evident AC-moves, we can eliminate some of the letters and relations and turn all $k_i$ into 1. Now we can refer to proven Theorem. $\square$

References

[1] J. Howie, Some remarks on a problem of J.H.C. Whitehead, Topology 22 (1983), 475–485.
[2] J. Howie, On the asphericity of ribbon disk complements, Trans. Amer. Math. Soc. 289 (1985), 281–302.
[3] S.V. Ivanov, On aspherical presentations of groups, Electron. Res. Announc. AMS 4 (1998), 109–114.
[4] S.V. Ivanov, Some remarks on the Whitehead asphericity conjecture, Illinois J. Math. 43 (1999), 793–799.
[5] S.V. Ivanov, The Whitehead asphericity conjecture and periodic groups, Internat. J. Algebra Comp. 9 (1999), 529–538.
[6] S.V. Ivanov and A.A. Klyachko, The asphericity and Freiheitssatz for certain LOT-presentations of groups, Internat. J. Algebra Comp. 11 (2001), 291–300.
[7] E. Luft, On 2-dimensional aspherical complexes and a problem of J.H.C. Whitehead, Math. Proc. Cambridge Phil. Soc. 119 (1996), 493–495.
[8] R.C. Lyndon and P.E. Schupp, Combinatorial group theory, Springer-Verlag, 1977.
[9] W. Magnus, J. Karrass, and D. Solitar, Combinatorial group theory, Interscience Pub., John Wiley and Sons, 1966.
[10] J.H.C. Whitehead, On adding relations to homotopy groups, Ann. Math. 42 (1941), 409–428.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, U.S.A.
E-mail address: ivanov@math.uiuc.edu