On large partial ovoids of symplectic and Hermitian polar spaces

Michela Ceria | Jan De Beule | Francesco Pavese | Valentino Smaldore

1Department of Mechanics, Mathematics and Management, Polytechnic University of Bari, Bari, Italy
2Department of Mathematics and Data Science, Vrije Universiteit Brussel, Brussel, Belgium
3Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Gent, Belgium
4Department of Mathematics, Computer Science and Economics, Contrada Macchia Romana, University of Basilicata, Potenza, Italy

Abstract
In this paper we provide constructive lower bounds on the sizes of the largest partial ovoids of the symplectic polar spaces \( \mathcal{W}(3, q) \), \( q \) odd square, \( q \not\equiv 0 \pmod{3} \), \( \mathcal{W}(5, q) \) and of the Hermitian polar spaces \( \mathcal{H}(4, q^2) \), \( q \) even or \( q \) odd square, \( q \not\equiv 0 \pmod{3} \), \( \mathcal{H}(6, q^2) \), \( \mathcal{H}(8, q^2) \).

KEYWORDS
Hermitian polar space, partial ovoid, symplectic polar space

MATHMATICAL SUBJECT CLASSIFICATION 2020
Primary 51E20, Secondary 05B25

1 | INTRODUCTION

Let \( \mathcal{P} \) be a finite classical polar space, that is, \( \mathcal{P} \) arises from a vector space of finite dimension over a finite field equipped with a nondegenerate reflexive sesquilinear or quadratic form. A projective subspace of maximal dimension contained in \( \mathcal{P} \) is called a generator of \( \mathcal{P} \). A (partial) ovoid \( \mathcal{O} \) of a polar space \( \mathcal{P} \) is a set of points of \( \mathcal{P} \) such that every generator of \( \mathcal{P} \) contains (at most) one point of \( \mathcal{O} \). A partial ovoid is said to be maximal if it is maximal with respect to set-theoretic inclusion.
Here we focus on the symplectic polar space $\mathcal{W}(2n - 1, q)$ and the Hermitian polar space $\mathcal{H}(2n, q^2)$, consisting of the absolute projective subspaces with respect to a nondegenerate symplectic polarity of $\text{PG}(2n - 1, q)$ and a nondegenerate unitary polarity of $\text{PG}(2n, q^2)$, respectively. It is well known that $\mathcal{W}(2n - 1, q)$ has ovoids if and only if $n = 2$ and $q$ is even, whereas $\mathcal{H}(2n, q^2)$ does not possess ovoids. Hence the question of the largest (maximal) partial ovoids of $\mathcal{W}(2n - 1, q)$, $(n, q) \neq (2, 2^h)$, and $\mathcal{H}(2n, q^2)$ naturally arises.

The main goal of the paper is to provide constructive lower bounds on the sizes of the largest partial ovoids of the symplectic polar spaces $\mathcal{W}(3, q)$, $q$ odd square, $q \equiv 0 \pmod{3}$, $\mathcal{W}(5, q)$ and of the Hermitian polar spaces $\mathcal{H}(4, q^2)$, $q$ even or $q$ odd square, $q \equiv 0 \pmod{3}$, $\mathcal{H}(6, q^2)$, $\mathcal{H}(8, q^2)$.

In Section 3, partial ovoids of symplectic polar spaces are studied. We show the existence of a partial ovoid of $\mathcal{W}(3, q)$, $q$ an odd square, $q \equiv 0 \pmod{3}$, of size $(q^{3/2} + 3q - q^{1/2} + 3)/3$. It is obtained by glueing together a twisted cubic $C$ of $\text{PG}(3, q)$ and an orbit of a subgroup of $\text{PSp}(4, q)$ isomorphic to $\text{PGL}(2, q^{1/2})$ stabilizing $C$. Next maximal partial ovoids of $\mathcal{W}(5, q)$ of size $q^2 + q + 1$ and of size $2q^2 - q + 1$, if $q$ is even, are exhibited.

In Section 4 we introduce the notion of tangent-set of a Hermitian variety $\mathcal{H}(2n - 1, q^2)$, that is, a set $T$ of points of $\text{PG}(2n - 1, q^2)$ such that every line that is either tangent or contained in $\mathcal{H}(2n - 1, q^2)$ has at least one point in common with $T$. In particular we show that starting from a partial ovoid of $\mathcal{W}(2n - 1, q)$ of size $x$, it is possible to obtain a tangent-set of $\mathcal{H}(2n - 1, q^2)$ of size roughly $qx$, which in turn gives rise to a partial ovoid of $\mathcal{H}(2n, q^2)$ of size roughly $xq^2$. Applying this construction technique to the known (partial) ovoids of $\mathcal{W}(2n - 1, q)$, $n \in \{2, 3, 4\}$, the following are obtained: maximal partial ovoids of $\mathcal{H}(4, q^2)$, $q$ even, of size $q^4 + 1$ and of size $q^4 - q^3 + q + 1$, a partial ovoid of $\mathcal{H}(4, q^2)$, $q$ an odd square, $q \equiv 0 \pmod{3}$, of size $(q^{7/2} + 3q^3 - q^{5/2} + 3q^2)/3$, partial ovoids of $\mathcal{H}(6, q^2)$ of size $q^4 + q^3 + 1$ and of size $2q^4 - q^3 + 1$, if $q$ is even, and a partial ovoid of $\mathcal{H}(8, q^2)$ of size $q^5 + 1$.

Tables 1 and 2 summarize old and new results regarding large partial ovoids of symplectic and Hermitian polar spaces in small dimensions. For more details about the known results on

| Table 1 | Large partial ovoids of $\mathcal{W}(2n - 1, q)$, $n \in \{2, 3, 4\}$ |
|---------|---------------------------------------------------|
| $\mathcal{W}(3, q)$, $q$ even | $q^2 + 1$ |
| $\mathcal{W}(3, q)$, $q$ odd | $(q^{3/2} + 3q - q^{1/2} + 3)/3$, $q = p^{2h}$, $p \neq 3$, $2q + 1$, $q = p^{2h+1}$, or $q = 3^h$ [13] |
| $\mathcal{W}(5, q)$ | $2q^2 - q + 1$, $q$, even $q^2 + q + 1$, $q$ odd $7, q = 2$ |
| $\mathcal{W}(7, q)$ | $q^3 + 1$ [2, 3], $q^4 - q^3 - q(q^{1/2} - 1)(q - q^{1/2} + 1) + 3$, $q > 2$ [8] $9, q = 2$ |
large partial ovoids of symplectic and Hermitian polar spaces, the reader is referred to the last part of Section 2.

2 | PRELIMINARY AND BACKGROUND

Let \( q \) be a prime power and let \( \mathbb{F}_q \) be the finite field of order \( q \). Let \( \text{PG}(n, q) \) be the \( n \)-dimensional projective space over \( \mathbb{F}_q \) equipped with homogeneous projective coordinates \((X_1, X_2, \ldots, X_{n+1})\). Denote by \( U_i \) the point having 1 in the \( i \)th position and 0 elsewhere. A nondegenerate symplectic or unitary polarity of \( \text{PG}(n, q^2) \) is induced by a nondegenerate alternating or Hermitian form on the underlying vector space.

The symplectic polar space \( \mathcal{W}(2n - 1, q) \) is formed by the projective subspaces that are absolute with respect to a nondegenerate symplectic polarity \( s \) of \( \text{PG}(2n - 1, q) \). It is left invariant by the group of projectivities \( \text{PSp}(4, q) \). It consists of all the points of \( \text{PG}(2n - 1, q) \) and of \((q + 1) \cdots (q^n + 1)\) generators. Through every point \( P \) of \( \text{PG}(2n - 1, q) \) there pass \((q + 1) \cdots (q^{n-1} + 1)\) generators and these generators all lie in a hyperplane. The hyperplane containing these generators is the polar hyperplane \( P_0 \) of \( P \) with respect to the symplectic polarity \( s \) defining \( \mathcal{W}(2n - 1, q) \).

A (nondegenerate) Hermitian variety \( \mathcal{H}(n, q^2) \) consists of the absolute points of a unitary polarity of \( \text{PG}(n, q^2) \). It has \((q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2 - 1)\) points. Subspaces of maximal dimension contained in \( \mathcal{H}(n, q^2) \) are \([n+1]/2\)-dimensional projective spaces and are called generators. There are either \((q + 1)(q^3 + 1) \cdots (q^{n-1} + 1)\) or \((q^3 + 1)(q^5 + 1) \cdots (q^{n+1} + 1)\) generators in \( \mathcal{H}(n, q^2) \), depending on whether \( n \) is odd or even, respectively. A line of \( \text{PG}(n, q^2) \) meets \( \mathcal{H}(n, q^2) \) in 1, \( q + 1 \) or, if \( n \geq 3 \), in \( q^2 + 1 \) points. The latter lines are the generators of \( \mathcal{H}(n, q^2) \) if \( n \in \{3, 4\} \); lines meeting \( \mathcal{H}(n, q^2) \) in one or \( q + 1 \) points are called tangent lines or secant lines, respectively. Through a point \( P \) of \( \mathcal{H}(n, q^2) \) there pass \((q^{n-1} + (-1)^{n-2})(q^{n-2} - (-1)^{n-2})/(q^2 - 1)\) generators and these generators are contained in a hyperplane. The hyperplane containing these generators is the polar hyperplane of \( P \) with respect to the unitary polarity of \( \text{PG}(n, q^2) \) defining \( \mathcal{H}(n, q^2) \) and it is also called the tangent hyperplane to \( \mathcal{H}(n, q^2) \) at \( P \). The tangent lines through \( P \) are precisely the remaining lines of its polar hyperplane that are incident with \( P \). If \( P \notin \mathcal{H}(n, q^2) \) then the polar hyperplane of \( P \) is a hyperplane of \( \text{PG}(n, q^2) \) meeting \( \mathcal{H}(n, q^2) \) in a nondegenerate Hermitian variety \( \mathcal{H}(n - 1, q^2) \) and it is said to be secant to \( \mathcal{H}(n, q^2) \). The stabilizer of \( \mathcal{H}(n, q^2) \) in \( \text{PGL}(n + 1, q^2) \) is

| Table 2 | Large partial ovoids of \( \mathcal{H}(2n, q^2) \), \( n \in \{2, 3, 4\} \) |
|---------|----------------------|
| \( \mathcal{H}(4, q^2) \) | \( q^4 + 1 \), \( q = 2^h \), or \( q = 3^h \) [11] |
| | \((q^{7/2} + 3q^3 - q^{5/2} + 3q^2)/3\), \( q = p^{2h}, p \ odd, p \neq 3 \) |
| | \( 2q^3 + q^2 + 1 \), \( q = p^{2h+1}, p \neq 2, 3 \) [4] |
| \( \mathcal{H}(6, q^2) \) | \( 2q^4 - q^3 + 1 \), \( q \), even |
| | \( q^4 + q^3 + 1 \), \( q \) odd |
| \( \mathcal{H}(8, q^2) \) | \( q^8 + 1 \) |
| | \( q^9 - q^8 + q^7 - q^5 - q^3 + q^2 + 1 \) [7] |
the group PGU\((n + 1, q^2)\). Recall that the symplectic polar space \(\mathcal{W}(2n - 1, q)\) can be embedded in \(\mathcal{H}(2n - 1, q^2)\).

In the remaining part of the section we summarize what is known about maximal partial ovoids of symplectic polar spaces \(\mathcal{W}(2n - 1, q)\) and of Hermitian polar spaces \(\mathcal{H}(2n, q^2)\). Constructions of maximal partial ovoids of \(\mathcal{W}(3, q)\), \(q\) even, of size an integer between about \(q^2/10\) and \(9q^2/10\) or of size \(q^2 - hq + 1\), \(1 \leq h \leq q/2\), have been provided in [12, 13]. The situation is somewhat different for \(q\) odd where the lack of examples is transparent. In [13] Tallini proved that a partial ovoid of \(\mathcal{W}(3, q)\), \(q\) odd, has size at most \(q^2 - q + 1\) and constructed a maximal partial ovoid of \(\mathcal{W}(3, q)\) of size \(2q + 1\). Regarding symplectic polar spaces in higher dimensions an upper bound on the size of the largest partial ovoid has been provided in [8] and, if \(q\) is even, a partial ovoid of an elliptic or hyperbolic quadric is also a partial ovoid of a symplectic polar space.

As for \(\mathcal{H}(2n, q^2)\), an upper bound on the size of the largest partial ovoid can be found in [7]. In particular, a partial ovoid has at most \(q^5 - q^4 + q^3 + 1\) points if \(n = 2\). The largest known example of a maximal partial ovoid of \(\mathcal{H}(2n, q^2)\), \(n = 2, 3\), occurs when \(q = 3^h\) and has size \(q^4 + 1\) [11]. A straightforward check shows that a nondegenerate plane section of \(\mathcal{H}(2n, q^2)\) is an example of maximal partial ovoid of \(\mathcal{H}(2n, q^2)\) of size \(q^3 + 1\). Other examples of maximal partial ovoids of \(\mathcal{H}(4, q^2)\) of size \(2q^3 + q^2 + 1\) have been constructed in [4] and of size \(q^3 + 1\) in [5, 11].

### 3. PARTIAL OVOIDS OF SYMPLECTIC POLAR SPACES

#### 3.1. \(\mathcal{W}(3, q)\), \(q\) odd square, \(q \not\equiv 0\) (mod 3)

Let \(\mathcal{W}(3, q)\), \(q \not\equiv 0\) (mod 3), be the symplectic polar space consisting of the subspaces of PG(3, q) induced by the totally isotropic subspaces of \(\mathbb{F}_q^4\) with respect to the nondegenerate alternating form \(\beta\) given by

\[
\begin{align*}
x_1 y_4 + x_2 y_3 - x_3 y_2 - x_4 y_1. & \tag{1}
\end{align*}
\]

Denote by \(\varepsilon\) the symplectic polarity of PG(3, q) defining \(\mathcal{W}(3, q)\). Let \(C\) be the twisted cubic of PG(3, q) consisting of the \(q + 1\) points \(\{P|t \in \mathbb{F}_q\} \cup \{(0, 0, 0, 1)\}\), where \(P = (1, -3t, t^2, t^3)\). It is well known that a line of PG(3, q) meets \(C\) in at most two points and a plane shares with \(C\) at most three points (i.e., \(C\) is a so-called \((q + 1)\)-arc). A line of PG(3, q) joining two distinct points of \(C\) is called a real chord and there are \(q(q + 1)/2\) of them. Let \(\bar{C} = \{P|t \in \mathbb{F}_q^2\} \cup \{(0, 0, 0, 1)\}\) be the twisted cubic of PG(3, q²) which extends \(C\) over \(\mathbb{F}_q^2\). The line of PG(3, q²) obtained by joining \(P\) and \(P_r\), with \(t \not\equiv \mathbb{F}_q\), meets the canonical Baer subgeometry PG(3, q) in the \(q + 1\) points of a line skew to \(C\). Such a line is called an imaginary chord and they are \(q(q - 1)/2\) in number. If \(r\) is a (real or imaginary) chord, then the line \(r^\varepsilon\) is called (real or imaginary) axis. Also, for each point \(P\) of \(C\), the line \(\ell_P = \langle P, P'\rangle\), where \(P'\) equals \((0, -3, 2t, 3t^2)\) or \(U_5\) if \(P = P_l\) or \(P = U_4\), respectively, is called the tangent line to \(C\) at \(P\). With each point \(P_l\) (resp., \(U_4\)) of \(C\) there corresponds the osculating plane \(P_l^\varepsilon\) (resp., \(U_4^\varepsilon\)) with equation \(t^2X_1 + t^2X_2 + 3tX_3 - X_4 = 0\) (resp., \(X_1 = 0\)), meeting \(C\) only at \(P_l\) (resp., \(U_4\)) and containing the tangent line. Hence the \(q + 1\) lines tangent to \(C\) are generators of \(\mathcal{W}(3, q)\) and they form a regulus \(\mathcal{R}\) if \(q\) even. Every point of PG(3, q) \(\setminus C\) lies on exactly one chord or a tangent of \(C\). For more properties and results on \(C\) the reader is referred to [9, Chap. 21]. Let \(G\) be the group of projectivities of PG(3, q) stabilizing \(C\). Then \(G \simeq \text{PGL}(2, q)\) whenever \(q \geq 5\), and elements of \(G\) are induced by the matrices
where $a, b, c, d \in \mathbb{F}_q$, $ad - bc \neq 0$. The group $G$ leaves invariant $\mathcal{W}(3, q)$ since $M_{a,b,c,d}^tJM_{a,b,c,d} = (ad - bc)^3J$, where

$$J = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}$$

is the Gram matrix of $\beta$.

Assume $q$ to be an odd square: Here we show the existence of a partial ovoid of $\mathcal{W}(3, q)$ obtained by glueing together the twisted cubic $C$ of $\text{PG}(3, q)$ and an orbit of size $\sqrt{q}(q - 1)/3$ of a subgroup $G_\epsilon$ of $\text{PSp}(4, q) \cap G$ isomorphic to $\text{PGL}(2, \sqrt{q})$ stabilizing $C$. In particular such a subgroup fixes a twisted cubic $C_\epsilon = C \cap \Lambda_\epsilon$ of a Baer subgeometry $\Lambda_\epsilon$ of $\text{PG}(3, q)$.

$$\sqrt{q} \equiv 1 \pmod{3}$$

Assume $\sqrt{q} \equiv 1 \pmod{3}$. Let $C_1 = \{P|t \in \mathbb{F}_{\sqrt{q}} \cup \{U_4\}. Thus $C_1 \subset C$ is a twisted cubic of the canonical Baer subgeometry $\Lambda_1 = \text{PG}(3, \sqrt{q})$ of $\text{PG}(3, q)$. The group $G_1$ of projectivities stabilizing $C_1$ is isomorphic to $\text{PGL}(2, \sqrt{q})$ and it is induced by the matrices

$$M_{a,b,c,d}^t, a, b, c, d \in \mathbb{F}_{\sqrt{q}}, ad - bc \neq 0.$$
If $bd \neq 0$, then $x = -\frac{a^2c}{bd} \in \mathbb{F}_{\sqrt{q}}$, a contradiction. Hence either $b = 0$ or $d = 0$. Taking into account that $ad - bc \neq 0$, if the former case occurs, then $c = 0$ and $d = \xi a$, with $\xi \in \mathbb{F}_{\sqrt{q}}$, $\xi^3 = 1$, whereas if the latter possibility arises, then $a = 0$ and $x^2 = \frac{c^3}{b} \in \mathbb{F}_{\sqrt{q}}$, a contradiction. Hence the stabilizer of $R$ in $G_1$ has order 3 and $|G_1| = \sqrt{q}(q - 1)$, by applying the Orbit–Stabilizer Theorem it follows that $O_1$ has the required size. To show that $O_1$ is a partial ovoid of $W(3, q)$, it is enough to see that the line joining $R$ and a further point $R^8$ of $O_1$ is not a generator of $W(3, q)$. Here $g \in G_1$ is induced by $M_{a,b,c,d}$. Assume by contradiction that this is not the case, then there are $a, b, c, d \in \mathbb{F}_{\sqrt{q}}$, with $ad - bc \neq 0$, such that $A(x) = 0$, where

$$A(x) = c^3 + xd^3 - xa^3 - x^2 b^3.$$  

Hence $A(x) + A(x)\sqrt{q} = A(x) - A(x)\sqrt{q} = 0$, that is,

$$2c^3 + (d^3 - a^3)(x + x\sqrt{q}) - (x^2 + x^2\sqrt{q})b^3 = 0,$$

$$d^3 - a^3 - (x + x\sqrt{q})b^3 = 0.$$  

If $b = 0$, then $a^3 = d^3$ and $c = 0$. Thus $g$ fixes $R$, that is, $R = R^8$. If $b \neq 0$, then the previous equations imply

$$x^{\sqrt{q}+1} = -\frac{c^3}{b^3},$$

that is a contradiction since $x^{\sqrt{q}+1}$ is not a cube in $\mathbb{F}_{\sqrt{q}}$. Indeed $x$ is not a cube in $\mathbb{F}_q$ and $(\sqrt{q} + 1, 3) = 1$.  

Assume $\sqrt{q} \equiv -1 \pmod{3}$. Let $C_{-1} = \{P \mid t \in \mathbb{F}_q, t^{\sqrt{q}+1} = 1\}$. Thus $C_{-1} \subseteq C$ is a twisted cubic of the Baer subgeometry

$$\Lambda_{-1} = \{(\alpha, -3\beta, \beta^q, \alpha^q) \mid \alpha, \beta \in \mathbb{F}_q, (\alpha, \beta) \neq (0, 0)\} \simeq \text{PG}(3, \sqrt{q})$$

of $\text{PG}(3, q)$. In this case the group $G_{-1}$ of projectivities stabilizing $C_{-1}$ is isomorphic to $\text{PGL}(2, \sqrt{q})$ and it is induced by the matrices

$$M_{a,b,c,d}, a, b, c, d \in \mathbb{F}_q, ad - bc \neq 0, ab^{\sqrt{q}} - cd^{\sqrt{q}} = 0, a^{\sqrt{q}+1} + b^{\sqrt{q}+1} - c^{\sqrt{q}+1} - d^{\sqrt{q}+1} = 0.$$  

Let $S = U_1 + xU_4$, for a fixed $x \in \mathbb{F}_q \setminus \mathbb{F}_{\sqrt{q}}$ where $x$ is not a cube in $\mathbb{F}_q$ and $x^{\sqrt{q}+1} \neq 1$. Set

$$O_{-1} = S^{G_{-1}} = \{(a^3 + xb^3, -3a^2c - 3xb^2d, ac^2 + xbd^2, c^3 + xd^3)a, b, c, d \in \mathbb{F}_q, ad - bc \neq 0,$$

$$ab^{\sqrt{q}} - cd^{\sqrt{q}} = 0, a^{\sqrt{q}+1} + b^{\sqrt{q}+1} - c^{\sqrt{q}+1} - d^{\sqrt{q}+1} = 0\}.$$

\[\ \square\]
Lemma 3.2. The set $\mathcal{O}_1$ is a partial ovoid of $\mathcal{W}(3, q)$ of size $\sqrt{q}(q - 1)/3$.

Proof. A projectivity of $G_{-1}$ fixing $S$ is induced by $M_{a,b,c,d}$, where $a, b, c, d \in \mathbb{F}_q$, $ad - bc \neq 0$, are such that $ab\sqrt{q} - cd\sqrt{q} = a\sqrt{q} + b\sqrt{q} + c\sqrt{q} + 1 - d\sqrt{q} + 1 = 0$ and satisfy (2)–(4). If $bd \neq 0$, by expliciting $x$ from (4) and substituting it in (3) gives $ac = 0$, which implies $x = 0$, a contradiction. Hence either $b = 0$ or $d = 0$. Since $ad - bc \neq 0$, if $d = 0$, then $a = 0$ and $x^2 = \frac{c^3}{b^3} \in \mathbb{F}_q$, contradicting the fact that $x$ is not a cube in $\mathbb{F}_q$; whereas if $b = 0$, then $c = 0$ and $d = \xi_4 a$, with $\xi \in \mathbb{F}_q$, $\xi^3 = 1$. In this case $a^3 = d^3$ and $a\sqrt{q} + 1 + b\sqrt{q} + 1 - c\sqrt{q} + 1 - d\sqrt{q} + 1 = 0$, such that $A(x) = 0$, where

$$A(x) = c^3 + xd^3 - xa^3 - x^2b^3.$$ 

If $b = 0$, then $c = 0$, since $cd\sqrt{q} = 0$ and $ad \neq 0$. Moreover $a\sqrt{q} + 1 = d\sqrt{q} + 1$ and $a^3 = d^3$, since $A(x) = 0$. Thus $g$ fixes $S$, that is, $S = S^g$. If $b \neq 0$, we may assume w.l.o.g. that $b = 1$. Then $a = cd\sqrt{q}$, $c(d\sqrt{q} + 1) \neq 0$, and $(1 - c\sqrt{q} + 1)(1 - d\sqrt{q} + 1) = 0$. Therefore $c\sqrt{q} + 1 = 1$ and $d\sqrt{q} + 1 \neq 1$. Moreover $A(x) = 0$ gives

$$d^3\sqrt{q} - \frac{d^3}{c^3} - \frac{c^3 - x^2}{c^3x} = 0.$$

By considering the last equation in the unknown $d^3$, by [10, Theorem 1.9.3], it admits solutions in $\mathbb{F}_q$ if and only

$$0 = \frac{c^3\sqrt{q} - x^2\sqrt{q}}{c^3\sqrt{q}x\sqrt{q}} + \frac{c^3 - x^2}{c^3x} \frac{1}{c^3\sqrt{q}} = \frac{x(1 - x\sqrt{q} + 1)(c^3\sqrt{q} + x\sqrt{q} - 1)}{c^3\sqrt{q}x\sqrt{q} + 1}.$$

Hence either $x = 0$ or $x\sqrt{q} + 1 = 1$ or $x\sqrt{q} - 1$ is a cube in $\mathbb{F}_q$. If the last possibility occurs then $x$ is a cube in $\mathbb{F}_q$ since $\sqrt{q} \equiv -1 \mod 3$. We infer that none of the three cases arises. $\square$

Theorem 3.3. Let $q$ be an odd square with $\sqrt{q} \equiv \varepsilon(\mod 3)$, $\varepsilon \in \{\pm 1\}$. Then the set $\mathcal{O}_\varepsilon \cup C$ is a partial ovoid of $\mathcal{W}(3, q)$, of size $(q^{3/2} + 3q - q^{1/2} + 3)/3$.

Proof. It is enough to show that there is no line of $\mathcal{W}(3, q)$ through $R$ or $S$ meeting $C$. Assume by contradiction that the line spanned by $R$ (or $S$) and $P_1 \in C$ belongs to $\mathcal{W}(3, q)$. Then $P_1 \in R^\varepsilon$ (or $S^\varepsilon$) and hence, by 1, $x = t^3$. A contradiction, since $x$ is not a cube in $\mathbb{F}_q$. Similarly the line joining $R$ (or $S$) with $U_4$ is not a line of $\mathcal{W}(3, q)$.

$\square$

Proposition 3.4. For $\varepsilon \in \{\pm 1\}$, the partial ovoid $\mathcal{O}_\varepsilon \cup C$ of $\mathcal{W}(3, q)$ is not maximal.

Proof. There are $\sqrt{q}(q - 1)/3$ points of $\mathcal{O}_\varepsilon$ not lying on an osculating plane of $\mathcal{O}_\varepsilon$, see [9, p. 235]. Let $N$ be one of these points. We claim that $\mathcal{O}_\varepsilon \cup C \cup \{N\}$ is a partial ovoid of $\mathcal{W}(3, q)$. To see this fact let $\ell$ be a line of $\mathcal{W}(3, q)$. If $\ell$ passes through a point $P$ of $\mathcal{O}_\varepsilon$, see [9, p. 235]. Let $N$ be one of these points. We claim that $\mathcal{O}_\varepsilon \cup C \cup \{N\}$ is a partial ovoid of $\mathcal{W}(3, q)$. To see this fact let $\ell$ be a line of $\mathcal{W}(3, q)$. If $\ell$ passes through a point $P$ of $\mathcal{O}_\varepsilon$,
then $P^s$ is the osculating plane of $C_z$ at $P$. Hence $\ell \subset P^s$. Denote by $\tau$ the Baer involution of $\text{PG}(3,q)$ fixing pointwise $\Lambda_c$. If $\ell$ contains a point $T$ of $C_\setminus \Lambda_c$, then $T^s \subset C_\setminus \Lambda_c$ and the line $r$ joining $T$ and $T^s$ meets $\Lambda_c$ in a subline skew to $C_c$. In particular such a subline is an imaginary chord of $C_c$ and $r_\ell \cap \Lambda_c$ is an imaginary axis of $C_c$. Hence $\ell$ lies in the plane spanned by $T$ and $T^s$; it follows that $\ell \cap \Lambda_c$ is a point lying on an imaginary axis of $C_c$. On the other hand every axis of $C_c$ is contained in an osculating plane of $C_c$. □

**Problem 3.5.** Find a maximal partial ovoid of $\mathcal{W}(3,q)$ containing $O_c \cup C$.

### 3.2 Maximal partial ovoids of $\mathcal{W}(5,q)$ of size $q^2 + q + 1$

Here we consider a symplectic polar space in $\text{PG}(5,q)$, where $\text{PG}(5,q)$ is embedded as a subgeometry in $\text{PG}(5,q^3)$. We denote by $N : x \in \mathbb{F}_q^3 \mapsto x^{q^2+q+1} \in \mathbb{F}_q$ the norm function of $\mathbb{F}_q^3$ over $\mathbb{F}_q$ and by $T : x \mapsto x + x^q + x^{q^2}$ the trace function of $\mathbb{F}_q^3$ over $\mathbb{F}_q$. Let $V$ be the six-dimensional $\mathbb{F}_q$-vector subspace of $\mathbb{F}_q^6$ given by

$$
\{P_{a,b} = (a, a^q, a^{q^2}, b^{q^2}, b, b) \mid a, b \in \mathbb{F}_q\}.
$$

Then $\text{PG}(V)$ is a $q$-order subgeometry of $\text{PG}(5,q^3)$. If $\mathcal{W}(5,q^3)$ is the symplectic polar space consisting of the subspaces of $\text{PG}(5,q^3)$ induced by the totally isotropic subspaces of $\mathbb{F}_q^6$ with respect to the nondegenerate alternating form given by

$$
b(x,y) = x_1y_6 + x_2y_5 + x_3y_4 - x_4y_3 - x_5y_2 - x_6y_1,
$$

then $\mathcal{W}(5,q^3)$ induces in $\text{PG}(V)$ a symplectic polar space, say $\mathcal{W}(5,q)$. It is straightforward to check that the cyclic group $K$ of order $q^2 + q + 1$ formed by the projectivities of $\text{PG}(V)$ induced by the matrices

$$
D_x = \begin{pmatrix}
x & 0 & 0 & 0 & 0 & 0 \\
0 & x^q & 0 & 0 & 0 & 0 \\
0 & 0 & x^{q^2} & 0 & 0 & 0 \\
0 & 0 & 0 & x^{-q^2} & 0 & 0 \\
0 & 0 & 0 & 0 & x^{-q} & 0 \\
0 & 0 & 0 & 0 & 0 & x^{-1}
\end{pmatrix}, \quad x \in \mathbb{F}_q, N(x) = 1,
$$

preserves $\mathcal{W}(5,q)$. The group $K$ fixes the two planes

$$
\pi_1 = \{P_{a,0} \mid a \in \mathbb{F}_q \setminus \{0\}\}, \quad \pi_2 = \{P_{0,b} \mid b \in \mathbb{F}_q \setminus \{0\}\}.
$$

**Lemma 3.6.** There are one or three $K$-orbits on points of $\pi_i$, $i = 1, 2$, according to $q \equiv 1 (\text{mod } 3)$ or $q \equiv 1 (\text{mod } 3)$. In the latter case their representatives are as follows

$$
\pi_1 : P_{z,0}, \quad \pi_2 : P_{0,z}, \quad i = 1, 2, 3,
$$

where $z = q^2 + q + 1$.
for some \( z \in \mathbb{F}_q^3 \) such that \( z^{q-1} = \xi \), where \( \xi \) is a fixed element in \( \mathbb{F}_q \) such that \( \xi^2 + \xi + 1 = 0 \).

The group \( K \) has \( q^3 - 1 \) orbits on points of \( \operatorname{PG}(V) \setminus (\pi_1 \cup \pi_2) \). Each of them has size \( q^2 + q + 1 \) and their representatives are the points

\[
P_{1,b}, \quad b \in \mathbb{F}_q \setminus \{0\}, \quad \text{if } q \equiv 1 \pmod{3},
\]

\[
P_{z^i,b}, \quad i = 1, 2, 3, \quad b \in \mathbb{F}_q \setminus \{0\}, \quad \text{if } q \equiv 1 \pmod{3}.
\]

**Proof.** The element of \( K \) induced by \( D_x \) fixes a point of \( \pi_1 \) (or of \( \pi_2 \)) if and only if \( x^{q-1} = x^{q^2 + q + 1} = 1 \), that is, \( x^q = x \), \( (x \in \mathbb{F}_q) \) and \( x^{q^2 + q + 1} = x^q x^q x = x^3 = 1 \). Therefore if \( q \equiv 1 \pmod{3} \), then \( K \) permutes in a single orbit the points of \( \pi_1 \) (or of \( \pi_2 \)). If \( q \equiv 1 \pmod{3} \), then the kernel of the action of \( K \) on both \( \pi_1 \) and \( \pi_2 \) consists of the subgroup induced by \( \langle D_\xi \rangle \), where \( \xi \) is a fixed element in \( \mathbb{F}_q^3 \) such that \( \xi^2 + \xi + 1 = 0 \). Such a subgroup has order three since \( \langle D_\xi \rangle \cong \mathbb{Z}/3 \mathbb{Z} \).

Let \( z \in \mathbb{F}_q^3 \) be such that \( z^{q-1} = \xi \). To see that the representatives of the \( K \)-orbits on points of \( \pi_1 \) (or of \( \pi_2 \)) are \( P_{z^i,0} \) (or \( P_{b,z^i} \)), \( i = 1, 2, 3 \), observe that

\[
\mathbb{F}_q \setminus \{0\} = \bigcup_{i=1}^{3} \{xz^i | x \in \mathbb{F}_q, x^{q^2 + q + 1} = 1 \}.
\]

Let \( R \) be a point of \( \operatorname{PG}(V) \setminus (\pi_1 \cup \pi_2) \). It is straightforward to check that no nontrivial element of \( K \) leaves \( R \) invariant, that is, \( |R^K| = q^2 + q + 1 \). Since the planes \( \pi_1 \) and \( \pi_2 \) are disjoint, there exists a unique line \( \ell \) of \( \operatorname{PG}(V) \) intersecting both \( \pi_1 \) and \( \pi_2 \). Hence if \( \ell \cap R^K \geq 2 \), then there exists a nontrivial element in \( K \) fixing \( \ell \) and hence stabilizing both \( \ell \cap \pi_1 \) and \( \ell \cap \pi_2 \). From the discussion above the existence of such a nontrivial element implies \( q \equiv 1 \pmod{3} \). Therefore, if \( q \not\equiv 1 \pmod{3} \), the representatives of the \( K \)-orbits on points of \( \operatorname{PG}(V) \setminus (\pi_1 \cup \pi_2) \) can be taken as the points on the \( q^2 + q + 1 \) lines through \( P_{1,0} \) and meeting \( \pi_2 \) in a point. Similarly, if \( q \equiv 1 \pmod{3} \), then these representatives can be chosen among the points of \( \operatorname{PG}(V) \setminus (\pi_1 \cup \pi_2) \) lying on the \( 3(q^2 + q + 1) \) lines through \( P_{z^i,0}, i = 1, 2, 3 \), and meeting \( \pi_2 \) in a point. In particular, in this case, \( P_{z^i,b} \) and \( P_{z^i,b'} \) lie in the same \( K \)-orbit if \( b = \xi b', i = 1, 2, 3 \). The result now follows. \( \square \)

**Theorem 3.7.** If \( c \in \mathbb{F}_q \setminus \{0\} \), the orbit \( O = P_{1,c}^K \) is a partial ovoid of \( \mathcal{W}(5, q) \) of size \( q^2 + q + 1 \).

**Proof.** Let \( P' \) be a point of \( P_{1,c}^K \setminus \{P_1,c\} \). Then there exists \( x \in \mathbb{F}_q^3 \), with \( N(x) = 1 \), \( x \not\equiv 1 \), such that

\[
P' = (x, x^q, x^{q^2}, cx^{-q^2}, cx^{-q}, cx^{-1}).
\]

If the line joining \( P_1,c \) with \( P' \) is a line of \( \mathcal{W}(5, q) \), then

\[
0 = cT(x^{-1} - x) = cT(x^{q+1} - x) = cN(1 - x)
\]

Therefore \( x = 1 \) and \( P_1,c = P', \) a contradiction. \( \square \)
To prove that $O$ is maximal we need to recall some results obtained by Culbert and Ebert in [6] in terms of Sherk surfaces. A Sherk surface $S(\alpha, \beta, \gamma, \delta)$, where $\alpha, \delta \in \mathbb{F}_q$, $\beta, \gamma \in \mathbb{F}_q^3$, can be seen as a hypersurface of the projective line $PG(1, q^3)$. More precisely
\[
S(\alpha, \beta, \gamma, \delta) = \{ x \in \mathbb{F}_q^3 \cup \{ \infty \} | \alpha N(x) + T(\beta^q x^q + 1) + T(\gamma x) + \delta = 0 \}.
\]
Furthermore, $\infty \in S(\alpha, \beta, \gamma, \delta)$ if and only if $\alpha = 0$.

**Lemma 3.8** (Culbert and Ebert [6]). Let $\alpha, \delta \in \mathbb{F}_q$, $\beta, \gamma \in \mathbb{F}_q^3$.
1. $|S(\alpha, \beta, \gamma, \delta)| \in \{ 1, q^2 - q + 1, q^2 + 1, q^2 + q + 1 \}$.
2. $|S(\alpha, \beta, \gamma, \delta)| = 1$ if and only if $(\alpha, \beta, \gamma, \delta) \in \{ (1, \beta, \beta^q + \gamma, N(\beta)), (0, 0, 0, 1) \}$.

Note that $\lambda S(\alpha, \beta, \gamma, \delta) + \mu S(\alpha, \beta, \gamma, \delta) = S(\lambda \alpha + \mu \alpha, \lambda \beta + \mu \beta, \lambda \gamma + \mu \gamma, \lambda \delta + \mu \delta)$ for any $\lambda, \mu \in \mathbb{F}_q$, $(\lambda, \mu) \neq (0, 0)$. Therefore any two distinct Sherk surfaces give rise to a pencil containing $q + 1$ distinct Sherk surfaces which together contain all the points of $PG(1, q^3)$. Moreover, any two distinct Sherk surfaces in a given pencil intersect in the same set of points, called the base locus of the pencil, which is the intersection of all the Sherk surfaces in that pencil.

**Lemma 3.9.** Let $\beta, \gamma \in \mathbb{F}_q^3$ not both zero. If one of the following is satisfied
\begin{itemize}
  \item $\beta \gamma = 0$,
  \item $N(\beta) \neq c^3$, for all $c \in \mathbb{F}_q \setminus \{0\}$,
  \item $N(\beta) = c^3$, for some $c \in \mathbb{F}_q \setminus \{0\}$, and $\gamma \beta \neq -c^2$,
\end{itemize}
then the base locus of the pencil generated by $S(1, 0, 0, -1)$ and $S(0, \beta, \gamma, 0)$ is not empty.

**Proof:** The pencil consists of $S(0, \beta, \gamma, 0)$ and $S(1, 0, 0, -1) + \lambda S(0, \beta, \gamma, 0) = S(1, \lambda \beta, \lambda \gamma, -1)$, where $\lambda \in \mathbb{F}_q$. Let $y_i$ be the number of Sherk surfaces in the pencil having $i$ points. Hence $y_i$ are nonnegative integer and $y_{q^2+q+1} \geq 1$ since $|S(1, 0, 0, -1)| = q^2 + q + 1$. By Lemma 3.8, it follows that $|S(0, \beta, \gamma, 0)| \neq 1$ and $|S(1, \lambda \beta, \lambda \gamma, -1)| \neq 1$, for $\lambda \in \mathbb{F}_q$, that is, $y_1 = 0$. Assume by contradiction that the base locus of the pencil is empty, then the following equations hold
\[
y_{q^2-q+1} + y_{q^2+1} + y_{q^2+q+1} = q + 1,
y_q^2 - q + 1)y_{q^2-q+1} + (q^2 + 1)y_{q^2+1} + (q^2 + q + 1)y_{q^2+q+1} = q^3 + 1.
\]
By substituting $y_{q^2-q+1} = q + 1 - y_{q^2+1} - y_{q^2+q+1}$ in the second equation, we have that
\[
y_{q^2+q+1} = -\frac{y_{q^2+1}}{2}.
\]
Therefore necessarily
\[ y_{q^2+q+1} = y_{q^2+1} = 0, y_{q^2-q+1} = q + 1, \]
which is a contradiction. □

**Theorem 3.10.** The partial ovoid \( \mathcal{O} \) of \( \mathcal{W}(5, q) \) is maximal.

**Proof.** It is enough to show that for each of the representatives \( R \) of the \( K \)-orbits on points of \( \text{PG}(V) \setminus \mathcal{O} \), there exists a point \( P = (x, x^q, x^{q^2}, cx^{-q^2}, cx^{-q}, cx^{-1}) \in \mathcal{O} \) such that the line joining \( R \) and \( P \) is a line of \( \mathcal{W}(5, q) \). If \( R \) is a point of \( \pi_1 \), then \( RP = 1, 0 \) if \( q \equiv 1 \pmod{3} \) or \( RP \in \{P_{x',0}^i | i = 1, 2, 3\} \), if \( q \equiv 1 \pmod{3} \). The line \( RP \) belongs to \( \mathcal{W}(5, q) \) if and only if there exists \( x \in F_q \), with \( N(x) = 1 \) such that
\[ cT(ux^{-1}) = 0, \tag{5} \]
where \( u = 1 \) or \( u \in \{z| i = 1, 2, 3\} \) according as \( q \not\equiv 1 \pmod{3} \) or \( q \equiv 1 \pmod{3} \). Since \( c \neq 0 \) and \( N(x) = 1 \), \( (5) \) is equivalent to \( T(u^q x^{q+1}) = 0 \), that is, \( x \in S(0, u, 0, 0) \). Hence the existence of a line \( RP \) of \( \mathcal{W}(5, q) \) is equivalent to the following
\[ |S(1, 0, 0, -1) \cap S(0, u, 0, 0)| \geq 1. \tag{6} \]

As before, let \( u = 1 \) or \( u \in \{z| i = 1, 2, 3\} \) according as \( q \not\equiv 1 \pmod{3} \) or \( q \equiv 1 \pmod{3} \). Let \( R \) be the point \( P_{0,u} \in \pi_2 \) or \( P_{a,b} \in \text{PG}(V) \setminus (\pi_1 \cup \pi_2 \cup \mathcal{O}) \), where \( b \in F_q \setminus \{0\} \), with \( b^3 \neq c^3 \), if \( u = 1 \), otherwise \( P_{1,b} \in \mathcal{O} \). Arguing in a similar way we get that the existence of a line \( RP \) of \( \mathcal{W}(5, q) \) is equivalent to
\[ |S(1, 0, 0, -1) \cap S(0, 0, u, 0)| \geq 1 \quad \text{or} \quad |S(1, 0, 0, -1) \cap S(0, cu, -b, 0)| \geq 1. \tag{7} \]

Since \( N(u) \) is a cube in \( F_q \) if and only if \( u = 1 \) and if \( u = 1 \), then \( b \neq c \), the inequalities in \( (6) \) and \( (7) \) are satisfied by Lemma 3.9. □

### 3.3 Maximal partial ovoids of \( \mathcal{W}(5, q) \), \( q \) even, of size \( 2q^2 - q + 1 \)

Assume \( q \) to be even and let \( \mathcal{W}(5, q) \) be the symplectic polar space of \( \text{PG}(5, q) \) induced by the totally isotropic subspaces of \( F_q^6 \) with respect to the nondegenerate alternating form given by
\[ b(x, y) = x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3 + x_5 y_6 + x_6 y_5. \]

Recall that \( s \) is the symplectic polarity of \( \text{PG}(5, q) \) associated with \( \mathcal{W}(5, q) \). Fix \( \delta_i \in F_q \) such that the polynomial \( X^2 + X + \delta_i \) is irreducible over \( F_q \), \( i = 1, 2 \). Let \( \Delta_1 \) and \( \Delta_2 \) be the three-n-space given by \( X_5 = X_6 = 0 \) and \( X_5 = X_4 = 0 \), respectively; consider the three-dimensional elliptic quadrics \( \mathcal{E}_i \subset \Delta_i \) defined as follows
\[ \mathcal{E}_1 : X_1X_2 + X_3^2 + X_3X_4 + \delta_1X_4^2 = 0, \]
\[ \mathcal{E}_2 : X_1X_2 + X_3^2 + X_5X_6 + \delta_2X_6^2 = 0. \]

Denote by \( \sigma \) the plane of \( \Delta_1 \) spanned by \( U_1, U_2, U_3 \); hence \( \mathcal{E}_1 \cap \sigma \) is a conic and \( U_2 \cap \Delta_1 = \sigma \). In the hyperplane \( U_2 : X_4 = 0 \), let \( \mathcal{P}_1 \) be the cone having as vertex the point \( U_3 \) and as base \( \mathcal{E}_2 \) and let \( \mathcal{P}_2 \) be the cone having as vertex the line \( \Delta_1^3 \) and as base the conic \( \mathcal{E}_1 \cap \sigma \). Set
\[ A = \mathcal{P}_1 \cap \mathcal{P}_2 = \{(1, c^2 + cd + \delta_2d^2, \sqrt{c^2 + cd + \delta_2d^2}, 0, c, d) | c, d \in \mathbb{F}_q \} \cup \{U_2\}. \]

Lemma 3.11. The point sets \( \mathcal{E}_1 \) and \( A \) are partial ovoids of \( \mathcal{W}(5, q) \) of size \( q^2 + 1 \).

Proof. The elliptic quadric \( \mathcal{E}_1 \) is an ovoid of \( \Delta_1 \cap \mathcal{W}(5, q) \). Hence it is a partial ovoid of \( \mathcal{W}(5, q) \). The set \( A \) is contained in \( U_2^3 \) and each of the lines is obtained by joining \( U_3 \) with a point of \( A \) with a point of \( \Delta_2 \) in a point of \( \mathcal{E}_2 \). In particular this gives a bijective correspondence between the points of \( A \) and those of \( \mathcal{E}_2 \). To see that \( A \) is a partial ovoid of \( \mathcal{W}(5, q) \) note that the elliptic quadric \( \mathcal{E}_2 \) is an ovoid of \( \mathcal{W}(5, q) \). \( \square \)

Theorem 3.12. The point set \( A \cup (\mathcal{E}_1 \setminus \sigma) \) is a maximal partial ovoids of \( \mathcal{W}(5, q) \) of size \( 2q^2 - q + 1 \).

Proof: Taking into account the previous lemma, it is enough to show that if \( P \) is a point of \( \mathcal{E}_1 \setminus \sigma \), then \( |P \cap A| = 0 \). By contradiction let \( P = (1, a^2 + ab + \delta_1b^2, a, b, 0, 0) \in \mathcal{E}_1 \setminus \sigma \) and \( R = (1, c^2 + cd + \delta_2d^2, \sqrt{c^2 + cd + \delta_2d^2}, 0, c, d) \in A \), for some \( a, b, c, d \in \mathbb{F}_q \), \( b \neq 0 \), with \( R \in P^3 \). Then \( \sqrt{c^2 + cd + \delta_2d^2} \in \mathbb{F}_q \) is a root of \( X^2 + bX + a^2 + ab + \delta_1b^2 \), which is irreducible over \( \mathbb{F}_q \), a contradiction.

To prove the maximality, let \( Q \) be a point not in \( A \cup (\mathcal{E}_1 \setminus \sigma) \) and assume that \( |Q \cap (\mathcal{E}_1 \setminus \sigma)| = 0 \). There are two possibilities: either \( Q \cap \Delta_1 \) coincides with \( \sigma \) or \( Q^3 \cap \Delta_1 \) is a plane meeting \( \mathcal{E}_1 \) in exactly one point of the conic \( \mathcal{E}_1 \cap \sigma \). In the former case \( Q^3 \) meets \( A \) in the points \( \{U_1, U_2\} \). If the latter possibility occurs, let \( R \) be the point of the conic \( \mathcal{E}_1 \cap \sigma \) contained in \( Q^3 \cap \Delta_1 \); since \( R^3 \cap \Delta_1 \subset Q^3 \), the point \( Q \) lies in the plane spanned by \( R \) and \( \Delta_1^3 \). Hence the line joining \( R \) and \( Q \) is a line of \( \mathcal{W}(5, q) \) and contains a unique point of \( A \). \( \square \)

Remark 3.13. Some computations performed with Magma [1] show that the largest partial ovoid of \( \mathcal{W}(5, q) \), has size 7 for \( q = 2 \) and 13 for \( q = 3 \). In particular, \( \mathcal{W}(5, 2) \) has a unique partial ovoid of size 7, up to projectivities, which coincides with the partial ovoids described above.

4 | PARTIAL OVOIDS OF HERMITIAN POLAR SPACES

Let \( \mathcal{H}(2n, q^2) \), \( n \geq 2 \), be a Hermitian polar space of \( \text{PG}(2n, q^2) \) and let \( \perp \) be its associated unitary polarity. Fix a point \( P \notin \mathcal{H}(2n, q^2) \) and a pointset \( T \subset P_{\perp} \) suitably chosen. In this section we explore the partial ovoids of \( \mathcal{H}(2n, q^2) \) that are constructed by taking the points of \( \mathcal{H}(2n, q^2) \) on the lines obtained by joining \( P \) with the points of \( T \). It turns out that to get a
partial ovoid, the set \( T \) has to intersect the lines that are tangent or contained in \( P^1 \cap \mathcal{H}(2n, q^2) \) in at most one point.

### 4.1 Tangent-sets of \( \mathcal{H}(2n - 1, q^2) \), \( n \in \{2, 3, 4\} \)

Let \( \mathcal{H}(2n - 1, q^2) \), \( n \geq 2 \), be a Hermitian polar space of \( \text{PG}(2n - 1, q^2) \). We introduce the notion of tangent-set of \( \mathcal{H}(2n - 1, q^2) \).

**Definition 4.1.** A tangent-set of a Hermitian polar space of \( \text{PG}(2n - 1, q^2) \) is a set \( \mathcal{T} \) of points of \( \text{PG}(2n - 1, q^2) \) such that every line that is either tangent or contained in \( \mathcal{H}(2n - 1, q^2) \) has at most one point in common with \( \mathcal{T} \). A tangent-set is said to be maximal if it is not contained in a larger tangent-set.

Here the main aim is to show that starting from a (partial) ovoid of \( \mathcal{W}(2n - 1, q) \) it is possible to obtain a tangent-set of \( \mathcal{H}(2n - 1, q^2) \). To do that some preliminary results are needed.

**Lemma 4.2.** An ovoid of \( \mathcal{W}(3, q) \subset \mathcal{H}(3, q^2) \) is a maximal partial ovoid of \( \mathcal{H}(3, q^2) \).

**Proof.** Let \( \mathcal{X} \) be an ovoid of \( \mathcal{W}(3, q) \subset \mathcal{H}(3, q^2) \). Then obviously \( \mathcal{X} \) is a partial ovoid of \( \mathcal{H}(3, q^2) \). To see that \( \mathcal{X} \) is maximal as a partial ovoid of \( \mathcal{H}(3, q^2) \), observe that every point of \( \mathcal{W}(3, q) \) lies on a unique extended line of \( \mathcal{H}(3, q^2) \).

Let us fix an element \( \iota \in F_q^2 \) such that \( \iota^2 = 0 \). Denote by \( \xi, \ldots, \xi_1 \) the elements of \( F_q^2 \) such that \( \iota \xi = \xi_i \). Then

\[
\{i : (\xi_i^q - \xi_i) = 0, \ldots, q\} = F_q.
\]

(8)

In particular we may assume \( \xi_1 = 0 \). Let \( \mathcal{H}_i \) be the Hermitian variety of \( \text{PG}(2n - 1, q^2) \) with equation

\[
t \left( X_i X_{i+1}^q - X_i^q X_{i+1} + \ldots + X_n X_{2n}^q - X_n^q X_{2n} + (\xi_i^q - \xi_i) X_{2n+1}^q \right) = 0, \quad i = 1, \ldots, q,
\]

and denote by \( \perp_i \) the polarity of \( \text{PG}(2n - 1, q^2) \) associated with \( \mathcal{H}_i \). Let \( \mathcal{H}_\infty : X_{2n+1}^q = 0 \). Then \( \mathcal{H}_\infty \) consists of the points of the hyperplane \( \Pi : X_{2n} = 0 \) and \( \mathcal{H}_i = \mathcal{H}_i + \lambda \mathcal{H}_\infty \), for some \( \lambda \in F_q \).

It follows that the set \( \{\mathcal{H}_i \mid i = 1, \ldots, q\} \cup \{\Pi\} \) is a pencil of Hermitian varieties. Since \( \Pi \cap \mathcal{H}_i = \mathcal{H}_\infty \cap \mathcal{H}_i = (\cap_{i=1}^q \mathcal{H}_i) \cap \mathcal{H}_\infty \), the base locus of the pencil is \( \Pi \cap \mathcal{H}_i \). Let \( \Sigma_i \) denote the Baer subgeometry of \( \text{PG}(2n - 1, q^2) \) consisting of the points

\[
\{(a_1, \ldots, a_{n-1}, a_n + \xi_i, a_{n+1}, \ldots, a_{2n-1}, 1) \mid a_i, \ldots, a_{2n-1} \in F_q\}
\]

\[
\cup\{(a_1, \ldots, a_{2n}, 0) \mid a_i, \ldots, a_{2n-1} \in F_q, (a_1, \ldots, a_{2n-1}) \neq (0, \ldots, 0)\},
\]

for \( i = 1, \ldots, q \). Note that if \( i \neq j \), then \( \Sigma_i \cap \Sigma_j = \Sigma_i \cap \Pi = \Sigma_j \cap \Pi \). It is easily seen that \( \Sigma_i \subset \mathcal{H}_i \), for \( i = 1, \ldots, q \). Hence \( \mathcal{H}_i \cap (\Sigma_i \cap \Pi) = \emptyset \), if \( i \neq 1 \). Moreover, since the unitary form
defining \( \mathcal{H}_i \), restricted to \( \Sigma_i \), is bilinear alternating, we have that \( \perp_{i|\Sigma_i} \) is a symplectic polarity of \( \Sigma_i \). Hence \( \mathcal{H}_i \) induces on \( \Sigma_i \) a symplectic polar space \( \mathcal{V}_i \) of \( \Sigma_i \) and the lines of \( \mathcal{H}_i \) having \( q + 1 \) points in common with \( \Sigma_i \) are those of \( \mathcal{V}_i \). Let \( H_i, i = 1, \ldots, q \), be the following \( 2n \times 2n \) matrix

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
-1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & -1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & -1 & 0 & 0 & \ldots & 0 & \xi^q_i - \xi_i \\
\end{pmatrix},
\]

and let \( K \) be the group of projectivities of \( \text{PG}(2n - 1, q^2) \) of order \( q^{2n-1} \) induced by the matrices

\[
M_a = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & a_1 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\alpha_{n+1} & -\alpha_{n+2} & \ldots & -\alpha_{2n-1} & 1 & a_1 & \ldots & a_{n-1} & a_n \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & a_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & a_{2n-1} \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \\
\end{pmatrix},
\]

where \( a = (a_1, ..., a_{2n-1}) \in \mathbb{F}_q^{2n-1} \). Since \( M_a^t H_i M_a^q = H_i \), it follows that \( K \) fixes \( \mathcal{H}_i \). Furthermore, if \( g \) is a projectivity of \( K \) induced by \( M_a \) and \( P \) is a point of \( \Sigma_i \), then \( P^g \in \Sigma_i \). Hence \( K \) stabilizes \( \Sigma_i \). We infer that \( K \) leaves invariant the symplectic polar space \( \mathcal{V}_i \) induced by \( \mathcal{H}_i \) on \( \Sigma_i \). The stabilizer of the point \( P^1_i = (0, \ldots, 0, \xi_i, 0, \ldots, 0, 1) \) in \( \Sigma_i \setminus \Pi \) is trivial. Hence the orbit \( P^K \) has size \( q^{2n-1} \). Since \( P^K \subset \Sigma_i \setminus \Pi \), it follows that \( P^K_i = \Sigma_i \setminus \Pi \), that is, \( K \) permutes in a single orbit the \( q^{2n-1} \) points of \( \Sigma_i \setminus \Pi \), \( i = 1, \ldots, q \).

Let \( s \) be the line joining \( U_n \) with \( P_1 \). If \( a \in \mathbb{F}_q \), then \((a + \xi_i - \xi_j)U_n + P_1 \) is a point of \( s \cap \Sigma_i \). Hence \( s \cap \Sigma_i = q + 1 \), \( i = 1, \ldots, q \). Since \( s \) has \( q \) points in common with \( \Sigma_i \setminus \Pi \) and \( K \) fixes \( U_n \) and acts transitively on the \( q^{2n-1} \) points of \( \Sigma_i \setminus \Pi \), it follows that \( s^K_i = q^{2n-2} \). In particular, the \( q^{2n} \) points of \( \bigcup_{i=1}^{q} \Sigma_i \setminus \Pi \) lie on the lines of \( s^K \). Denote by \( \mathcal{L} \) this set of \( q^{2n-2} \) lines. In the next lemma we restrict our attention to the case \( n = 2 \).

**Lemma 4.3.** Let \( \ell \in \mathcal{L} \) and let \( i \in \{2, \ldots, q\} \). The intersection of \( \Pi \) with the tangent lines to \( \mathcal{H}_i \) through a point of \( (\ell \cap \Sigma_i) \setminus \Pi \), is a nondegenerate Hermitian curve of \( \Pi \) belonging to the pencil determined by \( \ell^H_i \) and \( \Pi \cap \mathcal{H}_i \). Vice versa, if \( r \) is a line of \( \Pi \) with \( |r \cap \Sigma_i| = |r \cap \Pi| \) = \( q + 1 \) and \( \mathcal{U}_r \) is a nondegenerate Hermitian curve of \( \Pi \) belonging to the pencil determined by \( r \cap \Pi \) and \( \mathcal{H}_i \), then there exists \( k \) such that the lines joining a point of \((r^H_i \cap \Sigma_k) \setminus \Pi \) with a point of \( \mathcal{U}_r \) are tangent to \( \mathcal{H}_i \).
Proof. Let \( \ell \in \mathcal{L} \) and let \( i \in \{2, \ldots, q\} \). If \( P \) is a point of \((\ell \cap \Sigma_i) \setminus \Pi\), since \( K \) is transitive on \( \mathcal{L} \), we may assume \( \ell = s \) and hence \( P = (0, a_2 + \xi, 0, 1) \), \( \xi_i \neq 0 \), \( a_2 \in \mathbb{F}_q \). Then the points of \((x, y, z, 0)\) of \( \Pi \) lying on a line through \( P \) and tangent to \( \mathcal{H}_i \) satisfy

\[
\alpha^2 y^{q+1} + i\alpha(x^q z - xz^q) = 0,
\]

where \( \alpha = \ell\left(\xi - \xi_i\right) \). Hence they form a nondegenerate Hermitian curve of \( \Pi \) belonging to the pencil determined by \( s^{-1} : y^{q+1} = 0 \) and \( \Pi \cap \mathcal{H}_i : xz^q - x^q z = 0 \). Vice versa, let \( r \) be a line of \( \Pi \) with \( r \cap \Sigma_i = \emptyset \). Since \( \Sigma_i \) induces a polarity on \( \Sigma_1 \) and \( r \cap \Sigma_i = \emptyset \) implies \( U_2 \in r^{-1} \). Hence \( r^{-1} \) is a line of \( \mathcal{L} \). Since \( K \) is transitive on \( \mathcal{L} \), we may assume that \( r^{-1} = s \) and hence \( r \) is the line joining \( U_1 \) and \( U_2 \). It follows that \( U_\ell \) is given by (9), for some \( \alpha \in \mathbb{F}_q \setminus \{0\} \). Thus if \( \alpha = \ell\left(\xi - \xi_i\right) \), the lines joining a point of \((r^{-1} \cap \Sigma_k) \setminus \Pi\) with a point of \( \cup_{i=1}^q \) are tangent to \( \mathcal{H}_i \).

\[ \square \]

**Lemma 4.4.** Let \( \ell \) be a line that is either tangent to \( \mathcal{H}_i \) or contained in \( \mathcal{H}_i \). Then \( \ell \cap \left(\bigcup_{i=1}^q \Sigma_i\right) \in \{0, 1, q + 1\} \). In particular, if \( \ell \cap \left(\bigcup_{i=1}^q \Sigma_i\right) = q + 1 \) then there exists \( k \) such that \( \ell \cap \Sigma_k = q + 1 \) and \( \ell \subset \mathcal{H}_k \).

Proof. If \( \ell \) is contained in \( \mathcal{H}_i \), then there is nothing to prove. If \( \ell \) is tangent, let \( P_1, P_2 \) be two points belonging to \( \ell \cap \Sigma_i \) and \( \ell \cap \Sigma_j \), respectively, where \( i \neq 1 \). Since \( K \) is transitive on points of \( \Sigma_1 \setminus \Pi \), we may assume \( P_1 = (0, 0, \ldots, 0, \xi, 0, \ldots, 0, 1) \). If \( P_2 = (a_1, \ldots, a_{n-1}, a_n + \xi_j, a_{n+1}, \ldots, a_{2n-1}, 1) \) for some \( a_1, \ldots, a_{2n-1} \in \mathbb{F}_q \) not all zero, and \( \lambda \in \mathbb{F}_q \), then the point \( P_2 + \lambda P_1 \) belongs to \( \mathcal{H}_i \) if and only if \( \lambda \) is a root of

\[
\alpha\left(\xi - \xi_i\right)X^{q+1} + \beta\left(\xi - \xi_i + a_n\right)X + \gamma = 0,
\]

Some straightforward calculations show that \( \ell \) is tangent if and only if \( \xi_j = \xi_i - a_n \). If \( i \neq j \), then \( \left(\xi_j - \xi_i\right) = \beta\left(\xi - \xi_i\right) \), contradicting (8). Therefore \( i = j \) and \( a_n = 0 \). This means that \( \ell \cap \Sigma_i = q + 1 \). Moreover \( \ell \subset \mathcal{H}_i \). If \( P_2 = (a_1, \ldots, a_{2n-1}, 0) \) for some \( a_1, \ldots, a_{2n-1} \in \mathbb{F}_q \), then \( P_2 \in \Sigma_i \cap \Sigma_j \). Hence \( \ell \cap \Sigma_i = q + 1 \) and we may repeat the previous argument.

\[ \square \]

**Theorem 4.5.** Let \( O_i \) be a (partial) ovoid of \( \mathcal{W}_i \), \( i = 1, \ldots, q \), such that \( O_i \cap \Pi = O_1 \cap \Pi \), if \( i \neq j \). Then \( T = \bigcup_{i=1}^q O_i \) is a tangent-set of \( \mathcal{H}_i \). Moreover if \( n = 2 \) and \( O_i \) is an ovoid of \( \mathcal{W}_i \), for every \( i = 1, \ldots, q \), then \( T \) is a maximal tangent-set.

Proof. Assume by contradiction that \( T \) is not a tangent-set. If \( \ell \) is a line of \( \mathcal{H}_i \) and \( \ell \cap \Pi \geq 2 \), then \( \ell \cap T \subset O_1 \), since \( T \cap \mathcal{H}_1 = O_1 \). Hence \( \ell \) is an extended line of \( \mathcal{W}_i \) containing two points of \( O_1 \), contradicting the assumption that \( O_1 \) is a (partial) ovoid of \( \mathcal{W}_i \). Similarly, if \( \ell \) is tangent to \( \mathcal{H}_1 \) and \( \ell \cap \Pi \geq 2 \), then, by Lemma 4.4, there exists a \( k \) such that \( \ell \) is an extended line of \( \mathcal{W}_k \) and \( \ell \cap T \subset O_k \). Therefore \( O_k \) is not a (partial) ovoid of \( \mathcal{W}_k \), since \( \ell \cap O_k \geq 2 \), contradicting the hypotheses.
Assume now that \( n = 2 \) and \( O_i \) is an ovoid of \( \mathcal{W}_i \), for every \( i = 1, \ldots, q \). We claim that every point \( R \) of \( \text{PG}(3, q^2) \setminus T \) lies on at least one line that is either tangent or contained in \( \mathcal{H}_1 \) and contains one point of \( T \). If \( R \not\in \Pi \), then there exactly one \( \mathcal{H}_k \) with \( R \in \mathcal{H}_k \); hence there is at least one extended line, say \( r \) of \( \mathcal{W}_k \) containing \( R \). Such a line \( r \) has to contain exactly one point of \( O_k \) (see Lemma 4.2) and by Lemma 4.4 it is contained or tangent to \( \mathcal{H}_1 \) according as \( k = 1 \) or \( k \neq 1 \). Similarly if \( R \in \mathcal{H}_1 \cap \Pi \). Let \( R \) be a point of \( \Pi \setminus \mathcal{H}_1 \). There are two possibilities: either \( U_i \in O_i \) for every \( i = 1, \ldots, q \), and in this case the line joining \( R \) and \( U_i \) is tangent to \( \mathcal{H}_1 \) or \( U_i \not\in O_i \), for \( i = 1, \ldots, q \). If the latter possibility occurs, then there exists a line \( r \) of \( \Pi \) such that \( R \not\in r \), \( |I \cap O_i| = 0 \), and hence there is at least one extended line, say \( r \cap \Sigma_i | = r \cap \mathcal{H}_1 | = q + 1 \). Let \( \mathcal{U}_r \) be the (unique) nondegenerate Hermitian curve of \( \Pi \) belonging to the pencil determined by \( r \) and \( \Pi \cap \mathcal{H}_1 \) such that \( R \in \mathcal{U}_r \). By Lemma 4.3, there exists a \( k \) such that the lines joining a point of \( (r^k \cap \Sigma_k) \setminus \Pi \) with \( R \) are tangent to \( \mathcal{H}_1 \). The Baer subline \( r^{k-1} \cap \Sigma_k \) contains two points of \( O_k \) being the polar line of \( r \cap \Sigma_k \) with respect to the symplectic polarity of \( \Sigma_k \) associated with \( \mathcal{W}_k \). Therefore the line joining a point of \( r^{k-1} \cap \Sigma_k \cap O_k \) with \( R \) is tangent to \( \mathcal{H}_1 \).

**Proposition 4.6.** If there is a (partial) ovoid of \( \mathcal{W}(2n - 1, q) \) of size \( x + y \), with \( y \) points on a hyperplane, then there exists a tangent-set of \( \mathcal{H}(2n - 1, q^2) \) of size \( xq + y \).

The next result is reached by applying Theorem 4.5, where \( O_i \) is either an ovoid of \( \mathcal{W}(3, q) \), \( q \) even, or the partial ovoid of \( \mathcal{W}(3, q) \), obtained in Theorem 3.3, or the partial ovoids of \( \mathcal{W}(5, q) \) described in Theorem 3.7 and in Theorem 3.12, or the partial ovoid of \( \mathcal{W}(7, q) \) of size \( q^3 + 1 \) given in [2, 3].

**Corollary 4.7.** The following hold true.

- If \( q \) is even, there exist maximal tangent-sets of \( \mathcal{H}(3, q^2) \) of size \( q^3 + 1 \) and of size \( q^3 - q^2 + q + 1 \).
- If \( q \) is an odd square and \( q \neq 0(\text{mod} \ 3) \), there exists a tangent-set of \( \mathcal{H}(3, q^2) \) of size \( (q^{5/2} + 3q^2 - q^{3/2} + 3q)/3 \).
- If \( q \) is even, there exists a tangent-set of \( \mathcal{H}(5, q^2) \) of size \( 2q^3 - q^2 + 1 \).
- There exists a tangent-set of \( \mathcal{H}(5, q^2) \) of size \( q^3 + q^2 + 1 \).
- There exists a tangent-set of \( \mathcal{H}(7, q^2) \) of size \( q^3 + 1 \).

**Problem 4.8.** Construct larger tangent-sets of \( \mathcal{H}(2n - 1, q^2) \) and provide an upper bound on their size.

### 4.2 Partial ovoids of \( \mathcal{H}(2n, q^2) \) from tangent-sets of \( \mathcal{H}(2n - 1, q^2) \), \( n \in \{2, 3, 4\} \)

The next result shows that starting from a (maximal) tangent-set of \( \mathcal{H}(2n - 1, q^2) \) it is possible to obtain a (maximal) partial ovoid of \( \mathcal{H}(2n, q^2) \).

**Lemma 4.9.** Let \( P \) be a point of \( \text{PG}(2n, q^2) \setminus \mathcal{H}(2n, q^2) \) and let \( T \) be a (maximal) tangent-set of \( \mathcal{H}(2n - 1, q^2) = P^1 \cap \mathcal{H}(2n, q^2) \). Then
\[ \mathcal{O} = \left( \bigcup_{R \in \mathcal{T}} \langle P, R \rangle \right) \cap \mathcal{H}(2n, q^2) \]

is a (maximal) partial ovoid of \( \mathcal{H}(2n, q^2) \) of size \((q + 1) | T \setminus \mathcal{H}(2n - 1, q^2) | + | T \cap \mathcal{H}(2n - 1, q^2) | \).

**Proof.** Assume by contradiction that \( \mathcal{O} \) is not a partial ovoid, then there would exist a line \( \ell \) of \( \mathcal{H}(2n, q^2) \) containing two distinct points \( P_1 \) and \( P_2 \) of \( \mathcal{O} \). If at least one among \( PP_1, P_2 \) is not in \( \mathcal{P} \), then the line \( \overline{P_1P_2} \subset \mathcal{P} \) obtained by projecting \( \ell \) from \( P \) to \( \mathcal{P} \) is tangent to \( \mathcal{H}(2n - 1, q^2) = \mathcal{P} \cap \mathcal{H}(2n, q^2) \) and contains two points of \( T \), a contradiction. If both \( P_1 \) and \( P_2 \) are in \( \mathcal{P} \), then \( \ell \) is a line of \( \mathcal{H}(2n - 1, q^2) \) and contains two points of \( \mathcal{O} \), a contradiction.

As for maximality, note that if \( T \) is a point of \( \mathcal{H}(2n, q^2) \) such that \( \mathcal{O} \cup \{ T \} \) is a partial ovoid of \( \mathcal{H}(2n, q^2) \) and \( \bar{T} = \langle P, T \rangle \cap P \), then \( \mathcal{O} \cup \{ \bar{T} \} \) is a tangent-set of \( \mathcal{H}(2n - 1, q^2) \).

Combining Theorem 4.5 and Lemma 4.9, the following result arises.

**Theorem 4.10.** If there is a (partial) ovoid of \( \mathcal{W}(2n - 1, q) \) of size \( x + y \), with \( y \) points on a hyperplane, then there exists a partial ovoid of \( \mathcal{H}(2n, q^2) \) of size \( xq^2 + y \).

Note that if a partial ovoid of \( \mathcal{W}(2n - 1, q) \) is not maximal, then there exists a hyperplane disjoint from it. Hence, in this case it is possible to choose \( y = 0 \) in Theorem 4.10. It follows that starting from (partial) ovoids of \( \mathcal{W}(2n - 1, q) \), we get partial ovoids of \( \mathcal{H}(2n, q^2) \), \( n \in \{2, 3, 4\} \), as described below.

**Corollary 4.11.** The following hold true.

- If \( q \) is even, there exist maximal partial ovoids of \( \mathcal{H}(4, q^2) \) of size \( q^4 + 1 \) and of size \( q^4 - q^3 + q + 1 \).
- If \( q \) is an odd square and \( q \not\equiv 0 \pmod{3} \), there exists a partial ovoid of \( \mathcal{H}(4, q^2) \) of size \( (q^{7/2} + 3q^3 - q^{5/2} + 3q^2) / 3 \).
- If \( q \) is even, there exists a partial ovoid of \( \mathcal{H}(6, q^2) \) of size \( 2q^4 - q^3 + 1 \).
- There exists a partial ovoid of \( \mathcal{H}(6, q^2) \) of size \( q^4 + q^3 + 1 \).
- There exists a partial ovoid of \( \mathcal{H}(8, q^2) \) of size \( q^5 + 1 \).

In view of Lemma 4.9, it would be interesting to consider the following problem.

5 | CONCLUSIONS

In this paper we dealt with large partial ovoids of symplectic polar spaces and Hermitian polar spaces lying in projective spaces of even dimension. In general to obtain large partial ovoids seems to be a difficult task. Here, several subgroups of the related symplectic and unitary groups along with peculiar geometric settings are studied and used to construct examples of partial ovoids that, in some cases, are the currently best-known ones. However, as displayed in Tables 1 and 2, there is a large discrepancy between the known examples and the current best
theoretical upper bound. Even worse, this gap increases terribly in higher dimensions. This suggests that the following two problems require further investigations:

- construct large partial ovoids also by using different techniques (as for instance the probabilistic method);
- improve the theoretical upper bounds.

ACKNOWLEDGMENT
This study was supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA—INdAM).

ORCID
Jan De Beule http://orcid.org/0000-0001-5333-5224
Francesco Pavese http://orcid.org/0000-0002-8763-5329

REFERENCES
1. W. Bosma, J. Cannon, and C. Playoust, The magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), 235–265.
2. A. Cossidente, On twisted tensor product group embeddings and the spin representation of symplectic groups: The case q odd, ISRN Geom. 2011, (2011), 694605, 6pp.
3. A. Cossidente and O. H. King, Twisted tensor product group embeddings and complete partial ovoids on quadrics in PG(2t − 1, q), J. Algebra. 273 (2004), 854–868.
4. A. Cossidente and F. Pavese, On (0, α)-sets of generalized quadrangles, Finite Fields Appl. 30 (2014), 139–152.
5. A. Cossidente and A. Siciliano, Commuting polarities and maximal partial ovoids of H(4, q2), J. Combin. Des. 17 (2009), no. 4, 307–313.
6. C. Culbert and G. L. Ebert, Circle geometry and three-dimensional subregular translation planes, Innov. Incidence Geom. 1 (2005), 3–18.
7. J. De Beule, A. Klein, K. Metsch, and L. Storme, Partial ovoids and partial spreads in Hermitian polar space, Des. Codes Cryptogr. 47 (2008), 21–34.
8. J. De Beule, A. Klein, K. Metsch, and L. Storme, Partial ovoids and partial spreads in symplectic and orthogonal polar spaces, European J. Combin. 29 (2008), 1280–1297.
9. J. W. P. Hirschfeld, Finite projective spaces of three dimensions, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1985.
10. J. W. P. Hirschfeld, Projective geometries over finite fields, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1998.
11. F. Mazzocca, O. Polverino, and L. Storme, Blocking sets in PG(r, qt), Des. Codes Cryptogr. 44 (2007), 97–113.
12. C. Rößing and L. Storme, A spectrum result on maximal partial ovoids of the generalized quadrangle Q(4, q), q even, European J. Combin. 31 (2010), no. 1, 349–361.
13. G. Tallini, Fibrations through lines in a nonsingular quadric Q4,q of PG(4, q), Atti Accad. Peloritana Pericolanti. Cl. Sci. Fis. Mat. Natur. 66 (1988), 127–146 (1989).

How to cite this article: M. Ceria, J. D. Beule, F. Pavese, and V. Smaldore, On large partial ovoids of symplectic and Hermitian polar spaces, J. Combin. Des. (2023), 31, 5–22. https://doi.org/10.1002/jcd.21864