Dissipative Floquet Majorana modes in proximity-induced topological superconductors

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We study a realistic Floquet topological superconductor, a periodically driven system proximity to an equilibrium superconductor. Due to both strong energy and density fluctuations caused from the superconducting proximity effect, the superconductor is intrinsically poisoning and dissipative. We show that the Floquet band structure is still preserved in this dissipative system. In particular, we find that the external gap can protect topological characters: one of the Floquet Majorana modes, the Floquet Majorana zero mode, but not the Floquet Majorana $\tau$-mode, is preserved. The life time of the mode can be engineered by the external driving field.

**Introduction.**—Floquet engineering, which controls the quantum systems using periodic driving [1–3], is believed to provide a potentially accessible method to realize the topological nontrivial band structures and other exotic quantum states [4–24]. In addition, "Floquet engineering" also provides a simple platform to study nonequilibrium dynamics and statistical mechanics [1].

The non-interacting(or isolated) Floquet systems have been well studied. However, these systems are not realistic and fail to capture key physical properties. For example, a non-integrable interacting Floquet system usually approaches featureless infinite temperature states [25] and Floquet pre-thermal behaviors [26–28], and an open Floquet system usually shows complicated statistical behaviors depending on the details of system-bath couplings [29–32]. Therefore, it is crucial to consider more reasonable theoretical methods [29, 33–58], and ask if the Floquet band picture still works in the realistic conditions.

For some non-interacting models, the inappropriate application of Floquet band theory could result in contentious results. An example is the Floquet engineering in the presence of superconductivity (SC): the intrinsic SC is originated from certain interaction instabilities, which could be significantly modified by periodic driving force; proximity-induced SC is compatible with non-interacting assumption, but energy and particle fluctuations are unavoidable in the proximity effect [59]. This calls for a more realistic theoretical framework to study the Floquet engineered topological superconductivity and Floquet Majorana zero modes [8] due to proximity-induced SC. We can also see such a demand from the systems on which recent experimental progress in the realization and detection of Majorana zero modes [60–78] are based. Since the SC, or Cooper pair, is not intrinsic and comes from the tunneling of the SC substrate, a realistic framework is to regard the SC as an external bath, which not only provides the tunneling of Cooper pair but also acts as a dissipative source that renders the periodic driven nanowire reaching a non-equilibrium steady state. In such a realistic case, it is unclear that the Floquet band picture is still valid and Floquet Majorana modes can be preserved due to the interplay between the non-equilibrium conditions and strong dissipation. In principle, the dissipation effect can show different structures in different Floquet harmonic bands to destroy the Floquet band picture.

In this paper, we consider a realistic model based on a periodically driven system proximity to an equilibrium

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**FIG. 1.** The closed (or intrinsic) Floquet SC limit. (a) shows the setup. (b) and (c) shows the wave-functions of MZM with $E = 0$ and open boundary spectra/Floquet band structure of Eq. 3 with $\Sigma(\omega(k)) = -V^2\sigma_y\tau_y$. Note that the left black spectrum in (c) is for an open finite wire, while the right $k$–dependent spectrum is for a wire with spatially periodic boundary condition $N_P = 5$. The parameters are chosen $t = 1, \lambda = 1.5, \mu_0 = -2, V_z = 1.2, A = 3/2, \Omega = 6, V = 0.8, N_P = 5$. 

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superconductor. Using the Keldysh Green’s function calculation, we find that the \( \omega \)-dependent self-energy correction \( \Sigma_{sc}(k, \omega) \) due to the proximity SC plays a central role in the steady state and modifies the Floquet topological band structure: possibility of killing of the Floquet Majorana pi modes (FMPMs) and the poisoning of Floquet Majorana zero modes (FMZMs). We then introduce a simplified Floquet Majorana poisoning model, and show that the dissipation effect for different Floquet bands are the same and so does the decay in each Floquet harmonic band. We conclude that dissipation is inevitable for all the Floquet bands regardless they are inside or outside the external SC gap; and therefore, the Floquet band picture is preserved, but modes in each band acquire a finite lifetime. We also find that the lifetime can be engineered by the external driving field.

**Realistic Floquet proximity SC.**—We consider a periodic driven semiconductor nanowire coupled to a conventional \( s \)-wave SC as shown in Fig. 1 (a). The Hamiltonian has three parts and can be written as follows

\[
\hat{H}(t) = \hat{H}_{nw}(t) + \hat{H}_{sc} + \hat{H}_t, \tag{1}
\]

where \( \hat{H}_{nw}(t) = \sum_{k} \psi^\dagger_k (\hat{c}_{k\uparrow}(t) - \mu_0 + 2A \cos \Omega t) + V_{z} \sigma_z + \lambda \sin k \sigma_y) \psi_k \) with \( \psi_k = (\hat{c}_{k\uparrow}, \hat{c}_{k\downarrow})^\dagger \) is the Hamiltonian of nanowire driven by the external lead; \( \hat{H}_{sc} = \sum_q \phi^\dagger_q (\epsilon_q \sigma_z + \Delta \tau_y) \phi_q \) with \( \phi_q = (\hat{a}_{q\uparrow}, \hat{a}_{q\downarrow})^\dagger \) is the Hamiltonian of SC bath; \( \hat{H}_t = \sum_{p,q,\sigma} (V \hat{c}_{p\sigma} \hat{a}_{q\sigma} + V^* \hat{a}_{q\sigma}^\dagger \hat{c}_{p\sigma}^\dagger) \) is Hamiltonian describing the nanowire-bath coupling. Here the parameters \( V_z, \lambda, \Delta \) represent the Zeeman field, spin-orbit coupling strength, and SC order parameter of SC bath respectively. Without loss of generality, \( \Delta \) is assumed to be a positive number in the following discussion.

It is widely believed that an open Floquet system coupled to an external thermal bath will eventually reach a non-equilibrium steady state, in which the energy absorbed from the external driving field is balanced by the energy flowing out to the environment. Theoretically, the physical observables in the non-equilibrium steady state can be conveniently dealt within the framework of Keldysh formalism. To be more precise, the spectral properties and the distribution functions can be calculated from the retarded and Keldysh components of the Keldysh Green’s function. Since in this paper, we mainly focus on the quasi-particle lifetime, only the retarded component is needed. In the supplementary material [79], we show that when the external SC bath degrees of freedom are integrated out, the retarded component of the Keldysh Green’s function has the following form,

\[
G^R_{nw}(k, \omega) = \left[ \omega - H_{eff}(k, \omega) \right]^{-1}, \tag{2}
\]

where the underlines represent the Floquet space with Floquet dimension \( 2N_F + 1 \), and the \( \omega \)-dependent non-Hermitian effective Hamiltonian is

\[
H_{eff}(k, \omega) = \begin{pmatrix}
\cdots \\
H_{nw}(k) - \Omega + \Sigma_{sc}(\omega + \Omega) & A\sigma_0 \tau_z \\
A\sigma_0 \tau_z & H_{nw}(k) + \Sigma_{sc}(\omega - \Omega) \\
A\sigma_0 \tau_z & A\sigma_0 \tau_z & H_{nw}(k) + \Sigma_{sc}(\omega - \Omega) \\
\cdots
\end{pmatrix}. \tag{3}
\]

Here

\[
H_{nw}(k) = ((-2t \cos k - \mu_0) \sigma_0 + V_z \sigma_z + \lambda \sin k \sigma_y) \tau_z \tag{4}
\]

is the static Hamiltonian of the nanowire and

\[
\Sigma_{sc}(\omega) = V^2 [-(\omega + i\eta) - \Delta \sigma_y \tau_y] / \sqrt{-(\omega + i\eta)^2 + \Delta^2} \tag{5}
\]

with \( \eta = 0^+ \) is the self-energy correction of the nanowire induced by the SC bath[30]. The time-averaged momentum resolved density of states (DoS) \( \nu_k(\omega) \) can be calculated by the 00-Floquet-entry of the retarded component of Keldysh Green’s function, namely,

\[
\nu_k(\omega) = -\frac{1}{\pi} \text{Tr} \text{Im} \left[ G_{nw,k,\omega}^R \right]_{00}. \tag{6}
\]

From Eq. 2 - 5, one can notice that the informations of external bath are encoded from the self-energy correction. Inside the SC gap, \( \Sigma_{sc}(\omega) \) reduces to \( \Sigma_{sc}(|\omega| < \Delta) = V^2 [-\omega - \Delta \sigma_y \tau_y] / \sqrt{\Delta^2 - \omega^2} \), whose imaginary part is zero. In this case, the SC bath only provides the proximity Cooper pair to the nanowire due to the existence of \( \Delta \sigma_y \tau_y \) term. On the other hand, outside the SC gap, \( \Sigma_{sc}(\omega) \) reduces to \( \Sigma_{sc}(|\omega| > \Delta) = iV^2 [-|\omega| - \Delta \sigma_y \tau_y] / \sqrt{\omega^2 - \Delta^2} \), which is totally imaginary and also plays the role of effective dissipation. Physically, this can be understood by the fact that the SC bath outside the gap has nonzero DoS, which results the energy transfer between the system and the SC bath. When the self-energy correction \( \Sigma_{sc}(\omega) \) can be approximated to the zeroth order of \( \omega \) at \( \omega = 0 \), namely, \( \Sigma_{sc}(\omega) \approx \Sigma_{sc}(0) = -V^2 \sigma_y \tau_y \), the proximity SC is equivalent to the intrinsic SC with order parameter \( \Delta_t = -V^2 \), which is called intrinsic SC limit in this paper. This approximation has been widely applied in the discussion of static nanowire
systems. Since the celebrated topological band theory for equilibrium case is well established in this limit, we plot the corresponding Floquet band structure and open boundary spectra in Fig. 1 (c) for comparison. Both the FMZMs and FMPMs do exist in the open boundary spectra. The corresponding wave-function of the FMZMs with $E = 0$ is plotted in Fig. 1 (b). One can also notice that the FMZM is localized not only in the real space, but also in the Floquet space. Due to the translational symmetry of the Floquet Hamiltonian, the FMZMs with $E = n\Omega$ must be localized at the Floquet sites $n\Omega$ [80].

For the realistic Floquet proximity SC, the self-energy correction can not be approximated by a constant due to non-equilibrium excitations. The $\omega$ linear and non-linear terms will play a crucial role here. In order to investigate the role of the $\omega$-dependent self-energy correction $\Sigma_{sc}(\omega)$, we first apply the numerical calculation of $\nu_k(\omega)$ with spatially periodic boundary condition (PBC) as shown in Fig. 2 (a). Here, we consider the exact $\omega$-dependent self-energy in Eq. (5). When $\Delta$ is much larger than the driven frequency $\Omega$, $\nu_k(\omega)$ can be approximated described by the Floquet band theory as shown in (a1), which means the linear and non-linear self-energy effect can be ignored. With the decreasing of $\Delta$, the sharp features in the spectrum are continuously broadened due to the dissipation effect. More interestingly, when $2\Delta$ is smaller than $\Omega$, as shown in (a4), $\nu_k(\omega)$ even exhibits discontinuous behavior at $\omega = \pm \Delta$ due the singularity of $\Sigma_{sc}(\omega = \Delta)$. This is a strongly nonlinear self-energy effect, which will kill the FMPMs as shown later. It should be noted that, according to Eq. 5, the expansion

$$\Sigma_{sc}(\omega) \simeq -V^2\sigma_y^\tau_y - \omega V^2/\Delta + o(\omega^2)$$

implies the low energy theory around $\omega = 0$ for all different values of $\Delta$ have similar behaviors compared to those for the intrinsic SC limit in Fig. 1 (c); and this is consistent with our numerical results shown in Fig. 2 (a1)-(a4).

An important consequence of this observation is that the surviving and poisoning of FMZMs can be captured by the linear self-energy effect.

We now turn to the discussion of Floquet Majoranas. In order to characterize them, we numerically calculate the time-averaged local DoS ($\text{LDoS}$) at the end of the nanowire with open boundary condition, based on the recursive Green function method [81–83] (refer to SI [79] for the Floquet systems). As shown in Fig. 2 (b1)-(b4), the numerical results verify our observations from the band structure in (a1)-(a4), namely, the killing of FMPMs, and the surviving and poisoning of FMZMs. As shown in (b1), the peaks at $\omega = \pm \Omega/2$ correspond to Floquet Majoranas. When $\Delta$ decreases to $2\Delta < \Omega$, the FMPMs are destroyed as shown in (b1)-(b4). This can be understood by the gap closing at the BZ corners $\omega = \pm \Omega/2$ shown in (a4). In contrast, the FMZMs is not sensitive to the decreasing of $\Delta$. We finally note that in the numerical results in (b1)-(b4), for the FMZMs outside the SC gap, their peak heights are very tiny compared with the one inside the SC gap. It is important to answer whether the dissipation structures and the lifetime of the Floquet Majoranas in different Floquet BZs are identical or distinct. If the dissipation structures are identical, the dissipation-modified Floquet bands are still valid pictures along with a constant finite lifetime acquired due to competition between dissipation and non-equilibrium environments. On the other hand, the Floquet schemes are in dangerously trouble.

Floquet Majorana poisoning model.—In order to illustrate how the Floquet picture is modified by the dissipations provided by the SC, we propose a Floquet Majorana poisoning model, as shown in Fig. 3 (a). This models describes a boundary isolated Majorana coupled to a dissipative gapped bath and driven by an external field with frequency $\Omega$ and amplitude $A$. In this model, the undriven Majorana with zero energy is modified by the self-energy of the SC, namely, $\Sigma(\omega)$. Under external driven, the Majorana can hop to the other Floquet Brillouin (BZ) with absorbing or eliminating energy $\pm\Omega$, and is modified by the self-energy $\Sigma(\omega \mp \Omega)$. This gives the
FIG. 3. Floquet Majorana poisoning model. (a) shows the first order procedure. (b1)-(b3) show the exact, zeroth and first order numerical calculation of $\nu(\omega)$ in Eq. 6 with $\Omega = 2$, $\Delta = 3$, $A = 1/2$, $\eta = 1/1000$, and different values of $V$. (c) shows the comparison of the exact result and first order approximation. (d) shows the relation between the lifetime and $A$ with the same parameters shown above.

The following effective $\omega$ dependent Hamiltonian

$$H(\omega) = \begin{pmatrix} \cdots & \Sigma(\omega + \Omega) - \Omega & A & 0 \\ A & \Sigma(\omega) & A \\ 0 & A & \Sigma(\omega - \Omega) + \Omega \\ \cdots \end{pmatrix},$$

where $\Sigma(\omega) = -V^2(\omega + i\eta)/\sqrt{-(\omega + i\eta)^2 + \Delta^2}$, and $V$ is the Majorana-SC bath coupling strength. The time-averaged DoS $\nu(\omega)$ can be calculated by the 00-Floquet-entry of the retarded Green’s function

$$\nu(\omega) = -\frac{1}{\pi} \text{Im} \left[ G^{R}(\omega) \right]_{00},$$

where $G^{R}(\omega) = [\omega - H(\omega)]^{-1}$. As shown in Fig. 3 (a), although the zero energy Majorana mode is not directly coupled to the bath due to the SC gap, the mode can be excited and de-excited between the zero Floquet BZ and the higher Floquet BZs, which directly couples to the SC bath and cause finite dissipation. This means even for the Majoranas inside the SC gap, dissipations are inevitable due to the inter-Floquet-BZ coupling.

As shown in Fig. 3 (b) and (c), the exact results of $\nu(\omega)$ are plotted with the red lines. One can find at each Floquet BZ center $\omega = m\Omega$, there exist a Floquet Majorana peak. However, in the last section of Supplementary Materials, we show that there only exist a single pole of $1/|\omega - H(\omega)|$, which is at $\omega = i\eta$. This means the lifetimes of the Floquet Majoranas at $\omega = \pm m\Omega$ can not be defined using the traditional method. However, we find that these Floquet Majorana peaks at $\omega = m\Omega$ can be well fitted by the first order expansion of $\omega - H(\omega)$ around $\omega = m\Omega$. As shown in Fig. 3 (b)-(c), the first order results are shown with solid black lines. As a comparison, we also plot the zeroth order result in (b) with dashed black lines. One can noticed that the zeroth order breaks down, while the first order approximation works well.

The triumph of first order approximation and the absence of poles around $\omega = \pm m\Omega$ inspire us to modify the definition of quasi-particle lifetime in the open Floquet quantum systems. To be more precise, starting from the expansion of the inverse of the Floquet Green’s function around $\omega_0$,

$$\text{det} \left[ \omega - H(\omega) \right] \approx f_0(\omega_0) + f_1(\omega_0)\delta\omega + o(\delta\omega^2),$$

the lifetime of the quasiparticle around $\omega = \omega_0$ is

$$\Gamma_{\omega_0} = -\frac{1}{\text{Im}[f_0/f_1]}.$$  

Due to the discrete time translational symmetry, we have

$$\text{det} \left[ \omega - H(\omega) \right] = \text{det} \left[ (\omega + m\Omega) - H(\omega + m\Omega) \right].$$

This means the expansion of $\text{det} [\omega - H(\omega)]$ around $\omega = m\Omega$ must share the same expression. Therefore, their lifetimes must be the same. This is consistent with the intuition, since the total Hamiltonian does not break the discrete time translational symmetry. In addition, we can show that the differences between the Floquet Majoranas in different Floquet BZs comes from the Floquet space localization natural of the wavefunction instead of their lifetime difference (refer to the last part of SI [79] for more details).

**Discussion and conclusion.**—Our work reveals the crucial role of linear and non-linear terms of the self-energy correction in the Floquet proximity SC. It is widely believed that the zeroth order correction of the self-energy correction can be described by an effective non-Hermitian Hamiltonian[84–89]. The corresponding real and imaginary parts of the eigenvalues can be regarded as the renormalized band structure and quasiparticle lifetime. However, the linear correction of the self-energy is important for the definition of lifetime in the Floquet proximity SC. As shown in the Floquet poisoning model, the first order correction in Eq. 8 is not a diagonal matrix. This means the poles can not be calculated from direct diagonalization. This suggests that the Floquet Majorana lifetime can be engineered by the external field. As shown in Fig. 3 (d), one can notice that with the increasing of driving amplitude, the lifetime decreases dramatically.
In summary, we have shown that in reality Floquet proximity SCs, neither the FMZMs nor FMPMs can be described by the Floquet band theory. The difficulty comes from the linear and non-linear effects of the self-energy correction, which can poison or kill the FMZMs and FMPMs respectively. However, the dissipation structures in different Floquet bands are identical; and therefore, the Floquet picture is still valid in the presence of dissipation and non-equilibrium environments.

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## I. DERIVATION OF THE FLOQUET RETARDED GREEN’S FUNCTION

In this section, we will briefly show how to derive $G_{nw}^R(k; \omega)$. The Green’s function in our driven-dissipative system can be defined as

$$G_{nw}^R(k; t, t') = -i\theta(t - t') \langle \{ \psi(k, t), \psi^\dagger(k, t') \} \rangle,$$

where $\psi(k, t) = (c_\uparrow(k, t), c_\downarrow(k, t), c^\dagger_\uparrow(-k, t), c^\dagger_\downarrow(-k, t))^t$. The Fourier transformation for $\psi(k, t)$ in a periodical driving system can be expressed as

$$\psi(k, t) = \sum_m \int_{\Omega/2}^{\Omega/2} d\omega \frac{e^{-i(\omega + m\Omega)t}}{2\pi^2} \psi_m(k, \omega).$$

Then the Fourier transformation for the Green’s function $G_{nw}^R(k; t, t')$ will be:[1]:

$$G_{nw}^R(k; t, t') = \sum_{mn} \int_{-\Omega/2}^{\Omega/2} d\omega \frac{e^{-i(\omega + m\Omega)t + i(\omega + n\Omega)t'}}{2\pi^2} \left[ G_{nw}^R(k, \omega) \right]_{mn},$$

and

$$\left[ G_{nw}^R(k, \omega) \right]_{mn} = \frac{1}{T} \int_0^T dt_{av} \int_{-\infty}^{\infty} dt_{rel} e^{i(\omega + m\Omega)t - i(\omega + n\Omega)t'} G_{nw}^R(k; t, t'),$$

where $t_{av} = \frac{1}{2}(t + t')$ and $t_{rel} = t - t'$. After integrating out the SC bath degrees of freedom, the influence provided by the SC bath can be expressed by the self-energy $\Sigma_{sc}(t)$. Since the SC bath doesn’t be driven, the Fourier transformation of the self-energy provided by it is still the original one

$$\Sigma_{sc}(\omega) = \int dt e^{-i\omega t} \Sigma_{sc}(t),$$

where $\Sigma_{sc}(\omega) = V^2 \sigma_z \tau_0 [\hat{q}^R_{bath}(\omega)] \sigma_z \tau_0$ and $\hat{q}^R_{bath}(\omega)$ is the retarded quasi-classical Green’s function[2, 3] of the SC bath. Here we briefly show the derivation of $\hat{q}^R_{bath}(\omega)$.
The Nambu-Gor’kov Green’s function in the Matsubara formalism can be expressed as

\[ G(k, i\omega_n) = \frac{1}{(i\omega_n)^2 - (\epsilon_k^2 + \Delta^2)} \begin{pmatrix} i\omega_n + \epsilon_k & \Delta \\ \Delta & i\omega_n - \epsilon_k \end{pmatrix}. \]  

(6)

Analytical continuation \(\omega_n \rightarrow -i(\omega + i\eta)\) leads to the retarded Green’s function in the real frequency space

\[ G^R(k, \omega) = \frac{1}{(\omega + i\eta)^2 - (\epsilon_k^2 + \Delta^2)} \begin{pmatrix} \omega + i\eta + \epsilon_k & \Delta \\ \Delta & \omega + i\eta - \epsilon_k \end{pmatrix}. \]  

(7)

As the quasi-classical Green’s function is defined as \(\hat{G}^R_{\text{bath}}(\omega) = \rho_F \int d\epsilon_k G^R_{\text{bath}}(k, \omega)\), where \(\rho_F\) is the DoS in the vicinity of the Fermi surface, then residual theorem leads to

\[ \hat{G}^R_{\text{bath}}(\omega) = -i\pi\rho_F \frac{1}{\sqrt{(\omega + i\eta)^2 - \Delta^2}} \begin{pmatrix} \omega + i\eta & \Delta \\ \Delta & \omega + i\eta \end{pmatrix}. \]  

(8)

Matrices in our consideration are in BdG space, so we have

\[ \hat{G}^R_{\text{bath}}(\omega) = \pi\rho_F \frac{1}{\sqrt{-(\omega + i\eta)^2 + \Delta^2}} \begin{pmatrix} -(\omega + i\eta) & 0 & 0 & -\Delta \\ 0 & -(\omega + i\eta) & \Delta & 0 \\ 0 & \Delta & -(\omega + i\eta) & 0 \\ -\Delta & 0 & 0 & -(\omega + i\eta) \end{pmatrix} \]  

(9)

Since the value of \(\rho_F\) doesn’t matter in our problem, we can simply let \(\pi\rho_F = 1\). Therefore,

\[ \Sigma_{sc} = V^2 \frac{1}{\sqrt{-(\omega + i\eta)^2 + \Delta^2}} \begin{pmatrix} -(\omega + i\eta) & \Delta \end{pmatrix}. \]  

(10)

Together with the non-dissipative Floquet Hamiltonian, one can define the Floquet Green’s function in the frequency space as

\[ G^R_{nw}(k, \omega) = \frac{1}{\omega - H_{nw}(k) - \Sigma_{sc}(\omega)}, \]  

(11)

where the non-dissipative Floquet Hamiltonian reads as

\[ H_{nw}(k) = \begin{pmatrix} \cdots & \mathcal{H}_{nw}(k) - \Omega & A\sigma_0\tau_z & 0 \\ A\sigma_0\tau_z & \mathcal{H}_{nw}(k) & A\sigma_0\tau_z & 0 \\ 0 & A\sigma_0\tau_z & \mathcal{H}_{nw}(k) + \Omega & \cdots \end{pmatrix}, \]  

(12)

where \(\mathcal{H}_{nw}(k)\) reads

\[ \mathcal{H}_{nw}(k) = H_{nw}(k) + \Sigma_{sc}(\omega), \]  

\[ \left[H_{nw}(k)\right]_{mm'} = \frac{1}{2} \int_0^T dt e^{i(m-m')\Omega t} H_{nw}(k, t) \]  

and \(H_{nw}(k, t) = (-2t \cos k - \mu_0 + 2A \cos \Omega t)\sigma_0 + V_z\sigma_z + \lambda \sin k\sigma_y\tau_z\). Let \(\mathcal{H}_{eff}(k, \omega) = H_{nw}(k) + \Sigma_{sc}(\omega)\), we then get the expression of \(G^R_{nw}(k, \omega)\):

\[ G^R_{nw}(k, \omega) = \left[\omega - \mathcal{H}_{eff}(k, \omega)\right]^{-1}. \]  

(13)

Our numerical calculation is based on Eq.(13). We truncate \(\mathcal{H}_{eff}(k, \omega)\) and just leave a \((21 \times 4) \times (21 \times 4)\) matrix \((N_F = 21)\), then numerically calculate \(G^R_{nw}(k, \omega)\). Finally, calculating \(\nu_k(\omega)\) through Eq. (6) in the main body of this letter leads us to Fig. 2(a) in the main body.

II. RECURSIVE GREEN’S FUNCTION METHOD

This section will show how to get Fig. 2(b) in the main body through the recursive Green’s function method. For details of the recursive Green’s function method, we refer to Ref. [4–6]. In our problem, Green’s functions are more structured due to the introducing of Floquet and Keldysh space.
The key to using the recursive Green's function method is to regard the nanowire as a lattice model (see Fig. 1). We suppose that this nanowire has 200 sites. Since we just care about the energy spectrum and use the SC bath to produce an induced gap for the nanowire, the correlation in the bath can be ignored under Markovian approximation. In this case, and the influence on each site from the SC bath are the same, which equals to that we couple an identical SC bath to each site.

In order to detect MZMs localized in two edges of this wire, we use the following recursive equation:

$$ G_{n+1,n+1}^d(\omega) = [G_{n+1,n+1}^0(\omega) - V_{n+1,n} \cdot G_{n,n}^d(\omega) \cdot V_{n,n+1}]^{-1} \quad (14) $$

where $V_{n,n+1}$ denotes the hopping term between $n$th and $(n + 1)$th site in the Floquet $\otimes$ Keldysh $\otimes$ BdG space, the subscript $n$ denotes the $n$th site, and the superscript $d$ and 0 denote the dressed Green’s function of one site after considering the hopping term $V_{n,n+1}$ and the bare Green’s function of one site with only the onsite energy, respectively. Note that $G_{1,1}^d(\omega) = G_{1,1}^0(\omega)$. After 199 recursions, one will get $G_{200,200}^d(\omega)$. Then using Eq. (9) in the main body of this letter will lead us to Fig. 2(b) in the main body of this letter.

III. SOME ADDITIONAL NOTES OF THE MAJORANA POISONING MODEL

In this section, we provide some additional calculations and notes of the Majorana poisoning model. Starting from the effective Hamiltonian,

$$ H(\omega) = \begin{pmatrix} \cdots & \Sigma(\omega + \Omega) - \Omega & A & 0 \\ A & \Sigma(\omega) & A & \cdots \\ 0 & A & \Sigma(\omega - \Omega) + \Omega & \cdots \end{pmatrix}, \quad (15) $$

where

$$ \Sigma(\omega) = -V^2(\omega + i\eta)/\sqrt{-(\omega + i\eta)^2 + \Delta^2}. \quad (16) $$

Here $A$, $V$, $\Delta$ represent the driving amplitude, Majorana-bath coupling strength, and driving frequency, respectively. The major feature of this effective model is that the Hamiltonian is energy dependent, due to the existence of self-energy correction, $\Sigma(\omega)$. Since for a given $\omega$, $H(\omega)$ is non-Hermitian, it is convenient to introduce the biorthogonal basis,

$$ H(\omega)|u_m^R(\omega)\rangle = E_m(\omega)|u_m^R(\omega)\rangle, \quad \langle u_n^L(\omega)|H^\dagger(\omega) = E_n(\omega)\langle u_n^L(\omega)| $$

where $\langle u_n^L(\omega)|u_m^R(\omega)\rangle = \delta_{nm}$. Based on this basis, one can express the matrix element of the retarded Green’s function in the Floquet space,

$$ \left[G^R(\omega)\right]_{mn} = \langle m| \frac{1}{\omega - H(\omega)}|n\rangle = \sum_s \langle m|u_s^R(\omega)\rangle \langle u_s^L(\omega)|n\rangle \frac{1}{\omega - E_s(\omega)}, \quad (17) $$
where $m, n, s$ represent the Floquet sites. The time-averaged DoS is defined as the 00-entry of the retarded Green’s function defined in the Floquet space, namely,

$$
\nu(\omega) = -\frac{1}{\pi} \text{Im} \left[ G^R(\omega) \right]_{00},
$$

(18)

The corresponding propagator is defined as

$$
G^R(t, t') = \sum_{mn} \int_{-\Omega/2}^{\Omega/2} \frac{d\omega}{2\pi} e^{-i(\omega + m\Omega)t + i(\omega + n\Omega)t'} \left[ G^R(\omega) \right]_{mn}.
$$

(19)

Using $\left[ G^R(\omega) \right]_{(m+s)(n+s)} = \left[ G^R(\omega + s\Omega) \right]_{mn}$, one can obtain the propagator starting from $t_0 = 0$

$$
G^R(t, 0) = \sum_{n = -\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[ G^R(\omega) \right]_{n0}.
$$

(20)

Eq. 17, 18 and 20 are the main results of this section. In the following contents, we will analysis their behaviors.

A. The definition of lifetime

1. Review the definition of lifetime based on the Green’s function

We first review the definition of lifetime based on the Green’s function method. Consider a free electron with energy dispersion $E(k)$, the corresponding retarded Green’s function can be defined as

$$
G^R_0(k, \omega) = \frac{1}{\omega - E(k) + i\eta^+},
$$

(21)

where $\eta^+ = 0^+$. Therefore, the corresponding propagator becomes

$$
G_0(t) = \sum_k \int_{-\infty}^{\infty} \frac{\omega}{2\pi} e^{-i\omega t} G^R_0(k, \omega) \propto \sum_k e^{iE(k)t - \eta^+ t}.
$$

(22)

Here each $k$ corresponds to a quasiparticle mode. When $\eta^+ > 0$, the propagator will decay with time, and defines a corresponding lifetime. We note that in the non-interacting isolated systems, $\eta = 0^+$, which means the quasiparticle lifetime is infinity.

Generalizing the above argument to the interacting or open quantum systems, the retarded Green’s becomes

$$
G^R(k, \omega) = \frac{1}{\omega - E(k) - \Sigma(\omega)}.
$$

(23)

A standard statement of the lifetime is that the imaginary part of the pole corresponds to the lifetime. Namely, consider

$$
\omega - E(k) - \Sigma(\omega) = \prod_{n=1}^{m} \frac{[\omega - \varepsilon_n(k) + i\Gamma_n(k)]}{D(\omega)},
$$

(24)

the propagator becomes

$$
G(t) = \sum_k \int_{-\infty}^{\infty} \frac{\omega}{2\pi} e^{-i\omega t} G^R(k, \omega) \propto \sum_n \sum_k D[\varepsilon_n(k) - i\Gamma_n(k)] e^{i\varepsilon_n(k)t - \Gamma_n(k)t}.
$$

(25)

Here $D[\varepsilon_n(k) - i\Gamma_n(k)]$ describes the quasiparticle weight, and the quasiparticle lifetime is modified by the self-energy induced by interacting or external environment, and the additional degrees $n$ correspond to the additional quasiparticles provided by the external environment.

We note that the above argument implies $\omega - E(k) - \Sigma(\omega)$ is proportional to an algebraic equation. However, in general, the external environment degrees are infinity, which renders that the self-energy is not an algebraic polynomial (just like our Majorana poisoning model). In this case, can we generalize the above method directly?
Hamiltonian to express the Green’s function, for example,
\[ G(\omega) = \frac{1}{\omega - H(\omega)} \approx \frac{1}{-H(\omega) + \delta \omega - H^{(1)}(0, \delta \omega)} \]  

2. Some insights of the Majorana poisoning model

We first apply the above method to the Majorana poisoning model directly, and see what happens. In order to simplify the discussion, we assume \( N_F = 1 \). In this case,
\[ \omega - H(\omega) = \begin{pmatrix} \omega + \Omega - \Sigma(\omega + \Omega) & -A & 0 \\ -A & \omega - \Sigma(\omega) & -A \\ 0 & -A & (\omega - \Omega) - \Sigma(\omega - \Omega) \end{pmatrix} \]  

\((26)\)

\(\nu(\omega)\) is plotted in Fig. 2 (a) with the following parameters \( \Omega = 2, \Delta = 3, A = 1/2, \eta = 1/1000, V = 1 \). (b) and (c) shows the function \( \text{Abs}[\text{det}[\omega - H(\omega)]] \) around the peaks at \( \omega = 0 \) and \( \omega \approx 2.064 \).

Importantly, we numerically solve the following equation
\[ \text{det} \left[ \omega - H(\omega) \right] = 0 \]  

\((27)\)

and find that there only exist one solution, \( \omega = 0 + i \eta \). Another way to verify this is to plot the function \( \text{Abs}[\text{det}[\omega - H(\omega)]] \) around \( \omega = 0 \). We plot the function \( \text{Abs}[\text{det}[\omega - H(\omega)]] \) around them. As shown in Fig. 2 (b), we plot \( \text{Abs}[\text{det}[\omega - H(\omega)]] \) with \( \omega = 0 + i \gamma \). One can find it becomes zero at \( \gamma = -\eta \). However, when we plot the function \( \text{Abs}[\text{det}[\omega - H(\omega)]] \) around \( \omega = 2 \) in the complex \( \omega \) plane, namely, \( \Re[\omega] \in [2.06, 2.07] \) and \( \Im[\omega] \in [-0.0012, -0.008] \), the function \( \text{Abs}[\text{det}[\omega - H(\omega)]] \) does not become zero. We note that due to finite \( N_F \) namely, \( N_F = 1 \), the peaks in the second Floquet BZ does not at \( \omega = \pm 2 \) exactly, but around \( \omega = 2.064 \). That’s the reason why we chose the plot region in (c).

Although \( \text{det} \left[ \omega - H(\omega) \right] = 0 \) does not have poles around \( \omega = \pm 2 \) in the complex \( \omega \) plane, these peaks can be well fitted by the following method.

1. Expand the Hamiltonian \((26)\) around \( \omega = 0, \pm \Omega \) up to linear order. For example,
\[ \omega - H(\omega) \approx -H(0) + \frac{\delta \omega - H^{(1)}(0, \delta \omega)}{d \omega} \]
\[ = -H(0) + \begin{pmatrix} \delta \omega - \Sigma^{(1)}(\Omega) \delta \omega & 0 & 0 \\ 0 & -\Sigma^{(1)}(0) \delta \omega & 0 \\ 0 & 0 & -\Sigma^{(1)}(-\Omega) \delta \omega \end{pmatrix} \]

\((28)\)

where
\[ \Sigma^{(1)}(m\Omega) = \frac{d}{d \omega} \Sigma^{(1)}(\omega)|_{\omega = m\Omega} \]  

\((29)\)

Notice that the linear term of \( \delta \omega \) in the above matrix are not proportional to the identity matrix due to the fact that \( \Sigma^{(1)}(\pm \Omega) \neq \Sigma^{(1)}(0) \).

2. Using the expansion Hamiltonian to express the Green’s function, for example,
\[ G^R(\omega) = \frac{1}{\omega - H(\omega)} \approx \frac{1}{-H(0) + \delta \omega - H^{(1)}(0, \delta \omega)} \]

\((30)\)
FIG. 3. Floquet Majorana poisoning model. (a)-(c) show the exact, zeroth and first order numerical calculation of $\nu(\omega)$ in Eq. 18 with $\Omega = 2$, $\Delta = 3$, $A = 1/2$, $\eta = 1/1000$, $N_F = 10$, and different values of $V$.

3. Using the approximated Green’s function to calculate the DoS. For example,

$$\nu(\omega) = -\frac{1}{\pi} \text{Im} \left[ G_R(\omega) \right]_{00} \simeq -\frac{1}{\pi} \text{Im} \left[ \frac{1}{-H(0) + \left( \delta \omega - H^{(1)}(0, \delta \omega) \right)} \right]_{00}. \quad (31)$$

As shown in Fig. 2, the first order approximation are plotted with black lines. One can finds that they match the exact solutions very well.

We also note that in some discussion, the expansion around $\omega = 0$ is applied up to the zeroth order, and obtain an effective Hermitian or non-Hermitian Hamiltonian, namely,

$$H_{\text{eff}}(k) \simeq H_0(k) + \Sigma^{(0)}(k, \omega = 0). \quad (32)$$

The corresponding real and imaginary parts of the complex eigenvalues of $H_{\text{eff}}$ represent the renormalized band dispersion and quasiparticle lifetime. However, this approach also fails in our model. As shown in Fig. 3 (a)-(c), the exact/first order/second order results of the time-averaged DoS are plotted around $\omega = 0$ with the red/dashed black/sold black lines, respectively. One can noticed form Fig. 3 (a)-(c) that with the increasing of $V$, the zeroth order breaks down, while the first order approximation works well. When the expansion is applied around $\omega_0 = m \Omega$ up to linear order, one can use the same method to fix the order peaks as shown in Fig. 3 (c) the main text.

Based on the above analysis, although we find that the traditional method does not work here due to the absence of poles around $\omega = \pm \Omega$, the peaks around them can be approximated by the first order expansion.

3. Definition of lifetime

The triumph of the first order approximation is related to the definition of lifetime. As mentioned above, $\det[\omega - H(\omega)]$ cannot be expressed as a finite set of product of $\omega - E_n$ without any approximation. However, when we expand the Hamiltonian around $\omega_0$, the determinant of inverse of the Green’s function becomes an algebraic polynomial. As a result, one can find the corresponding poles around $\omega_0$. Based on this spirit, the corresponding lifetime is defined as the imaginary part of the poles around $\omega_0$. Now we summarize the procedure to calculate the lifetime.

1. Expand $\omega - H(\omega)$ around $\omega = m \Omega$,

$$\omega - H(\omega) \simeq \omega_0 - H^{(0)}(\omega_0) + \delta \omega - H^{(1)}(\omega_0, \delta \omega) - o(\delta \omega^2), \quad (33)$$

where $\delta \omega = \omega - \omega_0$, and $H^{(i)}(\omega_0, \delta \omega)$ is a matrix only containing $\delta \omega^i$ term.
2. Calculate the following equation
\[
\det \left[ \omega_0 - H^{(0)}(\omega_0) + \delta\omega - H^{(1)}(\omega_0, \delta\omega) \right] \simeq f_0 + f_1 \delta\omega + o(\delta\omega^2). \tag{34}
\]
3. Then the lifetime of the quasiparticle around \( \omega = \omega_0 \) is
\[
\Gamma_{\omega_0} = -\frac{1}{3rf_0/f_1}. \tag{35}
\]
Notice that the above procedure is equivalent to
\[
\det \left[ \omega - H(\omega) \right] \simeq f_0(\omega_0) + f_1(\omega_0)\delta\omega + o(\delta\omega^2). \tag{36}
\]
However, the direct calculation of \( \det[\omega - H(\omega)] \) is not easy in numerics.

B. The lifetime of the Floquet Majoranas in different Floquet BZs

Now we show that all the quasiparticles in different Floquet BZs have the same lifetime. We notice that due to the discrete time translational symmetry, we have
\[
\det \left[ \omega - H(\omega) \right] = \det \left[ (\omega + m\Omega) - H(\omega + m\Omega) \right]. \tag{37}
\]
Therefore, the expansion of \( \det[\omega - H(\omega)] \) around \( \omega = m\Omega \) must share the same expression. Therefore, their lifetimes are the same. This is consistent with the intuition, since the total Hamiltonian does not break the discrete time translational symmetry.

C. The differences between the peaks in different Floquet BZs

This section explains the differences between the FMZMs in different Floquet bands. We start from the \( mn \)-entry of the retarded Green’s function
\[
\left[ G^R(\omega) \right]_{mn} = \langle m | \frac{1}{\omega - H(\omega)} | n \rangle = \sum_s \langle m | u^R_s(\omega) \rangle \langle u^L_s(\omega) | n \rangle \frac{1}{\omega - E_s(\omega)}. \tag{38}
\]
Here \( H(\omega)|u^R_s(\omega)\rangle = E_s(\omega)|u^R_s(\omega)\rangle \), \( \langle u^L_s(\omega) | H^\dagger(\omega) = E_s(\omega)\langle u^L_s(\omega) | \), and \( \langle u^L_n(\omega) | u^R_m(\omega) \rangle = \delta_{nm} \) is the biorthogonal basis. The time-averaged DoS is defined by
\[
\nu(\omega) = -\frac{1}{\pi} \text{Im} \left[ G^R(\omega) \right]_{00} = -\frac{1}{\pi} \text{Im} \left[ \sum_s \frac{\langle 0 | u^R_s(\omega) \rangle \langle u^L_s(\omega) | 0 \rangle}{\omega - E_s(\omega)} \right]. \tag{39}
\]
As discussed in the main text, the wavefunctions \( \langle 0 | u^R_s(\omega) \rangle \) are localized in the central part of the Floquet base. Therefore, the DoS will decay in the higher Floquet BZs.

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