Figures of equilibrium of an inhomogeneous self-gravitating fluid

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Abstract. This paper is concerned with the figures of equilibrium of a self-gravitating ideal fluid with density stratification and a steady-state velocity field. As in the classical setting, it is assumed that the figures or their layers uniformly rotate about an axis fixed in space.

It is shown that the ellipsoid of revolution (spheroid) with confocal stratification, in which each layer rotates with inherent constant angular velocity, is at equilibrium. Expressions are obtained for the gravitational potential, change in the angular velocity and pressure, and the conclusion is drawn that the angular velocity on the outer surface is the same as that of the Maclaurin spheroid. We note that the solution found generalizes a previously known solution for piecewise constant density distribution. For comparison, we also present a solution, due to Chaplygin, for a homothetic density stratification.

We conclude by considering a homogeneous spheroid in the space of constant positive curvature. We show that in this case the spheroid cannot rotate as a rigid body, since the angular velocity distribution of fluid particles depends on the distance to the symmetry axis.

Keywords Self-gravitating fluid, confocal stratification, homothetic stratification, space of constant curvature

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**Introduction**

This paper is concerned with exact solutions to the problem of (axisymmetric) figures of equilibrium of a self-gravitating ideal fluid with density *stratification*. First of all, we briefly recall the well-known results along these lines.

For a *homogeneous* fluid, the following ellipsoidal equilibrium figures are well known for which the entire mass *uniformly rotates as a rigid body* about a fixed axis:

- the Maclaurin spheroid (1742),
- the Jacobi ellipsoid (1834),

In addition, in the case of a homogeneous fluid there also exist *figures of equilibrium with internal flows*:

- the Dedekind ellipsoid (1861),
- the Riemann ellipsoids (1861).

**Remark.** The discovery of the Dedekind and Riemann ellipsoids was inspired by the work of Dirichlet [13], where the dynamical equations for a liquid homogeneous self-gravitating ellipsoid were obtained (for this system all the above-mentioned figures of equilibrium are fixed points). For a recent review of dynamical aspects concerning liquid and gaseous self-gravitating ellipsoids and a detailed list of references, see [8]. We also note the integrability cases found in a related problem of gaseous ellipsoids [18].

While an enormous amount of research was devoted in the 19th and 20th centuries to asymmetric figures of equilibrium (see, e.g., references in [8, 9]), the Maclaurin spheroid remains the most important for applications to the theory of the figures of planets. However, it is well known that for all planets of the Solar System a real compression is different from the compression of the corresponding Maclaurin spheroid obtained from the characteristics of the planet. Usually this difference is attributed to the density stratification of the planet, which leads to the necessity of investigating inhomogeneous figures of equilibrium.

For a stratified fluid mass rotating as a rigid body with small angular velocity $\omega$, Clairaut [11] obtained the equation of a spheroid which is an equilibrium figure in the first order in $\omega^2$. Subsequently investigations of such figures were continued in the work of Laplace, Legendre and Lyapunov. Lyapunov obtained a final solution to this problem in the form of a power series in the small parameter $\omega^2$ and showed their convergence.

On the other hand, in [19, 39] and [33, Chapter 12] it was shown that for a stratified fluid mass rotating as a rigid body there exist no figures of equilibrium in the class of ellipsoids. We present here in modern formulation a theorem which was proved in these works.

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1. Relevant calculations can be easily performed using the formulae of Section 2.3 and astronomic data available from the Internet.
2. A. Clairaut took part in the first expeditions, which confirmed I. Newton’s viewpoint that the Earth is compressed from the poles.
Suppose the body consists of a self-gravitating, ideal, stratified fluid. Assume that

- the free surface of the fluid is an ellipsoid (it can be both three-axial and a spheroid),
- the density distribution \( \rho(r) \) is such that the level surfaces \( \rho(r) = \text{const} \) are ellipsoids coaxial with the outer surface.

Then such a fluid mass configuration cannot be the figure of equilibrium rotating as a rigid body about one of the principal axes.

Hamy proved this theorem for the case of a finite number of ellipsoidal layers with constant density, Volterra generalized this result to the case of continuous density distribution for a homothetic stratification of ellipsoids, and Pizzetti gave the simplest and most rigorous proof in the general case for both continuous and piecewise constant density distribution. Interestingly, there still appear papers (see, e.g., [23]) whose authors “discover” new “solutions” contradicting this theorem. Such publications show that there is still no complete understanding regarding the equilibrium figures of celestial bodies with stratified density. We also note that A. Veronnet [38] also tried to prove this theorem for the case of continuous density distribution but made some errors.

If one admits the possibility that the angular velocity of fluid particles is not constant for the entire fluid mass, then equilibrium figures for an arbitrary axisymmetric form of the surface and density stratification [33, Chapter 9] are possible. For example, in [10] S. A. Chaplygin explicitly showed a spheroidal equilibrium figure with a nonuniform distribution of angular velocities for the case of homothetic density stratification. It turns out that the surfaces with equal density \( \rho(r) = \text{const} \) do not coincide with the surfaces of equal angular velocity \( \omega(r) = \text{const} \). S. A. Chaplygin tried to use the resulting solution to explain the dependence of the angular velocity of rotation of the outer layers of the Sun on the latitude.

In [32] an explicit solution of another kind was found for which the equilibrium figure is a spheroid consisting of two fluid masses of different density \( \rho_1 \neq \rho_2 \) separated by the spheroidal boundary confocal to the outer surface, with each layer rotating at constant angular velocity such that \( \omega_1 \neq \omega_2 \). A generalization of this solution to the case of an arbitrary finite number of “confocal layers” was obtained in [15].

In this paper we obtain a generalization of this solution to the case of an arbitrary confocal (both continuous and piecewise constant) density stratification. For comparison, we also present Chaplygin’s solution for the homothetic stratification. In addition, we show that in the case of a space with constant curvature the homogeneous (curvilinear) spheroid is a figure of equilibrium only under the condition of a nonuniform distribution of the angular velocities of fluid particles \( \omega(r) \neq \text{const} \). In this case the solution can be represented as a power series in the space curvature.

\(^3\)This work was not published during the life-time of S. A. Chaplygin and appeared for the first time in his posthumous collected works prepared by L. N. Sretenskii.
1 Equations of motion and axisymmetric equilibrium figures

1.1 Equations of motion in curvilinear coordinates

In this case, to solve specific problems, it is convenient to use special curvilinear (nonorthogonal) coordinates, which we denote by $q_i = (q_1, q_2, q_3)$. Therefore, we first represent the equations describing this system in an appropriate form.

Suppose that an element of the fluid has coordinates $q_i$ at a given time $t$. Let $\dot{q}_i = (\dot{q}_1, \dot{q}_2, \dot{q}_3)$ denote the rates of change of its coordinates during the motion. They depend on both the coordinates $q_i$ of the chosen element and time $t$: $\dot{q}_i = \dot{q}_i(q, t)$ and the total derivative of any function $f$ of $q_i$ and $t$ is calculated from the formula

$$\frac{df}{dt} = \frac{\partial f}{\partial t} \sum_i \frac{\partial f}{\partial q_i} \dot{q}_i.$$  \hspace{1cm} (1)

Let $G = \|g_{ij}\|$ denote the metric tensor corresponding to these coordinates. In the case of orthogonal coordinates $G = \text{diag}(h_1^2, h_2^2, h_3^2)$, where $h_i$ are the Lamé coordinates.

As is well known [22], the equations of motion for a fluid in a potential field can be represented as

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} - \frac{1}{\rho} \frac{\partial p}{\partial q_i},$$  \hspace{1cm} (2)

where $\rho$ is the density, $p$ is the pressure, $U$ is the specific potential of external forces, and $T$ is the specific kinetic energy of the fluid calculated from the formula

$$T = \frac{1}{2} \sum_{i,j} g_{ij} \dot{q}_i \dot{q}_j.$$  

The continuity equations written in this notation become

$$\frac{\partial \rho}{\partial t} + \frac{1}{g} \sum_i \frac{\partial}{\partial q_i} (\rho g \dot{q}_i) = 0, \quad g = \sqrt{\det G}.$$  \hspace{1cm} (3)

In the case of a self-gravitating fluid the gravitational potential $U(q, t)$ can be calculated from the equation

$$\Delta U = 4\pi G \rho(q, t),$$  \hspace{1cm} (4)

where $G$ is the constant of gravitation and the Laplacian is given by the well-known relation

$$\Delta = \frac{1}{g} \sum_i \frac{\partial}{\partial q_i} \left( g \frac{\partial}{\partial q_i} \right), \quad \|g^{ij}\| = G^{-1},$$

assuming that outside the liquid body the density vanishes: $\rho = 0$. 

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In the absence of external influences at the free boundary $\partial B$ of the fluid mass the pressure vanishes:

$$p|_{\partial B} = 0,$$

and the gravitational potential and its normal derivative are continuous:

$$U_{\text{in}}|_{\partial B} = U_{\text{out}}|_{\partial B}, \quad \frac{\partial U_{\text{in}}}{\partial n}|_{\partial B} = \frac{\partial U_{\text{out}}}{\partial n}|_{\partial B}, \quad (5)$$

where the indices in and out denote the quantities inside and outside the body, respectively.

### 1.2 Steady-state axisymmetric flows

To explore possible figures of equilibrium, we choose curvilinear coordinates $\mathbf{q} = (r, \mu, \phi)$, which are related to the Cartesian coordinates as follows

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = Z(r, \mu).$$

Here the function $Z(r, \mu)$ is chosen so as to obtain a free surface of the fluid mass for one of the values $\mu = \mu_0$. Its specific form will be defined by an appropriate problem statement. The metric tensor is given by

$$G = \begin{pmatrix}
1 + Z_r^2 & Z_rZ_\mu & 0 \\
Z_rZ_\mu & Z_\mu^2 & 0 \\
0 & 0 & r^2
\end{pmatrix}, \quad g = \sqrt{\det G} = r Z_\mu,$$

where $Z_r = \frac{\partial Z}{\partial r}$, $Z_\mu = \frac{\partial Z}{\partial \mu}$.

**Remark.** P. Pizzetti [33] used usual cylindrical coordinates (i.e., he set $\mu = z$), with the equation for the free surface being $F(x, z) = 0$. From a practical point of view, this approach is inconvenient in searching for specific equilibrium figures of a stratified fluid.

We shall seek a steady-state solution of (2), for which the velocity distribution has the form

$$\dot{r} = 0, \quad \dot{\mu} = 0, \quad \dot{\phi} = \omega(r, \mu), \quad (6)$$

and the functions $U$, $p$, and $\rho$ do not depend on $\phi$. Then, substituting (6) into (2) and (3), we obtain the system of equations

$$\frac{\partial U}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = r \omega^2, \quad \frac{\partial U}{\partial \mu} + \frac{1}{\rho} \frac{\partial p}{\partial \mu} = 0,$$

$$\Delta r_\mu U = 4\pi G \rho(r, \mu),$$

$$\Delta r_\mu = \frac{1}{r Z_\mu} \frac{\partial}{\partial r} \left( rZ_\mu \frac{\partial}{\partial r} \right) + \frac{1}{r Z_\mu} \frac{\partial}{\partial \mu} \left( \frac{1+Z_r^2}{Z_\mu} \frac{\partial}{\partial \mu} \right) - \frac{1}{r Z_\mu} \left( \frac{\partial}{\partial r} \left( rZ_\mu \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \mu} \left( rZ_\mu \frac{\partial}{\partial \mu} \right) \right),$$

$$p(r, \mu)|_{\mu=\mu_0} = 0. \quad (7)$$

Note that the continuity equation (3) holds identically.

We choose the function $Z(r, \mu)$ defining the curvilinear coordinates in such a way that all coordinate surfaces $\mu = \text{const}$ are compact, and choose a value
of $\mu = \mu_0$ which corresponds to the boundary of the fluid and defines the distribution of density $\rho(r, \mu)$. Then, according to (7), after solving the equation for the potential one can always choose a distribution of pressure and of the squared angular velocity, which satisfy the first pair of equations:

$$p(r, \mu) = \int_{\mu_0}^{\mu} \rho \frac{\partial U}{\partial \mu} \, d\mu,$$

$$\omega^2(r, \mu) = \frac{1}{r \rho} \left( \rho_0 \frac{\partial U}{\partial r}(r, \mu_0) + \int_{\mu_0}^{\mu} \left( \frac{\partial U}{\partial \mu} \frac{\partial \omega}{\partial \mu} - \frac{\partial U}{\partial \mu} \frac{\partial \omega}{\partial r} \right) \, d\mu \right), \quad \rho_0 = \rho(r, \mu_0).$$

A possible obstruction to the existence of such equilibrium figures is that $\omega^2(r, \mu)$, defined from these equations, may turn out to be negative. The problem of equilibrium figures becomes more nontrivial when we impose some restrictions on the distribution of angular velocity.

For example, L. Lichtenstein and R. Wavre (see [26]) found sufficient conditions under which a body obviously possesses a plane of symmetry.

**Theorem.** Assume that for an inhomogeneous self-gravitating liquid body the following is satisfied:

1. the fluid is at relative equilibrium where all particles rotate about the fixed axis $Oz$, and their angular velocity depends only on the distance to the axis of rotation: $\omega = \omega(r^2)$,

2. the density is a piecewise continuous function,

3. the body consists of a finite number of bounded regions whose boundaries have the topological type of a sphere or a torus.

Then the body possesses a plane of symmetry perpendicular to the axis $Oz$.

It is also obvious that the center of mass lies on the intersection of the symmetry plane with the axis of rotation $Oz$.

## 2 Inhomogeneous figures with isodensity distribution of the angular velocity of layers

### 2.1 General equations for monotonic and piecewise constant density distribution

We now consider the case where the level surfaces of stratification of density $\rho$ coincide with the level surfaces of angular velocity $\omega$ (i.e., the fluids of equal density move with equal angular velocity); choosing them as coordinate lines $\mu = \text{const}$, we represent this condition as

$$\rho = \rho(\mu), \quad \omega = \omega(\mu). \quad (8)$$
Eliminating the pressure from the first pair of equations of the system (7) (multiplying them by \( \rho \) and differentiating the first one with respect to \( \mu \) and the second one with respect to \( r \) and subtracting one from the other), we obtain

\[
\rho'(\mu) \frac{\partial U(r, \mu)}{\partial r} = r \left( \rho(\mu) \omega^2(\mu) \right)',
\]

where the prime denotes the derivative with respect to \( \mu \).

1. We first consider the case where the density is nonconstant everywhere inside the body:

\[\rho'(\mu) \neq 0, \quad \text{inside.}\]

Then, according to (9), the potential \( U \) inside the body can be represented as

\[U(r, \mu) = \frac{1}{2} u(\mu) r^2 + v(\mu),\]

and from the first pair of equations (7) we obtain the unknowns \( p(r, \mu) \) and \( \omega(\mu) \) in the form

\[
p = -\frac{1}{2} P(\mu) r^2 - Q(\mu), \quad \omega^2(\mu) = u(\mu) - \frac{P(\mu)}{\rho(\mu)},
p = \int_{\mu_0}^{\mu} u'(\xi) \rho(\xi) \, d\xi, \quad Q(\mu) = \int_{\mu_0}^{\mu} v'(\xi) \rho(\xi) \, d\xi.
\]

Obviously,

\[
p(r, \mu) \bigg|_{\mu = \mu_0} = 0, \quad \frac{d\omega^2}{d\mu} \bigg|_{\mu = \mu_0} = 0.
\]

Hence, it follows that the figure of equilibrium of a fluid with stratification of density and angular velocity of the form (8) exists if and only if there exist functions \( Z(r, \mu) \) and \( u(\mu), v(\mu) \) satisfying the equation

\[
\Delta_{r, \mu} \left( \frac{1}{2} u(\mu) r^2 + v(\mu) \right) = 4\pi G \rho(\mu),\]

and the potential inside the fluid mass has the form (10).

2. We now consider a situation where in some layer the density takes a constant value:

\[\rho(\mu) = \rho_0 = \text{const}, \quad \mu \in (\mu_1, \mu_2),\]

then, according to (9), we conclude that the angular velocity of the entire layer is also constant:

\[\omega(\mu) = \omega_0 = \text{const}, \quad \mu \in (\mu_1, \mu_2).\]

Taking this into account, we integrate the first pair of equations (7) and obtain the following relation for the function \( U + \frac{p}{\rho_0} \) in the layer:

\[U + \frac{p}{\rho_0} = \frac{1}{2} \omega_0^2 r^2 + \Phi_0, \quad \Phi_0 = \text{const}.\]
Furthermore, at all points at the boundaries of the layer \( \mu = \mu_i, \ i = 1, 2 \) (see Fig. 1) the pressure inside and outside must be the same:

\[
p_{\text{in}}(r, \mu) \big|_{\mu=\mu_i} = p_{\text{out}}(r, \mu) \big|_{\mu=\mu_i}.
\] (14)

The potential in the layer also satisfies the Laplace equation

\[
\Delta r\mu U_{\text{in}}(r, \mu) = 4\pi G\rho_0,
\]
and at the boundaries the conditions (10) hold.

### 2.2 The family of confocal spheroids

Consider a particular case in which the sought-for solution exists. We shall show that in the case of confocal stratification of the density of a spheroid the gravitational potential is written as (10).

Choose the parameterization of confocal stratification in \( \mathbb{R}^3 \) as follows

\[
\frac{x^2 + y^2}{d^2(1 + \mu^2)} + \frac{z^2}{d^2\mu^2} = 1, \quad \mu \in [0, +\infty),
\]

where \( d \) is the focal distance of the meridional section (see Fig. 2). Thus, the parameter \( \mu \) defines the ratio between the small semi-axis of the spheroid and the focal distance, and the eccentricity \( e \) is expressed by the formula

\[
e = \frac{1}{\sqrt{1 + \mu^2}}.
\] (15)

Expressing \( z \), we find

\[
Z(r, \mu) = \pm \sqrt{d^2\mu^2 - r^2\frac{\mu^2}{\mu^2 + 1}}.
\] (16)

If the boundary of the spheroid filled with a fluid has semi-axes \( a \) and \( b \) (see Fig. 2), then the focal distance \( d \) and the coordinate of the boundary \( \mu_0 \) are defined by

\[
d = \sqrt{a^2 - b^2}, \quad \mu_0 = \frac{b}{\sqrt{a^2 - b^2}}.
\] (17)

**Remark.** It can be shown that for a prolate spheroidal stratification (i.e., for \( \frac{r^2}{a^2\mu^2} + \frac{z^2}{a^2(\mu^2 + 1)} = 1 \)) this solution leads to a negative square of the angular velocity of rotation of the layers \( \omega^2(\mu) < 0 \), therefore, we will not consider it.
Proposition 1. The gravitational potential for a spheroid with confocal stratification has the form

\[ U = \frac{k}{2} \left( \frac{r^2 \bar{u}(\mu)}{1 + \mu^2} + d^2 \bar{v}(\mu) \right), \quad k = 4\pi G. \] (18)

For the internal points

\[ \bar{u}^{\text{in}} = I_0(\mu)((1 + 3\mu^2) \arctg(\mu) - 3\mu) - I_1(\mu)(1 + 3\mu^2) \]
\[ \bar{v}^{\text{in}} = -I_0(\mu)((1 + \mu^2) \arctg(\mu) - \mu) + I_1(\mu)(1 + \mu^2) + 2I_2(\mu) \] (19)

\[ I_0(\mu) = \int_0^\mu \rho(\xi)(1 + 3\xi^2) \, d\xi, \quad I_1(\mu) = \int_0^\mu \rho(\xi)((1 + 3\xi^2) \arctg(\xi) - 3\xi) \, d\xi, \]
\[ I_2(\mu) = \int_0^\mu \xi \rho(\xi) \, d\xi. \]

For the external points

\[ \bar{u}^{\text{out}} = I_0(\mu_0)((1 + 3\mu^2) \arctg(\mu) - 3\mu), \quad \bar{v}^{\text{out}} = I_0(\mu_0)(\mu - (1 + \mu^2) \arctg(\mu)). \] (20)

Proof. We shall search for a potential in the form (18). Then Eq. (12) leads to two linear equations for the functions \( \bar{u}(\mu) \) and \( \bar{v}(\mu) \):

\[ \frac{d}{d\mu} \left( (1 + \mu^2) \frac{d\bar{u}}{d\mu} \right) - 6\bar{u} + 4\rho(\mu) = 0, \quad \frac{d}{d\mu} \left( (1 + \mu^2) \frac{d\bar{v}}{d\mu} \right) + 2\bar{u} - 2(1 + \mu^2) \rho(\mu) = 0. \] (21)

As is well known, the solution (21) is represented as the superposition

\[ \bar{u}(\mu) = \bar{u}_0(\mu) + \bar{u}_p(\mu), \quad \bar{v}(\mu) = \bar{v}_0(\mu) + \bar{v}_p(\mu), \] (22)

where \( \bar{u}_0 \) and \( \bar{v}_0 \) are a general solution of the homogeneous system (when \( \rho(\mu) = 0 \)), while \( \bar{u}_p \) and \( \bar{v}_p \) are a particular solution of the inhomogeneous system. In this case one can choose

\[ \bar{u}_0(\mu) = A_1(1 + 3\mu^2) + A_2 \left( (1 + 3\mu^2) \arctg(\mu) - 3\mu \right), \]
\[ \bar{v}_0(\mu) = -A_1\mu^2 + A_2 \left( (1 - \mu^2) \arctg(\mu) + \mu \right) + A_3 \arctg(\mu) + A_4. \] (23)

Using a modification of the method of variation of constants, the particular solution can be represented as single integrals:

\[ \bar{u}_p(\mu) = \left( (1 + 3\mu^2) \arctg(\mu) - 3\mu \right) \times \]
\[ \times \int_{\mu_\alpha}^\mu (1 + 3\xi^2) \rho(\xi) \, d\xi - (1 + 3\mu^2) \int_{\mu_\alpha}^\mu \left( (1 + 3\xi^2) \arctg(\xi) - 3\xi \right) \rho(\xi) \, d\xi, \] (24)
\[ \bar{v}_p(\mu) = \int_{\mu_\alpha}^\mu \left( \arctg(\xi) S(\xi) \right) \, d\xi - \arctg(\mu) \int_{\mu_\alpha}^\mu S(\xi) \, d\xi, \]
\[ 2S(\mu) = (\mu^2 + 1) \rho(\mu) - \bar{u}_p(\mu). \]
In the general case, for each of the integrals an arbitrary constant can be chosen as the lower bound \( \mu_s \) in (24).

The conditions which must be satisfied by the potential have the form

1. Away from the spheroid, the potential must tend to zero:

\[
\lim_{\mu \to \infty} \frac{\tilde{u}^{\text{out}}(\mu)}{1 + \mu^2} = 0, \quad \lim_{\mu \to \infty} \tilde{v}^{\text{out}}(\mu) = 0.
\]  

(25)

2. At the boundary of the spheroid \( \mu = \mu_0 \), the potential must be a smooth function:

\[
\begin{align*}
\tilde{u}^{\text{in}}(\mu_0) &= \tilde{u}^{\text{out}}(\mu_0), & \tilde{v}^{\text{in}}(\mu_0) &= \tilde{v}^{\text{out}}(\mu_0), \\
\tilde{u}^{\text{in}}(\mu_0) &= \tilde{u}^{\text{out}}(\mu_0), & \tilde{v}^{\text{in}}(\mu_0) &= \tilde{v}^{\text{out}}(\mu_0).
\end{align*}
\]  

(26)

3. As \( \mu \to 0 \), the potential on the section \( z = 0 \), \( r \in (0, d) \) must be a smooth function, i.e., the values of its derivatives must be the same at the points \( z_+ \) and \( z_- \) as \( \mu \to 0 \) (see Fig. [1]). This yields the condition

\[
\left. \tilde{u}^{\text{in}} \right|_{\mu=0} = 0, \quad \left. \tilde{v}^{\text{in}} \right|_{\mu=0} = 0.
\]  

(27)

Let us satisfy the first condition (25). To do so, we express the potential outside as a power series in \( \frac{1}{\mu} \):

\[
\frac{\tilde{u}^{\text{out}}(\mu)}{1 + \mu^2} = 3A_1^{\text{out}} + O\left(\frac{1}{\mu}\right), \quad \tilde{v}^{\text{out}} = -A_1^{\text{out}} \mu^2 + A_2^{\text{out}} + O\left(\frac{1}{\mu}\right)
\]

to give \( A_1^{\text{out}} = A_2^{\text{out}} = 0 \).

Next, we satisfy the condition (26). To simplify the system (26), we choose a particular solution in such a way that it vanishes on the surface. It is easily seen that this can be achieved by choosing \( \mu_s = \mu_0 \). Moreover, in this case Eqs. (26) are satisfied if we set \( A_1^{\text{in}} = A_3^{\text{in}} = 0 \), \( A_2^{\text{out}} = A_2^{\text{in}} \), \( A_3^{\text{out}} = A_3^{\text{in}} \).

From Eqs. (27) we find two remaining constants \( A_2^{\text{in}} \) and \( A_3^{\text{in}} \):

\[
A_2^{\text{in}} = \int_0^{\mu_0} (1 + 3\xi^2) \rho(\xi) d\xi, \quad A_3^{\text{in}} = -2 \int_0^{\mu_0} S(\xi) d\xi.
\]

Now, in order to obtain the relations (19), we only need to simplify the expression for \( A_3^{\text{in}} \):

\[
A_3^{\text{in}} = -2 \int_0^{\mu_0} (1 + \mu^2) \rho(\mu) d\mu + 2 \int_0^{\mu_0} \tilde{u}_\rho(\mu) d\mu,
\]

\[
2\tilde{u}_\rho(\mu) = 2 \left( \psi_1(\mu) \int_{\mu_0}^{\mu} \psi_2(\xi) \rho(\xi) d\xi - \psi_2(\mu) \int_{\mu_0}^{\mu} \psi_1(\xi) \rho(\xi) d\xi \right),
\]

\[
\psi_1(\mu) = (1 + 3\mu^2) \arctg \mu - 3\mu, \quad \psi_2(\mu) = 1 + 3\mu^2.
\]

We take the second integral in the expression for \( A_3^{\text{in}} \) by parts. To do this, we define the primitives

\[
\Psi_1(\mu) = \mu ((1 + \mu^2) \arctg \mu - \mu), \quad \Psi_1'(\mu) = \psi_1(\mu),
\]

\[
\Psi_2(\mu) = \mu(\mu + 1), \quad \Psi_2'(\mu) = \psi_2(\mu),
\]

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to give

\[
\int_0^{\mu_0} 2\tilde{\upsilon}_\mu(\mu) \, d\mu = \int_0^{\mu_0} (\psi_1(\mu)\Psi_2(\mu) - \psi_2(\mu)\Psi_1(\mu)) \rho(\mu) \, d\mu = -2 \int_0^{\mu_0} \mu^2 \rho(\mu) \, d\mu.
\]

Thus, we finally obtain

\[
A_{2n}^\text{in} = -2 \mu_0 \int_0^{\mu_0} (1 + 3\mu_0^2) \rho(\mu) \, d\mu.
\]

Writing the solution (24) with the known integration constants, taking the iterated integrals in \(\tilde{\upsilon}_\mu(\mu)\) by parts, as was done above, and reducing similar terms, we obtain (19) and (20).

Remark. If we make a change of the variable \(\mu = ix\) in Eqs. (21), they take the form of inhomogeneous Legendre equations with \(n = 2\) and \(n = 1\).

As a consequence of this representation of the potential, we obtain the well-known Maclaurin theorem [9] in the case of a spheroid.

**Theorem 1.** The gravitational potential that is produced by an inhomogeneous spheroid with confocal stratification and density \(\rho(\mu)\) is at the external point the same as the potential of a homogeneous spheroid with the density

\[
\langle \rho \rangle = \frac{1}{\mu_0(1 + \mu_0^2)} \int_0^{\mu_0} (1 + 3\xi^2) \rho(\xi) \, d\xi.
\]

According to Proposition 1, the family of confocal spheroids satisfies the condition (10), and hence the level surfaces of angular velocity are also confocal spheroids. After integrating \(P(\mu)\) by parts the final expression for the angular velocity of the layers can be represented as

\[
\frac{\omega(\mu)^2}{2\pi G} = I_0(\mu_0) \frac{\rho(\mu_0) (1 + 3\mu_0^2) \arctg(\mu_0) - 3\mu_0}{1 + \mu_0^2} - 2 \frac{\rho(\mu)}{\mu} \int_\mu^{\mu_0} \rho(\xi)((1 + 3\xi^2) \arctg(\xi) - 3\xi - I_1(\xi)(1 + 3\xi^2)) \, d\xi.
\]

(28)

From this relation, setting \(\mu = \mu_0\), we obtain the following result:

**Theorem 2.** For an arbitrary confocal stratification the angular velocity on the outer surface of the inhomogeneous spheroid is the same as the angular velocity of the Maclaurin spheroid with density \(\langle \rho \rangle\):

\[
\frac{\omega_0^2}{2\pi G \langle \rho \rangle} = \mu_0((1 + 3\mu_0^2) \arctg(\mu_0) - 3\mu_0),
\]

(29)

where \(\langle \rho \rangle\) is the average density of the spheroid.
2.3 The homogeneous Maclaurin spheroid

Let the density be constant everywhere inside some spheroid:

\[ \rho(\mu) = \begin{cases} 0, & \mu_0 < \mu, \\ \rho_0, & 0 < \mu \leq \mu_0, \end{cases} \]

where \( \mu_0 \) is defined by (17). In this case we find the gravitational potential from Proposition 1. Inside the spheroid it can be represented as

\[ U = \frac{2\pi G \rho_0}{\mu_0} \left( \frac{1}{2} \frac{r^2 u^{\text{in}}(\mu)}{1 + \mu^2} + d^2 v^{\text{in}}(\mu) \right), \]

where

\[ u^{\text{in}}(\mu) = \rho_0 (1 + 3\mu_0^2)(1 + \mu_0^2) \arctg \mu_0 - 2\mu_0^2), \]

\[ v^{\text{in}}(\mu) = \rho_0 (1 + \mu_0^2)(\mu_0^2 - \mu_0(1 + \mu_0^2) \arccot \mu_0). \]

Next, from (28) and taking into account the relationship (15) between \( \mu_0 \) and the eccentricity, we obtain the well-known expression for the angular velocity of the Maclaurin spheroid

\[ \frac{\omega_0^2}{2\pi G \rho_0} = \mu_0^2 \frac{(1 + 3\mu_0^2) \arccot \mu_0 - 3\mu_0}{1 + \mu_0^2} \equiv \frac{\sqrt{1 - e^2}}{e_3} \left( 3 - 2e^2 \right) \arcsin e - 3e \sqrt{1 - e^2}. \]

Using (11), we find the pressure for the Maclaurin spheroid:

\[ \frac{p}{2\pi G \rho_0^2} = \frac{\mu_0^2 - \mu^2}{1 + \mu^2} \left( d^2 (1 + \mu^2)(1 + \mu_0^2) - r^2 \right). \] (30)

It can be shown that the level surfaces (30) are homothetic spheroids. To do so, we use a relation defining the homothetic stratification, which in our case takes the form

\[ \frac{r^2}{d^2 (1 + \mu_0^2)} + \frac{z^2}{d^2 \mu_0^2} = m, \]

and (10), we find

\[ r = \frac{d^2 (1 + \mu_0^2)(1 + \mu_0^2)(m \mu_0^2 - \mu_0^2)}{\mu^2 - \mu_0^2}. \]

Then, substituting \( r \) into (30), we obtain

\[ \frac{p}{2\pi G \rho_0^2} = d^2 \mu_0^2 (1 + \mu_0^2) (1 - \mu_0 \arccot \mu_0) (1 - m). \]

2.4 A spheroid with piecewise constant density distribution

We now consider a spheroid with piecewise constant density, i.e., consisting of a sequence of embedded homogeneous layers with different densities. We will
number the outer layer, as before, by the index 0 and the last internal layer by the index \( n \). Thus, we obtain a spheroid consisting of \( n + 1 \) layers:

\[
\rho(\mu) = \begin{cases} 
0, & \mu_0 < \mu, \\
\rho_0, & \mu_1 < \mu < \mu_0, \\
\rho_1, & \mu_2 < \mu < \mu_1, \\
\ldots, & \ldots \\
\rho_n, & 0 < \mu < \mu_n.
\end{cases}
\]

The case of two layers of different density (in our notation \( n = 1 \)) is considered in [32], and the generalization of this case to an arbitrary number of layers is found in [15]. Interestingly, almost all calculations presented below are contained in [19], although he used them not to search for new figures of equilibrium but to prove the absence of inhomogeneous figures of equilibrium with rigid body rotation (see the Introduction).

From (13) we find that the pressure inside the \( k \)-th layer is given by

\[
\frac{p^{(k)}}{\rho_i} = \pi Gr^2 \left( \frac{\omega_i^2}{2 \pi G} - \frac{\tilde{u}_{\text{in}}(\mu)}{1 + \mu_i^2} \right) + 2 \pi G d^2 \tilde{v}_{\text{in}}(\mu) + \Phi_k, \quad k = 0, 1, \ldots, n.
\]

where \( \mu_k < \mu < \mu_{k+1} \).

Further, taking into account that the pressure at the outer boundary is zero and the potential and the pressure at the boundary between the layers change continuously, we obtain the following relations for unknown angular velocities:

\[
\frac{\Delta_0 \omega_0^2}{2 \pi G} = \Delta_0 \frac{\tilde{u}_{\text{in}}(\mu_0)}{1 + \mu_0^2}, \\
\ldots, \\
\frac{\Delta_n \omega_n^2}{2 \pi G} = \frac{\rho_n \omega_n^2}{2 \pi G} + \Delta_n \frac{\tilde{u}_{\text{in}}(\mu_n)}{1 + \mu_n^2}, \\
\Delta_0 = \rho_0, \quad \Delta_1 = \rho_1 - \rho_0, \quad \ldots, \quad \Delta_n = \rho_n - \rho_{n-1}.
\]

This yields the angular velocity for the \( k \)-th layer in the form

\[
\frac{\rho_k \omega_k^2}{2 \pi G} = \sum_{i=0}^{k} \Delta_i \frac{\tilde{u}_{\text{in}}(\mu_i)}{1 + \mu_i^2}.
\]

We obtain the expression for \( \tilde{u}_{\text{in}}(\mu_i) \) from (19):

\[
\tilde{u}_{\text{in}}(\mu_i) = I_0(\mu_i)((1 + 3\mu_i^2) \arctg \mu_i - 3\mu_i) - I_1(\mu_i)(1 + 3\mu_i^2).
\]

To calculate \( I_0(\mu_i) \) and \( I_1(\mu_i) \), we use the Heaviside function

\[
\theta(x) = \begin{cases} 
0, & x < 0, \\
1, & x \geq 0,
\end{cases}
\]
and represent the density of the spheroid under consideration as
\[ \rho(\mu) = \sum_{i=0}^{n} \Delta_i \theta(\mu_i - \mu). \]

Integrating, we find that
\[ I_0(\mu_i) = \sum_{j=i+1}^{n} \Delta_j \mu_j (1 + \mu_j^2) \]
\[ I_1(\mu_i) = \sum_{j=0}^{i} \Delta_j \left( \frac{2\mu_j^2}{1 + 3\mu_j^2} - \mu_j ((1 + \mu_j^2) \arctg \mu_j - \mu_j) \right). \]

As a result, we obtain an expression for the angular velocity of the \( k \)-th layer in the form
\[ \frac{\mu_k \omega_k^2}{2\pi G} = \sum_{i=0}^{k} \Delta_i \left( \frac{1 + 3\mu_i^2}{1 + \mu_i^2} \sum_{j=0}^{i} \Delta_j \left( \mu_j ((1 + \mu_j^2) \arctg \mu_j - \mu_j) - \frac{2\mu_j^2}{1 + 3\mu_j^2} \right) \right) + \left( (1 + 3\mu_i^2) \arctg \mu_i - 3\mu_i \right) \sum_{j=i+1}^{n} \Delta_j \mu_j (1 + \mu_j^2) \right). \]

(31)

2.5 A spheroid with continuous density distribution

To keep track of the dependence of the angular velocity of the layers on the change in density, we consider an inhomogeneous spheroid with different functions of density distribution of the following form:
\[ \rho(\mu) = \rho_0^{(n)} (1 - \alpha_n \mu^n), \quad n = 2, 4, 6, \]
\[ \rho_0^{(n)} \text{ and } \alpha_n \text{ are some constants (note that } \rho_0^{(n)} \text{ has the meaning of density at the center of the spheroid). We will determine their values from the given average density of the body } \langle \rho \rangle = \frac{1}{v} \rho dV \text{ and the given ratio between the density on the surface and the average density of the body } \varepsilon = \frac{\langle \rho \rangle}{\rho_0^{(n)}}, \]
\[ \alpha = \frac{(1 + n)(3 + n)(1 + \mu_0^n)(1 - \varepsilon)\mu_0^n}{(3 + n)(1 - \varepsilon(1 + n)(1 + \mu_0^n)) + 3(1 + n)\mu_0^n}, \]
\[ \rho_0 = \langle \rho \rangle \frac{(3 + n)(1 + n)(1 + \mu_0^n) - 1 - 3(1 + n)\mu_0^n}{n \varepsilon((1 + n)\mu_0^n + 3 + n)}. \]

As an example, assume that the eccentricity \( e_0 \) and \( \varepsilon \), which are the same as the data of the Earth [10]:
\[ e_0 = 0.08181, \quad \varepsilon = 2.5. \]
Figure 3 shows the dependence of \( \frac{\langle \rho \rangle}{\rho} \) on the coordinate of the layer \( \mu \) for a spheroid with density distribution described by (32). As we can see, the density increases most sharply at the center of the spheroid for \( n = 2 \) and then, as \( n \) increases, the density decreases.

To find the angular velocity, we substitute the density distributions (32) into (14) and obtain the dependence of the angular velocity on the layer. A graph of this dependence is shown in Fig. 4. (Since the explicit formulae for \( \omega(\mu) \) are unwieldy, we do not present them here.)

For the angular velocity with density distribution (32) one may draw the following conclusion from Fig. 4: the closer the center of the spheroid, the larger the angular velocity; specifically, the larger the value of density at the center of the spheroid (with \( n = 2 \)), the larger the increase in the angular velocity.

Next, we calculate the numerical value of the dependence of the period of revolution for each layer. If we assume the average density to be the same as that of the Earth \( \langle \rho \rangle = 5.51 \text{g/cm}^3 \), then we obtain the dependences of \( T(\mu) \) presented in Fig. 5.
3 The Chaplygin problem — a spheroid with homothetic density distribution

As is well known, the homothetic stratification is given by

$$\frac{z^2}{b^2} + \frac{r^2}{a^2} = \sigma, \quad \sigma \in [0, +\infty),$$

where, assuming that $a$ and $b$ are the principal semi-axes of a spheroid filled with a fluid (see Fig. 6), we obtain

$$\sigma_0 = 1, \quad Z(r, \sigma) = \pm b \sqrt{\sigma - \frac{r^2}{a^2}}.$$

Again we set

$$\rho = \begin{cases} \rho(\sigma) \text{ (does not depend on } r), & \sigma \leq 1, \\ 0, & \sigma > 1. \end{cases}$$

Using the second of Eqs. (7) and noting that $p|_{\sigma=1} = 0$, we obtain the pressure, which can be represented as

$$p(r, \sigma) = \rho_1 U(r, 1) - \rho(\sigma) U(r, \sigma) + \int_1^\sigma U(r, \sigma) \frac{\partial \rho}{\partial \sigma} d\sigma, \quad \rho_1 = \rho(1).$$

In a similar manner, substituting the pressure from the first of Eqs. (7), we obtain

$$\omega^2(r, \sigma) = \frac{1}{r \rho(\sigma)} \left( \rho_1 \frac{\partial U}{\partial r}(r, 1) + \int_1^\sigma \frac{\partial U}{\partial r}(r, \sigma) \frac{\partial \rho}{\partial \sigma} d\sigma \right). \quad (33)$$

![Fig. 6: Meridional sections of the surfaces $\sigma = \text{const}$ with homothetic stratification](image)
Thus, to complete the solution, we only need to find the potential from the equation
\[ \Delta r, \sigma U(r, \sigma) = 4\pi G\rho(\sigma). \]

In [17] a convenient integral representation of the potential for a (three-axial) ellipsoid with homothetic density stratification is obtained. Applying it to the case of the spheroid \( \sigma = 1 \) gives
\[
U^{\text{in}}(r, z) = \pi G a^2 b^2 \int_{s_0}^{\infty} \frac{f(1) - f \left( \frac{r^2}{a^2 + s} + \frac{z^2}{b^2 + s} \right)}{\Delta(s)} \, ds,
\]
\[
U^{\text{out}}(r, z) = \pi G a^2 b^2 \int_{s_0}^{\infty} \frac{f(1) - f \left( \frac{r^2}{a^2 + s} + \frac{z^2}{b^2 + s} \right)}{\Delta(s)} \, ds,
\]
\[ \Delta(s) = (a^2 + s)\sqrt{b^2 + s}, \]  \hspace{1cm} (34)

where the function \( f(\sigma) \) is related with the density of the fluid by
\[ \rho(\sigma) = \frac{df(\sigma)}{d\sigma}, \]
and the quantity \( s_0 \) for given \((r, z)\), which correspond to a point outside the liquid spheroid, is defined as the root of the equation
\[ \frac{r^2}{a^2 + s_0} + \frac{z^2}{b^2 + s_0} = 1. \]

As an example, we consider the density distribution of the form
\[ \rho(\sigma) = \rho_0(1 - \alpha\sigma^n), \quad n = 1, 2, 3. \] \hspace{1cm} (35)

Given the average density \( \langle \rho \rangle \) of the body and the ratio between the densities at the center and on the surface \( \eta = \frac{\rho_0}{\rho_1} \), we now define the constants \( \rho_0 \) and \( \alpha \):
\[ \alpha = \frac{\eta - 1}{\eta}, \quad \rho_0 = \frac{\eta(3 + 2n)\langle \rho \rangle}{3 + 2n\eta}. \] \hspace{1cm} (36)

Set
\[ \eta = 5, \quad \frac{b}{a} = \frac{1}{2}. \]

Further, we find the potential from (34) and obtain the angular velocity from (33). The meridional sections of the surfaces \( \frac{\omega_2^2}{2\pi G\rho} = \text{const} \) with equal spacings for different \( n = 1, 2, 3 \) are shown in Fig. 7. The graphs of change in the relation \( \frac{\omega_2^2}{2\pi G\rho} \) along the small semi-axis \( b \) is shown in Fig. 8.

For the densities from Figs. 7 and 8 one can draw the following conclusions:
1. The closer the center of the spheroid, the slower the change in the angular velocity.
Fig. 7: Meridional sections of the surfaces $\frac{\omega^2}{2\pi G(\rho)} = \text{const}$ with equal spacings

Fig. 8: The change of $\frac{\omega^2}{2\pi G(\rho)}$ along the small semi-axis $b$ for different $n$

2. For $n = 1$ the level surfaces near the center of the spheroid are concentric spheres. Further, as $n$ increases, the region in which the level lines are closed surfaces increases. For $n > 1$ these closed surfaces are no longer surfaces of the second order.

Let us consider in more detail the angular velocity at the boundary of the spheroid at the equator with densities of the form $\rho_0 e^2$, but now with an arbitrary $n$. From (33), changing the variable $s = a^2(t - 1)$, we obtain the angular velocity on the surface:

$$\frac{\omega_n^2(r, 1)}{2\pi G} = \rho_0 e^2 \sqrt{1 - e^2} \int_1^\infty \frac{t - 1}{t^2(t - e^2)^{3/2}} \times$$

$$\times \left(1 - \frac{\alpha t^{-n}}{(t - e^2)^n} (t - 1)e^2 + t(1 - e^2)\right)^n dt,$$
that is, for \( r = a \) we have

\[
\frac{\omega_n^2(a, 1)}{2\pi G} = \rho_0 e^2 \sqrt{1 - e^2} \int_1^\infty \frac{(t - 1)(1 - \alpha t^{-n})}{t^2(t - e^2)^{3/2}} dt.
\]

Explicitly integrating gives

\[
\frac{\omega_n^2(a, 1)}{2\pi G} = \rho_0 \omega_m^2 + \frac{2\alpha \rho_0 e^2}{3 + 2n} \times \left( \frac{\sqrt{1 - e^2}(2n(1 - e^2) + 3 - 2e^2)}{5 + 2n} F \left( \frac{3}{2}, n + \frac{5}{2}; n + \frac{7}{2}; e^2 \right) - 1 \right),
\]

where \( \omega_m^2 \) is the dimensionless angular velocity of the Maclaurin spheroid:

\[
\omega_m^2 = \frac{\sqrt{1 - e^2}}{e^3} \left( (3 - 2e^2) \arcsin e - 3e \sqrt{1 - e^2} \right).
\]

Substituting the expression (37) into the relation for the angular velocity, we obtain for two values of \( n \)

\[
\frac{\omega_0^2(a, 1)}{2\pi G \rho_0 (1 - \alpha)} = \frac{\omega_\infty^2(a, 1)}{2\pi G \rho_0} = \omega_m^2.
\]

Further, we shall define \( \rho_0 \) and \( \alpha \) from various known data for the Earth:

We are given the average density of the body \( \langle \rho \rangle = 5.51 \text{ g/cm}^3 \) and the ratio between the densities on the surface and at the center \( \frac{\rho_1}{\rho} = 5 \). In this case \( \rho_0 \) and \( \alpha \) are defined by (37), and the dependence of the period of revolution at the equator \( T \) on \( n \) is shown in Fig. 9.

As can be seen in Fig. 9 \( T(n) \) reaches the minimum at the point \( T(0.8675) = 24.1610 \text{ h} \).

We are given the average density of the body, \( \langle \rho \rangle = 5.51 \text{ g/cm}^3 \), and the ratio between the density on the surface and the average density, \( \frac{\rho_1}{\rho} = \varepsilon = 2.5 \).
The dependence of the period of revolution at the equator $T$ on $n$ is shown in Fig. 10

The dependence of the period of revolution $T$ on the polar radius $r$ on the surface is shown in Fig. 11

Fig. 10: The dependence of period $T$ on $n$ at the equator for $\langle \rho \rangle = 5.51 \text{g/cm}^3$ and $\varepsilon = 2.5$

Fig. 11: The dependence of period $T$ on the polar radius on the surface of the inhomogeneous spheroid $\langle \rho \rangle = 5.51 \text{g/cm}^3$ and $\varepsilon = 2.16$ for $n = 1$, $n = 2$ and $n = 3$. 

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4 Figures of equilibrium in $S^3$

One of the generalizations of the above results is that they are carried over to the spaces of constant curvature $S^3$ and $L^3$, by analogy with celestial mechanics of point masses [5, 21, 25, 36]. There is a vast classical and recent literature on the dynamics of gravitating point masses (see [1, 5, 6, 7], in which, for example, the well-known analogs of the Kepler law and those of the three-body problem were studied. However, a particular generalization of the theorems of Newtonian potential to $S^3$ and $L^3$ was performed only in [24]. As will be shown below, in this case the problem of equilibrium figures becomes considerably more complex. In particular, even in the case of homogeneous ellipsoids the rigid body rotation of a fluid mass is impossible (we recall that an ellipsoid in curved space is said to be a body obtained by the intersection of the sphere $S^3$ or the Lobachevsky space $L^3$, embedded in $\mathbb{R}^4$, with a conical quadric). One of the difficulties is due to the fact that although some generalizations of Ivory’s theorem on the potential of the elliptic layer [24] are possible, this and similar theorems cannot be completely extended to $S^3$ and $L^3$ (they are closely related to the homogeneity of plane space).

Remark. Generalizations of the problem of equilibrium figures to the relativistic case are also possible, see, e.g., the review [31]. Unfortunately, attempts to obtain explicit analytical exact solutions along these lines have yielded no results so far. This direction is a new research area.

4.1 Steady-state axisymmetric solutions in $S^3$

To explore possible figures of equilibrium in $S^3$, we choose curvilinear coordinates, as was done for the plane space $E^3$. For convenience, we assume $S^3$ to be embedded into $E^4$, then the transition to the coordinates under consideration has the form

\[ x_0 = \pm \sqrt{R^2 - r^2 - Z^2(r, \mu)}, \quad x_1 = Z(r, \mu), \quad x_2 = r \cos(\varphi), \quad x_3 = r \sin(\varphi), \]

where $Z(r, \mu)$ is defined, as before, by the specific problem statement. The metric tensor can be represented as

\[
\begin{pmatrix}
g_{11} & g_{12} & 0 \\
g_{12} & g_{22} & 0 \\
0 & 0 & r^2
\end{pmatrix},
\]

where

\[
g_{11} = 1 - Z \frac{(r + Z Z_r)^2}{R^2 - r^2 - Z^2}, \quad g_{12} = Z \frac{r Z + (R^2 - r^2) Z_r}{R^2 - r^2 - Z^2}, \quad g_{22} = \frac{(R^2 - r^2) Z^2}{R^2 - r^2 - Z^2}.
\]

We shall seek a steady-state solution for which the velocity distribution of fluid particles has the form

\[
\dot{r} = 0, \quad \dot{\mu} = 0, \quad \dot{\varphi} = \omega(r, \mu).
\]
As above, assuming that the density depends only on \( \mu \) and using the equations of Section 1.1, we obtain the system

\[
\frac{\partial U}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = r \omega^2, \quad \frac{\partial U}{\partial \mu} + \frac{1}{\rho} \frac{\partial p}{\partial \mu} = 0,
\]

\[
\Delta_{r\mu} U = 4\pi G \rho(\mu),
\]

\[
\Delta_{r\mu} = \frac{x_0}{r Z_\mu} \left( \frac{\partial}{\partial r} \left[ \frac{r Z_\mu}{x_0} \left( 1 - \frac{r^2}{R^2} \right) \frac{\partial}{\partial r} \right] + \frac{x_0}{Z_\mu} \frac{\partial}{\partial \mu} \left[ \frac{1}{x_0 Z_\mu} \left( 1 + Z_r^2 - \frac{(Z - r Z_r)^2}{R^2} \right) \right] \frac{\partial}{\partial \mu} \right)
\]

where \( x_0 = \sqrt{R^2 - r^2 - Z^2(r, \mu)} \) and it is assumed that the density \( \rho(\mu) \) vanishes everywhere outside the body (\( \mu_0 < \mu \)), and at the free boundary \( \mu = \mu_0 \) the pressure is zero as well:

\[
p(r, \mu)|_{\mu=\mu_0} = 0.
\]

As we can see, the hydrodynamical equations remain the same as in \( E^3 \). Therefore, as in Section 3 their solution inside the region (\( \mu \leq \mu_0 \)) filled with fluid can be represented as

\[
p(r, \mu) = \rho_0 U(r, \mu_0) - \rho(\mu) U(r, \mu) + \int_{\mu_0}^\mu U(r, \mu) \frac{d\rho(\mu)}{d\mu} d\mu, \quad \rho_0 = \rho(\mu_0),
\]

\[
\omega^2(r, \mu) = \frac{1}{r \rho(\mu)} \left( \rho_0 \frac{dU}{dr}(r, \mu_0) + \int_{\mu_0}^\mu \frac{dU}{dr}(r, \mu) \frac{d\rho(\mu)}{d\mu} d\mu \right).
\]

### 4.2 A homogeneous spheroid in \( S^3 \)

We now consider in more detail the case of a homogeneous spheroid, when for \( \mu \leq \mu_0 \) the density \( \rho(\mu) = \rho_0 = \text{const} \). The generalization of confocal stratification in \( S^3 \) is given as follows (cf. Section 2.2):

\[
\frac{x_0^2}{R^2 - d^2 \mu^2} - \frac{x_1^2}{d^2 \mu^2} - \frac{x_2^2 + x_3^2}{d^2(1 + \mu^2)} = 0, \quad \mu \in \left[ 0, \frac{R}{d} \right].
\]

Hence, we obtain

\[
Z(r, \mu) = \pm \sqrt{d^2 \mu^2 - r^2 \frac{R^2 + d^2}{R^2} \frac{\mu^2}{1 + \mu^2}}.
\]

As in the previous case (see Section 2.2), the parameter \( d \) and the boundary \( \mu_0 \) of a liquid spheroid with semi-axes \( a \) and \( b \) are given by

\[
d = \sqrt{a^2 - b^2}, \quad \mu_0 = \frac{b}{\sqrt{a^2 - b^2}}.
\]
According to (39), in the case of a homogeneous spheroid \( \frac{d\rho}{d\mu} = 0 \), therefore, the angular velocity of the fluid depends only on \( r \):

\[
\omega^2(r) = \frac{1}{r} \frac{\partial U}{\partial r}(r, \mu_0). \tag{40}
\]

We shall seek solutions to the equation for the potential in the form of a power series in the parameter \( \frac{d^2}{dx^2} \):

\[
U(r, \mu) = 2\pi Gd^2 \sum_{n=0}^\infty \left( \frac{d}{R} \right)^{2n} U_n(r, \mu).
\]

As can be shown, all terms of this series are polynomials in \( r \). It is convenient to represent them as

\[
U_n(r, \mu) = \sum_{n=0}^\infty \left( \frac{r}{d} \right)^{2m} \frac{u_{n,\mu}(\mu)}{2^m(1+\mu^2)^m}.
\]

The potential \( U_0(r, \mu) \) is equal (up to a multiplier) to the potential of the Maclaurin spheroid (see Section 2.3):

\[
U_0(r, \mu) = u_{0,0}(\mu) + \frac{r^2}{d^2} \frac{u_{0,1}(\mu)}{2(1+\mu^2)},
\]

inside the spheroid \( (\mu \leq \mu_0) \):

\[
u_{0,0}^{\text{in}}(\mu) = \rho_0(1+\mu_0^2)(\mu^2 - \mu_0(1+\mu^2) \arctan \mu_0),
\]

\[
u_{0,1}^{\text{in}}(\mu) = \rho_0(\mu_0(1+3\mu^2)((1+\mu_0^2) \arctan \mu_0 - \mu_0) - 2\mu^2).
\]

outside the spheroid \( (\mu_0 < \mu) \):

\[
u_{0,0}^{\text{out}}(\mu) = \rho_0 \mu_0(1+\mu_0^2)(\mu - (1+\mu^2) \arctan \mu),
\]

\[
u_{0,1}^{\text{out}}(\mu) = \rho_0 \mu_0(1+\mu_0^2)((1 + 3\mu^2) \arctan \mu - 3\mu).
\]

We shall assume that the space curvature is very small \( (R^2 \gg a^2) \) and, therefore, restrict ourselves to calculating the first correction

\[
U_1(r, \mu) = \frac{r^4}{d^4} \frac{u_{1,2}(\mu)}{4(1+\mu^2)^2} + \frac{r^2}{d^2} \frac{u_{1,1}(\mu)}{2(1+\mu^2)} + u_{1,0}(\mu),
\]

where the functions \( u_{1,0}(\mu) \), \( u_{1,1}(\mu) \), and \( u_{1,2}(\mu) \) satisfy the equations

\[
\frac{d}{d\mu} \left( (1+\mu^2) \frac{du_{1,2}}{d\mu} \right) - 20u_{1,2} + 16u_{0,1} = 0,
\]

\[
\frac{d}{d\mu} \left( (1+\mu^2) \frac{du_{1,1}}{d\mu} \right) - 6u_{1,1} - \mu(1+\mu^2) \frac{du_{0,1}}{d\mu} - 6(2 + \mu^2) u_{0,1} + 8u_{1,2} + 4\rho_0(1+\mu^2) = 0,
\]

\[
\frac{d}{d\mu} \left( (1+\mu^2) \frac{du_{0,0}}{d\mu} \right) - 2u_{1,1} - \mu(1+\mu^2) \frac{du_{0,0}}{d\mu} + 2\mu^2(u_{0,1} + \rho_0(1+\mu^2)) = 0.
\]

\( (41) \)
The functions $u_{1,0}$, $u_{1,1}$, and $u_{1,2}$ must also satisfy the following boundary conditions:

$$\left. \frac{du_{1,m}^{in}}{d\mu} \right|_{\mu=0} = 0, \quad m = 0, 1, 2.$$  
$$\left. u_{1,m}^{in} \right|_{\mu=\mu_0} = u_{1,m}^{out}, \quad m = 0, 1, 2.$$  
$$\left. \frac{du_{1,m}^{in}}{d\mu} \right|_{\mu=\mu_0} = \left. \frac{du_{1,m}^{out}}{d\mu} \right|_{\mu=\mu_0}, \quad m = 0, 1, 2.$$  
$$U_1(r, \mu)|_{\mu=\frac{a}{R}} = O(R^2).$$

Since the solution of the resulting system is rather unwieldy, we omit it here and confine ourselves to the expression for the angular velocity of the fluid, for which, according to \[10\], we find

$$\frac{\omega^2(r)}{2\pi G} = \frac{u_{1,1}^{in}(\mu_0)}{1 + \mu^2_0} + \frac{1}{R^2} \left( \frac{u_{1,2}^{in}(\mu_0)}{(1 + \mu^2_0)^2} r^2 + \frac{u_{1,1}^{in}(\mu_0)}{1 + \mu^2_0} d^2 \right) + O\left( \frac{d^4}{R^4} \right).$$

Substituting the solution for $u_{1,m}^{in}(\mu_0)$ and expressing $\mu_0$ in terms of the eccentricity of the boundary using the formula $e = \frac{1}{\sqrt{1 + \mu_0^2}}$, we obtain an explicit representation for the angular velocity in the form

$$\frac{\omega^2(r)}{2\pi G \rho_0} = \omega_{00} + \frac{1}{R^2} \left( \omega_{11} r^2 + \omega_{10} a^2 \right) + O\left( \frac{d^2}{R^4} \right),$$

$$\omega_{00} = \frac{\sqrt{1 - e^2}}{e} \left( 2 - \frac{3}{e^2} \right) \arcsin e - \frac{3}{e^2} (1 - e^2),$$

$$\omega_{11} = -\frac{\sqrt{1 - e^2}}{e} \left( 12 - \frac{30}{e^2} + \frac{35}{2e^4} \right) \arcsin e + \left( \frac{4}{3} - \frac{55}{3e^2} + \frac{35}{2e^4} \right) (1 - e^2),$$

$$\omega_{10} = \frac{\sqrt{1 - e^2}}{e} \left( 16 - \frac{27}{2e^2} + \frac{10}{e^4} \right) \arcsin e - \left( \frac{1}{3} - \frac{41}{6e^2} + \frac{10}{e^4} \right) (1 - e^2),$$

where we have also passed from the parameter $d$ (which tends to zero as $e \to 0$) to the value of the largest principal semi-axis $a$. The graphs of dependence of each of the corrections for the angular velocity on the eccentricity is presented in Fig. \[12\].

Thus, in the space of constant (positive) curvature the homogeneous liquid self-gravitating spheroid cannot rotate as a rigid body, and the angular velocity distribution of fluid particles depends only on the distance to the symmetry axis: $\omega = \omega(r)$.

**Remark.** For completeness we also present the equations which describe axisymmetric figures of equilibrium in curvilinear orthogonal coordinates $(\mu, \nu, \varphi)$ and are defined as follows:

$$\frac{x_0^2}{d^2} = (\delta - \mu)(\delta + \nu), \quad \frac{x_1^2}{d^2} = \mu\nu, \quad \delta = \frac{R^2}{d^2}.$$  
$$\frac{x_2^2}{d^2} = \frac{(1 + \mu)(1 - \nu)}{\delta + 1} \cos^2 \varphi, \quad \frac{x_3^2}{d^2} = \frac{(1 + \mu)(1 - \nu)}{\delta + 1} \sin^2 \varphi, \quad 0 < \mu < \delta, \quad 0 < \nu < 1.$$
Fig. 12: Dependences of $\omega_{00}$, $\omega_{11}$, and $\omega_{10}$ on the eccentricity $e$.

In this case the system \[ \frac{\partial U}{\partial \mu} + \frac{1}{\rho(\mu)} \frac{\partial p}{\partial \mu} = -\frac{\delta d^2}{2(\delta + 1)}(1 - \nu)\omega^2, \quad \frac{\partial U}{\partial \nu} + \frac{1}{\rho(\mu)} \frac{\partial p}{\partial \nu} = \frac{\delta d^2}{2(\delta + 1)}(1 + \mu)\omega^2, \] takes the form

\[ \Delta_{\mu\nu} U(\mu, \nu) = 4\pi G \rho(\mu), \]

\[ R^2 \Delta_{\mu\nu} = \frac{4}{\mu + \nu} \left( \sqrt{\mu(\delta - \mu)} \frac{\partial}{\partial \mu} \left( 1 + \mu \right) \sqrt{\mu(\delta - \mu)} \frac{\partial}{\partial \mu} \right) + \\
\quad + \sqrt{\nu(\delta + \nu)} \frac{\partial}{\partial \nu} \left( 1 - \nu \right) \sqrt{\nu(\delta + \nu)} \frac{\partial}{\partial \nu}. \]

This form of equations is preferable if it is necessary to obtain a solution in terms of quadratures (and not in the form of a power series).

5 Discussion

Thus, in this paper we have systematically analyzed the problem of inhomogeneous axisymmetric equilibrium figures of an ideal self-gravitating fluid. We have obtained the most general solution describing a stratified spheroid (the angular velocity of the fluid takes the same value on the layer with equal density, i.e., $\omega = \omega(\mu)$). This solution naturally yields the above-mentioned spheroids with piecewise constant density distribution [15, 32]. It is shown that the angular velocity of the outer surface of the spheroid with confocal stratification of density $\rho$ is the same as that of the homogeneous Maclaurin spheroid with density $\langle \rho \rangle$. Therefore, this model cannot be used to explain the deviation of the compression of planets from the compression of the Maclaurin spheroids rotating with the same angular velocity.

We have also presented a fairly detailed review (and a formulation in modern terms) of results in this vein. Of special note is Chaplygin’s work (previously
unpublished and found in archives) on spheroids with homofocal density stratification.

In the last section we have considered the problem of the conditions for equilibrium of a homogeneous spheroid in the spaces of constant curvature $S^3$ and shown that in this case the fluid cannot rotate as a rigid body and that the angular velocity of fluid particles depends only on the distance to the symmetry axis: $\omega = \omega(r)$.

We conclude by pointing out some open problems related to possible generalizations of the above results.

1. The stratified analogs of the Maclaurin spheroids raise the question of their stability. Of particular importance is here in all probability their secular stability, which was considered by Lyapunov [28] for the case of homogeneous fluid density. In his analysis of the perturbation of the free surface by spherical harmonics he concluded that the higher the order of a harmonic, the larger the value of eccentricity at which the loss of secular stability occurs. In the general case Lyapunov arrived at the conclusion that the secular stability of the Maclaurin spheroids is lost under arbitrary deformations if the eccentricity becomes equal to 0.8126 (for the special case of ellipsoidal perturbations this result was obtained by Dirichlet [13]).

As far as this problem is concerned, no finite-dimensional equations governing the dynamics of stratified ellipsoids have been obtained so far (see [13, 14, 35]). Because of this it is difficult to obtain all sufficient stability criteria determined by the finite dimensionality of the system (Lyapunov’s theorem, KAM theory).

2. Historically, attempts to derive the first equations of stratified ellipsoids go back to [3], but, as Tedone [37] noted, Betti made a mistake in his study. In this connection, the question of possible existence of three-axial inhomogeneous ellipsoids still awaits its solution.

3. The above solution for spheroids with confocal stratification is evidently the only solution possible, for which $\omega = \omega(\mu)$, but no proof of this fact has been found.

4. The problem of stability of the found figures of equilibrium with respect to both ellipsoidal and arbitrary perturbations is also an open question.

5. Another interesting problem is that of obtaining an explicit solution (not in the form of a power series) for a homogeneous spheroid in curved space and the search for other possible figures of equilibrium in the spaces of constant curvature.

6. We recall that for the Maclaurin and Jacobi ellipsoids there exists a “dynamical” generalization, due to Dirichlet, where the self-gravitating liquid ellipsoid retains an ellipsoidal shape but changes the directions and dimensions of the semi-axes during its motion. It is unknown whether there exists such a dynamical generalization for inhomogeneous figures of equilibrium.

Remark. A simple extension of Dirichlet’s method, for example, to a family with confocal density stratification (see Section [2]) is impossible since in Dirichlet’s solution the same fluid particles move in ellipsoids forming at each instant of time a homothetic (and not confocal) foliation.
7. Another possible generalization involves finding the figures of equilibrium of a stratified gas cloud. In this case, in order to close the system (7), one uses, as a rule, thermodynamical equations (for applications to fluid mass dynamics see the review [8] and references therein). In particular, one of the simplest assumptions used in [14] is that the temperature of the fluid/gas is constant along the entire volume $T(r, \mu) = T_0 = \text{const}$. In the case of an ideal gas this leads to a linear relation between density and pressure

$$p = \lambda \rho, \quad \lambda = RT_0,$$

where $R$ is the universal gas constant.

Assuming that $\rho = \rho(\mu)$, we obtain from (7) and (42) the system

$$\frac{\partial U}{\partial r} = r\omega^2, \quad \frac{\partial U}{\partial \mu} = \frac{\lambda \rho'(\mu)}{\rho(\mu)},$$

$$\Delta_{r\mu} U = 4\pi G \rho(\mu).$$

One of the unknowns in these equations is the function $Z(r, \mu)$ characterizing possible equilibrium figures of the cloud of an ideal gas.

Remark. To close the system, one can use, instead of the equation of state (42), the condition that the fluid flow be barotropic.

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References

[1] Albouy, A.: There is a Projective Dynamics. Eur. Math. Soc. Newsl. (89), 37–43 (2013)

[2] Appell, P.: Traité de Mécanique Rationnelle: T. 4-1. Figures d’Equilibre d’une Masse liquide Homogène en Rotation. Gautier-Villars, Paris (1921)

[3] Betti E. Sopra i moti he onservano la gura ellissoidale a una massa uida eterogenea. Annali di Matemati a Pura ed Appli ata, Serie II, X, 173187 (1881)

[4] Borisov, A. V., Mamaev, I. S.: Poisson Structures and Lie Algebras in Hamiltonian Mechanics. Izd. UdSU, Izhevsk (1999) (in Russian)

[5] Borisov, A. V., Mamaev, I. S.: The Restricted Two-Body Problem in Constant Curvature Spaces. Celestial Mech. Dynam. Astronom. 96(1), 1–17 (2006)

[6] Borisov, A. V., Mamaev, I. S.: Relations Between Integrable Systems in Plane and Curved Spaces. Celestial Mech. Dynam. Astronom. 99(4), 253–260 (2007)

[7] Borisov, A. V., Mamaev, I. S., Kilin, A. A.: Two-Body Problem on a Sphere. Reduction, Stochasticity, Periodic Orbits. Regul. Chaotic Dyn. 9(3), 265–279 (2004)

[8] Borisov, A. V., Mamaev, I. S., Kilin, A. A.: The Hamiltonian Dynamics of Self-gravitating Liquid and Gas Ellipsoids. Regul. Chaotic Dyn. 14(2), 179–217 (2009)

[9] Chandrasekhar, S.: Ellipsoidal Figures of Equilibrium. Yale University Press, New Haven (1969)
[10] Chaplygin, S. A.: Steady-State Rotation of a Liquid homogeneous spheroid In Collected works: Vol. 2. Hydrodynamics. Aerodynamics. Gosstehizdat, Moscow (1948)
[11] Clairaut, A. C.: Théorie de la Figure de la Terre: Tirée des Principes de l’Hydrostratique. Paris Courcier, Paris (1743)
[12] Craik, A. D. D.: James Ivory’s Last Papers on the ‘Figure of the Earth’ (with biographical additions). Notes Rec. R. Soc. Lond. 56(2), 187–204 (2002)
[13] Dirichlet, G. L.: Untersuchungen über ein Problem der Hydrodynamik (Aus dessen Nachlass hergestellt von Herrn R. Dedekind zu Zürich). J. Reine Angew. Math. (Crelle’s Journal) 58, 181–216 (1861)
[14] Dyson, F. J.: Dynamics of a Spinning Gas Cloud. J. Math. Mech. 18(1), 91–101 (1968)
[15] Esteban, E. P., Vasquez, S.: Rotating Stratified Heterogeneous Oblate Spheroid in Newtonian Physics. Celestial Mech. Dynam. Astronom. 81(4), 299–312 (2001)
[16] Fassò, F., Lewis, D.: Stability properties of the Riemann ellipsoids. Arch. Ration. Mech. Anal. 158, 259–292 (2001)
[17] Ferrers, N. M.: On the Potentials, Ellipsoids, Ellipsoidal Shells, Elliptic Laminae, and Elliptic Rings, of Variable Densities. Quart. J. Pure Appl. Math. 14, 1–23 (1875)
[18] Gaffet, B.: Spinning Gas Clouds: Liouville Integrability. J. Phys. A: Math. Gen. 34, 2097–2109 (2001)
[19] Hamy, M.: Étude sur la Figure des Corps Célestes. Ann. de l’Observatoire de Paris. Memories 19, 1–54 (1889)
[20] Jacobi, C. G. J.: Über die Figur des Gleichgewichts, Poggendorff Annalen der Physik und Chemie, 33, 229–238 (1834)
[21] Killing, H. W.: Die Mechanik in den nichteuklidischen Raumformen. J. Reine Angew. Math. XCVIII(1), 1–48 (1885)
[22] Kochin, N. E., Kibel, I. A., Rose, N. V. Theoretical Hydromechanics (in Russian), Vol. 1. Fizmatgiz, Moscow (1963)
[23] Kong, D., Zhang, K., Schubert, G.: Shapes of Two-Layer Models of Rotating Planets. J. Geophys. Res. 115(E12), doi:10.1029/2010JE003720 (2010)
[24] Kozlov, V. V.: The Newton and Ivory Theorems of Attraction in Spaces of Constant Curvature. (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. (5), 43–47 (2000)
[25] Kozlov, V. V., Harin, A. O.: Kepler’s Problem in Constant Curvature Spaces. Celestial Mech. Dynam. Astronom. 54(4) 393–399 (1992)
[26] Lichtenstein, L.: Gleichgewichtsfiguren Rotierender Flüssigkeiten. Springer, Berlin (1933)
[27] Liouville, J.: Sur la Figure d’une Masse Fluide Homogène, en Équilibre et Douée d’un Mouvement de Rotation. J. de l’École Polytechnique 14, 289–296 (1834)
[28] Lyapunov, A. M.: Collected Works, vol. 3. Moscow (1959)
[29] Lyttleton, R. A.: The Stability of Rotating Liquid Masses. Cambridge University Press, Cambridge (1953)
[30] MacLaurin, C.: A Treatise of Fluxions: In Two Books. Printed by T.W. & T. Ruddimans, Edinburgh (1742)
[31] Meinel, R., Ansorg, M., Kleinwachter, A., Neugebauer, G., Petroff, D.: Relativistic Figures of Equilibrium. Cambridge University Press, Cambridge (2008)
[32] Montalvo, D., Martínez, F. J., Cisneros, J.: On Equilibrium Figures of Ideal Fluids in the Form of Confocal Spheroids Rotating with Common and Different Angular Velocities. (1982)

[33] Pizzetti, P.: Principii della Teorii Meccanica della Figura dei Pianeti. Enrico Spoerri, Libraio-Editore, Pisa (1913)

[34] Rambaux, N., Van Hoolst, T., Dehant, V., Bois, E.: Inertial Core-Mantle Coupling and Libration of Mercury. Astronom. Astrophys. 468, 711–179 (2007)

[35] Riemann, B.: Ein Beitrag zu den Untersuchungen über die Bewegung eines Flüssigen gleichartigen Ellipsoides. Abh. d. Königl. Gesel 1. der Wiss. zu Göttingen (1861)

[36] Schrödinger, E.: A Method of Determining Quantum-Mechanical Eigenvalues and Eigenfunctions. Proc. Roy. Irish Acad. Sect. A 46, 9–16 (1940)

[37] Tedone O. Il moto di un ellissoide fluido se ondo l’ipotesi di Dirihlet. Annali del la S uola Normale Superiore di Pisa, 7, I–IV+1–100 (1895)

[38] Véronnet A.: Rotation de l’Ellipsoide Hétérogène et Figure Exacte de la Terre. J. Math. Pures et Appl., Sér. 6 8, 331–463 (1912)

[39] Volterra, V.: Sur la Stratification d’une Masse Fluide en Equilibre. Acta Math. 27(1), 105–124 (1903)

[40] Williams, D. R.: Earth Fact Sheet. Structural Geology of the Earth’s Interior: Proc. Natl. Acad. Sci. 76(9), (NASA (17 Nov 2010))