Quantum Hall Effect and Chaotic Motion in Phase Space

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Abstract
We discuss the relation between the Quantum Hall behaviour of charged carriers and their chaotic motion in phase space. It is shown that the quantum Hall diagram is comparable with the stepped diagram in phase space of a chaotic motion.
Recent developments in experimental and theoretical modeling of quantum Hall effects (QHE) show that charge carriers should have chaotic behaviour during their participation in QHE. We discuss here the relation between QHE and chaotic motion and show that QHE and specially its localization-delocalization phenomenology should be considered as a kind of chaotic motion in the phase space of system.

Let us first determine what we understand as an appropriate model of chaotic motion to be compared with the QHE. It is the so called snapshots model which contains as a limiting case the so called "kicked" models as well. It describes a classical motion in the phase space according to a Hamiltonian which is alternatively purely kinetic, or purely potential. The motion in phase space is a stepped motion for finite time intervals consisting of alternatively parallel lines to the momentum axis for the pure potential Hamiltonian and lines parallel to the position axis for purely kinetic motion.

Geometrically, such a potential behaviour is equivalent to a potential which is limited to differ from zero only in parts of a multiply connected region, where this one could be in general a time- or a space manifold. A quantization of the related motion then requires, in view of the well known flux quantization, that this potential should be a pure gauge potential. Thus, we have a potential which exists and disappears frequently in finite amount of time or in finite regions of space, i.e. in a multiply connected region.

Now the important point is that we have exactly this same potential behaviour in the QHE due to the disturbance and perturbations caused by impurities, so that the well known diagram of quantized Hall-conductivity or resistivity with plateaus is almost of the above mentioned stepped form. Therefore, we claim and show that the QHE can be considered as a realization of the "quantized" chaotic behaviour of the mentioned kind.

Coming back to the description of the mentioned chaotic motion let us remark that, in view of its well known non-regular behaviour, it can be considered partly as a "finite" limit of the usual classical continuous motion in phase space; or equivalently the usual classical regular motion can be considered as a continuum limit of the chaotic "finite" motion. The finiteness or the partial absence of derivative in chaotic motion can appear either with respect to the time- or with respect to the position variable depending on the experimental set up or on the theoretical model which describes the resulting chaotic
motion. Thereby, in view of the absence of derivatives at any point, the chaotic motion should be given by

\[ \Delta q = \frac{p}{\mu} \Delta t, \]
\[ \Delta p = -\frac{\Delta V}{\Delta q} \Delta t, \]

for finite quantities \( \{\Delta q, \Delta p, \Delta V, \Delta t\} \) with \( \mu \) representing the mass of the system, where the first equation describes the purely kinetik part of motion and the second one describes the purely potential part. They can be rewritten for differentiable potential \( V \) but for finite \( \Delta t = T \) and \( \Delta p = p_{n+1} - p_n \) in the following form which is more familiar from Chaos theory [3]:

\[ (a) \quad q_{n+1} - q_n = \frac{p_n T}{\mu} \]
\[ (b) \quad p_{n+1} - p_n = -\frac{\partial V}{\partial q} T, \]

where we used the definitions \( \Delta r := r_{n+1} - r_n \).

It is this finiteness of chaotic motion with finite quantities from the set \( \{\Delta q, \Delta p, \Delta E, \Delta t\} \) which is closely related to the quantum mechanics, if we recall that according to the uncertainty relation we have:

\[ \Delta p \cdot \Delta q = \Delta E \cdot \Delta t \geq \hbar \]  

Here also we have to do with finite magnitudes of momentum, position etc. which can not vanish [5]. The finite quantum structure arises here in view of the fact that \( \Delta E = E_{n+1} - E_n = E_0 = \frac{\hbar \omega}{2} \) where \( \Delta E \) defines the smallest amount of energy which is available quantum mechanically and which can not be undercut. Thus, it determines also the smallest quantum mechanical length and momentum which are considerable and which can not be undercut quantum mechanically. An example of such finite smallest quantum length is the well known magnetic length \( l_B \) which plays an important role just in the QHE [2]. It is the width of a "quantum ring" where the edge currents flow.
Accordingly, if one translates the QHE behaviour into the quantum chaotic terminology, then
\[ \Delta q = q_{n+1} - q_n \]
in the QHE case should be given by the magnetic length
\[
(\Delta q)^2 = l_B^2 = \frac{\hbar}{eB} = \frac{\nu}{2\pi n},
\]
where \( e, \nu \) and \( n \) are the elementary charge, filling factor and carrier concentration respectively.

It is the "unit" length or heights of the ideal stairs, i.e. parallel to the position or to the momentum axis in the phase space diagram according to the chaos theoretic description of QHE, which is given by the relation (4) according to the magnitude of the applied exterior magnetic field \( B \) or \( \frac{\nu}{n} \) in QHE samples.

Such a translation means on the other hand that the chaotic motion (2) is quantized according to the usual quantization methods in Chaos theory. Thus, the QHE behaviour can be translated according to the finiteness of quantum phase space cells into the mentioned finite "quantum" chaotic motion.

After this qualitative remarks we give a potential analysis of the localization-delocalization transition which is responsible for the typical step structure of the quantized Hall resistivity.

First of all let us mention that according to our understanding of IQHE, if one considers the ideal stepped diagram of quantum Hall resistivity or conductivity against the applied magnetic field strength, the plateaus mark positions where the field strength is not present and the vertical parts of diagram mark positions where it is present. It is the same situation as for a repeated Bohm-Aharonov set ups where the field strength is present in some parts but absent in other parts of space, whereby the difference of regions in QHE case is due to the impurities. As mentioned before, geometrically we have to do with a multiply connected region where there are "holes" which are comparable qualitatively with bounded regions of action of potentials or with dots and antidots structure. One can understand the IQHE situation directly from the circumstance that a present field strength or a present potential gives rise to a change of momentum, velocity or current density, whereas the absence of field strength or its potential results in the contanncy of the current density in view of the Ohm’s equations, as in relation (2(b)), in accordance with the above discussed chaotic motion diagram.
Moreover, according to our gauge field theoretical Chern-Simons-Schroedinger model of IQHE, it is the presence of electromagnetic potential in the primary version of current density relation of the classical Hall effect which results in the term proportional to the longitudinal conductivity \[^7\]. These positions in the diagram of longitudinal conductivity are according to the well known joint diagrams of Hall- and longitudinal conductivities in IQHE \[^2\] exactly the positions where the Hall conductivity is in vertical part of its diagram, i. e. in its purely potential domain according to the above discussed chaos terminology \[^1\]. On the other hand, in the plateau regions of Hall conductivity the longitudinal conductivity is almost vanishing \[^8\], which fits also in the terminology of the chaos theory, according to which these positions are the purely kinetic positions.

Thus, the integer quantum Hall conductivity diagram describes a motion in its own phase space which is similar to the motion described by the diagram of chaotic motion in the above discussed phase space, where in both cases the potential is alternatively present and absent in the Hamiltonian which results in the purely potential and purely kinetic motion respectively \[^1\]. To be complet, let us mention that in both cases there are the purely potential part of motion which causes the change in momentum resulting in lines which are parallel to the momentum axis in diagram. Moreover, in both cases there are the purely kinetic part which causes the change in the position resulting in lines which are parallel to the position axis or to the axis which represents the conjugate variable in phase space as a function of position variable, e. g. potential or field strength.

Furthermore, if the potential becomes in the kinetic domain absolutely zero, then we have the exact parallel lines to the position axis, whereas in quantum case the lines can become almost but never exactly parallel in view of the mentioned quantum residuens. Accordingly, in quantum mechanical cases like QHE we have always only almost purely potential or almost purely kinetic Hamiltonian or motion \[^8\] and the diagrams are also modified accordingly.

Of course the phase space of the IQHE system should be considered, also according to our model \[^7\], electrodynamically, i. e. with potential components as phase space variables \[^10\]. Nevertheless, in view of the invariance of the action functional with respect to the canonical transformations in phase space one should red up the phase space variables also from actions like \(e \int A_m dx^m = e \int \int Bds\) with \(m = 1, 2\) and \(s = area\), where \(A_m\) is the electromagnetic potential and \(B\) is the magnetic field strength.
Thereafter, the mentioned stepped diagram of IQHE can be considered, in view of the Ohm’s equations
\[ n e \frac{dx}{dt} = j_m = \epsilon_{nm} \sigma_H E_n = \epsilon_{nm} \sigma_H \frac{dA_n}{dt} \]
with \( \epsilon_{mn} = -\epsilon_{nm} = 1 \), as a chaotic motion between \( x := q \) and \( A := p \) variables which are related by \( \sigma_H \) in the phase space of IQHE or between other equivalent canonically conjugate phase spaces variables. An example of such equivalent set of phase space variables is \( \{ \sigma_H \left( \frac{e^2}{h} \right), \nu \} \) where the diagram in phase space, i.e. \( \sigma_H \) against \( \nu \) is of exact step form, whereby the joint diagram of longitudinal conductivity consists of vertical lines on the \( \nu \) axis [2e].

We mentioned already that from our gauge theoretical point of view [7] the IQHE diagrams could be understood according to the repeated Bohm-Aharonov effects [2] on the IQHE sample, i.e. in view of the presence and absence of potential or its field strength in different space regions, where this presence and absence depend on the constitution of region according to the distribution of the impurities. In other words, a simple model can be thought so that the potential is non-vanishing only in the impurity regions, whereas it is almost vanishing in other parts of sample [3]. However, its line integrals \( \oint A \) in these last parts of region is not zero [4], as it is also known from the Bohm-Aharonov effect. Moreover, the value of integral should increase with the length of circumference of the closed path which is related with the number of antidots inside the path [5]. Such a behaviour is, as already mentioned, the typical behaviour of an almost pure U(1) or electromagnetic gauge potential [4] [8].

It is interesting to mention that the variation of the action functional \( \oint (p - eA) \), which is suggested in Ref. [1] to explain the chaotic behaviour in IQHE, has the same almost pure electromagnetic gauge potential as its solution.

Furthermore, the quantum oscillations of longitudinal resistivity in the \( B \to 0 \) region reported in Ref.[1] which appear in "very low temperature” should arise just due to the quantum mechanical uncertainty relation between energy and time in the following manner, if we recall the above analysis of chaotic motion:

In the case of chaotic motion, if the time \( T \) tends to zero, i.e. \( T \to 0 \), then the succession of the horizontal and vertical lines approximates more and more the usual continuous trajectory which is associated with the regular continuous motion in the phase space arising from a conserved Hamiltonian [3]. However, if \( T \) is finite, then the stairs in the mentioned diagram become revealed in view of the fact that the pure potential and pure kinetic periods are remarkable. On the other hand, according to the uncertainty
relation $\Delta E \cdot \Delta t = \hbar$, very low energy $\Delta E \ll$ or very low temperature is related with a non-vanishing and relatively remarkable $T = \Delta t = \frac{\hbar}{\Delta E \ll}$. Thus, in view of our discussion the stairs which were negligible in the $T \rightarrow 0$ limit become revealed in the new situation. Accordingly, the longitudinal conductivity coming from purely potential positions in the step, i.e. in the vertical lines, becomes also revealed in the same manner as it is discussed above \[7\]. Recall furthermore that, in the ground state of QHE, very low energy is related with very low magnetic field $B$ according to $\Delta E = E_0 = \frac{\hbar e B}{2\mu}$, thus at very low magnetic field a very low energy implies non-vanishing and remarkable period of time where the potential acts before it becomes almost vanishing during the next finite time and so the stairs and the related resonances become revealed (see also Ref. \[3\]). Moreover, such a mechanism could be modeled in the way that the antidots become related with impurities or holes in the repeated Bohm-Aharonov context according to the following consideration:

Although $B$ is very low in the region under consideration, however the magnetic flux $\oint B ds = \oint A$ becomes larger, if the electronic orbits surround bigger areas, i.e. with more antidots. In view of the fact that not $B$ but only invariants like $\oint B ds = \oint A$ are relevant in quantum mechanical set ups as in QHE, one measures in QHE quantities which depend not only on $B$ but on the flux of $B$, as it is reported in Ref. \[1\]. Thus, the smallness of the $B$ value can be compensated by the largeness of the area in $\oint B$ and results in the revealed resonances \[1\].

Furthermore, let us mention that the potentials which are used in Refs. \[1\], \[3\] to model the hard wall billiard and the antidots respectively, are all variations of a U(1) (or electromagnetic) pure gauge potential in a multiply connected region as it is obvious from the action used in Ref. \[1\] mentioned above. It is the non-trivial homotopy of such a multiply connected $2 - D$ region which is also reflected by the non-trivial holonomy of the $U(1)$ bundle over such a region, which becomes manifested by the quantum effects on the mentioned $2 - D$ region and which can be described consistently only according to quantum mechanics.

It should be mentioned at last that just the use of KAM-theorem in QHE case in Ref. \[3\] is a hint in direction of a perturbation which is a quantum mechanical relict and not a true classical mechanical matter. We discuss this question in view of its general application separately \[12\].

Footnotes and references
References

[1] D. Weiss et al., Surface Science, 305, 408, (1994). See also references therein.

[2] For a general review on QHE and its experimental setting see:

[2a] K. v. Klitzing et al., Phys. Rev. Lett., 45, 145, (1980);

[2b] R.E. Prange and S.M. Girvin, ed., The quantum Hall effect, Graduate Texts in Contemporary Physics (Springer, New York, 1987);

[2c] A.H. Macdonald, ed., Quantum Hall effect: A Perspective, Perspectives in Condensed Matter Physics (Kluver Academic Publishers, 1989)

[2d] G. Morandi, The role of Topology in Classical and Quantum Physics, Lecture Notes in Physics m7 (Springer, New York 1992)

[2e] M. Janssen, et al, ed., J. Hajdu, Introduction to the Theory of the Integer Quantum Hall effect (VCH-verlag, Weinheim, New York, 1994)

[2f] V. J. Emery (editor), Correlated Electron Systems, (World Scientific, Singapore 1993)

[2g] J. Froehlich, T. Kerler, Nuc. Phys. B354 (1991) 369-417.

[3] See N. L. Balazs, Seminar 1 " Quantized maps" in: Chaos and Quantum Physics, Les Houches 1989-Session LII, editors: M.-J. Giannoni et al, ( North-Holland 1991).

[4] Considering the usual flux quantization according to $\oint \mathbf{A} = \int \int B ds = \frac{\hbar}{e} \Phi$ with $\Phi = 2\pi N$, $N \in \mathbb{Z}$ it is obvious that the potential is given by $A_m = \partial_m \Phi$ (see also Ref. [2]) with $A := A_m dx^m$. Of course, the $F_{mn}(A_m) = B$ is outsides of the area $s$ zero as in the Bohm-Aharonov case. Nevertheless, in a multiply connected region one has $\oint A \neq 0$, as it is known also from the Bohm-Aharonov case.

Furthermore, recall that 1) one should consider the multiply connected region in phase space which is, in general, not the same as the position space and 2) that the multiply connectedness of a region is related with the multiply valuedness of fuctions defined thereon.
[5] Recall also that one can extract from the relation (3) directly part of the above mentioned relations (1) for chaotic motion, if one consider \( \mu \Delta E = p \Delta p := \frac{1}{2} \Delta (p^2) \).

[6] See M. C. Gutzwiller in Ref. [3].

[7] The term mentioned in the text is the usual minimal coupling \( \frac{e}{m} \psi^* A_m \psi \) which becomes \( \sigma_L E_m \) in the \( A_m = E_m \tau \) gauge, if we use \( \sigma_L \approx \sigma_0 \) as usual in the \( \omega_c \tau \ll 1 \) limit. For further details see: F. Ghaboussi, "On the Integer Quantum Hall Effect", KN-UNI-preprint-95-1; "A Model of the Integer Quantum Hall Effect", KN-UNI-preprint-95-2, submitted for publication; See also "On the Hall-Effect and its Quantization", KN-UNI-preprint-95-3, submitted for publication; "The Edge Currents in IQHE", KN-UNI-preprint-95-4, submitted for publication; "On The Relation between Quantum Hall Effect and Superconductivity", KN-UNI-preprint-95-5, submitted for publication. All are available from quant-ph@xxx.lanl.gov.

[8] One must take into account that despite of classical physics in quantum physics one has in view of the uncertainty relations finite quantities, e.g. in ground states, which prevent related relevant quantities to become absolutely zero. To these relevant quantities in the QHE case it belongs the electromagnetic potential or its field strength. Equivalently, in quantum mechanics only global quantities \( \oint A = \int \int B ds \) are relevant but not their local components \( A \) or \( B \). For a constant non-vanishing value of a pure gauge potential the value of its closed path integral becomes proportional to the length of the circumference of the path.

Moreover, also the question of absolute zero temperature is related with this same quantum mechanical ground state circumstance.

[9] Recall also that, in accordance with the discussed diagram of chaotic motion, the field strength \( B = F_{mn} \) in the vertical parts of diagram have constant values as it is obvious in a diagram with \( B \) axis as its horizontal axis.

[10] See for potential components as phase space variables: E. Witten, Cumm. Math. Phys. 121 (1989) 351-399. See also F. Ghaboussi in Ref. [3].

[11] R. Fleischmann et al., Phys. Rev. Lett. 68, 1367, (1992); Europhys. Lett., 25(3), 219, (1994).
[12] under preparation.