0. Introduction. Let $X$ be a non-singular, connected, projective curve defined over an algebraically closed field $k$. Let $U^s$ denote the set of isomorphism classes of stable vector bundles on $X$ with given degree $d$ and rank $r$. In the sixties, C.S. Seshadri and D. Mumford ([14], [17] and [18]) supplied $U^s$ with a natural structure of quasi-projective variety, together with a natural compactification $U$, by adding semistable vector bundles at the boundary. The method used in the construction of such a structure was Mumford’s then recently developed Geometric Invariant Theory [8]. Roughly, the method consists of producing a variety $R$, and an action of a reductive group $G$ on $X$, linearized at some ample invertible sheaf $L$ on $R$, such that $U = R/G$ set-theoretically. Then, Geometric Invariant Theory (G.I.T.) tells us how to supply $R/G$ with a natural scheme structure, obtained from the $G$-invariant sections of tensor powers of $L$.

Up until recently, Seshadri’s and Mumford’s construction was the only purely algebraic construction available. In 1993, Faltings [7] showed how to construct $U^s$, and its compactification $U$, avoiding G.I.T.. His method, described also in [20], consisted in considering the so-called theta functions on $R$, naturally defined provided $R$ admits a family with the so-called local universal property. (We observe that the theta functions considered in this article are just those associated with vector bundles on $X$, as it will be clear from our definition in Sect. 2. Beauville [3, Sect. 2] has a more encompassing definition of theta functions than ours.) The theta functions are in fact $G$-invariant sections of tensor powers of a certain $G$-linear invertible sheaf $L_\theta'$ on $R$. Roughly speaking, using his first main lemma [20, Lemma 3.1, p. 166], Faltings showed that there are enough theta functions to produce a $G$-invariant morphism, $\theta: R \to \mathbb{P}^N$. By semistable reduction, the image,

$$U_\theta := \theta(R) \subseteq \mathbb{P}^N,$$

is a closed subvariety. Since $\theta$ is $G$-invariant, then $\theta$ factors (set-theoretically) through a map, $\pi: U = R/G \to U_\theta$. Then, using his second main lemma [20, Lemma 4.2, p. 174], Faltings showed that there is a bijection between the normalization of $U_\theta$ and $U$, through which $U$ acquires a natural structure of projective variety.

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There are interesting consequences of Faltings’ work; for instance, we can get from his first main lemma a cohomological characterization of semistable bundles (cf. Thm. 2 or [20, Thm. 6.2, p. 187]).

In his description of Faltings’ construction, Seshadri posed the following question ([20, Rmk. 6.1, p. 187]): Let $L_\theta$ denote the ample sheaf on $U$ lying below $L'_\theta$. The theta functions of powers of $L'_\theta$ descend to sections of powers of $L_\theta$, and span a graded subalgebra $A_\theta$ of

$$B_\theta := \bigoplus_{m \geq 0} \Gamma(U, L_\theta^\otimes m).$$

How close is $A_\theta$ to $B_\theta$? Lest the latter question is too difficult, we can think more geometrically, and rephrase it: how close is the normalization map,

$$\pi: U \rightarrow U_\theta,$$

to being an isomorphism, if we consider all theta functions? In other words, how much of the moduli space $U$ can theta functions describe? This question arises naturally from Faltings’ construction, and is relevant: if $U = U_\theta$, then we would have a better handle on a canonical projective embedding of $U$.

The goal of the present article is to provide a partial answer to Seshadri’s question. Our main technical lemma (Lemma 4) is a generalization of Faltings’ first main lemma. From Lemma 4, we obtain a cohomological characterization of stability (Thm. 6), allowing us to obtain a quick proof of the fact that $U$ is a fine moduli space if the degree $d$ and rank $r$ are coprime (Cor. 8). Our proof of this fact avoids completely G.I.T.. Using Lemma 4, we obtain separation lemmas (Lemma 9 and Lemma 11), allowing us to show that $\pi$ is bijective (Thm. 15) and (if the characteristic of the ground field $k$ is 0) an isomorphism over $U^s$ (Thm. 18).

Actually, motivated by the recent interest in the compactification of relative Jacobians over families of singular curves ([4], [16] and the more general [21]), and to provide support to [6], I have tried to be as general as possible. So, all the results of the present paper (with the exception of Thm. 18) apply to a general projective, connected, reduced curve $X$, with $n$ irreducible components, defined over an algebraically closed field $k$. In [19, Part 7], Seshadri constructed moduli spaces $U(a, m, \chi)$ of torsion-free sheaves with multirank $m = (m_1, \ldots, m_n)$ and Euler characteristic $\chi$ that are semistable with respect to a polarization $a = (a_1, \ldots, a_n)$. In this article, we consider theta functions on $U(a, m, \chi)$ and ask “Seshadri’s question”: how close is the rational map,

$$\pi: U(a, m, \chi) \rightarrow U_\theta(a, m, \chi),$$

obtained from all theta functions, to being an isomorphism? If $n = 1$ and $m = 1$, then $\pi$ has already been shown to be an isomorphism in [5]. In general, our best results to this extent are Thm. 15 and Thm. 16. In particular,
if \( \underline{m} = (1, \ldots, 1) \) (the compactified Jacobian case), then we obtain that \( \pi \) is bijective and an isomorphism over the stable locus, \( U^s(\underline{a}, \underline{m}, \chi) \subseteq U(\underline{a}, \underline{m}, \chi) \).

1. Preliminaries. Let \( X \) be a curve, that is, a projective, connected, reduced scheme of pure dimension 1 over an algebraically closed field \( k \). Let \( X_1, \ldots, X_n \) denote the irreducible components of \( X \). All schemes are assumed to be locally of finite type over \( k \). By a point we mean a closed point. By a vector bundle we mean a locally free sheaf of constant rank.

If \( H \) is a coherent sheaf on \( X \), we let \( r_H \) denote its multirank; more precisely, \( r_H := (r_1(H), \ldots, r_n(H)) \), where \( r_i(H) \) is the generic rank of \( H \) on \( X_i \) for every \( i \). We let \( r_H \) denote the maximum generic rank of \( H \). If \( E \) is a vector bundle on \( X \), we let \( d_E \) denote its multidegree, that is, \( d_E := (\deg_{X_1} E, \ldots, \deg_{X_n} E) \).

If \( H \) is a coherent sheaf, and \( E \) is a vector bundle on \( X \), then \( \chi(E \otimes H) = r_E \chi(H) + d_E \cdot \underline{r}_H \).

In particular, note that \( \chi(E \otimes H) \) depends only on the rank and multidegree of \( E \), rather than on \( E \) itself.

A torsion-free sheaf on \( X \) is a coherent sheaf with no embedded components. The dualizing sheaf of \( X \), denoted \( \omega \), is torsion-free [1, (6.5), p. 95]. Given any coherent sheaf \( H \), we let \( H^\omega := \text{Hom}_X(H, \omega) \). Since \( \omega \) is torsion-free, then so is \( H^\omega \). There is a natural surjective homomorphism \( H \rightarrow H^\omega \), whose kernel is the torsion subsheaf of \( H \).

Let \( S \) be a scheme. A coherent \( S \)-flat sheaf \( I \) on \( X \times S \) is called relatively torsion-free over \( S \) if the fibre \( I(s) \) is torsion-free for every \( s \in S \). Given any coherent sheaf \( H \) on \( X \times S \), we let \( H^\omega := \text{Hom}_{X \times S}(H, \omega \otimes \mathcal{O}_S) \).

If \( I \) is relatively torsion-free on \( X \times S \) over \( S \), then it follows from [1, (1.9), p. 59] and [1, (6.5), p. 95] that \( I^\omega \) is also relatively torsion-free, with \( I^\omega(s) = I(s)^\omega \) for every \( s \in S \). Anyhow, if \( \mathcal{H} \) is a coherent sheaf on \( X \times S \), it follows from loc. cit. that there is an open dense subscheme \( S' \subseteq S \) such that \( \mathcal{H}^\omega|_{X \times S'} \) is relatively torsion-free with \( \mathcal{H}^\omega(s) = \mathcal{H}(s)^\omega \) for every \( s \in S' \).

If \( L \) is an ample invertible sheaf on \( X \), we let \( P_H := P_H(T) \) denote the Hilbert polynomial of a coherent sheaf \( H \) on \( X \) with respect to \( L \). Using the dualizing properties of \( \omega \), we get that \( P_H(T) = -P_{H^\omega}(-T) \) for every coherent sheaf \( H \) on \( X \).

2. Theta functions. Let \( S \) be a scheme. Let \( \mathcal{F} \) be a coherent sheaf on \( X \times S \) that is flat over \( S \). The determinant of cohomology of \( \mathcal{F} \) over \( S \) is the invertible sheaf \( \mathcal{D}({\mathcal{F}}) \) on \( S \) constructed as follows: locally on \( S \) there is a complex,

\[
0 \rightarrow G^0 \xrightarrow{\lambda} G^1 \rightarrow 0,
\]
of free sheaves of finite rank such that, for every coherent sheaf \( M \) on \( S \), the cohomology groups of \( G^\bullet \otimes M \) are equal to the higher direct images of \( F \otimes M \) under the projection \( p : X \times S \to S \). The complex \( G^\bullet \) is unique (up to unique quasi-isomorphism). Hence, its determinant,

\[
\det G^\bullet := ( \bigwedge^{\text{rank } G^1} G^1 ) \otimes ( \bigwedge^{\text{rank } G^0} G^0 )^{-1},
\]

is unique (up to canonical isomorphism). The uniqueness allows us to glue together the local determinants to obtain the invertible sheaf \( D(F) \) on \( S \).

The most important properties of the determinant of cohomology are:

(a) **Additive property**: If

\[
\alpha : 0 \to F_1 \to F_2 \to F_3 \to 0
\]

is a short exact sequence of \( S \)-flat coherent sheaves on \( X \times S \), then there is a naturally associated isomorphism:

\[
D_\alpha : D(F_2) \cong D(F_1) \otimes D(F_3).
\]

(b) **Projection property**: If \( L \) is a line bundle on \( S \), and \( \chi(F(s)) = d \) for every \( s \in S \), then there is a naturally associated isomorphism:

\[
D_L : D(F \otimes L) \cong D(F) \otimes L^\otimes d.
\]

(c) **Base-change property**: If \( \nu : T \to S \) is any morphism, then there is a naturally associated base-change isomorphism:

\[
D_\nu : D((\text{id}_X, \nu)^* F) \cong \nu^* D(F).
\]

For a more systematic development of the theory of determinants, see \([11]\). It is also possible to adopt a more concrete approach to define \( D(F) \), like the one used in \([2, \text{Ch. IV, \S3}]\).

If \( F \) is an \( S \)-flat coherent sheaf on \( X \times S \) with \( \chi(F(s)) = 0 \) for every \( s \in S \), then there is a canonical global section \( \sigma_F \) of \( D(F) \) that is constructed as follows: since \( \chi(F(s)) = 0 \) for every \( s \in S \), then the ranks of \( G^0 \) and \( G^1 \) in the local complex \( G^\bullet \) are equal. By taking the determinant of \( \lambda \) we obtain a section of \( \det G^\bullet \). Since the complex \( G^\bullet \) is unique, such section is also unique, allowing us to glue the local sections to obtain \( \sigma_F \). The zero locus of \( \sigma_F \) on \( S \) parametrizes the points \( s \in S \) such that

\[
h^0(X, F(s)) = h^1(X, F(s)) = 0.
\]

(Another way of viewing \( \sigma_F \) is as a generator of the 0-th Fitting ideal of \( R^1 p_* F \).)

The global section \( \sigma_F \) satisfies properties compatible with those of \( D(F) \). For instance, we have the additive property: if

\[
\alpha : 0 \to F_1 \to F_2 \to F_3 \to 0
\]
is a short exact sequence of $S$-flat coherent sheaves on $X \times S$ of relative Euler characteristic 0 over $S$, then

$$\sigma_{F_2} = \sigma_{F_1} \otimes \sigma_{F_3}$$

under the identification given by $D_\alpha$. We leave it to the reader to state the projection and base-change properties of the global sections $\sigma_F$.

Let $E$ be a vector bundle on $X$. Let $S$ be a scheme, and $\mathcal{H}$ be an $S$-flat coherent sheaf on $X \times S$. Assume that $\chi(E \otimes H(s)) = 0$ for every $s \in S$. We define:

$$L_E(H) := D(E \otimes H) \quad \text{and} \quad \theta_E(H) := \sigma_E \otimes H.$$  

The line bundle $L_E(H)$ is called a theta line bundle, and $\theta_E(H)$ is called its theta function. We let $\Theta_E(H) \subseteq S$ denote the zero-scheme of $\theta_E(H)$, and call it a theta divisor.

**Lemma 1.** (Faltings) Let $S$ be a scheme. Let $I$ and $J$ be $S$-flat coherent sheaves on $X \times S$. Let $E$ and $F$ be vector bundles on $X$. Assume that:

(a) $\chi(I(s)) = \chi(J(s))$ and $r_{I(s)} = r_{J(s)}$ for every $s \in S$;
(b) $r_E = r_F$ and $\det E \cong \det F$.

Then, there is a canonical isomorphism,

$$\Phi_{F,E} : D(I \otimes F) \otimes D(J \otimes E) \cong D(I \otimes E) \otimes D(J \otimes F),$$

whose formation commutes naturally with base change. In addition, $\Phi_{F,E}$ is additive on $E$ and $F$, in the following sense: If

$$\alpha : 0 \to E_1 \to E_2 \to E_3 \to 0 \quad \text{and} \quad \beta : 0 \to F_1 \to F_2 \to F_3 \to 0$$

are short exact sequences of vector bundles on $X$ such that $r_{E_i} = r_{F_i}$ and $\det E_i \cong \det F_i$ for $i = 1, 2, 3$, then

$$\left(\Phi_{F_1,E_1} \otimes \Phi_{F_3,E_3}\right) \circ (D_{\beta \otimes I} \otimes D_{\alpha \otimes J}) = (D_{\alpha \otimes I} \otimes D_{\beta \otimes J}) \circ \Phi_{F_2,E_2}.$$  

**Proof.** It follows from (a) that the sheaves $I$ and $J$ are generically isomorphic, hence coincide generically in $K$-theory, in the sense defined by Faltings in [7, p. 509]. Thus, the lemma follows directly from Faltings’ [7, Thm. I.1, p. 509]. Though not explicitly stated in Faltings’ theorem, it follows from its proof that the homomorphism $\Phi_{F,E}$ is also additive on $I$ and $J$, in a sense that will be left to the reader to state. $\square$

Let $S$ be a scheme. Let $\mathcal{H}$ be an $S$-flat coherent sheaf on $X \times S$ with constant relative Euler characteristic and multirank over $S$. Let $H_0$ be a coherent sheaf on $X$ such that $\mathcal{L}_{H_0} = \mathcal{L}_{\mathcal{H}(s)}$ and $\chi(H_0) = \chi(\mathcal{H}(s))$ for every $s \in S$. Let $E$ be a vector bundle on $X$. Fix an isomorphism $D(H_0 \otimes E) \cong k$. It follows from Lemma
that, given a vector bundle $F$ on $X$, with $r_F = tr_E$ and $\det F \cong (\det E)^{\otimes t}$ for some $t > 0$, and an isomorphism $\mathcal{D}(H_0 \otimes F) \cong k$, there is a canonical isomorphism,

$$\mathcal{D}(\mathcal{H} \otimes F) \cong \mathcal{D}(\mathcal{H} \otimes E)^{\otimes t}. \quad (1.1)$$

Of course, the isomorphism in (1.1) depends on the choice of $H_0$.

Assume now that $\chi(E \otimes \mathcal{H}(s)) = 0$ for every $s \in S$. If $F$ is a vector bundle on $X$ with $r_F = tr_E$ and $\det F \cong (\det E)^{\otimes t}$ for some $t > 0$, then we may regard $\theta_F(\mathcal{H})$ as a section of $L_E(\mathcal{H})^{\otimes t}$ under the isomorphism in (1.1). The induced section $\theta_F(\mathcal{H}) \in H^0(S, L_E(\mathcal{H})^{\otimes t})$ is well defined modulo $k^*$. For every $t > 0$, let

$$V^t_E(\mathcal{H}) \subseteq H^0(S, L_E(\mathcal{H})^{\otimes t})$$

be the subvectorspace generated by the sections $\theta_F(\mathcal{H})$. We set $V^0_E(\mathcal{H}) := k$. It follows from the additive properties of the determinant of cohomology and its associated global section, and Lemma 1, that, if

$$0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$$

is a short exact sequence of vector bundles on $X$, such that $r_{F_i} = t_ir_E$ and $\det F_i \cong (\det E)^{\otimes t_i}$ for $i = 1, 2$, then

$$\theta_F(\mathcal{H}) = \theta_{F_1}(\mathcal{H}) \otimes \theta_{F_2}(\mathcal{H}) \quad (\text{modulo } k^*)$$

inside $H^0(S, L_E(\mathcal{H})^{\otimes (t_1+t_2)})$. Therefore, the tensor-product multiplication in

$$\Gamma_E(\mathcal{H}) := \bigoplus_{t \geq 0} H^0(S, L_E(\mathcal{H})^{\otimes t})$$

restricts to a multiplication in

$$V_E(\mathcal{H}) := \bigoplus_{t \geq 0} V^t_E(\mathcal{H}),$$

showing that $V_E(\mathcal{H})$ is a graded $k$-subalgebra of $\Gamma_E(\mathcal{H})$. The ring $V_E(\mathcal{H})$ is called a ring of theta functions.

3. The main lemma. Fix a vector bundle $E$ on $X$ of rank $r > 0$ and multidegree $d$. A non-zero torsion-free sheaf $I$ on $X$ is called semistable (resp. stable) with respect to $E$ if:

(a) $\chi(I \otimes E) = 0$;

(b) $\chi(K \otimes E) \leq 0$ (resp. $\chi(K \otimes E) < 0$) for every proper subsheaf $K \subsetneq I$.

The notions of stability and semistability are numerical, depending on the multislope $\mu := \frac{d}{r}$ of $E$, rather than on $E$ itself. We call $\mu$ the polarization. We shall fix $E$, and the ensuing polarization, for the remainder of the article.
If $X$ is irreducible, then a torsion-free sheaf $I$ on $X$ is semistable (resp. stable) in the usual sense if and only if $I$ is semistable (resp. stable) with respect to any (hence every) non-zero vector bundle $E$ on $X$ with $\chi(I \otimes E) = 0$.

Given any semistable sheaf $I$ on $X$, we can construct a Jordan-Hölder filtration,

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{q-1} \subsetneq I_q = I,$$

defined to be a filtration where each quotient $J_s := I_s/I_{s-1}$ is torsion-free and stable, for $s = 1, \ldots, q$. The above filtration is not unique, but its graded sheaf,

$$\text{Gr}(I) := J_1 \oplus J_2 \oplus \cdots \oplus J_q,$$

is unique by the Jordan-Hölder Theorem.

The following theorem can be thought of as a cohomological characterization of semistable sheaves.

**Theorem 2.** (Faltings) Let $I$ be a non-zero torsion-free sheaf on $X$. Then, $I$ is semistable if and only if there is a non-zero vector bundle $F$ on $X$ such that:

(a) $r_{dF} = r_{Fd}$;

(b) $h^0(X, I \otimes F) = h^1(X, I \otimes F) = 0$.

More precisely, if $I$ is semistable, and $\{L_t|\ t > 0\}$ is a sequence of invertible sheaves on $X$ with $d_{L_t} = t\underline{d}$ for every $t > 0$, then there is a vector bundle $F$ on $X$ such that, in addition to (a) and (b):

(c) $r_F = tr$ and $\det F \cong L_t$ for a certain $t > 0$.

**Proof.** As in [20, Lemma 3.1, p. 166] for the “only if” part of the first statement, and [20, Lemma 8.3, p. 195] for the “if” part. The last statement can be proved from the first using [5, Lemma 4] and the argument in the proof of [5, Thm. 5]. □

**Remark 3.** As it could be expected, there is an upper bound $R$ for the rank of $F$ in the statements of the above theorem, and this bound depends only on the numerical invariants attached to the statements. (Similar remark can be made regarding every statement in this article that asserts the existence of a vector bundle on $X$ with certain properties.) The existence of $R$ follows easily from the fact that the family of all semistable sheaves on $X$ is bounded. In the case where $X$ is non-singular, Le Potier [13] was the first to give an explicit bound $R$. His bound was later improved by Hein [10]. In the general case, one can expect the bound to depend on the kind of singularities, as it was the case for the compactified Jacobian (cf. [5]). Anyhow, these general bounds seem to be far from sharp (see Beauville’s survey article [3]).

A $n$-uple of integers $\underline{\epsilon}$ is called a deformation of the polarization given by $E$ if there is an integer $m$ with $m\underline{d} \equiv \underline{\epsilon} \mod r$. Equivalently, $\underline{\epsilon}$ is a deformation of the polarization if and only if there is a vector bundle $F$ on $X$ such that
We say that \( \varepsilon \) is non-negative if its components are non-negative. Note that, if \( \varepsilon \) is a deformation of the polarization, and \( I \) is a coherent sheaf on \( X \) with \( \chi(I \otimes E) = 0 \), then \( \varepsilon \cdot r_I \) is a multiple of \( r \).

Let \( S \) be a scheme, and \( \mathcal{F} \) an \( S \)-flat coherent sheaf on \( X \times S \). We say that \( \mathcal{F} \) is complete over \( S \) if the Kodaira-Spencer map,

\[
\delta_s: T_{S,s} \to \text{Ext}^1_X(F(s), \mathcal{F}(s)),
\]

of \( \mathcal{F} \) at \( s \) is surjective for every \( s \in S \). As remarked in the proof of [7, Thm. I.2, p. 514], it is easy to show that, given any vector bundle \( F \) on \( X \), there are a connected, non-singular scheme \( S \) and a vector bundle \( F \) on \( X \times S \) such that \( F \) is complete over \( S \), and \( F(s) \cong F \) for a certain \( s \in S \).

**Lemma 4.** Let \( I_1, \ldots, I_q \) be a finite collection of non-isomorphic stable sheaves on \( X \). Let \( \varepsilon \) be a deformation of the polarization such that \( \varepsilon \cdot r_{I_j} \geq 0 \) for \( j = 1, \ldots, q \). Then, there is a vector bundle \( F \) on \( X \) such that:

(a) \( \varepsilon = r_{F}d - rd_{F} \);
(b) \( \text{H}^0(X, I_j \otimes F) = 0 \) for \( j = 1, \ldots, q \);
(c) the natural homomorphism,

\[
\bigoplus_{j=1}^{q} (I_j \otimes \text{H}^1(X, I_j \otimes F)^*) \to F^* \otimes \omega,
\]

is injective with torsion-free cokernel.

**Proof.** Applying Thm. 2 to \( I_1 \oplus \cdots \oplus I_q \), we obtain a vector bundle \( G \) on \( X \) such that \( rd_{G} = rd_{F} \) and

\[
\text{H}^0(X, I_j \otimes G) = \text{H}^1(X, I_j \otimes G) = 0 \quad (4.1)
\]

for \( j = 1, \ldots, q \).

Let \( G_1 \) be a vector bundle on \( X \) with \( rd_{G_1} = rd_{G} \). Let \( L \) be an ample invertible sheaf on \( X \) such that \( G \otimes L \) and \( G_1 \otimes L \) are generated by global sections. Let \( S_1 \) be a smooth, connected scheme, and \( \mathcal{G}_1 \) be a vector bundle on \( X \times S_1 \), complete over \( S_1 \), such that \( \mathcal{G}_1(s) \cong G_1 \) for a certain \( s \in S_1 \). Replacing \( S_1 \) by an open dense subscheme, if necessary, we may assume that, for every \( s \in S_1 \), the sheaf \( \mathcal{G}_1(s) \otimes L \) is generated by global sections, and \( h^1_j := \text{h}^0(X, I_j \otimes \mathcal{G}_1(s)) \) is constant for every \( j \). Let \( G_2 := \mathcal{G}_1(s_1) \oplus G \), for any fixed \( s_1 \in S_1 \). Applying the above procedure to \( G_2 \), in the place of \( G_1 \), and then repeatedly, we obtain a sequence, \((S_m, \mathcal{G}_m)\), of smooth, connected schemes \( S_m \) and vector bundles \( \mathcal{G}_m \) on \( X \times S_m \) such that, for every \( m > 0 \), the sheaf \( \mathcal{G}_m \) is complete over \( S_m \);

\[
h^m_j := \text{h}^1(X, I_j \otimes \mathcal{G}_m(s))
\]

does not depend on \( s \in S_m \), for every \( j \), and \( \mathcal{G}_m(s) \otimes L \) is generated by global sections for each \( s \in S_m \). Actually, it follows from our construction and (4.1)
that \( h_j^{m+1} \leq h_j^m \) for every \( m > 0 \) and \( j = 1, \ldots, q \). Thus, reenumerating the
sequence \( (S_m, G_m) \), if necessary, we may assume that \( h_j := h_j^m \) does not depend
on \( m > 0 \). As in the proof of [20, Lemma 3.1, p. 166], we have that the bilinear
composition map,

\[
\text{Hom}_X(G_m^*(s), I_j) \times \text{Hom}_X(I_j, G_m^*(s) \otimes \omega) \rightarrow \text{Hom}_X(G_m^*(s), G_m^*(s) \otimes \omega), \quad (4.2)
\]
is zero for every \( m > 0 \), every \( s \in S_m \) and \( j = 1, \ldots, q \).

For every \( m > 0 \) and every \( j \), let \( V_j^m := R^1p_m^*(I_j \otimes G_m) \), where we denote
by \( p_m : X \times S_m \rightarrow S_m \) the projection map. Since \( S_m \) is reduced, it follows
from the properties of \( G_m \) that \( V_j^m \) is locally free of rank \( h_j \). (We may actually
assume that \( V_j^m \) is free.) Consider the canonical homomorphism,

\[
\lambda_m : \bigoplus_{j=1}^q (I_j \otimes (V_j^m)^*) \rightarrow G_m^* \otimes \omega,
\]
for every \( m > 0 \). Replacing \( S_m \) by an open dense subscheme, if necessary, we may assume that the cokernel \( C_m \) of \( \lambda_m \) is flat over \( S_m \). Let \( J_m \) denote the
image of \( \lambda_m \). Then, \( J_m \) is a relatively torsion-free quotient of \( \bigoplus(I_j \otimes (V_j^m)^*) \)
over \( S_m \). In fact, for every \( s \in S_m \), the sheaf \( J_m(s) \) is the smallest subfunctor
of \( G_m^*(s) \otimes \omega \) through which all homomorphisms \( I_j 
\rightarrow G_m^*(s) \otimes \omega \), for all \( j \),
factor. The sheaf \( C_m \) is not necessarily relatively torsion-free over \( S_m \), but we may assume that \( C_m^\omega \) is (see Sect. 1.1). Moreover, we may assume that \( C_m^\omega \)
is the largest torsion-free quotient of \( C_m(s) \) for every \( s \in S_m \). Let \( K_m \subseteq G_m^* \otimes \omega 
\)
denote the kernel of the composition,

\[
G_m^* \otimes \omega \rightarrow C_m \rightarrow C_m^\omega,
\]
for every \( m > 0 \). Then, \( K_m \) is a relatively torsion-free sheaf on \( X \times S_m \) over \( S_m \).
Moreover, it is clear that \( J_m \subseteq K_m \), with \( S_m \)-flat quotient \( K_m/J_m \) of relative
finite length over \( S_m \).

Let \( s \in S_m \). Since \( J_m(s) \) is a quotient of

\[
I := \bigoplus_{j=1}^q I_j^{\oplus h_j},
\]
then there is a lower bound \( a \), independent of \( m \) and \( s \in S_m \), for \( \chi(K_m(s)) \).
On the other hand, since \( K_m(s) \subseteq G_m^*(s) \otimes \omega \) with torsion-free quotient, and
\( G_m(s) \otimes L \) is generated by global sections, then there is an upper bound \( A \),
independent of \( m \) and \( s \in S_m \), for \( \chi(K_m(s)) \). To conclude, there are finitely
many Hilbert polynomials that \( K_m(s) \) can have. More precisely, the Hilbert
polynomial \( P_m(T) \) of \( K_m(s) \) with respect to \( L \) is of the form:

\[
P(T) = d + Td_L \cdot \tau, \quad \text{where } a \leq d \leq A \text{ and } 0 \leq \tau \leq \tau_L. \quad (4.3)
\]
Let \( Q \subseteq \text{Quot}_{f, \omega} \) denote the subscheme of Grothendieck's Quot-scheme which parametrizes quotients of \( I^\omega \) with Hilbert polynomial \( P(-T) - P_l(-T) \) with respect to \( L \), where \( P(T) \) ranges through the polynomials of the form (4.3). Let \( e := \dim Q \). Note that \( e \) depends only on \( I \) and \( L \). Since \( Y^m_j \) is free, choosing a basis of \( Y^m_j \) for every \( j \) we get that the induced embedding \( K^\omega_m \subset I^\omega \otimes O_{S_m} \) defines a morphism \( g_m : S_m \to Q \). As in the proof of [20, Lemma 3.1, p. 166], since \( G_m \) is complete over \( S_m \), we have that the image of the bilinear composition map, 

\[
\text{Hom}_X(G^*_m(s), K_m(s)) \times \text{Hom}_X(K_m(s), G^*_m(s) \otimes \omega) \to \text{Hom}_X(G^*_m(s), G^*_m(s) \otimes \omega),
\]

is contained in a subspace of dimension at most \( e \), for every \( s \in S_m \). Therefore, since \( K_m(s) \subseteq G^*_m(s) \otimes \omega \), then \( h^0(X, K_m(s) \otimes G_m(s)) \leq e \), and hence:

\[
\chi(K_m(s) \otimes G_m(s)) \leq e \tag{4.4}
\]

for every \( m > 0 \) and \( s \in S_m \). On the other hand, it follows from our construction of \((S_m, G_m)\) that

\[
\chi(K_m(s) \otimes G_m(s)) = (\text{rk } G_m/r) \chi(K_m(s) \otimes E) - (1/r) \ell \cdot \ell_{K_m(s)}. \tag{4.5}
\]

for every \( m > 0 \) and \( s \in S_m \). Combining (4.4) and (4.5), since \( \text{rk } G_m \to \infty \) as \( m \to \infty \), we get that \( \chi(K_m(s) \otimes E) = 0 \) for every \( s \in S_m \), if \( m >> 0 \). Since \( K_m(s) \supseteq J_m(s) \), and \( J_m(s) \) is a quotient of the semistable sheaf \( I \), then \( K_m(s) = J_m(s) \), and \( J_m(s) \) is semistable for every \( s \in S_m \), if \( m >> 0 \).

Fix \( s \in S_m \), for \( m >> 0 \). Let \( F := G_m(s) \) and \( J := J(s) \). Of course, \( F \) meets condition (a) in the statement of the lemma. Let (after a choice of bases):

\[
\lambda := \lambda_m(s) : I \to F^* \otimes \omega.
\]

Since \( J := \text{im}(\mu) \) is semistable, and \( I \) is a direct sum of stable sheaves, then (after a new choice of bases):

\[
J \cong \bigoplus_{j=1}^q I_j^{\otimes h'_j}
\]

for certain integers \( h'_j \leq h_j \), and the surjective homomorphism, \( \lambda : I \to J \), has the form:

\[
\lambda = \bigoplus_{j=1}^q (\text{id}_{I_j} \otimes \phi_j),
\]

where \( \phi_j : k^{\otimes h_j} \to k^{\otimes h'_j} \) is a linear surjective homomorphism, for \( j = 1, \ldots, q \). We claim that \( \phi_j \) is injective for \( j = 1, \ldots, q \). Indeed, if there were \( a \in k^{\otimes h_j} \) such that \( \phi_j(a) = 0 \), then the zero homomorphism would be written as a linear combination with coefficients \( a_1, \ldots, a_{h_j} \) of the homomorphisms forming a
basis of $\text{Hom}_X(I_j, F^* \otimes \omega)$. Consequently, $a = 0$. The upshot is that $\phi$ is an isomorphism, thus showing item (c) in the statement of the lemma.

Since $\lambda$ is injective, there is an embedding $I_j \hookrightarrow F^* \otimes \omega$ for $j = 1, \ldots, q$. It follows from the triviality of (4.2) that $H^0(X, I_j \otimes F) = 0$ for $j = 1, \ldots, q$. The proof of the lemma is complete. □

It follows from Thm. 2 and the proof of Lemma 4 that, if $\varepsilon \equiv 0 \mod r$, then we may choose the vector bundle $F$ in the statement of Lemma 4 with $r | r_F$.

4. Quasistable sheaves. Fix a vector bundle $E$ on $X$, of rank $r > 0$ and multidegree $d$, and the ensuing polarization. Let $I$ be a semistable sheaf on $X$. Let $\varepsilon$ be a deformation of the polarization. We say that $I$ is $\varepsilon$-quasistable (with respect to the polarization) if:

(a) $\varepsilon \cdot r_I > 0$;
(b) $\varepsilon \cdot r_J = 0$ for every proper semistable quotient $J$ of $I$.

We observe that an $\varepsilon$-quasistable sheaf is simple, that is, its automorphisms are homotheties.

It is clear that a stable sheaf $I$ is $\varepsilon$-quasistable for every deformation $\varepsilon$ of the polarization such that $\varepsilon \cdot r_I > 0$. Conversely, if a semistable sheaf $\tilde{I}$ is $(r, \ldots, r)$-quasistable, then $\tilde{I}$ is stable.

If $X$ is irreducible, then a semistable sheaf is quasistable if and only if it is stable. Thus, no new concept is being introduced in this case. On the other hand, if $X$ is reducible, then quasistable sheaves do not need to be stable. We will see in [6] that quasistable sheaves are useful in providing a fine (with universal sheaf) compactification of the (generalized) Jacobian of a reducible curve.

Proposition 5. Let $I$ be a semistable sheaf on $X$. Let:

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{q-1} \subsetneq I_q = I$$

be a Jordan-Hölder filtration of $I$. Let $\varepsilon$ be a deformation of the polarization such that $\varepsilon \cdot r_I > 0$. Then, the following statements are equivalent:

(a) $I$ is $\varepsilon$-quasistable.
(b) $I_s$ is $\varepsilon$-quasistable for $s = 1, \ldots, q$.
(c) $\varepsilon \cdot r_{I_1} = \cdots = \varepsilon \cdot r_{I_q}$, and the short exact sequence:

$$0 \to I_{s-1} \to I_s \to I_s/I_{s-1} \to 0$$

is not split for $s = 2, \ldots, q$.

Proof. We assume (a) and prove (b). By descending induction, it is enough to prove that $I_{q-1}$ is $\varepsilon$-quasistable. In fact, let $J$ be a proper semistable quotient of $I_{q-1}$. We must show that $\varepsilon \cdot r_J = 0$. Let $K := \ker(I_{q-1} \to J)$. Since $I_{q-1}$ and $J$ are semistable, then so is $K$. Since $I$ is semistable, then so is $J' := I/K$. 


Of course, we have a natural embedding \( J \subseteq J' \). Let \( J'' := J'/J \). Since \( K \) is a proper subsheaf of \( I \), and \( I \) is \( \varepsilon \)-quasistable, then:

\[
\varepsilon \cdot r_{J''} = \varepsilon \cdot r_{J'} = 0.
\]

Thus, \( \varepsilon \cdot r_J = 0 \), completing the proof of (b).

We assume (b) and prove (c). Since \( I_s \) is \( \varepsilon \)-quasistable for \( s = 1, \ldots, q \), it is clear that

\[
\varepsilon \cdot r_{I_s} = \cdots = \varepsilon \cdot r_{I_q}.
\]

In addition, if the exact sequence in (c) were split for a certain \( s > 1 \), then we would have that \( I_{s-1} \) is a proper semistable quotient of \( I_s \) with \( \varepsilon \cdot r_{I_{s-1}} > 0 \), contradicting the fact that \( I_s \) is \( \varepsilon \)-quasistable. The proof of (c) is complete.

We assume (c) and prove (a). By induction, we may assume that \( I_{q-1} \) is \( \varepsilon \)-quasistable. Let \( \lambda : I \to J \) be a proper surjective homomorphism, where \( J \) is semistable. Assume by contradiction that \( \varepsilon \cdot r_J \neq 0 \). Let \( J' := \lambda(I_{q-1}) \) and \( K := \ker(\lambda) \). Of course, \( J'/J' \) is a quotient of \( I/I_{q-1} \). Since \( I/I_{q-1} \) is stable, then either \( J = J' \) or \( J'/J' \cong I/I_{q-1} \). Either way, since \( \varepsilon \cdot r_J = \varepsilon \cdot r_{I_{q-1}} \), we have that \( \varepsilon \cdot r_{J'} = \varepsilon \cdot r_J \neq 0 \). Since \( I_{q-1} \) is \( \varepsilon \)-quasistable, then \( J' \cong I_{q-1} \). But then, the sequence

\[
0 \to I_{q-1} \to I \to I/I_{q-1} \to 0
\]

splits, contradicting the hypothesis of (c). The proof is complete. \( \square \)

The following theorem can be thought of as a cohomological characterization of \( \varepsilon \)-quasistability, for any deformation \( \varepsilon \) of the polarization. Therefore, the theorem provides a fortiori a characterization of stability.

**Theorem 6.** Let \( I \) be a semistable sheaf on \( X \). Let \( \varepsilon \) be a deformation of the polarization such that \( \varepsilon \cdot r_J > 0 \). If \( I \) is \( \varepsilon \)-quasistable, then there is a vector bundle \( F \) on \( X \) such that:

(a) \( r_F^\perp \neq 0 \);
(b) \( H^0(X, I \otimes F) = 0 \);
(c) the canonical homomorphism,

\[
I \otimes H^1(X, I \otimes F)^* \to F^* \otimes \omega,
\]

is injective.

The converse is true if \( \varepsilon \) is non-negative.

**Proof.** Assume that \( I \) is \( \varepsilon \)-quasistable. Let

\[
0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{q-1} \subseteq I_q = I
\]

be a Jordan-Hölder filtration of \( I \). Let \( J_s := I_s/I_{s-1} \) for \( s = 1, \ldots, q \). Since \( I \) is \( \varepsilon \)-quasistable, it follows from Prop. 5 that \( \varepsilon \cdot r_{J_s} = 0 \) for \( s = 2, \ldots, q \).
Consequently, $\xi \cdot \mathcal{L}_I = \xi \cdot \mathcal{L}_I > 0$. It follows from Lemma 4 that there is a vector bundle $F$ on $X$ such that $r_d F = r_F d - \xi$;

$$h^0(X, J_s \otimes F) = h^1(X, J_s \otimes F) = 0$$
for $s = 2, \ldots, q$;

$$h^0(X, J_1 \otimes F) = 0 \quad \text{and} \quad h^1(X, J_1 \otimes F) = \xi \cdot \mathcal{L}_I / r;$$

and the natural homomorphism,

$$\lambda_1 : I_1 \otimes H^1(X, I_1 \otimes F)^* \to F^* \otimes \omega,$$

is injective. It follows now from the long exact sequences in cohomology associated with the filtration (6.1) that $H^0(X, I \otimes F) = 0$. We shall prove by induction on $q$ that the canonical homomorphism (denoted henceforth by $\lambda$) of (c) is injective. The initial induction step ($q = 1$) is given by the injectivity of $\lambda_1$. We may now assume that the natural homomorphism,

$$\lambda_{q-1} : I_{q-1} \otimes H^1(X, I_{q-1} \otimes F)^* \to F^* \otimes \omega,$$

is injective. Note first that

$$H^1(X, I_1 \otimes F) = \cdots = H^1(X, I_{q-1} \otimes F) = H^1(X, I \otimes F). \quad (6.2)$$

Choose a basis for $H^1(X, I \otimes F)$. Then, we may view $\lambda$ in the form:

$$\lambda : I^c \to F^* \otimes \omega,$$

where $c := \xi \cdot \mathcal{L}_I / r$. It follows from (6.2) that $\lambda_{q-1}$ is the composition of the embedding $I^c_{q-1} \to I^c$ with $\lambda$. Hence, by induction hypothesis, $\lambda$ is injective on $I^c_{q-1}$. Let $K := \lambda(I^c)$ and $K' := \lambda(I^c_{q-1})$. Then, $\lambda$ induces a surjective homomorphism, $\rho : J^c q \to K / K'$. Since $J_q$ is stable, choosing a different basis for $H^1(X, I \otimes F)$, if necessary, we may assume that:

$$\rho = (\rho_1, \rho_2) : J^c_q \oplus J^c_{q-1} \to K / K',$$

where $\rho_1$ is an isomorphism, and $\rho_2 = 0$. Thus, $\lambda$ is injective on $I^c_q \oplus I^c_{q-1}$. We need only prove that $c_2 = 0$. Assume, by contradiction, that $c_2 \neq 0$. Then, there is a non-zero homomorphism $\mu : I \to F^* \otimes \omega$ such that $\mu(I) = \mu(I_{q-1})$. Since $\mu$ is injective on $I_{q-1}$, then the exact sequence,

$$0 \to I_{q-1} \to I \to J_q \to 0,$$

splits. We obtain a contradiction with Prop. 5. The proof of the first statement of the theorem is complete.
Assume now that there is a vector bundle $F$ on $X$ meeting the three conditions in the statement of the theorem. Let $J$ be a proper torsion-free quotient of $I$. It follows from (c) that $H^1(X, J \otimes F) = 0$. Hence, $\chi(J \otimes F) \geq 0$. On the other hand, it follows from (a) that

\[ r \chi(J \otimes F) = r_F \chi(J \otimes E) - \varepsilon \cdot \mu_J. \]

Combining the last two statements, we obtain that $\varepsilon \cdot \mu_J \leq r_F \chi(J \otimes E)$. Now, if $J$ is semistable, then $\chi(J \otimes E) = 0$ and, consequently, $\varepsilon \cdot \mu_J \leq 0$. Assuming that $\varepsilon$ is non-negative, it follows that $\varepsilon \cdot \mu_J = 0$. The proof of the theorem is complete.

Let $U$ denote the contravariant functor defined by:

\[ U(S) := \{\text{relatively torsion-free sheaves } I \text{ on } X \times S \text{ over } S\} / \sim \]

for every scheme $S$, where two relatively torsion-free sheaves $I$ and $J$ on $X \times S$ over $S$ are considered equivalent if there exists an invertible sheaf $N$ on $S$ such that $I \cong J \otimes N$. Fix an integer $\chi$, and an $n$-tuple $m$ such that $r \chi + \delta \cdot m = 0$. Let $\varepsilon$ be a deformation of the polarization such that $\varepsilon \cdot m > 0$. In this case, $\varepsilon \cdot m \geq r$, with equality only if the maximum common divisor of $\chi, m_1, \ldots, m_n$ is 1. Let

\[ U_\varepsilon(m, \chi) \subseteq U \]

denote the subfunctor parametrizing $\varepsilon$-quasistable sheaves with Euler characteristic $\chi$ and multirank $m$.

**Proposition 7.** The following statements are true:

(a) If $\varepsilon$ is non-negative, then the subfunctor $U_\varepsilon(m, \chi) \subseteq U$ is open.

(b) If $V \subseteq U_\varepsilon(m, \chi)$ is an open subfunctor of $U$, and $\varepsilon \cdot m = r$, then $V$ is representable by a scheme.

**Proof.** The first statement follows easily from the fact that properties (a), (b) and (c) in Thm. 6 are open. To show the second statement, we need only show that $V$ is locally representable by a scheme. Let $F$ be a vector bundle on $X$ with $r_{dF} = r_{dF} - \varepsilon$. Let $V_F \subseteq V$ denote the open subfunctor parametrizing torsion-free sheaves $I$ on $X$ such that $h^0(X, I \otimes F) = 0$, and the unique (modulo $k^*$) homomorphism, $I \rightarrow F^* \otimes \omega$, is injective. It follows from the hypothesis and Thm. 6 that $V$ is covered by open subfunctors of the form $V_F$, for $F$ running through all vector bundles with $r_{dF} = r_{dF} - \varepsilon$. Therefore, we need only show that $V_F$ is representable by a scheme. Let $W \subseteq \text{Quot}_{F^* \otimes \omega}$ be the open subscheme parametrizing quotients $q: F^* \otimes \omega \rightarrow C$ such that $\ker(q)$ has Euler characteristic $\chi$ and multirank $m$, and is represented in $V_F$. Let $W$ denote the functor of points of $W$. It should be clear now how to construct a map of functors $W \rightarrow V_F$, and show that it is an isomorphism. The proof is complete. □
Let $U^s(m, \chi) \subseteq U$ denote the subfunctor parametrizing stable sheaves with Euler characteristic $\chi$ and multirank $m$.

**Corollary 8.** The subfunctor $U^s(m, \chi) \subseteq U$ is open. If the maximum common divisor of $\chi, m_1, \ldots, m_n$ is 1, then $U^s(m, \chi)$ is representable by a scheme.

**Proof.** For the openness, we apply Prop. 7, item (a), with $\varepsilon = (r, \ldots, r)$. As for the representability, we chose a $n$-uple of integers $\delta$, and an integer $s$, such that $s\chi + \delta \cdot m = -1$. Let $\varepsilon := sd - r\delta$. It is clear that $\varepsilon$ is a deformation of the polarization. Moreover, $\varepsilon \cdot m = r$. Since $U^s(m, \chi) \subseteq U^\varepsilon(m, \chi)$, it follows from Prop. 7, item (b), that $U^s(m, \chi)$ is representable by a scheme. The proof is complete. □

**5. Separation lemmas.** Fix a vector bundle $E$ on $X$, of rank $r > 0$ and multidegree $d$, and the ensuing polarization.

**Lemma 9.** Let $I, J_1, \ldots, J_q$ be non-isomorphic stable sheaves on $X$. Then, there is a vector bundle $F$ on $X$, with $r_F = tr$ and $\det F \cong (\det E) \otimes t$ for some $t > 0$, such that

$$h^0(X, I \otimes F) = h^1(X, I \otimes F) \neq 0,$$

and

$$h^0(X, J_s \otimes F) = h^1(X, J_s \otimes F) = 0$$

for $s = 1, \ldots, q$.

**Proof.** Fix $l$ such that $c := r_l(I) \neq 0$. Let $c_s := r_l(J_s)$ for $s = 1, \ldots, q$. Let $\delta$ be the $n$-uple whose unique non-zero component is the $l$-th component, with value 1. According to Lemma 4 (and the observation thereafter), there is a vector bundle $G$ on $X$ such that:

(a) $r_G = t_0 r$ and $d_G = t_0 d - \delta$, for some $t_0 > 0$;

(b) $H^0(X, (I \oplus J_1 \oplus \cdots \oplus J_q) \otimes G) = 0$;

(c) the natural homomorphism (after choices of bases),

$$\lambda: I^{\oplus c} \oplus J_1^{\oplus c_1} \oplus \cdots \oplus J_q^{\oplus c_q} \to G^* \otimes \omega,$$

is injective.

Let $C := G^* \otimes \omega/\lambda(I^{\oplus c})$, and consider the injective homomorphism,

$$\mu = (\mu_1, \ldots, \mu_q): J_1^{\oplus c_1} \oplus \cdots \oplus J_q^{\oplus c_q} \hookrightarrow C,$$

induced by $\lambda$. Pick a non-singular point $p \in X$, contained in $X_l$, such that $\mu(p)$ is injective. If $c_1 + \cdots + c_q = 0$, let $\rho: C(p) \to k$ be any linear surjective
homomorphism. Otherwise, we proceed as follows: Let $e_{s,1},\ldots,e_{s,c_s}$ be a basis of $J_s(p)$, for $s=1,\ldots,q$. For every $s=1,\ldots,q$, and every pair $(j_1,j_2)$ with $1\leq j_1,j_2\leq c_s$, let

$$v_{s,(j_1,j_2)} := \mu_s(p)(0,\ldots,0,e_{s,j_1},0,\ldots,0),$$

where $e_{s,j_1}$ sits in the $j_2$-th position in the above $c_s$-uple. Since $\mu(p)$ is injective, then the vectors $v_{s,(j_1,j_2)} \in C(p)$ are linearly independent. Let $\rho: C(p) \to k$ be a linear homomorphism such that $\rho(v_{s,(j_1,j_2)}) = 0$ if $j_1 \neq j_2$, and $\rho(v_{s,(j,j)}) = 1$ for $s=1,\ldots,q$ and $j=1,\ldots,c_s$.

Put

$$F_0 := (\ker(G^* \to G^*(p) \to C(p) \to k))^*.$$ 

(We chose implicitly a trivialization of $\omega$ at $p$. The particular choice of trivialization is irrelevant.) Since $p \in X$ is non-singular, then $F_0$ is a vector bundle on $X$ with the same rank as $G$, and $\det F_0 \cong \det G \otimes \mathcal{O}_X(p)$. In particular, $d_{F_0} = t_0 d$. By construction, every homomorphism $I \to G^* \otimes \omega$ factors through $F_0^* \otimes \omega$. Hence,

$$h^0(X, I \otimes F_0) = h^1(X, I \otimes F_0) = r_l(I) \neq 0.$$ 

On the other hand, it follows from our choice of $\rho$ that all non-zero homomorphisms $J_s \to G^* \otimes \omega$, for $s=1,\ldots,q$, do not factor through $F_0^* \otimes \omega$. Hence,

$$h^0(X, J_s \otimes F_0) = h^1(X, J_s \otimes F_0) = 0$$

for $s=1,\ldots,q$.

According to Thm. 2, there is a vector bundle $F_1$ on $X$, with $r_{F_1} = t_1 r$ and $\det F_1 \cong (\det E)^{\otimes t_1 + t_0} \otimes (\det F_0)^{-1}$ for some integer $t_1 > 0$, such that

$$h^0(X, (I \oplus J_1 \oplus \cdots \oplus J_q) \otimes F_1) = h^1(X, (I \oplus J_1 \oplus \cdots \oplus J_q) \otimes F_1) = 0.$$ 

It is clear that $F := F_0 \oplus F_1$ meets the requirements of the lemma. The proof is complete. \hfill \qedsymbol

Let $A := k[\varepsilon]/\varepsilon^2$. Let $S := \text{Spec } (A)$, and let $s$ denote the unique point of $S$. Given a coherent sheaf $H$ on $X$, we say that an $S$-flat coherent sheaf $\mathcal{H}$ on $X \times S$ is a deformation of $H$ if $\mathcal{H}(s) \cong H$. We say that a deformation $\mathcal{H}$ of $H$ is trivial if $\mathcal{H} \cong H \otimes \mathcal{O}_S$.

**Lemma 10.** Let $\mathcal{H}$ be a deformation of a coherent sheaf $H$ on $X$. Let $F$ be a vector bundle on $X$ such that $\chi(H \otimes F) = 0$. Then, $\Theta_{F}(\mathcal{H}) = \{s\}$ scheme-theoretically if and only if the two conditions below are verified:

(a) $h^0(X, H \otimes F) = h^1(X, H \otimes F) = 1$;

(b) the unique (modulo $k^*$) non-zero homomorphism $H \to F^* \otimes \omega$ does not extend to a homomorphism $\mathcal{H} \to F^* \otimes \omega \otimes \mathcal{O}_S$ over $S$. 
Proof. Left to the reader. □

Lemma 11. Let $I$ be a stable sheaf on $X$. If $\mathcal{I}$ is a non-trivial deformation of $I$, then there is a vector bundle $F$ on $X$, with $r_F = tr$ and $\det F \cong (\det E)^{\otimes t}$ for some $t > 0$, such that $\Theta_F(\mathcal{I}) = \{ s \}$ scheme-theoretically.

Proof. Let $c_i := r_i(I)$ for $i = 1, \ldots, n$. Let $c := c_1 + \cdots + c_n$. According to Lemma 4 (and the observation thereafter), there is a vector bundle $G$ on $X$ such that:

(a) $r_G = t_0 r$ and $d_G = t_0 \underline{d} - (1, \ldots, 1)$, for some $t_0 > 0$;
(b) $H^0(X, I \otimes G) = 0$;
(c) the natural homomorphism (after a choice of basis),

$$\lambda: F^\oplus c \to G^* \otimes \omega,$$

is injective with torsion-free cokernel.

We will express $\lambda$ in the form:

$$\lambda = (\lambda_1, \ldots, \lambda_{c_1}, \lambda_2, \ldots, \lambda_{c_2}, \ldots, \lambda_1, \ldots, \lambda^n).$$

Since $H^0(X, I \otimes G) = 0$, then $\lambda$ extends to a homomorphism,

$$\tilde{\lambda}: \mathcal{I}^\oplus c \to G^* \otimes \omega \otimes \mathcal{O}_S,$$

with $S$-flat cokernel $\mathcal{C}$. Let $C := \mathcal{C}(s)$, and consider the “tangential” homomorphism,

$$\nu = (\nu_1, \ldots, \nu^n): F^\oplus c \to C,$$

induced by $\tilde{\lambda}$. Since the deformation $\mathcal{I}$ is non-trivial, then $\nu \neq 0$. Since $C$ is torsion-free, then there is a non-singular point $p \in X$ where $\nu(p) \neq 0$. We may assume that $\nu_1(p) \neq 0$. We may also assume that $p \in X_1$. Of course, we then have that $c_1 > 0$. Let $p_1 := p$, and pick non-singular points $p_2, \ldots, p_n \in X$ such that $p_i \in X_i$ for $i = 2, \ldots, n$. Fix trivializations of $\omega$ at the points $p_1, \ldots, p_n$. For $i = 1, \ldots, n$, let $e^i_1, \ldots, e^i_{c_i}$ be a basis of $I(p_i)$. We may assume that $\nu^i_1(p_i) | (e^i_1) \neq 0$. Let $v \in G^*(p_i)$ such that its image in $C(p_1)$ is $\nu^i_1(p_i) | (e^i_1)$. For $i_1, i_2 \in 1, \ldots, n$, $j_1 = 1, \ldots, c_1$ and $j_2 = 1, \ldots, c_2$, let $v_{j_1, j_2}^{i_1, i_2} := \lambda_{j_2}^{i_2}(p_i) | (e_{j_1}^{i_1})$. For each fixed $i$, the vectors $v_{j_1, j_2}^{i_1, i_2} \in G^*(p_i)$ are linearly independent. Moreover, since $v$ is not in the image of $\lambda(p_1)$, then $v$ is not a linear combination of the $v_{j_1, j_2}^{1, i_2} \in G^*(p_1)$. For $i = 1, \ldots, n$, let $\rho_i: G^*(p_i) \to k$ be a linear surjective homomorphism such that

$$\rho_i(v_{j_1, j_2}^{i_1, i_2}) = \begin{cases} 1, & \text{if } i_2 = i \text{ and } j_1 = j_2, \\ 0, & \text{otherwise}, \end{cases}$$

with the unique exception that $\rho_1(v_{1, 1}^{1, 1}) = 0$. Since $v$ is not a linear combination of the $v_{j_1, j_2}^{1, i_2} \in G^*(p_1)$, then we may also assume that $\rho_1(v) = 1$. 

Proof.
Put
\[ F_0 := \ker(G^* \to \bigoplus_{i=1}^{n} G^*(p_i) \to k^{\otimes n})^*. \]

Since \( p_1, \ldots, p_n \) are non-singular, then \( F_0 \) is a vector bundle on \( X \) with the same rank as \( G \), and
\[ \det F_0 \cong \det G \otimes \mathcal{O}_X(p_1 + \cdots + p_n). \]

In particular, \( d_{F_0} = t_0 d \). By construction, \( h^1(X, I \otimes F_0) = 1 \), where the unique (modulo \( k^* \)) homomorphism, \( \mu_1^1 : I \to F_0^* \otimes \omega \), is the factorization of \( \lambda_1^1 \). Let \( C' := \ker(\mu_1^1) \). Since \( \rho_1(v_{1,j_2}^{1}) = 0 \) for \( i_2 = 1, \ldots, n \) and \( j_2 = 1, \ldots, c_{i_2} \), and \( \rho_1(v) = 1 \), then \( \nu_1^{1} \) is not in the subspace of \( \text{Hom}_X(I, C) \) spanned by \( \text{Hom}_X(I, C') \) and \( \text{Hom}_X(I, G^* \otimes \omega) \). Therefore, \( \mu_1^1 \) does not extend to a homomorphism \( I \to F_0^* \otimes \omega \otimes \mathcal{O}_S \) over \( S \). Applying Lemma 9, we obtain that \( \Theta_{F_0}(I) = \{ s \} \) scheme-theoretically.

According to Thm. 2, there is a vector bundle \( F_1 \) on \( X \), with \( r_{F_1} = t_1 r \) and 
\[ \det F_1 \cong (\det E)^{\otimes t_1 + t_0} \otimes (\det F_0)^{-1} \text{ for some } t_1 > 0, \text{ such that} \]
\[ h^0(X, I \otimes F) = h^1(X, I \otimes F) = 0. \]

It is clear that \( F := F_0 \oplus F_1 \) meets the requirements of the lemma. The proof is complete. \( \square \)

6. The moduli spaces of semistable sheaves. Let \( a := (a_1, a_2, \ldots, a_n) \) be a \( n \)-uple of positive rational numbers with \( a_1 + a_2 + \cdots + a_n = 1 \). In [19, Part 7] Seshadri defined a non-zero torsion-free sheaf \( I \) on \( X \) to be \( a \)-semistable (resp. \( a \)-stable) if
\[ \chi(K) \leq \frac{a \cdot r_{K}}{a \cdot r_{I}} \chi(I) \quad (\text{resp. } \chi(K) < \frac{a \cdot r_{K}}{a \cdot r_{I}} \chi(I)) \]
for every proper subsheaf \( K \subseteq I \).

Let \( S(a, m, \chi) \) (resp. \( S'(a, m, \chi) \)) denote the set of isomorphism classes of \( a \)-semistable (resp. \( a \)-stable) torsion-free sheaves of Euler characteristic \( \chi \) and multirank \( m \) on \( X \). As in Sect. 3, there is a Jordan-Hölder filtration for any \( a \)-semistable sheaf \( I \), and we shall denote the associated graded sheaf by \( \text{Gr}_a(I) \). We say that two \( a \)-semistable sheaves \( I_1, I_2 \) are \( a \)-equivalent if \( \text{Gr}_a(I_1) = \text{Gr}_a(I_2) \).

Remark 12. We shall observe that Seshadri’s notion of stability is equivalent to ours. Let \( I \) be a non-zero torsion-free sheaf on \( X \) with Euler characteristic \( \chi \) and multirank \( m \). Assume that \( \chi > 0 \). Let \( a \) be a \( n \)-uple of positive rational numbers such that \( a_1 + \cdots + a_n = 1 \). Let \( A := (A_1, \ldots, A_n) \) be a \( n \)-uple of positive integers such that
\[ A_i a_j = A_j a_i \quad \text{for } 1 \leq i, j \leq n; \quad (12.1) \]
and $A := A \cdot m$ is a multiple of $\chi$. Since $a_1 + \cdots + a_n = 1$, then it follows from (12.1) that
\[ a_i = \frac{A_i}{A_1 + \cdots + A_n} \quad (12.2) \]
for $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, let $x_i^1, \ldots, x_i^{A_i} \in X_i$ be non-singular points of $X$. Let
\[ O_X(1) := O_X(\sum_{1 \leq i \leq n}^{1 \leq j \leq A_i} x_i^j). \]
Of course, $O_x(1)$ is an ample sheaf on $X$. Let
\[ E := (O_X^{r/t-1} \oplus O_X(-1))^\oplus t \quad (12.3) \]
for a certain integer $t > 0$, to be specified later, and $r := tA/\chi$. Since $d_E = -tA$, then
\[ \chi(I \otimes E) = \chi r + m \cdot d_E = 0. \]
If $K \subseteq I$ is a torsion-free subsheaf, then
\[ \chi(K \otimes E) = \chi(K)r - tA \cdot r_K = t((\chi(K)/\chi)A \cdot m - A \cdot r_K). \]
It follows that $\chi(K \otimes E) \leq 0$ (resp. $\chi(K \otimes E) < 0$) if and only if
\[ \chi(K) \leq \frac{A \cdot r_K}{A \cdot m} \chi \quad (\text{resp. } \chi(K) < \frac{A \cdot r_K}{A \cdot m} \chi). \]
Using (12.2), we see that $I$ is $a$-semistable (resp. $a$-stable) if and only if $I$ is semistable (resp. stable) with respect to $E$. Note that, if $I$ is $a$-semistable, then $\text{Gr}(I) = \text{Gr}_a(I)$.

**Theorem 13.** (Seshadri) There is a coarse moduli space for $S'(a, m, \chi)$, whose underlying scheme is a quasi-projective variety denoted by $U^s(a, m, \chi)$. Moreover, $U^s(a, m, \chi)$ has a natural projective compactification, to be denoted by $U(a, m, \chi)$. The set $U(a, m, \chi)$ is isomorphic to the quotient of $S(a, m, \chi)$ by the $a$-equivalence relation.

**Sketch of Proof.** We outline briefly the aspects we will need in Seshadri’s proof [19, Thm. 15, p. 155] of the theorem.

First note that a torsion-free sheaf $I$ on $X$ is $a$-semistable, or $a$-stable, if and only if $I \otimes L$ is, for every invertible sheaf $L$ on $X$. Thus, we may assume that $\chi >> 0$. Let
\[ Q := \text{Quot}^m \chi(O_X^{\oplus \chi}) \]
denote Grothendieck’s scheme of quotients of $O_X^{\oplus \chi}$ with Euler characteristic $\chi$ and multirank $m$. Let $R \subseteq Q$ be the open subscheme parametrizing the quotients $q: O_X^{\oplus \chi} \rightarrow I$ such that $I$ is torsion-free, and the induced homomorphism
$k \otimes \chi \to H^0(X, I)$ is an isomorphism. Let $R^{ss}$ (resp. $R^s$) denote the subset of $R$ parametrizing the quotients $q: \mathcal{O}_X^{\otimes \chi} \to I$ such that $I$ is $a$-semistable (resp. $a$-stable). Seshadri constructs $U(a, m, \chi)$ (resp. $U^s(a, m, \chi)$) as the good quotient (resp. geometric quotient) of $R^{ss}$ (resp. $R^s$) under the obvious action of $SL(\chi)$, as we describe below.

We shall assume the definitions and notations of Rmk. 12 in what follows. Let

$$Z := \prod_{i=1}^n A_i \prod_{j=1}^n Grass(\chi, m_i).$$

Define a morphism, $\tau: R \to Z$, by mapping a quotient, $q: \mathcal{O}_X^{\otimes \chi} \to I$, represented in $R$ to

$$(q(x_1^1), \ldots, q(x_1^{A_1}), \ldots, q(x_n^1), \ldots, q(x_n^{A_n})) \in Z,$$

where $q(x_j^i): k^{\otimes \chi} \to I(x_j^i)$ is the homomorphism induced by $q$ on $x_j^i$, for all $i, j$. It is clear that $\tau$ is an $SL(\chi)$-morphism, where $SL(\chi)$ acts on both $R$ and $Z$ in the obvious way. For $i = 1, \ldots, n$, let $V_i$ denote the tautological quotient sheaf of rank $m_i$ on $Grass(\chi, m_i)$. Consider the following invertible sheaf on $Z$:

$$M := (p_1^1)^* \bigwedge V_1 \otimes \cdots \otimes (p_{A_1}^1)^* \bigwedge V_1 \otimes \cdots \otimes (p_1^n)^* \bigwedge V_n \otimes \cdots \otimes (p_{A_n}^n)^* \bigwedge V_n,$$

where

$$id_Z = (p_1^1, \ldots, p_{A_1}^1, \ldots, p_1^n, \ldots, p_{A_n}^n): Z \to \prod_{i=1}^n A_i \prod_{j=1}^n Grass(\chi, m_i).$$

The group $SL(\chi)$ acts linearly on $M$. Thus, we may define stable and semistable points on $Z$ with respect to this action. Let $Z^{ss}$ (resp. $Z^s$) be the open subscheme of $Z$ parametrizing semistable (resp. stable) points on $Z$ with respect to the linear action of $SL(\chi)$ on $M$. By G.I.T. ([8, Thm. 1.10, p. 38] or [15, Thm. 3.21, p. 84]), there is a projective good quotient of $Z^{ss}$ for the action of $SL(\chi)$. In addition, some power of $M$ descends to an ample sheaf on the quotient.

If $\chi$ and $A$ are large enough, then Seshadri [19, Thm. 19, p. 158] shows that it is possible to choose the points $x_j^i$ in such a way that:

(a) $\tau$ is injective;
(b) $R^{ss} = \tau^{-1}(Z^{ss})$, hence $R^{ss} \subseteq R$ is open;
(c) $R^s = \tau^{-1}(Z^s)$, hence $R^s \subseteq R$ is open;
(d) the induced morphism, $\tau^{ss}: R^{ss} \to Z^{ss}$, is proper.

It follows from (a) and (b) that $\tau^{ss}$ is finite. Applying a result of Ramanathan’s [19, Prop. 26, p. 31], we get that there is a good quotient,

$$\phi: R^{ss} \to U(a, m, \chi),$$
for the action of $\text{SL}(\chi)$. In addition, there is an open subscheme,

$$U^s(\mathfrak{a}, \mathfrak{m}, \chi) \subseteq U(\mathfrak{a}, \mathfrak{m}, \chi),$$

such that $R^s = \phi^{-1}(U^s(\mathfrak{a}, \mathfrak{m}, \chi))$ and $\phi|_{R^s} : R^s \to U^s(\mathfrak{a}, \mathfrak{m}, \chi)$ is a geometric quotient of $R^s$ under the action of $\text{SL}(\chi)$. The varieties $U(\mathfrak{a}, \mathfrak{m}, \chi)$ and $U^s(\mathfrak{a}, \mathfrak{m}, \chi)$ are the ones mentioned in the statement of the theorem. The variety $U(\mathfrak{a}, \mathfrak{m}, d)$ is projective, and $\tau^*(M) \otimes t|_{R^{ss}}$ descends to an ample sheaf on $U(\mathfrak{a}, \mathfrak{m}, \chi)$ for some $t > 0$. Our sketch of Seshadri’s proof is complete. □

We shall retain the definitions and notations of Rmk. 12 and Thm. 13 in what follows. Let $q : \mathcal{O}_{X \times R}^\oplus \to \mathcal{I}$ be the restriction of the universal quotient on $X \times Q$ to $X \times R$. Clearly,

$$\tau^*(M) = \bigotimes_{1 \leq i \leq n} \bigotimes_{1 \leq j \leq A_i} M^i_j,$$

where $M^i_j := \mathcal{I}|_{x^j \times R}$ is regarded as a sheaf on $R$ under the canonical isomorphism $x^j \times R \cong R$, for all $i, j$. We choose $E$ as in (12.3), with an integer $t > 0$ such that $\tau^*(M) \otimes t|_{R^{ss}}$ descends to $U(\mathfrak{a}, \mathfrak{m}, \chi)$. Let $g : X \times R \to R$ denote the projection morphism. By definition of $R$, the adjoint homomorphism, $\mathcal{O}_R^\oplus : g_*\mathcal{I}$, induced by $q$ is an isomorphism, and $H^1(X, \mathcal{I}(s)) = 0$ for every $s \in R$. Thus, $\mathcal{D}(\mathcal{I}) = \mathcal{O}_R$, where $\mathcal{D}$ denotes the determinant of cohomology with respect to $g$. Using the exact sequence

$$0 \to \mathcal{I} \otimes \mathcal{O}_X(-1) \to \mathcal{I} \to \bigoplus_{1 \leq i \leq n} \mathcal{I}|_{x^j \times R} \to 0,$$

and the additive property of $\mathcal{D}$, we get that

$$\mathcal{D}(\mathcal{I} \otimes E) = \mathcal{D}(\mathcal{I}) \otimes (\mathcal{I} \otimes \mathcal{O}_X(-1)) \otimes t$$

$$= \mathcal{D}(\mathcal{I}) \otimes (\mathcal{I} \otimes \bigotimes_{1 \leq i \leq n} \bigotimes_{1 \leq j \leq A_i} M^i_j) \otimes t$$

$$= \tau^*(M) \otimes t.$$

Hence,

$$\mathcal{L}_E(\mathcal{I}|_{X \times R^{ss}}) \cong \tau^*(M) \otimes t|_{R^{ss}}.$$ 

As we observed in the “proof” of Thm. 13, the sheaf $\mathcal{L}_E(\mathcal{I}|_{X \times R^{ss}})$ descends to an ample sheaf on $U(\mathfrak{a}, \mathfrak{m}, \chi)$, henceforth denoted by $\mathcal{L}_E(\mathfrak{m}, \chi)$.

Fix a torsion-free sheaf $\mathcal{I}_0$ on $X$ with $\chi(\mathcal{I}_0) = \chi$ and $\mathfrak{m}_0 = \mathfrak{m}$. It can be verified from the functorial properties of theta functions, and Lemma 1, that the sections

$$\theta_F(\mathcal{I}|_{X \times R^{ss}}) \in H^0(R^{ss}, \mathcal{L}_E(\mathcal{I}|_{X \times R^{ss}})) \otimes t),$$
corresponding to vector bundles $F$ on $X$ with $r_F = tr$ and $\det F \cong (\det E)^{\otimes t}$, are $\SL(\chi)$-invariant. In other words,

$$V^t_E(I|_{X \times \RR_{ss}}) \subseteq H^0(R^{ss}, \mathcal{L}_E(I|_{X \times \RR_{ss}})^{\otimes t})^{\SL(\chi)}$$

for every $t > 0$. Thus, the sections $\theta_F(I|_{X \times \RR_{ss}})$ of $\mathcal{L}_E(I|_{X \times \RR_{ss}})^{\otimes t}$ descend to sections of $\mathcal{L}_E(m, \chi)^{\otimes t}$, which we shall denote by $\theta_F(m, \chi)$. The sections $\theta_F(m, \chi)$ are well defined modulo $k^*$. We denote by $\Theta_F(m, \chi)$ the zero-scheme of $\theta_F(m, \chi)$. Let

$$V_E(m, \chi) := \bigoplus_{t \geq 0} V^t_E(m, \chi) \subseteq \bigoplus_{t \geq 0} H^0(U(a, m, \chi), \mathcal{L}_E(m, \chi)^{\otimes t}) =: \Gamma_E(m, \chi)$$

denote the graded $k$-subalgebra generated by the sections $\theta_F(m, \chi)$. Let

$$U_{\theta}(a, m, \chi) := \Proj(V_E(m, \chi)).$$

Since $V_E(m, \chi) \subseteq \Gamma_E(m, \chi)$, and $U(a, m, \chi) = \Proj(\Gamma_E(m, \chi))$, then we have a rational map,

$$\pi: U(a, m, \chi) \rightarrow U_{\theta}(a, m, \chi).$$

**Remark 14.** We keep the definitions and notations used so far in this section. Let $I_0$ be a semistable sheaf on $X$ of Euler characteristic $\chi$ and multirank $m$. Suppose that there is a proper semistable subsheaf $I_1 \subseteq I_0$. Let $I_2 := I_0/I_1$. Let $\chi_i := \chi(I_i)$ and $m_i := r_{I_i}$ for $i = 1, 2$. Let

$$Q_i := \text{Quot}^{m_i, \chi_i}(\mathcal{O}_X^{\oplus \chi_i})$$

for $i = 1, 2$. Let

$$\nu: Q_1 \times Q_2 \rightarrow Q$$

be the morphism sending a pair of quotients,

$$([q_1: \mathcal{O}_X^{\oplus \chi_1} \rightarrow J_1], [q_2: \mathcal{O}_X^{\oplus \chi_2} \rightarrow J_2])$$

to

$$[q: \mathcal{O}_X^{\oplus \chi} = \mathcal{O}_X^{\oplus \chi_1} \oplus \mathcal{O}_X^{\oplus \chi_2} \rightarrow J_1 \oplus J_2].$$

It is clear that $\nu$ defines a monomorphism of the corresponding functors of points. It follows from [9, 8.11.5] that $\nu$ is a closed embedding.

For $i = 1, 2$, let $R_i \subseteq Q_i$ denote the open subscheme parametrizing quotients $q_i: \mathcal{O}_X^{\oplus \chi_i} \rightarrow J_i$ such that $J_i$ is torsion-free, and the induced homomorphism,

$$k^{\oplus \chi_i} \rightarrow H^0(X, J_i),$$
is an isomorphism. Of course, $\nu^{-1}(R) = R_1 \times R_2$. Let

$$\mu := \nu|_{R_1 \times R_2} : R_1 \times R_2 \to R$$

denote the induced closed embedding. Let $I_i$ denote the restriction of the universal quotient on $X \times Q_i$ to $X \times R_i$, for $i = 1, 2$. For $i = 1, 2$, let $R^{ss}_i$ be the open subscheme of $R_i$ parametrizing quotients $q_i : O_X^{\otimes \chi_i} \to J_i$ such that $J_i$ is $\sigma$-semistable. Of course, $\mu^{-1}(R^{ss}) = R^{ss}_1 \times R^{ss}_2$. Denote by

$$\mu^{ss} := \mu|_{R^{ss}_1 \times R^{ss}_2} : R^{ss}_1 \times R^{ss}_2 \to R^{ss}$$

the induced closed embedding. Since

$$(\id_X, \mu)^* I \cong I_1 \oplus I_2,$$

then it follows from properties of theta functions, and Lemma 1 (see its proof), that

$$(\mu^{ss})^* L_E(I|_{X \times R^{ss}}) \cong L_E(I_1|_{X \times R^{ss}}) \otimes L_E(I_2|_{X \times R^{ss}}),$$

and

$$(\mu^{ss})^* \theta_F(I|_{X \times R^{ss}}) = \theta_F(I_1|_{X \times R^{ss}}) \otimes \theta_F(I_2|_{X \times R^{ss}})$$

(modulo $k^*$) under the identification (14.1), for every vector bundle $F$ on $X$, with $r_F = tr$ and $\det F \cong (\det E)^{\otimes t}$ for some $t > 0$. Of course, $\mu$ is an $\text{SL}(\chi_1) \times \text{SL}(\chi_2)$-morphism. So, $\mu^{ss}$ induces a morphism,

$$\alpha : U(a, m_1, \chi_1) \times U(a, m_2, \chi_2) \to U(a, m, \chi).$$

Note that $[I]$ is contained in the image of $\alpha$. Under $\alpha$, there are relations similar to (14.1) and (14.2), namely:

$$\alpha^* L_E(m, \chi) \cong L_E(m_1, \chi_1) \otimes L_E(m_2, \chi_2),$$

and

$$\alpha^* \theta_F(m, \chi) = \theta_F(m_1, \chi_1) \otimes \theta_F(m_2, \chi_2)$$

(modulo $k^*$) under (14.3), for every vector bundle $F$ on $X$, with $r_F = tr$ and $\det F \cong (\det E)^{\otimes t}$ for some $t > 0$.

**Theorem 15.** The natural rational map,

$$\pi : U(a, m, \chi) \to U_0(a, m, \chi),$$

is defined everywhere and bijective.

**Proof.** It follows from Thm. 2 that $\pi$ is defined everywhere. Of course, $\pi$ is dominant. Since $U := U(a, m, \chi)$ is complete, then $\pi$ is surjective. We shall now show that $\pi$ is injective. Let $I$ and $J$ be semistable sheaves on $X$ with
Gr(I) \not\cong Gr(J). We need to show that \( \pi([I]) \neq \pi([J]) \). We may assume that

\( I \cong Gr(I) \) and \( J \cong Gr(J) \). Assume first that there is a stable summand of \( I \)

that does not occur as a stable summand of \( J \). Then, it follows from Lemma 9 that there is a vector bundle \( F \) on \( X \), with \( r_F = tr \) and \( \det F \cong (\det E)^{\otimes t} \)

for some \( t > 0 \), such that \( \theta_F(\tau, \chi)([I]) = 0 \), but \( \theta_F(\tau, \chi)([J]) \neq 0 \). Hence, \( \pi([I]) \neq \pi([J]) \). Assume now that that \( I \) and \( J \) have a common stable summand \( K \). Let \( \chi_1 := \chi(K) \) and \( m_1 := \tau_K \). Put \( \chi_2 := \chi - \chi_1 \) and \( m_2 := m - m_1 \). Write \( I = K \oplus I_2 \) and \( J = K \oplus J_2 \). Let \( U_i := U(\alpha, m_i, \chi_i) \) for \( i = 1, 2 \). The points \([I], [J] \in U \) lie in the image of the points \(([K], [I_2]), ([K], [J_2]) \in U_1 \times U_2 \), respectively, under the morphism

\[ \alpha: U_1 \times U_2 \to U \]

constructed in Rmk. 14. By an induction argument, we may assume that there is an integer \( t > 0 \), and vector bundles \( F, G \) on \( X \), with \( r_F = r_G = tr \) and

\( \det F \cong \det G \cong (\det E)^{\otimes t} \), such that:

(a) \( \theta_G(m, \chi)([I]) \neq 0 \) and \( \theta_G(m, \chi)([J]) \neq 0 \);

(b) \[
\frac{\theta_F(m_2, \chi_2)([I_2])}{\theta_G(m_2, \chi_2)([J_2])} \neq \frac{\theta_F(m_1, \chi_2)([I_2])}{\theta_G(m_1, \chi_2)([J_2])}.
\]

By modifying \( F \), if necessary, we may also assume that \( \theta_F(m_1, \chi)([K]) \neq 0 \). It follows from Rmk. 14 that

\[
\frac{\theta_F(m, \chi)([I])}{\theta_G(m, \chi)([J])} = \frac{\theta_F(m_1, \chi_1)([K])}{\theta_G(m_1, \chi_1)([J])} \frac{\theta_F(m_2, \chi_2)([I_2])}{\theta_G(m_2, \chi_2)([J_2])}
\]

\[
= \frac{\theta_F(m_1, \chi_1)([K])}{\theta_G(m_1, \chi_1)([J])} \frac{\theta_F(m_2, \chi_2)([I_2])}{\theta_G(m_2, \chi_2)([J_2])}
\]

\[
= \frac{\theta_F(m, \chi)([I])}{\theta_G(m, \chi)([J])}.
\]

Hence, \( \pi([I]) \neq \pi([J]) \). The proof is complete. \( \square \)

THEOREM 16. Assume that the maximum common divisor of \( \chi, m_1, \ldots, m_n \)

is 1. Then, the restriction,

\[ \pi^s : = \pi|_{U^s(\alpha, m, \chi)} : U^s(\alpha, m, \chi) \to U_\theta(\alpha, m, \chi), \]

is an open embedding.

Proof. Let \( U := U(\alpha, m, \chi) \) and \( U_\theta := U_\theta(\alpha, m, \chi) \). Since \( U \) is complete,

then \( \pi \) is proper. Since \( \pi \) is bijective, then \( \pi \) is a homeomorphism, and a finite morphism. To show that \( \pi^s \) is an open embedding, we need only show that the homomorphism of tangent spaces,

\[ d\pi_{[I]} : T_{[I], U} \to T_{\pi([I]), U_\theta}, \]
is injective for every stable sheaf $I$ represented in $U$. Let $v \in T_{[I],U}$ be a non-zero tangent vector. We need to show that $d\pi_{[I]}(v) \neq 0$. Since $U^s(a, m, \chi)$ is representable, by Cor. 8, then $v$ is represented by a deformation $\mathcal{I}$ of $I$. Since $v \neq 0$, then $\mathcal{I}$ is not the trivial deformation of $I$. It follows from Lemma 11 that there is a vector bundle $F$ on $X$, with $r_F = tr$ and $\det F \cong (\det E)^{\otimes t}$ for some $t > 0$, such that $\Theta_F(\mathcal{I}) = \{s\}$ scheme-theoretically. Since $\theta_F(m, \chi) \in V^t_E(m, \chi)$, it follows that $d\pi_{[I]}(v) \neq 0$. The proof is complete. □

Remark 17. If $X$ is irreducible, and the rank $m$ is coprime with the Euler characteristic $\chi$, then $U(m, \chi) = U^s(m, \chi)$. So, it follows from Thm. 16 that $\pi$ is an isomorphism in this case.

The hypothesis in the statement of Thm. 16 was used in order to identify a tangent vector of $U_\theta(a, m, \chi)$ at a point $[I]$, representing a stable sheaf $I$, with a deformation of $I$. It might be that we do not need that hypothesis for such identification, as it is the case of the next theorem.

Theorem 18. Assume that $X$ is irreducible, and the characteristic of the ground field $k$ is 0. Then, the restriction,

$$\pi^s: = \pi|_{U^s(m, \chi)} : U^s(m, \chi) \to U_\theta(m, \chi),$$

is an open embedding.

Proof. Apply the same proof of Thm. 16, now using [12, Thm. 8.14, p. 141] to guarantee that every tangent vector of $U^s(m, \chi)$ is represented by a deformation. The proof is complete. □

Remark 19. As we had mentioned in Rmk. 3, all the results we obtained in this article asserting the existence of a vector bundle $F$ on $X$ with certain properties could have been obtained within a certain range for the rank of $F$, depending only on numerical invariants. Thus, replacing $V_E(m, \chi)$ by the subalgebra,

$$V^t_E(m, \chi) \subseteq V_E(m, \chi),$$

generated by all theta functions $\theta_F(m, \chi)$ associated with vector bundles $F$ on $X$ with rank $r_F \leq tr$, and letting

$$U^{[t]}_\theta(a, m, \chi) := \text{Proj}(V^t_E(m, \chi)),$$

we have that Theorems 15, 16 and 18 hold for the rational map,

$$\pi^{[t]}: U(a, m, \chi) \to U^{[t]}_\theta(a, m, \chi),$$

as long as $t >> 0$. In particular, we have that $\pi^{[t]}$ is finite and scheme-theoretically surjective for $t >> 0$. Since $\pi$ has the same properties, and $U_\theta(a, m, \chi)$ is the inverse limit of $U^{[t]}_\theta(a, m, \chi)$ as $t \to \infty$, then it follows that

$$U_\theta(a, m, \chi) = U^{[t]}_\theta(a, m, \chi)$$
for every $t >> 0$.

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