Flattening the Inflaton’s Potential with Quantum Corrections

Ewan D. Stewart
Research Center for the Early Universe
University of Tokyo
Tokyo 113, Japan

June 14, 2021

Abstract

I show that a classical scalar potential with $V''/V \sim 1$ can be sufficiently flattened by quantum corrections to give rise to slow-roll inflation. This provides perhaps the simplest way to generate an inflationary potential without fine tuning. The most natural implementation of this idea produces an unviably small spectral index, but, for example, $n \sim 0.8$ can be obtained in other implementations.
1 Introduction

Slow-roll inflation \[1, 2\] requires an unusually flat scalar potential. This is quantified by the conditions

\[
\left( \frac{V'}{V} \right)^2 \ll 1 \tag{1}
\]

and

\[
\left| \frac{V''}{V} \right| \ll 1. \tag{2}
\]

The first condition is generally not difficult to achieve, but the second is more problematical \[3\]. In particular, it is straightforward to show that in supergravity \[4\] one generically gets

\[
\left| \frac{V''}{V} \right| \gtrsim 1. \tag{3}
\]

Most models of inflation resolve this problem by fine tuning. The only existing proposals for achieving Eq. (2) without fine tuning were made in \[3, 5\]. The purpose of this paper is to investigate another simpler method.

2 The idea

Any inflationary model must have a non-zero scalar potential energy density

\[ V = V_0. \tag{4} \]

In supergravity this will induce soft supersymmetry-breaking masses squared of order \( V_0 \) for all scalar fields\[3, 4\]

\[ V(\phi) = V_0 \left[ 1 - \frac{1}{2} A \phi^2 + \ldots \right] \tag{5} \]

with \( |A| \sim 1 \). The dots represent higher order terms which, in the case of \( A > 0 \), might naturally stabilise the potential at \( \phi \sim 1 \). Eq. (5) is our classical potential. It does not give rise to slow-roll inflation. In particular, \( |V''/V| \simeq |A| \sim 1 \).

If \( \phi \) has either gauge or Yukawa couplings, to vector or chiral superfields with soft supersymmetry-breaking masses squared of order \( V_0 \), quantum corrections

---

\[ 1 \] I set \( M_{\text{Pl}} = 1/\sqrt{8\pi G} = 1. \)

\[ 2 \] Natural inflation \[1\] naturally achieves a small \( V'' \) by assuming an approximate global \( U(1) \) symmetry, but does not naturally satisfy Eq. (2) because \( V \) vanishes in the limit where the symmetry is exact. A hybrid natural inflation model might avoid this problem though \[2\].

\[ 3 \] We assume that there are no other relevant larger masses, and so in particular that \( V_0 \lesssim M_s^4 \) where \( M_s \) is the scale of supersymmetry breaking not due to \( V_0 \). If \( M_s \) takes the same value as in our vacuum then we require \( V_0^{1/4} \gtrsim 10^{10} \) to \( 10^{11} \)GeV \( \sim 10^{-8} \) with \( V_0^{1/4} \sim 10^{-8} \) being the most obvious choice.

\[ 4 \] For simplicity we take the scalar fields to be real.
will renormalise \( \phi \)'s mass leading to a one-loop renormalisation group \([4]\) effective potential of the form

\[
V(\phi) = V_0 \left[ 1 - \frac{1}{2} f(\epsilon \ln \phi) \phi^2 \right]
\]

(6)

where \( \epsilon \ll 1 \) is the one-loop suppression factor, \( f(0) = A + \mathcal{O}(\epsilon) \) and the expression is valid for \( V_0 \ll \phi^2 \ll 1 \). The reader may care to consider \( f(x) = A + Bx \) as a toy model. A more realistic form is given in the Appendix.

Now

\[
\frac{V}{V_0} = 1 - \frac{1}{2} f \phi^2 ,
\]

(7)

\[
\frac{V'}{V_0} = - \left( f + \frac{\epsilon}{2} f' \right) \phi ,
\]

(8)

\[
\frac{V''}{V_0} = - \left( f + \frac{3\epsilon}{2} f' + \frac{\epsilon^2}{2} f'' \right) .
\]

(9)

Define \( \phi_* \) by

\[
f_* + \frac{\epsilon}{2} f_*' = 0
\]

(10)

where \( f_* \equiv f(\epsilon \ln \phi_*) \) etc. Then we can rewrite Eqs. (7), (8) and (9) as

\[
\frac{V}{V_0} = 1 - \frac{1}{2} \left[ \epsilon f_*' \left( \ln \frac{\phi}{\phi_*} - \frac{1}{2} \right) + \mathcal{O} \left( \epsilon^2 \ln^2 \frac{\phi}{\phi_*} \right) \right] \phi^2 ,
\]

(11)

\[
\frac{V'}{V_0} = - \left[ \epsilon \left( f_*' + \frac{\epsilon}{2} f_*'' \right) \ln \frac{\phi}{\phi_*} + \mathcal{O} \left( \epsilon^2 \ln^2 \frac{\phi}{\phi_*} \right) \right] \phi ,
\]

(12)

\[
\frac{V''}{V_0} = - \epsilon \left( f_*' + \frac{\epsilon}{2} f_*'' \right) - \epsilon \left( f_*' + \frac{3\epsilon}{2} f_*'' + \frac{\epsilon^2}{2} f_*''' \right) \ln \frac{\phi}{\phi_*} + \mathcal{O} \left( \epsilon^2 \ln^2 \frac{\phi}{\phi_*} \right).
\]

(13)

Note that although \( |V''/V| \sim 1 \) over most of the potential, the quantum corrections have flattened the potential in the vicinity of \( \phi_* \).

I assume that there exists one, and for simplicity only one, such \( \phi_* \) in the range \( V_0^{1/2} \ll \phi_* \ll 1 \). We can remove the ambiguity in the definition of \( f \) and \( \epsilon \) by setting

\[
\epsilon \ln \phi_* = -1 .
\]

(14)

To get \( \phi_* \gtrsim V_0^{1/2} \gtrsim 10^{-16} \approx e^{-37} \) then requires \( \epsilon \gtrsim 1/37 \). We see that the inflaton \( \phi \) should have unsuppressed couplings to other fields, as do for example the Minimal Supersymmetric Standard Model’s up Higgs \([4]\) or thermal inflation’s flaton \([8, 9]\).
3 Models

In order for the mechanism of Section 2 to work, we require that Eq. (10) has a solution, which for simplicity I assume to be unique, for $\phi < 1$. This requires that $f(0) \simeq A$ and $f'_{\ast}$ have the same sign. There are then two classes of potentials depending on whether $f'_{\ast}$ is greater or less than zero. If it is less than zero then $\phi_{\ast}$ is a minimum of the potential and so one must use a hybrid inflation type mechanism [7] to end inflation at some critical value $\phi_{c}$. However, as the potential is flat enough for slow roll inflation only in the neighbourhood of $\phi_{\ast}$, one must fine tune to get $\phi_{c}$ close to $\phi_{\ast}$. We do not consider these models further and from now on assume $A > 0$ and $f'_{\ast} > 0$.

If $f'_{\ast} > 0$ then $\phi_{\ast}$ is a maximum of the potential. There are then two possibilities depending on whether $\phi$ rolls towards the false vacuum at $\phi = 0$ or a true vacuum at $\phi \sim 1$. If $\phi$ rolls towards the false vacuum at $\phi = 0$ then one must again use a hybrid inflation type mechanism to end inflation at some critical value $\phi_{c}$. However this time one can take $\phi_{c} \ll \phi_{\ast}$ and so one does not have to fine tune $\phi_{c}$. The spectral index of such a model would change from $n < 1$ to $n > 1$, possibly over observable scales. However, it seems difficult to achieve the required initial conditions of $\phi \simeq \phi_{\ast}$. The unnaturalness of the initial conditions might though be compensated to some extent by the eternal inflation [1] that occurs at $\phi = \phi_{\ast}$.

In the case of the slow roll inflation occurring as $\phi$ rolls towards the true vacuum at $\phi \sim 1$, we have the natural initial condition $\phi = 0$. At first we have (eternal) old inflation with $\phi = 0$, then quantum fluctuations kick $\phi$ over the barrier at $\phi = \phi_{\ast}$ (or $\phi$ tunnels through the barrier), and slow roll inflation occurs as $\phi$ rolls off to the new vacuum at $\phi \sim 1$. In order to obtain a sufficiently high reheat temperature, at least in the case of $V_{0} \sim 10^{-32}$, we require that $\phi$ has new couplings to new superfields which become light at the new vacuum. This is a natural expectation in string (or M or F) theory. If $\phi$ corresponded to an Affleck-Dine field in the new vacuum, it might even be able to generate a large enough baryon number to survive the diluting effects of thermal inflation as its initial amplitude would be of the order of the Planck scale. However, the baryogenesis mechanism of [9] is more compelling. This model is theoretically healthy but, as we shall see in the next section, tends to give an uncomfortably small spectral index especially for the favoured $V_{0} \sim 10^{-32}$.

Indeed, it is perhaps the first truly natural and essentially complete model of slow roll inflation. Refs. [3] and [5], although extremely promising and potentially natural, can not yet be regarded as essentially complete as they require detailed knowledge of the as yet imprecisely understood high energy theory. The present model on the other hand uses $A$ of Eq. (5) to parameterize the unknown high energy physics and depends only on the qualitative features of the low energy vacuum, albeit a different low energy vacuum ($\phi = 0$) than our low energy vacuum ($\phi \sim 1$).
4 The spectral index

From Eq. (8) the equation of motion for $\phi$ is

$$\ddot{\phi} + 3H\dot{\phi} - V_0 \left( f + \frac{\epsilon}{2} f' \right) \phi = 0. \quad (15)$$

Define $3H_0^2 = V_0$, $x = \epsilon \ln \phi$ and $\tau = \epsilon H_0 t$, and take a dot to now represent differentiation with respect to $\tau$ rather than $t$ which it did before. Then, assuming $H \simeq H_0$ which will be a good approximation for $\phi \ll 1$, we get

$$\epsilon \ddot{x} + \dot{x}^2 + 3\dot{x} - 3f - \frac{3\epsilon}{2} f' = 0. \quad (16)$$

Neglecting the first term and taking the branch that connects to the slow roll trajectory gives

$$\dot{x} = -\frac{3}{2} \left( 1 - \sqrt{1 + \frac{4}{3} f + \frac{2\epsilon}{3} f'} \right). \quad (17)$$

Differentiating gives

$$\frac{\ddot{x}}{\dot{x}} = \frac{f' + \frac{\epsilon}{2} f''}{\sqrt{1 + \frac{4}{3} f + \frac{2\epsilon}{3} f'}} \quad (18)$$

and so the error in Eq. (17) is of order $\epsilon$ unless the square root becomes small. Integrating Eq. (17) gives

$$\tau = -\frac{2}{3} \int \frac{dx}{1 - \sqrt{1 + \frac{4}{3} f + \frac{2\epsilon}{3} f'}}. \quad (19)$$

The slow roll approximation corresponds to expanding in $x - x_*$. Working to lowest order in $\epsilon$ and second order in the slow roll approximation and using Eq. (10) gives

$$1 - \sqrt{1 + \frac{4}{3} f + \frac{2\epsilon}{3} f'} = -\frac{2}{3} f'_* (x - x_*) \left[ 1 + \frac{f'_*}{3} \left( \frac{3f''_*}{2f'^*_{\epsilon}} - 1 \right) (x - x_*) \right] \quad (20)$$

and so

$$\tau = \frac{1}{f'_*} \ln (x - x_*) - \frac{1}{3} \left( \frac{3f''_*}{2f'^*_{\epsilon}} - 1 \right) (x - x_*) \quad (21)$$

where the second term is of second order in the slow roll approximation. The number of $e$-folds to the end of inflation is given by $N = (\tau_2 - \tau)/\epsilon$ where subscript 2 denotes the end of inflation. Rewriting Eq. (21) and working to lowest order in the slow roll approximation at $\phi$, but not at $\phi_2$, gives

$$N = \frac{1}{\epsilon f'_*} \ln \left[ \frac{\ln (\phi_2/\phi_*)}{\ln (\phi/\phi_*)} \right] - C \quad (22)$$
where \( C \) is a constant given by

\[
C = \frac{1}{3} \left( \frac{3f''}{2f'^2} - 1 \right) \ln \frac{\phi_2}{\phi_*} + \mathcal{O} \left( \epsilon \ln^2 \frac{\phi_2}{\phi_*} \right).
\]  

(23)

\( C \) is calculated exactly in the Appendix for a semi-realistic example and found to be given by \( C = 2/3\epsilon \) for \( \phi_2 \sim 1 \). Inverting Eq. (22) and using Eq. (14) gives

\[
\ln \frac{\phi}{\phi_*} = \frac{1}{\epsilon} \left( 1 + \epsilon \ln \frac{\phi_2}{\phi_*} \right) e^{-\epsilon f'_* (N+1)}.
\]  

(24)

The COBE normalisation and Eq. (12) give

\[
\frac{V_1^{3/2}}{V_0^{1/2}} = \frac{V_0^{1/2}}{\epsilon f'_* \ln(\phi_1/\phi_*)} = 6 \times 10^{-4}
\]  

(25)

where subscript 1 denotes the time when COBE scales left the horizon. For \( \epsilon f'_* \ln(\phi_1/\phi_*) \sim 10^{-1} \) to \( 10^{-2} \) and \( V_0 \sim 10^{-32} \) this gives \( \phi_1 \sim 10^{-11} \sim e^{-25} \). Slow roll at \( \phi_1 \) requires \( \epsilon \ln(\phi_1/\phi_*) \ll 1 \) and so we get \( \epsilon \simeq 0.04 \). More generally, using Eqs. (14) and (24), it gives

\[
V_0^{1/2} = f'_* (1 + \epsilon \ln \phi_2) \exp \left[ -\frac{1}{\epsilon} - 7.4 + \frac{1}{\epsilon} (1 + \epsilon \ln \phi_2) e^{-\epsilon f'_* (N_1 + C)} \right].
\]  

(26)

The spectral index is, using Eqs. (13) and (24),

\[
n = 1 + 2 \frac{V''}{V} = 1 - 2\epsilon f'_* \left( \ln \frac{\phi}{\phi_*} + 1 \right)
\]  

(27)

\[
= 1 - 2\epsilon f'_* \left[ (1 + \epsilon \ln \phi_2) e^{-\epsilon f'_* (N+1) + \epsilon} \right].
\]  

(28)

A natural choice of parameters would be \( \phi_2 \sim 1, V_0 \sim 10^{-32} \) and the \( f \) of Eq. (31) with \( A = 1 \). Observable scales would then leave the horizon at \( N \sim 30 \), assuming thermal inflation. Unfortunately for these parameters, although slow roll does occur for sufficiently large \( N \), it has finished by \( N \sim 30 \) and we get an unacceptable spectral index. This situation can be improved in a number of ways:

1. Increase \( N \) by either abandoning thermal inflation or increasing \( V_0 \).

2. Increase \( \epsilon \) by increasing \( V_0 \).

3. Increase \( C \) by fiddling with \( f \).

4. Increase \( f'_* \) by fiddling with \( f \) or \( A \).

5. Decrease \( \phi_2 \), i.e. end inflation earlier, by for example using a hybrid inflation type mechanism [7].
6. Take $\phi_2 < \phi_\ast$ so that a partial cancellation occurs in $n$. This corresponds to the model of Section 3 with the problematic initial conditions.

To illustrate how these effects can help we take $\phi_2 \sim 1$, $\epsilon = 0.1$, $N = 40$, $C = 2/3\epsilon$ and $f'_x = 0.5$ corresponding to $V_0^{1/4} \sim 10^{14}\text{GeV}$, having thermal inflation, and the $f$ of Eq. (30) with $A = 1$. Eq. (28) then gives $n = 0.8$ which is the spectral index recently argued for in [10].

5 Conclusions

I have described a way to obtain slow-roll inflation which is completely natural from the particle physics point of view and should be realisable in string (or M or F) theory. The spectral index has the general form

$$n = 1 - \alpha e^{-\beta N} - 2\beta$$

(29)

and is likely to show an observable change over observable scales. Unfortunately, the most natural values of the parameters tend to give a spectral index that is too small to be consistent with observations. However, plausible values of the parameters can give the spectral index of $n \sim 0.8$ recently argued for in [10].

Acknowledgements

I would like to thank Lisa Randall for stimulating discussions and the Aspen Center for Physics for its hospitality when these discussions took place. I also thank David Lyth for many helpful discussions. I am supported by a JSPS Fellowship at RESCEU, and my work is supported by Monbusho Grant-in-Aid for JSPS Fellows No. 95209.

Appendix

As an example we take $f$ to be given by

$$f(x) = \frac{3}{4} \left[ \left( \frac{a+1}{a-x} \right)^2 - 1 \right]$$

(30)

with

$$a = \frac{3}{4A} \left( 1 + \sqrt{1 + \frac{4A}{3}} \right).$$

(31)

This is a realistic form except for the precise value of the coefficient $3/4$ which was chosen to make the integral in Eq. (19) analytically soluble. It corresponds to a gauge coupling which increases with the energy scale and negligible Yukawa couplings. As Eq. (17) has an error of order $\epsilon$, we work to lowest order in $\epsilon$. 
We have \( f(0) = A \) and \( f(-1) = 0 \) in accord with the definitions of Section 2. Also \( f'(-1) = (3/2)(a+1)^{-1}, f''(-1) = (9/2)(a+1)^{-2} \) and for \( A = 1 \) we have \( f'(-1) \approx 0.5 \). Substituting Eq. (30) into Eq. (19) gives

\[
\tau = \frac{1}{f'(-1)} \ln (1 + x) - \frac{2}{3} x
\]

(32)

and so for the above choice of \( f \) the \( C \) of Eq. (22) is given by

\[
C = \frac{2}{3\epsilon} (1 + \epsilon \ln \phi_2)
\]

(33)

Note that the second order slow roll formula gives the exact result in this case.

References

[1] A. D. Linde, Particle physics and inflationary cosmology (Harwood Academic, Switzerland, 1990).

[2] E. W. Kolb and M. S. Turner, The early universe (Addison-Wesley, New York, 1990).

[3] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, astro-ph/9401011, Phys. Rev. D 49 (1994) 6410; E. D. Stewart, hep-ph/9405389, Phys. Rev. D 51 (1995) 6847.

[4] H. P. Nilles, Phys. Rep. 110 (1984) 1; D. Bailin and A. Love, Supersymmetric gauge field theory and string theory (IOP, Bristol, 1994).

[5] M. K. Gaillard, H. Murayama and K. A. Olive, hep-ph/9504307, Phys. Lett. B 355 (1995) 71.

[6] K. Freese, J. A. Frieman and A. V. Olinto, Phys. Rev. Lett. 65 (1990) 3233; F. C. Adams, J. R. Bond, K. Freese, J. A. Frieman and A. V. Olinto, hep-ph/9207243, Phys. Rev. D 47 (1993) 426.

[7] A. D. Linde, Phys. Lett. B 259 (1991) 38; E. D. Stewart, astro-ph/9407040, Phys. Lett. B 345 (1995) 414; D. H. Lyth and E. D. Stewart, in preparation.

[8] D. H. Lyth and E. D. Stewart, hep-ph/9510204, Phys. Rev. D 53 (1996) 1784.

[9] E. D. Stewart, M. Kawasaki and T. Yanagida, hep-ph/9603324.

[10] M. White, P. T. P. Viana, A. R. Liddle and D. Scott, astro-ph/9605057.