Abstract

In this paper, we introduce the notion of $E$-Courant algebroids, where $E$ is a vector bundle. It is a kind of generalized Courant algebroid and contains Courant algebroids, Courant-Jacobi algebroids and omni-Lie algebroids as its special cases. We explore novel phenomena exhibited by $E$-Courant algebroids and provide many examples. We study the automorphism groups of omni-Lie algebroids and classify the isomorphism classes of exact $E$-Courant algebroids. In addition, we introduce the concepts of $E$-Lie bialgebroids and Manin triples.

1 Introduction

In recent years, Courant algebroids are widely studied from several aspects. They are applied in many mathematical objects such as Manin pairs and moment maps [1, 3, 16, 21], generalized

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0Keyword: $E$-Courant algebroids, $E$-Lie bialgebroids, omni-Lie algebroids, Leibniz cohomology.

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complex structures \[2, 11, 36\], \(L_\infty\)-algebras and symplectic supermanifolds \[31\], gerbes \[34\], BV algebras and topological field theories \[14, 32\].

We recall two notions closely related to Courant algebroids — Jacobi bialgebroids and omni-Lie algebroids. Jacobi bialgebroids and generalized Lie bialgebroids are introduced, respectively, in \[9\] and \[15\] to generalize Dirac structures from Poisson manifolds to Jacobi manifolds. More general geometric objects are generalized Courant algebroids \[29\] and Courant-Jacobi algebroids \[10\]. The notion of omni-Lie algebroids, a generalization of the notion of omni-Lie algebras introduced in \[39\], is defined in \[5\] in order to characterize all possible Lie algebroid structures on a vector bundle \(E\). An omni-Lie algebra can be regarded as the linearization of the exact Courant algebroid \(TM \oplus T^*M\) at a point and is studied from several aspects recently \[2, 14, 35, 38\]. Moreover, Dirac structures of omni-Lie algebroids are studied by the authors in \[6\].

In this paper, we introduce a kind of generalized Courant algebroid called \(E\)-Courant algebroids. The values of the anchor map of an \(E\)-Courant algebroid lie in \(D_E\), the bundle of differential operators. Moreover, its Dirac structures are necessarily Lie algebroids equipped with a representation on \(E\). The notion of \(E\)-Courant algebroids not only unifies Courant-Jacobi algebroids and omni-Lie algebroids, but also provides a number of interesting objects, e.g. the \(T^*M\)-Courant algebroid structure on the jet bundle of a Courant algebroid over \(M\) (Theorem 2.13).

Recall that an exact Courant algebroid structure on \(TM \oplus T^*M\) is a twist of the standard Courant algebroid by a closed 3-form \[34\]. This structure includes twisted Poisson structures and is related to gerbes and topological sigma models \[30, 37\]. In this paper, we are inspired to study exact \(E\)-Courant algebroids similar to the situation of exact Courant algebroids.

We also study the automorphism groups of omni-Lie algebroids, for which we need the language of Leibniz cohomologies \[25, 26\]. Moreover, we introduce the notion of \(E\)-Lie bialgebroids, which generalizes the notion of generalized Lie bialgebroids. We shall prove that, for an \(E\)-Lie bialgebroid, there induces on the underlying vector bundle \(E\) a Lie algebroid structure (\(\text{rank}(E) \geq 2\)), or a local Lie algebra structure (\(\text{rank}(E) = 1\)) (Theorem 6.6).

This paper is organized as follows. In Section 2 we introduce the notion of \(E\)-Courant algebroids. We prove that the jet bundle \(\mathcal{JC}\) of a Courant algebroid \(C\) over \(M\) admits a natural \(T^*M\)-Courant algebroid structure. In Section 3 we discuss the properties of \(E\)-dual pairs of Lie algebroids. In Section 4 we find the automorphism groups and all possible twists of omni-Lie algebroids. In Section 5 we study exact \(E\)-Courant algebroids and prove that every exact \(E\)-Courant algebroid with an isotropic splitting is isomorphic to an omni-Lie algebroid. In general, an exact \(E\)-Courant algebroid is a twist of the standard omni-Lie algebroid by a 2-cocycle in the Leibniz cohomology of \(\Gamma(D_E)\) with coefficients in \(\Gamma(J_E)\), which can also be treated as a 3-cocycle in the Leibniz cohomology of \(\Gamma(D_E)\) with coefficients in \(\Gamma(E)\). In Section 6 we study \(E\)-Lie bialgebroids. In Section 7 we extend the theory of Manin triples from the context of Lie bialgebroids to \(E\)-Lie bialgebroids and give some interesting examples.

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2 \(E\)-Courant algebroids

Let \(E \rightarrow M\) be a vector bundle and \(\mathcal{D}E\) the associated covariant differential operator bundle. Known as the gauge Lie algebroid of the frame bundle \(F(E)\) (see \[27, \text{Example 3.3.4}\]), \(\mathcal{D}E\) is a transitive Lie algebroid with the Lie bracket \([\cdot, \cdot]_\mathcal{D}\) (commutator). The corresponding Atiyah
sequence is as follows:

\[
0 \rightarrow \mathfrak{g}(E) \xrightarrow{i} \mathcal{D}E \xrightarrow{j} TM \rightarrow 0. \tag{1}
\]

In [5], the authors proved that the jet bundle \( \mathcal{J}E \) (see [7, 33] for more details about jet bundles) can be regarded as an \( E \)-dual bundle of \( \mathcal{D}E \), i.e.

\[
\mathcal{J}E \cong \{ \nu \in \text{Hom}(\mathcal{D}E, E) \mid \nu(\Phi) = \Phi \circ \nu(1_E), \ \forall \ \Phi \in \mathfrak{g}(E) \} \subset \text{Hom}(\mathcal{D}E, E).
\]

Associated to the jet bundle \( \mathcal{J}E \), the jet sequence of \( E \) is given by:

\[
0 \rightarrow \text{Hom}(TM, E) \xrightarrow{\rho} \mathcal{J}E \xrightarrow{\mathcal{D}} E \rightarrow 0. \tag{2}
\]

The operator \( d : \Gamma(E) \rightarrow \Gamma(\mathcal{J}E) \) is given by:

\[
d\mu(\mathfrak{d}) := \mathfrak{d}(u), \quad \forall \ \mu \in \Gamma(E), \ \mathfrak{d} \in \Gamma(\mathcal{D}E).
\]

The following formula is needed.

\[
d(fX) = df \otimes X + f dX, \quad \forall \ X \in \Gamma(C), \ f \in C^{\infty}(M). \tag{3}
\]

For a vector bundle \( K \) over \( M \) and a bundle map \( \rho : K \rightarrow \mathcal{D}E \), we denote the induced \( E \)-adjoint bundle map by \( \rho^* \), i.e.

\[
\rho^* : \text{Hom}(\mathcal{D}E, E) \rightarrow \text{Hom}(K, E), \quad \rho^*(\nu)(k) = \nu(\rho(k)), \ \forall \ k \in K, \ \nu \in \text{Hom}(\mathcal{D}E, E). \tag{4}
\]

The notion of Liebniz algebras is introduced by Loday [24, 25, 12]. A Leibniz algebra \( \mathfrak{g} \) is an \( R \)-module, where \( R \) is a commutative ring, endowed with a linear map \([\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}\) satisfying

\[
[g_1, [g_2, g_3]] = [[g_1, g_2], g_3] + [g_2, [g_1, g_3]], \quad \forall \ g_1, g_2, g_3 \in \mathfrak{g}.
\]

**Definition 2.1.** An \( E \)-Courant algebroid is a quadruple \( (\mathcal{K}, \langle \cdot, \cdot \rangle_E, \langle \cdot, \cdot \rangle_K, \rho) \), where

- \( \mathcal{K} \) is a vector bundle over \( M \) such that \( (\Gamma(\mathcal{K}), \langle \cdot, \cdot \rangle_{\mathcal{K}}) \) is a Leibniz algebra;

- \( \langle \cdot, \cdot \rangle_E : \mathcal{K} \otimes \mathcal{K} \rightarrow E \) is a symmetric nondegenerate \( E \)-valued pairing, which induces an embedding: \( \mathcal{K} \hookrightarrow \text{Hom}(\mathcal{K}, E) \);

- the anchor \( \rho : \mathcal{K} \rightarrow \mathcal{D}E \) is a bundle map.

such that the following properties hold for all \( X, Y, Z \in \Gamma(\mathcal{K}) \):

- (EC-1) \( \rho[X, Y]_{\mathcal{K}} = [\rho(X), \rho(Y)]_{\mathcal{D}} \); 

- (EC-2) \( [X, X]_{\mathcal{K}} = \rho^* d (X, X)_E \); 

- (EC-3) \( \rho(X)(Y, Z)_E = ([X, Y]_{\mathcal{K}}, Z)_E + (Y, [X, Z]_{\mathcal{K}})_E \); 

- (EC-4) \( \rho^*(\mathcal{J}E) \subset \mathcal{K} \), i.e. \( \rho^*(\mu)(X)_E = \frac{1}{2} \mu(\rho(X)), \ \forall \ \mu \in \mathcal{J}E \); 

- (EC-5) \( \rho \circ \rho^* = 0 \).

**Remark 2.2.** If the \( E \)-valued pairing \( \langle \cdot, \cdot \rangle_E : \mathcal{K} \otimes \mathcal{K} \rightarrow E \) is surjective, Properties (EC-4) and (EC-5) can be inferred from Property (EC-2). In particular, if \( E \) is a line bundle, any nondegenerate \( E \)-valued pairing \( \langle \cdot, \cdot \rangle_E \) is surjective.
Lemma 2.3. For any \( X, Y \in \Gamma(K) \) and \( f \in C^\infty(M) \), we have
\[
[Xf(Y)]_K = f[X,Y]_K + (i \circ \rho(X)f)Y, \\
[fX,Y]_K = f[X,Y]_K - (i \circ \rho(Y)f)X + 2\rho^*(df \otimes (X,Y)_E).
\] (5) (6)

Proof. By Property (EC-3), for all \( X, Y, Z \in \Gamma(K) \) and \( f \in C^\infty(M) \), we have
\[
\langle [Xf(Y)]_K, Z \rangle_E + \langle f[X,Y]_K, Z \rangle_E = \rho(X)\langle fY, Z \rangle_E = i \circ \rho(X)(f)\langle Y, Z \rangle_E + f\rho(X)\langle Y, Z \rangle_E = i \circ \rho(X)(f)\langle Y, Z \rangle_E + f \langle [X,Y]_K, Z \rangle_E + f \langle [X,Y]_K, Z \rangle_E.
\]

Since the pairing \( (\cdot, \cdot)_E \) is nondegenerate, it follows that
\[
[Xf(Y)]_K = i \circ \rho(X)(f)Y + f[X,Y]_K.
\]

By Property (EC-2), we have
\[
[Xf(Y)]_K + [fX,Y]_K = 2\rho^*(df \otimes (X,Y)_E) = 2f\rho^*(df \otimes (X,Y)_E) + 2\rho^*(df \otimes (X,Y)_E).
\]

Substitute \( [Xf(Y)]_K \) by \( \text{(5)} \) and apply Property (EC-2) again, we obtain \( \text{(6)}. \) \( \blacksquare \)

For a subbundle \( L \subset K \), denote by \( L^\perp \subset K \) the subbundle
\[
L^\perp = \{ e \in K | \langle e, l \rangle_E = 0, \forall l \in L \}.
\]

Definition 2.4. A Dirac structure of an \( E \)-Courant algebroid \( (K, (\cdot, \cdot)_E, [\cdot, \cdot], \rho) \) is a subbundle \( L \subset K \) which is closed under the bracket \( [\cdot, \cdot]_K \) and satisfies \( L = L^\perp \).

Evidently, \( L = L^\perp \) implies that \( L \) is maximal isotropic with respect to \( (\cdot, \cdot)_E \). In general, \( L \) being maximal isotropic with respect to \( (\cdot, \cdot)_E \) does not imply \( L = L^\perp \).

Example 2.5. Let \( K = \mathbb{R}^3 \) with the standard basis \( e_1, e_2, e_3 \). The \( \mathbb{R} \)-valued pairing \( (\cdot, \cdot)_\mathbb{R} \) is given by
\[
(e_1, e_3)_\mathbb{R} = (e_2, e_2)_\mathbb{R} = 1, \quad (e_1, e_1)_\mathbb{R} = (e_1, e_2)_\mathbb{R} = (e_2, e_3)_\mathbb{R} = (e_3, e_3)_\mathbb{R} = 0.
\]
Obviously, \( L = \mathbb{R}e_1 \) is maximal isotropic but \( L^\perp = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \neq L \).

Proposition 2.6. Any Dirac structure \( L \) has an induced Lie algebroid structure and is equipped with a Lie algebroid representation \( \rho_L : L \rightarrow \mathcal{D}E \).

Proof. Given a Dirac structure \( L \), by Property (EC-2), we have \( [X, Y]_K = 0 \), for all \( X \in \Gamma(L) \), which implies that \( [\cdot, \cdot]_K |_L \) is skew-symmetric. By \( \text{(5)} \), \( (L, [\cdot, \cdot], \rho_L) \) is a Jacobi Lie algebroid, which is, by definition, a Lie algebroid \( A \) together with a 1-cocycle \( \theta \in \Gamma(L^*) \) in the Lie algebroid cohomology \( \mathcal{H}^1(L^*) \).

Remark 2.7. If \( E \) is the trivial line bundle \( M \times \mathbb{R} \), then \( \mathcal{D}E \cong TM \oplus (M \times \mathbb{R}) \). Thus we can decompose \( \rho = a + \theta \), for some \( a : K \rightarrow TM \) and \( \theta : K \rightarrow M \times \mathbb{R} \). For a Dirac structure \( L \), since \( \rho_L \) is a representation of the Lie algebroid \( L \), it follows that \( \theta_L = \theta |_L \in \Gamma(L^*) \) is a 1-cocycle in the Lie algebroid cohomology of \( L \). Therefore, \( (L, \theta_L) \) is a Jacobi algebroid, which is, by definition, a Lie algebroid \( A \) together with a 1-cocycle \( \theta \in \Gamma(L^*) \) in the Lie algebroid cohomology \( \mathcal{H}^1(L^*) \).

One may refer to \( \text{[27]} \) for more general theories of Lie algebroids, Lie algebroid cohomologies and their representations. Now we briefly recall the notions of omni-Lie algebroids, generalized Courant algebroids, Courant-Jacobi algebroids and generalized Lie bialgebroids. We will see that \( E \)-Courant algebroids unify all these structures.
\begin{itemize}
  \item **Omni-Lie algebroids**

  The notion of omni-Lie algebroids is introduced in [5] to characterize Lie algebroid structures on a vector bundle. It is a generalization of Weinstein’s omni-Lie algebras. Recall that there is a natural symmetric nondegenerate $E$-valued pairing $\langle \cdot, \cdot \rangle_E$ between $\mathcal{J}E$ and $\Omega E$:

  \[ \langle \mu, \vartheta \rangle_E = \langle \vartheta, \mu \rangle_E \triangleq d\mu, \ \forall \mu = [u]_m \in \mathcal{J}E, \ u \in \Gamma(E), \ \vartheta \in \Omega E. \]

  Moreover, this pairing is $C^\infty(M)$-linear and satisfies the following properties:

  \[ \langle \mu, \Phi \rangle_E = \Phi \circ \rho(\mu), \ \forall \Phi \in \mathfrak{gl}(E), \ \mu \in \mathcal{J}E; \]

  \[ \langle \eta, \vartheta \rangle_E = \eta \circ j(\vartheta), \ \forall \eta \in \text{Hom}(TM, E), \ \vartheta \in \Omega E. \]

  Furthermore, $\Gamma(\mathcal{J}E)$ is invariant under any Lie derivative $\mathcal{L}_\vartheta$, $\vartheta \in \Gamma(\Omega E)$, which is defined by the Leibniz rule:

  \[ \langle \mathcal{L}_\vartheta \mu, \vartheta' \rangle_E \triangleq \vartheta \langle \mu, \vartheta' \rangle_E - \langle \mu, [\vartheta, \vartheta']_E \rangle_E, \ \forall \mu \in \Gamma(\mathcal{J}E), \ \vartheta' \in \Gamma(\Omega E). \quad (7) \]

  **Definition 2.8.** [5] Given a vector bundle $E$, the quadruple $(\mathcal{E}, \{\cdot, \cdot\}, \langle \cdot, \cdot \rangle_E, \rho)$ is called the omni-Lie algebroid associated to $E$, where $\mathcal{E} = \Omega E \oplus \mathcal{J}E$, the anchor $\rho$ is the projection from $\mathcal{E}$ to $\Omega E$, the bracket operation $\{\cdot, \cdot\}$ and the nondegenerate $E$-valued pairing $\langle \cdot, \cdot \rangle_E$ are given respectively by

  \[ \{\vartheta + \mu, \nu + \rho\} \triangleq \frac{1}{2} ([\vartheta, \nu]_E + \langle \nu, \mu \rangle_E), \quad (8) \]

  \[ \{\vartheta + \mu, \nu + \rho\} \triangleq [\vartheta, \nu]_E + \mathcal{L}_\vartheta \nu - \mathcal{L}_\nu \mu + d \langle \mu, \nu \rangle_E. \quad (9) \]

  If there is no risk of confusion, we simply denote the omni-Lie algebroid $(\mathcal{E}, \{\cdot, \cdot\}, \langle \cdot, \cdot \rangle_E, \rho)$ by $\mathcal{E}$. We call the $E$-valued pairing $\langle \cdot, \cdot \rangle_\mathcal{E}$ and the bracket $\{\cdot, \cdot\}_\mathcal{E}$, respectively, the standard pairing and the standard bracket on $\mathcal{E} = \Omega E \oplus \mathcal{J}E$. One may refer to [5] for more details of the property of omni-Lie algebroids. Evidently, the $E$-adjoint map $\rho^*$ is $1_{\mathcal{J}E}$, the identity map on $\mathcal{J}E$. It is easily seen that the omni-Lie algebroid $\mathcal{E}$ is an $E$-Courant algebroid. Its Dirac structures are studied by the authors in [6].

  \item **Generalized Courant algebroids (Courant-Jacobi algebroids)**

  The notion of generalized Courant algebroids is introduced in [29]. It is a pair $(\mathcal{K}, \mathcal{K}^*)$ subject to some compatibility conditions, where $\mathcal{K} \to M$ is a vector bundle equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a skew-symmetric bracket $[\cdot, \cdot]$ on $\Gamma(\mathcal{K})$ and a bundle map $\rho^* : \mathcal{K} \to TM \times \mathbb{R}$, which is also a first-order differential operator. We may write $\rho^*(X) = (\rho(X), (\theta, X))$, where $\rho : \mathcal{K} \to TM$ is linear and $\theta \in \Gamma(\mathcal{K}^*)$ satisfies

  \[ \theta([X, Y]) = \rho(X)\theta(Y) - \rho(Y)\theta(X), \ \forall X, Y \in \Gamma(\mathcal{K}). \]

  One should note that the skew-symmetric bracket $[\cdot, \cdot]$ does not satisfy the Jacobi identity. The notion of Courant-Jacobi algebroids is introduced in [10]. In [29], it is established the equivalence of generalized Courant algebroids and Courant-Jacobi algebroids. Roughly speaking, the difference between them is that the generalized Courant algebroid has a skew-symmetric bracket $[\cdot, \cdot]$ and a Courant-Jacobi algebroid has an operation $\circ$, which is also known as the Dorfman bracket [8]. The former does not satisfy the Jacobi identity, while the later satisfies the Leibniz rule. Moreover, $[\cdot, \cdot]$ can be realized as the skew-symmetrization of $\circ$. A generalized Courant algebroid reduces to a Courant algebroid if $\theta = 0$ (see [22]).

  Evidently, a generalized Courant algebroid is an $E$-Courant algebroid if we take $E = M \times \mathbb{R}$. It follows that all Jacobi algebroids and Courant algebroids are $M \times \mathbb{R}$-Courant algebroids.
• **Generalized Lie bialgebroids**

A Lie bialgebroid is a pair of vector bundles in duality, each of which is a Lie algebroid, such that the differential defined by one of them on the exterior algebra of its dual is a derivative of the Schouten bracket [13] [28]. A generalized Lie bialgebroid [15], or a Jacobi bialgebroid [9], is a pair \((A, φ_0), (A^*, X_0)\), where \(A\) and \(A^*\) are two vector bundles in duality, and, respectively, equipped with Lie algebroid structures \((A, [\cdot, \cdot], a)\) and \((A^*, [\cdot, \cdot], a_*)\). The data \(φ_0 ∈ Π(A^*)\) and \(X_0 ∈ Π(A)\) are 1-cocycles in their respective Lie algebroid cohomologies such that for all \(X, Y ∈ Π(A)\), the following conditions are satisfied:

\[
\begin{align*}
d_{X_0}[X, Y] &= [d_{X_0}X, Y]_{φ_0} + [X, d_{X_0}Y]_{φ_0}, \\
φ_0(X_0) &= 0, \quad a(X_0) = -a_*(φ_0), \quad Λ_{a_φ_0}X = Λ_{a_φ_0}X_0 = 0,
\end{align*}
\tag{10}
\]

where \(d_{X_0}\) is the \(X_0\)-differential of \(A\), \([\cdot, \cdot]_{φ_0}\) is the \(φ_0\)-Schouten bracket, Λ, and Λ are the usual Lie derivatives. For more information of these notations, please refer to [15]. For a Jacobi manifold \((M, X, A)\), \((\mathcal{T}M × R, (0, 0)), (\mathcal{T}^*M × R, (−X, 0)))\) is a generalized Lie bialgebroid. Furthermore, for a generalized Lie bialgebroid, there is an induced Jacobi structure on the base manifold \(M\). In particular, both \(((A, φ_0)\) and \((A^*, X_0)\)) are Jacobi algebroids. If \(φ_0 = 0\) and \(X_0 = 0\), a generalized Lie bialgebroid reduces to a Lie bialgebroid. It is known that for a generalized Lie bialgebroid \(((A, φ_0), (A^*, X_0))\), there is a natural generalized Courant algebroid \((A ⊕ A^*, φ_0 + X_0)\).

We give more examples of \(E\)-Courant algebroids.

**Example 2.9.** Let \(A\) be a Lie algebroid and \(ρ_A : A → Ξ E\) a representation of \(A\) on a vector bundle \(E\). Let \(K = A ⊕ (A^* ⊕ E)\). For any \(X, Y ∈ Γ(A)\), \(ξ ⊗ u, η ⊗ v ∈ Γ(A^* ⊕ E)\), we define the following operations:

\[
\begin{align*}
ρ(X + ξ ⊗ u) &= ρ_A(X), \quad [X + ξ ⊗ u, Y + η ⊗ v]_K = [X, Y] + Λ_X(η ⊗ v) - Λ_Y(ξ ⊗ u) + ρ^*_A(⟨Y, η⟩u), \\
(X + ξ ⊗ u, Y + η ⊗ v)_E &= \frac{1}{2}⟨(X, η)v + ⟨Y, ξ⟩u⟩.
\end{align*}
\]

Evidently, \(ρ^∗ = ρ^*_A : Ξ E → A^* ⊕ E\) and it is straightforward to check that \((A ⊕ (A^* ⊕ E), [\cdot, \cdot]_K, ⟨\cdot, \cdot⟩_E, ρ)\) is an \(E\)-Courant algebroid. In [29], the notion of \(AV\)-Courant algebroids is introduced in order to study generalized CR structures, which is closely related to this example but twisted by a 3-cocycle in the cohomology of the Lie algebroid representation \(ρ_A\).

**Example 2.10.** Consider an \(E\)-Courant algebroid \(K\) whose anchor \(ρ\) is zero. Thus \(ρ^∗ = 0\), and the bracket \([\cdot, \cdot]_K\) is skew-symmetric. So \(K\) is a bundle of Lie algebras. Property (EC-3) shows that there is an invariant \(E\)-valued pairing. We conclude that an \(E\)-Courant algebroid \(K\) whose anchor \(ρ\) is zero is equivalently a bundle of Lie algebras with an invariant \(E\)-valued pairing.

**Example 2.11.** An omni-Lie algebra \(gl(V) ⊕ V\) is a special omni-Lie algebroid whose base manifold is a point, hence a \(V\)-Courant algebroid. Moreover, one may consider a Lie algebra \((g, [\cdot, \cdot]_g)\) with faithful representation \(ρ_g : g → gl(V)\) on a vector space \(V\). This representation is called nondegenerate if for any \(v ∈ V\), there is some \(A ∈ g\) such that \(ρ_g(A)(v) ≠ 0\). Introduce a nondegenerate \(V\)-valued pairing \(⟨\cdot, \cdot⟩_V\) and a bilinear bracket \([\cdot, \cdot]\) on the space \(g ⊕ V\):

\[
\begin{align*}
(A + u, B + v)_V &= \frac{1}{2}[(ρ_g(A)(v) + ρ_g(B)(u)), \\
[A + u, B + v]_g &= [A, B]_g + ρ_g(A)(v), \quad ∀ A + u, B + v ∈ g ⊕ V,
\end{align*}
\]

where \(ρ : g ⊕ V → gl(V)\) is defined by \(ρ(A + u) = ρ_g(A)\) for \(A + u ∈ g ⊕ V\). Following from

\[
ρ^*(u)(B + v) = \frac{1}{2}ρ_g(B)(u) = (u, B)_V,
\tag{12}
\]
we have \( \rho^* = 1_V \), as a map \( \mathcal{J}V = V \rightarrow V \). Clearly, \( (\mathfrak{g} \oplus V, \cdot, \cdot, V, \cdot, \cdot, \rho) \) is a \( V \)-Courant algebroid.

The bracket defined above appeared in \([17]\). For any representation \( \rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \), we call \((\mathfrak{g} \oplus V, \cdot, \cdot, V, \cdot, \cdot, \rho)\) a \textit{hemisemidirect} product of \( \mathfrak{g} \) with \( V \). There is also a natural exact Courant algebra associated to any \( \mathfrak{g} \)-module \([2]\).

The above example can be generalized to the situation of Lie algebroids.

**Example 2.12.** Let \((A, \cdot, \cdot, a)\) be a Lie algebroid with a nondegenerate representation \( \rho_A : A \rightarrow \mathfrak{D}E \). On the vector bundle \( A \oplus \mathfrak{J}E \), define an \( E \)-valued pairing \( \langle \cdot, \cdot \rangle_E \) and a bracket \( \{\cdot, \cdot\} \) by

\[
\begin{align*}
\langle X + \mu, Y + \nu \rangle_E &= \frac{1}{2}\left( \langle \rho_A(X), \nu \rangle_E + \langle \rho_A(Y), \mu \rangle_E \right), \\
\{X + \mu, Y + \nu\} &= [X, Y] + \mathcal{L}_{\rho(X)}\nu - \mathcal{L}_{\rho(Y)}\mu + \varpi \langle \rho_A(Y), \mu \rangle_E,
\end{align*}
\]

for any \( X + \mu, Y + \nu \in \Gamma(A \oplus \mathfrak{J}E) \), and define \( \rho : A \oplus \mathfrak{J}E \rightarrow \mathfrak{D}E \) by \( \rho(X + \mu) = \rho_A(X) \). Similar to \([12]\), we have \( \rho^* = 1_{\mathfrak{J}E} \). Then, it is easily seen that \((A \oplus \mathfrak{J}E, \langle \cdot, \cdot \rangle_E, \{\cdot, \cdot\}, \rho)\) is an \( E \)-Courant algebroid.

- **The jet bundle of a Courant algebroid**

  At the end of this section, we prove that for any Courant algebroid \( \mathcal{J}C \), \( \mathcal{J}C \) is a \( T^*M \)-Courant algebroid. The original definition of a Courant algebroid is introduced in \([22]\). Here we use the alternative definition given by D. Roytenberg in \([31]\), that a Courant algebroid is a vector bundle \( C \rightarrow M \) together with some compatible structures — a nondegenerate bilinear form \( \langle \cdot, \cdot \rangle \) on the bundle, a bilinear operation \( \{\cdot, \cdot\} \) on \( \Gamma(E) \) and a bundle map \( a : C \rightarrow TM \) satisfying \( a \circ a^* = 0 \). In particular, \((\Gamma(C), \{\cdot, \cdot\})\) is a Leibniz algebra.

  On the jet bundle \( \mathcal{J}C \) of the vector bundle \( C \), we introduce the \( T^*M \)-valued pairing \( \langle \cdot, \cdot \rangle_* \), the bracket \( \{\cdot, \cdot\}_{\mathcal{J}C} \) and the anchor \( \rho : \mathcal{J}C \rightarrow \mathfrak{D}(T^*M) \) as follows.

  a) For any \( X, Y \in \Gamma(C) \), the \( T^*M \)-valued pairing \( \langle \cdot, \cdot \rangle_* \) of \( dX, dY \) is given by

  \[
  \langle dX, dY \rangle_* = d\langle X, Y \rangle.
  \]  

  By \([3]\), we get

  \[
  \langle dX, df \otimes Y \rangle_* = \langle X, Y \rangle df, \quad \langle df \otimes X, df \otimes Y \rangle_* = 0.
  \]

  b) For any \( X, Y \in \Gamma(C) \), the bracket \( \{\cdot, \cdot\}_{\mathcal{J}C} \) of \( dX, dY \) is given by

  \[
  [dX, dY]_{\mathcal{J}C} = d[X, Y].
  \]  

  By \([4]\), \([5]\) and \([6]\), we have

  \[
  \begin{align*}
  [dX, df \otimes Y]_{\mathcal{J}C} &= df \otimes [X, Y] + d(a(X)f) \otimes Y, \\
  [df \otimes Y, dX]_{\mathcal{J}C} &= df \otimes [Y, X] - d(a(X)f) \otimes Y + 2 \langle X, Y \rangle d\alpha^*(df), \\
  [df \otimes X, dg \otimes Y]_{\mathcal{J}C} &= a(X)(g)df \otimes Y - a(Y)(f)dg \otimes X.
  \end{align*}
  \]

  c) For any \( X \in \Gamma(C) \), \( \rho(dX) \in \Gamma(\mathfrak{D}(T^*M)) \) is given by

  \[
  \rho(dX)(\cdot) = \mathcal{L}_{a(X)}(\cdot).
  \]

  By \([3]\), we get

  \[
  \rho(df \otimes X) = a(X) \otimes df, \quad \forall f \in C^\infty(M).
  \]
For any $\xi \in \Omega^2(M)$, we have
\[
\rho(dX)(f\xi) = \mathcal{L}_{a(X)}(f\xi) = f\mathcal{L}_{a(X)}(\xi) + a(X)(f)\xi,
\]
which implies that $j \circ \rho \circ dX = a(X)$, where $j : \mathcal{D}(T^*M) \to TM$ is the anchor of $\mathcal{D}(T^*M)$ given in [1]. Furthermore, for any $g \in C^\infty(M)$, the fact that $\rho(df \otimes X)(g\xi) = g\rho(df \otimes X)(\xi)$ implies that $j \circ \rho(df \otimes X) = 0$.

We identify $\mathcal{C}$ with $C^*$ by the bilinear form. For any $f, g \in C^\infty(M)$, it is straightforward to obtain the following relations:
\[
\begin{align*}
\rho^*(ad(df)) &= dq \circ a^*(df), \\
\rho^*(d(df)) &= dq \circ a^*(df) + f(a^*dg).
\end{align*}
\]
These structures give rise to a $T^*M$-Courant algebroid.

**Theorem 2.13.** For any Courant algebroid $\mathcal{C}$, $(\mathfrak{J}\mathcal{C}, (\cdot, \cdot)_\mathcal{C}, \mathcal{J}\mathcal{C}, \rho)$ is a $T^*M$-Courant algebroid.

**Proof.** It is straightforward to see that the pairing $(\cdot, \cdot)_\mathcal{C}$ and $\rho$ are bundle maps and $(\Gamma(\mathfrak{J}\mathcal{C}), [\cdot, \cdot]_{T^*M}, \rho)$ is a Leibniz algebra. To show that the data $(\mathfrak{J}\mathcal{C}, (\cdot, \cdot)_\mathcal{C}, [\cdot, \cdot]_{T^*M}, \rho)$ satisfies the properties listed in Definition 2.11, it suffices to consider elements of the form $dX, dY, dZ, df \otimes X, dg \otimes Y, dh \otimes Z$, where $X, Y, Z \in \Gamma(C)$, $f, g, h \in C^\infty(M)$.

First we check Property (EC-1). Clearly, we have
\[
\rho([dX, dY]_{T^*M}) = \rho(ddf) = [\rho(df), \rho(dy)]_{T^*M}.
\]
Furthermore, since $a \circ a^* = 0$, we have
\[
\begin{align*}
\rho(df \otimes X, df \otimes Y)_{T^*M} &= \rho(df \otimes [X, Y] - d(a(Y)f) \otimes X + 2 \langle X, Y \rangle d(a^*(df))) \\
&= a([X, Y]) \otimes df - a(X) \otimes d(a(Y)f).
\end{align*}
\]
On the other hand,
\[
\begin{align*}
\rho(df \otimes X, df \otimes Y)_{T^*M}(\xi) &= \{a(X) \otimes df, \mathcal{L}_{a(Y)}\}(\xi) = (\mathcal{L}_{a(Y)}\xi, a(X))df - \mathcal{L}_{a(Y)}\langle a(X), \xi \rangle df \\
&= \langle a([X, Y]), \xi \rangle df - \langle a(X), \xi \rangle d(a(Y)f),
\end{align*}
\]
which implies
\[
\rho(df \otimes X, df \otimes Y)_{T^*M} = [\rho(df \otimes X), \rho(df \otimes Y)]_{T^*M}.
\]
Similarly, we have
\[
\begin{align*}
\rho(df \otimes X, df \otimes Y)_{T^*M} &= \rho(df \otimes X, df \otimes Y)_{T^*M} \\
&= a([X, Y]) \otimes df + a(Y) \otimes d(a(X)f), \\
\rho(df \otimes X, df \otimes Y)_{T^*M} &= \rho(df \otimes X, df \otimes Y)_{T^*M} \\
&= (a(X)g)a(Y) \otimes df - (a(Y)f)a(X) \otimes dg.
\end{align*}
\]
To see Property (EC-2), notice that $[dX, df \otimes X]_{T^*M} = 0$ and $(df \otimes X, df \otimes X)_* = 0$, so we have
\[
\begin{align*}
[dX, df \otimes X]_{T^*M} &= \rho^*d (df \otimes X, df \otimes X)_*.
\end{align*}
\]
Furthermore,
\[
\begin{align*}
[dX, df \otimes X]_{T^*M} &= d[dX, df \otimes X] = d \circ a^*(d \langle X, X \rangle) = \rho^*d \circ d \langle X, X \rangle = \rho^*d (dX, dX)_*, \\
[df \otimes X, df \otimes X]_{T^*M} &= 2 \langle X, Y \rangle \rho^*d df + 2df \otimes a^*(d \langle X, Y \rangle) \\
&= 2\rho^*d (\langle X, Y \rangle df) = 2\rho^*d (df \otimes X, df \otimes X)_*,
\end{align*}
\]
which implies that Property (EC-2) holds. It is straightforward to verify Property (EC-3). Property (EC-4) follows from [16]. Property (EC-5) follows from the fact that $a \circ a^* = 0$. 

---

8
3 The $E$-dual pair of Lie algebroids

Let $A$ be a vector bundle and $B$ a subbundle of $\text{Hom}(A, E)$. For any $\mu^k \in \text{Hom}(\wedge^k A, E)$, denote by $\mu^k_B$ the induced bundle map from $\wedge^{k-1} A$ to $\text{Hom}(A, E)$ such that

$$\mu^k_B(X_1, \ldots, X_{k-1})(X_k) = \mu^k(X_1, \ldots, X_{k-1}, X_k).$$

(17)

Introduce a series of vector bundles $\text{Hom}(\wedge^k A, E)_B, k \geq 0$ by setting $\text{Hom}(\wedge^0 A, E)_B = E$, $\text{Hom}(\wedge^1 A, E)_B = B$ and

$$\text{Hom}(\wedge^k A, E)_B \triangleq \{ \mu^k \in \text{Hom}(\wedge^k A, E) \mid \text{Im}(\mu^k) \subset B \}, \quad (k \geq 2).$$

(18)

If $B$ is a subbundle of $\text{Hom}(A, E)$, then $A$ is also a subbundle of $\text{Hom}(B, E)$. The notation $\text{Hom}(\wedge^k B, E)_A$ is thus clear.

**Definition 3.1.** Let $A$ and $E$ be two vector bundles over $M$. A vector bundle $B \subset \text{Hom}(A, E)$ is called an $E$-dual bundle of $A$ if the $E$-valued pairing $\langle \cdot, \cdot \rangle_E : A \times_M B \rightarrow E$, $\langle a, b \rangle_E \triangleq b(a)$ (where $a \in A, b \in B$) is nondegenerate.

Obtainedly, if $B$ is an $E$-dual bundle of $A$, then $A$ is an $E$-dual bundle of $B$. We call the pair $(A, B)$ an $E$-dual pair of vector bundles.

Assume that $(A, [\cdot, \cdot], \alpha)$ is a Lie algebroid and $B \subset \text{Hom}(A, E)$ is an $E$-dual bundle of $A$. A representation $\rho_A : A \rightarrow \mathfrak{d}E$ of $A$ on $E$ is said to be $B$-invariant if $\langle \text{Im}(\rho_A^\ast A), d_A \rangle$ is a subcomplex of $(\Omega^\ast(A, E), d_A)$, where $d_A$ is the coboundary operator associated to $\rho_A$. If $\rho_A$ is a $B$-invariant representation, we have $\rho_A^\ast(\mathfrak{d}E) \subset B$. In fact, by definition, one has

$$\rho_A^\ast(\mu)(X) = \langle \mu, \rho_A(X) \rangle_E, \quad \forall \mu \in \mathfrak{d}E, X \in A,$$

and it follows that $\rho_A^\ast : \mathfrak{d}E \rightarrow B$ is given by $\rho_A^\ast([u]_m) = (d_A u)_m$, for all $u \in \Gamma(E)$. Thus, $\rho_A^\ast(\mathfrak{d}E) \subset B$ is equivalent to the condition that $d_A(\Gamma(E)) \subset \Gamma(B)$.

Furthermore, for any representation $\rho_A : A \rightarrow \mathfrak{d}E$, there are two natural Lie derivative operations along $X \in \Gamma(A)$. The first one is

$$\mathfrak{L}_X : \Gamma(\text{Hom}(\wedge^k A, E)) \longrightarrow \Gamma(\text{Hom}(\wedge^k A, E)) = \Gamma(\wedge^k A^\ast \otimes E)$$

defined by

$$\mathfrak{L}_X(\omega \otimes u) = (\mathfrak{L}_X \omega) \otimes u + \omega \otimes \rho_A(X)u, \quad \forall \omega \in \Gamma(\wedge^k A^\ast), \quad u \in \Gamma(E).$$

The second one is

$$\mathfrak{L}_X : \Gamma(\text{Hom}(\wedge^k(A^\ast \otimes E), E)) \longrightarrow \Gamma(\text{Hom}(\wedge^k(A^\ast \otimes E), E)) = \Gamma(\wedge^k(A \otimes E^\ast) \otimes E)$$

defined by $\mathfrak{L}_X u = \rho_A(X)u$, for $u \in \Gamma(E)$, and

$$\mathfrak{L}_X \Xi(\varpi_1 \wedge \cdots \wedge \varpi_k) = \rho_A(X)(\Xi(\varpi_1 \wedge \cdots \wedge \varpi_k)) - \sum_{i=1}^{k} \Xi(\varpi_1 \wedge \cdots \wedge \mathfrak{L}_X \varpi_i \wedge \cdots \wedge \varpi_k), \quad (19)$$

for all $\Xi \in \Gamma(\text{Hom}(\wedge^k(A^\ast \otimes E), E))$, $\varpi_i \in \Gamma(A \otimes E)$. In particular, since $A \subset \text{Hom}(A^\ast \otimes E, E)$, we have

$$\mathfrak{L}_X Y = [X, Y], \quad \forall Y \in \Gamma(A).$$
Proposition 3.2. Let $A$ be a Lie algebroid together with a representation $\rho : A \to \mathcal{D}E$ and $B \subset \text{Hom}(A, E)$ a subbundle of $\text{Hom}(A, E)$ such that $(A, B)$ is an $E$-dual pair of vector bundles. Then the following statements are equivalent:

1. the representation $\rho_A : A \to \mathcal{D}E$ is $B$-invariant;
2. $d^4\Gamma(E) \subset \Gamma(B)$ and $d^4\Gamma(B) \subset \Gamma(\text{Hom}(\wedge^2A, E)_B)$;
3. $\Gamma(\text{Hom}(\wedge^kA, E)_B)$ is invariant under the operation $\mathcal{L}_X$ for any $X \in \Gamma(A)$;
4. $\Gamma(\text{Hom}(\wedge^kB, E)_A)$ is invariant under the operation $\mathcal{L}_X$ for any $X \in \Gamma(A)$.

Proof. The implication (1) $\implies$ (2) is obvious. We adopt an inductive approach to see the implication (2) $\implies$ (1). For any $n \geq 1$, $\mathcal{L}_X : \Gamma(\text{Hom}(\wedge^nA, E)_B) \to \Gamma(\text{Hom}(\wedge^nA, E)_B)$ is well defined and we have $i_Y\mathcal{L}_X - \mathcal{L}_X i_Y = i_{[Y, X]}$. Assume that $d^4\Gamma(\text{Hom}(\wedge^{n-1}A, E)_B) \subset \Gamma(\text{Hom}(\wedge^nA, E)_B)$ and $d^4\Gamma(\text{Hom}(\wedge^nA, E)_B) \subset \Gamma(\text{Hom}(\wedge^{n+1}A, E)_B)$ hold for all $\mu^{n+1} \in \Gamma(\text{Hom}(\wedge^{n+1}A, E)_B)$. To prove that $d^4\mu^{n+1} \in \Gamma(\text{Hom}(\wedge^{n+1}A, E)_B)$, it suffices to show that $i_X d^4\mu^{n+1} \in \Gamma(\text{Hom}(\wedge^nA, E)_B)$ holds for all $Y \in \Gamma(A)$. In fact,

$$
i_Yi_Xd^4\mu^{n+1} = (i_Y\mathcal{L}_X - \mathcal{L}_Xi_Y)d^4\mu^{n+1} = (i_Y\mathcal{L}_X - \mathcal{L}_X i_Y)d^4\mu^{n+1}
= (i_Y\mathcal{L}_X - \mathcal{L}_X i_Y)(X, i_Y)d^4\mu^{n+1} - i_Yd^4\mathcal{L}_X d^4\mu^{n+1}
= i_{[Y, X]}d^4\mu^{n+1} + \mathcal{L}_X i_Yd^4\mu^{n+1} - i_Yd^4\mathcal{L}_X d^4\mu^{n+1} \in \Gamma(\text{Hom}(\wedge^nA, E)_B).$$

So we conclude that $\Gamma(\text{Hom}(\wedge^kA, E)_B)$ is a subcomplex of $\Omega^*(A, E)$. This completes the proof of the equivalence of (1) and (2). The equivalence of (1) and (3) is obvious.

Next we prove the equivalence of (2) and (4). For any $X^k \in \Gamma(\text{Hom}(\wedge^kB, E)_A)$ and $\xi_i \in B$, we have

$$\langle i_{\xi_1, \ldots, \xi_{k-1}} \mathcal{L}_X X^k, \xi_k \rangle_E
= \langle \mathcal{L}_X X^k, \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{k-1} \wedge \xi_k \rangle
= \rho_A(X)(X^k(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{k-1})) - \sum_{i=1}^{k-1} X^k(\xi_1 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_i \wedge \cdots \wedge \xi_{k-1} \wedge \xi_k)
= \rho_A(X) \langle i_{\xi_1, \ldots, \xi_{k-1}} X^k, \xi_k \rangle_E - \sum_{j=1}^{k-1} \langle i_{\xi_1, \ldots, \xi_{j-1}, \xi_j, \ldots, \xi_{k-1}} X^k, \xi_k \rangle_E
= \left[ [X, i_{\xi_1, \ldots, \xi_{j-1}} X^k] - \sum_{j=1}^{k-1} i_{\xi_1, \ldots, \xi_{j-1}, \xi_j, \ldots, \xi_{k-1}} X^k, \xi_k \right]_E.$$

Since the $E$-valued pairing $\langle \cdot, \cdot \rangle_E$ is nondegenerate, we have

$$i_{\xi_1, \ldots, \xi_{k-1}} \mathcal{L}_X X^k = [X, i_{\xi_1, \ldots, \xi_{k-1}} X^k] - \sum_{j=1}^{k-1} i_{\xi_1, \ldots, \xi_{j-1}, \xi_j, \ldots, \xi_{k-1}} X^k,$$
which implies the equivalence of (2) and (4). $\blacksquare$

Definition 3.3. An $E$-dual pair of Lie algebroids $((A, \rho_A); (B, \rho_B))$ consists of two Lie algebroids $A$ and $B$ which are mutually $E$-dual vector bundles, a $B$-invariant representation $\rho_A : A \to \mathcal{D}E$ and an $A$-invariant representation $\rho_B : B \to \mathcal{D}E$. 

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Obviously, $\mathcal{J}E$ and $\mathfrak{D}E$ are mutually $E$-dual bundles. In the following, we show some properties of the bundle $\mathrm{Hom}(\wedge^k \mathfrak{D}E, E)_{\mathcal{J}E}$.

**Proposition 3.4.** If $k \geq 2$, for any $\mu^k \in \Gamma(\mathrm{Hom}(\wedge^k \mathfrak{D}E, E)_{\mathcal{J}E})$, there is a unique bundle map $\lambda_{\mu^k} \in \Gamma(\mathrm{Hom}(\wedge^{k-1}TM, E))$ such that

$$
\mu^k(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-1} \wedge \Phi) = \Phi \circ \lambda_{\mu^k}(j(\mathfrak{o}_1) \wedge \cdots \wedge j(\mathfrak{o}_{k-1})), \quad \forall \Phi \in \Gamma(\mathfrak{gl}(E)), \quad \mathfrak{o}_i \in \Gamma(\mathfrak{D}E).
$$

**Proof.** By definition, we have $\mu^k(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_k) = \left\langle \mu^k_1(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-1}), \mathfrak{o}_k \right\rangle_E$, which implies

$$
p \circ \mu^k_1(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-1}) = \mu^k(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-1} \wedge 1),
$$

We claim that $i_\Phi (p \circ \mu^k_1) = 0$, for all $\Phi \in \Gamma(\mathfrak{gl}(E))$. In fact, we have

$$
p \circ \mu^k_1(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-2} \wedge \Phi) = \mu^k(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-2} \wedge \Phi \wedge 1) = -\mu^k(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-2} \wedge 1 \wedge \Phi)
$$

$$
= -\left\langle \mu^k_1(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-2} \wedge 1), \Phi \right\rangle_E = -\Phi \circ p \circ \mu^k_1(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-2} \wedge 1)
$$

$$
= -\Phi \circ \mu^k(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-2} \wedge 1 \wedge 1)
$$

$$
= 0.
$$

Therefore, the bundle map $p \circ \mu^k_1 : \wedge^{k-1} \Gamma(\mathfrak{D}E) \to \Gamma(E)$ factors through $j$, i.e. there is a unique bundle map $\lambda_{\mu^k} : \Gamma(\wedge^{k-1}TM) \to \Gamma(E)$ such that

$$
p \circ \mu^k_1(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-1}) = \lambda_{\mu^k}(j(\mathfrak{o}_1) \wedge \cdots \wedge j(\mathfrak{o}_{k-1})).
$$

Therefore, for $k \geq 2$, the vector bundle $\mathrm{Hom}(\wedge^k \mathfrak{D}E, E)_{\mathcal{J}E}$ can be defined directly by

$$
\mathrm{Hom}(\wedge^k \mathfrak{D}E, E)_{\mathcal{J}E}
$$

$$
= \{ \mu^k \in \mathrm{Hom}(\wedge^k \mathfrak{D}E, E) \mid \exists ! \lambda_{\mu^k} \in \mathrm{Hom}(\wedge^{k-1}TM, E), \text{ s.t. } \forall \Phi \in \mathfrak{gl}(E), \quad \mathfrak{o}_i \in \mathfrak{D}E, \quad \mu^k(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_{k-1} \wedge \Phi) = \Phi \circ \lambda_{\mu^k}(j(\mathfrak{o}_1) \wedge \cdots \wedge j(\mathfrak{o}_{k-1})) \}\}
$$

We will write $\lambda_{\mu^k} = p^k(\mu^k)$ for $\mu^k$ given above. For any $\xi \in \mathrm{Hom}(\wedge^k TM, E)$, $k \geq 1$, we define $\varpi^k(\xi) \in \mathrm{Hom}(\wedge^k \mathfrak{D}E, E)_{\mathcal{J}E}$ by

$$
\varpi^k(\xi)(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_k) \triangleq \xi(j(\mathfrak{o}_1) \wedge \cdots \wedge j(\mathfrak{o}_k)), \quad \forall \mathfrak{o}_i \in \mathfrak{D}E.
$$

In addition, we regard $\mathrm{Hom}(\wedge^{-1}TM, E) = 0$, $\mathrm{Hom}(\wedge^0 TM, E) = E$ and $\mathrm{Hom}(\wedge^0 \mathfrak{D}E, E)_{\mathcal{J}E} = E$. Let $p^0 = 0$ and $\varpi^0 = 1_E$.

**Proposition 3.5.** For any $k \geq 0$, the following sequence is exact:

$$
0 \longrightarrow \mathrm{Hom}(\wedge^k TM, E) \xrightarrow{\varpi^k} \mathrm{Hom}(\wedge^k \mathfrak{D}E, E)_{\mathcal{J}E} \xrightarrow{p^k} \mathrm{Hom}(\wedge^{k-1}TM, E) \longrightarrow 0.
$$

**Proof.** If $k = 0, 1$, the result is clear. For $k \geq 2$, it is obvious that $\varpi^k$ is an injection and $p^k \circ \varpi^k = 0$. Now we prove that $p^k$ is surjective. For every $\lambda \in \Gamma(\mathrm{Hom}(\wedge^{k-1}TM, E))$, i.e. a bundle map from $\wedge^{k-1}TM$ to $E$, we define $\bar{\lambda} \in \Gamma(\mathrm{Hom}(\wedge^k \mathfrak{D}E, E)_{\mathcal{J}E})$ by

$$
\bar{\lambda}(\mathfrak{o}_1 \wedge \cdots \wedge \mathfrak{o}_k) \triangleq \sum_{i=1}^{k} (-1)^{i+1} j(\mathfrak{o}_i) \circ \lambda(j(\mathfrak{o}_1) \wedge \cdots \wedge \hat{j(\mathfrak{o}_i)} \wedge \cdots \wedge \mathfrak{o}_k))
$$

$$
+ \sum_{i<j} (-1)^{i+j} \lambda([j(\mathfrak{o}_i), j(\mathfrak{o}_j)] \wedge \mathfrak{o}_1 \wedge \cdots \wedge \hat{j(\mathfrak{o}_i)} \wedge \cdots \hat{j(\mathfrak{o}_j)} \wedge \mathfrak{o}_k],
$$

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for any \( \mathfrak{d}_i \in \Gamma(\mathcal{D}E), \ i = 1, \ldots, k \). It is straightforward to see that \( \tilde{\lambda} \) is a bundle map from \( \wedge^k \mathcal{D}E \) to \( E \). Since for any \( \Phi \in \Gamma(\mathfrak{gl}(E)) \),

\[
\tilde{\lambda}(\mathfrak{d}_1 \wedge \cdots \wedge \mathfrak{d}_{k-1} \wedge \mathfrak{d}_k) = (-1)^{k+1} \Phi \circ \lambda(\mathfrak{d}_1 \wedge \cdots \wedge \mathfrak{d}_{k-1}),
\]

we have \( \tilde{\lambda} \in \Gamma(\text{Hom}(\wedge^k \mathcal{D}E, E)_{\mathfrak{J}E}) \) and \( p^k \tilde{\lambda} = (-1)^{k+1} \lambda \).

Finally, if \( \mu^k \in \Gamma(\text{Hom}(\wedge^k \mathcal{D}E, E)_{\mathfrak{J}E}) \) satisfies \( p^k(\mu^k) = 0 \), then we have

\[
\mu^k(\mathfrak{d}_1 \wedge \cdots \wedge \mathfrak{d}_{k-1} \wedge \mathfrak{d}_k) = \Phi \circ \hat{\mu}^k(\mu^k(\mathfrak{d}_1 \wedge \cdots \wedge \mathfrak{d}_{k-1})) = 0,
\]

which implies that the map \( \mu^k \) factors through \( \hat{j} \), i.e. there is a unique \( \tilde{\xi} \in \text{Hom}(\wedge^k TM, E) \) such that

\[
\mu^k(\mathfrak{d}_1 \wedge \cdots \wedge \mathfrak{d}_k) = \tilde{\xi}(\hat{j}(\mathfrak{d}_1) \wedge \cdots \wedge \hat{j}(\mathfrak{d}_k)).
\]

Therefore, sequence (20) is exact. \( \blacksquare \)

In the sequel, we will omit the embedding \( \mathfrak{k} \) and directly regard \( \text{Hom}(\wedge^k TM, E) \) as a subbundle of \( \text{Hom}(\wedge^k \mathcal{D}E, E)_{\mathfrak{J}E} \). Hence we have

\[
p(i_\mathfrak{d} \mu^k) = i_\mathfrak{d} p^k \mu^k, \quad \forall \mathfrak{d} \in \mathcal{D}E, \ \mu^k \in \text{Hom}(\wedge^k \mathcal{D}E, E)_{\mathfrak{J}E}. \tag{21}
\]

Recall that \( \mathcal{D}E \) is a Lie algebroid and there is a natural representation \( 1_{\mathcal{D}E} \) on \( E \). For \( \mu^k \in \Gamma(\text{Hom}(\wedge^k \mathcal{D}E, E)_{\mathfrak{J}E}) \) and \( \mathfrak{d}_i \in \Gamma(\mathcal{D}E) \), the coboundary operator \( d : \bigwedge^{*+1}(\mathcal{D}E, E) \to \bigwedge^{*+1}(\mathcal{D}E, E) \) is given by

\[
d\mu^k(\mathfrak{d}_1 \wedge \cdots \wedge \mathfrak{d}_{k+1}) \triangleq \sum_{i=1}^{k+1} (-1)^{i+1} \mathfrak{d}_i \circ \mu^k(\mathfrak{d}_1 \wedge \cdots \mathfrak{d}_{i-1} \wedge \mathfrak{d}_{i+1} \wedge \mathfrak{d}_{k+1}) + \sum_{i<j} (-1)^{i+j} \mu^k([\mathfrak{d}_i, \mathfrak{d}_j] \mathcal{D} \wedge \mathfrak{d}_1 \wedge \cdots \mathfrak{d}_{i-1} \wedge \mathfrak{d}_{i+1} \wedge \cdots \mathfrak{d}_{k+1}). \tag{22}
\]

**Lemma 3.6.** The representation \( 1_{\mathfrak{D}E} \) of the gauge Lie algebroid \( \mathcal{D}E \) on \( E \) is \( \mathfrak{J}E \)-invariant, i.e. \( (\Gamma(\text{Hom}(\bigwedge^k \mathcal{D}E, E)_{\mathfrak{J}E}), d) \) is a subcomplex of \( (\Gamma(\text{Hom}(\bigwedge^k \mathcal{D}E, E)), d) \). More precisely, for any \( \mathfrak{d} \in \Gamma(\mathcal{D}E) \) and \( \mu^k \in \Gamma(\text{Hom}(\wedge^k \mathcal{D}E, E)_{\mathfrak{J}E}) \), \( k \geq 0 \), we have

\[
p^k(\mathfrak{L}_\mathfrak{d} \mu^k) = \mathfrak{L}_\mathfrak{d}(p^k(\mu^k)), \tag{23}
\]

\[
p^k d p^k \mu^k = (-1)^{k+1} d p^k \mu^k, \tag{24}
\]

\[
p^{k+1} d \mu^k = d p^k \mu^k + (-1)^k \mu^k. \tag{25}
\]

**Proof.** Assume that \( p^k \mu^k = \lambda \in \Gamma(\text{Hom}(\wedge^{k-1} TM, E)) \), i.e. for all \( \Phi \in \Gamma(\mathfrak{gl}(E)) \), \( \mathfrak{d}_i \in \Gamma(\mathcal{D}E) \),

\[
\mu^k(\mathfrak{d}_1 \wedge \cdots \wedge \mathfrak{d}_{k-1} \wedge \mathfrak{d}_k) = \Phi \circ \lambda(\mathfrak{d}_1) \wedge \cdots \wedge \mathfrak{d}_{k-1}).
\]
Theorem 3.9. For the cochain complex
\[ C_p = \cdots \rightarrow 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O} \rightarrow \cdots \]
we have
\[ H_p(C_p) = 0 \]
for all \( p = 0, 1, 2, \ldots \). In other words, there is a long exact sequence:
\[ 0 \rightarrow \Gamma(E) \rightarrow \cdots \rightarrow \Gamma(\mathcal{E}) \rightarrow H^k(C(E)) \rightarrow 0, \]
where \( n = \dim M + 1 \).

Proof. If \( k = \dim M + 2 \), we have \( \text{Hom}(\wedge^k TM, E) = \text{Hom}(\wedge^k+1 TM, E) = 0 \). By the exact sequence \( 20 \), \( \text{Hom}(\wedge^k+2 TM, E)_{\mathcal{E}} = 0 \). By \( 25 \) and if \( d\mu_k = 0 \), we have \( H^k(C(E)) = 0 \) and hence \( H^k(C(E)) = 0 \).

Example 3.10. If \( E \rightarrow V \rightarrow \mathcal{E} \rightarrow V \rightarrow 0 \). In fact, for any \( \phi \in \text{Hom}(\wedge^2 \mathfrak{gl}(V), V) \), \( A, B \in \mathfrak{gl}(V) \), we have
\[ \phi(AB) = B\phi(A) = -BA \phi(1) = 0. \]
Therefore, \( \phi = 0 \), which implies that \( \text{Hom}(\wedge^2 \mathfrak{gl}(V), V) = 0 \). On the other hand, for any \( v \in V \), \( d\phi = 0 \) and for any \( u \in V \), \( d\phi = 0 \). Thus, the first cohomology is trivial.
as a section of \( E^* \) satisfying \( 1^*(f) = f \), \( 1^*(X) = 1^*(\mu) = 0 \), for all \( f \in C^\infty(M) \), \( X \in \mathfrak{X}(M) \) and \( \mu \in \Gamma(\mathfrak{A}E) \). Since \( df = df + f1^* \), we have

\[
df = 0 \iff f = 0, \quad \forall f \in C^\infty(M),
\]

which implies that \( H^0(C(E)) = 0 \).

For any \( \lambda \in \Omega^1(M) \), by (25), one gets \( p^2(d\lambda) = -\lambda \). Furthermore, we have \( d(f1^*) = d(df) - df \), which implies that \( p^2(d(f1^*)) = df \). Therefore, for any \( \mu \in \Gamma(\mathfrak{A}E) \), we have

\[
d\mu = 0 \iff \mu = df + f1^* \iff \mu = df,
\]

for some \( f \in C^\infty(M) \), which implies that \( H^1(C(E)) = 0 \). For similar reasons, one has \( H^k(C(E)) = 0 \).

## 4 The automorphism groups of omni-Lie algebroids

In this section, we study the automorphism groups and the twists of omni-Lie algebroids. For \( i = 1, 2 \), it is subtle to define morphisms between two \( E_i \)-Courant algebroids with different base manifolds, which remains a topic in the future. As for general Lie algebroid morphisms and Courant algebroid morphisms, please refer to [4, 13, 3]. Here we only consider the automorphisms of \( E \)-Courant algebroids.

Given an automorphism \( \Phi : E \rightarrow E \) over the diffeomorphism \( \phi : M \rightarrow M \) of the base manifold, there induces a unique automorphism \( \text{Ad}_\phi \) of \( \mathfrak{D} \) such that:

\[
\text{Ad}_\phi(\varnothing)(u) = \Phi \circ \varnothing \circ (\Phi^{-1}(u)), \quad \forall \varnothing \in \Gamma(\mathfrak{D}E), \; u \in \Gamma(E).
\]

**Definition 4.1.** The automorphism group \( \text{Aut}(\mathcal{K}) \) of an \( E \)-Courant algebroid \( \mathcal{K} \) is the group of bundle automorphisms \( F : \mathcal{K} \rightarrow \mathcal{K} \) covering bundle automorphisms \( \Phi : E \rightarrow E \) such that:

1. \( F \) is orthogonal, i.e. \( (F(X), F(Y))_E = \Phi(X, Y)_E \);
2. \( F \) is bracket-preserving, i.e. \( F[X, Y]_\mathcal{K} = [F(X), F(Y)]_\mathcal{K} \);
3. \( F \) is compatible with the anchor, i.e. \( \rho \circ F = \text{Ad}_\phi \circ \rho \).

We usually denote such an automorphism by a pair \((F, \Phi)\). The set of all automorphisms \((F_1, \Phi_1)\) of the \( E \)-Courant algebroid \( \mathcal{K} \) is a normal subgroup of \( \text{Aut}(\mathcal{K}) \), similar to the \( B \)-field introduced in [11].

Now let us study the automorphism group of the omni-Lie algebroid \( \mathcal{E} = \mathfrak{D}E \oplus \mathfrak{A}E \) defined in Definition 2.8. For any automorphism \( \Phi : E \rightarrow E \), there is an induced map \( \Phi : \mathfrak{A}E \rightarrow \mathfrak{A}E \) defined by

\[
\tilde{\Phi}(\mu) = [\Phi(u)]_{\mathfrak{a}(m)}, \quad \forall \mu = [u]_m \in (\mathfrak{A}E)_m, \; u \in \Gamma(E).
\]

It is clear that the pair \((\text{Ad}_\phi + \tilde{\Phi}, \Phi)\) is an automorphism of the omni-Lie algebroid \( \mathcal{E} \) and it is totally determined by \( \Phi \).

There is another symmetry of the omni-Lie algebroid \( \mathcal{E} \), which we call the \( B \)-field transformation. Let us elaborate on this idea. For any \( b \in \Gamma(\text{Hom}(\mathfrak{D}E, \mathfrak{A}E)) \), there is a transformation \( e^b : \mathcal{E} \rightarrow \mathcal{E} \) defined by

\[
e^b \left( \begin{array}{c} \varnothing \\ \mu \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ b & 1 \end{array} \right) \left( \begin{array}{c} \varnothing \\ \mu \end{array} \right) = \left( \begin{array}{c} \varnothing \\ \mu + i_\varnothing b \end{array} \right).
\]

**Lemma 4.2.** For \( b \in \Gamma(\text{Hom}(\Lambda^2\mathfrak{D}E, \mathfrak{A}E)) \), the map \( e^b \) is an automorphism of the omni-Lie algebroid \( \mathcal{E} \) if and only if \( b \) is closed, i.e. \( db = 0 \).
Proof. Let $\mathfrak{d}, \mathfrak{r} \in \Gamma(\mathcal{D} E), \mu, \nu \in \Gamma(\mathcal{J} E)$. First, $b$ is skew-symmetric implies that $e^b$ preserves the standard pairing given in (3). We also have
\[
\{e^b(\mathfrak{d} + \mu), e^b(\mathfrak{r} + \nu)\} = \{\mathfrak{d} + \mu, \mathfrak{r} + \nu\} + \{\mathfrak{d}, \iota_b \mu\} + \{\iota_b \mathfrak{d}, \mathfrak{r}\} = \{\mathfrak{d} + \mu, \mathfrak{r} + \nu\} + \mathfrak{d} \iota_b \mu - i_b \mathfrak{d} \iota_b \mu = e^b(\{\mathfrak{d} + \mu, \mathfrak{r} + \nu\}) + i_b \mathfrak{d} \iota_b \mu. \tag{28}
\]
So $e^b$ is an automorphism of the omni-Lie algebroid $\mathcal{E}$ if and only if $i_b \mathfrak{d} \iota_b \mu = 0$ for all $\mathfrak{d}, \mathfrak{r} \in \Gamma(\mathcal{D} E)$, which happens if and only if $db = 0$. ■

The transformation $e^b$ defined by (27) will be called a B-field transformation, for any $b \in \Gamma(\text{Hom}(\wedge^2 \mathcal{D} E, \mathcal{E}))$ with $db = 0$.

Corollary 4.3. The abelian group of B-field transformations is isomorphic to $\Gamma(\text{Hom}(TM, E))$.

Proof. For any $b \in \Gamma(\text{Hom}(\wedge^2 \mathcal{D} E, \mathcal{E}))$ such that $db = 0$, assume that $\mu = p^b b \in \Gamma(\text{Hom}(TM, E))$, where $p^b$ is given in Sequence (29). Then, by (28), $b = -d\mu$. As a vector space, $\Gamma(\mathcal{J} E) \cong \Gamma(\text{Hom}(TM, E)) \oplus d\Gamma(E)$. Since $d^2 = 0$, it follows that
\[
d(\Gamma(\mathcal{J} E)) \cong d(\Gamma(\text{Hom}(TM, E))) \cong \Gamma(\text{Hom}(TM, E)).
\]

In fact, any automorphism of the omni-Lie algebroid $\mathcal{E}$ is a composition of an automorphism $\Phi$ of the vector bundle $E$ and a B-field transformation.

Theorem 4.4. Let $(F, \Phi)$ be an automorphism of the omni-Lie algebroid $\mathcal{E}$, where $\Phi$ is an automorphism of $E$ and $F : \mathcal{E} \rightarrow \mathcal{E}$ is an automorphism of $\mathcal{E}$. Then $F$ can be decomposed as a composition of an automorphism $\Phi$ of the vector bundle $E$ and a B-field transformation $e^b$.

Proof. Since $\Phi$ is an automorphism of $E$, $(\text{Ad}_\Phi + \Phi, \Phi)$ is an automorphism of the omni-Lie algebroid $\mathcal{E}$. Setting $G = \Phi^{-1} \circ F$, the pair $(G, 1_E)$ is again an automorphism of the omni-Lie algebroid $\mathcal{E}$. Since $G$ and $\rho$ are compatible, we can write
\[
G(\mathfrak{d} + \mu) = \mathfrak{d} + b(\mathfrak{d}) + \sigma(\mu), \quad \forall \mathfrak{d} + \mu \in \mathcal{E},
\]
where $b : \mathcal{D} E \rightarrow \mathcal{J} E$ and $\sigma : \mathcal{J} E \rightarrow \mathcal{J} E$ are two bundle maps. Then, by
\[
(G(\mathfrak{d} + \mu), G(\mathfrak{r} + \nu))_E = (\mathfrak{d} + \mu, \mathfrak{r} + \nu)_E, \quad \forall \mathfrak{d} + \mu, \mathfrak{r} + \nu \in \mathcal{E},
\]
we know that $\sigma = 1_{\mathcal{J} E}$ and $b$ is skew-symmetric:
\[
(b(\mathfrak{d}), \mathfrak{r})_E = -(b(\mathfrak{r}), \mathfrak{d})_E. \quad \forall \mathfrak{d}, \mathfrak{r} \in \mathcal{D} E.
\]
Using the equation
\[
\{G(\mathfrak{d} + \mu), G(\mathfrak{r} + \nu)\} = G\mathfrak{d} + \mu, \mathfrak{r} + \nu \}, \quad \forall \mathfrak{d} + \mu, \mathfrak{r} + \nu \in \Gamma(\mathcal{E}),
\]
we see that $b$ is closed with respect to the Lie algebroid cohomology of $\mathcal{D} E$. Thus $F = \Phi \circ e^b$, as required. ■

Corollary 4.5. The automorphism group $\text{Aut}(\mathcal{E})$ of the omni-Lie algebroid $\mathcal{E}$ is the semidirect product of $\text{Aut}(E)$ and $\Gamma(\text{Hom}(TM, E))$, i.e.
\[
\text{Aut}(\mathcal{E}) \cong \text{Aut}(E) \ltimes \Gamma(\text{Hom}(TM, E)), \tag{29}
\]
where the action of $\text{Aut}(E)$ on $\Gamma(\text{Hom}(TM, E))$, denoted by $\cdot$, is given by
\[
\Phi \cdot \eta = \Phi \circ \eta \circ \phi^{-1},
\]
where $\phi$ is the tangent of the map $\phi$ induced by $\Phi$ on the base manifold $M$. 15
Differentiating a 1-parameter family of automorphisms $F_t = \Phi_t \circ e^{tb}$, $F_0 = 1$, $b = -d\mu$, we see that the Lie algebra $\text{Der}(\mathcal{E})$ of infinitesimal symmetries of the omni-Lie algebroid $\mathcal{E}$ consists of pairs $(\delta, \mu) \in \Gamma(\mathcal{D}E) \oplus \Gamma(\text{Hom}(TM, \mathcal{E}))$. The pair $(\delta, \mu)$ acts on $\Gamma(\mathcal{E})$ via

$$(\delta + \mu) \cdot (\tau + \nu) = [\delta, \tau]_D + \mathcal{L}_\delta \nu - i_\delta d\mu, \quad \forall \tau + \nu \in \Gamma(\mathcal{E}). \quad (30)$$

From Theorem 4.4 we conclude:

**Proposition 4.6.** The Lie algebra $\text{Der}(\mathcal{E})$ of infinitesimal symmetries of an omni-Lie algebroid $\mathcal{E} = \mathcal{D}E \oplus \mathcal{J}E$ is isomorphic to the semidirect sum of $\Gamma(\mathcal{D}E)$ and $\Gamma(\text{Hom}(TM, \mathcal{E}))$, i.e.

$$\text{Der}(\mathcal{E}) \cong \Gamma(\mathcal{D}E) \ltimes \Gamma(\text{Hom}(TM, \mathcal{E})).$$

Moreover, all of these derivations are defined by the standard bracket $[\cdot, \cdot]_D$ of the omni-Lie algebroid $\mathcal{E}$ from the left hand side, and there is an exact sequence of Leibniz algebra morphism:

$$0 \longrightarrow \Gamma(\mathcal{E}) \stackrel{d}{{\longrightarrow}} \Gamma(\mathcal{E}) \stackrel{\text{ad}}{{\longrightarrow}} \text{Der}(\mathcal{E}) \longrightarrow 0. \quad (31)$$

**Proof.** By (30), for any derivation $\delta + \mu$, we have

$$(\delta + \mu) \cdot (\tau + \nu) = \{\delta + \mu, \tau + \nu\},$$

which implies that derivations of the omni-Lie algebroid $\mathcal{E}$ are defined by the standard bracket $[\cdot, \cdot]_D$. Also, $d^2 = 0$ implies that $d(\Gamma(\mathcal{E}))$ is the left center of the standard bracket $[\cdot, \cdot]_D$, i.e. the kernel of the map $\text{ad}$. Thus, Sequence (31) is exact. \[
\]

Similar to the fact that an exact Courant algebroid can be twisted by a closed 3-form, we consider the deformation of the omni-Lie algebroid $\mathcal{E}$. Given a linear map $\Theta : \Gamma(\mathcal{D}E \oplus \mathcal{J}E) \longrightarrow \Gamma(\mathcal{J}E)$, we define a new bracket $\{\cdot, \cdot\}_\Theta$ on $\Gamma(\mathcal{D}E \oplus \mathcal{J}E)$ by

$$\{\delta + \mu, \tau + \nu\}_\Theta = \{\delta + \mu, \tau + \nu\} + \Theta(\delta \otimes \tau).$$

To meet Property (EC-2), $\Theta$ must be skew-symmetric. To satisfy Property (EC-3), we need

$$0 = \langle \{\delta, \tau\}_\Theta, t \rangle_E + \langle \tau, \{\delta, t\}_\Theta \rangle_E = \frac{1}{2} \langle \{\Theta(\delta \wedge \tau), t\}_E + \{\Theta(\delta \wedge t), \tau\}_E \rangle_E$$

$$= \frac{1}{2} \langle \Theta(\delta \wedge \tau), t \rangle_E - \langle \Theta(t \wedge \delta), \tau \rangle_E.$$

Thus,

$$\langle \Theta(\delta \wedge \tau), t \rangle_E = \langle \Theta(t \wedge \delta), \tau \rangle_E, \quad \forall \delta, \tau, t \in \mathcal{D}E, \quad (32)$$

which implies that $\Theta \in \Gamma(\text{Hom}(\wedge^3 \mathcal{D}E, \mathcal{J}E))$. In this way, it is standard to prove that $\{\cdot, \cdot\}_\Theta$ defines a new $E$-Courant algebroid structure on $\mathcal{D}E \oplus \mathcal{J}E$ (using the standard pairing $[\cdot, \cdot]$ and the same anchor of the omni-Lie algebroid) if and only if $d\Theta = 0$. We call this $E$-Courant algebroid the $\Theta$-twisted omni-Lie algebroid $\mathcal{E}$. \[
\]

**Theorem 4.7.** Any twisted omni-Lie algebroid is isomorphic to the standard omni-Lie algebroid $\mathcal{E}$ in Definition 2.8.

**Proof.** By Theorem 4.4 there is some $b \in \Gamma(\text{Hom}(\wedge^3 \mathcal{D}E, \mathcal{J}E))$ such that $\Theta = db$. By (28), we have

$$e^b \{\delta + \mu, \tau + \nu\}_\Theta = \{\delta + \mu, \tau + \nu\}, \quad \forall \delta + \mu, \tau + \nu \in \Gamma(\mathcal{E}).$$

Furthermore, $b$ being skew-symmetric implies that $e^b$ preserves the standard pairing $[\cdot, \cdot]$. Therefore, the transformation $e^b$ is an isomorphism. \[
\]
5 Exact $E$-Courant algebroids

**Definition 5.1.** An $E$-Courant algebroid $(\mathcal{K}, (\cdot, \cdot)_E, [\cdot, \cdot]_\mathcal{K}, \rho)$ is said to be exact if the following sequence is exact:

$$0 \to \mathfrak{J}E \xrightarrow{\rho^*} \mathcal{K} \xrightarrow{\rho} \mathcal{D}E \to 0. \quad (33)$$

Obviously, omni-Lie algebroids are exact. In [31], it is shown that any exact Courant algebroid structure on $TM \otimes T^*M$ is a twist of the standard one by a closed 3-form. An important ingredient in the proof of this fact is that any exact Courant algebroid has an isotropic splitting, i.e., both $TM$ and $T^*M$ are isotropic subbundles. Unfortunately, this fact is no longer true for an exact $E$-Courant algebroid when $\text{rank}(E) \geq 2$. Therefore, we shall need the language of Leibniz algebra cohomologies.

Recall that any 2-chain $\Theta$ of $\Gamma(\mathcal{D}E)$ can be considered as a 3-chain $\hat{\Theta} \in \Gamma(\mathfrak{J}E)$. For any

$$\{\partial, \nu\} = \mathcal{L}_\partial \nu, \quad \{\mu, r\} = -\mathcal{L}_\mu d \langle \mu, r \rangle_E, \quad \forall \partial, \mu, \nu, r \in \Gamma(\mathcal{D}E), \quad \mu, \nu \in \Gamma(\mathfrak{J}E). \quad (34)$$

For any $b \in \Gamma(\text{Hom}(\mathfrak{J}E, \mathfrak{J}E))$, $\partial b = 0$ is equivalently saying that

$$\mathcal{L}_\partial b(r) - \mathcal{L}_r b(\partial) + d \langle b(\partial), r \rangle_E - b(\partial, r) = 0. \quad (35)$$

In the meantime, we treat the Lie algebra $\Gamma(\mathfrak{J}E)$ as a Leibniz algebra and define left and right actions of $\Gamma(E)$ on $\mathfrak{J}E$, respectively, by

$$[\partial, u] = \partial u, \quad [v, r] = -v \quad \forall \partial, u, v \in \Gamma(\mathfrak{J}E), \quad u, v \in \Gamma(E). \quad (36)$$

Note that any 2-chain $\Theta \in \Gamma(\text{Hom}(\mathfrak{J}E, \mathfrak{J}E))$ can be considered as a 3-chain $\hat{\Theta} \in \Gamma(\text{Hom}(\mathfrak{J}E, \mathfrak{J}E))$:

$$\hat{\Theta}(\partial \otimes r \otimes t) = \langle \Theta(\partial \otimes r), t \rangle_E. \quad (37)$$

The following lemma can be easily verified.
Lemma 5.2. The above notation being maintained, \( \Theta \in \Gamma(\text{Hom}(\oplus^2 \mathcal{DE}, \mathcal{J}E)) \) is closed in the cohomology of the Leibniz algebra \( \Gamma(\mathcal{DE}) \) with coefficients in \( \Gamma(\mathcal{J}E) \) if and only if \( \hat{\Theta} \in \Gamma(\text{Hom}(\oplus^2 \mathcal{DE}, \mathcal{J}E)) \) is closed in the cohomology of the Leibniz algebra \( \Gamma(\mathcal{DE}) \) with coefficients in \( \Gamma(\mathcal{E}) \).

Definition 5.3. Given a symmetric bundle map \( \omega : \mathcal{DE} \otimes \mathcal{DE} \to \mathcal{E} \) and \( \Theta \in \Gamma(\text{Hom}(\oplus^2 \mathcal{DE}, \mathcal{J}E)) \), the pair \((\omega, \Theta)\) is called an admissible pair of the omni-Lie algebroid \( \mathcal{E} \) if the following conditions are satisfied:

1) \( \Theta \) is a 2-cocycle of the Leibniz cohomology of \( \Gamma(\mathcal{DE}) \) with coefficients in \( \Gamma(\mathcal{J}E) \);
2) for any \( d \in \Gamma(\mathcal{DE}) \), \( \Theta(d \otimes d) = \text{ad}(\omega(d \otimes d)) \);
3) for any \( d, r \in \Gamma(\mathcal{DE}) \), \( \partial_b \omega(d \otimes r) = (\Theta(d \otimes r), r)_E + \omega([d, r]_D \otimes r) \).

Two admissible pairs \((\omega, \Theta)\) and \((\tilde{\omega}, \tilde{\Theta})\) are said to be equivalent if there is some \( b \in \Gamma(\text{Hom}(\mathcal{DE}, \mathcal{J}E)) \) such that

1) for any \( d, r \in \Gamma(\mathcal{DE}) \), \( \tilde{\omega}(d \otimes r) = \omega(d \otimes r) + \frac{1}{2}(b(d), r)_E + (b(r), d)_E \);
2) \( \tilde{\Theta} = \Theta + \partial b \).

For every \( b \in \Gamma(\text{Hom}(\mathcal{DE}, \mathcal{J}E)) \), \( \partial b \) is a 2-cocycle and we can define a symmetric bundle map \( \omega_b : \mathcal{DE} \otimes \mathcal{DE} \to \mathcal{E} \) by

\[
\omega_b(d \otimes r) = \frac{1}{2}((b(d), r)_E + (b(r), d)_E).
\]

It is straightforward to verify that the pair \((\omega_b, \partial b)\) is an admissible pair and it is equivalent to the admissible pair \((0, 0)\).

Theorem 5.4. There is a one-to-one correspondence between isomorphic classes of exact \( E \)-Courant algebroids and equivalence classes of admissible pairs of the omni-Lie algebroid \( \mathcal{E} = \mathcal{DE} \oplus \mathcal{J}E \).

More precisely, for any exact \( E \)-Courant algebroid \( (\mathcal{K}, (\cdot, \cdot)_E, [\cdot, \cdot]_{\mathcal{K}}, \rho) \), one may identify \( \mathcal{K} = \mathcal{DE} \oplus \mathcal{J}E \) and take \( \rho \) as the projection to \( \mathcal{DE} \), and then there is a corresponding admissible pair \((\omega, \Theta)\) such that

\[
(d + \mu, r + \nu)_E = \frac{1}{2}((\mu, \nu)_E + (r, \mu)_E) + \omega(d \otimes r), \quad \forall d + \mu, r + \nu \in \Gamma(\mathcal{DE} \oplus \mathcal{J}E),
\]

\[
[d + \mu, r + \nu]_\mathcal{K} = [d + \mu, r + \nu]_\Theta = d + \mu, r + \nu + \Theta(d \otimes r),
\]

where \([\cdot, \cdot]\) is the standard bracket of \( \mathcal{J}E \).

Conversely, for any admissible pair \((\omega, \Theta)\), \((\mathcal{DE} \oplus \mathcal{J}E, (\cdot, \cdot)_E, [\cdot, \cdot]_\mathcal{K}, \rho)\) is an exact \( E \)-Courant algebroid, where \((\cdot, \cdot)_E\) and \([\cdot, \cdot]_\mathcal{K}\) are given as above. Moreover, two exact \( E \)-Courant algebroids are isomorphic if and only if their corresponding admissible pairs are equivalent.

Proof. We split the proof into four steps. In Step 1, we prove that the \( E \)-valued pairing of the exact \( E \)-Courant algebroid \( \mathcal{K} \) is given by \((37)\). In Step 2, we prove that the bracket of \( \mathcal{K} \) is given by \((38)\). In Step 3, we prove that if we choose different splitting, we obtain equivalent admissible pairs. In Step 4, we give the proof of the reverse statement.

Step 1. By choosing a splitting \( s : \mathcal{DE} \to \mathcal{K} \) of the exact sequence \((33)\), we have \( \mathcal{K} \cong \mathcal{E} = \mathcal{DE} \oplus \mathcal{J}E \) and \( \rho \) is then the projection from \( \mathcal{E} \) to \( \mathcal{DE} \). By Properties (EC-4) and (EC-5), for all \( \mu, \nu \in \mathcal{J}E \), we have

\[
(r^* \mu, r^* \nu)_E = \frac{1}{2}((\mu, \rho \circ r^* \nu)_E = 0,
\]
which implies that $\rho^*\mathcal{J}E$ is isotropic under the pairing $(\cdot, \cdot)_E$. So if we transfer the pairing $(\cdot, \cdot)_E$ on $\mathcal{K}$ to a pairing $(\cdot, \cdot)_E$ on $\mathcal{D}E \oplus \mathcal{J}E$, $\mathcal{J}E$ is isotropic. For any $d \in \mathcal{D}E$, we have

$$(d, \nu)_E = (s(d), \rho^*(\nu))_E = \frac{1}{2} (d, \nu)_E.$$ \hfill (40)

Furthermore, for all $d, \tau \in \mathcal{D}E$, we have

$$(d, \tau)_E = (s(d), s(\tau))_E = \omega(d \otimes \tau),$$ \hfill (41)

where $\omega : \mathcal{D}E \otimes \mathcal{D}E \to E$ is a symmetric bundle map. By (39), (40) and (41), it follows that the pairing is given by (37).

**Step 2.** For any $d, \tau \in \Gamma(\mathcal{D}E)$, by Properties (EC-1) and (EC-2), we are able to write

$$[d, \tau]_E = [d, \tau]_D + \Theta(d \otimes \tau),$$ \hfill (42)

where $\Theta$ is a $\mathbb{R}$-linear mapping $\Gamma(\mathcal{D}E) \otimes \Gamma(\mathcal{D}E) \to \Gamma(\mathcal{J}E)$. By (35), we know that $\Theta$ is also $C^\infty(M)$-linear and hence $\Theta \in \Gamma(\text{Hom}(\otimes^2 \mathcal{D}E, \mathcal{J}E))$.

Again by Property (EC-1), there is a bi-linear map

$$\Delta : \Gamma(\mathcal{D}E) \times \Gamma(\mathcal{J}E) \to \Gamma(\mathcal{J}E),$$

such that

$$[d, \mu]_E = \Delta(d, \mu), \quad \forall \ d \in \Gamma(\mathcal{D}E), \ \mu \in \Gamma(\mathcal{J}E).$$

By Property (EC-3) and $\mathcal{J}E$ being isotropic, we have

$$d(\tau, \mu)_E = ([d, \tau]_E, \mu)_E + (\tau, [d, \mu]_E)_E = ([d, \tau]_D, \mu)_E + (\tau, \Delta(d, \mu))_E,$$

which implies that $\Delta(d, \mu) = \mathcal{L}_d\mu$, i.e.

$$[d, \mu]_E = \mathcal{L}_d\mu, \quad \forall \ d \in \Gamma(\mathcal{D}E), \ \mu \in \Gamma(\mathcal{J}E).$$ \hfill (43)

Furthermore, we have

$$d(\mu, \mu)_E = d(d + \mu, d + \mu)_E - d(d, d)_E = [d + \mu, d + \mu]_E - d(d, d)_E = [d, \mu]_E + [\mu, d]_E = \mathcal{L}_d\mu + [\mu, d]_E.$$

Therefore,

$$[\mu, d]_E = d(\mu, \mu)_E - \mathcal{L}_d\mu, \quad \forall \ \mu \in \Gamma(\mathcal{J}E), \ d \in \Gamma(\mathcal{D}E).$$ \hfill (44)

Again by Property (EC-3), we have

$$(\mu, [\nu, d]_E)_E + ([\mu, \nu]_E, d)_E = 0, \quad \forall \ \mu, \nu \in \Gamma(\mathcal{J}E), \ d \in \Gamma(\mathcal{D}E),$$

which implies

$$[\mu, \nu]_E = 0, \quad \forall \ \mu, \nu \in \Gamma(\mathcal{J}E).$$ \hfill (45)

By (42), (43), (44) and (45), we get

$$[d + \mu, \tau + \nu]_E = [d + \mu, \tau + \nu] + \Theta(d \otimes \tau), \quad \forall \ d + \mu, \tau + \nu \in \Gamma(\mathcal{E}).$$ \hfill (46)

Since the bracket $[\cdot, \cdot]_E$ satisfies the Leibniz rule, we have

$$\mathcal{L}_d\Theta(\tau, t) - \mathcal{L}_t\Theta(d, t) + \mathcal{L}_t\Theta(d, \tau) - d(\lambda(t, \Theta(d, \tau))E) + \Theta(d, [\tau, t]_D) - \Theta(\tau, [d, t]_D) = 0,$$
which implies that $\Theta$ is a 2-cocycle in the Leibniz cohomology of $\Gamma(\mathcal{D}E)$ with coefficients in $\Gamma(\mathcal{J}E)$ and the two actions are given in (33). Since the pair $(\omega, \Theta)$ comes from the $E$-Courant algebroid $\mathcal{K}$, it is straightforward to verify that it is an admissible pair.

**Step 3.** Suppose that we have two sections $s_1, s_2 : \mathcal{D}E \to \mathcal{K}$, then $\rho(s_1 - s_2) = 0$. Take $b = s_1 - s_2 : \mathcal{D}E \to \mathcal{J}E$. We then have $s_2(\mathcal{d}) = \mathcal{d} + b(\mathcal{d})$. By straightforward computations, we get

$$(\mathcal{d} + b(\mathcal{d}), \mathcal{r} + b(\mathcal{r})) = [\mathcal{d}, \mathcal{r}] + \mathcal{L}_\mathcal{d}^* \mathcal{r} + \mathcal{L}_\mathcal{d}^* b(\mathcal{r}) + \mathcal{L}_\mathcal{d}^* b(\mathcal{r}) + b([\mathcal{d}, \mathcal{r}]) + (\mathcal{d} + \theta b)(\mathcal{d} \otimes \mathcal{r}).$$

If we denote the new pairing by $\tilde{\omega}$, for any $\mathcal{d}, \mathcal{r} \in \Gamma(\mathcal{D}E)$, then

$$\tilde{\omega}(\mathcal{d}, \mathcal{r}) = \omega(\mathcal{d}, \mathcal{r}) + b(\mathcal{d}, \mathcal{r}) + (b(\mathcal{d}), \mathcal{r})_E .$$

Therefore, if we choose different admissible pairs, we obtain equivalent admissible pairs.

**Step 4.** Conversely, for any admissible pair $(\omega, \Theta)$, on $\mathcal{D}E \oplus \mathcal{J}E$, we define the pairing $(\cdot, \cdot)_E$ and the bracket $\{\cdot, \cdot\}_E$ by (37) and (38). It is straightforward to see that $(\mathcal{D}E \oplus \mathcal{J}E, (\cdot, \cdot)_E, \{\cdot, \cdot\}_E)$ is an $E$-Courant algebroid. If we choose different representative element $(\tilde{\omega}, \tilde{\Theta})$, assume that $\tilde{\Theta} = \Theta + \partial b$ for some $b \in \Gamma(\text{Hom}(\mathcal{J}E, \mathcal{J}E))$ and the corresponding $E$-Courant algebroid is $(\mathcal{D}E \oplus \mathcal{J}E, (\cdot, \cdot)'_E, \{\cdot, \cdot\}_E + \partial b, \rho)$. By some computations similar to (37), for any $\mathcal{d} + \mu$, $\mathcal{r} + \nu \in \Gamma(\mathcal{D}E \oplus \mathcal{J}E)$, we have

$$e^b(\mathcal{d} + \mu, \mathcal{r} + \nu) = \{e^b(\mathcal{d} + \mu), e^b(\mathcal{r} + \nu)\}_E .$$

It is also obvious that

$$(e^b(\mathcal{d} + \mu), e^b(\mathcal{r} + \nu)) = (\mathcal{d} + \mu, \mathcal{r} + \nu)'_E .$$

Therefore, the transformation $e^b$ is the isomorphism from $(\mathcal{D}E \oplus \mathcal{J}E, (\cdot, \cdot)'_E, \{\cdot, \cdot\}_E + \partial b, \rho)$ to $(\mathcal{D}E \oplus \mathcal{J}E, (\cdot, \cdot)_E, \{\cdot, \cdot\}_E, \rho)$. □

**Remark 5.5.** The extreme case that the induced symmetric bundle map $\omega : \mathcal{D}E \otimes \mathcal{D}E \to E$ is zero, i.e. the splitting is isotropic, already has been studied at the end of Section 3, which is in fact the twisted omni-Lie algebroid.

In some special cases, we can define an isotropic splitting as follows.

**Proposition 5.6.** Under the circumstances above, if the induced $E$-valued pairing $\omega : \mathcal{D}E \otimes \mathcal{D}E \to E$ on $\mathcal{D}E$ satisfies $\text{Im}(\omega_1) \subset \mathcal{J}E$, then there is an isotropic splitting $s(\mathcal{d}) = -\frac{1}{2}\omega_2(\mathcal{d})$. In particular, if $E$ is a line bundle, there always exist isotropic splittings.

**Proof.** By definition, we have

$$(\mathcal{d} + s(\mathcal{d}), \mathcal{r} + s(\mathcal{r})) = (\mathcal{d}, \mathcal{r})_E - \omega(\mathcal{d}, \mathcal{r}) = 0,$$

for all $\mathcal{d}, \mathcal{r} \in \mathcal{D}E$ and $\mu, \nu \in \mathcal{J}E$. Thus, $s(\mathcal{D}E)$ is isotropic and we proved the first claim.

Moreover, for any $\Phi \in \mathfrak{gl}(E)$,

$$\text{Im}(\omega_2) \subset \mathcal{J}E \iff \omega(\mathcal{d}, \Phi) = \Phi(\omega(\mathcal{d}, 1_E)).$$

If $E$ is a line bundle, the conclusion follows, because $\mathfrak{gl}(E)$ is then a trivial line bundle. □
6  E-Lie bialgebroids

In this section we introduce the notion of an E-Lie bialgebroid, whose double turns out to be an E-Courant algebroid. Conversely, any E-Courant algebroid which is the direct sum of two transverse Dirac structures provides an E-Lie bialgebroid. Similar to the fact that the base manifold of a Lie bialgebroid is a Poisson manifold, for an E-Lie bialgebroid, the underlining vector bundle E is a Lie algebroid (if rankE ≥ 2), or a local Lie algebra (if rankE = 1).

In the sequel, notations introduced in Section 3 are needed.

**Definition 6.1.** An E-dual pair ((A, ρA); (B, ρB)) is called an E-Lie bialgebroid if for all X, Y ∈ Γ(A), u, v ∈ Γ(E), the following conditions are satisfied:

1. \[ d^B[X, Y] = \xi_X(d^B Y) - \xi_Y(d^B X), \]
2. \[ \xi_{d^A u} X = -\xi_{d^B u} X, \]
3. \[ \langle d^B u, d^A u \rangle_E = 0. \]

When there is no confusion, we simply denote such an E-Lie bialgebroid by (A, B).

**Remark 6.2.** Condition (3) is equivalent to \( ρ_B \circ d^A = -ρ_A \circ d^B \).

Let us give some examples. Recall the properties of an omni-Lie algebroid which is exactly Condition (2) of Definition 6.1. Finally, substituting By (11), we have

\[ \xi_{d^A u} X = -\xi_{d^B u} X, \]

Furthermore, by definition, we have \( d^A = d^*_X a \).

**Proposition 6.3.** A generalized Lie bialgebroid ((A, φ0), (A*, X0)) is an E-Lie bialgebroid, where E is the trivial line bundle \( M \times \mathbb{R} \).

**Proof.** For any \( X \in Γ(A), \xi \in Γ(A^*) \), the representations \( ρ_A \) and \( ρ_A^* \) are given by

\[ ρ_A(X) = a(X) + φ_0(X), \quad ρ_{A^*}(ξ) = a_*(ξ) + X_0(ξ), \]

where \( a \) and \( a_* \) are, respectively, the anchors of \( A \) and \( A^* \). Evidently, we have \( d^{A^*} = d^*_X a_* \). Furthermore, by definition, we have

\[ [X, d_{X_0} Y]_{φ_0} = [X, d_{X_0} Y] - \langle φ_0, X \rangle d_{X_0} Y = \xi_X d_{X_0} Y, \quad \forall X, Y \in Γ(A). \]

Since ((A, φ0); (A*, X0)) is a generalized Lie bialgebroid, we have (10). By (48), Condition (1) of Definition 6.1 holds, i.e.

\[ d^{A^*}[X, Y] = \xi_X (d^{A^*} Y) - \xi_Y (d^{A^*} X). \]

To prove Condition (2) of Definition 6.1 one substitutes \( Y \) by \( fY \) in (49), where \( f \in C^∞(M) \), and gets

\[ \xi_{d^{A^*} f} X = -\xi_{d^{A^*} f} X + f(\xi_{φ_0} X + \xi_{X_0} X). \]

By (11), we have

\[ \xi_{d^{A^*} f} X = -\xi_{d^{A^*} f} X, \]

which is exactly Condition (2) of Definition 6.1. Finally, substituting \( X \) by \( fX \) in (50), we get

\[ ρ_{A^*} d^A(f) = -ρ_{A^*} d^{A_{f}} = -ρ_{A^*} d^{A_{f}} (f) + (a(X_0) + a_*(φ_0))(X). \]
Therefore, a generalized Lie bialgebroid \(((A, \phi_0), (A^*, X_0))\) is truly an $E$-Lie bialgebroid. 

Let \((A, [\cdot, \cdot], a)\) be a Lie algebroid and $\rho_A : A \to \mathfrak{d}E$ a $B$-invariant representation, where $B$ is an $E$-dual bundle of $A$. For any $u \in \Gamma(E)$, $X \in \Gamma(A)$ and $X^k \in \Gamma(\Lambda^k B, E)_A$, we define their Schouten brackets by

$$[u, X^k] = [X^k, u] = (-1)^{k+1}i_d\rho_A u X^k, \quad [X, X^k] = -[X^k, X] = \Sigma X X^k.$$  

The Schouten bracket $[H, K] \in \Gamma(\text{Hom}(\Lambda^3 B, E)_A)$ of $H$, $K \in \Gamma(\text{Hom}(\Lambda^2 B, E)_A)$ is defined by

$$[H, K](\xi, \eta, \zeta) = \langle \mathcal{L}_{K\xi} \eta, H\zeta \rangle_E + \langle \mathcal{L}_{H\xi} \eta, K\zeta \rangle_E + c.p., \quad \forall \xi, \eta, \zeta \in \Gamma(B).$$  

For any $\Lambda \in \Gamma(\text{Hom}(\Lambda^2 B, E)_A)$, we introduce a bracket $[\cdot, \cdot]_\Lambda$ on $\Gamma(B)$:

$$[\xi, \eta]_\Lambda = \mathcal{L}_\Lambda \xi \eta - \mathcal{L}_\Lambda \xi - d^A(\Lambda(\xi, \eta)), \quad \forall \xi, \eta \in \Gamma(B).$$  

By straightforward computations, we have the following formula:

$$[H\xi, H\eta] = H[\xi, \eta]_H + \frac{1}{2}[H, H](\xi, \eta),$$  

which implies that $[H, H] \in \Gamma(\text{Hom}(\Lambda^3 B, E)_A)$. Moreover, replacing $H$ by $H + K$, we know that $[H, K] \in \Gamma(\text{Hom}(\Lambda^2 B, E)_A)$.

Given $\Lambda \in \Gamma(\text{Hom}(\Lambda^2 B, E)_A)$, let $\Lambda_3 : B \to A$ and $[\Lambda, \Lambda]_3 : \Lambda^2 B \to A$ be the induced bundle maps defined by \((\ref{induced-map})\). Let us denote

$$a_B = a \circ \Lambda_3 : B \to TM \quad \text{and} \quad \rho_B = \rho_A \circ \Lambda_3 : B \to \mathfrak{d}E.$$  

**Proposition 6.4.** Under the circumstances above, $(B, [\cdot, \cdot]_\Lambda, a_B)$ is a Lie algebroid together with an $A$-invariant representation $\rho_B$ if and only if the following two conditions are satisfied:

1. $\rho_A \circ [\Lambda, \Lambda]_3 = 0$;
2. $\mathcal{L}_X [\Lambda, \Lambda] = 0, \quad \forall X \in \Gamma(A)$.

**Proof.** By \((\ref{induced-map})\), for any $\xi, \eta \in \Gamma(B)$, we have

$$\rho_B(\xi, \eta)_\Lambda = \rho_A \circ \Lambda_3(\xi, \eta)_\Lambda = [\rho_B \xi, \rho_B \eta] - \frac{1}{2} \rho_A \circ [\Lambda, \Lambda]_3(\xi, \eta).$$  

Therefore, $\rho_B$ is a homomorphism if and only if $\rho_A \circ [\Lambda, \Lambda]_3 = 0$. It is simple to see that for any $f \in C^\infty(M)$,

$$[\xi, f\eta]_\Lambda = a_B(\xi)(f) \eta + f[\xi, \eta]_\Lambda.$$  

Further, by \((\ref{induced-map})\), we have

$$a_B(\xi, \eta)_\Lambda = [a_B \xi, a_B \eta] - \frac{1}{2} \rho_A \circ [\Lambda, \Lambda]_3(\xi, \eta).$$  

Therefore, if $\rho_A \circ [\Lambda, \Lambda]_3 = 0$, $a_B$ is a homomorphism.

Let

$$J(\xi_1, \xi_2, \xi_3) = [[\xi_1, \xi_2]_\Lambda, \xi_3]_\Lambda + c.p., \quad \forall \xi_i \in \Gamma(B).$$  

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For any $X \in \Gamma(A)$, under the condition that $\rho_A \circ [\Lambda, \Lambda]\| = 0$ and by some similar calculations did in [23], one is able to get

$$
\langle J(\xi_1, \xi_2, \xi_3), X \rangle_E = -\frac{1}{2} \{[X, \Lambda\|], \xi_3\}\| + c.p. + \rho_A(X)([\Lambda, \Lambda](\xi_1, \xi_2, \xi_3))
$$

$$
= -\frac{1}{2} \{\mathcal{L}_X[\Lambda, \Lambda]\}(\xi_1, \xi_2, \xi_3).
$$

Thus, under the condition that $\rho_A \circ [\Lambda, \Lambda]\| = 0$, the bracket $[\cdot, \cdot]\|$ defined by (52) satisfies the Jacobi identity if and only if $\mathcal{L}_X[\Lambda, \Lambda]\|= 0$ for all $X \in \Gamma(A)$.

Finally, we show that the representation $\rho_B$ of the Lie algebroid $(B, [\cdot, \cdot]_A, a_B)$ is $A$-invariant. In fact, by straightforward computations, we have

$$
d^B u = -i_{\delta a\Lambda} \Lambda, \quad d^B X = -\mathcal{L}_X \Lambda, \quad \forall u \in \Gamma(E), \ X \in \Gamma(A),
$$

(55)
as required.

**Theorem 6.5.** Under the conditions of Proposition [6.2], $(A, B)$ is an $E$-Lie bialgebroid.

**Proof.** By [55], it is straightforward to check the compatibility conditions of an $E$-Lie bialgebroid. We omit the details.

The notion of local Lie algebra was introduced by Kirillov in [19], which is a vector bundle $E$ whose section space $\Gamma(E)$ has an $\mathbb{R}$-Lie algebra structure $[\cdot, \cdot]_E$ with the local property, $\text{supp}[u, v] \subset \text{supp}[u] \cap \text{supp}[v]$, for all $u, v \in \Gamma(E)$. In particular, $M$ is called a Jacobi manifold if the trivial bundle $M \times \mathbb{R}$ is a local Lie algebra, which is equivalent to that there is a pair $(\Lambda, X)$, where $\Lambda$ is a bi-vector field and $X$ is a vector field on $M$ such that $[\Lambda, \Lambda] = 2X \wedge \Lambda$ and $[\Lambda, X] = 0$. A Jacobi structure reduces to a Poisson structure if $X = 0$. Similar to the fact that the cotangent bundle of a Poisson manifold is a Lie algebroid, $T^*M \oplus \mathbb{R}$ enjoys a Lie algebroid structure for every Jacobi manifold. A local Lie algebra is not a Lie algebroid since there is no anchor map. For example, the trivial bundle $M \times \mathbb{R}$ with the Poisson bracket is only a local Lie algebra for every Poisson manifold $M$.

It is well known that a Lie bialgebroid $(A, A^\ast)$ gives a Poisson structure on the base manifold $M$ (28). If $(A, A^\ast)$ is a generalized Lie bialgebroid, there is an induced Jacobi structure on the base manifold $M$ (15). In the situation of an $E$-Lie bialgebroid $(A, B)$ as in Definition 6.1 we introduce a bracket $[\cdot, \cdot]_E$ on $\Gamma(E)$ as follows

$$[u, v]_E \triangleq [d^A u, d^B v]_E \quad (= \rho_B(d^A u)v), \quad \forall u, v \in \Gamma(E).
$$

(56)

**Theorem 6.6.** Let $(A, B)$ be an $E$-Lie bialgebroid,

1) If $\text{rank}(E) \geq 2$, $(E, [\cdot, \cdot]_E)$ is a Lie algebroid with the anchor $j \circ \rho_B \circ \rho_A^\ast \circ \text{ad}$.

2) If $\text{rank}E = 1$, $(E, [\cdot, \cdot]_E)$ is a local Lie algebra.

**Proof.** By Condition (3) of Definition 6.1 the bracket defined by (56) is skew-symmetric. To check the Jacobi identity, for all $u, v, w \in \Gamma(E)$, we have

$$[u, [v, w]]_E = \langle d^A u, d^B \langle d^A v, d^B w \rangle_E \rangle_E = \langle d^A u, d^B (i_{d^A v}d^B w) + i_{d^A v}d^B d^B w \rangle_E
$$

$$= -\langle d^A u, \mathcal{L}_{d^A v} d^B w \rangle_E = -\langle d^A u, [d^B v, d^B w] \rangle_E = -\rho_A([d^B v, d^B w]u)
$$

$$= -\rho_A(d^B v)\rho_A(d^B w)u + \rho_A(d^B w)\rho_A(d^B v)u
$$

$$= -\langle [u, w]_E, v \rangle_E + [[u, v]_E, w]_E.
$$
Moreover, we have an obvious expression:

\[
[u,fv]_E = \langle d^A u, f d^B v + \rho_A^*(df \otimes v) \rangle_E \\
= f[u,v]_E + \langle \rho_B(d^A u), df \otimes v \rangle_E \\
= f[u,v]_E + (j \circ \rho_B \circ \rho_A^*(du))(f)v.
\]

The map \( \rho_B \circ \rho_A^* : \mathcal{A}E \to \mathcal{D}E \) is skew-symmetric. In [5], it is shown that \( \text{rank}(E) \geq 2 \) implies that \( j \circ \rho_B \circ \rho_A^* \circ d : E \to TM \) is a bundle map. In this case, \( (E, [\cdot, \cdot]_E) \) is a Lie algebroid, whose anchor is \( j \circ \rho_B \circ \rho_A^* \circ d \). □

\section{7 Manin Triples}

In this section we develop the theory of Manin triples of \( E \)-Lie bialgebroids analogously to that of Lie bialgebroids. It is known that for a Lie bialgebroid \((A, A^*)\), we can endow \( A \oplus A^* \) with a Courant algebroid structure [22]. For a generalized Lie bialgebroid \((A, A^*)\), we can endow \( A \oplus A^* \) with a generalized Courant algebroid structure [29]. Similarly, for any \( E \)-Lie bialgebroid \((A, B)\), we can endow \( A \oplus B \) with an \( E \)-Courant algebroid structure. In fact, we define on \( A \oplus B \) an \( E \)-valued pairing \( (\cdot, \cdot)_E \) by

\[
(X_1 + \xi_1, X_2 + \xi_2)_E = \frac{1}{2}(\langle X_1, \xi_2 \rangle_E + \langle X_2, \xi_1 \rangle_E), \quad \forall X_i + \xi_i \in \Gamma(A \oplus B), \ i = 1, 2,
\]

and a bracket \( [\cdot, \cdot]_{A \oplus B} \) by

\[
[X_1 + \xi_1, X_2 + \xi_2]_{A \oplus B} = \langle X_1, X_2 \rangle_E + \mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1 + d^B \langle X_1, \xi_2 \rangle_E \\
+ [\xi_1, \xi_2] + \mathcal{L}_X \xi_2 - \mathcal{L}_X \xi_1 + d^A \langle X_2, \xi_1 \rangle_E.
\]

\begin{lemma}
Let \((A, B)\) be an \( E \)-Lie bialgebroid. Then, one has

\[
J(e_1, e_2, e_3) = -J_1(e_1, e_2, e_3) + \text{c.p.}(e_1, e_2, e_3) = J_2(e_1, e_2, e_3),
\]

for all \( e_i = X_i + \xi_i \in \Gamma(A \oplus B) \). Here

\[
J(e_1, e_2, e_3) = \{e_1, \{e_2, e_3\}\} - \{\{e_1, e_2\}, e_3\} - \{e_2, \{e_1, e_3\}\}.
\]

The notation \( \text{c.p.} \) means cyclic permutations. \( J_1(e_1, e_2, e_3) \) and \( J_2(e_1, e_2, e_3) \) are, respectively,

\[
J_1(e_1, e_2, e_3) = i_X (d^A \langle \xi_1, \xi_2 \rangle_E - \mathcal{L}_{\xi_1} d^A \xi_2 + \mathcal{L}_{\xi_2} d^A \xi_1) + i_\xi (d^B \langle X_1, X_2 \rangle_E - \mathcal{L}_X d^B X_2 + \mathcal{L}_X d^B X_1),
\]

\[
J_2(e_1, e_2, e_3) = \mathcal{L}_{d^A(\xi_1 \cdot X_1)} \xi_3 + [d^A(\langle \xi_2, X_1 \rangle_E, \xi_3] + \mathcal{L}_{d^B(\xi_2 \cdot X_1)} \xi_2 + [d^A(\langle \xi_3, X_1 \rangle_E, \xi_2] \\
+ \mathcal{L}_{d^A(\xi_3 \cdot X_2)} \xi_1 + [d^A(\langle \xi_3, X_2 \rangle_E, \xi_1].
\]

\textbf{Proof.} We need the following formula,

\[
i_X \mathcal{L}_\xi d^A \eta = [\xi, \mathcal{L}_X \eta] - \mathcal{L}_{\mathcal{L}_X \xi} \eta + [d^A(\langle \eta, X \rangle_E, \xi] + d^A(\rho_B(\xi)(\eta, X)_E) - d^A([\xi, \eta], X)_E,
\]

for all \( X \in \Gamma(A), \xi, \eta \in \Gamma(B) \). Then, the rest of the calculations are very similar to those in [22]. We omit the details. □

\begin{remark}
Equation [59] is different from Theorem 3.1 in [22] since the bracket \( [\cdot, \cdot]_{A \oplus B} \) given by [59] is not skew-symmetric.
\end{remark}
Lemma 7.3. For any $X \in \Gamma(A)$, $\xi \in \Gamma(B)$ and $u \in \Gamma(E)$, we have

$$[\rho_B(\xi), \rho_A(X)]_\mathcal{D}(u) = \rho_A(\mathcal{L}_X \xi(u) - \rho_B(\mathcal{L}_\xi X(u) + \rho_B \circ d^A \langle \xi, X \rangle_E(u) + \langle \mathcal{L}_{d^A u} \xi + [d^A u, \xi], X \rangle_E.$$ 

Proof.

$$\rho_B \circ d^A \langle \xi, X \rangle_E(u) = (\rho_A(d_B u)(\xi, X)_E) = (\mathcal{L}_{d_B u} \xi, X)_E + \langle \xi, [d_B u, X]_E \rangle$$

$$= (\mathcal{L}_{d^A u} \xi + [d^A u, \xi], X)_E + \langle \xi, [d^A u, X]_E \rangle - \langle \xi, \mathcal{L}_X d_B u \rangle_E$$

$$= [\rho_B(\xi), \rho_A(X)]_\mathcal{D}(u) - \rho_A(\mathcal{L}_X \xi(u) + \rho_B(\mathcal{L}_\xi X(u) + (\mathcal{L}_{d^A u} \xi + [d^A u, \xi], X \rangle_E).$$

Theorem 7.4. Given an E-Lie bialgebroid $(A, B)$ as above, the quadrable $(A \oplus B, (\cdot, \cdot)_E, [-, -]_E, \rho_A + \rho_B)$ is an E-Courant algebroid, where $(\cdot, \cdot)_E$ is given by (57) and $[-, -]_E$ is given by (58).

Proof. By Conditions (1), (2) of Definition 6.1 and Lemma 7.1, $(\Gamma(\mathcal{D} \oplus \mathcal{B}), (\cdot, \cdot)_E, [-, -]_E, \rho_A + \rho_B)$ is a Leibniz algebra. For any $X, Y \in \Gamma(A)$ and $\xi, \eta \in \Gamma(B)$, it is obvious that

$$\rho_A(X, Y)_{AB} = [\rho_A(X), \rho_A(Y)]_\mathcal{D}, \quad \rho_B(\xi, \eta)_{AB} = [\rho_B(\xi), \rho_B(\eta)]_\mathcal{D}.$$ 

By Lemma 7.3 and Conditions (2) and (3) of Definition 6.1, we have

$$[\rho_A(X), \rho_B(\xi)]_\mathcal{D} = \rho_B(\mathcal{L}_X \xi) - \rho_A(\mathcal{L}_\xi X) + \rho_A d^B(X, \xi)_E.$$

We also have

$$(\rho_A + \rho_B)[X, \xi]_{AB} = (\rho_A + \rho_B)(\mathcal{L}_X \xi - \mathcal{L}_\xi X + d_B(X, \xi)_E)$$

$$= \rho_B(\mathcal{L}_X \xi) - \rho_A(\mathcal{L}_\xi X) + \rho_A d^B(X, \xi)_E.$$ 

So we get Property (EC-1). Since $(\rho_A + \rho_B)^* = \rho_A^* + \rho_B^*$ and $\rho_A^*(du) = d^A u$, we have

$$[X + \xi, X + \xi]_{AB} = (d^A + d^B)(X, \xi)_E = (\rho_A + \rho_B)^* \circ d(X, \xi)_E,$$

and Property (EC-2) follows. Property (EC-3) is straightforward. Property (EC-4) follows from the fact that $\rho_A^* \mathcal{D} \subset \mathcal{D}$ and $\rho_B^* \mathcal{D} \subset \mathcal{D}$. Property (EC-5) follows from Condition (3) of Definition 6.1. This proves that $(A \oplus B, (\cdot, \cdot)_E, [-, -]_E, \rho_A + \rho_B)$ is an E-Courant algebroid. 

Conversely, for an E-Courant algebroid $K$, suppose that there are two transverse Dirac structures $A$ and $B$ in $K$ such that $K = A \oplus B$, we want to show that $(A, B)$ is an E-Lie bialgebroid. Evidently, $A$ and $B$ are mutually E-dual vector bundles, with the pairing

$$\langle X, \xi \rangle_E = 2(X, \xi)_E, \quad X \in \Gamma(A), \quad \xi \in \Gamma(B).$$ 

By Proposition 2.6 both $A$ and $B$ are Lie algebroids whose anchors are, respectively, $a = j \circ \rho |_A$ and $a_B = j \circ \rho |_B$. In the meantime, there are representations $\rho_A = \rho |_A$ of $A$ and $\rho_B = \rho |_B$ of $B$ on $E$. The associated coboundary operators $d^A : \Omega^*(A, E) \rightarrow \Omega^{*+1}(A, E)$ and $d^B : \Omega^*(B, E) \rightarrow \Omega^{*+1}(B, E)$ are standard.

Similar to the result in [22] for Courant algebroids, we have the E-Courant algebroid analogue:

Theorem 7.5. If an E-Courant algebroid has a decomposition $K = A \oplus B$, where $A$ and $B$ are transverse Dirac structures, then $(A, B)$ is an E-Lie bialgebroid.
One is able to check that the graph of \( \pi \)-broid equipped with a representation \( \rho \) is an \( \mathcal{E} \)-valued pairing implies that \( \mathfrak{d}(\mathfrak{u}, \mathfrak{x}) \) is given by (58). By Lemma 7.1 and the fact that \( \Gamma(\mathfrak{K}) \) is an \( \mathcal{E} \)-dual pair. Thus by Theorem 7.5, we have \( [\rho_B(\mathfrak{x}), \rho_A(\mathfrak{x})]_\mathcal{E} = \rho(\mathfrak{x}, \mathfrak{x})_\mathcal{E} = \rho_A(\mathfrak{x}) \rho_B(\mathfrak{x}) \rho_B \circ \mathcal{A}_\mathfrak{x} \rho_B(\mathfrak{x}) \). So Condition (3) of Definition 6.1 holds. In summary, \( (\mathcal{A}, \mathcal{B}) \) is an \( \mathcal{E} \)-Lie bialgebroid.

Finally, we give some examples of \( \mathcal{E} \)-Lie bialgebroids.

**The \( T^*M \)-Lie bialgebroid \( (\mathfrak{J}_A, \mathfrak{J}(\mathfrak{A}^*)) \)**

For any Lie bialgebroid \( (\mathcal{A}, \mathcal{B}) \), there associates a Courant bialgebroid structure on \( \mathcal{A} \oplus \mathcal{B} \). By Theorem 7.13, \( \mathfrak{J}(\mathfrak{A} \oplus \mathfrak{B}) = \mathfrak{J}(\mathfrak{A}) \oplus \mathfrak{J}(\mathfrak{B}) \) is a \( T^*M \)-Courant bialgebroid. It is easily seen that both \( \mathfrak{J}(\mathfrak{A}) \) and \( \mathfrak{J}(\mathfrak{B}) \) are transverse Dirac structures. Thus by Theorem 7.3, we have

**Proposition 7.6.** For any Lie bialgebroid \( (\mathfrak{A}, \mathfrak{B}) \), \( (\mathfrak{J}(\mathfrak{A}), \mathfrak{J}(\mathfrak{B})) \) is a \( T^*M \)-Lie bialgebroid.

**The \( \mathcal{E} \)-Lie bialgebroid \( (\mathfrak{E}, \mathfrak{J}(\mathfrak{E})) \) induced by a Lie algebroid \( (\mathfrak{E}, [\cdot , \cdot]_\mathfrak{E}, a) \)**

Recall Theorem 6.3, where it is shown that for any \( \mathcal{E} \)-Lie bialgebroid, there is a Lie algebra structure (local Lie algebra structure) on \( \mathcal{E} \) if \( \text{rank} \mathcal{E} \geq 2 \) (\( \text{rank} \mathcal{E} = 1 \)). There is also a canonical \( \mathcal{E} \)-Lie bialgebroid \( (\mathfrak{E}, [\cdot , \cdot]_\mathfrak{E}) \) coming from a given Lie algebroid \( (\mathfrak{E}, [\cdot , \cdot]_\mathfrak{E}, a) \). If \( \text{rank} \mathcal{E} = 1 \), similar conclusion holds for any local Lie algebroid \( (\mathfrak{E}, [\cdot , \cdot]_\mathfrak{E}) \).

In fact, if \( (\mathfrak{E}, [\cdot , \cdot]_\mathfrak{E}, a) \) is a Lie algebroid (or \( (\mathfrak{E}, [\cdot , \cdot]_\mathfrak{E}) \) is a local Lie algebra when \( \text{rank} \mathcal{E} = 1 \)), we can define a skew-symmetric bundle map \( \pi : \mathfrak{J}(\mathfrak{E}) \to \mathfrak{E} \) by setting

\[
\pi(du)(v) = [u, v]_\mathfrak{E}, \quad \forall u, v \in \Gamma(\mathfrak{E}).
\]

(This appeared in 5.) Moreover, we introduce a bracket \( [\cdot , \cdot]_\pi \) on \( \Gamma(\mathfrak{J}(\mathfrak{E})) \) by setting

\[
[m, n]_\pi \triangleq \mathfrak{L}_m(n) - \mathfrak{L}_n(m) - [m, n]_\mathfrak{E}.
\]

One is able to check that the graph of \( \pi \) is a Dirac structure and \( (\mathfrak{E}, [\cdot , \cdot]_\mathfrak{E}, \pi \circ \pi) \) is a Lie algebroid equipped with a representation \( \pi \). Therefore, we obtain an \( \mathcal{E} \)-Lie bialgebroid structure on \( (\mathcal{E}, \mathfrak{J}(\mathfrak{E})) \).
For a Poisson manifold \((M, \pi)\), there is a canonical Lie bialgebroid \((TM, T^*M)\). For a Jacobi manifold \((M, X, \Lambda)\), there is a canonical generalized Lie bialgebroid \((T \oplus \mathbb{R}, T^* \oplus \mathbb{R})\). Similarly, we have

**Proposition 7.7.** For any Lie algebroid \((E, \{\cdot, \cdot\}_E, a)\), there is a canonical \(E\)-Lie bialgebroid \((DE, J_E)\). If \(\text{rank} E = 1\), the conclusion holds for any local Lie algebra structure \((E, \{\cdot, \cdot\}_E)\).

**Dirac structures**

For an \(E\)-Lie bialgebroid \((A, B)\), suppose that \(H \in \Gamma(\text{Hom}(\wedge^2 B, E)_A)\). Treated as a bundle map \(H : B \rightarrow A\), \(H\) has its graph \(G_H = \{H\xi + \xi | \xi \in B\} \subset A \oplus B\).

**Theorem 7.8.** The graph \(G_H\) is a Dirac structure if and only if \(H\) satisfies the following Maurer-Cartan type equation:

\[
d^B H + \frac{1}{2}[H, H] = 0.
\]

**Proof.** The property that \(G_H\) is isotropic is equivalent to the condition that \(H \in \Gamma(\text{Hom}(\wedge^2 B, E)_A)\). By (58), we have

\[
[H\xi, \eta] + [\xi, H\eta] = [\xi, \eta]_H + \mathfrak{L}_\xi H\eta - \mathfrak{L}_\eta H\xi + d^B(H(\xi, \eta)).
\]

By (59), we have

\[
[H\xi + \xi, H\eta + \eta] = \mathfrak{L}_\xi H\eta - \mathfrak{L}_\eta H\xi + d^B(H(\xi, \eta)) + H[\xi, \eta]_H + \frac{1}{2}[H, H](\xi, \eta) + [\xi, \eta] + [\xi, \eta]_H.
\]

So \(\Gamma(G_H)\) is closed under the bracket \([\cdot, \cdot]\) if and only if for any \(\xi, \eta \in \Gamma(B)\),

\[
H[\xi, \eta] = \mathfrak{L}_\xi H\eta - \mathfrak{L}_\eta H\xi + d^B(H(\xi, \eta)) + \frac{1}{2}[H, H](\xi, \eta).
\]

(66)

On the other hand,

\[
(d^B H)(\xi, \eta) = \rho_B(\xi)(H\eta, \theta)_{E} - \rho_B(\eta)(H\xi, \theta)_{E} + \rho_B(\theta)(H\xi, \eta)_{E}
\]

\[
+ \langle H\theta, [\xi, \eta] \rangle_{E} - \langle H\eta, [\xi, \theta] \rangle_{E} + \langle H\xi, [\eta, \theta] \rangle_{E}
\]

\[
= \langle \mathfrak{L}_\xi H\eta - \mathfrak{L}_\eta H\xi - H[\xi, \eta] + d^B(H(\xi, \eta)), \theta \rangle_{E}.
\]

Therefore, (66) is equivalent to the condition that

\[
(d^B H)(\xi, \eta) + \frac{1}{2}[H, H](\xi, \eta) = 0,
\]

or equivalently,

\[
d^B H + \frac{1}{2}[H, H] = 0. \blacksquare
\]

In particular, if \(d^B = 0\) (i.e. \(B\) is a trivial Lie algebroid), the graph \(G_H\) is a Dirac structure if and only if \([H, H] = 0\), or \(H[\xi, \eta]_H = [H(\xi), H(\eta)]\).

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