UNIFORM CLOSE-TO-CONVEXITY RADIUS OF SECTIONS OF FUNCTIONS IN THE CLOSE-TO-CONVEX FAMILY

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Abstract. The authors consider the class $F$ of normalized functions $f$ analytic in the unit disk $D$ and satisfying the condition

$$\Re \left( 1 + z \frac{f''(z)}{f'(z)} \right) > -\frac{1}{2}, \quad z \in D.$$ 

Recently, Ponnusamy et al. [12] have shown that $1/6$ is the uniform sharp bound for the radius of convexity of every section of each function in the class $F$. They conjectured that $1/3$ is the uniform univalence radius of every section of $f \in F$.

In this paper, we solve this conjecture affirmatively.

1. Preliminaries and the Main Theorem

Let $A$ be the family of functions analytic in the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then the $n$-th section/partial sum of $f$, denoted by $s_n(f)(z)$, is defined to be the polynomial

$$s_n(f)(z) = z + \sum_{k=2}^{n} a_k z^k.$$ 

Let $S$ denote the class of functions in $A$ that are univalent in $D$. Finally, let $C$, $S^*$ and $K$ denote the usual geometric subclasses of functions in $S$ with convex, starlike and close-to-convex images, respectively (see [3]).

If $f \in S$ is arbitrary, then the argument principle shows that the $n$-th section $s_n(f)(z)$ is univalent in each fixed compact disk $|z| \leq r$ ($< 1$) provided that $n$ is sufficiently large. But then if we set $p_n(z) = r^{-1} s_n(f)(rz)$, then $p_n(z)$ is a polynomial that is univalent in the unit disk $D$. Consequently, the set of univalent polynomials is dense with respect to the topology of locally uniformly in $S$ (see [3]). Suffridge [19] showed that even the subclass of polynomials with the highest coefficient $a_n = 1/n$ is dense in $S$. Szegő [20] discovered that every section $s_n(f)$ is univalent in the disk $|z| < 1/4$ for all $f \in S$ and for each $n \geq 2$. The radius $1/4$ is best possible as the Koebe function $k(z) = z/(1 - z)^2$ shows. It is worth pointing out that the case $n = 3$ of Szegő's result is far from triviality.

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In [15], Ruscheweyh established a stronger result by showing that the partial sums \( s_n(f)(z) \) of \( f \) are indeed starlike in the disk \(|z| < 1/4\) for functions \( f \) belonging not only to \( S \) but also to the closed convex hull of \( S \). The following conjecture concerning the exact (largest) radius of univalence \( r_n \) of \( f \in S \) is still open (see [13] and [3, §8.2, p. 241–246]).

**Conjecture A.** If \( f \in S \), then \( s_n(f) \) is univalent in \(|z| < 1 - \frac{3}{n} \log n \) for all \( n \geq 5 \).

A surprising fact observed by Bshouty and Hengartner [2] is that the Koebe function is no more extremal for the above conjecture. On the other hand, this conjecture has been solved by using an important convolution theorem [16] for a number of geometric subclasses of \( S \), for example, the classes \( C \), \( S^* \) and \( K \). Indeed, for \( \phi(z) = z/(1-z) \), the sections \( s_n(\phi) \) are known to be convex in \(|z| < 1/4 \) (see [5]). Moreover for the Koebe function \( k(z) = z/(1-z)^2 \), \( s_n(k) \) is known to be starlike in \(|z| < 1 - \frac{2}{n} \log n \) for \( n \geq 5 \) and hence, for the convex function \( \phi(z) = z/(1-z) \), \( s_n(\phi) \) is convex in \(|z| < 1 - \frac{2}{n} \log n \) for \( n \geq 5 \). From a convolution theorem relating to the Pólya-Schoenberg conjecture proved by Ruscheweyh and Sheil-Small [16], it follows that all sections \( s_n(f) \) are convex (resp. starlike, close-to-convex) in \(|z| < 1/4 \) whenever \( f \in C \) (resp. \( f \in S^* \) and \( f \in K \)). Similarly, for \( n \geq 5 \), \( s_n(f) \) is convex (resp. starlike, close-to-convex) in \(|z| < 1 - \frac{2}{n} \log n \) whenever \( f \in C \) (resp. \( f \in S^* \) and \( f \in K \)). An account of history of this and related information may be found in [3, §8.2, p. 241–246] and also in the nice survey article of Iliev [6]. For further interest on this topic, we refer to [4, 14, 17, 18] and recent articles [8, 9, 10, 11].

One of the important criteria for an analytic function \( f \) defined on a convex domain \( \Omega \), to be univalent in \( \Omega \) is that \( \Re f'(z) > 0 \) on \( \Omega \) (see [3, Theorem 2.16, p. 47]). The following definition is a consequence of it.

A function \( f \in A \) is said to be close-to-convex (with respect to \( g \)), denoted by \( f \in K_g \), if there exists a \( g \in C \) such that

\[
\Re \left( e^{i\alpha} \frac{f'(z)}{g'(z)} \right) > 0, \quad z \in \mathbb{D},
\]

for some real \( \alpha \) with \(|\alpha| < \pi/2 \). More often, we consider \( K_g \) (with \( \alpha = 0 \) in (1)) and \( K = \bigcup_{g \in C} K_g \). For functions in \( K_g \), we have the following result of Miki [7].

**Theorem B.** Let \( f \in K_g \), where \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \). Then \( s_n(f) \) is close-to-convex with respect to \( s_n(g) \) in \(|z| < 1/4 \).

In a recent paper [1], the present authors proved the following.

**Theorem C.** Let \( f \in K \). Then every section \( s_n(f) \) of \( f \) belongs to the class \( K \) in the disk \(|z| < 1/2 \) for all \( n \geq 46 \).

Choosing different convex functions \( g \) in [1], the authors have found the value \( N(g) \in \mathbb{N} \) for \( f \in K_g \) such that \( s_n(f) \in K_g \) in a disk \(|z| < r \) for all \( n \geq N(g) \).
In [12], the authors consider the class $F$ of locally univalent functions $f$ in $A$ satisfying the condition
\begin{equation}
\text{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}.
\end{equation}

The importance of this class is outlined in [12] and it was also remarked that the class $F$ has a special role on certain problems on the class of harmonic univalent mappings in $D$ (see [12] and the references therein). It is worth remarking that functions in $F$ are neither included in $S^*$ nor includes $S^*$ nor $K$. It is well-known that $F \subsetneq K \subsetneq S$ and hence, it is obvious from an earlier observation that for $f \in F$, each $s_n(f)(z)$ is close-to-convex in $|z| < 1/4$. An interesting question is to determine the largest uniform disk with this property (see Conjecture 1 below). We now recall a recent result of Ponnusamy et al. [12].

**Theorem D.** Every section of a function in the class $F$ is convex in the disk $|z| < 1/6$. The radius $1/6$ cannot be replaced by a greater one.

In the same article the authors [12] observed that all sections functions of $F$ are close-to-convex in the disk $|z| < 1 - \frac{3}{n} \log n$ for $n \geq 5$. Consider
\begin{equation}
f_0(z) = \frac{z - z^2/2}{(1 - z)^2}.
\end{equation}
We see that $f_0 \notin S^*$, but $f_0 \in K$. Also, $f_0$ is extremal for many extremal problems for the class $F$. By investigating the second partial sum of $f_0 \in F$, the authors conjectured the following.

**Conjecture 1.** Every section $s_n(f)$ of $f \in F$ is close-to-convex in the disk $|z| < 1/3$ and $1/3$ is sharp.

In this article we solve this conjecture in the following form.

**Theorem 1.** Every section $s_n(f)$ of $f \in F$ satisfies $\text{Re}(s_n(f)'(z)) > 0$ in the disk $|z| < 1/3$. In particular every section is close-to-convex in the disk $|z| < 1/3$. The radius $1/3$ cannot be replaced by a greater one.

We remark that this result is much stronger than the original conjecture. The following lemma is useful in the proof of Theorem 1.

**Lemma E.** [12, Lemma 1] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F$, then the following estimates hold:

(a) $|a_n| \leq \frac{n + 1}{2}$ for $n \geq 2$. Equality holds for $f_0(z)$ given by (3) or its rotation.

(b) $\frac{1}{(1 + r)^3} \leq |f'(z)| \leq \frac{1}{(1 - r)^3}$ for $|z| = r < 1$. The bounds are sharp.

(c) If $f(z) = s_n(z) + \sigma_n(z)$, with $\sigma_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$, then for $|z| = r < 1$ we have
\begin{equation}
|\sigma_n'(z)| \leq \frac{n(n + 1)r^{n+2} - 2n(n + 2)r^{n+1} + (n + 1)(n + 2)r^n}{2(1 - r)^3}.
\end{equation}
2. PROOF OF THEOREM 1

Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F} \). We shall prove that each partial sum \( s_n(z) := s_n(f)(z) \) of \( f \) satisfies the condition \( \text{Re} \left( s'_n(z) \right) > 0 \) in the disk \( |z| < 1/3 \) for all \( n \geq 2 \).

Let us first consider the second section \( s_2(z) = z + a_2 z^2 \) of \( f \). A simple computation shows that

\[
\text{Re} \left( s'_2(z) \right) = 1 + \text{Re} \left( 2a_2 z \right).
\]

From Lemma E(a), we have \( |a_2| \leq 3/2 \) and as a consequence of it we get

\[
\text{Re} \left( s'_2(z) \right) \geq 1 - 2|a_2||z| \geq 1 - 3|z|
\]

which is positive provided \( |z| < 1/3 \). Thus, \( s_2(z) \) is close-to-convex in the disk \( |z| < 1/3 \). To show that the constant \( 1/3 \) is best possible, we consider the function \( f_0 \in \mathcal{F} \) given in (3), namely,

\[
f_0(z) = \frac{1}{2} \left[ \frac{1}{(1-z)^2} - 1 \right] = z + \sum_{n=2}^{\infty} \left( \frac{n+1}{2} \right) z^n.
\]

Let us denote by \( s_{2,0}(z) \), the second partial sum \( s_2(f_0)(z) \) of \( f_0(z) \) so that \( s_{2,0}(z) = z + (3/2)z^2 \). Then we get \( s'_{2,0}(z) = 1 + 3z \), which vanishes at \( z = -1/3 \). Thus the constant \( 1/3 \) is best possible.

Next, let us consider the case \( n = 3 \). Each \( f \in \mathcal{F} \) satisfies the analytic condition (2) and so we can write

\[
1 + \frac{2}{3} \frac{z f''(z)}{f'(z)} = p(z),
\]

where \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \) is analytic in \( \mathbb{D} \) and \( \text{Re} p(z) > 0 \) in \( \mathbb{D} \). From Carathéodory Lemma [3, p. 41] we get \( |p_n| \leq 2 \) for all \( n \geq 2 \). If we rewrite (4) in power series form, then

\[
1 + \frac{2}{3} \frac{z(2a_2 + 6a_3 z + 12a_4 z^2 + \cdots)}{1 + 2a_2 z + 3a_3 z^2 + \cdots} = 1 + p_1 z + p_2 z^2 + \cdots.
\]

Now comparing the coefficients of \( z \) and \( z^2 \) on both sides yields the relations

\[
p_1 = \frac{4}{3} a_2 \quad \text{and} \quad p_2 = \frac{4}{3} (3a_3 - 2a_2^2).
\]

As \( |p_1| \leq 2 \) and \( |p_2| \leq 2 \), we may rewrite the last two relations as

\[
a_2 = \frac{3}{2} \alpha \quad \text{and} \quad \frac{2}{3} (3a_3 - 2a_2^2) = \beta, \quad \text{i.e.} \quad a_3 = \frac{1}{2} (\beta + 3\alpha^2)
\]

for some \( |\alpha| \leq 1 \) and \( |\beta| \leq 1 \). Now we have to show that

\[
\text{Re} \left( s'_3(z) \right) = \text{Re} \left( 1 + 2a_2 z + 3a_3 z^2 \right) > 0
\]

in \( |z| < 1/3 \). Since the function \( \text{Re} \left( s'_3(z) \right) \) is harmonic in \( |z| \leq 1/3 \), it is enough to prove (6) for \( |z| = 1/3 \). By considering a suitable rotation of \( f \), it is enough to prove (6) for \( z = 1/3 \). Thus, it suffices to show that

\[
\text{Re} \left( 1 + \frac{2}{3} a_2 + \frac{1}{3} a_3 \right) > 0.
\]
By using the relations in (5) and the maximum principle, we see that the inequality (7) is equivalent to

\( \text{Re} \left( 1 + \alpha + \frac{\alpha^2}{2} + \frac{\beta}{6} \right) > 0, \)

where \( |\alpha| = 1 \) and \( |\beta| = 1 \). If we take \( \alpha = e^{i\theta} \) and \( \beta = e^{i\phi} \) \((0 \leq \theta, \phi < 2\pi)\), then in order to verify the inequality (8) it suffices to prove

\[ \min_{\theta, \phi} T(\theta, \phi) > 0, \]

where

\[ T(\theta, \phi) = 1 + \cos \theta + \frac{\cos 2\theta}{2} + \frac{\cos \phi}{6} \]

and \( \theta, \phi \) lies in \([0, 2\pi)\). Let

\[ g(\theta) = 1 + \cos \theta + \frac{\cos 2\theta}{2}, \quad \theta \in [0, 2\pi). \]

Then

\[ g'(\theta) = -\sin \theta (1 + 2 \cos \theta) \quad \text{and} \quad g''(\theta) = -[\cos \theta + 2 \cos 2\theta]. \]

The points at which \( g'(\theta) = 0 \) are \( \theta = 0, 2\pi/3, \pi \) and \( 4\pi/3 \). But \( g''(\theta) \) is positive for \( \theta = 2\pi/3 \) and \( \theta = 4\pi/3 \). Hence

\[ \min_{\theta} g(\theta) = g\left(\frac{2\pi}{3}\right) = g\left(\frac{4\pi}{3}\right) = \frac{1}{4}. \]

As the minimum value of \((\cos \phi)/6\) is \(-1/6\), it follows that

\[ \min_{\theta, \phi} T(\theta, \phi) = T\left(\frac{2\pi}{3}, \pi\right) = T\left(\frac{4\pi}{3}, \pi\right) = \frac{1}{12} > 0. \]

This proves the inequality (6) for \(|z| < 1/3\).

Now let us consider the case \( n \geq 4 \). Let \( f(z) = s_n(z) + \sigma_n(z) \), where \( \sigma_n(z) \) is as given in Lemma E(c). Then

\[ \text{Re} \left( s'_n(z) \right) = \text{Re} \left( f'(z) - \sigma'_n(z) \right) \geq \text{Re} \left( f'(z) \right) - |\sigma'_n(z)|. \]

By maximum principle it is enough to prove that \( \text{Re} \left( s'_n(z) \right) > 0 \) for \(|z| = 1/3\). Now let us estimate the values of \( \text{Re} \left( f'(z) \right) \) and \(|\sigma'_n(z)|\) on \(|z| = 1/3\).

As in the proof of Lemma E(b) in [12], we have the subordination relation for \( f \in \mathcal{F}, \)

\[ f'(z) \prec \frac{1}{(1 - z)^3}, \quad z \in \mathbb{D}. \]

We need to find the image of the circle \(|z| = r\) under the transformation \( w(z) = 1/(1 - z)^3 \). As the bilinear transformation \( T(z) = 1/(1 - z) \) maps the circle \(|z| = r\) onto the circle

\[ \left| T - \frac{1}{1 - r^2} \right| = \frac{r}{1 - r^2}, \quad \text{i.e.,} \quad T(z) = \frac{1 + re^{i\theta}}{1 - r^2}, \]
a little computation shows that the image of the circle \(|z| = r\) under the transformation \(w = 1/(1 - z)^3\) is a closed curve described by

\[
w = \frac{(1 + re^{i\theta})^3}{(1 - r^2)^3} = \frac{1 + r^3e^{3i\theta} + 3r^2e^{2i\theta} + 3re^{i\theta}}{(1 - r^2)^3}, \quad \theta \in [0, 2\pi).
\]

From this relation, the substitution \(r = 1/3\) gives that

\[
Re w = \left(\frac{9}{8}\right)^3 \left[1 + \cos \theta + \frac{\cos 2\theta}{3} + \frac{\cos 3\theta}{27}\right] = h(\theta) \text{ (say)}.
\]

If we write \(h(\theta)\) in powers of \(\cos \theta\), then we easily get

\[
h(\theta) = \left(\frac{9}{8}\right)^3 \left[\frac{2}{3} + \frac{8}{9} \cos \theta + \frac{2}{3} \cos^2 \theta + \frac{4}{27} \cos^3 \theta\right].
\]

If we let \(x = \cos \theta\), then we can rewrite \(h(\theta)\) in terms of \(x\) as

\[
p(x) = \left(\frac{9}{8}\right)^3 \left[\frac{2}{3} + \frac{8}{9} x + \frac{2}{3} x^2 + \frac{4}{27} x^3\right],
\]

where \(-1 \leq x \leq 1\). In order to find the minimum value of \(h(\theta)\) for \(\theta \in [0, 2\pi]\), it is enough to find the minimum value of \(p(x)\) for \(x \in [-1, 1]\). A computation shows that

\[
p'(x) = \frac{81(x + 2)(x + 1)}{128} \quad \text{and} \quad p''(x) = \frac{81(3 + 2x)}{128}.
\]

In the interval \([-1, 1]\), \(p'(x) = 0\) implies \(x = -1\) is the only possibility. Also \(p''(-1) > 0\) and so the minimum value of the function \(p(x)\) in \([-1, 1]\) occurs at \(x = -1\). The above discussion implies that

\[
\min_{\theta} h(\theta) = h(\pi) = \frac{27}{64}.
\]

Moreover, from the subordination relation (10), we deduce that

\[
\min_{|z|=1/3} \text{Re } (f'(z)) \geq \min_{|z|=1/3} \text{Re } \left(\frac{1}{(1 - z)^3}\right) = \frac{27}{64}.
\]

Images of the disks \(|z| < r\) for \(r = 1/3, 1/2, 3/4, 4/5\), under the function \(H(z) = 1/(1 - z)^3\) are drawn in Figures 1(a)-(d). From Lemma E(c), we have for \(|z| = 1/3\)

\[
-|\sigma_n(z)| \geq \frac{-1}{8 \times 3^{n-1}} \left[2n^2 + 8n + 9\right] = k(n) \text{ (say)}.
\]

Now

\[
k'(n) = \frac{-1}{8 \times 3^{n-1}} \left[\log \left(\frac{1}{3}\right) \left(2n^2 + 8n + 9\right) + 4n + 8\right].
\]

For \(n \geq 4\), \(k'(n) > 0\) and hence \(k(n)\) is an increasing function of \(n\). Thus for all \(n \geq 4\), we have \(k(n) \geq k(4) = -73/216\).

Finally, from the relations (9), (11) and (12) it follows that

\[
\text{Re } (s'_n(z)) > \frac{27}{64} - \frac{73}{216} = \frac{145}{1728} > 0 \text{ for all } n \geq 4.
\]

The proof is complete.
Sections of functions in the close-to-convex family

We end the paper with the following conjecture.

Conjecture 2. Every section $s_n(f)$ of $f \in \mathcal{F}$ is starlike in the disk $|z| < 1/3$.

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