A posteriori error estimators suitable for moving finite element methods under anisotropic meshes

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Abstract. In this paper, we give a new type of a posteriori error estimators suitable for moving finite element methods under anisotropic meshes for general second-order elliptic problems. The computation of estimators is simple once corresponding Hessian matrix is recovered. Wonderful efficiency indices are shown in numerical experiments.

Keywords. a posteriori error estimator; moving finite element method; anisotropic mesh.

AMS subject classification. 65N15, 65N30

1 Introduction

Nowadays adaptive algorithms have been an indispensable tool for most finite element simulations. They basically consist of the ingredients “Solve – Estimate error – Refine mesh” which are repeated until the desired accuracy is achieved. Generally, they can be classified into three types: \( h \)-, \( r \)- and \( hp \)-version. In this paper we consider the second ingredient (Estimate error) for \( r \)-version (or moving finite element method) under anisotropic meshes.

Then, what does “anisotropic mesh” mean? Denote by \( h_K \) the diameter of the finite element \( K \), and by \( s_K \) the supremum of the diameters of all balls contained in \( K \). It is assumed in the classical finite element theory that

\[ h_K \lesssim s_K. \tag{1.1} \]

(The notation \( \lesssim \) means smaller than up to a constant.) Elements which satisfy \( (1.1) \) are called isotropic elements.

Many physical problems exhibit a common anisotropic feature that their solutions change more significantly in one direction than the others. Examples include those having boundary layers, shock waves, interfaces, and edge singularities, etc. In such cases
it is advantageous to reflect this anisotropy in the discretization by using meshes with anisotropic elements (sometimes also called elongated elements). These elements have a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction. That is to say, anisotropic elements do not satisfy condition (1.1). Conversely they are characterized by

$$\frac{h_K}{\varrho_K} \to \infty$$

(1.2)

where the limit can be considered as \( h \to 0 \) (near edges) or \( \epsilon \to 0 \) (in layers) where \( \epsilon \) is some (small perturbation) parameter of the problem. Indeed anisotropic meshes have been used successfully in many areas, for example in singular perturbation and flow problems [2, 6, 7, 16, 21, 31] and in adaptive procedures [9, 12, 24, 27]. For problems with very different length scales in different spatial directions, long and thin triangles turn out to be better choices than shape regular ones if they are properly used. This motivated an intensive study on the error analysis for anisotropic meshes in the finite element method. For instance, Apel [4] described an error estimate in terms of the length scales \( h_1 \) and \( h_2 \) along the \( x \) and \( y \) direction, respectively. Berzins [8] developed a mesh quality indicator measuring the correlation between the anisotropic features of the mesh and those of the solutions. Kunert [20] introduced the concept of “matching function” which measures the correspondence between an anisotropic mesh and a given function. Using this concept he gave three types of a posteriori error estimator for anisotropic meshes under the assumption that the anisotropic mesh \( T_h \) is ‘adapted’ to the anisotropic solution. Formaggia and Perotto [15] used the spectral properties of the affine mapping from a reference triangle to obtain a full information about the orientation, dimension and aspect ratio of a given element. After that they proposed a posteriori estimators for elliptic problems under anisotropic meshes. Picasso [25] combined the method in [15] and a Zienkiewicz-Zhu error estimator to approach the error gradient. Cao [11] revealed the precise relation between the error of linear interpolation on a general triangle and the geometric characters of the triangle. This list is certainly incomplete, but from the papers we can find the interpolation error depends on the solution and the size and shape of the elements in the mesh.

In the mesh generation community, the error estimate is often studied for the model problem of interpolating quadratic functions. This model is a reasonable simplification of the cases involving general functions, since quadratic functions are the leading terms in the local expansion of the linear interpolation errors. For instance, Nadler [23] derived an exact expression for the \( L^2 \)-norm of the linear interpolation error in terms of the three sides \( \ell_1, \ell_2, \) and \( \ell_3 \) of the triangle \( K \),

$$\|u - u_I\|_{L^2(K)}^2 = \frac{|K|}{180} \left( d_1 + d_2 + d_3 \right)^2 + d_1 d_2 + d_2 d_3 + d_1 d_3 ,$$

(1.3)

where \(|K|\) is the area of the triangle, \( d_i = \ell_i \cdot H \ell_i \) with \( H \) being the Hessian matrix of \( u \). Bank and Smith [5] gave a formula for the \( H^1 \)-seminorm of the linear interpolation error

$$\|\nabla (u - u_I)\|_{L^2(K)}^2 = \frac{1}{4} d \cdot B d,$$

(1.4)
where \( \mathbf{d} = [d_1, d_2, d_3]^T \),

\[
B = \frac{1}{48|K|} \begin{pmatrix}
|\ell_1|^2 + |\ell_2|^2 + |\ell_3|^2 & 2\ell_1 \cdot \ell_2 & 2\ell_1 \cdot \ell_3 \\
2\ell_1 \cdot \ell_2 & |\ell_1|^2 + |\ell_2|^2 + |\ell_3|^2 & 2\ell_2 \cdot \ell_3 \\
2\ell_1 \cdot \ell_3 & 2\ell_2 \cdot \ell_3 & |\ell_1|^2 + |\ell_2|^2 + |\ell_3|^2
\end{pmatrix}
\]

In this paper we’ll develop the formula for \( H^1 \)-seminorm of the linear interpolation error

\[
\|\nabla (u - u_I)\|_{L^2(K)}^2 \approx \frac{1}{48|K|} \sum_{i=1}^3 d_i^2 |\ell_i|^2,
\]

where \( d_i = \ell_{i+1} \cdot H_{\ell_{i+2}} \), and that of discretization error

\[
\|\nabla (u - u_h)\|_{L^2(\Omega)}^2 \approx -\frac{1}{24} \sum_{K \in T_h} \sum_{i=1}^3 \left( f_K + |\ell_i| \partial_n u_h \right) d_i.
\]

The quality of an a posteriori error estimator is often measured by its efficiency index, i.e., the ratio of the true error and the estimated error (in some norm). An error estimator is called efficient if its efficiency index together with its inverse remain bounded for all mesh-sizes. It is called asymptotically exact if its efficiency index tends to one when the mesh-size converges to zero. From numerical results we see our estimators are often asymptotically exact although we couldn’t prove it rigorously.

The paper is organized as follows. In section 2 we give some preliminary results, especially the error expansions for \( u - u_I \) and \( \nabla (u - u_I) \). In section 3 these error expansions are used to derive a posteriori error estimators for the interpolation error and discretization error, respectively. Section 4 contains “efficient index” tables and pictures from numerical experiments for some second-order elliptic problems which yield anisotropic solutions. The results show remarkable agreement with the theoretical predictions. Finally, in section 5 we state our conclusions and direction for further research.

2 Preliminaries

Consider the following model problem. Find \( u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) such that

\[
\begin{align*}
Lu &= -\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + bu = f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( b = b(\mathbf{x}) \geq 0 \) a.e. in \( \Omega \) and \( a_{ij} = a_{ij}(\mathbf{x}) \) are given functions. The domain \( \Omega \) is an open, bounded subset of \( \mathbb{R}^2 \) and the operator \( L \) is elliptic and self-adjoint. The corresponding variational formulation seeks \( u \in H^1_0(\Omega) \) such that

\[
a(u, v) = (f, v) \quad \forall v \in V \equiv H^1_0(\Omega),
\]
where
\[
a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + buv \right) dx
\]
and
\[
(f, v) \equiv \int_{\Omega} f v dx.
\]

We shall use the standard notations in [14] for the Sobolev spaces \(H^s(\Omega)\) and their associated inner products \((\cdot, \cdot)_s\), norms \(|| \cdot ||_s\), and seminorms \(| \cdot |_s\) for \(s \geq 0\).

By \(F = \{ T_h \}\) we denote a family of triangulations \(T_h\) of \(\Omega\). Let \(V_h\) be the space of continuous, piecewise linear functions over \(T_h\), and \(V_{0,h} \equiv V_h \cap H^1_0(\Omega)\). The finite element approximation problem of (2.2) seeks \(u_h \in V_{0,h}\) such that
\[
a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_{0,h}.
\]
(2.3)

The three vertices of an arbitrary triangle \(K \in T_h\) are denoted by \(a_1 = (x_1, y_1)^T\), \(a_2 = (x_2, y_2)^T\) and \(a_3 = (x_3, y_3)^T\). Additionally we define the edge vectors \(\ell_1 = a_3 - a_2\), \(\ell_2 = a_1 - a_3\) and \(\ell_3 = a_2 - a_1\) (Figure 1). Denote by \(u_I\) the linear interpolation of \(u\) at the three vertices of \(K\). Let \(\{ \lambda_i(x) \}^{3}_{i=1}\) be the barycentric coordinates of \(K\). From [13][26] we know for a quadratic function \(u\) over \(K\) the following formulas hold:
\[
u(x) - u_I(x) = -\frac{1}{2} \sum_{i=1}^{3} \lambda_i(x) \left[ (x - a_i) \cdot H(x - a_i) \right] \quad \forall x \in K,
\]
(2.4)
\[
\nabla \left( u(x) - u_I(x) \right) = -\frac{1}{2} \sum_{i=1}^{3} \nabla \lambda_i(x) \left[ (x - a_i) \cdot H(x - a_i) \right] \quad \forall x \in K,
\]
(2.5)
where \(H\) is the Hessian matrix of \(u\).
3 A posteriori error estimates

Many authors have discussed the interpolation error to derive their adaptive algorithm\([4, 5, 8, 15, 19]\). However, the interpolation error is different from the discretization error in most cases. In this section we first discuss the former and then the latter. Finally we will analyze their relationship using the concept “superapproximation”.

3.1 An a posteriori error estimator for the interpolation error

**Theorem 3.1.** Let \( u \) be a quadratic function and \( u_I \) is the Lagrangian linear finite element interpolation of \( u \). Denote by \( H \) the Hessian matrix of \( u \). The following relationship holds:

\[
\| \nabla (u - u_I) \|_{L^2(K)}^2 = \frac{1}{48|K|^3} \sum_{i=1}^{3} c_i^2 |\ell_i|^2,
\]

(3.1)

where \( c_i = \ell_{i+1} \cdot H \ell_{i+2} \). Here we prescribe \( i + 3 = i, i - 3 = i \).

**Proof.** From (2.5), we have

\[
\| \nabla (u - u_I) \|_{L^2(K)}^2 = \frac{1}{4} \int_K \left| \sum_{i=1}^{3} \nabla \lambda_i(x) \left[ (x - a_i) \cdot H(x - a_i) \right] \right|^2 dx.
\]

(3.2)

Due to the properties of the barycentric coordinates it is known that

\[
\left| \sum_{i=1}^{3} \nabla \lambda_i(x) \left[ (x - a_i) \cdot H(x - a_i) \right] \right|^2 \in P_2(K).
\]

(3.3)

We use the second-order quadrature scheme which is exact for polynomial of degree less or equal to 2, i.e.

\[
\int_K \varphi(x) dx = \frac{1}{4} |K| \left[ \varphi(a_{12}) + \varphi(a_{23}) + \varphi(a_{31}) \right], \quad \forall \varphi \in P_2(K),
\]

(3.4)

where \( a_{ij} \) is the midpoint of the segment \( \overline{a_i a_j} \). Notice that

\[
\nabla \lambda_i(x) = \left( \frac{y_{i+1} - y_{i+2}}{2|K|}, \frac{x_{i+1} - x_{i+2}}{2|K|} \right)^T,
\]

after a simple calculation we get (3.1).

Here we set

\[
\eta_I = \sqrt{\sum_{K \in T_h} \frac{1}{48|K|^3} \sum_{i=1}^{3} c_i^2 |\ell_i|^2}
\]

(3.5)

as the a posteriori estimator for \( \| \nabla (u - u_I) \|_{L^2(K)} \).

**Remark 1** Using the same technique we can also get the corresponding estimator for \( \| u - u_I \|_{0, \Omega} \) denoted by \( \eta_I \).

**Remark 2** The estimators \( \eta_I \) and \( \eta_I \) have been given in different forms, for example, in \([5, 11, 23]\), e.t.c.
3.2 An a posteriori error estimator for the discretization error

In this subsection an a posteriori error estimator for the discretization error of problem (2.2) will be given.

**Theorem 3.2.** Assume \( L \) be an elliptic and adjoint operator, \( c(x) \) and \( a_{ij}(x) \) be zero and constant functions, respectively. We have the following estimate:

\[
\int_{\Omega} \left| \nabla (u - u_h) \right|^2 \, dx \approx -\frac{1}{24} \sum_{K \in T_h} \sum_{i=1}^{3} \left( f_K + |\ell_i| [\partial_n u_h]_{\ell_i} \right) d_i,
\]

(3.6)

where \( f_K = \int_K f(x) \, dx \) and \([\partial_n u_h]_{\ell_i}\) is the jump of the conormal derivative of \( u_h \)

\[
\partial_n u_h = \sum_{i,j=1}^{2} a_{ij} \frac{\partial u_h}{\partial x_j} n_i
\]

across the edge \( \ell_i \), with \( n = (n_1, n_2)^T \) the unit outward normal vector.

**Proof.** Using the Galerkin orthogonality, we have

\[
\int_{\Omega} \left| \nabla (u - u_h) \right|^2 \, dx = a(u - u_h, u - u) = a(u - u_h, u - u_I)
\]

\[
= \sum_{K \in T_h} \int_K \left[ \sum_{i,j=1}^{2} a_{ij} \frac{\partial (u - u_h)}{\partial x_j} \frac{\partial}{\partial x_i} (u - u_I) \right] \, dx
\]

\[
= \sum_{K \in T_h} \left\{ -\int_K \left[ \sum_{i,j=1}^{2} \partial_{x_i} \left( a_{ij} \frac{\partial (u - u_h)}{\partial x_j} \right) (u - u) \right] \, dx + \int_{\partial K} \partial_n (u - u_h)(u - u_I) \, ds \right\}
\]

\[
= \sum_{K \in T_h} \left\{ \int_K \frac{f_K}{|K|} (u - u_I) \, dx + \frac{1}{2} \int_{\partial K} [\partial_n u_h](u - u_I) \, ds \right\} + \sum_{K \in T_h} \int_K \left( f - \frac{f_K}{|K|} \right) (u - u_I) \, dx
\]

\[
\approx \sum_{K \in T_h} \left\{ \int_K \frac{f_K}{|K|} (u - u_I) \, dx + \frac{1}{2} \int_{\partial K} [\partial_n u_h](u - u_I) \, ds \right\}
\]

\[
\approx -\frac{1}{24} \sum_{K \in T_h} \left( f_K \sum_{i=1}^{3} d_i + \sum_{i=1}^{3} |\ell_i| [\partial_n u_h]_{\ell_i} d_i \right) = -\frac{1}{24} \sum_{K \in T_h} \sum_{i=1}^{3} \left( f_K + |\ell_i| [\partial_n u_h]_{\ell_i} \right) d_i,
\]

where we use the error expansion (2.4) and the second-order quadrature scheme on \( K \) and \( \partial K \), respectively.

**3.3 Discussion of the estimators**

From the Theorem 3.2 we get easily an a posteriori error estimator for the discretization error:

\[
\eta = \sqrt{-\frac{1}{24} \sum_{K \in T_h} \sum_{i=1}^{3} \left( f_K + |\ell_i| [\partial_n u_h]_{\ell_i} \right) d_i}.
\]
Obviously this estimator can be computed easily provided that $H$ is properly given. A number of numerical recovery approaches have been proposed in the literature for second-order derivatives \cite{11,22,29,30,32}. Comparisons of these techniques have also been made in \cite{10,28}. Particularly the authors \cite{28} compared four methods for reconstructing the second-order derivatives of a piecewise linear function: DLF(Double linear fitting), SLF(Simple linear fitting), QF(Quadratic fitting) and DL2P(Double $L^2$-projection). In this paper we will recover $H$ using the quadratic fitting method elaborated by Zhang \cite{29}.

To end this section, it is advantageous to discuss the relationship between the interpolation error $\|\nabla (u - u_I)\|_{0, \Omega}$ and the discretization error $\|\nabla (u - u_h)\|_{0, \Omega}$.

Denote by $N$ the number of elements in $T_h$. Assume $\|\nabla (u - u_I)\|_{0, \Omega} \approx CN^{1/2}$ and $\|\nabla (u - u_h)\|_{0, \Omega} \approx CN^{1/2}$. Then, by simple calculus, we have

$$\|\nabla (u - u_I)\|_{0, \Omega}^2 - \|\nabla (u - u_h)\|_{0, \Omega}^2 = \int_\Omega \nabla (u_I + u_h - 2u) \cdot (\nabla u_I - \nabla u_h) \, dx. \quad (3.7)$$

From (3.7) we conclude that if

$$\|\nabla (u_I - u_h)\|_{0, \Omega} \leq CN^{1/2 - \gamma}, \quad (3.8)$$

where $\gamma > 0$ (this phenomena is called superapproximation \cite{6,21}), then

$$\|\nabla (u - u_I)\|_{0, \Omega}^2 - \|\nabla (u - u_h)\|_{0, \Omega}^2 = O(N^{-1 - \gamma}). \quad (3.9)$$

Assume $\eta_I$ be an asymptotically exact estimator of $\|\nabla (u - u_I)\|_{0, \Omega}$, that is

$$\lim_{N \to \infty} \frac{\eta_I}{\|\nabla (u - u_I)\|_{0, \Omega}} = 1.$$ 

Then $\eta_I$ can also be used as the estimator of $\|\nabla (u - u_h)\|_{0, \Omega}$ because

$$\lim_{N \to \infty} \frac{\eta_I^2}{\|\nabla (u - u_h)\|_{0, \Omega}^2} = \lim_{N \to \infty} \frac{\eta_I^2}{\|\nabla (u - u_I)\|_{0, \Omega}^2 + O(N^{-1 - \gamma})} = 1. \quad (3.10)$$

However, the superapproximation can be proved only in some structured meshes such as uniform and uniform Chevron triangular meshes in \cite{21}, and $O(h^{2\sigma})$ irregular triangular meshes in \cite{6}, under the assumption that $u$ is very smooth. When the solution doesn’t have superapproximation property we couldn’t replace the discretization error by the interpolation error. Fortunately, from numerical experiments in section 4 we guess that the superapproximation always holds during the adaptive procedure.

### 3.4 Problem for general coefficients

For the discussion above we assume that $b(x)$ and $a_{ij}(x)$ are zero and constant functions, respectively. In fact, we can get the corresponding results for the general smooth functions
This technique works. From this point we will verify if there exists a positive number such that

\[ a \]

where

\[ H \]

First we give some definitions,

4 Numerical experiments

We induce the exact Hessian for comparison in examples 4.1-4.4 and 4.6, while in example 4.5 we just use the recovered \( H_r \) where the true solution doesn’t belong to \( H^2(\Omega) \).

Because we use the Hessian recovery technique in [29], it is advantageous to show how this technique works. From this point we will verify if there exists a positive number \( \delta \) such that \( ||H - H_r||_{0, \Omega} < CN^{-\delta} \) in our numerical experiment.

Example 4.1 This example is to solve the boundary value problem of Poisson’s equation

\[ -\Delta u = f, \quad x \in \Omega \equiv (0, 1) \times (0, 1), \]

where the Dirichlet boundary condition and the right-hand side term are chosen such that the exact solution is given by

\[ u(x) = \left(1 + e^{\frac{x_1 + x_2 - 0.85}{0.5}} \right)^{-1} \]
with \( \epsilon \) being taken to be 0.005 (taken from [19]). Here we use the Delauney mesh generator to get the nearly uniform mesh, where \( nu \) is the number of initial points on the boundary. See Table 1 and Figure 2 for more details.

Figure 2: (a) Four estimators and (b) the initial mesh of the example 4.1.

| \( nu \) | \( N \) | \( E \) | \( E_r \) | \( EI \) | \( EI_r \) | \( ||H - H_r|| \) | \( \delta \) |
|-------|------|------|------|------|------|--------------|------|
| 40    | 3744 | 0.395664 | 0.361983 | 0.475616 | 0.250096 | 172.773 | -    |
| 80    | 8664 | 0.701157 | 0.584010 | 0.621573 | 0.434262 | 92.8695 | 1.48 |
| 160   | 15154| 0.799865 | 0.701711 | 0.726035 | 0.560066 | 62.6707 | 1.41 |
| 320   | 23674| 0.875311 | 0.795383 | 0.815560 | 0.680712 | 40.1777 | 1.99 |
| 640   | 34108| 0.910547 | 0.845726 | 0.862312 | 0.746308 | 29.0403 | 1.78 |

From example 4.2 to 4.6 meshes are generated using a c++ code BAMG (Bidimensional Anisotropic Mesh Generator) developed by Hecht [17] (r-version adaptive procedure).

**Example 4.2** The same problem as in example 4.1. In fact the solution exhibits a sharp layer on line \( x_1 + x_2 - 0.85 = 0 \). The result is shown in the following table, where \( p \) stands for the step of the adaptive procedure. Results are listed in Table 2 and Figure 3.

| \( p \) | \( N \) | \( E \) | \( E_r \) | \( EI \) | \( EI_r \) | \( ||H - H_r|| \) | \( \delta \) |
|-------|------|------|------|------|------|--------------|------|
| 1     | 94   | -0.015593 | 0.136566 | 0.812635 | 0.127626 | 265.946 | -    |
| 2     | 113  | 0.035375 | 0.176612 | 0.849958 | 0.179425 | 233.315 | 1.42 |
| 3     | 189  | 0.668308 | 0.540584 | 0.601845 | 0.414778 | 153.646 | 1.62 |
| 4     | 272  | 0.950449 | 0.846033 | 0.929412 | 0.738008 | 62.0580 | 4.98 |
| 5     | 278  | 0.994030 | 0.916821 | 1.018023 | 0.850781 | 34.0592 | 55.0 |

**Example 4.3** This example is to solve the boundary value problem of Poisson’s equation

\[-\Delta u = f, \quad \mathbf{x} \in \Omega \equiv (0, 1) \times (0, 1), \tag{4.3}\]
where the Dirichlet boundary condition and the right-hand side term are chosen such that the exact solution is given by

\[ u(x) = e^{x_1^2 - 0.8}. \]  

This is an extreme example for anisotropic behavior where the function \( u \) only depends on one variable \( x_1 \) or \( x_2 \). Such functions are the real challenge in the a posteriori error analysis since one is not allowed to use this knowledge. It is obvious our four estimators perform very well. See Table 3 and Figure 4 for more details.

\begin{table}[h]
\centering
\caption{Four estimators and \( \delta \) in example 4.3}
\begin{tabular}{llllllll}
\hline
step & \( N \) & \( E \) & \( E_r \) & \( E_I \) & \( E_{Ir} \) & \( \|H - H_r\| \) & \( \delta \) \\
\hline
1 & 26 & 0.863888 & 0.659135 & 1.14815 & 0.700543 & 1.02272 & - \\
2 & 26 & 0.899800 & 0.695822 & 1.13359 & 0.721021 & 0.976705 & - \\
3 & 32 & 0.984515 & 0.804626 & 1.13656 & 0.835906 & 0.808743 & 1.82 \\
4 & 43 & 0.993768 & 0.938219 & 1.08240 & 0.967953 & 0.402943 & 4.72 \\
5 & 66 & 0.990844 & 0.960663 & 1.04390 & 0.970201 & 0.190728 & 3.49 \\
\hline
\end{tabular}
\end{table}
Example 4.4 This example is to solve the boundary value problem of Poisson’s equation

\[- \triangle u = f, \quad x \in \Omega \equiv (-1, 1) \times (-1, 1), \tag{4.5}\]

where the Dirichlet boundary condition and the right-hand side term are chosen such that the exact solution is given by

\[u(x) = x_1^2 x_2 + x_2^3 + \tanh(10(\sin(5x_2) - 2x_1)). \tag{4.6}\]

The solution is anisotropic along the zigzag curve \(\sin(5x_2) - 2x_1 = 0\) and changes sharply in the direction normal to this curve (taken from [18, 22]). For more details see Table 4 and Figure 5.

Table 4: Four estimators and \(\delta\) in example 4.4

| step | \(N\) | \(E\) | \(E_{L}\) | \(E_{I}\) | \(E_{II}\) | \(\|H - H_{\ast}\|\) | \(\delta\) |
|------|-----|-----|-----|-----|-----|------|-----|
| 1    | 146 | -0.644460 | 0.106387 | 0.842915 | 0.082528 | 520.885 | 0.000000 |
| 2    | 325 | 0.014871  | 0.180764 | 0.534598 | 0.113211 | 490.853 | 0.15  |
| 3    | 756 | 0.642053  | 0.538530 | 0.559720 | 0.380420 | 289.914 | 1.25  |
| 4    | 1515| 0.940350  | 0.854614 | 0.935262 | 0.766470 | 105.952 | 2.90  |
| 5    | 2826| 0.993081  | 0.913258 | 1.08005 | 0.896111 | 75.6472 | 1.08  |

Figure 4: (a) Four estimators, (b) the final mesh and (c) \(u_h\) of the example 4.3.
Example 4.5 This example is to solve the boundary value problem of

\[- \Delta u = 0, \quad x \in \Omega \equiv (-0.5, 0.5) \times (0, 0.5) \cup (-0.5, 0) \times (-0.5, 0).\]  

(4.7)

The Dirichlet boundary condition is chosen such that the exact solution is given by

\[u = r^2 \sin\left(\frac{2}{3}\theta\right),\]

where \((r, \theta) \in \Omega\) are the usual polar coordinates. It is well known that the exact solution \(u \in H^{\frac{5}{3}-\epsilon}(\Omega) (\forall \epsilon > 0)\). So we expect our estimators can be extended to more problems especially for those with low regularity. See Figure 6 for more details.

Example 4.6 This example is to solve the boundary value problem

\[- \epsilon \Delta u + u = f, \quad x \in \Omega \equiv (0, 1) \times (0, 1),\]  

(4.8)

where the Dirichlet boundary condition and the right-hand side term are chosen such that the exact solution is the same as example 4.1(taken from [19]). Note that

\[E = \frac{\eta^2}{\epsilon^{-1} \|u - u_h\|^2_{0,\Omega} + \|\nabla(u - u_h)\|^2_{0,\Omega}}\]
Figure 6: (a) Two estimators, (b) the final mesh and (c) \( u_h \) of the example 4.5. (\( E_r, EI \) and \( EI_r \) are defined similarly). See Table 6 and Figure 7 for more details.

| step | \( N \) | \( E_r \) | \( E_I \) | \( EI_r \) | \( \|H - H_r\|\) | \( \delta \) |
|------|--------|--------|--------|--------|----------------|--------|
| 1    | 94     | 0.033119 | 0.158037 | 0.779738 | 0.143494 | 264.624 | - |
| 2    | 113    | 0.049065 | 0.216479 | 0.811994 | 0.206575 | 229.060 | 1.57 |
| 3    | 189    | 0.666480 | 0.568354 | 0.584863 | 0.448080 | 148.367 | 1.69 |
| 4    | 272    | 0.958780 | 0.867167 | 0.918925 | 0.756519 | 58.9264 | 5.07 |
| 5    | 278    | 0.999348 | 0.929424 | 1.006420 | 0.863337 | 32.6764 | 54.05 |

From experiments above we conclude that our a posteriori error estimators \( \eta \) and \( \eta_I \) are always asymptotically exact under various isotropic and anisotropic meshes. So we may guess that the superapproximation always holds during the adaptive process.
5 Conclusions

In the previous sections we have developed a new type of a posteriori error estimators suitable for moving mesh methods under general meshes (especially anisotropic meshes). In our next paper we want to design adaptive algorithms using the estimators, i.e., to give a new metric tensor for moving mesh method.

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