Composable Coresets for Constrained Determinant Maximization and Beyond

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Abstract

We study the task of determinant maximization under partition constraint, in the context of large data sets. Given a point set $V \subset \mathbb{R}^d$ that is partitioned into $s$ groups $V_1, \ldots, V_s$, and integers $k_1, \ldots, k_s$ where $k = \sum_i k_i$, the goal is to pick $k_i$ points from group $i$ such that the overall determinant of the picked $k$ points is maximized. Determinant Maximization and its constrained variants have gained a lot of interest for modeling diversity, and have found applications in the context of fairness and data summarization.

We study the design of composable coresets for the constrained determinant maximization problem. Composable coresets are small subsets of the data that (approximately) preserve optimal solutions to optimization tasks and enable efficient solutions in several other large data models including the distributed and the streaming settings. In this work, we consider two regimes. For the case of $k > d$, we show a peeling algorithm that gives us a composable coreset of size $kd$ with an approximation factor of $d\tilde{O}(d)$. We complement our results by showing that this approximation factor is tight. For the case of $k \leq d$, we show that a simple modification of the previous algorithms results in an optimal coreset verified by our lower bounds.

Further, we extend the above results in two directions: In particular, first we extend the result to all strongly Rayleigh distributions, and several other experimental design problems. Second, we show coreset construction algorithms under the more general laminar matroid constraints. Moreover, our approach provides a generalization of the directional height lemma of [Indyk-Mahabadi-OveisGharan-Rezaei–ICML’19], and also the exchange inequality lemma of [Anari-Liu-OveisGharan-Vinzant-Vuong–STOC’21]. Finally, as an application of our result, we improve the runtime of the algorithm for determinantal maximization under partition constraint in the recent work of [Anari-Vuong–COLT’22] from $O(n^{2k^2})$ to $O(n\text{poly}(k))$, making it only linear in the number of points $n$.

1 Introduction

Given a set of $n$ vectors $V = \{v_1, \ldots, v_n\}$ in $\mathbb{R}^d$, and a parameter $k$, the goal of the determinant maximization problem is to choose a subset $S \subseteq V$ of $k$ vectors such that $\det(\sum_{v \in S} v v^T)$ is maximized. Geometrically, if $k \leq d$, this determinant is equal to the volume squared of the parallelepiped spanned by the points in $S$. The best approximation factor for the problem in this regime is due to the work of [Nik15] who shows a factor of $e^d$, and it is known that an exponential dependence on $k$ is necessary [CM13]. On the other hand if $k > d$, the problem is known as the D-optimal design.
problem, and the best known approximation algorithm for it achieves a factor of \( \min \{ e^k, (\frac{k}{d})^d \} \), which is always at most \( \leq 2^d \) and becomes a constant when \( k \geq d^2 \) [Mad+19]. Determinant maximization and its variants have applications in computational geometry, experimental design, machine learning, and data summarization. In particular, the determinant of a subset of points is one way to measure the diversity of the subset, and thus they have been studied extensively over the last decade in this context [MJK17; Gon+14; KT+12; Cha+15; KT11; Yao+16; Lee+16].

The determinant maximization as well as other diversity maximization problems have been considered under partition and more generally matroid constraints [Mad+20; NS16; AMT13; MMM20; Add+22]. In the simpler case of partition constraint, the data set \( V \) is partitioned into \( s \) groups \( V_1, \cdots, V_s \) and we are provided with \( s \) numbers \( k_1, \cdots, k_s \), and the goal is to pick \( k_i \) points from group \( i \) such that the overall determinant (or more generally diversity) of the chosen \( k = \sum_i k_i \) points is maximized. This can be used to control the contribution of each group in a summary: e.g. bound the number of movies from each genre in a recommendation system, and has further applications in the context of fair data summarization. More generally, given a matroid \( ([n], \mathcal{I}) \) of rank \( k \), the problem of finding a basis of the matroid that maximizes the determinant admits a \( \min \{ k^{O(k)}, O(d)^d \} \) approximation, and it improves to \( \min \{ e^{O(k)}, O(d)^d \} \) for the estimation problem when the goal is to only estimate the value of the optimal solution [Mad+20; NS16; Bro+22].

As most applications of determinant maximization needs to deal with large amounts of data, there has been a large body of work on solving this problem in large data models of computation [MJK17; WIB14; Pan+14; Mir+13; Mir+15; MZ15; Bar+15]. One such strong model that we focus on in this work, is composable coreset. Coreset is generally defined as a small subset of the data that approximately preserves the optimal solution to an optimization problem [AHV+05]. Composable coresets [Ind+14] are coresets with an additional composability property: union of multiple coresets should contain a good solution for the union of the original data sets (see Section 2.3 for a formal definition). Having a composable coreset for an optimization task automatically gives a solution for the same task in several massive data models including distributed/parallel algorithms and streaming algorithms.

Composable coresets have been designed for determinant maximization problem, where for \( k \leq d \), one can get an \( k^{O(k)} \)-approximate coreset of size \( O(k) \) which is also known to be tight [Ind+20; Mah+19]. Further for \( k \geq d \), if the solution is allowed to pick vectors from \( V \) with repetition, then one can get a coreset of size \( \tilde{O}(d) \) with an approximation factor of \( \tilde{O}(d)^d \). In this work we study construction of coresets for constraint determinant maximization in the remaining setting of “without repetition”, as well as getting coresets for the problem in all regimes under the partition and laminar matroid constraints.

### 1.1 Our results

In this work, we show efficient construction of composable coreset for determinant maximization under partition and laminar matroid constraints as shown in Table 1, and verified in Theorem 28. Our results hold more generally for Strongly Rayleigh distributions (see Theorem 26), and are obtained via showing an improved exchange inequality for determinant when \( k > d \).

In Theorem 31, we further show application of our results to get composable coresets for other experimental design problems in the “without repetition” settings for all regular objective functions (this generalizes [Ind+20]’s result for experimental design in the with replacement setting).
We complement our results with the following lower bounds that are shown in Section 6.

- We show (in Lemma 32) that for $k \leq d$, any composable coreset for the problem under partition constraint that has finite approximation factor, must have size at least $\Omega(sk)$. This shows that our results on partition constraint is essentially tight. This is because our approximation factor matches that of [Ind+20] for the unconstrained version of the problem which is known to be tight.

- We show (in Lemma 33) that for $k \geq d$, any composable coreset with a finite approximation factor must have a size of at least $k + d(d-1)$.

- Lastly, we show (in Theorem 35) that for $d \leq k \leq \text{poly}(d)$ and coreset of size polynomial in $k$, the approximation of $d^{O(d)}$ is essentially the best possible.

Finally, we show application of our results to the recent result of [AV21] for determinant maximization under partition constraint and improve its runtime from $O(n^2k^2)$ to $O(n\text{poly}(k))$, making it only linear in the number of points $n$ (see Lemma 29).

### 1.2 Overview of the techniques

Let us give a brief overview of our approach. As mentioned earlier, for $k \leq d$, the determinant of any subset $T \subseteq V$ of $k$ selected vectors, i.e., $\det(\sum_{v \in T} vv^T)$ corresponds to the (square of) the volume spanned by vectors in $T$. Mahabadi, Indyk, Gharan, and Rezaei [Mah+19] shows that in this case, any local maximum $U \subseteq V$ of size $k$ for the $\det(\cdot)$ approximately preserves the $k$-directional height of the vectors $V$, meaning that for any set $S \subseteq \mathbb{R}^d$ of $k$ vectors and any $v \in S \cap V$, one can replace $v$ with some $u \in U$ so that the distance $d(u, H)$ from $u$ to the $(k-1)$-dimensional subspace $H$ spanned by $S \setminus \{v\}$ is at least $\frac{1}{k}$ the distance from $v$ to $H$, thus

$$k \det(uu^T + \sum_{w \in S \setminus \{v\}} ww^T) \geq \det(\sum_{w \in S} ww^T).$$

This immediately implies that $U$ is also a $k^{O(k)}$ composable coreset w.r.t $\det(\cdot)$ [see Mah+19, for details]. We generalize the notion of directional height to the setting when $k \geq d$, by showing that for $U$ of size $d$ being a local maximum w.r.t. $\det(\cdot)$, then for $S \subseteq \mathbb{R}^d$ of size $k$ and $v \in S \cap V$, there exists $u \in U$ s.t.

$$\det(duu^T + \sum_{w \in S \setminus \{v\}} ww^T) \geq \det(\sum_{w \in S} ww^T).$$

| $k \leq d$ | $O(k)$ | $O(k)^{2k}$ | cardinality | [Ind+20; Mah+19] |
| $k \geq d$ | $O(kd)$ | $O(d)^{kd}$ | cardinality (with repetition) | [Ind+20] |
| $k \geq d$ | $kd$ | $d^{2d}$ | cardinality (without repetition) | This work |
| $k \leq d$ | $sk$ | $k^{2k}$ | partition | This work |
| $k \geq d$ | $kd$ | $k^{2k}$ | partition | This work |
| $k \leq d$ | $k^{2d}$ | | laminar | This work |
| $k \geq d$ | $(kd)^k$ | | laminar | This work |

Table 1: The table shows our upper bound results on composable coresets for determinant maximization. Here $s$ is the number of groups in the partition constraints. Note that the third row follows from our results on partition constraint but we spell it out to compare to the previous result.
This already implies that $U$ is a composable coreset for determinant maximization in the with repetition setting, i.e., when the selected subset is allowed to be a multi-set. However, for the without repetition setting, we need to additionally ensure that $(S \setminus \{v\}) \cup \{u\}$ is a proper subset, i.e., $u \notin S \setminus \{v\}$. We apply the idea of peeling coreset, which was previously applied to construct robust coresets, i.e., coresets that can handle outliers [AHY08; Abb+13]. Our construction repeatedly peels away local optimum solutions from the input set, and takes the union of all the peeled local optimums to be the final coreset. By pigeon hole principle, this ensures that any given set $S$ who is not contained in the coreset must have no intersection with at least one of the peeled-away local optimums, thus we can replace an element of $S$ by an element inside this local optimum without creating a multiset.

For determinant maximization under partition matroid constraint, our coreset construction is simple and intuitive when $k \leq d$: it is the union of the coresets for each part of the partition constraint. For $k \geq d$, we obtain a coreset of size $kd$ and approximation factor $d^{O(d)}$ by letting the coreset to be the union of the peeling coresets for each part $P_i$ of the partition.

For the laminar matroid constraint case, we apply the peeling coreset idea to ensure that for any subset $S$ satisfying the laminar constraint, there exists one peeled-away subset $U$ s.t. replacing an element of $S$ by an element of $U$ will not violate the laminar constraint.

Our construction also applies to the experimental design problem with respect to other objective functions. When replacing the base-level building blocks, i.e., the local optimum w.r.t $\det(\cdot)$, in our construction with spectral spanners [Ind+20], we can ensure that the (union of) the coresets contains a feasible fractional solution as a combination of input vectors that achieves a good value. However, the algorithm to round the fractional solution to an integral solution under matroid constraint only exists in limited cases for objective function other than the determinant.

For lower bound on the size of composable coreset for determinant maximization under partition constraint, for $k \leq d$, we show that any coreset that achieves a finite approximation factor needs to contain at least $k$ vectors in each part of the partition. For $k \geq d$, we show an analogous result, that any coreset that achieves finite approximate factor needs to contain at least $d$ vectors in at least $d$ parts of the partition. The proof that the approximation factor of $d^{O(d)}$ is the best possible when $k \geq d$ is similar to [Ind+20]'s proof for the case $k \leq d$.

2 Preliminaries

Let $[n]$ denote the set $\{1, \cdots, n\}$. For a set $U$, we use $\binom{U}{k}$ to denote the family of all size-$k$ subsets of $U$. For a matrix $L \in \mathbb{R}^{n \times n}$ and $S \subseteq [n]$, we use $L_S$ to denote the principal submatrix of $L$ whose rows and columns are indexed by $S$. For sets $U, W$, we use $U + W$ and $U - W$ to denote $U \cup W$ (union) and $U \setminus W$ (set-exclusion) respectively. For singleton subsets, we abuse notation and write $U - e$ ($U + e$ resp.) for $U - \{e\}$ ($U + \{e\}$ resp.).

We use $S_d^{+}$ to denote the set of all positive semi-definite symmetric matrices in $\mathbb{R}^{d \times d}$.

**Definition 1** (Local optima). For a function $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ and $\zeta \geq 1$, we say $U$ is an $\zeta$-approximate local optima of $\mu$ iff $\zeta \mu(U) \geq \max_{e \in U, f \in [n] \setminus U} \mu(U - e + f)$.

When $\zeta = 1$, we simply refer to $U$ as a local optima.
2.1 Matroids

We say a family of sets $B \subseteq \binom{[n]}{k}$ is the family of bases of a matroid if $B$ satisfies the basis exchange axiom: for any two bases $B_1, B_2 \in B$ and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $B_1 - x + y \in B$. We call $k$ the rank of the matroid, and $[n]$ the ground set of the matroid. We let the family of independent sets of the matroid be $I = \{ I \in 2^{[n]} \mid \exists B \in B : I \subseteq B \}$.

The family $\binom{[n]}{k}$ of all size-$k$ subsets of $[n]$ form the set of bases of the uniform matroid of rank $k$ over $[n]$.

We define two simple classes of matroids that are widely used in applications.

**Definition 2 (Partition matroid).** Given a partition of $[n]$ into $P_1, \cdots, P_s$ and integers $k_1, \cdots, k_s$, the associated partition matroid is defined by: a set $I \subseteq [n]$ is independent iff $|I \cap P_i| \leq k_i$, for $\forall i \in [s]$. The rank of the matroid is $k := \sum_{i=1}^{s} k_i$.

**Definition 3 (Laminar matroid).** A family $F$ of subsets of $[n]$ is laminar iff for any $F_1, F_2 \in F$ either $F_1$ and $F_2$ are disjoint or $F_1$ contains $F_2$ or $F_2$ contains $F_1$. Given a laminar family $F$ and integers $k_F$ for each $F \in F$, the associated laminar matroid is defined by: a set $I \subseteq [n]$ is independent iff $|I \cap F| \leq k_F$, for $\forall F \in F$. The maximal independent sets have the same cardinality $k$, and they form the bases of the laminar matroid.

We assume that $k_F > 0$, for $\forall F \in F$, otherwise we can remove the set $F$ from $F$ and all elements in $F$ from the ground set. For two sets $F_1, F_2$ in $F$ s.t. $F_1 \subseteq F_2$, we can assume $k_{F_2} > k_{F_1}$, otherwise the constraint on $I \cap F_1$ is redundant and $F_1$ can be removed from the laminar family. We call such a laminar family non-redundant.

2.2 Determinant maximization and experimental design problems

Given vectors $v_1, \cdots, v_n \in \mathbb{R}^d$ and a parameter $k$, determinantal point processes (DPPs) samples a size-$k$ subset $S$ of $[n]$ s.t.

$$\mathbb{P}[S] \sim \det(\sum_{i \in S} v_i v_i^T).$$

This distribution favors diversity, since sets of vectors that are different from each other are assigned higher probabilities. The fundamental optimization problem associated with DPPs, and probabilistic model in general, is to find a “most diverse” subset by computing $\arg \max_S \mathbb{P}[S]$ i.e. solving the maximum a priori (MAP) inference problem.

When $k \leq d$, $\mathbb{P}[S]$ is proportional to the squared volume of the parallelepiped spanned by the elements of $S$, thus MAP-inference for DPPs is also known as the volume maximization problem.

**Experimental design** In the experimental design problem (under matroid constraint), we are additionally given a matroid $M = ([n], B)$ with set of bases $B$, and the goal is to find a set $S \in B$ so that $\sum_{i \in S} v_i v_i^T$ optimizes a given objective function $f : S^d \rightarrow \mathbb{R}$. When the matroid constraint is the uniform matroid we simply refer to the problem as experimental design under cardinality constraint, or simply the experimental design problem. The most popular and well-studied objective functions include:

- D(eterminant)-design: $f(A) = \det(A)^{1/d}$.
- A(verage)-design: $f(A) = -\text{Tr}(A^{-1})/d$
• E(eigen)-design: $f(A) = -\|A^{-1}\|_2$
• T(race)-design: $f(A) = d/\text{Tr}(A)$

The above objective functions are regular (see Definition 4). [ALW17] shows that under uniform matroid i.e. cardinality constraint, any fractional feasible solution of a regular function can be rounded into an integral solution while incurring only $O(1)$ loss in the objective function. For laminar matroids, [LZ21] shows the same results for $D$-design and $A$-design when $k_F \geq Cd$ for $\forall F \in \mathcal{F}$ and for some absolute constant $C$. For general matroid and $f(\cdot) = \det(\cdot)$, [Mad+20] shows a fractional feasible solution can be rounded into an integral solution while suffering a $d^{O(d)}$ loss in expectation.

**Definition 4.** A function $f : S_d^+ \rightarrow \mathbb{R}$ is regular if it satisfies the following properties

- Monotonicity: for any $A, B \in S_d^+$, if $f(A) \leq f(B)$ for $A \preceq B$.
- Concavity: for $A, B \in S_d^+$ and $t \in [0,1]$, we have $f(tA + (1-t)B) \geq tf(A) + (1-t)f(B)$.

In particular, this implies the existence of an efficient algorithm that solves the continuous relaxation

$$\max_{s_1,\ldots,s_n} f(\sum_{i=1}^n s_i v_i v_i^\top) \text{ s.t. } s_i \in [0,1] \text{ and } \sum_{i=1}^n s_i \leq k.$$  

- Reciprocal linearity: for any $A \in S_d^+$ and $t \in (0,1)$, $f(tA) = t^{-1}f(A)$.

People have also studied the setting where $S$ is allowed to be a multiset. This is known as the experimental design problem with repetition [ALW17; Mad+19], as opposed to the without repetition setting where $S$ needs to be a proper subset. The with repetition setting is generally easier: it can be reduced to the without repetition setting by duplicating each vector $k$ times. Composable coreset for experimental design problem (under cardinality constraint) in the with repetition setting has been studied in [Ind+20]. In this work we generalize [Ind+20]’s result to the without repetition setting.

### 2.3 Composable coresets

In the context of the optimization problem on $\mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0}$, a function $c$ that maps any set $V \subseteq [n]$ to one of its subsets is called an $\alpha$-composable coreset ([Ind+14]) if it satisfies the following condition: given any integer $m$ and any collection of sets $V_1, \ldots, V_m \subseteq [n]$

$$\alpha \cdot \max \left\{ \mu(S) \bigg| S \subseteq \bigcup_{i=1}^m c(V_i) \right\} \geq \max \left\{ \mu(S) \bigg| S \subseteq \bigcup_{i=1}^m V_i \right\}.$$  

We also say $c$ is a coreset of size $t$ if $|c(V)| \leq t$ for all sets $V$. Composable coresets are very versatile; when a composable coreset is designed for a task, they automatically imply efficient streaming and distributed algorithms for the same task.

### 2.4 Directional height

**Definition 5** (Directional height and $k$-directional height [Mah+19]). For a set $V \subseteq \mathbb{R}^d$ of vectors and a unit vector $x$, the directional height of $V$ w.r.t $x$ is $h(V, x) = \max_{v \in V} |\langle v, x \rangle|$.

The $k$-directional height w.r.t a $(k-1)$-dimensional subspace $H$ is $d(V, H) = \max_{v \in V, x \in H^\perp} |\langle v, x \rangle|$ where $H^\perp$ is the $(d-k+1)$-dimensional subspace perpendicular to $H$.  


**Theorem 6** (Coreset for \( k \)-directional height [Mah+19]). For \( k \leq d \) and \( V \subseteq \mathbb{R}^d \), any size \( k \) local optimum \( U \) w.r.t \( \det(\cdot) \) inside \( V \) approximately preserves the \( k \)-directional height i.e. for any \((k-1)\)-dimensional subspace \( H \)

\[
d(U, H) \geq \frac{1}{k} d(V, H).
\]

where for a point set \( P \), we define \( d(P, H) = \max_{p \in P} d(p, H) \).

### 2.5 Spectral spanner

**Definition 7** ([Ind+20]). For a set of vectors \( V \subseteq \mathbb{R}^d \), a subset \( U \subseteq V \) is a \( \alpha \)-spectral spanner of \( V \) iff for any \( v \in V \), there exists a distribution \( \mu_v \) of vectors in \( U \) s.t.

\[
v v^\top \preceq \alpha \mathbb{E}_{u \sim \mu_v}[uu^\top]
\]

**Theorem 8** ([Ind+20, Proposition 4.2, Lemma 4.6]). Given \( V \subseteq \mathbb{R}^d \), there exists an efficient algorithm that constructs \( \tilde{O}(d) \)-spectral spanner of size \( \tilde{O}(d) \).

### 2.6 Strongly Rayleigh distribution and exchange inequalities

For a distribution \( v : \binom{[n]}{\ell} \rightarrow \mathbb{R}_{\geq 0} \), its generating polynomial is

\[
g_{v}(z_1, \ldots, z_n) = \sum_{S \subseteq \binom{[n]}{\ell}} \mu(S) \prod_{i \in S} z_i.
\]

**Definition 9.** A distribution \( v : \binom{[n]}{\ell} \rightarrow \mathbb{R}_{\geq 0} \) is strongly Rayleigh or real-stable if its generating polynomial \( g_v \) does not have roots in the upper-half of the complex plane i.e. \( g_v(z_1, \ldots, z_n) \neq 0 \) if \( \text{Im}(z_1), \ldots, \text{Im}(z_n) \geq 0 \).

Strongly Rayleigh distributions satisfy the following exchange inequality, which implies that for any local optimum subset \( U \) w.r.t a strongly Rayleigh distribution \( v \), and a given set \( W \in \text{supp}(v) \), we can replace an element of \( S \) for an element of \( U \) while approximately preserving the evaluation in \( v(\cdot) \).

**Lemma 10.** Exchange inequality [Ana+20, Lemma 26] Let \( v : \binom{[n]}{\ell} \rightarrow \mathbb{R}_{\geq 0} \) be strongly Rayleigh. Let \( V \subseteq [n] \) be an arbitrary subset. For \( \zeta \geq 1 \), let \( U \) be a \( \zeta \)-local-optimum (w.r.t. \( v \)) inside \( V \), with \( v(U) \neq 0 \). For \( W \in \binom{[n]}{\ell}, \) and \( e \in W \setminus U \)

\[
v(W)v(U) \leq \ell \sum_{j \in U \setminus W} v(W - e + j)v(U + e - j)
\]

In particular, if \( e \in V \) then by approximate local optimality of \( U \) inside \( V \), we have

\[
v(W) \leq \zeta \ell \sum_{j \in U} v(W - e + j) \leq (\zeta \ell)^2 \max_{j \in U} v(W - e + j)
\]

where we implicitly understand that if \( j \in W - e \) then \( W - e + j \) is not a proper set, and \( v(W - e + j) = 0 \).

In the context of determinant maximization with given input vectors \( v_1, \ldots, v_n \in \mathbb{R}^d \), for any \( k \leq d \), \( v(S) = \det(\sum_{i \in S} v_i v_i^\top) \) defines a strongly Rayleigh distribution over \( \binom{[n]}{k} \) [BBL09; AGR16].
When the number of chosen vectors $k$ is larger than the dimension $d$, by Cauchy-Binet formula
\[
\det(\sum_{i \in S} v_i v_i^T) = \sum_{W \subseteq \binom{[d]}{k}} \det(\sum_{i \in W} v_i v_i^T).
\]

Let $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ be defined by $\mu(S) = \det(\sum_{i \in S} v_i v_i^T)$ then $\mu$ is strongly Rayleigh (see Proposition 11). Moreover, $\mu(S) = \sum_{W \subseteq \binom{[d]}{k}} v(W) \nu(S)$ where $\nu$ is itself strongly Rayleigh. In this setting, we can obtain stronger exchange inequality for $\mu$ that leads to an improved approximation factor for composability coreset w.r.t $\mu$ (see Section 3).

**Proposition 11.** For $\nu : \binom{[n]}{\ell} \to \mathbb{R}_{\geq 0}$ is strongly Rayleigh and $k \geq \ell$, define $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ by $\mu(S) = \sum_{W \subseteq \binom{[\ell]}{k}} v(W)$ then $\mu$ is strongly Rayleigh.

**Proof.** Consider the elementary symmetric polynomial
\[
e_{k-\ell}(z_1, \ldots, z_n) = \sum_{L \subseteq \binom{[n]}{k-\ell}} \prod_{i \in L} z_i
\] then $e_{k-\ell}$ is real stable i.e. has no roots in the upper half plane. Since the same is true for $g_\nu$, the product $g_\nu e_{k-\ell}$ also has no root on the upper half plane [see e.g. BBL09, Proposition 3.1 for a proof]. Consider the linear map $\phi : \mathbb{R}[z_1, \ldots, z_n] \to \mathbb{R}[z_1, \ldots, z_n]$ that maps monomial $z_1^{a_1} \cdots z_n^{a_n}$ to itself if $a_i \leq 1$ for $\forall i$, and to 0 otherwise. This map preserves real-stability of polynomial [BB09], and $\phi(g_\nu e_{k-\ell}) = g_\mu$, thus $g_\mu$ is real stable, and $\mu$ is strongly Rayleigh.

## 3 Improved exchange inequalities and the peeling coreset

As mentioned in the overview of the techniques, if $U$ is a composable coreset for $V$ with respect to the function $\mu(\cdot)$, then for any set $S$, we can replace an element in $S \cap V$ with an element of $U$ while not reducing $\mu(\cdot)$ by much. We formalize that intuition with the following definition.

**Definition 12.** Given $\tilde{\mu} : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ and $V \subseteq [n]$, we say $U \subseteq V$ is *value-preserving* with respect to $\tilde{\mu}$ if for any $S \in \binom{[n]}{k}$ and $e \in S \cap V$, there exists $f \in U \setminus (S - e)$ s.t. $\tilde{\mu}(S) \leq \tilde{\mu}(S - e + f)$.

The following lemma shows the relationship between value-preserving sets and coresets.

**Lemma 13.** Suppose functions $\mu, \tilde{\mu} : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ satisfies that $\mu(S) \leq \tilde{\mu}(S) \leq \alpha \mu(S)$ for $\forall S$, and coreset map $c$ satisfy $U := c(V) \subseteq V$ is value-preserving with respect to $\tilde{\mu}$ then $c$ gives $\alpha$-composability coreset with respect to $\mu$.

**Proof.** Consider a collection of sets $V_1, \ldots, V_m$, let $U_i := c(V_i)$ for $\forall i \in [m]$, and $S$ be an arbitrary size-$k$ subset of $\bigcup_{i=1}^m V_i$. Since $U_i$ is value-preserving w.r.t. $\tilde{\mu}$, for any $e \in S \cap V_i$, we can replace $e$ with $f \in U_i \setminus (S - e)$ while keeping $\tilde{\mu}(\cdot)$ non-decreasing. Thus, we successively replace $E := S \setminus \bigcup_{i=1}^m U_i$ with $L \subseteq \bigcup_{i=1}^m U_i$ while ensuring that
\[
\tilde{\mu}(S) \leq \tilde{\mu}(S - E + L)
\]
Moreover, $S - E + L \subseteq \bigcup_{i=1}^m U_i$ and
\[
\mu(S) \leq \tilde{\mu}(S) \leq \tilde{\mu}(S - E + L) \leq \alpha \mu(S - E + L).
\]
By choosing $S$ s.t. $\mu(S) = \max \{\mu(S) \mid S \subseteq \bigcup_{i=1}^m V_i\}$ we get the desired conclusion. \qed
We show that an (approximate) local optima with respect to \( \det(\cdot) \) is value preserving for suitably chosen functions. When \( k \leq d \), [Mah+19] shows that any size-\( k \) local optimum \( U \) with respect to \( \det(\cdot) \) approximately preserves \( k \)-directional height (see Theorem 6), and hence is a value-preserving set with respect to \( \tilde{\mu} \), where \( \tilde{\mu} \) is defined by

\[
\tilde{\mu}(S) = \det(\sum_{i \in S} k^{[i \in U]} v_i v_i^T).
\]

This is easy to see since for \( |S| = k \leq d \), \( \mu(S) = \det(\sum_{v \in S} v v^T) \) is precisely the square of the volume spanned by vectors in \( S \) i.e. \( \mu(S) = \text{Vol}^2(\{v \in S\}) \) and \( \tilde{\mu}(S) = k^2 |U| \text{Vol}^2(\{v \in S\}) \).

Below, we show that for \( k \geq d \), a size-\( d \) local optimum \( U \) is value-preserving with respect to \( \tilde{\mu} \) defined by

\[
\tilde{\mu}(S) = \det(\sum_{i \in S} d^{2 \times 1[i \in U]} v_i v_i^T).
\]  

(1)

Observe that, by Cauchy-Binet’s formula, for \( k \geq d \) and \( |S| = k \)

\[
\det(\sum_{i \in S} v_i v_i^T) = \sum_{W \in \binom{S}{d}} \det(\sum_{i \in W} v_i v_i^T).
\]

We can generalize this setting by considering \( v : (\mathbb{Z}_+^n) \to \mathbb{R}_{\geq 0} \) and \( \mu : (\mathbb{R}_{\geq 0}^n) \to \mathbb{R}_{\geq 0} \) by

\[
\mu(S) = \sum_{W \in \binom{S}{d}} v(W).
\]

(2)

We will assume that \( v \) is strongly Rayleigh, and consequently satisfies the exchange inequality in Lemma 10. Because Lemma 10 only applies to local optima with \( v(S) > 0 \), it will be more convenient to work with a full-support distribution i.e. \( v(S) > 0 \) for \( \forall S \in (\mathbb{R}_+^n) \). Fortunately, we can approximately any strongly Rayleigh distribution \( v \) with a full-support strongly Rayleigh distribution a.k.a. strictly real stable distribution. In other words, for any \( \epsilon > 0 \) there exists a strongly Rayleigh distribution \( \tilde{v} : (\mathbb{R}_+^n) \to \mathbb{R}_{\geq 0} \) s.t. \( \forall S : \tilde{v}(S) > 0 \) and \( |\tilde{v}(S) - v(S)| \leq \epsilon \). Moreover, \( \tilde{v} \) can be efficiently computed given \( v \) (see the main theorem of [Nui68], [BH19, Proof of Proposition 2.2] and [Brà20, page 7]). We include a formal statement and a simple proof for completeness.

**Proposition 14.** Let \( v : (\mathbb{R}_+^n) \to \mathbb{R}_{\geq 0} \) be strongly Rayleigh. For any \( \epsilon > 0 \), there exists strongly Rayleigh \( \tilde{v} : (\mathbb{R}_+^n) \to \mathbb{R}_{\geq 0} \) that approximates \( v \) i.e. \( |v(S) - \tilde{v}(S)| \leq \epsilon \).

**Proof.** By [Nui68], for \( i, j \in [n] \) and \( s \in \mathbb{R}_{\geq 0} \), the following operator preserves strongly Rayleigh/real-stability of polynomials

\[
T_{i,j,s} g = g + s z_j \frac{\partial g}{\partial i}
\]

for \( g \in \mathbb{R}[z_1, \ldots, z_n] \).

If \( v(S) = 0 \), for \( \forall S \), then we can let \( \tilde{v}(S) = \epsilon \) for \( \forall S \). W.l.o.g. assume \( v(S) \neq 0 \) for \( S = \{1, \ldots, \ell\} \). Let \( g \) be the generating polynomial for \( v \) and let

\[
f = \prod_{i \in [n], j \in [n]} T_{i,j,s} g = T_{1,1,s} \circ \cdots \circ T_{1,n} \circ T_{2,1,s} \circ \cdots T_{2,n,s} \circ \cdots \circ T_{n,n,s} g
\]

\(^1\text{Think of } \ell \text{ as being equal to } d^\epsilon\)
then $f$ is strongly Rayleigh. It is easy to see that for small enough $s$, $f$ approximates $g$. One practical choice for $s$ is $s = \epsilon (\sum_j v(S))^{-c}$ for some $c > 1$; computing the partition function $\sum_j v(S)$ can be done efficiently and even in $\tilde{O}(1)$-parallel time for distributions of interest e.g. DPP. The map from $f$ to the multi-affine part $f^{\text{MAP}}$ of $f$ preserves real stability [BBL09] (recall that for $f(z_1, \ldots, z_n) = \sum_{(a_i)} z_{a_i} c(\vec{a}) \prod_{i=1}^n z_i$, the multi-affine part of $f$ is $f^{\text{MAP}}(z_1, \ldots, z_n) = \sum_{(a_i)} z_{a_i} c(\vec{a}) \prod_{i=1}^n z_i$. Now we only need to check that $f^{\text{MAP}}$ has positive coefficients. Indeed, for $S = \{i_1, \ldots, i_\ell\}$ consider the coefficient of the monomial $z^S = \prod_{i \in S} z_i$ in $f$ and $f^{\text{MAP}}$: it is a sum which includes the term

$$\prod_{j=1}^\ell (sz_j \frac{\partial}{\partial z_j}) v([\ell]) z^\ell = z^S s^\ell v([\ell]) > 0$$

thus the coefficient of $z^S$ in $f^{\text{MAP}}$ is positive. \hfill \Box

Remark 15. We remark that a $O(1)$-approximate local optima of size $\ell$ w.r.t $\det(\cdot)$ can be found in time $O(n \text{poly}(\ell)^4)$ using a combination of simple heuristic such as greedy and local search (also known as Fedorov exchange algorithm). The same algorithmic result holds more generally for all strongly Rayleigh distribution $v : [n] \rightarrow \mathbb{R}_{\geq 0}$ [see AV21; Ana+20, for details].

In the remaining of the paper, we will consider the problem of maximizing the function $\mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0}$ under matroid constraint where $\mu(S) = \sum_{W \in \binom{[n]}{\ell}} v(W)$ and $v : [n] \rightarrow \mathbb{R}_{\geq 0}$.

Remark 16. This set-up of $\mu$ encompasses determinant maximization for both cases of $k \leq d$ and $k \geq d$. More concretely, for the former case, we set $\ell = k$ and for the latter case we set $\ell = d$. We will explain this in more details in Theorem 28.

For some constant $\xi \geq 1$ to be specified later, let $\phi(W) = (\xi \ell)^{2|W \cap U|}$. We define $\bar{\mu} : [n] \rightarrow \mathbb{R}_{\geq 0}$ by:

$$\bar{\mu}(S) = \sum_{W \in \binom{[n]}{\ell}} \phi(W) v(W)$$

This is the proper generalization of Eq. (1). We observe the following simple fact.

Fact 17.

$$\mu(S) \leq \bar{\mu}(S) \leq (\xi \ell)^{2\ell} \mu(S).$$

Lemma 18. For $V \subseteq [n]$, let $U$ be a $\xi$-approximate local optimum inside $V$ with respect to $v$, for $\xi = O(1)$. For any $e \in (V \cap S) \setminus U$, there exists $f \in U$ s.t.

$$\bar{\mu}(S) \leq \bar{\mu}(S - e + f).$$

Proof. Consider $W \in \binom{[n]}{\ell}$ with $e \in W$. Using Lemma 10,

$$v(W) \leq \ell' \sum_{f \in U} v(W - e + f)$$

with $\ell' = \xi \ell$. 

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Summing over all such $W$ we have

$$
\sum_{W \in \binom{\hat{S}}{\ell'}} \phi(W)\nu(W) \leq \ell' \sum_{f \in U} \sum_{W \in \binom{\hat{S}}{\ell'}} \phi(W)\nu(W - e + f) \\
\leq (\zeta\ell')^2 \max_{f \in U} \sum_{W \in \binom{\hat{S}}{\ell'}} \phi(W)\nu(W - e + f) \\
= \sum_{W \in \binom{\hat{S}}{\ell'}} \phi(W - e + f^*)\nu(W - e + f^*)
$$

with $f^*$ being the maximizer of the second line. The desired inequality then follows by adding $\sum_{W \in \binom{\hat{S}}{\ell'}} \phi(W)\nu(W)$ to both sides. \hfill \Box

We have just shown how to exchange $e \in (S \cap V) \setminus U$ for $f \in U$ while keeping $\hat{\mu}$ non-decreasing. However, we still need to ensure that $S - e + f$ is a proper set i.e. ensure that $f \notin S - e$. To do so, we need a slightly more elaborate coreset construction.

**Definition 19 (Peeling-coreset).** Given $V \subseteq [n]$, and a number $k_V \geq 1$, define the $(V, k_V, \zeta)$-peeling coreset $U$ as follows:

- Let $U_0 = \emptyset$.
- For $i = 1, \ldots, k_V$, let $V_i := V \setminus \bigcup_{j=0}^{i-1} U_j$, and let $U_i \subseteq V_i$ be a $\zeta$-approximate local optimal w.r.t. $\nu$ inside $V_i$.
- Let $U = \bigcup_{i=1}^{k_V} U_i$.

Note that the $U_i$’s are disjoint and $|U| \leq k_V\ell$.

**Lemma 20.** The $(V, k_V)$-peeling coreset $U$ constructed in **Definition 19** is a value-preserving subset of $V$ with respect to $\hat{\mu} : [n]^k \rightarrow \mathbb{R}_{\geq 0}$ s.t.

$$
\hat{\mu}(S) = 1[S \in \binom{[n]}{k}] \land |S \cap V| \leq k_V\hat{\mu}(S)
$$

**Proof.** Fix $S \in \binom{[n]}{k}$. Since $S \cap V \leq k_V$ and $e \in (S \cap V) \setminus U$. Since $S$ has at most $k_V - 1$ elements inside $U = \bigcup_{j=1}^{k_V} U_j$ there exists $j \in [k_V]$ where $S \cap U_j = \emptyset$. Note that $e \in (S \cap V) \setminus U \subseteq (S \cap V_j) \setminus U_j$, thus there exists $f \in U_j$ s.t. $\hat{\mu}(S) \leq \hat{\mu}(S - e + f)$. Since $S \cap U_j = \emptyset$, $f$ is not in $S - e$. \hfill \Box

**Lemma 21.** For $V \subseteq [n]$ and $\zeta \geq 1$, the $\zeta$-approximate local optimum $U$ w.r.t $\mu$ is a value-preserving subset of $V$ w.r.t $\hat{\mu}$ defined by $\hat{\mu}(S) = (\zeta k)^2|U \cap S|\mu(S)$.

**Proof.** We use the fact that $\mu$ is strongly Rayleigh, and **Lemma 10**. For any $S \in \binom{[n]}{k}$ and $e \in (S \cap V) \setminus U$, there exists $j \in U \setminus S$ s.t.

$$
\mu(S) \leq (\zeta k)^2 \mu(S - e + j)
$$

Multiplying both sides by $(\zeta k)^2|U \cap S|$ and using the fact that $|(S - e + j) \cap U| = |S \cap U| + 1$ we have

$$
\hat{\mu}(S) \leq \hat{\mu}(S - e + j)
$$

$^2|S \cap U| \leq |(S \cap V) \setminus e| \leq k_V - 1.$
Lemma 22 (Composability of value-preserving subsets). Consider disjoint $V_1, \ldots, V_m$ with $U_i$ being a value preserving subset of $V_i$ w.r.t. $\mu$, then $U := \bigcup_{i=1}^m U_i$ is a value-preserving subset of $V := \bigcup_{i=1}^m V_i$ w.r.t. $\tilde{\mu}$.

Proof. Consider $S \subseteq [n]$ and $e \in (S \cap V) \setminus U$. Clearly, $e \in (S \cap V_i) \setminus U_i$ for some $i \in [m]$. Since $U_i$ is value-preserving w.r.t. $\tilde{\mu}$, there exists $j \in U_i \subseteq U$ s.t. $\tilde{\mu}(S) \leq \tilde{\mu}(S - e + f)$.

4 Composable coresets for partition and laminar matroids

We construct composable coresets for determinant maximization under laminar matroid constraint. To build intuition, we first describe composable coresets for the simpler case of partition matroids. The idea is to build a peeling coreset of suitable size for each part of the partition which define the partition matroid.

As in section Section 3, given a matroid $\mathcal{M}$ with the set of bases $B$, we consider the problem of maximizing $\mu(S)$ (under matroid constraint) where $\mu(S) = \sum_{W \in \binom{[n]}{s}} v(W)$ and $v : \binom{[n]}{s} \to \mathbb{R}_{\geq 0}$ is strongly Rayleigh. Let $\mu_{\mathcal{M}}$ be the restriction of $\mu$ to the set of bases of $\mathcal{M}$ i.e. $\mu_{\mathcal{M}}(S) = 1|S \in B(\mathcal{M})|\mu(S)$.

Definition 23. Consider a partition matroid $\mathcal{M} = ([n], \mathcal{I})$ defined by the partition $P_1, \ldots, P_s$ of $[n]$ and $k_1, \ldots, k_s \in \mathbb{N}$. Fix constant $\zeta \geq 1$. For $V \subseteq [n]$, the composable coreset $U$ for $V$ w.r.t. $\mu_{\mathcal{M}}$ is constructed as follows:

- When $k > \ell : U$ is the union of $(V \cap P_i, k_i, \zeta)$-peeling coresets for each $i \in [s]$, thus $|U| = k\ell$.
- When $k = \ell : U$ is the union over $i \in [s]$ of the $\zeta$-approximate local optimum w.r.t. $\nu$ in $V \cap P_i$, thus $|U| = s\ell = sk$.

Lemma 24. The coreset constructed in Definition 23 has an approximation factor of $(\zeta\ell)^{2/3}$.

Proof. Note that in both cases, by Lemmas 20 and 21, $U$ is the union of value-preserving subsets $U_i$ of $V \cap P_i$ w.r.t. $\tilde{\mu}_{\mathcal{M}}(S) = 1|S \in B| \sum_{W \in \binom{[n]}{s}} (\zeta\ell)^2 |W \cap U| v(W)$. Thus, by Lemma 22, $U$ is a value-preserving subset of $V$ w.r.t. $\tilde{\mu}_{\mathcal{M}}$. Fact 17 and Lemma 13 together imply that $U$ is $\ell^{2/3}$-composable coreset w.r.t. $\mu_{\mathcal{M}}$.

We generalize the above construction to all laminar matroids.

Definition 25. Consider a laminar matroid over the ground set $[n]$ defined by a laminar family $\mathcal{F}$ and the associated integers $(k_s)_{s \in \mathcal{F}}$. Fix constant $\zeta \geq 1$. For $V \subseteq [n]$, the coreset for $V$ is constructed as follows:

1. For each maximal set $F \in \mathcal{F}$, construct a coreset $U_F \subseteq V \cap F$ by:
   - Let $D_0 = \emptyset$, $V_0 = V \cap F$.
   - For $i = 1, \ldots, k_s$, let $U_i$ be the $\zeta$-approximate local optimal w.r.t. $\nu$ in $V_i = V_{i-1} \setminus D_{i-1}$. For $e \in U_i$, let $F^e \in \mathcal{F}$ be the maximal proper subset of $F$ containing $e$ or $\{e\}$ if no such $F^e$ exists. Let $D_i := \bigcup_{e \in U_i} F^e$. For each $e \in U_i$ with $F^e \neq \{e\}$, recursively construct a
corest $U_F \subseteq V \cap F'$. Observe that if no proper subset of $F$ is inside $F$, then the corest $U_F$ is precisely the $(V \cap F, k_F)$ peeling-corest.

- The corest for $F$ is the union of all $U_i$ and $U_{F'}$ for $e \in U_i$.

2. The corest $U$ of $V$ is the union of all coresets $U_F$ for maximal sets $F \in \mathcal{F}$ and all elements $e \in V$ that do not belong to any set $F \in \mathcal{F}$.

**Theorem 26.** Consider $v : \binom{[n]}{\ell} \to \mathbb{R}_{\geq 0}$ that is strongly Rayleigh and $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ s.t.

$$\mu(S) = \sum_{W \in \binom{[n]}{\ell}} v(W).$$

Consider laminar matroid constraint $\mathcal{M}$ of rank $k$ defined by non-redundant family $\mathcal{F}$ with cover number $r$ i.e. $r := \max_{e \in [n]} \left| \{ F \in \mathcal{F} : e \in F \} \right|$. **Definition 25** gives a $(\zeta \ell)^{2 \ell}$-composable coreset w.r.t $\mu$ under matroid constraint $\mathcal{M}$ of size at most $(\zeta \ell)^k$.

**Proof of the approximation factor.** Let $\hat{\mu}(S) = \sum_{W \in \binom{[n]}{\ell}} (\zeta \ell)^{2 |W \cap U|} v(W)$. $U$ is a value-preserving subset of $V$ w.r.t the restriction $\hat{\mu}_\mathcal{M}$ of $\hat{\mu}$ to the set of bases of the laminar matroid i.e. $\hat{\mu}_\mathcal{M}(S) = \mathbb{1}[S \in B(\mathcal{M})] \hat{\mu}(S)$. This combined with **Lemma 13** immediately imply that $U$ is a $\ell^{2\ell}$-composable coreset w.r.t $\hat{\mu}_\mathcal{M}$.

We only need to show that $U_F$ is value preserving for each $F \in \mathcal{F}$. Fix $S \in B$ and $h \in (S \cap V \cap F) \setminus U_F$. We claim that there exists $f \in U_F$ s.t. $S - h + f \in B$ and $\hat{\mu}(S) \leq \hat{\mu}(S - h + f)$.

We prove this by induction on $F$. For the base case when $F$ has no proper subset inside $F$, then $U_F$ is the $(V \cap F, k_F)$ peeling-corest, and the claim follows from **Lemma 20**. If $h \in D_i$ for some $i$, then $h$ must be contained in a proper subset $F' \in \mathcal{F}$ of $F$ where $e \in U_i$ and $F' \neq \{ e \}^3$ and we can use the induction hypothesis. Now, assume $h \notin D_i$ for all $i \in [k_F]$. In particular, this means $h \in V_{k_F} \subseteq \cdots \subseteq V_1 = V \cap F$ and $D_i$ and $U_i$ are non-empty for all $i \in [k_F]$. Note that since $D_i$’s are disjoint, and $S$ contains at most $k_F - 1$ elements inside $F$, $S \cap D_i = \emptyset$ for some $i$. In particular, $S \cap U_i = \emptyset$ and $h \notin (V_i \cap S) \setminus U_i$, so **Lemma 18** implies that there exists $f \in U_i$ s.t. $\hat{\mu}(S) \leq \hat{\mu}(S - h + f)$. Replacing $h$ with $f$ only affects the constraints for sets $F' \in \mathcal{F}$ containing $f$. Consider such a set $F'$. $F'$ must be contained inside $D_i$ by the definition of $D_i$, thus $S \cap F' = \emptyset$, and $|(S - h + f) \cap F'| \leq 1 \leq k_F$. We just verify that $S - h + f$ is also a base of the laminar matroid, thus

$$\hat{\mu}(S - h + f) = \hat{\mu}(S - h + f) \geq \hat{\mu}(S) = \hat{\mu}(S).$$

\[\square\]

**Proof of upper bound on the size of the coreset.** For a set $H$ let $r_H := \max_{e \in H} \left| \{ F \in \mathcal{F} : F \subseteq H \wedge e \in F \} \right|$. We show that $|U_F| \leq (k_F \ell)^r_F$ for each $F \in \mathcal{F}$ by induction on $r_F$. For the base case $r_F = 1$, we have $|U_F| = k_F \ell$, by **Definition 19**. Fix $F \in \mathcal{F}$ with $r_F \geq 2$ and suppose the induction hypothesis holds for $r < r_F$. Using the definition of $U_F$, we can bound

$$|U_F| \leq \sum_{i=1}^{k_F} |U_i| + \sum_{e \in U_{i+1} \setminus U_i} |U_{F'}| \leq (1) k_F d + (k_F \ell)(d \max_{F \subseteq F', F \in \mathcal{F}} k_F)^{r_F - 1} \leq (2) (k_F \ell)^{r_F}$$

\[3\text{if } F' = \{ e \} \text{ then } h = e \in U_i \subseteq U_{F'}, \text{ a contradiction}\]
where in (1) we use the fact that \( r_F' < r_F \) since \( F' \) is a proper subset of \( F \), and in (2) we use
\[
(\max_{F' \subseteq F, F \in \mathcal{F}} k_{F'})^r - 1 \leq (k_F - 1)^r - 1 \leq k_F^r - 1.
\]
Thus the induction hypothesis holds for all \( r \).

Suppose the maximal set(s) in \( \mathcal{F} \) are \( F_1, \ldots, F_t \), and let \( R := [n] \setminus \bigcup_{i=1}^t F_i \). Then the rank of the laminar matroid is \( k = |R| + \sum_{i=1}^t k_{F_i} \), and
\[
|U| = |R \cup \sum_{i=1}^t U_{F_i}| \leq |R| + \sum_{i=1}^t (k_{F_i} \ell)^r \leq (k \ell)^r.
\]

\[
\square
\]

**Remark 27.** For any laminar family \( \mathcal{F} \) of rank \( k \), we can construct a \( d^{O(d)} \)-coreset of size \( |\mathcal{F}|dk \) by taking the union of all value-preserving subsets of \( V \cap (F \setminus \bigcup_{F \subseteq F', F \notin \mathcal{F}} F') \). However, the size of the coreset might be as bad as linear in \( n \). Indeed, consider the laminar family defined by: \( F_i = \{2i + 1, 2i + 2\} \), \( k_{F_i} = 1 \) for \( \forall i \in \lfloor n/2 \rfloor \) and \( F_0 = [n] \), \( k_{F_0} = k \) then Definition 25 gives a coreset of size \( d \leq k^2d^2 \) whereas the naive construction gives a coreset of size \( \geq (n/2)d \).

We immediately obtain the following corollary about determinant maximization under matroid constraints.

**Theorem 28.** For the determinant maximization matroid constraints with input vectors \( v_1, \ldots, v_n \in \mathbb{R}^d \), we obtain the following results:

1. **Partition matroid defined by partition \( P_1, \ldots, P_s \) of \([n]\),** Definition 23 gives:
   - For \( k \leq d \): \( k^{2k} \)-composable coreset of size \( O(sk) \).
   - For \( k \geq d \): \( d^{2d} \)-composable coreset of size \( O(kd) \).

2. **Laminar matroid:**
   - For \( k \leq d \): \( k^{2k} \)-composable coreset of size \( O(k^{2k}) \).
   - For \( k \geq d \): \( d^{2d} \)-composable coreset of size \( O((kd)^k) \).

**Proof.** We show how to adapt the setting of \( v : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0} \) and \( \mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0} \) where \( \mu(S) = \sum_{W \in \binom{[n]}{\ell}} v(W) \) with \( v \) being strongly Rayleigh to the determinant maximization setting.

- For \( k \leq d \) we let \( \ell = k \) and \( \mu(S) = v(S) = \det(\sum_{i \in S} v_i v_i^\top) \) for \( |S| = k \). By replacing \( \ell \) with \( k \) we get the stated result.

- For \( k \geq d \) we let \( \ell = d \), \( v(W) = \det(\sum_{i \in W} v_i v_i^\top) \) for \( |W| = \ell \) and \( \mu(S) = v(S) = \det(\sum_{i \in S} v_i v_i^\top) \) for \( |S| = k \).

\[
\square
\]

Recall that \( O(1) \)-approximate local optima can be found in time \( O(n \text{poly}(k)) \) (see Remark 15). Thus, our coreset construction is highly efficient: it takes time \( O(n \text{poly}(k)) \) for the case of partition matroid constraint. As a corollary, we obtain a quasilinear algorithm for MAP-inference for DPP under partition matroid constraint.

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4We can improve the bound to \( kd^2 \) by a more careful analysis.
Lemma 29. Consider a partition matroid $\mathcal{M} = ([n], \mathcal{I})$ of rank $k$ defined by the partition $P_1, \ldots, P_s$ of $[n]$ and $k_1, \ldots, k_s \in \mathbb{N}$. Given input vectors $v_1, \ldots, v_n \in \mathbb{R}^d$, there exists an $O(\text{poly}(k))$ algorithm that outputs a $\min\{k^{O(k)}, d^{O(d)}\}$-approximation for the determinant maximization under partition matroid constraint $\mathcal{M}$.

Proof. W.l.o.g., we can assume $k_i \geq 1 \forall i$. We construct coreset $U$ as in Theorem 28. Note that since $k_1 + \cdots + k_s = k$ and $k_i \geq 1$, we have that $s \leq k$ thus the size of $U$ is $O(k^2)$ for both cases $k \leq d$ and $k \geq d$. We can restrict the ground set to $U$ and use the existing efficient algorithms [Bro+22] to get a $\min\{k^{O(k)}, d^{O(d)}\}$-approximation for constrained determinant maximization with input vectors from $U$, which is also a $\min\{k^{O(k)}, d^{O(d)}\}$-approximation for the original constrained determinant maximization problem.

\section{Other experimental design problems}

In this section, we generalize our composable coreset construction to other experimental design problems such as $\Lambda$-design, $E$-design, etc.

The main idea is to replace the local optimum in the coreset construction with an $\alpha$-spectral spanner. By replacing local optimum with spectral spanner, we can ensure that the coreset contains a high-valued feasible fractional $x$ in the convex hull $P(\mathcal{M}) \subseteq [0, 1]^n$ of the matroid polytope of $\mathcal{M}$, which can be rounded to an integral solutions for uniform matroid constraint and certain class of laminar matroid constraint.

Theorem 30 (Rounding for experimental design, [Mad+20]). Consider the experimental design problem with objective function $f(\cdot)$ and input vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ under matroid constraint $\mathcal{M}$ of rank $k$. For any fractional $x \in P(\mathcal{M}) \subseteq [0, 1]^n$, there exists $z \in B(\mathcal{M}) \subseteq \{0, 1\}^n$ s.t.

- When $f(A) = \det(A)$:

$$\min\{d^{O(d)}, 2^{O(k)}\} f\left(\sum_{i=1}^{n} z_i v_i v_i^\top\right) \geq f\left(\sum_{i=1}^{n} x_i v_i v_i^\top\right)$$

The factor $d^{O(d)}$ can be improved to $2^{O(d)}$ when $\mathcal{M}$ is a partition matroid.

- When $k \geq d$, $\mathcal{M}$ is the uniform matroid and $f$ is regular:

$$O(1) f\left(\sum_{i=1}^{n} z_i v_i v_i^\top\right) \geq f\left(\sum_{i=1}^{n} x_i v_i v_i^\top\right)$$

- When $k \geq d$, $\mathcal{M}$ is a laminar matroid defined by the laminar family $\mathcal{F}$ and $(k_F)_{F \in \mathcal{F}}$ with $k_F \geq C d$ for $\forall F \in \mathcal{F}$ for some large absolute constant $C$, and $f(A) = -\text{Tr}(A^{-1})/d$:

$$O(1) f\left(\sum_{i=1}^{n} z_i v_i v_i^\top\right) \geq f\left(\sum_{i=1}^{n} x_i v_i v_i^\top\right)$$

We show $\tilde{O}(d)$-composable coreset of size $\tilde{O}(dk)$ for experimental design problems in the without repetition setting.
Theorem 31. Given input vectors $v_1, \ldots, v_n \in \mathbb{R}^d$, $V \subseteq \{v_1, \ldots, v_n\}$ and a number $k_V \geq 1$, the $(V, k_V)$-spectral peeling coreset $U$ is defined by the same procedure as in Definition 19, but replacing the local optimal $U_i$ by a $O(d)$-spectral spanner $U_i$ of $V_i$ (see Definition 7 and Theorem 8). Then $|U| \leq \tilde{O}(k_V d)$. For any $S$ with $|S \cap V| \leq k_V$, there exists a distribution $\mu_v$ for $v \in S \cap V$ with disjoint supports s.t. $\text{supp}(\mu_v) \subseteq U$ and for any regular objective function $f$:

$$f(\sum_{v \in S} \nu(v)) \leq f(\sum_{v \in S} \mathbb{E}_{u \sim \mu_v}[duu^T])$$

Consequently, for $k_V = k$, $U$ is a $\tilde{O}(d)$-composable coreset for the experimental design problem under cardinality constraint $k$ w.r.t $f$.

Proof. Consider the set $(S \cap V) \setminus U$. Since $|S \cap V| \leq k_V$, there is an injective map $\pi : (S \cap V) \setminus U \rightarrow [k_V]$ s.t. $S \cap U_{\pi(v)} = \emptyset$ for each $v \in (S \cap V) \setminus U$. For each $v \in (S \cap V) \setminus U$, since $v \in V_{\pi(v)}$, we can use the fact that $U_{\pi(v)}$ is a spectral spanner of $V_i$ to deduce that there exists $\mu_v$ supported on $U_{\pi(v)}$ where $\nu(v) \leq d \mathbb{E}_{u \sim \mu_v}[uu^T]$. Note that $\mu_v$ are disjoint by injectivity of $\pi$. The claim then follows from the monotonicity of $f$.

For sets $V_1, \ldots, V_m$ let $U'_i$ be the $(V_i, k)$-peeling coreset for $V_i$. Let $V' := \bigcup_{i=1}^m V_i$ and $U' := \bigcup U'_i$.

Let $S \in \binom{V'}{k}$ be a subset that maximizes $f(\sum_{v \in S} \nu(v))$. Using the above argument, we obtain that $U'$ contains a fractional solution $s \in [0, 1]^{U'}$ s.t. $\sum s_i = k$ and

$$f(\sum_{v \in S} \nu(v)) \leq f\left(\sum_{i \in U'} s_i(v_i \nu(v)^T)\right) \leq O(d) f(\sum_{u \in S} uu^T)$$

for some $S \in \binom{U'}{k}$, where the second inequality follows from Theorem 30. \hfill \Box

Using similar construction and proof technique, we obtain $\tilde{O}(d)$-composable coreset of size $\tilde{O}(dk)$ and $\tilde{O}((dk)^k)$ respectively for $A$-design under certain laminar and partition matroid constraint $\mathcal{M}$ where $k_F \geq C d$ for all $F \in \mathcal{F}$.

6 Lower bound

In this section, we show that the coreset we constructed essentially attains the best possible size and approximation factor. We first show that for determinant maximization in $\mathbb{R}^d$ when $k \leq d$ under partition matroid constraint, our coreset size is optimal.

Lemma 32. Suppose $k \leq d$. Consider a partition matroid $\mathcal{M} = ([n], \mathcal{I})$ defined by a partition $P_1, \ldots, P_s$ and constraint $k_1, \ldots, k_s$. Let $k := \text{rank}(\mathcal{M}) = \sum_{i=1}^s k_i$. Any $\alpha$-composable coreset for the determinant maximization problem under partition matroid constraint $\mathcal{M}$ with size $t < sk$ must incur an infinite approximation factor.

Proof. Consider a partitioning of $n$ vectors in $\mathbb{R}^d$ into two sets $V, V'$ such that each $V \cap P_i$ consists of $d$ vectors that are identical to the standard basis for $\mathbb{R}^d$ i.e. $V \cap P_i = \{e_1, \ldots, e_d\}$ for each $i$. We need to show that for any subset $U \subseteq V$ of size $t < sk$, we can choose the vectors in $V'$ s.t. $\text{OPT}(V \cup V') \gg \text{OPT}(U \cup V')$. Indeed, fix one such subset $U$, there must exists $i \in [s]$ s.t. $|U \cap P_i| \leq k - 1$. W.l.o.g., we can assume that $U \cap P_1 \subseteq \{e_1, \ldots, e_{k-1}\}$. Choose $V'$ s.t. $V' \cap P_1 = \emptyset$,
and \( V' \cap P_i = \{ M_\sum_{j=1}^{k} e_j, \ldots, M_\sum_{j=1}^{k-1} e_j \} \) for some arbitrarily large \( M > 0 \). Consider \( S \subseteq V \cup V' \) s.t. \( S \cap P_1 = \{ e_1, \ldots, e_{k-1}, e_k \} \) and \( S \cap P_i = V' \cap P_i \) for \( i = 2, \ldots, s \), then \( S \in \mathcal{M} \) and \( \mu(S) := \det(\sum_{i \in S} v_i v_i^T) = M^2 \sum_{j=2}^{k_j} \), thus \( \OPT(V \cup V') \geq M^2 \sum_{j=2}^{k_j} \). On the other hand, for any \( S' \in (U \cup V') \), either:

- \( S' \cap P_i \subseteq V' \forall i \geq 2 \) : in this case \( \mu(S') = 0 \) because all the vectors in \( S' \) are contained in the \((k-1)\)-dimensional subspace spanned by \( e_1, \ldots, e_{k-1} \).

- \( S' \cap P_i \nsubseteq V' \) for some \( i \geq 2 \) : in this case \( \mu(S') \leq M^2 \sum_{j=2}^{k_j} \) since there are at most \( \sum_{j=2}^{k_j} \) vectors in \( S' \) that are from \( V' \) and thus have norm \( M \), while the remaining vectors are from \( V \) and have norm 1.

In either case, we have \( \OPT(U \cup V') \leq M^2 \sum_{j=2}^{k_j} \) \( \OPT(V \cup V') / M^2 \), thus \( \OPT(V \cup V') \) can be arbitrarily large compared to \( \OPT(U \cup V') \). \(\square\)

For \( k \geq d \), using similar arguments, we can show that any \( \alpha \)-coreset for determinant maximization under partition constraint with finite approximation factor \( \alpha \) must have size \( t \geq k + d(d-1) \).

**Lemma 33.** Suppose \( k \geq d \). Consider the partition matroid \( \mathcal{M} = ([n], \mathcal{I}) \) of rank \( k \) defined by a partition \( P_1, \ldots, P_k \) and constraint \( k_1 = \cdots = k_k = 1 \). Any composable coreset for the determinant maximization problem under partition matroid constraint \( \mathcal{M} \) with size \( t < k + d(d-1) \) must incur an infinite approximation factor.

**Proof.** The construction is similar to the proof of Lemma 32. Let \( V \) be s.t. \( V \cap P_i \) consists of \( \{ M_i e_1, \ldots, M_i e_d \} \) with \( M_1 \geq M_2 \geq M_3 \gg M_{k+1} \cdots \geq M_k \) to be chosen later. Let \( U \) be a coreset for \( V \) with finite approximation factor. Clearly, \( |V \cap P_i| \geq 1 \). We will show that \( |U \cap P_i| = d \) for \( i = 1, \cdots, d \), and thus conclude that \( |U| \geq (k-d) + d^2 = k + d(d-1) \).

For the base case of \( i = 0 \), the claim holds trivially. Suppose that the claim holds for \( i - 1 \) with \( i \geq 1 \). Then we show that it holds for \( i \). We assume for contradiction that \( |U \cap P_i| \leq d - 1 \). W.l.o.g., assume \( U \cap P_i \subseteq M_i e_1, \ldots, M_i e_1, M_i e_{i+1}, \ldots, M_i e_d \). Indeed, define \( V' \) where \( V' \cap P_i = \{ M_i e_t \} \) for \( t \in \{ 1, \ldots, i-1, i+1, \cdots, d \} \) and \( V' \cap P_i = \emptyset \) otherwise. By choosing \( M \gg M_1 \) and \( M_i \gg M_{d+1} \), we can ensure that the optimal instance in \( V \cup V' \) must contain \( d - 1 \) vectors in \( V' \) and \( M_i e_t \in V \cap P_i \), thus

\[
\OPT(V \cup V') \geq (M^{d-1} M_i)^2.
\]

On the other hand, for any \( S \subseteq U \cup V' \), either

- \( |S \cap V'| \leq d - 1 \) : in this case \( \mu(S) \leq \binom{d}{d-1}(M^{d-2} M_i)^2 \) since any \( W \in \binom{d}{d-1} \) must contain at most \( d - 2 \) vectors of norm \( M \) from \( V' \), and the remaining vectors have norm at most \( M_1 \).

- \( |S \cap V'| = d - 1 \) : since \( U \cap P_i \) is in the span of \( S \cap V' \), any \( W \in \binom{d}{d} \) with \( \det(\sum_{i \in W} v_i v_i^T) \neq 0 \) must consist of at most \( d - 1 \) vectors of norm \( M \) from \( V' \), and the remaining vectors must have norm at most \( M_{d+1} \), thus \( \mu(S) \leq \binom{d}{d}(M^{d-1} M_{d+1})^2 \)

In either case, \( \OPT(U \cup V') \) can be arbitrarily small compared to \( \OPT(V \cup V') \). \(\square\)

Finally, we show that for \( k \geq d \), the approximation factor of \( d^{O(d)} \) is optimal. For \( k \leq d \), [Ind+20] shows that approximation factor of \( k^{O(k)} \) is optimal.

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5Even when we replace \( \mu \) by a full-support \( \tilde{\mu} \) that approximate \( \mu \) within distance \( \epsilon \), we will have \( \tilde{\mu}(S) < \epsilon < \OPT(V \cup V') / M^2 \) if we choose \( \epsilon \) small enough.
The following construction is from [Ind+20, section 7.1]. We include this for completeness.

**Definition 34** (Hard input for composable coreset). Let $\beta = o(d / \log^2 d)$, $m = d / \log d$ so that $d^{d/m} = O(1)$. Consider $G \subseteq \mathbb{R}^{m+1}$ of $d^{\beta+2}$ vectors s.t. for every two vectors $p, q \in G$, we have $\langle p, q \rangle \leq O(\sqrt[2]{\beta \log^2 d}/\sqrt{d})$.

For $i = 1, \cdots, d - m$, construct $X_i$ as follows: pick a random index $\pi_i \in [n]$. Embed $G$ into the subspace spanned by $\{e_i, \cdots, e_m, e_{m+i}\}$ s.t. the $\pi(i)^{th}$ vectors in $G$ is mapped into $e_{m+i}$.

Choose a random rotation matrix $Q$, and return $QX_1, \cdots, QX_{d-m}$ and $QY_1, \cdots, QY_m$ with $Y_i = \{Me_i\}$ for a large enough $M$.

**Theorem 35.** For $d \leq k \leq d^\gamma$ and $\gamma' = o(d / \log^2 d)$, any composable coreset of size $k'\gamma'$ must incur an approximation of $\left(\frac{d}{k}\right)^{d(1-o(1))}\gamma'$. For example, the theorem applies when $\gamma, \gamma'$ are constant, i.e. $d \leq k \leq \text{poly}(d)$ and the coreset has size $\text{poly}(k)$.

**Proof.** For a set $V$ of vectors, let $V^{\times t}$ be the set where each vector in $V$ is duplicated $t$ times. Let $\beta = \gamma \gamma'$. We use the construction in **Definition 34** where every vector is duplicated $t = k/d$ times. Let $QX_1^{\times t}, \cdots, QX_{d-m}^{\times t}, QY_1^{\times t}, \cdots, QY_m^{\times t}$ be the input sets. Let $S = \{Me_1, \cdots, Me_m, e_{m+1}, \cdots, e_d\}$ then $S^{\times t}$ has value $\mu(S) \geq (k/d)^{d(M^m)^2}$.

On the other hand, let $c(QX_i^{\times t})$ be an arbitrary coreset of size $k' \gamma' \leq d^\beta$ for $QX_i^{\times t}$.

As observed in [Ind+20, Lemma 7.2], the probability that $C_i := c(QX_i^{\times t})$ contains $Qe_{m+i}$ is bounded by $|c(QX_i^{\times t})|/|QX_i^{\times t}| \leq 1/d^2$. Thus, with probability $\geq 1 - 1/d$, we have $Qe_{m+i} \notin c(QX_i^{\times t})$ for all $i \in [d - m]$. Assume that this happens. Then for any $u \in C_i$

$$\langle \sum_{i=m+1}^{d} e_i e_i^T, uu^T \rangle \leq O\left(\frac{\beta \log^2 d}{d}\right)$$

thus for any $u_1, \cdots, u_m$ in $C := \bigcup_{i=1}^{m} c(QX_i^{\times t}) \cup \bigcup_{i=m}^{d-m} c(QX_i^{\times t})$

$$\det(\sum_{i=m+1}^{d} (Me_i)(Me_i)^T + \sum_{i=1}^{m} u_i u_i^T) \leq M^{2m}(\max\langle \sum_{i=m+1}^{d} e_i e_i^T, uu^T \rangle)^{d-m} \leq M^{2m}\left(\frac{O(\sqrt{\beta}) \log d}{d}\right)^{2(d-m)}.$$

Hence, with probability at least $1 - 1/d$, any size-$d$ subset $W$ in $C$ has $\det(\sum_{v \in W} vv^T) \leq M^{2m}\left(\frac{O(\sqrt{\beta}) \log d}{d}\right)^{2(d-m)}$, thus by Cauchy Binet, any size-$k$ subset $S$ in $C$ has

$$\mu(S) \leq \left(\frac{k}{d}\right) M^{2m}\left(\frac{O(\sqrt{\beta}) \log d}{d}\right)^{2(d-m)}.$$

Thus the approximation factor is at least $\frac{1}{d}(O(\sqrt{\beta}) \log d)^{2d-m}$ with $m = o(d)$.

\qed
References

[Abb+13] Sofiane Abbar, Sihem Amer-Yahia, Piotr Indyk, Sepideh Mahabadi, and Kasturi R Varadarajan. “Diverse near neighbor problem”. In: Proceedings of the twenty-ninth annual symposium on Computational geometry. 2013, pp. 207–214.

[Add+22] Raghavendra Addanki, Andrew McGregor, Alexandra Meliou, and Zafeiria Moumoulidou. “Improved Approximation and Scalability for Fair Max-Min Diversification”. In: arXiv preprint arXiv:2201.06678 (2022).

[AGR16] Nima Anari, Shayan Oveis Gharan, and Alireza Rezaei. “Monte Carlo Markov chain algorithms for sampling strongly Rayleigh distributions and determinantal point processes”. In: Conference on Learning Theory. PMLR. 2016, pp. 103–115.

[AHV+05] Pankaj K Agarwal, Sariel Har-Peled, Kasturi R Varadarajan, et al. “Geometric approximation via coresets”. In: Combinatorial and computational geometry 52.1 (2005).

[AHY08] Pankaj K Agarwal, Sariel Har-Peled, and Hai Yu. “Robust shape fitting via peeling and grating coresets”. In: Discrete & Computational Geometry 39.1 (2008), pp. 38–58.

[ALW17] Zeyuan Allen-Zhu, Yuanzhi Li, and Yining Wang. “Near-Optimal Discrete Optimization for Experimental Design: A Regret Minimization Approach”. In: Mathematical Programming 186 (Nov. 2017). doi: 10.1007/s10107-019-01464-2.

[AMT13] Zeinab Abbassi, Vahab S Mirrokni, and Mayur Thakur. “Diversity maximization under matroid constraints”. In: Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining. 2013, pp. 32–40.

[Ana+20] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. “Log-Concave Polynomials IV: Exchange Properties, Tight Mixing Times, and Faster Sampling of Spanning Trees”. In: CoRR abs/2004.07220 (2020). arXiv: 2004.07220.

[AV21] Nima Anari and Thuy-Duong Vuong. “From Sampling to Optimization on Discrete Domains with Applications to Determinant Maximization”. In: arXiv preprint arXiv:2102.05347 (2021).

[Bar+15] Rafael Barbosa, Alina Ene, Huy Nguyen, and Justin Ward. “The power of randomization: Distributed submodular maximization on massive datasets”. In: International Conference on Machine Learning. 2015, pp. 1236–1244.

[BB09] Julius Borcea and Petter Brändén. “The Lee-Yang and Pólya-Schur programs. II. Theory of stable polynomials and applications”. In: Communications on Pure and Applied Mathematics 62.12 (Dec. 2009), pp. 1595–1631. doi: 10.1002/cpa.20295.

[BBL09] Julius Borcea, Petter Brändén, and Thomas Liggett. “Negative dependence and the geometry of polynomials”. In: Journal of the American Mathematical Society 22.2 (2009), pp. 521–567.

[BH19] Petter Brändén and June Huh. Lorentzian polynomials. 2019. doi: 10.48550/ARXIV.1902.03719.

[Brä20] Petter Brändén. Spaces of Lorentzian and real stable polynomials are Euclidean balls. 2020. doi: 10.48550/ARXIV.2012.04531.

[Bro+22] Adam Brown, Aditi Laddha, Madhusudhan Pittu, Mohit Singh, and Prasad Tetali. Determinant Maximization via Matroid Intersection Algorithms. 2022. doi: 10.48550/ARXIV.2207.04318.

[Cha+15] Wei-Lun Chao, Boqing Gong, Kristen Grauman, and Fei Sha. “Large-Margin Determinantal Point Processes.” In: UAI. 2015, pp. 191–200.

[CM13] Ali Civril and Malik Magdon-Ismail. “Exponential inapproximability of selecting a maximum volume sub-matrix”. In: Algorithmica 65.1 (2013), pp. 159–176.

[Gon+14] Boqing Gong, Wei-Lun Chao, Kristen Grauman, and Fei Sha. “Diverse sequential subset selection for supervised video summarization”. In: Advances in Neural Information Processing Systems. 2014, pp. 2069–2077.
[Ind+14] Piotr Indyk, Sepideh Mahabadi, Mohammad Mahdian, and Vahab S Mirrokni. “Composable core-sets for diversity and coverage maximization”. In: Proceedings of the 33rd ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems. ACM. 2014, pp. 100–108.

[Ind+20] Piotr Indyk, Sepideh Mahabadi, Shayan Oveis Gharan, and Alireza Rezaei. “Composable core-sets for determinant maximization problems via spectral spanners”. In: Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM. 2020, pp. 1675–1694.

[KT+12] Alex Kulesza, Ben Taskar, et al. “Determinantal point processes for machine learning”. In: Foundations and Trends® in Machine Learning 5.2–3 (2012), pp. 123–286.

[Lee+16] Donghoon Lee, Geonho Cha, Ming-Hsuan Yang, and Songhwai Oh. “Individualness and determinantal point processes for pedestrian detection”. In: European Conference on Computer Vision. Springer. 2016, pp. 330–346.

[Mah+19] Sepideh Mahabadi, Piotr Indyk, Shayan Oveis Gharan, and Alireza Rezaei. “Composable core-sets for determinant maximization: A simple near-optimal algorithm”. In: International Conference on Machine Learning. PMLR. 2019, pp. 4254–4263.

[Mir+13] Baharan Mirzasoleiman, Amin Karbasi, Rik Sarkar, and Andreas Krause. “Distributed submodular maximization: Identifying representative elements in massive data”. In: Advances in Neural Information Processing Systems. 2013, pp. 2049–2057.

[Mir+15] Baharan Mirzasoleiman, Amin Karbasi, Ashwinkumar Badanidiyuru, and Andreas Krause. “Distributed submodular cover: Succinctly summarizing massive data”. In: Advances in Neural Information Processing Systems. 2015, pp. 2881–2889.

[MJK17] Baharan Mirzasoleiman, Stefanie Jegelka, and Andreas Krause. “Streaming Non-monotone Submodular Maximization: Personalized Video Summarization on the Fly”. In: arXiv preprint arXiv:1706.03583 (2017).

[MMP20] Zafeiria Moumoulidou, Andrew McGregor, and Alexandra Meliou. “Diverse Data Selection under Fairness Constraints”. In: arXiv preprint arXiv:2010.09141 (2020).

[MZ15] Vahab Mirrokni and Morteza Zadimoghaddam. “Randomized composable core-sets for distributed submodular maximization”. In: Proceedings of the forty-seventh annual ACM symposium on Theory of computing. ACM. 2015, pp. 153–162.

[Nik15] Aleksandar Nikolov. “Randomized rounding for the largest simplex problem”. In: Proceedings of the forty-seventh annual ACM symposium on Theory of computing. 2015, pp. 861–870.

[NS16] Aleksandar Nikolov and Mohit Singh. “Maximizing determinants under partition constraints”. In: Proceedings of the forty-eighth annual ACM symposium on Theory of Computing. 2016, pp. 192–201.

[Nui68] Waa Nuij. “A Note on Hyperbolic Polynomials.” In: Mathematica Scandinavica 23 (1968), pp. 69–72.
[Pan+14] Xinghao Pan, Stefanie Jegelka, Joseph E Gonzalez, Joseph K Bradley, and Michael I Jordan. “Parallel double greedy submodular maximization”. In: Advances in Neural Information Processing Systems. 2014, pp. 118–126.

[WIB14] Kai Wei, Rishabh Iyer, and Jeff Bilmes. “Fast multi-stage submodular maximization”. In: International conference on machine learning. 2014, pp. 1494–1502.

[Yao+16] Jin-ge Yao, Feifan Fan, Wayne Xin Zhao, Xiaojun Wan, Edward Y Chang, and Jianguo Xiao. “Tweet Timeline Generation with Determinantal Point Processes.” In: AAAI. 2016, pp. 3080–3086.