RATIONAL CURVES ON DEL PEzzo MANIFOLDS

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ABSTRACT. We exploit an elementary specialization technique to study some properties of rational curves on index $n-1$ Fano $n$-folds. We prove a simple formula for counting rational curves passing through a suitable number of points in the case $n = 3$. The arguments have immediate deformation theoretic consequences which translate to properties of the normal bundle of such curves.

1. Introduction

The index of a smooth Fano variety $X$ is the largest integer by which $-K_X$ is divisible in $\text{Pic} X$. It can be proved that the index is at most $\dim X + 1$, and moreover, it is equal to $\dim X + 1$ only for projective spaces, respectively to $\dim X$ only for smooth quadric hypersurfaces. We will be concerned with the study of the rational curves on $X$. Since $\mathbb{P}^n$ and $\mathbb{Q}^n$ are convex, meaning that $\mathcal{M}_{0,0}(X, \beta)$ is a smooth stack, and easily understood from the point of view of quantum cohomology, it is natural to go one step further and study rational curves on varieties of index $\dim X - 1$. In this paper, we will use an elementary specialization technique specific to the index $\dim X - 1$ case, to give simple answers to some questions related to the deformation and enumeration of such curves.

A del Pezzo manifold is a smooth Fano algebraic variety $X$ such that $-K_X$ is divisible by $n - 1$ in $\text{Pic} X$, where $n$ is the complex dimension of $X$. The polarization $\mathcal{O}(1)$ is defined by $-K_X = \mathcal{O}(n - 1)$. We call $\mathcal{O}(1)$ the hyperplane class. The degree $d$ of $X$ is the $n$-fold self-intersection of $\mathcal{O}(1)$. Del Pezzo manifolds have been completely classified by Fujita and Iskovskikh, cf. [Fu80, Fu90, Isk78, Isk80]. For the reader’s convenience we reproduce here their classification in dimension three and larger.

1. $d = 1$, then $X$ is a sextic hypersurface in $\mathbb{P}[3,2,1,...,1]$;
2. $d = 2$, then $X$ is a double cover of $\mathbb{P}^2$ ramified along a quartic;
3. $d = 3$, then $X$ is a cubic hypersurface in $\mathbb{P}^{n+1}$;
4. $d = 4$, then $X$ is a complete intersection of two quadrics in $\mathbb{P}^{n+2}$;
5. $d = 5$, then $X$ is a linear section of the Grassmannian $G(1,4) \subset \mathbb{P}^9$;
6. $d = 6$, then $X$ is either (6a) $\mathbb{P}T_{\mathbb{P}^2}$, (6b) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, or (6c) $\mathbb{P}^2 \times \mathbb{P}^2$;
7. $d = 7$, then $X$ is the blowup of $\mathbb{P}^3$ at one point;
8. $d = 8$, then $X$ is $\mathbb{P}^3$.

In the last case, the polarizing line bundle $\mathcal{O}(1)$ is what we would normally call $\mathcal{O}_{\mathbb{P}^3}(2)$ and the index is 4 rather than 2. In this paper, we will mainly be concerned with the classes of Picard rank one and base point free polarization, which amounts to $d \in \{2,3,4,5,8\}$, to avoid excessive bookkeeping when $d \in \{6,7\}$ respectively some technical issues when $d = 1$. However, most of the arguments still go through.

We denote a del Pezzo manifold of dimension $n$ and type $\theta$ by $d\mathbb{P}[n, \theta]$, where $\theta$ is one of the 10 types listed above. Furthermore, if $d\mathbb{P}[m, \theta]$ is a smooth plane section of $d\mathbb{P}[n, \theta]$, which is still del Pezzo by adjunction, we denote the inclusion map by $j^m_n[\theta] : d\mathbb{P}[m, \theta] \rightarrow d\mathbb{P}[n, \theta]$.

\footnotesize
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\end{itemize}
The purpose of the paper is to study the rational curves on $dP[n,\theta]$ using a certain elementary specialization technique.

Let us describe the the specialization technique in the case of a cubic hypersurface $X$, the extension to other classes being straightforward. Consider the incidence correspondence

$$\Sigma = \{(C, \xi_1, ..., \xi_e) | \xi_i \in C\}$$

between degree $e$ rational curves on $X$ and $e$-tuples of points on $X$ such that all $e$ points lie on the curve. We are being imprecise regarding the compactification of $R_e(X)$. The idea is to specialize the $e$ points to $e$ distinct points on $E = dP[1, (3)]$, a smooth plane cubic curve on $X$ obtained by cutting $X$ with a 2-plane. Then the curves containing the $e$ special points are forced to lie on cubic surfaces obtained by cutting $X$ with a 3-plane. Moreover, the divisor class of the rational curve on the corresponding cubic surface restricts on $E$ to $O_E(\xi_1 + ... + \xi_e)$. These observations allows us to give a sufficiently explicit geometric description of the special fiber $\alpha^{-1}(\xi_1, ..., \xi_e)$ where $(\xi_1, ..., \xi_e) \in E$ which can be exploited to prove the following results.

**Theorem 1.1.** Assume that $j_2^{[\theta]} : dP[2, \theta] \to dP[n, \theta]$ is a sufficiently general plane section of a del Pezzo manifold of degree $d \in \{2, 3, 4, 5, 8\}$. Let $f : \mathbb{P}^1 \to dP[2, \theta]$ be an unramified degree $e$ morphism such that $f_*[\mathbb{P}^1] \notin c_1(dP[2, \theta])\mathbb{Q}$. Then the normal bundle $N_{j_2^{[\theta]}f} \otimes \mathcal{O}_dP[n, \theta]$ of $f$ relative to $dP[n, \theta]$ is isomorphic to $\mathcal{O}(e-1) \oplus \mathcal{O}(e)^{(n-3)}$.

This statement generalizes an old result of Harris, Hulek and Eisenbud and Van de Ven [EVV81, Hu81] which says that a type $(1, k)$ rational curve on a smooth quadric surface has balanced normal bundle relative to the ambient $\mathbb{P}^3$ if $k > 1$. An immediately corollary is that, with few obvious exceptions, for a homology class $\beta \in H^2(dP[n, \theta]; \mathbb{Z})$ and $e = \deg \beta$. Then we have

$$\langle [pt]^e \rangle_{0, \beta}^{dP[n, \theta]} = \frac{1}{d(9-d)} \sum_{j_2^{[\theta]} \gamma = \beta} ((K \cdot \gamma)^2 - (K^2)(\gamma^2)) \langle [pt]^{e-1} \rangle_{0, \gamma}^{dP[2, \theta]}$$

where $K$ is the canonical divisor of the del Pezzo surface $dP[2, \theta]$.

Some comments are in order. The enumerative invariants above can very well be computed using various enumerative methods. However, the existence of such simple

\footnote{Irreducibility is actually known to hold for the cubic hypersurface case $n \geq 4$ [CS09] and it is totally conceivable that similar arguments work for other classes.}

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identities could be a slight surprise. The most mysterious feature of this identity is the constant $d(9 - d)$, which is theoretically obtained by doing intersection theory on a moduli space, but in effect we will only be able to prove its existence and then compute it retroactively by plugging in a special $\beta$.

There is an analogous identity in the case of $\mathbb{P}^3$, which was found\footnote{or rather conjectured, as the author acknowledges [Co83]} [Co83] long before the development of modern enumerative geometry. The argument in loc. cit. is different. Very recently, the identity was proved by Brugallé and Georgieva [BG15], following an approach suggested by Kollár [Ko14], which is similar to the one used here.

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2. Preliminaries

2.1. Genus Zero Maps to log K3 Pairs. In this section and the next, we prove some basic preliminary results which are more convenient to state independently of the rest of the argument. Let $\Sigma = d\mathbb{P}[2, \theta]$ be a complex del Pezzo surface and $E \subset \Sigma$ a smooth anticanonical divisor. By adjunction, $E$ has genus one. We will show, roughly, that there exist only finitely rational curves on $\Sigma$ which cut $E$ in a predetermined collection of points. For a homology class $\beta \in H_2(\Sigma; \mathbb{Z})$, let $M_{0,e}(\Sigma, \beta)$ be the space of genus zero stable maps to $\Sigma$ with $e$ marked points, where we will choose $e = (E \cdot \beta)$.

There is an evaluation map $ev : M_{0,e}(\Sigma, \beta) \times \Sigma^{e} \{ (\xi_1, \ldots, \xi_e) \}$ sweeps out a locus of dimension (at most) one on $\Sigma$.

If, moreover, for every proper subset $I \subset [e] = \{1, 2, \ldots, e\}$, we have

$$\mathcal{O}_E \left( \sum_{i \in I} \xi_i \right) \notin \text{Im}(\text{Pic}(\Sigma) \to \text{Pic}(E))$$

then $M_{0,e}(\Sigma, \beta) \times \Sigma^{e} \{ (\xi_1, \ldots, \xi_e) \}$ has dimension zero if nonempty. Moreover, the sources of all the stable maps it parametrized are smooth.

Proof. If this was not the case, by properness, this family of curves would have to sweep out the whole surface $\Sigma$. Let $\xi'$ be a point on $E$, distinct from all the $\xi_i$. Let $(C, f, p_1, \ldots, p_e) \in M_{0,e}(\Sigma, \beta) \times \Sigma^{e} \{ (\xi_1, \ldots, \xi_e) \}(\mathbb{C})$ whose image hits the point $\xi'$. The pullback of the Cartier divisor $E$ on $\Sigma$ vanishes on at least $e + 1$ points in different fibers of $f$: $p_1, \ldots, p_e$ and another point $p'$ mapping to $\xi'$, so, by degree considerations, it must vanish on some irreducible component of $C$ which is not contracted by $f$. Hence $C$ has an irreducible component $C_0$ which maps nonconstantly to $E$, which is impossible since $C_0$ has genus zero.

For the second part, note that the condition implies that for any $(C, f, p_1, \ldots, p_e) \in M_{0,e}(\Sigma, \beta) \times \Sigma^{e} \{ (\xi_1, \ldots, \xi_e) \}(\mathbb{C})$, there is at most one irreducible component of $C$ which is
not contracted by \( f \). (Here we are implicitly using that \( \mathcal{O}_\Sigma(E) \) is ample, so \( E \) intersects any divisor on \( \Sigma \).) Since \( C \) has arithmetic genus zero, it is not hard to infer that \( C \) has no contracted components, so in particular it must be irreducible. Moreover, since \( \xi_1, \ldots, \xi_e \) are distinct, \( f \) cannot factor as a multiple cover. Then finiteness follows easily from the previous result. \( \square \)

**Remark 2.2.** Without further hypotheses, \( \overline{M}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{ (\xi_1, \ldots, \xi_e) \} \) may fail to be reduced, e.g. when it contains ramified maps.

Note that in order for \( \overline{M}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{ (\xi_1, \ldots, \xi_e) \} \) to be nonempty, it is necessary that the unique line bundle on \( \Sigma \) whose first Chern class is Poincaré dual to \( \beta \) restricts to \( \mathcal{O}_E(\xi_1 + \ldots + \xi_e) \) on \( E \). If that is the case, then the forgetful map

\[
\overline{M}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{ (\xi_1, \ldots, \xi_e) \} \longrightarrow \overline{M}_{0,e-1}(\Sigma, \beta) \times_{\Sigma^{e-1}} \{ (\xi_1, \ldots, \xi_{e-1}) \}
\]

is an isomorphism, since the image of any map in the second space necessarily intersects \( E \) transversally at \( \xi_e \). Therefore, if additionally the condition in the second part of lemma 2.1 is satisfied, then

\[
\deg \left[ \overline{M}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{ (\xi_1, \ldots, \xi_e) \} \right] = \langle [pt]^{e-1} \rangle_0^\Sigma.
\]

This simple fact will be used later in the proof of formula 1.2.

**2.2. Lines on Fano Threefolds of Index 2.** Let \( X = dP[3, \theta] \) be a smooth Fano threefold of index 2, Picard rank one and base point free polarization, i.e. \( d \in \{2, 3, 4, 5\} \). We will need to know the number of lines (i.e. curves of degree 1 relative to the polarization by one-half the canonical class) through a general point \( x \in X \).

There are several ways to carry out this count, here we sketch a combinatorial approach. Consider the morphism \( X \to \mathbb{P}^{d+1} \). This is an embedding if \( d \geq 3 \), respectively a ramified covering if \( d = 2 \). A general hyperplane section of \( X \) is a del Pezzo surface of degree \( d \) and it can be described as the blowup of \( \mathbb{P}^2 \) at \( 9 - d \leq 7 \) points, no three of which are collinear and no five of which lie on the same conic. If, \( d \geq 3 \) let \( H \subset \mathbb{P}^{d+1} \) be a general hyperplane containing the projective tangent space to \( X \). If \( d = 2 \), we assume that \( x \) lies in the ramification locus of \( X \to \mathbb{P}^2 \) and choose \( H \) to be the plane tangent to the branch locus at the image of \( x \). The case \( x \in X \) general will still follow by a semi-continuity argument, which we skip. The pullback \( X \times_{\mathbb{P}^{d+1}} H \) has a simple double point at \( x \) and is smooth elsewhere. Moreover, it will contain any line on \( X \) passing through \( x \). Resolving this singularity, we obtain a surface \( S \) with a \((-2)\)-curve. The lines on \( X \) through \( x \) correspond to lines, i.e. \((-1)\)-curves, on \( S \) intersecting the \((-2)\)-curve.

The surface \( S \) also can be described as the blowup of \( \mathbb{P}^2 \) at \( 9 - d \) points, but with the feature (for instance) that 3 of the points have become collinear. Of course, the \((-2)\)-curve is simply the proper transform of the line through these 3 points. Let \( A, B, C \in \mathbb{P}^2 \) be the three collinear points and \( P_1, \ldots, P_{6-d} \in \mathbb{P}^2 \) the remaining \( 6 - d \) points. The only \((-1)\)-curves intersecting the \((-2)\)-curve are the three exceptional divisors of the blow up at \( A, B \) and \( C \), the proper transforms of the lines \( P_i P_j \), \( i < j \), and the proper transforms of conics passing through one of the points \( A, B, C \) and 4 of the \( P_i \), so the answer to our question is

\[
3 \binom{6 - d}{4} + \binom{6 - d}{2} + 3
\]

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which is 12 for \( d = 2, 6 \) for \( d = 3, 4 \) for \( d = 4 \) respectively 3 for \( d = 5 \). Note, in particular, that these counts are compatible with the formula in theorem 1.2.

3. Homology Classes on Surfaces in a Pencil

We have a smooth Fano threefold \( X \) of index 2 and degree \( d \), with polarization \( \mathcal{O}_X(1) \). Assume that \( \text{Pic}(X) = \mathbb{Z} \) and the polarization is base-point free, which amounts to \( d \in \{2, 3, 4, 5\} \). Consider a general pencil of sections of the polarizing line bundle with base locus \( E \) and total space \( \rho : W \to \mathbb{P}^1 \), where \( W \) is the blowup of \( X \) along \( E \). The members of the pencil are generically smooth del Pezzo surfaces of the same degree \( d \) as \( X \). Let \( \mathbb{P}^0 \subset \mathbb{P}^1 \) parametrize smooth del Pezzo fibers and \( W^0 \subset W \) its preimage. Choose a closed point \( b \in \mathbb{P}^0 \) and \( \beta_b \in H_2(W_b, \mathbb{Z}) \). The Poincaré dual of \( \beta_b \) is the first Chern class of a (uniquely determined) line bundle \( L_b \) on the surface \( W_b \).

First, some notation. We will consider objects \((N, \langle \cdot, \cdot \rangle, \nu)\) consisting of the following data: (1) a finitely generated free abelian group \( N \); (2) a bilinear map \( \langle \cdot, \cdot \rangle : N \otimes \mathbb{Z} N \to \mathbb{Z} \); and (3) a distinguished element \( \nu \in N \). A morphism between two such objects \((N, \langle \cdot, \cdot \rangle_1, \nu)\) and \((M, \langle \cdot, \cdot \rangle_2, \mu)\) requires a map \( \varphi : N \to M \) such that \( \varphi(\nu) = \mu \) and \( \langle \varphi(v), \varphi(w) \rangle_2 = \langle v, w \rangle_1 \). We denote the resulting category by \( \mathcal{D} \). The purpose of this category is merely to simplify language.

Set \( r = 9 - d \). Let \((H_r, \langle \cdot, \cdot \rangle, \omega) \in \text{Ob}(\mathcal{D})\) defined by \( H_r = \mathbb{Z}l_0 \oplus \mathbb{Z}l_1 \oplus \ldots \oplus \mathbb{Z}l_r \),

\[
\langle l_i, l_j \rangle = \begin{cases} 
0 & \text{if } i \neq j, \\
1 & \text{if } i = j = 0, \\
-1 & \text{if } i = j \geq 1 
\end{cases}
\]  

and \( \omega = -3l_0 + l_1 + l_2 + \ldots + l_r \). Note that this is isomorphic to \( H^2(\Sigma, \mathbb{Z}) \) of a degree \( d \) del Pezzo surface with the intersection form and canonical class. The abelian subgroup \( \omega^\perp = \{ v : \langle v, \omega \rangle = 0 \} \) with the negative pairing is the lattice \( E_r \), if \( d \in \{2, 3, 4, 5\} \). We are abusing notation by writing \( E_r \) for \( r = 4, 5 \) instead of \( A_4 \) and \( D_5 \) respectively. The complexification \( E_r \otimes \mathbb{C} = \omega^\perp_{\mathbb{C}} \) is the (dual) Cartan subalgebra \( h^\vee \) of the corresponding Lie algebra and the restriction of the pairing is the Killing form. Of course, the Killing form identifies \( h^\vee \) and \( h \), so we will simply write \( h \) despite the fact that we sometimes mean the dual. Recall that the automorphism group \( \text{Aut}_\mathcal{D}(H_r, \langle \cdot, \cdot \rangle, \omega) \) is the Weyl group \( G = W(E_r) \). Another folklore fact which will be particularly important to us is the following: up to scalars, the Killing form is the only \( G \)-invariant bilinear map on \( h \).

Consider the monodromy action \( \pi_1(\mathbb{P}^0, b) \to \text{Aut}_\mathcal{D}(H^2(W_b, \mathbb{Z}), \cup, c_1(T_{W_b}^\vee)) \). Since \( W \to \mathbb{P}^1 \) is a Lefschetz pencil, it is known that the image of the monodromy homomorphism is the full Weyl group \( G \). Now, let \( B^0 \) parametrize pairs \((t, \varphi_t)\) consisting of a closed point \( t \in \mathbb{P}^0 \) and a \( \mathcal{D} \)-isomorphism of \((H^2(W_t, \mathbb{Z}), \cup, c_1(T_{W_t}^\vee))\) with \((H_r, \langle \cdot, \cdot \rangle, \omega)\). Then \( B^0 \to \mathbb{P}^0 \) is finite and étale. Let \( W_B \) be the pullback of \( W \) to \( B^0 \). Then, by construction, the relative Picard scheme \( \text{Pic}(W_B/B^0) \) is simply \( H_r \times B^0 \), so every \( \beta \in H_r \) naturally induces a section \( \sigma(\beta) : B^0 \to \text{Pic}(W_B/B^0) \). Before stating the main enumerative problem of this section, we prove a lemma which will prove very useful in dealing away with potential multiplicities later on.

**Lemma 3.1.** The composition \( B^0 \to \text{Pic}(W_B/B^0) \to \text{Pic}(E) \) of \( \sigma(\beta) \) with the restriction map is constant if and only if \( \beta \) is a multiple of \( \omega \).
Proof. The if direction is trivial since \(-K_{W_t} = \mathcal{O}_{W_t}(1)\) restricts to \(\mathcal{O}_E(1)\) on \(E \subset W_t\).

To prove the converse, we use monodromy. Let \(\Lambda \subset H_r\) be the set of all \(\lambda \in H_r\) for which the composition in the statement of the lemma is actually constant. Of course, \(\Lambda \neq \emptyset\) since it contains \(\omega\). It is not hard to check that \(\Lambda\) has the following properties: (1) \(\Lambda\) is a subgroup of \(H_r\), (2) \(\Lambda\) is \(G\)-invariant and (3) if \(m \neq 0\) is an integer and \(m\lambda \in \Lambda\), then \(\lambda \in \Lambda\).

An easy exercise in lattice theory proves that the properties above imply that either \(\Lambda\) consists precisely of the multiples of \(\omega\), or \(\Lambda = H_r\). We claim that the latter is impossible. Indeed, we will show that \(l = l_1 \notin \Lambda\). Geometrically, \(l\) is the class of a line on a del Pezzo surface of degree \(d\). In section 2.2, we’ve shown that there exist lines passing through each point of the original threefold \(X\). Applying this to points of \(E\) and noting that any line on \(X\) intersecting \(E\) has to lie inside some \(W_t\), we conclude that \(l \notin \Lambda\). \(\square\)

We will mainly use the infinitesimal version of the lemma: the differential of the composition above is generically nonzero. Now fix \(L_E \in \text{Pic}^c(E)\) and an element \(\beta \in H_r\) corresponding to the chosen \(\beta_b \in H_2(W_b, \mathbb{Z})\) under a suitable isomorphism. In this section, we want to address the following enumerative problem.

**Question 3.2.** Assuming sufficiently general choices, how many pairs \((t, L_t)\) consisting of a closed point \(t \in \mathbb{P}^r\) and \(L_t \in \text{Pic}(W_t) \cong H^2(W_t; \mathbb{Z})\) such that

- \(L_t\) restricts on \(E\) to the line bundle \(L_E\); and
- there exists a \(\mathcal{O}\)-isomorphism \((H^2(W_t; \mathbb{Z}), \cup, K_{W_t}) \cong (H_r, \langle \cdot, \cdot \rangle, \omega)\) mapping \(L_t \mapsto \beta\) are there?

The way we will answer 3.2 is similar to the way most enumerative questions are answered: by doing intersection theory on a moduli space. Consider the functor \(\text{Sch}_{/\mathbb{C}}^{op} \to \text{Set}\) mapping a scheme \(S\) over \(\mathbb{C}\) to the set of homogeneous \(S\)-group scheme homomorphisms \(H_r \otimes \mathcal{O}_S \to \text{Pic}(E \times S/S)\) which send \(\omega \otimes 1\) to \(\mathcal{O}_S(-1)\) constant over \(S\). We require this to be homogeneous with respect to the grading on \(H_r\) given by pairing with \(-\omega\) and the natural grading by degree on \(\text{Pic}(E)\). The map on arrows is defined in the usual manner by pulling back such families along a morphism \(S' \to S\).

It is not hard to prove that the functor defined above is represented by an abelian variety \(A\) of complex dimension \(r\). The tangent bundle of \(A\) is naturally isomorphic to \(\mathfrak{h} \otimes \mathcal{O}_A = E_r \otimes \mathbb{Z} \mathcal{O}_A\). If we defined a similar functor without the requirement that \(\omega \otimes 1 \mapsto \mathcal{O}_E(-1)\), we’d have obtained an abelian variety of dimension \(r + 1\) which we’ll denote by \(V\). Note that we are not dropping the homogeneity requirement. It is clear that \(A\) sits naturally inside \(V\).

**Fact 3.3.** Let \(T\) be the a real torus, i.e a power of the circle. It is easy to describe the cohomology ring of a torus \(T\): for any commutative ring \(R\), there is an isomorphism

\[
(H^*(T; R), \cup) \cong \left( \wedge \star H^1(T; R), \wedge \right)
\]

(3.2)

of graded-commutative \(R\)-algebras. The isomorphism is most explicit if we take \(R\) to be the field of real or complex numbers and interpret cohomology as deRham cohomology. Assume now that \(T\) is a complex torus. For \(R = \mathbb{C}\), there are natural Hodge structures on both sides of (3.2): on the right hand side, the Hodge structure is obtained by algebraically taking exterior powers of the Hodge structure on \(H^1(T; \mathbb{C})\), while on the
left hand side, it is simply the Hodge structure on the cohomology ring of \( T \). The two structures coincide under the isomorphism above. In particular, we have a canonical isomorphism \( H^{1,1}(T; \mathbb{C}) = H^{1,0}(T; \mathbb{C}) \otimes H^{0,1}(T; \mathbb{C}) \), which we are going to use later.

Consider the quotient \( q : A \to [A/G] \). This quotient is the moduli space on which we will do intersection theory to answer question 3.2. The main point is that the family \( E \times \mathbb{P}^0 \subset W^\circ \to \mathbb{P}^0 \) induces a morphism \( \mathbb{P}^0 \to [A/G] \). By properness, we can extend the morphism back to the whole projective line, call it \( \alpha : \mathbb{P}^1 \to [A/G] \).

Let \( D(\beta, L_E) \) be the divisor on \( A \) parametrizing abelian group homomorphism \( H_p \to \text{Pic}(E) \) with the required properties, which are additionally required to send \( \beta \) to \( L_E \).

We have a theoretical answer to question 3.2:

\[
\frac{(q_*[D(\beta, L_E)] \cdot \alpha_*[\mathbb{P}^1])}{\#\text{Stab}_G(\beta)},
\]

the reason for the denominator being that the restriction of \( q \) to \( D(\beta, L_E) \) is \( \#\text{Stab}_G(\beta) \)-to-one onto its image. To obtain a numerical answer, we need to evaluate the intersection number above. This will be done, but indirectly.

**Lemma 3.4.** The cohomology class

\[
\frac{q_*[D(\beta, L_E)]}{(\omega \cdot \beta)^2 - (\omega^2)(\beta^2)} \in H^2([A/G], \mathbb{Q})
\]

is independent of \( \beta \in H_p \). Call this class \( y \).

**Remark 3.5.** A consequence of the Hodge index theorem is that the denominator is non-negative and it vanishes only when \( \beta \) is a multiple of \( \omega \). In the latter case, we simply require that the numerator also vanishes, which is clear.

Consider the Gysin map on cohomology \( q_* : H^{1,1}(A) \to H^2([A/G], \mathbb{C}) \). The domain of definition is canonically isomorphic to \( \mathfrak{h} \otimes \overline{\mathfrak{h}} \), so we obtain a \( G \)-invariant map

\[
q_* : \mathfrak{h} \otimes \overline{\mathfrak{h}} \to H^2([A/G], \mathbb{C}).
\]

The crucial observation is that by the uniqueness up to scalars of \( G \)-invariant bilinear maps, \( q_* \) factors through the map \( v \otimes \overline{w} \mapsto (v \cdot w) \), where the inner product is the Killing form. Note that we are actually using the general fact stated above for sesquilinear maps, which is equivalent since the action of \( G \) on \( \mathfrak{h} \) is real, i.e. it preserves the \( \mathbb{R} \)-span of \( E_r \).

Let \( \eta \in H^2([A/G], \mathbb{C}) \) be the image of \( 1 \in \mathbb{C} \) under the residual map. It follows that

\[
q_*(v \otimes \overline{w}) = (v \cdot w)\eta,
\]

so we’ve boiled down the proof of the lemma to computing \( [D(\beta, L_E)] \in H^{1,1}(A) \).

**Lemma 3.6.** Let \( \beta^\perp \) be the projection of \( \beta \) to the orthogonal complement of \( \omega \). Then

\[
[D(\beta, L_E)] = \beta^\perp \otimes \beta^\perp
\]

under the identification of \( H^{1,1}(A) \) with \( \mathfrak{h} \otimes \overline{\mathfrak{h}} \).
Proof. The evaluation of a homogeneous homomorphism $H_r \to \Pic(E)$ at $\beta$ is a morphism of abelian varieties $\hat{\beta} : A \to \Pic^\epsilon(E)$. Consider the pullback maps

$$\hat{\beta}^*_{i,j} : H^{i,j}(\Pic^\epsilon(E)) \to H^{i,j}(A)$$

for all $i,j \in \{0,1\}$. Note that the domain is canonically $\mathbb{C}$ in all cases and the class of $D(\beta, L_E)$ is simply $\hat{\beta}^*_{1,1}(1)$. Consider the dual differential

$$d\hat{\beta}^\vee : \hat{\beta}^*\Omega^1\Pic^\epsilon(E) \to \Omega^1(A).$$

This map is in fact simply the morphism $\hat{\theta}_A \to \hat{\theta}_A \otimes \mathfrak{h}$ given by $s \mapsto s \otimes \beta^\perp$. This will be proved shortly, let’s just assume it for now. Taking global sections of $d\hat{\beta}^\vee$ we recover the map $\hat{\beta}^*_{1,0}$, which was therefore the unique linear map $\mathbb{C} \to \mathfrak{h}$ such that $1 \mapsto \beta^\perp$. Having understood $\hat{\beta}^*_{1,0}$, we extrapolate to $\hat{\beta}^*_{0,1}$ by conjugation and then to $\hat{\beta}^*_{1,1}$ by tensoring, obtaining the desired conclusion. □

Claim 3.7. The map $d\hat{\beta}^\vee$ is just $\hat{\theta}_A \to \hat{\theta}_A \otimes \mathfrak{h}$, $s \mapsto s \otimes \beta^\perp$.

Proof. Note that there is a related map $d\hat{\beta}^\vee : \hat{\beta}^*\Omega^1\Pic^\epsilon(E) \to \Omega^1(V)$. This is easily seen to be simply tensoring with $\beta$. To obtain the original $d\hat{\beta}^\vee$, we need to restrict to $A$ and compose this map with the restriction $\Omega^1(V) \otimes \hat{\theta}_A \to \Omega^1(A)$, which is simply the sheafified version of the projection map $H_r \otimes \mathbb{C} \to \mathfrak{h}$ to the orthogonal complement of $\omega$.

□

Proof of Lemma 3.4. From (3.3) and (3.4), we have

$$q_*[D(\beta, L_E)] = q_*(\beta^\perp \otimes \beta^\perp) = \left(\beta^\perp \cdot \beta^\perp\right)\eta = \left[\frac{(\omega \cdot \beta)^2}{(\omega^2)} - (\beta^2)\right] \eta,$$

as desired. □

Let us return to the main problem. Denote $\Delta(\omega, \beta) = (\omega \cdot \beta)^2 - (\omega^2)(\beta^2)$. Lemma 3.4 implies that

$$\frac{(q_*[D(\beta, L_E)] \cdot \alpha_s[\mathbb{P}^1])}{\#\text{Stab}_G(\beta)} = \frac{\Delta(\omega, \beta)}{\#\text{Stab}_G(\beta)(y \cdot \alpha_s[\mathbb{P}^1])},$$

which is an answer to question 3.2, up to the computation of the constant $C = (y \cdot \alpha_s[\mathbb{P}^1])$. This computation seems to be in principle difficult, but we will return to it at the end of the paper, where we will show a way to circumvent a direct approach.

Finally, we will rearrange the enumerative problem we’ve just solved in a more convenient form. Let $S(L_E, \beta)$ be the set of solutions to question 3.2. As a scheme, this lives on $\Pic(W^\circ/\mathbb{P}^o)$ for general $L_E$, since there are only countably many divisor classes on $E$ obtained by restricting divisor classes on the singular fibers of $W \to \mathbb{P}^1$. Moreover, it is reduced by Lemma 3.1 and its infinitesimal version, since no $L_t$ can be a multiple of $K_W$, if $L_E(-m)$ is not torsion for all integers $m$. In the curve counting problem, we will encounter a $G$-constant counting function $N : H_r \to \mathbb{N}$. (In that case $N$ counts the
number of class β genus 0 stable maps through \( e - 1 = \deg(h) - 1 \) points on a degree \( d \) del Pezzo surface \( \Sigma \).) Then, as purely combinatorial statement, we have

\[
\sum_{\beta \in H^r} \#S(L_E, \beta)N(\beta) = \sum_{\beta \in H^r} C \frac{\Delta(\omega, \beta)}{\# \text{Stab}_G(\beta)} N(\beta) \frac{1}{\# G} = C \frac{\# G}{\sum_{\beta \in H^r} \Delta(\omega, \beta)N(\beta)},
\]

where \( H^r \) is the degree \( e \) piece of \( H_r \). We denote the constant in front of the sum by \( \kappa \); this is the final notation for the constants we had to carry along.

4. Interpolating Points on a Genus One Curve

Let \( X \) be an arbitrary del Pezzo variety of dimension \( n, \beta \in H_2(X; \mathbb{Z}) \) and \( \overline{M}_{0,0}(X; \beta) \) the space of class \( \beta \) genus 0 stable maps to \( X \). It is easy to check that that \( H_2(X; \mathbb{Z}) \) is torsion free. The hyperplane class \( \Theta(1) \in \text{Pic} \ X \) is uniquely determined by the property \( -K_X = \Theta(n - 1) \). Let \( e = (\beta \cdot \Theta(1)) \) be the degree of \( \beta \).

We will analyze the incidence correspondence between \( m \)-tuples of points on \( X \) and rational curves containing them, as described in the introduction. In the stable map compactification, the correspondence translates as the evaluation map \( \text{ev} : \overline{M}_{0,m}(X, \beta) \rightarrow X^m \). Roughly, the evaluation map is dominant if and only if it is possible to interpolate \( m \) general points on \( X \) with a rational curve of class \( \beta \). This is expected to happen when \( \dim \overline{M}_{0,m}(X, \beta) \geq \dim X^m \), or equivalently,

\[
m \leq \frac{(-K_X \cdot \beta) - 2}{\dim X - 1} + 1 = \frac{(n - 1)e - 2}{n - 1} + 1,
\]

where \( \dim \overline{M}_{0,m}(X, \beta) = \dim X - (K_X \cdot \beta) + m - 3 \), so the largest integer \( m = m_{\text{max}} \) for which the inequality \((2.1)\) holds is

\[
m_{\text{max}} = \begin{cases} 
  e & \text{if } n \geq 3, \\
  e - 1 & \text{if } n = 2.
\end{cases}
\]

Before outlining the approach, we introduce some notation.

Let \( X \) be any smooth variety, \( \beta \in H_2(X; \mathbb{Z})/\text{torsion}, Y \) a closed subscheme of \( X \) and \( m \leq n \). We write

\[
\overline{M}^{[m]}_{g,n}(X, Y; \beta) = \overline{M}_{g,n}(X; \beta) \times_X Y^m,
\]

where the map \( \overline{M}_{g,n}(X; \beta) \rightarrow X^m \) is \((\text{ev}_1, ..., \text{ev}_m)\). If \( U \subset Y^m \) is open, let

\[
\overline{M}^{[m]}_{g,n}(X, Y; \beta)|_U = \overline{M}_{g,n}(X, Y; \beta) \times_{Y^m} U.
\]

If \( m = n \), we drop the superscript. If \( n = m + 1 \), we can think of \( \overline{M}^{[m]}_{g,m+1}(X, Y; \beta) \) as the universal curve over \( \overline{M}_{g,m}(X, Y; \beta) \) and we denote it by \( \mathcal{C}_{g,m}(X, Y; \beta) \). If \( g = 0 \), which is the only case treated in this paper, we drop the genus index.

Let us return to the problem. Let \( E \subset X \) be a section of \( X \) by \( n - 1 \) general hyperplanes. By adjunction and Bertini, the property of being del Pezzo is preserved at each step, so \( E \) is a smooth genus one curve. Set

\[
V = H^0(X; \mathcal{I}_E/X \otimes \Theta(1)) \subset H^0(X; \Theta(1)).
\]
Roughly, the main observation is the following: a curve of class $\beta$ which meets $E$ at $d$ distinct points is forced to lie in a surface on $X$ containing $E$, obtained by cutting $X$ with $n - 2$ hyperplanes. To use this observation, we have to formalize it in families. Let $S$ be any finite type scheme over $C$ and

$$(C, \pi, f, p_1, p_2, \ldots, p_e) \in \overline{M}_e(X, E; \beta)|_{\Delta^e}(S),$$

where $\Delta^e = E^e \setminus \Delta$ and $\Delta$ is the big diagonal of $E^e$. Denote by $D_i \subset C$ the image of the closed embedding $p_i : S \to C$ and set $D = \sum D_i$. We start by proving the observation for of stable maps, then proceed with the formalism in families. Much of the formalism below is forced by the possibility that $S$ is not reduced. Since we can’t say a priori that $\overline{M}_e(X, E; \beta)$ is generically reduced (i.e. that it is not contained in the ramification locus of the evaluation map), this is a difficulty we are forced to face. The generic smoothness of $\overline{M}_e(X, E; \beta)$ will be a corollary of the subsequent analysis.

**Lemma 4.1.** If $S = \text{Spec } C$, then $f^*\mathcal{O}(1) \cong \mathcal{O}_C(D)$. Moreover, there is no irreducible component of $C$ mapped constantly to a point on $E$.

**Proof.** Let $M$ be a finite set indexing all maximal connected curves of arithmetic genus zero $C_\mu \subset C$ which are contracted by $f$ to a point on $E$ and let $\overline{C} = \bigsqcup_{\mu \in M} C_\mu \cup C_0$ with $C_0$ possibly disconnected. The dual graph $\Gamma_\mu$ of each $C_\mu$ is a tree. We further decorate each dual graph with "legs" for each intersection point with $C_0$. For all $\mu \in M$, let $\nu_\mu$ be the number of $i$ such that $p_i \in C_\mu$ and $\lambda_\mu$ the number of legs in the dual graph $\Gamma_\mu$. Since $f(p_i) \neq f(p_j)$ for $i \neq j$, $\nu_\mu \leq 1$ for all $\mu \in M$. By stability, this implies that no vertex of any $\Gamma_\mu$ can be a leaf of the dual graph of $C$. Therefore, there is at least one leg attached to each leaf of $\Gamma_\mu$, so $\lambda_\mu \geq 2$. In particular, $\lambda_\mu > \nu_\mu$. Similarly, we let $\nu_0$ be the number of $i$ such that $p_i \in C_0$ and $\lambda_0 = \sum \lambda_\mu$. Let $D_0$ be the restriction of $D$ to $C_0$ and $D_\lambda$ the divisor on $C_0$ consisting of the $\lambda_0$ "bridge points" to the union of the components on which $f$ maps constantly to $E$. Clearly, $D_0$ and $D_\lambda$ are reduced and have disjoint supports. Let $f_0$ be the restriction of $f$ to $C_0$. We claim that the line bundle

$$L_0 := f_0^*\mathcal{O}(1) \otimes \mathcal{O}_{C_0}(-D_0 - D_\lambda)$$

admits sections with finitely many zeroes. We may argue using the pullback map on global sections $H^0(X; \mathcal{I}_{E/X}(1)) \to H^0(C_0; L_0)$; indeed, all we have to do is pick a section of $\mathcal{I}_{E/X}(1)$ which is not identically zero on the image of any component of $C_0$ and correspondingly map it to a section of $L_0$ with finitely many zeroes. From here, we get the inequality $e - \nu_0 - \lambda_0 \geq 0$, or $\nu_0 + \mu_0 \leq e$. Since $\nu_0 + \sum \nu_\mu = e$, it follows that

$$\sum \lambda_\mu = \lambda_0 \leq \sum \nu_\mu,$$

which contradicts $\lambda_\mu > \nu_\mu$ for all $\mu \in M$, unless $M = \emptyset$. Regardless, the line bundle $L_0$, now $f^*\mathcal{O}(1) \otimes \mathcal{O}_C(-D)$, still has sections with finitely many zeroes and visibly has degree 0, so it must be trivial. □

**Lemma 4.2.** As above, let $L = f^*\mathcal{O}(1) \otimes \mathcal{O}_C(-D)$. Then $\pi_*L$ is invertible. Moreover, if

$$\varphi_S : V \otimes \mathcal{O}_S \longrightarrow \pi_*L$$
is the natural $\mathcal{O}_S$-modules map, there exists a unique morphism $\psi_S : S \to \mathbb{P}V$ such that $\ker \varphi_S = \psi_S^* \mathcal{U}_{\mathbb{P}V}$, where $\mathcal{U}_{\mathbb{P}V}$ is the tautological subbundle of $V \otimes \mathcal{O}_{\mathbb{P}V}$.

**Proof.** First, let us spell out the construction of the map $\varphi_S$ in the statement. The $\mathcal{O}_X$-modules homomorphism $H^0(\mathcal{O}(1)) \otimes \mathcal{O}_X \to \mathcal{O}(1)$ pulls back via $f$ to a map $H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C \to f^*(\mathcal{O}(1))$. By the adjoint property, we get an $\mathcal{O}_S$-modules homomorphism

$$H^0(\mathcal{O}(1)) \otimes \mathcal{O}_S \to \pi_* f^*(\mathcal{O}(1)).$$

This map composed further with $\pi_* f^*(\mathcal{O}(1)) \to \pi_*(f^*(\mathcal{O}(1))|_D)$ vanishes on $V \otimes \mathcal{O}_S$, so it induces an $\mathcal{O}_S$-modules homomorphism $\varphi_S : V \otimes \mathcal{O}_S \to \pi_*(f^*(\mathcal{O}(1))\otimes \mathcal{O}_C(-D)) = \pi_* L$. For any closed point $s \in S$, the map

$$V \otimes_{\mathbb{C}} \kappa(s) \mapsto \pi_* L \otimes_{\mathcal{O}_S} \kappa(s) \mapsto H^0(C_s, L_s)$$

is nonzero; otherwise, the image of $f_s$ would be contained inside $E$, which is impossible. However, $L_s$ is trivial by Lemma 4.1 meaning that the rightmost term above is 1-dimensional, so the composed map above is surjective. Furthermore, the second map in the composition has to be surjective as well, so by the cohomology and base change theorem, it is actually an isomorphism. By the same theorem, this property extends automatically to the non-closed points. The $\pi$-pushforward of any torsion-free sheaf on $C$ is torsion free as well, so $\pi_* L$ is torsion free. Together with

$$\dim_{\kappa(s)} (\pi_* L \otimes_{\mathcal{O}_S} \kappa(s)) = 1$$

for closed $s \in S$, this proves that $\pi_* L$ is invertible. Indeed, the corresponding stalk of $\pi_* L$ at $s$ is generated by a single element as an $\mathcal{O}_{s,S}$-module by Nakayama’s lemma and, since it is torsion free, it has to be free of rank one. Finally, since $(\varphi_S)_s : V \otimes_{\mathbb{C}} \kappa(s) \mapsto \pi_* L \otimes_{\mathcal{O}_S} \kappa(s)$ is surjective for closed points $s \in S$, we can define a morphism of schemes $\psi_S : S \to \mathbb{P}V$ such that $\ker \varphi_S = \psi_S^* \mathcal{U}_{\mathbb{P}V}$, where $\mathcal{U}_{\mathbb{P}V}$ is the tautological subbundle of $V \otimes \mathcal{O}_{\mathbb{P}V}$. □

**Corollary 4.3.** The sheaf $(\psi_S \circ \pi)^* \mathcal{U}_{\mathbb{P}V}$, regarded as an $\mathcal{O}_C$-submodule of $H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C$, is contained in the kernel of $H^0(X; \mathcal{O}(1)) \otimes \mathcal{O}_C \to f^* \mathcal{O}(1)$.

**Proof.** By the definition of $\varphi_S$, $\ker \varphi_S$ lies inside the kernel of the map $H^0(\mathcal{O}(1)) \otimes \mathcal{O}_S \to \pi_* f^* \mathcal{O}(1)$. First, $\pi^*$ is left exact because $\pi$ is flat, so $\pi^* \ker \varphi_S = (\psi_S \circ \pi)^* \mathcal{U}_{\mathbb{P}V}$ can be regarded as an $\mathcal{O}_C$-submodule of $H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C$. Moreover, it is contained in the kernel of the map

$$H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C \to \pi^* \pi_* f^* \mathcal{O}(1).$$

Composing the last map with $\pi^* \pi_* f^* \mathcal{O}(1) \to f^* \mathcal{O}(1)$, we recover the obvious pullback homomorphism $H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C \to f^* \mathcal{O}(1)$. □
formal statement will be given later in proposition 4.5. We introduce \( W \), the blowup of \( X \) along \( E \). As it is always the case with blowups, the graded \( \mathcal{O}_X \)-algebras homomorphism 

\[
\text{Sym}^* V \otimes \mathcal{O}_X \longrightarrow \bigoplus_{k=0}^{\infty} \mathcal{I}_k^{E/X}
\]

induces the natural morphism of schemes \( \tau = \tau_X \times \tau_{P^V} : W \rightarrow X \times P^V \). Note that there is a natural map \( w : \text{proj}_{P^V}^* \mathcal{G}_{P^V} \rightarrow \text{proj}_X^* \mathcal{O}_X(1) \), where \( \text{proj}_X \) and \( \text{proj}_{P^V} \) denote the projections to the respective factors of \( X \times P^V \).

**Claim 4.4.** The scheme-theoretic vanishing locus of \( w \), regarded as a section of the vector bundle \( \text{Hom}(\text{proj}_{P^V}^* \mathcal{G}_{P^V}, \text{proj}_X^* \mathcal{O}_X(1)) = \mathcal{O}_X(1) \otimes \mathcal{G}_{P^V} \), is precisely \( W \).

**Proof.** Let \( \chi : X \rightarrow \mathbb{P}H^0(X, \mathcal{O}_X) \) be the embedding associated with \( \mathcal{O}_X(1) \) and \( P \) the blowup of \( \mathbb{P}H^0(X, \mathcal{O}_X) \) along the projectivization of the cokernel of 

\[
V = H^0(X, \mathcal{J}_{E/X}(1)) \rightarrow H^0(X, \mathcal{O}_X(1)).
\]

Again, \( P \) sits naturally inside \( \mathbb{P}H^0(X, \mathcal{O}_X(1)) \times \mathbb{P}V \). Replacing \( X \) with \( \mathbb{P}H^0(X, \mathcal{O}_X) \), there is an analogous way to define a natural section \( p \) of \( \mathcal{O}_X(1) \otimes \mathcal{G}_{P^V} \). The analogous statement, that the vanishing locus of \( p \) is \( P \), is clear. Since \( w \) is the pullback of \( p \) to \( X \times P^V \), the claim follows simply by pulling back via \( \chi \times \text{Id}_{P^V} \).

**Proposition 4.5.** There exists a lift \( \tilde{f} : C \rightarrow W \) of \( f \) along \( \tau_X \) such that \( \psi_S \circ \pi = \tau_{P^V} \circ \tilde{f} \), i.e. we require

\[
\begin{array}{ccc}
C & \xrightarrow{\tilde{f}} & W \\
\downarrow \psi_S & & \uparrow \tau_X \\
S & \xrightarrow{\psi_S} & P^V \\
\end{array}
\]

to be a commutative diagram.

**Proof.** If we show that the morphism \( f \times (\psi_S \circ \pi) : C \rightarrow X \times P^V \) factors through \( \tau \), we may take \( \tilde{f} \) to be the "quotient" morphism. The important point is that pullback by \( f \times (\psi_S \circ \pi) \) kills \( w \). Indeed, the pullback map \( (f \times (\psi_S \circ \pi))^* w : (\psi_S \circ \pi)^* \mathcal{G}_{P^V} \rightarrow f^* \mathcal{O}_X(1) \) is the restriction to \( (\psi_S \circ \pi)^* \mathcal{G}_{P^V} \) of the homomorphism \( H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_C \rightarrow f^* \mathcal{O}_X(1) \), which is zero, by Corollary 4.3. Then 4.4 shows that \( f \times (\psi_S \circ \pi) : C \rightarrow X \times P^V \) indeed factors through \( \tau \), completing the proof.

The crucial step is to understand the image of \( \Phi_S \). The idea is very simple: if some \( W_t \) contains some rational curve of some class \( \hat{\beta} \) (lift of \( \hat{\beta} \)) through all \( \xi_i \), then the rational curve will cut the copy of \( E \) inside \( W_t \) precisely in the divisor \( \xi_1 + \ldots + \xi_e \) by degree
considerations. The existence of divisor classes on $W_t$ restricting to a predetermined divisor class on $E$ imposes one condition on $t$.

Let $(C, \pi, f, p_1, p_2, \ldots, p_e) \in M_e(X, E; \beta)|_{\Delta^c}(S)$ such that the map $\pi$ is smooth, and $W_S := W \times_S \mathbb{P}V \to S$ is smooth. The square in the diagram of proposition 4.5 induces an $S$-morphism $\tilde{f} : C \to W_S$. Note that, since we’re over the complement of $\Delta$, any component of the source of any individual map is either contracted, or mapped birationally onto its image. Let $\mathcal{X}$ be the kernel of $\mathcal{O}_{W_S} \to \tilde{f}_*\mathcal{O}_C$. The fact that $C$ and $W_S$ are flat over $S$ easily implies that $\mathcal{X}$ is also flat over $S$. However, since the restriction of $\mathcal{X}$ to $W_s$ for closed points $s \in S$ is invertible by obvious geometric considerations, it follows by 5.2 (the reader is referred to the appendix) that $\mathcal{X}$ is itself invertible. The section of $\mathcal{H}om(\mathcal{X}, \mathcal{O}_{W_S})$ corresponding to the inclusion $\mathcal{X} \to \mathcal{O}_{W_S}$ restricts on $E \times S$ to the section of $\mathcal{H}om(I_{p(S)}, \mathcal{O}_{E \times S}) \cong \mathcal{O}_{E \times S}(p(S))$ corresponding to the inclusion $I_{p(S)} \to \mathcal{O}_{E \times S}$, so we have a commutative diagram

$$
\begin{array}{ccc}
S & \longrightarrow & \text{Pic}(W_S/S) \\
\downarrow & & \downarrow \\
E^e & \longrightarrow & \text{Pic}(E)
\end{array}
$$

in which the right vertical map is the pullback, while the lower horizontal map is simply the map sending a divisor on $E$ to its class.

By well-known properties, $H_2(W; \mathbb{Z}) \cong H_2(X; \mathbb{Z}) \oplus \mathbb{Z}$. Moreover, there exists a unique $(\tau_X)_*\ast$-lift, $\hat{\beta} \in H_2(W; \mathbb{Z})$ of $\beta$, such that $(\tau_{\mathbb{P}V})_*\hat{\beta} = 0$. It is not hard to see that the mapping defined in Proposition 4.5 $(C, \pi, f, p_1, p_2, \ldots, p_e) \mapsto (C, \pi, \tilde{f}, p_1, p_2, \ldots, p_e)$ induces a morphism of DM-stacks

$$
\Phi : \overline{M}_e(X, E; \beta)|_{\Delta^c} \to \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta}).
$$

Since the obvious map $j : \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta}) \to \overline{M}_e(X, E; \beta)$ is inverse to $\Phi$, the morphism $\Phi$ induces an isomorphism $\overline{M}_e(X, E; \beta)|_{\Delta^c} \to \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_{\Delta^c \times (\mathbb{P}V)^c}$. To conclude, there is a commutative diagram

$$
\begin{array}{ccc}
\overline{C}_e(X, E; \beta)|_{\Delta^c} & \xrightarrow{\Phi} & \overline{C}_e(W, E \times \mathbb{P}V; \hat{\beta}) \\
\downarrow & & \downarrow \\
\overline{M}_e(X, E; \beta)|_{\Delta^c} & \xrightarrow{\Phi} & \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta}) \\
\downarrow & & \downarrow \\
E^e & \xrightarrow{ev_{X,E}} & (E \times \mathbb{P}V)^c
\end{array}
$$

and $\Phi$ is an open embedding. Let $\xi \in E^c$ be a general $e$-tuple of closed points on $E$, so $\xi \in \Delta^e$. The diagram above shows that

$$
ev_{X}^{-1}(\xi) = ev_{X,E}^{-1}(\xi) \cong ev_{W,E \times \mathbb{P}V}^{-1}(\xi \times \mathbb{P}V).$$
Let $M_\xi$ denote the last space and $C_\xi$ its universal family, constructed in the obvious way by restricting the universal family above.

Let $U \subset \mathbb{P}V$ be an open subset over which $W \to \mathbb{P}V$ is smooth. By a slight abuse of notation we write $\overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_U$ where we actually mean restriction to $\Psi^{-1}(U) \cap (\Delta^e \times \mathbb{P}V^e)$. Denote by $\overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_U$ the open locus where the source curve is smooth. By the discussion above, there is a map $\omega : \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_U \to \text{Pic}(W_U/U)$, where $W_U = \tau^{-1}_U(U)$. Let $\rho$ be the restriction map $\rho : \text{Pic}(W_U/U) \to U \times \text{Pic} E$. Similar to the construction in section 3, let $U'$ parametrize pairs $(t, \varphi_t)$, where $t \in U$ is a closed point and $\varphi_t$ is a $\varpi$-isomorphism of $(H^2(W_t, \mathbb{Z}), \cup, c_1(T_{W_t}^\varpi))$ with $(H_r, \langle \cdot, \cdot \rangle, \omega)$. Then $U' \to U$ is an étale morphism. The data is summarized in the following diagram.

$$
\begin{array}{ccc}
U' \times_U \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_U & \longrightarrow & \text{Pic}(W_{U'/U'}) \\
\downarrow & & \downarrow \\
\overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_U & \longrightarrow & \text{Pic}(W_U/U) \longrightarrow \text{Pic}(E) \times U \\
\omega & & \rho \\
\downarrow & & \downarrow \\
E^e & \longrightarrow & \text{Pic}(E)
\end{array}
$$

Finally, we remark that the analogue of lemma 3.1, namely the statement that the section $U' \to \text{Pic}(W_{U'/U'})$ corresponding to an element $\beta \in H_r$ which is not a multiple of $\omega \in H_r$ composed with the map $\text{Pic}(W_{U'/U'}) \to \text{Pic}(E)$ is nonconstant, follows from 3.1 itself simply by restricting to a line $\mathbb{P}^1 \subset \mathbb{P}V$.

**Proof of 1.1.** We use the current notation. Let $(C, \tilde{f}, p_1, ..., p_e) \in M_\xi(C)$, which is the $\Phi$-image of some stable map $(C, f, p_1, ..., p_e) \in \text{ev}_{X,E}^{-1}(\xi) \subset \overline{M}_e(X, E; \beta) \subset \overline{M}_e(X; \beta)$. Let

$$
N_{f,X} = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus ... \oplus \mathcal{O}(a_{n-1})
$$

with $a_1 \geq a_2 \geq ... \geq a_{n-1}$. Then, as we’ve seen, $f$ maps to the surface $W_t \hookrightarrow X$. We denote the tangent space to $M_\xi$ at $(C, \tilde{f}, p_1, ..., p_e)$ by $\text{Def}_{W_t,\xi}(\tilde{f})$. Similarly, $\text{Def}_{W_t,\xi}^l(\tilde{f})$ denotes the space of first order deformations of $\tilde{f}$ which remain inside $W_t$ and $\text{Def}_{X,\xi}^l(f)$ denotes the space of first order deformations of $f$, subject to the condition that the marked points map to $\xi$. By the observation preceding the proof and the assumption that $W_t$ is general, we have an exact sequence

$$
0 \longrightarrow \text{Def}_{W_t,\xi}^l(\tilde{f}) \longrightarrow \text{Def}_{W_t,\xi}(\tilde{f}) \longrightarrow T_t\mathbb{P}V \longrightarrow \mathbb{C} \longrightarrow 0.
$$

It follows that $\dim \text{Def}_{W_t,\xi}^l(\tilde{f}) = \dim \text{Def}_{W_t,\xi}(\tilde{f}) + \dim \mathbb{P}V - 1$. We claim that $\text{Def}_{W_t,\xi}^l(\tilde{f}) = 0$. Recall that $\tilde{f}$ is unramified, so the normal bundle $N_{f,W_t}$ of $f$ relative to $W_t$ is locally free of rank one. From the standard sequence

$$
0 \longrightarrow T_C \longrightarrow \tilde{f}^*T_{W_t} \longrightarrow N_{f,W_t} \longrightarrow 0,
$$

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we infer that $c_1(N_{f,W}) = \tilde{f}^*c_1(W) - c_1(T_C)$, so $\deg N_{f,W} = e - 2$. Therefore,

$$\text{Def}_{W,\xi}(\tilde{f}) = H^0(N_{f,W} \otimes O_C(-p_1 - \ldots - p_e)) = 0,$$

as desired. Therefore, $\dim \text{Def}_{W,\xi}(\tilde{f}) = n - 3$, so $\text{Def}_{X,\xi}(f) = n - 3$ as well, since $\Phi$ is an open immersion. Using once more the interplay between first order deformations and normal bundles, we conclude that $h^0(N_{f,X}(-p_1 - \ldots - p_e)) = n - 3$. This says

$$\dim \bigoplus_{i=1}^{n-1} H^0(\Theta'(a_i - e)) = n - 3,$$

and combining with the constrain $a_1 + a_2 + \ldots + a_{n-1} = \deg c_1(N_{f,X}) = (n-1)e - 2$, we can infer that $a_{n-1} \geq e - 1$.

Let $e_{e+1} \in X \setminus E = W \setminus E \times P^1$, say $e_{e+1} \in W_t$. We consider stable maps with one additional marked point. Let $e_1, \ldots, e_e$ as before, $\xi' = (\xi_1, \ldots, \xi_e, e_{e+1})$ and $ev_{X}^X(\xi')$ the space of stable maps $f \to X$ such that $f(p_i) = \xi_i$. We will only sketch this part of the argument, being similar to the analysis above. The condition $f(p_{e+1}) = e_{e+1}$ forces $f$ to map to $W_t$, so $\text{Def}_{X,\xi'}(f)$ is now isomorphic to $\text{Def}_{W_t,\xi'}(\tilde{f})$, but, as above, $\text{Def}_{W_t,\xi'}(\tilde{f}) = 0$, so $\text{Def}_{X,\xi'}(f) = 0$. In terms of normal bundles, $h^0(N_{f,X}(-p_1 - \ldots - p_{e+1})) = 0$, implying that $a_1 \leq e$. Combining with the previous inequality, we conclude that $a_1 = a_2 = e$ and $a_3 = \ldots = a_{n-1} = e - 1$, as desired. □

**Proof of 1.2.** We are in the situation analyzed in section 3, $X = dP[3,\theta]$ a Fano threefold of index 2 and degree $d \in \{2,3,4,5\}$ and $W \to P^1$ a Lefschetz pencil. Let $L := \mathcal{O}_E(\xi_1 + \ldots + \xi_e) \in \text{Pic}^e(E)$. As in section 3, let $S(\xi) = S(L)$ be the set of pairs $(t,L_t)$ such that $t \in P^0 = U$ and $L_t$ restricts to $L_E$ on $E$. To each such $L_t$, we can associate $\tilde{\xi}_t \in H^2(W_t;\mathbb{Z})$, the Poincaré dual to $c_1(L_t)$. The condition in the second part of lemma 2.1 is satisfied for a general choice of $\xi$. Indeed, for $e \geq 2$, we may move the $e$ points around preserving $L_E$ (and therefore $S(\xi)$) to avoid the finitely many prohibited situations. For $e = 1$, the condition is satisfied vacuously.

Of course, $S(\xi)$ splits as a disjoint union of $S(L_E,\gamma)$ consisting of those pairs $(t,L_t)$ such that $\tilde{\xi}_t$ corresponds to $\gamma$ under a suitable $\mathcal{O}$-isomorphism, where $\gamma$ varies over a set of representatives of $H^2_t/G$. From the discussion at the end of section 3, $S(L_E,\gamma)$ is reduced and we have

$$M_{\xi} \cong \bigsqcup_{(t,L_t) \in S(\xi)} \overline{\mathcal{M}_{e-1}(W_t,\tilde{\xi}_t)} \times_{W_t} \{\xi\} \cong \bigsqcup_{(t,L_t) \in S(\xi)} \overline{\mathcal{M}_{e-1}(W_t,\tilde{\xi}_t)} \times_{W_t} \{e_1,\ldots,e_{e-1}\}.$$

Taking the degrees of these 0-cycles, we obtain

$$\deg [M_{\xi}] = \sum_{(t,L_t) \in S(\xi)} \deg \left[ \overline{\mathcal{M}_{e-1}(W_t,\tilde{\xi}_t)} \times_{W_t} \{e_1,\ldots,e_{e-1}\} \right]$$

$$\Rightarrow \langle [pt] \rangle_{0,\tilde{\xi}}^{X} = \sum_{(t,L_t) \in S(\xi)} \langle [pt] \rangle_{e-1}^{W_t} = \sum_{\gamma \in H^2_t/G} \sum_{(t,L_t) \in S(L_E,\gamma)} \langle [pt] \rangle_{e-1}^{W_t}.\]$$

However, all $W_t$ are deformation equivalent to any degree $d$ del Pezzo surface $\Sigma$, so $\langle [pt] \rangle_{e-1}^{W_t} = \langle [pt] \rangle_{e-1}^{\Sigma}$ for $(t,L_t) \in S(L_E,\gamma)$). By (3.5), we conclude that

$$\langle [pt] \rangle_{0,\tilde{\xi}}^{X} = \sum_{\gamma \in H^2_t/G} \#S(L_E,\gamma) \langle [pt] \rangle_{e-1}^{\Sigma} = \kappa \sum_{\gamma \in H^2_t} \Delta(\omega,\gamma) \langle [pt] \rangle_{0,\tilde{\xi}}^{\Sigma}.$$
completing the proof up to the computation of the constant $\kappa$. To compute $\kappa$, we simply take $\beta = [\text{line}] \in H_2(X; \mathbb{Z})$, so

$$\kappa = \frac{\langle [pt] \rangle^X_{0, [\text{line}]} }{\Delta(\omega, [\text{line}]) \cdot \#(\text{lines on } \Sigma)}.$$ 

Using the ad hoc computation of $\langle [pt] \rangle^X_{0, [\text{line}]}$ in section 2.2, $\Delta(\omega, [\text{line}]) = d + 1$ and the well known counts for lines on del Pezzo surfaces, we compute $\kappa$ obtaining $\kappa^{-1} = d(9 - d)$ in all cases. □

**Remark 4.6.** It can be checked that for $d \in \{3, 4\}$ and $e \leq 3$, our numbers agree with those computed in [Be95]. It is worth noting that the index of the algebraic variety also plays a central role in the arguments in loc. cit.

5. Appendix: Some Commutative Algebra

Here we state and prove a lemma in commutative algebra which was invoked in the previous section.

**Fact 5.1.** Let $f : (A, m) \to (B, n)$ be a flat local homomorphism of local Noetherian rings. If $I \subseteq m^e$ is an ideal of $B$ such that $B/I$ is flat as an $A$-module, then $I = (0)$.

**Proof.** First, tensoring the short exact sequence $0 \to m \to A \to A/m \to 0$ with $B$ we obtain an exact sequence of $A$-modules

$$0 \to m \otimes_A B \to A \otimes_A B \to A/m \otimes_A B \to 0.$$ 

The middle term is $B$ and the next term is $B/mB = B/m^e$, so flatness of $B$ over $A$ implies that the $A$-module homomorphism $B \otimes_A m \to m^e$ is actually an isomorphism. Second, tensoring the short exact sequence $0 \to m \to A \to A/m \to 0$ with $B/I$ we obtain an exact sequence of $A$-modules

$$0 \to m \otimes_A B/I \to A \otimes_A B/I \to A/m \otimes_A B/I \to 0.$$ 

First, $A/m \otimes_A B/I$ is $(B/I)/m(B/I) = (B/I)/m^e(B/I)$. The kernel of the surjective composition $B \to B/I \to (B/I)/m^e(B/I)$ is just $m^e$ hence $A/m \otimes_A B/I \cong B/m^e$. We therefore have an isomorphism $m \otimes_A B/I \cong m^e/I$.

However, we have a well defined surjective $A$-module homomorphism $m \otimes_A B/I \to m^e/m^eI$, which fits in the following commutative diagram of $A$-modules.

$$\begin{array}{ccc}
\mathbb{m}^e & \longrightarrow & \mathbb{m} \\
\downarrow & & \downarrow \\
m^e/m^eI & \longrightarrow & \mathbb{m}/I \\
\end{array}$$
The vertical lateral maps are simply quotients. The lower left horizontal map is surjective and the all other horizontal maps are isomorphisms. It follows that $I = m^e I$. Inductively, $I = (m^e)^k I$ for all positive integers $k$, hence

$$I \subseteq \bigcap_{k \geq 0} (m^e)^k = (0),$$

by Krull’s intersection theorem, so $I = (0)$, as desired. □

**Corollary 5.2.** As above, let $f: (A, m) \to (B, n)$ be a flat local homomorphism of local Noetherian rings. Let $M$ be a $B$-module such that:

- $M$ is flat as an $A$-module; and
- $B/m^e \otimes_B M$ is a free rank one module over $B/m^e$.

Then $M$ is a free rank one module over $B$.

**Proof.** This follows immediately from 5.1 and Nakayama’s lemma. □

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