The planar double box integral for top pair production with a closed top loop to all orders in the dimensional regularisation parameter

Luise Adams, Ekta Chaubey and Stefan Weinzierl
PRISMA Cluster of Excellence, Institut für Physik, Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Germany

(Dated: April 30, 2018)

We compute systematically for the planar double box Feynman integral relevant to top pair production with a closed top loop the Laurent expansion in the dimensional regularisation parameter \( \varepsilon \). This is done by transforming the system of differential equations for this integral and all its sub-topologies to a form linear in \( \varepsilon \), where the \( \varepsilon^0 \)-part is strictly lower triangular. This system is easily solved order by order in the dimensional regularisation parameter \( \varepsilon \). This is an example of an elliptic multi-scale integral involving several elliptic sub-topologies. Our methods are applicable to similar problems.

INTRODUCTION

The physics of heavy elementary particles like the Higgs boson, the top quark or the W- and Z-bosons plays an important role at the LHC and future colliders. Precision particle physics at these colliders relies crucially on our abilities to perform higher-order perturbative calculations and in particular on our abilities to compute the relevant Feynman integrals. The method of differential equations\(^{[1]}\)\(^{[2]}\) has been used successfully for many Feynman integrals which evaluate to multiple polylogarithms\(^{[10]}\)\(^{[11]}\). For a large number of scattering processes with massless particles this is sufficient. However, as soon as massive particles enter the game, it is known that starting at two loops multiple polylogarithms will not be sufficient to express the Feynman integrals. The simplest example of a Feynman integral not expressible in terms of multiple polylogarithms is the two-loop sunrise integral with equal non-zero internal masses\(^{[13]}\)\(^{[34]}\). This integral is related to an elliptic curve and can be expressed to all orders in the dimensional regularisation parameter \( \varepsilon \) in iterated integrals of modular forms of \( \Gamma_1(6) \). Integrals, which do not evaluate to multiple polylogarithms are now an active field of studies in particle physics\(^{[35]}\)\(^{[52]}\) and string theory\(^{[53]}\)\(^{[58]}\).

In this letter we report on a more involved computation. We consider the planar double box integral relevant to top-pair production with a closed top loop. This integral enters the next-to-next-to-leading order (NNLO) contribution for the process \( pp \rightarrow t\bar{t} \). Up to now, it is not known analytically. The existing NNLO calculation for this process uses numerical approximations for this integral\(^{[54]}\)\(^{[60]}\). Our inability to compute this integral analytically has been a show-stopper for further progress on the analytical side. In this letter we show how to compute analytically this integral. Our methods are applicable to similar problems.

The planar double box integral depends on two scales (for example \( s/m^2 \) and \( t/m^2 \), where \( s \) and \( t \) are the usual Mandelstam variables and \( m \) the mass of the heavy particle). It involves the sunrise graph as a sub-topology. Therefore, we do not expect this integral to evaluate to multiple polylogarithms. Phrased differently, we expect to see elliptic generalisations of multiple polylogarithms.

An obvious question is: Which elliptic curve? To some surprise, there is not a single elliptic curve associated to this integral, but three different ones. We show in this letter how to extract the elliptic curves from the maximal cuts of the (sub-) topologies. From these elliptic curves we obtain their periods.

In the next step we bring the system of differential equations to a form linear in \( \varepsilon \), where the \( \varepsilon^0 \)-part is strictly lower triangular. We introduce kinematic variables \( x \) and \( y \), which rationalise the square roots in the polylogarithmic case (i.e. for \( t = m^2 \)). The transformation of the basis of master integrals is not rational in \( x \) and \( y \), however we find a transformation which is rational in \( x \), \( y \), the periods of the three elliptic curves and their \( y \)-derivatives. Note that a system of differential equations linear in \( \varepsilon \), where the \( \varepsilon^0 \)-part is strictly lower triangular, can easily transformed to an \( \varepsilon \)-form (i.e. without any \( \varepsilon^0 \)-part) by introducing primitives for the terms occurring in the \( \varepsilon^0 \)-part. Both systems are equivalent and both are easily solved order by order in the dimensional regularisation parameter \( \varepsilon \). For the case at hand the required primitives are usually transcendental functions. We prefer to work with a system linear in \( \varepsilon \), where in the transformation matrix only the periods and their derivatives occur as transcendental functions.

There are two interesting cases, where the solution for the Feynman integrals simplify: for \( t = m^2 \) the solution degenerates to multiple polylogarithms, for \( s = \infty \) the solution degenerates to iterated integrals of modular forms for \( \Gamma_1(6) \).

THE INTEGRAL

We consider the planar double box integral shown in fig. 1. This integral is relevant to the NNLO corrections for \( t\bar{t} \)-production at the LHC. In fig. 1 the solid lines
correspond to propagators with a mass $m$, while dashed lines correspond to massless propagators. All external momenta are out-going and on-shell: $p_1^2 = p_2^2 = 0$ and $p_3^2 = p_4^2 = m^2$.

We are interested in the dimensional regulated integral

$$\omega_p = \int_0^{\infty} \frac{d\lambda_1}{\lambda_1} \int_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2} \cdots \int_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k} f(\lambda_1, \lambda_2, \ldots, \lambda_k) \cdot (6)$$

Multiple polylogarithms are iterated integrals, where all differential one-forms are of the form

$$\omega_j = \frac{d\lambda}{\lambda - c_j} \cdot (7)$$

If $f(\tau)$ is a modular form, we simply write with a slight abuse of notation $f$ instead of $2\pi i f d\tau$ in the arguments of iterated integrals.

**The Kinematic Variables for the Multiple Polylogarithms**

The Feynman integral is a function of two kinematic ratios, say $s/m^2$ and $t/m^2$. A significant fraction of the sub-topologies depends only on $s/m^2$, but not on $t/m^2$. These integrals are expressible in terms of multiple polylogarithms and their system of differential equations can be transformed to an $\varepsilon$-form. This introduces square roots, which are absorbed by a change of kinematic variables. The square root $\sqrt{-s(4m^2 - s)}$ is typical for massive Feynman integrals, however there are also sub-topologies, which lead to the square root $\sqrt{-s(-4m^2 - s)}$ (note the sign in front of $4m^2$). An example is shown in fig. 2. A transformation, which absorbs both square roots simultaneously is given by

$$\frac{s}{m^2} = \frac{(1 + x^2)^2}{x(1 - x^2)} \cdot \frac{t}{m^2} = y \cdot (8)$$

This defines the variables $x$ and $y$. The variable $y$ is not needed for integrals depending only on $s/m^2$. For the integrals depending only on $s/m^2$ we introduce five differential one-forms

$$\omega_0 = \frac{ds}{s} = \frac{2dx}{x - i} + \frac{2dx}{x + i} - \frac{dx}{x - 1} - \frac{dx}{x + 1} - \frac{dx}{x}.$$
and define the modulus and the complementary modulus

\[
\omega_1 = \frac{ds}{s - 4m^2} = \frac{2dx}{x - (1 + \sqrt{2})} + \frac{2dx}{x - (1 - \sqrt{2})},
\]

\[
\omega_{-1} = \frac{ds}{s + 4m^2} = \frac{2dx}{x - (-1 + \sqrt{2})} + \frac{2dx}{x - (-1 - \sqrt{2})},
\]

\[
\omega_{0,4} = \frac{ds}{\sqrt{-s (4m^2 - s)}} = \frac{dx}{x - 1} - \frac{dx}{x + 1} + \frac{dx}{x},
\]

\[
\omega_{-4,0} = \frac{ds}{\sqrt{s (-4m^2 - s)}} = -\frac{dx}{x - 1} + \frac{dx}{x + 1} + \frac{dx}{x},
\]

Then all sub-topologies, which depend only on \(s/m^2\), can be expressed as iterated integrals with letters given by these five differential one-forms. From eq. (9) it is clear that they are expressible in terms of multiple polylogarithms.

**ELLIPTIC CURVES**

Let us consider an elliptic curve defined by the quartic equation

\[
E : w^2 = (z - z_1)(z - z_2)(z - z_3)(z - z_4). \tag{10}
\]

We set

\[
Z_1 = (z_2 - z_1)(z_4 - z_3),
\]

\[
Z_2 = (z_3 - z_2)(z_4 - z_1),
\]

\[
Z_3 = (z_3 - z_1)(z_4 - z_2)
\]

and define the modulus and the complementary modulus

\[
k^2 = \frac{Z_1}{Z_3}, \quad \bar{k}^2 = \frac{Z_2}{Z_3} \tag{12}
\]

Our standard choice for the periods is

\[
\psi_1 = \frac{4K(k)}{Z_3^2}, \quad \psi_2 = \frac{4iK(\bar{k})}{Z_3^2}, \tag{13}
\]

where \(K(x)\) denotes the complete elliptic integral of the first kind. For the double box integral we have to consider three elliptic curves \(E^{(a)}\), \(E^{(b)}\) and \(E^{(c)}\), which occur for the first time in the three Feynman graphs shown in fig. 3. The equations of the elliptic curves are extracted from the maximal cuts of these Feynman integrals \[62-69\], specifically from the maximal cuts of

\[
I_{1001001}(2 - 2\varepsilon), \quad I_{1112001}(4 - 2\varepsilon), \quad I_{2001111}(4 - 2\varepsilon).
\]

We find for all three curves

\[
z_{1}^{(a,b,c)} = \frac{t}{m^2} - 4, \quad z_{4}^{(a,b,c)} = \frac{t}{m^2}, \tag{14}
\]

They differ in the values for the roots \(z_2\) and \(z_3\). We have

\[
z_{2,3}^{(a)} = -1 \mp 2\sqrt{\frac{t}{m}}, \tag{15}
\]

\[
z_{2,3}^{(b)} = -1 \mp 2\sqrt{\frac{t}{m^2} + \frac{(m^2 - t)^2}{m^2 s}},
\]

\[
z_{2,3}^{(c)} = -\frac{(s + 4t)}{(s - 4m^2)} \mp \frac{2}{s - 4m^2} \sqrt{\frac{s}{m^2} (st + (m^2 - t)^2)}.
\]

It is easily checked by computing the \(j\)-invariants that the three curves are not isomorphic. However, the curves \(E^{(b)}\) and \(E^{(c)}\) degenerate to curve \(E^{(a)}\) in the limit \(s \to \infty\). Associated to the curve \(E^{(a)}\) are modular forms of \(\Gamma_1(6)\). We set

\[
g_{n,r} = -\frac{1}{2} y (y - 1) (y - 9) \left(\frac{\psi_1^{(a)}}{\pi}\right)^n,
\]

\[
p_{n,s} = -\frac{1}{2} y (y - 1)^{1+s} (y - 9) \left(\frac{\psi_2^{(a)}}{\pi}\right)^n, \tag{16}
\]

Relevant to the problem is the set

\[
\{g_{2,0}, g_{2,1}, g_{2,9}, g_{3,1}, g_{3,3}, g_{3,0}, g_{4,0}, g_{4,1}, g_{4,9}, p_{4,0}, p_{4,4}\}. \tag{17}
\]

These are modular forms of \(\Gamma_1(6)\) in the variable \(\tau_6 = \psi_2^{(a)}/(6\psi_1^{(a)})\), which we may substitute for the variable \(y\).
The main result of this letter is that there exists a trans-
plicated topologies. For a given set of master integrals
However, the reductions disagree for the three most com-
tions into account, all programs give 45 master integrals.
In general, this system is not yet linear in \( \varepsilon \)
\( x \), \( y \), \( \psi_1^{(a)} \), \( \psi_1^{(b)} \), \( \psi_1^{(c)} \), \( \partial_y \psi_1^{(a)} \), \( \partial_y \psi_1^{(b)} \), \( \partial_y \psi_1^{(c)} \). (23)
We constructed this matrix by analysing the Picard-
Fuchs operators in the diagonal blocks \([74, 75]\) and by us-
fail the integrability check. Still, all three programs cor-
rectly implement the Laporta algorithm \([73]\). However,
At first sight, the results of two of three programs above
agree and the integrability condition is satisfied. In addition, we verified numerically
the first few terms in the \( \varepsilon \)-expansion of this relation.
In this letter we are interested in the integral \( \int_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_0 \nu_7} \)
and \( A^{(0)} \) is strictly lower triangular (i.e. \( A_{ij}^{(0)} = 0 \) for \( j \geq i \)). The system of differential equations is linear in \( \varepsilon \)
easily solved order by order in \( \varepsilon \) in terms of iterated integrals. The transformation matrix is rational in
\( \varepsilon , x , y , \psi_1^{(a)} , \psi_1^{(b)} , \psi_1^{(c)} , \partial_y \psi_1^{(a)} , \partial_y \psi_1^{(b)} , \partial_y \psi_1^{(c)} \).

\[ J_{24} = \varepsilon^3 \frac{(1 + x^2)^2}{x (1 - x^2)} \frac{\pi}{\psi_1^{(a)}} I_{1112001}, \]
\[ J_{25} = \varepsilon^3 (1 - 2 \varepsilon) \frac{(1 + x^2)^2}{x (1 - x^2)} I_{1111001} + R_{25,24} \frac{\psi_1^{(b)}}{\pi} J_{24}, \]
\[ J_{26} = \frac{6 (\psi_1^{(b)})^2}{\varepsilon 2 \pi i W_y^{(b)} dy} J_{24} + R_{26,24} \left( \frac{\psi_1^{(b)}}{\pi} \right)^2 J_{24} \]
\[ - \frac{\varepsilon^2}{24} (y^2 - 30 y - 27) \frac{\psi_1^{(a)}}{\pi} D^{-1} I_{1001001}, \]
where \( R_{25,24} \) and \( R_{26,24} \) are rational functions in \( (x, y) \), \( D^{-1} \)
denotes the dimension shift operator \( D \to D - 2 \) and \( W_y^{(b)} \) the Wronskian
\[ W_y^{(b)} = \psi_1^{(a)} \partial_y \psi_2^{(b)} - \psi_2^{(a)} \partial_y \psi_1^{(b)}. \]

As in the sunrise sector \([40]\), one integral is divided by a
period \( J_{24} \), while a second integral is given as a derivative
plus additional terms \( J_{26} \). This pattern applies to all
elliptic sectors.
The matrix \( A^{(0)} \) in eq. (22) vanishes for \( x = 0 \) or \( y = 1 \).
The occurrence of \( \varepsilon \)-terms in the differential equations is expected from the study of the sunrise integral with unequal masses \([29, 33]\). For \( y = 1 \) the entries of \( A^{(1)} \) reduce
to the differential one-forms of eq. (3), for \( x = 0 \) they reduce to the modular forms of eq. (17).

CONCLUSIONS

In this letter we analysed the planar double box in-
tegral relevant to top pair production with a closed top
loop. This integral depends on two scales and involves
several elliptic sub-sectors. This integral has not been
known analytically and impedes further progress on the
analytic computation of higher-loop Feynman integrals with massive particles. In this letter we reported that we may transform the system of differential equations to a form linear in $\epsilon$, where the $\epsilon^0$-term is strictly lower-triangular. With such a linear form the solution in terms of iterated integrals is immediate. Our techniques open the door for more complicated Feynman integrals.

**Acknowledgements**

L.A. and E.C. are grateful for financial support from the research training group GRK 1581. S.W. would like to thank the Hausdorff Research Institute for Mathematics for hospitality, where part of this work was carried out.

[1] A. V. Kotikov, Phys. Lett. **B254**, 158 (1991).
[2] A. V. Kotikov, Phys. Lett. **B267**, 123 (1991).
[3] E. Remiddi, Nuovo Cim. **A110**, 1435 (1997), hep-th/9711188.
[4] T. Gehrmann and E. Remiddi, Nucl. Phys. **B580**, 485 (2000), hep-ph/9912329.
[5] M. Argeri and P. Mastrolia, Int. J. Mod. Phys. **A22**, 4375 (2007), arXiv:0707.4037.
[6] S. Bauberger, M. Böhm, G. Weiglein, F. A. Berends, and R. Scharf, Z. Phys. **C63**, 227 (1994).
[7] S. Bauberger, M. Böhm, G. Weiglein, F. A. Berends, and M. Buza, Nucl. Phys. **B434**, 383 (1995), arXiv:hep-ph/9409388.
[8] S. Bauberger and M. Böhm, Nucl. Phys. **B445**, 25 (1995), arXiv:hep-ph/9501201.
[9] M. Caffo, H. Czyz, S. Laporta, and E. Remiddi, Nuovo Cim. **A111**, 365 (1998), arXiv:hep-th/9805118.
[10] S. Laporta and E. Remiddi, Nucl. Phys. **B704**, 349 (2005), hep-ph/0406160.
[11] A. B. Goncharov, Math. Res. Lett. **1**, A48 (2004), math.AG/0312568.
[12] L. Adams, E. Chaubey, and S. Weinzierl, Phys. Rev. **D88**, 032304 (2013), arXiv:1212.4389.
[13] J. Vollinga and S. Weinzierl, J. Math. Phys. **54**, 052303 (2013), arXiv:1302.7004.
[14] S. Bloch and P. Vanhove, J. Numb. Theor. **148**, 177 (2015), arXiv:1309.5865.
[15] L. Adams, C. Bogner, and S. Weinzierl, J. Phys. **55**, 102301 (2014), arXiv:1405.5640.
[16] L. Adams, C. Bogner, and S. Weinzierl, J. Math. Phys. **57**, 072303 (2016), arXiv:1504.03255.
[17] L. Adams, C. Bogner, and S. Weinzierl, J. Math. Phys. **57**, 032304 (2016), arXiv:1512.05630.
[18] E. Remiddi and L. Tancredi, Nucl. Phys. **B880**, 343 (2014), arXiv:1311.3342.
[19] S. Bloch, M. Kerr, and P. Vanhove, Adv. Theor. Math. Phys. **21**, 1373 (2017), arXiv:1601.08181.
[20] S. Groote, J. R. Körner, and A. Pivovarov, Nucl. Phys. **B922**, 528 (2017), arXiv:1705.08952.
[21] L. Adams and S. Weinzierl, Phys. Lett. **B781**, 270 (2018), arXiv:1802.05020.
[22] M. Sgaard and Y. Zhang, Phys. Rev. **D91**, 032304 (2015), arXiv:1412.5577.
[23] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, (2017), arXiv:1802.05020.
[24] J. Broedel, C. Duhr, B. Penante, and M. Wilhelm, Phys. Rev. Lett. **121**, 052302 (2018), arXiv:1712.07089.
[25] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, (2018), arXiv:1803.10256.
[26] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, (2018), arXiv:1803.10256.
[55] J. Broedel, N. Matthes, G. Richter, and O. Schlotterer, (2017), arXiv:1704.03449.
[56] E. D’Hoker, M. B. Green, G. Grdogan, and P. Vanhove, Commun. Num. Theor. Phys. 11, 165 (2017), arXiv:1512.06779.
[57] J. Broedel, O. Schlotterer, and F. Zerbini, (2018), arXiv:1803.00527.
[58] K.-T. Chen, Bull. Amer. Math. Soc. 83, 831 (1977).
[59] P. Baikov, Nucl. Instrum. Meth. A389, 347 (1997), arXiv:hep-ph/9611449.
[60] R. N. Lee, Nucl. Phys. B830, 474 (2010), arXiv:0911.0252.
[61] D. A. Kosower and K. J. Larsen, Phys. Rev. D85, 045017 (2012), arXiv:1108.1180.
[62] S. Caron-Huot and K. J. Larsen, JHEP 1210, 026 (2012), arXiv:1205.0801.
[63] A. Primo and L. Tancredi, Nucl. Phys. B916, 94 (2017), arXiv:1610.08397.
[64] H. Frellesvig and C. G. Papadopoulos, JHEP 04, 083 (2017), arXiv:1701.07356.
[65] J. Bosma, M. Sogaard, and Y. Zhang, JHEP 08, 051 (2017), arXiv:1704.04255.
[66] M. Harley, F. Moriello, and R. M. Schabinger, JHEP 06, 049 (2017), arXiv:1705.03478.
[67] A. von Manteuffel and C. Studerus, (2012), arXiv:1201.4330.
[68] P. Maierhoefer, J. Usovitsch, and P. Uwer, (2017), arXiv:1705.05610.
[69] A. V. Smirnov, Comput. Phys. Commun. 189, 182 (2015), arXiv:1408.2372.
[70] S. Laporta, Int. J. Mod. Phys. A15, 5087 (2000), hep-ph/0102033.
[71] C. Meyer, JHEP 04, 006 (2017), arXiv:1611.01087.
[72] C. Meyer, Comput. Phys. Commun. 222, 295 (2018), arXiv:1705.06252.
[73] L. Adams, E. Chaubey, and S. Weinzierl, (2018), in preparation.