Interpolating Discriminant Functions in High-Dimensional Gaussian Latent Mixtures

Xin Bing∗  Marten Wegkamp†

Abstract

This paper considers binary classification of high-dimensional features under a postulated model with a low-dimensional latent Gaussian mixture structure and non-vanishing noise. A generalized least squares estimator is used to estimate the direction of the optimal separating hyperplane. The estimated hyperplane is shown to interpolate on the training data. While the direction vector can be consistently estimated as could be expected from recent results in linear regression, a naive plug-in estimate fails to consistently estimate the intercept. A simple correction, that requires an independent hold-out sample, renders the procedure minimax optimal in many scenarios. The interpolation property of the latter procedure can be retained, but surprisingly depends on the way the labels are encoded.

Keywords: High-dimensional classification, latent factor model, generalized least squares, interpolation, benign overfitting, overparametrization, discriminant analysis, minimax optimal rate of convergence.

1 Introduction

We consider binary classification of a high-dimensional feature vector. That is, we are given a \( n \times p \) data-matrix \( X \) with \( n \) independent \( p \)-dimensional rows, with \( p \gg n \), and a response vector \( y \in \{0,1\}^n \) of corresponding labels, and the task is to predict the labels of new \( p \)-dimensional features. Complex models such as kernel support vector machines (SVM) and deep neural networks have been observed to have surprisingly good generalization performance despite overfitting the training data. Towards understanding such benign overfitting phenomenon, one popular example that has gained increasing attention is the estimator \( \hat{\theta} = X^+ y \) in the regression context. Here \( X^+ \) represents the Moore-Penrose inverse of \( X \). It can be shown that \( \hat{\theta} \) has the minimal \( \ell_2 \)-norm among all least squares estimators, that is,

\[
\|\hat{\theta}\|_2 = \min \left\{ \|\theta\|_2 : \|y - X\theta\|_2 = \min_{u \in \mathbb{R}^p} \|y - Xu\|_2 \right\}.
\]

∗Department of Statistical Sciences, University of Toronto. E-mail: xin.bing@utoronto.ca
†Department of Mathematics and Department of Statistics and Data Science, Cornell University. E-mail: marten.wegkamp@cornell.edu.
Since often $\mathbf{X}\hat{\theta} = \mathbf{y}$ holds, for instance, when $\mathbf{X}$ has full row-rank $n < p$, the estimator $\hat{\theta}$ is referred to as the minimum-norm interpolator. It serves as the prime example to illustrate the phenomenon that overfitting in linear regression can be benign, in that $\hat{\theta}$ can still lead to good prediction results. See, for instance, Belkin et al. (2018); Bartlett et al. (2020); Hastie et al. (2022); Bunea et al. (2022) and the references therein.

More recently, the minimum-norm interpolator further finds its importance in binary classification problems. Specifically, $\hat{\theta}$ is shown to coincide with the solution of the hard margin SVM under the over-parametrized Gaussian mixture models (Muthukumar et al., 2019; Wang and Thrampoulidis, 2021) and beyond (Hsu et al., 2021). In the over-parametrized logistic regression model, $\hat{\theta}$ is also closely connected to the solution of maximizing the log-likelihood, obtained by gradient descent with sufficiently small step size (Soudry et al., 2018; Cao et al., 2021). In the over-parametrized setting $p \gg n$, the hyperplane $\{ x \mid x^T\hat{\theta} = 0 \}$ separates the training data perfectly, leading to a classifier that has zero training error. There is a growing literature (Cao et al., 2021; Wang and Thrampoulidis, 2021; Chatterji and Long, 2021; Minsker et al., 2021) that shows that interpolating classifiers $\bar{g}(\mathbf{x})$ can also have vanishing misclassification error $\mathbb{P}\{ \bar{g}(\mathbf{X}) \neq Y \}$ in (sub-)Gaussian mixture models. We extend these works in this paper motivated by the following observations.

First, we notice that the separating hyperplane $\{ x \mid x^T\hat{\theta} = 0 \}$ considered in the above mentioned literature has no intercept. Under symmetric Gaussian mixture models, that is, $\mathbb{P}\{ Y = 1 \} = \mathbb{P}\{ Y = 0 \}$ and $\mathbf{X} \mid Y = k \sim \mathcal{N}_p((2k - 1)\mu, \Sigma)$ for $k \in \{0, 1\}$, the optimal (Bayes) rule is indeed based on a hyperplane through the origin (no intercept). However, this is no longer true in the asymmetric setting where the class probabilities differ, $\mathbb{P}\{ Y = 1 \} \neq \mathbb{P}\{ Y = 0 \}$, rendering the usage of $\bar{g}(\mathbf{x})$ questionable. Although Wang and Thrampoulidis (2021) shows that in the asymmetric setting $\mathbb{P}\{ \bar{g}(\mathbf{X}) \neq Y \}$ still tends to zero if the separation between $\mathbf{X} \mid Y = 0$ and $\mathbf{X} \mid Y = 1$ diverges, the rate of this convergence is unfortunately exponentially slower than the optimal rate, in part due to not using an intercept. This motivates us to propose an improved linear classifier based on $\hat{\theta}$ that includes an intercept, formally introduced in (1.1) below. Finding a meaningful intercept under the interpolation of $\hat{\theta}$ requires extra care, as standard approaches, such as the empirical risk minimization, can not be used when the hyperplane $\{ x \mid x^T\hat{\theta} = 0 \}$ separates the training data perfectly.

Second, the aforementioned works all focus on the misclassification risk $\mathbb{P}\{ \bar{g}(\mathbf{X}) \neq Y \}$. In particular, they show that $\mathbb{P}\{ \bar{g}(\mathbf{X}) \neq Y \}$ vanishes only if the separation between the two mixture distributions diverges. In general, the excess risk - the difference between the misclassification error and the Bayes error – is a more meaningful criterion, because the Bayes error $\inf_h \mathbb{P}\{ h(\mathbf{X}) \neq Y \}$ is the smallest possible misclassification error among all classifiers and generally does not vanish. For this reason, we focus on analyzing the excess risk of the proposed classifier, and our results are informative even if the separation between the two mixture distributions does not diverge (and therefore the Bayes risk does not vanish).

Summarizing, the existing results on interpolating classifiers are not satisfactory as they only consider stylized examples that do not address the more realistic scenarios when the mixture probabilities are asymmetric and the Bayes error does not vanish. In fact, we will argue that these interpolation methods without intercept in the literature actually fail in the asymmetric setting when the conditional distributions are not asymptotically
distinguishable - which is the statistically more challenging case. In this work, we instead analyze the interpolating classifier with a judiciously chosen intercept under a recently proposed latent, low-dimensional statistical model (Bing and Wegkamp, 2022), and our results reveal that its excess risk has minimax-optimal rate of convergence in the over-parametrized setting even when the separation between the two mixture distributions does not diverge. Together with the interpolation property of the proposed classifier, we thus provide a concrete instance of the interesting phenomenon that overfitting and minimax-optimal generalization performance can coexist in a more realistic statistical setting, against traditional statistical belief.

1.1 Our contributions

Concretely, in this paper we study the linear classifier

\[ \tilde{g}(x) = 1 \{ x^\top \hat{\theta} + \tilde{\beta}_0 > 0 \}, \quad \text{for any } x \in \mathbb{R}^p, \]  

(1.1)

based on the minimum-norm interpolator \( \hat{\theta} \) and some estimated intercept \( \tilde{\beta}_0 \in \mathbb{R} \). Following Bing and Wegkamp (2022), we assume that each feature consists of linear combinations of hidden low-dimensional Gaussian components, obscured by independent, possibly non-Gaussian, noise. The low-dimensional Gaussian component suggests to take a Linear Discriminant Analysis (LDA) approach, reducing the problem to find (i) the unknown low-dimensional space, (ii) its dimension \( K \), with \( K \ll n \), and (iii) the optimal hyperplane \( \{ z \in \mathbb{R}^K \mid z^\top \beta + \beta_0 = 0 \} \) in this latent space. The formal model and expression of the optimal hyperplane are described in Section 2. Existing literature on LDA in high-dimensional classification problems often imposes sparsity on the coefficients of the hyperplane (Tibshirani et al., 2002; Fan and Fan, 2008; Witten and Tibshirani, 2011; Shao et al., 2011; Cai and Liu, 2011; Mai et al., 2012; Cai and Zhang, 2019). In this work, we take a different route and do not assume that the high-dimensional features are Gaussian, nor rely on any sparsity assumption.

The recent work Bing and Wegkamp (2022) successfully utilized Principal Component Regression (PCR) to estimate \( z^\top \beta \) with its low-dimension estimated via the method developed in Bing and Wegkamp (2019). The classifier in (1.1) estimates \( z^\top \beta \) by \( x^\top \hat{\theta} \) via the minimum-norm interpolator \( \hat{\theta} \) instead. This estimator can be viewed as a limit case of PCR, with the number of retained principal components equal to \( p \) (not \( K \)), and as such an extension of Bing and Wegkamp (2022). The practical advantage of this extension is to avoid estimation of the latent dimension \( K \), meanwhile it sheds light on the robustness of the PCR-based classifiers of Bing and Wegkamp (2022) against misspecification of \( K \). From a theoretical perspective, it is surprising to see that \( x^\top \hat{\theta} \) adapts to the low-dimensional structure in \( z^\top \beta \), as explained below.

Section 4 provides theoretical guarantees for our proposed classifier. Theorem 8 in Section 4 states that \( x^\top \hat{\theta} \) consistently estimates \( z^\top \beta \) by adapting to the low-dimensional structure, and the rate of this convergence is often minimax-optimal in over-parametrized setting. Establishing Theorem 8 is our main technical challenge and its proof occupies a large part of the paper and is delegated to Section 6.1. Although this convergence is in line with current developments in regression that \( \hat{\theta} \) surprisingly succeeds, our analysis is more complicated as \( y \) is no longer linearly related with \( X \). In Section 4.1 we also
explain that similar arguments explored in Bing and Wegkamp (2022) cannot be used here. A key step in our proof is to recognize and characterize the implicit regularization of \( \hat{\theta} = X^*y \) in high-dimensional factor models. In our proof, we generalize existing analyses of factor models (Bai, 2003; Bai and Ng, 2008; Fan et al., 2013; Stock and Watson, 2002) by relaxing the stringent conditions that require all singular values of the latent components of \( X \) to grow at the same rate, proportional to the dimension \( p \).

Given the success in estimating \( z^T \beta \) via \( x^T \hat{\theta} \), a rather difficult \( p \)-dimensional problem, one would expect that consistent estimation of \( \beta_0 \) is much easier. Surprisingly, Proposition 4 in Section 3.2 shows that this is not the case. The natural plug-in estimate of \( \beta_0 \) based on \( \hat{\theta} \) and standard non-parametric estimates of the conditional means and label probabilities, always takes the value \(-1/2\), regardless of the true value of \( \beta_0 \). The same is true for an estimate based on empirical risk minimization. Simulations confirm that this problematic behavior leads to an inferior classifier. In Section 3.2 we offer a simple rectification and propose to estimate \( \beta_0 \) using an independent hold-out sample. In Proposition 11 of Section 4, we derive the consistency of our proposed estimator of \( \beta_0 \). Finally, in Theorem 12 we establish the rate of convergence of the excess-classification risk of our proposed classifier and discuss its minimax-optimal properties in Remark 3.

In view of the optimal guarantees of the proposed classifier in over-parametrized setting, we also find an interesting observation on its interpolation property. Specifically, Lemma 3 in Section 3.1 and our discussion in Section 3.3 reveal that its interpolation property crucially depends on the way we encode the labels. For instance, interpolation always happens if we encode \( Y \) as \( \{-1, 1\} \), whereas this is not always the case for the \( \{0, 1\} \) encoding scheme unless the majority class is encoded as 0. This suggests that interpolation is a rather arbitrary property.

The paper is organized as follows. In Section 2 we formally introduce the statistical model. We discuss the interpolation property of the proposed classifier and introduce the estimator of the intercept in Section 3. Section 4 is devoted to study the rate of convergence of the excess risk of the proposed classifier. Section 5 contains simulation studies. The main proofs are deferred to Section 6 while auxiliary lemmas are stated in Section 7.

1.2 Notation

We use the common notation \( \varphi(x) = \exp(-x^2/2)/\sqrt{2\pi} \) for the standard normal density, and denote by \( \Phi(x) = \int \varphi(t)1\{t \leq x\} \, dt \) its c.d.f.

For any positive integer \( d \), we write \([d] := \{1, \ldots, d\} \). For two numbers \( a \) and \( b \), we write \( a \wedge b = \min\{a, b\} \) and \( a \vee b = \max\{a, b\} \). For any two sequences \( a_n \) and \( b_n \), we write \( a_n \preceq b_n \) if there exists some constant \( C \) such that \( a_n \leq C b_n \). The notation \( a_n \asymp b_n \) stands for \( a_n \preceq b_n \) and \( b_n \preceq a_n \). We often write \( a_n \ll b_n \) if \( a_n = o(b_n) \) and \( a_n \gg b_n \) for \( b_n = o(a_n) \).

For any vector \( v \), we use \( \|v\|_q \) to denote its \( \ell_q \) norm for \( 0 \leq q \leq \infty \). We also write \( \|v\|^2_Q = v^T Q v \) for any commensurate, square matrix \( Q \). For any real-valued matrix \( M \in \mathbb{R}^{p \times q} \), we use \( M^+ \) to denote the Moore-Penrose inverse of \( M \), and \( \sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_{\min(r,q)}(M) \) to denote the singular values of \( M \) in non-increasing order. We define the operator norm \( \|M\|_{op} = \sigma_1(M) \). For a symmetric positive semi-definite matrix \( Q \in \mathbb{R}^{p \times p} \),
we use $\lambda_1(Q) \geq \lambda_2(Q) \geq \cdots \geq \lambda_p(Q)$ to denote the eigenvalues of $Q$ in non-increasing order. We use $I_d$ to denote the $d \times d$ identity matrix and use $1_d (0_d)$ to denote the vector with all ones (zeroes). For $d_1 \geq d_2$, we use $O_{d_1 \times d_2}$ to denote the set of all $d_1 \times d_2$ matrices with orthonormal columns.

2 Background

Suppose our training data consists of independent copies of the pair $(X,Y)$ with features $X \in \mathbb{R}^p$ according to

$$X = AZ + W$$

(2.1)

and labels $Y \in \{0, 1\}$. Here $A$ is a deterministic, unknown $p \times K$ loading matrix, $Z \in \mathbb{R}^K$ are unobserved, latent factors and $W$ is random noise. We assume throughout this study the following set of assumptions:

(i) $W$ is independent of both $Z$ and $Y$

(ii) $\mathbb{E}[Z] = 0_K$, $\mathbb{E}[W] = 0_p$

(iii) $A$ has rank $K$

(iv) $Z$ is a mixture of two Gaussians

$$Z \mid Y = k \sim N_k(\alpha_k, \Sigma_{Z|Y}), \quad \mathbb{P}(Y = k) = \pi_k, \quad k \in \{0, 1\}$$

(2.2)

with different means $\alpha_0 := \mathbb{E}[Z|Y = 0]$ and $\alpha_1 := \mathbb{E}[Z|Y = 1]$, but with the same, strictly positive definite covariance matrix

$$\Sigma_{Z|Y} := \text{Cov}(Z|Y = 0) = \text{Cov}(Z|Y = 1)$$

(2.3)

(v) $W = \Sigma_1^{1/2}V$, for some semi-positive definite matrix $\Sigma_W$, with $\mathbb{E}[V] = 0_p, \mathbb{E}[VV^\top] = I_p$ and $\mathbb{E}[\exp(u^\top V)] \leq \exp(\sigma^2/2)$ for all $\|u\|_2 = 1$, for some $0 < \sigma < \infty$.

(vi) $\pi_0 = \mathbb{P}\{Y = 0\}$ and $\pi_1 = \mathbb{P}\{Y = 1\}$ are fixed and strictly positive.

We emphasize that the distributions of $X$ given $Y$ are not necessarily Gaussian. This mathematical framework allows for a substantial dimension reduction in classification for $K \ll p$ as is evident from the inequality

$$R^*_x := \inf_g \mathbb{P}\{g(X) \neq Y\} \geq R^*_z := \inf_h \mathbb{P}\{h(Z) \neq Y\}$$

(2.4)

in terms of the Bayes’ misclassification errors, see Bing and Wegkamp (2022, Lemma 1). As in Bing and Wegkamp (2022), we first change the classification problem into a regression problem by drawing a connection of the Bayes rule to a quantity that can be identified via regressing $Y$ onto $Z$. We denote by $\Sigma_Z = \mathbb{E}[ZZ^\top]$ the unconditional covariance matrix of $Z$ and we define

$$\beta = \pi_0 \pi_1 \Sigma_Z^{-1}(\alpha_1 - \alpha_0),$$

$$\beta_0 = -\frac{1}{2}(\alpha_0 + \alpha_1)\top \beta + [1 - (\alpha_1 - \alpha_0)\top \beta] \pi_0 \pi_1 \log \frac{\pi_1}{\pi_0}. \quad (2.5)$$

$$\beta_0 = -\frac{1}{2}(\alpha_0 + \alpha_1)\top \beta + [1 - (\alpha_1 - \alpha_0)\top \beta] \pi_0 \pi_1 \log \frac{\pi_1}{\pi_0}. \quad (2.6)$$
Proposition 1. Let \( \beta, \beta_0 \) be defined in (2.5). The (Bayes) rule
\[
g^*_z(z) = 1\{z^\top \beta + \beta_0 > 0\} \tag{2.7}
\]
minimizes the misclassification error \( P\{g(Z) \neq Y\} \) over all \( g : \mathbb{R}^K \to \{0,1\} \). Furthermore,
\[
\beta = \Sigma_Z^{-1} \mathbb{E}[ZY].
\]

Proof. See Bing and Wegkamp (2022, Proposition 4). \( \square \)

Note that \( g_z^* \) in (2.7) has different form from the canonical LDA rule based on \( \Sigma_Z^{-1}(\alpha_1 - \alpha_0) \) with \( \Sigma_Z \) being the conditional covariance matrix. The advantage of expressing \( g_z^* \) in terms of \( \beta \) lies in the fact that \( \beta \) can be obtained by simply regressing \( Y \) on \( Z \), hence there is no need to estimate \( \Sigma_Z^{-1} \). If \( Z = (Z_1, \ldots, Z_n)^\top \in \mathbb{R}^{n \times K} \) were observed, it is natural to use the least squares estimator \( Z^+ = (Z^\top Z)^{-1}Z^\top Y \) to estimate \( \beta \). However, we only have access to the \( n \times p \) data-matrix
\[
X = (X_1, \ldots, X_n)^\top
\]
based on \( n \) independent observations \( X_i \in \mathbb{R}^p \) from (2.1) and the vector
\[
y = (Y_1, \ldots, Y_n)^\top \in \{0,1\}^n
\]
of labels \( Y_i \) corresponding to the rows \( X_i \) of \( X \). For a new feature \( x \in \mathbb{R}^p \) generated from model (2.1), the inner-product \( z^\top \beta \) is estimated by \( x^\top \hat{\theta} \) based on the minimum-norm interpolator, also generally termed as the generalized least squares (GLS) estimator,
\[
\hat{\theta} = X^+y = (X^\top X)^{-1}X^\top y. \tag{2.8}
\]

In Bing and Wegkamp (2022), the author used Principal Component Regression (PCR) instead of the GLS-estimate \( \hat{\theta} \) to estimate \( z^\top \beta \). The intuition of PCR lies in approximating the span of \( Z \) by that of the first \( K \) principal components of \( X \). Thus PCR is a more complicated method because of estimating the latent dimension \( K \), but is often minimax optimal (sometimes using a slight, yet necessary modification involving data-splitting). The GLS-estimate \( \hat{\theta} \) on the other hand does not require selection of \( K \), and is free of tuning parameters. Because of this, it is far from clear whether the span of \( X \) approximates that of \( Z \). It is therefore of great interest to see whether the method based on \( \hat{\theta} \) works, and if so, whether it is minimax optimal.

The classifier that we study in this paper has the form in (1.1). We refer to this classifier as the GLS-based classifier. Estimation of the intercept \( \beta_0 \) is discussed in Section 3.2. Our goal is to analyze its misclassification error relative to the oracle risk \( R^*_z \) in (2.4).

Remark 1. For the oracle risk, we have the explicit expression
\[
R^*_z = 1 - \pi_1 \Phi \left( \frac{\Delta}{2} + \frac{\log \frac{\pi_1}{\pi_0}}{\Delta} \right) - \pi_0 \Phi \left( \frac{\Delta}{2} - \frac{\log \frac{\pi_1}{\pi_0}}{\Delta} \right), \tag{2.9}
\]
see, for instance, Izenman (2008, Section 8.3, pp 241–244), based on the Mahalanobis distance
\[
\Delta^2 := (\alpha_1 - \alpha_0)^\top \Sigma_Z^{-1}(\alpha_1 - \alpha_0) = \|\alpha_1 - \alpha_0\|^2_{\Sigma_Z^{-1}}. \tag{2.10}
\]
In particular, when \( \pi_0 = \pi_1 \), the expression in (2.9) simplifies to
\[
R^*_z = 1 - \Phi \left( \Delta/2 \right).}

6
3 Interpolation and estimation of the intercept

We first review the interpolation property of the minimum-norm interpolator $\hat{\theta}$ and discuss the interpolation of the classifier $1\{x^T\hat{\theta} + \beta_0 > 0\}$ based on the true intercept. We then show that a natural plug-in estimator of $\beta_0$ is surprisingly inconsistent, which leads us to propose a different estimator of $\beta_0$. Finally, we show that the interpolation property of the classifier is connected with the way we encode our labels. This reveals that the interpolation property is a rather arbitrary artifact.

3.1 Interpolation

The theoretical performance of the GLS estimator (2.8) including its interpolation property is now well understood in linear regression settings $y = X\theta + \varepsilon$ when the feature dimension $p$ is much larger than the sample size $n$ (see, for instance, Belkin et al. (2018); Bartlett et al. (2020); Hastie et al. (2022); Bunea et al. (2022) and the references therein). The following result shows that with high probability, $\text{rank}(X) = n$ whence $\hat{\theta}$ interpolates the training data, provided that $K \leq n$, $\text{tr}(\Sigma_W) > 0$ and $r_e(\Sigma_W) := \text{tr}(\Sigma_W)/\|\Sigma_W\|_{\text{op}} \gg n$.

Proposition 2. Assume $n \geq K$. Then, there exist finite, positive constants $C, c$ depending on $\sigma$ only, such that, provided $r_e(\Sigma_W) \geq Cn$,

$$\mathbb{P}\left\{\sigma_n^2(X) \geq \frac{1}{8}\text{tr}(\Sigma_W)\right\} \geq 1 - 3\exp(-c n)$$

and thus, if in addition $\text{tr}(\Sigma_W) > 0$, $\hat{\theta}$ interpolates: $\mathbb{P}\{|\hat{\textbf{X}}\hat{\theta} - \textbf{y}| = 0\} \geq 1 - 3\exp(-c n)$.

Proof. See Bunea et al. (2022, Proposition 14).

From Proposition 2, $\hat{\theta}$ interpolates the training data provided that $K \leq n$ and $r_e(\Sigma_W) \gg n$. The latter is connected to the over-parametrization. To see this, suppose that $\Sigma_W$ has bounded eigenvalues, that is, for some absolute constants $0 < c \leq C < \infty$,

$$c \leq \lambda_p(\Sigma_W) \leq \lambda_1(\Sigma_W) \leq C. \quad (3.1)$$

We see that $r_e(\Sigma_W) \asymp p$, whence $r_e(\Sigma_W) \gg n$ reduces to the over-parametrized setting $p \gg n$.

Given the interpolation property of $\hat{\theta}$, we immediately see that $X_i^T\hat{\theta} + \hat{\beta}_0 = Y_i + \hat{\beta}_0 > 0$ if and only if $Y_i = 1$, for all $i \in [n]$, as long as $\hat{\beta}_0 \in (-1, 0]$. Hence, for any $\beta_0 \in (-1, 0]$ (including zero intercept advocated in the recent literature), the classifier $1\{x^T\hat{\theta} + \beta_0 > 0\}$ would perfectly classify the training data. A natural question is to see whether or not the classifier $1\{x^T\hat{\theta} + \beta_0 > 0\}$ that uses the true intercept $\beta_0$ would yield zero training error. This is equivalent with verifying if $\beta_0 \in (-1, 0]$. The following lemma provides the answer, which surprisingly depends on the way we encode.

Lemma 3. The intercept $\beta_0$ in (2.6) satisfies

$$\text{sign}(\beta_0) = \text{sign}\left(\frac{1}{2} - \pi_0\right), \quad |\beta_0| \leq \left|\frac{1}{2} - \pi_0\right|. \quad (7)$$
Proof. By the identity (3.1), the definition (2.10) of \( \Delta \), the identity
\[
\Sigma^{-1}_Z(\alpha_1 - \alpha_0) = \frac{1}{1 + \pi_0 \pi_1 \Delta^2} \Sigma^{-1}_Z \| \alpha_1 - \alpha_0 \|
\]
(see, the proof of Lemma 14 in Bing and Wegkamp (2022)) and the fact that \( \mathbb{E}[Z] = \pi_0 \alpha_0 + \pi_1 \alpha_1 = 0_K \), we have, after a bit of simple algebra,
\[
\beta_0 = \left[ \frac{1}{2} - \pi_0 - \frac{1}{\Delta^2} \left( \frac{1}{2} - \pi_0 - \pi_0 \pi_1 \log \frac{\pi_1}{\pi_0} \right) \right].
\] (3.2)

It is readily seen that \( \text{sign}(\beta_0) = \text{sign} \left( \frac{1}{2} - \pi_0 \right) \).

For the second claim, suppose \( 0 < \pi_0 < 1/2 \). After rearranging terms, we find
\[
\beta_0 = \left[ \frac{1}{2} - \pi_0 - \frac{1}{\Delta^2} \left( \frac{1}{2} - \pi_0 - \pi_0 \pi_1 \log \frac{\pi_1}{\pi_0} \right) \right].
\]

Since the term in parenthesis is positive, \( 1/2 - \pi_0 - \pi_0 \pi_1 \log(\pi_1/\pi_0) > 0 \), we conclude that \( 0 < \beta_0 < (1/2 - \pi_0) \). A similar argument can be used to prove \( (1/2 - \pi_0) < \beta_0 < 0 \) for the case \( 1/2 \leq \pi_0 < 1 \).

Lemma 3 has the curious consequence that only if we encode the majority class as 0, does the rule \( 1\{x^\top \hat{\theta} + \beta_0 > 0\} \) have zero training error under interpolation \( X\hat{\theta} = y \). In Section 3.3 we will show that the interpolation property also depends on the values we use to encode the labels, and is therefore rather arbitrary.

### 3.2 Estimation of the intercept

Given \( \hat{\theta} = X^+y \), assuming \( x^\top \hat{\theta} \) is close to \( z^\top \beta \), we can naively estimate \( \beta_0 \) in (2.6) by the following plug-in estimator,
\[
\hat{\beta}_0 := -\frac{1}{2} (\hat{\mu}_0 + \hat{\mu}_1)^\top \hat{\theta} + \left[ 1 - (\hat{\mu}_1 - \hat{\mu}_0)^\top \hat{\theta} \right] \hat{n}_0 \hat{n}_1 \log \frac{\hat{n}_1}{\hat{n}_0}
\] (3.3)

based on standard non-parametric estimates
\[
\hat{n}_k = \sum_{i=1}^n \mathbb{1}\{Y_i = k\}, \quad \hat{\pi}_k = \frac{\hat{n}_k}{n}, \quad \hat{\mu}_k = \frac{1}{n_k} \sum_{i=1}^{n_k} X_i \mathbb{1}\{Y_i = k\}, \quad k \in \{0, 1\}.
\] (3.4)

This leads to the naive classifier
\[
\hat{g}(x) := 1\{x^\top \hat{\theta} + \hat{\beta}_0 > 0\}.
\] (3.5)

The following lemma shows that \( \hat{\beta}_0 = -1/2 \), irrespective of the true value of \( \beta_0 \), whenever \( \hat{\theta} \) interpolates. On the one hand, this means that the naive classifier \( \hat{g}(x) \) always interpolates as \( \hat{\beta}_0 \in (-1, 0] \). On the other hand, it shows that \( \hat{\beta}_0 \) clearly is an inconsistent estimate of \( \beta_0 \) in general.

**Proposition 4.** Let \( \hat{\beta}_0 \) be defined in (3.3). On the event \( \{X\hat{\theta} = y\} \) where \( \hat{\theta} \) interpolates, we have \( \hat{\beta}_0 = -1/2 \).
Proof. We define the vectors \( v, w \in \mathbb{R}^n \) as
\[
  v = \frac{1}{n_1} (1\{Y_1 = 1\}, \ldots, 1\{Y_n = 1\})^\top + \frac{1}{n_0} (1\{Y_1 = 0\}, \ldots, 1\{Y_n = 0\})^\top,
\]
\[
  w = \frac{1}{n_1} (1\{Y_1 = 1\}, \ldots, 1\{Y_n = 1\})^\top - \frac{1}{n_0} (1\{Y_1 = 0\}, \ldots, 1\{Y_n = 0\})^\top.
\]
By (3.4), we can write
\[
  \hat{\mu}_1 + \hat{\mu}_0 = X^\top v, \quad \hat{\mu}_1 - \hat{\mu}_0 = X^\top w
\]
and hence
\[
  \hat{\beta}_0 = -\frac{1}{2} (\hat{\mu}_0 + \hat{\mu}_1)^\top \hat{\theta} + \left[ 1 - (\hat{\mu}_1 - \hat{\mu}_0)^\top \hat{\theta} \right] \hat{\pi}_0 \hat{\pi}_1 \log \frac{\hat{\pi}_1}{\hat{\pi}_0}
\]
\[
  = -\frac{1}{2} v^\top X \hat{\theta} + \left[ 1 - w^\top X \hat{\theta} \right] \hat{\pi}_0 \hat{\pi}_1 \log \frac{\hat{\pi}_1}{\hat{\pi}_0}
\]
Use \( X \hat{\theta} = y \) and \( v^\top y = w^\top y = 1 \) to obtain
\[
  \hat{\beta}_0 = -\frac{v^\top y}{2} + \left[ 1 - w^\top y \right] \hat{\pi}_0 \hat{\pi}_1 \log \frac{\hat{\pi}_1}{\hat{\pi}_0} = -\frac{1}{2}.
\]
This proves the our claim. \( \square \)

Proposition 4 implies that the naive classifier \( \hat{g}(x) \) from (3.5) cannot be consistent in general due to the inconsistency of \( \hat{\beta}_0 \), for the same reason that taking no intercept \( (\hat{\beta}_0 = 0) \) is inconsistent. From the proof of Proposition 4, we see that this phenomenon still exists in the classical LDA setting, where \( X \mid Y \) is Gaussian, should any interpolating regression estimate such as \( \hat{\theta} \) be employed and plugged in (3.3). The inconsistency of GLS-based LDA is in sharp contrast to its magical performance in factor regression models, see Bunea et al. (2022); Bing et al. (2021). This phenomenon is corroborated in our simulation study in Section 5.

While the estimator \( \hat{\beta}_0 \) in (3.3) based on the GLS estimator \( \hat{\theta} \) is clearly inconsistent for estimating the intercept \( \beta_0 \), we will show in Theorem 8 of the next section that \( \hat{\theta} \) does estimate the direction \( \beta \) consistently. In other words, failure of consistently estimating the intercept \( \beta_0 \) is the only cause for the subpar misclassification rate of \( \hat{g}(x) \). In the symmetric case \( \pi_0 = 0, \pi_1 = 1/2 \), we have \( \beta_0 = 0 \) and we prove in Corollary 10 of the next section that the classifier \( 1 \{ x^\top \hat{\theta} + 0 > 0 \} \) is consistent, often even minimax optimal.

In general, when \( \pi_0 \neq \pi_1 \), we should choose a different estimate for \( \beta_0 \). Our solution is to use an independent hold-out sample \( (X'_i, Y'_i), i \in [n'] \), for some integer \( n' > 0 \), to estimate \( \beta_0 \) by
\[
  \tilde{\beta}_0 := -\frac{1}{2} (\tilde{\mu}_0 + \tilde{\mu}_1)^\top \tilde{\theta} + \left[ 1 - (\tilde{\mu}_1 - \tilde{\mu}_0)^\top \tilde{\theta} \right] \tilde{\pi}_0 \tilde{\pi}_1 \log \frac{\tilde{\pi}_1}{\tilde{\pi}_0}
\]
with \( \tilde{\theta} \) from (2.8), \( \tilde{\pi}_k \) from (3.4) and
\[
  \tilde{\mu}_k = \frac{1}{\tilde{n}_k} \sum_{i=1}^{n'} X'_i 1\{Y'_i = k\}, \quad \tilde{n}_k = \sum_{i=1}^{n'} 1\{Y'_i = k\}, \quad k \in \{0, 1\}.
\]
This simple modification ensures that \( \tilde{\beta}_0 \) is a consistent estimator of \( \beta \), as shown in Proposition 11 of the next section. Furthermore, the corresponding classification rule

\[
1 \{ x^\top \hat{\theta} + \tilde{\beta}_0 > 0 \}
\]

is consistent, and even minimax-optimal, in many scenarios (see Remark 3), although it no longer necessarily classifies the training data perfectly.

**Remark 2 (Alternative estimation of \( \beta_0 \)).** It is essential for establishing consistency of \( \tilde{\beta}_0 \) that the estimates \( \tilde{\mu}_0, \tilde{\mu}_1 \) and \( \hat{\theta} \) are statistically independent. Alternatively, we could use the hold-out sample to estimate the intercept via minimizing the empirical risk

\[
\sum_{i}^n 1 \{ (2Y_i - 1)(\hat{\theta}^\top X_i + \beta_0) < 0 \}
\]

over \( \beta_0 \in \mathbb{R} \). It is clear that we need to use the hold-out sample in (3.8) as well, since minimizing over the same training data would lead to interpolation and any value within \((-1, 0]\) whenever \( \hat{\theta} \) interpolates. In our simulation of Section 5, we found that not only can we compute \( \tilde{\beta}_0 \) in (3.6) much faster comparing to (3.8), it also leads to better classification performance as well.

### 3.3 Effect of label encoding on interpolation

In this section we discuss how the (in sample) interpolation property depends on the way we encode our labels. Let \( b > a \) be any scalars and consider the encoding \( Y \in \{a, b\} \). In the following lemma we show that the optimal decision boundary in the latent space is independent of the particular encoding. This reassures us that the optimal classification rule does not depend on the encoding.

**Lemma 5.** Given the encoding \( Y \in \{a, b\} \) with any \( a < b \), the Bayes rule is

\[
(b - a)1 \{ (b - a)(z^\top \beta + \beta_0) > 0 \} + a
\]

with \( \beta \) and \( \beta_0 \) defined in (2.5) and (2.6), respectively. In particular, the optimal boundary \( \{ z \mid (b - a)(z^\top \beta + \beta_0) = 0 \} \) is invariant to any pair \( (a, b) \) with \( b > a \).

**Proof.** From the proof of Proposition 4 in Bing and Wegkamp (2022), we can deduce that the optimal hyperplane is \( \{ z \mid z^\top \beta^{(a,b)} + \beta_0^{(a,b)} = 0 \} \) where

\[
\beta^{(a,b)} = (b - a)\pi_0\pi_1\Sigma_Z^{-1}(\alpha_1 - \alpha_0),
\]

\[
\beta_0^{(a,b)} = -\frac{1}{2}(\alpha_0 + \alpha_1)^\top \beta + [b - a - (\alpha_1 - \alpha_0)^\top \beta] \pi_0\pi_1 \log \frac{\pi_1}{\pi_0}.
\]

This proves our claim. \( \square \)

However, it is a different story for the interpolation property with

\[
(b - a)1 \{ x^\top \hat{\theta} + (b - a)\beta_0 > 0 \} + a.
\]

(3.10)
Note this is the classifier corresponding to \( 1 \{ x^\top \hat{\theta} + \beta_0 > 0 \} \) under the encoding \( Y \in \{0, 1\} \). Given that \( X \hat{\theta} = y \), the classifier in (3.10) has zero training error if and only if
\[
(b - a)\beta_0 \in (-b, -a].
\] (3.11)
Indeed, for any \( i \in [n] \), observe that \( X_i^\top \hat{\theta} + (b - a)\beta_0 > 0 \) if and only if \( Y_i + (b - a)\beta_0 > 0 \). Whether the classifier in (3.10) interpolates is thus equivalent to whether (3.11) holds. Below we use Lemma 3 to compare our \{0, 1\} encoding with another popular encoding \{-1, 1\}.

- As noted earlier, if we use the \{0, 1\} encoding, (3.11) becomes \( \beta_0 \in (-1, 0] \) and the classifier in (3.10) has zero training classification error if and only if we encode the majority class as 0.

- If we use the encoding \{-1, 1\}, then (3.11) becomes \( \beta_0 \in (-1/2, 1/2] \) which holds in view of Lemma 3. Therefore, the classifier in (3.10) always has zero training classification error. Furthermore, we expect that all rules based on (3.10) with \( \beta_0 \) replaced by a consistent estimate to classify the training data perfectly.

4 Rates of convergence for the excess risk

In this section, we analyze the excess risk of the classifier
\[
\tilde{g}(x) = 1 \{ x^\top \hat{\theta} + \tilde{\beta}_0 > 0 \}
\] (4.1)
for \( \hat{\theta} \) defined in (2.8) and \( \tilde{\beta}_0 \) defined in (3.6). We define the excess risk of this classifier as
\[
P \{ \tilde{g}(X) \neq Y \} - R^*_z \text{ with } R^*_z \text{ given in (2.4) (see, also, (2.9))}.\]
Following Bing and Wegkamp (2022, Theorem 5), we have, for all \( t > 0 \),
\[
P \{ \tilde{g}(X) \neq Y \} - R^*_z \leq P \left\{ |X^\top \hat{\theta} - Z^\top \beta + \tilde{\beta}_0 - \beta_0| > t \right\} + P(\Delta, t) \] (4.2)
where, with \( c_\Delta := \Delta^2 + (\pi_0\pi_1)^{-1} \),
\[
P(\Delta, t) = t \min \left\{ \frac{1}{\Delta}, \frac{1}{\Delta \pi_0} \right\} P \left\{ -t < Z^\top \beta + \beta_0 < 0 \mid Y = 0 \right\} + \pi_1 P \left\{ 0 < Z^\top \beta + \beta_0 < t \mid Y = 1 \right\}. \]
We see that the excess risk depends on:
(a) the probabilistic behavior of the boundary of the optimal hyperplane, and
(b) the quality of our estimate of the optimal hyperplane in \( \mathbb{R}^K \).

Part (a) is expressed in the quantity \( P(\Delta, t) \) and reflects the intrinsic difficulty of the classification problem. As in Bing and Wegkamp (2022), we can distinguish four cases: Let \( c \) and \( c' \) be some absolute positive constants. For any \( t > 0 \),
\[
P(\Delta, t) = \begin{cases} 
  t^2 & \text{if } \Delta \asymp 1; \\
  t^2 \exp \{-[c + o(1)]\Delta^2\} & \text{if } \Delta \to \infty \text{ and } t \to 0; \\
  t^2 \exp \{-[c' + o(1)]/\Delta^2\} & \text{if } \Delta \to 0, \pi_0 \neq \pi_1 \text{ and } t \to 0; \\
  t \min \{1, t/\Delta\} & \text{if } \Delta \to 0 \text{ and } \pi_0 = \pi_1. 
\end{cases} \] (4.3)
The case $\Delta \to \infty$ can be considered an easy case as the Bayes error $R^*_z$ vanishes exponentially fast in $\Delta^2$. In this case, $P(\Delta, t)$ also vanishes exponentially fast. We also note that the proof of Lemma 3 reveals that $|\beta_0 - (1/2 - \pi_0)| \to 0$ in this case; that is, the true intercept is easier to estimate as well. The case $\Delta \to 0$ and $\pi_0 \neq \pi_1$ has a trivial Bayes risk $R^*_z \to \min\{\pi_0, \pi_1\}$, hence is easy to classify since the optimal Bayes rule classifies only according to the largest unconditional class probability $\pi_k$, irrespective of the covariate $x$. In this case the intrinsic difficulty $P(\Delta, t)$ goes to zero exponentially fast in $1/\Delta^2$. The case $\Delta \to 0$ and $\pi_0 = \pi_1$ is impossible to classify as $R^*_z \to 1/2$, corresponding to random guessing. For this reason, we concentrate on the intermediate case $\Delta \approx 1$ in this work.

For part (b), we need to control
\[
|X^T \hat{\theta} - Z^T \beta| \leq |Z(A^T \hat{\theta} - \beta)| + |W^T \hat{\theta}|,
\]
the error of predicting the ‘direction’ $Z^T \beta$, as well as $|\tilde{\beta}_0 - \beta_0|$, the error of estimating the intercept. The following proposition provides upper bounds of the two terms on the right of (4.4).

**Proposition 6.** For every $\delta > 0$, we have
\[
P\left[|W^T \hat{\theta}| \geq \sigma \sqrt{2\|\hat{\theta}\|_{\Sigma_w} \log(1/\delta)}\right] \leq 2\delta
\]
and
\[
P\left[|Z^T (A^T \hat{\theta} - \beta)| \geq \left(\frac{1}{2\pi_0 \pi_1} + \sqrt{\log(1/\delta)}\right) \|A^T \hat{\theta} - \beta\|_{\Sigma_z}\right] \leq 4\delta.
\]

**Proof.** The bounds follow from the independence of $\hat{\theta}, Z$ and $W$, the Gaussian assumption (iv) of $Z \mid Y = k$ and subGaussian distribution (v) of $W$. See, for instance, the proof of Proposition 6 in Bing and Wegkamp (2022).

For the later analysis, we will apply Proposition 6 with $\delta = 1/n^c$ for some absolute constant $c > 0$, which will result in a multiplicative $\log(n)$ term for the corresponding rates. From Proposition 6, it is clear that we need to bound $\|\hat{\theta}\|_{\Sigma_w}$ and $\|A^T \hat{\theta} - \beta\|_{\Sigma_z}$.

**Proposition 7.** Assume $n \geq K$. Then, there exist finite, positive constants $C, c$ depending on $\sigma$ only, such that, provided $r_e(\Sigma_w) \geq Cn$,
\[
P\left\{\|\hat{\theta}\|_{\Sigma_w}^2 \leq \frac{8n}{r_e(\Sigma_w)}\right\} \geq 1 - 3 \exp(-c n).
\]

**Proof.** By definition,
\[
\|\Sigma_{1/2} \hat{\theta}\|_2^2 = \|\Sigma_{1/2} X^+ y\|_2^2 \leq \|\Sigma_{1/2} y\|_{\text{op}} \frac{\|y\|_2^2}{\sigma_n^2(X)}.
\]
The result follows from the inequality $\|y\|_2 \leq \sqrt{n}$ and Proposition 2.
From Proposition 7, \( \| \hat{\theta} \|_{\Sigma_W} \to 0 \), with overwhelming probability, is ensured if \( n/r_e(\Sigma_W) \to 0 \) as \( n \to \infty \). The latter holds in the over-parametrized setting \( p \gg n \) with condition (3.1).

Bounding \( \| A^T \hat{\theta} - \beta \|_{\Sigma_Z} \) from above on the other hand is the main difficulty in our analysis. To present our result, we write the non-zero eigenvalues of \( A \Sigma_Z A^T \) as \( \lambda_1 \geq \cdots \geq \lambda_K \) and its condition number \( \lambda_1/\lambda_K \) as \( \kappa \). We further define 

\[
\xi := \frac{\lambda_K}{\| \Sigma_W \|_{\text{op}}} \tag{4.5}
\]

as the signal-to-noise ratio of predicting the signal \( Z \) from \( X = AZ + W \) in the presence of the noise \( W \). At the sample level, the noise level \( \| \Sigma_W \|_{\text{op}} \) gets inflated in the sense that

\[
P\left\{ \frac{1}{n} \| W^T W \|_{\text{op}} \leq 12\sigma^2 \| \Sigma_W \|_{\text{op}} \left( 1 + \frac{r_e(\Sigma_W)}{n} \right) \right\} = 1 - \exp(-n), \tag{4.6}
\]

see, Lemma 20 in the Appendix. Finally, we set

\[
\omega_n = \sqrt{\frac{K \log(n)}{n}} + \sqrt{\frac{K \log(n)}{n}} \frac{\kappa n}{r_e(\Sigma_W)} + \sqrt{\frac{n}{r_e(\Sigma_W)}} + \frac{r_e(\Sigma_W) \sqrt{\kappa}}{\xi} + \frac{\kappa}{\sqrt{\xi}}. \tag{4.7}
\]

**Theorem 8.** Assume the following holds as \( n \to \infty \),

\[
\frac{K \log(n)}{n} \to 0, \quad \frac{n}{r_e(\Sigma_W)} \to 0 \quad \text{and} \quad \frac{\kappa}{\xi} \left( 1 + \frac{r_e(\Sigma_W)}{n} \right) \to 0. \tag{4.8}
\]

Then, for any constant \( c > 0 \), there exists a constant \( C = C(\sigma) < \infty \) such that

\[
P\left\{ \| A^T \hat{\theta} - \beta \|_{\Sigma_Z} \leq C \omega_n \right\} = 1 - O\left( n^{-c} \right)
\]

We refer to Section 6.1 for the proof and explain the main difficulties of the analysis in Section 4.1 below. When \( \kappa = O(1) \), the set of assumptions (4.8) is needed to ensure \( \omega_n \to 0 \) in (4.7). The first condition puts a restriction on the latent dimension relative to the sample size, the second condition holds in the over-parametrized setting \( p \gg n \) with (3.1), while the third one requires the signal \( \lambda_K \) of predicting \( Z \) from \( X \) to exceed the sample level noise (cf. (4.6)). In Remark 3 we simplify the expression of \( \omega_n \) and provide an interpretation for each term.

**Corollary 9.** Under condition (4.8), for any constant \( c > 0 \), there exists a constant \( C = C(\sigma) < \infty \) such that

\[
P\left\{ |X^T \hat{\theta} - Z^T \beta| \geq C \omega_n \sqrt{\log(n)} \right\} = O(n^{-c}).
\]

**Proof.** Combination of Proposition 6, Proposition 7 and Theorem 8 immediately yields the result.

In case \( \pi_0 = \pi_1 = 1/2 \), the intercept \( \beta_0 \) does not need to be estimated as \( \beta_0 = 0 \). Coupled with (4.2) and (4.3), Corollary 9 immediately gives a bound on the excess risk of the classifier \( \hat{g}(x) = 1\{ x^T \hat{\theta} > 0 \} \) that uses a hyperplane through the origin.
Corollary 10. Under condition (4.8), assume that $\Delta \asymp 1$ and $\pi_0 = \pi_1 = 1/2$. The classifier $\hat{g}(x) = 1\{x^\top \hat{\theta} > 0\}$ satisfies

$$
\mathbb{P}\{\hat{g}(X) \neq Y\} - R^*_z \lesssim \omega_n^2 \log(n).
$$

Having successfully bounded $|X^\top \hat{\theta} - Z^\top \beta|$, it remains to bound $|\tilde{\beta}_0 - \beta_0|$ in order to apply the excess risk bound (4.2). Recall that $\tilde{\beta}_0$ is given by (3.7).

Proposition 11. Assume $n' \gtrsim n$ and $n/r_e(\Sigma_W) \to 0$ as $n \to \infty$. Then, for any $c > 0$, there exists a $C = C(\sigma) < \infty$ such that

$$
\mathbb{P}\left\{|\tilde{\beta}_0 - \beta_0| \leq C\sqrt{\frac{\log(n)}{n}} + C\|A^\top \hat{\theta} - \beta\|_{\Sigma_Z}\right\} = 1 - O(n^{-c}).
$$

Proof. See Appendix 6.2.

For ease of presentation, we state our results for $n' \gtrsim n$. Tracking our proof reveals that for any $n' \to \infty$, the statement above continues to hold when we replace $n$ by $n' \wedge n$.

Finally, we can state our main result:

Theorem 12. Assume (4.8), $n' \gtrsim n$ and $\Delta \asymp 1$. Then $\tilde{g}(x)$ in (4.1) satisfies

$$
\mathbb{P}\{\tilde{g}(X) \neq Y\} - R^*_z \lesssim \omega_n^2 \log(n).
$$

Proof. The result follows from the excess risk bound (4.2) and (4.3) with $t = \omega_n\sqrt{\log(n)}$.

Corollary 9 and Proposition 11.

Remark 3 (Simplified excess risk bound and minimax optimality). We now discuss the case when $p \gg n \gg K$, $\Delta \asymp 1$, $n \gtrsim n'$, $r_e(\Sigma_W) \asymp p$ and $\kappa \asymp 1$. In this scenario, we will argue that our classifier $\tilde{g}(x)$ in (4.1) is minimax-optimal, provided both the ambient dimension $p$ and the signal-to-noise ratio $\xi = \lambda_K/\|\Sigma_W\|_{\text{op}}$ are large. We first observe that we have $n/r_e(\Sigma_W) \asymp n/p \to 0$ and $\omega_n$ in (4.7) can be simplified to

$$
\omega_n^2 \asymp \frac{K \log(n)}{n} + \frac{n}{p} + \left(\frac{p}{n \xi}\right)^2 + \frac{1}{\xi}.
$$

We can follow the discussion after Theorem 3 in Bing and Wegkamp (2022) to summarize the first, third and fourth terms. The first term $K \log(n)/n$ is the optimal rate of the excess risk when the latent factors $Z$ and $Z$ were observable, hence it reflects the benefit of having a hidden, low-dimensional structure with $K \ll n$. The last term is essentially $R^*_z - R^*_z$ given by (2.4), representing the irreducible error of predicting $Z$ from $X$ at the population level (see (4.5)), while the third term can be interpreted as the error of predicting $Z$ from $X$ at the sample level (see (4.6)). Different from Bing and Wegkamp (2022), is the second term $n/p$ in (4.9), which is due to using $X^\top \hat{\theta}$ to predict $Z^\top \beta$ instead of the PCR method advocated in Bing and Wegkamp (2022). It reveals the benefit of over-parametrization.
In view of (4.9), consistency of the classifier \( \tilde{g} \) in (4.1) requires the signal-to-noise ratio \( \xi \) to be sufficiently large in the precise sense that \( \xi \gg p/n \). Furthermore, under the stronger assumption of \( \xi \gtrsim (p/n) \cdot (n/K)^{1/2} \), we find

\[
\mathbb{P}\{ \tilde{g}(X) \neq Y \} - R^*_\mathcal{X} \lesssim \omega^2_n \log(n) \asymp \frac{K}{n} \log^2(n) + \frac{n}{p} \log(n).
\]

(4.10)

We emphasize that this condition on \( \xi \) is a much weaker requirement than the condition \( \xi \gtrsim p \) commonly made in the existing literature of high-dimensional factor models. See, for instance, Bai (2003); Fan et al. (2013); Stock and Watson (2002). Finally, Bing and Wegkamp (2022, Theorem 1) proves that the minimax optimal rate is proportional to \((K/n) + (1/ξ)\), and we find that the above rate (4.10) coincides with the minimax optimal rate, up to the logarithmic factor in \( n \), in the high-dimensional setting \( p \gg n^2/K \).

Remark 4 (Rates of the excess risk for \( \Delta \to \infty \) and \( \Delta \to 0 \)). Since \( \Delta \approx 1 \) is the most realistic and interesting case as discussed after display (4.3), we state our main result in Theorem 8 in this regime. Nevertheless, the results for \( \Delta \to \infty \) and \( \Delta \to 0 \) can be easily obtained by combining the excess risk bound (4.2), display (4.3), Proposition 6, Theorem 12 in this regime. Nevertheless, the results for \( \Delta \to \infty \) and \( \Delta \to 0 \) can be easily obtained by combining the excess risk bound (4.2), display (4.3), Proposition 6, Theorem 8 and Proposition 11, in conjunction with (4.3) for \( t = \omega_n \sqrt{\log n} \).

4.1 Technical difficulties in the proof of Theorem 8

On the event \( \mathcal{E}(Z, X) = \{ \sigma^2_{\Sigma}(Z \Sigma^{-1/2}_Z) \geq n/2, \sigma^2_n(X) \geq \text{tr}(\Sigma_W)/8 \} \), which can be shown to hold with overwhelming probability (see, Proposition 2 in Section 3.1 and Lemma 16 in Section 7), the identities \( Z^+Z = I_K \) and \( XX^+ = I_n \) lead to the following chain of identities

\[
A^\top \hat{\beta} = Z^+ZA^\top X^+y \\
= Z^+(X - W)X^+y \\
= Z^+y - Z^+WX^+y
\]

so that

\[
\|A^\top \hat{\beta} - \beta\|_{\Sigma_Z} \leq \|Z^+y - \beta\|_{\Sigma_Z} + \|Z^+WX^+y\|_{\Sigma_Z}.
\]

The first term is relatively easy to analyze and it can be bounded as \( \mathcal{O}_p(\sqrt{K \log(n)/n}) \). The second term is technically challenging to analyze because commonly used arguments render meaningless bounds. To appreciate the difficulty of the problem, let us consider three types of arguments in the simplified case \( \Sigma_W = I_p \) and \( r_n(\Sigma_W) = \text{tr}(\Sigma_W) = p \).

(i) By the identity \( \Sigma^{-1/2}_Z = (\Sigma^{-1/2}_Z Z^\top Z \Sigma^{-1/2}_Z)^{1/2} + \Sigma^{-1/2}_Z Z^\top \Sigma^{-1/2}_Z \), we have on \( \mathcal{E}(Z, X) \),

\[
\|Z^+WX^+y\|_{\Sigma_Z} \leq \frac{2}{n} \|\Sigma^{-1/2}_Z Z^\top W\|_{\text{op}} \|X^+y\|_2.
\]

Since \( \|X^+y\|_2 \leq \sqrt{\|y\|_2/\sigma_n(X)} \leq \sqrt{8n/p} \) and standard concentration arguments ensure that \( \|\Sigma^{-1/2}_Z Z^\top W\|_{\text{op}} \) is at least of order \( \mathcal{O}_p(\sqrt{n/p}) \), we end up with a trivial bound \( \|Z^+WX^+y\|_{\Sigma_Z} = \mathcal{O}_p(1) \). Note that since \( X = ZA^\top + W \) depends on both \( W \) and \( y \), the above arguments do not appear to be loose.
(ii) In fact, by $X = ZA^T + W$ and $X^+ = X^+(XX^T)^+$, we also have

$$
\|Z^+WX^+y\|_{\Sigma_Z} = \|Z^+W(ZA^T + W)^+(XX^T)^+y\|_{\Sigma_Z} \\
\leq \|Z^+WAZ^T(XX^T)^+y\|_{\Sigma_Z} + \|Z^+WW^T(XX^T)^+y\|_{\Sigma_Z}.
$$

By similar arguments, the second term could be bounded by

$$
\|Z^+WW^T\|_{op}(XX^T)^+y \leq \frac{2}{n}\|\Sigma_Z^{-1/2}Z^TW\|_{op}\sqrt{\frac{8n}{p}}.
$$

As $\|\Sigma_Z^{-1/2}Z^TW\|_{op}$ has the order of $O(p\sqrt{n})$, we again obtain a trivial bound.

(iii) Since $X = ZA^T + W$ and especially for $p \gg n \gg K$, intuitively, we would expect that $\|Z^+WX^+y\|_{\Sigma_Z}$ contains, or is mainly about, the term

$$
\|Z^+WW^+y\|_{\Sigma_Z} = \|Z^+I_ny\|_{\Sigma_Z} = \|Z^+y\|_{\Sigma_Z}.
$$

As we have argued that $\|Z^+y\|_{\Sigma_Z}$ concentrates around $\|\beta\|_{\Sigma_Z} = \pi_0\pi_1\Delta$ in attempt (i), it seems hopeless for the rate of $\|Z^+WW^+y\|_{\Sigma_Z}$ to converge to zero for non-vanishing $\Delta$.

Despite these failed attempts, the situation can be salvaged via a more delicate argument. This is done by splitting $\|Z^+WX^+y\|_{\Sigma_Z}$ into two parts,

$$
\|Z^+WX^+y\|_{\Sigma_Z} \leq \|Z^+WV_KV_K^TX^+y\|_{\Sigma_Z} + \|Z^+WV_{-K}V_{-K}^TX^+y\|_{\Sigma_Z}
$$

based on $V_K$, the first $K$ right-singular vectors of $X$, and $V_{-K}$, the last $(p - K)$ right-singular vectors of $X$. The key is to recognize and capture the implicit regularization of $\hat{\theta} = X^+y$ in the second term in the high-dimensional regime (see, Sections 6.1.1 and 6.1.2). This is highly nontrivial even in the ideal case where $V_KV_K^T$ is close to the projection onto the column space of $A$. We use a key observation made in Bai (2003) that $XV_{-K}$ estimates $ZQ$ well for a certain $K \times K$ transformation matrix $Q$. We sharpen this result in Lemmas 14 & 15 by relaxing the stringent condition $\lambda_1(4\Sigma_ZA^T) \times \lambda_K(4\Sigma_ZA^T) \times p$ imposed by Bai (2003).

## 5 Simulation Study

In this section we first verify the inconsistency of the naive classifier that uses the naive plug-in estimator of $\beta_0$ and contrast with other consistent classifiers. We then evaluate the performance of our propose classifier in terms of its misclassification error as well as its estimation errors of $\beta$ and $\beta_0$. We also examine their dependence on the dimensions $p$ and $K$ as well as the signal-to-noise ratio $\xi$.

We generated the data as follows: We set $\pi_0 = \pi_1 = 0.5$, $\alpha_0 = -\alpha_1$, $\alpha_1 = 1_K\sqrt{2/K}$ and $\Sigma_ZY = I_K$ such that $\Delta^2 = 8$. The entries of $W$ and $A$ are independent realizations of $N(0, 1)$ and $N(0, 0.3^2)$, respectively.
Inconsistency of the Naive Classifier

We refer as GLS-Naive the classifier \( \hat{g}(x) = 1 \{ x^T \hat{\theta} + \hat{\beta}_0 > 0 \} \) with \( \hat{\beta}_0 \) being the naive plug-in estimator in (3.3), while GLS-Oracle, GLS-Plugin and GLS-ERM represent the classifiers \( 1 \{ x^T \hat{\theta} + \bar{\beta}_0 > 0 \} \) with \( \bar{\beta}_0 \) chosen as the true \( \beta_0 \), the plug-in estimate (3.6) based on data splitting, and the estimate (3.8) based on empirical risk minimization in Remark 2, respectively. Besides the optimal Bayes classifier (Bayes), we also choose the oracle procedure (Oracle-LS) that uses both \( Z \) and \( Z \) to estimate \( \beta \) and \( \beta_0 \) in (2.7) as our benchmark.

In the left panel of Figure 1, we plot the performance of all classifiers on 200 test data points by fixing \( K = 5 \) and \( n = 100 \), while varying \( p \in \{ 300, 600, 1000, 2000, 4000, 6000 \} \). Each setting is repeated 100 times and the averaged results are reported. For GLS-Plugin and GLS-ERM, we additionally generate 100 data points as the validation set. Clearly, GLS-Naive is inconsistent while the other three GLS-based classifiers get closer to the Oracle-LS as \( p \) increases. Moreover, the performance of GLS-Plugin is as good as GLS-Oracle (that uses the true \( \beta_0 \)) and better than GLS-ERM.

We also plot the training misclassification errors of all classifiers in the right panel of Figure 1. As expected from Proposition 4, GLS-Naive interpolates the training data despite its inconsistency. As discussed after Lemma 3, by recalling that \( \pi_0 = 1/2 \) hence \( \beta_0 = 0 \), GLS-Oracle also interpolates the training data. On the other hand, neither GLS-Plugin nor GLS-ERM interpolates. This is because their estimates of \( \beta_0 \) are centered around zero and only the non-positive ones lead to interpolation according to our discussion after Lemma 3. Furthermore, if we encode \( Y \in \{ -1, 1 \} \), simulation shows that both GLS-Plugin and GLS-ERM also interpolate the training data in addition to GLS-Oracle and GLS-Naive.

Figure 1: The averaged misclassification errors of each algorithm for various choices of \( p \). The left panel depicts the misclassification errors of the training data while the right one shows the test misclassification errors.
Performance of the Proposed Classifier

We evaluate the performance of our proposed classifier, GLS-Plugin, and examine its dependence on $p$, $K$ and $\xi$ by varying them one at a time. We consider three metrics: the misclassification error on 200 test data points, the estimation error of $\beta$, $\|\beta - A^\top \hat{\theta}\|_{\Sigma_Z}$, as analyzed in Theorem 8, and the estimation error of $\beta_0$, $|\hat{\beta}_0 - \beta_0|$. The sample size is fixed as $n = 100$ and we use a validation set with 100 data points to compute $\tilde{\beta}_0$. To vary the signal-to-noise ratio $\xi$, we choose the standard deviation $\sigma_A$ of each entry of $A$ from $\{0.01, 0.05, 0.1, 0.2\}$. Note that a larger $\sigma_A$ implies a larger $\xi$.

We repeat each setting 100 times and the averaged metrics as well as the standard errors are reported in Table 1. In line with Theorem 8, Proposition 11 and Theorem 12, all three metrics decrease in $p$ and $\xi$, while they increase in $K$.

Table 1: The averaged metrics of GLS-Plugin over 100 repetitions (the numbers within parentheses are the standard errors).

| Setting           | Misclassification errors | Errors of estimating $\beta$ | Errors of estimating $\beta_0$ |
|-------------------|--------------------------|-----------------------------|-------------------------------|
| $K = 5, \sigma_A = 0.3$ |
| $p = 300$        | 0.256 (0.046)            | 0.144 (0.052)               | 0.040 (0.031)                |
| $p = 600$        | 0.198 (0.037)            | 0.127 (0.046)               | 0.034 (0.023)                |
| $p = 1000$       | 0.156 (0.032)            | 0.117 (0.041)               | 0.029 (0.021)                |
| $p = 2000$       | 0.132 (0.034)            | 0.115 (0.039)               | 0.029 (0.024)                |
| $p = 4000$       | 0.116 (0.027)            | 0.112 (0.032)               | 0.027 (0.020)                |
| $p = 1000, \sigma_A = 0.3$ |
| $K = 3$          | 0.152 (0.033)            | 0.091 (0.039)               | 0.028 (0.020)                |
| $K = 5$          | 0.161 (0.029)            | 0.117 (0.039)               | 0.032 (0.022)                |
| $K = 10$         | 0.178 (0.036)            | 0.180 (0.036)               | 0.033 (0.027)                |
| $K = 15$         | 0.186 (0.038)            | 0.219 (0.040)               | 0.030 (0.022)                |
| $p = 1000, K = 5$ |
| $\sigma_A = 0.01$ | 0.479 (0.038)            | 0.397 (0.004)               | 0.048 (0.039)                |
| $\sigma_A = 0.05$ | 0.282 (0.039)            | 0.239 (0.024)               | 0.034 (0.026)                |
| $\sigma_A = 0.1$  | 0.187 (0.035)            | 0.124 (0.037)               | 0.029 (0.019)                |
| $\sigma_A = 0.24$ | 0.161 (0.033)            | 0.109 (0.034)               | 0.029 (0.022)                |

6 Main proofs

6.1 Proof of Theorem 8

Let $X = ZA^\top + W$ be the matrix version of model (2.1) based on independent observations and write the singular value decomposition of $(np)^{-1/2}X$ as

$$(np)^{-1/2}X = \sum_{k=1}^{n} d_k u_k v_k^\top = UDV^\top = U_K D_K V_K^\top + U_{-K} D_{-K} V_{-K}^\top$$  (6.1)
with \( D = (D_K; D_{-K}) = \text{diag}(d_1, \ldots, d_n), D_K = \text{diag}(d_1, \ldots, d_K), D_{-K} = \text{diag}(d_{K+1}, \ldots, d_n), \)
\( U = (U_K; U_{-K}) = (u_1, \ldots, u_n), U_K = (u_1, \ldots, u_K), U_{-K} = (u_{K+1}, \ldots, u_n), \) and \( V = (V_K; V_{-K}) = (v_1, \ldots, v_n), V_K = (v_1, \ldots, v_K), V_{-K} = (v_{K+1}, \ldots, v_n). \) Define
\[
\delta_W := \|\Sigma_W\|_{\text{op}} \left( 1 + \frac{r_c(\Sigma_W)}{n} \right). \tag{6.2}
\]
We define the events
\[
\mathcal{E}_Z := \left\{ \frac{1}{2} \leq \frac{1}{n} \sigma_K^2(Z\Sigma_Z^{-1/2}) \leq \frac{1}{n} \sigma_1^2(Z\Sigma_Z^{-1/2}) \leq 2 \right\}
\]
\[
\mathcal{E}_W := \left\{ \frac{1}{n} \sigma_1^2(W) \leq 12\sigma^2\delta_W \right\}
\]
\[
\mathcal{E}_X := \left\{ \sigma_2^2(X) \geq \frac{1}{8}r(\Sigma_W) \right\}
\]
In the sequel, we work on the event
\[
\mathcal{E} := \mathcal{E}_Z \cap \mathcal{E}_W \cap \mathcal{E}_X.
\]
This event holds with probability greater than \( 1 - O(n^{-c}) \), see Proposition 2 and Lemmas 16 and 20. Observe that we need at this point our assumptions \( n/r_c(\Sigma_W) \to 0 \) and \( K \log(n)/n \to 0 \) in (4.8).

Since \( Z^+Z = I_K \) and \( XX^+ = I_n \) on the event \( \mathcal{E}_Z \cap \mathcal{E}_X \), we find
\[
A^\top \hat{\theta} = Z^+ZA^\top X^+y = Z^+(X - W)X^+y = Z^+y - Z^+WX^+y \tag{6.3}
\]
so that
\[
\|\Sigma_Z^{1/2}(A^\top \hat{\theta} - \beta)\|_2 \leq \|\Sigma_Z^{1/2}(Z^+y - \beta)\|_2 + \|\Sigma_Z^{1/2}Z^+WX^+y\|_2 \tag{6.4}
\]
\[
\leq \|\Sigma_Z^{1/2}(Z^+y - \beta)\|_2 + \|\Sigma_Z^{1/2}Z^+WV_KV_K^\top X^+y\|_2 + \|\Sigma_Z^{1/2}Z^+WV_{-K}V_{-K}^\top X^+y\|_2.
\]
We bound the first term \( \|\Sigma_Z^{1/2}(Z^+y - \beta)\|_2 \) in (6.4) by \( C\sqrt{K\log(n)/n} \) in Lemma 18 below.
The third term, \( \|\Sigma_Z^{1/2}Z^+WV_{-K}V_{-K}^\top X^+y\|_2 \), turns out to be more difficult to bound and we analyze it separately in the next section 6.1.1. We first bound the second term \( \|\Sigma_Z^{1/2}Z^+WV_KV_K^\top X^+y\|_2 \) in (6.4). Let \( V_A \in \mathcal{O}_{p \times K} \) be the matrix with the left-singular vectors of \( A \) as its columns. We have
\[
\|\Sigma_Z^{1/2}Z^+WV_KV_K^\top X^+y\|_2
\]
\[
= \frac{1}{\sqrt{n}}\|\Sigma_Z^{1/2}Z^+WV_KV_K^\top V_KD_K^{-1}U_K^\top y\|_2
\]
\[
\leq \frac{2}{n\sqrt{p}}\|WV_KV_K^\top\|_{\text{op}}\|D_K^{-1}U_K^\top y\|_2
\]
\[
\leq \frac{2}{n\sqrt{p}} (\|WV_AV_A^\top V_K\|_{\text{op}} + \|W(V_A - V_KV_K^\top)V_K\|_{\text{op}}) \|D_K^{-1}U_K^\top y\|_2
\]
\[
\leq \frac{2}{n\sqrt{p}} (\|WV_A\|_{\text{op}} + \|W\|_{\text{op}}\|V_KV_K^\top - V_AV_A^\top\|_{\text{op}}) \|D_K^{-1}U_K^\top y\|_2.
\]
In first inequality, we used \( \| \Sigma_Z^{1/2}Z^+ \|_{op} = 1/\sigma_K(\Sigma_Z^{-1/2}) \leq 2 \) on \( \mathcal{E}_Z \). The last inequality uses \( \| V_A \|_{op} \leq 1 \) and \( \| V_K \|_{op} \leq 1 \) since \( V_K, V_A \in \mathcal{O}_{p \times K} \). Lemma 14 ensures
\[
\frac{\| D_K^{-1}U_K^T y \|_2}{\sqrt{n^p}} \leq \frac{\| y \|_2}{d_K}\frac{1}{\sqrt{n^p}} \leq \frac{1}{d_K} \leq \sqrt{\frac{1}{\lambda_K}}.
\]
Note that we need \( \lambda_K \gg \kappa \delta_W \) in (4.8) in order to apply this lemma. Invoke Lemma 13 and Lemma 19 (that require \( K \log(n) \ll n \) and \( \delta_W \ll \lambda_K \)), and conclude
\[
\| \Sigma_Z^{1/2}Z^+ W V_K V_K^T X^+ y \|_2 \leq \left( \frac{\| \Sigma_W \|_{op}}{\lambda_K} + \frac{\delta_W}{\lambda_K} \right)^{1/2} \delta_K. \tag{6.5}
\]
The proof of Theorem 8 is completed by collecting the bounds in Lemma 18, inequality (6.5) and inequality (6.6) in Appendix 6.1.1. \qed

### 6.1.1 Bound of \( \| \Sigma_Z^{1/2} Z^+ W V_{-K} V_{-K}^T X^+ y \|_2 \)

We will prove that, for any \( c > 0 \), there exists a finite \( C > 0 \), independent of \( n \), such that
\[
C^{-1} \| \Sigma_Z^{1/2} Z^+ W V_{-K} V_{-K}^T X^+ y \|_2 \leq \frac{\kappa \sqrt{n K \log n}}{r_e(\Sigma_W)} + \sqrt{n \frac{1}{r_e(\Sigma_W)}} + \kappa \left( \frac{\| \Sigma_W \|_{op}}{\lambda_K} \right) \tag{6.6}
\]
holds with probability larger than \( 1 - \mathcal{O}(n^{-c}) \). By \( X^+ = X^T (XX^T)^+ \) and the definition of the event \( \mathcal{E}_Z \), we have
\[
\| \Sigma_Z^{1/2} Z^+ W V_{-K} V_{-K}^T X^+ y \|_2 \leq \frac{2}{n} \| \Sigma_Z^{-1/2} Z^T W V_{-K} V_{-K}^T X^T (XX^T)^+ y \|_2
\]
\[
= \frac{2}{n} \| \Sigma_Z^{-1/2} Z^T W X^T U_{-K} U_{-K}^T (XX^T)^+ y \|_2
\]
\[
\leq I + II
\]
on the event \( \mathcal{E}_Z \), where
\[
I = \frac{2}{n} \| \Sigma_Z^{-1/2} Z^T W A Z^T U_{-K} U_{-K}^T (XX^T)^+ y \|_2,
\]
\[
II = \frac{2}{n} \| \Sigma_Z^{-1/2} Z^T W W^T U_{-K} U_{-K}^T (XX^T)^+ y \|_2.
\]

We bound I and II in the sequel separately. Following Bai (2003), we define
\[
\hat{Z} = \sqrt{n} U_K \in \mathbb{R}^{n \times K}
\]
and
\[
H = \frac{1}{np} A^T A Z^T \hat{Z} D_K^{-2} \in \mathbb{R}^{K \times K},
\]
such that \( Z H = (np)^{-1} Z A^T A Z^T \hat{Z} D_K^{-2} \) is invariant to different parametrizations of \( A \) and \( \Sigma_Z Y \). Note that, on the event \( \mathcal{E}_Z \cap \mathcal{E}_W \) and under the set of assumptions (4.8), the results of Lemma 14 hold, and, in particular,
\[
\sigma_K \left( \Sigma_Z^{1/2} H \right) \gtrsim \sqrt{\frac{\lambda_K}{\lambda_1}}, \quad \sigma_1 \left( H^{-1} A^T \right) \lesssim \sqrt{\lambda_1}. \tag{6.7}
\]
Bound of I: Notice that
\[
I \leq \frac{1}{n} \| \Sigma_Z^{-1/2} Z^T W A (H^T)^{-1} \|_{op} \| H^T Z^T U_{-K} U_{-K}^T (X X^T)^+ y \|_2. \tag{6.8}
\]
We find that, on the event $\mathcal{E}_X$,
\[
\| H^T Z^T U_{-K} U_{-K}^T (X X^T)^+ y \|_2 \\
= \| (Z H - \hat{Z})^T U_{-K} U_{-K}^T (X X^T)^+ y \|_2 \quad \text{since } \hat{Z}^T = \sqrt{n} U_K \\
\leq \| \hat{Z} - Z H \|_{op} \| (X X^T)^+ y \|_2 \quad \text{since } U_{-K} U_{-K}^T \text{ is a projection} \\
\leq \| \hat{Z} - Z H \|_{op} \frac{\| y \|_2}{\sigma_2^2(X)} \quad \text{(6.9)}
\]
on $\mathcal{E}_X$
\[
\leq \sqrt{\frac{\| \Sigma_W \|_{op}}{\lambda_K}} \sqrt{\frac{\lambda_1}{\lambda_K}} \frac{n}{\text{tr}(\Sigma_W)} \quad \text{by Lemma 15}
\]
Lemma 17 and (6.7) ensure that
\[
\frac{1}{n} \| \Sigma_Z^{-1/2} Z^T W A (H^T)^{-1} \|_{op} \leq \frac{1}{n} \| \Sigma_Z^{-1/2} Z^T W V A \|_{op} \| A (H^T)^{-1} \|_{op} \\
= \frac{1}{n} \| \Sigma_Z^{-1/2} Z^T W V A \|_{op} \| A (H^T)^{-1} \|_{op} \tag{6.10}
\]
with probability $1 - O(n^{-c})$. From (6.8), (6.9) and (6.10), we conclude
\[
I \leq C \frac{\lambda_1}{\lambda_K} \frac{\| \Sigma_W \|_{op}}{\text{tr}(\Sigma_W)} \sqrt{n K \log n} = C \frac{\kappa}{r_e(\Sigma_W)} \sqrt{n K \log n} \tag{6.11}
\]
with probability $1 - O(n^{-c})$.

Bound of II: By adding and subtracting $\text{tr}(\Sigma_W) I_n$, we have $I_2 \leq I_1 + I_2$ where
\[
I_1 = \frac{1}{n} \| \Sigma_Z^{-1/2} Z^T (W W^T - \text{tr}(\Sigma_W) I_n) U_{-K} U_{-K}^T (X X^T)^+ y \|_2, \\
I_2 = \frac{\text{tr}(\Sigma_W)}{n} \| \Sigma_Z^{-1/2} Z^T U_{-K} U_{-K}^T (X X^T)^+ y \|_2.
\]
Next, we invoke the definitions of the events $\mathcal{E}_X$, $\mathcal{E}_Z$, and apply Lemma 13 to obtain
\[
I_1 \lesssim \frac{1}{\sqrt{n}} \| W W^T - \text{tr}(\Sigma_W) I_n \|_{op} \| y \|_2 \leq C \sqrt{n \| \Sigma_W \|_{op} \frac{\| y \|_2}{\sigma_2^2(X)}} = C \sqrt{n \frac{\| \Sigma_W \|_{op}}{\text{tr}(\Sigma_W)}} \tag{6.12}
\]
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with probability \(1 - \mathcal{O}(n^{-c})\). Regarding II\(_2\), we observe that
\[
\begin{align*}
\Pi_2 & \leq \frac{\text{tr}(\Sigma_W)}{n} \| \Sigma_{Z}^{-1/2}(H^T)^{-1} \|_{\text{op}} \| H^T Z^T U_{-K} U_{-K}^T (XX^T)^+ y \|_2 \\
& \lesssim \frac{\text{tr}(\Sigma_W)}{n} \sqrt{\lambda_1} \sqrt{\| \Sigma_W \|_{\text{op}}} \sqrt{\frac{\lambda_1}{\lambda_K} \frac{n}{\lambda_K \text{tr}(\Sigma_W)}} \\
& = \kappa \sqrt{\frac{\| \Sigma_W \|_{\text{op}}}{\lambda_K}}. 
\end{align*}
\]
using (6.7) and (6.9)
\[
(6.13)
\]
Collecting the bounds in (6.11), (6.12) and (6.13) yields the desired result. \(\square\)

### 6.1.2 Technical lemmas used in the proof of Theorem 8

**Lemma 13.** Assume \(K \leq n\). With probability \(1 - \exp(-n)\), we have
\[
\frac{1}{n} \sigma_1^2(WV_A) \leq 14\sigma^2 \| \Sigma_W \|_{\text{op}}
\]
Assume \(r_e(\Sigma_W) \geq n\). With probability \(1 - 2\exp(-n)\), we have for some positive, universal constant \(C\),
\[
\| WW^T - \text{tr}(\Sigma_W)I_n \|_{\text{op}} \leq C\sigma^2 \sqrt{n\| \Sigma_W \|_{\text{op}} \text{tr}(\Sigma_W)}.
\]

**Proof.** To prove the first result, recall that \(W = \tilde{W} \Sigma_W^{1/2}\). An application of Lemma 20 gives
\[
\mathbb{P}\left\{ \frac{1}{n} \| WV_A V_A^T W^T \|_{\text{op}} \leq \sigma^2 \left( \sqrt{\frac{\text{tr}(M)}{n}} + \sqrt{6\| M \|_{\text{op}}} \right)^2 \right\} \geq 1 - \exp(-n).
\]
Here \(M = \Sigma_W^{1/2} V_A V_A^T \Sigma_W^{1/2}\). The first claim now follows from \(\text{tr}(M) \leq K \| M \|_{\text{op}}, \| M \|_{\text{op}} \leq \| \Sigma_W \|_{\text{op}}\), the inequality \((x + y)^2 \leq 2x^2 + 2y^2\) and our assumption \(K \leq n\).

For the proof of the second claim, we use a standard discretization argument to obtain
\[
\| WW^T - \text{tr}(\Sigma_W)I_n \|_{\text{op}} = \sup_{u \in \mathcal{S}^{n-1}} u^T (WW^T - \text{tr}(\Sigma_W)I_n) u
\]
\[
\leq 2 \max_{u \in \mathcal{N}_n(1/4)} u^T (WW^T - \text{tr}(\Sigma_W)I_n) u
\]
Here \(\mathcal{N}_n(1/4)\) is a minimal \((1/4)\)-net of \(\mathcal{S}^{n-1}\). It has cardinality \(|\mathcal{N}_n(1/4)| \leq 9^n\) (see, for instance, Lemma 5.4 of Vershynin (2012)). For any fixed \(u \in \mathcal{N}_n(1/4)\), we apply the Hanson-Wright inequality (see, Rudelson and Vershynin (2013)) to find, for any \(t \geq 0\),
\[
\mathbb{P}\left\{ | u^T (WW^T - \text{tr}(\Sigma_W)I_n) u | > t \right\} \leq 2 \exp\left\{ -c \min\left\{ \frac{t^2}{\sigma^4 \text{tr}(\Sigma_W)}, \frac{t}{\sigma^2 \| \Sigma_W \|_{\text{op}}} \right\} \right\}
\]
Here \( c \leq 1 \) is some universal constant. Next, we choose

\[
t = C \sigma^2 \sqrt{n \|W\|_{\text{op}} \text{tr}(W)}
\]

with \( C = \log(9e)/c \geq 1 \) and we take a union bound over \( u \in \mathcal{N}_n(1/4) \) to conclude

\[
\|WW^\top - \text{tr}(W)I_n\|_{\text{op}} \leq \frac{\log(9e)}{c} \sigma^2 \sqrt{n \|W\|_{\text{op}} \text{tr}(W)}
\]

with probability at least

\[
1 - 2|\mathcal{N}_{1/4}| \exp\left\{-c \min\left( nC^2, C \sqrt{\frac{n \text{tr}(W)}{\|W\|_{\text{op}}}} \right) \right\} \geq 1 - 2 \cdot 9^n \exp(-cCn) \
\geq 1 - 2 \exp(-n)
\]

We used our assumption \( r_u(W) \geq n \) in the first inequality. \( \square \)

The following lemma states the rates of the first \( K \) singular values of \( X \) and provides a lower bound for \( \sigma_K(\Sigma_{1/2}^1 H) \).

**Lemma 14.** Assume \( \lambda_K \geq 48\sigma^2 \kappa \delta_W \). On the event \( E_Z \cap E_W \), we have

\[
\frac{1}{2} \sqrt{\lambda_k/p} \leq d_k \leq 4 \sqrt{\lambda_k/p} \quad \forall k \in [K]
\]

\[
\sigma^2_K \left( \Sigma_{1/2}^1 H \right) \geq \frac{\lambda_K}{2\lambda_1} = \frac{1}{2\kappa}
\]

\[
\sigma^2_1 \left( H^{-1} A^\top \right) \leq 4\lambda_1
\]

**Proof.** We work on \( E_z \cap E_W \). For the first claim. For any \( k \in [K] \), we have

\[
d_k = \frac{1}{\sqrt{np}} \sigma_k(X) \geq \frac{1}{\sqrt{np}} [\sigma_k(ZA^\top) - \sigma_1(W)] \quad \text{using Weyl's inequality}
\]

\[
\geq \sqrt{\frac{1}{2p} \sigma_k(\Sigma_{1/2}^1 Z^\top A^\top) - \sqrt{\frac{12\sigma^2_2 \delta_W}{p}}} \quad \text{on } E_Z \cap E_W
\]

\[
\geq \frac{1}{2} \sqrt{\frac{\lambda_k}{p}} \quad \text{since } \lambda_K \geq 48\sigma^2 \delta_W.
\]

Similarly, we also have

\[
d_k \leq \frac{1}{\sqrt{np}} [\sigma_k(ZA^\top) + \sigma_1(W)] \leq 4 \sqrt{\frac{\lambda_k}{p}}.
\]

We bound from below \( \sigma_K(H) \) as follows:

\[
\sigma^2_K(\Sigma_{1/2}^1 H) = \frac{1}{n^2 p^2} \lambda_K \left( D_K^{-2} \hat{Z} \Sigma_Z^{-1/2} \left( \Sigma_{1/2}^1 A^\top A \Sigma_{1/2}^1 \right) \Sigma_Z^{-1/2} \hat{Z} D_K^{-2} \right)
\]

\[
\geq \frac{\lambda_K}{n^2 p^2} \lambda_K \left( D_K^{-2} \hat{Z} A^\top A \hat{Z} D_K^{-2} \right)
\]

\[
= \frac{\lambda_K}{n^2 p^2} \sigma^2_K(D_K^{-2} \hat{Z} A^\top)
\]

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Since
\[ \sigma_K(D_K^{-2}\hat{Z}^\top ZA^\top) \geq \sigma_K(D_K^{-2}\hat{Z}^\top X) - \sigma_1(D_K^{-2}\hat{Z}^\top W) \]
by Weyl’s inequality
\[ \geq n\sqrt{p} \sigma_K(D_K^{-1}) - \sigma_1(W)\sigma_1(D_K^{-2})\sigma_1(\hat{Z}) \]
by \( \hat{Z} = \sqrt{n} U_K \)
\[ \geq \frac{np}{\sqrt{\lambda_1}} - np \sqrt{\frac{12\sigma_2\delta W}{\lambda_K^2}} \]
since \( \sigma_1(\hat{Z}) = \sqrt{n} \)
\[ \geq \frac{1}{2} \frac{np}{\sqrt{\lambda_1}} \]
by \( \lambda_K^2 \geq 48\sigma_2^2\lambda_1\delta W \). (6.14)

We used the first result in the last two steps.

Finally, by analogous arguments, the last result follows from
\[ \sigma_1^2(H^{-1}A^\top) = \lambda_i \left( (\Sigma_Z^{1/2}H)^{-1}\Sigma_Z^{1/2}A\Sigma_Z^{1/2}(H^\top\Sigma_Z^{1/2})^{-1} \right) \]
\[ = \left[ \lambda_K \left( H^\top\Sigma_Z^{1/2}(\Sigma_Z^{1/2}A\Sigma_Z^{1/2})^{-1}\Sigma_Z^{1/2}H \right) \right]^{-1} \]
\[ = n^2p^2 \left[ \lambda_K \left( D_K^{-2}\hat{Z}^\top ZA^\top AZ^\top D_K^{-2} \right) \right]^{-1} \]
\[ = n^2p^2 \left[ \sigma_2^2 \left( D_K^{-2}\hat{Z}^\top ZA^\top \right) \right]^{-1} \]
\[ \leq 4\lambda_1 \]
by (6.14).

This completes our proof. \( \square \)

**Lemma 15.** Assume \( \lambda_K \geq 8\delta_W(1 \vee 6\sigma_2^2\kappa) \), \( K \leq n \) and \( \mathbb{E}(\Sigma_W) \geq n \). On the event \( \mathcal{E}_Z \cap \mathcal{E}_W \), we have, with probability \( 1 - 3\exp(-n) \),
\[ \| \hat{Z} - ZH \|_{\text{op}} \lesssim \sqrt{\frac{n\|\Sigma_W\|_{\text{op}}}{\lambda_K}} \sqrt{\frac{\lambda_1}{\lambda_K}}. \]

**Proof.** We work on the event \( \mathcal{E}_Z \cap \mathcal{E}_W \). First, by the SVD of \( X = UD^\top \sqrt{np} \), we find the following identity
\[ \frac{1}{np}XX^\top \hat{Z} = UD^2U^\top U_K \sqrt{n} = \sqrt{n} U_K D_K^2 = \hat{Z} D_K^2. \]

Further observe that
\[ \frac{1}{np} \left( XX^\top - \text{tr}(\Sigma_W)I_n \right) \hat{Z} = \hat{Z} \left( D_K^2 - \frac{\text{tr}(\Sigma_W)}{np} I_K \right). \]

Define the matrix
\[ J = D_K^2 - \frac{\text{tr}(\Sigma_W)}{np} I_K \]
and note that, by our assumption \( \lambda_K \geq 8\delta_W \),
\[ \sigma_K(J) \geq d_K^2 - \frac{\text{tr}(\Sigma_W)}{np} \]
by the definition of \( J \)
\[ \geq \frac{1}{p} \left( \frac{\lambda_K}{4} - \frac{\text{tr}(\Sigma_W)}{n} \right) \]
using Lemma 14 with \( \lambda_K \geq 48\sigma_2^2\kappa \delta W \)
\[ \geq \frac{\lambda_K}{8p} \]
since \( \lambda_K \geq 8\delta_W \geq 8\text{tr}(\Sigma_W)/n \)
Plugging in $X = ZA^T + W$ and rearranging terms yield

$$\hat{Z} - ZH = \frac{1}{np} \left[ ZA^TW^T + WAZ^T + (WW^T - \text{tr}(\Sigma W)I_n) \right] \hat{Z}J^{-1}.$$  

The previous two displays and the inequality $\|\hat{Z}\|_{op} \leq \sqrt{n}$ (as $U_K \in O_{p \times n}$), further imply

$$\|\hat{Z} - ZH\|_{op} \leq \frac{8}{\lambda_K \sqrt{n}} \left( 2\|ZA^TW\|_{op} + \|WW^T - \text{tr}(\Sigma W)I_n\|_{op} \right).$$

On the event $E_Z$, Lemma 13 yields, with probability $1 - 3\exp(-n)$,

$$\frac{1}{\sqrt{n}}\|Z^T\|_{op} \leq 2\|WA\|_{op} \sigma_1(A^2) \leq 2\sqrt{14\sigma^2 n \lambda_1 \|\Sigma W\|_{op}}$$

and

$$\frac{1}{\sqrt{n}} \|WW^T - \text{tr}(\Sigma W)I_n\|_{op} \leq C\sigma^2 \sqrt{\|\Sigma W\|_{op} \text{tr}(\Sigma W)}.$$  

The result follows after we use $\lambda_1 \geq \lambda_K \geq \delta_W / 8 \geq \text{tr}(\Sigma W)/(8n)$ and collect terms.

### 6.2 Proof of Proposition 11

**Proof.** Set $\Delta\bar{\mu} = (\bar{\mu}_0 + \bar{\mu}_1)/2$ and $\bar{\alpha} = (\alpha_0 + \alpha_1)/2$. We first recall Bing and Wegkamp (2022, Fact 1 in Appendix C) that, for any semi-positive definite matrix $M$, we have

$$\alpha_0^\top M \alpha_0 + \alpha_1^\top M \alpha_1 \leq \max(\pi_0, \pi_1)(\alpha_1 - \alpha_0)^\top M(\alpha_1 - \alpha_0)$$  

(6.15)

and the identity (Bing and Wegkamp, 2022, (A.11) in Appendix A)

$$(\alpha_1 - \alpha_0)^\top \Sigma_Z^{-1}(\alpha_1 - \alpha_0) = \frac{\Delta^2}{1 + \pi_0 \pi_1} \leq \frac{1}{\pi_0 \pi_1}$$  

(6.16)

which is mainly a consequence of Woodbury’s formula. By the triangle inequality, we find

$$|\beta_0 - \beta| \leq |\bar{\alpha}^\top (A^\top \hat{\theta} - \beta) + \beta_0 - \beta| + |\bar{\alpha}^\top (A^\top \hat{\theta} - \beta)|$$

$$\leq |\bar{\alpha}^\top (A^\top \hat{\theta} - \beta) + \beta_0 - \beta| + \frac{1}{\sqrt{\pi_0 \pi_1}} \|A^\top \hat{\theta} - \beta\|_{\Sigma_Z}$$  

(6.17)

The second inequality used

$$|\bar{\alpha}^\top (A^\top \hat{\theta} - \beta)| \leq \frac{1}{2} \left( \|\Sigma_Z^{-1/2} \alpha_1\| + \|\Sigma_Z^{-1/2} \alpha_0\| \right) \|A^\top \hat{\theta} - \beta\|_{\Sigma_Z} \quad \text{by Cauchy-Schwarz}$$

$$\leq \frac{1}{2} \left( \sqrt{\pi_1} + \sqrt{\pi_0} \right) \|\alpha_1 - \alpha_0\|_{\Sigma_Z^{-1/2}} \|A^\top \hat{\theta} - \beta\|_{\Sigma_Z} \quad \text{by (6.15)}$$

$$\leq \frac{1}{\sqrt{\pi_0 \pi_1}} \|A^\top \hat{\theta} - \beta\|_{\Sigma_Z} \quad \text{by (6.16)}.$$
For the first term on the right of (6.17), we notice that \( \alpha^\top (A^\top \hat{\theta} - \beta) + \beta_0 - \beta_0 = R_1 + R_2 \) with
\[
R_1 = (A\alpha - \Delta \tilde{\mu})^\top \hat{\theta} \\
R_2 = \left\{ 1 - (\tilde{\mu}_1 - \tilde{\mu}_0)^\top \hat{\theta} \right\} \pi_0 \pi_1 \log \frac{\pi_0}{\pi_1} - \left\{ 1 - (\alpha_1 - \alpha_0)^\top \beta \right\} \pi_0 \pi_1 \log \frac{\pi_1}{\pi_0}
\]

For \( R_2 \), we notice, using the same tedious calculation as in Bing and Wegkamp (2022, Proof of Lemma 15), that
\[
|R_2| \leq |(\tilde{\mu}_0 - \tilde{\mu}_1)^\top \hat{\theta} - (\alpha_0 - \alpha_1)^\top \beta| + \frac{3}{\pi_0 \wedge \frac{\pi_1}{\pi_0}} \left| \pi_0 - \pi_0 \right|.
\]

For the first term on the right, we have
\[
|(\alpha_0 - \alpha_1)^\top \beta - (\tilde{\mu}_0 - \tilde{\mu}_1)^\top \hat{\theta}|
\leq |(\alpha_0 - \alpha_1)^\top (\beta - A^\top \hat{\theta})| + |(\alpha_0 - \alpha_1)^\top A^\top \hat{\theta} - (\tilde{\mu}_0 - \tilde{\mu}_1)^\top \hat{\theta}|
\leq \|\alpha_1 - \alpha_0\|_{\Sigma_Z^{-1}} \|A^\top \hat{\theta} - \beta\|_{\Sigma_Z} + |(\alpha_0 - \alpha_1)^\top A^\top \hat{\theta} - (\tilde{\mu}_0 - \tilde{\mu}_1)^\top \hat{\theta}|
\leq \frac{1}{\sqrt{\pi_0 \pi_1}} \|A^\top \hat{\theta} - \beta\|_{\Sigma_Z} + |R_1|.
\]

Hence,
\[
|\hat{\alpha}^\top (A^\top \hat{\theta} - \beta) + \beta_0 - \beta_0| \leq \frac{\|A^\top \hat{\theta} - \beta\|_{\Sigma_Z} + 2 |(A\alpha - \Delta \tilde{\mu})^\top \hat{\theta}| + \frac{3|\tilde{\pi}_0 - \pi_0|}{\pi_0 \wedge \frac{\pi_1}{\pi_0}}}{\sqrt{\pi_0 \pi_1}} + 2t.
\]

We have bounded the first term \( \|A^\top \hat{\theta} - \beta\|_{\Sigma_Z} \) in Theorem 8 above. By Hoeffding’s inequality, \( \tilde{\pi}_k \geq \pi_k/2 \) with probability larger than \( 1 - \exp(-n\pi_k^2/8) \), for \( k \in \{0, 1\} \). By further applying Hoeffding’s inequality, we bound the third term on the right in (6.18) by
\[
\frac{6}{\pi_0 \wedge \pi_1} + \frac{2t}{\sqrt{n}}
\]

with probability \( 1 - 2 \exp(-t^2/2) - 2 \exp(-n(\pi_0 \wedge \pi_1)^2/8) \).

We bound the second term on the right in (6.18) as follows. For each \( k \in \{0, 1\} \),
\[
\tilde{\mu}_k = \frac{1}{n_k} \sum_{i=1}^{n'} X_i' \mathbb{1}\{Y_i' = k\} = \frac{1}{n_k} \sum_{i=1}^{n'} (AZ_i' + W_i') \mathbb{1}\{Y_i' = k\} := A\tilde{\alpha}_k + W_k',
\]

hence we can write
\[
|(A\alpha_k - \tilde{\mu}_k)^\top \hat{\theta}| \leq |(\alpha_k - \tilde{\alpha}_k)^\top \beta| + |(\tilde{\alpha}_k - \alpha_k)^\top (A^\top \hat{\theta} - \beta)| + \left| (W_k')^\top \hat{\theta} \right|.
\]

Next, we use the normal assumption (iv), the independence of \( \tilde{\alpha}_k \) and \( \hat{\theta} \) and the fact that \( \Sigma_Z - \Sigma_{Z|Y} = \pi_0 \pi_1 (\alpha_1 - \alpha_0)(\alpha_1 - \alpha_0)^\top \) is positive definite to deduce that the first two terms are bounded by
\[
\frac{t}{\sqrt{n_k}} \left( \|\beta\|_{\Sigma_Z} + \|A^\top \hat{\theta} - \beta\|_{\Sigma_Z} \right)
\]
with probability $1 - 4 \exp(-t^2/2)$. Since $\overline{W}'_{(k)}$ is subGaussian with parameter $\sigma(\hat{\theta}^\top \Sigma \hat{\theta}/\overline{n}_k)^{1/2}$ and is independent of $\hat{\theta}$ and the labels $Y'_1, \ldots, Y'_n$, we have

$$
|\left(\overline{W}'_{(k)}\right)^\top \hat{\theta}| \leq t\sigma\|\hat{\theta}\|_{\Sigma_w}/\sqrt{\overline{n}_k}.
$$

with probability $1 - 2 \exp(-t^2/2)$. Now use the inequality

$$
\|\beta\|_{\Sigma_Z}^2 = (\pi_0 \pi_1)^2 \|\alpha_1 - \alpha_0\|_{\Sigma_Z^{-1}}^2 \leq \pi_0 \pi_1 \leq 1/4,
$$

using (6.16), and the bound

$$
\|\hat{\theta}\|_{\Sigma_w}^2 \leq \|\Sigma_w\|_{op}\|X^+ y\|^2 \leq \|\Sigma_w\|_{op}\|y\|^2/\sigma^2_{\Sigma}(X) \leq \frac{8n}{r_e(\Sigma_W)} \to 0
$$

The last inequality holds with probability at least $1 - 3 \exp(-C'n)$, by Proposition 7, for some $C' > 0$. By Hoeffding’s inequality, $\overline{n}_k \leq n' \pi_k/2$ with probability larger than $1 - \exp(-n' \pi_k^2/8)$. We conclude that for any $c > 0$, the combination of above bounds with $n' \asymp n$, $t = C(\log n)^{1/2}$ for some finite $C = C(c, \pi_0, \pi_1, \sigma)$ large enough, yields

$$
P \left\{ \left| (A\alpha_k - \tilde{\mu}_k)^\top \hat{\theta} \right| \geq C\sqrt{\frac{\log n}{n}} \left( 1 + \|A^\top \tilde{\theta} - \beta\|_{\Sigma_Z} \right) \right\} \lesssim n^{-c} \quad (6.20)
$$

Finally, (6.17), (6.18), (6.19) and (6.20) prove our result.

7 Auxiliary lemmas

We restate the following lemmas which are proved in Bing and Wegkamp (2022).

**Lemma 16.** (Bing and Wegkamp, 2022, Lemma 31) Under assumptions (ii) and (iv) and $K \log n \ll n$, for any constant $c > 0$,

$$
P \left\{ \frac{1}{2} \leq \frac{1}{n} \sigma_K^2(Z \Sigma_Z^{-1/2}) \leq \frac{1}{n} \sigma^2_{\Sigma}(Z \Sigma_Z^{-1/2}) \leq 2 \right\} = 1 - O(n^{-c}).
$$

**Lemma 17.** (Bing and Wegkamp, 2022, Lemma 32) Under assumptions (i) – (v), for any $c > 0$, there exists a $C < \infty$ such that

$$
P \left\{ \frac{1}{n} \|Z^{-1/2}Z^\top W V A\|_{op} \leq C \sqrt{\|\Sigma_W\|_{op} \sqrt{K \log n / n}} \right\} = 1 - O(n^{-c}).
$$

**Lemma 18.** (Bing and Wegkamp, 2022, Lemma 18) Under assumptions (ii) and (iv) and $K \log n \ll n$, for any $c > 0$, there exists a $C < \infty$ such that

$$
P \left\{ \|Z^+ y - \beta\|_{\Sigma_Z} \leq C \sqrt{K \log n / n} \right\} = 1 - O(n^{-c}).
$$

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Lemma 19. (Bing and Wegkamp, 2022, Lemma 21) Under assumptions (i) – (v), $K \log n \ll n$ and $\delta_W \ll \lambda_K$, for any $c > 0$, there exists a $C < \infty$ such that

$$\mathbb{P} \left\{ \left\| V_K V_K^\top - V_A V_A^\top \right\|_{op} \leq C \sqrt{\kappa \delta_W / \lambda_K} \wedge 1 \right\} = 1 - O(n^{-c}).$$

The following lemma provides an upper bound on the operator norm of $G H G^\top$ where $G \in \mathbb{R}^{n \times d}$ is a random matrix and its rows are independent sub-Gaussian random vectors. It is proved in Lemma 22 of Bing et al. (2021).

Lemma 20. Let $G$ be a $n \times d$ matrix with rows that are independent $\sigma$ sub-Gaussian random vectors with identity covariance matrix. Then for all symmetric positive semi-definite matrices $H$,

$$\mathbb{P} \left\{ \frac{1}{n} \| G H G^\top \|_{op} \leq \sigma^2 \left( \sqrt{\frac{1}{n} \text{tr}(H)} + \sqrt{6 \| H \|_{op}} \right)^2 \right\} \geq 1 - \exp(-n).$$

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