An Invariant of Algebraic Curves from the Pascal Theorem*

Zhongxuan Luo†

School of Software, School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China

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Abstract

In 1640’s, Blaise Pascal discovered a remarkable property of a hexagon inscribed in a conic - Pascal Theorem, which gave birth of the projective geometry. In this paper, a new geometric invariant of algebraic curves is discovered by a different comprehension to Pascals mystic hexagram or to the Pascal theorem. Using this invariant, the Pascal theorem can be generalized to the case of cubic (even to algebraic curves of higher degree), that is, For any given 9 intersections between a cubic \( \Gamma_3 \) and any three lines \( a, b, c \) with no common zero, none of them is a component of \( \Gamma_3 \), then the six points consisting of the three points determined by the Pascal mapping applied to any six points (no three points of which are collinear) among those 9 intersections as well as the remaining three points of those 9 intersections must lie on a conic. This generalization differs quite a bit and is much simpler than Chasles’s theorem and Cayley-Bacharach theorems.

Keywords: Algebraic curve; Pascal theorem; Characteristic ratio; Characteristic mapping; Characteristic number; spline.

1 Introduction

Algebraic curve is a classical and an important subject in algebraic geometry. An algebraic plane curve is the solution set of a polynomial equation \( P(x, y) = 0 \), where \( x \) and \( y \) are real or complex variables, and the degree of the curve is the degree of the polynomial \( P(x, y) \). Let \( \mathbb{P}^2 \) be the projective plane and \( \mathbb{P}_n \) be the space of all homogeneous polynomials in homogeneous coordinates \( (x, y, z) \) of total degree \( \leq n \). An algebraic curve \( \Gamma_n \) in the projective plane is defined by the solution set of a homogeneous polynomial equation \( P(x, y, z) = 0 \) of degree \( n \).

In 1640, Blaise Pascal discovered a remarkable property of a hexagon inscribed in a circle, shortly thereafter Pascal realized that a similar property holds for a hexagon inscribed in an ellipse even a conic. As the birth of the projective geometry, Pascal theorem assert: If six points on a conic section is given and a hexagon is made out of them in an arbitrary order, then the points of intersection of opposite sides of this hexagon will all lie on a single line. The generalizations of Pascal’s theorem have a glorious history. It has been a subject of active and exciting research. As generalizations of the Pascal theorem, Chasles’s theorem and Cayley-Bacharach theorems in various versions received a great attention both in algebraic geometry and in multivariate interpolation. A detailed introduction to Cayley-Bacharach theorems as well as conjectures can be found in \[16, 19, 30\].

The Pascal theorem can be comprehended in the following several aspect: first, it is easy to verify that Pascal’s theorem can be proved by Chasles’s theorem \[16\] and therefore, probably, Chasles’s theorem has been regarded as a generalization of Pascal’s theorem in the literature. However, the Chasles’s theorem and Cayley-Bacharach theorems have not formally inherited the appearance of the Pascal theorem, that is the three points joined a line are obtained from intersections of three pair of

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†Corresponding author: zxluo@dlut.edu.cn
lines in which each line was determined by two points lying on a conic; Secondly, the Pascal theorem can be used to (geometrically) judge whether or not any six points simultaneously lie on a conic. Another interesting observation to the Pascal theorem is that it plays a key role in revealing the instability of a linear space $S^1_2(\Delta_{MS})$ (the set of all piecewise polynomials of degree 2 with global smoothness 1 over Morgan-Scott triangulation, see figure 5.1 and refer to Appendix 5.1). That is, the Pascal theorem gives an equivalent relationship between the algebraic and geometric conditions to the instability of $S^1_2(\Delta_{MS})$ (refer to Appendix 5.1).

Actually, readers will see that the Pascal theorem contains a geometric invariant of algebraic curves, which is exactly reason that we take up the Pascal theorem in this paper. In order to get this new invariant of algebraic curves, one must observe the Pascal theorem from a different viewpoint in which “arbitrary six points are given by intersections of a conic and any three lines without no common zero” instead of “six points on a conic section is given” (a historical viewpoint) in the Pascal theorem. This slight different comprehension to the Pascal theorem makes us easily generalize the Pascal theorem to algebraic curves of higher degrees and discover an invariant of algebraic curves. Similar to the source of this paper in which all involved points are the set of intersections between lines and a curve, [20] has given some interesting results to the special case of the following classical problem: Let $X$ be the intersection set of two plane algebraic curves $D$ and $E$ that do not share a common component. If $d$ and $e$ denote the degrees of $D$ and $E$, respectively, then $X$ consists of at most $d \cdot e$ points (a week form of Bezout’s theorem [34]). When the cardinality of $X$ is exactly $d \cdot e$, $X$ is called a complete intersection. How does one describe polynomials of degree at most $k$ that vanish on a complete intersection $X$ or on its subsets? The case in which both plane curves $D$ and $E$ are simply unions of lines and the union $D \cup E$ is the $(d \times e)$-cage in question.

Our main results in this paper are enlightened by studying the instability of spline space and are proved by spline method and the “principle of duality ” in the projective plane. This paper is organized as follows: In section 2, some basic preliminaries of the projective geometry are given. In section 3, some new concepts such as characteristic ratio, characteristic mapping and characteristic number of algebraic curve are introduced by discussing the properties of a line and a conic. Section 4 gives our main results for the invariant of cubic and presents a generalization of the Pascal type theorem to cubic. Moreover, some corresponding conclusions to the case of algebraic curves of higher degrees ($n > 3$) are also stated in this section without proofs. The basic theory of bivariate spline , a series of results on the singularity of spline space and the proof of the main result of this paper are given in Appendix in the end of the paper.

2 Preliminaries of Projective Geometry

It is well known that the “homogeneous coordinates” and the “principle of duality”[1] are the essential tools in the projective geometry. A point is the set of all triads equivalent to given triad $(x) = (x_1, x_2, x_3)$, and a line is the set of all triads equivalent to given triad $[X] = [X_1, X_2, X_3]$. By a suitable multiplication (if necessary), any point in the projective plane can be expressed in the form $(x_1, x_2, 1)$, which can be shortened to $(x_1, x_2)$, and the two numbers $x_1$ and $x_2$ are called the affine coordinates. In other words, if $x_3 \neq 0$, the point $(x_1, x_2, x_3)$ in the projective plane can be regarded as the point $(x_1/x_3, x_2/x_3)$ in the affine plane. The “principle of duality” in the projective plane can be seen clearly from the following result: “three points $(u), (v)$ and $(w)$ in $P^2$ are collinear” is equivalent to ”three lines $[u], [v]$ and $[w]$ in $P^2$ are concurrent”. In fact, the necessary and sufficient condition for the both statements is: there are numbers $\lambda, \mu, \nu$, not all zero, such that $\lambda u_i + \mu v_i + \nu w_i = 0 (i = 1, 2, 3)$, namely,

$$
\begin{vmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
 w_1 & w_2 & w_3
\end{vmatrix} = 0.
$$

If $(u), (v)$ are distinct points, $\nu \neq 0$. Hence the general point collinear with $(u)$ and $(v)$ can be formed a linear combination of $(u)$ and $(v)$. In other word, a point $(u) = (u_1, u_2, u_3) \in P^2$ corresponds uniquely

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[1] Poncelet claimed this principle as his own discovery, but its nature was more clearly understood by another Franchman, J. D. Gergonne(1771-1859) [11].

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to a line \([u] = [u_1, u_2, u_3] : u_1x + u_2y + u_3z = 0\), while a line \([u] = [u_1, u_2, u_3] : u_1x + u_2y + u_3z = 0\) corresponds uniquely a point \((u) = (u_1, u_2, u_3)\). We say that a point \((u)\) and the corresponding line \([u]\) are dual to each other - which is the two-dimensional “principle of duality”. Under this duality, it follows the following definition.

**Definition 2.1 (Duality of planar figure).** Let \(\Delta\) be a planar figure consisting of lines and points in the projective plane. A planar figure obtained by the corresponding dual lines and points of the points and lines in \(\Delta\) respectively is called the Dual figure of \(\Delta\), denotes by \(\Delta^*\).

For instance, the dual figure of Fig. 1 is shown in Fig. 2 where \([·]\) represents the corresponding dual line of the point \((·)\) in Fig. 1.

3 New Definitions

In what follows, we shall use \(u\) to represent a point \((u)\) or a line \([u]\) when no ambiguities exist, \(u = <a, b>\) for the intersection point of lines \(a\) and \(b\), and \(a = (u, v)\) for the line which joins the points \(u\) and \(v\).

First, we review the following properties of a line and a conic. Suppose a line \(l\) be cut by any three lines \(a, b\) and \(c\) with no common zero (see Fig. 3). Let \(u = <c, a>\), \(v = <a, b>\) and \(w = <b, c>\), \(P = <l, a>\), \(Q = <l, b>\) and \(R = <l, c>\). Obviously, there exist numbers \(a_i, b_i\) \((i = 1, 2, 3)\) such that \(P = a_1u + b_1v, Q = a_2v + b_2w, R = a_3w + b_3u\), provided in turn \(u, v, w\), then we have

**Proposition 3.1.**

\[
\frac{b_1}{a_1} \cdot \frac{b_2}{a_2} \cdot \frac{b_3}{a_3} = -1.
\]

**Proof.** Without loss of generality, we assume that \(u = (1, 0, 0), v = (0, 1, 0)\) and \(w = (0, 0, 1)\). Since \(P, Q\) and \(R\) are collinear, hence

\[
\begin{vmatrix}
0 & a_1 & b_1 \\
0 & a_2 & b_2 \\
0 & a_3 & b_3
\end{vmatrix} = 0.
\]

It follows that \(\frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3} = -1\).
With the same notations, it follows that the necessary and sufficient condition for \( P, Q \) and \( R \) to be collinear is \( \frac{b_1}{a_1} \cdot \frac{b_2}{a_2} \cdot \frac{b_3}{a_3} = -1 \).

Now let us replace the line \( l \) in proposition 3.1 by a conic \( \Gamma \). There are two intersections between \( \Gamma \) and each \( a, b, c \). Let \( \{ p_1, p_2 \} = < \Gamma, a > \), \( \{ p_3, p_4 \} = < \Gamma, b > \) and \( \{ p_5, p_6 \} = < \Gamma, c > \). Consequently, there are real numbers \( \{ a_i, b_i \}_{i=1}^6 \) such that

\[
\begin{align*}
\begin{cases}
p_1 = a_1u + b_1v \\
p_2 = a_2u + b_2v \\
p_3 = a_3v + b_3w \\
p_4 = a_4v + b_4w \\
p_5 = a_5w + b_5u \\
p_6 = a_6w + b_6u \end{cases}
\end{align*}
\]

(3.1)

We have

**Theorem 3.2.** Let a conic be cut by any three lines with no common zero. Under the notations above, we have

\[
\frac{b_1b_2}{a_1a_2} \cdot \frac{b_3b_4}{a_3a_4} \cdot \frac{b_5b_6}{a_5a_6} = 1.
\]

(3.2)

**Proof.** Let \( u = < c, a >, v = < a, b >, w = < b, c > \). Notice that the duality of the figure composed of the points \( \{ p_i \}_{i=1}^6, u, v, w \) and the lines \( a, b, c \) turns out a planar figure with a structure of Morgan-Scott triangulation with inner edges consists of the dual lines of the points \( \{ p_i \}_{i=1}^6 \) and each \( u, v, w \) (see Fig. 6). Note that the six points \( \{ p_i \}_{i=1}^6 \) lie on a conic, it is shown from Theorem 5.4 (see appendix 5.1) that the spline space \( S^2_1(\Delta_{MS}) \) (the set of all piecewise polynomial of degree 2 with smoothness 1 over Morgan-Scott triangulation \( \Delta_{MS} \)) is singular, that is \( \dim S^2_1(\Delta_{MS}) = 7 \). Which implies from Theorem 5.5 (see appendix 5.1) that Theorem 3.3 thus follows.

On the other hand, Theorem 3.2 can be used to tell whether or not any six points simultaneously lie on a conic. In fact, let \( p_i \in \mathbb{P}^2(i = 1, 2, \ldots, 6) \) be any six distinct points without any three points are collinear, \( a = (p_1, p_2), b = (p_3, p_4), c = (p_5, p_6) \), and \( u = < c, a >, v = < a, b >, w = < b, c > \). Using the same notations as in (3.2), it follows from the proof of Theorem 3.2 that

**Proposition 3.3.** For any given six points \( p_1, p_2, \ldots, p_6 \) without no three points are collinear, (3.2) is a necessary and sufficient condition for those six points to be lying on a conic.

Actually, Theorem 3.2 is equivalent to the Pascal theorem.

**[Proof of Pascal theorem.]** Let \( p_i \in \mathbb{P}^2(i = 1, 2, \ldots, 6) \) be any six distinct points without any three points are collinear. Denoted by \( a = (p_1, p_2), b = (p_3, p_4), c = (p_5, p_6) \) and \( u = < c, a >, v = < a, b >, w = < b, c > \). Without loss of generality, we assume \( u = (1, 0, 0), v = (0, 1, 0) \) and \( w = (0, 0, 1) \). Since the 6 points \( p_i \in \mathbb{P}^2(i = 1, 2, \ldots, 6) \) lie on a conic, (3.2) holds. It is clear that

\[
\begin{align*}
q_1 &= < (p_1, p_2), (p_4, p_5) > = (b_1b_5, -a_4a_5, 0) = b_4b_5u - a_4a_5w, \\
q_2 &= < (p_2, p_3), (p_5, p_6) > = (a_2a_3, 0, -b_2b_3) = b_2b_3v + a_2a_3u, \\
q_3 &= < (p_3, p_4), (p_1, p_6) > = (0, -b_1b_6, a_1a_6) = -b_1b_6w + a_1a_6v,
\end{align*}
\]

and (3.2) is equivalent to

\[
\left( -\frac{b_1b_5}{a_1a_6} \right) \cdot \left( -\frac{b_2b_3}{a_2a_3} \right) \cdot \left( -\frac{b_4b_5}{a_4a_5} \right) = -1.
\]

(3.3)

By Proposition 3.1, three points \( \{ q_1, q_2, q_3 \} \) must be collinear. This is the conclusion of the Pascal theorem.

Notice that Proposition 3.1 and Theorem 3.2, -1 and 1 are invariants of line and conic respectively. We therefore introduce the following definitions.

**Definition 3.4 (Characteristic ratio).** Let \( u, v \in \mathbb{P}^2 \) be two distinct points (or lines), \( p_1, p_2, \ldots, p_k \) be points (or lines) on the line \( (u, v) \) (or passing through \( < u, v > \)), then there are numbers \( a_i, b_i \) such that \( p_i = a_iu + b_iv, i = 1, 2, \ldots, k \). The ratio

\[
[u, v; p_1, \ldots, p_k] := \frac{b_1b_2 \cdots b_k}{a_1a_2 \cdots a_k}
\]

is called the **Characteristic ratio** of \( p_1, p_2, \ldots, p_k \) with respect to the basic points (or lines) \( u, v \). If there are multiple points in the intersection points, the corresponding characteristic ratio is defined by their limit form.
Remark 3.5. For four collinear points \(u, v, p_1, p_2\), while the Characteristic ratio of \(p_1, p_2\) with respect to \(u, v\) is \(\frac{b_{p_1}}{a_{p_2}}\), the cross ratio in the projective geometry is defined as \(\frac{a_{p_1}}{a_{p_2}}\).

Definition 3.6 (Characteristic mapping). Let \(u\) and \(v\) be two distinct points, and the line \((u, v)\) join the points \(p\) and \(q\). We call \(q\) (or \(p\)) the characteristic mapping point of \(p\) (or \(q\)) with respect to the basic points \(u\) and \(v\) if
\[
[u, v; p, q] = 1,
\]
and denote \(q = \chi_{(u,v)}(p)\) (or \(p = \chi_{(u,v)}(q)\)).

Apparently if \(q\) is the characteristic mapping point (or line) of \(p\), then \(p\) is the characteristic mapping of \(q\) as well. That is, the characteristic mapping is reflexive, i.e., \(\chi_{(u,v)} \circ \chi_{(u,v)} = I\) (identity mapping). Geometrically, \(\chi_{(u,v)}(p)\) and \(p\) are symmetric with respect to the mid-point of \(u\) and \(v\).

From the definition of the characteristic mapping, Proposition 3.1 and Theorem 3.2, the property of the characteristic mapping can be shown in the following corollaries.

Corollary 3.7. Any three points \(P, Q\) and \(R\) in the projective plane \(\mathbb{P}^2\) are collinear if and only if their characteristic mapping points \(\chi_{(u,v)}(P), \chi_{(v,u)}(Q)\) and \(\chi_{(w,u)}(R)\) are collinear.

Corollary 3.8. Any six distinct points \(p_i \in \mathbb{P}^2(i = 1, 2, \cdots, 6)\) lie on a conic if and only if the image of their characteristic mapping \(\chi_{(u,v)}(p_1), \chi_{(u,v)}(p_2), \chi_{(v,u)}(p_3), \chi_{(v,u)}(p_4), \chi_{(w,u)}(p_5)\) and \(\chi_{(w,u)}(p_6)\) lie on a conic as well.

Bezout’s theorem [35] says that two algebraic curves of degree \(r\) and \(s\) with no common components have exactly \(r \cdot s\) points in the projective complex plane. In particular, a line \(L\) and an algebraic curve \(C\) of degree \(n\) without the component \(L\) meet in exactly \(n\) points in the projective complex plane.

Definition 3.9 (Characteristic number). Let \(\Gamma_n\) be an algebraic curve of degree \(n\), and \(a, b, c\) be any three distinct lines (without common zero) where none of them is a component of \(\Gamma_n\). Suppose that there exist \(n\) intersections between the each line and \(\Gamma\), and denoted by \(\{p_i^{(a)}, p_i^{(b)}, p_i^{(c)}\}_{i=1}^{n}\) the intersections between \(\Gamma_n\) and the lines \(a, b, c\), respectively. Let \(u = < c, a >, v = < a, b >, w = < b, c >\). The number
\[
K_n(\Gamma_n) := [u, v; p_1^{(a)}, \cdots, p_n^{(a)}) \cdot [v, w; p_1^{(b)}, \cdots, p_n^{(b)}] \cdot [w, u; p_1^{(c)}, \cdots, p_n^{(c)}],
\]
independent of \(a, b\) and \(c\) (See Theorem 4.4 below), is called the characteristic number of algebraic curve \(\Gamma_n\) of degree \(n\).

It is obvious from the Definition 3.9 that if \(\Gamma_n\) is a reducible curve of degree \(n\) and has components \(\Gamma_{n_1}\) and \(\Gamma_{n_2}\), \(n = n_1 + n_2\), then \(K_n(\Gamma_n) = K_{n_1}(\Gamma_{n_1}) \cdot K_{n_2}(\Gamma_{n_2})\). From the discussion below the Characteristic number is a global invariant of algebraic curves.

By Definition 3.9, the characteristic numbers of line and conic are \(-1\) and \(+1\) respectively.

Definition 3.10 (Pascal mapping). For any 6 points \(p_1, p_2, \cdots, p_6\) without any three points are collinear in the projective plane, first define \(\Phi\) by
\[
\Phi(\{p_1, p_2, \cdots, p_6\}) = \{q_1, q_2, q_3\},
\]
where \(q_1 = < (p_1, p_2), (p_4, p_5) >, q_2 = < (p_2, p_3), (p_5, p_6) >\) and \(q_3 = < (p_3, p_4), (p_6, p_1) >\) (i.e. \(\{q_i\}_{i=1}^{3}\) are the three pairs of the continuations of opposite side of the hexagon determined by \(\{p_i\}_{i=1}^{6}\)). Then the Pascal mapping \(\Psi\) to \(\{p_1, p_2, \cdots, p_6\}\) is defined by
\[
\Psi(p_1, p_2, \cdots, p_6) := \chi \circ \Phi(p_1, p_2, \cdots, p_6) := \{\chi_{(u,v)}(q_1), \chi_{(u,v)}(q_2), \chi_{(v,u)}(q_3)\},
\]
where \(u = < (p_1, p_2), (p_5, p_6) >, v = < (p_1, p_2), (p_3, p_4) >\) and \(w = < (p_3, p_4), (p_5, p_6) >\).

Notice that the Pascal mapping on \(p_1, p_2, \cdots, p_6\) giving above depends on the order of \(\{p_i\}_{i=1}^{6}\). One can also define the Pascal mapping on \(p_1, p_2, \cdots, p_6\) by \(u = < (p_2, p_3), (p_1, p_5) >, w = < (p_4, p_5), (p_1, p_6) >\) and \(v = < (p_2, p_3), (p_1, p_6) >\) instead, which will not affect the result of the Pascal theorem (Theorem 3.11) giving below. But for the case of higher degrees as stated below, we must insist on \(u, v\) and \(w\) being defined as in the definition above.
\[ \Gamma = \{ a, b >, w = \} \]

Proof. \( a \) conic.

Theorem 4.3.

In this section, we show our main results on the characteristic number to algebraic curves. Consequently, a generalization of the Pascal theorem to curves of higher degree is given by the “principle of duality” and the spline method.

4 Invariant and Pascal Type Theorem

In this section, we show our main results on the characteristic number to algebraic curves. Consequently, a generalization of the Pascal theorem to curves of higher degree is given by the “principle of duality” and the spline method.

For cubic, we have

**Theorem 4.1.** The characteristic number of cubic is \(-1\), that is, \( \mathcal{K}_3(\Gamma_3) = -1 \).

**Proof.** See Appendix 5.2. \( \Box \)

Let \( a, b \) and \( c \) be any three distinct lines with no common zero in the projective plane, denoted by \( u = c, a >, v = a, b >, w = b, c > \). Assume that \( p_1, p_2, p_3 \) are three points on \( a, p_4, p_5, p_6 \) are on \( b, \) and \( p_7, p_8, p_9 \) are on \( c, \) then there exist real numbers \( a_i, b_i, i = 1, 2, \ldots, 9 \) such that

\[
\begin{align*}
\{ p_1 & = a_1 u + b_1 v \\
\{ p_2 & = a_2 u + b_2 v \\
\{ p_3 & = a_3 u + b_3 v \\
\{ p_4 & = a_4 v + b_4 w \\
\{ p_5 & = a_5 v + b_5 w \\
\{ p_6 & = a_6 v + b_6 w \\
\{ p_7 & = a_7 w + b_7 u \\
\{ p_8 & = a_8 w + b_8 u \\
\{ p_9 & = a_9 w + b_9 u,
\end{align*}
\]

(4.1)

Similar to Proposition 3.3, one can easily show, following the proof of Theorem 4.1, that

**Proposition 4.2.** The nine points \( p_1, p_2, \ldots, p_9 \) lie on a cubic which differs from \( a \cdot b \cdot c = 0 \) if and only if

\[
\frac{b_1 b_2 b_3}{a_1 a_2 a_3} \cdot \frac{b_4 b_5 b_6}{a_4 a_5 a_6} \cdot \frac{b_7 b_8 b_9}{a_7 a_8 a_9} = -1,
\]

(4.2)

holds.

Now, from Proposition 4.2, we provide in the following a new generalization of the Pascal theorem to cubic.

**Theorem 4.3.** For any given 9 intersections between a cubic \( \Gamma_3 \) and any three lines \( a, b, c \) with no common zero, none of them is a component of \( \Gamma_4 \), then the six points consisting of the three points determined by the Pascal mapping applied to any six points (no three points of which are collinear) among those 9 intersections as well as the remaining three points of those 9 intersections must lie on a conic.

**Proof.** Let \( \{ p_1, p_2, p_7 \} = \Gamma_3 \cap a, \{ p_3, p_4, p_8 \} = \Gamma_3 \cap b, \{ p_5, p_6, p_9 \} = \Gamma_3 \cap c, \) and \( u = c, a >, v = a, b >, w = b, c > \). Without loss of generality, we assume that \( u = (1, 0, 0), v = (0, 1, 0) \) and \( w = (0, 0, 1) \). It is shown in Theorem 4.1 that those 9 points \( \{ p_i \}_{i=1}^9 \in \mathbb{P}^2 \) lying on a cubic implies \( \mathcal{K}_3(\Gamma_3) = -1 \), or equivalently, (4.2) holds. Notice that Fig. 4 illustrates the Pascal mapping and the Pascal theorem that the three points \( \chi_{(u, v)}(q_1) \cdot \chi_{(u, w)}(q_2) \cdot \chi_{(v, w)}(q_3) \) are derived by applying the Pascal mapping to \( p_1, p_2, \ldots, p_6 \). From Corollary 3.7, we state the following version of Pascal theorem in order to generalize it to cubic.

**Theorem 3.11** (Pascal theorem). For given 6 points \( p_1, p_2, \ldots, p_6 \) on a conic section, the 3 points of image of the Pascal mapping on these six points, \( \Psi(\{ p_i \}_{i=1}^6) \), will all lie on a single line.
\[ q_1 = \langle p_1, p_2 \rangle, (p_4, p_5) = (b_4 b_5, -a_4 a_5, 0) = b_4 b_5 u - a_4 a_5 v, \]
\[ q_2 = \langle (p_2, p_3) \rangle, (p_5, p_6) = (a_2 a_3, 0, -b_2 b_3) = -b_2 b_3 w + a_2 a_3 u, \]
\[ q_3 = \langle (p_1, p_6) \rangle, (p_3, p_4) = (0, -b_1 b_6, a_1 a_6) = -b_1 b_6 v + a_1 a_6 w, \]

So applying the Pascal mapping on \( p_1, p_2, p_3, p_4, p_5, p_6 \), we have
\[
\{ \chi(u, v)(q_1), \chi(w, u)(q_2), \chi(v, w)(q_3) \} = \{-a_4 a_5 u + b_4 b_5 v, a_2 a_3 w - b_2 b_3 u, a_1 a_6 v - b_1 b_6 w\}.
\]

Since (4.2) is equivalent to
\[
\left( \frac{b_2 b_5}{-a_4 a_5} \right) \cdot \frac{b_2}{a_7} \cdot \frac{-b_1 b_6}{a_8} \cdot \frac{b_4 b_5}{a_9} = 1. \tag{4.3}
\]
Thus by Theorem 3.2 and Proposition 3.3, the six points \( \{ \chi(u, v)(q_1), \chi(w, u)(q_2), \chi(v, w)(q_3), p_7, p_8, p_9 \} \) must lie on a conic.

![Figure 5: Generalization of Pascal Theorem](image)

Theorem 4.3 implies that if \( p_1, p_2, \ldots, p_9 \) are intersection points between a cubic and any three distinct lines where none of them is a component of the cubic (see Fig. 5), the three points \( \chi(u, v)(q_1), \chi(w, u)(q_2), \chi(v, w)(q_3) \) along with \( p_7, p_8, p_9 \) will lie on a conic. Obviously, this is an intrinsic property of cubic!

Here, let us give an example to illustrate the Pascal type theorem 4.3. Let a cubic \( \Gamma_3 \) be given by
\[
-1120x^3 + 560x^2 y - 60xy^2 + 1008y^3 - 450xyz + 1200yz^2 + 580xz^2 - 1514yz^2
-729z^3 = 0,
\]
and three lines \( a : x + z = 0, b : -y + z = 0 \) and \( c : -x + z = 0 \) be given. Then the 9 intersections between \( \Gamma_3 \) and \( a, b, c \) are
\[
p_1 = (-4, -1, 4), \quad p_2 = (-1, -\frac{3}{2}, 1), \quad p_3 = (\frac{1}{4}, 1, 1),
\]
\[
p_4 = (\frac{1}{4}, 1, 1), \quad p_5 = (1, -\frac{3}{2}, 1), \quad p_6 = (1, -\frac{3}{4}, 1),
\]
\[
p_7 = (2, -1, -2), \quad p_8 = (\frac{1}{2}, 1, 1), \quad p_9 = (1, \frac{47}{42}, 1),
\]
and \( u = (0, -1, 0), v = (-1, 1, 1), w = (1, 1, 1) \). By direct computation, we have

\[
q_1 = (p_1, p_2), (p_4, p_5) = (-1, 1/2, 1),
q_2 = (p_2, p_3), (p_5, p_6) = (1, 1/2, 1),
q_3 = (p_1, p_6), (p_3, p_4) = (-6, 1, 1)
\]

and consequently

\[
\chi(u, v)(q_1) = (-1, 1/3, 1), \chi(w, u)(q_2) = (1, 1/3, 1), \chi(v, w)(q_3) = (6, 1, 1).
\]

It is easy to verify that the six points \( \chi(u, v)(q_1), \chi(v, u)(q_2), \chi(w, v)(q_3) \) as well as \( p_7, p_8, p_9 \) lie on a conic:

\[
4x^2 + 39xy - 126y^2 - 65xz + 312yz - 174z^2 = 0.
\]

In general, for algebraic curves of degree \( n(n \geq 3) \), we have proved the invariant of algebraic curves and the Pascal type theorem to higher degrees. They are listed in the paper without proofs.

**Theorem 4.4.** For any algebraic curve \( \Gamma_n \) of degree \( n \), its characteristic number \( K_n(\Gamma_n) \) is always equal to \((-1)^n\).

With this invariant, we may formulate a Pascal type Theorem for algebraic curves of higher degrees:

**Theorem 4.5** (Pascal type Theorem). Let \( a, b, c \) be any three distinct lines in the projective plane, and \( \{ p_i^{(a)} \}_{i=1}^n, \{ p_i^{(b)} \}_{i=1}^n, \{ p_i^{(c)} \}_{i=1}^n \) be given \( n \) points lying on \( a, b \) and \( c \), respectively. Then those \( 3n \) points \( \{ p_i^{(a)} \}_{i=1}^n, \{ p_i^{(b)} \}_{i=1}^n, \{ p_i^{(c)} \}_{i=1}^n \) lie on an algebraic curve of degree \( n \) if and only if the \( 3(n-1) \) points consisting of the three points determined by the Pascal mapping applied to any \( 3n \) points (no three points of which are collinear) among those \( 3n \) intersections as well as the remaining \( 3(n-2) \) points of those \( 3n \) intersections must lie on an algebraic curve of degree \( n-1 \) as well.

In view of the simplicity of the invariant, some known results of algebraic curves (see [35], pp.123) can be easily understood from our invariant (the characteristic number).

**Theorem 4.6.** If a line cuts a cubic in three distinct points, the residual intersections of the tangents at these three points are collinear.

**Proof.** Let \( l \) be a line cutting a given cubic \( \Gamma_3 \) at three points \( p_1, p_2, p_3 \) and \( l_1, l_2, l_3 \) be the three tangents at these points respectively. Denote by \( q_1, q_2, q_3 \) the residual intersections between \( \Gamma_3 \) and \( l_1, l_2, l_3 \), respectively. Let \( u = (l_2, l_3), v = (l_1, l_2), w = (l_3, l_1) \). Then there are real numbers \( \{a_i, b_i, c_i, d_i\}^n_{i=1} \) such that \( p_1 = a_1v + b_1w, p_2 = a_2u + b_2v, p_3 = a_3w + b_3u \) and \( q_1 = c_1v + d_1w, q_2 = c_2u + d_2v, q_3 = c_3w + d_3u \). From Theorem 4.1 and Proposition 4.2, we have

\[
\frac{b_1}{a_1} \cdot \frac{c_1}{d_1} \cdot \frac{b_2}{a_2} \cdot \frac{c_2}{d_2} \cdot \frac{b_3}{a_3} \cdot \frac{c_3}{d_3} = -1.
\]

Since \( p_1, p_2, p_3 \) are collinear, then \( \frac{b_1}{a_1} \cdot \frac{b_2}{a_2} \cdot \frac{b_3}{a_3} = -1 \). Hence, we have \( \frac{c_1}{d_1} \cdot \frac{c_2}{d_2} \cdot \frac{c_3}{d_3} = -1 \), and those three points \( q_1, q_2, q_3 \) are collinear.

**Theorem 4.7.** A line joining two flexes of a cubic passes through a third flexes.

**Proof.** Let \( p_1, p_2, p_3 \) be three flexes of a cubic, and \( l_1, l_2, l_3 \) be the three tangents at these points. Let \( u = (l_2, l_3), v = (l_1, l_2), w = (l_3, l_1) \). Then there are real numbers \( \{a_i, b_i\}^n_{i=1} \) such that \( p_1 = a_1v + b_1w, p_2 = a_2u + b_2v, p_3 = a_3w + b_3u \). From Theorem 4.1, we have \( \frac{b_1}{a_1} \cdot \frac{b_2}{a_2} \cdot \frac{b_3}{a_3} = -1 \). Hence \( \frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} = -1 \). Which implies \( p_1, p_2, p_3 \) are collinear.

Similar to the proofs of Theorem 4.6 and Theorem 4.7, the following theorem can be also easily proved by using the invariant that we found.

**Theorem 4.8.** If a conic is tangent to a cubic at three distinct points, the residual intersections of the tangents at these points are collinear.
5 Appendix

5.1 Bivariate Spline Space over Triangulations

It is well known that spline is an important approximation tool in computational geometry, and it is widely used in CAGD, scientific computations and many fields of engineering. Splines, i.e., piecewise polynomials, forms linear spaces that have a very simple structure in univariate case. However, it is quite complicated to determine the structure of a space of bivariate spline over arbitrary triangulation.

Bivariate spline is defined as follows [37]:

**Definition 5.1.** Let \( \Omega \) be a given planar polygonal region and \( \Delta \) be a triangulation or partition of \( \Omega \), denoted by \( T_i, i = 1, 2, \ldots, V \), called cells of \( \Delta \). For integer \( k > \mu \geq 0 \), the linear space

\[
S_k^\mu(\Delta) := \{ s | s|_{T_i} \in \mathbb{P}_k, s \in C^\mu(\Omega), \forall T_i \in \Delta \}
\]

is called the spline space of degree \( k \) with smoothness \( \mu \), where \( \mathbb{P}_k \) is the polynomial space of total degree less than or equal to \( k \).

From the Smoothing Cofactor method [37], the fundamental theorem on bivariate splines was established.

**Theorem 5.2.** \( s(x, y) \in S_k^\mu(\Delta) \) if and only if the following conditions are satisfied:

1. For each interior edge of \( \Delta \), which is defined by \( \Gamma_i : l_i(x, y) = 0 \), there exists a so-called smoothing cofactor \( q_i(x, y) \), such that

\[
p_{i1}(x, y) - p_{i2}(x, y) = l_i^{\mu+1}(x, y)q_i(x, y),
\]

where the polynomials \( p_{i1}(x, y) \) and \( p_{i2}(x, y) \) are determined by the restriction of \( s(x, y) \) on the two cells \( \Delta_{i1} \) and \( \Delta_{i2} \) with \( \Gamma_i \) as the common edge and \( q_i(x, y) \in \mathbb{P}_{k-(\mu+1)} \).

2. For any interior vertex \( v_j \) of \( \Delta \), the following conformity conditions are satisfied

\[
\sum (l_i^{(j)}(x, y))^{\mu+1}q_i^{(j)}(x, y) \equiv 0,
\]

where the summation is taken on all interior edges \( \Gamma_i^{(j)} \) passing through \( v_j \), and the sign of the smoothing cofactors \( q_i^{(j)} \) are refixed in such a way that when a point crosses \( \Gamma_i^{(j)} \) from \( \Delta_{i1} \) to \( \Delta_{i2} \), it goes around \( v_j \) counter-clockwisely.

From Theorem 5.2, the dimension of the space \( S_k^\mu(\Delta) \) can be expressed as

\[
\dim S_k^\mu(\Delta) = \left( \begin{array}{c} k + 2 \\ 2 \end{array} \right) + \tau,
\]

where \( \tau \) is the dimension of the linear space defined by the conformity conditions (5.1).

However, for an arbitrary given triangulation, the dimension of these spaces depends not only on the topology of the triangulation, but also on the geometry of the triangulation. In general cases, no dimension formula is known. We say that a triangulation is singular to \( S_k^\mu(\Delta) \) if the dimension of the spline space depends on, in additional to the topology of the triangulation, the geometric position of the vertices of \( \Delta \), and \( S_k^\mu(\Delta) \) is singular when its dimension increases according to the geometric property of \( \Delta \). Hence, the singularity of multivariate spline spaces is an important object that is inevitable in the research of the structure of multivariate spline spaces. For example, Morgan and Scott’s triangulation \( \Delta_{MS} \) [27], see Fig. 6) is singular to \( S_2^1(\Delta_{MS}) \). That is to say that the dimension of the space \( S_2^1(\Delta_{MS}) \) is 6 in general but it increases to 7 when the position of the inner vertices satisfy certain conditions.

Now, we take an example for \( S_2^1(\Delta_{MS}) \) to intuitively understand Theorem 5.2. Let \( l_i : \alpha_ix + \beta_iy + \gamma_iz = 0 \ (i = 1, 2, \ldots, 6) \), \( u : \alpha_u x + \beta_uy + \gamma_uz = 0 \), \( v : \alpha_v x + \beta_vy + \gamma_vz = 0 \) and \( w : \alpha_w x + \beta_wy + \gamma_wz = 0 \).
While the singularity of multivariate spline over any triangulation has not been completely settled, many results on the structure of multivariate spline space in the past 30 years can be found in many of references [1–5, 7–10, 12–14, 23, 28, 29, 36, 37]. For Morgan-Scott’s triangulation, Shi [31] and Diener [14] independently obtained the geometric significance of the necessary and sufficient condition of \( \dim S^1_2(\Delta_{MS}) = 7 \), respectively, and an equivalent geometric necessary and sufficient condition of singularity of \( S^1_2(\Delta_{MS}) \) from the viewpoint of projective geometry was obtained in [15].

Figure 6: Morgan-Scott triangulation

in Morgan-Scott triangulation shown in Fig. 6. From Theorem 5.2, the global conformality condition in \( S^1_2(\Delta_{MS}) \) is

\[
\begin{align*}
\lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 v^2 + \lambda_4 w^2 &= 0, \\
\lambda_3 t_3^2 + \lambda_4 t_4^2 - \lambda_5 v^2 + \lambda_6 w^2 &= 0, \\
\lambda_5 t_5^2 + \lambda_6 t_6^2 - \lambda_7 v^2 + \lambda_8 w^2 &= 0,
\end{align*}
\]

where all letters of \( \lambda \)'s are undetermined real constants. Then the \( \dim S^1_2(\Delta_{MS}) = 6 + \tau \), where \( \tau \) is the dimension of the linear space defined by (5.2). However, the structure of \( S^1_2(\Delta_{MS}) \) depends on the geometric positions of the inner vertices \( a, b \) and \( c \), which can be obviously shown from the following conclusions.

**Theorem 5.3 (31).** The spline space \( S^1_2(\Delta_{MS}) \) is singular (i.e. \( \dim S^1_2(\Delta_{MS}) = 7 \)) if and only if \( Aa, Bb, Cc \) are concurrent, otherwise \( \dim S^1_2(\Delta_{MS}) = 6 \) (see Fig.6).

**Theorem 5.4 (15).** Let \( l_i(x,y,z) = a_i x + b_i y + c_i z = 0 \) (i = 1, 2, ···, 6), then the spline space \( S^1_2(\Delta_{MS}) \) is singular (i.e. \( \dim S^1_2(\Delta_{MS}) = 7 \)) if and only if 6 points \( \{(a_i, b_i, c_i)\}_{i=1}^6 \) lie on a conic, otherwise \( \dim S^1_2(\Delta_{MS}) = 6 \).

Using the principle of duality, an interesting fact is that the equivalent relations in Theorem 5.3 and Theorem 5.4 hold because of the Pascal theorem!

More Precisely, for the Morgan-Scott triangulation, let

\[
\begin{align*}
l_1 &= a_1 u + b_1 v, \\
l_2 &= a_2 u + b_2 v, \\
l_3 &= a_3 v + b_3 w, \\
l_4 &= a_4 v + b_4 w, \\
l_5 &= a_5 w + b_5 u, \\
l_6 &= a_6 w + b_6 u,
\end{align*}
\]

where all \( a_i \)'s and \( b_i \)'s are constants, then by solving the system of equations in (5.2), we have

**Theorem 5.5 (22, 31).** The spline space \( S^1_2(\Delta_{MS}) \) is singular (i.e. \( \dim S^1_2(\Delta_{MS}) = 7 \)) if and only if

\[
\frac{b_1 b_2}{a_1 a_2} \cdot \frac{b_3 b_4}{a_3 a_4} \cdot \frac{b_5 b_6}{a_5 a_6} = 1.
\]

In general, for \( \mu \geq 3 \), Luo & Chen [24] gave an equivalent condition in an algebraic form to the singularity of \( S^1_2(\Delta_{MS}^\mu) \) (\( \mu \geq 3 \)) as follows: for a given triangulation \( \Delta_{MS}^\mu \) (see Fig. 9), suppose

\[
\begin{align*}
l_i &= a_i u + b_i v, & i = 1, 2, \ldots, \mu + 1, \\
l_j &= a_j v + b_j w, & j = \mu + 2, \mu + 3, \ldots, 2\mu + 2, \\
l_k &= a_k w + b_k u, & k = 2\mu + 3, 2\mu + 4, \ldots, 3\mu + 3,
\end{align*}
\]

then

\[
\begin{align*}
\frac{b_1 b_2}{a_1 a_2} \cdot \frac{b_3 b_4}{a_3 a_4} \cdot \frac{b_5 b_6}{a_5 a_6} = 1.
\end{align*}
\]
Remark 5.6. In fact, there also exists the singularity in the simplest spline space $S^1_1(\Delta)$ consisting of continuous piecewise linear polynomials over arbitrary partition $\Delta$. For instance, let $\Delta$ be a partition shown in Fig. 7, the dual figure of $\Delta$ is in Fig. 3. Using the same notations in Proposition 3.1, it is easy to verify through the duality principle that

$\dim S^1_1(\Delta) = 4$ when $\frac{b_1}{a_1} \cdot \frac{b_2}{a_2} \cdot \frac{b_3}{a_3} = -1$, otherwise $\dim S^1_1(\Delta) = 3$.

Theorem 5.7 ([24]). The spline space $S^\mu_{\mu+1}(\Delta^\mu_{MS})$ is singular if and only if

$$\frac{a_1 \ldots a_{\mu+1}}{b_1 \ldots b_{\mu+1}} = \frac{a_{\mu+2} \ldots a_{2\mu+2}}{b_{\mu+2} \ldots b_{2\mu+2}} = \frac{a_{2\mu+3} \ldots a_{3\mu+3}}{b_{2\mu+3} \ldots b_{3\mu+3}} = (-1)^{\mu+1}. \quad (5.6)$$

For the geometric condition of the singularity of $S^2_{3}(\Delta^2_{MS})$, it was analyzed in [23] from projective geometry point of view and the following result was obtained.

Let $l_i: \alpha_i x + \beta_i y + \gamma_i z = 0 \ (i = 1, 2, \ldots, 9)$, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, and $c = (c_1, c_2, c_3)$ in $\Delta^2_{MS}$ triangulation (see Fig.5.3). Let $l_a = a_1 x + a_2 y + a_3 z$, $l_b = b_1 x + b_2 y + b_3 z$ and $l_c = c_1 x + c_2 y + c_3 z$. We define $\mathbb{P}_3$ to be the cubic polynomial subspaces spanned by any nine monomials of $\{x^3, y^3, z^3, x^2 y, x y^2, y^2 z, y z^2, x z^2, x y z\}$ as in [23].

Theorem 5.8 ([23]). The spline space $S^2_3(\Delta^2_{MS})$ is singular (i.e. $\dim S^2_3(\Delta^2_{MS}) = 11$) if and only if $p_i = (\alpha_i, \beta_i, \gamma_i), (i = 1, 2, \ldots, 9)$ lie on a plane curve, which differs from $l_a \cdot l_b \cdot l_c = 0$, in $\mathbb{P}_3$.

Proof. Embedding $\Delta^2_{MS}$ to $\mathbb{P}^2$ by the map: $(x, y) \mapsto [x, y, 1]$. Suppose the lines $bc, ca$ and $ab$ are given by $u = 0, v = 0$ and $w = 0$, respectively. There are real numbers $a_i, b_i (i = 1, 2, \ldots, 9)$ such that

$$\begin{align*}
&l_1 = a_1 u + b_1 v, \\
l_2 = a_2 u + b_2 v, \\
l_3 = a_3 u + b_3 v,
&l_4 = a_4 v + b_4 w, \\
l_5 = a_5 v + b_5 w, \quad \text{and} \quad l_7 = a_7 w + b_7 u, \\
l_6 = a_6 v + b_6 w, \\
l_8 = a_8 w + b_8 u, \\
l_9 = a_9 w + b_9 u.
\end{align*}
$$

(5.7)

Let $\lambda_i (i = 1, 2, \ldots, 9)$ be the corresponding smoothing cofactors and let $p_i = (\alpha_i, \beta_i, \gamma_i), (i = 1, 2, \ldots, 9)$. Then the global conformity conditions in $S^2_3(\Delta^2_{MS})$ become

$$\begin{align*}
&\lambda_1 l_1^3 + \lambda_2 l_2^3 + \lambda_3 l_3^3 + \lambda_a u^3 + \lambda_w v^3 = 0, \\
&\lambda_4 l_4^3 + \lambda_5 l_5^3 + \lambda_6 l_6^3 - \lambda_a v^3 + \lambda_w w^3 = 0, \\
&\lambda_7 l_7^3 + \lambda_8 l_8^3 + \lambda_9 l_9^3 - \lambda_a w^3 - \lambda_w u^3 = 0.
\end{align*}
$$

(5.8)
Let \( \psi : (\lambda_1, \lambda_2, \cdots, \lambda_9, \lambda_u, \lambda_v, \lambda_w) \mapsto (\lambda_1, \lambda_2, \cdots, \lambda_9) \), then \( \psi \) is an injective linear map from the solution spaces of (5.7) to the solution space of

\[
\sum_{i=1}^{9} \lambda_i l_i^2(x, y, z) = 0.
\]

(5.9)

Similar to [13], we intend to prove that the map \( \psi \) is also bijective. For this purpose, let \( \Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_9) \) be a solution of (5.9). Taking \( u = 0, w = 0 \) and \( v = 0 \) in (5.9) respectively, we have

\[
\begin{align*}
\lambda_4 l_1^2 + \lambda_5 l_2^2 + \lambda_6 l_3^2 + k_1 w^3 + k_2 v^3 &= 0, \\
\lambda_7 l_1^2 + \lambda_8 l_2^2 + \lambda_9 l_3^2 + k_3 u^3 + k_4 w^3 &= 0,
\end{align*}
\]

and

\[
\lambda_7 l_1^2 + \lambda_8 l_2^3 + \lambda_9 l_3^3 + k_5 u^3 + k_6 w^3 = 0,
\]

where all \( k_i (i = 1, 2, \cdots, 6) \) are real numbers determined by \( \Lambda \) and the coefficients in (5.9). Since \( \Lambda \) is a solution of (5.9), it follows that

\[
k_1 w^3 + k_2 v^3 + k_3 u^3 + k_4 w^3 + k_5 u^3 + k_6 w^3 = 0,
\]

and \( k_1 = -k_6, k_2 = -k_4, k_3 = -k_5 \). Consequently, \( \tilde{\Lambda} := (\lambda_1, \cdots, \lambda_9, k_3, k_2, k_1) \) is a solution of (5.8).

Hence, \( \dim S_3^3(\Delta_M) = 11 \) (or \( S_3^3(\Delta_M) \) is singular) if and only if there exists a nonzero solution of equation (5.9). Now expand (5.9) with respect to \( x, y, z \), will result in a system of linear equations:

\[
MA := \begin{pmatrix}
\alpha_1^3 & \alpha_2^3 & \cdots & \alpha_8^3 & \alpha_9^3 \\
\alpha_1^2 \beta_1 & \alpha_2^2 \beta_2 & \cdots & \alpha_8^2 \beta_8 & \alpha_9^2 \beta_9 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\beta_1 \gamma_1 & \beta_2 \gamma_2 & \cdots & \beta_8 \gamma_8 & \beta_9 \gamma_9 \\
\gamma_1 & \gamma_2 & \cdots & \gamma_8 & \gamma_9
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_9
\end{pmatrix} = 0
\]

(5.10)

Notice that \( p_i = (\alpha_i, \beta_i, \gamma_i) (i = 1, 2, \cdots, 9) \) lie on the cubic \( C_3 := l_a \cdot l_b \cdot l_c = 0 \), obviously the row vectors of the coefficient matrix are linearly dependent. Since no four points in \( \{p_i = (\alpha_i, \beta_i, \gamma_i)\}_{i=1}^{9} \) are collinear, it can be shown from a classical results of algebraic geometry that \( \text{rank}(M) \geq 8 \). Hence, (5.10) has a non-zero solution \( \Lambda \) if and only if the rank of the coefficient matrix of (5.10) is equal to 8, implying that those nine points \( p_i = (\alpha_i, \beta_i, \gamma_i) (i = 1, 2, \cdots, 9) \) lie on a cubic in \( \mathbb{P}_3 \). Conversely, let \( \{p_i = (\alpha_i, \beta_i, \gamma_i)\}_{i=1}^{9} \) lie on a cubic \( \Gamma_3 \) in \( \mathbb{P}_3 \), and \( \Gamma_3 \) differ from \( C_3 \). Without loss of generality, suppose

\[
\Gamma_3 : \ a_1 x^3 + a_2 x^2 y + a_3 xy^2 + a_4 y^3 + a_5 xyz + a_6 x^2 z + a_7 y^2 z + a_8 yz^2 + a_9 y^2 z^2 = 0
\]

(no \( z^3 \) term), then we claim that \( C_3 \) must contain a \( z^3 \) term. Otherwise, by simple computation, there exists constant \( d \) such that a cubic \( \Gamma_3 = \Gamma_3 + dC_3 \), composed some 8 basis elements in \( \{x^3, y^3, x^2 y, xy^2, y^2 z, yz^2, x^2 z, x z^2, y^2 z^2, x^2 z^3, x z^2 y, y z^2 x, z^3 x, z^2 y, z y x, y x z, x y z, z^3 y, z^2 y, z y^2, y^2 z, z^3 x, z^2 y, z y^2, y z^2 \} \), passes through the nine points \( \{p_i = (\alpha_i, \beta_i, \gamma_i)\}_{i=1}^{9} \). Thus the rank of the coefficient matrix \( M \) must be less than 8, which is contradictory. Since \( p_i (i = 1, 2, \cdots, 9) \) lie on \( \Gamma_3 \),

\[
\begin{pmatrix}
\alpha_1^3 & \alpha_2^3 & \cdots & \alpha_8^3 & \alpha_9^3 \\
\alpha_1^2 \beta_1 & \alpha_2^2 \beta_2 & \cdots & \alpha_8^2 \beta_8 & \alpha_9^2 \beta_9 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1 \beta_1 \gamma_1 & \alpha_2 \beta_2 \gamma_2 & \cdots & \alpha_8 \beta_8 \gamma_8 & \alpha_9 \beta_9 \gamma_9 \\
\beta_1 \gamma_1 & \beta_2 \gamma_2 & \cdots & \beta_8 \gamma_8 & \beta_9 \gamma_9
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_9
\end{pmatrix} = 0.
\]

(5.11)

Obviously, the system of linear equations:

\[
\begin{pmatrix}
\alpha_1^3 & \alpha_2^3 & \cdots & \alpha_8^3 & \alpha_9^3 \\
\alpha_1^2 \beta_1 & \alpha_2^2 \beta_2 & \cdots & \alpha_8^2 \beta_8 & \alpha_9^2 \beta_9 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1 \beta_1 \gamma_1 & \alpha_2 \beta_2 \gamma_2 & \cdots & \alpha_8 \beta_8 \gamma_8 & \alpha_9 \beta_9 \gamma_9 \\
\beta_1 \gamma_1 & \beta_2 \gamma_2 & \cdots & \beta_8 \gamma_8 & \beta_9 \gamma_9
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_9
\end{pmatrix} = 0
\]

(5.12)
has a non-zero solution. The condition of $C_3$ containing a term $z^3$ and passing through $p_i(i = 1, 2, \cdots, 9)$ show that the vector $(\gamma_1^3, \gamma_2^3, \cdots, \gamma_9^3)$ can be expressed by the linear combination of the 9 row vectors in (5.12). Therefore, the non-zero solution $A$ of (5.12) is also solution of (5.9) and (5.8). This completes the proof.

**Remark 5.9.** In fact, it can be easily seen from the process of the proof of Theorem 5.8 or from the Chasles’s Theorem that Theorem 5.8 can be improved as: The spline space $S^3_{MS}(\Delta^2_{MS})$ is singular (i.e. $\dim S^3_{MS}(\Delta^2_{MS}) = 11$) if and only if $p_i = (\alpha_i, \beta_i, \gamma_i), (i = 1, 2, \ldots, 9)$ lie on a cubic, which differs from $l_a \cdot l_b \cdot l_c = 0$.

### 5.2 Proof of Theorem 4.1

Let $a, b$ and $c$ be any three distinct lines in the projective plane $P^2$, denoted by $u = < c, a >, v = < a, b >$ and $w = < b, c >$, and $\Gamma_3$ be a cubic in $P^2$. Assume that $p_1, p_2, p_3$ are three intersection points of $a$ and $\Gamma_3$, $p_4, p_5, p_6$ are intersection points of $b$ and $\Gamma_3$, and $p_7, p_8, p_9$ are intersection points of $c$ and $\Gamma_3$. Then there are real numbers $\{a_i, b_i\}$ such that

$$
\begin{cases}
    p_1 = a_1u + b_1v \\
    p_2 = a_2u + b_2v \\
    p_3 = a_3u + b_3v \\
    p_4 = a_4v + b_4w \\
    p_5 = a_5v + b_5w \quad \text{and} \quad p_7 = a_7w + b_7u \\
    p_6 = a_6v + b_6w \quad \text{and} \quad p_8 = a_8w + b_8u \\
    p_9 = a_9w + b_9u.
\end{cases}
$$

Using Definition 1.3, the duality of the figure composed by the lines $a, b$ and $c$, the points $u, v, w$ and $\{p_i\}_{i=1}^9$ turns precisely out the Morgan-Scott type partition $\Delta^2_{MS}$ (in which $\mu = 2$) as shown in Fig.5.3, where $l_i = \alpha_i x + \beta_i y + \gamma_i z, i = 1, 2, \ldots, 9$. From Theorem 5.8, we see that the spline space $S^3_{MS}(\Delta^2_{MS})$ is singular, that is, $\dim S^3_{MS}(\Delta^2_{MS}) = 11$. Consequently, it follows from Theorem 5.7 ($\mu = 2$) that the characteristic number of a cubic is always equal to $(-1)^3 = -1$. Which completes the proof of our main result.

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