The unramified two-dimensional Langlands correspondence

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Abstract. We describe the unramified Langlands correspondence for two-dimensional local fields and construct a categorical analogue of the unramified principal series representation and study its properties. The main tool for this description is the construction of a certain central extension. For this and other central extensions, we prove non-commutative reciprocity laws (that is, splitting of the central extensions over certain subgroups) for arithmetic surfaces and projective surfaces over finite fields. These reciprocity laws connect central extensions constructed locally and globally.

Keywords: 2-vector spaces, two-dimensional local fields, higher adèles, generalized Langlands programme, two-dimensional non-commutative reciprocity laws.

Dedicated to I. R. Shafarevich on the occasion of his 90th birthday

§ 1. Introduction

In 1995 Kapranov [1] proposed a very hypothetical generalization of the Langlands programme to the two-dimensional case. The classical Langlands correspondence deals with a one-dimensional base: with number fields or function fields of algebraic curves defined over a finite field. It was asked in [1] whether there is an analogue of this programme for arithmetic surfaces or algebraic surfaces defined over a finite field. The paper [1] contained no constructions but treated the case of the abelian two-dimensional (local) Langlands correspondence.

The main idea suggested in [1] is to associate with any $n$-dimensional representation of the group $\text{Gal}(K^{\text{sep}}/K)$ some (categorical) representation of the group $\text{GL}_{2n}(K)$ when $K$ is a two-dimensional local field,\(^1\) or of the group $\text{GL}_{2n}(A)$ when $K$ is the field of rational functions of an arithmetic surface or a projective algebraic surface over a finite field, in a 2-vector space. (Here $A$ is the ring of two-dimensional Parshin–Beilinson adèles [3] of the arithmetic surface or projective algebraic surface over a finite field. In the case of an arithmetic surface, this ring must take into account the fibres over infinite (Archimedean) points of the base; see [4], Example 11, [5].)

\(^1\)In this paper, a two-dimensional local field $K$ will denote a complete discrete valuation field whose residue field $\overline{K}$ is a one-dimensional local field, that is, a complete discrete valuation field with finite residue field $\mathbb{F}_q$. (A basic survey of the key notions of two-dimensional local fields and two-dimensional adèles is contained in [2].)

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The existence of a one-dimensional (classical) Langlands correspondence for $n = 1$ is the content of one-dimensional (ordinary) class field theory. A two-dimensional class field theory was developed by Parshin, Kato and others (see the survey [6]). In the local case, it reduces to a description of the abelian Galois group of a two-dimensional local field. This description is based on a construction of the reciprocity map

$$K_2(K) \rightarrow \text{Gal}(K^{ab}/K),$$

(1)

where $K^{ab}$ is the maximal abelian extension of a two-dimensional local field $K$.

We note that Parshin [5] has recently stated a conjecture on the direct image of automorphic forms. This conjecture connects the two-dimensional abelian Langlands correspondence with the classical Langlands correspondence in dimension 1. The classical Hasse–Weil conjecture on the existence of a functional equation for $L$-functions of arithmetic surfaces follows from Parshin’s conjecture.

In this paper we describe the unramified Langlands correspondence for two-dimensional local fields, construct a categorical analogue of the unramified principal series representations of the group $GL_{2n}(K)$ (where $K$ is a two-dimensional local field and $GL_{2n}(K)$ acts on an abelian $\mathbb{C}$-linear category) and study the properties of the resulting categorical representations. Our main tool is a construction of a certain central extension of the group $GL_2(K)$ by the group $\mathbb{R}_+^*$. For this and other central extensions, we prove the non-commutative reciprocity law (that is, the splitting of these central extensions over certain subgroups) for arithmetic surfaces and projective surfaces over finite fields. Such reciprocity laws connect central extensions that are constructed locally and globally.

The paper is organized as follows.

In § 2 we recall the abelian case of the two-dimensional local Langlands correspondence. In § 2.1 we connect one-dimensional 2-representations of a group with central extensions of this group. In § 2.2 we calculate the second cohomology group of $GL_n(K)$ for an infinite field $K$.

In § 3 we consider central extensions constructed from $C_2$-spaces. In § 3.1 we recall the definition of the category $C_2^{\text{fin}}$. Examples of the objects of this category are two-dimensional local fields and adelic rings of arithmetic surfaces or algebraic surfaces over finite fields. In § 3.2 we construct central extensions of the group $GL_n(\mathbb{A}_\Delta)$, where $\mathbb{A}_\Delta$ is a subring of the adelic ring of an arithmetic surface or algebraic surface over a finite field. In § 3.3 we study these central extensions in the case when $\mathbb{A}_\Delta$ is a finite product of two-dimensional local fields. In § 3.4 we construct and study some central extensions of the groups $GL_n(\mathbb{R}((t)))$ and $GL_n(\mathbb{C}((t)))$. These central extensions are used to construct central extensions of the group $GL_n(\mathbb{A}_X^{\text{ar}})$, where $\mathbb{A}_X^{\text{ar}}$ is the arithmetical adelic ring of an arithmetic surface $X$ (that is, the adelic ring that takes the Archimedean fibres into account). In § 3.5 we prove the non-commutative reciprocity laws: Theorem 1 asserts that the central extensions constructed from the whole adelic ring $\mathbb{A}$ of a projective surface over finite field or an arithmetic surface split over certain subgroups of $GL_n(\mathbb{A})$. These subgroups play an important role in the description of the semilocal situation on the scheme under consideration. They are related either to closed points or to integral one-dimensional subschemes of this scheme.
In §4 we consider the unramified Langlands correspondence for two-dimensional local fields. In §4.1 we recall the classical construction of this correspondence for one-dimensional local fields. In §4.2 we recall some known facts (to be used later) about group actions on $k$-linear categories, where $k$ is a field. In §4.3 we construct a categorical analogue of the unramified principal series representations of the group $GL_{2n}(K)$, where $K$ is a two-dimensional local field. We also define the notion of a smooth action of $GL_l(K)$ on a $k$-linear abelian category (this category is referred to as a generalized 2-vector space). Our categorical analogue of the principal series representation is smooth (we have $k = \mathbb{C}$ in this case). This and other properties of these representations are proved in Theorem 2. To construct the categorical principal series representations of $GL_{2n}(K)$, we use an analogue of the induced representation (for categories) along with a central extension of $GL_{2n}(K)$, which is constructed in §3 and is connected with the unramified class field theory for $K$. In §4.4 we discuss a conjecture on the relation between smooth spherical actions of $GL_{2n}(K)$ on an arbitrary $\mathbb{C}$-linear abelian category and the categorical principal series representations (constructed in §4.3) of $GL_{2n}(K)$.

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§2. The abelian case of the two-dimensional Langlands correspondence

2.1. Central extensions and one-dimensional 2-vector spaces. We recall that a finite-dimensional 2-vector space $C$ over a field $k$ is a semisimple abelian $k$-linear category in which the set of isomorphism classes of simple objects is finite and we have $\text{Hom}_{C}(E, E) \simeq k$ for every simple object $E \in C$ (see [7]). (By a $k$-linear category we mean a category $C$ where the sets $\text{Hom}_{C}(A, B)$ are $k$-vector spaces for all objects $A, B$ and the composition of morphisms is bilinear. A category is said to be semisimple if every object is a finite direct sum of simple objects.) The number of isomorphism classes of simple objects is called the dimension of a 2-vector space. It is easy to see that any $n$-dimensional 2-vector space is equivalent to the category $(\text{Vect}_k^{\text{fin}})^n$, where $\text{Vect}_k^{\text{fin}}$ is the category of finite-dimensional $k$-vector spaces.

Consider a one-dimensional 2-vector space $C$. All simple objects in $C$ are isomorphic, and every $k$-linear endofunctor $F: C \to C$ is given (up to isomorphism) by a finite-dimensional vector space over $k$:

$$V = \text{Hom}_{C}(E, F(E)),$$

where $E$ is a simple object in $C$. Conversely, a one-dimensional 2-vector space is equivalent to the category $\text{Vect}_k^{\text{fin}}$. Therefore any finite-dimensional vector space $V \in \text{Vect}_k^{\text{fin}}$ determines a $k$-linear endofunctor on the category $\text{Vect}_k^{\text{fin}}$ by the rule

$$X \mapsto V \otimes_k X, \quad X \in \text{Vect}_k^{\text{fin}}.$$

Let $G$ be a group. A representation of $G$ on a 2-vector space $C$ (of any dimension) is a strongly monoidal functor from the discrete category of $G$ (whose objects are elements of $G$ and all morphisms are identity morphisms) to the monoidal
category of functors which are \( k \)-linear equivalences of the category \( C \) (see [8], §5.3 from [1], as well as § 4.2 below). Therefore any 2-representation of \( G \) on a one-dimensional 2-vector space is determined (up to equivalence) by specifying 1-dimensional \( k \)-vector spaces \( V_g \) for every element \( g \) of \( G \) along with the natural isomorphisms

\[
V_{g_1} \otimes_k V_{g_2} \longrightarrow V_{g_1 g_2}
\]

which satisfy the associativity conditions for all elements \( g_1, g_2, g_3 \) of \( G \). These data correspond to a central extension of \( G \):

\[
1 \longrightarrow k^* \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1,
\]

where the \( k^* \)-torsor \( \pi^{-1}(g) \) is equal to \( V_g \setminus 0 \). Hence we conclude that the set of equivalence classes of one-dimensional 2-representations of \( G \) coincides with the set \( H^2(G, k^*) \).

**Remark 1.** The category of representations of an arbitrary group (more generally, an arbitrary 2-group) on a finite-dimensional 2-vector space is studied in detail in [9].

### 2.2. The second cohomology of \( \text{GL}_n(K) \)

Let \( A \) be any abelian group and \( K \) an arbitrary infinite field. Following [10], §1 (see also [11]), we shall construct an explicit map

\[
\text{Hom}(K_2(K), A) \longrightarrow H^2(\text{GL}_2(K), A).
\]

Consider the universal central extension

\[
1 \longrightarrow K_2(K) \longrightarrow \text{St}(K) \longrightarrow \text{SL}(K) \longrightarrow 1,
\]

where \( \text{St}(K) \) is the Steinberg group of the field \( K \) and \( \text{SL}(K) = \lim_{\rightarrow n} \text{SL}_n(K) \). We also consider the central extension which is the pullback of the central extension (3) under the inclusion of \( \text{SL}_2(K) \) in \( \text{SL}(K) \):

\[
1 \longrightarrow K_2(K) \longrightarrow \tilde{\text{SL}}_2(K) \longrightarrow \text{SL}_2(K) \longrightarrow 1.
\]

We have

\[
K_2(K) = H_2(\text{SL}(K), \mathbb{Z}) = H_2(\text{SL}_2(K), \mathbb{Z})_{K^*}.
\]

Note that \( \text{GL}_2(K) = \text{SL}_2(K) \rtimes K^* \), where the group \( K^* = \text{GL}(1, K) \hookrightarrow \text{GL}_2(K) \) (embedded in the upper-left corner) acts on \( \text{SL}_2(K) \) by conjugations which are inner automorphisms of \( \text{GL}_2(K) \). The group \( K^* \) also acts by conjugation on \( \text{SL}(K) \). Since the central extension (3) is universal, we get a lift of the action of \( K^* \) on \( \text{St}(K) \), and its restriction to the kernel \( K_2(K) \) is trivial by (4). Hence we get a central extension

\[
1 \longrightarrow K_2(K) \longrightarrow \tilde{\text{GL}}_2(K) \longrightarrow \text{GL}_2(K) \longrightarrow 1,
\]

which splits over \( K^* \), and \( \tilde{\text{GL}}_2(K) = \tilde{\text{SL}}_2(K) \rtimes K^* \). The universal symbol \((x, y) \in K_2(K)\) is obtained in the following way (see [10], §1.9):

\[
(x, y) = \left\langle \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \right\rangle,
\]

as follows (see [10], §1.9):
where the element $\langle A, B \rangle = [\hat{A}, \hat{B}]$ in $K_2(K)$ is well defined for all commuting matrices $A$ and $B$ in $GL(2, K)$, that is, it does not depend on the choice of the elements $\hat{A}$ and $\hat{B}$ of $GL_2(K)$ such that $\pi(\hat{A}) = A$ and $\pi(\hat{B}) = B$. Applying an element of $\text{Hom}(K_2(K), A)$ to the kernel of the central extension (5), we obtain an explicit description of the map (2).

On the other hand, the Hochschild–Serre spectral sequence yields an exact sequence

$$0 \rightarrow H^2(K^*, A) \rightarrow H^2(GL_2(K), A) \xrightarrow{\alpha} H^2(SL_2(K), A)^{K^*}.$$  

(Note that $H^1(SL_2(K), A) = \text{Hom}(SL_2(K), A) = 0$ since the group $SL_2(K)$ is perfect.)

Since $SL_2(K)$ is perfect, we also obtain from the universal coefficient formula that

$$H^2(SL_2(K), A) = \text{Hom}(H_2(SL_2(K), \mathbb{Z}), A).$$

Therefore,

$$H^2(SL_2(K), A)^{K^*} = \text{Hom}(H_2(SL_2(K), \mathbb{Z})_{K^*}, A) = \text{Hom}(K_2(K), A).$$

Moreover, the map (2) is a section of $\alpha$. Summing up, we get the following assertion.

**Proposition 1.** Let $A$ be an abelian group and $K$ an infinite field. Then

$$H^2(GL_2(K), A) = H^2(K^*, A) \oplus \text{Hom}(K_2(K), A).$$

**Remark 2.** One can similarly prove the following more general assertion for any $n \geq 2$ and any infinite field $K$:

$$H^2(GL_n(K), A) = H^2(K^*, A) \oplus \text{Hom}(K_2(K), A).$$  

(7)

Combining the result of Proposition 1 with the reciprocity map (1), we obtain a map from the characters of the Galois group of a two-dimensional local field $K$ to the one-dimensional 2-representations of $GL(2, K)$. Hence, taking into account the topology on the group $K_2(K)$, we obtain a complete description of the abelian two-dimensional Langlands correspondence.

§ 3. $C_2$-spaces and central extensions

**3.1. The category $C_2^\text{fin}$.** The categories $C_n^\text{fin}$, $n \geq 0$, were constructed in [12] and [4], § 3.1. We are interested in the category $C_2^\text{fin}$. It is constructed by induction.

The category $C_0^\text{fin}$ is the category of finite abelian groups.

The objects of $C_1^\text{fin}$ are filtered abelian groups $(I, F, V)$. Here $V$ is an abelian group, $I$ is a partially ordered set such that for all $i, j \in I$ there are $k, l \in I$ with $k \leq i \leq l$ and $k \leq j \leq l$, and $F$ is a function from $I$ to the set of subgroups of $V$ such that $F(i) \subseteq F(j)$ whenever $i \leq j$. We also demand that $\bigcap_{i \in I} F(i) = 0$ and $\bigcup_{i \in I} F(i) = V$.

The inductive step is expressed by the requirement that the group $F(j)/F(i)$ must be finite for all $j \geq i \in I$ (that is, this group must be an object of $C_0^\text{fin}$).
Morphisms between the objects constructed copy the definition of continuous morphisms in one-dimensional local fields. More precisely, if $E_1 = (I_1, F_1, V_1)$ and $E_2 = (I_2, F_2, V_2)$ are objects in $C^\text{fin}_1$, then $\text{Hom}_{C^\text{fin}_1}(E_1, E_2)$ consists of all homomorphisms $f : V_1 \to V_2$ (between abelian groups) that satisfy the following condition. For every $j \in I_2$ there is $i \in I_1$ such that $f(F_1(i)) \subset F_2(j)$. Examples of objects in $C^\text{fin}_1$ are one-dimensional local fields, adelic rings of curves over finite fields, and the rings of finite adèles of number fields.

The objects of $C^\text{fin}_2$ are filtered abelian groups $(I, F, V)$ such that all the groups $F(j)/F(i)$, $i \leq j \in I$, are endowed with the structure of objects in $C^\text{fin}_1$ satisfying some compatibility conditions for $i \leq j \leq k \in I$ (see [4], Definition 4, for an exact definition). Morphisms between objects in $C^\text{fin}_2$ are defined by induction in terms of the morphisms in $C^\text{fin}_1$ between the quotient groups of the filtrations (see [4], Definition 5, for an exact definition). Examples of objects in $C^\text{fin}_2$ are two-dimensional local fields, adelic rings of algebraic surfaces over finite fields, and adelic rings of arithmetic surfaces (excluding the fibres over Archimedean places); see [12], Theorem 2.1, [4], Example 1.

3.2. The central extension $\widehat{\text{GL}_n(A_\Delta)}_{\mathbb{R}_+}$. Let $K$ be a one-dimensional local field with finite residue field $\mathbb{F}_q$. Then there is a homomorphism

$$\text{GL}(n, K) \to \mathbb{R}_+^*, \quad A \mapsto q^{\nu(\det(A))},$$

where $\nu$ is a discrete valuation on the field $K$. We see that this homomorphism comes from a homomorphism $K^* \to \mathbb{Z}$, that is, from an element of $H^1(K^*, \mathbb{Z})$. There is also a homomorphism $\text{GL}(n, A^\text{fin}) \to \mathbb{R}_+^*$, where $A^\text{fin}$ is the ring of finite adèles of a curve over a finite field or of a number field. This homomorphism is obtained by multiplying the local maps (8).

Let $X$ be an integral two-dimensional normal scheme of finite type over $\mathbb{Z}$ (for example, a surface over a finite field or an arithmetic surface). Let $\Delta$ be a set of pairs of the form $x \in C$, where $x \in X$ is a closed point and $C$ is an integral one-dimensional subscheme on $X$ containing $x$. We define the following subrings of the ring $\prod_{x \in C} K_{x, C}$:

$$A_\Delta = A_X \cap \prod_{\{x \in C\} \in \Delta} K_{x, C}, \quad O_{A_\Delta} = A_X \cap \prod_{\{x \in C\} \in \Delta} O_{K_{x, C}},$$

where $A_X$ is the adelic ring of $X$, the ring $\prod_{\{x \in C\} \in \Delta} K_{x, C}$ is the finite product (constructed from the pair $x \in C$) of the two-dimensional local fields $K_i$, and $O_{K_{x, C}} = \prod_i O_{K_i}$, where $O_{K_i}$ is the discrete valuation ring of rank 1 of $K_i$. Note that if $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \emptyset$, then

$$A_{\Delta_1} = A_{\Delta_1} \times A_{\Delta_2}, \quad O_{A_{\Delta}} = O_{A_{\Delta_1}} \times O_{A_{\Delta_2}},$$

$$\text{GL}_n(A_{\Delta}) = \text{GL}_n(A_{\Delta_1}) \times \text{GL}_n(A_{\Delta_2}).$$

For every $n \geq 1$ we shall construct a central extension

$$1 \to \mathbb{R}_+^* \to \widehat{\text{GL}_n(A_{\Delta})}_{\mathbb{R}_+} \to \text{GL}_n(A_\Delta) \to 1.$$  \[9\]

2This ring is described in detail at the beginning of §3.3 below.
We consider the following $C_2^{\text{fin}}$-structure $(I, F, A^*_\Delta)$ on $A^*_\Delta$, where the set
\[ I = \{ O_{A\Delta}-\text{submodules } T \subset A^*_\Delta \mid T = gO^n_{A\Delta} \text{ for some } g \in \text{GL}_n(A\Delta) \} \]
is ordered by inclusions of submodules and the function $F$ maps each element of $I$ to the corresponding submodule in $A^*_\Delta$. For any elements $i \leq j \in I$, the $O_{A\Delta}$-module $F(j)/F(i)$ is a locally compact abelian group whose topology is obtained as the quotient of the topology induced from the topological group $A^*_\Delta$. (The group $A\Delta$ is by construction endowed with the natural topology of iterated inductive and projective limits.) We define a filtration on $F(j)/F(i)$ by open compact subgroups. This filtration endows $F(j)/F(i)$ with the structure of an object in $C_2^{\text{fin}}$. These structures are compatible in the sense of $C_2^{\text{fin}}$-structure for all $i \leq j \leq k \in I$.

For any locally compact abelian group $E$, let $\mu(E)$ be the canonical $\mathbb{R}_+^*$-torsor of non-zero Haar measures on $E$. For any elements $i, j \in I$ we consider an $\mathbb{R}_+^*$-torsor $\mu(F(i) \mid F(j))$ which is canonically defined by the following conditions:
1) $\mu(F(i) \mid F(j)) \otimes \mu(F(j) \mid F(k)) = \mu(F(i) \mid F(k))$ for all $i, j, k \in I$;
2) $\mu(F(i) \mid F(j)) = \mu(F(j)/F(i))$ for all $i \leq j \in I$.

Any element $g \in \text{GL}_n(A\Delta)$ transforms the $\mathbb{R}_+^*$-torsor $\mu(F(i) \mid F(j))$ into the $\mathbb{R}_+^*$-torsor $\mu(gF(i) \mid gF(j))$. We define a group
\[ \widetilde{\text{GL}}_n(A\Delta)_{\mathbb{R}_+} = \{(g, \mu) \mid g \in \text{GL}_n(A\Delta), \mu \in \mu(O^n_{A\Delta} \mid gO^n_{A\Delta})\}. \]
The multiplication law in this group is defined by the formula
\[ (g_1, \mu_1)(g_2, \mu_2) = (g_1g_2, \mu_1 \otimes g_1(\mu_2)) \]
with obvious identity and inverse elements. The map $\theta: (g, \mu) \mapsto g$ determines the central extension (10).

We have $\text{GL}_n(A\Delta) = \text{SL}_n(A\Delta) \rtimes A^*_\Delta$, where the group $A^*_\Delta$ is embedded in the upper left corner of the group $\text{GL}_n(A\Delta)$; see also §2.2. The central extension (10) lifts the action of $A^*_\Delta$ uniquely to an action on the group $\theta^{-1}(\text{SL}_n(A\Delta))$ by means of inner automorphisms of the group $\widetilde{\text{GL}}_n(A\Delta)_{\mathbb{R}_+}$. This action becomes trivial after restriction to the kernel $\mathbb{R}_+^*$. We define a group
\[ \widetilde{\text{GL}}_n(A\Delta)_{\mathbb{R}_+} = \theta^{-1}(\text{SL}_n(A\Delta)) \rtimes A^*_\Delta. \]
This yields a central extension
\[ 1 \longrightarrow \mathbb{R}_+^* \longrightarrow \widetilde{\text{GL}}_n(A\Delta)_{\mathbb{R}_+} \longrightarrow \text{GL}_n(A\Delta) \longrightarrow 1, \quad (11) \]
which splits over the subgroup $A^*_\Delta$ of $\text{GL}_n(A\Delta)$.

**Remark 3.** The central extension $\widetilde{\text{GL}}_n(A\Delta)_{\mathbb{R}_+}$ is not isomorphic to the central extension $\widetilde{\text{GL}}_n(A\Delta)_{\mathbb{R}_+}$ since the former splits over the subgroup $A^*_\Delta$ of $\text{GL}_n(A\Delta)$ while the latter does not. This non-splitting can be seen, for example, by explicitly calculating the commutator of the lifts of two elements of the commutative
The central extensions constructed above possess the following properties. It clearly suffices to prove the corresponding assertions for the central extensions $GL_n(\hat{A}_\Delta)$ will be canonically isomorphic to the central extension $\hat{GL}_n(\hat{A}_\Delta)$ of the same group. To prove this, we assume for simplicity that $n = 2$. Fix a matrix $A = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ which conjugates the matrices $\text{diag}(b, 1)$ and $\text{diag}(1, b)$ in the group $GL_2(\hat{A}_\Delta)$. Let $\phi_A$ be the inner automorphism of the group $\theta^{-1}(SL_2(\hat{A}_\Delta))$ given by the lift $\hat{A}$ of the element $A$ to this group (the automorphism $\phi_A$ is independent of the choice of lifting). The homomorphisms $\hat{A}_\Delta^* \to \text{Aut}(\theta^{-1}(SL_2(\hat{A}_\Delta)))$ constructed from different embeddings of $\hat{A}_\Delta^*$ in the diagonal of $GL_2(\hat{A}_\Delta)$ differ from each other by conjugation by the element $\phi_A$ in the group $\text{Aut}(\theta^{-1}(SL_2(\hat{A}_\Delta)))$. Hence the map $\{x, b\} \mapsto \{\phi_A(x), b\}$, where $\{x, b\} \in \theta^{-1}(SL_2(\hat{A}_\Delta)) \times \hat{A}_\Delta^*$, is an isomorphism between the two semidirect products constructed from the different embeddings of $\hat{A}_\Delta^*$ in the diagonal. This isomorphism induces an inner automorphism (determined by the element $A$) of the group $GL_2(\hat{A}_\Delta)$. It remains to note that every inner automorphism of a group induces a canonical automorphism of every central extension of this group.

We mention the following direct corollary of the construction of central extensions. Two central extensions $\hat{GL}_n(\hat{A}_\Delta)_{\mathbb{R}_+^*}$ and $\hat{GL}_n(\hat{A}_\Delta)_{\mathbb{R}^*_+}$ split canonically over the subgroup $\hat{GL}_n(O_{A_\Delta})$ of $GL_n(\hat{A}_\Delta)$.

**Proposition 2.** The central extensions constructed above possess the following properties.

1) If $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \emptyset$, then the central extension $\hat{GL}_n(\hat{A}_\Delta)_{\mathbb{R}_+^*}$ is the Baer sum of the central extensions $\hat{GL}_n(\hat{A}_{\Delta_1})_{\mathbb{R}_+^*}$ and $\hat{GL}_n(\hat{A}_{\Delta_2})_{\mathbb{R}_+^*}$ (with respect to the projections onto the direct summands in (9)).

2) The central extension $\hat{GL}(\hat{A}_\Delta)_{\mathbb{R}^*_+}$ of the group $GL(\hat{A}_\Delta) = \lim_{\longleftarrow n} GL_n(\hat{A}_\Delta)$ is well defined by the central extensions $\hat{GL}_n(\hat{A}_\Delta)_{\mathbb{R}_+^*}$, $n \geq 1$.

**Proof.** It clearly suffices to prove the corresponding assertions for the central extensions (and groups) $\hat{GL}_n(\hat{A}_\Delta)_{\mathbb{R}_+^*}$. Property 1) follows from the construction of the central extension (10) and the following properties. If

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0$$

is a short exact sequence of locally compact abelian groups, where all morphisms are continuous, and $V_1 \leftarrow V$ is a closed embedding, then there is a canonical isomorphism of $\mathbb{R}^*_+$-torsors

$$\mu_{V_1, V_2} : \mu(V_1) \otimes \mu(V_2) \rightarrow \mu(V).$$

If $V = V_1 \oplus V_2$, then for any elements $v_1 \in \mu(V_1)$, $v_2 \in \mu(V_2)$ we have

$$\mu_{V_1, V_2}(v_1 \otimes v_2) = \mu_{V_2, V_1}(v_2 \otimes v_1). \quad (12)$$

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$^3$If we assume that $V_i$, $1 \leq i \leq 2$, are finite-dimensional vector spaces over a field and replace $\mu(V_i)$ by $\det V_i = \wedge^{\text{max}}(V_i)$, then the analogue of (12) holds only up to a (possible) change of sign.
To prove property 2), it suffices to show that if $1 \leq n_1 \leq n_2$, then the central extension $GL_{n_1}(\mathbb{A}_\Delta)_{R^*_+}$ is obtained from the central extension $GL_{n_2}(\mathbb{A}_\Delta)_{R^*_+}$ by restriction in the image of $\theta$ to the subgroup $GL_{n_1}(\mathbb{A}_\Delta)$ of $GL_{n_2}(\mathbb{A}_\Delta)$. This follows from the construction of the central extension. $\square$

3.3. The case when $\Delta$ is a singleton. If $\Delta$ consists of a single element $\{x \in C\}$, then $\mathbb{A}_\Delta = K_{x,C}$ is a finite product of two-dimensional local fields. The local ring $O_x$ of the closed point $x$ on $X$ and its completion $\hat{O}_x$ at the maximal ideal are normal rings without divisors of zero.$^4$ The curve $C$ determines a prime ideal of height 1 in $O_x$. We denote this ideal by $\eta_C$. Let $\eta_i, 1 \leq i \leq m$, be the prime ideals of height 1 in the ring $\hat{O}_x$ that contain the ideal $\eta_C\hat{O}_x$. We define a two-dimensional local field $K_i, 1 \leq i \leq m$, as the completion of the field $\text{Frac}(\hat{O}_x)$ over the discrete valuation associated with $\eta_i$. Then we have

$$\mathbb{A}_\Delta = K_{x,C} = \prod_{i=1}^m K_i.$$ (13)

Let $K$ be any of the two-dimensional local fields that appear as factors in (13). Let $\mathbb{F}_q$ be the last residue field of $K$. By considering corresponding filtrations on the group $K^n$ and on its subquotients, we easily see that the $R^*_+$-torsors occurring in the construction of the central extension (10) (after its restriction to $GL_n(K)$) come from $q^\mathbb{Z}$-torsors. Therefore the restrictions of the central extensions (10) and (11) to $GL_n(K)$ arise from elements$^5$ of $H^2(GL_n(K), \mathbb{Z})$ after applying the map $\mathbb{Z} \to R^*_+, a \mapsto q^a$. Note that, in contrast with the case of one-dimensional local fields, for two-dimensional local fields $K$ the central extensions (10) and (11) for $n > 1$ are not obtained from the case $n = 1$ using the map det: $GL_n(K) \to K^*$. 

Remark 4. If $X$ is a surface over a finite field $\mathbb{F}_q$ and $\Delta$ is any set of pairs, then the central extensions (10) and (11) are also obtained from the elements of $H^2(GL_n(\mathbb{A}_\Delta), \mathbb{Z})$ as above. Indeed, to construct these central extensions, it suffices to consider the filtrations by $\mathbb{F}_q$-spaces on the space $\mathbb{A}_\Delta^n$ and its subfactors (see also [13]).

We denote the restrictions of the central extensions (10) and (11) to the subgroup $GL_n(K)$ of $GL_n(\mathbb{A}_\Delta)$ by $GL_n(K)_{R^*_+}$ and $GL_n(\mathbb{A}_\Delta)_{R^*_+}$ respectively. (Note the following direct consequence of the construction. If $\mathbb{A}_\Delta = \prod_{i=1}^m K_i$, then $GL_n(\mathbb{A}_\Delta) = \prod_{i=1}^m GL_n(K_i)$, and $GL_n(\mathbb{A}_\Delta)_{R^*_+}$ and $GL_n(\mathbb{A}_\Delta)_{R^*_+}$ are the Baer sums over $1 \leq i \leq m$ of $GL_n(K_i)_{R^*_+}$ and $GL_n(K_i)_{R^*_+}$ respectively.) We seek the element of $\text{Hom}(K_2(K), R^*_+)$ that corresponds to $GL_2(K)_{R^*_+}$ for a two-dimensional local field $K$ under the isomorphism in Proposition 1. We define a map $\nu_K(\cdot, \cdot): K^* \times K^* \to \mathbb{Z}$ by the formula

$$\nu_K(f, g) = \nu_K(\frac{f_{\nu_K}(g)}{g_{\nu_K}(f)}),$$ (14)

---

$^4$We recall that $X$ is assumed to be an integral two-dimensional normal scheme of finite type over $\mathbb{Z}$.

$^5$A more precise assertion will be obtained in Proposition 3.
where \( \nu_K : K^* \to \mathbb{Z} \) and \( \nu_{\overline{K}} : \overline{K}^* \to \mathbb{Z} \) are the discrete valuations. Since the expression in brackets is a tame symbol without sign, we see that \( \nu_K(\cdot, \cdot) \) is a map from \( K_2(K) \) to \( \mathbb{Z} \).

**Proposition 3.** Let \( K \) be a two-dimensional local field with last residue field \( \mathbb{F}_q \). Then \( \hat{\text{GL}}_2(K)_{\mathbb{R}_+^*} \) is obtained from \( \hat{\text{GL}}_2(K) \) (see the exact sequence (5)) by the map \( q^{-\nu_K(\cdot, \cdot)} : K_2(K) \to \mathbb{R}_+^* \).

**Proof.** By (6) it suffices to verify the following formula for all \( x, y \in K^* \):

\[
\langle \text{diag}(y, 1), \text{diag}(1, x) \rangle = q^{\nu_K(y, x)} = q^{-\nu_K(x, y)},
\]

where \( \langle \cdot, \cdot \rangle \) is the commutator of the lifts of two commuting elements of \( \text{GL}_2(K) \) to \( \hat{\text{GL}}_2(K)_{\mathbb{R}_+^*} \). Indeed, we have

\[
\langle \text{diag}(y, 1), \text{diag}(1, x) \rangle = \langle \text{diag}(y, 1), \text{diag}(x, 1) \text{diag}(x^{-1}, x) \rangle = \langle \text{diag}(y, 1), \text{diag}(x, 1) \rangle \langle \text{diag}(y, 1), \text{diag}(x^{-1}, x) \rangle = \langle \text{diag}(y, 1), \text{diag}(x^{-1}, x) \rangle.
\]

Here we use the bimultiplicativity of the map \( \langle \cdot, \cdot \rangle \) (see, for example, [13], Proposition 6). Moreover, \( \langle \text{diag}(y, 1), \text{diag}(x, 1) \rangle = 1 \) since the central extension \( \hat{\text{GL}}_2(K)_{\mathbb{R}_+^*} \) splits over the subgroup \( K^* \) of \( \text{GL}_2(K)_{\mathbb{R}_+^*} \). Since \( \hat{\text{GL}}_2(K)_{\mathbb{R}_+^*} \) is a semidirect product by construction, one can calculate \( \langle \text{diag}(y, 1), \text{diag}(x^{-1}, x) \rangle \) in \( \hat{\text{GL}}_2(K)_{\mathbb{R}_+^*} \) (the answer will be the same). It is clear from (12) and the construction of the central extension (10) that the commutator of the lifts of diagonal matrices can be calculated componentwise (at each place on the diagonal). Then we take the product over both components to get the answer. Therefore,

\[
\langle \text{diag}(y, 1), \text{diag}(x^{-1}, x) \rangle = \langle y, x^{-1} \rangle,
\]

where \( \langle y, x^{-1} \rangle \) is calculated in \( \hat{\text{GL}}_1(K)_{\mathbb{R}_+^*} \). The formula

\[
\langle y, x^{-1} \rangle = q^{\nu_K(y, x)}
\] (15)

can be verified using the bimultiplicativity of the map \( \langle \cdot, \cdot \rangle \) and the decomposition \( K^* = t^\mathbb{Z} \cdot \mathcal{O}_K^* \), where \( t \) is a local parameter in \( K \) (compare also with the proof of Theorem 1 in [13]). \( \square \)

**Remark 5.** For \( n \geq 2 \) we easily see that passage from \( \hat{\text{GL}}_n(K)_{\mathbb{R}_+^*} \) to \( \hat{\text{GL}}_n(K)_{\mathbb{R}_+^*} \) (see the construction in \( \S\ 3.2 \)) corresponds, at the level of second cohomology groups, to projection onto the second direct summand on the right-hand side of (7). We shall make more concrete calculations at the level of \( K \)-groups in the proof of Theorem 1 below.
3.4. Central extensions on arithmetic surfaces with Archimedean valua-
tions taken into account. Every number field has points at infinity: Archimedean valuations. The full (arithmetic) adelic ring of a number field takes these valuations into account.

Let $X$ be a two-dimensional normal integral scheme of finite type over $\mathbb{Z}$ such that there is a proper surjective morphism onto $\text{Spec} \mathbb{Z}$. Then we call $X$ an arithmetic surface. Let $X_\mathbb{Q} = X \otimes_{\text{Spec} \mathbb{Z}} \text{Spec} \mathbb{Q}$ be the generic fibre of the morphism. For every closed point $p \in X_\mathbb{Q}$ we define rings

$$K_p \hat{\otimes} \mathbb{R} = (\mathbb{Q}(p) \otimes \mathbb{R})((t_p)), \quad \mathcal{O}_{K_p} \hat{\otimes} \mathbb{R} = (\mathbb{Q}(p) \otimes \mathbb{R})[[t_p]],$$

where $\mathbb{Q}(p)$ is the residue field of the point $p$ on the curve $X_\mathbb{Q}$ and $t_p$ is a local parameter at the point $p$ on $X_\mathbb{Q}$ (note that the local ring of the point $p$ on the one-
dimensional scheme $X_\mathbb{Q}$ is a discrete valuation ring). We now define the arithmetic adelic ring $\mathbb{A}_{X}^\mathbb{Q}$ (see also [4], Example 11) as

$$\mathbb{A}_{X}^\mathbb{Q} = \mathbb{A}_{X} \times \mathbb{A}_{X, \infty}, \quad \mathbb{A}_{X, \infty} = \prod_{p \in X_\mathbb{Q}} (K_p \hat{\otimes} \mathbb{R}), \quad (16)$$

where the ring $\mathbb{A}_{X, \infty}$ is the restricted product with respect to the subrings $\mathcal{O}_{K_p} \hat{\otimes} \mathbb{R}$. We note that the closed points $p \in X_\mathbb{Q}$ are in a one-to-one correspondence with the integral one-dimensional subschemes of $X$ that are mapped surjectively onto $\text{Spec} \mathbb{Z}$ (the horizontal arithmetic curves on $X$). Moreover, $\mathbb{Q}(p) \otimes \mathbb{R} = \prod_i L_i$, where the product is taken over all equivalence classes of Archimedean valuations of the field $\mathbb{Q}(p)$, and each field $L_i$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. (We also note that $\mathbb{A}_{X, \infty}$ is a subring of the adelic ring of the curve $X_\mathbb{R} = X_\mathbb{Q} \otimes \mathbb{R}$, and the non-zero components of this subring correspond exactly to those components of the adelic ring of $X_\mathbb{R}$ which come from algebraic (non-transcendental) points of $X_\mathbb{R}$, that is, from those closed points of $X_\mathbb{R}$ whose images under the natural map $X_\mathbb{R} \to X_\mathbb{Q}$ are closed points.) Therefore the definition of $\mathbb{A}_{X}^\mathbb{Q}$ may be interpreted as adding one of the fields $\mathbb{R}((t_p))$ or $\mathbb{C}((t_p))$ to the scheme adèles of $X$. These fields correspond to the equivalence classes of Archimedean valuations on irreducible horizontal arithmetic curves and satisfy the adelic condition along $X_\mathbb{Q}$.

Let $\Delta$ be any set of closed points on $X_\mathbb{Q}$. We put

$$\mathbb{A}_{\Delta, \infty} = \prod_{p \in \Delta} (K_p \hat{\otimes} \mathbb{R}), \quad \mathcal{O}_{\mathbb{A}_{\Delta, \infty}} = \prod_{p \in \Delta} (\mathcal{O}_{K_p} \hat{\otimes} \mathbb{R}).$$

Similarly to the constructions in § 3.2, we define central extensions\(^6\) $\tilde{\text{GL}}_n(\mathbb{A}_{\Delta, \infty})_{\mathbb{R}^+_\mathbb{A}}$ and $\text{GL}_n(\mathbb{A}_{\Delta, \infty})_{\mathbb{R}^+_\mathbb{A}}$ of $\text{GL}_n(\mathbb{A}_{\Delta, \infty})$ by $\mathbb{R}^+_\mathbb{A}$. By construction, these central extensions

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\(^6\)Note that $\tilde{\text{GL}}_n(\mathbb{A}_{\Delta, \infty})_{\mathbb{R}^+_\mathbb{A}}$ can also be obtained from the central extension of $\text{GL}_n(\mathbb{A}_{\Delta, \infty})$ by $\mathbb{R}^*_\mathbb{A}$ constructed by Arbarello, de Concini and Katz [14]. This is done by applying the map $\mathbb{R}^* \to \mathbb{R}^*_+, \ x \mapsto |x|^{-1}$, to the kernel of this central extension in the following way. We regard $\mathbb{A}_{\Delta, \infty}$ as a vector space over $\mathbb{R}$ with the canonical action of $\text{GL}_n(\mathbb{A}_{\Delta, \infty})$. Then, for every finite-dimensional $\mathbb{R}$-vector space $V$, there is a canonical identification of the $\mathbb{R}^*_+$-torsors $|\det(V)^* \setminus 0|$ and $\mu(V)$ by integration of differential forms, where the first $\mathbb{R}^*_+$-torsor is obtained from the $\mathbb{R}^*$-torsor $\det(V)^* \setminus 0$ by applying the norm map between the structure groups.
split canonically over the subgroup \( \text{GL}_n(\mathcal{O}_{\mathbb{A}_{\nabla}}, \mathbb{R}) \) of \( \text{GL}_n(\mathbb{A}_{\nabla}, \mathbb{R}) \). Proposition 2 and Remark 5 also hold for these central extensions for similar reasons. Here is an analogue of Proposition 3.

**Proposition 4.** The central extensions \( \text{GL}_2(\mathbb{R}(t))_{\mathbb{R}^+} \) and \( \text{GL}_2(\mathbb{C}(t))_{\mathbb{R}^+} \) are obtained from the central extensions (5) by applying the maps

\[
K_2(\mathbb{R}(t)) \to \mathbb{R}^+, \quad (f, g) \mapsto \frac{f^\nu(g)}{g^\nu(f)}_{\mathbb{R}},
\]

\[
K_2(\mathbb{C}(t)) \to \mathbb{R}^+, \quad (f, g) \mapsto \left| \frac{f^\nu(g)}{g^\nu(f)} \right|^2_{\mathbb{C}}
\]

respectively, where \( \nu \) is the discrete valuation, and \( |\cdot|_{\mathbb{R}}, |\cdot|_{\mathbb{C}} \) are the usual absolute values on the fields \( \mathbb{R}, \mathbb{C} \).

**Proof.** This is similar to the proof of Proposition 3. \( \square \)

We define a central extension \( \text{GL}_n(\mathbb{A}_{\nabla X})_{\mathbb{R}^+} \) of \( \text{GL}_n(\mathbb{A}_{\nabla X}) \) by \( \mathbb{R}^+ \) as the Baer sum of the central extensions \( \text{GL}_n(\mathbb{A}_{\nabla X})_{\mathbb{R}^+} \) and \( \text{GL}_n(\mathbb{A}_{\nabla X, \infty})_{\mathbb{R}^+} \) with respect to (16). We similarly define the central extension \( \text{GL}_n(\mathbb{A}_{\nabla X})_{\mathbb{R}^+} \).

### 3.5. Non-commutative reciprocity laws.

Let \( X \) be a two-dimensional normal integral scheme of finite type over \( \mathbb{Z} \). For every closed point \( x \in X \) we define a ring \( K_x \) as the localization of the ring \( \hat{O}_x \) with respect to the multiplicative system \( \mathcal{O}_x \setminus 0 \) (recall that \( \mathcal{O}_x \) is the local ring of \( x \) on \( X \) and \( \hat{O}_x \) is the completion of \( \mathcal{O}_x \) by the maximal ideal). For every integral one-dimensional subscheme \( C \) on \( X \) we define a field \( K_C \) as the completion of the field of rational functions on \( X \) over the discrete valuation given by the curve \( C \). The rings \( K_x \) and \( K_C \) appear naturally in the theory of two-dimensional adèles when one describes the semilocal situation on \( X \) (see [15]).

There is a diagonal embedding of the ring \( K_x \) in the ring \( \mathbb{A}_{\nabla X} \) via the embeddings in all the two-dimensional local fields arising from the integral one-dimensional subschemes of \( X \) that pass through \( x \). We also have a diagonal embedding of the field \( K_C \) in the ring \( \mathbb{A}_{\nabla X} \) (or in \( \mathbb{A}_{\nabla X}^{\text{ar}} \) if \( X \) is an arithmetic surface) via the embeddings in all the two-dimensional local fields arising from the points of \( C \) (in the case of an arithmetic surface \( X \) and a horizontal arithmetic curve \( C \) we must also embed \( K_C \) in the Archimedean part \( \mathbb{A}_{\nabla X, \infty} \), which takes the Archimedean points of \( C \) into account). Moreover, the field of rational functions \( \mathbb{F}_q(X) \) (for surfaces \( X \) over \( \mathbb{F}_q \)) or the field of rational functions \( \mathbb{Q}(X) \) (for arithmetic surfaces \( X \)) is diagonally embedded in the ring \( \mathbb{A}_{\nabla X} \) or the ring \( \mathbb{A}_{\nabla X}^{\text{ar}} \) respectively. We denote this field of rational functions by \( K_X \).

**Theorem 1.** Let \( X \) be an integral two-dimensional normal scheme of finite type over \( \mathbb{Z} \) which is either a projective surface over \( \mathbb{F}_q \) or an arithmetic surface. Let \( \mathbb{A} \) be the ring \( \mathbb{A}_{\nabla X} \) (for surfaces over \( \mathbb{F}_q \)) or \( \mathbb{A}_{\nabla X}^{\text{ar}} \) (for arithmetic surfaces). Then the following non-commutative reciprocity laws hold. For every \( n \geq 1 \) the central extension \( \text{GL}_n(\mathbb{A})_{\mathbb{R}^+} \) of \( \text{GL}_n(\mathbb{A}) \) by \( \mathbb{R}^+ \) splits canonically over the subgroups \( \text{GL}_n(K_x), \text{GL}_n(K_C) \), where \( x \) is any closed point on \( X \) and \( C \) is any integral one-dimensional subscheme of \( X \).
Proof. We first prove that $\GL_n(\hat{A}_{x})_{R^*_+}^*$ splits over $\GL_n(K_x)$.

Consider the ring of adèles $\hat{A}_{x}$ of the scheme $\text{Spec} \mathcal{O}_x$. This adelic ring contains only those two-dimensional local fields that arise from integral one-dimensional subschemes containing $x$ on $X$. Note that we actually only need $\hat{A}_{x}$ to construct the restriction of the central extension to $\GL_n(K_x)$. The ring of adèles $\hat{A}_{x}$ of the scheme $\text{Spec} \hat{\mathcal{O}}_x$ contains $\hat{A}_{x}$ as a direct factor and also contains the two-dimensional local fields obtained from the prime ideals $\eta$ of height 1 in $\hat{\mathcal{O}}_x$ such that $\eta \cap \mathcal{O}_x = 0$. We note that if $K_\eta$ is a two-dimensional local field of this kind, then $K_x \subset \mathcal{O}_{K_\eta}$ (recall that $\mathcal{O}_{K_\eta}$ is the discrete valuation ring of the field $K_\eta$).

The central extensions $\GL_n(\hat{A}_{x})_{R^*_+}^*$ and $\GL_n(\hat{A}_{x})_{R^*_+}^*$ of $\GL_n(\hat{A}_{x})_{R^*_+}^*$ are constructed as in § 3.2. We first prove that the latter splits over the subgroup $\GL_n(\text{Frac}(\hat{\mathcal{O}}_x))$, where $\text{Frac}(\hat{\mathcal{O}}_x)$ is the field of fractions of the ring $\hat{\mathcal{O}}_x$. Since the group $\text{SL}_n(\text{Frac}(\hat{\mathcal{O}}_x))$ is perfect, it follows from the construction of the central extension that it suffices to prove the splitting of $\GL_n(\hat{A}_{x})_{R^*_+}^*$ over $\text{SL}_n(\text{Frac}(\hat{\mathcal{O}}_x))$.

We denote the restriction of the central extension $\GL_n(\hat{A}_{x})_{R^*_+}^*$ to the subgroup $\text{SL}_n(\text{Frac}(\hat{\mathcal{O}}_x))$ by $\text{SL}_n(\text{Frac}(\hat{\mathcal{O}}_x))_{R^*_+} = \text{SL}_n(\text{Frac}(\hat{\mathcal{O}}_x))_{R^*_+}$. The resulting central extensions give a well-defined central extension $\text{SL}(\text{Frac}(\hat{\mathcal{O}}_x))_{R^*_+} = \text{SL}(\text{Frac}(\hat{\mathcal{O}}_x))_{R^*_+}$ of $\text{SL}(\text{Frac}(\hat{\mathcal{O}}_x))_{R^*_+}$ by $\mathbb{R}^*_+$. Since the central extension (3) is universal, the central extension $\text{SL}(\text{Frac}(\hat{\mathcal{O}}_x))_{R^*_+}$ is obtained from $\text{St}(\text{Frac}(\hat{\mathcal{O}}_x))$ by means of some map $K_2(\text{Frac}(\hat{\mathcal{O}}_x)) \to \mathbb{R}^*_+$. To calculate this map, it suffices to compute the expression $\langle \text{diag}(u, u^{-1}, 1), \text{diag}(v, 1, v^{-1}) \rangle$ in $\text{SL}(\text{Frac}(\hat{\mathcal{O}}_x))_{R^*_+}$ for any element $(u, v) \in K_2(\text{Frac}(\hat{\mathcal{O}}_x))$ (see [16], Remark after Corollary 11.3).

Since the commutator of the lifts of diagonal matrices can be calculated by treating each place at the diagonal separately and then taking the product, it follows from the construction of the central extension that

$$\langle \text{diag}(u, u^{-1}, 1), \text{diag}(v, 1, v^{-1}) \rangle = \langle u, v \rangle,$$

where $\langle u, v \rangle$ is computed in the group $\GL_1(\text{Frac}(\hat{\mathcal{O}}_x))_{R^*_+} \subset \GL_1(\hat{A}_{x})_{R^*_+}$. To calculate $\langle u, v \rangle$, we split the ring $\hat{A}_{x}$ in a direct sum of two subrings:

$$\hat{A}_{x} = A_1 \oplus A_2,$$

\[17\]

\[7\] Indeed, the group $\GL_n(\hat{A}_{x})_{R^*_+}^*$ is a semidirect product of the group obtained by restriction of the central extension $\GL_n(\hat{A}_{x})_{R^*_+}^*$ to the subgroup $\text{SL}_n(\hat{A}_{x})$ and the group $\hat{A}_{x}^*$. Since $\text{SL}_n(\text{Frac}(\hat{\mathcal{O}}_x))$ is perfect, any section of the central extension over $\text{SL}_n(\text{Frac}(\hat{\mathcal{O}}_x))$ is invariant under the automorphism determined by an element of $\text{Frac}(\hat{\mathcal{O}}_x)^*$ that occurs in the definition of the semidirect product. Therefore the composite of this section and the identity section over $\text{Frac}(\hat{\mathcal{O}}_x)^*$ gives a section of the central extension $\GL_n(\hat{A}_{x})_{R^*_+}^*$ over the subgroup $\GL_n(\text{Frac}(\hat{\mathcal{O}}_x)) = \text{SL}_n(\text{Frac}(\hat{\mathcal{O}}_x)) \times \text{Frac}(\hat{\mathcal{O}}_x)^*$.

\[8\] Compare also the proof of Theorem 2 in [13].
where \( \mathbb{A}_1 \) is a finite product of two-dimensional local fields such that any field in this product is endowed with the discrete valuation which vanishes on \( u \) or \( v \) and \( \mathbb{A}_1 \) contains all the two-dimensional local fields with this property, and \( \mathbb{A}_2 \) is the adelic product of all other two-dimensional local fields in \( \hat{\mathbb{A}}_x \). An analogue of part 1) of Proposition 2 for (17) yields that \( \langle u, v \rangle = \langle u, v \rangle_1 \cdot \langle u, v \rangle_2 \), where the commutator \( \langle u, v \rangle_i \), \( 1 \leq i \leq 2 \), is calculated in \( \text{GL}_1(\mathbb{A}_i)_{\mathbb{R}_+^*} \). Since \( u, v \in O_{\mathbb{A}_2}^* \) and \( \text{GL}_1(\hat{\mathbb{A}}_2)_{\mathbb{R}_+^*} \) splits over the subgroup \( O_{\mathbb{A}_2}^* \), we have \( \langle u, v \rangle_2 = 1 \). Again using the analogue of part 1) of Proposition 2, we see that \( \langle u, v \rangle_1 \) is a finite product of separately calculated commutators over all two-dimensional local fields in \( \mathbb{A}_1 \).

We now use formula (15) in the proof of Proposition 3 and the equation \( \nu_K(v, u) = 0 \) for \( u, v \in O_K^* \), where \( K \) is a two-dimensional local field, to deduce that

\[
\langle u, v \rangle = \prod_K q_K^{\nu_K(v, u)},
\]

where \( K \) ranges over the two-dimensional local fields which are components of \( \hat{\mathbb{A}}_x \) and \( q_K \) is the number of elements in the residue field of \( K \).

Since the map \( \nu_K(\cdot, \cdot) \) is a composite of boundary maps in Milnor \( K \)-theory, an analogue of the Gersten–Quillen complex for Milnor \( K \)-theory (see [17], Proposition 1) yields the reciprocity law around the point \( x \):

\[
\sum_K \nu_K(v, u) \log_{q_x}(q_K) = 0,
\]

where \( q_x \) is the number of elements in the residue field of \( x \). Therefore the central extension \( \text{GL}_n(\hat{\mathbb{A}}_x)_{\mathbb{R}_+^*} \) splits over the subgroup \( \text{GL}_n(\text{Frac}(\hat{\mathbb{O}}_x)) \).

By construction, the restriction of \( \text{GL}_n(\hat{\mathbb{A}})_{\mathbb{R}_+^*} \) to \( \text{GL}_n(K_x) \) coincides with the restriction of \( \text{GL}_n(\hat{\mathbb{A}}_x)_{\mathbb{R}_+^*} \) to the same subgroup. We have just proved that the latter splits over this subgroup because \( \text{GL}_n(K_x) \subset \text{GL}_n(\text{Frac}(\hat{\mathbb{O}}_x)) \). Hence \( \text{GL}_n(\hat{\mathbb{A}})_{\mathbb{R}_+^*} \) splits over \( \text{GL}_n(K_x) \).

The proof of the splitting of \( \text{GL}_n(\hat{\mathbb{A}})_{\mathbb{R}_+^*} \) over \( \text{GL}_n(K_C) \) is similar to that in the previous case. It reduces to the proof of the splitting of \( \text{SL}(K_C)_{\mathbb{R}_+^*} \) over \( \text{SL}(K_C) \). For this purpose we use the group \( \text{St}(K_C) \) and compute \( \langle u, v \rangle \) in the group \( \text{GL}_1(K_C)_{\mathbb{R}_+^*} \) for \( (u, v) \in K_2(K_C) \). Using the bimultiplicativity of \( \langle \cdot, \cdot \rangle \), the property \( \langle t, t \rangle = (-1, t) \) (which follows from the Steinberg property) and the construction of \( \text{GL}_1(K_C)_{\mathbb{R}_+^*} \), we easily see that the only non-trivial case to calculate is that of \( \langle a, t \rangle \), where \( t \) is a local parameter of the field \( K_C \) and \( \nu_t(a) = 0 \) (\( \nu_t \) is a discrete valuation in the field \( K_C \)). In this case we easily see that \( \langle a, t \rangle \) coincides with the product over all normalized absolute values of the field of rational functions on the one-dimensional subscheme \( C \) applied to the image of \( a \) in this field (compare with Propositions 3, 4). Then the product formula implies that \( \langle a, t \rangle = 1 \). Hence \( \langle u, v \rangle = 1 \) for all \( (u, v) \in K_2(K_C) \). This proves that \( \text{GL}_n(\hat{\mathbb{A}})_{\mathbb{R}_+^*} \) splits over \( \text{GL}_n(K_C) \).
The proof of the splitting of $\widehat{\text{GL}}_n(A)_{\mathbb{R}^+}$ over $\text{GL}_n(K_X)$ is similar to that in the case just analyzed. We use the group $\text{St}(K_X)$ and prove the equation $\langle u, v \rangle = 1$ for $(u, v) \in K_2(K_X)$ using the product formulae over the normalized absolute values of the fields of rational functions of finitely many one-dimensional integral subschemes on $X$ (namely, the subschemes along which the functions $u$ and $v$ have zeros or poles). □

Remark 6. It follows from Theorem 1 that the central extension $\widehat{\text{GL}}_n(A)_{\mathbb{R}^+}$ splits over the subgroups $\text{GL}_n(K_x)$, $\text{GL}_n(K_C)$ and $\text{GL}_n(K_X)$. (In case of $\text{GL}_n(K_x)$ one must first consider the central extensions over the group $\text{GL}_n(\text{Frac}(\hat{O}_x))$ as in the proof of Theorem 1.) Indeed, Remarks 2, 5 reduce the proof to the case $n = 1$. Using the same reciprocity laws as above (around a point and along a one-dimensional subscheme), we see that the group in which we compute the commutator and prove the splitting is abelian. We further note that $\text{Ext}_1^Z(\cdot, \mathbb{R}^+_+) = 0$ since $\mathbb{R}^+_+$ is a divisible group.

§ 4. The unramified Langlands correspondence for two-dimensional local fields

4.1. The unramified Langlands correspondence for one-dimensional local fields. We first recall the construction for a one-dimensional local field $K$ (see the surveys [18] and [19]).

Let $\hat{K}$ be a one-dimensional local field with finite residue field $\mathbb{F}_q$, and let $K^{nr}$ be the maximal unramified extension of $K$. Then the Galois group $\text{Gal}(K^{nr}/K) = \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) = \hat{\mathbb{Z}}$ is topologically generated by the Frobenius automorphism $\text{Fr}$. The reciprocity map is given by

$$K^* \to \text{Gal}(K^{nr}/K), \quad f \mapsto \text{Fr}^{\nu_K(f)},$$

where $\nu_K$ is the discrete valuation on $K$.

Let $\rho$ be an unramified $n$-dimensional semisimple complex representation of the Weil group $W_K \subset \text{Gal}(K^{ab}/K)$. Then this representation is a semisimple representation of the group $\mathbb{Z}$ and, up to isomorphism, is determined by the $n$-tuple of non-zero complex numbers $\alpha_1, \ldots, \alpha_n$ which are the eigenvalues of the lift of the Frobenius automorphism $\text{Fr}$.

On the other hand, consider the group $\text{GL}_n(K)$. Let $B \subset \text{GL}_n(K)$ be the Borel subgroup of upper triangular matrices. We define a character $\chi_{\alpha_1, \ldots, \alpha_n} : B \to \mathbb{C}^*$ by

$$\chi\left(\begin{pmatrix} b_1 & * & * & * \\ b_2 & * & * & * \\ \vdots & \vdots & \ddots & \vdots \\ b_n \end{pmatrix}\right) = \alpha_1^{\nu_K(b_1)} \cdots \alpha_n^{\nu_K(b_n)} = (q^{-\nu_K(b_1)})^{a_1} \cdots (q^{-\nu_K(b_n)})^{a_n},$$

where the element $(a_1, \ldots, a_n) \in \mathbb{C}^n/(2\pi i/\ln q)\mathbb{Z}^n$ is given by $q^{-a_i} = \alpha_i$, $1 \leq i \leq n$.

For every $m \geq 1$ we define a normal subgroup $C_m$ of $\text{GL}_n(\mathcal{O}_K)$:

$$C_m = 1 + t_K^m M_n(\mathcal{O}_K).$$
We used a non-normalized induction in the definition of the representation. One can rewrite the definition of the induced representation as follows:

\[ \pi(\alpha_1, \ldots, \alpha_n) = \text{Ind}_B^{GL_n(K)}(\chi_{\alpha_1, \ldots, \alpha_n}), \]

where the space of the induced representation \( \text{Ind}_B^{GL_n(K)}(\chi_{\alpha_1, \ldots, \alpha_n}) \) is

\[ \left\{ f : GL_n(K) \to \mathbb{C} \mid \begin{array}{l} 1) f(bg) = \chi_{\alpha_1, \ldots, \alpha_n}(b)f(g) \ \forall b \in B, \ \forall g \in GL_n(K) \\ 2) \exists n_f \geq 1 : f(hu) = f(h) \ \forall h \in GL_n(K), \ \forall u \in C_n \end{array} \right\}. \]

The group \( GL_n(K) \) acts on this space by right translations. In other words, given any \( g \in GL_n(K) \) and \( f \in \text{Ind}_B^{GL_n(K)}(\chi_{\alpha_1, \ldots, \alpha_n}) \), we have \( (gf)(x) = f(xg) \) for all \( x \in GL_n(K) \).

**Remark 7.** We used a non-normalized induction in the definition of the representation \( \pi(\alpha_1, \ldots, \alpha_n) \). For a normalized induction, one usually adds to the character \( \chi_{\alpha_1, \ldots, \alpha_n} \) a character which takes into account the non-unimodularity of the group \( B \).

**Remark 8.** One can rewrite the definition of the induced representation as follows (this will be used below). For every \( m \geq 0 \) we define an infinite-dimensional \( \mathbb{C} \)-vector space \( V_m = \prod_{g \in GL_n(K)/C_m} \mathbb{C} \). The vectors in \( V_m \) are any functions \( f \) from the set \( GL_n(K)/C_m \) of left cosets to the field \( \mathbb{C} \). Then the group \( B \) acts on \( V_m \) by

\[ b(f)(x) = \chi_{\alpha_1, \ldots, \alpha_n}(b)f(b^{-1}x), \]

where \( b \in B, \ f \in V_m, \ x \in GL_n(K)/C_m \). It is now clear that

\[ \pi(\alpha_1, \ldots, \alpha_n) = \bigcup_{m \geq 0} V_m^B, \]

where \( V_m^B \) is the space of \( B \)-invariant elements. The group \( GL_n(K) \) acts on the space \( \pi(\alpha_1, \ldots, \alpha_n) \) by the rule \( g(f)(x) = f(xg) \), where \( f \in V_m^B, \ g, x \in GL_n(K), \ g(f) \in V_l \) for some \( l \geq 0 \) (depending on \( m \geq 0 \) and \( g \)).

A representation of the group \( GL_n(K) \) on a vector space \( W \) is said to be smooth if \( W = \bigcup_{m \geq 0} W^{C_m} \).

An irreducible representation of the group \( GL_n(K) \) on a vector space \( W \) is said to be spherical (or unramified) if \( W^{GL_n(O_K)} \neq 0 \).

We note that, by construction, \( \pi(\alpha_1, \ldots, \alpha_n) \) is a smooth representation of \( GL_n(K) \). Moreover,

\[ \dim \mathbb{C} \pi(\alpha_1, \ldots, \alpha_n)^{GL_n(O_K)} = 1. \]

The representation \( \pi(\alpha_1, \ldots, \alpha_n) \) of \( GL_n(K) \) always admits a unique spherical subquotient, which becomes an isomorphic subquotient under any permutation of the complex numbers \( \alpha_1, \ldots, \alpha_n \). (Moreover, one can always permute \( \alpha_1, \ldots, \alpha_n \) in such a way that this subquotient will be a quotient.) The unramified Langlands correspondence 729
correspondence associates the representation \( \rho \) (see the beginning of this section) with the quotient representation (described above) of \( \text{GL}_n(K) \). All admissible\(^9\) spherical representations of \( \text{GL}_n(K) \) are obtained in this way.

We note that the irreducible admissible representations of \( \text{GL}_n(K) \) are nothing but the irreducible smooth representations of the same group (see [20], Theorem 3.25). Therefore, in the previous paragraph, one can speak about smooth spherical representations of \( \text{GL}_n(K) \) instead of admissible spherical representations.

### 4.2. Group actions on a \( k \)-linear category

We recall the definition of an action of a group \( G \) on a \( k \)-linear category \( \mathcal{B} \), where \( k \) is a field (see also [8]).

**Definition 1.** An action of a group \( G \) on a \( k \)-linear category \( \mathcal{B} \) consists of the following data.

1) For every element \( g \in G \) there is a \( k \)-linear functor \( \tau(g): \mathcal{B} \to \mathcal{B} \).

2) For every pair of elements \( g, h \in G \) there is an isomorphism of functors \( \psi_{g,h}: \tau(g) \circ \tau(h) \Rightarrow \tau(gh) \).

3) There is an isomorphism of functors \( \psi_1 : \tau(1) \Rightarrow \text{Id}_\mathcal{B} \).

Moreover, the following conditions must be satisfied.

(i) For all \( g, h, k \in G \) we have associativity \( \psi_{g,h,k}(\psi_{g,h} \circ \tau(k)) = \psi_{g,hk}(\tau(g) \circ \psi_{h,k}) \).

(ii) For every \( g \in G \) we have \( \psi_{1,g} = \psi_1 \circ \tau(g) \) and \( \psi_{g,1} = \tau(g) \circ \psi(1) \).

To specify the category being considered, we shall use the symbols \( \tau_\mathcal{B}(g) \), \( \psi_{\mathcal{B},g,h} \) and \( \psi_{\mathcal{B},1} \).

**Definition 2.** Let \( \mathcal{B} \) and \( \mathcal{C} \) be \( k \)-linear categories with a \( G \)-action. A \( G \)-linear functor \( T = (T, \varepsilon) \) from \( \mathcal{B} \) to \( \mathcal{C} \) consists of the following data.

1) There is a \( k \)-linear functor \( T: \mathcal{B} \to \mathcal{C} \).

2) For every \( g \in G \) there is an isomorphism of functors \( \varepsilon_g : T \circ \tau_\mathcal{B}(g) \Rightarrow \tau_\mathcal{C}(g) \circ T \).

Moreover, the following conditions must be satisfied.

(i) The following diagram is commutative for all \( g, h \in G \):

\[
\begin{array}{ccc}
T \circ \tau_\mathcal{B}(g) \circ \tau_\mathcal{B}(h) & \xrightarrow{T \circ \psi_{\mathcal{B},g,h}} & T \circ \tau_\mathcal{B}(gh) \\
\varepsilon_g \circ \tau_\mathcal{B}(h) & \downarrow & \varepsilon_{gh} \\
\tau_\mathcal{C}(g) \circ T \circ \tau_\mathcal{B}(h) & \xrightarrow{T \circ \psi_{\mathcal{C},g,h} \circ T} & \tau_\mathcal{C}(g) \circ T \circ \tau_\mathcal{C}(h) \circ T
\end{array}
\]

(ii) We have \( (\psi_{\mathcal{C},1} \circ T)\varepsilon_1 = T \circ \psi_{\mathcal{B},1} \).

**Remark 9.** The \( G \)-linear functors from \( \mathcal{B} \) to \( \mathcal{C} \) are the objects of a natural \( k \)-linear category, which is denoted by \( \mathcal{H}_\text{om} \). If \( (L, \varepsilon) \) and \( (M, \varepsilon) \) are objects of \( \mathcal{H}_\text{om} \), then the \( k \)-vector space \( \text{Hom}_{\mathcal{H}_\text{om}}((L, \varepsilon), (M, \varepsilon)) \) consists by definition of the morphisms of functors \( \varphi: L \Rightarrow M \) such that the following diagram

\[^9\text{We do not recall here the definition of an admissible representation.}\]
is commutative for every element \( g \in G \):

\[
L \circ \tau_B(g) \xrightarrow{\varepsilon_g} \tau_C(g) \circ L \\
\varphi \circ \tau_B(g) \downarrow \quad \downarrow \tau_C(g) \circ \varphi \\
M \circ \tau_B(g) \xrightarrow{\varepsilon_g} \tau_C(g) \circ M
\]

If \( B, C, D \) are \( k \)-linear categories with a \( G \)-action and \( (L, \varepsilon) \in \text{Ob}(\text{Hom}_G(B, C)) \), \( (N, \zeta) \in \text{Ob}(\text{Hom}_G(C, D)) \), then

\[
(N, \zeta) \circ (L, \varepsilon) = (N \circ L, \eta) \in \text{Ob}(\text{Hom}_G(B, D)),
\]

where, for every \( g \in G \), the morphism of functors \( \eta_g : N \circ L \circ \tau_B(g) \Rightarrow \tau_D(g) \circ N \circ L \) is defined as the composite of morphisms of functors

\[
N \circ L \circ \tau_B(g) \xrightarrow{N \circ \varepsilon_g} N \circ \tau_C(g) \circ L \xrightarrow{\zeta_g \circ L} \tau_D(g) \circ N \circ L.
\]

We can now say that two \( k \)-linear categories \( B \) and \( C \) with \( G \)-action are \( G \)-equivalent if one can find a \( G \)-linear functor \( \mathfrak{L} \) from \( B \) to \( C \) and a \( G \)-linear functor \( \mathfrak{N} \) from \( C \) to \( B \) such that the \( G \)-linear functors \( \mathfrak{N} \circ \mathfrak{L} \) and \( (\text{Id}_B, \text{Id}) \) are isomorphic as objects of \( \text{Hom}_G(B, B) \), and the \( G \)-linear functors \( \mathfrak{L} \circ \mathfrak{N} \) and \( (\text{Id}_C, \text{Id}) \) are isomorphic as objects of \( \text{Hom}_G(C, C) \).

Clearly, to establish the \( G \)-equivalence of \( B \) and \( C \), it suffices to construct a \( G \)-linear functor \( (L, \varepsilon) : B \rightarrow C \) and verify that the functor \( L \) is an equivalence of categories.

**Definition 3.** Let \( B \) be a \( k \)-linear category with \( G \)-action. The category \( \mathcal{B}^G \) of \( G \)-equivariant objects is defined as follows.

1) An object of \( \mathcal{B}^G \) consists of an object \( E \) of \( B \) and a system of isomorphisms \( \theta_g : E \rightarrow \tau(g)(E) \) (for every \( g \in G \)) such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\theta_g} & \tau(g)(E) \\
\theta_{gh} \downarrow & & \downarrow \tau(g)(\theta_h) \\
\tau(gh)(E) & \xleftarrow{\psi_{g,h,E}} & \tau(g)(\tau(h)(E))
\end{array}
\]  \tag{18}

is commutative for all elements \( g, h \in G \).

2) A morphism \( (E, \theta) \rightarrow (D, \vartheta) \) of equivariant objects is a morphism \( f \in \text{Hom}_B(E, D) \) such that \( \tau(g)(f) \theta_g = \vartheta_g f \).

**Remark 10.** If \( (E, \theta) \in \text{Ob}(\mathcal{B}^G) \), then every object \( D \) isomorphic to \( E \) has an obvious \( G \)-equivariant structure \( \vartheta \).

**Remark 11.** Let \( (E, \theta) \) and \( (D, \vartheta) \) be objects of \( \mathcal{B}^G \). Then the group \( G \) acts on the \( k \)-vector space \( \text{Hom}_B(E, D) \) by the rule \( f \mapsto \vartheta^{-1}_g \tau(g)(f) \theta_g \). (This is indeed a group action by diagram (18).) We now easily see that \( \text{Hom}_B(E, D)^G = \text{Hom}_G((E, \theta), (D, \vartheta)) \).
4.3. A categorical analogue of the unramified principal series representations. Up to the end of the paper, $K$ is a two-dimensional local field with last finite residue field $F_q$. Let $\mathcal{O}_K$ be the discrete valuation ring of rank 1 of the field $K$, and let $t_K$ be a local parameter with respect to this discrete valuation. For every $m \geq 1$ we define by analogy with § 4.1 the following normal subgroups (congruence subgroups) of $GL_l(\mathcal{O}_K)$, $l \geq 1$:

$$\mathbb{C}_m = 1 + t_K^m M_l(\mathcal{O}_K).$$

We also put $\mathbb{C}_0 = GL_l(\mathcal{O}_K)$.

**Definition 4.** A category $\mathcal{B}$ is called a generalized 2-vector space over a field $k$ if $\mathcal{B}$ is a $k$-linear abelian category. We say that the category $\mathcal{B}$ has finite dimension $r$ if $\mathcal{B}$ is equivalent to the category $(\text{Vect}_k^{\text{fin}})^r$.

We introduce a categorical analogue of a smooth representation.

**Definition 5.** An action of the group $GL_l(K)$ on a generalized 2-vector space $\mathcal{B}$ is said to be smooth if the following condition holds. For any objects $E, D$ in $\mathcal{B}$ and any morphism $f \in \text{Hom}_\mathcal{B}(E, D)$ one can find an integer $m \geq 0$ and objects $(E, \theta)$ and $(D, \vartheta)$ of the category $\mathcal{B}^{\mathbb{C}_m}$ such that $f \in \text{Hom}_{\mathcal{B}^{\mathbb{C}_m}} ((E, \theta), (D, \vartheta))$.

In particular, applying Definition 5 to the identity morphism, we see that every object of $\mathcal{B}$ has a $\mathbb{C}_m$-equivariant structure for some $m \geq 0$.

For a two-dimensional local field $K$ with last residue field $F_q$, let $K^\text{nr}$ be the maximal unramified extension of $K$ as a two-dimensional local field. Namely, the field $K^\text{nr}$ is unramified with respect to the discrete valuation on $K$, and the residue field $\overline{K}^\text{nr}$ of $K^\text{nr}$ is unramified with respect to the discrete valuation on the residue field $\overline{K}$ of $K$. Then the Galois group $\text{Gal}(K^\text{nr}/K) = \hat{\mathbb{Z}}$ is topologically generated by the Frobenius automorphism $\text{Fr}$. The two-dimensional unramified reciprocity map (see [20], Theorem 1) is given by

$$K_2(K) \to \text{Gal}(K^\text{nr}/K), \quad (f, g) \longmapsto \text{Fr}^{\nu_K(f,g)},$$

where the map $\nu_K(\cdot, \cdot)$ is defined by (14).

We define the Weil group$^{10}$ $W_K$ of a two-dimensional local field $K$ as the group which is the image of the whole reciprocity map $K_2(K) \to \text{Gal}(K^{ab}/K)$. Then every $n$-dimensional unramified semisimple complex representation $\rho$ of $W_K$ factors through the group $\mathbb{Z}$. Hence $\rho$ is determined (up to isomorphism) by the $n$-tuple $\alpha_1, \ldots, \alpha_n$ of non-zero complex numbers. The element $(a_1, \ldots, a_n) \in \mathbb{C}^n/(2\pi i/\ln q) \mathbb{Z}^n$ is now defined by putting $q^{-a_i} = \alpha_i$, $1 \leq i \leq n$.

Regard any ordered $n$-tuple $(a_1, \ldots, a_n)$ as an element of $\mathbb{C}^n/(2\pi i/\ln q) \mathbb{Z}^n$. Let $A$ be an arbitrary $\mathbb{R}_+^*$-torsor and $b$ any element of $\mathbb{C}$. We construct a one-dimensional $\mathbb{C}$-vector space

$$A^b = (A \otimes_{\mathbb{R}_+^*} \mathbb{C}^*) \cup 0,$$

---

$^{10}$By two-dimensional local class field theory, $W_K$ is a dense subgroup of the profinite group $\text{Gal}(K^{ab}/K)$. The image of $W_K$ under the map $\text{Gal}(K^{ab}/K) \to \text{Gal}(K^\text{nr}/K) = \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) = \hat{\mathbb{Z}}$ is the group $\mathbb{Z}$. 

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where the $\mathbb{C}^*$-torsor $A \otimes_{\mathbb{R}_+^*} \mathbb{C}^*$ is defined by means of the homomorphism
\[
\mathbb{R}_+^* \to \mathbb{C}^*, \quad x \mapsto x^b.
\]

In §3 we constructed the central extension
\[
1 \longrightarrow \mathbb{R}_+^* \longrightarrow \text{GL}_2(K)_{\mathbb{R}_+^*} \xrightarrow{\pi} \text{GL}_2(K) \longrightarrow 1.
\] (20)

By construction, this central extension splits canonically over the subgroup $\text{GL}_2(O_K)$. For every element $g \in \text{GL}_2(K)$ we define an $\mathbb{R}_+^*$-torsor $A_g = \pi^{-1}(g)$.

For every element $b \in \mathbb{C}$ and any elements $g, h \in \text{GL}_2(K)$, the central extension (20) yields a canonical isomorphism of one-dimensional $\mathbb{C}$-vector spaces
\[
(A_g)^b \otimes_{\mathbb{C}} (A_h)^b \to (A_{gh})^b
\] (21)
which satisfies the associativity condition for all $g, h, k \in \text{GL}_2(K)$.

For every integer $n \geq 1$ we consider the standard parabolic subgroup $P$ of block upper-triangular matrices in $\text{GL}_{2n}(K)$:
\[
P = \left\{ g = \begin{pmatrix} g_1 & * & * & \cdots & * \\
* & g_2 & * & \cdots & * \\
& \cdots & \cdots & \cdots & \cdots \\
& & & \cdots & \cdots \\
g_n & & & & 
\end{pmatrix} : g_i \in \text{GL}_2(K) \right\}.
\]

Given an element $(a_1, \ldots, a_n) \in \mathbb{C}^n/(2\pi i/\ln q)\mathbb{Z}^n$, we define the following action of the group $P$ on the $\mathbb{C}$-linear category $\text{Vect}^\text{fin}_\mathbb{C}$. For every $g \in P$ we define a $\mathbb{C}$-linear functor $\tau_{a_1,\ldots,a_n}(g) : \text{Vect}^\text{fin}_\mathbb{C} \to \text{Vect}^\text{fin}_\mathbb{C}$ by putting
\[
\tau_{a_1,\ldots,a_n}(g)(Y) = (A_{g_1})^{a_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (A_{g_n})^{a_n} \otimes_{\mathbb{C}} Y, \quad Y \in \text{Vect}^\text{fin}_\mathbb{C}.
\] (22)

The isomorphisms of functors $\tau_{a_1,\ldots,a_n}(g) \circ \tau_{a_1,\ldots,a_n}(h) \Rightarrow \tau_{a_1,\ldots,a_n}(gh)$ (where $g, h \in P$) and $\tau_{a_1,\ldots,a_n}(1) \Rightarrow \text{Id}_{\text{Vect}^\text{fin}_\mathbb{C}}$ are obtained from the maps (21) in the obvious way. Then all the conditions of Definition 1 hold for these isomorphisms.

Remark 12. This action of $P$ on $\text{Vect}^\text{fin}_\mathbb{C}$ depends only on the element of $\mathbb{C}^n/(2\pi i/\ln q)\mathbb{Z}^n$ (and not on the element of $\mathbb{C}^n$) because the central extension (20) arises from an element of $H^2(\text{GL}_2(K), \mathbb{Z})$ by applying the homomorphism $\mathbb{Z} \to \mathbb{C}^*$, $a \mapsto q^a$ (see §3.3).

We now construct a categorical analogue $\mathcal{V}_{a_1,\ldots,a_n}$ of the principal series representations of $\text{GL}_{2n}(K)$. For every $m \geq 0$ we consider a $\mathbb{C}$-linear category
\[
\mathcal{V}_m = \prod_{g \in \text{GL}_{2n}(K)/C_m} \text{Vect}^\text{fin}_\mathbb{C}
\]
whose objects are all maps $f$ from the set $\text{GL}_{2n}(K)/C_m$ of left cosets to the objects of the category $\text{Vect}^\text{fin}_\mathbb{C}$, and the morphisms in $\mathcal{V}_m$ are obvious. There is an action $\xi_{a_1,\ldots,a_n}$ of the group $P$ on the category $\mathcal{V}_m$:
\[
\xi_{a_1,\ldots,a_n}(p)(f)(x) = \tau_{a_1,\ldots,a_n}(p)(f(p^{-1}x)),
\] (23)
where \( p \in P, \ x \in \text{GL}_{2n}(K)/\mathbb{C}_m \) and \( f \in \text{Ob}(\mathcal{V}_m) \). (This action is also defined on the morphisms of \( \mathcal{V}_m \) in the obvious way.) We note that for any integers \( m_2 \geq m_1 \geq 0 \) there is a natural \( P \)-linear functor \( \mathcal{V}_{m_1} \to \mathcal{V}_{m_2} \) which arises from the map of sets \( \text{GL}_{2n}(K)/\mathbb{C}_{m_2} \to \text{GL}_{2n}(K)/\mathbb{C}_{m_1} \). This functor induces a natural functor \( Q_{m_1,m_2} : (\mathcal{V}_{m_1})^P \to (\mathcal{V}_{m_2})^P \) between the categories of \( P \)-equivariant objects. (In our situation there is also a strict equality of functors \( Q_{m_2,m_3} \circ Q_{m_1,m_2} = Q_{m_1,m_3} \) for all \( 0 \leq m_1 \leq m_2 \leq m_3 \).) We now define a \( \mathbb{C} \)-linear category

\[
\mathcal{V}_{a_1,...,a_n} = \lim_{m \to 0} (\mathcal{V}_m)^P. \tag{24}
\]

The objects of \( \mathcal{V}_{a_1,...,a_n} \) are all pairs \((m,f)\), where \( m \geq 0 \) is an integer and \( f \in \text{Ob}((\mathcal{V}_m)^P) \), and the morphisms are given by

\[
\text{Hom}_{\mathcal{V}_{a_1,...,a_n}} ((m_1,f_1),(m_2,f_2)) = \lim_{m \to \max(m_1,m_2)} \text{Hom}_{\mathcal{V}_m} (Q_{m_1,m}(f_1),Q_{m_2,m}(f_2)).
\]

(The definition of the categorical direct limit \( \lim_{m \to 0} \), which is used in (24), is taken from [22], Appendix.)

**Theorem 2.** Take an element \((a_1,\ldots,a_n) \in \mathbb{C}^n/(2\pi i/\ln q)\mathbb{Z}^n\). Then the following assertions hold.

1) The category \( \mathcal{V}_{a_1,...,a_n} \) is a generalized 2-vector space over \( \mathbb{C} \). There is a natural smooth action of the group \( \text{GL}_{2n}(K) \) on the category \( \mathcal{V}_{a_1,...,a_n} \).

2) The generalized 2-vector space \( (\mathcal{V}_{a_1,...,a_n})^{\text{GL}_{2n}((\mathbb{C})_K)} \) contains a category of dimension 1 (that is, a category equivalent to \( \text{Vect}_{\mathbb{C}}^{\text{fin}} \)) as a full subcategory.

3) Suppose that \( n = 1 \) and consider the following action of the group \( \text{GL}_2(K) \) on the category \( \text{Vect}_{\mathbb{C}}^{\text{fin}} \). This action depends on a parameter \( a_1 \in \mathbb{C}/(2\pi i/\ln q)\mathbb{Z} \) and arises from an unramified character of the group \( \text{Gal}(K^{\text{sep}}/K) \) (as described by formula (19) and in §2). Then there is a \( \text{GL}_2(K) \)-linear functor \( \mathcal{F} = (T,\varepsilon) \) from \( \text{Vect}_{\mathbb{C}}^{\text{fin}} \) to \( \mathcal{V}_{a_1} \) such that the functor \( T \) is fully faithful.

**Remark 13.** The functors from \( \text{Vect}_{\mathbb{C}}^{\text{fin}} \) which are mentioned in parts 2), 3) of Theorem 2 are exact because the category \( \text{Vect}_{\mathbb{C}}^{\text{fin}} \) is semisimple.

**Proof of Theorem 2.** The category \( \mathcal{V}_{a_1,...,a_n} \) is abelian for the following reasons. Since the category \( \prod_{g \in X} \text{Vect}_{\mathbb{C}}^{\text{fin}} \) is abelian for any set \( X \), the categories \( (\mathcal{V}_m)^P \) of \( P \)-equivariant objects are abelian for all \( m \geq 0 \). Moreover, the categorical direct limit \( \lim_{m \to 0} \) of any set of abelian categories with exact transition functors is an abelian category.

We define the following action \( \sigma \) of the group \( \text{GL}_{2n}(K) \) on the category \( \mathcal{V}_{a_1,...,a_n} \). (This action is a categorical analogue of the corresponding action for one-dimensional local fields; see Remark 8.) Suppose that

\[
(m,\tilde{f}) \in \text{Ob}(\mathcal{V}_{a_1,...,a_n}), \ g \in \text{GL}_{2n}(K).
\]

Taking into account the map \( \text{GL}_{2n}(K) \to \text{GL}_{2n}(K)/\mathbb{C}_m \), we consider \( \tilde{f} = (f,\theta) \), where \( f \) is a map from the set \( \text{GL}_{2n}(K) \) to the objects of the category \( \text{Vect}_{\mathbb{C}}^{\text{fin}} \) and \( \theta = \{\theta_{p,x}\} \ (p \in P, \ x \in \text{GL}_{2n}(K)) \) determines the structure of a \( P \)-equivariant
object on $f$ under the action of $P$ by formula (23). Then, by definition, we put
\[ \sigma(g)(m,(f,\theta)) = (l,(g(f),g(\theta))) \]
where $g(\{\theta_{x,g}\}) = \{\theta_{p,xg}\}$, $g(f)(x) = f(xg)$
for any $x \in \text{GL}_{2n}(K)$ and $l \geq 0$ is the smallest non-negative integer such that
$gC_{m}g^{-1} \supset C_{l}$. (The existence of such an integer $l$ follows, for example, from the
Cartan decomposition of the group $\text{GL}_{2n}(K)$ over the discrete valuation field $K$.)
Here the map $g(f)$ may be regarded as a well-defined map from the set $\text{GL}_{2n}(K)/C_{l}$
to the objects of $\text{Vect}_{\mathbb{C}}^{\text{fin}}$, and $g(\theta)$ determines the structure of a $P$-equivariant object
on $g(f)$. The action of the functor $\sigma(g)$ on the morphisms of the category $\mathcal{V}_{a_{1},...,a_{n}}$
is defined in the obvious way. Clearly, we obtain a well-defined action $\sigma$ of the
group $\text{GL}_{2n}(K)$ on the category $\mathcal{V}_{a_{1},...,a_{n}}$.

The smoothness of $\sigma$ follows directly from Definition 5 and the construction
of $\mathcal{V}_{a_{1},...,a_{n}}$ as a 2-inductive limit. (For every $m \geq 0$ the group $\mathbb{C}_{m}$ acts as the
identity on the category $(\mathcal{V}_{m})^{P}$. Therefore any object of this category always has the
$\mathbb{C}_{m}$-equivariant structure given by the identity morphisms.) This proves part 1) of
the theorem.

We now construct a functor $L$ from the category $\text{Vect}_{\mathbb{C}}^{\text{fin}}$ to the category
$(\mathcal{V}_{a_{1},...,a_{n}})^{\text{GL}_{2n}(O_{K})}$. It follows from the Iwasawa decomposition of the group
$\text{GL}_{2n}(K)$ over the discrete valuation field $K$ that $\text{GL}_{2n}(K) = P \cdot \text{GL}_{2n}(O_{K})$.

For every coset $x \in \text{GL}_{2n}(K)/\text{GL}_{2n}(O_{K})$ we fix a representative $g_{x} \in \text{GL}_{2n}(K)$
and a decomposition $g_{x} = p_{x}h_{x}$, where $p_{x} \in P$, $h_{x} \in \text{GL}_{2n}(O_{K})$. Given any
$V \in \text{Ob}(\text{Vect}_{\mathbb{C}}^{\text{fin}})$, we define $f_{V} \in \text{Ob}(\mathcal{V}_{0})$ by the formula
$f_{V}(x) = \tau_{a_{1},...,a_{n}}(p_{x})(V)$ (see (22)). There is a canonical splitting of the central extension (20)
over the subgroup $\text{GL}_{2}(O_{K})$. This splitting gives a canonical trivialization of the action $\tau_{a_{1},...,a_{n}}$
(restricted to the group $P \cap \text{GL}_{2n}(O_{K})$) on the category $\text{Vect}_{\mathbb{C}}^{\text{fin}}$. This trivialization
provides a unique lift of the object $f_{V}$ to an object $\widetilde{f}_{V}$ of the category $(\mathcal{V}_{0})^{P}$. Since the
group $\text{GL}_{2n}(O_{K})$ acts as the identity on the category $(\mathcal{V}_{0})^{P}$, there is an obvious
lift of the object $(0,\widetilde{f}_{V})$ from the category $\mathcal{V}_{a_{1},...,a_{n}}$ to an object $L(V)$ of the
category $(\mathcal{V}_{a_{1},...,a_{n}})^{\text{GL}_{2n}(O_{K})}$ by means of the identity morphisms. One defines $L$ on the
morphisms of the category $\text{Vect}_{\mathbb{C}}^{\text{fin}}$ in the obvious way. Clearly, $L$ is a fully faithful
functor from the category $\text{Vect}_{\mathbb{C}}^{\text{fin}}$ to the abelian category $(\mathcal{V}_{a_{1},...,a_{n}})^{\text{GL}_{2n}(O_{K})}$. This
proves part 2) of the theorem.

The functor $T$ in part 3) is the composite of the functor $L$ constructed in the
proof of part 2) and the forgetful functor (which forgets the $\text{GL}_{2}(O_{K})$-equivariant
structure). Namely, for every $V \in \text{Ob}(\text{Vect}_{\mathbb{C}}^{\text{fin}})$ we construct $T(V) = (0,\widetilde{f}_{V}) \in \text{Ob}(\mathcal{V}_{a_{1}})$. Note that $T$ is fully faithful because $L$ and the forgetful functor are fully faithful (for the forgetful functor, this follows since the $\text{GL}_{2}(O_{K})$-equivariant
structure is given by the identity morphisms).

It follows from the description of the reciprocity map for the two-dimensional
local field $K$ (see (19)), the construction of $\S 2$ and Proposition 3 that the action
of $\text{GL}_{2}(K)$ on $\text{Vect}_{\mathbb{C}}^{\text{fin}}$ (depending on the parameter $a_{1} \in \mathbb{C}/(2\pi i/\ln q)\mathbb{Z}$ and arising
from an unramified character of the group $\text{Gal}(K^{\text{sep}}/K)$ is described by the
functors $\tau_{a_{1}}(g)$ by formula (22), where $g \in P = \text{GL}_{2}(K)$. We now construct
a $\text{GL}_{2}(K)$-linear structure $\varepsilon$ on $T$ using the canonical splitting of the central extension (20)
over the subgroup $\text{GL}_{2}(O_{K})$. $\square$
Remark 14. We do not obtain an equivalence of categories in parts 2), 3) of Theorem 2 for the following reason. In part 2) we use the fact that $GL_{2n}(O_K)$ acts as the identity on $(\mathcal{V}_0)^P$ and take the identity morphisms in the $GL_{2n}(O_K)$-equivariant structure. In this case, besides the $GL_{2n}(O_K)$-equivariant structure given by the identity morphisms, there are many other $GL_{2n}(O_K)$-equivariant structures on the objects of $(\mathcal{V}_0)^P$, and these structures are not isomorphic to the identity structure. Similar problems appear in part 3) because $P \cap C_m \neq \{1\}$ for every $m \geq 0$.

Remark 15. The category $\mathcal{V}_{a_1,\ldots,a_n}$ with the $GL_{2n}(K)$-action is a categorical analogue of the induced representation in §4.1. For finite groups $H \subset G$ and a category $\mathcal{B}$ with an $H$-action, the category of the induced $G$-representation $\text{ind}|^G_H(\mathcal{B})$ was constructed and studied in [8] as a category of $H$-equivariant objects in a certain category. (See also [23], where the study of $G$-equivariant objects was continued.)

Remark 16. One can define analogous induced representations (depending on $(a_1,\ldots,a_n) \in \mathbb{C}^n$) for the group $GL_{2n}(\hat{A}_\Delta)$, where $\hat{A}_\Delta$ is a subring in the adelic ring $A_X$ or $\hat{A}_X^{\text{ab}}$ of a two-dimensional normal integral scheme $X$ of finite type over $\mathbb{Z}$ (see §3). Here one must use the central extension $\overline{GL_2(\hat{A}_\Delta)}_{\mathbb{R}_+^\times}$ (constructed in §3) of $GL_2(\hat{A}_\Delta)$ by $\mathbb{R}_+^\times$. Instead of the groups $\mathbb{C}_m$, one must take the following subgroups $\mathbb{C}_{\{m_C\}}$ of $\prod_{x \in C} GL_{2n}(K_{x,C})$:

$$\mathbb{C}_{\{m_C\}} = GL_{2n}(\hat{A}_\Delta) \cap \left(1 + \prod_{C \subset X} t_C^{m_C} \prod_{x \in C} M_{2n}(O_{K_{x,C}})\right),$$

where $C$ ranges over all one-dimensional integral subschemes of $X$, $t_C$ is a local parameter in the field $K_C$, and the set $\{m_C\}$ consists of integers $m_C \geq 0$ such that $m_C = 0$ for almost all $C$ (but not all $m_C$ are equal to zero). If $m_C = 0$ for any integral one-dimensional subscheme $C$, then

$$\mathbb{C}_{\{0\}} = GL_{2n}(\hat{A}_\Delta) \cap \prod_{x \in C} GL_{2n}(O_{K_{x,C}}).$$

When $\hat{A}_\Delta$ coincides with a two-dimensional local field $K$, the resulting categorical representation of $GL_{2n}(K)$ is exactly the categorical representation $\mathcal{V}_{a_1,\ldots,a_n}$ constructed above.

Remark 17. By Remark 11, for every subgroup $H \subset GL_{2n}(K)$ there is an action of $H$ on the $\mathbb{C}$-vector space $\text{Hom}_{\mathcal{V}_{a_1,\ldots,a_n}}(E, D)$ for any fixed objects $(E, \theta)$, $(D, \vartheta)$ of the category of $(\mathcal{V}_{a_1,\ldots,a_n})^H$. It would be interesting to compare these actions with those used in [24], in which the representation theory of reductive groups over two-dimensional local fields was constructed in the category $\text{Pro Vect}_\mathbb{C}$, where $\text{Vect}_\mathbb{C}$ is the category of all vector spaces over $\mathbb{C}$.

4.4. A conjecture. In analogy with §4.1 we make a conjecture about smooth spherical actions of $GL_{2n}(K)$ on generalized 2-vector spaces over $\mathbb{C}$, where $K$ is a two-dimensional local field.

Let $\mathcal{B}$ be an abelian $k$-linear category with a $G$-action, where $G$ is a group and $k$ is a field.
A $G$-subrepresentation of $B$ is a pair $(A, \Sigma)$, where $A$ is an abelian $k$-linear category with $G$-action and $\Sigma = (T, \varepsilon)$ is a $G$-linear functor from $A$ to $B$ such that the functor $T$ is fully faithful and exact.

The action of $G$ on $B$ is said to be irreducible if, for every $G$-subrepresentation $(A, \Sigma)$ of $B$, either $A$ is equivalent to a category consisting of only zero objects, or $T$ is essentially surjective (in the latter case, $T$ is an equivalence between $A$ and $B$, whence $A$ and $B$ are $G$-equivalent; see Remark 9).

A $G$-quotient representation of $B$ is a pair $(C, \Phi)$, where $C$ is an abelian $k$-linear category with $G$-action and $\Phi = (P, \varepsilon)$ is a $G$-linear functor from $B$ to $C$. Here we also demand that $P = Q \circ R$, where $R$ is the functor from $B$ to $B/A$ for some Serre subcategory $A$ of $B$, and $Q$ is an equivalence of the categories $B/A$ and $C$. (Note that the functor $P$ is exact.)

The definition of a spherical (or unramified) action must involve more than just the irreducibility of the action of $GL_{2n}(K)$ (where $n \geq 1$ and $K$ is a two-dimensional local field) on $B$ with the condition that the abelian category $B^{GL_{2n}(O_K)}$ is not equivalent to a category consisting of only zero objects. This may be explained as follows. The subring $O_K \hookrightarrow K$ is independent of the discrete valuation on the residue field $K$. But our unramified extensions of $K$ must also be unramified with respect to this valuation. Therefore we must take into account the discrete valuation on $K$.

The correct definition of a spherical (unramified) action of $GL_{2n}(K)$ would enable us to make the following conjecture.

**Conjecture.** Any smooth spherical action of $GL_{2n}(K)$ on a $C$-linear abelian category $B$ can be obtained as a subquotient representation (that is, a quotient of a subrepresentation) of the category $V_{a_1,\ldots,a_n}$ for some $(a_1,\ldots,a_n) \in \mathbb{C}^n/(2\pi i/\ln q)\mathbb{Z}^n$. (Here we do not mention the $GL_{2n}(K)$-linear functors which determine the quotient representation and the subrepresentation.)

We now consider the case when $n = 1$ and $B$ is equivalent to the category $\text{Vect}_{\mathbb{C}}^\text{fin}$ (in other words, $B$ is a one-dimensional 2-vector space). Then we easily see that any action of $GL_2(K)$ on $B$ is irreducible. By §2.1, up to equivalence, the actions of $GL_2(K)$ on $B$ correspond to central extensions (and this correspondence is one-to-one)

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \hat{G} \longrightarrow GL_2(K) \longrightarrow 1. \quad (25)$$

Clearly, the condition that the action of $GL_2(K)$ on $B$ is spherical, must match the condition that the central extension (25) is obtained from the central extension

$$1 \longrightarrow K_2(K) \longrightarrow \widehat{GL_2(K)} \longrightarrow GL_2(K) \longrightarrow 1$$

by means of a map

$$K_2(K) \rightarrow \mathbb{C}^*, \quad (f, g) \mapsto q^{-a_{\nu,K}(f,g)}, \quad (26)$$

where $q$ is the number of elements in the last residue field of $K$ and $a$ belongs to $\mathbb{C}/(2\pi i/\ln q)\mathbb{Z}$. When this condition holds, the action of $GL_2(K)$ on $B$ will be smooth because the central extension $\hat{G}$ splits over the subgroup $GL_2(O_K)$. 

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(Indeed, if this condition holds, then Proposition 3 yields that \( \hat{G} \) is obtained from \( \text{GL}_2(K)_{\mathbb{R}_+^2} \) by means of the map \( \mathbb{R}_+^* \to \mathbb{C}^* \), \( x \mapsto x^a \). By construction, \( \text{GL}_2(K)_{\mathbb{R}_+^2} \) splits over \( \text{GL}_2(O_K) \).) Moreover, by part 3) of Theorem 2, \( B \) is a \( \text{GL}_2(K) \)-subrepresentation of the category \( V_a \) (here we do not mention the \( \text{GL}_2(K) \)-linear functor that determines this subrepresentation).

We now consider a two-dimensional local field \( K \) which is isomorphic to one of the following fields (see also [2], § 2, for a description of how these fields arise from algebraic surfaces and arithmetic surfaces):

\[
\mathbb{F}_q(((u))((t))), \quad L((t)), \quad L\{\{u\}\},
\]

where \( L \) is a one-dimensional local field of characteristic zero and \( L\{\{u\}\} \) is the completion of the field \( \text{Frac}(O_L[[u]]) \) with respect to the discrete valuation given by the prime ideal \( m_L O_L[[u]] \) of height 1 in the ring \( O_L[[u]] \) (\( m_L \) is the maximal ideal of \( O_L \)). In accordance with (27) we define a subring \( B \) of \( K \) as one of the following expressions:

\[
\mathbb{F}_q[[u]][(t)], \quad O_L((t)), \quad L \cdot O_L[[u]].
\]

Note that \( B \) depends on the choice of an isomorphism (27). Therefore we fix such a subring \( B \) of \( K \).

Let \( E \) be a simple object of \( B \). We suppose that

\[
(E, \theta) \in \text{Ob}(B^{GL_2(O_K)}), \quad (E, \vartheta) \in \text{Ob}(B^{GL_2(B)}), \quad (E, \upsilon) \in \text{Ob}(B^{K^*})
\]

for some \( \theta, \vartheta \) and \( \upsilon \), where \( K^* \hookrightarrow \text{GL}_2(K) \) (embedding in the upper left corner). Then we easily see that \( \theta, \vartheta \) and \( \upsilon \) split the group \( \hat{G} \) over the subgroups \( \text{GL}_2(O_K), \text{GL}_2(B) \) and \( K^* \) respectively. This assumption will give us the condition of sphericity for the action of \( \text{GL}_2(K) \) on \( B \), where \( B \) is a one-dimensional 2-vector space. This is the content of the following proposition.

**Proposition 5.** Let \( K \) be a two-dimensional local field isomorphic to one of the fields of the form (27). A central extension \( \hat{G} \) is obtained from the central extension \( \text{GL}_2(K) \) by means of the map (26) (for some element \( a \in \mathbb{C}/(2\pi i/\ln q)\mathbb{Z} \)) if and only if \( \hat{G} \) splits over the subgroups \( \text{GL}_2(O_K), \text{GL}_2(B) \) and \( K^* \) of \( \text{GL}_2(K) \) (\( K^* \) is embedded in the upper-left corner).

**Remark 18.** Although the subring \( B \) depends on the choice of an isomorphism (27), the resulting description of the central extension \( \hat{G} \) in terms of the map (26) is independent of the choice of this isomorphism.

**Proof of Proposition 5.** Assume that \( \hat{G} \) is obtained from \( \text{GL}_2(K) \) by means of the map \( K_2(K) \to \mathbb{C}^*, \quad (f, g) \mapsto q^{-a \nu_K(f, g)}. \) Since \( \text{GL}_2(K) \) splits (by construction) over the subgroup \( K^* \) of \( \text{GL}_2(K) \), so does \( \hat{G} \). It follows from Proposition 3 that \( \hat{G} \) is obtained from \( \text{GL}_2(K)_{\mathbb{R}_+^2} \) by means of the map \( \mathbb{R}_+^* \to \mathbb{C}^*, \quad x \mapsto x^a \). (To construct the central extension \( \text{GL}_2(K)_{\mathbb{R}_+^2} \), one must obtain the field \( K \) from the scheme \( \text{Spec} R \), where the ring \( R \) is one of the \( \mathbb{F}_q[[u, t]], O_L[[t]], O_L[[u]] \).) Hence it suffices to
prove that $\widetilde{\text{GL}_2(K)}_{R^*_+}$ splits over the subgroups $\text{GL}_2(O_K)$ and $\text{GL}_2(B)$ of $\text{GL}_2(K)$. In the case of $\text{GL}_2(O_K)$, this follows immediately from the construction of the central extension.

We now prove that the central extension $\widetilde{\text{GL}_2(K)}_{R^*_+}$ splits over the subgroup $\text{GL}_2(B)$ of $\text{GL}_2(K)$. To begin with, we claim that $\widetilde{\text{GL}_2(K)}_{R^*_+}$ (see §3.2) splits over $\text{GL}_2(B)$. Indeed, for every $g \in \text{GL}_2(K)$ we can uniquely define an element $\mu_{B,g} \in \mu(O_K^2 \mid gO_K^2)$ by putting $\mu_{B,g} = \mu_1^{-1} \otimes \mu_2$, where $\mu_1 \in \mu(O_K^2/hO_K^2_k)$ and $\mu_2 \in \mu(gO_K^2/hO_K^2_k)$, and the element $h \in \text{GL}_2(K)$ satisfies $hO_K^2 \subset O_K^2$ and $hO_K^2 \subset gO_K^2$. The elements $\mu_1$ and $\mu_2$ are defined by the following rules:

$$\mu_1\left(\frac{B^2 \cap O_K^2}{B^2 \cap hO_K^2_k}\right) = 1, \quad \mu_2\left(\frac{B^2 \cap gO_K^2}{B^2 \cap hO_K^2_k}\right) = 1$$

(this definition makes sense since the spaces in the brackets for $\mu_1$ and $\mu_2$ are open compact subgroups of the locally compact groups $O_K^2/hO_K^2_k$ and $gO_K^2/hO_K^2_k$ respectively). The resulting element $\mu_{B,g}$ is independent of the choice of $h \in \text{GL}_2(K)$. We now see from this construction that the section $g \mapsto (g, \mu_{B,g})$, where $g \in \text{GL}_2(B)$, splits $\widetilde{\text{GL}_2(K)}_{R^*_+}$ over $\text{GL}_2(B)$ (use the equation $gB^2 = B^2$ for all $g \in \text{GL}_2(B)$).

Since $\text{GL}_2(B) = \text{SL}_2(B) \times B^*$, it follows that $\widetilde{\text{GL}_2(K)}_{R^*_+}$ splits over $\text{GL}_2(B)$. (The action of $B^*$ does not change the constructed section over $\text{SL}_2(B)$. This is obvious if one also fixes the constructed section over $B^*$.)

We now assume that $\hat{G}$ splits over the subgroups $\text{GL}_2(O_K)$, $\text{GL}_2(B)$ and $K^*$ of $\text{GL}_2(K)$. Since $\hat{G}$ splits over $K^*$, it follows from Proposition 1 that $\hat{G}$ is obtained from $\text{GL}_2(K)$ by means of some map $\phi \in \text{Hom}(K_2(K), C^*)$. By the description of $\nu_K(\cdot, \cdot) : K_2(K) \to \mathbb{Z}$ as a composite of boundary maps in Milnor $K$-theory, we see that the group $\text{Ker} \nu_K(\cdot, \cdot)$ is generated by the elements $(f, g)$, where $f$ and $g$ are in $O_K^*$, and the elements $(f, g)$, where $f$ and $g$ are in $B^*$. (Indeed, $\nu_K(\cdot, \cdot) = \partial_1 \circ \partial_2$, and the sequences

$$1 \longrightarrow K_2(O_K) \longrightarrow K_2(K) \longrightarrow \overline{K^*} \longrightarrow 1,$$

$$1 \longrightarrow O_K^* \longrightarrow \overline{K^*} \longrightarrow \mathbb{Z} \longrightarrow 0$$

are exact, where $\partial_2$ is the tame symbol associated with the discrete valuation on the field $K$, $\partial_1 = \nu_K$.)

It follows from formula (6) that if $\hat{G}$ splits over $\text{GL}_2(O_K)$, then the elements $(f, g)$, where $f$ and $g$ are in $O_K^*$, belong to the group $\text{Ker} \phi$. We similarly see from the splitting of $\hat{G}$ over $\text{GL}_2(B)$ that the elements $(f, g)$, where $f$ and $g$ are in $B^*$, belong to the group $\text{Ker} \phi$. Thus we obtain that $\phi = \varphi \circ \nu_K(\cdot, \cdot)$ for some homomorphism $\varphi$ from the group $\mathbb{Z}$ to the group $C^*$. We find $a \in \mathbb{C}/(2\pi i/\ln q) \mathbb{Z}$ such that $q^{-a} = \varphi(1)$. Then $\phi = q^{-a}\nu_K(\cdot, \cdot)$. $\square$
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