THE SET OF LOCAL $A$-PACKETS CONTAINING A GIVEN REPRESENTATION

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Abstract. In this paper, we give an algorithm to determine all local $A$-packets containing a given irreducible representation of a $p$-adic classical group. Especially, we can determine whether a given irreducible representation is of Arthur type or not.

1. Introduction

In a magnificent work [1], Arthur gave a classification of the discrete spectrum of square integral automorphic forms on a quasi-split special orthogonal group $SO_{2n+1}(F)$ or a symplectic group $Sp_{2n}(F)$ over a number field. This classification says that the discrete spectrum of the space of square integrable automorphic forms is divided into disjoint subsets called global $A$-packets. An element in a global $A$-packet is defined as the tensor product of elements of local $A$-packets. In other words, the local $A$-packets classify the local factors of square integrable automorphic representations. The purpose of this paper is to give an algorithm to determine all local $A$-packets which contain a given irreducible representation. This result might be regarded as an ultimate form of the globalization of irreducible representations of $p$-adic classical groups.

Let us describe our results. Fix a non-archimedean local field $F$ of characteristic zero. Let $G_n$ be a split special odd orthogonal group $SO_{2n+1}(F)$ or a symplectic group $Sp_{2n}(F)$ of rank $n$ over $F$. A (local) $A$-parameter for $G_n$ is a homomorphism

$$\psi: W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to \hat{G}_n$$

such that $\psi(W_F)$ is bounded, where $W_F$ is the Weil group of $F$ and $\hat{G}_n$ is the complex dual group of $G_n$. Associated to an $A$-parameter $\psi$ for $G_n$, Arthur [1, Theorem 1.5.1] defined the (local) $A$-packet $\Pi_\psi$, which is a multi-set over $\text{Irr}_{\text{unit}}(G_n)$. Here, $\text{Irr}_{\text{unit}}(G_n)$ denotes the set of equivalence classes of irreducible (unitary) representations of $G_n$. In fact, Mœglin [7] showed that the $A$-packet $\Pi_\psi$ is multiplicity-free, i.e., $\Pi_\psi$ is a subset of $\text{Irr}_{\text{unit}}(G_n)$. We say that an irreducible representation $\pi$ of $G_n$ is of Arthur type if $\pi$ belongs to $\Pi_\psi$ for some $A$-parameter $\psi$ for $G_n$.

Unlike $L$-packets, $A$-packets do not give a classification of $\text{Irr}_{\text{unit}}(G_n)$. Indeed, there is an irreducible unitary representation $\pi$ which is not of Arthur type. Moreover, even if $\psi_1 \not\sim \psi_2$, the intersection $\Pi_{\psi_1} \cap \Pi_{\psi_2}$ is not necessarily empty. Therefore, the following problem occurs.

Problem 1.1. Let $\pi$ be an irreducible representation of $G_n$.

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(1) Determine whether \( \pi \) is of Arthur type or not.
(2) If \( \pi \) is of Arthur type, list all \( A \)-parameters \( \psi \) such that \( \pi \in \Pi_\psi \).

By a result of Mœglin ([5, Theorem 6], [8, Proposition 8.1]), this problem is reduced to the good parity case. Here, we say that an irreducible representation \( \pi \) of \( G_n \) is of good parity if \( \pi \) can be embedded into a parabolically induced representation \( \rho_1 \times \cdots \times \rho_r \rtimes \sigma \) such that
- \( \rho_i \) is an irreducible cuspidal representation of \( \text{GL}_{d_i}(F) \) for \( i = 1, \ldots, r \);
- \( \sigma \) is an irreducible cuspidal representation of a classical group \( G_{n_0} \) of the same type as \( G_n \);
- for each \( i = 1, \ldots, r \), there exists an integer \( m_i \) such that the parabolically induced representation \( \rho_i \mid \cdot \mid m_i \rtimes \sigma \) is reducible.

As a refinement of Mœglin’s explicit construction of \( A \)-packets, in [2], we introduced a notion of extended multi-segments \( E \) for \( G_n \). See Definition 3.1. An extended multi-segment \( E \) for \( G_n \) explicitly gives
- a representation \( \pi(E) \) of \( G_n \), which is zero or an irreducible representation of good parity; and
- an \( A \)-parameter \( \psi_E \) for \( G_n \) of good parity.

Here, we say that an \( A \)-parameter \( \psi \) for \( G_n \) is of good parity if all irreducible components of \( \psi \) are self-dual of the same type as \( \psi \). Moreover, for an \( A \)-parameter \( \psi \) for \( G_n \) of good parity, (after fixing an auxiliary datum) the \( A \)-packet \( \Pi_\psi \) is given as

\[
\Pi_\psi = \{ \pi(E) \mid \psi_E \cong \psi \} \setminus \{0\}.
\]

Therefore, Problem 1.1 for irreducible representations of good parity can be reformulated as follows:

**Problem 1.2.** Let \( \pi \) be an irreducible representation of \( G_n \) of good parity.

(1) Determine whether there is an extended multi-segment \( E \) such that \( \pi(E) \cong \pi \).
(2) In the affirmative case, determine all extended multi-segments \( E \) such that \( \pi(E) \cong \pi \).

To solve this problem, we need two key notions. The one is derivatives and the other is extended cuspidal supports. Fix an irreducible unitary cuspidal representation \( \rho \) of \( \text{GL}_d(F) \) and a real number \( x \). For an irreducible representation \( \pi \) of \( G_n \), define the \( \rho \mid \cdot \mid x \)-derivative \( D_{\rho \mid \cdot \mid x} (\pi) \) as a semisimple representation satisfying that

\[
[Jac_P(\pi)] = \rho \mid \cdot \mid x \otimes D_{\rho \mid \cdot \mid x} (\pi) + \text{(others)},
\]

where \( [Jac_P(\pi)] \) is the semisimplification of the Jacquet module of \( \pi \) along a standard parabolic subgroup \( P \) of \( G_n \) with Levi \( \text{GL}_d(F) \times G_{n-d} \). In [2, Section 5], a composition of derivatives of \( \pi(E) \) was described in terms of extended multi-segments. On the other hand, the extended cuspidal support \( \text{ex.supp}(\pi) \) of an irreducible representation \( \pi \) of \( G_n \) is a refinement of the usual cuspidal support of \( \pi \) (see Definition 2.2). Unlike the usual cuspidal support, members of an \( A \)-packet \( \Pi_\psi \) share an extended cuspidal support, which is determined by the diagonal restriction \( \psi_d \) of \( \psi \) (see Proposition 2.3).

Now we can roughly state our first main result, which solves Problem 1.2 (1).

**Algorithm 1.3** (Algorithm 3.3). Let \( \pi \) be an irreducible representation of \( G_n \) of good parity.
Step 1\(\pm\): Suppose that there exist an irreducible unitary cuspidal representation \(\rho\) of \(GL_d(F)\) and \(x \geq 1\) or \(x < 0\) such that \(D_{\rho,\pm}^d(\pi) \neq 0\). In this case, one can construct an irreducible representation \(\pi^\pm\) of \(G_{n,\pm}\) of good parity with \(n^\pm < n\) by Definition 2.5. Then \(\pi\) is of Arthur type if and only if there exists an extended multi-segment \(E^\pm\) for \(G_{n,\pm}\) satisfying certain conditions such that \(\pi(E^\pm) \cong \pi^\pm\). In this case, we can explicitly define \(E\) from \(E^\pm\) such that \(\pi(E) \cong \pi\).

Step 2: Otherwise, the extended cuspidal support \(\text{ex.supp}(\pi)\) of \(\pi\) gives at most one \(A\)-parameter \(\psi\) for \(G_n\) such that \(\pi\) is of Arthur type if and only if \(\pi \in \Pi_\psi\).

To apply this algorithm, we need to solve Problem 1.2 (2). The following is the second main result.

**Theorem 1.4** (Theorem 3.5). Let \(E_1\) and \(E_2\) be two extended multi-segments for \(G_n\). Suppose that \(\pi(E_1) \neq 0\). Then \(\pi(E_1) \cong \pi(E_2)\) if and only if \(E_2\) can be obtained from \(E_1\) by a finite chain of three operations \((C), (UI), (P)\) defined in Definition 3.4 and their inverses.

This paper is organized as follows. In Section 2, we review \(A\)-parameters, \(A\)-packets, extended cuspidal supports, Langlands classification, and derivatives. After recalling results in the previous paper [2], we state main results and give some examples in Section 3. Finally, in Section 4, we prove the main results.

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**Notation.** Let \(F\) be a non-archimedean local field of characteristic zero. The normalized absolute value is denoted by \(|\cdot|\), which is also regarded as a character of \(GL_d(F)\) via composing with the determinant map.

Let \(G_n\) be a split special odd orthogonal group \(SO_{2n+1}(F)\) or a symplectic group \(Sp_{2n}(F)\) of rank \(n\) over \(F\). The set of equivalence classes of irreducible smooth representations of a group \(G\) is denoted by \(\text{Irr}(G)\). Let \(\text{Irr}_{\text{unit}}(G_n)\) (resp. \(\text{Irr}_{\text{temp}}(G_n)\)) be the subset of \(\text{Irr}(G_n)\) consisting of equivalence classes of irreducible unitary (resp. tempered) representations of \(G_n\).

The Weil group of \(F\) is denoted by \(W_F\). The group \(SL_2(\mathbb{C})\) has a unique irreducible algebraic representation of dimension \(a\), which is denoted by \(S_a\). We denote by \(\widehat{G_n}\) the complex dual group of \(G_n\). Namely, \(\widehat{G_n} = Sp_{2n}(\mathbb{C})\) if \(G_n = SO_{2n+1}(F)\), and \(\widehat{G_n} = SO_{2n+1}(\mathbb{C})\) if \(G_n = Sp_{2n}(F)\).

The set of equivalence classes of irreducible cuspidal representations of \(GL_d(F)\) is denoted by \(\text{Cusp}(GL_d(F))\). By the local Langlands correspondence for \(GL_d(F)\), we identify \(\rho \in \text{Cusp}(GL_d(F))\) with an irreducible \(d\)-dimensional representation of \(W_F\). The subset of \(\text{Cusp}(GL_d(F))\) consisting of unitary (resp. self-dual) elements is denoted by \(\text{Cusp}^\text{unit}_{\text{unit}}(GL_d(F))\) (resp. \(\text{Cusp}^\text{unit}_{\text{temp}}(GL_d(F))\)).

We will often extend the set theoretical language to multi-sets. Namely, we write a multi-set as \(\{x, \ldots, x, y, \ldots, y, \ldots\}\). When we use a multi-set, we will mention it.

## 2. Preliminary

In this section, we review several results on local \(A\)-packets.
2.1. \(A\)-parameters. Recall that an \(A\)-parameter for \(G_n\) is the \(\hat{G}_n\)-conjugacy class of an admissible homomorphism

\[
\psi : W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \hat{G}_n
\]

such that the image of \(W_F\) is bounded. By composing with the standard representation of \(\hat{G}_n\), we can regard \(\psi\) as a representation of \(W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})\). We write

\[
\psi = \bigoplus_{\rho} \left( \bigoplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \right),
\]

where \(\rho\) runs over \(\sqcup_{d \geq 1} \text{Cusp}_{\text{unit}}(\text{GL}_d(F))\).

For \(\psi\) as above, we say that \(\psi\) is of good parity if \(\rho \boxtimes S_{a_i} \boxtimes S_{b_i}\) is self-dual of the same type as \(\psi\) for any \(\rho\) and \(i \in I_\rho\), i.e.,

- \(\rho \in \text{Cusp}_d^\perp(\text{GL}_d(F))\) is orthogonal and \(a_i + b_i \equiv 0 \mod 2\) if \(G_n = \text{Sp}_{2n}(F)\) (resp. \(a_i + b_i \equiv 1 \mod 2\) if \(G_n = \text{SO}_{2n+1}(F)\)); or
- \(\rho \in \text{Cusp}_d^\perp(\text{GL}_d(F))\) is symplectic and \(a_i + b_i \equiv 1 \mod 2\) if \(G_n = \text{Sp}_{2n}(F)\) (resp. \(a_i + b_i \equiv 0 \mod 2\) if \(G_n = \text{SO}_{2n+1}(F)\)).

Let \(\Psi(G_n)\) be the set of \(A\)-parameters. The subset of \(\Psi(G_n)\) consisting of \(A\)-parameters of good parity is denoted by \(\Psi_{\text{gp}}(G_n)\). Also, we denote by \(\Phi_{\text{temp}}(G_n)\) the subset of \(\Psi(G_n)\) consisting of tempered \(A\)-parameters, i.e., \(A\)-parameters \(\phi\) which are trivial on the second \(\text{SL}_2(\mathbb{C})\). Finally, we set \(\Phi_{\text{gp}}(G_n) = \Psi_{\text{gp}}(G_n) \cap \Phi_{\text{temp}}(G_n)\).

2.2. \(A\)-packets. To an \(A\)-parameter \(\psi \in \Psi(G_n)\), Arthur [1] Theorem 1.5.1 (a)) associated an \(A\)-packet \(\Pi_\psi\), which is a finite multi-set over \(\text{Irr}_{\text{unit}}(G_n)\). In fact, Mœglin [7] showed that \(\Pi_\psi\) is multiplicity-free, i.e., a subset of \(\text{Irr}_{\text{unit}}(G_n)\). We say that \(\pi \in \text{Irr}(G_n)\) is of Arthur type if \(\pi \in \Pi_\psi\) for some \(\psi \in \Psi(G_n)\).

Recall that for a representation \(\psi_1\) of \(W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})\), we have an irreducible unitary representation \(\tau_{\psi_1}\) of \(\text{GL}_m(F)\) with \(m = \dim(\psi)\), which is a product of unitary Speh representations.

**Proposition 2.1 (Mœglin ([5] Theorem 6], [8] Proposition 8.11)).** Any \(\psi \in \Psi(G_n)\) can be decomposed as

\[
\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\perp,
\]

where

- \(\psi_0 \in \Psi_{\text{gp}}(G_{na})\);
- \(\psi_1\) is a direct sum of irreducible representations of \(W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})\) which are not self-dual of the same type as \(\psi\).

For \(\pi_0 \in \Pi_{\psi_0}\), the parabolically induced representation \(\tau_{\psi_1} \times \pi_0\) is irreducible and independent of the choice of \(\psi_1\). Moreover,

\[
\Pi_\psi = \{ \tau_{\psi_1} \times \pi_0 \mid \pi_0 \in \Pi_{\psi_0} \}.
\]

By [1] Theorem 1.5.1 (b)], if \(\phi \in \Phi_{\text{temp}}(G_n)\) is a tempered \(A\)-parameter, then \(\Pi_\phi\) is a subset of \(\text{Irr}_{\text{temp}}(G_n)\) and

\[
\text{Irr}_{\text{temp}}(G_n) = \bigcup_{\phi \in \Phi_{\text{temp}}(G_n)} \Pi_\phi \quad \text{(disjoint union)}.
\]
Moreover, after fixing a Whittaker datum for $G_n$, the tempered $A$-packet $\Pi_{\phi}$ is parametrized by $\widehat{S}_{\phi}$, which is the Pontryagin duals of the component group $S_{\phi}$. When $\phi = \oplus_{i \in I_{\rho}} \rho \boxtimes S_{a_i} \in \Phi_{gd}(G_n)$, an element $\varepsilon \in \widehat{S}_{\phi}$ is characterized by $\varepsilon(\rho \boxtimes S_{a_i}) \in \{ \pm 1 \}$ for $\rho$ and $i \in I_{\rho}$ such that

- if $a_i = a_j$, then $\varepsilon(\rho \boxtimes S_{a_i}) = \varepsilon(\rho \boxtimes S_{a_j})$;
- $\prod_{i \in I_{\rho}} \varepsilon(\rho \boxtimes S_{a_i}) = 1$.

We denote the element of $\Pi_{\phi}$ corresponding to $\varepsilon \in \widehat{S}_{\phi}$ by $\pi(\phi, \varepsilon)$.

### 2.3. Extended cuspidal support

One of key notions in this paper is as follows.

**Definition 2.2** (Mœglin). Let $\pi \in \text{Irr}(G_n)$. Take $\rho_i \in \text{Cusp}(\text{GL}_{d_i}(F))$ for $1 \leq i \leq r$ and an irreducible cuspidal representation $\sigma$ of $G_{n_0}$ such that

$$\pi \mapsto \rho_1 \times \cdots \times \rho_r \times \sigma.$$ Write $\sigma = \pi(\phi, \varepsilon)$ with

$$\phi = \bigoplus_{j=1}^t \rho_j \boxtimes S_{a_j} \in \Phi_{\text{temp}}(G_{n_0}).$$

Then we define an extended cuspidal support $\text{ex supp}(\pi)$ as the multi-set

$$\text{ex supp}(\pi) = \{ \rho_1, \ldots, \rho_r, \rho_1^\vee, \ldots, \rho_r^\vee \}$$

over $\sqcup_{d \geq 1} \text{Cusp}(\text{GL}_{d}(F))$. Here, $\rho^\vee$ denotes the contragredient of $\rho$.

Let $\psi \in \Psi(G_n)$. Recall that $\psi$ is a homomorphism from $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \widehat{G}_n$. Define the diagonal restriction of $\psi$ by $\psi_\Delta = \psi \circ \Delta$, where

$$\Delta: W_F \times \text{SL}_2(\mathbb{C}) \to W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}), (w, \alpha) \mapsto (w, \alpha, \alpha).$$

The following is a key proposition.

**Proposition 2.3** ([6 4.1 Proposition]). Let $\psi \in \Psi(G_n)$. Write $\psi_d = \oplus_{i=1}^r \rho_i \boxtimes S_{a_i}$. Then for any $\pi \in \Pi_{\psi}$, the extended cuspidal support of $\pi$ is given by

$$\text{ex supp}(\pi) = \bigcup_{i=1}^r \pi \rho_i \frac{a_{i+1}-1}{2}, \pi \rho_i \frac{a_{i+1}-3}{2}, \ldots, \pi \rho_i \frac{a_{i+1}}{2}.$$ As a consequence, Mœglin showed that if $\Pi_{\psi_1} \cap \Pi_{\psi_2} \neq \emptyset$, then $\psi_{1, d} \cong \psi_{2, d}$ ([6 4.2 Corollaire]).

### 2.4. Langlands classification

A *segment* is a finite set consisting of cuspidal representations of $\text{GL}_{d}(F)$ of the form

$$[x, y]_{\rho} = \{ \rho^x \rho^{x-1} \ldots \rho \rho^y \rho^{y-1} \ldots \rho \},$$

where $\rho \in \text{Cusp}_{\text{unit}}(\text{GL}_{d}(F))$ and $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}$ and $x \geq y$. For a segment $[x, y]_{\rho}$ as above, we have a Steinberg representation $\Delta_{\rho}[x, y]$ of $\text{GL}_{d}(x-y+1)(F)$, which is a unique irreducible subrepresentation of parabolically induced representation

$$\rho \rho^x \rho^{x-1} \cdots \rho \rho^y \rho^{y-1} \cdots \rho \rho^y \rho^{y-1} \ldots \rho \rho^y.$$
By the Langlands classification for $G_n$, any $\pi \in \text{Irr}(G_n)$ is a unique irreducible subrepresentation of $\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi(\phi, \varepsilon)$, where

- $\rho_1, \ldots, \rho_r \in \text{Cuspunit}(\text{GL}_{d_k}(F))$;
- $x_1 + y_1 \leq \cdots \leq x_r + y_r < 0$;
- $\phi \in \Phi_{\text{temp}}(G_{n_0})$ and $\varepsilon \in \widehat{S}_\phi$.

In this case, we write

$$\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi(\phi, \varepsilon)),$$

and call $(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi(\phi, \varepsilon))$ the Langlands data for $\pi$.

We say that an irreducible representation $\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi(\phi, \varepsilon))$ is of good parity if

- $x_1, \ldots, x_r \in (1/2)\mathbb{Z}$;
- $\rho_i \boxtimes S_2[x_i+1]$ is self-dual representation of $W_F \times \text{SL}_2(\mathbb{C})$ of the same type as $\phi$ for $i = 1, \ldots, r$; and
- $\phi \in \Phi_{\text{gp}}(G_{n_0})$.

This definition is equivalent to the one in Section 1. We denote the set of equivalence classes of irreducible representations of $G_n$ of good parity by $\text{Irr}_{\text{gp}}(G_n)$. Note that if $\psi \in \Psi_{\text{gp}}(G_n)$, then $\Pi_\psi \subset \text{Irr}_{\text{gp}}(G_n)$. Moreover, by Proposition 2.1 (together with [3, Theorem 5.3]), Problem 1.1 is reduced to the case where $\pi \in \text{Irr}_{\text{gp}}(G_n)$.

### 2.5. Derivatives

As in [2], we use the following notion.

**Definition 2.4.** Fix $\rho \in \text{Cusp}(\text{GL}_d(F))$. Let $\pi$ be a smooth representation of $G_n$ of finite length.

1. The $k$-th $\rho$-derivative of $\pi$ is a semisimple representation $D^{(k)}_{\rho}(\pi)$ satisfying

$$[\text{Jac}_{P_{dk}}(\pi)] = \rho^k \otimes D^{(k)}_{\rho}(\pi) + \sum_{i \in I} \tau_i \otimes \pi_i,$$

where

- $[\text{Jac}_{P_{dk}}(\pi)]$ is the semisimplification of Jacquet module of $\pi$ along the standard parabolic subgroup $P_{dk}$ with Levi part $\text{GL}_{dk}(F) \times G_{n-dk}$;
- $\rho^k = \rho \times \cdots \times \rho$ ($k$-times);
- $\tau_i \in \text{Irr}(\text{GL}_{dk}(F))$ such that $\tau_i \not\sim \rho^k$.

2. For simplicity, we write $D_{\rho}(\pi) = D^{(1)}_{\rho}(\pi)$.

3. When $D^{(k)}_{\rho}(\pi) \neq 0$ but $D^{(k+1)}_{\rho}(\pi) = 0$, we call $D^{(k)}_{\rho}(\pi)$ the highest $\rho$-derivative of $\pi$.

In this paper, we use derivatives as the following forms.

**Definition 2.5.** Fix $\rho \in \text{Cuspunit}(\text{GL}_d(F))$. Let $\pi \in \text{Irr}(G_n)$. Define

$$D^+_{\rho}(\pi) = \left( (x, k_0), (x + 1, k_1), \ldots, (x + t - 1, k_{t-1}); \pi^+ \right),$$

$$D^-_{\rho}(\pi) = \left( (x, k_0), (x - 1, k_1), \ldots, (x - t + 1, k_{t-1}); \pi^- \right)$$

as follows.

1. If $D^+_{\rho(z);t}(\pi) = 0$ for any $z \in \mathbb{R}$, we set $D^+_{\rho}(\pi) = D^-_{\rho}(\pi) = \{\pi\}$ (so that $t = 0$ and $\pi^+ = \pi$).
(2) Otherwise, for $\epsilon \in \{\pm\}$, set
\[
x = \begin{cases} \max\{z \in \mathbb{R} \mid D_{\mathcal{D}_{t, x}}(\pi) \neq 0\} & \text{if } \epsilon = +, \\ \min\{z \in \mathbb{R} \mid D_{\mathcal{D}_{t, x}}(\pi) \neq 0\} & \text{if } \epsilon = -.
\end{cases}
\]
Define $t > 0$ and $(k_0, \ldots, k_{t-1})$ so that
\[
\pi^\epsilon := D_{\mathcal{D}_{t, x}}^{(k_{t-1})} \circ \cdots \circ D_{\mathcal{D}_{t, x}}^{(k_0)}(\pi) \neq 0,
\]
but for $j = 0, \ldots, t$, we have
\[
D_{\mathcal{D}_{t, x}}^{(k_{j+1})} \circ \left( D_{\mathcal{D}_{t, x}}^{(k_{j})} \circ \cdots \circ D_{\mathcal{D}_{t, x}}^{(k_0)} \right)(\pi) = 0
\]
with $k_t := 0$.

Recall that if $\rho \in \text{Cusp}^L(\text{GL}_d(F))$ and $x \in \mathbb{R}$ with $x \neq 0$, for $\pi \in \text{Irr}(G_n)$, its highest $\rho|_{\cdot|^{\epsilon}}$-derivative $D_{\mathcal{D}_{t, x}}^{(k)}(\pi)$ is also irreducible. Moreover, if we know the Langlands data for $\pi$, one can compute the Langlands data for $D_{\mathcal{D}_{t, x}}^{(k)}(\pi)$ (see [4, Sections 6, 7]). In particular, for $\epsilon \in \{\pm\}$, if $D_{\mathcal{D}_{t, x}}^{(\epsilon)}(\pi) = [(x, k_0), \ldots, (x + \epsilon(t - 1), k_{t-1}); \pi^\epsilon]$ with $\epsilon x > 0$, we can compute the Langlands data for $\pi^\epsilon$.

3. Main results and Examples

In this section, we state our main results and give some examples.

3.1. Extended multi-segments. To describe $A$-packets $\Pi_\psi$ for $\psi \in \Psi_{\text{gp}}(G_n)$, in [2], we introduced the following notions.

**Definition 3.1.**
(1) An extended segment is a triple $([A, B], l, \eta)$, where
- $[A, B]_{\rho} = \{\rho|_{\cdot|^{A}}, \ldots, \rho|_{\cdot|^{B}}\}$ is a segment;
- $l \in \mathbb{Z}$ with $0 \leq l \leq \frac{b}{2}$, where $b := \#(A, B)_{\rho} = A - B + 1$;
- $\eta \in \{\pm\}$.

(2) An extended multi-segment for $G_n$ is a weak equivalence class of multi-sets of extended segments
\[
\mathcal{E} = \bigcup_{\rho} \{([A_i, B_i], l_i, \eta_i)\}_{i \in \tau(\rho, \rho)}
\]
such that
- $\rho$ runs over $\sqcup_{d \geq 1} \text{Cusp}^L(\text{GL}_d(F))$;
- $I_\rho$ is a totally ordered finite set with a fixed order $> \text{satisfying}$
\[
A_i < A_j, B_i < B_j \implies i < j,
\]
which is called an admissible order;
- $A_i + B_i \geq 0$ for all $\rho$ and $i \in I_\rho$;
- as a representation of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$,
\[
\psi_{\mathcal{E}} := \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}
\]
belongs to $\Psi_{\text{gp}}(G_n)$, where $a_i := A_i + B_i + 1$ and $b_i := A_i - B_i + 1$;
Algorithm 3.3.

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whether given \( \pi \).

Specifying of Arthur type representations.

(1) Theorem 3.2. It is irreducible or zero. The following properties were proven in \([2, \text{Theorems 1.2, 1.3, 1.4}]\):

\( E \in [2] \), to an extended multi-segment \( \pi \).

Two extended segments \(( [A, B]_{\rho}, l, \eta) \) and \(( [A', B']_{\rho'}, l', \eta') \) are equivalent if

- \([A, B]_{\rho} = [A', B']_{\rho'}
- \( l = l' \)
- \( \eta = \eta' \)

whenever \( l = l' < \frac{b}{2} \).

Similarly, \( E = \cup_{\rho} \{ ([A_{i}, B_{i}]_{\rho}, l, \eta_{i}) \}_{i \in (I_{\rho}, >)} \) and \( E' = \cup_{\rho} \{ ([A'_{i}, B'_{i}]_{\rho}, l', \eta'_{i}) \}_{i \in (I_{\rho}, >)} \) are weak equivalent if \(( [A_{i}, B_{i}]_{\rho}, l, \eta_{i}) \) and \(( [A'_{i}, B'_{i}]_{\rho}, l', \eta'_{i}) \) are equivalent for all \( \rho \) and \( i \in I_{\rho} \).

(4) An admissible order > on \( I_{\rho} \) is called very admissible if it satisfies a stronger condition

\[
B_{i} < B_{j} \implies i < j.
\]

We say that \( E \) has very admissible orders if the admissible order > on \( I_{\rho} \) is very admissible for any \( \rho \).

In [2], to an extended multi-segment \( E \) for \( G_{n} \), we associate a representation \( \pi(E) \) of \( G_{n} \).

The following properties were proven in [2, Theorems 1.2, 1.3, 1.4]:

**Theorem 3.2.** (1) For \( \psi = \oplus_{\rho \searrow I_{\rho}, \rho} T S_{A_{i}} \otimes S_{B_{i}} \) in \( \Psi_{\text{gp}}(G_{n}) \), after fixing a very admissible order > on \( I_{\rho} \) for each \( \rho \), we have

\[
\Pi_{\psi} = \{ (\pi(E) \mid \psi \in (E) \} \setminus \{0\}.
\]

(2) There exists a non-vanishing criterion for \( \pi(E) \).

3.2. Specifying of Arthur type representations. Now we give an algorithm to determine whether given \( \pi \in \text{Irr}_{\text{gp}}(G_{n}) \) is of Arthur type or not. The following is the first main result.

**Algorithm 3.3.** Let \( \pi \in \text{Irr}_{\text{gp}}(G_{n}) \). Assume that the Langlands data for \( \pi \) is given.

**Step 1**: Suppose that there exist \( \rho \in \text{Cusp}^{+}(\text{GL}_{d}(F)) \) and \( z \geq 1 \) such that \( D_{\rho, z}(\pi) \neq 0 \). Write

\[
D^{+}_{\rho}(\pi) = [(x, k_{0}), (x + 1, k_{1}), \ldots, (x + t - 1, k_{t-1}); \pi^{+}]
\]

with \( \pi^{+} \in \text{Irr}_{\text{gp}}(G_{n+}) \). Note that \( t > 0 \) and \( x \geq 1 \). Then \( \pi \) is of Arthur type if and only if there exists an extended multi-segment \( E^{+} \) for \( G_{n+} \) such that

- \( \pi^{+} \cong \pi(E^{+}) \);
- \( \max \{ \rho \mid ([A, B]_{\rho}, l, \eta) \in E^{+} \} = x - 1 \);
- \( E^{+} \) contains extended segments of the form \( ([x + j - 2, x - 1]_{\rho}, * , *) \) with at least \( k_{j-1} - k_{j} \) times for \( 1 \leq j \leq t \) with \( k_{t} := 0 \).

In this case, let \( E \) be given from \( E^{+} \) by replacing \( ([x + j - 2, x - 1]_{\rho}, l, \eta_{j}) \) with \( ([x + j - 1, x]_{\rho}, l, \eta_{j}) \) exactly \( k_{j-1} - k_{j} \) times for \( 1 \leq j \leq t \). Then \( \pi \cong \pi(E) \).

**Step 2**: Suppose that there exist \( \rho \in \text{Cusp}^{+}(\text{GL}_{d}(F)) \) and \( z < 0 \) such that \( D_{\rho, z}(\pi) \neq 0 \). Write

\[
D^{-}_{\rho}(\pi) = [(x, k_{0}), (x - 1, k_{1}), \ldots, (x - t + 1, k_{t-1}); \pi^{-}]
\]

with \( \pi^{-} \in \text{Irr}_{\text{gp}}(G_{n-}) \). Note that \( t > 0 \) and \( x < 0 \). Then \( \pi \) is of Arthur type if and only if there exists an extended multi-segment \( E^{-} \) for \( G_{n-} \) such that

\[
\prod_{\rho} \prod_{i \in I_{\rho}} (-1)^{\frac{l}{2} + l_{i} \cdot b_{i}} = 1
\]
\[ \begin{align*} & \bullet \pi^- \cong \pi(E^-); \\
& \bullet \min\{B \mid ([A, B]_\rho, l, \eta) \in E^-\} = x + 1; \\
& \bullet E^- \text{ contains extended segments of the form } ([−x + j − 2, x + 1]_\rho, *, *) \text{ with at least } k_{j−1} − k_j \text{ times for } 1 \leq j \leq t \text{ with } k_t := 0. \text{ Here, when } x = −1/2, \text{ we omit this condition for } j = 1 \text{ since } [−1/2, 1/2]_\rho = \emptyset. \]

In this case, let \( E \) be given from \( E^- \) by replacing \( ([−x + j − 2, x + 1]_\rho, l_j, \eta_j) \) with \( ([−x + j − 1, x]_\rho, l_j + 1, \eta_j) \) exactly \( k_{j−1} − k_j \) times for \( 1 \leq j \leq t. \) Here, when \( x = −1/2, \) we understand this operation for \( j = 1 \) as adding \( ([1/2, −1/2]_\rho, 1, +1) \) exactly \( k_0 − k_1 \) times whose indices are less than any element in \( I_\rho. \) Then \( \pi \cong \pi(E). \)

**Step 2:** Otherwise, i.e., suppose that for \( \rho \in \text{Cusp}^\perp(\text{GL}_d(F)) \) and \( z \in \mathbb{R}, \)

\[ D_{\rho|z}(\pi) \neq 0 \implies z \in \{0, 1/2\}. \]

Write \( \pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi(\phi, z)) \) as in the Langlands classification. For \( \rho \in \text{Cusp}^\perp(\text{GL}_d(F)) \) and \( z \in (1/2)\mathbb{Z} \) with \( z \geq 0, \) set \( k_{\rho, z} := \#\{i \in \{1, \ldots, r\} \mid \rho_i \cong \rho, \text{ and, } x_i = z \text{ or } y_i = −z\} + m_\phi(\rho \boxtimes S_{z+1}), \)

where \( m_\phi(\rho \boxtimes S_a) \) is the multiplicity of \( \rho \boxtimes S_a \) in \( \phi. \) Then unless \( k_{\rho, z} \geq k_{\rho, z+1} \) for any \( \rho \) and \( z \geq 0, \) then \( \pi \) is not of Arthur type. In this case, set

\[ \psi = \bigoplus_{\rho} \bigoplus_{z \in (1/2)\mathbb{Z}_{\geq 0}} (\rho \boxtimes S_{z+1}^{d_\rho} \boxtimes S_{z+1−d_\rho})^{\oplus(k_{\rho, z}−k_{\rho, z+1})} \]

with \( d_\rho \in \{0, 1/2\} \) such that \( z + d_\rho \in \mathbb{Z}. \) Then \( \pi \) is of Arthur type if and only if \( \pi \in \Pi_\psi. \)

The claims in this algorithm will be proven in Section 3.2 below. To apply this algorithm, in Step 1±, we need to know all extended multi-segments \( E^\pm \) such that \( \pi^\pm \cong \pi(E^\pm). \) It is done by the result in the next subsection. An example of Algorithm 3.3 is given in Section 3.5.

### 3.3. Strongly equivalence classes.

Recall that for \( \psi, \psi' \in \Psi_{\text{sp}}(G_n), \) even if \( \psi \not\cong \psi', \) the A-packets \( \Pi_\psi \) and \( \Pi_{\psi'} \) can have an intersection. In other words, for two extended multi-segments \( E \) and \( E' \) for \( G_n, \) even if \( E \not\cong E', \) one might have \( \pi(E) \cong \pi(E') \neq 0. \) In this subsection, we determine this situation.

**Definition 3.4.** Let \( E_1 \) and \( E_2 \) be two extended multi-segments for \( G_n. \) We say that \( E_1 \) and \( E_2 \) are strongly equivalent if \( E_2 \) can be obtained from \( E_1 \) by a finite chain of the following three operations and their inverses: Write

\[ E = \bigcup_{\rho} \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)} \]

and let \( i < j \) be adjacent elements in \( I_\rho. \)

**Changing admissible orders (C):** Suppose that \([A_i, B_i]_\rho \subset [A_j, B_j]_\rho. \) Let \( >' \) be another admissible order on \( I_\rho \) given from \( > \) by changing \( i >' j. \) Then we define \( E \mapsto E' \) by replacing \((I_\rho, >) \) with \((I_\rho, >') \) and

\[ \{([A_i, B_i]_\rho, l_i, \eta_i), ([A_j, B_j]_\rho, l_j, \eta_j)\} \]

Other than the above, we are interested in the case that \( A_i = A_j \) and \( B_i = B_j. \)
with
\[
\{([A_i, B_i], l_i, \eta_i), ([A_j, B_j], l_j, \eta_j)\}
\]

where \(l_i', \eta_i', l_j', \eta_j'\) are given explicitly in [2] Theorem 1.3 (see also [2, Section 4.2]).

**Union-Intersection (UI):** Suppose that \(B_i < B_j, A_i < A_j\) and that one of conditions (1)-(3) in [2] Section 5.2 holds. Then we define \(E \to E'\) by replacing
\[
\{([A_i, B_i], l_i, \eta_i), ([A_j, B_j], l_j, \eta_j)\}
\]
with
\[
\{([A_j, B_i], l_i', \eta_i'), ([A_i, B_j], l_j', \eta_j')\}
\]
where \(l_i', \eta_i', l_j', \eta_j'\) are given explicitly in [2] Theorem 5.2. Note that \([A_j, B_i] = [A_i, B_i] \cup [A_i, B_j] = [A_i, B_i] \cap [A_j, B_j]\). Here, when \(B_j = A_i + 1\) so that \([A_i, B_j] = \emptyset\), we remove \(([A_j, B_i], l_i', \eta_i')\).

**Phantom (dis)appearing (P):** Formally add \(([l - 1, -l], l, +1)\) for an integer \(l > 0\) or \(([l - 1/2, -l - 1/2], l, +1)\) for an integer \(l \geq 0\) to \(E\), whose index \(i_0\) is the minimum in \(I^0\cup \{i_0\}\).

The second main result is as follows.

**Theorem 3.5.** Let \(E_1\) and \(E_2\) be two extended multi-segments for \(G_n\). Suppose that \(\pi(E_1) \neq 0\). Then \(\pi(E_1) \cong \pi(E_2)\) if and only if \(E_1\) and \(E_2\) are strongly equivalent.

This theorem is proven in Section 3.3 below. By this theorem, if \(\pi \in \text{Irr}_{\text{sp}}(G_n)\) is known to be of Arthur type, then one can list all \(\psi \in \Psi_{\text{sp}}(G_n)\) such that \(\pi \in \Pi_\psi\).

### 3.4. Example of Theorem 3.5

In this and the next subsections, we set \(\rho = 1_{\text{GL}_1(F)}\) and drop \(\rho\) from the notation. Moreover, we only consider \(A\)-parameters of the form \(\psi = \oplus_{i \in I} S_{a_i} \boxtimes S_{b_i}\) and extended multi-segments \(E = \{([A_i, B_i], l_i, \eta_i)\}_i \in \{l \geq 0\}\). When \(\phi = S_{2x_1+1} + \cdots + S_{2x_r+1}\) and \(\varepsilon(S_{2x_i+1}) = \varepsilon_i \in \{\pm 1\}\), we write \(\pi(\phi, \varepsilon) = \pi(x_1^{t_1}, \ldots, x_r^{t_r})\).

As in [2] Section 3.1, we regard \(E\) as the following symbol. When \(E = \{([A, B], l, \eta)\}\) is a singleton, we write
\[
E = \left(\begin{array}{cccccc}
B & & & & & \\
\land & \land & \cdots & \land & \land & \\
& B+l-1 & & & & \\
\land & \land & \cdots & \land & \land & \\
A-l & & & & & \\
\land & \land & \cdots & \land & \land & \\
A-l+1 & & & & & \\
\land & \land & \cdots & \land & \land & \\
A & & & & & \\
\end{array}\right)
\]

where \(\odot\) is replaced with \(\oplus\) and \(\ominus\) alternately, starting with \(\oplus\) if \(\eta = +1\) (resp. \(\ominus\) if \(\eta = -1\)). In general, we put each symbol vertically.

Now we consider \(\pi = \pi(0^+, 1^+, 2^-) \in \text{Irr}(\text{Sp}_8(F))\). It is known that \(\pi\) is supercuspidal. Let us construct all \(A\)-parameters \(\psi \in \Psi_{\text{sp}}(\text{Sp}_8(F))\) such that \(\pi \in \Pi_\psi\).

1. Define
\[
\varepsilon_1 = \left(\begin{array}{ccc}
0 & 1 & 2 \\
\oplus & \oplus & \oplus \\
\end{array}\right)
\]

Clearly, \(\pi(\varepsilon_1) = \pi\). The associated \(A\)-parameter is \(\psi_1 = \psi_{\varepsilon_1} = S_1 + S_3 + S_5 \in \Phi_{\text{sp}}(\text{Sp}_8(F))\).
(2) By using (UI) for the first and second lines of $\mathcal{E}_1$, we obtain

\[ \mathcal{E}_2 = \begin{pmatrix} 0 & 1 & 2 \\ \oplus & \ominus & \ominus \end{pmatrix}. \]

The associated $A$-parameter is $\psi_2 = \psi_{\mathcal{E}_2} = S_2 \boxtimes S_2 + S_5$.

(3) After adding $([0, -1], 1, 1)$ to $\mathcal{E}_2$ by (P), we use (UI). Then we obtain

\[ \mathcal{E}_3 = \begin{pmatrix} -1 & 0 & 1 & 2 \\ < & \ominus & \ominus & \ominus \\ \ominus & \ominus & \ominus \end{pmatrix}. \]

The associated $A$-parameter is $\psi_3 = \psi_{\mathcal{E}_3} = S_1 \boxtimes S_3 + S_1 + S_5$.

(4) By using (UI) for the second and third lines of $\mathcal{E}_1$, we obtain

\[ \mathcal{E}_4 = \begin{pmatrix} 0 & 1 & 2 \\ \ominus & \ominus & \ominus \end{pmatrix}. \]

The associated $A$-parameter is $\psi_4 = \psi_{\mathcal{E}_4} = S_1 \boxtimes S_2$.

(5) By using (UI) for $\mathcal{E}_2$ or $\mathcal{E}_4$, we obtain

\[ \mathcal{E}_5 = \begin{pmatrix} 0 & 1 & 2 \\ \ominus & \ominus & \ominus \end{pmatrix}. \]

The associated $A$-parameter is $\psi_5 = \psi_{\mathcal{E}_5} = S_3 \boxtimes S_3$.

(6) After adding $([0, -1], 1, 1)$ to $\mathcal{E}_5$ by (P), we use (UI). Then we obtain

\[ \mathcal{E}_6 = \begin{pmatrix} -1 & 0 & 1 & 2 \\ < & \ominus & \ominus & \ominus \\ \ominus & \ominus \end{pmatrix}. \]

The associated $A$-parameter is $\psi_6 = \psi_{\mathcal{E}_6} = S_1 \boxtimes S_4 + S_1$.

(8) After adding $([1, -2], 2, 1)$ to $\mathcal{E}_5$ by (P), we use (UI). Then we obtain

\[ \mathcal{E}_8 = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ < & < & \ominus & \ominus & \ominus \\ \ominus & \ominus \end{pmatrix}. \]

The associated $A$-parameter is $\psi_8 = \psi_{\mathcal{E}_8} = S_1 \boxtimes S_5 + S_2 \boxtimes S_2$.

(7) By using the inverse of (UI) for the second line of $\mathcal{E}_8$, we obtain

\[ \mathcal{E}_7 = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ < & < & \ominus & \ominus & \ominus \\ \ominus \end{pmatrix}. \]

The associated $A$-parameter is $\psi_7 = \psi_{\mathcal{E}_7} = S_1 \boxtimes S_5 + S_1 + S_3$. 
(9) After adding \((1, -2, 2, 1)\) to \(E_6\) by \((P)\), we use \((UI)\). Then we obtain

\[
E_9 = \left(\begin{array}{cccc}
-2 & -1 & 0 & 1 \\
\langle & \langle & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus
\end{array}\right)\n\]

The associated \(A\)-parameter is \(\psi_9 = \psi_{E_9} = S_1 \boxtimes S_5 + S_1 \boxtimes S_3 + S_1\). Note that we can obtain \(E_9\) from \(E_8\) by using \((P)\), \((C)\) and \((UI)\).

By Theorem 3.5 we can check that \(E_1, \ldots, E_9\) are all of the extended multi-segments \(E\) such that \(\pi(E) \cong \pi\). The relation among \(\{E_1, \ldots, E_9\}\) can be written as follows:

\[
\begin{array}{cccc}
E_1 \rightarrow E_2 \rightarrow E_3 \\
E_4 \rightarrow E_5 \rightarrow E_6 \\
E_7 \rightarrow E_8 \rightarrow E_9
\end{array}
\]

In \cite{2} Definition 6.1, Theorem 6.2, we defined an explicit map \(E \mapsto \hat{E}\) such that \(\pi(\hat{E})\) is the Aubert dual of \(\pi(E)\). Since \(\pi\) is supercuspidal, it is fixed by the Aubert duality. Hence the set \(\{E_1, \ldots, E_9\}\) is stable under \(E \mapsto \hat{E}\). Indeed, it is easy to check that \(\hat{E}_i = E_{10-i}\) for \(1 \leq i \leq 9\).

3.5. Example of Algorithm 3.3. Here, we give an example of applying Algorithm 3.3.

For \(\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2) \in \{\pm\}^3\) with \(\varepsilon_0 \varepsilon_1 \varepsilon_2 = 1\), consider

\[
\pi^\varepsilon = L \left(\Delta\left[-\frac{1}{2}, -\frac{5}{2}\right], \cdot; -\frac{1}{2}, \Delta\left[\frac{3}{2}, -\frac{5}{2}\right]; \pi((\frac{1}{2})^{\varepsilon_1}, (\frac{3}{2})^{\varepsilon_2}, (\frac{5}{2})^{\varepsilon_3})\right) \in \text{Irr}(SO_{31}(F)).
\]

Then

\[
\Pi = \{\pi^\varepsilon \mid \varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2) \in \{\pm\}^3, \varepsilon_0 \varepsilon_1 \varepsilon_2 = 1\}
\]

is a non-tempered \(L\)-packet of \(SO_{31}(F)\). Let us determine whether \(\pi_\varepsilon\) is of Arthur type or not.

First of all, we apply Step 1 of Algorithm 3.3. Then \(D^-(\pi^\varepsilon) = \left[(-\frac{1}{2}, 2), (-\frac{3}{2}, 1), (-\frac{5}{2}, 1); \pi_1^\varepsilon\right]\) with

\[
\pi_1^\varepsilon = L \left(\Delta\left[\frac{3}{2}, -\frac{5}{2}\right]; \pi((\frac{1}{2})^{\varepsilon_1}, (\frac{3}{2})^{\varepsilon_2}, (\frac{5}{2})^{\varepsilon_3})\right).
\]

Next, we apply Step 1 of Algorithm 3.3.

(1) Suppose that \(\varepsilon = (+, +, +)\). Then \(D^+(\pi_1^{(+,+,+)}) = \left[\left(\frac{3}{2}, 2\right), \left(\frac{5}{2}, 2\right); \pi_2^{(+,+,+)}\right]\) with

\[
\pi_2^{(+,+,+)} = L \left(\Delta\left[\frac{1}{2}, -\frac{3}{2}\right]; \pi((\frac{1}{2})^+, (\frac{1}{2})^+, (\frac{3}{2})^+)\right).
\]

Note that \(\pi_2^{(+,+,+)}\) is in the situation of Step 2 in Algorithm 3.3. According to this step, consider

\[
\psi = S_2 \boxtimes S_1 + S_3 \boxtimes S_2 + S_3 \boxtimes S_2.
\]
Then one can check that $\pi_2^{(+,+,+)} \in \Pi_\psi$. In fact, we have

$$\pi_2^{(+,+,+)} \cong \pi \begin{pmatrix} 1/2 & 3/2 \\ 1/2 & 3/2 & 5/2 \end{pmatrix}.$$ 

By Step 1+ in Algorithm 3.3 we have

$$\pi_1^{(+,+,+)} \cong \pi \begin{pmatrix} 1/2 & 3/2 & 5/2 \\ 1/2 & 3/2 & 5/2 \end{pmatrix}.$$ 

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the above extended multi-segments so that $\pi_1^{(+,+,+)} \cong \pi(\mathcal{E}_1) \cong \pi(\mathcal{E}_2)$. By Theorem 3.5 we see that if $\mathcal{E}$ satisfies that $\pi_1^{(+,+,+)} \cong \pi(\mathcal{E})$, then $\mathcal{E} \in \{\mathcal{E}_1, \mathcal{E}_2\}$, or $\mathcal{E}$ is given from $\mathcal{E}_2$ by (C). Since both $\mathcal{E}_1$ and $\mathcal{E}_2$ do not satisfy the conditions of Step 1- in Algorithm 3.3 we conclude that $\pi^{(+,+,+)}$ is not of Arthur type.

(2) Suppose that $\varepsilon = (-,-,+)$. Then $\mathcal{D}^+(\pi_1^{(-,-,+)}) = [((\frac{2}{3}, 1); \pi_2^{(-,-,+)}]$ with

$$\pi_2^{(-,-,+)} = \pi((\frac{1}{2})^-, (\frac{2}{3})^-, (\frac{3}{2})^-, (\frac{5}{2})^+, (\frac{5}{2})^+).$$

Moreover, $\mathcal{D}^+(\pi_2^{(-,-,+)}) = [(\frac{3}{2}, 3), (\frac{5}{2}, 1); \pi_3^{(-,-,+)}]$ with

$$\pi_3^{(-,-,+)} = \pi((\frac{1}{2})^-, (\frac{1}{2})^-, (\frac{1}{2})^-, (\frac{3}{2})^+, (\frac{3}{2})^+).$$

Note that $\pi_3^{(-,-,+)}$ is in the situation of Step 2 in Algorithm 3.3. According to this step, consider

$$\psi = (S_2 \boxtimes S_1) \boxtimes 3 + S_3 \boxtimes S_2.$$ 

Then one can check that $\pi_3^{(-,-,+)} \in \Pi_\psi$. In fact, we have

$$\pi_3^{(-,-,+)} \cong \pi \begin{pmatrix} 1/2 & 3/2 \\ 1/2 & 3/2 & 5/2 \end{pmatrix}.$$ 

By Step 1+ in Algorithm 3.3 we have

$$\pi_2^{(-,-,+)} \cong \pi \begin{pmatrix} 1/2 & 3/2 & 5/2 \\ 1/2 & 3/2 & 5/2 \end{pmatrix}.$$
(3) Suppose that \( E \) is given from Step 1. Let \( E \) satisfy the conditions of Step 1. Moreover, then one can check that

\[ \psi = (S_2 \boxtimes S_1) \oplus S_4 \boxtimes S_3. \]

Then one can check that \( \pi_3^{(-,+,-)} \in \Pi_\psi \). In fact, we have

\[ \pi_3^{(-,+,-)} \cong \pi \begin{pmatrix} 1/2 & 3/2 & 5/2 \\ \oplus & \oplus & \oplus \\ \oplus & \oplus & \oplus \end{pmatrix} \cong \pi \begin{pmatrix} 1/2 & 3/2 & 5/2 \\ \oplus & \oplus & \oplus \\ \oplus & \oplus & \oplus \end{pmatrix}. \]

By Step 1 in Algorithm 3.3, we conclude that \( \pi_3^{(-,+,-)} \) is not of Arthur type.

By Theorem 3.3, we see that if \( E \) satisfies that \( \pi_1^{(-,-,+)} \cong \pi(E) \), then \( \pi \in \{ E_1, E_2 \} \) or \( E \) is given from \( E_1 \) or \( E_2 \) by (C). Since both \( E_1 \) and \( E_2 \) do not satisfy the conditions of Step 1 in Algorithm 3.3, we conclude that \( \pi^{(-,-,+)} \) is not of Arthur type.
4.1. Derivatives of $\pi(\mathcal{E})$. For an irreducible representation $\pi$ of $G_n$ and for $\epsilon \in \{\pm\}$, we have defined $\mathcal{D}_\rho^+(\pi)$ in Definition 2.5. The description of $\mathcal{D}_\rho^+(\pi(\mathcal{E}))$ was given in [2] Section 5.

**Theorem 4.1.** Let $\mathcal{E} = \cup_{\rho}\{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_{\rho}, >)}$ be an extended multi-segment for $G_n$ such that $\pi(\mathcal{E}) \neq 0$. Suppose that the order $> \bigcirc$ on $I_{\rho}$ is very admissible for any $\rho$.

1. **Write**

$$\mathcal{D}_\rho^+(\pi(\mathcal{E})) = [(x, k_0), (x + 1, k_1), \ldots, (x + t - 1, k_{t-1}); \pi^+] .$$

Assume that $t > 0$ and $x \geq 1$. Then we can construct $\mathcal{E}^*$ from $\mathcal{E}$ by a finite chain of operations (C), (UI) and their inverses such that

- $\pi(\mathcal{E}^*) \cong \pi(\mathcal{E})$;
- $\max\{B \mid ([A, B]_\rho, l, \eta) \in \mathcal{E}^\ast\} = x$;
In particular, extended cuspidal support $\{[A, B]_\rho \mid ([A, B]_\rho, l, \eta) \in \mathcal{E}^* \}, \ B = x$ is exactly equal to

$$
\bigcup_{j=1}^t \{[x+j-1,x]_\rho, \ldots, [x+j-1,x]_\rho\}
$$

with $k_1 := 0$;

- if we define $\mathcal{E}^+$ from $\mathcal{E}^*$ by replacing each $([x+j-1,x]_\rho, l_j, \eta_j)$ with $([x+j-2,x-1]_\rho, l_j, \eta_j)$ for $1 \leq j \leq t$, then $\pi^+ \cong \pi(\mathcal{E}^+)$.  

(2) Write

$$
\mathcal{D}_{\rho}^{-}(\pi(\mathcal{E})) = \{(x,k_0),(x-1,k_1),\ldots,(x-t+1,k_{t-1});\pi^-\}.
$$

Assume that $t > 0$ and $x < 0$. Then we can construct $\mathcal{E}^*$ from $\mathcal{E}$ by a finite chain of operations (C), (UI), (P) and their inverses such that

- $\pi(\mathcal{E}^*) \cong \pi(\mathcal{E})$;
- $\min\{B \mid ([A,B]_\rho, l, \eta) \in \mathcal{E}^*\} = x$;
- the multi-set $\{[A, B]_\rho \mid ([A, B]_\rho, l, \eta) \in \mathcal{E}^* \}, \ B = x$ is exactly equal to

$$
\bigcup_{j=1}^t \{[-x+j-1,x]_\rho, \ldots, [-x+j-1,x]_\rho\}
$$

with $k_1 := 0$;

- if we define $\mathcal{E}^-$ from $\mathcal{E}^*$ by replacing each $([-x+j-1,x]_\rho, l_j, \eta_j)$ with $([-x+j-2,x+1]_\rho, l_j-1, \eta_j)$ for $1 \leq j \leq t$, then $\pi^- \cong \pi(\mathcal{E}^-)$.

Proof. This is Theorem 5.1 together with Algorithms 5.5 and 5.6 in [2].

4.2. Proof of Algorithm 3.3. First, we show the claims in Algorithm 3.3. 

Proof of claims in Algorithm 3.3 Step 1+ and Step 1− are Theorem 4.1. So we consider Step 2.

Let $\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi(\phi, \varepsilon))$ and

$$
k_{\rho,z} := \#\{i \in \{1, \ldots, r\} \mid \rho_i \cong \rho, \text{ and, } x_i = z \text{ or } y_i = -z\} + m_\phi(\rho \boxtimes S_{z+1})
$$

be as in Step 2. By the assumption, we have $\min_{1 \leq i \leq r}\{x_i\} \in \{0,1/2\}$ so that $x_i \geq 0$ for $1 \leq i \leq r$. Since $x_i \geq y_i$ and $x_i + y_i < 0$, we have $y_i < 0$ for $1 \leq i \leq r$. Therefore, for $\rho \in \text{Cusp}^+(\text{GL}_d(F))$ and $x \in \{1/2\}\mathbb{Z}$ with $x \geq 0$, the multiplicity $M_{\rho,x}$ of $\rho \cdot |z|^x$ in the extended cuspidal support $\text{ex.supp}(\pi)$ is given by

$$
M_{\rho,x} = \sum_{\substack{z \in \{1/2\}\mathbb{Z} \\ z \geq x}} k_{\rho,z}.
$$

In particular,

$$
k_{\rho,x} = M_{\rho,x} - M_{\rho,x+1}.
$$

Now suppose that $\pi$ is of Arthur type. Then by Theorem 4.1 (or, more precisely, by Theorem 5.1 together with Algorithms 5.5 and 5.6 in [2]) and the assumption in Step 2, we obtain an extended multi-segment $\mathcal{E}^*$ with $\pi \cong \pi(\mathcal{E}^*)$ such that

$$
([A, B]_\rho, l, \eta) \in \mathcal{E}^* \implies B \in \{0,1/2\}.
$$
In conclusion, if $\pi$ is of Arthur type, we must have
$$ρ|·|^2 \Psi_{\pi} = 0$$
for all $ρ \in \text{Cusp}^+(\text{GL}_d(F))$ and $x \in (1/2)Z$ with $x \geq 0$. In this case, the $A$-parameter $ψ$ is nothing but the one in Step 2. This completes the proof. \hfill \Box

4.3. Proof of Theorem 3.5. Next we prove Theorem 3.5. The “if” part follows from [9, Theorem 1.3], [2, Theorem 5.2] and the definition of $π(\mathcal{E})$ (see [2, Section 3.2]).

We prove the “only if” part by induction on the rank $n$ of $G_n$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be extended multi-segments for $G_n$ such that $π(\mathcal{E}_1) \cong π(\mathcal{E}_2) \neq 0$.

**Case 1:** Consider the case where $\mathcal{E}_1$ and $\mathcal{E}_2$ are non-negative, i.e., for $i \in \{1, 2\}$,
$$([A, B]_ρ, l, η) \in \mathcal{E}_i \implies B \geq 0.$$

Then we show that $\mathcal{E}_2$ can be obtained from $\mathcal{E}_1$ by a finite chain of the operations (C), (UI) and their inverses.

If $D_{ρ|·|^2}(π(\mathcal{E}_1)) = 0$ for all $ρ \in \text{Cusp}^+(\text{GL}_d(F))$ and $x > 0$, then $D_{ρ|·|^2}(π(\mathcal{E}_1)) = 0$ for all $ρ \in \text{Cusp}^+(\text{GL}_d(F))$ and $x \in R$ with $x \neq 0$. In this case, by replacing $\mathcal{E}_1$ and $\mathcal{E}_2$ using Theorem 4.1, we may assume that for $i \in \{1, 2\}$,
$$([A, B]_ρ, l, η) \in \mathcal{E}_i \implies B = 0.$$

Then by considering the extended cuspidal support of $π(\mathcal{E}_1)$, we can see that $ψ_{\mathcal{E}_1} \cong ψ_{\mathcal{E}_2}$. See the argument in Section 4.2. Hence we have $\mathcal{E}_1 = \mathcal{E}_2$.

Suppose that $D_{ρ|·|^2}(π(\mathcal{E}_1)) \neq 0$ for some $ρ \in \text{Cusp}^+(\text{GL}_d(F))$ and $x > 0$. Write
$$D_{ρ|·|^2}(π) = [(x, k_0), (x + 1, k_1), \ldots, (x + t - 1, k_{t-1}); π^+].$$
Then $t > 0$ and $x > 0$. By replacing $\mathcal{E}_1$ and $\mathcal{E}_2$ using Theorem 4.1, we may assume that for $i \in \{1, 2\}$,  
- $\max\{B | ([A, B]_{\rho}, l, \eta) \in \mathcal{E}_i\} = x$; and 
- the multi-set $\{[A, B]_{\rho} | ([A, B]_{\rho}, l, \eta) \in \mathcal{E}_i, \ B = x\}$ is exactly equal to 
\[
\bigcup_{j=1}^{t} \{[x+j-1, x]_{\rho}, \ldots, [x+1, x]_{\rho}\}
\]
with $k_i \coloneqq 0$. 
Let $\mathcal{E}'_{i}$ be defined from $\mathcal{E}_i$ by removing all extended segments of the form $([x+j-1, x]_{\rho}, l_j, \eta_j)$ for $1 \leq j \leq t$. Then by definition of $\pi(\mathcal{E})$, we see that $\pi(\mathcal{E}_1) \cong \pi(\mathcal{E}_2) \neq 0$ implies that $\pi(\mathcal{E}'_1) \cong \pi(\mathcal{E}'_2) \neq 0$. By the inductive hypothesis, $\mathcal{E}'_2$ can be given from $\mathcal{E}'_1$ by an operator which is a finite chain of the operations (C), (UI) and their inverses. Then $\mathcal{E}'_2$ can be given from $\mathcal{E}_1$ by the same operator. 

**Case 2:** We prove the general case. For $i \in \{1, 2\}$ and $z \in \mathbb{Z}$ with $z > 0$, let $\mathcal{E}_{i,z}$ be given from $\mathcal{E}_i$ by replacing each element $([A, B]_{\rho}, l, \eta) \in \mathcal{E}_i$ with $([A+z, B+z]_{\rho}, l, \eta)$. Take $z \gg 0$ so that $\mathcal{E}_{1,z}$ and $\mathcal{E}_{2,z}$ are both non-negative. Note that $\pi(\mathcal{E}_{i,z}) \neq 0$ for $i \in \{1, 2\}$. Write 
\[
\pi(\mathcal{E}_{1,z}) = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_1}[x_r, y_r]; \pi(\phi, \varepsilon z)), \\
\pi(\mathcal{E}_{2,z}) = L(\Delta_{\rho'_1}[x'_1, y'_1], \ldots, \Delta_{\rho'_r}[x'_{r'}, y'_{r'}]; \pi(\phi', \varepsilon' z)).
\]
Then by [2, Theorem 3.6],  
\[
\pi(\mathcal{E}_1) = L(\Delta_{\rho_1}[x_1 - z, y_1 + z], \ldots, \Delta_{\rho_1}[x_r - z, y_r + z]; \pi(\phi, \varepsilon)), \\
\pi(\mathcal{E}_2) = L(\Delta_{\rho'_1}[x'_1 - z, y'_1 + z], \ldots, \Delta_{\rho'_{r'}}[x'_{r'} - z, y'_{r'} + z]; \pi(\phi', \varepsilon')),
\]
where 
- $\phi$ is given from $\phi_z$ by replacing each $\rho \boxtimes S_a \subset \phi_z$ with $\rho \boxtimes S_{a-2z}$; 
- $\varepsilon(\rho \boxtimes S_{a-2z}) = \varepsilon_z(\rho \boxtimes S_a)$; and 
- $(\phi', \varepsilon')$ is given from $(\phi'_z, \varepsilon'_z)$ by the same way. 
Since $\pi(\mathcal{E}_1) \cong \pi(\mathcal{E}_2)$, we see that 
- if we consider the multi-sets $X = \{\Delta_{\rho_1}[x_1 - z, y_1 + z], \ldots, \Delta_{\rho_1}[x_r - z, y_r + z]\}$ and $X' = \{\Delta_{\rho'_1}[x'_1 - z, y'_1 + z], \ldots, \Delta_{\rho'_{r'}}[x'_{r'} - z, y'_{r'} + z]\}$, any element in $X \setminus (X \cap X')$ or $X' \setminus (X \cap X')$ is of the form $\Delta_{\rho}[-l+z, -l+1+z]$ or $\Delta_{\rho}[-l+1/2+z, -l+1/2-z]$ for some $l \in \mathbb{Z}$ with $l > 0$; 
- as a virtual representation, $\phi_z - \phi'_z$ is a sum of representations of the form $\rho \boxtimes S_{2z}$ (with possibly negative multiplicities). 
Moreover, by definition and [2, Theorem 1.3], for $i \in \{1, 2\}$, if $([A, B]_{\rho}, l, \eta) \in \mathcal{E}_i$, then 
- $A + B \geq 0$; 
- $B + l \geq -1/2$; 
- if $B + l = -1/2$, then $\eta \in \{\pm 1\}$ is uniquely determined.  
Therefore, for $i \in \{1, 2\}$, one can define $\mathcal{E}'_{i,z}$ from $\mathcal{E}_{i,z}$ by repeatedly adding $([l-1+z, l+1]_{\rho}, l, \pm 1)$ for suitable $l > 0$ or $([l-1/2+z, l-1/2+z]_{\rho}, l, \pm 1)$ for suitable $l \geq 0$ such that $\pi(\mathcal{E}'_{1,z}) \cong \pi(\mathcal{E}'_{2,z})$. Let $\mathcal{E}'_i$ be given from $\mathcal{E}'_{i,z}$ by replacing each element $([A, B]_{\rho}, l, \eta) \in \mathcal{E}'_{i,z}$ with $([A-z, B-z]_{\rho}, l, \eta)$. Then by construction, $\mathcal{E}'_i$ is given from $\mathcal{E}_i$ by applying the operation (P) several times. Since $\pi(\mathcal{E}'_{1,z}) \cong \pi(\mathcal{E}'_{2,z})$, by Case 1,
\( \mathcal{E}'_2, z \) is given from \( \mathcal{E}'_1, z \) by a finite chain of the operations (C), (UI) and their inverses. Then \( \mathcal{E}'_2 \) can be given from \( \mathcal{E}'_1 \) by the same operator. In conclusion, \( \mathcal{E}'_2 \) can be given from \( \mathcal{E}'_1 \) by a finite chain of the operations (C), (UI), (P) and their inverses.

This completes the proof of Theorem 3.5.

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