Boundedness Estimates for Commutators of Riesz Transforms Related to Schrödinger Operators

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Abstract. Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n (n \geq 3)$, where the non-negative potential $V$ belongs to reverse Hölder class $RH_{q_1}$ for $q_1 > \frac{n}{2}$. Let $H^p_{\mathcal{L}}(\mathbb{R}^n)$ be the Hardy space associated with $\mathcal{L}$. In this paper, we consider the commutator $[b, T_\alpha]$, which associated with the Riesz transform $T_\alpha = V_\alpha (-\Delta + V)^{-\alpha}$ with $0 < \alpha \leq 1$, and a locally integrable function $b$ belongs to the new Campanato space $\Lambda^\theta_\beta(\rho)$. We establish the boundedness of $[b, T_\alpha]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < q_1/\alpha$ with $1/q = 1/p - \beta/n$. We also show that $[b, T_\alpha]$ is bounded from $H^p_{\mathcal{L}}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $n/(n+\beta) < p \leq 1,1/q = 1/p - \beta/n$. Moreover, we prove that $[b, T_\alpha]$ maps $H^{n/\alpha}_{\mathcal{L}}(\mathbb{R}^n)$ continuously into weak $L^1(\mathbb{R}^n)$.

Key Words: Riesz transform, Schrödinger operator, commutator, Campanato space, Hardy space.

AMS Subject Classifications: 42B30, 42B25, 35J10

1 Introduction and results

Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, where $n \geq 3$. The function $V$ is nonnegative, $V \neq 0$, and belongs to a reverse Hölder class $RH_{q_1}$ for some $q_1 > n/2$, that is to say, $V$ satisfies the reverse Hölder inequality

$$\left( \frac{1}{|B|} \int_B V(y)^{q_1} dy \right)^{1/q_1} \leq C \frac{|B|}{|B|} \int_B V(y) dy$$

for all ball $B \subset \mathbb{R}^n$. We consider the Riesz transform $T_\alpha = V_\alpha (-\Delta + V)^{-\alpha}$, where $0 < \alpha \leq 1$.

Many results about $T_\alpha = V_\alpha (-\Delta + V)^{-\alpha}$ and its commutator have been obtained. Shen [1] established the $L^p$-boundedness of $T_1$ and $T_{1/2}$, Liu and Tang [2] showed that $T_1$ and $T_{1/2}$ are bounded on $H^p_{\mathcal{L}}(\mathbb{R}^n)$ for $\frac{n}{n+\beta} < p \leq 1$. For $0 < \alpha \leq 1$, Sugano [3] studied the $L^p$-boundedness and Hu and Wang [4] obtained the $H^p_{\mathcal{L}}(\mathbb{R}^n)$ boundedness. When $b \in BMO$, Guo,

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Li and Peng [5] obtained the $L^p$-boundedness of commutators $[b,T]_1$ and $[b,T_{1/2}]$, Li and Peng in [6] proved that $[b,T_1]$ and $[b,T_{1/2}]$ map continuously $H^1_1(R^n)$ into weak $L^1(R^n)$. When $b \in BMO_0(\rho)$ and $0 < \alpha \leq 1$, the $L^p$- boundedness of $[b,T_\alpha]$ was investigated in [7] and the boundedness from $H^1_1(R^n)$ into weak $L^1(R^n)$ given in [4].

In this paper, we are interested in the boundedness of $[b,T_\alpha]$ when $b$ belongs to the new Campanato class $\Lambda^\theta_\beta(\rho)$. Let us recall some concepts.

As in [1], for a given potential $V \in RH_{q_1}$ with $q_1 > n/2$, we define the auxiliary function

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-\theta}} \int_{B(x,r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n.$$  

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

Let $\theta > 0$ and $0 < \beta < 1$, in view of [8], the new Campanato class $\Lambda^\theta_\beta(\rho)$ consists of the locally integrable functions $b$ such that

$$\frac{1}{|B(x,r)|^{1+\beta/n}} \int_{B(x,r)} |b(y) - b_B| dy \leq C \left( 1 + \frac{r}{\rho(x)^\theta} \right)$$

for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda^\theta_\beta(\rho)$, denoted by $[b]_{\Lambda^\theta_\beta(\rho)}$, is given by the infimum of the constants in the inequalities above.

Note that if $\theta = 0$, $\Lambda^\theta_\beta(\rho)$ is the classical Campanato space; If $\beta = 0$, $\Lambda^\theta_\beta(\rho)$ is exactly the space $BMO_0(\rho)$ introduced in [9].

We recall the Hardy space associated with Schrödinger operator $L$, which had been studied by Dziubański and Zienkiewicz in [10,11]. Because $V \in L^{q_1}_{loc}(\mathbb{R}^n)$, the Schrödinger operator $L$ generates a (C_0) contraction semigroup $\{T^b_s : s > 0\} = \{e^{-sL} : s > 0\}$. The maximal function associated with $\{T^b_s : s > 0\}$ is defined by $M^b f(x) = \sup_{s > 0} |T^b_s f(x)|$, we always denote $\delta' = \min\{1, 2 - n/q_1\}$. For $\frac{n}{n+\delta'} < p \leq 1$, We say that $f$ is an element of $H^p_{\delta'}(\mathbb{R}^n)$ if the maximal function $M^b f$ belongs to $L^p(\mathbb{R}^n)$. The quasi-norm of $f$ is defined by $\|f\|_{H^p_{\delta'}(\mathbb{R}^n)} = \|M^b f\|_{L^p(\mathbb{R}^n)}$.

We now formulate our main results as follows.

**Theorem 1.1.** Let $V \in RH_{q_1}$ with $q_1 > n/2$, and let $b \in \Lambda^\theta_\beta(\rho)$. If $0 < \alpha \leq 1$ and $\frac{a_1}{q_1 - a} < p < \infty$, then

$$\|[b,T^a_\alpha]\|_{L^p(\mathbb{R}^n)} \leq C[b]_{\Lambda^\theta_\beta(\rho)} \|f\|_{L^p(\mathbb{R}^n)},$$

where $1/q = 1/p - \beta/n$, and $T^a_\alpha = (-\Delta + V)^{-\alpha} V^a$.

We immediately deduce the following result by duality.

**Corollary 1.1.** Let $V \in RH_{q_1}$ with $q_1 > n/2$, and let $b \in \Lambda^\theta_\beta(\rho)$. If $0 < \alpha \leq 1$ and $1 < p < q_1/a$, then

$$\|[b,T_\alpha]\|_{L^{q_1/a}(\mathbb{R}^n)} \leq C[b]_{\Lambda^\theta_\beta(\rho)} \|f\|_{L^{q_1/a}(\mathbb{R}^n)},$$

where $1/q = 1/p - \beta/n$. 

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Theorem 1.2. Let \( V \in RH_{q_1} \) with \( q_1 > n/2 \), and let \( 0 < \alpha \leq 1 \). Suppose \( b \in \Lambda_\beta^\theta (\rho) \) and \( 0 < \beta < \delta' \). If \( \frac{n}{n+\beta} < p \leq 1, \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n} \) and \( q < \frac{q_1}{\alpha} \), then the commutator \([b, T_\alpha]\) is bounded from \( H^p L^q (\mathbb{R}^n) \) into \( L^q (\mathbb{R}^n) \).

Theorem 1.3. Let \( V \in RH_{q_1} \) with \( q_1 > n/2 \), and let \( 0 < \alpha \leq 1 \). Suppose \( b \in \Lambda_\theta^\beta (\rho) \), \( 0 < \beta < \delta' \). Then the commutator \([b, T_\alpha]\) is bounded from \( H^\frac{n}{n+\beta} L^1(\mathbb{R}^n) \) into weak \( L^1 (\mathbb{R}^n) \).

We shall use the symbol \( A \lesssim B \) to indicate that there exists a universal positive constant \( C \), independent of all important parameters, such that \( A \leq CB \). \( A \approx B \) means that \( A \lesssim B \) and \( B \lesssim A \).

2 Some preliminaries

We recall some important properties concerning the auxiliary function.

Proposition 2.1 (see [1]). Let \( V \in RH_{n/2} \). For the function \( \rho \) there exist \( C \) and \( k_0 \geq 1 \) such that
\[
C^{-1} \rho(x) \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq C \rho(x) \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{k_0}
\]
for all \( x, y \in \mathbb{R}^n \).

Assume that \( Q = B(x_0, \rho(x_0)) \), for any \( x \in Q \), Proposition 2.1 tell us that \( \rho(x) \approx \rho(y) \), if \( |x-y| < C \rho(x) \). It is easy to get the following result from Proposition 2.1.

Lemma 2.1. Let \( k \in \mathbb{N} \) and \( x \in 2^{k+1} B(x_0, r) \setminus 2^kB(x_0, r) \). Then we have
\[
\frac{1}{ \left( 1 + \frac{2^k r}{\rho(x)} \right)^N } \lesssim \frac{1}{ \left( 1 + \frac{2^k r}{\rho(x_0)} \right)^{N/(k_0+1)} }.
\]

Lemma 2.2 (see [11]). Suppose \( V \in RH_{q_1}, q_1 \geq n/2 \). Then there exists constants \( C > 0 \) and \( l_0 > 0 \) such that
\[
\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq C \left( 1 + \frac{r}{\rho(x)} \right)^{l_0}.
\]

The following finite overlapping property given by Dziubański and Zienkiewicz in [10].

Proposition 2.2. There exists a sequence of points \( \{ x_k \}_{k=1}^\infty \) in \( \mathbb{R}^n \), so that the family of critical balls \( Q_k = B(x_k, \rho(x_k)) \), \( k \geq 1 \), satisfies
(i) \( \bigcup_k Q_k = \mathbb{R}^n \).
Lemma 2.4 (see [4, 12]). Let $K$ be the kernel of balls as in Proposition 2.2, then

$$B_n = K_{\rho,\alpha} g(x) = \sup_{x \in B \subset B_\rho} \frac{1}{|B|} \int_B |g(y)| dy,$$

$$B_n' = K_{\rho,\alpha} g(x) = \sup_{x \in B \subset B_\rho} \frac{1}{|B|} \int_B |g(y) - g_\theta| dy,$$

where $B_{\rho,\alpha} = \{ B(z, r) : z \in \mathbb{R}^n \text{ and } r \leq a \rho(y) \}$. We have the following Fefferman-Stein type inequality.

Proposition 2.3 (see [9]). For $1 < p < \infty$, there exist $\delta$ and $\beta$ such that if $\{Q_k\}_k$ is a sequence of balls as in Proposition 2.2, then

$$\int_{\mathbb{R}^n} |M_{\rho,\alpha} g(x)|^p dx \lesssim \int_{\mathbb{R}^n} |M_{\rho,\beta} g(x)|^p dx + \sum_k |Q_k| \left( \frac{1}{|Q_k|} \int_{2Q_k} |g| \right)^p$$

for all $g \in L^1_{\text{loc}}(\mathbb{R}^n)$.

We have an inequality for the function $b \in \Lambda^\rho_\beta(\rho)$.

Lemma 2.3 (see [8]). Let $1 \leq s < \infty$, $b \in \Lambda^\rho_\beta(\rho)$, and $B = B(x, r)$. Then

$$\left( \frac{1}{|2^k B|} \int_{2^k B} |b(y) - b_\beta|^s dy \right)^{1/s} \leq C |b|^{\rho}_\beta \left( 2^k r \right)^\beta \left( 1 + \frac{2^k r}{\rho(x)} \right)^{\theta'},$$

for all $k \in \mathbb{N}$, where $\theta' = (k_0 + 1)\theta$ and $k_0$ is the constant appearing in Proposition 2.1.

Let $K_\alpha$ be the kernel of $(-\Delta + V)^{-\alpha}$. The following results give the estimates on the kernel $K_\alpha(x, y)$.

Lemma 2.4 (see [4, 12]). Suppose $V \in RH_{q_1}$ with $q_1 \geq \frac{n}{2}$.

(i) For every $N > 0$, there exists a constant $C$ such that

$$|K_\alpha(x, y)| \lesssim \frac{1}{(1 + |x - y|/\rho(x))^N} \frac{1}{|x - y|^{n - 2\alpha}}.$$

(ii) For every $0 < \delta < \delta'$ there exists a constant $C$ such that for every $N > 0$, we have

$$|K_\alpha(x, y) - K_\alpha(x, z)| + |K_\alpha(y, x) - K_\alpha(z, x)|$$

$$\lesssim \frac{1}{(1 + |x - y|/\rho(x))^N} \frac{|y - z|^\delta}{|x - y|^{n + \delta - 2\alpha}},$$

where $|y - z| \leq |x - y|/4$. 

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Proposition 2.4 (see [13]). Suppose that $V \in RH_{q_1}$ with $q_1 > \frac{n}{2}$. Let $0 < \beta_2 \leq \beta_1 \leq 1$, $1 < \left(\frac{q_1}{p_2}\right)' < p_1 < \frac{n}{2(\beta_1 - \beta_2)}$ and $\frac{1}{p_2} = \frac{1}{p_1} - 2(\beta_1 - \beta_2) n$. Then

$$\left|(-\Delta + V)^{-\beta_1} \psi^{\beta_2}(f)(x)\right| \lesssim M_{2(\beta_1 - \beta_2) \frac{q_1}{p_2}} f(x).$$

Let $\beta_1 = \beta_2 = a$, by Proposition 2.4 and duality we get

Corollary 2.1. Suppose that $V \in RH_{q_1}$ with $q_1 > \frac{n}{2}$. Let $0 < \alpha \leq 1$

(i) For $0 < \frac{q_1}{q_1 - a} < p \leq \infty$, $T_A^a$ is bounded on $L^p(\mathbb{R}^n)$;

(ii) For $1 \leq p < \frac{q_1}{\alpha}$, $T_A^a$ is bounded on $L^p(\mathbb{R}^n)$.

3 The $L^p$-boundedness of $[b, T_A^a]$

To prove Theorem 1.1, we need the following Lemmas.

Lemma 3.1. Suppose $V \in RH_{q_1}$ with $q_1 > \frac{n}{2}$, and $b \in \Lambda_{\beta}^p(\rho)$. If $\frac{q_1}{q_1 - a} < s < \infty$, then for all $f \in L^s_{\text{loc}}(\mathbb{R}^n)$ and every critical ball $Q = B(x_0, \rho(x_0))$, we have

$$\frac{1}{|Q|} \int_Q |[b, T_A^a] f(y)| \, dy \lesssim \|b\|_{\beta} \inf_{x \in Q} M_{\beta,s}(f)(x),$$

where

$$M_{\beta,s}(f)(x) = \sup_{B \subset B(x)} \left( \frac{1}{|B|^{1-\beta s/n}} \int_B |f(y)|^s \, dy \right)^{1/s}.$$

Proof. Since

$$[b, T_A^a] f(y) = (b(y) - b_Q) T_A^a f(y) - T_A^a (b(y) - b_Q) f(y),$$

then

$$\frac{1}{|Q|} \int_Q |[b, T_A^a] f(y)| \, dy \leq \frac{1}{|Q|} \int_Q |(b(y) - b_Q) T_A^a f(y)| \, dy + \frac{1}{|Q|} \int_Q |T_A^a ((b(y) - b_Q) f(y))| \, dy = I_1 + I_2.$$

By Hölder’s inequality and Lemma 2.3 we have

$$I_1 \leq \left( \frac{1}{|Q|} \int_Q |b(y) - b_Q|^s \, dy \right)^{1/s'} \left( \frac{1}{|Q|} \int_Q |T_A^a f(y)|^s \, dy \right)^{1/s} \lesssim [b]_{\beta}^p \rho(x_0)^{\beta} \left( \frac{1}{|Q|} \int_Q |T_A^a f_1(y)|^s \, dy \right)^{1/s} + \left( \frac{1}{|Q|} \int_Q |T_A^a f_2(y)|^s \, dy \right)^{1/s}$$

$$= I_{11} + I_{12},$$
Thus, taking $f = f_1 + f_2$ with $f_1 = f \chi_{2Q}$.

By the $L^s$-boundedness of $T^s_\alpha$ (Corollary 2.1), we have

$$I_{11} \lesssim [b]^\beta \rho(x_0)^\beta \left( \frac{1}{|Q|} \int_{2Q} |f(y)|^s dy \right)^{1/s} \lesssim [b]^\beta \inf_{x \in Q} M_{\beta,s}(f)(x).$$

By Lemma 2.4,

$$|T^s_\alpha f_2(y)| \lesssim \int_{(2Q)^c} |K_\alpha(y,z) f(z)| V(z)^\alpha dz \lesssim \int_{(2Q)^c} \frac{|f(z)| V(z)^\alpha dz}{\left( 1 + \frac{|y-z|}{\rho(y)} \right)^N |y-z|^{n-2\alpha}}.$$  

For any $y \in Q$ and $z \in (2Q)^c$, we have $\rho(y) \approx \rho(x_0)$, and $|y-z| \approx |x_0-z|$. So, decomposing $(2Q)^c$ into annuli $2^k Q \setminus 2^{k-1} Q, k \geq 2$, we get

$$|T^s_\alpha f_2(y)| \lesssim \sum_{k \geq 2} 2^{-kN} \int_{2Q} |f(z)| V(z)^\alpha dz.$$  

Since $\frac{q_1}{n-\alpha} < s < \infty$, we can choose $t < q_1$ such that $\frac{1}{s} + \frac{t}{s} = 1$. By Hölder’s inequality and Lemma 2.2, we get

$$\frac{1}{|2^k Q|} \int_{2^k Q} V(z)^\alpha |f(z)| dz$$

$$\lesssim \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f(z)|^s dz \right)^{1/s} \left( \frac{1}{|2^k Q|} \int_{2^k Q} V(z)^\alpha dz \right)^{\frac{t}{s}}$$

$$\lesssim \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f(z)|^s dz \right)^{1/s} \left( \frac{1}{|2^k Q|} \int_{2^k Q} V(z)^\beta dz \right)^{\frac{t}{s}}$$

$$\lesssim \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f(z)|^s dz \right)^{1/s} \left( \frac{2^k \rho(x_0) - 2}{|2^k Q|^{1-2/n} \int_{2^k Q} V(z) dz} \right)^{\alpha}$$

$$\lesssim 2^{k\alpha} (2^k \rho(x_0))^{-2\alpha} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f(z)|^s dz \right)^{1/s}.$$  

Then

$$T^s_\alpha f_2(y) \lesssim \sum_{k \geq 2} 2^{-k(N-l_0\alpha)} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f(z)|^s dz \right)^{1/s}.$$  

Thus, taking $N > l_0\alpha$ we get

$$I_{12} \lesssim [b]^\beta \inf_{x \in Q} M_{\beta,s}(f)(x).$$
The estimate for \( I_2 \) can be proceeded in the same way of \( I_1 \). The decomposition \( f = f_1 + f_2 \) gives

\[
I_2 \leq \frac{1}{|Q|} \int_Q |T_a^s((b-b_Q)f_1)(y)|dy + \frac{1}{|Q|} \int_Q |T_a^s((b-b_Q)f_2)(y)|dy
\]

\[
= I_{21} + I_{22}.
\]

Let \( \frac{\theta}{q_1 - \alpha} < \delta < s \). By Hölder’s inequality, \( L^\delta \)-boundedness of \( T_a^s \) and Lemma 2.3, for some \( u > 1 \) we have

\[
I_{21} \lesssim \left( \frac{1}{|Q|} \int_{2Q} |(b-b_Q)f_1(y)|^\delta dy \right)^{1/\delta}
\]

\[
\lesssim \left( \frac{1}{|Q|} \int_{2Q} |(b-b_Q)f_1(y)| dy \right)^{1/\delta}
\]

\[
\lesssim \left( \frac{1}{|Q|} \int_{2Q} |f(y)|^\delta dy \right)^{1/\delta} \left( \frac{1}{|Q|} \int_{2Q} |b(y) - b_Q|^u dy \right)^{1/u}
\]

\[
\lesssim |b|_p^\delta \inf_{x \in Q} M_{\beta, \delta}(f)(x).
\]

The estimate \( I_{22} \lesssim |b|_p^\delta \inf_{x \in Q} M_{\beta, \delta}(f)(x) \) can be obtained by the similar approach to ones of \( I_{12} \) and \( I_{21} \). Then we omit the details here.

\[\square\]

**Lemma 3.2.** Let \( B = B(x_0, r) \) with \( r \leq \gamma \rho(x_0) \) and let \( x \in B \), then for any \( y, z \in B \) we have

\[
\int_{(2B)_\rho} |K_a(y, u) - K_a(z, u)| |b(u) - b_B| |f(u)| |V(u)|^a du \lesssim |b|_p^\delta M_{\beta, \delta}(f)(x).
\]

**Proof.** Setting \( Q = B(x_0, \gamma \rho(x_0)) \), due to the fact \( \rho(y) \approx \rho(z) \approx \rho(x_0) \) and \( |y - u| \approx |z - u| \approx |x_0 - u| \), then by Lemma 2.4 we get

\[
\int_{(2B)_\rho} |K_a(y, u) - K_a(z, u)| |b(u) - b_B| |f(u)| |V(u)|^a du \lesssim K_1 + K_2,
\]

where

\[
K_1 = r^\delta \int_{Q \setminus 2B} \frac{|f(u)(b(u) - b_B)| |V(u)|^a}{|x_0 - u|^{|n+\delta-2\alpha|}} du
\]

and

\[
K_2 = r^\delta \rho(x_0)^N \int_{Q^\rho} \frac{|f(u)(b(u) - b_B)| |V(u)|^a}{|x_0 - u|^{|n+N+\delta-2\alpha|}} du.
\]

Let \( j_0 \) be the least integer such that \( 2^{j_0} \geq \gamma \rho(x_0)/r \). Splitting into annuli, we have

\[
K_1 \leq \sum_{j=2}^{j_0} \frac{1}{2^{j-1}} \int_{2^j B} |f(u)||b(u) - b_B| |V|^a(u) du.
\]
By \( \frac{q_1}{q_1 - a} < s < \infty \), we have \( \frac{1}{2} + \frac{a}{q_1} < 1 \). We choose \( t < q_1 \) and \( \nu > 1 \) such that \( \frac{1}{2} + \frac{a}{q_1} + \frac{1}{\nu} = 1 \). Then by Hölder’s inequality and Lemma 2.3, we have

\[
\begin{align*}
\frac{1}{2B} \int_{2B} |f(u)||b(u) - b_B| V(u)^a du &\leq \left( \frac{1}{2B} \int_{2B} |f(u)|^s du \right)^{1/s} \left( \frac{1}{2B} \int_{2B} |b(u) - b_B|^\nu du \right)^{1/\nu} \times \left( \frac{1}{2B} \int_{2B} V(u)^t du \right)^{a/t} \\
&\lesssim |b|_p^a \left( \frac{2r}{\rho(x_0)} \right)^{-2a} \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\theta' + l_0} \left( \frac{1}{2B} \int_{2B} |f(u)|^s du \right)^{1/s}.
\end{align*}
\]

Note \( \frac{2r}{\rho(x_0)} \leq \gamma \), we get

\[
K_1 \lesssim |b|_p^a M_{\beta,s}(f)(x).
\]

For \( K_2 \), splitting into annuli,

\[
K_2 \lesssim \rho(x_0)^N \sum_{j \geq j_0} 2^{-j^2} \left( \frac{2r}{\rho(x_0)} \right)^{-2a} \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\theta' + l_0} \left( \frac{1}{2B} \int_{2B} |f(u)|^s du \right)^{1/s}.
\]

Since \( \frac{2r}{\rho(x_0)} \geq \gamma \), taking \( N \geq \theta' + l_0 \), we get

\[
K_2 \lesssim |b|_p^a M_{\beta,s}(f)(x).
\]

Thus, we complete the proof. \( \square \)

**Lemma 3.3.** Let \( \frac{q_1}{q_1 - a} < s < \infty \), let \( B = B(x_0, r) \) with \( r \leq \gamma \rho(x_0) \) and let \( x \in B \). Then

\[
M_{\beta,\gamma}^s([b,T^*_a]f)(x) \lesssim |b|_p^a (M_{\beta,s}(f)(x) + M_{\beta,s}(T^*_a f)(x)).
\]

**Proof.** Write

\[
\begin{align*}
&\frac{1}{|B|} \int_B |[b,T^*_a]f(y) - ([b,T^*_a]f)_B| dy \\
&\leq \frac{2}{|B|} \int_B |(b(y) - b_B)T^*_a f(y)| dy + \frac{2}{|B|} \int_B |T^*_a((b - b_B)f_1)(y)| dy \\
&\quad + \frac{1}{|B|} \int_B |T^*_a((b - b_B)f_2)(y) - (T^*_a((b - b_B)f_2))_B| dy \\
&= J_1 + J_2 + J_3,
\end{align*}
\]
where \( f = f_1 + f_2 \) with \( f_1 = f \chi_{2B} \).

Since \( r \geq \gamma \rho(x_0) \) and \( \rho(x) \approx \rho(x_0) \), by Hölder’s inequality and Lemma 2.3, we get

\[
J_1 \leq \left( \frac{1}{|B|} \int_B |b(y) - b_B|^{s'} dy \right)^{1/s'} \left( \frac{1}{|B|} \int_B |T_a^s f(y)|^s dy \right)^{1/s} \\
\lesssim [b]_{\beta}^{\theta} \left( \frac{1}{|B|} \int_B |T_a^s f(y)|^s dy \right)^{1/s} \lesssim [b]_{\beta}^{\theta} M_{\beta,s}(T_a^s f)(x)
\]

Select \( r_0 \) so that \( \frac{q_1}{q_1 - \alpha} < r_0 < s < \infty \), then by Hölder’s inequality and Lemma 2.3,

\[
J_2 \lesssim \left( \frac{1}{|B|} \int_B |T_a^s ((b - b_B)f_1)(y)|^{r_0} dy \right)^{1/r_0} \\
\lesssim \left( \frac{1}{|B|} \int_{2B} |(b(y) - b_B)f(y)|^{r_0} dy \right)^{1/r_0} \\
\lesssim \left( \frac{1}{|B|} \int_{2B} |b(y) - b_B|^{u} dy \right)^{1/u} \left( \frac{1}{|B|} \int_{2B} |f(y)|^s \right)^{1/s} \\
\lesssim [b]_{\beta}^{\theta} M_{\beta,s}(f)(x)
\]

By Lemma 3.2,

\[
J_3 \leq \frac{1}{|B|^2} \int_B \int_B \int_{(2B)^c} \left| K_a(y,u) - K_a(z,u) \right| |b(u) - b_B||f(u)|V(u)^a dudz \text{dy} \\
\lesssim \int_{(2B)^c} \left| K_a(y,u) - K_a(z,u) \right| |b(u) - b_B||f(u)|V(u)^a du \\
\lesssim [b]_{\beta}^{\theta} M_{\beta,s}(f)(x)
\]

So, we complete the proof. \( \square \)

We now come to prove Theorem 1.1. By Proposition 2.3, Lemma 3.1 and Lemma 3.3 we have

\[
\| [b, T_a^s] f \|_{L^q(\mathbb{R}^n)}^q \\
\leq \int_{\mathbb{R}^n} |M_{\rho,s}([b, T_a^s] f)(x)|^q dx \\
\leq \int_{\mathbb{R}^n} |M_{\rho,s}([b, T_a^s] f)(x)|^q dx + \sum_k |Q_k| \left( \frac{1}{|Q_k|} \int_{2Q_k} |[b, T_a^s] f(x)| dx \right)^q \\
\leq \int_{\mathbb{R}^n} |M_{\rho,s}([b, T_a^s] f)(x)|^q dx + \sum_k |Q_k| \left( \inf_{y \in 2Q_k} M_{\beta,s}(f)(y) \right)^q \\
\lesssim ([b]_{\beta}^{\theta})^q \int_{\mathbb{R}^n} |M_{\beta,s}(f)(x) + M_{\beta,s}(T_a^s f)(x)|^q dx + ([b]_{\beta}^{\theta})^q \sum_k \int_{2Q_k} |M_{\beta,s}(f)(x)|^q dx
\]
\[
\lesssim (b^0) \frac{q}{p} \left( \int_{\mathbb{R}^n} |M_{\beta, \delta} (f)(x)|^q dx + \int_{\mathbb{R}^n} |M_{\beta, \delta} (T_\alpha f)(x)|^q dx \right)
\]
\[
\lesssim (b^0) \frac{q}{p} \| f \|_{L^q(\mathbb{R}^n)}^q,
\]
where we have used the finite overlapping property given by Proposition 2.2.

## 4 The \( H^p_{L^q} \)-boundedness of \([b, T_\alpha]\)

We have the following atomic characterization of Hardy space.

**Definition 4.1.** Let \( \frac{np}{n+np} < p \leq q \leq \infty \). A function \( a \in L^2(\mathbb{R}^n) \) is called an \( H^p_{L^q} \)-atom if \( r < \rho(x_0) \) and the following conditions hold:

(i) \( \text{supp } a \subset B(x_0, r) \),

(ii) \( \| a \|_{L^q(\mathbb{R}^n)} \leq |B(x_0, r)|^{1/q-1/p} \),

(iii) \( \| a \|_{L^q(\mathbb{R}^n)} \leq (B(x_0, r/2) \setminus B(x_0, r)) \) if and only if \( f \) can be written as \( f = \sum \lambda_j a_j \), where \( a_j \) are \( H^p_{L^q} \)-atoms, \( \sum |\lambda_j|^p < \infty \), and the sum converges in the \( H^p_{L^q} \)-norm. Moreover

\[
\| f \|_{H^p_{L^q}(\mathbb{R}^n)} \approx \inf \left\{ \left( \sum_j |\lambda_j|^p \right)^{1/p} \right\},
\]

where the infimum is taken over all atomic decompositions of \( f \) into \( H^p_{L^q} \)-atoms.

Let us prove Theorems 1.2. Choose \( \tau \) such that \( 1 < \tau < \frac{n}{p} \). By Proposition 4.1, we only need to show that for any \( H^{p, \tau} \)-atom \( a \),

\[
\| [b, T_\alpha] a \|_{L^q(\mathbb{R}^n)} \leq C
\]

holds, where \( C \) is a constant independent of \( a \).

Suppose \( \text{supp } a \subset B = B(x_0, r) \) with \( r < \rho(x_0) \). Then

\[
\| [b, T_\alpha] a \|_{L^q(\mathbb{R}^n)} \leq \left( \int_{2B} |[b, T_\alpha] a(x)|^q dx \right)^{1/q} + \left( \int_{(2B)^c} |[b, T_\alpha] a(x)|^q dx \right)^{1/q} = A_1 + A_2.
\]

Let \( 1/t = 1/\tau - \beta/n \). By Corollary 1.1 and the size condition of atom \( a \), we have

\[
A_1 \leq \left( \int_{2B} |[b, T_\alpha] a(x)|^\tau dx \right)^{1/t} \lesssim (2r)^{\frac{\alpha}{n} - \frac{\tau}{n}}
\]

\[
A_2 \leq \left( \int_{(2B)^c} |a(x)|^\tau dx \right)^{1/t} \lesssim (2r)^{\frac{\alpha}{n} - \frac{\tau}{n}} \lesssim 1.
\]
For $A_2$, we consider two cases, that is $r < \rho(x_0)/4$ and $\rho(x_0)/4 \leq r < \rho(x_0)$.

**Case I:** When $r < \rho(x_0)/4$, by the vanishing condition of $a$, we have

$$A_2 \leq \left( \int_{(2B)^c} |b(x) - b_B|^q V(x)^{aq} \left( \int_B |K_a(x,y) - K_a(x,x_0)||a(y)|dy \right)^q dx \right)^{1/q}$$

$$+ \left( \int_{(2B)^c} V(x)^{aq} \left( \int_B |K_a(x,y)(b(y) - b_B)a(y)|dy \right)^q dx \right)^{1/q}$$

$$= A_{21} + A_{22}.$$

Note that

$$\int_B |a(y)|dy \lesssim r^{n - \frac{n}{p}}. \quad (4.1)$$

By $1 < q < \frac{n}{\alpha}$, we choose $s$ so that $aq < s < q_1$. Then by Hölder’s inequality, Lemma 2.3 and Lemma 2.2 we have

$$\lesssim \left( \frac{1}{|2^kB|} \int_{2^kB} |b(x) - b_B|^q V(x)^{aq} dx \right)^{1/q}$$

$$\lesssim \left( \frac{1}{|2^kB|} \int_{2^kB} |b(x) - b_B|^u dx \right)^{1/u} \left( \frac{1}{|2^kB|} \int_{2^kB} V(x)^s dx \right)^{\alpha/s}$$

$$\lesssim [b_{\tilde{p}}(2^k\rho)^\beta \left( 1 + \frac{2^kr}{\rho(x_0)} \right)^q \left( \frac{1}{|2^kB|} \int_{2^kB} V(x)^q dx \right)^{\alpha/q}$$

$$\lesssim [b_{\tilde{p}}(2^k\rho)^{\beta - 2a} \left( 1 + \frac{2^kr}{\rho(x_0)} \right)^{\theta' + b\alpha} \quad (4.2)$$

When $x \in 2^{k+1}B(x_0,r) \setminus 2^kB(x_0,r)$, and $y \in B$, by Lemma 2.4 and Lemma 2.1, we can take $0 < \beta < \delta < \delta'$ such that

$$|K_a(x,y) - K_a(x,x_0)| \lesssim \frac{1}{\left( 1 + \frac{|2^kr|}{\rho(x_0)} \right)^{N/(k_0 + 1)}} \left( \frac{2^kr}{\rho(x_0)} \right)^{n - \delta - 2\alpha}.$$

Notice $(n - \frac{n}{p})q + n = (n - \beta)q$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ and $q > \frac{n}{n - \beta} > \frac{n}{n + \delta - \rho}$, then we get

$$(A_{21})^q \lesssim r^{(n - \frac{n}{p})q} \sum_{k \geq 1} \left( 1 + \frac{2^kr}{\rho(x_0)} \right)^{Nq/(k_0 + 1)} \left( \frac{2^kr}{\rho(x_0)} \right)^{(n + \delta - 2\alpha)q} \int_{2^kB} |b(x) - b_B|^q V(x)^{aq} dx$$

$$\lesssim [b_{\tilde{p}}(2^k\rho)^{(n + \delta - \frac{n}{p})q} \sum_{k \geq 1} \left( 1 + \frac{2^kr}{\rho(x_0)} \right)^{Nq/(k_0 + 1)} \left( \frac{2^kr}{\rho(x_0)} \right)^{(n + \delta - \beta)q} \left( \frac{2^kr}{\rho(x_0)} \right)^n$$

$$\lesssim [b_{\tilde{p}}(2^k\rho)^{n + \delta - \rho} \sum_{k \geq 1} \left( 2^k \right)^{n + \delta - \beta} \lesssim [b_{\tilde{p}}(2^k\rho)^{n + \delta - \beta} \lesssim (b_{\tilde{p}})^q].$$
For \( x \in 2^{k+1}B \setminus 2^kB, y \in B \), we have \(|x-y| \approx 2^k r\). Then by Lemma 2.4 and Lemma 2.1,

\[
|K_\alpha(x,y)| \lesssim \frac{1}{1 + \frac{2^k r}{\rho(x_0)}} \frac{1}{(2^k r)^n - 2^k r}.
\] (4.3)

Choosing \( s \) such that \( aq < s < q_1 \), we get

\[
\left( \frac{1}{|2^kB|} \int_{2^kB} V(x)^{aq} dx \right)^{1/q} \lesssim \left( \frac{1}{|2^kB|} \int_{2^kB} V(x)^{s} dx \right)^{a/s}
\]

\[
\lesssim \left( \frac{1}{|2^kB|} \int_{2^kB} V(x)^{s} dx \right)^{a} \lesssim (2^k r)^{-2^k r} \left( 1 + \frac{2^k r}{\rho(x_0)} \right)^{l_0 \alpha}.
\]

By Hölder’s inequality and Lemma 2.3 we get

\[
\int_B |b(y) - b_B| |a(y)| dy \lesssim \left( \int_B |a(y)| \tau dy \right)^{1/\tau} \left( \int_B |b(y) - b_B| \tau dy \right)^{1/\tau'}
\]

\[
\lesssim |b|^{\theta} r^{n + a + \beta} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta'}
\]

\[
\lesssim |b|^{\theta} r^{n + a + \beta} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta'}.
\]

Then, by Minkowski’s inequality and taking \( N > l_0 \alpha (k_0 + 1) \), we get

\[
A_{22} \lesssim \int_B |b(y) - b_B| |a(y)| dy \left( \sum_{k \geq 1} \int_{2^{k+1}B \setminus 2^kB} |K_\alpha(x,y)|^{\theta} V(x)^{aq} dx \right)^{1/q}
\]

\[
\lesssim |b|^{\theta} r^{n + a + \beta} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta'} \left( \sum_{k \geq 1} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^{Nq/(a+1)}} \frac{1}{(2^k r)^{n - 2^k r}} \int_{2^{k+1}B} V(x)^{aq} dx \right)^{1/q}
\]

\[
\lesssim |b|^{\theta} r^{n + a + \beta} \left( \sum_{k \geq 1} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^{Nq/(a+1)}} (2^k r)^{n - q} \right)^{1/q} \lesssim |b|^{\theta}.
\]

**Case II:** When \( \rho(x_0)/4 \leq r < \rho(x_0) \), this means \( r \approx \rho(x_0) \). The atom \( a \) does not satisfy the
vanishing condition. By Minkowksi’s inequality,
\[
A_2 \leq \left\{ \int_{(2B)^c} |b(x) - b_B|^q V(x)^{aq} \left| \int_B K_a(x,y)a(y)dy \right|^q dx \right\}^{1/q} + \left\{ \int_{(2B)^c} V(x)^{aq} \left| \int_B |K_a(x,y)(b(y) - b_B)a(y)|dy \right|^q dx \right\}^{1/q}
\]
\[= A_{21} + A_{22}.\]

Note \( r \approx \rho(x_0) \), then by (4.1), (4.2) and (4.3) we get
\[
(A_{21})^q \lesssim \sum_{k \geq 1} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^{\frac{nq}{n - \alpha}}} \frac{(2^k r)^n}{(2^k r)^{(n - 2\alpha)q}} \left( \int_B |a(y)|dy \right)^q \frac{1}{|2^k B|} \int_{2^k B} |b(x) - b_B|^q V(x)^{aq} dx
\]
\[\lesssim \sum_{k \geq 1} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^{\frac{nq}{n - \alpha}}} \frac{(2^k r)^n}{(2^k r)^{(n - 2\alpha)q}} \int_B |a(y)|dy \int_{2^k B} |b(x) - b_B|^q V(x)^{aq} dx
\]
\[\lesssim \sum_{k \geq 1} \frac{1}{(2^k)^{\frac{qN}{n} - \alpha} (1 + \frac{2^k r}{\rho(x_0)})^\alpha} \lesssim ([b]_q^{\alpha})^q.
\]

The estimate of \( A_{22} \) is exactly the same as \( A_{22} \), we omit the detail of the proof. \( \square \)

**Proof of Theorem 1.3.** Let \( f \in H^p_{\mathbb{R}^n} \), we write \( f = \sum_{j=-\infty}^{\infty} \lambda_j a_j \), where each \( a_j \) is an \( H^p_{\mathbb{R}^n} \)-atom, \( 1 < l < \frac{q}{\alpha} \) and
\[
\left( \sum_{j=-\infty}^{\infty} |\lambda_j|^{\frac{q}{lp}} \right)^{\frac{lp}{q}} \leq 2 \|f\|_{H^p_{\mathbb{R}^n}}.
\]

Suppose that \( \text{supp} a_j \subset B_j = B(x_j, r_j) \) with \( r_j < \rho(x_j) \). Write
\[
[b, T_a]f(x) = \sum_{j=-\infty}^{\infty} \lambda_j [b, T_a]a_j(x) \chi_{8B_j}(x)
\]
\[+ \sum_{j | r_j \geq \rho(x_j)/4} \lambda_j (b(x) - b_{B_j})T_a a_j(x) \chi_{(8B_j)^c}(x)
\]
\[+ \sum_{j | r_j < \rho(x_j)/4} \lambda_j (b(x) - b_{B_j})T_a a_j(x) \chi_{(8B_j)^c}(x)
\]
\[- \sum_{j=-\infty}^{\infty} \lambda_j T_a ((b - b_{B_j})a_j)(x) \chi_{(8B_j)^c}(x)
\]
\[= \sum_{i=1}^{4} \sum_{j=-\infty}^{\infty} \lambda_j A_{ij}(x).\]
Note that
\( \left( \int_{B_j} |a_j(x)|^t \, dx \right)^{1/t} \lesssim |B_j|^{1 - \frac{n+\beta}{n}}. \)

Choose \( t \) such that \( \frac{1}{t} = \frac{1}{r} - \frac{\beta}{n} \). By Hölder’s inequality and Corollary 1.1 we get
\[
\| A_{1,j} \|_{L^1(\mathbb{R}^n)} \lesssim \left( \int_{\partial B_j} |[b, T_a] a_j(x)|^t \, dx \right)^{1/t} r_j^\beta \\
\lesssim |b|_\beta^\theta \left( \int_{B_j} |a_j(x)|^t \, dx \right)^{1/t} r_j^\beta \\
\lesssim |b|_\beta^\theta |B_j|^{\frac{1}{r} - 1} \approx |b|_\beta^\theta.
\]

Note \( 0 < \frac{n}{n+\beta} < 1 \), we get
\[
\left\| \sum_{j = -\infty}^\infty \lambda_j A_{1,j} \right\|_{L^1(\mathbb{R}^n)} \lesssim \left( \sum_{j = -\infty}^\infty |\lambda_j| \| A_{1,j} \|_{L^1(\mathbb{R}^n)} \right)^{\frac{1}{n+\beta}} \\
\lesssim |b|_\beta^\theta \sum_{j = -\infty}^\infty |\lambda_j| \lesssim |b|_\beta^\theta \left( \sum_{j = -\infty}^\infty |\lambda_j|^{\frac{n}{n+\beta}} \right)^{\frac{n}{n+\beta}} \lesssim |b|_\beta^\theta \| f \|_{H^{\frac{n}{n+\beta}}_N}.
\]

Then
\[
\left\{ x \in \mathbb{R}^n : \left| \sum_{j = -\infty}^\infty \lambda_j A_{1,j} \right| > \frac{\lambda}{4} \right\} \lesssim \frac{1}{|b|_\beta^\theta} \| f \|_{H^{\frac{n}{n+\beta}}_N}.
\]

Since \( x \in B_j, y \in 2^{k+1} B_j \setminus 2^k B_j \), we have \(|x-y| \approx |x-x_j| \approx 2^k r_j\), and by Lemma 2.1 we get
\[
\frac{1}{(1 + \frac{|x-y|}{\rho(x)})^{Nt}} \lesssim \frac{1}{(1 + \frac{2^k r_j}{\rho(x_j)})^{\frac{Nt}{l}}}. \]

Since \( \alpha < \frac{n}{2} < q_1 \), we select \( s \) such that \( \alpha < s < q_1 \). Then
\[
\frac{1}{|2^{k+1} B_j|} \int_{2^{k+1} B_j} |b(x) - b_B| V(x)^a \, dx \\
\lesssim \left( \frac{1}{|2^{k+1} B_j|} \int_{2^{k+1} B_j} |b(x) - b_B|' \, dx \right)^{1/s'} \left( \frac{1}{|2^{k+1} B_j|} \int_{2^{k+1} B_j} V(x)^{s} \, dx \right)^{a/s'} \\
\lesssim |b|_\beta^\theta (2^k r_j)^\beta \left( \frac{1}{|2^{k+1} B_j|} \int_{2^{k+1} B_j} V(x) \, dx \right)^a \\
\lesssim |b|_\beta^\theta (2^k r_j)^\beta \left( 1 + \frac{2^k r_j}{\rho(x_j)} \right)^{l_0 a}.
\]
Note $\int_{B_j} |a_j(y)| dy \leq r_j^{-\beta}$, and $r_j/\rho(x_j) \geq 1/4$. Then

$$\|A_{2,j}(x)\|_{L^1(\mathbb{R}^n)} = \sum_{k \geq 3} \int_{2^{k+1}B_j \setminus 2^kB_j} |b(x) - b_{B_j}| V(x)^{\alpha} \int_{B_j} \left[ \frac{1}{1 + 2^{k}r_j \rho(x_j)} \right] \frac{1}{N} (2^{k}r_j)^{n-2\alpha} |a_j(y)| dy dx \leq \sum_{k \geq 3} \left( \frac{1}{1 + 2^{k}r_j \rho(x_j)} \right)^{\frac{N}{n+\beta}} \frac{1}{(2^{k}r_j)^{n-2\alpha}} \int_{2^{k+1}B_j} |b(x) - b_{B_j}| V(x)^{\alpha} dx \int_{B_j} |a_j(y)| dy \lesssim |b|^\beta \sum_{k \geq 1} \frac{1}{2^k (\frac{N}{n+\beta} - \delta - \theta - \rho_{B_j})} \lesssim |b|^\beta.$$  

Then

$$\left\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} \lambda_j A_{2j} > \frac{\lambda}{4} \right\} \lesssim |b|^\beta \|f\|_{H^\frac{n}{n+\beta}}.$$

When $x \in 2^{k+1}B_j \setminus 2^kB_j$, and $y \in B_j$, by Lemma 2.4 and Lemma 2.1, we have

$$|K_{\alpha}(x,y) - K_{\alpha}(x,x_j)| \lesssim \frac{1}{(1 + 2^{k}r_j \rho(x_j))^{\frac{N}{n+\beta}}} \frac{r_j^{\delta}}{(2^{k}r_j)^{n+\beta - 2\alpha}}.$$

Thus, by the vanishing condition of $a_j$ and $0 < \beta < \delta < \delta'$ we have

$$\|A_{3,j}(x)\|_{L^1(\mathbb{R}^n)} = \sum_{k \geq 3} \int_{2^{k+1}B_j \setminus 2^kB_j} |b(x) - b_{B_j}| V(x)^{\alpha} \int_{B_j} |K_{\alpha}(x,y) - K_{\alpha}(x,x_j)||a_j(y)| dy dx \leq \sum_{k \geq 3} \left( \frac{1}{1 + 2^{k}r_j \rho(x_j)} \right)^{\frac{N}{n+\beta}} \frac{1}{(2^{k}r_j)^{n+\beta - 2\alpha}} \int_{2^{k+1}B_j} |b(x) - b_{B_j}| V(x)^{\alpha} dx \int_{B_j} |a_j(y)| dy \lesssim |b|^\beta \sum_{k \geq 1} \frac{1}{2^k (\delta - \beta)} \lesssim |b|^\beta.$$  

Therefore

$$\left\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} \lambda_j A_{3j} > \frac{\lambda}{4} \right\} \lesssim |b|^\beta \|f\|_{H^\frac{n}{n+\beta}}.$$
Note that
\[
\|(b - b_{B_j}a_j)\|_{L^1} \leq \left( \int_{B_j} |b(x) - b|^{\rho'} \, dx \right)^{1/\rho'} \left( \int_{B_j} |a_j(x)|^l \, dx \right)^{1/l} \\
\lesssim |b|_{\rho'}^{\frac{n-\beta}{\rho'} + \frac{n}{l'}} \left( 1 + \frac{r_j}{\rho(x_j)} \right)^{\theta'} \lesssim |b|_{\rho'},
\]
and
\[
|A_{4j}(x)| \leq \sum_{j=\infty}^{\infty} |\lambda_j|^{|T_a(|(b - b_{B_j})a_j|)(x)\chi_{(8B_j)^c}(x)|} \leq T_a \left( \sum_{j=\infty}^{\infty} |\lambda_j|^{|(b - b_{B_j})a_j|} \right)(x).
\]
By Corollary 2.1, we know that $T_a$ is bounded from $L^1(\mathbb{R}^n)$ to $W^{1,1}(\mathbb{R}^n)$, then
\[
\left| \left\{ x \in \mathbb{R}^n : \sum_{j=\infty}^{\infty} |\lambda_j|^{|A_{4j}|} > \frac{\lambda}{4} \right\} \right| \\
\leq \left| \left\{ x \in \mathbb{R}^n : T_a \left( \sum_{j=\infty}^{\infty} |\lambda_j|^{|(b - b_{B_j})a_j|} \right)(x) > \frac{\lambda}{4} \right\} \right| \\
\lesssim \frac{1}{\lambda} \left\| \sum_{j=\infty}^{\infty} |\lambda_j|^{|(b - b_{B_j})a_j|} \right\|_{L^1} \\
\lesssim \frac{1}{\lambda} \sum_{j=\infty}^{\infty} |\lambda_j|^{|(b - b_{B_j})a_j|} \right\|_{L^1} \\
\lesssim \frac{|b|_{\rho'} |f|_{H^\beta_{L^p}}}{\lambda} \left( \sum_{j=\infty}^{\infty} |\lambda_j| \right) \lesssim \frac{|b|_{\rho'} |f|_{H^\beta_{L^p}}}{\lambda}.
\]
Thus,
\[
\left| \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{4} \sum_{j=\infty}^{\infty} |\lambda_j A_{ij}| > \lambda \right\} \right| \\
\lesssim \sum_{i=1}^{4} \left| \left\{ x \in \mathbb{R}^n : \sum_{j=\infty}^{\infty} |\lambda_j A_{ij}| > \frac{\lambda}{4} \right\} \right| \\
\lesssim \frac{|b|_{\rho'} |f|_{H^\beta_{L^p}}}{\lambda} \sum_{j=\infty}^{\infty} |\lambda_j| \lesssim \frac{|b|_{\rho'} |f|_{H^\beta_{L^p}}}{\lambda}.
\]
Thus, we complete the proof of Theorem 1.3
\]

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