The number of gauge singlets in supersymmetric
Yang-Mills quantum mechanics

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Abstract

We compute generating functions for number of $U(N)(SU(N))$ singlets in Fock space in several space dimensions. The motivation to find the explicit form of the functions is from the numerical approach to supersymmetric Yang-Mills quantum mechanics, based on Fock space. Incidentally the functions give many important insights into the quantum mechanical models based $U(N)(SU(N))$ gauge group.

1 Introduction

Yang Mills theories in zero volume limit provide a natural arena where quantum mechanics with the singlet constraint emerge due to the Gauss law [1]. The gauge fields $A_i^a$, where $i$ and $a$ are spatial and color indices respectively, become now the coordinate operators $x_i^a$ in emerging Yang-Mills quantum mechanics (YMQM). The resulting system is extremely difficult to solve and accordingly there are no exact solutions in the literature.

The model and its supersymmetric extension (SYMQM) play an important role in quantum mechanical description of the membrane [2] and the supermembrane [3], i.e. they give a regularized description of a (super)membrane. The $D = 9 + 1$ dimensional SYMQM model [4] became famous due to the BFSS conjecture [5], relating the later to

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M-theory, and since then has been studied in number of papers (we refer to the existing reviews [6]). However there are no nonperturbative calculations of e.g. the spectrum of BFSS matrix model because of the high complexity (of e.g. the Fock space) that makes any numerical approach difficult.

A particularly useful approach is the cutoff method [7], which consists of representing the hamiltonian in truncated Fock space and diagonalizing the resulting finite matrix. In the number of papers [7, 8, 9, 10, 11] the method, applied to the $D = 2, 4, SU(2)$ system, proved to be very fruitful giving the nonperturbative values of the spectra, wave functions, the Witten index etc. The Fock space approach can also be applied to the systems with large number of colors $N$ [12, 13, 14, 15, 16] where the nonperturbative results may shed some new light into the area of large $N$ quantum field theory.

An important step in applying the cutoff method is the construction of the $SU(N)$ invariant basis in Fock space. This however proved to be very time consuming in some cases [17]. We focus in this paper on computing the number of $SU(N)$ singlets in Fock space analytically hence facilitating the numerical considerations. The difficulty in computing this number arises due to many identities (in fact an infinite number of them, emerging from the Cayley-Hamilton theorem) among $SU(N)$ singlets that make them linearly dependent. There are, of course, very well known algorithms to produce the singlet states [18]. One simply takes the trace operators, e.g. $Tr(x_i x_j \ldots)$, $x_i = x^a_i T_a$ where $T_a$’s are $SU(N)$ generators in the fundamental representation, and act with an arbitrary number of products of trace operators on e.g. the Fock vacuum. Such set of states certainly spans the whole Fock space however they are in general linearly dependent. One can use the Cayley-Hamilton theorem to find the linearly independent ones but this is in general tedious when the number of $x_i$ matrices becomes large.

There is however a way to circumvent this problem when $N = \infty$ since, in that case, there are no additional identities among singlets. Furthermore, if one is interested only in single trace states then the famous Polya theorem [19] can be applied to compute the number of singlets explicitly. The first computation of this kind is by Sundborg [20], Polyakov [21] and Aharony et al. [22] in the context of weakly coupled Yang Mills theories and by Semenoff et al. [23] in the context of BFSS matrix model.
In quantum mechanical systems the property, that the single trace states are the most important ones in large $N$ limit, is not always valid. One can verify [12] that in the case of the anharmonic oscillator the single trace states do not reproduce the correct large $N$ spectrum. A different example is the free hamiltonian [25] where it was explicitly shown that all the $1/N^k$ terms play an important role in the large $N$ limit due to the distinguished role of the bilinear operators $Tr(x^2)$.

In this paper we neither assume that $N$ is large nor do we consider only the single trace states. The method that we will exploit in details in sections 2 and 3 is the very well known character method in group theory.$\textsuperscript{1}$ In this framework we compute the generating functions for numbers of singlets in sectors with given number of bosonic/fermionic quanta and therefore give the exact number of singlets. In section 2 we discuss only the $D = 2$ SYMQM. The generating function of number of gauge singlets can be related, in this case, to the Witten index of particular models. In this way we obtain some physical constraints, on seemingly unrelated group theory numbers, which we find very interesting. In section 3 and 4 we do the analogous computation only for $D = 4, 6, 10$ and relate our generating functions with the partition function of the weakly coupled Yang-Mills theories on $R \times S^1$.

While this work was done a related paper [26] by F. Dolan was published. There is a significant overlap between [26] and our paper although the motivation as well as the detailed calculations are in different spirit.

2 The character method for $D = 1 + 1$ SYMQM

Let us consider the most general state in $D = 1 + 1$ SYMQM

$$|s\rangle = T_{b_1...b_n B \ e_1...e_n F} a^{b_1} \ldots a^{b_n} B f^{c_1} \ldots f^{c_n} F |0\rangle.$$ 

Here $|0\rangle$ is the Fock vacuum while $a^{\dagger \ b}$ and $f^{\dagger \ b}$ are bosonic and fermionic creation operators being in the adjoint representation of the Lie group G. The operators obey the commutation and anticommutation relations $[a^{\ b}, a^{\dagger \ c}] = \delta^{bc}$, $\{f^{\ b}, f^{\dagger \ c}\} = \delta^{bc}$. The tensor $T_{b_1...b_n B \ e_1...e_n F}$ is a group invariant tensor so that the state $|s\rangle$ is a group singlet. The

$\textsuperscript{1}$I thank R. Janik for the idea.
number of bosonic quanta in $|\psi\rangle$ is $n_B$ and the number of fermionic quanta is $n_F$. In this case we say that $|\psi\rangle$ is in the $(n_B, n_F)$ sector.

For the Lie group $G$ we take $U(N)$ (or $SU(N)$). The adjoint representation of $U(N)$ will be denoted by $R$ hence the state $|\psi\rangle$ is in the representation $Sym(\otimes_{i=1}^{n_B} R) \times Alt(\otimes_{i=1}^{n_F} R)$ where $Alt$ and $Sym$ stand for the anti-symmetrization and symmetrization of the tensor product respectively. Let us denote the number of $U(N)$ singlets in $(n_B, n_F)$ sector by $D_{n_B, n_F}^{U(N)}$. Using the orthogonality of group characters, we have

$$D_{n_B, n_F}^{U(N)} = \int d\mu_{U(N)} \chi_{Sym}^{(n_B)}(R) \chi_{Alt}^{[n_F]}(R),$$

with the group invariant measure $d\mu_{U(N)}$. The symmetric and antisymmetric powers of $R$, $\chi_{Sym}^{(n_B)}(R)$, $\chi_{Alt}^{[n_F]}(R)$ and the characters $\chi$ can be readily constructed using the Frobenius formula (the complete calculation is given in the Appendix A).

The direct evaluation of $D_{n_B, n_F}^{U(N)}$ is difficult, however we can evaluate it indirectly introducing the following generating functions

$$G_{n_B, n_F}^{U(N)}(a, b) = \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{\infty} D_{n_B, n_F}^{U(N)} a^{n_B} (-b)^{n_F}, \quad |a| < 1, \quad b \in \mathbb{R}. \quad (1)$$

The sum over $n_F$ is in fact finite (due to the Pauli principle, i.e. $\chi_{Alt}^{[k]}(R) = 0$ for $k > N^2$ for $U(N)$) but it is more convenient to work with infinite sum. In Appendix A we derive the following integral representation

$$G_{n_B, n_F}^{U(N)}(a, b) = \left( \frac{1 - b}{1 - a} \right)^N \int_0^{2\pi} \prod_i \frac{d\alpha_i}{2\pi} \prod_{i \neq j} \left( 1 - \frac{z_i}{z_j} \right) \frac{1 - b \frac{z_i}{z_j}}{1 - a \frac{z_i}{z_j}}, \quad z_j = e^{i\alpha_j}. \quad (2)$$

A similar expression was derived by Skagerstam [27] in the context of a singlet ideal gas in flat space.

Above integrals can be calculated explicitly for $b = 0$. We have (see Appendix B)

$$G_{n_B, 0}^{U(N)}(a, 0) = \prod_{k=1}^{N} \frac{1}{(1 - a^k)} = \sum_{n_B=0}^{\infty} a^{n_B} p_N(n_B), \quad (3)$$

where $p_N(n_B)$ is the number of partitions of $n_B$ into numbers $1, 2, \ldots, N$, i.e. the number of natural solutions of the equation $\sum_{k=1}^{N} k_i k = n_B$. 

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In short the calculation goes as follows. First we use the Cauchy determinant formula and rewrite the expression under the integral (2) as a determinant of a certain matrix. Then we express the determinant as the sum over cycles and integrate each cycle separately. Finally, we use the Cayley identity to write the result in a compact form.

According to Eqn. (2), \( G^U(N)(a, b) \) is a polynomial in variable \( b \). The coefficient next to \( b^0 \) is equal \( G^U(N)(a, 0) \). We now write (2) as

\[
G^U(N)(a, b) = \left( \prod_{k=1}^{N} \frac{1}{1 - a^k} \right) \sum_{i=0}^{N^2} (-1)^i b^i c^U_i(a),
\]

where \( c^U_i(a) \) are polynomials in variable \( a \) and \( c^U_0(a) = 1 \). \(^2\)

The case of \( SU(N) \) group is analogous. We have (see Appendix B)

\[
G^{SU(N)}(a, b) = \left( \prod_{k=2}^{N} \frac{1}{1 - a^k} \right) \sum_{i=0}^{N^2-1} (-1)^i b^i c^{SU}_i(a),
\]

\[
G^{SU(N)}(a, 0) = \prod_{k=2}^{N} \frac{1}{1 - a^k} = \sum_{n_B=0}^{\infty} a^{n_B} q_N(n_B),
\]

where \( q_N(n_B) \) is the number of partitions of \( n_B \) into numbers 2, 3, \ldots, \( N \) and \( c^{SU}_i(a) \) are polynomials in variable \( a \).

The determination of \( c^U_i(a) \) or \( c^{SU}_i(a) \) is difficult for arbitrary \( N \), however we can get an idea about their structure by simply counting the constructed gauge singlets.

### 2.1 Explicit construction of gauge invariant states

The formula (3) can be derived by directly counting the \( U(N) \) singlets. If we introduce the matrices \( a^\dagger = a^\dagger b T_b \) where \( T_b \) are \( U(N) \) generators \(^3\) then all the singlets are linear combination of the following states

\[
| i_1, i_2, \ldots, i_N \rangle = (a^\dagger)^{i_1} (a^\dagger)^{i_2} \ldots (a^\dagger)^{i_N} | 0 \rangle,
\]

\(^2\)At this stage it is not evident that one can factor out the term \( G^U(N)(a, 0) \) from \( G^U(N)(a, b) \) leaving \( c^U_i(a) \)'s in the form of polynomials. However it is indeed the case as we shall see in the next subsection.

\(^3\)We use the following conventions for \( U(N) \) and \( SU(N) \) generators

\[
T_a T_b = \frac{1}{2N} \delta_{ab} 1 + (d_{abc} + if_{abc}) T_c,
\]

where \( d_{abc} \) and \( f_{abc} \) are corresponding structure tensors.
where we used the shorthand notation \((A) = \text{Tr}(A)\) where \(A\) is an arbitrary matrix. Higher powers of \(a^\dagger\) do not appear due to the Cayley-Hamilton theorem. Vectors \(5\) are already linearly independent therefore they form a basis in the space of \(U(N)\) singlets. For a given number of quanta \(n_B\) the number of vectors \(5\) is equal to the number of natural solutions of the equation \(\sum_{k=1}^{N} k i_k = n_B\), i.e. there are \(p_N(n_B)\) such vectors. Therefore the generating function is exactly \(3\).

The advantage of the explicit integration over characters is that we can compute the number of singlets in all fermionic sectors. The direct construction of such states is possible introducing the matrices \(f^\dagger = f^\dagger b T_b\). It follows that all the singlets can be obtained by acting with \((a^{i_1} f^\dagger j \ldots)\) operators on \(|0\rangle\). There are however new identities among such vectors that make many of them linearly dependent. The process of choosing the independent ones is in general tedious therefore the polynomials \(c_i^{U(N)}(a)\) are difficult to obtain by simply counting the constructed singlets. On the contrary, using the character method this is straightforward (for given \(N\)).

From \(2\) we see that

\[
G^{U(N)}(a, b) = b^{N^2} G^{U(N)}(a, 1/b),
\]

therefore

\[
c_i^{U(N)}(a) = c_i^{U(N)}(a),
\]

hence

\[
D^{U(N)}_{n_B, n_F} = D^{U(N)}_{n_B, N^2 - n_F^*}.
\]

Equation \(6\) is the simplest example of constraints on \(D^{U(N)}_{n_B, n_F}\). It follows that the number of gauge invariant states in \((n_B, n_F)\) sector is equal to the number of gauge invariant states in \((n_B, N^2 - n_F)\) sector. The identity is related to the particle-hole symmetry, i.e. the invariance of the system under the transformation \(f^\dagger a \rightarrow f^a\), \(f^a \rightarrow f^\dagger a\) (the empty fermionic states are replaced by filled fermionic ones).

Examples of \(c_i^{U(N)}(a)\) and \(c_i^{SU(N)}(a)\) for \(N = 2, 3, 4\) are in the Appendix C. We can determine these polynomials for low values of \(N\) by explicitly constructing the singlet states. Let us start with the case of \(SU(2)\). According to character integrals we have (
and the corresponding states are

\[ |i\rangle = (a^{12})^i |0\rangle, \quad (a^{1\dagger} f^{\dagger}) |i\rangle, \quad (a^{1\dagger} f^{\dagger} f^{\dagger}) |i\rangle, \quad (f^{\dagger} f^{\dagger} f^{\dagger}) |i\rangle. \]

There are no other, independent singlet states which (for \(SU(2)\) group) can be seen from the following simple argument. The \(SU(2)\) tensor \(T_{a \ldots b \ldots}\) in (2) can only be made out of linear combinations of products and contractions between the \(SU(2)\) primitive tensors \(\epsilon_{abc}\) and \(\delta_{ab}\). However products (and in particular contractions) of \(\epsilon_{abc}\)'s can be expressed as linear combinations of products of \(\delta_{ab}\)'s. Therefore, \(T_{a \ldots b \ldots}\) is either a product of \(\delta_{ab}\)'s or a product of \(\delta_{ab}\)'s and one \(\epsilon_{abc}\). Since for \(SU(2)\) group \(\epsilon_{abc} \propto (T_a T_b T_c)\) we arrive at the following states

\[ |i\rangle = (a^{12})^i |0\rangle, \quad (a^{1\dagger} f^{\dagger}) |i\rangle, \quad (a^{1\dagger} f^{\dagger} f^{\dagger}) |i\rangle, \quad (f^{\dagger} f^{\dagger} f^{\dagger}) |i\rangle. \]

(according to the fermion exclusion principle we have \((f^{\dagger} f^{\dagger}) = 0\) and \((a^{1\dagger} f^{\dagger})^2 = 0\) hence there are no operators of this form). The last state is in fact proportional to the last but one. To see this we write

\[ (a^{1\dagger} f^{\dagger})(a^{1\dagger} f^{\dagger} f^{\dagger}) \propto a_1^{1\dagger} f_1^{\dagger} + a_2^{1\dagger} f_2^{\dagger} + a_3^{1\dagger} f_3^{\dagger} \propto (a^{1\dagger} a^{1\dagger})(f^{\dagger} f^{\dagger} f^{\dagger}) \]

The above arguments are difficult to generalize for \(SU(N > 2)\) due to the additional completely symmetric tensors \(d_{abc}\) that have to be considered and the fact that the products of \(f_{abc}\) tensors cannot be expressed in terms of products of \(\delta_{ab}\)'s. It is more convenient use another method, which takes advantage of the Cayley-Hamilton theorem and which can be generalized for \(SU(N > 2)\). In section 2.1.1. we give the details of this approach for the \(SU(3)\) group.

The case of \(U(2)\) is more complicated. We have

\[ c_0^{U(2)} = 1, \quad c_1^{U(2)} = a, \quad c_2^{U(2)} = 2a, \quad c_3^{U(2)} = 1 + a, \quad c_4^{U(2)} = 1, \]

and the corresponding singlets are

\[ |i, j\rangle = (a^{1\dagger})^i (a^{1\dagger})^j |0\rangle \quad \longleftrightarrow \quad c_0^{U(2)} = 1, \]
$$(f^\dagger) |i, j\rangle, \quad (a^\dagger f^\dagger) |i, j\rangle \longleftrightarrow c_1^{U(2)} = 1 + a,$$

$$(f^\dagger)(a^\dagger f^\dagger) |i, j\rangle, \quad (a^\dagger f^\dagger f^\dagger) |i, j\rangle \longleftrightarrow c_2^{U(2)} = 2a,$$

$$(f^\dagger f^\dagger f^\dagger) |i, j\rangle, \quad (a^\dagger f^\dagger)(f^\dagger) |i, j\rangle \longleftrightarrow c_3^{U(2)} = 1 + a,$$

$$(f^\dagger)(f^\dagger f^\dagger f^\dagger) |i, j\rangle \longleftrightarrow c_4^{U(2)} = 1.$$ 

The only difference between the $SU(2)$ case are the additional operators $(a^\dagger)$ and $(f^\dagger)$ that have to be considered when constructing the singlet state.

The construction of independent states for $SU(3)$ or $U(3)$, in all fermion sectors, is already very nontrivial and will be discussed in section 2.1.1. However, for $n_F = 0, 1$ we can construct them for arbitrary $U(N)$ and $SU(N)$. They are

$$|i_1, i_2, \ldots, i_N\rangle \longleftrightarrow c_0^{U(N)} = 1,$$

$$(f^\dagger a^k) |i_1, i_2, \ldots, i_N\rangle, \quad k = 0, \ldots, N - 1 \longleftrightarrow c_1^{U(N)} = 1 + a + \ldots + a^{N-1},$$

$$|0, i_2, \ldots, i_N\rangle \longleftrightarrow c_0^{SU(N)} = 1,$$

$$(f^\dagger a^k) |0, i_2, \ldots, i_N\rangle, \quad k = 1, \ldots, N - 1 \longleftrightarrow c_1^{SU(N)} = a + \ldots + a^{N-1}.$$ 

From the explicit results for $N = 2, 3, 4$ we see that there is a relation between the generating functions for $U(N)$ and $SU(N)$ namely

$$G^{U(N)}(a, b) = \frac{1-b}{1-a} G^{SU(N)}(a, b).$$

It can be understood in terms of gauge singlets just constructed. For the case of $U(N)$ we have additional trace operators $(a^\dagger)$ and $(f^\dagger)$ (comparing to the $SU(N)$ case) therefore to obtain all the $U(N)$ singlets we have to multiply the $SU(N)$ singlets by $(a^\dagger)^k, k \geq 0$ and by $(f^\dagger)^k, k = 0, 1$. In terms of generating functions this corresponds to multiplying $G^{SU(N)}(a, b)$ by $\frac{1-b}{1-a}$.

### 2.1.1 The construction of $SU(3)$ invariant states

The explicit evaluation of the integral (2) for $N = 3$ gives

$$c_0^{SU(3)} = 1, \quad c_1^{SU(3)} = a + a^2, \quad c_2^{SU(3)} = a + a^2 + 2a^3,$$
\[ c_3^{SU(3)} = 1 + a + 2a^2 + 3a^3 + a^4, \quad c_4^{SU(3)} = 2a + 4a^2 + 2a^3 + 2a^4, \quad c_i^{SU(3)} = c_{8-i}^{SU(3)}. \]

It follows that \( D_{n_B,n_F}^{SU(3)} \) with \( n_F > 0 \) can be obtained by linear combinations of \( D_{n_B,0}^{SU(3)} = q_3(n_B) \). According to the above results we have

\[
D_{n_B,0}^{SU(3)} = q_3(n_B),
\]

\[
D_{n_B,1}^{SU(3)} = q_3(n_B - 1) + q_3(n_B - 2),
\]

\[
D_{n_B,2}^{SU(3)} = q_3(n_B - 1) + q_3(n_B - 2) + 2q_3(n_B - 3),
\]

\[
D_{n_B,3}^{SU(3)} = q_3(n_B) + q_3(n_B - 1) + 2q_3(n_B - 2) + 3q_3(n_B - 3) + q_3(n_B - 4),
\]

\[
D_{n_B,4}^{SU(3)} = 2q_3(n_B - 1) + 4q_3(n_B - 2) + 2q_3(n_B - 3) + 2q_3(n_B - 4).
\]

(7)

We now construct the states in Fock space corresponding to these numbers. The construction is done separately in sectors with given number of fermionic quanta.

The bases for \( n_F = 0 \) and \( n_F = 1 \) sectors are

\[
| i, j \rangle = (a^\dagger a^\dagger)^i (a^\dagger a^\dagger)^j | 0 \rangle \longleftrightarrow 1,
\]

(8)

and

\[
(a^\dagger f^\dagger) | i, j \rangle \longleftrightarrow a,
\]

\[
(a^\dagger a^\dagger f^\dagger) | i, j \rangle \longleftrightarrow a^2.
\]

(9)

The number of vectors (8) with given number of quanta \( n_B \) is exactly \( q_3(n_B) \) and the number of vectors (9) with given number of quanta \( n_B \) is precisely \( q_3(n_B - 1) + q_3(n_B - 2) \), i.e. the number of states \( (a^\dagger f^\dagger) | i, j \rangle \) with \( n_B \) bosons is \( q_3(n_B - 1) \) and the number of states \( (a^\dagger f^\dagger f^\dagger) | i, j \rangle \) with \( n_B \) bosons is \( q_3(n_B - 2) \). This result agrees with \( D_{n_B,0}^{SU(3)} \) and \( D_{n_B,1}^{SU(3)} \) in (7).

Our strategy to determine the basis in sectors with \( n_F > 1 \) is the following. First we list all possible trace operators with given number of fermions. Then, with use of the Cayley-Hamilton theorem and some symmetry arguments, we choose only the linearly independent ones. The resulting trace operators acting on states \( | i, j \rangle \) are our candidates for the basis. In order to find out whether they really form a basis we compare the number of singlets constructed in this way with \( D_{n_B,n_F}^{SU(3)} \) in (7) (or equivalently associate the corresponding polynomial \( c_i^{SU(3)}(a) \)).
In $n_F = 2$ sector the possible trace operators are

$$(a^\dagger f^\dagger f^\dagger), \quad n_B = 1,$$

$$(a^\dagger a^\dagger f^\dagger f^\dagger), (a^\dagger f^\dagger a^\dagger f^\dagger) = 0, (a^\dagger f^\dagger)(a^\dagger f^\dagger) = 0, \quad n_B = 2,$$

$$(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger), (a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger), \quad n_B = 3,$$

$$(a^\dagger a^\dagger f^\dagger a^\dagger a^\dagger f^\dagger) = 0, (a^\dagger a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger) = 0, \quad n_B = 4.$$

The other trace operators involve at least one $a^\dagger a^\dagger$ hence they are linearly dependent. We have already taken into consideration the cyclicity of the trace. Therefore, there are four families of $SU(3)$ invariant states in this sector

$$(a^\dagger f^\dagger f^\dagger) | i, j ⟷ a$$

$$(a^\dagger a^\dagger f^\dagger f^\dagger) | i, j ⟷ a^2$$

$$(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) | i, j), (a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger) i, j) ⟷ 2a^3. \quad (10)$$

In the sector with $n_F = 3$ the possible trace operators are

$$(f^\dagger 3), \quad n_B = 0,$$

$$(a^\dagger f^\dagger 3), \quad n_B = 1,$$

$$(a^\dagger f^\dagger 2 f^\dagger 3), (a^\dagger f^\dagger 3)(a^\dagger f^\dagger 2), \quad n_B = 2,$$

$$(a^\dagger 2 f^\dagger a^\dagger f^\dagger 2), (a^\dagger 2 f^\dagger a^\dagger f^\dagger 2), (a^\dagger f^\dagger)(a^\dagger 2 f^\dagger), (a^\dagger 2 f^\dagger)(a^\dagger f^\dagger), \quad n_B = 3,$$

$$(a^\dagger 2 f^\dagger a^\dagger 2 f^\dagger 2), (a^\dagger 2 f^\dagger a^\dagger 2 f^\dagger 2), (a^\dagger f^\dagger)(a^\dagger 2 f^\dagger 2), (a^\dagger 2 f^\dagger)(a^\dagger a^\dagger f^\dagger), \quad n_B = 4,$$

$$(a^\dagger 2 f^\dagger a^\dagger 2 f^\dagger a^\dagger f^\dagger), (a^\dagger 2 f^\dagger)(a^\dagger 2 f^\dagger a^\dagger f^\dagger), \quad n_B = 5,$$

$$(a^\dagger 2 f^\dagger a^\dagger 2 f^\dagger a^\dagger a^\dagger f^\dagger), \quad n_B = 6.$$

Again, other trace operators involve at least one $a^\dagger 3$ hence they are linearly dependent. From the 17 operators listed above only 8 are linearly independent. The linear dependence is due to the identity for $SU(3)$ generators

$$T_{(a T_b T_c)} = \delta_{(a b} T_{c)} + 4d_{abc} 1. \quad \text{(4)}$$

The complete symmetrization over indices is

$$T_{(a T_b T_c)} = T_a T_b T_c + \text{permutations},$$

without the conventional $\frac{1}{3!}$ factor.
This identity is equivalent to the Cayley-Hamilton theorem for $3 \times 3$ traceless matrices. If we contract its left and right hand side with $a_i^\dagger a_i^\dagger b f_c^\dagger$, we will obtain an analogue of Cayley-Hamilton theorem (with fermionic matrices)

$$a_i^\dagger a_i^\dagger f^\dagger + f_i^\dagger a_i^\dagger + a_i^\dagger f_i^\dagger a_i^\dagger = \frac{1}{2}(a_i^\dagger a_i^\dagger f^\dagger + (a_i^\dagger f_i^\dagger) a_i^\dagger + (a_i^\dagger a_i^\dagger f_i^\dagger)).$$

All the linearly dependent trace operators can be derived from the last identity (see Appendix D) and one finds that there are only 8 families of independent vectors. They are

$$(f_i^\dagger^3) | i, j \rangle \leftrightarrow 1,$$

$$(a_i^\dagger f_i^\dagger^3) | i, j \rangle \leftrightarrow a,$$

$$(a_i^\dagger a_i^\dagger f_i^\dagger^3) | i, j \rangle, \quad (a_i^\dagger f_i^\dagger)(a_i^\dagger f_i^\dagger^2) | i, j \rangle \leftrightarrow 2a^2,$$

$$(a_i^\dagger^2 f_i^\dagger a_i^\dagger f_i^\dagger) | i, j \rangle, \quad (a_i^\dagger^2 f_i^\dagger^2)(a_i^\dagger f_i^\dagger) | i, j \rangle, \quad (a_i^\dagger f_i^\dagger)(a_i^\dagger^2 f_i^\dagger^2) | i, j \rangle \leftrightarrow 3a^3,$$

$$(a_i^\dagger^2 f_i^\dagger)(a_i^\dagger^2 f_i^\dagger^2) | i, j \rangle \leftrightarrow a^4. \quad (11)$$

The $n_F = 4$ sector is the most complicated one. There are 52 trace operators that one can construct however only 10 of them are independent. They are

$$(a_i^\dagger f_i^\dagger^3) | i, j \rangle, \quad (a_i^\dagger f_i^\dagger)(f_i^\dagger^3) | i, j \rangle \leftrightarrow 2a,$$

$$(a_i^\dagger^2 f_i^\dagger^4) | i, j \rangle, \quad (a_i^\dagger^2 f_i^\dagger)(f_i^\dagger^3) | i, j \rangle, \quad (a_i^\dagger f_i^\dagger)(a_i^\dagger^2 f_i^\dagger^3) | i, j \rangle, \quad (a_i^\dagger f_i^\dagger)(a_i^\dagger^2 f_i^\dagger^2) | i, j \rangle \leftrightarrow 4a^2,$$

$$(a_i^\dagger^2 f_i^\dagger)(a_i^\dagger f_i^\dagger^3) | i, j \rangle, \quad (a_i^\dagger^2 f_i^\dagger^2)(a_i^\dagger^2 f_i^\dagger^2) | i, j \rangle \leftrightarrow 2a^3,$$

$$(a_i^\dagger f_i^\dagger)(a_i^\dagger^2 f_i^\dagger^2)(a_i^\dagger f_i^\dagger^2) | i, j \rangle, \quad (a_i^\dagger f_i^\dagger^2)(a_i^\dagger^2 f_i^\dagger^2 a_i^\dagger f_i^\dagger) | i, j \rangle \leftrightarrow 2a^4.$$

The method to extract these 10 independent ones is the same as in the $n_F = 3$ case.

### 2.2 Physical constraints on $D_{n_B,n_F}^{U(N)}$

There is a class of identities analogous to (6) which also have a physical interpretation. For example if we put $b = 1$ in (2) we obtain

$$G^{U(N)}(a, 1) = 0,$$

therefore

$$\sum_{n_F \text{ even}} D_{n_B,n_F}^{U(N)} = \sum_{n_F \text{ odd}} D_{n_B,n_F}^{U(N)}.$$ \hspace{1cm} (12)
The above equation is suggestive of supersymmetry since the number of fermionic states (the sum over odd $n_F$) equals to the number of bosonic degrees of freedom (the sum over even $n_F$). However, since we did not specify any hamiltonian, Eq. (12) tells us that the bosonic and fermionic states match, as they should, i.e. the Hilbert space of singlets is already "prepared" for supersymmetry.

An interesting observation is given by equation (1), when $a = b$. The result is of a form of the Witten index. Indeed, if we consider supersymmetric harmonic oscillator

$$H = \{Q, Q^\dagger\} = a^\dagger b a^b + f^\dagger b f^b, \quad Q = a^\dagger b f^b,$$

then the energy is proportional to the number of quanta hence there is only one vacuum state and naturally it is the Fock vacuum. Therefore, the Witten index is 1. We confirm that by explicitly computing $G^{U(N)}(a,a)$ using Eqn. (2) from which the identity $G^{U(N)}(a,a) = 1$ follows. We also note that if we put $b = a$ in (1) we obtain

$$G^{U(N)}(a,a) = \sum_{n_B, n_F} (-1)^{n_F} D^{U(N)}_{n_B,n_F} a^{n_B+n_F} = 1 + \sum_{k>0} \sum_{n_B+n_F=k} (-1)^{n_F} D^{U(N)}_{n_B,n_F} a^{n_B+n_F},$$

decreasing

$$\sum_{n_B+n_F=\text{const.}>0} (-1)^{n_F} D^{U(N)}_{n_B,n_F} = 0.$$ 

The origin of the above identities lies in the dynamics of the supersymmetric harmonic oscillator although it is perhaps not evident at first sight.

Another example of a hamiltonian which brings physical meaning to some identities including $D^{U(N)}_{n_B,n_F}$'s is the hamiltonian given by the supercharge (12)

$$Q = f^b a^\dagger b + gd_{abc} a^\dagger a f^a b f^c.$$

One can show (15) that the in the limit of strong 't Hooft coupling $\lambda = Ng^2 \to \infty$ the energies are proportional to $n_B + 2n_F$ and that the supercharges act in the subspace of vectors such that $n_B + 2n_F$ is fixed. Therefore the contribution to the Witten index in terms of generating function is now $G^{U(N)}(a,a^2)$. This quantity is not necessarily a constant in variable $a$ since there may be other vacua in fermion sectors. However there is a finite number of vacua hence $G^{U(N)}(a,a^2)$ is at most a polynomial in $a$. We confirm this by explicitly calculating $G^{U(N)}(a,a^2)$ and $G^{SU(N)}(a,a^2)$ for the lowest values of $N$. We have
\[ G^{U(2)}(a, a^2) = 1 + a + a^2 + a^5, \quad G^{U(3)}(a, a^2) = 1 + a + a^2 + a^3 + a^5 + a^6 + a^7 - a^9 + a^{10} + a^{11}, \]

and

\[ G^{SU(2)}(a, a^2) = 1 + a^2 - a^3 + a^4, \quad G^{SU(3)}(a, a^2) = 1 + a^2 + a^5 + a^7 - a^8 + a^{10}. \]

The coefficients of polynomials \(G^H(a, a^2)\) where \(H = U(N), SU(N)\) give us the difference between the number of bosonic and fermionic vacua. In general, since \(G^H(a, a^2)\) is a polynomial, the constraint for \(D^H_{n_B, n_F}\)'s, coming from Eqn. [1], is now

\[ \sum_{n_B+2n_F=k} (-1)^{n_F} D^H_{n_B, n_F} = 0, \]

for \(k\) greater then some \(k_0\). It also seems that

\[ \sum_{n_B+2n_F=k} (-1)^{n_F} D^H_{n_B, n_F} = 0, 1, -1, \]

for \(k \leq k_0\).

We find it very interesting that although \(D^H_{n_B, n_F}\)'s are just some group theory numbers, they are constrained by the dynamics of the properly chosen supersymmetric hamiltonian.

3 The character method for \(D = 3 + 1, 5 + 1, 9 + 1\) SYMQM

The generalization of the \(D = 2\) case to \(D = 4, 6, 10\) cases is now straightforward. The bosonic and fermionic creation operators \(a_i^b, f_\alpha^b\) are now labeled by color index \(b\), spatial index \(i = 1, \ldots, d\) and spinor index \(\alpha = 1, \ldots, 8\) for \(D=9+1\).

\[ \alpha = 1, 2 \quad \text{for} \quad D=3+1, \quad \alpha = 1, 2, 3, 4 \quad \text{for} \quad D=5+1, \quad \alpha = 1, \ldots, 8 \quad \text{for} \quad D=9+1. \]

The state with \(n_B\) bosons and \(n_F\) fermions is now

\[ a_{i_1}^{b_1} \ldots a_{i_n}^{b_n} f_\alpha^{c_1} \ldots f_\alpha^{c_{n_F}} | 0 \rangle, \]

and the number of \(U(N)\) singlets is
\[ D_{n_B, n_F}^{U(N), d} = \int d\mu(U(N)) \chi_{\text{sym}}(R_B^{n_B}) \chi_{\text{Alt}}(R_F^{n_F}), \]

where

\[ \chi(R_B) = d \chi_{d=1}(R), \quad \chi(R_F) = (d-1) \chi_{d=1}(R). \]

The above equations for \( \chi(R_B) \) and \( \chi(R_F) \) are in fact the only difference between the \( d = 1 \) case. We can introduce the generating functions analogous to (1) and perform the same manipulations to find that the corresponding generating functions are

\[ G^{U(N), d}(a, b) = \frac{(1 - b)^N (d-1)}{(1 - a)^N d} \int_0^{2\pi} d\alpha_i \prod \frac{1}{2\pi} \prod (1 - \frac{z_i}{z_j})(1 - \frac{b z_i}{z_j})^{d-1}(1 - \frac{a z_i}{z_j})^d. \] (13)

From (13) we identify the particle-hole symmetry

\[ D_{n_B, n_F}^{U(N), d} = D_{n_B, N^2 d - n_F}, \]

and supersymmetry

\[ \sum_{n_F \text{ even}} D_{n_B, n_F}^{U(N), d} = \sum_{n_F \text{ odd}} D_{n_B, n_F}^{U(N), d}. \]

Taking \( b = a \) and using the results from previous section we obtain

\[ G^{U(N), d}(a, a) = \frac{1}{\prod_{k=1}^N (1 - a^k)}, \]

therefore

\[ \sum_{n_{F+B} = k} (-1)^{n_F} D_{n_B, n_F}^{U(N), d} = p_N(k). \]

The result does not depend on \( d \) which is surprising but possible since \( G^{U(N), d}(a, a) \) is the generating function for the differences between bosonic and fermionic gauge invariant states. However, \( G^{U(N), d}(a, a) \) cannot be interpreted as the contribution to the Witten index for supersymmetric harmonic oscillator in \( d + 1 \) dimensions. This is because the supersymmetric harmonic oscillator with gauge degrees of freedom does not exist in \( d + 1 > 2 \) dimensions together with the singlet constraint. Such system cannot exist since the number of bosonic and fermionic degrees of freedom do not match. For example there are \( d \) states with \( n_B = 1 \) and \( n_F = 0 \), they are \( (a_i^\dagger), i = 1, \ldots, d \). On the other hand there are \( d - 1 \) states with \( n_B = 0 \) and \( n_F = 1 \), they are \( (f_\alpha^\dagger), \alpha = 1, \ldots, d - 1 \). The difference is precisely equal \( p_N(1) = 1. \)
The generating function can be computed explicitly for arbitrary value of $N$ although the general $N$ dependence is difficult to obtain even for $b = 0$. The case $N = 2$ is particularly easy to evaluate, we have

$$G^{SU(2),d}(a,0) = -\frac{1}{2} \int \frac{dz_1}{2\pi i} z_1^{d-2} \frac{(z_1 - 1)^2}{(z_1 - a)^d(1 - az_1)^d} = -\frac{1}{2(d - 1)!} \frac{d^d}{dz^d} \left( \frac{z^{d-2}(z - 1)^2}{(1 - az)^d} \right) \bigg|_{z=a}. \tag{14}$$

The cases with $N = 3, 4, d = 3, 5, 9$ are presented in Appendix E while the values of $D_{n_B,n_F}^{SU(2),9}$ for $n_F \leq 12, n_B \leq 10$ are presented in Tables 1, 2 and 3.

Some of these values have been already obtained earlier \[17\] with considerable numerical effort by constructing the singlets directly. We see that the numbers of singlets grow extremely fast in this case, e.g. $D_{8,12}^{SU(2),9} \approx 2.5 \cdot 10^{11}$. It follows that the direct numerical approaches (e.g. the cutoff method \[7\]) to SYMQM in $d = 9$ dimensions is difficult to deal with even for the fastest computers. The $d = 9$ model is particularly troublesome because the fermion number is not conserved hence one cannot diagonalize the hamiltonian in each fermion sector separately.

If the hamiltonian has additional $SO(d)$ symmetry then it is convenient to work with

| $n_F$ | 0 | 1 | 2 | 3 | 4 |
|------|---|---|---|---|---|
| $n_B$ | 0 | 1 | 72 | 28 | 120 | 406 |
|      | 1 | 0 | 288 | 324 | 2016 | 9072 |
|      | 2 | 45 | 3240 | 3816 | 21024 | 89838 |
|      | 3 | 84 | 12960 | 23652 | 150360 | 692874 |
|      | 4 | 1035 | 74520 | 144000 | 882720 | 4049640 |
|      | 5 | 2772 | 270864 | 662436 | 4331880 | 20528802 |
|      | 6 | 16215 | 1119096 | 2906448 | 18805104 | 89459160 |
|      | 7 | 46530 | 3635280 | 10912572 | 72993096 | 353298330 |
|      | 8 | 189288 | 12260160 | 38914524 | 259803720 | 1263689658 |

Table 1: Number of $SU(2)$ singlets for $D = 10$ spacetime dimensions in sectors with $0 \leq n_F \leq 4$ and $0 \leq n_B \leq 8$. 
| $n_F$ | 5   | 6   | 7   | 8   |
|------|-----|-----|-----|-----|
| 0    | 1512 | 4060 | 8856 | 17605 |
| 1    | 29232 | 81648 | 192528 | 374544 |
| 2    | 321048 | 907452 | 2121192 | 4230801 |
| 3    | 2426928 | 6998040 | 16742544 | 33436080 |
| 4    | 14752080 | 42942060 | 103041000 | 208064475 |
| 5    | 74701368 | 220014900 | 533991024 | 1081967760 |
| 6    | 331885680 | 984408096 | 2399008272 | 4891876599 |
| 7    | 1314510120 | 3928885884 | 9640642968 | 19721394891 |
| 8    | 4754606472 | 14285876220 | 35181176976 | 35181176976 |

Table 2: Number of $SU(2)$ singlets for $D = 10$ spacetime dimensions in sectors with $5 \leq n_F \leq 8$ and $0 \leq n_B \leq 8$.

| $n_F$ | 9   | 10  | 11  | 12  | $\Sigma$ |
|------|-----|-----|-----|-----|---------|
| 0    | 29512 | 41392 | 51520 | 56056 | 211068 |
| 1    | 626040 | 908460 | 1126944 | 1205568 | 4556448 |
| 2    | 7158600 | 10328580 | 12886776 | 13896792 | 51966252 |
| 3    | 56800008 | 82741428 | 103339320 | 111140484 | 414495042 |
| 4    | 355678200 | 518416380 | 649288080 | 700074900 | 2597347530 |
| 5    | 1856261448 | 2718108792 | 3408546960 | 3673243476 | 13592436138 |
| 6    | 8426386704 | 12356524344 | 15522106992 | 16746545508 | 61770199986 |
| 7    | 34066733976 | 50095041876 | 62995900968 | 67964640282 | 250166120589 |
| 8    | 72270195525 | 125190973512 | 184350316788 | 232102914120 | 920270148237 |

Table 3: Number of $SU(2)$ singlets for $D = 10$ spacetime dimensions in sectors with $9 \leq n_F \leq 12$ and $0 \leq n_B \leq 8$. $\Sigma$ gives the cumulative size up to $n_B$. 

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gauge and \(SO(d)\) singlets. The method presented here can be applied also in this case by computing the generating function for \(SU(N)\) invariant states with given spin (see Appendix F). Fortunately all the SYMQM have the \(SO(d)\) symmetry. It is then possible that the numerical analysis of these models is within our reach once we work in sectors with given \(SO(9)\) angular momentum.

4 The partition functions for free Yang-Mills theories

The generating functions \(G^{U(N),d}(a,b)\) have an interesting application in, seemingly unrelated, problem of computing the partition function of free Yang-Mills theories on \(S^1\times \text{time}\).

We show in this section that the partition functions of such theories can be expressed in terms of \(G^{U(N),d}(a,b)\)’s in a rather simple way. Following Sundborg [20] and Aharony et al. [22] we write the partition function of the free Yang-Mills theory with \(n_S, n_V\) and \(n_F\) number of scalar vector and fermion fields respectively as

\[
Z(x) = \int d\mu_G \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} [Z_B(x^m) + (-1)^m Z_F(x^m)] \chi(R^m) \right),
\] (15)

where \(Z_B(x)\) and \(Z_F(x)\) are bosonic and fermionic single partition functions given explicitly, for \(D=2\), by

\[
Z_B(x) = n_S z_S(x) + n_V z_V(x), \quad Z_F(x) = n_F z_F(x),
\]

\[
z_S(x) = \frac{1 + x}{1 - x}, \quad z_V(x) = x^2, \quad z_F(x) = \frac{4\sqrt{x}}{1 - x}.
\]

It is clear that \(Z(x)\) is very similar to \(G^{U(N),d}(a,b)\). The only difference is that there are no single particle partition function \(z_B(x)\) and \(z_F(x)\) in \(G^{U(N),d}(a,b)\), instead there are two generating parameters \(a\) and \(b\). In two dimensions scalar fields have the scaling dimension zero hence \(z_S(0) = 1\) and the partition function is divergent. To avoid this we take \(n_S = 0\). Next, since \(z_F(x)\) is a rational function, we Taylor expand \(z_F(x)\) and substitute it to (15). Using the formulas and conventions from Appendix A, (15) becomes

\[
Z(x) = \int_{|z_j|<1} \prod_{j=1}^{N} \frac{dz_j}{2\pi i z_j} \prod_{i\neq j} (1 - \frac{z_j}{z_i}) \prod_{k=0}^{\infty} \prod_{i,j} \frac{(1 + x^{k+\frac{1}{2}} \frac{z_i}{z_j})^{4n_F}}{(1 - x^{2k+1} \frac{z_i}{z_j})^{n_V}}.
\]
The above formula is more complicated than (13) due to the infinite product over $k$. However, the product does not appear when we take $n_F = 0$. In this case $Z(x)$ becomes

$$Z(x) = G^{U(N),n_V}(x^2, 0).$$

(16)

It is amusing that the partition function of free Yang-Mills theory on $S^1 \times \mathbb{R}$ with $n_V$ vector fields is given directly by $G^{U(N),n_V}(x^2, 0)$. On the other hand two dimensional gauge theories have many exceptional features hence their partition functions may resemble some simplicity. Indeed, Yang-Mills theories on compact, orientable surfaces are exactly solvable [28] and their partition functions are known to be simple expressions depending on group theory parameters.

5 Summary

In this paper we focused on calculating the number of $U(N)$ ($SU(N)$) singlets motivated by the numerical approach based on the cut Fock space. Once the basis is known the cutoff method may be used. However, the very construction of such a basis is far from easy and proved [7, 9, 17] to be very time consuming when symbolic programs are involved. The results presented here give us an algorithm to construct such basis theoretically thereby facilitating the numerical considerations. In particular we hope that the results presented here applied to the $j = 0$ sector of $D = 9 + 1$ SYMQM will help to obtain the nonperturbative spectra of this highly complicated system.

It is interesting that the generating functions $G^{U(N),d}(a, b)$ have other, seemingly independent, applications, i.e. they give rise to the Witten index in a class of models as well as they can be used to compute the partition functions of free Yang-Mills theories on $S^1 \times \mathbb{R}$.

6 Acknowledgments

I thank R. Janik, G. Veneziano and J. Wosiek for many discussions. I also thank referees for bringing to my attention references [26, 27] and for many important comments regarding this manuscript. This work was supported by the the grant of Polish Ministry of
Appendix A - Group theory conventions and the integral representation of $G^{U(N)}(a,b)$

Here we give the conventions used in integrals over characters and derive (2). They can be found in, e.g. ref. [29]. The methods used in this appendix are similar to the ones used in [22].

The $U(N)$ invariant measure is

$$d\mu_{U(N)} = \frac{1}{N!} \prod_{i=1}^{N} \frac{d\alpha_i}{2\pi} |M|^2,$$

and the $SU(N)$ measure is

$$d\mu_{SU(N)} = \frac{1}{N!} \prod_{i=1}^{N} \frac{d\alpha_i}{2\pi} \delta_P(\sum_{i=1}^{N} \alpha_i) |M|^2, \quad \alpha_i \in [0, 2\pi],$$

where $\delta_P$ is a periodic Dirac delta with period $2\pi$

$$\delta_P(x) = \sum_k \delta(x - 2\pi k),$$

the measure factor $M$ is given by Vandermonde determinant

$$M = Det(z_j^{(N-i)}) = \prod_{i<j}(z_i - z_j), \quad z_j = e^{i\alpha_j},$$

the symmetric and antisymmetric powers of $R$, $\chi^{(nB)}_{Sym}(R)$ and $\chi^{[a_F]}_{Alt}(R)$, are given by Frobenius formulas

$$\chi^{(nB)}_{Sym}(R) = \sum_{\sum_k i_k = nB} \prod_{k=1}^{nB} \frac{\chi^{R^k}(i_k)}{i_k!} k^{i_k},$$

$$\chi^{[a_F]}_{Alt}(R) = \sum_{\sum_k i_k = nF} (-1)^{\sum_k i_k} \prod_{k=1}^{nF} \frac{\chi^{R^k}(i_k)}{i_k!} k^{i_k},$$

and the characters $\chi$ are given by Weyl determinant formula

$$\chi(R) \equiv \chi(\{\alpha_i\}_{i=1}^{N}) = \frac{Det(z_j^{(N-i+l_i)})}{Det(z_j^{(N-l_i)})}, \quad \chi(R^k) = \chi(\{k\alpha_i\}_{i=1}^{N}).$$

The numbers $l_i$ enumerate the representation in which the character is calculated. In our case it is the adjoint representation of $U(N)$ (or $SU(N)$) therefore $(l_1, l_2, \ldots, l_N) = (2, 1, \ldots, 1, 0)$. In this representation the characters simplify into

$$\chi_{U(N)}(\{\alpha_i\}) = \sum_{i,j} \frac{z_i}{z_j}.$$
\[ \chi_{SU(N)}(\{\alpha_i\}) = \sum_{i,j} \frac{z_i}{z_j} - 1. \]

In order to derive (2) let us introduce

\[
F_{\text{Sym}}(a, \{\alpha_i\}_{i=1}^N) = \sum_{n_B=0}^{\infty} a^{n_B} \chi_{\text{Sym}}(R^{n_B}), \quad |a| < 1,
\]

\[
F_{\text{Alt}}(b, \{\alpha_i\}_{i=1}^N) = \sum_{n_F=0}^{\infty} (-1)^{n_F} b^{n_F} \chi_{\text{Alt}}(R^{n_F}),
\]

\[
G^H(a, b) = \int_{[0,2\pi]^N} d\mu G F_{\text{Sym}}(a, \{\alpha_i\}) F_{\text{Alt}}(b, \{\alpha_i\}),
\]

\[
D^H_{n_B,n_F} = \frac{1}{n_B!} \frac{(-1)^{n_F}}{n_F!} \partial^{n_B}_{a^n_B} \partial^{n_F}_{b^n_F} G^G(a, b).
\]

The last sum is in fact finite since for \( U(N) \) (or \( SU(N) \)), \( \chi_{\text{Alt}}(R^{n_F}) = 0 \) when \( n_F > N^2 \) (or \( n_F > N^2 - 1 \)). It is however more convenient to work with infinite sum as we will see in the following. The \( b \) variable is not bounded.

Using the standard manipulations we obtain

\[
F_{\text{Sym}}(a, \{\alpha_i\}_{i=1}^N) = \exp \left( \sum_{k=1}^{\infty} \frac{a^k}{k} \chi(\{k\alpha_i\}_{i=1}^N) \right),
\]

\[
F_{\text{Alt}}(b, \{\alpha_i\}_{i=1}^N) = \frac{1}{F_{\text{Sym}}(b, \{\alpha_i\}_{i=1}^N)}.
\]

The generating function can be calculated explicitly for arbitrary \( U(N) \) and \( SU(N) \). We have

\[
F_{\text{Sym}}^{U(N)}(a, \{\alpha_i\}_{i=1}^N) = \prod_{i,j} \left( 1 - \frac{b \alpha_i}{\alpha_j} \right),
\]

\[
F_{\text{Sym}}^{SU(N)}(a, \{\alpha_i\}_{i=1}^N) = (1 - a) F_{\text{Sym}}^{U(N)}(a, \{\alpha_i\}_{i=1}^N),
\]

therefore the generating functions are

\[
G^{U(N)}(a, b) = \frac{1}{N!} \left( \frac{1 - b}{1 - a} \right)^N \int_0^{2\pi} \prod_i \frac{d\alpha_i}{2\pi} \prod_{i \neq j} (1 - z_i) \frac{(1 - b \alpha_i)}{(1 - a \alpha_i)};
\]

\[
G^{SU(N)}(a, b) = \frac{1}{N!} \left( \frac{1 - b}{1 - a} \right)^{N-1} \int_0^{2\pi} \prod_i \frac{d\alpha_i}{2\pi} \delta(\alpha_N) \prod_{i \neq j} (1 - z_i) \frac{(1 - b \alpha_i)}{(1 - a \alpha_i)};
\]

where in the last integral we changed variables \( z_i \to z_i \prod_{j=1}^{N} z_j, z_N \to \prod_{j=1}^{N} z_j \).
Here we evaluate the integral \( (2) \) explicitly for \( b = 0 \). It can be done with use of the
Cauchy determinant formula
\[
\det \left( \frac{1}{z_i - az_j} \right) = \frac{a^{N(N-1)/2}}{(a-1)^N} \prod_i \frac{1}{z_i} \prod_{i<j} \frac{(z_i - z_j)^2}{(z_i - az_j)(z_j - az_i)},
\]
(17)
which for \( U(N) \) gives
\[
G^{U(N)}(a, 0) = \frac{1}{N!} \left( -1 \right)^N \int_{|z_i| = 1} \prod_{i=1}^{N} \frac{dz_i}{2\pi i} \det \left( \frac{1}{z_i - az_j} \right). \quad (18)
\]
The determinant under the integral \((18)\) can be expressed as a sum over cycles. The
integration over each cycle can be done separately and it gives the factor \( \frac{1}{1-a^k} \), i.e.
\[
\int_{|z_i| = 1} \prod_{i=1}^{k} \frac{dz_i}{2\pi i} \frac{1}{z_i - az_j} \frac{1}{z_2 - az_3} \cdots \frac{1}{z_{k-1} - az_k} \frac{1}{z_k - az_1} = \frac{1}{1-a^k},
\]
therefore we obtain
\[
\int_{|z_i| = 1} \prod_{k=1}^{N} \frac{dz_k}{2\pi i} \det \left( \frac{1}{z_i - az_j} \right) = \sum_{i_1+i_2+\ldots+N i_N = N} \left( -1 \right)^{\sum_{k=1}^{N} i_k} L_{i_1\ldots i_N} \prod_{k=1}^{N} \frac{1}{(1-a^k)^{i_k}}, \quad (19)
\]
where \( L_{i_1\ldots i_N} \) is the number of different permutations with the same cycle structure given
by the partition \((1^{i_1} \ldots N^{i_N})\) i.e. \( L_{i_1\ldots i_N} = N! / \prod_{k=1}^{N} k^{i_k} i_k! \). The right hand side of \((19)\)
is in fact very simple due to the Cayley identity
\[
\sum_{i_1+i_2+\ldots+N i_N = N} \left( -1 \right)^{\sum_{k=1}^{N} i_k} L_{i_1\ldots i_N} \prod_{k=1}^{N} \frac{1}{(1-a^k)^{i_k}} = \left( -1 \right)^{N} a^{N(N-1)/2} N! \prod_{k=1}^{N} \frac{1}{(1-a^k)}, \quad (20)
\]
It can be proven most efficiently with use of the Bell polynomials \([30]\). Therefore we finally
obtain \((3)\).

\[5\] The general form of Cauchy determinant formula is
\[
\det \left( \frac{1}{z_i - x_j} \right) = \prod_{i<j} \frac{(z_i - z_j)(x_i - x_j)}{\prod_{i,j} (z_i - x_j)},
\]
which for \( x_i = az_i \) yields \((17)\).

\[6\] The Cayley identity by definition is
\[
\sum_{i_1+i_2+\ldots+N i_N = N} L_{i_1\ldots i_N} \prod_{k=1}^{N} \frac{1}{(1-a^k)^{i_k}} = N! \prod_{k=1}^{N} \frac{1}{(1-a^k)},
\]
which for \( a \to 1/a \) yields \((20)\).
For $SU(N)$ the only difference is that $Tr(A) = 0$ hence in Eqn. (3) there is no $1/(1-a)$ factor, i.e.

$$G^{SU(N)}(a, 0) = (1-a)G^{U(N)}(a, 0) = \sum_{n_B=0}^{\infty} a^{n_B} q_N(n_B),$$

where $q_N(n_B)$ is the number of partitions of $n_B$ into numbers $2, 3, \ldots, N$.

The generating function $G^{SU(N)}(a, b)$ clearly have the form

$$G^{SU(N)}(a, b) = \left( \prod_{k=2}^{N} \frac{1}{1-a^k} \right)^{N^2-1} \sum_{i=0}^{N^2-1} (-1)^i b^i c_i^{SU(N)}(a),$$

where $c_i^{SU(N)}(a)$ are polynomials in variable $a$.

9 Appendix C - Examples of polynomials $c^{U(N)}$ and $c^{SU(N)}$

Here we list the polynomials $c^{U(N)}$ and $c^{SU(N)}$ for $N = 2, 3, 4$. They can be obtained from equation (2) using some symbolic program, e.g. Mathematica, to evaluate the corresponding residues. For $N = 2$ they are

$$c^{U(2)}_0 = 1, \quad c^{U(2)}_1 = 1 + a, \quad c^{U(2)}_2 = 2a, \quad c^{U(2)}_i = c^{U(2)}_{4-i}.$$

and

$$c^{SU(2)}_0 = 1, \quad c^{SU(2)}_1 = a, \quad c^{SU(2)}_2 = c^{SU(2)}_{3-i}.$$

For $N = 3$ they are

$$c^{U(3)}_0 = 1, \quad c^{U(3)}_1 = 1 + a + a^2, \quad c^{U(3)}_2 = 2a + 2a^2 + 2a^3, \quad c^{U(3)}_3 = 1 + 2a + 3a^2 + 5a^3 + a^4, \quad c^{U(3)}_4 = 1 + 3a + 6a^2 + 5a^3 + 3a^4, \quad c^{U(3)}_5 = c^{U(3)}_{6-i}.$$

and

$$c^{SU(3)}_0 = 1, \quad c^{SU(3)}_1 = a + a^2, \quad c^{SU(3)}_2 = a + 2a^2 + 3a^3, \quad c^{SU(3)}_3 = 1 + a + 2a^2 + 3a^3 + a^4, \quad c^{SU(3)}_4 = 2a + 4a^2 + 2a^3 + 2a^4 \quad c^{SU(3)}_5 = c^{SU(3)}_{7-i}.$$

For $N = 4$ they are

$$c^{U(4)}_0 = 1, \quad c^{U(4)}_1 = 1 + a + a^2 + a^3, \quad c^{U(4)}_2 = 2a + 2a^2 + 4a^3 + 2a^4 + 2a^5.$$
\[ c_3^{U(4)} = 1 + 2a + 4a^2 + 8a^3 + 8a^4 + 7a^5 + 5a^6 + a^7, \quad c_4^{U(4)} = 1 + 3a + 9a^2 + 13a^3 + 19a^4 + 17a^5 + 18a^6 + 7a^7 + 3a^8, \]
\[ c_5^{U(4)} = 1 + 4a + 11a^2 + 22a^3 + 33a^4 + 38a^5 + 34a^6 + 23a^7 + 11a^8 + 3a^9, \]
\[ c_6^{U(4)} = 1 + 5a + 12a^2 + 33a^3 + 45a^4 + 62a^5 + 55a^6 + 45a^7 + 22a^8 + 11a^9 + a^{10}, \]
\[ c_7^{U(4)} = 1 + 5a + 16a^2 + 37a^3 + 59a^4 + 75a^5 + 77a^6 + 60a^7 + 37a^8 + 17a^9 + 4a^{10}, \]
\[ c_8^{U(4)} = 2 + 4a + 18a^2 + 36a^3 + 68a^4 + 78a^5 + 86a^6 + 64a^7 + 46a^8 + 18a^9 + 6a^{10}. \]

and
\[ c_0^{SU(4)} = 1, \quad c_1^{SU(4)} = a + a^2 + a^3, \quad c_2^{SU(4)} = a + a^2 + 3a^3 + 2a^4 + 2a^5, \]
\[ c_3^{SU(4)} = 1 + a + 3a^2 + 5a^3 + 6a^4 + 5a^5 + 5a^6 + a^7, \quad c_4^{SU(4)} = 2 + 6a + 8a^2 + 13a^3 + 12a^4 + 13a^5 + 6a^6 + 3a^7 + 3a^8, \]
\[ c_5^{SU(4)} = 1 + 2a + 5a^2 + 14a^3 + 20a^4 + 27a^5 + 21a^6 + 17a^7 + 8a^8 + 3a^9, \]
\[ c_6^{SU(4)} = 3a + 7a^2 + 19a^3 + 25a^4 + 36a^5 + 34a^6 + 28a^7 + 14a^8 + 8a^9 + a^{10}, \]
\[ c_7^{SU(4)} = 1 + 2a + 9a^2 + 18a^3 + 34a^4 + 39a^5 + 43a^6 + 32a^7 + 23a^8 + 9a^9 + 3a^{10}. \]

10 Appendix D - Linear dependence of the trace operators for the SU(3) group

Here we derive the linear dependence of 17 trace operators listed in subsection 2.1.1. Our starting point is the equation

\[ a^\dagger a^f f^\dagger + f^\dagger a^a a^\dagger + a^f a^f a^\dagger = \frac{1}{2} (a^\dagger a^f f^\dagger + (a^\dagger f^\dagger) a^\dagger + (a^f a^f f^\dagger)). \tag{21} \]

It is the source of the following relations.

Multiplying (21) from the right hand side by \( f^\dagger f^\dagger \) and taking the trace gives

\[ (a^f f^\dagger a^a f^\dagger) = \frac{1}{2} (a^\dagger a^f f^\dagger f^\dagger f^\dagger) + (a^\dagger f^\dagger)(a^f f^\dagger) - 2(a^a f^\dagger f^\dagger f^\dagger f^\dagger). \]

Therefore, we may neglect, e.g. \((a^a f^\dagger a^f f^\dagger)\).

Multiplying (21) from the right hand side by \( f^\dagger f^\dagger \) and from the left hand side by \( a^\dagger \) and then taking the trace gives

\[ (a^\dagger a^f f^\dagger a^a f^\dagger) + (a^f f^\dagger a^a f^\dagger) = (a^\dagger f^\dagger)(a^a a^f f^\dagger) + (a^\dagger a^f)(a^f f^\dagger) - \frac{1}{3} (a^\dagger a^a)(f^\dagger f^\dagger f^\dagger), \]

\[ (a^\dagger a^f f^\dagger a^a f^\dagger) = (a^\dagger f^\dagger)(a^a a^f f^\dagger) + (a^\dagger a^f)(a^f f^\dagger) - \frac{1}{3} (a^\dagger a^a)(f^\dagger f^\dagger f^\dagger), \]
where the Cayley-Hamilton theorem for $a^\dagger$ matrices was also used.

Multiplying $f^\dagger a f^\dagger$ from the right hand side by $f^\dagger a^\dagger f^\dagger$ and taking the trace gives

$$(a^\dagger f^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger a^\dagger f^\dagger) = \frac{1}{2}(a^\dagger f^\dagger)(a^\dagger f^\dagger) - \frac{1}{3}(a^\dagger a^\dagger f^\dagger f^\dagger).$$

From the two above equations it follows that $(a^\dagger f^\dagger a^\dagger f^\dagger)$ can be expressed in terms of multiple trace operators. Also, we may neglect, e.g. $(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)$.

Multiplying $f^\dagger a^\dagger f^\dagger$ from the right hand side by $f^\dagger a^\dagger f^\dagger$ and taking the trace gives

$$(a^\dagger f^\dagger a^\dagger f^\dagger) = \frac{1}{2}(a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger f^\dagger) + \frac{1}{2}(a^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger a^\dagger f^\dagger) - \frac{1}{3}(a^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger f^\dagger f^\dagger) - \frac{1}{2}(a^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger a^\dagger f^\dagger),$$

where the Cayley-Hamilton theorem for $a^\dagger$ matrices was also used. Therefore we neglect $(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)$.

Multiplying $f^\dagger a^\dagger f^\dagger$ from the right hand side by $f^\dagger a^\dagger f^\dagger$ and taking the trace gives

$$2(a^\dagger f^\dagger a^\dagger f^\dagger) + (a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) = \frac{1}{2}(a^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger a^\dagger f^\dagger) + \frac{1}{2}(a^\dagger f^\dagger)(a^\dagger f^\dagger a^\dagger f^\dagger) - (a^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger f^\dagger f^\dagger),$$

where the Cayley-Hamilton theorem for $a^\dagger$ matrices was again used. From the two above equations it follows that we can also neglect $(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)$.

Multiplying $f^\dagger a^\dagger f^\dagger$ from the right hand side by $a^\dagger f^\dagger a^\dagger f^\dagger$ and taking the trace gives

$$2(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) + (a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) = \frac{1}{2}(a^\dagger a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) + \frac{1}{2}(a^\dagger f^\dagger)(a^\dagger f^\dagger a^\dagger f^\dagger) + (a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger),$$

where the Cayley-Hamilton theorem for $a^\dagger$ matrices was used. The operator $(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)$ was already excluded before hence, from the above identity, it follows that there is a relation between the multiple trace operators. Therefore we can neglect one, e.g. $(a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)$.

Multiplying $f^\dagger a^\dagger f^\dagger$ from the right hand side by $a^\dagger f^\dagger a^\dagger f^\dagger$ and taking the trace gives

$$(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) = -\frac{1}{3}(a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger f^\dagger) - 2(a^\dagger a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger a^\dagger f^\dagger).$$
where the Cayley-Hamilton theorem for $a^\dagger$ matrices was used. Therefore, we neglect the operator $(a^\dagger a^\dagger a^\dagger a^\dagger f^\dagger)\) from the right hand side by $a^\dagger f^\dagger a^\dagger f^\dagger$ and taking the trace gives

\[
2(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) + \frac{1}{2}(a^\dagger a^\dagger)(f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) + \frac{1}{3}(a^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger f^\dagger) = \frac{1}{2}(a^\dagger a^\dagger)(f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)
\]

where the Cayley-Hamilton theorem for $a^\dagger$ matrices was used. The $(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)$ is already excluded therefore we can neglect one multitrace operator, e.g. $(a^\dagger a^\dagger f^\dagger)(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)$.

Multiplying (21) from the right hand side by $a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger$ and taking the trace gives

\[
(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) + \frac{1}{2}(a^\dagger a^\dagger)(a^\dagger f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger) + \frac{1}{3}(a^\dagger a^\dagger)(a^\dagger f^\dagger f^\dagger a^\dagger f^\dagger)
\]

where the Cayley-Hamilton theorem for $a^\dagger$ matrices was also used. It follows that the operator $(a^\dagger a^\dagger f^\dagger a^\dagger f^\dagger a^\dagger f^\dagger)$ can be excluded as well.

There are no other independent relations following from (21) except from the ones where the Cayley-Hamilton theorem for $a^\dagger$ matrices is used. We excluded 9 operators as being linearly dependent. The remaining 8 trace operators are indicated in subsection 2.1.1.

11 Appendix E - Examples of generating functions $G^{SU(N),d}(a, b)$

Below we list the generating functions $G^{SU(N),d}(a, b)$, for $d = 3, 5, 9$, $N = 2, 3$. The case of $U(N)$ group is obtained from the identity

\[
G^{U(N),d}(a, b) = \frac{(1-b)^{d-1}}{(1-a)^d} G^{SU(N),d}(a, b).
\]
For \( d = 3 \), \( N = 2 \) we have

\[
G^{SU(2),d=3}(a, b) = \frac{1}{(1-a)(1-a^2)^3} \sum_{i=0}^{6} b^i c_i^{SU(2),d=3},
\]

where \( c_i^{SU(2),d=3} = c_0^{SU(2),d=3} \).

\[ c_0^{SU(2),d=3} = 1 - a + a^2, \quad c_1^{SU(2),d=3} = 6a, \quad c_2^{SU(2),d=3} = 1 + 8a + 7a^2 + 2a^3 - 2a^4 - a^5, \]

\[ c_3^{SU(2),d=3} = 4 + 2a + 16a^2 + 4a^3 - 4a^4 - 2a^5. \]

For \( d = 5 \), \( N = 2 \) we have

\[
G^{SU(2),d=5}(a, b) = \frac{1}{(1-a)^3(1-a^2)^9} \sum_{i=0}^{12} (-1)^i c_i^{SU(2),d=5} b^i,
\]

\[ c_0^{SU(2),d=5} = 1 - 3a + 9a^2 - 9a^3 + 9a^4 - 3a^5 + a^6. \]

We will not list the rest of \( c_i \)'s since there are many of them and they become more complicated.

For \( d = 9 \), \( N = 2 \) we have

\[
G^{SU(2),d=9}(a, b) = \frac{1}{(1-a)^7(1-a^2)^{17}} \sum_{i=0}^{24} (-1)^i c_i^{SU(2),d=9} b^i,
\]

\[ c_0^{SU(2),d=9} = 1 - 7a + 49a^2 - 147a^3 + 441a^4 - 735a^5 + 1225a^6 - 1225a^7 + 1225a^8 - 735a^9 + 441a^{10} - 147a^{11} + 49a^{12} - 7a^{13} + a^{14}. \]

For \( d = 3 \), \( N = 3 \) we have

\[
G^{SU(3),d=3}(a, b) = \frac{1}{(1-a)(1-a^2)^8(1-a^3)^7} \sum_{i=0}^{16} (-1)^i c_i^{SU(3),d=3} b^i,
\]

\[ c_0^{SU(3),d=3} = 1 - a - 2a^2 + 6a^3 + 6a^4 - 9a^5 + a^6 + 17a^7 + a^8 - 9a^9 + 6a^{10} + 6a^{11} - 2a^{12} - a^{13} + a^{14}. \]

For \( d = 5 \), \( N = 3 \) we have

\[
G^{SU(3),d=5}(a, b) = \frac{1}{(1-a)^3(1-a^2)^{16}(1-a^3)^{13}} \sum_{i=0}^{32} (-1)^i c_i^{SU(3),d=5} b^i,
\]

\[ c_0^{SU(3),d=5} = 1 - 3a + 2a^2 + 34a^3 - 4a^4 - 18a^5 + 421a^6 + 624a^7 + 251a^8 + 2107a^9 + 5377a^{10} + 4766a^{11} + 6384a^{12} + 16031a^{13} + 19327a^{14} + 14592a^{15} + 21381a^{16} + 29839a^{17} + 21381a^{18} + 14592a^{19} + 19327a^{20} + 16031a^{21} + 6384a^{22} + 4766a^{23} + 5377a^{24} + 2107a^{25} + 251a^{26} + 624a^{27} + 421a^{28} - 18a^{29} - 4a^{30} + 34a^{31}. \]
For $d = 9$, $N = 3$ we have
\[
G^{SU(3), d=9}(a, b) = \frac{1}{(1-a)^6(1-a^2)^3(1-a^3)^3} \sum_{i=0}^{64} (-1)^i c_i^{SU(3), d=9} b^i,
\]
where $c_0^{SU(3), d=9}$ is of order 74.

12 Appendix F - Number of gauge singlets with given angular momentum

Here we discuss the character method applied to sectors with fixed angular momentum.

The projection to sectors with fixed angular momentum $j$ is due to the decomposition
\[
V = Sym(\otimes_{k=1}^{n_B} A_k^{j=1}) \times Alt(\otimes_{l=1}^{n_F} F_l^{j=1/2}),
\]
where $A_k^{j=1}$, $F_l^{j=1/2}$ are vector spaces spanned by $a_k^{j=1} \mid 0 \rangle$, and $f_l^{j=1/2} \mid 0 \rangle$ where operators $a_k^{j=1}$, $f_l^{j=1/2}$ are assumed to carry $SO(3)$ spin 1 and 1/2 respectively. Therefore the dimensions of subspaces with angular momentum $j$ are
\[
D_{n_B, n_F}^{U(N), d, j} = \int d\mu_{SO(d)} | \chi^{SO(d), j} | \int d\mu_{U(N)} \chi_{Sym}(R_B^{j=1}) | \chi_{Alt}(R_F^{j=1/2}),
\]
where $d\mu_{SU(N)}$ and $d\mu_{SO(d)}$ are $SU(N)$ and $SO(d)$ invariant measures. We will restrict to the $d = 3$ case hence we take
\[
d\mu_{SO(3)} = \frac{1}{\pi} \frac{\sin \beta}{\sin \frac{\beta}{2}} d\beta, \quad \beta \in [0, 2\pi], \quad \int d\mu_{SO(3)} = 1.
\]

$R_B^{j=1}$, $R_B^{j=1/2}$ are the adjoint representation of $SU(N)$ and $j = 1$, $j = 1/2$ representations of $SO(d)$ respectively, i.e.
\[
\chi(R_B^{j=1}) = \chi(R^{SO(d), j=1}) \chi(R^{SU(N), j=1}),
\]
\[
\chi(R_B^{j=1/2}) = \chi(R^{SO(d), j=1/2}) \chi(R^{SU(N), j=1}).
\]

For $d = 3$ we have
\[
\chi^{SO(3), j}(\alpha) = \frac{\sin(j + \frac{1}{2})\alpha}{\sin \frac{\alpha}{2}} = \sum_{k=-j}^{k=j} t^k, \quad t = e^{i\alpha},
\]

27
therefore for $SU(2)$ gauge group we obtain

$$\chi(R_{B}^{j=1}) = \chi(R_{SO(3)}^{j=1}) \chi(R_{SU(2)}^{j=1}) = (1 + 2 \cos \beta)(1 + 2 \cos \alpha),$$

$$\chi(R_{F}^{j=1/2}) = \chi(R_{SO(3)}^{j=1/2}) \chi(R_{SU(2)}^{j=1}) = 2 \cos \frac{\beta}{2}(1 + 2 \cos \alpha).$$

The explicit calculation of the generating function for $D_{n_{B} n_{F}}^{SU(2), d=3, j}$ is now straightforward. Below we perform the calculation for the purely bosonic sector, i.e. we evaluate

$$G(a, c) = \sum_{n_{B}=0}^{\infty} \sum_{j=0}^{\infty} D_{n_{B} n_{F}}^{SU(2), d=3, j} a^{n_{B}} c^{j}.$$

Using the same conventions and techniques as in Appendix A we perform the sum over $n_{B}$. We have

$$G(a, c) = \sum_{j=0}^{\infty} c^{j} \int d\mu_{SO(3)} d\mu_{SU(2)} \left( \sum_{k=-j}^{k=j} t^{k} \right) F(a, z; c, t), \quad z = e^{i\alpha}, \quad t = e^{i\beta},$$

where

$$F(a, z; c, t) = \frac{t^{3} z^{3}}{(1 - a)(1 - at)(1 - a z)(1 - atz)(a - t)(a - t z)(a - z)(a - t z)(t - a z)}.$$

Now the sum over $j$ is also possible and the evaluation of the resulting integral gives

$$G(a, c) = \frac{1 - a^{2}c + a^{4}c^{2}}{(1 - a^{2})(1 - a^{4})(1 - a^{2}c)(1 - a^{2}c^{2})}.$$

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