Complex manifolds in $q$-convex boundaries

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Abstract We consider a $C^\infty$ boundary $b\Omega \subset \mathbb{C}^n$ which is $q$-convex in the sense that its Levi-form has positive trace on every complex $q$-plane. We prove that $b\Omega$ is tangent of infinite order to the complexification of each of its submanifolds which is complex tangential and of finite bracket type. This generalizes Diederich and Fornaess (Ann Math 107:371–384, 1978) from pseudoconvex to $q$-convex domains. We also readily prove that the rows of the Levi-form are $\frac{1}{2}$-subelliptic multipliers for the $\bar{\partial}$-Neumann problem on $q$-forms (cf. Ho in Math Ann 290:3–18, 1991). This allows to run the Kohn algorithm of Acta Math 142:79–122 (1979) in the chain of ideals of subelliptic multipliers for $q$-forms. If $b\Omega$ is real analytic and the algorithm gets stuck on $q$-forms, then it produces a variety of holomorphic dimension $q$, and in fact, by our result above, a complex $q$-manifold which is not only tangent but indeed contained in $b\Omega$. Altogether, the absence of complex $q$-manifolds in $b\Omega$ produces a subelliptic estimate on $q$-forms.

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1 Complex $q$-manifolds in the boundary and the Kohn algorithm on $q$-forms

Let $\Omega$ be a smooth domain in $\mathbb{C}^n$ defined by $r = 0$ with $\partial r \neq 0$, and $M$ a smooth CR submanifold of $b\Omega$ of CR dimension $q$ and CR codimension $p$. We assume that $M$ is “complex tangential” to $b\Omega$ in the sense that...
Condition (1.1) is familiar in the ambient of peak-interpolation sets. If \( M \) is minimal in the sense of Tumanov, that is, it does not contain a proper submanifold with the same CR dimension, then it is endowed with a “wedge complexification” of dimension \( q + p \), that is, a complex \((q + p)\)-manifold \( \mathcal{W} \) of wedge type with edge \( M \) (cf. [8]). When \( b\Omega \) is pseudoconvex, then \( \mathcal{W} \subset b\Omega \); this refines Bedford and Fornaess [1] which is in turn a development of Diederich and Fornaess [4]. In fact, according to [8], \( \mathcal{W} \) is made out of analytic discs attached to \( M \). The pseudoconvexity of \( b\Omega \) forces the discs to be inside \( \bar{\Omega} \) and their complex tangency to \( b\Omega \), which follows from (1.1), forces them to be in \( b\Omega \). Otherwise, take a sub-disc with a boundary point inside \( \Omega \): there exists an infinitesimal deformation of the subdisc (cf. [8]) transversal to \( b\Omega \) from \( C^n \setminus \Omega \) at any point of initial tangency. However, the presence of such a disc would imply the extension of holomorphic functions contradicting pseudoconvexity.

There is a more classical argument (cf. [1]) which uses the fact that \(-(-r)^\eta\) is plurisubharmonic in \( \Omega \) for \( \eta \) arbitrarily close to 1 in a suitable neighborhood of each boundary point (Remark (b) p. 133 in [3]). Then, applying the Hopf Lemma to this function would show contact of \( b\Omega \) with the sub-disc of order \( q < 2 \) in contrast to the hypothesis of tangency. This reproves [4]. We weaken the hypothesis of pseudoconvexity and assume that \( b\Omega \) is \( q \)-convex, that is, for a choice of the Hermitian metric, the trace of the Levi form \( L_{b\Omega} = \partial \bar{\partial} r |_{T^C_{b\Omega}} \) is positive on every complex \( q \)-plane of \( T^C_{b\Omega} \), the complex tangent bundle to \( b\Omega \).

We strengthen the hypothesis of minimality and assume that \( M \) is of “finite bracket type”, that is, the subsequent brackets of \( C^\infty \) vector fields with values in \( T^C M \) generate the whole tangent bundle \( TM \). Note that when \( M \) is real analytic, finite type and minimality coincide (Nagano Theorem).

**Theorem 1.1** Let \( b\Omega \) be \( q \)-convex, let \( M \subset b\Omega \) be a CR submanifold which is complex tangential of finite bracket type, and let \( \mathcal{W} \) be the wedge complexification of \( M \). Then \( \mathcal{W} \) is tangent to \( b\Omega \) of infinite order along \( M \).

The proof will be given in Sect. 2.

The holomorphic dimension of a real subvariety \( V \subset b\Omega \) at \( z_o \) is defined by Kohn [7] by

\[
\text{hol dim}_{z_o} V = \inf_{U_{z_o}} \sup_{z \in U \cap V} \dim_{\mathbb{C}}(T^C_{z} V \cap \text{Ker} L_{b\Omega}(z)),
\]

for \( U_{z_o} \) ranging through the family of neighborhoods of \( z_o \). We set \( q := \text{hol dim}_{z_o} V \). Note that \( T V \cap \text{Ker} L_{b\Omega} \) is involutive; moving from \( z_o \) to a nearby point where \( V \) is a CR manifold such that \( \dim_{\mathbb{C}}(T^C_{z} V \cap \text{Ker} L_{b\Omega}) \equiv q \) and \( \dim(T V \cap \text{Ker} L_{b\Omega}) \equiv \text{const} \), we may apply Frobenius Theorem and produce a foliation by smooth leaves of CR-dimension \( \geq q \). We select a leaf \( M \), denote by \( \mathcal{L} \) the Lie span of \( T^C M \), and observe that \( \mathcal{L} \subset \text{Ker} L_{b\Omega} \subset T^C b\Omega \). By redefining \( z_o \), if necessary, we may assume that rank \( \mathcal{L} \equiv \text{const} \). We still denote by \( M \) an integral leaf of \( \mathcal{L} \); thus \( M \) is complex tangential and of finite bracket type. Altogether, we have obtained
Corollary 1.2 (i) Let $b\Omega$ be a $q$-convex boundary and let $V \subset b\Omega$ be a real subvariety with holomorphic dimension $q$ at $z_o$. Then, there is a CR manifold $M \subset V$ of CR-dimension $\geq q$ whose wedge complexification $W$ is tangent of infinite order to $b\Omega$.

(ii) If, moreover, $b\Omega$ and $V$ are real analytic, then $W$ is contained in $b\Omega$ and is a (complex) manifold not just a wedge manifold.

Our purpose is now to run the Kohn algorithm in a $q$-convex domain and to show that, when it goes through, it produces a subelliptic estimate for $q$-forms. This requires a minor effort in adapting the proof by Kohn [7] in which the domain is pseudoconvex in the usual sense.

We choose an orthonormal basis $\omega_1, \omega_2, \ldots, \omega_n = \partial r$ of $(1, 0)$ forms, and the dual basis $L_j$ of $(1, 0)$ vector fields. In this basis, we denote by $(\tilde{\partial} \partial r)$ and by $u = \sum_{|J|=k} u_J \tilde{\partial} J$ an antiholomorphic $q$ form with summation being taken over ordered multiindices $|J| = q$. The form is assumed to belong to the domain $D_{\tilde{\partial} \partial}^\ast \cap C^\infty_c (\tilde{\Omega} \cap U)^k$, for any $u \in D_{\tilde{\partial} \partial}^\ast \cap C^\infty_c (\tilde{\Omega} \cap U)^k$, $k \geq q$. (1.3)

We express (1.3) by saying that each row of $\partial \tilde{\partial} r$ is a $\frac{1}{2}$-subelliptic row-multiplier on $k$-form. We use the notation $Q(u, u)$ for the energy of the $\tilde{\partial}$-Neumann problem, that is, the term in the right of (1.3).

**Proof** We follow [7] Proposition 4.7, point (C). We show that for any $v \in C^\infty_c (U' \cap \Omega)^k$, for $U' \supset U$, and for any derivative $D$, we have

$$
\left| \sum_{|K|=k} \sum_{i} \int_{\Omega} r_{ij} u_i K D \tilde{v}_j K dV \right| \lesssim Q(u, u) + \sum_j |L_j (v)|^2 + ||v||^2
$$

$$
+ \sum_{|K|=k} \sum_{i} \int_{b\Omega} r_{ij} v_i K \tilde{v}_j K dS.
$$

(1.4)

For $D = L_k$, (1.4) follows from the Schwarz inequality. For $D = \tilde{L}_k$, $k < n$, it follows from integration by parts, the Schwarz inequality, and the basic estimate for $u$. Finally, for $D = \tilde{L}_n$, we write

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Using again the Schwarz inequality on $b\Omega$ for the positive 2-form \( \sum_{ij=1,\ldots,n} r_{ij} u_i K \bar{v} j K \) over \( k \)-vectors \( u \), we get

\[
\left| \sum_{|K|=k-1} \sum_{ij} r_{ij} u_i K \bar{v} j K \right| \leq \left( \sum_{|K|=k-1} \sum_{ij} r_{ij} u_i K \bar{v} j K \right)^{\frac{1}{2}} \left( \sum_{|K|=k-1} \sum_{ij} r_{ij} u_i K \bar{v} j K \right)^{\frac{1}{2}};
\]

if we plug the last inequality into (1.5) and use the basic estimate we get (1.4). Inequality (1.4), which corresponds to (4.17) of [7], is the only point which is worth checking in our new hypothesis of \( q \)-convexity. The proof then continues as in [7]; for the convenience of the reader we give a sketch of it. We use (1.4) for \( v_j K = \sum r_{kj} S^0 u_k K \) where \( \xi \) is a cut-off in \( C^\infty_c(U') \) with \( \xi | \bar{U} \equiv 1 \) and \( S^0 \) is a tangential operator of order 0. Reasoning as in [7] p. 97, we get, for any tangential derivative \( D_m, m = 1, \ldots, 2n-1 \),

\[
\left| \sum_{|K|=k-1} \sum_{ijk} \int_{\Omega} r_{ij} u_i K D_m S^0(r_{kj} \bar{u} K K)dV \right| \leq Q(u, u),
\]

where \( \xi \) has disappeared since it is 1 on the support of \( u \). We apply (1.6) for \( S^0 = -D_m \Lambda^{-1} \) where \( \Lambda \) is the tangential standard elliptic pseudodifferential operator of order 1, take summation over \( m \), observe that \(- \sum_m D_m^2 \Lambda^{-1} = \Lambda^1 - \Lambda^{-1} \), use the microlocal factorization \( \Lambda^1 = \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \), and get

\[
\left| \sum_{|K|=k-1} \sum_{ijk} \left| \sum_{j=1,\ldots,n} r_{ij} \bar{u} j K \right|^2 \right|^{\frac{1}{2}} = \sum_{|K|=k-1} \sum_{ijk} \int_{\Omega} r_{ij} u_i K \Lambda^1(r_{kj} \bar{u} K K)dV \right| \leq Q(u, u),
\]

that is, (1.3).

We recall that Kohn [7] developed an algorithmic procedure for constructing a chain of ideals of subelliptic multipliers \( I^q_1 \subset I^q_2 \subset \cdots \subset I^q_h \) over the ring of germs of \( C^\infty \) functions at \( z_0 \). Here is its outline. Define \( I^q \) to be the real radical of the ideal generated by \( r \) and by determinants of \( n-q+1 \) rows (cf. [7] Definition 4.6) of the Levi
form $\partial \bar{\partial} r \wedge \partial r \wedge \bar{\partial} r$. For $h \geq 2$, we define $I^q_h$ to be the radical of the ideal generated by $I^q_{h-1}$ and by determinats of $n - q + 1$ rows which are either rows of the Levi form or complex gradients of multipliers of $I^q_{h-1}$ wedged with $\partial r$ and $\bar{\partial} r$. By Proposition 1.3, and by Gårding inequality, $I^q_h$ is an ideal of $\frac{1}{2}$-subelliptic multipliers over $q$-forms. By [7] Proposition 4.7, the full chain of $I^q_h$’s is made out of subelliptic multipliers over $q$-forms. The proof of this point remains unchanged from pseudoconvex to $q$-convex domains.

We draw our conclusions. If $1 \in I^q_h$ for some $h$, then we have a subelliptic estimate (for some $\epsilon$ depending on the number $h$ of steps and on the operation of radical) on $q$-forms and, in fact, on $k$-forms for any $k \geq q$. If, instead, $I^q_{h+1} = I^q_h$ (and $I^q_h$ does not capture 1), this reveals under the extra assumption $b\Omega \in C^\omega$, that $V = V(I^q_h)$, the zero-set of $I^q_h$, has holomorphic dimension $\geq q$. By Corollary 1.2 this implies the existence of a complex $q$-manifold in $b\Omega$. Putting it all together, we have proven.

**Theorem 1.4** Assume that in a neighborhood of $z_0$, $b\Omega$ is real analytic, $q$-convex, and contains no germ of holomorphic manifold of dimension $\geq q$. Then a subelliptic estimate in degree $k \geq q$ for the $\bar{\partial}$-Neumann problem holds in a neighborhood $U$ of $z_0$, that is, for some $\epsilon$ we have

$$||u||_2^2 \leq Q(u, u) \quad \text{for any } u \in D_{\bar{\partial}^*} \cap C^\infty_c(\Omega \cap U)^k.$$

**Example 1.5** In $\mathbb{C}^3$, consider the domain $\Omega$ defined by

$$x_3 > -|z_1|^2|z_2|^2 + \left(\frac{1}{4}|z_1|^4 + \frac{3}{4}|z_2|^4\right).$$

Here $b\Omega$ is real analytic, there are no complex 2-manifolds at 0 but just the complex curve defined by $z_1 = z_2$. Also, if we compute the Levi form of $b\Omega$ in the metric in which $\pi^{-1}_z(1, 0, 0)$ and $\pi^{-1}_z(0, 1, 0)$ (for $\pi_z : T_z b\Omega \to \mathbb{C}^2 \times \mathbb{R}$ being the projection along the $x_3$-axis) is an orthonormal system for $T_z^C b\Omega$, we have

$$L_{b\Omega} = \begin{bmatrix} -|z_2|^2 + |z_1|^2 & -\bar{z}_1z_2 \\ -z_1\bar{z}_2 & -|z_1|^2 + 3|z_2|^2 \end{bmatrix}.$$

It follows

$$\text{trace } L_{b\Omega} = 2|z_2|^2 \geq 0.$$

Thus we have a subelliptic estimate in degree 2 according to Theorem 1.4. Note that this example could not be explained neither by usual pseudoconvexity nor by strong 2-pseudoconvexity. In fact

$$\det L_{b\Omega} = (|z_1|^2 - |z_2|^2)(3|z_2|^2 - |z_1|^2) - |z_1|^2|z_2|^2$$

$$= -|z_1|^4 - 3|z_2|^4 + 3|z_1|^2|z_2|^2$$

$$\leq 0.$$
Thus,

- \( b \Omega \) is not pseudoconvex (because \( \det L_{b \Omega} \leq 0 \) implies that there are eigenvalues of opposite sign),
- \( b \Omega \) does not satisfy \( Z(2) \) (in the sense of [5]) because there are no positive eigenvalues at 0.

2 Proof of Theorem 1.1

When \( q = 1 \), that is, \( \Omega \) is pseudoconvex, we have proposed in Sect. 1 two alternate simple proofs one using the infinitesimal deformation of a disc and the other the existence of a plurisubharmonic Hölder exhaustion function for \( \Omega \) with index close to 1. Both bypass the original argument by Diederich and Fornaess [4] which uses an asymptotic analysis at \( M \) of a graphing function of \( b \Omega \) over \( W \). However, for the case \( q > 1 \) on which we are mainly interested, only this latter appears usable. The reason is that \( q \)-convexity does not imply the complexification wedge \( \mathcal{W} \) is contained in \( \tilde{\Omega} \) and also the complex tangency of its edge \( M \) does not mean it lies in \( b \Omega \). Thus we adapt the proof of [4] Proposition 3 to the new situation in which \( b \Omega \) is no longer pseudoconvex but just \( q \)-convex. We move to a nearby point that we still denote by \( z_0 \) at which the “multitype” in the sense of (i)–(v) below is minimal (in the lexicographic order). We observe that the wedge complexification \( \mathcal{W} \) can be (non-uniquely) continued to a smooth manifold without boundary \( W \) of real dimension \( 2(q + p) \). Since \( \mathcal{W} \) is holomorphic, then \( W \) is “approximately holomorphic” at \( M \). By a linear unitary coordinate change we can assume that \( z_0 = 0 \), \( T_{z_0}M = \mathbb{C}^q \times \mathbb{R}^p \times \{0\} \) and \( T_{z_0}W = \mathbb{C}^q \times \mathbb{C}^p \times \{0\} \) and \( T_{z_0}b\Omega = \mathbb{C}^{n-1} \times i\mathbb{R} \). We observe that the projection \( \pi \) along the \( z_n \)-axis is transversal to \( W \); thus \( \pi(W) \) and \( \pi^{-1}(W) \) are real manifolds of dimension \( 2(q + p) \) and \( 2(q + p + 1) \) respectively. We use the notation \( t := n - (p + q + 1) \). We suppose that \( \pi^{-1}(W) \) is defined by real equations \( \mu_j(z') = 0, j = 1, \ldots, 2t \) such that setting \( f_j =: \mu_j + i\mu_{t+j} \), \( j \leq t \), we have \( \tilde{\partial} f_j = O_M^\infty \), and \( W \) is graphed over \( \pi(W) \) by \( z_n = h + ig \) with \( \tilde{\partial}(h + ig) = O_M^\infty \); here \( O_M^\infty \) denotes a zero of infinite order at \( M \). Clearly \( M \) is defined by \( x_n - h = 0 \), \( y_n - g = 0 \), \( \mu_j = 0 \), \( j = 1, \ldots, 2t \) and by a system of additional equations \( \rho_j = 0 \), \( j = 1, \ldots, p \). Note that [4] assumes for simplicity \( \pi^{-1}(W) = \mathbb{C}^n \) so that the \( f^* \)'s are missing and \( M \subset \{ z : z_n = 0 \} \) (i.e. \( M \) is defined by \( z_n = 0 \), \( \rho = 0 \)). We cover the most general situation and give the suitable explanations in (2.4) and (2.6) below.

We consider the Hermitian metric on \( \mathbb{C}^n \) in which \( \Omega \) is \( q \)-convex and the induced Euclidian metric on \( \mathbb{R}^{2n} \). In this metric, we choose an orthonormal basis \( \{ X_{0,i} \}_{i=1}^{p_0} \) of \( T^C M \) and a completion to a full basis of \( TM \)

\[
\{ X_{0,i} \}_{i=1}^{p_0}, \{ X_{1,i} \}_{i=1}^{p_1}, \ldots, \{ X_{s,i} \}_{i=1}^{p_s} \quad \text{with} \quad p_0 = 2q \quad \text{and} \quad \sum_{j=1}^{s} p_j = p. \quad (2.1)
\]

We may assume that

(i) any \( j \)-iterated bracket of the \( X_{0,i} \)'s is in the span of the \( X_{h,i} \)'s for \( h \leq j \),

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(ii) $X_{j,i} = [X_{0,v}, X_{j-1,\mu}]$ modulo $\text{Span}(X_{j,i})_{j \leq j-1}$ for suitable $X_{0,v} \in \text{Span}(X_{0,i})$ and $X_{j-1,\mu} \in \text{Span}(X_{h,i})_{h \leq j-1}$ when $j \geq 1$.

This is an immediate consequence of Jacobi identity. We put $L^0 = T^C M$, write inductively, $L^j = \text{Span}(L^{j-1}, [X_{0,v}, X_{j-1,\mu}])_{v, \mu}$ and decompose

$$TM = L^0 \oplus \frac{L^1}{L^{j-1}} \oplus \cdots \oplus \frac{L^s}{L^{s-1}}.$$

We can assume that our linear unitary tranformation gives $\frac{L_j}{L^{j-1}}|_{z_0} = \{0\} \times \mathbb{R}^p \times \{0\}$. Also, we can choose our basis so that, in addition to (i)–(ii) we also have

(iii) each group $\{X_{j,i}\}_{i=1,\ldots, p_j}$ is orthogonal one to another for different $j$.

(iv) in a basis $z_{0,1}, \ldots, z_{0,q}, z_{1,1}, \ldots, z_{1,p_1}, \ldots$ of $\mathbb{C}^{q+p}$ we have $X_{0,i}|_{z_0} = \partial_{x_i}$, $X_{0,i+q}|_{z_0} = \partial_{y_i}$, $i \leq q$, and $X_{j,i}|_{z_0} = \partial_{z_i}$, for $j \geq 1$.

(v) $M$ is the intersection of $W$ with the set defined by $\rho_{j,i} = 0$, $j \geq 1$, where the $\rho_{j,i}$'s are functions on $\pi(W)$ with $\text{Span}\{\Re \partial \rho_{j,i}\} = \text{Span}\{\Re \partial \rho_{j,i}\}$ and with $\langle \partial \rho_{h,i}, L_{j,i} \rangle = 0$ for any $h \geq j + 1$.

In particular, note that (v) implies that $\partial \bar{\partial} \rho_{h}(L_{j,i}, \bar{L}_{j',i'}) = 0$ for any $j, j' \leq h - 2$.

We identify the $X_{j,i} \in TM$ to the real or imaginary parts of vector fields $L_{j,i} \in \mathbb{C}(TM + JTM) \cap T^{1,0} \mathbb{C}^n$ defined by

$$\begin{cases}
L_{0,i} := \frac{1}{\sqrt{2}}(X_{0,i} + iX_{0,q+i}) \\
L_{1,i} := \frac{1}{\sqrt{2}}(X_{1,i} + iJX_{1,i}) \\
\cdots \\
L_{s,i} := \frac{1}{\sqrt{2}}(X_{s,i} + iJX_{s,i})
\end{cases} \quad (2.2)$$

Since $\mathbb{C}(TM + JTM) \cap T^{1,0} \mathbb{C}^n \subset T^{1,0} b\Omega|_M$, we extend the $L = L_{j,i}$ from $M$ to the whole $b\Omega$ as sections of $T^{1,0} b\Omega$ keeping unchanged their notation. Since $\pi^{-1} \pi(b\Omega)$ and $\pi^{-1} \pi(W)$ (defined by $\mu = 0$) have contact of infinite order along $M$, then the extended vector fields $L$ satisfy $\langle L, \mu \rangle = O_M^\infty$. Also, remember that $L_{0,i}|_M$ is an orthonormal system and that $L_{j,i}|_M$, $j \geq 1$, are orthogonal to $L_{0,i}|_M$ for any $i$. It is clear that we have full freedom to use our extensions so that all these orthogonality conditions are preserved.

Recall that for the equation $z_n = h + ig$ of $W$, we have supposed $\bar{\partial}(h + ig) = O_M^\infty$ and thus, in particular, $\partial \bar{\partial} h = O_M^\infty$. Thus, if $b\Omega$ is graphed by $x_n = h + \sigma$ (which serves as a definition of $\sigma$), we have $L_{b\Omega} = \partial \bar{\partial} \sigma|_{T^{1,0} b\Omega} + O_M^\infty$. We also denote by $r := x_n - (h + \sigma)$ a defining function for $b\Omega$. Note that $\sigma = 0$ on $M$; we want to prove that

$$\sigma = O(\rho^\infty) \quad \text{when} \quad y_n - g = 0 \quad \text{and} \quad \mu = 0,$$

and hence $W$ is tangent of infinite order to $b\Omega$ along $M$. We expand

$$\sigma = \sum_{|I|=k} a_I \rho^I + O(\rho^{k+1}) + \mathcal{E} + \mathcal{E}_1,$$

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where $I$ is a multi bi-index in the $(j, i)$'s and where $\mathcal{E} = O(y_n - g)$ and $\mathcal{E}_1 = O(\mu)$. We observe that

$$
\partial \bar{\partial} \mathcal{E}(L, \tilde{L}) = O(\langle \partial (y_n - g), L \rangle \langle \partial \rho, L \rangle) + |\langle \partial (y_n - g), L \rangle|^2 + O(y_n - g). \tag{2.4}
$$

We recall that $\langle \partial r, L_{j,i} \rangle = 0$ and $\partial (\tilde{\varepsilon}_n - (h - \rho g)) = O_{M}^{\infty}$ which implies $\langle \partial (y_n - g), L_{j,i} \rangle = L_{j,i}(O(\rho^k)) + O(\rho, \mu, y_n - g)$ and, specifically for $j = 0$, $\langle \partial (y_n - g), L_{0, i} \rangle = O(\rho^k) + O(\mu, y_n - g)$. This, combined with $\langle \partial \rho, L_{0, i} \rangle = O(\rho)$, yields for $y_n - g = 0$ and $\mu = 0$

$$
\partial \bar{\partial} \mathcal{E}(L_{j,i}, \tilde{L}_{j,i'}) = O\left(\rho^{k-1}\right), \quad \partial \bar{\partial} \mathcal{E}(L_{0,i}, \tilde{L}_{0,i'}) = O\left(\rho^k\right),
$$

$$
\partial \bar{\partial} \mathcal{E}(L_{0,i}, \tilde{L}_{0,i'}) = O\left(\rho^{k+1}\right). \tag{2.5}
$$

As for $\partial \bar{\partial} \mathcal{E}_1$, recalling also $\langle \partial \mu, L \rangle = O_{M}^{\infty}$, we have

$$
\partial \bar{\partial} \mathcal{E}_1(L, \tilde{L}) \sim \partial \bar{\partial} \mu(L, \tilde{L}) + |\langle \partial \mu, L \rangle| (|\langle \partial \rho, L \rangle| + |\langle \partial (y_n - g), L \rangle|) + O(\mu)
$$

$$
= O_{M}^{\infty} + O(\rho^{\infty}) + O(\mu). \tag{2.6}
$$

We call $k$ the first integer for which there is in (2.3) a non-trivial occurrence $a_I$ for $|I| = k$; we wish to show that $k$ cannot be finite. First, the inclusion $TW|_M \subset T^{\mathbb{C}}b\Omega|_M$ implies $k \geq 2$. We first show that $k$ cannot be odd. In fact, by a choice of $L = X + iJX, X \in L^j, j \geq 1$ such that $\partial \bar{\partial} \sigma(L, \tilde{L})$ is obtained by differentiating two factors once, we get

$$
\partial \bar{\partial} \sigma(L, \tilde{L}) = \sum_{|I'| = k-2} a_{I'} \rho^{I'} + O(\rho^{k-1}) + \partial \bar{\partial} \mathcal{E}(L, \tilde{L}) + O_{M}^{\infty} + O(\mu)
$$

$$
= \sum_{|I'| = k-2} a_{I'} \rho^{I'} + O(\rho^{k-1}) + O_{M}^{\infty} + O(\mu), \tag{2.7}
$$

with $a_{I'} \neq 0$ for at least one $I'$. Thus the form in the right of (2.7), having odd order, changes sign. (Recall again that [4] does not discuss $\mathcal{E}$ and $\mathcal{E}_1$ because there it is assumed that $g = 0$ and $\mu = 0$ respectively.)

On the other hand,

$$
\partial \bar{\partial} \sigma(L_{0,i}, \tilde{L}_{0,i'}) = O(\rho^{k-1}). \tag{2.8}
$$

Define a $q$-plane by $Q_q := \text{Span}[L, L_{0,i}]$; (for any choice of $q - 1$ among the indices $i$); we can conclude that $\text{trace}_{Q_q} \partial \bar{\partial} \sigma$ changes sign, which violates the $q$-convexity of $b\Omega$. Thus $k$ cannot be odd.

We show that $k$ cannot be even either. We first remove any possible term with a factor of $\rho_{1,i}$ in the homogeneous expression of degree $k$ of $\sigma$, that is, $\sum a_{(1,i)} I' \rho^{(1,i)} I'$. Springer
We have

$$\ddbar{\sigma}_{|C\mathcal{L}^0} = \sum_{|I'|=k-1} \left( \sum_i a_{(1,i)I'} \ddbar{\rho}_{1,i}_{|C\mathcal{L}^0} \right) \rho^{I'} + O(\rho^k) + \ddbar{\mathcal{E}}_{|C\mathcal{L}^0}. \quad (2.9)$$

By the third expression of (2.5), the last term in (2.9) can be neglected. If, for some \( |I'_o| = k - 1 \), we have \( \text{trace}_{C\mathcal{L}^0} \left( \sum_i a_{(1,i)I'_o} \ddbar{\rho}_{1,i} \right) \neq 0 \), then \( \text{trace}_{C\mathcal{L}^0}(\ddbar{\sigma}) \) changes sign since the \( \rho^{I'} \) vary independently. Otherwise, assume

$$\text{trace}_{C\mathcal{L}^0} \left( \sum_i a_{(1,i)I'_o} \ddbar{\rho}_{1,i} \right) = 0 \quad \text{for any } I'.$$ \quad (2.10)

Recall that the commutators of the \( L'_{0,i} \)'s span a space of dimension \( p_1 \); by Cartan formula, this is equivalent to saying that the Levi matrices \( \ddbar{\rho}_{1,i}_{|C\mathcal{L}^0} \), \( i = 1, \ldots, p_1 \) are independent. Thus, from \( \sum_i a_{(1,i)I'_o} \rho_{1,i} \neq 0 \) for some \( I'_o \), we get for some vector of \( C\mathcal{L}^0 \), say \( L_{0,1} \),

$$\sum_i a_{(1,i)I'_o} \ddbar{\rho}_{1,i}(L_{0,1}, L_{0,1}) \neq 0. \quad (2.11)$$

Define \( L_t = \frac{(1-t)L_{0,1}+t^2L_{1,i}}{c_t} \) (any \( i \)) where \( c_t \) is a factor which normalizes \( |L_t| = 1 \). We deform \( C\mathcal{L}^0 \) to

$$Q_q = \text{Span}\{L_t, L_{0,2}, \ldots, L_{0,q}\}.$$ Combining (2.10) and (2.11) we get

$$\text{trace}_{Q_q} \left( \sum_i a_{(1,i)I'_o} \ddbar{\rho}_{1,i} \right) = tc_{I'_o} \quad \text{for } c_{I'_o} \neq 0.$$ Then, using (2.4), we have for the trace of the full \( \sigma = \sum_{|I| \geq k} a_I \rho^I \)

$$\text{trace}_{Q_q} \sigma = tc_{I'_o} \rho^{I'_o} + \sum_{I' \neq I'_o \atop |I'|=k-1} c_I' \rho^{I'} + t^4 O(\rho^{k-2}) + t^2 O(\rho^{k-1}) + O(\rho^k) + \ddbar{\mathcal{E}}_{|Q_q};$$ \quad (2.12)

observing that by (2.5) we have \( \ddbar{\mathcal{E}}_{|Q_q} = t^4 O(\rho^{k-1}) + t^2 O(\rho^k) + O(\rho^{k+1}) \), we see that this term can be neglected. By taking restriction to a suitable region of the plane \( \mathbb{R}' \times \mathbb{R} \) of \((\rho_{j,i}, t)\), all terms in the right of (2.12) are negligible comparing to the first: thus, again, \( \text{trace}_{Q_q}(\ddbar{\sigma}) \) changes sign.
At last, we have to consider the case when \( \sum |I| = k \) contains factors \( \rho_j, i \) which start from \( j_0 > 1 \). For fixed \( h \), each group of matrices \( \bar{\partial} \rho_{h,i} \), \( i = 1, \ldots, p_h \), are independent. Thus, for a pair of vectors, say \( L_{0,1} \in \mathbb{C}L^0 \) and \( L_{j_0-1,1} \in \mathbb{C}L^{j_0-1} \), and for some \( |I_o| = k - 1 \), we have \( (\sum_i a_{(j_0,i)} I_o \bar{\partial} \rho_{j_0,i})(L_{0,1}, \bar{L}_{j_0-1,1}) \neq 0 \). But then, under the choice \( L := \frac{t^{-1}L_{0,1} + L_{j_0-1,1}}{c_t}, \ t << 1 \), (for a normalization factor \( c_t \)) we have

\[
\left( \sum_i a_{(j_0,i)} I_o \bar{\partial} \rho_{j_0,i} \right)(L, \bar{L}) = c_{I_o} \neq 0. \tag{2.13}
\]

We then complete \( L \) by \( q - 1 \) vectors in \( \mathbb{C}L^0 \) to an orthonormal basis of a \( q \)-space \( Q_q \) thus obtaining

\[
\text{trace}_{Q_q}(\bar{\partial} \bar{\partial} \sigma) = tc_{I_o} \rho_{I_o}^t + t \sum_{I' \neq I_o} c_{I'} \rho_{I'} + t^2 O(\rho^{k-1}) + O(\rho^k), \tag{2.14}
\]

where \( t O(\rho^{k-1}) \) comes from differentiation once with respect to \( L \) different terms \( \rho_{j_0,i} \) in \( (k + 1) \)-powers and where we have controlled the term \( \bar{\partial} \bar{\partial} E \) by \( t^2 O(\rho^{k-1}) + O(\rho^k) + O(\rho^{k+1}) \). Again, we can make negligible on the right of (2.14) the terms which follow the first and conclude that the trace changes sign, a contradiction.

In conclusion, \( k \) cannot be either odd or even and therefore \( \sigma \) vanishes of infinite order along \( M \).

\[\square\]

References

1. Bedford, E., Fornaess, J.E.: Complex manifolds in pseudoconvex boundaries. Duke Math. J. 38(1), 279–288 (1981)
2. Catlin, D.: Boundary invariants of pseudoconvex domains. Ann. Math. 120, 529–586 (1984)
3. Diederich, K., Fornaess, J.E.: Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions. Invent. Math. 39, 129–141 (1977)
4. Diederich, K., Fornaess, J.E.: Pseudoconvex domains with real analytic boundary. Ann. Math. 107(3), 371–384 (1978)
5. Folland, G.B., Kohn, J.J.: The Neumann problem for the Cauchy–Riemann complex. In: Annals of Mathematics Studies, vol. 75. Princeton University Press, Princeton (1972)
6. Ho, L.H.: \( \bar{\partial} \)-problem on weakly \( q \)-convex domains. Math. Ann. 290, 3–18 (1991)
7. Kohn, J.J.: Subellipticity of the \( \bar{\partial} \)-Neumann problem on pseudoconvex domains: sufficient conditions. Acta Math. 142, 79–122 (1979)
8. Tumanov, A.: Extending CR functions on a manifold of finite type over a wedge. Mat. Sb. 136, 129–140 (1988)