FURTHER EXAMPLES OF NON-GEOMETRIC
SECTIONS OF ARITHMETIC FUNDAMENTAL GROUPS

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Abstract. We show the existence of group-theoretic sections of certain geometrically pro-nilpotent by abelian arithmetic fundamental groups of hyperbolic curves over $p$-adic local fields which are non-geometric, i.e., which do not arise from rational points. Among these quotients is the geometrically metabelian arithmetic fundamental group.

§0. Introduction/Statement of the Main Result. Grothendieck’s anabelian section conjecture predicts that sections of arithmetic fundamental groups of hyperbolic curves over finitely generated fields over $\mathbb{Q}$ arise from rational points (cf. [Saïdi] for a more precise formulation of the conjecture). Accordingly, sections of arithmetic fundamental groups of hyperbolic curves over $p$-adic local fields; which are defined over number fields, and which arise from global sections, should arise from rational points. In this context it is tempting to predict a $p$-adic analog of Grothendieck’s anabelian section conjecture. In [Saïdi1] we investigated such analog, and exhibited two necessary and sufficient conditions for a section of the arithmetic fundamental group of a hyperbolic curve over a $p$-adic local field to be geometric, i.e., to arise from a rational point (cf. loc. cit. Theorem 4.5).

For the time being there are no examples of sections of (the full) arithmetic fundamental groups of hyperbolic curves over $p$-adic local fields which are non-geometric, and one can still hope the validity of a $p$-adic analog of the section conjecture. On the other hand, recent examples were found of group-theoretic sections of certain (geometrically characteristic) quotients of arithmetic fundamental groups of curves over $p$-adic local fields which are non-geometric. Hoshi constructed examples of sections of the geometrically pro-$p$ quotient of arithmetic fundamental groups of curves over $p$-adic local fields which are non-geometric (cf. [Hoshi]). (Actually, Hoshi’s example arises from group-theoretic sections of geometrically pro-$p$ fundamental groups of hyperbolic curves over number fields (cf. loc. cit.).) In [Saïdi] we constructed examples of group-theoretic sections of geometrically prime-to-$p$ fundamental groups of hyperbolic curves over $p$-adic local fields which are non-geometric (cf. loc. cit. §3). Further, in [Saïdi2] we provided examples of group-theoretic sections of the ”étale by geometrically abelian” fundamental group of hyperbolic curves over $p$-adic local fields which are non-geometric. The existence of these examples is crucial for our understanding of the $p$-adic section conjecture. Indeed, if the $p$-adic version of the section conjecture holds true then it may possibly hold true even for smaller quotients of the arithmetic fundamental group, and one would like to know these quotients in this case. On the other hand, more elaborate examples
of non-geometric sections as above may lead to a counterexample for the $p$-adic version of the section conjecture.

In this note we provide further examples of sections of certain quotients of arithmetic fundamental groups of curves over $p$-adic local fields which are non-geometric. These quotients include the geometrically metabelian and certain geometrically pronilpotent by abelian quotients.

Next, we fix notations and state our main results.

- Let
  $$1 \to H' \to H \xrightarrow{pr} G \to 1$$
be an exact sequence of profinite groups. We will refer to a continuous homomorphism $s : G \to H$ satisfying $pr \circ s = \text{id}_G$ as a (group-theoretic) section, or splitting, of the above sequence, or simply a section of the projection $pr : H \to G$.

- Given a profinite group $H$, and a prime integer $\ell$, we will denote by $H^\ell$ the maximal pro-$\ell$ quotient of $H$, $H^{ab}$ the maximal abelian quotient of $H$, and $H^{ab,\ell}$ its maximal abelian pro-$\ell$ quotient. Thus $H^{ab,\ell} = (H^\ell)^{ab}$.

Let $p \geq 2$ be a prime integer, and $k$ a $p$-adic local field; meaning $k/\mathbb{Q}_p$ is a finite extension, with ring of integers $\mathcal{O}_k$, and residue field $F$. Thus $F$ is a finite field of characteristic $p$. Let $X \to \text{Spec } k$ be a proper, smooth, and geometrically connected hyperbolic (i.e., genus($X \geq 2$) curve) over $k$. Let $\eta$ be a geometric point of $X$ above its generic point, which determines an algebraic closure $\overline{k}$ of $k$, and a geometric point $\overline{\eta}$ of $\overline{X} \overset{\text{def}}{=} X \times_k \overline{k}$. There exists a canonical exact sequence of profinite groups (cf. [Grothendieck], Exposés IX, Théorème 6.1)

$$1 \to \pi_1(\overline{X}, \overline{\eta}) \to \pi_1(X, \eta) \to G_k \to 1.$$ 

Here, $\pi_1(X, \eta)$ denotes the arithmetic étale fundamental group of $X$ with base point $\eta$, $\pi_1(\overline{X}, \overline{\eta})$ the étale fundamental group of $\overline{X} \overset{\text{def}}{=} X \times_k \overline{k}$ with base point $\overline{\eta}$, and $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$ the absolute Galois group of $k$.

- Let $\Pi$ be a quotient of $\pi_1(X, \eta)$ such that the projection $\pi_1(X, \eta) \to G_k$ factors as $\pi_1(X, \eta) \to \Pi \to G_k$, and which is geometrically non-trivial; meaning $\text{Ker}(\Pi \to G_k)$ is non-trivial. Given a section $s : G_k \to \Pi$ of the projection $\Pi \to G_k$, we say that $s$ is geometric if $s(G_k)$ is contained in (hence equal to) the decomposition group $D_x \subset \Pi$ associated to a rational point $x \in X(k)$. In this case we say $s$ arises from the rational point $x$. We say that the section $s$ is non-geometric if $s$ is not geometric in the above sense, i.e., $s(G_k)$ is not contained in the decomposition group associated to a rational point $x \in X(k)$. (Note that in the above discussion the decomposition group $D_x$ is only defined up to conjugation by elements of $\text{Ker}(\Pi \to G_k)$.)

- We assume $X(k) \neq \emptyset$. We fix a $k$-rational point $x \in X(k)$, and $s \overset{\text{def}}{=} s_x : G_k \to \pi_1(X, \eta)$ a section of the projection $\pi_1(X, \eta) \to G_k$ associated to $x$. Thus $s$ is defined only up to conjugation by $\pi_1(\overline{X}, \overline{\eta})$. Note that the section $s$ induces a structure of $G_k$-group on any characteristic quotient of $\pi_1(\overline{X}, \overline{\eta})$.

- Let $\Delta$ be a quotient of $\pi_1(\overline{X}, \overline{\eta})$ which fits in an exact sequence

$$1 \to \tilde{H} \to \pi_1(\overline{X}, \overline{\eta}) \to \Delta \to 1,$$

where $\tilde{H} \overset{\text{def}}{=} \text{Ker}(\pi_1(\overline{X}, \overline{\eta}) \to \Delta)$. We consider the following condition $(\star)$ on $\Delta$.
Condition (⋆).

(i) $\Delta$ is pro-nilpotent, and is a characteristic quotient of $\pi_1(X, \bar{\eta})$.
(ii) $H^0(U, \Delta) = 0$ for every open subgroup $U$ of $G_k$.
(iii) The quotient $\pi_1(X, \bar{\eta}) \twoheadrightarrow \pi_1(X, \bar{\eta})_{ab}$ factors as
$$\pi_1(X, \bar{\eta}) \twoheadrightarrow \Delta \twoheadrightarrow \pi_1(X, \bar{\eta})_{ab}.$$
(iv) Let $H \overset{\text{def}}{=} \tilde{H}_{ab}$, and $\Gamma \overset{\text{def}}{=} \pi_1(X, \bar{\eta}) / \text{Ker}(\tilde{H} \twoheadrightarrow H)$. We have a push-out diagram
$$
\begin{array}{cccccc}
1 & \longrightarrow & \tilde{H} & \longrightarrow & \pi_1(X, \bar{\eta}) & \longrightarrow & \Delta & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & \\
1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & \Delta & \longrightarrow & 1
\end{array}
$$
where the middle and left vertical maps are surjective. There exists a prime integer $\ell \neq p$, such that the natural surjective map $\Gamma^\ell \twoheadrightarrow \Delta^\ell$ is not an isomorphism.

Let $\Pi \overset{\text{def}}{=} \pi_1(X, \eta) / \text{Ker}(\pi_1(X, \bar{\eta}) \twoheadrightarrow \Gamma)$, and $\tilde{\Pi} \overset{\text{def}}{=} \pi_1(X, \eta) / \text{Ker}(\pi_1(X, \bar{\eta}) \twoheadrightarrow \Delta)$. We have the following push-out diagrams
$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(X, \bar{\eta}) & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \tilde{\Pi} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}
$$
where the vertical maps are surjective. Thus $\text{Ker}(\Pi \twoheadrightarrow \tilde{\Pi}) = \text{Ker}(\Gamma \twoheadrightarrow \Delta) = H$.

• Examples of quotients $\Delta$ satisfying the condition (⋆) are: $\Delta = \pi_1(X, \bar{\eta})_{ab}$ is the maximal abelian quotient of $\pi_1(X, \bar{\eta})$ (cf. [Saïdi3], Lemma 1.3, for the condition (⋆)(ii)); in this case $\tilde{\Pi} \overset{\text{def}}{=} \pi_1(X, \eta)_{(ab)}$ is the geometrically abelian quotient of $\pi_1(X, \eta)$, $\Gamma$ is the maximal metabelian quotient of $\pi_1(X, \bar{\eta})$, and $\Pi$ is the geometrically metabelian quotient of $\pi_1(X, \eta)$. More generally, any pro-nilpotent characteristic quotient $\Delta$ of $\pi_1(X, \bar{\eta})$ which satisfies conditions (⋆)(ii), (⋆)(iii), and for which there exists a prime integer $\ell \neq p$ such that the natural projection $\pi_1(X, \bar{\eta})^\ell \twoheadrightarrow \Delta^\ell$ is not an isomorphism, satisfies the condition (⋆).

Given a finite extension $k' / k$ (all finite extensions of $k$ we consider are contained in $k$), and the corresponding open subgroup $G_{k'} \subseteq G_k$, we will denote by $\Pi_{k'}$ the pullback of the group extension $\Pi$ by $G_{k'} \hookrightarrow G_k$. Thus we have a commutative diagram of exact sequences
$$
\begin{array}{cccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi_{k'} & \longrightarrow & G_{k'} & \longrightarrow & 1 \\
\parallel & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & G_k & \longrightarrow & 1
\end{array}
$$
where the right square is cartesian. Likewise we write \( \bar{\Pi}_{k'} \overset{\text{def}}{=} \bar{\Pi} \times_{G_k} G_{k'} \). Note that \( \Pi_{k'} \) is a quotient of \( \pi_1(X_{k'}, \eta) \), where \( X_{k'} \overset{\text{def}}{=} X \times_{\text{Spec} \bar{k}} \text{Spec} k' \) and \( \eta \) is naturally induced by the above geometric point \( \eta \). Our first main result in this paper is the following.

**Theorem A.** We use notations as above. Let \( X \) be a proper, smooth, and geometrically connected hyperbolic curve over the \( p \)-adic local field \( k \). Assume that \( X(k) \neq \emptyset \), and \( X \) has potentially good reduction. Let \( \Delta \) be a quotient of \( \pi_1(X, \bar{\eta}) \) satisfying the condition \((*), \) and \( \Pi \) the corresponding quotient of \( \pi_1(X, \eta) \) as in the above discussion which fits in the exact sequence \( 1 \to \Gamma \to \Pi \to G_k \to 1 \). Then there exists a finite extension \( \bar{k}/k \) such that the following holds. For every finite extension \( k'/\bar{k} \), there exists a section \( s : G_k \to \Pi_{k'} \) of the projection \( \Pi_{k'} \to G_k \) which is non-geometric.

As a corollary of Theorem A we obtain the following (cf. examples discussed after introducing the condition \((*)\)).

**Corollary B.** There exist non-geometric sections of geometrically metabelian arithmetic fundamental groups of hyperbolic curves over \( p \)-adic local fields.

Let \( m \geq 1 \) be an integer. With the notations above, let \( \Delta_m \overset{\text{def}}{=} \Delta_{m,X} \) be the maximal \( m \)-step solvable pro-\( p \) quotient of \( \pi_1(X, \bar{\eta}) \), and \( \Pi_m \overset{\text{def}}{=} \Pi_{m,X} \) the geometrically \( m \)-step solvable pro-\( p \) quotient of \( \pi_1(X, \eta) \) which sits in the exact sequence

\[
1 \to \Delta_m \to \Pi_m \to G_k \to 1
\]

(cf. [Saïdi3], §1). Note that \( \Delta_m \) doesn’t satisfy condition \((*)\)(iii). It is plausible, in light of Hoshi’s example in [Hoshi] (cf. above discussion), that there exist non-geometric sections of \( \Pi_m \) for a suitable \( X/k \) as above (this is easily seen if \( m = 1 \), using the Kummer exact sequence associated to \( \text{Pic}_0^0 \)). In this context we prove the following.

**Theorem C.** We use notations as above. There exists an integer \( N \geq 2 \), such that the following holds. For every prime integer \( p \geq N \) there exists a proper, smooth, and geometrically connected hyperbolic curve \( X \) over a \( p \)-adic local field \( k \), an integer \( m \geq 2 \), and a section \( s : G_k \to \Pi_{m,X} \) of the projection \( \Pi_{m,X} \to G_k \) which is non-geometric.

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**§1. Proof of Theorem A.** We use the notations introduced in §0, as well as the notations and assumptions in Theorem A. Thus \( X \) is a proper, smooth, and geometrically connected hyperbolic curve over the \( p \)-adic local field \( k \), \( X(k) \neq \emptyset \), and we assume (without loss of generality) that \( X \) has good reduction over \( \mathcal{O}_k \). Further, \( \Delta \) is a characteristic quotient of \( \pi_1(X, \bar{\eta}) \) satisfying the condition \((*)\), and \( \Pi \) is the corresponding quotient of \( \pi_1(X, \eta) \) as above which fits in the exact sequence \( 1 \to \Gamma \to \Pi \to G_k \to 1 \) (cf. §0). We have the following commutative diagram of exact sequences.
Recall $x \in X(k)$ is a $k$-rational point, and $s \overset{\text{def}}{=} s_x : G_k \to \pi_1(X, \eta)$ a section of the projection $\pi_1(X, \eta) \twoheadrightarrow G_k$ associated to $x$. Further, $s$ induces sections: $s_1 \overset{\text{def}}{=} s_{1,x} : G_k \to \tilde{\Pi}$ of the projection $\tilde{\Pi} \twoheadrightarrow G_k$, and $s_2 \overset{\text{def}}{=} s_{2,x} : G_k \to \Pi$ of the projection $\Pi \twoheadrightarrow G_k$, which fit in a commutative diagram

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta & \longrightarrow & \tilde{\Pi} & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & 1 & & 1 & & 1 & & 1
\end{array}
\] (1.1)

where the right vertical map is the one in diagram (1.1).

The profinite group $\Delta$ is \textbf{finitely generated}, as follows from the well-known finite generation of the profinite group $\pi_1(X, \eta)$ which projects onto $\Delta$. Let $\{\Delta^i\}_{i \geq 1}$ be a countable system of \textbf{characteristic open} subgroups of $\Delta$ such that

\[
\Delta^{i+1} \subseteq \Delta^i, \quad \Delta^1 = \Delta, \quad \text{and} \quad \bigcap_{i \geq 1} \Delta^i = \{1\}.
\]

Write $\Delta_i \overset{\text{def}}{=} \Delta / \Delta^i$. Thus $\Delta_i$ is a \textit{finite characteristic} quotient of $\Delta$, and we have a push-out diagram of exact sequences

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \tilde{\Pi} & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta_i & \longrightarrow & \Pi_i & \longrightarrow & G_k & \longrightarrow & 1 \\
\end{array}
\] (1.2)

which defines a \textit{(geometrically finite)} quotient $\Pi_i$ of $\tilde{\Pi}$. The section $s_1$ induces a section

\[
\rho_i : G_k \to \Pi_i
\]

of the projection $\Pi_i \twoheadrightarrow G_k, \forall i \geq 1$. Write

\[
\tilde{\Pi}^i \overset{\text{def}}{=} \tilde{\Pi}^i[s_1] \overset{\text{def}}{=} \Delta^i.s_1(G_k).
\]
Note that $\tilde{\Pi}^i \subseteq \tilde{\Pi}$ is an open subgroup which contains the image $s_1(G_k)$ of $s_1$. Write $\Pi^i$ for the inverse image of $\tilde{\Pi}^i$ in $\pi_1(X, \eta)$. Thus $\Pi^i \subseteq \pi_1(X, \eta)$ is an open subgroup corresponding to an étale cover

$$X_i \to X_1 \overset{\text{def}}{=} X$$

defined over $k$ (since $\Pi^i$ maps onto $G_k$ via the natural projection $\pi_1(X, \eta) \to G_k$, by the very definition of $\Pi^i$).

Note that the étale cover $\overline{X}_i \overset{\text{def}}{=} X_i \times_{\text{Spec} \, k} \text{Spec} \, \overline{k} \to \overline{X}$ is Galois with Galois group $\Delta_i$, and we have a commutative diagram of étale covers

$$\overline{X}_i \longrightarrow \overline{X}$$
$$\downarrow \quad \downarrow$$
$$X_i \longrightarrow \ X$$

where $\overline{X}_i \to \overline{X}$ is Galois with Galois group $\Pi_i$, and $\overline{X}_i \to X_i$ is Galois with Galois group $\rho_i(G_k)$. We have a commutative diagram of exact sequences

$$1 \longrightarrow \tilde{\Delta}^i = \pi_1(\overline{X}_i, \overline{\eta}) \longrightarrow \Pi^i = \pi_1(X_i, \eta) \longrightarrow G_k \longrightarrow 1$$
$$\downarrow \quad \downarrow$$
$$\downarrow$$

$$1 \longrightarrow \pi_1(\overline{X}, \overline{\eta}) \longrightarrow \pi_1(X, \eta) \longrightarrow G_k \longrightarrow 1$$

where $\tilde{\Delta}^i$ is the inverse image of $\Delta^i$ in $\pi_1(\overline{X}, \overline{\eta})$, and the equalities $\tilde{\Delta}^i = \pi_1(\overline{X}_i, \overline{\eta})$, $\Pi^i = \pi_1(X_i, \eta)$, are natural identifications; the base points $\eta$ (resp. $\overline{\eta}$) of $X_i$ (resp. $\overline{X}_i$) are those induced by the base points $\eta$ (resp. $\overline{\eta}$) of $X$ (resp. $\overline{X}$). Note that $\Pi^{i+1} \subseteq \Pi^i$, and $\tilde{\Delta}^{i+1} \subseteq \tilde{\Delta}^i$, as follows from the various definitions.

**Lemma 1.1.** With the above notations and those in §0, the following holds:

$$\overline{H} = \bigcap_{i \geq 1} \tilde{\Delta}^i.$$ 

**Proof.** Follows from the various definitions. □

We take this opportunity to correct a mistake that occurred in [Saïdi2], Lemma 1.1. The claim there that $I_X = \bigcap_{i \geq 1} \Pi_i$ is false, however this doesn’t affect the validity of the results or other assertions made in loc. cit..

For each integer $i \geq 1$, consider the push-out diagram

$$1 \longrightarrow \tilde{\Delta}^i = \pi_1(\overline{X}_i, \overline{\eta}) \longrightarrow \Pi^i = \pi_1(X_i, \eta) \longrightarrow G_k \longrightarrow 1$$
$$\downarrow \quad \downarrow$$
$$\downarrow$$

$$1 \longrightarrow \tilde{\Delta}^{i, \text{ab}} \longrightarrow \Pi^{(i, \text{ab})} \longrightarrow G_k \longrightarrow 1$$
where $\tilde{\Delta}_{ab}^i$ is the maximal abelian quotient of $\tilde{\Delta}^i$, and $\Pi^{(i,ab)}$ is the \textbf{geometrically abelian} fundamental group of $X_i$. Consider the commutative diagram

\[
\begin{array}{ccccccc}
1 & \longrightarrow & H & \longrightarrow & \mathcal{H} = \mathcal{H}[s_1] & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow s_1 & & \downarrow & & \downarrow \\
1 & \longrightarrow & H & \longrightarrow & \Pi & \longrightarrow & \tilde{\Pi} & \longrightarrow & 1 \\
\end{array}
\]

(1.3)

where the right square is cartesian. Thus (the group extension) $\mathcal{H}$ is the pull-back of (the group extension) $\Pi$ via the section $s_1 : G_k \to \tilde{\Pi}$.

**Lemma 1.2.** We have natural identifications $H \tilde{\simeq} \varprojlim_{i \geq 1} \tilde{\Delta}_{ab}^i$, and $\mathcal{H} \tilde{\simeq} \varprojlim_{i \geq 1} \Pi^{(i,ab)}$.

**Proof.** Similar to the proof of Lemma 1.3 in [Saïdi2]. \qed

The section $s_2 : G_k \to \Pi$, which lifts the section $s_1$, induces a section $s_2 : G_k \to \mathcal{H}$ of the projection $\mathcal{H} \to G_k$ (since $s_2(G_k) \subset \mathcal{H}$). We fix the section $s_2 : G_k \to \mathcal{H}$ as a base point of the torsor of splittings of the upper sequence in diagram (1.3). Thus the set of splittings of the group extension $\mathcal{H}$, modulo conjugation by elements of $H$, is a torsor under $H^1(G_k, H)$; the $G_k$-module structure of $H$ being deduced from diagram (1.3). The splitting $s_2 : G_k \to \mathcal{H}$ thus corresponds to $0 \in H^1(G_k, H)$. Note that the set of splittings $\tilde{s} : G_k \to \mathcal{H}$ of the group extension $\mathcal{H}$ is in one-to-one correspondence with the set of sections $\tilde{s} : G_k \to \Pi$ of the projection $\Pi \to G_k$ which lift the section $s_1$.

Let $\tilde{s} : G_k \to \mathcal{H}$ be a section of the group extension $\mathcal{H}$, which induces a section $\tilde{s} : G_k \to \Pi$ of the projection $\Pi \to G_k$ which lift the section $s_1$. Let $[\tilde{s}]$ be the class of $\tilde{s}$ in $H^1(G_k, H)$ (cf. above discussion).

**Fact 1.3.** The section $\tilde{s} : G_k \to \Pi$ is geometric if and only if $[\tilde{s}] = 0$. In this case the section $\tilde{s}$ is associated to the rational point $x \in X(k)$.

**Proof.** First, assume that the section $\tilde{s} : G_k \to \Pi$ is geometric and arises from a rational point $\tilde{x} \in X(k)$. Both sections $\tilde{s} : G_k \to \Pi$, and $s_2 : G_k \to \Pi$, induce splittings $\tilde{s}_{ab} : G_k \to \pi_1(X, \eta)^{(ab)}$, and $s_2^{ab} : G_k \to \pi_1(X, \eta)^{(ab)}$ of the group extension $1 \to \pi_1(X, \eta)^{(ab)} \to \pi_1(X, \eta)^{(ab)} \to G_k \to 1$, where $\pi_1(X, \eta)^{(ab)}$ is the geometrically abelian quotient of $\pi_1(X, \eta)$. Further one has $\tilde{s}_{ab} = s_2^{ab}$ (cf. condition (i)(iii), and the fact that both $\tilde{s}$ and $s_2$ lift the section $s_1$). A standard argument, resorting to the Kummer exact sequence associated to the jacobian $\text{Pic}^0_X$ of $X$, shows that $\tilde{x} = x$ (cf. [Tamagawa], Proposition 2.8).

Next, we claim $[\tilde{s}] = 0$. Indeed, the classes of $\tilde{s}$ and $s_2$ in $H^1(G_k, \Gamma)$ coincide as both sections are geometric and associated to the same rational point $x$, hence $\tilde{s}$ and $s_2$ are conjugate by an element of $\Gamma$. Here we view the set of splittings of the group extension $\Pi$ (of $\Gamma$ by $G_k$) as a torsor under $H^1(G_k, \Gamma)$, with base point the class of the section $s_2$. Further the natural map $H^1(G_k, H) \to H^1(G_k, \Gamma)$ of pointed cohomology sets is injective as follows from the condition (i)(ii) (cf. [Serre], I.§5, Proposition 38, and diagram (1.1)). (Here the $G_k$-module structure on $H$ (resp. $G_k$-group structure on $\Gamma$) is induced by the section $s_1$ (resp. $s_2$) (cf. diagram (1.1)).) Thus $[\tilde{s}] = 0$.

Conversely, if $[\tilde{s}] = 0$, then $\tilde{s}$ is conjugate to $s_2$ by an element of $H$ hence is geometric and associated to the rational point $x$. \qed

As a consequence we obtain the following.
Lemma 1.4. Let $k'/k$ be a finite extension. There exists a section $\bar{s} : G_{k'} \to \Pi_{k'}$ of the projection $\Pi_{k'} \to G_{k'}$, which lifts the section $s_{1,k'} : G_{k'} \to \bar{\Pi}_{k'}$ of the projection $\bar{\Pi}_{k'} \to G_k$ induced by $s_1$, and which is non-geometric, if and only if $H^1(G_{k'}, H) \neq 0$.

Thus proving Theorem A reduces to proving the following.

Proposition 1.5. There exists a finite extension $\bar{k}/k$ such that $H^1(G_{k'}, H) \neq 0$ for every finite extension $k'/\bar{k}$.

The rest of this section is devoted to proving Proposition 1.5. Let $\ell \neq p$ be a prime integer such that the map $\Gamma^\ell \to \Delta^\ell$ is not an isomorphism (cf. condition (\star)(iv)). Write $\pi_1(\bar{X}, \bar{\eta})^\ell$ for the maximal pro-$\ell$ quotient of $\pi_1(\bar{X}, \bar{\eta})$, and

$$\pi_1(X, \eta)^{(\ell)} \overset{\text{def}}{=} \pi_1(X, \eta)/\text{Ker}(\pi_1(\bar{X}, \bar{\eta}) \to \pi_1(\bar{X}, \bar{\eta})^\ell)$$

for the geometrically pro-$\ell$ quotient of $\pi_1(X, \eta)$, which fits in the exact sequence

$$1 \to \pi_1(\bar{X}, \bar{\eta})^\ell \to \pi_1(X, \eta)^{(\ell)} \to G_k \to 1.$$

Let $s^\ell = s_x^\ell : G_k \to \pi_1(X, \eta)^{(\ell)}$ be the section of the projection $\pi_1(X, \eta)^{(\ell)} \to G_k$ induced by the section $s = s_x$. This section induces a representation

$$\rho^\ell : G_k \to \text{Aut}(\pi_1(\bar{X}, \bar{\eta})^\ell)$$

which factors as $G_k \to G_F \to \text{Aut}(\pi_1(\bar{X}, \bar{\eta})^\ell)$, where $G_F$ is the quotient of $G_k$ by its inertia subgroup, since $X$ has good reduction over $\mathcal{O}_k$. Further the image of the representation $\rho^\ell$ is almost pro-$\ell$, i.e., $\rho^\ell(G_k)$ possesses an open subgroup which is pro-$\ell$. In particular, there exists a finite extension $\bar{k}/k$ such that the restriction $\rho^\ell_k : G_k \to \text{Aut}(\pi_1(\bar{X}, \bar{\eta})^\ell)$ of $\rho^\ell$ to $G_k$ has a pro-$\ell$ image. In order to prove Proposition 1.5 we will, without loss of generality, assume that the image of $\rho^\ell$ is pro-$\ell$, and will show $H^1(G_k, H) \neq 0$.

Let $\Delta^\ell$ be the maximal pro-$\ell$ quotient of $\Delta$, and $\bar{\Pi}^{(\ell)} \overset{\text{def}}{=} \bar{\Pi}/\text{Ker}(\Delta \to \Delta^\ell)$ the geometrically pro-$\ell$ quotient of $\bar{\Pi}$. We have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \bar{\Pi} & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta^\ell & \longrightarrow & \bar{\Pi}^{(\ell)} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

where the left and middle vertical maps are surjective. For $i \geq 1$, let $N^i$ be the image of $\Delta^i$ in $\Delta^\ell$, and $\bar{N}^i$, $\bar{\Delta}^i$, the pre-images of $N^i$ in $\pi_1(\bar{X}, \bar{\eta})$, and $\pi_1(\bar{X}, \bar{\eta})^\ell$; respectively. Note that $\bar{N}^i$ is a characteristic subgroup of $\pi_1(\bar{X}, \bar{\eta})$, and $\bar{N}^i$ is stable by the action of $s^\ell(G_k)$.

Let $\bar{U}^i \overset{\text{def}}{=} \bar{N}^i.s(G_k)$, and $\bar{U}^i \overset{\text{def}}{=} \bar{N}^i.s^\ell(G_k)$, for $i \geq 1$. Thus $\bar{U}^i$ is an open subgroup of $\pi_1(X, \eta)$ corresponding to an étale cover $\bar{Y}_i \to X$, and the étale cover $X_i \to X$ factorises as

$$X_i \to \bar{Y}_i \to X$$
\[ \pi_1(\pi_i) \subset \pi_1(Y_i, \eta) \] as follows from the various definitions], where \( X_i \to Y_i \) is an étale cover of degree prime-to-\( \ell \), since \( \Delta \) (hence also \( \Delta_i \), for \( i \geq 1 \)) is pro-nilpotent (see condition (\*)\((i)\)). Further \( \tilde{U}^i, \tilde{U}^\ell \), and \( \tilde{U}^\ell \), are naturally identified with \( \pi_1(Y_i, \eta) \), and \( \pi_1(Y_i, \eta)^{\ell(\ell)} \); respectively, \( \forall i \geq 1 \). Here \( \pi_1(Y_i, \eta)^{\ell(\ell)} \) is the geometrically pro-\( \ell \) quotient of \( \pi_1(Y_i, \eta) \), and sits in an exact sequence

\[ 1 \to \pi_1(\tilde{Y}_i, \tilde{\eta})^\ell \to \pi_1(Y_i, \eta)^{\ell(\ell)} \to G_k \to 1, \]

where \( \tilde{Y}_i \) def \( Y \times_k \overline{k} \).

The natural action of \( G_k \) on \( \pi_1(\tilde{X}, \tilde{\eta})^\ell \), and which factorises through \( G^\ell_{\pi} \) by our assumption on the representation \( \pi \) quotient of \( \pi \) is surjective, \( \forall i \geq 1 \).

There is a surjective homomorphism (recall Lemma 1.2)

\[ H^1(G_k, H) = \lim_{i \geq 1} H^1(G_k, \tilde{\Delta}_i^{ab, \ell}) \to \lim_{i \geq 1} H^1(G_k, \tilde{\Delta}_i^{ab, \ell}). \]

(Indeed, \( \lim_{i \geq 1} H^1(G_k, \tilde{\Delta}_i^{ab, \ell}) \to \lim_{i \geq 1} H^1(G_k, \tilde{\Delta}_i^{ab, \ell}), \) where the product is over all prime integers \( l \) and the above homomorphism is the projection onto the \( \ell \)-th factor.) Further the étale covers \( \{X_i \to Y_i\}_{i \geq 1} \) induce a homomorphism (recall \( \tilde{\Delta} = \pi_1(\tilde{X}, \tilde{\eta}) \))

\[ \lim_{i \geq 1} H^1(G_k, \tilde{\Delta}_i^{ab, \ell}) \to \lim_{i \geq 1} H^1(G_k, \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell}), \]

which is surjective. More precisely, the map \( H^1(G_k, \tilde{\Delta}_i^{ab, \ell}) \to H^1(G_k, \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell}) \) is surjective, \( \forall i \geq 1 \), as follows easily from a restriction-restriction argument using the fact that the degree of the cover \( X_i \to Y_i \) is prime-to-\( \ell \) (observe the maps on cohomology induced by the natural maps \( \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell} \to \tilde{\Delta}_i^{ab, \ell} \), \( \tilde{\Delta}_i^{ab, \ell} \) factors through \( \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell} \) arising from the morphisms \( \text{Pic}^0(Y_i) \to \text{Pic}^0(X_i) \to \text{Pic}^0(Y_i) \), where the first one is the pull-back map of line bundles and the second is the norm map). Thus in order to prove Proposition 1.5 it suffices to show the following.

**Proposition 1.6.** With the above notations, it holds that

\[ \lim_{i \geq 1} H^1(G_k, \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell}) \neq 0. \]

**Proof.** As discussed above the natural action of \( G_k \) on \( \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell} \) factors through \( G^\ell_{\pi} \) (which is isomorphic to \( \mathbb{Z}_\ell \)). There is an injective inflation map

\[ \inf : \lim_{i \geq 1} H^1(G^\ell_{\pi}, \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell}) \to \lim_{i \geq 1} H^1(G_k, \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell}). \]

Further

\[ (\lim_{i \geq 1} \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell})_{G^\ell_{\pi}} = H^1(G^\ell_{\pi}, \lim_{i \geq 1} \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell}) = H^1(G^\ell_{\pi}, \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell}) \]

where the notation \( (\ )_{G^\ell_{\pi}} \) stands for the co-invariant module, the first equality follows from the fact that \( G^\ell_{\pi} \) is procyclic, and the second follows from [Neukirch-Schmidt-Winberg] (2.3.5) Corollary.

There is a natural isomorphism

\[ (\ker (\Gamma^\ell \to \Delta^\ell)) \cong \lim_{i \geq 1} \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell}. \]

(Proof similar to the proof of Lemma 1.2.) Thus \( \lim_{i \geq 1} \pi_1(\tilde{Y}_i, \tilde{\eta})^{ab, \ell} \neq 0 \) (cf. condition (\*)\((iii)\), and our choice of \( \ell \)). The proof of Proposition 1.6 follows from the following.
Lemma 1.7. Let $T$ be an abelian pro-$\ell$ group, and $P$ an infinite pro-$\ell$ cyclic group. Assume $T$ is a continuous $P$-module. Then the co-invariant module $(T)_P = \{0\}$ is trivial if and only if $T = \{0\}$ itself is trivial.

Proof. Let $T^\wedge$ be the Pontryagin dual of $T$ which is an $\ell$-primary torsion group. The dual of $(T)_P$ is the invariant group $(T^\wedge)^P$. It suffices to show that $(T^\wedge)^P$ is trivial if and only if $(T^\wedge)$ is trivial. The action of $P$ on $T^\wedge$ is discrete, in particular $T^\wedge$ is the union of finite $\ell$-groups which are stable $P$-submodules. We can thus reduce to the case where $T^\wedge$ and $P$ are finite. Suppose $(T^\wedge)$ is finite, and non-trivial, then $(T^\wedge)^P$ is non-trivial since its order is divisible by $\ell$, and $(T^\wedge)^P$ contains 0. □

This finishes the proof of Proposition 1.6, hence the proof of Proposition 1.5, and the proof of Theorem A. □

§2. Proof of Theorem C. The rest of this paper is devoted to proving Theorem C. We use the notations as introduced in §0, and the statement of Theorem C.

Let $K$ be a number field (finite extension of $\mathbb{Q}$), and $\overline{K}$ an algebraic closure of $K$. Let $X$ be a proper, smooth, and geometrically connected hyperbolic curve over $K$. Write $J = \text{Pic}^0_X$ for the jacobian of $X$. Assume $X(K) \neq \emptyset$. Fix a rational point $x \in X(K)$, and consider the embedding $\iota : X \hookrightarrow J$ defined by $\iota(x) = 0$. For any field extension $\overline{K} \subset L$, with $L$ algebraically closed, let $J^{\text{tor}} = J(L)^{\text{tor}} = J(\overline{K})^{\text{tor}}$ be the torsion subgroup of $J$. The intersection $X \cap J^{\text{tor}}$ is finite by [Raynaud]. Let $M$ be the cardinality of the subgroup of $J^{\text{tor}}$ generated by $X \cap J^{\text{tor}}$. We assume $M \geq 2$ (this $M$ will be the integer $N$ required in theorem C).

Let $p > M$ be a prime integer, $k$ a $p$-adic completion of $K$, $\overline{k}$ an algebraic closure of $k$, $X_k = X \times_K k$, and $X_{\overline{k}} = X \times_K \overline{k}$. Recall the exact sequence of fundamental groups (cf. §0)

$$1 \to \pi_1(X_{\overline{k}}, \bar{\eta}) \to \pi_1(X_k, \eta) \to G_k \to 1.$$ 

Let $\Delta$ be the maximal pro-$p$ quotient of $\pi_1(X_{\overline{k}}, \bar{\eta})$, and

$$\Pi \overset{\text{def}}{=} \pi_1(X_k, \eta) / \text{Ker}(\pi_1(X_{\overline{k}}, \bar{\eta}) \to \Delta)$$

the geometrically pro-$p$ arithmetic fundamental group of $X_k$.

For an integer $m \geq 1$, let $\Delta_m$ be the maximal $m$-step solvable pro-$p$ quotient of $\pi_1(X_{\overline{k}}, \bar{\eta})$, and

$$\Pi_m \overset{\text{def}}{=} \pi_1(X_k, \eta) / \text{Ker}(\pi_1(X_{\overline{k}}, \bar{\eta}) \to \Delta_m)$$

the geometrically $m$-step solvable pro-$p$ arithmetic fundamental group of $X_k$. 

We have a commutative diagram of exact sequences

\[
\begin{array}{c}
1 & \rightarrow & \Delta[m] & \rightarrow & \Pi_m & \rightarrow & G_k & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \Delta_{m+1} & \rightarrow & \Pi_{m+1} & \rightarrow & G_k & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \Delta_m & \rightarrow & \Pi_m & \rightarrow & G_k & \rightarrow & 1 \\
\end{array}
\]

(2.1)

where \(\Delta[m] \overset{\text{def}}{=} \ker(\Delta_{m+1} \rightarrow \Delta_m) = \ker(\Pi_{m+1} \rightarrow \Pi_m)\) (cf. [Saïdi3], §1, for more details). Further we have natural identifications

\[
\Delta = \varprojlim_{m \geq 1} \Delta_m, \quad \text{and} \quad \Pi = \varprojlim_{m \geq 1} \Pi_m.
\]

Let \(s_x : G_k \rightarrow \Pi\) be a section of the projection \(\Pi \rightarrow G_k\) associated to the \(k\)-rational point (image in \(X_k\) of) \(x\), which induces sections \(s_{x,m} : G_k \rightarrow \Pi_m\) of the projections \(\Pi_m \rightarrow G_k\), \(\forall m \geq 1\). (Thus \(s_x\) is defined up to conjugation by \(\Delta\).) We fix the section \(s_{x,1}\) as a base point of the torsor of splittings of the exact sequence \(1 \rightarrow \Delta_1 \rightarrow \Pi_1 \rightarrow G_k \rightarrow 1\) (\(\Pi_1\) is the geometrically abelian pro-\(p\) arithmetic fundamental group of \(X_k\)), which is a torsor under \(H^1(G_k, \Delta_1)\).

Let \(y \in (X \cap J^{\text{tor}})(K) \setminus \{0_J\}\) (the existence of \(y\) follows from our assumption \(M \geq 2\)), \(s_y : G_k \rightarrow \Pi\) a section of the projection \(\Pi \rightarrow G_k\) associated to the \(k\)-rational point (image in \(X_k\) of) \(y\), which induces sections \(s_{y,m} : G_k \rightarrow \Pi_m\) of the projections \(\Pi_m \rightarrow G_k\), \(\forall m \geq 1\). The classes \([s_{x,1}] = 0\), and \([s_{y,1}]\), of the sections \(s_{x,1}\) and \(s_{y,1}\); respectively, in \(H^1(G_k, \Delta_1)\) coincide. Indeed this follows easily from the (pro-\(p\)) Kummer exact sequence associated to \(J\), and the fact that \(t(y)\) is a torsion point of order prime-to-\(p\) (recall \(p > M\)).

More generally, for \(m \geq 1\), consider the following commutative diagram

\[
\begin{array}{c}
1 & \rightarrow & \Delta[m+1] & \rightarrow & E[m+1] & \rightarrow & G_k & \rightarrow & 1 \\
\| & & \downarrow & & \downarrow & \overset{s_{x,m}}{\downarrow} & & \downarrow \\
1 & \rightarrow & \Delta[m+1] & \rightarrow & \Pi_{m+1} & \rightarrow & \Pi_m & \rightarrow & 1 \\
\end{array}
\]

(2.2)

where the right square is cartesian. Thus the group extension \(E[m+1]\) is the pull-back of the group extension \(\Pi_{m+1}\) via the section \(s_{x,m}\).

The upper exact sequence in diagram (2.2) splits. Indeed this follows from the existence of the section \(s_{x,m+1} : G_k \rightarrow \Pi_{m+1}\) which lifts the section \(s_{x,m}\), and induces a splitting \(s_{x,m+1} : G_k \rightarrow E[m+1]\) of the group extension \(E[m+1]\). We fix the section \(s_{x,m+1}\) as a base point for the torsor of splittings of the group extension
$E[m+1]$, which is a torsor under $H^1(G_k, \Delta[m+1])$; the $G_k$-module structure of $\Delta[m+1]$ is deduced from diagram (2.2). If $z \in X(k)$, and $s_{z,m} = s_{x,m} : G_k \to \Pi_m$, then the section $s_{z,m+1}$ gives rise to a splitting $s_{z,m+1} : G_k \to E[m+1]$ of the upper exact sequence in diagram (2.2), hence to a class $[s_{z,m+1}] \in H^1(G_k, \Delta[m+1])$.

Define $S_m$ to be the set of rational points $z \in X(k)$ such that $s_{x,m}(G_k)$ coincide with a decomposition group of $\Pi_m$ associated to $z$. We have the following inclusions

$$
\cdots \subseteq S_{m+1} \subseteq S_m \subseteq \cdots \subseteq S_2 \subseteq S_1 = X(k) \cap J^{tor,p'} \subseteq X \cap J^{tor}.
$$

The equality $S_1 = X(k) \cap J^{tor,p'}$ follows from the (pro-$p$) Kummer exact sequence associated to $J$, and the well-known structure of $J(k)$.

**Lemma 2.1.** The equality $\bigcap_{m \geq 1} S_m = \{x\}$ holds.

**Proof.** Follows from [Mochizuki], Theorem C, and a limit argument using the fact that $\Pi = \varprojlim_{m \geq 1} \Pi_m$. □

It follows from Lemma 2.1, and the above discussion, that there exists $m \geq 1$ such that

$$
\{x\} \subsetneq S_m, \quad \text{and} \quad \{x\} = S_{m+1}.
$$

Let

$$
A \overset{\text{def}}{=} \{[s_{z,m+1}] : z \in S_m\} \subset H^1(G_k, \Delta[m+1]).
$$

Note that $\{0\} \subsetneq A$; which follows from the facts that $\{x\} \subsetneq S_m$ and $\{x\} = S_{m+1}$. Further $\text{Card}(A) \leq \text{Card}(S_m) \leq M < p$. In particular,

$$
\exists \alpha \in H^1(G_k, \Delta[m+1]) \setminus A,
$$

since $H^1(G_k, \Delta[m+1])$ is $p$-primary. Thus $\alpha$ corresponds to a section $\alpha : G_k \to \Pi_{m+1}$ of the projection $\Pi_{m+1} \twoheadrightarrow G_k$, which lifts the section $s_{x,m}$.

**Lemma 2.3.** The section $\alpha : G_k \to \Pi_{m+1}$ is non-geometric.

**Proof.** Follows from the various definitions, and the fact that $\alpha \notin A$. □

This finishes the proof of Theorem C. □

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