Resolving ambiguities in model-independent partial-wave analysis of three-body decays

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October 30, 2017

Abstract

Partial wave analysis (PWA) is an important tool for analyzing large data sets in hadronic decays of light and heavy mesons. It commonly relies on the isobar model, which assumes multihadron final states originate from successive two-body decays of well-known undisturbed intermediate states. Recently, analyses of heavy-meson decays and diffractively produced states have attempted to overcome the strong model dependences of the isobar model. These analyses have overlooked that model-independent, or freed-isobar, PWA can break the orthogonality of partial waves and introduce mathematical ambiguities into results. We show how these ambiguities arise and present general techniques for identifying their presence and for correcting for them. We demonstrate these techniques with specific examples in both heavy-meson decay and pion–proton scattering.

1 Introduction

Physicists have used spectroscopy to precisely determine the masses, widths and other parameters of light mesons and to discover new mesonic states in even very faint signals. In analysing multibody decays of heavy mesons (for example, B, D, and heavy quarkonia), they have used spectroscopy to measure both strong and weak phases, allowing for measurement of CP asymmetries.

In a fixed-target experiment, an incoming particle interacts with a target to produce a superposition of states that decays to a set of final-state mesons. All resonant states with quantum numbers allowed by the conservation laws of the initial interaction contribute to the superposition. In heavy meson decay, there is only one decaying state—the heavy meson itself. Both light- and heavy-meson spectroscopy commonly use partial-wave analysis (PWA)—often referred to as Dalitz-plot analysis—which expands the amplitude for the production of

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the final state into a sum of contributions from partial waves: one for each possible combination of quantum numbers for all states. Each contribution is factorized into components whose forms are dictated by quantum mechanics—spin-dependent amplitudes—and components whose forms are not—dynamic amplitudes, or so-called mass-dependent line shapes. The dynamic amplitudes parameterize the dependence of the partial-wave amplitude on the masses of intermediary states.

Dynamic amplitudes are commonly decomposed into sums of contributions from known resonances. For example, for two pions in a state of spin, parity, and charge-conjugation parity $^1_{1}^{−−}$ with total isospin 1, the dynamic amplitude can be a sum of contributions from $\rho(770)$ and $\rho'(1450)$. Each resonance has its own dynamic amplitude model—for example the Breit-Wigner line shape—and an accompanying complex constant parameterizing its admixture into the total dynamic amplitude. This decomposition is commonly called the isobar model, with the individual resonances called isobars.

The results of a partial-wave analysis are strongly dependent on the quality of the analysis model: for example, on the assumptions of what resonances to include; how to model their dynamic amplitudes; and what parameter values to use in those models. PWA with the isobar model suffers other problems: Models quickly become very complicated when we include all possible quantum numbers and all known resonances—even for the production of only three final-state particles. Resonances with identical quantum numbers that significantly overlap in invariant mass—like the $\rho(770)$ and $\rho'(1450)$—often lead to unphysical modeling. And dynamic amplitude models often ignore the strong interactions that can occur between resonances and other particles in the decay.

These assumptions lead to problems increasingly present in the analyses of the large data sets provided by current and recent experiments. In section 2 we present a method to determine dynamic amplitudes directly from the data without models. Several heavy-meson analyses have used this technique in recent years [1–4] and often refer to it as model-independent PWA—we call it freed-isobar analysis. We demonstrate its applicability to scattering analyses. In section 3 we demonstrate there exist potentially fatal mathematical ambiguities that have not been pointed out in previous analyses. And in section 4 we present several ways to resolve these ambiguities in both heavy meson and scattering contexts.

2 Model-independent partial-wave analysis

We can remove model dependencies of the isobar model by determining dynamic amplitude forms from the data. Instead of decomposing a dynamic amplitude into contributions from intermediary resonances—each with its own dynamic amplitude model—we parameterize it as a complex step function.

Before detailing the formalism of these freed isobars, we briefly review the standard PWA formalism using the isobar model. To make some of the formulas concrete and to simplify notation, we refer to the example of three-pion production, from both pion-proton scattering,

$$\pi^- p \rightarrow \pi^- \pi^+ \pi^- p,$$

(1)

We denote these quantum number throughout in this order.
and D meson decay,
\[ D^- \rightarrow \pi^- \pi^+ \pi^- . \] (2)
A fully generic description of the isobar model is available in reference [5].

2.1 Partial-wave analysis

In partial-wave analysis, we assume events are distributed according to the square of the sum of the partial-wave amplitudes—the intensity of the model. Each amplitude describes a unique transition through intermediary states with well-defined quantum numbers to the final state. We assume the transition proceeds through two-body decays. In our three-pion examples, the quantum numbers defining a partial wave are the spin \((J)\), spin projection \((M)\), parity \((P)\), and charge-conjugation parity \((C)\) of the three-pion system, \(X\); the spin of the intermediary two-pion system \((S)\); and the angular momentum \((L)\) between the two-pion system and the third pion, referred to as the spectator pion. Since there are two possible combinations of pions to form the intermediary system, the sum over amplitudes also sums over the two combinations—this is commonly called Bose symmetrization, with each combination referred to as a symmetrization.

Each partial-wave amplitude has a spin-dependent amplitude component, \(\psi(m_{3\pi}, \vec{\tau})\), which is dependent on the mass of the initial system, \(m_{3\pi}\), and the coordinate \(\vec{\tau}\) in phase space of the final-state particles, \(\vec{\tau}\). This component is fully specified by the quantum numbers of the wave; but its exact form is formalism dependent.

Each partial-wave amplitude also has a dynamic amplitude component for the production of the wave, \(\Delta^{\text{prod}}(m_{3\pi})\), and one for the production of the intermediary state, \(\Delta(m_{2\pi})\). Each is dependent only on the mass of the state whose production it parameterizes—not on any other phase-space coordinates. Commonly one also includes angular-momentum barrier factors and form factors, which are dependent on both \(m_{3\pi}\) and \(m_{2\pi}\)—we denote the product of such terms by \(F(m_{3\pi}, m_{2\pi})\).

The total amplitude, summing over partial waves, \(a\), and symmetrizations is
\[ A(m_{3\pi}, \vec{\tau}) \equiv \sum_a \Delta^{\text{prod}}_a(m_{3\pi}) \psi_a(m_{3\pi}, \vec{\tau}) F_a(m_{3\pi}, \hat{m}_{2\pi}) \Delta_a(\hat{m}_{2\pi}) , \] (3)
where we denote symmetrization-dependent functions and variables (and the sum itself) by hats—we use this notation throughout the paper.

For scattering experiments, data are conventionally analyzed independently in bins of \(m_{3\pi}\), so that the dynamic amplitudes for the production of the three-pion states are learned empirically. In this case, \(\Delta^{\text{prod}}_a(m_{3\pi})\) is a set of complex parameters, one for each three-pion mass bin, \(b\)—which we call production amplitudes. We rewrite equation (3)—now applying independently for

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2For three-particle decay, one needs five independent coordinates to specify \(\vec{\tau}\). For scattering, these are usually two Gottfried-Jackson angles; two angles, each one between two final-state momenta; and the invariant mass of one pair of final-state particles. For heavy-meson decay, these are usually two invariant masses of pairs of final-state particles; and three Euler angles describing the overall orientation of the decay plane. If the heavy meson is spinless, as in our example, the Euler angles can be omitted since the decay is isotropic in them.
each three-pion mass bin—as a sum over partial waves each with a production amplitude, $\Delta^{(b)}_a$:

$$A^{(b)}(\vec{\tau}) \equiv \sum_a \Delta^{(b)}_a(\vec{\tau}) \hat{\psi}_a(\vec{\tau}) \Delta_a(\hat{m}_{2\pi});$$

(4)

since within one three-pion mass bin $F(m_{3\pi},\hat{m}_{2\pi})$ is dependent only on the two-pion mass, we absorb it into $\Delta_a(\hat{m}_{2\pi})$. To simplify notation, since it is sufficient to discuss the formalism within a single three-pion mass bin, we omit the mass-bin index in the remainder of the paper.

In heavy-meson decay, there is only one initial state—the heavy meson itself—with a fixed mass, so we can parameterize $\Delta^{\text{prod}}_a(m_{3\pi})$ by a single complex variable and also use equation (4).

2.2 Model-dependent isobars

There are many ways to formulate dynamic amplitudes, none of which are dictated by first principles. The most common way is the isobar model, in which the dynamic amplitude for the two-pion state is a sum of contributions from known resonances, $\xi$, with the quantum numbers of the two-pion state in the wave:

$$\Delta_a(m) = \sum_\xi \alpha_\xi \Delta_\xi(m).$$

(5)

Each resonance is parameterized by an individual dynamic amplitude, $\Delta_\xi(m)$, and a complex admixture variable, $\alpha_\xi$. There are myriad ways to formulate the resonance dynamic amplitudes. One of the most common is the relativistic Breit-Wigner shape:

$$\Delta^{\text{BW}}_\xi(m) \equiv \frac{m_\xi \Gamma_\xi}{m^2_\xi - m^2 - im_\xi \Gamma_\xi},$$

(6)

which ascribes a mass, $m_\xi$, and width, $\Gamma_\xi$, to the resonance particle.

We can absorb $\Delta^{(b)}_a$ into the $\alpha_\xi$, and rewrite equation (4) as a sum over waves and the resonances contributing to each wave:

$$A(\vec{\tau}) = \sum_a \sum_\xi \hat{\psi}_a(\vec{\tau}) \alpha_\xi^a \Delta_\xi(\hat{m}_{2\pi}),$$

(7)

where each complex-valued $\alpha_\xi^a$ parameterizes the production of a resonance and the spectator pion in the total spin configuration of the wave: for example, a three-pion state with quantum numbers $0^{-+}$ decaying into $f_0\pi^-$ in a relative S wave, where $f_0$ is a $\pi^-\pi^+$ resonance with quantum numbers $0^{++}$ and an assumed dynamic amplitude model (and parameter values therein); or a three-pion state with quantum numbers $2^{-+}$ decaying into $\rho\pi^-$ in a relative P wave, where the $\rho$ is a $\pi^-\pi^+$ resonance with quantum numbers $1^{--}$ and an assumed dynamic amplitude model.

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3The numerator $m_\xi \Gamma_\xi$ normalizes the shape so that $\Delta_\xi = i$ at $m = m_\xi$.

4States with spin greater than zero should also have a spin projection specified, which we leave off here for brevity.
The isobar model has been very useful in analysis of scattering experiments and heavy-meson decays. But it requires assumptions concerning what resonances are present and what forms their dynamic amplitudes have. These assumptions may bias analyses and ignore small structures not easily modeled, which can distort fit results. These possible substructures—increasingly more visible in the larger and larger data sets of modern experiments—may arise from new resonant states or from final-state interactions.

2.3 Model-independent isobars

In freed-isobar PWA, we remove the model dependency inherent in the isobar model by parameterizing dynamic amplitudes as complex step functions:

\[ \Delta_a(m) = \sum_\beta \omega_\beta \mathbb{1}_\beta(m), \]

where the \( \beta \) are disjoint bins of the two-pion mass range; \( \omega_\beta \), the complex values of the dynamic amplitude in those ranges; and \( \mathbb{1}_\beta \), the indicator function,

\[ \mathbb{1}_\beta(m) \equiv \begin{cases} 1, & \text{if } m \in \beta, \\ 0, & \text{if } m \notin \beta. \end{cases} \]

The division of the mass range into bins is independent for each wave, but is identical for all symmetrizations of a wave. With this model-independent dynamic amplitude, the description of an isobar is freed from assumptions on both what resonances comprise it and how to formulate the dynamic amplitudes of those resonances.

Substituting into equation (4) and absorbing \( \Delta^{(b)}_a \) into the \( \omega_\beta \), we have

\[ A(\vec{\tau}) = \sum_a \sum_\beta \hat{\psi}_a(\vec{\tau}) \omega_\beta \mathbb{1}_\beta(\hat{m}_{2\pi}) \]

This has a form identical to equation (7), with each two-pion mass bin in each wave appearing as an intermediate state with an indicator function for a dynamic amplitude. So we can use the computational techniques (and software) used for model-dependent PWA.

3 Zero modes in freed-isobar PWA

If we allow dynamic amplitudes for isobars to have arbitrary forms, mathematical ambiguities may arise: there may exist functions, \( \tilde{\Delta}_a \), whose combined amplitude vanishes—

\[ \sum_a \hat{\psi}_a(\vec{\tau}) \tilde{\Delta}_a(\hat{m}_{2\pi}) = 0 \]

—no matter what values are taken for parameters of the functions, including an overall scaling of them. Since the amplitude is zero, the parameters of these functions are superfluous degrees of freedom in the total PWA amplitude. We refer to each set of \( \tilde{\Delta}_a \) as a zero mode. For such a zero mode to exist, there must be at least two terms in the sum of equation (11); that is, there must be at least two symmetrizations or two waves in the sum.
The binned functions of model-independent PWA introduce enough freedom to the dynamic amplitudes that the sum in equation (11) can be approximately zero—we can have modes that contribute very weakly to the overall amplitude. These weakly contributing modes can complicate analyses and obscure underlying results. Since it is these approximately zero modes that show up in model-independent PWA, we refer to them also simply as zero modes when there is no possibility of confusion.

We can decompose the $\omega^a_{\beta}$ of equation (10) into a contribution describing nature, $\alpha^a_{\beta}$, and a contribution from zero modes:

$$\omega^a_{\beta} = \alpha^a_{\beta} + \sum_z \tilde{\eta}_z z^a_{\beta},$$

(12)

where each zero mode has complex scaling factor $\tilde{\eta}_z$ and values in the two-pion mass bins, $z^a_{\beta}$, that approximate the $\tilde{\Delta}^a(m)$:

$$\tilde{\Delta}^a(m) \approx \tilde{\eta}^a \sum_{\beta} z^a_{\beta} \Pi_{\beta}(m).$$

(13)

Without loss of generality we can define a single zero mode as a collection of real functions with one common complex scaling factor—so that the $z^a_{\beta}$ are real.

3.1 A concrete example

Let us demonstrate the presence of a zero mode in our three-pion examples: The final state particles are all spinless, therefore $S$ is equal to the spin of the two-pion state. Let us also consider a spinless three-pion state—which is possible in pion–proton scattering and is always the case in D decay—for which $L$ equals $S$. Since there are no doubly-charged mesons, we consider only the $\pi^+\pi^-$ intermediary states, of which there are two symmetrizations. We label waves by the spin of the $\pi^+\pi^-$ system; and label the pions as $\pi_1\pi_2\pi_3$ and the two $\pi^+\pi^-$ symmetrizations as 12 and 23.

We will fulfil equation (11) with functions in the S and P waves:

$$\psi_{12}^S(\vec{r}) \tilde{\Delta}^S(m_{12}) + \psi_{12}^P(\vec{r}) \tilde{\Delta}^P(m_{12}) + \psi_{23}^S(\vec{r}) \tilde{\Delta}^S(m_{23}) + \psi_{23}^P(\vec{r}) \tilde{\Delta}^P(m_{23}) = 0.$$  

(14)

The S-wave spin-dependent amplitudes are unity. To find explicit forms for the zero-mode dynamic amplitudes, we must assume a formalism for the P-wave spin-dependent amplitudes. We use the Zemach tensor formalism of [6] because it is simple and common:

$$\psi_{12}^L(\vec{r}) = |\vec{p}_1|^L |\vec{p}_3|^L P^L_L(\hat{p}_1 \cdot \hat{p}_3),$$

(15)

where $P^L_L$ is the $L$’th-order Legendre polynomial and the momenta are in the $\pi_1\pi_2$ rest frame. For the P wave,

$$\psi_{12}^P(\vec{r}) = \frac{1}{4} \left( m^2_{12} + 2m^2_{23} - m^2_{3\pi} - 3m^2_{\pi} \right),$$

(16)

$\psi_{23}^P$ is formed by swapping $m_{12}$ and $m_{23}$. If we set $\tilde{\Delta}^P(m)$ constant, then the P-wave contribution in equation (14) only has terms dependent on at most one
two-pion-symmetrization mass. These terms can be canceled by the S-wave dynamic amplitude. The explicit zero-mode dynamic amplitudes are

\[ \tilde{\Delta}^P(m) = 4\tilde{\eta} \]
\[ \tilde{\Delta}^S(m) = \tilde{\eta}(m_{3\pi}^2 + 3m^2 - 3m^2) \]

The zero mode has two degrees of freedom, those of the arbitrary complex coefficient \(\tilde{\eta}\).

### 3.2 Numerically determining zero modes

Most zero modes are not as simple as the example above. Those for higher-spin decays are particularly more complicated. But we can numerically determine their shapes in the freed-isobar formulation.

In model-independent PWA, a zero mode satisfies

\[ \sum_{a,\beta} \hat{\psi}_a(\vec{\tau}) z^a_{\alpha \beta}(m_{2\pi}) \approx 0 \]

where the \(z^a_{\alpha \beta}\) are the values in each two-pion mass bin of each freed wave such that the sum is very small—they are real since we have defined a zero mode as real in equation (13). The standard mathematical tool to solve for the \(z^a_{\alpha \beta}\) that fulfill equation (19) is to look for the eigenvectors of the Gram matrix of the freed isobars that have vanishingly small eigenvalues. This matrix is

\[ I_{a\beta,b\delta} \equiv \frac{1}{N_{a\beta} N_{b\delta}} \int \left( \sum_{\alpha} \hat{\psi}_\alpha(\vec{\tau}) \mathbb{I}_{\alpha \beta} (m_{2\pi}) \right) \left( \sum_{\gamma} \hat{\psi}_\gamma(\vec{\tau}) \mathbb{I}_{\gamma \delta} (m_{2\pi}) \right) d\vec{\tau} \]  

where \(a\) and \(b\) are wave labels, \(\beta\) and \(\delta\) two-pion-mass bins in each, respectively, and the sums are over possible symmetrizations of each wave. The normalization constants are chosen such that the diagonal elements of the matrix are unity:

\[ N_{a\beta}^2 \equiv \int \left| \sum_{\alpha} \hat{\psi}_\alpha(\vec{\tau}) \mathbb{I}_{\alpha \beta} (m) \right|^2 d\vec{\tau}; \]

they mitigate spurious effects from two-pion mass bins that hang over an edge of phase space. It is enlightening to connect this mathematical tool back to a physical interpretation: This is the overlap integral matrix for our two-pion mass bins. An eigenvector of it with a very small eigenvalue is a set of dynamic amplitude values in each bin of each wave that contribute negligibly to the overall intensity. The set of freed-isobar dynamic amplitudes forming a zero mode are therefore

\[ \tilde{\Delta}^z_a(m) = \tilde{\eta} z^a_{\alpha \beta}(m), \quad z^a_{\alpha \beta} = \tilde{z}^a_{\beta} N_{a\beta}^{-1} \]

where \(\tilde{z}^a_{\beta}\) are the elements of the eigenvector.

We use numerical integration techniques to construct the overlap-integral matrix. It will have \(\text{dim}(I)\) eigenvectors; those corresponding to zero modes will not only be small, but will have values that decrease, owing to the construction of the Gram matrix, quadratically with the (average) width of the two-pion-mass bins.
We demonstrate this technique with the decay $D^- \to \pi^- \pi^+ \pi^-$, with the S and P waves freed. We expect to find the zero mode of equations (17) and (18). The top-left plot in figure 1 shows the eigenvalue spectrum of the integral matrix. There is one significantly small eigenvalue. The top-right plot in the figure shows the dependence of this eigenvalue and the next-largest one on the average bin width. The smallest eigenvalue quadratically depends on the bin width, but the next-largest one is constant. The bottom-left plot in the figure shows the zero mode formed from the eigenvector alongside that of equations (17) and (18)—the two are nearly indistinguishable\footnote{The shapes of the zero mode also include barrier factors and form factors. If these are put in to an analysis model explicitly, the shapes of the zero mode change accordingly.}. They deviate from each other only in the P-wave high-mass region: At high masses, since the S-wave exact solution varies rapidly, the single value allowed by the step function in each bin is a worse approximation than it is in the low-mass region, where the zero-mode...
varies more slowly. The P-wave steps at high mass deviate from the expected form to compensate for the discrepancy in the S wave. When we reduce the bin widths, this discrepancy disappears.

4 Resolving zero-mode ambiguities

When we fit a freed-isobar PWA model to data, zero modes will contribute to the dynamic amplitudes the fitter finds—the $\omega_{a\beta}$ of equation (10)—with their complex $\tilde{\eta}_z$ taking on arbitrary values as artifacts of the fitting process. We can correct for their presence using knowledge of their shapes and some assumption about the true underlying amplitudes and recover the underlying physical values, the $\alpha_{a\beta}$, that are the goal of an analysis. We refer to the fit that determines the $\omega_{a\beta}$ as the fit to data and further steps (including additional fits) that determine the $\alpha_{a\beta}$ as the zero-mode correction.

There are many possible assumptions we can make about the true underlying physical amplitudes; we describe several examples below, which each assume a model for some part of the freed isobar amplitudes. This gives us expectations for the underlying physical values, which we label $\epsilon_{a\beta}$. Equation (12) tells us that the difference between the underlying value and our expectation is

$$\alpha_{a\beta} - \epsilon_{a\beta} = \omega_{a\beta} - \sum_z \tilde{\eta}_z z_{a\beta} - \epsilon_{a\beta};$$  \hspace{1cm} (23)

since we can learn the $z_{a\beta}$ using the method described in section 3.2, the only unknowns are the $\tilde{\eta}_z$. We can fit for the $\tilde{\eta}_z$ and correct for them, yielding the true underlying values. We do this by minimizing

$$\chi^2 \equiv \sum_{a,\beta,b,\delta} \left( \omega_{a\beta} - \epsilon_{a\beta} - \sum_z \tilde{\eta}_z z_{a\beta} \right) C_{a\beta,b\delta}^{-1} \left( \omega_{b\delta} - \epsilon_{b\delta} - \sum_z \tilde{\eta}_z z_{b\delta} \right),$$  \hspace{1cm} (24)

where $a, \beta, b, \delta$ are as defined for equation (20) and $C_{a\beta,b\delta}$ is the matrix of covariances of the $\omega_{a\beta}$ determined by the fit to data.

This step is not equivalent to having assumed a model from the very start: The model-independent approach has many more degrees of freedom than model-dependent approaches. The zero-mode correction step only reduces the number of degrees of freedom in the analysis by the number of free parameters in equation (24), leaving still much more freedom than in a model-dependent analysis.

4.1 Zero-mode correction examples

We demonstrate zero-mode correction using simulated data of three-pion states produced by D-meson decay and pion–proton scattering. We test three types of assumptions to correct for the zero mode: a model that predicts a value in every two-pion-mass bin in every freed wave with the $\tilde{\eta}_z$ the only free parameters; a model for only a subset of bins, with the $\tilde{\eta}_z$ the only free parameters; and a model with additional free parameters beyond those of the zero mode.

For $D \rightarrow \pi^- \pi^+ \pi^-$, we generated one million events for each study according to a model containing $f_0(980)\pi^-$ in the S wave, modeled using the Flatté lineshape [7], and $\rho(770)\pi^-$ in the P wave, modeled using the relativistic Breit-Wigner lineshape [8]. For the spin-dependent amplitudes, we used the Zemach formalism of [6].
Table 1: Freed waves in $\pi^- p \rightarrow \pi^- \pi^+ \pi^- p$, grouped by three-pion quantum numbers, with presence of a zero mode indicated in the last column.

| $J^{PC}M^\prime$ | $[\pi\pi]_j \pi L$          |     |
|-------------------|-----------------------------|-----|
| $0^{-+}0^+$       | $[\pi\pi]_S \pi S$, $[\pi\pi]_P \pi P$ | ✓   |
| $1^{++}0^+$       | $[\pi\pi]_S \pi P$, $[\pi\pi]_P \pi S$ | ✓   |
| $1^{++}1^+$       | $[\pi\pi]_P \pi S$         |     |
| $2^{-+}0^+$       | $[\pi\pi]_S \pi D$, $[\pi\pi]_P \pi D$, $[\pi\pi]_P \pi F$, $[\pi\pi]_D \pi S$ | ✓   |
| $2^{++}1^+$       | $[\pi\pi]_P \pi D,$ |     |

Our D-decay fit model frees both the S and P waves. The steps in both freed waves are contiguous and cover the full mass range from $2m_\pi$ to $m_D - m_\pi$; they are 20 MeV wide. With freed S and P waves, as we demonstrated above, there will be a zero mode with one complex degree of freedom.

For $\pi^- p \rightarrow \pi^- \pi^+ \pi^- p$, we generated 260,000 events for each study in the three-pion mass range 1.50 GeV–1.54 GeV according to the model used by the COMPASS collaboration in [1] with the parameters they extracted from data. We used the helicity formalism of [9–11]—to be consistent with the analysis in [1]. The COMPASS model contains 88 partial waves. Each is a unique combination of quantum numbers for the three-pion state, a dynamic amplitude model for the two-pion isobar (denoted by $\xi$), and an orbital angular momentum between the isobar and the spectator pion. The three-pion quantum numbers are formulated in the reflectivity basis [12], with reflectivity $\epsilon$. This is all abbreviated as $J^{PC}M^\prime \xi \pi L$.

Our $\pi\pi$-scattering fit model frees the dynamic amplitudes in eleven different waves distinguished by the quantum numbers of the three-pion system; the spin, $j$, of the two-pion isobar, which we write $[\pi\pi]_j$; and the angular momentum between the isobar and the spectator pion. These eleven freed waves, listed in table 1, replace fifteen waves of the COMPASS model: waves with freed $[\pi\pi]_S$ replace those with the $0^{++}$ resonances $f_0(500)$ and $f_0(980)$ (and $f_0(1500)$ in the $0^{-+}0^+$ wave); waves with freed $[\pi\pi]_P$ replace those with the $1^{--}$ resonance $\rho(770)$; and the wave with freed $[\pi\pi]_D$ replaces one with the $2^{++}$ resonance $f_2(1270)$. The remaining 73 waves are included in their model-dependent formulation. Any combination of model-independent and model-dependent waves is possible within the freed-isobar approach; here we have freed the most prominent waves. The two-pion mass bins in each freed wave are contiguous and cover the full mass range from $2m_\pi$ to $m_{3\pi} - m_\pi$; near the regions of the $\rho(770)$ and $f_2(1270)$, they are 20 MeV wide; near the $f_0(980)$, they are 10 MeV wide; and elsewhere they are 40 MeV wide.

Using the technique described in section 3.2, we find the zero modes of this scattering fit model. They do not connect waves with differing initial-state quantum numbers. We find three zero modes in this model—they are indicated in table 1; each has one complex degree of freedom.

Figure 2 shows an example result from fitting this model to simulated data. The fit result is shown in red and the generating model in gray. The left plot shows the intensity as a function of two-pion mass bin; and the right
plot shows the dynamic amplitude in the complex plane. All other plots in this section are identically structured. The freed-isobar shown is that of the $2^{++}$ $\pi\pi$ $P\pi$ D wave, which has no zero mode. The fit result agrees well with the data-generation model.

4.1.1 Complete-model constraint

With our freed D-decay model, we determine the complex dynamic amplitude value in each two-pion mass bin via a fit to simulated data. In figure 3, the red points show the fit results of the S- and P-wave freed waves. The grey lines show the data-generation model. The fit result has peaks corresponding to those of the generating model, but also considerable intensity elsewhere that does not match the generating model.

To correct for the zero mode, we fit for its complex parameter, $\tilde{\eta}$, by minimizing the $\chi^2$ of equation (24). We predict $\epsilon_{ab}$ in all bins of both the S and P waves using a model that contains both the $f_0(980)$ and $\rho(770)$—that is, our original data-generation model. However, for our prediction we change the masses and widths of the resonances by more than the width of a two-pion mass bin.

In figure 3, the green points show the freed waves with the zero mode subtracted given the value of $\tilde{\eta}$ found in the second fit:

$$\alpha_{\beta}^a = \omega_{\beta}^a - \tilde{\eta}\tilde{z}_{\beta}^a. \quad (25)$$

This result very closely resembles the generating model. Though we predicted the $\epsilon_{ab}$ with shifted values for the masses and widths, our final result recovers the correct values. This demonstrates that we do not need detailed and accurate expectations for the zero-mode correction; nor can we coax a result out of the fit via our expectation. This is in contrast to model-dependent PWA, which is very sensitive to the fit model.
Figure 3: Intensities (left) and dynamic amplitudes in the complex plane (right) of the simulated $D^- \rightarrow \pi^- \pi^+ \pi^-$ model (grey), the results of the fit to data ($\omega_a^\beta$, red), and the results after zero-mode correction ($\alpha_a^\beta$, green).

4.1.2 Partial-model constraint

With our freed $\pi p$-scattering model, we determine the complex dynamic amplitude value in each two-pion mass bin via a fit to simulated data. In figure 3, the red points show the fit results of the $0^- + 0^+[\pi p]_S \pi S$ and $0^- + 0^+[\pi p]_P \pi P$ freed waves. The grey lines show the data-generation model. The disagreement between the fit result and the generating model is due to the zero mode.

Again, to correct for the zero mode, we fit for its complex parameter, $\tilde{\eta}$, by minimizing $\chi^2$. Since the zero modes in our freed $\pi p$-scattering model are contained entirely in waves with the same initial-state quantum numbers, we need not assume a model for the entire process—which, with 84 partial waves, is very complicated—but only for the waves in which the zero mode arises. We need only predict $\epsilon_\eta^a$ for the relevant waves, and accordingly only sum over those waves in equation (24). In figure 3, the blue points show the freed waves with the zero mode subtracted given the value of $\tilde{\eta}$ found in the second fit. They agree well with the generating model.
Since the zero mode links the $[\pi\pi]_S$ $\pi$ $S$ wave and the $[\pi\pi]_P$ $\pi$ $P$ wave, it is even enough to fit for $\tilde{\eta}$ in only one of them. We restrict our prediction of $\epsilon^a_\beta$ further to only the mass bins of the $[\pi\pi]_P$ $\pi$ $P$ wave, and accordingly sum only over those bins in equation (24). In figure 4 the green points show both freed waves with the zero mode subtracted given the values of $\tilde{\eta}$ found in a fit to only one wave. The result very closely resembles the generating model and agrees well with the zero-mode correction that used both waves. This demonstrates that one can correct for a zero mode with only a minimal assumption about a model.

4.1.3 Constraints with additional parameters

In both zero-mode corrections above, the only free parameters were the $\tilde{\eta}$—additional parameters used to predict the $\epsilon^a_\beta$, such as masses and widths of resonances in our assumed model, were fixed. But a common goal of PWA is to measure such parameters. It would be superfluous and error prone to first
correct for the zero mode and then fit the zero-mode-corrected results for such parameters. Instead, we should fit for them and the contribution of the zero mode simultaneously.

We demonstrate this with a zero-mode-correction fit to the results in the $0^{-}0^{+} [\pi\pi]_P \pi P$ wave of the $\pi p$-scattering model (as determined, again, from a fit to simulated data). We fit for both $\tilde{\eta}$ and the mass and width of the $\rho(770)$ (in a Breit-Wigner lineshape) that is contained in this wave. We recover a zero-mode corrected result identical to that shown already in figure 4; and we find

$$m_\rho = (766.7 \pm 1.8) \text{ MeV, \quad \text{and} \quad} \Gamma_\rho = (154.6 \pm 4.0) \text{ MeV},$$

which agree within their uncertainties with the simulated values 769.0 MeV and 150.9 MeV.

5 Conclusion

Freed-isobar partial-wave analysis, by using empirical step functions to describe dynamic amplitudes, allows us to overcome the limits of model-dependent analysis. It can be useful for studies of light-meson spectroscopy and heavy-meson and $\tau$ decays. In particular, for CP-asymmetry measurements, freed-isobar PWA could be a robust alternative to the common schemes of measuring asymmetries in bins of phase space. Using the technique, we could instead determine asymmetries in mass bins in distinct projections of the isobar quantum numbers.

Several analyses have used freed-isobar PWA in limited ways, both as a central analysis tool and as a cross check of model-dependent analyses. But more expanded use of the technique—to fit with many freed waves, whether initially or through a bootstrapping procedure—has failed to produce meaningful results. This is due to the presence of zero modes and their arbitrary degrees of freedom.

We have demonstrated how to correct for these zero modes and remove arbitary degrees of freedom using minimal assumptions. And we have provided examples using simulated data of three-pion production via both pion–proton scattering and D-meson decay. Our enhanced freed-isobar PWA techniques may be useful for determining two-body dynamic amplitudes that are consistent across a large variety of strong-interaction problems.

Acknowledgements

We acknowledge the support of the Cluster of Excellence Universe, Exc153, Computational Centre for Astro- and Particle Physics (C2PAP), Transregional Research Centre TR110, all funded by the Deutsche Forschungsgemeinschaft. We would like to thank M. Pennington (Sussex), A. Szepaniak (U. Indiana), J. Pelaez (Madrid), and W. Ochs (MPP) for many discussions in the course of this project.

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