Fuzzy decision implications: interpretation within fuzzy decision context

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Abstract: Fuzzy decision implication is an extension of decision implication in the fuzzy setting, serving to uncover the dependencies of fuzzy attributes. This study presents the interpretation of fuzzy decision implication in the fuzzy decision context. Specially, they will show that from fuzzy decision contexts one can obtain a closed fuzzy set of fuzzy decision implications, and the semantical characteristic of the obtained fuzzy set can be interpreted by fuzzy decision context and can be represented by some operators of fuzzy decision context. Conversely, starting from a fuzzy set of fuzzy decision implications, they can form a fuzzy decision context, from which the given fuzzy set can be derived. The result actually implies that they have constructed a correspondence between closed fuzzy sets of fuzzy decision implications and fuzzy decision contexts, and thus shows the equivalence of two interpretations of fuzzy decision implications.

1 Introduction

Decision-making is of importance in all science-based professions, where specialists apply their knowledge to make valuable decisions. This kind of knowledge can be captured by a decision implication \( A \Rightarrow B \), a term in formal concept analysis [1–5], expressing that when all conditions in \( A \) occur, one should take the decisions in \( B \) [6–10].

Similar to attribute implications [1, 11], the logical study of decision implications can be divided into two parts [7], the semantical aspect and the syntactical aspect. The semantical aspect accounts for the following questions on decision implications:

- The soundness of decision implications: how to determine whether a decision implication is valid?
- Redundancy of decision implications: does there exist a decision implication that can be deduced from the other decision implications?
- Completeness of decision implications: how to obtain a compact set of valid decision implications from the given set of decision implications without loss of information?
- Decision implication basis [8–10]: how to derive a non-redundant complete set of decision implications?

In the syntactical aspect, one starts with a set of decision implications and some inference rules [6, 12], and then deduce new decision implications from the given set by repeatedly applying some inference rules. This process brings forth the following questions concerning the semantical aspect:

- The soundness of inference rules: is any deduced decision implication valid, provided that any decision implication taken from the given set is valid?
- Completeness of inference rules: when the given set is complete, can one obtain all valid decision implications only by repeatedly applying some inference rules?
- Redundancy of inference rules: can one obtain one inference rule from the others?

Considering fuzzy values also existing in real datasets, Zhai et al. [13], Zhai et al. [14] proposed fuzzy decision implication canonical basis and proved that the canonical basis is complete, non-redundant and contains the least number of fuzzy decision implications among all complete sets of fuzzy decision implications, just as canonical basis with respect to attribute implications [1] and decision implication canonical basis with respect to decision implications [8, 9]. Thus, a fuzzy decision implication canonical basis can be regarded as the most compact set of decision information without any information loss.

All the results, however, were obtained in a logical way and are not applied to data tables. Though the logical studies of fuzzy decision implications have provided a deep and theoretical insight into fuzzy decision implications, the studies of fuzzy decision implications with respect to data tables can link the properties of data tables with fuzzy decision implications and provide a practical insight into fuzzy decision implications [15]. Thus, this study aims to provide an interpretation of fuzzy decision implications within data tables, i.e. fuzzy decision contexts. We will show how to derive fuzzy decision implications from a fuzzy decision context and describe how to connect fuzzy decision implications with a fuzzy decision context. The results actually imply the equivalence of two interpretations of fuzzy decision implications, i.e. the logical way and the data-driven way.

This paper will be organised as follows. Section 2 provides an overview of basic notions, in particular, some basic properties of a complete residuated lattice. We will introduce fuzzy decision implications and fuzzy decision contexts in Section 3. Afterwards, we will connect fuzzy decision implications with fuzzy decision contexts in Sections 4 and 5. Section 6 concludes the paper and presents some further work.

2 Preliminaries

A complete residuated lattice [16] is an algebra \((L, \land, \lor, \otimes, \to, 0, 1)\) such that

(i) \((L, \land, \lor, 0, 1)\) is a complete lattice with 0 and 1 being the least and greatest elements of \(L\), respectively;
(ii) \(\otimes\) is commutative, associative, and \(a \otimes 1 = 1 \otimes a = a\), for each \(a \in L\);
(iii) \(\otimes\) and \(\to\) satisfy the adjointness property: \(a \otimes b \leq c\) if and only if \(a \leq b \to c\), for each \(a, b, c \in L\),
A truth-stressing hedge (hedge for short): \(*L \to L*\) such that for each \(a, b \in L, 1^* \leq a, (a \to b)^* \leq a^* \to b^*\) and \(a^{**} = a^*\).

As for hedges, two commonly used ones are

(i) Identity, i.e. for each \(a \in L, a^* = a\).
(ii) Globalisation, i.e.

\[
\begin{align*}
    a^* &= 1 & \text{if } a = 1 \\
    & 0 & \text{otherwise}
\end{align*}
\]

Several properties of the residuated lattice are listed here for later use; for \(a, b, c, y \in L\), we have

\[
\begin{align*}
    a \leq b &\iff a \to b = 1 \quad (1) \\
    a \leq b &\iff a \otimes c \leq a \to c \quad (2) \\
    (a \otimes b) \to c & = (a \to b \to c) \quad (3) \\
    a \to \land y & = \land (a \to y) \quad (4) \\
    1 \to a & = a \quad (5) \\
    a \leq (b \to (a \otimes b)) & \quad (6)
\end{align*}
\]

As a special case of complete residuated lattice, \(L = \{0, 1\}\) is commonly used, with 0 and 1 being the least and greatest elements, respectively, and \(\land\) and \(\lor\) being minimum and maximum, respectively. Several important pairs of adjoint operators are

\[
\begin{align*}
    \text{Łukasiewicz:} & \quad \begin{align*}
        a \otimes b & = \max (a + b - 1, 0) \\
        a \to b & = \min (1 - a + b, 1)
    \end{align*} \\
    \text{Gödel:} & \quad \begin{align*}
        a \otimes b & = \min (a, b) \\
        a \to b & = \begin{cases} 
            1, & \text{if } a \leq b \\
            0, & \text{otherwise}
        \end{cases}
    \end{align*} \\
    \text{Goguen:} & \quad \begin{align*}
        a \otimes b & = a \land b \\
        a \to b & = \begin{cases} 
            1, & \text{if } a \leq b \\
            b, & \text{otherwise}
        \end{cases}
    \end{align*}
\end{align*}
\]

With \(L\) being the structure of truth degrees, we define a fuzzy set \(A\) (or \(L\)-set) as a mapping from a universe \(U\) to \(L, A: U \to L\), whose value \(A(u)\) is the degree to which \(u\) is contained in \(A\). As usual, we also denote a fuzzy set \(A\) by \(\{a_i/u_i, \ldots, a_n/u_n\}\), where \(a_i = A(u_i)\). By \(L^U\), we denote the set of all \(L\)-sets in \(U\). Several useful operators can be defined for \(L\)-sets \(A\) and \(B\) accordingly, such as

\[
(A \cap B)(u) = \min (A(u), B(u))
\]

and

\[
(A \cup B)(u) = \max (A(u), B(u)).
\]

Other notions can, therefore, be adopted based on the two above operators; e.g. we say \(A \subseteq B\) if, for each \(u \in U\), we have \(A(u) \leq B(u)\). For \(A \in L^U\) and \(A \in L^{U'}\), two special \(L\)-sets are given by \(a \otimes A\) and \(a \to A\) with \(a \otimes A(u) = a \otimes A(u)\) and \(a \to A(u) = a \to A(u)\).

As an extension of the classical subset-weak relation \(\subseteq\), the subset-weak degree is given by

\[
S(A, B) = \land_{u \in U} (A(u) \to B(u)).
\]

### 3 Fuzzy decision implications and fuzzy decision context

In this section, we will recall some basic notions and present some results of fuzzy decision implications and a fuzzy decision context \([13, 14]\).

#### 3.1 Fuzzy decision implications

Let \(C, D\) be two finite universes and \(L^C, L^D\) be two systems of \(L\)-sets in \(C\) and \(D\), respectively. A fuzzy decision implication on \(C\) and \(D\) is of the expression \(A \Rightarrow B\), where \(A \in L^C\) and \(B \in L^D\). Here \(A\) is the premise of the implication and \(B\) the consequence of the implication. All fuzzy decision implications on \(C\) and \(D\) are denoted by \(\Rightarrow\).

For a fuzzy set \(T \subseteq L^{C \cup D}\), the degree to which \(T\) respects \(A \Rightarrow B\) is defined by

\[
\|A \Rightarrow B\|_T = S(A, T \cap C) \Rightarrow S(B, T \cap D)
\]

where \(*\) is a hedge. The degree to which \(A \Rightarrow B\) holds in a set \(T = \{T_1, T_2, \ldots, T_n\}\) is defined by

\[
\|A \Rightarrow B\|_T = \land_{T \subseteq T} \|A \Rightarrow B\|_T.
\]

For a fuzzy set \(C\) of fuzzy decision implications, the set of models of \(C\) is given by

\[
\text{Mod}(C) = \{T \subseteq L^{C \cup D} | \forall \psi \in T, \|\psi\|_T \leq \|\psi\|_T \}
\]

where \(\psi\) is the membership degree of \(\psi\) in the fuzzy set \(\psi\). Then the degree to which \(A \Rightarrow B\) semantically follows from \(\psi\) is defined by

\[
\|A \Rightarrow B\|_{\text{Mod}(C)}.
\]

A fuzzy set \(C\) of fuzzy decision implications is closed if, for each \(A \Rightarrow B\), we have \(\|A \Rightarrow B\|_C = \land_{T \subseteq T} \|A \Rightarrow B\|_T\). A fuzzy set \(D \subseteq C\) is complete with respect to a closed fuzzy set \(C\), if we have \(\|A \Rightarrow B\|_D = \|A \Rightarrow B\|_C\), for each \(A \Rightarrow B\). A fuzzy set \(D \subseteq C\) is non-redundant if no proper subset of \(D\) is complete with respect to \(C\).

#### 3.2 Fuzzy decision context

In this subsection, we will introduce a fuzzy decision context as an extension of the decision context under the setting of fuzzy attributes.

**Definition 1:** Given a complete residuated lattice \(L\), a fuzzy decision context is a quadruple \(K = (G, C, D, I)\), where \(G\) is a set of objects, \(C\) is a set of condition attributes, \(D\) is a set of decision attributes, and \(I\) is a set of decision attributes and \(I = I_C \cup I_D\) with \(I_C\) assigning a degree \(I_C(g, m) \in L\) to \(g \in G\) and \(m \in C\), and \(I_D\) assigning a degree \(I_D(g, m) \in L\) to \(g \in G\) and \(m \in D\).

**Example 1:** An illustrative example is presented in Table 1, where

|   | Small | Large | Far | Near |
|---|-------|-------|-----|------|
| Mercury | 1     | 0     | 0   | 1    |
| Venus   | 1     | 0     | 0   | 1    |
| Earth   | 1     | 0     | 0   | 1    |
| Mars    | 1     | 0.5   | 0   | 1    |
| Jupiter | 0     | 1     | 1   | 0.5  |
| Saturn  | 0     | 1     | 1   | 0.5  |
| Uranus  | 0.5   | 0.5   | 0   | 1    |
| Neptune | 0.5   | 0.5   | 0   | 1    |
| Pluto   | 1     | 0     | 0   | 1    |

and entries indicate the degrees to which objects have attributes. For
instance, $I_g$(Uranus, small) = 0.5 says that the degree to which Uranus is small is equal to 0.5, while $I_g$(Uranus, large) = 0.5 says that the degree to which Uranus is large is also equal to 0.5, which means that Uranus is not so small and is not so large either. Given a hedge *, the following notations from [13] are used in the study: for $A \in L^G$ and $m \in C$

\[ A^C(m) = \bigwedge_{g \in G} (A(g)^* \rightarrow I(g, m)) \]

and, for $A \in L^G$ and $m \in D$

\[ A^D(m) = \bigwedge_{g \in G} (A(g)^* \rightarrow I(g, m)) \]

and, for $B \in L^C$ and $g \in G$

\[ B^C(g) = \bigwedge_{m \in C} (B(m)^* \rightarrow I(g, m)) \]

and, for $B \in L^D$ and $g \in G$

\[ B^D(g) = \bigwedge_{m \in D} (B(m)^* \rightarrow I(g, m)). \]

In the sequel, for $g \in G$, we use $C^g \subseteq L^C$ and $D^g \subseteq L^D$ to denote the fuzzy sets $I_1(g, m)$, $m \in C$ and $I_2(g, m)$, $m \in D$, respectively. Now, we introduce **-consistent fuzzy decision contexts.

**Definition 2:** A fuzzy decision context $K$ is **-consistent if for any $g$, $h \in G$, we have $S(C^g, C^h)^* \leq S(D^g, D^h)$, where $*$ is a hedge.

This limitation in Definition 2 ensures that stronger conditions ($S(C^g, C^h)$) will produce stronger consequences ($S(D^g, D^h)$). This will be clearer when Definition 2 degenerates to the crisp case. Recall that in the crisp case, a decision context is consistent [6] if for any $g$, $h \in G$, $C^g = C^h$ implies $D^g = D^h$. Observe that the condition of **-consistent is an extension of consistent. In fact, $C^g = C^h$ yields $S(C^g, C^h) = 1$ and $S(C^g, C^h)^* = 1$, which implies $S(D^g, D^h) = 1$. Thus, we have $D^g \subseteq D^h$, $D^h \supseteq D^g$, and $D^g = D^h$.

**Example 2:** Table 1 is not **-consistent for any hedge * since, for the objects, Mercury and Pluto, we have

\[ S((1/\text{small}, 0/\text{large}), (1/\text{small}, 0/\text{large}))^* = 1 \]

but $S((0/\text{far}, 1/\text{near}), (0/\text{far}, 0/\text{near}) = 0 \rightarrow 1 \land 1 \rightarrow 0 = 0$.

4 From fuzzy decision context to fuzzy decision implication

Now we show how to extract fuzzy decision implications from fuzzy decision contexts.

**Definition 3:** A fuzzy decision implication $A \Rightarrow B$ holds in $K = (G, C, D, I)$ to the degree given by

\[ \|A \Rightarrow B\|_K = \bigwedge_{g \in G} \|A \Rightarrow B\|_G \cup D^K. \]

Obviously, $\|A \Rightarrow B\|_K$ is just the degree to which $A \Rightarrow B$ holds in the set $I_g = [C^g \cup D^g \in G]$. We denote the fuzzy set of fuzzy decision implications by $K(A \Rightarrow B) = \|A \Rightarrow B\|_K$.

Since we have formed the fuzzy set $K$ of fuzzy decision implications in Definition 3, it is natural to apply the notions such as complete, model, and non-redundant to the case of fuzzy decision context, if the fuzzy set $K$ is closed.

**Theorem 1:** The fuzzy set $K$ is closed, i.e. $K(A \Rightarrow B) = \|A \Rightarrow B\|_K$, for any $A \Rightarrow B$.

**Proof:** Observing that $\|A \Rightarrow B\|_K = \bigwedge_{g \in G} \|A \Rightarrow B\|_K$, and $K(A \Rightarrow B) = \|A \Rightarrow B\|_K$ for each $T \in Mod(K)$, we have $K(A \Rightarrow B) = \bigwedge_{g \in G} \|A \Rightarrow B\|_K$. To prove the other inequality $K(A \Rightarrow B) \geq \|A \Rightarrow B\|_K$, we need to prove that

\[ \|A \Rightarrow B\|_K = \bigwedge_{g \in G} \|A \Rightarrow B\|_K \geq \bigwedge_{g \in G} \|A \Rightarrow B\|_K. \]

Thus, it suffices to show $I_g \subseteq Mod(K)$. This is correct because, for each $I_g \subseteq I_g$, we have

\[ K(A \Rightarrow B) = \|A \Rightarrow B\|_K = \bigwedge_{g \in G} \|A \Rightarrow B\|_K \leq \|A \Rightarrow B\|_K \]

where $I_g$ is a model of Mod(K).

By Theorem 1, the fuzzy set $K$ is closed; therefore, all the notations such as complete, model, and non-redundant, which are defined with respect to a closed fuzzy set, can now be applied to $K$ and regarded as the notations with respect to fuzzy decision contexts. Now we say that a fuzzy set $T \in I^C \cup D^K$ is a model of $K$, if for any $A \Rightarrow B$ we have $\|A \Rightarrow B\|_K \leq \|A \Rightarrow B\|_K$. We denote the set of all models of $K$ (i.e. the fuzzy set $K$) by Mod($K$). A fuzzy set $\mathcal{L}$ is complete with respect to $K$ if it is complete with respect to $K$; $\mathcal{L}$ is non-redundant if no proper subset of $\mathcal{L}$ is complete.

A special set of fuzzy decision implications can be formed by using only one row of fuzzy decision context one time, i.e. $I = \{ C^g \Rightarrow D^h | g \in G \}$. At first glance, all of the fuzzy decision implications from $I$ should fully hold in $K$, i.e. $K(C^g \Rightarrow D^h) = 1$. However, it is not true in general.

**Example 3:** Take the object, Mercury in Table 1, as a counterexample. For $h = Pluto$, we have

\[ \|1/\text{small}, 0/\text{large} \Rightarrow 1/\text{far}, 0/\text{near}\|_K = \bigwedge_{I_g \subseteq I_g} \|1/\text{small}, 0/\text{large} \Rightarrow 1/\text{far}, 0/\text{near}\|_I \]

\[ \leq \|1/\text{small}, 0/\text{large} \Rightarrow 1/\text{far}, 0/\text{near}\|_I \]

\[ = S((0/\text{far}, 1/\text{near}), (0/\text{far}, 0/\text{near})) = 0 \]

which means that the fuzzy decision implication \{1/\text{small}, 0/\text{large} \Rightarrow 1/\text{far}, 0/\text{near}\} does not hold in Table 1.

In fact, for the general case, we have the following result.

**Theorem 2:** $I \subseteq K$ if and only if $K$ is *-consistent.

**Proof:** It is easily seen that $I \subseteq K$ if and only if $K(C^g \Rightarrow D^h) = 1$ for each $g \in G$. Now we have

\[ K(C^g \Rightarrow D^h) = 1 \]

\[ \Leftrightarrow \|C^g \Rightarrow D^h\|_K = 1 \]

\[ \Leftrightarrow \bigwedge_{I_g \subseteq I_g} \|C^g \Rightarrow D^h\|_I = 1 \]

\[ \Leftrightarrow \|C^g \Rightarrow D^h\|_I = 1, \text{ for } I_g \subseteq I_g \]

\[ \Leftrightarrow S(C^g, C^h)^* \leq S(D^g, D^h), \text{ for } I_g \subseteq I_g. \]

The last inequality is just the definition of *-consistent.
Now we recall the notion of ‘unite closure’, which was first proposed in [13] and plays an important role in the semantical structures of decision implications [7], fuzzy decision implications [13], and variable decision implications [17].

For a fuzzy set \( L \) of fuzzy decision implications and an L-set \( A \in L^C \), the closure of \( A \) with respect to \( L \) is an L-set defined by

\[
  A^C = \bigcup \{ L(A_i \Rightarrow B_i) \otimes S(A_i, A^*) \otimes B_i | A_i \in L^C, B_i \in L^D \}.
\]

The unite closure of \( A \) with respect to \( L \) is given by \( A \cup A^C \). Closure and unite closure have the following properties [13].

**Lemma 1:** For any L-set \( A \) and fuzzy set \( L \) of fuzzy decision implications, we have

(i) \( A \cup A^C \in \text{Mod}(L) \);
(ii) \( \|A \Rightarrow A^C\|_L = 1 \)

The first result of Lemma 1 shows that for each given L-set \( A \in L^C \), we can obtain a model of \( L \), which is just the unite closure of \( A \), whereas the second says that each fuzzy decision implication with the form \( A \Rightarrow A^C \) can be fully followed from \( L \).

Concerning the fuzzy decision context, we now obtain the following result.

**Theorem 3:** For any \( A \in L^C \), we have \( A^C = A^D \).

**Proof:** For \( A \in L^C \), by the properties of resitted lattice, we have

\[
  A^K \subseteq A^CD
\]

\[
\Leftrightarrow \forall m \in D, (A^K(m) \rightarrow A^CD(m)) = 1 \]

\[
\Leftrightarrow \forall m \in D, (A^K(m) \rightarrow (\bigwedge_{g \in E} A^g(m)) = 1 \]

\[
\Leftrightarrow \forall m \in D, (A^K(m) \rightarrow (A^K(m) \rightarrow (g(m)))) = 1 \]

\[
\Leftrightarrow \forall m \in D, (A^K(m) \rightarrow (g(m))) = 1 \]

\[
\Leftrightarrow \forall m \in D, (A^K(m) \rightarrow (A^K(m))) = 1 \]

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\Leftrightarrow \forall m \in D, (A^K(m) \rightarrow (A^K(m))) = 1 \]

\[
\Leftrightarrow \forall m \in D, (A^K(m) \rightarrow (A^K(m))) = 1 \]

The last equation is true by Theorem 1 and (2) of Lemma 1. On the other hand, we have \( A^K \subseteq A^CD \) by the literature [16], i.e., \( (\bigwedge_{g \in E} A^g(m)) \rightarrow A^K(m) \subseteq A^K(m) \rightarrow (A^CD(m)) \) for any \( g \in G \), which equals \( S(A^K, C^K) \subseteq S(A^K, D^K) \). Thus, we have \( \|A \Rightarrow A^CD\|_K = 1 \) for any \( g \in G \), which implies \( \Leftrightarrow \forall m \in D, (A^K(m) \rightarrow A^K(m)) = 1 \). We know that \( A \cup A^C \) is a model of \( K \) by Lemma 1, i.e., \( \forall A_i \Rightarrow B_i \), implies \( \forall A_i \Rightarrow B_i \). Particularly, \( K(A \Rightarrow B) \leq \|A \Rightarrow B\|_{K_{L \Rightarrow D}} \), i.e., \( 1 = \forall m \in D, (A^K(m) \rightarrow A^K(m)) \leq \|A \Rightarrow B\|_{K_{L \Rightarrow D}} = S(A^K, A^K) \rightarrow S(A^K, D^K) \). Thus, we have \( S(A^K, D^K) \subseteq A^K \) and then \( A^CD \subseteq A^K \), which completes the proof.

This is the main result of the study, serving to connect fuzzy sets of fuzzy decision implications with fuzzy decision contexts, and showing that one can obtain closure and unite closure not only by means of the fuzzy set of fuzzy decision implications but also by using the operators \( C \) and \( D \) in fuzzy decision context. In other words, given a fuzzy decision context \( K \) and an L-set \( A \), one can compute \( A^K \) by the operators \( C \) and \( D \) in \( K \); one can also achieve this by (i) extracting all the fuzzy decision implications from \( K \) and forming the fuzzy set \( K \), and (ii) computing the maximal consequence \( A^K \) of premise \( A \) according to \( K \). This is correct since Theorem 3 shows that the maximal consequence \( A^K \) is equal to \( A^CD \).

By Theorem 3, many results from the logical study of fuzzy decision implications can be transferred to the data-driven study. For example, the following result shows how to represent models by means of the operators \( C \) and \( D \).

**Theorem 4:** Let \( K \) be a fuzzy decision context. Then we have

\[
\text{Mod}(K) = \{ A \cup \bar{A} | A \in L^C, \bar{A} \in L^D \text{ and } A^CD \subseteq \bar{A} \}.
\]

**Proof:** By Theorem 3, we only need to show

\[
\text{Mod}(K) = \{ A \cup \bar{A} | A \in L^C, \bar{A} \in L^D \text{ and } A^K \subseteq \bar{A} \}.
\]

By Lemma 1, it is easy to see that \( \text{Mod}(K) \supseteq \{ A \cup \bar{A} | A \in L^C \} \). Furthermore, for any \( A \rightarrow B \), we have \( S(B, A^K) \leq S(B, A) \) since \( A^K \subseteq A \), which yields \( \|A \Rightarrow B\|_{K_{L \Rightarrow D}} \leq \|A \Rightarrow B\|_{K_{L \Rightarrow D}} \), i.e. \( A \cup \bar{A} \in \text{Mod}(K) \).

Conversely, let \( T \in \text{Mod}(K) \). Then, since we have \( K(T \cap C) \rightarrow (T \cap C) \leq 1 \) by Theorem 1 and Lemma 1, and since \( T \) is a model of \( K \), we obtain \( K(T \cap C) \rightarrow (T \cap C) \leq 1 \) by Theorem 1 and Lemma 1, which yields \( \|T \cap C \Rightarrow (T \cap C)\|_K = 1 \), i.e. \( (T \cap C, T \cap C) \rightarrow S(T \cap C, T \cap D) \in \{S(T \cap C, T \cap D) = 1 \), \( T \cap D) \). Thus, we obtain \( (T \cap C) \subseteq \bar{A} \subseteq T \subseteq \bar{A} \).

This completes the proof.

Theorem 4 shows that the set of models of \( K \) can be represented by the operators \( C \) and \( D \). This result also implies that for any \( A \in L^C \), \( A \rightarrow A^K \) is the least set to make \( A \cup \bar{A} \) be a model of \( K \).

In the case of \( * \)-consistent, the unite closures of the special fuzzy sets, \( \{C^g | g \in G \} \), have simpler forms.

**Theorem 5:** For a \( * \)-consistent fuzzy decision context and \( g \in G \), we have \( (C^g)^K = (C^g)^{CD} = D^g \).

**Proof:** By Theorem 3, it suffices to show \( (C^g)^K = D^g \). By definition of \( K \), we have \( (C^g)^K \subseteq \cup_{A \in B}(K(A \Rightarrow B) \otimes S(A, C^g)^* \otimes B) \supseteq K(C^g) \subseteq (D^g) \otimes (S(C^g, C^g)^* \otimes D^g) \). By Theorem 2, we know \( K(C^g) \subseteq (D^g) \). Thus and (C^g)^K \subseteq K(C^g) \subseteq (D^g) \otimes (S(C^g, C^g)^* \otimes D^g) \).

On the other hand, by the definition of \( C^g \), we have \( (C^g)^K \subseteq D^g \) if and only if \( \cup_{A \in B}(K(A \Rightarrow B) \otimes S(A, C^g)^* \otimes B) \subseteq (D^g) \), if and only if for any \( A \in B \) and \( B \in D \), \( K(A \Rightarrow B) \otimes S(A, C^g)^* \otimes B) \subseteq (D^g) \), if and only if only if \( K(A \Rightarrow B) \otimes S(A, C^g)^* \otimes B) \subseteq (D^g) \). Thus, we need to prove that \( K(A \Rightarrow B) \otimes S(A, C^g)^* \otimes A(K(A \Rightarrow B)) \).
Lemma 4: $\text{mod}(D)$ with respect to a closed set. For example, by Lemma 2, it is easy to check, because for each $g \in G$, $(C^g)^c = (C^g)^D = D$. Then $\text{mod}(D)$, the closures of the fuzzy sets $\{D^g\}$, $g \in G$.

5 From fuzzy decision implication to the fuzzy decision context

We have shown how to obtain a closed fuzzy set of fuzzy decision implications from a fuzzy decision context. In this section, we want to form a fuzzy decision context from a given fuzzy set of fuzzy decision implications and prove that the obtained fuzzy decision context has the same closed fuzzy set of fuzzy decision implications as that of the given fuzzy set. By doing this, one can study the fuzzy decision implications by studying the corresponding fuzzy decision context.

A sufficient and necessary condition for completeness based on the set of models is first adopted from [13] for later use.

Lemma 2: Let $L$ be a closed fuzzy set of fuzzy decision implications. Then $D \subseteq L$ is complete if and only if $\text{mod}(D) = \text{mod}(L)$.

Lemma 2 provides a convenient way to check if a subset is complete with respect to a closed set. For example, by Lemma 2, it is easy to see that for any complete fuzzy sets $D_1$, $D_2$, we have $\text{mod}(D_1 \cap D_2) = \text{mod}(D_1 \cap D_2)$.

Lemma 3: Let $L$ be a fuzzy set of fuzzy decision implications and set $\text{mod}(D \rightarrow B) = \|D \rightarrow B\|_L$. Then $L$ is a closed fuzzy set and $L$ is complete with respect to $L$.

Lemma 3 [13] shows that one needs only one step to get a closed fuzzy set of fuzzy decision implications, i.e. setting $\text{mod}(D \rightarrow B) = \|D \rightarrow B\|_L$, moreover, the closed set given is complete with respect to the closed fuzzy set obtained.

Now, for a fuzzy set $L$ of fuzzy decision implications on $C$ and $D$, we denote

$$D = \{A \Rightarrow A^c | A \in L^c\}.$$  

Example 4: ([14]): Let $C = \{x, y\}$, $D = \{z\}$ and $L = \{0, 0.5, 1\}$. Then $L$ is complete with respect to $L$.

$$L = \{1/0.5/x, 1/y\} \Rightarrow 0.5/z, 1/1/x, 0.5/y\} \Rightarrow 1/z\}.$$  

one can compute the unite closures for each $A \in L^c$ and list the set $D$ as follows:

$$D = \{0/x, 0/y\} \Rightarrow 0z, \{0/x, 0.5/y\} \Rightarrow 0z, \{0.5/x, 0/y\} \Rightarrow 0z, \{0.5/x, 0.5/y\} \Rightarrow 0z, \{0.5/x, 1/y\} \Rightarrow 0.5/z, \{1/x, 0.5/y\} \Rightarrow 0z, \{1/x, 1/y\} \Rightarrow 1/z\}.$$  

Lemma 4: $D$ is complete with respect to $L$.

By Lemma 4 [13], $D$ is a complete subset of $L$; in other words, one can use the subset $D$ in place of $L$ while keeping all the information needed.

Thus, given a fuzzy set $L$ of fuzzy decision implications, by the complete set $\triangleleft D$, we can form a fuzzy decision context $K_D = (G_D, C_D, D)$ as follows:

- $g \in G_D$ if and only if there exists $A \Rightarrow A^c \in D$ with $L_{c}(g, m) = l_{c}(m) = A(m)$ for $m \in C$ and $I_{c}(g, m) = A^c(m)$ for $m \in D$.

Theorem 6: $K_D$ is $*$-consistent.

Proof: According to the definition of $K_D$, we only need to prove that $\text{mod}(A_1, A_2)^c \leq \text{mod}(A_1^c, A_2^c)$ for any $A, A_2 \in L^c$. In fact, we have

$$\text{mod}(A_1, A_2)^c \leq \text{mod}(A_1^c, A_2^c) \Rightarrow \text{mod}(A_1, A_2)^c \leq \text{mod}(A_1^c, A_2^c) \Rightarrow \text{mod}(A_1, A_2)^c \leq \text{mod}(A_1^c, A_2^c).$$

Since $A_1^c(m) = \forall_{A \Rightarrow B} (L(A \Rightarrow B) \Rightarrow (S(A, A^c) \Rightarrow B(m)))$ and

$$A_2^c(m) = \forall_{A \Rightarrow B} (L(A \Rightarrow B) \Rightarrow (S(A, A^c) \Rightarrow B(m))),$$

we only need to prove $S(A_1, A_2)^c \leq \text{mod}(A_1^c, A_2^c) \Rightarrow \text{mod}(A_1^c, A_2^c) \Rightarrow \text{mod}(A_1^c, A_2^c) \Rightarrow \text{mod}(A_1^c, A_2^c)$. This is true since, by (6), $A_1^c(m) = \text{mod}(A_1, A_2)^c \Rightarrow \text{mod}(A_1^c, A_2^c) \Rightarrow \text{mod}(A_1^c, A_2^c).$

Theorem 6 shows that the formed fuzzy decision context $K_D$ is $*$-consistent. Thus, by Theorem 2, all the fuzzy decision implications in $L$ hold in $K_D$; in fact, by the construction process of $K_D$, we have $L = \triangleleft D$. By Theorem 5, we have that for each $g \in G$, $(C^g)^c = (C^g)^D = D$. This is also easy to check, because for each $g \in G$, by the construction process of $K_D$, there exists $A \Rightarrow A^c \in D$ such that $A = C^g$ and $A^c = D^g$, and by Theorem 3, we obtain $(\triangleleft D)^c = A^c = D^g = \triangleleft D$. As a result, we can derive the closed fuzzy set $L$ from the obtain fuzzy decision context.

Theorem 7: We have $K_D = L$, where $L$ is given by Lemma 3.

Proof: By Theorem 3, if one can show that $A^D = A^c$ for any $A \in L^c$, then it follows that $A^D = A^c$, which yields $\text{mod}(K_D) = \text{mod}(L)$ by Theorem 4. Moreover, since both $K_D$ and $L$ are closed, by Lemma 2 we have $K_D = L$.

Now we show that $A^D = A^c$ for any $A \in L^c$. For $m \in D$, we reframe $A^D(m)$ as follows:

$$A^D(m) = \lambda_{g \in G_D} (\lambda_{x,y} (A_1(x, y) \Rightarrow I(g, m)))$$

$$= \lambda_{g \in G_D} (\lambda_{x,y} (A_1(n) \Rightarrow I(g, n) \Rightarrow I(g, m)))$$

From the definition of $K_D$, it follows that there exists $h \in G_D$ such that $C^g = A$ and $D^c = A^c$, so we know

$$A^D(m) = S(A, A)^c \Rightarrow I(h, m) \Lambda \{S(A, C^g)^c \Rightarrow I(g, m) | g \in G_D, g \neq h\} \leq S(A, A)^c \Rightarrow I(h, m) = A^c(m)$$

which implies $A^D \subseteq A^c$.

To prove $A^D \supseteq A^c$, i.e. $A^D(m) \geq A^c(m)$, it suffices to show that $A^c(m) \leq \lambda_{g \in G_D} (\lambda_{x,y} (A_1(x, y) \Rightarrow I(g, m)) | g \in G_D, g \neq h)$. Now
To the closed fuzzy set $\overline{L}$ of $L$ (Theorem 7). This implies that the fuzzy decision context obtained preserves all information from the given fuzzy set $L$ and that one can study the fuzzy decision context instead of the given fuzzy set of fuzzy decision implications.

### 6 Conclusion and further work

The study intended to interpret fuzzy decision implications within fuzzy decision contexts. Thus, one can extract a fuzzy set of fuzzy decision implications from a fuzzy decision context and form a fuzzy decision context from the given fuzzy set of fuzzy decision implications. This actually establishes a correspondence between fuzzy decision contexts and fuzzy sets of fuzzy decision implications and furthermore implies the equivalence of the logic way and the data-driven way of interpreting fuzzy decision implications.

Important further works include: (i) using fuzzy concept lattice [16] to analyze the relationship between condition sub-context and decision sub-context; (ii) clarifying the effects of various hedges on the connections between fuzzy decision implications and fuzzy decision contexts; (iii) considering the data-driven way in variable decision implications [17].

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**Table 2** Fuzzy decision context $K_{D}$

| $x$ | $y$ | $z$ |
|-----|-----|-----|
| $g_1$ | 0   | 0   | 0   |
| $g_2$ | 0   | 0.5 | 0   |
| $g_3$ | 0   | 1   | 0   |
| $g_4$ | 0.5 | 0   | 0.5 |
| $g_5$ | 0.5 | 0.5 | 0.5 |
| $g_6$ | 0.5 | 1   | 0.5 |
| $g_7$ | 1   | 0.5 | 0   |
| $g_8$ | 1   | 0.5 | 1   |
| $g_9$ | 1   | 1   | 1   |

We show that for each $g \in G_{D}$, $A^{*}(m) \leq S(A, C^{y}) \rightarrow I(g, m)$. In fact, for $g \in G_{D}$, there exists $B \Rightarrow B^{*} \in D$ such that $C^{y} = B$ and $D^{*} = B^{*}$; thus we need to show that $A^{*}(m) \leq S(A, B^{*}) \rightarrow B^{*}(m)$. Then, by definition of $\overline{L}$, we have

$$B^{*}(m) = \forall L(A_{1} \Rightarrow B_{1}) \otimes S(A_{1}, B) \otimes B_{1}(m)$$

$$\leq L(A \Rightarrow A^{*}) \otimes S(A, B^{*}) \otimes A^{*}(m),$$

and thus

$$S(A, B^{*}) \rightarrow B^{*}(m) \leq S(A, B) \rightarrow (L(A \Rightarrow A^{*}) \otimes S(A, B^{*}) \otimes A^{*}(m)).$$

Thus, we need to prove

$$A^{*}(m) \leq S(A, B^{*}) \rightarrow (L(A \Rightarrow A^{*}) \otimes S(A, B^{*}) \otimes A^{*}(m))$$

i.e. $A^{*}(m) \leq S(A, B^{*}) \rightarrow (L(A \Rightarrow A^{*}) \otimes S(A, B^{*}) \otimes A^{*}(m))$. This is true since, by (6), we have $S(A, B^{*}) \rightarrow (S(A, B^{*}) \otimes A^{*}(m)) \leq A^{*}(m)$. Thus we have $K_{D} = L$. □

Example 5: (continued Example 4): By the set $D$, one can form the corresponding fuzzy decision context $K_{D} = (G_{D}, C, D, I_{D})$, as shown in Table 2, each row of which is generated by one of the fuzzy decision implications in $D$.

Now, by Theorem 6, $K_{D}$ is *-consistent, where * is the identity hedge because $D$ is computed by using Lukasiewicz adjoint pair and identity hedge. This result can also be checked by the definition of *-consistent. For example, for the objects $g_3$ and $g_4$, we have

$$S(C^{y}, C^{x}) = ((0 \rightarrow 0.5) \wedge (1 \rightarrow 0)) = (0 \leftrightarrow 0) = 0$$

$$L(D^{0}, D^{y}) = 0 \leq S(C^{y}, C^{x}) \leq (0 \rightarrow 0.5) \wedge (1 \rightarrow 0)$$. Then, we have

$$K_{D}(A \Rightarrow B) = L(A \Rightarrow B)_{K_{D}} = \bigwedge_{g \in G} L(A \Rightarrow B)_{K_{D}, D} = 0.5 \otimes 0.5 \otimes 0.5 \otimes 1 \otimes 0.5 \otimes 1 \otimes 1 \otimes 1 = 0.5.$$