Cavity broadcasting via Raman scattering

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Abstract

Abstract. The notion of broadcasting is extended to include the case where an arbitrary input density state of a two-mode radiation field gives rise to an output state with identical marginal states for the respective modes, albeit different from the input state. The initial unknown input density state is unitarily related to the output state but is not equal to the two identical output marginal states. This extended notion of broadcasting suggests a possible way of discriminating between two noncommuting quantum states.
1. Introduction

The no-cloning theorem for pure states, discovered in the early 1980s, is one of the earliest results of quantum computation and quantum information. The no-cloning theorem states that it is impossible to make a copy of an unknown, pure quantum state \[1\]. The theorem places a limit on the ability to manipulate quantum information. It follows from the no-cloning theorem that no copying machine can make perfect copies of all incoming states, if they are not all orthogonal to each other. However, the whole idea of quantum cloning is to produce at the output of the cloning machine two identical pure states, which are unitarily connected with the input pure state. This less restrictive condition for cloning was recently realized in the exact cloning of nonorthogonal coherent states in a two-mode cavity via Raman scattering \[2\].

The no-cloning theorem for pure states has been generalized and extended to the case of mixed states \[3\]. The notion of broadcast is introduced to include the possibility that although an arbitrary mixed state may not be cloned nevertheless it may be marginally reproduced. Thus cloning of mixed states would represent a strong form of broadcasting. It should be remarked that the notion of cloning or broadcasting considered \[3\] refers to the case where the cloned or broadcasted output states are precisely the same as the input state to be cloned or broadcasted.

2. Raman cavity dynamics for pure states

In Ref. \[2\], the dynamics of a two-level atom, singly injected into a two-mode, high-Q cavity was studied. The atom-fields interaction is given by the Raman-coupled Hamiltonian \(H_{\text{int}}\). The time evolution of an arbitrary state of the atom-fields system is determined in the interaction picture by \(|\Psi(t)\rangle = U(t)|\Psi\rangle\), where \(U(t) = e^{-iH_{\text{int}}t/\hbar}\). Steady states can be achieved inside the cavity by singly injecting atoms that are in a coherent superposition of their upper and lower states, where the admixture of the atomic states \(|1\rangle\) and \(|2\rangle\) represents the tunable parameter of the experiment. The successive iterations give the following reduced fields density matrix \(\rho_l\) after \(l\) such atoms have singly traversed the cavity

\[
\rho_l = \text{tr}_a[U(\tau)\rho_{l-1}\rho_aU^\dagger(\tau)],
\]

where the trace is over the atomic states \(\rho_a = (\alpha|1\rangle + \lambda|2\rangle)(\langle 1|\alpha^* + \langle 2|\beta^*)\), which is the state of the injected atoms, and \(\tau\) is the interaction time. If the limit of the iterations
exists, then the resulting state is called a steady state, which is a fixed point of the map (1).

The state

$$\Psi_{\text{steady}} = \sum_{n_1, n_2=0}^{\infty} C_{n_1, n_2} |n_1, n_2 > \otimes (\alpha|1> + \beta|2>),$$

(2)

where

$$C_{n_1, n_2} = \left(-\frac{\alpha}{\beta r}\right)^{n_2} \sqrt{\frac{(n_1 + n_2)!}{n_1! n_2!}} C_{n_1+n_2,0},$$

(3)

is a fixed point of the map (1) and so constitutes a steady state [2].

The dynamics governed by the Raman Hamiltonian conserves the total number of photons in the two modes. Accordingly, if the cavity fields in the two modes are initially disentangled with field mode one in a coherent state $|\gamma>\equiv |\gamma_1>\otimes |\gamma_2>\otimes |\gamma_3>$ and field mode two in the vacuum state $|0>$, then the steady state of the electromagnetic fields inside the cavity is a two-mode coherent state and the effect of atom-fields interactions inside the cavity is to give rise to the following relationship between the initial input state and the output steady state [2]

$$|\gamma_1>\otimes |0>_2 \rightarrow \left|\frac{\gamma}{\sqrt{1 + |\alpha/\beta|^2}}\right>_1 \otimes \left|\frac{-\alpha/\beta \gamma}{\sqrt{1 + |\alpha/\beta|^2}}\right>_2,$$

(4)

where the subscripts indicate the modes of the radiation fields and $r$ is the ration of the atom-photon coupling constants.

The cavity transformation (4) is effected by the unitary, quantum beam splitter operator

$$S(\lambda) = e^{\lambda a_1^+ a_2 - \lambda^* a_2^+ a_1},$$

(5)

where $\lambda = |\theta|e^{-i\phi}$ with $\alpha/\beta r = e^{i\phi}\tan|\theta|$ and $a_i^+ (a_i), i = 1, 2$, represents the creation (annihilation) operators for the respective fields. The cloning of the coherent state $|\gamma>$ is achieved with input state $|\sqrt{2}\gamma>_1 \otimes |0>_2$ when the tuning parameter $\alpha/\beta r = -1$. This value for the tuning parameter corresponds to a 50/50 quantum beam splitter. There is an important distinction between an optical 50/50 beam splitter and the quantum beam splitter generated by the cavity transformation (4). Both beam splitters, effected by operator (5), conserve the total number of photons between the input and output states. The optical beam splitter conserves energy owing to the equality of the input and output photon frequencies. However, in the cavity transformation (4), modes one and two have, in general, different frequencies and so one must consider the energy gained or lost by the atoms traversing the cavity in order to conserve the total energy. Transformation (4) resembles a parametric down-conversion whereby the fraction $|\alpha \beta^{-1} r^{-1}|^2/(1 + |\alpha \beta^{-1} r^{-1}|^2)$ of
photons in the input coherent state with the higher frequency \( \omega_1 \) are converted into photons in the output coherent state with the lower frequency \( \omega_2 \). Total energy conservation is established if the energy of the traversing atoms are taken into account since the traversing atoms make transitions from the atomic ground state \( |1\rangle \) to the atomic excited state \( |2\rangle \) and vice versa via Raman scattering. Optimal universal cloning can be realized using parametric down-conversion and also via stimulated emission [4]. Our quantum beam splitter also resembles a “parametric up-conversion” when the initial input state is given by \( |0\rangle_1 \otimes |\gamma\rangle_2 \). The fraction \( \frac{1}{1 + |\alpha\beta r|^{-2}} \) of photons in the input state at the lower frequency \( \omega_2 \) are converted in the output state into photons with the higher frequency \( \omega_1 \).

3. Raman cavity dynamics for mixed states

Consider the general case where initially inside the cavity mode one is in an arbitrary mixed state while mode two is in the vacuum state. One can write the initial state \( \rho_i \) inside the cavity in the coherent state representation and so

\[
\rho_i = \int d^2\gamma \ P(\gamma, \gamma^*) |\gamma > \langle \gamma| \otimes |0 >\langle 0|,
\]

where \( P(\gamma, \gamma^*) \) is some real function of \( \gamma \) and \( \gamma^* \) which is normalized to unity and the subscripts indicate the modes of the radiation field.

The resulting steady state or equivalently the state \( \rho_s = S(\lambda)\rho_i S^\dagger(\lambda) \) is

\[
\rho_s = \int d^2\gamma \ P(\gamma, \gamma^*) \left| \frac{\gamma}{\sqrt{1 + |\alpha r\gamma|^2}} \right\rangle \left\langle \frac{\gamma}{\sqrt{1 + |\alpha r\gamma|^2}} \right| \otimes \left| \frac{-\alpha \beta r \gamma}{\sqrt{1 + |\alpha \beta r\gamma|^2}} \right\rangle \left\langle \frac{-\alpha \beta r \gamma}{\sqrt{1 + |\alpha \beta r\gamma|^2}} \right|,
\]

where the corresponding marginal density operators for modes one and two are given by

\[
\rho_1 = \int d^2\gamma \ P(\gamma, \gamma^*) \left| \frac{\gamma}{\sqrt{1 + |\alpha r\gamma|^2}} \right\rangle \left\langle \frac{\gamma}{\sqrt{1 + |\alpha r\gamma|^2}} \right|
\]

and

\[
\rho_2 = \int d^2\gamma \ P(\gamma, \gamma^*) \left| \frac{-\alpha \beta r \gamma}{\sqrt{1 + |\alpha \beta r\gamma|^2}} \right\rangle \left\langle \frac{-\alpha \beta r \gamma}{\sqrt{1 + |\alpha \beta r\gamma|^2}} \right|,
\]

respectively. If the tuning parameter \( \alpha/\beta r = -1 \), then the arbitrary input state \( \int d^2\gamma \ P(\gamma, \gamma^*) |\gamma > \langle \gamma| \) has an output state where both modes of the field are in the same,
precise marginal state given by
\[ \rho_b = \int d^2 \gamma \, P(\gamma, \gamma^*) |\gamma\rangle <\langle \gamma|/\sqrt{2} \langle \gamma| \sqrt{2} |\gamma\rangle. \] (10)

This result indicates that arbitrary states may be broadcasted, where the reduced density matrices of the two field modes of the output state are identical but differ from the input state and are unitarily connected to it. Therefore, if the arbitrary state \[ \int d^2 \gamma \, P(\gamma, \gamma^*) |\gamma\rangle <\langle \gamma| \]
is to be broadcasted, then the cavity must have one mode initially in the vacuum state and the other in the state \[ \frac{1}{2} \int d^2 \gamma \, P(\gamma, \gamma^*) |\gamma\rangle <\langle \gamma|. \]

Note, however, that the input state \[ |\sqrt{2}\gamma\rangle_1 \otimes |0\rangle_2 \]
is the only initial state [2] that can lead to a disentangled pure output state, viz. \[ |\gamma\rangle_1 \otimes |\gamma\rangle_2, \] and thus to the exact cloning of \[ |\gamma\rangle. \] This is achieved for \[ P(\gamma', \gamma'^*) = \delta^2 (\gamma' - \sqrt{2}\gamma). \]

It is important to remark that in a Gaussian quantum-cloning machine [5], which can be viewed as the continuous counterpart of the universal qubit cloner, one has to prepare an initial coherent state \[ |\alpha\rangle \] as input and its two outputs are a mixture of coherent states characterized by a density matrix with fidelity 2/3. In the present work, instead, one has to prepare the initial coherent state \[ |\sqrt{2}\alpha\rangle_1 \] in order to produce the pure output state \[ |\alpha\rangle_1 \otimes |\alpha\rangle_2. \] The universal cloner gives rise to a mixed output state in the cloning of a coherent state while the present cloner gives rise to two identical coherent states as output. Our cloning machine is a “universal cloning machine” for coherent states since the cloning process is input independent.

The mean electric-field strength in a coherent state \[ |\xi\rangle \] looks like the electric-field strength of a coherent, classical radiation with mode amplitude \[ \xi. \] In addition, the relative noise of the electric-field strength in the coherent state \[ |\xi\rangle \] is inversely proportional to \[ |\xi| \] and so the relative noise decreases for increasing mean photon number. Accordingly, coherent states \[ |\xi\rangle \] with larger mean photon numbers are closer to their classical, coherent wave counterpart. The laser field well above threshold is described as being in a coherent state. The mean laser intensity in the steady-state grows in proportion to the pump parameter while the relative mean squared light intensity fluctuation goes down with increasing value of the pump parameter. The quantum state of the laser field can be close to a coherent state, where the photon occupation number can be exceedingly large and the radiation produced by the laser comes close to being classical. Accordingly, coherent states \[ |\xi\rangle \] with large mean photon numbers are closer to their classical, coherent wave counterpart and thus are easier
to produce. A weak coherent state can be obtained in turn from a more intense coherent state [6].

Note that Eq. (6) need not represent a classically correlated density matrix [7]. The real function $P(\gamma, \gamma^*)$ may assume negative values and be more singular than a delta function and thus possess no classical analog. Note also that even if $P(\gamma, \gamma^*)$ behaved like a true probability density, i.e., it is non-negative and not more singular than a delta function, it would still not describe probabilities owing to the non-orthogonality of the coherent states.

4. Non-orthogonal mixed state discrimination

The results presented here also suggest a possible way to achieve a probabilistic error-free discrimination, that is, one which sometimes fails, but when successful never gives an erroneous result. This procedure is referred to as an unambiguous discrimination [8]. If the initial input state is the arbitrary state (6), then the output reduced density matrix of mode one is obtained from (7) and so

$$\rho_A = \int d^2\gamma P(\gamma, \gamma^*)|A\gamma >_1<A\gamma|,$$

where

$$A = \frac{1}{\sqrt{1 + |\alpha/\beta|^2}}.$$  

(11)

Similarly, for the initial state

$$\sigma_i = \int d^2\lambda Q(\lambda, \lambda^*)|\lambda >_1<\lambda| \otimes |0 >_2<0|,$$

with output reduced density matrix

$$\sigma_A = \int d^2\lambda Q(\lambda, \lambda^*)|A\gamma >_1<A\gamma|.$$  

(13)

(14)

In the limit $|\alpha/\beta| \to \infty$, i.e., when the input atoms are essentially in the atomic ground state $|1 >$, $A \to 0$ and so the contributions to $\rho_A$ and $\sigma_A$ come mainly from the vacuum $|0 >$ and the one photon state $|1 >$ hence

$$\rho_A = \left[1 - A^2 < |\gamma|^2 > \right] |0 > < 0| + A < \gamma^* > |0 > < 1|$$

$$+ A < \gamma > |1 > < 0| + A^2 < |\gamma|^2 > |1 > < 1|,$$

(15)
where $\langle |\gamma|^2 \rangle = \text{tr}(\rho_i a_1^\dagger a_1)$ and $\langle \gamma \rangle = \text{tr}(\rho_i a_1)$ for the initial state $\rho_i$. Similar equations hold for $\sigma_A$ where the expectation values are, instead, in terms of $\sigma_i$. The diagonal terms in (15) are given by

$$
\langle n | \rho_A | n \rangle = \frac{A^{2n}}{n!} \int |\gamma|^{2n} e^{-A^2|\gamma|^2} \, P(\gamma, \gamma^*) \, d^2\gamma
$$

and so the many-photon contributions in (15) are vanishingly small for arbitrarily small values of $A$ and such terms and their corresponding off-diagonal matrix elements have been neglected in (15).

For arbitrary $\langle \gamma \rangle$ and $\langle |\gamma|^2 \rangle$, the density matrix (15) represents a mixed state. Therefore, the supports of $\rho_A$ and $\sigma_A$ are equal and so, in general, the two states cannot be discriminated [8]. However, if

$$
\langle |\gamma|^2 \rangle - |\gamma| \langle |\gamma|^2 \rangle = A^2 (\langle |\gamma|^2 \rangle)^2 \ll \langle |\gamma|^2 \rangle,
$$

then $\rho_A = |\Phi_\rho \rangle \langle \Phi_\rho |$ is a pure state with

$$
|\Phi_\rho \rangle = \left[ 1 - A^2 \langle |\gamma|^2 \rangle \right]^{1/2} \left[ |0 \rangle + A \frac{\langle |\gamma|^2 \rangle}{\langle \gamma^* \rangle} |1 \rangle \right].
$$

Similar results hold for $\sigma_A$ with $\sigma_A = |\Phi_\sigma \rangle \langle \Phi_\sigma |$ and $|\Phi_\sigma \rangle$ is given by (18) with expectation values taken with respect to $\sigma_i$ rather than $\rho_i$. One may perform measurements in order to optimally distinguish [9] between the two nonorthogonal states $|\Phi_\rho \rangle$ and $|\Phi_\sigma \rangle$. Such measurements are described by a positive operator-valued measure (POVM) $\{E_\rho, E_\sigma, E_?\}$ [10], where $E_\rho = C_\rho |\Phi_\rho^\text{orth} \rangle \langle \Phi_\rho^\text{orth}|$, $E_\sigma = C_\sigma |\Phi_\sigma^\text{orth} \rangle \langle \Phi_\rho^\text{orth}|$, $E_? = 1 - E_\rho - E_\sigma$, and $\langle \Phi_\rho | \Phi_\rho^\text{orth} \rangle = \langle \Phi_\sigma | \Phi_\sigma^\text{orth} \rangle = 0$. Positive constants $C_\rho$ and $C_\sigma$ can always be found such that $E_?$ is a positive operator. The measurement procedure can have up to three possible outcomes, associated with identifying the state $\rho_i$, identifying the state $\sigma_i$, and failing to identify the state conclusively [8]. The discrimination between the mixed states $\rho_i$ and $\sigma_i$ is optimized for appropriate values of $C_\rho$ and $C_\sigma$ consistent with the positivity of the operator $E_?$. The successful unambiguous discrimination [8, 9] gives the maximum probability $P_{\text{max}} = 1 - | \langle \Phi_\sigma | \Phi_\rho \rangle |$ for the case of equal *a priori* probabilities. A practical proposal for preparing chosen superpositions of the vacuum and the one-photon states [11] may be used for the above POVM. Also, optimal unambiguous weak coherent states discrimination may be realized using polarization beam splitters [12]. Weak coherent states are easy to generate from strong coherent states [6]. In an effort to increase the maximum unambiguous discrimination between the attenuated states $|\Phi_\rho \rangle$ and $|\Phi_\sigma \rangle$, one can consider weaker
attenuation and thus include higher photon number states in (15). This would require optimum measurements for three or more distinct outcomes [13].

The above approach to discriminate between non-orthogonal mixed states represents an optical attenuation of an arbitrary mixed states into the vacuum and the one-photon subspace. Note that $P^\text{max} = O(A^2)$ and so the success probability for unambiguous discrimination would be rather small but increases significantly for mixed quantum states with small mean photon numbers. Our procedure does not represent a “quantum scissors” for density matrices since the vacuum and the one-photon matrix elements of the attenuated density matrix differs from the vacuum and the one-photon matrix elements of the original density operator. Quantum entanglement and the nonlocality of a single photon has been used to truncate the number state expansion of a pure optical state thus leaving only the vacuum and one-photon components [11]. However, at present there is no known scheme to project an arbitrary density matrix onto the zero- and one-photon subspace.

The class of mixed states that yield to unambiguous discrimination are determined by (17) and are close to a coherent state and includes, for instance, superposition of two or more coherent states. Of course, the procedure of identifying mixed states, by reducing the support of the mixed states $\rho_i$ and $\sigma_i$ to the vacuum and the one photon state, was considered for states for which $\text{tr}(\rho a) = \langle \gamma \rangle \neq 0$. Mixed classical states such as the randomly phased laser model, thermal light, etc., where $P(\gamma, \gamma^*) = P(|\gamma|) \geq 0$ and so $\langle \gamma \rangle = 0$, do not yield to an unambiguous discrimination since for these cases the states $\rho_A$ and $\sigma_A$ commute and so the original mixed states $\rho_i$ and $\sigma_i$ cannot be discriminated. The latter is expected since such type of mixed states are diagonal in the number of particle representation with non-zero diagonal matrix elements and thus possess the same support.

5. Summary and Conclusion

In closing, the no-cloning theorem [1] and the no-broadcasting theorem [3] impose fundamental quantum mechanical restrictions on the ability to copy or broadcast arbitrary states. Both theorems are combined into one by proving that noncommuting mixed states cannot be broadcast and that cloning represents a strong form of broadcasting [3]. In Ref. [2], the cloning of coherent states by a cavity-cloning machine suggests a weaker operational definition of cloning or copying. In the former, the cloning machine produces two identical pure states as output [2] that are unitarily related to a different input state. In the latter
the input and the two output states are required to be identical \cite{WoottersZurek82}.

In this work, the meaning of broadcasting \cite{BarnumCavesFuchsJozsaSchumacher96} is weakened to mean that the output density state for the two-mode radiation field are in the same marginal state, which differs from the input state but is obtained from it by a unitary transformation. Such more modest, operational definition of quantum copier has also been considered previously \cite{SimonWeihsZeilinger00}. The ability of the unambiguous discrimination of two mixed states ought to be of interest in quantum cryptography where the usual procedure is to encode information into noncommuting mixed states in order to prevent eavesdropping.

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