Symmetrization Selection Rules

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Abstract

We introduce strong and electromagnetic interaction selection rules for the two–body decay and production of exotic $J^{PC} = 0^{+}, 1^{−+}, 2^{++}, 3^{++} \ldots$ hybrid mesons, four–quark states and glueballs. The rules arise from symmetrization in states. Examples include various decays to $\eta \eta'$, $\eta \pi$, $\eta' \pi$ and $\pi^\pm \pi^0$. The symmetrization rules can discriminate between hybrid and four–quark interpretations of a $1^{−+}$ signal.

Selection rules valid for $SU(3)$ flavour symmetry were first noted using the Wigner–Eckart theorem [1], and later were recognized as being valid for the decay of hybrids $1^{−+} \to \eta \pi$, $\eta' \pi$ [2, 3] within the context of isospin symmetry. We offer an approach in which all possible rules of the same kind can be classified. $SU(3)$ flavour symmetry will not be assumed, and our selection rules do not trivially follow from the reduction of $SU(3)$ to isospin $SU(2)$ symmetry. The $1^{−+} \to \eta \pi$, $\eta' \pi$ rules follow as a specific example. We obtain a novel and substantially enlarged list of processes to which selection rules apply. Both the necessary and sufficient conditions for the validity of the rules are clearly indicated. We also demonstrate that non–trivial rules arise even in the absence of assuming isospin symmetry.

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In the following we shall be interested in fully relativistic two–body strong and electromagnetic decay and production $A \leftrightarrow BC$ processes in the rest frame of $A$. Since strong and electromagnetic interactions are considered, we assume charge conjugation $C$ and parity $P$ conservation, but not in general isospin symmetry. For simplicity we shall usually refer to the decay process $A \rightarrow BC$, but the statements will be equally valid for the production process $A \leftarrow BC$. We shall restrict the states $A$, $B$ and $C$ to some assumed leading combination of “valence” quarks with arbitrary gluonic content, except when sea components are explicitly considered. The strong interactions include all interactions described by QCD. The quarks and antiquarks in $A$ are assumed to travel in all possible complicated paths going forward and backward in time and emitting and absorbing gluons until they emerge in $B$ and $C$. We shall restrict $B$ and $C$ to angular momentum $J = 0$ states with valence $Q\bar{Q}$ quark content and arbitrary gluonic excitation, i.e. to hybrid or conventional mesons. $B$ and $C$ can be radial excitations or ground states, with $J^P = 0^−$ or $0^+$. If $C$–parity is a good quantum number, $J^{PC} = 0^{−+}, 0^{++}, 0^{−+}$ or $0^{++}$ are allowed. Since $0^{−+}$ ground state meson states $B$ and $C$ are most likely to be allowed by phase space, they are used in the examples. We assume that states $B$ and $C$ are identical in all respects except, in principle, their flavour and their equal but opposite momenta $p_B \equiv p$ and $p_C \equiv −p$. Hence $B$ and $C$ have the same parity, $C$–parity, radial and gluonic excitation, as well as the same internal structure.

The three symmetrization selection rules for various topologies are clearly stated in the next section, where we proceed to derive the rules.

1 Symmetrization selection rules

For the leading theory of the strong interactions, QCD, a decay or production amplitude is a linear combination of products of colour $C$ and flavour $F$ overlaps, and the “remaining” overlap $J$. For reasons that will soon become evident, we shall be interested in the exchange properties of these overlaps when the labels (e.g. parity, $C$–parity, radial and gluonic excitation, and internal structure) that specify the states $B$ and $C$ are formally exchanged, denoted by $B \leftrightarrow C$. For example, $C_{B\leftrightarrow C}$ denotes the effect of exchanging the colour labels of $B$ and $C$.

We are only interested in decays where $B$ and $C$ have the same colour content, i.e. the way the quarks and gluons couple to form the total colour singlet state required by QCD is identical. For a conventional meson the quarks and antiquarks are in $3$ and $\bar{3}$ representations. In an adiabatic picture \([4, 5]\) the same holds when $B$ and $C$ are hybrid mesons. In a constituent gluon picture hybrid mesons $B$ and $C$ have the colour coupling of an $8$ gluon with $3$ quarks and $\bar{3}$ antiquarks. As long as the colour content of $B$ and $C$ is identical we trivially have $C_{B\leftrightarrow C} = C$.

When we exchange $p \rightarrow −p$, we equivalently exchange $p_B \leftrightarrow p_C$. But since all other aspects (other than flavour) of $B$ and $C$ are the same, it is in fact equivalent to exchanging labels $B \leftrightarrow C$.
for every property in the remainder of the state. So \( p \to -p \) is equivalent to \( J \leftrightarrow J_{B \leftrightarrow C} \).

We shall be interested in processes where the amplitude is in principle the sum of two parts (or “diagrams”), i.e.

\[
A_{\text{tot}}(p) = A(p) + A_{B \leftrightarrow C}(p) = C \otimes F \otimes J(p) + C_{B \leftrightarrow C} \otimes F_{B \leftrightarrow C} \otimes J_{B \leftrightarrow C}(p) \quad (1)
\]

The amplitude is the sum of two parts for the coupling of (hybrid) mesons and four–quark states shown in Figs. 1, 2, 4 – 7, since there is always either the possibility that a quark \( Q \) in \( A \) would end up in the particle with momentum \( p \) and the possibility that it would end up in the particle with momentum \( -p \), corresponding to \( A \) and \( A_{B \leftrightarrow C} \) respectively.

Under \( p \to -p \) (or equivalently \( J \leftrightarrow J_{B \leftrightarrow C} \))

\[
A_{\text{tot}}(p) \to C \otimes F \otimes J(-p) + C_{B \leftrightarrow C} \otimes F_{B \leftrightarrow C} \otimes J_{B \leftrightarrow C}(-p) = C \otimes F \otimes J_{B \leftrightarrow C}(p)
\]

\[
+ C_{B \leftrightarrow C} \otimes F_{B \leftrightarrow C} \otimes J(p) = f \{ C_{B \leftrightarrow C} \otimes F_{B \leftrightarrow C} \otimes J_{B \leftrightarrow C}(p) + C \otimes F \otimes J(p) \} = f A_{\text{tot}}(p) \quad (2)
\]

where we used \( C_{B \leftrightarrow C} = C \) and defined \( F_{B \leftrightarrow C} \equiv f F \). We shall only be interested in cases where \( f = \pm 1 \), and where both \( F \) and \( F_{B \leftrightarrow C} \) are non–zero. If \( f = (-1)^{L+1} \), where \( L \) is the partial wave between \( B \) and \( C \), it follows from Eq. (2) that \( p \to -p \) implies that \( A_{\text{tot}} \to (-1)^{L+1} A_{\text{tot}} \). Since in \( L \)-wave under \( p \to -p \) we have by analyticity that \( A_{\text{tot}}(p) \to (-1)^L A_{\text{tot}}(p) \), it follows that \( A_{\text{tot}}(p) \) vanishes. This is the symmetrization selection rule, arizing due to symmetrization in states \( B \) and \( C \).

We now find necessary and sufficient conditions for the requirement \( f = (-1)^{L+1} \). Since \( B \) and \( C \) are identical (except possibly in flavour) they have the same parity, and we conclude that for a parity allowed process, \( P_A = (-1)^L \). We shall show in subsections 1.1 – 1.4 that for various flavour scenarios \( f = C_A^0 \). For a neutral state, \( C_A^0 \) is just the C–parity of the state. For charged states (with no C–parity), we assume that at least one of the states in the isomultiplet it belongs to has a well–defined C–parity, denoted by \( C_A^0 \). Hence

\[
P_A = (-1)^L = -(-1)^{L+1} = -f = -C_A^0
\]

i.e. state \( A \) is CP odd. Since states \( B \) and \( C \) both have \( J = 0 \), it follows by conservation of angular momentum that an \( L \)-wave decay would necessitate \( J_A = L \). Hence states \( A \) have \( J^{PC} = 0^{+-}, 1^{-+}, 2^{+-}, 3^{-+}, \ldots \), which are exotic \( J^{PC} \) not found in the quark model. So \( A \) is not a conventional meson.

We now show that \( f = C_A^0 \).

1\(^{\text{When B and C have } J = 0, \text{ helicity and partial wave amplitudes are identical.}}\)
1.1 Indistinguishable flavours

The simplest case is when states B and C have indistinguishable flavours, e.g. \( Q\bar{Q} Q\bar{Q} \) states B and C. This does not have to be satisfied for the full flavour content of B and C, but the decay must be such that indistinguishable flavour components of B and C are always selected. Then \( f = 1 \). The only interesting cases arise when B and C are neutral; and since their C–parities are identical, we have \( C_A = 1 \) by conservation of C–parity. So \( f = 1 = C_A \). The preceding argument is independent of the decay topology, even though specific examples are determined by the topology, and are given in the following subsections.

**Symmetrization selection rule I:** (Indistinguishable flavours) Decay and production in topologies 1 – 9 (see Fig. 1) to two \( J = 0 \) hybrid or conventional mesons B and C which are identical in all respects, except possibly flavour, vanish. This only applies to \( J^P C = 1^− , 3^+ , \ldots \) hybrid, four–quark state or glueball A coupling to flavour components of B and C that are indistinguishable, e.g. \( Q\bar{Q} \) or \( Q\bar{Q} q\bar{q} \rightarrow Q\bar{Q} Q\bar{Q} \). Isospin symmetry is not assumed.

The remaining cases for which \( f = \) \( C_A^0 \) are discussed in the following subsections. The corresponding symmetrization selection rules are then stated.

1.2 Connected hybrid meson coupling

The possible hybrid decay topologies are 1 – 3, but we focus here on the connected topology 1.

**Indistinguishable flavours:** Examples: \( \Pi^0 \rightarrow \pi^0 \eta, \pi^0 \eta' \); \( \Omega, s\bar{s} \rightarrow \eta \eta' \), where \( \Pi^0 \) denotes a \( u, d \) quark neutral isovector state (e.g. \( u\bar{u} - d\bar{d} \)), and \( \Omega \) an isoscalar state (e.g. \( u\bar{u} + d\bar{d} \)). In the examples listed the decay topology always has the effect of selecting indistinguishable \( u\bar{u} u\bar{u}, d\bar{d} d\bar{d} \) or \( s\bar{s} s\bar{s} \) subcomponents of the states B and C, as required.

**With isospin symmetry:** If we assume isospin symmetry for \( u, d \), then by G–parity conservation \( G_A = G_B G_C \). Since \( G_H = C_H^0 (-1)^I_H \), we obtain \( C_A^0 = (-1)^{I_A + I_B + I_C} \) because the C–parities of B and C are identical. It can, however, by explicit calculation be verified that \( f = (-1)^{I_A + I_B + I_C} \) (see Appendix). Hence \( f = (-1)^{I_A + I_B + I_C} = C_A^0 \).

**Examples:** \( \Pi^\pm \rightarrow \pi^\pm \pi^0, \pi^\pm \eta, \pi^\pm \eta' \).

1.3 Connected four–quark coupling

We now discuss a four–quark state A, which is not a molecular bound state of two mesons. The possible four–quark decay topologies are 4 – 8. Here we focus on the connected topologies 4 – 6.

**Indistinguishable flavours:** Examples: \( \Omega, s\bar{s}s\bar{s}, KK, DD, BB, D_s D_s, B_s B_s \rightarrow \eta \eta \),
Figure 1: Decay topologies. For each diagram state A is on the left–hand side, and states B and C on the right–hand side. Quark flavours are labelled by $Q$, $q$ and $P$. 
where e.g. $K\bar{K}$ includes $K^+K^-$ and $K^0\bar{K}^0$. Another application is to flavour components of state $A$ e.g. $u\bar{u}u\bar{u}, d\bar{d}d\bar{d} \rightarrow \pi^0\eta, \pi^0\eta'$.

With isospin symmetry: If we assume isospin symmetry for $u,d$, the arguments are similar to §1.2, noting that $f = (-1)^{I_A + I_B + I_C}$ (see Appendix) \[6\].

Examples: $\Pi^\pm, \Omega^\pm \rightarrow \pi^\pm \pi^0, \Omega^0 \rightarrow \pi^0\eta, \pi^0\eta'$.

where $\Omega$ denotes an isotensor.

Symmetrization selection rule II: (With isospin symmetry) Connected decay and production in topology 1 and 4 - 6 to two $J = 0$ hybrid or conventional mesons $B$ and $C$ which are identical in all respects, except possibly flavour, vanish. The processes should involve only $u,d$ quarks, and isospin symmetry is assumed. $A$, $B$ or $C$ may be charged, but the hybrid or four–quark state $A$ should have a neutral isopartner with $J^{PC} = 0^+, 1^+, 2^+, 3^+, \ldots$ \[6\].

Four–quark rules can also be applied to meson sea components, i.e. to the connected meson sea topologies 4 – 6. Assuming isospin symmetry and states $B$ and $C$ that are the same in all respects except possibly flavour, $u,d$ isoscalar sea corrections to the connected meson decay topology 1 vanish, as long as the corresponding four–quark decays vanish. Non–vanishing decays arise if $s\bar{s}$ sea components are allowed in a $u,d$ meson, or $u\bar{u}, \bar{d}d$ sea in $s\bar{s}$, e.g. in the channels $\eta', \eta'\pi$ or $\eta\pi$. In this case the dominant decay is expected from the quark rearrangement topology 5 ($u\bar{u}s\bar{s} \rightarrow u\bar{u}s\bar{s}$ or $d\bar{d}s\bar{s} \rightarrow d\bar{d}s\bar{s}$), because symmetrization arguments are invalid for this topology.

1.4 Non–connected coupling

We now study the non–connected topologies 2 and 7.

Indistinguishable flavours: Examples: In $\Pi^0 \rightarrow \pi^0\eta, \pi^0\eta'$; $\Omega, s\bar{s} \rightarrow \eta'\eta$ contributions from $u\bar{u}, d\bar{d}d\bar{d}, s\bar{s}s\bar{s}$ components vanish. In $\Omega \rightarrow \eta'\eta$ contributions from $\eta'_{u\bar{u}+d\bar{d}} \eta_{u\bar{u}+d\bar{d}}$ components vanish.

For the topologies 3 and 8, and the glueball topology 9, it naively appears that $A_{B\leftrightarrow C}$ is not topologically distinct from $A$, invalidating the application of symmetrization arguments. Although there exists diagrams in perturbative QCD with this property, the majority of diagrams have $A_{B\leftrightarrow C}$ topologically distinct from $A$. For the latter diagrams we proceed to apply symmetrization arguments. Symmetrization selection rules where states $B$ and $C$ are in a “half doughnut” topology (as in topology 3b) can be shown to apply only for decays already known to vanish by CP conversion, so we only proceed to consider states $B$ and $C$ in a “raindrop” topology (as in topology 3a). From the Appendix $f = 1$. Since states $B$ and $C$ have identical $C$–parities, we have $C_A = 1$. So $f = 1 = C_A$.

Examples: Neutral isoscalar hybrids ($\Omega, s\bar{s}, c\bar{c}, b\bar{b}$), four–quark states ($\Omega, s\bar{s}s\bar{s}, K\bar{K}, D\bar{D}$,
Symmetrization selection rule III: Non–connected decay and production in topologies 3 and 8 – 9 of a $J^{PC} = 1^{-+}, 3^{-+}, \ldots$ hybrid, four–quark state or glueball to two $J = 0$ hybrid or conventional mesons B and C which are identical in all respects, except possibly flavour, vanish. The statement only holds when the $B \leftrightarrow C$ exchanged diagram is topologically distinct from the original diagram. Isospin symmetry is not assumed.

2 Breaking of symmetrization selection rules

If we do not assume isospin symmetry, the possibility of different strengths of pair creation for different flavours does not break the selection rules.

Suppose that the states B and C have some factorizable property $F_H$, which can be factored in front of the amplitude as $F_B F_C$. The arguments in Eq. 2 would still be valid, since $F_B F_C$ is invariant under $B \leftrightarrow C$, even if $F_B \neq F_C$. Particularly, states have energy dependence $F_H = \exp iE_H t$ due to time translational invariance. Hence different energies or masses for B and C does not explicitly break the validity of the arguments.

It is clear that states B and C with different internal structure, indirectly related to them having different masses and energies, would break the symmetrization selection rules. Corrections of this nature are found to be small in models [5, 7, 8] as long as the $J = 0$ states B and C have the same radial excitation, and substantial otherwise. This is accord with expectations since we expect different radial excitations in B and C to invalidate the selection rules. When off–shell states B and C are allowed, breaking of the rules could be more substantial [5], enabling off–shell meson exchange as a potentially significant exotic hybrid, four–quark or glueball production mechanism, e.g in $\pi N \rightarrow J^{PC} N$ with low energy $\pi$ exchange.

3 Summary of symmetrization selection rules

For connected topologies production and decay of neutral exotic $1^{-+}, 3^{-+}\ldots$ hybrid mesons and four–quark states to two $J = 0$ states (hybrid or conventional mesons), e.g. pseudoscalar mesons, which are identical in all respects except possibly flavour, vanish. The same is true for charged and neutral $0^{+-}, 1^{-+}, 2^{+-}, 3^{-+}\ldots$ states A in decays involving only $u, d$ quarks if isospin symmetry is invoked [3]. For non–connected topologies, vanishing production and decay result for $1^{-+}, 3^{-+}, \ldots$ hybrid mesons, four–quark states and glueballs to two $J = 0$ states, which are identical in all respects, except possibly flavour. For topologies 2 and 7 this only applies to certain flavour

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2 These must be interactions clearly happening within states B and C, and hence associated with B and C, distinct from the remainder of the interaction topology.
components of the two $J = 0$ states, and for topologies 3 and 8–9 there are conditions on the diagrams. All symmetrization rules are broken if the internal structure of B and C differs, but do not depend explicitly on the energy and mass differences between B and C.

A special case is decays of hybrid or four–quark $0^+, 1^- , \ldots$ $\Pi^\pm, \Xi^\pm \to \pi^\pm \pi^0$ which vanish by isospin symmetry in all possible topologies to which it contributes (topologies 1, 4 and 6), including isoscalar sea components in the case of hybrid A [3].

The selection rules derived in this letter go beyond well–known selection rules, because they depend upon the specific flavour content of the states, and on the production or decay topology. For ground state pseudoscalar mesons B and C, selection rules are found for $\eta', \eta, \eta \pi, \eta' \pi$ [3, 8], $\pi^\pm \pi^0, \eta_0 \eta, \eta_0 \eta', \eta_b \eta, \eta_b \eta'$ and $\eta_c \eta_b$. We found three categories of symmetrization selection rules. Firstly, in the absence of isospin symmetry selection rules result when B and C have indistinguishable flavour components, e.g. $Q \bar{Q} Q \bar{Q}$. Secondly, in the case of isospin symmetry, selection rules are found to apply to states B and C containing a neutral flavour–mixed hybrid or conventional meson with flavour content $u \bar{u} + d \bar{d}$ or $u \bar{u} - d \bar{d}$. The selection rules result from cancellations of amplitudes containing either the $u \bar{u}$ component or the $d \bar{d}$ component of the neutral flavour–mixed meson. In this way the relative sign between the $u \bar{u}$ and $d \bar{d}$ components is sampled. When we sum the amplitude $A$ and the $B \leftrightarrow C$ amplitude $A_{B \leftrightarrow C}$, the one amplitude picks the $u \bar{u}$ component and the other the $d \bar{d}$ component. Thirdly, for non–connected “raindrop” topologies we found in the absence of isospin symmetry that the flavour overlap is always invariant under $B \leftrightarrow C$, leading to selection rules for e.g. $\eta_c \eta_b$ states B and C.

4 Comments and Phenomenology

Assuming the same internal structure for $\eta, \eta'$ and $\pi$, we predict the connected decays of valence and $u,d$ sea components in hybrid $1^- \to \eta \pi, \eta' \pi$ to be negligible [3]. If the OZI suppression of non–connected decays of mesons can be extrapolated to hybrids, a small non–connected contribution is expected. It is significant that QCD sum rule calculations consistently predict a tiny $\eta \pi$ mode, e.g. $\sim 0.3$ MeV (versus 600 MeV for $\rho \pi$ and 300 MeV for $K^* K$) [10] and small $\eta' \pi$ of 4 MeV [10] or 3 MeV (versus $\rho \pi$ of 270 MeV and $K^* K$ of 8 MeV) [10]. The relative size of $\eta \pi$ and $\eta' \pi$ is consistent with a selection rule based on SU(3) flavour symmetry [1, 11]. Ref. [12] also notes that $\eta \pi$, $\eta' \pi$ are “suppressed” relative to $\rho \pi$ of 10 – 100 MeV. In addition, ref. [11] notes that $\pi \pi$ is “suppressed”, consistent with the claim of this letter that within isospin symmetry, this connected decay should vanish even for sea components [1].

We note that if $0^+, 1^- , \ldots$ light $u,d$ four–quark systems exist, not only are their (domi-
nant) quark rearrangement topologies 4 and 5 to pseudoscalars suppressed, but also topology 6 [3]. Hence decays only happen through (suppressed) non-connected topologies, confirmed by a model calculation [8]. We hence naively expect e.g. the $\eta\pi$ mode of a $1^{-+}$ exotic be similar whether it is a hybrid or four-quark state. It has, however, been noted [9] that $u\bar{u}$, $d\bar{d}$ components of a four-quark state can in perturbation theory be expected to mix substantially via single gluon exchange with $s\bar{s}$, although flavour mixing of this kind has been found to be $\lesssim 10\%$ in a model calculation [13]. Presence of $s\bar{s}$ components would allow quark rearrangement decay via topology 5 which is not forbidden by symmetrization rules. Measurement of $\eta\pi$, $\eta'\pi$ decay hence samples the strength of the $s\bar{s}$ component in a $u,d$ four-quark state. A four-quark state would on general grounds be expected to have a larger total width than a hybrid due to quark rearrangement topologies to non-pseudoscalars. Thus a wide $1^{-+}$ wave could be interpreted as a four-quark state. If the $s\bar{s}$ component of a four-quark state is small, the state may have a typical mesonic width, otherwise it is expected to be wide. A candidate state $\hat{\rho}(1405)$ with width $180 \pm 20$ MeV, possibly decaying to $\eta\pi$ but absent in $\rho\pi$ has been reported [14]. There is recent preliminary evidence [15] for a resonance with similar width and decay patterns. If the $1^{-+}$ state is indeed significantly produced, the $\eta\pi$ mode may discriminate against the hybrid interpretation, since only the (suppressed) non-connected topology 2 contributes. However, the mode may be due to a $s\bar{s}$ component in a four-quark state. Since the $s\bar{s}$ components in $\eta$ and $\eta'$ are nearly the same, and due to P-wave phase space, we expect $\eta'\pi < \eta\pi$ [3].

A subset of the rules has explicitly been shown to arise in QCD field theory [2]. It would be a challenge for lattice gauge theory and Dyson-Schwinger techniques to see if agreement is found with this result, and to estimate the size of non-connected topologies when unquenching the calculation.

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A Appendix: Flavour Overlaps

The flavour state is

$$|H\rangle = \sum_{hh} H_{hh}|h\rangle |\bar{h}\rangle \quad \text{where} \quad H_{hh} = (I_H I_H \frac{1}{2} h \bar{h} - \bar{h} \bar{h} (-1)^{\frac{1}{2} h - \bar{h}}$$

Assuming a small $\rho\pi$ coupling. Modes to $K^*K$, $\rho\eta$, $f_2\pi$, $f_1\pi$, $b_1\pi$ are near the edge of phase space.

The connected decay $1^{-+} \to \eta\pi$ vanishes in quenched Euclidean QCD with isospin symmetry, assuming no final state interactions and a $t \to \infty$ limiting procedure which isolates only the ground state $1^{-+}$. 

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and \( |\frac{1}{2}\rangle = u, \ | - \frac{1}{2}\rangle = d, \ |\frac{1}{2}\rangle = \bar{u}, \ | - \frac{1}{2}\rangle = \bar{d} \). This just yields the usual \( I = 1 \) flavour 
\(-ud, \ \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), \ d\bar{u} \) for \( I^z = 1, 0, -1 \) and \( \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \) for \( I = 0 \). The advantage of this way of identifying flavour is that any pair creation or annihilation that takes place do so with \( I = 0 \) pairs \( \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) = \frac{1}{\sqrt{2}} \sum_n \phi_n |\phi_n\rangle |\bar{\phi}_n\rangle \) being formed out of the vacuum, making the operator trivial.

For the connected decay of topology 1 the flavour overlap \( \mathcal{F} \) is

\[
\sum_{aab} A_{ab}B_{ab}\delta_{bb}C_{ba} = \sum_{aab} (-1)^{\frac{1}{2} - b} \langle I_A I_A^z | \frac{1}{2}^{1/2} - a \rangle \langle I_B I_B^z | \frac{1}{2}^{1/2} - b \rangle \langle I_C I_C^z | \frac{1}{2}^{1/2} - a \rangle
\]

which can easily be shown under \( B \leftrightarrow C \) to give the sign \( f = (-1)^{I_A + I_B + I_C} \).

For four–quark states \( A \) we are free to decompose the four quarks in two different ways in terms of two quark–antiquark pairs. The flavour can be decomposed as

\[
\text{Topology 4 & 6 :} \quad \sum_{I_Q I_{\bar{Q}} I_{\bar{Q}}} \langle I_A I_A^z | I_{Q\bar{Q}} I_{Q\bar{Q}} I_{Q\bar{Q}} I_{Q\bar{Q}} \rangle | A^Q\bar{Q} \rangle | A^Q\bar{Q} \rangle
\]

\[
\text{Topology 5 :} \quad \sum_{I_{\bar{Q}} I_{\bar{Q}}} \langle I_A I_A^z | I_{Q\bar{Q}} I_{Q\bar{Q}} I_{Q\bar{Q}} I_{Q\bar{Q}} \rangle | A^Q\bar{Q} \rangle | A^Q\bar{Q} \rangle
\]

where we summed over all isospin projections. In the quark rearrangement topologies 4 and 5 it is convenient to choose a flavour decomposition for \( A \) which makes the overlap with \( B \) and \( C \) trivial. We obtain the flavour overlaps \( \mathcal{F} \)

\[
\text{Topology 4 :} \quad \sum_{I_Q I_{\bar{Q}} I_{\bar{Q}}} \langle I_A I_A^z | I_{Q\bar{Q}} I_{Q\bar{Q}} I_{Q\bar{Q}} I_{Q\bar{Q}} \rangle \delta_{IQI}_{\bar{B}} \delta_{IQI}_{\bar{C}} \delta_{IQI}_{\bar{C}} \delta_{IQI}_{\bar{C}}
\]

\[
\text{Topology 5 :} \quad \sum_{I_{\bar{Q}} I_{\bar{Q}}} \langle I_A I_A^z | I_{Q\bar{Q}} I_{Q\bar{Q}} I_{Q\bar{Q}} I_{Q\bar{Q}} \rangle \delta_{IQI}_{\bar{B}} \delta_{IQI}_{\bar{C}} \delta_{IQI}_{\bar{C}} \delta_{IQI}_{\bar{C}}
\]

\[
\text{Topology 6 :} \quad \sum_{I_{QP} I_{QP}} \langle I_A I_A^z | I_{QP} I_{QP} I_{QP} I_{QP} \rangle \sum_{aabbcc} A_{ab}P_{ba}C_{ca} \sum_{abc} \langle I_P I_P | \frac{1}{2}^{1/2} - b \rangle \langle I_B I_B | \frac{1}{2}^{1/2} - b \rangle \langle I_C I_C | \frac{1}{2}^{1/2} - b \rangle
\]

The four–quark states in Eqs. 3 and 4 are characterized by \( I_A, I_A^z \) and the isospins of the two quark–antiquark pairs, generically referred to as \( I_X \) and \( I_Y \). In Eq. 3 denote \( I_{Q\bar{Q}}, I_{QP} \) by \( I_X \) and \( I_{Q\bar{Q}}, I_{QP} \) by \( I_Y \) in Eq. 4. Write the four–quark state as \( |I_A I_A I_X I_Y \rangle \). It can be seen by explicit computation that if \( I_X = I_Y \), then \( f = (-1)^{I_A + I_B + I_C} \) under \( B \leftrightarrow C \) for each of the expressions in Eq. 3. When \( I_A = 0 \), the physical state is a linear combination of \( |0 0 0 0 \rangle \) and \( |0 0 1 1 \rangle \). For \( I_A = 2 \), the physical state is \( |2 I_A^z 1 1 \rangle \). So in both cases \( I_X = I_Y \). When \( I_A = 1 \), the
physical state is a linear combination of $|1 I_z^A 1 1\rangle$, $|1 I_z^A 1 0\rangle$ and $|1 I_z^A 0 1\rangle$. For each of the expressions in Eq. 8 it can be shown that $|1 I_z^A 1 0\rangle \rightarrow (-1)^{I_A^+ I_B^+ I_C^+} |1 I_z^A 0 1\rangle$ under $B \leftrightarrow C$. Defining new states $|\pm\rangle \equiv |1 I_z^A 1 0\rangle \pm |1 I_z^A 0 1\rangle$, we see that under $B \leftrightarrow C$, $|\pm\rangle \rightarrow \pm (-1)^{I_A^+ I_B^+ I_C^+} |\pm\rangle$. The above statements about the behaviour of $F$ under $B \leftrightarrow C$ are true in any decomposition of $A$, hence also in the “diquonium” decomposition, where pairs of two quarks and two antiquarks are used. The advantage of this decomposition is that neutral states $|\pm\rangle$ would be eigenfunctions of charge conjugation [13], as required. Hence, when $I_A = 1$, a physical state is either the “positive” linear combination of $|1 I_z^A 1 1\rangle$ and $|+\rangle$, or the “negative” linear combination of $|1 I_z^A 1 1\rangle$ and $|-\rangle$. The former gives $f = (-1)^{I_A^+ I_B^+ I_C^+}$ under $B \leftrightarrow C$, and the latter has no proper symmetry under $B \leftrightarrow C$. In summary, for $I_A = 0$ or 2, or for “positive” $I_A = 1$ states, $f = (-1)^{I_A^+ I_B^+ I_C^+}$ under $B \leftrightarrow C$.

For the “raindrop” configurations in topologies 3 and 8 – 9 the part of the flavour overlap $F$ containing reference to $B$ and $C$ is

$$\sum_{aa} A_{aa} \delta_{aa} \sum_{bb} B_{bb} \delta_{bb} = Tr(A) Tr(B)$$

which under $B \leftrightarrow C$ gives $f = 1$.

References

[1] C.A. Levinson, H.J. Lipkin, S. Meshkov, N. Cim. 32 (1964) 1376.
[2] F. Iddir et al. Phys. Lett. B207 (1988) 325.
[3] H.J. Lipkin, Phys. Lett. 219 (1989) 99.
[4] C. Michael et al., Nucl. Phys. B347 (1990) 854; Liverpool Univ. report LTH-286 (1992), Proc. of the Workshop on QCD : 20 Years Later (Aachen, 1992).
[5] F.E. Close, P.R. Page, Nucl. Phys. B443 (1995) 233; ibid. Phys. Rev. D52 (1995) 1706.
[6] As explained in the Appendix, this sentence does not hold for “negative” $I_A = 1$ four–quark states $A$.
[7] Yu.S. Kalashnikova, Z. Phys. C62 (1994) 323.
[8] A. Le Yaouanc et al. Phys. Lett. B205 (1988) 564; Phys. Lett. B79 (1978) 459.
[9] F.E. Close, H.J. Lipkin, Phys. Lett. B196 (1987) 245.
[10] J.I. Latorre et al., Z. Phys. C34 (1987) 347.
[11] S. Narison, “QCD spectral sum rules”, Lecture Notes in Phys. Vol. 26 (1989), p. 369.
[12] J. Govaerts, F. de Viron, Phys. Rev. Lett. 53 (1984) 2207.
[13] C. Semay, B. Silvestre-Brac, Phys. Rev. D51 (1995) 1258.

[14] Particle Data Group, Phys. Rev. D54 (1996) 1.

[15] S.-U. Chung, private communication; N.M. Cason et al. (E852 Collab.) Proc. of HADRON’95 (Manchester, 1995), eds. M.C. Birse et al., p. 55.

[16] M. Tanimoto, Phys. Rev. D27 (1983) 2648.