The Work of Laurent Lafforgue

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Laurent Lafforgue has been awarded the Fields Medal for his proof of the Langlands correspondence for the full linear groups $GL_r$ ($r \geq 1$) over function fields.

What follows is a brief introduction to the Langlands correspondence and to Lafforgue’s theorem.

1. The Langlands correspondence

A global field is either a number field, i.e. a finite extension of $\mathbb{Q}$, or a function field of characteristic $p > 0$ for some prime number $p$, i.e. a finite extension of $\mathbb{F}_p(t)$ where $\mathbb{F}_p$ is the finite field with $p$ elements. The global fields constitute a primary object of study in number theory and arithmetic algebraic geometry.

The conjectural Langlands correspondence, which was first formulated by Robert Langlands in 1967 in a letter to André Weil, relates two fundamental objects which are naturally attached to a global field $F$:

- its Galois group $\text{Gal}(\overline{F}/F)$, where $\overline{F}$ is an algebraic closure of $F$, or more accurately its motivic Galois group of $F$ which is by definition the tannakian group of the tensor category of Grothendieck motives over $F$,
- the ring $\mathcal{A}$ of adèles of $F$, or more precisely the collection of Hilbert spaces $L^2(G(F)\backslash G(\mathcal{A}))$ for all reductive groups $G$ over $F$.

Roughly speaking, for any (connected) reductive group $G$ over $F$, Langlands introduced a dual group $^L G = \hat{G} \rtimes \text{Gal}(\overline{F}/F)$, the connected component $\hat{G}$ of which is the complex reductive group whose roots are the co-roots of $G$ and vice versa. And he predicted that a large part of the spectral decomposition of the Hilbert space $L^2(G(F)\backslash G(\mathcal{A}))$, equipped with the action by right translations of $G(\mathcal{A})$, is governed by representations of the motivic Galois group of $F$ with values in $^L G$.

Of special importance is the group $G = GL_r$, the Langlands dual of which is simply the direct product $^L GL_r = GL_r(\mathbb{C}) \times \text{Gal}(\overline{F}/F)$. Indeed, any complex reductive group $\hat{G}$ may be embedded into $GL_r(\mathbb{C})$ for some $r$.

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The particular case $G = GL_1$ of the Langlands correspondence is the abelian class field theory of Teiji Takagi and Emil Artin which was developed in the 1920s as a wide extension of the quadratic reciprocity law.

The Langlands correspondence embodies a large part of number theory, arithmetic algebraic geometry and representation theory of Lie groups. Small progress made towards this conjectural correspondence had already amazing consequences, the most striking of them being the proof of Fermat’s last theorem by Andrew Wiles. Famous conjectures, such as the Artin conjecture on $L$-functions and the Ramanujan-Petersson conjecture, would follow from the Langlands correspondence.

2. Lafforgue’s main theorem

Over number fields, the Langlands correspondence in its full generality seems still to be out of reach. Even its precise formulation is very involved. In the function field case the situation is much better. Thanks to Lafforgue, the Langlands correspondence for $G = GL_r$ is now completely understood.

From now on, $F$ is a function field of characteristic $p > 0$. We also fix some auxiliary prime number $\ell \neq p$.

As Alexandre Grothendieck showed, any algebraic variety over $F$ gives rise to $\ell$-adic representations of $\text{Gal}(\overline{F}/F)$ on its étale cohomology groups and the irreducible $\ell$-adic representations of $\text{Gal}(\overline{F}/F)$ are good substitutes for irreducible motives over $F$. Therefore, the Langlands correspondence may be nicely formulated using $\ell$-adic representations.

Let $r$ be a positive integer. On the one hand, we have the set $G_r$ of isomorphism classes of rank $r$ irreducible $\ell$-adic representations of $\text{Gal}(\overline{F}/F)$ the determinant of which is of finite order. To each $\sigma \in G_r$, Grothendieck attached an Eulerian product $L(\sigma, s) = \prod_x L_x(\sigma, s)$ over all the places $x$ of $F$, which is in fact a rational function of $p^{-s}$ and which satisfies a functional equation of the form $L(\sigma, s) = \epsilon(\sigma, s)L(\sigma^\vee, 1-s)$ where $\sigma^\vee$ is the contragredient representation of $\sigma$ and $\epsilon(\sigma, s)$ is some monomial in $p^{-s}$. If $\sigma$ is unramified at a place $x$, we have

$$L_x(\sigma, s) = \prod_{i=1}^r \frac{1}{1 - z_i p^{-s} \deg(x)}$$

where $z_1, \ldots, z_r$ are the Frobenius eigenvalues of $\sigma$ at $x$ and $\deg(x)$ is the degree of the place $x$.

On the other hand, we have the set $\mathcal{A}_r$ of isomorphism classes of cuspidal automorphic representations of $\text{GL}_r(\mathbb{A})$ the central character of which is of finite order. Thanks to Langlands’ theory of Eisenstein series, they are the building blocks of the spectral decomposition of $L^2(\text{GL}_r(F) \backslash \text{GL}_r(\mathbb{A}))$. To each $\pi \in \mathcal{A}_r$, Roger Godement and Hervé Jacquet attached an Eulerian product $L(\pi, s) = \prod_x L_x(\pi, s)$ over all the places $x$ of $F$, which is again a rational function of $p^{-s}$, satisfying a functional equation $L(\pi, s) = \epsilon(\pi, s)L(\pi^\vee, 1-s)$. If $\pi$ is unramified at a place $x$, we have

$$L_x(\pi, s) = \prod_{i=1}^r \frac{1}{1 - z_i p^{-s} \deg(x)}$$
where $z_1, \ldots, z_r$ are called the Hecke eigenvalues of $\pi$ at $x$.

**Theorem** (i) (The Langlands Conjecture) There is a unique bijective correspondence $\pi \rightarrow \sigma(\pi)$, preserving $L$-functions in the sense that $L_x(\sigma(\pi), s) = L_x(\pi, s)$ for every place $x$, between $A_r$ and $G_r$.

(ii) (The Ramanujan-Petersson Conjecture) For any $\pi \in A_r$ and for any place $x$ of $F$ where $\pi$ is unramified, the Hecke eigenvalues $z_1, \ldots, z_r \in \mathbb{C}^\times$ of $\pi$ at $x$ are all of absolute value 1.

(iii) (The Deligne Conjecture) Any $\sigma \in G_r$ is pure of weight zero, i.e. for any place $x$ of $F$ where $\sigma$ is unramified, and for any field embedding $\iota: \overline{\mathbb{Q}_\ell} \rightarrow \mathbb{C}$, the images $\iota(z_1), \ldots, \iota(z_r)$ of the Frobenius eigenvalues of $\sigma$ at $x$ are all of absolute value 1.

As I said earlier, in rank $r = 1$, the theorem is a reformulation of the abelian class field theory in the function field case. Indeed, the reciprocity law may be viewed as an injective homomorphism with dense image

$$F^\times \backslash \mathbb{A}^\times \rightarrow \text{Gal}(\overline{F}/F)^{ab}$$

from the idèle class group to the maximal abelian quotient of the Galois group.

In higher ranks $r$, the first breakthrough was made by Vladimir Drinfeld in the 1970s. Introducing the fundamental concept of shtuka, he proved the rank $r = 2$ case. It is a masterpiece for which, among others works, he was awarded the Fields Medal in 1990.

### 3. The strategy

The strategy that Lafforgue is following, and most of the geometric objects that he is using, are due to Drinfeld. However, the gap between the rank two case and the general case was so big that it took more than twenty years to fill it.

Lafforgue considers the $\ell$-adic cohomology of the moduli stack of rank $r$ Drinfeld shtukas (see the next section) as a representation of $\text{GL}_r(\mathbb{A}) \times \text{Gal}(\overline{F}/F) \times \text{Gal}(\overline{F}/F)$. By comparing the Grothendieck-Lefschetz trace formula (for Hecke operators twisted by powers of Frobenius endomorphisms) with the Arthur-Selberg trace formula, he tries to isolate inside this representation a subquotient which decomposes as

$$\bigoplus_{\pi \in A_r} \pi \otimes \sigma(\pi)^\vee \otimes \sigma(\pi).$$

Such a comparison of trace formulas was first made by Yasutaka Ihara in 1967 for modular curves over $\mathbb{Q}$. Since, it has been extensively used for Shimura varieties and Drinfeld modular varieties by Langlands, Robert Kottwitz and many others. There are two main difficulties to overcome to complete the comparison:

- to prove suitable cases of a combinatorial conjecture of Langlands and Diana Shelstad, which is known as the **Fundamental Lemma**,  
- to compare the contribution of the “fixed points at infinity” in the Grothendieck-Lefschetz trace formula with the weighted orbital integrals of James Arthur which occur in the geometric side of the Arthur-Selberg trace formula.
For the moduli space of shtukas, the required cases of the Fundamental Lemma were proved by Drinfeld in the 1970s. So, only the second difficulty was remaining after Drinfeld had completed his proof of the rank 2 case. This is precisely the problem that Lafforgue has solved after seven years of very hard work. The proof has been published in three papers totalling about 600 pages.

4. Drinfeld shtukas

Let $X$ be “the” smooth, projective and connected curve over $\mathbb{F}_p$ whose field of rational functions is $F$. It plays the role of the ring of integers of a number field. Its closed points are the places of $F$. For any such point $x$ we have the completion $F_x$ of $F$ at $x$ and its ring of integers $\mathcal{O}_x \subset F_x$.

Let $\mathcal{O} = \prod_x \mathcal{O}_x \subset \mathbb{A}$ be the maximal compact subring of the ring of ad` eles. Weil showed that the double coset space

$$GL_r(F) \backslash GL_r(\mathbb{A}) / GL_r(\mathcal{O})$$

can be naturally identified with the set of isomorphism classes of rank $r$ vector bundles on $X$.

Starting from this observation, with the goal of realizing a congruence relation between Hecke operators and Frobenius endomorphisms, Drinfeld defined a rank $r$ shtuka over an arbitrary field $k$ of characteristic $p$ as a diagram

$$\tau \mathcal{E} \overset{\sim}{\longrightarrow} \mathcal{E}'' \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}$$

where $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ are rank $r$ vector bundles on the curve $X_k$ deduced from $X$ by extending the scalars to $k$, where $\mathcal{E} \hookrightarrow \mathcal{E}'$ is an elementary upper modification of $\mathcal{E}$ at some $k$-rational point of $X$ which is called the pole of the shtuka, where $\mathcal{E}'' \hookrightarrow \mathcal{E}'$ is an elementary lower modification of $\mathcal{E}'$ at some $k$-rational point of $X$ which is called the zero of the shtuka, and where $\tau \mathcal{E}$ is the pull-back of $\mathcal{E}$ by the endomorphism of $X_k$ which is the identity on $X$ and the Frobenius endomorphism on $k$.

Drinfeld proved that the above shtukas are the $k$-rational points of an algebraic stack over $\mathbb{F}_p$ which is equipped with a projection onto $X \times X$ given by the pole and the zero. More generally, he introduced level structures on rank $r$ shtukas and he constructed an algebraic stack $\text{Sht}_r$ parametrizing rank $r$ shtukas equipped with a compatible system of level structures. This last algebraic stack is endowed with an algebraic action of $GL_r(\mathbb{A})$ through the Hecke operators.
5. Iterated shtukas

The geometry at infinity of the moduli stack $\text{Sht}_r$ is amazingly complicated. The algebraic stack $\text{Sht}_r$ is not of finite type and one needs to truncate it to obtain manageable geometric objects. Bounding the Harder-Narasimhan polygon of a shtuka, Lafforgue defines a family of open substacks $(\text{Sht}_{\leq P}^r)_P$ which are all of finite type and whose union is the whole moduli stack. But in doing so, he loses the action of the Hecke operators which do not stabilize those open substacks.

In order to recover the action of the Hecke operators, Lafforgue enlarges $\text{Sht}_r$ by allowing specific degenerations of shtukas that he has called *iterated shtukas*.

More precisely, Lafforgue lets the isomorphism $\varphi : \mathcal{E} \longrightarrow \mathcal{E}''$ appearing in the definition of a shtuka, degenerate to a *complete homomorphism* $\mathcal{E} \Rightarrow \mathcal{E}''$, i.e. a continuous family of complete homomorphisms between the stalks of the vector bundles $\mathcal{E}$ and $\mathcal{E}''$.

Let me recall that a complete homomorphism $V \Rightarrow W$ between two vector spaces of the same dimension $r$ is a point of the partial compactification $\tilde{\text{Hom}}(V, W)$ of $\text{Isom}(V, W)$ which is obtained by successively blowing up the quasi-affine variety $\text{Hom}(V, W) - \{0\}$ along its closed subsets

$$\{f \in \text{Hom}(V, W) - \{0\} \mid \text{rank}(f) \leq i\}$$

for $i = 1, \ldots, r - 1$. If $V = W$ is the standard vector space of dimension $r$, the quotient of $\tilde{\text{Hom}}(V, W)$ by the action of the homotheties is the Procesi-De Concini compactification of $\text{PGL}_r$.

In particular, Lafforgue obtains a smooth compactification, with a normal crossing divisor at infinity, of any truncated moduli stack of shtukas without level structure.

6. One key of the proof

Lafforgue proves his main theorem by an elaborate induction on $r$. Compared to Drinfeld’s proof of the rank 2 case, a very simple but crucial novelty in Lafforgue’s proof is the distinction in the $\ell$-adic cohomology of $\text{Sht}_r$ between the $r$-negligible part (the part where all the irreducible constituents as Galois modules are of dimension $< r$) and the $r$-essential part (the rest). Lafforgue shows that the difference between the cohomology of $\text{Sht}_r$ and the cohomology of any truncated stack $\text{Sht}_{\leq P}^r$ is $r$-negligible. He also shows that the cohomology of the boundary of $\text{Sht}_{\leq P}^r$ is $r$-negligible. Therefore, the $r$-essential part, which is defined purely by considering the Galois action and which is naturally endowed with an action of the Hecke operators, occurs in the $\ell$-adic cohomology of any truncated moduli stack $\text{Sht}_{\leq P}^r$ and also in their compactifications.

At this point, Lafforgue makes an extensive use of the proofs by Richard Pink and Kazuhiro Fujiwara of a conjecture of Deligne on the Grothendieck-Lefschetz trace formula.
7. Compactification of thin Schubert cells

In proving the Langlands conjecture for functions fields, Lafforgue tried to construct nice compactifications of the truncated moduli stacks of shtukas with arbitrary level structures. A natural way to do that is to start with some nice compactifications of the quotients of $\text{PGL}_{r}^{n+1}/\text{PGL}_{r}$ for all integers $n \geq 1$, and to apply a procedure similar to the one which leads to iterated shtukas.

Lafforgue constructed natural compactifications of $\text{PGL}_{r}^{n+1}/\text{PGL}_{r}$. In fact, he remarked that $\text{PGL}_{r}^{n+1}/\text{PGL}_{r}$ is the quotient of $\text{GL}_{r}^{n+1}/\text{GL}_{r}$ by the obvious free action of the torus $\mathbb{G}_{m}^{n+1}/\mathbb{G}_{m}$ and that $\text{GL}_{r}^{n+1}/\text{GL}_{r}$ may be viewed as a thin Schubert cell in the Grassmannian variety of $r$-planes in a $r(n + 1)$-dimensional vector space. And, more generally, he constructed natural compactifications of all similar quotients of thin Schubert cells in the Grassmannian variety of $r$-planes in a finite-dimensional vector space.

Let me recall that thin Schubert cells are by definition intersections of Schubert varieties and that Israel Gelfand, Mark Goresky, Robert MacPherson and Vera Serganova constructed natural bijections between thin Schubert cells, matroids and certain convex polyhedra which are called polytope matroids.

For $n = 1$ and arbitrary $r$, Lafforgue’s compactification of $\text{PGL}_{r}^{2}/\text{PGL}_{r}$ coincides with the Procesi-De Concini compactification of $\text{PGL}_{r}$. It is smooth with a normal crossing divisor at infinity.

For $n = 2$ and arbitrary $r$, Lafforgue proves that his compactification of $\text{PGL}_{r}^{3}/\text{PGL}_{r}$ is smooth over a toric stack, and thus can be desingularized.

For $n \geq 3$ and $r \geq 3$, the geometry of Lafforgue’s compactifications is rather mysterious and not completely understood.

Gerd Faltings linked the search of good local models for Shimura varieties in bad characteristics to the search of smooth compactifications of $\mathbb{G}_{m}^{n+1}/G$ for a reductive group $G$. He gave another construction of Lafforgue’s compactifications of $\text{PGL}_{r}^{n+1}/\text{PGL}_{r}$ and he succeeded in proving that Lafforgue’s compactification of $\text{PGL}_{r}^{n+1}/\text{PGL}_{r}$ is smooth for $r = 2$ and arbitrary $n$.

8. Conclusion

I hope that I gave you some idea of the depth and the technical strength of Lafforgue’s work on the Langlands correspondence for which we are now honoring him with the Fields Medal.
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