We consider a qubit that is driven along its logical $z$ axis, with noise along the $z$ axis in the driving field $\Omega$ proportional to some function $f(\Omega)$, as well as noise along the logical $x$ axis. We establish that whether or not errors due to both types of noise can be canceled out, even approximately, depends on the explicit functional form of $f(\Omega)$ by considering a power law form, $f(\Omega) \propto \Omega^k$. In particular, we show that such cancellation is impossible for $k = 0, 1$, or any even integer. However, any other odd integer value of $k$ besides 1 does permit cancellation; in fact, we show that both types of errors can be corrected with a sequence of four square pulses of equal duration. We provide sets of parameters that correct for errors for various rotations and evaluate the error, measured by the infidelity, for the corrected rotations versus the naïve rotations. We also consider a train of four trapezoidal pulses, which take into account the fact that there will be, in real experimental systems, a finite rise time, again providing parameters for error-corrected rotations that employ such pulse sequences. Our dynamical decoupling error correction scheme works for any qubit platform as long as the errors are quasistatic.

I. INTRODUCTION

Error correction is one of the most important topics in the field of quantum computation. In fact, the whole subject of quantum computation as a practical field exists simply because quantum error correction is theoretically possible provided the error is not too large. Unlike the bits in a classical computer, which can only take one of two states, 0 and 1, a qubit in a quantum computer can take on an uncountably infinite number of different states of the form, $\alpha |0 \rangle + \beta |1 \rangle$, where $\alpha$ and $\beta$ are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. As a result, even small errors in the state of a qubit will adversely affect the results of a computation. Since decoherence-related errors arising from external noise in the environment (even in the absence of any explicit errors in gate operations) are unavoidable in a quantum system, carrying out practical quantum error corrections is the key roadblock in building effective quantum computing circuits. The development of techniques for correcting for errors is therefore of utmost importance. In fact, much research has been devoted to just this topic, taking various approaches. One such approach is to build in error resistance via engineering of the physical system and error correction using ancilla qubits. For example, in semiconductor-based electron spin qubits, magnetic noise due to the presence of magnetic isotopes (i.e., non-zero nuclear spin in the environment) is a major source of error. This may be reduced by such techniques as isotopic purification (if possible) and polarization of the atomic nuclei. Another important error in semiconductor (and other solid state) qubits is the charge noise in the environment. One method that employs ancilla qubits to perform error correction is the surface code technique. Topological qubits, such as the Majorana qubit, are, to an extent, immune to error (i.e., some, though not all, operations are protected). Error cancellation methods that use designed pulses include the NMR-inspired Hahn echo method and its generalization, the Carr-Purcell-Meiboom-Gill (CPMG) technique. Bayesian estimation of parameters and dynamical decoupling through specially-designed pulse sequences. In fact, carrying out error correction protocols through various alternative techniques, employing both software and hardware methods, is the single most active area in the subject of quantum computation, and progress in general has been slow in the sense that no qubit platform has yet experimentally achieved one single error-free logical qubit.

Our current work will focus on error correction via pulse engineering through the dynamical decoupling technique. We generalize the recent work of Ref. 13 which considered a qubit driven only along the logical $z$ axis subject to a noise term along its logical $x$ axis (orthogonal-to-driving-field noise), with a Hamiltonian of the form

$$H = \frac{i}{\hbar} \Omega(t) \sigma_z + \delta \beta \sigma_x,$$

where $\Omega(t)$ is the driving field and the $\delta \beta$ is a noise-induced error term. We summarize below the main finding of Ref. 13 relevant for our work. In Ref. 13 which itself is a special case of the general geometric method for dynamical decoupling developed in Ref. 14. It is shown that the conditions for correcting for this error term along the $x$ axis (with the control driving field along the $z$ axis) without any error in the driving field may be cast into a geometric picture; specifically, it is shown that the condition that results in cancellation of errors due to $\delta \beta$ to first order is equivalent to the condition that the curve traced out in the complex plane by

$$x(t) + iy(t) = \int_0^t e^{i\phi(t')} dt',$$

where $\phi(t)$, the rotation angle at time $t$, given by

$$\phi(t) = \frac{1}{\hbar} \int_0^t \Omega(t') dt',$$
must be closed, i.e.,
\[ \int_0^T e^{i\phi(t)} \, dt = 0, \tag{4} \]
where \( T \) is the duration of the gate. The driving field is simply proportional to the curvature of this curve,
\[ \frac{\Omega(t)}{\hbar} = \frac{\dot{x} y - \dot{y} x}{(x^2 + y^2)^{3/2}}. \tag{5} \]
Conditions for cancellation of higher-order terms are also found, imposing further restrictions on this curve. We will generalize these results to the case in which noise in the driving field is also present, i.e., we consider the Hamiltonian,
\[ H = \frac{1}{2} [\Omega(t) + f(\Omega) \delta\epsilon] \sigma_z + \delta\beta \sigma_x, \tag{6} \]
where \( f(\Omega) \) is some function of the driving field. In this case, \( \delta\epsilon \) represents fluctuations in some parameter that controls the driving field. We refer the reader to Ref. [13] for the details and the background on the geometric methods for dynamical decoupling and for a general review of the literature in this context, focusing entirely on our work on how to correct for errors existing along both the \( x \) and \( z \) axes as shown explicitly in Eq. (6) above.
Throughout this work, we will make the quasistatic approximation, i.e., we will assume that \( \delta\epsilon \) and \( \delta\beta \) remain constant for the entirety of a given gate. Thus, the applied pulses are thought to be fast compared with the dynamical time scale of the error terms in Eq. (6), which is a reasonable approximation for many types of “slow noise” arising in practical qubits. Also, this “slow-noise” approximation could always be satisfied, in principle, by making the external pulses and drives faster. We will also be considering power law forms for \( f(\Omega) \), i.e., \( f(\Omega) \propto \Omega^k \).

Our main goal is to determine whether or not correction for arbitrary rotations is impossible—in the former case, it is not possible to correct any rotation, and in the latter, only the identity operation, for which \( \phi = 0 \), can be corrected. This comes from the fact that the integral is just proportional to \( \phi \) when \( k = 1 \). Furthermore, even values of \( k \) do not allow for this condition to be satisfied; the integrand on the left-hand side will always be non-negative, and thus the integral can only be zero if \( \Omega(t) = 0 \) for all times \( t \). The only values of \( k \) for which error correction may be possible are therefore odd values other than 1. We show, in fact, that error correction is possible for these cases. This is simply due to the fact that Eq. (7) yields a distinct condition on the driving field from the condition that the operation result in a rotation by a specific angle \( \phi \) about the \( z \) axis,
\[ \frac{1}{\hbar} \int_0^T \Omega(t) \, dt = \phi, \tag{8} \]
rather than a redundant \((k = 1 \text{ and } \phi = 0) \) or contradictory \((k = 0 \text{ or } 1 \text{ and } \phi \neq 0) \) condition. We will consider the case of four square pulses of equal duration. Our choice of this sequence comes from the fact that there are four real-valued conditions, along with recent results indicating that the fastest pulses that cancel errors due to the \( \delta\beta \) term are square pulse[14]. The four real-valued conditions are just the two error-cancellation conditions, Eqs. (4) and (7), along with the simple requirement that the gate perform a rotation about the \( z \) axis by a specific angle. We provide parameters for this four-pulse sequence that cancel both types of error for \( k = -3, -1, \ldots \)
3, and 5, though similar parameters can be found for other suitable values of \( k \) as well.

While these sequences made up of square pulses provide a proof of principle demonstrating that error correction is possible for the forms of \( f(\Omega) \) considered, such perfect square pulses are not realizable in actual experiments. They assume that the driving field can be increased arbitrarily rapidly, which is not the case—there is always a finite rise time. As a result, we are interested in considering pulses that include such finite rise times for a practical implementation of our proposal. In particular, we consider a train of trapezoidal pulses, which consist of a linear rise or fall, followed by a constant segment. This choice is motivated by the fact that, for such a pulse sequence, the equations giving the constraints, while complicated, can be obtained in analytic form, albeit in terms of non-elementary functions (Fresnel integrals). We consider a train of four such pulses, plus a final ramp down to \( \Omega = 0 \), and repeat the above analysis, this time only for \( f(\Omega) \propto \Omega^3 \). We once again show that error correction can be done, this time for a more realistic pulse sequence.

The rest of this article is organized as follows. We derive the general conditions for cancellation of errors to first order in Sec. II. We then specialize to the case of four square pulses in Sec. III and that of four trapezoidal pulses in Sec. IV, providing pulse parameters for each case and evaluating the performance of these pulses in terms of the fidelity of the operations. Finally, we give our conclusions in Sec. V.

II. ERROR CANCELLATION CONDITIONS

Let us begin by deriving the error cancellation conditions, Eqs. (4) and (7). We will consider here a qubit with the Hamiltonian given in Eq. (6). We make the quasistatic approximation, i.e., we assume that \( \delta \beta \) and \( \delta \epsilon \) are independent of time. Furthermore, we assume that the error in the driving field is proportional to some function \( f(\Omega) \) of said driving field.

Without the error terms, we can solve for the evolution operator exactly; it is simply

\[
U_0(t) = \begin{bmatrix} e^{-i\phi(t)/2} & 0 \\ 0 & e^{i\phi(t)/2} \end{bmatrix},
\]

where

\[
\phi(t) = \frac{1}{\hbar} \int_0^t \Omega(t') \, dt'.
\]

We now add back in the error terms and attempt a perturbative solution in powers of \( \delta \beta \) and \( \delta \epsilon \). We decompose the full time evolution operator into two parts, \( U(t) = U_0(t)U_p(t) \). If we substitute this into the time-dependent Schrödinger equation for the system, we find that

\[
\hbar \frac{dU_p}{dt} = H_{\text{eff}} U_p,
\]

where

\[
H_{\text{eff}} = U_0^\dagger [H - \frac{1}{2} \Omega(t) \sigma_x] U_0
= \frac{1}{2} f(\Omega) \sigma_z \delta \epsilon + \sigma_x - \sin \phi(t) \sigma_y \delta \beta.
\]

Because \( U_p(t) \) is a \( 2 \times 2 \) unitary operator, it may be written in the general form,

\[
U_p(t) = \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix}.
\]

If we now substitute this into the effective Schrödinger equation, we obtain the following equations:

\[
\begin{align*}
\hbar \frac{du}{dt} &= \frac{1}{2} f(\Omega) u \, \delta \epsilon + e^{i\phi} v \, \delta \beta, \\
\hbar \frac{dv}{dt} &= -\frac{1}{2} f(\Omega) v \, \delta \epsilon + e^{-i\phi} u \, \delta \beta.
\end{align*}
\]

We now expand \( u \) and \( v \) in power series in \( \delta \beta \) and \( \delta \epsilon \):

\[
\begin{align*}
u(t) &= \sum_{k,l=0}^\infty g_{kl}(t) \delta \beta^k \delta \epsilon^l, \\
v(t) &= \sum_{k,l=0}^\infty h_{kl}(t) \delta \beta^k \delta \epsilon^l.
\end{align*}
\]

If we substitute these expansions into the above equations and evaluate the performance of these pulses in terms of the fidelity of the operations. Finally, we give our conclusions in Sec. V.
Since \( g_{01}(t) \) and \( h_{10}(t) \) are already identical to zero for all times \( t \), the condition for the cancellation of error due to orthogonal-to-driving-field noise to first order, represented by the \( \delta \beta \) term, is
\[
\int_0^T e^{-i\phi(t)} dt = 0, \tag{26}
\]
while that for driving field noise, represented by the \( \delta \epsilon \) term, is
\[
\int_0^T f[\Omega(t)] dt = 0. \tag{27}
\]
These are just Eqs. (4) and (7), respectively. The first of these conditions is the same as that which was previously obtained in Ref. \( \text{[13]} \) in the case in which only \( \delta \beta \) was present, i.e., assuming no noise in the driving field. The second simply imposes a further condition on the driving field. Note that, not unexpectedly, the existence of the longitudinal noise \( \delta \epsilon \) along the drive direction does not affect the condition for correcting the transverse error \( \delta \beta \).

We may show that, in fact, if \( \delta \beta = 0 \), then Eq. (7) will result in exact (i.e., to all orders) cancellation of errors due to driving field noise. To see this, we simply note that, if \( \delta \beta = 0 \), then we can solve for the evolution operator exactly, obtaining
\[
U(t) = \begin{bmatrix} e^{-r\Phi(t)/2} & 0 \\ 0 & e^{r\Phi(t)/2} \end{bmatrix}, \tag{28}
\]
where \( \Phi(t) = \phi(t) + \delta \phi(t) \) and
\[
\delta \phi(t) = \frac{1}{\hbar} \int_0^t f[\Omega(t')] dt' \delta \epsilon. \tag{29}
\]
We can thus cancel the error exactly if we require that \( \delta \phi(T) = 0 \), which gives us Eq. (7). Of course, we could also cancel the error by letting \( \delta \phi(T) = 2\pi n \), with \( n \) any integer, but a nonzero \( n \) would require a precise knowledge of \( \delta \epsilon \), thus making such error cancellation very impractical.

We now consider two simple cases—\( f(\Omega) = C \), where \( C \) is an arbitrary dimensionless constant, and \( f(\Omega) = \frac{\Omega}{\Omega_0} \), where \( \Omega_0 \) is an arbitrary constant with units of energy. In the first case, if we substitute the form of \( f(\Omega) \) into the left-hand side of Eq. (7), we get \( CT \). This condition can only be satisfied if \( C = 0 \) or \( T = 0 \), i.e., we either have no driving field noise at all or we simply do not perform an operation. If \( C \neq 0 \), we see that it is impossible to cancel the error for any operation. For the second case, in which the noise depends linearly on the driving field, we find that the left-hand side of Eq. (7) is just
\[
\frac{1}{\Omega_0} \int_0^T \Omega(t) dt. \tag{30}
\]
This is just proportional to \( \phi(T) \):
\[
\frac{1}{\Omega_0} \int_0^T \Omega(t) dt = \frac{\hbar}{\Omega_0} \phi(T). \tag{31}
\]

This tells us that, to cancel the driving field error, we would need to be simply performing an identity operation, i.e., \( \phi(T) = 0 \). Therefore, if the noise in the driving field is proportional to the driving field, then it is not possible to devise a means to cancel the driving field noise, even just to first order, for any non-trivial (i.e., other than the identity) operation.

We will show, however, that other forms of \( f(\Omega) \) do allow for cancellation of driving field noise for arbitrary operations. One such form is a power law dependence, \( f(\Omega) = \left( \frac{\Omega}{\Omega_0} \right)^{2n+1} \), with an odd power, i.e., for integer \( n \). We note that the power must be odd (and, as just shown, not equal to 1), since, if the power were even, then \( f(\Omega) \) will always be non-negative, and thus we cannot satisfy the driving field error cancellation condition, Eq. (7), unless \( \Omega(t) = 0 \) for all times \( t \).

## III. TRAIN OF SQUARE PULSES

We will demonstrate this by showing that a sequence of four square pulses can cancel both types of error if \( f(\Omega) \) is of such an odd (including negative integers) power law form. In fact, for such forms, we will find that the parameters do not depend on the constant \( \Omega_0 \), but only on the power \( 2n + 1 \). We require four pulses because we have four (real) conditions to satisfy in order to perform a rotation by an angle \( \phi \) and to cancel out error due to driving field and orthogonal-to-driving-field noise. We will assume throughout, without any loss of generality, that the four pulses are of equal duration, i.e., they each last for a time \( \frac{T}{4} \), and will label the value of the driving field for each segment \( \Omega_k \), with \( k \) running from 1 to 4. Doing this, the condition that we perform a rotation by \( \phi \) becomes
\[
\frac{T}{4\hbar} (\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4) = \phi. \tag{32}
\]
Let us now introduce the dimensionless quantities, \( \omega_k = \Omega_k T/\hbar \). We may then write
\[
\frac{T}{4} (\omega_1 + \omega_2 + \omega_3 + \omega_4) = \phi. \tag{33}
\]
Similarly, we may write the condition for canceling driving field noise as
\[
\omega_1^{2n+1} + \omega_2^{2n+1} + \omega_3^{2n+1} + \omega_4^{2n+1} = 0. \tag{34}
\]
Finally, the condition for canceling error from the \( \delta \beta \) term is
\[
\frac{e^{-i\omega_1/4} - 1}{\omega_1} + \frac{e^{-i\omega_1/4}(e^{-i\omega_2/4} - 1)}{\omega_2} + \frac{e^{-i(\omega_1 + \omega_2)/4} (e^{-i\omega_3/4} - 1)}{\omega_3} + \frac{e^{-i(\omega_1 + \omega_2 + \omega_3)/4} (e^{-i\omega_4/4} - 1)}{\omega_4} = 0. \tag{35}
\]
We will now need to solve for the values of the \( \omega_k \) numerically. We have obtained values for the cases, \( n = -2, -1, 1, \) and 2, corresponding to \( f(\Omega) \propto \Omega^{-3}, \) \( \Omega^{-1}, \) \( \Omega^{3}, \) and \( \Omega^{5}, \) respectively, and for arbitrary rotations, thus showing that it is in fact possible to correct for both driving field noise and orthogonal-to-driving-field noise. We present plots of the parameters that we obtain below as a function of the rotation angle \( \phi \) for \( 0 \leq \phi \leq 2\pi \) in Fig. 1, as well as tables of values for some of these rotations in Tables II–IV. These are by no means the only solutions to the equations given above; they are provided as examples to show that it is in fact possible to satisfy all of the required constraints for the values of \( n \) considered. Also, we note that these parameters are independent of the proportionality constant, as it cancels out of the equations.

![Plot of the dimensionless pulse parameters \( \omega_k \) as a function of the rotation angle \( \phi \) for \( f(\Omega) \propto \Omega^{n+1} \) and for \( n = -2 \) (a), -1 (b), 1 (c), and 2 (d).](image)

**TABLE I: Pulse parameters for different rotation angles \( \phi \) for \( f(\Omega) \propto \Omega^{-3}. \)**

| \( \phi \) | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) | \( \omega_4 \) |
|-----------|-------------|-------------|-------------|-------------|
| \( \pi/4 \) | 75.8215      | -223.357    | -75.8627    | 226.54      |
| \( \pi/2 \) | 76.3572      | -220.639    | -76.4437    | 227.008     |
| \( 3\pi/4 \) | 77.1698      | -218.215    | -77.3077    | 227.777     |
| \( \pi \)   | 78.4934      | -216.345    | -78.6919    | 229.11      |
| \( 5\pi/4 \) | 80.3912      | -215.143    | -80.6635    | 231.124     |
| \( 3\pi/2 \) | 82.5698      | -214.433    | -82.9295    | 233.642     |
| \( 7\pi/4 \) | 84.4567      | -214.117    | -84.9088    | 236.56      |
| \( 2\pi \)  | 81.5261      | -216.137    | -81.9458    | 241.689     |

**TABLE II: Pulse parameters for different rotation angles \( \phi \) for \( f(\Omega) \propto \Omega^{-1}. \)**

| \( \phi \) | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) | \( \omega_4 \) |
|-----------|-------------|-------------|-------------|-------------|
| \( \pi/4 \) | 286.42      | 109.271     | 290.092     | -109.801    |
| \( \pi/2 \) | 284.197     | 105.476     | 291.465     | -106.461    |
| \( 3\pi/4 \) | 282.446     | 102.662     | 293.266     | -104.057    |
| \( \pi \)   | 281.248     | 100.883     | 295.603     | -102.672    |
| \( 5\pi/4 \) | 280.291     | 99.7883     | 298.176     | -101.966    |
| \( 3\pi/2 \) | 279.218     | 99.0761     | 300.637     | -101.646    |
| \( 7\pi/4 \) | 277.66      | 98.6883     | 302.633     | -101.67     |
| \( 2\pi \)  | 274.699     | 98.8635     | 303.306     | -102.337    |

**TABLE III: Pulse parameters for different rotation angles \( \phi \) for \( f(\Omega) \propto \Omega^{3}. \)**

| \( \phi \) | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) | \( \omega_4 \) |
|-----------|-------------|-------------|-------------|-------------|
| \( \pi/4 \) | 8.36239     | -17.3526    | -1.66872    | 16.8005     |
| \( \pi/2 \) | 10.0363     | -18.837     | -2.77338    | 17.8573     |
| \( 3\pi/4 \) | 11.2749     | -20.0533    | -0.58396    | 18.7871     |
| \( \pi \)   | 11.9997     | -21.0243    | 1.96658     | 19.6244     |
| \( 5\pi/4 \) | 12.1933     | -21.8642    | 4.95667     | 20.4222     |
| \( 3\pi/2 \) | 12.2262     | -22.8384    | 8.26825     | 21.1937     |
| \( 7\pi/4 \) | 12.8317     | -24.1614    | 11.422      | 21.8898     |
| \( 2\pi \)  | 14.2087     | -25.753     | 14.2087     | 22.4684     |

**TABLE IV: Pulse parameters for different rotation angles \( \phi \) for \( f(\Omega) \propto \Omega^{5}. \)**

| \( \phi \) | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) | \( \omega_4 \) |
|-----------|-------------|-------------|-------------|-------------|
| \( \pi/4 \) | 9.1941      | -16.9967    | -5.90998    | 16.8542     |
| \( \pi/2 \) | 10.5926     | -18.3902    | -4.07217    | 18.153      |
| \( 3\pi/4 \) | 11.4567     | -19.5346    | -1.75394    | 19.2557     |
| \( \pi \)   | 11.7197     | -20.4685    | 1.10502     | 20.2102     |
| \( 5\pi/4 \) | 11.4261     | -21.3171    | 4.47574     | 21.1233     |
| \( 3\pi/2 \) | 11.0611     | -22.3193    | 7.94993     | 22.1579     |
| \( 7\pi/4 \) | 11.4561     | -23.655     | 10.7573     | 23.4328     |
| \( 2\pi \)  | 12.7367     | -25.2161    | 12.7367     | 24.8754     |

**A. Evaluation of error**

We now evaluate the error in our error-corrected pulse sequences against the corresponding naive operations. We will use the state-averaged infidelity as our measure of the error. The fidelity of a gate as a function of the initial state \( |\psi\rangle, F(\psi), \) is simply the probability that one will measure the qubit in the state that the gate was intended to evolve it into:

\[
F(\psi) = \left| \langle \psi \mid U^\dagger R \mid \psi \rangle \right|^2,
\]

where \( R \) is the ideal gate and \( U \) is the actual gate. We define the infidelity as \( 1 - F(\psi). \) The state-averaged in-
fidelity $1 - \bar{F}$ is simply the infidelity averaged over all distinct qubit states, parametrized as
\[
|\psi(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle.
\] (37)

The state-averaged infidelity may be written as
\[
1 - \bar{F} = 1 - \frac{1}{4\pi} \int d\Omega F[\psi(\theta,\phi)].
\] (38)

Evaluating the integral, we obtain
\[
1 - \bar{F} = \frac{2}{3} - \frac{1}{6} |\text{Tr}(U^\dagger R)|^2.
\] (39)

We evaluate the infidelity numerically for rotations by $\frac{\pi}{2}$ and for $f(\Omega) \propto \Omega^{2n+1}$ with $n = -2, -1, 1, 2$, and provide plots of the results for both the naïve and corrected sequences for two cases, $\delta \epsilon = 0$ and $\delta \epsilon = \delta \beta$ in Figs. 2–5. Note that we do not present plots for $\delta \beta = 0$, since, as argued earlier, the corrected sequences result in exact cancellation of error, and thus the infidelity for these sequences would be exactly zero for all values of $\delta \epsilon$. We choose to present just the plots for rotations by $\frac{\pi}{2}$ because they are representative of the general features found for all rotations. We see that, indeed, the corrected pulse sequences result in lower infidelities by several orders of magnitude than the naïve sequences. In some cases, the decrease in infidelity is especially large; this is likely connected to the fact that, if $\delta \beta$ were to be zero, then these sequences would exactly cancel noise-induced error. We note that the infidelity, in some cases, oscillates for larger values of $\delta \beta$. A similar effect can be seen in the results of Ref. [19] (which considered two-qubit gates), and is an interference effect.

\[\begin{align*}
\text{FIG. 2: Plot of the infidelity in rotations by } & \frac{\pi}{2} \text{ for } \delta \epsilon = 0 \text{ and as a function of } \delta \beta \text{ (left) and for } \delta \epsilon = \delta \beta \text{ and as a function of } \\
& \sqrt{(\delta \beta)^2 + (\delta \epsilon)^2} \text{ (right) for } f(\Omega) = \left(\frac{\Omega}{\Omega_0}\right)^3 \text{ and } \Omega_0 = 0.05 \frac{\hbar}{T}.
\end{align*}\]

The blue curves represent the naïve pulse sequences, while the red curves represent the corrected sequences.

\[\begin{align*}
\text{FIG. 3: Plot of the infidelity in rotations by } & \frac{\pi}{2} \text{ for } \delta \epsilon = 0 \text{ and as a function of } \delta \beta \text{ (left) and for } \delta \epsilon = \delta \beta \text{ and as a function of } \\
& \sqrt{(\delta \beta)^2 + (\delta \epsilon)^2} \text{ (right) for } f(\Omega) = \left(\frac{\Omega}{\Omega_0}\right)^3 \text{ and } \Omega_0 = 0.05 \frac{\hbar}{T}.
\end{align*}\]

The blue curves represent the naïve pulse sequences, while the red curves represent the corrected sequences.

\[\begin{align*}
\text{FIG. 4: Plot of the infidelity in rotations by } & \frac{\pi}{2} \text{ for } \delta \epsilon = 0 \text{ and as a function of } \delta \beta \text{ (left) and for } \delta \epsilon = \delta \beta \text{ and as a function of } \\
& \sqrt{(\delta \beta)^2 + (\delta \epsilon)^2} \text{ (right) for } f(\Omega) = \left(\frac{\Omega}{\Omega_0}\right)^3 \text{ and } \Omega_0 = 8 \frac{\hbar}{T}.
\end{align*}\]

The blue curves represent the naïve pulse sequences, while the red curves represent the corrected sequences.

\[\begin{align*}
\text{FIG. 5: Plot of the infidelity in rotations by } & \frac{\pi}{2} \text{ for } \delta \epsilon = 0 \text{ and as a function of } \delta \beta \text{ (left) and for } \delta \epsilon = \delta \beta \text{ and as a function of } \\
& \sqrt{(\delta \beta)^2 + (\delta \epsilon)^2} \text{ (right) for } f(\Omega) = \left(\frac{\Omega}{\Omega_0}\right)^3 \text{ and } \Omega_0 = 8 \frac{\hbar}{T}.
\end{align*}\]

The blue curves represent the naïve pulse sequences, while the red curves represent the corrected sequences.

\[\begin{align*}
\text{IV. TRAPEZOIDAL PULSES}
\end{align*}\]

We now consider the case of trapezoidal pulses. Such pulses are similar to the square pulses just considered, but include ramp up or down segments as well. We consider such a shape because, in reality, experimental setups cannot produce perfect square pulses—there will always be a finite rise or fall time. The basic building
blocks of these sequences will be of the form,
\[ \Omega(t) = \begin{cases} \Omega_n - \Omega_{n-1} t + \Omega_{n-1}, & 0 \leq t < \alpha T', \\ \Omega_n, & \alpha T' \leq t \leq T'. \end{cases} \quad (40) \]

This pulse represents a linear ramp up/down of the driving field from \( \Omega_{n-1} \) to \( \Omega_n \) over a time \( \alpha T' \), where \( T' \) is the duration of this segment of the pulse train and \( 0 < \alpha < 1 \) is an arbitrary constant determining what fraction of the segment is spent on the ramp up/down, followed by holding the field at a constant value \( \Omega_n \) for the remainder of the duration.

We will build up our corrected pulse sequences from four of these segments, plus a final ramp down to zero.

We also start the first segment from zero (i.e., ramp up from zero to a constant). For simplicity, we will assume that the four segments are all of equal length, and that all ramps up and down take the same amount of time (i.e., the four segments all have duration \( T' \), and all ramps up and down, including the final one, have duration \( \alpha T' \)).

We can express the formulas giving the error correction conditions in analytic form as follows. We first calculate the contributions to the left-hand sides of these conditions for a single segment of the form given above. Doing this, we find that the contributions to the (ideal) rotation angle about the \( z \) axis, to the left-hand side of Eq. \( 7 \) and to the real and imaginary parts of the left-hand side of Eq. \( 4 \) are, respectively,

\[
\phi_n = \omega'_n - \frac{1}{2}(\omega'_n - \omega'_n)\alpha, \quad (41)
\]

\[
\int_0^1 [\omega'_n(\tau)]^k d\tau = \frac{\omega'_n - \omega'_n}{\omega'_n - \omega'_n} \] \frac{\alpha}{k + 1} + (\omega')^n(1 - \alpha), \quad (42)
\]

\[
\text{Re} \int_0^1 e^{-i\phi_n(\tau)} d\tau = \frac{2}{\omega'_n} \cos \left( \frac{\omega'_n - \omega'_n}{\omega'_n - \omega'_n} \right) \sin \left[ \omega'_n(1 - \alpha) \right] + \sqrt{\frac{\pi \alpha}{\omega'_n - \omega'_n - 1}} \left\{ \cos \left( \left( \omega'_n - 1 \right)^2 \right) \left( \omega'_n - \omega'_n - 1 \right) \right\} + \sin \left( \frac{\alpha}{2} \omega'_n(1 - \alpha) \right) \left[ S \left( \frac{\alpha}{\omega'_n - \omega'_n - 1} \right) - S \left( \frac{\alpha}{\omega'_n - \omega'_n - 1} \right) \right], \quad (43)
\]

\[
\text{Im} \int_0^1 e^{-i\phi_n(\tau)} d\tau = -\frac{2}{\omega'_n} \sin \left( \frac{\omega'_n - \omega'_n}{\omega'_n - \omega'_n} \right) \sin \left[ \omega'_n(1 - \alpha) \right] + \frac{\pi \alpha}{\omega'_n - \omega'_n - 1} \left\{ \sin \left( \left( \omega'_n - 1 \right)^2 \right) \left( \omega'_n - \omega'_n - 1 \right) \right\} - \cos \left( \frac{\alpha}{2} \omega'_n(1 - \alpha) \right) \left[ S \left( \frac{\alpha}{\omega'_n - \omega'_n - 1} \right) - S \left( \frac{\alpha}{\omega'_n - \omega'_n - 1} \right) \right], \quad (44)
\]

where \( \tau = \frac{t}{T'} \), \( \omega'_n = \frac{T'}{T} \Omega_n \), and \( C(x) \) and \( S(x) \) are the Fresnel integrals,

\[
C(x) = \int_0^x \cos \left( \frac{\pi t^2}{2} \right) dt, \quad (45)
\]

\[
S(x) = \int_0^x \sin \left( \frac{\pi t^2}{2} \right) dt. \quad (46)
\]

We also derive similar expressions for the final ramp down:

\[
\phi_f = \frac{1}{2} \omega'_f \alpha, \quad (47)
\]

\[
\int_0^1 [\omega'_f(\tau)]^k d\tau = \frac{\omega'_f}{\omega'_f} \] \frac{\alpha}{k + 1} + \omega'_f(1 - \alpha), \quad (48)
\]

\[
\text{Re} \int_0^1 e^{-i\phi_f(\tau)} d\tau = \sqrt{\frac{\pi \alpha}{|\omega'_f|}} \left\{ \cos \left( \left( \frac{\alpha}{2} \right) \right) \left( \frac{\alpha}{\omega'_f} \right) \right\} \left[ S \left( \frac{\alpha}{\omega'_f} \right) - S \left( \frac{\alpha}{\omega'_f} \right) \right], \quad (49)
\]

\[
\text{Im} \int_0^1 e^{-i\phi_f(\tau)} d\tau = \text{sgn}(\omega'_f) \sqrt{\frac{\pi \alpha}{|\omega'_f|}} \left\{ \sin \left( \left( \frac{\alpha}{2} \right) \right) \right\} \}
\]

where \( \omega'_f = \frac{T'}{T} \Omega_f \) is the driving field at the beginning of this final ramp down. In order to obtain the error cancellation conditions, we can simply add the contributions given above to the intended rotation angle and to the driving field noise cancellation condition. However, one cannot simply add together the contributions to the \( \delta \beta \) cancellation conditions. The contributions to the overall rotation angle and to the left-hand side of Eq.
only involve a simple integral of a function of \(\Omega(t)\)—
\(\Omega(t)\) itself and \([\Omega(t)]^k\), respectively, to be exact—and thus
the contribution of each segment simply adds to that of
the previous segment(s). However, the contributions to
the left-hand side of Eq. (4) are more complicated—they
involve an integral of a function that itself involves an
integral of \(\Omega(t)\), namely, \(\exp\left[-i\int_0^T \Omega(t) dt\right]\). We may
still use the above expressions to build up the full error
cancellation conditions by noting that the contribution
from the \(n\)th segment acquires an extra phase factor of
\(\exp\left(-i\sum_{j=1}^{n-1} \phi_j\right)\) in the full expression for the \(\delta\beta\) error
cancellation condition.

As before, we solve the resulting equations numerically.
We consider here the case, \(f(\Omega) \propto \Omega^3\) and \(\alpha = 0.05\). We
find that, once again, it is possible to correct rotations
by arbitrary angles, and we give the parameters that we
find that, once again, it is possible to correct rotations
for certain rotations in Table V. It is straightforward, if
 tedious, to obtain similar parameters for other
powers (i.e., values of \(k\) different from 3).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\phi & \omega_1' & \omega_2' & \omega_3' & \omega_4' \\
\hline
\pi/4 & 2.36859 & -4.34927 & -1.39592 & 4.16199 \\
\pi/2 & 2.66739 & -4.73666 & -0.80657 & 4.46644 \\
3\pi/4 & 2.9033 & -5.03395 & -0.197971 & 4.68482 \\
\pi & 3.00887 & -5.26353 & 0.495858 & 4.90039 \\
5\pi/4 & 2.9789 & -5.47004 & 1.30658 & 5.11155 \\
3\pi/2 & 2.96033 & -5.73813 & 2.17134 & 5.31884 \\
7\pi/4 & 3.14526 & -6.10634 & 2.94228 & 5.51659 \\
2\pi & 3.52834 & -6.53153 & 3.59846 & 5.69151 \\
\hline
\end{array}
\]

TABLE V: Pulse parameters for different rotation angles \(\phi\) for
\(f(\Omega) \propto \Omega^3\) and using the trapezoidal pulses with \(\alpha = 0.05\).

A. Evaluation of error

We now evaluate these pulse sequences to show that
they indeed cancel noise-induced errors to first order. We
once again adopt the state-averaged infidelity, Eq. (39),
as our metric. In this case, we cannot solve for the evolu-
tion operator for the ramp up/down portions of the pulse
sequences analytically, so we must do so numerically. We
use as our “naïve” pulse sequence a single trapezoidal
pulse that begins and ends with \(\Omega = 0\) and has the same
duration as our corrected pulse sequence. We determine
the infidelity for the same two cases as for the rectangular
pulses, \(\delta\epsilon = 0\) and \(\delta\epsilon = \delta\beta\), and plot our results in Fig.

We see that, once again, the pulse sequences work as
intended—they cancel noise-induced errors to first order,
and reduce infidelity by many orders of magnitude if the
noise is sufficiently small.

V. CONCLUSION

We have determined conditions under which it is pos-
sible to cancel both (longitudinal) driving field noise and
(transverse) orthogonal-to-driving-field noise in a qubit
driven along its logical z axis. We considered the Hamil-
tonian, Eq. (10), which we repeat here for convenience:

\[
H = \frac{1}{2}[\Omega(t) + f(\Omega) \delta\epsilon] \sigma_z + \delta\beta \sigma_x,
\]

where \(f(\Omega)\) is some function of the driving field \(\Omega(t)\).
We specifically set out to determine what forms of this
function allow us to cancel the effects of the noise terms,
the driving field noise \(\delta\epsilon\) and orthogonal-to-driving-field
noise \(\delta\beta\). We considered power law forms for \(f(\Omega)\), i.e.,
\(f(\Omega) \propto \Omega^k\). We began by deriving the conditions for can-
celing the effects of these two error terms to first order.
We showed that the condition for canceling the effects of the \(\delta\beta\) term to first order was identical to that obtained in
Ref. [18] in the case that this was the only term present.
Therefore, the corresponding condition for the \(\delta\epsilon\) term
simply imposes a further constraint on the driving field.
This second constraint, unfortunately, does not permit
cancellation of driving field noise-induced errors in arbi-
trary rotations for the power law form of \(f(\Omega) \propto \Omega^k\) if the
exponent \(k = 0\) or 1. More specifically, cancellation of
these errors is only possible for the identity operation
if \(k = 1\), and is not possible for any operations if \(k = 0\).
Furthermore, it does not allow for cancellation if \(k\) is any
even integer. However, we find that, if \(k\) is any odd in-
teger besides 1, then we can correct noise-induced errors
in the presence of both driving field and orthogonal-to-
driving-field noise. We in fact determine four-part se-
quences of square pulses of equal duration (correspond-
ing to the number of real-valued constraints that we must
satisfy) that do just that. Our choice of square pulses is
motivated by the work of Ref. [17] which shows that,
for the system that we consider, square pulses minimize
the time duration of all possible error-correcting pulses.
We then evaluate the effectiveness of these sequences at combating errors by determining the infidelity of both the naïve and corrected versions of sequences for implementing rotations by $\frac{\pi}{2}$.

We should emphasize that the condition under which error correction is possible only applies to the system that we considered, in which the only intentionally applied field is the driving field along the logical $z$ axis, i.e., there are no fields applied along, say, the logical $x$ axis. For example, all of the work on SUPCODE and its generalizations\textsuperscript{11} considered a system in which there are driving fields along two axes (specifically a singlet-triplet qubit), i.e.,

$$H = \frac{1}{2}[\Omega(t) + f(\Omega) \delta \epsilon] \sigma_z + (\beta + \delta \beta) \sigma_x,$$  \hspace{1mm} \text{(52)}

assumes that $f(\Omega) \propto \Omega$, which we have just shown does not permit error cancellation for the Hamiltonian considered in this work. In fact, the specific platform being considered in those works is restricted to positive values of $\Omega(t)$, and thus error correction should be possible for any power law form of $f(\Omega)$. An important extension of the present work and of the work that inspired it\textsuperscript{13,14} would be to consider the case in which we include an intentionally applied field along the logical $x$ axis. This, however, would be a much more difficult problem to solve, as there is no known exact solution for arbitrary pulse shapes to the resulting Hamiltonian. This remains an important future problem to solve, but our work shows that, at least when the driving field is only along the logical $\sigma_z$ direction, geometrical dynamical decoupling techniques can indeed correct for errors arising from quasistatic noise existing along both $x$ and $z$ directions provided certain well-defined conditions are satisfied by the longitudinal noise term. Our results apply to all physical qubits obeying Eq. (6).

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1 A. G. Fowler, M. Mariantoni, J. M. Martinis, A. N. Cleland, Phys. Rev. A 86, 032324 (2012).
2 S. Das Sarma, M. Freedman, and C. Nayak, npj Quantum Information 1, 15001 (2015).
3 W. M. Witzel and S. Das Sarma, Phys. Rev. Lett. 98, 077601 (2007).
4 W. M. Witzel and S. Das Sarma, Phys. Rev. B 76, 241303 (2007).
5 B. Lee, W. M. Witzel, and S. Das Sarma, Phys. Rev. Lett. 100, 160505 (2008).
6 M. Shulman, S. P. Harvey, J. M. Nichol, S. D. Bartlett, A. C. Doherty, V. Umansky, and A. Yacoby, Nature Commun. 5, 5156 (2014).
7 A. Sergeevich, A. Chandran, J. Combes, S. D. Bartlett, and H. M. Wiseman, Phys. Rev. A 84 052315 (2011).
8 M. D. Shulman, O. E. Dial, S. P. Harvey, H. Bluhm, V. Umansky, and A. Yacoby, Science 336, 202 (2012).
9 H. Bluhm, S. Foletti, D. Mahalu, V. Umansky, A. Yacoby, Phys. Rev. Lett. 105, 216803 (2010).
10 H. Bluhm, S. Foletti, I. Neder, M. Rudner, D. Mahalu, V. Umansky, and A. Yacoby, Nat. Phys. 7, 109113 (2011).
11 J. T. Muhonen, J. P. Dehollain, A. Laucht, F. E. Hudson, T. Sekiguchi, K. M. Itoh, D. N. Jamieson, J. C. McCallum, A. S. Dzurak, and A. Morello, Nat. Nanotechnol. 9, 986 (2014).
12 F. K. Malinowski, F. Martins, P. D. Nissen, E. Barnes, L. Cywiński, M. S. Rudner, S. Fallahi, G. C. Gardner, M. J. Manfra, C. M. Marcus, and F. Kuemmeth, Nat. Nanotechnol. 12, 16 (2017).
13 J. Zeng, X.-H. Deng, A. Russo, and E. Barnes, New Journal of Physics 20, 033011 (2018).
14 E. Barnes, X. Wang and S. Das Sarma, Sci. Rep. 5 12685 (2015).
15 F. H. L. Koppens, C. Buizert, K. J. Tielrooij, I. T. Vink, K. C. Nowack, T. Meunier, L. P. Kouwenhoven, and L. M. K. Vandersypen, Nature 442, 766 (2006).
16 J. R. Petta, A. C. Johnson, J. M. Taylor, E. A. Laird, A. Yacoby, M. D. Lukin, C. M. Marcus, M. P. Hanson, A. C. Gossard, Science 309, 2180 (2005).
17 J. Zeng and E. Barnes, Phys. Rev. A 98, 012301 (2018).
18 M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK, 2000).
19 X. Wang, E. Barnes, and S. Das Sarma, npj Quantum Information 1, 15003 (2015).
20 X. Wang, L. S. Bishop, J. P. Kestner, E. Barnes, K. Sun, and S. Das Sarma, Nat. Commun. 3, 997 (2012).
21 X. Wang, L. S. Bishop, E. Barnes, J. P. Kestner, and S. Das Sarma, Phys. Rev. A 89, 022310 (2014).
22 R. E. Throckmorton, C. Zhang, X.-C. Yang, X. Wang, E. Barnes, and S. Das Sarma, Phys. Rev. B 96, 195424 (2017).