Background cosmological dynamics in $f(R)$ gravity and observational constraints

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In this paper, we carry out a study of viable cosmological models in $f(R)$-gravity at the background level. We use observable parameters like $\Omega$ and $\gamma$ to form autonomous system of equations and show that the models under consideration exhibit two different regimes in their time evolution, namely, a phantom phase followed by a quintessence like behavior. We employ statefinder parameters to emphasize a characteristic discriminative signature of these models.

I. INTRODUCTION

One of the most challenging problems of modern cosmology today is associated with the attempts to understand the late time acceleration of Universe which is supported by cosmological observations of complimentary nature such as Supernovae Ia, Cosmic Microwave Background anisotropies, Large Scale Structure formation, baryon oscillations and weak lensing observations. Many theoretical approaches have been employed to explain the phenomenon of late time cosmic acceleration. The standard lore assumes the presence of an exotic fluid known as dark energy. The simplest dark energy model based upon cosmological constant dubbed $\Lambda$CDM model suffers from extreme fine-tuning and coincidence problems. Scalar fields minimally coupled to gravity, called quintessence, with generic features might allow to alleviate these problems. Many other possibilities have been proposed, including a scalar field with a non-standard kinetic term (k-essence), or simply an arbitrary barotropic fluid with a pre-determined form for $p(\rho)$, such as the Chaplygin gas and its various generalizations.

As an alternative to dark energy, the large scale modifications of gravity could account for the current acceleration of universe. We know that gravity is modified at short distance and there is no guarantee that it would not suffer any correction at large scales where it is never verified directly. Large scale modifications might arise from extra dimensional effects or can be inspired by fundamental theories. They can also be motivated by phenomenological considerations such as $f(R)$ theories of gravity (see for a recent review). However, any large scale modification of gravity should reconcile with local physics constraints and should have potential of being distinguished from cosmological constant.

Most of the $f(R)$ gravity models proposed in the literature either ruled out by cosmological constraints imposed by the history or fail to meet the local gravity constraints. The viable $f(R)$ models can be distinguished from the $\Lambda$CDM by studying the the evolution of the growth of matter density perturbations. In this paper, we explore the possibility of discriminating the $\Lambda$CDM from viable models of $f(R)$ at the background level (see also Ref. on the related theme).

II. $f(R)$ COSMOLOGY

In what follows, it would be convenient to us to write $f(R)$ gravity action in the form

$$S = \frac{1}{16\pi G} \int \text{d}^4x \sqrt{-g} \left[ R + \epsilon(R) \right] + S_m(g_{\mu\nu}, \Psi_m)$$

where $G$ is the bare gravitational constant, $\epsilon(R)$ is a function of the curvature scalar $R$ only and $S_m$ is a functional of some matter fields $\Psi_m$ and metric $g_{\mu\nu}$.

In case of flat homogenous and isotropic universe

$$\text{ds}^2 = -\text{d}t^2 + a(t)^2 \left( \text{d}x^2 + \text{d}y^2 + \text{d}z^2 \right)$$

the action gives rise to the following evolution equations (we use the unit, $8\pi G = 1$)

$$3H^2 = \rho_m + \frac{R}{2} - 3H\dot{\epsilon}_R - 3H^2\epsilon_R$$

$$-2\dot{H} = \rho_m + \ddot{\epsilon}_R - H\dot{\epsilon}_R + 2H\epsilon_R$$

$$\dot{\rho}_m + 3H\rho_m = 0$$

The equations can be rewritten with the definition of the density of dark energy

$$3H^2 = \rho_m + \rho_{DE}$$

$$-2\dot{H} = \rho_m + \rho_{DE}(1 + w_{DE})$$

where $w_{DE} = P_{DE}/\rho_{DE}$.

In order to study the late time evolution in the $f(R)$ models under consideration, we introduce the set of variable $(\Omega, \gamma, R)$ ($\Omega$ is the ratio of the dark energy density and the critical density) and $\gamma = 1 + w_{DE}$

$$\Omega' = 3\Omega(1 - \Omega)(1 - \gamma)$$

$$\gamma' = -\frac{1}{\Omega} + (3\gamma - 1)(\gamma - 1) - \frac{1 - 3\Omega(\gamma - 1)}{3\Omega} \frac{R'}{R}$$

$$R' = -\frac{1}{\epsilon_{RR}} \left[ \Omega + \epsilon_R + \frac{\pi - \epsilon_R}{2} (1 - 3\Omega(\gamma - 1)) \right]$$

where a prime indicates differentiation with respect to $\ln a$. One can see this an autonomous system of equations involving the observable cosmological parameters.
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\[ \begin{align*}
\Omega &= 0.0 \\
\gamma &= 0.0
\end{align*} \]

(a) (b)

Figure 1: The left panel (a) shows the time evolution of model (1) for different values of the parameters \((n, \lambda)\) in the phase space \((\Omega, \gamma)\). For all parameters, the model is close to ΛCDM in the past followed by a phantom phase and the amplitude of this phase depends on the parameters of the model. For large values of \((n, \lambda)\), the model is indistinguishable from the ΛCDM model. The transition occurs between the phantom and non-phantom phases around \(\Omega < 0\) (for the range of parameters studied) before the matter-DE equivalence. The epoch of this transition depends on the parameters of the model. For large values of these parameters, the transition shifts towards the large values of \(\Omega\) corresponding to small values of redshift.

The right panel (b) shows the time evolution of the model (2) for various values of \(\lambda\). For all models, the de Sitter point \((\Omega, \gamma) = (1, 0)\) is an attractor.

In what follows, we shall be interested in the f(R) models which contain a standard matter phase \([27]\) to be compatible with the early universe physics (BBN) \((\epsilon(R) \rightarrow \text{Constant in the past})\) and give rise to the de-Sitter attractor at late times. Consistency also demands that these models should satisfy the stability criteria \(((1 + \epsilon_R > 0); (\epsilon''(R) > 0)[43])\) and be consistent with the local gravity constraints \([42]\).

Bearing in mind the aforesaid, we consider the following models:

- (A) \(\epsilon(x) = -\lambda R_c x^{2n}/(x^{2n} + 1) [39]\),
- (B) \(\epsilon(x) = -\lambda R_c (1 - (1 + x^2)^{-n}) [40]\),
- (C) \(\epsilon(x) = -\lambda R_c (1 - e^{-x}) [38]\),
- (D) \(\epsilon(x) = -\lambda R_c \tanh(x) [41]\),

where \(x = R/R_c\) and \(R_c\) of the order of the observed cosmological constant.

Investigation reveals that models (A), (B) and (C), (D) are cosmologically duplicate. Thus in the analysis to follow, we shall focus on models (A), (C) and refer to them as model (1) and model (2) respectively.

III. COSMOLOGICAL EVOLUTION

We have investigated models (1) & (2) numerically. To do this, we have assumed that initially the models are closed to ΛCDM with \(\gamma_i = 0\) and \(\Omega\) is negligibly small. We find that for all viable models belonging to these two classes, we have the same evolution in the \((\gamma, \Omega)\) plane. Fig. (1) shows that for both models, we have a phantom phase in the past \((\Omega < 0\). For the models (1), with smaller values of \(n\), the models deviate from ΛCDM whereas for models (2), the same is true for parameter \(\lambda\).

IV. STATEFINDER ANALYSIS

As demonstrated in \([37]\), it is possible to discriminate different models of dark energy from each other using the statefinder parameters \((r, s) [36]\),

\[ r = \frac{\ddot{a}}{aH^3} = 3 + \frac{1 - 3\Omega(\gamma - 1)}{2} \left( \frac{R'}{R} - 1 \right) \] \(\text{(11)}\)
where $r$ is the jerk parameter and $s$ is a function of the jerk and the decelerating parameter ($q$)

$$s = \frac{r - 1}{3(q - 1/2)} \quad (12)$$

The statefinder parameters are a natural next step beyond the Hubble function $H = \dot{a}/a$. By adding more derivative of the scale factor, Sahni et al. constructed geometrical parameters (defined using the metric only) which can discriminate various models. This method can be used to distinguish our models from ΛCDM model, characterized by $(r, s) = (1, 0)$.

For both types of models (1) and (2), we found the same evolution in the $(s, r)$ and $(q, r)$ planes. The models under consideration are close to the ΛCDM model in the past (see Fig.2) which appears clearly in the $(r, s)$ plane and shows that the system is close to the critical point $(r = 1, s = 0)$. The violation of the weak energy condition, which corresponds to the early evolution of the system, defined a loop in the upper left of the $(r, s)$-plane. In fact for a simple model like a power law evolution of the scale factor $a(t) \approx t^{3/3\gamma}$, we have $r = (1 - 3\gamma)(1 - 3\gamma/2)$ and $s = \gamma$. Then a violation of the weak energy condition ($\gamma < 0$) is characterized by $r > 1$ and $s < 0$. The system crosses the phantom line (Fig.1) given by $\gamma = 0$ which corresponds to the ΛCDM point in the $(r, s)$-plane. An another loop is exhibited by the model for the quintessence evolution of the system which corresponds to $r < 1$ and $s > 0$ in the simplest power law model.

It must be emphasized that the aforesaid features of cosmological evolution appear for all the viable models in $f(R)$ that we have studied. These models generically exhibit two different dynamical regimes, a phantom evolution in the past followed by quintessence like phase at late time.

V. OBSERVATIONAL CONSTRAINTS

We constrain the free parameters of the models studied by using Supernovae data and the BAO data. We used the compiled Constitution set [31] of 397 type Ia supernovae for which the $\chi^2$ is defined by

$$\chi^2_{SN1a} = \sum_i \frac{(\mu_{th,i} - \mu_{obs,i})^2}{\sigma_i^2} \quad (13)$$

with

Figure 2: The left panel (a) shows the time evolution of the statefinder pair \{s, r\} for the model (2). We have the same evolution for various values of the free parameter $\lambda$. All models outline loops around the ΛCDM model. The upper left part of this plane corresponds to the past evolution of the system where the model shows a phantom phase and the lower right part is the non-phantom phase.

The right panel (b) shows the time evolution of the pair \{q, r\}. The solid line is the time evolution of the ΛCDM model which divides the surface into 2 planes. The upper part is the phantom evolution of the model while the lower half is the non-phantom phase of the model. For all models the de-Sitter (dS) point, $(s, r) = (0, 1)$ or equivalently $(q, r) = (-1, 1)$, is an attractor.
Figure 3: 68%, 95% and 99.9% confidence intervals for the model (1). In the left panel (a) we imposed $\Omega_{m,0} = 0.2$ and $\Omega_{m,0} = 0.3$ for the right panel (b).

Figure 4: 68%, 95% and 99.9% confidence intervals for the model (2). The local gravity constraints and the stability of the de-Sitter phase impose $\lambda > 1$.

$\mu_{th, i} = 5 \log(d_L(z_i)) + \mu_0 + \frac{15}{4} \log\left(\frac{G_{\text{eff}}(z_i)}{G_{\text{eff}}(z = 0)}\right)$  \hspace{1cm} (14)

where $\mu_0 = 25 + 5 \log\left(\frac{cH_0^{-1}}{M_{\text{Pl}}^{-1}}\right)$ is marginalised \cite{33, 34} and $d_L$ is the luminosity-distance.

The addition of the last term in (14) takes into account a varying gravitational constant \cite{35}. We will not include this term in the numerical analysis. In fact it is negligible in viable $f(R)$-gravities models because of the thin-shell effect.

We also used the BAO distance ratio $D_v(z = 0.35)/D_v(z = 0.2) = 1.736 \pm 0.065$ \cite{35}, where

$$D_v(z) = \left[\frac{z}{H(z)} \left(\int_0^z \frac{dz'}{H(z')}\right)^2\right]^{1/3}$$  \hspace{1cm} (15)

In case of model (1) we fixed the value of $\Omega_{m,0}$ (Fig. 3) and find that the model is strongly sensitive to this parameter. For $\Omega_{m,0} = 0.2$, the small values of the parameters $(n, \lambda)$ are preferred. This is the range of the scalar regime \cite{20} which is crucially different from $\Lambda$CDM model. While the model is totally unconstrained for $\Omega_{m,0} = 0.3$.

We observe that the model 2 is unconstrained by the data (Fig. 4). The density of matter today is constrained around the concordance value and $\lambda$ appears like a free parameter.

We also use information criteria (IC) to assess the strength of models. These statistics favors models that give a good fit with fewer parameters. We use the Bayesian information criterion (BIC) and Akaike information criterion (AIC) to select the best fit models.
AIC and BIC are defined as

\[ AIC = -2\ln L + 2k \]  
\[ BIC = -2\ln L + k\ln N \]

Where \( L \) is the maximum likelihood, \( k \) is the number of parameters and \( N \) is the number of data used in the fit. For gaussian errors, \( \chi^2 = -2\ln L \), we plot the best fit values of the AIC and BIC as the function of \( \Omega_{m,0} \) for the standard model based on cosmological constant and model 2 respectively (Fig. 5) (For model (1) this has been already done in [44]. We can see from the figure that while the cosmological constant gives slightly better fit, when larger or smaller values of \( \Omega_{m,0} \) are considered the AIC and BIC tests shows model 2 is slightly favoured over \( \Lambda \)CDM. For model 2, \( \Delta AIC = 2 \) and \( \Delta BIC = 6 \).

VI. CONCLUSION

In this paper, we have examined the cosmological dynamics of two different classes of viable \( f(R) \)-gravity models. To begin with, we have formed an autonomous system of equations involving observable cosmological parameters like \( \gamma \) and \( \Omega \) together with the Ricci scalar \( R \). This is interesting set of equations and can give interesting results upon phase-plane analysis.

We then use this system equations to study the cosmological behaviour of the models. For this we assume that in the early time the models are close to \( \Lambda \)CDM. As a generic feature, these models exhibit two distinct regimes in the time-evolution of the system. In the past, the models violate the weak energy condition a la the phantom phase followed by a quintessence like behavior at the present epoch. A simple analysis at small redshift can lead to wrong conclusion that the quintessence model fits the data perfectly. Fitting our models with SNIa data as well as with the BAO, we see that except for small \( \Omega_m \) where there is strong bound on \( n \) for models (1), there is no significant constraints on both the models from cosmological observations.

We emphasize that a comprehensive analysis should be conducted independently in two different ranges of the redshift, i.e., after and before the equivalence between dark-energy and matter. In our opinion, this could be a smoking gun for models which with different evolutionary phases of dark energy.

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