INTRODUCTION TO CHTOUCAS FOR REDUCTIVE GROUPS AND TO THE GLOBAL LAGNGLANDS PARAMETERIZATION

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INTRODUCTION

This is a translation in English\footnote{This is an experiment, see \url{http://www.math.jussieu.fr/~vlafforg/bilingual.pdf} for general ideas about bilingual versions.} of version 2 of [Laf14]. Readers who understand French are advised to read [Laf14].

We explain the result of [Laf12] and give all the ideas of the proof. This article corresponds essentially to the introduction of [Laf12] (slightly expanded) and to subsections 11.1 and 11.2 (shortened). Compared to version 4 of [Laf12], it brings an improvement that will be incorporated in the introduction of the version 5 of [Laf12] (namely that the properties a), b), c) of proposition 1.3 below are true and not only heuristic).

We show the direction “automorphic to Galois” of the global Langlands correspondence \cite{Lan70} for all reductive group $G$ over a function field. Moreover we construct a canonical decomposition of the space of cuspidal automorphic forms, indexed by global Langlands parameters. We do not obtain any new result when $G = GL_r$ since everything was already known by Drinfeld \cite{Dri78, Dri87, Dri88, Dri89} for $r = 2$ and Laurent Lafforgue \cite{Laf02a} for $r$ arbitrary.

The method is completely independent of Arthur-Selberg trace formulas. It uses the following ingredients

- classifying stacks of chtoucas, introduced by Drinfeld for $GL_r$ \cite{Dri78, Dri87} and generalized to all reductive groups by Varshavsky \cite{Var04}
- the geometric Satake equivalence of Lusztig, Drinfeld, Ginzburg, and Mirkovic–Vilonen \cite{Lus82, Gin95, BD99, MV07}.

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Contents. In section 1 we state the main result (for \( G \) split), the idea of the proof and two intermediate statements which give the structure of the proof. Section 2, which is a summary of subsections 11.1 and 11.2 of [Laf12], discusses the non split case, some complements and a number of open problems. Section 3 introduces the stacks of choucas. Section 4 shows that the “Hecke-finite” part of their cohomology satisfies some properties which were stated in proposition 1.3 of section 1. With the help of these properties we show in section 5 the main theorem, which had been stated in section 1. In sections 6 to 9 we show some properties of the cohomology of the stacks of choucas, which had been admitted in section 4 and at the end of section 5. Section 10 explains the link with the geometric Langlands program. Lastly section 11 discusses the relation with previous works.

Depending on the time at his disposal, the reader may restrict himself to section 1, to sections 1 and 2, or to sections 1 to 5. Sections 1, 2 and 5 do not require any background in algebraic geometry.

1. Statement and main ideas

1.1. Preliminaries. Let \( \mathbb{F}_q \) be a finite field. Let \( X \) be a smooth projective geometrically irreducible curve over \( \mathbb{F}_q \) and let \( F \) be its field of functions. Let \( G \) be a connected reductive group over \( \mathbb{F}_q \). Let \( \ell \) be a prime number not dividing \( q \).

To state the main theorem we assume that \( G \) is split (the non split case will be explained in section 2.1). We denote by \( \hat{G} \) the Langlands dual group of \( G \), considered as a split group over \( \mathbb{Q}_\ell \). Its roots and weights are the coroots and coweights of \( G \), and reciprocally (see [Bor79] for more details).

For topological reasons we have to work with finite extensions of \( \mathbb{Q}_\ell \) instead of \( \mathbb{Q}_\ell \). Let \( E \) be a finite extension of \( \mathbb{Q}_\ell \) containing a square root of \( q \) and let \( \mathcal{O}_E \) be its ring of integers.

Let \( v \) be a place of \( X \). We denote by \( \mathcal{O}_v \) the completed local ring at \( v \) and by \( F_v \) its field of fractions. We have the Satake isomorphism \( [V] \mapsto h_{V,v} \) from the ring of representations of \( \hat{G} \) (with coefficients in \( E \)) to the Hecke algebra \( C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), E) \) (see [Sat63, Car79, Gro98]). In fact the \( h_{V,v} \) for \( V \) irreducible form a basis over \( \mathcal{O}_E \) of \( C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathcal{O}_E) \). We denote by \( A = \prod_{v \in \mathcal{X}} F_v \) the ring of adeles of \( F \) and we write \( \mathcal{O} = \prod_{v \in \mathcal{X}} \mathcal{O}_v \). Let \( N \) be a finite subscheme of \( X \). We denote by \( \mathcal{O}_N \) the ring of functions of \( N \), and by

\[
K_N = \text{Ker}(G(\mathcal{O}) \to G(\mathcal{O}_N))
\]

the compact open subgroup of \( G(\mathcal{A}) \) associated to the level \( N \). We fix a lattice \( \Xi \subset Z(F)\backslash Z(\mathcal{A}) \) (where \( Z \) is the center of \( G \)). A function \( f \in C_c(G(F)\backslash G(\mathcal{A})/K_N\Xi, E) \) is said to be cuspidal if for all parabolic subgroup \( P \subset G \), of Levi \( M \) and of unipotent radical \( U \), the constant term
obtain a canonical decomposition

Theorem 1.1. We will construct the following "excursion operators". Let $I$ be a finite set, $f$ be a function over $\hat{G}(\mathcal{F})/\hat{G}$ (the coarse quotient of $G(\mathcal{F})$ by the left and right translations by the diagonal $\hat{G}$), and $(\gamma_i)_{i \in I} \in (\text{Gal}(\mathcal{F}/\mathcal{F}))^I$. We will construct the "excursion operator"

$$S_{I,f,(\gamma_i)_{i \in I}} \in \text{End}(C_c(K_N \backslash G(\mathcal{A})/K_N, E))(C^c_c(G(F) \backslash G(\mathcal{A})/K_N, E)),$$

We will show that these operators generate a commutative subalgebra of $B$.

We do not know if $B$ is reduced but by spectral decomposition we nevertheless obtain a canonical decomposition

$$C^c_c(G(F) \backslash G(\mathcal{A})/K_N, \mathcal{F}) = \bigoplus_{\nu} \mathcal{H}_\nu,$$

where the direct sum in the RHS is indexed by the characters $\nu$ of $B$, and where $\mathcal{H}_\nu$ is the generalized eigenspace associated to $\nu$. We will show later that to every character $\nu$ of $B$ corresponds a unique Langlands parameter $\sigma$ (in the sense of the following theorem), characterized by (1.4) below. Setting $\mathcal{H}_\sigma = \mathcal{H}_\nu$, we will deduce the following theorem.

**Theorem 1.1.** We have a canonical decomposition of $C_c(K_N \backslash G(\mathcal{A})/K_N, \mathcal{F})$-

$$C^c_c(G(F) \backslash G(\mathcal{A})/K_N, \mathcal{F}) = \bigoplus_{\sigma} \mathcal{H}_\sigma,$$

where the direct sum in the RHS is indexed by global Langlands parameters, i.e. $\hat{G}(\mathcal{F})$-conjugacy classes of morphisms $\sigma : \text{Gal}(\mathcal{F}/\mathcal{F}) \rightarrow \hat{G}(\mathcal{F})$ defined over a finite extension of $\mathcal{F}$, continuous, semisimple and unramified outside $N$.

This decomposition is characterized by the following property: $\mathcal{H}_\sigma$ is equal to the generalized eigenspace $\mathcal{H}_\nu$ associated to the character $\nu$ of $B$ defined by

$$\nu(S_{I,f,(\gamma_i)_{i \in I}}) = \nu((\sigma(\gamma_i))_{i \in I}).$$

It is compatible with the Satake isomorphism at every place $v$ of $X \backslash N$, i.e. for every irreducible representation $V$ of $\hat{G}$, $T(h_{V,v})$ acts on $\mathcal{H}_\sigma$ by multiplication by the scalar $\chi_V(\sigma(\text{Frob}_v))$, where $\chi_V$ is the character of $V$ and $\text{Frob}_v$ is an arbitrary lifting of a Frobenius element at $v$. It is also compatible with the limit over $N$.

The compatibility with the Satake isomorphism at the places of $X \backslash N$ shows that this theorem realizes the global Langlands "correspondence" in the direction "from automorphic to Galois". In fact, except in the case of $GL_r$ [Laf02a], the conjectures of Langlands rather consist of

- a parameterization, obtained in the theorem above,
the Arthur multiplicity formulas for the $\mathcal{F}_\sigma$, which we are not able to handle with the methods of this article.

1.3. **Main ideas of the proof.** To construct the excursion operators and prove this theorem the strategy will be the following. The stacks of chtoucas, which play a role analogous to the Shimura varieties over number fields, exist in a much greater generality. Indeed, while the Shimura varieties are defined over an open subscheme of the spectrum of the ring of integers of a number field and are associated to a minuscule coweight of the dual group, we possess for every finite set $I$, every level $N$ and every irreducible representation $W$ of $(\hat{G})^I$ a stack of chtoucas $\text{Cht}_{N,I,W}$ which is defined over $(X \setminus N)^I$.

We will then construct a $E$-vector space $H_{I,W}$ (where the letter $N$ is omitted to shorten the formulas) as a subspace of the intersection cohomology of the fiber of $\text{Cht}_{N,I,W}$ over a geometric generic point of $(X \setminus N)^I$. More precisely we will take this intersection cohomology with compact support, with coefficients in $E$ and in degree 0 (for the perverse normalization). Using the "partial Frobenius morphisms" introduced by Drinfeld, we will endow $H_{I,W}$ with an action of $\text{Gal}(\overline{F}/F)^I$.

**Remark 1.2.** In this article we will define this subspace $H_{I,W}$ by a technical condition (of finiteness under the action of the Hecke operators) but in a project with Yakov Varshavsky we hope to prove that $H_{I,W}$ is the "cuspidal" part of the intersection cohomology and that it is of finite dimension.

Using the fact that the geometric Satake equivalence is canonical and tensorial w.r.t. the fusion product, we will show that the $H_{I,W}$ come from canonical functors $W \mapsto H_{I,W}$, equipped with the data of the isomorphisms (1.5) below.

**Proposition 1.3.** The $H_{I,W}$ satisfy the following properties:

a) for every finite set $I$,

\[ W \mapsto H_{I,W}, \ u \mapsto \mathcal{H}(u) \]

is a $E$-linear functor from the category of finite-dimensional $E$-linear representations of $(\hat{G})^I$ to the category of inductive limits of continuous finite-dimensional $E$-linear representations of $\text{Gal}(\overline{F}/F)^I$.

b) for every map $\zeta: I \to J$, we possess an isomorphism

\[ \chi_\zeta: H_{I,W} \xrightarrow{\sim} H_{J,W^\zeta}, \tag{1.5} \]

which is

- functorial in $W$, where $W$ is a representation of $(\hat{G})^J$ and $W^\zeta$ denotes the representation of $(\hat{G})^I$ on $W$ obtained by composition with the diagonal morphism

\[ (\hat{G})^J \to (\hat{G})^I, (g_j)_{j \in J} \mapsto (g_{\zeta(i)})_{i \in I} \]

- $\text{Gal}(\overline{F}/F)^I$-equivariant, where $\text{Gal}(\overline{F}/F)^J$ acts on the LHS by the diagonal morphism

\[ \text{Gal}(\overline{F}/F)^J \to \text{Gal}(\overline{F}/F)^I, (\gamma_j)_{j \in J} \mapsto (\gamma_{\zeta(i)})_{i \in I}. \tag{1.6} \]
– and compatible with the composition, i.e. \( \chi \zeta \circ \zeta' = \chi \circ \chi' \).

c) for \( I = \emptyset \) and \( W = 1 \), we have an isomorphism

\[
H_{\emptyset,1} = C_c^{\text{cusp}}(G(F) \backslash G(A)/K_N \Xi, E).
\]

Moreover the \( H_{I,W} \) are modules over \( C_c(K_N \backslash G(A)/K_N, E) \), in a way compatible with the properties a), b), c) above.

**Remark 1.4.** For every finite set \( J \), applying b) to the obvious map \( \zeta : \emptyset \to J \), we get an isomorphism \( \chi \zeta : H_{\emptyset,1} \sim H_{J,1} \) (where 1 is the trivial representation) and therefore the action of \( \text{Gal}(\overline{F}/F)^J \) over \( H_{J,1} \) is trivial.

Thanks to c) the decomposition (1.3) we are looking for is equivalent to a decomposition

\[
(1.7) \quad H_{\emptyset,1} = \bigoplus_{\sigma} \mathcal{H}_\sigma
\]

(where, increasing \( E \) if necessary, we assume that the \( \sigma \) and \( \mathcal{H}_\sigma \) are defined over \( E \)).

The following definition, where we construct the excursion operators, will be repeated in section 5. The reader may consult now, if he wishes, the subsection 1.4 below for a heuristic description of the \( H_{I,W} \), which enlightens a posteriori the definition of the excursion operators.

We need to consider a set with one element and we denote it by \( \{0\} \). For every finite set \( I \) we denote by \( \zeta_I : I \to \{0\} \) the obvious map, so that \( W^{\zeta_I} \) is nothing but \( W \) equipped with the diagonal action of \( \hat{G} \).

**Definition 1.5.** For every function \( f \in \mathcal{O}(\hat{G} \backslash (\hat{G})^I / \hat{G}) \) we can find a representation \( W \) of \( (\hat{G})^I \), and \( x \in W \) and \( \xi \in W^* \) invariant by the diagonal action of \( \hat{G} \), such that

\[
(1.8) \quad f((g_i)_{i \in I}) = \langle \xi, (g_i)_{i \in I} \cdot x \rangle.
\]

Then the endomorphism \( S_{f,I,(\gamma_i)_{i \in I}} \) of

\[
H_{\{0\},1} \xrightarrow{\chi_{\phi}^{-1}} H_{\emptyset,1} = C_c^{\text{cusp}}(G(F) \backslash G(A)/K_N \Xi, E)
\]

is defined as the composition

\[
(1.9) \quad H_{\{0\},1} \xrightarrow{3(\xi)} H_{\{0\},W^{\zeta_I}} \xrightarrow{\chi_{\phi}^{-1}} H_{I,W} \xrightarrow{(\gamma_i)_{i \in I}} H_{I,W} \xrightarrow{\chi_{\phi}^{-1}} H_{\{0\},W^{\zeta_I}} \xrightarrow{3(\xi)} H_{\{0\},1}
\]

where \( x : 1 \to W^{\zeta_I} \) and \( \xi : W^{\zeta_I} \to 1 \) are considered here as morphisms of representations of \( \hat{G} \).

We will show in section 5, using properties a) and b) of proposition 1.3, that \( S_{f,I,(\gamma_i)_{i \in I}} \) does not depend on the choice of \( W, x, \xi \) satisfying (1.8) and is therefore well-defined. We will show then that these excursion operators satisfy the following properties (which are the expected ones because they are tautologically satisfied by the RHS of (1.4)).
Proposition 1.6. The excursion operators $S_{I,f,(\gamma_i)_{i \in I}}$ satisfy the following properties:

(i) for every $I$ and $(\gamma_i)_{i \in I} \in \text{Gal}(\overline{F}/F)^I$,

$$f \mapsto S_{I,f,(\gamma_i)_{i \in I}}$$

is a morphism of commutative algebras $\mathcal{O}(\hat{G}\backslash(\hat{G})^I/\hat{G}) \to \mathcal{B}$,

(ii) for every map $\zeta : I \to J$, every function $f \in \mathcal{O}(\hat{G}\backslash(\hat{G})^I/\hat{G})$ and every $(\gamma_j)_{j \in J} \in \text{Gal}(\overline{F}/F)^J$, we have

$$S_{I,f,(\gamma_i)_{i \in I}} = S_{I,f,(\gamma_{\zeta(i)})_{i \in I}}$$

where $f^\zeta \in \mathcal{O}(\hat{G}\backslash(\hat{G})^J/\hat{G})$ is defined by

$$f^\zeta((g_j)_{j \in J}) = f((g_{\zeta(i)})_{i \in I}),$$

(iii) for every $f \in \mathcal{O}(\hat{G}\backslash(\hat{G})^I/\hat{G})$ and $(\gamma_i)_{i \in I}, (\gamma'_i)_{i \in I}, (\gamma''_i)_{i \in I} \in \text{Gal}(\overline{F}/F)^I$ we have

$$S_{I,f,(\gamma_i)_{i \in I}} = S_{I,f,(\gamma_i(\gamma_i')^{-1}\gamma''_i)_{i \in I}}$$

where $I \cup I \cup I$ is a disjoint union and $\tilde{f} \in \mathcal{O}(\hat{G}\backslash(\hat{G})^{I\cup I\cup I}/\hat{G})$ is defined by

$$\tilde{f}((g_i)_{i \in I} \times (g_i')_{i \in I} \times (g_i'')_{i \in I}) = f((g_i(\gamma_i')^{-1}g_i'')_{i \in I}).$$

• (iv) for every $I$ and $f$, the morphism

$$(1.10) \quad \text{Gal}(\overline{F}/F)^I \to \mathcal{B}, \quad (\gamma_i)_{i \in I} \mapsto S_{I,f,(\gamma_i)_{i \in I}}$$

is continuous, when $\mathcal{B}$ is equipped with the $E$-adic topology,

• (v) for every place $v$ of $X \setminus N$ and every irreducible representation $V$ of $\hat{G}$, the Hecke operator

$$T(h_{V,v}) \in \text{End}(C_c^{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K_N \Xi, E))$$

is equal to the excursion operator $S_{1,2,(\text{Frob}_v,1)}$, where $f \in \mathcal{O}(\hat{G}\backslash(\hat{G})^2/\hat{G})$ is given by $f(g_1, g_2) = \chi_V(g_1g_2^{-1})$, and $\text{Frob}_v$ is an arbitrary lifting of a Frobenius element at $v$.

In fact properties (i), (ii), (iii) and (iv) will follow formally from proposition 1.3 and (v) will be obtained by a geometric argument (the computation of the composition of two cohomological correspondences between stacks of chitoucas).

In section 5 we will deduce quite easily from the previous proposition that to every character $\nu$ of $\mathcal{B}$ corresponds a Langlands parameter $\sigma$ satisfying (1.4), unique up to conjugation by $\hat{G}(\overline{Q}_\ell)$. Indeed the knowledge of $\nu(S_{I,f,(\gamma_i)_{i \in I}})$ (which has to be equal to $f((\sigma(\gamma_i))_{i \in I})$) for every function $f$ determines the image of the $I$-uplet $(\sigma(\gamma_i))_{i \in I} \in (\hat{G}(\overline{Q}_\ell))^I$ as a point defined over $\overline{Q}_\ell$ of the coarse quotient $\hat{G}\backslash(\hat{G})^I/\hat{G}$. Taking $I = \{0, \ldots, n\}$ we see that

$$(\hat{G})^n \# \hat{G} \cong (\hat{G}\backslash(\hat{G})^{[0,\ldots,n]}/\hat{G}, (g_1, \ldots, g_n) \mapsto (1, g_1, \ldots, g_n)$$

is an isomorphism, where $(\hat{G})^n \# \hat{G}$ denotes the coarse quotient of $(\hat{G})^n$ by diagonal conjugation. Therefore for every integer $n$ and every $n$-uplet $(\gamma_1, \ldots, \gamma_n) \in$
Gal($\overline{F}/F$)$^n$, the knowledge of $\nu$ determines the image of $(\sigma(\gamma_1), ..., \sigma(\gamma_n))$ as a point defined over $\overline{\mathbb{Q}}_l$ of the coarse quotient $(\hat{G})^n/\hat{G}$. Thanks to results of [Ric88] based on geometric invariant theory, this means that $\nu$ determines $(\sigma(\gamma_1), ..., \sigma(\gamma_n)) \in (\hat{G}(\overline{\mathbb{Q}}_l))^n$ up to semisimplification and diagonal conjugation. Since we require $\sigma$ to be semisimple, it is clear (choosing $n$ and $(\gamma_1, ..., \gamma_n)$ such that the subgroup generated by $\sigma(\gamma_1), ..., \sigma(\gamma_n)$ is Zariski dense in the image of $\sigma$) that these data determine $\sigma$ up to conjugation. Conversely the relations (i), (ii), (iii) and (iv) satisfied by the excursion operators will allow, in proposition 5.7, to prove the existence of $\sigma$ satisfying (1.4) and property (v) will ensure the compatibility with the Satake isomorphism at the places of $X \setminus N$.

1.4. A heuristic remark. This subsection suggests a conjectural description of the $H_{I,W}$, which justifies a posteriori the definition of the excursion operators given in (1.9). Of course this conjectural description of the $H_{I,W}$ will never appear in the arguments and this subsection will be used nowhere in the rest of the article.

We conjecture that there exists a finite set $\Sigma$ (depending on $N$) of semisimple Langlands parameters (well defined up to conjugation), and that, increasing $E$ if necessary, we have for each $\sigma \in \Sigma$ a $E$-linear representation $A_\sigma$ of the centralizer $S_\sigma$ of the image of $\sigma$ in $\hat{G}$ (trivial on $Z(\hat{G})$), in such a way that for every $I$ and $W$

$$
(1.11) \quad H_{I,W} = \bigoplus_{\sigma \in \Sigma} \left( A_\sigma \otimes_E W_{\sigma I} \right)^{S_\sigma},
$$

where $W_{\sigma I}$ denotes the representation of $\text{Gal}(\overline{F}/F)^I$ obtained by composition of the representation $W$ with the morphism $\sigma^I : \text{Gal}(\overline{F}/F)^I \to (\hat{G}(E))^I$. Moreover $A_\sigma$ should be a module over $C_c(K_N \backslash G(\mathbb{A})/K_N, E)$, and (1.11) should be an isomorphism of $C_c(K_N \backslash G(\mathbb{A})/K_N, E)$-modules. In the particular case where $I = \emptyset$ and $W = 1$, (1.11) should be the decomposition (1.7) and one should have $\delta_{I,\sigma} = (A_\sigma)^{S_\sigma}$.

There conjectures are well known by experts, by extrapolation of the conjectures of [Kot90] on the multiplicities in the cohomology of Shimura varieties. In the case of $GL_r$ we expect that $\Sigma$ is the set of irreducible representations of dimension $r$ of $\pi_1(X \setminus N, \overline{\gamma})$ and that for every $\sigma \in \Sigma$, $S_\sigma = \mathbb{G}_m = Z(\hat{G})$ and $A_\sigma = (\pi_\sigma)^{K_N}$ where $\pi_\sigma$ is the cuspidal automorphic representation corresponding to $\sigma$ (see [Laf02a] and the conjecture 2.35 of [Var04]). In general if $\sigma$ is associated to an elliptic Arthur parameter $\psi$ (as in subsection 2.2 below), $A_\sigma$ should be induced from a finite dimensional representation of the subgroup of $S_\sigma$ generated by the centralizer of $\psi$ and the diagonal $\mathbb{G}_m \subset SL_2$ (because we consider only the cohomology in degree 0).

We assume moreover that (1.11) is functorial in $W$ and that for every map $\zeta : I \to J$ it intertwines $\chi_\zeta$ with

$$
\text{Id} : \bigoplus_{\sigma} \left( A_\sigma \otimes_E W_{\sigma I} \right)^{S_\sigma} \to \bigoplus_{\sigma} \left( A_\sigma \otimes_E (W^{\zeta})_{\sigma J} \right)^{S_\sigma}
$$

where
(since $W_{\sigma}$ and $(W^\xi)_{\sigma}$ are both equal to $W$ as $E$-vector spaces, the meaning of $\text{Id}$ is clear). Under these hypotheses, the composition (1.9) (which defines $S_{I,F,(\gamma_i)_{i\in I}}$) acts on $\mathcal{H}_\sigma = (A_\sigma)^{S_\sigma} \subset H_{[0,1]}$ by the composition

$$
(A_\sigma)^{S_\sigma} \xrightarrow{\text{Id}_{A_\sigma} \otimes x} (A_\sigma \otimes E W_\sigma)^{S_\sigma} \xrightarrow{\text{Id}_{A_\sigma} \otimes (\gamma_i)_{i\in I}} (A_\sigma \otimes E W_\sigma)^{S_\sigma} \xrightarrow{\text{Id}_{A_\sigma} \otimes \xi} (A_\sigma)^{S_\sigma}
$$

i.e. by the product by the scalar $\langle \xi, (\sigma(\gamma_i))_{i\in I} \cdot x \rangle = f((\sigma(\gamma_i))_{i\in I})$. This justifies a posteriori the definition of the $S_{I,F,(\gamma_i)_{i\in I}}$ (and suggests that these operators are diagonalizable, but we do not know how to prove it).

The conjecture (1.11) is not proven but Drinfeld has suggested that properties a) and b) of proposition 1.3 (and perhaps also the hypothesis that the $H_{I,W}$ are finite-dimensional, which Yakov Varshavsky and I hope to establish in a joint project) would be enough to imply a decomposition in the style of (1.11) (but more difficult to state, because the data of $\Sigma$ and the $A_\sigma$ would be replaced by a “$O$-module on the stack of global Langlands parameters”).

This subsection was heuristic and from now on we forget conjecture (1.11).

2. **Non split case, complements and open questions**

2.1. **Case where $G$ is not necessarily split.** Here we give only the statements and we refer to chapter 11 of [Laf12] for the proofs, which do not require new ideas compared to the split case. Let $G$ be a connected reductive group over $F$. Let $\widetilde{F}$ be a finite extension of $F$ splitting $G$ and $L^G = \widetilde{G} \rtimes \text{Gal}(\widetilde{F}/F)$ (where the semi-direct product is taken for the action of $\text{Gal}(\widetilde{F}/F)$ by automorphisms of $\widetilde{G}$ preserving a splitting). Let $U$ be an open subscheme of $X$ over which $G$ is reductive. At each point of $X \smallsetminus U$ we choose a Bruhat-Tits parahoric model [BT84] for $G$, so that $G$ is a smooth group scheme over $X$. It is convenient to assume that the level $N$ is big enough so that $X \smallsetminus N \subset U$. We denote by $\text{Bun}_{G,N}$ the (smooth [Hei10]) stack classifying the $G$-torsors over $X$ with structure of level $N$, in other words for every scheme $S$ over $\mathbb{F}_q$, $\text{Bun}_{G,N}(S)$ is the groupoid classifying the $G$-torsors $\mathcal{G}$ over $X \times S$ equipped with a trivialization of $\mathcal{G}|_{N \times S}$.

We define $K_N$ as previously in (1.1). We have

$$
\text{Bun}_{G,N}(\mathbb{F}_q) = \bigcup_{\alpha \in \ker^1(F,G)} G_\alpha(F) \backslash G_\alpha(\widehat{A}) / K_N
$$

where the union is disjoint, $\ker^1(F,G)$ is finite and $G_\alpha$ is the pure inner form of $G$ obtained by torsion by $\alpha$. For every $\alpha \in \ker^1(F,G)$ we have $G_\alpha(\widehat{A}) = G(\widehat{A})$ and therefore the quotient by $K_N$ in the RHS makes sense.

We fix a lattice $\Xi \subset Z(\widehat{A}) / Z(F)$. We define

$$
C_{c}^{\text{cusp}}(\text{Bun}_{G,N}(\mathbb{F}_q)/\Xi, E) = \bigoplus_{\alpha \in \ker^1(F,G)} C_{c}^{\text{cusp}}(G_\alpha(F) \backslash G_\alpha(\widehat{A}) / K_N \Xi, E).
$$

Then the excursion operators are endomorphisms

$$
S_{I,F,(\gamma_i)_{i\in I}} \in \text{End}_{C_c(K_N \backslash G(\widehat{A}) / K_N \Xi, E)}(C_{c}^{\text{cusp}}(\text{Bun}_{G,N}(\mathbb{F}_q)/\Xi, E))
$$
where $I$ is a finite set, $(\gamma_i)_{i \in I} \in \text{Gal}(\overline{F}/F)^I$ and $f$ is a function over the coarse quotient $\hat{G}\backslash(L^1G)^I/\hat{G}$. The method to construct them is the same as in the split case, thanks to a twisted variant over $X \setminus N$ of the geometric Satake equivalence (the unramified case of [Ric12, Zhu14]). In this variant $L^1G$ intervenes because the splitting of $\hat{G}$ appears naturally in the fiber functor of Mirkovic-Vilonen (indeed this fiber functor is given by the total cohomology and the splitting is determined by the graduation by the cohomological degree and by the cup-product by the $c_1$ of a very ample line bundle over the affine grassmannian). The excursion operators generate a commutative subalgebra $B$ and by spectral decomposition with respect to the characters of $B$ we get a decomposition

$$C_{c}^{\text{cusp}}(\text{Bun}_{G,N}(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_{\sigma} \mathcal{H}_\sigma.$$  

(2.3)

The direct sum in the RHS is indexed by global Langlands parameters, i.e. the $\hat{G}(\overline{\mathbb{Q}_\ell})$-conjugacy classes of morphisms $\sigma : \text{Gal}(\overline{F}/F) \to L^1G(\overline{\mathbb{Q}_\ell})$ defined over a finite extension of $\mathbb{Q}_\ell$, continuous, semisimple, unramified outside $N$ and giving rise to the commutative diagram

$$\begin{array}{ccc}
\text{Gal}(\overline{F}/F) & \xrightarrow{\sigma} & L^1G(\overline{\mathbb{Q}_\ell}) \\
\downarrow & & \downarrow \\
\text{Gal}(\overline{F}/F) & & \\
\end{array}$$  

(2.4)

As in theorem 1.1 the decomposition (2.3) is characterized by (1.4), and it is compatible with the (twisted) Satake isomorphism [Sat63, Car79, Bor79, BR94] at all places of $X \setminus N$.

**Remark 2.1.** Usually (for instance in the Arthur multiplicity formulas) we take the quotient of the set of morphisms $\sigma$ by a weaker equivalence relation, which, in addition to the conjugation by $\hat{G}(\overline{\mathbb{Q}_\ell})$, allows to twist $\sigma$ by elements of $\ker^1(F, Z(\hat{G})(\overline{\mathbb{Q}_\ell}))$. By Kottwitz [Kot84, Kot86] (and theorem 2.6.1 of Nguyen Quoc Thang [NQT11] for the adaptation to characteristic $p$), this finite group is the dual of $\ker^1(F, G)$. Therefore it has the same cardinal and from the point of view of Arthur multiplicity formulas our finer equivalence relation on $\sigma$ compensates exactly the fact that our space is a sum indexed by $\alpha \in \ker^1(F, G)$. For example, if $G$ is a torus, the subspaces $\mathcal{H}_\sigma$ of (2.3) are of dimension 1.

**Remark 2.2.** When $G$ is split, $\ker^1(F, Z(\hat{G})(\overline{\mathbb{Q}_\ell}))$ is 0 by the theorem of Tchebotarev, hence $\ker^1(F, G)$ is 0 also and $\text{Bun}_{G,N}(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/K_N$. This is why the quotient $G(F)\backslash G(\mathbb{A})/K_N\Xi$ appeared in section 1, and will appear again in section 3 and afterwards (where we will come back to the split case to simplify the redaction).

We refer to proposition 11.4 of [Laf12] for the fact that the decomposition (2.3) is compatible with the isogenies of $G$ (and more generally with all the morphisms $G \to G'$ whose image is normal).
We hope that the decomposition (2.3) is compatible with all the known cases of functoriality given by an explicit kernel. In particular we should be able to prove it is compatible with the theta correspondence, thanks to the geometrization of the theta kernel by Lysenko [Lys06, Lys11] and to the link between our construction and [BV06] (explained in section 10).

2.2. Arthur parameters. We would like to prove that the Langlands parameters \( \sigma \) which appear in the decomposition (1.3) (or (2.3) in the non split case) come from elliptic Arthur parameters. We recall that an Arthur parameter is a \( \hat{G}(\mathbb{Q}_\ell) \)-conjugacy class of morphism

\[
\psi : \text{Gal}(\overline{F}/F) \times \text{SL}_2(\mathbb{Q}_\ell) \to \hat{L}(\mathbb{Q}_\ell) \text{ (algebraic over } SL_2(\mathbb{Q}_\ell)),
\]

whose restriction to \( \text{Gal}(\overline{F}/F) \) takes its values in a finite extension of \( \mathbb{Q}_\ell \), is semisimple, pure of weight 0, continuous and makes diagram (2.4) commute. Moreover \( \psi \) is said to be elliptic if the centralizer of \( \psi \) in \( \hat{G}(\mathbb{Q}_\ell) \) is finite modulo \( (\hat{G}(\mathbb{Q}_\ell))_{\text{Gal}(\overline{F}/F)} \).

The Langlands parameter associated to \( \psi \) is \( \sigma_\psi : \text{Gal}(\overline{F}/F) \to L(\mathbb{Q}_\ell) \) defined by

\[
\sigma_\psi(\gamma) = \psi\left(\gamma, \begin{pmatrix} |\gamma|^{1/2} & 0 \\ 0 & |\gamma|^{-1/2} \end{pmatrix}\right)
\]

where \( |\gamma|^{1/2} \) is well defined thanks to the choice of a square root of \( q \). We conjecture that every Langlands parameter \( \sigma \) occurring in the decomposition (1.3) (or (2.3) in the non split case) is of the form \( \sigma_\psi \) with \( \psi \) an elliptic Arthur parameter. By [Kos59] the \( \hat{G}(\mathbb{Q}_\ell) \)-conjugacy class of \( \psi \) is uniquely determined by the \( \hat{G}(\mathbb{Q}_\ell) \)-conjugacy class of \( \sigma \). To prove the previous conjecture we could hope to use the actions of the Lefschetz \( SL_2 \) on the cohomology of compactifications of the stacks of chtoucas (the compactifications seem necessary in order to apply the hard Lefschetz theorem).

We would like to obtain a canonical decomposition as (1.3) (or (2.3) in the non split case) for the whole discrete part (and not only the cuspidal part) and this decomposition should be indexed by elliptic Arthur parameters.

2.3. Meaning of the decomposition. The decomposition (1.3) is finer in general than the one obtained by diagonalization of the Hecke operators at the unramified places. Even the isomorphism classes of representations of \( C_\tau(K_N \setminus G(\mathbb{A}_e)/K_N, \mathbb{Q}_\ell) \) do not allow to determine in general the decomposition (1.3), and although the Arthur multiplicity formulas are stated with a sum indexed by Arthur parameters, such a canonical decomposition seems unknown in general in the case of number fields. Indeed after Blasius [Bla94], Lapid [Lap99] and Larsen [Lar94, Lar96], for some groups \( G \) (including split ones) the same representation may occur in different subspaces \( \mathcal{H}_\sigma \), because of the following phenomenon. There are examples of finite groups \( \Gamma \) and of morphisms \( \tau, \tau' : \Gamma \to \hat{G}(\mathbb{Q}_\ell) \) such that \( \tau \) and \( \tau' \) are not conjugated but that for every \( \gamma \in \Gamma \), \( \tau(\gamma) \) and \( \tau'(\gamma) \) are conjugated. We expect then there could exist a surjective everywhere unramified morphism \( \rho : \text{Gal}(\overline{F}/F) \to \Gamma \) and a representation
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$(H_\pi, \pi)$ of $G(\mathbb{A})$ such that $(H_\pi)^{K_N}$ occurs in both $\mathcal{H}_r$ and $\mathcal{H}_r'$. The examples of Blasius and Lapid are for $G = SL_r$, $r \geq 3$ (in fact in this case we can recover \textit{a posteriori} the decomposition (1.3) thanks to the embedding $SL_r \hookrightarrow GL_r$). But for some groups (for instance $E_8$) we do not know how to recover the decomposition (1.3) by other means than the methods of the present article, which work only for function fields.

2.4. Independence of $\ell$. We hope that the decomposition (1.3) is defined over $\overline{\mathbb{Q}}$, independent of $\ell$ (and of the embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$), and indexed by motivic Langlands parameters. This conjecture seems out of reach for the moment. We refer to conjecture 11.11 of [Laf12] for a more precise statement.

2.5. Case of number fields. It is obviously out of reach to apply the methods of this article to number fields. However one can hope a decomposition analogous to the canonical decomposition (1.3) (or (2.3) in the non split case) and an Arthur multiplicity formula for each of the spaces $\mathcal{H}_\sigma$. When $F$ is a function field as in this article, the limit $\lim_{\leftarrow N} \text{Bun}_{G,N}(\mathbb{F}_q)$ is equal to $(G(\mathbb{F}) \backslash G(\mathbb{A} \otimes \mathbb{F}))^{\text{Gal}(\mathbb{F}/F)}$. This expression still makes sense for number fields and to understand its topology we just notice it is equal

- to $(G(\tilde{F}) \backslash G(\mathbb{A} \otimes F \tilde{F}))^{\text{Gal}(\tilde{F}/F)}$ where $\tilde{F}$ is a finite Galois extension of $F$
- to $\bigcup_{\alpha \in \ker^1(F,G)} G_\alpha(F) \backslash G_\alpha(\mathbb{A})$ where $\ker^1(F,G)$ is finite and $G_\alpha$ is a inner form of $G$.

We can hope that if $F$ is a number field and if $\Xi$ is a lattice in $Z(F) \backslash Z(\mathbb{A})$, the discrete part

$$L^2_{\text{disc}} \left( (G(\bar{F}) \backslash G(\mathbb{A} \otimes \bar{F}))^{\text{Gal}(\bar{F}/F)}/\Xi, \mathbb{C} \right)$$

admits a \textit{canonical} decomposition indexed by the $\hat{G}(\mathbb{C})$-conjugacy classes of elliptic Arthur parameters. The particular case which is the most similar to the case of function fields is the case of cohomological automorphic forms. Indeed the cohomological part of (2.5) is defined over $\overline{\mathbb{Q}}$ and we can hope that it admits a \textit{canonical} decomposition over $\overline{\mathbb{Q}}$ indexed by equivalence classes of elliptic motivic Langlands parameters (with the subtleties of [BG11] about the difference between $L$-algebraicity and $C$-algebraicity).

2.6. Importance of the sum over $\ker^1$. In the non split case we don’t know if the inclusion

$$C^c_{\text{cusp}}(G(F) \backslash G(\mathbb{A})/K_N \Xi, \overline{\mathbb{Q}}_\ell) \subset C^c_{\text{cusp}}(\text{Bun}_{G,N}(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}}_\ell)$$

(corresponding to the case $\alpha = 0$ in the RHS of (2.1)) is compatible with the decomposition (1.3), even after we regroup together the $\sigma$ which differ by an element of $\ker^1(F, Z(\hat{G})(\overline{\mathbb{Q}}_\ell))$. So, except in the case when the sum (2.1) is reduced to one term (i.e. when $\ker^1(F, Z(\hat{G})(\overline{\mathbb{Q}}_\ell)) = 0$, and in particular when $G$ is an inner form of a split group), we do not obtain a canonical decomposition of the space $C^c_{\text{cusp}}(G(F) \backslash G(\mathbb{A})/K_N \Xi, \overline{\mathbb{Q}}_\ell)$. 

2.7. **Coefficients in finite fields.** Thanks to the fact that the geometric Satake equivalence is defined over \( \mathfrak{O}_E \) [MV07, Gai07], we can prove that when the function \( f \) is defined over \( \mathfrak{O}_E \), \( SI,f, (\gamma_i)_{i \in I} \), \( \gamma_i \in I \) preserves \( C^c_c(Bun_{G,N}(F_q)/\Xi, \mathfrak{O}_E) \). By spectral decomposition of the reduction of these operators modulo the maximal ideal of \( \mathfrak{O}_E \) we get a decomposition of \( C^c_c(Bun_{G,N}(F_q)/\Xi, \mathfrak{O}_E) \) indexed by the classes of \( \widehat{G}(\mathbb{F}_\ell) \)-conjugation of morphisms \( \sigma : \text{Gal}(\overline{F}/F) \to L^G(\mathbb{F}_\ell) \) defined over a finite field, continuous, making the diagram analogous to (2.4) commute, and completely reducible in the sense of Jean-Pierre Serre [Ser05, BMR05] (i.e. if the image is included in a parabolic subgroup of \( L^G \), it is included in an associated Levi subgroup). We refer to chapter 12 of [Laf12] for more details.

2.8. **Local parameters.** The local Langlands parameterization (up to semisimplification) and the local-global compatibility will be proven in a work in progress with Alain Genestier [GL14]. They will be deduced from the following statement: if \( v \) is a place of \( X \) and if all the \( \gamma_i \) belong to \( \text{Gal}(\overline{F}_v/F_v) \), then \( S_{I,f,(\gamma_i)_{i \in I}} \) is equal to the action of an element of the \( (\ell \text{-adic completion of the}) \) Bernstein center of \( G(F_v) \) (obviously the interesting case is when \( v \in N \) because the unramified case is completely solved by the compatibility with the Satake isomorphism in theorem 1.1).

2.9. **Multiplicities.** This work does not say anything about the multiplicities, and in particular it does not say for which Langlands parameters \( \sigma \) the space \( H_\sigma \) is non zero.

2.10. **Case of metaplectic groups.** All the results of this article continue to hold for metaplectic groups, thanks to the metaplectic version of the geometric Satake equivalence established in [FL10]. We refer to chapter 13 of [Laf12] for more details.

3. **Chtoucas of Drinfeld for the reductive groups, after Varshavsky**

In the rest of this article \( G \) is split (the non split case, which was discussed in subsection 2.1 above, is handled in chapter 11 of [Laf12] but no new idea is necessary). The geometric ingredients of our construction are explained in this section and in section 6. Here is short overview. The intersection cohomology with compact support of the stacks of chtoucas provides for every finite set \( I \), for every level \( N \) and for every representation \( W \) of \((\widehat{G})^I \) an inductive system
\[
\lim_{\mu \to \mu} \mathcal{H}_{N,I,W}^{\leq \mu} \text{ of constructible } E\text{-sheaves over } (X \setminus N)^I.
\]
The goal of this section is to construct this inductive system, functorially in \( W \), and to equip it with actions of the Hecke operators and of the partial Frobenius morphisms \( F_{(i)} \), and to establish the coalescence isomorphisms (3.13) which describe its restriction by a diagonal morphism \((X \setminus N)^I \to (X \setminus N)^I \) (associated to an arbitrary map \( I \to J \)). In section 6 we will show that the Hecke operators at unramified places can be rewritten with the help of coalescence isomorphisms and partial Frobenius morphisms. It is this property that will ensure the compatibility of our construction with the Satake isomorphism at unramified places. It will also
play a fundamental technical role by allowing to extend the Hecke operators to morphisms of sheaves over the whole \((X \setminus N)^I\), and by providing the Eichler-Shimura relations. These relations (which will be stated in proposition 6.4 below) claim that for each place of \(X \setminus N\) and for each \(i \in I\) the restriction at \(x_i = v\) of the partial Frobenius morphism \(F_{(i)}\) is killed by a polynomial whose coefficients are Hecke operators at \(v\) (with coefficients in \(O_E\)). They will be used in section 7 to prove that the property of finiteness under the action of the Hecke operators (which gives the definition of the \(H_{I, W}\)) implies a property of finiteness under the action of the partial Frobenius morphisms, whence, thanks to a fundamental lemma of Drinfeld, the action of \((\text{Gal} \langle \overline{F} / F \rangle)^I\) over \(H_{I, W}\). We warn the reader that this use of the Eichler-Shimura relations is completely unusual.

The chtwoucas were introduced by Drinfeld [Dri78, Dri87] for \(GL_r\) and generalized to arbitrary reductive groups (and arbitrary coweights) by Varshavsky in [Var04] (meanwhile the case of division algebras was considered by Laumon-Rapoport-Stuhler, Laurent Lafforgue, Ngô Bao Châu and Eike Lau, see the references at the beginning of section 11). Let \(I\) be a finite set and \(W\) be an \(E\)-linear irreducible representation of \((\hat{G})^I\). We write \(W = \bigotimes_{i \in I} W_i\) where \(W_i\) is an irreducible representation of \(\hat{G}\). The stack \(\text{Ch}_{N, I, W}^{(I)}\) classifying the \(G\)-chtoucas with structure of level \(N\), was studied in [Var04]. Contrary to [Var04] we require it to be reduced in the following definition (of course this does not matter for étale cohomology).

**Notation.** For every scheme \(S\) over \(\mathbb{F}_q\) and for every \(G\)-torsor \(\mathcal{G}\) over \(X \times S\) we write \(\tau \mathcal{G} = (\text{Id}_X \times \text{Frob}_S)^*(\mathcal{G})\).

**Definition 3.1.** We define \(\text{Ch}_{N, I, W}^{(I)}\) as the reduced Deligne-Mumford stack whose points over a scheme \(S\) over \(\mathbb{F}_q\) classify

- points \(x_i \in I : S \to (X \setminus N)^I\),
- a \(G\)-torsor \(\mathcal{G}\) over \(X \times S\),
- an isomorphism

\[
\phi : \big|_{(X \times S) \setminus (\bigcup_{i \in I} \Gamma_{x_i})} \mathcal{G} \sim \big|_{(X \times S) \setminus (\bigcup_{i \in I} \Gamma_{x_i})} \tau \mathcal{G}
\]

where \(\Gamma_{x_i}\) denotes the graph of \(x_i\), such that the relative position at \(x_i\) is bounded by the dominant coweight of \(G\) corresponding to the dominant weight \(\omega_i\) of \(W_i\),
- a trivialization of \((\mathcal{G}, \phi)\) over \(N \times S\).

This definition will be generalized in definition 3.5 below, and the condition about the relative position will be made more precise in remark 3.7.

We write \(\text{Ch}_{I, W}^{(I)}\) when \(N\) is empty and we note that \(\text{Ch}_{N, I, W}^{(I)}\) is a \(G(O_N)\)-torsor over \(\text{Ch}_{I, W}^{(I)}\).

**Remark 3.2.** Readers who know the geometric Langlands program will recognize that \(\text{Ch}_{N, I, W}^{(I)}\) is the intersection of a Hecke stack (considered as a correspondence from \(\text{Bun}_{G, N}\) to itself) with the graph of the Frobenius morphism of \(\text{Bun}_{G, N}\).
The $x_i$ will be called the legs of the chtouca. We will denote by

$$p_{N,I,W}^{(I)} : \text{Ch}_{N,I,W}^{(I)} \to (X \setminus N)^I$$

the corresponding morphism.

For every dominant coweight $\mu$ of $G^{nd}$ we denote by $\text{Ch}_{N,I,W}^{(I),\leq \mu}$ the open substack of $\text{Ch}_{N,I,W}^{(I)}$ defined by the condition the the Harder-Narasimhan polygon of $\mathcal{G}$ is $\leq \mu$. We denote by $p_{N,I,W}^{(I),\leq \mu}$ the restriction of $p_{N,I,W}^{(I)}$ to $\text{Ch}_{N,I,W}^{(I),\leq \mu}$. We fix a lattice $\Xi \subset Z(F) \setminus Z(\mathbb{A})$. Then $\text{Ch}_{N,I,W}^{(I),\leq \mu}/\Xi$ is a Deligne-Mumford stack of finite type.

We denote by $\mathcal{IC}_{\text{Ch}_{N,I,W}^{(I),\leq \mu}/\Xi}$ the IC sheaf of $\text{Ch}_{N,I,W}^{(I),\leq \mu}/\Xi$ with coefficients in $E$, normalized relatively to $(X \setminus N)^I$. The following definition will be made more canonical in definition 3.9.

**Definition 3.3.** We set

$$\mathcal{H}_{N,I,W}^{\leq \mu} = R^0(p_{N,I,W}^{(I),\leq \mu})_!(\mathcal{IC}_{\text{Ch}_{N,I,W}^{(I),\leq \mu}/\Xi}).$$

Compared to version 4 of [Laf12] we have shortened the notations, by removing the indices which recalled that $\mathcal{H}_{N,I,W}^{\leq \mu}$ is a cohomology with compact support, taken in degree 0 (for the perverse normalization), with coefficients in $E$ and that it depends on the choice of $\Xi$. The cohomology is taken in the sense of [LMB99, LO08] but in fact the only background we need is the étale cohomology of schemes. Indeed, as soon as the degree of $N$ is big enough in function of $\mu$, $\text{Ch}_{N,I,W}^{(I),\leq \mu}/\Xi$ is a scheme of finite type. Therefore for any open subscheme $U \subset X \setminus N$ such that $U \subset X$ (in order to be able to increase $N$ without changing $U$), $\text{Ch}_{N,I,W}^{(I),\leq \mu}/\Xi|_U$ is the quotient of a scheme of finite type by a finite group.

When $I$ is empty and $W = 1$, we have

$$(3.1) \quad \lim_{\mu} \mathcal{H}_{N,\emptyset,1}^{\leq \mu} \bigg|_{\mathbb{F}_q} = C_c(G(F) \setminus G(\mathbb{A})/K_N \Xi, E)$$

because $\text{Ch}_{N,\emptyset,1}$ is the discrete stack $\text{Bun}_{G,N}(\mathbb{F}_q)$, considered as a constant stack over $\mathbb{F}_q$, and moreover $\text{Bun}_{G,N}(\mathbb{F}_q) = G(F) \setminus G(\mathbb{A})/K_N$ (here we use the hypothesis that $G$ is split, in general by (2.1), $\text{Bun}_{G,N}(\mathbb{F}_q)$ would be a finite union of adélic quotients of $G$).

**Remark 3.4.** More generally for every $I$ and $W = 1$, the stack $\text{Ch}_{N,1,I}/\Xi$ is simply the constant stack $G(F) \setminus G(\mathbb{A})/K_N \Xi$ over $(X \setminus N)^I$.

We consider $\lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu}$ as an inductive system of constructible $E$-sheaves over $(X \setminus N)^I$. We will now define the actions of the partial Frobenius morphisms and the Hecke operators over this inductive system (we warn the reader that these actions increase $\mu$). For every subset $J \subset I$ we denote by

$$\text{Frob}_J : (X \setminus N)^I \to (X \setminus N)^I$$
the morphism which sends \((x_i)_{i \in I}\) to \((x'_i)_{i \in I}\) with
\[
x'_i = \text{Frob}(x_i) \quad \text{if} \quad i \in J \quad \text{and} \quad x'_i = x_i \quad \text{otherwise}.
\]

Then we have

- for \(\kappa\) big enough and for every \(i \in I\), a morphism
\[
F_{(i)} : \text{Frob}^*_{(i)}(\mathcal{H}^{\leq \mu}_{N,I,W}) \to \mathcal{H}^{\leq \mu + \kappa}_{N,I,W}
\]

of constructible sheaves over \((X \setminus N)^I\), in such a way that the \(F_{(i)}\) commute with each other and their product for \(i \in I\) is the natural action of the total Frobenius morphism of \((X \setminus N)^I\) on the sheaf \(\mathcal{H}^{\leq \mu}_{N,I,W}\).

- for every \(f \in C_c(K_N \setminus G(A)/K_N, E)\) and for \(\kappa\) big enough, a morphism
\[
T(f) : \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{(X \setminus \mathfrak{P} )^I} \to \mathcal{H}^{\leq \mu + \kappa}_{N,I,W} \big|_{(X \setminus \mathfrak{P} )^I}
\]

of constructible sheaves over \((X \setminus \mathfrak{P} )^I\) where \(\mathfrak{P}\) is a finite set of places containing \(|N|\) and outside of which \(f\) is trivial.

The morphisms \(T(f)\) are called “Hecke operators” although they are morphisms of sheaves. They are obtained thanks to the obvious construction of Hecke correspondences between the stacks of chtoucas. We will see after proposition 6.3 that \(T(f)\) can be extended naturally to a morphism of sheaves over \((X \setminus N)^I\), but this is not trivial. Of course when \(I = \emptyset\) and \(W = 1\), the morphisms \(T(f)\) are the usual Hecke operators on \((3.1)\).

To construct the actions (3.2) of the partial Frobenius morphisms, we need a small generalization of the stacks \(\text{Cht}^{(I)}_{N,I,W}\) where we ask a factorization of \(\phi\) as a composition of several modifications. Let \((I_1, \ldots, I_k)\) be an (ordered) partition of \(I\).

**Definition 3.5.** We define \(\text{Cht}^{(I_1, \ldots, I_k)}_{N,I,W}\) as the reduced Deligne-Mumford stack whose points over a scheme \(S\) over \(\mathbb{F}_q\) classify
\[
((x_i)_{i \in I}, (\mathcal{G}_0, \psi_0) \xrightarrow{\phi_1} (\mathcal{G}_1, \psi_1) \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} (\mathcal{G}_{k-1}, \psi_{k-1}) \xrightarrow{\phi_k} (\tau \mathcal{G}_0, \tau \psi_0))
\]

with

- \(x_i \in (X \setminus N)(S)\) for \(i \in I\),
- for \(i \in \{0, \ldots, k - 1\}\, (\mathcal{G}_i, \psi_i) \in \text{Bun}_{G,N}(S)\) (i.e. \(\mathcal{G}_i\) is a \(G\)-torsor over \(X \times S\) and \(\psi_i : \mathcal{G}_i|_{N \times S} \xrightarrow{\sim} G|_{N \times S}\) is a trivialization over \(N \times S\)) and we note \((\mathcal{G}_k, \psi_k) = (\tau \mathcal{G}_0, \tau \psi_0)\)
- for \(j \in \{1, \ldots, k\}\)
  \[
  \phi_j : \mathcal{G}_{j-1}|_{(X \times S) \setminus (\bigcup_{i \in I_j} \Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}_{j}|_{(X \times S) \setminus (\bigcup_{i \in I_j} \Gamma_{x_i})}
  \]
  is an isomorphism such that the relative position of \(\mathcal{G}_{j-1}\) w.r.t. \(\mathcal{G}_j\) at \(x_i\) (for \(i \in I_j\)) is bounded by the dominant coweight of \(G\) corresponding to the dominant weight of \(W_i\),
- the \(\phi_j\), which induce isomorphisms over \(N \times S\), respect the level structures, i.e. \(\psi_j \circ \phi_j|_{N \times S} = \psi_{j-1}\) for every \(j \in \{1, \ldots, k\}\).
The condition on the relative position will be made more precise in remark 3.7. We denote by \( \text{Cht}^{(I_1,\ldots,I_k)}_{N,I,W} \) the indstack obtained when we forget this condition.

Moreover we denote by \( \text{Cht}^{(I_1,\ldots,I_k)\leq\mu}_{N,I,W} \) the open substack of \( \text{Cht}^{(I_1,\ldots,I_k)}_{N,I,W} \) defined by the condition that the Harder-Narasimhan polygon of \( \mathcal{G}_0 \) is \( \leq \mu \). We denote by

\[
\varphi^{(I_1,\ldots,I_k)}_{N,I,W} : \text{Cht}^{(I_1,\ldots,I_k)}_{N,I,W} \rightarrow (X \setminus N)^I
\]

the morphism which associates to a chtouca the family of its legs.

**Exemple.** When \( G = GL_r \), \( I = \{1,2\} \) and \( W = \text{St} \otimes \text{St}^* \), the stacks \( \text{Cht}^{(1)}_{N,I,W} \), \( \text{resp.} \) \( \text{Cht}^{(2)}_{N,I,W} \) are the stacks of right, \( \text{resp.} \) left chtoucas introduced by Drinfeld (and used also in [Laf02a]), and \( x_1 \) and \( x_2 \) are the pole and the zero.

We construct now a smooth morphism (3.8) from \( \text{Cht}^{(I_1,\ldots,I_k)}_{N,I,W} \) to the quotient of a closed stratum of a Beilinson-Drinfeld affine grassmannian by a smooth group scheme (in addition this will allow us in remark 3.7 to formulate more precisely the condition on the relative positions in definition 3.5). Readers familiar with Shimura varieties may consider this morphism as a “local model” provided they note

- that we are in a situation of good reduction since the \( x_i \) belong to \( X \setminus N \),
- and that however this local model is not smooth (except if all the \( I_j \) are singletons and all the coweights are minuscule).

**Definition 3.6.** The Beilinson-Drinfeld affine grassmannian is the indscheme \( \text{Gr}^{(I_1,\ldots,I_k)}_I \) over \( X^I \) whose \( S \)-points classify

\[
((x_i)_{i \in I}, \mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} \mathcal{G}_{k-1} \xrightarrow{\phi_k} \mathcal{G}_k \xrightarrow{\theta} G_{X \times S})
\]

where the \( \mathcal{G}_i \) are \( G \)-torsors over \( X \times S \), \( \phi_i \) is an isomorphism over \( (X \times S) \setminus (\bigcup_{i \in I_j} \Gamma_{x_i}) \) and \( \theta \) is a trivialization of \( \mathcal{G}_k \). The closed stratum \( \text{Gr}^{(I_1,\ldots,I_k)}_I \) is the reduced closed subscheme of \( \text{Gr}^{(I_1,\ldots,I_k)}_I \) defined by the condition that the relative position of \( \mathcal{G}_{j-1} \) w.r.t. \( \mathcal{G}_j \) at \( x_i \) (for \( i \in I_j \)) is bounded by the dominant coweight of \( G \) corresponding to the dominant weight \( \omega_i \) of \( W_i \). More precisely over the open subscheme \( U \) of \( X^I \) where the \( x_i \) are pairwise distinct, \( \text{Gr}^{(I_1,\ldots,I_k)}_I \) is a product of usual affine grassmannians and

- we define the restriction of \( \text{Gr}^{(I_1,\ldots,I_k)}_I \) over \( U \) as the product of the usual closed strata (denoted \( \overline{\text{Gr}}_{\omega_i} \) in [BG02, MV07]),
- then we define \( \text{Gr}^{(I_1,\ldots,I_k)}_I \) as the Zariski closure (in \( \text{Gr}^{(I_1,\ldots,I_k)}_I \)) of its restriction over \( U \).

By Beauville-Laszlo [BL95] (see also the first section of [Laf12] for additional references in [BD99]), \( \text{Gr}^{(I_1,\ldots,I_k)}_I \) can also be defined as the indscheme whose \( S \)-points classify

\[
((x_i)_{i \in I}, \mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} \mathcal{G}_{k-1} \xrightarrow{\phi_k} \mathcal{G}_k \xrightarrow{\theta} G_{\bigotimes_{x_i} \infty})
\]
where the $\mathcal{G}_i$ are $G$-torsors over the formal neighborhood $\Gamma \sum \infty x_i$ of the union of the graphs of the $x_i$ in $X \times S$, $\phi_i$ is an isomorphism over $\Gamma \sum \infty x_i \backslash (\bigcup_{i \in I} \Gamma x_i)$ and $\theta$ is a trivialization of $\mathcal{G}_k$. The restriction à la Weil $G \sum \infty x_i$ of $G$ from $\Gamma \sum \infty x_i$ to $S$ acts therefore on $Gt_i(I_{1, \ldots, I_k})$ and on $Gr_{I, W}(I_{1, \ldots, I_k})$ by change of the trivialization $\theta$.

We have a natural morphism

$$
(3.7) \quad \text{Ch}_{N, I, W}^i(I_{1, \ldots, I_k}) \to Gr_{I, W}^i(I_{1, \ldots, I_k}) / G \sum \infty x_i
$$

which associates to a chtouca (3.4) the $G \sum \infty x_i$-torsor $\mathcal{G}_k|_{\Gamma \sum \infty x_i}$ and, for any trivialization $\theta$ of it, the point of $Gr_{I, W}^i(I_{1, \ldots, I_k})$ equal to (3.6).

**Remark 3.7.** The best way to formulate the condition on the relative positions in the definition 3.5 is to define $\text{Ch}_{N, I, W}^i(I_{1, \ldots, I_k})$ as the inverse image of $Gr_{I, W}^i(I_{1, \ldots, I_k}) / G \sum \infty x_i$ by the morphism $\text{Ch}_{N, I, W}^i(I_{1, \ldots, I_k}) \to Gr_{I, W}^i(I_{1, \ldots, I_k}) / G \sum \infty x_i$ constructed as in (3.7).

For $(n_i)_{i \in I} \in \mathbb{N}^I$ we denote by $\Gamma \sum n_i x_i$ the closed subscheme of $X \times S$ associated to a Cartier divisor $\sum n_i x_i$ which is effective and relative over $S$. We denote by $G \sum n_i x_i$ the smooth group scheme over $S$ obtained by restriction à la Weil of $G$ from $\Gamma \sum n_i x_i$ to $S$. Then if the integers $n_i$ are big enough in function of $W$, the action of $G \sum \infty x_i$ on $Gr_{I, W}^i(I_{1, \ldots, I_k})$ factorizes through $G \sum n_i x_i$. Then the morphism (3.7) provides a morphism

$$
(3.8) \quad \text{Ch}_{N, I, W}^i(I_{1, \ldots, I_k}) \to Gr_{I, W}^i(I_{1, \ldots, I_k}) / G \sum n_i x_i
$$

(which associates to a chtouca (3.4) the $G \sum n_i x_i$-torsor $\mathcal{G}_k|_{\Gamma \sum n_i x_i}$ and, for any trivialization $\lambda$ of it, the point of $Gr_{I, W}^i(I_{1, \ldots, I_k})$ extending $\lambda$ of $\Gamma \sum n_i x_i$ to $\Gamma \sum \infty x_i$).

We show in proposition 2.8 of [Laf12] that the morphism (3.8) is smooth of dimension $\dim G \sum n_i x_i = \sum_{i \in I} n_i \dim G$ (the idea is the following : it is enough to prove it in the case where $N$ is empty and then it is an easy consequence of the fact that the Frobenius morphism of $\text{Bun}_G$ has a zero derivative).

We deduce from this that the morphism (forgetting intermediate modifications) (3.9)

$$
\text{Ch}_{N, I, W}^i(I_{1, \ldots, I_k}) \to \text{Ch}_{N, I, W}^i(I)
$$

which sends (3.4) to $((x_i)_{i \in I}, (\mathcal{G}_0, \psi_0) \xrightarrow{\phi_k\cdots\phi_1} (\mathcal{G}_0, \tau \psi_0))$, is small. Indeed it is known that the analogous morphism

$$
(3.10) \quad \text{Gr}_{I, W}^i(I_{1, \ldots, I_k}) \to \text{Gr}_{I, W}^i(I) \quad \text{which sends (3.6) to } ((x_i)_{i \in I}, \mathcal{G}_0 \xrightarrow{\phi_k\cdots\phi_1} \mathcal{G}_k \xrightarrow{\theta} G \Gamma \sum \infty x_i)
$$

is small, and by the way this fact plays an essential role in [MV07]. Moreover the inverse image of $\text{Ch}_{N, I, W}^i(I_{1, \ldots, I_k}) \leq \mu$ by (3.9) is exactly $\text{Ch}_{N, I, W}^i(I_{1, \ldots, I_k}) \leq \mu$ since the truncatures were defined with the help of the Harder-Narasimhan polygon of $\mathcal{G}_0$. Therefore

$$
\mathcal{H}_{N, I, W}^{\leq \mu} = R^0(\mathcal{P}_{N, I, W}^{(I_{1, \ldots, I_k}) \leq \mu}) \cdot \left(\text{IC}_{\text{Ch}_{N, I, W}^i(I_{1, \ldots, I_k}) \leq \mu} / \mathcal{E}\right)
$$
for every partition \((I_1, \ldots, I_k)\) of \(I\) (whereas definition 3.3 used the coarse partition \((I)\)).

The partial Frobenius morphism

\[
\text{Fr}_{I_1}^{(I_1, \ldots, I_k)} : \text{Ch}_{N,I,W}^{(I_1, \ldots, I_k)} \to \text{Ch}_{N,I,W}^{(I_2, \ldots, I_k)}
\]
defined by

\[
\text{Fr}_{I_1}^{(I_1, \ldots, I_k)} \left( (x_i)_{i \in I}, (\mathcal{G}_0, \psi_0) \right) \phi_1 (\mathcal{G}_1, \psi_1) \phi_2 \cdots \phi_{k-1} (\mathcal{G}_{k-1}, \psi_{k-1}) \phi_k (\tau \mathcal{G}_0, \tau \psi_0) = (\text{Frob}_{I_1} \left( (x_i)_{i \in I}, (\mathcal{G}_1, \psi_1) \right), (\mathcal{G}_2, \psi_2) \phi_2 \cdots \phi_k (\tau \mathcal{G}_0, \tau \psi_0) \tau \phi_k (\tau \mathcal{G}_1, \tau \psi_1))
\]
lies over the morphism \(\text{Frob}_{I_1} : (X \setminus N)^I \to (X \setminus N)^I\). Since \(\text{Fr}_{I_1}^{(I_1, \ldots, I_k)}\) is a completely radical local homeomorphism, we have a canonical isomorphism

\[
\left( \text{Fr}_{I_1}^{(I_1, \ldots, I_k)} \right)^* \left( \text{IC}_{\text{Ch}_{N,I,W}^{(I_2, \ldots, I_k)}} \right) = \text{IC}_{\text{Ch}_{N,I,W}^{(I_1, \ldots, I_k)}}.
\]

The proper base change isomorphism then provides a morphism

\[
\text{F}_{I_1} : \text{Frob}_{I_1}^* (\mathcal{F}^{\leq \mu}_{N,I,W}) \to \mathcal{H}^{\leq \mu + \kappa}_{N,I,W}
\]

for \(\kappa\) big enough (indeed, with the notations above, if the Harder-Narasimhan polygon of \(\mathcal{G}_1\) is \(\leq \mu\), since the modification between \(\mathcal{G}_0\) and \(\mathcal{G}_1\) is bounded in function of \(W\), the Harder-Narasimhan polygon of \(\mathcal{G}_0\) is \(\leq \mu + \kappa\) where \(\kappa\) depends on \(W\)). By taking any partition \((I_1, \ldots, I_k)\) such that \(I_1 = \{i\}\) we get \(\text{F}_{(i)}^*\) in (3.2).

For the moment we have defined \(\mathcal{F}^{\leq \mu}_{N,I,W}\) for the isomorphisms classes of irreducible representations \(W\) of \((\hat{G})^I\). We will refine this construction into the construction of a canonical \(E\)-linear functor

\[
W \mapsto \mathcal{H}^{\leq \mu}_{N,I,W}
\]

from the category of finite dimensional \(E\)-linear representations of \((\hat{G})^I\) to the category of constructible \(E\)-sheaves over \((X \setminus N)^I\). In particular for every morphism \(u : W \to W'\) of \(E\)-linear representations of \((\hat{G})^I\) we will denote by

\[
\mathcal{H}(u) : \mathcal{H}^{\leq \mu}_{N,I,W} \to \mathcal{H}^{\leq \mu}_{N,I,W'}
\]

the associated morphism of constructible sheaves.

The functor (3.11) will be compatible with the coalescence of legs, in the following sense. In all this article we call coalescence the situation where legs are glued to each other. We could have used the word fusion instead of coalescence but we prefered to use the word coalescence for the legs (which are just points over the curve) while keeping the word fusion for the fusion product (which occurs in the geometric Satake equivalence and involves the perverse sheaves over the affine grassmannians of Beilinson-Drinfeld). Let \(\zeta : I \to J\) be any map. We denote by \(W^\zeta\) the representation of \((\hat{G})^J\) which is the composition of the representation \(W\) with the diagonal morphism

\[
\hat{G}^J \to \hat{G}^I, (g_j)_{j \in J} \mapsto (g_{\zeta(j)})_{i \in I}.
\]
We denote by
\[
\Delta_\zeta : X^J \to X^I, \quad (x_j)_{j \in J} \mapsto (x_{\zeta(i)})_{i \in I}
\]
the diagonal morphism. We will construct, after [Var04] and [BV06], a canonical isomorphism of constructible sheaves over \((X \setminus N)^J\), called the coalescence isomorphism:
\[
\chi_\zeta : \Delta_\zeta^* (\mathcal{H}_N^{\leq \mu}) \to \mathcal{H}_N^{\leq \mu}.
\]
(3.13)
Now let us explain the construction of the functor (3.11) and of the coalescence isomorphism (3.13).

When \(W\) is not irreducible we denote by \(G_t^{(I_1, \ldots, I_k)}\) the union of the \(G_t^{(I_1, \ldots, I_k)} \subset G_t^{(I_1, \ldots, I_k)}\) for irreducible constituents \(V\) of \(W\). We do the same with \(\text{Ch}_t^{(I_1, \ldots, I_k)}\).

We recall now the geometric Satake equivalence, of Lusztig, Drinfeld, Ginzburg and Mirkovic-Vilonen. For references we quote [Lus82, Gin95, BD99, MV07, Ga01, Gai07, Ric12, Zhu14]. Here we will use it in the form explained by Gaitsgory in [Gai07]. Usually the geometric Satake equivalence can be formulated in the following way. For every field \(k\) of characteristic prime to \(\ell\), the category of \(G(k[[z]])\)-equivariant perverse sheaves over the affine grassmannian \(G(k((z))/G(k[[z]])\) is equipped with a tensor structure by the fusion product (or the convolution product) and with a fiber functor. The fiber functor is given by the total cohomology, up to modifying the rule of signs (and also if \(k\) is not algebraically closed we must introduce a Tate twist, i.e. tensor by \(E(\frac{\pi}{2})\) the part of cohomological degree \(i\) for every \(i \in \mathbb{Z}\)). Thus this tensor category is equivalent to the category of representations of an algebraic group, which happens to be isomorphic to \(\hat{G}\) (and equipped with a canonical splitting). Moreover these perverse sheaves are naturally equivariant under the action of the automorphism group of \(k[[z]]\). This allows to replace \(\text{Spf}(k[[z]])\) by an arbitrary formal disk, and in particular a formal disk moving on a curve.

Here we use only one direction of this equivalence, namely the functor from the category of representations of \(\hat{G}\) to the category of \(G(k[[z]])\)-equivariant perverse sheaves over the affine grassmannian. On the other hand we state it with the help of the affine grassmannian of Beilinson-Drinfeld. It is extremely important for us to know that this functor is canonical. This is a consequence of the existence of fiber functor. We recall that this fiber functor is given by the total cohomology, for example, in the notations of theorem below, \(W\) is equal to the total cohomology of \(S_t^{(I_1, \ldots, I_k)}\) in the fibers of \(G_t^{(I_1, \ldots, I_k)}\) over \(X^I\) (with the previous warnings about the rule of signs and the Tate twists). We note that in this article the fiber functor is only used through the fact that it guarantees the canonicality of the functor.

**Theorem 3.8.** (one direction of the geometric Satake equivalence [BD99, MV07, Gai07]). We have for every finite set \(I\) and for every partition \((I_1, \ldots, I_k)\) a canonical \(E\)-linear functor
\[
W \mapsto S_t^{(I_1, \ldots, I_k)}
\]
from the category of finite dimensional \(E\)-linear representations of \((\hat{G})^I\) to the category of \(G_{\sum_{\infty x_i}}\)-equivariant perverse \(E\)-sheaves over \(G_t^{(I_1, \ldots, I_k)}\) (for the perverse
normalization relative to $X^I$). Moreover $S_{I,W}^{(I_1,\ldots,I_k)}$ is supported by $Gr_{I,W}^{(I_1,\ldots,I_k)}$ and uniformly locally acyclic relatively to $X^I$. These functors satisfy the following properties.

a) Compatibility with the morphisms which forget the intermediate modifications: $S_{I,W}^{(I)}$ is canonicaly isomorphic to the direct image of $S_{I,W}^{(I_1,\ldots,I_k)}$ by the forgetful morphism $Gr_{I}^{(I_1,\ldots,I_k)} \to Gr_{I}^{(I)}$ (defined in (3.10)).

b) Compatibility with convolution: if $W = \boxtimes_{j \in \{1,\ldots,k\}} W_j$ where $W_j$ is a representation of $(\hat{G})^I_j$, $S_{I,W}^{(I_1,\ldots,I_k)}$ is canonicaly isomorphic to the inverse image of $\boxtimes_{j \in \{1,\ldots,k\}} S_{I_j,W_j}^{(I_j)}$ by the morphism

$$Gr_{I}^{(I_1,\ldots,I_k)}/G_{\sum_{i \in I} \times_{x_i}} \to \prod_{j=1}^{k} \left( Gr_{I_j}^{(I_j)}/G_{\sum_{i \in I_j} \times_{x_i}} \right)$$

$$(S_0 \to S_1 \to \cdots \to S_k) \mapsto \left( (S_{j-1}|_{\Gamma_{\sum_{i \in I_j} \times_{x_i}}} \to S_{j}|_{\Gamma_{\sum_{i \in I_j} \times_{x_i}}}) \right)_{j=1,\ldots,k}$$

where the $S_i$ are $G$-torsors over $\Gamma_{\sum_{i \in I} \times_{x_i}}$.

c) Compatibility with fusion: let $I,J$ be finite sets and $\zeta : I \to J$ be any map. Let $(J_1,\ldots,J_k)$ be a partition of $J$. Its inverse image $(\zeta^{-1}(J_1),\ldots,\zeta^{-1}(J_k))$ is a partition of $I$. We denote by

$$\Delta_\zeta : X^J \to X^I, \quad (x_j)_{j \in J} \mapsto (x_{\zeta(i)})_{i \in I}$$

the diagonal morphism associated to $\zeta$. We denote again by $\Delta_\zeta$ the inclusion

$$Gr_{I}^{(J_1,\ldots,J_k)} = Gr_{I}^{(\zeta^{-1}(J_1),\ldots,\zeta^{-1}(J_k))} \times_{X^I} X^J \hookrightarrow Gr_{I}^{(\zeta^{-1}(J_1),\ldots,\zeta^{-1}(J_k))}.$$ Let $W$ be a finite dimensional $E$-linear representation of $\hat{G}^I$. We denote by $W^\zeta$ the representation of $\hat{G}^J$ which is the composition of the representation $W$ with the diagonal morphism

$$\hat{G}^J \to \hat{G}^I, \quad (g_j)_{j \in J} \mapsto (g_{\zeta(i)})_{i \in I}.$$ Then we have a canonical isomorphism

$$(3.14) \quad \Delta_\zeta^* \left( S_{I,W}^{(\zeta^{-1}(J_1),\ldots,\zeta^{-1}(J_k))} \right) \simeq S_{I,W^\zeta}^{(J_1,\ldots,J_k)}$$

which is functorial in $W$ and compatible with the composition for $\zeta$.

d) When $W$ is irreducible, the perverse sheaf $S_{I,W}^{(I_1,\ldots,I_k)}$ over $Gr_{I,W}^{(I_1,\ldots,I_k)}$ is isomorphic to its IC-sheaf (with the perverse normalization relative to $X^I$).

The properties a) and b) could have been stated with more general partitions, but at the cost of cumbersome notations.

In the previous theorem $S_{I,W}^{(I_1,\ldots,I_k)}$ is supported by $Gr_{I,W}^{(I_1,\ldots,I_k)}$ and therefore we can consider it as a perverse sheaf (up to a shift) over $Gr_{I,W}^{(I_1,\ldots,I_k)}/G_{\sum_{i \in I} \times_{x_i}}$ (if the integers $n_i$ are big enough).

Here is the construction of the functor (3.11).
Definition 3.9. We define the perverse sheaf (with the normalization relative to \((X \setminus N)^{\nu}\)) \(\mathcal{F}_{N,I,W}^{(I_1,\ldots,I_k)}\) on \(\text{Cht}_{N,I,W}^{(I_1,\ldots,I_k)} / \Xi\) as the inverse image of \(\mathcal{S}_{I,W}^{(I_1,\ldots,I_k)}\) by the morphism (3.8). Then we define the functor (3.11) by setting

\[
\mathcal{H}_{N,I,W}^{\leq \mu} = R^0(\mathcal{F}_{N,I,W}^{(I_1,\ldots,I_k)}) \left|_{\text{Cht}_{N,I,W}^{(I_1,\ldots,I_k)} / \Xi} \right.
\]

for any partition \((I_1,\ldots,I_k)\) of \(I\).

Thanks to a) of the previous theorem, the definition (3.15) does not depend on the choice of the partition \((I_1,\ldots,I_k)\).

When \(W\) is irreducible, the smoothness of the morphism (3.8) and the computation of its dimension imply that \(\mathcal{F}_{N,I,W}^{(I_1,\ldots,I_k)}\) is isomorphic to the IC-sheaf of \(\text{Cht}_{N,I,W}^{(I_1,\ldots,I_k)} / \Xi\) (with the perverse normalization relative to \((X \setminus N)^{\nu}\)). Thus the previous definition is compatible with definition 3.3 (which it refines and makes more canonical).

The action of the Hecke operators and the partial Frobenius morphisms can be rewritten with the help of the definition (3.15). However the canonicity of definition (3.15) is mostly crucial for the construction of the coalescence isomorphisms (3.13), which will be explained now, because the source and target spaces are not the same.

Definition 3.10. The canonical isomorphism (3.13) is defined (thanks to the proper base change theorem) by the canonical isomorphism between \(\mathcal{F}_{N,J,W}^{(J_1,\ldots,J_k)}\) and the inverse image of \(\mathcal{F}_{N,I,W}^{(\zeta^{-1}(J_1),\ldots,\zeta^{-1}(J_k))}\) by the inclusion

\[
\text{Cht}_{N,J}^{(J_1,\ldots,J_k)} = \text{Cht}_{N,I}^{(\zeta^{-1}(J_1),\ldots,\zeta^{-1}(J_k))} \times_{(X \setminus N)^{\nu}} (X \setminus N)^{J} \hookrightarrow \text{Cht}_{N,I}^{(\zeta^{-1}(J_1),\ldots,\zeta^{-1}(J_k))}
\]

which comes from the isomorphism (3.14) in c) of theorem 3.8.

The previous definition is independent on the choice of the partition \((J_1,\ldots,J_k)\).

Remark 3.11. The fact that we took an arbitrary partition \((J_1,\ldots,J_k)\) allows us to prove the compatibility between the coalescence isomorphism (3.13) and the partial Frobenius morphisms, namely that for every \(j \in J\), \(\Delta^{\ast}_x(F_{\zeta^{-1}(\{j\})})\) and \(F_{\{j\}}\) correspond to each other by the isomorphism \(\chi_{\zeta}\) of (3.13).

4. PROOF OF PROPOSITION 1.3

We call a geometric point \(\overline{x}\) of a scheme \(Y\) the data of an algebraically closed field \(k(\overline{x})\) and a morphism \(\text{Spec}(k(\overline{x})) \to Y\). In this article \(k(\overline{x})\) will always be an algebraic closure of the residue field \(k(x)\) of the point \(x \in Y\) below \(\overline{x}\). We denote by \(Y_{\overline{x}}\) the strict localization (or strict henselianization) of \(Y\) at \(\overline{x}\). In other words \(Y_{\overline{x}}\) is the spectrum of the ring \(\lim_{\rightarrow} \Gamma(U, \mathcal{O}_U)\), where the inductive limit is taken over the \(\overline{x}\)-pointed étale neighborhoods of \(x\). It is a local henselian ring whose residue field is the separable closure of \(k(x)\) in \(k(\overline{x})\). If \(\overline{x}\) and \(\overline{y}\) are two geometric points of \(Y\), we call a specialization arrow \(\text{sp} : \overline{x} \to \overline{y}\) a morphism \(Y(\overline{x}) \to Y(\overline{y})\), or equivalently a morphism \(\overline{x} \to Y(\overline{y})\) (such an arrow exists if and only if \(\overline{y}\) is in the Zariski closure of \(x\)). By section 7 of [SGA4-2-VIII] a specialization arrow
\( \text{sp} : \overline{\tau} \to \overline{y} \) induces for every sheaf \( \mathcal{F} \) over a open subscheme of \( Y \) containing \( y \) a specialization homomorphism \( \text{sp}^* : \mathcal{F}_{\overline{y}} \to \mathcal{F}_{\overline{\tau}} \).

We fix an algebraic closure \( \overline{F} \) of \( F \) and we denote by \( \overline{\eta} = \text{Spec}(\overline{F}) \) the corresponding geometric point over the generic point \( \eta \) of \( X \).

We denote by \( \Delta : X \to X \) the diagonal morphism. We fix a geometric point \( \overline{\eta}^I \) over the generic point \( \eta \) of \( X \), and a specialization arrow \( \text{sp} : \overline{\eta}^I \to \Delta(\overline{\eta}) \).

The role of \( \text{sp} \) is to make the fiber functor at \( \overline{\eta}^I \) more canonical, and in particular compatible with the coalescence of legs (the last claim is clear when \( \text{sp}^* \) is an isomorphism and in practice we will be in this situation).

A fundamental result of Drinfeld (theorem 2.1 of [Dri78] and proposition 6.1 of [Dri89]) is recalled in the following lemma (see chapter 8 of [Laf12] for other references, especially [Lau04]). We will always denote the \( \mathcal{O}_E \)-modules and the \( \mathcal{O}_E \)-sheaves by gothic letters.

**Lemma 4.1.** (Drinfeld) If \( \mathcal{E} \) is a smooth constructible \( \mathcal{O}_E \)-sheaf over a dense open subscheme of \( (X \setminus N)^I \), equipped with an action of the partial Frobenius morphisms, i.e. with isomorphisms

\[
F_{\{i\}} : \text{Frob}_{\{i\}}^*(\mathcal{E})|_{\eta^I} \to \mathcal{E}|_{\eta^I}
\]

commuting to each other and whose composition is the natural isomorphism \( \text{Frob}^*(\mathcal{E}) \cong \mathcal{E} \), then it extends to a smooth sheaf over \( U^I \), where \( U \) is a small enough open dense subscheme of \( X \setminus N \), and the fiber \( \mathcal{E}|_{\Delta(\overline{\eta})} \) is equipped with an action of \( \pi_1(U, \overline{\eta}) \). Moreover, if we fix \( U \), the functor \( \mathcal{E} \mapsto \mathcal{E}|_{\Delta(\overline{\eta})} \) provides an equivalence

- from the category of smooth constructible \( \mathcal{O}_E \)-sheaves over \( U^I \) equipped with an action of the partial Frobenius morphisms
- to the category of continuous representations of \( \pi_1(U, \overline{\eta}) \) on \( \mathcal{O}_E \)-modules of finite type,

in a way compatible with the permutations of \( I \), and with the coalescence.

**Remark 4.2.** In the situation of the previous lemma, \( \text{sp}^* : \mathcal{E}|_{\Delta(\overline{\eta})} \to \mathcal{E}|_{\overline{\eta}^I} \) is an isomorphism, hence \( \mathcal{E}|_{\overline{\eta}^I} \) is also equipped with an action of \( \pi_1(U, \overline{\eta}) \) but this action depends on the choice of \( \text{sp} \).

We cannot apply directly this lemma, because the action of the partial Frobenius morphisms increases \( \mu \), and on the other hand the inductive limit \( \lim_{\rightarrow \mu} \mathcal{H}^\mu_{N,I,W} |_{\overline{\tau}} \) is not constructible (because its fibers are of infinite dimension). Nevertheless we will be able to apply Drinfeld’s lemma to the “Hecke-finite” part, in the following sense.

**Definition 4.3.** Let \( \overline{\tau} \) be a geometric point of \( (X \setminus N)^I \). An element of \( \lim_{\rightarrow \mu} \mathcal{H}^\mu_{N,I,W} |_{\overline{\tau}} \) is said to be Hecke-finite if it belongs to a \( \mathcal{O}_E \)-submodule of finite type of \( \lim_{\rightarrow \mu} \mathcal{H}^\mu_{N,I,W} |_{\overline{\tau}} \) which is stable by \( T(f) \) for every \( f \in C_c(K_N \setminus G(\mathbb{A})/K_N, \mathcal{O}_E) \).
We denote by \( \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \right)_{\text{Hf}} \) the set of all the Hecke-finite elements. It is an \( E \)-vector subspace of \( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta} \) and it is stable by \( \pi_1(x, \overline{x}) \) and \( C_c(K_N \setminus G(\mathbb{A})/K_N, E) \).

**Remark 4.4.** The definition above will be applied only with \( \overline{x} \) equal to \( \Delta(\overline{\eta}) \) or \( \overline{\eta}^f \). In this case the action of the Hecke operators \( T(f) \) on \( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta} \) is obvious (and does not require their extension to morphisms of sheaves over \( (X \setminus N)^f \) which will be obtained after proposition 6.4).

We have the specialization homomorphism

\[
\text{sp}^*: \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\Delta(\overline{\eta})} \to \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^f}
\]

where both sides are considered as \( E \)-vector spaces (inductive limits of \( E \)-vector spaces of finite dimension).

We admit now two results, which will be justified in sections 6, 7 and 8.

**First result that we admit for the moment** (lemma 7.1 and proposition 7.2). The space \( \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^f} \right)_{\text{Hf}} \) is a union of \( \mathcal{O}_E \)-submodules of finite type stable by the partial Frobenius morphisms and by Drinfeld’s lemma it is endowed with an action of \( \text{Gal}((\overline{F}/F)^f) \) (depending on the choice of \( \text{sp} \)), more precisely \( \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^f} \right)_{\text{Hf}} \) is an inductive limit of finite dimensional continuous representations of \( \text{Gal}((\overline{F}/F)^f) \).

**Second result that we admit for the moment** (corollary 8.4). The restriction of the homomorphism \( \text{sp}^* \) of (4.1) to the Hecke-finite parts is an isomorphism

\[
\left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\Delta(\overline{\eta})} \right)_{\text{Hf}} \xrightarrow{\text{sp}^*} \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^f} \right)_{\text{Hf}}.
\]

Thanks to these two results we can now define the \( E \)-vector spaces \( H_{I,W} \) (we omit \( N \) in the notation \( H_{I,W} \) to reduce the size of the diagrams in the next section).

**Definition 4.5.** We define \( H_{I,W} \) as the LHS of (4.2).

The action of \( \text{Gal}((\overline{F}/F)^f) = \pi_1(\eta, \overline{\eta})^f \) on \( H_{I,W} \) does not depend on the choice of \( \text{sp} \). Indeed by the first admitted result there exist

- an increasing union (indexed by \( \lambda \in \mathbb{N} \)) of constructible \( \mathcal{O}_E \)-subsheaves \( \mathfrak{F}_{\lambda} \subset \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^f} \) stable by the partial Frobenius morphisms (to which we can apply Drinfeld’s lemma)
- a decreasing sequence of open subschemes \( U_{\lambda} \subset X \setminus N \) such that \( \mathfrak{F}_{\lambda} \) extends to a smooth sheaf over \( (U_{\lambda})^f \).
in such a way that \( \bigcup_{\lambda \in \mathbb{N}} \mathfrak{F}_\lambda|_{\eta^I} = \left( \lim_{\mu \to \eta^I} \mathcal{H}_{N,I,W}^{\leq \mu}|_{\eta^I} \right)^{Hf} \). Then the second admitted result implies that the natural morphism

\[
H_{I,W} = \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu}|_{\Delta(\eta)} \right)^{Hf} \to \bigcup_{\lambda \in \mathbb{N}} \mathfrak{F}_\lambda|_{\Delta(\eta)}
\]

(4.3)

(which comes from the smoothness of \( \mathfrak{F}_\lambda \) over \( (U_\lambda)^I \ni \Delta(\eta) \)) is an isomorphism. But the action of \( \text{Gal}(\mathcal{F}/F)^I \) on the RHS of (4.3), which is given by Drinfeld’s lemma, does not depend on the choice of \( \mathfrak{g}^* \), and therefore the action of \( \text{Gal}(\mathcal{F}/F)^I \) on the LHS does not depend on it either.

**Remark 4.6.** In a work in project with Yakov Varshavsky we hope to prove that \( H_{I,W} \) is of finite dimension. In this article we only know that \( H_{I,W} \) is an inductive limit of \( E \)-vector spaces of finite dimension equipped with continuous representations of \( \text{Gal}(\mathcal{F}/F)^I \) (note however that c) of proposition 1.3, which will be proven at the end of this section, implies that \( H_{\emptyset,1} \) is of finite dimension).

For every map \( \zeta : I \to J \), the coalescence isomorphism (3.13) obviously respects the Hecke-finite parts and therefore it induces an isomorphism

\[
H_{I,W} = \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu}|_{\Delta(\eta)} \right)^{Hf} \sim \left( \lim_{\mu} \mathcal{H}_{N,J,W}^{\leq \mu}|_{\Delta(\eta)} \right)^{Hf} = H_{J,W}\zeta
\]

(4.4)

where \( \Delta \) denotes the diagonal morphism \( X \to X_I \) or \( X \to X_J \).

**Definition 4.7.** We define the isomorphism

\[
\chi_\zeta : H_{I,W} \sim \to H_{J,W}\zeta
\]

occuring in b) of the proposition 1.3 to be (4.4).

The isomorphism (4.4) is \( \text{Gal}(\mathcal{F}/F)^J \)-equivariant, where \( \text{Gal}(\mathcal{F}/F)^J \) acts on the LHS by the diagonal morphism

\[
\text{Gal}(\mathcal{F}/F)^J \to \text{Gal}(\mathcal{F}/F)^J, \ (\gamma_j)_{j \in J} \mapsto (\gamma_\zeta(i))_{i \in I}
\]

(4.5)

Indeed, if \( \Delta_\zeta : X^J \to X^I \) is the diagonal morphism (3.12) and if the sequence \( (\mathfrak{F}_\lambda)_{\lambda \in \mathbb{N}} \) is as above relatively to \( I \) and \( W \), then the sequence \( (\Delta_\zeta^*(\mathfrak{F}_\lambda))_{\lambda \in \mathbb{N}} \) satisfies the same properties relatively to \( J \) and \( W_\zeta \), and thus

\[
\chi_\zeta : H_{I,W} = \bigcup_{\lambda \in \mathbb{N}} \mathfrak{F}_\lambda|_{\Delta(\eta)} = \bigcup_{\lambda \in \mathbb{N}} \Delta_\zeta^*(\mathfrak{F}_\lambda)|_{\Delta(\eta)} = H_{J,W}\zeta
\]

is \( \text{Gal}(\mathcal{F}/F)^J \)-equivariant.

**Proof of proposition 1.3.** The properties a) and b) were already explained. Note that applying b) to the obvious map \( \zeta_{\emptyset} : \emptyset \to \{0\} \), we get an isomorphism

\[
\chi_{\emptyset} : H_{\emptyset,1} \sim \to H_{\{0\},1}
\]

(4.6)

that we already knew by remark 3.4 (applied to \( I = \{0\} \)).
Property c) comes from the fact that the Hecke-finite part of (3.1) consists exactly of cuspidal automorphic forms, i.e.

\[
(\mathcal{C}_c^\text{cusp}(G(F)\backslash G(\mathbb{A})/K_N\Xi, E))^{\text{Hf}} = \mathcal{C}_c^\text{cusp}(G(F)\backslash G(\mathbb{A})/K_N\Xi, E).
\]

**Proof of \(\supset\).** Any cuspidal function is Hecke-finite because the \(\mathcal{O}_E\)-module \(\mathcal{C}_c^\text{cusp}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \mathcal{O}_E)\) is of finite type and is stable by all the Hecke operators \(T(f)\) for \(f \in \mathcal{C}_c(K_N \backslash G(\mathbb{A})/K_N, \mathcal{O}_E)\).

**Proof of \(\subset\).** We assume by contradiction that a Hecke-finite function \(f\) is not cuspidal. Then there exists a parabolic \(P \subset G\), of Levi \(M\) and radical unipotent \(U\), such that the constant term \(f_P : g \mapsto \int_{U(F) \backslash U(\mathbb{A})} f(ug)\) is non zero. Let \(v\) be a place of \(X \setminus N\). Since the ring of representations of \(\widehat{M}\) is a module of finite type over the ring of representations of \(\widehat{G}\), \(f_P\) is also Hecke-finite w.r.t. the Hecke operators for \(M\) at \(v\). Since the quotient of the center of \(M\) by the center of \(G\) contains at least \(\mathbb{G}_m\), this contradicts the fact that \(f_P\) has compact support over \(U(\mathbb{A})M(F)\backslash G(\mathbb{A})/K_N\Xi\).

We refer to proposition 8.11 of [Laf12] for a more detailed proof of c). \(\Box\)

5. **Proof of theorem 1.1 with the help of proposition 1.3**

Here is an overview of the idea: thanks to (4.6) and to c) of the proposition 1.3, we have

\[
H_{(0),1} = \mathcal{C}_c^\text{cusp}(G(F)\backslash G(\mathbb{A})/K_N\Xi, E).
\]

To get the decomposition (1.3) it is thus equivalent to construct (increasing \(E\) if necessary) a canonical decomposition

\[
H_{(0),1} = \bigoplus_\sigma \mathcal{S}_\sigma.
\]

This decomposition will be obtained by spectral decomposition of a commutative family of endomorphisms of \(H_{(0),1}\), called excursion operators, that we will now construct and study with the help of properties a) and b) of proposition 1.3.

Let \(I\) be a finite set and \(W\) be a \(E\)-linear representation of \((\widehat{G})^I\). We denote by \(\zeta_I : I \to \{0\}\) the obvious map, so that \(W^{\zeta_I}\) is just \(W\) equipped with the diagonal action of \(\hat{G}\). Let \(x : 1 \to W^{\zeta_I}\) and \(\xi : W^{\zeta_I} \to 1\) be morphisms of representations of \(\hat{G}\) (in other words \(x \in W\) and \(\xi \in W^*\) are invariant under the diagonal action of \(\hat{G}\)). Let \((\gamma_i)_{i \in I} \in \text{Gal}(\overline{F}/F)^I\).

**Definition 5.1.** We define the operator

\[
S_{I,W,x,\xi, (\gamma_i)_{i \in I}} \in \text{End}(H_{(0),1})
\]

as the composition

\[
H_{(0),1} \xrightarrow{\gamma_i(x)} H_{(0),1}^{\zeta_I} \xrightarrow{\chi_\zeta^{-1}} H_{I,W} \xrightarrow{\gamma_i(1)} H_{I,W} \xrightarrow{\chi_\zeta} H_{(0),W^{\zeta_I}} \xrightarrow{\gamma_i(\xi)} H_{(0),1}.
\]

This operator will be called an “excursion operator”. Paraphrasing (5.2) it is the composition of
where most of the notations are obvious,\( (5.3) \)

\[ S_{I,W,x',u(x')} = S_{I,W',u(x)} \]

where \( u : W \to W' \) is a \( \widehat{G} \)-equivariant morphism and \( x \in W \) and \( x' \in (W')^* \) are \( \widehat{G} \)-invariant,

\[ S_{I,W,x,\xi,(\gamma_\ell i)_{i \in I}} = S_{I,W,\xi,(\gamma_i^{(\ell)})_{i \in I}} \]

\[ S_{I,\cup W,\xi,(\gamma_\ell i)_{i \in I}} = S_{I,\cup W,\xi,(\gamma_i^{(\ell)})_{i \in I}} \]

where most of the notations are obvious, \( I_1 \cup I_2 \) and \( I \cup I \cup I \) denote disjoint unions, and \( \delta_w : 1 \to W \otimes W^* \) and \( \text{ev}_W : W^* \otimes W \to 1 \) are the natural morphisms.

**Proof of (5.3).** We set \( x' = u(x) \in W' \) and \( \xi = u'(\xi') \in W^* \). The diagram

\[
\begin{array}{ccc}
H_{\{0\}, (W')^*} & \xrightarrow{\chi_{\ell i}} & H_{\{0\}, (W')^*} \\
\downarrow{\pi(u)} & & \downarrow{\pi(u)} \\
H_{\{0\}, W^*} & \xrightarrow{\chi_{\ell i}} & H_{\{0\}, W^*} \\
\end{array}
\]

is commutative. But the lower line is equal to \( S_{I,W,x,\xi,(\gamma_\ell i)_{i \in I}} \) and the upper line is equal to \( S_{I,W',x',\xi',(\gamma_\ell i)_{i \in I}} \).

**Proof of (5.4).** The diagram

\[
\begin{array}{ccc}
H_{\{0\}, (W')^*} & \xrightarrow{\chi_{\ell i}} & H_{\{0\}, (W')^*} \\
\downarrow{\pi(u)} & & \downarrow{\pi(u)} \\
H_{\{0\}, W^*} & \xrightarrow{\chi_{\ell i}} & H_{\{0\}, W^*} \\
\end{array}
\]

is commutative. But the lower line is equal to \( S_{I,W,x,\xi,(\gamma_\ell i)_{i \in I}} \) and the upper line is equal to \( S_{I,W',x',\xi',(\gamma_\ell i)_{i \in I}} \).

**Proof of (5.5).** The obvious map \( \{0\} \cup \{0\} \to \{0\} \) gives an isomorphism \( H_{\{0\} \cup \{0\}, 1} \cong H_{\{0\}, 1} \). If we denote by \( \zeta_1 : I_1 \to \{0\} \) and \( \zeta_2 : I_2 \to \{0\} \) the
obvious maps, the LHS of (5.5) is equal to the composition

\[
H_{\{0\} \cup \{0\}, 1} \xrightarrow{\gamma \chi \xi_{\{0\} \cup \{0\}}} H_{\{0\} \cup \{0\}, W_1^{1} \otimes W_2^{1}} \xrightarrow{\chi \gamma_{\{0\} \cup \{0\}}} H_{\{0\} \cup \{0\}, W_1^{1} \otimes W_2^{1}} \xrightarrow{\gamma_i} H_{\{0\} \cup \{0\}, 1}.
\]

Putting together \(x_1, \chi_{\{0\} \cup \{0\}}, (\gamma_1)_i \in I_1, \chi_{\{0\} \cup \{0\}}, \xi_1\) on one side and \(x_2, \chi_{\{0\} \cup \{0\}}, (\gamma_2)_i \in I_2, \chi_{\{0\} \cup \{0\}}, \xi_2\) on the other side we get the RHS. We are allowed to do this because in the following diagram (where we write \(\gamma_1 = (\gamma_1)_i \in I_1\) and \(\gamma_2 = (\gamma_2)_i \in I_2\) all squares and triangles commute.

**Proof of (5.6).** For every \((g_i)_{i \in I} \in \tilde{(G)^I}\),

- \(\xi \otimes \ev_W\) is invariant by \((1)_{i \in I} \times (g_i)_{i \in I} \times (g_i)_{i \in I}\)
- \(\delta_W \otimes x\) is invariant by \((g_i)_{i \in I} \times (g_i)_{i \in I} \times (1)_{i \in I}\).

Therefore for every \((\alpha_i)_{i \in I}\) and \((\beta_i)_{i \in I}\) in \(\text{Gal}(\overline{F}/F)^I\), the RHS of (5.6) is equal to

\[
S_{I, I \cup I, W \otimes W \otimes W} \xrightarrow{\delta_W \otimes \xi \otimes \ev_W} (\gamma_i, (\gamma_1')_{i \in I} \times (\alpha_i, (\gamma_2')_{i \in I})_{i \in I} \times (\alpha_i, (\gamma_3')_{i \in I})_{i \in I}.
\]

To prove it in a formal way we factorize the RHS of (5.6) through

\[
H_{1,1} \xrightarrow{\gamma \delta_W} H_{1, (W \otimes W)^*} \xrightarrow{\chi} H_{1, (W \otimes W)^*} \xrightarrow{\gamma \delta_W} H_{1,1},
\]

where \(\zeta : I \cup I \to I\) is the obvious map, and we use the fact that \(\text{Gal}(\overline{F}/F)^I\) acts trivially on \(H_{1,1} \simeq H_{0,0}\). We take \(\alpha_i = \gamma_i (\gamma_1')^{-1}\) and \(\beta_i = (\gamma_2')^{-1}\gamma''_i\). Then (5.7) is equal to

\[
S_{I, I \cup I, W \otimes W \otimes W, \delta_W \otimes \xi \otimes \ev_W} \xrightarrow{\gamma \delta_W} \xrightarrow{\chi} \xrightarrow{\gamma \delta_W} H_{1,1},
\]

where \(\zeta : I \cup I \cup I \to I\) is the obvious map, and we use the fact that \(\text{Gal}(\overline{F}/F)^I\) acts trivially on \(H_{1,1} \simeq H_{0,0}\). We take \(\alpha_i = \gamma_i (\gamma_1')^{-1}\) and \(\beta_i = (\gamma_2')^{-1}\gamma''_i\). Then (5.7) is equal to

\[
S_{I, I \cup I, W \otimes W \otimes W, \delta_W \otimes \xi \otimes \ev_W} \xrightarrow{\gamma \delta_W} \xrightarrow{\chi} \xrightarrow{\gamma \delta_W} H_{1,1},
\]

Applying (5.4) to the obvious map \(\zeta : I \cup I \cup I \to I\), we see that (5.8) is equal to

\[
S_{I, W \otimes W \otimes W, \delta_W \otimes \xi \otimes \ev_W} \xrightarrow{\gamma \delta_W} \xrightarrow{\chi} \xrightarrow{\gamma \delta_W} H_{1,1},
\]

Lastly one shows that (5.9) is equal to the LHS of (5.6) by applying (5.3) to the \((\tilde{G})^I\)-linear injection

\[
u : W = 1 \otimes W \xrightarrow{\delta_W \otimes \text{Id}_W} W \otimes W^* \otimes W,
\]
which satisfies $\delta_W \otimes x = u(x)$ and $t^i u(\xi \otimes \text{ev}_W) = \xi$, since the composition
\[ W \xrightarrow{\delta_W \otimes \text{Id}_W} W \otimes W^* \otimes W \xrightarrow{\text{Id}_W \otimes \text{ev}_W} W \]
is equal to $\text{Id}_W$. \hfill \qed

We denote by $\mathcal{B} \subset \text{End}_{C_c(K_N \backslash G(k)/K_N,E)}(H_{(0),1})$ the subalgebra generated by all the excursion operators $S_{I,W,x,\xi,(\gamma_i)_{i \in I}}$. By (5.5), $\mathcal{B}$ is commutative.

In the rest of this section we consider $\hat{G}$ as a group scheme defined over $E$.

**Observation 5.3.** The functions
\[(5.10) \quad f : (g_i)_{i \in I} \mapsto \langle \xi, (g_i)_{i \in I} \cdot x \rangle \]
that we get by making $W$, $x$, and $\xi$ vary, are exactly the regular functions on the coarse quotient of $(\hat{G})^I$ by the left and right translations by $\hat{G}$ diagonal, that we denote by $\hat{G} \backslash (\hat{G})^I / \hat{G}$.

**Lemma 5.4.** The operator $S_{I,W,x,\xi,(\gamma_i)_{i \in I}}$ depends only on $I$, $f$, and $(\gamma_i)_{i \in I}$, where $f$ is given by (5.10).

**Proof.** Let $W$, $x$, $\xi$ be as above and let $f \in \mathcal{O}(\hat{G} \backslash (\hat{G})^I / \hat{G})$ be given by (5.10). We denote by $W_f$ the finite dimensional $E$-vector subspace of $\mathcal{O}(\hat{G})^I / \hat{G}$ generated by the left translates of $f$ by $(\hat{G})^I$. We set $x_f = f \in W_f$ and we denote by $\xi_f$ the linear form on $W_f$ given by the evaluation at $1 \in (\hat{G})^I / \hat{G}$. Then $W_f$ is a subquotient of $W$: if $W_x$ is the $E$-linear $(\hat{G})^I$-subrepresentation of $W$ generated by $x$, $W_f$ is the quotient of $W_x$ by the biggest $E$-linear $(\hat{G})^I$-subrepresentation on which $\xi$ vanishes. Thus we have the diagrams
\[
\begin{array}{cccc}
W & \xleftarrow{\alpha} & W_x & \xrightarrow{\beta} \quad W_f, \\
x & \xleftarrow{\alpha} & x & \xrightarrow{\beta} x_f, \\
\xi & \xleftarrow{\xi} & \xi & \xleftarrow{\xi_f} \xi_f
\end{array}
\]
of $(\hat{G})^I$-representations, of $\hat{G}$-invariant vectors and of $\hat{G}$-invariant linear forms. Applying (5.3) to $u = \alpha$ and $u = \beta$, we get
\[ S_{I,W,x,\xi,(\gamma_i)_{i \in I}} = S_{I,W_x,x,\xi,(\gamma_i)_{i \in I}} = S_{I,W_f,x_f,\xi_f,(\gamma_i)_{i \in I}}. \]
This shows that $S_{I,W,x,\xi,(\gamma_i)_{i \in I}}$ depends only on $I$, $f$, and $(\gamma_i)_{i \in I}$. \hfill \qed

**Definition 5.5.** For every function $f \in \mathcal{O}(\hat{G} \backslash (\hat{G})^I / \hat{G})$ we set
\[(5.11) \quad S_{I,f,(\gamma_i)_{i \in I}} = S_{I,W,x,\xi,(\gamma_i)_{i \in I}} \in \mathcal{B} \]
where $W$, $x$, $\xi$ are such that $f$ satisfies (5.10).

This new notation allows a more synthetic formulation of properties (5.4), (5.5) and (5.6), in the form of properties (i), (ii), (iii) and (iv) of proposition 1.6 that we are now able to justify.

**Proof of properties (i), (ii), (iii) and (iv) of proposition 1.6.** We deduce (ii) from (5.4). To prove (i) we use (5.5) with $I_1 = I_2 = I$, and we apply (5.4) to the obvious map $\zeta : I \cup I \rightarrow I$. Property (iii) comes from (5.6), since
\[
\langle \xi \otimes \text{ev}_W, ((g_i)_{i \in I} \boxtimes (g_i')_{i \in I} \boxtimes (g_i'')_{i \in I}) \cdot (\delta_W \boxtimes x) \rangle = \langle \xi, (g_i(g_i')^{-1}g_i'')_{i \in I} \cdot x \rangle.
\]
Lastly (iv) comes from the fact that $H_{I,W}$ is an inductive limit of finite dimensional continuous representations of $(\text{Gal}(\overline{F}/F))^I$.

**Remark 5.6.** In addition to (iv) we can show easily that for every $I$ and for every $f$ there exists an open subscheme $U$ of $X$ such that the morphism (1.10) factorizes through $\pi_1(U, \overline{\eta})^I$. Indeed (4.3) implies that $H_{I,W}$ is an inductive limit of continuous representations of $(\text{Gal}(\overline{F}/F))^I$ admitting such a factorization property. In this introductory article we do not state these properties because they are not necessary for the arguments and we wait for section 9 to prove the properties of non-ramification over $X \setminus N$ that we need anyway.

We do not know if $\mathcal{B}$ is reduced. Nevertheless we get a spectral decomposition (i.e. a decomposition into generalized eigenspaces)

\begin{equation}
H_{\{0\},1} = \bigoplus_{\nu} \mathcal{S}_\nu,
\end{equation}

where in the RHS the direct sum is indexed by the characters $\nu$ of $\mathcal{B}$. Increasing $E$ if necessary, we assume that all the characters of $\mathcal{B}$ are defined over $E$.

The following proposition allows to obtain decomposition (5.1) from (5.12) because it associates to every character $\nu$ a Langlands parameter $\sigma$.

**Proposition 5.7.** For every character $\nu$ of $\mathcal{B}$ there exists a morphism $\sigma : \text{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ such that

(C1) $\sigma$ takes its values in $\widehat{G}(E')$, where $E'$ is a finite extension of $E$, and it is continuous,

(C2) $\sigma$ is semisimple, i.e. the Zariski closure of its image is reductive,

(C3) for every $I$ and $f \in \mathcal{O}(\widehat{G} \setminus \widehat{G})^I$, we have

$$\nu(S_{I,f, (\gamma_i)_{i \in I}}) = f((\sigma(\gamma_i))_{i \in I}).$$

Moreover $\sigma$ is unique up to conjugation by $\widehat{G}(\overline{\mathbb{Q}_\ell})$.

**Proof.** We refer to the proof of proposition 10.7 of [Laf12] for some additional details. The proof uses only properties (i), (ii), (iii) and (iv) of proposition 1.6. Let $\nu$ be a character of $\mathcal{B}$.

For every $n \in \mathbb{N}$ we denote by $(\widehat{G})^n/\widehat{G}$ the coarse quotient of $(\widehat{G})^n$ by the action of $\widehat{G}$ by diagonal conjugation, i.e.

$$h.(g_1, ..., g_n) = (hg_1h^{-1}, ..., hg_nh^{-1}).$$

Then the morphism

$$(\widehat{G})^n \to (\widehat{G})^{\{0, ..., n\}}, (g_1, ..., g_n) \mapsto (1, g_1, ..., g_n)$$

induces an isomorphism

$$\beta : (\widehat{G})^n/\widehat{G} \cong (\widehat{G}\setminus (\widehat{G})^{\{0, ..., n\})/\widehat{G},$$

whence an algebra isomorphism

$$\mathcal{O}((\widehat{G})^n/\widehat{G}) \cong \mathcal{O}(\widehat{G}\setminus (\widehat{G})^{\{0, ..., n\})/\widehat{G}), \quad f \mapsto f \circ \beta^{-1}.$$
We introduce

$$
\Theta^n_\nu : \mathcal{O}((\widehat{G})^n/\widehat{G}) \to C(\text{Gal}(\overline{F}/F)^n, E)
$$

$$
f \mapsto [(\gamma_1, \ldots, \gamma_n) \mapsto \nu(S_l f_{\sigma}^{-1}, (1, \gamma_1, \ldots, \gamma_n))]
$$

Condition (C3) that $\sigma$ must satisfy can be reformulated in the following way: for every $n$ and for every $f \in \mathcal{O}((\widehat{G})^n/\widehat{G})$,

$$
\Theta^n_\nu(f) = [(\gamma_1, \ldots, \gamma_n) \mapsto f((\sigma(\gamma_1), \ldots, \sigma(\gamma_n)))]
$$

We deduce immediately from properties (i), (ii), (iii) and (iv) of proposition 1.6 that the sequence $(\Theta^n_\nu)_{n \in \mathbb{N}^*}$ satisfies the following properties

- for every $n$, $\Theta^n_\nu$ is an algebra morphism,
- the sequence $(\Theta^n_\nu)_{n \in \mathbb{N}^*}$ is functorial relatively to all the maps between the sets $\{1, \ldots, n\}$, i.e. for $m, n \in \mathbb{N}^*$,

$$
\zeta : \{1, \ldots, m\} \to \{1, \ldots, n\}
$$

arbitrary, $f \in \mathcal{O}((\widehat{G})^m/\widehat{G})$ and $(\gamma_1, \ldots, \gamma_n) \in \text{Gal}(\overline{F}/F)^n$, we have

$$
\Theta^n_\nu(f^\zeta)((\gamma_j)_{j \in \{1, \ldots, n\}}) = \Theta^n_\nu(f)((\gamma_{\zeta(i)})_{i \in \{1, \ldots, m\}})
$$

where $f^\zeta \in \mathcal{O}((\widehat{G})^n/\widehat{G})$ is defined by

$$
f^\zeta((g_j)_{j \in \{1, \ldots, n\}}) = f((g_{\zeta(i)})_{i \in \{1, \ldots, m\}}),
$$

- for $n \geq 1$, $f \in \mathcal{O}((\widehat{G})^n/\widehat{G})$ and $(\gamma_1, \ldots, \gamma_{n+1}) \in \text{Gal}(\overline{F}/F)^{n+1}$ we have

$$
\Theta^n_{n+1}(f)_{\gamma_1, \ldots, \gamma_{n+1}) = \Theta^n_\nu(f)(\gamma_1, \ldots, \gamma_{n}\gamma_{n+1})
$$

where $f \in \mathcal{O}((\widehat{G})^{n+1}/\widehat{G})$ is defined by

$$
f(g_1, \ldots, g_n, g_{n+1}) = f(g_1, \ldots, g_n, g_{n+1}).
$$

To justify the last property, we apply property (iii) of proposition 1.6 to $I = \{0, \ldots, n+1\}$, $(\gamma_i)_{i \in I} = (1, \ldots, 1, \gamma)$, $(\gamma'_i)_{i \in I} = (1)_{i \in I}$, $(\gamma''_i)_{i \in I} = (1, \gamma_1, \ldots, \gamma_n, \gamma')$ and we use (ii) to delete all the 1 except the first one in $(\gamma_i)_{i \in I} \times (\gamma'_i)_{i \in I} \times (\gamma''_i)_{i \in I}$.

We will see that these properties of the sequence $(\Theta^n_\nu)_{n \in \mathbb{N}^*}$ imply the existence and the unicity of $\sigma$ satisfying (C1), (C2) and (C3) (i.e. (5.13)).

For $G = GL_r$ the sequence $(\Theta^n_\nu)_{n \in \mathbb{N}^*}$ is determined by $\Theta^1_\nu(\text{Tr})$ (which must be the character of $\sigma$) and $\Lambda^{r+1} \text{St} = 0$ implies the pseudo-character relation whence the existence of $\sigma$ by [Tay91]. We refer to remark 10.8 of [Laf12] for further details.

For general $G$ we use results of [Ric88]. We say that a $n$-uplet $(g_1, \ldots, g_n) \in \widehat{G}(\overline{F}_\ell)^n$ is semisimple if every parabolic subgroup containing it admits an associated Levi subgroup containing it. Since $\overline{F}_\ell$ is of characteristic 0 this is equivalent to the condition that the Zariski closure $< g_1, \ldots, g_n >$ of the subgroup $< g_1, \ldots, g_n >$ generated by $g_1, \ldots, g_n$ is reductive. By theorem 3.6 of [Ric88] the $\widehat{G}$-orbit (by conjugation) of $(g_1, \ldots, g_n)$ is closed in $(\widehat{G})^n$ if and only if $(g_1, \ldots, g_n)$ is semisimple. Therefore the points over $\overline{F}_\ell$ of the coarse quotient $(\widehat{G})^n/\widehat{G}$ (which correspond
to the closed $\hat{G}$-orbits defined over $\overline{Q}_\ell$ in $(\hat{G})^n$ are in bijection with the $\hat{G}(\overline{Q}_\ell)$-conjugacy classes of semisimple $n$-uplets $(g_1, \ldots, g_n) \in \hat{G}(\overline{Q}_\ell)^n$.

For any $n$-uplet $(\gamma_1, \ldots, \gamma_n) \in \text{Gal}(\overline{F}/F)^n$ we denote by $\xi_n(\gamma_1, \ldots, \gamma_n)$ the point defined over $\overline{Q}_\ell$ of the coarse quotient $(\hat{G})^n/\hat{G}$ associated to the character

$$O((\hat{G})^n/\hat{G}) \to \overline{Q}_\ell, \ f \mapsto \Theta_n'(f)(\gamma_1, \ldots, \gamma_n).$$

We denote by $\xi_n^{ss}(\gamma_1, \ldots, \gamma_n)$ the conjugacy class of semisimple $n$-uplets corresponding to $\xi_n(\gamma_1, \ldots, \gamma_n)$ by the result of [Ric88] stated above.

The relation (5.13) is equivalent to the condition that for every $n$ and for every $(\gamma_1, \ldots, \gamma_n)$, $(\sigma(\gamma_1), \ldots, \sigma(\gamma_n)) \in (\hat{G}(\overline{Q}_\ell))^n$ (which is not always semisimple) lies over $\xi_n(\gamma_1, \ldots, \gamma_n)$.

**Unicity of $\sigma$ (up to conjugation).** We choose $n$ and $(\gamma_1, \ldots, \gamma_n)$ such that

$$\sigma(\gamma_1), \ldots, \sigma(\gamma_n)$$

generate a Zariski dense subgroup in $\text{Im}(\sigma)$. Since $\sigma$ is assumed to be semisimple, $(\sigma(\gamma_1), \ldots, \sigma(\gamma_n))$ is semisimple. We fix $(g_1, \ldots, g_n)$ in $\xi_n^{ss}(\gamma_1, \ldots, \gamma_n)$. Then $(\sigma(\gamma_1), \ldots, \sigma(\gamma_n))$ is conjugated to $(g_1, \ldots, g_n)$ and by conjugating $\sigma$ we can assume the latter is equal to the former. Then $\sigma$ is uniquely determined because for every $\gamma$, $\sigma(\gamma)$ belongs to the Zariski closure of the subgroup generated by $(g_1, \ldots, g_n)$ and $(g_1, \ldots, g_n, \sigma(\gamma)) \in \xi_n^{ss}(\gamma_1, \ldots, \gamma_n, \gamma)$, therefore the knowledge of $\xi_{n+1}(\gamma_1, \ldots, \gamma_n, \gamma)$ determines $\sigma(\gamma)$ uniquely.

**Existence of $\sigma$.** For every $n$ and every $(\gamma_1, \ldots, \gamma_n) \in \text{Gal}(\overline{F}/F)^n$ we choose $(g_1, \ldots, g_n) \in \xi_n^{ss}(\gamma_1, \ldots, \gamma_n)$ (well defined up to conjugation). Then we choose $n$ and $(\gamma_1, \ldots, \gamma_n) \in \text{Gal}(\overline{F}/F)^n$ such that

- (H1) the dimension of $\langle g_1, \ldots, g_n \rangle$ is the greatest possible
- (H2) the centralizer $C(g_1, \ldots, g_n)$ of $\langle g_1, \ldots, g_n \rangle$ is the smallest possible (minimal dimension and then minimal number of connected components).

We fix $(g_1, \ldots, g_n) \in \xi_n^{ss}(\gamma_1, \ldots, \gamma_n)$ for the rest of the proof and we construct a map

$$\sigma : \text{Gal}(\overline{F}/F) \to \hat{G}(\overline{Q}_\ell)$$

by asking that for every $\gamma \in \text{Gal}(\overline{F}/F)$, $\sigma(\gamma)$ is the unique element $g$ of $\hat{G}(\overline{Q}_\ell)$ such that $(g_1, \ldots, g_n, g) \in \xi_{n+1}^{ss}(\gamma_1, \ldots, \gamma_n, \gamma)$. The existence and the unicity of such a $g$ are justified in the following way.

- **A) Existence of $g$** : for $(h_1, \ldots, h_n, h) \in \xi_{n+1}^{ss}(\gamma_1, \ldots, \gamma_n, \gamma)$, $(h_1, \ldots, h_n)$ is necessarily semisimple (because $(h_1, \ldots, h_n)$ is over $\xi_n(\gamma_1, \ldots, \gamma_n)$ and $(g_1, \ldots, g_n) \in \xi_n^{ss}(\gamma_1, \ldots, \gamma_n)$ thus by theorem 5.2 of [Ric88],

  $$\langle h_1, \ldots, h_n \rangle$$

admits a Levi subgroup isomorphic to $\langle g_1, \ldots, g_n \rangle$, but

  $$\dim(\langle h_1, \ldots, h_n \rangle) \leq \dim(\langle g_1, \ldots, g_n \rangle)$$

by (H1)), thus conjugating $(h_1, \ldots, h_n, h)$ we can assume that $(h_1, \ldots, h_n) = (g_1, \ldots, g_n)$ and then we take $g = h$.

- **B) Unicity of $g$** : we have $C(g_1, \ldots, g_n, g) \subset C(g_1, \ldots, g_n)$ and equality holds by (H2), therefore $g$ commutes with $C(g_1, \ldots, g_n)$ and since it was well defined up to conjugation by $C(g_1, \ldots, g_n)$ it is unique.

Then we show that the map $\sigma$ we have just constructed is a morphism of groups. Indeed let $\gamma, \gamma' \in \text{Gal}(\overline{F}/F)$. The same argument as in A) above shows
that there exist \(g, g'\) such that
\[(5.14) \quad (g_1, \ldots, g_n, g, g') \in \xi_{n+2}^{ss}(\gamma_1, \ldots, \gamma_n, \gamma, \gamma').\]
Thanks to the properties satisfied by the sequence \((\Theta^n_n)_{n \in \mathbb{N}^*}\), we see that \(\xi_{n+1}(\gamma_1, \ldots, \gamma_n, \gamma')\) is the image of \(\xi_{n+2}(\gamma_1, \ldots, \gamma_n, \gamma, \gamma')\) by the morphism
\[(\hat{G})^{n+2} \parallel \hat{G} \to (\hat{G})^{n+1} \parallel \hat{G}, (h_1, \ldots, h_n, h, h') \mapsto (h_1, \ldots, h_n, hh').\]
From this we deduce that \((g_1, \ldots, g_n, gg')\) is over \(\xi_{n+1}(\gamma_1, \ldots, \gamma_n, \gamma')\). Moreover \((g_1, \ldots, g_n, gg')\) is semisimple by the same argument as in A), because
\[
\dim(<g_1, \ldots, g_n, gg'>) \leq \dim(<g_1, \ldots, g_n, g, g'>) \leq \dim(<g_1, \ldots, g_n>)
\]
(where the last inequality comes from (H1)). Therefore \((g_1, \ldots, g_n, gg')\) belongs to \(\xi_{n+1}^{ss}(\gamma_1, \ldots, \gamma_n, \gamma')\) and \(gg' = \sigma(\gamma')\). The same arguments show that
\[(5.15) \quad g = \sigma(\gamma) \quad \text{and} \quad g' = \sigma(\gamma').\]
Endly we showed that \(\sigma(\gamma') = \sigma(\gamma)\sigma(\gamma')\).

Thus \(\sigma\) is a group morphism with values in \(\hat{G}(E')\) (where \(E'\) is a finite extension of \(E\) such that \(g_1, \ldots, g_n\) belong to \(\hat{G}(E')\)). The argument to prove that \(\sigma\) is continuous is the following. We know that for every function \(f\) over \((\hat{G}_{E'})^{n+1} \parallel \hat{G}_{E'}\),
\[
\gamma \mapsto f(g_1, \ldots, g_n, \sigma(\gamma)) = \Theta_{n+1}^\nu(f)(\gamma_1, \ldots, \gamma_n, \gamma)
\]
is continuous. But the morphism
\[
\mathcal{O}((\hat{G}_{E'})^{n+1} \parallel \hat{G}_{E'}) \to \mathcal{O}(\hat{G}_{E'} \parallel C(g_1, \ldots, g_n))
\]
\[
f \mapsto [g \mapsto f(g_1, \ldots, g_n, g)]
\]
is surjective (because \((g_1, \ldots, g_n)\) is semisimple, therefore its orbit by conjugation is an affine closed subvariety of \((\hat{G}_{E'})^n\), isomorphic to \(\hat{G}_{E'}/C(g_1, \ldots, g_n)\)). Moreover, if we denote by \(D(g_1, \ldots, g_n)\) the centralizer of \(C(g_1, \ldots, g_n)\) (which contains the image of \(\sigma\)), the restriction morphism
\[
\mathcal{O}(\hat{G}_{E'} \parallel C(g_1, \ldots, g_n)) = \mathcal{O}(\hat{G}_{E'})^C(g_1, \ldots, g_n) \to \mathcal{O}(D(g_1, \ldots, g_n))
\]
is surjective because the restriction \(\mathcal{O}(\hat{G}_{E'}) \to \mathcal{O}(D(g_1, \ldots, g_n))\) is obviously surjective, and remains so when we take the invariants by the reductive group \(C(g_1, \ldots, g_n)\), and that \(C(g_1, \ldots, g_n)\) acts trivially on \(\mathcal{O}(D(g_1, \ldots, g_n))\). Thus for every function \(h \in \mathcal{O}(D(g_1, \ldots, g_n))\), \(\gamma \mapsto h(\sigma(\gamma))\) is continuous, and we proved that \(\sigma\) is continuous.

It remains to prove (5.13), i.e. for \(m \in \mathbb{N}^*\), \(f \in \mathcal{O}((\hat{G})^m \parallel \hat{G})\) and \((\delta_1, \ldots, \delta_m) \in \text{Gal}(\overline{F}/F)^m\), we have
\[
f(\sigma(\delta_1), \ldots, \sigma(\delta_m)) = (\Theta_m^\nu(f))(\delta_1, \ldots, \delta_m).
\]
By the same arguments as for (5.14) and (5.15) we show that
\[
(g_1, \ldots, g_n, \sigma(\delta_1), \ldots, \sigma(\delta_m)) \in \xi_{n+m}^{ss}(\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m).
\]
Therefore \((\sigma(\delta_1), \ldots, \sigma(\delta_m))\) is over \(\xi_m(\delta_1, \ldots, \delta_m)\). \(\square\)
Thus we have obtained the decomposition (1.3). This concludes the proof of theorem 1.1, provided we admit the following result, which will be justified in section 9.

**Result admitted for the moment** (proposition 9.4). Any $\sigma$ occurring in the decomposition (1.3) is unramified outside $N$ and the decomposition (1.3) is compatible with the Satake isomorphism at the places of $X \setminus N$.

6. CREATION AND ANNIHILATION MORPHISMS

From now one and until section 9 we will justify
- the two results admitted in the section 4 and which were used for the proof of proposition 1.3
- and the result admitted at the end of the previous section and which was used to finish the proof of theorem 1.1.

In this section our goal is to use the coalescence isomorphisms (3.13) to construct creation and annihilation morphisms, and then to rewrite the Hecke operators at the places of $X \setminus N$ as the composition
- of a creation morphism,
- of the action of a partial Frobenius morphism,
- of an annihilation morphism,

and to use this to extend the Hecke operators (3.3) to morphisms of sheaves over the whole $(X \setminus N)^I$ and to obtain the Eichler-Shimura relations.

Let $I$ and $J$ be finite sets. We will now define the creation and annihilation morphisms, in the following way. The legs indexed by $I$ will remain unchanged and we will create (or annihilate) the legs indexed by $J$ at the same point of the curve (indexed by a set with one element, that we will denote by $\{0\}$).

We have obvious maps
$$\zeta_J : J \to \{0\}, \quad \zeta_J^I = (\text{Id}_I, \zeta_J) : I \cup J \to I \cup \{0\} \quad \text{and} \quad \zeta_J^0 = (\text{Id}_J, \zeta_0) : I \to I \cup \{0\}.$$  

Let $W$ and $U$ be finite dimensional $E$-linear representations of $(\hat{G})^I$ and $(\hat{G})^J$ respectively. We recall that $U^\zeta_J$ is the representation of $\hat{G}$ obtained by restricting $U$ to the diagonal $\hat{G} \subset (\hat{G})^J$. Let $x \in U$ and $\xi \in U^*$ be invariant under the diagonal action of $\hat{G}$. Then $W \boxtimes U$ is a representation of $(\hat{G})^{I \cup \{0\}}$ and $W \boxtimes U^\zeta_J$ are representations of $(\hat{G})^{I \cup \{0\}}$ linked by the morphisms
$$W \boxtimes 1 \xrightarrow{\text{Id}_W \boxtimes x} W \boxtimes U^\zeta_J \quad \text{and} \quad W \boxtimes U^\zeta_J \xrightarrow{\text{Id}_W \boxtimes \xi} W \boxtimes 1.$$

We denote by $\Delta : X \to X^J$ the diagonal morphism and we write $E_{X \setminus N}$ for the constant sheaf over $X \setminus N$.

**Definition 6.1.** We define the creation morphism $C^I_x$ as the composition
$$\mathcal{H}^\leq_{N,I,W} \boxtimes E_{(X \setminus N)} \xrightarrow{\chi^I_{\xi_J}} \mathcal{H}^\leq_{N,I \cup \{0\},W \boxtimes 1}$$
$$\xrightarrow{\mathcal{H}(\text{Id}_W \boxtimes \xi)} \mathcal{H}^\leq_{N,I \cup \{0\},W \boxtimes U^\zeta_J} \xrightarrow{\chi^{-1}_{\xi_J}} \mathcal{H}^\leq_{N,I \cup \{0\},W \boxtimes U^\zeta_J} \xrightarrow{\Delta_{(X \setminus N)^I \times (X \setminus N)}}$$
where $\chi^{\ell}_{ij}$ and $\chi^{\ell}_{ij}$ are the coalescence isomorphisms of (3.13). Similarly we define the annihilation morphism $C_0^\xi$ as the composition

$$
\mathcal{H}^{\leq \mu}_{N,J,W \otimes U \mid (X \setminus N)^I \times \Delta(X \setminus N)} \xrightarrow{\chi^{\ell}_{ij}} \mathcal{H}^{\leq \mu}_{N,J \cup \{0\}, W \otimes U \otimes \xi^j}
$$

$$
\mathcal{H}^{\leq \mu}_{N,J \cup \{0\}, W \otimes \xi^j} \xrightarrow{\chi^{\ell}_{ij}^{-1}} \mathcal{H}^{\leq \mu}_{N,I,W} \boxtimes E(X \setminus N).
$$

All the morphisms above are morphisms of sheaves over $(X \setminus N)^I \times (X \setminus N)$. Now we will use these morphisms with $J = \{1, 2\}$. Let $v$ be a place in $|X| \setminus |N|$. We consider $\nu$ also as a subscheme of $X$ and we denote by $E_v$ the constant sheaf over $v$. Let $V$ be an irreducible representation of $\hat{G}$. As previously we denote by $1 \xrightarrow{\delta_v} V \otimes V^*$ and $V \otimes V^* \xrightarrow{ev_v} 1$ the natural morphisms.

For $\kappa$ big enough (in function of $\deg(v), V$), we define $S_{V,v}$ as the composition

$$(6.1) \quad \mathcal{H}^{\leq \mu}_{N,I,W} \boxtimes E_v$$

$$(6.2) \quad \mathcal{H}^{\leq \mu}_{N,I \cup \{1,2\}, W \otimes V \otimes V^* \mid (X \setminus N)^I \times \Delta(v)}$$

$$(6.3) \quad \mathcal{H}^{\leq \mu + \kappa}_{N,I \cup \{1,2\}, W \otimes V \otimes V^* \mid (X \setminus N)^I \times \Delta(v)}$$

$$(6.4) \quad \mathcal{H}^{\leq \mu + \kappa}_{N,I,W} \boxtimes E_v.$$

In other words we create two new legs at $v$ with the help of $\delta_v : 1 \to V \otimes V^*$, we apply the partial Frobenius morphism (to the power $\deg(v)$) to the first one, and then we annihilate them with the help of $ev_v : V \otimes V^* \to 1$.

As a morphism of constructible sheaves over $(X \setminus N)^I \times v$, $S_{V,v}$ commutes with the natural action of the partial Frobenius morphism on $E_v$ in (6.1) and (6.4), since

- the creation and annihilation morphisms intertwine this action with the action of $F_{\{1,2\}}$ over (6.2) and (6.3), by remark 3.11,
- $F_{\{1\}}$ and therefore $F_{\{1\}}^{\deg(v)}$ commute with $F_{\{1,2\}} = F_{\{1\}}F_{\{2\}}$.

**Definition 6.2.** By abuse of notations we still write

$$S_{V,v} : \mathcal{H}^{\leq \mu}_{N,I,W} \to \mathcal{H}^{\leq \mu + \kappa}_{N,I,W}$$

for the morphism of sheaves over $(X \setminus (N \cup v))^I$ obtained by descent (i.e. by taking the invariants by the natural action of the partial Frobenius morphism on $E_v$ in (6.1) and (6.4)).

**Proposition 6.3.** The restriction of $S_{V,v}$ to $(X \setminus (N \cup v))^I$ is equal, as a morphism of sheaves over $(X \setminus (N \cup v))^I$, to the Hecke operator

$$T(h_{V,v}) : \mathcal{H}^{\leq \mu}_{N,I,W \mid (X \setminus (N \cup v))^I} \to \mathcal{H}^{\leq \mu + \kappa}_{N,I,W \mid (X \setminus (N \cup v))^I}.$$
It is enough to prove this when \( V \) and \( W \) are irreducible. The proof is of geometric nature. We sketch it here in a simple case, where it is reduced to the intersection of two smooth substacks in a smooth Deligne-Mumford stack and where this intersection happens to be transverse. The proof is more complicated in general because of the singularities. We refer to the proof of proposition 6.2 of [Laf12] for the general case (but an alternative solution could consist to reduce to the situation of a smooth transverse intersection with the help of Bott-Samelson resolutions).

**Proof when \( V \) is minuscule and \( \deg(v) = 1 \).** We recall that an irreducible representation of \( \hat{G} \) is said to be minuscule if all its weights are conjugated by the Weyl group. This is equivalent to the property that the corresponding orbit in the affine grassmannian is closed (and therefore it implies that the corresponding closed stratum is smooth).

Thanks to the hypothesis that \( \deg(v) = 1 \) we can delete \( \Box E_v \) everywhere. We consider the Deligne-Mumford stack

\[
Z^{((1),(2),I)} = \text{Ch}_{N,I\cup\{1,2\},W\hat{\mathbb{G}}\hat{\mathbb{V}}V^*}^*|(X\setminus(N\cup v))I\times\Delta(v),
\]

We will construct two closed substacks \( Y_1 \) and \( Y_2 \) in \( Z^{((1),(2),I)} \), equipped with morphisms \( \alpha_1 \) and \( \alpha_2 \) towards

\[
Z^{(I)} = \text{Ch}_{N,I,W}^*|(X\setminus(N\cup v))I,
\]

in such a way that

- **A** the restriction to \( (X\setminus(N\cup v))I \) of the composition \((6.1)\to(6.2)\to(6.3)\) of the creation morphism and of the action of the partial Frobenius morphism is realized by a cohomological correspondence supported by the correspondence \( Y_2 \) from \( Z^{(I)} \) to \( Z^{((1),(2),I)} \), and whose restriction to the open smoothness locus is determined by its support,

- **B** the restriction to \( (X\setminus(N\cup v))I \) of the annihilation morphism \((6.3)\to(6.4)\) is realized by a cohomological correspondence supported by the correspondence \( Y_1 \) from \( Z^{((1),(2),I)} \) to \( Z^{(I)} \), and whose restriction to the open smoothness locus is determined by its support.

Therefore \( S_{V,v} \) will be realized by a cohomological correspondence supported by the product \( Y_1 \times_{Z^{((1),(2),I)}} Y_2 \) of these correspondences. We will see

- that the product \( Y_1 \times_{Z^{((1),(2),I)}} Y_2 \) is nothing but the Hecke correspondence \( \Gamma^{(I)} \) of \( Z^{(I)} \) to itself (which is a finite correspondence, and even étale)
- that \( S_{V,v} \), which is therefore a cohomological correspondence supported by \( \Gamma^{(I)} \) is in fact equal to the obvious cohomological correspondence supported by \( \Gamma^{(I)} \) (which realizes \( T(h_{V,v}) \) since \( V \) is minuscule).

Thanks to the hypothesis that \( V \) is minuscule it will suffice to do this computation over the open smoothness locus, and the computation will be immediate because we will see that over this open smoothness locus the intersection \( Y_1 \times_{Z^{((1),(2),I)}} Y_2 \) is a transverse intersection of two smooth substacks.
We construct now all these objects. The Hecke correspondence $\Gamma^{(I)}$ is the stack classifying the data of $(x_i)_{i \in I}$ and of a diagram

\[
\begin{array}{ccc}
(\mathcal{G}', \psi') & \xrightarrow{\phi'} & (\tau \mathcal{G}', \tau \psi') \\
\downarrow \kappa & & \downarrow \tau \kappa \\
(\mathcal{G}, \psi) & \xrightarrow{\phi} & (\tau \mathcal{G}, \tau \psi)
\end{array}
\]

such that

- the lower line $((x_i)_{i \in I}, (\mathcal{G}, \psi) \xrightarrow{\phi} (\tau \mathcal{G}, \tau \psi))$ and the upper line $((x_i)_{i \in I}, (\mathcal{G}', \psi') \xrightarrow{\phi'} (\tau \mathcal{G}', \tau \psi'))$ belong to $\mathcal{Z}^{(I)},$

- $\kappa : \mathcal{G}|_{(X \times v) \times S} \xrightarrow{\sim} \mathcal{G}'|_{(X \times v) \times S}$ is an isomorphism such that the relative position of $\mathcal{G}$ w.r.t. $\mathcal{G}'$ at $v$ is equal to the dominant weight of $V$ (we recall that $V$ is minuscule),

- the restriction of $\kappa$ to $N \times S,$ which is an isomorphism, intertwines the level structures $\psi$ and $\psi'.$

Moreover the two projections $\Gamma^{(I)} \to \mathcal{Z}^{(I)}$ are the morphisms which keep the lower and upper lines of (6.5).

Since the legs indexed by $I$ vary in $X \setminus (N \cup v)$ and remain disjoint from the legs 1 and 2 fixed at $v,$ we can replace the partition $\{(1), \{2\}, I\}$ by $\{(1), \{2\}\}$ and therefore we have

\[
\mathcal{Z}^{(1),\{2\},I} = \text{Cht}^{(1),\{1\}I,\{2\}}_{N,I\cup\{1,2\},W/\text{EV}}|_{(X \setminus (N \cup v))^I \times \Delta(v)}.
\]

In other words the stack $\mathcal{Z}^{(1),\{2\},I}$ classifies the data of $(x_i)_{i \in I}$ and of a diagram

\[
\begin{array}{ccc}
(S_0, \psi_0) & \xrightarrow{\phi_1} & (S_1, \psi_1) \\
\downarrow \phi_2 & & \downarrow \phi_5 \\
(S_2, \psi_2) & \xrightarrow{\phi_3} & (\tau S_0, \tau \psi_0)
\end{array}
\]

with

\[
((x_i)_{i \in I}, (S_0, \psi_0) \xrightarrow{\phi} (S_1, \psi_1) \xrightarrow{\phi_2} (S_2, \psi_2) \xrightarrow{\phi_5} (\tau S_0, \tau \psi_0))
\]

\[
\in \text{Cht}^{(1),\{1\}I,\{2\}}_{N,I\cup\{1,2\},W/\text{EV}}|_{(X \setminus (N \cup v))^I \times \Delta(v)}
\]

and

\[
((x_i)_{i \in I}, (S_0, \psi_0) \xrightarrow{\phi_1} (S_1, \psi_1) \xrightarrow{\phi_2} (S_2, \psi_2) \xrightarrow{\phi_5} (\tau S_0, \tau \psi_0))
\]

\[
\in \text{Cht}^{(1)I,\{2\}}_{N,I\cup\{1,2\},W/\text{EV}}|_{(X \setminus (N \cup v))^I \times \Delta(v)}.
\]

The oblique, vertical and horizontal arrow of the diagram (6.6) are respectively the modifications associated to the leg 1, to the leg 2 and to the legs indexed by $I.$
We denote by $\mathcal{Y}_1 \xrightarrow{\xi} \mathcal{Z}^{(1),\{2\},I}$ the closed substack defined by the condition that in the diagram (6.6), $\phi_2 \phi_1 : \mathcal{Y}_0|_{(X-v) \times S} \rightarrow \mathcal{Y}_2|_{(X-v) \times S}$ extends to an isomorphism over $X \times S$. We have a morphism

$$\alpha_1 : \mathcal{Y}_1 \rightarrow \mathcal{Z}^{(I)}$$

which sends

$$(\mathcal{G}_1, \psi_1) \xrightarrow{\phi_2} (\mathcal{G}_2, \psi'_2) \xrightarrow{\tau \phi_1} (\mathcal{G}_1, \tau \psi_1)$$

$$(\mathcal{G}_0, \psi_0) \xrightarrow{\sim} (\mathcal{G}_2, \psi'_2) \xrightarrow{\phi_3} (\mathcal{G}_0, \tau \psi_0)$$

to the lower line, i.e.

$$(x_i)_{i \in I}, (\mathcal{G}_0, \psi_0) \xrightarrow{\phi_3(\phi_2 \phi_1)} (\mathcal{G}_0, \tau \psi_0).$$

The assertion B) above comes from a similar statement involving the Mirkovic-Vilonen sheaves. Indeed

- by a) of theorem 3.8 the direct image of $\mathcal{S}^{(1,\{2\})}_{\{1,2\},V_{\mathbb{Z}}V^*}$ (which is the constant sheaf $E$ with a cohomological shift) by the morphism (which forgets the intermediate modification) $\text{Gr}^{(1,\{2\})}_{\{1,2\},V_{\mathbb{Z}}V^*} \rightarrow \text{Gr}^{(1,\{2\})}_{\{1,2\},V_{\mathbb{Z}}V^*}$ is equal to $\mathcal{S}^{(1,\{2\})}_{\{1,2\},V_{\mathbb{Z}}V^*}$,

- by c) of theorem 3.8 the restriction of $\mathcal{S}^{(1,\{2\})}_{\{1,2\},V_{\mathbb{Z}}V^*}$ over the diagonal (and $a$ fortiori over $\Delta(v)$) is equal to $\mathcal{S}^{(0,\{0\})}_{\{0\},V_{\mathbb{Z}}V^*}$ that we send to the skyscraper sheaf $\mathcal{S}^{(0,\{0\})}_{\{0\},1}$ by $\text{ev}_V : V \otimes V^* \rightarrow 1$

and by the proper base change theorem this gives rise to a cohomological correspondence between $\text{Gr}^{(1,\{2\})}_{\{1,2\},V_{\mathbb{Z}}V^*}|_{\Delta(v)}$ and the point, and one can check that it is the obvious cohomological correspondence supported by the smooth closed subscheme of $\text{Gr}^{(1,\{2\})}_{\{1,2\},V_{\mathbb{Z}}V^*}|_{\Delta(v)}$ consisting of $(\mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \mathcal{G}_2 \xrightarrow{\sim} G)$ such that $\phi_2 \phi_1$ is an isomorphism.

We denote by $\mathcal{Y}_2 \xrightarrow{\xi} \mathcal{Z}^{(1),\{2\},I}$ the closed substack defined by the condition that in the diagram (6.6), $\tau \phi_1 \phi'_3 : \mathcal{Y}_2|_{(X-v) \times S} \rightarrow \tau \mathcal{G}_1|_{(X-v) \times S}$ extends to an isomorphism over $X \times S$. We have a morphism

$$\alpha_2 : \mathcal{Y}_2 \rightarrow \mathcal{Z}^{(I)}$$

which sends

$$(\mathcal{G}_1, \psi_1) \xrightarrow{\phi_2} (\mathcal{G}_2, \psi'_2) \xrightarrow{\tau \phi_1} (\mathcal{G}_1, \tau \psi_1)$$

$$(\mathcal{G}_0, \psi_0) \xrightarrow{\sim} (\mathcal{G}_2, \psi'_2) \xrightarrow{\phi_3} (\mathcal{G}_0, \tau \psi_0)$$
to the upper line, i.e.
\[
(6.9) \quad ((x_i)_{i \in I}, (S_1, \psi_1) \xrightarrow{(\tau_\phi \phi_i') \phi'_2} (\tau S_1, \tau \psi_1)).
\]

The justification of assertion A) above is given

- by an argument similar to the argument used to justify B) but involving
  \( \delta_V : 1 \to V \otimes V^* \) and the stack \( \text{Ch}_{N,I\cup\{1,2\},W\otimes V^*} \)
- by the fact that the restriction to \( (X \setminus (N \cup v))^I \times \Delta(v) \) of the partial Frobenius morphism
  \( \text{Fr}^{(I),\{1,2\}} : \text{Ch}_{N,I\cup\{1,2\},W\otimes V^*} \to \text{Ch}_{N,I\cup\{1,2\},W\otimes V^*} \)
  sends \( (6.6) \) to

\[
\begin{array}{ccc}
(S_1, \psi_1) & \xrightarrow{\phi'_2} & (S'_2, \psi'_2) \\
& \phi'_3 \downarrow & \phi_3 \swarrow \\
& (\tau S_1, \tau \psi_1) & \tau S_0, \tau \psi_0
\end{array}
\]

On the other hand we have a canonical isomorphism
\[
(6.10) \quad \Gamma^{(I)} \simeq Y_1 \times_{Z^{(1),\{1,2\},I}_I} Y_2.
\]

Indeed a point of \( Z^{(1),\{1,2\},I} \) belonging to \( Y_1 \) and to \( Y_2 \) is given by a diagram

\[
\begin{array}{ccc}
(S_1, \psi_1) & \xrightarrow{\phi'_2} & (S'_2, \psi'_2) \\
\phi_1 \downarrow & & \phi_3 \swarrow \\
(S_0, \psi_0) & \xrightarrow{\sim} & (S_2, \psi_2) \\
& \phi'_3 \downarrow & \phi_3 \swarrow \\
& (\tau S_1, \tau \psi_1) & \tau S_0, \tau \psi_0
\end{array}
\]

Therefore it is nothing but a point of \( \Gamma^{(I)} \), because the contraction of the two isomorphisms in the previous diagram produces the diagram

\[
\begin{array}{ccc}
(S_1, \psi_1) & \xrightarrow{(\tau_\phi \phi_i') \phi'_2} & (\tau S_1, \tau \psi_1) \\
\phi_1 \uparrow & & \phi_1 \uparrow \\
(S_0, \psi_0) & \xrightarrow{\phi_3 (\phi_2 \phi_1)} & (\tau S_0, \tau \psi_0)
\end{array}
\]

that we identify to the diagram \( (6.5) \).

We have natural morphisms from the stacks \( Z^{(1),\{1,2\},I}_I, Z^{(I)}_I, Y_1, Y_2 \) and \( \Gamma^{(I)} \) to \( \text{Gr}_{I,W}^{(I)}/G_{\sum n_i x_i} \). Since \( V \) is minuscule these morphisms are smooth. Therefore the open smoothness loci \( \partial \text{Gr}_{I,W}^{(I)}/G_{\sum n_i x_i} \) are the inverse images of \( \partial \text{Gr}_{I,W}^{(I)} \) under \( \text{Gr}_{I,W}^{(I)}/G_{\sum n_i x_i} \), where \( \partial \text{Gr}_{I,W}^{(I)} \) denotes the open smoothness locus of \( \text{Gr}_{I,W}^{(I)} \).

A computation of tangent spaces shows that \( \partial Y_1 \) and \( \partial Y_2 \) are smooth substacks in the smooth Deligne-Mumford stack \( \mathcal{L}_{1,\{1,2\},I} \) and that their intersection is transverse, and moreover \( (6.10) \) implies that their intersection is equal to \( \partial \Gamma^{(I)} \). Thus we obtained an equality of cohomological correspondences between \( S_{V,v} \) and \( T(h_{V,v}) \) over \( \partial \Gamma^{(I)} \) but since \( \Gamma^{(I)} \) is an étale correspondence between \( Z^{(I)} \) and itself.
the equality holds everywhere (indeed a morphism from the perverse sheaf $\text{IC}_{\Gamma(I)}$ to itself is determined by its restriction to $^*\Gamma(I)$).

A consequence of proposition 6.3 is that we have for every $f \in C_c(K_N \backslash G(\mathbb{A})/K_N, E)$ and $\kappa$ big enough, a natural extension of the morphism $T(f)$ (introduced in (3.3)) to a morphism $T(f) : \mathcal{H}_{N,I,W}^{\leq \mu} \to \mathcal{H}_{N,I,W}^{\leq \mu + \kappa}$ of constructible sheaves over the whole $(X \setminus N)^I$, in a way compatible with the composition of Hecke operators. Indeed, if we write $K_N = \prod K_{N,v}$, it is enough to do it for each place $v$ and for every $f \in C_c(K_{N,v} \backslash G(F_v)/K_{N,v}, E)$. There is nothing to do if $v \notin N$. If $v \in N$ it is enough to consider the case where $f = h_{V,v}$ and then the extension is given by $S_{V,v}$ thanks to proposition 6.3. For further details, we refer to corollary 6.5 of [Laf12].

For the Shimura varieties over number fields such extensions were defined in many cases, in a modular way, by Zariski closure or with the help of nearby cycles (see [Del71, FC90, GT05]).

Since $S_{V,v}$ is the extension of $T(h_{V,v})$, the following proposition is exactly the Eichler-Shimura relation. We use again $\{0\}$ to denote a set with one element (indexing the leg to which the Eichler-Shimura relation applies).

**Proposition 6.4.** Let $I, W$ be as above and $V$ be an irreducible representation of $\hat{G}$. Then

$$F_{\{0\}}^{\deg(v)} : \lim_{\mu} \mathcal{H}_{N,I,\{0\},W^{\mathbb{Z}}V}^{\leq \mu}(X \setminus N)^I \times v \to \lim_{\mu} \mathcal{H}_{N,I,\{0\},W^{\mathbb{Z}}V}^{\leq \mu}(X \setminus N)^I \times v$$

is killed by a polynomial of degree $\dim(V)$ whose coefficients are the restrictions to $(X \setminus N)^I \times v$ of the morphisms $S_{\Lambda^V,v}$. More precisely we have

$$\sum_{i=0}^{\dim V} (-1)^i (F_{\{0\}}^{\deg(v)})^i \circ S_{\Lambda^{\dim V - i},v}(X \setminus N)^I \times v = 0.$$

We recall that $S_{\Lambda^V,v}$ extends the Hecke operator $T(h_{\Lambda^V,v})$

from $(X \setminus (N \cup v))^{I\cup\{0\}}$ to $(X \setminus N)^{I\cup\{0\}}$

and we note that this extension is absolutely necessary in order to take its restriction to $(X \setminus N)^I \times v$. Thanks to the definition of the morphisms $S_{\Lambda^V,v}$ by (6.1)-(6.4), the proof of proposition 6.4 is a simple computation of tensor algebra (inspired by a proof of the Hamilton-Cayley theorem, and based uniquely on the fact that $\Lambda^{\dim V + 1} = 0$). We refer to chapter 7 of [Laf12] for this proof.

7. **CONSTRUCTIBLE SUBSHEAVES STABLE UNDER THE ACTION OF THE PARTIAL FRObenius MOrPHISMS**

The goal of this section is to prove lemma 7.1 and proposition 7.2, which had been admitted in section 4. We refer to chapter 8 of [Laf12] for more details.

We recall that the we have fixed a geometric point $\overline{\eta} = \text{Spec}(\overline{F})$ over the generic point $\eta$ of $\hat{X}$. Let $I$ be a finite set and $W = \bigoplus_{i \in I} W_i$ be an irreducible representation of $(\hat{G})^I$. We denote by $\Delta : X \to X^I$ the diagonal morphism. We
recall that we have fixed a geometric point $\eta'$ over the generic point $\eta'$ of $X'$ and a specialization arrow $\text{sp} : \eta' \rightarrow \Delta(\eta)$.

**Lemma 7.1.** The space $\left( \lim_{\rightarrow \mu} \mathcal{H}^{\leq \mu}_{N,I,W} \right)^{\text{Hf}}|_{\eta'}$ is the union of $\mathcal{O}_E$-submodules $\mathcal{M} = \mathfrak{S}|_{\eta'}$ where $\mathfrak{S}$ is a constructible $\mathcal{O}_E$-subsheaf of $\left( \lim_{\rightarrow \mu} \mathcal{H}^{\leq \mu}_{N,I,W} \right)|_{\eta'}$ stable under the action of the partial Frobenius morphisms.

**Proof.** We refer to the proof of proposition 8.14 of [Laf12] for more details. For every family $(v_i)_{i \in I}$ of closed points of $X \setminus N$, we denote $x_{i \in I} v_i$ their product, which is a finite union of closed points of $(X \setminus N)'$. Let $\mathcal{M}$ be a $\mathcal{O}_E$-submodule of finite type of $\left( \lim_{\rightarrow \mu} \mathcal{H}^{\leq \mu}_{N,I,W} \right)|_{\eta'}$ stable by $\pi_1(\eta', \eta')$ and $C_c(K_N \setminus G(\mathbb{A})/K_N, \mathcal{O}_E)$. We will construct $\mathcal{M} \supset \mathcal{M}$ satisfying the properties of the statement of the lemma. Since $\mathcal{M}$ is of finite type, we can find $\mu_0$ such that $\mathcal{M}$ is included in the image of $\mathcal{H}^{\leq \mu_0}_{N,I,W}|_{\eta'}$ in $\left( \lim_{\rightarrow \mu} \mathcal{H}^{\leq \mu}_{N,I,W} \right)|_{\eta'}$. Increasing $\mu_0$ if necessary, we can assume that $\mathcal{M}$ is a $\mathcal{O}_E$-submodule of $\mathcal{H}^{\leq \mu_0}_{N,I,W}|_{\eta'}$. Let $\Omega_0$ be a dense open subscheme of $X'$ over which $\mathcal{H}^{\leq \mu_0}_{N,I,W}$ is smooth. We have a unique smooth $\mathcal{O}_E$-subsheaf $\mathfrak{S} \subset \mathcal{H}^{\leq \mu_0}_{N,I,W}|_{\Omega_0}$ over $\Omega_0$ such that $\mathfrak{S}|_{\eta'} = \mathcal{M}$. We choose $(v_i)_{i \in I}$ such that $x_{i \in I} v_i$ is included in $\Omega_0$. For every $i$, the Eichler-Shimura relation (proposition 6.4) implies that

$$(7.1) \quad (F_{\{i\}}^{\text{deg}(v_i)})^{\dim W_i} \left( \mathfrak{S}|_{x_{i \in I} v_i} \right) \subset \sum_{\alpha=0}^{\dim W_i-1} (F_{\{i\}}^{\text{deg}(v_i)})^\alpha (S_{\Lambda^\dim W_i - \alpha W_i, v_i}^{\text{dim} W_i} \left( \mathfrak{S}|_{x_{i \in I} v_i} \right))$$

in $\left( \lim_{\rightarrow \mu} \mathcal{H}^{\leq \mu}_{N,I,W} \right)|_{x_{i \in I} v_i}$. Thanks to the smoothness of $(F_{\{i\}}^{\text{deg}(v_i)})^{\dim W_i} \left( \mathfrak{S}|_{\eta'} \right)$ at $x_{i \in I} v_i$, the inclusion (7.1) propagates to $\eta'$, i.e.

$$F_{\{i\}}^{\text{deg}(v_i)} \dim W_i \left( (F_{\{i\}}^{\text{deg}(v_i)} \dim W_i)|^{\eta'}(\mathfrak{S}|_{\eta'}) \right) \subset \sum_{\alpha=0}^{\dim W_i-1} F_{\{i\}}^{\text{deg}(v_i)}(F_{\{i\}}^{\text{deg}(v_i)})^\alpha (S_{\Lambda^\dim W_i - \alpha W_i, v_i}(\mathfrak{S}|_{\eta'}))$$

in $\left( \lim_{\rightarrow \mu} \mathcal{H}^{\leq \mu}_{N,I,W} \right)|_{\eta'}$. But $\mathfrak{S}|_{\eta'}$ is stable by $S_{\Lambda^\dim W_i - \alpha W_i, v_i} = T(h_{\Lambda^\dim W_i - \alpha W_i, v_i})$ since

$$h_{\Lambda^\dim W_i - \alpha W_i, v_i} \in C_c(G(\mathcal{O}_v) \setminus G(F_v)/G(\mathcal{O}_v), \mathcal{O}_E) \subset C_c(K_N \setminus G(\mathbb{A})/K_N, \mathcal{O}_E).$$

Consequently

$$F_{\{i\}}^{\text{deg}(v_i)} \dim W_i \left( (F_{\{i\}}^{\text{deg}(v_i)})^{\dim W_i}|^{\eta'}(\mathfrak{S}|_{\eta'}) \right) \subset \sum_{\alpha=0}^{\dim W_i-1} F_{\{i\}}^{\text{deg}(v_i)}(F_{\{i\}}^{\text{deg}(v_i)})^\alpha (\mathfrak{S}|_{\eta'}).$$
\[ \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^I}. \] Therefore
\[
\mathcal{G} = \sum_{(n_i)_{i \in I} \in \prod_{i \in I} \{0, \ldots, \text{deg}(n_i) \text{ dim}(W_i) - 1\}} \prod_{i \in I} F_{(i)}^{n_i} \left( \prod_{i \in I} \text{Frob}^{n_i}_{(i)} \right)^{*} \left( \mathcal{G} \right) \mid_{\eta^I}
\]
is a constructible \( \mathcal{O}_E \)-subsheaf of \( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^I} \) which is stable under the action of the partial Frobenius morphisms. Since \( \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^I} \right)^{Hf} \) is the union of \( \mathcal{O}_E \)-submodules \( \mathfrak{M} \) as at the beginning of the proof and since \( \mathfrak{M} = \mathcal{G} \mid_{\eta^I} \) contains \( \mathfrak{M} \) we get the statement of the lemma. \( \square \)

**Proposition 7.2.** The space \( \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^I} \right)^{Hf} \) is equipped with a natural action of \( \pi_1(\eta, \overline{\eta})^I \) (depending on the choice of \( \mathfrak{sp} \)). More precisely it is a union of \( E \)-vector subspaces equipped with a continuous action of \( \pi_1(\eta, \overline{\eta})^I \).

**Proof.** For every constructible \( \mathcal{O}_E \)-subsheaf \( \mathcal{G} \) of \( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^I} \) stable under the action of partial Frobenius morphisms, Drinfeld’s lemma provides a continuous action of \( \pi_1(\eta, \overline{\eta})^I \) on \( \mathfrak{M} = \mathcal{G} \mid_{\eta^I} \) (depending on the choice of \( \mathfrak{sp} \)). By lemma 7.1, \( \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^I} \right)^{Hf} \) is the union of such \( \mathfrak{M} \). \( \square \)

**Remark 7.3.** The action of \( \pi_1(\eta, \overline{\eta})^I \) on \( \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^I} \right)^{Hf} \) is uniquely determined by the actions of \( \pi_1(\eta^I, \overline{\eta}^I) \) and of the partial Frobenius morphisms (and by the choice of \( \mathfrak{sp} \)). This is a consequence of lemma 4.1 but we can see it also in the following way (for more details we refer to chapter 8 of [Laf12]). Following [Dri78], we shall define a group \( \text{Weil}^F(\eta^I, \overline{\eta}^I) \)

- which is a extension of \( \mathbb{Z}^I \) by \( \text{Ker}(\pi_1(\eta^I, \overline{\eta}^I) \to \hat{\mathbb{Z}}) \),
- and which coincides, when \( I \) is a singleton, with the usual Weil group \( \text{Weil}(\eta, \overline{\eta}) = \pi_1(\eta, \overline{\eta}) \times_{\hat{\mathbb{Z}}} \mathbb{Z} \).

We denote by \( F^I \) the field of functions of \( X^I \), by \( (F^I)_{\text{perf}} \) its perfectization and by \( \overline{F^I} \) the algebraic closure of \( F^I \) such that \( \overline{\eta}^I = \text{Spec}(\overline{F^I}) \). Then we define
\[
\text{Weil}^F(\eta^I, \overline{\eta}^I) = \{ \varepsilon \in \text{Aut}_{\overline{F^I}}((F^I)), \exists (n_i)_{i \in I} \in \mathbb{Z}^I, \varepsilon \mid_{(F^I)_{\text{perf}}} = \prod_{i \in I} (\text{Frob}_{(i)})^{n_i} \}.
\]
The choice of \( \mathfrak{sp} \) provides an inclusion \( \overline{F} \otimes_{\overline{F^I}} \cdots \otimes_{\overline{F^I}} \overline{F} \subset \overline{F^I} \). By restriction of the automorphisms, this gives a surjective morphism
\[
(7.2) \quad \text{Weil}^F(\eta^I, \overline{\eta}^I) \to (\text{Weil}(\eta, \overline{\eta}))^I
\]
(dependent on the choice of \( \mathfrak{sp} \)). The statement of proposition 7.2 can be reformulated by saying that the natural action of \( \text{Weil}^F(\eta^I, \overline{\eta}^I) \) on \( \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta^I} \right)^{Hf} \) factorizes through the morphism (7.2), and even through \( (\pi_1(\eta, \overline{\eta}))^I \).
8. Specialization homomorphisms and Hecke-finite cohomology

The goal of this section is to prove corollary 8.4, which had been admitted in section 4. Let $W = \bigotimes_{i \in I} W_i$ be an irreducible $E$-linear representation of $(\hat{G})^I$.

**Proposition 8.1.** The image of the specialization homomorphism

$$\sp^* : \varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\Delta(\eta)} \to \varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\eta'}$$

contains $\left( \varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\eta'} \right)^H$.

**Proof.** We refer to the proof of proposition 8.18 of [Laf12] for more details. By lemma 7.1, $\left( \varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\eta'} \right)^H$ is the union of $\mathcal{O}_E$-submodules $\mathcal{M} = \mathcal{G}|_{\eta'}$ where $\mathcal{G}$ is a constructible $\mathcal{O}_E$-subsheaf of $\varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\eta'}$. Let $\Omega_0$ be a dense open subscheme of $X^I$ such that $\mathcal{H}^{\leq \mu}_{N,I,W}|_{\Omega_0}$ is smooth. Then $\mathcal{G}$ extends to a smooth $\mathcal{O}_E$-subsheaf of $\mathcal{H}^{\leq \mu}_{N,I,W}|_{\Omega_0}$. By the proof of lemma 9.2.1 of [Lau04], the set of the $\prod_{i \in I} \text{Frob}^{n_i}_{\{i\}}(\Delta(\eta))$ for $(n_i)_{i \in I} \in \mathbb{N}^I$ is Zariski dense in $X^I$. Therefore we can find $(n_i)_{i \in I}$ such that $\prod_{i \in I} \text{Frob}^{n_i}_{\{i\}}(\Delta(\eta))$ belongs to $\Omega_0$. Then $\mathcal{G}|_{\prod_{i \in I} \text{Frob}^{n_i}_{\{i\}}(\eta')}$ is included in the image of

$$\tilde{\sp}^* : \varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\prod_{i \in I} \text{Frob}^{n_i}_{\{i\}}(\eta')} \to \varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\prod_{i \in I} \text{Frob}^{n_i}_{\{i\}}(\eta')}$$

for every specialization arrow $\tilde{\sp} : \prod_{i \in I} \text{Frob}^{n_i}_{\{i\}}(\eta') \to \prod_{i \in I} \text{Frob}^{n_i}_{\{i\}}(\Delta(\eta))$, and in particular for the image of $\sp$ by $\prod_{i \in I} \text{Frob}^{n_i}_{\{i\}}$. Since $\mathcal{G}$ is stable under the action of the partial Frobenius morphisms, we conclude that $\mathcal{M} = \mathcal{G}|_{\eta'}$ is included in the image of (8.1).

The following proposition is new compared to version 4 of [Laf12].

**Proposition 8.2.** The specialization homomorphism

$$\sp^* : \varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\Delta(\eta)} \to \varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\eta'}$$

is injective.

**Proof.** Let $a$ be in the kernel of (8.3). We choose $\mu_0$ and $\tilde{a} \in \mathcal{H}^{\leq \mu_0}_{N,I,W} \big|_{\Delta(\eta)}$ such that $a$ is the image of $\tilde{a}$ in $\varprojlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \big|_{\Delta(\eta)}$. Let $\Omega_0$ be a dense open subscheme of $X \setminus N$ over which $\Delta^*(\mathcal{H}^{\leq \mu_0}_{N,I,W})$ is smooth. Let $v \in |\Omega_0|$. We set $d = \deg(v)$ to shorten the formulas. Let $\overline{v}$ be a geometric point over $v$. Let $\sp_v : \eta \to \overline{v}$ be a specialization arrow. We still denote by $\sp_v : \Delta(\eta) \to \Delta(\overline{v})$ the specialization.
Lemma 8.3. a) For every \( j \in I \) and for every \((n_i)_{i \in I} \in \mathbb{N}^I\),

\[
\sum_{\alpha=0}^{\dim W_j} (-1)^\alpha S_{\alpha W_j}^{\dim W_j - \alpha} (a(n_i + \alpha \delta_{i,j})) = 0
\]

in \( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \big|_{\Delta(\sigma)} \).
b) For every \((n_i)_{i \in I} \in \mathbb{N}^I\) such that \(\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}}(\Delta(\overline{\eta})) \in \Omega_1\), we have \(a_{(n_i)_{i \in I}} = 0\) in \(\lim_{\mu} \mathcal{H}_{\leq \mu}^{\leq \mu}_{N,I,W}|_{\Delta(\overline{\eta})}\).

**Proof of a).** The \(b_{(n_i)_{i \in I}}\) satisfy a relation identical to (8.6) (in \(\lim_{\mu} \mathcal{H}_{\leq \mu}^{\leq \mu}_{N,I,W}|_{\Delta(\overline{\eta})}\)), namely the Eichler-Shimura relation for the leg \(j\) (proposition 6.4). Then (8.6) is obtained by applying \(\text{sp}_v^*\) to this relation (we are allowed to do this because the \(S_{\alpha}^{\dim W_j-\alpha}W_j\) are morphisms of sheaves).

**Proof of b).** Let \((n_i)_{i \in I}\) satisfy the hypotheses of b). Since (8.4) is a morphism of sheaves over \((X \setminus N)^I\), we can inverse the order of the specialization homomorphisms and the partial Frobenius morphisms. In other words we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W}|_{\Delta(\overline{\eta})} = (\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}})^*(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W})|_{\Delta(\overline{\eta})} & \xrightarrow{\Pi_{i \in I} F_{\{i\}}^{d_{ni}}} & \lim_{\mu} \mathcal{H}_{\leq \mu}^{\leq \mu}_{N,I,W}|_{\Delta(\overline{\eta})} \\
(\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}})^*(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W})|_{\Delta(\overline{\eta})} & \xrightarrow{\Pi_{i \in I} F_{\{i\}}^{d_{ni}}} & \lim_{\mu} \mathcal{H}_{\leq \mu}^{\leq \mu}_{N,I,W}|_{\Delta(\overline{\eta})} \\
\end{array}
\]

where the notation \(\text{sp}_v^*_{\{i\},(n_i)_{i \in I}}\) indicates that the specialization homomorphism associated to the arrow \(\text{sp}_v : \Delta(\overline{\eta}) \rightarrow \Delta(\overline{v})\) is applied to the sheaf \((\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}})^*(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W})\) (and not to \(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W}\)). The previous diagram gives rise to

\[
\begin{array}{ccc}
\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W}|_{\Delta(\overline{\eta})} = (\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}})^*(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W})|_{\Delta(\overline{\eta})} & \xrightarrow{\Pi_{i \in I} F_{\{i\}}^{d_{ni}}} & \lim_{\mu} \mathcal{H}_{\leq \mu}^{\leq \mu}_{N,I,W}|_{\Delta(\overline{\eta})} \\
(\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}})^*(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W})|_{\Delta(\overline{\eta})} & \xrightarrow{\Pi_{i \in I} F_{\{i\}}^{d_{ni}}} & \lim_{\mu} \mathcal{H}_{\leq \mu}^{\leq \mu}_{N,I,W}|_{\Delta(\overline{\eta})} \\
\end{array}
\]

Therefore to prove \(a_{(n_i)_{i \in I}} = 0\) (and finish the proof of b)) it suffices to prove that

\[
(8.7) \quad \text{sp}_v^*_{\{i\},(n_i)_{i \in I}}(\overline{b}) \in \mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W}|_{(\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}})^*(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W})}|_{\Delta(\overline{\eta})} = (\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}})^*(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W})|_{\Delta(\overline{\eta})}
\]

is zero. But (8.7) may also be considered as the image of \(\overline{b}\) by a specialization homomorphism for the sheaf \(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W}\) associated to a specialization arrow \((\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}})^*(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W})|_{\Delta(\overline{\eta})}\). Therefore (8.7) is zero because

- \(\prod_{i \in I} \text{Frob}_{\{i\}}^{d_{ni}}(\Delta(\overline{\eta}))\) belongs to \(\Omega_1\) by hypothesis
- for every geometric point \(\overline{x}\) of \(\Omega_1\) and every specialization arrow \(\text{sp}_{\overline{x}} : \overline{x} \rightarrow \Delta(\overline{\eta}), \text{sp}_{\overline{x}}^*_{\overline{x}}(\overline{b})\) vanishes in \(\mathcal{H}_{\leq \mu_1}^{\leq \mu_1}_{N,I,W}|_{\overline{x}}\).
This last assertion comes from the fact that \( \mathcal{H}_{N,I,W}^{\leq \mu} \) is smooth over \( \Omega_1 \) and that the image of \( \hat{b} \) by every specialization homomorphism to \( \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta} \) is zero (since it is the case of \( \text{sp}^*(\text{sp}_c^*(b)) \) and \( \pi_1(\eta', \eta') \) acts transitively on the specialization arrows from \( \eta' \) to \( \Delta(\pi) \).

**End of the proof of proposition 8.2.** Since \( \prod_{i \in I} \text{Frob}_{(i)} \) is the total Frobenius, \( \prod_{i \in I} F_{(i)}^{dn} \) acts bijectively on \( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\Delta(\eta)} \) and sends \( a_{(n_i)_{i \in I}} \) to \( a_{(n_i+n_i)_{i \in I}} \). From this and from a) of lemma 8.3 we deduce easily that to prove that \( a = a_{(0)_{i \in I}} \) is zero (and even that the whole sequence \( a_{(n_i)_{i \in I}} \) vanishes) it suffices to find \( (n_i)_{i \in I} \in \mathbb{N}^I \) such that

\[
a_{(n_i+n_i)_{i \in I}} = 0 \quad \text{for every} \quad (\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \ldots, \dim W_i - 1\}.
\]

This is possible by b) of lemma 8.3, because the density of the open subscheme \( \Omega_1 \) implies that we can find \( (n_i)_{i \in I} \in \mathbb{N}^I \) such that

\[
\prod_{i \in I} \text{Frob}_{(i)}(\Delta(\eta)) \in \Omega_1 \quad \text{for every} \quad (\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \ldots, \dim W_i - 1\}.
\]

This ends the proof of proposition 8.2. \( \square \)

Propositions 8.1 and 8.2 imply the following corollary.

**Corollary 8.4.** *The specialization homomorphism*

\[
(8.8) \quad \text{sp}^*: \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\Delta(\eta)} \right)^{\text{HF}} \rightarrow \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\eta} \right)^{\text{HF}}
\]

*is a bijection.*

**Proof.** The injectivity comes from proposition 8.2. Here is the proof of the surjectivity. Let \( c \in \left( \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\Delta(\eta)} \right)^{\text{HF}} \). By proposition 8.1 we can find \( a \in \lim_{\mu} \mathcal{H}_{N,I,W}^{\leq \mu} \mid_{\Delta(\eta)} \) such that \( \text{sp}^*(a) = c \). The injectivity of \( \text{sp}^* \) that we proved in proposition 8.2 implies that \( a \) is Hecke-finite. \( \square \)

9. *Compatibility with the Satake isomorphism at unramified places*

The goal of this section is to prove the two following lemmas, as well as proposition 9.4 which was admitted at the end of section 5.

**Lemma 9.1.** *Any parameter \( \sigma \) occurring in (1.3) is unramified over \( X \setminus N \).*

**Proof.** Let \( v \) be a place of \( X \setminus N \). We fix an embedding \( \overline{F} \subset \overline{F}_v \) whence an inclusion \( \text{Gal}(\overline{F}_v/F_v) \subset \text{Gal}(\overline{F}/F) \). Let \( I_v = \text{Ker}(\text{Gal}(\overline{F}_v/F_v) \rightarrow \mathbb{Z}) \) be the inertia subgroup at \( v \). Then for \( I,W,x,\xi \) as in (5.2), the image of the composition \( H_{(0),1} \xrightarrow{\mathcal{H}(x)} H_{(0),W,x} \xrightarrow{\xi^{-1}} H_{I,W} \) (which is the beginning of (5.2)) consists of elements invariant by \( (I_v)^I \), since the creation operators are morphisms
of sheaves over the whole $\Delta(X \setminus N)$ (and in particular over $\Delta(v)$). Thus for $(\gamma_i)_{i \in I} \in \text{Gal}(\overline{F}/F)^I$ and $(\delta_i)_{i \in I} \in (I_v)^I$ we have
\begin{equation}
S_{I,W,x,\xi,(\gamma_i)_{i \in I}} = S_{I,W,x,\xi,(\gamma_i)_{i \in I}}.
\end{equation}

Thanks to the proof of the unicity of $\sigma$ included in the proof of proposition 5.7, the relation (9.1) implies that for every $\sigma$ corresponding to a character $\nu$ of $\mathbb{B}_i$ we have $I_v \subset \text{Ker} \sigma$ and therefore $\sigma$ is unramified at $v$. □

The following lemma shows assertion (v) of proposition 1.6, namely that the Hecke operators at unramified places are particular cases of excursion operators.

Let $v$ be a place in $X \setminus N$. We fix an embedding $\overline{F} \subset \overline{F}_v$. As previously $1 \xrightarrow{\delta_v} V \otimes V^*$ and $V \otimes V^* \xrightarrow{\text{ev}_v} 1$ are the natural morphisms.

**Lemma 9.2.** For every $d \in \mathbb{N}$ and every $\gamma \in \text{Gal}(\overline{F}_v/F_v) \subset \text{Gal}(\overline{F}/F)$ such that $\text{deg}(\gamma) = 1$, $S_{(1,2),V^*\xi_\gamma,\delta_v,\text{ev}_v,(\gamma,1)}$ depends only on $d$, and if $d = 1$ it is equal to $T(h_{V,v})$.

**Proof.** We fix a geometric point $\mathfrak{p}$ over $v$ and a specialization arrow $\text{sp}_v : \mathfrak{p} \to \mathfrak{p}$. We still denote by $\text{sp}_v$ the specialization arrow $\Delta(\mathfrak{p}) \to \Delta(\mathfrak{p})$ equal to its image by $\Delta$. In order to reduce the size of the following diagram we set $I = \{1, 2\}$ and $W = V \boxtimes V^*$. The diagram

$$
C^{cusp}_c(G(F) \setminus G(\mathbb{A})/K_N^\infty, E) \xrightarrow{\varepsilon^{\gamma}_{\Delta(\mathfrak{p})}} \left(\lim_{\gamma \to \mu} \mathcal{H}^{\leq \mu}_{N,I,W}|_{\Delta(\mathfrak{p})}\right)^{\text{Hf}} \xrightarrow{\text{sp}_v^*} \left(\lim_{\gamma \to \mu} \mathcal{H}^{\leq \mu}_{N,I,W}|_{\Delta(\mathfrak{p})}\right)^{\text{Hf}} \xrightarrow{\sim} \left(\lim_{\gamma \to \mu} \mathcal{H}^{\leq \mu}_{N,I,W}|_{\mathfrak{p}^\infty}\right)^{\text{Hf}}
$$

is commutative (the commutativity of the big rectangle is justified with the help of lemma 4.1, or of remark 7.3). But $S_{(1,2),V^*\xi_\gamma,\text{ev}_v,(\gamma,1)}$ is equal by definition to the composition by the rightmost path. Thus it is equal to the composition given by the left column. Consequently it depends only on $d$. When $d = 1$ the composition given by the left column is equal by definition to $S_{V,v}$, and thus to $T(h_{V,v})$ by proposition 6.3. □

**Remark 9.3.** We computed the composition given by the left column only for $d = 1$ because we can prove that for other values of $d$ it is equal to a combination of $S_{W,v}$ with $W$ irreducible representation of $\tilde{G}$, and therefore it does not bring anything new.

The following proposition claims that the decomposition (1.3) is compatible with the Satake isomorphism at the places of $X \setminus N$. 

Proposition 9.4. Let $\sigma$ be a parameter occurring in (1.3) and let $v$ be a place of $X \setminus N$. Then $\sigma$ is unramified at $v$ and for every irreducible representation $V$ of $\hat{G}$, $T(h_{V,v})$ acts on $\mathcal{H}_{\sigma}$ by multiplication by the scalar $\chi_V(\sigma(\text{Frob}_v))$, where $\chi_V$ is the character of $V$ and $\text{Frob}_v$ is an arbitrary lifting of a Frobenius element at $v$.

**Proof.** The fact that $\sigma$ is unramified at $v$ was already established in lemma 9.1. We adopt again the notations of lemma 9.2. Since $\langle ev_V, (\sigma(\gamma), 1), \delta_V \rangle = \chi_V(\sigma(\gamma))$ this lemma implies that for every $\gamma \in \text{Gal}(\overline{F_v}/F_v)$ with $\deg(\gamma) = 1$, and every irreducible representation $V$ of $\hat{G}$, $\mathcal{H}_{\sigma}$ is included in the generalized eigenspace of $T(h_{V,v})$ for the eigenvalue $\chi_V(\sigma(\gamma))$. But we know that the Hecke operators at unramified places are diagonalizable (they are normal operators on the hermitian space of cuspidal automorphic forms with coefficients in $\mathbb{C}$). Therefore $T(h_{V,v})$ acts on $\mathcal{H}_{\sigma}$ by multiplication by the scalar $\chi_V(\sigma(\gamma))$. □

This ends the proof of theorem 1.1.

10. **Link with the geometric Langlands program**

Il is obvious that coalescence and permutation of legs are linked to factorization structures introduced by Beilinson and Drinfeld [BD04] and indeed our article uses in an essential way the fusion product on the affine grassmannian of Beilinson-Drinfeld in the geometric Satake equivalence [MV07, Gai07]. Moreover the idea of spectral decomposition is familiar in the geometric Langlands program, see [Bei06] and especially corollary 4.5.5 of [Gai13] which claims (in the setting of the geometric Langlands program for $D$-modules where the curve $X$ is defined over an algebraically closed field of characteristic 0) that the DG-category of $D$-modules on $\text{Bun}_G$ "lies over" the stack of $\hat{G}$-local systems (curiously we remark that we cannot formulate an analogous statement with the $\ell$-adic sheaves when $X$ is over $\mathbb{F}_q$, and however our article may be considered as a "classical" or rather "arithmetical" version of this statement). We shall see now that the link with the geometric Langlands program is even more direct.

Indeed the preprint [BV06] provides a very enlightening explanation of our approach if we admit the conjectures of the geometric Langlands program. We take here $N$ empty, i.e. $K_N = G(\mathbb{Q})$ but we could consider any level $N$ (and even non-split reductive groups).

The conjectures of the geometric Langlands program involve the following Hecke functors: for every representation $W$ of $(\hat{G})^I$ the Hecke functor $\phi_{I,W} : D_c^b(\text{Bun}_G, \overline{\mathbb{Q}_\ell}) \to D_c^b(\text{Bun}_G \times X^I, \overline{\mathbb{Q}_\ell})$ is given by $\phi_{I,W}(\mathcal{F}) = q_1_!(q_0^!(\mathcal{F}) \otimes \mathcal{F}_{I,W})$ where $\text{Bun}_G \xrightarrow{\text{Hecke}_{I,W}^{(I)}} \text{Bun}_G \times X^I$ is the Hecke correspondence and

- when $W$ is irreducible, $\mathcal{F}_{I,W}$ is equal, up to a shift, to the IC-sheaf of $\text{Hecke}_{I,W}^{(I)}$.
- in general it is defined as the inverse image of $S_{I,W}^{(I)}$ by the smooth natural morphism $\text{Hecke}_{I,W}^{(I)} \to \text{Gr}_{I,W}^{(I)} / G_{\Sigma i}$ (where the $n_i$ are big enough).
Let $\mathcal{E}$ be a $\hat{G}$-local system over $X$. Then $\mathcal{F} \in D^b_c(Bun_G, \overline{\mathbb{Q}}_\ell)$ is said to be an eigensheaf for $\mathcal{E}$ if we have, for every finite set $I$ and every representation $W$ of $(\hat{G})^I$, an isomorphism $\phi_{I,W}(\mathcal{F}) \cong \mathcal{F} \boxtimes W_\mathcal{E}$, functorial in $W$, and compatible with exterior products and with fusion (i.e. with the inverse image by the diagonal morphism $X^J \to X^I$ associated to every map $I \to J$). The conjectures of the geometric Langlands program imply the existence of an eigensheaf $\mathcal{F}$ for $\mathcal{E}$ (which satisfies an additional Whittaker normalization condition which prevents it in particular to be zero). In the geometric Langlands program $X$ and $Bun_G$ are usually defined over an algebraically closed field but here we work over $\overline{\mathbb{F}}_q$.

Let $\mathcal{F}$ be an eigensheaf for $\mathcal{E}$. We denote by $f \in C(Bun_G(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_\ell)$ the function associated to $\mathcal{F}$, i.e. for $x \in Bun_G(\overline{\mathbb{F}}_q)$, $f(x) = \text{Tr}(\text{Frob}_x, \mathcal{F}|_x)$. Let $\Xi \subset Z(F) \setminus Z(\mathcal{A})$ be a lattice. We assume that $\mathcal{F}$ is $\Xi$-equivariant, so that $f \in C(Bun_G(\overline{\mathbb{F}}_q)/\Xi, \overline{\mathbb{Q}}_\ell)$ (decreasing $\Xi$ if necessary this is implied by an adequate condition on $\mathcal{E}$, in fact on its image by the morphism from $\hat{G}$ to its abelianization).

It is well-known that $f$ is an eigenvector w.r.t. Hecke operators: for every place $v$ and every irreducible representation $V$ of $\hat{G}$, $T(h_{V,v})(f) = \text{Tr}(\text{Frob}_v, V_\mathcal{E}|_v)f$, where $\text{Frob}_v$ is any lifting of a Frobenius element at $v$.

The following proposition (which relies on a result that has not yet been written) formulates the compatibility between the geometric Langlands program and the decomposition (1.3).

**Proposition 10.1.** Let $\mathcal{F}$ be a $\Xi$-equivariant eigensheaf for $\mathcal{E}$ such that the function $f$ associated to $\mathcal{F}$ is cuspidal. Then $f$ belongs to $\mathcal{H}_\sigma$ where $\sigma : \pi_1(X, \overline{\eta}) \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ is the Galois representation corresponding to the local system $\mathcal{E}$.

**Proof.** In [BV06], Braverman and Varshavsky use a very general trace morphism, and the fact that Ch_{I,W} is the intersection of the Hecke correspondence with the graph of the Frobenius endomorphism of Bun_G, to construct a morphism of sheaves over $X^I$

$$\pi^{F,E}_{I,W} : \lim_{\mu} \mathcal{H}^{\leq \mu}_{N,I,W} \to W_\mathcal{E}. \quad (10.1)$$

These morphisms are functorial in $W$, and compatible with the coalescence of legs and with the action of the partial Frobenius morphisms (this last point has not yet been written). Moreover, $\pi^{F,E}_{\emptyset,1} : C_c(Bun_G(\overline{\mathbb{F}}_q)/\Xi, \overline{\mathbb{Q}}_\ell) \to \overline{\mathbb{Q}}_\ell$ is nothing but $h \mapsto \int_{Bun_G(\overline{\mathbb{F}}_q)/\Xi} f h$. The properties of these morphisms $\pi^F_{I,W}$ imply that for every $I, W, x, \xi, (\gamma_i)_{i \in I}$, we have

$$S_{I,W,x,\xi,(\gamma_i)_{i \in I}}(f) = \langle \xi, (\sigma(\gamma_i))_{i \in I}, x \rangle f.$$

This finishes the proof of proposition 10.1. \qed

### 11. Relation with previous works

The methods used in this work are completely different from the methods based on the trace formulas which were developed notably by Drinfeld [Dri78, Dri87, Dri88, Dri89], Laumon, Rapoport and Stuhler [LRS93], Laumon [Lau96,
Laurent Lafforgue [Laf97, Laf98, Laf02a, Laf02b], Ngô Bao Châu [NBC99, NBC06a], Eike Lau [Lau04, Lau07], Ngo Dac Tuan [NDT07, NDT09, NDT11], Ngô Bao Châu and Ngo Dac Tuan [NN08], Kazhdan and Varshavsky [KV13, Var09] and Badulescu and Roche [BR13].

Nevertheless the action on the cohomology of the permutation groups of the legs of the chtoucas occurs already in the works of Ngô Bao Châu, Ngo Dac Tuan and Eike Lau that we have just quoted. These actions of the permutation groups also play an essential role in the geometric Langlands program, and notably in the proof of the vanishing conjecture by Gaitsgory [Gai04]. Moreover the coalescence of legs appears in the thesis of Eike Lau [Lau04] and it is also used in the preprint [BV06] of Braverman and Varshavsky (in order to prove the non-vanishing of the morphisms (10.1)). The article [Var04] of Varshavsky about the stacks of $G$-chtoucas and the very enlightening preprint [BV06] of Braverman and Varshavsky, were essential for us. Lastly we repeat the link, already mentioned in the previous section, with the corollary 4.5.5 of [Gai13] (which by the way generalizes the vanishing conjecture).

**Remark 11.1.** Contrary to the methods based on the trace formulas, our methods do not allow us to compute the multiplicities in the spaces of automorphic forms or in the cohomology of the classifying stacks of chtoucas. For example the much more difficult methods of [Laf02a] (and [Dri88] for $GL_2$), which are based on trace formulas and compactifications, show essentially that if $\pi$ is cuspidal and $\sigma$ is the associated Galois representation, $\pi \otimes \sigma \otimes \sigma^*$ occurs in $H_{\{1,2\},S\times S^*}$ with multiplicity exactly equal to 1. We note also that the construction of these compactifications, which represents a considerable effort and is extended to other groups and other coweights in [NDT07] and in forthcoming works of Ngo Dac Tuan and Varshavsky, will likely be necessary to understand the Arthur parameters (see the discussion in section 2.2).

In the rest of this section we explain a few additional arguments which enable us to give a new proof of the inductive step of [Laf02a] as a consequence of theorem 1.1. We take care to avoid any circularity and do not use the results which are now well-known but are consequences of [Laf02a]. What follows does not bring any new result and can be skipped by the reader.

**Lemma 11.2.** Let $\sigma$ be a parameter occuring in decomposition (1.3). Let $V$ be an irreducible representation of $\bar{G}$ and $V_{\sigma} = \bigoplus \tau \otimes \frak{U}_{\tau}$ be the decomposition of the semisimple representation $V_{\sigma}$ indexed by the isomorphism classes of irreducible representations $\tau$ of $\pi_1(\eta, \eta)$. Then if $\frak{U}_{\tau} \neq 0$, $\tau \boxtimes \tau^*$ occurs as a subquotient of the representation

$$H_{\{1,2\}, V \boxtimes V^*} = \left( \lim_{\mu} \mathcal{H}^{<\mu}_{N,\{1,2\}, V \boxtimes V^*} \bigg|_{\{1,2\}} \right)^{Hf}$$

of $(\pi_1(\eta, \eta))^2$. Moreover $\tau$ is $i$-pure for every isomorphism $i : \overline{Q}_l \xrightarrow{\sim} \mathbb{C}$.

**Remark 11.3.** Of course the previous assertion is not a new result because theorem VII.6 of [Laf02a] implies that every irreducible representation (defined
over a finite extension of $\mathbb{Q}_\ell$ and continuous) of $\pi_1(X \setminus N, \bar{\eta})$ is $\iota$-pure for every $\iota$.

**Proof.** Increasing $E$ if necessary, we assume that $\sigma$ and $\mathfrak{H}_\sigma$ are defined over $E$. Let $h \neq 0$ be in the subspace of $\mathfrak{H}_\sigma$ over which $\mathcal{B}$ acts by the character $\nu$ associated to $\sigma$ by (1.4) (we recall that we do not know whether $\mathcal{B}$ is reduced). Let $\tilde{h} \in C_c^{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, E)$ be such that

\begin{equation}
(11.2) \quad \int_{G(F)\backslash G(\mathbb{A})/K_N\Xi} \tilde{h} h = 1.
\end{equation}

We denote by

- $f$ the element of (11.1) equal to the image of $h$ by the composition

$$C_c^{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, E) = H_{(0), 1} \xrightarrow{\chi_1} H_{(0), V \otimes V^*} \xrightarrow{\chi_1^{-1}} H_{(1,2), V \otimes V^*};$$

- $\tilde{f}$ the linear form over (11.1) equal to the composition of

$$H_{(1,2), V \otimes V^*} \xrightarrow{\chi_1} H_{(0), V \otimes V^*} \xrightarrow{\mathcal{H}(\nu)} H_{(0), 1} = C_c^{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, E)$$

and of the linear form

$$C_c^{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, E) \to E, \quad g \mapsto \int_{G(F)\backslash G(\mathbb{A})/K_N\Xi} \tilde{h} g.$$

Then $f$ and $\tilde{f}$ are invariant under the diagonal action of $\pi_1(\eta, \bar{\eta})$. For every $(\gamma, \gamma') \in (\pi_1(\eta, \bar{\eta}))^2$ we have

$$\langle \tilde{f}, (\gamma, \gamma') \cdot f \rangle = \int_{G(F)\backslash G(\mathbb{A})/K_N\Xi} \tilde{h} S_{(1,2), V \otimes V^*, \delta_V, ev_V, (\gamma, \gamma')}(h)$$

$$= \nu(S_{(1,2), V \otimes V^*, \delta_V, ev_V, (\gamma, \gamma')}) = \chi_V(\sigma(\gamma^{-1} \gamma')) = \chi_{V_\iota}(\gamma^{-1} \gamma'),$$

where

- the first equality comes from the definition of the excursion operators given in (5.2),
- the second equality comes from the hypothesis that $h$ is an eigenvector for $\mathcal{B}$ w.r.t. the character $\nu$, and from (11.2),
- the third equality comes from the fact that $\nu$ is associated to $\sigma$ by (1.4).

The quotient of the representation of $(\pi_1(\eta, \bar{\eta}))^2$ generated by $f$ by the biggest subrepresentation on which $\tilde{f}$ vanishes is then isomorphic to the subrepresentation generated by $\chi_{V_\iota}$ in $C(\pi_1(\eta, \bar{\eta}), E)$ equipped with the action by left and right translations by $(\pi_1(\eta, \bar{\eta}))^2$. By [Bou12] chapter 20.5 theorem 1, this representation is isomorphic to $\bigoplus_{\tau, \xi_\iota \neq 0} \tau \otimes \tau^*$. We have shown that if $\mathfrak{M}_\tau \neq 0$, $\tau \otimes \tau^*$ is a quotient of a subrepresentation of (11.1). From this we deduce that $\tau$ is $\iota$-pure. Since $\tau \otimes \tau^*$ is a subquotient of (11.1), it results from Weil II [Del80] that $\tau \otimes \tau^*$ is $\iota$-pure of weight $\leq 0$ as a representation of $\pi_1((X \setminus N)^2, \Delta(\bar{\eta}))$. Therefore for almost all place $v$ the eigenvalues of $\tau(\text{Frob}_v)$ have equal $\iota$-weights, and they are determined by the $\iota$-weight of $\det(\tau)$. \qed
In the rest of this section we consider the case where $G = GL_r$. Of course we do not obtain any new result. We recall that in [Laf02a] the Langlands correspondence is obtained by induction on $r$, with the help of the “induction principle” of Deligne, which combines

- the functional equations of the $L$-functions due to Grothendieck [SGA5],
- the product formula of Laumon [Lau87],
- the multiplicity one theorems [Pia75, Sha74] and the converse theorems of Hecke, Weil, Piatetski-Shapiro and Cogdell [CPS94].

The induction is explained in section 6.1 and appendix B of [Laf02a] and the induction step is

(11.3) \[ \text{the hypothesis of proposition VI.11 (ii) of [Laf02a]} \]

namely that to every cuspidal automorphic representation $\pi$ for $GL_r$ of level $N$ we can associate $\sigma : \pi_1(X \setminus N, \mathfrak{G}) \to GL_r(\mathbb{Q}_\ell)$ defined over a finite extension of $\mathbb{Q}_\ell$, continuous, pure of weight $0$ and corresponding to $\pi$ in the sense of Satake at all the places of $X \setminus N$. Our theorem 1.1 provides a new proof of (11.3), thanks to the following lemma.

Lemma 11.4. We take $G = GL_r$. We assume that the Langlands correspondence is known for $GL_{r'}$ for every $r' < r$. Then every $\sigma$ occurring in the decomposition (1.3) is irreducible and pure of weight $0$.

Remark 11.5. This lemma does not bring any new result because a posteriori it results from [Laf02a].

Proof (extracted from [Laf02a]). Let $(H_\pi, \pi)$ be a cuspidal automorphic representation such that $(H_\pi)^{K_N}$ appears in $\mathcal{H}_\pi$. We denote by

(11.4) $\sigma = \bigoplus_\tau \tau \otimes \mathfrak{Y}_\tau$

the decomposition of the semisimple representation $\sigma$ indexed by equivalence classes of irreducible representations $\tau$ of $\pi_1(X \setminus N, \mathfrak{G})$. We assume by contradiction that $\sigma$ is not irreducible. Any representation $\tau$ such that $\mathfrak{Y}_\tau \neq 0$ is therefore of rank $r_\tau < r$ and, by the induction hypothesis included in the statement, there exists a cuspidal automorphic representation $\pi_\tau$ for $GL_{r_\tau}$ associated to $\tau$ by the Langlands correspondence for $GL_{r_\tau}$. We choose a finite set $S$ of places outside of which the representations $\pi, \sigma$, and $\tau, \pi_\tau$ such that $\mathfrak{Y}_\tau \neq 0$ (whose number is finite) are unramified and correspond to each other by the Satake isomorphism.

By the method of Rankin-Selberg for $GL_r \times GL_{r'}$ (due to Jacquet, Piatetski-Shapiro, Shalika [JS81, JPS83]), $L(\pi \times \pi_\tau, Z)$ is a polynomial in $Z$ hence a fortiori $L_S(\pi \times \pi_\tau, Z)$ is a polynomial in $Z$ (since the local factors may have poles but never zeros). Thus

$L_S(\pi \times \pi, Z) = L_S(\pi \times \sigma, Z) = \prod_\tau L_S(\pi_\tau \times \tau, Z)^{\dim \mathfrak{Y}_\tau} = \prod_\tau L_S(\pi_\tau \times \pi_\tau, Z)^{\dim \mathfrak{Y}_\tau}$

has no pole. However by theorem B.10 of [Laf02a] (due to Jacquet, Shahidi, Shalika), $L_S(\pi \times \pi, Z)$ has a pole at $Z = q^{-1}$, and we get a contradiction. We have proven that $\sigma$ is irreducible. For every $\iota$ we know by lemma 11.2 that $\sigma$ is
\(\iota\)-pure and the knowledge of its determinant (by class field theory) implies then that the \(\iota\)-weight is zero. Thus \(\sigma\) is pure of weight 0.

Remark 11.6. We mention for the reader that the already very important consequences of the Langlands correspondence for \(GL_r\), explained in the chapter VII of [Laf02a], were widened in recent works of Deligne and Drinfeld [Del12, EK12, Dri12] about rationality of the Frobenius traces and about the independence of \(\ell\) for the \(\ell\)-adic sheaves over smooth varieties over \(\mathbb{F}_q\).

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