Semisimple algebraic tensor categories

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For a field $k$ a monoidal $k$-linear category abelian category $T$ is an abelian $k$-linear category with biadditive tensor functor $\otimes : T \times T \to T$, $k$-linear and exact in each variable, with associativity and commutativity constraints and unit element $1_T$ satisfying the axioms ACU of [SR]. Then $T$ is called rigid, if every object $X$ has a dual $X^*$ with morphisms

$$\delta_X : 1_T \to X \otimes X^* , \quad ev_X : X^* \otimes X \to 1_T$$

so that $(id_X \otimes ev_X) \circ (\delta_X \otimes id_X) = id_X$ and $(\delta_X \otimes id_{X^*}) \circ (id_{X^*} \otimes ev_X) = id_{X^*}$.

See [CP] or [SR] for a detailed exposition.

Under the assumptions above, if $T$ is a small category such that $End_T(1_T) \cong k$, the category $T$ is called a ‘categorie $k$-tensorielle ’in [D]. If in addition $T$ is generated by one of its objects $V$ as a tensor category, such that for some integer $N$ the length $l_T(V \otimes r)$ in $T$ is bounded by $N^r$ for all $r$, the category $T$ will be called an algebraic tensor category over $k$.

The typical example for an algebraic tensor category over $k$ (see [D], p.228) is the category of finite dimensional $k$-linear $\varepsilon$-super representations

$$T = Rep_k(G, \varepsilon)$$

of a super-affine groupscheme $G$ over $k$. The main result on algebraic tensor categories is the following

**Theorem 1.** ([D]) *Suppose $k$ is algebraically closed of characteristic zero. Then any algebraic tensor category over $k$ is of the form $Rep_k(G, \varepsilon)$.***

So let $k$ be algebraically closed of characteristic zero. Under this assumption it is then interesting to know the cases where the category $Rep_k(G, \varepsilon)$ is a semisimple abelian category. It is very easy to see that this only depends on the super-affine groupscheme $G$ and not on the additional twist $\varepsilon$. In other words $Rep_k(G, \varepsilon)$ is semisimple if and only if the category $Rep_k(G)$ of all $k$-linear finite dimensional super representations of $G$ is semisimple. More or less by definition $Rep_k(G)$ coincides with the tensor category $CoRep_k(A)$ of $k$-finite dimensional
$A$-comodules, where $A$ is the super-affine Hopf algebra over $k$ defined the coordinate ring $\mathcal{O}(G)$ of $G$. If these categories are semisimple, we say $G$ is reductive.

For a super-affine groupscheme $G$ over a field $k$ of characteristic zero $k$ the reduced groupscheme of $G$ is an algebraic group $G$ over $k$. The left-invariant super derivations of the underlying Hopf algebra $A$ corresponding to $G$ define a finite dimensional Lie superalgebra $g = \text{Lie}(G)$ over $k$. A Lie superalgebra $g$ over $k$ will be called reductive if modulo its supercenter it is isomorphic to a direct sum of simple Lie superalgebras over $k$ of the classical types $A_n$ ($n \geq 1$), $B_n$ ($n \geq 3$), $C_n$ ($n \geq 2$), $D_n$ ($n \geq 3$), $E_6$, $E_7$, $E_8$, $G_2$, $F_4$ and of the orthosymplectic simple supertypes $BC_r$ ($r \geq 1$). We then show

**Theorem 2.** $G$ is reductive if and only its reduced group $G$ is a reductive algebraic group over $k$ and its Lie superalgebra $\text{Lie}(G)$ is reductive over $k$.

In particular $G$ is reductive if and only if its connected component $G^0$ with respect to the Zariski topology is reductive. In the connected case we show that $G$ is reductive if and only if etale unramified coverings are connected.

For the proof of theorem 2 we pass from super-affine groupschemes $G$ over $k$ defined by their super-affine Hopf coordinate algebra $A$ over $k$, to their associated supergroups $(G, g_-, Q)$. Here $G$ is the reduced group of $G$. The even part $g_+$ of $\text{Lie}(G) = g_+ \oplus g_-$ is the Lie algebra of $G$. The odd part $g_-$ is an algebraic $G$-module, and the Lie superbracket defines a $G$-equivariant symmetric map $Q : g_- \times g_- \to g_+$. Together these data give rise to a triple $(G, g_-, Q)$ called a supergroup or a Harish-Chandra triple. For a suitable notion of representations for supergroups then the following holds

**Theorem 3.** The categories of $k$-finite dimensional super representations $\text{Rep}_k(G)$ and $\text{Rep}_k(G, g_-, Q)$ are equivalent as algebraic tensor categories over $k$.

Theorem 3 allows us to reduce the proof of theorem 2 to the classical results on the reductivity of semisimple Lie superalgebras obtained by Djokovic and Hochschild [DH].
Affine super Hopf algebras

Let $k$ be field of $\text{char}(k) \neq 2$ and $A$ be a Hopf algebra with comultiplication, counit and antipode $(m^*_A, e^*_A, \iota^*_A)$ over the field $k$. Suppose $A$ is super-affine, i.e. suppose that as a ring $A$ is a finitely generated super-commutative $k$-algebra such that $(m^*_A, e^*_A, \iota^*_A)$ are morphisms in the category $(\text{salg})$ of super-commutative $k$-algebras.

Remark. The tensor product $\otimes^\varepsilon$ of the category $(\text{salg})$ is the ordinary tensor product $\otimes_k$ except that it carries an induced grading with additional sign rules for certain structures like the tensor product of super $k$-algebras etc. For a detailed exposition of this we refer to [DM].

For the $\mathbb{Z}/2\mathbb{Z}$-grading $A = A_+ \oplus A_-$ defined by the super structure the super-commutativity rule $xy = (-1)^{|x||y|}yx$ implies $x^2 = 0$ for $x \in A_-$. Thus $A_-$ and the ideal $J$ generated by $A_-$ in $A$ are nilpotent. We call $J$ the super radical of $A$. $J$ is a Hopf ideal, i.e. $\iota^*_A(J) \subset J$, $e^*_A(J) = 0$ and

$$m^*_A(J) \subset J \otimes^\varepsilon A + A \otimes^\varepsilon J$$

as an immediate consequence of

$$J = A_- + (A_-)^2$$

and $m^*_A(A_-) \subset (A \otimes^\varepsilon A)_- \subset A_- \otimes^\varepsilon A + A \otimes^\varepsilon A_-$. Surjective Hopf algebra homomorphisms $\pi : A \rightarrow A'$ are in 1-1 correspondence with Hopf ideals $I = \text{Kern}(\pi)$ of $A$. Since $A/J$ is even, the quotient

$$\pi : A \rightarrow B = A/J$$

defines an commutative affine Hopf algebra quotient $B$ for which therefore

$$G = \text{Spec}(B)$$

is a group scheme of finite type over $k$. We say $A$ is connected, if $G$ is connected in the Zariski topology. Similar for the notion of being simply connected. If $\text{char}(k) = 0$, then $G$ is automatically reduced by a result of Cartier. In this case the super radical $J$ is the nilradical of $A$. 

3
$A$-comodules

An $A$-comodule $(V, \Delta_V)$ is a $k$-super vector space $V$ together with a $k$-superlinear map

$$\Delta_V : V \rightarrow V \otimes^e A$$

satisfying the axioms (Modass) and (Modun) as in [S], p.30, i.e. the commutativity of

$$\begin{array}{ccc}
V & \xrightarrow{\Delta_V} & V \otimes^e A \\
\Delta_V & \downarrow & \Delta_V \otimes^e id_A \\
V \otimes^e A & \xrightarrow{id_V \otimes^e m_A^*} & V \otimes^e A \otimes^e A
\end{array}$$

The notion of $A$-comodule only depends on the cogebra structure of $A$. With the obvious notion of $A$-comodule homomorphism (see [S], p.31) the category of $A$-comodules is an abelian category. Any $A$-comodule is a union of its $k$-finite dimensional $A$-submodules. The category $CoRep_k(A)$ of $k$-finite dimensional $A$-comodules is a $k$-linear rigid abelian (monoidal) tensor category (see [CP], p.141).

Example a). $(A, m_A^*)$ itself is an $A$-comodule by the Hopf algebra axioms

$$\begin{array}{ccc}
A & \xrightarrow{m_A^*} & A \otimes^e A \\
m_A^* & \downarrow & \left(id_A \otimes^e m_A^*\right) \\
A \otimes^e A & \xrightarrow{m_A^* \otimes^e id_A} & (A \otimes^e A) \otimes^e A
\end{array}$$

Example b). For a $k$-super subvectorspace $V \subset A$ such that $m_A^*(V) \subset V \otimes^e A$ the restriction $\Delta_V = m_A^*|V$ defines an $A$-comodule $(V, \Delta_V)$, a subcomodule of $A$.

Example c). Any Hopf algebra quotient $\pi : A \rightarrow B$ map makes $A$-comodules $(V, \Delta_V)$ into $B$-comodules $(V, \Delta)$ with respect to

$$\Delta = (id_V \otimes^e \pi) \circ \Delta_V : V \rightarrow V \otimes^e B.$$ 

This is a consequence of $(\pi \otimes^e \pi) \circ m_A^* = m_B^* \circ \pi$ and $e_A^* \circ \pi = e_B^*$. 

4
Suppose for an $A$-comodule $(V, \Delta_V)$ that the super vectorspace $V = k^{r|s}$ is finite dimensional with basis $e_i$ for $i = 1 \ldots r + s$. Then $\Delta_V(e_i) = \sum_j e_j \otimes^{e} f_{ji}$ for certain $f_{ji} \in A$. The axiom (Modass) implies $m_A^\ast (f_{ki}) = \sum_k f_{kj} \otimes^{e} f_{ji}$. Thus the coefficients $f_{ji}$ define a homomorphism of super Hopf algebras

$$O(\text{Gl}(V)) \rightarrow A$$

from the super Hopf algebra $A' = O(\text{Gl}(V))$ of the general linear group of the super vector space $V$ to $A$. Indeed as $k$-algebra $A' = k[X_{ij}, \text{det}_1^{-1}, \text{det}_2^{-1}]$ is generated by elements $X_{kj}$ and the inverse of the determinants $\text{det}_1, \text{det}_2$ of the $X_{ij}$ for $i, j \leq r$ resp. $i, j > r$ subject to the rule $m_A^\ast (X_{ki}) = \sum_k X_{kj} \otimes^{e} X_{ji}$. The elements $X_{ij}$ are even iff $i, j \leq r$ or $i, j > r$. In other words, this defines a super representation of $\text{Spec}^{\epsilon}(A)$, i.e. a homomorphism of super group schemes

$$\text{Spec}^{\epsilon}(A) \rightarrow \text{Gl}(V).$$

Conversely, it is easy to see that this defines a 1-1 correspondence between $k$-finite dimensional $A$-comodules $V$ and finite $k$-linear dimensional super representations $V$ of the Lie super group scheme $\text{Spec}^{\epsilon}(A)$. The category $\text{Rep}_k(A)$ of such $k$-finite dimensional super representations of $\text{Spec}^{\epsilon}(A)$ is an algebraic tensor category over $k$. The following is well known (see [D])

**Lemma 1.** This correspondence induces a tensor-equivalence between the algebraic tensor categories $\text{CoRep}_k(A)$ and $\text{Rep}_k(A)$ over $k$.

**The functor of invariants** $V \mapsto V^G$

For the $k$-groupscheme $G = \text{Spec}(B)$ consider the left-exact functor

$$V \mapsto V^G = Hom_{B-\text{comod}}(k, V)$$

from the category of $B$-comodules to the category of $k$-vectorspaces. The $k$-vectorspace $V^G \subseteq V$ can be identified with the maximal trivial $B$-subcomodule of $V$ of all elements $v$ in $V$ for which

$$\Delta_M(v) = v \otimes 1_B.$$
We say a $B$-comodule $V$ is free, if it is isomorphic to a $B$-comodule of the form $V = V_0 \otimes B = B^d$. Here $V_0$ is a $k$-vectorspace and $d = \dim_k(V_0)$. $B$-comodules will be called almost free, if they have a finite filtration by $B$-subcomodules whose successive quotients are free $B$-comodules. Notice $B^G = k \cdot 1_B$, since $v = (e_B^* \otimes id_B)(m_B^*(v)) = (e_B^* \otimes id_B)(\Delta_V(v)) = (e_B^*(v) \otimes 1_B) \in k \cdot 1_B$ for $v \in B^G$. Hence for free $V = V_0 \otimes B$

$$(V_0 \otimes B)^G = V_0.$$ 

Using bar-resolutions (see [DG], p.233ff) one can define derived functors $H^i(G, -)$ such that $H^0(G, V) = V^G$. In other words a short exact sequence of $B$-comodules gives rise to a long exact sequence of $k$-vectorspaces using the derived functors $H^i(G, -)$. By [DG], lemma 3.4

$H^i(G, B) = 0$ , $i \geq 1$

for any free $B$-comodule. Obviously $H^1(G, V) = 0$ for almost free $B$-comodules $V$. Hence

**Lemma 2.** *On the Grothendieck group of almost free $B$-comodules $V$  

$$\text{rang}_k(V) = \dim_k(V^G)$$

defines a homomorphism.*

**The Hopf ideals defined by $J$**

Let $A$ be a super-affine Hopf algebra over $k$. Then its super radical $J$ is generated as an $A$-module by finitely many elements in $A_-$. If $J$ is generated by $s$ elements then it is easy to see that $J^{s+1} = 0$. Hence there exists a finite descending filtration by $A$-right (and left) ideals

$$0 \subset J^s \subset J^{s-1} \subset .. \subset J^2 \subset J \subset A$$

whose successive quotients

$$V_i = J^i / J^{i+1}$$

are right (and left) $B = A/J$-modules. Although the $J^i$ are not $B$-modules a priori, they are $B$-subcomodules of the $B$-comodule $(A, \Delta)$ with structure map

$$\Delta = (id_A \otimes \pi) \circ m_A^*$$
using the examples a), b) and c) above. There is a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{m_A^*} & A \otimes^\varepsilon B \\
\downarrow & & \downarrow \\
J^i & \xrightarrow{\Delta} & J^i \otimes^\varepsilon B
\end{array}
\]

since the image of \( m_A^*(J^i) \subset m_A^*(J)^i \subset (J \otimes^\varepsilon A + A \otimes^\varepsilon J)^i \subset \sum_{a+b=i} J^a \otimes^\varepsilon J^b \) in \( A \otimes^\varepsilon B \), under \( \text{id}_A \otimes^\varepsilon \pi \), is contained in \( J^i \otimes^\varepsilon B \). Thus \( J^i \) becomes a \( B \)-comodule. The \( V_i \) then are quotient \( B \)-comodules of the \( J^i \) in the obvious way.

**Lemma 3.** \( V_i \cong B^{d_i} \) is finite free both as a \( B \)-right module and a \( B \)-comodule.

**Proof.** The \( k \)-linear structure map \( \Delta : A \to A \otimes^\varepsilon B \) of the \( B \)-comodule \( A \) is \( A \)-linear in the following sense: For \( a \in A \) and \( x \in A \) of course \( x \cdot a \in A \). Since \( \pi \) is \( A \)-linear

\[
\Delta(x \cdot a) = (\text{id}_A \otimes^\varepsilon \pi)(m_A^*(x) \cdot m_A^*(a)) = \Delta(x) \cdot m_A^*(a)
\]

where \( A \otimes^\varepsilon B \) is viewed as a \( A \otimes^\varepsilon A \)-right module in the obvious way. In other words \( m_A^*(a) = \sum a_v \otimes a'_v \) acts on \( y = a \otimes^\varepsilon b \) via \( y \cdot m_A^*(a) = \sum a_v(-1)^{|a_v||b|} a_v \otimes^\varepsilon b \cdot a'_v \). Since \( \Delta(J^i) \subset (J^i) \) the map \( \Delta \) induces a quotient map

\[
\Delta_V : V \to V \otimes^\varepsilon B
\]
on \( V = V_i = J^i / J^{i+1} \) making it to a \( B \)-comodule. The right action of \( A \) on \( V \) factors over the quotient ring \( B \). Similarly the right action of \( A \otimes^\varepsilon A \) on \( V \otimes^\varepsilon B \) factors over the quotient ring \( A / J \otimes^\varepsilon B = B \otimes^\varepsilon B \), so that now (*)

\[
\Delta_V(x \cdot b) = \Delta_V(x) \cdot m_B^*(b)
\]
is obvious: The composition of \( m_A^* \) with the projection \( A \otimes^\varepsilon A \to A / J \otimes^\varepsilon B \) is equal to \( m_B^* \circ \pi \).

It is the property (*) which makes the right \( B \)-module and right \( B \)-comodule \( V \) into a \( B \)-right Hopf module in the sense of [S], p.83. Since \( B \) is an ordinary Hopf algebra we can immediately apply [S], theorem 4.1.1. It states that

\[
M \cong M^G \otimes B = B^d, \quad d = \text{dim}_k(M^G)
\]
as a Hopf right $B$-module and comodule for any Hopf right $B$-module and co-module $M$. Applied for $M = V$ we now use the fact that $J$, hence also $V$, are finitely generated $B$-right modules. Hence $d = d_i < \infty$ in our case. This proves our claim. QED

Therefore $A$ is an almost free $B$-comodule. By lemma 2 this implies

**Corollary 1.** $\dim_k(A^G) = \sum_{i=0}^{s} d_i$ for $d_i = \text{rank}_B(J^i/J^{i+1})$.

**Remark.** We will see later in corollary 5 that for $A$ affine super group scheme over $k$ we have $d_i = (s_i)^{\pm}$. This will imply

$$\dim_k(A^G) = 2^s.$$ 

**Lemma 4.** $A^G$ is a finite dimensional $k$-subalgebra of $A$.

**Proof.** $\Delta(v) = v \otimes 1_B$ and $\Delta(v') = v' \otimes 1_B$ imply $\Delta(v \cdot v') = \Delta(v) \cdot \Delta(v') = (v \otimes 1_B) \cdot (v' \otimes 1_B) = (v \cdot v') \otimes 1_B$. QED

### Superderivations

Let $(A, m^*_A, e_A^*, i_A^*)$ be an super-affine Hopf algebra over $k$. Let $m = m(A) = \ker(e_A^*)$ be the maximal ideal of $A$ at the identity. Target vectors $X \in (m/m^2)_\pm$ extend to even or odd $k$-linear superderivations $d_X : A \rightarrow k$ by composing $X : m/m^2 \rightarrow k$ with the projection $A = k \cdot 1 \oplus m \rightarrow m \rightarrow m/m^2$. Define $k$-super derivations

$$d_X : A \rightarrow A$$

by the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \\
\downarrow_{D_X} & & \downarrow_{\text{id}_A \otimes^\varepsilon d_X} \\
A & & A
\end{array}
\]

Then $d_X = e_A^* \cdot D_X$ by definition. The $k$-superderivations $D_X : A \rightarrow A$ so constructed are left-invariant, i.e. for $D = D_X$ there exists a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \\
\downarrow_{D} & & \downarrow_{\text{id}_A \otimes^\varepsilon D} \\
A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A
\end{array}
\]
by the coassociativity law \((id_A \otimes^e m_A^*) \circ m_A^* = (m_A^* \otimes^e id_A) \circ m_A^* \) and \(A \otimes^e (A \otimes^e A) = (A \otimes^e A) \otimes^e A\). Indeed, if we apply \(id_A \otimes^e (id_A \otimes^e d_X) = (id_A \otimes^e id_A) \otimes^e d_X\) on the left side of the coassociativity law, this becomes \((id_A \otimes^e D_X) \circ m_A^*\). On the right side of the coassociativity law it becomes \(m_A^* \circ D_X\).

**Lemma 5.** There exists a canonical isomorphism \(X \mapsto D_X\) of \(k\)-vectorspaces \((m/m^2)^* \rightarrow \text{Lie}(A)\) between the tangent space at the identity element and the \(k\)-vector space \(\text{Lie}(A)\) of all left-invariant \(k\)-superderivations of \(A\).

**Proof.** The inverse map is \(\text{Lie}(A) \ni D \mapsto d = e_A^* \circ D\). Since \(d : A \rightarrow k\) is a \(k\)-superderivation, it must vanish on \(m^2\) and on \(k \cdot 1\). Hence \(d = d_X\) for some \(X \in (m/m^2)^*\). The left-invariant \(k\)-superderivation \(D : A \rightarrow A\) is uniquely determined by its restriction \(d = e_A^* \circ D\), since \(d\) determines \(D\) via the right vertical arrow \(id_A \otimes e^*\) of the composed commutative diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{m_A^*} & A \otimes^e A \\
\downarrow D & & \downarrow id_A \otimes^e D \\
A & \xrightarrow{m_A^*} & A \otimes^e A \\
\downarrow id_A & & \downarrow id_A \otimes^e e^* \\
A & & A
\end{array}
\]

For any Hopf ideal \(J\) of \(A\) with quotient map \(A \rightarrow B = A/J\) the cotangent space \(m(A)/m(A)^2\) surjects onto the cotangent space \(m(B)/m(B)^2\). Hence \(\text{Lie}(B)\) injects into \(\text{Lie}(A)\). QED

**Lemma 6.** The image of the natural injection \(\text{Lie}(B) \hookrightarrow \text{Lie}(A)\) is the space of left-invariant \(k\)-derivations \(D\) of \(A\) (as in the last lemma) with the property \(D(J) \subset J\).

**Proof.** Such \(D\) induce left-invariant derivations on the quotient \(B = A/J\). So it suffices that \(X \in \text{Lie}(B)\) implies \(D_X(J) \subset J\). For this let \(x : A \rightarrow B\) be the quotient map with kernel \(J\), considered as a \(B\)-valued point of \(A\). For \(f \in A\) by definition \(D_X(f)(x) = (x \otimes^e d_X)(m_A^*(f))\). Now \((x \otimes^e d_X)(m_A^*(f)) \subset (x \otimes^e d_X)(A \otimes^e J + J \otimes^e A)\) for \(f \in J\) since \(J\) is a Hopf ideal. But \(x(J) = 0\). On
the other hand \( d_X(J) = 0 \) for \( X \in \text{Lie}(B) \), since \( \text{Lie}(B) \) is the space of linear forms \( d_X : m(A)/m(A)^2 \rightarrow k \) trivial on the image of \( J \). Hence \( D_X(f)(x) = 0 \) or \( D_X(f) \in J \). QED

**The Lie algebra.** The supercommutator \([D, D'] = D \circ D' - (-1)^{|D||D'|} D' \circ D\) of two \( k \)-super derivations \( D, D' \) is a \( k \)-superderivations. Since the super commutator of left-invariant derivations is left invariant, the finite dimensional \( k \) super-vector space

\[
g = \text{Lie}(A) = (m(A)/m(A)^2)^* \]

defined by \((A, m_A^*, e_A^*)\) becomes a Lie \( k \)-superalgebra with \( g \cong k^{r|s} \) as a super vectorspace

\[
g = g_+ \oplus g_- .
\]

**The super radical \( J \).** Notice \( J = A_+ + (A_-)^2 \) implies \( J^2 = (A_-)^2 + (A_-)^3 \). Hence the quotient \( J/J^2 = A_-/(A_-)^3 \) is odd. Since \( J \) is nilpotent, we have \( J \subset m(A) \) and \( m(B) = m(A)/J \). Clearly the quotient \( A_-/(A_-)^3 = J/J^2 \rightarrow J/(J \cap m(A)^2) \) again is odd. Since \( B \) is even, also \( m(B)/m(B)^2 \) is even with \( g_+ = \text{Lie}(B) = (m(B)/m(B)^2)^* \) even. Hence the exact sequence

\[
0 \rightarrow J/(m(A)^2 \cap J) \rightarrow m(A)/m(A)^2 \rightarrow m(B)/m(B)^2 \rightarrow 0
\]
gives rise to a splitting of the super-vectorspace \( \text{Lie}(A) \) with \( \text{Lie}(G) \) even

\[
\text{Lie}(A) = \text{Lie}(G) \oplus g_-
\]
and with \( g_- \cong (J/(J \cap m(A)^2))^* \) odd.

Fix a basis \( \tilde{\theta}_i \) of \((V_1)^G = (J/J^2)^G \) and representatives \( \theta_i \in J^G \) of the elements \( \tilde{\theta}_i \). Then \( J/J^2 = \bigoplus_{i=1}^{d_1} \tilde{\theta}_i \cdot B \) as a \( B \) right-module. Consider the exact sequence of odd \( k \)-vectorspaces

\[
0 \rightarrow K \rightarrow J/J^2 \rightarrow J/(J \cap m(A)^2) \rightarrow 0 .
\]

We claim \( K = \bigoplus_{i=1}^{d_1} \tilde{\theta}_i \cdot m(B) \).

Since \( \theta_i \in J \subset m(A) \), the right hand side is contained in \( K \). Conversely elements \( k \in K \) have odd representatives \( x \) in \( J \cap m(A)^2 \), or hence in \( A_- \cap m(A)^2 \). Notice \( A_- \cap m(A)^2 = (A_- \cap m(A))(A_+ \cap m(A)) \) by a case by case verification and the definition of the super graded ring structure on \( A \). Since \( m(A) \cap A_- \subset J \),
hence $A_\cap m(A)^2 \subset J \cdot (A_\cap m(A))$. As $m(A)$ acts on $J/J^2$ via its quotient $m(B)$ therefore the image $k$ of $x$ is contained in $\bigoplus_{i=1}^{d_1} \tilde{\theta}_i \cdot m(B)$. This proves the claim. As a consequence

$$(J/J^2)^G = \bigoplus_{i=1}^{d_1} \tilde{\theta}_i \cdot k \cong (J/J^2)/K \cong J/(J \cap m(A)^2) \cong (g_-)^*.$$ 

Together with the lemma 6 this implies

**Corollary 2.** The left-invariant derivations $D_X$ for $X \in \text{Lie}(G) \subset \text{Lie}(A)$ respect the exact sequence defined by the super radical $J$

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0.$$ 

A left-invariant superderivation $D_X \in \text{Lie}(A)$ preserves the super radical $J$ if and only if $X \in \text{Lie}(G)$. Furthermore

$$\text{dim}_k(g_-) = \text{rank}_B(J/J^2) = d_1.$$ 

**Homomorphisms.** A homomorphism $\Phi^* : A \rightarrow A'$ between super-affine $k$-Hopf algebras induces a map between the tangent spaces at the identity element, hence a $k$-linear map

$$\text{Lie}(\Phi) : \text{Lie}(A') \rightarrow \text{Lie}(A).$$

$\text{Lie}(\Phi)$ is a homomorphism of $k$-super Lie algebras, since $\Phi^* \circ D_X = D_{X'} \circ \Phi^*$ for $X' = \text{Lie}(\Phi)(X)$. [Reduce to $(\Phi^* \otimes^e \Phi^*) \circ (id \otimes^e d_X) = (id \otimes^e d_{X'}) \circ \Phi^*$, hence to $\Phi^* \circ d_X = d_{X'}$.]

**Adjoint action.** The interior automorphism $\Phi^* = (\text{Int}_x)^*$ defined by a $k$-valued point of $\text{Spec}(A/I)$ induces a Lie algebra homomorphism $Ad(x) = \text{Lie}(\text{Int}_x)$ from $\text{Lie}(A)$ to $\text{Lie}(A)$. Obviously $Ad(x) \circ Ad(y) = Ad(xy)$. Hence $Ad(x)$ defines a $k$-linear representation on $\text{Lie}(A)$ of the underlying algebraic group $G$

$$Ad : G(k) \rightarrow \text{Gl}_k(\text{Lie}(A)).$$

This adjoint action respects the super structure, hence decomposes into representations $Ad_\pm$ of $G$ on $g_+$ and $g_-$ respectively. $Ad_+$ is the usual adjoint action of $G(k)$ on its Lie algebra $g_+ = \text{Lie}(G)$. 

11
**Left versus right**

Similar to left-invariant superderivations define right-invariant superderivations of a Hopf algebra $A$. The Lie superalgebra of the left-invariant and right-invariant superderivations are isomorphic (use the antipode). Left-invariant superderivations $D$ and right-invariant superderivations $D'$ of $A$ supercommute. Use

$$(-1)^{|D||D'|} m_A^*(DD') = (D' \otimes^e D)(m_A^*(x)) = m_A^*(D'D)$$

to show that their supercommutator $[D, D']$ is a derivation with $m_A^*( [D, D'])(x) = 0$. Hence $[D, D'] = 0$ by applying the counit $e_A^*$.

**Lemma 7.** For quotients $B = A/I$ by a Hopf ideal $I$ and $X \in \text{Lie}(B) \subset \text{Lie}(A)$ the left-invariant superderivations $D_X$ of $A$ preserve $B$-subcomodules $V$ of $A$.

**Proof.** The commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{m_A^*} & A \otimes^e A \xrightarrow{id_A \otimes^e \pi} A \otimes^e B \\
& \downarrow D_X & \downarrow id_A \otimes^e D_X \\
V & \xrightarrow{m_A^*} & A \otimes^e A \xrightarrow{id_A \otimes^e \pi} A \otimes^e B \\
\end{array}
$$

for $D_X$ and $X \in \text{Lie}(B) \subset \text{Lie}(A)$ gives $\Delta \circ D_X = (id \otimes^e D_X) \circ \Delta$ for the structure map $\Delta = (id_A \otimes^e \pi) \circ m_A^*$ of the $B$-comodule $A$.

Next notice $(id_A \otimes e_B^*) \circ \Delta = id_A$ and the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{m_A^*} & A \otimes^e A \xrightarrow{id_A \otimes^e \pi} A \otimes^e B \\
& \downarrow id_A \otimes^e e_A^* & \downarrow id_A \otimes^e e_B^* \\
A & \xrightarrow{id_A \otimes^e e_A^*} & A \otimes^e B \\
\end{array}
$$

Hence for $v \in V \subset A$ and $\Delta(v) = \sum v_i \otimes^e b_i$ with $v_i \in V, b_i \in B$ the element

$$D_X(v) = (id_A \otimes^e e_B^*) \circ \Delta(D_X(v))$$

is $(id_A \otimes^e e_B^*)(\sum v_i \otimes^e D_X(b_i))) = \sum v_i \cdot d_X(b_i)$ using left-equivariance of $D_X$ as in first diagram above. Thus $D_X(v) \in V$ and $D_X(V) \subset V$. QED.

For $\Delta(v) = v \otimes^e 1_B$ in particular $D_X(v) = 0$, since $d_X(1_B) = 0$. 

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Corollary 3. $D_X(A^G) = 0$ for all $D_X, X \in \text{Lie}(G)$ where $G = \text{Spec}^\varepsilon(A/I)$.

Corollary 4. $A^G$ is stable under all right-invariant superderivations $D'_X$ in $\text{Lie}(A)$.

Proof. This follows from the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{m_A^*} & A \otimes^\varepsilon A \xrightarrow{id_A \otimes^\varepsilon \pi} A \otimes^\varepsilon B \\
& & \\
D'_X & & D'_X \otimes^\varepsilon id_A \\
& & id_A \otimes^\varepsilon D'_X \\
V & \xleftarrow{m^*_A} & A \otimes^\varepsilon A \xrightarrow{id_A \otimes^\varepsilon \pi} A \otimes^\varepsilon B
\end{array}
\]

which implies $\Delta \circ D'_X = (D'_X \otimes^\varepsilon id_B) \circ \Delta$ for the structure map $\Delta$ of the $B$-comodule $A$. For $v \in A^G$ by definition $\Delta(v) = v \otimes^\varepsilon 1_B$. Hence $\Delta(D'_X(v)) = (D'_X \otimes^\varepsilon id_B) \circ \Delta(v) = D'_X(v) \otimes 1_B$. This shows $D'_X(A^G) \in A^G$. QED

The subring $A^G \subset A$

For the super radical $J$ of $A$ put $G = spec(B)$ and $B = A/J$ as before. Since $(J/J^2)$ is odd and almost free, the quotient map $(J_-)^G \to (J/J^2)^G$ is surjective so that we can choose representatives $\theta_1, ..., \theta_s \in (J_-)^G$ of a $k$-basis in $(J/J^2)^G$ so that the $\theta_i$ are also a $B$-basis of $J/J^2$ by lemma 2. Then by recursion modulo the $J^n$

\[J = \theta_1 \cdot A + \cdots + \theta_s \cdot A.\]

The $\theta_i$ are odd. Hence by supercommutativity

\[\theta_i \theta_j = -\theta_j \theta_i.\]

For $I \subset \{1, ..., s\}$ define $\theta_I = \theta_{i_1} \cdots \theta_{i_n}$ if $I = \{i_1, ..., i_n\}$ and $i_1 < ... < i_n$. With these notations $J^n$ is generated as an $A$-right module by the $\theta_I$ with $|I| = n$. Hence for the elements $\tilde{\theta}_I = \theta_I \mod J^{n+1}$ in $(J^n/J^{n+1})^G$ we get

\[J^n/J^{n+1} = \sum_{|I|=n} \tilde{\theta}_I \cdot B.\]

We may replace by a $B$-right linear independent subset of $T_n$ of the set of all the $\tilde{\theta}_I$, since we already know that $J^n/J^{n+1}$ is a free $B$-right module generated by a $k$-basis of $(J^n/J^{n+1})^G$. Therefore

\[J^n/J^{n+1} = \bigoplus_{I \in T_n} \tilde{\theta}_I \cdot B\]
and
\[(J^n)^G / (J^{n+1})^G \cong (J^n / J^{n+1})^G \cong k^\#T_n.\]

Since \(\theta_i \in A^G\), recursively now any element in \(A^G\) can be written as a superpolynomial in the elements \(\theta_1, \ldots, \theta_s\) by induction modulo the \(A^G \cap J^n = (J^n)^G\). This defines a surjective \(k\)-algebra homomorphism \(f : S_e^e(k^{0|s}) \to A^G\) mapping the generators of the superpolynomial ring \(S_e^e(k^{0|s})\) to the \(\theta_i\).

**Lemma 8.** \(A^G\) is a superpolynomial ring \(S_e^e(k^{0|s})\) over \(k\) in the odd variables \(\theta_i\).

**Proof.** Recall \((J / J^2) \cong (g_-)^*\). This means that we can find \(s\) odd right-invariant superderivations \(D'_i\) in \(g_- \subset \text{Lie}(A)\) such that \(\epsilon_A^s(D'_i(\theta_j)) = d'_i(\theta_j) = \delta_{ij}\) in \(k\). In other words
\[D'_i(\theta_j) \equiv \delta_{ij} \mod m(A)\.

Since \(D'_i(A^G) \subset A^G\) and since \(m(A) \cap A^G = J^G\)
\[D'_i(\theta_j) = \delta_{ij} + Q_{ij}(\theta)\]

for certain superpolynomials \(Q_{ij}\) in the variables \(\theta_i\), whose minimal nonvanishing Taylor coefficient has degree \(\geq 1\). Suppose \(P \neq 0\) is an element in \(I = \text{Kern}(f)\) with minimal nonvanishing Taylor coefficient say of degree \(d\), such that this \(d\) is minimal among all \(0 \neq P \in I\). If \(d = 0\), then \(P\) is a unit in the superpolynomial ring and the quotient \(A^G\) would be zero in contradiction to \(1_A \in A^G\). Hence \(d > 0\).

Let \(\theta_i\) be a variable which occurs nontrivially in the Taylor coefficient of \(P\) of degree \(d\). Then apply the derivative \(D'_i(P)\). Obviously \(D'_i(P)\) has a nonvanishing Taylor coefficient of degree \(d - 1\). On the other hand \(D'_i(I) \subset I\), hence \(D'_i(P) \in I\). This gives a contradiction unless the kernel vanishes \(I = 0\). QED

Then by an obvious counting argument lemma 8 implies

**Corollary 5.** \(d_n = \#T_n = \binom{s}{n}\) for all \(n\).

**Choice of bases.** Up to a scalar \(\eta = \theta_1\) for \(I = \{1, \ldots, s\}\) is independent of the choice of the basis \(\theta_i\), since it is a generator of the one dimensional \(k\) vectorspace \((J^s)^G\). Hence \(\eta\) is an eigenvalue of the right-invariant operators \(D' \in \text{Lie}(G)\) corresponding to the character \(\det(J / J^2) = \det(g_-)^{-1}\) of \(G\). \(\eta\) generates \(A^G\) as a \(U\)-right module for the universal enveloping algebra \(U = U(\text{Lie}(A))\).

For the odd superderivations \(D'_i\) dual to the \(\tilde{\theta}_i \in g_-\) for \(i = 1, \ldots, s\) define
\[\kappa_A = D'_s \circ \cdots \circ D'_1(\eta) \in A^G\].

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Since $\text{Lie}(G)$ acts on $k \cdot D'_0 \circ \cdots \circ D'_1$ by the character $\det(g_-)$, it is easy to see that $\kappa_A$ is annihilated by all right-invariant derivations $D'_X$ for $X$ in $\text{Lie}(G)$. Furthermore $\kappa_A = 1$ modulo $A^G \cap J = J^G$ or

$$e^*_A(\kappa_A) = 1.$$  

**A global splitting**

The even derivation $D = D_\theta : A \to A$ defined by the Euler operator

$$D(x) = \sum_{i=1}^s \theta_i \cdot D'_i(x)$$

obviously satisfies $D(A) \subset J$ (with notations as in the last section). Hence as a derivation $D(J^\nu) \subset J^\nu$ for all $\nu \geq 1$. The map

$$E_\nu : \frac{J^\nu}{J^{\nu+1}} \to \frac{J^\nu}{J^{\nu+1}}$$

induced by $D$ is $B$-linear. So it suffices to compute $E_\nu$ on the basis elements $\tilde{\theta}_I$.

[For $x \in J^\nu$ and $a \in A$ use that $D(xa) = xD(a) + D(x)a = D(x)a \mod J^{\nu+1}$ and $D(A) \in J$ implies $D(xa) = D(x)a \mod J^{\nu+1}$.] Therefore, as an immediate consequence of $D(\theta_j) = \theta_j$ modulo $J^2 \cap A^G$, this shows

$$E_\nu(\tilde{\theta}_I) = \nu \cdot \tilde{\theta}_I, \quad E_\nu = \nu \cdot \text{id}_{J^\nu/J^{\nu+1}}.$$

**Lemma 9.** For $\text{char}(k) = 0$ or $\text{char}(k) > s$ the even derivation $D : A \to J$ induces an $k$-linear isomorphism

$$D : J \cong J.$$  

**Proof.** For large enough $\nu$ we have $J^{\nu+1} = 0$. The diagram

$$
\begin{array}{ccccccccc}
0 & \to & J^{\nu+1} & \to & J^\nu & \to & J^{\nu}/J^{\nu+1} & \to & 0 \\
& & D & & D & & \nu\cdot \text{id} \downarrow & & \\
0 & \to & J^{\nu+1} & \to & J^\nu & \to & J^{\nu}/J^{\nu+1} & \to & 0
\end{array}
$$

commutes. Hence by downward induction $D : J^\nu \to J^\nu$ is an $k$-linear isomorphism for all $\nu \geq 1$ using the snake lemma. QED
The kernel
\[ \tilde{B} = \text{kernel}(D : A \to A) \]
of the derivation \( D \) is a \( k \)-subalgebra of \( A \). In the situation of the last lemma the snake lemma for
\[
\begin{array}{ccc}
0 & \to & J \\
\downarrow & & \downarrow \\
D & \to & A \\
\downarrow & & \downarrow \\
0 & \to & J \\
\end{array}
\]
implies that the restriction of the quotient homomorphism \( \pi : A \to B \) to \( \tilde{B} \subset A \) is bijective. This inverse of the isomorphism \( \tilde{B} \cong B \) then defines a splitting of \( \pi : A \to B \). Hence we get

**Splitting theorem.** Suppose \( \text{char}(k) = 0 \) or \( \text{char}(k) > s \). Then \( \tilde{B} \cong B \) is even and there exists an isomorphism of \( k \)-superalgebras
\[ A = A^G \otimes^\varepsilon \tilde{B} \cong k[\theta_1, \ldots, \theta_s] \otimes^\varepsilon \tilde{B} . \]

**Supergroups**

An affine algebraic group \( G \) acts on its Lie algebra \( g_+ \) by the adjoint representation. Let \( g_- \) be any finite dimensional algebraic representation of \( G \) over \( k \) with action denoted by \( Ad_- \). Then \( g_+ \) acts on \( g_- \) by derivations \( ad_- = \text{Lie}(Ad_-) \). Consider \( G \)-equivariant quadratic maps
\[ Q : g_- \to g_+ \]
with respect to these actions of \( G \) (i.e. arising from a symmetric \( k \)-bilinear form on \( g_- \) with values in \( g_+ \)). A triple \( \mathbf{G} = (G, g_-, Q) \) as above will be called a supergroup (over \( k \)) provided
\[ ad_-(Q(v)) v = 0 \]
holds for all \( v \in g_- \). An associated Lie algebra \( \text{Lie}(\mathbf{G}) \) considered as a \( \mathbb{Z}_2 \)-graded Lie algebra structure is defined on \( g_+ \oplus g_- \) in the obvious way by the Lie bracket induced by the group structure of \( G \), the action of \( G \) on \( g_- \) and the map \( Q \) (super commutator). See [DM], p.59.

**Example 1.** If \( \mathbf{G} = (G, g_-, Q) \) is a supergroup, then also its connected component in the Zariski topology \( \mathbf{G}^0 = (G^0, g_-, Q) \).
Example 2. Super-affine Hopf algebra $A$ define supergroups

$$(\text{Spec}(A/J), g_-, Q),$$

where $Q$ is the restriction of the Lie bracket on $g_-$ to the diagonal.

Example 3. As a special of example 2 for a finite dimensional super vector space $V = V_+ \oplus V_-$ over $k$ the standard supergroup $Gl(V)$ is defined by $G = Gl(V_+) \times Gl(V_-)$ together with $g_- = \text{Hom}_k(V_+, V_-) \oplus \text{Hom}_k(V_-, V_+)$ and $Q(A \oplus B) = \{A, B\}$ for the super commutator $\{A, B\} = A \circ B + B \circ A$. Here we used the obvious identification $\text{Lie}(Gl(V_\pm)) = \text{End}_k(V_\pm)$.

Center. For a super group $G = (G, g_-, Q)$ let the center $Z(G)$ be the maximal central subgroup of $G$, which acts trivial on $g_-$. Morphisms. A homomorphism $(G, g_-, Q) \rightarrow (G', g'_-, Q')$ between supergroups is a pair $\Phi = (\phi, \varphi)$, where $\phi : G \rightarrow G'$ is a group homomorphism between algebraic groups over $k$ and where $\varphi : g_- \rightarrow g'_-$ is a $k$-linear $\phi$-equivariant map such that $Q'(\varphi(X)) = \text{Lie}(\phi)(Q(X))$. Representations. A representation $(V, \Phi)$ of a supergroup $G = (G, g_-, Q)$ is a finite dimensional $k$ super vector space $V$ together with a homomorphism of supergroups $\Phi : (G, g_-, Q) \rightarrow Gl(V)$. The category of such representations, also denoted $G$-modules, is a $k$-linear abelian rigid (monoidal) tensor category

$$\text{Rep}_k(G)$$

with the forget functor $(V, \Phi) \mapsto V$ as a super fibre functor. This fiber functor factorizes over the functor

$$\text{Lie} : \text{Rep}_k(G) \rightarrow \text{Rep}_k(\text{Lie}(G)).$$

The category $\text{Rep}_k(\text{Lie}(G))$ of super representations of the Lie superalgebra $\text{Lie}(G)$ again is a $k$-linear abelian rigid (monoidal) tensor category. Notation: Let $\sigma$ be an automorphism of the supergroup $G$. If $(V, \Phi)$ is a $G$-module, then also $(V, \Phi \circ \sigma)$. 17
An equivalence of representation categories

Suppose \( k = \mathbb{C} \). Let \( \mathcal{H} \) be the opposite of the category of affine super Hopf algebras over \( k \). Let \( \mathcal{HC} \) be the category of supergroups \( G = (G, g_-, Q) \). Recall \( G \) is an affine algebraic groups over \( k \), and morphisms in \( \mathcal{HC} \) are algebraic with respect to the first component of the triples. There is an obvious forget functor

\[ \mathcal{H} \to \mathcal{HC}. \]

There is a similar forget functor from the category \( \mathcal{H}_\infty \) of differentiable Lie supergroups (as in [DM]) to the category \( \mathcal{HC}_\infty \) of differentiable Harish Chandra triples. Objects now are \( G_\infty = (G_\infty, g_-, Q) \) for classical Lie groups \( G_\infty \). According to [DM] p.79, [CF], [K] p. 232 this forget functor is a quasi-equivalence of categories in the \( C^\infty \)-case. Consider the following commutative diagram of forget functors

\[
\begin{array}{ccc}
\mathcal{HC} & \longrightarrow & \mathcal{HC}_\infty \\
\downarrow & & \downarrow \\
\mathcal{H} & \longrightarrow & \mathcal{H}_\infty
\end{array}
\]

Since an algebraic morphism is determined by its associated \( C^\infty \) map, the functor \( \mathcal{H} \to \mathcal{HC} \) is faithful by going over the top of the diagram. We now show

**Theorem 4.** The functor \( \mathcal{H} \to \mathcal{HC} \) is fully faithful.

This immediately implies theorem 3 or the equivalent

**Corollary 6.** For a super-affine Hopf algebra \( A \) over \( k = \mathbb{C} \) with its associated supergroup \( G \) there exists a tensor-equivalence of algebraic tensor categories over \( k \)

\[ \text{Rep}_k(A) \sim \text{Rep}_k(G). \]

**Proof of theorem.** For \( A, A' \) in \( \mathcal{H} \) with associated triples \( Y' = (\text{Spec}(B'), g'_-, Q') \) and \( Y = (\text{Spec}(B), g_-, Q) \) in \( \mathcal{HC} \) and a morphism

\[ \Phi : Y' \to Y \]

in \( \mathcal{HC} \) we have to construct a homomorphism of super Hopf algebras \( \Phi^* : A \to A' \) inducing \( \Phi \). By the diagram above the corresponding differentiable morphism \( \Phi_\infty \) exists in \( \mathcal{H}_\infty \).
By construction $\Phi_\infty$ is ‘reduced algebraic’, i.e. the underlying morphism of Lie groups $G'_\infty \to G_\infty$ is induced from an algebraic morphism $\Phi_{\text{red}} : G' \to G$ between the underlying reduced algebraic groups. Hence it suffices, if reduced algebraic morphisms $\Phi_\infty$ of $\mathcal{H}_\infty$ are induced from algebraic scheme morphisms $\Phi^*$, The algebraic scheme morphism then automatically respects the additional structures comultiplication, antipode and augmentation; this is obvious, since by assumption the $C^\infty$ morphism $\Phi_\infty$ induced from it has this property.

To construct $\Phi^*$ from a reduced algebraic $\Phi_\infty$ consider its graph $\Psi_\infty = (\text{id}, \Phi_\infty)$

\[(id, \Phi_\infty) : (G_\infty, g, Q) \to (H_\infty, h, Q_H) = (G_\infty, g, Q) \times (G'_\infty, g', Q'),\]

which again is reduced algebraic. By projection onto the second factor it suffices to show that $\Psi_\infty$ is algebraic. Thus it is enough to consider reduced algebraic morphisms $\Psi_\infty$ which are closed immersions. This means that the underlying Hopf algebra morphism $\Psi_{\text{red}} : B \otimes^e B' \longrightarrow B$

is surjective, and that the map $\text{Lie}(H_\infty) \hookrightarrow \text{Lie}(G_\infty)$ induced by $\Psi_\infty$ is injective.

**Construction of $\Phi^*$**. We may assume that $\Phi_\infty$ is a locally algebraic closed immersion. How to find $\Phi^*$? By the splitting theorem it suffices to find a right vertical ring homomorphism $\varphi : A^G \to (A')^G$

\[
A \cong \tilde{B} \otimes A^G \\
\Phi^* \quad \Phi^*_{\text{red}} \quad \varphi \\
\downarrow \quad \downarrow \quad \downarrow \\
A' \cong \tilde{B}' \otimes (A')^G
\]

such that the morphism of super schemes $\Phi^*$ induced on the left extends to the given $\Phi_\infty$ in the differentiable category. Such $\varphi$ of course exists if an only if the pullback $\Phi_{\text{red}}^*$ of superfunctions in the differentiable sense satisfies the algebraicity condition

\[
\Phi_{\text{red}}^*(A^G) \subset (A')^G'.
\]

Now use $\text{Lie}(H) = \text{Lie}(H_\infty)$ and $\text{Lie}(G) = \text{Lie}(G_\infty)$, being defined by left-invariant derivations $D_X$ on the super ring of algebraic resp. differentiable functions. For $X \in g'_+ \subset g_+$ there is a commutative diagram

\[
\begin{array}{ccc}
C^\infty(Y_\infty) & \xrightarrow{D_X} & C^\infty(Y_\infty) \\
\Phi_{\text{red}}^* \downarrow & & \Phi_{\text{red}}^* \\
C^\infty(Y'_\infty) & \xrightarrow{D_X} & C^\infty(Y'_\infty)
\end{array}
\]
Since $\text{Lie}(G'_\infty) \hookrightarrow \text{Lie}(G_{\infty})$ the kernel $C^\infty(Y)^G$ of all $D_X$, $X \in g_+$ derivations on $C^\infty(Y)$ (being contained in the kernel of all $D_X$ for $X \in g'_+$) pulls back to the kernel $C^\infty(Y')^G'$ of all $D_{X'}$, $X' \in g'_+$ on $C^\infty(Y')$. Thus the desired existence of $\varphi$ is evident, if the natural injection

$$A'^G \hookrightarrow C^\infty(Y')^G'$$

is a bijection. Notice $g'_+$ is a Lie algebra, hence integrable! Thus $\dim_k(C^\infty(Y')^G') = 2^s$ for $s = \dim_k(g_-)$ as a consequence of the Frobenius theorem. See [DM], p.75 and [K], p. 230. Therefore

$$\dim_k(A'^G) = 2^s = \dim_k(C^\infty(Y')^G').$$

This implies $A'^G = C^\infty(Y')^G'$ and proves the claim. QED

**Semisimple tensor categories**

For a $k$-linear abelian rigid (monoidal) tensor category $T$ with unit object $1_T$ and $\text{End}_T(1_T) = k$ the object $1_T$ is simple (see [DMi], prop 1.17). Furthermore

**Lemma 10.** $T$ is semisimple iff $1_T$ is injective or projective or $\text{Hom}_T(1_T, -)$ is exact or $\text{Ext}_T^1(L, 1_T) = 0$ holds for all simple objects $L$ in $T$.

**Proof.** $T$ is semisimple iff $\text{Hom}_T(N, M) = \text{Hom}_T(1_T, N^* \otimes M) = \text{Hom}_T(N \otimes M^*, 1_T)$ is exact in $N, M$. This is equivalent to $\text{Ext}_T^1(L, 1_T) = 0$ for all (simple) objects $L$ in $T$. QED

For tensor categories $T$ and $T'$ as above let $R : T \to T'$ be an exact covariant functor with an isomorphism $\iota : 1_{T'} \cong R(1_T)$. Assume $I : T' \to T$ is a left-exact covariant functor. Let $p$ be an epimorphism in $T$

$$p : I(1_{T'}) \to 1_T.$$

Suppose there exists a natural transformation

$$\nu : id \to R \circ I$$

such that $R(p) \circ \nu_{1_{T'}} = \iota$. 

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Example. $R$ exact tensor functor with left adjoint $I$. Then $id \in \text{Hom}_T(I(W), I(W))$ defines $\nu_W \in \text{Hom}_T(W, RI(W))$ and let $p \in \text{Hom}_T(I(1_{T'}), 1_T)$ correspond to $\iota \in \text{Hom}_T(I_{T'}, R(1_{T'}))$ for $\iota : 1_{T'} \cong R(1_T)$. Then the above properties hold.

Lemma 11. a) In the situation above $T'$ is semisimple, if $T$ is semisimple. b) If $I$ is adjoint to $R$ and $\text{End}_{T'}(1_{T'}) = k$, then $T$ is semisimple iff $T'$ is semisimple and $p$ splits in $T$.

Proof. b) Suppose $T'$ is semisimple. Then $\text{Hom}_T(I(1_{T'}), -) = \text{Hom}_T(1_{T'}, R(-))$ is exact. If $p$ splits in $T$, then $1_T \oplus I^+ = I(1_{T'})$. Hence also $\text{Hom}_T(1_T, -)$ is exact. Hence $T$ is semisimple. Conversely if $T$ is semisimple, $p$ splits.

a) Suppose $T$ is semisimple. If $T'$ is not semisimple, then by the lemma 10 there exists a simple object $L$ and a nonsplit extension $E$ in $T'$

$$0 \to 1_{T'}, \begin{array}{c} a \cr b \end{array} \to E \to L \to 0 .$$

Since $\nu_{1_{T'}} : 1_{T'} \hookrightarrow RI(1_{T'})$ and $RI(a) : RI(1_{T'}) \hookrightarrow RI(E)$ by our assumptions, $a(1_{T'}) \subset E$ is not in the kernel of $\nu_E : E \to RI(E)$. Hence $b : \ker(\nu_E) \to L$ is a monomorphism. Then $\ker(\nu_E) \neq 0$ implies $\ker(\nu_E) \cong L$, since $L$ is simple. Since this would split $E$ this proves

$$\nu_E : E \hookrightarrow RI(E) .$$

Since $T$ is semisimple, $I(a) : I(1_{T'}) \hookrightarrow I(E)$ has a section $s : I(E) \to I(1_{T'})$. Then

$$c : 1_{T'} \to R(1_T)$$

defined by $c = R(p) \circ R(s) \circ \nu_E \circ a$ is nonzero. [Otherwise $R(s) \circ R(I(a)) = id$, from $s \circ I(a) = id$, would give $\iota = R(p) \circ \nu_{1_{T'}} = R(p) \circ R(s) \circ R(I(a)) \circ \nu_{1_{T'}} = R(p) \circ R(s) \circ \nu_E \circ a = 0$ by the naturality $\nu_E \circ a = RI(a) \circ \nu_{1_{T'}}$ of $\nu$. Hence $c$ is an isomorphism as $R(1_T) \cong 1_{T'}$ is simple, using $\text{End}_{T'}(1_{T'}) = k$. Then $c^{-1} \circ R(p \circ s)$ splits $\nu_E(E)$

$$\begin{array}{c}
0 \to \text{kernel} \to \nu_E(E) \xrightarrow{R(p \circ s)} R(1_T) \to 0 \\
\uparrow \cong \\

\nu_E(a(1_{T'})) \xrightarrow{c}
\end{array}$$

Since $\nu_E(E) \cong E$ this splits $a(1_{T'})$ in $E$. Contradiction! Hence $T'$ is semisimple.
Semisimple representation categories

Let $G = (G, g, Q)$ be a supergroup over $k = \mathbb{C}$. The obvious covariant exact restriction functor $R : \text{Rep}_k(G) \to \text{Rep}_k(G)$ satisfies $R(k) \cong k$. There exists a covariant induction functor

$I : \text{Rep}_k(G) \to \text{Rep}_k(G)$

which, for $V$ in $\text{Rep}_k(G)$ and $g = \text{Lie}(G)$, is defined by

$I(V) = U(g) \otimes_{U(g_+)} V$.

The action of $g_+ = \text{Lie}(G)$ on $I(V)$ comes from an algebraic action of $G$ on $I(V)$ by the $g_+$-module isomorphism $I(V) \cong \Lambda^*(g_-) \otimes^\epsilon V$. Hence $I(V) \in \text{Rep}_k(G)$. It is easy to see that $I$ is exact and left adjoint to $R$, i.e. Frobenius reciprocity $\text{Hom}_G(I(V), W) = \text{Hom}_G(V, R(W))$.

Since $k$ has characteristic zero $\text{Rep}_k(G)$ is semisimple if and only if $G$ is a reductive algebraic group over $k$. Therefore lemma 11 b) implies

**Theorem 5.** $\text{Rep}_k(G)$ is semisimple if and only if (a) $G$ is reductive and (b) the surjection of $G$-modules defined by the adjunction morphism

$$ad : I(k) \to k$$

has a splitting in the category $\text{Rep}_k(G)$.

**Remark.** By $\text{char}(k) = 0$ condition a) holds iff $g_+ = \text{Lie}(G)$ is a reductive Lie algebra over $k$. Condition b) says that the restriction

$$ad : I(k)^G \to k$$

to the space of $G$-invariant subspace of $I(k)$ is surjective. For $g = \text{Lie}(G)$ then $I(k)^G = (I(k)^g)^G = (I(k)^g)^{\pi_0(G)}$ by [DG], prop. 2.1(c), p.309. The group of connected components $\pi_0(G)$ of $G$ in the Zariski topology is finite. Since $\text{char}(k) = 0$ the functor of $\pi_0(G)$-invariants is exact by Maschke’s theorem. Hence conditions a) resp b) are equivalent to the following conditions

a’) The Lie algebra $g_+$ is reductive.
b’) The restriction of $ad : I(k) \to k$ to $I(k)^g$ is surjective.

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**Definition.** If \( g \) satisfies these two properties, we say the Lie superalgebra \( g \) is *reductive*. \( \text{Rep}_k(G) \) is semisimple if and only if \( g = \text{Lie}(G) \) is reductive (by theorem 5). In this case we say \( G \) is reductive.

**Connected component.** As already explained

1. \( G \) is reductive if and only if its connected component \( G^0 \) is reductive.

**Etale coverings.** Similarly we may replace \( G \) by a finite etale covering \( G' \to G \). We say that the supergroup \( G' = (G', g, Q) \) attached to \( G = (G, g, Q) \) is a finite etale (central) cover of \( G \). Then of course

2. \( G \) is reductive if and only if the etale cover \( G' \) is reductive.

Then \( G = G'/F \) for a finite subgroup \( F \) of the center \( Z(G') \) of \( G' \).

**Central quotients.** Finally if \( Z \) is a closed subgroup of the center \( Z(G) \) of \( G \), then \( G/Z = (G/Z, g, Q) \) again a supergroup called a central quotient. Obviously

3. If \( G \) is reductive, then any central quotient \( G/Z \) is also reductive since \( \text{Rep}_k(G/Z) \) is a full subcategory of \( \text{Rep}_k(G) \), if \( \text{Rep}_k(G) \) is semisimple!

**Reductive supergroups**

The main classification statement involves the orthosymplectic supergroups

\[
\text{Spo}(1, 2r) = (Sp(2r, J), k^{2r}, Q).
\]

Fix a nondegenerate antisymmetric \( 2r \times 2r \)-matrix \( J' = -J \) so that \( g \in Sp(2r, k) \Leftrightarrow g'Jg = J \). This identifies \( sp(2r, J) \) with the matrices \( X \) for which \( JX \) is symmetric. For the standard action \( Ad_- \) of \( Sp(2r, J) \) on \( k^{2r} \) the map \( Q : k^{2r} \to sp(2r, J) \)

\[
Q(v)_{\alpha\beta} = \sum_{\gamma=1}^{2r} v_\alpha v_\gamma J_{\gamma\beta}
\]

for \( v = (v_1, \ldots, v_{2r}) \in k^{2r} \) is well defined and equivariant such that \( Q(v)v = 0 \). So this defines a supergroup. Different choices of \( J \) yield isomorphic supergroups.

**Proposition 1.** A supergroup is reductive over \( k = \mathbb{C} \) if and only if its connected component admits a finite etale central covering, which as a supergroup is a direct product of super groups of the following type
1. A classical central $k$-torus
2. Simple connected simply connected classical $k$-groups
3. Simple supergroups of orthosymplectic type $S\text{po}(1, 2r)$ for integers $r \geq 1$.

Similarly a Lie superalgebra is reductive if and only if, modulo the center, it is a direct sum of simple Lie superalgebras of classical type or of the orthosymplectic types $BC_r = sp(1, 2r)$ corresponding to the super groups $S\text{po}(1, 2r)$.

Proof. A product of reductive supergroups is reductive. We leave this as an exercise. So in one direction it suffices that the supergroups $S\text{po}(1, 2r)$ are reductive. In fact $Rep_k(S\text{po}(1, 2r)) = Rep_k(sp(1, 2r))$ because $Sp(2r, J)$ is simply connected, and this reduces to [DH], theorem 4.1.

Now for the converse. By our preliminary remarks in the last section we may replace $G^0$ by an etale finite covering $G'$, where $G' = T \times S$ for a $k$-torus $T$ and a product $S$ of connected simple and simply connected $k$-groups. Then we can divide $G'$ by its maximal central torus $Z$. The new supergroup $G''$ is reductive, if $G$ is reductive. This allows to reduce the proof to the case $G = (G, g_-, Q)$ without central torus so that in addition $G$ is connected and a product of a torus $T$ and a simple simply connected $k$-group $S$. If these conditions hold and $Rep_k(G)$ is semisimple, we say $G$ is good. So assume $G$ is good. Then by [DG], page 309ff and theorem 4 it suffices to prove that $g = \text{Lie}(G)$ is a product of simple Lie superalgebras of the classical type and types $BC_r$. Using condition b') this immediately would follow from [DH], theorem 4.1 for semisimple $g_+$. 

We already know $g_+$ is reductive. To show that $g_+$ is semisimple we claim that $g$ is a direct sum of Lie superalgebras $g_\nu$ with $(g_\nu)_+ \neq 0$ and either $(g_\nu)_- = 0$ or $(g_\nu)_-$ is an irreducible $(g_\nu)_+$-module with $(g_\nu)_+ = [(g_\nu)_-, (g_\nu)_-]$. This is easy: For $g_- = s \oplus t$ and an irreducible $g_+$-submodule $s$

$\mathfrak{h}_+ = [s, s]$

is an ideal in $g_+$ by the Jacobi identity $[g_+, [s, s]] \subset [s, [g_+, s]] + [[g_+, s], s] \subset [s, s]$. Hence either $\mathfrak{h} = \mathfrak{h}_+$ in case $\mathfrak{h}_+$ commutes with $g_-$, or otherwise

$\mathfrak{h} = \mathfrak{h}_+ \oplus s$,

is an ideal in $g$ with the desired property. (As a $G$-module, thus as a $g$-module) $g = \mathfrak{h} \oplus \mathfrak{h}'$ splits into ideals by the semisimplicity of $Rep_k(G)$. The ideal property
[h, h'] \subset h \cap h' = 0$ decomposes $\epsilon$. Since condition b') easily implies $h_+ \neq 0$ for $s \neq 0$ (see [DH], prop.2.2) our claim follows by induction.

To show that $h = h_+ \oplus s$ is an ideal for $[h_+, g_-] \neq 0$, notice $[h_+, t] = 0$. Indeed $[h_+, t] \subset [g_+, t] \subset t$ and $[h_+, t] = [[s, s], t] \subset [s, t] \subset [g_+, s] \subset s$. Thus $[h_+, s] = [h_+, g_-] \neq 0$. Therefore $s = [h_+, s]$, since $[h_+, s]$ is a $g_+$-submodule of $s$. Obvious are $[g_+, s] \subset s$ and $[g_+, h_+] \subset h_+$ and similarly $[g_-, h_+] = [g_-, [s, s]] \subset [[g_-, s], s] \subset [g_+, s] \subset s$. To show $[g_-, s] = [t, s] + [s, s] \subset h_+$ use $[t, s] = [t, [h_+, s]] \subset [[h_+, t], s] + [t, [s, h_+]] = [t, [s, h_+]] \subset [g_+, h_+] \subset h_+$.

If $(g_\nu)_- \neq 0$ is an irreducible $(g_\nu)_+$-module, the center $z_\nu$ of $(g_\nu)_+$ acts by a character $\chi_\nu$. By the equivariance and surjectivity (!) of the Lie bracket

\[
(g_\nu)_- \times (g_\nu)_- \to (g_\nu)_+ \neq 0
\]

the trivial action of $z_\nu$ on $(g_\nu)_+$ forces $2\chi_\nu = 0$, hence $\chi_\nu = 0$. Thus $z_\nu$ is in the center of $g$, therefore trivial by our assumption that $G$ is good. Hence the reductive Lie algebra $g_+$ is semisimple. QED

**The categories $\text{Rep}_k(G, \epsilon)$**

For a supergroup $G = (G, g_-, Q)$ suppose $\epsilon$ is in the center of $G(k)$ such that $\epsilon^2 = 1$ and $\text{Ad}_-(\epsilon) = -\text{id}_{g_-}$. Let $T = \text{Rep}_k(G, \epsilon)$ be the full subcategory of $\text{Rep}_k(G)$ defined by the super representations $(V, \phi, \varphi)$ for which $\phi(\epsilon) = \sigma_V$ is the super parity automorphism $\sigma_V$ of $V$. $T$ is an algebraic tensor category over $k$ (see [DJ]).

Not every supergroup $G = (G, g_-, Q)$ admits twisting elements $\epsilon$ as above. But the extended supergroup $G^{ext} = (G \times \mu_2, g_-, Q)$, where $\text{Ad}_-(g, \pm 1) = \pm \text{Ad}_-(g)$, always has the twisting element $\epsilon^{ext} = (1, -1) \in G^{ext} = G \times \mu_2$. The forget functor defines a tensor-equivalence

\[
\text{Rep}_k(G^{ext}, \epsilon^{ext}) = \text{Rep}_k(G),
\]

since $(V, \phi, \varphi) \in \text{Rep}_k(G)$ extends uniquely to $(V, \phi^{ext}, \varphi) \in \text{Rep}_k(G^{ext}, \epsilon^{ext})$ for $\phi^{ext}(g, \pm 1) = \sigma_V \phi(g) = \phi(g) \sigma_V$.

**Lemma 12.** $\text{Rep}_k(G, \epsilon)$ is semisimple if and only if $\text{Rep}_k(G)$ is semisimple.
Since this is a statement on the underlying abelian categories, we may ignore the tensor structures on these categories. On the underlying abelian categories the parity change \( \Pi(V) = V \otimes^s \bar{\mathbb{T}} \), defined by the trivial super representation \( \bar{\mathbb{T}} = \Pi(1) \) on \( k^{0|1} \), induces a functor \( \Pi : \text{Rep}_k(G) \to \text{Rep}_k(G) \) which in general does not preserve the subcategory \( \text{Rep}_k(G, \varepsilon) \). However

\[ \Pi : \text{Rep}_k(G^{\text{ext}}) \to \text{Rep}_k(G^{\text{ext}}) \]

preserves the subcategory \( \text{Rep}_k(G^{\text{ext}}, \varepsilon^{\text{ext}}) \).

**Proof of lemma 12.** In the extended supergroup \( G^{\text{ext}} \) we have two twisting elements \( \varepsilon \) and \( \varepsilon^{\text{ext}} \). This defines an element \( z = \varepsilon \varepsilon^{\text{ext}} = (\varepsilon, -1) \in G \times \mu_2 \) in the center of the supergroup \( G^{\text{ext}} \), i.e. \( z \) is in the center of \( G^{\text{ext}} \) with trivial action on \( g_- \), and \( z \) commutes with \( \varepsilon \) and \( \varepsilon^{\text{ext}} \). The eigenspace decomposition with respect to \( z \) decomposes the category

\[ \text{Rep}_k(G^{\text{ext}}) = \text{Rep}_k^+(G^{\text{ext}}) \oplus \text{Rep}_k^-(G^{\text{ext}}) \]

and also its subcategories \( \text{Rep}_k^+(G^{\text{ext}}, \varepsilon^{\text{ext}}) \) and \( \text{Rep}_k^+(G^{\text{ext}}, \varepsilon) \). Then by definition

\[ \text{Rep}_k^+(G^{\text{ext}}, \varepsilon^{\text{ext}}) = \text{Rep}_k^+(G^{\text{ext}}, \varepsilon) \quad \text{and} \quad \text{Rep}_k^-(G^{\text{ext}}, \varepsilon^{\text{ext}}) = \Pi(\text{Rep}_k^+(G^{\text{ext}}, \varepsilon^{\text{ext}})) , \]

since \( \varepsilon \) has trivial action and \( \varepsilon^{\text{ext}} \) acts by \( -1 \) on \( \bar{\mathbb{T}} \in \text{Rep}_k(G^{\text{ext}}, \varepsilon^{\text{ext}}) \). Ignoring tensor structures \( T = \text{Rep}_k(G, \varepsilon) = \text{Rep}_k^+(G^{\text{ext}}, \varepsilon) = \text{Rep}_k^+(G^{\text{ext}}, \varepsilon^{\text{ext}}) \) and \( \text{Rep}_k(G^{\text{ext}}, \varepsilon^{\text{ext}}) = \text{Rep}_k^+(G^{\text{ext}}, \varepsilon^{\text{ext}}) \oplus \Pi(\text{Rep}_k^+(G^{\text{ext}}, \varepsilon^{\text{ext}})) \) give

\[ \text{Rep}_k(G^{\text{ext}}, \varepsilon^{\text{ext}}) = T \bigoplus \Pi(T) , \quad T = \text{Rep}_k(G, \varepsilon) . \]

Hence \( T \) is semisimple iff \( \text{Rep}_k(G^{\text{ext}}, \varepsilon^{\text{ext}}) = \text{Rep}_k(G) \) is semisimple. QED

**Remarks on** \( G = \text{Spo}(1, 2r) \)

We discuss the representations of the orthosymplectic group over \( k = \mathbb{C} \). The category \( \text{Rep}_k(G) \) of super representations of a supergroup \( G \) contains the trivial even representation \( 1 \) on \( k = k^{1|0} \) and the odd trivial representation \( \bar{\mathbb{T}} \) on \( k^{0|1} \) such that \( \bar{\mathbb{T}} \otimes^s \bar{\mathbb{T}} = 1 \).

For \( G = \text{Spo}(1, 2r) \) the center of \( G = \text{Sp}(2r, J) \) is \( \mu_2 \). The center of \( G \) is trivial. Hence \( \varepsilon = -id \) gives a unique choice for a twisting element \( \varepsilon \) to define a category

\[ \text{Rep}_k(G) = T \bigoplus \Pi(T) . \]
Also notice $\Pi(W) = W \otimes^r \mathbb{T}$.

The standard representation $V$. Consider the following representation $(V, \phi, \varphi) \in \text{Rep}_k(G, \varepsilon)$ of the supergroup on $G = Sp(1, 2r)$. As a $G$-module $V = V_+ \oplus V_- = k \oplus g_-$ with trivial action on $V_+ = k$ and with the standard representation of $G$ on $V_-$. This defines $\phi(X) \in \text{End}(V)_+$ for $X \in g_+$. We identify $V_-$ with $g_-$. The odd elements $v \in g_-$ act on $V$ by $\varphi(v) \in \text{End}(V)_-$ defined by the annihilation and creation operators

$$\varphi(v)w = \frac{1}{2}v^\prime Jw \in V_+, \quad w \in V_-$$

$$\varphi(v)\lambda = \lambda \cdot v \in V_-, \quad \lambda \in V_+.$$

Then $\phi(Q(v)) = [\varphi(v), \varphi(v)]$ for $v \in g_-$. We call $V$ the orthosymplectic standard representation. It is easy to see that $V$ is an irreducible super representation.

Invariant form $b$. The orthosymplectic standard representation $V$ admits a nondegenerate supersymmetric $G$-invariant form

$$b : V \otimes^\varepsilon V \rightarrow k^{1|0}$$

where $b$ is the orthogonal direct sum of the symmetric form $b(\lambda_1, \lambda_2) = \lambda_1\lambda_2$ on $V_+ = k$ and the antisymmetric form $b(v_1, v_2) = -\frac{1}{2}v_1^\prime Jv_2$ on $V_-$. In fact the orthosymplectic supergroup $G$ is the automorphism group of this supersymmetric form $b$ on $V$. In particular: The standard representation $V$ is an „orthogonal self dual“ faithful representation of $G$. Hence $V$ is a tensor generator of

$$T = \text{Rep}_k(G, \varepsilon) = \langle V \rangle.$$

See [Sh] for an explicit decomposition of the tensor powers $V^{\otimes r}$. See [RS] for a connection of $T$ with the representation category of the group $SO(2r + 1)$.

Lemma 13. All irreducible representations in $T$ are „orthogonal self dual“. All representations in $\Pi(T)$ are „symplectic self dual“.

Proof. If $W$ is „orthogonal self dual“ then $\Pi(W)$ is „symplectic self dual“ and vice versa. Since $\text{Rep}_k(G) = T \oplus \Pi(T)$ it therefore suffices that $T$ contains all „orthogonal self dual“ irreducible representations. Tensor products of „orthogonal self dual“ representations are „orthogonal self dual“, hence any multiplicity one representation contain in it is again „orthogonal self dual“.
of highest weight vectors any irreducible representation $W$ in $Rep_k(G)$ appears with multiplicity one in a tensor power of irreducible fundamental representations $V_i, i = 1, \ldots, r$ of $G$ up to parity shift. For these $(T^\otimes i \otimes \varepsilon V_i)_+ = \Lambda^i(g_-)$ and $(T^\otimes i \otimes \varepsilon V_i)_- = \Lambda^{i-1}(g_-)$. See [Dj], p.31 and p.36. Obviously $V_i \in Rep_k(G, \varepsilon)$. The $V_i$ are self dual, therefore ‘orthogonal self dual’ by considering their restriction to $G$, which contains the highest weight representation with multiplicity one as an ‘orthogonal self dual’ representation of $G$. QED

We claim

**Lemma 14.** For $G = Sp(1, 2r)$ the tensor subcategory of $Rep_k(G)$ generated by the standard representation $V = k^{1|2r}$ of $G$ is $Rep_k(G, \varepsilon)$. The tensor subcategory generated by $\Pi(V)$ is the full category $Rep_k(G)$.

**Proof.** It suffices to find $\mathcal{T} = \Pi(1)$ in a tensor power of $\Pi(V)$. Then $V = \Pi(V) \otimes^\varepsilon \mathcal{T}$ generates $T$ and $T \bigoplus (T \otimes^\varepsilon \mathcal{T}) = Rep_k(G)$. We claim

$$\Pi(1) \hookrightarrow \Pi(I(1)) \cong \Lambda^{2r+1}(\Pi(V))$$

for the induced module $I(k) = I(1)$. By Frobenius reciprocity the dimension of

$$\text{End}_G(I(k)) \cong \text{Hom}_G(k, I(k)) \cong (\Lambda^\bullet(g_-))^G$$

is $r + 1$ by the classical invariant theory of the group $G = Sp(2r)$. A basis for the invariants are the powers $\omega^i$ of the symplectic form $\omega \in \Lambda^2(g_-)$. Indeed

$$I(k) = \bigoplus_{i=0}^{r} V_i \in Rep_k(G, \varepsilon)$$

for $V_0 = 1$ and the different fundamental representations $V_1, \cdots V_r$ of $G$ (see [Dj], p.36). By Frobenius reciprocity also the dimension of

$$\text{Hom}_G(I(k), \Lambda^{2r+1}(\Pi(V)) \otimes^\varepsilon \mathcal{T}) = \text{Hom}_G(k, \Lambda^{2r+1}(\Pi(V)) \otimes^\varepsilon \mathcal{T})$$

is equal to $r + 1$ using

$$\Lambda^{2r+1}(\Pi(V))^G = \bigoplus_{j=0}^{2r+1} \Lambda^j(g_-)^G \otimes^\varepsilon \mathcal{T}^{\otimes(2r+1-j)} \cong \bigoplus_{j=0}^{r} \mathcal{T}^{\otimes(2r+1-2j)} \ .$$
Then $I(k) \cong \Lambda^{2r+1}(\Pi(V)) \otimes^\varepsilon \mathbf{T}$, provided \( \Lambda^{2r+1}(\Pi(V)) = \Pi(Sym^{2r+1}(V)) \) has at least \( r+1 \) nonisomorphic irreducible constituents. For this (with the convention $I$ of [DM], p.49 and p.62f) consider the superpolynomial ring $S^e(V) = Sym^e(V)$

\[
S^e(V) = Sym^e(V_+) \otimes^\varepsilon \Lambda^e(V_-).
\]

Multiplication with the invariant form $b \in Sym^2(V)$ is injective inducing a filtration $F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_r$ of $F_r$ by $G$-modules $F_i = Sym^{2i+1}(V) \otimes^\varepsilon b^{\otimes (r-i)}$. Notice $F_i \cong F_{i+1}$ for $i \geq r$. The highest weight submodules of the $G_i = F_i/F_{i-1}$ for $i = 0, \ldots, r$ define $r+1$ nonisomorphic $G$-modules, since $(G_i)_- \cong \Lambda^{2i+1}(V_-)$ and $(G_i)_+ \cong \Lambda^{2i}(V_-)$ as $G$-modules. Hence

\[
I(1) \cong Sym^{2r+1}(V).
\]

Therefore the $G_i$ must have been irreducible $G$-modules. Considering highest weights a comparison shows $G_i \cong V_{2i+1}$ for $0 \leq i < \frac{r}{2}$ and $G_{r-i} \cong V_{2i}$ for $0 \leq i \leq \frac{r}{2}$. Hence all the representations $V_i$ for $i = 0, \ldots, r$ are constituents of the tensor power $V^{\otimes (2r+1)}$. QED

We remark that there is a dual filtration $F'_i = Sym^{2i}(V) \otimes^\varepsilon b^{\otimes (r-i)}$ on $Sym^{2r}(V)$ with $G'_i \cong G_{r-i}$, and again $I(1) \cong Sym^{2r}(V)$.

**Lemma 15.** Let $T$ be a semisimple algebraic tensor category over an algebraically closed field $k$ of characteristic zero. For a simple object $W \neq 0$ of $T$ the categorial rank $rk_k(W)$ does not vanish.

**Proof.** Since $rk_k(W) = sdim_k(W)$, this follows from [Ka], p.619 formula (2.6) with $B(0, n) = spo(2n, 1)$ in the notations of loc. cit. QED

**Structure Theorem**

Assume $k = \mathbb{C}$. Then according to proposition 1 a connected reductive supergroup $G$ is of the form $G = (G' \times H)/F$ where $G' = \prod_{r \geq 1} Spo(1, 2r)^{n_r}$ is a product of orthosymplectic supergroups and where $H$ is a reductive algebraic $k$-group. Since $F$ is a finite central subgroup of $G' \times H$ and since the center of $G'$ is trivial, this implies $F \subset H$. Hence $G = G' \times H'$ for $H' = H/F$. Hence

**Lemma 16.** A connected reductive supergroup $G$ is isomorphic to a product $G' \times H$ where $H$ is a reductive algebraic $k$-group and where $G' = \prod_{r \geq 1} Spo(1, 2r)^{n_r}$ is a product of orthosymplectic supergroups.
For $G = Sp(1, 2r)$ and $G = Sp(2r)$ one has $Aut(G) = G_{ad}$ and therefore $Aut(G) = G$. In other words, any automorphism of $G$ is an inner automorphism $Int(g)$ for a unique element $g \in G$. Let $G$ be a reductive supergroup. Then the group $\pi_0(G) = \pi_0(G)$ acts on $G'$. For $g \in \pi_0(G)$ we can choose a representative $g \in G$, by a suitable modification with an element in $G' = \prod_{r \geq 1} Sp(2r)^{nr}$, such that $g$ acts by a strict permutation of the factors on $G'$. The group of such $g \in G$ defines a canonical subgroup $G_1 \subset G$ such that $G_1 \cap G' = 1$. Hence $G_1 \subset H$. Hence any $g \in \pi_0(G) = \pi_0(H)$ has a representative in $G_1 \subset H$. We get a canonical homomorphism

$$p : H \to \prod_{r \geq 1} \Sigma_{n_r}$$

into the product of symmetric permutation groups $\Sigma_{n_r}$ whose kernel is $G_1$. Conversely given such a homomorphism $p : H \to \prod_{r \geq 1} \Sigma_{n_r}$ for a reductive algebraic $k$-group $H$ one can construct the semidirect product supergroup $G = G' \rtimes H$ obtained from the permutation action of $H$ on $G' = \prod_{r \geq 1} Sp(1, 2r)^{nr}$. Obviously in our case therefore

**Theorem 6.** Any reductive supergroup $G$ over an algebraically closed field $k$ of characteristic zero is isomorphic to a semidirect product $G' \rtimes H$ of a reductive algebraic $k$-group $H$ with a product $G' = \prod_{r \geq 1} Sp(1, 2r)^{nr}$ of simple supergroups of $BC$-type, where the semidirect product is defined by an abstract group homomorphism

$$p : \pi_0(H) \to \prod_{r \geq 1} \Sigma_{n_r}.$$

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