ON THE EFFECTS OF BOHM’S POTENTIAL ON A STATIONARY MACROSCOPIC SYSTEM OF SELF-INTERACTING PARTICLES

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Abstract. We consider a macroscopic model describing a system of self-gravitating particles. We study the existence and uniqueness of non-negative stationary solutions and allude the differences to results obtained from classical gravitational models. The problem is considered on a bounded domain up to three space dimension, subject to Neumann boundary condition for the particle density, and Dirichlet boundary condition for the self-interacting potential. Finally, we show numerical simulations that affirm our findings.

Key words. Second order elliptic systems, Bohm’s potential, self-interaction.

AMS subject classifications.

1. Introduction. Consider a stationary macroscopic system of self-interacting particles with Bohm’s potential, which describe the normalized density \( n \geq 0 \),

\[
-\epsilon^2 \sqrt{n}^{-1} \Delta \sqrt{n} + \log n - \sigma \Phi = F \quad \text{in } \Omega, \quad \partial_n \sqrt{n} = 0 \quad \text{on } \Gamma,
\]

the quasi Fermi-level \( F \),

\[
-\text{div} (n \nabla F) = 0 \quad \text{in } \Omega, \quad n \partial_n F = 0 \quad \text{on } \Gamma,
\]

and the potential \( \Phi \) due to self-interaction in a particle system,

\[
-\Delta \Phi = n \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \Gamma,
\]

where \( \nu \) is the outer normal to the convex, bounded domain \( \Omega \subset \mathbb{R}^d \), \( d \leq 3 \) with Lipschitz boundary \( \Gamma \), \( \epsilon > 0 \) is the scaled Planck constant and \( |\sigma| \in [0, \infty) \) is the mass of the system of self-interacting particles, where \( \text{sign}(\sigma) \) dictates the nature of the interaction involved. In this case, positive mass \( \sigma > 0 \) would indicate the presence of self-attraction, while negative mass \( \sigma < 0 \) indicates self-repulsion. By passing to the limit \( \epsilon \to 0 \), we formally recover either the classical drift-diffusion equations \( \sigma < 0 \) [1] or a model for a system of self-gravitating particles \( \sigma > 0 \) [3].

A simple observation of (1b) suggests that for any positive density \( n > 0 \), we obtain a solution \( F \in \mathbb{R} \). In fact, any constant function is a solution. However, we shall see below that this constant solution is fixed due to the normality of \( n \).

In order to bring this system of equations into a system similar to that of the classical equations for self-interacting particles (c.f. [1, 3]), we introduce the quasi potential \( u := \log n - F \). Assuming \( n > 0 \), we insert \( F = \log n - u \) into (1b) to obtain

\[
-\Delta n + \text{div} (n \nabla u) = 0 \quad \text{in } \Omega, \quad \partial_n n - n \partial_n u = 0 \quad \text{on } \Gamma.
\]

At this point, one directly sees the resemblance to the classical equations for self-gravitating particles if we simply set \( u = \Phi \) in (2) and couple it with equation (1c) for the potential \( \Phi \). For this reason, we call \( u \) the quasi potential. Clearly, we may further rewrite (2) in the equivalent form

\[-\text{div} (e^u \nabla (ne^{-u})) = 0 \quad \text{in } \Omega, \quad \partial_n (ne^{-u}) = 0 \quad \text{on } \Gamma.\]
This allows us to relate \( n \) and \( u \) via \( n = \alpha e^u \) for some constant \( \alpha > 0 \). Since \( n \) is normalized, i.e., \( \int_\Omega n \, dz = 1 \), we deduce that \( \alpha = \| e^u \|^{-1}_{L^1(\Omega)} \). Notice that, by fixing \( \alpha \), we also fix the quasi Fermi-level, which is explicitly given by \( F = \log(\alpha) \).

Introducing this into (1) leads to the coupled system for \((u, \Phi)\) given by an elliptic equation with natural gradient growth for \( u \)

\[
(3a) \quad -\frac{\epsilon^2}{2} \Delta u + u = \frac{\epsilon^2}{4} |\nabla u|^2 + \sigma \Phi \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \Gamma,
\]

and the equation for the potential \( \Phi \),

\[
(3b) \quad -\Delta \Phi = \| e^u \|^{-1}_{L^1(\Omega)} e^u \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \Gamma.
\]

Note that system (3) is equivalent to system (1) if \( n > 0 \), or equivalently, if \( u \) is an essentially bounded function. The existence of bounded weak solutions \((u, \Phi)\) will be shown for (3), thereby implying the existence of solutions for (1).

We introduce the short hand \( \mathcal{X} \) to denote the space

\[
\mathcal{X} := H^1(\Omega) \cap L^\infty(\Omega).
\]

**Theorem 1.** Let \( d \in \{2, 3\} \) and the mass \( |\sigma| \in [0, +\infty) \) be given. Then problem (3) has a solution \((u, \Phi)\) in \( \mathcal{X} \times \mathcal{X} \). Consequently, \((n, F, \Phi)\) with

\[
\sqrt{n} = \| e^u \|^{-\frac{1}{2}}_{L^1(\Omega)} e^{u/2} \in \mathcal{X} \quad \text{and} \quad F = \log n - u = - \log \| e^u \|_{L^1(\Omega)} \in \mathbb{R},
\]

is a solution of (1). Moreover, there exists \( \theta \in (0, 1) \), such that \( \theta \leq n \leq 1/\theta \).

Observe that \( \sigma > 0 \) can be chosen arbitrarily large as opposed to classical self-gravitating particles, where a threshold for existence exists. In this sense, system (1) can be thought of as a regularization of the classical self-gravitating system.

Let us discuss the techniques used to show existence of solutions for system (3). The solvability of (3a) and its variants with homogeneous Dirichlet boundary data were shown in the papers [7, 8, 12]. Moreover, if \( \Phi \in L^p(\Omega) \) with \( p > d/2 \), then \( u \in \mathcal{X} \). Adopting the methods used in [7], we show in Section 2 that this holds true also for homogeneous Neumann boundary data. In this case, systems (1) and (3) are equivalent. Furthermore, we obtain from [6] the following sharp estimates for functions in the space \( W^{2,1}_{\Delta,0} = \overline{D(\Omega)}^{\| \Delta \|_{L^1(\Omega)}} \).

**Proposition 2.** For any \( f \in W^{2,1}_{\Delta,0} \) with \( f \geq 0 \) in \( \Omega \) we have the estimates

\[
\| f \|_{L^{\exp}(\Omega)} \leq (8\pi)^{-\frac{1}{2}} \| \Delta f \|_{L^1(\Omega)}, \quad d = 2,
\]

\[
\| f \|_{L^{\frac{d}{d-2},\infty}(\Omega)} \leq (2\gamma_d)^{-\frac{1}{2}} \| \Delta f \|_{L^1(\Omega)}, \quad d \geq 3,
\]

with \( \gamma_d = \omega_{d-1}^{d/2}/d \omega_{d-1}^{d} \), where \( \omega_{d-1} \) is the measure of the unit sphere in \( \mathbb{R}^d \). The constants given above are the best possible, independently of the domain.

Here we used \( L^{\exp}(\Omega) \) to denote the Zygmund space, whose elements \( f \) satisfy \( \int_\Omega e^{\lambda f} \, dz < \infty \) for some \( \lambda = \lambda(f) > 0 \), and \( L^{p,\infty}(\Omega) \) to denote the classical weak-\( L^p \) space (see [2]). These spaces are related to each other and to the \( L^p \) spaces by the following continuous embeddings for \( 1 < p < \infty \),

\[
L^\infty(\Omega) \hookrightarrow L^{\exp}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^{p,\infty}(\Omega) \hookrightarrow L^1(\Omega).
\]
The estimates in Proposition 2 necessarily implies that the $L^q$-norm of $\Phi$, for any $q \in (1, \frac{d}{d-2})$, is uniformly bounded for $d \geq 2$. In particular, we have

$$\|\Phi\|_{L^q(\Omega)} \leq c_q \mu_d^{-1}, \quad \text{with} \quad \mu_d = \begin{cases} 8\pi, & d = 2, \\ 2\gamma d, & d \geq 3, \end{cases}$$

where $c_q$ is the embedding constant obtained from (4).

Therefore, an application of the Schauder fixed point theorem on a self-mapping for the potential $\Phi$ leads to its existence in $L^q(\Omega)$ for some $q \in (\frac{d}{2}, \frac{d}{d-2})$, and consequently also for $u$ and $n$, which yields the existence result. For sufficiently $\sigma$ we further obtain uniqueness of solutions, given by the following result.

**Theorem 3.** Let $\theta \in (0, 1)$ and $n \in \mathcal{X}$ with $\theta \leq n \leq 1/\theta$. There exist constants $c_0(c_0(d, \Omega) > 0$ and $c_1 = c_1(d, \Omega, \theta) > 0$ such that for

$$|\sigma| < \mu_d \sqrt{c_0^2 + 2\epsilon^2 c_1^2},$$

the solutions of (1) are equal almost everywhere in $\Omega$.

To our knowledge, this estimate for uniqueness appears to be new for both attractive and repulsive potentials. For $\sigma < 0$, it is known that in the classical case $\epsilon = 0$, uniqueness depends on the smallness of the applied voltage. As a matter of fact, the performance of many semi-conductor devices (thyristors) depends on the existence of multiple solutions (c.f. [14] and references therein).

Unfortunately, this estimate is not sharp, since for $d = 2$ and $\sigma > 0$, it is known that uniqueness is valid for $\sigma < \mu_2$ (c.f. [5]). Nevertheless, the above estimate provides a convenient relationship between uniqueness and the Bohm potential ($\epsilon > 0$).

**2. Elliptic equation with natural gradient growth.** We begin by providing results for the subproblem (3a) required to prove Theorem 1.

**Definition 4.** A function $u \in \mathcal{X}$ is said to be a solution of (3a) if it satisfies

$$\frac{\epsilon^2}{2} \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega u \varphi \, dx = \frac{\epsilon^2}{4} \int_\Omega (\nabla u)^2 \varphi \, dx + \sigma \int_\Omega \Phi \varphi \, dx$$

for every function $\varphi \in \mathcal{X}$.

**Theorem 5.** Suppose $\Phi \in L^p(\Omega)$, $p > d/2$. Then there is a solution $u \in \mathcal{X}$ of problem (6). Furthermore, we have $e^{n/2} \in \mathcal{X}$.

Before proving the theorem, we state two results regarding the regularity of the solution $u$, whose proofs can be found in the appendix.

**Lemma 6.** Let $u$ be solution of (3a) with $\Phi \in L^p(\Omega)$, $p > d/2$. Then

1. for any $\lambda \geq 0$ there exist constants $K_1, K_2 = K_2(\lambda) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq K_1 \quad \text{and} \quad \|e^{\lambda u}\|_{L^\infty(\Omega)} \leq K_2.$$

In particular, $e^{\lambda u} \in L^\infty(\Omega)$ for every $\lambda \geq 0$.

2. there exist constants $M_1, M_2 > 0$ such that

$$\|u\|_{H^1(\Omega)} \leq M_1 \quad \text{and} \quad \|e^{u/2}\|_{H^1(\Omega)} \leq M_2.$$

**Proof of Theorem 5.** Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(s) = (e^{s} - 1) \text{sign}(s)$ and the cut-off function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = \max\{-k, \min\{s, k\}\}$ for some $k \in \mathbb{R}$. We begin by considering the auxiliary problem for $u_k \in \mathcal{X}$:

$$\frac{\epsilon^2}{2} \int_\Omega \nabla u_k \cdot \nabla \varphi \, dx + \int_\Omega u_k \varphi \, dx = \frac{\epsilon^2}{4} \int_\Omega (\nabla u_k)^2 \varphi \, dx + \sigma \int_\Omega T_k(\Phi) \varphi \, dx.$$
Since the right-hand side is bounded, the existence of a bounded solution for (9), $k \in \mathbb{N}$, may be deduced from classical results (see for example [11] for the existence and [10] for the boundedness). Due to the fact that $T_k(|\nabla u_k|^2) \leq |\nabla u_k|^2$ and $|T_k(\Phi)| \leq |\Phi|$ along with Lemma 6, there exists a function $u \in \mathcal{X}$ such that

$$u_k \rightharpoonup u \text{ in } H^1(\Omega) \quad \text{and} \quad u_k \rightarrow^* u \text{ in } L^\infty(\Omega).$$

In order to pass to the limit in (9), we still need to show that $\nabla u_k \rightharpoonup \nabla u$ in $L^2(\Omega)$, i.e., the strong convergence of the gradients of $u_k$ in $L^2(\Omega)$. To do so, we test (9) with $\varphi_k = \phi(u_k - u)$ to obtain

$$\frac{\epsilon^2}{2} \int_\Omega \nabla u_k \cdot \nabla (u_k - u) \varphi'_k \, dx + \int_\Omega u_k \varphi_k \, dx \leq \frac{\epsilon^2}{4} \int_\Omega |\nabla u_k|^2 |\varphi_k| \, dx + \sigma \int_\Omega |\Phi||\varphi_k| \, dx.$$  

For the first term on the left-hand side we have

$$\int_\Omega \nabla u_k \cdot \nabla (u_k - u) \varphi'_k \, dx = \int_\Omega |\nabla (u_k - u)|^2 \varphi'_k \, dx + \int_\Omega \nabla u \cdot \nabla (u_k - u) \varphi'_k \, dx.$$  

As for the second term on the left hand-side, we have

$$\int_\Omega u_k \varphi_k \, dx = \int_\Omega |u_k - u|(e^{\varphi_k} - u) \, dx + u \varphi_k \, dx \geq \int_\Omega |u_k - u|^2 \, dx + \int_\Omega u \varphi_k \, dx.$$  

For the first term on the right-hand side we have

$$\int_\Omega |\nabla u_k|^2 |\varphi_k| \, dx = \int_\Omega \nabla u_k \cdot \nabla (u_k - u) |\varphi_k| \, dx + \int_\Omega \nabla u_k \cdot \nabla u |\varphi_k| \, dx$$

$$= \int_\Omega |\nabla (u_k - u)|^2 |\varphi_k| \, dx + \int_\Omega \nabla u \cdot \nabla (u_k - u) |\varphi_k| \, dx + \int_\Omega \nabla u_k \cdot \nabla u |\varphi_k| \, dx.$$  

Due to the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we get a convergent subsequence, denoted again by $\{u_k\}$, such that $u_k \rightharpoonup u$ in $L^2(\Omega)$. Consequently we obtain yet another subsequence, denoted again by $\{u_k\}$, such that $u_k(x) \rightarrow u(x)$ for a.e. $x \in \Omega$, which implies the almost everywhere convergences

$$|\varphi_k(x)| \rightarrow 0 \quad \text{and} \quad \varphi'_k(x) \rightarrow 1 \quad \text{for a.e. } x \in \Omega.$$  

From Lebesgue’s dominated convergence for these sequences and their boundedness in $L^\infty(\Omega)$, we have the strong convergences

$$\nabla u |\varphi_k| \rightarrow 0 \quad \text{and} \quad \nabla u \varphi'_k \rightarrow \nabla u \text{ in } L^2(\Omega), \quad u |\varphi_k|, |\Phi||\varphi_k| \rightarrow 0 \text{ in } L^1(\Omega).$$  

Passing to the limit in (10) yields

$$\int_\Omega |\nabla (u_k - u)|^2 \varphi'_k \, dx + \int_\Omega |\nabla (u_k - u)|^2 \, dx \rightarrow 0,$$

which necessarily implies that $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)$. Therefore, passing to the limit in (9) yields the solution $u \in \mathcal{X}$ satisfying (6). The fact that $e^{\lambda u} \in \mathcal{X}$ for every $\lambda \geq 0$ follows directly from Lemma 6. \hfill \square
3. Proof of Theorem 1. As drafted out above, we use the Schauder fixed point theorem (c.f. [9, Corollary 11.2]) to facilitate the proof. We define the closed, convex and bounded subset of $L^q(\Omega)$

$$M = \{ w \in L^q(\Omega) \mid \|w\|_{L^q(\Omega)} \leq c_q \mu^{-1}_d, \ w \geq 0 \},$$

for some $q \in \left( \frac{d}{2}, \frac{d}{d-2} \right)$ and with $\mu_d$ as given in (5).

For a given $w \in M$, we consider the auxiliary problem for $\Phi$ given by

$$-\frac{\epsilon}{2} \Delta u + u = \frac{\epsilon}{4} |\nabla u|^2 + \sigma w \text{ in } \Omega, \quad \partial_{\nu} u = 0 \text{ on } \Gamma,$$

(11a)

$$-\Delta \Phi = \|e^u\|_{L^1(\Omega)}^{-1} e^u \text{ in } \Omega, \quad \Phi = 0 \text{ on } \Gamma.$$

(11b)

This induces a compact mapping

$$H: L^q(\Omega) \to L^q(\Omega); \ w \mapsto \Phi,$$

simply due to the continuity of the solution operators between their respective spaces and the compact embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$. Indeed, since $w \in L^q(\Omega)$, we obtain a solution $u \in X$ of (11a) as a result of Theorem 5. Inserting $u$ into (11b) and solving for $\Phi$ gives us $\Phi \in X$ with $\Phi \geq 0$ due to standard theory of elliptic equations.

Furthermore, we have $H: M \to M$ simply due to the estimates in (5). A direct application of the Schauder fixed point theorem concludes the proof.

4. Proof of Theorem 3. Let $A: X \to H^1(\Omega)^*$ denote the operator defined by

$$\langle A(n), \varphi \rangle = \int_{\Omega} \left( -\frac{\epsilon^2}{2} \Delta \sqrt{n}^{-1} \Delta \sqrt{n} + \log n \right) \varphi \ dx \quad \forall \varphi \in H^1(\Omega).$$

Then the weak formulation of (1a) can be written as

$$n \in X: \quad \langle A(n) - \sigma \Phi, \varphi \rangle = \langle F, \varphi \rangle \quad \forall \varphi \in H^1(\Omega).$$

With similar arguments to those by Pinnau, Unterreiter in [14, Theorem 26] the operator $A$ is well-defined. Moreover, for a fixed $\varphi \in H^1(\Omega)$, the Gateaux derivative of $\langle A(\cdot), \varphi \rangle: X \to \mathbb{R}$ at a point $n \in X$ in any direction $h \in X$ exists and is given by

$$\langle A'(n)[h], \varphi \rangle = -\frac{\epsilon^2}{2} \int_{\Omega} \left( \frac{\Delta h}{n} \frac{\Delta n}{n^2} h - \frac{\nabla n \cdot \nabla h}{n^2} + \frac{|\nabla n|^2}{n^3} h \right) \varphi \ dx + \int_{\Omega} \left( \frac{h}{n} \right) \varphi \ dx.$$

Now let $(n_i, F_i, \Phi_i) \in [X]^3, \ i = 1, 2$, be two solutions of (1) and set the difference to be $(\delta n, \delta F, \delta \Phi) := (n_1 - n_2, F_1 - F_2, \Phi_1 - \Phi_2) \in [X]^2$. In particular we have

$$n_i \in X: \quad \langle A(n_i), \varphi \rangle = \langle \sigma \Phi_i + F_i, \varphi \rangle \quad \forall \varphi \in H^1(\Omega)$$

(12)

Setting $\varphi = \delta n$ we obtain by subtraction from (12)

$$\langle A(n_1) - A(n_2), \delta n \rangle = \sigma \langle \delta \Phi, \delta n \rangle + \langle \delta F, \delta n \rangle = \sigma \langle \delta \Phi, \delta n \rangle,$$

where we used the fact that $\delta F \in \mathbb{R}$ and $\int_{\Omega} \delta n \ dx = 0$. 

Let \( n_\tau = n_1 - \tau \delta n, \tau \in [0,1] \) be the convex combination of \( n_1 \) and \( n_2 \). Since the function \([0,1] \ni \tau \mapsto \langle A(n_\tau), \delta n \rangle \in \mathbb{R} \) is differentiable, we obtain, by the mean value theorem, a \( \tau \in (0,1) \) such that

\[
\epsilon^2 \int \Omega \left| \nabla \left( \frac{\delta n}{n_\tau} \right) \right|^2 \, dx + \int \Omega \left| \frac{\delta n}{n_\tau} \right|^2 \, dx = \langle A'(n_\tau)[\delta n], \delta n \rangle = \sigma \int \Omega \delta \Phi \delta n \, dx.
\]

Now, in order to estimate the first term on the left-hand side from below, we use a variant of the result obtained in [14, Lemma 24].

**Proposition 7.** Let assumption (A) hold. Then there exists for any \( \beta \in \mathbb{R} \) and \( \theta \in (0,1) \) a constant \( c_K = c_K(\Omega, \theta, s) > 0 \) such that for any \( n \in X \) with \( \theta \leq n \leq 1/\theta \) and any \( \varphi \in X \) with \( \int_\Omega \varphi \, dx = 0 \):

\[
\int_\Omega n \left| \nabla \left( \frac{\varphi}{n} \right) \right|^2 \, dx \geq c_K^2 \| \varphi \|_{L^s(\Omega)}^2,
\]

where \( s \in [1, \infty) \) such that the Sobolev embedding \( H^1(\Omega) \hookrightarrow L^s(\Omega) \) holds.

For the second term on the left-hand side, we apply Hölder’s inequality to obtain

\[
\int_\Omega |\delta n| \, dx = \int_\Omega \left( \left| \delta n \right| \right)^{\frac{1}{2}} \left( \left| n_\tau \right| \right)^{-\frac{1}{2}} \, dx \leq \left( \int_\Omega \left| \delta n \right| \, dx \right)^{\frac{1}{2}} \left( \int_\Omega \left| n_\tau \right| \, dx \right)^{-\frac{1}{2}},
\]

where we used the fact that \( \int_\Omega n_\tau \, dx = 1 \). Altogether we have the estimate

\[
\langle A'(n_\tau)[\delta n], \delta n \rangle \geq c^2 c_K^2 \| \delta n \|_{L^s(\Omega)}^2 + \| \delta n \|_{L^1(\Omega)}^2
\]

for some \( s \in [1, \infty) \) satisfying the requirements of Proposition 7.

Note that by subtraction, \( \delta \Phi \in X \) solves the problem

\[ -\Delta \delta \Phi = \delta n \quad \text{in} \quad \Omega, \quad \delta \Phi = 0 \quad \text{on} \quad \Gamma. \]

As before, the estimates in Proposition 2 yield for \( q \in \left( \frac{4d}{\pi^2}, \frac{d}{\pi^2} \right) \) the inequality

\[
\| \delta \Phi \|_{L^q(\Omega)} \leq c_q \mu_d^{-1} \| \delta n \|_{L^1(\Omega)}.
\]

Due to the constraint placed on \( q \), we have the Sobolev embedding \( H^1(\Omega) \hookrightarrow L^{q'}(\Omega) \), where \( q' = \frac{q}{q-1} \), thereby allowing us to choose \( s = q' \) in (14).

Putting together all the inequalities obtained above, we have

\[
\epsilon^2 c_K^2 \| \delta n \|_{L^{q'}(\Omega)}^2 + \| \delta n \|_{L^1(\Omega)}^2 \leq c_q \mu_d^{-1} |\sigma| \| \delta n \|_{L^1(\Omega)} \| \delta n \|_{L^{q'}(\Omega)}.
\]

Applying Young’s inequality we obtain

\[
\epsilon^2 c_K^2 \| \delta n \|_{L^{q'}(\Omega)}^2 + \| \delta n \|_{L^1(\Omega)}^2 \leq \frac{1}{2} \| \delta n \|_{L^1(\Omega)}^2 + \frac{(c_q \mu_d^{-1} |\sigma|)^2}{2} \| \delta n \|_{L^{q'}(\Omega)}^2.
\]

Using the continuous embedding \( L^{q'}(\Omega) \hookrightarrow L^1(\Omega) \), we finally arrive at

\[
\left( \frac{2\epsilon^2 c_K^2 - c_q \mu_d^{-2} |\sigma|^2}{c_q^2} + 1 \right) \| \delta n \|_{L^1(\Omega)}^2 \leq 0.
\]

In conclusion, for sufficiently small mass

\[
|\sigma| < \mu_d \sqrt{(c_{q'}/c_q)^2 + 2\epsilon^2 (c_K/c_q)^2} = \mu_d \sqrt{\frac{c_0^2}{c_0^2} + 2\epsilon^2 c_1^2},
\]

we obtain uniqueness for \( n \), and consequently for \( \Phi \). \( \square \)
5. The semi-classical limit. This result is well-known for the case \( \sigma < 0 \), i.e., for the quantum drift-diffusion equations (c.f. [1]). Therefore, we restrict ourselves to the case \( \sigma > 0 \). Furthermore, we consider only the case \( d = 2 \).

According to [4] the free energy functional

\[
\mathcal{E}_0(n) = \int_\Omega n(\log n - 1) \, dx - \frac{\sigma}{2} \int_\Omega n \Phi \, dx, \quad \sigma < \mu_2 = 8\pi,
\]

where \( \Phi \) is a solution of the Poisson problem (1c) attains a minimum \( n_0 \) in the set
\[
\mathcal{P} = \left\{ n \in L^1(\Omega) \mid n \geq 0, \int_\Omega n \, dx = 1, \int_\Omega n \log n \, dx \in L^1(\Omega) \right\}.
\]

Moreover, the minimizer is unique (c.f. [5, Theorem 3.2]). We denote its associated potential by \( \Phi_0 \) and recall the relationship

\[
-\Delta \Phi_0 = n_0 = \| e^{\sigma \Phi_0} \|_{L^1(\Omega)} e^{\sigma \Phi_0},
\]

which solves the classical stationary system of self-gravitating particles (c.f. [3, 15]),

\[
\begin{align*}
-\Delta n + \sigma \text{div}(n \nabla \Phi) &= 0 \quad \text{in } \Omega, \\
\partial_\nu n - \sigma n \partial_\nu \Phi &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

\[
\begin{align*}
-\Delta \Phi &= n \quad \text{in } \Omega, \\
\Phi &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

Now consider the energy functional associated to (1)

\[
\mathcal{E}_\varepsilon(n) = \varepsilon^2 \mathcal{F}(n) + \mathcal{E}_0(n), \quad \sigma < \mu_2,
\]

where \( \mathcal{F} \) is the Fisher information given by

\[
\mathcal{F}(n) = \int_\Omega |\nabla \sqrt{n}|^2 \, dx.
\]

It is easy to see that \( \mathcal{E}_\varepsilon \) is weakly lower semicontinuous, strictly convex and coercive on \( \mathcal{P}_\varepsilon = \{ n \in \mathcal{P} \mid \sqrt{n} \in H^1(\Omega) \} \). Indeed, \( \mathcal{E}_0 \) is equivalent to the functional

\[
\mathcal{G}(\Phi) = \frac{\sigma}{2} \int_\Omega |\nabla \Phi|^2 - \log \left( \int_\Omega e^{\sigma \Phi} \, dx \right) - 1, \quad \sigma < \mu_2,
\]

which is uniformly bounded from below for all \( \Phi \in H^1_0(\Omega) \) due to Moser [13]. Therefore, it attains a unique minimum \( n_\varepsilon \) in the set \( \mathcal{P}_\varepsilon \), and in particular in \( \mathcal{P} \).

**Theorem 8.** Let \( n_0 \) and \( n_\varepsilon \) be solutions of the problems

\[
\min_{n \in \mathcal{P}} \mathcal{E}_0(n) \quad \text{and} \quad \min_{n \in \mathcal{P}} \mathcal{E}_\varepsilon(n)
\]

respectively. Then there exists \( (n_*, \Phi_*, F_*) \in \mathcal{X} \times \mathcal{X} \times \mathbb{R} \), which solves the classical self-gravitation system (16) such that the following convergences hold for \( \varepsilon \to 0_+ \):

\[
\sqrt{n_\varepsilon} \to \sqrt{n_*} \quad \text{in } H^1(\Omega), \quad \Phi_\varepsilon \to \Phi_* \quad \text{in } H^1(\Omega), \quad u_\varepsilon \to \sigma \Phi_* \quad \text{in } L^2(\Omega),
\]

\[
F_\varepsilon \to F_* = -\log \| e^{\sigma \Phi_*} \|_{L^1(\Omega)} \quad \text{in } \mathbb{R}.
\]

Furthermore, if \( n_0 \) is a unique minimizer of \( \mathcal{E}_0 \), then \( n_* \equiv n_0 \) and \( \Phi_* \equiv \Phi_0 \).

**Proof.** We begin by showing that \( \{ \sqrt{n_\varepsilon} \} \subset H^1(\Omega) \) is bounded. Indeed, since \( n_\varepsilon \) is a minimum of \( \mathcal{E}_\varepsilon \), we have

\[
\varepsilon^2 \mathcal{F}(n_\varepsilon) + \mathcal{E}_0(n_\varepsilon) = \mathcal{E}_\varepsilon(n_\varepsilon) \leq \mathcal{E}_\varepsilon(n_0) = \varepsilon^2 \mathcal{F}(n_0) + \mathcal{E}_0(n_0),
\]
but \( \mathcal{E}_0(n_0) \leq \mathcal{E}_0(n_\epsilon) \) since \( n_0 \) is a minimum of \( \mathcal{E}_0 \). Hence, \( \mathcal{F}(n_\epsilon) \leq \mathcal{F}(n_0) \) for all \( \epsilon > 0 \), which was to be shown. We can then extract a subsequence, denoted again by \( n_\epsilon \), such that \( \sqrt{n_\epsilon} \rightharpoonup \sqrt{n_*} \) in \( H^1(\Omega) \) and \( n_\epsilon \to n_* \) in \( L^2(\Omega) \) for some \( n_* \in \mathcal{P} \), where the second convergence follows from the compact Sobolev embedding \( H^1(\Omega) \hookrightarrow L^4(\Omega) \). Furthermore, we have

\[
\mathcal{E}_0(n_0) \leq \liminf_{\epsilon \to 0^+} \mathcal{E}_0(n_\epsilon) \leq \liminf_{\epsilon \to 0^+} \mathcal{E}_\epsilon(n_\epsilon) \leq \limsup_{\epsilon \to 0^+} \mathcal{E}_\epsilon(n_\epsilon) \leq \mathcal{E}_0(n_0),
\]

which implies that \( \mathcal{E}_0(n_0) = \lim_{\epsilon \to 0^+} \mathcal{E}_\epsilon(n_\epsilon) \). On the other hand, by the weakly lower \( L^2(\Omega) \)-semicontinuity of the functional \( \mathcal{E}_0 \),

\[
\mathcal{E}_0(n_*) \leq \liminf_{\epsilon \to 0^+} \mathcal{E}_0(n_\epsilon) \leq \limsup_{\epsilon \to 0^+} \mathcal{E}_\epsilon(n_\epsilon) = \mathcal{E}_0(n_0).
\]

Therefore, \( n_* \) is a minimizer of \( \mathcal{E}_0 \). The strong convergence \( \Phi_\epsilon \to \Phi_* := (-\Delta_0)^{-1} n_* \) in \( H^1(\Omega) \) follows easily from the strong convergence \( n_\epsilon \to n_* \) in \( L^2(\Omega) \), due to the continuity of the solution operator \( (-\Delta_0)^{-1} \).

Now consider the Euler-Lagrange equation associated to \( \mathcal{E}_\epsilon \), i.e., the variational formulation of (1a), given by

\[
\epsilon^2 \int_\Omega \nabla \sqrt{n_\epsilon} \cdot \nabla \varphi \, dx + \int_\Omega (\log n_\epsilon - \sigma \Phi_\epsilon) \varphi \sqrt{n_\epsilon} \, dx = \int_\Omega F_\epsilon \varphi \sqrt{n_\epsilon} \, dx \quad \forall \varphi \in H^1(\Omega).
\]

Similarly we have the Euler-Lagrange equation associated to \( \mathcal{E}_0 \), given by

\[
\int_\Omega (\log n_* - \sigma \Phi_*) \varphi \, dx = \int_\Omega F_* \varphi \, dx \quad \forall \varphi \in L^1(\Omega),
\]

where \( F_* \in \mathbb{R} \) is the Lagrange multiplier for the constraint \( \int_\Omega n_* \, dx = 1 \) with

(17) \quad \quad \quad \quad F_* = -\log \| e^{\sigma \Phi_*} \|_{L^1(\Omega)}, \quad \text{since} \quad \log n_* = \sigma \Phi_* + F_* \in L^\infty(\Omega).

Therefore, by testing the former variational formulation with \( \varphi = \sqrt{n_\epsilon} \) and the latter with \( \varphi = n_\epsilon \), and taking the difference of the resulting equations, we obtain

\[
\epsilon^2 F_\epsilon(n_\epsilon) + \int_\Omega \log(n_\epsilon/n_*) n_\epsilon \, dx + \sigma \int_\Omega (\Phi_* - \Phi_\epsilon) n_\epsilon \, dx = (F_\epsilon - F_*),
\]

where we have used the fact that \( F_\epsilon, F_* \in \mathbb{R} \) and \( \int_\Omega n_\epsilon \, dx = 1 \). Due to the convergences derived above, we conclude that \( F_\epsilon \to F_* \) in \( \mathbb{R} \). At this point, it is easy to see from (17) that the pair \( (u_*, \Phi_* ) \) solves (16a). To see that it also solves (16b), we notice that \( \Phi_* \) is a minimizer of \( \mathcal{G} \) and that (16b) is simply the Euler-Lagrange equation associated to \( \mathcal{G} \). Hence, \( (n_*, \Phi_* ) \in \mathcal{X} \times \mathcal{X} \) is indeed a solution of (16).

Furthermore, from the representation \( u_\epsilon = \log n_\epsilon - F_\epsilon \) we obtain the strong convergence \( u_\epsilon \to \sigma \Phi_* \) in \( L^2(\Omega) \). Indeed, by taking the difference of the two representations, multiplying the resulting equation with \( (u_\epsilon - \sigma \Phi_*) \) and integrating over \( \Omega \), we obtain

\[
\int_\Omega \| u_\epsilon - \sigma \Phi_* \|^2 \, dx = \int_\Omega \log(n_\epsilon/n_*) (u_\epsilon - \sigma \Phi_* ) \, dx + \int_\Omega (F_\epsilon - F_*) (u_\epsilon - \sigma \Phi_* ) \, dx \\
\leq \left( \| \log(n_\epsilon/n_*) \|_{L^2(\Omega)} + |\Omega|^{1/2} |F_\epsilon - F_*| \right) \| u_\epsilon - \sigma \Phi_* \|_{L^2(\Omega)},
\]

which clearly yields the required convergence.

Finally, if the minimizer \( n_0 \) is unique, then \( n_* \equiv n_0 \) and consequently \( \Phi_* \equiv \Phi_0 \).
6. Numerical Simulations. In this section we show two numerical simulations that validate the theoretical results obtained above. In both cases, we considered the unit disk $D \subset \mathbb{R}^2$ with scaled Planck constant $\epsilon = 1 \times 10^{-3}$. An adaptive finite element method was used to solve the coupled problem (3) for $(u, V)$ iteratively. The outer iteration consist of a Picard iteration procedure for the potential $V$ and the inner iteration consists of Newton’s method to solve (3a) for the quasi potential $u$.

**Case 1:** The first case pertains to the existence of stationary states for system (3) with a large mass ($\sigma = 10\pi$), which clearly exceeds the threshold ($\sigma = 8\pi$) of existence in the classical setting. The numerical results are shown in Figure 6. One clearly observes the similarity of $u$ and $V$. This similarity shows that the quasi potential $u$ is a slight perturbation of the potential $V$ when the scaled Planck constant is small.

![Quasi potential u](image1)

![Potential V](image2)

**Fig. 6.1.** A stationary solution $(u, V)$ for $\sigma = 10\pi$ with $F = -20.188$

**Case 2:** The second case corresponds to the non-uniqueness of stationary states when their quasi Fermi-levels $F$ are identical and their mass $\sigma$ exceeds the threshold given in Theorem 3. As in Case 1, we set $\sigma = 10\pi$. By shifting the position of the starting value for the iteration procedure in Case 1, we obtained another solution with the same quasi Fermi-level. This solution is in fact just a shift in position of the solution we obtained in Case 1.

![Quasi potential u](image1)

![Potential V](image2)

**Fig. 6.2.** Another stationary solution $(u, V)$ for $\sigma = 10\pi$ with $F = -20.188$

**Appendix A. Proof of Lemma 6.**

**A.1. Proof of Lemma 6.1.** As in Theorem 5 we define the function $\phi: \mathbb{R} \to \mathbb{R}$ by $\phi(s) = (e^{s} - 1)\text{sign}(s)$ and let

$$G_k(s) = s - T_k(s) = s - \max\{-k, \min\{s, k\}\} = \min\{k - s, \max\{0, s - k\}\}.$$ 

Set $A_k = \{|u| > k\}$. Using $\varphi_k = \phi(G_k(u))$, $k \geq k_0 \geq 1$ as a test function in (6),

$$\frac{c^2}{4} \int_{A_k} |\nabla G_k(u)|^2 |\varphi_k'| \, dx + \frac{c^2}{4} \int_{A_k} |\nabla G_k(u)|^2 \, dx + \int_{A_k} |u||\varphi_k| \, dx \leq \int_{A_k} |V||\varphi_k| \, dx.$$
Note that by definition $\psi(s) = \int_0^{|s|} \sqrt{\phi(t)} \, dt = 2(e^{|s|/2} - 1)$. Furthermore, it is easy to see that there exist constants $\mu_1, \mu_2 > 0$ such that

$$\mu_1 |\psi(s)|^2 \leq |\phi(s)|, \quad |s| \geq 0 \quad \text{and} \quad \mu_2 |\psi(s)|^2, \quad |s| \geq 1.$$  

Writing the left-hand side of the inequality above in terms of $\psi(G_k(u))$ and using the first inequality in (18) for half of the third term, we get

$$\frac{1}{4} \min\{\epsilon^2, 2\mu_1\} \|\psi(G_k(u))\|^2_{H^1(A_k)} + \frac{\epsilon^2}{4} \|\nabla G_k(u)\|^2_{L^2(A_k)}$$

$$+ \frac{1}{2} \int_{A_k} |u| |\varphi_k| \, dx \leq \int_{A_k} |V| |\varphi_k| \, dx.$$  

Now decompose the right-hand side as follows

$$\int_{A_k} |V| |\varphi_k| \, dx \leq \int_{(A_k \setminus A_{k+1}) \cap \{|V| > 1\}} |V| |\varphi_k| \, dx$$

$$+ \int_{A_{k+1} \cap \{|V| > 1\}} |V| |\varphi_k| \, dx + \int_{A_k \cap \{|V| \leq 1\}} |\varphi_k| \, dx$$

$$= J_1 + J_2 + J_3.$$  

For $k_0 \geq 4$, we can absorb $J_3$ into the left-hand side of (19). As for $J_1$, we have

$$|J_1| \leq |\phi(1)| \int_{A_k \cap \{|V| > 1\}} |V| \, dx \leq |\phi(1)||V|_{L^p((|V| > 1))}|A_k|^{1/p}.$$  

Since $p' \in (1, \frac{2d}{d-2})$ we can use the Hölder, interpolation and Young inequalities, along with the Sobolev embedding $H^1 \hookrightarrow L^{\frac{2d}{d-2}}$ to obtain

$$|J_2| \leq \|V\|_{L^p(|V| > 1)} \|\varphi_k\|_{L^{p'}(A_{k+1})}$$

$$\leq \|V\|_{L^p(|V| > 1)} \|\varphi_k\|_{L^1(A_{k+1})}^{1 - \frac{d}{2p'}} \|\varphi_k\|_{L^{\frac{2d}{d-2}}(A_{k+1})}$$

$$\leq c_1 \|\psi(G_k(u))\|^2_{H^1(A_k)} + c_2 \|V\|_{L^{p'}(|V| > 1)} \|\varphi_k\|_{L^1(A_k)}$$

$$\leq c_1 c_3 \|\psi(G_k(u))\|^2_{H^1(A_k)} + c_2 \|V\|_{L^{p'}(|V| > 1)} \|\varphi_k\|_{L^1(A_k)}.$$  

So by choosing $c_1$ sufficiently small and $k_0$ sufficiently large, $J_2$ may also be absorbed into the left-hand side of (19), leaving us with

$$\delta_1 \|\psi(G_k(u))\|^2_{H^1(A_k)} + \delta_2 \|G_k(u)\|^2_{H^1(A_k)} \leq \delta_3 |A_k|^{1/p'},$$

for suitable constants $\delta_i > 0$, $i = 1, 2$. Here we used $|s| \leq |\phi(s)|$ for $|s| \geq 0$ and the fact that $|G_k(u)| = |u| - k$ a.e. on $A_k$.

Now if $h > k > k_0$, then $A_k \subset A_h$ and for arbitrary $m \in \mathbb{N}$,

$$\left( \int_{A_k} (|u| - k)^m \, dx \right)^{2/m} \geq \left( \int_{A_k} (h - k)^m \, dx \right)^{2/m} = (h - k)^2 |A_h|^{2/m}.$$  

Hence, by the Sobolev embedding $H^1 \hookrightarrow L^{\frac{2d}{d-2}}$

$$c_3 \delta_2 (h - k)^2 |A_h|^{(d-2)/d} \leq c_3 \delta_2 \|G_k(u)\|^2_{L^{\frac{2d}{d-2}}(A_k)} \leq \delta_3 |A_k|^{1/p'}.$$
Defining $\zeta(h) := |A_h|^{(d-2)/d}$ and $\beta := d/p'(d-2) > 1$, we finally obtain

$$
\zeta(h) \leq \frac{\delta}{(h-k)^2} \zeta(k)^\beta, \quad \text{for } h > k \geq k_0,
$$

with $\delta = \delta_\varepsilon/c_3\delta$. From a lemma of Kinderlehrer and Stampacchia (c.f. [10, II. Lemma B1]), it follows that $\zeta(k) = 0$ for every $k \geq K_1 \geq k_0$, where

$$
K_1 = k_0 + 2^{\beta/(\beta-1)} \zeta(k_0)^{(\beta-1)/2} - \frac{\delta}{\sqrt{\beta}}.
$$

From the definition of $\zeta$ we finally obtain the uniform bound $|u| \leq K_1$ a.e. on $\Omega$, which gives the bounds required. \(\square\)

**A.2. Proof of Lemma 6.2.** Following the proof of Lemma 6.1 we consider $\varphi_k = \phi(G_k(u))$ for $k \geq k_0(h) = \max\{1, 4h\}$, with a suitable $h$ chosen later, as a test function in (6) to obtain

$$
\frac{1}{4} \min\{\epsilon^2, 2\mu_1\} \|\psi(G_k(u))\|^2_{H^1(A_k)} + \frac{\epsilon^2}{4} \|\nabla G_k(u)\|^2_{L^2(A_k)} + \frac{1}{2} \int_{A_k} |u|\varphi_k^2 \, dx \leq \int_{A_k} |V|\varphi_k \, dx.
$$

This time we decompose the right-hand side as follows

$$
\int_{A_k} |V|\varphi_k \, dx = \int_{(A_k \backslash A_{k+1}) \cap \{|V| > h\}} |V|\varphi_k \, dx + \int_{A_{k+1} \cap \{|V| > h\}} |V|\varphi_k \, dx + h \int_{A_k \cap \{|V| \leq h\}} |\varphi_k| \, dx = J_1 + J_2 + J_3.
$$

For each of the $J_i$, $i \in \{1, 2, 3\}$, we have the bounds

$$
|J_1| \leq |\phi(1)| \int_{\{|V| > h\}} |V| \, dx \leq |\phi(1)| h^{\frac{d+2}{2}} \|V\|_{L^2(\{|V| > h\})}^{\frac{d}{2}}
$$

$$
|J_2| \leq \|V\|_{L^2(\{|V| > h\})} \|\phi(G_k(u))\|_{L^2(A_k)} \leq c_1 \mu_2 \|V\|_{L^2(\{|V| > h\})} \|\psi(G_k(u))\|_{H^1(A_k)}^{\frac{d}{2}}
$$

$$
|J_3| \leq \frac{1}{4} \int_{A_k} k_0 |\varphi_k| \, dx \leq \frac{1}{4} \int_{A_k} |u| |\varphi_k| \, dx,
$$

where we used the Sobolev embedding $H^1 \hookrightarrow L^{\frac{2d}{d-2}}$ and the second inequality in (18) for $J_2$. By absorbing $J_2$ and $J_3$ into the left-hand side, we obtain

$$
\delta \|\psi(G_k(u))\|_{H^1(A_k)}^{\frac{d}{2}} + \frac{\epsilon^2}{4} \|G_k(u)\|_{H^1(A_k)}^{\frac{d}{2}} \leq c_4(h, V),
$$

where, for $h$ large enough,

$$
\delta = \frac{1}{4} \left( \min\{\epsilon^2, 2\mu_1\} - 4c_1 \mu_2 \|V\|_{L^2(\{|V| > h\})}^{\frac{d}{2}} \right) > 0.
$$

Hence $\psi(G_k(u)) \in H^1(\Omega)$ and $\nabla G_k(u) \in [L^2(\Omega)]^2$ for any $k \geq k_0(h)$.
Now we fix $h$ such that (21) holds and use $\varphi_{k_0} = \phi(T_{k_0}(u))$ as a test function in the weak formulation (6), to obtain
\[
\frac{\epsilon^2}{2} \int_\Omega |\nabla T_{k_0}(u)|^2 |\varphi_{k_0}| \, dx + \int_\Omega |u| |\varphi_{k_0}| \, dx \leq \frac{\epsilon^2}{4} \int_\Omega |\nabla T_{k_0}(u)|^2 |\varphi_{k_0}| \, dx
\]
\[
+ |\phi(k_0(h))| \left[ \frac{\epsilon^2}{4} \int_\Omega |\nabla G_{k_0}(u)|^2 \, dx + \int_\Omega |V| \, dx \right],
\]
which leads to
\[
\frac{\epsilon^2}{4} \int_\Omega |\nabla \psi(T_{k_0}(u))|^2 \, dx + \frac{\epsilon^2}{4} \int_\Omega |\nabla T_{k_0}(u)|^2 \, dx + \int_\Omega |u| |\varphi_{k_0}| \, dx \leq c_5(h, V).
\]
Combining (20) and (22), we obtain a uniform bound for the gradient term $\nabla u$,
\[
\int_\Omega |\nabla u|^2 \, dx = \int_{\{|u| \leq k_0\}} |\nabla T_{k_0}(u)|^2 \, dx + \int_{\{|u| > k_0\}} |\nabla G_{k_0}(u)|^2 \, dx \leq c_6(h, V)
\]
and uniform bound for the gradient term $\nabla e^{|u|/2}$,
\[
\int_\Omega |\nabla e^{|u|/2}|^2 \, dx \leq \frac{1}{4} \left[ \int_{\{|u| \leq k_0\}} |\nabla \psi(T_{k_0}(u))|^2 \, dx + e^{k_0} \int_{\{|u| > k_0\}} |\nabla \psi(G_{k_0}(u))|^2 \, dx \right]
\]
\[
\leq c_7(h, V).
\]
Hence, Lemma 6.1 and the estimates above give us constants $M_1, M_2 > 0$ such that
\[
\|u\|_{H^1(\Omega)} \leq M_1 \quad \text{and} \quad \|\nabla e^{|u|/2}\|_{H^1(\Omega)} \leq M_2,
\]
which concludes the proof. \(\square\)

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