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Abstract

This paper presents a systematic study for harmonic analysis of metaplectic wave-packet representations on the Hilbert function space $L^2(\mathbb{R}^d)$. The abstract notions of symplectic wave-packet groups and metaplectic wave-packet representations will be introduced. We then present an admissibility condition on closed subgroups of the real symplectic group $\text{Sp}(\mathbb{R}^d)$, which guarantees the square-integrability of the associated metaplectic wave-packet representation on $L^2(\mathbb{R}^d)$.

Keywords: symplectic group, multivariate metaplectic wave-packet representations, symplectic wave-packet group, metaplectic wave-packet transform, square-integrable representations

1. Introduction

Many interesting applications of mathematical analysis in theoretical physics (e.g. paraxial optic, quantum mechanics, etc) prompt particular forms of multivariate metaplectic (Shale-Weyl) representation [14–16, 25, 41] under various names; quadratic-phase transforms, linear canonical transforms [10, 36], Fresnel transforms, fractional Fourier transforms [54], Gaussian integral [51]. In the following article, we shall approach the topic from the classical theory of coherent state transforms [3].

The abstract theory of covariant/coherent state transforms is the mathematical basis of modern high frequency approximation techniques and time-frequency (resp. time-scale) analysis [37, 44, 48, 49]. Over the last decades, abstract and computational aspects of covariant/coherent state transforms have achieved significant popularity in mathematical and theoretical physics, see [3, 5, 37, 47] and references therein. Coherent state transforms are classically obtained by a given coherent function systems. Then admissibility conditions on the coherent system imply analyzing of functions with respect to the system by the inner product evaluation...
From harmonic and functional analysis aspects such coherent structures are classically originated from square-integrable representations of locally compact groups, see [33, 46, 50, 59] and references therein. Commonly used coherent states transforms in theoretical physics, computational science and engineering are wavelet transform [49], Gabor transform [37], wave-packet transform [27–30, 32].

The mathematical theory of Gabor analysis is based on the coherent state generated by modulations and translations of a given window function [4, 6, 31, 34]. Wavelet analysis is a time-scale analysis which is based on the continuous affine group as the group of dilations and translations [9]. Abstract harmonic analysis extensions of wavelet analysis are studied in [7, 49]. The theory of wave packet transform over the real line has been extended for higher dimensions by several authors, see [11]. The mathematical theory of classical wave-packet analysis on the real line is originated from classical dilations, translations, and modulations of a given window function. The mathematical theory of wave-packet analysis as a coherent state analysis has been recently abstracted in the setting of locally compact Abelian groups in [28]. In a nutshell, wave-packet analysis which is also well-known as Gabor-wavelet analysis is a shrewd extensions of the two most prominent coherent states analysis, namely Gabor and wavelet analysis.

The following paper consists of abstract aspects of nature of metaplectic wave-packet transforms over $\mathbb{R}^d$. This paper aims to introduce the notion of metaplectic wave-packet transform over the Hilbert function space $L^2(\mathbb{R}^d)$. We shall address analytic aspects of metaplectic wave-packet transforms over $L^2(\mathbb{R}^d)$ using tools from representation theory of locally compact groups and abstract harmonic analysis.

This article contains 6 sections. Section 2 is devoted to fix notations and a summary of classical Fourier analysis on $\mathbb{R}^d$ and classical harmonic analysis on projective representations and square-integrable representations over locally compact groups. In section 3 we present a brief study of harmonic analysis over the real symplectic group $\text{Sp}(\mathbb{R}^d)$. We introduce the abstract notion of symplectic wave-packet groups associated to closed subgroups of $\text{Sp}(\mathbb{R}^d)$. We shall also show that the group structure of symplectic wave-packet groups canonically determines an irreducible projective (unitary) group representation of the group, which is called as metaplectic wave-packet representation. We then present an admissibility criterion on closed subgroups of $\text{Sp}(\mathbb{R}^d)$ to guarantee the square-integrability of the associated metaplectic wave-packet representation on $L^2(\mathbb{R}^d)$. As an application of our results we study analytic aspects of metaplectic wave-packet transforms associated to closed subgroups of the real symplectic group $\text{Sp}(\mathbb{R}^d)$. It is also shown that, if $\mathbb{H}$ is a compact subgroup of $\text{Sp}(\mathbb{R}^d)$, for all non-zero window functions we can continuously reconstruct any $L^2$-function from metaplectic wave-packet coefficients. Finally, we will illustrate application of these techniques in the case of well-known compact subgroups of the real symplectic group $\text{Sp}(\mathbb{R}^d)$.

2. Preliminaries and notations

Let $G$ be a locally compact group and $\mathcal{H}$ be a Hilbert space. Let $\mathcal{U}(\mathcal{H})$ be the multiplicative group of all unitary operators on $\mathcal{H}$. A projective group representation of $G$ on $\mathcal{H}$ is a mapping $\Gamma : G \to \mathcal{U}(\mathcal{H})$ which satisfies

$$\Gamma(gg') = z(g, g')\Gamma(g)\Gamma(g') \quad \text{for all } g, g' \in G$$
where \( z(g, g') \) are unimodular numbers. The projective group representation \( \Gamma \) is called irreducible on \( \mathcal{H} \), if \( \{0\} \) and \( \mathcal{H} \) are the only closed \( \Gamma \)-invariant subspaces of \( \mathcal{H} \).

A projective group representation \((\Gamma, \mathcal{H})\) is called left square integrable if there exists a non-zero vector \( \zeta \in \mathcal{H} \) such that

\[
\int_{\mathcal{G}} |\langle \zeta, \Gamma(g)\zeta \rangle|^2 \, dm_G(g) < \infty,
\]

for some left Haar measure \( m_G \) of \( G \). Similarly, it is called right square integrable if there exists a non-zero vector \( \zeta \in \mathcal{H} \) such that

\[
\int_{\mathcal{G}} |\langle \zeta, \Gamma(g)\zeta \rangle|^2 \, dm_G(g) < \infty,
\]

for some right Haar measure \( n_G \) of \( G \).

Since \( \mathbb{R}^d \) is an LCA (locally compact Abelian) group, according to the Schur’s lemma, all irreducible representations of \( \mathbb{R}^d \) are one-dimensional. Thus any irreducible unitary representation \((\pi, \mathcal{H}_x)\) of \( \mathbb{R}^d \) satisfies \( \mathcal{H}_x = \mathbb{C} \) and hence there exists a continuous homomorphism \( \omega \) of \( \mathbb{R}^d \) into the circle group \( \mathbb{T} \), such that for each \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and \( z \in \mathbb{C} \) we have \( \pi(x)(z) = \omega(x)z \). Such homomorphisms are called characters of \( \mathbb{R}^d \) and the set of all such characters of \( \mathbb{R}^d \) is denoted by \( \mathbb{R}_d^\ast \). If \( \mathbb{R}^d \) equipped with the topology of compact convergence on \( \mathbb{R}^d \) which coincides with the \( w^* \)-topology that \( \mathbb{R}^d \) inherits as a subset of \( L^\infty(\mathbb{R}^d) \), then \( \mathbb{R}^d \) with respect to the product of characters is an LCA group which is called the dual (character) group of \( \mathbb{R}^d \). The character group \( \mathbb{R}_d^\ast \), that is the multiplicative group of all continuous additive homomorphisms of \( \mathbb{R}^d \) into the circle group \( \mathbb{T} \), can be parametrized by \( \mathbb{R}^d \) via the following duality notation with \( \mathbb{R}^d \) via

\[
\omega(x) = (x, \omega) = e^{2\pi i x \cdot \omega}
\]

for each \( \omega \in \mathbb{R}_d^\ast \). The linear map \( \mathcal{F}_{\mathbb{R}^d} : L^1(\mathbb{R}^d) \rightarrow C(\mathbb{R}_d^\ast) \) defined by \( f \mapsto \mathcal{F}_{\mathbb{R}^d}(f) = \hat{f} \) via

\[
\mathcal{F}_{\mathbb{R}^d}(f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} f(s) \omega(s) \, dm_{\mathbb{R}^d}(s), \tag{2.1}
\]

is called the Fourier transform on \( \mathbb{R}^d \). It is a norm-decreasing \( * \)-homomorphism from \( L^1(\mathbb{R}^d) \) into \( C_0(\mathbb{R}_d^\ast) \) with a uniformly dense range in \( C_0(\mathbb{R}_d^\ast) \). If a Haar measure \( m_{\mathbb{R}^d} \) on \( \mathbb{R}^d \) is given and fixed then there is a Haar measure \( m_{\mathbb{R}_d^\ast} \) on \( \mathbb{R}_d^\ast \), which is called the normalized Plancherel measure associated to \( m_{\mathbb{R}^d} \), such that the Fourier transform (2.1) is an isometric isomorphism on \( L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) and hence it can be extended uniquely to a unitary isomorphism from \( L^2(\mathbb{R}^d) \) onto \( L^2(\mathbb{R}_d^\ast) \), see [24]. Then each \( f \in L^1(\mathbb{R}^d) \) with \( \hat{f} \in L^1(\mathbb{R}_d^\ast) \) satisfies the following Fourier inversion formula

\[
f(s) = \int_{\mathbb{R}_d^\ast} \hat{f}(\omega) \omega(s) \, dm_{\mathbb{R}^d}(\omega) \text{ for a.e. } s \in \mathbb{R}^d. \tag{2.2}
\]

For \( x \in \mathbb{R}^d \) and \( f \in L^2(\mathbb{R}^d) \), the translation of \( f \) by \( x \) is defined by \( T_xf(y) = f(y - x) \) for \( y \in \mathbb{R}^d \). The translation \( T_x : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \) is a unitary operator. For \( \omega \in \mathbb{R}_d^\ast \) and \( f \in L^2(\mathbb{R}^d) \), the modulation of \( f \) by \( \omega \) is defined by \( M_\omega f(y) = \omega(y)f(y) \) for \( s \in \mathbb{R}^d \). The modulation operator \( M_\omega : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \) is unitary as well. The modulation and translation operators are connected via the Fourier transform by
\( \mathcal{M}_{\omega} f = T_{-\omega} f, \quad \mathcal{T}_{k} f = M_{k} f, \) \hspace{1cm} (2.3)

for all \( f \in L^2(\mathbb{R}^d), \omega \in \mathbb{R}^d, \) and \( k \in \mathbb{R}^d, \) see [24, 38, 52].

From now on and in this article, for a fixed Haar (Lebesgue) measure \( m_2 \) on \( \mathbb{R}^d, \) we mean the induced product measure on \( \mathbb{R}^d \), see [24, 38, 52].

For all \( \lambda \in \mathbb{R}^d , \omega \in \mathbb{R}^d \), the time-frequency shift operator \( \pi_\lambda \): \( L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \) is defined by \( \pi_\lambda f = M_{\omega} T_{\lambda} f. \) Then, it is well-known as the Moyal’s formula, that

\[
\int_{\mathbb{R}^d} |(f, \pi_\lambda g)_{L^2(\mathbb{R}^d)}|^2 d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) = \| f \|_{L^2(\mathbb{R}^d)}^2 \| g \|_{L^2(\mathbb{R}^d)}^2,
\]

for all \( f, g \in L^2(\mathbb{R}^d), \) see [37] and classical references therein.

3. Harmonic analysis over symplectic groups

Throughout this section, we briefly present basics of classical harmonic analysis over the real symplectic group \( \text{Sp}(\mathbb{R}^d) \), for a complete picture of this matrix group we referee the readers to [18–20, 44–46] and the comprehensive list of classical references therein.

For \( d \geq 1 \), let \( \Omega : M_{d \times d}(\mathbb{C}) \rightarrow M_{d \times 2d}(\mathbb{R}) \) be the linear map given by

\[
\Omega(A + iB) := \begin{pmatrix} A & -B \\ B & A \end{pmatrix},
\]

for all \( A, B \in M_{d \times d}(\mathbb{R}). \)

A matrix \( S \in M_{d \times 2d}(\mathbb{R}) \) is called symplectic if and only if \( S^TJS = JS^TJ = J, \) with \( J = \begin{pmatrix} 0 & I_{d \times d} \\ -I_{d \times d} & 0 \end{pmatrix} \) where \( I_{d \times d} \) is \( d \times d \) identity matrix. The group consists of all symplectic matrices is called the (real) symplectic group which is denoted by \( \text{Sp}(\mathbb{R}^d). \) It is a simple non-compact finite-dimensional real Lie group. In block-matrix notation, the symplectic group \( \text{Sp}(\mathbb{R}^d) \) consists of all real \( 2d \times 2d \) matrices in block form

\[
S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \quad A, B, C, D \in M_{d \times d}(\mathbb{R}),
\]
such that \( A^T = C^T, B^T D = D^T B, \) and \( A^T D - C^T B = I_{d \times d}. \)

The real symplectic group \( \text{Sp}(\mathbb{R}^d) \) satisfies the following decomposition, namely Iwasawa (Gram-Schmidt) decomposition, \( \text{Sp}(\mathbb{R}^d) = K_{d}AN, \) where [55, 56]

\[
K_d := \left\{ \Omega(A + iB) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : A + iB \in \text{U}(d, \mathbb{C}) \right\},\]

(3.1)

and

\[
A := \{ \text{diag}(h_1, ..., h_d, h_1^{-1}, ..., h_d^{-1}) : h_1, ..., h_d > 0 \},
\]

(3.2)

and

\[
N := \left\{ \begin{pmatrix} A & B \\ 0 & (A^{-1})^T \end{pmatrix} : A \text{ is unit upper triangular}, \ AB^T = BA^T \right\}.
\]

(3.3)
If we regard elements of $\text{Sp}(\mathbb{R}^d)$ as linear transformations over the vector space (time-frequency phase space) $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$, then the symplectic group $\text{Sp}(\mathbb{R}^d)$ is precisely the group of all linear automorphisms of $\mathbb{R}^d \times \mathbb{R}^d$ which preserve the canonical (symplectic) form. Also, it is easy to check that, if $\mu_{\mathbb{R}^d \times \mathbb{R}^d}$ is the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$, then

$$d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(S \cdot \lambda) = d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda),$$

for all $S \in \text{Sp}(\mathbb{R}^d)$.

A metaplectic operator on $L^2(\mathbb{R}^d)$ is a unitary operator $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ which satisfies the following intertwining identity

$$U\pi(\lambda)^{-1} = \alpha(\lambda)\pi(S \cdot \lambda), \quad (\lambda \in \mathbb{R}^d \times \mathbb{R}^d)$$

(3.4)

for some $S \in \text{Sp}(\mathbb{R}^d)$ and a second degree character $\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T}$.

In coordinate terms, a metaplectic operator on $L^2(\mathbb{R}^d)$ is a unitary operator $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ which satisfies the following intertwining identity

$$UM_tT_xU^{-1} = \alpha(x, \omega)M_{C_{x+d}+T_{A_x+B_x}}((x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d)$$

(3.5)

for some $S \in \text{Sp}(\mathbb{R}^d)$ and a second degree character $\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T}$. In this case, the operator $U$ is called as the metaplectic operator on $L^2(\mathbb{R}^d)$ associated to the symplectic matrix $S$.

For $H \in \text{GL}(d, \mathbb{R})$, the dilation operator $D_H : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is given by

$$D_H f(t) := |\det H|^{-1/2} f(H^{-1} \cdot t),$$

for all $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$.

For $C \in M_{d \times d}(\mathbb{R})$ with $C = C^*$, the chirp multiplication operator $E_C : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by

$$E_C f(t) := \exp(\pi i \cdot t^T C t) f(t),$$

for all $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$.

The following proposition [43] shows that the Fourier transform, dilations, and chirp multiplications can be considered as metaplectic operators.

**Proposition 3.1.** Let $H \in \text{GL}(d, \mathbb{R})$ and $C \in M_{d \times d}(\mathbb{R})$ with $C^T = C$. Then

1. The Fourier transform $\mathcal{F}_{\mathbb{R}^d} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a metaplectic operator on $L^2(\mathbb{R}^d)$ associated to the symplectic matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and satisfies the following intertwining identity

$$\mathcal{F}_{\mathbb{R}^d}\pi(x, \omega)\mathcal{F}_{\mathbb{R}^d}^{-1} = e^{2\pi i x^T \pi(\omega, -x)}$$

2. The dilation operator $D_H : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a metaplectic operator on $L^2(\mathbb{R}^d)$ associated to the symplectic matrix $\begin{pmatrix} H & 0 \\ 0 & (H^T)^{-1} \end{pmatrix}$ and satisfies the following intertwining identity

$$D_H \pi(x, \omega)D_H^{-1} = \pi(H \cdot x, (H^T)^{-1} \cdot \omega)$$

3. The chirp multiplication operator $E_C : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a metaplectic operator on $L^2(\mathbb{R}^d)$ associated to the symplectic matrix $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$ and satisfies the following intertwining identity
Then the following [43] result gives us a unified and also explicit construction of meta-
plectic operators on $L^2(\mathbb{R}^d)$ by splitting them into simple operators given in proposition 3.1.

**Theorem 3.2.** Let $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(\mathbb{R}^d)$ be given. Let $\mathbb{I}_A \subseteq \mathbb{N}_d$ be such that the columns of $A$ indexed by $\mathbb{I}_A$ form a basis for $\mathbb{R}(A)$ and $\Lambda \in M_{d \times d}(\mathbb{Z})$ be the diagonal matrix whose diagonal is 0 at $\mathbb{I}_A$ and 1 at the complementary set $\mathbb{N}_d \setminus \mathbb{I}_A$. Let $H := A + B \Lambda$ and $Q := C + D \Lambda$. Then $H \in \text{GL}(d, \mathbb{R})$ and the unitary operator

$$U_S := E_{QH}^{-1} D_H^{-1} F^{-1} E_{-H}^{-1} y_F F_{\mathbb{R}^d} E_{-\Lambda}$$

is the metaplectic operator associated to the symplectic matrix $S$.

**4. Multivariate metaplectic wave packet representations**

In this section we present the abstract structure of multivariate symplectic wave-packet groups associated to closed subgroups of the real symplectice group $\text{Sp}(\mathbb{R}^d)$. Then we introduce the associated multivariate metaplectic wave-packet representation. We shall also study classical properties of these representations.

For a closed subgroup $\mathbb{H}$ of the real symplectic group $\text{Sp}(\mathbb{R}^d)$, the underlying manifold

$$\mathcal{G}(d, \mathbb{H}) := H \times \mathbb{R}^d \times \mathbb{R}^d = H \times \mathbb{R}^d \times \mathbb{R}^d,$$

equipped with operations given by

$$(S, \lambda) \times (S', \lambda') := (SS', S'^{-1} \cdot \lambda + \lambda'), \quad (4.1)$$

$$(S, \lambda)^{-1} := (S^{-1}, -S \cdot \lambda), \quad (4.2)$$
is a group with the identity element $(1, 0, 0)$.

We call this group as symplectic wave-packet group associated to the subgroup $\mathbb{H}$ over $\mathbb{R}^d$. For simplicity, we may use $\mathcal{G}(\mathbb{H})$ instead of $\mathcal{G}(d, \mathbb{H})$, at times. The groups $\mathcal{H}$ and $\mathbb{R}^d \times \mathbb{R}^d$ can be considered as closed subgroups of $\mathcal{G}(\mathbb{H})$.

Then we present the following theorem concerning basic properties of the symplectic wave-packet group $\mathcal{G}(\mathbb{H})$ in the framework of harmonic analysis.

**Theorem 4.1.** Let $\mathbb{H}$ be a closed subgroup of the symplectic group $\text{Sp}(\mathbb{R}^d)$ with the modular function $\Delta_\mathbb{H}$ and $m_\mathbb{H}$ (resp. $n_\mathbb{H}$) be a left (resp. right) Haar measure of $\mathbb{H}$. Then, $\mathcal{G}(\mathbb{H})$ is a locally compact group with a left Haar measure given by $d\mathcal{G}(\mathbb{H})(S, \lambda) := d\mathbb{H}(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda)$, and a right Haar measure given by $d\mathcal{G}(\mathbb{H})(S, \lambda) := d\mathbb{H}(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda)$.

**Proof.** It can readily be checked that the mapping $\tau : \mathbb{H} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ given by $(S, \lambda) \rightarrow S \cdot \lambda$ is continuous. This automatically implies that the symplectic wave-packet group $\mathcal{G}(\mathbb{H})$ is a locally compact group. Let $F \in \mathcal{C}_c(\mathcal{G}(\mathbb{H}))$ and $g = (S, \lambda) \in \mathcal{G}(\mathbb{H})$. Since the Lebesgue measure $\mu_{\mathbb{R}^d \times \mathbb{R}^d}$ is translation invariant and also $m_\mathbb{H}$ is a left Haar measure on $\mathbb{H}$, we have
\[
\int_{G(\mathbb{H})} F(g \cdot g') \, d\mu_{G(\mathbb{H})} = \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F((S, \lambda) \times (S', \lambda')) \, d\mu_{\mathbb{H}(S')}(\lambda') \, d\mu_{\mathbb{H}(S)}(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F((SS', S^{-1} \cdot \lambda + \lambda') \, d\mu_{\mathbb{H}(S)}(\lambda') \, d\mu_{\mathbb{H}(S)}(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F((SS', S^{-1} \cdot \lambda + \lambda') \, d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S')}(\lambda')
\]
\[
= \int_{\mathbb{H}^n} \left( \int_{\mathbb{R}^n} F(SS', \lambda) \, d\mu_{\mathbb{H}(S)}(\lambda) \right) \, d\mu_{\mathbb{H}(S')}(\lambda')
\]
\[
= \int_{\mathbb{H}^n} \left( \int_{\mathbb{R}^n} F(S', \lambda) \, d\mu_{\mathbb{H}(S')}(\lambda') \right) \, d\mu_{\mathbb{H}(S)}(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F(S', \lambda) \, d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S')}(\lambda')
\]
\[
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F(S', \lambda) \, d\mu_{\mathbb{H}(S')}(\lambda') \, d\mu_{\mathbb{H}(S)}(\lambda) = \int_{G(\mathbb{H})} F(g') \, d\mu_{G(\mathbb{H})}(g'),
\]
which implies that \(d\mu_{G(\mathbb{H})}(S, \lambda) := d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S)}(\lambda)\) is a left Haar measure for \(G(\mathbb{H})\). Similarly, using (3.4), Fubini's theorem and also since the Lebesgue measure \(\mu_{\mathbb{R}^n} \times \mathbb{R}^n\) is translation invariant, we get
\[
\int_{G(\mathbb{H})} F(g' \cdot g) \, d\mu_{G(\mathbb{H})}(g') = \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F((S', \lambda) \times (S, \lambda)) \, d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S')}'(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F(SS', S^{-1} \cdot \lambda + \lambda') \, d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S')}'(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F(SS', S^{-1} \cdot \lambda + \lambda') \, d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S')}'(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \left( \int_{\mathbb{R}^n} F(SS', \lambda) \, d\mu_{\mathbb{H}(S)}(\lambda) \right) \, d\mu_{\mathbb{H}(S')}'(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \left( \int_{\mathbb{R}^n} F(S', \lambda) \, d\mu_{\mathbb{H}(S')}'(\lambda) \right) \, d\mu_{\mathbb{H}(S)}(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F(S', \lambda) \, d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S')}'(\lambda)
\]
\[
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} F(S', \lambda) \, d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S')}'(\lambda) = \int_{G(\mathbb{H})} F(g) \, d\mu_{G(\mathbb{H})}(g'),
\]
 implying that \(d\mu_{G(\mathbb{H})}(S, \lambda) := d\mu_{\mathbb{H}(S)}(\lambda) \, d\mu_{\mathbb{H}(S)}(\lambda)\) is a right Haar measure for \(G(\mathbb{H})\). \qed
Next we deduce the following consequences.

**Corollary 4.2.** Let \( \mathbb{H} \) be a closed subgroup of the symplectic group \( \text{Sp}(\mathbb{R}^d) \) with the modular function \( \Delta_{\mathbb{H}} \) and \( m_{\mathbb{H}} \) (resp. \( n_{\mathbb{H}} \)) be a left (resp. right) Haar measure of \( \mathbb{H} \). Then

1. The modular function \( \Delta_{\mathbb{G}(\mathbb{H})} : \mathbb{G}(\mathbb{H}) \to (0, \infty) \) is given by \( \Delta_{\mathbb{G}(\mathbb{H})}(S, \lambda) := \Delta_{\mathbb{H}}(S) \). In particular, the symplectic wave-packet group \( \mathbb{G}(\mathbb{H}) \) is unimodular if and only if \( \mathbb{H} \) is unimodular.
2. The closed subgroup \( \mathbb{H} \) is normal in \( \mathbb{G}(\mathbb{H}) \) if and only if \( \mathbb{H} = \{1\} \).
3. The closed subgroup \( \mathbb{R}^d \times \mathbb{R}^d / (\mathbb{Z}) \) is a normal Abelian subgroup of \( \mathbb{G}(\mathbb{H}) \).

**Proof.**

(1) Let \( F \in \mathcal{C}_c(\mathbb{G}(\mathbb{H})) \) be a non-zero and positive function. Also, let \( (S, \lambda) \in \mathbb{G}(\mathbb{H}) \). Then we can write

\[
\Delta_{\mathbb{G}(\mathbb{H})}(S, \lambda)^{-1} \int_{\mathbb{G}(\mathbb{H})} F(S', \lambda') dm_{\mathbb{G}(\mathbb{H})}(S', \lambda') = \int_{\mathbb{G}(\mathbb{H})} F(S', \lambda) \times (S, \lambda) dm_{\mathbb{G}(\mathbb{H})}(S', \lambda)
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(S', \lambda) \times (S, \lambda) dm_{\mathbb{G}(\mathbb{H})}(S') d\mu_{\mathbb{R}^d}(\lambda)
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(S' S^{-1} \cdot \lambda' + \lambda) dm_{\mathbb{G}(\mathbb{H})}(S') d\mu_{\mathbb{R}^d}(\lambda)
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(S S^{-1} \cdot \lambda' + \lambda) dm_{\mathbb{G}(\mathbb{H})}(S') d\mu_{\mathbb{R}^d}(\lambda)
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(S S^{-1} \cdot \lambda) dm_{\mathbb{G}(\mathbb{H})}(S') d\mu_{\mathbb{R}^d}(\lambda)
\]

\[
= \Delta_{\mathbb{G}(\mathbb{H})}(S)^{-1} \int_{\mathbb{R}^d} F(S' \cdot \lambda) dm_{\mathbb{G}(\mathbb{H})}(S') d\mu_{\mathbb{R}^d}(\lambda)
\]

implying that \( \Delta_{\mathbb{G}(\mathbb{H})}(S, \lambda) = \Delta_{\mathbb{H}}(S) \) for all \( (S, \lambda) \in \mathbb{G}(\mathbb{H}) \).

(2) and (3) are straightforward from structure of the symplectic wave-packet group \( \mathbb{G}(\mathbb{H}) \).

**Remark 4.3.** From now on, once the left (resp. right) Haar measure \( m_{\mathbb{H}} \) (resp. \( n_{\mathbb{H}} \)) over \( \mathbb{H} \) is fixed, we call the associated left (resp. right) Haar measure on the symplectic wave-packet group \( \mathbb{G}(\mathbb{H}) \), which is constructed via theorem 4.1, as left (resp. right) Haar measure induced by \( m_{\mathbb{H}} \) (resp. \( n_{\mathbb{H}} \)).

For \( g = (S, \lambda) = (A, x, \omega) \in \mathbb{G}(\mathbb{H}) \), define the linear operator \( \Gamma_{\mathbb{G}(\mathbb{H})}(g) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) by

\[
\Gamma_{\mathbb{G}(\mathbb{H})}(g) := U_3 \pi(\lambda) = U_3 T_{\omega}.
\]

(4.3)
The following theorem shows that \( g \mapsto \Gamma_\mathcal{H}(g) \) given by (4.3), defines an irreducible projective group representation of the symplectic wave-packet group \( \mathcal{G}(\mathbb{H}) \) on the Hilbert function space \( L^2(\mathbb{R}^d) \).

**Theorem 4.4.** Let \( \mathbb{H} \) be a closed subgroup of the symplectic group \( \text{Sp}(\mathbb{R}^d) \) and \( \mathcal{G}(\mathbb{H}) \) be the associated symplectic wave-packet group. Then \( \Gamma_\mathcal{H} : \mathcal{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d)) \) given by \( g \mapsto \Gamma_\mathcal{H}(g) \) is an irreducible projective group representation of the locally compact group \( \mathcal{G}(\mathbb{H}) \) on the Hilbert function space \( L^2(\mathbb{R}^d) \).

**Proof.** Plainly, we have \( \Gamma_\mathcal{H}(1, 0, 0) = I_{L^2(\mathbb{R}^d)} \) where \( I : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is the identity operator. Let \((S, \lambda), (S', \lambda') \in \mathcal{G}(\mathbb{H})\). Invoking definition of \( \Gamma_\mathcal{H}(S, \lambda) \) it is evident to check that \( \Gamma_\mathcal{H}(S, \lambda) \) is a unitary operator, because it is the composition of two unitary operators, namely \( U_S \) and \( \pi(\lambda) \). Let \( \beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T} \) be a second degree character such that the intertwining identity (3.5) holds for \( \mathcal{S}' \). Hence, we get

\[
U_S \pi(S' \cdot \lambda) = \beta(S' \cdot \lambda) \pi(S' \cdot (S' \cdot \lambda)) U_S = \beta(S' \cdot \lambda) \pi(\lambda) U_S.
\]

Also, the operator \( U_S U' \) is a metaplectic operator associated to \( \mathcal{S}' \). Thus, there exists a complex number \( z(S, S') \in \mathbb{T} \) such that

\[
U_S \pi(S' \cdot \lambda + \lambda') = z(S, S') U_S U' \pi(S' \cdot \lambda + \lambda')
\]

\[
= z(S, S') U_S U' \pi(S' \cdot \lambda) \pi(\lambda') = z(S, S') \beta(S' \cdot \lambda) U_S \pi(\lambda) U_S \pi(\lambda').
\]

Therefore, we get

\[
\Gamma_\mathcal{H}(S, \lambda) \otimes (S', \lambda') = \Gamma_\mathcal{H}(S S', S' \cdot \lambda + \lambda')
\]

\[
= U_S \pi(S' \cdot \lambda + \lambda')
\]

\[
= z(S, S') \beta(S' \cdot \lambda) U_S \pi(\lambda) U_S \pi(\lambda') = z(S, S') \beta(S' \cdot \lambda) \Gamma_\mathcal{H}(S', \lambda'),
\]

which implies that \( \Gamma_\mathcal{H} : \mathcal{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d)) \) is a projective group representation of the locally compact group \( \mathcal{G}(\mathbb{H}) \) on the Hilbert function space \( L^2(\mathbb{R}^d) \). Since restriction of \( \Gamma_\mathcal{H} \) to the closed subgroup \( \mathbb{R}^d \times \mathbb{R}^d \) is equivalent with the projective Shrödinger representation of the subgroup \( \mathbb{R}^d \times \mathbb{R}^d \) on \( L^2(\mathbb{R}^d) \), we deduce that \( \Gamma_\mathcal{H} \) is irreducible on \( L^2(\mathbb{R}^d) \) as well. \( \square \)

**Remark 4.5.**

(i) The restriction of the metaplectic wave-packet representation to the closed subgroup \( \mathbb{R}^d \times \mathbb{R}^d \) is unitarily equivalent to the projective Schrödinger representation of \( \mathbb{R}^d \times \mathbb{R}^d \) on \( L^2(\mathbb{R}^d) \), see [37] and references therein.

(ii) Let \( \mathbb{H} \) be a closed subgroup of the symplectic group \( \text{Sp}(\mathbb{R}^d) \) which contains \( \text{GL}(d, \mathbb{R}) \).

Then the restriction of the metaplectic wave-packet representation to the closed subgroup \( \text{GL}(d, \mathbb{R}) \times \mathbb{R}^d \times \mathbb{R}^d \) is unitarily equivalent to the classic wave-packet representation associated to the action of the multiplicative matrix group \( \text{GL}(d, \mathbb{R}) \) on the time-frequency plan \( \mathbb{R}^d \times \mathbb{R}^d \), see [28, 42, 57, 58] and the comprehensive list of references therein.
5. Square-integrability of multivariate metaplectic wave-packet representations

Throughout this section, we study the square-integrability of multivariate metaplectic wave-packet representations. We still assume that $\mathbb{H}$ is a closed subgroup of the symplectic group $\text{Sp}(\mathbb{R}^d)$.

It should be mentioned that in the framework of classical voice/coherent state transforms [59], the problem of admissibility conditions for subgroups of the symplectic group studied from an algebraic perspective in [1, 2, 12, 13, 17, 21].

Let $\psi \in L^2(\mathbb{R}^d)$ be a window function. The metaplectic wave-packet transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function $\psi$ is given by the voice transform associated to the metaplectic wave-packet representation, that is

$$V_{\psi}f(S, x, \omega) := \langle f, \Gamma_\mathbb{H}(S, x, \omega)\psi \rangle_{L^2(\mathbb{R}^d)} = \langle f, U_{S}T_{M_x}\psi \rangle_{L^2(\mathbb{R}^d)},$$

(5.1)

for $(S, x, \omega) \in \mathbb{H} \times \mathbb{R}^d \times \mathbb{R}^d$.

**Remark 5.1.**

(i) The restriction of the metaplectic wave-packet transform to the closed subgroup $\mathbb{R}^d \times \mathbb{R}^d$ is the continuous Gabor (short-time Fourier) transform over $L^2(\mathbb{R}^d)$, see [37] and references therein.

(ii) Let $\mathbb{H}$ be a closed subgroup of $\text{Sp}(\mathbb{R}^d)$ which contains $\text{GL}(d, \mathbb{R})$. Then the restriction of the metaplectic wave-packet transform to the closed subgroup $\text{GL}(d, \mathbb{R}) \times \mathbb{R}^d \times \mathbb{R}^d$ is the classic wave-packet transform induced by the action of the multiplicative matrix group $\text{GL}(d, \mathbb{R})$ on the time-frequency plane $\mathbb{R}^d \times \mathbb{R}^d$, see [28] and the comprehensive list of references therein.

The following theorem can be considered as a constructive topological criterion on the closed subgroup $\mathbb{H}$, which guarantees the square-integrability of the associated metaplectic wave-packet representation $\Gamma_\mathbb{H}$ on the Hilbert function space $L^2(\mathbb{R}^d)$.

**Theorem 5.2.** Let $\mathbb{H}$ be a closed subgroup of the real symplectic group $\text{Sp}(\mathbb{R}^d)$ and $G(\mathbb{H})$ be the associated symplectic wave-packet group. Then, the metaplectic wave-packet representation $\Gamma_{\mathbb{H}} : L^2(\mathbb{R}^d) \rightarrow (L^2(\mathbb{R}^d))$ is left (resp. right) square-integrable over the symplectic wave-packet group $G(\mathbb{H})$ if and only if $\mathbb{H}$ is compact. In this case, all non-zero functions in the Hilbert function space $L^2(\mathbb{R}^d)$ are square-integrable over $G(\mathbb{H})$ with respect to $\Gamma_{\mathbb{H}}$.

**Proof.** Let $m_\mathbb{H}$ be a left Haar measure for $\mathbb{H}$. Then by theorem 4.1, the positive Radon measure $m_{G(\mathbb{H})}$ given by $\text{dm}_{G(\mathbb{H})}(S, \lambda) = m_{\mathbb{H}}(S)\text{dm}_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda)$ is a left Haar measure for the symplectic wave-packet group $G(\mathbb{H})$. Now, suppose that the metaplectic wave-packet representation $\Gamma_{\mathbb{H}}$ be left square-integrable over $G(\mathbb{H})$. Then, there exists a non-zero function $\psi \in L^2(\mathbb{R}^d)$ such that

$$\int_{G(\mathbb{H})} |\langle \psi, \Gamma_{\mathbb{H}}(g)\psi \rangle_{L^2(\mathbb{R}^d)}|^2 \text{dm}_{G(\mathbb{H})}(g) < \infty.$$ 

Then, using Fubini’s theorem and also the Moyal’s formula (2.4), we get
\[ \int_{G(\mathbb{H})} |\langle \psi, \Gamma_G(g) \psi \rangle_{L^2(\mathbb{R}^d)}|^2 \, dm_{G(\mathbb{H})}(g) = \int_{G(\mathbb{H})} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |\langle \psi, \Gamma_G(S, \lambda) \psi \rangle_{L^2(\mathbb{R}^d)}|^2 \, dm_{G(\mathbb{H})}(S) \right) \, dm_{G(\mathbb{H})}(g). \]

Since metaplectic operators are unitary on \(L^2(\mathbb{R}^d)\), we deduce that

\[ \|\psi\|_{L^2(\mathbb{R}^d)}^2 \left( \int_{\mathbb{R}^d} dm_{G(\mathbb{H})} \right) = \|\psi\|_{L^2(\mathbb{R}^d)}^2 \left( \int_{\mathbb{R}^d} \|U^* \psi\|_{L^2(\mathbb{R}^d)}^2 dm_{G(\mathbb{H})} \right) \]

which implies the square-integrability of the metaplectic wave-packet representation \(\Gamma_{\mathbb{H}}\) over the symplectic wave-packet group \(G(\mathbb{H})\).

Thus \(m_{G(\mathbb{H})} < \infty\) and hence \(\mathbb{H}\) is compact. Conversely, let \(\mathbb{H}\) be a compact subgroup of \(\text{Sp}(\mathbb{R}^d)\) with the probability Haar measure \(\sigma_{\mathbb{H}}\), that is the unique positive Radon measure \(\sigma_{\mathbb{H}}\) which is both left and right Haar measure of \(\mathbb{H}\) with \(\sigma_{\mathbb{H}}(\mathbb{H}) = 1\). Then, each non-zero function \(\psi \in L^2(\mathbb{R}^d)\) satisfies

\[ \int_{G(\mathbb{H})} |\langle \psi, \Gamma_{\mathbb{H}}(g) \psi \rangle_{L^2(\mathbb{R}^d)}|^2 \, dm_{G(\mathbb{H})}(g) < \infty. \]

As a consequence of theorem 5.2, we deduce the following orthogonality relation concerning the metaplectic wave-packet transforms.

**Corollary 5.3.** Let \(\mathbb{H}\) be a compact subgroup of the real symplectic group \(\text{Sp}(\mathbb{R}^d)\) with the probability Haar measure \(\sigma_{\mathbb{H}}\) and \(G(\mathbb{H})\) be the associated metaplectic wave-packet group with the induced Haar measure \(m_{G(\mathbb{H})}\) by \(\sigma_{\mathbb{H}}\). Also, let \(\psi, \varphi \in L^2(\mathbb{R}^d)\) be non-zero window functions and \(f, g \in L^2(\mathbb{R}^d)\). Then, we have

\[ \langle \varphi, f \rangle_{L^2(\mathbb{R}^d, m_{G(\mathbb{H})})} = \langle \varphi, \psi \rangle_{L^2(\mathbb{R}^d)} \langle f, \psi \rangle_{L^2(\mathbb{R}^d)}. \]

**Proof.** The same argument used in theorem 5.2 implies that

\[ ||\mathcal{V}_{\varphi}||_{L^2(\mathbb{R}^d, m_{G(\mathbb{H})})}^2 = ||\psi||_{L^2(\mathbb{R}^d)}^2 ||f||_{L^2(\mathbb{R}^d)}^2. \]
Then (5.4) and also twice applying the Polarization identity guarantees (5.3). □

Next result is an inversion (reconstruction) formula for the metaplectic wave-packet transform defined by (5.1).

**Theorem 5.4.** Let $H$ be a compact subgroup of the real symplectic group $\text{Sp}(\mathbb{R}^d)$ with the probability Haar measure $\sigma_H$ and $G(H)$ be the associated symplectic wave-packet group with the induced Haar measure $m_{G(H)}$ by $\sigma_H$. Also, let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. Then, each function $f \in L^2(\mathbb{R}^d)$ can be recovered continuously in the weak sense of the Hilbert function space $L^2(\mathbb{R}^d)$, from metaplectic wave-packet coefficients generated by $\psi$, via the following resolution of the identity formula:

$$f = \|\psi\|_{L^2(\mathbb{R}^d)}^2 \int_\mathbb{H} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{V}_V f(S, \lambda) \Gamma_{H}(S, \lambda) \psi \, d\sigma_H(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda).$$

(5.5)

**Proof.** Let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. For $f \in L^2(\mathbb{R}^d)$, define

$$f(\psi) : = \int_\mathbb{H} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{V}_V f(S, \lambda) \Gamma_{H}(S, \lambda) \psi \, d\sigma_H(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda),$$

in the weak sense of the Hilbert function space $L^2(\mathbb{R}^d)$. Using (5.3), for all $g \in L^2(\mathbb{R}^d)$, we have

$$\langle f(\psi), g \rangle_{L^2(\mathbb{R}^d)} = \int_\mathbb{H} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{V}_V f(S, \lambda) \Gamma_{H}(S, \lambda) \psi \, d\sigma_H(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) \langle g, h \rangle_{L^2(\mathbb{R}^d)}$$

$$= \int_\mathbb{H} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{V}_V f(S, \lambda) \Gamma_{H}(S, \lambda) \psi \, d\sigma_H(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda)$$

$$= \int_\mathbb{H} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{V}_V f(S, \lambda) \Gamma_{H}(S, \lambda) \psi \, d\sigma_H(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda)$$

$$= \langle \mathcal{V}_V f, \mathcal{V}_V g \rangle_{L^2(G(H), m_{G(H)})} = \|\psi\|_{L^2(\mathbb{R}^d)}^2 \langle f, g \rangle_{L^2(\mathbb{R}^d)}.$$

Then $f(\psi) \in L^2(\mathbb{R}^d)$ and $f(\psi) = \|\psi\|_{L^2(\mathbb{R}^d)}^2 f$ in $L^2(\mathbb{R}^d)$, which equivalently implies the reconstruction formula (5.5) in the weak sens of the Hilbert function space $L^2(\mathbb{R}^d)$. □

Then we can present the following reproducing property for the metaplectic wave-packet representations.

**Corollary 5.5.** Let $H$ be a compact subgroup of the real symplectic group $\text{Sp}(\mathbb{R}^d)$ with the probability Haar measure $\sigma_H$ and $G(H)$ be the associated symplectic wave-packet group with the induced Haar measure $m_{G(H)}$ by $\sigma_H$. Let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function and $H$ be range of the metaplectic wave-packet transform $\mathcal{V}_V : L^2(\mathbb{R}^d) \rightarrow L^2(G(H), m_{G(H)})$. Then

(1) $H$ is a closed subspace of $L^2(G(H)), m_{G(H)})$.

(2) $H$ is the unique reproducing kernel Hilbert space (RKHS) over $G(H)$ associated to the positive definite kernel $K_\psi : G(H) \times G(H) \rightarrow \mathbb{C}$ given by

$$K_\psi(S, \lambda, (S', \lambda')) := \langle U_\psi \pi(\lambda) \psi, U_{S'} \pi(\lambda') \psi \rangle_{L^2(\mathbb{R}^d)},$$

for all $(S, \lambda), (S', \lambda') \in G(H)$.

Next corollary summarizes our recent results in terms of continuous frame theory [8, 53].

**Corollary 5.6.** Let $H$ be a compact subgroup of the real symplectic group $\text{Sp}(\mathbb{R}^d)$ and $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. Then the multivariate wave-packet system
\[ \mathfrak{A}(\mathbb{H}, \psi) := \{ \Gamma_{\mathbb{H}}(S, \lambda) \psi : (S, \lambda) \in G(\mathbb{H}) \}, \]

is a continuous tight frame for the Hilbert space \( L^2(\mathbb{R}^d) \).

6. Analysis of multivariate metaplectic wave-packet representations over compact subgroups of the real symplectic group \( \text{Sp}(\mathbb{R}^d) \)

Throughout this section, we study analytic aspects of compact subgroups of the real symplectic group \( \text{Sp}(\mathbb{R}^d) \) in the framework of coherent state metaplectic wave-packet analysis.

As it is proved in theorem 5.2, just compact subgroups of the real symplectic group \( \text{Sp}(\mathbb{R}^d) \) are interesting from the \( L^2 \)-theory and reproducing property of metaplectic wave-packet representations. Roughly speaking, only compact subgroups of \( \text{Sp}(\mathbb{R}^d) \) are highly important in the framework of coherent state metaplectic wave-packet analysis over the Hilbert function space \( L^2(\mathbb{R}^d) \), since they guarantee that the associated metaplectic wave-packet transforms over \( L^2(\mathbb{R}^d) \) satisfy resolution of the identity formulas which are valid in the weak sense of the Hilbert function space \( L^2(\mathbb{R}^d) \).

6.1. The case \( d = 1 \)

In this case [26], the real symplectic group \( \text{Sp}(\mathbb{R}) \) is precisely the special linear group \( \text{SL}(2, \mathbb{R}) \), that is the the multiplicative matrix group, consists of all real \( 2 \times 2 \) matrices with determinant one. That is,

\[ \text{SL}(2, \mathbb{R}) := \left\{ S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}. \]

It is a simple real 3-dimensional Lie group. The special linear group \( \text{SL}(2, \mathbb{R}) \) satisfies the following decomposition, namely Iwasawa (Gram-Schmidt) decomposition, \( \text{SL}(2, \mathbb{R}) = KAN \) where \( K = \text{SO}(2) \) is the special orthogonal group consists of all \( 2 \times 2 \)-orthogonal matrices with real entries and the subgroups \( A, N \) are given by

\[ A = \left\{ D(h) := \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} : h > 0 \right\}, \quad N = \left\{ N(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}. \]

The group \( \text{SL}(2, \mathbb{R}) \) is non-compact but unimodular. A Haar measure of \( \text{SL}(2, \mathbb{R}) \) is given by

\[ \phi \mapsto \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\infty} \phi \left( \begin{pmatrix} x \sqrt{y} & \cos \theta \sin \theta \\ 0 & \cos \theta \end{pmatrix} \right) d\theta dy^{-\frac{1}{2}} d\lambda, \]

for all \( \phi \in C_c(\text{SL}(2, \mathbb{R})) \).

6.1. Continuous compact subgroups of \( \text{SL}(2, \mathbb{R}) \). The subgroup \( \mathbb{H} = \text{SO}(2) \) is the most significant compact subgroup of \( \text{SL}(2, \mathbb{R}) \). The compact subgroup \( \text{SO}(2) \) is the multiplicative matrix group consists of all \( 2 \times 2 \)-orthogonal matrices with unit determinant. That is, \( \text{SO}(2) = \{ H(\theta) : 0 < \theta \leq 2\pi \} \), where

\[ H(\theta) := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \]
The subgroup $\text{SO}(2)$ is isomorphic, as a real Lie group, to the circle group, also known as $\mathbb{T} = \text{U}(1)$, via the canonical Lie group isomorphism which sends the complex number $e^{i\theta}$ of absolute value 1, to the special orthogonal matrix $H(\theta)$. From now on, we may call $\text{SO}(2)$ as the circle group, at times. It can be readily checked that, any closed subgroup of $\text{SL}(2, \mathbb{R})$ conjugated to $\text{SO}(2)$ is also compact in $\text{SL}(2, \mathbb{R})$. In addition, the circle group $\text{SO}(2)$ is a maximal compact subgroup of the multiplicative matrix Lie group $\text{SL}(2, \mathbb{R})$, which means that $\text{SO}(2)$ is a compact subgroup and it is maximal among such subgroups as well. Thus, any continuous (non-discrete) and compact subgroup is one-dimensional. Then by proposition 3.2 of [45], it is conjugated to the compact subgroup $\text{SO}(2)$.

\textbf{(i) The circle group.} By the above argument and theoretical motivation, first we shall focus on analytic and constructive analysis of metaplectic wave-packet representations over the compact subgroup $\text{SO}(2)$.

The normalized Haar measure $\sigma_{\text{SO}(2)}$ of the circle group $\text{SO}(2)$ is given by

$$
\int_{\text{SO}(2)} \phi(S) d\sigma_{\text{SO}(2)}(S) = (2\pi)^{-1} \int_0^{2\pi} \phi(H(\theta)) d\theta, \quad (6.1)
$$

for all $\phi \in C(\text{SO}(2))$.

The following theorem characterizes analytic aspects of the metaplectic wave-packet representation associated to the compact subgroup $\text{SO}(2)$.

\textbf{Theorem 6.1.} Let $0 < \theta \leq 2\pi$ and $U_\theta := U_{H(\theta)}$ be the associated metaplectic operator to $H(\theta)$.

\begin{enumerate}
\item For $\theta = \pi/2, 3\pi/2$, we have $U_\theta = E_{-\tan a}D_{\cos \theta}E_{-\tan \theta} E_{\tan \theta} E_{\tan \theta}$.
\item For $\theta = \pi/2$, we have $U_{\pi/2} = E_{-1}E_{\tan \theta} E_{\tan \theta} E_{-1}$.
\item For $\theta = 3\pi/2$, we have $U_{3\pi/2} = E_{-1}D_1 E_{-1}E_{-1}E_{-1}$.
\end{enumerate}

\textbf{Proof.}

\begin{enumerate}
\item Let $0 < \theta \leq 2\pi$ with $\theta = \pi/2, 3\pi/2$. Then $a := \cos \theta = 0$. Hence, using theorem 3.2 with $a = d$ and $b := \sin \theta = -e$, we get $U_\theta = E_{-a^{-1}b}E_{-e^{-1}b}E_{-a^{-1}b} E_{-e^{-1}b} E_{-e^{-1}b} E_{-e^{-1}b}$.
\item and (3) are straightforward from theorem 3.2.
\end{enumerate}

Also, we can deduce the following result.

\textbf{Proposition 6.2.} $\mathcal{G}(\text{SO}(2))$ is a non-Abelian, non-compact, and unimodular group with a Haar measure given by

$$
\int_{\mathcal{G}(\text{SO}(2))} F(S, \lambda) dm_{\mathcal{G}(\text{SO}(2))}(S, \lambda) = (2\pi)^{-1} \int_0^{2\pi} \int_{\mathbb{R}} F(H(\theta), \lambda) d\theta d\mu_{\mathbb{R} \times \mathbb{R}}(\lambda),
$$

for all $F \in \mathcal{C}(\mathcal{G}(\text{SO}(2)))$.

Let $\psi \in L^2(\mathbb{R})$ be a non-zero window function. The metaplectic wave-packet transform can be regarded as $\mathcal{V}_0 : \mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2((0, 2\pi] \times \mathbb{R} \times \mathbb{R})$ given by $f \mapsto \mathcal{V}_0 f$, where

$$
\mathcal{V}_0 f(\theta, x, \omega) := \langle f, U_\theta M_{T_x} \psi \rangle_{L^2(\mathbb{R})}, \quad (6.2)
$$

for all $(\theta, x, \omega) \in (0, 2\pi] \times \mathbb{R} \times \mathbb{R}$.

The Plancherel formula for (6.2) reads as follows;
\[
\int_{0}^{2\pi} \int_{\mathbb{R} \times \mathbb{R}} |\langle f, U_{\mathcal{M}_{x}T_{\psi}} \rangle|_{L^2(\mathbb{R})}^2 \, d\theta d\mu_{\mathbb{R} \times \mathbb{R}}(x, \omega) = (2\pi) \cdot \|f\|_{L^2(\mathbb{R})}^2 \cdot \|\psi\|_{L^2(\mathbb{R})}^2. \tag{6.3}
\]

Then (6.3) guarantees the following reconstruction formula;
\[
f = (2\pi)^{-1} \cdot \|\psi\|_{L^2(\mathbb{R})}^2 \cdot \int_{\mathbb{R} \times \mathbb{R}} V_{\psi} f(\theta, x, \omega) U_{\theta \mathcal{M}_{x}T_{\psi}} \, d\theta d\mu_{\mathbb{R} \times \mathbb{R}}(x, \omega). \tag{6.4}
\]

### 6.1.2. Finite subgroups of $\mathbf{SL}(2, \mathbb{R})$

Since every subgroup of the circle group is either dense or finite, we deduce that any closed proper subgroup of the circle group is finite.

Let $N \in \mathbb{N}$ be a positive integer and $\mathbb{T}_N := \{ z \in \mathbb{T} : z^N = 1 \}$. Then $\mathbb{T}_N$ is a finite subgroup of $\mathbb{T}$ of order $N$. One can also check that, $SO_N(2) := \{ H(2\pi k/N) : k = 0, \ldots, N - 1 \}$, is a finite subgroup of $SO(2)$ of order $N$. Also, it is easy to check that any finite subgroup of $SL(2, \mathbb{R})$ of order $N$ is conjugated to $SO_N(2)$.

**(i) Finite circle groups** Let $N \in \mathbb{N}$ be a positive integer. The normalized Haar measure of $SO_N(2)$ is given by
\[
\int_{SO_N(2)} \phi(S) d\sigma_{SO_N(2)}(S) := \frac{1}{N} \sum_{k=0}^{N-1} \phi(H(2\pi k/N)),
\]
for all $\phi : SO_N(2) \rightarrow \mathbb{C}$.

**Proposition 6.3.** Let $N \in \mathbb{N}$ be a positive integer. Then $\mathbb{G}(SO_N(2))$ is a non-Abelian, non-compact, and unimodular group with a Haar measure given by
\[
\int_{\mathbb{G}(SO_N(2))} F(S, \lambda) d\mu_{\mathbb{G}(SO_N(2))}(S, \lambda) = \frac{1}{N} \sum_{k=0}^{N-1} \int_{\mathbb{R} \times \mathbb{R}} F(H(2\pi k/N), \lambda) d\mu_{\mathbb{R} \times \mathbb{R}}(\lambda),
\]
for all $F \in \mathcal{C}_c(\mathbb{G}(SO_N(2)))$.

Let $\psi \in L^2(\mathbb{R})$ be a non-zero window function. The metaplectic wave-packet transform can be regarded as $V_{\psi} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{Z}_N \times \mathbb{R} \times \mathbb{R})$ given by $f \mapsto V_{\psi} f$, where
\[
V_{\psi} f(k, x, \omega) := \langle f, U_{2\pi k/N \mathcal{M}_{x}T_{\psi}} \rangle_{L^2(\mathbb{R})}, \tag{6.5}
\]
for all $(k, x, \omega) \in \mathbb{Z}_N \times \mathbb{R} \times \mathbb{R}$.

The Plancherel formula for (6.5) reads as follows;
\[
\sum_{k=0}^{N-1} \int_{\mathbb{R} \times \mathbb{R}} |\langle f, U_{2\pi k/N \mathcal{M}_{x}T_{\psi}} \rangle|_{L^2(\mathbb{R})}^2 d\mu_{\mathbb{R} \times \mathbb{R}}(x, \omega) = N \cdot \|f\|_{L^2(\mathbb{R})}^2 \cdot \|\psi\|_{L^2(\mathbb{R})}^2. \tag{6.6}
\]

Then (6.6) guarantees the following reconstruction formula;
\[
f = N^{-1} \cdot \|\psi\|_{L^2(\mathbb{R})}^2 \sum_{k=0}^{N-1} \int_{\mathbb{R} \times \mathbb{R}} V_{\psi} f(k, x, \omega) U_{2\pi k/N \mathcal{M}_{x}T_{\psi}} \, d\mu_{\mathbb{R} \times \mathbb{R}}(x, \omega). \tag{6.7}
\]

### 6.2. The case $d > 1$

It is well-known that $K_d$ is the maximal compact subgroup of the real symplectic group $Sp(\mathbb{R}^d)$, see [18–20, 45] and the classical list of references therein. Also, it can readily be check that
Theorem 6.4. Let \( S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K_d \) be given. Let \( I_A \subseteq \mathbb{N}_d \) be such that the columns of \( A \) indexed by \( I_A \) form a basis for \( \mathcal{R}(A) \) and \( \Lambda \in M_{d \times d}(\mathbb{Z}) \) be the diagonal matrix whose diagonal is 0 at \( I_A \) and 1 at the complementary set \( \mathbb{N}_d \setminus I_A \). Let \( H := A - B\Lambda \) and \( Q := B + A\Lambda \). Then \( H \in \text{GL}(d, \mathbb{R}) \) and the unitary operator

\[
U_S := E_{QH}^{-1}D_H^{-1}E_{-H^{-1}B}E_{-\Lambda} \tag{6.8}
\]

is the metaplectic operator associated to the symplectic matrix \( S \).

Next we can also present the following characterizations.

Corollary 6.5. Let \( d > 1 \) and \( S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K_d \).

1. If \( A \in \text{GL}(d, \mathbb{R}) \) we have \( U_S = E_{BA}^{-1}S^{-1}E_{A^{-1}B}E_{-\Lambda} \).
2. If \( A = 0 \) then \( B \in \text{O}(d, \mathbb{R}) \) and we have \( U_S = E_{H}D\_B^{-1}E_{-1}F_{\mathbb{R}^d}E_{-1} \).
3. If \( B = 0 \) then \( A \in \text{O}(d, \mathbb{R}) \) and we have \( U_S = D_H \).

Proof. Let \( d > 1 \) and \( S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K_d \).

1. Let \( A \in \text{GL}(d, \mathbb{R}) \). Then \( \Lambda = 0 \) and hence \( H = A \) and \( Q = B \). Thus, using theorem 6.4, we deduce that

\[
U_S = E_{QH}^{-1}D_H^{-1}E_{-H^{-1}B}E_{-\Lambda} = E_{BA}^{-1}S^{-1}E_{A^{-1}B}E_{-\Lambda}.
\]

2. Let \( A = 0 \). Then \( \Lambda = I \). Also, since \( A^T + BB^T = I \) and \( A^T A + B^T B = I \), we get \( B^T B = BB^T = I \). Hence, \( B \in \text{O}(d, \mathbb{R}) \) and \( -H = Q = B \). Thus, using theorem 6.4, we deduce that

\[
U_S = E_{QH}^{-1}D_H^{-1}E_{-H^{-1}B}E_{-\Lambda} = E_{BA}^{-1}S^{-1}E_{A^{-1}B}E_{-\Lambda}.
\]

3. Let \( B = 0 \). Since \( A^T + BB^T = I \) and \( A^T A + B^T B = I \), we get \( A^T A = AA^T = I \). Therefore, \( A \in \text{O}(d, \mathbb{R}) \) and hence \( \Lambda = 0 \). Then, \( H = A \) and \( Q = 0 \). Thus, using theorem 6.4, we deduce that

\[
U_S = E_{QH}^{-1}D_H^{-1}E_{-H^{-1}B}E_{-\Lambda} = D_H.
\]

6.2.1. The maximal compact subgroup \( K_d \). Let \( H = K_d \) be the maximal compact subgroup of the real symplectic group \( \text{Sp}(\mathbb{R}^d) \) and \( \sigma_{K_d} \) be the probability measure over the compact group \( K_d \). In this case, the associated multivariate symplectic wave-packet group \( G(H) \) is the underlying manifold \( K_d \times \mathbb{R}^d \times \mathbb{R}^d \), equipped with the following group law

\[(S, \lambda) \rtimes (S', \lambda') = (SS', S'^{-1}\lambda + \lambda'),\]

for all \((S, \lambda), (S', \lambda') \in G(H)\). Then \( d\mu_{G(H)}(S, \lambda) = d\sigma_{O(d)}(S)d\mu_{\mathbb{R}^d}E_{\mathbb{R}^d}(\lambda) \) is a Haar measure for the symplectic wave-packet group \( G(H) \). The multivariate symplectic wave-packet representation \( \Gamma_H : G(H) \to \mathcal{U}(L^2(\mathbb{R}^d)) \) is given by \( \Gamma_H(S, \lambda) = U_S\pi(\lambda) \) for all \((S, \lambda) \in G(H)\).
The multivariate metaplectic wave-packet transform of \( f \in L^2(\mathbb{R}^d) \) with respect to the window function \( \psi \), is given by
\[
\mathcal{V}_\psi f(S, \lambda) = \langle f, \Gamma_\psi(S, \lambda) \psi \rangle_{L^2(\mathbb{R}^d)} = \langle f, U_\lambda \pi(\psi) \rangle_{L^2(\mathbb{R}^d)},
\]
for all \((S, \lambda) \in \mathbb{G}(\mathbb{H})\). Then, corollary 5.3 guarantees the following Plancherel formula
\[
\int_{K_d} \int_{\mathbb{R}^d} |\langle f, \Gamma_\psi(S, \lambda) \psi \rangle_{L^2(\mathbb{R}^d)}|^2 \, d\sigma_{K_d}(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) = \| \psi \|_{L^2(\mathbb{R}^d)}^2 \| f \|_{L^2(\mathbb{R}^d)}^2,
\]
which is equivalent to the following reconstruction formula in the sense of the Hilbert space \( L^2(\mathbb{R}^d) \):
\[
f = \| \psi \|^2_{L^2(\mathbb{R}^d)} \int_{K_d} \int_{\mathbb{R}^d} \mathcal{V}_\psi f(S, \lambda) \Gamma_\psi(S, \lambda) \psi \, d\sigma_{K_d}(S) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda).
\]

### 6.2.2. Compact subgroups of \( K_d \) generated by compact subgroups of \( \text{GL}(d, \mathbb{R}) \)

Let \( K \) be a compact subgroup of the general linear group \( \text{GL}(d, \mathbb{R}) \). Then \( K \) is a compact subgroup of the real symplectic group \( \text{Sp}(\mathbb{R}^d) \). Also, it is easy to check that \( U_H = D_H \) for all \( H \in K \), see [27].

The subgroup \( K = \text{O}(d, \mathbb{R}) \) is the most significant compact subgroup of \( \text{GL}(d, \mathbb{R}) \). The compact subgroup \( O(d, \mathbb{R}) \), or simply just \( O(d) \), is the multiplicative matrix group consists of all \( d \times d \)-orthogonal matrices. That is,
\[
\mathcal{O}(d, \mathbb{R}) := \{ A \in M_{d \times d}(\mathbb{R}) : A^T A = I_{d \times d} \}.
\]

The compact group \( O(d) \) is a \( \frac{d(d-1)}{2} \)-dimensional real Lie group and it is non-connected. The probability (normalized Haar) measure over \( O(d) \) is given by
\[
\int_{O(d)} \phi(H) d\sigma_{O(d)}(H) = \int_{S^{d-1}} \tilde{\phi}(y) d\nu_{d-1}(y),
\]
where \( \nu_{d-1} \) is the normalized surface measure on \( S^{d-1} \), that is the standard unit sphere in \( \mathbb{R}^d \), and the function \( \tilde{\phi} : S^{d-1} \to \mathbb{C} \) is given by \( \tilde{\phi}(Hx) := \phi(H) \) for all \( A \in O(d) \) and a fixed point \( x \in S^{d-1} \).

Let \( K \) be a compact subgroup of \( \text{GL}(d, \mathbb{R}) \) with the probability Haar measure \( \sigma_K \). Then \( \langle \cdot, \cdot \rangle_K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) given by
\[
\langle x, y \rangle_K := \int_K \langle Hx, Hy \rangle d\sigma_K(H),
\]
for all \( x, y \in \mathbb{R}^d \), is a positive and symmetric bilinear from on \( \mathbb{R}^d \). Also, it is a \( K \)-invariant form, that is
\[
\langle Hx, Hy \rangle_K = \langle x, y \rangle_K,
\]
for all \( x, y \in \mathbb{R}^d \) and \( H \in K \). Thus, there exists a positive definite matrix \( D \in M_{d \times d}(\mathbb{R}) \) such that
\[
\langle x, y \rangle_K = \langle x, Dy \rangle, \quad \forall x, y \in \mathbb{R}^d.
\]
Let \( \mathbf{D} = B^T B \) be the Cholesky factorization of \( D \) with \( B \) invertible. Then we deduce that \( \mathbb{K} \subset B^{-1} \mathbf{O}(d), \) or equivalently \( \mathbb{K} \subset B^{-1} \mathbf{O}(d). \) This implies that, up to conjugation, \( \mathbf{O}(d) \) is the maximal compact subgroup of \( \mathbf{GL}(d, \mathbb{R}). \)

(i) The orthogonal group. By the above argument and theoretical motivation, first we shall focus on analytic and constructive analysis of multivariate metaplectic wave-packet representations over the block diagonal compact subgroups of \( \mathcal{K}_{d} \) generated by \( \mathbb{K} = \mathbf{O}(d) \).

In this case, the associated multivariate symplectic wave-packet group \( G(\mathbb{H}) \) is isomorphic with the underlying manifold \( \mathbf{O}(d) \times \mathbb{R}^d \times \mathbb{R}^d = \mathbf{O}(d) \times \mathbb{R}^d \times \mathbb{R}^d, \) equipped with the following group law
\[
(H, x, \omega) \times (H', x', \omega') = (HH', HH^{-1} x + x', HH' \omega + \omega'),
\]
for all \((H, x, \omega), (H', x', \omega') \in \mathbf{O}(d) \times (\mathbb{R}^d \times \mathbb{R}^d).\) Then \( dm_{G(\mathbb{H})}(\widetilde{H}, \lambda) = d\sigma_{\mathbf{O}(d)}(H)d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) \) is a Haar measure for the symplectic wave-packet group \( G(\mathbb{H}). \) The multivariate symplectic wave-packet representation \( \Gamma_{\mathbb{H}} : G(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d)) \) is given by \( \Gamma_{\mathbb{H}}(H, x, \omega) = D_H T_M \) for all \((H, x, \omega) \in G(\mathbb{H}). \)

The multivariate metaplectic wave-packet transform of \( f \in L^2(\mathbb{R}^d) \) with respect to the window function \( \psi, \) is given by
\[
\mathcal{V}_f(\widetilde{H}, x, \omega) = \langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, x, \omega) \psi \rangle_{L^2(\mathbb{R}^d)} = \langle f, D_H T_M \psi \rangle_{L^2(\mathbb{R}^d)},
\]
for all \((\widetilde{H}, x, \omega) \in G(\mathbb{H}). \)

Then, corollary 5.3 guarantees the following Plancherel formula
\[
\int_{\mathbf{O}(d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, \lambda) \psi \rangle_{L^2(\mathbb{R}^d)}|^2 d\sigma_{\mathbf{O}(d)}(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) = \|f\|^2_{L^2(\mathbb{R}^d)} \|\psi\|^2_{L^2(\mathbb{R}^d)},
\]
which is equivalent to the following reconstruction formula in the sense of the Hilbert space \( L^2(\mathbb{R}^d); \)
\[
f = \|\psi\|^2_{L^2(\mathbb{R}^d)} \int_{\mathbf{O}(d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{V}_f(\widetilde{H}, \lambda) \Gamma_{\mathbb{H}}(\widetilde{H}, \lambda) \psi d\sigma_{\mathbf{O}(d)}(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda).
\]

(ii) The special orthogonal group. For \( d > 2, \) the special orthogonal \( \mathbb{K} := \mathbf{SO}(d, \mathbb{R}) \) or just \( \mathbf{SO}(d) \) is given by
\[
\mathbf{SO}(d) := \{A \in \mathbf{O}(d) : \det A = 1\}.
\]
It is a connected and compact real Lie group.

In this case, the associated multivariate symplectic wave-packet group \( G(\mathbb{H}) \) is isomorphic with the underlying manifold \( \mathbf{SO}(d) \times \mathbb{R}^d \times \mathbb{R}^d = \mathbf{SO}(d) \times \mathbb{R}^d \times \mathbb{R}^d, \) which is equipped with the following group law
\[
(H, x, \omega) \times (H', x', \omega') = (HH', HH^{-1} x + x', HH' \omega + \omega'),
\]
for all \((H, x, \omega), (H', x', \omega') \in \mathbf{SO}(d) \times (\mathbb{R}^d \times \mathbb{R}^d).\) Then \( dm_{G(\mathbb{H})}(\widetilde{H}, \lambda) = d\sigma_{\mathbf{SO}(d)}(H)d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) \) is a Haar measure for the multivariate symplectic wave-packet group \( G(\mathbb{H}). \) The metaplectic wave-packet representation \( \Gamma_{\mathbb{H}} : G(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d)) \) is given by \( \Gamma_{\mathbb{H}}(H, x, \omega) = D_H T_M \) for all \((H, x, \omega) \in G(\mathbb{H}). \)

The multivariate metaplectic wave-packet transform of \( f \in L^2(\mathbb{R}^d) \) with respect to the window function \( \psi, \) is given by
\[
\mathcal{V}_f(\widetilde{H}, x, \omega) = \langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, x, \omega) \psi \rangle_{L^2(\mathbb{R}^d)} = \langle f, D_H T_M \psi \rangle_{L^2(\mathbb{R}^d)},
\]
for all \((\widetilde{H}, x, \omega) \in G(\mathbb{H}). \)

Then, corollary 5.3 guarantees the following Plancherel formula
\[
\int_{\mathbf{SO}(d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, \lambda) \psi \rangle_{L^2(\mathbb{R}^d)}|^2 d\sigma_{\mathbf{SO}(d)}(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) = \|f\|^2_{L^2(\mathbb{R}^d)} \|\psi\|^2_{L^2(\mathbb{R}^d)},
\]
which is equivalent to the following reconstruction formula in the sense of the Hilbert space \( L^2(\mathbb{R}^d); \)
\[
f = \|\psi\|^2_{L^2(\mathbb{R}^d)} \int_{\mathbf{SO}(d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{V}_f(\widetilde{H}, \lambda) \Gamma_{\mathbb{H}}(\widetilde{H}, \lambda) \psi d\sigma_{\mathbf{SO}(d)}(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda).
\]
The Plancherel formula is
\[
\int \int |\Gamma_\psi(r)|^2 d\sigma_{SO}(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) = \|\psi\|_{L^2(\mathbb{R}^d)}^2 \|f\|_{L^2(\mathbb{R}^d)}^2,
\]
which is equivalent to the following reconstruction formula in the sense of the Hilbert space \(L^2(\mathbb{R}^d)\):
\[
f = \|\psi\|_{L^2(\mathbb{R}^d)}^2 \int d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) \int \int \mathcal{V}_\psi(\widetilde{H}, x, \omega) \Gamma_{\widetilde{H}}(\widetilde{H}, \lambda) \psi d\sigma_{SO}(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda).
\]

(**iii** The maximal tori. A circle group is a linear (matrix) group isomorphic to \(S^1\). A torus (tori) is a direct sum of circle groups. Thus any torus is a compact connected Abelian Lie group. A maximal torus (tori) is a torus in a linear (matrix) group which is not contained in any other torus. The rank of a maximal tori \(T\) is the number \(r\) such that \(T = \oplus_{j=1}^{r} S^1\).

The following proposition [39, 40] characterizes structure of a maximal tori of the special orthogonal group \(SO(d)\).

**Proposition 6.6.** Let \(d > 2\) and \(T\) be a maximal tori of \(SO(d)\). Then,

1. if \(d = 2r\) with \(r \in \mathbb{N}\), then \(T = \oplus_{j=1}^{r} S^1\).
2. if \(d = 2r + 1\) with \(r \in \mathbb{N}\), then \(T = (\oplus_{j=1}^{r} S^1) \oplus \{1\}\).

In this case, the associated multivariate symplectic wave-packet group \(G(T)\) is isomorphic with the underlying manifold \(T \times \mathbb{R}^d \times \mathbb{R}^d = T \times \mathbb{R}^d \times \mathbb{R}^d\), which is equipped with the following group law

\[
(H, x, \omega) \times (H', x', \omega') = (HH', H^{-1}x + x', H'\omega + \omega'),
\]

for all \((H, x, \omega), (H', x', \omega') \in T \times \mathbb{R}^d \times \mathbb{R}^d\). Then \(dm_{G(T)}(H, \lambda) = d\sigma_T(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda)\) is a Haar measure for the multivariate symplectic wave-packet group \(G(T)\). The multivariate metaplectic wave-packet representation \(\Gamma_{\widetilde{H}} : G(T) \rightarrow \mathcal{U}(L^2(\mathbb{R}^d))\) is given by \(\Gamma_{\widetilde{H}}(H, x, \omega) = D_H T_{L} M_{\omega}\) for all \((H, x, \omega) \in G(T)\).

The multivariate metaplectic wave-packet transform of \(f \in L^2(\mathbb{R}^d)\) with respect to the window function \(\psi\), is given by

\[
\mathcal{V}_\psi(f)(\widetilde{H}, x, \omega) = \langle f, \Gamma_{\widetilde{H}}(\widetilde{H}, x, \omega) \psi \rangle_{L^2(\mathbb{R}^d)} = \langle f, D_H T_{L} M_{\omega} \psi \rangle_{L^2(\mathbb{R}^d)},
\]

for all \((\widetilde{H}, x, \omega) \in G(T)\).

Then, corollary 5.3 guarantees the following Plancherel formula

\[
\int \int \int \int |\Gamma_{\widetilde{H}}(\widetilde{H}, x, \omega, \psi)_{L^2(\mathbb{R}^d)}|^2 d\sigma_{\mathbb{R}^d}(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda) = \|\psi\|_{L^2(\mathbb{R}^d)}^2 \|f\|_{L^2(\mathbb{R}^d)}^2,
\]

which is equivalent to the following reconstruction formula in the sense of the Hilbert space \(L^2(\mathbb{R}^d)\):

\[
f = \|\psi\|_{L^2(\mathbb{R}^d)}^2 \int \int \int \mathcal{V}_\psi(\widetilde{H}, x, \omega) \Gamma_{\widetilde{H}}(\widetilde{H}, x, \omega) \psi d\sigma_{\mathbb{R}^d}(H) d\mu_{\mathbb{R}^d \times \mathbb{R}^d}(\lambda).
\]
Concluding Remarks. The main purpose of this article was dedicated to presenting a constructive admissibility criterion on closed subgroups of the real symplectic group $\text{Sp}(\mathbb{R}^d)$ which guarantees square integrability of the associated multivariate metaplectic wave-packet representations and hence a valid resolution of the identity operator in the sense of the Hilbert function space $L^2(\mathbb{R}^d)$.

Invoking topological and geometric structure of the real Lie group $\text{Sp}(\mathbb{R}^d)$, there is a high degree of freedom in selecting an admissible subgroup $H$ of $\text{Sp}(\mathbb{R}^d)$. Among all closed subgroups of $\text{Sp}(\mathbb{R}^d)$, just compact ones are admissible and hence they guarantee a square-integrable multivariate metaplectic wave-packet representation and valid reconstruction formula.

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