EQUIDISTRIBUTION IN MEASURE-PRESERVING ACTIONS
OF SEMISIMPLE GROUPS : CASE OF $SL_2(\mathbb{R})$

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Abstract. We prove pointwise convergence for the semi-radial averages on $G = SL_2(\mathbb{R})$ given by $\int_t^{t+1} m_K * \delta_{a_t} ds$ (and similar variants), acting on $K$-finite $L^p$-functions in a probability-measure-preserving action of the group, for $p > 1$.

1. Pointwise convergence for semi-radial averages on $SL_2(\mathbb{R})$

Let $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, and let $K = SO(2, \mathbb{R}) \subset SL(2, \mathbb{R})$. Let $(X, \mu)$ be a standard Borel measure space on which $SL(2, \mathbb{R})$ acts, preserving the probability measure $\mu$ on $X$. We let $\pi_X$ denote the unitary representation of $G$ in $L^2(X)$, given by $\pi_X(g)f(x) = f(g^{-1}x)$. Recall that $f \in L^2(X)$ is called a $K$-finite function if the linear span $\{\pi_X(k)f; k \in K\}$ is finite dimensional.

Assume $\eta \in C_c(\mathbb{R})$ is a non-negative bump function of unit integral, and define the averaging operators

$$M^n_\eta f(x) = \int_{-\infty}^{\infty} \eta(t - s) \left( \int_K f(a_s k x) dm_K(k) \right) ds$$

where $m_K$ is normalized Haar probability measure on $K$.

Theorem 1.1. Assume $f \in L^{1+\kappa}(X, \mu)$ for some $\kappa > 0$, and $f$ is a $K$-finite function. If $\mu$ is an ergodic measure, then for $\mu$-almost every $x$ in $X$,

$$\lim_{t \to \infty} M^n_\eta f(x) = \int_X f(y) d\mu(y).$$

We remark that the same pointwise convergence result holds for any (not necessarily ergodic) invariant measure $\mu$, with the limit being the conditional expectation on the sub-$\sigma$-algebra of $G$-invariant sets. Furthermore, the same conclusion holds when $t \to -\infty$ as well.

We note that a mean ergodic theorem for semi-radial averages $m_K * \delta_{a_t}$ is proved in greater generality in [Ve1]. The pointwise convergence result stated in Theorem 1.1 is utilized in [EM1], which provided the motivation for its formulation. Another application of semi-radial averages can be found in [EMM].

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2. Ergodic theorems for character-spherical averages on $SL_2(\mathbb{R})$

2.1. The maximal inequality for character-spherical operators. Before proceeding to the proof of Theorem 1.1 we begin with some preliminaries. First, let $\psi_n(k_\theta) = \exp(in\theta)$ denote the characters of the circle group $SO(2, \mathbb{R}) = K = \{ k_\theta \mid 0 \leq \theta \leq 2\pi \}$. Let $\chi_n$ denote the (complex) Radon measure on $K$ whose density w.r.t. normalized Haar measure on $K$ is $\overline{\psi}_n$. To any bounded Borel measure $\nu$ on $G$, there corresponds in any strongly continuous unitary representation $\pi$, the operator $\pi(\nu)v = \int_G \pi(g)vd\nu(g)$. This correspondence is a bounded homomorphism of the convolution algebra, namely $\pi(\nu_1 \ast \nu_2) = \pi(\nu_1) \circ \pi(\nu_2)$, and $\|\pi(\nu)\| \leq \|\nu\|$ (where $\|\nu\|$ is the total variation norm).

In particular the operator $\pi(\chi_n)$ is equal, in any strongly continuous unitary representation $(\pi, \mathcal{H}^\pi)$ of $K$, to the orthogonal projection onto the closed linear subspace $\mathcal{H}^\pi_n = \{v \in \mathcal{H}^\pi \mid \pi(k_\theta)v = \overline{\psi}_n(k_\theta)v\}$. In particular, $\pi(\chi_n)$ is a self-adjoint idempotent operator.

Now consider the (complex) measure $\sigma_t(n, m) = \chi_n \ast \delta_{at} \ast \chi_m$, where $\delta_{at}$ denotes the delta measure at the element $a_t \in A$. The corresponding operator $\pi(\sigma_t(n, m))$ has norm bounded by 1, but is not self-adjoint, in general. If $\eta \in C_c(\mathbb{R})$, satisfies $\eta \geq 0$ and $\int_{-\infty}^{\infty} \eta(s)ds = 1$, the measure $\gamma_t^n(n, m) = \int_{-\infty}^{\infty} \eta(t-s)\sigma_t(n, m)ds$, is a convex combination of the measures $\sigma_s(n, m)$, where $s$ ranges over $t - B_\eta$ with $B_\eta = \text{supp} \eta$. In general, the measures of the form $\chi_n \ast \omega \ast \chi_m$ are precisely the complex measures $\omega$ on $G$ satisfying the equation $\chi_n \ast \omega \ast \chi_m = \omega$, and will be called character-spherical measures.

Note that if $f \in L^2(X)$ is $K$-finite, then there is a finite set $F_f \subset \mathbb{Z}$ of characters of the circle group $K$, such that the finite-dimensional space $V(f)$ spanned by the $K$-translates of $f$ is a finite direct sum $\bigoplus_{n \in F_f} V_n(f)$, where $V_n(f)$ is finite-dimensional, and $\pi(k_\theta)$ acts on $V_n(f)$ as scalar multiplication by $\psi_n(k_\theta) = e^{in\theta}$. In particular, $\pi(\chi_n)f$ is the orthogonal projection of $f$ on $V_n(f)$ for every $n \in F_f$, and $\sum_{n \in F_f} \pi(\chi_n)f = f$. Conversely, every function $f \in L^2(X)$ satisfying that $\sum_{n \in F} \pi(\chi_n)f = f$ for some finite set $F$ of characters of $K$, is a $K$-finite function.

The pointwise convergence stated in Theorem 1.1 is based on the following maximal inequality.

**Theorem 2.1.** In any probability-measure-preserving action $(X, \mu)$ of $SL_2(\mathbb{R})$, the operators $\gamma_t^n(n, m)$ (for any $n, m$) satisfy the strong type $(p, p)$ global maximal inequality in every $L^p$, $1 < p \leq \infty$, namely:

$$\|\sup_{t \in \mathbb{R}} |\pi(\gamma_t^n(n, m))f|\|_p \leq C_p(\eta)\|f\|_p$$

The constant $C_p(\eta)$ is independent of $f$, and also of $m$ and $n$.

**Proof.** Clearly, for almost every $x \in X$:

$$|\pi(\chi_n \ast \delta_{a_t} \ast \chi_m)f(x)| = \left|\int_K \int_K f(k_{\theta_1}^{-1}a_{s}^{-1}k_{\theta_2}^{-1})\overline{\psi}_n(k_{\theta_1})\overline{\psi}_m(k_{\theta_2})dm_K(k_{\theta_1})dm_K(k_{\theta_2})\right| \leq \pi(m_K \ast \delta_{a_t} \ast m_K)|f(x)|$$
Denoting $\sigma_t = \sigma_t(0,0) = m_K \ast \delta_{ut} \ast m_K$ and $\gamma^n_t = \gamma^n_t(0,0) = \int_\mathbb{R} \eta(t-s)\sigma_s ds$, we therefore have

$$|\pi(\sigma_t(n,m))f(x)| \leq |\pi(\sigma_t)|f(x)| \text{ and } |\pi(\gamma^n_t(n,m))f(x)| \leq |\pi(\gamma^n_t)|f(x)|.$$ 

The maximal inequality for $\gamma^n_t$ in every $L^p(X)$, $1 < p \leq \infty$ is a straightforward consequence of the maximal inequality for the operators $\gamma_t = \int_{t+1}^{t+1} \sigma_s ds$ that was established in [NS, §3, Prop.3]. Indeed, in the reference cited the operator considered was $\sup_{t \geq 1} \gamma_t$, but the intervals $[t, t+1]$ can be replaced by intervals of any fixed length. Furthermore the local operator $\sup_{0 \leq t \leq 1} \gamma_t$ certainly satisfies the maximal inequality, by the local transfer principle. Recalling that $\sigma_t = m_K \ast \delta_{ut} \ast m_K = m_K \ast \delta_{a,t} \ast m_K = \sigma_{-t}$, we can consider arbitrary intervals in $\mathbb{R}$. Finally, the maximal inequalities for averages which are uniform on intervals clearly imply the corresponding results averages defined by non-negative compactly supported bump functions. \hfill \Box

2.2. Pointwise convergence of character-spherical operators. We now turn to the proof of the following pointwise convergence result, that will be utilized in the proof of Theorem 2.1 below.

**Theorem 2.2.** In any finite-measure-preserving action $(X, \mu)$ of $SL_2(\mathbb{R})$, and for any $m \in \mathbb{Z}$, the sequence $\pi(\gamma^n_t(0,m))f(x)$ converges, as $t \to \infty$ and $t \to -\infty$, for every $f \in L^p(X)$, $1 < p < \infty$, pointwise almost every (and in the $L^p$-norm). The limit is zero, unless $m = 0$, in which case the limit is $E f(x)$, where $E$ is the projection of $f$ on the space of $G$-invariant functions.

As is well known, given the strong maximal inequality stated in Theorem 2.1 in order to prove Theorem 2.2 it suffices to establish the existence of a dense subspace of functions $f \in L^2(X)$ for which $\pi(\gamma_t(0,m))f(x)$ converges almost everywhere to the stated limit. Indeed, if $\|f - f_k\|_p \to 0$, where $f_k$ belong to the dense subspace then

$$|\pi(\gamma_t(0,m))f(x) - \pi(\gamma_s(0,m))f(x)| \leq |\pi(\gamma_t(0,m))(f - f_k)(x)| + |(|\pi(\gamma_t(0,m))| - |\pi(\gamma_s(0,m))|)f_k(x)| + |\pi(\gamma_s(0,m))(f_k - f)(x)|$$

so that

$$\limsup_{t,s \to \infty} |\pi(\gamma_t(0,m))f(x) - \pi(\gamma_s(0,m))f(x)| \leq 2 \sup_{t > 0} |\pi(\gamma_t(0,m))(f - f_k)(x)| + 0$$

and hence

$$\left\|\limsup_{t,s \to \infty} |\pi(\gamma_t(0,m))f(x) - \pi(\gamma_s(0,m))f(x)|\right\|_p \leq 2C_p \|f - f_k\|_p \to 0.$$ 

To establish the existence of such a subspace we will consider the spectral decomposition of the unitary representation $\pi_X$, and use the decay estimates of $K$-finite functions and their first derivatives in irreducible unitary representations of $G$.

We begin with some preliminaries. First, let $(\tau, \mathcal{H}_\tau)$ be an irreducible non-trivial strongly continuous unitary representation of $G = SL_2(\mathbb{R})$. Then $\mathcal{H}_\tau = \sum_{n \in \mathbb{Z}} \mathcal{H}_n^\tau$, where $\mathcal{H}_n^\tau$ is the closed subspace that affords the representation with character $\psi_n$ of the circle group $K$. As is well known (see e.g. [HT, Ch.3]), the dimension of $\mathcal{H}_n^\tau$ is either zero or one, depending on $\tau$. Let $v_n^\tau$ denote a unit vector in $\mathcal{H}_n^\tau$.
if such a vector exists, and the zero vector otherwise. Given \( v \in \mathcal{H}_\tau \), we write
\[
v = \sum_{k \in \mathbb{Z}} \langle v, v_k^\tau \rangle v_k^\tau,
\]
so that \( \|v\|^2 = \sum_{k \in \mathbb{Z}} |\langle v, v_k^\tau \rangle|^2 \). Recall that \( \tau(\chi_m) = (\tau(\chi_m))^* \) is the orthogonal projection operator on \( \mathcal{H}_m \), so that \( \tau(\chi_m)v = \langle v, v_m^\tau \rangle v_m^\tau \), and
\[
|\langle v, v_m^\tau \rangle|^2 = \|\tau(\chi_m)v\|^2.
\]
Since \( \sigma_t(n, m) = \chi_n * \delta_t * \chi_m \), Bessel identity implies
\[
\|\tau(\sigma_t(n, m))v\|^2 = \sum_{k \in \mathbb{Z}} |\langle \tau(\sigma_t(n, m))v, v_k^\tau \rangle|^2 = \sum_{k \in \mathbb{Z}} |\langle \tau(a_t)\tau(\chi_m)v, \tau(\chi_m)v_k^\tau \rangle|^2
\]
\[
= |\langle v, v_m^\tau \rangle|^2 |\langle \tau(a_t)v_m^\tau, v_m^\tau \rangle|^2 = |\langle v, v_m^\tau \rangle|^2 |\Phi_{m,n}^\tau(a_t)|^2,
\]
where the matrix coefficient \( \Phi_{m,n}^\tau(g) = \langle \tau(g)v_m^\tau, v_n^\tau \rangle \) is called the \((m, n)\)-spherical function associated with the representation \( \tau \).

The same argument shows that
\[
\|\tau(\sigma_t(n, m))v - \tau(\sigma_s(n, m))v\|^2 = \|\langle v, v_m^\tau \rangle \cdot \langle \tau(a_t)v_m^\tau, v_m^\tau \rangle - \langle v, v_m^\tau \rangle \cdot \langle \tau(a_s)v_m^\tau, v_m^\tau \rangle\|^2 = \|\langle v, v_m^\tau \rangle|^2 \left| \Phi_{m,n}^\tau(a_t) - \Phi_{m,n}^\tau(a_s) \right|^2
\]
Taking \( s = t + h \), dividing by \( h \) and letting \( h \) tend to zero, we conclude that if \( v \in \mathcal{H}_\tau \) is a \( C^\infty \)-vector of the representation \( \tau \), then
\[
\left\| \frac{d}{dt} \tau(\sigma_t(n, m))v \right\|^2 = \left\langle v, v_m^\tau \right\rangle \left\| \frac{d}{dt} \Phi_{m,n}^\tau(a_t) \right\|^2
\]
which is an explicit form for the multiplier operator corresponding to the differentiation operator given by \( \frac{d}{dt}(\chi_n * \delta_t \ast \chi_m) \).

2.3. Estimates of \( K \)-finite functions. We turn to stating the estimates of generalized spherical functions and their derivatives that we will use below. We denote the unitary spectrum of \( SL_2(\mathbb{R}) \) by \( \Sigma \), and consider the character-spherical functions
\[
\Phi_{m,n}^\tau(a_t) = \langle \tau(a_t)v_m^\tau, v_n^\tau \rangle,
\]
for \( \tau \in \Sigma \setminus \{1\} \) (here 1 denotes the trivial representation). We will give a simple direct argument for the spectral estimates we will actually use, following the exposition in [HT, Ch. V, §3.1]. First consider the function space of smooth even functions on \( \mathbb{R}^2 \setminus \{0\} \), homogeneous of degree \( z \) :
\[
S^{z,+} = \left\{ f \in C^\infty(\mathbb{R}^2 \setminus \{0\}) ; f(ts) = |t|^z f(s) \right\}
\]
and the corresponding space of odd functions :
\[
S^{z,-} = \left\{ f \in C^\infty(\mathbb{R}^2 \setminus \{0\}) ; f(ts) = \text{sgn}(t) |t|^z f(s) \right\}
\]
\( SL_2(\mathbb{R}) \) acts on these function spaces via the usual (linear) action on \( \mathbb{R}^2 \setminus \{0\} \), and we denote this action by \( \rho(g)f = f \circ g^{-1} \). The (even) principal series of unitary representations is realized on the spaces \( S^{z,+} \) corresponding to \( z = -1 + i \mathbb{R} \), and the (odd) principal series on the spaces \( S^{z,-} \) with \( z = -1 + i(\mathbb{R} \setminus \{0\}) \). The complementary series representation is realized on \( S^{z,+} \) with \( z \in (-2, 0) \). Write \( z = -1 - \pi \), and define a bilinear form on \( S^{-1-\pi,\pm} \times S^{-1+\alpha,\pm} \) by integrating the function values on the unit circle
\[
\langle f, h \rangle = \int_0^{2\pi} f(\cos \theta, \sin \theta) h(\cos \theta, \sin \theta) d\theta.
\]
This bilinear form is invariant under the action \( \rho \) of \( SL_2(\mathbb{R}) \) on the two function spaces involved, namely for \( f \in S^{-1-\pi,\pm} \) and \( h \in S^{-1+\alpha,\pm} \)
\[
\langle \rho(g)f, \rho(g)h \rangle = \langle f, h \rangle.
\]
The associated matrix coefficients are given along \( A \) by

\[
\langle f, \rho(a_t)h \rangle = \int_0^{2\pi} f(\cos \theta, \sin \theta) h(\cos \beta, \sin \beta) d\theta
\]

where \( \beta \) is a function of \( t \) and \( \theta \). If follows immediately that if \(|f| = |h| = 1\) on the unit circle, then

\[
|\langle f, \rho(a_t)h \rangle| \leq \int_0^{2\pi} (e^{-2t \cos^2 \theta} + e^{2t \sin^2 \theta})^{\frac{1}{2}(1+\Re \Pi)} d\theta
\]  

(2.1)

The standard estimate of the positive-definite spherical functions on \( SL_2(\mathbb{R}) \) will be recalled presently. But beforehand, note that we will also require in our analysis estimates for the derivative of the character-spherical functions, and we will develop here the estimates that we will actually use below by a simple direct argument.

Let us choose now \( h \) to be the \( K \)-invariant vector in the representation space (when it exists), whose restriction to the unit circle is the constant 1, and also choose \( f \) as the vector affording the representation \( \psi_n \) (when it exists), so that its restriction to the unit circle is \( e^{in\theta} \). The resulting expression is

\[
\langle e^{in\theta}, \rho(a_t)1 \rangle = \int_0^{2\pi} e^{in\theta} (e^{-2t \cos^2 \theta} + e^{2t \sin^2 \theta})^{\frac{1}{2}(1+\Re \Pi)} d\theta
\]

When the parameter \( z \) is such that the representation on \( S^2 \) (and its dual) is a unitary representation \( \tau \) (namely even or odd principal series, or complementary series), the preceding expression is precisely \( \langle \rho(a_t)1, e^{in\theta} \rangle = \Phi_{0,n}(a_t) \). Note also that

\[
\langle 1, \rho(a_t)(e^{in\theta}) \rangle = \Phi_{0,n}^\tau(a_t) = \langle \rho(a_{-t})1, (e^{in\theta}) \rangle = \Phi_{0,n}^\tau(a_{-t})
\]

Thus \( \Phi_{0,n}^\tau(a_t) = \Phi_{0,n}^\tau(a_{-t}) \).

The parametrization of the irreducible non-trivial unitary representations of \( SL_2(\mathbb{R}) \) that we will use below is given in [HT] Ch. III, §1.3, Thm. 1.3.1, as follows.

The representation \( \rho \) on the space \( S^{(-1+i\lambda, \pm)} \), \( \lambda \in \mathbb{R} \) will be denoted by \( \pi_\lambda^\pm \), and constitutes the even and odd principal series, where \( \lambda \in \mathbb{R} \) in the first case, and \( \lambda \in \mathbb{R} \setminus \{0\} \) in the second case. The representation \( \rho \) on \( S^{-1-s} \), \( 0 < s < 1 \) will be denoted by \( \pi_s \), and constitutes the complementary series. We note that \( \pi_s \) is also the representation realized on \( S^{-1+s} \), \( 0 < s < 1 \). Finally, we denoted by \( \pi^{(k, \pm)} \), \( k \geq 1 \) the representations of the discrete series \( (k \geq 2) \) and limits of discrete series \( (k = 1) \).

We let \( \Xi \) denote the Harish Chandra \( \Xi \)-function, given by \( \Xi = \Phi_{0,0}^{\pi_0^+}(a_t) \). Recall that \( \Xi \)-function satisfies the estimate : \( 0 < \Xi(t) \leq C(1+t) \exp(-t) \).

Before stating the spectral estimates which will be relevant to our discussion, let us note (see [HT] Ch. III, Prop. 1.2.6]) that the odd principal series representations \( \pi_\lambda^- \) do not contain a \( K \)-invariant unit vector, so that the matrix coefficients \( \Phi_{n,0}^{\pi_\lambda^-} \) and \( \Phi_{0,n}^{\pi_\lambda^-} \) defined above are equal to zero. It is well-known (see e.g. [HT] Ch. III]) that
the discrete series representations and the limit of discrete series representations also do not contain a $K$-invariant unit vectors, so that the same conclusion applies.

We now note the following fact.

**Lemma 2.3.** There exists a constant $B$ such that the following holds.

Let $z = -1 + i\lambda$ ($\lambda \in \mathbb{R}$) be a principal series parameter, and $\pi_\lambda^+$ be the associated irreducible unitary representation of $SL_2(\mathbb{R})$. Then

\begin{enumerate}
    \item \[|\Phi_{0,n}^{\pi}(a_t)| \leq \Xi(a_t) \leq B(1 + t) \exp(-t).\]
    \item \[\left| \frac{d}{dt} \Phi_{0,n}^{\pi}(a_t) \right| \leq B(1 + |\lambda|)(1 + t) \exp(-t).
    \]
    \end{enumerate}

Let $z = -1 - s$ ($s \in (0,1)$) be a complementary series parameter, and let $\pi_s$ be the associated irreducible unitary representation. Then

\begin{enumerate}
    \item \[|\Phi_{0,n}^{\pi}(a_t)| \leq B \cdot \exp(-(1 - s)t)\]
    \item \[\left| \frac{d}{dt} \Phi_{0,n}^{\pi}(a_t) \right| \leq B \cdot \exp(-(1 - s)t)\]
    \end{enumerate}

The same estimates apply to the functions $\Phi_{n,0}^{\pi}$ with $\pi$ a principal series or complementary series representation.

\begin{enumerate}
    \item All other estimates apply to the functions $\Phi_{n,0}^{\pi}$ and $\Phi_{0,n}^{\pi}$ are identically zero.
\end{enumerate}

**Proof.** Consider first the representations of the even principal series $\pi_\lambda^+$ or the complementary series $\pi_s$, which we denote temporarily by $\tau_z$. As noted above, to cover all cases it suffices to estimate the derivative of

\[\Phi_{0,n}^{\pi}(a_t) = \langle \rho(a_t)1, e^{i\theta} \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} (e^{-2t\cos^2\theta} + e^{2t\sin^2\theta})^{\frac{1}{2}(1 + \text{Re}(\overline{\tau}))} d\theta\]

Differentiating, we obtain:

\[\left| \frac{d}{dt} \Phi_{0,n}^{\pi}(a_t) \right| = \frac{1}{2} (-1 + \text{Re}(\overline{\tau})) \cdot 2 \int_0^{2\pi} e^{-i\theta} (e^{-2t\cos^2\theta} + e^{2t\sin^2\theta})^{\frac{1}{2}(1 + \text{Re}(\overline{\tau}))} \cdot \frac{\sin^2\theta e^{2t} - \cos^2\theta e^{-2t}}{\sin^2\theta e^{2t} + \cos^2\theta e^{-2t}} d\theta\]

\[\leq (1 + |\alpha|) \int_0^{2\pi} (e^{-2t\cos^2\theta} + e^{2t\sin^2\theta})^{\frac{1}{2}(1 + \text{Re}(\overline{\tau}))} d\theta = (1 + |\alpha|) \Phi_{0,0}^{\pi}(a_t),\]

using

\[\left| \frac{\sin^2\theta e^{2t} - \cos^2\theta e^{-2t}}{\sin^2\theta e^{2t} + \cos^2\theta e^{-2t}} \right| \leq 1.\]

Thus the proof of parts (1)-(4) is complete upon recalling the estimate of Prop. 3.1.5 of [HT] Ch. V,3.1 for the standard spherical function, which asserts the desired exponential decay estimates.

We conclude:

**Proposition 2.4.** For every irreducible non-trivial unitary representation $\tau$ of $SL_2(\mathbb{R})$ with parameter $(-1 - \overline{\tau}, \pm)$, the following estimates hold, for all $n \in \mathbb{Z}$, with $B$ an absolute constant.
In particular it commutes with all the operators \(\pi\) together with the foregoing spectral estimates, and conclude:

\[
|\Phi^\tau_{0,n}(a_t)| \leq B(1 + |t|) \exp(-\varepsilon \tau |t|),
\]

with \(\varepsilon > 1\) for \(\tau = \pi^\lambda (\lambda \in \mathbb{R})\) and \(\varepsilon = 1 - s\) for \(\tau = \pi_s, (0 < s < 1)\). We denote \(C(\tau) = B(1 + |\alpha|)\).

2.4. **Proof of Theorem 2.2.** Given a (cyclic) unitary representation \(\pi\) of \(SL_2(\mathbb{R})\), we consider a direct integral decomposition of \(\pi\), writing \(\pi = \int_{\pi}^\oplus \tau d\zeta_\pi(\tau)\), where \(\Sigma\) denotes the unitary spectrum of the group and \(\zeta_\pi\) the spectral measure. Given \(n\) and \(\varepsilon > 0\), we define the following subset of the spectrum \(\Sigma\), using the parameters \(\varepsilon\) and \(C(\tau)\) defined in Proposition 2.4

\[
\Sigma_\varepsilon = \left\{ \tau \in \Sigma; \varepsilon > \varepsilon, C(\tau) < \frac{1}{\varepsilon} \right\}.
\]

We let \(P_\varepsilon\) denote the orthogonal projection on the subspace \(\mathcal{H}_\varepsilon^\pi\) of vectors \(v \in \mathcal{H}^\pi\) whose spectral measure is supported in the set \(\Sigma_\varepsilon\). \(P_\varepsilon\) commutes with all the operators \(\pi(g), g \in G\), and hence also with \(\pi(\nu)\) for any complex measure \(\nu\) on \(G\). In particular it commutes with all the operators \(\pi(\sigma_t(n, m)), \pi(\gamma_t(n, m))\) as well as their derivatives.

Let \(v \in C^\infty(\mathcal{H}_\varepsilon^\pi)\), namely \(v\) is a \(C^\infty\) vector belonging to the subspace of vectors whose spectral support is in \(\mathcal{H}_\varepsilon^\pi\).

Let \(v = \int_{\pi}^\oplus v^\tau d\zeta_\pi,\nu(\tau)\) denote the direct integral decomposition of \(v\). We use the expression found in §2.2 for the spectral multiplier of the derivative operator together with the foregoing spectral estimates, and conclude:

\[
\left\| \frac{d}{dt} \pi(\sigma_t(0,n))v \right\|^2 = \int_{\Sigma_\varepsilon} \left| \langle v^\tau, v_n^\tau \rangle \right|^2 \left\| \frac{d}{dt} \Phi^\tau_{0,0}(a_t) \right\|^2 d\zeta_\pi,\nu(\tau)
\leq \frac{1}{\varepsilon^2}(1 + |t|)^2 \exp(-2 |t| \varepsilon) \cdot \|v\|^2.
\]

In the next subsection, using the argument of [N1 §7.1], the pointwise convergence of \(\pi(\sigma_t(0,n))f(x)\) to zero for almost every \(x \in X\) will be established below for every \(f \in \cup_{\varepsilon > 0} \mathcal{D}(\mathcal{H}_\varepsilon^\pi)\), where \(\mathcal{D}(\mathcal{H}_\varepsilon^\pi)\) is a subspace of \(C^\infty(\mathcal{H}_\varepsilon^\pi)\), norm-dense in \(\mathcal{H}_\varepsilon^\pi\). Given this fact, pointwise almost sure convergence holds of course also for the operators \(\pi_X(\gamma_t(0,n))f(x)\). The union of the subspaces \(\mathcal{D}(\mathcal{H}_\varepsilon^\pi)\) for \(\varepsilon > 0\) is norm-dense in \(L_0^2(X)\) (assuming the action is ergodic). Hence, using the maximal inequality for \(\gamma_t^\nu(0,n)\) in \(L_2^2(X)\) stated in Theorem 2.1 we conclude that \(\pi_X(\gamma_t^\nu(0,n))f(x) \rightarrow 0\) almost everywhere for all \(f \in L_0^2(X)\). Clearly the same holds on the space of constants, unless \(n = 0\), in which case the limit is \(\int_X f dm\).

Now, since \(L_\infty(X) \subset L_2^2(X)\) is \(L^p\)-norm-dense in every \(L^p(X)\), we conclude that \(L^p(X)\) has a norm-dense subspace on which pointwise almost sure convergence holds for \(\pi_X(\gamma_t^\nu(0,n))f\). Using the strong type maximal inequality for \(1 < p \leq \infty\) for these operators, pointwise almost sure convergence holds for for every \(f \in L^p(X)\). The same now follows for any p.m.p. action of \(G\) using standard facts about ergodic decompositions.
To complete the proof of Theorem 1.1, it remains to establish pointwise convergence for the operators $\mathcal{M}_t^p = \int_\mathbb{R} \eta(t-s)\pi_X(m_K * \delta_{a-s})ds$, acting in $L^p(X)$, $1 < p < \infty$. Note that the operators $\pi(\chi_n)$ are defined on $L^p$ and constitute projections of norm at most one, satisfying $\pi(\chi_n)\pi(\chi_n) = \pi(\chi_n * \chi_n) = \pi(\chi_n)$, and also $\pi(\chi_n)\pi(\chi_m) = \pi(\chi_n * \chi_m) = 0$ if $n \neq m$. By definition, if $f \in L^p(X)$ is $K$-finite then the linear span of its translates under $K$ is a finite-dimensional space $V(f)$, which is obviously $K$-invariant. The representation of $K$ in $V(f)$ is equivalent to a unitary representation, and $V(f)$ decomposes to a finite direct sum $\oplus_{n \in F} V_n(f)$, where the $K$-representation on $V_n(f)$ is via scalar multiplication by $\psi_n$, and $F$ is finite. Clearly, for every $n \in F_n$, each operator $\pi_X(\chi_n)$ acts as the identity on $V(f)$, and $\sum_{n \in F} \pi_X(\chi_n)$ acts as the identity on $V(f)$. By definition $f = \sum_{n \in F} \pi_X(\chi_n)f$. Therefore, when $1 < p \leq \infty$, for almost every $x \in X$

$$\mathcal{M}_t^p f = \mathcal{M}_t^p \left( \sum_{n \in F} \pi_X(\chi_n)f \right) = \sum_{n \in F} \int_\mathbb{R} \eta(t-s)\pi_X(m_K * \delta_{a-s})\pi_X(\chi_n)f(x)ds$$

$$= \sum_{n \in F} \pi_X(\gamma_{\eta_t}(0,n))f \rightarrow \int_X f(y)d\mu(y)$$

where the final pointwise convergence result follows from Theorem 2.2 applied to the operators $\gamma_{\eta_t}(0,n)$ with $\eta'(s) = \eta(-s)$, and $t \rightarrow \pm \infty$. As to the identification of the limit, if $0 \notin F_f$, then the limit is $0 = \int_X f d\mu$, and if $0 \in F_f$, the limit is $\int_X \pi_X(\chi_0)f d\mu = \int_X f d\mu$, as stated.

2.5. **Pointwise convergence on a dense subspace.** We now turn to last remaining step, namely the construction of a dense subspace of functions in $L^2(X)$ for which $\pi_X(\sigma_t(\nu_n))f(x)$ converges for almost all $x \in X$, using the arguments of [N1, §7.1]. First, for a smooth bump function $a(g)$ on $G$, let $\nu_a$ denote the absolutely continuous measure whose density w.r.t. Haar measure on $G$ is $a$. Consider spectral subspaces $\mathcal{H}_\varepsilon$ defined above, and define the subspace $\mathcal{D}(\mathcal{H}_\varepsilon) = \{\pi_X(\nu_a)f = \int_G a(u)\pi_X(u)f du : a \in C_c^\infty(G), f \in \mathcal{H}_\varepsilon\}$. Note that for $f \in \mathcal{H}_\varepsilon$ we have the following norm estimate:

$$\|\pi_X(\sigma_t(0, n))f\|_2^2 = \int_{t \in \Sigma_n} |\langle f^{\tau}, \nu_n^{\tau} \rangle|^2 |\Phi_{\nu_n,0}(a_t)|^2 d\zeta_{\pi,f}(\tau)$$

$$\leq B^2(1 + |t|)^2 e^{-2\varepsilon|t|} \|f\|_2^2$$

For $h = \pi_X(\nu_a)f \in \mathcal{D}(\mathcal{H}_\varepsilon)$, we can consider

$$\frac{d}{dt}\pi_X(\sigma_t(0, n))h = \lim_{r \rightarrow 0} \pi_X(\sigma_{t+r}(0, n))h - \pi_X(\sigma_t(0, n))h$$

where the limit is taken in the $L^2$-norm. The derivative is also contained in $\mathcal{H}_\varepsilon$, since $\mathcal{H}_\varepsilon$ is a closed $\pi_X(G)$-invariant subspace. In fact the derivative at the point $t$ is equal to $\int_G D_t^s(a(u)\pi_X(u)f(x)du = \pi_X(\nu_{D_t^s(a(u)})f$, where $D_t^s(a(u)$ is the smooth bump function $\frac{d}{dt}(\sigma_t(0, n) * \nu_a)(u)$. In particular, if $h$ belongs to $\mathcal{D}(\mathcal{H}_\varepsilon)$ its derivative $\frac{d}{dt}\pi_X(\sigma_t(0, n))h$ is contained in $\mathcal{H}_\varepsilon$ also.

Furthermore, we claim that $t \mapsto \pi_X(\sigma_t(0, n))h(x) = \int_G \pi_X(\sigma_t(0, n) * \nu_a)f(x)du$ is continuous in $t$ for $\mu$-almost all $x \in X$. Indeed if $\alpha(u)$ is a compactly supported
bounded function on $G$, then the following standard inequality holds (using e.g. Jensen’s inequality):

$$\int_X \left| \int_G \alpha(u) f(u^{-1}x)du \right|^2 d\mu(x) \leq \int_X \left( \int_G |\alpha(u)||f(u^{-1}x)|^2 du \right) d\mu(x)$$

$$= \|\alpha\|_{L^1(G)} \|f\|_{L^2(X)}^2 < \infty$$

Choosing $\alpha$ as the characteristic function of a large fixed ball $B$ in $G$, it follows that the function $u \mapsto |f(u^{-1}x)|^2$ restricted to $B$ is an $L^1$ function on $B$, for almost every $x \in X$. Hence if $\alpha_r$ are uniformly bounded continuous functions supported in $B$ such that $r \mapsto \alpha_r$ is continuous in the $L^1$-norm on $B$, then for almost every $x$ (by Cauchy-Schwartz inequality)

$$\left| \int_B (\alpha_t(u) - \alpha_s(u)) f(u^{-1}x)du \right|$$

$$\leq \left( \int_B |\alpha_t(u) - \alpha_s(u)| du \right) \left( \int_B |\alpha_t(u) - \alpha_s(u)||f(u^{-1}x)|^2 du \right)$$

so that $t \mapsto \int_B \alpha_t(u)f(u^{-1}x)$ is continuous in $t$ for almost every $x$. This holds of course for $\alpha_t$ defined as the density of the measure $\sigma_t(0,n) \ast a$.

Fix $f \in \mathcal{D}(\mathcal{H}_x)$, and let $w$ denote an arbitrary vector in $L^2$. The function $y_w(t) = \langle \pi(\sigma_t(0,n)) f, w \rangle$ is differentiable (in fact $C^\infty$) in $t$, and since $f$ is a differentiable vector, by the fundamental theorem of calculus:

$$\int_s^t \left< \frac{d}{du} \pi(\sigma_u(0,n)) f, w \right> du = \int_s^t \frac{d}{du} \left< \pi(\sigma_u(0,n)) f, w \right> du$$

$$= \int_s^t \frac{d}{du} y_w(u) du = y_w(t) - y_w(s) = \left< \int_s^t \frac{d}{du} \pi(\sigma_u(0,n)) f du, w \right> .$$

Since the equality is valid for all $w \in L^2$, we obtain by Fubini’s theorem, for $\mu$-almost all $x \in X$:

$$\int_s^t \left( \frac{d}{du} \pi(\sigma_u(0,n)) f \right)(x) du = \pi(\sigma_t(0,n)) f(x) - \pi(\sigma_s(0,n)) f(x)$$

Hence, if $t, s \geq N$:

$$|\pi(\sigma_t(0,n)) f(x) - \pi(\sigma_s(0,n)) f(x)| \leq \int_N^\infty \left| \frac{d}{du} \pi(\sigma_u(0,n)) f(x) \right| du$$

Therefore, using the continuity of $t \mapsto \pi(\sigma_t(0,n)) f(x)$ for almost all $x$, for any $N \geq 0$ we have

$$\limsup_{t,s \to \infty} \left| \pi(\sigma_t(0,n)) f(x) - \pi(\sigma_s(0,n)) f(x) \right| \leq \int_N^\infty \left| \frac{d}{du} \pi(\sigma_u(0,n)) f(x) \right| du$$

It follows that the set

$$\left\{ x : \limsup_{t,s \to \infty} \left| \pi(\sigma_t(0,n)) f(x) - \pi(\sigma_s(0,n)) f(x) \right| \geq \delta \right\}$$
is contained in the set \( \{ x : \int_0^\infty |\frac{d}{du} \pi(\sigma_u(0,n))f(x)| du \geq \delta \} \). The measure of the latter set is estimated by integrating over \( X \). We obtain, for any \( \delta > 0 \) and \( N \geq 0 \), the bound:

\[
\frac{1}{\delta} \int_X \int_0^\infty \left| \frac{d}{du} \pi(\sigma_u(0,n))f(x) \right| du \mu(x) = \\
\leq \frac{1}{\delta} \int_0^\infty (1 + e^{-\epsilon u}) du \| f \|_2 \\
\leq \frac{1}{\delta} e^{-\epsilon N} \| f \|_2 \quad \text{as} \quad N \to \infty.
\]

It follows that the set where \( \pi(\sigma_t(0,n))f(x) \) does not converge is a null set. Consequently, for \( f \in \mathcal{D}(\mathcal{H}_t) \), \( \lim_{t \to \infty} \pi(\sigma_t(0,n))f(x) \) exists for \( \mu \)-almost all \( x \in X \) and is equal to \( \int_X f(x) d\mu = 0 \), by the mean ergodic theorem for these operators (which is an immediate consequence of convergence in norm). Finally, we just have to note that indeed \( \bigcup_{\epsilon > 0} \mathcal{D}(\mathcal{H}_\epsilon) \) is norm-dense in \( \{ f \in L^2(X) \mid \int_X f d\mu = 0 \} \), by its construction. Thus \( \sigma_t(0,n)f \) converge pointwise for \( f \) in a norm-dense subspace.

As already noted above it follows that \( \gamma_t((0,n)f) \) converges pointwise for \( f \) in the same dense subspace, and the proof of the pointwise ergodic theorem for \( \gamma_t(0,n) \) in \( L^2(X) \) now follows from the maximal inequality. \( \square \)

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