Cluster structure of light nuclei

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Abstract. Matter and charge densities of $k\alpha$ structures with $k=2$ ($^8\text{Be}$), $k=3$ ($^{12}\text{C}$) and $k=4$ ($^{16}\text{O}$) calculated within the framework of the algebraic cluster model (ACM) are briefly reviewed and explicitly displayed. Their parameters are determined from a comparison with electron scattering data.

1. Introduction

The cluster structure of light nuclei has been the subject of many investigations since the seminal work of Wheeler [1]. In 1965, Brink [2],[3] suggested specific geometric configurations for nuclei composed of $k\alpha$-particles, here referred as $k\alpha$ nuclei. In particular, the suggested configurations of the ground state were for $k=2$ ($^8\text{Be}$) a dumbbell configuration with $Z_2$ symmetry, for $k=3$ ($^{12}\text{C}$) an equilateral triangle with $D_{3h}$ symmetry, and for $k=4$ ($^{16}\text{O}$) a tetrahedron with $T_d$ symmetry.

2. Algebraic cluster model

In order to understand what are the signatures of specific geometric configurations, and to provide explicit analytic formulas which can be easily compared with experiment, we have recently developed an algebraic approach to clustering based on the algebraic theory of molecules introduced in 1981 [4]. For $k\alpha$ structures, the approach amounts to a bosonic quantization of the Jacobi variables in terms of the algebra of $U(3k-2)$, where $k$ is the number of constituents. An explicit construction of the algebra has been completed for $k=2$, 3, 4 and is summarized in table 1.

| $k$ | Nucleus | $U(3k-2)$ | Symmetry | Jacobi variables |
|-----|---------|-----------|-----------|------------------|
| 2   | $^8\text{Be}$ | $U(4)$ [4-5] | $Z_2$ | $\rho$ |
| 3   | $^{12}\text{C}$ | $U(7)$ [6-7] | $D_3$ | $\rho, \lambda$ |
| 4   | $^{16}\text{O}$ | $U(10)$ [8-11] | $T_d$ | $\rho, \lambda, \eta$ |
2.1. Matter and charge densities. Analytic expressions for energies and electromagnetic transition rates have been reported previously [12], [13]. Here we report charge and matter densities of \( k \alpha \)-clusters with \( k=2, 3, 4 \) and from these derive analytic expressions for form factors in electron scattering. Densities are obtained by assuming that each cluster has a matter and charge distribution of Gaussian form

\[
\rho_a(r) = \left( \frac{\alpha}{\pi} \right)^{3/2} e^{-a r^2}.
\]

The matter and charge density of \( k \alpha \)-clusters located at a distance \( \beta \) from the center of mass with spherical coordinates \((\beta, \theta, \phi)\) is then given by [14]

\[
\rho(r) = \left( \frac{\alpha}{\pi} \right)^{3/2} \sum_{i=1}^{k} \exp\left[ -\alpha (r-r_i)^2 \right]
\]

\[
= \left( \frac{\alpha}{\pi} \right)^{3/2} e^{-a(r^2+b^2)} 4\pi \sum_{\lambda \mu} i_\lambda j_\lambda (2\alpha \beta r) Y_{\lambda \mu} (\theta, \phi) \sum_{i=1}^{k} Y^*_{\lambda \mu} (\theta_i, \phi_i)
\]

where \( i_\lambda = j_\lambda(ix)/i^k \) is the modified spherical Bessel function. The charge density is obtained by multiplying equation (2) by \( Z e/k \), where \( Z \) is the total charge, and the matter density by \( A/k \), where \( A \) is the total mass. The densities of equation (2) are shown in figures 1-3, respectively.

![Figure 1](image1.png)  
**Figure 1.** Densities of a \( k=2 \) \( \alpha \)-cluster as given in equation (2). The value of \( \alpha=0.56 \) fm\(^2\). The color scale is in fm\(^3\). Reproduced with permission from [14].

![Figure 2](image2.png)  
**Figure 2.** Densities of a \( k=3 \) \( \alpha \)-cluster as given in Eq.(2). The value of \( \alpha=0.56 \) fm\(^2\). The color scale is in fm\(^3\). Reproduced with permission from [14].

One may note that the densities given in figures 1-3 describe clusters with the appropriate symmetry, \( k=2 \) (Z\(_2\)), \( k=3 \) (D\(_{3h}\)), and \( k=4 \) (T\(_d\)), not only in the intermediate region, \( \beta \sim 2 \) fm, but also in the asymptotic region, \( \beta \rightarrow \infty \).
2.2. Form factors in electron scattering. From the charge densities one can evaluate the form factors in electron scattering. For the \( \alpha \)-particle, the elastic form factor is given by the Fourier transform of equation (1),

\[
F_\alpha(q^2) = e^{q^2/4\alpha^2},
\]

conventionally normalized to 1 for \( q^2 = 0 \). By fitting the elastic electron scattering data [15] one obtains \( \alpha = 0.56(2) \text{ fm}^2 \). For \( k\alpha \) clusters, one can derive analytic expressions for both elastic and inelastic scattering to members of the ground state rotational band. They are given by

\[
F_i(0 \rightarrow L) = e^{q^2/4\alpha^2}c_L(q\beta),
\]

which are the convolution of the form factor of the \( \alpha \)-particle with that of the cluster, again normalized to 1 for \( L=0 \) and \( q^2=0 \). Here \( j_L \) is the spherical Bessel function and the coefficients \( c_L \) depend on the symmetry of the cluster. Explicit analytic expressions for the coefficients \( c_L \) are:

\[
Z_2 \quad c_L^2 = \frac{2L+1}{4} \left[ 2 + 2P_L(-1) \right] \quad (5a)
\]

\[
D_{3h} \quad c_L^2 = \frac{2L+1}{9} \left[ 3 + 6P_L \left( -\frac{1}{2} \right) \right] \quad (5b)
\]

\[
T_d \quad c_L^2 = \frac{2L+1}{16} \left[ 4 + 12P_L \left( -\frac{1}{3} \right) \right] \quad (5c)
\]

where \( P_L \) is a Legendre polynomial. The value of \( \beta \) in the density can then be obtained by fitting the elastic form factor data at small \( q^2 \) (charge radii), except for \( ^8\text{Be} \) where electron scattering cannot be done since this nucleus is unstable. In this case the value of \( \beta \) is obtained from the observed moment of inertia [16], which is given by

\[
I = \frac{Am}{\beta^2 + \frac{1}{\alpha}}.
\]

The values so obtained are summarized in table 2.

**Table 2.** Density parameters of \( k\alpha \) structures, \( k=2, 3, 4 \).

| \( k \) | Nucleus | \( \beta [\text{fm}] \) | \( \alpha [\text{fm}^2] \) |
|---|---|---|---|
| 2 | \(^8\text{Be}\) | 1.82(4) | 0.56(2) |
| 3 | \(^{12}\text{C}\) | 1.74(4) | 0.52(4) |
| 4 | \(^{16}\text{O}\) | 2.07(4) | 0.56(4) |
The value of $\alpha$ in $^{12}\text{C}$ appears to be slightly different from the free value due to polarization effects. It should be noted that the parameters in table 2 correspond to close packing of the $\alpha$-particles.

With these values of $\beta$ and $\alpha$ one can calculate all properties of $k\alpha$ nuclei. As an example, in figure 4, the elastic $0^+_1 \rightarrow 0^+_1$ and inelastic $0^+_1 \rightarrow 3^-_1, 0^+_1 \rightarrow 4^+_1$ form factors of $^{16}\text{O}$ are shown as a function of momentum transfer $q$. The agreement with experimental data is excellent, especially for $q \leq 3 \text{ fm}^{-1}$. For larger values of $q$ other effects come into play originating from both meson exchange corrections and from the fact that the $\alpha$ form factor of equation (3) should be corrected as discussed in [15]. It should be noted that while the elastic form factor contains a parameter $\beta$ which is adjusted as discussed above, the inelastic form factors are predictions of the cluster ACM model and do not contain additional adjustable parameters.

**Figure 4.** Comparison between the experimental form factors $|F_L(0 \rightarrow L)|^2$ of $^{16}\text{O}$ for the final states $0^+_1, 3^-_1$ and $4^+_1$ and those obtained with the ACM. Figure adapted from [11].
From the values of $\alpha$ and $\beta$ one can also calculate the charge radii, given, for all cases $k=2, 3$ and 4, by the expression

$$\langle r^2 \rangle^{1/2} = \sqrt{\frac{3}{2\alpha} + \beta^2}$$

(7)

and shown in table 3.

Table 3. Charge radii of $k\alpha$ nuclei

| $k$ | Nucleus | $\langle r^2 \rangle^{1/2}$ (fm) Th. | $\langle r^2 \rangle^{1/2}$ (fm) Exp. |
|-----|---------|-----------------------------------|----------------------------------|
| 2   | $^8$Be   | 2.45                              | ---                             |
| 3   | $^{12}$C | 2.47                              | 2.468(12)                       |
| 4   | $^{16}$O | 2.71                              | 2.710(15)                       |

B(EL) values can also be evaluated in analytic form by making use of

$$B(EL; 0 \to L) = \left(\frac{Ze}{2}\right)^2 \frac{(2L+1)!!}{4\pi} \lim_{q \to \infty} \frac{F_q(0 \to L)}{q^2},$$

(8)

where the normalization of the charge density has been explicitly introduced. From equation (4), one obtains the expressions

$$Z_2 B(EL; 0 \to L) = \left(\frac{Ze\beta_2}{2}\right)^2 \frac{(2L+1)!!}{4\pi} \left[2 + 2p_1 (-1)\right]$$

(9a)

$$D_{3h} B(EL; 0 \to L) = \left(\frac{Ze\beta_3}{3}\right)^2 \frac{(2L+1)!!}{4\pi} \left[3 + 6p_1 \left(-\frac{1}{2}\right)\right]$$

(9b)

$$T_d B(EL; 0 \to L) = \left(\frac{Ze\beta_4}{4}\right)^2 \frac{(2L+1)!!}{4\pi} \left[4 + 12p_1 \left(-\frac{1}{3}\right)\right]$$

(9c)

The comparison with experiments is shown in tables 2-4 of [12] and will not be repeated here.

The density of equation (2), and the expressions equations (4)-(9) which can be derived from it, form the basis for a treatment of structures composed of $k\alpha$-particles plus x-neutrons or protons. By convoluting the density equation (2) with the nucleon-alpha interaction, one can construct the potential generated by the clusters. By considering the form factors equation (4) and the B(EL) values equation (9), one can evaluate the cluster contribution to $k\alpha + x$ neutrons (or protons) structures. The corresponding results are summarized in the accompanying contribution [17].

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References
[1] Wheeler JA 1937 Phys. Rev. 52 1083
[2] Brink DM 1965 Proc. Int. School of Physics “Enrico Fermi”, Course XXXVI, 247
[3] Brink DM, Friedrich H, Weiguny A and Wong CW 1970 Phys. Lett. B 33 143
[4] Iachello F 1981 Chem. Phys. Lett. 78 581
[5] Iachello F 1981 Phys. Rev. C 23 2778 (R)
[6] Bijker R and Iachello F 2000 Phys. Rev. C 61 067305
[7] Bijker R and Iachello F 2002 Ann. Phys. (N.Y.) 298 334
[8] Bijker R 2010 AIP Conf. Proc. 1323 28
[9] Bijker R 2012 J. Phys. Conf. Series 380 012003
[10] Bijker R and Iachello F 2014 *Phys. Rev. Lett.* **112** 152501
[11] Bijker R and Iachello F 2017 *Nucl. Phys. A* **957** 154
[12] Iachello F 2017 *IOP Conf. Series* **863** 012003
[13] Bijker R 2017 *IOP Conf. Series* **863** 012009
[14] Della Rocca V, Bijker R and Iachello F 2017 *Nucl. Phys. A* **966** 158
[15] Sick I, McCarthy JS and Whitney RR 1976 *Phys. Lett. B* **64** 33
[16] Tilley DR et al. 2004 *Nucl. Phys. A* **745** 155
[17] Della Rocca V and Iachello F 2017 These Proceedings