Differential Equations for $\mathbb{F}_q$-Linear Functions, II: Regular Singularity

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Abstract

We study some classes of equations with Carlitz derivatives for \( F_q \)-linear functions, which are the natural function field counterparts of linear ordinary differential equations with a regular singularity. In particular, an analog of the equation for the power function, the Fuchs and Euler type equations, and Thakur’s hypergeometric equation are considered. Some properties of the above equations are similar to the classical case while others are different. For example, a simple model equation shows a possibility of existence of a non-trivial continuous locally analytic \( F_q \)-linear solution which vanishes on an open neighbourhood of the initial point.

Key words: \( F_q \)-linear function; Carlitz derivative; regular singularity; Fuchs equation; Euler equation; hypergeometric equation
1 INTRODUCTION

Let $K$ be the field of formal Laurent series $t = \sum_{j=N}^{\infty} \xi_j x^j$ with coefficients $\xi_j$ from the Galois field $\mathbb{F}_q$, $\xi_N \neq 0$ if $t \neq 0$, $q = p^\nu$, $v \in \mathbb{Z}_+$, where $p$ is a prime number. It is well known that any non-discrete locally compact field of characteristic $p$ is isomorphic to such $K$. The absolute value on $K$ is given by $|t| = q^{-N}$, $|0| = 0$. The ring of integers $O = \{t \in K : |t| \leq 1\}$ is compact in the topology corresponding to the metric $\text{dist}(t, s) = |t - s|$. Let $\overline{K}_c$ be a completion of an algebraic closure of $K$. The absolute value $| \cdot |$ can be extended in a unique way onto $\overline{K}_c$.

A function defined on a $\mathbb{F}_q$-subspace $K_0$ of $K$, with values in $\overline{K}_c$, is called $\mathbb{F}_q$-linear if $f(t_1 + t_2) = f(t_1) + f(t_2)$ and $f(\alpha t) = \alpha f(t)$ for any $t, t_1, t_2 \in K, \alpha \in \mathbb{F}_q$. Many interesting functions studied in analysis over $K$, like analogs of the exponential, logarithm, Bessel, and hypergeometric functions (see e.g. [2, 3, 9, 10, 19, 20, 18, 12, 13]) are $\mathbb{F}_q$-linear and satisfy some differential equations with polynomial coefficients, in which the role of a derivative is played by the operator

$$d = q^{-\nu} \circ \Delta, \quad (\Delta u)(t) = u(xt) - xu(t),$$

where $x$ is a prime element in $K$. The meaning of a polynomial coefficient in the function field case is not a usual multiplication by a polynomial, but the action of a polynomial in the operator $\tau$, $\tau u = u^\varphi$.

This paper is a continuation of the article [14], in which a general theory of such equations was initiated. In particular, we considered regular equations (or systems) of the form

$$du(t) = P(\tau)u(t) + f(t)$$

where for each $z \in (\overline{K}_c)^m$, $t \in K$,

$$P(\tau)z = \sum_{k=0}^{\infty} \pi_k z^k, \quad f(t) = \sum_{j=0}^{\infty} \varphi_j t^q^j,$$

(2)

$\pi_k$ are $m \times m$ matrices with elements from $\overline{K}_c$, $\varphi_j \in (\overline{K}_c)^m$, and the series (2) have positive radii of convergence. The action of the operator $\tau$ upon a vector or a matrix is defined component-wise, so that $z^k(\tau) = (z_1^k, \ldots, z_m^k)$ for $z = (z_1, \ldots, z_m)$. Similarly, if $\pi = (\pi_{ij})$ is a matrix, we write $\tau^k(\pi) = (\pi_{ij}^k)$. $\tau$ is an automorphism of the ring of matrices over $\overline{K}_c$.

It was shown in [14] that for any $u_0 \in (\overline{K}_c)^m$ the equation (1) has a unique local analytic $\mathbb{F}_q$-linear solution satisfying the initial condition

$$\lim_{t \to 0} t^{-1} u(t) = u_0,$$

(3)

Singular higher order scalar equations were also considered (here the singularity means that the leading coefficient is a non-constant analytic function of the operator $\tau$), and it was shown that, in contrast to the classical analytic theory of differential equations, any formal power series solution has a positive radius of convergence. Note however that in general a singular equation need not possess a formal power series solution.
In this paper we study an analog of the class of equations with regular singularity, the most thoroughly studied class of singular equations (see [11] or [4] for the classical theory of differential equations over \(\mathbb{C}\); the case of non-Archimedean fields of characteristic zero was studied in [5, 17]).

A typical class of systems with regular singularity at the origin \(\zeta = 0\) over \(\mathbb{C}\) consists of systems of the form

\[
\zeta y' = \left( B + \sum_{k=1}^{\infty} A_k \zeta^k \right) y
\]

where \(B, A_j\) are constant matrices, and the series converges on a neighbourhood of the origin. Such a system possesses a fundamental matrix solution of the form \(W(\zeta)\zeta^C\) where \(W(\zeta)\) is holomorphic on a neighbourhood of zero, \(C\) is a constant matrix, \(\zeta^C = \exp(C \log \zeta)\) is defined by the obvious power series. Under some additional assumptions regarding the eigenvalues of the matrix \(B\), one can take \(C = B\). For similar results over \(\mathbb{C}_p\) see Sect. III.8 in [5].

In order to investigate such a class of equations in the framework of \(\mathbb{F}_q\)-linear analysis over \(K\), one has to go beyond the class of functions represented by power series. An analog of the power function need not be holomorphic, and cannot be defined as above. Fortunately, we have another option here – instead of power series expansions we can use the expansions in Carlitz polynomials on the compact ring of integers \(O \subset K\). It is important to stress that our approach would fail if we consider equations over \(K_c\) instead of \(K\) (our solutions may take their values from \(K_c\), but they are defined over subsets of \(K\)). In this sense our techniques are different from the ones developed for both the characteristic zero cases.

We begin with the simplest model scalar equation

\[
\tau du = \lambda u, \quad \lambda \in \overline{K}_c, \quad (4)
\]

whose solution may be seen as a function field counterpart of the power function \(t \mapsto t^\lambda\). If \(|\lambda| < 1\), the equation (4) has a non-trivial continuous \(\mathbb{F}_q\)-linear solution \(u\) on the ring of integers \(O \subset K\). This solution is analytic on \(O\) if and only if \(\lambda = [j], j = 0, 1, 2, \ldots\); we use Carlitz’s notation

\[
[j] = x^{q^j} - x.
\]

In this case \(u(t) = ct^{[j]}, c \in \overline{K}_c\). If \(\lambda \neq [j]\) for any \(j\), then the solution of (4) is locally analytic on \(O\) if and only if \(\lambda = -x\), and in the latter case \(u(t) \equiv 0\) for \(|t| \leq q^{-1}\). This paradoxical fact is a good illustration to the violation of the principle of analytic continuation in the non-Archimedean case. It is also interesting that the nonlinear equation \(du = \lambda u\) is much simpler than the linear equation \(\tau du = \lambda u\).

The construction of a solution of the equation (4) is generalized to the case of systems of equations where \(\lambda\) is a matrix. This makes it possible to study the system

\[
\tau du - P(\tau)u = 0 \quad (5)
\]

where \(P(\tau)\) is a matrix-valued analytic function of \(\tau\) of the form (2). We construct a matrix-valued solution of (5) which is written as \(W(g(t))\), where \(g(t)\) is a solution of the equation

\[
\tau dg = \pi_0 g, \quad W(s) = \sum_{k=0}^{\infty} w_k s^{q^k}
\]

has a non-zero radius of convergence. In contrast to similar
results for equations over \( \mathbb{C} \) \cite{4, 11}, here we have a composition of matrix-functions instead of their multiplication. As an example, an Euler-type scalar higher-order equation is considered.

Finally, we study a problem set by Thakur \cite{20}. Within his theory of hypergeometric functions on \( K \), Thakur introduced \cite{19, 20} an analog of the Gauss hypergeometric function \( _2F_1(a, b; c; t) \) and the corresponding differential equation. He constructed two families of analytic solutions which coincide if \( c = 1 \). Classically (over \( \mathbb{C} \)) in the latter case there is another solution with a logarithmic singularity near the origin, and a natural question is about the kind of singularity of a non-analytic solution in our function field case. Note that the function field hypergeometric equation is nonlinear (it is only \( \mathbb{F}_q \)-linear), and the set of solutions is not parametrized by parameters from \( \overline{K}_c \). We prove that in the case \( c = 1 \) a generic solution of the hypergeometric equation is defined only for \( t \in \mathbb{F}_q[x] \), and its formal Fourier-Carlitz series cannot be extended to non-polynomial arguments.

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\section{A MODEL SCALAR EQUATION}

Let us consider the equation (4). We shall look for a continuous \( \mathbb{F}_q \)-linear solution

\[ u(t) = \sum_{i=0}^{\infty} c_i f_i(t), \quad t \in O, \]

(6)

where \( \overline{K}_c \ni c_i \to 0 \) as \( i \to \infty \), \( \{f_i(t)\} \) is the sequence of orthonormal Carlitz polynomials, that is \( f_i(t) = D_i^{-1} e_i(t) \),

\[ D_0 = 1, \quad D_i = [\overline{1}]_i [\overline{1}]_{i-1} \cdots [\overline{1}]_1 \cdot [\overline{1}], \quad e_0(t) = t, \]

\[ e_i(t) = \prod_{\omega \in \mathbb{F}_q[x] \atop \deg \omega < i} (t - \omega), \quad i \geq 1 \]

(7)

(see \cite{2}). The orthonormality means that

\[ \sup_{t \in O} |u(t)| = \sup_i |c_i|. \]

In fact, \( \{f_i\} \) is a basis of the space of continuous \( \mathbb{F}_q \)-linear functions on \( O \) with values in \( \overline{K}_c \) – if \( u \) is such a function, then it can be represented by a convergent series (6) where the coefficients \( c_i \to 0 \) are determined uniquely. Conversely, a series (6) with \( c_i \to 0 \) defines a continuous \( \mathbb{F}_q \)-linear function.

The polynomials \( f_i \) are the characteristic \( p \) analogs of the binomial coefficients forming the Mahler basis of the space of continuous functions on \( \mathbb{Z}_p \), the elements \( D_i \) are the counterparts of the factorials \( i! \).

The rate of decay of the coefficients \( c_i \) corresponds to the smoothness properties of a function \( u \) (see \cite{13, 21}). In particular, \( u(t) \) is locally analytic if and only if

\[ \gamma = \lim_{n \to \infty} \inf \left\{ -q^{-n} \log_q |c_n| \right\} > 0, \]

(8)
and if (8) holds, then \( u(t) \) is analytic on any ball of the radius \( q^{-l} \), \( l = \max(0, \lceil -(\log(q - 1) + \log \gamma)/\log q \rceil + 1) \) (see [21]). Note, in order to avoid confusion, that the formula (8) looks different from the corresponding formula in Sect. 4 of [21]. The reason is that Yang [21] considers expansions of arbitrary continuous functions on \( O \) (not just \( \mathbb{F}_q \)-linear ones) with respect to a certain basis \( \{G_n\} \), such that \( f_n = G_{q^n} \). Therefore \( q^{-n} \) appears in (8), instead of \( n^{-1} \) in the formula from [21].

Since the operator \( \Delta = \tau d \) is linear, we have \( \Delta f_i = D_i^{-1} \Delta e_i \), \( i \geq 1 \); clearly \( \Delta f_0 = 0 \). It is known [9] that

\[
\Delta e_i = \frac{D_i}{D_i^q} e_{i-1}^q, \quad e_{i-1}^q = e_i + D_{i-1}^{q-1} e_{i-1}.
\]

Since \( D_i = [i] D_i^q \), we find that

\[
\Delta f_i = [i] f_i + f_{i-1}, \quad i \geq 1.
\] (9)

It follows from (6) and (9) that

\[
\Delta u(t) = \sum_{j=0}^{\infty} (c_{j+1} + [j] c_j) f_j(t).
\]

Substituting into (4) and using uniqueness of the Carlitz expansion we find a recurrence relation

\[
c_{j+1} + [j] c_j = \lambda c_j, \quad j = 0, 1, 2, \ldots,
\]

whence, given \( c_0 \), the solution is determined uniquely by

\[
c_n = c_0 \prod_{j=0}^{n-1} (\lambda - [j]).
\]

Suppose that \( |\lambda| \geq 1 \). Since \( |[j]| = q^{-1} \) for \( j \geq 1 \), we see that \( |c_n| = |c_0| \cdot |\lambda|^n \not\to 0 \) if \( c_0 \neq 0 \). This contradiction shows that the equation (4) has no continuous solutions if \( |\lambda| \geq 1 \). Therefore we shall assume that \( |\lambda| < 1 \). Let \( u(t, \lambda) \) be the solution of (4) with \( c_0 = 1 \); note that the fixation of \( c_0 \) is equivalent to the initial condition \( u(1, \lambda) = c_0 \). The function \( u(t, \lambda) \) is a function field counterpart of the power function \( t^\lambda \).

**Theorem 1.** The function \( t \mapsto u(t, \lambda), |\lambda| < 1 \), is continuous on \( O \). It is analytic on \( O \) if and only if \( \lambda = [j] \) for some \( j \geq 0 \); in this case \( u(t, \lambda) = u(t, [j]) = t^q^j \). If \( \lambda \neq [j] \) for any integer \( j \geq 0 \), then \( u(t, \lambda) \) is locally analytic on \( O \) if and only if \( \lambda = -x \), and in that case \( u(t, -x) = 0 \) for \( |t| \leq q^{-1} \). The relation

\[
u(t^q^m, \lambda) = u(t, \lambda^q^m + [m]), \quad t \in O,
\] (10)

holds for all \( \lambda, |\lambda| < 1 \), and for all \( m = 0, 1, 2, \ldots \).

**Proof.** If \( u(t) = t^q^j \), \( j \geq 0 \), then

\[
\Delta u(t) = (xt)^q^j - xt^q^j = (x^q^j - x) t^q^j = [j] u(t),
\]

\[
t \in O.
\]
so that \( u(t, [j]) = t^{q^j} \).

Suppose that \( \lambda \neq [j], j = 0, 1, 2, \ldots \) Then \( |c_n| \leq \{\max(|\lambda|, q^{-1})\}^n \to 0 \) as \( n \to \infty \), so that \( u(t, \lambda) \) is continuous. More precisely, if \( \lambda \neq -x \), then \( |\lambda + x| = q^{-\nu} \) for some \( \nu > 0 \),

\[
|\lambda - [j]| = |(\lambda + x) - x^{q^j}| = q^{-\nu}, \quad j \geq j_0,
\]

if \( j_0 \) is large enough. This means that for some positive constant \( C \)

\[
|c_n| = Cq^{-n\nu}, \quad n \geq j_0. \tag{11}
\]

On the other hand, if \( \lambda = -x \), then \( |\lambda - [j]| = q^{-q^j} \),

\[
|c_n| = q^{-\frac{n(q^{q^j})}{q-1}}, \tag{12}
\]

If \( \lambda \neq -x \), then by (11) \( \gamma = 0 \), so that \( u(t, \lambda) \) is not locally analytic. If \( \lambda = -x \), we see from (8) and (12) that \( \gamma = (q - 1)^{-1}, l = 1 \), and \( u(t, -x) \) is analytic on any ball of the radius \( q^{-1} \). We have

\[
u(t, -x) = \sum_{n=0}^{\infty} (-1)^n x^{\frac{q^n}{q-1}} f_n(t),
\]

and \( u(t, -x) \) is not the identical zero on \( O \) due to the uniqueness of the Fourier-Carlitz expansion.

At the same time, since \( u(t, -x) \) is analytic on the ball \( \{|t| \leq q^{-1}\} \), we can write

\[
u(t, -x) = \sum_{m=0}^{\infty} a_m t^{q^m}, \quad |t| \leq q^{-1}.
\]

Substituting this into the equation (4) with \( \lambda = -x \), we find that \( a_m = 0 \) for all \( m \), that is \( u(t, -x) = 0 \) for \( |t| \leq q^{-1} \).

In order to prove (10), note first that (10) holds for \( \lambda = [j], j = 0, 1, 2, \ldots \). Indeed,

\[
u(t^{q^m}, [j]) = (t^{q^m})^{q^j} = t^{q^{m+j}}
\]

and

\[ [j]^{q^m} + [m] = (x^{q^j} - x)^{q^m} + x^{q^m} - x = [m + j]. \]

Let us fix \( t \in O \). We have

\[
u(t, \lambda) = \sum_{n=0}^{\infty} \left\{ \prod_{j=0}^{n-1} (\lambda - [j]) \right\} f_n(t),
\]

and the series converges uniformly with respect to \( \lambda \in \overline{F}_{r} \) where

\[
\overline{F}_{r} = \{ \lambda \in \overline{K}_c : |\lambda| \leq r \},
\]

for any positive \( r < 1 \). Thus \( u(t, \lambda) \) is an analytic element on \( \overline{F}_{r} \) (see Chapter 10 of [3]). Similarly, \( u(t^{q^m}, \lambda) \) and \( u(t, \lambda^{q^m} + [m]) \) are analytic elements on \( \overline{F}_{r} \) (for the latter see Theorem
11.2 from [6]). Suppose that $q^{-1} \leq r < 1$. Since both sides of (10) coincide on an infinite sequence of points $\lambda = [j]$, $j = 0, 1, 2, \ldots$, they coincide on $P_r$ (see Corollary 23.8 in [6]). This implies their coincidence for $|\lambda| < 1$. ■

Similarly, if in (4) $\lambda$ is a $m \times m$ matrix with elements from $\mathcal{K}_c$ (we shall write $\lambda \in M_m(\mathcal{K}_c)$), and we look for a solution $u \in M_m(\mathcal{K}_c)$, then we can find a continuous solution (6) with matrix coefficients

$$c_i = \left\{ \prod_{j=0}^{i-1} (\lambda - [j]I_m) \right\} c_0, \quad i \geq 1$$

(13)

($I_m$ is a unit matrix) if $|\lambda| \overset{\text{def}}{=} \max |\lambda_{ij}| < 1$. Note that $c_0 = u(1)$, so that if $c_0$ is an invertible matrix, then $u$ is invertible on a certain neighbourhood of 1.

3 FIRST ORDER SYSTEMS

Let us consider a system (5) with the coefficient $P(\tau)$ given in (2). We assume that $|\pi_k| \leq \gamma$, $\gamma > 0$, for all $k$, $|\pi_0| < 1$. Denote by $g(t)$ a solution of the equation $\tau dg = \pi_0 g$. Let $\lambda_1, \ldots, \lambda_m \in \mathcal{K}_c$ be the eigenvalues of the matrix $\pi_0$.

Theorem 2. If

$$\lambda_i - \lambda_j^k \neq [k], \quad i, j = 1, \ldots, m; \quad k = 1, 2, \ldots, \quad (14)$$

then the system (5) has a matrix solution

$$u(t) = W(g(t)), \quad W(s) = \sum_{k=0}^{\infty} w_k s^k, \quad w_0 = I_m, \quad (15)$$

where the series for $W$ has a positive radius of convergence.

Proof. Substituting (15) into (5), using the fact that $\Delta = \tau d$ is a derivation of the composition ring of $\mathbf{F}_q$-linear series, and that $\Delta(t^g) = [k] t^g$, we come to the identity

$$\sum_{k=0}^{\infty} [k] w_k \tau^k(g(t)) + \sum_{k=0}^{\infty} w_k \tau^k(\pi_0) \tau^k(g(t)) - \sum_{j=0}^{\infty} \pi_j \tau^j \left( \sum_{k=0}^{\infty} w_k \tau^k(g(t)) \right) = 0. \quad (16)$$

If the series for $W$ has indeed a positive radius of convergence (which will be proved later), then all the expressions in (16) make sense for a small $|t|$, since $g(t) \to 0$ as $|t| \to 0$. Since $w_0 = I_m$, the first summand in the second sum in (16) and the summand with $j = k = 0$ in the third sum are cancelled. Changing the order of summation we find that (16) is equivalent to the system of equations

$$w_k ( [k] I_m + \tau^k(\pi_0) ) - \pi_0 w_k = \sum_{j=1}^{k} \pi_j \tau^j(w_{k-j}), \quad k = 1, 2, \ldots, \quad (17)$$

with respect to the matrices $w_k$. 8
The system (17) is solved step by step – if the right-hand side of (17) with some \( k \) is already known, then \( w_k \) is determined uniquely, provided the spectra of the matrices \([k]I_m + \tau^k(\pi_0)\) and \( \pi_0 \) have an empty intersection ([1], Sect. VIII.1). This condition is equivalent to (14), and it remains to prove that the series for \( W \) has a non-zero radius of convergence.

Let us transform \( \pi_0 \) to its Jordan normal form. We have \( U^{-1}\pi_0U = A \) where \( U \) is an invertible matrix over \( \mathbb{K}_c \), and \( A \) is block-diagonal:

\[
A = \bigoplus_{\alpha=1}^l (\lambda^{(\alpha)} I_{d_\alpha} + H^{(\alpha)})
\]

where \( \lambda^{(\alpha)} \) are eigenvalues from the collection \( \{\lambda_1, \ldots, \lambda_m\} \), \( H^{(\alpha)} \) is a Jordan cell of the order \( d_\alpha \) having zeroes on the principal diagonal and 1’s on the one below it. Denote \( \mu_k^{(\alpha)} = (\lambda^{(\alpha)})^{q^k} + [k] \).

If \( V_k = \tau^k(U) \), then

\[
V_k^{-1} \left( [k]I_m + \tau^k(\pi_0) \right) V_k = \bigoplus_{\alpha=1}^l \left( \mu_k^{(\alpha)} I_{d_\alpha} + H^{(\alpha)} \right).
\]

If \( B_k \) is the matrix (18), and \( C_k \) is the matrix in the right-hand side of (17), then (17) takes the form

\[
w_k V_k B_k V_k^{-1} - UAU^{-1}w_k = C_k
\]

or, if we use the notation \( \tilde{w}_k = U^{-1}w_k V_k \),

\[
\tilde{w}_k B_k - A\tilde{w}_k = \tilde{C}_k, \quad k = 1, 2, \ldots, \quad (19)
\]

where

\[
\tilde{C}_k = U^{-1}C_k V_k = U^{-1} \left( \sum_{j=1}^k \pi_j \tau_j \left( U\tilde{w}_{k-j}V_{k-j}^{-1} \right) \right) V_k = U^{-1} \sum_{j=1}^k \pi_j \tau_j (U) \tau_j (\tilde{w}_{k-j}),
\]

\( \tilde{w}_0 = I_m \). We may assume that \(|U| < 1\), \(|U^{-1}| \leq \rho\), \( \rho > 0 \).

In accordance with the quasi-diagonal form of the matrices \( A \) and \( B_k \) we can decompose the matrix \( \tilde{w}_k \) into \( d_\alpha \times d_\beta \) blocks

\[
\tilde{w}_k = \begin{pmatrix} \tilde{w}_k^{(\alpha\beta)} \end{pmatrix}, \quad \alpha, \beta = 1, \ldots, l.
\]

Similarly we write \( \tilde{C}_k = \begin{pmatrix} \tilde{C}_k^{(\alpha\beta)} \end{pmatrix} \). Then the system (19) is decoupled into a system of equations for each block:

\[
\left( \mu_k^{(\beta)} - \lambda^{(\alpha)} \right) \tilde{w}_k^{(\alpha\beta)} - H^{(\alpha)} \tilde{w}_k^{(\alpha\beta)} + \tilde{w}_k^{(\alpha\beta)} H^{(\beta)} = \tilde{C}_k^{(\alpha\beta)}.
\]

The equation (20) can be considered as a system of scalar equations with respect to elements of the matrix \( \tilde{w}_k^{(\alpha\beta)} \). Let us enumerate these elements \( \begin{pmatrix} \tilde{w}_k^{(\alpha\beta)} \end{pmatrix} \) lexicographically (in i,j) with the inverse enumeration order of the second index \( j \). The product \( H^{(\alpha)} \tilde{w}_k^{(\alpha\beta)} \) is obtained from \( \tilde{w}_k^{(\alpha\beta)} \) by the shift of all the rows one step upwards, the last row being filled by zeroes. Similarly,
the product \( \tilde{w}_k^{(\alpha \beta)} H^{(\beta)} \) is the result of shifting all the columns of \( \tilde{w}_k^{(\alpha \beta)} \) one step to the right and filling the first column by zeroes (\[4\], Sect. I.3). This means that the system (2) (with fixed \( \alpha, \beta \)) with the above enumeration of the unknowns is upper triangular. Indeed, the latter is equivalent to the fact that each equation contains, together with some unknown, only the unknowns with larger numbers, and this property is the result of the above shifts.

Therefore the determinant \( D_k^{(\alpha \beta)} \) of the system (20) equals \((\mu_k^{(\beta)} - \lambda^{(\alpha)})^{d_\alpha d_\beta}\). By our assumption \( |\pi_0| < 1 \), and if \( \lambda^{(\alpha)} \) is an eigenvalue of \( \pi_0 \) with an eigenvector \( f \neq 0 \), then \( |\lambda^{(\alpha)}| \cdot |f| = |\pi_0 f| < |f| \), so that \( |\lambda^{(\alpha)}| < 1 \). This means that all the coefficients on the left in (2) have the absolute values \( \leq \rho \) with \( \rho \) fixed \( \alpha, \beta \) and filling the first column by zeroes (\[7\], Sect. I.3). This means that the system (2) (with unknowns with larger numbers, and this property is the result of the above shifts.

It follows from (14) that \( \lambda_i \neq -x, i = 1, \ldots, n \). As \( k \to \infty \), \( \mu_k^{(\beta)} = (\lambda^{(\beta)})^{q^k} + x^{q^k} - x \to -x \). Thus \( \left| \mu_k^{(\beta)} - \lambda^{(\alpha)} \right| \geq \sigma_1 > 0 \) for all \( k \), whence \( \left| D_k^{(\alpha \beta)} \right| \geq \sigma_2 > 0 \) where \( \sigma_2 \) does not depend on \( k \).

Now we obtain an estimate for the solution of the system (19),

\[
|\tilde{w}_k| \leq \rho_1 |\tilde{C}_k|, 
\]

with \( \rho_1 > 0 \) independent of \( k \).

Looking at (21) we find that

\[
|\tilde{w}_k| \leq \rho_2 \max_{1 \leq j \leq k} |\tilde{w}_{k-j}|^{q^j}
\]

where \( \rho_2 \) does not depend on \( k \). We may assume that \( \rho_2 \geq 1 \). Now we find that

\[
|\tilde{w}_k| \leq \rho_2^{q^k-1+q^{k-2}+\ldots+1}, \quad k = 1, 2, \ldots 
\]

Indeed, (22) is obvious for \( k = 1 \). Suppose that we have proved the inequalities

\[
|\tilde{w}_j| \leq \rho_2^{q^{j-1}+q^{j-2}+\ldots+1}, \quad 1 \leq j \leq k - 1.
\]

Then

\[
|\tilde{w}_k| \leq \rho_2 \max \left( 1, |\tilde{w}_1|^{q^{k-1}}, \ldots, |\tilde{w}_{k-1}|^{q^k} \right)
\]

\[
\leq \rho_2 \max \left( 1, \rho_2^{q^{k-1}}, \rho_2^{q^{(q+1)q^{k-2}}}, \ldots, \rho_2^{q^{(q^{k-2}+\ldots+1)q}} \right) = \rho_2^{q^{k-1}+q^{k-2}+\ldots+1},
\]

and (22) is proved.

Therefore, since \( w_k = U \tilde{w}_k \tau^k (U^{-1}) \), we have

\[
|w_k| \leq \rho_3^{k} \cdot \rho_2^{q^{k-1}+\ldots+1} \leq \rho_3^{\frac{k+1}{q}}, \quad \rho_3 > 0,
\]

which means that the series in (15) has a positive radius of convergence. \( \blacksquare \)

Remarks. 1). If \( \varphi \in F_q^m \), then, as usual, \( v = u \varphi \) is a vector solution of the system \( \tau dv - P(\tau)v = 0 \), since the system is \( F_q \)-linear. However, the system is nonlinear over \( K_c \), so that we cannot obtain a vector solution in such a way for an arbitrary \( \varphi \in K_c \).

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2). Analogs of the condition (14) occur also in the analytic theory of differential equations over \( \mathbb{C} \) (see Corollary 11.2 in [11]) and \( \mathbb{Q}_p \) (Sect. III.8 in [5]). For systems over \( \mathbb{C} \), it is requested that differences of the eigenvalues of the leading coefficient \( \pi_0 \) must not be non-zero integers. Over \( \mathbb{Q}_p \), in addition to that, the eigenvalues must not be non-zero integers themselves. In both the characteristic zero cases it is possible to get rid of such conditions by using special changes of variables called shearing transformations. For example, let \( m = 1 \), and the equation over \( \mathbb{Q}_p \) has the form

\[
\zeta u'(\zeta) = nu(\zeta) + \left( \sum_{k=1}^{\infty} \pi_k \zeta^k \right) u(\zeta), \quad n \in \mathbb{N}.
\]

Then the change of variables \( u(\zeta) = \zeta v(\zeta) \) gives the transformed equation

\[
\zeta v'(\zeta) = (n - 1)v(\zeta) + \left( \sum_{k=1}^{\infty} \pi_k \zeta^k \right) v(\zeta).
\]

Repeating the transformation, we remove the term violating the condition. A modification of this approach works for systems of equations.

In our case the situation is different. Let us consider again the scalar case \( m = 1 \). Here the condition (14) is equivalent to the condition \( \pi_0 \neq -x \) (the general solution of the equation \( \pi_0 - \pi_q^k = [k] \) has the form \( \pi_0 = -x + \xi \) where \( \xi - \xi^q = 0 \), that is either \( \xi = 0 \), or \( |\xi| = 1 \); the latter contradicts our assumption \( |\pi_0| < 1 \). If, on the contrary, \( \pi_0 = -x \), then, as we saw in Theorem 1, \( g(t) = 0 \) for \( |t| \leq q^{-1} \), and the construction (15) does not make sense. On the other hand, a formal analog of the shearing transformation for this case is the substitution \( u = \tau(v) \). However it is easy to see that \( v \) satisfies an equation with the same principal part, as the equation for \( u \).

### 4 Euler Type Equations

Classically, the Euler equation has the form

\[
\zeta^m u^{(m)}(\zeta) + \beta_{m-1}\zeta^{m-1} u^{(m-1)}(\zeta) + \cdots + \beta_0 u(\zeta) = 0
\]

where \( \beta_0, \beta_1, \ldots, \beta_{m-1} \in \mathbb{C} \). It can be reduced to a first order linear system with a constant matrix. The solutions are linear combinations of functions of the form \( \zeta^k (\log \zeta)^k \). Of course, such functions have no direct \( \mathbb{F}_q \)-linear counterparts, and our study of the Euler-type equations will again be based on expansions in the Carlitz polynomials.

Let us consider a linear equation

\[
\tau^m d^m u + b_{m-1}\tau^{m-1} d^{m-1} u + \cdots + b_0 u = 0
\]

(23)

where \( b_0, b_1, \ldots, b_{m-1} \in \overline{\mathbb{K}}_c \). In order to reduce (23) to a first order system, it is convenient to set

\[
\varphi_k = \tau^{k-1} d^{k-1} u, \quad k = 1, \ldots, m.
\]

Since \( d^{k-1} - \tau^{k-1} d = [k - 1]^{1/q} \tau^{k-2} \) (see [14]), we have

\[
\tau d\varphi_k = \tau^k d^k u + [k - 1]^{1/q} \tau^{k-1} d^{k-1} u = \varphi_{k+1} + [k - 1]^{1/q} \varphi_k,
\]

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\[ k = 1, \ldots, m - 1. \] Next, by (23),
\[ \tau \partial \varphi_m = \tau^{m} \partial^m u + [m - 1] \tau^{m-1} \partial^{m-1} u = ([m - 1] - b_{m-1}) \varphi_m - b_{m-2} \varphi_{m-1} - \cdots - b_0 \varphi_1. \]

Thus the equation (23) can be written as a system
\[ \tau \partial \varphi = B \varphi, \quad \varphi = (\varphi_1, \ldots, \varphi_m), \quad (24) \]
where
\[
B = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & [1] & 1 & 0 & \ldots & 0 \\
0 & 0 & [2] & 1 & \ldots & 0 \\
0 & 0 & 0 & [3] & \ldots & 0 \\
& \cdots & & & \cdots & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 \\
-b_0 & -b_1 & -b_2 & -b_3 & \ldots & [m - 1] - b_{m-1}
\end{pmatrix}
\]

This time we cannot use directly the above results, since \(|B| \geq 1\). However in some cases it is possible to proceed in a slightly different way.

Suppose that all the eigenvalues of the matrix \(B\) lie in the open disk \(|\lambda| < 1\). Transforming \(B\) to its Jordan normal form we find that
\[ B = X^{-1}(B_0 + N)X \]
where \(X\) is an invertible matrix, \(B_0\) is a diagonal matrix, \(|B_0| = \mu < 1\), \(N\) is nilpotent, that is \(N^{\kappa} = 0\) for some natural number \(\kappa\), and \(N\) commutes with \(B_0\). If \(\Psi\) is a matrix solution of the system
\[ \tau \partial \Psi = (B_0 + N)\Psi, \]
then \(\Phi = X^{-1}\Psi X\) is a matrix solution of (24).

On the other hand, we can obtain \(\Psi\) just as in the case \(N = 0\) considered in Sect. 2, writing
\[ \Psi(t) = \sum_{i=0}^{\infty} c_i f_i(t), \quad (25) \]
\[ c_i = \left\{ \prod_{j=0}^{i-1} (B_0 + N - [j]I_m) \right\} c_0. \quad (26) \]

Indeed, the product in (26) is the sum of the expressions \((-[j])^{\nu_1} B_0^{\nu_2} N^{\nu_3}\) where \(\nu_1 + \nu_2 + \nu_3 = i\), \(\nu_3 < \kappa\). Therefore in (25)
\[ |c_i| \leq |c_0| \cdot |N|^\kappa \left\{ \max(\mu, q^{-1}) \right\}^{i-\kappa} \rightarrow 0, \quad i \rightarrow \infty. \]

Let us consider in detail the case where \(m = 2\). Our equation is
\[ \tau^2 \partial^2 u + b_1 \tau \partial u + b_0 u = 0. \quad (27) \]

Now we have the system (24) with
\[
B = \begin{pmatrix}
0 & 1 \\
-b_0 & [1] - b_1
\end{pmatrix}.
\]
The characteristic polynomial of \( B \) is \( D_2(\lambda) = \lambda^2 + \lambda(b_1 - [1]) + b_0 \), with the discriminant \( \delta = (b_1 - [1])^2 - 4b_0 \). We assume that the eigenvalues are such that \( |\lambda_1|, |\lambda_2| < 1 \). This condition is satisfied, for example, if \( p \neq 2, |b_0| < 1, |b_1| < 1 \).

The greatest common divisor of the first order minors of \( B - \lambda I_2 \) is 1. This means that \( B \) is diagonalizable if and only if \( \lambda_1 \neq \lambda_2 \), that is if \( \delta \neq 0 \) (see e.g. [9]). In this case

\[
B = X^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X
\]

for some invertible matrix \( X \), and our system has a matrix solution \( \Phi \), such that

\[
X\Phi(t)X^{-1} = tI_2 + \sum_{n=1}^{\infty} \text{diag}\left\{ \prod_{j=0}^{n-1} (\lambda_1 - [j]), \prod_{j=0}^{n-1} (\lambda_2 - [j]) \right\} f_n(t) \overset{\text{def}}{=} \text{diag}\{ \psi_1(t), \psi_2(t) \}.
\]

It is easy to see that \( \psi_1(t) \) and \( \psi_2(t) \) are solutions of the equation (27). Indeed, if \( X^{-1} = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \), then

\[
\Phi(t)X^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\xi_{11}\psi_1(t), \xi_{21}\psi_2(t)),
\]

and (for \( \psi_1 \)) it is sufficient to show that \( \xi_{11} \neq 0 \). However \( X^{-1}X = I_2 \), and if \( \xi_{11} = 0 \), then writing \( X = (\chi_{11}^{12} \chi_{22}^{12}) \) we find that \( \xi_{12}\chi_{22} = 0, \xi_{12}\chi_{21} = 1 \). At the same time, by (28), \( \xi_{12}\lambda_2\chi_{21} = 0, \) and \( \xi_{12}\lambda_2\chi_{22} = 1 \), and we come to a contradiction. A similar reasoning works for \( \psi_2(t) \). It follows from the uniqueness of the Fourier-Carlitz expansion that \( \psi_1 \) and \( \psi_2 \) are linearly independent.

If \( \lambda_1 = \lambda_2 \overset{\text{def}}{=} \lambda \), then \( B \) is similar to the Jordan cell

\[
N = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.
\]

It is proved by induction that

\[
\prod_{j=0}^{n-1} (N - [j]I_2) = \begin{pmatrix} \prod_{j=0}^{n-1} (\lambda - [j]) \sum_{j=0}^{n-1} \prod_{i \neq j}^{n-1} (\lambda - [i]) \\ 0 \prod_{j=0}^{n-1} (\lambda - [j]) \end{pmatrix}.
\]

In this case we have the following two linearly independent solutions of (27):

\[
\psi_1(t) = t + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} (\lambda - [j]) \right\} f_n(t),
\]

\[
\psi_2(t) = t + \sum_{n=1}^{\infty} \left\{ \sum_{j=0}^{n-1} \prod_{i \neq j}^{n-1} (\lambda - [i]) \right\} f_n(t).
\]

Thus, for the case of the eigenvalues from the disk \( \{ |\lambda| < 1 \} \), we have given an explicit construction of solutions for the Euler type equations.
5 DISCONTINUOUS SOLUTIONS

For all the above equations, the solutions were found as Fourier-Carlitz expansions (6), and we had to impose certain conditions upon coefficients of the equations, in order to guarantee the uniform convergence of the series (6) on $O$. However, formally we could write the series for the solutions without those conditions. Thus it is natural to ask whether the corresponding series (6) converge at some points $t \in O$. Note that (6) always makes sense for $t \in \mathbb{F}_q[x]$ (for each such $t$ only a finite number of terms is different from zero). The question is whether the series (6) converges on a wider set; if the answer is negative, such a formal solution will be called strongly singular.

We shall need the following property of the Carlitz polynomials.

**Lemma 1.**

$$|f_i(x^n)| = \begin{cases} 0, & \text{if } n < i, \\ q^{i-n}, & \text{if } n \geq i. \end{cases}$$

**Proof.** If $n < i$, it follows from (7) that $f_i(x^n) = 0$. Let $n \geq i$. Then $|x^n - \omega| = |\omega|$ for all $\omega \in \mathbb{F}_q[x]$, $\deg \omega < i$. Writing

$$e_i(t) = t \prod_{\substack{0 \neq \omega \in \mathbb{F}_q[x] \\ \deg \omega < i}} (t - \omega)$$

we find that

$$|e_i(x^n)| = |x^n| \prod_{\substack{\deg \omega < i \\ \omega \neq 0}} |\omega| = |x^n| \cdot \lim_{t \to 0} \frac{|e_i(t)|}{t}.$$ 

It is known [2, 3, 10] that

$$e_i(t) = \sum_{j=0}^{i} (-1)^{i-j} \frac{D_i}{D_j L_{i-j}^n} t^{q^j}$$

where $L_i = [i][i-1] \cdots [1]$, $i \geq 1$, and $L_0 = 1$. Therefore

$$\lim_{t \to 0} \frac{e_i(t)}{t} = (-1)^i \frac{D_i}{L_i}$$

whence

$$|f_i(x^n)| = \frac{q^{-n}}{|L_i|} = q^{i-n}$$

as desired. 

Now we get a general sufficient condition for a function (6) to be strongly singular.

**Theorem 3.** If $|c_i| \geq \rho > 0$ for all $i \geq i_0$ (where $i_0$ is some natural number), then the function (6) is strongly singular.
Proof. In view of the convergence criterion for series in a non-Archimedean field (see Sect. 1.1.8 in [1]), it is sufficient to find, for any \( t \neq F_q[x] \), such a sequence \( i_k \to \infty \) that \( |f_{i_k}(t)| = 1 \), \( k = 1, 2, \ldots \).

In fact, if \( t \in O \setminus F_q[x] \), then \( t = \sum_{n=0}^{\infty} \xi_n x^n \), \( \xi_n \in F_q \), with \( \xi_{i_k} \neq 0 \) for some sequence \( i_k \to \infty \).

We have (by Lemma 1)

\[
 f_{i_k}(t) = \sum_{n=i_k}^{\infty} \xi_n f_{i_k}(x^n)
\]

where \( |f_{i_k}(x^{i_k})| = 1 \), \( |\xi_{i_k}| = 1 \), \( |f_{i_k}(x^{i_k})| = q^{i_k-n} \leq q^{-1} \) for \( n > i_k \). Thus \( |f_{i_k}(t)| = 1 \) for all \( k \). □

It follows from Theorem 3 and the discussion preceding Theorem 1 that non-trivial formal solutions of the equation (4) with \( |\lambda| \geq 1 \) are strongly singular. A more complicated example of an equation with such solutions will be given in the next section.

6 HYPERGEOMETRIC EQUATION

The equation for Thakur’s function field analog of the hypergeometric function \( _2F_1(a, b; 1; t) \) has the form

\[
 (\Delta - [-a])(\Delta - [-b])u = d\Delta u
\]

where \( a, b \in \mathbb{Z} \).

A corresponding classical equation over \( \mathbb{C} \) has the form

\[
 \left( \zeta \frac{d}{d\zeta} + a \right) \left( \zeta \frac{d}{d\zeta} + b \right) u = \frac{d}{d\zeta} \left( \zeta \frac{d}{d\zeta} \right). \]

Its holomorphic solution near the origin is \( _2F_1(a, b; 1; \zeta) \); the second solution has a logarithmic singularity (see [1], Chapter 5, §§10, 11).

As before, we look for a solution of the form (6) defined at least for \( t \in F_q[x] \); note that the operators \( \Delta \) and \( d \) are well-defined on functions of \( t \in F_q[x] \). Using the relation (8) and the fact that \( df_0 = 0 \), \( df_i = f_{i-1} \) (\( i \geq 1 \)) we obtain a recursive relation

\[
 \left( c_{i+2}^{1/q} - c_{i+2} \right) + c_{i+1}^{1/q} [i + 1]^{1/q} - c_{i+1} ([i] + [i + 1] - [-a] - [-b]) - c_i ([i] - [-a])([i] - [-b]) = 0, \quad i = 0, 1, 2, \ldots
\]

(30)

Taking arbitrary initial coefficients \( c_0, c_1 \in K_c \) we obtain a solution \( u \) defined on \( F_q[x] \). On each step we have to solve the equation

\[
 z^{1/q} - z = v.
\]

(31)

If \( |c_i| \leq 1 \) and \( |c_{i+1}| \leq 1 \), then in the equation for \( c_{i+2} \) we have \( |v| < 1 \).

Lemma 2. The equation (31) with \( |v| < 1 \) has a unique solution \( z_0 \in K_c \), for which \( |z_0| \leq |v| \), and \( q-1 \) other solutions \( z, |z| = 1 \).
Proof. It is convenient to investigate the equivalent equation

\[ z^q - z = w, \quad |w| < 1. \]

Consider the polynomial \( \varphi(z) = z^q - z - w, \varphi \in O[z] \). Since \( \varphi'(z) \equiv -1 \) and \( \varphi(w^{1/q}) = -w^{1/q} \), we have \( |\varphi(w^{1/q})| = |w|^{1/q} < 1 = \left| \left\{ \varphi'(w^{1/q}) \right\} \right|^2 \). Thus we are within the conditions of the version of the Hensel lemma for a field with a possibly non-discrete valuation, given in Chapter XII of [15]. By that result the polynomial \( \varphi \) has such a root \( z_0 \in K_c \) that \( |z_0 - w^{1/q}| \leq |w|^{1/q} \), whence

\[ |z_0| \leq \max \left( |z_0 - w^{1/q}|, |w|^{1/q} \right) = |w|^{1/q}. \]

Taking \( w = -v^q \) we obtain the required solution of (31).

Other solutions of (31) have the form \( z = z_0 + \theta, 0 \neq \theta \in F_q \). Obviously \( |z| = 1 \). ■

It follows from Lemma 2 that if \( |c_i|, |c_{i+1}| \leq 1 \) for some \( i \), then \( |c_n| \leq 1 \) for all \( n \geq i \).

The relation (30), Lemma 2, and Theorem 3 imply the following property of solutions of the equation (29). It is natural to call a solution generic if, starting from a certain step of finding the coefficients \( c_n \), we always take the most frequent option corresponding to a solution of (31) with \( |z| = 1 \).

**Theorem 4.** A generic solution of the equation (29) is strongly singular.

Of course, in some special cases the recursion (30) can lead to more regular solutions, in particular, to the holomorphic solutions found by Thakur [19, 20].

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