Formal equivariant $\hat{A}$ class, splines and multiplicities of the index of transversally elliptic operators

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Abstract. Let $G$ be a connected compact Lie group acting on a manifold $M$ and let $D$ be a transversally elliptic operator on $M$. The multiplicity of the index of $D$ is a function on the set $\hat{G}$ of irreducible representations of $G$. Let $T$ be a maximal torus of $G$ with Lie algebra $t$. We construct a finite number of piecewise polynomial functions on $t^*$, and give a formula for the multiplicity in terms of these functions. The main new concept is the formal equivariant $A$ class.

Keywords: equivariant index, equivariant $K$-theory, splines.

§ 1. Introduction

Let $G$ be a compact Lie group acting on a manifold $M$ of dimension $d$. As described in the monograph [1], Atiyah–Singer have associated to any $G$-transversally elliptic symbol $\sigma$ on $M$ a virtual trace class representation of $G$. Let $\text{Index}_G(\sigma)(g)$ be its trace:

$$\text{Index}_G(\sigma)(g) = \sum_{\lambda \in \hat{G}} \text{mult}_G(\sigma)(\lambda) \chi_\lambda(g).$$

Thus $\text{Index}_G(\sigma)(g)$ is a $G$-invariant (generalized) function on $G$, and the right hand side of the above formula is its Fourier expansion in terms of the traces $\chi_\lambda$ of the unitary irreducible representations $V_\lambda$ of $G$. If $D$ is a transversally elliptic operator with principal symbol $\sigma$, we write indifferently $\text{Index}_G(D)$ or $\text{Index}_G(\sigma)$, $\text{mult}_G(D)$ or $\text{mult}_G(\sigma)$. The computation of $\text{mult}_G(\sigma)(\lambda)$ is important. For example, if $D$ is a transversally elliptic operator with principal symbol $\sigma$, the multiplicity of the trivial representation in $\text{Index}_G(\sigma)$ is the (virtual) dimension of the space of $G$-invariant (virtual) solutions of $D$.

In this article, we will restrict ourselves to the case when $G$ is connected.

Let $g$ be the Lie algebra of $G$, and $g^*$ its dual vector space. Our aim is to construct a canonical $G$-invariant function $m_G(\sigma)$ on $g^*$ which extends the multiplicity function $\text{mult}_G(\sigma)$ on $\hat{G} \subset g^*/G$.

The first instance of such a relation between the multiplicity function on $\hat{G}$ and functions on $g^*$ is the formula for the Kostant partition function in terms of derivatives of spline functions (that is, piecewise polynomial functions) [9], [8]. Similarly, Heckman’s result [17] on branching rules relates asymptotically multiplicities to spline functions. For example, if $T$ is the maximal torus of $G$, the asymptotic
$T$-multiplicity function of the irreducible representation $V_{k\lambda}$ of $G$ suitably normalized converges when $k$ tends to $\infty$ to the Duistermaat–Heckman measure, a piece-wise polynomial measure on $t^*$ supported on the projection of $G\lambda$ on $t^*$. Since then, much more precise results have been obtained, in the spirit of the $[Q, R] = 0$ theorem, for special elliptic symbols [19], [21].

Here we consider a general transversally elliptic symbol. The value of our function $m_G(\sigma)$ at $\lambda$ is given by a double integral formula reminiscent of Witten non-abelian localization formula [27]. Our present result does not provide another proof of the $[Q, R] = 0$ theorems as, for general $\sigma$, we do not have a geometric interpretation of $m_G(\sigma)(\lambda)$ in terms of the index of an elliptic operator on a reduced space. However, we believe that our formula provides a unifying framework for all ‘known cases’ of geometric multiplicity formulae and we hope to justify in a future work this statement.

First recall the definition of a transversally elliptic symbol. We denote by $(x, \xi)$, with $x \in M$ and $\xi \in T^*_xM$, a point of the cotangent bundle $T^*M$, and by $p: T^*M \to M$ the projection. Let $T_G^*M \subset T^*M$ be the set of $(x, \xi) \in T^*M$ such that $\xi$ is orthogonal to the tangent space to the $G$-orbit through $x$. We say that $\xi$ is a transverse direction. Let $\mathcal{E}_\pm$ be two $G$-equivariant complex vector bundles over $M$. A $G$-equivariant bundle map $\sigma: p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ will be called a symbol. The support $\text{supp}(\sigma) \subset T^*M$ of $\sigma$ is the set of elements $(x, \xi) \in T^*M$ such that the linear map $\sigma(x, \xi): \mathcal{E}_x^+ \to \mathcal{E}_x^-$ is not invertible. A symbol $\sigma$ is said to be elliptic if its support is compact. A symbol $\sigma$ is said to be transversally elliptic if its support intersected with $T_G^*M$ is compact. In other words, $\sigma$ is ‘elliptic’ in the directions transverse to the orbits.

Let us give some basic examples of transversally elliptic symbols:

(i) if $D: \Gamma(M, \mathcal{E}^+) \to \Gamma(M, \mathcal{E}^-)$ is a $G$-invariant elliptic differential operator on a compact $G$-manifold $M$, acting on the spaces $\Gamma(M, \mathcal{E}^\pm)$ of smooth sections of $\mathcal{E}^\pm$, the principal symbol $\sigma$ of $D$ is elliptic, thus transversally elliptic, and $\text{Index}_G(\sigma)(g) = \text{Tr}_{\text{Ker}(D)}(g) - \text{Tr}_{\text{Coker}(D)}(g)$;

(ii) if $M$ is homogeneous, any symbol is transversally elliptic.

Let us describe our result. In this introduction, we will make the simplifying hypothesis that all stabilizers of the action of $G$ on $M$ are connected.

If $G$ is a torus with Lie algebra $\mathfrak{g}$, we parameterize $\hat{G}$ by the lattice $\Lambda \subset \mathfrak{g}^*$ of weights of $G$. We denote by $g^\lambda$ the character of $G$ indexed by $\lambda$. Given a transversally elliptic symbol $\sigma$, we construct a particular piecewise polynomial function $m_G(\sigma)$ on $\mathfrak{g}^*$. Under the hypothesis that stabilizers of the action of $G$ on $M$ are connected, the function $m_G(\sigma)$ is continuous and extends the function $\lambda \mapsto \text{mult}_G(\sigma)(\lambda)$ on $\Lambda$.

A trivial example is when $G$ acts on $G = M$ by left translations and $D$ is the operator $0$, with index function the trace of the regular representation of $G$, that is, the $\delta$-function at $1$:

$$\delta_1(g) = \sum_{\lambda \in \Lambda} g^\lambda.$$  

Then the function $\lambda \mapsto \text{mult}_G(D)(\lambda)$ is identically equal to $1$ on $\Lambda$, and is extended by the constant function $m_G(D) = 1$.

In the case when $G$ is any compact connected Lie group, we parameterize $\hat{G}$ by the set of admissible regular coadjoint orbits (see §4). We construct a $G$-invariant
function \( m_G(\sigma) \) on \( \mathfrak{g}^* \), and we determine the multiplicity \( \text{mult}_G(\sigma)(\lambda) \) of the irreducible representation \( V_\lambda \) parameterized by \( G\lambda \) in \( \text{Index}_G(\sigma) \) in terms of the value of \( m_G(\sigma) \) at \( \lambda \).

Our formula for \( m_G(\sigma) \) involves equivariant cohomology classes on the \( G \)-manifold \( T^*M \), namely the equivariant Chern character of \( \sigma \), and the equivariant truncated \( \hat{A} \) class. Let \( N \) be a \( G \)-manifold and let \( \mathcal{A}(N) \) be the space of differential forms on \( N \), graded by its exterior degree. Following [5] and [26], an equivariant form is a \( G \)-invariant smooth function \( \alpha: \mathfrak{g} \to \mathcal{A}(N) \), thus \( \alpha(X) \) is a differential form on \( N \) depending differentially on \( X \in \mathfrak{g} \). Consider the operator

\[
d_\mathfrak{g} \alpha(X) = d\alpha(X) - \iota(v_X)\alpha(X),
\]

where \( \iota(v_X) \) is the contraction by the vector field \( v_X \) generated by the action of \(-X\) on \( N \). Then \( d_\mathfrak{g} \) is an odd operator with square 0, and the equivariant cohomology \( \mathcal{H}^*_G(N) \) is defined to be the cohomology space of \( d_\mathfrak{g} \). It is important to note that the dependence of \( \alpha \) on \( X \) may be \( C^\infty \). If the dependence of \( \alpha \) on \( X \) is polynomial, we denote by \( H^*_G(N) \) the corresponding \( \mathbb{Z} \)-graded algebra. By definition, the grading of \( P(X) \otimes \mu, P \) a homogeneous polynomial and \( \mu \) a differential form on \( N \), is the exterior degree of \( \mu \) plus twice the polynomial degree in \( X \).

When \( \sigma \) is elliptic, \( \sigma \) determines an element of the topological equivariant \( K \)-group \( K^0_G(T^*M) \), and its equivariant Chern character \( \text{Ch}(\sigma)(X) \) is a compactly supported equivariant cohomology class (with \( C^\infty \) coefficients). When \( \sigma \) is only transversally elliptic, \( \sigma \) determines an element of \( K^0_G(T^*_GM) \) and we have described in [13] a Cartan model for the equivariant cohomology with compact support of \( T^*_GM \). Thus a representative of \( \text{Ch}(\sigma)(X) \) is a compactly supported equivariant form on \( T^*M \) (depending on \( X \) in a \( C^\infty \) way), and equivariantly closed on a small neighborhood of \( T^*_GM \) in \( T^*M \).

Let \( J(A) = \det_{\mathbb{R}^d}(e^A - 1)/A \), an invariant function of \( A \in \text{End}(\mathbb{R}^d) \). Then, \( J(0) = 1 \). Consider \( 1/J(A) \) and its Taylor expansion at 0:

\[
\frac{1}{J(A)} = \det_{\mathbb{R}^d}(e^A - 1) = \sum_{k=0}^{\infty} B_k(A).
\]

Each function \( B_k(A) \) is an invariant polynomial of degree \( k \) on \( \text{End}(\mathbb{R}^d) \) and by the Chern–Weil construction, \( B_k \) determines an equivariant characteristic class \( B_k(M)(X) \) on \( M \) of homogeneous degree 2\( k \).

We define the formal series of equivariant cohomology classes:

\[
[B(M)](X) = \sum_{k=0}^{\infty} B_k(M)(X).
\]

For \( X \) small enough and \( M \) compact, the series is convergent, and its sum \( B(M)(X) \) is the equivariant \( \hat{A} \) class of \( T^*M \), that is, the square of the equivariant \( \hat{A} \) class of \( M \). Choose an integer \( K \), and define

\[
B(M, K)(X) = \sum_{k=0}^{K} B_k(M)(X).
\]
Thus \( B(M, K) \) defines an equivariant cohomology class with polynomial coefficients on \( M \). We call it the truncated \( \hat{A} \) class of \( T^*M \), and \( K \) is the order of truncation.

Let \( j_\sigma(X) = \det_\sigma(e^{\text{ad}_X} - 1) \) and \( j_\sigma^{1/2}(X) \) its square root, a \( G \)-invariant analytic function on \( \mathfrak{g} \).

Assume first that \( \sigma \) is elliptic. We define the \( C^\infty \) function \( I_G(\sigma, K)(X) \) on \( \mathfrak{g} \) by

\[
I_G(\sigma, K)(X) = j_\sigma^{1/2}(X) \left( \frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \text{Ch}(\sigma)(X) B(M, K)(X) \right).
\]

Consider its Fourier transform \( S_G(\sigma, K) \), a generalized function of \( y \in \mathfrak{g}^* \). In the sense of generalized functions, \( S_G(\sigma, K)(y) \) is given by the double integral

\[
S_G(\sigma, K)(y) = \frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \int_{\mathfrak{g}} j_\sigma^{1/2}(X) \text{Ch}(\sigma)(X) B(M, K)(X) e^{-i(y,X)} dX.
\]

When \( \sigma \) is only transversally elliptic, introduce the canonical Liouville 1-form \( \ell \) on \( T^*M \). Let \( \omega = -\ell \). The infinitesimal index (as defined in [13]) allows us to still define the double integral \( S_G(\sigma, K) \) above by multiplying the integrand by the oscillatory factor \( e^{isd_\sigma \omega(X)} \) (which is congruent to 1 in equivariant cohomology).

That is:

\[
S_G(\sigma, K)(y) = \lim_{s \to \infty} \frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \int_{\mathfrak{g}} e^{isd_\sigma \omega(X)} j_\sigma^{1/2}(X) \text{Ch}(\sigma)(X) B(M, K)(X) e^{-i(y,X)} dX.
\]

We denote the corresponding limit of the double integral by

\[
\frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \int_{\mathfrak{g}} j_\sigma^{1/2}(X) \text{Ch}(\sigma)(X) B(M, K)(X) e^{-i(y,X)} dX.
\]

Assume first that \( G \) is a torus. In this case, the function \( j_\sigma \) is identically equal to 1. We prove:

1) The generalized function \( S_G(\sigma, K) \) has singularities on a union \( \mathcal{H} \) of a finite number of affine hyperplanes.

2) The restriction of \( S_G(\sigma, K) \) to each connected component \( c \) of the complement of \( \mathcal{H} \) is a polynomial function. This polynomial function is independent of \( K \) if \( K \geq \dim M \).

This allows us to define, for \( y \in \mathfrak{g}^* \setminus \mathcal{H} \),

\[
m_G(\sigma)(y) = \frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \int_{\mathfrak{g}} \text{Ch}(\sigma)(X) \sum_{k=0}^{\infty} B_k(M)(X) e^{-i(y,X)} dX
\]

\[
= \frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \int_{\mathfrak{g}} \text{Ch}(\sigma)(X) [B(M)](X) e^{-i(y,X)} dX
\]

as the terms associated to \( k > \dim M \) vanish.

The function \( m_G(\sigma)(y) \) is a piecewise polynomial function on \( \mathfrak{g}^* \). In the language of approximation theory [10], a spline function is a piecewise polynomial function satisfying some further continuity properties.
Theorem 1.1. The function $m_G(\sigma)$ defined on $\mathfrak{g}^* \setminus \mathcal{H}$ extends to a continuous function on $\mathfrak{g}^*$, still denoted by $m_G(\sigma)$.

If $\lambda$ belongs to the lattice $\Lambda$ of weights of $G$, we have the equality

$$\text{mult}_G(\sigma)(\lambda) = m_G(\sigma)(\lambda).$$

Thus we have constructed a canonical spline function on $\mathfrak{g}^*$, extending the multiplicity function $\lambda \mapsto \text{mult}_G(\sigma)(\lambda)$.

Let $G$ be a connected compact Lie group. Let $G\lambda \subset \mathfrak{g}^*$ be a regular coadjoint orbit and let $\text{vol}(G\lambda)$ be its symplectic volume. Similarly, the value of $S_G(\sigma, K)(y)$ when $y$ is generic and tends to $\lambda$ is well defined, independent of $K$, provided $K \geq \dim M$, and

$$\text{mult}_G(\sigma)(\lambda) = \text{vol}(G\lambda) \lim_{y \to \lambda} S_G(\sigma, K)(y).$$

In the general case when stabilizers are not necessarily connected, we define similar functions associated to the action of $G$ on $\mathcal{M}_g$, where $g$ varies over a finite number of special elements of $G$. Here $\mathcal{M}_g$ is the fixed point submanifold for the action of $g \in G$ on $M$ and $G^g$ is the centralizer of $g$. The multiplicity is described with the help of these functions as a piecewise quasi-polynomial function.

We now explain the motivation of this formula. Let $M$ be a compact $G$-manifold, where $G$ is a torus. Consider $D$ an elliptic $G$-invariant differential operator, with principal symbol $\sigma$. The function $\text{Index}_G(D)(g) = \sum_{\lambda \in \mathcal{G}} \text{mult}_G(\sigma)(\lambda) \chi_\lambda(g)$ is then an analytic function on $G$. Recall the formula: for $X \in \mathfrak{g}$ small,

$$\text{Index}_G(D)(\exp X) = \frac{1}{(2i\pi)^{\dim M}} \int_{T^* M} \text{Ch}(\sigma)(X)B(M)(X),$$

obtained in ([7], [6]), a ‘delocalized’ version of Atiyah–Bott–Segal–Singer [2] equivariant index formula. The function $X \mapsto B(M)(X)$ is defined only when $X$ is sufficiently small. Inspired by the inversion formulae of box splines [10], we have replaced the equivariant class $X \mapsto B(M)(X)$ by its approximation $B(M, K)(X)$, which is a polynomial function of $X$. Under the hypothesis that stabilizers are connected, when $K$ is sufficiently large, the value of the Fourier transform of

$$\frac{1}{(2i\pi)^{\dim M}} \int_{T^* M} \text{Ch}(\sigma)(X)B(M, K)(X)$$

at an integral point $\lambda \in \Lambda$ coincides with the Fourier coefficient $\text{mult}_G(\sigma)(\lambda)$ of the periodic function $\text{Index}_G(D)(g)$.

Let us compare the formulae for $\text{mult}_G(\sigma)$, a function on $\Lambda \subset \mathfrak{g}^*$, and for $m_G(\sigma)$, a function on $\mathfrak{g}^*$. The number $\text{mult}_G(\sigma)(\lambda)$ is the Fourier coefficient of the periodic function $\text{Index}_G(D)(g)$, thus it is given by

$$\text{mult}_G(\sigma)(\lambda) = \int_G \text{Index}_G(D)(g) g^{-\lambda} dg.$$

Replace formally $\text{Index}_G(D)(\exp X)$ by

$$\frac{1}{(2i\pi)^{\dim M}} \int_{T^* M} \text{Ch}(\sigma)(X)[B(M)](X).$$
and integrate on \( g \) instead of \( G \). This gives the similar double integral formula:

\[
m_G(\sigma)(y) = \frac{1}{(2\pi)^{\dim M}} \int_{T^* M} \int_g \text{Ch}(\sigma)(X)[B(M)](X)e^{-i\langle y,X \rangle} dX.
\]

We have the miraculous equality

\[
\text{mult}_G(\sigma)(\lambda) = m_G(\sigma)(\lambda)
\]

at all integral points \( \lambda \in \Lambda \).

This double integral formula for \( m_G(\sigma)(\lambda) \) is reminiscent of Witten non-abelian localization formula for computing characteristic numbers of symplectic reductions. In particular, Jeffrey–Kirwan [18] used the truncated Todd class of \( M \) in order to compute Riemann–Roch numbers on symplectic reductions. As pointed out by Guillemin [15], the Witten non-abelian localization formula gives an heuristic proof of the \([Q,R] = 0\) theorem. This certainly acted as an inspiration for our work. However, our context is different. First remark that we work on \( T^* M \) and \( \sigma \) is any transversally elliptic symbol. Our formula for \( m_G(\sigma) \) is clearly additive on symbols. For general symbols, we do not have a geometric interpretation of the double integral defining \( m_G(\sigma)(y) \). So further work will have to be done for reinterpreting \( m_G(\sigma)(\lambda) \) in terms of an index on a reduced space, for the special symbols of Dirac operators twisted by line bundles, as in [19], [21]. Clearly the symbols have to be very special to obtain such a geometric interpretation. Think of the case when \( D \) is the twisted Dolbeaut–Dirac operator by a sum of two line bundles \( L_1, L_2 \), then the multiplicity function \( m \) is the sum of the multiplicity functions \( m_1, m_2 \). Each one of the functions \( m_1, m_2 \) has a geometric interpretation in terms of reduced spaces, but the geometric meaning for the sum \( m \) is not clear.

Let us now give some indications of the proofs. Following the method of Atiyah–Singer as described in the monograph [1], we reduce to the case of the maximal torus \( T \) of \( G \). In particular, if \( \sigma \) is a \( T \)-transversally elliptic symbol, we can define the \( T \)-multiplicity function \( m_T(\sigma)(y) \), and our formula for the generalized function \( m_G(\sigma) \) is nothing else than the usual formula

\[
m_G(\sigma)(y) = \sum_w \varepsilon(w)m_T(\sigma)(y+w\rho),
\]

an alternate sum over the Weyl group. When \( M \) is a vector space with a linear action of a torus \( G \), the construction of \( m_G(\sigma) \) was the object of the articles [11], [12]. Our proof is very similar. We use Atiyah–Singer description of a set of generators of \( K_G^0(T_G^* M) \). This reduces (almost) our study to the case when \( M = P \times V \) where \( P \) is a space with a free action of \( G \), and \( V \) is a linear representation of \( G \). Then the miraculous equality (1.3) follows from a generalized Dahmen–Michelli inversion formula for box splines (see [25]), a Riemann–Roch formula in approximation theory. Simple examples of the inversion formula for box splines are given in Example 2.14 and §3.6.

It may be useful to compute our formula for the spline function \( m_G(\sigma)(\xi) \) on the very simple example of the projective space.

**Example 1.2** (The projective space). Let \( M = P_1(\mathbb{C}) \) with action of \( S^1 \) acting on homogeneous coordinates \([z_1, z_2]\) by \([tz_1, z_2]\). Let \( E \) be the element of \( \text{Lie}(S^1) \) such that \( \exp(xE) = e^{ix} \in S^1 \). We identify the lattice of characters of \( S^1 \) with \( \mathbb{Z} \).
Let \( 0 \to L \to \mathbb{C}^2 \to Q \to 0 \) be the exact sequence of vector bundles on \( M \), where \( L \) is the tautological bundle and \( Q \) its quotient bundle. Let \( a, b \) be integers, and consider the Dolbeaut operator \( \overline{\partial}_{a,b} \) acting on sections of the graded vector bundle \( L^a \otimes Q^b \otimes \Lambda^\bullet \Omega^{0,1}(M) \). Let \( \sigma_{a,b} \) be the principal symbol of \( \overline{\partial}_{a,b} \). As shown by Atiyah–Bott, the index of \( \overline{\partial}_{a,b} \) depends only on the two line bundles \( \mathcal{L}^+ = L^a \otimes Q^b \) and \( \mathcal{L}^- = L^a \otimes Q^b \otimes \Omega^{0,1}(M) \). The supertrace of the action of \( \exp(xE) \) on the fiber of the bundle \( \mathcal{L}^+ \oplus \mathcal{L}^- \) at the fixed point \([1,0]\) is \( e^{i(ax)}(1 - e^{-ix}) \) while it is \( e^{ibx}(1 - e^{ix}) \) at the fixed point \([0,1]\). Atiyah–Bott–Lefschetz formula is

\[
\text{Index}_{S^1}(\overline{\partial}_{a,b})(e^{ix}) = \frac{e^{i(ax)}(1 - e^{-ix})}{(1 - e^{ix})(1 - e^{-ix})} + \frac{e^{ibx}(1 - e^{ix})}{(1 - e^{ix})(1 - e^{ix})}.
\]

(1.4)

If \( a \leq b \), we obtain

\[
\text{Index}_{S^1}(\overline{\partial}_{a,b})(e^{ix}) = \sum_{k=a}^{b} e^{ikx}.
\]

So the multiplicity function \( k \mapsto \text{mult}_{S^1}(\sigma_{a,b})(k) \) on \( \mathbb{Z} \) is \( \text{mult}_{S^1}(\sigma_{a,b})(k) = 1 \) if \( 0 \leq a \leq k \leq b \), or 0 otherwise.

Let us now compute our formula. Here the equivariant class \([B(M)]\) is the Taylor series of \( x^2/((1 - e^{ix})(1 - e^{-ix})) \):

\[
[B(M)](x) = \sum_{k=0}^{\infty} c_k x^k = 1 + \frac{1}{12} x^2 + \frac{1}{240} x^4 + \cdots.
\]

Thus

\[
I_{S^1}(\sigma_{a,b}, K)(xE) = \left( \sum_{k=0}^{K} c_k x^k \right) \left( \frac{1}{2i\pi} \int_{T^* M} \text{Ch}(\sigma_{a,b})(xE) \right).
\]

Using the localization formula in compactly supported equivariant cohomology on \( T^* M \), we obtain

\[
\frac{1}{2i\pi} \int_{T^* M} \text{Ch}(\sigma_{a,b})(xE) = \frac{e^{i(ax)}(1 - e^{-ix})}{x^2} + \frac{e^{ibx}(1 - e^{ix})}{x^2}.
\]

By Fourier transform, we obtain

\[
\frac{1}{2i\pi} \int_{T^* M} \text{Ch}(\sigma_{a,b})(xE) = \int_{\mathbb{R}} e^{-iyx} s_{a,b}(y) \, dy.
\]

Here \( s_{a,b}(y) \) is the continuous spline function on \( \mathbb{R} \) represented (for \( a = 0, b = 3 \)) by the graph of Fig. 1.

Thus the Fourier transform \( S_{S^1}(\sigma_{a,b}, K) \) of \( I_{S^1}(\sigma_{a,b}, K) \) is easy to compute:

\[
S_{S^1}(\sigma_{a,b}, K)(y) = s_{a,b}(y) + \frac{1}{12} \left( \delta_a(y) - \delta_{a-1}(y) + \delta_{b}(y) - \delta_{b+1}(y) \right)
- \frac{1}{240} \frac{\partial^2}{\partial y^2} \left( \delta_a(y) - \delta_{a-1}(y) + \delta_{b}(y) - \delta_{b+1}(y) \right) + \cdots.
\]
We see that $S_{S^1}(\sigma_{a,b}, K)$ coincides with the continuous spline function $s_{a,b}$ on $\mathbb{R}\setminus\{a-1, a, b, b+1\}$ as soon as $K \geq 1$. We thus have

$$m_{S^1}(\sigma_{a,b}) = s_{a,b}.$$ 

Remark that the value of $s_{a,b}$ at an integer $k$ is 0, except if $a \leq k \leq b$, where the value is 1. Our theorem is verified:

$$\text{mult}_{S^1}(\sigma)(k) = m_{S^1}(\sigma_{a,b})(k).$$

Consider now the group $SU(2)$ and let $Z$ be its center. Let $G = SU(2)/Z$. Then $G$ acts on $M = P_1(\mathbb{C})$ with connected stabilizers. We take again the $G$-equivariant line bundle $L_{a,b}$ with $a \leq b$ of the same parity (in order that the corresponding representation of $G$ factorize by $Z$). Then the index of $\overline{\partial}_{a,b}$ operator is the irreducible representation of $G$, of odd dimension $(b-a)+1$.

Let us now compute our formula. Let $g$ be the Lie algebra of $SU(2)$. We identify both $g$ and $g^\ast$ with $\mathbb{R}^3$, with

$$g = \left\{ \begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix} \right\}, \quad g^\ast = \left\{ \begin{pmatrix} iy_1 & y_2 + iy_3 \\ -y_2 + iy_3 & -iy_1 \end{pmatrix} \right\}.$$

The standard scalar product is $G$-invariant. A coadjoint orbit is the sphere $S_r = \{y_1^2 + y_2^2 + y_3^2 = r^2\}$. The Liouville measure on $S_r$ is such that $\text{vol}(S_r) = r$, and the admissible coadjoint orbits for $G$ are the spheres of odd radius. The irreducible representation of $G$ of dimension $k$ is parameterized by the coadjoint orbit $S_k$.

Let

$$H_\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then $\exp(xH_\alpha)$ transforms $[z_1, z_2] \in P_1(\mathbb{C})$ in $[e^{2ix}z_1, z_2]$.

Consider

$$I_G(\sigma_{a,b}, K)(X) = j_g^{1/2}(X) \int_{T^*M} \text{Ch}(\sigma_{a,b})(X)B(M, K)(X),$$

a $G$-invariant function on $g$. Let $R = b - a$. Following the same computation as in the preceding case, we obtain

$$I_G(\sigma_{a,b}, K)(xH_\alpha) = \left(\frac{e^{ix} - e^{-ix}}{2ix}\right) V(R, K)(x).$$
with
\[ V(R, K)(x) = \left( \sum_{k=0}^{K} c_k(2x)^k \right) \frac{e^{iRx}(1 - e^{2ix}) + e^{-iRx}(1 - e^{-2ix})}{4x^2}. \]

Let \( r = (y_1^2 + y_2^2 + y_3^2)^{1/2} \) and let \( v \) be the \( G \)-invariant function on \( \mathfrak{g}^* \) such that
\[
v(y_1, y_2, y_3) = \begin{cases} 
  \frac{r - (R - 1)}{2r} & \text{if } R - 1 \le r \le R + 1, \\
  \frac{(R + 3) - r}{2r} & \text{if } R + 1 \le r \le R + 3, \\
  0 & \text{otherwise.}
\end{cases}
\]

We see that \( v \) is a continuous function on \( \mathfrak{g}^* \).

The Fourier transform \( S_G(\sigma_{a,b}, K) \) of \( I_G(\sigma_{a,b}, K) \) coincides with \( v \) outside spheres of radius \( R - 1, R + 1, R + 3 \), as soon as \( K > 1 \). We thus have \( m_G(\sigma) = v \).

Recall that \( R = b - a \) is even. We see that \( v \) vanishes on all spheres of odd radius, except on the sphere of radius \( R + 1 \), parameterizing our representation \( \text{Index}_G(\bar{\sigma}_{a,b}) \), where its value is \( 1/(R + 1) \). Thus our formula is verified:

\[ \text{mult}_G(\lambda) = \text{vol}(G\lambda)v(\lambda). \]

\section{2. Preliminaries}

\subsection{2.1. Formal series.}
Let \( E \) be a vector space. We introduce \( E[[q]] \) as the space of formal series \( f[q] = \sum_{k=0}^{\infty} q^k f_k \) with \( f_k \in E \). If the series is convergent (\( E \) being a topological vector space) at \( q = 1 \), we write \( f[1] \) for the sum \( \sum_{k=0}^{\infty} f_k \). If \( E \) is an algebra and \( f_0 \) invertible, we can define \( 1/f[q] \) in \( E[[q]] \).

If \( \theta(q, X) \) is a smooth function of \( q \in \mathbb{C} \) defined in a neighborhood of \( q = 0 \) and depending on a parameter \( X \), we denote by
\[
\theta([q])(X) = \sum_{k=0}^{\infty} q^k \theta_k(X)
\]
its Taylor series at \( q = 0 \). This is a formal series of functions of \( X \).

Let \( V \) be a real vector space. Let \( A \in \text{End}(V) \). Consider the function
\[
J_V(A) = \det_V\left( \frac{e^A - 1}{A} \right).
\]

If \( V \) is a complex vector space, we define
\[
J_V^C(A) = \det_V^C\left( \frac{e^A - 1}{A} \right),
\]

where the determinant is the complex determinant. If \( V \) is a Euclidean vector space, and \( A \) antisymmetric, one has
\[
J_V(A) = J_V^C(A) J_V^C(-A).
\]
Introduce a variable $q$, and consider $J_V(q, A) = J_V(qA)$. Then

$$J_V([q])(A) = \sum_{k=0}^{\infty} q^k T_k(A) = 1 + q \frac{\text{Tr}(A)}{2} + \cdots,$$

where $T_k$ is an invariant homogeneous polynomial of degree $k$ on $\text{End}(V)$. We may also write $J_V([q]A)$ instead of $J_V([q])(A)$.

If $N$ is a real vector space and $s \in \text{End}(N)$ is a transformation of $N$, we denote by $\text{GL}(s)$ the group of invertible linear transformations of $N$ commuting with $s$. We consider

$$D_N(q, s)(A) = \det_N(1 - se^{qA}),$$

an analytic function on $\text{End}(N)$. Write the Taylor series

$$D_N([q], s)(A) = \sum_{k=0}^{\infty} q^k D_k^s(A).$$

Then $D_k^s(A)$ is a homogeneous polynomial of degree $k$ on $\text{End}(N)$, invariant under conjugation by $\text{GL}(s)$. If $1 - s$ is invertible, we can define $1/D_N([q], s)(A)$.

2.2. Transversally elliptic symbols. Let $G$ be a compact Lie group. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. If $G$ acts on a $G$-manifold $N$, and $X \in \mathfrak{g}$, we denote by

$$v_X(n) = \left. \frac{d}{d\varepsilon} \exp(-\varepsilon X) \cdot n \right|_{\varepsilon=0}$$

the vector field on $N$ generated by $-X$.

Let us consider two $G$-equivariant Hermitian bundles $F^\pm$ on $N$ and let $\sigma: F^+ \to F^-$ be a $G$-equivariant morphism. The support $\text{supp}(\sigma)$ of $\sigma$ is the set of points $n \in N$ where $\sigma_n$ is not invertible. We will say that $\sigma$ is elliptic if $\text{supp}(\sigma)$ is compact. Then $\sigma$ determines an element of the $G$-equivariant topological $K$-group $K^0_G(N)$, still denoted by $\sigma$.

Recall the definition of the Bott symbol. Let $N$ be a Hermitian vector space with Hermitian form $(\cdot, \cdot)$ and complex structure $J$. We denote by $U$ the unitary group of transformations of $N$, and by $U(1) \subset U$ the subgroup formed the homotheties $n \mapsto e^{i\theta} n$. Consider $S = \bigwedge N$, graded in even and odd degree. Let $c: N \to \text{End}(S)$ be the Clifford action. The map $c$ is equivariant, interchanges $S^\pm$ and $c(n)^2 = -\|n\|^2 \text{Id}_S$. If $N = \mathbb{C}$, then

$$c(z) = \begin{pmatrix} 0 & -\overline{z} \\ z & 0 \end{pmatrix}.$$ 

**Definition 2.1.** The Bott symbol $\text{Bott}(N, J) \in \Gamma(N, \text{Hom}(S^+, S^-))$ is defined by

$$\text{Bott}(N, J)(z) = c(z) : \bigwedge_{J \text{ even}} N \longrightarrow \bigwedge_{J \text{ odd}} N, \quad z \in N.$$ 

It is an elliptic symbol, with support $\{0\}$, and equivariant with respect to $U$.

If we take the opposite complex structure on $N$, one has the relation

$$\text{Bott}(N, -J) = (-1)^{\dim c} N \chi \text{Bott}(N, J).$$
in $K^0_U(N)$, where $\chi$ is the character $\chi(g) = (\det_N^G(g))^{-1}$ of $U$. This is easily seen by restriction to $\{0\} \to N$.

Let $M$ be a $G$-manifold, $N = T^*M$ its cotangent bundle. We denote by $(x, \xi)$, with $x \in M$ and $\xi \in T^*_x M$, a point of the cotangent bundle $T^*M$, and by $p: T^*M \to M$ the projection. Let $\ell$ be the Liouville 1-form: for $x \in M$, $\xi \in T^*_x M$ and $V$ a tangent vector at the point $(x, \xi) \in T^*M$, $\ell_{x, \xi}(V) = \langle \xi, p_* V \rangle$. Then $\Omega = -d\ell$ is the symplectic form of $T^*M$, and we use the corresponding orientation of $T^*M$ to compute integrals on $T^*M$ of differential forms with compact support. Denote by $T^*_G M \subset T^*M$ the union of the space of covectors conormal to the $G$-orbits. Let $\mu: T^*M \to g^*$ be the moment map $\langle \mu(x, \xi), X \rangle = \langle \xi, v_X \rangle$. Then $T^*_G M = \mu^{-1}(0)$ is the zero fiber of the moment map $\mu$.

Let $E^\pm$ be two $G$-equivariant complex vector bundles over $M$. So a $G$-equivariant bundle map $\sigma: p^*E^+ \to p^*E^-$ is a symbol. A symbol $\sigma$ is said to be transversally elliptic if its support intersected with $T^*_G M$ is compact. If $\sigma$ is transversally elliptic, it determines an element of $K^0_G(T^*_G M)$, still denoted by $\sigma$. Atiyah–Singer (see the monograph [1]) have associated to any element $\sigma \in K^0_G(T^*_G M)$ a virtual trace class representation of $G$. Let $\text{Index}_G(\sigma)(g)$ be its trace:

$$\text{Index}_G(\sigma)(g) = \sum_{\lambda \in \hat{G}} m_G(\sigma)(\lambda) \chi^G_\lambda(g).$$

Here $\chi^G_\lambda$ is the trace of the unitary irreducible representation $V^G_\lambda$ of $G$. When $G$ is given, we might write simply $\chi_\lambda$ or $V_\lambda$ instead of $\chi^G_\lambda$ or $V^G_\lambda$. We might also write

$$\text{Index}_G(\sigma) = \bigoplus_{\lambda \in \hat{G}} m_G(\sigma)(\lambda) V^G_\lambda,$$

an infinite sum of irreducible representations with finite multiplicities $m_G(\sigma)(\lambda) \in \mathbb{Z}$.

Recall examples of transversally elliptic symbols.

**Example 2.2** (Elliptic symbols). Any elliptic symbol is transversally elliptic. If $M$ is an even-dimensional compact manifold and is oriented, then any element of $K^0_G(T^*M)$ is the symbol of a twisted Dirac operator.

**Example 2.3** (Branching rule). Let $M = G$ and let $H \subset G$ be a compact subgroup of $G$. Consider the action of $G \times H$ on $M$ by left and right translations. Let $E^+ = M \times \mathbb{C}$ be the trivial vector bundle, and let $E^- = M \times \{0\}$. Then the 0-symbol $\sigma_0$ is transversally elliptic with respect to the action of $G \times H$ on $M$, and

$$\text{Index}_{G \times H}(\sigma_0)(g, h) = \sum_{\lambda \in \hat{G}, \mu \in \hat{H}} m_{G \times H}(\sigma_0)(\lambda, \mu) \overline{\chi^G_\lambda(g)} \chi^H_\mu(h),$$

where $m_{G, H}(\sigma_0)(\lambda, \mu)$ is the multiplicity of the irreducible representation $V^H_\mu$ of $H$ in the irreducible representation $V^G_\lambda$ of $G$. The function $m_{G, H}(\sigma_0)(\lambda, \mu)$ is thus the branching function.

**Example 2.4** (Orbifold index). Let $M$ be a compact manifold and let $G$ be a compact group acting on $M$. Assume that all stabilizers $G_m$ of points of $M$ are finite. Then $M/G$ is an orbifold. Let $\sigma$ be a $G$-transversally elliptic symbol on $M$, and write $\text{Index}_G(\sigma)(g) = \sum_{\lambda \in \hat{G}} m_G(\sigma)(\lambda) \chi_\lambda(g)$. If $\lambda_0$ is the trivial representation
of $G$, $m_G(\sigma)(\lambda_0)$ is the index of the elliptic symbol on the orbifold $M/G$ associated to $\sigma$.

**Example 2.5** (Atiyah symbol). Let $N$ be an even-dimensional vector space with a linear action of a torus $G$. We choose a $G$-invariant Euclidean product $(\cdot, \cdot)$. This allows us to identify $T^*N$ with $N \oplus N$. Let us choose a $G$-invariant complex structure $J$ on $N$ preserving the inner product and let us consider $\text{Sym}(N) = \bigoplus_{k=0}^{\infty} \text{Sym}^k(N)$ where $\text{Sym}^k(N)$ is the subspace of $\otimes^k N$ formed of symmetric tensors. Here $N$ is considered as a complex vector space via the complex structure $J$.

We can consider the Bott symbol $\text{Bott}(N, s)$ of $N$.

**Definition 2.6.** The Atiyah symbol at$(N)$ is the reciprocal image of $\text{Bott}(N, -J)$ associated to the opposite complex structure. Consider the equivariant map $q: N \oplus N \rightarrow N$ defined by $(x, \xi) \mapsto \xi + Jx$.

The following proposition is proved in [1] (see also [7]).

**Proposition 2.7.** $\text{Index}^N_U(\text{at}(N)) = \text{Sym}(N)$.

Similarly, if $N \rightarrow M$ is a Hermitian vector bundle on $M$, using a connection, we can define a symbol $\text{at}(N)$ on $T^*N$.

### 2.3. Equivariant cohomology and infinitesimal index.

We recall the definitions of the equivariant cohomology given in the introduction. Let $N$ be a $G$-manifold and let $\mathcal{A}(N)$ be the space of differential forms on $N$, graded by its exterior degree. Consider the operator (equation (1.1))

$$d_g: C^\infty(\mathfrak{g}, \mathcal{A}(N))^G \rightarrow C^\infty(\mathfrak{g}, \mathcal{A}(N))^G.$$  

The equivariant cohomology $H^*_G(N)$ is the cohomology space of $d_g$.

If the dependence of $\alpha$ on $X$ is polynomial, this model for equivariant cohomology is equivalent to the Cartan model for topological equivariant cohomology of the $G$-space $N$ (see [16]). We denote by $H^*_G(N)$ the corresponding $\mathbb{Z}$-graded algebra.

The equivariant integration $\int_N \alpha(X)$ associates to an equivariant cohomology class with compact support on $N$ a $G$-invariant $C^\infty$ function on $\mathfrak{g}$.

If $N \rightarrow M$ is an oriented $G$-equivariant vector bundle over $M$, the equivariant Thom class $\text{Thom}(N)(X)$ of $N$ is the equivariant class with polynomial coefficients, and compact support along the fiber, such that its integral along the fiber is identically equal to 1.

In our article, we need to take Fourier transforms of equivariant integrals. To this purpose, we consider (as in [13]) the subcomplex of equivariant forms $\alpha(X)$, with compact support on $N$, which can be written as $\alpha(X) = \int_{\mathfrak{g}^*} e^{i(y, X)} q(y)$, where $q(y)$ is a distribution with compact support on $\mathfrak{g}^*$ with values in the space of differential forms with compact support on $N$. We denote by $\mathcal{H}^m_G(N)$ the corresponding cohomology space (the letter $m$ is for moderate growth). The space $\mathcal{H}^m_G(N)$ is a module for $H^*_G(N)$.

Let us consider two $G$-equivariant Hermitian bundles $\mathcal{F}^\pm$ on $N$ and let $\sigma: \mathcal{F}^+ \rightarrow \mathcal{F}^-$ be a symbol. We recall the definition of the equivariant Chern character of $\sigma$. We choose $G$-invariant Hermitian connections $\nabla^\pm$ on $\mathcal{F}^\pm$ with curvature $R^\pm$. Then $R^\pm$ is a 2-form on $N$ with values endomorphisms of $\mathcal{F}^\pm$. Let $\mu^{\mathcal{F}^\pm}(X)$ be the corresponding moment maps determined by the Kostant equation

$$\mathcal{L}^{\mathcal{F}^\pm}(X) = \nabla^\pm_{v_X} + \mu^{\mathcal{F}^\pm}(X)$$
and let $R^\pm(X) = \mu F^\pm(X) + R^\pm$ be the corresponding equivariant curvatures of $F^\pm$. We assume that, outside a small neighborhood of the support of $\sigma$, the connections $\nabla^\pm$ are transformed to each other by the isomorphism $\sigma$. We say that $\nabla^\pm$ are adapted to $\sigma$. It follows that the closed equivariant form $\text{Ch}(\sigma, \nabla)(X) := \text{Tr}(e^{R^+}(X)) - \text{Tr}(e^{R^-}(X))$ is a differential form on $N$ supported on a neighborhood of $\text{supp}(\sigma)$.

Let $g$ be an element of $G$, and let $N^g$ be the fixed point submanifold of the action of $g$ on $N$. Then $g$ acts fiberwise on the bundles $F^\pm$ restricted to $N^g$. Let $G^g$ be the centralizer of $g$, and $\mathfrak{g}^g$ its Lie algebra. We restrict the equivariant curvatures $R^\pm(X)$ of $F^\pm$ to $N^g$. Then, for $X \in \mathfrak{g}^g$, we define $\text{Ch}(g, \sigma, \nabla)(X) := \text{Tr}(ge^{R^+}(X)) - \text{Tr}(ge^{R^-}(X))$. This is a closed $G^g$-equivariant differential form on $N^g$ supported on a neighborhood of $\text{supp}(\sigma) \cap N^g$.

Assume that $\sigma$ is elliptic, then $\text{Ch}(\sigma, \nabla)(X)$ is a differential form on $N$ with compact support. Furthermore, as the dependence of $R^\pm(X)$ on $X$ is through the linear map $\mu F^\pm(X)$ with values skew Hermitian endomorphisms on $F^\pm$, it is easy to see (see [12], Lemma 5.3) that we can write

$$\text{Ch}(\sigma, \nabla)(X) = \int_{\mathfrak{g}^g} e^{i(y, X)} q(y, n)$$

where, for each $n \in N$, $q(y, n)$ is a distribution with compact support on $\mathfrak{g}^*$. So $\text{Ch}(\sigma, \nabla)$ determines a class $\text{Ch}(\sigma)$ in $\mathcal{H}^{m}_{G, c}(N)$, depending only on the class of $\sigma$ in $K^0_{G}(N)$. This is the equivariant Chern character. Similarly $\text{Ch}(g, \sigma, \nabla)$ determines a class $\text{Ch}(g, \sigma)$ in $\mathcal{H}^{m}_{G^g, c}(N^g)$, called the twisted equivariant Chern character.

Let us give a simple example (see the proof of Proposition 5.9 in [12]) of the equivariant Chern character of an elliptic symbol. Let $N = \mathbb{C}$ be a one-dimensional complex vector space, with action of $G = U(1)$. We identify $\mathfrak{g}$ with $\mathbb{R}$ by choosing a basis $E$ of $\mathfrak{g}$ so that $\exp(\theta E) = e^{i\theta}$. We consider the Bott symbol $\sigma$ on $N$ (2.1). Thus $F^+ = N \times \mathbb{C}$ is the trivial vector bundle on $N$, and $F^- = N \times N$, with morphism $\sigma(z) : F^+_z \to F^-_z$ the multiplication by $z$. Let us compute its equivariant Chern character. Let $\chi$ be a function on $\mathbb{R}$ with compact support contained in $|t| \leq 1$ and identically 1 near 0. Let $\beta = (\chi(|z|^2) - 1) dz/z$. This is a well-defined 1-form on $\mathbb{C}$, invariant under the action of $U(1)$. The connections $\nabla^+ = d$ and $\nabla^- = d + \beta$ are $G$-invariant connections on $F^+$, $F^-$ and for $|z| > 1$, we have $\nabla^- \sigma = \sigma \nabla^+$. We see that

$$\text{Ch}(\sigma, \nabla)(\theta) = 1 - e^{i\theta x(|z|^2)} + e^{i\theta x(|z|^2)} \chi'(|z|^2) dz \wedge d\bar{z}$$

is a compactly supported equivariant form on $\mathbb{C}$. For each $z$, the Fourier transform of $\text{Ch}(\sigma, \nabla)(\theta)$ is supported at the point $-\chi'(|z|^2)$. The integral $\int_{\mathbb{C}} \text{Ch}(\sigma, \nabla)(\theta)$ is easily computed in polar coordinates. We obtain

$$\frac{1}{2i\pi} \int_{\mathbb{C}} \text{Ch}(\sigma, \nabla)(\theta) = \frac{e^{i\theta} - 1}{i\theta}.$$

Its Fourier transform is the characteristic function of the interval $[-1, 0]$. 

M. Vergne
We have the following proposition (see [12], Proposition 5.10).

**Proposition 2.8.** Let \( N \) be a Hermitian vector space, and let \( G \) be a group acting unitarily on \( N \). Then, for any \( g \in G \), we have the equality

\[
\text{Ch}(g, \text{Bott}(N, J))(X) = (2i\pi)^{\dim_c N^g} \det_{N^g}^C \left( e^X - 1 \right) \det_{N/N^g}^C (1 - ge^X) \text{Thom}(N^g)(X)
\]

in \( \mathcal{H}^m_{G,c}(N^g) \).

**Remark 2.9.** When \( N \) is a vector bundle over \( M \), and \( \sigma \) an elliptic symbol on \( N \), we may privilege a representative of \( \text{Ch}(\sigma)(X) \) via rapidly decreasing differential forms on the fibers of \( N \). This is the choice we made in [6]. It is easy to see that the construction of equivariant Chern characters via adapted connections differs from BV construction via a boundary which is also rapidly decreasing at \( \infty \) on the fibers.

If \( Z \) is a \( G \)-invariant closed subset of \( N \), we have defined in [13] a Cartan model for the space \( \mathcal{H}_G(Z) \) of equivariant cohomology. A representative is an equivariant form \( \alpha : \mathfrak{g} \to \mathcal{A}(N) \) such that \( d_\mathfrak{g} \alpha = 0 \) in a neighborhood of \( Z \). The dependence of \( \alpha \) on \( X \) is \( C^\infty \). We denote by \( 1_Z \in \mathcal{H}_G(Z) \) the class represented by a \( G \)-invariant function \( \text{One}_Z \) identically equal to 1 in a neighborhood of \( Z \), and supported on a small neighborhood of \( Z \).

We similarly consider the subcomplex of equivariant forms \( \alpha(X) \) with compact support on \( N \) such that \( d_\mathfrak{g} \alpha = 0 \) in a neighborhood of \( Z \), and whose Fourier transform in \( X \) is supported on a compact subset of \( \mathfrak{g}^* \), and we denote the corresponding cohomology group by \( \mathcal{H}^m_{G,c}(Z) \).

Let \( \sigma \) be a symbol on \( N \) and assume that the support of \( \sigma \) intersected with \( Z \) is compact. Thus \( \sigma \) defines a class in \( K^0_G(Z) \) still denoted by \( \sigma \). We then define a representative of \( \text{Ch}(\sigma)(X) \) to be the equivariant cohomology class \( \text{One}_Z \text{Ch}(\sigma, \nabla)(X) \). Here \( \nabla \) and \( \text{One}_Z \) are chosen such that the support of the differential form \( \text{One}_Z \text{Ch}(\sigma, \nabla)(X) \) is compact. Similarly, for \( g \in G \), we obtain the twisted Chern character \( \text{Ch}(g, \sigma) \in \mathcal{H}^m_{G,c}(Z^g) \).

We now recall the definition of infinitesimal index [13].

Let \( \omega \) be a \( G \)-invariant 1-form on \( N \), and let \( \mu(n)(X) = -\langle \omega, v_X \rangle \) be the corresponding moment map \( N \to \mathfrak{g}^* \). Let \( d_\mathfrak{g} \omega(X) = \mu(X) + d\omega \). The main example we will consider is when \( N = T^*M \), and \( \omega = -\ell \) is minus the Liouville form. So \( (d_\mathfrak{g} \omega)(X) = \mu(X) + \Omega \) is the equivariant symplectic form of \( T^*M \).

Assume that \( Z = \mu^{-1}(0) \). Let \( \alpha(X) \) be an equivariant form with compact support on \( N \), whose Fourier transform in \( X \) is supported on a compact subset \( K \) of \( \mathfrak{g}^* \). Then

\[
I(s) = \int_{\mathfrak{g}} e^{isd_\mathfrak{g} \omega(X)} \alpha(X) dX
\]

is a differential form on \( N \) with support contained in the set of elements \( n \) such that \( s\mu(n) \in K \). So, when \( s \) tends to \( \infty \), \( I(s) \) is supported in a neighborhood of \( Z \). Consider an element \( \theta \in \mathcal{H}^m_{G,c}(Z) \) and denote still by \( \theta \) a representative of the class \( \theta \). The infinitesimal index \( \text{Inf} \text{dex}^2_G(\theta) \) of \( \theta \) is the distribution on \( \mathfrak{g}^* \) such
that, for a test function $f$ on $\mathfrak{g}^*$,
\[
\langle \text{Infdex}_G^\omega(\theta), f \rangle = \lim_{s \to \infty} \int_N \left( \int_{\mathfrak{g}} e^{is \omega(X)} \theta(X) \widehat{f}(X) dX \right).
\]

This distribution depends only on the class of $\theta$ in $H_{G,c}^m(Z)$ and not on its representative. It however depends on $\omega$. We will denote the limit
\[
\lim_{s \to \infty} \int_N \left( \int_{\mathfrak{g}} e^{is \omega(X)} \theta(X) \widehat{f}(X) dX \right)
\]
by $\int_N \int_{\mathfrak{g}} \theta(X) \widehat{f}(X) dX$. Thus we write
\[
\langle \text{Infdex}_G^\omega(\theta), f \rangle = \int_N \int_{\mathfrak{g}} \theta(X) \widehat{f}(X) dX.
\]

Here
\[
\widehat{f}(X) = \int_{\mathfrak{g}^*} e^{-i(y,X)} f(y) dy
\]
and the measure $dX$ is chosen such that
\[
f(y) = \int_{\mathfrak{g}} e^{i(y,X)} \widehat{f}(X) dX.
\]

Let us give an example of infinitesimal index (see [12], proof of Theorem 4.21). Let us consider $N = \mathbb{R}^2 = \mathbb{C}$, with action of $G = U(1)$. Let $Z = \{0\}$, and $\alpha = \frac{1}{2} (x_1 dx_2 - x_2 dx_1)$ with moment map $\frac{1}{2} (x_1^2 + x_2^2)$.

**Lemma 2.10.** Let $1_Z \in H_{G,c}(Z)$ and let $f$ be a test function on $\mathfrak{g}^*$. Then
\[
\frac{1}{2i\pi} \langle \text{Infdex}_G^\omega(1_Z), f \rangle = \int_0^\infty f(y) dy.
\]

**2.4. Formal series of equivariant classes.** Let $M$ be our $G$-manifold. Let $V \to M$ be a real or complex $G$-equivariant vector bundle on $M$ with typical fiber a real or complex vector space $V$. The Chern–Weil map $W$ associates to a $\text{GL}(V)$-invariant polynomial $f$ on $\text{End}(V)$ an equivariant characteristic class $W(f)$ in $H_G^*(M)$. If $f$ is homogeneous of degree $k$, then $W(f)$ is homogeneous of degree $2k$.

Our conventions for the Chern–Weil homomorphism $W$ are as in [4].

Let $A \in \text{End}(V)$. Introduce a variable $q$, and consider the Taylor expansion
\[
J_V([q]A) = \det_V \left( \frac{e^{[q]A} - 1}{[q]A} \right) = \sum_{k=0}^{\infty} q^k T_k(A).
\]

Our main new concept is the introduction of the following formal equivariant characteristic class of $M$.

**Definition 2.11.** The formal $J$-class of $M$ is the series of elements of $H_G^*(M)$ defined by
\[
J([q], M) = \sum_{k=0}^{\infty} q^k W(T_k)
\]
obtained by applying the Chern–Weil map for the real vector bundle \( \mathcal{V} = TM \to M \) to the series
\[
\det_{\mathcal{V}} \left( e^{[q]A} - 1 \right) = \sum_{k=0}^{\infty} q^k T_k(A).
\]

Here \( W(T_k) \) is homogeneous of degree \( 2k \).

We can thus consider \( B([q], M) = 1/J([q], M) \) in the ring of formal series of equivariant cohomology classes with polynomial coefficients. We write \( B([q], M) = \sum_{k=0}^{\infty} q^k B_k(M) \). In the introduction, we have introduced \( \left[ B(M) \right] = \sum_{k=0}^{\infty} B_k(M) \) and the truncated class \( B(M, K) = \sum_{k=0}^{K} B_k(M) \). It is more convenient not to fix an order of truncation, and to work with the full formal series \( 1/J([q], M) \).

When \( G = \{1\} \), then \( p^*J([1], M) = J(M) \) is the inverse of the usual Todd class of the tangent bundle to \( T^*M \) (considered as an almost complex manifold). Furthermore, if \( \sigma \) is elliptic, Atiyah–Singer formula \([3]\) for \( \text{Index}(\sigma) \in \mathbb{Z} \) is
\[
\text{Index}(\sigma) = \frac{1}{(2i\pi)\dim M} \int_{T^*M} \frac{\text{Ch}(\sigma)}{J(M)}.
\]

Consider
\[
D_N([q], s)(A) = \det_N \left( 1 - se^{[q]A} \right) = \sum_{k=0}^{\infty} q^k D_k^s(A).
\]

Consider the normal bundle \( \mathcal{N} \to M^g \). Thus \( g \) produces an invertible linear transformation of \( \mathcal{N}_x \) at any \( x \in M^g \). The Chern–Weil homomorphism for the real vector bundle \( \mathcal{N} \) (with structure group \( \text{GL}(g) \)) produces a series \( D([q], g, M/M^g) := \sum_{k=0}^{\infty} q^k W(D_k^g) \) of closed equivariant differential forms on \( M^g \). The coefficient in \( q^0 \) of this series is just the function \( x \mapsto \det_{\mathcal{N}_x}(1 - g) \), a function which is a non-zero constant on each connected component of \( M^g \).

We thus obtain a formal series of \( G^g \)-equivariant classes on \( M^g \) by considering the form \( J([q], M^g)(X)D([q], g, M/M^g)(X) \). We can also invert this series in the ring of formal series.

### 2.5. Piecewise polynomial functions.
In this subsection, \( G \) is a torus. Let \( V = \mathfrak{g}^* \) equipped with the lattice \( \Lambda \subset \mathfrak{g}^* \) of weights of \( G \). If \( g = \exp X \), we write \( g^\lambda = e^{i\lambda \cdot X} \). The function \( g \mapsto g^\lambda \) is a character (a one-dimensional representation) of \( G \).

Using the Lebesgue measure \( dy \) associated to \( \Lambda \), we identify generalized functions on \( \mathfrak{g}^* \) and distributions on \( \mathfrak{g}^* \). If \( h \) is a generalized function on \( \mathfrak{g}^* \), we denote by \( \int_{\mathfrak{g}^*} h(y)f(y) \, dy \) its value on the test function \( f \).

Let \( \mathcal{H} \) be a finite collection of rational affine hyperplanes in \( \mathfrak{g}^* \). An element of \( \mathcal{H} \) will be called an admissible wall. For \( H \) an affine hyperplane, let \( \text{lin}(H) \) be the hyperplane parallel to \( H \). An element \( v \in V \) is said to be \( \mathcal{H} \)-generic if \( v \) is not on any hyperplane of the collection \( \{ \text{lin}(H), H \in \mathcal{H} \} \). We just say that \( v \) is generic. A tope \( \zeta \) is a connected component of the complement of all admissible walls (thus \( \zeta \) is an open convex subset of \( V \)) and we denote by \( V_{\text{reg}} \) the union of topes. If \( v \in V \), and \( \varepsilon \) is a generic vector, then \( v + t\varepsilon \) is in \( V_{\text{reg}} \) if \( t > 0 \) and sufficiently small. Assume that \( f \) is a function on \( V_{\text{reg}} \), given on each tope by the restriction of an analytic function of \( y \in V \) (depending on the tope). We say that \( f \) is a piecewise analytic function on \( V_{\text{reg}} \).
Definition 2.12. Let \( v \in V \), and let \( f \) be a piecewise analytic function. Let \( \varepsilon \) be a generic vector. Define
\[
(\lim_{\varepsilon} f)(v) = \lim_{t > 0, t \to 0} f(v + t\varepsilon).
\]

A piecewise polynomial function is a function on \( V_{\text{reg}} \) which is given by a polynomial formula on each tope. We denote by \( PW \) the space of piecewise polynomial functions. Consider \( f \in PW \) (defined on \( V_{\text{reg}} \)) as a locally \( L^1 \)-function on \( V \), thus \( f \) defines a generalized function on \( V \). An element of \( PW \), considered as a generalized function on \( V \), will be called a piecewise polynomial generalized function.

Definition 2.13. The space \( S \) is the space of generalized functions on \( V \) generated by the action of constant coefficients differential operators on piecewise polynomial generalized functions.

For example, the Heaviside function on \( \mathbb{R} \) is a piecewise polynomial generalized function. Its derivative in the sense of generalized functions is the Dirac function at 0, and belongs to \( S \).

Introduce formal series \( m([q]) = \sum_{k=0}^{\infty} q^k m_k \) of generalized functions on \( V \). Then, if \( f \) is a test function,
\[
\int_{\mathfrak{g}^*} m([q])(y) f(y) \, dy = \sum_{k=0}^{\infty} q^k \int_{\mathfrak{g}^*} m_k(y) f(y) \, dy
\]
is a formal power series in \( q \). It may be evaluated at \( q = 1 \) if the preceding series is finite (or convergent). If \( \varepsilon \) is generic, \( v \in V \), and the functions \( m_k \) restricted to \( V_{\text{reg}} \) piecewise analytic, we may define
\[
\lim_{\varepsilon} m([q])(v) = \sum_{k=0}^{\infty} q^k \lim_{\varepsilon} m_k(v).
\]
This is a formal series in \( \mathbb{C}[[q]] \).

An important piecewise polynomial function is the box spline [10]. Let \( \Phi = [\phi_1, \phi_2, \ldots, \phi_N] \) be a list of non-zero elements of \( \Lambda \). We assume that the set \( \Phi \) spans \( V \). Let \( Z(\Phi) = \{ \sum_{i=1}^{N} t_i \phi_i; 0 \leq t_i \leq 1 \} \) be the zonotope determined by \( \Phi \). The box spline \( B_\Phi(y) \) is the measure on \( V \) supported on \( Z(\Phi) \) obtained by convolutions of the intervals \( [0,1]\phi_k \):
\[
\int_V B_\Phi(y) f(y) \, dy = \int_0^1 \cdots \int_0^1 f \left( \sum_{k=1}^{N} t_k \phi_k \right) \, dt_1 \cdots dt_N.
\]
The Fourier transform of \( B_\Phi \) is the function
\[
J_\Phi(X) = \prod_{\phi \in \Phi} \frac{1 - e^{-i\langle \phi, X \rangle}}{i\langle \phi, X \rangle}
\]
of \( X \in \mathfrak{g} \).

Let \( \mathcal{H}_0 \) be the set of hyperplanes of \( V \) generated by \( \dim V - 1 \) linearly independent elements of \( \Phi \). Let \( F \) be the set of sums \( \phi_I = \sum_{i \in I} \phi_i \) of elements of \( \Phi \), where \( I \) is
a sublist of $\Phi$. We consider the finite set $\mathcal{H}$ of affine hyperplanes of the form $p + H$ with $H \in \mathcal{H}_0$, $p \in F$. Consider the set $V_{\text{reg}}$ of elements $v$ of $V$ such that $v$ does not belong to any affine hyperplane $H$ with $H \in \mathcal{H}$. Then $B_\Phi$ is a locally polynomial function with respect to $\mathcal{H}$.

**Example 2.14** (The box spline for $\Phi = [1, 1, -1, -1]$). Consider the function

$$j_\Phi(x) = \left(\frac{e^{ix} - 1}{ix}\right)^2 \left(\frac{1 - e^{-ix}}{ix}\right)^2$$

of $x \in \mathbb{R}$. Its Fourier transform (the convolution of the intervals $[0, 1], [0, 1], [-1, 0], [-1, 0]$) is the measure $b_4(y)$ given by

$$b_4(y) = \begin{cases} 
\frac{1}{6}(y + 2)^3 & \text{if } -2 < y < -1, \\
\frac{2}{3} - y^2 - \frac{1}{2}y^3 & \text{if } -1 \leq y \leq 0, \\
\frac{2}{3} - y^2 + \frac{1}{2}y^3 & \text{if } 0 \leq y \leq 1, \\
-\frac{1}{6}(y - 2)^3 & \text{if } 1 < y < 2.
\end{cases}$$

Consider the equation $(1/j_\Phi(x))j_\Phi(x) = 1$. Replace $1/j_\Phi(x)$ by its Taylor series

$$\frac{1}{j_\Phi(x)} = 1 + \frac{1}{6}x^2 + \cdots .$$

The inversion formula for the box spline (due to Schoenberg [23] for splines in one variable) states that the Fourier transform $Db_4$ of the function $(1 + \frac{1}{6}x^2)j_\Phi(x)$ is continuous and its value at any integer $k$ is equal to 0, except for $k = 0$, where the value is 1. The function $b_4$ and the function $Db_4 = b_4 - \frac{1}{6}(\frac{d}{dy})^2b_4$ are plotted in Fig. 2 (a more elaborate example is given in §3.6).

![Figure 2](image-url)
Assume that the cone \( \text{Cone}(\Phi) \) generated by \( \Phi \) is a salient cone in \( V \). Then we can define the spline \( T(\Phi) \). This is the measure on \( V \) supported on \( \text{Cone}(\Phi) \) obtained by convolutions of the half lines \([0, \infty) \phi_k\):

\[
\int_V T(\Phi)(y) f(y) \, dy = \int_0^\infty \cdots \int_0^\infty f \left( \sum_{k=1}^N t_k \phi_k \right) \ dt_1 \ldots dt_N.
\]

It is a locally polynomial measure. More generally, if \( y \in \mathbb{C}^N \), we introduce the spline function \( T(\Phi, y) \) with parameters \( y = (y_1, y_2, \ldots, y_N) \) as the convolution of the distributions \( \int_0^\infty f(t \phi_k) e^{ty_k} \, dt \). This function is piecewise analytic.

Let \( N \) be a Hermitian vector space with a linear representation of \( G \). We write \( N = \bigoplus_{\phi \in \Phi} N_\phi \), as a sum of one-dimensional representation spaces of \( G \). If \( \Phi \) spans \( V = g^* \), the action of \( G \) on \( N \) is with finite generic stabilizer. We still assume that \( \Phi \) spans a salient cone \( \text{Cone}(\Phi) \) in \( V \). Then the action of \( G \) in \( \text{Sym}(N) \) is with finite multiplicity. We write

\[
\text{Sym}(N) = \bigoplus_{\lambda \in \mathcal{G}} \text{Sym}_\lambda(N),
\]

its decomposition with respect to \( G \). Thus, under the action of \( G \), we have

\[
\text{Tr}_{\text{Sym}(N)}(g) = \sum_{\lambda \in \mathcal{G}} \dim(\text{Sym}_\lambda(N)) g^\lambda
\]

and the function \( \lambda \mapsto \dim(\text{Sym}_\lambda(N)) \) is the so-called Kostant partition function (for \( \Phi \)). Let us give Brion–Szenes–Vergne ([8], [24]) formula (slightly modified) for \( \dim(\text{Sym}_\lambda(N)) \) in the form of an integral formula. Let \( g \in G \), and let \( N^g \) be the fixed subspace of \( N \) by the action of \( g \). If the action of \( G \) on \( N^g \) is with generic finite stabilizer, we say that \( g \) is a vertex of \( \Phi \). This is equivalent to saying that the list \( \Phi^g \) of elements \( \phi \) of \( \Phi \) such that \( g \phi = 1 \) still spans \( V \). The set \( \mathcal{V}(\Phi) \) of vertices is a finite subset of \( G \). The set of vertices \( \mathcal{V}(\Phi) \) is reduced to the identity element if and only if the system \( \Phi \) is unimodular.

We define

\[
A^C(g)(X) = \det^C_{N^g} \left( \frac{\exp(X) - 1}{X} \right) \det^C_{N/N^g} \left( 1 - g \exp(X) \right),
\]

\[
A^R(g)(X) = \det^R_{N^g} \left( \frac{\exp(X) - 1}{X} \right) \det^R_{N/N^g} \left( 1 - g \exp(X) \right).
\]

Here \( \det^C \) is the complex determinant, while \( \det^R \) is the real determinant.

Consider

\[
Z(q, g)(X) = \frac{A^C(g^{-1})(-X)}{A^R(g)(qX)}
\]

so

\[
Z(q, g)(X) = \frac{\det^C_{N^g} \left( (1 - \exp(-X))/X \right) \det^C_{N/N^g} \left( 1 - g^{-1} \exp(-X) \right)}{\det^R_{N^g} \left( ((\exp(qX) - 1)/(qX)) \right) \det^R_{N/N^g} \left( 1 - g \exp(qX) \right)}.
\]

For \( q = 1 \),

\[
Z(1, g)(X) = \frac{1}{\det^C_{N^g} \left( (\exp(X) - 1)/X \right) \det^C_{N/N^g} \left( 1 - g \exp(X) \right)}.
\]
Remark that $Z(q, g)(X)$ is holomorphic at $q = 0$. Indeed
\[ Z(0, g)(X) = \frac{A^C(g^{-1})(-X) \det_R^N/N}{(1 - g)}. \]
Thus $Z([q], g)(X)$ is a series of analytic functions of $X \in \mathfrak{g}$. Define the series $m([q], g)(y)$ of distributions on $\mathfrak{g}^*$ by the formula
\[ \langle m([q], g), f \rangle = \frac{1}{(2i\pi)^{\dim_{\mathbb{C}} N^g}} \int_{N^g} \left( \int_\mathfrak{g} e^{\langle Xv, v \rangle - \langle dv, dv \rangle} Z([q], g)(X) \hat{f}(X) dX \right). \]

Our set of admissible walls is defined to be $\mathcal{H} := \{ p + H \}$ where $p$ is in the set $F = \{-\phi_I\}$ and $H$ is a hyperplane generated by elements of $\Phi$. Recall that the set $V_{\text{reg}}$ is the complement of the union of admissible walls.

**Theorem 2.15.** Write $m([q], g)(y) = \sum_{k=0}^{\infty} q^k m_k(g)(y)$.
1) The distributions $m_k(g)(y)$ belong to the space $S$ of derivatives of piecewise polynomial generalized functions.
2) If $g$ is not a vertex of $\Phi$, the restriction of $m_k(g)$ to $V_{\text{reg}}$ is equal to 0.
3) The restriction of $m_k(g)(y)$ to each connected component $c$ of $V_{\text{reg}}$ is a polynomial function of $y$. It vanishes if $k \geq 2|\Phi|$.
4) For any $\lambda \in \Lambda$, and any generic vector $\varepsilon$,
\[ \dim(\text{Sym}_{\lambda}(N)) = \sum_{g \in V(\Phi)} g^{-\lambda} \lim_{\varepsilon} m([1], g)(\lambda). \]

The relation with the usual formulation of the value of the Kostant partition function $\dim(\text{Sym}_{\lambda}(N))$ at $\lambda$ via a sum of values of spline functions obtained by differentiating $T(\Phi)$ is as follows. The Fourier transform of the integral
\[ I(X) = \int_{N^g} e^{\langle Xv, v \rangle - \langle dv, dv \rangle} \]
is the spline $T(\Phi^g)$. Then, the Fourier transform of $A^C(g^{-1})(-X)I(X)$ is the convolution of $T(\Phi^g)$ with the box spline $\text{Box}(-\Phi^g)$, followed by a series of translations. We obtain again a piecewise polynomial function on $V$. Then to obtain the Fourier transform of $1/(A^R(g)([q], X))A^C(g^{-1})(-X)I(X)$, we apply an infinite series of constant coefficient differential operators to this piecewise polynomial function.

Remark that, in the last assertion of Theorem 2.15, $\varepsilon$ is any generic vector. So we tend to $\lambda$ coming from any direction. So this theorem is not exactly Brion–Szenes–Vergne theorem (there it was required that $y$ tends to $\lambda$ along directions in the cone $\text{Cone}(\Phi)$). Here we have (as in the spirit of this article and of [11]) considered the representation of $G$ in $T^*N$ and written:
\[ \prod_{\phi} \frac{1}{1 - e^{i\phi}} = \prod_{\phi} \frac{(1 - e^{-i\phi})}{(1 - e^{-i\phi})(1 - e^{i\phi})} \]
and expanded
\[ \Theta = \frac{1}{\prod_{\phi}(1 - e^{-i\phi})(1 - e^{i\phi})} \]
as a Fourier series supported in $\text{Cone}(\Phi)$. The theorem results then from the inversion formula for box splines [9] (see also [12], Theorem 2.29, and [25]). Indeed,
as the zonotope generated by $\Phi$ and $-\Phi$ contains 0 in its interior, the values of the Fourier coefficients of $\Theta$ are given by a piecewise analytic function on $V_{\text{reg}}$ which is continuous at any point of $\Lambda$.

Furthermore if the set of vertices $\mathcal{V}(\Phi)$ is reduced to $\{1\}$, then the function $m([1], 1)(y)$ gives by restriction to $V_{\text{reg}}$ a piecewise polynomial function which extends continuously to $V$ (see [12], Remark 3.15).

Let $u \in U(N)$ be a unitary transformation of $N$ commuting with the action of $G$, and let $R \in \text{End}(N)$ be a complex endomorphism of $N$ commuting with the action of $G$ and with $u$. The transformation $ue^R$ leaves stable the finite-dimensional space $\text{Sym}_\Lambda(N)$. We consider the function $\lambda \mapsto \text{Tr}_{\text{Sym}_\Lambda(N)}(ue^R)$. Then, for $R$ small, we can compute the function $\lambda \mapsto \text{Tr}_{\text{Sym}_\Lambda(N)}(ue^R)$ as a limit of a piecewise analytic function on $V_{\text{reg}}$ (depending on $u, R$). Let us give the formula.

We define the set $\mathcal{V}(\Phi, u) \subset G$ as the set of elements $g \in G$ such that the action of $g$ on $N^g_u$ has a generic finite stabilizer.

Let

\[ A^C(g, u, R)(X) = \det_{N^g_u}^C \left( \frac{\exp(X + R) - 1}{X + R} \right) \det_{N/N^g_u}^C (1 - gu \exp(X + R)), \]

\[ A^R(g, u, R)(X) = \det_{N^g_u}^R \left( \frac{\exp(X + R) - 1}{X + R} \right) \det_{N/N^g_u}^R (1 - gu \exp(X + R)). \]

Let

\[ Z(q, g, u, R)(X) = \frac{A^C(g^{-1}, u^{-1}, -R)(-X)}{A^R(g, u, qR)(qX)}. \]

So

\[ Z(q, g, u, R)(X) = \frac{\det_{N^g_u}^C \left( (1 - \exp(-(X + R)))/(X + R) \right) \det_{N/N^g_u}^C (1 - (gu)^{-1} \exp(-(X + R)))}{\det_{N^g_u}^R \left( (\exp(q(X + R)) - 1)/q(X + R) \right) \det_{N/N^g_u}^R (1 - gu \exp(q(X + R)))}. \]

We have

\[ Z(1, g, u, R)(X) = \frac{1}{\det_{N^g_u}^C \left( (\exp(X + R) - 1)/(X + R) \right) \det_{N/N^g_u}^C (1 - gu \exp(X + R))}. \]

Define the series $m([q], g, u, R)(y)$ of distributions on $g^*$ by the formula

\[ \langle m([q], g, u, R), f \rangle = \frac{1}{(2\pi)^{\dim C N^g_u}} \int_{N^g_u} \left( \int_g e^{\langle (X + R)v, v \rangle - \langle dv, dv \rangle} Z([q], g, u, R)(X) \hat{f}(X) dX \right). \]

**Theorem 2.16.** Write $m([q], g, u, R)(y) = \sum_{k=0}^{\infty} q^k m_k(g, u, R)(y)$.

1) If $g$ is not in $\mathcal{V}(\Phi, u)$, the restriction of $m_k(g, u, R)$ to $V_{\text{reg}}$ is equal to 0.

2) The restriction of $m_k(g, u, R)(y)$ to each connected component $c$ of $V_{\text{reg}}$ is given by the restriction to $c$ of an analytic function of $y \in V$.

If $y \in c$, the series

\[ m([1], g, u, R)(y) = \sum_{k=0}^{\infty} m_k(g, u, R)(y) \]

is convergent if $R$ is sufficiently small.
3) If $R$ is nilpotent and $R^K = 0$, the restriction of $m_k(g,u,R)(y)$ to each connected component $c$ of $V_{\text{reg}}$ is a polynomial function of $y$, $R$, and vanishes if $k \geq K + 2|\Phi|$.

4) If $R$ is sufficiently small, then for any $\lambda \in \Lambda$, and any generic vector $\varepsilon$,

$$\text{Tr}_{\text{Sym}_\lambda(N)}(ue^R) = \sum_{g \in V(\Phi,u)} g^{-\lambda} \lim_{\varepsilon} m([1], g, u, R)(\lambda)$$

as a convergent series.

5) If $R$ is nilpotent, then for any $\lambda \in \Lambda$, and any generic vector $\varepsilon$,

$$\text{Tr}_{\text{Sym}_\lambda(N)}(ue^R) = \sum_{g \in V(\Phi,u)} g^{-\lambda} \lim_{\varepsilon} m([1], g, u, R)(\lambda)$$

as a finite sum.

Remark that when $X$, $R$ are small, we have the identities of meromorphic functions

$$Z(1, g, u, R)(X) \left( \int_{N^u} e^{(X+R)v,w} e^{-\langle dv,dv \rangle} \right)$$

$$= \det_{N^u}^C \left( \frac{X + R}{1 - \exp(X + R)} \right) \det_{N/N^u}^C \left( \frac{1}{1 - gu \exp(X + R)} \right) \left( \frac{1}{\det_{N^u}^C (X + R)} \right)$$

$$= \det_N^C \left( \frac{1}{1 - gu \exp(X + R)} \right) = \text{Tr}_{\text{Sym}(N)}(gue^{X+R}).$$

As $g \in G$, and $X \in \mathfrak{g}$, this is

$$\sum_{\lambda} g^\lambda e^{i\langle \lambda, X \rangle} \text{Tr}_{\text{Sym}_\lambda(N)}(ue^R)$$

which gives a vague plausibility to the formula.

This theorem can be proved as in [24] by carefully reducing the computation to a one-dimensional computation.

We give some simple examples of the corresponding formula.

1) Let $N$ be the complex one-dimensional space with action of $S^1$ by $e^{i\theta}$. Let $u$ be a complex number of modulus 1, and $r \in \mathbb{C}$. Then we have, for $\lambda$ a non-negative integer,

$$\text{Tr}_{\text{Sym}_\lambda(N)}(ue^r) = u^\lambda e^{\lambda r}.$$

Let us compute our formula. Here we have just one vertex $g = 1/u$. We compute the distribution $m([1], g, u, r)(y)$, for $g = 1/u$, restricted to $V_{\text{reg}}$. This is a piecewise analytic distribution on $\mathbb{R}$ given by the following formula:

$$m([1], g, u, r)(y) = \begin{cases} e^{ry} & \text{if } y \geq 0, \\ (y+1)e^{ry} & \text{if } -1 \leq y \leq 0, \\ 0 & \text{if } y \leq -1. \end{cases}$$

So $m([1], g, u, r)(y)$ extends to a continuous function of $y$. Furthermore we see that the limit when $y \to \lambda$ from left or right of $g^\lambda m([1], g, r)(y)$ is zero on all strictly negative integers, and is equal to $u^\lambda e^{\lambda y}$, when $\lambda$ is a non-negative integer.
2) Let \( N \) be the complex two-dimensional space with action of \( S^1 \) by \( (e^{i\theta}, e^{2i\theta}) \). Let \( u \in S^1 \) acting by homothety and let \( R = 0 \). Then

\[
\text{Tr}_{\text{Sym}_\lambda(N)}(u) = \begin{cases} 
\frac{u^\lambda}{1 - u^{-1}} + \frac{u^{\lambda/2}}{1 - u} & \text{if } \lambda \text{ is a non-negative even integer}, \\
\frac{u^\lambda}{1 - u^{-1}} + \frac{u^{(\lambda+1)/2}}{1 - u} & \text{if } \lambda \text{ is a positive odd integer}, \\
0 & \text{if } \lambda \text{ is a strictly negative integer}.
\end{cases}
\]

Let us now compute our formula. There are 3 vertices, \( g = 1/u \) and \( g_1, g_2 \) the two elements such that \( g_i^2 = 1/u \).

We compute the distribution \( m([1], 1/u, u, 0)(y) \) (restricted to \( V_{\text{reg}} \)). This is a piecewise polynomial distribution on \( \mathbb{R} \) given by the following formula:

\[
m([1], 1/u, u, 0)(y) = \begin{cases} 
\frac{1}{(1 - u^{-1})} & \text{if } y \geq 0, \\
\frac{y}{(1 - u^{-1})(1 - u)} + \frac{u^2 + 3}{(1 - u^{-1})(1 - u)^2} & \text{if } -1 \leq y \leq 0, \\
\frac{1}{(1 - u^{-1})^2} & \text{if } -2 \leq y \leq -1, \\
\frac{y}{(1 - u^{-1})^2} - \frac{u - 3}{(1 - u)(1 - u^{-1})^2} & \text{if } -3 \leq y \leq -2, \\
0 & \text{if } y \leq -3.
\end{cases}
\]

Consider the vertices \( g_1, g_2 \). We compute the distribution \( m([1], g_i, u, 0)(y) \) (restricted to \( V_{\text{reg}} \)). This is a piecewise polynomial distribution on \( \mathbb{R} \) given by the following formula:

\[
m([1], g_i, u, 0)(y) = \begin{cases} 
\frac{1}{2(1 - g_i u)} & \text{if } y \geq 0, \\
\frac{y}{4 (1 - g_i u)(1 - (g_i u)^{-1})} + \frac{3u - 3g_i u + 2}{4 (1 - g_i u)^3} & \text{if } -1 \leq y \leq 0, \\
\frac{1}{4 (1 - g_i u)} & \text{if } -2 \leq y \leq -1, \\
\frac{y}{4 (1 - g_i u)} - \frac{1}{2} & \text{if } -3 \leq y \leq -2, \\
0 & \text{if } y \leq -3.
\end{cases}
\]

One verifies that the limit when \( y \to \lambda \) from left or right of

\[
\sum_{g \in V(\Phi, u)} g^{-\lambda} m([1], g, u, 0)(y)
\]

is equal to zero on all strictly negative integers, and is equal to 1 when \( \lambda = 0 \). If \( \lambda \) is a strictly positive integer, we have:

\[
\sum_{g \in V(\Phi, u)} g^{-\lambda} m([1], g, u, 0)(\lambda) = \frac{u^\lambda}{1 - u^{-1}} + g_1^{-\lambda} \frac{1}{2(1 - g_1 u)} + g_2^{-\lambda} \frac{1}{2(1 - g_2 u)}
\]

and this is indeed the right formula for \( \text{Tr}_{\text{Sym}_\lambda(N)}(u) \).
§ 3. The multiplicity index formula for a torus action

Let $G$ be a torus acting on our manifold $M$, and let $V = g^*$. We assume that $M$ is connected and that the generic infinitesimal stabilizer of the action of $G$ on $M$ is equal to $\{0\}$. It is immediate to reduce to this case. Furthermore, we assume that our manifold $M$ can be embedded $G$-equivariantly in a vector space with a linear action of $G$. This is to ensure that the set of stabilizers of points of $M$ is a finite set of subgroups of $G$.

Let $S^1 = \{\mathbb{R}U_a\}$ be the set of infinitesimal stabilizers for the action of $G$ on $M$ which are of dimension 1. This is a finite set of lines $\mathbb{R}U_a$ in $g$.

**Definition 3.1.** We consider the following set of hyperplanes in $g^*$:

$$\mathcal{H}_0(M) = \{U_a^\perp; \mathbb{R}U_a \in S^1\}.$$  

If the action of $G$ on $M$ is locally free, that is, if all stabilizers of points of $M$ are finite subgroups, this set of hyperplanes is empty.

Let $\sigma$ be a transversally elliptic symbol on $M$. Let $g \in G$ and let $\text{Ch}(g, \sigma)(X)$ be the twisted Chern character of $\sigma$. If $\alpha \in H^*_G(M^g)$ is an equivariant cohomology class with polynomial coefficients, we can define a generalized function $h(g, \sigma, \alpha)$ on $g^*$ by the formula

$$\int_{T^*M^g} \int_g \text{Ch}(g, \sigma)(X)\alpha(X)\hat{f}(X) \, dX = \int_{g^*} h(g, \sigma, \alpha)(y) f(y) \, dy$$

for any test function $f$ on $g^*$.

We say that $g \in G$ is a vertex of the action of $G$ on $M$ if there exists a point $m \in M^g$ with finite stabilizer. Then the set $V(M)$ of vertices is a finite subset of $G$. If all stabilizers are connected, the set $V(M)$ is reduced to the identity element.

We have constructed the formal series $J([q], M^g)D([q], g, M/M^g)$ of equivariant classes with polynomial coefficients on $M^g$. Its inverse is also a series of equivariant classes with polynomial coefficients.

**Definition 3.2.** Define the series of generalized functions $m([q], g, \sigma)$ of generalized functions on $g^*$ such that

$$\int_{T^*M^g} \int_g (2i\pi)^{-\dim M^g} \frac{\text{Ch}(g, \sigma)(X)}{J([q], M^g)(X)D([q], g, M/M^g)(X)} \hat{f}(X) \, dX$$

$$= \int_{g^*} m([q], g, \sigma)(y) f(y) \, dy$$

for any test function $f(y)$ on $g^*$.

We write $m([q], g, \sigma)(y) = \sum_{k=0}^{\infty} q^k m_k(g, \sigma)(y)$, a series of generalized functions on $g^*$.

The following lemmas and theorem will be proved in the next subsections.

**Lemma 3.3.** There exists a finite subset $\mathcal{H}(\sigma)$ of affine walls such that all distributions $m_k(g, \sigma)$ are derivatives of piecewise (with respect to $\mathcal{H}(\sigma)$) polynomial distributions. Furthermore an affine hyperplane of the collection $\mathcal{H}(\sigma)$ is parallel to a hyperplane in the family $\mathcal{H}_0(M)$.  

Denote by $V_{\text{reg}}$ the complement of the union of affine walls in $\mathcal{H}(\sigma)$. Let $\epsilon$ be a connected component of $V_{\text{reg}}$. It follows from Lemma 3.3 that for each $k \geq 0$ and $g \in \mathcal{V}(M)$, $m_k(g, \sigma)$ is given by a polynomial formula on each tope $\epsilon$.

**Lemma 3.4.** If $k \geq \dim M$, the distribution $m_k(g, \sigma)$ vanishes on $V_{\text{reg}}$.

Thus we can define the piecewise polynomial function $m(g, \sigma)$ on $V_{\text{reg}}$ by writing $m(g, \sigma) = \sum_{k=0}^{\infty} m_k(g, \sigma)$.

**Theorem 3.5.** For any $\lambda \in \Lambda$, and any generic vector $\varepsilon$, we have

$$\text{mult}_G(\sigma)(\lambda) = \sum_{g \in \mathcal{V}(M)} g^{-\lambda} \lim_{\varepsilon} m(g, \sigma)(\lambda).$$

This theorem is particularly nice when stabilizers of the action of $G$ on $M$ are connected. In this case, there is only one vertex $g = 1$, and the theorem reads

$$\text{mult}_G(\sigma)(\lambda) = \frac{1}{(2i\pi)^{\dim M}} \int_{T^* M} \int_X \frac{\text{Ch}(\sigma)(X)}{J(M)[X]} e^{-i\langle \lambda, X \rangle} dX.$$

Here the notation $[X]$ in an equivariant cohomology class means that we have replaced the cohomology class by its formal expansion in homogeneous classes with polynomial coefficients. The value at $\lambda$ is obtained by limits from values in $V_{\text{reg}}$ and the limit can be taken from any direction. Furthermore, in this case, the piecewise polynomial function $m(1, \sigma)$ on $V_{\text{reg}}$ extends continuously on $\mathfrak{g}^*$. This is the function $m_G(\sigma)$ that we considered in the introduction.

This formula is particularly suggestive when $\lambda = 0$. Indeed, if $D$ is a transversally elliptic operator with principal symbol $\sigma$, the dimension of $[\text{Index}_G(\sigma)]^G$ is the virtual dimension of the space of $G$-invariant solutions of $D$. It is given by $\int_G \text{Index}_G(D)(g) \, dg$. On the other hand, for $g$ near $1$,

$$\text{Index}_G(\exp X) = \frac{1}{(2i\pi)^{\dim M}} \int_{T^* M} \frac{\text{Ch}(\sigma)(X)}{J(M)[X]} dX.$$

Our ‘miraculous formula’ reads

$$\int_G \text{Index}_G(D)(g) \, dg = \frac{1}{(2i\pi)^{\dim M}} \int_{T^* M} \frac{\text{Ch}(\sigma)(X)}{J(M)[X]} dX.$$

Note the dichotomy in the formula: it is very important that $\text{Ch}(\sigma)(X)$ is computed as a class with $C^\infty$ coefficients, and not in the completion of $H^*_G(N)$. In contrast, $1/(J(M)[X])$ is in the completion of the ring $H^*_G(N)$ of equivariant characteristic classes with polynomial coefficients.

**3.1. Generators for $G$-transversally elliptic symbols.** Following Atiyah–Singer (see the monograph [1]), we describe a set of generators for $K^0_G(T^*_G M)$.

Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$ arising as a stabilizer $\mathfrak{g}_m$. The set of such subalgebras is finite. We denote by $M((\mathfrak{s}))$ the strata of $M$ consisting of elements $m$ such that $\mathfrak{g}_m = \mathfrak{s}$. Thus $M((\mathfrak{s}))$ is a locally closed submanifold of $M$, eventually with several connected components denoted by $C_\mathfrak{s}$. If $S$ is the subtorus of $G$ with Lie algebra $\mathfrak{s}$, the action of $G/S$ on $C_\mathfrak{s}$ is infinitesimally free. We identify a tubular neighborhood $U_\mathfrak{s}$ of $C_\mathfrak{s}$ in $M$ with the normal bundle $\mathcal{N}_\mathfrak{s}$ of $C_\mathfrak{s}$ in $M$. Then $S$ acts
fiberwise in $\mathcal{N}_s$, and the set of fixed points $(\mathcal{N}_s)^S$ for the action of $S$ is reduced to the zero section. Thus the real vector bundle $\mathcal{N}_s$ is of even rank. Furthermore, we can choose a $G$-invariant complex structure $J$ on the fibers of the bundle $\mathcal{N}_s$. Using $J$, consider $\mathcal{N}_s$ as a complex vector bundle. As explained in [22], we can then define a map $\text{At}_J : K_G^0(T_G^*C_s) \to K_G^0(T_G^*U_s)$ using the lozenge product of a $G/S$ transversally elliptic symbol on $C_s$ with the Atiyah symbol $\text{at}(\mathcal{N}_s)$.

The following theorem is proved in [1]. See also [20] for a detailed study of the $R(G)$-module $K_G(T_G^*M)$.

**Theorem 3.6** (Atiyah–Singer). The module $K_G^0(T_G^*M)$ is generated over $R(G)$ by the submodules $\text{At}_J(K_G^0(T_G^*C_s))$, where $s$ runs over the infinitesimal stabilizers of the action of $G$ on $M$, and $C_s$ over the connected components of $M(\{s\})$.

**3.2. The proof for an infinitesimally free action.** Let us first make a remark on Fourier inversion. Assume that $\Theta$ is a generalized function on $G$ such that the support $\mathcal{V}$ of $\Theta$ is finite. Then, for each $v \in \mathcal{V}$ and $X \in \mathfrak{g}$ small,

$$\Theta(v \exp X) = \int_{\mathfrak{g}^*} e^{i(y,X)} m_v(y) \, dy$$

where $m_v(y)$ is a polynomial function on $\mathfrak{g}^*$. This equality is in the sense of generalized functions. Then it is easy to verify (using for example Poisson summation formula) that, as a generalized function on $G$,

$$\Theta(g) = \sum_{\lambda \in \Lambda} \left( \sum_{v \in \mathcal{V}} v^{-\lambda} m_v(\lambda) \right) g^\lambda.$$

We thus see that the Fourier coefficients of $\Theta$ on $\hat{G} = \Lambda$ are given by the quasi-polynomial function $\lambda \mapsto \sum_{v \in \mathcal{V}} v^{-\lambda} m_v(\lambda)$.

Let $M$ be a manifold with an infinitesimally free action of $G$. The set $\mathcal{V}(M)$ of vertices of the action is thus the set of elements $g \in G$ such that $M^g \neq \emptyset$.

The space $T^*_G M$ is a vector bundle over $M$, and using a $G$-invariant metric, we can write a direct sum decomposition of the bundle $T^* M$ as $T^* M = T^*_G M \oplus M \times \mathfrak{g}^*$. Thus we obtain a projection of $T^* M$ on $T^*_G M$ by $(m, \xi + \eta) \mapsto \xi$, and the moment map $\mu$ is $\mu(m, \xi + \eta) = \eta$. Here $m \in M$, $\xi \in (T^*_G M)_m$ and $\eta \in \mathfrak{g}^*$ (and $T^*_G M$ is the set of zeroes of the moment map).

Recall the description of the equivariant cohomology of a manifold $M$ with infinitesimally free action. We say that a differential form $\nu$ on $M$ is basic if $\nu$ is invariant and if $\iota(v_X)\nu = 0$ for all $X \in \mathfrak{g}$. If $G$ acts infinitesimally freely on $M$, the complex $\mathcal{A}^\text{basic}(M)$ of basic forms is stable under the de Rham differential and the equivariant cohomology of $M$ is the cohomology of the complex of basic forms. This is due to H. Cartan in the polynomial case (see [16]). See [14] for extending the proof to the $C^\infty$-case. In particular we have $H^*_G(M) = 0$ if $k > \dim M - \dim G$. Thus any series $\sum_{k=0}^\infty q^k \alpha_k$ of equivariant classes with polynomial coefficients is finite.

If $\mathcal{F}$ is a $G$-equivariant vector bundle on $M$, we can always choose a $G$-invariant basic connection $\nabla$ on $\mathcal{F}$. That is, the horizontal space on $\mathcal{F}$ determined by $\nabla$ contains the vector fields $v_X$ with $X \in \mathfrak{g}$. Thus the moment map determined by the connection is identically 0, the equivariant curvature is just $R^\mathcal{F}$, and the equivariant Chern character of $\mathcal{F}$ is the basic form $\text{Tr}(e^{R^\mathcal{F}})$. 

Let $\sigma$ be a transversally elliptic symbol on $M$. When the action is free, the symbol $\sigma$ is simply the pullback of an elliptic symbol on the manifold $M/G$, and the distribution $\text{Index}_G(\sigma)(g)$ is supported at $g = 1$. More generally, in the case of an infinitesimally free action, it is easy to see [1] that the support of the generalized function $\text{Index}_G(\sigma)(g)$ is contained in the set of the vertices $\mathcal{V}(M)$. Thus, by the remark above, the Fourier coefficients of $\text{Index}_G(\sigma)(g)$ are given by a quasi-polynomial function on $\Lambda$. The fact that this quasi-polynomial function coincides with our formula will follow from the BV formula [7], [6].

Provided $X$ is small enough, BV general formula asserts that, for any $g \in G$,

$$\text{Index}_G(\sigma)(g \exp X) = \int_{T^*M^g} \frac{1}{(2i\pi)^{\dim M^g}} e^{i d_g \omega(X)} \frac{\text{ChBV}(g, \sigma)(X)}{J(M^g)(X) D(g, M/M^g)(X)}.$$

In BV formula, $\text{ChBV}(g, \sigma)$ is a representative of the twisted equivariant Chern character chosen to be slowly increasing on the fibers of $T^*M$ and rapidly decreasing in the transverse directions $\xi$. The integral is shown to be convergent in the generalized sense. As the action is infinitesimally free, we can choose the differential form $\text{ChBV}(g, \sigma)$ on $T^*M$ to be the reciprocal image of a basic form on $T^*_G M$ (by the projection $T^*M \rightarrow T^*_G M$), and rapidly decreasing on the fibers of $T^*_G M$. So in particular $\text{ChBV}(g, \sigma)$ is independent of $X$. The equivariant class $J(M^g) D(g, M/M^g)$ can be represented by an invertible basic form on $M$, so is independent of $X$. Thus

$$I_g(X) = \int_{T^*M^g} \frac{1}{(2i\pi)^{\dim M^g}} e^{i d_g \omega(X)} \frac{\text{ChBV}(g, \sigma)}{J(M^g) D(g, M/M^g)}$$

is defined for any $X \in \mathfrak{g}$, in the generalized sense. As $d_g \omega(X) = \langle \mu(m), X \rangle + \Omega$, it is easy to see that $I_g(X)$ is the Fourier transform of a polynomial distribution $h(g, \sigma)(y)$:

$$\int_{\mathfrak{g}} I_g(X) \hat{f}(X) dX = \int_{\mathfrak{g}^*} h(g, \sigma)(y) f(y) dy$$

for any test function $f$ on $\mathfrak{g}^*$. We have to show that $h(g, \sigma)(y) = m([1], g, \sigma)(y)$.

In this work, we have defined our distribution $m([1], g, \sigma)(y)$ using the infinitesimal index. We have thus chosen $\text{Ch}(g, \sigma)$ to be represented by a differential form supported near $\text{supp}(\sigma)$. We can choose this differential form to be the reciprocal image by projection of $T^*M$ on $T^*_G M$ of a basic form with compact support on $T^*_G M$. As in Remark 2.9, the two choices $\text{ChBV}(g, \sigma)$ and $\text{Ch}(g, \sigma)$ differ by a de Rham differential of a basic form on $T^*_G M$ which is rapidly decreasing on the fibers of the bundle $T^*_G M \rightarrow M$. We have chosen a function $\text{One}_Z$ equal to 1 in the neighborhood of $T^*_G M$. We might choose $\text{One}_Z(m, \xi + \eta)$ (where $\xi \in (T^*_G M)_m$ and $\eta \in \mathfrak{g}^*$) to be a function of $\eta$ compactly supported on $\mathfrak{g}^*$ and equal to 1 in a neighborhood of 0. Our series

$$\frac{1}{J([q], M^g)(X) D([q], g, M/M^g)(X)}$$
is a finite series, and its value at \( q = 1 \) is the basic form \( 1/(J(M^g)D(g, M/M^g)) \). Thus we see as in Lemma 2.10 (see [12], proof of Theorem 4.21) that

\[
\int_{T^*M^g} \int_g e^{i d_g \omega(X)} \frac{ChBV(g, \sigma)}{J(M^g)D(g, M/M^g)} \hat{f}(X) dX
= \lim_{s \to -\infty} \int_{T^*M^g} \int_g e^{i s d_g \omega(X)} \text{One}_Z \frac{Ch(g, \sigma)}{J(M^g)D(g, M/M^g)} \hat{f}(X) dX.
\]

With the help of the remark above, our theorem holds thus for infinitesimally free action.

3.3. The case of the Atiyah symbol on a Hermitian vector space. Consider a Hermitian vector space \( N \). Let \( G \) be a torus acting on \( N \), and we assume that \( N = \bigoplus_{\phi \in \Phi} N_\phi \) where the list \( \Phi \) spans a salient cone. We use the notation of Example 2.6 and § 2.5. The set \( \mathcal{V}(N) \) is the set of vertices \( \mathcal{V}(\Phi) \). The set of hyperplanes \( \mathcal{H}_0(N) \) is the set of hyperplanes generated by elements of \( \Phi \).

Then, the Atiyah symbol \( \sigma = \text{at}(N) \) is \( G \)-transversally elliptic, and

\[
\text{Index}_G(\sigma)(g) = \text{Tr}_{\text{Sym}(N)}(g) = \sum_\lambda \dim(\text{Sym}_\lambda(N)) g^\lambda.
\]

Let us prove that our multiplicity index formula is true for the symbol \( \sigma = \text{at}(N) \).

Let \( g \in \mathcal{V}(N) \). The equivariant form \( J(N^g)(X) \) is just equal to \( \det_{N^g}((e^X - 1)/X) \) while the form \( D(g, N/N^g)(X) = \det_{N/N^g}(1 - g e^X) \) (here determinants are real determinants). Consider the series of distributions \( m([q, g, \sigma](y)) \) on \( g^* \) such that

\[
\int_{T^*N^g} \int_g \frac{Ch(g, \sigma)(X)}{J([q, N^g](X))D([q, g, N/N^g](X))} \hat{f}(X) dX = \int_{g^*} m([q, g, \sigma](y)) f(y) dy.
\]

Let \( \alpha \) be the 1-form \( \text{Im}((v, dv)) \) on the Hermitian space \( N \). We identify \( T^*N \) with \( N \oplus N \). Now we use the new variables \( u = \xi - Jx, v = \xi + Jx \) on \( N \oplus N \), and compute as in Appendix 1 of [7]. Using the relation between the Atiyah symbol, the Bott symbol and the Thom form, we obtain

\[
\int_{T^*N^g} \int_g \frac{Ch(g, \sigma)(X)}{J([q, N^g](X))D([q, g, N/N^g](X))} \hat{f}(X) dX
= \lim_{s \to -\infty} \int_{N^g} \int_g e^{i s d_g \alpha(X)} \frac{\det_{N^g}(1 - e^{-X})/X}{\det_{N/N^g}(1 - g^{-1}e^{-X})} \frac{\det_{N/N^g}(1 - g e^X)}{J([q, N^g](X))D([q, g, N/N^g](X))}.
\]

We then use Theorem 2.16, and we obtain our result. Here the finite set of admissible walls is the set \( p + H \) where \( p \) varies in \( F = \{-\phi_I\} \) and \( H \) varies in \( \mathcal{H}_0(N) \).

3.4. The case of a vector bundle over a manifold with infinitesimally free action. Let \( P \) be a manifold with an infinitesimally free action of a torus \( G_1 \). Let us consider a \( G_1 \)-equivariant Hermitian complex vector bundle \( N \to P \), with complex structure \( J \). Let \( G_2 \) be a torus acting trivially on \( P \) and fiberwise on \( N \), preserving the Hermitian structure, and commuting with the \( G_1 \)-action. We consider \( G = G_1 \times G_2 \). So \( N \) is a \( G \)-equivariant Hermitian vector bundle over \( P \). We denote
by \( N \) the typical fiber of \( N \). This is a Hermitian vector space provided with an action of the torus \( G_2 \).

We write \( N = \bigoplus_{i=1}^k N_{\phi_i} \) where \( \phi_i \) are characters of \( G_2 \), and \( N_{\phi} \) is the subbundle of \( N \) where \( G_2 \) acts via \( g \cdot n = \phi(g)n \). Each bundle \( N_{\phi_i} \) is \( G_1 \)-equivariant. We assume that the list \( \Phi = [\phi_1, \phi_2, \ldots, \phi_k] \) is contained in a half space. Thus the bundle

\[
\text{Sym}(N) = \bigoplus_{\lambda \in \hat{G}_2} \text{Sym}_\lambda(N)
\]

is a sum of finite-dimensional \( G_1 \)-equivariant vector bundles over \( P \).

Let \( \sigma \) be a \( G_1 \) transversally elliptic symbol on \( P \). Thus, for each \( \lambda \in \hat{G}_2 \), the symbol \( \sigma \otimes \text{Sym}_\lambda(N) \) is \( G_1 \)-transversally elliptic.

Let \( \text{at}(N) \) be the Atiyah symbol. We consider the \( G \)-transversally elliptic operator \( \text{At}_J(\sigma) = \sigma \hat{\otimes} \text{at}(N) \) on the total space of \( N \). We need to prove that our formula for the multiplicity index holds for the transversally elliptic symbol \( \text{At}_J(\sigma) \). Recall (see [21], Subsection 6.3) that we have

\[
\text{Index}_G(\text{At}_J(\sigma))(g_1, g_2) = \sum_{\lambda_2 \in \hat{G}_2} \text{Index}_{G_1}(\sigma \otimes \text{Sym}_{\lambda_2}(N))(g_1)g_2^{\lambda_2}. 
\]

Thus we obtain, for \( \lambda_1 \in \hat{G}_1 \) and \( \lambda_2 \in \hat{G}_2 \),

\[
\text{mult}_G(\text{At}_J(\sigma))(\lambda_1, \lambda_2) = \text{mult}_{G_1}(\sigma \otimes \text{Sym}_{\lambda_2}(N))(\lambda_1).
\]

Let us use the multiplicity formula obtained in §3.2 for the infinitesimally free action of \( G_1 \) on \( P \). If \( v_1 \) is a vertex of the action of \( G \) on \( P \), then \( v_1 \) produces a fiberwise transformation on \( N \) restricted to \( P^{v_1} \). We obtain

\[
\text{mult}_{G_1}(\sigma \otimes \text{Sym}_{\lambda_2}(N))(\lambda_1)
\]

\[
= \sum_{v_1} v_1^{-\lambda_1} \int_{T^* P^{v_1}} \int_{G_1} \frac{\text{Ch}(v_1, \sigma \otimes \text{Sym}_{\lambda_2}(N))}{J(P^{v_1})D(v_1, P/P^{v_1})} e^{-i(\lambda_1, X_1)} dX_1.
\]

We have written \( J(P^{v_1})D(v_1, P/P^{v_1}) \) for the invertible basic class representing the equivariant cohomology class \( J(P^{v_1})(X)D(v_1, P/P^{v_1})(X) \). We have

\[
\text{Ch}(v_1, \sigma \otimes \text{Sym}_{\lambda_2}(N)) = \text{Ch}(v_1, \sigma) \text{Ch}(v_1, \text{Sym}_{\lambda_2}(N)).
\]

The value at \( \lambda_1 \) has to be computed by limit.

Let us be more explicit on \( \text{Ch}(v_1, \text{Sym}_{\lambda_2}(N)) \). As the action of \( G_1 \) on \( P \) is infinitesimally free, we can choose a \( G \)-invariant Hermitian connection \( \nabla \) on \( N \), basic with respect to \( G_1 \). Thus its equivariant curvature \( R(X_1, X_2) \) is \( X_2 + R \) where we still denote by \( X_2 \) the fiberwise action of \( X_2 \) on \( N \), while \( R \) is the curvature of \( \nabla \). Thus \( R \) is an endomorphism of \( N_p \) with coefficients 2-forms on \( P^{v_1} \) and commutating with the action of \( v_1 \) and \( G_2 \). We thus have

\[
\text{Ch}(v_1, \text{Sym}_{\lambda_2}(N)) = \text{Tr}_{\text{Sym}_{\lambda_2}(N_p)}(v_1 e^R).
\]
Consider now our formula. We use the notation of §2.5. It is not difficult to see that the vertices \((v_1, v_2)\) for the action of \(G = G_1 \times G_2\) on \(\mathcal{N}\) are the couples \(v = (v_1, v_2)\) where \(v_1 \in \mathcal{V}(P)\) while \(v_2 \in \mathcal{V}(v_1, \Phi)\). Write, in the sense of generalized functions,

\[
\int_{T^*\mathcal{N}^\nu} \int_{\mathfrak{g}} \frac{\text{Ch}(v, \text{At}_J(\sigma))(X_1, X_2)}{J([q], \mathcal{N}^\nu)(X_1, X_2)D([q], v, \mathcal{N}/\mathcal{N}^\nu)(X_1, X_2)} e^{-i(y_1, X_1)} e^{-i(y_2, X_2)} dX = \sum_{k=0}^\infty m([q], v)(y_1, y_2).
\]

Here \(X = (X_1, X_2) \in \mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2\) and \(dX = dX_1 dX_2\). Our wanted formula is

\[
\text{mult}_G(\text{At}_J(\sigma))(\lambda_1, \lambda_2) = \sum_{(v_1, v_2)} v_1^{-\lambda_1} v_2^{-\lambda_2} \lim_{\varepsilon \to 0} m([1], v)(\lambda_1, \lambda_2).
\]

We have, for \(v = (v_1, v_2)\) acting by \(u = v_1 v_2\) on \(\mathcal{N}_p = \mathcal{N}\),

\[
\text{Ch}(v, \text{At}_J(\sigma))(X_1, X_2) = \text{Ch}(v_1, \sigma) \text{Ch}(u, at(\mathcal{N}))(X_2 + R).
\]

Similarly

\[
J([q], \mathcal{N}^\nu)(X_1, X_2) = J([q], P^{v_1})\det_{\mathcal{N}^\nu} \left( \frac{\exp([q](X_2 + R)) - 1}{[q](X_2 + R)} \right),
\]

\[
D([q], v, \mathcal{N}/\mathcal{N}^\nu)(X_1, X_2) = D([q], v_1, P/P^{v_1})\det_{\mathcal{N}/\mathcal{N}^\nu} (1 - \exp([q](X_2 + R))).
\]

Fix \(v_1\) a vertex for the action of \(G_1\) on \(P\), and consider the integration over the fiber \(T^*\mathcal{N}^\nu\) of the bundle \(T^*\mathcal{N} \to T^*P^{v_1}\). Thus it is sufficient to prove that

\[
\text{Tr}_{\text{Sym}_{\lambda_2}(\mathcal{N}_p)}(v_1 e^R) = \sum_{v_2} v_2^{-\lambda_2} \int_{T^*\mathcal{N}^\nu} \int_{\mathfrak{g}_2} \frac{\text{Ch}(u, at(\mathcal{N}))(X_2 + R)}{J_{\mathcal{N}^\nu}([q](X_2 + R))\det_{\mathcal{N}^\nu}(u)([q](X_2 + R))} e^{-i(y_2, X_2)} dX_2.
\]

Using the relation between the Atiyah symbol, the Bott symbol and the Thom form, we are reduced to Theorem 2.16. We remark that as \(R\) is a matrix with values 2-forms on \(P^{v_1}\), certainly the nilpotency index of \(R\) is smaller that the dimension of \(P\), and our distributions vanish for \(k \geq \dim \mathcal{N}\).

Here our finite set of directions of hyperplanes is the set of the admissible hyperplanes of the form \(\mathfrak{g}_1^* \oplus H\) where \(H \subset \mathfrak{g}_2^*\) is an admissible hyperplane for \(\Phi\). We see that these hyperplanes are contained in \(\mathcal{H}_0(\mathcal{N})\).

We thus obtain our theorem for \(\text{At}_J(\sigma)\).

### 3.5. The general case for a torus

Let \(G\) be a torus with weight lattice \(\Lambda\). Let \(M\) be a \(G\)-manifold and let \(\nu\) be a transversally elliptic symbol on \(M\). Let \(a \in \Lambda\), and let \(L_a\) be the corresponding one-dimensional representation of \(G\). If \(\nu_a = \nu \otimes L_a\) is the twisted symbol of \(\nu\), we have \(\text{Index}(\nu_a)(g) = g^a \text{Index}(\nu)(g)\). Remark that our multiplicity index formula is just \(\text{mult}_G(\nu_a)(\lambda) = \text{mult}_G(\nu)(\lambda - a)\). Thus it is sufficient to prove our formula for generators of \(K_G^0(T^*_GM)\) over \(R(G)\).
Let $s$ be a subalgebra of $g$ of the form $g_m$, with $m \in M$. Take a decomposition $G = G_1 \times G_2$, with $G_2$ the subtorus of $G$ with Lie algebra $s$. Then $G_2$ acts trivially on $M((s))$, and $G_1$ infinitesimally freely. From §3.4, our theorem holds for transversally elliptic symbols $\nu = \text{At}_J(\sigma)$. Our proof of Theorem 3.5 follows from Theorem 3.6.

3.6. An amusing example. Let us consider the flag manifold $M$ of $\mathbb{C}^3$ with the action of the adjoint torus $T$ in the adjoint group $K$ of $U(3)$. This is a case where all stabilizers are connected. Let $\Lambda$ be the root lattice of $A_2$. This is also identified with $\hat{T}$. We consider the elliptic operator $\partial$ on $M$, with index the trivial representation of $T$. Let $\sigma$ be its symbol. So the multiplicity index of $\sigma$ is the $\delta$ function on $\Lambda$. It takes value 0 at all points of $\Lambda$, except at $\lambda = 0$ where the value is 1. We promised to obtain (naturally) this multiplicity index as the restriction to $\Lambda$ of a continuous spline on $t^*$. Let $\Phi$ be the root system $A_2$, consisting of $\pm \alpha$, $\pm \beta$, $\pm (\alpha + \beta)$. Our formula is obtained by integration on $T^*(K/T)$. Using a similar computation as in the introduction, we see that

$$B(X) = \int_{T^*(K/T)} \text{Ch}(\sigma)(X) = \prod_{\phi \in \Phi} \frac{e^{i\langle \phi, X \rangle} - 1}{i \langle \phi, X \rangle}.$$  

This is the Fourier transform of the box spline $B_{A_2}$ of the system $A_2$. The box spline is supported on the convex hull of the points in $2\Phi$. We draw in Fig. 3 the picture of the box spline $B_{A_2}$. It is a locally polynomial measure, where each local polynomial is of degree 4. The value of $B_{A_2}$ at 0 is 1/2.

![Figure 3. The box spline for the root system $A_2$](image)

Now we consider the Taylor series $[B]$ up to order 4 of the analytic function $B(X)$. Formally, our function $m(\sigma)$ on $t^*$ is the Fourier transform of $B(X)/[B(X)]$ a function with Taylor series at $X = 0$ equal to 1 (however the Taylor series of $1/B(X)$ is not convergent!!). Thus $m(\sigma)$ is obtained by differentiating $B_{A_2}$ by the infinite series of differential operators $[B](i\partial)^{-1}$. We draw the corresponding function $DB_{A_2}$ on $t^*_{\text{reg}}$ in Fig. 4.
It is possible to verify directly that the function $\text{DBA}_2$ is continuous and vanishes at all points of the root lattice $\Lambda$, except $\lambda = 0$, where its value is 1. This is fortunate, as this is the consequence of the inversion formula for box splines (see for example [25] for a proof).

§ 4. The case of a connected compact group

We now give a formula for a general compact connected Lie group $G$. We assume that the infinitesimal stabilizer of the action of the center of $G$ on $M$ is reduced to 0. We can always reduce easily to this case.

If $g \in G$, we denote by $G^g$ the centralizer of $g$ in $G$, and $\mathfrak{g}^g$ its Lie algebra. We say that $g$ is a vertex of the $G$-action if the infinitesimal stabilizer of the action of $G^g$ on $M^g$ is reduced to 0. Let $T$ be a maximal torus of $G$ with Lie algebra $\mathfrak{t}$ and let $\mathcal{V}(M)$ be the set of vertices for the $T$-action on $M$. Then $g$ is a vertex if and only if $g$ is conjugated to an element of $\mathcal{V}(M)$.

We parameterize the set of irreducible representations of $G$ as follows. We consider $\Lambda \subset \mathfrak{t}^*$ the lattice of weights of $T$. We choose a system of positive roots $\Delta^+ \subset \mathfrak{t}^*$. Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$, and let $\mathfrak{t}^*_{>0}$ be the positive (open) Weyl chamber. Let $\tilde{\Lambda} = \rho + \Lambda$ and write $\Lambda_{>0} = \tilde{\Lambda} \cap \mathfrak{t}^*_{>0}$. For $\lambda \in \tilde{\Lambda}_{>0}$, we denote by $V_{\lambda}$ the irreducible representation of $G$ of highest weight $\lambda - \rho$.

We consider the $G$-invariant function $d_G(y)$ on $\mathfrak{g}^*$, where $d_G(y)$ is the volume of the coadjoint function $Gy$. For $\lambda \in \tilde{\Lambda}_{>0}$, $d_G(\lambda)$ is the dimension of the representation $V_{\lambda}$.

Let $M$ be a $G$-manifold and let $\sigma \in K^0_G(T^*_{\mathfrak{g}} M)$. We write

$$\text{Index}_G(\sigma) = \sum_{\lambda \in \tilde{\Lambda}_{>0}} \text{mult}_{G}(\sigma)(\lambda)V_{\lambda}.$$ 

We now express $\text{mult}_{G}(\sigma)(\lambda)$ in terms of the equivariant Chern character of $\sigma$.

Let us consider the function $j_{\mathfrak{g}}(X) = \det_{\mathfrak{g}}((e^{\text{ad}X} - 1)/(\text{ad}X))$. Recall that $j_{\mathfrak{g}}(X)$ has an analytic square root on $\mathfrak{g}$. We also defined $D_{\mathfrak{g}/\mathfrak{g}^g}(g)(X) = \det_{\mathfrak{g}/\mathfrak{g}^g}(1 - ge^{\text{ad}X})$. 
If \( \rho \) is in the weight lattice \( \Lambda \), we choose the square root \( D^{1/2}_{g^\rho} (g)(X) \) such that

\[
g^{-\rho} D^{1/2}_{g^\rho} (0) = i^{\dim g^\rho / 2} \prod_{\alpha > 0} (1 - g^{-\alpha}).
\]

Thus, for \( \lambda \in \tilde{\Lambda} \), \( g^{-\lambda} D^{1/2}_{g^\rho} (g)(X) \) is well defined, without conditions on \( \rho \).

Let \( g \in G \). We have associated a series \( J([q], M^g)(X) D([q], g, M/M^g)(X) \) of \( G^g \)-equivariant cohomology classes with polynomial coefficients on \( M^g \). We define the series of distributions \( m([q], g) \) on \( (g^g)^* \) by the formula

\[
(m([q], g), f) = \int_{T^* M^g} \int_{g^g} j^{1/2}_{g^g} (X) D^{1/2}_{g^\rho} (g)(X) \text{Ch}(g, \sigma)(X) \hat{f}(X) dX.
\]

We write \( m([q], g)(y) = \sum_{k=0}^{\infty} q^k m_k(g)(y) \). Let us first give some indication on the nature of the generalized function \( m_k(g)(y) \) on \( (g^g)^* \). If \( g \in T \), \( t \) is also a Cartan subalgebra of \( g^g \). Thus, by \( G^g \) invariance, we can restrict the function \( d_{G^g}(y)m_k(g)(y) \) to the interior \( t^*_\varepsilon \) of the Weyl chamber (for \( G \)). We obtain a generalized function on \( t^*_\varepsilon \). In the course of the proof we will see that there exists a system \( \mathcal{H} \) of affine hyperplanes in \( t^* \) such that for any connected component \( \varepsilon \) of the complement of \( \mathcal{H} \), the function \( d_{G^g}(y)m_k(g)(y) \) is given by a polynomial function on \( \varepsilon \cap t^*_\varepsilon \). Furthermore, when \( k \) is sufficiently large, \( m_k(g)(y) \) restricts to 0 on \( \varepsilon \cap t^*_\varepsilon \). Thus, for \( \varepsilon \) generic, and \( \lambda \in \tilde{\Lambda}_{>0} \), \( d_{G^g}(\lambda) \lim_{\varepsilon} m([1], g)(\lambda) \) is well defined.

**Theorem 4.1.** We have

\[
\text{mult}_G(\sigma)(\lambda) = \sum_{g \in V(M)} g^{-\lambda} d_{G^g}(\lambda) \lim_{\varepsilon} m([1], g)(\lambda).
\]

The theorem is particularly nice when stabilizers of the action of \( G \) on \( M \) are connected. In this case, there is only one vertex \( g = 1 \), and the theorem reads

\[
\text{mult}_G(\sigma)(\lambda) = \int_{T^* M} \int_{g} j_{g} (X)^{1/2} \frac{\text{Ch}(\sigma)(X)}{J(M)[X]} e^{-i\langle \lambda, X \rangle} dX,
\]

a double integral formula reminiscent of Witten non-abelian localization formula.

**Example 4.2.** Let \( G \) act on itself by left translations, and consider \( \sigma \) the 0-symbol. Its index is the \( \delta \) distribution on \( G \).

\[
\text{Index}_G(\sigma) = \sum_{\lambda \in \tilde{\Lambda}_{>0}} d_G(\lambda)V_\lambda.
\]

Let us compute our formula. The set of vertices is reduced to 1, and the equivariant class \( J(M)(X) \) is identically 1. For any \( s > 0 \), the distribution

\[
I(X) = \int_{T^* G} e^{isd_\rho \omega(X)}
\]

is easily seen to be the \( \delta \) function on \( g \).
Thus our distribution is
\[ \langle m(\sigma), f \rangle = \lim_{s \to \infty} \int_{T^*G} \int_\g j_\g(X)^{1/2} e^{is\omega(X)} \hat{f}(X) \, dX = \int_\g^* f(y) \, dy. \]

So \( m([q], \sigma)(y) \) is identically equal to the function 1 on \( \g^* \), and we obtain our formula.

We now give the proof. It is based on Theorem 5.21 in [12] which relates the infinitesimal index \( \text{Index}_G^\omega \) defined on \( \mathcal{H}^m_{G,c}(T^*_G M) \) and the infinitesimal index \( \text{Index}_T^\omega \) defined on \( \mathcal{H}^m_{T,c}(T^*_T M) \).

We may assume in the proof that \( G \) is simply connected. We recall that Atiyah–Singer associates to the \( G \)-transversally elliptic symbol \( \sigma \) a \( T \)-transversally elliptic symbol \( A(\sigma) \) on \( M \) such that
\[ \text{mult}_G(\sigma)(\lambda) = \text{mult}_T(A(\sigma))(\lambda). \]

If \( \sigma \) is itself \( T \) transversally elliptic, we have, for \( X \in t \),
\[ \text{Index}_T(A(\sigma))(\exp X) = \prod_{\alpha > 0} (e^{i(\alpha/2,X)} - e^{-i(\alpha/2,X)}) \text{Index}_G(\sigma)(\exp X). \]

More precisely, let \( \mu: T^*M \to \g^* \) be the moment map. Write \( \g = t \oplus q \). Let \( \mu^q: T^*M \to q^* \) be the projection of \( \mu \) on \( q^* \). We consider \( T^*_T M \), relative to the action of \( T \) on \( M \). Thus \( T^*_T M = T^*_G M \cap (\mu^q)^{-1}(0) \). The space \( q^* \) is provided with the structure of a Hermitian vector space. It thus has a Bott symbol \( \text{Bott}(q^*) \in K^0_T(q^*) \) such that its restriction to \( \{0\} \in q^* \) is \( \prod_{\alpha > 0} (1 - t^{-\alpha}) \). Then, as a \( T \)-transversally elliptic symbol,
\[ A(\sigma) = \sigma \otimes (\mu^q)^* \text{Bott}(q^*) \otimes L_\rho. \]

Thus we compute \( \text{mult}_G(\sigma)(\lambda) \) by using our preceding theorem for \( \text{mult}_T(A(\sigma))(\lambda) \).

For \( g \in \mathcal{V}(M) \), we need to compare the series \( m([q], g)(y) \) of distributions on \( \g^* \), and \( n([q], g)(y) \) on \( t^* \) associated respectively to \( \sigma \) and \( A(\sigma) \).

We do it for \( g = 1 \). By definition, \( m([q], 1) \) is the infinitesimal index of
\[ \Theta(X) = j_\g^{1/2}(X) \frac{\text{Ch}(\sigma)(X)}{J([q], M)(X)} \]
and \( n([q], 1) \) is the infinitesimal index of
\[ \theta(X) = \frac{\text{Ch}(A(\sigma))(X)}{J([q], M)(X)}. \]

Using the relation between the Bott symbol and the Thom class, we see thus that
\[ \theta(X) = \frac{\text{Ch}(\sigma)(X)}{J([q], M)(X)} (2i\pi)^{\dim q} (\mu^q)^*(\text{Thom}(q^*))(X). \]

By \( G \)-invariance, a \( G \)-invariant distribution restricts to \( t^*_{>0} \) (as a distribution).
**Lemma 4.3.** Let $P(X) \in \mathcal{H}_{G,c}^m(T^*G,M)$ and let

$$p(X) = (2i\pi)^{\dim \mathfrak{q}} P(X)(\mu^q)^*(\text{Thom}(\mathfrak{q}^*))(X).$$

Then on $t^*_> 0$, we have

$$\text{Ind}_G^ω(p)(y) = d_G(y) \text{Ind}_G^ω(P)(y).$$

**Proof.** It is easy to see that this is true when $P(X)$ is in the cohomology with compact support of $T^*M$. Then we can replace $(2i\pi)^{\dim \mathfrak{q}} \text{Thom}(\mathfrak{q}^*)(X)$ by the function $\text{Pf}(X) = \prod_{\alpha > 0} \langle \alpha, X \rangle$. The infinitesimal index is then just the Fourier transform on $\mathfrak{g}$ of $I(X) = \int_{T^*M} P(X)$. The formula follows from Harish-Chandra relation between Fourier transform on $g$ and $t$ of $G$-invariant functions.

In the general case, we use [12], Theorem 5.21. We have

$$\langle \text{Ind}_G^ω(P), f \rangle = \langle \text{Ind}_T^q, \nu, f \rangle$$

with $\nu = \text{Ind}_G^ω(\text{Pf}(X)p(X))$ and

$$\langle \text{Ind}_T^q, \nu, f \rangle = \int_t \nu(y) \left( \int_{q^*} f(y + q) \, dq \right) \, dy.$$

It is easy to see that this implies Lemma 4.3 by using Fourier transform.

The computation is similar for other $g \in V(M)$.

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