INTRINSIC GEOMETRY AND BOUNDARY STRUCTURE OF PLANE DOMAINS
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Abstract: Given a nonempty compact set \( E \) in a proper subdomain \( \Omega \) of the complex plane, we denote the diameter of \( E \) and the distance from \( E \) to the boundary of \( \Omega \) by \( d(E) \) and \( d(E, \partial \Omega) \), respectively. The quantity \( d(E)/d(E, \partial \Omega) \) is invariant under similarities and plays an important role in geometric function theory. In case \( \Omega \) has the hyperbolic distance \( h_\Omega(z, w) \), we consider the infimum \( \kappa(\Omega) \) of the quantity \( h_\Omega(E)/\log(1 + d(E)/d(E, \partial \Omega)) \) over compact subsets \( E \) of \( \Omega \) with at least two points, where \( h_\Omega(E) \) stands for the hyperbolic diameter of \( E \). Let the upper half-plane be \( \mathbb{H} \). We show that \( \kappa(\Omega) \) is positive if and only if the boundary of \( \Omega \) is uniformly perfect and \( \kappa(\Omega) \leq \kappa(1) \) for all \( \Omega \), with equality holding precisely when \( \Omega \) is convex.

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1. Introduction

Let \( \Omega \) be a domain in the complex plane \( \mathbb{C} \) with the hyperbolic metric \( \rho_\Omega(z)|dz| \) of Gaussian curvature \(-1\) [1]. The celebrated Uniformization Theorem [2, p. 81] guarantees the existence of \( \rho_\Omega \) for a domain \( \Omega \) when the boundary \( \partial \Omega \) of \( \Omega \) contains at least three points. Such domain is called hyperbolic. Here and in what follows, the boundary of a domain is taken with respect to the Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \).

The function \( \rho_\Omega(z) \) is sometimes called the hyperbolic density of \( \Omega \). For instance, for the unit disk \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \) and the upper half-plane \( \mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\} \), the hyperbolic densities are given by \( \rho_\mathbb{D}(z) = 2/(1 - |z|^2) \) and \( \rho_\mathbb{H}(z) = 1/\text{Im } z \). Let \( h_\Omega(z_1, z_2) \) denote the hyperbolic distance induced by \( \rho_\Omega(z)|dz| \) and \( d(z, \partial \Omega) \) the Euclidean distance from \( z \in \Omega \) to \( \partial \Omega \). Then \( \rho_\Omega(z) \leq 2/d(z, \partial \Omega) \) for each \( z \in \Omega \) as a simple consequence of Schwarz’ Lemma [3, (2.1)]. On the other hand, \( \rho_\Omega(z) \geq 1/(2d(z, \partial \Omega)) \) for a simply connected domain \( \Omega \) [3, (2.2); 1, p. 35, Theorem 8.6; 4, p. 34, (3.2.1)].

The distance on \( \Omega \) induced by the continuous Riemannian metric \( |dz|/d(z, \partial \Omega) \) is called the quasihyperbolic distance and denoted by \( k_\Omega(z_1, z_2) \) [5]. Note that \( h_\Omega(z_1, z_2) \leq 2k_\Omega(z_1, z_2) \) for a general domain \( \Omega \) and \( h_\Omega(z_1, z_2) \geq k_\Omega(z_1, z_2)/2 \) for a simply connected domain \( \Omega \). These two inequalities are very handy, because there are many estimates for quasihyperbolic distances whereas hyperbolic distances are not easy to estimate because the density function \( \rho_\Omega(z) \) depends on the local boundary structure in the vicinity of \( z \) in a subtle manner [3; 2, p. 241, Theorem 14.5.2; 6]. It should be noticed that the second estimate does not apply to general domains, because the hyperbolic distance is not bounded below by a constant multiple of the quasihyperbolic distance, for instance, if the domain has isolated boundary points. To measure the similarity between \( h_\Omega \) and \( k_\Omega \), the domain functional [7]

\[
c(\Omega) = \inf_{z \in \Omega} \rho_\Omega(z)d(z, \partial \Omega) = \inf_{z_1, z_2 \in \Omega, z_1 \neq z_2} \frac{h_\Omega(z_1, z_2)}{k_\Omega(z_1, z_2)}
\]

is useful, where the second equality will be proven in the next section. By the above, \( c(\Omega) \leq 2 \) for a general domain \( \Omega \) and \( c(\Omega) \geq 1/2 \) for a simply connected domain \( \Omega \). But more is known about this domain constant:

\[c(\Omega) \leq \kappa(\Omega) \leq \kappa(1)\]

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**Theorem A.** Let \( \Omega \) be a hyperbolic domain in \( \mathbb{C} \). Then \( c(\Omega) \leq 1 \) with equality if and only if \( \Omega \) is convex. Furthermore, \( c(\Omega) > 0 \) if and only if \( \partial \Omega \) is uniformly perfect.

The general inequality \( c(\Omega) \leq 1 \) is due to Harmelin and Minda [7] and the equality condition is due to Mejía and Minda [8]. The last assertion is due to Beardon and Pommerenke [3]. Here a closed set \( E \) in \( \mathbb{C} \) with \( \text{card}(E) \geq 2 \) is said to be uniformly perfect if there is a constant \( 0 < \alpha < 1 \) such that the closed annulus \( \alpha r \leq |z - a| \leq r \) meets \( E \) whenever \( a \in E \) and \( 0 < r < d(E) \). Here and hereafter, \( c(E) \) denotes the cardinality of \( E \) and \( d(E) \) is the Euclidean diameter of \( E \). In other words, \( d(E) = \sup_{z, w \in E} |z - w| \). We set \( d(E) = +\infty \) when \( \infty \in E \). As regards uniformly perfect sets, we refer to [9; 10, pp. 343–345; 11–14]. Uniform perfectness has many applications in potential theory, metric spaces, Kleinian groups, and complex dynamics as well as geometric function theory; in addition to the above references see, for instance, [9; 15–17].

In their work about the quasihyperbolic metric, Gehring and Palka [5] introduced the distance-ratio metric

\[
j_\Omega(z_1, z_2) = \log \left( 1 + \frac{|z_1 - z_2|}{\min \{d(z_1, \partial \Omega), d(z_2, \partial \Omega)\}} \right)
\]

for \( z_1, z_2 \in \Omega \); see also [18, p. 61]. They proved that \( j_\Omega(z_1, z_2) \leq k_\Omega(z_1, z_2) \) holds always. It is known as well that \( j_\Omega \) satisfies the triangle inequality on \( \Omega \) [18, p. 59, Lemma 4.6]. The opposite inequality characterizes the so-called uniform domains: a domain \( \Omega \) is uniform if and only if there exists a constant \( b > 0 \) such that

\[
k_\Omega(z_1, z_2) \leq b j_\Omega(z_1, z_2);
\]

see Gehring and Osgood [19; 18, p. 84]. These domains are ubiquitous in geometric function theory [4].

It is a natural and interesting question to ask what can be said if we replace \( k_\Omega \) by \( h_\Omega \). Our answer is the following result:

**Theorem 1.2.** Let \( \Omega \) be a hyperbolic domain in \( \mathbb{C} \). There is a constant \( c > 0 \) such that \( cj_\Omega(z_1, z_2) \leq h_\Omega(z_1, z_2) \) for all \( z_1, z_2 \in \Omega \) if and only if the boundary of \( \Omega \) in \( \mathbb{C} \) is uniformly perfect.

In conjunction with the Gehring–Osgood Theorem [19, pp. 59–60], we have

**Corollary 1.3.** Let \( \Omega \) be a hyperbolic domain in \( \mathbb{C} \). Then the hyperbolic metric \( h_\Omega \) is comparable with the distance-ratio metric \( j_\Omega \) if and only if \( \Omega \) is uniform and has uniformly perfect boundary.

Indeed, if we have

\[
c_1 j_\Omega(z_1, z_2) \leq h_\Omega(z_1, z_2) \leq c_2 j_\Omega(z_1, z_2) \quad \text{for} \quad z_1, z_2 \in \Omega
\]

for some constants \( 0 < c_1 \leq c_2 \), then we see first that \( \partial \Omega \) is uniformly perfect. Hence, \( h_\Omega \) is comparable with \( k_\Omega \) by Theorem A. We now conclude that \( \Omega \) is uniform by the Gehring–Osgood Theorem. The converse follows readily from Theorem 1.2 and the Gehring–Osgood Theorem.

Given a subset \( E \) of \( \Omega \) with \( \text{card}(E) \geq 2 \), we define the set functionals

\[
h_\Omega(E) = \sup_{z_1, z_2 \in E} h_\Omega(z_1, z_2) \quad \text{and} \quad J_\Omega(E) = \log \left( 1 + \frac{d(E)}{d(E, \partial \Omega)} \right).
\]

Here and hereafter, \( d(E, F) \) denotes the Euclidean distance between \( E \) and \( F \). For a singleton \( E = \{z\} \), we write \( d(\{z\}, F) = d(z, F) = d(F, z) \). We will frequently use the following monotonicity property in the sequel: \( h_\Omega(E) \leq h_\Omega(E') \) and \( J_\Omega(E) \leq J_\Omega(E') \) for \( E \subset E' \subset \Omega \). We note that \( h_\Omega(E) \) is the hyperbolic diameter of \( E \) in \( \Omega \) and that \( J_\Omega(E) \) is important in connection with the capacity estimates of \( E \) (see, for instance, [20]). We now consider the domain constant

\[
k(\Omega) = \inf_E \frac{h_\Omega(E)}{J_\Omega(E)},
\]

where \( E \) runs over all compact subsets of \( \Omega \) with \( \text{card}(E) \geq 2 \). As the following result tells, the two domain constants \( c(\Omega) \) and \( k(\Omega) \) are comparable.
Theorem 1.4. Let $\Omega$ be a hyperbolic domain in $\mathbb{C}$. Then
\[ \frac{c(\Omega)}{2} \leq \kappa(\Omega) \leq c(\Omega). \]
In particular, $\kappa(\Omega) > 0$ if and only if $\partial \Omega$ is uniformly perfect.

It is a little surprising that $\kappa(\Omega)$ behaves like $c(\Omega)$ in the following sense (compare the sequel with Theorem A):

Theorem 1.5. Let $\Omega$ be a hyperbolic domain in $\mathbb{C}$. Then $\kappa(\Omega) \leq \kappa(\mathbb{H})$, with equality holding if and only if $\Omega$ is convex.

In view of the above theorem, we are curious about the value of $\kappa(\mathbb{H})$. However, it seems difficult to evaluate $\kappa(\mathbb{H})$ in simple form. Since $c(\mathbb{H}) = 1$, the first part of Theorem 1.4 implies $1/2 \leq \kappa(\mathbb{H}) \leq 1$. We will prove later that $\kappa(\mathbb{H}) < 1$ and give a numerical approximation of the value of $\kappa(\mathbb{H})$ in Theorem 4.18, thus answering the problem in [18, p. 455, item (12)].

The existence of an extremal configuration of $E$ for the functional $h_{\Omega}(E)/J_{\Omega}(E)$ is more subtle. We will prove the following result in Section 5. We note that a convex domain in $\mathbb{C}$ carries the hyperbolic metric unless the domain is $\mathbb{C}$ itself.

Theorem 1.6. Let $\Omega$ be a convex proper subdomain of $\mathbb{C}$. There exists a compact subset $E$ in $\Omega$ satisfying $\kappa(\Omega) = h_{\Omega}(E)/J_{\Omega}(E)$ if and only if $\Omega$ is a half-plane.

When $\Omega$ is the upper half-plane $\mathbb{H}$, there exists a three-point set $E^*$ of the form $\{i, z_1, z_2\}$ constituting a hyperbolic equilateral triangle with $\kappa(\mathbb{H}) = h_{\mathbb{H}}(E^*)/J_{\mathbb{H}}(E^*)$, $1 < \mathfrak{Re} z_j (j = 1, 2)$, and $z_1 = \overline{z_2}$. Moreover, such an extremal three-point set is unique up to similarities keeping $\mathbb{H}$ invariant.

In view of the application in Section 5, it is important to have a lower bound for $\kappa(\Omega)$ when $\Omega$ is simply connected. We consider the number
\[ \kappa_0 = \inf_{\Omega} \kappa(\Omega), \quad (1.7) \]
where $\Omega$ runs over all simply connected proper subdomains of $\mathbb{C}$. From Theorem 1.4 and the well-known estimate $c(\Omega) \geq 1/2$, we obtain that $\kappa_0 \geq 1/4$. On the other hand, when $\Omega$ is the slit domain $\Omega_0 = \mathbb{C} \setminus (-\infty, 0]$, numerically we have $\kappa(\Omega_0) \leq h_{\Omega_0}(E)/J_{\Omega_0}(E) = 0.4251604 \ldots$ for $E = \{w_0, w_1, w_2\}$, $w_0 = 1$, $w_1 = 2.121820474 + 1.198476681i$, and $w_2 = \overline{w_1}$. Note that $h_{\Omega_0}(w_0, w_1) = h_{\Omega_0}(w_0, w_2) \approx h_{\Omega_0}(w_1, w_2)$.

Thus, we have

Corollary 1.8. $1/4 \leq \kappa_0 < 0.4251605$.

It is an open problem to determine $\kappa_0$.

The organization of this paper is as follows: In Section 2, we give some preliminary results about the domain constant $\kappa(\Omega)$ and prove Theorems 1.2 and 1.4. Section 3 deals with the proof of Theorem 1.5. We determine the extremal configurations of three-point sets $E$ with respect to the set functional $h_{\mathbb{H}}(E)/J_{\mathbb{H}}(E)$ and prove Theorem 1.6 in Section 4. We also give numerical observations on $\kappa(\mathbb{H})$. We will apply our results to lower estimation of the capacity of a condenser in the final section.

2. Preliminaries

In this section, we prove several simple preliminary results. We begin with the proof of the second equality in (1.1). To distinguish between the two sides of (1.1), for a while, we write
\[ c(\Omega) = \inf_{z \in \Omega} \rho_{\Omega}(z)d(z, \partial \Omega) \quad \text{and} \quad c'(\Omega) = \inf_{z, w \in \Omega} \frac{h_{\Omega}(z, w)}{k_{\Omega}(z, w)}. \]
We will prove that $c(\Omega) = c'(\Omega)$. Since $\rho_{\Omega}(z) \geq c(\Omega)/d(z, \partial \Omega)$, we easily find
\[ h_{\Omega}(z_1, z_2) \geq c(\Omega)k_{\Omega}(z_1, z_2). \]
Hence, \( c'(\Omega) \geq c(\Omega) \). On the other hand, using the formula
\[
\lim_{w \to z} \frac{h_\Omega(z, w)}{k_\Omega(z, w)} = \lim_{w \to z} \frac{h_\Omega(z, w)}{|z - w|} \cdot \frac{|z - w|}{k_\Omega(z, w)} = \frac{\rho_\Omega(z)}{1/d(z, \partial \Omega)} = \rho_\Omega(z)d(z, \partial \Omega),
\]
we infer that \( c(\Omega) \geq c'(\Omega) \). Thus, we are done.

For the analysis of domain constants, we introduce some variants of the domain constant \( \kappa(\Omega) \). First, we replace \( h \) with \( k \) and define the domain constant
\[
\hat{\kappa}(\Omega) = \inf_E \frac{k_\Omega(E)}{\log(1 + d(E)/d(E, \partial \Omega))},
\]
where the infimum is taken over all compact subsets \( E \) of \( \Omega \) with \( \text{card}(E) \geq 2 \). Here \( k_\Omega(E) \) denotes the quasihyperbolic diameter of \( E \). We also define the auxiliary domain constants for integers \( n \geq 2 \):
\[
\kappa_n(\Omega) = \inf_{E \subset \Omega, \text{card}(E) = n} \frac{h_\Omega(E)}{\log(1 + d(E)/d(E, \partial \Omega))}
\]
and
\[
\tilde{\kappa}_n(\Omega) = \inf_{E \subset \Omega, \text{card}(E) = n} \frac{k_\Omega(E)}{\log(1 + d(E)/d(E, \partial \Omega))}.
\]
Given \( E = \{z_1, \ldots, z_n\} \) and letting \( z_n \to z_{n-1} \), we observe that
\[
\kappa_2(\Omega) \geq \kappa_3(\Omega) \geq \cdots \geq \kappa(\Omega)
\]
and
\[
\tilde{\kappa}_2(\Omega) \geq \tilde{\kappa}_3(\Omega) \geq \cdots \geq \tilde{\kappa}(\Omega).
\]
For these domain constants, we have the following results: In particular, we see that \( \kappa_n(\Omega) = \kappa(\Omega) \) and \( \tilde{\kappa}_n(\Omega) = \tilde{\kappa}(\Omega) \) for every \( n \geq 3 \).

**Lemma 2.1.** (i) \( \tilde{\kappa}_2(\Omega) \geq 1 \).
(ii) \( \kappa_3(\Omega) = \kappa(\Omega) \) and \( \tilde{\kappa}_3(\Omega) = \tilde{\kappa}(\Omega) \).
(iii) \( \kappa_2(\Omega) \leq 2\kappa_3(\Omega) \) and \( \tilde{\kappa}_2(\Omega) \leq 2\tilde{\kappa}_3(\Omega) \).

**Proof.** Item (i) is clear from the Gehring–Palka inequality \( k_\Omega(z_1, z_2) \geq j_\Omega(z_1, z_2) \). Let \( E \) be an arbitrary compact set in \( \Omega \) with \( \text{card}(E) \geq 2 \). Take \( z_0, z_1, z_2 \in E \) so that \( d(E) = |z_1 - z_2| \) and \( d(E, \partial \Omega) = d(z_0, \partial \Omega) \) and let \( E_0 = \{z_0, z_1, z_2\} \). (Note that one of the points \( z_1 \) and \( z_2 \) may be the same as \( z_0 \) or \( z_0 \) may be the same as \( z_2 \).) Then
\[
h_\Omega(E) \geq h_\Omega(E_0) \geq \kappa_3(\Omega) \log(1 + d(E_0)/d(E_0, \partial \Omega))
\]
\[
= \kappa_3(\Omega) \log(1 + |z_1 - z_2|/d(z_0, \partial \Omega))
\]
\[
= \kappa_3(\Omega) \log(1 + d(E)/d(E, \partial \Omega)).
\]
(2.2)
Taking infimum over compact subsets \( E \) of \( \Omega \), we obtain the inequality \( \kappa(\Omega) \geq \kappa_3(\Omega) \). Since \( \kappa(\Omega) \leq \kappa_3(\Omega) \) as we noted above, we conclude that \( \kappa(\Omega) = \kappa_3(\Omega) \). In the same way, we can verify that \( \tilde{\kappa}(\Omega) = \tilde{\kappa}_3(\Omega) \).

Finally, we prove item (iii). Let \( E \subset \Omega \) with \( \text{card}(E) = 3 \) and choose \( z_0 \in E \) so that \( d(E, \partial \Omega) = d(z_0, \partial \Omega) \). Also choose \( z_1, z_2 \in E \) so that \( d(E) = |z_1 - z_2| \). Then
\[
\log(1 + d(E)/d(E, \partial \Omega)) = \log(1 + |z_1 - z_2|/d(z_0, \partial \Omega))
\]
\[
\leq \log(1 + (|z_1 - z_0| + |z_2 - z_0|)/d(z_0, \partial \Omega))
\]
\[
\leq \log(1 + |z_1 - z_0|/d(z_0, \partial \Omega)) + \log(1 + |z_2 - z_0|/d(z_0, \partial \Omega))
\]
\[
\leq \kappa_2(\Omega)^{-1}(h_\Omega(z_1, z_0) + h_\Omega(z_2, z_0)) \leq 2h_\Omega(E)/\kappa_2(\Omega),
\]
which implies \( \kappa_2(\Omega) \leq 2\kappa_3(\Omega) \). In the same way, we can prove the other inequality. \( \square \)
Lemma 2.3. \( \kappa_2(\Omega) \leq c(\Omega) \) for a hyperbolic domain \( \Omega \) in \( \mathbb{C} \).

PROOF. Noting the formula

\[
\lim_{w \to z} \frac{h_\Omega(z, w)}{d(z, \partial \Omega)} = \rho_\Omega(z)d(z, \partial \Omega),
\]

we have

\[
\kappa_2(\Omega) = \inf_{z \neq w} \frac{h_\Omega(z, w)}{d(z, \partial \Omega)} \leq \inf_{z \neq w} \rho_\Omega(z) = c(\Omega). \quad \square
\]

PROOF OF THEOREM 1.4. Using Lemma 2.3 and the inequality \( h_\Omega(x, y) \geq c(\Omega)k_\Omega(x, y) \), for an arbitrary compact set \( E \) in \( \Omega \) we have

\[
\frac{h_\Omega(E)}{\log(1 + d(E)/d(\Omega, \partial \Omega))} \geq \frac{c(\Omega)k_\Omega(E)}{\log(1 + d(E)/d(\Omega, \partial \Omega))} \geq c(\Omega)\kappa(\Omega) \geq c(\Omega)\hat{\kappa}_2(\Omega) \geq \frac{c(\Omega)}{2}.
\]

Hence \( \kappa(\Omega) \geq c(\Omega)/2 \). The other inequality follows from Lemma 2.3:

\[
\kappa(\Omega) \leq \kappa_2(\Omega) \leq c(\Omega). \quad \square
\]

PROOF OF THEOREM 1.2. Assume that \( c_\Omega(z_1, z_2) \leq h_\Omega(z_1, z_2) \) for \( z_1, z_2 \in \Omega \). Then \( \kappa_2(\Omega) \geq c \).

Using Lemma 2.1 and Theorem 1.4, we obtain

\[
c(\Omega) \geq \kappa(\Omega) \geq \frac{1}{2}\kappa_2(\Omega) \geq \frac{c}{2} > 0.
\]

Thus, \( \partial \Omega \) is uniformly perfect. Conversely, if \( \partial \Omega \) is uniformly perfect; then \( \kappa_2(\Omega) \geq \kappa(\Omega) \geq c(\Omega)/2 > 0 \). Thus, \( c_\Omega(z_1, z_2) \leq h_\Omega(z_1, z_2) \) with \( c = \kappa_2(\Omega) > 0 \). \( \square \)

3. Proof of Theorem 1.5

In this section, we will prove Theorem 1.5 step by step. We begin with the following result:

Lemma 3.1. \( \kappa(\Omega) \leq \kappa(\mathbb{D}) \) for every hyperbolic domain \( \Omega \) in \( \mathbb{C} \).

PROOF. By definition, for a given \( \varepsilon > 0 \), there is a compact subset \( E \) of \( \mathbb{D} \) such that

\[
h_{\mathbb{D}}(E) \leq \frac{\kappa(\mathbb{D}) + \varepsilon}{J_{\mathbb{D}}(E)}.
\]

Moreover, by rotating \( E \), if need be, we may further assume that the nearest point of the boundary \( \partial \mathbb{D} \) to \( E \) is 1. Namely, \( d(E, \partial \mathbb{D}) = d(E, 1) \).

Let \( \Omega \) be an arbitrary hyperbolic domain in \( \mathbb{C} \). Given an arbitrary point \( z_0 \in \Omega \), choose \( \zeta_0 \in \partial \Omega \) so that \( d(z_0, \partial \Omega) = |z_0 - \zeta_0| \). Since \( \kappa(\Omega) \) is invariant under similarities, we may assume that \( z_0 = 0 \) and \( \zeta_0 = 1 \). Then \( \mathbb{D} \subset \Omega \). By the domain monotonicity of the hyperbolic metric, \( h_\Omega(E) \leq h_{\mathbb{D}}(E) \). On the other hand,

\[
d(E, \partial \Omega) = d(E, 1) = d(E, \partial \mathbb{D})
\]

so that \( J_\Omega(E) = J_{\mathbb{D}}(E) \). Hence,

\[
\kappa(\mathbb{D}) + \varepsilon > \frac{h_{\mathbb{D}}(E)}{J_{\mathbb{D}}(E)} \geq \frac{h_\Omega(E)}{J_\Omega(E)} \geq \kappa(\Omega).
\]

Since \( \varepsilon > 0 \) is arbitrary, we obtain the required inequality \( \kappa(\mathbb{D}) \geq \kappa(\Omega) \). \( \square \)

Remark 3.2. Note that the set functional \( J_D(E) \) in the above proof is not the same thing as the diameter of \( E \) in the \( j_D \) metric

\[
j_D(E) = \sup\{j_D(x, y) : x, y \in E\}.
\]

It is easy to see that

\[
J_D(E)/2 \leq j_D(E) \leq J_D(E)
\]

for all \( E \subset D \), with equality in the second inequality if \( E \) is a disk, \( \text{card}(E) = 2 \), or \( \text{card}(E) = 3 \) and the triangle with vertices \( E \) is either equilateral or a so-called Reuleaux triangle.

Moreover, for a half-plane, we have the following result:
Lemma 3.3. Let $H$ be an open half-plane in $\mathbb{C}$. Then $\kappa(\mathbb{D}) = \kappa(H)$.

Proof. By Lemma 3.1, it suffices to prove that $\kappa(H) \geq \kappa(\mathbb{D})$. We choose the right half-plane \( \{ z : \text{Re } z > 0 \} \) as $H$. Given $\varepsilon > 0$, we can find a compact subset $E$ of $H$ such that

\[
\frac{h_H(E)}{J_H(E)} < \kappa(H) + \varepsilon.
\]

Let $\zeta_0$ be the nearest boundary point to $E$. For simplicity, we assume that $\zeta_0 = 0$. For $R > 0$, we denote the disk \( \{ z : |z - R| < R \} \) by $\Delta_R$. If $R$ is large enough, then $E \subset \Delta_R$ and $d(E, \partial \Delta_R) = d(E, 0) = d(E, \partial H)$ so that $J_H(E) = J_{\Delta_R}(E)$. On the other hand, since $\rho_{\Delta_R}(z) = \frac{2R}{R^2 - |z - R|^2} = \frac{1}{\text{Re } z - |z|^2/(2R)} \to \frac{1}{\text{Re } z} = \rho_H(z)$ locally uniformly on $\mathbb{H}$, we obtain $h_{\Delta_R}(E) \to h_H(E)$ as $R \to +\infty$. Noting that

\[
h_{\Delta_R}(E)/J_{\Delta_R}(E) \geq \kappa(\Delta_R) = \kappa(\mathbb{D}),
\]

we have

\[
\frac{h_H(E)}{J_H(E)} = \lim_{R \to +\infty} \frac{h_{\Delta_R}(E)}{J_{\Delta_R}(E)} \geq \kappa(\mathbb{D}).
\]

Hence, $\kappa(H) + \varepsilon > \kappa(\mathbb{D})$. Since $\varepsilon > 0$ was arbitrary, we obtain the inequality $\kappa(H) \geq \kappa(\mathbb{D})$, as required. □

Lemma 3.4. Let $\Omega$ be a convex domain in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$. Then $\kappa(\Omega) = \kappa(\mathbb{D})$.

Proof. Let $E$ be a compact subset of $\Omega$. Take $\zeta_0 \in \partial \Omega$ so that $d(E, \partial \Omega) = d(E, \zeta_0)$. Since $\Omega$ is convex, there is a supporting line, say, $L$ at the point $\zeta_0$. Let $H$ be the connected component of $\mathbb{C} \setminus L$ including $\Omega$. Then $\Omega \subset H$ and $\zeta_0 \in \partial H = L$. Since $d(E, \partial H) = d(E, \zeta_0) = d(E, \partial \Omega)$, we obtain

\[
\frac{h_\Omega(E)}{J_\Omega(E)} \geq \frac{h_H(E)}{J_H(E)} \geq \kappa(H) = \kappa(\mathbb{D}).
\]

Here we used Lemma 3.3. Taking the infimum over $E$, we see that $\kappa(\Omega) \geq \kappa(\mathbb{D})$. Recalling Lemma 3.1, we have the desired relation. □

To deduce the equality condition is the most subtle part in the proof of Theorem 1.5. The key ingredient is Keogh’s Lemma about nonconvex domains; see Fig. 1.

![Fig. 1](image-url)
Lemma 3.5 (Keogh [21]). Suppose that a domain $\Omega$ in $\mathbb{C}$ is not convex. Then there are two open disks $\Delta_1$ and $\Delta_2$ whose boundaries intersect perpendicularly such that $G = \Delta_1 \setminus \overline{\Delta_2}$ lies in $\Omega$ and the midpoint $\zeta_0$ of the concave boundary arc $\Delta_1 \cap \partial \Delta_2$ of $G$ belongs to the boundary $\partial \Omega$ of $\Omega$.

We are now ready to prove the following result which is the last piece of the proof of Theorem 1.5.

Lemma 3.6. Let $\Omega$ be a nonconvex domain in $\mathbb{C}$. Then $\kappa(\Omega) < \kappa(\mathbb{D})$.

Proof. We find open disks $\Delta_1$ and $\Delta_2$ as in Keogh’s Lemma so that $G = \Delta_1 \setminus \overline{\Delta_2} \subset \Omega$ and the midpoint $\zeta_0$ of the concave boundary arc of $G$ belongs to $\partial \Omega$. We may assume that $\Delta_1 = \mathbb{D}$ and $\zeta_0 = a \in (0, 1)$ so that the center of $\Delta_2$ lies on the real axis. Then the second disk $\Delta_2$ is the image of the right half-plane $H$ under the Möbius transformation $T(z) = \frac{z + a}{1 + az}$.

Thus, $G = T(\mathbb{D}_-)$, where $\mathbb{D}_-$ is the left half $\{z \in \mathbb{D} : \text{Re } z < 0\}$ of the unit disk. We now construct a conformal map $f$ of the upper half-plane $\mathbb{H}$ onto $G$ as follows: We denote the analytic automorphism $(1 + z/2)/(1 - z/2)$ of $\mathbb{H}$ by $M$. Note that $M$ maps the positive imaginary axis $i \mathbb{R}_+ = \{iy : 0 < y < +\infty\}$ onto the upper half of the unit circle $|z| = 1$. The function $S(\zeta) = \sqrt{\zeta}$ maps $\mathbb{H}$ onto the first quadrant $D = \{w : \text{Re } w > 0, \text{Im } w > 0\}$. Then the Möbius transformation $L(w) = i(w - 1)/(w + 1)$ maps $D$ onto the left half $\mathbb{D}_-$ of $\mathbb{D}$. Hence, the function $f = T \circ L \circ S \circ M$ maps $\mathbb{H}$ onto $G$ in such a way that $f(i \mathbb{R}_+) = (-1, a)$. More concretely, $f$ is expressed by

$$f(z) = T\left(i\frac{\sqrt{1 + z/2} - \sqrt{1 - z/2}}{\sqrt{1 + z/2} + \sqrt{1 - z/2}}\right).$$

In view of this form, we see that $f(z)$ is analytic on $|z| < 1$. (This follows also from the Schwarz reflection principle.) Therefore, we can expand $f(z)$ about $z = 0$ as follows:

$$f(z) = a + a_1 z + a_2 z^2 + \cdots \quad (|z| < 1).$$

A straightforward computation yields

$$a_1 = \frac{i}{4}(1 - a^2), \quad a_2 = \frac{a}{16}(1 - a^2)$$

and so

$$A := \frac{a_2}{a_1} = \frac{a}{4i} \quad (3.7)$$

Let $E_x := x E^* = \{xz_j : j = 0, 1, 2\}$ for $0 < x < 1$, where $E^* = \{z_0, z_1, z_2\} \subset \mathbb{H}$ with $z_0 = i$ is the set in Theorem 1.6 and thus $\kappa(\mathbb{H}) = h_b(E^*)/J_b(E^*)$. Let $w_j = f(x z_j)$ and put $E'_x = E_x \{w_j : j = 0, 1, 2\}$. Since $f(x z) = a + a_1 x z + O(x^3)$ as $x \to 0$ locally uniformly in $z$, $d(E'_x) = |w_1 - w_2|$ and $d(E'_x, \partial G) = d(w_0, \Delta_1 \cap \partial \Delta_2)$ for a small enough $x > 0$. Note here that $w_0 = f(x z_0) = f(ix) \in (0, a)$ because $f(i \mathbb{R}_+) = (-1, a)$. Hence, $d(E'_x, \partial G) = d(w_0, \Delta_1 \cap \partial \Delta_2) = d(w_0, a)$. We now look at the quantity

$$F(x) = \frac{d(E'_x)}{d(E'_x, \partial G)} = \frac{|w_1 - w_2|}{|w_0 - a|} = \frac{|w_1 - w_2|}{|w_0 - a|}.$$

We observe that

$$W = \frac{w_1 - w_2}{w_0 - a} = \frac{f(x z_1) - f(x z_2)}{f(x z_0) - f(0)}$$

is even analytic in $x \in \mathbb{D}$ and we compute

$$W = \frac{a_1 x (z_1 - z_2) + a_2 x^2 (z_1^2 + z_2^2) + O(x^3)}{a_1 x z_0 + a_2 x^2 z_0^2 + O(x^3)}$$

$$= \frac{z_1 - z_2}{z_0} \cdot \frac{1 + A x (z_1 + z_2) + O(x^2)}{1 + A x z_0 + O(x^2)}$$

$$= \frac{z_1 - z_2}{z_0} \cdot [1 + A x (z_1 + z_2) + O(x^2)],$$

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where $A = a_2/a_1 = a/(4i)$ by (3.7). Hence $F(x) = |W|$ is real analytic in $-1 < x < 1$ and

$$F(x) = \frac{|z_1 - z_2|}{|z_0|} \{1 + \Re[Ax(z_1 + z_2 - z_0)] + O(x^2)\}$$

$$= \frac{|z_1 - z_2|}{|z_0|} \left\{1 + \frac{ax}{4} \Im(z_1 + z_2 - z_0) + O(x^2)\right\}$$

as $x \to 0$. Since $\Im z_j = d(z_j, \partial H) > d(z_0, \partial H)$ for $j = 1, 2$, we have

$$F(0) = \frac{|z_1 - z_2|}{|z_0|} = \frac{d(E^*)}{d(E^*, \partial H)} \quad \text{and} \quad F'(0) = \frac{a|z_1 - z_2|}{4|z_0|} \Im(z_1 + z_2 - z_0) > 0.$$

In particular, $F(x)$ is strictly increasing at $x = 0$ and thus $F(x) > F(0)$ for small enough $x > 0$. Since $G \subset \Omega$; therefore, $h_{\Omega}(E'_x) \leq h_G(E'_x)$. We also note that

$$d(E'_x, \partial \Omega) \geq d(E'_x, \partial G) = d(w_0, a) \geq d(E_x, \partial \Omega),$$

because $a \in \partial \Omega$, and so $d(E'_x, \partial \Omega) = d(E'_x, \partial G)$ so that $J_{\Omega}(E'_x) = J_G(E'_x)$. Moreover, since the hyperbolic distance is conformally invariant,

$$h_G(E'_x) = h_G(f(E_x)) = h_{\Omega}(E_x) = h_{\Omega}(E^*).$$

Hence, for a small enough $x > 0$,

$$\kappa(\Omega) \leq \frac{h_{\Omega}(E'_x)}{J_{\Omega}(E'_x)} \leq \frac{h_G(E'_x)}{J_G(E'_x)} = \frac{h_{\Omega}(E^*)}{\log(1 + F(x))} < \frac{h_{\Omega}(E^*)}{\log(1 + F(0))} = \frac{h_{\Omega}(E^*)}{J_{\Omega}(E^*)} = \kappa(\Omega).$$

The proof is finished. \(\square\)

Now Theorem 1.5 follows from Lemmas 3.1, 3.4, and 3.6.

### 4. Extremal Configuration of Three Points in $\mathbb{H}$

In this section, we work to find extremal configurations of three-point sets $E$ in the upper half-plane for the functional $h_{\Omega}(E)/J_{\Omega}(E)$. Since $h_{\Omega}(E)$ and $J_{\Omega}(E)$ are invariant under the affine mappings of the form $z \mapsto az + b$ with $a > 0$ and $b \in \mathbb{R}$, we may restrict consideration to the family $\mathcal{E}$ of three-point subsets $E$ of $\mathbb{H}$ containing $i = \sqrt{-1}$ with $d(E, \partial \mathbb{H}) = d(i, \partial \mathbb{H}) = 1$. Namely, the infimum in the definition of $\kappa_3(\mathbb{H})$ may be limited to $\mathcal{E}$; i.e.,

$$\kappa_3(\mathbb{H}) = \inf_{E \in \mathcal{E}} \frac{h_{\Omega}(E)}{J_{\Omega}(E)} = \inf_{E \in \mathcal{E}} \frac{h_{\Omega}(E)}{\log(1 + d(E))}.$$

Our goal in this section is to determine the extremal sets $E$ at which the above infimum is attained, and to compute (at least numerically) the value of $\kappa_3(\mathbb{H})$. First, we note the following fact for the upper half-plane $\mathbb{H}$. Though the result is essentially known (e.g., [18, Lemma 4.9(2)]), we give a short proof for convenience of the reader.

**Lemma 4.1.** \(\kappa_2(\mathbb{H}) = \inf_{z_1, z_2 \in \mathbb{H}} \frac{h_{\Omega}(z_1, z_2)}{J_{\Omega}(z_1, z_2)} = 1.\)

**Proof.** Note that $\rho_{\Omega}(z) = 1/\Re z = 1/d(z, \partial \mathbb{H})$. Hence, $h_{\Omega}(z, w) = k_{\Omega}(z, w)$ for $z, w \in \mathbb{H}$. Thus, $j_{\Omega}(z, w) \leq h_{\Omega}(z, w)$ is nothing but the Gehring–Palka inequality [5]. Hence, $\kappa_2(\mathbb{H}) \geq 1$. On the other hand, by Lemma 2.3 $\kappa_2(\mathbb{H}) \leq c(\mathbb{H}) \leq 1$, where the last inequality follows from Theorem A. \(\square\)

We will write

$$\Delta(z_0, r) = \{z \in \mathbb{H} : h_{\Omega}(z, z_0) < r\} = \{z : |z - z_0| < \rho |z - z_0|\}$$

for the open hyperbolic disk in $\mathbb{H}$ centered at $z_0 \in \mathbb{H}$ with hyperbolic radius $r > 0$, where $\rho = \tanh(r/2) = (e^r - 1)/(e^r + 1) \in (0, 1)$ and denote the closure of the disk by $\overline{\Delta}(z_0, r)$. We need the following elementary fact for the proof of Lemma 4.15, which will be a key result below:

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Lemma 4.2. Let $C$ be the boundary circle of the hyperbolic disk $\Delta(z_0, r)$ in $\mathbb{H}$.

(i) The Euclidean distance $|z - z_0|$ between $z \in C$ and $z_0$ attain maximum at the top of $C$ and minimum at the bottom of $C$.

(ii) The Euclidean diameter of $C$ is $2(\text{Im } z_0) \sinh r$.

(iii) The hyperbolic distance of the endpoints of an arbitrary diameter of $C$ is at least equal to $\varphi(r)$ given in (4.3).

Proof. We write $z_0 = x_0 + iy_0$. It is well known (see, e.g., [18, (4.11)]) that the boundary of $\Delta(z_0, r)$ is the Euclidean circle $|z - c| = R$, where

$$c = x_0 + iy_0 \cosh r \quad \text{and} \quad R = y_0 \sinh r.$$ 

Since $\text{Re } z_0 = \text{Re } c$ and $\text{Im } z_0 < \text{Im } c$, it is evident that $|z - z_0|$ is maximized at $z = c + iR$ and minimized at $z = c - iR$ on $C$. The proof of (i) is now complete. Assertion (ii) is clear because the Euclidean diameter of $C$ is $2R$. It is clear that the diameter of $C$ with the minimal hyperbolic diameter is $[c - R, c + R]$. We now compute the hyperbolic distance

$$h_{\mathbb{H}}(c + R, c - R) = h_{\mathbb{H}}(i \cosh r + \sinh r, i \cosh r - \sinh r)$$

$$= 2 \text{artanh} \frac{\sinh r}{\sqrt{\cosh 2r + \sinh r}} = \log \frac{\sqrt{\cosh 2r + \sinh r}}{\sqrt{\cosh 2r - \sinh r}}$$

$$= 2 \log \frac{\sqrt{\cosh 2r + \sinh r}}{\cosh r} = \varphi(r). \quad (4.3)$$

Then (iii) follows. □

Remark 4.4. By geometry, we see that $|c + i \text{Re}^{1+i\theta} - z_0|$ is strictly decreasing in $0 < \theta < \pi$, which will be needed in the proof of Lemma 4.15.

We remark also that the sharp upper bound of the hyperbolic distance of the endpoints of a diameter of $C$ is $h_{\mathbb{H}}(c + iR, c - iR) = 2r$. By the form of $\varphi(r)$, we also see that $\varphi(r) \to \log \frac{\sqrt{2} + 1}{\sqrt{2} - 1} = 2 \log(\sqrt{2} + 1) = 1.762\ldots$ as $r \to +\infty$.

In order to find the extremal configuration, we divide the family $\mathcal{E}$ into one-parameter subfamilies. More concretely, for $u > 0$, let $\mathcal{E}(u)$ be the subfamily of $\mathcal{E}$ consisting of sets $E$ with $h_{\mathbb{H}}(E) = 2u$. Then

$$k_3(\mathbb{H}) = \inf_{0 < u < +\infty} \inf_{E \in \mathcal{E}(u)} \frac{2u}{J_{\mathbb{H}}(E)} = \inf_{0 < u < +\infty} \frac{2u}{\log(1 + M(u))}, \quad (4.5)$$

where

$$M(u) = \sup_{E \in \mathcal{E}(u)} d(E). \quad (4.6)$$

Our task is to find the extremal configuration of $E \in \mathcal{E}(u)$ for the functional $d(E)$. We first define a candidate of the extremal set. Given $u > 0$, we choose $t > 0$ and $\theta \in (0, \pi/2)$ such that

$$h_{\mathbb{H}}(i e^{t+i\theta}, i e^{-i\theta}) = h_{\mathbb{H}}(i e^{t+i\theta}, i) = 2u.$$ 

In other words, we choose $t$ and $\theta$ so that $E^*(u) = \{i, i e^{t+i\theta}, i e^{-i\theta}\}$ forms the vertices of a hyperbolic equilateral triangle with sidelength $2u$. We now give formulas describing $t$ and $\theta$ in terms of $u$. Since $h_{\mathbb{H}}(i e^{t+i\theta}, i e^{t}) = u$, we see that $u = 2 \text{artanh}(\tan(\theta/2))$, and so

$$\theta = 2 \text{arctan}(\tanh(u/2)). \quad (4.7)$$

Moreover, by the hyperbolic cosine formula for a hyperbolic right triangle [22, Theorem 7.11.1, p. 146], we have

$$\cosh t = \cosh h_{\mathbb{H}}(i e^{t}, i) = \frac{\cosh h_{\mathbb{H}}(i e^{t+i\theta}, i)}{\cosh h_{\mathbb{H}}(i e^{t+i\theta}, i e^{t})} = \frac{\cosh 2u}{\cosh u}.$$
Hence,

\[ t = \operatorname{arcosh}(\cosh 2u) / \cosh u. \]  

(4.8)

We now compute

\[ |ie^{t+i\theta} - ie^{t-i\theta}| = 2e^t \sin \theta = \chi(u), \]

where

\[
\chi(u) = 2e^{\operatorname{arcosh}(\cosh 2u) / \cosh u} \sin[2 \arctan \tanh(u/2)]
\]

\[
= \frac{\cosh 2u + \sqrt{(\cosh^2 2u) - (\cosh^2 u)}}{\cosh u} \cdot \tanh u
\]

\[
= \frac{2 \sinh u}{1 + \sinh^2 u} \left[ 1 + 2 \sinh^2 u + \sinh u \sqrt{3 + 4 \sinh^2 u} \right]. \tag{4.9}
\]

Note that \( \chi(u) \leq d(\mathcal{E}^*(u)) \). In the same way, we compute

**Lemma 4.10.** The set \( \mathcal{E}^*(u) \) of the vertices of the hyperbolic equilateral triangle in \( \mathbb{H} \) with side-length \( 2u \) constructed above belongs to \( \mathcal{E}'(u) \) for every \( u > 0 \).

We make further preparatory observations.

**Lemma 4.11.** If \( 0 < u \leq \log(11/4) \approx 1.0116 \), then \( d(\mathcal{E}^*(u)) = \chi(u) \) and

\[
\frac{2u}{\log(1 + M(u))} < 1.
\]

**Proof.** We will prove that

\[
\frac{2u}{\log(1 + \chi(u))} < 1 \tag{4.12}
\]

for \( 0 < u \leq \log(11/4) \). Since \( \mathcal{E}^*(u) \in \mathcal{E}'(u) \) by Lemma 4.10, we have \( M(u) \geq d(\mathcal{E}^*(u)) \geq \chi(u) \). Thus, the second assertion follows from (4.12).

By using the elementary inequality \( \sqrt{3 + 4 \sinh^2 u} > \sqrt{3 + 3 \sinh^2 u} = \sqrt{3} \cosh u \) for \( u > 0 \), we obtain the estimate

\[
\chi(u) > \frac{2 \sinh u}{1 + \sinh^2 u} \left[ 1 + 2 \sinh^2 u + \sqrt{3} \sinh u \cosh u \right].
\]

Thus,

\[
\chi(u) + 1 - e^{2u} \geq \frac{2 \sinh u}{1 + \sinh^2 u} \left[ 1 + 2 \sinh^2 u + \sqrt{3} \sinh u \cosh u \right] + 1 - e^{2u}
\]

\[
= \frac{(\sqrt{3} - 1)(e^u + 1)(e^u - 1)^2 P(e^u - 1)}{e^u(e^{2u} + 1)^2},
\]

where \( P(T) \) is the polynomial

\[
P(T) = 4 + (7 + \sqrt{3})T + 4T^2 - \sqrt{3}T^3 - \frac{1 + \sqrt{3}}{2} T^4.
\]
We now estimate \( P(T) \) for \( T \geq 0 \) from below:

\[
P(T) \geq 4 + 8T + 4T^2 - 2T^3 - 2T^4 = 2(1 + T)(2 + 2T - T^3).
\]

Since \( Q(T) = 2 + 2T - T^3 \) is concave on \([0, +\infty)\), we have

\[
Q(T) \geq \min\{Q(0), Q(7/4)\} = 964 > 0 \quad \text{for} \quad 0 \leq T \leq 7/4.
\]

Hence, we proved that \( e^{2u} < 1 + \chi(u) \) and thus (4.12) holds for \( 0 < u < \log(11/4) \).

Finally, we prove that \( d(E^*(u)) = \chi(u) \) for such \( u \). Indeed, the inequality

\[
|ie^{t+i\theta} - ie^{t-i\theta}| < |ie^{t+i\theta} - i|
\]

would hold otherwise. Then the two-point subset \( E = \{i, ie^{t+i\theta}\} \) of \( E^*(u) \) satisfies \( h_\partial(E) = 2u \), \( d(E) = d(E^*(u)) \), and \( d(E, \partial\Omega) = d(E^*(u), \partial\Omega) = 1 \). Thus, we would have

\[
\frac{2u}{\log(1 + \chi(u))} > \frac{2u}{J_\partial(E^*(u))} = \frac{2u}{J_\partial(E)} \geq \kappa_2(\Omega) = 1
\]

by Lemma 4.1. This contradicts (4.12). In this way, we have proved that \( d(E^*(u)) = \chi(u) \). \( \Box \)

**Lemma 4.13.** Let \( 0 < u < +\infty \). The condition \( \varphi(2u) \geq 2u \) holds if and only if \( u \leq u_0 \), where \( \varphi \) is given in (4.3) and \( u_0 \approx 0.831443 \) is the positive solution to the equation \( 4 \cosh^4 u = \cosh 4u \).

**Proof.** We see for \( u > 0 \) that

\[
\varphi(2u) = 2 \text{artanh} \left( \frac{\sinh 2u}{\sqrt{\cosh 4u}} \right) < 2u \implies \frac{\sinh 2u}{\sqrt{\cosh 4u}} = \frac{2 \sinh u \cosh u}{\sqrt{\cosh 4u}} < \tanh u = \frac{\sinh u}{\cosh u} \iff 4 < \frac{\cosh 4u}{\cosh^4 u}.
\]

Since \( (\cosh 4u)/\cosh^4 u \) increases from 1 to 8 when \( u \) moves from 0 to \( +\infty \), there is a unique \( u_0 > 0 \) satisfying the relation \( 4 = (\cosh 4u_0)/\cosh^4 u_0 \). We now see that \( \varphi(2u) < 2u \) if and only if \( u > u_0 \). \( \Box \)

The following elementary result is also needed later.

**Lemma 4.14.** The function \( f(x) = x/\log(1 + 2 \sinh x) \) strictly increases from 1/2 to 1 as \( x \) moves from 0 to \( +\infty \).

**Proof.** Because \( f(x) = x/\log(e^x - e^{-x} + 1) \), differentiation yields

\[
f'(x) = h(x)/\left[ \log(e^x - e^{-x} + 1) \right]^2, \quad \text{where} \quad h(x) = \log(e^x - e^{-x} + 1) - \frac{x(e^x + e^{-x})}{e^x - e^{-x} + 1}.
\]

Further,

\[
h'(x) = -\frac{x(e^x - e^{-x})}{e^x - e^{-x} + 1} + \frac{x(e^x + e^{-x})^2}{(e^x - e^{-x} + 1)^2} = \frac{x(e^{-x} - e^x + 4)}{(e^x - e^{-x} + 1)^2} = \frac{2x(2 - \sinh x)}{(1 + 2 \sinh x)^2}.
\]

We now see that \( h'(x) > 0 \) for \( 0 < x < \arcsinh 2 \) and \( h'(x) < 0 \) for \( \arcsinh 2 < x \). Since \( h(0) = 0 \) and

\[
h(x) = x + \log(1 + e^{-x} - e^{-2x}) - x \frac{1 + e^{-2x}}{1 + e^{-x} - e^{-2x}} = O(xe^{-x}) = o(1)
\]

as \( x \to +\infty \), the function \( h(x) \) is positive for all \( x > 0 \). Hence, \( f'(x) > 0 \) for all \( x > 0 \), which implies that \( f(x) \) is strictly increasing in \( x > 0 \). It is easy to see that \( f(x) \to 1/2 \) as \( x \to 0 \) and that \( f(x) \to 1 \) as \( x \to +\infty \). \( \Box \)

We are ready to prove our result.
Lemma 4.15. Let $u > 0$. Then the quantity $M(u)$ defined in (4.6) is evaluated as
\[
M(u) = \begin{cases} 
\chi(u) & \text{if } 0 < u < u_0, \\
2 \sinh u & \text{if } u_0 \leq u,
\end{cases}
\]
where $\chi(u)$ is given in (4.9) and $u_0 \approx 0.831443$ is the positive solution to the equation $4 \cosh^4 u = \cosh 4u$. Moreover, when $0 < u < u_0$, a set $E \in \mathcal{E}(u)$ satisfies $d(E) = M(u)$ if and only if $E = E^*(u)$.

**Proof.** We will denote the circle $\partial \Delta(i, 2u)$ by $C$. Since every $E \in \mathcal{E}(u)$ lies in the closed disk $\overline{\Delta}(i, 2u)$, the diameter $d(E)$ is at most $2 \sinh 2u$ by Lemma 4.2(ii). Hence, we observe that
\[
M(u) \leq 2 \sinh 2u, \quad u > 0.
\]

We assume first that $u \geq u_0$; equivalently by Lemma 4.13 $\varphi(2u) \leq 2u$. Let $z_1$ and $z_2$ be the endpoints of the horizontal diameter of the boundary circle $C = \partial \Delta(i, 2u)$. Note that $\text{Im } z_j = \cosh 2u > 1$. Then, by Lemma 4.2(iii), $h_{\Delta}(z_1, z_2) = \varphi(2u) \leq 2u$. Thus, $E = \{i, z_1, z_2\} \in \mathcal{E}(u)$ which implies $d(E) = 2 \sinh 2u \leq M(u)$. Consequently, we have proved that $M(u) = 2 \sinh 2u$. Note that the extremal set $E$ is not necessarily unique when $\varphi(2u) < 2u$ (for instance, we can rotate the diameter a little about the Euclidean center of $C$).

Next, we assume that $u < u_0$; namely, $\varphi(2u) > 2u$. We prove that there is a set $E_0 \in \mathcal{E}(u)$ providing the supremum in (4.6); namely, $M(u) = d(E_0)$. Indeed, we can find a sequence of sets $E_k$ in $\mathcal{E}(u)$ such that $d(E_k) \to M(u)$ as $k \to \infty$. Since each $E \in \mathcal{E}(u)$ lies in the closed hyperbolic disk $\overline{\Delta}(i, 2u)$, by passing to a subsequence, if need be, we may assume that $E_k = \{i, z_k, w_k\}$ and $z_k \to z_\infty$ and $w_k \to w_\infty$ as $k \to \infty$ for some $z_\infty, w_\infty \in \overline{\Delta}(i, 2u)$. By continuity, $d(E_\infty) = M(u)$ for $E_\infty = \{i, z_\infty, w_\infty\}$. We have to check that $E_\infty$ belongs to $\mathcal{E}(u)$. If $E_\infty$ consists only of two points, by Lemma 4.1,
\[
\log(1 + M(u)) \leq J_{\Delta}(E_\infty) \leq h_{\Delta}(E_\infty) = 2u,
\]
which contradicts Lemma 4.11 because $u \leq u_0 < \log(11/4)$. We have proved the claim.

Assume now that $E_0 = \{i, z_0, w_0\} \in \mathcal{E}(u)$ satisfies $d(E_0) = M(u)$. By assumption, $z_0 \in \overline{\Delta}(i, 2u) \cap \Delta(w_0, 2u)$. Observe that $z_0 \in \partial \Delta(i, 2u) = C$ in the present situation. In fact, let $r = h_{\Delta}(z_0, w_0)$ and suppose that $h_{\Delta}(z_0, i) < 2u$. Then $z_0$ can be moved along the circle $\partial \Delta(w_0, r)$ upwards a bit to get a new point $z'_0$ in such a way that
\[
\text{Im } z_0 < \text{Im } z'_0, \quad h_{\Delta}(z'_0, i) < 2u, \quad h_{\Delta}(z'_0, w_0) = r, \quad \text{and } |z_0 - w_0| < |z'_0 - w_0|
\]
by Lemma 4.2 and Remark 4.4. Hence we would have $h_{\Delta}(E'_0) = h_{\Delta}(E_0)$ and $d(E_0) < d(E'_0)$ for $E'_0 = \{i, z'_0, w_0\}$. This, however, violates the initial assumption that $d(E_0) = M(u)$. Therefore, we conclude that $h_{\Delta}(z_0, i) = 2u$. In the same way, we find that $h_{\Delta}(w_0, i) = 2u$. We can further prove, as before (cf. the proof of Lemma 4.11) that $|z_0 - w_0| = d(E_0)$.

The remaining task is now to determine the configuration of the points $z_0$ and $w_0$ on the circle $C$ maximizing $|z_0 - w_0|$ under the constraints $h_{\Delta}(z_0, w_0) \leq 2u$ and $\min\{|\text{Im } z_0, \text{Im } w_0| \geq 1$. We recall that the hyperbolic distance of the endpoints of an arbitrary Euclidean diameter of $C$ is at least $\varphi(2u)$ by Lemma 4.2(iii). We suppose first that $\varphi(2u) < 2u$. Let $C_0$ be the shorter component of $C$ \ $\{z_0, w_0\}$. It is evident that the chord $|z_0 - w_0|$ is shortest if (and only if) $z_0$ and $w_0$ are situated symmetrically with respect to the imaginary axis. Therefore,
\[
E_0 = E^*(u) \quad \text{and} \quad M(u) = 2u/\log(1 + d(E^*(u))) = \xi(u).
\]

By the above proof, the uniqueness of the extremal set for $0 < u \leq u_0$ is clear. Thus, the proof is now complete. \(\square\)

**Remark 4.16.** In view of Lemmas 4.11 and 4.14, as a corollary of the last lemma, we see that
\[
\inf_{E \in \mathcal{E}(u)} \frac{2u}{J_{\Delta}(E)} = \frac{2u}{\log(1 + M(u))} < 1
\]
for every $u > 0$.
Theorem 4.17. There is a zero \( u = u^* \) of the derivative \( \xi'(u) \) of the function
\[
\xi(u) = \frac{2u}{\log(1 + \chi(u))}
\]
in the interval \( 0 < u < u_0 \approx 0.83 \) such that
\[
\kappa(H) = \frac{h_{z^*}(z^*, w^*)}{\log(1 + |z^* - w^*|)} = \frac{h_{E^*}(E^*)}{\log(1 + d(E^*)/d(E^*, \partial H))},
\]
where \( u_0 \) is given in Lemma 4.13, \( E^* = E^*(u^*) = \{i, z^*, w^*\} \), \( z^* = i e^{t^* + i \theta^*} \), and \( w^* = i e^{t^* - i \theta^*} \), while \( t^* \) and \( \theta^* \) are given in (4.8) and (4.7), respectively, for \( u = u^* \). Moreover, if \( \kappa(H) = h_{E^*}/\log(1 + d(E)/d(E, \partial H)) \) for a three-point set \( E \) in \( H \), then there are reals \( a \) and \( b \) with \( a > 0 \) such that \( E = aE^* + b \).

**Proof.** Lemma 4.15 implies that for \( u \geq u_0 = 0.831 \ldots \)
\[
\frac{2u}{\log(1 + M(u))} = \frac{2u}{\log(1 + 2 \sinh 2u)}.
\]

Since the function \( x/\log(1 + 2 \sinh x) \) increases in \( 0 < x < +\infty \) by Lemma 4.14, we can restrict the range of the infimum in (4.5) to \( (0, u_0] \); i.e.,
\[
\kappa(H) = \kappa_3(H) = \inf_{0 < u \leq u_0} \frac{2u}{\log(1 + M(u))} = \inf_{0 < u \leq u_0} \frac{2u}{\log(1 + \chi(u))} = \inf_{0 < u \leq u_0} \xi(u),
\]
where \( \chi(u) \) is given in (4.9). By the form of \( \chi(u) \) in (4.9), we observe that \( \chi(u) = 2u + 2\sqrt{3}u^2 + O(u^3) \) as \( u \to 0^+ \). Thus,
\[
\xi(u) \geq 2u/\log(1 + \chi(u)) = 1 - (\sqrt{3} - 1)u + O(u^2) \quad \text{as} \quad u \to 0^+.
\]
In particular, \( \xi(0^+) = 1 \) and \( \xi'(0^+) = 1 - \sqrt{3} < 0 \). Since \( \xi'(u_0) = 0.1917 \ldots > 0 \), the above infimum of \( \xi(u) \) is attained at its critical point in \( (0, u_0) \).

The last assertion easily follows from the uniqueness of the extremal set in Lemma 4.15. The proof is now complete. \( \square \)

See Fig. 2 for the graph of the function \( 2u/\log(1 + M(u)) \). From numerical computations we obtain that \( u^* \approx 0.432335123777, t^* \approx 0.727535978839, \theta^* \approx 0.419463976058, \) and \( \kappa(H) = \xi(u^*) \approx 0.8750987500145 \). Note that by Theorem 1.5 \( \kappa(H) = \kappa(\Omega) \) for a convex hyperbolic domain \( \Omega \). In conclusion, we have the following theorem:

**Theorem 4.18.** \( \kappa(\Omega) \approx 0.875098750014 \) for every convex hyperbolic domain \( \Omega \).

Fig. 2. The graph of \( 2u/\log(1 + M(u)) \) (the thick line); the upper curve indicates the graph of \( \xi(u) \) and the lower one does the graph of \( 2u/\log(1 + 2 \sinh 2u) \).
Proof of Theorem 1.6. It remains to prove the first assertion. Let \( \Omega \subseteq \mathbb{C} \) be a convex domain and suppose that \( \kappa(\Omega) = h_{\Omega}(E)/J_{\Omega}(E) \) for a compact subset \( E \) of \( \Omega \). As in the proof of Lemma 2.1, we take points \( z_0, z_1, z_2 \in E \) so that \( d(E) = |z_1 - z_2| \) and \( d(E, \partial \Omega) = d(z_0, \partial \Omega) \) and let \( E_0 = \{ z_0, z_1, z_2 \} \). (Since \( \kappa(\Omega) = \kappa(\tilde{\Omega}) < 1 \), the set \( E_0 \) contains exactly three points.) By Lemma 2.1, \( \kappa(\Omega) = \kappa(\tilde{\Omega}) \). Thus, in the chain of inequalities (2.2), the last term is the same as the initial term. Thus, \( h_{\Omega}(E) = h_{\Omega}(E_0) \).

Hence \( \kappa(\Omega) = h_{\Omega}(E_0)/J_{\Omega}(E_0) \).

Let \( \zeta_0 \in \partial \Omega \) be such that \( d(E_0, \partial \Omega) = d(z_0, \partial \Omega) = |z_0 - \zeta_0| \).

Take a half-plane \( H \) as in the proof of Lemma 3.4 such that \( \Omega \subset H \) and \( z_0 \in \partial H \). Then \( J_{\Omega}(E_0) = J_H(E_0) \) and \( h_{\Omega}(E_0) \geq h_H(E_0) \). If \( \Omega \) is a proper subdomain of \( H \), then we would have \( h_H(E_0) < h_{\Omega}(E_0) \). Thus, \( \kappa(H) \leq h_H(E_0)/J_H(E_0) < h_{\Omega}(E_0)/J_{\Omega}(E_0) = \kappa(\Omega) \).

On the other hand, Theorem 1.5 yields \( \kappa(H) = \kappa(\Omega) \), which is a contradiction. Thus, \( \Omega \) equals \( H \), a half-plane. \( \Box \)

5. Application to Capacity Estimation

Finally, we apply the above results to capacity estimation. First, we recall some basic notions.

Definition 5.1 [18, Definition 9.2, p. 150]. A pair \( (\Omega, E) \) of a domain \( \Omega \) in \( \mathbb{C} \) and a nonempty compact subset \( E \) of \( \Omega \) is called a condenser. The capacity of this condenser is defined to be

\[
\text{cap}(\Omega, E) = \inf_u \iint_{\mathbb{C}} |\nabla u(z)|^2 \, dx \, dy \quad (z = x + iy),
\]

where the infimum is taken over the family of all nonnegative functions \( u \) in the Sobolev space \( W^{1,2}_{\text{loc}}(\mathbb{C}) \) with compact support in \( \Omega \) such that \( u(z) \geq 1 \) for \( z \in E \).

If \( \Omega \) is a simply connected proper subdomain of \( \mathbb{C} \) and \( E \) is a (nondegenerate) continuum in \( \Omega \) such that \( R = \Omega \setminus E \) is a doubly connected domain (a ring), then its modulus is known to be \( 2\pi / \text{cap}(\Omega, E) \).

We define the homeomorphism \( \mu : (0, 1) \to [0, \infty) \) by the formula (see, e.g., [18, 7.4.1, p. 122])

\[
\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1 - r^2})}{\mathcal{K}(r)},
\]

where \( \mathcal{K}(r) \) is Legendre’s complete elliptic integral of the first kind defined by

\[
\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - r^2 x^2)}}.
\]

It is known that \( \mu(r) \) represents the modulus of the Grötzsch ring \( \mathbb{D} \setminus [0, r] \). In particular, \( \mu(r) \) decreases from \( +\infty \) to 0 as \( r \) moves from 0 to 1. We note that \( 2\pi / \mu(r) \) is the capacity of \( \mathbb{D} \setminus [0, r] \). For later convenience, we put

\[
\Phi(x) = \frac{2\pi}{\mu(\tanh(x/2))}, \quad 0 < x < \infty.
\]

Note that \( \Phi(x) \) increases from 0 to \( +\infty \) as \( x \) moves from 0 to \( +\infty \). We are ready to give the main result in this section. Recall that \( J_{\Omega}(E) = \log(1 + d(E)/d(E, \partial \Omega)) \).
Theorem 5.2. Let $E$ be a continuum in a simply connected domain $\Omega \subseteq \mathbb{C}$. Then

(i) The inequality

$$\text{cap}(\Omega, E) \geq \Phi(\kappa(\Omega) J_{\Omega}(E)) \geq \Phi(\kappa_0 J_{\Omega}(E))$$

holds, where $\kappa_0$ is given in (1.7).

(ii) If $\Omega$ is convex,

$$\text{cap}(\Omega, E) \geq \Phi(\kappa_1 J_{\Omega}(E)),$$

where $\kappa_1 = \kappa(\mathbb{D}) > 0.87509875$.

Proof. Let $f : \Omega \to \mathbb{D}$ be a conformal homeomorphism and set $E' = f(E)$. Since the capacity and the hyperbolic distance are conformally invariant, we find

$$\text{cap}(\Omega, E) = \text{cap}(\mathbb{D}, E') \geq \Phi(h_\mathbb{D}(E')) = \Phi(h_\Omega(E)),$$

where we used a consequence of the circular symmetrization (see [18, Lemma 9.20, p. 163]). Other parts follow from Corollary 1.8 and Theorem 4.18. □

Example 5.3. Consider the example where $\Omega = \{ z : -1 < \text{Im} z < 1 \}$ and $E = [1, 2]$. Because $\Omega$ is convex, it follows from Theorem 5.2 that

$$\text{cap}(\Omega, E) \geq \Phi(\kappa_1 J_{\Omega}(E)) \approx 2 \pi \mu(0.43754937 \log 2) > 2.4288.$$

By applying the circular (spherical) symmetrization (see [18, 9.1, pp. 155–157]) with the origin as a center and $x$-axis as the symmetrization axis. Observe first that the negative $x$-axis is contained in the complement of the symmetrized condenser whereas $[1, 2]$ remains invariant and hence

$$\text{cap}(\Omega, E) \geq \tau_2(1) = 2,$$

where $\tau_2(t)$ denotes the capacity of the Teichmüller ring $\mathbb{C} \setminus ((-1, 0] \cup [t, +\infty))$ for $t > 0$ (see [18, 7.3, p. 120]), which is a weaker lower bound for the capacity than what we proved above. On the other hand, if we take it into account that the whole left half-plane is contained in the complement of the symmetrized condenser, we obtain

$$\text{cap}(\Omega, E) \geq \Phi(\kappa_1 J_{\Omega}(E)) \approx 2.55852.$$

Hence the value of our bound given in Theorem 5.2 lies between these two bounds obtained by symmetrization. Finally, let us find the exact value of $\text{cap}(\Omega, E)$. Obviously, $\text{cap}(\Omega, E) = \text{cap}(\Omega, E_0)$, where $E_0 = [0, 1]$. Note that the function $f(z) = \frac{2}{\pi} \log \frac{1 + z}{1 - z}$ maps the unit disk $\mathbb{D}$ onto $\Omega$ and that $f^{-1}(E_0) = [0, \tanh(\pi/4)]$.

Thus,

$$\text{cap}(\Omega, E) = \text{cap}(\Omega, E_0) = \text{cap}(\mathbb{D}, [0, \tanh(\pi/4)]) = \frac{2\pi}{\mu(\tanh(\pi/4))} = \Phi(\frac{\pi}{2}) \approx 3.75108.$$

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