Finite element simulation of fibrous composites strain with stress concentrators

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Abstract. In the paper the problems of numerical modeling of nonlinear physical processes of elements stress-strain state of the construction materials are considered. The effect and mutual influence of stress concentrators in the form of a circular cavity, vertically located two cavities and a horizontally located system of two cavities on the strain of construction is studied.

1. Introduction
Elastoplastic environment of inhomogeneous solid material consisting of reinforcing elements (fiber) and matrix (or binder) which enables combined work of reinforcing elements is investigated. It is known that fiber material and transversely isotropic medium are equivalent concepts. Therefore, to solve the problem of physically nonlinear strain of fibrous composites the theory of small elastoplastic strains for a transversely isotropic environment is applied. It is noted that while considering reinforced composite the stiffness of reinforcing elements of which is substantially greater than the stiffness of the binder, it is possible to use the simplified strain theory of plasticity.

A simplified theory allows application of small elastoplastic strains theory for solving specific applied problems. The essence of simplification is to assume that in simple tension of composite in the direction of transverse isotropy axis and the direction perpendicular thereto, plastic strain does not occur. In consequence of it the intensity of the stresses and strain is determined separately along a major axis of transverse isotropy and in a plane situated perpendicularly.

2. Methods
Formulation of boundary problem of elasticity for anisotropic bodies includes:

- equilibrium equations
  \[
  \sigma_{j,i} + X_i = 0, \quad x_i \in V,
  \]
  \[\tag{1}\]

- the generalized Hooke's law
  \[
  \sigma_{j,i} = C_{ijkl} \varepsilon_{kl},
  \]
  \[\tag{2}\]

- Cauchy relations
  \[
  \varepsilon_{kl} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right)
  \]
  \[\tag{3}\]

and boundary conditions
\[
 u_i \big|_{\Sigma_1} = u_i^0, \quad x_i \in \Sigma_1, \quad \sum_{j=1}^3 \sigma_{j,i} n_j \big|_{\Sigma_2} = S_i^0, \quad x_i \in \Sigma_2,
\]
\[\tag{4}\]
where \( u_i \) is components of displacement vector; 
\( S_i, X_i \) are surface and volume forces; 
\( \Sigma_1, \Sigma_2 \) are parts of the surface \( \Sigma \) of \( V \) volume; 
\( n_j \) is external normal to surface \( \Sigma_2 \) of \( V \) volume; 
\( C_{ijkl} \) is tensor of elastic constants.

In case of strain theory of transversely isotropic environments plasticity application, the determined relation (2) is replaced by the relation, which is in the form of decomposition of the stress tensor on the spherical \( (\tilde{\sigma}, \sigma_{33}) \) and deviatory \( (P_q, Q_q) \) parts [1]:

\[
\sigma_y = \tilde{\sigma}(\delta_y - \delta_{ij}\delta_{j3}) + \sigma_{22}\delta_{ij}\delta_{j3} + \frac{P_u}{P_u} p_y + \frac{Q_u}{Q_u} q_y ,
\]

where

\[
\tilde{\sigma} = (\lambda_2 + \lambda_3)\bar{\sigma} + \lambda_3\epsilon_{33}, \quad \sigma_{33} = \lambda_3\theta + \lambda_3\epsilon_{33},
\]

\[
P_u = 2\lambda_3(1 - \pi)P_u, \quad Q_u = 2\lambda_3(1 - \chi)Q_u ,
\]

\( \lambda_i \) is elastic constants of transversely isotropic materials, 
\( \pi, \chi \) are experimentally determined functions, 
\( P_q, q_q \) are decomposition components of deviatory parts of transversely isotropic strain tensor, 
\( (P_u, p_u) \) and \( (Q_u, q_u) \) are the intensity of stress and strain tensor, respectively, on a plane, and the axis of isotropy:

\[
P_u = \sqrt{\frac{1}{2} p_y p_y} = \sqrt{\frac{2}{2}(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} , \quad Q_u = \sqrt{\frac{1}{2} Q_y Q_y} = \sqrt{\sigma_{13}^2 + \sigma_{23}^2} ,
\]

\[
p_u = \sqrt{\frac{1}{2} p_y p_y} = \sqrt{\frac{2}{2}(\epsilon_{11} - \epsilon_{22})^2 + 4\epsilon_{12}^2} , \quad q_u = \sqrt{\frac{1}{2} q_y q_y} = \sqrt{\epsilon_{13}^2 + \epsilon_{23}^2}
\]
at that

\[
p_y = \epsilon_{ij} + \tilde{\bar{\sigma}}(\delta_{ij}\delta_{j3} - \delta_{ij}) + \epsilon_{33}\delta_{ij}\delta_{j3} - (\epsilon_{ij}\delta_{j3} + \epsilon_{33}\delta_{ij})
\]

\[
q_y = \epsilon_{33}\delta_{ij} + \epsilon_{33}\delta_{j3} - 2\epsilon_{33}\delta_{ij}\delta_{j3}, \quad \tilde{\bar{\sigma}} = \epsilon_{11} + \epsilon_{22},
\]

\[
P_y = \sigma_y + \tilde{\bar{\sigma}}(\delta_{ij}\delta_{j3} - \delta_{ij}) + \sigma_{33}\delta_{ij}\delta_{j3} - (\sigma_{ij}\delta_{j3} + \sigma_{33}\delta_{ij}),
\]

\[
Q_y = \sigma_y + \tilde{\bar{\sigma}}(\delta_{ij}\delta_{j3} - \delta_{ij}) + 2\sigma_{33}\delta_{ij}\delta_{j3}, \quad \tilde{\bar{\sigma}} = \sigma_{11} + \sigma_{22}/2.
\]

For simplified transversely isotropic plasticity theory the generalized Hooke's law (2) takes the following form

\[
\tilde{\sigma} = (\lambda_2 + \lambda_3)\bar{\sigma} + \lambda_3\epsilon_{33}, \quad \sigma_{33} = \lambda_3\theta + \lambda_3\epsilon_{33},
\]

\[
P_y = \frac{P_u}{P_u} p_y, \quad Q_y = \frac{Q_u}{Q_u} q_y ,
\]

where

\[
P_u = 2\lambda_3(1 - \pi(P_u))P_u, \quad Q_u = 2\lambda_3(1 - \chi(q_u))Q_u ,
\]

\( \pi(p) \) and \( \chi(q) \) – functions of plasticity of Ilyushin's type the values of which in the elastic zone are equal to zero. Plastic strain zones are defined on the basis of Mises criterion.

Mechanical parameters of transversely isotropic material are associated with modules \( \lambda_i \) by the following relationships:
\[
\begin{align*}
\lambda_1 &= E(1-\nu)/l, \quad \lambda_2 = E(\nu + kv^3)/(1 + \nu), \quad \lambda_3 = E' v'/l, \\
\lambda_4 &= G = E /[2(1 + \nu)], \quad \lambda_5 = G', \quad l = 1 - \nu - 2v^3 k, \quad k = E / E'.
\end{align*}
\] (11)

Here \( \nu \) and \( \nu' \) – effective Poisson's ratios, \( E \) and \( E' \) – the effective elastic modules, respectively, on the plane and on isotropy axis of the transversely isotropic material.

In general, by presenting the relationship between the stress tensor \( \sigma_{ij} \) and strain tensor \( \varepsilon_{kl} \) as \( \sigma_{ij} = F(\varepsilon_{kl}) \), the Cauchy’s function of correlations and the displacement vector of each particle in \( Ox_1x_2x_3 \) coordinate system as \( \bar{u}(u_1, u_2, u_3) \), the nonlinear relationship between the stress tensor and the displacement vector \( u_i \) can be represented as:

\[
\sigma_{ij} = F(\varepsilon_{ij}(\bar{u})) = \sigma_{ij}(\bar{u}).
\]

Then the equilibrium equation (1) defines system of three partial differential equations with respect to three components of the displacement vector. For this system of equations three types of boundary conditions could be put: in displacements, in stresses or mixed type (4). Thus, the strain process of the solid, which is in equilibrium under the action of external forces, can be reduced to the determination of the displacement vector \( \bar{u} \). On the basis of the solution of boundary value problems the components of the displacement vector are determined and tensor components of strain and stress are calculated.

3. Results

The process of stress-strain state reduction by changing the shape of the circuit with a minimally distorted stress state is considered. Through structural changes, it is possible to achieve an improvement in the distribution of stresses and to achieve an increase in structural strength. We note that the above effect was described in [2] using the example of increasing fatigue strength for round bars with transverse holes. The effect of discharge slots on the stress state of the rock mass in the vicinity of the mine is described in [3].

The elastoplastic stress-strain state of a fibrous plate made of boron aluminum is studied. It is stretched uniaxial in the direction of the fiber. The plate, for purely structural reasons, is provided with a round hole in the center. The dimensions of the rectangular plate: height – 1.0, width – 0.5, thickness – 0.1 (hereinafter all linear dimensions are given relative to the unit size plate). The volume content of boron fibers is 35\%, the radius of the hole is \( R = 0.05 \) and the external load is \( P_{xx} = 950 \) MPa. Taking into account the symmetry, \( 1/4 \) part of the structure is considered (figure 1a).

**Figure 1.** 1/4 part of structure with isolated and vertically located two holes

Strain state of unit-length quadratic plate with isolated or vertically located two holes is considered. The fibers are arranged in the direction of the \( OZ \)-axis. Volume content of boron fibers \( \nu = 35\% \).
Structure is stretched in the direction of the fibers ($P_{zz} = 950$ MPa). Thickness of plate $l = 0.1$ and radius of holes $R = 0.05$. In this studied problem, in vicinities of an isolated hole increasing of strain values is observed (figure 2a).

It’s assumed that there is a second additional hole along the vertical. Taking into account the symmetry, $\frac{1}{4}$ part of the structure is considered in figure 1. To research the influence of distance between two vertically spaced holes computational experiment is performed. Additional hole causes an increasing of stress in the vicinities of holes, but the presence of mutual influence of the holes leads to a reduction of overall stress. Analysis of the strain intensity indicates to the presence of mutual influence of horizontally located holes (figure 2b). The vicinities of holes are unloaded and no plastic zone is presented. The increasing values of strain are observed along the axis of holes’ system.

In this case the values of parameters of the stress-strain state are smaller than in the case of with isolated idle hole. Thus, the value of the strain intensity $P_u$ at a distance between the centers of the holes $h = 0.2$ is decreased in the most removed from each other points of the hole contour by 7.7% (table 1), while in the less removed points – by 26.7% (table 2).

| Hole spacing | Elastic problem | Elastoplastic problem |
|--------------|----------------|-----------------------|
|              | $p_u\times10^3$ | $P_u[MPa]$            | $p_u\times10^3$ | $P_u[MPa]$            |
| isolated     | 5.2423          | 451.99                | 4.4860          | 322.72                |
| h=0.2        | 4.8918          | 421.77                | 4.1370          | 307.68                |
| h=0.3        | 5.2054          | 448.81                | 4.5117          | 323.83                |
| h=0.4        | 5.5988          | 482.73                | 5.0660          | 347.72                |

It is interesting to note that for the elastic problem, these values are, respectively, 6.7% and 32.7% [4].

It is defined that when $h = 0.2$, two vertically arranged holes form a single stress concentrator and it leads to unloading of construction’s stress state. As increasing distance between holes, i.e. when $h = 0.3$ and 0.4, their mutual influence disappears (tables 1 and 2).

| Hole spacing | Elastic problem | Elastoplastic problem |
|--------------|----------------|-----------------------|
|              | $p_u\times10^3$ | $P_u[MPa]$            | $p_u\times10^3$ | $P_u[MPa]$            |
| isolated     | 5.2423          | 451.99                | 4.4860          | 322.72                |
| 0.2          | 3.5285          | 304.22                | 3.2486          | 269.38                |
| 0.3          | 4.6796          | 403.48                | 4.3282          | 315.92                |
| 0.4          | 5.2662          | 454.05                | 4.9074          | 340.89                |
This phenomenon could be explained using the idea of the power stream: external forces create a stream spreading along the construction. Line of pressure (power stream) is rejected by the second hole. The influence of the hole after the passing power flow is rejected can no longer increase [2].

Thus, analyses of the results of computational experiments allow designing a rational structure of fiber composites, to determine placement of the structural holes and to reduce the concentration of stress in constructions.

4. Conclusions and contributions
In the course of the research produced the following results.

1. On the basis of finite element method and the theory of small elastoplastic strains for a transversely isotropic environment the numerical model for three-dimensional problems of physically nonlinear strain of fibrous materials are developed.

2. On the basis of numerical modeling and computational experiments three-dimensional elastoplastic problems connected with the study influence of stress concentrators on the strain of fibre composite materials are solved.

3. The presence of doubly periodic system of spherical cavities in an infinite homogeneous thick plate provides the appearance of a narrow strip in the plastic strain zone, encircling the neighbourhood diametrical cross section of the cavities, and reduces the intensity of strain by 11.8%, and the weight of the structure by 6.74%.

4. On the basis of the computational experiments revealed regularities of strain relief effect demonstration in fibre composite structures: the presence of additional holes at a distance of one diameter of the design reduces the stress state in the vicinity system of holes.

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