Complex Geometry and Supersymmetry

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I stress how the form of sigma models with (2,2) supersymmetry differs depending on the number of manifest supersymmetries. The differences correspond to different aspects/formulations of Generalized Kähler Geometry.
1. Introduction

In this brief presentation I report on aspects of the relation between two-dimensional $N = (2,2)$ sigma models and complex geometry that I find remarkable: To each superspace formulation of the sigma model, be it $N = (2,2)$, $N = (2,1)$, $N = (1,2)$ or $N = (1,1)$, there is always a natural corresponding formulation of the Generalized Kähler Geometry on the target space. I first introduce the relevant formulations of Generalized Kähler Geometry and then the sigma models. The results are collected from a number of papers where we have used sigmamodels as tools to probe the geometry: [1]-[14]. See also [15], [16] for related early discussions.

2. Formulations of Generalized Kähler Geometry

Generalized Kähler Geometry was defined by Gualtieri [18] in his PhD thesis on Generalized Complex Geometry. The latter subject was introduced by Hitchin in [19]. In [18] it is also described how GKG is a reformulation of the bihermitean geometry of [23], which we now turn to.

2.1 Generalized Kähler Geometry I; Bihermitean Geometry.

Bihermitean geometry is the set $(M, g, J_{(\pm)}, H)$, i.e., a manifold $M$ equipped with a metric $g$, two complex structures $J_{(\pm)}$ and a closed three-form $H$. The defining properties may be summarized as follows:

$$J_{(\pm)}^2 = -I, \quad J_{(\pm)}^* g J_{(\pm)} = g, \quad \nabla_{(\pm)} J_{(\pm)} = 0$$

$$\Gamma^{(\pm)} = \Gamma^0 \pm \frac{1}{2} g^{-1} H, \quad H = dB.$$

Table 1: Bihermitean 1

In words, the metric is hermitean with respect to both complex structures and these, in turn, are covariantly constant with respect to connections which are the sum of the Levi-Civita connection and a torsion formed from the closed three-form. Locally, the three-form may be expressed in terms of a potential two-form $B$. This $B$-field, or NSNS two-form, is conveniently combined with the metric into one tensor $E$:

$$E := g + B.$$ (2.1)

A reformulation of the data in Table.1 more adapted to Generalized Complex Geometry is as the set $(M, g, J_{(\pm)})$ supplemented with (integrability)conditions according to
Complex Geometry and Susy  

\[ J^2_{(\pm)} = -\mathbb{I}, \quad J_{(\pm)} g J_{(\pm)} = g, \quad \omega_{(\pm)} := g J_{(\pm)} \]

\[ d^c(\omega_{(\pm)}) + d^c(\omega_{(-)}) = 0, \quad dd^c(\omega_{(\pm)}) = 0, \]

\[ H := d^c(\omega_{(\pm)}) = -d^c(\omega_{(-)}) \]

**Table 2**: Bihermitean 2

Here \( \omega_{\pm} \) are the generalizations of the Kähler forms for the two complex structures, \( d^c \) is the differential which reads \( i(\bar{\partial} - \partial) \) in local coordinates where the complex structure is diagonal, and we see that the three-form is defined in terms of the basic data.

### 2.2 Generalized Kähler Geometry II; Description on \( T \oplus T^* \)

Generalized Complex Geometry [19], [18], is formulated on the sum of the tangent and cotangent bundles \( T \oplus T^* \) equipped with an endomorphism which is a (generalized) almost complex structure, i.e., a map

\[ \mathcal{J} : T \oplus T^* \to T \oplus T^* : \quad \mathcal{J}^2 = -\mathbb{I}. \]  

(2.2)

The further requirements that turn \( \mathcal{J} \) into a generalized complex structure is first that it preserves the natural pairing on \( T \oplus T^* \)

\[ \mathcal{J} I \mathcal{J} = I, \quad I := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

(2.3)

where the matrix expression refers to the coordinate basis \( (\partial_\mu, dx^\nu) \) in \( T \oplus T^* \), and second the integrability condition

\[ \pi_{\pm}[\pi_{\pm}X, \pi_{\pm}Y]_C = 0, \quad X, Y \in T \oplus T^*. \]  

(2.4)

Here \( C \) denotes the Courant bracket [21], which for \( X = x + \xi, Y = y + \eta \in T \oplus T^* \) reads

\[ [X, Y]_C := [x, y] + \mathcal{L}_x \eta - \mathcal{L}_y \xi - \frac{1}{2} d(\iota_x \eta - \iota_y \xi), \]

(2.5)

with the Lie bracket, Lie derivative and contraction of forms with vectorfields appearing on the right hand side. Generalized Kähler Geometry [18] requires the existence of two commuting such Generalized Complex Structures, i.e.,

\[ \mathcal{J}^2_{(1,2)} = -\mathbb{I}, \quad [\mathcal{J}_{(1)}, \mathcal{J}_{(2)}] = 0, \quad \mathcal{J}_{(1,2)}^I \mathcal{J}_{(1,2)} = I, \quad \mathcal{G} := -\mathcal{J}_{(1)} \mathcal{J}_{(2)}, \]

(2.6)

with both GCSs satisfying (2.4) and the last line defines an almost product structure \( \mathcal{G} \):

\[ \mathcal{G}^2 = \mathbb{I}. \]  

(2.7)
When formulated in $T \oplus T^*$, Kähler geometry satisfies these condition, and so does bihermitean geometry. In fact the Gualtieri map \[18\] gives the precise relation\(^1\) to the data in Table 2:

$$
\mathcal{J}_{(1,2)} = \begin{pmatrix}
1 & 0 \\
B & 1
\end{pmatrix}
\begin{pmatrix}
J_+ & -\omega_+^{-1} + \omega_-^{-1} \\
\omega_+ & \omega_-
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-B & 1
\end{pmatrix}
$$

(2.8)

2.3 Generalized Kähler Geometry III; Local Symplectic Description

Bihermitean geometry emphasizes the complex aspect of generalized Kähler Geometry. There is another formulation where the (local) symplectic structure is in focus.

Given the bi-complex manifold $(M, J_{(\pm)})$, there exists locally defined non-degenerate “symplectic” two-forms $\mathcal{F}_{(\pm)}$ such that $d\mathcal{F}_{(\pm)} = 0$ and \[12\]

$$
\mathcal{F}_{(\pm)}(v, J_{(\pm)}v) > 0 \quad d(\mathcal{F}_{(+)}J_+ - J_+^t \mathcal{F}_{(-)}) = 0.
$$

**Table 3:** Conditions on $\mathcal{F}$

In the first condition $v$ is an arbitrary contravariant vector field and the condition says that $\mathcal{F}_{\pm}$ tames the complex structures $J_{(\pm)}$. The bihermitean data is recovered from

$$
\mathcal{F}_{(\pm)} = \frac{1}{2}i(B^{(2,0)}_{(\pm)} - B^{(0,2)}_{(\pm)}) \pm \omega_{(\pm)}
$$

$$
\mathcal{F}_{(+)} = -\frac{1}{2}E^t_{(+)}J_+ , \quad \mathcal{F}_{(-)} = -\frac{1}{2}J_+^tE^t_{(-)}
$$

(2.9)

where, e.g., $B^{(2,0)}_{(\pm)}$ refers to the holomorphic property of $B$ under $J_{(\pm)}$.

2.4 Summary

As we have seen, the geometric data representing Generalized Kähler Geometry may be packaged in various equivalent ways as, e.g., $(M, g, H, J_{(\pm)})$, as $(M, g, J_{(\pm)})$ or as $(M, \mathcal{F}_{(\pm)}, J_{(\pm)})$. In each case, there is a complete description in terms of a Generalized Kähler potential $K$ \[4\]\(^2\). Unlike the Kähler case, the expressions are non-linear in second derivatives of $K$. E.g., restricting attention to the situation $[J_+, J_-] \neq 0$, the left complex structure is given by

$$
J_+ = \begin{pmatrix}
J & 0 \\
(K_{LR})^{-1}[J, K_{LL}] & (K_{LR})^{-1}JK_{LR}
\end{pmatrix}
$$

(2.10)

\(^1\)The derivation from sigma models is given in \[6\].

\(^2\)The description is complete away from irregular points of certain poisson structures
where we introduced local coordinates \((X^L, X^R)\), \(L := \ell, \bar{\ell}\), \(R := r, \bar{r}\), and \(K_{LR}\) is shorthand for the matrix
\[
K_{LR} := \begin{pmatrix}
\frac{\partial^2 K}{\partial X^\ell \partial X^r} & \frac{\partial^2 K}{\partial X^\ell \partial X^{\bar{r}}} \\
\frac{\partial^2 K}{\partial X^{\bar{\ell}} \partial X^r} & \frac{\partial^2 K}{\partial X^{\bar{\ell}} \partial X^{\bar{r}}}
\end{pmatrix}.
\]

(2.11)

The metric is
\[
g = \Omega[J_\pm, J_\mp],
\]

(2.12)

and the local symplectic structures have potential one-forms \(\lambda_\pm\). E.g.,
\[
\mathcal{F}_\pm = d\lambda_\pm, \quad \lambda_\pm|_\ell = iK_R(K_{LR})^{-1}K_L|_\ell, \ldots
\]

(2.13)

The relations may be extended to the whole manifold in terms of gerbes \([12]\).

3. Sigma Models

The \(d = 2, N = (2, 2)\) supersymmetry algebra of covariant derivatives is
\[
\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial =
\]

(3.1)

The covariant derivatives can be used to constrain superfields. We shall need chiral, twisted chiral and left and right semichiral superfields \([17]\):
\[
\bar{\mathbb{D}}_\pm \phi^a = 0, \\
\mathbb{D}_+ \chi^d = \mathbb{D}_- \chi^d = 0, \\
\bar{\mathbb{D}}_+ X^{\ell} = 0, \\
\bar{\mathbb{D}}_- X^r = 0,
\]

(3.2)

and their complex conjugate. The collective index notation is taken to be; \(c := a, \bar{a}\), \(t := a', \bar{a}'\), and, as before, \(L := \ell, \bar{\ell}, R := r, \bar{r}\).

3.1 Superspace I

The \((2, 2)\) formulation of the \((2, 2)\) sigma model uses the generalized Kähler Potential \(K\) directly:
\[
S = \int \mathbb{D}_+ \bar{\mathbb{D}}_+ \mathbb{D}_- \bar{\mathbb{D}}_- K(\phi^c, \chi^t, X^L, X^R)
\]

(3.3)

Note that \(K\) has many roles: as a Lagrangian as in (3.3), as a potential for the geometry, \((2.10), (2.11)\), as a “prepotential” for the local symplectic form \(\mathcal{F}\), \((2.13)\), and, as shown in \([4]\), as a generating function for symplectomorphisms between coordinates where \(J_\pm\) and coordinates where \(J_-\) are canonical.
3.2 Superspace II

To discuss reduction of the action (3.3) to (2, 1) superspace [14], we restrict the potential to \( K(\mathcal{X}^L, \mathcal{X}^R) \) to simplify the expressions.

The reduction entails representing the (2, 2) right derivative as a sum of (2, 1) derivative and a generator of supersymmetry:

\[
\mathbb{D}_- = D_- - iQ_-, \quad (3.4)
\]

and defining the (2, 1) components of a (2, 2) superfield as

\[
\mathcal{X}| = X, \quad Q_-X^L| = \Psi_-^L. \quad (3.5)
\]

The action (3.3) then reduces as

\[
S = \int \mathbb{D}_+\bar{\mathbb{D}}_- ( K_L \Psi_-^L + K_R J_- \mathcal{X}^R ). \quad (3.6)
\]

Here \( \Psi \) is a Lagrange multiplier field enforcing \( \bar{\mathbb{D}}_+ K_\ell = 0 \) and its c.c., which are the (2, 1) components of the (2, 2) \( \mathcal{X}_\ell^\ell \) and \( \mathcal{X}_\ell^\ell \) equations. We solve this by going to (2, 2) coordinates \( (\mathcal{X}_L, \mathcal{Y}_L) \) [8], [14], whose (2, 1) components will now both be chiral. The action then reads

\[
S = i \int \mathbb{D}_+\bar{\mathbb{D}}_+ (\lambda_\alpha^{(0)} D_- \phi^\alpha + c.c.) \quad (3.7)
\]

with \( \phi^\alpha \in (\mathcal{X}_\ell^\ell, Y_\ell) \) and \( \bar{\mathbb{D}}_+ \phi^\alpha = 0 \). This is the standard form of a (2, 1) sigma model [22] but with the vector potential now identified (up to factors) with \( \lambda_\alpha^{(0)} \) in (2.13), \( (\mathcal{F}_\alpha^{(0)} = d\lambda_\alpha^{(0)}) \). Of the two complex structures \( J_{(\pm)} \) only \( J_{(+)} \) is now manifest. The complex structure \( J_{(-)} \) instead appears in the non-manifest supersymmetry

\[
\delta \phi^\alpha = \bar{\mathbb{D}}_+ (\epsilon J_{(-)}^\alpha \mathbb{D}_- \phi^\prime) , \quad \{ \phi^i \} = \{ \phi^\alpha, \bar{\phi}^\alpha \} \quad (3.8)
\]

Similarly, reduction of (3.3) to (1, 2) yields a model in which \( J_{(-)} \) is the remaining manifest complex structure. It is found from the (2, 1) model by the replacement \(+ \rightarrow -\), and \( L \rightarrow R \).

3.3 Superspace III

We may reduce the action (3.3) to (1, 1) superspace directly or via the (2, 1) formulation. The resulting action now involves the metric and \( B \)-fields in the combination (2.1) as geometric objects:

\[
S = \int D_+D_- ( D_+XED_-X ) , \quad (3.9)
\]

where we have suppressed the indices. Starting from (2, 1) superspace and the action (3.6), the reduction goes via

\[
\mathbb{D}_+ =: D_+ - iQ_+ , \quad Q_+X^R| =: \Psi_+^R , \quad (3.10)
\]

and both the auxiliary spinors \( \Psi_-^L \) and \( \Psi_-^R \) have been eliminated\(^3\). Both complex structures are now non-manifest and arise in the extra supersymmetry transformations as explained in [23].

\(^3\)Note that these spinors have the role of Lagrange multipliers in the (2, 1) and (1, 2) formulations, but the role of auxiliary fields with algebraic field equations in the (1, 1) formulation.
3.4 Summary

The various sigma models have different formulations of Generalized Kähler Geometry manifest. Thus the (2,2) sigma model is written directly in terms of the generalized Kähler potential. The (2,1) or (1,2) model involves the one form $\lambda_{(+)}$ or $\lambda_{(-)}$ respectively, which connects it to the local symplectic formulation. The (1,1) sigma model, finally, is expressed directly in terms of the metric and $B$-field, making that aspect of the geometry manifest. These are also the objects that determine the (0,0) formulation, i.e., the component formulation of the sigma model.

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