A system of state-dependent delay differential equation modelling forest growth I: semiflow properties

Pierre Magal and Zhengyang Zhang
Univ. Bordeaux, IMB, UMR 5251, F-33076 Bordeaux, France
CNRS, IMB, UMR 5251, F-33400 Talence, France
July 4, 2018

Abstract: In this article we investigate the semiflow properties of a class of state-dependent delay differential equations which is motivated by some models describing the dynamics of the number of adult trees in forests. We investigate the existence and uniqueness of a semiflow in the space of Lipschitz and $C^1$ weighted functions. We obtain a blow-up result when the time approaches the maximal time of existence. We conclude the paper with an application of a spatially structured forest model.

Keywords: State-dependent delay differential equations, forest population dynamics, semiflow, time of blow-up.

AMS Subject Classification: 34K05, 37L99, 37N25.

1 Introduction

Let $\Omega$ be a compact subset of $\mathbb{R}^n$ (with $n \geq 1$). Denote for simplicity that $C(\Omega) := C(\Omega, \mathbb{R})$, $C(\Omega^2) := C(\Omega^2, \mathbb{R})$ and $C_+(\Omega) := C(\Omega, [0, +\infty))$. In this article we consider the following class of state-dependent delay differential equation:

\[
\begin{align*}
\partial_t A(t, x) &= F(A(t, .), \tau(t, .), A(t - \tau(t))(.,.))(x), \\
\int_{-\tau(t, x)}^{0} f(A(t + s, .))(s)ds &= \int_{-\tau_0(x)}^{0} f(\varphi(s, .))(s)ds,
\end{align*}
\]

(1.1)

where $F : C(\Omega)^2 \times C(\Omega^2) \to C(\Omega)$ and $f : C(\Omega) \to C(\Omega)$ and $A(t-\tau(t)) \in C(\Omega^2)$ is the map defined by

\[
A(t - \tau(t))(x, y) := A(t - \tau(t), x, y)
\]

(1.2)

with the initial condition

\[
A(t, x) = \varphi(t, x), \forall t \leq 0 \text{ and } \tau(0) = \tau_0 \in C_+(\Omega),
\]
and the initial distribution $\varphi$ belongs to

$$\text{Lip}_\alpha := \{ \phi \in C((\infty, 0], C(\Omega)) : t \to e^{-\alpha|t|}\phi(t, \cdot) \text{ is bounded and Lipschitz continuous from } (-\infty, 0) \to C(\Omega) \}, \alpha \geq 0.$$ 

Recall that the space $\text{Lip}_\alpha$ is a Banach space endowed with the norm

$$\| \phi \|_{\text{Lip}_\alpha} := \| \phi_\alpha \|_\infty + \| \phi_\alpha \|_{\text{Lip}}$$

where $\phi_\alpha : (-\infty, 0] \to C(\Omega)$ is defined by

$$\phi_\alpha(t, x) := e^{-\alpha|t|}\phi(t, x), \forall t \in (-\infty, 0], \forall x \in \Omega. \quad (1.3)$$

and

$$\| \phi_\alpha \|_\infty := \sup_{t \leq 0} \| \phi_\alpha(t, \cdot) \|_\infty$$

and for fixed $t$,

$$\| \phi_\alpha(t, \cdot) \|_\infty := \sup_{x \in \Omega} |\phi_\alpha(t, x)|$$

and

$$\| \phi_\alpha \|_{\text{Lip}} := \sup_{t, s \leq 0: t \neq s} \frac{\| \phi_\alpha(t, \cdot) - \phi_\alpha(s, \cdot) \|_\infty}{|t - s|}.$$ 

In the rest of the paper the product space $\text{Lip}_\alpha \times C(\Omega)$ will be endowed with the usual product norm

$$\| (\phi, r) \|_{\text{Lip}_\alpha \times C(\Omega)} := \| \phi \|_{\text{Lip}_\alpha} + \| r \|_\infty, \forall \phi \in \text{Lip}_\alpha, \forall r \in C(\Omega).$$

We will make the following assumptions throughout this paper.

**Assumption 1.1** We assume that the map $F : C(\Omega)^2 \times C(\Omega^2) \to C(\Omega)$ is Lipschitz continuous on bounded sets, that is to say that for each constant $M > 0$, there exists a constant $L(M) > 0$ satisfying

$$\| F(u, v, w) - F(\tilde{u}, \tilde{v}, \tilde{w}) \|_\infty \leq L(M) \| u - \tilde{u} \|_\infty + \| v - \tilde{v} \|_\infty + \| w - \tilde{w} \|_\infty$$

whenever $\| u \|_\infty, \| \tilde{u} \|_\infty, \| v \|_\infty, \| \tilde{v} \|_\infty, \| w \|_\infty, \| \tilde{w} \|_\infty \leq M$.

We also assume that the map $f : C(\Omega) \to C(\Omega)$ is Lipschitz continuous and there exists a real number $M > 0$ such that

$$0 < f(\varphi)(x) \leq M, \forall x \in \Omega \text{ and } \forall \varphi \in C(\Omega),$$

and $f$ is monotone non-increasing, that is to say that

$$\varphi(x) \leq \tilde{\varphi}(x), \forall x \in \Omega \Rightarrow f(\varphi)(x) \geq f(\tilde{\varphi})(x), \forall x \in \Omega.$$

Examples of state-dependent delay differential equations of this form have been considered first by Smith [12, 13, 14, 15]. This idea has been successfully used in [4, 7] (see also the references therein). Our motivation to consider such a class of state-dependent delay differential equations comes from forest modelling. In [9, 10] such state-dependent delay differential equations have been used to model the competition for light between trees.
Example 1.2 (Finite number of species) The $m$-species case corresponds to the case $n = 1$ and the domain $\Omega$ contains exactly $m$ elements. We can choose for example

$$\Omega = \{1, 2, \ldots, m\}$$

and for $x = 1, \ldots, m$,

$$F(A(t, \cdot), \tau(t, \cdot), A(t - \tau(t))(\cdot))(x) = G(x, A(t, \cdot), \tau(t, x), A(t - \tau(t, x))(\cdot))$$

where $G : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is a map (see [9] for more details).

Example 1.3 (Spatially structured case) For the spatially structured case, we can choose

$$\Omega = [0, x_{\text{max}}] \times [0, y_{\text{max}}].$$

Moreover assume (for simplicity) that we have a single species, then we can choose

$$F(A(t, \cdot), \tau(t, \cdot), A_1(t, \cdot))(x, y) := e^{-\mu J_\tau(t, x, y) f(A(t, x, y)) f(A_1(t, x, y)) (I - \varepsilon \Delta)^{-1} [\beta A_1(t, \cdot)](x, y)}$$

where $\Delta$ is the Laplacian operator on the domain $\Omega$ with periodic boundary conditions. This model corresponds to the spatially structured model in [9].

Let $A \in C((-\infty, r], C(\Omega))$ (for some $r \geq 0$) be given. Then for each $t \leq r$, we will use the standard notation $A_t \in C((-\infty, 0], C(\Omega))$, which is the map defined by

$$A_t(\theta, \cdot) = A(t + \theta, \cdot), \forall \theta \leq 0.$$

For clarity we will specify the notion of a solution.

Definition 1.4 Let $r \in (0, +\infty]$. A solution of the system (1.1) on $[0, r)$ is a pair of continuous maps $A : (-\infty, r) \to C(\Omega)$ and $\tau : [0, r) \to C_+(\Omega)$ satisfying

$$A(t, x) = \begin{cases} \varphi(0, x) + \int_0^t F(A(l, \cdot), \tau(l, \cdot), A(l - \tau(l))(\cdot))(x)dl, & \forall t \in [0, r), \forall x \in \Omega, \\ \varphi(t, x), & \forall t \leq 0, \forall x \in \Omega, \end{cases}$$

and

$$\int_{t - \tau(t, x)}^t f(A(s, \cdot))(x)ds = \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x)ds, \forall t \in [0, r), \forall x \in \Omega.$$

In this problem the initial distribution is $(\varphi, \tau_0)$. The semiflow generated by (1.1) is

$$U(t)(\varphi(\cdot, x), \tau_0(x)) := (A_t(\cdot, x), \tau(t, x)),$$

where $(A, \tau)$ is the solution of (1.1) with the initial distribution $(\varphi, \tau_0)$.

In order to clarify the notion of semiflow in this context, we introduce the following definition.
Definition 1.5 Let \((M, d)\) be a metric space. Let \(U : D_U \subset [0, +\infty) \times M \to M\) be a map defined on the domain
\[
D_U := \{(t, x) \in [0, +\infty) \times M : 0 \leq t < T_{BU}(x)\},
\]
where \(T_{BU} : M \to (0, +\infty]\) is a lower semi-continuous map (the blow-up time). We will use the notation
\[
U(t)x := U(t, x), \forall (t, x) \in D_U.
\]
We say that \(U\) is a maximal semiflow on \(M\) if the following properties are satisfied:

(i) \(T_{BU}(U(t)x) + t = T_{BU}(x), \forall x \in M, \forall t \in [0, T_{BU}(x))\);
(ii) \(U(0)x = x, \forall x \in M\);
(iii) \(U(t)U(s)x = U(t + s)x, \forall t, s \in [0, T_{BU}(x))\) with \(t + s < T_{BU}(x)\);
(iv) If \(T_{BU}(x) < +\infty\), then
\[
\lim_{t \to T_{BU}(x)} d(U(t)x, y) = +\infty
\]
for some \(y \in M\).

We will say that the semiflow \(U\) is state variable continuous if for each \(t \geq 0\) the map \(x \mapsto U(t)x\) is continuous around each point where \(U(t)\) is defined. We will say that the semiflow \(U\) is locally uniformly state variable continuous if for each \(r \in [0, T_{BU}(x))\),
\[
\lim_{x \to x_0} \sup_{t \in [0, r]} d(U(t)x, U(t)x_0) = 0 \quad (1.4)
\]
whenever the map \(U(t)\) is defined at \(x\) and \(x_0\) and for each \(t \in [0, r]\).

We will say that the semiflow \(U\) is continuous if the map \((t, x) \mapsto U(t)x\) is continuous from \(D_U\) into \(M\).

Actually the semiflow of the state-dependent delay differential equation (1.1) is not always continuous in time. Assume for example that \(\alpha = 0, \Omega = \{1\}\) and \(\forall u, v \in C(\Omega), \forall w \in C(\Omega^2), \forall x \in \Omega,\)
\[
F(u, v, w)(x) \equiv 1 \text{ and } f(u)(x) \equiv 1.
\]
Consider \((A(t), \tau(t))\) (we omit the \(x\) variable since there is only one element in \(\Omega\)) the solution of (3.1) with the initial distribution
\[
(\varphi, \tau_0) = (0_{\text{Lip}_\alpha}, \tau_0).
\]
This solution can be solved explicitly:
\[
A(t) = \begin{cases} t, \forall t > 0, \\ 0, \forall t \leq 0, \tau(t) = \tau_0, \forall t \geq 0. \end{cases}
\]
And the semiflow will be defined by $U(t)(L_{\alpha}, \tau_0) = (A_t, \tau)$. Notice that the map $t \mapsto A(t)$ is differentiable almost everywhere and

$$A'(t) = \begin{cases} 
1, & \text{if } t > 0, \\
0, & \text{if } t < 0.
\end{cases}$$

Therefore for each $\hat{t} \geq 0$,

$$\lim_{t \to \hat{t}} \|A_t - A_{\hat{t}}\|_{Lip} = \lim_{t \to \hat{t}} \|A'(t) - A'(\hat{t})\|_{L^\infty(-\infty, 0)} = 1.$$ 

Therefore due to the discontinuity of $A'(t)$ at time $t = 0$, the semiflow is not continuous in time.

The following theorem is the main result of this section.

**Theorem 1.6** There exists a maximal semiflow $U : D_U \subset [0, +\infty) \times Lip_\alpha \times C_+(\Omega) \to Lip_\alpha \times C_+(\Omega)$ and its corresponding blow-up time $T_{BU} : Lip_\alpha \times C_+(\Omega) \to (0, +\infty]$ such that for each initial distribution $(\phi, \tau_0) \in Lip_\alpha \times C_+(\Omega)$, there exists a unique solution $A : (-\infty, T_{BU}(\phi, \tau_0)) \to C_+(\Omega)$ and $\tau : [0, T_{BU}(\phi, \tau_0)) \to C_+(\Omega)$ of (1.1) satisfying

$$U(t)(\phi, \tau_0)(x) = (A_t(., x), \tau(t, x)), \forall t \in [0, T_{BU}(\phi, \tau_0)), \forall x \in \Omega.$$ 

Moreover if $T_{BU}(\phi, \tau_0) < +\infty$, then

$$\limsup_{t \to T_{BU}(\phi, \tau_0)} \|A(t, .)\|_{\infty} = +\infty.$$ 

Furthermore the semiflow $U$ has the following properties:

(i) The map $T_{BU}$ is lower semi-continuous and $D_U$ is relatively open in $[0, +\infty) \times Lip_\alpha \times C_+(\Omega)$.

(ii) The semiflow $U$ is locally uniformly state variable continuous in $Lip_\alpha \times C_+(\Omega)$.

In the sequel we will use the notation

$$BUC_\alpha := \{\phi \in C((-\infty, 0], C(\Omega)) : \phi_\alpha \in BUC((-\infty, 0], C(\Omega))\}, \alpha \geq 0$$

where

$$\phi_\alpha(t, x) := e^{-\alpha|t|}\phi(t, x)$$

and $BUC((-\infty, 0], C(\Omega))$ denotes the space of bounded uniformly continuous maps from $(-\infty, 0]$ to $C(\Omega)$. The space $BUC_\alpha$ is again a Banach space endowed with the norm

$$\|\phi\|_{BUC_\alpha} = \sup_{t \leq 0} \|\phi_\alpha(t, .)\|_\infty.$$ 

We will also use the notation

$$BUC^1_\alpha := \{\phi \in C^1((-\infty, 0], C(\Omega)) : \phi_\alpha \in BUC((-\infty, 0], C(\Omega))$$

and $\partial_t(\phi_\alpha) \in BUC((-\infty, 0], C(\Omega))\}.$
and the space $BUC^1_\alpha$ is again a Banach space endowed with the norm
\[
\|\phi\|_{BUC^1_\alpha} := \|\phi_\alpha\|_\infty + \|\partial_t(\phi_\alpha)\|_\infty = \|\phi_\alpha\|_\infty + \|\phi_\alpha\|_{\text{Lip}}.
\]
Now we consider the following set $D_\alpha$ containing the couple $(\phi, \tau_0)$ satisfying a compatibility condition:
\[
D_\alpha := \{ (\phi, \tau_0) \in BUC^1_\alpha \times C_+(\Omega) : \partial_t \phi(0, x) = F(\phi(0, \cdot), \tau_0(\cdot), \phi(-\tau_0(\cdot), \cdot))(x), \forall x \in \Omega \}.
\]
One can note that $D_\alpha$ is a closed subset of $BUC^1_\alpha \times C_+(\Omega)$. Therefore $D_\alpha$ is a complete metric space endowed with the distance
\[
d_{D_\alpha}((\phi, \tau_0), (\hat{\phi}, \hat{\tau}_0)) := \|\phi - \hat{\phi}\|_{BUC^1_\alpha} + \|\tau_0 - \hat{\tau}_0\|_\infty.
\]
We also have
\[
D_\alpha \subset BUC^1_\alpha \times C_+(\Omega) \subset \text{Lip}_\alpha \times C_+(\Omega),
\]
and the topology of $BUC^1_\alpha \times C_+(\Omega)$ and $\text{Lip}_\alpha \times C_+(\Omega)$ coincide on $D_\alpha$. We have the following results.

**Theorem 1.7** The sub-domain $D_\alpha$ is dense in $BUC_\alpha \times C_+(\Omega)$, namely
\[
\overline{D_\alpha}^{BUC_\alpha \times C(\Omega)} = BUC_\alpha \times C_+(\Omega).
\]
Moreover we have following properties:

(i) The subdomain $D_\alpha$ is positively invariant by the semiflow $U$, that is to say that for each $(\phi, \tau_0) \in D_\alpha$,
\[
U(t)(\phi, \tau_0) \in D_\alpha, \forall t \in [0, T_{BU}(\phi, \tau_0)).
\]

(ii) The semiflow $U$ restricted to $D_\alpha$ is a continuous semiflow when $D_\alpha$ is endowed with the metric $d_{D_\alpha}$.

Particularly, from (ii), we know that we can choose two different state space for $A_t$ ($\text{Lip}_\alpha$ or $BUC^1_\alpha$), but only in the case of $BUC^1_\alpha$ can we get a continuous (in time) semiflow.

In system (1.1), we can see from the second equation that the delay $\tau(t, x)$ is a solution of an integral equation. In the following (Lemma 3.2) we will see that the delay $\tau(t, x)$ can be seen as the solution of a partial differential equation, too. In Lemma 3.6, we will see that the delay $\tau(t, x)$ can be also regarded as a functional of $A_t$ and $(\phi, \tau_0)$, which shows that it is actually a state-dependent delay. Specifically speaking, let $\delta_0 \in C_+(\Omega)$ be fixed, then we can define the map $\tau : D(\tau) \subset \text{Lip}_\alpha \to [0, +\infty)$ as the solution of
\[
\int_{-\tau(\phi, \cdot)}^{0} f(\phi(s, \cdot))(x) ds = \delta_0(x) \quad (1.5)
\]
with

\[ D(\tau) = \left\{ \phi \in \text{Lip}_\alpha : \delta_0(x) < \int_{-\infty}^0 f(\phi(s,.))(x)ds, \forall x \in \Omega \right\}. \]

Then we will see that

\[ \tau(A_t, x) = \tau(t, x), \forall t \geq 0, \]

and the first equation in (1.1) can be rewritten as

\[ \partial_t A(t, x) = F(A(t,.), \tau(t, .), A(t - \tau(A_t, .), .))(x), \forall t \geq 0. \]

State-dependent delay differential equations have been used in the study of population dynamics of species [1, 2, 6, 7]. We refer in addition to [3, 5] and the references therein for a nice survey on this topic. Moreover, the semiflow properties of a general class of state-dependent delay differential equations have been recently studied by Walther [16] in \( D_\alpha \).

As an illustration, let us consider for example the following system

\[
\begin{align*}
\partial_t A(t, x) &= F(A(t, .))(x), \forall t \geq 0, \forall x \in \Omega, \\
\int_{-\tau(x)}^{0} f(A(t+s, .))(x)ds &= \int_{-\tau_0(x)}^{0} f(\varphi(s,.))(x)ds, \forall t \geq 0, \forall x \in \Omega,
\end{align*}
\]

and the map \( F : BUC^{1}_\alpha \to \mathbb{R} \) is defined by

\[ F(\varphi) := \varphi(-\tau(\varphi)) \]

where \( \tau(\varphi) \) is defined as above in (1.5). Assume in addition that \( f \) is continuously differentiable, then by Lemma 3.4, the state-dependent delay \( \tau : BUC_\alpha \to C(\Omega) \) is \( C^1 \). Then for \( \varphi_0 \in BUC^{1}_\alpha \), we have

\[
F(\psi + \varphi_0) - F(\varphi_0) = (\psi + \varphi_0)(-\tau(\psi + \varphi_0)) - \varphi_0(-\tau(\varphi_0)) = \psi(-\tau(\psi + \varphi_0)) + \varphi_0(-\tau(\psi + \varphi_0)) - \varphi_0(-\tau(\varphi_0)),
\]

from which we deduce the derivative

\[ DF(\varphi_0)\psi = \psi(-\tau(\varphi_0)) + \varphi'_0(-\tau(\varphi_0)) \cdot \partial_\varphi \tau(\varphi_0)\psi, \]

which satisfies the assumption (E) in Walther [16].

In this article, we consider the pair \( (A_t, \tau(t, .)) \) as the state variable, and in this case we can also apply the result by Walther in [16] to the delay differential equation

\[
\begin{align*}
\partial_t A(t, x) &= F(A(t, .), \tau(t, .), A(t - \tau(A_t, .), .))(x), \\
\partial_t \tau(t, x) &= 1 - \frac{f(A(t,.))(x)}{f(A(t - \tau(t,.), .))(x)}. \tag{1.6}
\end{align*}
\]

Nevertheless the existence of a maximal semiflow as well as the blow-up time has been considered by Walther [16].
The article is organized as follows. In section 2 we prove that $D_\alpha$ is dense in $BUC_\alpha \times C_+(\Omega)$. In section 3 we prove some results regarding the delay $\tau(t,x)$. In sections 4 and 5 we will investigate the uniqueness and local existence of solutions, and the properties of semiflows. In the last section of the article, we will illustrate our results by proving the global existence of solutions for a spatially structured forest model.

2 Density of the domain

In this preliminary section we will prove the first result of Theorem 1.7, namely the density of $D_\alpha$ in the space $BUC_\alpha \times C_+(\Omega)$.

Proof. Fix $\tau_0 \in C_+(\Omega)$. Consider the space

$$X := C(\Omega) \times BUC_\alpha$$

which is a Banach space endowed with the usual product norm. Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} \varphi \\ 0 \end{pmatrix} := \begin{pmatrix} -\partial_t \varphi(0,.) \\ \partial_t \varphi \end{pmatrix}, \quad \forall \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \in D(A),$$

with

$$D(A) := \{0_{C(\Omega)}\} \times BUC^1_\alpha.$$  \hfill (2.1)

Then it is not difficult to prove that

$$D(A) = \{0_{C(\Omega)}\} \times BUC_\alpha.$$  \hfill (2.1)

Moreover, the linear operator $A$ is a Hille-Yosida operator (see [8]). More precisely, we have $(0,\infty) \subset \rho(A)$ and for each $\lambda \in (0,\infty)$,

$$(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix},$$

$$\Leftrightarrow \psi(\theta,x) = \frac{1}{\lambda} e^{\lambda \theta} [\alpha + \varphi(0,x)] + \int_0^\theta e^{\lambda (\theta - l)} \varphi(l,x) dl.$$  \hfill (2.3)

The linear operator $A$ is Hille-Yosida since we have the following estimation from [8]

$$\|(\lambda I - A)^{-n}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda^n}, \forall n \geq 1, \forall \lambda > 0.$$  \hfill (2.2)

By using (2.1) and (2.2) and by the fact that

$$\lambda(\lambda I - A)^{-1} - I = A(\lambda I - A)^{-1},$$

it follows that for each $\psi \in BUC_\alpha$,

$$\lim_{\lambda \to +\infty} \|\lambda(\lambda I - A)^{-1} \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} - \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix}\|_{X} = 0.$$  \hfill (2.3)
We define the nonlinear map $\mathcal{F} : \overline{D(\mathcal{A})} \to \mathcal{X}$,
\[
\mathcal{F} \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi \end{pmatrix} \right) := \left( F(\varphi(0,.), \tau_0(., \varphi(-\tau_0(.,.)))) \right)_{\mathcal{BUC}_\alpha}, \forall \varphi \in \mathcal{BUC}_\alpha.
\]

We observe that $(\varphi, \tau_0) \in D_\alpha \iff (\mathcal{A} + F)(0_{\mathcal{C}(\Omega)} \varphi) \in D(\mathcal{A}) \text{ with } (0_{\mathcal{C}(\Omega)} \varphi) \in D(\mathcal{A})$, \forall $\lambda > 0$.

Let \( \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \psi \end{pmatrix} \right) \in \{0_{\mathcal{C}(\Omega)}\} \times \mathcal{BUC}_\alpha \) be fixed. Then for each $\lambda > 0$, consider
\[
(I - \lambda \mathcal{A} - \lambda \mathcal{F}) \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi_\lambda \end{pmatrix} \right) = \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \psi \end{pmatrix} \right) \text{ with } \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi_\lambda \end{pmatrix} \right) \in D(\mathcal{A}),
\]
which is equivalent to the fixed point problem
\[
\left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi_\lambda \end{pmatrix} \right) = \lambda^{-1} \left( \lambda^{-1} I - \mathcal{A} \right)^{-1} \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \psi \end{pmatrix} \right) + \left( \lambda^{-1} I - \mathcal{A} \right)^{-1} \mathcal{F} \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi_\lambda \end{pmatrix} \right).
\]

Define the map $\Phi_\lambda \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi \end{pmatrix} \right) := \lambda^{-1} \left( \lambda^{-1} I - \mathcal{A} \right)^{-1} \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \psi \end{pmatrix} \right) + \left( \lambda^{-1} I - \mathcal{A} \right)^{-1} \mathcal{F} \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi \end{pmatrix} \right)$.

Then $r > 0$ being fixed, by using the fact that $\mathcal{F}$ is Lipschitz on bounded sets and $\mathcal{A}$ is a Hille-Yosida operator, one can prove that there exists $\eta = \eta(r) > 0$ such that
\[
\Phi_\lambda(B_{\psi,r}) \subset B_{\psi,r}, \forall \lambda \in (0, \eta)
\]
and $\Phi_\lambda$ is a strict contraction on $B_{\psi,r}$, where
\[
B_{\psi,r} := B \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \psi \end{pmatrix} \right), r
\]
is the ball with center $\left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \psi \end{pmatrix} \right)$ and radius $r$ in $\overline{D(\mathcal{A})} = \{0_{\mathcal{C}(\Omega)}\} \times \mathcal{BUC}_\alpha$. Thus by the Banach fixed point theorem, $\forall \lambda \in (0, \eta]$, there exists $\left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi_\lambda \end{pmatrix} \right) \in B_{\psi,r}$ satisfying
\[
\Phi_\lambda \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi_\lambda \end{pmatrix} \right) = \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi_\lambda \end{pmatrix} \right).
\]

Finally, since $\left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \psi \end{pmatrix} \right) \in \overline{D(\mathcal{A})}$ and by using (2.2) and (2.3), we have
\[
\lim_{\lambda \to 0^+} \left\| \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \varphi_\lambda \end{pmatrix} - \left( \begin{pmatrix} 0_{\mathcal{C}(\Omega)} \\ \psi \end{pmatrix} \right) \right\|_{\mathcal{X}} = 0.
\]
\[
\lim_{\lambda \to 0^+} \left\| (\lambda^{-1} (\lambda^{-1} I - \mathcal{A})^{-1} - I) \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} + (\lambda^{-1} I - \mathcal{A})^{-1} \mathcal{F} \begin{pmatrix} 0_{C(\Omega)} \\ \varphi_{\lambda} \end{pmatrix} \right\|_{X} = 0,
\]
which completes the proof. ■

3 Properties of the integral equation for \( \tau(t, x) \)

In this section we will make the following assumption.

**Assumption 3.1** Let \((\varphi, \tau_{0}) \in C((-\infty, 0], C(\Omega)) \times C_{+}(\Omega)\). Let \( A \in C((-\infty, r), C(\Omega)) \) (with \( r \in (0, +\infty] \)) be given and satisfy

\[ A(t, .) = \varphi(t, .), \forall t \leq 0. \]

**Lemma 3.2** There exists a uniquely determined map \( \tau : [0, r) \to C(\Omega) \) satisfying

\[
\int_{t-\tau(t, x)}^{t} f(A(s, .))(x) ds = \int_{-\tau_{0}(x)}^{0} f(\varphi(s, .))(x) ds, \forall t \in [0, r), \forall x \in \Omega. \tag{3.1}
\]

Moreover this uniquely determined map \( t \mapsto \tau(t, x) \) is continuously differentiable and satisfies the following equation

\[
\begin{cases}
\partial_{t} \tau(t, x) = 1 - \frac{f(A(t, .))(x)}{f(A(t - \tau(t, x), .))(x)}, \forall t \in [0, r), \forall x \in \Omega, \\
\tau(0, x) = \tau_{0}(x).
\end{cases} \tag{3.2}
\]

Conversely if \( t \mapsto \tau(t, x) \) is a \( C^{1} \) map satisfying the above ordinary differential equation (3.2), then it also satisfies the above integral equation (3.1).

**Remark 3.3** By using equation (3.2), it is easy to check that

\[ \tau_{0}(x) > 0 \Rightarrow \tau(t, x) > 0, \forall t \in [0, r), \forall x \in \Omega, \]

and

\[ \tau_{0}(x) = 0 \Rightarrow \tau(t, x) = 0, \forall t \in [0, r), \forall x \in \Omega. \]

**Proof.** Step 1 (Existence of \( \tau(t, x) \)): By Assumption 1.1, \( f \) is strictly positive, so fix \( t \in [0, r) \) and \( x \in \Omega \), and by considering the function \( \tau \mapsto \int_{t-\tau}^{t} f(A(s, .))(x) ds \) and observing that

\[
\begin{align*}
\int_{t-0}^{t} f(A(s, .))(x) ds &= 0 \leq \int_{-\tau_{0}(x)}^{0} f(\varphi(s, .))(x) ds, \\
\int_{t-(t+\tau_{0}(x))}^{t} f(A(s, .))(x) ds &\geq \int_{-\tau_{0}(x)}^{0} f(\varphi(s, .))(x) ds,
\end{align*}
\]

10
it follows by the intermediate value theorem that there exists a unique \( \tau(t, x) \in [0, t + \tau_0(x)] \) satisfying (3.1).

**Step 2 (The map \( t \mapsto t - \tau(t, x) \) is increasing):** First we prove that the function \( t \mapsto t - \tau(t, x) \) is increasing. Indeed, assume by contradiction that \( t_1 \leq t_2 \) while \( t_1 - \tau(t_1, x) \geq t_2 - \tau(t_2, x), \forall x \in \Omega \), namely we have

\[
t_2 - \tau(t_2, x) \leq t_1 - \tau(t_1, x) \leq t_1 \leq t_2.
\]

Then by (3.1) we have

\[
\int_{t_1 - \tau(t_1, x)}^{t_1} f(A(s, .))(x)ds = \int_{t_2 - \tau(t_2, x)}^{t_2} f(A(s, .))(x)ds
\]

\[
= \int_{t_1 - \tau(t_1, x)}^{t_2 - \tau(t_2, x)} f(A(s, .))(x)ds + \int_{t_1}^{t_1 - \tau(t_1, x)} f(A(s, .))(x)ds + \int_{t_1}^{t_2} f(A(s, .))(x)ds,
\]

thus

\[
\int_{t_2 - \tau(t_2, x)}^{t_1 - \tau(t_1, x)} f(A(s, .))(x)ds + \int_{t_1}^{t_2} f(A(s, .))(x)ds = 0,
\]

which is impossible since the function \( f \) is strictly positive.

**Step 3 (The continuity of the map \( x \mapsto \tau(t, x) \)):** Next we will prove the continuity of the map \( x \mapsto \tau(t, x) \). By step 2 we have for each \( t \in [0, r) \),

\[
t - \tau(t, x) \leq -\tau_0(x) \iff 0 \leq \tau(t, x) \leq t - \tau_0(x), \forall x \in \Omega.
\]

Now the boundedness of the function \( x \mapsto \tau(t, x) \) follows from the boundedness of \( \tau_0(x) \). Then for any \( t \in [0, r) \), we know that

\[
\tau^\infty(t) := \sup_{x \in \Omega} \tau(x, t) < +\infty.
\]

Let \( \xi(x) := \int_{-\tau_0(x)}^{0} f(\varphi(s, .))(x)ds, \forall x \in \Omega \). Then fix \( x_0 \in \Omega \), for any \( x \in \Omega \), we have

\[
|\xi(x_0) - \xi(x)| = \left| \int_{-\tau_0(x_0)}^{0} f(\varphi(s, .))(x_0)ds - \int_{-\tau_0(x)}^{0} f(\varphi(s, .))(x)ds \right|
\]

\[
\leq \left| \int_{-\tau_0(x_0)}^{-\tau_0(x)} f(\varphi(s, .))(x_0)ds \right| + \int_{-\tau_0(x)}^{0} |f(\varphi(s, .))(x_0) - f(\varphi(s, .))(x)|ds
\]

\[
\leq |\tau_0(x_0) - \tau_0(x)| f(m_1)(x_0) + \tau_0^\infty \sup_{s \in [-\tau_0^\infty, 0]} [f(\varphi(s, .))(x_0) - f(\varphi(s, .))(x)],
\]

where \( m_1 \) is the constant function defined with the constant value

\[
m_1(x) = \inf_{s \in [-\tau_0^\infty, 0]} \|\varphi(s, .)\|_{\infty}, \forall x \in \Omega
\]
and \(\tau_0^\infty := \sup_{x \in \Omega} \tau_0(x)\). Then the continuity of \(\xi(x)\) in \(x\) follows from
\[
\lim_{x \to x_0} |\xi(x_0) - \xi(x)| = 0
\]
by the continuity of \(\tau_0(x)\) in \(x\) and \(f(\varphi)\) in \(\varphi\). Now for each \(t \in [0, r]\) fixed, by (3.1) we have
\[
\xi(x_0) - \xi(x) = \int_{t-\tau(t,x_0)}^{t} f(A(s,))(x_0)ds - \int_{t-\tau(t,x)}^{t} f(A(s,))(x)ds
\]
\[
= \int_{t-\tau(t,x_0)}^{t-\tau(t,\gamma)} f(A(s,))(x_0)ds + \int_{t-\tau(t,\gamma)}^{t} (f(A(s,))(x_0) - f(A(s,))(x))ds,
\]
thus
\[
\int_{t-\tau(t,x_0)}^{t-\tau(t,\gamma)} f(A(s,))(x_0)ds \leq |\xi(x_0) - \xi(x)| + \tau^\infty(t) \sup_{s \in [t-\tau^\infty(t), t]} |f(A(s,))(x_0) - f(A(s,))(x)|.
\]
On the other hand, we have
\[
\int_{t-\tau(t,\gamma)}^{t} (f(A(s,))(x_0) - f(A(s,))(x))ds \leq |\tau(t, x_0) - \tau(t, x)| f(M_1)(x_0),
\]
where \(M_1\) is the constant function defined by
\[
M_1(x) := \sup_{s \in [-\tau_0^\infty, t]} \|A(s,)||_\infty, \forall x \in \Omega.
\]
Then there exists a constant \(\eta := \eta(t, x_0)\) such that
\[
|\tau(t, x_0) - \tau(t, x)| \leq \eta \left( |\xi(x_0) - \xi(x)| + \sup_{s \in [t-\tau^\infty(t), t]} |f(A(s,))(x_0) - f(A(s,))(x)| \right).
\]
Then the continuity of \(\tau(t, x)\) in \(x\) follows from
\[
\lim_{x \to x_0} |\tau(t, x_0) - \tau(t, x)| = 0
\]
by the continuity of \(\xi(x)\) in \(x\) and \(f(A)\) in \(A\).

**Step 4 (Differentiability of the map \(t \mapsto \tau(t, x)\))**: By applying the implicit function theorem to the map \(\psi : (0, r) \times C(\Omega) \to C(\Omega)\) defined by
\[
\psi(t, \gamma)(x) = \int_{\gamma(x)}^{t} f(A(s,))(x)ds - \int_{-\tau_0(x)}^{0} f(\varphi(s,))(x)ds
\]
(which is possible since \(\frac{\partial \psi(t, \gamma)}{\partial \gamma}(\hat{\gamma})(x) = -f(A(\gamma(x),))(x)\hat{\gamma}(x)\) and by Assumption 1.1, \(f\) is strictly positive), we deduce that \(t \mapsto t - \tau(t, x)\) is continuously
differentiable on $[0, r)$. Since the above formula of the derivative is also valid at $t = 0$ and $t = r$, the map $t \mapsto t - \tau(t, x)$ is continuously differentiable on $[0, r]$. By calculating the time derivative on both sides of (3.1), we get that $\tau(t, x)$ is a solution of (3.2).

**Step 5 (**(3.2)$\Rightarrow$(3.1)):** Conversely, assume that $\tau(t, x)$ is a solution of (3.2). Then

$$
\begin{align*}
f(A(t, \cdot))(x) &= \left(1 - \frac{\partial \tau(t, x)}{\partial t}\right) f(A(t - \tau(t, x), \cdot))(x), \forall t \in [0, r), \forall x \in \Omega.
\end{align*}
$$

Integrating both sides with respect to $t$, we have

$$
\int_0^t f(A(s, \cdot))(x)ds = \int_0^t f(A(s - \tau(s, x), \cdot))(x) \left(1 - \frac{\partial \tau(s, x)}{\partial s}\right) ds.
$$

Make the change of variable $l = s - \tau(s, x)$, we have $\forall t \in [0, r), \forall x \in \Omega$,

$$
\begin{align*}
\int_0^t f(A(s, \cdot))(x)ds &= \int_{l_0(x)}^{l(t, x)} f(A(l, \cdot))(x)dl \\
\Leftrightarrow\int_{l_0(x)}^{l(t, x)} f(A(s, \cdot))(x)ds + \int_0^{l(t, x) - l_0(x)} f(A(s, \cdot))(x)ds &= \int_{l_0(x)}^{l(t, x)} f(A(s, \cdot))(x)ds \\
\Leftrightarrow\int_{l_0(x)}^{l(t, x)} f(A(s, \cdot))(x)ds &= \int_{l_0(x)}^{l(t, x)} f(A(s, \cdot))(x)ds + \int_0^{l(t, x) - l_0(x)} f(A(s, \cdot))(x)ds \\
\Leftrightarrow\int_{l_0(x)}^{l(t, x)} f(A(s, \cdot))(x)ds &= \int_0^{l(t, x) - l_0(x)} f(\phi(\cdot))(x)ds,
\end{align*}
$$

so $\tau(t, x)$ also satisfies the equation (3.1). \hfill \blacksquare

In order to see that the delay $\tau$ is also a functional of $A_t$ and $(\varphi, \tau_0)$, we define the following functional. We define the map $\tilde{\tau}: D(\tilde{\tau}) \subset BUC_\alpha \times C(\Omega) \to C(\Omega)$ as the solution of

$$
\int_{-\tilde{\tau}(\phi, \delta)(x)}^0 f(\tilde{\phi}(s, \cdot))(x)ds = \delta(x) \tag{3.3}
$$

where

$$
\tilde{\phi}(s, x) := \begin{cases} 
\phi(s, x), & \text{if } s \leq 0 \\
\phi(0, x), & \text{if } s \geq 0.
\end{cases}
$$

Since by Assumption 1.1 the map $f(\phi(0, \cdot))(x) > 0$, then if $\delta(x) \leq 0$ we have $\tilde{\tau}(\phi, \delta)(x) \leq 0$ and

$$
\int_{-\tilde{\tau}(\phi, \delta)(x)}^0 f(\phi(0, \cdot))(x)ds = \delta(x) \Leftrightarrow \tilde{\tau}(\phi, \delta)(x) = \frac{\delta(x)}{f(\phi(0, \cdot))(x)}.
$$

We define the domain $D(\tilde{\tau})$ by

$$
D(\tilde{\tau}) = \left\{ (\phi, \delta) \in BUC_\alpha \times C(\Omega) : \delta(x) < \int_{-\infty}^0 f(\phi(s, \cdot))(x)ds \text{ if } \delta(x) > 0 \right\}.
$$

For clarity we prove the following lemma.
Lemma 3.4 For each \((\phi, \delta) \in D(\tilde{\tau})\) there exists \(\tilde{\tau}(\phi, \delta) \in C(\Omega)\).

Proof. Let \((\phi, \delta) \in D(\tilde{\tau})\) be fixed.

Step 1 (Existence of \(\tilde{\tau}(\phi, \delta)(x)\)): Let \(x \in \Omega\) be fixed. If \(\delta(x) \leq 0\), we have

\[ \tilde{\tau}(\phi, \delta)(x) = \frac{\delta(x)}{f(\phi(0,.))(x)}. \]

If \(\delta(x) > 0\), by the definition of the domain \(D(\tilde{\tau})\) we have

\[ \delta(x) < \int_{-\infty}^{0} f(\phi(s,.))(x)ds, \]

therefore by the intermediate value theorem, we can find \(\tilde{\tau}(\phi, \delta)(x) \in \mathbb{R}\) such that

\[ \delta(x) = \int_{-\tilde{\tau}(\phi, \delta)(x)}^{0} f(\phi(s,.))(x)ds. \]

Step 2 (Boundedness of \(\tilde{\tau}(\phi, \delta)(x)\)): Assume by contradiction that \(x \mapsto \tilde{\tau}(\phi, \delta)(x)\) is unbounded. Since \(\Omega\) is compact, we can find a converging sequence \(x_n \rightarrow x \in \Omega\) as \(n \rightarrow +\infty\) such that

\[ \lim_{n \rightarrow +\infty} \tilde{\tau}(\phi, \delta)(x_n) = +\infty. \]

It is sufficient to consider the case \(\delta(x_n) > 0\), since the case \(\delta(x_n) \leq 0\) is explicit. By the continuity of the function \(\delta\) we can assume that \(\delta(x) \geq 0\). By the definition of the domain \(D(\tilde{\tau})\) we have

\[ \delta(x) < \int_{-\infty}^{0} f(\phi(s,.))(x)ds. \]

So we can find a constant \(M > 0\) such that

\[ \delta(x) < \int_{-M}^{0} f(\phi(s,.))(x)ds, \]

and by continuity we can find a neighborhood \(U\) of \(x\) such that

\[ \delta(x) < \int_{-M}^{0} f(\phi(s,.))(x)ds, \forall x \in U. \]

It follows that for all integer \(n\) large enough,

\[ \tilde{\tau}(\phi, \delta)(x_n) \leq M \]

a contradiction.

Step 3 (Continuity of the map \(x \mapsto \tilde{\tau}(\phi, \delta)(x)\)): From the previous part, we know that

\[ \tilde{\tau}^\infty := \sup_{x \in \Omega} |\tilde{\tau}(\phi, \delta)(x)| < +\infty. \]
Fix \( x_0 \in \Omega \). If \( \delta(x_0) < 0 \) there is nothing to prove. Let us assume that
\[
\delta(x_0) \geq 0.
\]

Let \( x_n \to x_0 \) be a converging sequence. If \( \delta(x_0) = 0 \) and there exists a subsequence \( \{x_{n_k}\} \subset \{x_n\} \) satisfying \( \delta(x_{n_k}) < 0 \), \( \forall k \in \mathbb{N} \), then it is clear that \( \hat{\tau}(\phi, \delta)(x_{n_k}) \to \hat{\tau}(\phi, \delta)(x_0) = 0 \). Assume that \( \delta(x_n) \geq 0 \) for each integer \( n \geq 0 \). By (3.3) we have
\[
\delta(x_0) - \delta(x) = \int_{-\hat{\tau}(\phi, \delta)(x)}^{0} f(\phi(s,.))(x_0)ds - \int_{-\hat{\tau}(\phi, \delta)(x)}^{0} f(\phi(s,.))(x)ds
\]
\[
= \int_{-\hat{\tau}(\phi, \delta)(x_0)}^{-\hat{\tau}(\phi, \delta)(x)} f(\phi(s,.))(x_0)ds + \int_{-\hat{\tau}(\phi, \delta)(x)}^{0} f(\phi(s,.))(x_0)ds
\]
\[
- \int_{-\hat{\tau}(\phi, \delta)(x)}^{0} f(\phi(s,.))(x)ds.
\]

Assume without loss of generality that \( \hat{\tau}(\phi, \delta)(x_0) > \hat{\tau}(\phi, \delta)(x) \), then we have
\[
\int_{-\hat{\tau}(\phi, \delta)(x)}^{-\hat{\tau}(\phi, \delta)(x_0)} f(\phi(s,.))(x_0)ds
\]
\[
= (\delta(x_0) - \delta(x)) + \int_{-\hat{\tau}(\phi, \delta)(x)}^{0} (f(\phi(s,.))(x) - f(\phi(s,.))(x_0))ds
\]
\[
\leq |\delta(x_0) - \delta(x)| + \hat{\tau}(\phi, \delta) \sup_{s \in [-\hat{\tau}(\phi, \delta), 0)} |f(\phi(s,.))(x) - f(\phi(s,.))(x_0)|,
\]
and
\[
\int_{-\hat{\tau}(\phi, \delta)(x_0)}^{-\hat{\tau}(\phi, \delta)(x)} f(\phi(s,.))(x_0)ds \geq |\hat{\tau}(\phi, \delta)(x_0) - \hat{\tau}(\phi, \delta)(x)|f(M_1)(x_0),
\]

where \( M_1 \) is the constant function assigned with the single value \( \sup_{s \in [-\hat{\tau}(\phi, \delta), 0]} ||\phi(s,.)||_\infty \).

Thus there exists a constant \( \eta := \eta(x_0) \) such that
\[
|\hat{\tau}(\phi, \delta)(x_0) - \hat{\tau}(\phi, \delta)(x)| \leq \eta \left( |\delta(x_0) - \delta(x)| + \sup_{s \in [-\hat{\tau}(\phi, \delta), 0]} |f(\phi(s,.))(x) - f(\phi(s,.))(x_0)| \right).
\]

Then the result follows by the continuity of the functions \( \delta \) and \( f(\phi) \) in \( x \). 

**Lemma 3.5** Assume in addition that \( f \) is continuously differentiable. Then the domain \( D(\hat{\tau}) \) is an open subset of \( BUC_\alpha \times C(\Omega) \) and the map \( \hat{\tau} : D(\hat{\tau}) \subset BUC_\alpha \times C(\Omega) \to C(\Omega) \) is continuously differentiable.

**Proof.** Define the map \( \Gamma : BUC_\alpha \times C(\Omega) \times C(\Omega) \to C(\Omega) \) by
\[
\Gamma(\phi, \delta, \gamma)(x) := \int_{-\hat{\tau}(x)}^{0} f(\phi(s,.))(x)ds - \delta(x).
\]

...
Since by Assumption 1.1 the map $f$ is $C^1$, so is the map $\Gamma$. By (3.3), we have $\Gamma(\phi, \delta, \hat{\gamma}(\phi, \delta))(x) = 0$ and
\[
\partial_\gamma \Gamma(\phi, \delta, \gamma)(\hat{\gamma})(x) = f(\phi(-\gamma(x), \cdot)\hat{\gamma}(x).
\]
Since $f$ is strictly positive, it follows that $\partial_\gamma \Gamma(\phi, \delta, \gamma)$ is invertible. The result follows by applying the implicit function theorem.

Lemma 3.6 Set
\[\delta_0(x) := \int_{-\tau_0(x)}^0 f(\phi(s, .))(x)ds, \forall x \in \Omega.\]
Then we have the following equality
\[\hat{\tau}(A_t, \delta_0)(x) = \tau(t, x), \forall t \in (0, r),\]
where $A$ is given by Assumption 3.1 and $\tau(t, x)$ is the solution of (3.1).

Proof. It is sufficient to observe that $\forall t \in [0, r)$ and $x \in \Omega$,
\[
\int_{-\tau(A_t, \delta_0)(x)}^0 f(A_t(s, .))(x)ds = \delta_0(x) = \int_{-\tau_0(x)}^0 f(\phi(s, .))(x)ds
\]
\[= \int_{-\tau(t, x)}^0 f(A(t + s, .))(x)ds.\]

For simplicity, we will write $\tau(\phi, x)$ instead of $\hat{\tau}(\phi, \delta_0)(x)$ if the function $\delta_0$ is defined as in Lemma 3.6.

Lemma 3.7 Let $\phi, \hat{\phi} \in BUC_\alpha$. If $\phi \leq \hat{\phi}$, namely $\phi(s, x) \leq \hat{\phi}(s, x), \forall s \leq 0, x \in \Omega$, then $\tau(\phi, x) \leq \tau(\hat{\phi}, x), \forall x \in \Omega$.

Proof. By (3.3) we have
\[
\int_{-\tau(\phi, x)}^0 f(\phi(s, .))(x)ds = \delta_0(x) = \int_{-\tau(\hat{\phi}, x)}^0 f(\hat{\phi}(s, .))(x)ds.
\]
Assume by contradiction that there exists $x \in \Omega$ such that $\tau(\phi, x) > \tau(\hat{\phi}, x)$, then
\[
0 = \int_{-\tau(\phi, x)}^0 f(\phi(s, .))(x)ds - \int_{-\tau(\hat{\phi}, x)}^0 f(\hat{\phi}(s, .))(x)ds
\]
\[= \int_{-\tau(\phi, x)}^0 f(\phi(s, .))(x)ds + \int_{-\tau(\hat{\phi}, x)}^0 [f(\phi(s, .)) - f(\hat{\phi}(s, .)))(x)ds.\]
Since by assumption \( f(\phi(s,.))(x) > 0 \) and \( \tau(\phi, x) > \tau(\hat{\phi}, x) \), we have
\[
\int_{-\tau(\hat{\phi},x)}^{-\tau(\phi,x)} f(\phi(s,.))(x) ds > 0.
\]
By assumption we also have \( f(\phi(s,.))(x) \geq f(\hat{\phi}(s,.))(x) \) when \( \tau(\phi, x) > \tau(\hat{\phi}, x) \), then
\[
\int_{-\tau(\hat{\phi},x)}^{0} [f(\phi(s,.)) - f(\hat{\phi}(s,.))](x) ds \geq 0.
\]
Then we have
\[
0 = \int_{-\tau(\hat{\phi},x)}^{0} f(\phi(s,.))(x) ds + \int_{-\tau(\hat{\phi},x)}^{0} [f(\phi(s,.)) - f(\hat{\phi}(s,.))](x) ds > 0 + 0 = 0,
\]
which is a contradiction.

In the following lemma we obtain some a priori estimates for the delay.

**Lemma 3.8** Let Assumption 3.1 be satisfied. Assume that there exists a constant \( M > 0 \) such that
\[
\sup_{t \in [0, r]} \| A(t, .) \|_{\infty} \leq M. \tag{3.4}
\]
Then
\[
\tau_{\text{min}} \leq \tau(A_{t}, x) \leq \tau_{\text{max}}, \forall t \in [0, r), \forall x \in \Omega,
\]
where the constants \( \tau_{\text{min}} \) and \( \tau_{\text{max}} \) are defined as follows:
\[
0 \leq \tau_{\text{min}} := \inf_{x \in \Omega} \left[ \tau_{0}(x) f(\varphi_{\text{max}}(x)) \right] \leq \tau_{\text{max}} := \sup_{x \in \Omega} \left[ \tau_{0}(x) f(-\varphi_{\text{max}}(x)) \right]
\]
with \( M_{1} \) and \( \varphi_{\text{max}} \) being defined as constant functions as follows
\[
M_{1}(x) := \max\{M, \varphi_{\text{max}}\} \quad \text{and} \quad \varphi_{\text{max}}(x) := \sup_{t \in [-\tau_{\infty}, 0]} \| \varphi(t, .) \|_{\infty}, \forall x \in \Omega \tag{3.5}
\]
and
\[
\tau_{\infty} := \sup_{x \in \Omega} \tau_{0}(x).
\]

**Proof.** For any \( x \in \Omega \) and \( t \in [0, r) \), since by Assumption 1.1, the map \( f \) is decreasing, it follows that
\[
\int_{-\tau_{0}(x)}^{0} f(\varphi(s,.))(x) ds = \int_{-\tau(A_{t},x)}^{0} f(A(t + s,.))(x) ds
\]
\[
= \int_{-\tau(A_{t},x)}^{t} f(A(l,.))(x) dl
\]
\[
\leq \int_{-\tau(A_{t},x)}^{t} f(-M_{1})(x) ds = \tau(A_{t}, x) \sup_{x \in \Omega} f(-M_{1})(x).
\]
Then $\forall x \in \Omega$, $\forall t \in [0, r)$,

$$
\tau(A_t, x) \geq \int_{-\tau_0(x)}^{0} f(\varphi(s,.))(x) ds \geq \inf_{x \in \Omega} \left[ \tau_0(x) f(\varphi_{\max})(x) \right] \sup_{x \in \Omega} f(-M_1)(x)
$$

The derivation of the estimation from above for $\tau(t, x)$ is similar.

**Lemma 3.9** Let $(\varphi, \tau_0), (\tilde{\varphi}, \tilde{\tau}_0), A, \tilde{A}$ satisfy Assumption 3.1. Assume that

$$
M := \max \left\{ \sup_{t \in [0, r)} \|A(t, .)\|_{\infty}, \sup_{t \in [0, r)} \|\tilde{A}(t, .)\|_{\infty} \right\} < +\infty.
$$

Then there exists a constant $L_r > 0$ such that $\forall t \in [0, r)$, $\forall x \in \Omega$,

$$
|\check{T}(A_t, \delta_0)(x) - \check{T}(\tilde{A}_t, \tilde{\delta}_0)(x)| \leq L_r \left[ \sup_{s \in [-\tau_0^r, r)} \|A(s, .) - \tilde{A}(s, .)\|_{\infty} + \|\delta_0 - \tilde{\delta}_0\|_{\infty} \right],
$$

where $\delta_0$ and $\tilde{\delta}_0$ are defined as in Lemma 3.6 respectively with $(\varphi, \tau_0)$ and $(\tilde{\varphi}, \tilde{\tau}_0)$ and

$$
\tilde{\tau}_0^\infty := \max \left\{ \sup_{x \in \Omega} \tau_0(x), \sup_{x \in \Omega} \tilde{\tau}_0(x) \right\}.
$$

**Proof.** Let $t \in [0, r)$ and $x \in \Omega$. Recall from Lemma 3.6 that

$$
\delta_0(x) = \int_{-\tau_0(x)}^{0} f(\varphi(s,.))(x) ds \quad \text{and} \quad \tilde{\delta}_0(x) = \int_{-\tilde{\tau}_0(x)}^{0} f(\tilde{\varphi}(s,.))(x) ds, \forall x \in \Omega.
$$

Without loss of generality we may assume that $\check{T}(A_t, \delta_0)(x) \geq \check{T}(\tilde{A}_t, \tilde{\delta}_0)(x) > 0$.

Then we have

$$
\delta_0(x) - \tilde{\delta}_0(x) = \int_{-\tau_0(x)}^{0} f(\varphi(s,.))(x) ds - \int_{-\tilde{\tau}_0(x)}^{0} f(\tilde{\varphi}(s,.))(x) ds
$$

$$
= \int_{-\check{T}(A_t, \delta_0)(x)}^{t} f(A(s,.))(x) ds - \int_{-\check{T}(\tilde{A}_t, \tilde{\delta}_0)(x)}^{t} f(\tilde{A}(s,.))(x) ds
$$

$$
= \int_{-\check{T}(A_t, \delta_0)(x)}^{t} f(A(s,.))(x) ds + \int_{-\check{T}(A_t, \delta_0)(x)}^{t} \left[ f(A(s,.))(x) - f(\tilde{A}(s,.))(x) \right] ds.
$$

Since by Assumption 1.1, $f$ is Lipschitz continuous, then

$$
\int_{-\check{T}(A_t, \delta_0)(x)}^{t} f(A(s,.))(x) ds
$$

$$
= \int_{-\check{T}(A_t, \delta_0)(x)}^{t} \left[ f(A(s,.))(x) - f(A(s,.))(x) \right] ds + (\delta_0(x) - \tilde{\delta}_0(x))
$$

18
4 Existence and uniqueness of solutions

We start this section with two technical lemmas.

Lemma 4.1 Let \( a < b \) be two real numbers. Let \( \chi \in \text{Lip}([a,b], C(\Omega)) \). Then for each \( c \in (a,b) \) we have the following estimation

\[
\| \chi \|_{\text{Lip}([a,b], C(\Omega))} \leq \| \chi \|_{\text{Lip}([a,c], C(\Omega))} + \| \chi \|_{\text{Lip}([c,b], C(\Omega))}.
\]

where

\[
\| \chi \|_{\text{Lip}(t,C(\Omega))} := \sup_{t, s \in I : t \neq s} \| \chi(t, .) - \chi(s, .) \|_{\infty} / |t - s|.
\]

Proof. Let \( t, s \in [a, b] \) with \( t > s \). Define the function \( \rho : [0, 1] \to \mathbb{R} \) by

\[
\rho(h) := \| \chi((t-s)h + s, .) - \chi(s, .) \|_{\infty}, \forall h \in [0, 1].
\]

Then we have \( \forall h, \hat{h} \in [0, 1], \)

\[
|\rho(h) - \rho(\hat{h})| \leq \| \chi((t-s)h + s, .) - \chi(s, .) \|_{\infty} - \| \chi((t-s)\hat{h} + s, .) - \chi(s, .) \|_{\infty}.
\]

\[
\leq \| \chi((t-s)h + s, .) - \chi((t-s)\hat{h} + s, .) \|_{\infty}.
\]

\[
\leq \| \chi \|_{\text{Lip}([a,b], C(\Omega))} |t-s| |h - \hat{h}|,
\]

thus \( \rho \) is Lipschitz continuous. Denote

\[
\text{Lip}(\rho)(h) := \limsup_{\varepsilon \to 0^+} \frac{\rho(h + \varepsilon) - \rho(h)}{\varepsilon},
\]

then

\[
\text{Lip}(\rho)(h) \leq \begin{cases} \\
\| \chi \|_{\text{Lip}([a,c], C(\Omega))} |t-s|, & \text{if } (t-s)h + s \in [a,c], \\
\| \chi \|_{\text{Lip}([c,b], C(\Omega))} |t-s|, & \text{if } (t-s)\hat{h} + s \in [c,b].
\end{cases}
\]
Since $\rho$ is Lipschitz continuous, by using Theorem 8.17 in page 158 of Rudin \[11\], we deduce that $\rho$ is differentiable everywhere on a subset of the form $[0, 1] \setminus N$ (where $N$ has null Lebesgue measure) and

$$\rho(t) = \rho(0) + \int_0^t \rho'(l)dl, \forall t \in [0, 1].$$

By using the definition of $\text{Lip}(\rho)(t)$ we deduce that

$$\rho'(t) \leq \text{Lip}(\rho)(t) \leq C, \forall t \in [0, 1] \setminus N,$$

where $C := \max \{ \|A\|_{\text{Lip}([a,c], C(\Omega))} + \|A\|_{\text{Lip}([c,b], C(\Omega))} \}$. Therefore we obtain

$$\|\chi(t,.) - \chi(s,.)\|_{\infty} = \rho(1) - \rho(0) = \int_0^1 \rho'(l)dl \leq \int_0^1 Cdl$$

which completes the proof.

**Lemma 4.2** Let $t \geq 0$. Assume that $A \in C((-\infty, t], C(\Omega))$ and $A_0 = \phi$. Define for each $(\theta, x) \in (-\infty, t] \times \Omega$,

$$A_{t,\alpha}(\theta, x) := e^{-\alpha|\theta|}A_t(\theta, x) \text{ and } \varphi_{\alpha}(\theta, x) := e^{-\alpha|\theta|}\phi(\theta, x).$$

Then we have the following estimations

$$\|A_{t,\alpha}\|_{\infty} \leq \sup_{\theta \in [0, t]} \|e^{\alpha(\theta-t)}A(\theta, .)\|_{\infty} + e^{-\alpha t}\|\varphi_{\alpha}\|_{\infty}, \quad (4.1)$$

$$\|A_{t,\alpha}\|_{\text{Lip}((-\infty,0], C(\Omega))} \leq \|A_{t,\alpha}\|_{\text{Lip}([-t,0], C(\Omega))} + e^{-\alpha t}\|\varphi_{\alpha}\|_{\text{Lip}((-\infty,0], C(\Omega))} \quad (4.2)$$

and

$$\|A_t\|_{\text{Lip}_{\alpha}} \leq \sup_{\theta \in [0,t]} \|e^{\alpha(\theta-t)}A(\theta, .)\|_{\infty} + \|A_{t,\alpha}\|_{\text{Lip}([-t,0], C(\Omega))} + e^{-\alpha t}\|\varphi_{\alpha}\|_{\text{Lip}_{\alpha}}. \quad (4.3)$$

**Proof.** We have for the supremum norm

$$\|A_{t,\alpha}\|_{\infty} = \sup_{\theta \leq 0} \|e^{\alpha\theta}A(t + \theta, .)\|_{\infty} = e^{-\alpha t} \sup_{\theta \leq 0} \|e^{\alpha(t+\theta)}A(t + \theta, .)\|_{\infty} \leq \sup_{s \in [0,t]} \|e^{\alpha(s-t)}A(s, .)\|_{\infty} + e^{-\alpha t}\|\varphi_{\alpha}\|_{\infty}.$$

The result follows by using similar arguments combined with Lemma 4.1 for the Lipschitz semi-norm.

**Lemma 4.3 (Uniqueness of solutions)** Let $\varphi \in \text{Lip}_{\alpha}$ and $\tau_0 \in C_+(\Omega)$ satisfy

$$\|\varphi\|_{\text{Lip}_{\alpha}} + \|\tau_0\|_{\infty} \leq M_0,$$

where $M_0 > 0$ is a given real number. Let $r \in (0, +\infty)$ be given. Then the equation (1.1) admits at most one solution $(A, \tau) \in C((-\infty, r], C(\Omega)) \times C([0, r], C(\Omega))$. 

20
Proof. Suppose that there exist \((A^1, \tau^1), (A^2, \tau^2) \in C((−∞, r], C(Ω)) \times C([0, r], C(Ω))\) two solutions of (1.1) on \((−∞, r]\) with

\[
(A^1_0, \tau^1(0, .)) = (A^2_0, \tau^2(0, .)) = (\varphi, t_0).
\]

Define

\[
t_0 = \sup \left\{ t \in [0, r] : A^1(s, x) = A^2(s, x), \tau^1(s, x) = \tau^2(s, x), \forall s \in [0, t], \forall x \in Ω \right\}.
\]

Assume that \(t_0 < r\). We first observe that since \(r\) is finite, we have

\[
\tilde{K}_0 := \sup_{s \in [0, r]} \|A^1(s, .)\|_∞ + \sup_{s \in [0, r]} \|A^2(s, .)\|_∞ < +∞.
\]

By Lemma 3.8, \(\tau^1(t, x)\) and \(\tau^2(t, x)\) are also bounded from above (by \(\tau_{max}^1\) and \(\tau_{max}^2\) respectively) on \(t \in [0, r]\). Since by Assumption 1.1, \(F : C(Ω)^2 \times C(Ω^2) \rightarrow C(Ω)\) is Lipschitz on bounded sets and for each \(i = 1, 2\), each \(t \in [0, r]\) and each \(x \in Ω\),

\[
A^i(t, x) = \varphi(0, x) + \int_0^t F(A^i(l, .), \tau^i(l, .), A^i(l - \tau^i(l))(.))(x) dl,
\]

it follows by using Lemma 3.8 that for each \(i = 1, 2\),

\[
\sup_{s \in [0, r]} \|F(A^i(s, .), \tau^i(s, .), A^i(s - \tau^i(s))(.))\|_∞ < +∞,
\]

and

\[
K_L := \|A^1\|_{Lip([0, r], C(Ω))} + \|A^2\|_{Lip([0, r], C(Ω))} < +∞.
\]

Set

\[
K_0 := 2(\tilde{K}_0 + \varphi_{max}) + \tau_{max}^1 + \tau_{max}^2,
\]

where \(\varphi_{max}\) is defined in (3.5). For each \(t \in [t_0, r]\) and each \(x \in Ω\) we have

\[
A^1(t, x) - A^2(t, x) = \int_0^t \left[ F(A^1(l, .), \tau^1(l, .), A^1(l - \tau^1(l))(.))(x)
- F(A^2(l, .), \tau^2(l, .), A^2(l - \tau^2(l))(.))(x) \right] dl,
\]

thus by using the fact that \(F\) is Lipschitz on bounded sets, we obtain

\[
\|A^1(t, .) - A^2(t, .)\|_∞ \leq (t - t_0) L(K_0) \sup_{s \in [t_0, t]} \|A^1(s, .) - A^2(s, .)\|_∞
+ \|A^1(s - \tau^1(s))(. .) - A^2(s - \tau^2(s))(. .)\|_∞
+ \|\tau^1(s, .) - \tau^2(s, .)\|_∞.
\]

Define

\[
\|A^1_t - A^2_t\|_∞ := \sup_{s \leq 0} \|A^1(t + s, .) - A^2(t + s, .)\|_∞.
\]

21
Thus

\[ \|\tau^1(t,\cdot) - \tau^2(t,\cdot)\|_\infty = \|\tau(A^1_{s,\cdot}) - \tau(A^2_{s,\cdot})\|_\infty \leq L_\tau \|A^1_t - A^2_t\|_\infty \leq L_\tau \|A^1_t - A^2_t\|_\infty. \]

By Lemma 3.9, for each \( s \in [t_0, t] \) we have

\[ \|\tau^1(s,\cdot) - \tau^2(s,\cdot)\|_\infty = \|\tau(A^1_{s,\cdot}) - \tau(A^2_{s,\cdot})\|_\infty \leq L_\tau \|A^1_t - A^2_t\|_\infty \leq L_\tau \|A^1_t - A^2_t\|_\infty. \]

Thus

\[ \|A^1(s - \tau^1(s)) - A^2(s - \tau^2(s))\|_\infty \]
\[ \leq \|A^1(s - \tau^1(s)) - A^1(s - \tau^2(s))\|_\infty + \|A^1(s - \tau^2(s)) - A^2(s - \tau^2(s))\|_\infty \]
\[ \leq K_L \|\tau^1(s,\cdot) - \tau^2(s,\cdot)\|_\infty + \|A^1_t - A^2_t\|_\infty, \]

hence

\[ \|A^1(s - \tau^1(s)) - A^2(s - \tau^2(s))\|_\infty \leq (K_LL_\tau + 1)\|A^1_t - A^2_t\|_\infty. \]

So we obtain for each \( t \in [t_0, r], \)

\[ \|A^1_t - A^2_t\|_\infty \leq (t - t_0)L(K_0)((K_L + 1)L_\tau + 2)\|A^1_t - A^2_t\|_\infty. \]

It follows that we can find \( \varepsilon \in (0, r - t_0) \) such that

\[ \|A^1_t - A^2_t\|_\infty = 0, \forall t \in [t_0, t_0 + \varepsilon], \]

which contradicts with the definition of \( t_0. \) Thus \( t_0 = r. \)

\[ \text{Theorem 4.4 (Local existence of solutions)} \]

Let \( M_0 > 0 \) be fixed. Then for \( M > M_0, \) there exists a time \( r = r(M_0, M) > 0 \) such that for each \( (\varphi, \tau_0) \in \text{Lip}_\alpha \times C_+(\Omega) \)
satisfying

\[ \|\varphi\|_{\text{Lip}_\alpha} + \|\tau_0\|_\infty \leq M_0, \]

system (1.1) admits a unique solution \( (A, \tau) \in C((-\infty, r], C(\Omega)) \times C([0, r], C(\Omega)). \)

Moreover,

\[ \|A(t, \cdot)\|_\infty \leq M, \forall t \in [0, r]. \]

Proof. Step 1 (Fixed point problem): We start by defining the fixed point problem. Let \( M_0 > 0 \) be fixed. Let \( \varphi \in \text{Lip}_\alpha \) and \( \tau_0 \in C_+(\Omega) \) satisfy

\[ \|\varphi\|_{\text{Lip}_\alpha} + \|\tau_0\|_\infty \leq M_0. \]

Let \( M > M_0 \) be fixed. Define

\[ E_\varphi := \{ A \in C((-\infty, r], C(\Omega)) : A_0 = \varphi \text{ and } \sup_{t \in [0, r]} \|A(t, \cdot)\|_\infty \leq M \} \]

where \( r \) will be determined later on.

Let \( \Phi : E_\varphi \to C((-\infty, r], C(\Omega)) \) be the map defined as follows: for each \( t \in [0, r] \) and \( x \in \Omega, \)

\[ \Phi(A)(t)(x) := \varphi(0, x) + \int_0^t F(A(l, \cdot), \tau(A_l, \cdot), A(l - \tau(A_l, \cdot), \cdot))(x)dl \quad (4.4) \]

22
where $\tau(A_t, x)$ is the unique solution of the integral equation (3.1), and for $t \leq 0$ and $x \in \Omega$,
\[ \Phi(A)(t)(x) := \varphi(t, x). \]

Set
\[ \tilde{M} := \max\{M, \varphi_{\max}\} \tag{4.5} \]
where $\varphi_{\max}$ is defined in (3.5). For any $A \in E_{\varphi}$ and $t \in [0, r]$, we have
\[ \|\Phi(A)(t)\|_{\infty} \leq \|\varphi(0, .)\|_{\infty} + \int_{0}^{t} \|F(A(l, .), \tau(A_l, .), A(l - \tau(A_l, .), .))\|_{\infty} dl \]
\[ \leq M_0 + \int_{0}^{t} \|F(A(l, .), \tau(A_l, .), A(l - \tau(A_l, .), .)) - F(0, 0, 0)\|_{\infty} dl \]
\[ + \int_{0}^{t} \|F(0, 0, 0)\|_{\infty} dl \]
\[ \leq M_0 + r[(2\tilde{M} + \tau_{\max})L(2\tilde{M} + \tau_{\max}) + \|F(0, 0, 0)\|_{\infty}]. \]
Since $M > M_0$ we can find $r_1 > 0$ such that for each $r \in (0, r_1]$, $M_0 + r[(2\tilde{M} + \tau_{\max})L(2\tilde{M} + \tau_{\max}) + \|F(0, 0, 0)\|_{\infty}] \leq M$, and it follows that $\Phi(E_{\varphi}) \subset E_{\varphi}$.

Step 2 (Lipschitz estimation): Set
\[ M_L := \frac{M - M_0}{r} \geq (2\tilde{M} + \tau_{\max})L(2\tilde{M} + \tau_{\max}) + \|F(0, 0, 0)\|_{\infty}. \tag{4.6} \]
For each $t, s \in [0, r]$ with $t \geq s$ and $A \in E_{\varphi}$ we have
\[ \frac{\|\Phi(A)(t) - \Phi(A)(s)\|_{\infty}}{|t - s|} \leq \int_{s}^{t} \frac{\|F(A(l, .), \tau(A_l, .), A(l - \tau(A_l, .), .))\|_{\infty} dl}{|t - s|} \]
\[ \leq \int_{s}^{t} \frac{\|F(A(l, .), \tau(A_l, .), A(l - \tau(A_l, .), .)) - F(0, 0, 0)\|_{\infty} dl}{|t - s|} + \int_{s}^{t} \frac{\|F(0, 0, 0)\|_{\infty} dl}{|t - s|}, \]
thus
\[ \|\Phi(A)\|_{\text{Lip}([0, r], C(\Omega))} \leq M_L, \forall A \in E_{\varphi}. \]

Step 3 (Iteration procedure): Consider the sequence $\{A^n\}_{n \in \mathbb{N}} \subset E_{\varphi}$ defined by iteration as follows: for each $(t, x) \in (-\infty, r] \times \Omega$,
\[ A^0(t, x) = \begin{cases} \varphi(0, x), & \text{if } t \in [0, r], \\ \varphi(t, x), & \text{if } t \leq 0, \end{cases} \]
and for each integer $n \geq 0$,
\[ A^{n+1}(t, x) = \begin{cases} \Phi(A^n)(t)(x), & \text{if } t \in [0, r], \\ \varphi(t, x), & \text{if } t \leq 0. \end{cases} \]
From the step 2 and the definition of $A^0$, we know that for each integer $n \geq 0$, \[ A^n \in \text{Lip}([-\tau_0^n, r], C(\Omega)). \]
and
\[ \|A^n\|_{\text{Lip}([-\tau_0^n, r], C(\Omega))} \leq \max\{ML, \|\varphi\|_{\text{Lip}([-\tau_0^n, 0], C(\Omega))}\} =: \tilde{M}_L. \]
For each integer $n, p \geq 0$, the maps $A^n$ and $A^p$ coincide for negative time $t$, therefore we can define
\[ \|A^n - A^p\|_{\infty} := \sup_{t \leq r} \|A^n(t, \cdot) - A^p(t, \cdot)\|_{\infty}. \]
Next, we have $\forall n \in \mathbb{N}$,
\[ \|A^{n+1} - A^n\|_{\infty} = \sup_{t \in [0, r]} \|\Phi(A^n)(t, \cdot) - \Phi(A^{n-1})(t, \cdot)\|_{\infty} \]
\[ \leq \int_0^r \|F(A^n(l, \cdot), \tau(A^n_l, \cdot), A^n(l - \tau(A^n_l, \cdot))) - F(A^{n-1}(l, \cdot), \tau(A^{n-1}_{l-1}, \cdot), A^{n-1}(l - \tau(A^{n-1}_{l-1}, \cdot), \cdot))\|_{\infty} dl \]
\[ \leq rL(\tilde{M} + \tau_{\max}) \left[ \|A^n - A^{n-1}\|_{\infty} + \sup_{t \in [0, r]} \|\tau(A^n_{t, l}, \cdot) - \tau(A^{n-1}_{t, l}, \cdot)\|_{\infty} \right], \]
\[ \leq rL(\tilde{M} + \tau_{\max}) \left[ \|A^n - A^{n-1}\|_{\infty} + (1 + \|A^n\|_{\text{Lip}([-\tau_0^n, r], C(\Omega))}) \cdot \sup_{t \in [0, r]} \|\tau(A^n_{t, l}, \cdot) - \tau(A^{n-1}_{t, l}, \cdot)\|_{\infty} \right]. \]
By Lemma 3.9, for each integer $n \geq 1$ we obtain
\[ \|A^{n+1} - A^n\|_{\infty} \leq \tau C \|A^n - A^{n-1}\|_{\infty} \]
with $C := \left(2 + L_r(1 + \tilde{M}_L)\right) L(\tilde{M} + \tau_{\max})$.
Now we can find $r_2 \in (0, r_1]$ such that $r_2 C < 1/2$, and for each $r \in (0, r_2]$ we have
\[ \|A^{n+1} - A^n\|_{\infty} \leq \frac{1}{2^n} \|A^1 - A^0\|_{\infty}, \forall n \geq 1. \]
It follows that \{ $A^n\}_{[0, r]}$ is a Cauchy sequence in the space $C([0, r], C(\Omega))$ and \{ $A^n$ \} coincides with $\varphi$ for negative $t$. Define
\[ A(t, x) := \begin{cases} \lim_{n \to +\infty} A^n(t, x), & \text{if } t \in [0, r], x \in \Omega, \\ \varphi(t, x), & \text{if } t \leq 0, x \in \Omega. \end{cases} \]
Then we have
\[ \lim_{n \to +\infty} \|A^n - A\|_{\infty} = 0. \]
By using again Lemma 3.9, we have
\[ \sup_{t \in [0,r]} \| \tau(A^n_t, \cdot) - \tau(A_t, \cdot) \|_\infty \leq L \| A^n - A \|_\infty, \forall n \geq 1. \]

Finally by taking the limit on both sides of the equation
\[ A^{n+1}(t, x) := \varphi(0, x) + \int_0^t F(A^n(l, \cdot), \tau(A^n_l, \cdot), A^n(l - \tau(A^n_l, \cdot), \cdot))(x) dl, \]
we deduce that the couple \((A, \tau(A_t, \cdot))\) is a solution of equation (1.1).

**Step 4: (Estimation of the solution)** We observe that
\[ \| A^{n+1} - A^0 \|_\infty \leq \| A^{n+1} - A^n \|_\infty + \| A^n - A^{n-1} \|_\infty + \ldots + \| A^1 - A^0 \|_\infty \leq \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k \| A^1 - A^0 \|_\infty, \]
therefore
\[ \| A^{n+1} - A^0 \|_\infty \leq 2 \| A^1 - A^0 \|_\infty, \]
and by taking the limit when \( n \) goes to infinity we obtain
\[ \| A - A^0 \|_\infty \leq 2 \| A^1 - A^0 \|_\infty. \]

From the definition of \( A^1 \) and \( A^0 \) we have
\[ \| A^1 - A^0 \|_\infty \leq \int_0^r \| F(A^0(l, \cdot), \tau(A^0 l, \cdot), A^0(l - \tau(A^0 l, \cdot), \cdot)) \|_\infty dl, \]
so
\[ \| A^1 - A^0 \|_\infty \leq r C_1, \]
where \( C_1 = \sup_{t \in [0,r]} \| F(A^0(l, \cdot), \tau(A^0 l, \cdot), A^0(l - \tau(A^0 l, \cdot), \cdot)) \|_\infty. \) It follows that for each \( t \in [0,r], \)
\[ \| A(t, \cdot) \|_\infty \leq \| A(t, \cdot) - \varphi(0, \cdot) \|_\infty + \| \varphi(0, \cdot) \|_\infty \leq \| A - A^0 \|_\infty + M_0, \]
thus
\[ \| A(t, \cdot) \|_\infty \leq M_0 + 2r C_1, \]
and by choosing \( r \) small enough we obtain \( M_0 + 2r C_1 \leq M. \) The proof is completed.

From step 2 in the above proof combined with Lemma 4.2, we have the following corollary.

**Corollary 4.5** With the same notation as in Theorem 4.4, we have
\[ A_t \in \text{Lip}_\alpha, \forall t \in [0, r], \]
and there exists \( \hat{M} := \hat{M}(M, \tau_{\text{max}}, \alpha) > M \) such that
\[ \| A_t \|_{\text{Lip}_\alpha} \leq \hat{M}, \forall t \in [0, r]. \]
5 Properties of the semiflow

For each initial distribution $W_0 = \left(\phi \tau_0\right) \in \text{Lip}_\alpha \times C_+(\Omega)$, define

$$T_{BU}(W_0) = \sup\{t > 0 : \text{there exists a solution of (3.1) on the interval } [0,t] \text{ with the initial distribution } W_0\},$$

Observe by Theorem 4.4 that we must have $T_{BU}(W_0) > 0$.

In this section we investigate the semiflow properties of the map $W : D(W) \subset [0, +\infty) \times \text{Lip}_\alpha \times C(\Omega) \to \text{Lip}_\alpha \times C(\Omega)$ defined for each initial distribution $W_0$ as

$$W(t, W_0)(x) = \left(\left(A_t(., x) \tau(t, x)\right)\right),$$

where $\{(A_t, \tau(t, .))\}_{t \in [0, T_{BU}(W_0))]$ is the solution of (1.1) with initial distribution $W_0$ which can be defined up to the maximal time of existence $T_{BU}(W_0)$. The domain is

$$D(W) = \bigcup_{W_0 \in \text{Lip}_\alpha \times C(\Omega)} [0, T_{BU}(W_0)) \times \{W_0\}.$$

Proof. (First part of Theorem 1.6) Suppose that $W_0 \in \text{Lip}_\alpha \times C(\Omega)$ satisfies $\|W_0\|_{\text{Lip}_\alpha \times C(\Omega)} \leq M_0$, where $M_0$ is a positive constant, and define $U$ by

$$U(t)W_0(x) := \left(\left(A_t(., x) \tau(t, x)\right)\right),$$

where the map $t \mapsto (A_t, \tau(t, .))$ belongs to $C([0, T_{BU}(W_0)]), \text{Lip}_\alpha \times C(\Omega)$ is the maximal solution of (1.1) with the initial value $W_0$. In this proof, we will verify that $U$ satisfies the properties (i)-(iv) of Definition 1.5.

The property (ii) of Definition 1.5 is trivially satisfied, since by construction $U(0)W_0 = W_0$.

Step 1 ((i) and (iii) of Definition 1.5): By Lemma 3.2, the map $t \mapsto (A(t, .), \tau(t, .))$ is the solution of (1.1) if and only if for each $t \in [0, T_{BU}(W_0))$ and each $x \in \Omega$ the equations

$$\begin{cases}
A(t, x) = \varphi(0, x) + \int_0^t F(A(l, .), \tau(l, .), A(l - \tau(l))(., .))(x)dl, \\
\tau(t, x) = \tau_0(x) + \int_0^t \left[1 - \frac{f(A(l, .))(x)}{f(A(l - \tau(l))(., .))(x)}\right]dl,
\end{cases} \quad (5.1)$$

are satisfied together with the initial condition

$$(A_0, \tau(0, .)) = W_0.$$
and for each \( t \in [0, T_{BU}(U(s)W_0)) \),

\[
U(t + s)W_0 = U(t)U(s)W_0. \tag{5.2}
\]

For each \( t \in [0, T_{BU}(U(s)W_0)) \),

\[
U(t)\mathcal{U}(s)(W_0) = U(t) \left( \begin{array}{c} A_s \\ \tau(s, .) \end{array} \right) = \left( \begin{array}{c} \tilde{A}_t \\ \tilde{\tau}(t) \end{array} \right)
\]

where \( t \mapsto (A(t, .), \tau(t, .)) \) is the solution of (1.1) with initial condition \( W_0 \) and \( t \mapsto (\tilde{A}(t, .), \tilde{\tau}(t, .)) \) is the solution of (1.1) with initial condition \( U(s)(W_0) \). Then we have for each \( t \in [0, T_{BU}(U(s)W_0)) \),

\[
\left\{ \begin{array}{l}
A(t, x) = A(s, x) + \int_0^t F(A(l, .), \tilde{\tau}(l, .), \tilde{A}(l) - \tilde{\tau}(l)(\cdot))(x)dl \\
\tilde{\tau}(t, x) = \tau(s, x) + \int_0^t \left[ 1 - \frac{f(A(l, .))(x)}{f(A(l - \tilde{\tau}(l)(\cdot))(\cdot))(x)} \right] dl 
\end{array} \right. \tag{5.3}
\]

with initial condition

\[
\tilde{A}_0 = A_s, \tilde{\tau}(0, x) = \tau(s, x).
\]

Now by setting

\[
(A_t, \tilde{\tau}(t, .)) := \left\{ \begin{array}{ll}
(A_{t-s}, \tilde{\tau}(t-s, .)), & \text{if } t \in [s, s + T_{BU}(U(s)W_0)), \\
(A_t, \tau(t, .)), & \text{if } t \in [0, s],
\end{array} \right.
\]

then by using (5.1) and (5.3) we deduce that \( t \mapsto (\tilde{A}(t, .), \tilde{\tau}(t, .)) \) is a solution of (1.1) on the time interval \([0, s + T_{BU}(U(s)W_0))\) with initial condition \( W_0 \). It follows that

\[
s + T_{BU}(U(s)W_0) \leq T_{BU}(W_0).
\]

Now assume that \( t \mapsto (A(t, .), \tau(t, .)) \) is a solution of (1.1) on the time interval \([0, T_{BU}(W_0))\) with initial condition \( W_0 \). Let \( s \in [0, T_{BU}(W_0)) \). Then by using (5.1) it follows that \( t \mapsto (A(t + s, .), \tau(t + s, .)) \) defined on \([0, T_{BU}(W_0) - s]\) is a solution of (1.1) with initial condition \( U(s)W_0 \). It follows that

\[
T_{BU}(W_0) - s \leq T_{BU}(U(s)W_0)
\]

and the properties (i) and (iii) of Definition 1.5 follow.

Step 2 (iv) of Definition 1.5: Assume that \( T_{BU}(W_0) < +\infty \). Suppose that \( \|U(t)W_0\|_{\text{Lip}_x \times C(\Omega)} \) does not go to \( +\infty \) when \( t \not\geq T_{BU}(W_0) \). Then there exists a constant \( M_0 > 0 \) and a sequence \( \{t_n\} \subset [0, T_{BU}(W_0)) \) such that \( \lim_{n \to +\infty} t_n = T_{BU}(W_0) \), and for any \( n \in \mathbb{N} \),

\[
\|U(t_n)W_0\|_{\text{Lip}_x \times C(\Omega)} = \left\| \left( \begin{array}{c} \Lambda_{t_n} \\ \tau(t_n, .) \end{array} \right) \right\|_{\text{Lip}_x \times C(\Omega)} \leq M_0.
\]
Let \( W_{0,n} := \left( \begin{array}{c} A_{t_{n}} \\ \tau(t_{n},.) \end{array} \right) \). Then by the local existence Theorem 4.4 and Corollary 4.5 we can find a constant \( \hat{M} > M_0 \) and a time \( r = r(M_0, \hat{M}) > 0 \) such that for any \( n \in \mathbb{N} \),

\[
T_{BU}(W_{0,n}) \geq r.
\]

Moreover, from the previous part of the proof we have

\[
T_{BU}(W_0) = t_n + T_{BU}(W_{0,n}) \geq t_n + r,
\]

and when \( n \to +\infty \) we obtain

\[
T_{BU}(W_0) \geq T_{BU}(W_0) + r
\]

a contradiction since \( r > 0 \). Thus we have

\[
\lim_{t \to T_{BU}(W_0)} \|U(t)W_0\|_{Lip_{a} \times C(\Omega)} = +\infty. \tag{5.4}
\]

**Proof.** (Second part of Theorem 1.6) Let us prove that if \( T_{BU}(W_0) < +\infty \) then

\[
\lim_{t \to T_{BU}(W_0)} \sup_{s \leq t} \|A(t,.)\|_{\infty} = +\infty.
\]

Assume that \( T_{BU}(W_0) < +\infty \) and assume by contradiction that

\[
\lim_{t \to T_{BU}(W_0)} \sup_{s \leq t} \|A(t,.)\|_{\infty} < +\infty.
\]

Since the map \( t \mapsto t - \tau(t,x) \) is increasing, we have

\[-\tau_0(x) \leq t - \tau(t,x) \leq t < T_{BU}(W_0), \forall x \in \Omega,
\]

therefore

\[
0 \leq \tau(t,x) \leq T_{BU}(W_0) + \tau_0(x).
\]

And by assumption \( T_{BU}(W_0) < +\infty \), then

\[
\lim_{t \to T_{BU}(W_0)} \|\tau(t,.)\|_{\infty} < +\infty.
\]

Moreover, for each \( t \in [0, T_{BU}(W_0)) \),

\[
\partial_t A(t,x) = F(A(t,.),\tau(t,.),A(t - \tau(t))(.,.))(x),
\]

and since \( F \) is Lipschitz on bounded sets and by Lemma 4.2 we deduce that

\[
\lim_{t \to T_{BU}(W_0)} \|e^{\alpha t} A(.)\|_{Lip} < +\infty,
\]

which contradicts (5.4). \( \blacksquare \)
Lemma 5.1 We have the following results:

(i) The map $W_0 \mapsto T_{BU}(W_0)$ is lower semi-continuous on $\text{Lip}_\alpha \times C_+(\Omega)$.

(ii) For every $x \in \Omega$, for every $W_0 \in \text{Lip}_\alpha \times C_+(\Omega)$, $\hat{T} \in (0, T_{BU}(W_0))$, and every sequence $\{W_0^{(n)}\}_{n \in \mathbb{N}} \subset \text{Lip}_\alpha \times C_+(\Omega)$ satisfying

$$\lim_{n \to +\infty} W_0^{(n)} = W_0 \text{ in } \text{Lip}_\alpha \times C_+(\Omega),$$

we have

$$\lim_{n \to +\infty} \sup_{t \in [0, \hat{T}]} \left\| U(t)W_0^{(n)} - U(t)W_0 \right\|_{\text{Lip}_\alpha \times C(\Omega)} = 0. \quad (5.5)$$

Proof. Step 1 (Fixed point problem): Let $t \mapsto (\bar{A}(t, .), \bar{\tau}(t, .))$ be a solution of system (1.1) which exists up to the maximal time of existence $T_{BU}(W_0)$ with the initial distribution $\overline{W}_0 = \left( \begin{array}{c} \bar{\varphi} \\ \bar{\tau}_0 \end{array} \right) \in \text{Lip}_\alpha \times C_+(\Omega)$.

Let $t^* \in (0, T_{BU}(\overline{W}_0))$ be fixed. By construction the map $t \mapsto (\bar{A}_t, \bar{\tau}(t, .))$ is continuous from $[0, T_{BU}(W_0))$ to $BUC_\alpha \times C_+(\Omega)$. Therefore

$$\sup_{t \in [0, t^*]} [\|\bar{A}_t\|_{BUC_\alpha} + \|\bar{\tau}(t, .)\|_\infty] < +\infty,$$

and since $\bar{A}(t, .)$ satisfies the equation (1.1) for positive time $t$, it follows that

$$\bar{M} := \sup_{t \in [0, t^*]} \|\bar{A}_t\|_{\text{Lip}_\alpha} < +\infty.$$

Let $t_0 \in [0, t^*]$ and $r > 0$ with $t_0 + r < t^*$ where $r$ will be determined later on.

Let $\varepsilon > 0$ be fixed. Let $W_0 = \left( \begin{array}{c} \varphi \\ \tau_0 \end{array} \right) \in \text{Lip}_\alpha \times C_+(\Omega)$ satisfy

$$\|\varphi - \bar{A}_{t_0}\|_{\text{Lip}_\alpha} \leq \varepsilon \text{ and } \|\tau_0 - \bar{\tau}(t_0, .)\|_\infty \leq \varepsilon.$$

Let $M > \varepsilon$ be fixed. Define the space

$$E_{\varphi, t_0} := \{A \in BUC_\alpha((-\infty, r], C(\Omega)) : A_0 = \varphi \text{ and } \sup_{t \in [0, r]} \|A(t, .) - \bar{A}_{t_0}(t, .)\|_\infty \leq M\}.$$

Let $\Phi : E_{\varphi, t_0} \to C((-\infty, r], C(\Omega))$ be the map defined by

$$\Phi(A)(t)(x) := \varphi(0, x) + \int_0^t F(A(l, .), \bar{\tau}(A_l, \delta_0), A(l - \hat{\tau}(A_l, \delta_0), .))(x)dl \quad (5.6)$$

whenever $t \in [0, r]$ and $x \in \Omega$, and

$$\Phi(A)(t)(x) := \bar{A}_{t_0}(t, x)$$

whenever $t \leq 0$ and $x \in \Omega$. 29
In the formula (5.6) the delay \( \hat{\tau}(A_t, \delta_0)(x) \) is the unique solution of

\[
\int_{-\hat{\tau}(A_t, \delta_0)(x)}^{0} f(A(t + s, .))(x)ds = \delta_0(x)
\]

where

\[
\delta_0(x) := \int_{-\tau_0(x)}^{0} f(\varphi(s, .))(x)ds.
\]

For any \( A \in E_{\varphi, t_0} , t \in [0, r] \) and \( x \in \Omega \), we have

\[
|\Phi(A)(t)(x) - \bar{A}_{t_0}(t, x)|
\]

\[
= |\varphi(0, x) + \int_{0}^{t} F(A(l, .), \hat{\tau}(A_l, \delta_0), A(l - \hat{\tau}(A_l, \delta_0), .))(x)dl
- \bar{A}_{t_0}(0, x) - \int_{0}^{t} F(\bar{A}_{t_0}(l, .), \hat{\tau}(\bar{A}_{t_0+l}, \tilde{\delta}_0), \bar{A}_{t_0}(l - \hat{\tau}(\bar{A}_{t_0+l}, \tilde{\delta}_0), .))(x)dl|
\]

\[
\leq \varepsilon + \int_{0}^{t} |F(A(l, .), \hat{\tau}(A_l, \delta_0), A(l - \hat{\tau}(A_l, \delta_0), .))(x)
- F(\bar{A}_{t_0}(l, .), \hat{\tau}(\bar{A}_{t_0+l}, \tilde{\delta}_0), \bar{A}_{t_0}(l - \hat{\tau}(\bar{A}_{t_0+l}, \tilde{\delta}_0), .))(x)|dl,
\]

where

\[
\delta_0(x) = \int_{-\hat{\tau}(t_0, x)}^{0} f(\bar{A}(t_0 + s, .))(x)ds = \int_{-\tau_0(x)}^{0} f(\bar{\varphi}(s, .))(x)ds.
\]

By Lemma 3.9, we can find a constant \( L_\tau > 0 \) such that

\[
\|\hat{\tau}(A_t, \delta_0) - \bar{\tau}(\bar{A}_{t_0+t}, \tilde{\delta}_0)\|_\infty \leq L_\tau \left[ \sup_{s \in [-\max\{\tau_0^\infty, \tau_0^\infty\}, r]} \|A(s, .) - \bar{A}_{t_0}(s, .)\|_\infty + \|\delta_0 - \tilde{\delta}_0\|_\infty \right]
\]

where \( \tau_0^\infty \) and \( \tau_0^\infty \) are defined as in Lemma 3.8.

By using the definition of \( \delta_0 \) and \( \tilde{\delta}_0 \) we have

\[
\|\delta_0 - \tilde{\delta}_0\|_\infty \leq \sup_{x \in \Omega} \left| \int_{-\tau_0(x)}^{0} f(\varphi(s, .))(x)ds \right|
\]

\[
+ \sup_{x \in \Omega} \left| \int_{-\hat{\tau}(t_0, x)}^{0} [f(\varphi(s, .))(x) - f(\bar{A}(t_0 + s, .))(x)]ds \right|
\]

\[
\leq \|\tau_0(. - \bar{\tau}(t_0, .))\|_\infty \sup_{x \in \Omega} f(-\varphi_{\text{max}})(x)
+ \|\tau_0\|_\infty \|\bar{f}\|_{\text{Lip}} \sup_{s \in [-\tau_0^\infty, 0]} \|\varphi(s, .) - \bar{A}(t_0 + s, .)\|_\infty
\]

\[
\leq \varepsilon \left( \sup_{x \in \Omega} f(-\varphi_{\text{max}})(x) + \tau_0^\infty e^{\alpha \tau_0^\infty} \|\bar{f}\|_{\text{Lip}} \right)
\]

where \( \varphi_{\text{max}} \) is defined as in Lemma 3.8.
From the above estimations, it follows that there exists a constant $M_1 > 0$ such that
\[ \|\widehat{T}(A_t, \delta_0) - \widehat{T}(\bar{A}_{t_0 + t}, \bar{\delta}_0)\|_\infty \leq M_1. \]

Now, similarly as in step 3 of the proof of Theorem 4.4 to evaluate $\|A^{n+1} - A^n\|_\infty$, we deduce that there exists a constant $r_1 > 0$ such that for each $r \in (0, r_1]$, we have $\Phi(E_{\varphi,t_0}) \subset E_{\varphi,t_0}$.

Step 2 (Lipschitz estimation): Similarly as in step 2 of the proof of Theorem 4.4, we can deduce that there exists a constant $\hat{M}_L > 0$ such that
\[ \|\Phi(A)\|_{\text{Lip}([0,r],C(\Omega))} \leq \hat{M}_L, \forall A \in E_{\varphi,t_0}. \]

Step 3 (Iteration procedure): Consider the sequence $\{A^n\}_{n \in \mathbb{N}} \subset E_{\varphi,t_0}$ defined by iteration as follows: for each $(t, x) \in (-\infty, r] \times \Omega$,
\[ A^0(t, x) = \bar{A}_{t_0}(t, x), \]
and for each integer $n \geq 0$,
\[ A^{n+1}(t, x) := \begin{cases} 
\Phi(A^n)(t)(x), & \text{if } t \in [0, r], \\
\varphi(t, x), & \text{if } t \leq 0.
\end{cases} \]

From step 2, we deduce that there exists a constant $\hat{M}_L > 0$ such that for each integer $n \geq 0$,
\[ \|A^n\|_{\text{Lip}([t_0, t],C(\Omega))} \leq \hat{M}_L. \]

By using the same argument as in step 3 of the proof of Theorem 4.4, we can find a constant $r_2 \in (0, r_1]$ such that $\forall r \in (0, r_2]$, 
\[ \|A^{n+1} - A^n\|_\infty \leq \frac{1}{2^n}\|A^1 - A^0\|_\infty, \forall n \geq 1. \]

It follows that $\{A^n\}_{0 \leq r}$ is a Cauchy sequence in the space $C([0, r], C(\Omega))$. Define
\[ A(t, x) := \begin{cases} 
\lim_{n \to +\infty} A^n(t, x), & \text{if } t \in [0, r], x \in \Omega, \\
\varphi(t, x), & \text{if } t \leq 0, x \in \Omega.
\end{cases} \]

Then we have
\[ \lim_{n \to +\infty} \|A^n - A\|_\infty = 0, \]
and we deduce that $(A, \widehat{T}(A_l, \delta_0))$ is a solution of (1.1) with the initial distribution $(\varphi, \tau_0)$.

Step 4 (Estimation of the solution): As in step 4 of the proof of Theorem 4.4, we also have
\[ \|A - A^0\|_\infty \leq 2\|A^1 - A^0\|_\infty. \]

Since we have
\[ \|A^1 - A^0\|_\infty \]
\[
\sup_{t \in [0,r]} \sup_{x \in \Omega} \left| \phi(0, x) + \int_0^t F(A^0(l, \cdot), \tilde{\tau}(A^0, \delta_0), A^0(l - \tilde{\tau}(A^0, \delta_0), \cdot))(x) dl - \tilde{A}_{t_0}(t, x) \right|
\]
\[
\leq \sup_{t \in [0,r]} \left[ \left\| \phi(0, \cdot) - \tilde{A}_{t_0}(0, \cdot) \right\|_{\infty}
+ \sup_{x \in \Omega} \left| \tilde{A}_{t_0}(0, \cdot) + \int_0^t F(A^0(t, \cdot), \tilde{\tau}(A^0, \delta_0), A^0(t - \tilde{\tau}(A^0, \delta_0), \cdot))(x) dl - \tilde{A}_{t_0}(t, x) \right| \right]
\]

and since \( \tilde{A}(t, x) \) is a solution, the last term in the above inequality is null, it follows that
\[
\| A^1 - A^0 \|_{\infty} \leq \varepsilon.
\]

Then we obtain
\[
\| A(t, \cdot) - \tilde{A}_{t_0}(t, \cdot) \|_{\infty} \leq 2\varepsilon, \forall t \in [t_0, t_0 + r].
\]

**Step 5 (Convergence result):** Fix \( r = \frac{t^*}{n} \leq r_2 \) for some integer \( n \geq 1 \). Choose an initial value satisfying
\[
\| \phi - \tilde{A}_{t_0} \|_{\text{Lip}_\alpha} \leq \frac{\varepsilon}{2n+1} \quad \text{and} \quad \| \tau_0 - \tilde{\tau}(t_0, \cdot) \|_{\infty} \leq \frac{\varepsilon}{2n+1}.
\]

By using the above prove result when \( t_0 = 0 \), we deduce that
\[
\| A(t, \cdot) - \tilde{A}(t, \cdot) \|_{\infty} \leq \frac{\varepsilon}{2n+1}, \forall t \in [0, r].
\]

and by induction \( t_0 = kr \) for \( k = 0, ..., n \) we obtain
\[
\| A(t, \cdot) - \tilde{A}(t, \cdot) \|_{\infty} \leq \varepsilon, \forall t \in [0, t^*],
\]

the result follows.

The part of of Theorem 1.7: time continuity of the semiflow in \( BUC^1_\alpha \) is left to the reader.

### 6 Application to the forest model with space

For \( x \in \Omega := [0,1] \) and \( t \geq 0 \) we consider the forest spatial model with one species (see [9] for more details)

\[
\partial_t A(t, x) = e^{-\mu_J r(t, x)} \frac{f(A(t, x))}{f(A(t - \tau(t, x), x))} B(t - \tau(t, x), x) - \mu_A A(t, x), \quad (6.1)
\]

where the birth is defined by
\[
B(t, x) := (I - \varepsilon \Delta)^{-1}[\beta A(t, \cdot)](x),
\]

where \( \Delta \) is the Laplacian operator on the domain \( \Omega \) with periodic boundary conditions, and the state-dependent delay satisfies
\[
\int_{t - \tau(t, x)}^t f(A(\sigma, x)) d\sigma = \int_{\tau_0(x)}^0 f(\phi(\sigma, x)) d\sigma,
\]

32
and \( A(t, x) \) satisfies the initial condition
\[
A(t, x) = \varphi(t, x), \forall t \leq 0, x \in \Omega
\]
with
\[
\varphi \in \text{Lip}_\alpha \quad \text{and} \quad \varphi \geq 0.
\]
Then it is well known that \((I - \varepsilon \Delta)^{-1}\) is a positive operator, i.e.
\[
(I - \varepsilon \Delta)^{-1}C_+(\Omega) \subset C_+(\Omega)
\]
and
\[
\|(I - \varepsilon \Delta)^{-1}\|_{\mathcal{L}(C(\Omega))} = 1.
\]

### 6.1 Positivity

Assume that
\[
\varphi(t, x) \geq 0, \forall (t, x) \in (-\infty, 0] \times \Omega.
\]
Assume that the solution starting from this initial distribution exists up to the time \( T_{BU}(\varphi, \tau_0) > 0 \). Then we have for each \( t \in [0, T_{BU}(\varphi, \tau_0)) \),
\[
A(t, x) = e^{-\mu A t} \varphi(0, x) + \int_0^t e^{-\mu A (t-s)} e^{-\mu J \tau(s, x)} \frac{f(A(s, x))}{f(A(s - \tau(s, x), x))} ds.
\]
Since the operator \((I - \varepsilon \Delta)^{-1}\) preserves the positivity of the distribution and by Assumption 1.1, \( f \) is strictly positive, then the positivity of the solution follows by using fixed point arguments on \( A(t, x) \) in the above integral equation.

### 6.2 Global existence

Consider the spatial density of juveniles
\[
J(t, x) := \int_{t-\tau(t, x)}^t e^{-\mu J (t-s)} \beta (I - \varepsilon \Delta)^{-1}(A(s, .))(x) ds
\]
for each \( t \in [0, T_{BU}(\varphi, \tau_0)) \). It is clear that
\[
J(t, x) \geq 0, \forall t \in [0, T_{BU}(\varphi, \tau_0)). \tag{6.2}
\]
Moreover we have
\[
\partial_t J(t, x) = \beta (I - \varepsilon \Delta)^{-1}(A(t, .))(x) - e^{-\mu J \tau(t, x)} \frac{f(A(t, x))}{f(A(t - \tau(t, x), x))}.
\]
By summing equation (6.1) and the above equation we obtain
\[
\partial_t [A(t, x) + J(t, x)] = \beta (I - \varepsilon \Delta)^{-1}(A(t, .))(x) - \mu A(t, x) - \mu J(t, x). \tag{6.3}
\]
Set 
\[ U(t, x) := A(t, x) + J(t, x), \]
then we have
\[ \partial_t U(t, x) \leq \beta (I - \varepsilon \Delta)^{-1}(U(t, .))(x) - \mu U(t, x), \]
where \( \mu := \min\{\mu_A, \mu_J\} \). By using a comparison argument we deduce that
\[ U(t, x) \leq e^{[\beta(t - \varepsilon \Delta)^{-1} - \mu]t}(U(0, .))(x), \forall t \in [0, T_{BU}(\varphi, \tau_0)). \]
Therefore by using (6.2), we deduce that
\[ A(t, x) \leq e^{[\beta(t - \varepsilon \Delta)^{-1} - \mu]t}(U(0, .))(x), \forall t \in [0, T_{BU}(\varphi, \tau_0)), \]
and by using Theorem 1.6, we must have \( T_{BU}(\varphi, \tau_0) = +\infty \).

References

[1] W. G. Aiello, H. I. Freedman and J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay, SIAM Journal on Applied Mathematics, 52(3) (1992), 855-869.

[2] J. F. M. Al-Omari and S. A. Gourley, Dynamics of a stage-structured population model incorporating a state-dependent maturation delay, Nonlinear Analysis: Real World Applications, 6 (2005), 13-33.

[3] O. Arino, E. Sanchez and A. Fathallah, State-dependent delay differential equations in population dynamics: modeling and analysis, Topics in Functional Differential and Difference Equations, Fields Institute Communications, Vol. 29, 19-36, American Mathematical Society, 2001.

[4] H. Brunner, S.A. Gourley, R. Liu and Y. Xiao, Pauses of larval development and its consequences for stage-structured populations, SIAM Journal on Applied Mathematics (to appear).

[5] F. Hartung, T. Krisztin, H.O. Walther and J. Wu, Functional differential equations with state-dependent delays: Theory and applications, Handbook Of Differential Equations: Ordinary Differential Equations, Vol. 3, 435-545, Elsevier, 2006.

[6] M. L. Hbid, M. Louihi and E. Sanchez, A threshold state-dependent delayed functional equation arising from marine population dynamics: modelling and analysis, Journal of Evolution Equations, 10(4) (2010), 905-928.

[7] M. Kloosterman, S. A. Campbell and F. J. Poulin, An NPZ model with state-dependent delay due to size-structure in juvenile zooplankton, SIAM Journal on Applied Mathematics, 76(2) (2016), 551-577.
[8] Z. Liu and P. Magal, Functional differential equation with infinite delay in a space of exponentially bounded and uniformly continuous functions, *in submitted*.

[9] P. Magal and Z. Zhang, Competition for light in a forest population dynamic model: from computer model to mathematical model, *Journal of Theoretical Biology, 419* (2017), 290-304.

[10] P. Magal and Z. Zhang, Numerical simulations of a population dynamic model describing parasite destruction in a wild type pine forest, *Ecological Complexity (to appear)*.

[11] W. Rudin, *Real and Complex Analysis, Third Edition*, Tata McGraw-Hill Education, 1987.

[12] H. L. Smith, Reduction of structured population models to threshold-type delay equations and functional differential equations: A case study, *Mathematical Biosciences, 113* (1993), 1-23.

[13] H. L. Smith, A structured population model and a related functional differential equation: global attractors and uniform persistence, *Journal of Dynamics and Differential Equations, 6* (1) (1994), 71-99.

[14] H. L. Smith, Existence and uniqueness of global solutions for a size-structured model of an insect population with variable instar duration, *Rocky Mountain Journal of Mathematics, 24*(1) (1994), 311-334.

[15] H. L. Smith, Equivalent dynamics for a structured population model and a related functional differential equation, *Rocky Mountain Journal of Mathematics, 25*(1) (1995), 491-499.

[16] H.-O. Walther, Differential equations with locally bounded delay, *Journal of Differential Equations, 252* (2012), 3001-3039.