On the truncated operator trigonometric moment problem.

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1 Introduction.

The truncated operator trigonometric moment problem consists of finding a non-decreasing \([H]\)-valued function \(F(t), t \in [0, 2\pi]\), \(F(0) = 0\), which is strongly left-continuous in \((0, 2\pi]\) and such that

\[
\int_{0}^{2\pi} e^{int} dF(t) = S_n, \quad n = 0, 1, \ldots, d, \tag{1}
\]

where \(\{S_n\}_{n=0}^{d}\) is a prescribed sequence of bounded operators on \(H\) (moments). Here \(H\) is a fixed Hilbert space and \(d \in \mathbb{Z}_+\) is a fixed number. The operator Stieltjes integrals in (1) are understood as limits of the corresponding integral sums in the strong operator topology. Set \(S_{-k} = S_k^*, k = 1, 2, \ldots, d\). The following condition:

\[
\sum_{k,l=0}^{d} (S_{l-k}h_l, h_k)_H \geq 0, \tag{2}
\]

where \(\{h_k\}_{0}^{d}\) are arbitrary elements of \(H\), is necessary and sufficient for the solvability of the moment problem (1) (e.g. [1]). The solvable moment problem (1) is said to be determinate if it has a unique solution and indeterminate in the opposite case. The truncated operator trigonometric moment problem was studied in papers [1], [4] (in slightly different statements). The conditions of the solvability were obtained in [1]. In the case of the strict positivity of the corresponding Toeplitz operator, all solutions to the moment problem were described in [1]. For the history of scalar and matrix truncated trigonometric moment problems we refer to [8], [9].

Our aim here is to describe all solutions to the solvable moment problem (1) without any additional conditions. For this purpose we shall develop the operator approach of Szőkefalvi-Nagy and Koranyi in [6], [7]. Also our approach is close to the approach of Krein and Krasnoselskii in [5]. All solutions of the moment problem are described by a Nevanlinna-type parameterization. In the case of moments acting in a separable Hilbert space, the matrices of the operator coefficients in the Nevanlinna-type formula are
calculated by the prescribed moments. Conditions for the determinacy of the moment problem are given.

**Notations.** As usual, we denote by \( \mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+ \), the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}; \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}; \mathbb{T}_e = \{ z \in \mathbb{C} : |z| \neq 1 \}. \) By \( k \in 0, \rho \) we mean that \( k \in \mathbb{Z}_+ : 0 \leq k \leq \rho \) if \( \rho \in \mathbb{Z}_+ \), and \( k \in \mathbb{Z}_+ \), if \( \rho = +\infty \).

In this paper Hilbert spaces are not necessarily separable, operators in them are supposed to be linear.

If \( H \) is a Hilbert space then \((\cdot, \cdot)_H\) and \( \| \cdot \|_H \) mean the scalar product and the norm in \( H \), respectively. Indices may be omitted in obvious cases. For a linear operator \( A \) in \( H \), we denote by \( D(A) \) its domain, by \( R(A) \) its range, and \( A^* \) means the adjoint operator if it exists. If \( A \) is invertible then \( A^{-1} \) means its inverse. \( \overline{A} \) means the closure of the operator, if the operator is closable. If \( A \) is bounded then \( \| A \| \) denotes its norm. For a set \( M \subseteq H \) we denote by \( \overline{M} \) the closure of \( M \) in the norm of \( H \). For an arbitrary set of elements \( \{x_n\}_{n \in I} \) in \( H \), we denote by \( \text{Lin}\{x_n\}_{n \in I} \) the set of all linear combinations of elements \( x_n \), and \( \text{span}\{x_n\}_{n \in I} := \overline{\text{lin}\{x_n\}_{n \in I}} \). Here \( I \) is an arbitrary set of indices. By \( E_H \) we denote the identity operator in \( H \), i.e. \( E_H x = x, x \in H \). In obvious cases we may omit the index \( H \). If \( H_1 \) is a subspace of \( H \), then \( P_{H_1} = P_{H_1}^H \) is an operator of the orthogonal projection on \( H_1 \) in \( H \). By \( [H] \) we denote a set of all bounded operators on \( H \). For a closed isometric operator \( V \) in \( H \) we denote: \( M_\zeta(V) = (E_H - \zeta V)D(V), N_\zeta(V) = H \ominus M_\zeta(V), \zeta \in \mathbb{C}; M_{\infty}(V) = R(V), N_{\infty}(V) = H \ominus R(V) \).

By \( S(D; N, N') \) we denote a class of all analytic in a domain \( D \subseteq \mathbb{C} \) operator-valued functions \( F(z) \), which values are linear non-expanding operators mapping the whole \( N \) into \( N' \), where \( N \) and \( N' \) are some Hilbert spaces.

2 The solvability and a description of solutions for the moment problem.

Suppose that the moment problem \( (1) \) is given and it is solvable. For arbitrary elements \( \{h_k\}_0^d \) of \( \mathcal{H} \) we may write:

\[
\sum_{k,l=0}^d (S_{l-k}h_l, h_k)_\mathcal{H} = \sum_{k,l=0}^d \left( \int_0^{2\pi} e^{i(l-k)t}dF(t)h_l, h_k \right)_\mathcal{H} = \sum_{k,l=0}^d \left( \lim_{\delta \rightarrow 0^+} \sum_{r=0}^N e^{i(l-k)t_r} (F(t_{r+1}) - F(t_r))h_l, h_k \right)_\mathcal{H}
\]
Here $\delta$ is the diameter of a partition of $[0, 2\pi]$ and $\{t_r\}_{0}^{N}$ are points of this partition. Thus, condition (2) is satisfied.

Conversely, suppose that the moment problem (1) with $d \in \mathbb{N}$ is given and condition (2) is satisfied. Like it was done in [7] we consider abstract symbols $\epsilon_j, j = 0, 1, \ldots, d$, and form a formal sum $h$:

$$h = \sum_{j=0}^{d} h_j \epsilon_j,$$

where $h_j \in \mathcal{H}$. If $\alpha \in \mathbb{C}$, then we set $\alpha h = \sum_{j=0}^{d} (\alpha h_j) \epsilon_j$. If

$$g = \sum_{j=0}^{d} g_j \epsilon_j,$$

where $g_j \in \mathcal{H}$, then we set $h + g = \sum_{j=0}^{d} (h_j + g_j) \epsilon_j$. A set of all formal sums of type (3) becomes a complex linear vector space $\mathcal{B}$. Let, $h, g \in \mathcal{B}$ have the form as in (3), (4). Let

$$\Phi(h, g) = \sum_{j,k=0}^{d} (S_{j-k} h_j, g_k)_{\mathcal{H}}.$$

The functional $\Phi$ is sesquilinear and it has the properties $\overline{\Phi(h,g)} = \Phi(g,h)$, $\Phi(h,h) \geq 0$. If $\Phi(h - g, h - g) = 0$, we put elements $h$ and $g$ to the same equivalence class denoted by $[h]$ or $[g]$. A set of all equivalent classes we denote by $\mathcal{L}$. By the completion of $\mathcal{L}$ we obtain a Hilbert space $H$. Set

$$x_{h,j} := [h \epsilon_j], \quad h \in \mathcal{H}, \ 0 \leq j \leq d.$$

Observe that

$$(x_{h,j}, x_{g,k})_H = (S_{j-k} h_j, g_k)_{\mathcal{H}}, \quad h, g \in \mathcal{H}, \ 0 \leq j, k \leq d.$$

Set

$$D_0 = \operatorname{Lin}\{x_{h,j}\}_{h \in \mathcal{H}, \ 0 \leq j \leq d-1} = \{x_{h_0,0} + x_{h_1,1} + \ldots + x_{h_{d-1},d-1}\}_{h_0, h_1, \ldots, h_{d-1} \in \mathcal{H}}.$$
Consider a linear operator $A_0$ with $D(A_0) = D_0$:

$$A_0 \sum_{j=0}^{d-1} x_{h_j, j} = \sum_{j=0}^{d-1} x_{h_j, j+1}, \quad h_j \in \mathcal{H}. \tag{8}$$

Let us check that $A_0$ is well-defined. Suppose that an element $h \in D_0$ has two representations:

$$h = \sum_{j=0}^{d-1} x_{h_j, j} = \sum_{j=0}^{d-1} x_{g_j, j}, \quad h_j, g_j \in \mathcal{H}.$$

Then

$$\left\| \sum_{j=0}^{d-1} x_{h_j, j+1} - \sum_{j=0}^{d-1} x_{g_j, j+1} \right\|_H = \left( \sum_{j=0}^{d-1} x_{h_j - g_j, j+1}, \sum_{k=0}^{d-1} x_{h_k - g_k, k+1} \right)_H = 0,$$

Thus, $A_0$ is well-defined. If

$$h = \sum_{j=0}^{d-1} x_{h_j, j}, \quad f = \sum_{k=0}^{d-1} x_{f_k, k}, \quad h_j, g_k \in \mathcal{H},$$

then

$$(A_0 h, A_0 f)_H = \sum_{j,k=0}^{d-1} (x_{h_j, j+1}, x_{f_k, k+1})_H = \sum_{j,k=0}^{d-1} (S_{j-k} h_j, f_k)_H = 0.$$

Therefore $A_0$ is isometric. Set $\tilde{A} = A_0$. By the induction argument it may be checked that

$$x_{h, j} = A^j x_{h, 0}, \quad 0 \leq j \leq d.$$

Let $\tilde{A} \supseteq A$ be a unitary operator in a Hilbert space $\tilde{H} \supseteq H$, and $\{ E_t \}_{t \in [0, 2\pi]}$ be its strongly left-continuous orthogonal resolution of the identity. We may write

$$(S_j h, g)_H = (x_{h, j}, x_{g, 0})_H = (A^j x_{h, 0}, x_{g, 0})_H = (\tilde{A}^j x_{h, 0}, x_{g, 0})_{\tilde{H}} = \cdots$$
Consider the following operator $I : H \rightarrow H$:

$$Ih = x_{h,0}, \quad h \in H.$$  \hfill (10)

It is readily checked that $I$ is linear. Moreover, since

$$\|Ih\|^2_H = (x_{h,0}, x_{h,0})_H = (S_0h, h)_H \leq \|S_0\|\|h\|^2_H,$$

then $I$ is bounded. By (9) we may write

$$\langle S_jh, g \rangle_H = \left( P_{H}^{\bar{H}} \int_{0}^{2\pi} e^{ijt} d\bar{E}_t Ih, Ig \right)_H = \left( I^{*} P_{H}^{\bar{H}} \int_{0}^{2\pi} e^{ijt} d\bar{E}_t Ih, g \right)_H = \left( \int_{0}^{2\pi} e^{ijt} d \left( I^{*} P_{H}^{\bar{H}} \bar{E}_t I \right) h, g \right)_H, \quad 0 \leq j \leq d, \ h, g \in H.$$  

Therefore

$$S_j = \int_{0}^{2\pi} e^{ijt} d (I^{*}E_t I), \quad 0 \leq j \leq d,$$ \hfill (11)

where $E_t$ is a strongly left-continuous spectral function of $A$ (corresponding to $\widetilde{A}$). Thus,

$$F(t) = I^{*}E_t I, \quad t \in [0, 2\pi],$$ \hfill (12)

is a solution of the moment problem (11). We conclude that each strongly left-continuous spectral function of $A$ generates a solution of the moment problem (11) by relation (12).

Let $\tilde{F}(t)$ be an arbitrary solution of the moment problem (11). We shall check that $\tilde{F}(t)$ can be constructed by relation (12). By $C_{00}(H; [0, 2\pi])$ we denote a set of all strongly continuous $H$-valued functions $f(t), \ t \in [0, 2\pi]$, which take their values in finite-dimensional subspaces of $H$ (depending on $f$). For arbitrary $f, g \in C_{00}(H; [0, 2\pi])$ we set (see [2])

$$\Psi(f, g) = \int^{2\pi}_{0} \left( d\tilde{F}(t)f(t), g(t) \right)_H = \lim_{\delta \rightarrow +0} \sum_{k=1}^{N} \left( \tilde{F}(\Delta_k) f(t_k), g(t_k) \right)_H,$$ \hfill (13)

where $\delta$ is the diameter of a partition $\{\Delta_k\}_{k=1}^{N},$, $\Delta_k = [a_k, b_k]$ of $[0, 2\pi]$ and $t_k \in \Delta_k$. Here, as usual, the limit does not depend on the choice of partitions and points $t_k$. It is easy to see that in the case of $f, g \in C_{00}(H; [0, 2\pi])$ the limit in (13) exists and reduces to a finite sum of scalar Stieltjes-type integrals.
Introducing classes of the equivalence with respect to $\Psi$ and by the completion we obtain a Hilbert space $L_2 = L_2(\mathcal{H}; [0, 2\pi]; d\hat{F}(t))$. Consider two operator polynomials of the following form:

$$p(t) = \sum_{j=0}^{d} e^{ij't}h_j, \quad q(t) = \sum_{k=0}^{d} e^{ikt}g_k, \quad h_j, g_k \in \mathcal{H}. \quad (14)$$

Since $p, q \in C_00(\mathcal{H}; [0, 2\pi])$ then the corresponding classes $[p], [q]$ belong to $L_2(\mathcal{H}; [0, 2\pi]; d\hat{F}(t))$. As usual in such situations, we shall say that $p, q$ belong to $L_2(\mathcal{H}; [0, 2\pi]; d\hat{F}(t))$. Then

$$(p, q)_{L_2(\mathcal{H}; [0, 2\pi]; d\hat{F}(t))} = \int_{0}^{2\pi} (d\hat{F}(t)p(t), q(t))_{\mathcal{H}} =$$

$$= \sum_{j, k=0}^{d} \int_{0}^{2\pi} e^{ij-(j-k)t}d \left( \hat{F}(t)h_j, g_k \right)_{\mathcal{H}} = \sum_{j, k=0}^{d} (S_{j-k}h_j, g_k)_{\mathcal{H}} =$$

$$= \sum_{j, k=0}^{d} (x_{h_j, j}, x_{g_k, k})_{\mathcal{H}} = \left( \sum_{j=0}^{d} x_{h_j, j}, \sum_{k=0}^{d} x_{g_k, k} \right)_{\mathcal{H}} \quad (15)$$

Denote by $P(\mathcal{H}; [0, 2\pi]; d\hat{F}(t))$ a set of all (classes of the equivalence which contain) polynomials of type (14) from $L_2(\mathcal{H}; [0, 2\pi]; d\hat{F}(t))$. Set $L_{2, 0}(\mathcal{H}; [0, 2\pi]; d\hat{F}(t)) = P(\mathcal{H}; [0, 2\pi]; d\hat{F}(t))$. Consider the following transformation:

$$W_0 \left[ \sum_{j=0}^{d} e^{ij't}h_j \right] = \sum_{j=0}^{d} x_{h_j, j}, \quad h_j \in \mathcal{H},$$

which maps $P(\mathcal{H}; [0, 2\pi]; d\hat{F}(t))$ on the whole $\mathcal{L}(\subseteq \mathcal{H})$. Let us check that $W_0$ is well-defined. In fact, suppose that $p, q$ from $[14]$ belong to the same class of the equivalence. Then

$$0 = \left\| \sum_{j=0}^{d} e^{ij'(h_j - g_j)} \right\|_{L_2} = \left( \sum_{j=0}^{d} e^{ij'(h_j - g_j)}, \sum_{k=0}^{d} e^{ikt}(h_k - g_k) \right)_{L_2} =$$

$$= \left( \sum_{j=0}^{d} x_{h_j - g_j, j}, \sum_{k=0}^{d} x_{h_k - g_k, k} \right)_{\mathcal{H}} = \left\| \sum_{j=0}^{d} x_{h_j, j} - \sum_{j=0}^{d} x_{g_j, j} \right\|^2_{\mathcal{H}}.$$
Therefore $W_0$ is well-defined. Moreover, $W_0$ is linear and relation (15) shows that $W_0$ is isometric. By the continuity we extend $W_0$ to a unitary transformation $W$ which maps $L_{2,0}(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t))$ on the whole $H$. Denote by $U_0$ an operator in $L_2(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t))$ which put into correspondence for a class of the equivalence which has a representative $f(t) \in C_{00}(\mathcal{H}; [0, 2\pi])$ the class $[e^{it}f(t)]$. It is readily checked that this operator is well-defined, linear and isometric. We extend it by the continuity to a unitary operator $U$ on $L_2(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t))$. Observe that

$$A_0x_{h,j} = x_{h,j+1} = W[e^{i(j+1)t}h] = WU[e^{ijt}h] = WUW^{-1}x_{h,j},$$

for arbitrary $h \in \mathcal{H}$, $0 \leq j \leq d - 1$. Therefore

$$WUW^{-1} \supset A.$$ (16)

Set

$$L_{2,1}(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t)) = L_2(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t)) \oplus L_{2,0}(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t)),$$

and $W = W \oplus E_{L_{2,1}(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t))}$. Notice that $W$ is a unitary transformation which maps $L_2(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t))$ on $H_1 := H \oplus L_{2,1}(\mathcal{H}; [0, 2\pi]; d\tilde{F}(t))$. Set $\tilde{A} := WUW^{-1}$. By (16) we see that $\tilde{A}$ is a unitary extension of $A$. Let $\{\tilde{E}_t\}_{t \in [0, 2\pi]}$ be a strongly left-continuous resolution of the identity of $\tilde{A}$. Denote by $E_t$ ($t \in [0, 2\pi]$) the corresponding strongly left-continuous spectral function of $A$. For arbitrary $h, g \in \mathcal{H}$, $\zeta \in \mathbb{T}_e$ we may write

$$\int_0^{2\pi} \frac{1}{1 - \zeta e^{it}}d(I^*E_tIh, g)_H = \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}}d(\tilde{E}_tx_{h,0}, x_{g,0})_{H_1} =$$

$$= \left( (E_{H_1} - \zeta \tilde{A})^{-1}W[h], W[g] \right)_{H_1} = \left( W^{-1}(E_{H_1} - \zeta \tilde{A})^{-1}W[h], W[g] \right)_{L_2} =$$

$$= \left( (E_{L_2} - \zeta U)^{-1}[h], [g] \right)_{L_2} = \int_0^{2\pi} \left( d\tilde{F}(t) \frac{1}{1 - \zeta e^{it}}h, g \right)_{\mathcal{H}} =$$

$$= \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}}d(\tilde{F}(t)h, g)_{\mathcal{H}}.$$

Therefore

$$\int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}}d(I^*E_tIh, g)_H = 2 \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}}d(I^*E_tIh, g)_H -$$
\[- \int_0^{2\pi} d(I^* E_t I^*_t, g)_H = 2 \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} d(I^* E_t I^*_t, g)_H - (S_0 h, g)_H =
\]
\[= 2 \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} d(\hat{F}(t) h, g)_H - (\hat{F}(2\pi) h, g)_H = \int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} d(\hat{F}(t) h, g)_H.\]

By the well-known inversion formula we conclude that \( \hat{F}(t) = I^* E_t I \).

**Theorem 1** Let the truncated operator trigonometric moment problem (1) with \( d \in \mathbb{N} \) be given and condition (2) hold. Let an operator \( A_0 \) in a Hilbert space \( H \) be constructed as in (8), \( A = A_0 \). All solutions of the moment problem have the following form:

\[F(t) = I^* E_t I, \quad t \in [0, 2\pi],\]

where \( I \) is defined by (10) and \( E_t \) is a strongly left-continuous spectral function of \( A \). On the other hand, each strongly left-continuous spectral function of \( A \) generates by (17) a solution of the moment problem. Moreover, the correspondence between all strongly left-continuous spectral functions of \( A \) and all solutions of the moment problem (1) is one-to-one.

**Proof.** It remains to check that different left-continuous spectral functions of \( A \) generate different solutions of the moment problem (1). Set \( L_0 := \{x_{h,0}\}_{h \in H} \). Choose an arbitrary element \( x \in \mathcal{L}, \ x = \sum_{j=0}^{d} x_{h_j, j}, h_j \in H \). For arbitrary \( \zeta \in T_e \setminus \{0\} \) there exists the following representation:

\[x = v + y, \quad v \in M_\zeta(A), \ y \in L_0.\]

Here \( v \) and \( y \) may depend on the choice of \( \zeta \). In fact, for an arbitrary element \( u \in D(A_0), u = \sum_{j=0}^{d-1} x_{g_j, j}, g_j \in H \), we may write

\[(E_H - \zeta A)u = \sum_{j=0}^{d-1} x_{g_j, j} - \zeta \sum_{j=0}^{d-1} x_{g_{j+1}, j} = \sum_{j=0}^{d-1} x_{g_j, j} - \zeta \sum_{j=1}^{d} x_{g_{j-1}, j} =
\]

\[= x_{g_0, 0} + \sum_{j=1}^{d-1} x_{g_j - \zeta g_{j-1}, j} + x - \zeta g_{d-1}.d.\]

Consider the following system of equations:

\[
\left\{
\begin{align*}
g_j - \zeta g_{j-1} &= h_j, & 1 \leq j \leq d - 1 \\
-\zeta g_{d-1} &= h_d
\end{align*}
\right.
\]
We can find $g_{d-1}$, then $g_{d-2}$, ..., $g_0$. Consider $u$ with this choice of $g_j$ and set $v := (E_H - \zeta A)u \in M_\zeta(A)$. By (19), (20) we see that

$$x - v = x_{h_0,0} - x_{g_0,0} = x_{h_0 - g_0,0} =: y \in L_0.$$ 

Then relation (18) holds.

Suppose to the contrary that two different strongly left-continuous spectral functions $E_{j,t}$, $j = 1, 2$, generate the same solution of the moment problem: $I^*E_{1,t}I = I^*E_{2,t}I$. For arbitrary $f, g \in \mathcal{H}$ we have:

$$(E_{1,t}x_{f,0}, x_{g,0})_H = (E_{2,t}x_{f,0}, x_{g,0})_H.$$ 

Multiplying by $\frac{1}{1 - \zeta e^{it}}$ and integrating we get

$$(R_{1,\zeta}x_{f,0}, x_{g,0})_H = (R_{2,\zeta}x_{f,0}, x_{g,0})_H, \quad f, g \in \mathcal{H}, \quad \zeta \in \mathbb{T}_e,$$ 

where $R_{j,\zeta}$ is a generalized resolvent corresponding to $E_{j,t}$, $j = 1, 2$. Let $R_{j,\zeta}$ is generated by a unitary extension $\tilde{A}_j$ of $A$ in a Hilbert space $\tilde{H}_j \supseteq H$, $j = 1, 2$. Since for arbitrary $f \in D(A), \zeta \in \mathbb{T}_e$ and $j = 1, 2$ we have

$$\left(E_{\tilde{H}_j} - \zeta \tilde{A}_j\right)^{-1} (E_H - \zeta A) f = \left(E_{\tilde{H}_j} - \zeta \tilde{A}_j\right)^{-1} \left(E_{\tilde{H}_j} - \zeta \tilde{A}_j\right) f = f \in H,$$ 

then

$$R_{1,\zeta}u = R_{2,\zeta}u, \quad u \in M_\zeta(A), \quad \zeta \in \mathbb{T}_e.$$ 

Choose an arbitrary $\zeta \in \mathbb{T}_e \setminus \{0\}$. We may write:

$$(R_{j,\zeta}x_{f,0}, u) = (x_{f,0}, R_{j,\zeta}^*u) = \left(x_{f,0}, u - R_{j,\zeta}^{-1} u\right),$$

where $f \in \mathcal{H}, u \in M_\zeta(A), j = 1, 2$. By (22) we get

$$(R_{1,\zeta}x_{f,0}, u) = (R_{2,\zeta}x_{f,0}, u), \quad u \in M_\zeta(A), \quad f \in \mathcal{H}, \quad \zeta \in \mathbb{T}_e \setminus \{0\}. \quad (23)$$

Choose an arbitrary element $w \in \mathfrak{L}$ and $\zeta \in \mathbb{T}_e \setminus \{0\}$. By (18) we may write: $w = v + y$, where $v \in M_\zeta(A), y \in L_0$. By (21), (23) we get $(R_{1,\zeta}x_{f,0}, w) = (R_{2,\zeta}x_{f,0}, w), f \in \mathcal{H}$. Therefore

$$R_{1,\zeta}x = R_{2,\zeta}x, \quad x \in L_0, \quad \zeta \in \mathbb{T}_e \setminus \{0\}. \quad (24)$$

For an arbitrary $w \in \mathfrak{L}$ using (18) we may write: $w = v' + y'$, where $v' \in M_\zeta(A), y' \in L_0, \zeta \in \mathbb{T}_e \setminus \{0\}$. By (22), (24) we get $R_{1,\zeta}w = R_{2,\zeta}w,$
\[ \zeta \in \mathbb{T}\setminus\{0\}. \] Therefore \( E_{1,t} = E_{2,t} \). This contradiction completes the proof. \[ \Box \]

Notice that relation (17) is equivalent to the following relation:

\[ (\mathcal{F}(t)g, h)_{\mathcal{H}} = (E_t x_{g,0}, x_{h,0}), \quad g, h \in \mathcal{H}. \] (25)

From the latter relation it follows that

\[ \int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} d(\mathcal{F}(t)g, h)_{\mathcal{H}} = 2(R_{\zeta} x_{g,0}, x_{h,0}) - (S_0 g, h)_{\mathcal{H}}, \quad g, h \in \mathcal{H}, \quad \zeta \in \mathbb{T}. \] (26)

By virtue of Chumakin’s formula for the generalized resolvents of an isometric operator (see [3]) we conclude that the following formula:

\[ \int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} d(\mathcal{F}(t)g, h)_{\mathcal{H}} = 2 \left( [E_H - \zeta (A \oplus \Phi_{\zeta})]^{-1} x_{g,0}, x_{h,0} \right)_{\mathcal{H}} - (S_0 g, h)_{\mathcal{H}}, \quad g, h \in \mathcal{H}, \quad \zeta \in \mathbb{D}. \] (27)

establishes a one-to-one correspondence between all functions \( \Phi \in S(\mathbb{D}; H \ominus D(A), H \ominus R(A)) \) and all solutions of the moment problem (1).

We shall need the following proposition which is close to Frobenius’s inversion formula for matrices.

**Proposition 1** Let \( M \) be a linear bounded operator in a Hilbert space \( \mathcal{H} \), \( D(M) = \mathcal{H} \). Suppose that

\[ \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \] (28)

where \( \mathcal{H}_1, \mathcal{H}_2 \) are subspaces of \( \mathcal{H} \), and the operator \( M \) has the following block representation:

\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \] (29)

where \( A = P_{\mathcal{H}_1} M|_{\mathcal{H}_1}, B = P_{\mathcal{H}_1} M|_{\mathcal{H}_2}, C = P_{\mathcal{H}_2} M|_{\mathcal{H}_1}, D = P_{\mathcal{H}_2} M|_{\mathcal{H}_2} \). Suppose that \( A \) has a bounded inverse which is defined on the whole \( \mathcal{H}_1 \). Then the following assertions hold.

(i) If the operator \( \mathcal{H} = \mathcal{D} - CA^{-1}B \) has a bounded inverse which is defined on the whole \( \mathcal{H}_2 \), then \( M \) has a bounded inverse which is defined on the whole \( \mathcal{H} \), and \( M^{-1} \) has the following block representation with respect to decomposition (28):

\[ M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B\mathcal{H}^{-1}CA^{-1} & -A^{-1}B\mathcal{H}^{-1} \\ -\mathcal{H}^{-1}CA^{-1} & \mathcal{H}^{-1} \end{pmatrix}. \] (30)
(ii) If the operator $M$ has a bounded inverse which is defined on the whole $\mathcal{H}$, then the operator $\mathcal{H} = D - CA^{-1}B$ has a bounded inverse which is defined on the whole $\mathcal{H}_2$, and $M^{-1}$ has the block representation \( (30) \) with respect to decomposition \( (28) \).

**Proof.** (i): in this case the operator which is defined by the block representation in \( (30) \), is defined on the whole $\mathcal{H}$ and it is bounded. The fact that it is the inverse of $M$ is verified by a direct block multiplication.

(ii): let us check that $\mathcal{H}$ is invertible. To the contrary, suppose that there exists a non-zero element $h$ in $\mathcal{H}_2$: $\mathcal{H}h = 0$. Denote $u = -A^{-1}Bh$. Then

$$\begin{pmatrix} A & B \\ 0 & \mathcal{H} \end{pmatrix} \begin{pmatrix} u \\ h \end{pmatrix} = 0. \tag{31}$$

Notice that

$$\begin{pmatrix} A & B \\ 0 & \mathcal{H} \end{pmatrix} = \begin{pmatrix} E_{H_1} & 0 \\ -CA^{-1} & E_{H_2} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and the operator $\begin{pmatrix} E_{H_1} & 0 \\ -CA^{-1} & E_{H_2} \end{pmatrix}$ has a bounded inverse which is defined on the whole $\mathcal{H}$, equal to $\begin{pmatrix} E_{H_1} & 0 \\ CA^{-1} & E_{H_2} \end{pmatrix}$. Therefore an operator $\begin{pmatrix} A & B \\ 0 & \mathcal{H} \end{pmatrix}$ has a bounded inverse which is defined on the whole $\mathcal{H}$. This contradicts to relation \( (31) \). Thus, the operator $\mathcal{H}$ is invertible. Since $R\left( \begin{pmatrix} A & B \\ 0 & \mathcal{H} \end{pmatrix} \right) = \mathcal{H}$, then for an arbitrary element $\tilde{h} \in \mathcal{H}_2$ there exist elements $u_1 \in H_1$, $h_2 \in H_2$, such that:

$$\begin{pmatrix} 0 \\ \tilde{h} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & \mathcal{H} \end{pmatrix} \begin{pmatrix} u_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} Au_1 + Bh_2 \\ \mathcal{H}h_2 \end{pmatrix}.$$

Therefore $R(\mathcal{H}) = \mathcal{H}_2$. Since an operator $\mathcal{H}^{-1}$ is closed and defined on the whole $\mathcal{H}_2$, then $\mathcal{H}^{-1}$ is bounded. Applying assertion (i) we conclude that $M^{-1}$ has a block representation \( (30) \) with respect to decomposition \( (28) \). □

Return to our constructions for the solvable moment problem \( (1) \) with $d \in \mathbb{N}$. Let us apply Proposition \( (1) \) to the operator $M := E_H - \zeta(A \oplus \Phi\zeta)$, $\zeta \in \mathbb{D}$, $\Phi \in S(\mathbb{D})$, $H \oplus D(A)$, $H \oplus R(A)$, in a Hilbert space $H = \mathcal{H}$ with $H_1 = D(A)$, $H_2 = H \oplus D(A)$. In this case the operators $A, B, C, D$ in \( (29) \) have the following form:

$$A = E_{D(A)} - \zeta P^H_{D(A)}A, \quad B = -\zeta P^H_{D(A)}\Phi\zeta,$$
\[ C = -\zeta P^H_{H \ominus D(A)} A, \quad D = E_{H \ominus D(A)} - \zeta P^H_{H \ominus D(A)} \Phi \zeta, \quad \zeta \in \mathbb{D}. \quad (32) \]

By (27) we conclude that the following relation

\[
\int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} d(F(t)g, h)_H =
\]

\[
eq 2 \left( \left\{ A^{-1} + A^{-1}B^H \Phi^{-1}CA^{-1} \right\} x_{g,0}, x_{h,0} \right)_H - (S_0g, h)_H, \quad g, h \in H, \quad \zeta \in \mathbb{D}, \quad (33) \]

establishes a one-to-one correspondence between all functions \( \Phi \in S(\mathbb{D}; H \ominus D(A), H \ominus R(A)) \) and all solutions of the moment problem \[(1). \]

By the substitution of expressions from (32) we obtain that relation (33) takes the following form:

\[
\int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} d(F(t)g, h)_H =
\]

\[
eq 2 \left( \left\{ A(\zeta) + B(\zeta) \Phi \zeta \left( E_{H \ominus D(A)} + C(\zeta) \Phi \zeta \right)^{-1} D(\zeta) \right\} x_{g,0}, x_{h,0} \right)_H -
\]

\[- (S_0g, h)_H, \quad g, h \in H, \quad \zeta \in \mathbb{D}, \quad (34) \]

where

\[
A(\zeta) = \left( E_{D(A)} - \zeta P^H_{D(A)} A \right)^{-1}, \quad B(\zeta) = -\zeta A(\zeta) P^H_{D(A)},
\]

\[
D(\zeta) = -\zeta P^H_{H \ominus D(A)} AA(\zeta),
\]

\[
C(\zeta) = -\zeta P^H_{H \ominus D(A)} + \zeta D(\zeta) P^H_{D(A)}, \quad \zeta \in \mathbb{D}. \quad (35) \]

Finally, we obtain that the following relation

\[
\int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} dF(t) =
\]

\[
= A(\zeta) + B(\zeta) \Phi \zeta \left( E_{H \ominus D(A)} + C(\zeta) \Phi \zeta \right)^{-1} D(\zeta), \quad \zeta \in \mathbb{D}, \quad (36) \]

establishes a one-to-one correspondence between all functions \( \Phi \in S(\mathbb{D}; H \ominus D(A), H \ominus R(A)) \) and all solutions of the moment problem \[(1). \]

Here

\[
A(\zeta) = 2I^* A(\zeta) I - S_0, \quad B(\zeta) = 2I^* B(\zeta) \big|_{H \ominus R(A)},
\]

\[
C(\zeta) = C(\zeta) \big|_{H \ominus R(A)}, \quad D(\zeta) = D(\zeta) I, \quad \zeta \in \mathbb{D}. \quad (37) \]
**Theorem 2** Let the truncated operator trigonometric moment problem (1) with \( d \in \mathbb{N} \) be given and condition (2) hold. Let an operator \( A_0 \) in a Hilbert space \( H \) be constructed as in (8), \( A = A_0 \). Relation (36) establishes a one-to-one correspondence between all functions \( \Phi \in S(\mathbb{D}; H \ominus D(A), H \ominus R(A)) \) and all solutions of the moment problem (1).

**Proof.** The proof follows from the preceding considerations. \( \square \)

**Corollary 1** Let the truncated operator trigonometric moment problem (1) with \( d \in \mathbb{N} \) be given and condition (2) hold. Let an operator \( A_0 \) in a Hilbert space \( H \) be constructed as in (8), \( A = A_0 \). The moment problem is determinate if and only if (at least) one of the defect numbers of \( A \) is zero.

**Proof.** If one of the defect numbers of \( A \) is zero, then the only function in \( S(\mathbb{D}; H \ominus D(A), H \ominus R(A)) \) is zero.

On the other hand, if both defect numbers of \( A \) are non-zero than we may choose unit nonzero vectors \( h \in H \ominus D(A), g \in H \ominus R(A) \). Set \( \Phi_1(\zeta) = 0, \Phi_2(\zeta) = (\cdot, h)g \), \( \zeta \in \mathbb{D} \). Functions \( \Phi_1, \Phi_2 \) generate different solutions of the moment problem. \( \square \)

Let the truncated operator trigonometric moment problem (1) with \( d \in \mathbb{N} \) be given. Suppose that condition (2) holds and the Hilbert space \( \mathcal{H} \) is separable, \( \mathcal{H} \neq \{0\} \). Let \( \mathcal{C} = \{f_\omega \}_{k=0}^{\omega-1} \), \( 1 \leq \omega \leq +\infty \), is an orthonormal basis in \( \mathcal{H} \). Let us calculate the matrices of operators \( A, B, C, D \) in (37) with respect to some proper bases. As a consequence, we can obtain the matrix of the operator appearing on the right of (36) with respect to \( \mathcal{C} \) using the prescribed moments.

Observe that \( H \) is separable. In fact, an arbitrary element \( x \) of \( \mathcal{L} \) has the following form: \( x = \sum_{j=0}^{d} x_{h_j,j}, h_j \in \mathcal{H} \). Let \( \mathcal{W} \) be a dense subset of \( \mathcal{H} \) (which is not supposed to be countable). Choose an arbitrary \( \varepsilon > 0 \). There exist elements \( y_j \in \mathcal{W}, 0 \leq j \leq d \), such that

\[
\|h_j - y_j\|_{\mathcal{H}} \leq \frac{\varepsilon}{(d + 1)\|S_0\|^{1/2}}, \quad 0 \leq j \leq d.
\]

Then

\[
\left\| x - \sum_{j=0}^{d} x_{h_j,j} \right\|_{H} \leq \sum_{j=0}^{d} \|x_{h_j,j} - x_{y_j,j}\|_{H} =
\]

\[
= \sum_{j=0}^{d} \sqrt{(x_{h_j,j} - y_j, x_{h_j,j} - y_j)} \leq \sum_{j=0}^{d} \sqrt{(S_0(h_j - y_j), h_j - y_j)} \mathcal{H} \leq
\]

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Therefore a set \( \tilde{W} := \{ \sum_{j=0}^{d} x_{y_j,j}, \quad y_j \in W \} \) is dense in \( H \). If \( W \) is chosen countable then \( \tilde{W} \) is countable, as well. Thus, \( H \) is separable. On the other hand, if we choose \( W = \text{Lin} C \) then we obtain that \( \text{Lin}\{ x_{f_k,j} \}_{k\in[0,\infty), 0 \leq j \leq d} \) is dense in \( H \).

At first we suppose that the moment problem is indeterminate. If we would have \( D(A) = \{0\} \), then \( (S_0 h, g)_H = (x_{h,0}, x_{g,0})_H = 0, \quad h, g \in H \). This would meant that \( S_0 = 0 \) and the moment problem is determinate. This contradicts to our assumptions. Therefore \( D(A) \neq \{0\} \). Observe that

\[
D(A) = \text{span}\{ x_{f_k,j} \}_{k\in[0,\infty), 0 \leq j \leq d-1}. \tag{38}
\]

Let us numerate the elements of the set \( \Omega := \{ x_{f_k,j} \}_{k\in[0,\infty), 0 \leq j \leq d-1} \) by a unique index: \( \Omega := \{ y_n \}_{n\in[0,\infty), 0 \leq \kappa \leq +\infty} \). Apply the Gram-Schmidt orthogonalization procedure to the following sequence:

\[
\{ y_n \}_{n\in[0,\infty)}, \tag{39}
\]

removing linear dependent elements, if they appear. After the orthogonalization we shall obtain an orthonormal basis \( \mathfrak{A}_1 = \{ u_k \}_{k=0}^{\tau-1}, 1 \leq \tau \leq +\infty \) in \( D(A) \). Apply the Gram-Schmidt orthogonalization procedure to the following sequence:

\[
\left\{ x_{f_k,d} - P_{D(A)}^H x_{f_k,d} \right\}_{k\in[0,\infty)}, \tag{40}
\]

removing linear dependent elements, if they appear. Observe that the elements \( P_{D(A)}^H x_{f_k,d} \) in \( \text{Lin}(\mathfrak{A}_1) \) can be constructed using \( \mathfrak{A}_1 \). After this orthogonalization we shall obtain an orthonormal basis \( \mathfrak{A}_2 := \{ u'_j \}_{j=0}^{\delta-1}, 1 \leq \delta \leq +\infty \) in \( H \ominus D(A) \).

Observe that \( \mathfrak{A}'_1 := \{ A_0 u_k \}_{k=0}^{\tau-1} \) is an orthonormal basis in \( R(A) \). Apply the Gram-Schmidt orthogonalization procedure to the following sequence:

\[
\left\{ x_{f_k,0} - P_{R(A)}^H x_{f_k,0} \right\}_{k\in[0,\infty)}, \tag{41}
\]

removing linear dependent elements, if they appear. Observe that the elements \( P_{R(A)}^H x_{f_k,0} \) in \( \text{Lin}(\mathfrak{A}'_1) \) can be constructed using \( \mathfrak{A}'_1 \). By the orthogonalization we shall obtain \( \mathfrak{A}'_2 := \{ v'_k \}_{k=0}^{\rho-1}, 1 \leq \rho \leq +\infty \). Notice that \( \mathfrak{A}'_2 \) is an orthonormal basis in \( H \ominus R(A) \).
The above orthonormal bases can be used to construct the matrices of operators on the right in (36). Here relation (17) will be used intensively.

In the case of the determinate moment problem the right-hand side of (36) is equal to $A(\zeta)$. Thus, in this case we can use $C$ and an orthonormal basis $\mathcal{A}$, constructed by the orthogonalization of $\{x f_{k,j}\}_{k \in \mathbb{Z}, \omega - 1, 0 \leq j \leq d}$.

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On the truncated operator trigonometric moment problem.

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In this paper we study the truncated operator trigonometric moment problem. All solutions of the moment problem are described by a Nevanlinna-type parameterization. In the case of moments acting in a separable Hilbert space, the matrices of the operator coefficients in the Nevanlinna-type formula are calculated by the prescribed moments. Conditions for the determinacy of the moment problem are given, as well.