Matrix Roots of Imprimitive Irreducible Nonnegative Matrices

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Abstract

Using matrix function theory, Perron-Frobenius theory, combinatorial matrix theory, and elementary number theory, we characterize, classify, and describe in terms of the Jordan canonical form the matrix $p$th-roots of imprimitive irreducible nonnegative matrices. Preliminary results concerning the matrix roots of reducible matrices are provided as well.

Keywords: matrix function, eventually nonnegative matrix, Jordan form, irreducible matrix, matrix root, Perron-Frobenius theorem

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1. Introduction

A real matrix $A$ is said to be \textit{eventually nonnegative} (positive) if there exists a nonnegative integer $p$ such that $A^k$ is entrywise nonnegative (positive) for all $k \geq p$. If $p$ is the smallest such integer, then $p$ is called the \textit{power index of $A$} and is denoted by $p(A)$.

Eventual nonnegativity has been the subject of study in several papers \cite{3, 4, 5, 10, 14, 15, 19, 20} and it is well-known that the notions of eventual positivity and nonnegativity are associated with properties of the eigenspace corresponding to the spectral radius.

A real matrix $A$ is said to possess the \textit{Perron-Frobenius property} if its spectral radius is a positive eigenvalue corresponding to an entrywise nonnegative eigenvector. The \textit{strong Perron-Frobenius property} further requires that the spectral radius is simple; that it dominates in modulus every other eigenvalue of $A$; and that it has an entrywise positive eigenvector.
Several challenges regarding the theory and applications of eventually nonnegative matrices remain unresolved. For example, eventual positivity of $A$ is equivalent to $A$ and $A^\top$ possessing the strong Perron-Frobenius property, however, the Perron-Frobenius property for $A$ and $A^\top$ is a necessary but not sufficient condition for eventual nonnegativity of $A$.

A matrix $p$th-root or matrix root of a matrix $A$ is any matrix that satisfies the equation $X^p - A = 0$. An eventually nonnegative (positive) matrix with power index $p = p(A)$ is, by definition, a $p$th-root of the nonnegative (positive) matrix $A^p$. As a consequence, in order to gain more insight into the powers of an eventually nonnegative (positive) matrix, it is natural to examine the roots of matrices that possess the (strong) Perron-Frobenius property. In [13], the matrix-roots of eventually positive matrices were classified. In this research, we classify the matrix roots of irreducible imprimitive nonnegative matrices; in particular, the main results in Section 3 provide necessary and sufficient conditions for the existence of an eventually nonnegative matrix $p$th-root of such a matrix. In addition, our proofs demonstrate how to construct these roots given a Jordan canonical form of such a matrix.

2. Background

Denote by $i$ the imaginary unit, i.e., $i := \sqrt{-1}$. When convenient, an indexed set of the form \( \{x_i, x_{i+1}, \ldots, x_{i+j}\} \) is abbreviated to \( \{x_k\}_{k=i}^{i+j} \).

For $h$ a positive integer greater than one, let
\[
R(h) := \{0, 1, \ldots, h-1\}, \quad \omega = \omega_h := \exp(2\pi i/h) \in \mathbb{C}, \quad \Omega_h := \{\omega^k\}_{k=0}^{h-1} = \{\omega^k\}_{k=0}^{h-1} \subseteq \mathbb{C}.
\]
\[
\Omega_h^* := (1, \omega, \ldots, \omega^{h-1}) \in \mathbb{C}^n. \tag{2.1}
\]

With $\omega$ as defined, it is well-known that for $\alpha, \beta \in \mathbb{Z}$,
\[
\alpha \equiv \beta \mod h \implies \omega^\alpha = \omega^\beta. \tag{2.3}
\]

Denote by $M_n(\mathbb{C})$ ($M_n(\mathbb{R})$) the algebra of complex (respectively, real) $n \times n$ matrices. Given $A \in M_n(\mathbb{C})$, the spectrum of $A$ is denoted by $\sigma(A)$, the spectral radius of $A$ is denoted by $\rho(A)$, and the peripheral spectrum, denoted by $\pi(A)$, is the multi-set given by
\[
\pi(A) = \{\lambda \in \sigma(A) : |\lambda| = \rho(A)\}.
\]

The direct sum of the matrices $A_1, \ldots, A_k$, where $A_i \in M_n(\mathbb{C})$, denoted by $A_1 \oplus \cdots \oplus A_k$, or $\bigoplus_{i=1}^k A_i$, or $\text{diag}(A_1, \ldots, A_k)$, is the $n \times n$ matrix
\[
\begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & A_k
\end{bmatrix}.
\]
where \( n = \sum_{i=1}^{k} n_i \).

For \( \lambda \in \mathbb{C} \), \( J_n(\lambda) \) denotes the \( n \times n \) Jordan block with eigenvalue \( \lambda \). For \( A \in M_n(\mathbb{C}), \) denote by \( J = Z^{-1}AZ = \bigoplus_{i=1}^{s} J_{n_i}(\lambda_i) = \bigoplus_{i=1}^{t} J_{n_i}, \) where \( \sum n_i = n \), a Jordan canonical form of \( A \). Denote by \( \lambda_1, \ldots, \lambda_s \) the distinct eigenvalues of \( A \), and, for \( i = 1, \ldots, s \), let \( m_i \) denote the index of \( \lambda_i \), i.e., the size of the largest Jordan block associated with \( \lambda_i \).

For \( z = r \exp (i\theta) \in \mathbb{C}, \) where \( r > 0, \) and an integer \( p > 1, \) let
\[
z^{1/p} := r^{1/p} \exp (i\theta/p),
\]
and, for \( j \in R(p), \) let
\[
f_j(z) := z^{1/p} \exp (i2\pi j/p) = r^{1/p} \exp (i(\theta + 2\pi j)/p),
\]
i.e., \( f_j \) denotes the \((j + 1)\)st-branch of the \( p \)th-root function.

### 2.1. Combinatorial Structure

For notation and definitions concerning the combinatorial structure of a matrix, i.e., the location of the zero-nonzero entries of a matrix, we follow [2] and [7]; for further results concerning combinatorial matrix theory, see [2] and references therein.

A directed graph (or simply digraph) \( \Gamma = (V, E) \) consists of a finite, nonempty set \( V \) of vertices, together with a set \( E \subseteq V \times V \) of arcs. For \( A \in M_n(\mathbb{C}), \) the directed graph (or simply digraph) of \( A \), denoted by \( \Gamma = \Gamma(A) \), has vertex set \( V = \{1, \ldots, n\} \) and arc set \( E = \{(i, j) \in V \times V : a_{ij} \neq 0\} \). If \( R, C \subseteq \{1, \ldots, n\} \), then \( A[R|C] \) denotes the submatrix of \( A \) whose rows and columns are indexed by \( R \) and \( C \), respectively.

A digraph \( \Gamma \) is strongly connected or strong if for any two distinct vertices \( u \) and \( v \) of \( \Gamma \), there is a walk in \( \Gamma \) from \( u \) to \( v \) (following [2], we consider every vertex of \( V \) as strongly connected to itself). For a strongly connected digraph \( \Gamma \), the index of imprimitivity is the greatest common divisor of the the lengths of the closed walks in \( \Gamma \). A strong digraph is primitive if its index of imprimitivity is one, otherwise it is imprimitive.

For \( n \geq 2, \) a matrix \( A \in M_n(\mathbb{C}), \) is reducible if there exists a permutation matrix \( P \) such that
\[
P^\top AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},
\]
where \( A_{11} \) and \( A_{22} \) are nonempty square matrices and \( 0 \) is a rectangular zero block. If \( A \) is not reducible, then \( A \) is called irreducible. The connection between reducibility and the digraph of \( A \) is as follows: \( A \) is irreducible if and only if \( \Gamma(A) \) is strongly connected (see, e.g., [2, Theorem 3.2.1] or [8, Theorem 6.2.24]).

\(^{1}\)Following [2], vertices are strongly connected to themselves so we take this result to hold for all \( n \in \mathbb{N}. \)
For \( h \geq 2 \), a digraph \( \Gamma = (V, E) \) is cyclically \( h \)-partite if there exists an ordered partition \( \Pi = (\pi_1, \ldots, \pi_h) \) of \( V \) into \( h \) nonempty subsets such that for each arc \((i, j) \in E\), there exists \( \ell \in \{1, \ldots, h\} \) such that \( i \in \pi_\ell \) and \( j \in \pi_{\ell+1} \) (where, for convenience, we take \( V_{h+1} := V_1 \)). For \( h \geq 2 \), a strong digraph \( \Gamma \) is cyclically \( h \)-partite if and only if \( h \) divides the index of imprimitivity (see, e.g., [2, p. 70]). A matrix \( A \in M_n(\mathbb{C}) \) is called \( h \)-cyclic if \( \Gamma(A) \) is cyclically \( h \)-partite and if \( \Gamma(A) \) is cyclically \( h \)-partite with ordered partition \( \Pi \), then \( A \) is said to be cyclically \( h \)-partite with ordered partition \( \Pi \) or that \( \Pi \) describes the cyclic structure of \( A \). The ordered partition \( \Pi = (\pi_1, \ldots, \pi_h) \) is consecutive if \( \pi_1 = \{1, \ldots, i_1\}, \pi_2 = \{i_1+1, \ldots, i_2\}, \ldots, \pi_h = \{i_{h-1}+1, \ldots, n\} \). If \( A \) is cyclically \( h \)-partite with consecutive ordered partition \( \Pi \), then \( A \) has the block form

\[
\begin{bmatrix}
0 & A_{12} & 0 & \cdots & 0 \\
0 & 0 & A_{23} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & A_{(h-1)h} \\
A_{hh} & 0 & 0 & \cdots & 0
\end{bmatrix}
\] (2.5)

where \( A_{i,i+1} = A^{[\pi_i|\pi_{i+1}]} \) (2 p. 71]). For any \( h \)-cyclic matrix \( A \), there exists a permutation matrix \( P \) such that \( P^TAP \) is \( h \)-cyclic with consecutive ordered partition. The cyclic index or index of cyclicity of \( A \) is the largest \( h \) for which \( A \) is \( h \)-cyclic.

An irreducible nonnegative matrix \( A \) is primitive if \( \Gamma(A) \) is primitive, and the index of imprimitivity of \( A \) is the index of imprimitivity of \( \Gamma(A) \). If \( A \) is irreducible and imprimitive with index of imprimitivity \( h \geq 2 \), then \( h \) is the cyclic index of \( A \). \( \Gamma(A) \) is cyclically \( h \)-partite with ordered partition \( \Pi = (\pi_1, \ldots, \pi_h) \), and the sets \( \pi_i \) are uniquely determined (up to cyclic permutation of \( \pi_i \)) (see, for example, [2, p. 70]). Furthermore, \( \Gamma(A^h) \) is the disjoint union of \( h \) primitive digraphs on the sets of vertices \( \pi_i, i = 1, \ldots, h \) (see, e.g., [2, §3.4]).

Following [1], given an ordered partition \( \Pi = (\pi_1, \ldots, \pi_h) \) of \( \{1, \ldots, n\} \) into \( h \) nonempty subsets, the cyclic characteristic matrix, denoted by \( \chi_{\Pi} \), is the \( n \times n \) matrix whose \((i, j)\)-entry is 1 if there exists \( \ell \in \{1, \ldots, h\} \) such that \( i \in \pi_\ell \) and \( j \in \pi_{\ell+1} \), and 0 otherwise. For an ordered partition \( \Pi = (\pi_1, \ldots, \pi_h) \) of \( \{1, \ldots, n\} \) into \( h \) nonempty subsets, note that

1. \( \chi_{\Pi} \) is \( h \)-cyclic and \( \Gamma(\chi_{\Pi}) \) contains every arc \((i, j)\) for \( i \in \pi_\ell \) and \( j \in \pi_{\ell+1} \);

and

2. \( A \in M_n(\mathbb{C}) \) is \( h \)-cyclic with ordered partition \( \Pi \) if and only if \( \Gamma(A) \subseteq \Gamma(\chi_{\Pi}) \).

We recall the Perron-Frobenius Theorem for irreducible, imprimitive matrices.

**Theorem 2.1** (see, e.g., [1] or [8]). Let \( A \in M_n(\mathbb{R}), n \geq 2, \) and suppose that \( A \) is irreducible, nonnegative, imprimitive and suppose that \( h > 1 \) is the cyclic index of \( A \). Then
(a) \( \rho > 0 \);
(b) \( \rho \in \sigma (A) \);
(c) there exists a positive vector \( x \) such that \( Ax = \rho x \);
(d) \( \rho \) is an algebraically (and hence geometrically) simple eigenvalue of \( A \); and
(e) \( \pi (A) = \{ \rho \exp (i2\pi k/h) : k \in \mathbb{R}(h) \} \).
(f) \( \omega^k \sigma (A) = \sigma (A) \) for \( k \in \mathbb{R}(h) \).

2.2. Matrix Functions

For background material concerning matrix functions, we follow [6]; for more results concerning matrix functions, see, e.g., [6], [9, Chapter 6], or [11, Chapter 9]. Herein it is assumed that the complex matrix \( A \) has \( s \leq t \) distinct eigenvalues.

**Definition 2.2.** Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a function and let \( f^{(k)} \) denote the \( k \)th derivative of \( f \). The function \( f \) is said to be defined on the spectrum of \( A \) if the values

\[
    f^{(k)}(\lambda_i), \quad k = 0, \ldots, m_i - 1, \quad i = 1, \ldots, s,
\]

called the values of the function \( f \) on the spectrum of \( A \), exist.

**Definition 2.3** (Matrix function via Jordan canonical form). If \( f \) is defined on the spectrum of \( A \in M_n(\mathbb{C}) \), then

\[
    f(A) := Z f(J) Z^{-1} = Z \left( \bigoplus_{i=1}^t f(J_{n_i}) \right) Z^{-1},
\]

where

\[
    f(J_{n_i}) := \begin{bmatrix}
        f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{f^{(n_i-1)}(\lambda_i)}{(n_i-1)!} \\
        f(\lambda_i) & \ddots & \vdots & \ddots \\
        \vdots & \ddots & f'(\lambda_i) & f(\lambda_i)
    \end{bmatrix}, \quad (2.6)
\]

**Theorem 2.4** (see, e.g., [6, §1.9] or [17, Theorem 2.2.5]). Let \( f \) be defined on the spectrum of a nonsingular matrix \( A \in M_n(\mathbb{C}) \) and suppose that \( f'(\lambda_i) \neq 0 \) for \( i = 1, \ldots, t \). If \( J = \bigoplus_{i=1}^t J_{n_i}(\lambda_i) = Z^{-1}AZ \) is a Jordan canonical form of \( A \), then

\[
    J_f := \bigoplus_{i=1}^t J_{n_i}(f(\lambda_i))
\]

is a Jordan canonical form of \( f(A) \).
Theorem 2.5 ([18, Theorems 2.1 and 2.2]). If \( A \in M_n(\mathbb{C}) \) is nonsingular, then \( A \) has precisely \( p^s \) \( p \)-th-roots that are expressible as polynomials in \( A \), given by

\[
X_j = Z \left( \bigoplus_{i=1}^{t} f_{j_i}(J_{m_i}) \right) Z^{-1}, \tag{2.7}
\]

where \( j = (j_1, \ldots, j_t) \), \( j_i \in \{0, 1, \ldots, p-1\} \), and \( j_i = j_k \) whenever \( \lambda_i = \lambda_k \).

If \( s < t \), then \( A \) has additional \( p \)-th-roots that form parameterized families

\[
X_j(U) = ZU \left( \bigoplus_{i=1}^{t} f_{j_i}(J_{m_i}) \right) U^{-1}Z^{-1}, \tag{2.8}
\]

where \( U \) is an arbitrary nonsingular matrix that commutes with \( J \) and, for each \( j \), there exist \( i \) and \( k \), depending on \( j \), such that \( \lambda_i = \lambda_k \), while \( j_i \neq j_k \).

In the theory of matrix functions, the roots given by (2.7) are called the primary roots of \( A \), and the roots given by (2.8), which exist only if \( A \) is derogatory (i.e., some eigenvalue appears in more than one Jordan block), are called the nonprimary roots [6, Chapter 1].

2.3. Other results

Theorem 2.6 ([16, Theorem 13]). Let \( f_k \) be defined as in (2.4). If

\[
\mathcal{B} := \{ j = (j_0, j_1, \ldots, j_{h-1}) : j_k \in R(p), \forall k \in R(h) \},
\]

and

\[
(\Omega_h)^{1/p}_j := \{ f_{j_k}(\omega^k) \}_{k=0}^{h-1}, \quad j \in \mathcal{B},
\]

then there exists a unique \( j \in \mathcal{B} \) such that \( (\Omega_h)^{1/p}_j = \Omega_h \) if and only if \( \gcd(h, p) = 1 \).

Remark 2.7. For every \( k \in R(h) \) and \( j = (j_1, \ldots, j_{h-1}) \in \mathcal{B} \), it is easy to verify that \( f_{j_k}(\omega^k) = \omega^{(k+hj_k)/p} \), whence it follows that

\[
(\Omega_h)^{1/p}_j = \left\{ \omega^{(k+hj_k)/p} \right\}_{k=0}^{h-1}.
\]

In [16] it is shown that if \( j \) is the unique \( h \)-tuple as specified in Theorem 2.6 then the set

\[
\mathcal{E}_j := \left\{ \frac{k + hj_k}{p} \right\}_{k=0}^{h-1}
\]

is integral and a complete residue system modulo \( h \), i.e., the map \( \phi : \mathcal{E}_j \to R(h) \), where

\[
(k + hj_k)/p \to ((k + hj_k)/p) \mod h,
\]
is bijective. Moreover, because $0 \leq (k + hj_k)/p \leq h - 1$ (the lower-bound is trivial and the upper bound follows because for any $k$,
\[
\frac{k + hj_k}{p} \leq \frac{(h - 1) + h(p - 1)}{p} = \frac{hp - 1}{p} = h - \frac{1}{p} < h
\]
it follows that
\[E_j = R(h). \tag{2.9}\]

If $(\Omega^* h^j)_1^1/p := (f_{j_0}(1), f_{j_1}(\omega), \ldots, f_{j_{h-1}}(\omega^{h-1})))$, then, following (2.9), $(\Omega^* h^j)_1^1/p$ corresponds to a permutation of $\Omega^*_h$.

Because \( \gcd(h, p) = 1 \), it follows that \((k + hj_1)/p) \equiv ((k + hj_k)/p) \mod h\), which, along with (2.3), yields
\[(\omega^{(1+hj_1)/p})^k = \omega^{(k+hj_1)/p} = \omega^{(k+hj_k)/p}, \quad k \in \{2, \ldots, h - 1\}. \]

Hence
\[(\Omega^* h^j)_1^1/p = (1, (\omega^{(1+hj_1)/p})/p, \ldots, (\omega^{(h-1)+hj_{h-1}})/p) = (1, (\omega^{(1+hj_1)/p})^1, \ldots, (\omega^{(1+hj_1)/p})^{h-1}). \]

Note that
\[(\Omega^* h^q) := (1, \omega^q, \ldots, \omega^{qh-1}) = (1, (\omega^q)^1, \ldots, (\omega^q)^{h-1}) \]
is a permutation of the elements in $\Omega^*_h$ if and only if $\gcd(h, q) = 1$ [16, Corollary 7]: following (2.3), for every $h$ there are $\varphi(h)$ such permutations, where $\varphi$ denotes Euler’s totient function. Thus, if $\gcd(h, p) = 1$, then there exists $q \in \mathbb{N}$, $\gcd(h, q) = 1$, such that
\[(\Omega^* h^j)_1^1/p = (\Omega^* h^q). \tag{2.10}\]

Next, we state results concerning the structure of the Jordan chains of $h$-cyclic matrices. It is assumed that $A \in M_n(\mathbb{C})$ is nonsingular, $h$-cyclic with ordered-partition $\Pi$, and has the form (2.5).

**Corollary 2.8** [12, Corollary 3.2]. If $J_r(\lambda)$ is a Jordan block of $J$, then $J_r(\lambda^k)$ is a Jordan block of $J$ for $k \in R(h)$.

**Remark 2.9.** With $\Omega^*_h$ as defined in (2.2) and
\[J(\lambda \Omega^*_h, r) := \begin{bmatrix} J_r(\lambda) & J_r(\lambda\omega) & \cdots & J_r(\lambda\omega^{h-1}) \end{bmatrix} \in M_{rh}(\mathbb{C}), \tag{2.11} \]
it follows that a Jordan form of a nonsingular $h$-cyclic matrix $A$ has the form
\[Z^{-1}AZ = J = \bigoplus_{i=1}^{\ell'} J(\lambda_i \Omega^*_h, r_i), \quad \ell' \in \mathbb{Z}. \]
Lemma 2.10 ([12, Theorem 3.6]). For \( i = 1, \ldots, t' \), if

\[
A_{\lambda_i} := Z \text{diag} \left( 0, \ldots, 0, J(\lambda_i \Omega_h^*, r_i), 0, \ldots, 0 \right) Z^{-1} \in M_n(\mathbb{C}), \quad (2.12)
\]

then \( \Gamma(A_{\lambda_i}) \subseteq \Gamma(\chi \Pi) \), \( A_{\lambda_i} \) commutes with \( A \), and \( A_{\lambda_i} A_{\lambda_j} = A_{\lambda_j} A_{\lambda_i} = 0 \) for \( i \neq j, \ j = 1, \ldots, t' \).

Corollary 2.11. If \( x \) is a strictly nonzero right eigenvector and \( y \) is a strictly nonzero left eigenvector of \( A \) corresponding to \( \lambda \in \mathbb{C} \), then \( A_{\lambda} \) has cyclic index \( h \) and \( \Gamma(A_{\lambda}) = \Gamma(\chi \Pi) \).

In particular, we examine the case when \( A \) is a nonnegative, irreducible, imprimitive, nonsingular matrix with index of cyclicity \( h \). Without loss of generality, it is assumed that \( \rho(A) = 1 \). Following [Theorem 2.1] and [Corollary 2.8] note that the Jordan form of \( A \) is

\[
Z^{-1} A Z = \begin{bmatrix}
J(\Omega_h^*, 1) & & \\
 & J(\lambda_2 \Omega_h^*, r_2) & \\
& & \ddots \\
& & & J(\lambda_t \Omega_h^*, r_t')
\end{bmatrix}.
\]

Consider the matrix

\[
A_1 := Z \left[ J(\Omega_h^*, 1) \ 0 \ 0 \right] Z^{-1}.
\]

Following [Corollary 2.11] \( \Gamma(A_1) = \Gamma(\chi \Pi) \), \( AA_1 = A_1 A \), and, following [Theorem 2.1] there exist positive vectors \( x \) and \( y \) such that \( Ax = x \) and \( y^\top A = y^\top \). If \( x \) and \( y \) are partitioned conformably with \( A \) as

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_h \end{bmatrix} \quad \text{and} \quad y^\top = \begin{bmatrix} y_1^\top \\ y_2^\top \\ \cdots \\ y_h^\top \end{bmatrix},
\]

then

\[
A_1 = h \begin{bmatrix}
0 & x_1 y_2^\top & \cdots & \cdots & 0 \\
0 & 0 & x_2 y_3^\top & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & x_{h-1} y_h^\top \\
x_h y_1^\top & 0 & \cdots & 0 & 0
\end{bmatrix} \geq 0.
\]
3. Main Results

Unless otherwise noted, we assume $A \in M_n(\mathbb{R})$ is a nonnegative, nonsingular, irreducible, imprimitive matrix with $\rho(A) = 1$. Before we state our main results, we introduce additional concepts and notation: following Friedland [5], for a multi-set $\sigma = \{\lambda_i\}_{i=1}^n \subseteq \mathbb{C}$, let $\rho(\sigma) := \max_i\{|\lambda_i|\}$ and $\bar{\sigma} := \{\bar{\lambda}_i\}_{i=1}^n \subseteq \mathbb{C}$. If $\sigma = \bar{\sigma}$, we say that $\sigma$ is self-conjugate. Clearly, $\sigma$ is self-conjugate if and only if $\bar{\sigma}$ is self-conjugate.

The (multi-)set $\sigma$ is said to be a Frobenius (multi-)set if, for some positive integer $h \leq n$, the following properties hold:

(i) $\rho(\sigma) > 0$;

(ii) $\sigma \cap \{z \in \mathbb{C} : |z| = \rho(\sigma)\} = \rho(\sigma) \Omega_h$; and

(iii) $\sigma = \omega \sigma$, i.e., $\sigma$ is invariant under rotation by the angle $2\pi/h$.

Clearly, the set $\Omega_h$ as defined in (2.1) is a self-conjugate Frobenius set.

The importance of Frobenius multi-sets becomes clear in view of the following result, which was introduced and stated without proof in [5, §4, Lemma 1] and proven rigorously in [20, Theorem 3.1].

**Lemma 3.1.** Let $A$ be an eventually nonnegative matrix. If $A$ is not nilpotent, then the spectrum of $A$ is a union of self-conjugate Frobenius sets.

Let $\lambda = r \exp(i\theta) \in \mathbb{C}$, $\Im(\lambda) \neq 0$, $\lambda \Omega_h := \{\lambda, \lambda \omega_h, \ldots, \lambda \omega_h^{h-1}\}$, $\varphi = 2\pi k/h$, and assume $\gcd(h, p) = 1$. With $f_j$ as defined in (2.4), a tedious but straightforward calculation shows that

$$f_i(\lambda) f_{j_k}(\omega^k) = f_{j_k(i)}(\hat{\lambda}),$$

where $\hat{\lambda} = r \exp(i(\theta + \varphi))$ and $j_k^{(i)} = (i + j_k) \mod p \in \{0, 1, \ldots, p - 1\}$ (17, pp. 62–63).

Hence, following Theorem 2.6, there exists a unique $j \in \mathcal{B}$ such that, for all $i \in R(p)$, the set

$$f_i(\lambda) (\Omega_h)^{1/p} := \{f_i(\lambda) f_{j_0}(1), f_i(\lambda) f_{j_1}(\omega_h), \ldots, f_i(\lambda) f_{j_{h-1}}(\omega_h^{h-1})\}$$

is a self-conjugate Frobenius set; moreover, following (3.1), there exists $j^{(i)} = (j_0^{(i)}, j_1^{(i)}, \ldots, j_{h-1}^{(i)}) \in \mathcal{B}$ such that

$$f_i(\lambda) (\Omega_h)^{1/p} = \{f_{j_0^{(i)}}(\lambda), f_{j_1^{(i)}}(\lambda \omega_h), \ldots, f_{j_{h-1}^{(i)}}(\lambda \omega_h^{h-1})\}.$$  

As a consequence of Theorem 2.6 and Lemma 3.1 it should be clear that $A$ cannot possess an eventually nonnegative $p$th-root if $\gcd(h, p) > 1$; however, more can be ascertained.
Partition the Jordan form of $A$ as
\[
\begin{bmatrix}
J_+ & J_- \\
& J_C
\end{bmatrix}
\tag{3.4}
\]
where:
(1) $J (\lambda \Omega^*_h, r)$ is a submatrix of $J_+$ if and only if $\sigma (J (\lambda \Omega^*_h, r)) \cap \mathbb{R}^+ \neq \emptyset$ and $\sigma (J (\lambda \Omega^*_h, r)) \cap \mathbb{R}^- = \emptyset$;
(2) $J (\lambda \Omega^*_h, r)$ is a submatrix of $J_-$ if and only if $\sigma (J (\lambda \Omega^*_h, r)) \cap \mathbb{R}^- \neq \emptyset$; and
(3) $J (\lambda \Omega^*_h, r)$ is a submatrix of $J_C$ if and only if $\sigma (J (\lambda \Omega^*_h, r)) \cap \mathbb{R} = \emptyset$.
(4) $J (\lambda \Omega^*_h, r)$ is defined as in (2.11), i.e.,
\[
J (\lambda \Omega^*_h, r) := \begin{bmatrix}
J_r(\lambda) \\
J_r(\lambda \omega) \\
\vdots \\
J_r(\lambda \omega^{h-1})
\end{bmatrix} \in M_{rh}(\mathbb{C}),
\]
Suppose $J_+$ has $r_1$ distinct blocks of the form $J (\lambda \Omega^*_h, r)$, $J_-$ has $r_2$ distinct blocks of the form $J (\lambda \Omega^*_h, r)$, and $J_C$ has $c$ distinct blocks of the form $J (\lambda \Omega^*_h, r)$.

We are now ready to present our main results.

**Theorem 3.2.** Let $A \in M_n(\mathbb{R})$, and suppose that $A \geq 0$, nonsingular, irreducible, and imprimitive. Let $h > 1$ be the cyclic index of $A$, let $\Pi$ describe the $h$-cyclic structure of $A$, let the Jordan form of $A$ be partitioned as in (3.4), and suppose that $\gcd(h, p) = 1$.

If $p$ is even, and
(a) $r_2 = 0$, then $A$ has $2^{r_1-1}p^c$ eventually nonnegative primary $p$th-roots; or
(b) $r_2 > 0$, then $A$ has no eventually nonnegative primary $p$th-roots.

If $p$ is odd, then $A$ has $p^c$ eventually nonnegative primary $p$th-roots.

**Proof.** For $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $j = (j_0, j, \ldots, j_{h-1}) \in \mathcal{B}$, let
\[
F_j (J (\lambda \Omega^*_h, r)) := \begin{bmatrix}
f_{j_0} (J_r(\lambda)) \\
f_{j_1} (J_r(\lambda \omega)) \\
\vdots \\
f_{j_{h-1}} (J_r(\lambda \omega^{h-1}))
\end{bmatrix}.
\]
We form an eventually nonnegative root $X$ by carefully selecting a root for each submatrix of every block appearing in (3.4).

Case 1: $p$ is even. We consider the blocks appearing in (3.4):
(i) \(J_+:\) Following Theorem 2.1, \(J(\Omega^*_h, 1)\) is a submatrix of \(J_+\) (note that \(h\) must be odd) and, following Theorem 2.6, there exists \(j = (j_0,j_1,\ldots,j_h) \in \mathcal{B}\) such that \(\sigma(F_j(J(\Omega^*_h, 1))) = \Omega_h\). With that specific choice of \(j\), it is clear that
\[
\rho(F_j(J(\Omega^*_h, 1))) = 1.
\]

For any other submatrix \(J(\lambda\Omega^*_h, r)\) of \(J_+\), without loss of generality, we may assume that \(\lambda \in \mathbb{R}^+\). For every such \(\lambda, k\) must be chosen such that \(f_k(\lambda)\) is real (see [13, Corollary 2.16]) and \(f_k(\lambda)\) is real if \(k = 0\) or \(k = p/2\); hence, there are two choices such that \(f_k(\lambda)\) is real and, following (3.2) and (3.3), there are two choices such that \(\sigma(F_{j(k)}(J(\lambda\Omega^*_h, r)))\) is a self-conjugate Frobenius set.

(ii) \(J_-:\) If \(r_2 > 0\), then \(A\) does not have a real primary root so that, a fortiori, it can not have an eventually nonnegative primary root.

(iii) \(J_C:\) For any submatrix \(J(\lambda\Omega^*_h, r)\) of \(J_C\), following (3.2) and (3.3), for the same \(j\) chosen for \(F_j(J(\Omega^*_h, 1))\) in (ii), there exists \(j^{(k)} = (j_0^{(k)}, j_1^{(k)},\ldots,j_{r-1}^{(k)}) \in \mathcal{B}\)

such that \(\sigma(F_{j^{(k)}}(J(\lambda\Omega^*_h, r)))\) is a self-conjugate Frobenius set for all \(k \in R(p)\). Thus, there are \(p^c\) ways to choose roots for blocks in \(J_C\).

Following the analysis contained in (i) (iii) note that there are \(2^{r_1-1}p^c\) ways to form a root in this manner.

Case 2: \(p\) is odd. Following Theorem 2.6 and properties of the \(p\)th-root function, if \(p\) is odd, then for any submatrix \(J(\lambda\Omega^*_h, r)\) of \(J_+\) or \(J_-\), there is only one choice \(j \in \mathcal{B}\) such that \(\sigma(F_j(J(\lambda\Omega^*_h, r)))\) is a self-conjugate Frobenius set. For submatrices \(J(\lambda\Omega^*_h, r)\) of \(J_C\), the analysis is the same as in (iii). In this manner, there are \(p^c\) possible selections.

Partition the Jordan form of \(A\) as
\[
\begin{bmatrix}
J(\Omega^*_h, 1) & \tilde{j} \\
\end{bmatrix}.
\]

With the above partition in mind, consider the matrix \(p\)th-root of \(A\) given by
\[
X = Z \begin{bmatrix} F_j(J(\Omega^*_h, 1)) & \\ F(\tilde{j}) & \end{bmatrix} Z^{-1}
\]
where \(j\) is selected as in Theorem 2.6 and \(F(\tilde{j})\) is a \(p\)th-root of \(\tilde{j}\) containing blocks of the form \(F_{j^{(k)}}(J(\lambda\Omega^*_h, r))\), chosen as in Case 1 or Case 2.

The matrix \(X\) must be irreducible because every power of a reducible matrix is reducible, and if \(\tilde{h}\) is the cyclic index of \(X\), then \(1 \leq \tilde{h} \leq h\) (if \(\tilde{h} > h\), then \(X\) would have \(\tilde{h}\) eigenvalues of maximum modulus and consequently so would \(A\), contradicting the maximality of \(h\)) and \(\tilde{h}\) must divide \(h\). However, we claim
that \( \tilde{h} = h \). For contradiction, assume \( \tilde{h} < h \) and consider the matrix \( A_1 \) given by

\[
A_1 = Z \begin{bmatrix}
  J(\Omega^*_h, 1) & 0 \\
  0 & 0
\end{bmatrix} Z^{-1}.
\]

Following Corollary 2.11, \( A_1 \geq 0 \), irreducible, \( h \)-cyclic, and \( \Pi \) describes the \( h \)-cyclic structure of \( A_1 \). Next, consider the matrix \( X_1 \) given by

\[
X_1 = Z \begin{bmatrix}
  F_j (J(\Omega^*_h, 1)) & 0 \\
  0 & 0
\end{bmatrix} Z^{-1},
\]

which is a matrix \( p \)-th root of \( A_1 \). Following Corollary 2.11, the cyclic index of \( X_1 \) is \( \tilde{h} \). However, following the remarks leading up to (2.10), there exists \( q \in \mathbb{N} \) relatively prime to \( h \) such that \( X_1 = A_1^q \geq 0 \) (along with \( X_1 \) being a \( p \)-th root of \( A_1 \), this implies \( X_1 = X_1^{pq} \)). Thus, \( X_1 \) is a nonnegative, irreducible, \( h \)-cyclic matrix, contradicting the maximality of \( h \). Hence, \( \tilde{h} = h \) and \( X \) is \( h \)-cyclic.

Before we continue with the proof, note that \( \Gamma (X) \subseteq \Gamma (X_1) = \Gamma (\chi^{q}_{II}) \).

Following Theorem 2.4 there exists \( \tilde{Z} \in M_n(\mathbb{C}) \) such that

\[
\tilde{Z}^{-1} X \tilde{Z} = \begin{bmatrix}
  F_j (J(\Omega^*_h, 1)) & J \left( f_{j_2} (\lambda_2) \right) \left( \Omega^*_h \right)^{1/p} , r_2 \\
  & \ddots \\
  & & J \left( f_{j_t} (\lambda_t) \right) \left( \Omega^*_h \right)^{1/p} , r_t
\end{bmatrix}.
\]

By construction of \( \tilde{Z} \) (Lemma 1.3.2), note that

\[
X_1 = Z \begin{bmatrix}
  F_j (J(\Omega^*_h, 1)) & 0 \\
  0 & 0
\end{bmatrix} Z^{-1} = Z \begin{bmatrix}
  F_j (J(\Omega^*_h, 1)) & 0 \\
  0 & 0
\end{bmatrix} \tilde{Z}^{-1}.
\]

Let

\[
X_2 := \tilde{Z} \begin{bmatrix}
  0 & J \left( f_{j_2} (\lambda_2) \right) \left( \Omega^*_h \right)^{1/p} , r_2 \\
  & \ddots \\
  & & J \left( f_{j_t} (\lambda_t) \right) \left( \Omega^*_h \right)^{1/p} , r_t
\end{bmatrix} \tilde{Z}^{-1}.
\]

Following Lemma 2.10, \( X_2 \) is \( h \)-cyclic, \( \Gamma (X_2) \subseteq \Gamma (\chi^{q}_{II}) \), and \( X_1 X_2 = X_2 X_1 = 0 \). Thus, for \( k \in \mathbb{N} \), \( X_k = X_1^k + X_2^k \). Because \( \rho (X_2) < 1 \), \( \lim_{k \to \infty} X_2^k = 0 \). The matrices \( X_1 \) and \( X_2 \) are \( h \)-cyclic, \( \text{rank} (X_1^k) = \text{rank} (X_1) \), \( \text{rank} (X_2^k) = \text{rank} (X_2) \), and \( \Gamma (X_1) \subseteq \Gamma (X_2) = \Gamma (\chi^{q}_{II}) \), thus, following Theorem 2.7, note that \( \Gamma (X_2^k) \subseteq \Gamma (X_1^k) = \Gamma (\chi^{q}_{II}) \). Thus, there exists \( p \in \mathbb{N} \) such that \( X_1^k > X_2^k \) for all \( k \geq p \), i.e., \( X \) is eventually nonnegative.
Example 3.3. We demonstrate Theorem 3.2 via an example: consider the matrix
\[
A = \frac{1}{3} \begin{bmatrix}
0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 \\
2 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0
\end{bmatrix} \in M_6(\mathbb{R}).
\]
One can verify that \(A = ZDZ^{-1}\), where
\[
Z = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 \\
1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\
1 & \omega & \omega^2 & -1 & -\omega & -\omega \\
1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\
1 & \omega^2 & \omega & -1 & -\omega & -\omega
\end{bmatrix},
\]
and
\[
D = \text{diag} \left( 1, \omega, \omega^2, \frac{1}{3}, \frac{1}{3} \omega, \frac{1}{3} \omega^2 \right).
\]
Because \(A\) has six distinct eigenvalues, following Theorem 2.5 it has \(2^6 = 64\) primary square roots (and no nonprimary roots).

The matrices
\[
\hat{D} = \text{diag} \left( 1, \omega, \omega^2, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \omega, \frac{\sqrt{3}}{3} \omega^2 \right)
\]
and
\[
\tilde{D} = \text{diag} \left( 1, \omega^2, \omega, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \omega, -\frac{\sqrt{3}}{3} \omega^2 \right)
\]
are square roots of \(D\), and, following Theorem 3.2 the matrices
\[
\hat{X} = Z\hat{D}Z^{-1} \approx \begin{bmatrix}
0 & 0 & 0 & 0 & 0.7887 & 0.2113 \\
0 & 0 & 0 & 0 & 0.2113 & 0.7887 \\
0.7887 & 0.2113 & 0 & 0 & 0 & 0 \\
0.2113 & 0.7887 & 0 & 0 & 0 & 0 \\
0 & 0.7887 & 0.2113 & 0 & 0 & 0 \\
0 & 0 & 0.2113 & 0.7887 & 0 & 0
\end{bmatrix}
\]
and
\[
\tilde{X} = Z\tilde{D}Z^{-1} \approx \begin{bmatrix}
0 & 0 & 0 & 0 & 0.2113 & 0.7887 \\
0 & 0 & 0 & 0 & 0.7887 & 0.2113 \\
0.2113 & 0.7887 & 0 & 0 & 0 & 0 \\
0.7887 & 0.2113 & 0 & 0 & 0 & 0 \\
0 & 0.2113 & 0.7887 & 0 & 0 & 0 \\
0 & 0 & 0.7887 & 0.2113 & 0 & 0
\end{bmatrix}
\]
are the only eventually nonnegative square roots of $A$.

**Remark 3.4.** Following the notation of [Theorem 3.2] note that all primary roots of $A$ are given by

\[
X = Z \begin{bmatrix}
F_{j_1} (J(\Omega^*_h, 1)) & & \\
& F_{j_2} (J(\lambda^*_2 \Omega^*_h, r_2)) & \\
& & \ddots \\
& & & F_{j_t} (J(\lambda^*_t \Omega^*_h, r_t))
\end{bmatrix} Z^{-1}
\]

where $j_k \in B$ for $k = 1, \ldots, t'$, and subject to the constraint that $j_k = j_\ell$ (because $j_k$ and $j_\ell$ are ordered $h$-tuples, equality is meant entrywise) if $\lambda_k = \lambda_\ell$.

If $A$ is *deregatory*, i.e., some eigenvalue appears in more than one Jordan block in the Jordan form of $A$, then $A$ has additional roots of the form

\[
X(U) = ZU \begin{bmatrix}
F_{j_1} (J(\Omega^*_h, 1)) & & \\
& F_{j_2} (J(\lambda^*_2 \Omega^*_h, r_2)) & \\
& & \ddots \\
& & & F_{j_t} (J(\lambda^*_t \Omega^*_h, r_t))
\end{bmatrix} U^{-1} Z^{-1}
\]

where $U$ is any matrix that commutes with $J$ and subject to the constraint that $j_k \neq j_\ell$, if $\lambda_k = \lambda_\ell$. Select branches $j_1, j_2, \ldots, j_t$ following the proof of [Theorem 3.2] Following [11, Theorem 1, §12.4], note that the matrix $U$ must be of the form

\[
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_h
\end{bmatrix}
\]

and $X(U)$ is real if $U$ is selected to be real. The matrix $X(U)$ is irreducible because $X(U)^p = A$. Moreover,

\[
X_1(U) := ZU \begin{bmatrix}
F_{j_1} (J(\Omega^*_h, 1)) & 0 \\
0 & 0
\end{bmatrix} U^{-1} Z^{-1} = X_1,
\]

where $X_1$ is defined as in the proof of [Theorem 3.2]. Thus, $X_1(U)$ is a non-negative, irreducible, $h$-cyclic matrix that commutes with $X(U)$, so that the argument demonstrating the eventual nonnegativity of $X$ is also valid for $X(U)$.

**Corollary 3.5.** Let $A \in M_n(\mathbb{R})$ and suppose that $A \geq 0$, irreducible, and imprimitive with index of cyclicity $h$. Then $A$ possesses an eventually nonnegative $p$th-root if and only if $A$ possesses a real $p$th-root and $\gcd(h, p) = 1$.

**Theorem 3.6.** Let $A \in M_n(\mathbb{R})$ and suppose that $A \geq 0$, irreducible, and imprimitive. If $Z^{-1} AZ = J = J_0 \oplus J_1$ is a Jordan canonical form of $A$, where
$J_0$ collects all the singular Jordan blocks and $J_1$ collects the remaining Jordan blocks, and $A$ possesses a real root, then all eventually nonnegative $p$th-roots of $A$ are given by $A = Z(X_0 \oplus X_1)Z^{-1}$, where $X_1$ is any $p$th-root of $J_1$ characterized by Theorem 3.2 or Remark 3.4 and $X_0$ is a real $p$th-root of $J_0$.

Although the following result is known (see [7, Algorithm 3.1 and its proof]), our work provides another proof.

**Corollary 3.7.** If $A \in M_n(\mathbb{R})$ is irreducible with index of cyclicity $h$, where $h > 1$, and $\Pi$ describes the $h$-cyclic structure of $A$, then $A$ is eventually nonnegative if and only if there exists a nonsingular matrix $Z$ such that

$$Z^{-1}AZ = \begin{bmatrix} J(\Omega^*_h, 1) & \hat{J} \\ & \end{bmatrix},$$

and, associated with $\rho(A) = 1 \in \sigma(A)$, is a positive left eigenvector $x$ and right eigenvector $y$.

**Proof.** If $A$ is eventually nonnegative, select $p$ relatively prime to $h$ such that $A^p \geq 0$. Then $A$ is a $p$th-root of $A^p$ and the result follows from Theorem 3.6.

The converse is clear by setting $X_1 = Z\begin{bmatrix} J(\Omega^*_h, 1) & 0 \\ 0 & 0 \end{bmatrix}Z^{-1}$ and $X_2 = Z\begin{bmatrix} 0 & 0 \\ 0 & \hat{J} \end{bmatrix}Z^{-1}$.

**Remark 3.8.** Theorem 3.6 remains true if the assumption of nonnegativity is replaced with eventual nonnegativity.

**Remark 3.9.** For $A \in M_n(\mathbb{C})$ with no eigenvalues on $\mathbb{R}^-$, the principal $p$th-root, denoted by $A^{1/p}$, is the unique $p$th-root of $A$ all of whose eigenvalues lie in the segment $\{z : -\pi/p < \arg(z) < \pi/p\}$ [6, Theorem 7.2]. A nonnegative matrix $A$ is **stochastic** if $\sum_j a_{ij} = 1$ for all $i = 1, \ldots, n$. Following Theorem 2.6 the principal $p$th-root of an imprimitive irreducible stochastic matrix is never stochastic.

3.1. Reducible Matrices

Identifying the eventually nonnegative matrix roots of nonnegative reducible matrices poses many obstacles, chief of which is controlling the entries of the off-diagonal blocks in the Frobenius normal form. Moreover, the assumption that $\gcd(h, p) = 1$ is not necessarily required; for instance, consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
and note that the matrix
\[
B := \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} = A^2,
\]
is a reducible 2-cyclic matrix, \(\sigma(B) = \Omega_2 \cup \Omega_2\), \(B\) obviously possesses an irreducible nonnegative square root, but \(\gcd(2, 2) = 2 > 1\).

**Definition 3.10.** A matrix \(A \in M_n(C)\) is said to be **completely reducible** if there exists a permutation matrix \(P\) such that
\[
P^\top AP = \bigoplus_{i=1}^{k} A_{ii} = \begin{bmatrix}
A_{11} & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A_{kk}
\end{bmatrix},
\]
where \(k \geq 2\) and \(A_{11}, \ldots, A_{kk}\) are square irreducible matrices.

**Remark 3.11.** Following the definition, if \(A\) is completely reducible, then, without loss of generality, we may assume \(A\) is in the form of the matrix on the right-hand side of (3.5). Furthermore, it is clear that \(A\) is eventually nonnegative if and only if \(A_{11}, \ldots, A_{kk}\) are eventually nonnegative.

The following is corollary to **Corollary 3.5** and **Theorem 3.6**.

**Corollary 3.12.** Let \(A\) be eventually nonnegative, nonsingular, and completely reducible. Let \(h_i\) denote the cyclicity of \(A_{ii}\) for \(i = 1, \ldots, k\). If each \(A_{ii}\) possesses a real pth-root, then \(A\) possesses an eventually nonnegative pth-root if and only if \(\gcd(h_i, p) = 1\) for all \(i\).

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