Abstract

We study stochastic inflation in the presence of a dynamical gravitational constant. We describe the Arnowitt–Deser–Misner formalism for Jordan–Brans–Dicke theory of gravity with an inflaton field. The inflaton and dilaton scalar fields can be separated into coarse-grained background fields and quantum fluctuations. We compute the amplitude of the perturbations generated by those quantum fluctuations in JBD theory with an arbitrary potential for the inflaton field. The effect of the quantum fluctuations on the background fields is equivalent to a Brownian motion of the scalar fields, which can be described with the use of a Fokker–Planck diffusion equation. The probability to find a given value of the fields in the comoving frame can be written as a Gaussian distribution centered on their classical trajectory, with decreasing dispersion along both field directions. We also calculate the condition for the Universe to enter a self-regenerating inflationary phase. The probability distribution in the physical frame, which takes into account the expansion of the proper volume of the inflationary domains, will be concentrated at the Planck boundary and will move along it towards large values of the fields.

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1 Introduction

The early inflationary scenarios could not solve but only postpone the problem of initial conditions for the Big Bang. The first models assumed that the universe started in a very hot state that supercooled in a metastable vacuum, which then decayed to the true vacuum through a first order phase transition or just rolled down through a second order phase transition. Linde’s chaotic inflation opened the possibility of arbitrary initial conditions for arbitrary effective potentials for the inflaton field. It was soon realized that the most natural initial conditions for inflation were close to Planck scale, where quantum fluctuations of the space-time metric become important.

With the inclusion of the fields’ quantum fluctuations, it was understood that the whole Universe might be in eternal self-reproduction of inflationary domains, while our own observable universe could be just a late subproduct of one of such domains. The Brownian motion of the inflaton field under the effect of its own quantum fluctuations can be described with the help of diffusion equations, in the context of the so-called stochastic inflation formalism. This formalism describes the interplay between quantum fluctuations and coarse-grained background fields, and allows one to study the very large scale structure of the Universe, much beyond our observable universe. It is also a consistent framework for the description of the origin of density perturbations during inflation, that may have given rise to the recently observed temperature fluctuations in the cosmic background radiation and, furthermore, to the density inhomogeneities for galaxy formation.

Stochastic inflation stresses the fact that inflation is a random process, in which quantum fluctuations of the fields act as stochastic forces on their own background values as a result of the expansion in de Sitter space. The universe is divided into causally independent inflationary domains, in which the fields acquire different values. The global description of this stochastic process uses diffusion equations for the probability distribution to find a given value of the fields in a given inflationary domain. The probabilistic description of stochastic inflation opened the possibility of a connection with quantum cosmology, in the sense that it may provide an interpretation of the wave function of the universe as the probability amplitude for an ensemble of inflationary domains, instead as for the whole Universe.

It is generally assumed in stochastic inflation that the gravitational dynamics of the whole Universe is correctly described by general relativity. However, it seems a strong assumption to extrapolate the description of the gravitational phenomena at our local and low energy scales to the very large scales beyond our observable universe. In fact, it is believed that the theory of general relativity is just a low energy effective theory of the gravitational interaction at the quantum level. So far the only consistent but by no means definite, since we lack the experimental observations needed to confirm it, theory of quantum gravity is string theory. String theory contains in its massless gravitational sector a dilaton scalar field as well as the graviton. The low energy effective theory from strings has the form of a scalar-tensor theory, with non-trivial couplings of the dilaton to matter. Therefore, it is expected that the description of gravitational phenomena close to Planck scale should also include this extra scalar field.

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The simplest scalar-tensor theory is Jordan–Brans–Dicke (JBD) theory \[14\] with a constant \( \omega \) parameter, where the Brans–Dicke scalar field is in this case the string dilaton, which acts like a dynamical gravitational constant. Jordan–Brans–Dicke theory of gravity was introduced in the inflationary universe scenario as extended inflation \[15\], in an attempt to solve the graceful-exit problem of old inflation. However, only small values of the BD parameter were compatible with observations of the cosmic background radiation, in contradiction with the large values coming from post-Newtonian experiments. A possible solution could be a variable \( \omega \) parameter \[16\], see however \[17\]. Extended Inflation has also been considered in the context of string effective actions \[18\]. A different proposal is to consider JBD theory together with chaotic inflation \[19\].

In this paper we study the stochastic inflation formalism in the context of JBD theory of gravity with an arbitrary chaotic potential for the inflaton. The presence of a scalar field in the gravitational sector of the theory complicates the picture. There are new effects associated with the quantum fluctuations of both the inflaton and dilaton fields; the former defines the effective dynamics of inflation and the latter the fluctuations of the gravitational constant \[20\]. The main idea of stochastic inflation is to solve the equations for the inhomogeneous fields in de Sitter space by separating both the gravitational and scalar fields in short-distance quantum fluctuations, which oscillate on scales smaller than the Hubble radius, and long-distance fluctuations which are treated as classical fields. It is the special character of de Sitter space and its horizon which allows this distinction. As the inflationary domains expand, short wavelengths cross the horizon and communicate with the long wavelength classical background fields, acting as a stochastic noise term. Therefore we can separate the evolution of the long wavelength field as a two-step process, first a random kick from short-distance quantum fluctuations (diffusion) followed by classical evolution (drift). This process can be understood as a Brownian motion of the fields and thus can be described with the Langevin and Fokker–Planck equations.

In Section 2 we describe the Arnowitt–Deser–Misner (ADM) \[21\] formalism for JBD theory with an inflaton scalar field, see also \[22\]. In this Section we follow closely the work of Salopek and Bond \[23\] and generalize it to JBD. We find the energy and momentum constraints for the inhomogeneous fields, the gravitational and the scalar fields’ equations. These non-linear equations are extremely difficult to solve and one usually expands in small perturbations with respect to a homogeneous background. Fortunately, in the context of stochastic inflation, one is allowed to separate the short-distance behavior from the long-wavelength background fields, which can then be described by a coarse-grained homogeneous field. The background field equations are then obtained by neglecting large-scale gradients. There is a natural scale in de Sitter space given by the Hubble radius. Since we are interested in structures larger than the horizon scale it is natural to expand in powers of \( k/(aH) \) where \( k \) is the comoving wave number of the perturbation. This is done in Section 3, where we arrive at a consistent set of background field equations. In Section 4 we solve these equations in the synchronous gauge and in the slow-roll approximation, where we assume that the scalar fields evolve slower than the expansion of the universe. This simplifies significantly the equations and we obtain approximate solutions for the background fields for generic chaotic potentials of the type \( V(\sigma) = \frac{\lambda}{2n} \sigma^{2n} \). In Section 5 we describe the formalism of small perturbations on a homogeneous background, using linear perturbation theory of the metric and scalar fields in the longitudinal gauge \[24\]. We find the amplitude of perturbations in the slow-roll approximation for the Einstein frame, in terms of both the dilaton and
inflaton fields. Those perturbations have a quantum mechanical origin in the short-wavelength fluctuations of the scalar fields. We compute the amplitude of those quantum fluctuations in a de Sitter universe and find the Gibbons–Hawking temperature [25] for both the inflaton and dilaton in certain well defined limits. We then obtain an expression for the amplitude of density perturbations in the cosmic background radiation that should be compared with the one coming from general relativity.

In Section 6 we describe the stochastic inflation formalism [5, 4, 6] for Jordan–Brans–Dicke inflation, see also [20]. The Brownian motion of the scalar fields in de Sitter space can be written as a Langevin equation for the coarse-grained fields with an effective white noise generated by quantum fluctuations. The associated Fokker–Planck equation for the probability distribution of finding a given value of the scalar fields in a given point of space-time can be derived from the Langevin equation and written in a form which is manifestly time-reparameterization invariant. The proper way to find stationary solutions is to solve the diffusion equation subject to certain well defined boundary conditions [9]. We show that the probability distribution far from the Planck boundary behaves like a Gaussian with fields centered at their classical trajectories and we calculate its dispersion coefficients. The probability distribution in the physical frame takes into account the exponential growth of each inflationary domain. The condition for self-reproduction of the universe is then calculated. Those inflationary domains with fields inside the region of self-reproduction will dominate the proper volume of the universe and will tend to diffuse towards the Planck boundary.

In Section 7 we study the stationary solutions of the Fokker–Planck equation in the physical frame by reexpressing the evolution equation as a Schrödinger equation in two dimensions with an effective potential. We find that the distribution will very quickly settle at the Planck boundary. However, since the Planck boundary is a line it will slide along it, unless we impose extra boundary conditions [20]. In Section 8 we present our conclusions.

2 ADM formalism for JBD Cosmology

In the Arnowitt–Deser–Misner formalism [21] the four dimensional metric $g_{\mu\nu}$ is parametrized by the three-metric $h_{ij}$ and the lapse and shift functions $N$ and $N^i$, which describe the evolution of time-like hypersurfaces [20, 27],

$$g_{00} = -N^2 + h^{ij} N_i N_j, \quad g_{0i} = g_{i0} = N_i, \quad g_{ij} = h_{ij}. \quad (1)$$
The action for the Jordan–Brans–Dicke theory plus an inflaton scalar field with potential $V(\sigma)$ in the ADM formalism has the form \[ S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2(\phi)}{16\pi} R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} (\partial\sigma)^2 - V(\sigma) \right] \]

\[ = \int d^4x \ N \sqrt{\kappa} \left[ \frac{M_p^2(\phi)}{16\pi} \left( (3)R + K_{ij}K^{ij} - K^2 + \frac{4}{\phi} K \Pi^\phi \right) + \frac{1}{2} \left( (\Pi^\phi)^2 - \phi_{i|\phi} \phi^i \right) + \frac{1}{2} \left( (\Pi^\sigma)^2 - \sigma_{i|\sigma} \phi^i \right) - V(\sigma) \right], \]

where $M_p^2(\phi) = \frac{2\pi}{\omega} \phi^2$ is the dilaton-dependent Planck mass and $\Pi^\phi$, $\Pi^\sigma$ are the scalar-fields' momenta

\[ \Pi^\phi = \frac{1}{N}(\dot{\phi} - N^i \phi_i), \]

\[ \Pi^\sigma = \frac{1}{N}(\dot{\sigma} - N^i \sigma_i). \]

Vertical bars denote three-space-covariant derivatives with connections derived from $h_{ij}$, $(3)R$ is the three-space curvature associated with the metric $h_{ij}$, and $K_{ij}$ is the extrinsic curvature three-tensor

\[ K_{ij} = \frac{1}{2N} (N_{ij} + N_j^i - \dot{h}_{ij}), \]

where a dot denotes differentiation with respect to the time coordinate. The traceless part of a tensor is denoted by an overbar. In particular,

\[ \bar{K}_{ij} = K_{ij} - \frac{1}{3} Kh_{ij}, \quad \bar{K} = \bar{K}^i_i. \]

The trace $K$ is a generalization of the Hubble parameter, as will be shown below. Variation of the action with respect to $N$ and $N^i$ yields the energy and momentum constraint equations

\[ -(3)R + \bar{K}_{ij} \bar{K}^{ij} - \frac{2}{3} K^2 + \frac{4}{\phi} K \Pi^\phi + \frac{16\pi}{M_p^2(\phi)} \epsilon = 0, \]

\[ \bar{K}^i_j - \frac{2}{3} K^i_i + \frac{2}{\phi} (\phi_{ij} \bar{K}^j_i + \frac{1}{3} \phi_{|i|K}^i) + \frac{2}{\phi^2} (\phi \Pi^\phi)_{|i} + \frac{8\pi}{M_p^2(\phi)} (\Pi^\phi_{|i} + \Pi^\sigma_{|ii}) = 0. \]

Variation with respect to $h_{ij}$ gives the dynamical gravitational field equations, which can be separated into the trace and traceless parts

\[ \dot{K} - N^i K_{|i} = - N_{|i}^i + N \left( \frac{1}{4} c (3)R + \frac{3}{4} \bar{K}_{ij} \bar{K}^{ij} + \frac{1}{2} K^2 + \frac{4\pi}{M_p^2(\phi)} T \right) \]

\[ - \frac{2}{\phi} K N \Pi^\phi + \frac{3}{\phi^2} (\Pi^\phi - N^i \Pi^\phi_{|i}) + \frac{3N}{\phi^2} (\Pi^\phi)^2 \]

\[ \dot{K}^i_j - N^k K^i_{jk} + N^i_k K^k_j - N^i_j K^k_k = -N_{|i}^i + \frac{1}{3} N^i_k \delta^i_{|j} \]

\[ - \frac{2}{\phi} \bar{K}^i_j N \Pi^\phi + N \left( (3)\bar{R}^i_j + \bar{K} \bar{K}^i_j - \frac{8\pi}{M_p^2(\phi)} T^i_j \right). \]
Variation with respect to $\phi$ and $\sigma$ gives the scalar-fields’ equations of motion

\[ \frac{1}{N}(\dot{\Pi}^\phi - N^i \Pi^\phi_{\dot{\phi}^i}) - K \Pi^\phi - \frac{1}{N} N_{\dot{i}} \phi_{\dot{\phi}^i} - \phi_{\dot{\phi}^i} + \frac{M_p^2(\phi)}{4\pi\phi N}(\dot{K} - N^i \dot{K}_{\dot{i}}) \]

\[ - \frac{M_p^2(\phi)}{8\pi\phi} \left( (3)R + \bar{K}_{ij} \bar{K}^{ij} + \frac{4}{3} K^2 \right) = 0, \]

\[ \frac{1}{N}(\dot{\Pi}^\sigma - N^i \Pi^\sigma_{\dot{\sigma}^i}) - K \Pi^\sigma - \frac{1}{N} N_{\dot{i}} \sigma_{\dot{\sigma}^i} - \sigma_{\dot{\sigma}^i} + \frac{\partial V}{\partial \sigma} = 0. \]

The energy density on a constant-time hypersurface is

\[ \epsilon = \frac{1}{2} \left[ (\Pi^\phi)^2 + \phi_{\dot{\phi}^i} \phi^{\dot{\phi}^i} \right] + \frac{1}{2} \left[ (\Pi^\sigma)^2 + \sigma_{\dot{\sigma}^i} \sigma^{\dot{\sigma}^i} \right] + V(\sigma), \]

and the stress three-tensor

\[ T_{ij} = \phi_{\dot{\phi}^i} \phi_{\dot{\phi}^j} + \sigma_{\dot{\sigma}^i} \sigma_{\dot{\sigma}^j} + h_{ij} \left[ \frac{1}{2} \left[ (\Pi^\phi)^2 - \phi_{\dot{k}} \phi^{\dot{k}} \right] + \frac{1}{2} \left[ (\Pi^\sigma)^2 - \sigma_{\dot{k}} \sigma^{\dot{k}} \right] - V(\sigma) \right]. \]

It is extremely difficult to solve these highly nonlinear coupled equations in a cosmological scenario without making some approximations. The usual approach is to assume homogeneity of the fields to give a background solution and then linearize the equations to study deviations from spatial uniformity. The smallness of cosmic microwave background anisotropies \[10\] gives some justification for this perturbative approach at least in our local part of the Universe. However, there is no reason to believe it will be valid on much larger scales. In fact, the stochastic approach to inflation suggests that the Universe is extremely inhomogeneous on very large scales \[2\]. Fortunately, in this framework one can coarse-grain over a horizon distance and separate the short- from the long-distance behavior of the fields, where the former communicates with the latter through stochastic forces. The equations for the long-wavelength background fields are obtained by neglecting large-scale gradients, leading to a self-consistent set of equations, as we will discuss in the next section.

### 3 Spatial gradient expansion

It is reasonable to expand in spatial gradients whenever the forces arising from time variations of the fields are much larger than forces from spatial gradients. In linear perturbation theory one solves the perturbation equations for evolution outside of the horizon: a typical time scale is the Hubble time $H^{-1}$, which is assumed to exceed the gradient scale $a/k$, where $k$ is the comoving wave number of the perturbation. Since we are interested in structures on scales larger than the horizon, it is reasonable to expand in $k/(aH)$. In particular, for inflation this is an appropriate parameter of expansion since spatial gradients become exponentially negligible after a few $e$-folds of expansion beyond horizon crossing, $k = aH$.

It is therefore useful to split the fields $\phi$ and $\sigma$ into coarse-grained long-wavelength background fields $\phi(t, x^j)$ and residual short-wavelength fluctuating fields $\delta\phi(t, x^j)$. There is a preferred
timelike hypersurface within the stochastic inflation approach in which the splitting can be made consistently, but the definition of the background field will depend on the choice of hypersurface, i.e. the smoothing is not gauge invariant. For stochastic inflation the natural smoothing scale is the comoving Hubble length \((aH)^{-1}\) and the natural hypersurfaces are those on which \(aH\) is constant. In that case a fundamental difference between between \(\phi\) and \(\delta\phi\) is that the short-wavelength components are essentially uncorrelated at different times, while long-wavelength components are deterministically correlated through the equations of motion.

In order to solve the equations for the background fields, we will have to make suitable approximations. The idea is to expand in the spatial gradients of \(\phi\) and to treat the terms that depend on the fluctuating fields as stochastic forces describing the connection between short- and long-wavelength components. In this Section we will neglect the stochastic forces due to quantum fluctuations of the scalar fields and will derive the approximate equation of motion for the background fields. We retain only those terms that are at most first order in spatial gradients, neglecting such terms as \(\phi_i|_i\), \(\phi_i\phi^i\), \((3)R\), \((3)R^i_i\), and \(\bar{T}^j_i\).

We will also choose the simplifying gauge \(N^i = 0\). The evolution equation (8) for the traceless part of the extrinsic curvature is then 
\[
\dot{K}^i_j = NK_j^j - \frac{2}{\phi} K_j^j \dot{\phi}.
\]
Using \(NK = -\frac{\partial}{\partial t} \ln \sqrt{h}\) from (3), we find the solution 
\[
K^i_j \propto \phi^{-2} h^{-1/2},
\]
where \(h\) is the determinant of \(h_{ij}\). During inflation \(h^{-1/2} \equiv a^{-3}\), with \(a\) the overall expansion factor, therefore \(K^i_j\) decays extremely rapidly and can be set to zero in the approximate equations. The most general form of the three-metric with vanishing \(\bar{K}^i_j\) is
\[
h_{ij} = a^2(t, x^k) \gamma_{ij}(x^k), \quad a(t, x^k) \equiv \exp[\alpha(t, x^k)],
\]
where the time-dependent conformal factor is interpreted as a space-dependent expansion factor. The time-independent three-metric \(\gamma_{ij}\), of unit determinant, describes the three-geometry of the conformally transformed space. Since \(a(t, x^k)\) is interpreted as a scale factor, we can substitute the trace \(K\) of the extrinsic curvature for the Hubble parameter
\[
H(t, x^i) \equiv \frac{1}{N(t, x^i)} \dot{\alpha}(t, x^i) = -\frac{1}{3} K(t, x^i).
\]
Furthermore, since the Brans–Dicke parameter \(\omega\) is bounded by post-Newtonian experiments \[28\] and primordial nucleosynthesis \[29\] to be very large, \(\omega > 500\), we will use the approximation \(\omega \gg 1\) in equations (6, 7). The energy and momentum constraint equations (6) can now be written as
\[
H^2 = \frac{8\pi}{3M_p^2(\phi)} \left(\frac{1}{2}(\Pi^\phi)^2 + \frac{1}{2}(\Pi^\sigma)^2 + V(\sigma)\right) - \frac{2}{\phi} H \Pi^\phi,
\]
\[
H_{ij} = -\frac{4\pi}{M_p^2(\phi)} (\Pi^\phi \phi_{ij} + \Pi^\sigma \sigma_{ij}) + \frac{H}{\phi} \phi_{ij},
\]

\[14\]

\[
-\frac{1}{N} \dot{H} = \frac{3}{2} H^2 + \frac{4\pi}{3M_p^2(\phi)} T - \frac{1}{\phi} H \Pi^\phi,
\]

\[15\]
where \( T = 3 \left( \frac{1}{2}(\Pi^\phi)^2 + \frac{1}{2}(\Pi^\sigma)^2 - V(\sigma) \right) \).

In general, \( H \) is a function of the scalar fields and time, \( H(t, x^i) \equiv H(\phi(t, x^i), \sigma(t, x^i), t) \). From the momentum constraint (14) we find that the scalar-fields’ momenta must obey

\[
\Pi^\phi = -\frac{M_p^2(\phi)}{4\pi} \left[ \left( \frac{\partial H}{\partial \phi} \right)_t - \frac{H}{\phi} \right], \\
\Pi^\sigma = -\frac{M_p^2(\phi)}{4\pi} \left( \frac{\partial H}{\partial \sigma} \right)_t.
\] (16)

Contrary to the situation in Einstein theory [23], \( H \) has an explicit time dependence. Comparing equation (15) with the time derivative of \( H(\phi, \sigma, t) \),

\[
\frac{1}{N} \left( \frac{\partial H}{\partial t} \right)_x = \Pi^\phi \left( \frac{\partial H}{\partial \phi} \right)_t + \Pi^\sigma \left( \frac{\partial H}{\partial \sigma} \right)_t + \frac{1}{N} \left( \frac{\partial H}{\partial t} \right)_\phi \sigma,
\]

we find \( \frac{1}{N} \left( \frac{\partial H}{\partial t} \right)_\phi \sigma = \frac{3}{\phi} H \Pi^\phi \). It is only in the slow-roll approximation, when we neglect \( \Pi^\phi \) versus \( H \), that the Hubble parameter becomes time-independent.

On the other hand, the scalar fields’ equations (11) can be written to first order in spatial gradients as

\[
\frac{1}{N} \hat{\Pi}^\phi + 3H \Pi^\phi = \frac{3M_p^2(\phi)}{2\pi} H^2, \\
\frac{1}{N} \hat{\Pi}^\sigma + 3H \Pi^\sigma = -\frac{\partial V}{\partial \sigma}.
\] (18)

This set of equations are still too complicated to solve for arbitrary potentials \( V(\sigma) \). In the next section we will find solutions to them in the slow-roll approximation.

4 Jordan–Brans–Dicke Inflationary Cosmology

We are now interested in the classical behavior of the long-wavelength quasi-homogeneous background fields, and later will study the effect of the stochastic perturbations on these background fields. Furthermore, we will chose the synchronous gauge \( N = 1 \), together with \( N^i = 0 \), and solve the homogeneous part of the inflationary equations of motion (18), (14) in the slow-roll approximation,

\[
H^2 = H_{SR}^2 \equiv \frac{8\pi V(\sigma)}{3M_p(\phi)},
\] (19)
together with \( \dot{\phi} \ll H \dot{\phi} \ll H^2 \phi \). The field equations then look like

\[
\dot{\phi} = -\frac{M_p^2(\phi)}{2\pi} \frac{\partial H}{\partial \phi} = \frac{\dot{\phi}}{\omega} H, \\
\dot{\sigma} = -\frac{M_p^2(\phi)}{4\pi} \frac{\partial H}{\partial \sigma} = -\frac{V'(\sigma)}{3H}, \\
\dot{\alpha} = H
\]

Let us now study the behavior of the approximate inflationary solutions for a generic chaotic potential, \( V(\sigma) = \frac{\lambda}{2n} \sigma^{2n} \).

\[
\dot{\phi} = \frac{2}{n} \left( \frac{\frac{n\lambda}{6\omega}}{\sigma} \right)^{1/2} \sigma^n, \\
\dot{\sigma} = -\frac{\dot{\phi}}{\sigma} \left( \frac{\frac{n\lambda}{6\omega}}{\sigma} \right)^{1/2} \sigma^n.
\]

For these theories, \( \phi \) and \( \varphi \equiv \sqrt{\frac{2}{n}} \sigma \) move along a circumference of constant radius in the plane \((\varphi, \phi)\). We can parametrize the classical trajectory by polar coordinates \((\phi(t), \varphi(t)) = (r \sin(\theta(t)), r \cos(\theta(t)))\), with constant radius \(r\), and angular velocity given by

\[
\dot{\theta}(t) = \left( \frac{\lambda}{3\omega} \right)^{1/2} \left( \frac{n}{2} \right)^{\frac{n-1}{2}} [r \cos(\theta(t))]^{n-1}.
\]

For \( n = 1 \) we find a constant angular velocity and the solutions can be written as

\[
\phi(t) = r \sin \left( \theta_o + \frac{mt}{\sqrt{3}\omega} \right), \\
\sigma(t) = \sqrt{\frac{r}{2}} \cos \left( \theta_o + \frac{mt}{\sqrt{3}\omega} \right), \\
a(t) = a_o \left( \frac{\sin(\theta_o + \frac{mt}{\sqrt{3}\omega})}{\sin \theta_o} \right)^{\omega} \sim t^{\omega}, \quad \theta_o \frac{\sqrt{3}\omega}{m} < t < \frac{\pi \sqrt{3}\omega}{2m},
\]

with the usual power-law behavior of extended inflation. For \( n \geq 2 \), the angular velocity decreases and the classical solutions are more complicated, although still power-law. These solutions are actually attractors of the complete equations of motion, for all \( n \).

In the chaotic inflation scenario, the most natural initial conditions for inflation are set at the Planck boundary, \( V(\sigma_p) \simeq M_p^4(\phi_p) \), beyond which a classical space–time has no meaning and the energy gradient of the inhomogeneities produced during inflation becomes greater than the potential energy density, thus preventing inflation itself. The initial conditions for inflation are thus defined at the curve \( \phi_p^2 = \frac{\omega}{2\pi} V(\sigma)^{\frac{n}{2}} \). On the other hand, inflation will end when the kinetic energy density of the scalar fields becomes comparable with the potential energy density, \( \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\sigma}^2 \simeq V(\sigma) \) or \( \phi_e = \frac{n}{2\sqrt{3}\pi} M_p(\phi_e) = \frac{n}{\sqrt{6}\omega} \phi_e \).
In the absence of any potential for $\phi$, the dilaton remains approximately constant after inflation, and therefore the Planck mass today is given by its value at the end of inflation, 

$$M_p \simeq \sqrt{\frac{2\pi}{\omega}} \phi_e,$$

while the total amount of inflation is approximately 

$$\frac{a_e}{a_p} \simeq \left(\frac{\phi_e}{\phi_p}\right)\omega.$$

Now that we have the background solutions we would like to describe the formalism of small perturbations on the homogeneous background, to account for the quantum fluctuations in de Sitter space.

## 5 Perturbations on a homogeneous background

In order to compute the perturbations on a homogeneous long-wavelength background, we use linear perturbation theory of the metric and scalar fields in the longitudinal gauge and apply it to our case of Brans-Dicke theory with an inflaton field. We follow the notation of [24], and define

$$\begin{align*}
\phi(t, x^i) &= \phi(t) + \delta \phi(t, x^i) \\
\sigma(t, x^i) &= \sigma(t) + \delta \sigma(t, x^i) \\
N(t, x^i) &= 1 + \Phi(t, x^i), \quad N^i(t, x^i) = 0 \\
h_{ij}(t, x^i) &= [1 - 2\Phi(t, x^i)] a^2(t) \delta_{ij}.
\end{align*}$$

(24)

Note that we are calling $\phi$ to the homogeneous part, to avoid carrying unnecessary subindices, and $\delta \phi$ to the inhomogeneous perturbation. The extrinsic curvature (4) and the three-curvature scalar take simple expressions,

$$K = -3H(1 - \Phi) + 3\dot{\Phi}, \quad \bar{K}_{ij} = 0, \quad (3)R = \frac{4}{a^2} \Phi_{,ij}^{,i},$$

(25)

while the scalar fields’ momenta are $\Pi^\phi = \dot{\phi} + \delta \dot{\phi} - \dot{\phi} \Phi$. The homogeneous part of the momentum constraint (3) is trivial, but that of the energy constraint is not,

$$H^2 = \frac{8\pi}{3M_p^2(\phi)} \left(\frac{1}{2} \dot{\sigma}^2 + \frac{1}{2} \dot{\phi}^2 + V(\sigma)\right) = 2H \dot{\phi}.$$

(26)

The inhomogeneous parts of the energy and momentum constraints have complicated expressions,

$$\begin{align*}
\frac{1}{a^2} \Phi_{,ij}^{,i} - 3H^2 \Phi - 3H \dot{\Phi} &= 3H \frac{\dot{\phi}}{\phi} \left(2\Phi + \frac{\delta \phi}{\phi} - \frac{\dot{\phi}}{\phi}\right) + 3\frac{\dot{\phi}}{\phi} \\
+ \frac{4\pi}{M_p^2(\phi)} \left(\dot{\phi} \delta \phi + \dot{\sigma} \delta \sigma - \dot{\phi}^2 \Phi - \dot{\sigma}^2 \Phi - \phi^2 \frac{\delta \phi}{\phi} - \phi^2 \frac{\dot{\phi}}{\phi} + \frac{\partial V}{\partial \sigma} \delta \sigma - 2V \frac{\delta \phi}{\phi}\right),
\end{align*}$$

(27)

$$\begin{align*}
\left(\dot{\Phi} + H\Phi + H \frac{\delta \phi}{\phi}\right)_{,i} &= \frac{4\pi}{M_p^2(\phi)} \left(\dot{\phi} \delta \phi + \dot{\sigma} \delta \sigma\right)_{,i}.
\end{align*}$$

(28)
The homogeneous part of the gravitational equation (7) is

\[ 2\dot{H} + 3H^2 = -\frac{8\pi}{M_p^2(\phi)} \left( \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\sigma}^2 - V(\sigma) \right) + 2H\dot{\phi}/\phi. \] (29)

while the inhomogeneous part can be written as

\[
\ddot{\Phi} + 4H\dot{\Phi} + 2H\Phi + 3H^2\Phi = H\frac{\dot{\phi}}{\phi} \left( 2\Phi + a\frac{\dot{\delta\phi}}{\phi} \right) + \frac{\dot{\phi}}{\phi} \left( \frac{1}{a^2}\dot{\delta\phi} \right)_i \left( \frac{4\pi}{M_p^2(\phi)} \right) \left( \dot{\delta\phi} + \dot{\sigma}\delta\sigma - \dot{\phi}^2\Phi - \dot{\sigma}^2\Phi - \dot{\delta\phi}^2 - \dot{\sigma}^2\delta\phi - \frac{\partial V}{\partial \sigma}\delta\sigma + 2\frac{\dot{\delta\phi}}{\phi} \right),
\] (30)

Putting together (26) and (29), we find

\[ \dot{H} = \frac{\ddot{a}}{a} - H^2 = -\frac{4\pi}{M_p^2(\phi)} \left( \dot{\phi}^2 + \dot{\sigma}^2 \right) + 4H\frac{\dot{\phi}}{\phi}. \] (31)

Furthermore, the scalar field equations can be separated into homogeneous and inhomogeneous parts

\[
\ddot{\phi} + 3H\dot{\phi} = \frac{3\phi}{\omega}H^2,
\]
\[
\ddot{\sigma} + 3H\dot{\sigma} = -\frac{\partial V}{\partial \sigma}.
\]

\[
\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{a^2}\dot{\delta\phi} = -\frac{3}{\omega} \left( \frac{\dot{\phi}}{12a^2}\Phi_i + \frac{H^2\delta\phi}{\phi} - 2\phi H\dot{\Phi} \right),
\]
\[
\ddot{\delta\sigma} + 3H\dot{\delta\sigma} - \frac{1}{a^2}\dot{\delta\sigma} = -\frac{\partial^2 V}{\partial \sigma^2}\delta\sigma - 2\frac{\partial V}{\partial \sigma}\Phi.
\] (33)

Subtracting (30) from (27) and using equations (28), (31), and (32), one finally finds a long wavelength solution very similar to that of reference [24],

\[ \Phi + \frac{\delta\phi}{\phi} = A \left( 1 - \frac{\dot{a}}{a^2} \int \! dt \right), \] (34)

where \( A \) is a constant that can be calculated during inflation, using (28) and (31) in the slow-roll approximation, as

\[ A = \frac{a}{\int \! dt} \left( \frac{\dot{\phi}\delta\phi + \dot{\sigma}\delta\sigma}{\phi^2 + \dot{\sigma}^2} \right). \] (35)

In the Einstein frame \( \bar{g}_{\mu\nu} = \phi^2 g_{\mu\nu} \), the density contrast of perturbations that enter the horizon during the matter era is related to both \( \Phi \) and \( \delta\phi \) by [24]

\[ \frac{\delta\rho}{\bar{\rho}} = -2 \left( \Phi + \frac{\delta\phi}{\phi} \right) = -\frac{6}{5} H\frac{\dot{\phi}\delta\phi + \dot{\sigma}\delta\sigma}{\phi^2 + \dot{\sigma}^2}. \] (36)

\[ ^2\text{This expression was first obtained in [31] in the Einstein frame. See also [30].} \]
The energy density perturbations we observe in the cosmic background radiation could have originated during inflation in our model (2), as quantum fluctuations of both scalar fields that first left the horizon during inflation and later reentered during the radiation or matter dominated eras. The amplitude of energy density perturbations (36) associated with a given wavelength should be evaluated during inflation at the time in which that wavelength first crossed the horizon.

5.1 Quantum Fluctuations of Scalar Fields

We will now compute the amplitude $\delta \phi$ and $\delta \sigma$ of the quantum fluctuations of the scalar fields in de Sitter space, whose wavelengths are stretched beyond the horizon and act on the quasi–homogeneous background fields like a stochastic force. In order to calculate this effect we coarse–grain over a horizon distance and separate the scalar fields into long–wavelength classical background fields $\phi(x)$ and $\sigma(x)$ plus short–wavelength quantum fluctuations $\delta \phi(x)$ and $\delta \sigma(x)$ with physical momenta $k/a > H$,

$$\delta \phi(\vec{x}, t) = \int d^3k \, \theta(k - \varepsilon aH) \left[ a_k u_k(x) + a_k^\dagger u_k^*(x) \right],$$
$$\delta \sigma(\vec{x}, t) = \int d^3k' \, \theta(k' - \varepsilon aH) \left[ b_{k'} v_{k'}(x) + b_{k'}^\dagger v_{k'}^*(x) \right],$$

where $\varepsilon$ is an arbitrary parameter that shifts the scale for coarse–graining [5]. Physical results turn out to be independent of the choice of $\varepsilon$. The quantum fluctuations are assumed to satisfy the following commutation relations

$$[a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = \delta^3(\vec{k} - \vec{k'}), \quad [a_k, b_{k'}^\dagger] = 0 .$$

The approximate solutions to the perturbation equations (33) in de Sitter space with $V(\sigma) = \frac{1}{2} m^2 \sigma^2$ are given by [32]

$$u_k(x) = \frac{e^{i\vec{k} \cdot \vec{x}}}{(2\pi)^{3/2}} \frac{H \eta}{2} \sqrt{\pi \eta} \, H^{(2)}_{\mu}(k \eta), \quad \mu^2 = \frac{9}{4} + \frac{3}{\omega} \approx \frac{9}{4} ,$$
$$v_{k'}(x) = \frac{e^{i\vec{k'} \cdot \vec{x}}}{(2\pi)^{3/2}} \frac{H \eta}{2} \sqrt{\pi \eta} \, H^{(2)}_{\nu}(k' \eta), \quad \nu^2 = \frac{9}{4} - \frac{m^2}{H^2} \approx \frac{9}{4} ,$$

where $\eta = -(aH)^{-1}$ is the conformal time, and $H^{(2)}_{3/2}(x) = -e^{-x}(1 + \frac{1}{x}) \exp(ix)$.

The amplitude of the quantum fluctuations of $\phi$ and $\sigma$ can then be computed as

$$\delta \phi = \left( 4\pi k^3 |u_k|^2 \right)^{1/2} = \frac{H}{2\pi} ,$$
$$\delta \sigma = \left( 4\pi k^3 |v_{k'}|^2 \right)^{1/2} = \frac{H}{2\pi} ,$$

which coincides with the Gibbons–Hawking temperature [25]. It should be noted that we have made two approximations when deriving this formula, first that $\mu$ and $\nu$ are both $3/2$, see [33],
and second that $H$ is constant, when in general this is not true. We should have corrections to (40), but they are expected to be small [23].

Substituting the amplitude of quantum fluctuations of the scalar fields (40) into the energy density perturbations (36) and using the equations of motion (20), we find for arbitrary $V(\sigma)$

$$\frac{\delta \rho}{\rho} \cong \frac{24 H(\sigma, \phi) V(\sigma)}{5 M_p^2(\phi) V'(\sigma)} \left( \frac{1 + \dot{\phi}/\dot{\sigma}}{1 + (\dot{\phi}/\dot{\sigma})^2} \right)_{N_\lambda},$$

where $N_\lambda$ stands for the number of e-folds before the end of inflation associated with the horizon crossing of a particular wavelength. For perturbations of the size of the present horizon, we must compute the last expression at $N_\lambda \sim 65$ [2]. Note that in the large $\omega$ limit, $\dot{\phi} \ll \dot{\sigma}$ during the last stages of inflation, which ensures the approximate equivalence of the Einstein and Jordan frame. We then recover the usual expression [2]

$$\frac{\delta \rho}{\rho} \cong \frac{24 H(\sigma, \phi) V(\sigma)}{5 M_p^2(\phi) V'(\sigma)}_{N_\lambda},$$

where $M_p$ is now $\phi$–dependent. For theories with potentials of the type $\frac{\lambda}{2n} \sigma^{2n}$, it behaves like

$$\frac{\delta \rho}{\rho} \approx \frac{6\omega}{5n\pi} \left( \frac{2\omega \lambda}{3n} \right)^{1/2} \frac{\sigma^{n+1}}{\phi^3} \bigg|_{N_\lambda}.$$

In the case of the theory $\lambda \sigma^4/4$, the density perturbation (13) takes the usual $\sqrt{\lambda}$ dependence. However, in the case $m^2 \sigma^2/2$, the perturbation can be written as

$$\frac{\delta \rho}{\rho} \approx \frac{2 e^{3N_\lambda}}{5\sqrt{3\pi}} \left[ 1 + 3\omega \left( 1 - e^{-\frac{2N_\lambda}{\omega}} \right) \right] \frac{m}{M_p(\phi_e)}.$$

Therefore, we note that the larger is the Planck mass at the end of inflation in a given region of the Universe, the smaller is the density perturbation in this region. For a detailed discussion of its cosmological implications see ref. [20].

6 Stochastic Inflation

We can now put together the results of previous sections and study the Brownian motion of the background fields under the stochastic impulses of their own quantum fluctuations. We could write the evolution of the coarse–grained fields in the form of Langevin equations

$$\Delta \phi = \Pi^\phi N \Delta t + \Delta S_\phi,$$

$$\Delta \sigma = \Pi^\sigma N \Delta t + \Delta S_\sigma,$$

$$\Delta \alpha = H N \Delta t + \Delta S_\alpha,$$

(45)
where $\Delta S_{\phi_k}$ are the stochastic noise terms acting on the long-wavelength background. They are Gaussian distributed with zero mean and variance

$$
\langle \Delta S_{\phi}(t) \Delta S_{\phi}(t') \rangle = \delta_{ij} \left( \frac{H(\phi_k)}{2\pi} \right)^2 H(\phi_k) N \Delta t \delta(t - t'),
$$

which corresponds to the Brownian motion of the background scalar fields $\phi$ and $\sigma$ with step \(46\). It is then straightforward to derive the corresponding Fokker–Planck equation, see e.g. the appendix in [23],

$$
\frac{\partial P_c}{\partial t} = \frac{\partial}{\partial \sigma} \left( \frac{M_p^2}{4\pi} \frac{\partial H}{\partial \sigma} P_c \right) + \frac{(NH)^{1/2}}{8\pi^2} \frac{\partial}{\partial \sigma} \left( (NH)^{1/2} P_c \right) \approx - \frac{\partial J_\phi}{\partial \phi} - \frac{\partial J_\sigma}{\partial \sigma},
$$

where we have chosen the Stratonovich version of stochastic processes. $P_c(\sigma, \phi; t)$ represents the probability distribution of finding a given field configuration $(\sigma, \phi)$ at a given point (of size $H^{-1}$) in comoving space. Equation \(47\) can be interpreted as the continuity equation $\frac{\partial P}{\partial t} + \nabla \cdot J = 0$, associated with the conservation of probability. The first terms of each current correspond to the classical drift forces \(20\) associated with the scalar fields, while the second terms correspond to the quantum diffusion due to short–wavelength fluctuations \(40\).

It can be seen by inspection that there is an exact time reparametrization invariant solution given by

$$
P_c(\sigma, \phi; t) \sim (NH)^{-1/2} \exp \left( \frac{3M_p^4(\phi)}{8V(\sigma)} \right),
$$

which is proportional to the square of the Hartle–Hawking wave function of the Universe [33], for an arbitrary potential $V(\sigma)$ and a variable gravitational constant. Unfortunately, this function is not normalizable for potentials that vanish at their minimum. But the real problem is that the inflationary regime ends much before the fields reach the minimum of the potential, and thus equation \(47\) is not valid there. Therefore, there are no stationary solutions for $P_c$ in realistic inflationary models. The proper way to find stationary solutions during the inflationary phase would be to consider the distribution $P_p$ and solve the corresponding diffusion equation subject to certain boundary conditions [1]. This was done explicitly in ref. [20]; we will return to this question in the next section.

Instead of looking for stationary solutions of equation \(47\) we will try to understand the general behavior of the probability distribution $P_c(\sigma, \phi; t)$ in the synchronous gauge $N = 1$, for generic potentials $V(\sigma) = \frac{1}{2n} \sigma^{2n}$. $P_c$ then satisfies the equation

$$
\frac{\partial P_c}{\partial t} = \frac{\partial}{\partial \sigma} \left( \frac{1}{2\omega} \left( \frac{2\omega \lambda}{3n} \right)^{1/2} \phi^2 \frac{\partial h}{\partial \sigma} P_c + \frac{1}{8\pi^2} \left( \frac{2\omega \lambda}{3n} \right)^{3/2} h^{3/2} \frac{\partial}{\partial \sigma} \left( h^{3/2} P_c \right) \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\omega} \left( \frac{2\omega \lambda}{3n} \right)^{1/2} \phi^2 \frac{\partial h}{\partial \phi} P_c + \frac{1}{8\pi^2} \left( \frac{2\omega \lambda}{3n} \right)^{3/2} h^{3/2} \frac{\partial}{\partial \phi} \left( h^{3/2} P_c \right) \right),
$$

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In order to follow the evolution of the scalar fields under classical drift forces and quantum diffusion processes, it will be useful to introduce some redefinitions,

$$h = \frac{\sigma^n}{\phi}, \quad u = \frac{1}{8\pi^2} \left(\frac{2\omega\lambda}{3n}\right)^{3/2} t.$$  \hfill (50)

We will follow closely the work of Goncharov and Linde \[6\], generalized to JBD. We define the new functions $A$ and $B$ as

$$A_\sigma = -\frac{3n}{2\lambda} \left(\frac{2\pi}{\omega}\right)^2 \phi^2 \frac{\partial h}{\partial \sigma}(\sigma, \phi), \quad A_\phi = -\frac{3n}{\lambda} \left(\frac{2\pi}{\omega}\right)^2 \phi^2 \frac{\partial h}{\partial \phi}(\sigma, \phi), \quad B = h^3(\sigma, \phi),$$  \hfill (51)

and write the evolution equation for the probability distribution,

$$\frac{\partial P_c}{\partial u} = \frac{\partial}{\partial \sigma} \left[-A_\sigma(\sigma, \phi)P_c(\sigma, \phi) + B^{1/2}(\sigma, \phi) \frac{\partial}{\partial \sigma} \left(B^{1/2}(\sigma, \phi)P_c(\sigma, \phi)\right)\right]$$

$$+ \frac{\partial}{\partial \phi} \left[-A_\phi(\sigma, \phi)P_c(\sigma, \phi) + B^{1/2}(\sigma, \phi) \frac{\partial}{\partial \phi} \left(B^{1/2}(\sigma, \phi)P_c(\sigma, \phi)\right)\right].$$ \hfill (52)

We will now consider the case in which $V(\sigma) \ll M^4(\phi)$, where quantum diffusion is much smaller than classical drift,

$$\frac{\partial}{\partial \sigma} B(\sigma, \phi) \ll |A_\sigma(\sigma, \phi)|, \quad \frac{\partial}{\partial \phi} B(\sigma, \phi) \ll |A_\phi(\sigma, \phi)|,$$  \hfill (53)

and the classical solutions to the equations of motion satisfy $\dot{\sigma}(u) = A_\sigma(\sigma_c, \phi_c)$ and $\dot{\phi}(u) = A_\phi(\sigma_c, \phi_c)$ respectively. Under a further change of variables, $x = \sigma - \sigma_c(s)$, $y = \phi - \phi_c(s)$ and $s = u$, the evolution equation (52) can be written as

$$\frac{\partial P_c}{\partial s} \simeq - \left(\frac{\partial A_\sigma}{\partial \sigma_c}(\sigma_c, \phi_c)\right) \frac{\partial}{\partial x} [xP_c(x, y)] + B(\sigma_c, \phi_c) \frac{\partial^2 P_c}{\partial x^2}$$

$$- \left(\frac{\partial A_\phi}{\partial \phi_c}(\sigma_c, \phi_c)\right) \frac{\partial}{\partial y} [yP_c(x, y)] + B(\sigma_c, \phi_c) \frac{\partial^2 P_c}{\partial y^2},$$ \hfill (54)

where $A_\sigma(\sigma, \phi) \simeq x \frac{\partial A_\sigma}{\partial \sigma_c} + y \frac{\partial A_\sigma}{\partial \phi_c}$ and $A_\phi(\sigma, \phi) \simeq y \frac{\partial A_\phi}{\partial \phi_c} + x \frac{\partial A_\phi}{\partial \sigma_c}$. The cross-terms have been neglected since they contribute with vanishing correlators.

Under the initial condition $P(x, y; 0) = \delta(x - x_o)\delta(y - y_o)$, the dispersion coefficients can be calculated as

$$\Delta^2_\sigma = 2A^2_\sigma \int_{\sigma_o}^{\sigma} \frac{B}{A^2_\sigma} d\sigma = \frac{\lambda}{3n^2} \left(\frac{\omega}{2\pi}\right)^2 \sigma_c^2 - \frac{\lambda}{3n^2} \left(\frac{\omega}{2\pi}\right)^2 \sigma_c^4 \phi_c^4$$

$$\Delta^2_\phi = 2A^2_\phi \int_{\phi_o}^{\phi} \frac{B}{A^2_\phi} d\phi = \frac{\lambda}{3n} \left(\frac{\omega}{2\pi}\right)^2 \sigma_c^2 \left(\frac{1}{\phi_o^2} - \frac{1}{\phi_c^2}\right) \phi_c^4.$$  \hfill (55)
The probability distribution $P_c$ then looks like a Gaussian with decreasing dispersions along both field directions, centered on the classical field trajectory in the plane $(\sigma, \phi)$,

$$P_c(\sigma, \phi; t) \sim \exp \left\{ -\frac{(\sigma - \sigma_c(t))^2}{2\Delta_\sigma^2(t)} - \frac{(\phi - \phi_c(t))^2}{2\Delta_\phi^2(t)} \right\}.$$  \hspace{1cm} (56)

This distribution describes the diffusion of the scalar fields in comoving space after a given initial condition, and shows how the fields follow the classical trajectory with variable dispersion. However, in this description we are not taking into account the expansion of physical space.

### 6.1 Self-reproduction of the inflationary universe

If all the inflationary domains contained fields that followed their classical trajectory with a given dispersion, all would have ended inflation by now. However, it was realized in ref. [4] that those few domains that jump opposite to the classical trajectory contribute with a larger physical space and therefore dominate the physical volume of the universe. Those domains will split into smaller domains, some with lower values of the scalar fields, some with higher. As a consequence of the diffusion process, there will always be domains which are still inflating, and this corresponds to what is known as the self-reproduction of the inflationary universe.

We are thus interested not in the probability distribution in the comoving frame $P_c$, but on the distribution $P_p(\sigma, \phi; t)$ over the physical volume, which takes into account the different quasi–exponential growth of the proper volume in different parts of the universe,

$$P_p(\sigma, \phi; t) \simeq P_c(\sigma, \phi; t) e^{3\Delta t(H - H_c)}$$  \hspace{1cm} (57)

where $P_c(\sigma, \phi; t)$ is given in (56). We can now study the onset of the self–reproduction of the inflationary universe. The bifurcation line is defined by the values of the scalar fields for which the maximum of the probability distribution $P_p$ \textcolor{red}{(57)} starts increasing, or equivalently, where the quantum diffusion of the fields dominates its classical motion. Its expression is best found in polar coordinates, in which the classical motion takes the form,

$$z \equiv \frac{\dot{\phi}}{\varphi} \rightarrow \frac{\dot{z}}{z} = \frac{H}{\omega} \left(1 + z^2\right),$$
$$r^2 \equiv \phi^2 + \varphi^2 \rightarrow \dot{r} = 0,$$  \hspace{1cm} (58)

while the quantum diffusion steps can be written as

$$\delta z/z = \left(\frac{\delta \phi}{\phi} + \frac{\delta \varphi}{\varphi}\right)^{1/2} = \frac{H}{2\pi \phi} \left(1 + z^2\right)^{1/2},$$
$$r \delta r = \left((\phi \delta \phi)^2 + (\varphi \delta \varphi)^2\right)^{1/2} = \frac{H}{2\pi} r.$$  \hspace{1cm} (59)

It is clear that along the radial direction quantum diffusion will always dominate classical motion, since $\dot{r} = 0$. However, along the angular direction the condition that quantum diffusion dominates...
classical motion during the time interval $\Delta t = H^{-1}$ ensures the self-reproduction of the universe only in the range \[5\]

\[
\frac{3M^4_p(\phi)}{4\omega} \left(1 + \frac{\phi^2}{\varphi^2}\right) < V(\sigma) < M^4_p(\phi).
\] (60)

Depending on the value of $\omega$, this could be a very wide range of values for the scalar fields. Once an inflationary domain enters this regime it will create more domains with growing values of the scalar fields. In particular, the inflaton field will tend to diffuse towards larger and larger values, which favor a larger volume of physical space. A reasonable question would be to ask if there is a stationary solution to the upward drive of the inflaton due to its quantum diffusion. This will be examined in the next section.

7 Stationary Solutions

The best way to analyze the possibility of a stationary regime to the upward diffusion of the inflaton is to look for stationary solutions of the Fokker–Planck equation for the probability distribution in the physical frame,

\[
\frac{\partial P_p}{\partial t} = \frac{\partial}{\partial \sigma} \left[ \frac{M^2_p(\phi)}{4\pi} \frac{\partial H}{\partial \sigma} P_p + \frac{H^{3/2}}{8\pi^2} \frac{\partial}{\partial \sigma} \left(H^{3/2} P_p\right) \right] + \frac{\partial}{\partial \phi} \left[ \frac{M^2_p(\phi)}{2\pi} \frac{\partial H}{\partial \phi} P_p + \frac{H^{3/2}}{8\pi^2} \frac{\partial}{\partial \phi} \left(H^{3/2} P_p\right) \right] + 3HP_p.
\] (61)

Due to the term $3HP_p$ there are no simple stationary solutions. However, with the use of the identity $D^2 g - 2D(Df g) = e^f D^2(e^{-f} g) - [(Df)^2 + D^2 f]g$, we can write the stationary probability distribution for an arbitrary potential as

\[
P_p(\sigma, \phi; t) \propto e^{Et} H(\sigma, \phi)^{-3/2} \exp \left( \frac{3M^4_p(\phi)}{16V(\sigma)} \right) \Psi(\sigma, \phi),
\] (62)

where the new function $\Psi(\sigma, \phi)$ satisfies a two-dimensional Schrödinger–like equation,

\[
\left( H^{3/2} \frac{\partial}{\partial \sigma} \right)^2 \Psi + \left( H^{3/2} \frac{\partial}{\partial \phi} \right)^2 \Psi - V(\sigma, \phi) \Psi = 8\pi^2 E\Psi,
\] (63)

with an effective potential

\[
V(\sigma, \phi) = \frac{16\pi^4}{9} H^{-5}(\sigma, \phi)V'(\sigma)^2 + \frac{4\pi^2}{3} H^{-1}(\sigma, \phi) \left( \frac{5}{4} V'(\sigma)^2 - V''(\sigma) \right) + \frac{64\pi^4}{3\omega} H^{-3}(\sigma, \phi)V(\sigma) + \frac{6\pi^2}{\omega} H(\sigma, \phi) - 24\pi^2 H(\sigma, \phi).
\] (64)

For general theories of the type $V(\sigma) = \frac{1}{2n}\sigma^{2n}$, the Schrödinger potential \([64]\) can be written as

\[
\left( \frac{2\omega}{3n} \right)^{3/2} V(\sigma, \phi) = \left( \frac{2\pi}{\omega} \right)^4 \left( \frac{3n^2}{4\lambda} \right)^2 \phi^5 \sigma^{n+2} + \left( \frac{2\pi}{\omega} \right)^2 \frac{3n^2(n+2)}{8\lambda} \phi \sigma^{2-n}
+ \left( \frac{2\pi}{\omega} \right)^4 \left( \frac{3n^2}{2\lambda} \right)^2 \sigma^n + \left( \frac{2\pi}{\omega} \right)^2 \left( \frac{18\pi n}{\lambda} \right)^2 \sigma^n.
\] (65)
Even in this case it is extremely difficult to find exact analytical solutions to the Schrödinger equation (63), due to the non-linear coupling of the two scalar fields. In ref. [20] we performed a numerical simulation that gave us insight into the behavior of the distribution. We will try to give here a general idea by analyzing a simple generic case, \( n = 2 \). By transforming into polar variables (58), we notice that the Universe will enter the regime of self-reproduction for \( z < (\lambda \omega^3/12 \pi^2)^{1/4} \ll 1 \) and therefore we can approximate the diffusion equation by
\[
\frac{r^{3/2}}{z^{3/2}} \frac{\partial}{\partial r} \Psi + \frac{r^{1/2}}{z^{3/2}} \frac{\partial}{\partial z} \Psi + \frac{72\pi^2 r}{\lambda \omega z} \Psi = 8\pi^2 E \left( \frac{3}{\lambda \omega} \right)^{3/2} \Psi.
\]
Equation (66) does not have a stationary solution. It is well known from quantum mechanics that a potential proportional to \(-1/s^2\) is too strong and produces singular solutions at \( s = 0 \) (\( r = \infty \)). The probability distribution will move forever towards large \( r \), i.e. it is non-stationary. This result is generic, it is true for all potentials of the type \( \lambda \sigma^{2n} \). Those probability distributions were named “runaway solutions” in ref. [20].

Runaway solutions are a special feature of Jordan–Brans–Dicke, since the evolution in general relativity is equivalent to the motion along the \( z \)-direction, which is stationary. The dynamics of the gravitational constant introduces the \( r \)-motion along the Planck boundary, that is non-stationary unless we impose further boundary conditions. For instance, it was shown numerically in ref. [20] that a sharp increase in the inflaton potential could effectively simulate a boundary condition in \( \sigma \), and therefore produce global stationary solutions.

8 Conclusions

In this paper we have described the mathematical formalism for stochastic inflation in the context of Jordan–Brans–Dicke theory of gravity. If string theory is the correct description for the gravitational interaction at the quantum level, it is important to know the behavior of the massless gravitational sector of strings in the low energy effective theory, below the Planck scale. Such a sector contains a dilaton scalar field as well as the graviton, and therefore it is worthwhile studying the cosmology of scalar-tensor theories of gravity. The dilaton acts like a dynamical gravitational constant, the Brans–Dicke field. Even though it is argued that the present experimental observations put severe constraints on such a scalar component of gravity, we cannot disregard its effects close to the Planck scale, where quantum fluctuations of the fields become important.
Since chaotic inflation, it was realized that the inflationary regime went all the way up to the Planck scale. The quantum fluctuations of the fields then became important and their effect on their own background fields was described with the use of diffusion equations. Stochastic inflation is the mathematical formalism that describes the Brownian motion of those coarse-grained background fields during inflation. It is also a very powerful tool for the description of the very large scale structure of the Universe, much beyond our observable universe. Its probabilistic description opened the possibility of a connection with quantum cosmology. In order to incorporate the effect of the string dilaton in the description of the inflationary universe close to the Planck scale, we have studied the stochastic inflation formalism in the context of the simplest scalar-tensor theory, Jordan–Brans–Dicke theory of gravity.

For that purpose we described the Arnowitt–Deser–Misner formalism for JBD theory with an inflaton scalar field, in order to include in a consistent way the non-linear evolution and backreaction of the scalar fields and the metric. We then separated the fields into long-wavelength background fields and short-wavelength quantum fluctuations, thanks to the natural scale of the de Sitter horizon in inflationary domains, and obtained a self-consistent set of equations for the background fields by expanding in spatial gradients. Solutions to these equations are easily found in the slow-roll approximation, describing the classical evolution of the scalar fields.

Quantum fluctuations of the scalar fields during inflation give rise to adiabatic energy density perturbations that could be responsible for the temperature fluctuations recently observed by COBE in the cosmic background radiation. We describe a consistent way of computing the amplitude of the perturbations on a homogeneous background in terms of the quantum fluctuations of both dilaton and inflaton scalar fields. For theories of the type $\lambda \sigma^4$ we find the usual scale-invariant $\sqrt{\lambda}$ dependence. However, for theories with a mass term for the inflaton, the amplitude of adiabatic density perturbations is proportional to the ratio of such a fundamental scale (assumed given) and the effective value of the Planck scale at the end of inflation.

We then studied the new stochastic phenomena that appear when the dilaton is included in the diffusion equations for the probability distribution of finding a certain value of the scalar fields both in the comoving and the physical frame. While the probability distribution in the comoving frame followed the classical trajectories of the scalar fields, with some time-dependent dispersion, the description in the physical frame takes into account the quasi-exponential growth of the inflationary domains. Those few domains that jump opposite to the classical trajectory inflate more and thus contribute with a larger physical space. There is a bifurcation point (in fact a line in JBD inflation) at which the contribution to the proper volume of the Universe of a given inflationary domain dominates the tendency of the fields to follow their classical trajectories, and the Universe enters a regime of self-reproduction. In that regime some domains end inflation and enter the radiation and matter dominated eras, while others continue inflating and producing new inflationary domains, overwhelmingly dominating the total proper volume of the Universe. In that sense, our own observable universe might be an insignificant subproduct of one of those domains that ended inflation. We compute the bifurcation line in the case of JBD stochastic inflation and find that the range of values of the scalar fields in the self-reproduction regime depends on the value of the Brans–Dicke parameter.
The main question to be addressed is the existence of stationary solutions for the probability distribution in the physical frame. With the introduction of the dilaton field, the diffusion space is now bidimensional and the self-reproduction of the Universe occurs on an infinite wedge in this space, see eq. (60), bounded between the bifurcation line and the Planck boundary although open for arbitrary large values of the scalar fields. We have found, see also ref. [20] for a more detailed description based on numerical simulations, that in general we will obtain runaway solutions: non-stationary probability distributions concentrated close to the Planck boundary but moving with different speeds towards large values of the scalar fields, in particular to an infinitely large Planck mass. Perhaps this is an indication that we need to introduce a dynamical cutoff in the inflaton potential. In ref. [20] we discussed the cosmological consequences of this result.

In the case of general relativity, the energy eigenvalue of the stationary distributions was related to the fractal dimension of the global inflationary universe [34]. In our case, the existence of runaway solutions indicate a very complicated fractal structure of the universe close to the Planck scale. Further work should be done in this direction.

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