Harish-Chandra modules over the high rank 
W-algebra $W(2,2)$

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Abstract: In this paper, using the theory of $A$-cover developed in [1,2], we completely classify all simple Harish-Chandra modules over the high rank $W$-algebra $W(2,2)$. As a byproduct, we obtain the classification of simple Harish-Chandra modules over the classical $W$-algebra $W(2,2)$ studied in [7,12,17].

Key words: high rank $W$-algebra $W(2,2)$, Harish-Chandra module, weight module.

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1 Introduction

In the representation theory of infinite-dimensional Lie algebras, there is a very important class of weight modules called Harish-Chandra modules (namely, weight modules with finite-dimensional weight spaces). The classification of simple Harish-Chandra modules over the Virasoro algebra (also called $N=0$ superconformal algebra), conjectured by Kac (see [13]), was given in [20]. Combined [23] with [27], a new method was presented to obtain this classification. After that, a lot of general versions of the Virasoro algebra have been investigated by some authors. Those include, but are not limited to, the generalized Virasoro algebra (see, e.g., [11,19,22,25,28,31]), the generalized Heisenberg-Virasoro algebra (see [10,15,18]), the $W$-algebra $W(2,2)$ (see [7,12,17]), the loop-Virasoro algebra (see [12]), and so on. To classify all simple Harish-Chandra modules over the Lie algebra $W_n$ of vector fields on $n$-dimensional torus, Billig and Futorny developed a new technique called $A$-cover theory in [1,2]. The result gained here was a generalized version of Mathieu’s classification theorem for the Virasoro algebra. From then on, the $A$-cover theory was used in some other Lie (super)algebras (see, e.g., [3,4,7,32]).

The $W$-algebra $W(2,2)$ was introduced in [33] by Zhang and Dong for investigated the classification of simple vertex operator algebras generated by two weight 2 vectors. The centerless $W$-algebra $W(2,2) \overline{W}[Z]$ can be obtained from the point of view of non-relativistic analogues of the conformal field theory. By using the “non-relativistic limit” on a pair of commuting algebras vect$(S^1) \oplus vect(S^1)$ (see [26]) via a group contraction, one has the following generators

$$L_m = -t^{m+1} \frac{d}{dt} - (m + 1)t^m y \frac{d}{dy} - (m + 1)\sigma t^m - m(m + 1)\eta t^{m-1}y,$$

$$W_m = -t^{m+1} \frac{d}{dy} - (m + 1)\eta t^m,$$

where $m \in Z$, $\sigma$ and $\eta$ are respectively the scaling dimension and a free parameter. Then the centerless $W$-algebra $W(2,2)$ is a Lie algebra with the basis $\{L_m, W_m \mid m \in Z\}$ and the
non-vanishing commutators as follows

\[ [L_m, L_m'] = (m' - m)L_{m+m'}, \quad [L_m, W_{m'}] = (m' - m)W_{m+m'}, \]

for \( m, m' \in \mathbb{Z} \). It is an infinite-dimensional extension of an algebra called either non-relativistic or conformal Galilei algebra \( \text{CGA}(1) \cong \langle L_{\pm 1, 0}, W_{\pm 1, 0} \rangle \) (see [9]). Basically, the relationship with conformal algebras makes it widely studied in string theory (see [6]). Furthermore, \( \mathcal{W}[\mathbb{Z}] \) can be realized by the semidirect product of the Witt algebra \( \mathcal{V}[\mathbb{Z}] \) and the \( \mathcal{V}[\mathbb{Z}] \)-module \( A_{0,-1} \) of the intermediate series in [14], that is, \( \mathcal{W}[\mathbb{Z}] \cong \mathcal{V}[\mathbb{Z}] \rtimes A_{0,-1} \). What we concern most is the approach of realizing \( \mathcal{W}[\mathbb{Z}] \) from a truncated loop-Witt algebra (see, e.g., [11, 16]). The detailed description on it will be shown in Section 3, which is closely associated with the usage of \( \mathcal{A} \)-cover theory. The aim of this paper is to present a completely classification of simple Harish-Chandra modules over the high rank \( W \)-algebra \( W(2,2) \), which reobtains the classification result of classical \( W \)-algebra \( W(2,2) \) (when \( k = 1 \)) studied in [7, 12, 17].

The paper is organized as follows. In Section 2, we introduce some notations and definitions related to the high rank \( W \)-algebra \( W(2,2) \) and Harish-Chandra modules. We also recall some known classification theorems over several related Lie algebras for later use. In Section 3, we give a classification of simple cuspidal modules over the higher rank \( W \)-algebra \( W(2,2) \) in Theorem 3.14. In Section 4, we present a classification of simple Harish-Chandra modules over the higher rank \( W \)-algebra \( W(2,2) \) in Theorem 4.2.

Throughout the present article, we denote by \( \mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N} \) and \( \mathbb{Z}^+ \) the sets of complex numbers, real numbers, integers, nonnegative integers and positive integers, respectively. All vector spaces and Lie algebras are over \( \mathbb{C} \). All simple modules are considered to be non-trivial. For a Lie algebra \( \mathfrak{L} \), we use \( U(\mathfrak{L}) \) to denote the universal enveloping algebra.

## 2 Preliminaries

### 2.1 The high rank \( W \)-algebra \( W(2,2) \) and its cuspidal module

The *high rank \( W \)-algebra \( W(2,2) \)* is an infinite dimensional Lie algebra

\[
\mathcal{W}[\mathbb{Z}^k] = \bigoplus_{\alpha \in \mathbb{Z}^k} \mathbb{C}L_\alpha \oplus \bigoplus_{\alpha \in \mathbb{Z}^k} \mathbb{C}W_\alpha \oplus \mathbb{C}C,
\]

which satisfies the following Lie brackets

\[
[L_\alpha, L_\beta] = (\beta - \alpha)L_{\alpha+\beta} + \delta_{\alpha+\beta,0} \frac{\alpha^3 - \alpha}{12} C,
\]

\[
[L_\alpha, W_\beta] = (\beta - \alpha)W_{\alpha+\beta} + \delta_{\alpha+\beta,0} \frac{\alpha^3 - \alpha}{12} C,
\]

\[
[W_\alpha, W_\beta] = [\mathcal{W}[\mathbb{Z}^k], C] = 0,
\]

(2.1)
where $\alpha, \beta \in \mathbb{Z}^k, k \in \mathbb{Z}_+$. Clearly, $\mathcal{W}[\mathbb{Z}^k]$ has an infinite-dimensional Lie subalgebra $\mathcal{V}[\mathbb{Z}^k] := \text{span}\{L_\alpha, C \mid \alpha \in \mathbb{Z}^k\}$, which is called high rank Virasoro algebra. Note that $\mathbb{C}C$ is the center of $\mathcal{W}[\mathbb{Z}^k]$. The quotient algebras $\overline{\mathcal{W}}[\mathbb{Z}^k] = \mathcal{W}[\mathbb{Z}^k]/\mathbb{C}C$ and $\overline{\mathcal{V}}[\mathbb{Z}^k] = \mathcal{V}[\mathbb{Z}^k]/\mathbb{C}C$ are respectively called centerless high rank $W$-algebra $W(2, 2)$ and high rank Witt algebra. When $k = 1$, we say that $\mathcal{V}[\mathbb{Z}]$ and $\mathcal{W}[\mathbb{Z}]$ are respectively classical Virasoro algebra and classical $W$-algebra $W(2, 2)$. For any $\alpha \in \mathbb{Z}^k \setminus \{0\}$, we know that $\mathcal{V}[\mathbb{Z}\alpha]$ and $\mathcal{W}[\mathbb{Z}\alpha]$ are respectively isomorphic to the classical Virasoro algebra and classical $W$-algebra $W(2, 2)$.

Now we recall some definitions related to the weight module. Consider a non-trivial module $M$ over $\mathcal{V}[\mathbb{Z}^k]$ or $\mathcal{W}[\mathbb{Z}^k]$. We set that the action of the central element $C$ is a scalar $c$. The module $M$ is said to be trivial if the action of whole algebra on $M$ is trivial. Denote $M_\lambda = \{v \in M \mid L_\alpha v = \lambda v\}$, which is called a weight space of weight $\lambda \in \mathbb{Z}^k$. We call that $M$ is a weight module if $M = \bigoplus_{\lambda \in \mathbb{Z}^k} M_\lambda$. Set $\text{Supp}(M) = \{\lambda \mid M_\lambda \neq 0\}$, which is called the support (or called the weight set) of $M$. The indecomposable weight module $M$ with all weight spaces one-dimensional is called the intermediate series module.

**Definition 2.1.** Let $M$ be a weight module over $\mathcal{W}[\mathbb{Z}^k]$.

1. If $\dim(M_\lambda) < +\infty$ for all $\lambda \in \text{Supp}(M)$, then $M$ is called Harish-Chandra module.

2. If there exists some $K \in \mathbb{Z}_+$ such that $\dim(M_\lambda) < K$ for all $\lambda \in \text{Supp}(M)$, then $M$ is called cuspidal (or uniformly bounded).

We define a class of cuspidal $\mathcal{W}[\mathbb{Z}^k]$-modules as follows, which are exactly intermediate series modules for $\mathcal{W}[\mathbb{Z}^k]$.

**Definition 2.2.** For any $g, h \in \mathbb{C}$, the $\mathcal{W}[\mathbb{Z}^k]$-module $M(g, h; \mathbb{Z}^k)$ has a basis $\{v_\beta \mid \beta \in \mathbb{Z}^k\}$ and the $\mathcal{W}[\mathbb{Z}^k]$-action:

$$L_\alpha v_\beta = (g + \beta + h\alpha)v_{\alpha + \beta}, \quad W_\alpha v_\beta = Cv_\beta = 0.$$

It is clear that the modules $M(g, h; \mathbb{Z}^k)$ are isomorphic to the intermediate series modules of $\mathcal{V}[\mathbb{Z}^k]$. By \cite{31}, we see that the modules $M(g, h; \mathbb{Z}^k)$ are reducible if and only if $g \in \mathbb{Z}^k$ and $h \in \{0, 1\}$. We use $\overline{M}(g, h; \mathbb{Z}^k)$ to denote the unique non-trivial simple subquotient of $M(g, h; \mathbb{Z}^k)$. Then $\text{Supp}(\overline{M}(g, h; \mathbb{Z}^k)) = g + \mathbb{Z}^k$ or $\text{Supp}(\overline{M}(g, h; \mathbb{Z}^k)) = \mathbb{Z}^k \setminus \{0\}$. We also define $\overline{M}(g, h; \mathbb{Z}^k)$ as intermediate series modules of $\mathcal{W}[\mathbb{Z}^k]$.

### 2.2 Generalized highest weight modules

In this section, a general class of Lie algebras are considered. Assume that $\mathcal{H} = \sum_{\alpha \in \mathbb{Z}^k} \mathcal{H}_\alpha$ is a $\mathbb{Z}^k$-graded Lie algebra such that $\mathcal{H}_0$ is abelian. And the gradation of $\mathcal{H}$ is the root space decomposition with respect to $\mathcal{H}_0$. 
Let \( g \) be a subgroup of \( \mathbb{Z}^k \) such that \( \mathbb{Z}^k = g \oplus \mathbb{Z}\mu \) for some \( \mu \in \mathbb{Z}^k \). We define the subalgebra of \( \mathcal{H} \) as follows

\[
\mathcal{H}_g = \bigoplus_{\alpha \in g} \mathcal{H}_\alpha, \quad \mathcal{H}_g^+ = \bigoplus_{\alpha \in g, m \in \mathbb{Z}_+} \mathcal{H}_{\alpha + m\mu}, \quad \mathcal{H}_g^- = \bigoplus_{\alpha \in g, m \in \mathbb{Z}_+} \mathcal{H}_{\alpha - m\mu}.
\]

Let \( \mathcal{K} \) be a simple \( \mathcal{H}_g \)-module. Then \( \mathcal{K} \) can be extended to an \( (\mathcal{H}_g^+ + \mathcal{H}_g^-) \)-module by defining \( \mathcal{H}_g^+ \mathcal{K} = 0 \). Now we can define the generalized Verma module \( V_{g,\mu,\mathcal{K}} \) for \( \mathcal{H} \) as

\[
V_{g,\mu,\mathcal{K}} = \text{Ind}^{\mathcal{H}}_{\mathcal{H}_g^+ + \mathcal{H}_g^-} \mathcal{K} = U(\mathcal{H}) \otimes_{U(\mathcal{H}_g^+ + \mathcal{H}_g^-)} \mathcal{K}.
\]

It is easy to know that \( V_{g,\mu,\mathcal{K}} \) has a unique simple quotient module for \( \mathcal{H} \) and we write it as \( P_{g,\mu,\mathcal{K}} \). Then \( P_{g,\mu,\mathcal{K}} \) is called a simple highest weight module. As far as we know, the generalized Verma module (or generalized highest weight module) was introduced and investigated in some other references (see, e.g., [5, 8, 21]).

Fix a basis of \( \mathbb{Z}^k \). Assume that \( M \) is a weight module for \( \mathcal{H} \). Then \( M \) is called dense if \( \text{Supp}(M) = \lambda + \mathbb{Z}^k \) for some \( \lambda \in \mathcal{H}_0^* \). On the other hand, if there exist \( \lambda \in \text{Supp}(M), \tau \in \mathbb{R}^k \setminus \{0\} \) and \( \beta \in \mathbb{Z}^k \) such that \( \text{Supp}(M) \subseteq \lambda + \beta + \mathbb{Z}^k(\tau) \leq 0 \), where \( \mathbb{Z}^k(\tau) = \{\alpha \in \mathbb{Z}^k \mid (\tau|\alpha) \leq 0\} \) and \( (\tau|\alpha) \) is the usual inner product in \( \mathbb{R}^k \), then \( M \) is called cut. Obviously, the modules \( P_{g,\mu,\mathcal{K}} \) defined above are cut modules. If there exist a \( \mathbb{Z} \)-basis \( \{\epsilon_1, \ldots, \epsilon_k\} \) of \( \mathbb{Z}^k \) and \( K \in \mathbb{Z}_+ \) such that \( \mathcal{H}_\alpha v = 0 \) for all \( \alpha = \Sigma_{i=1}^k \alpha_i \epsilon_i \) with \( \alpha_i > K, i \in \{1, \ldots, k\} \), then the element \( v \in M \) is called a generalized highest weight vector.

The following general result of cut \( \mathcal{H} \)-modules appeared in Theorem 4.1 of [21].

**Theorem 2.3.** Let \( [\mathcal{H}_\alpha, \mathcal{H}_\beta] = \mathcal{H}_{\alpha + \beta} \) for all \( \alpha, \beta \in \mathbb{Z}^k, k \in \mathbb{Z}_+ \) with \( \alpha \neq \beta \). Assume that \( M \) is a simple weight module over \( \mathcal{H} \), which is neither dense nor trivial. If \( M \) contains a generalized highest weight vector, then \( M \) is a cut module.

**2.3 The know results**

The classification theorems of simple Harish-Chandra modules over the classical \( W \)-algebra \( W(2,2) \) and high rank Virasoro algebra will be recalled in this section.

The following result for the high rank Virasoro algebra appeared in [19].

**Theorem 2.4.** Let \( k > 1 \). Any non-trivial simple Harish-Chandra module for \( \mathcal{V}[\mathbb{Z}^k] \) is either a module of intermediate series or isomorphic to \( P_{g,\mu,K}^{\mathcal{V}[\mathbb{Z}^k]} \) for some \( \mu \in \mathbb{Z}^k \setminus \{0\} \), a subgroup \( g \) of \( \mathbb{Z}^k \) with \( \mathbb{Z}^k = g \oplus \mathbb{Z}\mu \) and a non-trivial simple intermediate series \( \mathcal{V}[g] \)-module \( \mathcal{K} \).
The classification of Harish-Chandra modules over the classical $W$-algebra $W(2, 2)$ was given in [17], which was reobtained in [7, 12] by some new ideas.

**Theorem 2.5.** Any non-trivial simple Harish-Chandra module over $W[Z]$ is either a module of intermediate series, or a highest/lowest weight module.

### 3 Cuspidal module

In this section, we determine the simple cuspidal module for the higher rank $W$-algebra $W(2, 2)$.

Let $Z^k = \bigoplus_{i=1}^k Z\epsilon_i$, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ is a $Z$-basis of $Z^k \subseteq \mathbb{C}$. Given $\alpha \in Z^k$, we set $\alpha = \sum_{i=1}^k \alpha_i \epsilon_i$ for $\alpha_i \in Z$. For any $\alpha, \beta \in Z^k$ with $\alpha_i, \beta_i \in \mathbb{N}, i \in \{1, \ldots, k\}$, we denote

$$\alpha^\beta := \alpha_1^{\beta_1} \cdots \alpha_k^{\beta_k} \quad \text{and} \quad \beta! = \beta_1! \cdots \beta_k!.$$

Conveniently, we denote $\partial := \frac{d}{dt}$. The high rank $W$-algebra $W(2, 2)$ can be realized from truncated high rank loop-Witt algebra $\mathcal{W}[Z^k] \otimes (\mathbb{C}[x]/\langle x^2 \rangle)$ (see, e.g., [12, 17]), namely,

$$L_{\alpha} = t^{\alpha+1} \partial \otimes 1, \quad W_{\alpha} = t^{\alpha+1} \partial \otimes x,$$

where $\alpha \in Z^k$. Denote $A = \text{span}\{t^\alpha \otimes 1 \mid \alpha \in Z^k\}$, which is a unital associative algebra with multiplication $(t^\alpha \otimes 1)(t^\beta \otimes 1) = t^{\alpha+\beta} \otimes 1$ for $\alpha, \beta \in Z^k$. For convenience, we write $t^{\alpha+1} \partial = t^{\alpha+1} \partial \otimes 1$ and $t^{\alpha} = t^{\alpha} \otimes 1$ for $\alpha \in Z^k$.

#### 3.1 $\mathcal{A\mathcal{W}}[Z^k]$-module

We describe the structure of cuspidal $\mathcal{W}[Z^k]$-modules that admit a compatible action of the commutative unital algebra $A$.

**Definition 3.1.** (see [2]) A module $M$ is called an $\mathcal{A\mathcal{W}}[Z^k]$-module if it is a module for both $\mathcal{W}[Z^k]$ and the commutative unital algebra $A = \mathbb{C}[t^{\pm 1}] \otimes 1$ with these two structures being compatible:

$$y(fv) = (yf)v + f(ys) \quad \text{for} \quad f \in A, \ y \in \mathcal{W}[Z^k], \ v \in M.$$

Let $M$ be a weight module over $\mathcal{A\mathcal{W}}[Z^k]$. From (3.1), we see that the action of $A$ is compatible with the weight grading of $M$:

$$A_{\alpha}M_{\lambda} \subset M_{\alpha+\lambda} \quad \text{for} \ \alpha, \lambda \in Z^k.$$

We suppose that $\mathcal{A\mathcal{W}}[Z^k]$-module $M$ has a weight space decomposition, and one of the weight spaces is finite-dimensional. According to all non-zero homogeneous elements of $A$
are invertible, we know that all weight spaces of $M$ have the same dimension. Then $M$ is also a free $A$-module of a finite rank. It is clear that $\mathcal{A}\mathbb{W}[\mathbb{Z}^k]$-module $M$ is cuspidal (as $\mathbb{W}[\mathbb{Z}^k]$-modules).

Assume that $M$ is a cuspidal $\mathcal{A}\mathbb{W}[\mathbb{Z}^k]$-module. Let $W = M_g$ for $g \in \mathbb{Z}^k$ and $\dim(W) < \infty$. From $M$ is a free $A$-module, we can write

$$M \cong A \otimes W.$$

**Lemma 3.2.** Let $M$ defined as above. For any $\alpha, n \in \mathbb{Z}^k$, we have $W_\alpha(t^n v) = t^n(W_\alpha v)$ for $v \in M$.

**Proof.** For any $\beta, n \in \mathbb{Z}^k$, by (3.1), we have

$$W_\beta(t^n v) = (W_\beta t^n)v + t^n(W_\beta v).$$

Note that $W_\beta t^n = nt^{n+\beta} \otimes x = n(t^{n+\beta} \otimes 1)(1 \otimes x)$. It is clear that $[1 \otimes x, \mathbb{W}[\mathbb{Z}^k] \oplus A] = 0$. Then there is a homomorphism of algebras $\chi : 1 \otimes x \to \mathbb{C}$ such that $1 \otimes x$ acts on $M$ as $\chi(1 \otimes x) \in \mathbb{C}$. So the action of $W_\beta t^n$ on $M$ can be written as $nt^ {n+\beta}$, where $\mu \in \mathbb{C}$. Now from

$$0 = [W_\alpha, W_\beta](t^n v) = n\mu^2(\beta - \alpha)t^{\alpha+\beta+n}v,$$

we get $\mu = 0$ by taking $n \neq 0, \alpha \neq \beta$, namely, $(W_\beta t^n)v = 0$. Putting this into (3.2), one has $W_\alpha(t^n v) = t^n(W_\alpha v)$ for $v \in M$. The lemma has been proved. \qed

**Remark 3.3.** We note that the $\mathcal{A}\mathbb{W}[\mathbb{Z}^k]$-module is a module for the semidirect product Lie algebras $\mathbb{W}[\mathbb{Z}^k] \ltimes A$ (the action of $A$ as a unital commutative associative algebra). The Lie brackets between $\mathbb{W}[\mathbb{Z}^k]$ and $A$ are given by $[L_m, t^n] = nt^m, [W_m, t^n] = 0$ for $m, n \in \mathbb{Z}^k$.

For $m \in \mathbb{Z}^k$, we consider the following operator

$$\mathfrak{D}(m) : W \to W.$$

It can be defined as the restriction to $W$ of the composition $t^{-m} \circ (t^{m+1}\partial)$ regarded also as an operator on $M$. Note that $\mathfrak{D}(0) = g \text{Id}$.

According to (3.1), Lemma 3.2 and the finite-dimensional operator $\mathfrak{D}(m)$, we get the action on $M$ as follows

$$L_m(t^n v) = (t^{m+1}\partial)(t^n v) = nt^{m+n}v + t^{m+n}\mathfrak{D}(m)v, \quad W_m(t^n v) = t^n(W_m v),$$

where $m, n \in \mathbb{Z}^k, v \in W$. Based on (2.1) and (3.3), it is easy to derive the Lie bracket (also see Lemma 3.2 in [1]):

$$[\mathfrak{D}(s), \mathfrak{D}(m)] = (m - s)\mathfrak{D}(s + m) - m\mathfrak{D}(m) + s\mathfrak{D}(s).$$

Next, we show that $\mathfrak{D}(m)$ can be expressed as a polynomial in $m = (m_1, \ldots, m_k)$. 
Theorem 3.4. Assume that \( M \) is a cuspidal \( \mathcal{AW}[\mathbb{Z}^k] \)-module, \( M = A \otimes W \), where \( W = M_g, g \in \mathbb{Z}^k \). Then the action of \( \mathcal{W}[\mathbb{Z}^k] \) on \( M \) is presented as

\[
L_m(t^nv) = nt^{m+n}v + t^{m+n}D(m)v, \quad W_m(t^n v) = t^n(W_m v),
\]

\( m, n \in \mathbb{Z}^k, v \in W \), where the family of operators \( D(m) : W \to W \) can be shown as an \( \text{End}(W) \)-valued polynomial in \( m = (m_1, \ldots, m_k) \) with the constant term \( D(0) = g\text{Id} \), and \( \text{Id} \) is the identification endomorphism of \( W \).

Proof. By \( m \in \mathbb{Z}^k \), we can write

\[
m = \sum_{i=1}^{k} m_i \epsilon_i,
\]

where \( m_i \in \mathbb{Z}, \epsilon_1, \epsilon_2, \ldots, \epsilon_k \) is a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^k \). According to Theorem 2.2 in [1], we obtain that \( D(m_i \epsilon_i) \) is a polynomial in \( m_i \in \mathbb{Z} \) with coefficients in \( \text{End}(W) \) for all \( i \in \{1, \ldots, k\} \). Now suppose that \( D(\sum_{i=1}^{j-1} m_i \epsilon_i) \) is a polynomial in \( \alpha_1, \ldots, \alpha_{j-1} \) for some \( 1 < j \leq k \). For \( m_j \in \mathbb{Z} \), it follows from (3.4) that

\[
\sum_{i=1}^{j-1} m_i \epsilon_i - m_j \epsilon_j D\left(\sum_{i=1}^{j} m_i \epsilon_i\right) = D(m_j \epsilon_j) D\left(\sum_{i=1}^{j} m_i \epsilon_i\right) - \sum_{i=1}^{j-1} (m_i \epsilon_i) D\left(\sum_{i=1}^{j} m_i \epsilon_i\right) - (m_j \epsilon_j) D(\alpha_j \epsilon_j).
\]

Consider \( m_j \neq 0 \). Then from the linearly independence of \( \epsilon_1, \ldots, \epsilon_j \), one has \( \sum_{i=1}^{j-1} m_i \epsilon_i - m_j \epsilon_j \neq 0 \). By the induction assumption, we conclude that \( D(\sum_{i=1}^{j} m_i \epsilon_i) \) is a polynomial in \( m_1, \ldots, m_j \), where \( 1 < j \leq k \). Choosing \( j = k \), one can see that \( D(m) \) is a polynomial in \( m_1, \ldots, m_k \). By the definition of operator \( D(m) \), one has \( D(0) = g\text{Id} \) for \( g \in \mathbb{C} \). We complete the proof. \( \square \)

We can write \( D(m) \) in the form (also see [1][3][5])

\[
\sum_{\tilde{i} \in \mathbb{N}^k} \frac{m_{\tilde{i}}}{\tilde{i}!} D(\tilde{i}), \tag{3.5}
\]

where \( \tilde{i}! = \prod_{j=1}^{k} \tilde{i}_j! \) and only has a finite number of the nonzero operators \( D(\tilde{i}) \in \text{End}(W) \).
Note that $\mathcal{D}(0) = D^{(0)}$. For $m, s \in \mathbb{Z}^k$, by (3.4), we have

$$\sum_{\tilde{i}, \tilde{j} \in \mathbb{N}^k} \frac{s^\tilde{i} m^\tilde{j}}{\tilde{i}! \tilde{j}!} [D^{\tilde{i}}, D^{\tilde{j}}]$$

$$= (\sum_{\tilde{i} \in \mathbb{N}^k} \frac{s^\tilde{i}}{\tilde{i}!} D^{\tilde{i}}) (\sum_{\tilde{j} \in \mathbb{N}^k} \frac{m^\tilde{j}}{\tilde{j}!} D^{\tilde{j}}) - (\sum_{\tilde{j} \in \mathbb{N}^k} \frac{m^\tilde{j}}{\tilde{j}!} D^{\tilde{j}}) (\sum_{\tilde{i} \in \mathbb{N}^k} \frac{s^\tilde{i}}{\tilde{i}!} D^{\tilde{i}})$$

$$= [\mathcal{D}(s), \mathcal{D}(m)] = (s - m) \mathcal{D}(s + m) - s \mathcal{D}(s) + m \mathcal{D}(m)$$

$$= \sum_{l=1}^{k} \left( \sum_{\tilde{i}, \tilde{j} \in \mathbb{N}^k} \frac{s^\tilde{i} m^\tilde{j}}{\tilde{i}! (\tilde{j} - \epsilon_l)!} D^{\tilde{i} + \tilde{j} - \epsilon_l} \right) - \sum_{l=1}^{k} \left( \sum_{\tilde{i}, \tilde{j} \in \mathbb{N}^k} \frac{s^\tilde{i} m^\tilde{j}}{\tilde{i}! (\tilde{i} - \epsilon_l)! \tilde{j}!} D^{\tilde{i} + \tilde{j} - \epsilon_l} \right). \quad (3.6)$$

Comparing the coefficients of $\frac{s^\tilde{i} m^\tilde{j}}{\tilde{i}! \tilde{j}!}$ in (3.6), we check that

$$[D^{\tilde{i}}, D^{\tilde{j}}] = \begin{cases} \sum_{l=1}^{k} (\tilde{j}_l - \tilde{i}_l) \epsilon_l D^{\tilde{i} + \tilde{j} - \epsilon_l} & \text{if } \tilde{i}, \tilde{j} \in \mathbb{N}^k \setminus \{0\}, \\ 0 & \text{if } \tilde{i} = 0 \text{ or } \tilde{j} = 0. \end{cases} \quad (3.7)$$

From (3.7), the operators $\mathcal{G} = \text{span}\{D^{\tilde{i}} \mid \tilde{i} \in \mathbb{N}^k \setminus \{0\}\}$ yield a Lie algebra. For $\tilde{i} \in \mathbb{N}^k$, let $|\tilde{i}| = \tilde{i}_1 + \tilde{i}_2 + \cdots + \tilde{i}_k$. Denote

$$\mathcal{G}_p = \text{span}\{D^{\tilde{i}} \mid \tilde{i} \in \mathbb{N}^k, |\tilde{i}| - 1 = p\} \quad \text{for } p \in \mathbb{N}.$$ 

Then $\mathcal{G} = \bigoplus_{p \in \mathbb{N}} \mathcal{G}_p$ is a $\mathbb{Z}$-graded Lie algebra (also see [1, 3, 15]). From (3.7), we know that $\mathcal{G}_0 = \text{span}\{D^{(\epsilon_i)} \mid i = 1, \ldots, k\}$ is a subalgebra of $\mathcal{G}$, whose Lie algebra structure is presented as

$$[D^{(\epsilon_i)}, D^{(\epsilon_j)}] = \epsilon_j D^{(\epsilon_i)} - \epsilon_i D^{(\epsilon_j)},$$

where $i, j \in \{1, 2, \ldots, k\}$. It is easy to get that $[\mathcal{G}_0, \mathcal{G}_0]$ is nilpotent. Hence, $\mathcal{G}_0$ is a solvable Lie algebra. Now we define the following one-dimensional $\mathcal{G}$-module $V(h) = \mathbb{C}v \neq 0$ for any $h \in \mathbb{C}$:

$$D^{(\epsilon_i)} v = h \epsilon_i v \quad \text{for } i \in \{1, \ldots, k\}. \quad (3.8)$$

The following lemma was proved in [15].

**Lemma 3.5.** Assume that $T$ and $W$ are finite-dimensional simple modules over $\mathcal{G}_0$ and $\mathcal{G}$, respectively. Then

(a) $T \cong V(h)$ for $h \in \mathbb{C}$.

(b) $D^{(\tilde{i})} W = 0$ for any $\tilde{i} \in \mathbb{N}^k$ with $|\tilde{i}|$ sufficiently large.
(c) \( G_p W = 0 \) for all \( p \in \mathbb{Z}_+ \) and \( W \cong V(h) \) as a \( G_0 \)-module for some \( h \in \mathbb{C} \).

**Theorem 3.6.** Any simple cuspidal \( \mathcal{A} \overline{W} [\mathbb{Z}^k] \)-module is isomorphic to a module of intermediate series \( \overline{M}(g, h; \mathbb{Z}^k) \) for some \( g, h \in \mathbb{C} \).

**Proof.** Let \( M \) be a simple cuspidal \( \mathcal{A} \overline{W} [\mathbb{Z}^k] \)-module, \( M = \mathcal{A} \otimes W \), where \( W = M_g, g \in \mathbb{Z}^k \). For \( n, \alpha \in \mathbb{Z}^k, v \in W \), based on Theorem 3.4, Lemma 3.5 and (3.5), we check that

\[
L_\alpha (t^n v) = nt^{\alpha + n}v + t^{\alpha + n}(\mathcal{D}(\alpha)v)
= t^{\alpha + n}((n + \mathcal{D}(0) + \sum_{i \in \mathbb{N} \setminus \{0\}} \frac{\alpha_i}{i!} D^{(i)}(\mathcal{D}(\alpha))v)
= t^{\alpha + n}((n + \mathcal{D}(0) + \sum_{i=1}^k \alpha_i D^{(\epsilon_i)}(\mathcal{D}(\alpha))v)
= t^{\alpha + n}((n + g \text{Id} + \sum_{i=1}^k h \alpha_i \epsilon_i)v)
= (n + g + h \alpha)(t^{\alpha + n}v),
\]

where \( g, h \in \mathbb{C} \). Using (3.9) and \((\beta - \alpha)W_{\alpha + \beta}(t^n v) = (L_{\alpha}W_{\beta} - W_{\beta}L_{\alpha})(t^n v)\), we obtain

\[
(\beta - \alpha)t^n(W_{\alpha + \beta}v) = \beta t^{\alpha + n}(W_{\beta}v).
\]

Taking \( \beta = 0 \) in (3.10), we immediately get \( t^n(W_{\alpha}v) = 0 \) for \( \alpha \neq 0 \). Considering \( \alpha = -\beta \neq 0 \) in (3.10) again, one checks \( t^n(W_0v) = 0 \). Then we conclude \( t^n(W_{\alpha}v) = 0 \) for \( \alpha, n \in \mathbb{Z}^k \), that is to say, \( W_{\alpha}(t^n v) = 0 \). This completes the proof. \( \square \)

### 3.2 \( \mathcal{A} \)-cover of a cuspidal \( \overline{W} [\mathbb{Z}^k] \)-module

We first recall the definitions of coinduced module and \( \mathcal{A} \)-cover (see \[2\]).

**Definition 3.7.** A module coinduced from a \( \overline{W} [\mathbb{Z}^k] \)-module \( M \) is the space \( \text{Hom}(\mathcal{A}, M) \) with the actions of \( \overline{W} [\mathbb{Z}^k] \) and \( \mathcal{A} \) as follows

\[
(a \varphi)(f) = a(\varphi(f)) - \varphi(a(f)), \quad (y \varphi)(f) = \varphi(yf),
\]

where \( \varphi \in \text{Hom}(\mathcal{A}, M), a \in \overline{W} [\mathbb{Z}^k], f, y \in \mathcal{A} \).

**Definition 3.8.** An \( \mathcal{A} \)-cover of a cuspidal module \( M \) over \( \overline{W} [\mathbb{Z}^k] \) is an \( \mathcal{A} \overline{W} [\mathbb{Z}^k] \)-submodule

\[
\hat{M} = \text{span}\{\phi(a, w) | a \in \overline{W} [\mathbb{Z}^k], w \in M\} \subset \text{Hom}(\mathcal{A}, M),
\]

where \( \phi(a, w) : \mathcal{A} \to M \) is defined as

\[
\phi(a, w)(f) = (fa)(w).
\]
The action of $\mathcal{A}\mathcal{W}[\mathbb{Z}^k]$ on $\hat{M}$ is given by

$$b\phi(a, w) = \phi([b, a], w) + \phi(a, bw),$$
$$f\phi(a, w) = \phi(fa, w) \quad \text{for } a, b \in \mathcal{W}[\mathbb{Z}^k], w \in M, f \in \mathcal{A}.$$ 

Let

$$\mathfrak{R}(M) = \left\{ \sum_{\alpha \in \mathbb{Z}^k} a_\alpha \otimes w_\alpha \in \mathcal{W}[\mathbb{Z}^k] \otimes M \mid \sum_{\alpha \in \mathbb{Z}^k} (fa_\alpha)w_\alpha = 0, \quad \forall f \in \mathcal{A} \right\}.$$ 

Then $\mathfrak{R}(M)$ is an $\mathcal{A}\mathcal{W}[\mathbb{Z}^k]$-submodule of $\mathcal{W}[\mathbb{Z}^k] \otimes M$. The $\mathcal{A}$-cover $\hat{M}$ can also be constructed as a quotient $\mathcal{A}\mathcal{W}[\mathbb{Z}^k]$-module

$$(\mathcal{W}[\mathbb{Z}^k] \otimes M)/\mathfrak{R}(M),$$

where $\mathcal{W}[\mathbb{Z}^k] M = M$. Clearly, the following linear map

$$\Theta : \quad \hat{M} \rightarrow \mathcal{W}[\mathbb{Z}^k] M$$
$$a \otimes w + \mathfrak{R}(M) \mapsto aw$$

is a $\mathcal{W}[\mathbb{Z}^k]$-module epimorphism.

**Lemma 3.9.** (see [2]) Let $M$ be a cuspidal module for $\mathcal{W}[\mathbb{Z}^k]$. Then there exists $l \in \mathbb{Z}_+$ such that for all $\alpha, \beta, \gamma \in \mathbb{Z}^k$ the operator $\Omega_{\alpha, \beta}^{(l, \gamma)} = \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha-i\gamma}L_{\beta+i\gamma} \text{ annihilates } M.$

**Lemma 3.10.** Let $M$ be a cuspidal $\mathcal{W}[\mathbb{Z}^k]$-module. Then there exists $r \in \mathbb{Z}_+$ such that for all $\alpha, \beta, \gamma \in \mathbb{Z}^k$ the following two operators

$$\Omega_{\alpha, \beta}^{(r, \gamma)} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_{\alpha-i\gamma}L_{\beta+i\gamma} \quad \text{and} \quad \tilde{\Omega}_{\alpha, \beta}^{(r, \gamma)} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} W_{\alpha-i\gamma}L_{\beta+i\gamma}$$

annihilate $M$.

**Proof.** Note that $M$ is also a cuspidal module for $\mathcal{V}[\mathbb{Z}^k]$. According to Lemma 3.9, there exists $l \in \mathbb{Z}_+$ such that $\Omega_{\alpha, \beta}^{(l, \gamma)} M = 0$ for all $\alpha, \beta, \gamma \in \mathbb{Z}^k$. It follows from this that

$$0 = \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} \left( L_{\alpha-(i-1)\gamma}L_{\beta+(i-1)\gamma} - 2L_{\alpha-i\gamma}L_{\beta+i\gamma} + L_{\alpha-(i+1)\gamma}L_{\beta+(i+1)\gamma} \right) \right) M$$
$$= \left( \sum_{i=0}^{l+2} (-1)^i \binom{l+2}{i} L_{\alpha-(i-1)\gamma}L_{\beta+(i-1)\gamma} \right) M. \quad (3.11)$$
Setting $r = l + 2$ in (3.11), one gets $\Omega_{\alpha,\beta}^{(r,\gamma)}M = 0$ for all $\alpha, \beta, \gamma \in \mathbb{Z}^k$. For any $s \in \mathbb{Z}^k$, from Lemma 3.9, we immediately get $[\Omega_{\alpha,\beta}^{(l,\gamma)}, W_s]M = 0$ for all $\alpha, \beta, \gamma \in \mathbb{Z}^k$. Now for any $\alpha, \beta, s \in \mathbb{Z}^k$ and $\gamma \in \mathbb{Z}^k \setminus \{0\}$, we can compute that

$$0 = \left( [\Omega_{\alpha,\beta}^{(l,\gamma)}, W_{s+\gamma}] - 2[\Omega_{\alpha,\beta}^{(l,\gamma)}, W_s] + [\Omega_{\alpha,\beta}^{(l,\gamma)}, W_{s-\gamma}] - [\Omega_{\alpha,\beta}^{(l,\gamma)}, W_s] \right) M$$

$$= \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha-i\gamma} L_{\beta-\gamma+i\gamma}, W_{s+i\gamma} \right) - 2\left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha-i\gamma} L_{\beta+i\gamma}, W_s \right)$$

$$+ \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha-i\gamma} L_{\beta+i\gamma}, W_{s-\gamma} \right) - \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha+i\gamma} L_{\beta+\gamma+i\gamma}, W_s \right)$$

$$+ 2\left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha+i\gamma} L_{\beta+\gamma+i\gamma}, W_{s-\gamma} \right) \right) M$$

$$= \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} \left( (s + (2 - i)\gamma - \beta)L_{\alpha-i\gamma} W_{\beta+s+i\gamma} \right.$$

$$+ (s + (i + 1)\gamma - \alpha)W_{\alpha+s+(1-i)\gamma} L_{\beta+(i-1)\gamma}$$

$$- 2((s - \beta - i\gamma)L_{\alpha-i\gamma} W_{\beta+s+i\gamma} + (s - \alpha + i\gamma)W_{\alpha+s-i\gamma} L_{\beta+i\gamma})$$

$$+ (s - \beta - (i + 2)\gamma)L_{\alpha-i\gamma} W_{\beta+s+i\gamma} + (s - \alpha + (i - 1)\gamma)W_{\alpha+s-(i+1)\gamma} L_{\beta+(i+1)\gamma}$$

$$- (s - \beta - (i - 1)\gamma)L_{\alpha+(i-1)\gamma} W_{\beta+s+(i-1)\gamma} - (s - \alpha + (i - 1)\gamma)W_{\alpha+s+(1-i)\gamma} L_{\beta+(i-1)\gamma}$$

$$+ 2((s - \beta - (i + 1)\gamma)L_{\alpha+(i-1)\gamma} W_{\beta+s+(i-1)\gamma} + (s - \alpha + (i - 2)\gamma)W_{\alpha+s-i\gamma} L_{\beta+i\gamma})$$

$$- (s - \beta - (i + 3)\gamma)L_{\alpha+(i-1)\gamma} W_{\beta+s+(i-1)\gamma}$$

$$- (s - \alpha + (i + 3)\gamma)L_{\alpha+(i-1)\gamma} W_{\beta+s+(i+1)\gamma} \left. \right) M$$

$$= \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} \left( W_{\alpha+s-(i-1)\gamma} L_{\beta+(i-1)\gamma} - 2W_{\alpha+s-i\gamma} L_{\beta+i\gamma} + W_{\alpha+s-(i+1)\gamma} L_{\beta+(i+1)\gamma} \right) \right) M$$

$$= \left( \sum_{i=0}^{l+2} (-1)^i \binom{l+2}{i} W_{\alpha+s-(i-1)\gamma} L_{\beta+(i-1)\gamma} \right) M.$$

Moreover, the module $M$ can be annihilated by the operator $\bar{\Omega}_{\alpha,\beta}^{(l+2,0)}$. Then we conclude that

$$\bar{\Omega}_{\alpha,\beta}^{(r,\gamma)}M = \left( \sum_{i=0}^{r} (-1)^i \binom{r}{i} W_{\alpha-i\gamma} L_{\beta+i\gamma} \right) M = 0,$$

where $r = l + 2$ and all $\alpha, \beta, \gamma \in \mathbb{Z}^k$. The lemma holds.

**Proposition 3.11.** Let $M$ be a cuspidal module over $\overline{W}[\mathbb{Z}^k]$. Then the $\mathcal{A}$-cover $\hat{M}$ of $M$ is also a cuspidal $\mathcal{A}\overline{W}[\mathbb{Z}^k]$-module.
Proof. Let $M_\lambda$ be a weight space with weight $\lambda \in \mathbb{Z}^k$. For $\alpha \in \mathbb{Z}^k$, we denote

$$\phi(L_\alpha \cup W_\alpha, M_\lambda) = \left\{ \phi(L_\alpha, w), \phi(W_\alpha, w) \mid w \in M_\lambda \right\} \subset \hat{M}.$$ 

By considering the weight spaces of $M$, the space $\phi(L_\alpha \cup W_\alpha, M_\lambda)$ is finite-dimensional.

Since $\hat{M}$ is an $\mathcal{A}$-module, we see that one of its weight spaces is finite-dimensional. For a fixed weight $\beta \in \mathbb{Z}^k$, we will show that $\hat{M}_\beta$ is finite-dimensional. Obviously, the space $\hat{M}_\beta$ is spanned by the set

$$\left( \bigcup_{\gamma \in \mathbb{Z}^k} \phi(L_{\beta-\gamma}, M_{\gamma}) \right) \cup \left( \bigcup_{\gamma \in \mathbb{Z}^k} \phi(W_{\beta-\gamma}, M_{\gamma}) \right).$$

Define a norm on $\mathbb{Z}^k$ as follows

$$\|\alpha\| = \sum_{i=1}^{k} |\alpha_i|,$$

where $\alpha = \sum_{i=1}^{k} \alpha_i \epsilon_i$. By Lemma 3.10 there exists $r \in \mathbb{N}$ such that for all $\alpha, \beta, \gamma \in \mathbb{Z}^k$ the following two operators

$$\Omega_{\alpha,\beta}^{(r,\gamma)} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_{\alpha-i\gamma} L_{\beta+i\gamma}$$

and

$$\tilde{\Omega}_{\alpha,\beta}^{(r,\gamma)} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} W_{\alpha-i\gamma} L_{\beta+i\gamma}$$

annihilate $M$, namely, $\Omega_{\alpha,\beta}^{(r,\gamma)} v = \tilde{\Omega}_{\alpha,\beta}^{(r,\gamma)} v = 0$ for all $\alpha, \beta, \gamma \in \mathbb{Z}^k, v \in M$. Hence, $\Omega_{\alpha,\beta}^{(r,\gamma)} v$ and $\tilde{\Omega}_{\alpha,\beta}^{(r,\gamma)} v$ are both in $\mathcal{R}(M)$.

Claim 1. For any $\alpha, \beta \in \mathbb{Z}^k$, $\hat{M}_{\alpha+\beta}$ is equal to

$$\text{span} \left\{ \phi(L_{\alpha+\beta} \cup W_{\alpha+\beta}, M_0), \phi(L_{\alpha-\gamma}, M_{\beta+\gamma}), \phi(W_{\alpha-\gamma}, M_{\beta+\gamma}) \mid \gamma \neq -\beta, \|\gamma\| \leq \frac{kr}{2} \right\}.$$

For all $q \in \mathbb{Z}^k$ and $w \in M_{\beta+q}$, we have $\phi(L_{\alpha-q}, w)$ and $\phi(W_{\alpha-q}, w)$ in $\hat{M}_{\alpha+\beta}$. Now we prove this claim by induction on $\|q\|$. If $|q_i| \leq \frac{r}{2}$ for all $i \in \{1, \ldots, k\}$, the result clears. On the other hand, suppose $|q_i| > \frac{r}{2}$ for some $i \in \{1, \ldots, k\}$. We may assume $q_i < -\frac{r}{2}$, and the other case $q_i > -\frac{r}{2}$ can be proved by the similar method. It is easy to see that $\|q+je_i\| < \|q\|$ for all $j \in \{1, \ldots, r\}$. We only need to give the proof for $\beta + q \neq 0$. From the action of $L_0$ on $M_{\beta+q}$ is a nonzero scalar, we can write $w = L_0 v$, where $v \in M_{\beta+q}$. We will verify that

$$\sum_{j=0}^{r} (-1)^{j} \binom{r}{j} \phi(L_{\alpha-q-j\epsilon_i}, L_{j\epsilon_i} v) = \sum_{j=0}^{r} (-1)^{j} \binom{r}{j} \phi(W_{\alpha-q-j\epsilon_i}, L_{j\epsilon_i} v) = 0$$

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Based on Definition 3.8 and Lemma 3.10, for any $m \in \mathbb{Z}^k$ we deduce
\[
\sum_{j=0}^{r} (-1)^j \binom{r}{j} \phi(L_{\alpha-q-j\epsilon_i}, L_{j\epsilon_i}v)(t^m)
= \sum_{j=0}^{r} (-1)^j \binom{r}{j} L_{\alpha+m-j\epsilon_i} L_{j\epsilon_i}v = \Omega^{(r,\epsilon_i)}_{\alpha+m-q,0} v = 0
\]
and
\[
\sum_{j=0}^{r} (-1)^j \binom{r}{j} \phi(W_{\alpha-q-j\epsilon_i}, L_{j\epsilon_i}v)(t^m)
= \sum_{j=0}^{r} (-1)^j \binom{r}{j} W_{\alpha+m-j\epsilon_i} L_{j\epsilon_i}v = \tilde{\Omega}^{(r,\epsilon_i)}_{\alpha+m-q,0} v = 0.
\]
Thus, one has
\[
\phi(L_{\alpha-q}, w) = -\sum_{j=1}^{r} (-1)^j \binom{r}{j} \phi(L_{\alpha-q-j\epsilon_i}, L_{j\epsilon_i}v), \quad (3.12)
\]
\[
\phi(W_{\alpha-q}, w) = -\sum_{j=1}^{r} (-1)^j \binom{r}{j} \phi(W_{\alpha-q-j\epsilon_i}, L_{j\epsilon_i}v). \quad (3.13)
\]
By induction assumption the right hand sides of (3.12) and (3.13) belong to $\widehat{M}_{\alpha+\beta}$, and so do $\phi(L_{\alpha-q}, w), \phi(W_{\alpha-q}, w)$. This proves the claim. Therefore, $\widehat{M}_{\alpha+\beta}$ is finite-dimensional. The proposition follows.

The Claim 1 can also be described as follows.

**Remark 3.12.** For $\alpha, \beta, q \in \mathbb{Z}^k$ and $\beta + q \neq 0, w \in M_{\beta+q}$, we get
\[
\phi(L_{\alpha-q}, w) \in \sum_{\|\gamma\| \leq \frac{\beta}{2}} \phi(L_{\alpha-\gamma}, M_{\beta+\gamma}) + \mathfrak{H}(M),
\]
\[
\phi(W_{\alpha-q}, w) \in \sum_{\|\gamma\| \leq \frac{\beta}{2}} \phi(W_{\alpha-\gamma}, M_{\beta+\gamma}) + \mathfrak{H}(M).
\]

Now we give a classification for all simple cuspidal $\mathcal{W}[\mathbb{Z}^k]$-modules.

**Theorem 3.13.** Any simple cuspidal $\mathcal{W}[\mathbb{Z}^k]$-module is isomorphic to a module of intermediate series $\widehat{M}(g, h; \mathbb{Z}^k)$ for some $g, h \in \mathbb{C}$. 

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Proof. Assume that $M$ is a simple cuspidal $\mathcal{W}[\mathbb{Z}^k]$-module. It is clear that $\mathcal{W}[\mathbb{Z}^k]M = M$. Then there exist an $\mathcal{A}$-cover $\hat{M}$ of $M$ with a surjective homomorphism $\Theta : \hat{M} \to M$. It follows from Proposition 3.11 that $\hat{M}$ is a cuspidal $\mathcal{AW}[\mathbb{Z}^k]$-module. Hence, we can consider the composition series

$$0 = \hat{M}_0 \subset \hat{M}_1 \subset \cdots \subset \hat{M}_c = \hat{M}$$

with the quotients $\hat{M}_{i+1}/\hat{M}_i$ being simple $\mathcal{AW}[\mathbb{Z}^k]$-modules. Let $d$ be the smallest integer such that $\Theta(\hat{M}_d) \neq 0$. Then by the simplicity of $M$, we obtain $\Theta(\hat{M}_d) = M$ and $\Theta(\hat{M}_{d-1}) = 0$. So we get an $\mathcal{AW}[\mathbb{Z}^k]$-epimorphism

$$\Theta : \hat{M}_d/\hat{M}_{d-1} \to M.$$ 

Now from Theorem 3.13, $\hat{M}_d/\hat{M}_{d-1}$ is isomorphic to a module of intermediate series $\overline{M}(g, h; \mathbb{Z}^k)$ for some $g, h \in \mathbb{C}$. We complete the proof. \qed

Based on the representation theory of $\mathcal{V}[\mathbb{Z}^k]$ studied in [28, 29], we see that the action of $C$ on any simple cuspidal $\mathcal{W}[\mathbb{Z}^k]$-modules is trivial. Therefore, the category of simple cuspidal modules over $\mathcal{W}[\mathbb{Z}^k]$ is equivalent to the category of simple cuspidal modules over $\mathcal{W}[\mathbb{Z}^k]$. Then Theorem 3.13 can be described as follows.

**Theorem 3.14.** Let $M$ be a simple cuspidal module over $\mathcal{W}[\mathbb{Z}^k]$. Then $M$ is isomorphic to a simple quotient of intermediate series module $M(g, h; \mathbb{Z}^k)$ for some $g, h \in \mathbb{C}$.

## 4 Non-cuspidal modules

In this section, we determine the simple weight modules with finite-dimensional weight spaces which are not cuspidal for the higher rank $\mathcal{W}$-algebras $\mathcal{W}(2, 2)$. These modules are called generalized highest weight modules and defined in Section 2.2.

The result of high rank Virasoro algebras of Theorem 2.4 plays a key role in the following proof.

**Theorem 4.1.** Let $M$ be a simple Harish-Chandra module over $\mathcal{W}[\mathbb{Z}^k]$. Then $M$ is either a cuspidal module, or isomorphic to some $P_{g, \mu, K}^{\mathcal{W}[\mathbb{Z}^k]}$, where $\mu \in \mathbb{Z}^k \setminus \{0\}$, $g$ is a subgroup of $\mathbb{Z}^k$ such that $\mathbb{Z}^k = g \oplus \mathbb{Z}\mu$ and $K$ is a non-trivial simple cuspidal $\mathcal{W}[\mathbb{Z}^k]_g$-module.

**Proof.** Suppose that $M$ is not a cuspidal module over $\mathcal{W}[\mathbb{Z}^k]$. Let us recall that $M = \bigoplus_{\alpha \in \mathbb{Z}^k} M_{\alpha}$ where $M_{\alpha} = \{w \in M \mid L_0 w = (g + \alpha)w\}$. By Theorem 2.5, we see that the statement holds for any $\mathbb{Z}^k$ of $k = 1$. 

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Now suppose \( k \geq 2 \). View \( M \) as a \( V[\mathbb{Z}^k] \)-module. Then based on Theorem 2.4 we obtain that the action of the central element \( C \) on \( M \) is trivial. Hence \( M \) can be seen as a \( \overline{W}[\mathbb{Z}^k] \)-module. We fix a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^k \), which is also suitable for \( \mathbb{Z}^k \). For any \( \sigma \in \mathbb{R}^k \) and \( g \in \mathbb{Z}^k \), we have the inner product \( \langle \sigma | g \rangle \). It follows from \( \overline{W}[\mathbb{Z}^k]_\alpha, \overline{W}[\mathbb{Z}^k]_\beta = \overline{W}[\mathbb{Z}^k]_{\alpha+\beta} \) that Theorem 2.3 can be applied to \( \mathbb{Z}^k \).

Since \( M \) is not cuspidal, we can find some rank \( k - 1 \) direct summand \( \overline{\mathbb{Z}}^k \) of \( \mathbb{Z}^k \) such that \( M_{\overline{\mathbb{Z}}^k} \) is not cuspidal. Without loss of generality, we may assume that \( \overline{\mathbb{Z}}^k \) is spanned by \( \{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\} \setminus \{\epsilon_j\} \), where \( j \in \{1, 2, \ldots, k\} \). Then there exists some \( \tilde{\alpha} \in \overline{\mathbb{Z}}^k \) such that

\[
\dim(M_{\overline{\mathbb{Z}}^{k}}) > 2k\left(\dim(M_{\epsilon_j}) + \sum_{i=1, i \neq j}^{k} \dim(M_{\epsilon_j + \epsilon_i})\right).
\]

(4.1)

For simplicity, we denote \( \xi_j = \tilde{\alpha} + \epsilon_j \) and \( \xi_i = \tilde{\alpha} + \epsilon_j + \epsilon_i \) for any \( i \in \{1, 2, \ldots, k\} \setminus \{j\} \). Then it is easy to check that the linear transformation sending each \( \epsilon_i \) to \( \xi_i \) for any \( i \in \{1, \ldots, k\} \), has determinant 1 and therefore \( \{\xi_i \mid i = 1, \ldots, k\} \) is also a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^k \). According to (4.1), there exists some nonzero element \( w \in M_{\overline{\mathbb{Z}}^{k}} \) such that \( L_{\xi_i}w = W_{\xi_i}w = 0 \) for all \( i \in \{1, \ldots, k\} \). Thus, \( w \) is a generalized highest weight vector associated with the \( \mathbb{Z} \)-basis \( \{\xi_i \mid i = 1, \ldots, k\} \).

It is clear that \( M \) is neither dense nor trivial. From Theorem 2.3 there exist some \( \beta \in \mathbb{Z}^k \) and \( \tau \in \mathbb{R}^k \setminus \{0\} \) such that \( \text{Supp}(M) \subseteq g + \beta + \mathbb{Z}^k_{<0} \). Consider \( M \) as a \( V[\mathbb{Z}^k] \)-module. Then \( M \) has a simple non-trivial \( V[\mathbb{Z}^k] \)-subquotient, and we denote it by \( \overline{M}^\gamma \), which is not cuspidal.

By Theorem 2.4 we know that \( \overline{M}^\gamma \cong P_{\psi, \mu, \nu}^{[\mathbb{Z}^k]} \) for some nonzero \( \mu \in \mathbb{Z}^k \), subgroup \( \mathfrak{g} \) of \( \mathbb{Z}^k \) with \( \mathbb{Z}^k = \mathfrak{g} \oplus \mathbb{Z} \mu \) and \( \mathcal{K} \) being a simple intermediate series module over \( V[\mathfrak{g}] \). Write \( \mathbb{Z}_\tau^k = \{\alpha \in \mathbb{Z}^k \mid \langle \tau | \alpha \rangle = 0\} \). In particular, we have

\[
g - \tilde{\alpha} \mu + \mathfrak{g} \subseteq \text{Supp}(\overline{M}^\gamma) \subseteq \text{Supp}(M) \subseteq g + \beta + \mathbb{Z}^k_{<0}
\]

for sufficiently large \( \tilde{c} \in \mathbb{Z}_+ \). This gives \( \mathfrak{g} = \mathbb{Z}_\tau^k \) and \( \langle \tau | \mu \rangle > 0 \).

We set that \( \tilde{c} \alpha \in \mathbb{Z} \) is the maximal number such that \( \mathcal{K} = M_{\psi + \tilde{c} \alpha \mu + \mathfrak{g}} \neq 0 \). Hence, \( \mathcal{W}[\mathbb{Z}^k]^+ \mathcal{K} = 0 \). Then it follows from the simplicity of \( \mathcal{W}[\mathbb{Z}^k] \)-module \( M \) that the simple \( \mathcal{W}[\mathfrak{g}] \)-module \( \mathcal{K} \) and \( M = P_{\psi, \mu, \nu}^{[\mathbb{Z}^k]} \). At last, note that \( \mathcal{K} \) is non-trivial and cuspidal. This proves the theorem.

Based on Theorems 3.14 and 4.1 we give a classification of simple Harish-Chandra modules over the higher rank \( W \)-algebra \( W(2,2) \).

**Theorem 4.2.** Assume that \( M \) is a non-trivial simple Harish-Chandra module over \( \mathcal{W}[\mathbb{Z}^k] \) for some \( k \in \mathbb{Z}_+ \).
(1) If $M$ is cuspidal, then $M$ is isomorphic to some $\mathcal{M}(g, h; \mathbb{Z}^k)$ for some $g, h \in \mathbb{C}$;

(2) If $M$ is not cuspidal, then $M$ is isomorphic to $P_{\mathfrak{g}, \mu, K}^{W[\mathbb{Z}^k]}$ for some $\mu \in \mathbb{Z}^k \setminus \{0\}$, a subgroup $\mathfrak{g}$ of $\mathbb{Z}^k$ with $\mathbb{Z}^k = \mathfrak{g} \oplus \mathbb{Z}\mu$ and a non-trivial simple intermediate series $W[\mathfrak{g}]$-module $K$.

Note that Theorem 2.5 is a special case of Theorem 4.2 for $k = 1$.

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