Observational Constraints on Inflation with the Nonlinear Sigma-Fields in Light of Planck 2015

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We study an inflation model with a nonlinear sigma field which has $SO(3)$ symmetry. The background solution of the nonlinear sigma field is linearly proportional to the space coordinates while keeping the homogeneous and isotropic background spacetime. We calculate the observable quantities, including the power spectra of the scalar and tensor modes, the spectral indices, the tensor-to-scalar ratio, and the running of the spectral indices, and then constrain our model with the Planck 2015 observational data.

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I. INTRODUCTION

Since the accidental detection of cosmic microwave background (CMB) radiation by Penzias and Wilson [1], CMB observations by WMAP [2] and recent ones by the Planck satellite [3, 4] have provided cosmological data with high precision. Recent Planck data give $n_s \approx 0.968$, $r < 0.11$ [4] and $\alpha_s = dn_s/d\ln k = -0.003$ [5] at a 95% confidence level (CL). These cosmological data seem to favor an inflationary scenario as the solution to the standard big bang problems. Inflation predicts an almost Gaussian and nearly scale invariant spectrum and provides the seeds for large-scale structure formation and primordial gravitational waves. With the coming precision cosmology era, discriminating inflationary models [6] and obtaining information around the Planck time may be possible.

The inflation model studied in this paper is based on the nonlinear sigma model, which has $SO(3)$ symmetry. An interesting feature of this model is the spatial coordinate dependent background solutions. Though the space-dependent background solutions appear to breaking the cosmological principle, i.e., homogeneity and isotropy, if the solutions are linearly proportional to the spatial coordinate, the cosmological principle is preserved. The ansatz of the spatially-linear background solution was used for compactifying extra dimensions [7] and for giving masses to gravitons through the Higgs mechanism of gravity [8,9]. The space-dependent background solutions were also studied before in Ref. [10,11] and may be obtained with a two-form gauge field that is dual to a pseudo-scalar field [12].

This model can provide the physical mechanism for the suppression of the CMB spectrum at large angular scales [13,14] if the sigma field is not coupled to an inflaton. The comoving horizon stays constant in the early phase of the inflation due to the existence of the space-dependent term in the solution. This implies the existence of minimum $k$ mode, $k_{min}$. Because the modes satisfying $k < k_{min}$ do not contribute to the power spectrum, this loss of $k$ modes results the lack of both the power spectrum of the tensor and scalar modes [15,16]. This is a generic feature of this model. In light of recent Planck data [5], we constrain the model parameters for this inflation model.

This paper is organized as follows: In Sec. II, we briefly introduce the inflation model with the nonlinear sigma fields. In Sec. III, the background solutions for $V \sim \phi^n$ are provided using the iteration method and assuming $\xi$ to be small enough. With these solutions, the $e$-folding number is obtained. The power spectra for the scalar and the tensor modes are presented in Sec. IV; then we calculate the observation variables in Sec. V. Also, the observation variables are constrained in light of Planck 2015 data in Sec. V. Finally, we summarise...
our results in Sec. VI.

II. BACKGROUND EVOLUTION

In this section, we describe the background evolution of the inflation model with the nonlinear sigma fields [16, 17]. The dynamics of a scalar field, \( \phi \), with an additional triad of scalar fields, \( \sigma^a \), is governed by the action

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right. \\
\left. - \frac{1}{2} g^{\mu\nu} \delta_{ab} \partial_\mu \sigma^a \partial_\nu \sigma^b \right],
\]

(1)

where \( \sigma^a \)’s have an \( SO(3) \) symmetry, the indices \( a \) and \( b \) run from 1 to 3 and \( \kappa^2 = 8\pi G \). Varying the action in Eq. (1) with respect to the metric \( g_{\mu\nu} \) yields the Einstein equation

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\kappa^2}{3} T_{\mu\nu},
\]

(2)

where the energy-momentum tensor is given by

\[
T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \partial_\mu \sigma^a \partial_\nu \sigma^a - g_{\mu\nu} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \sigma^a \partial_\beta \sigma^a + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right],
\]

(3)

and with respect to the scalar fields \( \phi \) and \( \sigma^a \) yields the equations of motion for both \( \phi \) and \( \sigma^a \):

\[
\partial_\mu \partial^\mu \phi - V_\phi = 0,
\]

(4)

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \sigma^a \right) = 0,
\]

(5)

where \( V_\phi \equiv dV/d\phi \). Assuming the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric for \( g_{\mu\nu} \),

\[
ds^2 = -dt^2 + a^2(x^i \delta_{ij} dx^i dx^j),
\]

(6)

where \( a(t) \) is a scale factor, and requiring the cosmological principle of homogeneity and isotropy to be preserved, we choose a spatially linear background solution for \( \sigma^a \) [8,10,11,16–18]:

\[
\sigma^a = \xi x^i \delta^a_i,
\]

(7)

where \( \xi \) is an arbitrary constant parameter of \( [M_p^2] \) dimensions, which satisfies the equations of motion in Eq. (5).

1 We use \( \xi \) instead of \( \alpha \), which was used in Ref. [17], to avoid confusion with the running spectral indices, \( \alpha \), that we will use later.

The background dynamics of this system yields the following equations of motion for the homogeneous scalar field, \( \phi \), and the scale factor \( a(t) \):

\[
H^2 = \frac{\kappa^2}{3} \left( \frac{1}{2} \dot{\phi}^2 + V + \frac{3}{2a^2} \xi^2 \right),
\]

(8)

\[
\dot{H} = -\frac{\kappa^2}{2} \left( \dot{\phi}^2 + \frac{\xi^2}{a^2} \right),
\]

(9)

\[
\ddot{\phi} + 3H \dot{\phi} + V_\phi = 0,
\]

(10)

where \( H \equiv \dot{a}/a \). The energy density and pressure from Eq. (3) with Eq. (7) are given by

\[
\rho = \frac{1}{2} \dot{\phi}^2 + V + \rho_\sigma,
\]

(11)

\[
p = \frac{1}{2} \dot{\phi}^2 - V + p_\sigma,
\]

(12)

where

\[
\rho_\sigma = \frac{3\xi^2}{2a^2} \quad \text{and} \quad p_\sigma = -\frac{\xi^2}{2a^2}.
\]

(13)

In the early phase of the evolution of the Universe, the \( \xi \)-dependent term can dominate over the kinetic and the potential terms due to the factor of \( 1/a^2 \). Then, the resulting equation of state parameter in that \( \xi \)-dominant phase from Eq. (13), \( w_\sigma \equiv p_\sigma/\rho_\sigma = -\frac{4}{3} \), shows that inflation does not occur in the early phase of the evolution of the Universe. As the Universe evolves, the \( \xi \)-dependent term decays away very quickly; then, the potential term of the scalar field \( \phi \) begins to dominate such that the spacetime experiences an accelerated expansion.

In this work, in order to have an inflationary solution, we assume that the \( \xi \) parameter is small enough; hence, we can neglect the pre-inflation phase. Therefore, during the phase of our interests, the initial state for the quantum fluctuations stays in the Bunch-Davies vacuum state, and the potential term dominates the \( \xi \)-dependent term such that usual slow-roll inflation takes place, although, a small contribution from the \( \xi \)-dependent term still remains [17].

III. SLOW-ROLL INFLATION MODEL WITH \( V \sim \phi^N \) POTENTIAL

As we learned in previous section, the spacetime experiences an accelerated expansion, \( \ddot{a} > 0 \), if and only if the \( \xi \) is small enough such that the potential energy of the scalar field dominates its kinetic energy:

\[
\dot{\phi}^2 \ll V.
\]

(14)

The accelerated expansion can be sustained for a sufficiently long period of time if the second-order time derivatives of \( \phi \) are small enough:

\[
|\ddot{\phi}| \ll |3H \dot{\phi}|, |V_\phi|.
\]

(15)
Therefore, in this section, we consider the slow-roll inflation model with the power-law potential

\[ V(\phi) = \lambda M_p^4 \left( \frac{\phi}{M_p} \right)^n, \tag{16} \]

where \( \lambda \) is an arbitrary dimensionless parameter. The background equations of motion, Eqs. (8) and (10), with the slow-roll approximations, Eqs. (14) and (15), and with the inflaton potential, in Eq. (16), yield

\[ H^2 = \frac{\kappa^2}{3} \left( \lambda M_p^4 \left( \frac{\phi}{M_p} \right)^n + \frac{3\xi^2}{2a^2} \right), \tag{17} \]

\[ 3H \dot{\phi} + n \lambda M_p^2 \left( \frac{\phi}{M_p} \right)^{n-1} \simeq 0. \tag{18} \]

Following Ref. [17], we define the slow-roll parameters to reflect the slow-roll conditions, in Eqs. (14) and (15), as

\[ \epsilon \equiv \frac{\kappa^2 \dot{\phi}^2}{H^2}, \quad \eta \equiv \frac{V_{\phi\phi}}{3H^2}. \tag{19} \]

and inflation requires smallness of the slow-roll parameters. Inflation ends when the slow-roll conditions are violated \( \epsilon(\phi_e) = 1 \): where \( \phi_e \) is the field value at the end of inflation. We compute the e-folding number before the end of inflation as

\[ N = \kappa^2 \int_{\phi_e}^{\phi_i} \left( \frac{V}{V_{\phi\phi}} + \frac{3\xi^2}{2a^2V_{\phi}} \right) d\phi \]

\[ = \frac{\kappa^2}{n} \int_{\phi_e}^{\phi_i} \frac{\phi d\phi}{\epsilon + \xi^2 \phi + n} \int_{\phi_e}^{\phi_i} \frac{1}{a^2 \phi^{n-1}} d\phi. \tag{20} \]

Because \( \xi \) is small, which is consistent with observations [17], we can find the background analytic solutions through the iteration method. We can expand \( a \) and \( \phi \) up to \( \xi^2 \) order:

\[ a(t) = a_0(t) + \xi^2 a_1(t) + \mathcal{O}(\xi^4), \tag{21} \]

\[ \phi(t) = \phi_0(t) + \xi^2 \phi_1(t) + \mathcal{O}(\xi^4). \tag{22} \]

Because the second term in Eq. (20) is already order of \( \xi^2 \), only the 0th order solutions for both \( \phi(t) \) and \( a(t) \) need to be considered. At zeroth order, we obtain for general \( n \neq 4 \)

\[ \phi_0(t) = 2\pi \left\{ (n - 4)(\beta t - c_1) \right\}^{1/(n-4)}, \tag{23} \]

\[ a_0(t) = a_0 e^{-\frac{4}{n} (\phi_0^2 - \phi_1^2)}, \tag{24} \]

and for \( n = 4 \)

\[ \phi_0(t) = \phi_1 e^{-\beta t}, \tag{25} \]

where \( \beta = \sqrt{n^2 \lambda M_p^4 - n}/(3n^2) \).

Then the e-folding number, Eq. (20), up to \( \xi^2 \) order becomes

\[ N = \frac{\kappa^2}{2n} (\phi_0^2 - \phi_1^2) + \frac{\xi^2 n}{2\beta^2} \frac{1}{a_0^2 e^{\pi^2 \phi^2_0/4}} \times \left\{ -\frac{1}{2} \phi_0^{-2-n} \left( \frac{\kappa^2 \phi_0^2}{n} \right)^{n/2-1} \Gamma \left( 1 - \frac{n}{2} - \frac{\kappa^2 \phi_0^2}{n} \right) \right\}, \tag{26} \]

where \( \Gamma(a,z) \) is the incomplete Gamma function. If we take \( \phi_0 \sim 15 M_p \), then \( \kappa^2 \phi_0^2/n \sim 8\pi(15 M_p)^2/n M_p^2 \sim 10^4/2n \), we can approximate \( x \equiv -\kappa^2 \phi_0^2/n \to \infty \). Using the asymptotic property of the incomplete Gamma function as \( x \to \infty \),

\[ \Gamma(s, x) \to x^{s-1} e^{-x} \text{ as } x \to \infty, \]

we use asymptotically

\[ \Gamma \left( 1 - \frac{n}{2} - \frac{\kappa^2 \phi_0^2}{n} \right) \simeq \left( -\frac{\kappa^2 \phi_0^2}{n} \right)^{-\frac{n}{2}} e^{\pi^2 \phi_0^2/4}. \tag{27} \]

After substituting Eq. (22) and Eq. (27) into Eq. (26), we obtain

\[ N \simeq \frac{\kappa^2}{2n} \left[ \phi_0^2 + \frac{2\xi^2}{M_p^2} \left( M_p^4 \phi_0 \phi_1 + \frac{3n}{4 \kappa^2 \lambda M_p^4} a_0^2 \phi_0^2 \right) \right], \tag{28} \]

where we have neglected \( \phi_1^2 \) because we assume that the value of the scalar field at the end of inflation is much smaller than that at the beginning, \( \phi_e \ll \phi_0 \).

For simplicity, we consider only the \( n = 2 \) case in this work. The next-leading-order solution for \( \phi_1(t) \) can be obtained as

\[ \phi_1(t) = \frac{3 M_p^2}{4 \lambda M_p^4} \frac{1}{a_0^2 \phi_0^2} \left[ 1 - \sqrt{\frac{2\kappa^2 \phi_0}{F(x)}} F \left( \frac{\sqrt{\frac{2\kappa^2 \phi_0}{F(x)}}}{2} \right) \right] \approx 0, \tag{29} \]

where \( F(x) \) is a Dawson function and the last "\(~\)" holds for \( \sqrt{\kappa^2/2 \phi_0} \to \infty \) because \( F'(x) = 1 - 2x F(x) \to 0 \) as \( x \to \infty \). Therefore, the e-folding number, Eq. (28), becomes

\[ N \approx \frac{\kappa^2}{4} \left( \phi_0^2 + \frac{M_p^4 \delta}{a_0^2 \phi_0^2} \right), \tag{30} \]

where we introduce a dimensionless variable \( \delta \equiv \frac{3 \phi_0^2}{2 \kappa^2 M_p^2} \), which is much smaller than unity. Because \( \delta \ll 1 \), we expand \( \varphi_N \equiv \phi_0^2 \) as a function of \( N \) as

\[ \varphi_N = \varphi_0(N) + \delta \varphi_1(N) + \mathcal{O}(\delta^2). \tag{31} \]

Then, Eq. (30) yields, up to linear order in \( \delta \),

\[ N \approx \frac{\kappa^2}{4} \left[ \varphi_0(N) + \delta \varphi_1(N) + \frac{1}{a_0^2 \phi_0^2 \phi_1} e^{-\frac{\pi^2}{4} \phi_0^2/4} \right]. \tag{32} \]
We obtain the following expressions: for the zeroth order in $\delta$,
\[ N = \frac{\kappa^2}{4} \varphi^0(N), \tag{33} \]
and for the first order in $\delta$,
\[ 0 = \varphi^1(N) + \frac{1}{a^2 e^{\frac{\xi}{2} \phi^0 e^{-\frac{\varphi^0}{2} \phi^0}}}. \tag{34} \]
By substituting Eqs. (33) and (34) into Eq. (31), we obtain the scalar field as function of $N$, the number of e-folds:
\[ \phi^2_0 = \frac{1}{\kappa^2} N - \frac{4}{\kappa^2} \frac{\delta}{a^2 e^{\frac{\xi}{2} \phi^0 e^{-\frac{\varphi^0}{2} \phi^0}}}. \tag{35} \]

We re-express the slow-roll parameters, Eq. (19), in terms of the inflaton potential for $n = 2$ in Eq. (16) as
\[ \epsilon = \frac{V^2}{2M^2 V^2} \left( 1 - \frac{3\xi^2}{a^2 V} \right) = - \frac{2}{\kappa^2 \phi^2} \left( 1 - \frac{3\xi^2}{\lambda M_p^2 a^2 \phi^2} \right), \tag{36} \]
\[ \eta = \frac{V_{\phi\phi}}{\kappa^2 V} \left( 1 - \frac{3\xi^2}{a^2 V} \right) = - \frac{2}{\kappa^2 \phi^2} \left( 1 - \frac{3\xi^2}{2\lambda M_p^2 a^2 \phi^2} \right). \tag{37} \]

If we use the same approximation for $n = 2$ as in Eq. (29), we can replace $\phi$ with $\phi_0$ which is zeroth order in $\xi^2$.
\[ \epsilon = \frac{2}{\kappa^2 \phi_0^2} \left( 1 - \frac{3\xi^2}{\lambda M_p^2 a^2 \phi_0^2} e^{\frac{\xi}{2} \phi_0 e^{-\frac{\varphi^0}{2} \phi_0}} \right), \tag{38} \]
\[ \eta = \frac{2}{\kappa^2 \phi_0^2} \left( 1 - \frac{3\xi^2}{2\lambda M_p^2 a^2 \phi_0^2} e^{\frac{\xi}{2} \phi_0 e^{-\frac{\varphi^0}{2} \phi_0}} \right). \tag{39} \]
In terms of $N$, Eqs. (38) and (39) can be written as
\[ \epsilon = \frac{1}{2N} \left( 1 - \frac{3\kappa^2 \xi^2 (8N - 1)}{32\lambda M_p^2 a^2 e^{\frac{\xi}{2} \phi_0 e^{-\frac{\varphi^0}{2} \phi_0}} e^{-N N^2}} \right), \tag{40} \]
\[ \eta = \frac{1}{2N} \left( 1 - \frac{3\kappa^2 \xi^2 (4N - 1)}{32\lambda M_p^2 a^2 e^{\frac{\xi}{2} \phi_0 e^{-\frac{\varphi^0}{2} \phi_0}} e^{-N N^2}} \right). \tag{41} \]

### IV. LINEAR PERTURBATIONS IN SLOW-ROLL INFLATION

In this section, we will briefly review the linear perturbations of both the scalar and the tensor modes for our model (see the details in Ref. [17] with $f(\phi) = 1$). We do the linear perturbations in the context of slow-roll inflation by assuming the model parameter to be small, but non-vanishing, $\xi \ll 1$, and by assuming the slow-roll approximation given in Eqs. (14) and (15). The linearly perturbed metric in a conformal Newtonian gauge is given by
\[ ds^2 = -a^2(\tau)[(1 + 2A)dt^2 + \{(1 - 2\psi)\delta_{ij} + h_{ij}\}dx^i dx^j], \tag{42} \]
where $\tau$ is a conformal time, $dt = a(\tau)d\tau$, and $h_{ij}$ is given by
\[ h_{ij} = \frac{1}{2} \left( \xi + \dot{\delta} + \dot{\phi} + \ddot{\phi} + \dddot{\phi} \right), \tag{43} \]
They satisfy the following equations of motion in the slow-roll approximation with $\xi \ll 1$: \[ \ddot{Q}_\phi + 3H\dot{Q}_\phi + \left( \frac{k^2}{a^2} + \frac{\dot{\phi}_V}{M_p^2 H} + V_{\phi\phi} \right) Q_\phi \approx 0, \tag{44} \]
\[ \ddot{Q}_u + 3H\dot{Q}_u + \left( \frac{k^2}{a^2} + \frac{2\xi^2}{M_p^2 a^2} \right) Q_u \approx 0. \tag{45} \]
The comoving curvature perturbations, $\mathcal{R}$, and the isocurvature perturbations, $S$, in the slow-roll approximation with a small, non-vanishing value of $\xi$, are defined as [17]
\[ \mathcal{R} \approx \frac{\sqrt{\pi}}{\sqrt{\kappa} M_p} \left( 1 - \frac{\xi^2}{2\lambda M_p^2 a^2 H^2} \right) Q_\phi + \frac{\xi}{2\lambda M_p k^2 H} Q_u, \tag{46} \]
\[ S \approx \frac{\xi^2}{12\sqrt{\pi} M_p^2 a^2 H^2} Q_\phi + \frac{4\sqrt{\pi}}{3\sqrt{\kappa} M_p H} \left( 1 - \frac{\xi^2}{3\lambda M_p^2 a^2 H^2} \right) \dot{Q}_\phi \tag{47} \]
\[ - \frac{\xi k}{9M_p k a^2 H} Q_u. \]
In our current work, we are interested in obtaining constraints on the model parameters from the Planck data by using $n_s$, $r$, and $\alpha_s$; we will ignore the isocurvature modes, $S$, in this work. Because $\sigma^a \propto O(\xi)$, we expect $\delta a^2 = \partial^a u = \partial^a Q_u \propto O(\xi^2)$. If we keep only $\xi^2$ order, curvature perturbations become
\[ \mathcal{R} \approx \frac{\sqrt{\pi}}{\sqrt{\kappa} M_p} \left( 1 - \frac{\xi^2}{2\lambda M_p^2 a^2 H^2} \right) Q_\phi. \tag{48} \]
This can also be justified from the resulting power spectrum of the curvature perturbations in which the $Q_u$ term contributes $O((\xi/M_p k)^2)$ in the range $\xi/(M_p k) < 1$ [16,17]. As we will see below, this range is the range of interest in this work.
After Fourier transforming $Q_\phi$ and $h_{ij}$ to momentum space,

$$Q_\phi(\tau) = \frac{1}{a} \int d^3k u_k(\tau)e^{i k \cdot x},$$

$$h_{ij}(\tau) = 2\frac{8\pi G}{a} \int d^3k u_k(\tau)e^{i k \cdot x},$$

we can obtain the Sasaki-Mukhanov equation for the scalar and the tensor modes as

$$u_k'' + \left( k^2 - \frac{\mu^2}{a^2} \right) u_k = 0,$$

$$v_k'' + \left( k^2 - \frac{\mu^2}{a^2} \right) v_k = 0,$$

where

$$k_s^2 = k^2 - \frac{\xi^2}{6M_p^2}, \quad k_t^2 = k^2 + \frac{11\xi^2}{6M_p^2},$$

$$\mu_s = \frac{3}{2} + 3\epsilon - \eta, \quad \mu_t = \frac{3}{2} - \epsilon. $$

In order to obtain Eqs. (51) and (52), we use background equations, Eqs. (17) and (18), in the slow-roll approximation and keep the terms linear in the slow-rolls parameters $\epsilon$, $\eta$, and $\xi^2$. Choosing the Bunch-Davies vacuum for an initial state at $\tau \to -\infty$ by taking the positive frequency modes, we obtain the exact solutions for the scalar and the tensor modes:

$$u_k = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}(\mu_s + \frac{1}{2})\sqrt{-\tau}H_\mu(1)(-k_s\tau)},$$

$$v_k = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}(\mu_t + \frac{1}{2})\sqrt{-\tau}H_\mu(1)(-k_t\tau)},$$

where we assume that the slow-roll parameters are constants and that $k_s$ should be real to have a well-defined quantum state. Thus, $k_s^2$ also should be positive. Then, $k_s^2$ can be constrained from the expression of $k_s^2$ in Eq. (53):

$$k_s^2 \geq \frac{\xi^2}{6M_p^2}. $$

The existence of the cutoff scale, $k_{min} = \xi/\sqrt{6}M_p$, implies a lack of power in the power spectrum [16] and results in the suppression of the angular power spectrum in the CMB. The cut-off scale for the scalar modes is shown explicitly in Eq. (53). However, another cutoff scale exists in the power spectrum for the tensor modes, but is not shown explicitly [15]. This cutoff originates from the fact that the comoving horizon is given by $aH \sim \kappa \xi/\sqrt{2}$ in Eq. (17) in the early phase when the $\xi$ term is dominant. Therefore, the model satisfying $k < \kappa \xi/\sqrt{2}$ cannot contribute to the observed power spectra for the scalar and the tensor modes. This cutoff scale seems to be a very generic feature of this inflation scenario which is motivated by the nonlinear sigma model.

$$u_k \approx e^{i\frac{\pi}{4}(\mu_s + \frac{1}{2})\sqrt{-\tau}H_\mu(1)(-k_s\tau)}$$

$$v_k \approx e^{i\frac{\pi}{4}(\mu_t + \frac{1}{2})\sqrt{-\tau}H_\mu(1)(-k_t\tau)},$$

where we have used

$$H_\mu(1) \approx \frac{-2}{1 - e^{2\mu\pi} - e^{\mu\pi} \Gamma(1 - \mu)} \left( \frac{\tau}{2} \right)^{-\mu}. $$

The power spectra in the large-scale limit of the curvature perturbation $R$ and the tensor modes yield

$$P_R(k) = \frac{k^3}{2\pi^2} |R_k|^2 = \frac{2k^3}{\pi \epsilon M_p a^2} \left( 1 - \frac{\xi^2}{\epsilon M_p^2 a^2 H^2} \right) |u_k|^2$$

$$\approx \frac{H^2}{\pi \epsilon M_p^2} \left( 1 + 2C\delta - 2\epsilon - \frac{\xi^2}{\epsilon M_p^2 a^2 H^2} \right) \frac{k}{aH}^{-6\epsilon + 2\eta},$$

$$P_T(k) = 2P_R = \frac{k^3}{2\pi^2} \frac{64\pi}{M_p a^2} |u_k|^2$$

$$\approx \frac{16H^2}{\pi M_p^2} \left( 1 + (2C - 2)\epsilon - \frac{33\xi^2}{12M_p^2 a^2 H^2} - \frac{\xi^2}{3M_p^2 a^2 H^2} \right)$$

$$\times \left( \frac{k}{aH} \right)^{-2\epsilon},$$

where the numerical factor “2” in Eq. (62) comes from the two polarization states, $\delta = 3\epsilon - \eta$, $C = 2 - \gamma - \ln 2$, and $\gamma \approx 0.5772$ is an Euler-Mascheroni constant. We have also used

$$\tau = -\frac{1}{aH} \left( 1 + \epsilon + \frac{\xi^2}{6M_p^2 a^2 H^2} \right).$$

Note that Eq. (61) is valid only in the range of $\xi^2$ given by

$$\xi^2 \leq \epsilon^2 k^2 M_p^2. $$

V. OBSERVATIONAL CONSTRAINT IN LIGHT OF PLANK 2015

In this section, we consider the observational constraint on our model with the potential given in Eq. (16) where we only consider the $n = 2$ case. From Eqs. (61) and (62), we compute the spectral indices, the tensor-to-scalar ratio, and the running spectral indices at the.
horizon crossing time at which $k = aH$ holds. We obtain the following results up to leading order in the slow-roll parameters $\epsilon$ and $\eta$, and $\xi^2$

$$n_s - 1 = \frac{d\ln P_R(k)}{d\ln k} \bigg|_{k=aH} = 2\eta - 6\epsilon + \frac{2\xi^2}{M_p^2 k^2},$$

$$\alpha_s = \frac{dn_s}{d\ln k} \bigg|_{k=aH} = -4\xi^2 M_p^2 k^2,$$

$$n_t = \frac{d\ln P_T(k)}{d\ln k} \bigg|_{k=aH} = -2\epsilon + \frac{31\xi^2}{6M_p^2 k^2},$$

$$\alpha_t = \frac{dn_T}{d\ln k} \bigg|_{k=aH} = -\frac{31\xi^2}{6M_p^2 k^2},$$

$$r = \frac{P_T}{P_R} \bigg|_{k=aH} = 16\epsilon \left(1 - 4C - 2\right) + 2C\eta + \frac{\xi^2}{2 M_p^2 k^2}.$$  

(65) (66) (67) (68) (69)

Because $\xi^2/\epsilon^2 M_p^2 k^2 > 0$, $\alpha_s$ and $\alpha_t$ take on negative values, and if we take into account of Eq. (64), they are on the orders of $\epsilon$ and $\epsilon^2$, respectively. Note that in standard single-field inflation, $\alpha_s$ is on the order of $O(\epsilon^2)$. For the numerical results below, we use the slow-roll parameters obtained in Eqs. (40) and (41) and set $\Lambda = 0.5 \times 10^{-12}$, $\phi_i = 16M_p$, $a_i = 1$ and $M_p^2 = 1$. We plot $n_s$ vs. $r$ in Fig. 1 for two different pivot scales: $k = 0.05\text{Mpc}^{-1}$ (Planck) on the left and $k = 0.002\text{Mpc}^{-1}$ (WMAP) on the right profile. In Fig. 1, and throughout this paper, the circles represent $N = 60$ while the triangles correspond to $N = 50$.

The usual single-field inflation predictions correspond to $\xi = 0$, and the model expectation for $N = 50$ is situated outside the $2\sigma$ contour while it is situated at the edge of $2\sigma$ contour for $N = 60$ [5]. As we turn on the $\xi$ value, our results in Eqs. (65)–(69) depend on the comoving scale $k$ as well as the model parameter $\xi$. In Fig. 1(a) where we set the pivot scale $k = 0.05\text{Mpc}^{-1}$, the model expectation is situated inside the $2\sigma$ contour, and $\xi$ takes values in the interval $10^{-6} < \xi < 2.4 \times 10^{-5}$ for $N = 60$. Similarly in Fig. 1(b) where we set $k = 0.002\text{Mpc}^{-1}$, the parameter range of $\xi$, which is situated inside the $2\sigma$ contour, is $10^{-7} < \xi < 8 \times 10^{-6}$ for $N = 60$. From Fig. 1, we see that the model expectation for $N = 60$ shows better consistency with the data than that for $N = 50$, for which our results are situated outside the $2\sigma$ contour, close the model parameter becomes $\xi \leq 10^{-6}$ for $k = 0.05\text{Mpc}^{-1}$ and $\xi \leq 10^{-7}$ for $k = 0.002\text{Mpc}^{-1}$, our result converges to that of the standard single-field inflation where $\xi = 0$. As is seen in Fig. 1, we obtain the parameter ranges of $\xi$ for a fixed pivot scale $k$ such that the model expectation is consistent with the observational data [5] up to the 95% confidence level.

Because our results in Eqs. (65)–(69) show a scale dependence, along with a model parameter dependence, we plot $k$ vs. $\xi$ in Fig. 2 for fixed $\epsilon$-folding number, $N = 60$, and for the observable quantities, Eqs. (65), (66) and (69).

We add grid lines in the background of Fig. 2 to see the corresponding values between $k$ and $\xi$ and to obtain the parameter range in which our results are consistent with the observational data obtained by Planck2015 [5]. For the background grid lines, we set the model parameters as follows: the blue solid line is for $\xi = 2.4 \times 10^{-5}$, the blue dashed line is for $\xi = 2.2 \times 10^{-5}$, the red solid line is for $\xi = 1.3 \times 10^{-5}$, the blue dotted line is for $\xi = 4.5 \times 10^{-6}$, the red dashed line is for $\xi = 3.5 \times 10^{-6}$, the green solid line is for $\xi = 10^{-6}$, the blue dot-
Fig. 2. (Color online) Plots of $k$ vs. $\xi$ for observable parameters (65), (66) and (69). We set $\lambda = 0.5 \times 10^{-12}$, $\phi_i = 16 M_p$, $a_i = 1$ and $M_p^2 = 1$. For the background grid lines, we set the model parameters as follows: the blue solid line is for $\xi = 2.4 \times 10^{-7}$, the blue dashed line is for $\xi = 2.2 \times 10^{-5}$, the red solid line is for $\xi = 1.3 \times 10^{-3}$, the blue dotted line is for $\xi = 4.5 \times 10^{-5}$, the red dashed line is for $\xi = 3.5 \times 10^{-6}$, the green solid line is for $\xi = 10^{-6}$, the blue dot-dashed line is for $\xi = 4.5 \times 10^{-7}$, the red dotted line is for $\xi = 3.5 \times 10^{-7}$ and the green dashed line is for $\xi = 10^{-7}$ from the top.

(a) A pivot scale set at $k = 0.05\text{Mpc}^{-1}$

(b) A pivot scale set at $k = 0.002\text{Mpc}^{-1}$

Fig. 3. (Color online) The $n_s$ vs. $\alpha_s$ plot for $\lambda = 0.5 \times 10^{-12}$, $\phi_i = 16$, $a_i = 1$ and $M_p^2 = 1$, where the circles represent $N = 60$ while the triangles indicate $N = 50$. The dashed line is for $\xi = 4.5 \times 10^{-7}$, the red dotted line is for $\xi = 3.5 \times 10^{-7}$, and the green dashed line is for $\xi = 10^{-7}$ from the top. Figure 2 shows the plot range of each observable parameter, in a bar legend, chosen in such a way that the result would be consistent with the observational data [5]. The observational data favor the range $0.96 \leq n_s \leq 0.98$, $10^{-3} \leq r < 0.3$, and $-0.017 \leq \alpha_s \leq -0.0002$. As Fig. 2 shows, in order to be consistent with the observational data, the model parameter $\xi$ should decrease as the $k$-value decreases and vice versa. Figure 3 shows the plot of $n_s$ vs. $\alpha_s$ for the same numerical values of $\xi$ and $k$ that were used in Fig. 1.

Figure 1 showed that our result was situated outside the $2\sigma$ contour for $N = 50$. However, for the running scalar spectral indices for the scalar mode, the model expectation is now consistent with the data and is even situated well inside the $2\sigma$ contour for all chosen $\xi$ values.
VI. CONCLUSION AND DISCUSSION

We have studied slow-roll inflation with the nonlinear sigma fields that have $SO(3)$ symmetry. Motivated from the compactification in higher dimension theory, we use the linearly spatial-coordinate-dependent ansatz, as a solution, for the nonlinear sigma fields. This model has two interesting features. First, the $\xi$-term-dominant phase due to the nonlinear sigma fields in the Friedmann equation is followed by a standard inflation phase. The equation-of-state parameter for the nonlinear sigma fields are dominant. This pre-inflation phase would provide or constrain the initial condition to the observed power spectrum. The cutoff scale gives $-1/3$, so inflation does not occur when the nonlinear sigma fields are dominant. This pre-inflation phase would provide or constrain the initial condition to the observed power spectrum [16]. Second, a generic cutoff scale exists in the power spectrum. This cutoff scale originates from the fact that the comoving horizon is given by $aH \sim \kappa \xi \sqrt{2}$ in the nonlinear sigma field dominant phase. This implies the mode satisfying $k < k_{\text{min}} \sim \kappa \xi \sqrt{2}$ cannot contribute to the observed power spectrum [15].

In light of Planck 2015 observational data, we constrain our model with $V(\phi) \sim \phi^n$, especially for $n = 2$. We use the iteration method to obtain the background solution for $\phi$ and $a$ and then express the slow-roll parameters in terms of the $e$-folding number in Eqs. (40) and (41). For the $n_a - r$ plot, our model shows different results depending on $\xi$ as well as $k$. For example, if the model parameter $\xi$ takes value within the interval $10^{-6} < \xi \leq 2.4 \times 10^{-5}$ for fixed pivot scale $k = 0.05$ Mpc $^{-1}$ and $e$-folding number $N = 60$, the model predictions reside inside a $2 \sigma$ contour. If $\xi \leq 10^{-6} \sim 10^{-7}$, the model predictions give the same results with usual single field inflation ($\xi = 0$). We have found in Fig. 2 that as the $k$-value decreases, $\xi$-value decreases as well to be consistent with the observational data. Although we restrict our analysis to the $\phi^2$ potential in this paper, considering different types of potentials and finding the parameter range for $\xi$, in light of the observation, would be interesting as considering inflation with the nonlinear sigma fields that are coupled with the inflaton.

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