CLUSTER PRODUCTION WITH COALESCENCE AND BREAKUP

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Abstract

The problem of hadronic cluster production in quark-hadron phase transition in heavy-ion collisions is studied by cellular automata. Previous result on the scaling behavior is extended to include variation in the drift speed. It is also shown that coalescence is more important than growth in generating scaling. A new set of rules is adopted to free the clusters from being rigid. It is found that the scaling exponent is independent of not only the shapes of the clusters, but also the probability of breakup of the clusters. The universality of the scaling behavior is now extended to a wide range of physical properties characterizing the geometry and dynamics of the phase transition process.

1 Introduction

The problem of hadronic cluster production in the mixed region in heavy-ion collisions has been investigated during the past year, yielding the intriguing result that the distribution of cluster sizes exhibits universal scaling behavior \cite{1, 2}. During the mixed phase the quark-hadron system evolves by itself to self-organized criticality (SOC) \cite{3}. The method of investigation is by use of cellular automata. Simple rules are adopted to describe the dynamical process of formation, growth, coalescence, breakup and emission of the hadronic clusters in the quark-gluon plasma undergoing quark-hadron phase transition. The breakup process is considered in the 1-dimensional (1D) problem \cite{1}, because it is easy to implement, but not in 2D \cite{4} since there are too many possibilities that are difficult to cover by a set of simple rules. In this paper we modify the cellular automaton so that not only is breakup made feasible simply, but also is the deformation of the clusters allowed as they evolve.

Despite the drastic simplification of the complicated nonequilibrium dynamical process by the adoption of a set of simple rules, an essential aspect of SOC that
renders the result interesting physically is that the scaling behavior is independent of
the parameters in the rules. Moreover, the result should be independent of the details
of the rules themselves so that the power law obtained is a general consequence of the
physical system and not of the particular ways in which the dynamics is transcribed
into cellular automata. For that reason a change of rules, as considered in this paper,
is an important part of the overall program to determine the generic properties of
SOC that the quark-gluon system possesses.

There are three areas of investigations in this paper. First, in the framework of
the original cellular automaton \[2\], we study the dependences on the drift speed and
the relative importance of growth versus coalescence. Second, we modify the rules to
allow for deformation of the cluster shapes. Third, using the new cellular automaton
we consider the breakup of clusters.

## 2 Drift, Growth and Coalescence

We begin with a brief summary of the modelling of the problem and the original rules
of the cellular automaton used in Ref. \[4\]. Starting with the conventional picture
of heavy-ion collision, there is a cylinder of quark-matter, expanding mostly in the
longitudinal direction, and at a lower rate transversely. Consider a cross section of
the cylinder at midrapidity some time after the collision so that in the ideal case
there is a circular core region containing quarks and gluons. Hereafter, for brevity
we shall just call them quarks with gluons implied, and refer to the core as the \(Q\)
region. Surrounding the \(Q\) region is the \(M\) region, an annular ring in which quarks
and hadrons coexist in the mixed phase. Due to radial expansion the hadrons move
outward and leave the \(M\) region, while the quarks remain the \(M\) region because of
confinement. The \(Q\) region reduces in size as quarks are fed into the \(M\) region; the \(M\)
region also shrinks in time as hadronic clusters are emitted. A wedge of the \(M\) region
is mapped to a \(L \times L\) lattice initially, where the left boundary represents the boundary
between the \(Q\) and \(M\) regions, and is fixed. The right boundary moves semilocally
to the left depending on where the clusters are emitted. Periodic boundary condition
is imposed on the upper and lower sides.

There are three parameters in the problem: \(L\), the initial lattice size in units of
some hadronic scale; \(S_0\), the nucleation size in units of the same scale; and \(p\), the
nucleation probability in each time step. The rules of the cellular automaton are in
essence the following.

- **Nucleation.** An unoccupied site represents the quark phase, while an occupied
  site represents the hadron phase. At each time step an unoccupied site has prob-
  ability \(p\) of becoming an occupied site, which, if it is an isolate side, constitutes
  a single-site nucleation with \(S_0 = 1\).

- **Growth.** If the newly nucleated site is a near neighbor of an existing occupied
site, it is regarded as a growth process and the sites are bonded to form a cluster. Further growth of the cluster can take place in the same way in later steps.

c Drift. A cluster moves to the right one step plus a random step to any one of its four immediate neighbors.

d Coalescence. When two clusters collide, i.e., when their constituent sites overlap, they are bonded to form a larger cluster with one of the overlapping sites moved to the nearest unoccupied site.

e Emission. Upon reaching the right boundary a cluster is removed from the \( M \) region, whose new boundary is reset at one layer to the left for the same number of vertical sites as the number of sites \( S \) contained in the cluster emitted. When the right boundary reaches the left boundary, the transition process is terminated.

According to the above rules the distribution of the sizes of the emitted clusters is found to satisfy the scaling law \[ P(S) \propto S^{-\gamma}, \] where \[ \gamma = 1.86 \pm 0.18. \]

Furthermore, it is found that \( \gamma \) is independent of \( L \) and \( S_0 \) (provided that \( L \) is large enough, e.g., 16 and 32), and depends very little on \( p \) (provided \( p \) is in the range \( 0.05 \leq p \leq 0.5 \)). In that sense we regard the result as exhibiting universal scaling behavior.

There are two questions that can be clarified without changing the rules. One is the dependence of the above result on the average drift velocity. The other is the relative importance of growth versus coalescence in their contributions to the formation of large clusters. For the former we double the average drift step to 2, keeping \( S_0 = 1 \) and \( L = 16 \), and obtain the result as shown in Fig. 1. We see that \( P(S) \) fails to develop scaling behavior for small \( p \), but for \( p > 0.1 \) there is scaling with essentially the same slope as before. Evidently, as the clusters move more rapidly out of the \( M \) region, they have less time to grow and coalesce with the consequence that for \( p \leq 0.05 \) the formation of large clusters is suppressed. When we double \( L \) to 32, then scaling is restored for \( p \geq 0.05 \) as indicated in Fig. 2, thus verifying the importance of having enough steps in the \( M \) region for the large clusters to form.

Comparing Fig. 2 to the result of the “standard” case \( S_0 = 1, L = 16, \) drift step = 1, considered in [2], and reproduced in Fig. 3, one finds that \( P(S) \) are nearly identical for each value of \( p \). Note that this is not merely a consequence of scaling up the lattice spacing. The random-walk step size has not been changed. Thus Fig.
2 corresponds to the standard case with half the step size in random walk. That reveals another feature of the universality, i.e., the independence on the temperature (if thermal description is appropriate) at which the transition takes place. But scaling follows only when the $M$ region is large enough (relative to the drift step) which is a conclusion already reached in Ref. [3], when $L$ is reduced to 8.

Concerning the relative importance of growth and coalescence, we consider in the standard case when only growth is allowed (Fig. 4) and then when only coalescence is allowed (Fig. 5). It is clear from these two figures that coalescence is the principal cause of the scaling behavior. Without coalescence there are no large clusters no matter how large $p$ is.

3 Modified Rules

According to the rules thus far considered, the clusters develop irregular shapes because the bondings are rigid. Since dendritic structure has more unoccupied neighboring sites, it favors growth and coalescence. One may question whether that contributes to a bias in favor of developing larger clusters. If the surface tension is negative, the clusters should have irregular surfaces. But if it is positive and large, then they have circular shape in 2D. It is difficult to require specific shapes for the clusters on a lattice by adding new rules to the cellular automaton used above. We propose to modify the relevant part of the rules fundamentally.

Our new approach is to stack a cluster vertically on only one site and to assign an effective radius $R$ to that site with

$$R = \alpha \sqrt{S/\pi} \quad (3)$$

where $\alpha$ is a parameter that quantifies the irregularity of the cluster shape. If $\alpha = 1$, the shape is circular; if $\alpha > 1$, then it can be dendritic or even porous, depending on how large $\alpha$ is. The stack is always placed at a lattice site, whether or not the effective circle corresponding to (3) covers $S$ discrete sites. In drift and random walk the stack moves in discrete steps as before. Growth and coalescence can now be treated on the same footing. Let the distance between two clusters, $S_1$ and $S_2$, located at $(x_1, y_1)$ and $(x_2, y_2)$, respectively, be

$$d_{12} = \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{1/2} \quad (4)$$

If the two sites satisfy the condition

$$d_{12} \leq R_1 + R_2 \quad , \quad (5)$$

then coalescence occurs. If one of the sites is a newly occupied position with size $S_0$, then it is a growth [when (5) is satisfied with the corresponding $R$ being given by (3) with $S = S_0$]. In either case we put the combined cluster of size $S_1 + S_2$ at the
site closest to the center-of-mass of the two original sites. If more than two clusters coalesce simultaneously, we stack them all at the new site closest to the overall cm position. Since we impose periodic boundary condition on the upper and lower edges of the lattice, there are always two vertical distances for every pair of \( y_1 \) and \( y_2 \):

\[ |y_1 - y_2| \text{ and } L - |y_1 - y_2|. \]

In (4) we use the lesser of these two for \((y_1 - y_2)\).

The result obtained with this new cellular automaton is shown in Fig. 6 for the standard case: \( L = 16 \), \( S_0 = 1 \) and \( p = 0.2 \). For \( \alpha = 1.0 \) the scaling exponent is nearly identical to the result of the earlier cellular automaton, shown in Fig. 3. Thus our clusters produced previously \(^2\) are not very different from circular clusters. More significantly, the result does not depend sensitively on the specific rules adopted. This is of great importance if the scaling behavior that emerges is to be regarded as a general consequence of the physical system, not of the particular cellular automaton used. Fig. 6 also shows the two other cases with \( \alpha = 1.2 \) and 1.4. There is a slight increase of the slopes, but very little. Note that with \( \alpha = 1.4 \) the effective area of a cluster is double that for \( \alpha = 1.0 \). This enlargement causes crowding of the \( M \) region and increases the emission rate, thus reducing the probability of incubating large clusters. Since the increase of the slope is so small, we should regard the result as essentially indicating approximate independence on \( \alpha \), or the shapes of the clusters.

4 Breakup

In the original cellular automaton of Sec. 2, it is difficult to introduce breakup, since no simple rule can be added to take care of all possible ways a big cluster of irregular shape fragments. With our new cellular automaton of Sec. 3, the process is easy to implement.

Our procedure is to regard breakup as a consequence of spontaneous fission \( S \rightarrow S_1 + S_2 \). Breakup in a collision is a combination of coalescence and fission: \( S_1 + S_2 \rightarrow S_3 \rightarrow S_4 + S_5 \). If one of \( S_4 \) and \( S_5 \) further fissions in the next time step then the overall reaction may be regarded as a \( 2 \rightarrow 3 \) cluster production process. In each time step we have all four processes: nucleate, walk, coalesce, and break up, in that order.

Since a cluster is now a stack at one site with an effective radius of extension, breakup is a simple partition of the stack \( S \) into two stacks \( S_1 \) and \( S_2 \). The only essential input is the probability of occurrence for various channels of partitioning. Our guidance is the minimization of the circumference-to-area ratio. Thus we parameterize the breakup probability by

\[
B(S, S_1, S_2) = \beta \left( \sqrt{S_1/S} + \sqrt{S_2/S} \right)^{-b}
\]

with \( S_1 + S_2 = S; \beta \) and \( b \) are both positive. The positions of the \( S_1 \) and \( S_2 \) clusters are specified as follows. If the original site of \( S \) is at \((x, y)\), then \( S_1 \) and \( S_2 \) are placed at \((x, y + y_1)\) and \((x, y - y_2)\), respectively, where \( y_{1,2} \) are the largest integers less than
\( \alpha \sqrt{S_2,1/\pi} + 1 \). According to this rule the effective areas of \( S_1 \) and \( S_2 \) will never overlap. If either one of those areas overlaps with the effective area of any existing cluster, then the breakup is forbidden from taking place, since overcrowding will lead to immediate coalescence and further breakup, \textit{ad infinitum}.

In our simulation we have set \( b = 1 \) as a representative value, and varied \( \beta \) from 0 to 1.0 with the other parameters set at the standard values: \( L = 16, S_0 = 1, p = 0.01 \) to 0.2, \( \alpha = 1 \). In Fig. 7 we show the result for \( p = 0.01 \). Recall from Fig. 3 for no breakup that scaling cannot be achieved at the low nucleation probability of \( p = 0.01 \). Now with breakup it is even more difficult as indicated by the curves for \( \beta = 0.5 \) and 1.0 in Fig. 7. Clearly, large clusters cannot easily be formed when there are few existing clusters to coalesce, and even when a large cluster comes into being, it breaks up when \( \beta \) is large. The situation changes as \( p \) is increased, as can be seen in Fig. 8 and 9. When \( p = 0.05 \), more growth and coalescence can occur to give scaling \( P(S) \), if \( \beta = 0 \). But as \( \beta \) increases, breakup still has a small effect in disrupting the formation of large clusters, as is evident in Fig. 8. However, when \( p = 0.2 \), Fig. 9 shows that breakup has no more effect. There is sufficiently many clusters in the \( M \) region that breakup and coalescence occur at equal rate to result in negligible difference for any \( \beta \).

From these results we may conclude that when scaling is fully developed for \( 0.05 < p < 0.5 \) the cluster distribution is essentially independent of whether the clusters can break up. Thus the universality of the scaling behavior is broadened in one additional aspect of the dynamics of cluster formation.

5 Conclusion

The complicated process of quark-hadron phase transition cannot be treated reliably by any analytical method, especially when thermal equilibrium is not assumed. The use of cellular automata, on the other hand, could also be subject to the criticism that the process has been approximated by a scheme of unknown reliability. However, complications have not been ignored; they have been taken into account by simple rules. It is when the result exhibits independence on the details of the rules that one develops confidence in the meaningfulness of the general implication of the result.

In Ref. [2] we have shown that the scaling behavior of \( P(S) \) is insensitive to the lattice size \( L \), the nucleation radius \( S_0 \), and the nucleation probability \( p \). Here, in this paper we have demonstrated its insensitivity to the drift speed and fluctuations, and have shown that the primary mechanism responsible for scaling is coalescence, not growth. By adopting a set of simpler rules, we have further shown the independence on the factors (such as surface tension) that govern the shape of the clusters. Moreover, the scaling exponent is independent of whether the clusters break up when there is sufficient nucleation. The universality of the scaling behavior is truly amazing.

The independence on the various factors examined suggests that so long as the
region in which hadrons are formed while in the environment of quarks and gluons is large enough, universal scaling will ensue whatever the dynamical details may be. This is of tremendous significance to the search of signatures of phase transition when the confinement process is not well understood. Of course, the realistic experimental signature still awaits further study on how the hadronic clusters produced at high temperature (considered here) are transformed to cold hadrons at the detector. Our result provides the motivation to examine that problem both theoretically and experimentally. The result also suggests the possibility that the problem of self-organized criticality studied here may in some way be related to percolation in nuclear fragmentation. To set the stage for connecting quark-hadron phase transition in heavy-ion collision with self-organized criticality is in itself perhaps the most interesting development that this line of investigation has initiated.

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References

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Figure Captions

**Fig. 1** Cluster-size distribution $P(S)$ for $L = 16$ when the drift step-size is doubled.

**Fig. 2** Same as Fig. 1, but for $L = 32$.

**Fig. 3** $P(S)$ for the standard case of $S_0 = 1$, $L = 16$ when the drift step-size is 1 unit.

**Fig. 4** The standard case with growth only, without coalescence.

**Fig. 5** The standard case with coalescence only, without growth.

**Fig. 6** $P(S)$ for various effective radii parameterized by $\alpha$.

**Fig. 7** $P(S)$ for $p = 0.01$ when there is breakup parameterized by $\beta$.

**Fig. 8** Same as Fig. 7, but for $p = 0.05$.

**Fig. 9** Same as Fig. 7, but for $p = 0.2$. 
Fig. 1
Fig. 2
Fig. 3
Growth only $S_0=1$

$L=16$

- $p=0.01$
- $p=0.05$
- $p=0.1$
- $p=0.2$
- $p=0.5$

Fig. 4
Fig. 5
Fig. 7
Fig. 8
Fig. 9