Mapping Hawking into Unruh for global embeddings

Wen-Yuan Ai, Hua Chen, Jian-Bo Deng

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, P. R. China

E-mail: aiwy11@lzu.edu.cn, hchen12@lzu.edu.cn, dengjb@lzu.edu.cn

Abstract: We study the mechanism of global embeddings into the Minkowski space-time(GEMS) with the Hawking into Unruh mapping. We find a constraint that the extrinsic acceleration of the static observer in the Riemann space must satisfy for such embeddings. Thus the question raised by Paston in Ref. [1], that is, when does the Hawking into Unruh mapping for global embeddings work, is partly addressed. We also calculate the potential barrier of a scalar field to reach $r \to \infty$ in the ambient space. The results show that the potential barrier is finite, hence the static observer at $r \to \infty$ can indeed detect the radiation caused by the Unruh effect from the embedding view. However the potential barriers calculated in both the Riemann background and the Minkowski background are not coincident, therefore the GEMS approach is not complete and the Hawking effect can be distinguished from the Unruh effect of the GEMS in principle.

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\(^1\)Corresponding author.
1 Introduction

In 1970s Hawking discovered that, using the quantum field theory (QFT) in curved spacetime, a black hole emits thermal radiation with characteristic Hawking temperature

$$T_H = \frac{\kappa}{2\pi}$$  \hspace{1cm} (1.1)

as seen by asymptotic observers, where $\kappa$ is the surface gravity of the black hole [2]. The local temperature measured by a fiducial observer at a finite distance from a black hole is then described by the Tolman temperature [3]

$$T = \frac{1}{2\pi} \frac{\kappa}{\sqrt{g_{00}}}$$  \hspace{1cm} (1.2)

where $x^0$ is the time-like Killing vector of a detector in its rest frame.

Soon after Hawking’s discovery, Unruh realized that many features present in the Hawking effect could be well understood in a simpler context of Minkowski spacetime [4]. The Unruh effect says that the Minkowski vacuum, i.e., the quantum state associated with the nonexistence of particles according to the inertial observers, corresponds to a thermal bath of elementary particles at temperature

$$T_U = \frac{a}{2\pi}$$  \hspace{1cm} (1.3)

as measured by uniformly accelerated observers with proper acceleration $a$. It reflects that the particle content of spacetime is an observer dependent quantity. The key to understand the Hawking effect via the Unruh effect lies in the principle of equivalence. A freely falling observer near the horizon would observe locally around him a flat Minkowski spacetime. And in his vicinity a static observer would be as a constant accelerating observer with a
acceleration, say $a$. Now the static observer would see a thermal flux at an Unruh temperature $\pi a^2$. And a static observer at infinity would at last have the Hawking temperature with a redshift.

Even so, there are differences between the Hawking effect and the Unruh effect. For example, the temperature measured by a fiducial observer at infinity is nonvanishing in the Hawking effect and vanishing in the Unruh effect. This difference is due to the different potential barriers of fields in a black hole background and a Rindler background [5]. The potential barrier in a black hole is cut off when the distance from the event horizon becomes bigger, while in the Rindler case the potential barrier increases without bound. Also the Unruh observer is more closely related to the observer’s proper acceleration.

A more unified description of the Hawking effect and the Unruh effect, based on an earlier work [6], was first put forward by Deser and Levin [7]. This is called the global embedding Minkowski spacetime approach, also named GEMS approach. In this approach, the Hawking detector and its event horizon are mapped to the Rindler detector in the corresponding flat higher dimensional embedding spacetime [8] and its Rindler horizon. Then identifying the acceleration of the Unruh detector, the Hawking radiation can be interpreted as the Unruh effect in the flat ambient spacetime. Subsequently, the corresponding results were obtained for other types of black holes, black strings, wormholes, as well as de Sitter and anti-de Sitter spaces [7, 9–14]. The GEMS approach to some other motions has also been discussed [15–17]. It has been noted in [18] that the Hawking into Unruh mapping for black holes also takes place if we embed the (t-r) sector of the Riemann space only.

Since the Hawking effect results from the QFT in curved spacetime and the Unruh effect in flat spacetime. The existence of the Hawking into Unruh mapping may mean a close relationship between the QFT in curved spacetime and in its flat ambient spacetime. Such relationship may support the suggestion that the gravity, usually described in the framework of General Relativity, can be considered as an embedding theory where the 4-dimensional spacetime is a surface in a flat higher dimensional space. This approach may provide new constructions of a correct quantum theory of gravity. There are various works on this embedding theory done in [19–25]. In the work [26] the gravity and matter fields are, on the same footing, formulated as field theories in a flat ambient space.

Though there are many successful embeddings with the Hawking into Unruh mapping, Paston has pointed out that for some embeddings the Hawking into Unruh mapping is invalid [1]. Thus great care is needed when the GEMS approach is used. And a natural question is raised by Paston: when does the Hawking into Unruh mapping for global embeddings work?

The aim of the present work is to study the mechanism of the GEMS approach with the Hawking into Unruh mapping. We discuss in what cases, the thermal Hawking temperature can be obtained by use of the GEMS approach. A specific condition will be found for such cases. When the condition is satisfied, we also calculate the potential barrier of free scalar fields in the flat ambient spacetime.

In section 2, we discuss the mechanism of the GEMS approach. The required condition when the thermal Hawking temperature can be calculated by use of the Unruh effect in the ambient spacetime is obtained. In sections 3 and 4, we explain in detail why for the
embeddings discussed the Hawking into Unruh mapping works or does not work. Finally, in section 5, we calculate the potential barrier of a massless scalar field in the flat background spacetime. And comparison is made with the potential barrier in a black hole background to get some conclusions.

2 The mechanism of Hawking into Unruh mapping for global embeddings

To apply the GEMS approach, we need to isometrically embed a Riemann space, $\mathcal{M} = (M^n, g)$, into a Minkowski space, $\bar{\mathcal{M}} = (\bar{M}^m, \eta)$, where $m > n$. For future use, let us first consider the covariant derivative $\bar{\nabla}_X \bar{Y}$, where $X, Y$ are restricted in $T(M)$, i.e., $\bar{X} = X$, $\bar{Y} = Y$. We can decompose $\bar{\nabla}_X$ into tangential and normal components [8]

\[
\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y).
\] (2.1)

This defines the symmetric, bilinear map $\alpha : T(M) \times T(M) \rightarrow N(M)$, which is called the second fundamental form of $M \subset \bar{M}$.

To obtain the thermal Hawking temperature, we need to further identify the static observer as a constant accelerating observer in the ambient space. However it is not sure that we can obtain the same temperatures. The Unruh temperature, which is closely related to the proper acceleration of the detector, is

\[
T_U = \frac{\bar{a}}{2\pi}.
\] (2.2)

Here $\bar{a} = |\nabla_\xi \xi|/|\xi|^2$ is the observer’s proper acceleration in the Minkowski spacetime and $\xi$ is the observer’s time-like Killing vector. This is not the same for the Hawking temperature which is described by eq. (1.2). The surface gravity is defined as [27]

\[
\kappa^2 = -\frac{1}{2} (\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu),
\] (2.3)

where the right side is to be evaluated at the horizon. At first sight, the Hawking temperature has no relation with the observer’s proper acceleration. But we can extract the observer’s proper acceleration from the surface gravity and define

\[
b^2 = \frac{\kappa^2}{g_{00}} - a^2,
\] (2.4)

where $a = |\nabla_\xi \xi|/|\xi|^2$.

Now letting $X = Y = \xi$ in eq. (2.1), one can obtain

\[
\bar{\nabla}_\xi \xi = \nabla_\xi \xi + \alpha(\xi, \xi).
\] (2.5)

Let us define the extrinsic acceleration as

\[
a_e = |\alpha(\xi, \xi)|/|\xi|^2.
\] (2.6)
Then we have
\[ a^2 = a^2 + a_c^2. \] (2.7)

Making the Hawking temperature equal to the Unruh temperature, we immediately find that \( b^2 = a_c^2 \) or \( \kappa = \sqrt{g_{00}}a_c \), or explicitly written as
\[ \kappa |\xi|^2 = \sqrt{g_{00}}|\nabla \xi|. \] (2.8)

This is the condition we get for global embeddings with the thermal Hawking temperature into the Unruh temperature mapping.

Here we would like to give some comments on the condition. The quantity \( b^2 \) becomes the extrinsic acceleration from the embedding view while has no explicit meaning from the intrinsic view. The Hawking effect is also closely related to the observer’s acceleration with contributions from both intrinsic acceleration and a new quantity, which emerges from the GEMS approach, the extrinsic acceleration.

3 Embeddings with mapping

In this section, we will take some successful embeddings to examine the condition (2.8).

For the Schwarzschild spacetime with the metric
\[ ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \] (3.1)

it has been found in [7] that for a Fronsdal’s isometric embedding [28] in a flat 6-dimensional space with the (+ - - - - -) signature
\[
\begin{align*}
y^0 &= 4M \sqrt{1 - 2M/r} \sinh(t/4M), \\
y^1 &= 4M \sqrt{1 - 2M/r} \cosh(t/4M), \\
y^2 &= \int \sqrt{(2Mr^2 + 4M^2 r + 8M^2)/r^3} dr, \\
y^3 &= r \cos \theta, \\ y^4 &= r \sin \theta \cos \varphi, \\ y^5 &= r \sin \theta \sin \varphi,
\end{align*}
\] (3.2)

the Hawking into Unruh thermal properties mapping work. Here the event horizon maps to the Rindler horizon and the static observer maps to the constant accelerating observer. The surface gravity for Schwarzschild spacetime is \( 1/4M \). And the acceleration in the GEMS for embedding (3.2) is \( 1/(4M \sqrt{1 - 2M/r}) \). Thus we can see the condition (2.8) is indeed satisfied.

The above case can be generalized to Schwarzschild-AdS spaces using a D=7 GEMS with an additional timelike dimension \( y^6 \) [7].
\[
\begin{align*}
y^0 &= \kappa^{-1} \sqrt{1 - 2M/r + r^2/R^2} \sinh(\kappa t), \\
y^1 &= \kappa^{-1} \sqrt{1 - 2M/r + r^2/R^2} \cosh(\kappa t), \\
y^2 &= \int \frac{R^3 + R^2_H + r^2_H + r^3_H}{R^2 + 3r^2_H} \sqrt{\frac{r^2 r_H + r^2_H + r^3_H}{r^3(r^2 + r r_H + r_H^2 + R^2)}} dr, \\
y^6 &= \int \sqrt{\left( R^4 + 10R^2 r^2_H + 9r^4_H \right)(r^2 + r r_H + r_H^2)} \frac{dr}{r^2 + r r_H + r_H^2 + R^2}. \\
\end{align*}
\] (3.3)
and \((y^3, y^4, y^5)\) remain the same as in eq. (3.2), where \(\kappa = (R^2 + 3r_H^2)/2r_H R^2\) is the surface gravity at the root \(r_H\) of \((1 - 2M/r + r^2/R^2) = 0\). The acceleration in GEMS is \(\ddot{a} = \kappa/\sqrt{1 - 2M/r + r^2/R^2}\). Hence conditon (2.8) is also satisfied.

For a charged black hole

\[
d s^2 = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) d t^2 - \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} d r^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

(3.4)

there is an embedding [7]

\[
y^0 = \kappa^{-1}\sqrt{1 - 2M/r + e^2/r^2} \sinh(\kappa t),
\]

\[
y^1 = \kappa^{-1}\sqrt{1 - 2M/r + e^2/r^2} \cosh(\kappa t),
\]

\[
y^2 = \int \left(\frac{r^2(r_+ + r_-) + r_+^2(r + r_+)}{r^2(r - r_-)}\right)^{1/2} d r,
\]

\[
y^6 = \int \left(\frac{4r_+^5r_-}{r^4(r_+ - r_-)^2}\right)^{1/2} d r
\]

(3.5)

with \((y^1, y^2, y^3)\) as in eq. (3.2), \(\kappa = (r_+ - r_-)/2r_+^2\), and \(y^6\) is timelike. Clearly one can check that eq. (2.8) is satisfied and deduce that the Hawking into Unruh mapping works for this embedding [7].

Obviously the analysis goes on for other embeddings. Let us end this section with an analysis of Fronsdal-type embeddings in which timelines are hyperbolas. In this case, the embedding function depends on time as

\[
y^0 = f(r) \sinh \alpha t,
\]

\[
y^1 = \pm f(r) \cosh \alpha t,
\]

(3.6)

where \(f(r) \geq 0\). To make \(y^0\) and \(t\) have the same direction, we set \(\alpha \geq 0\). The acceleration in the GEMS is \(\ddot{a} = 1/f(r)\) and \(g_{00} = \alpha^2 f(r)^2\). According to condition (2.8), we have

\[
\kappa = \alpha.
\]

(3.7)

This explains why we choose \(\alpha = 1/4M\) for the embeddings of Schwarzschild spacetimes.

4 Embeddings without mapping

Paston has pointed out that there are some new embeddings for which the thermal temperature can not be obtained through the GEMS approach [1]. As mentioned before, to get the thermal temperature, we need to map the static observer to the constant accelerating observer in the GEMS. However we find that there is no such mapping for the embeddings given in [1]. To see this, we just need to calculate the intrinsic metric for those embeddings.
4.1 Schwarzschild metric embeddings without mapping

In the work [29], three new 6-dimensional isometric embeddings of the Schwarzschild space-time are found. All have the same signature and components \((y^3, y^4, y^5)\) as the embedding (3.2).

The first one is the cubic embedding

\[
y^0 = \frac{\xi^2}{6} t'^3 + \left(1 - \frac{M}{r}\right) t' + u(r), \\
y^1 = \frac{\xi^2}{6} t'^3 - \frac{M}{r} t' + u(r), \\
y^2 = \frac{\xi}{2} t'^2 + \frac{1}{2\xi} \left(1 - \frac{2M}{r}\right),
\]

where \(t'\) is the time for some falling coordinates, and \(\xi\) is a parameter of the embedding. After some simple calculations, we get the intrinsic metric

\[
ds^2 = \left(1 - \frac{2M}{r}\right) dt'^2 + \frac{2}{\xi^2} \left(1 - \frac{2M}{r}\right) d\Omega,
\]

where \(u_r\) is the derivative respect to \(r\), and \(d\Omega = r^2(d\theta^2 + \sin^2\theta d\phi^2)\). It is clear that the timeline does not correspond to the static observer. Therefore the thermal temperature can not be obtained by the GEMS approach in general.

The second one is the Davidson-Paz embedding

\[
y^0 = \frac{M}{\beta \sqrt{r_e r}} \left(e^{\beta t'+u(r)} - \frac{r - r_c}{2M} e^{-\beta t'-u(r)}\right), \\
y^1 = \frac{M}{\beta \sqrt{r_e r}} \left(e^{\beta t'+u(r)} + \frac{r - r_c}{2M} e^{-\beta t'-u(r)}\right), \\
y^2 = \hat{\gamma} t',
\]

where \(\beta, \hat{\gamma}, r_c\) are the parameters of the embedding. After some algebra, we obtain

\[
ds^2 = \left(\frac{2M(r - r_c)}{r_c r} - \hat{\gamma}^2\right) dt'^2 + \frac{4Mu_r(r - r_c)}{\beta r_c r} dt'dr - \frac{1}{4r^2} \left(\frac{2M}{r} + \frac{1}{\beta^2 r_c r}\right) dr^2 - d\Omega.
\]

This is not the case for which timeline is the static observer’s worldline.

There remains the asymptotically flat embedding

\[
y^0 = t', \\
y^1 = \frac{(6M)^{3/2}}{\sqrt{r}} \sin \left(\frac{t'}{3^{3/2}2M} - \sqrt{\frac{2M}{r}} \left(1 + \frac{r}{6M}\right)^{3/2}\right), \\
y^2 = \frac{(6M)^{3/2}}{\sqrt{r}} \cos \left(\frac{t'}{3^{3/2}2M} - \sqrt{\frac{2M}{r}} \left(1 + \frac{r}{6M}\right)^{3/2}\right). 
\]
We get the intrinsic metric for this embedding

\[
\begin{align*}
 ds^2 &= \left(1 - \frac{2M}{r}\right) dt^2 + \frac{12\sqrt{6}M^{5/2}}{r} \sqrt{\left(\frac{1}{r} + \frac{1}{6M}\right) \left(1 - \frac{1}{r} - \frac{1}{6M}\right)} dtdr \\
 &\quad - \frac{(6M)^3}{2r} \left[ \frac{1}{2r^2} + \left(\frac{M}{r} + \frac{1}{6}\right) \left(1 - \frac{1}{r} - \frac{1}{6M}\right)^2 + \frac{2r}{(6M)^3} \right] dr^2 - d\Omega.
\end{align*}
\] (4.6)

It is reasonable that the Hawking into Unruh thermal properties mapping does not work for such an embedding.

### 4.2 Reissner-Nordström metric embeddings without mapping

For Reissner-Nordström spacetimes there are three new 6-dimensional embeddings found in the work [30]. These three embeddings have signature \((+ + - - - -)\) and components \((y^3, y^4, y^5)\) as the embedding (3.2). In the following, \(t'\) is the time for some falling coordinates, \(\gamma, \beta, \alpha, b\) are the parameters of the embeddings. We will calculate the intrinsic metric directly.

For the cubic embedding

\[
\begin{align*}
 y^0 &= \frac{2M^3}{q^4} t'^2 - \frac{q^4}{8M^3} \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right), \\
 y^1 &= \frac{4M^6}{3q^8} t'^3 - \frac{1}{4} \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) t' - t' + u \left(\frac{2Mr}{q^2}\right), \\
 y^2 &= \frac{4M^6}{3q^8} t'^3 - \frac{1}{4} \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) t' + t' + u \left(\frac{2Mr}{q^2}\right),
\end{align*}
\] (4.7)

we get the metric as

\[
\begin{align*}
 ds^2 &= \left(1 - \frac{2M}{r} - \frac{q^2}{r^2}\right) dt^2 - 2u_r dtdr - \left[1 - \frac{1}{4r^2} \left(\frac{M}{r^2} - \frac{q^2}{r^3}\right)^2\right] dr^2 - d\Omega.
\end{align*}
\] (4.8)

The exponential embedding is

\[
\begin{align*}
 y^0 &= \gamma t', \\
 y^1 &= \frac{1}{2\beta} \left(e^{-\beta t' - u(2Mr/q^2)} - \left(1 - \frac{2M}{r} + \frac{q^2}{r^2} - \gamma^2\right) e^{\beta t' + u(2Mr/q^2)}\right), \\
 y^2 &= \frac{1}{2\beta} \left(e^{-\beta t' - u(2Mr/q^2)} + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2} - \gamma^2\right) e^{\beta t' + u(2Mr/q^2)}\right).
\end{align*}
\] (4.9)

The metric we obtain for this embedding is

\[
\begin{align*}
 ds^2 &= \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left[\frac{2M}{\beta r^2} + \frac{2q^2}{\beta r^3} + \frac{2u_r}{\beta} \left(1 - \frac{2M}{r} + \frac{q^2}{r^2} - \gamma^2\right)\right] dtdr \\
 &\quad - \left[1 - \frac{u_r^2}{\beta^2} \left(1 - \frac{2M}{r} + \frac{q^2}{r^2} - \gamma^2\right) - \frac{1}{\beta^2} \left(\frac{2M}{r^2} + \frac{2q^2}{r^3}\right)\right] dr^2 - d\Omega.
\end{align*}
\] (4.10)
At last, for the spiral embedding

\[
y^0 = \sqrt{\left(Mr - q^2\right)^2 + b^2r^2} \frac{\alpha qr}{M - q^2} \sin \left(\alpha t' + u \left(\frac{2Mr}{q^2}\right)\right),
\]

\[
y^1 = \sqrt{\left(Mr - q^2\right)^2 + b^2r^2} \frac{\alpha qr}{M - q^2} \cos \left(\alpha t' + u \left(\frac{2Mr}{q^2}\right)\right),
\]

\[
y^3 = \sqrt{\frac{b^2 + m^2 - q^2}{q} t'},
\]

we get the metric

\[
ds^2 = \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \frac{2u_r}{\alpha q^2 r^2} \left(M^2 r^2 + q^4 - 2Mq^2 r + b^2 r^2\right) dt dr
\]

\[
- \left[1 - \frac{q^2 (Mr - q^2)^2}{\alpha^2 r^4 (M^2 r^2 + q^4 - 2Mq^2 r + b^2 r^2)} - \frac{u^2_r (M^2 r^2 + q^4 - 2Mq^2 r + b^2 r^2)}{\alpha^2 q^2 r^2}ight] dr^2
\]

\[
- d\Omega.
\]

Clearly, similar analysis on the embeddings of de Sitter spaces without the Hawking into Unruh mapping given in [1] can be carried out.

5 The potential barrier for scalar fields in the ambient space

Up to now, we see that when the condition (2.8) is satisfied, the Hawking temperature is coincident with the Unruh temperature. In this section, we will check that is there any difference remained for the GEMS approach.

We know that the Hawking effect results from the QFT in curved spacetime. For example, the radiation can be described by a massless scalar field satisfying the covariant equation [31]

\[
\nabla_\mu \nabla^\mu \varphi(x) = 0.
\]

The Unruh radiation can be described by the QFT in flat spacetime with, for example, a massless scalar field satisfying

\[
\partial_a \partial^a \varphi(y) = 0.
\]

Since the behaviors of quantum fields near horizons are same for the Hawking effect and the Unruh effect. The main difference between the Hawking radiation and the Unruh radiation is the different potential barriers of quantum fields to reach infinity. Thus it is quite natural to calculate the potential barrier of the Hawking radiation using the GEMS approach.

Without loss of generality, we will consider a massless scalar field in the GEMS of Schwarzschild spacetimes. For a massless scalar field, the action is [31]

\[
S = \frac{1}{2} \int d^n x \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi,
\]

\[
-8-
\]
The metric in the higher dimensional Minkowski space is
\[ ds^2 = dy_0^2 - dy_1^2 - dy_2^2 - dy_3^2 - dy_4^2 - dy_5^2. \] (5.4)

Following Unruh [4], we write it in a Rindler form
\[ ds^2 = \rho^2 d\omega^2 - d\rho^2 - dy_\perp^2, \] (5.5)
where \( y^0 = \rho \sinh \omega, \) \( y^1 = \rho \cosh \omega \) and \( y_\perp \) denotes \( (y^2, y^3, y^4, y^5) \). By introducing
\[ u = \log \rho, \] (5.6)
the metric becomes
\[ ds^2 = e^{2u}(d\omega^2 - du^2) - dy_\perp^2. \] (5.7)
And the scalar action becomes
\[ S = \frac{1}{2} \int dud\omega dy_\perp \left[ (\partial_\omega \phi)^2 - (\partial_u \phi)^2 - e^{2u}(\partial_\perp \phi)^2 \right]. \] (5.8)

Now decompose \( \phi \) into transverse plane waves with transverse wave vector \( k_\perp \)
\[ \phi = \int d^4k_\perp e^{ik_\perp y_\perp} \phi(u, \omega, k_\perp). \] (5.9)
The action for a given wave number \( k_\perp \) is
\[ S = \frac{1}{2} \int d\omega du \left[ (\partial_\omega \phi)^2 - (\partial_u \phi)^2 - k_\perp^2 e^{2u} \phi^2 \right]. \] (5.10)

It can be read directly that the potential is
\[ V(k_\perp, u) = k_\perp^2 e^{2u} = k_\perp^2 \rho^2. \] (5.11)

From eq. (3.2) we find that
\[ \rho = 4M \sqrt{1 - 2M/r}. \] (5.12)

Thus \( \rho \to 4M \) when \( r \to \infty \). On the other hand, the Rindler horizon is really at \( r = 2M \).
We conclude that the maximum potential barrier for a massless scalar particle to reach \( r \to \infty \)
in the ambient space is
\[ V_{\text{max}} = 16M^2 k_\perp^2. \] (5.13)
The potential barrier is finite, hence the fiducial observer at \( r \to \infty \) can indeed detect the
particles caused by the Unruh effect in the ambient spacetime.

To compare this result with the case in the Riemann space, we note that the maximum potential barrier of a massless scalar in Schwarzschild backgrounds is [5]
\[ \tilde{V}_{\text{max}} = \left( 1 - \frac{2M}{r_{\text{max}}} \right) \left( \frac{l(l+1)}{r_{\text{max}}^2} + \frac{2M}{r_{\text{max}}^3} \right), \] (5.14)
where \( l \) is the angular momentum and
\[ r_{\text{max}} = 3M \left( \frac{1}{2} \left( 1 + \sqrt{1 + \frac{14l^2 + 14l + 9}{9l^2(l+1)^2}} \right) - \frac{1}{2l(l+1)} \right). \] (5.15)
The transverse wave vector $k_\perp$ has contributions from $(y^2, y^3, y^4, y^5)$. If we neglect the contribution from $y^2$, then $k_\perp$ becomes the wave vector in space $(y^3, y^4, y^5)$, which is exactly the space $(x, y, z)$ in the Schwarzschild spacetime. Furthermore, if we take the approximation $1 = |k_\perp|r = 2M|k_\perp|$, we have

$$V_{\text{max}} = 4l^2. \quad (5.16)$$

It is obviously different from eq.(5.14).

This shows that though the Hawking temperature coincides with the Unruh temperature in the GEMS approach when the condition (2.8) is satisfied, the potential barriers in the Riemann background and the higher dimensional Minkowski background is different from each other. Therefore the GEMS approach is incomplete and the Hawking effect can be distinguished from the Unruh effect of the ambient space in principle. Also we suggest that a rigorous analysis of ‘brane-world’ type is required to study the GEMS approach.

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