SCHWARZ LEMMA FOR MAPPINGS SATISFYING
BIHARMONIC EQUATIONS

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ABSTRACT. In this paper, we establish some Schwarz type lemmas for mappings \( \Phi \) satisfying the inhomogeneous biharmonic Dirichlet problem \( \Delta(\Delta(\Phi)) = g \) in \( \mathbb{D} \), \( \Phi = f \) on \( T \) and \( \partial_n \Phi = h \) on \( T \), where \( g \) is a continuous function on \( \mathbb{D} \), \( f, h \) are continuous functions on \( T \), where \( \mathbb{D} \) is the unit disc of the complex plane \( \mathbb{C} \) and \( T = \partial \mathbb{D} \) is the unit circle. To reach our aim, we start by investigating some properties of \( T_2 \)-harmonic functions. Finally, we prove a Landau-type theorem.

1. Preliminaries and Main Results

Let \( \mathbb{C} \) denote the complex plane and \( \mathbb{D} \) the open unit disk in \( \mathbb{C} \). Let \( T = \partial \mathbb{D} \) be the boundary of \( \mathbb{D} \), and \( \overline{\mathbb{D}} = \mathbb{D} \cup T \), the closure of \( \mathbb{D} \). Furthermore, we denote by \( C^m(\Omega) \) the set of all complex-valued \( m \)-times continuously differentiable functions from \( \Omega \) into \( \mathbb{C} \), where \( \Omega \) stands for a domain of \( \mathbb{C} \) and \( m \in \mathbb{N} \). In particular, \( C(\Omega) := C^0(\Omega) \) denotes the set of all continuous functions in \( \Omega \).

For a real \( 2 \times 2 \) matrix \( A \), we use the matrix norm
\[
||A|| = \sup\{ |Az| : |z| = 1 \},
\]
and the matrix function
\[
\lambda(A) = \inf\{ |Az| : |z| = 1 \}.
\]
For \( z = x + iy \in \mathbb{C} \), the formal derivative of a complex-valued function \( \Phi = u + iv \) is given by
\[
D_{\Phi} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},
\]
so that
\[
||D_{\Phi}|| = |\Phi_z| + |\Phi_{\overline{z}}| \quad \text{and} \quad \lambda(D_{\Phi}) = ||\Phi_z| - |\Phi_{\overline{z}}||,
\]
where
\[
\Phi_z = \frac{1}{2}(\Phi_x - i\Phi_y) \quad \text{and} \quad \Phi_{\overline{z}} = \frac{1}{2}(\Phi_x + i\Phi_y).
\]
We use
\[
J_\Phi := \det D_{\Phi} = |\Phi_z|^2 - |\Phi_{\overline{z}}|^2.
\]

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The main objective of this paper is to establish a Schwarz-type lemma for the solutions to the following inhomogeneous biharmonic Dirichlet problem (briefly, IBDP):

\[
\begin{align*}
\Delta^2 \Phi &= g \quad \text{in} \quad \mathbb{D}, \\
\Phi &= f \quad \text{on} \quad \partial \mathbb{D}, \\
\partial_n \Phi &= h \quad \text{on} \quad \partial \mathbb{D},
\end{align*}
\]  
(1.1)

where

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},
\]
denotes the standard Laplacian and \( \partial_n \) denotes the differentiation in the inward normal direction, \( g \in C(\mathbb{D}) \) and the boundary data \( f \) and \( h \in C(\partial \mathbb{D}) \).

We would like to mention that in [10] and [11], the authors have considered similar inhomogeneous biharmonic equations but with different boundary conditions.

In order to state our main results, we introduce some necessary terminologies. For \( z, w \in \mathbb{D} \), let

\[
G(z, w) = |z - w|^2 \log \left| \frac{1 - \overline{z}w}{z - w} \right|^2 - (1 - |z|^2)(1 - |w|^2),
\]
and

\[
P(z) = \frac{1 - |z|^2}{|1 - z|^2},
\]
denote the biharmonic Green function and the harmonic Poisson kernel, respectively.

By [22, Theorem 1.1], we see that all solutions of IBDP (1.1) are given by

\[
\Phi(z) = F_0[f](z) + H_0[h](z) - G[g](z),
\]
where

\[
F_0[f](z) = \frac{1}{2\pi} \int_0^{2\pi} F_0(ze^{-i\theta})f(e^{i\theta})d\theta, \quad H_0[h](z) = \frac{1}{2\pi} \int_0^{2\pi} H_0(ze^{-i\theta})h(e^{i\theta})d\theta,
\]
and

\[
G[g](z) = \frac{1}{16} \int_{\mathbb{D}} G(z, \omega)g(\omega)dA(\omega)
\]
where \( dA(\omega) \) denotes the Lebesgue area measure in \( \mathbb{D} \). Here the kernels \( H_0 \) and \( F_0 \) are given by

\[
F_0(z) = H_0(z) + K_2(z),
\]

\[
H_0(z) = \frac{1}{2}(1 - |z|^2)P(z),
\]

\[
K_2(z) = \frac{1}{2} \frac{(1 - |z|^2)^3}{|1 - z|^4}.
\]
Thus, the solutions of the equation (1.1) are given by

\[ \Phi(z) = \frac{1}{2}(1 - |z|^2)P[f + h](z) + K_2[f](z) - G[g](z). \]

Obviously \( P[f + h] \) is a bounded harmonic function, and Heinz [16] proved the following result, which is called the Schwarz lemma for planar harmonic functions: If \( \Phi \) is a harmonic mapping from \( \mathbb{D} \) into itself with \( \Phi(0) = 0 \), then for \( z \in \mathbb{D} \),

\[ |\Phi(z)| \leq \frac{4}{\pi} \arctan |z|. \]

Hethcote [17] and Pavlović [27, Theorem 3.6.1] improved Heinz’s result, by removing the assumption \( \Phi(0) = 0 \), and proved the following.

**Theorem A.** Let \( \Phi : \mathbb{D} \to \mathbb{D} \) be a harmonic function from the unit disc to itself, then

\[ |\Phi(z) - \frac{1 - |z|^2}{1 + |z|^2} \Phi(0)| \leq \frac{4}{\pi} \arctan |z|, \quad z \in \mathbb{D}. \]  

(1.2)

Remark that \( K_2[f] \) is a bounded \( T_2 \)-harmonic which is a special type of biharmonic functions. So naturally our first aim is to study the class of \( T_2 \)-harmonic functions. Let us recall first the definition of \( T_\alpha \)-harmonic functions.

**Definition 1.** [26] Let \( \alpha \in \mathbb{R} \), and let \( f \in C^2(\mathbb{D}) \). We say that \( f \) is \( T_\alpha \)-harmonic if \( f \) satisfies

\[ T_\alpha(f) = 0 \quad \text{in} \quad \mathbb{D}, \]

where the \( T_\alpha \)-Laplacian operator is defined by

\[ T_\alpha = -\frac{\alpha^2}{4}(1 - |z|^2)^{-(\alpha+1)} + \frac{1}{2}L_\alpha + \frac{1}{2}L_\alpha, \]

with the weighted Laplacian operator \( L_\alpha \) is defined by

\[ L_\alpha = \frac{\partial}{\partial z} \frac{1}{1 - |z|^2} \frac{\partial}{\partial z}. \]

**Remark 1.1.** Let \( f \) be a \( T_\alpha \)-harmonic function.

1. If \( \alpha = 0 \), then \( f \) is harmonic.
2. If \( \alpha = 2(n-1) \), then \( f \) is \( n \)-harmonic, where \( n \in \{1, 2, \ldots\} \), see [11, 22, 25, 26].

The following result is the homogeneous expansion of \( T_\alpha \)-harmonic functions.

**Theorem B.** [25] Let \( \alpha \in \mathbb{R} \) and \( f \in C^2(\mathbb{D}) \). Then \( f \) is \( T_\alpha \)-harmonic if and only if it has a series expansion of the form

\[ f(z) = \sum_{k=0}^{\infty} c_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; |z|^2\right) z^k + \sum_{k=1}^{\infty} c_{-k} F\left(-\frac{\alpha}{2}, k + \frac{\alpha}{2}; k + 1; |z|^2\right) \overline{z}^k \]  

(1.3)
for some sequence \( \{c_k\} \) of complex numbers satisfying
\[
\limsup_{|k| \to \infty} |c_k|^{\frac{1}{|k|}} \leq 1.
\]

\( F \) is the hypergeometric function defined by the power series
\[
F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!}, \quad |x| < 1,
\]
for \( a, b, c \in \mathbb{R} \), with \( c \neq 0, -1, -2, \ldots \), where \((a)_0 = 1\) and \((a)_n = a(a+1)\ldots(a+n-1)\) for \( n = 1, 2, \ldots \) are the Pochhammer symbols.

We refer the reader to [25], where Olofsson gave a Poisson type integral representation theorem for \( T_\alpha \)-harmonic mappings, for \( \alpha > -1 \).

It is well known that the Schwarz lemma is one of the most influential results in many branches of mathematical research for more than a hundred years. We refer the reader to [3, 10, 18, 19, 23, 24] for generalizations and applications of this lemma.

In this context, we establish a Schwarz type lemma for \( T_2 \)-harmonic functions.

**Theorem 1.** Let \( u : \mathbb{D} \to \mathbb{D} \) be a \( T_2 \)-harmonic function, then
\[
\left| u(z) - \left( \frac{(1-|z|^2)^3}{(1+|z|^2)^2} u(0) \right) \right| \leq \frac{2}{\pi} \left( (1+|z|^2) \arctan |z| + \frac{|z|(1-|z|^2)}{1+|z|^2} \right),
\]
for all \( z \in \mathbb{D} \).

Next, we prove a Schwarz-Pick lemma for \( T_2 \)-harmonic functions.

**Theorem 2.** Let \( u : \mathbb{D} \to \mathbb{D} \) be a \( T_2 \)-harmonic function, then
\[
\|D_u(z)\| \leq \frac{(2+5|z|)(1+|z|^2)}{1-|z|^2}, \quad \text{for all } z \in \mathbb{D}.
\]
Moreover, the inequality (1.4) is sharp.

Next we establish a Landau type theorem for \( T_2 \)-harmonic functions.

**Theorem 3.** Let \( u \in \mathcal{C}^2(\mathbb{D}) \) be a \( T_2 \)-harmonic function satisfying \( u(0) = |J_u(0)| - 1 = 0 \) and \( \sup_{z \in \mathbb{D}} |f(z)| \leq M \), where \( M > 0 \) and \( J_u \) is the Jacobian of \( u \). Then \( u \) is univalent on \( D_{r_0} \), where \( r_0 \) satisfies the following equation
\[
\frac{\pi}{4M} - \frac{4}{\pi} \frac{Mr_0}{(1-r_0)^3} \left( -r_0^3 + 3r_0^2 - 3r_0 + 3 \right) = 0.
\]
Moreover, \( u(\mathbb{D}_{r_0}) \) contains a univalent disk \( D_{R_0} \) with

\[
R_0 \geq \frac{4M r_0^2}{\pi (1 - r_0)^3} \left[ -\frac{4}{5} r_0^3 + \frac{9}{4} r_0^2 - 2 r_0 + \frac{3}{2} \right].
\]

Now we are in the position to prove our main results

**Theorem 4.** Let \( g \in C(\overline{\mathbb{D}}) \), \( f, h \in C(\mathbb{T}) \) and suppose that \( \Phi \in C^4(\mathbb{D}) \cap C(\mathbb{D}) \) satisfies (1.7). Then for \( z \in \mathbb{D} \),

\[
\begin{align*}
|\Phi(z) - \frac{1}{2} (1 - |z|^2)^3 P[f](0) - \frac{1}{2} (1 - |z|^2)^2 P[f + h](0)| & \leq \left[ \frac{2}{\pi} (1 - |z|^2) \arctan |z| \right] \|f + h\|_{\infty} \\
& + \frac{2}{\pi} \left[ (1 + |z|^2) \arctan |z| + |z| \frac{1 - |z|^2}{1 + |z|^2} \right] \|f\|_{\infty} \\
& + \frac{1 - |z|^2}{64} \|g\|_{\infty}.
\end{align*}
\]

where \( \|f\|_{\infty} = \sup_{z \in \mathbb{T}} |f(\zeta)| \), \( \|f + h\|_{\infty} = \sup_{z \in \mathbb{T}} |f(\zeta) + h(\zeta)| \) and \( \|g\|_{\infty} = \sup_{\zeta \in \mathbb{D}} |g(\zeta)| \).

**Remark 1.2** Under the hypothesis of Theorem 4 if \( g \equiv 0 \), then \( \Phi \) is biharmonic mapping and the estimate (1.6) can be written in the following form

\[
\begin{align*}
|\Phi(z) - \frac{1}{2} (1 - |z|^2)^3 P[f](0) - \frac{1}{2} (1 - |z|^2)^2 P[f + h](0)| & \leq \left[ \frac{2}{\pi} (1 - |z|^2) \arctan |z| \right] \|f + h\|_{\infty} \\
& + \frac{2}{\pi} \left[ (1 + |z|^2) \arctan |z| + |z| \frac{1 - |z|^2}{1 + |z|^2} \right] \|f\|_{\infty} \\
& + \frac{1 - |z|^2}{64} \|g\|_{\infty}.
\end{align*}
\]

**Theorem 5.** Let \( g \in C(\overline{\mathbb{D}}) \), \( f \) and \( h \in C(\mathbb{T}) \). Suppose that \( \Phi \in C^4(\mathbb{D}) \) is satisfying (1.7). Then for all \( z \in \mathbb{D} \)

\[
\|D\Phi(z)\| \leq \frac{2 + 5|z|}{1 - |z|^2} (1 + |z|^2) \|f\|_{\infty} + \frac{2}{\pi} |f + h| \|f\|_{\infty} + \frac{23}{48} \|g\|_{\infty}.
\]

Moreover at \( z = 0 \), we have

\[
\|D\Phi(0)\| \leq \frac{4}{\pi} \|f\|_{\infty} + \frac{2}{\pi} \|f + h\|_{\infty} + \frac{23}{48} \|g\|_{\infty}.
\]

The classical Schwarz lemma at the boundary is as follows.

**Theorem C.** Suppose \( f : \mathbb{D} \rightarrow \mathbb{D} \) is a holomorphic function with \( f(0) = 0 \), and further, \( f \) is analytic at \( z = 1 \) with \( f(1) = 1 \). Then, the following two conditions hold:

(a) \( f'(1) \geq 1 \);

(b) \( f'(1) = 1 \) if and only if \( f(z) = z \).
Which is known as the Schwarz lemma on the boundary, and its generalizations have important applications in geometric theory of functions (see, [15, 21, 28]). Among the recent papers devoted to this subject, for example, Burns and Krantz [3], Krantz [19], Liu and Tang [23] explored many versions of the Schwarz lemma at the boundary point of holomorphic functions, Dubinin also applied this latter for algebraic polynomials and rational functions (see [13, 14]). In the present paper, we refine the Schwarz type lemma at the boundary for $\Phi$ satisfies (1.1) as an application of Theorem 4.

**Theorem 6.** Suppose that $\Phi \in C^4(\mathbb{D}) \cap C^{\overline{\mathbb{D}}}$ satisfies (1.1), where $g \in C(\mathbb{D})$ and $f, h \in C(T)$ such that $\|f\|_{\infty} \leq 1$, and $\|f + h\|_{\infty} \leq 1$. If $\lim_{r \to 1} |\Phi(r\eta)| = 1$ for $\eta \in T$, then

$$\liminf_{r \to 1} \frac{|\Phi(\eta) - \Phi(r\eta)|}{1 - r} \geq 1 - \|f + h\|_{\infty}.$$ 

In particular if $\|f + h\|_{\infty} = 0$, then

$$\liminf_{r \to 1} \frac{|\Phi(\eta) - \Phi(r\eta)|}{1 - r} \geq 1,$$

and this estimate is sharp.

Denote by $\mathcal{H}(\mathbb{D})$ the set of all holomorphic functions $\Phi$ in $\mathbb{D}$ satisfying the standard normalization: $\Phi(0) = \Phi'(0) - 1 = 0$. In the early 20th century, Landau [20] showed that there is a constant $r > 0$, independent of elements in $\mathcal{H}(\mathbb{D})$, such that $\Phi(\mathbb{D})$ contains a disk of radius $r$. Later, the Landau theorem has become an important tool in geometric function theory. To establish analogs of the Landau type theorem for more general classes of functions, it is necessary to restrict our focus on certain subclasses (cf. [1], [4], [5], [6], [7], [8], [9]).

For convenience, we make a notational convention: for $g \in C(\mathbb{D})$ and $h \in C(T)$, let $\mathcal{B}_F_{g,h}(\mathbb{D})$ denote the class of all complex-valued functions $\Phi \in C^4(\mathbb{D}) \cap C^{\overline{\mathbb{D}}}$ satisfying (1.1) with the normalization $\Phi(0) = J_{\Phi}(0) - 1 = 0$.

We establish the following Landau-type theorem for $\Phi \in \mathcal{B}_F_{g,h}(\mathbb{D})$. In particular, if $g \equiv 0$, then $\Phi \in \mathcal{B}_F_{g,h}(\overline{\mathbb{D}})$ is biharmonic. In this sense, the following result is a generalization of [1] Theorem 1 and [11] Theorem 2.

**Theorem 7.** Suppose that $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$ are constants, and suppose that $\Phi \in \mathcal{B}_F_{g,h}(\mathbb{D})$ satisfies the following conditions:

$$\sup_{z \in T} |f(z)| \leq M_1, \quad \sup_{z \in T} |f(z) + h(z)| \leq M_2, \quad \text{and} \quad \sup_{z \in \mathbb{D}} |g(z)| \leq M_3.$$ 

Then $\Phi$ is univalent in $\mathbb{D}_{r_0}$, and $\Phi(\mathbb{D}_{r_0})$ contains a univalent disk $\mathbb{D}_{R_0}$, where $r_0$ satisfies the following equation:

$$\left(\frac{4}{\pi} M_1 + \frac{2}{\pi} M_2 + \frac{23}{48} M_3\right) \sigma(r_0) = 1,$$

where $\sigma(r)$ is the spherical mean of $\Phi$. In particular, if $g \equiv 0$, then $\Phi \in \mathcal{B}_F_{g,h}(\overline{\mathbb{D}})$ is biharmonic.
with
\[ \sigma(|z|) := (M_1 + M_2 + \frac{101}{120}M_3)|z| + \frac{2M_2|z|}{\pi}\left[\frac{(2 - |z|)(1 + |z|^2)}{(1 - |z|)^2} + |z|\right] \]
\[ + \frac{4M_1|z|}{\pi(1 - |z|)^3}(-|z|^3 + 3|z|^2 - 3|z| + 3). \]

and
\[ R_0 \geq \frac{r_0}{\frac{2}{\pi} M_1 + \frac{4}{\pi} M_2 + \frac{24}{25} M_3}. \]

2. Schwarz and Landau Type Lemmas for $T_2$-harmonic functions

2.1. Schwarz Type Lemma for $T_2$-harmonic functions.

The main purpose of this section is to prove Schwarz type lemma for $T_2$-harmonic functions.

Proof of Theorem 1. Let $0 \leq r = |z| < 1$. As $u$ is a $T_2$-harmonic function, then

\[ u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \frac{(1 - r^2)^3}{|1 - z e^{-i\theta}|^4} u^*(e^{i\theta}) d\theta, \]

where $u^* \in L^\infty(T)$. Thus

\[ \left| u(z) - \frac{(1 - r^2)^3}{(1 + r^2)^2} u(0) \right| \leq \frac{1}{4\pi} \int_0^{2\pi} \left| \frac{(1 - r^2)^3}{(1 + r^2 - 2r \cos \theta)^2} - \frac{(1 - r^2)^3}{(1 + r^2)^2} \right| d\theta \]
\[ = \frac{1}{2\pi} \left[ \int_0^{\pi/2} \frac{(1 - r^2)^3}{(1 + r^2 - 2r \cos \theta)^2} - \frac{(1 - r^2)^3}{(1 + r^2)^2} d\theta \right. \]
\[ - \left. \int_{\pi/2}^{\pi} \frac{(1 - r^2)^3}{(1 + r^2 - 2r \cos \theta)^2} - \frac{(1 - r^2)^3}{(1 + r^2)^2} d\theta \right] \]
\[ = \frac{1}{2\pi} \left[ \int_0^{\pi/2} \frac{(1 - r^2)^3}{(1 + r^2 - 2r \cos \theta)^2} d\theta - \int_{\pi/2}^{\pi} \frac{(1 - r^2)^3}{(1 + r^2 - 2r \cos \theta)^2} d\theta \right] \]
\[ = \frac{1}{2\pi} \left[ 2J(\pi/2) - J(\pi) \right], \]

where
\[ J(\theta) := \int_0^\theta \frac{(1 - r^2)^3}{(1 + r^2 - 2r \cos \varphi)^2} d\varphi. \]

Easy but tedious computations show that
Lemma 1. For $0 \leq \theta < \pi$, and $r \in [0, 1)$, we have

$$J(\theta) = \frac{2r(1 - r^2)}{1 + r^2 - 2r \cos \theta} \sin \theta + 2(1 + r^2) \arctan \left( \frac{(1 + r) \tan \theta/2}{1 - r} \right),$$

and $J(\pi) = \lim_{\theta \to \pi} J(\theta) = \pi(1 + r^2)$.

Then by Lemma 1, and using the fact that $\arctan \left( \frac{1 + r}{1 - r} \right) - \frac{\pi}{4} = \arctan r$, we have

$$\left| u(z) - \frac{(1 - r^2^3)}{(1 + r^2)^2} u(0) \right| \leq \frac{1}{2\pi} \left( 2J(\pi/2) - J(\pi) \right) \leq \frac{1}{2\pi} \left[ 4\frac{r(1 - r^2)}{1 + r^2} + 4(1 + r^2) \arctan \left( \frac{1 + r}{1 - r} \right) - \pi(1 + r^2) \right] = \frac{2}{\pi} \left[ \frac{r(1 - r^2)}{1 + r^2} + (1 + r^2) \arctan r \right].$$

□

To prove Theorem 2, we need the following lemma

Lemma D. \[22\] For any $z \in \mathbb{D}$, we have

$$I_\alpha(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^{2\alpha}} = \sum_{n=0}^{+\infty} \left( \frac{\Gamma(n + \alpha)}{n!\Gamma(\alpha)} \right)^2 |z|^{2n},$$

where $\alpha > 0$ and $\Gamma$ denotes the Gamma function.

Thus

$$I_2(z) = \sum_{n=0}^{\infty} (n + 1)^2 |z|^{2n} = \frac{1 + |z|^2}{(1 - |z|^2)^3}. \quad (2.1)$$

Proof of Theorem 2\[3\] As $u$ is a bounded $T_2$-harmonic function, then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} K_2(z e^{-i\theta}) u^*(e^{i\theta}) d\theta,$$

where $u^* \in L^\infty(\mathbb{T})$. Elementary computations show that

$$u_z(z) = \frac{1}{2\pi} \int_0^{2\pi} [K_2(z e^{-i\theta})]_z u^*(e^{i\theta}) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left( 1 - |z|^2 \right) [2e^{-i\theta} (1 - |z|^2) - 3\bar{z} \left( 1 - ze^{-i\theta} \right)] u^*(e^{i\theta}) d\theta. \quad (2.2)$$
Hence
\[ |u_z(z)| \leq \frac{(1 - |z|^2)^2}{2} \left[ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{2e^{-i\theta}(1 - |z|^2)}{(1 - ze^{i\theta})^2(1 - ze^{-i\theta})^3} \right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{3z(1 - ze^{-i\theta})}{(1 - ze^{i\theta})^2(1 - ze^{-i\theta})^3} \right| d\theta \right] \]
\[ \leq \frac{(1 - |z|^2)^2}{2} \left[ 2(1 - |z|^2)I_{5/2}(z) + 3|z|I_2(z) \right] \]
\[ \leq \frac{(1 - |z|^2)^2}{2}(2 + 5|z|)I_2(z). \]

The last inequality follows from the following estimate \( I_{5/2}(z) \leq \frac{I_2(z)}{1 - |z|} \).

Using the explicit expression of \( I_2 \), see (2.1), we obtain
\[ |u_z(z)| \leq \frac{(2 + 5|z|)(1 + |z|^2)}{2(1 - |z|^2)}. \]

Similarly,
\[ |u_{z\bar{z}}(z)| \leq \frac{(2 + 5|z|)(1 + |z|^2)}{2(1 - |z|^2)}. \]

Thus
\[ \|D_u(z)\| = |u_z(z)| + |u_{z\bar{z}}(z)| \leq \frac{(2 + 5|z|)(1 + |z|^2)}{1 - |z|^2}. \]

Next let us show the estimate (1.5). From Theorem 1, we deduce that near 0
\[ |u(z) - u(0)| \leq \frac{4}{\pi} |z| + O(|z|^2). \] (2.3)

Indeed, near 0,
\[ (1 + r^2 \arctan r + \frac{1 - r^2}{1 + r^2}) = 2r + O(r^2), \]
and
\[ \frac{(1 - r^2)^3}{(1 + r^2)^2} = 1 + O(r^2). \]

Hence from (2.3), we get
\[ \|D_u(0)\| \leq \frac{4}{\pi}. \]

To show that the last estimate is sharp. Let us consider the \( T_2 \)-harmonic mapping defined by
\[ U(z) = K_2[\chi_{T^+} - \chi_{T^-}](z), \]
where \( \chi_{T^+} \) (resp. \( \chi_{T^-} \)) denotes the characteristic function of the upper (resp. lower) unit circle \( T \).

By (2.2), we have
\[ U_z(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta}(\chi_{T^+} - \chi_{T^-})(\theta) d\theta = \frac{1}{2\pi} \int_0^\pi e^{-i\theta} d\theta - \frac{1}{2\pi} \int_\pi^{2\pi} e^{-i\theta} d\theta = -\frac{2i}{\pi}. \]
Hence

\[ |\nabla U(0)| = 2|U_z(0)| = \frac{4}{\pi}. \]

\[ \square \]

2.2. Landau type theorem for \( T_2 \)-harmonic functions.

We will use the following theorem provides some estimates on the coefficients of \( T_2 \)-harmonic mappings.

**Theorem E.** \[1\] For \( \alpha > -1 \), let \( u \in C^2(\mathbb{D}) \) be a \( T_2 \)-harmonic function with the series expansion of the form (1.3) and \( \sup_{z \in \mathbb{D}} |u(z)| \leq M \), where \( M > 0 \). Then, for \( k \in \{1, 2, \ldots\} \),

\[ |c_k F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; 1)| + |c_{-k} F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; 1)| \leq \frac{4M}{\pi}, \]  

(2.4)

and

\[ |c_0 F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1)| \leq M. \]  

(2.5)

**Proof of Theorem E** As \( u \) is \( T_2 \)-harmonic function on the unit disk with \( u(0) = 0 \), then by plugging \( \alpha = 2 \) in (1.3), we have \( c_0 = 0 \) and

\[
\begin{align*}
u(z) &= \sum_{k=1}^{\infty} c_k F(-1, k - 1; k + 1; |z|^2)z^k + \sum_{k=1}^{\infty} c_{-k} F(-1, k - 1; k + 1; |z|^2)\overline{z}^k. \\
&= \sum_{k=1}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} \overline{z}^k.
\end{align*}
\]

An elementary computation shows that the function \( F(-1, k - 1; k + 1; \cdot) \) is given by

\[
F(-1, k - 1; k + 1; w) = 1 - \frac{k - 1}{k + 1}w, \quad w \in [0, 1).
\]

(2.6)

Clearly, the function \( w \mapsto F(-1, k - 1; k + 1; w) \) is decreasing on \([0, 1]\) for \( k \geq 1 \). Thus

\[
\frac{2}{k + 1} \leq F(-1, k - 1; k + 1; w) \leq 1 \quad \text{for } k \geq 1 \text{ and } w \in [0, 1].
\]

(2.7)

Combining (2.4) for \( \alpha = 2 \) and (2.7), we obtain

\[
|c_k| + |c_{-k}| \leq \frac{2M}{\pi}(k + 1) \quad \text{for } k \geq 1.
\]

(2.8)

Using (2.3), we see that the mapping \( u \) is given by

\[
u(z) = \sum_{k=1}^{\infty} c_k z^k + c_{-k} \overline{z}^k - \sum_{k=1}^{\infty} \frac{k - 1}{k + 1} \left( c_k z^{k+1} + c_{-k} \overline{z}^{k+1} \right).
\]

Therefore
\[ u_z(z) - u_z(0) = \sum_{k=2}^{\infty} kc_k z^{k-1} - \sum_{k=2}^{\infty} \frac{k-1}{k+1} \left( (k+1)c_k z^k + c_{-k} z^{k+1} \right). \quad (2.9) \]

Similarly
\[ u_{\pi}(z) - u_{\pi}(0) = \sum_{k=2}^{\infty} kc_{-k} z^{k-1} - \sum_{k=2}^{\infty} \frac{k-1}{k+1} \left( (k+1)c_{-k} z^k + c_k z^{k+1} \right). \quad (2.10) \]

Applying (2.9), (2.10) and (2.8), we obtain
\[
\begin{align*}
|u_z(z) - u_z(0)| + |u_{\pi}(z) - u_{\pi}(0)| &\leq \sum_{k=2}^{\infty} k \left( |c_k| + |c_{-k}| \right) |z|^{k-1} \\
&+ \sum_{k=2}^{\infty} (k-1) \left( |c_k| + |c_{-k}| \right) |z|^{k+1} \\
&+ \sum_{k=2}^{\infty} \frac{k-1}{k+1} \left( |c_k| + |c_{-k}| \right) |z|^{k+1} \\
&\leq \frac{2M}{\pi} \left( \sum_{k=2}^{\infty} k(k+1)|z|^k + \sum_{k=2}^{\infty} (k-1)(k+1)|z|^{k+1} \right) \\
&= \frac{2M|z|}{\pi(1-|z|)^2} \left( 2|z|^2 - 3|z| + 3 \right) + \frac{|z|^2(3-|z|)}{1-|z|} + |z|^2.
\end{align*}
\]

That is
\[
|u_z(z) - u_z(0)| + |u_{\pi}(z) - u_{\pi}(0)| \leq \frac{4M|z|}{\pi(1-|z|)^3} (-|z|^3 + 3|z|^2 - 3|z| + 3). \quad (2.11)
\]

Moreover, the RHS of (2.11) is strictly increasing as a function of $|z|$.

Applying Theorem 2 (1.5), we get
\[
1 = J_u(0) = \|D_u(0)\| = \lambda(D_u(0)) \leq \frac{4M}{\pi} \lambda(D_u(0)),
\]

which gives that
\[
\lambda(D_u(0)) \geq \frac{\pi}{4M}. \quad (2.12)
\]

We will show that $u$ is univalent in $\mathbb{D}_{r_0}$, where $r_0$ satisfies the following equation
\[
\frac{\pi}{4M} - \frac{4M r_0}{\pi (1 - r_0)^3} (-r_0^3 + 3r_0^2 - 3r_0 + 3) = 0.
\]
Indeed, let $z_1$ and $z_2$ be two distinct points in $D_{r_0}$ and let $[z_1, z_2]$ denote the line segment from $z_1$ to $z_2$.

By (2.11), (2.12) we have

\[
|u(z_1) - u(z_2)| = \left| \int_{[z_1, z_2]} u_z(z) \, dz + u_{\bar{z}}(z) \, d\bar{z} \right|
\geq \left| \int_{[z_1, z_2]} u_z(0) \, dz + u_{\bar{z}}(0) \, d\bar{z} \right|
- \left| \int_{[z_1, z_2]} (u_z(z) - u_z(0)) \, dz + (u_{\bar{z}}(z) - u_{\bar{z}}(0)) \, d\bar{z} \right|
\geq \lambda(D_u(0)) |z_1 - z_2|
- \left| \int_{[z_1, z_2]} (|u_z(z) - u_z(0)| + |u_{\bar{z}}(z) - u_{\bar{z}}(0)|) |dz| \right|
\geq |z_2 - z_1| \left\{ \frac{\pi}{4M} - \frac{4}{\pi} \frac{Mr_0}{(1 - r_0)^3} \left( r_0^3 + 3r_0^2 - 3r_0 + 3 \right) \right\}
= 0.
\]

This implies that $u$ is univalent on $D_{r_0}$.

Let $\xi = r_0 e^{i\theta} \in \partial D_{r_0}$. Then we infer from (2.11) that

\[
|u(\xi) - u(0)| \geq \lambda(D_\Phi(0))r_0 - \int_{[0, \xi]} (|u_z(z) - u_z(0)| + |u_{\bar{z}}(z) - u_{\bar{z}}(0)|) |dz|
\geq \frac{\pi r_0}{4M} - \frac{4M}{\pi} \frac{r_0^2}{(1 - r_0)^3} \int_0^1 (-r_0^3 t^4 + 3r_0^2 t^3 - 3r_0 t^2 + 3t) dt
= \frac{4Mr_0^2}{\pi(1 - r_0)^3} \left( -r_0^3 + 3r_0^2 - 3r_0 + 3 - \left( -\frac{r_0^3}{5} + \frac{3}{4}r_0^2 - r_0 + \frac{3}{2} \right) \right)
= \frac{4Mr_0^2}{\pi(1 - r_0)^3} \left( -\frac{4}{5}r_0^3 + \frac{9}{4}r_0^2 - 2r_0 + \frac{3}{2} \right).
\]

Hence $u(D_{r_0})$ contains an univalent disk $D_{R_0}$ with

\[
R_0 \geq \frac{4Mr_0^2}{\pi(1 - r_0)^3} \left( -\frac{4}{5}r_0^3 + \frac{9}{4}r_0^2 - 2r_0 + \frac{3}{2} \right).
\]

□
3. Schwarz-Type Lemmas for Solutions to Inhomogeneous Biharmonic Equations

Proof of Theorem 4 The solution of (1.1) can be written in the following form
\[
\Phi(z) = \frac{1}{2}(1 - |z|^2)P[f + h](z) + K_2[f](z) - G[g](z).
\]
As \(z \mapsto K_2[f](z)\) is \(T_2\)-harmonic function, then by Theorem 1, we have
\[
\left| K_2[f](z) - \frac{(1 - |z|^2)^2}{(1 + |z|^2)^2}K_2[f](0) \right| \leq \frac{2}{\pi} \left( (1 + |z|^2) \arctan |z| + \frac{|z|(1 - |z|^2)}{1 + |z|^2} \right) \|f\|_\infty. \tag{3.1}
\]
On the other hand, using the estimate (1.2) for the harmonic mapping \(P[f + h]\), we get
\[
\left| P[f + h](z) - \frac{1 - |z|^2}{1 + |z|^2}P[f + h](0) \right| \leq \frac{4}{\pi} \arctan |z| \|f + h\|_\infty. \tag{3.2}
\]
Using (10) inequality 2.3, we obtain
\[
|G[g](z)| \leq \frac{(1 - |z|^2)^2}{64} \|g\|_\infty. \tag{3.3}
\]
Finally as \(K_2[f](0) = \frac{1}{2}P[f](0)\), then if follows from (3.1) (3.3) that
\[
\left| \Phi(z) - \frac{1}{2}(1 - |z|^2)^3 \right| \|P[f](0) - \frac{1}{2}(1 - |z|^2)^2P[f + h](0) \right| \leq \frac{2}{\pi} (1 - |z|^2) \arctan |z| \|f + h\|_\infty + \frac{2}{\pi} \left( |z|^2 + 1 \right) \arctan |z| + \frac{|z|(1 - |z|^2)}{1 + |z|^2} \|f\|_\infty + \frac{(1 - |z|^2)^2}{64} \|g\|_\infty.
\]
Hence, the proof is complete. \(\square\)

Proof of Theorem 5 The solution of (1.1) can be written in the following form
\[
\Phi(z) = \frac{1}{2}(1 - |z|^2)P[f + h](z) + K_2[f](z) - G[g](z).
\]
Thus
\[
\Phi_z(z) = \frac{1}{2} \left[ (1 - |z|^2)[P[f + h](z)]_z - \overline{z}P[f + h](z) \right] + K_2[f]_z(z) - G[g]_z(z),
\]
\[
\Phi_\pi(z) = \frac{1}{2} \left[ (1 - |z|^2)[P[f + h](z)]_\pi - zP[f + h](z) \right] + K_2[f]_\pi(z) - G[g]_\pi(z).
\]
Therefore
\[
\|D_\Phi(z)\| \leq \frac{1}{2} (1 - |z|^2)\|DP[f + h](z)\| + |z|\|P[f + h](z)\| + \|DK_2[f](z)\| + \|DG[g](z)\|.
\]
By Colonna [12], we have

$$\|DP_{f+h}(z)\| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2} \|f + h\|_\infty. \quad (3.4)$$

It follows from [22, Lemma 2.5], that

$$\|DG_{g}(z)\| \leq \frac{23}{48} \|g\|_\infty, \quad (3.5)$$

since

$$\int_D |G(z,\omega)g(\omega)| dA(\omega) \leq \frac{23}{6} \|g\|_\infty \quad \text{and} \quad \int_D |G(z,\omega)g(\omega)| dA(\omega) \leq \frac{23}{6} \|g\|_\infty.$$

Therefore, combining (3.4)-(3.5), we obtain

$$\|D\Phi(z)\| \leq \frac{1}{2} (1 - |z|^2) \|DP_{f+h}(z)\| + |z| \|P[f+h](z)\| + \|DG_{g}(z)\| + \|DK_{z}[f](z)\| + \|DG_{g}(z)\|$$

$$\leq \frac{2}{\pi} \|P[f+h]\|_\infty + |z| \|f + h\|_\infty + \frac{(2 + 5|z|)(1 + |z|^2)}{1 - |z|^2} \|f\|_\infty + \frac{23}{48} \|g\|_\infty$$

$$\leq \left(\frac{2}{\pi} + |z|\right) \|f + h\|_\infty + \frac{(2 + 5|z|)(1 + |z|^2)}{1 - |z|^2} \|f\|_\infty + \frac{23}{48} \|g\|_\infty.$$

□

**Proof of Theorem [12]** Suppose that $|z| = r$, it follows from Theorem [11] that

$$|\Phi(\eta) - \Phi(r\eta)| \geq 1 - \frac{1}{2} (1 - r^2) \|f + h\|_\infty - \frac{2}{\pi} \left[ (r^2 + 1) \arctan r + \frac{r(1 - r^2)}{1 + r^2} \right]$$

$$\quad - \frac{\|g\|_\infty}{64} (1 - r^2)^2 - \frac{1}{2} (1 - r^2)^3 \|P[f](0)\| - \frac{1}{2} \frac{(1 - r^2)^2}{1 + r^2} \|P[f+h](0)\|$$

Divide by $1 - r$ and used the Hospital rule, we obtain

$$\liminf_{r \to 1} \frac{|\Phi(\eta) - \Phi(r\eta)|}{1 - r} \geq \lim_{r \to 1} \frac{1 - \frac{2}{\pi} (r^2 + 1) \arctan r}{1 - r} - \frac{2}{\pi} \frac{r(1 - r^2)}{(1 - r)(1 + r^2)}$$

$$\quad - \frac{1}{2} \lim_{r \to 1} (1 + r) \|f + h\|_\infty$$

$$\quad = [\varphi'(r)]_{r=1} - \frac{2}{\pi} - \|f + h\|_\infty,$$

where

$$\varphi(r) = \frac{2}{\pi} (r^2 + 1) \arctan r.$$ 

Then

$$\varphi'(r) = \frac{2}{\pi} \left[ 2r \arctan r + 1 \right], \quad \text{so} \quad \varphi'(1) = 1 + \frac{2}{\pi}.$$
Hence
\[ \liminf_{r \to 1} \left| \frac{\Phi(\eta) - \Phi(r\eta)}{1 - r} \right| \geq 1 - \|f + h\|_{\infty}. \]

4. A Landau-Type Theorem for Solutions to Inhomogeneous Biharmonic Equations

First, let us recall the following result.

**Theorem F** ([8], Lemma 1). Suppose \( f \) is a harmonic mapping of \( \mathbb{D} \) into \( \mathbb{C} \) such that \( |f(z)| \leq M \) for all \( z \in \mathbb{D} \) and
\[
 f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n. 
\]
Then \( |a_0| \leq M \) and for all \( n \geq 1 \),
\[
 |a_n| + |b_n| \leq \frac{4M}{\pi}. 
\]

**Proof of Theorem F**. The solution of (1.1) can be written in the following form
\[
 \Phi(z) = H_0[f + h](z) + K_2[f](z) - G[g](z),
\]
where
\[
 H_0[f + h](z) = \frac{1}{2}(1 - |z|^2)P[f + h](z). \tag{4.1}
\]
Since \( P[f + h](z) \) is harmonic in \( \mathbb{D} \), we have
\[
 H_0[f + h](z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n 
\]
Since \( |P[f + h](z)| \leq M_2 \) for all \( z \in \mathbb{D} \), by Theorem F, we have
\[
 |a_n| + |b_n| \leq \frac{4M_2}{\pi} \quad \text{for } n \geq 1. \tag{4.2}
\]
By (4.1) and (4.2), we have
\[
 [H_0[f + h](z)]_z = \frac{1}{2}(1 - |z|^2)P[f + h]_z(z) - \frac{1}{2}zP[f + h](z),
\]
and
\[
 [H_0[f + h](z)]_z = \frac{1}{2}(1 - |z|^2)P[f + h]_\overline{z}(z) - \frac{1}{2}\overline{z}P[f + h](z). 
\]
Thus
\[
\left| H_0[f + h]z(z) - H_0[f + h]z(0) \right| + \left| H_0[f + h]\bar{z}(z) - H_0[f + h]\bar{z}(0) \right|
\leq \left| \left[ z \right] P[f + h](z) \right| + \frac{1}{2} \left( \left| P[f + h]z(z) - P[f + h]z(0) \right| + \left| P[f + h]\bar{z}(z) - P[f + h]\bar{z}(0) \right| \right)
+ \frac{1}{2} |z|^2 \left( \left| P[f + h]z(z) \right| + \left| P[f + h]\bar{z}(z) \right| \right)
\leq M_2 |z| + \frac{1}{2} \sum_{n \geq 2} n(|a_n| + |b_n|) |z|^{n-1} + \frac{1}{2} |z|^2 \sum_{n \geq 1} n(|a_n| + |b_n|) |z|^{n-1}
\leq M_2 |z| + \frac{1}{2} (1 + |z|^2) \sum_{n \geq 2} n(|a_n| + |b_n|) |z|^{n-1} + \frac{1}{2} |z|^2 (|a_1| + |b_1|)
\leq M_2 |z| + \frac{2M_2 |z|}{\pi} \left[ \frac{(2 - |z|)(1 + |z|^2)}{(1 - |z|)^2} + |z| \right].
\] (4.3)

Since $K_2$ is $T_2$-harmonic, then

\[
K_2(z) = \sum_{k=0}^{\infty} c_k F(-1, k - 1; k + 1; |z|^2) z^k + \sum_{k=1}^{\infty} c_{-k} F(-1, k - 1; k + 1; |z|^2) \bar{z}^k.
\]

Let us denote

\[
K_2^0[f](z) := K_2(z) - c_0 F(-1, -1; 1; |z|^2) = K_2[f](z) - c_0(1 + |z|^2).
\]

Hence

\[
K_2[f] = K_2^0[f](z) + c_0(1 + |z|^2).
\]

\[
\left| K_2[f]z(z) - K_2[f]z(0) \right| + \left| K_2[f]\bar{z}(z) - K_2[f]\bar{z}(0) \right|
\leq \left| K_2^0[f]z(z) - K_2^0[f]z(0) \right| + \left| K_2^0[f]\bar{z}(z) - K_2^0[f]\bar{z}(0) \right| + 2|c_0||z|
\]

By (2.5), we have

\[2|c_0| \leq M_1.\]

On the other hand, as $K_2^0(f)$ is a $T_2$-harmonic function with $K_2^0(0) = 0$, it yields

\[
|K_2^0[f]z(z) - K_2^0[f]z(0)| + |K_2^0[f]\bar{z}(z) - K_2^0[f]\bar{z}(0)| \leq \frac{4M_1 |z|}{\pi(1 - |z|)^3} \left( -|z|^{3} + 3|z|^2 - 3|z| + 3 \right).
\]

Thus

\[
\left| K_2[f]z(z) - K_2[f]z(0) \right| + \left| K_2[f]\bar{z}(z) - K_2[f]\bar{z}(0) \right| \leq \frac{4M_1 |z|}{\pi(1 - |z|)^3} \left( -|z|^{3} + 3|z|^2 - 3|z| + 3 \right) + M_1 |z|. \] (4.4)
Let
\[ \psi_1(z) = \left| \frac{1}{16\pi} \int_{\mathbb{D}} g(\omega)(G_z(z, \omega) - G_z(0, \omega))dA(\omega) \right| \]
and
\[ \psi_2(z) = \left| \frac{1}{16\pi} \int_{\mathbb{D}} g(\omega)(G_z(z, \omega) - G_z(0, \omega))dA(\omega) \right|. \]
Then by [10, Inequality (3.6)], we have
\[ \psi_1(z) \leq \left( \frac{1 - |z|^2}{16} + \frac{43}{120} \right) \|g\|_{\infty}|z|, \] (4.5)
and
\[ \psi_2(z) \leq \left( \frac{1 - |z|^2}{16} + \frac{43}{120} \right) \|g\|_{\infty}|z|. \]
Now, it follows from (4.3), (4.4) and (4.5) that
\[
|\Phi_z(z) - \Phi_z(0)| + |\Phi_{\overline{z}}(z) - \Phi_{\overline{z}}(0)| \\
\leq M_2|z| + \frac{2M_2|z|}{\pi} \left[ (2 - |z|)(1 + |z|^2) + |z| \right] + \\
+ \frac{4M_1|z|}{\pi(1 - |z|)^3} \left( -|z|^3 + 3|z|^2 - 3|z| + 3 \right) + M_1|z| + \psi_1(z) + \psi_2(z) \\
\leq \sigma(z), \] (4.6)
where
\[ \sigma(|z|) := (M_1 + M_2 + \frac{101}{120}M_3)|z| + \frac{2M_2|z|}{\pi} \left[ (2 - |z|)(1 + |z|^2) + |z| \right] + \frac{4M_1|z|}{\pi(1 - |z|)^3} \left( -|z|^3 + 3|z|^2 - 3|z| + 3 \right) \]
Remark that not only \( \sigma(|z|) \) is increasing but also \( \frac{\sigma(|z|)}{|z|} \) is increasing with respect to \( |z| \) in \([0, 1)\).

By Theorem 5, we obtain that
\[ 1 = J_\Phi(0) = \|D_\Phi(0)\|\lambda(D_\Phi(0)) \leq \lambda(D_\Phi(0)) \left( \frac{4}{\pi}M_1 + \frac{2}{\pi}M_2 + \frac{23}{48}M_3 \right) \]
yields
\[ \lambda(D_\Phi(0)) \geq \frac{1}{\frac{4}{\pi}M_1 + \frac{2}{\pi}M_2 + \frac{23}{48}M_3}. \] (4.7)
We will prove \( \Phi \) is univalent in \( \mathbb{D}_{r_0} \), where \( r_0 \) satisfies the following equation:
\[ \left( \frac{4}{\pi}M_1 + \frac{2}{\pi}M_2 + \frac{23}{48}M_3 \right)\sigma(r_0) = 1. \] (4.8)
We choose two points \( z_1 \neq z_2 \in D_{r_0} \), and by (4.6)-(4.8), we obtain

\[
|\Phi(z_1) - \Phi(z_2)| = \left| \int_{[z_1, z_2]} \Phi_z(z)dz + \Phi_{0}z(z)dz \right|
\]

\[
\geq \left| \int_{[z_1, z_2]} \Phi_z(0)dz + \Phi_{0}(0)dz \right|
\]

\[
- \left| \int_{[z_1, z_2]} (\Phi_z(z) - \Phi_z(0))dz + (\Phi_{0}(z) - \Phi_{0}(0))dz \right|
\]

\[
\geq \lambda(D_{\Phi}(0))|z_1 - z_2|
\]

\[
- \int_{[z_1, z_2]} (|\Phi_z(z) - \Phi_z(0)| + |\Phi_{0}(z) - \Phi_{0}(0)|)|dz|
\]

\[
> |z_1 - z_2| \left( \frac{1}{\pi\pi M_1} + \frac{1}{\pi M_2} + \frac{23}{48}M_3 - \sigma(r_0) \right)
\]

\[
= 0.
\]

Thus, from the arbitrariness of \( z_1 \) and \( z_2 \), the univalence of \( \Phi \) follows.

Now, we will prove \( \Phi(D_{r_0}) \) contains an univalent disk \( D_{R_0} \) To reach this goal, let \( \xi = r_0 e^{i\theta} \in \partial D_{r_0} \). As the mapping \( \frac{\sigma(|z|)}{|z|} \) is increasing, we deduce

\[
\int_{[0,\xi]} \sigma(|z|)|dz| \leq \frac{\sigma(r_0)r_0}{2}.
\]

Therefore,

\[
|\Phi(\xi) - \Phi(0)| = \left| \int_{[0,\xi]} \Phi_z(z)dz + \Phi_{0}(z)dz \right|
\]

\[
\geq \left| \int_{[0,\xi]} \Phi_z(0)dz + \Phi_{0}(0)dz \right|
\]

\[
- \left| \int_{[0,\xi]} (\Phi_z(z) - \Phi_z(0))dz + (\Phi_{0}(z) - \Phi_{0}(0))dz \right|
\]

\[
\geq \lambda(D_{\Phi}(0))r_0 - \int_{[0,\xi]} \sigma(z)|dz|
\]

\[
\geq \sigma(r_0)r_0 - \frac{\sigma(r_0)r_0}{2}
\]

\[
= \frac{\sigma(r_0)r_0}{2}
\]

\[
= \frac{r_0}{\pi M_1 + \frac{4}{\pi}M_2 + \frac{23}{24}M_3}.
\]
Hence $\Phi(D_{r_0})$ contains an univalent disk $D_{R_0}$ with the radius $R_0$ satisfying

$$R_0 \geq \frac{r_0}{\frac{2}{\pi} M_1 + \frac{4}{\pi} M_2 + \frac{24}{24} M_3}.$$

\[\square\]

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