The problem of \emph{d-Path Vertex Cover}, \emph{d-PVC} lies in determining a subset \( F \) of vertices of a given graph \( G = (V,E) \) such that \( G \setminus F \) does not contain a path on \( d \) vertices. The paths we aim to cover need not to be induced. It is known that the \emph{d-PVC} problem is NP-complete for any \( d \geq 2 \). When parameterized by the size of the solution \( k \), \emph{5-PVC} has direct trivial algorithm with \( O(5^k n^{O(1)}) \) running time and, since \emph{d-PVC} is a special case of \emph{d-Hitting Set}, an algorithm running in \( O(4.0755^k n^{O(1)}) \) time is known. In this paper we present an iterative compression algorithm that solves the \emph{5-PVC} problem in \( O(4^k n^{O(1)}) \) time.

1 Introduction

The problem of \emph{d-Path Vertex Cover}, \emph{d-PVC} lies in determining a subset \( F \) of vertices of a given graph \( G = (V,E) \) such that \( G \setminus F \) does not contain a path on \( d \) vertices (even not a non-induced one). The problem was first introduced by Brešar et al. \cite{Brešar2009}, but its NP-completeness for any \( d \geq 2 \) follows already from the meta-theorem of Lewis and Yannakakis \cite{Lewis1980}. The 2-PVC problem corresponds to the well known \emph{Vertex Cover} problem and the 3-PVC problem is also known as \emph{Maximum Dissociation Set}. The \emph{d-PVC} problem is motivated by the field of designing secure wireless communication protocols \cite{Liu2014} or in route planning and speeding up shortest path queries \cite{Liu2017}.

Since the problem is NP-hard, any algorithm solving the problem exactly is expected to have exponential running time. If one measures the running time solely in terms of the input size, then several efficient (faster than trivial enumeration) exact algorithms are known for \( 2 \)-PVC and \( 3 \)-PVC. In particular, \( 2 \)-PVC (\emph{Vertex Cover}) can be solved in \( O(1.1996^n) \) time and polynomial space due to Xiao and Nagamochi \cite{Xiao2011} and \( 3 \)-PVC can be solved in \( O(1.4656^n) \) time and polynomial space due to Xiao and Kou \cite{Xiao2013}.

In this paper we aim on the parameterized analysis of the problem, that is, to confine the exponential part of the running time to a specific parameter of the input, presumably much smaller than the input size. The problem is called \emph{fixed-parameter tractable} if there exists an algorithm (called a \emph{fixed-parameter algorithm}) that runs in \( f(k)n^{O(1)} \) time, where \( k \) is the parameter. The class of problems containing all fixed-parameter tractable problems is called \emph{FPT}. See Cygan et al. \cite{Cygan2015} for a broader introduction to parameterized algorithms.

When parameterized by the size of the solution \( k \), the \emph{d-PVC} problem is directly solvable by a trivial FPT algorithm that runs in \( O^*(d^k) \) time\footnote{\emph{O}*(\() notation suppresses all factors polynomial in the input size.}. However, since \emph{d-PVC} is a special case of \emph{d-Hitting Set}, it was shown by Fomin...
et al. [8] that for any $d \geq 4$ we have an algorithm solving $d$-PVC in $O^*((d - 0.9245)^k)$. In order to find more efficient solutions, the problem has been extensively studied in a setting where $d$ is a small constant. For the 2-PVC (VERTEX COVER) problem, the algorithm of Chen, Kanj, and Xia [4] has the currently best known running time of $O^*(1.2738^k)$. For 3-PVC, Tu [18] used iterative compression to achieve a running time $O^*(2^k)$. This was later improved by Katrenič [12] to $O^*(1.8127^k)$, by Xiao and Kou [21] to $O^*(1.7485^k)$ by using a branch-and-reduce approach and it was further improved by Tsur [16] to $O^*(1.713^k)$. For the 4-PVC problem, Tu and Jin [19] again used iterative compression and achieved a running time $O^*(3^k)$ and Tsur [17] gave the current best algorithm that runs in $O^*(2.619^k)$ time. For $d = 5, 6, 7$ Tsur [15] claimed algorithms for $d$-PVC with running times $O^*(3.945^k)$, $O^*(4.947^k)$, and $O^*(5.951^k)$, respectively. Recently, the authors of this paper claimed to have developed a procedure that generates even faster algorithms for $d$-PVC for some $d$ [3].

In this paper, we present an algorithm that solves the 5-PVC problem parameterized by the size of the solution $k$ in $O^*(4^k)$ time by employing the iterative compression technique. Using the result of Fomin et al. [9] this also yields $O(1.7501^n)$ time algorithm improving upon previously known $O(1.7547^n)$ time algorithm.

**Organization of this paper.** We introduce the notation and define the 5-PVC problem in Section 2. Our disjoint compression routine for iterative compression is exposed in Section 3. We conclude this paper with a few open questions.

## 2 Preliminaries

We use the $O^*$ notation as described by Fomin and Kratsch [10], which is a modification of the big-O notation suppressing all factors bounded by a polynomial of the input size. We use the notation of parameterized complexity as described by Cygan et al. [3]. We use standard graph notation and consider simple and undirected graphs unless otherwise stated. Vertices of graph $G$ are denoted by $V(G)$, edges by $E(G)$. By $G[X]$ we denote the subgraph of $G$ induced by vertices of $X \subseteq V(G)$. By $N(v)$ we denote the set of neighbors of $v \in V(G)$ in $G$. Analogically, $N(X) = \bigcup_{x \in X} N(x)$ denotes the set of neighbors of vertices in $X \subseteq V(G)$. The degree of vertex $v$ is denoted by $deg(v) = |N(v)|$. For simplicity, we write $G \setminus v$ for $v \in V(G)$ and $G \setminus X$ for $X \subseteq V(G)$ as shorthands for $G[V(G) \setminus \{v\}]$ and $G[V(G) \setminus X]$, respectively.

A $k$-path, denoted as an ordered $k$-tuple $P_k = (p_1, p_2, \ldots, p_k)$, is a path on $k$ vertices $\{p_1, p_2, \ldots, p_k\}$. A path $P_k$ starts at vertex $x$ when $p_1 = x$. A $k$-cycle is a cycle on $k$ vertices. A triangle is a 3-cycle. A $P_3$-free graph is a graph that does not contain a $P_3$ as a subgraph (the $P_3$ need not to be induced). The 5-PATH VERTEX COVER problem is formally defined as follows:

**5-PATH VERTEX COVER, 5-PVC**

| INPUT:          | A graph $G = (V, E)$, an integer $k \in \mathbb{Z}^+$ |
|-----------------|--------------------------------------------------|
| OUTPUT:         | A set $F \subseteq V$, such that $|F| \leq k$ and $G \setminus F$ is a $P_3$-free graph |

**Definition 1.** A star is a graph $S$ with vertices $V(S) = \{s\} \cup \{l_1, \ldots, l_k\}$, $k \geq 3$ and edges $E(S) = \{\{s, l_i\} \mid i \in \{1, \ldots, k\}\}$. Vertex $s$ is called a center, vertices $l_1, \ldots, l_k$ are called leaves.

**Definition 2.** A star with a triangle is a graph $S^\triangle$ with vertices $V(S^\triangle) = \{s, t_1, t_2\} \cup \{l_1, \ldots, l_k\}$, $k \geq 1$ and edges $E(S^\triangle) = \{\{s, t_1\}, \{s, t_2\}, \{t_1, t_2\}\} \cup \{\{s, l_i\} \mid i \in \{1, \ldots, k\}\}$. Vertex $s$ is called a center, vertices $T = \{t_1, t_2\}$ are called triangle vertices and vertices $L = \{l_1, \ldots, l_k\}$ are called leaves.

**Definition 3.** A di-star is a graph $D$ with vertices $V(D) = \{s, s'\} \cup \{l_1, \ldots, l_k\} \cup \{l'_1, \ldots, l'_m\}$, $k \geq 1, m \geq 1$ and edges $E(D) = \{\{s, s'\}\} \cup \{\{s, l_i\} \mid i \in \{1, \ldots, k\}\} \cup \{\{s', l'_j\} \mid j \in \{1, \ldots, m\}\}$. Vertices $s, s'$ are called centers, vertices $L = \{l_1, \ldots, l_k\}$ and $L' = \{l'_1, \ldots, l'_m\}$ are called leaves.

**Lemma 4.** If a connected graph is $P_3$-free and has more than 5 vertices, then it is a star, a star with a triangle, or a di-star.

**Proof.** Suppose we have a $P_3$-free graph $G$ on at least 5 vertices. Firstly, $G$ does not contain a $k$-cycle, $k \geq 5$ as a subgraph, since $P_3$ is a subgraph of such a $k$-cycle. Secondly, $G$ does not contain a 4-cycle as a subgraph, since $G$ has at least 5 vertices and it is connected which implies that there is at least one vertex connected to the 4-cycle which in turn implies a $P_5$ in $G$. Finally, $G$ does not contain two edge-disjoint triangles as a subgraph, since $G$ is connected, the two triangles are either sharing a vertex or are connected by some path, which in both cases implies a $P_3$ in $G$. Consequently, $G$ contains either exactly one triangle or is acyclic.
Consider the first case where $G$ contains exactly one triangle. Label the vertices of the triangle with \{ $t_1, t_2, t_3$ \}. Then we claim that all vertices outside the triangle are connected by an edge to exactly one vertex of that triangle, let that vertex be $t_1$. Indeed, for contradiction suppose they are not. Since we have at least 5 vertices in $G$, label the two existing vertices outside the triangle $x$ and $y$. Then we either have $x$ and $y$ connecting to two different vertices of the triangle, let them be $t_1, t_2$, which immediately implies a $P_5 = (x, t_1, t_3, t_2, y)$ in $G$, or we have a $P_3 = (x, y, t_1)$ connected to the triangle, which again implies a $P_5 = (x, y, t_1, t_2, t_3)$. Hence, if $G$ contains a triangle, then it is a star with a triangle.

Consider the second case where $G$ is acyclic. Then we claim that there is a dominating edge in $G$, i.e. an edge $e = \{x, y\}$ such that $V(G) = N(\{x, y\})$. Indeed, for contradiction suppose that there is no such edge. Then we have that for each edge $e = \{x, y\}$ in $G$ there must be a vertex $v$ that is adjacent neither to $x$, nor to $y$. Assume that $v$ is connected to $y$ through some vertex $u$. The same also holds for the edge $\{y, u\}$, so assume that there is a vertex $v' \neq x$ that is connected to $u$ through some vertex $u' \neq y$. But then we have a $P_5 = (x, y, u, u', v')$ in $G$.

Label the dominating edge $e = \{s, s'\}$. Here, if only one of the vertices $s, s'$ has degree greater than one, we have a star, otherwise we have a di-star. \hfill \square

### 3 5-PVC with $P_5$-free bipartition

We employ the generic iterative compression framework as described by Cygan et al.\cite{5} pages 80–81. We skip the generic steps and only present the disjoint compression routine (see also Subsection 3.11 for a brief discussion of the whole iterative compression algorithm). That is, we assume that we are given a solution to the problem and search for another solution which is strictly smaller than and disjoint from the given one. Moreover, if the graph induced by the solution contains a $P_5$, then we can directly answer no. Hence our routine *DISJOINT_R* restricts itself to a problem called 5-PVC with $P_5$-FREE BIPARTITION and we need it to run in $O^*(3^k)$ time.

A $P_5$-free bipartition of graph $G = (V, E)$ is a pair $(V_1, V_2)$ such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and $G[V_1], G[V_2]$ are $P_5$-free. The 5-PVC with $P_5$-FREE BIPARTITION problem is formally defined as follows:

| INPUT: | A graph $G = (V, E)$ with $P_5$-free bipartition $(V_1, V_2)$, an integer $k \in \mathbb{Z}_0^+$. |
| OUTPUT: | A set $F \subseteq V_2$, such that $|F| \leq k$ and $G \setminus F$ is a $P_5$-free graph. |

Throughout this paper the vertices from $V_1$ will be also referred to as “red” vertices and vertices from $V_2$ will be also referred to as “blue” vertices. The same colors will also be used in figures with the same meaning.

#### 3.1 Algorithm Outline

Our algorithm is a recursive procedure $\text{DISJOINT}_R(G, V_1, V_2, F, k)$, where $G$ is the input graph, $V_1, V_2$ are the partitions of the $P_5$-free bipartition of $G$, $F$ is the solution being constructed, and $k$ is the maximum number of vertices we can still add to $F$. The procedure repeatedly tries to apply a series of rules with a condition that a rule (RI) can be applied only if all rules that come before (RF) cannot be applied. It is paramount that in every call of $\text{DISJOINT}_R$ at least one rule can be applied. The main work is done in rules of two types: *redaction rules* and *branching rules*.

A *redaction rule* is used to simplify a problem instance, i.e. remove some vertices or edges from $G$ and possibly add some vertices to a solution, or to halt the algorithm. A *branching rule* splits the problem instance into at least two subinstances. The branching is based on subsets of vertices that we try to add to a solution and by adding them to the solution we also remove them from $G$.

The notation we use to denote the individual branches of a branching rule is as follows: $\langle X_1 | X_2 | \ldots | X_l \rangle$. Such a rule has $l$ branches and $X_1, X_2, \ldots, X_l$ are subsets of $V_2$ which we try to add to the solution. This rule is translated into the following $l$ calls of the procedure:

$$\text{DISJOINT}_R(G \setminus X_i, V_1, V_2 \setminus X_i, F \cup X_i, k - |X_i|) \text{ for } i \in \{1, \ldots, l\}$$

A rule is *applicable* if the conditions of the rule are satisfied and none of the previous rules is applicable. A redaction rule is *correct* if it satisfies that the problem instance has a solution if and only if the simplified problem instance has a solution. A branching rule is *correct* if it satisfies that if the problem instance has a solution, then at least one of the branches of the rule will return a solution.
When we say we \textit{delete} a vertex, we mean that we remove it from \(G\) and also add it to the solution \(F\). When we say we \textit{remove} a vertex, we mean that we remove it from \(G\) and \textit{do not} add it to the solution \(F\).

For the rest of this paper assume that the parameters of the current call of \textsc{disjoint} are \(G, V_1, V_2, F, k\).

3.2 Preprocessing

\textbf{Reduction rule (R0).} This rule stops the recursion of \textsc{disjoint}. It has three stopping conditions:

1. If \(k < 0\), return \textit{no solution};
2. else if \(G\) is \(P_5\)-free, return \(F\);
3. else if \(k = 0\), return \textit{no solution}.

\textbf{Reduction rule (R1).} Let \(v \in V(G)\) be a vertex such that there is no \(P_5\) in \(G\) that uses \(v\). Then remove \(v\) from \(G\).

\textit{Proof of correctness.} Let \(v \in V(G)\) be a vertex that is not used by any \(P_5\) in \(G\) and let \(F\) be a solution to the \(5\)-PVC\w B instance \((G \setminus v, V_1 \setminus \{v\}, V_2 \setminus \{v\}, k)\). Then \(F\) is also a solution to \((G, V_1, V_2, k)\) since \(v\) is not used by any \(P_5\) in \(G\).

If \((G \setminus v, V_1 \setminus \{v\}, V_2 \setminus \{v\}, k)\) does not have a solution, then we claim that \((G, V_1, V_2, k)\) also does not have a solution. Indeed, adding vertices can only create new \(P_5\) paths.

\textbf{Reduction rule (R2).} Let \(u, v \in V_2\) be two vertices such that \(u\) is contained in every \(P_5\) in \(G\) which contains \(v\). Then move the vertex \(v\) to \(V_1\) (make it red).

\textit{Proof of correctness.} Let \(u, v \in V_2\) be two vertices such that \(u\) is contained in every \(P_5\) in \(G\) which contains \(v\) and let \(F\) be a solution to the \(5\)-PVC\w B instance \((G, V_1 \setminus \{v\}, V_2 \cup \{v\}, k)\). Then \(F\) is also a solution to \((G, V_1, V_2, k)\). Similarly, if \(F\) is a solution to \((G, V_1, V_2, k)\) which does not contain \(v\), then it is also a solution to \((G, V_1 \setminus \{v\}, V_2 \cup \{v\}, k)\).

Finally, if \(F\) is a solution to \((G, V_1, V_2, k)\) which contains \(v\), then \(F' = (F \setminus v) \cup \{u\}\) is also a solution to \((G, V_1, V_2, k)\):

\begin{itemize}
  \item \textbf{Branching rule (R3).} Let \(P\) be a \(P_5\) in \(G\) with \(X = V(P) \cap V_2\) such that \(|X| \leq 3\). Then branch on \(\langle x_1 \mid x_2 \mid \ldots \rangle, x_i \in X\), i.e. branch on the blue vertices of \(P\).
\end{itemize}

\textit{Proof of correctness.} We have to delete at least one blue vertex in \(P\), thus branching on the blue vertices of \(P\) is correct.

\textbf{Lemma 5.} Assume that Rules \((\text{R0}) - (\text{R3})\) are not applicable. Then for each vertex \(v \in V(G)\) there exists a \(P_5\) in \(G\) that uses \(v\); every \(P_5\) in \(G\) uses exactly one red vertex; and there are only isolated vertices in \(G[V_1]\).

\textit{Proof.} If Rule (R1) is not applicable, then for each vertex \(v \in V(G)\) there exists a \(P_5\) in \(G\) that uses \(v\). If Rule (R3) is not applicable, then every \(P_5\) in \(G\) uses at most one red vertex and since \((V_1, V_2)\) is a \(P_5\)-free bipartition we cannot have a \(P_5\) in \(G\) that uses no red vertex.

To prove that there are only isolated vertices in \(G[V_1]\), assume for contradiction that there is an edge \(e = \{v_1, v_2\}\) in \(G[V_1]\). Since Rule (R1) and Rule (R3) are not applicable, there must be a path \(P_1 = (u_1, u_2, u_3, u_4, u_5)\) in \(G\) that uses \(v_1\) and not uses \(v_2\) and \(P_2 = (w_1, w_2, w_3, w_4, w_5)\) in \(G\) that uses \(v_2\) and not uses \(v_1\). Paths \(P_1\) and \(P_2\) are not necessarily disjoint.

Now consider the following cases for \(v_1\) and \(P_1\). If \(v_1 = u_1\), then there is a path \(P' = (v_2, v_1, u_2, u_3, u_4)\) contradicting Rule (R3) not being applicable. Similarly, if \(v_1 = u_2\), then there is a path \(P' = (v_2, u_1, u_3, u_4, u_5)\) contradicting Rule (R3) not being applicable.

The same arguments apply for the cases where \(v_1 = u_5\) and \(v_1 = u_4\) respectively and the same logic applies also when considering \(v_2\) and \(P_2\).

Thus we have that \(v_1 = u_3\) and \(v_2 = u_3\). Now it suffices to see that either \(w_2 \notin \{u_1, u_2\}\) or \(w_2 \notin \{u_4, u_5\}\). In the first case we get a path \(P' = (u_1, u_2, u_1, v_2, u_2)\) and in the second case we get a path \(P' = (u_5, u_4, v_1, v_2, u_2)\). In both cases we get a contradiction with Rule (R3) not being applicable.

\textbf{3.3 Dealing with isolated vertices in \(G[V_2]\)}

\textbf{Lemma 6.} Assume that Rules \((\text{R0}) - (\text{R3})\) are not applicable. Let \(v\) be an isolated vertex in \(G[V_2]\) and let \(F\) be a solution to 5-PVC\w B which uses vertex \(v\). Then there exists a solution \(F'\) that does not use vertex \(v\) and \(|F'| \leq |F|\).
Proof. From Lemma 5 we get that each $P_5$ in $G$ which contains $v$ must also start in $v$, otherwise it would imply a $P_5$ that uses more than one red vertex. In particular, $v$ has at most one red neighbor. Suppose that there exists a path $P = (v, w, x, y, z)$ where $w$ is a red vertex and $\{x, y, z\} \cap F = \emptyset$ (see Figure 1). If there is no such $P$, then, as each vertex is in at least one $P_5$ due to Rule (R1) not being applicable, we have that each $P_5$ starting in $v$ has at least one of the vertices $x, y, z$ in $F$. In that case, we can put $F' = F \setminus \{v\}$ and the lemma holds.

There cannot exist another path $P' = (v, w, x', y', z')$ such that $x' \neq x$ and $\{x', y', z'\} \cap F = \emptyset$, otherwise we would have a $P_5 = (x', w, x, y, z)$ in $G$ that is not hit by $F$. Consequently, each $P_5$ that is hit only by vertex $v$ also contains vertex $x$, which implies that $F' = (F \setminus \{v\}) \cup \{x\}$ is a solution and $|F'| \leq |F|$, thus the lemma holds.

\[ \square \]

Reduction rule (R4). Let $v$ be an isolated vertex in $G[V_2]$. Then move $v$ to $V_1$.

Proof of correctness. Let $v$ be an isolated vertex in $G[V_2]$. If $F$ is a solution to the 5-PVCwB instance $(G, V_1 \setminus \{v\}, V_2 \cup \{v\}, k)$, then $F$ is also a solution to $(G, V_1, V_2, k)$. Similarly, if $F$ is a solution to $(G, V_1, V_2, k)$, then by Lemma 6 we can assume that it does not contain $v$ and, hence, it is also a solution to $(G, V_1 \setminus \{v\}, V_2 \cup \{v\}, k)$.

Observation 7. Assume that Rules [R0] – [R4] are not applicable. Then there are no isolated vertices in $G[V_2]$.

3.4 Dealing with isolated edges in $G[V_2]$

Lemma 8. Assume that Rules [R0] – [R4] are not applicable. Let $v$ be a blue vertex to which at least two red vertices are connected and let $C_v$ be a connected component of $G[V_2]$ which contains $v$. Then for each red vertex $w$ connected to $v$ we have that $N(w) \subseteq V(C_v)$.

Proof. Let $w_1, w_2$ be red vertices connected to $v$. For contradiction assume that $w_1$ is connected to some vertex $v'$ in $G[V_2]$ such that $v' \notin V(C_v)$. From Observation 7 we know that $v'$ has degree at least one in $G[V_2]$. Label some neighbor of $v'$ in $G[V_2]$ as $w'$. We obtained a $P_5 = (w, v', w_1, v, w_2)$ which contradicts Lemma 5.

Lemma 9. Assume that Rules [R0] – [R4] are not applicable. Let $e = \{u, v\} \subseteq V_2$ be a blue edge to which at least two red vertices are connected ($|N(e) \cap V_1| \geq 2$) in a way that to both $u$ and $v$ there is at least one red vertex connected ($|N(u) \cap V_1| \geq 1$, $|N(v) \cap V_1| \geq 1$). Let $C_e$ be a connected component of $G[V_2]$ which contains $e$. Then for each red vertex $w$ connected to $e$ we have that $N(w) \subseteq V(C_e)$.

Proof. Let $w_1, w_2$ be red vertices connected to $e$ and assume that $w_1$ is connected to $u$ and $w_2$ is connected to $v$. For contradiction assume that $w_1$ is connected to some vertex $v'$ in $G[V_2]$ such that $v' \notin V(C_e)$. We obtain a $P_5 = (v', w_1, u, v, w_2)$ which contradicts Lemma 5.

Lemma 10. Let $X$ be a subset of $V_2$ such that $N(X) \cap V_1 = \emptyset$ and $|N(X) \cap V_2| = 1$. Then Rule (R2) applies. In particular, if $v \in V_2$ has degree one, then its neighbor is in $V_1$, or Rule (R2) applies.

Proof. Assume that $N(X) \cap V_2 = \{v\}$. Then each $P_5$ that uses some vertex in $X$ must also use vertex $v$, otherwise it would be contained in $X$ which contradicts $G[V_2]$ being $P_5$-free.

Definition 11. We say that two nodes $x, y$ are twins if $N(x) \setminus \{y\} = N(y) \setminus \{x\}$.

Lemma 12. Let $x, y$ be blue vertices that are twins. Let $F$ be a solution and $x \in F$. Then at least one of the following holds:

1. $y \in F$.
2. $F' = (F \setminus \{x\}) \cup \{y\}$ is a solution.
Figure 2: Configuration in Rule (R5)

Proof. Assume that there is a vertex connected to $x$ that is hit only by $w$ and $y$. Since $x, y$ are twins, for each path $P = (p_1, p_2, p_3, p_4, p_5)$ with $p_i = x$ and $y$, there also exists a path $P' = (p'_1, p'_2, p'_3, p'_4, p'_5)$ such that $p'_j = p_j$ for $j \in \{1, 2, 3, 4, 5\} \setminus \{i\}$ and $p'_1 = y$. Firstly, if there is no $P_5$ containing $x$, then trivially (2) holds. Secondly, if all $P_5$ paths that contain $x$ are hit by some other vertex $z, z \neq x, z \in F$, then again (2) holds. So suppose that there exists a $P_5$ path $P$ that is hit only by $x$. If $y \notin F$, then we know that there is a path $P'$ as described above and we get a contradiction with $F$ being a solution since $P'$ is not hit by $F$ and (1) must hold. Otherwise, all $P_5$ paths that contain $x$ also contain $y$ and (2) holds.

**Lemma 13.** Assume that Rules (R0)–(R4) are not applicable. If there is an isolated edge $e = \{u, v\} \in G[V_2]$, then there is exactly one red vertex $w$ connected to $e$. Moreover $N(w) \setminus \{u, v\} \neq \emptyset$.

Proof. For contradiction assume that Rules (R0)–(R4) are not applicable and there is an isolated edge $e = \{x, y\}$ in $G[V_2]$. If there is no $P_5$ that uses vertices from $e$, then Rule (R1) is applicable on $e$. Hence there are red vertices connected to $e$. If there are at least two red vertices connected to $e$, then from Lemma 8 and 9 we know that these red vertices are not connected to any other vertices outside $e$ and there again cannot be a $P_5$ that uses vertices from $e$ and Rule (R1) is applicable on $e$. So, assume that there is exactly one red vertex connected to $e$ and label that vertex $w$. If $N(w) \subseteq \{u, v\}$, then from Lemmata 12 we know that these vertices $u, v, w$ are not part of any $P_5$ and Rule (R1) is applicable on them. Therefore, $N(w) \setminus \{u, v\} \neq \emptyset$ and the lemma holds.

**Branching rule (R5).** Let $e = \{u, v\}$ be an isolated edge in $G[V_2]$. Let there be a red vertex $w$ connected to at least one vertex in $e$. Assume that $x$ is some vertex to which $w$ connects outside $e$ and let $y$ be a neighbor of $x$ in $G[V_2]$. Then branch on $\langle v \mid x \mid y \rangle$.

**Proof of correctness.** By Lemma 13 $w$ is the only red vertex connected to $e$. Firstly, assume that $w$ is connected only to one vertex of $e$. Then Rule (R2) applies. Secondly, assume that $w$ is connected to both vertices of $e$. Since $u, v$ are twins, from Lemma 12 it follows that we can try deleting only one of them. Thus branching on $\langle v \mid x \mid y \rangle$ is correct.

**Observation 14.** Assume that Rules (R0)–(R5) are not applicable. Then there are no isolated edges in $G[V_2]$.

Proof. If there is an isolated edge in $G[V_2]$, then by Lemma 13 there is exactly one red vertex connected to it and Rule (R5) is applicable.

3.5 Dealing with isolated $P_3$ paths in $G[V_2]$

**Lemma 15.** Assume that Rules (R0)–(R5) are not applicable. Let $P$ be a $P_3 = (t, u, v)$ that forms a connected component in $G[V_2]$. There is only one red vertex $w$ connected to $P$. In particular, $w$ is connected to $t$ and $v$, to some component of $G[V_2] \setminus P$, and possibly to $u$.

Proof. For contradiction assume that Rules (R0)–(R5) are not applicable and there is an isolated $P_3$ path $P = (t, u, v)$ in $G[V_2]$. If there is no $P_5$ that uses vertices from $P$, then Rule (R1) is applicable on $P$. Hence there are red vertices connected to $P_3$. Suppose there are at least two red vertices connected to $P_3$. If they are connected to vertices $t, v$, then Rule (R3) is applicable, since there is a $P_5$ that uses at least two red vertices. So suppose the red vertices are connected to a single vertex or a single edge in $P$. Then from Lemmata 8 and 9 we know that those red vertices are not connected to any other vertices outside $P$. Consequently, there cannot be a $P_5$ that uses vertices from $P$ and again Rule (R1) is applicable on $P$.

So suppose that there is a $P_5$ that uses vertices from $P$ and there is only one red vertex $w$ connected to $P$. If $w$ is not connected to $t$ or $v$, then Rule (R2) is applicable.

**Branching rule (R6).** Let $P$ be a $P_3 = (t, u, v)$ that forms a connected component in $G[V_2]$ and $w$ be the only red vertex connected to $P$. Assume that $x$ is some vertex to which $w$ connects outside $P$ and let $y$ be a neighbor of $x$ in $G[V_2]$ (see Figure 3). Then branch on $\langle u \mid v \mid x \rangle$. 
Proof of correctness. By Lemma 15, vertex $w$ is connected to $t, v$ in $P$ and $w$ can be also connected to $u$ in $P$. If we do not delete vertex $x$, then we have to delete something in $P$. In both cases, when $w$ is connected to $u$ and when not, $t, v$ are twins and from Lemma 12 we know that we have to try only one of $t, v$. Thus branching on $\langle u \mid v \mid x \rangle$ is correct.

Observation 16. Assume that Rules $[R0] - [R6]$ are not applicable. Then there are no isolated $P_3$ paths in $G[V_2]$.

Proof. For contradiction assume that Rules $[R0] - [R5]$ are not applicable and there is an isolated $P_3$ in $G[V_2]$. Then, by Lemma 15, Rule $[R6]$ applies.

3.6 Dealing with isolated triangles in $G[V_2]$

Lemma 17. Assume that Rules $[R0] - [R6]$ are not applicable. Let $T$ be a $K_3 = \{t, u, v\}$ that forms a connected component in $G[V_2]$. There is only one red vertex $w$ connected to $T$. Furthermore, $w$ is connected to at least two vertices of $T$ and to some component of $G[V_2]$ other than $T$.

Proof. For contradiction assume that Rules $[R0] - [R6]$ are not applicable and there is an isolated triangle $T = \{t, u, v\}$ in $G[V_2]$. If there is no $P_3$ that uses vertices from $T$, then Rule $[R3]$ is applicable on $T$. Hence there are red vertices connected to $T$. Suppose there are at least two red vertices connected to $T$. If the red vertices are not connected to a single vertex in $T$, then Rule $[R3]$ is applicable, since there is a $P_3$ that uses at least two red vertices. So suppose the red vertices are connected to a single vertex in $T$. Then from Lemma 8, we know that those red vertices are not connected to any other vertices outside $T$. Consequently, there cannot be a $P_3$ that uses red vertices from $T$ again. So suppose that there is a $P_3$ that uses vertices from $T$ and there is only one red vertex $w$ connected to $T$. If $w$ is connected to only one vertex of $T$, then Rule $[R2]$ is applicable on the other vertices of $T$.

Branching rule $[R7]$. Let $T$ be a $K_3 = \{t, u, v\}$ that forms a connected component in $G[V_2]$ and $w$ the only red vertex connected to $T$. Suppose that vertex $w$ is connected to at least two vertices in $T$, let those vertices be $u, v$. Assume that $x$ is some vertex to which $w$ connects outside $T$ and let $y$ be a neighbor of $x$ in $G[V_2]$ (see Figure 4). Then branch on $\langle t \mid v \mid x \rangle$.

Proof of correctness. If we do not delete vertex $x$, then we have to delete something in $T$. Since $u, v$ are twins, from Lemma 12, we know that we have to try only one of $u, v$. Thus branching on $\langle t \mid v \mid x \rangle$ is correct.

Observation 18. Assume that Rules $[R0] - [R7]$ are not applicable. Then there are no isolated triangles in $G[V_2]$.

Proof. For contradiction assume that Rules $[R0] - [R6]$ are not applicable and there is an isolated triangle $T = \{t, u, v\}$ in $G[V_2]$. Then, by Lemma 17, Rule $[R7]$ is applicable.
3.7 Dealing with $4$-cycles in $G[V_2]$

**Lemma 19.** Let $C$ be a connected component of $G[V_2]$ and $X = V(C) \cap N(V_1)$. Let $F$ be a solution that deletes at least $|X|$ vertices in $C$. Then $F' = (F \setminus V(C)) \cup X$ is also a solution and $|F'| \leq |F|$.

**Proof.** Each $P_5$ that uses some vertex in $C$ must also use some vertex $x \in X$, otherwise it would be contained in $C$ which contradicts $G[V_2]$ being $P_5$-free. Consequently, any $P_5$ that is hit by a vertex from $C$ in the solution $F$ can be also hit by some vertex $x \in X$ and thus $F' = (F \setminus V(C)) \cup X$ is also a solution and $|F'| \leq |F|$.

**Lemma 20.** Assume that Rules [R0]–[R7] are not applicable. Let $Q$ be a connected component in $G[V_2]$ which contains a 4-cycle. Then $Q$ is a subgraph of $K_4$ and there is exactly one red vertex connected to $Q$ and it must be connected to at least two vertices in $Q$ and to some component of $G[V_2]$ other than $Q$.

**Proof.** Assume that Rules [R0]–[R7] are not applicable and there is a component $Q$ in $G[V_2]$ that contains a 4-cycle as a subgraph, label the vertices of the 4-cycle $(v_1, v_2, v_3, v_4)$. Observe that $Q$ is a subgraph of $K_4$, as otherwise there would be a $P_5$ in $G[V_2]$.

If there is no $P_5$ that uses vertices from $Q$, then Rule (R1) is applicable on $Q$. Hence there are red vertices connected to $Q$. Suppose that there are at least two red vertices connected to $Q$. If the red vertices are not connected to a single vertex or a single edge in $Q$, then Rule (R3) is applicable, since there is a $P_5$ that uses at least two red vertices. So suppose the red vertices are connected to a single vertex or a single edge in $Q$. Then from Lemmata 8 and 9 we know that those red vertices are not connected to any other vertices outside $Q$. Then, we have that every $P_5$, which uses some vertices from $Q$, actually uses all of the vertices from $Q$ and exactly one red vertex connected to $Q$. But then, Rule (R2) is applicable on $Q$. Hence, we have that there is exactly one red vertex connected to $Q$, label it $w$. Again, if $w$ is not connected to some component of $G[V_2]$ other than $Q$, then Rule (R2) applies on $Q$. Now assume, that $w$ is connected to only one vertex of $Q$. Then Rule (R2) is applicable on that vertex and the other vertices of $Q$. Therefore, vertex $w$ must be connected to at least two vertices of $Q$ and the lemma holds.

Let $Q$ be a connected component such that a 4-cycle is a subgraph of $Q$, label the vertices of the 4-cycle $(v_1, v_2, v_3, v_4)$. We will call pairs of vertices $\{v_1, v_3\}$ and $\{v_2, v_4\}$ diagonal, all other pairs will be called non-diagonal.

**Branching rule (R8).** Let $Q$ be a connected component in $G[V_2]$ such that $Q \subseteq K_4$ and a 4-cycle $(v_1, v_2, v_3, v_4)$ is a subgraph of $Q$. Assume that there is only one red vertex $w$ connected to $Q$ and $X = V(Q) \cap N(w)$. Set $X$ contains at least one diagonal pair, let that pair be $\{v_1, v_3\}$ (see Figure 5a). Then branch on $\{v_1 | v_2 | v_4\}$.

**Proof of correctness.** We have to delete something in $Q$. Since $v_1, v_3$ are twins, from Lemma 12 we know that we have to try only one of $v_1, v_3$. Thus branching on $\{v_1 | v_2 | v_4\}$ is correct.

**Branching rule (R9).** Let $Q$ be a connected component in $G[V_2]$ such that $Q \subseteq K_4$ and a 4-cycle $(v_1, v_2, v_3, v_4)$ is a subgraph of $Q$. Assume that there is only one red vertex $w$ connected to $Q$ and $X = V(Q) \cap N(w)$. Suppose that set $X$ is of size 2 and forms a non-diagonal pair, let that pair be $\{v_1, v_2\}$. Assume that there is a vertex $x$ to which $w$ connects outside $Q$ and let $y$ be a neighbor of $x$ in $G[V_2]$ (see Figure 5b). Then branch on $\{v_1, v_2\} | x | y$.

**Proof of correctness.** If none of the vertices $x, y$ is deleted, then we have to delete at least two vertices in $Q$. From Lemma 19 we know that we only have to try deleting vertices $\{v_1, v_2\}$. Thus branching on $\{v_1, v_2\} | x | y$ is correct.
Lemma 21. Assume that Rules (R0) – (R9) are not applicable. Then there is no component of \(G[V_2]\) that contains a 4-cycle as a subgraph.

Proof. For contradiction assume that Rules (R0) – (R9) are not applicable and there is a component \(Q\) in \(G[V_2]\) that contains a 4-cycle as a subgraph, label the vertices of the 4-cycle \((v_1, v_2, v_3, v_4)\). By Lemma 20, \(Q\) is a subgraph of \(K_4\) and there is exactly one red vertex \(w\) connected to \(Q\) and \(w\) is connected to at least two vertices in \(Q\) and to some component of \(G[V_2]\) other than \(Q\).

Let \(X = V(Q) \cap N(w)\). As \(w\) is connected to at least two vertices in \(Q\), \(X\) either contains at least one diagonal pair of \(Q\), or \(X\) is of size 2 and forms a non-diagonal pair. In the case \(X\) contains a diagonal pair, Rule (R8) applies. In the case \(X\) forms a non-diagonal pair, Rule (R9) applies.

3.8 Dealing with stars in \(G[V_2]\)

Lemma 22. Assume that Rules (R0) – (R9) are not applicable. Suppose that there is a connected component \(S\) of \(G[V_2]\) which is isomorphic to a star with at least 4 vertices. Then there is exactly one red vertex \(w\) connected to \(S\) and \(w\) is connected to all the leaves of \(S\).

Proof. Assume that Rules (R0) – (R9) are not applicable and there is a star \(S\) in \(G[V_2]\) with at least 4 vertices.

If there is no \(P_5\) that uses vertices from \(S\), then Rule (R1) is applicable on \(S\). Hence there are red vertices connected to \(S\). Suppose there are at least two red vertices connected to \(S\). If the red vertices are not connected to a single vertex or a single edge in \(S\), then Rule (R3) is applicable, since there is a \(P_5\) that uses at least two red vertices. So suppose the red vertices are connected to a single vertex or a single edge in \(S\). Then from Lemmata 8 and 9 we know that those red vertices are not connected to any other vertices outside \(S\). Consequently, there cannot be a \(P_5\) that uses vertices from \(S\) and again Rule (R1) is applicable on \(S\).

So suppose that there is a \(P_5\) that uses vertices from \(S\) and there is only one red vertex \(w\) connected to \(S\). If there is a leaf \(l\) of \(S\) not connected to \(w\), then Rule (R2) applies to \(l\).

Branching rule (R10). Let \(S\) be a connected component of \(G[V_2]\) isomorphic to a star with at least 4 vertices, \(l\) an arbitrary one of its leaves and \(s\) its center. Then branch on \(⟨l \mid s⟩\).

Proof of correctness. We have to delete something in \(S\), since there is a path \(P_5 = (l_1, w, l_2, s, l_3)\) for some three leaves \(l_1, l_2, l_3\) of \(S\).

Since all leaves are twins, from Lemma 12 we know that we have to try only one of them. Therefore branching on \(⟨l \mid s⟩\) is correct.

Observation 23. Assume that Rules (R0) – (R10) are not applicable. Then there are no stars in \(G[V_2]\).

Proof. For contradiction assume that Rules (R0) – (R10) are not applicable and there is a star \(S\) in \(G[V_2]\). By Lemmata 7, 14 and 16 it has at least 4 vertices. Then, by Lemma 22, Rule (R10) is applicable.

3.9 Dealing with stars with a triangle in \(G[V_2]\)

Lemma 24. Assume that Rules (R0) – (R10) are not applicable. Suppose that there is a connected component \(S^{△}\) of \(G[V_2]\) which is isomorphic to a star with a triangle with at least 4 vertices. Then there is exactly one red vertex \(w\) connected to \(S^{△}\) and \(w\) is connected to all the leaves of \(S^{△}\) and at least one of its triangle vertices.
Proof. Assume that Rules [R0]–[R10] are not applicable and there is a star with a triangle $S^\triangle$ in $G[V_2]$ with at least 4 vertices.

If there is no $P_5$ that uses vertices from $S^\triangle$, then Rule (R1) is applicable on $S^\triangle$. Hence there are red vertices connected to $S^\triangle$. If there is no red vertex connected to any of the triangle vertices of $S^\triangle$, then Rule (R2) applies to both the triangle vertices.

So suppose that there is a red vertex connected to some of the triangle vertices, label it $w$. If there is a leaf $l$ of $S^\triangle$ not connected to a red vertex, then Rule (R2) applies to $l$. If the center or some leaf is connected to a different red vertex than $w$, then there is a $P_5$ that uses at least two red vertices and Rule (R3) applies. Hence $w$ is the red vertex connected to all the leaves and if there is a red vertex connected to the center, then it is also $w$. It follows, that there is no red different than $w$ connected to the triangle vertices of $S^\triangle$, as otherwise there is a $P_5$ that uses at least two red vertices and Rule (R3) applies. Therefore, $w$ is the only red vertex connected to $S^\triangle$.

**Lemma 25.** Assume that Rules [R0]–[R10] are not applicable. Let $S^\triangle$ be a star with a triangle in $G[V_2]$, let $s$ be its a center and $t_1, t_2$ its triangle vertices. Let $w$ be a red vertex connected to $S^\triangle$ such that $t_1$ is connected to $w$ and $t_2$ is not connected to $w$ (see Figure 7). If $F$ is a solution that contains $t_2$, then at least one of the following holds:

1. $t_1 \in F$,
2. $F' = (F \setminus \{ t_2 \}) \cup \{ t_1 \}$ is a solution.

**Proof.** If there is no $P_5$ containing $t_2$, then (2) trivially holds. Suppose that every $P_5$ that contains $t_2$ also contains $t_1$, then again (2) trivially holds. So assume that there is a $P_5$ labeled $P$ that contains $t_2$ but does not contain $t_1$. If for each such $P$ there is some vertex $x$ such that $x \neq t_2$ and $x \in F$, then (2) holds, since $t_2$ is not needed in the solution. Finally assume that $V(P) \cap F = \{ t_2 \}$, then, since $P$ does not contain $t_1$, $P$ must start at $t_2$ and $P = (t_2, d, p_1, p_2, p_3)$. But then there also exists a path $P' = (t_1, s, p_2, p_3, p_4)$ and $P'$ is not hit, which is a contradiction with $F$ being a solution and (1) must hold.

**Branching rule (R11).** Let $S^\triangle$ be a star with a triangle in $G[V_2]$ with at least 4 vertices, let $s$ be its center, $t_1, t_2$ its triangle vertices and $L$ the set of leaves. Let $w$ be a red vertex connected to $S^\triangle$ such that $|\{ t_1, t_2 \} \cap N(w) | \geq 1$, assume that $w$ is connected to $t_1$ (see Figure 7). Then branch on $\langle t_1 \mid s \mid L \rangle$.

**Proof of correctness.** We have to delete something in $S^\triangle$ since there is a $P_5 = (w, t_1, t_2, s, l)$ for some leaf $l$ of $S^\triangle$. If we do not delete any vertex from $\{ s, t_1, t_2 \}$, then the only thing we can do is to delete each vertex in $L$.

So assume that we did not delete all vertices from $L$, label some remaining vertex from $L$ as $x$. If we do not delete anything in $\{ s, t_1, t_2 \}$, then we have to delete $s$, otherwise a path $(w, t_1, t_2, s, x)$ would remain.

Finally, if $\{ t_1, t_2 \} \subseteq N(w)$, then $t_1$ and $t_2$ are twins and from Lemma 12 we know that we have to try only one of them. If $t_2$ is not connected to $w$, then from Lemma 25 we see that deleting only $t_1$ is sufficient. Thus branching on $\langle t_1 \mid s \mid L \rangle$ is correct.

**Observation 26.** Assume that Rules [R0]–[R11] are not applicable. Then there are no stars with a triangle in $G[V_2]$.

**Proof.** For contradiction assume that Rules [R0]–[R11] are not applicable and there is a star with a triangle $S^\triangle$ in $G[V_2]$. By Observation 18 it has at least 4 vertices. Then, by Lemma 24 Rule (R11) is applicable.
3.10 Dealing with di-stars in $G[V_2]$

Lemma 27. Assume that Rules [R0]–[R11] are not applicable. Let $D$ be a connected component of $G[V_2]$ which is isomorphic to a di-star and let $V(D) = \{s, s'\} \cup L \cup L'$, $|L| \geq 1$, $|L'| \geq 1$, where $s, s'$ are the centers, and $L$ and $L'$ are the leaves connected to $s$ and $s'$, respectively. Then there is a red vertex $w$ such that every leaf $l \in L$ is connected to $w$ and a red vertex $w'$ (possibly $w = w'$) such that every leaf $l' \in L'$ is connected to $w'$. Furthermore,

- if $w \neq w'$ then neither $w$ nor $w'$ is connected to any of $s$ and $s'$, $w$ is not connected to $l'$ and $w'$ is not connected to $l$;
- there are no red vertices other than $w$ and $w'$ connected to $D$;
- if $|L| = |L'| = 1$, then both $w$ and $w'$ are also connected to some component of $G[V_2]$ other than $D$.

Proof. If there is an $l \in L$ such that no red vertex is connected to $l$, then Rule (R2) applies. If there were two leaves $l_1$ and $l_2$ in $L$ such that $l_1$ is connected to red vertex $w_1$ and $l_2$ is connected to red vertex $w_2$, then there is a $P_5 = (w_1, l_1, s, l_2, w_2)$ and Rule (R3) applies. Hence there is a $w$ such that every leaf $l \in L$ is connected to $w$, and, if $|L| \geq 2$, then no other red vertex is connected to leaves in $L$. A symmetric argument shows that there is a red vertex $w'$ (possibly $w = w'$) such that every leaf $l' \in L'$ is connected to $w'$, and, if $|L'| \geq 2$, then no other red vertex is connected to leaves in $L'$.

If $w \neq w'$ and $w'$ was connected to $l \in L$, then we have a $P_5 = (w, l, w', l', s')$ for some $l' \in L'$ and Rule (R3) applies. A symmetric argument shows that if $w \neq w'$, then $w$ is not connected to any $l' \in L'$. Moreover, the same argument shows that if $w = w'$, then no other red vertex is connected to $L \cup L'$.

If there is some red vertex $w_1 \neq w$ connected to $s$, then we have $P_5 = (w_1, s, s', l', w')$ for some $l' \in L'$ and Rule (R3) applies. If $w$ is connected to $s$ and $w \neq w'$, then we have a $P_5 = (w, l, s, w', l')$ for some $l \in L$ and $l' \in L'$ and Rule (R3) applies. Hence, only $w$ can be connected to $s$ and only in case $w = w'$. A symmetric argument shows that the same holds for $s'$. In particular, we already showed, that in case $w = w'$ there is no other red vertex connected to $D$.

Hence, if there is some red vertex $w_1$ other than $w$ and $w'$ connected to $D$, then $w \neq w'$ and $w_1$ is connected to some $l \in L$ or some $l' \in L'$. If $w_1$ is connected to both $l \in L$ and $l' \in L'$, then we have a $P_5 = (w, l, w_1, l', w')$ and Rule (R3) applies. If there is some $w_1 \neq w$ connected to some $l \in L$, then we have $|L| = 1$ and by Lemma 8 all red vertices connected to $l$ are connected to any other component of $G[V_2]$, in particular, by the previous arguments, they are only connected to $l$. Hence, then Rule (R2) applies to $l$ and $s$, as $s$ is contained in any $P_5$ containing $l$. Hence, there are no other vertices connected to $L$. The same argument shows that Rule (R2) applies if $w \neq w'$ and $w$ is not connected to any other component of $G[V_2]$. A symmetric argument shows that there are no other vertices connected to $L'$ and if $w \neq w'$, then $w'$ is also connected to some component of $G[V_2]$ other than $D$. Hence, there are no other red vertices connected to $D$.

Finally, assume that $|L| = |L'| = 1$, $w = w'$ and $w$ is not connected to any component of $G[V_2]$ other than $D$. Then all vertices of $D$ are contained in the same set of $P_5$'s and Rule (R2) applies.

Branching rule (R12). Let $D$ be a di-star in $G[V_2]$ and let $V(D) = \{s, s'\} \cup L \cup L'$, $|L| \geq 1$, $|L'| \geq 1$, where $s, s'$ are the centers, and $L$ and $L'$ are the leaves connected to $s$ and $s'$, respectively. Let $|L| \geq 2$ or $|L'| \geq 2$. If $|L| \geq 2$, then let $l_1 \in L$ (see Figure 8), otherwise let $l_1 \in L'$. Branch on $\langle l_1 \mid s \mid s' \rangle$.

Proof of correctness. Assume that $|L| \geq 2$, the other case is symmetric. By Lemma 27 there is a red vertex $w$ connected to all leaves in $L$, let $l_1, l_2 \in L$, $l_1 \neq l_2$ (see Figure 8). Moreover, there is no other red vertex connected to $L$. We have to delete something in $\{l_1, l_2, s, s'\}$ and since $l_1, l_2$ are twins, from Lemma 12 we know that we have to try only one of them, thus branching on $\langle l_1 \mid s \mid s' \rangle$ is correct.

Observation 28. Assume that Rules (R0)–(R12) are not applicable. Then every di-star in $G[V_2]$ is a $P_4$.

Proof. If there is a di-star which is not a $P_4$, then Rule (R12) applies.

Branching rule (R13). Let $P_4 = (l, s, s', l')$ be a connected component of $G[V_2]$. Assume that there is a red vertex $w$ connected to $l$, $l'$, and a component of $G[V_2]$ other than $D$, and not connected to any of $s, s'$. Let $x$ be the vertex $w$ connects to outside $D$ and let $y$ be a neighbor of $x$ in $G[V_2]$ (see Figure 9). Then branch on $\langle x \mid y \mid \{l, l'\} \rangle$.

Proof of correctness. If none of the vertices $x, y$ is deleted, then we have to delete at least two vertices in $\{l, s, s', l'\}$ and from Lemma 19 we know that we only have to try to delete $\{l, l'\}$. Therefore branching on $\langle x \mid y \mid \{l, l'\} \rangle$ is correct.
Branching rule (R14). Let $P_4 = (l, s, s', l')$ be a connected component of $G[V_2]$. Assume that there is a red vertex $w$ connected to $l, l'$, a component of $G[V_2]$ other than $D$, and at least one of $s, s'$. Let $x$ be the vertex $w$ connects to outside $D$ and let $y$ be a neighbor of $x$ in $G[V_2]$ (see Figure 10). Then branch on $\langle x \mid y \mid \{l, s'\} \mid \{s, l'\} \mid \{s, s'\} \rangle$.

Proof of correctness. If none of the vertices $x, y$ is deleted, then we have to delete at least two vertices in $D$. Assume that we want to delete only two vertices in $D$. Out of six possible pairs of vertices only $\{l, s'\}, \{s, l'\}, \{s, s'\}$ lead to a solution. Deleting more than two vertices in $D$ also deletes at least one of the pairs $\{l, s'\}, \{s, l'\}, \{s, s'\}$. Thus branching on $\langle x \mid y \mid \{l_1, s'\} \mid \{s, l'_2\} \mid \{s, s'\} \rangle$ is correct.

Lemma 29. Assume that Rules (R0) – (R14) are not applicable. Then every connected component of $G[V_2]$ is $P_4 = (l, s, s', l')$ and there are exactly two red vertices $w, w'$ connected to it. Vertex $w$ is connected to $l$ and to some other component of $G[V_2]$ but not to $s, s'$, and $l'$. Vertex $w'$ is connected to $l'$ and to some other component of $G[V_2]$ but not to $l, s, and s'$.

Proof. From Lemma 4 together with Observations 7, 14, 16, 18, 23, and 26 and Lemma 21 all components of $G[V_2]$ are di-stars. By Observation 28 all di-stars are actually $P_4$’s. Let $P_4 = (l, s, s', l')$ be one such component. By Lemma 27 there are at most two red vertices connected to it, $w$ connected to $l$ and $w'$ connected to $l'$. Moreover, both $w$ and $w'$ are also connected to some other component of $G[V_2]$. If $w = w'$ and $w$ is connected to neither $s$ nor $s'$, then Rule (R13) applies. If $w = w'$ and $w$ is connected to at least one of $s$ nor $s'$, then Rule (R14) applies. Hence, $w \neq w'$ and by Lemma 27 neither $w$ nor $w'$ is connected to any of $s$ and $s'$, $w$ is not connected to $l'$ and $w'$ is not connected to $l$.

Branching rule (R15). Let $P_4 = (l, s, s', l')$ be a connected component of $G[V_2]$. Let there be two red vertices $w, w'$ connected to leaves $l \in L$ and $l' \in L'$, respectively, and both $s, s'$ have degree exactly two in $G$ (see Figure 11). Then branch on $\langle l \mid l' \rangle$.

Proof of correctness. By Lemma 29 each connected component of $G[V_2]$ is a $P_4$ with two red vertices connected. Let $F$ be a solution. Label the di-star components of $G[V_2]$ as $D_1, D_2, \ldots, D_r$. Observe that $F$ deletes at least one vertex in each di-star component $D_i$. 
Firstly, we construct a directed graph $G'$ such that $V(G') = V_1$ and there is an edge $e_i = (x, y)$ in $G'$ if and only if $F$ deletes exactly one vertex in $D_i$ and the deleted vertex is either $s_i^y$ or $l_i^y$ where $l_i^y$ is a leaf $y$ connects to in $D_i$ and $s_i^y$ is the center of $D_i$ to which $l_i^y$ is connected.

We claim that each vertex in $G'$ has outdegree at most one. Indeed, for contradiction assume that vertex $w$ has outdegree at least two, which means that there are two di-star components $D_i, D_j$ connected to $w$ such that $F$ does not contain the leaves $w$ is connected to in $D_i, D_j$, let them be $l_i^w, l_j^w$ and the centers to which these leaves are connected, let them be $s_i^w, s_j^w$, respectively. But that implies a $P_5 = (s_i^w, l_i^w, w, l_j^w, s_j^w)$ in $G$ and $F$ would not be a solution, which is a contradiction.

Secondly, we construct a set $F'$ in the following way: (1) for each di-star component $D_i$ where $F$ deletes at least two vertices, add to $F'$ the two leaves of $D_i$ and (2) for each edge $e_j = (x, y)$ in $G'$ add to $F'$ a leaf connected to $y$ in $D_j$.

Finally, $F'$ is also a solution because in the di-star $D_i$ where $F$ deleted at least two vertices we know from Lemma 19 that it suffices to delete only the leaves of $D_i$ and we claim that in the graph $G \setminus F'$ there is no $P_5$. Indeed, for contradiction assume that there is a $P_5$ in $G \setminus F'$. But that could only happen if there was a vertex $w$ in $G'$ with outdegree at least two, which is a contradiction.

Therefore $F'$ is a solution that uses only leaves of the di-stars in $G$ and from construction of $G'$ and $F'$ we have that $|F'| \leq |F|$. Thus branching on $\langle l_1 \mid l_1' \rangle$ is correct.

Observation 30. If $G[V_2]$ is non-empty, then at least one of the at least one of Rules $(R0) - (R15)$ is applicable.

Proof. By Lemma 29 if none of Rules $(R0) - (R14)$ is applicable, then every connected component of $G[V_2]$ is a $P_5 = (l, s, s', l')$ with two red vertices $w, w'$ connected to leaves $l \in L$ and $l' \in L'$, respectively, and both $s, s'$ have degree exactly two in $G$. But then Rule $(R15)$ applies.

3.11 Final remarks

From Observation 30 we know that there is always at least one rule applicable. It remains to analyze the running time of the disjoint compression routine $\text{DISJOINT}_R$.

Theorem 31. The $\text{DISJOINT}_R$ procedure solves the 5-PVCWB problem in $O^*(3^k)$ time.

Proof. We use the technique of analysis of branching algorithms as described by Fomin and Kratsch [10].
Let $T(k)$ be the maximum number of leaves in any search tree of a problem instance with parameter $k$. We analyze each branching rule separately and finally use the worst-case bound on the number of leaves over all branching rules to bound the number of leaves in the search tree of the whole procedure.

Let $\langle X_1 \mid X_2 \mid \ldots \mid X_l \rangle$ be the branching rule to be analyzed. We have that $l \geq 2$ and $|X_i| \geq 1$. This implies the linear recurrence

$$T(k) \leq T(k - |X_1|) + T(k - |X_2|) + \cdots + T(k - |X_l|).$$

It is well known that the base solution of such linear recurrence is of the form $T(k) = \lambda^k$ where $\lambda$ is a complex root of the polynomial

$$\lambda^k - \lambda^{k-|X_1|} - \lambda^{k-|X_2|} - \cdots - \lambda^{k-|X_l|} = 0$$

and the worst-case bound on the number of leaves of the branching rule is given by the unique positive root of the polynomial. This positive root $\lambda$ is called a branching factor.

The worst-case upper bound of the number of leaves in the search tree of the whole procedure is the maximal branching factor among the branching factors of all the branching rules. Be advised that the branching factor does not necessarily correspond to the number of branching calls, e.g., Rule (R14) generates 5 branching calls, but 3 of them delete more than one vertex, which results in branching factor of 3 rather than 5. In our case, the worst-case branching factor is 3 (see Table 1 for the branching factors), therefore the upper bound of the number of leaves in the search tree is $O^*(3^k)$.

Now we have to upper bound the number of inner nodes in the search tree. We claim that each path from the root to some leaf of the search tree has at most $O(|V(G)|)$ vertices. Indeed, each rule removes at least one vertex from $G$. Therefore the upper bound of the number of inner nodes in the search tree is $O^*(3^k)$.

Since the running time of each rule (the work that is done in each node of the search tree) is polynomial in $|V(G)|$, we get that the worst-case running time of the whole procedure is $O^*(3^k)$.

To understand the key ideas behind iterative compression algorithms and how the DISJOINT_R routine is involved, we briefly describe the iterative compression algorithm (for in-depth description see Cygan et al. [5] pages 80–81)).

We start with an empty vertex set $V' = \emptyset$ and empty solution $F = \emptyset$ and work with the graph $G'[V']$. Surely, an empty set $F$ is a solution for a currently empty graph $G'[V']$. We add vertices $v \in V \setminus V'$ one by one to both $V'$ and $F$ until $V' = V$ and if at any time the solution becomes too large, i.e. if $|F| = k + 1$, then we start the compression routine.

The compression routine takes $F$ and goes through every partition of $F$ into two sets $X, Y$ such that $Y \neq \emptyset$. Here, $X$ is the part of $F$ that we want to keep in the solution and $Y$ is the part of $F$ that we want to replace with vertices from $V' \setminus F$. Since $X$ are vertices we already decided to keep in the solution, we remove them from $G'[V']$, i.e. we continue with $G' = G'[V'] \setminus X$. Now the problem is to find a solution $F'$ for $G'$ such that $|F'| \leq |Y| - 1$ and $F'$ is disjoint from $Y$. We consider this partition only if $G'[Y]$ is $P_3$-free. Indeed, we require that $F'$ is disjoint from $Y$ so we cannot have any $P_3$ paths in $G'[Y]$. To find this smaller disjoint solution $F'$ for $G'$ we use the disjoint compression routine which in our case is the DISJOINT_R procedure. The smaller solution for $G'[V']$ is then constructed as $\hat{F} = X \cup F'$ and it follows from construction of $\hat{F}$ that $|\hat{F}| \leq k$.

If after going through all partitions of $F$ we did not find a smaller solution for $G'[V']$, then we know that $F$ was optimal in size and signalize that there is no solution.

The complexity of the whole iterative compression algorithm is then computed as follows. The compression routine is called at most $|V(G)|$ times and the worst case running time of one run of the compression routine can be computed as

$$\sum_{X \subseteq F} O^*(3^k - |X|) = \sum_{i=0}^{k} \binom{k+1}{i} O^*(3^k-i) = O^*(4^k),$$

which finally gives us the following corollary.

**Corollary 32.** The iterative compression algorithm solves the 5-PVC problem and runs in $O^*(4^k)$ time.

4 Conclusion

We conclude this paper with a few open questions.

Firstly, we again kindly remind the reader of our recent work on generating efficient algorithms for $d$-PVC [3]. As the generated algorithms consist of thousands (and in some cases hunders of thousands) branching rules, we ask, whether there exist significantly simpler algorithms with comparable running times.
Table 1: Branching factors $\lambda$ of the branching rules.

| Rule | $\lambda$ | Rule | $\lambda$ | Rule | $\lambda$ |
|------|-----------|------|-----------|------|-----------|
| (R3) | 3         | (R8) | 3         | (R13)| 2.415     |
| (R4) | 3         | (R9) | 2.415     | (R14)| 3         |
| (R5) | 3         | (R10)| 2         | (R15)| 2         |
| (R6) | 3         | (R11)| 3         |      |           |
| (R7) | 3         | (R12)| 3         |      |           |

Secondly, as the $d$-Hitting SET algorithm of Fernau [7] gets closer to the running time of $O^*((d-1)^k)$ with increasing $d$, we ask, whether one can find a general $d$-PVC algorithm with running time $O^*((d-1)^k)$ or if it is possible to go below the $d-1$ base of the exponential.

Finally, in our recent work on kernels for $d$-PVC we give a kernel with $O(k^3)$ edges for 4-PVC and 5-PVC we have a kernel with $O(k^2)$ edges. As Dell and Melkebeek [6] have shown that for VERTEX COVER it is not possible to achieve a kernel with $O((k^2-\epsilon)^3)$ edges unless coNP is in NP/poly (which would imply a collapse of the polynomial hierarchy) and as their result extends to $d$-PVC, the best we can hope for is a kernel with $O(k^2)$ edges for $d$-PVC. And therefore we ask, can we bridge the gap between our $O(k^3)$ kernel towards the $O(k^2)$ kernel?

References

[1] Boštjan Brešar, František Kardoš, Ján Katrenič, and Gabriel Semanišin. Minimum k-path vertex cover. Discrete Applied Mathematics, 159(12):1189–1195, 2011. doi:10.1016/j.dam.2011.04.008

[2] Radovan Červený and Ondřej Suchý. Faster FPT algorithm for 5-path vertex cover. In Peter Rossmanith, Pinar Heggernes, and Joost-Pieter Katoen, editors, 44th International Symposium on Mathematical Foundations of Computer Science, MFCS 2019, August 26-30, 2019, Aachen, Germany, volume 138 of LIPIcs, pages 32:1–32:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. doi:10.4230/LIPIcs.MFCS.2019.32

[3] Radovan Červený and Ondřej Suchý. Generating faster algorithms for d-path vertex cover. CoRR, abs/2111.05896, 2021. URL: https://arxiv.org/abs/2111.05896, arXiv:2111.05896

[4] Jianer Chen, Iyad A. Kanj, and Ge Xia. Improved upper bounds for vertex cover. Theor. Comput. Sci., 411(40-42):3736–3756, 2010. doi:10.1016/j.tcs.2010.06.026

[5] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015. doi:10.1007/978-3-319-21275-3

[6] Holger Dell and Dieter van Melkebeek. Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses. In Leonard J. Schulman, editor, Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, Cambridge, Massachusetts, USA, 5-8 June 2010, pages 251–260. ACM, 2010. doi:10.1145/1806689.1806725

[7] Henning Fernau. Parameterized algorithms for d-hitting set. Int. J. Comput. Math., 87(14):3157–3174, 2010. URL: https://doi.org/10.1080/00207160903176868

[8] Fedor V. Fomin, Serge Gaspers, Dieter Kratsch, Mathieu Liedloff, and Saket Saurabh. Iterative compression and exact algorithms. Theor. Comput. Sci., 411(7-9):1045–1053, 2010. doi:10.1016/j.tcs.2009.11.012

[9] Fedor V. Fomin, Serge Gaspers, Daniel Lokshtanov, and Saket Saurabh. Exact algorithms via monotone local search. J. ACM, 66(2):8:1–8:23, 2019. doi:10.1145/3284176

[10] Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2010. doi:10.1007/978-3-642-16533-7

[11] Stefan Funke, André Nussner, and Sabine Storandt. On k-path covers and their applications. VLDB J., 25(1):103–123, 2016. doi:10.1007/s00778-015-0392-3

[12] Ján Katrenič. A faster FPT algorithm for 3-path vertex cover. Inf. Process. Lett., 116(4):273–278, 2016. doi:10.1016/j.ipl.2015.12.002

[13] John M. Lewis and Mihalis Yannakakis. The node-deletion problem for hereditary properties is NP-complete. J. Comput. Syst. Sci., 20(2):219–230, 1980. doi:10.1016/0022-0000(80)90060-4

15
[14] Marián Novotný. Design and analysis of a generalized canvas protocol. In Information Security Theory and Practices. Security and Privacy of Pervasive Systems and Smart Devices, 4th IFIP WG 11.2 International Workshop, WISTP 2010, Passau, Germany, April 12-14, 2010. Proceedings, pages 106–121, 2010. URL: https://doi.org/10.1007/978-3-642-12368-9_8

[15] Dekel Tsur. l-path vertex cover is easier than l-hitting set for small l. CoRR, abs/1906.10523, 2019. URL: http://arxiv.org/abs/1906.10523, arXiv:1906.10523.

[16] Dekel Tsur. Parameterized algorithm for 3-path vertex cover. Theoretical Computer Science, 783:1–8, 2019. URL: http://www.sciencedirect.com/science/article/pii/S0304397519301665, doi: https://doi.org/10.1016/j.tcs.2019.03.013

[17] Dekel Tsur. An $O^*(2.619^k)$ algorithm for 4-path vertex cover. Discret. Appl. Math., 291:1–14, 2021. URL: https://doi.org/10.1016/j.dam.2020.11.019

[18] Jianhua Tu. A fixed-parameter algorithm for the vertex cover $P_3$ problem. Inf. Process. Lett., 115(2):96–99, 2015. doi:10.1016/j.ipl.2014.06.018

[19] Jianhua Tu and Zemin Jin. An FPT algorithm for the vertex cover $P_4$ problem. Discrete Applied Mathematics, 200:186–190, 2016. doi:10.1016/j.dam.2015.06.032

[20] Mingyu Xiao and Shaowei Kou. Exact algorithms for the maximum dissociation set and minimum 3-path vertex cover problems. Theor. Comput. Sci., 657:86–97, 2017. doi:10.1016/j.tcs.2016.04.043

[21] Mingyu Xiao and Shaowei Kou. Kernelization and parameterized algorithms for 3-path vertex cover. In Theory and Applications of Models of Computation - 14th Annual Conference, TAMC 2017, Bern, Switzerland, April 20-22, 2017, Proceedings, pages 654–668, 2017. URL: https://doi.org/10.1007/978-3-319-55911-7_47, doi:10.1007/978-3-319-55911-7_47

[22] Mingyu Xiao and Hiroshi Nagamochi. Exact algorithms for maximum independent set. Inf. Comput., 255:126–146, 2017. doi:10.1016/j.ic.2017.06.001