The \( p \)-spectral radius of the Laplacian

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August 13, 2018

Abstract

The \( p \)-spectral radius of a graph \( G = (V,E) \) with adjacency matrix \( A \) is defined as
\[
\lambda(p)(G) = \max \{ x^T Ax : \|x\|_p = 1 \}.
\]
This parameter shows remarkable connections with graph invariants, and has been used to generalize some extremal problems. In this work, we extend this approach to the Laplacian matrix \( L \), and define the \( p \)-spectral radius of the Laplacian as
\[
\mu(p)(G) = \max \{ x^T Lx : \|x\|_p = 1 \}.
\]
We show that \( \mu(p)(G) \) relates to invariants such as maximum degree and size of a maximum cut. We also show properties of \( \mu(p)(G) \) as a function of \( p \), and a upper bound on \( \max_{G: |V(G)|=n} \mu(p)(G) \) in terms of \( n = |V| \) for \( p \geq 2 \), which is attained if \( n \) is even.

Keywords: Laplacian Matrix, \( p \)-spectral radius

1 Introduction and main results

Let \( G = (V,E) \) be a simple \( n \)-vertex graph at least one edge with adjacency matrix \( A \) and Laplacian matrix \( L \). We recall that \( L = D - A \), where \( D \) is the diagonal matrix of vertex degrees.

It is well known that obtaining the least and the largest eigenvalues (\( \lambda_1 \) and \( \lambda_n \), respectively) of a real symmetric matrix \( M \in \mathbb{R}^{n \times n} \) can be viewed as an optimization problem using the Rayleigh-Ritz Theorem [\( \Pi \) Theorem 4.2.2]:
\[
\lambda_1(M) = \min_{\|x\|=1} x^T M x \leq \frac{x^T M x}{x^T x} \leq \max_{\|x\|=1} x^T M x = \lambda_n,
\]
where \( x \in \mathbb{R}^n \). Using the fact that \( x^T A x = 2 \sum_{ij \in E} x_i x_j \), Keevash, Lenz and Mubayi [2] replaced the Euclidean norm \( \|x\| \) by the \( p \)-norm \( \|x\|_p \), where \( p \in [1, \infty] \), and defined the \( p \)-spectral radius \( \lambda(p)(G) \):
\[
\lambda_p(G) = \max_{\|x\|_p=1} 2 \sum_{ij \in E} x_i x_j.
\]
This parameter shows remarkable connections with some graph invariants. For instance, \( \lambda_1(G) \) is equal to the Lagrangian \( \mathcal{L}_G \) of \( G \), which was defined by Motzkin and Straus \[3\] and satisfies \( 2\mathcal{L}_G - 1 = 1/\omega(G) \), where \( \omega(G) \) is the clique number of \( G \). Obviously \( \lambda_2(G) \) is the usual spectral radius, and it can be shown that \( \lambda_\infty(G)/2 \) is equal to the number of edges of \( G \).

An interesting result involving this parameter is about \( K_r \)-free graphs, that is, graphs that do not contain a complete graph with \( r \) vertices as a subgraph. Turán \[6\] proved that, for all positive integers \( n \) and \( r \), the balanced complete \( r \)-partite graph, known as a Turán graph \( T_r(n) \), is the only graph with maximum number of edges among all \( K_{r+1} \)-free graphs of order \( n \). Kang and Nikiforov \[4\] proved that, for \( p \geq 1 \), the graph \( T_r(n) \) is also the only graph that maximizes \( \lambda(p)(G) \) over \( K_{r+1} \)-free graphs of order \( n \), thus generalizing Turán’s result (which is the case \( p = \infty \)). Other results were obtained and extended to hypergraphs \[5\].

This motivates us to extend this approach to the Laplacian matrix \( L \), replacing the Euclidean norm by the \( p \)-norm. As \( x^T L x = \sum_{ij \in E} (x_i - x_j)^2 \), we define the \( p \)-spectral radius of the Laplacian as follows:

**Definition 1.** Let \( G = (V,E) \). The \( p \)-spectral radius of the Laplacian matrix of \( G \) is given by

\[
\mu(p)(G) = \max_{\|x\|_p = 1} \sum_{ij \in E} (x_i - x_j)^2.
\]

According to Mohar \[7\], the Laplacian matrix is considered to be more natural than the adjacency matrix. It is a discrete analog of the Laplace operator, which is present in many important differential equations. The Kirchhoff Matrix-Tree theorem is a early example of the use of \( L \) in Graph Theory. The largest eigenvalue (spectral radius) of \( L \) has been associated, for example, with degree sequences of a graph \[8–11\]. The second smallest eigenvalue and its associated eigenvectors have also been studied since the seminal work by Fiedler \[12\], which has been used in graph partitioning and has led to an extensive literature in spectral clustering. For more information about this area, see the survey \[14\] and the references therein.

Therefore we hope that the definition of \( \mu(p) \) will shed some light on classical parameters of graph theory. In fact, we show that, in the same fashion as \( \lambda(p)(G) \), the parameter \( \mu(p)(G) \) relates to graph invariants, such as the maximum degree and the size of a maximum cut. We also show some properties of \( \mu(p)(G) \) as a function of \( p \). The main results are:

**Theorem 1.** Let \( G = (V,E) \) be a graph with at least one edge. Then

(a) \( \mu(1)(G) \) is equal to the maximum degree of \( G \);

(b) \( \mu(\infty)(G)/4 \) is equal to the size of a maximum cut of \( G \);

(c) The function \( f_G : [1, \infty) \to \mathbb{R} \) defined by \( f_G(p) = \mu(p)(G) \) is strictly increasing, continuous and converges when \( p \to \infty \);

It seems to be the case that, by varying \( p \), the vector \( x \) that achieves \( \mu(p)(G) \) defines a maximum cut of the graph under different restrictions. For instance, \( \mu(1)(G) \) leads to a maximum cut with the constraint that one of the classes is a singleton, while \( \mu(\infty)(G) \) gives a maximum cut with no additional constraint. A rigorous basis for this statement remains a question for further investigation.
From the computational complexity point of view, it is interesting to note that computing $\mu^{(1)}(G)$ is easy (can be done in linear time), while computing $\mu^{(\infty)}(G)$ is an NP-complete problem, it is equivalent to finding the size of a maximum cut of $G$. For $\lambda^{(p)}$, the opposite happens: finding $\lambda^{(1)}(G)$ is NP-complete (equivalent to finding the clique number of $G$), while $\lambda^{(\infty)}(G)$ can be found in linear time.

We also present an upper bound on $\mu^{(p)}(G)$ if $p \geq 2$, which is attained for even $n$.

**Theorem 2.** Let $G = (V, E)$ be a graph with $n = |V|$. Then for $p \geq 2$,

$$\mu^{(p)}(G) \leq n^{2-2/p}.$$

If $n$ is even, equality holds if and only if $G$ contains $K_{n/2,n/2}$ as subgraph.

Note that this means that, for even $n$, the value of $\mu^{(p)}(K_n)$ is the same as the value for the balanced complete bipartite graph with $n$ vertices. We conjecture that this holds for all $n$.

This paper is organized as follows. In the remainder of the section we introduce some notation. In sections 2 and 3 we prove Theorems 1 and 2, respectively. In section 4 we present some additional remarks, conjectures and questions for future research.

Before proving our results, we set the notation used throughout the paper. The objective function of the optimization problems is

$$F_G(x) = x^T L x = \sum_{ij \in E(G)} (x_i - x_j)^2.$$  

We may drop the subscript of $F_G$ if $G$ is clear from context. It can be readily seen that $F_{G'}(x) \leq F_G(x)$ for a subgraph $G'$ of $G$, and so $F_G(x) \leq F_{K_n}(x)$ for any $n$-vertex graph $G$. Furthermore, $F_G(x) = 0$ if $x$ is constant in each connected component of $G$.

Finally, given an $n$-vertex graph $G = (V, E)$ and a vector $x \in \mathbb{R}^n$, the vertex sets $P, N$ and $Z$ are those on which $x_i$ is positive, negative, or equal to zero, respectively. We write $d_i$ for the degree of vertex $i$, and $d_{ij}$ is the number of edges between vertices $i$ and $j$ ($0$ or $1$). The all-ones vector in $\mathbb{R}^n$ is $e$ and the $i$-th vector of the canonical basis of $\mathbb{R}^n$ is $e_i$.

## 2 Proof of Theorem 1

In this section, we prove Theorem 1 which relates $\mu^{(p)}(G)$ relates to graph invariants and gives properties of $\mu^{(p)}(G)$ as a function of $p$. Item (a) states that $\mu^{(1)}(G)$ is equal to the maximum degree of $G$. In order to prove it, we need two lemmas.

**Lemma 2.1.** Let $x \in \mathbb{R}^n$ such that $\|x\|_1 = 1$ and $F_G(x) = \mu^{(1)}(G)$. Then at most one entry of $x$ or $-x$ is positive.

**Proof.** Let $x$ be as above. Without loss of generality, suppose $a, b \in P$ and define $x'$ and $x''$ as

$$x'_k = \begin{cases} x_a + x_b & \text{if } k = a; \\ 0 & \text{if } k = b; \\ x_k & \text{otherwise.} \end{cases}$$

and

$$x''_k = \begin{cases} 0 & \text{if } k = a; \\ x_a + x_b & \text{if } k = b; \\ x_k & \text{otherwise.} \end{cases}$$
Consider the differences $\Delta' = F(x') - F(x)$ e $\Delta'' = F(x'') - F(x)$.

$$\Delta' = (d_a - d_{ab})(2x_a x_b + x_b^2) - 2x_b \sum_{a_j \in E, j \neq b} x_j - d_b x_b^2 + 2x_b \sum_{b_j \in E, j \neq a} x_j + 4d_{ab}x_a x_b$$

The expression for $\Delta''$ can be readily obtained switching the roles of $a$ and $b$. As $x_a, x_b > 0$ we can take

$$\frac{\Delta'}{x_b} + \frac{\Delta''}{x_a} = (d_a + d_b + d_{ab})(x_a + x_b) > 0,$$

so that at least one of the differences $\Delta'$ and $\Delta''$ is positive. This contradicts the maximality of $x$. □

So we can assume that $|P| = |N| = 1$.

**Lemma 2.2.** Let $x \in \mathbb{R}^n$ such that $\|x\|_1 = 1$, $P = \{a\}$, $N = \{b\}$ and $d_a \geq d_b$. Then $d_a = F(e_a) \geq F(x)$, with equality if and only if $d_a = d_b = d_{ab}$.

**Proof.** Note that $x_a^2 + x_b^2 < 1$, because $|x_a| + |x_b| = 1$. Then

$$F(x) = d_a x_a^2 + d_b x_b^2 + d_{ab}(1 - x_a^2 - x_b^2) \leq d_a (x_a^2 + x_b^2) + d_{ab}(1 - x_a^2 - x_b^2) \leq d_a = F(e_a).$$

The first and second inequalities become equalities if and only if $d_a = d_b$ and $d_a = d_{ab}$, respectively. □

So $\mu^{(1)}(G)$ is obtained for a vector $e_a$ for a vertex $a$ with maximum degree. That proves item (a) of Theorem 1. Note that the solutions are always of this form if the maximum degree is at least 2, because the equality situation of Lemma 2.2 is of interest only if the maximum degree is one. For instance, for $G = K_2$, any feasible vector attains the maximum.

Now we proceed to prove item (b), which states that $\mu^{(\infty)}(G)/4$ is equal to the size of a maximum cut of $G$. In this case, the problem is of the form

$$\mu^{(\infty)}(G) = \max_{\text{max } |x_i| = 1} \sum_{i \in E} (x_i - x_j)^2.$$

**Lemma 2.3.** Let $x \in \mathbb{R}^n$ such that $\max_i |x_i| = 1$ and $F_G(x) = \mu^{(\infty)}(G)$. Then $|x_i| = 1$, for all $i \in V$.

**Proof.** Let $x$ be as stated above. Suppose that there is $a \in V$ with $-1 < x_a < 1$. Define $x', x'' \in \mathbb{R}^n$ as

$$x'_i = \begin{cases} 1 & \text{if } i = a; \\ x_i & \text{otherwise.} \end{cases} \quad \text{and} \quad x''_i = \begin{cases} -1 & \text{if } i = a; \\ x_i & \text{otherwise.} \end{cases}$$

Consider the differences $\Delta' = F(x') - F(x)$ and $\Delta'' = F(x'') - F(x)$. Then

$$\Delta' = d_a (1 - x_a^2) - 2(1 - x_a) \sum_{a_j \in E} x_j$$

and

$$\Delta'' = d_a (1 - x_a^2) - 2(1 - x_a) \sum_{a_j \in E} x_j$$
and similarly
\[ \Delta'' = d_a (1 - x_a^2) + 2(1 + x_a) \sum_{a_j \in E} x_j, \]
and therefore
\[ \frac{\Delta'}{1 - x_a} + \frac{\Delta''}{1 + x_a} = 2d_a > 0. \]
So at least one of the differences \( \Delta' \) and \( \Delta'' \) is positive. This contradicts the maximality of \( x \).  

Now for a vector \( x \) in the form given by Lemma 2.3 let 
\( S = \{ i \in V : x_i = 1 \} \) and 
\( T = \{ i \in V : x_i = -1 \}. \) So
\[ F(x) = \sum_{i \in S, j \in T} (x_i - x_j)^2 = 4\text{cut}(S,T). \]
Then of course \( F_G(x) = \mu(\infty)(G) \) if \( \text{cut}(S,T) \) is a maximum cut. That proves item (a) of Theorem 1. Also, the maximum among graphs of order \( n \)
is
\[ \mu(\infty)(K_n) = \mu(\infty)(K_{[n/2],[n/2]}) = \begin{cases} n^2 & \text{if } n \text{ is even;} \\ n^2 - 1 & \text{if } n \text{ is odd.} \end{cases} \]
Finally we prove item (c), which shows properties of the function \( f_G : [1, \infty) \rightarrow \mathbb{R} \) defined by \( f_G(p) = \mu^{(p)}(G) \). Namely, the function is strictly increasing (Lemma 2.6), continuous (Lemma 2.7) and converges when \( p \rightarrow \infty \) (Lemma 2.8). First we state two technical lemmas that will be useful.

**Lemma 2.4.** Let \( q \geq p \geq 1 \). Then for \( x \in \mathbb{R}^n \),
\[ \|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q. \]
Furthermore, \( x^*_i = n^{-1/q}\|x\|_q \) holds for a nonzero vector \( x^* \) that attains the upper bound.

**Proof.** Without loss of generality, we can consider that \( x \) has positive entries and \( \|x\|_q = 1 \). The lower bound holds because the \( p \)-norm is decreasing on \( p \), and it is attained by \( e_i \). By applying the power mean inequality to the entries of \( nx \), we can see that the upper bound is attained if and only if all entries are \( n^{-1/q} \).

**Lemma 2.5.** Let \( G = (V,E) \) be a graph and \( x \in \mathbb{R}^n \) with \( \|x\|_p = 1 \) and \( p \geq 2 \). Then 
\[ F_G(x) \leq n^{1-2/p} \mu^{(2)}(G). \]

**Proof.** By Rayleigh-Ritz theorem, we have \( F_G(x) \leq \|x\|_p^2 \mu^{(2)}(G) \) for \( x \neq 0 \in \mathbb{R}^n \). Using Lemma 2.4 we obtain 
\[ F_G(x) \leq \|x^*\|_p^2 \mu^{(2)}(G) = n^{1-2/p} \mu(G). \]

The proof will be broken down in three lemmas, one for each result.

**Lemma 2.6.** For a graph \( G \) and \( p \geq 1 \), \( \mu^{(p)}(G) \) is strictly increasing in \( p \).
Proof. Let \( x \in \mathbb{R}^n \) such that \( \|x\|_p = 1 \) and \( F(x) = \mu^{(p)}(G) \), and \( p' > p > 1 \). Define \( x' := x/\|x\|_{p'} \). As \( \|x\|_{p'} \leq 1 \), we have

\[
\mu^{(p)}(G) \geq F(x') = \frac{1}{\|x\|_{p'}^2} F(x) \geq \mu^{(p)}(G).
\]  

(2.1) \{inc_quota\}

As \( G \) has at least one edge \( ij \), \( \mu^{(p)}(G) > 0 \); pick \( x \) such that \( x_i = -x_j = 2^{-1/p} \), and \( x_i = 0 \) otherwise. Equality holds in equation (2.1) if and only if \( x = e_i \) for some \( i \). We argue now that for \( p > 1 \), \( e_i \) never attains the maximum, so that \( \mu^{(p)}(G) \) is strictly increasing.

For \( p > 1 \), the stationarity conditions of the problem are \( Lx = \lambda \nabla_x (|x_1|^p + \cdots + |x_n|^p - 1) \).

Note that \( x \to |x|^p \) is differentiable for \( p > 1 \). The \( j \)-th equation is

\[
d_jx_j - \sum_{j \in E} x_k = \begin{cases} p|x_j|^{p-1} \text{sign}(x_j), & \text{if } x_j \neq 0; \\ 0, & \text{if } x_j = 0. \end{cases}
\]  

(2.2) \{lagrange_j\}

Without loss of generality, assume \( G \) has no isolated vertices (as they don’t contribute to the sum in \( F \)). Let \( i \in V \) and \( j \) a neighbor of \( i \). Taking \( x = e_i \), then \( x_k = 0 \) if \( k \neq i \); in particular, \( x_j = 0 \). Then the right hand side of (2.2) is 0, and the left hand side is \( d_jx_j - \sum_{j \in E} x_k = 0 - x_i = -1 \). Therefore, \( e_i \) doesn’t satisfy the optimality conditions of the problem, that is, for any \( i \in V \), \( F(e_i) < \mu^{(p)}(G) \) for \( p > 1 \).

With this last statement in mind, recall that, by the proof of item a of Theorem 1, \( \mu^{(1)}(G) = F(e_i) \) for \( i \) with maximum degree. Therefore, we conclude that \( \mu^{(1)}(G) < \mu^{(p)}(G) \) for \( p > 1 \). This completes the proof.

\( \square \)

Lemma 2.7. For any graph \( G \) and \( p \geq 1 \), the function \( p \to \mu^{(p)}(G) \) is continuous.

Proof. Let \( x' \in \mathbb{R}^n \) such that \( \|x'\|_{p'} = 1 \) and \( F(x') = \mu^{(p')}(G) \), and \( p' > p \geq 1 \). By Lemma 2.4 that \( \|x'\|_p \leq n^{1 - \frac{1}{p'}} \|x'\|_{p'} \). Define \( x := x'/\|x'\|_p \). Then

\[
\mu^{(p')}(G) = F(x') = \|x'\|^2_2 F(x) \leq n^{\frac{2}{p'}} \|x'\|_{p'} \mu^{(p)}(G) = n^{\frac{2}{p'}} \mu^{(p)}(G)
\]

By Lemma 2.6, we know that \( \mu^{(p')}(G) > \mu^{(p)}(G) > 0 \). We also know from spectral graph theory (check for example [15]) that \( \mu^{(2)}(G) \leq \mu^{(2)}(K_n) = n \). Combining this with Lemma 2.5, we have \( \mu^{(p)}(G) \leq n^{2-2/p} \) para \( p \geq 2 \); as \( \mu^{(p)}(G) \) is strictly increasing in \( p \) (Lemma 2.6), this bound holds for \( p \geq 1 \). So

\[
\mu^{(p')} - \mu^{(p)}(G) \leq n^{\frac{2}{p'} - \frac{2}{p}} \mu^{(p)}(G) - \mu^{(p)}(G) \\
\leq \left( n^{\frac{2}{p'} - \frac{2}{p}} - 1 \right) n^{2-2/p} \\
< \left( n^{2(p'-p) - 1} \right) n^2.
\]

So we have \( \mu^{(p')} - \mu^{(p)}(G) < \epsilon \) if \( p' - p < \frac{1}{2} \log_{n}(\epsilon/n^2 + 1) \).

\( \square \)

Lemma 2.8. For any graph \( G \),

\[
\lim_{p \to \infty} \mu^{(p)}(G) = \mu^{(\infty)}(G).
\]
Proof. For a given $p$, let $x$ such that $\|x\|_p = 1$ and $F(x) = \mu^{(p)}(G)$. By the proof of Lemma 2.6 we know that $x \neq e_i$, so $\max_i |x_i| < 1$. Define $x' := x/\max |x_i|$. We can choose $N = N(x') \in \mathbb{N}$ such that

$$\mu^{(p)}(G) = F(x) = (\max |x_i|)^2 F(x') > (\max |x_i|)^N \mu^{(\infty)}(G),$$

so that $0 < \mu^{(\infty)}(G) - \mu^{(p)}(G) < (1 - (\max |x_i|)^N) \mu^{(\infty)}(G)$. One can check that $\max |x_i| \geq n^{-1/p}$. The proof concludes noting that

$$0 < \mu^{(\infty)}(G) - \mu^{(p)}(G) < (1 - n^{-N/p}) \mu^{(\infty)}(G),$$

and $n^{-N/p} \rightarrow 1$ when $p \rightarrow \infty$. \hfill \Box

3 Proof of Theorem 2

In this section we prove Theorem 2 which establishes the upper bound $\mu^{(p)}(G) \leq n^{2-2/p}$ for $p \geq 2$, as well as a necessary and sufficient condition for equality. We denote $G = (S, T, E)$ a bipartite graph with vertex classes $S$ and $T$. First we state three auxiliary lemmas.

Lemma 3.1. Let $G = (S, T, E)$ be a bipartite graph, and $x \in \mathbb{R}^n$ such that $\|x\|_p = 1$ and $F(x) = \mu^{(p)}(G)$. Then for $x$ or $-x$ we have $P = S$ and $N = T$.

Proof. Let $x$ be as stated above. Note that we can freely invert the entry signs preserving feasibility. Without loss of generality, if we invert the signs of negative entries in $S$ and positive entries in $T$, we are replacing, in the sum of $F$, terms of the form $(|x_i| - |x_j|)^2$ by $(|x_i| + |x_j|)^2$, thus increasing $F$. \hfill \Box

Lemma 3.2. Let $G = (S, T, E)$ be a bipartite graph, and $x \in \mathbb{R}^n$ such that $\|x\|_p = 1$ and $F(x) = \mu^{(p)}(G)$. Then for $p \geq 2$, if $i$ and $j$ are in the same class, then $x_i = x_j$.

Proof. Suppose $x$ as stated above has entries with $i, j \in S(=P)$ without loss of generality, by Lemma 3.1 with $x_i \neq x_j$. So

$$F(x) = \sum_{j \in T} \sum_{i \in E} (x_i - x_j)^2.$$ 

Let $M_p$ denote the power mean of $\{x_i : i \in S\}$. We exchange each $x_i$ by $M_p$. One can check that feasibility is preserved. For fixed $j$, it is sufficient to check the variation of $\sum_i x_i^2 + 2 \sum_i x_i x_j$:

$$|S|M_p^2 + 2S|M_p x_j | |S|M_2^2 + 2S|M_1 x_j | = \sum_i x_i^2 + 2 \sum_i x_i x_j.$$

The inequality holds by the power mean inequality. So the exchange increases $F$, contradicting the maximality of $x$. \hfill \Box

This allows us to obtain a formula for complete bipartite graphs.
**Lemma 3.3.** Let \( G = (S, T, E) \) be a complete bipartite graph. For \( p \geq 2 \),
\[
\mu^{(p)}(G) = |S||T|(a + b)^2,
\]
where
\[
a = \left( |S| + |T| \right)^{1/p}, \quad b = \left( \frac{|S|}{|T|} \right)^{1/p} a.
\]

**Proof.** By Lemma 3.2, we can assume \( x_i = a \) for \( i \in S \) and \( x_i = -b \) for \( i \in T \). Then apply Lagrange method to the function \( g(a, b) = |S||T|(a + b)^2 \) constrained by \( h(a, b) = |S|a^p + |T|b^p = 1 \).

In the proof of the item (c) of Theorem 1, the balanced complete bipartite graph attains the maximum for \( \mu^{(\infty)} \) among graphs of order \( n \). The same holds for \( \mu^{(p)} \) if \( 2 \leq p < \infty \) if \( n \) is even.

**Proof of Theorem 2.** As \( \mu^{(2)}(K_n) = n \), the bound \( \mu^{(p)}(G) \leq n^{2-2/p} \) is a direct consequence of Lemma 2.5. By Lemma 3.3, one can check that \( \mu^{(p)}(K_{n/2,n/2}) = n^{2-2/p} \). Furthermore, if \( K_{n/2,n/2} \subseteq G \), the inequality is trivial, because \( F_G(x) \) won’t decrease if we add edges to \( G \).

Now let \( G \) and \( x \in \mathbb{R}^n \) such that \( F_G(x) = \mu^{(p)}(G) = n^{2-2/p} \). Note that \( |x_i| = |x_j|, \forall i, j \in V \); otherwise, as \( \mu^{(2)}(G) \leq \mu^{(2)}(K_n) = n \) and by Lemma 2.5, we would have \( F_G(x) \leq n^{2-2/p} \).

Also, \( K_{|P|,|N|} = (P, N, E') \) is a subgraph of \( G \); otherwise there would be \( a \in P \) and \( b \in N \) such that \( \{a, b\} \notin E(G) \) and \( F_G(\{a, b\}) \geq F_G(x) = n^{2-2/p} \), in contradiction with Lemma 2.5.

Therefore, \( F_{K_{|P|,|N|}}(x) = F_G(x) = n^{2-2/p} \), because the edges induced by \( P \) or \( N \) do not contribute to \( F_G(x) \). Observe that, by Lemma 3.3, \( |x_i| = |x_j| \) if and only if \( |P| = |N| \), therefore \( |P| = |N| = n/2 \).

Although we conjecture that the equality condition of Theorem 2 also holds for odd \( n \) (of course with a different quota given by 3.3), the reasoning used in the proof does not work in this case, because then the balanced complete bipartite graph does not attain the bound given by Lemma 2.5.

4 Concluding remarks

As already mentioned in the introduction, we seem to obtain maximum cuts under different restrictions in the graph by varying \( p \). That motivates the following broad question for further investigation:

**Question 4.1.** For \( p \geq 1 \), which relation possibly exists between \( \mu^{(p)}(G) \) and cuts (or other parameters) of \( G \)?

Also, we proved that computing \( \mu^{(1)}(G) \) can be done in linear time, while computing \( \mu^{(\infty)}(G) \) is an NP-complete problem. As finding the maximum degree of \( G \) can be trivially reduced in linear time to finding the size of a maximum cut of \( G \), it might be the case that, by increasing \( p \), we obtain a problem that is at least as hard. This motivates the following conjecture:
Conjecture 4.2. Let $q > p \geq 1$. The problem of finding $\mu^{(p)}(G)$ can be reduced to the problem of finding $\mu^{(q)}(G)$ in polynomial time.

There are other approaches that seek to generalize eigenvalues via the introduction of the $p$-norm. Amghibech [16] introduced a non-linear operator, which he called the $p$-Laplacian $\Delta_p$, that induces a functional of the form $\langle x, \Delta_p x \rangle = \sum_{ij \in E} |x_i - x_j|^p$ instead of the quadratic form of the Laplacian. This functional is unbounded for $p = \infty$ over the $p$-norm unit ball, and the case $p = 1$ cannot be treated directly. However, the eigenvalue formulation used allows to explore eigenvalues other than the largest and the smallest: $\lambda$ is said to be a $p$-eigenvalue of $M$ if there is a vector $v \in \mathbb{R}^n$ such that

$$(\Delta_p x)_i = \lambda \phi_p(v_i), \quad \phi_p(x) = |x|^{p-1}\text{sign}(x).$$

The vector $v$ is called a $p$-eigenvector of $M$ associated to $\lambda$. Using this formulation, Bühler and Hein [17] proved that the cut obtained by “thresholding” (partitioning according to entries greater than a certain constant) an eigenvector associated to the second smallest eigenvalue of $\Delta_p$ converges to the optimal Cheeger cut when $p \to 1$; in practice, the case $p = 2$ is used to obtain an approximation to this cut [13][14].

It may be possible to adapt this method to the standard Laplacian operator, which would allow us to explore a $p$-norm version of the second smallest eigenvalue of $L$, which could potentially also lead to different cuts according to the value of $p$.

Acknowledgments This work was partially supported by CAPES Grant PROBRAI 408/13 - Brazil and DAAD PROBRAI Grant 56267227 - Germany.

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