SEGREGATED VECTOR SOLUTIONS FOR LINEARLY COUPLED NONLINEAR SCHRÖDINGER SYSTEMS

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ABSTRACT. We consider the following system linearly coupled by nonlinear Schrödinger equations in \( \mathbb{R}^3 \)
\[
\begin{cases}
-\Delta u_j + u_j = u_j^3 - \varepsilon \sum_{i \neq j}^N u_i, & x \in \mathbb{R}^3, \\
\quad u_j \in H^1(\mathbb{R}^3), & j = 1, \ldots, N,
\end{cases}
\]
where \( \varepsilon \in \mathbb{R} \) is a coupling constant. This type of system arises in particular in models in nonlinear \( N \)-core fiber.

We examine the effect of the linear coupling to the solution structure. When \( N = 2, 3 \), for any prescribed integer \( \ell \geq 2 \), we construct a non-radial vector solutions of segregated type, with two components having exactly \( \ell \) positive bumps for \( \varepsilon > 0 \) sufficiently small.

We also give an explicit description on the characteristic features of the vector solutions.

1. Introduction

We consider the following nonlinear Schrödinger systems which are linearly coupled by \( N \) equations
\[
\begin{cases}
-\Delta u_j + u_j = u_j^3 - \varepsilon \sum_{i \neq j}^N u_i, & x \in \mathbb{R}^3, \\
\quad u_j \in H^1(\mathbb{R}^3), & j = 1, \ldots, N,
\end{cases}
\]
(1.1)

where \( \varepsilon \in \mathbb{R} \). These systems arise when one considers stationary pulselike (standing wave) solutions of the time-dependent \( N \)-coupled Schrödinger systems of the form
\[
\begin{cases}
-i \frac{\partial}{\partial t} \Phi_j = \Delta \Phi_j - \Phi_j + |\Phi_j|^2 \Phi_j - \varepsilon \sum_{i \neq j}^N \Phi_i, & \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\
\quad \Phi_j = \Phi_j(x, t) \in \mathbb{C}, t > 0, & j = 1, \ldots, N.
\end{cases}
\]
(1.2)

This type of system arises in nonlinear optics. For example, the propagation of optical pulses in nonlinear \( N \)-core directional coupler can be described by \( N \) linearly coupled nonlinear Schrödinger equations. Here \( \Phi_j \) \((j = 1, \ldots, N)\) are envelope functions and \( \varepsilon \), which is the normalized coupling coefficient between the cores, is equal to the linear coupling coefficient times the dispersion length. The sign of \( \varepsilon \) determines whether the interactions of fiber couplers are repulsive or attractive. In the attractive case the components of a vector solution tend to go along with each other leading to synchronization, and in the repulsive case the components tend to segregate with each other leading to phase separations. These phenomena have been documented in numeric simulations (e.g., \[1\] and references therein).
Nonlinear Schrödinger equations have been broadly investigated in many aspects, such as existence of solitary waves, concentration and multi-bump phenomena for semiclassical states (see e.g. [10], [24] and the references therein). The study on system of Schrödinger equations began quite recently. Mathematical work on systems with the nonlinearly coupling terms (e.g. the term $\sum_{i \neq j} u_i$ in (1.1) being replaced by $u_j \sum_{i \neq j} u_i^2$) has been studied extensively in recent years, for example, [6, 8, 9, 11, 12, 14, 16, 17, 20, 21, 22, 23] and references therein, where phase separation or synchronization has been proved in several cases.

However, for the linearly coupled system (1.1), as far as the authors know, it seems that there are few results. In [5], when $N = 2$, solitons of linearly coupled systems of semilinear non-autonomous equations were studied by using concentration compactness principle, and existence of both positive ground and bound states was proved under some decay assumptions on the potentials at infinity. In [2], this type of non-autonomous systems was also considered by using a perturbation argument. Concerning on autonomous systems, we also mention some results. If $N = 2$ and the dimension is one, for $\varepsilon < 0$, (1.1) has in addition to the semi-trivial solutions $(\pm U, 0)$, $(0, \pm U)$, two types of soliton like solutions, given by

$$(U_{1+\varepsilon}, U_{1+\varepsilon}), (-U_{1+\varepsilon}, -U_{1+\varepsilon}), \text{ for } -1 \leq \varepsilon \leq 0, \text{ (symmetric states)},$$

$$(U_{1-\varepsilon}, -U_{1-\varepsilon}), (-U_{1-\varepsilon}, U_{1-\varepsilon}), \text{ for } \varepsilon \leq 0, \text{ (anti-symmetric states)},$$

where, for $\lambda > 0$, $U_\lambda$ is the unique solution of

$$\begin{cases}
-u'' + \lambda u = u^3, & u > 0, \text{ in } \mathbb{R}, \\
u(0) = \max_{x \in \mathbb{R}} u(x), & u(x) \in H^1(\mathbb{R}).
\end{cases}$$

By using numerical methods, a bifurcation diagram is reported in [1] where it is indicated that for $\varepsilon \in (-1, 0)$, there exists a family of new solutions for (1.1), bifurcating from the branch of the anti-symmetric state at $\varepsilon = -1$. This kind of results was rigorously verified in [3] for small value of the parameter $\varepsilon < 0$. More precisely, in [3], it was proved that a solution with one 2-bump component having bumps located near $\pm |\ln(-\varepsilon)|$, while the other component having one negative peaks exists. This type of results was generalized recently in an interesting paper [4] to two and three dimensional cases. In [4], it was proved that if $P$ denotes a regular polytope centered at the origin of $\mathbb{R}^d$ ($d = 2, 3$) such that its side is larger than the radius of the circumscribed circle or sphere, then there exists a solution with one multi-bump component having bumps located near the vertices of $\ln(-\varepsilon)P$, while the other component has one negative peak as $\varepsilon \to 0^-$. So in [4], the first component of the solutions has more than one bump, while the second component is negative and has only one bump. We emphasize here that the solutions obtained in [4] bifurcate also from the branch of anti-symmetric state at $\varepsilon = 0$. Furthermore, as pointed out in [3], for $\varepsilon < 0$, vector solutions with one component being multi-bump do not exist near symmetric states, but only near the anti-symmetric ones. Hence, an interesting problem is: can we find solutions bifurcating from the symmetric state if $\varepsilon > 0$? In this paper, our main purpose is to prove that, for any prescribed integer $\ell \geq 2$, (1.1) has new solutions, different from the previous
ones, with the feature that two components have exactly $\ell$ positive bumps when $\varepsilon > 0$ is sufficiently small.

To state our main results, we introduce some notations.

The Sobolev space $H^1(\mathbb{R}^3)$ is endowed with the standard norm

$$\|u\|_{\mathbb{R}^3} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \right)^{\frac{1}{2}}.$$

Denote by $U$ the unique solution of the following problem

$$\begin{cases}
-\Delta u + u = u^3, & u > 0, \text{ in } \mathbb{R}^3, \\
u(0) = \max_{x \in \mathbb{R}^3} u(x), & u(x) \in H^1(\mathbb{R}^3).
\end{cases} \tag{1.3}$$

It is well known that $U(x) = U(|x|)$ satisfies

$$\lim_{|x| \to +\infty} |x| e^{\varepsilon|x|} U = A > 0, \text{ and } \lim_{|x| \to +\infty} \frac{U'(|x|)}{U(x)} = -1.$$ 

Moreover, $U(x)$ is non-degenerate, that is,

$$\text{Kernel}(\mathbb{L}) = \text{span}\left\{ \frac{\partial U(x)}{\partial x_i} : i = 1, 2, 3 \right\},$$

where $\mathbb{L}$ is the linearized operator

$$\mathbb{L} : H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad \mathbb{L}(u) = : \Delta u - u + 3U^2u.$$

Let

$$x^j = \left(r \cos \frac{2(j-1)\pi}{\ell}, r \sin \frac{2(j-1)\pi}{\ell}, 0 \right) : = (x'^j, 0), \; j = 1, \ldots, \ell, \tag{1.4}$$

and

$$y^j = \left(\rho \cos \frac{(2j-1)\pi}{\ell}, \rho \sin \frac{(2j-1)\pi}{\ell}, 0 \right) : = (y'^j, 0), \; j = 1, \ldots, \ell, \tag{1.5}$$

where $r, \rho \in [r_0, \ln \varepsilon], \; r_1 \ln \varepsilon]$ for some $r_1 > r_0 > 0$.

In this paper, for any function $W : \mathbb{R}^3 \to \mathbb{R}$ and $\xi \in \mathbb{R}^3$, we define $W_\xi = W(x - \xi)$.

We first consider the following problem linearly coupled by two nonlinear Schrödinger equations

$$\begin{cases}
-\Delta u + u = u^3 - \varepsilon v, & x \in \mathbb{R}^3, \\
-\Delta v + v = v^3 - \varepsilon u, & x \in \mathbb{R}^3.
\end{cases} \tag{1.6}$$

The main result can be stated as follows

**Theorem 1.1.** For any integer $\ell \geq 2$, there exists $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0)$, problem (1.6) has a solution $(u, v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ satisfying

$$u^\varepsilon \sim \sum_{j=1}^{\ell} U_{x^j}, \; v^\varepsilon \sim \sum_{j=1}^{\ell} U_{y^j},$$
where \( x^j_\varepsilon \) and \( y^j_\varepsilon \) are respectively defined by (1.4) and (1.5) with
\[
\begin{align*}
  r_\varepsilon &= \frac{2 \sin \frac{\pi}{\ell}}{2 \sin \frac{\pi}{\ell} - \sqrt{2(1 - \cos \frac{\pi}{\ell})}} |\ln \varepsilon| + o(|\ln \varepsilon|), \\
  \rho_\varepsilon &= \frac{2 \sin \frac{\pi}{\ell}}{2 \sin \frac{\pi}{\ell} - \sqrt{2(1 - \cos \frac{\pi}{\ell})}} |\ln \varepsilon| + o(|\ln \varepsilon|).
\end{align*}
\]
Moreover, as \( \varepsilon \to 0^+ \),
\[
\|u^\varepsilon(\cdot) - v^\varepsilon(T_\ell \cdot)\|_{H^1} + \|u^\varepsilon(\cdot) - v^\varepsilon(T_\ell \cdot)\|_{L^\infty} \to 0.
\]
Here \( T_\ell \in SO(3) \) is the rotation on the \((x_1, x_2)\) plane of \( \frac{\pi}{\ell} \).

Theorem 1.2 says that \( |x^j_\varepsilon - y^j_\varepsilon|/|\ln \varepsilon| \to a_{i,j} > 0 \) \((i, j) = 1, \cdots, \ell\) as \( \varepsilon \to 0 \). Hence Theorem 1.2 gives segregated types of solutions for system (1.6) with the essential support of the two components being segregated for \( \varepsilon \) sufficiently small.

We also construct segregated vector solutions for the following three coupled systems, which arise when one considers the propagation of pulses in a 3-core couplers with circular symmetry:
\[
\begin{align*}
-\Delta u + u &= u^3 - \varepsilon(v + \omega), & x \in \mathbb{R}^3, \\
-\Delta v + v &= v^3 - \varepsilon(u + \omega), & x \in \mathbb{R}^3, \\
-\Delta \omega + \omega &= \omega^3 - \varepsilon(u + v), & x \in \mathbb{R}^3.
\end{align*}
\]

**Theorem 1.2.** For any integer \( \ell \geq 2 \), there exists \( \varepsilon_0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \), problem (1.7) has a solution \( (u^\varepsilon, v^\varepsilon, \omega^\varepsilon) \in (H^1(\mathbb{R}^3))^3 \) satisfying
\[
\begin{align*}
  u^\varepsilon &\sim \sum_{j=1}^\ell U_{x^j_\varepsilon}, & v^\varepsilon &\sim \sum_{j=1}^\ell U_{y^j_\varepsilon}, & \omega^\varepsilon &\sim U,
\end{align*}
\]
where \( x^j_\varepsilon \) and \( y^j_\varepsilon \) are the same as those of Theorem 1.1 if \( \ell > 2 \), but for \( \ell = 2 \)
\[
  r_\varepsilon = |\ln \varepsilon| + o(|\ln \varepsilon|), & \rho_\varepsilon = |\ln \varepsilon| + o(|\ln \varepsilon|).
\]
Moreover, as \( \varepsilon \to 0^+ \),
\[
\|u^\varepsilon(\cdot) - v^\varepsilon(T_\ell \cdot)\|_{H^1} + \|u^\varepsilon(\cdot) - v^\varepsilon(T_\ell \cdot)\|_{L^\infty} \to 0.
\]

**Remark 1.3.** The segregation nature of these solutions are demonstrated from the \( L^\infty \) estimates in the theorems and will be more clear in Propositions 3.1 and 4.1 stated later after we find a good approximate solution and fix the notations. Roughly speaking, as \( \varepsilon \to 0 \), the segregated solutions may have a large number of bumps near infinity while the locations of the bumps for \( u \) and \( v \) have an angular shift.

**Remark 1.4.** In [4], to guarantee the existence of the solutions, the side of the polytope should be greater than the radius, which implies that the number of the solutions cannot be very large (at least in two dimensional case). In our results, the number of the bumps can be very large, and the energy of the solutions can become so large as we expected. Moreover, all the bumps are positive, which implies that these solutions bifurcate from the symmetric state at \( \varepsilon = 0 \). Hence our results are in striking contrast with those of [4].
Remark 1.5. Our argument also works well for the following more general problems in various dimensional case:

\[
\begin{cases}
-\Delta u_j + u_j = |u_j|^{p-2}u_j - \varepsilon \sum_{i \neq j}^N u_i, & x \in \mathbb{R}^d, \\
\end{cases}
\]

Here \( N = 2, 3, \) \( d > 1, \) and \( 2 < p < 2^*, \) where \( 2^* = \frac{2d}{d-2} \) if \( d \geq 3 \) and \( 2^* = +\infty \) if \( d = 2. \) We point out that our results are most likely wrong for \( d = 1, \) which is verified by the numerical computation in [1].

To prove the main results, we will employ the well-known Lyapunov-Schmidt reduction (see, e.g., [19]) to glue the functions \( U_{\epsilon j} \) (or \( U_{\mu j} \)) \( (j = 1, \cdots, \ell). \) In performing this technique, to find critical points of the reduced functionals, a basic requirement is that the error terms of the functionals, which come from the finite dimensional reduction, should be of higher order small data of the main terms in the reduced functionals. However, in our linearly coupled systems, different from the nonlinearly coupled ones (see, e.g., [13] and [18]), if we choose \((U, U)\) as an approximate solution, the error terms from the linear coupling dominate the main terms (which are generated from the interaction between the neighbor bumps) of the reduced functionals. To overcome this difficulty, we should modify another approximate solution \((U, 0)\). This idea is essentially from [4], where an approximate solution \((U_\epsilon, V_\epsilon)\) bifurcates from \((U, 0).\) However, comparing with [4], we encounter two more problems. Firstly, we need a new approximate solution and a precise estimate on it. To this end, we will make a modification on \((U, 0)\) carefully by using the reduction technique (see section 2). This procedure provides us a more accurate approximate solution with required estimate. Secondly, after performing a second reduction, we need to solve a two-dimensional critical point problem, which requires us to choose a very delicate domain and make a precise analysis on the reduced functionals. So, we need a very accurate estimate on the energy of the reduced functional, which also needs the help of the approximate solution. Hence, here we will perform the reduction twice and deal with more complicated reduced functionals.

To find vector solutions with two components having the prescribed number of bumps, we will employ the idea proposed by Wei and Yan in [24], where infinitely many positive solutions were constructed for single Schrödinger equations. This idea is also effective in finding infinitely many non-radial positive solutions for semilinear elliptic problems with critical or super-critical Sobolev growth (see, for example, [25, 26, 27]) and Schrödinger systems with nonlinear coupling (see, for example, [18]).

This paper is organized as follows. In section 2, we will perform a reduction argument for the first time and modify the vector function \((U, 0)\) so that we can get an accurate approximate solution and a precise estimate on it. In section 3, using the approximate solution, we will formulate a more precise version of the main results which give more precise descriptions about the segregated character of the solutions. We will also carry out the reduction for the second time to a finite two-dimensional setting and prove Theorem 1.1.
Schrödinger equations will be briefly discussed in section 4 by using our framework of methods. We conclude with the energy expansion in the appendix.

2. An approximate solution

In this section, to look for a proper approximate vector solution, we need to modify \((U, 0)\). Let \(H^1_r(\mathbb{R}^3)\) and \(L^2_r(\mathbb{R}^3)\) denote the corresponding spaces of radial functions. For \((u, v) \in H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3)\), we define \(\| (u, v) \| = \| u \|_{H^1_r(\mathbb{R}^3)} + \| v \|_{H^1_r(\mathbb{R}^3)}\).

Solving in \(H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3)\) the equations

\[
\begin{cases}
-\Delta u_1 + u_1 - 3U^2 u_1 = 0, & x \in \mathbb{R}^3, \\
-\Delta \tilde{v}_1 + \tilde{v}_1 = -U, & x \in \mathbb{R}^3,
\end{cases}
\]

we get \(u_1 = 0\) and \(\tilde{v}_1 < 0\). Let \(c(x) \in H^1_r(\mathbb{R}^3)\) satisfy

\[-\Delta c(x) + c(x) = \tilde{v}_1^3,\]

then \(v_1 = \tilde{v}_1 + \varepsilon^2 c(x)\) solves

\[-\Delta v_1 + v_1 = -U + \varepsilon^2 \tilde{v}_1^3.\]

Now for \(k \geq 2\), by the Fredholm Alternative Theorem we can define \((u_k, v_k) \in H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3)\) by solving

\[
\begin{cases}
-\Delta u_k + u_k - 3U^2 u_k = -k u_{k-1}, & x \in \mathbb{R}^3, \\
-\Delta v_k + v_k = -k u_{k-1}, & x \in \mathbb{R}^3.
\end{cases}
\]

(2.1)

We can also see that \(v_2 = 0\).

Remark 2.1. Here we execute the second modification by defining \(v_1\) so that the norm of the error terms in \(H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\) can be dominated by \(C\varepsilon^4\) (see Proposition 2.2 later).

We want to find suitable \((w(x), h(x)) \in H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3)\) such that

\[
(U_\varepsilon, v_\varepsilon) := \left( U + \sum_{i=1}^4 \frac{\varepsilon^i}{i!} u_i + \varepsilon^4 w, \sum_{i=1}^4 \frac{\varepsilon^i}{i!} v_i + \varepsilon^4 h \right)
\]

(2.2)

solves problem (1.6).

Inserting (2.2) into (1.6) and employing (2.1), we find

\[
\begin{cases}
-\Delta w + w - 3U^2 w = \frac{H_\varepsilon(u_2, u_3, u_4, v_4, U)}{\varepsilon^4} + l_\varepsilon(h, w) + \frac{R_\varepsilon(\varepsilon^4 w)}{\varepsilon^4}, \\
-\Delta h + h = \frac{\tilde{H}_\varepsilon(v_1, v_3, v_4, u_4)}{\varepsilon^4} + \tilde{l}_\varepsilon(h, w) + \frac{\tilde{R}_\varepsilon(\varepsilon^4 h)}{\varepsilon^4},
\end{cases}
\]

(2.3)

where

\[
H_\varepsilon(u_2, u_3, u_4, v_4, U) = \left( U + \sum_{i=2}^4 \frac{\varepsilon^i}{i!} u_i \right)^3 - U^3 - 3U^2 \sum_{i=2}^4 \frac{\varepsilon^i}{i!} u_i - \frac{\varepsilon^5}{4!} v_4.
\]
\[ l_\varepsilon(w, h) = 3 \left( U + \sum_{i=2}^{4} \frac{\varepsilon^i}{i!} u_i \right)^2 - U^2 \right) w - \varepsilon h, \]

\[ R_\varepsilon(\varepsilon^4w) = 3 \left( U + \sum_{i=2}^{4} \frac{\varepsilon^i}{i!} u_i \right) (\varepsilon^4w)^2 + (\varepsilon^4w)^3, \]

\[ \bar{H}_\varepsilon(v_1, v_3, v_4, u_4) = \left( \sum_{i=1}^{4} \frac{\varepsilon^i}{i!} v_i \right)^3 - \varepsilon^3 v_1^3 - \frac{\varepsilon^5}{4!} u_4, \]

\[ \bar{l}_\varepsilon = -\varepsilon w + \left( \sum_{i=1}^{4} \frac{\varepsilon^i}{i!} v_i \right)^2 h, \]

\[ \bar{R}_\varepsilon(\varepsilon^4h) = 3 \left( \sum_{i=1}^{4} \frac{\varepsilon^i}{i!} v_i \right) (\varepsilon^4h)^2 + (\varepsilon^4h)^3. \]

Direct calculation yields that
\[
\left| \frac{H_\varepsilon(u_2, u_3, u_4, v_4, U)}{\varepsilon^4} \right| \leq C,
\]

\[
\frac{\bar{H}_\varepsilon(v_1, v_3, v_4, u_4)}{\varepsilon^4} = \left( \sum_{i=1}^{4} \frac{\varepsilon^i}{i!} v_i \right)^3 - \varepsilon^3 v_1^3 - \frac{\varepsilon^5}{4!} u_4 \bigg/ \varepsilon^4,
\]

\[
= \left( \sum_{i=1}^{4} \frac{\varepsilon^i}{i!} v_i \right)^3 - \varepsilon^3 v_1^3 + \varepsilon^3 (v_1^3 \bar{v}_1^3) - \frac{\varepsilon^5}{4!} u_4 \bigg/ \varepsilon^4
\]

\[
= O(\varepsilon),
\]

where we have used the fact \( v_2 = 0 \) and \(|v_1 - \bar{v}_1| = O(\varepsilon^2)\).

Since the kernel of operator
\[
\mathbb{L} \left( \begin{array} {c} w \\ h \end{array} \right) = \left( \begin{array} {c} -\Delta w + w - 3U^2w \\ -\Delta h + h \end{array} \right) : H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3) \rightarrow L^2_r(\mathbb{R}^3) \times L^2_r(\mathbb{R}^3)
\]

is \( \{0, 0\} \) in \( H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3) \), we know that the operator \( \mathbb{L} \) has bounded inverse in \( H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3) \).

Define
\[
\left( \begin{array} {c} \bar{w} \\ \bar{h} \end{array} \right) = \mathbb{L}^{-1} \left( \begin{array} {c} \frac{H_\varepsilon(u_2, u_3, u_4, v_4, U)}{\varepsilon^4} + l_\varepsilon(h, w) + \frac{R_\varepsilon(\varepsilon^4w)}{\varepsilon^4} \\ \frac{\bar{H}_\varepsilon(v_1, v_3, v_4, u_4)}{\varepsilon^4} + \bar{l}_\varepsilon(h, w) + \frac{\bar{R}_\varepsilon(\varepsilon^4h)}{\varepsilon^4} \end{array} \right) =: \mathbb{A} \left( \begin{array} {c} w \\ h \end{array} \right)
\]

and the set
\[
\mathbb{S} = \{ (w, h) \in H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3) : \| (w, h) \| \leq |\varepsilon|^{-\sigma} \},
\]

where \( \sigma > 0 \) is sufficiently small.
Then by direct calculation, we find for \((w, h), (w_1, h_1), (w_2, h_2) \in S,\)
\[
\|(\bar{w}, \bar{h})\| \leq C(1 + \varepsilon) \leq |\varepsilon|^{-\sigma},
\]
\[
\|(\bar{w}_1 - \bar{w}_2, \bar{h}_1 - \bar{h}_2)\| = \|A(w_1 - w_2, h_1 - h_2)\|
\leq |\varepsilon|\|(w_1 - w_2, h_1 - h_2)\| < \frac{1}{2}\|(w_1 - w_2, h_1 - h_2)\|.
\]
Therefore, the operator \(A\) maps \(S\) into \(S\) and is a contraction map. So, by the contraction mapping theorem, there exists \((w, h) \in S,\) such that \((w, h) = A(w, h).\) Direct computation yields
\[
\left| \int_{\mathbb{R}^3} \frac{H_\varepsilon(u_2, u_3, u_4, v_4, U)}{\varepsilon^4} \varphi + \frac{\bar{H}_\varepsilon(v_1, v_3, v_4, u_4)}{\varepsilon^4} \psi \right| 
\leq C\|(\varphi, \psi)\|, \forall (\varphi, \psi) \in H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3).
\]
As a result, we see
\[
\|(w, h)\| \leq Ce^4. \quad (2.4)
\]
Now we consider the asymptotic behavior of \(u_i, v_i, (i = 1, \ldots, 4)\) at infinity. We claim that for any fixed small \(\tau > 0,\) there exists a positive constant \(C\) depending on \(\tau, u_i, v_i, (i = 1, \ldots, 4)\) such that
\[
|u_i(r)| + |v_i(r)| \leq Ce^{-(1-\tau)r}, \quad (i = 1, \ldots, 4), \forall r > 1. \quad (2.5)
\]
Indeed, by induction, we suppose \(|v_{i-1}| \leq C_{i-1}e^{-(1-\tau)r}.\) Since
\[
-\Delta e^{-(1-\tau)r} + e^{-(1-\tau)r} - 3U^2e^{-(1-\tau)r}
= \left(1 - (1 - \tau)^2 + \frac{N - 1}{r} - 3U^2\right)e^{-(1-\tau)r},
\]
we can choose \(C_i, R_i\) depending on \(u_i, \tau, i\) and \(C_{i-1}\) such that \(\bar{C}_i e^{-(1-\tau)r}\) is a super-solution of the first equation of (2.1) on \(\mathbb{R}^3 \setminus B_{R_i}(0).\) By comparison theory of elliptic equations, we conclude
\[
u_i \leq \bar{C}_i e^{-(1-\tau)r}, \forall r \geq R_i.
\]
With the same argument, we can also prove that
\[
u_i \geq -\bar{C}_i e^{-(1-\tau)r}, \forall r \geq R_i.
\]
Hence, we can choose \(C_i\) depending on \(u_i, \tau, i, C_{i-1}\) such that
\[
u_i(r) \leq C_i e^{-(1-\tau)r}, \forall r > 1.
\]
Similarly, we can prove that \(v_1 \leq Ce^{-(1-\tau)r}, |c(x)| \leq Ce^{-(1-\tau)r}\) and also \(|v_i| \leq C_i e^{-(1-\tau)r}\) for \(r > 1.\)

The above results can be summarized as

**Proposition 2.2.** There exists \(\varepsilon_0 > 0\) such that for \(\varepsilon \in (-\varepsilon_0, \varepsilon_0),\) problem (1.6) has a solution \((U_\varepsilon, v_\varepsilon) \in H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3)\) satisfying \(U_\varepsilon \to U, v_\varepsilon \to 0\) in \(H^1_r\) as \(\varepsilon \to 0.\) Moreover,
\[
U_\varepsilon = U + \sum_{i=2}^4 \frac{\varepsilon^i}{i!} u_i + w, \quad v_\varepsilon = \sum_{i=1}^4 \frac{\varepsilon^i}{i!} v_i + h. \quad (2.6)
\]
Here
\[ \| (w, h) \| \leq \tilde{C} \varepsilon^4, \] (2.7)
\( \tilde{C} > 0 \) is independent of \( \varepsilon \). \( u_i \) and \( v_i \) satisfy
\[ |u_i(r)| + |v_i(r)| \leq C e^{-(1-r)r}, \quad \forall \ r > 1, \] (2.8)
where \( \tau > 0 \) is any fixed small constant, \( C \) depends on \( \tau, u_i, v_i, \ (i = 1, \cdots, 4) \).

With the same argument we can also construct a solution for problem (1.7) which is linearly coupled by three equations.

The main result is

**Proposition 2.3.** There exists \( \varepsilon_0 > 0 \) such that for \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), problem (1.7) has a solution \( (U_\varepsilon, v_\varepsilon, \omega_\varepsilon) \in (H^1_r(\mathbb{R}^3))^3 \) satisfying \( U_\varepsilon \to U, \ v_\varepsilon \to 0 \) and \( \omega_\varepsilon \to 0 \) in \( H^1_r \) as \( \varepsilon \to 0 \).

Moreover,
\[ U_\varepsilon = U + \frac{4 \varepsilon}{i!} u_i + w, \quad v_\varepsilon = \frac{4 \varepsilon}{i!} v_i + h, \quad \omega_\varepsilon = \frac{4 \varepsilon}{i!} \omega_i + g. \] (2.9)

Here
\[ \| (w, h, g) \| =: \| w \|_{H^1(\mathbb{R}^3)} + \| h \|_{H^1(\mathbb{R}^3)} + \| g \|_{H^1(\mathbb{R}^3)} \leq \tilde{C} \varepsilon^4, \] (2.10)
\( \tilde{C} > 0 \) is independent of \( \varepsilon \). \( u_i, v_i \) and \( \omega_i \) satisfy
\[ |u_i(r)| + |v_i(r)| + |\omega_i(r)| \leq C e^{-(1-r)r}, \quad \forall \ r > 1, \] (2.11)
where \( \tau > 0 \) is any fixed small constant, \( C \) depends on \( \tau, u_i, v_i, \omega_i \ (i = 1, \cdots, 4) \).

**Proof.** Solve
\[ \begin{align*}
-\Delta u_1 + u_1 - 3U^2 u_1 &= 0, \quad x \in \mathbb{R}^3, \\
-\Delta \tilde{v}_1 + \tilde{v}_1 &= -U, \quad x \in \mathbb{R}^3, \\
-\Delta \tilde{\omega}_1 + \tilde{\omega}_1 &= -U, \quad x \in \mathbb{R}^3,
\end{align*} \]
then \( u_1 = 0, \ \tilde{v}_1 \in H^1_r(\mathbb{R}^3), \ \tilde{\omega}_1 \in H^1_r(\mathbb{R}^3) \). Let \( c(x), d(x) \in H^1_r(\mathbb{R}^3) \) satisfy
\[ -\Delta c(x) + c(x) = \tilde{v}_1^3, \quad -\Delta d(x) + d(x) = \tilde{\omega}_1^3, \]
we see that \( v_1 = \tilde{v}_1 + \varepsilon^2 c(x) \) and \( \omega_1 = \tilde{\omega}_1 + \varepsilon^2 d(x) \) solve
\[ -\Delta v_1 + v_1 = -U + \varepsilon^2 \tilde{v}_1^3, \quad -\Delta \omega_1 + \omega_1 = -U + \varepsilon^2 \tilde{\omega}_1^3. \]

For \( k \geq 2 \), we can define \( (u_k, v_k, \omega_k) \in (H^1_r(\mathbb{R}^3))^3 \) by solving
\[ \begin{align*}
-\Delta u_k + u_k - 3U^2 u_k &= -k(u_{k-1} + \omega_{k-1}), \quad x \in \mathbb{R}^3, \\
-\Delta v_k + v_k &= -k(u_{k-1} + \omega_{k-1}), \quad x \in \mathbb{R}^3, \\
-\Delta \omega_k + \omega_k &= -k(u_{k-1} + v_{k-1}), \quad x \in \mathbb{R}^3, \quad (2.12)
\end{align*} \]
Proceeding as we prove Proposition 2.2, we can find \( (w, h, g) \in (H^1_r(\mathbb{R}^3))^3 \) such that (2.10) and (2.11) hold true and \( (U_\varepsilon, v_\varepsilon, \omega_\varepsilon) \) defined by (2.9) satisfies problem (1.7). \( \square \)
3. Segregated vector solutions for 2 coupled Schrödinger system

We will use \((U_\varepsilon, v_\varepsilon)\) to construct multi-bump solutions for (1.6). It follows from Proposition 2.2 that \((U_\varepsilon, v_\varepsilon)\) has the form

\[
U_\varepsilon = U + \varepsilon^2 p_\varepsilon(r) + w, \quad v_\varepsilon = \varepsilon q_\varepsilon(r) + h,
\]

where

\[
p_\varepsilon(r) \leq C e^{-(1-\tau)r}, \quad q_\varepsilon(r) \leq C e^{-(1-\tau)r}, \quad \|(w, h)\| \leq C \varepsilon^4.
\]

Here \(C\) is independent of \(\varepsilon\), and \(\tau > 0\) is defined in (2.8).

For any integer \(\ell \geq 2\), set

\[
m = 2 \sin \frac{\pi}{\ell}, \quad n = \sqrt{2 \left(1 - \cos \frac{\pi}{\ell}\right)}.
\]

Then it can be easily check that

\[
m > n > 0, \quad 2 < \frac{m}{m - n} < 4.
\]

Let \(x^j\) and \(y^j\) be defined by (1.4) and (1.5) respectively. In this section, we assume

\[
(r, \rho) \in D_\varepsilon \times D_\varepsilon =: \left[ \frac{|\ln \varepsilon|}{m - n + \mu |\ln \varepsilon|}, \frac{|\ln \varepsilon|}{m - n} \right] \times \left[ \frac{|\ln \varepsilon|}{m - n + \mu |\ln \varepsilon|}, \frac{|\ln \varepsilon|}{m - n} \right],
\]

(3.2)

where the constant \(\mu > m - n\). For any function \(W : \mathbb{R}^3 \to \mathbb{R}\) and \(\xi \in \mathbb{R}^3\), we define

\[
W_\xi = W(x - \xi).
\]

Set

\[
U_{\varepsilon,r} = \sum_{i=1}^{\ell} U_{\varepsilon,x^i}, \quad v_{\varepsilon,r} = \sum_{i=1}^{\ell} v_{\varepsilon,x^i}, \quad U_{\varepsilon,\rho} = \sum_{i=1}^{\ell} U_{\varepsilon,y^i}, \quad v_{\varepsilon,\rho} = \sum_{i=1}^{\ell} v_{\varepsilon,y^i},
\]

and

\[
Y_{\varepsilon,j} = \frac{\partial U_{\varepsilon,x^j}}{\partial r}, \quad Z_{\varepsilon,j} = \frac{\partial U_{\varepsilon,y^j}}{\partial \rho}, \quad j = 1, \ldots, \ell.
\]

Define

\[
H_s = \{ u : u \in H^1(\mathbb{R}^3), u \text{ is even in } x_h, h = 2, 3, \}
\]

\[
u(r \cos \theta, r \sin \theta, x') = u(r \cos(\theta + \frac{2\pi j}{\ell}), r \sin(\theta + \frac{2\pi j}{\ell}), x'),
\]

and

\[
E = \left\{ (u, v) \in H_s \times H_s, \sum_{j=1}^{\ell} \int_{\mathbb{R}^3} U_{\varepsilon,x^j}^2 Y_{\varepsilon,j} u = 0, \sum_{j=1}^{\ell} \int_{\mathbb{R}^3} U_{\varepsilon,y^j}^2 Z_{\varepsilon,j} v = 0 \right\}.
\]

(3.3)

To prove Theorem 1.1, it suffices to prove

**Proposition 3.1.** For any integer \(\ell \geq 2\), there exists \(\varepsilon_0 > 0\) such that for \(\varepsilon \in (0, \varepsilon_0)\), problem (1.6) has a solution \((u, v)\) with the form

\[
u = U_{\varepsilon,r} + v_{\varepsilon,\rho} + \varphi_\varepsilon, \quad v = U_{\varepsilon,\rho} + v_{\varepsilon,r} + \psi_\varepsilon,
\]

where \((\varphi_\varepsilon, \psi_\varepsilon) \in E\) satisfies \(\| (\varphi_\varepsilon, \psi_\varepsilon) \| = o(\varepsilon^{\frac{m}{m - n}})\).
Let
\[ I(u, v) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2 + |\nabla v|^2 + v^2) \]
\[ - \frac{1}{4} \int_{\mathbb{R}^3} (u^4 + v^4) + \varepsilon \int_{\mathbb{R}^3} uv, \quad (u, v) \in H_s \times H_s, \]
and
\[ J(\varphi, \psi) = I(U_{\varepsilon,r} + v_{\varepsilon,\rho} + \varphi, U_{\varepsilon,\rho} + v_{\varepsilon,r} + \psi). \]

Expand \( J(\varphi, \psi) \) as follows:
\[ J(\varphi, \psi) = J(0, 0) - l(\varphi, \psi) + \frac{1}{2} \tilde{L}(\varphi, \psi) - R(\varphi, \psi), \quad (\varphi, \psi) \in \mathbb{E}, \quad (3.4) \]
where
\[ l(\varphi, \psi) = \int_{\mathbb{R}^3} ((U_{\varepsilon,r} + v_{\varepsilon,\rho})^3 - \sum_{j=1}^{\ell} U_{\varepsilon,xj}^3 - \sum_{j=1}^{\ell} v_{\varepsilon,yj}^3) \varphi \]
\[ + \int_{\mathbb{R}^3} ((U_{\varepsilon,\rho} + v_{\varepsilon,r})^3 - \sum_{j=1}^{\ell} U_{\varepsilon,yj}^3 - \sum_{j=1}^{\ell} v_{\varepsilon,xj}^3) \psi \]
\[ \tilde{L}(\varphi, \psi) = \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + \varphi^2 - 3(U_{\varepsilon,r} + v_{\varepsilon,\rho})^2 \varphi^2) \]
\[ + \int_{\mathbb{R}^3} (|\nabla \psi|^2 + \psi^2 - 3(U_{\varepsilon,\rho} + v_{\varepsilon,r})^2 \psi^2) + 2\varepsilon \int_{\mathbb{R}^3} \varphi \psi, \]
and
\[ R(\varphi, \psi) = \int_{\mathbb{R}^3} ((U_{\varepsilon,\rho} + v_{\varepsilon,r}) \varphi^3 + (U_{\varepsilon,\rho} + v_{\varepsilon,r}) \psi^3) + \frac{1}{4} \int_{\mathbb{R}^N} (\varphi^4 + \psi^4). \]

**Remark 3.2.** Here, in the expression of the linear part \( l(\varphi, \psi) \), there are no terms from the coupled term \( \varepsilon \int_{\mathbb{R}^3} uv \) since we use \((U_{\varepsilon}, v_{\varepsilon})\) to construct the vector solutions. We will see later in the proof of Proposition 3.1 that this choice of the approximate solution guarantees that the error terms of the reduced functional are dominated by \( \varepsilon^{m+\sigma-\eta} \), which is of higher order small datum of the main terms. However, if we use \((U, U)\) as an approximate solution, then in the expression of \( l(\varphi, \psi) \), the terms from the coupling like \( \varepsilon \int_{\mathbb{R}^N} (\sum_{j=1}^{\ell} U_{xj}) \psi + \varepsilon \int_{\mathbb{R}^3} (\sum_{j=1}^{\ell} U_{yj}) \varphi \) will appear, which implies \( \|l(\varphi, \psi)\| = O(\varepsilon) \). Hence the error terms of the reduced functional are of order \( O(\varepsilon^2) \), which will dominate the main terms and we have no way to solve the reduced functional.

It is easy to check that \( \tilde{L}(\varphi, \psi) \) can be generated by a bounded linear operator \( L \) from \( \mathbb{E} \) to \( \mathbb{E} \), which is defined as
\[
\langle L(u, v), (\varphi, \psi) \rangle = \int_{\mathbb{R}^3} \left( \nabla u \nabla \varphi + u \varphi - 3(U_{\varepsilon,r} + v_{\varepsilon,\rho})^2 u \varphi \right) \]
\[ + \int_{\mathbb{R}^3} \left( \nabla v \nabla \psi + v \psi - 3(U_{\varepsilon,\rho} + v_{\varepsilon,r})^2 v \psi \right) + \varepsilon \int_{\mathbb{R}^3} (u \psi + v \varphi). \]

Now, we discuss the invertibility of \( L \).
Lemma 3.3. There exists $\varepsilon_0 > 0$, such that for $\varepsilon \in (0, \varepsilon_0)$, there is a constant $\rho > 0$, independent of $\varepsilon$, satisfying that for any $(r, \rho) \in D_\varepsilon \times D_\varepsilon$, \[
\|L(u, v)\| \geq \rho \|(u, v)\|, \quad (u, v) \in E.
\]

Proof. Suppose to the contrary that there are $\varepsilon_n \to 0^+$ (as $n \to +\infty$), $(r_n, \rho_n) \in D_{\varepsilon_n} \times D_{\varepsilon_n}$, and $(u_n, v_n) \in E$, with
\[
\langle L(u_n, v_n), (\varphi, \psi) \rangle = o_n(1)\|(u_n, v_n)\|\|\varphi, \psi\|, \quad \forall (\varphi, \psi) \in E. \tag{3.5}
\]
We may assume that $\|(u_n, v_n)\| = 1$. We see from (3.5),
\[
\begin{align*}
\int_{\mathbb{R}^3} (\nabla u_n \nabla \varphi + u_n \varphi - 3(U_{\varepsilon_n, r_n} + v_{\varepsilon_n, \rho_n})^2 u_n \varphi) \\
+ \int_{\mathbb{R}^3} (\nabla v_n \nabla \psi + v_n \psi - 3(U_{\varepsilon_n, \rho_n} + v_{\varepsilon_n, r_n})^2 v_n \psi) \\
+ \varepsilon_n \int_{\mathbb{R}^3} (u_n \psi + v_n \varphi) = o_n(1)\|\varphi, \psi\|, \quad \forall (\varphi, \psi) \in E. \tag{3.6}
\end{align*}
\]
In particular,
\[
\begin{align*}
\int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2 - 3(U_{\varepsilon_n, r_n} + v_{\varepsilon_n, \rho_n})^2 u_n^2) \\
+ \int_{\mathbb{R}^3} (|\nabla v_n|^2 + v_n^2 - 3(U_{\varepsilon_n, \rho_n} + v_{\varepsilon_n, r_n})^2 v_n^2) + 2\varepsilon_n \int_{\mathbb{R}^3} u_n v_n = o_n(1), \tag{3.7}
\end{align*}
\]
and
\[
\int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2 + |\nabla v_n|^2 + v_n^2) = 1. \tag{3.8}
\]
Let
\[
\bar{u}_n(x) = u_n(x - x^1), \quad \bar{v}_n(x) = v_n(x - y^1).
\]
We may assume the existence of $u$, such that as $n \to +\infty$,
\[
\bar{u}_n \to u, \quad \text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \quad \bar{v}_n \to u, \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3).
\]
Moreover, $u$ is even in $x_h$, $h = 2, 3$.

By symmetry, we see
\[
\int_{\mathbb{R}^3} U_{\varepsilon_n, \bar{x}_h}^2 Y_{\varepsilon_n, 1} u_n = 0.
\]
It follows from $(U_{\varepsilon_n}, v_{\varepsilon_n}) \to (U, 0)$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ that
\[
\left| \int_{\mathbb{R}^3} U_{\varepsilon_n, \bar{x}_h}^2 Y_{\varepsilon_n, 1} u_n - \int_{\mathbb{R}^3} U_{\varepsilon_n, \bar{x}_h}^2 \frac{\partial U_{\varepsilon_n, \bar{x}_h}}{\partial x_n} u_n \right| \leq \left| \int_{\mathbb{R}^3} (U_{\varepsilon_n, \bar{x}_h}^2 - U_{\varepsilon_n, \bar{x}_h}^2) Y_{\varepsilon_n, 1} u_n \right| + \int_{\mathbb{R}^3} U_{\varepsilon_n, \bar{x}_h}^2 \left( Y_{\varepsilon_n, 1} - \frac{\partial U_{\varepsilon_n, \bar{x}_h}}{\partial x_n} \right) u_n \to 0, \quad (n \to +\infty).
\]
Hence
\[
\int_{\mathbb{R}^3} U_{\varepsilon_n, \bar{x}_h}^2 \frac{\partial U_{\varepsilon_n, \bar{x}_h}}{\partial x_n} u_n \to 0,
\]
which implies
\[ \int_{\mathbb{R}^3} U^2 \frac{\partial U}{\partial x_1} u = 0. \] (3.9)

Let \( \varphi \in C_0^\infty(B_R(0)) \) be even in \( x_h, h = 2, 3 \). Define \( \varphi_n(x) := \varphi(x - x^1) \in C_0^\infty(B_R(x^1)) \). We may identify \( \varphi_n(x) \) as elements in \( H_s \) by redefining the values outside \( B_R(x^1) \) with the symmetry.

From the fact that \( U_{\varepsilon n} \to U \) and \( v_{\varepsilon n} \to 0 \) in \( H^1(\mathbb{R}^3) \), we deduce
\[
\int_{\mathbb{R}^3} (U_{\varepsilon n, r_n} + v_{\varepsilon n, \rho_n})^2 u_n \varphi_n
= \int_{\mathbb{R}^3} (U_{\varepsilon n, r_n} + 2U_{\varepsilon n, r_n} v_{\varepsilon n, r_n}^2 + v_{\varepsilon n, \rho_n}^2) u_n \varphi_n
= \int_{\mathbb{R}^3} U^2 u \varphi + o_n(1). \] (3.10)

Then choosing \((\varphi, \psi) = (\varphi_n, 0)\) in (3.6) and considering (3.10), we can use the argument in [24], to prove that \( u \) solves
\[ -\Delta u + u - 3U^2 u = 0, \quad x \in \mathbb{R}^3. \] (3.11)

Since we work in the space of functions which are even in \( x_2 \) and \( x_3 \), we see \( u = c \frac{\partial U}{\partial x_1} \) for some \( c \), which implies that \( u = 0 \) since \( u \) satisfies (3.9).

To deal with \( v_n \), we first claim that for any \( v(x) \in H_s \), \( v(x) \) is even with respect to the ray with an angle of \( \pi/\ell \).

Indeed, suppose that \( |(x_1, x_2)| = a \), then
\[
v(x) = v(a \cos\left(\frac{\pi}{\ell} + \theta\right), a \sin\left(\frac{\pi}{\ell} + \theta\right), x_3)
= v(a \cos\left(\frac{\pi}{\ell} + \theta\right), -a \sin\left(\frac{\pi}{\ell} + \theta\right), x_3)
= v(a \cos\left(-\frac{\pi}{\ell} - \theta\right), a \sin(-\frac{\pi}{\ell} - \theta), x_3)
= v(a \cos\left(-\frac{\pi}{\ell} - \theta\right), a \sin\left(\frac{\pi}{\ell} - \theta\right), x_3).\]

Now as we deal with \( u_n \), we can check
\( \bar{v}_n \to 0 \), weakly in \( H^1_{\text{loc}}(\mathbb{R}^3) \), \( \bar{v}_n \to 0 \), strongly in \( L^2_{\text{loc}}(\mathbb{R}^3) \).

Similar to (3.10), using the fact that \( U_{\varepsilon n} \to U \) and \( v_{\varepsilon n} \to 0 \) in \( H^1(\mathbb{R}^3) \) as \( n \to +\infty \), we deduce that
\[
\int_{\mathbb{R}^3} (U_{\varepsilon n, r_n} + v_{\varepsilon n, \rho_n})^2 u_n^2 + \int_{\mathbb{R}^3} (U_{\varepsilon n, r_n} + v_{\varepsilon n, r_n})^2 v_n^2
= \int_{\mathbb{R}^3} \left( \sum_{j=1}^{\ell} U_{x_j}^2 \right) u_n^2 + \int_{\mathbb{R}^3} \left( \sum_{j=1}^{\ell} U_{y_j}^2 \right) v_n^2 + o_n(1). \] (3.12)
Hence we find
\[
\begin{align*}
o_n(1) &= \int_{\mathbb{R}^3} \left( |\nabla u_n|^2 + u_n^2 - 3(U_{\varepsilon_n} r_n + v_{\varepsilon_n} \rho_n)^2 u_n^2 \right) \\
&\quad + \int_{\mathbb{R}^3} \left( |\nabla v_n|^2 + v_n^2 - 3(U_{\varepsilon_n} \rho_n + v_{\varepsilon_n} r_n)^2 v_n^2 \right) + 2\varepsilon_n \int_{\mathbb{R}^N} u_n v_n \\
&= 1 + Ce^{-R},
\end{align*}
\]
which is impossible for large \(n\) and large \(R\).

As a result, we complete the proof.

Lemma 3.4. There is a constant \(C > 0\), independent of \(\varepsilon\), such that
\[
\|R(\varphi, \psi)\| \leq C\| (\varphi, \psi) \|^3, \quad \|R'(\varphi, \psi)\| \leq C\| (\varphi, \psi) \|^2, \quad \|R''(\varphi, \psi)\| \leq C\| (\varphi, \psi) \|.
\]

Proof. The proof can be completed by direct calculation and we omit it.

Now we perform the finite-dimensional reduction procedure.

Proposition 3.5. There exists \(\varepsilon_0 > 0\) such that for \(\varepsilon \in (0, \varepsilon_0)\), there is a \(C^1\) map from \(\mathcal{D}_\varepsilon \times \mathcal{D}_\varepsilon\) to \(H_s \times H_s\): \((\varphi, \psi) = (\varphi(r, \rho), \psi(r, \rho))\), satisfying \((\varphi, \psi) \in \mathbb{E}\), and
\[
J'_{(\varphi, \psi)}(\varphi, \psi) = 0, \quad \text{on } \mathbb{E}.
\]

Moreover, there is a constant \(C > 0\) independent of \(\varepsilon\), such that
\[
\| (\varphi, \psi) \| \leq C \left( \frac{e^{-|x_1 - x_2|^2}}{|x_1 - x_2|^2} + \frac{e^{-|y_1 - y_2|^2}}{|y_1 - y_2|^2} + \varepsilon e^{-1-\sigma} |x_1 - y_1| + \varepsilon^4 \right).
\]

Proof. It follows from the proof of Lemma 3.6 below, that \(l(\varphi, \psi)\) is a bounded linear functional in \(\mathbb{E}\). Thus, there is an \(f_\varepsilon \in \mathbb{E}\), such that
\[
l(\varphi, \psi) = \langle f_\varepsilon, (\varphi, \psi) \rangle.
\]

Thus, finding a critical point for \(J(\varphi, \psi)\) in \(\mathbb{E}\) is equivalent to solving
\[
f_\varepsilon - L(\varphi, \psi) + R'(\varphi, \psi) = 0.
\]

By Lemma 3.3, \(L\) is invertible. Thus, (3.15) can be rewritten as
\[
(\varphi, \psi) = A(\varphi, \psi) =: L^{-1}(f_\varepsilon + R'(\varphi, \psi)).
\]

Set
\[
D = \left\{ (\varphi, \psi) : (\varphi, \psi) \in \mathbb{E}, \|(\varphi, \psi)\| \leq \frac{e^{-(1-\sigma)|x_1 - x_2|^2}}{|x_1 - x_2|^2} + \frac{e^{-(1-\sigma)|y_1 - y_2|^2}}{|y_1 - y_2|^2} + \varepsilon^{1-\sigma} e^{-1-\sigma |x_1 - y_1|} + \varepsilon^{4-\sigma} \right\},
\]
where \(\sigma > 0\) is small.
From Lemma 3.4 and Lemma 3.6 below, for \( \varepsilon \) small,
\[
\|A(\varphi, \psi)\| \leq C\|f_\varepsilon\| + C\|(\varphi, \psi)\|^2
\]
\[
\leq \frac{e^{-(1-\sigma)|x^1-x^2|}}{|x^1-x^2|} + \frac{e^{-(1-\sigma)|y^1-y^2|}}{|y^1-y^2|} + \varepsilon e^{-|x^1-y^1|} + \varepsilon^4,
\]
(3.17)
and
\[
\|A(\varphi_1, \psi_1) - A(\varphi_2, \psi_2)\| = \|L^{-1}R'(\varphi_1, \psi_1) - L^{-1}R'(\varphi_2, \psi_2)\|
\]
\[
\leq C\left(\|\varphi_1, \psi_1\| + \|\varphi_2, \psi_2\|\right)\|\varphi_1 - \varphi_2\| + \|\psi_1 - \psi_2\|
\]
\[
\leq \frac{1}{2}\|\varphi_1 - \varphi_2\|.
\]
Therefore, \( A \) maps \( D \) into \( D \) and is a contraction map. So, there exists \((\varphi, \psi) \in \mathbb{E}\), such that \((\varphi, \psi) = A(\varphi, \psi)\). Moreover by (3.16), we have
\[
\|\varphi, \psi\| \leq C\left(\frac{e^{-(1-\sigma)|x^1-x^2|}}{|x^1-x^2|} + \frac{e^{-(1-\sigma)|y^1-y^2|}}{|y^1-y^2|} + \varepsilon^4\right).
\]
\[\Box\]

**Lemma 3.6.** There is a constant \( C > 0 \) independent of \( \varepsilon \), such that
\[
\|f_\varepsilon\| \leq C\left(\frac{e^{-(1-\sigma)|x^1-x^2|}}{|x^1-x^2|} + \frac{e^{-(1-\sigma)|y^1-y^2|}}{|y^1-y^2|} + \varepsilon e^{-|x^1-y^1|} + \varepsilon^4\right).
\]

**Proof.** We see
\[
\int_{\mathbb{R}^3} ((U_{\varepsilon,r} + v_{\varepsilon,\rho})^3 - \sum_{j=1}^{\ell} U_{\varepsilon,xj}^3 - \sum_{j=1}^{\ell} v_{\varepsilon,yj}^3) \varphi
\]
\[
= \int_{\mathbb{R}^3} \left(\sum_{j=1}^{\ell} U_{\varepsilon,xj}^3 - \sum_{j=1}^{\ell} U_{\varepsilon,xj}^3 + \sum_{j=1}^{\ell} v_{\varepsilon,yj}^3 - \sum_{j=1}^{\ell} v_{\varepsilon,yj}^3 + 3U_{\varepsilon,r}^2 v_{\varepsilon,\rho} + 3U_{\varepsilon,r} v_{\varepsilon,\rho}^2\right) \varphi
\]
\[
= \int_{\mathbb{R}^3} \left(3 \sum_{j \neq i} U_{\varepsilon,xj}^2 U_{\varepsilon,xj} + 3 \sum_{j \neq i} v_{\varepsilon,yj} v_{\varepsilon,yj} + 3U_{\varepsilon,r} v_{\varepsilon,\rho} + 3U_{\varepsilon,r} v_{\varepsilon,\rho}^2\right) \varphi.
\]
It follows from Proposition 2.2, Proposition A.1 and Hölder inequality that for \( i \neq j \)
\[
\left| \int_{\mathbb{R}^3} U_{\varepsilon,xj}^2 U_{\varepsilon,xj} \varphi \right|
\]
\[
= \left| \int_{\mathbb{R}^3} ((U_{xj} + \varepsilon^2 p_\varepsilon(|x - x^i|) + w(x - x^i)xj + \varepsilon^2 p_\varepsilon(|x - x^i|) + w(x - x^i)j) \varphi \right|
\]
\[
\leq C\left(\frac{e^{-(1-\sigma)|x^1-x^2|}}{|x^1-x^2|} + \varepsilon e^{-(1-\sigma)|x^1-x^2|} + \varepsilon^4\right) \|\varphi\|_{H^1(\mathbb{R}^3)},
\]
\[
\left| \int_{\mathbb{R}^3} v_{\varepsilon,yj}^2 v_{\varepsilon,yj} \varphi \right|
\]
\[
\begin{align*}
&= C \left| \int_{\mathbb{R}^3} (\varepsilon^2 q^2 |x - y|^2 + h^2 (|x - y|^2)) (\varepsilon q e |x - y|) \varphi \right| \\
&\leq C (\varepsilon^3 e^{-(1-3\tau)|y i - y j|} + \varepsilon^4) \| \varphi \|_{H^1(\mathbb{R}^3)},
\end{align*}
\]
and
\[
\left| \int_{\mathbb{R}^3} (3U_{\varepsilon,r}^2 v_{\varepsilon,\rho} + 3U_{\varepsilon,\rho} v_{\varepsilon,r}^2) \varphi \right| \leq C \sum_{i,j=1}^\ell (\varepsilon e^{-(1-\tau)|x i - y j|} + \varepsilon^4) \| \varphi \|_{H^1(\mathbb{R}^3)}.
\]

Therefore,
\[
\left| \int_{\mathbb{R}^3} \left( (U_{\varepsilon,r} + v_{\varepsilon,\rho})^3 - \sum_{j=1}^\ell U_{\varepsilon,x}^3 - \sum_{j=1}^\ell v_{\varepsilon,y}^3 \right) \varphi \right|
\leq C \left( \sum_{i\neq j} \frac{e^{-|x i - x j|}}{|x i - x j|^2} + \varepsilon \sum_{i,j=1}^\ell \frac{e^{-(1-\tau)|x i - y j|}}{|y i - y j|} + \varepsilon^4 \right) \| \varphi \|_{H^1(\mathbb{R}^3)}.
\]

Similarly,
\[
\left| \int_{\mathbb{R}^3} \left( (U_{\varepsilon,\rho} + v_{\varepsilon,r})^3 - \sum_{j=1}^\ell U_{\varepsilon,y}^3 - \sum_{j=1}^\ell v_{\varepsilon,x}^3 \right) \psi \right|
\leq C \left( \sum_{i\neq j} \frac{e^{-|y i - y j|}}{|y i - y j|^2} + \varepsilon \sum_{i,j=1}^\ell \frac{e^{-(1-\tau)|x i - y j|}}{|x i - y j|} + \varepsilon^4 \right) \| \psi \|_{H^1(\mathbb{R}^3)}.
\]

As a result, we complete the proof. \(\square\)

Now we are ready to prove Proposition 3.1. Let \((\varphi_{r,\rho}, \psi_{r,\rho}) = (\varphi(r, \rho), \psi(r, \rho))\) be the map obtained in Proposition 3.5. Define
\[
F(r, \rho) = I(U_{\varepsilon,r} + v_{\varepsilon,\rho} + \varphi_{r,\rho}, U_{\varepsilon,\rho} + v_{\varepsilon,r} + \psi_{r,\rho}), \quad \forall (r, \rho) \in \mathcal{D}_\varepsilon \times \mathcal{D}_\varepsilon.
\]

With the same argument in [10, 19], we can easily check that for \(\varepsilon\) sufficiently small, if \((r, \rho)\) is a critical point of \(F(r, \rho)\), then \((U_{\varepsilon,r} + v_{\varepsilon,\rho} + \varphi_{r,\rho}, U_{\varepsilon,\rho} + v_{\varepsilon,r} + \psi_{r,\rho})\) is a critical point of \(I\).

**Proof of Proposition 3.1.** The boundedness of \(L\) in \(H_s \times H_s\) and Lemma 3.4 imply that
\[
\| L(\varphi_{r,\rho}, \psi_{r,\rho}) \| \leq C \| (\varphi_{r,\rho}, \psi_{r,\rho}) \|, \quad |R(\varphi_{r,\rho}, \psi_{r,\rho})| \leq C \| (\varphi_{r,\rho}, \psi_{r,\rho}) \|^3.
\]
So, Proposition 3.5 and Lemma 3.6 combined by Proposition A.2 give

\[ F(r, \rho) = I(U_{x, r} + v_{x, r}, U_{x, \rho} + v_{x, r}) - l(\varphi_{r, \rho}, \psi_{r, \rho}) + \frac{1}{2} \left(L(\varphi_{r, \rho}, \psi_{r, \rho}), (\varphi_{r, \rho}, \psi_{r, \rho})\right) - R(\varphi_{r, \rho}, \psi_{r, \rho}) \]

\[ = I(U_{x, r} + v_{x, r}, U_{x, \rho} + v_{x, r}) + O\left(\|f\varepsilon\| L(\varphi_{r, \rho}, \psi_{r, \rho}) + \|\varphi_{r, \rho}, \psi_{r, \rho}\|\right)^2 \]

\[ = \sum_{j=1}^{\ell} I(U_{x,j}, v_{x,j}) + \sum_{j=1}^{\ell} I(U_{y,j}, v_{y,j}) - \sum_{i<j} C_{ij} e^{-|x_i - x_j|^2} - \sum_{i<j} C_{ij} e^{-|y_i - y_j|^2} + \varepsilon \sum_{i,j=1}^{\ell} \bar{C}_{ij} e^{-|x_i - y_j|} \]

\[ + O\left(\varepsilon e^{-|y_i - y_j|^2} + \varepsilon^2 e^{-|y_i - y_j|} + \varepsilon e^{-|y_i - y_j|^2} + \varepsilon^4\right) \]

\[ + O\left(\frac{e^{-|x_i - x_j|^2}}{|x_i - x_j|^2} + \frac{e^{-|y_i - y_j|^2}}{|y_i - y_j|^2} + \varepsilon e^{-|y_i - y_j|^2}\right)^2, \]

where \(\bar{C}_{ij}\) and \(C_{ij}\) are those in Proposition A.2.

Recalling

\[ r, \rho \in \mathcal{D}_{\varepsilon} = \left[\frac{\ln \varepsilon}{m - n + \mu \ln \ln \varepsilon}, \frac{\ln \varepsilon}{m - n}\right], \]

where \(m = 2 \sin \frac{\pi}{\ell}, n = \sqrt{2(1 - \cos \frac{\pi}{\ell})}, \mu > m - n > 0\), and noting

\[ \frac{1}{m - n + \mu \ln \ln \varepsilon} = \frac{1}{m - n} - \frac{\mu \ln \ln \varepsilon}{(m - n)^2 \ln \varepsilon} + O\left(\frac{\ln \ln \varepsilon}{\ln \varepsilon}\right)^2, \]

we can check

\[ O\left(\varepsilon e^{-|y_i - y_j|^2} + \varepsilon^2 e^{-|y_i - y_j|} + \varepsilon e^{-|y_i - y_j|^2} + \varepsilon^4\right) \]

\[ + O\left(\frac{e^{-|x_i - x_j|^2}}{|x_i - x_j|^2} + \frac{e^{-|y_i - y_j|^2}}{|y_i - y_j|^2} + \varepsilon e^{-|y_i - y_j|^2}\right)^2 = O\left(\varepsilon^{m-n+\sigma}\right), \]

where \(\sigma > 0\) is a small number such that \(2 < \frac{m}{m-n} + \sigma < 4\) for \(\ell \geq 2\).

Hence, considering the symmetry again, we find

\[ F(r, \rho) = \sum_{j=1}^{\ell} I(U_{x,j}, v_{x,j}) + \sum_{j=1}^{\ell} I(U_{y,j}, v_{y,j}) \]

\[ - \sum_{i<j} C_{ij} e^{-|x_i - x_j|^2} - \sum_{i<j} C_{ij} e^{-|y_i - y_j|^2} + \varepsilon \sum_{i,j=1}^{\ell} \bar{C}_{ij} e^{-|x_i - y_j|} + O\left(\varepsilon^{m-n+\sigma}\right) \]

\[ = C_{\varepsilon} + \ell \left(\bar{C}_{\varepsilon} e^{-\sqrt{r^2 + \rho^2 - 2r\rho \cos \frac{\pi}{\ell}}} - \frac{C}{mr} e^{-mr} - \frac{C}{m\rho} e^{-mp}\right) + O\left(\varepsilon^{m-n+\sigma}\right), \]

where \(C_{\varepsilon}\) depends on \(\varepsilon\) but is independent of \(r\) and \(\rho\), \(C = C_{12}, \bar{C} = \bar{C}_{11}\).
Now we prove that the maximizer of $F(r, \rho)$ in $D_{\epsilon} \times D_{\epsilon}$ is an interior point of $D_{\epsilon} \times D_{\epsilon}$.

To this end, we consider the function

$$G(r, \rho) = \bar{C}\varepsilon - \sqrt{r^2 + \rho^2 - 2r\rho \cos \frac{\pi}{\ell} - \frac{C}{mr}e^{-mr} - \frac{C}{m\rho}e^{-m\rho}}, \quad r, \rho \in D_{\epsilon}.$$  

In order to check that $G(r, \rho)$ achieves maximum at some point $(r_0, \rho_0)$ in the interior of $D_{\epsilon} \times D_{\epsilon}$, we need to estimate both the value of $G(r, \rho)$ on the boundary of $D_{\epsilon} \times D_{\epsilon}$ and the value of $G(r_0, \rho_0)$.

Set

$$\hat{\epsilon} = \frac{|\ln \varepsilon|}{m - n + \frac{\mu \ln(\ln \varepsilon)}{|\ln \varepsilon|}}, \quad \hat{\rho} = \frac{|\ln \varepsilon|}{m - n + \frac{\mu \ln(\ln \varepsilon)}{|\ln \varepsilon|}},$$

and define

$$\rho_\theta = \frac{|\ln \varepsilon|}{m - n + \frac{\mu \ln(\ln \varepsilon)}{|\ln \varepsilon|} \theta}, \quad \theta \in [0, 1].$$

Then $\hat{\epsilon} \leq \rho_\theta \leq \hat{\rho}$ for $\theta \in [0, 1]$, and

$$\sqrt{\hat{\epsilon}^2 + \rho_\theta^2 - 2\hat{\epsilon}\rho_\theta \cos \frac{\pi}{\ell}} = \frac{n}{m - n} |\ln \varepsilon| - \frac{\mu \theta n}{2(m - n)^2} |\ln \varepsilon| + O\left(\frac{|\ln \varepsilon|^2}{|\ln \varepsilon|}\right). \quad (3.20)$$

Hence

$$G(\hat{\epsilon}, \rho_\theta) = -C\varepsilon^{\frac{m}{m-n}} |\ln \varepsilon| - c\varepsilon^{\frac{m}{m-n}} |\ln \varepsilon|^{\frac{\mu n m}{(m-n)^2}} + \bar{C}\varepsilon^{\frac{m}{m-n}} |\ln \varepsilon|^{\frac{\mu n m}{(m-n)^2}} + O\left(\varepsilon^{\frac{m}{m-n}} |\ln \varepsilon|^{\frac{\mu n m}{2(m-n)^2}} + \varepsilon^{\frac{m}{m-n}} |\ln \varepsilon|^{\frac{\mu n m}{2(m-n)^2}}\right). \quad (3.21)$$

where $C, c$ and $\bar{C}$ are positive constants independent of $\varepsilon$.

Set

$$f(\theta) = \frac{\mu \theta m}{(m-n)^2} - 1 - \frac{\mu \theta n}{2(m-n)^2}.$$  

Since $\mu > m - n > 0$, we see

$$f(1) = \frac{\mu m}{(m-n)^2} - 1 - \frac{\mu n}{2(m-n)^2} > \frac{\mu m}{(m-n)^2} - 1 - \frac{\mu n}{2(m-n)^2} = \frac{\mu}{m - n} - 1 > 0.$$  

Considering $f(0) = -1 < 0$, there exists a unique $\bar{\theta} \in (0, 1)$ such that

$$\frac{\mu \bar{\theta} m}{(m-n)^2} - 1 = \frac{\mu \bar{n} n}{2(m-n)^2}. \quad (3.22)$$

Moreover, if $\theta \in (\bar{\theta}, 1]$, then

$$\frac{\mu \theta m}{(m-n)^2} - 1 > \frac{\mu \theta n}{2(m-n)^2}. $$
which implies $G(\hat{r}, \rho_\theta) < 0$. But, if $\theta \in [0, \bar{\theta})$, then
\[
\frac{\mu \theta m}{(m - n)^2} - 1 < \frac{\mu \theta n}{2(m - n)^2},
\]
which means $G(\hat{r}, \rho_\theta) > 0$ and
\[
G(\hat{r}, \rho_\theta) < c_1 \varepsilon^{\frac{m}{m-n}} |\ln\varepsilon|^{\frac{\mu \theta n}{2(m-n)^2}}
\]
for some constant $c_1 > 0$ independent of $\varepsilon$.

Therefore, we get that for $\varepsilon$ sufficiently small
\[
G(\hat{r}, \rho_\theta) \leq 2c_1 \varepsilon^{\frac{m}{m-n}} |\ln\varepsilon|^{\frac{\mu \theta n}{2(m-n)^2}} \quad \forall \theta \in [0, 1],
\]
which says that
\[
\max_{\rho \in D_\varepsilon} G(\hat{r}, \rho) \leq 2c_1 \varepsilon^{\frac{m}{m-n}} |\ln\varepsilon|^{\frac{\mu \theta n}{2(m-n)^2}}.
\]
Similarly,
\[
\max_{r \in D_\varepsilon} G(r, \hat{r}) \leq 2c_1 \varepsilon^{\frac{m}{m-n}} |\ln\varepsilon|^{\frac{\mu \theta n}{2(m-n)^2}}.
\]

**Remark 3.7.** It can be verified from (3.22) and (3.24) that for $\theta \in (0, \bar{\theta})$, there exists a constant $c_2 > 0$ independent of $\varepsilon$ such that for $\varepsilon$ sufficiently small,
\[
\max_{\rho \in D_\varepsilon} G(\hat{r}, \rho) \geq c_2 \varepsilon^{\frac{m}{m-n}} |\ln\varepsilon|^{\frac{\mu \theta n}{2(m-n)^2}}.
\]

Now we estimate $\max_{\rho \in D_\varepsilon} G(\hat{r}, \rho)$.

Since for $\varepsilon > 0$ sufficiently small,
\[
\sqrt{\hat{r}^2 + \rho^2 - 2\hat{r}\rho \cos\frac{\pi}{\ell}} \geq n\hat{r},
\]
it follows from (3.18) and the fact $\mu > m - n > 0$ that for $\varepsilon$ sufficiently small,
\[
G(\hat{r}, \rho) \leq -\frac{C}{m\hat{r}} e^{-m\hat{r}} + \bar{C}_1 \varepsilon e^{-n\hat{r}}
\]
\[
\leq -\frac{C}{m\hat{r}} e^{-m\hat{r}} + \bar{C}_1 \varepsilon e^{-n\hat{r}}
\]
\[
\leq -C_1 \varepsilon^{\frac{m}{m-n}} |\ln\varepsilon|^{\frac{m}{(m-n)^2}} + \bar{C}_1 \varepsilon^{\frac{m}{m-n}} |\ln\varepsilon|^{\frac{m}{(m-n)^2}}
\]
\[
< 0,
\]
where $C_1$ and $\bar{C}_1$ are positive constants independent of $\varepsilon$. Hence,
\[
\max_{\rho \in D_\varepsilon} G(\hat{r}, \rho) \leq 0.
\]
The same argument yields
\[
\max_{r \in D_\varepsilon} G(r, \hat{r}) \leq 0.
\]
At last, we estimate $G(r_0, \rho_0)$. Taking $\theta = \bar{\theta}$ in (3.20), we find for $\varepsilon$ sufficiently small
\[ G(r_0, \rho_0) \geq G(\rho_\bar{\theta}, \rho_\bar{\theta}) \]
\[ = \bar{C}\varepsilon e^{-n\rho_\bar{\theta}} - \frac{2C}{m\rho_\bar{\theta}} e^{-m\rho_\bar{\theta}} \]
\[ = \bar{C}\varepsilon \frac{m}{m-n} |\ln\varepsilon|^\frac{\mu n}{(m-n)^2} - \frac{2(m-n)C}{m} \varepsilon \frac{m}{m-n} |\ln\varepsilon|^\frac{\mu m}{(m-n)^2} - 1 + o\left( \varepsilon \frac{m}{m-n} |\ln\varepsilon|^\frac{\mu n}{(m-n)^2} \right) \]
\[ \geq \left\{ \begin{array}{c}
\bar{C}\varepsilon \frac{m}{m-n} |\ln\varepsilon|^\frac{\mu n}{(m-n)^2}, \\
\end{array} \right. \]

since by (3.23), $\mu \ln \bar{\theta} > (m-n)^2 \ln \varepsilon$. The above estimate and (3.25)-(3.28) show that for $\varepsilon > 0$ sufficiently small, $(r_0, \rho_0)$ is in the interior of $D_\varepsilon \times D_\varepsilon$. Comparing the above estimate on $G(r_0, \rho_0)$ and (3.25)-(3.28) with (3.19), we conclude that $F(r, \rho)$ achieves (local) maximum also in the interior of $D_\varepsilon \times D_\varepsilon$. As a consequence, we complete the proof.

4. Segregated solutions for system coupled by three equations

In this section, we consider the following system linearly coupled by three equations
\[ \begin{align*}
-\Delta u + u &= u^3 - \varepsilon(v + \omega), \quad x \in \mathbb{R}^3, \\
-\Delta v + v &= v^3 - \varepsilon(u + \omega), \quad x \in \mathbb{R}^3, \\
-\Delta \omega + \omega &= \omega^3 - \varepsilon(u + v), \quad x \in \mathbb{R}^3.
\end{align*} \]

Let $(U_\varepsilon, v_\varepsilon, \omega_\varepsilon) \in (H^1_r(\mathbb{R}^3))^3$ be the solution of (4.1) obtained in Proposition 2.3. In this part, we will use the same notations as those in previous sections. Define
\[ \omega_{\varepsilon,r} = \sum_{j=1}^{\ell} \omega_{\varepsilon,x}^j, \quad \omega_{\varepsilon,\rho} = \sum_{j=1}^{\ell} \omega_{\varepsilon,y}^j \]
and
\[ E = \{ (\varphi, \psi, \phi) \in (H^1_r(\mathbb{R}^3))^3 : (\varphi, \psi) \in E, \phi \in H^s \} \]
where $E$ is defined as (3.23), $r, \rho \in D_\varepsilon$ and $D_\varepsilon$ is defined by (3.2) for $\ell > 2$ but by
\[ D_\varepsilon = \left[ \frac{|\ln\varepsilon|}{1 + \mu |\ln\varepsilon| |\ln\varepsilon|}, |\ln\varepsilon| \right], \quad (\mu > 1) \]
for $\ell = 2$.

To prove Theorem 1.2, we only need to verify

**Proposition 4.1.** For any integer $\ell \geq 2$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, problem 4.1 has a solution $(u, v, \omega)$ with the form
\[ \begin{align*}
u = U_{\varepsilon,r} + v_{\varepsilon,\rho} + \omega_{\varepsilon} + \varphi, \\
v = \omega_{\varepsilon,r} + U_{\varepsilon,\rho} + v_{\varepsilon} + \psi, \\
\omega = v_{\varepsilon,r} + \omega_{\varepsilon,\rho} + U_{\varepsilon} + \phi.
\end{align*} \]
Proof. The proof is similar to that of Proposition 3.1, we only give a sketch here.

Define

$$I(u, v, \omega) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2 + |\nabla v|^2 + v^2 + |\nabla \omega|^2 + \omega^2)$$

$$-\frac{1}{4} \int_{\mathbb{R}^3} (u^4 + v^4 + \omega^4) + \varepsilon \int_{\mathbb{R}^3} (uv + uw + vw), \quad \forall \ (u, v, \omega) \in (H_\delta)^3,$$

and

$$J(\varphi, \psi, \phi) = I(U_{\varepsilon, r} + v_{\varepsilon, \rho} + \omega_\varepsilon + \varphi, \omega_{\varepsilon, r} + U_{\varepsilon, \rho} + v_\varepsilon + \psi, v_{\varepsilon, r} + \omega_{\varepsilon, \rho} + U_\varepsilon + \phi),$$

$$\forall \ (\varphi, \psi, \phi) \in \mathbf{E}.$$

Proceeding as we prove Proposition 3.5, we find that for \( \varepsilon \) sufficiently small, there is a \( C^1 \) map from \( (D_\varepsilon)^2 \) to \( \mathbf{E} \): \( (\varphi, \psi, \phi) = (\varphi(r, \rho), \psi(r, \rho), \phi(r, \rho)) \), satisfying

$$J(\varphi, \psi, \phi) = 0, \quad \text{on} \ \mathbf{E},$$

and

$$||(\varphi, \psi, \phi)|| = O\left( \frac{e^{-|x^1-x^2|}}{|x^1-x^2|} + \frac{e^{-|y^1-y^2|}}{|y^1-y^2|} + \varepsilon e^{-(1-r)|x^1-x^2|} + \varepsilon e^{-(1-r)|y^1-y^2|} + \varepsilon e^{-(1-r)|x^1|} + \varepsilon e^{-(1-r)|y^1|} + \varepsilon^4 \right). \quad (4.2)$$

We should point out here that when we carry out the finite dimensional reduction, we do not impose an orthogonal decomposition on \( \phi \) (see the definition of \( \mathbf{E} \)), since the kernel of the operator \( \Delta - (1 - 3U^2)I \) in \( H_\delta \) is \( \{0\} \).

It follows from Proposition A.3 and (4.2) that

$$F(r, \rho) = J(\varphi(r, \rho), \psi(r, \rho), \phi(r, \rho))$$

$$= \sum_{j=1}^\ell I(U_{\varepsilon, x_j} + v_{\varepsilon, x_j} + \omega_{\varepsilon, x_j}) + \sum_{j=1}^\ell I(U_{\varepsilon, y_j} + v_{\varepsilon, y_j} + \omega_{\varepsilon, y_j}) + \sum_{j=1}^\ell I(U_{\varepsilon, \omega_j} + v_{\varepsilon, \omega_j} + \omega_{\varepsilon, \omega_j})$$

$$- \sum_{i<j} C_{ij} e^{-|x^i-x^j|} - \sum_{i<j} C_{ij} e^{-|y^i-y^j|} + \sum_{i,j=1}^\ell \tilde{C}_{ij} e^{-|x^i-x^j|} + \sum_{i,j=1}^\ell \tilde{C}_{ij} e^{-|y^i-y^j|}$$

$$+ O(\varepsilon e^{-(1-r)|y^1-y^2|} + \varepsilon e^{-(1-r)|x^1-x^2|} + \varepsilon^2 (e^{-(1-r)|x^1|} + e^{-(1-r)|y^1|} + e^{-(1-r)|x^1-y^1|}) + \varepsilon^4$$

$$+ O\left( \frac{e^{-|x^1-x^2|}}{|x^1-x^2|} + \frac{e^{-|y^1-y^2|}}{|y^1-y^2|} + \varepsilon e^{-(1-r)|x^1-y^1|} + \varepsilon e^{-(1-r)|y^1|} + \varepsilon e^{-(1-r)|x^1|} + \varepsilon e^{-(1-r)|y^1|} \right)^2$$

$$= C_\varepsilon + \ell \left( \tilde{C}_\varepsilon e^{-r} + \tilde{C}_\varepsilon e^{-\rho} \right) - \frac{C}{mr} e^{-mr} - \frac{C}{mp} e^{-mp} + O(\varepsilon^{m-n+\sigma}),$$

where \( C_\varepsilon > 0 \) depends on \( \varepsilon \) but is independent of \( r \) and \( \rho \). \( \tilde{C}, \tilde{C}, C \) are positive constants independent of \( \varepsilon, r \) and \( \rho \).
Define function
\[
\tilde{G}(r, \rho) = \tilde{C}_\varepsilon e^{-\sqrt{r^2 + \rho^2 - 2r\rho \cos \frac{\pi}{\varepsilon}}} + \tilde{C}_\varepsilon (e^{-r} + e^{-\rho}) - \frac{C}{mr} e^{-mr} - \frac{C}{m\rho} e^{-m\rho}, \quad r, \rho \in \mathcal{D}_\varepsilon.
\]

We want to verify that \(\tilde{G}(r, \rho)\) achieves maximum at some point \((r_0, \rho_0)\) which is in the interior of \(\mathcal{D}_\varepsilon \times \mathcal{D}_\varepsilon\). We have three cases: (1) \(\ell = 2\); (2) \(\ell = 3\); (3) \(\ell > 3\).

**Case (1): \(\ell = 2\).**

In this situation, \(m = 2, n = \sqrt{2}, |x^1 - y^1| = \sqrt{r^2 + \rho^2 - 2r\rho \cos \frac{\pi}{\varepsilon}} > (1 + \sigma) \max\{r, \rho\}\) for some \(\sigma > 0\). Hence, without loss of generality, we suppose
\[
\tilde{G}(r, \rho) = (\tilde{C}_\varepsilon e^{-r} - \frac{C}{2r} e^{-2r}) + (\tilde{C}_\varepsilon e^{-\rho} - \frac{C}{2\rho} e^{-2\rho}) =: G(r) + G(\rho), \quad r, \rho \in \mathcal{D}_\varepsilon.
\]

Therefore we need to modify \(\mathcal{D}_\varepsilon\) as
\[
\mathcal{D}_\varepsilon = \left[\frac{|\ln \varepsilon|}{1 + \mu |\ln |\ln \varepsilon||}, |\ln \varepsilon|\right], \quad \mu > 1.
\]

Using the argument to prove Proposition 3.1 (see Remark 3.7), we can find \(\bar{r}_0\) which is interior points of \(\mathcal{D}_\varepsilon\) such that
\[
G(\bar{r}_0) = \max_{r \in \mathcal{D}_\varepsilon} G(r) \geq C_1 \varepsilon^2 |\ln \varepsilon|^{\bar{\theta}} \geq C_1 \varepsilon^2 \geq C_1 \varepsilon^{\frac{2}{1-\sqrt{2}}}
\]
for some \(\bar{\theta} > 0\) and \(C_1 > 0\).

Suppose that \(\tilde{G}(r, \rho)\) achieves maximum at \((r_0, \rho_0) \in \mathcal{D}_\varepsilon \times \mathcal{D}_\varepsilon\), then
\[
\tilde{G}(r_0, \rho_0) \geq 2G(\bar{r}_0) \geq 2C_1 \varepsilon^2 |\ln \varepsilon|^{\bar{\theta}}.
\]  \hspace{1cm} (4.3)

On the other hand, there exist \(\sigma > 0\) and \(C_2 > 0\) such that
\[
\tilde{G}(\bar{r}, \rho) \leq -C_2 \varepsilon^2 |\ln \varepsilon|^{2\mu-1} + G(\bar{r}_0) < (1 - \sigma) \tilde{G}(r_0, \rho_0), \quad \forall \rho \in \mathcal{D}_\varepsilon,
\]
\[
\tilde{G}(r, \bar{\rho}) \leq C_2 \varepsilon^2 + G(\bar{r}_0) < (1 - \sigma) \tilde{G}(r_0, \rho_0), \quad \forall \rho \in \mathcal{D}_\varepsilon,
\]
\[
\tilde{G}(r, \bar{\rho}) \leq G(\bar{r}_0) - C_2 \varepsilon^2 |\ln \varepsilon|^{2\mu-1} < (1 - \sigma) \tilde{G}(r_0, \rho_0), \quad \forall r \in \mathcal{D}_\varepsilon,
\]
\[
\tilde{G}(r, \bar{\rho}) \leq G(\bar{r}_0) + C_2 \varepsilon^2 < (1 - \sigma) \tilde{G}(r_0, \rho_0), \quad \forall r \in \mathcal{D}_\varepsilon,
\]
where \(\bar{r}\) and \(\bar{\rho}\) are modified respectively as
\[
\bar{r} = \frac{|\ln \varepsilon|}{1 + \mu |\ln |\ln \varepsilon||}, \quad \bar{\rho} = |\ln \varepsilon|.
\]

Therefore, \((r_0, \rho_0)\) is an interior point of \(\mathcal{D}_\varepsilon \times \mathcal{D}_\varepsilon\). Comparing (4.4) with \(\tilde{F}(r, \rho)\) and (4.3), we conclude that \(\tilde{F}(r, \rho)\) has (local) maximizer in the interior of \(\mathcal{D}_\varepsilon \times \mathcal{D}_\varepsilon\).

**Case (2): \(\ell = 3\).**

In this case, \(m = \sqrt{3}, n = 1\), and it is possible that \(r \sim \rho \sim |x^1 - y^1| = \sqrt{r^2 + \rho^2 - 2r\rho \cos \frac{\pi}{\varepsilon}}\). So we consider
\[
\tilde{G}(r, \rho) = \tilde{C}_\varepsilon e^{-\sqrt{r^2 + \rho^2 - r\rho}} + \tilde{C}_\varepsilon (e^{-r} + e^{-\rho}) - \frac{C}{\sqrt{3}r} e^{-\sqrt{3}r} - \frac{C}{\sqrt{3}\rho} e^{-\sqrt{3}\rho}, \quad r, \rho \in \mathcal{D}_\varepsilon.
\]
Now we analyze $\bar{G}(r, \rho)$ on $\partial(D_\epsilon \times D_\epsilon)$.

Firstly, again using (3.18), we see

\[
\bar{G}(\hat{r}, \rho) \leq \hat{C} \epsilon e^{-\sqrt{\rho}} < 0, \forall \rho \in D_\epsilon,
\]

\[
\bar{G}(r, \hat{r}) \leq \hat{C} \epsilon e^{-\sqrt{3\rho}} < 0, \forall r \in D_\epsilon,
\]

where $\hat{C}$ and $C$ are positive constants independent of $\epsilon$.

On the other hand, suppose that, in $D_\epsilon$, $\bar{G}(\hat{r}, \rho)$ achieves maximizer $\bar{\rho} \in (\hat{r}, \hat{r})$. Arguing as we prove Proposition 3.1 (see Remark 3.7), we find

\[
\bar{G}(\hat{r}, \bar{\rho}) \geq C_3 \epsilon \sqrt[\epsilon \sqrt{3} - 1] {\ln |\epsilon|} \theta_0
\]

for some $\theta_0 > 0$ and $C_3 > 0$.

Since

\[
\hat{C} \epsilon e^{-\sqrt{\rho}} - \frac{C}{\sqrt{3\rho}} e^{-\sqrt{3\rho}} = O(\epsilon^{\sqrt{\theta}})
\]

and

\[
e^{-\sqrt{r^2 + \rho^2 - \epsilon\rho}} < e^{-\bar{\rho}},
\]

we see

\[
C_3 \epsilon \sqrt[\epsilon \sqrt{3} - 1] {\ln |\epsilon|} \theta_0 \leq \bar{G}(\hat{r}, \bar{\rho}) < \bar{C} \epsilon e^{-\bar{\rho}} + O(\epsilon^{\sqrt{1}}) + \bar{C} \epsilon e^{-\bar{\rho}} - \frac{C}{\sqrt{3\rho}} e^{-\sqrt{3\rho}}.
\]

Hence, there exists $\sigma > 0$ such that

\[
\bar{G}(\bar{\rho}, \bar{\rho}) = \bar{C} \epsilon e^{-\bar{\rho}} + 2(\bar{C} \epsilon e^{-\bar{\rho}} - \frac{C}{\sqrt{3\rho}} e^{-\sqrt{3\rho}}) > \bar{G}(\hat{r}, \bar{\rho}) + \bar{C} \epsilon e^{-\bar{\rho}} - \frac{C}{\sqrt{3\rho}} e^{-\sqrt{3\rho}} > (1 + \sigma) \bar{G}(\hat{r}, \bar{\rho}),
\]

which implies

\[
\max_{r, \rho \in D_\epsilon} \bar{G}(r, \rho) \geq \bar{G}(\bar{\rho}, \bar{\rho}) > (1 + \sigma) \max_{\rho \in D_\epsilon} \bar{G}(\hat{r}, \bar{\rho}).
\]

Similarly,

\[
\max_{r, \rho \in D_\epsilon} \bar{G}(r, \rho) \geq (1 + \sigma) \max_{\hat{r} \in D_\epsilon} \bar{G}(\hat{r}, \bar{\rho}).
\]

These two estimates and (4.5) imply that $\bar{F}(r, \rho)$ has (local) maximizer in the interior of $D_\epsilon \times D_\epsilon$.

**Case (3):** $\ell > 3$.

In this situation, $\sqrt{r^2 + \rho^2 - 2r\rho \cos \frac{\pi}{\ell}} = |x^1 - y^1| < (1 - \sigma) \min\{r, \rho\}$ for some $\sigma > 0$. Then

\[
\bar{G}(r, \rho) = \bar{C} \epsilon e^{-\sqrt{r^2 + \rho^2 - 2r\rho \cos \frac{\pi}{\ell}}} - \frac{C}{m^r} e^{-m^r} - \frac{C}{m^\rho} e^{-m^\rho}, \, r, \rho \in D_\epsilon.
\]

This is exactly $G(r, \rho)$ in the proof of Proposition 3.1 and we omit the rest of the proof.
As a result, we complete the proof.

APPENDIX A. ENERGY EXPANSIONS

In this section, we will expand the energy \( I(U_\varepsilon, r + v_\varepsilon, \rho, v_\varepsilon, r + U_\varepsilon, \rho) \), which is defined as

\[
I(u, v) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2 + |\nabla v|^2 + v^2) \quad - \frac{1}{4} \int_{\mathbb{R}^3} (u^4 + v^4) + \varepsilon \int_{\mathbb{R}^3} uv, \quad (u, v) \in H_s \times H_s.
\]

Recall that \((U_\varepsilon, v_\varepsilon)\) has the form

\[
U_\varepsilon = U + \varepsilon^2 p_\varepsilon(r) + w, \quad v_\varepsilon = \varepsilon q_\varepsilon(r) + h,
\]

where

\[
p_\varepsilon(r) \leq C e^{-(1-\tau)r}, \quad q_\varepsilon(r) \leq C e^{-(1-\tau)r}, \quad \|(w, h)\| \leq C \varepsilon^4.
\]

The following estimates can be found in [4].

**Proposition A.1.** Suppose that \(u(x), v(x) \in H^1_s(\mathbb{R}^N) (N \geq 1)\) satisfy

\[
u(r) \sim r^\alpha e^{-\beta r}, \quad v(r) \sim r^\gamma e^{-\eta r}, \quad (r \to +\infty),
\]

where \(\alpha, \gamma \in \mathbb{R}, \beta > 0, \eta > 0\). Let \(y \in \mathbb{R}^N\) with \(|y| \to +\infty\). We have

(i) if \(\beta < \eta\), then

\[
\int_{\mathbb{R}^N} u_y v \sim |y|^\alpha e^{-\beta |y|}.
\]

(ii) if \(\beta = \eta\), suppose for simplicity, that \(\alpha \geq \gamma\). Then

\[
\int_{\mathbb{R}^N} u_y v \sim \begin{cases} e^{-\beta |y|}|y|^\alpha e^{\frac{1+N}{2}} & \text{if } \gamma > -\frac{1+N}{2}, \\ e^{-\beta |y|}|y|^\alpha \ln |y| & \text{if } \gamma = -\frac{1+N}{2}, \\ e^{-\beta |y|}|y|^\alpha & \text{if } \gamma < -\frac{1+N}{2}. \end{cases}
\]

**Proposition A.2.** We have

\[
I(U_\varepsilon, r + v_\varepsilon, \rho, v_\varepsilon, r + U_\varepsilon, \rho)
= \sum_{j=1}^\ell I(U_\varepsilon, r^j, v_\varepsilon, r^j) + \sum_{j=1}^\ell I(U_\varepsilon, r^j, v_\varepsilon, y^j)
- \sum_{i<j}^\ell C_{ij} e^{-|x^i-x^j|} - \sum_{i<j}^\ell C_{ij} e^{-|y^i-y^j|} + \varepsilon \sum_{i,j=1}^\ell \tilde{C}_{ij} e^{-|x^i-y^j|}
+ O(\varepsilon e^{-(1-\tau)|y^i-y^j|} + \varepsilon^2 e^{-(1-\tau)|x^i-y^j|} + \varepsilon e^{-(1-\tau)|x^i-x^j|} + \varepsilon^4),
\]

where \(C_{ij}, \tilde{C}_{ij} (i,j = 1, \cdots, \ell)\) are positive constants independent of \(\varepsilon, r\) and \(\rho\).
Proof. Write

\[ I(U_{\varepsilon,r} + v_{\varepsilon,\rho}, v_{\varepsilon,r} + U_{\varepsilon,\rho}) \]

\[ = I(U_{\varepsilon,r}, v_{\varepsilon,r}) + I(U_{\varepsilon,\rho}, v_{\varepsilon,\rho}) \]

\[ - \frac{1}{4} \int_{\mathbb{R}^3} \left( (U_{\varepsilon,r} + v_{\varepsilon,\rho})^4 - U_{\varepsilon,r}^4 - v_{\varepsilon,\rho}^4 - 4 \sum_{i,j=1}^{\ell} U_{\varepsilon,x_i} v_{\varepsilon,y_j} \right) \]

\[ - \frac{1}{4} \int_{\mathbb{R}^3} \left( (U_{\varepsilon,\rho} + v_{\varepsilon,r})^4 - U_{\varepsilon,\rho}^4 - v_{\varepsilon,r}^4 - 4 \sum_{i,j=1}^{\ell} U_{\varepsilon,y_i} v_{\varepsilon,x_j} \right) \]

\[ + \varepsilon \int_{\mathbb{R}^3} \left( (U_{\varepsilon,r} + v_{\varepsilon,\rho})(v_{\varepsilon,r} + U_{\varepsilon,\rho}) - U_{\varepsilon,r} v_{\varepsilon,r} - U_{\varepsilon,\rho} v_{\varepsilon,\rho} - 2 \sum_{i,j=1}^{\ell} v_{\varepsilon,x_i} v_{\varepsilon,y_j} \right) \]

\[ =: I(U_{\varepsilon,r}, v_{\varepsilon,r}) + I(U_{\varepsilon,\rho}, v_{\varepsilon,\rho}) - I_1 - I_2 + \varepsilon I_3. \]

(A.3)

Now we estimate each term in (A.3).

For \( I_1 \), from (A.2) we see

\[ I_1 = \int_{\mathbb{R}^3} \left[ 4 \left( \sum_{i=1}^{\ell} U_{\varepsilon,x_i} \right)^3 \sum_{i=1}^{\ell} v_{\varepsilon,y_i} - 4 \sum_{i,j=1}^{\ell} U_{\varepsilon,x_i} v_{\varepsilon,y_j} + 4 \sum_{i=1}^{\ell} U_{\varepsilon,x_i} \left( \sum_{i=1}^{\ell} v_{\varepsilon,y_i} \right)^3 \right] \]

\[ + O \left( \int_{\mathbb{R}^3} \left( \sum_{i=1}^{\ell} U_{\varepsilon,x_i}^2 \right)^2 \left( \sum_{i=1}^{\ell} v_{\varepsilon,y_i}^2 \right)^2 \right) \]  

\[ = O \left( \int_{\mathbb{R}^3} \sum_{i \neq j}^{\ell} U_{\varepsilon,x_i} U_{\varepsilon,x_j} \sum_{i=1}^{\ell} v_{\varepsilon,y_i} + O(\varepsilon^3 e^{-(1-\tau)|x^1-y^1|} + \varepsilon^2 e^{-(1-\tau)|x^1-y^1|} + \varepsilon^4) \right) \]

\[ = O(\varepsilon e^{-(1-\tau)|x^1-x^2|} + \varepsilon^4) + O(\varepsilon e^{-(1-\tau)|x^1-y^1|}). \]  

(A.4)

Similarly,

\[ I_2 = O(\varepsilon e^{-(1-\tau)|y^1-y^2|} + \varepsilon^4) + O(\varepsilon^2 e^{-(1-\tau)|x^1-y^1|}). \]  

(A.5)

Calculating \( I_3 \), we obtain

\[ I_3 = \int_{\mathbb{R}^3} \left( \sum_{i,j=1}^{\ell} U_{\varepsilon,x_i} U_{\varepsilon,y_j} - \sum_{i,j=1}^{\ell} v_{\varepsilon,x_i} v_{\varepsilon,y_j} \right). \]
On the other hand, by (A.2) and Proposition A.1 we see
\[ \int_{\mathbb{R}^3} U_{\varepsilon,x^i} U_{\varepsilon,y^j} = \int_{\mathbb{R}^3} \left( U_{x^i} + \varepsilon^2 p_{\varepsilon}(|x - x^j|) + w(|x - x^j|) \right) \times \left( U_{y^j} + \varepsilon^2 p_{\varepsilon}(|x - y^j|) + w(|x - y^j|) \right) \]
\[ = \int_{\mathbb{R}^3} U_{x^i} U_{y^j} + O(\varepsilon e^{-(1-\tau)|x^1-y^1|} + \varepsilon) \]
\[ = \bar{C}_{ij} e^{-|x^i-y^j|} + O(\varepsilon e^{-(1-\tau)|x^1-y^1|} + \varepsilon), \tag{A.6} \]
and similarly,
\[ \int_{\mathbb{R}^3} \sum_{i,j=1}^{\ell} v_{\varepsilon,x^i} v_{\varepsilon,y^j} = O(\varepsilon^2 e^{-(1-2\tau)|x^1-y^1|} + \varepsilon^5). \]

Hence,
\[ I_3 = \sum_{i,j=1}^{\ell} \bar{C}_{ij} e^{-|x^i-y^j|} + O(\varepsilon e^{-(1-\tau)|x^1-y^1|} + \varepsilon^4). \tag{A.7} \]

At last, we estimate \( I(U_{\varepsilon,x}, v_{\varepsilon,x}) + I(U_{\varepsilon,\rho}, v_{\varepsilon,\rho}) \). We find
\[ I(U_{\varepsilon,x}, v_{\varepsilon,x}) = \sum_{j=1}^{\ell} I(U_{\varepsilon,x^j}, v_{\varepsilon,x}) - \frac{1}{4} \int_{\mathbb{R}^3} \left[ \left( \sum_{j=1}^{\ell} U_{\varepsilon,x^j} \right)^4 - \sum_{j=1}^{\ell} U_{\varepsilon,x^j}^4 - 4 \sum_{i<j}^{\ell} U_{\varepsilon,x^j} U_{\varepsilon,x^i} \right] \]
\[ - \frac{1}{4} \int_{\mathbb{R}^3} \left[ \left( \sum_{j=1}^{\ell} v_{\varepsilon,x^j} \right)^4 - \sum_{j=1}^{\ell} v_{\varepsilon,x^j}^4 - 4 \sum_{i<j}^{\ell} v_{\varepsilon,x^j} v_{\varepsilon,x^i} \right] \]
\[ = \sum_{j=1}^{\ell} I(U_{\varepsilon,x^j}, v_{\varepsilon,x}) \]
\[ - \int_{\mathbb{R}^3} \left( \sum_{i<j}^{\ell} U_{\varepsilon,x^j} U_{\varepsilon,x^i} + 3 \sum_{i=1}^{\ell} U_{\varepsilon,x^i}^2 U_{\varepsilon,x^1} + \sum_{i<j}^{\ell} v_{\varepsilon,x^j} v_{\varepsilon,x^i} + 3 \sum_{i,j=1}^{\ell} v_{\varepsilon,x^j} v_{\varepsilon,x^i} \right). \]

Similar to (A.6), we have for \( i \neq j \)
\[ \int_{\mathbb{R}^3} \sum_{i<j}^{\ell} U_{\varepsilon,x^j} U_{\varepsilon,x^i} = \sum_{i<j}^{\ell} C_{ij} e^{-|x^i-x^j|} + O(\varepsilon e^{-(1-\tau)|x^1-x^2|} + \varepsilon^4), \]
\[ \int_{\mathbb{R}^3} \sum_{i,j=1}^{\ell} U_{\varepsilon,x^j} U_{\varepsilon,x^i}^2 = \sum_{i<j}^{\ell} C_{ij} e^{-2|x^1-x^2|} + O(\varepsilon e^{-(1-\tau)|x^1-x^2|} + \varepsilon^4), \]
and
\[ \int_{\mathbb{R}^3} \left( \sum_{i<j}^{\ell} v_{\varepsilon,x^j} v_{\varepsilon,x^i} + 3 \sum_{i,j=1}^{\ell} v_{\varepsilon,x^j} v_{\varepsilon,x^i}^2 \right) = O(\varepsilon^4). \]
Therefore,

\[ I(U_{\varepsilon,r},v_{\varepsilon,r}) = \sum_{j=1}^{\ell} I(U_{\varepsilon,x_j},v_{\varepsilon,x_j}) + \sum_{i<j} C_{ij} e^{-|x^i-x^j|} + O(\varepsilon e^{-(1-\tau)|x^1-x^2|} + \varepsilon^4). \tag{A.8} \]

With the same argument, we check

\[ I(U_{\varepsilon,\rho},v_{\varepsilon,\rho}) = \sum_{j=1}^{\ell} I(U_{\varepsilon,y_j},v_{\varepsilon,y_j}) + \sum_{i<j} C_{ij} e^{-|y^i-y^j|} + O(\varepsilon e^{-(1-\tau)|x^1-x^2|} + \varepsilon^4). \tag{A.9} \]

Now, inserting (A.4), (A.5), (A.7), (A.8) and (A.9) into (A.3), we complete the proof.

\[ \square \]

**Proposition A.3.** We have

\[ \bar{I}(U_{\varepsilon,r} + v_{\varepsilon,r} + \omega_{\varepsilon,r} + U_{\varepsilon,\rho} + v_{\varepsilon,\rho} + \omega_{\varepsilon,\rho} + U_\varepsilon) \]

\[ = \sum_{j=1}^{\ell} \bar{I}(U_{\varepsilon,x_j},v_{\varepsilon,x_j},\omega_{\varepsilon,x_j}) + \sum_{j=1}^{\ell} \bar{I}(U_{\varepsilon,y_j},v_{\varepsilon,y_j},\omega_{\varepsilon,y_j}) + \bar{I}(U_{\varepsilon},v_{\varepsilon},\omega_{\varepsilon}) \]

\[ - \sum_{i<j} C_{ij} e^{-|x^i-x^j|} - \sum_{i<j} C_{ij} e^{-|y^i-y^j|} + \varepsilon \sum_{i,j=1}^{\ell} \tilde{C}_{ij} e^{-|x^i-y^j|} + \varepsilon \sum_{j=1}^{\ell} \tilde{C}_j (e^{-|x^j|} + e^{-|y^j|}) \]

\[ + O(\varepsilon e^{-(1-\tau)|y^1-y^2|} + \varepsilon e^{-(1-\tau)|x^1-x^2|} + \varepsilon^2 (e^{-(1-\tau)|x^1|} + e^{-(1-\tau)|y^1|} + e^{-(1-\tau)|x^1-y^1|} + \varepsilon^4), \]

where \( \tilde{C}_{ij}, \tilde{C}_j \) and \( C_{ij} (i, j = 1, \ldots, \ell) \) are positive constants independent of \( \varepsilon, r \) and \( \rho \).

**Proof.** The proof is similar to that of Proposition A.2 and we omit it here.

\[ \square \]

**Acknowledgment.** The authors are grateful to Shusen Yan for helpful discussion. S. Peng thanks Taida Institute for Mathematical Sciences for the warm hospitality during his visit.

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