Lepton flavor-changing Scalar Interactions and Muon $g - 2$

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Abstract

A systematic investigation on muon anomalous magnetic moment and related lepton flavor-violating process such as $\mu \to e\gamma$, $\tau \to e\gamma$ and $\tau \to \mu\gamma$ is made at two loop level in the models with flavor-changing scalar interactions. The two loop diagrams with double scalar exchanges are studied and their contributions are found to be compatible with the ones from Barr-Zee diagram. By comparing with the latest data, the allowed ranges for the relevant Yukawa couplings $Y_{ij}$ in lepton sector are obtained. The results show a hierarchical structure of $Y_{\mu e, \tau e} \ll Y_{\mu\tau} \simeq Y_{\mu\mu}$ in the physical basis if $\Delta a_{\mu}$ is found to be $> 50 \times 10^{-11}$. It deviates from the widely used ansatz in which the off diagonal elements are proportional to the square root of the products of related fermion masses. An alternative Yukawa coupling matrix in the lepton sector is suggested to understand the current data. With such a reasonable Yukawa coupling ansatz, the decay rate of $\tau \to \mu\gamma$ is found to be near the current experiment upper bound.

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I. INTRODUCTION

Recently, the Muon $g - 2$ Collaboration at BNL reported their improved result on the measurement of muon anomalous magnetic moment ($g - 2$) [1]. Combining with the early measurements in CERN and BNL, the new average value of muon $g - 2$ is as follows

$$a_{\mu}^{\text{exp}} = (116592030 \pm 80) \times 10^{-11}$$  \hspace{1cm} (1)

This result confirmed the earlier measurement[2] with a much higher precision. With this new result the difference between experiment and the Standard Model (SM) prediction is enlarged again. The most recent analysis by different groups are given by

$$\Delta a_{\mu} \equiv a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = \begin{cases} (303.3 \pm 106.9) \times 10^{-11} & \text{(ex) [3]} \\ (297.0 \pm 107.2) \times 10^{-11} (in) & \text{[4]} \\ (357.2 \pm 106.4) \times 10^{-11} (ex) & \text{[4]} \end{cases}$$  \hspace{1cm} (2)

As the large muon $g - 2$ may imply the existence of new physics beyond the SM, in the recent years large amount of work has been done in checking the new physics contributions to muon $g - 2$ by using model dependent[5, 6] and independent approaches [7].

In this work, we would like to focus on a general discussion on the models with lepton flavor-changing scalar interactions where the new physics contributions mainly arise from additional Yukawa couplings. Such models may be considered as the simple extension of the standard model (SM) with more than one Higgs doublet $\phi_i \ (i > 1)$ but without imposing any discrete symmetry. For example the extension of SM with two Higgs doublets (S2HDM)[8] motivated from the spontaneous CP violation[9, 10].

The general form of Yukawa interaction reads

$$L_Y = \bar{\psi}_i Y^a_{ij} \psi^a_j \phi_a,$$  \hspace{1cm} (3)

where $Y^a_{ij} \ (i, j = 1, 2, 3)$ is the Yukawa coupling matrices. The index $a = 1, 2, \cdots$ labels the Higgs doublets. The behavior of the Yukawa interactions depends on the texture of Yukawa coupling matrices. In general there are two kind of ansatz on Yukawa coupling matrices in mass eigenstates: 1. Yukawa coupling matrices of the scalar interactions are diagonal due to some discrete global symmetry [11]. 2. Yukawa coupling matrices contain non-zero off-diagonal elements which are naturally suppressed by the light quark masses [12, 13]. In the following sections (section II and III) we discuss at two loop level the constraints on those Yukawa coupling matrix elements under the above two ansatz and will mainly focus on the latter one. One kind of two loop diagrams with double scalar exchanges are studied in detail and their contributions to muon $g - 2$ are found to be compatible with the one from Barr-Zee diagram. In section IV, combined constraints from muon $g - 2$ and several lepton flavor violating (LFV) processes are obtained. We note that unlike other experiments which often impose upper bounds of parameters in the new physics models, the current data on muon $g - 2$ may provide nontrivial lower bounds. It is found that a small lower bound of $\Delta a_{\mu} > 50 \times 10^{-11}$ will significantly modify the texture of Yukawa coupling matrix and make it deviate from the widely used ansatz in which the off diagonal elements are proportional to the square root of the products of related fermion masses.

II. MUON $g - 2$ FROM DIAGONAL YUKAWA COUPLINGS

The ansatz of zero off-diagonal matrix elements is often used to avoid the flavor-changing neutral current (FCNC) at tree level which was originally suggested from the kaon physics,
such as $K \to \mu^+ \mu^-$ decay and $K^0 - \bar{K}^0$ mixing. Such a texture structure of the Yukawa couplings can be obtained by imposing some kind of discrete symmetries\[11\]. The minimal SUSY Standard Model (MSSM) and the two-Higgs-doublet model (2HDM) of type I and II can be cataloged into this type. In such models, the Yukawa interactions are flavor conserving and the couplings are proportional to the related fermion masses

$$Y_{ii} = \frac{g m_i}{2m_W} \xi_i \quad \text{and} \quad Y_{ij} = 0 \ (i \neq j) \quad (4)$$

where $g$ is the weak coupling constant and $m_W$ is the mass of $W$ boson. $\xi_i$ is the rescaled coupling constant. In the minimal SUSY model and the 2HDM of type II, $\xi_i = \tan \beta (\cot \beta)$ for down (up) type fermions. The corresponding Feynman diagrams contributing to muon $g - 2$ at one loop level which is shown in Fig. 1a, which have been recently discussed and compared with the current data in Ref.\[14, 15, 16\].

FIG. 1: One loop diagram contribution to muon $g - 2$. The dashed curves represent the scalar or pseudo-scalar propagator. (a) Flavor conserving Yukawa interactions. (b) Flavor changing Yukawa interactions in which $\mu$ changes into $\tau$ in the loop.

As the muon lepton mass is small, i.e., $m_\mu \ll m_\phi$, where $m_\phi$ is the mass of scalar ($\phi = h$) or pseudo-scalar ($\phi = A$), the one loop contribution to muon $g - 2$ can be written as\[17\]

$$\Delta a_\mu = \pm \frac{1}{8\pi^2} \frac{m_\mu^2}{m_\phi^2} \ln \left( \frac{m_\phi^2}{m_\mu^2} \right) Y_{ii}^2 \quad (5)$$

where the sign “+ (-)” is for scalar ($\phi = h$) (pseudo-scalar $\phi = A$) exchanges. It can be seen from the above equation that the one loop scalar contribution is not large enough to
explain the current data. Even for a large value of \( \xi_\mu = \tan \beta \sim 50 \), one still needs a very light mass of the scalar \( M_h \sim 5 \text{ GeV} \), which seems not favored by the LEP experiment. The situation will be even worse when both the scalar and pseudo-scalar are included as their contributions have opposite signs.

The situation may be quite different if one goes to two loop level. From the well known Barr-Zee mechanism (see Fig. 2) in which the scalar or pseudo-scalar couples to a heavy fermion loop. As the Yukawa couplings are no longer suppressed by the light fermion mass, the two loop contributions could be considerable. Taking the top quark loop as an example, the two loop Barr-Zee diagram contribution to muon \( g - 2 \) is given by

\[
\Delta a_{\mu}^h = \frac{N_c q_t^2 m_\mu m_t}{\pi^2 m_\phi^2} F \left( \frac{m_t^2}{m_\phi^2} \right) Y_{tt} Y_{\mu \mu}
\]

where \( N_c = 3 \) and \( q_t = 2/3 \) are the color number and the charge of top quark respectively. The integral function \( F(z) \) has the following form [18]

\[
F(z) = \begin{cases} 
-\frac{1}{2} \int_0^1 dx \frac{1-2x(1-z)}{x(1-x)-z} \ln \frac{x(1-z)}{z} & \text{for scalar} \\
\frac{1}{2} \int_0^1 dx \frac{1}{x(1-x)-z} \ln \frac{x(1-z)}{z} & \text{for pseudo-scalar}
\end{cases}
\]

FIG. 2: Two loop Barr-Zee diagram contribution to muon \( g - 2 \).

It is noticed that the contributions from Barr-Zee diagram through scalar and pseudo-scalar exchanges have also different sign, negative for scalar and positive for pseudo-scalar,
FIG. 3: Contribution to muon $g - 2$ from the two loop Barr-Zee diagrams. The three solid curves (from down to up) correspond to $Y_{\mu \mu}(\xi_\mu) = 2 \times 10^{-2} (48.3), 4 \times 10^{-2} (97.6)$ and $1 \times 10^{-1} (273.9)$ respectively. The horizontal lines represent the 1$\sigma$ allowed range from Ref. [3] which is just opposite to the one loop case. Thus there exists a cancellation between one and two loop diagram contributions. It was found in Refs. [19, 20] that the pseudo-scalar exchanging Barr-Zee diagram can overwhelm its negative one loop contributions and results in a positive contribution to $g - 2$. For a sufficient large value of the coupling $\xi_\mu = \tan \beta \sim 50$, its contribution can reach the 2$\sigma$ experimental bound with $m_\phi \leq 70$ GeV. To avoid the cancellation between scalar and pseudo-scalar exchange, the mass of the scalar boson has to be pushed to be very heavy (typically greater than 500 GeV). In Fig. 3, the numerical calculation of Barr-Zee diagram contribution to muon $g - 2$ is presented, which agrees with those results.

III. MUON $g - 2$ FROM OFF-DIAGONAL YUKAWA COUPLINGS

When imposing the strict discrete symmetries to Yukawa interaction, the off-diagonal elements of Yukawa coupling matrix are all zeros. This is the simplest way to prevent the theory from tree level FCNC. However, to meet the constraints from the data on $K^0 - \bar{K}^0$ mixing and $K \rightarrow \mu^+ \mu^-$ the off-diagonal elements do not necessarily to be zero. An alternative way is to impose some approximate symmetries such as global family symmetry $\hat{S}$ on the Lagrangian. This results in the second ansatz of the Yukawa matrices in which small off-diagonal matrix elements are allowed, which leads to an enhancement for many flavor changing processes. As the constraints from $K^0 - \bar{K}^0$ mixing are strong, the corresponding
off-diagonal matrix elements should be very small. However, up to now there is no such strong experimental constraints on the FCNC processes involving heavier flavors such as c and b quarks. The possibility of off-diagonal elements associated with the second and the third generation fermions are not excluded. One of the widely used ansatz of the Yukawa matrix basing on the hierarchical fermion mass spectrum $m_u, d \ll m_{c,s} \ll m_{t,b}$ was proposed by Cheng and Sher [12, 13]. In this ansatz, the off-diagonal matrix element has the following form:

$$Y_{ij} = \frac{g \sqrt{m_i m_j}}{2m_W} \xi_{ij}$$

where $\xi_{ij}$'s are the rescaled Yukawa couplings which are roughly of the same order of magnitudes for all $i,j$'s. In this ansatz, the scalar or pseudo-scalar mediating $d-s$ transition is strongly suppressed by small factor $\sqrt{m_d m_s} / (2m_W) \simeq 4 \times 10^{-4}$, which easily satisfies the constraints from $\Delta m_K, \epsilon_K$ and $\Gamma(K \rightarrow \mu^+ \mu^-)$. As the couplings grow larger for heavier fermions, the tree level FCNC processes may give considerable contributions in $B_0 - \bar{B}_0$ mixing, $\mu^+ \mu^- \rightarrow tc, \mu \tau$ and several rare $B$ and $\tau$ decay modes[21, 22, 23, 24].

Unlike the flavor-conserving one loop diagrams, the flavor-changing one loop diagrams (see Fig.1b) with internal heavy fermions can give large contribution to muon $g-2$. The reason is that the loop integration yield an enhancement factor of $\sim m_i \ln(m_i^2/m_\phi^2) / (m_\mu \ln(m_\mu^2/m_\phi^2))$. For the internal $\tau$ loop, it is a factor of $O(10)$. If one uses the scaled coupling $\xi_{\mu\tau}$ and takes $\xi_{\mu\tau} \simeq \xi_{\mu}$ as in the “Cheng-Sher” ansatz, the value of the enhancement factor can reach $O(10^2)$. In the following discussion, for simplicity we only take one loop diagram with internal $\tau$ loop into consideration as it is dominated over other fermion loops in the case that the Yukawa couplings are of the same order of magnitudes.

The expression of one loop flavor changing diagram contribution to muon $g-2$ is given by [25]

$$\Delta a_\mu = \pm \frac{1}{8\pi^2} \frac{m_\mu m_\tau}{m_\phi^2} \left( \frac{m_\phi^2}{m_\tau^2} - \frac{3}{2} \right) Y^{2}_{\mu\tau}$$

where the sign “+ (-)” is for scalar ($\phi = h$) (pseudo-scalar $\phi = A$) exchanges. For detailed discussion on one loop flavor changing diagram, we refer to the Refs.[26, 27].

As the two loop contribution to muon $g-2$ is more considerable via the Barr-Zee mechanism in flavor-conserving case, it is nature to go further to consider the same diagram with flavor-changing couplings. However, in the case of muon $g-2$, as the initial and final states are all muons, it is easy to see that the Barr-Zee diagram with flavor-changing coupling can not contribute. It can only appear in flavor changing process such as $\mu \rightarrow e\gamma$. The non-trivial two loop diagrams which give non-negligible contribution to $g-2$ are those diagrams (as shown in Fig.9) which have two internal scalars with both of them coupling to a heavy fermion loop.

It is known that large Yukawa couplings between scalar and heavy fermions can compensate the loop suppressing factor $g^2/16\pi^2$ and make the Barr-Zee diagram to be sizable. The same mechanism also enhances the two-loop double scalar exchanging diagrams. Further more, in the flavor changing case, the $\mu$ lepton can go into heavier lepton $\tau$ in the lower loop, this may provide an additional enhancement in loop integration. Taking the internal $t$-quark loop as an example, the ratio between the contribution to muon $g-2$ from two-loop double scalar diagrams relative to the one from Barr-Zee type diagrams can be roughly estimated by the ratio between the couplings, which gives $\sim \xi_t \xi_{\mu\tau}^2 m_t m_\tau / (4 \xi_\mu^2 m_W^2 \sin^2 \theta_W)$, where $\theta_W$
is Weinberg angle with the value $\sin^2 \theta_W \simeq 0.23$. For the typical values of $\xi_t = 1$ and $\xi_{\mu\tau} \simeq 30$ the ratio is of order 1. Thus this kind of two-loop double scalar-exchanging diagram is compatible with the one of Barr-Zee type. In the large $m_t$ limit, the contribution to muon $g-2$ from two loop double scalar (pseudo-scalar) exchanging diagram has the following form

$$
\Delta a_\mu = \mp \frac{N_C m_\tau m_\mu m_t^2}{16\pi^2 m_\phi^2} \left( -\frac{5}{2} + \ln \frac{m_\phi^2}{m_\tau^2} \right) Y^2 Y^2_{\mu\tau}
$$

The details of the two loop calculations can be found in the appendix at the end of this paper. Comparing with the one-loop flavor-changing diagram in the same way, one can see that the contribution from this diagram could be sizable. For a comparison, the contribution to muon $g-2$ from two-loop double pseudo scalar-exchanging diagrams and Barr-Zee diagrams with pseudo-scalar are shown in Fig.4.

![Graph](image)

**FIG. 4:** Comparison between two loop Barr-Zee pseudo-scalar and double pseudo-scalar exchanging diagram in contribution to muon $g - 2$. The contribution to muon $g - 2$ is plotted as function of $Y = Y_{\mu\tau} = Y_{\mu\mu}$. Three solid curves (from up to down) correspond to double scalar exchanging diagram contribution with scalar (pseudo-scalar) mass $m_A = 100, 150, 200\text{GeV}$ respectively. Three dashed curves indicate the ones from two loop Barr-Zee diagrams with pseudo-scalar exchange. The horizontal lines represent the $1\sigma$ allowed range from Ref.3.

To make the two kind of contributions comparable, we take $Y_{\mu\mu} = Y_{\mu\tau} \equiv Y$. It can be seen that the contribution from the former highly depends on the coupling $Y$ and the scalar mass. In the range $0.05 \leq Y \leq 0.15$, the contribution from double scalar-exchanging
diagram is much larger than the one from Barr-Zee diagram when $m_A$ is about $100 \sim 150$ GeV. It decreases with $m_A$ increasing and becomes quite small when $m_A \sim 200$ GeV. In Fig. 5, the contribution to muon $g - 2$ from two-loop double scalar-exchanging diagrams is compared with the one from the corresponding flavor-changing one loop diagrams. Note that just like the case of Barr-Zee diagram, the two-loop double scalar-(pseudo-scalar) exchanging diagrams give negative (positive) contributions, which have the opposite signs as the one from one loop scalar (pseudo-scalar)-exchanging diagram. The reason is that a closed fermion loop always contributes a minus sign. It results in a strong cancellation between one and two loop diagram contributions in the case of flavor changing couplings with real Yukawa coupling constants. The allowed range of the scalar mass will be strongly constrained. Taking $Y_{\mu\tau} = 0.08(\xi_{\mu\tau} \simeq 50)$, $Y_{tt} = 0.67(\xi_t \simeq 1)$ and $\Delta a_\mu > 50 \times 10^{-11}$ as an example, the mass of scalar $m_h$ lies in a narrow window of $\sim 100 \leq m_h \leq 200$ GeV.

FIG. 5: Comparison between one loop and two loop double scalar exchange diagrams in contribution to muon $g - 2$. The contribution to muon $g - 2$ is plotted as function of scalar mass. The two dashed curves represent the contribution at one loop with $Y_{\mu\tau}(\xi_{\mu\tau}) = 0.12(70.6)$ (up) and $0.08(47)$ (down) respectively. The two dotted curves correspond to the one from two loop double scalar diagram with the same couplings. (Note that their contribution are negative) The solid curves are the total contribution to $g - 2$ from the both diagrams. The horizontal lines represent the $1\sigma$ allowed range from Ref. [3].
IV. LEPTON FLAVOR VIOLATION PROCESSES AND THE TEXTURE OF YUKAWA MATRIX

The flavor changing Yukawa couplings will unavoidably lead to the enhancement of decay rates of lepton flavor violating processes, such as $\mu \rightarrow e\gamma$, $\tau \rightarrow \mu(e)\gamma$, $\mu \rightarrow e^-e^+e^+$ and $\tau \rightarrow e^-e^-e^+(\mu^-\mu^+)$. The current experimental data especially the data of $\mu \rightarrow e\gamma$ will impose strongest constraints on the related Yukawa couplings. From the current data the upper bound of the decay $\mu \rightarrow e\gamma$ is $\Gamma(\mu \rightarrow e\gamma) \leq 3.6 \times 10^{-30}$ GeV [28]. It constrains the coupling $Y_{e\tau(\mu)}$ to be extremely small. In the models with flavor changing scalar interactions, the leading contributions to $\mu \rightarrow e\gamma$ come from the one loop flavor changing diagram, the two loop double scalar exchanging diagram and the two loop flavor changing Barr-Zee diagrams.

The effective vertex for one loop flavor changing scalar interaction reads [29]

$$\Gamma_{\mu}^{one} = \frac{1}{2(4\pi)^2}\frac{m_\tau}{m_\phi^2} \left( \ln \frac{m_\phi^2}{m_\tau^2} - \frac{3}{2} \right) Y_{\mu \tau} Y_{\tau e} \bar{\ell} i \sigma_{\mu\nu} \ell q^\nu \tag{11}$$

while the one for two loop loop double scalar exchanging is

$$\Gamma_{\mu}^{twoloop} = \frac{N_C m_\tau m_f^2}{32\pi^4 m_\phi^2} \left( \ln \frac{m_\phi^2}{m_\tau^2} - \frac{5}{2} \right) Y_{ff}^2 Y_{\mu \tau}^2 \bar{\ell} i \sigma_{\mu\nu} \ell q^\nu \tag{12}$$

In Fig 5 the decay rate from the sum of the first two diagrams are presented as function of the scalar mass. In the calculation we take the value of coupling $Y_{tt} = 0.67$ (or $\xi_t \approx 1$). The value of $Y_{\mu \tau}$ is taken to be $0.08$ (or $\xi_{\mu \tau} \approx 50$) which is the typical allowed value from the current data on $g - 2$. It can be seen from the figure that the decay rate of $\mu \rightarrow e\gamma$ constrain the value of $Y_{\tau e}$ to be no more than $10^{-6} \sim 10^{-5}$ for $100 \leq m_h \leq 200$GeV. Similarly, the value of coupling $Y_{\mu e}$ is also constrained to be very small by the decay rate $\mu \rightarrow e\gamma$. The reason is that $Y_{\mu e}$ is associated with the flavor changing Barr-Zee diagram in which muon goes into tau in the lower loop. If there is no accidental cancellation with other diagrams the upper bound of $Y_{\mu e}$ can be obtained by assuming that the flavor changing Barr-Zee diagram is dominant. The decay rate of $\mu \rightarrow e\gamma$ from this diagram alone can be obtained from Eq. (13) and is given by

$$\Gamma^{BZ}(\mu \rightarrow e\gamma) = 8\alpha m_\mu^5 \left| \frac{N_C q_f^2 m_\mu m_t}{\pi^2 m_\phi^2} F \left( \frac{m_t^2}{m_\phi^2} \right) Y_{tt} Y_{\mu \mu} \right|^2 \tag{13}$$

The numerical result is represented in Fig 7 which shows that the upper bound of $Y_{\mu e}$ is also of the order $10^{-6} \sim 10^{-5}$ for $100 \leq m_h \leq 200$GeV.

With the above constraints on the values of Yukawa couplings in the lepton sector, let us discuss the possible texture of Yukawa coupling matrix. In the SM with one Higgs doublet, it is well known that by assuming the Yukawa matrix to be of the Fritsch form [30, 31] in flavor basis, i.e.

$$Y \simeq \begin{pmatrix} 0 & \sqrt{m_1 m_2} & \sqrt{m_1 m_3} & 0 \\ \sqrt{m_1 m_2} & 0 & \sqrt{m_2 m_3} & m_3 \\ \sqrt{m_1 m_3} & \sqrt{m_2 m_3} & 0 & m_3 \\ 0 & \sqrt{m_1 m_3} & \sqrt{m_2 m_3} & 0 \end{pmatrix} \tag{14}$$
one can reproduce not only correct quark masses in mass eigenstates but also, in a good approximation, some of the mixing angles. In the models with multi-Higgs doublets, one can simply extend this Fritzsch parameterization to all the other Yukawa matrices including the leptons\[12\]. This results in the ansatz as in Eq.(8) with all $\xi_{ij}$ being of the same order of magnitude.

It is not difficult to see that such an ansatz may be challenged by current experiment data in the lepton sector. This is because in order to explain the possible large muon $g-2$, the off-diagonal elements connecting the second and third families should be enhanced, while to meet the constraints from $\mu \to e\gamma$, the ones connecting the first and second or the first and third families should be greatly suppressed.

Taking the value of $\Delta a_\mu > 50 \times 10^{-11}$, $m_h \sim 150\text{GeV}$ and $m_A \gg m_h$ as an example, in the case of muon $g-2$, if the flavor-conserving Barr-Zee Diagram is playing the major role, the rescaled coupling $\xi_\mu$ should be as large as 50 (see Fig.3). If one assumes that the flavor-changing coupling is responsible for the large muon $g-2$, $\xi_{\mu\tau}(Y_{\mu\tau})$ should be about 10(0.02). On the other hand, due to the strong constraint from $\mu \to e\gamma$, for the flavor-changing contribution dominated case, $\xi_{\tau e}$ has to be less than 0.08 when $\xi_{\mu\tau}(Y_{\mu\tau})$ is taken a typical value of 17.6(0.03). In the case of flavor changing Barr-Zee diagram dominant, the

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FIG. 6: Contribution to decay $\mu \to e\gamma$ from the sum of one loop and two loop double scalar diagrams. The three solid curves (from down to up) correspond to $Y_{\tau e} = 1 \times 10^{-6}, 3 \times 10^{-6}$ and $1 \times 10^{-5}$ respectively. The coupling $Y_{\mu\tau}$ is taken to be 0.08. The horizontal line indicates the experimental upper bound of $\mu \to e\gamma$. \[20\]
FIG. 7: Contribution to decay $\mu \to e \gamma$ from the two loop flavor changing Barr-Zee diagrams. The three solid curves (from down to up) correspond to $Y_{\mu e} = 3 \times 10^{-6}, 7 \times 10^{-6}$ and $1 \times 10^{-5}$ respectively. The horizontal line indicates the experimental upper bound of $\mu \to e \gamma$.

Yukawa coupling $\xi_{\mu e}$ has to be less than 0.24. Thus one finds that

$$\begin{align*}
\xi_{\mu} &\sim \xi_{\mu \tau} \simeq \mathcal{O}(10) \\
\xi_{\tau e} &\sim \xi_{\mu e} \simeq \mathcal{O}(10^{-1})
\end{align*}$$

(15)

which clearly indicates that the rescaled couplings $\xi_{ij}$ are not in the same order of magnitude.

In the case of light pseudo-scalar mass $m_A \simeq 150$ GeV and $m_h \gg m_A$ the results are similar.

From these considerations, it is suggested that the Yukawa matrices associated with the physical scalar bosons may take the following form in the mass eigenstate

$$Y \simeq l^2 \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\lambda^n) & \mathcal{O}(\lambda^n) \\ \mathcal{O}(\lambda^n) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\lambda^n) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}$$

(16)

where $l \approx 0.22$ is roughly of the same order of the Wolfenstein parameter $l$, and $n \simeq 2 \sim 3$. With such a parameterization, one is able to understand all the current experimental data concerning both muon $g - 2$ and lepton flavor-changing processes.

If one takes the Yukawa matrix of the form in Eq. (16), the decay rate of $\tau \to \mu \gamma$ could be predicted. In a good approximation, the decay rate can be obtained by replacing $Y_{\mu \tau}Y_{\tau e}$ into $Y_{\tau \tau}Y_{\tau \mu}$ in Eqs. (11) and (12). Assuming $\tau$ lepton dominance in the loop, the contributions to $\tau \to \mu \gamma$ and shown in Fig. 8. The current upper bound on $\tau \to \mu \gamma$ is $3.5 \times 10^{-19}$ GeV.\[28\]
FIG. 8: Prediction of decay rate $\tau \to \mu \gamma$ from the sum of one loop and two loop double scalar diagrams. The three solid curves (from down to up) correspond to $Y_{\tau \tau}$: 0.003, 0.01 and 0.03 respectively. The coupling $Y_{\mu \tau}$ is taken to be 0.08. The horizontal line indicates the experimental upper bound of $\tau \to \mu \gamma$.

It is found that, the predicted decay rate could reach the current experimental bound. A modest improvement in the precision of the present experiment for $\tau \to \mu \gamma$ may yield a first evidence of lepton family number non-conservation.

In summary, we have studied the muon $g - 2$ and several lepton flavor violation processes in the models with flavor-changing scalar interactions. The two loop diagrams with double scalar exchanges have been investigated and their contribution to muon $g - 2$ is found to be compatible with the one from Barr-Zee diagram. The constraints on Yukawa coupling constants have been resulted from the current data of muon $g - 2$ and several lepton flavor violation processes. The results have shown a very strong constraints on the flavor-changing couplings associated with the first generation lepton. The early ansatz that the flavor changing couplings are proportional to the square root of the products of related fermion masses may not be suitable for the lepton sector if the $\Delta q_\mu$ is found to be $> 50 \times 10^{-11}$. This indicates that both experimental and theoretical uncertainties need to be further reduced in order to explore the existence of new physics from muon $g - 2$. It has been shown that an alternative simple parameterization given in Eq. (16) is more attractive to understand the current experimental data. With such a parameterization, the decay rate of $\tau \to \mu \gamma$ is found to be close to the current experiment upper bound.
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APPENDIX A: TWO LOOP DOUBLE SCALAR DIAGRAMS IN MUON $g-2$ AND $\mu \rightarrow e\gamma$

From the Yukawa interaction shown in Eq.(3), the $\bar{q}q\phi$ vertex has the following form in $d$ dimension.

$$ig\mu^{\epsilon/2}(Y_1 + Y_2\gamma_5)$$  \hspace{1cm} (A1)

where $\mu$ is renormalization scale and $\epsilon/2 = 2 - d/2$. The total amplitude can be written as the product of lower and upper parts as follows

$$\Gamma_\mu = M \cdot I_\mu$$  \hspace{1cm} (A2)

The amplitude $M$ for upper loop is given by

\begin{equation}
M = -g^2\mu^\epsilon \cdot 2N_C \int \frac{d^d l}{(2\pi)^d} \left[ (A_1 + A_2\gamma_5)(l + \not{q} + m_f)(B_1 + B_2\gamma_5)(l + m_f) \right] \frac{1}{(l + q)^2 - m_\tau^2} \frac{1}{l^2 - m_f^2} \tag{A3}
\end{equation}
where \( N_C \) and \( m_f \) are the color number and mass of fermion \( f \). For \( t \) quark \( f = t \) and \( N_C = 3 \). \( A_{1,2} \) and \( B_{1,2} \) are the couplings of vertex \( A \) and \( B \).

The amplitude \( I_\mu \) for the lower loop is given by

\[
I_\mu = -g^2 \mu^\varepsilon \bar{\ell}(p_2)(C_1 + C_2 \gamma_5)(\not{p}_2 - \not{q} + m_\tau)\gamma_\mu(\not{q}_1 - \not{q} + m_\tau)(D_1 + D_2 \gamma_5)\ell(p_1)
\]

\[
\times \frac{1}{(q - p_2)^2 - m_\tau^2} \cdot \frac{1}{(q - p_1)^2 - m_\tau^2} \cdot \frac{1}{(q^2 - m_0^2)^2}
\]

(A4)

where \( C_{1,2} \) and \( D_{1,2} \) are the couplings for vertex \( C \) and \( D \).

After integrating over the lower loop and isolating the poles from Feynman integration, we obtain

\[
\Gamma_\mu = \frac{-8 \cdot 2N_C g^4 (A_1 B_1 - A_2 B_2)m_\tau m_\mu}{(4\pi)^4} \int_0^1 dx x(1-x) \int_0^1 dy \int_0^{1-y} dz (y+z)(1-y-z)
\]

\[
\left( \left( \frac{2}{\ell} - 2\gamma_E + 2 \ln 4\pi - \ln x(1-x) + \frac{1}{2} \right) \cdot f_{1,\text{div}} + 2 \cdot f_{1,\text{con}} \right)
\]

\[
+ \frac{1}{2} C_{ab} R \cdot \left( \left( \frac{2}{\ell} - 2\gamma_E + 2 \ln 4\pi - \ln x(1-x) \right) \cdot f_{2,\text{div}} + 2 \cdot f_{2,\text{con}} \right)
\]

\[
\bar{\ell}(C_1 D_1 + C_2 D_2 + (C_1 D_2 + C_2 D_1) \gamma_5) \frac{i\sigma^{\mu\nu} k_\nu}{2m_\mu} \ell
\]

(A5)

with \( \Delta' = (y+z)m_\tau^2 + (1-y-z)m_\bar{\sigma}^2 \), \( R = m_f^2/|x(1-x)| \) and \( C_{ab} = 2A_2 B_2/(A_1 B_1 - A_2 B_2) \).

In large \( m_f \) limit, i.e. \( m_f^2 \gg \frac{1}{4} m_\tau^2 \gg m_\bar{\sigma}^2 \), the functions \( f_{1,\text{div(con)}} \) and \( f_{2,\text{div(con)}} \) have the following forms

\[
f_{1,\text{div}} \rightarrow \frac{R}{\Delta'^2}, \quad f_{2,\text{div}} \rightarrow -\frac{1}{\Delta'^2}
\]

\[
f_{1,\text{con}} \rightarrow \frac{R}{2\Delta'^2} \left[ 1 - \ln \left( \frac{\Delta' R}{\mu^4} \right) \right], \quad f_{2,\text{con}} \rightarrow \frac{1}{2\Delta'^2} \left[ 1 + \ln \left( \frac{\Delta' R}{\mu^4} \right) \right]
\]

(A6)

After the renormalization in \( \overline{\text{MS}} \) scheme for the upper loop, one finds

\[
\Gamma_\mu = \frac{-8 \cdot 2N_C g^4 m_\tau m_\mu (A_1 B_1 - A_2 B_2)}{(4\pi)^4} \int_0^1 dx x(1-x) \int_0^1 dy \int_0^{1-y} dz (y+z)(1-y-z)
\]

\[
\left[ \frac{3\Delta' + R}{\Delta'^2} (-\ln x(1-x) + \frac{1}{2} + \mathcal{F}(x) + \ln \frac{\Delta'}{\mu^2}) + 2 \cdot f_{1,\text{con}}
\right.
\]

\[
- \frac{1}{\Delta'^2} C_{ab} R \left[ (-\ln x(1-x) + \frac{1}{2} + \mathcal{F}(x) + \ln \frac{\Delta'}{\mu^2}) \right] + \frac{1}{2} C_{ab} R (2 \cdot f_{2,\text{con}} - \frac{1}{2}) + \frac{3\Delta' - 2R}{2\Delta'^2}
\]

\[
\bar{\ell}(p_2)(C_1 D_1 + C_2 D_2 + (C_1 D_2 + C_2 D_1) \gamma_5) i\sigma^{\mu\nu} k_\nu \ell(p_1)
\]

(A7)

with

\[
\mathcal{F}(x) = \ln \frac{m_f^2 - x(1-x)m_\tau^2}{\mu^2}
\]

(A8)
In the limit of $m_f^2 >> \frac{1}{4}m_\phi^2 >> m_\tau^2$, The above equation can be simplified as

$$\Gamma_\mu = -\frac{N_C g^4 m_\tau m_\mu^2}{16\pi^4 m_\phi^4} \left( -\frac{5}{2} + \ln \frac{m_\phi^2}{m_\tau^2} \right) \left( (C_1 D_1 + C_2 D_2) \bar{e}(p_2) \frac{i\sigma_{\mu\nu}k^\nu}{2m_\mu} \ell(p_1) + (C_1 D_2 + C_2 D_1) \bar{\ell}(p_2) \frac{i\sigma_{\mu\nu}k^\nu\gamma_5}{2m_\mu} \ell(p_1) \right)$$  \hspace{1cm} (A9)

Therefore its contribution to muon $g - 2$ is as follows

$$\Delta a_\mu = -\frac{N_C g^4 m_\tau m_\mu^2}{16\pi^4 m_\phi^4} \left( -\frac{5}{2} + \ln \frac{m_\phi^2}{m_\tau^2} \right) (A_1 B_1 + A_2 B_2)(C_1 D_1 + C_2 D_2)$$ \hspace{1cm} (A10)

In the real coupling case, for scalar exchange, one has

$$g A_1 = g B_1 = Y_{f f}, \quad g C_1 = g D_1 = Y_{\mu\tau}$$ \hspace{1cm} (A11)

and others are zero. Similarly for pseudoscalar exchange the couplings are

$$g A_2 = g B_2 = i Y_{f f}, \quad g C_2 = g D_2 = i Y_{\mu\tau}$$ \hspace{1cm} (A12)

Therefore, the two loop double scalar (pseudo-scalar) diagram’s contribution to $\Delta a_\mu$ is

$$\Delta a_\mu = \pm \frac{N_C m_\tau m_\mu m_f^2}{16\pi^4 m_\phi^4} \left( -\frac{5}{2} + \ln \frac{m_\phi^2}{m_\tau^2} \right) Y_{f f}^2 Y_{\mu\tau}^2$$ \hspace{1cm} (A13)

For the decay $\mu \rightarrow e\gamma$, the effective vertex is

$$\Gamma(\mu \rightarrow e\gamma) = \frac{1}{16\pi m_\mu} \sum |e\Gamma(\mu \rightarrow e\gamma)_{\ell\mu}|^2$$ \hspace{1cm} (A15)

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