HIGGS BUNDLES AND GEOMETRIC STRUCTURES ON SURFACES

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Dedicated to Nigel Hitchin on the occasion of his sixtieth birthday.

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INTRODUCTION

In the late 1980’s Hitchin [51] and Simpson [71] discovered deep connections between representations of fundamental groups of surfaces and algebraic geometry. The fundamental group \( \pi = \pi_1(\Sigma) \) of a closed orientable surface \( \Sigma \) of genus \( g > 1 \) is an algebraic object governing the topology of \( \Sigma \). For a Lie group \( G \), the space of conjugacy classes of representations \( \pi \to G \) is a natural algebraic object \( \text{Hom}(\pi,G)/G \) whose geometry, topology and dynamics intimately relates the topology of \( \Sigma \) and the various geometries associated with \( G \). In particular \( \text{Hom}(\pi,G)/G \) arises as a moduli space of locally homogeneous geometric structures as well as flat connections on bundles over \( \Sigma \).

Giving \( \Sigma \) a conformal structure profoundly affects \( \pi \) and its representations. This additional structure induces further geometric and analytic structure on the...
deformation space $\text{Hom}(\pi, G)/G$. Furthermore this analytic interpretation allows Morse-theoretic methods to compute the algebraic topology of these non-linear finite-dimensional spaces.

For example, when $G = U(1)$, the space of representations is a torus of dimension $2g$. Give $\Sigma$ a conformal structure — denote the resulting Riemann surface by $X$. The classical Abel-Jacobi theory identifies representations $\pi_1(X) \to U(1)$ with topologically trivial holomorphic line bundles over $X$. The resulting Jacobi variety is an abelian variety, whose structure strongly depends on the Riemann surface $X$. However the underlying symplectic manifold depends only on the topology of $\Sigma$, and indeed just the fundamental group $\pi$.

Another important class of representations of $\pi$ arises from introducing the local structure of hyperbolic geometry to $\Sigma$. Giving $\Sigma$ a Riemannian metric of curvature $-1$ determines a representation $\rho$ in the group $G = \text{Isom}^+(H^2) \cong \text{PSL}(2, \mathbb{R})$. These representations, which we call Fuchsian, are characterized as embeddings of $\pi$ onto discrete subgroups of $G$. Equivalence classes of Fuchsian representations comprise the Fricke-Teichmüller space $\mathfrak{F}(\Sigma)$ of marked hyperbolic structures on $\Sigma$, which embeds in $\text{Hom}(\pi, G)/G$ as a connected component. This component is a cell of dimension $6g - 6$ upon which the mapping class group acts properly.

The theory of Higgs bundles, pioneered by Hitchin and Simpson, provides an analytic approach to studying surface group representations and their deformation space. The purpose of this paper is to describe the basic examples of this theory, emphasizing relations to deformation and rigidity of geometric structures. In particular we report on some very recent developments when $G$ is a real Lie group, either a split real semisimple group or an automorphism group of a Hermitian symmetric space of noncompact type.

In the twenty years since the appearance of Hitchin’s and Simpson’s work, many other developments directly arose from this work. These relate to variations of Hodge structures, spectral curves, integrable systems, Higgs bundles over noncompact Riemann surfaces and higher-dimensional Kähler manifolds, and the finer topology of the deformation spaces. None of these topics are discussed here. It is an indication of the power and the depth of these ideas that so mathematical subjects have been profoundly influenced by the pioneering work of Hitchin and Simpson.

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1. Representations of the fundamental group

1.1. Closed surface groups. Let $\Sigma = \Sigma_g$ be a closed orientable surface of genus $g > 1$. Orient $\Sigma$, and choose a smooth structure on $\Sigma$. Ignoring base points, denote the fundamental group $\pi_1(\Sigma)$ of $\Sigma$ by $\pi$. The familiar decomposition of $\Sigma$ as a $4g$-gon with $2g$ identifications (depicted in Figures 1 and 2) of its sides leads to a presentation

$$\pi = \langle A_1, B_1, \ldots, A_g, B_g \mid [A_1, B_1] \ldots [A_g, B_g] = 1 \rangle$$

where $[A, B] := ABA^{-1}B^{-1}$.

Figure 1. The pattern of identifications for a genus 2 surface. The sides of an octagon are pairwise identified to construct a surface of genus 2. The 8 vertices identify to a single 0-cell in the quotient, and the 8 sides identify to four 1-cells, which correspond to the four generators in the standard presentation of the fundamental group.

Figure 2. The genus 2 surface as an identification space.
1.2. The representation variety. Denote the set of representations \( \pi \to G \) by \( \text{Hom}(\pi, G) \). Evaluation on a collection \( \gamma_1, \ldots, \gamma_N \in \pi \) defines a map:

\[
\text{Hom}(\pi, G) \to G^N
\]

(1.2.1)

\[
\rho \mapsto \begin{bmatrix}
\rho(\gamma_1) \\
\vdots \\
\rho(\gamma_N)
\end{bmatrix}
\]

which is an embedding if \( \gamma_1, \ldots, \gamma_N \) generate \( \pi \). Its image consists of \( N \)-tuples \((g_1, \ldots, g_N) \in G^N\) satisfying equations

\[
R(g_1, \ldots, g_N) = 1 \in G
\]

where \( R(\gamma_1, \ldots, \gamma_N) \) are defining relations in \( \pi \) satisfied by \( \gamma_1, \ldots, \gamma_N \). If \( G \) is a linear algebraic group, these equations are polynomial equations in the matrix entries of \( g_i \). Thus the evaluation map (1.2.1) identifies \( \text{Hom}(\pi, G) \) as an algebraic subset of \( G^N \). The resulting algebraic structure is independent of the generating set. In particular \( \text{Hom}(\pi, G) \) inherits both the Zariski and the classical topology. We consider the classical topology unless otherwise noted.

In terms of the standard presentation (1.1.1), \( \text{Hom}(\pi, G) \) identifies with the subset of \( G^{2g} \) consisting of

\[
(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)
\]

satisfying the single \( G \)-valued equation

\[
[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1.
\]

1.3. Symmetries. The product \( \text{Aut}(\pi) \times \text{Aut}(G) \) acts naturally by left- and right-composition, on \( \text{Hom}(\pi, G) \): An element

\[
(\phi, \alpha) \in \text{Aut}(\pi) \times \text{Aut}(G)
\]

transforms \( \rho \in \text{Hom}(\pi, G) \) to the composition

\[
\pi \xrightarrow{\phi^{-1}} \pi \xrightarrow{\rho} G \xrightarrow{\alpha} G.
\]

The resulting action preserves the algebraic structure on \( \text{Hom}(\pi, G) \).

1.4. The deformation space. For any group \( H \), let \( \text{Inn}(H) \) denote the normal subgroup of \( \text{Aut}(H) \) comprising inner automorphisms. The quotient group \( \text{Aut}(H)/\text{Inn}(H) \) is the outer automorphism group, denoted \( \text{Out}(H) \).

We will mainly be concerned with the quotient

\[
\text{Hom}(\pi, G)/\text{Inn} := \text{Hom}(\pi, G)/(\{1\} \times \text{Inn}(G)),
\]

which we call the deformation space. For applications to differential geometry, such as moduli spaces of flat connections (gauge theory) or locally homogeneous geometric structures, it plays a more prominent role than the representation variety \( \text{Hom}(\pi, G) \). Although \( \text{Inn}(G) \) preserves the algebraic structure, \( \text{Hom}(\pi, G)/\text{Inn} \) will generally not admit the structure of an algebraic set.

Since the \( \text{Inn}(G) \)-action on \( \text{Hom}(\pi, G) \) absorbs the \( \text{Inn}(\pi) \)-action on \( \text{Hom}(\pi, G) \), the outer automorphism group \( \text{Out}(\pi) \) acts on \( \text{Hom}(\pi, G)/\text{Inn} \). By a theorem of M. Dehn and J. Nielsen (compare Nielsen [64] and Stillwell [73]), \( \text{Out}(\pi) \) identifies with the mapping class group

\[
\text{Mod}(\Sigma) := \pi_0(\text{Diff}(\Sigma)).
\]
One motivation for this study is that the deformation spaces provide natural objects upon which mapping class groups act \cite{AF}.

2. **Abelian groups and rank one Higgs bundles**

The simplest groups are commutative. When $G$ is abelian, then the commutators $[\alpha, \beta] = 1$ and the defining relation in (1.1.1) is vacuous. Thus

$$\text{Hom}(\pi, G) \longleftrightarrow G^{2g}$$

Furthermore $\text{Inn}(G)$ is trivial so

$$\text{Hom}(\pi, G)/G \longleftrightarrow G^{2g}$$

as well.

2.1. **Symplectic vector spaces.** Homological machinery applies. By the Hurewicz theorem and the universal coefficient theorem,

$$\text{Hom}(\pi, G) \cong \text{Hom}(\pi/[\pi, \pi], G) \cong \text{Hom}(H_1(\Sigma), G) \cong H^1(\Sigma, G)$$

(or $H^1(\pi, G)$ if you prefer group cohomology). In particular when $G = \mathbb{R}$, then $\text{Hom}(\pi, G)/G$ is the real vector space

$$H^1(\Sigma, \mathbb{R}) \cong \mathbb{R}^{2g}$$

which is naturally a **symplectic vector space** under the cup-product pairing

$$H^1(\Sigma, \mathbb{R}) \times H^1(\Sigma, \mathbb{R}) \to H^2(\Sigma, \mathbb{R}) \cong \mathbb{R},$$

the last isomorphism corresponding to the orientation of $\Sigma$.

Similarly when $G = \mathbb{C}$, the representation variety and the deformation space

$$\text{Hom}(\pi, G)/G = \text{Hom}(\pi, G) \longleftrightarrow H^1(\pi, \mathbb{C}) \cong H^1(\Sigma, \mathbb{C})$$

is a **complex-symplectic vector space**, that is, a complex vector space with a complex-bilinear symplectic form.

The mapping class group action factors through the action on homology of $\Sigma$, or equivalently the abelianization of $\pi$, which is the homomorphism

$$\text{Mod}(\Sigma) \to \text{Sp}(2g, \mathbb{Z}).$$

2.2. **Multiplicative characters**: $G = \mathbb{C}^*$. Representations $\pi \to \mathbb{C}^*$ correspond to **multiplicative characters**, and are easily understood using the universal covering

$$\mathbb{C} \to \mathbb{C}^*$$

$$z \mapsto \exp(2\pi iz)$$

with kernel $\mathbb{Z} \subset \mathbb{C}$. Such a representation corresponds to a **flat complex line bundle over $\Sigma$**. The deformation space $\text{Hom}(\pi, G)$ identifies with the quotient

$$H^1(\Sigma, \mathbb{C})/H^1(\Sigma, \mathbb{Z}).$$

Restricting to unit complex numbers $G = U(1) \subset \mathbb{C}^*$, identifies $\text{Hom}(\pi, G)$ with the $2g$-dimensional **torus**

$$H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}),$$

the quotient of a real symplectic vector space by an integer lattice, $\text{Mod}(\Sigma)$-acts on this torus by **symplectomorphisms**.
2.3. The Jacobi variety of a Riemann surface. The classical Abel-Jacobi theory (compare for example Farkas-Kra [30]), identifies unitary characters \( \pi_1(X) \rightarrow U(1) \) of the fundamental group of a Riemann surface \( X \) with topologically trivial holomorphic line bundles over \( X \). In particular \( \text{Hom}(\pi, G) \) identifies with the Jacobi variety \( \text{Jac}(X) \).

While the basic structure of \( \text{Hom}(\pi, G) \) is a \( 2g \)-dimensional compact real torus with a parallel symplectic structure, the conformal structure on \( X \) provides much stronger structure. Namely, \( \text{Jac}(X) \) is a principally polarized abelian variety, a projective variety with the structure of an abelian group. Indeed this extra structure, by Torelli’s theorem, is enough to recover the Riemann surface \( X \).

In particular the analytic/algebraic structure on \( \text{Jac}(X) \) is definitely not invariant under the mapping class group \( \text{Mod}(\Sigma) \). However the symplectic structure on \( \text{Hom}(\pi, G) \) is independent of the conformal structure \( X \) and is invariant under \( \text{Mod}(\Sigma) \).

The complex structure on \( \text{Jac}(X) \) is the effect of the complex structure on the tangent bundle \( TX \) (equivalent to the Hodge \( * \)-operator). The Hodge theory of harmonic differential forms finds unique harmonic representatives for cohomology classes, which uniquely extend to holomorphic differential forms. Higgs bundle theory is nonabelian Hodge theory (Simpson [72]) in that it extends this basic technique from ordinary 1-dimensional cohomology classes to flat connections.

When \( G = \mathbb{C}^* \), then \( \text{Hom}(\pi, G) \) acquires a complex structure \( J \) coming from the complex structure on \( \mathbb{C}^* \). This depends only on the topology \( \Sigma \), in fact just its fundamental group \( \pi \). Cup product provides a holomorphic symplectic structure \( \Omega \) on this complex manifold, giving the moduli space the structure of a complex-symplectic manifold.

As for the \( U(1) \)-case above, Hodge theory on the Riemann surface \( X \) determines another complex structure by \( I \); then these two complex structures anti-commute:

\[
IJ + JI = 0,
\]

generating a quaternionic action on the tangent bundle with \( K := IJ \). The symplectic structure arising from cup-product is not holomorphic with respect to \( I \); instead it is Hermitian (of Hodge type \((1,1)\)) with respect to \( I \), extending the Kähler structure on \( \text{Jac}(X) \). Indeed with the structure \( I \), \( \text{Hom}(\pi, \mathbb{C}^*) \) identifies with the cotangent bundle \( T^*\text{Jac}(X) \) with Kähler metric defined by

\[
g(X,Y) := \Omega(X, IY)
\]

The triple \((\Omega, I, J)\) defines a hyper-Kähler structure refining the complex-symplectic structure. If one thinks of a complex-symplectic structure as a \( G \)-structure where \( G = \text{Sp}(2g, \mathbb{C}) \), then a hyper-Kähler refinement is a reduction of the structure group to the maximal compact \( \text{Sp}(2g, \mathbb{C}) \supset \text{Sp}(2g) \). The more common definition of a hyper-Kähler structure involves the Riemannian metric \( g \) which is Kählerian with respect to all three complex structures \( I, J, K \); alternatively it is characterized as a Riemannian manifold of dimension \( 4g \) with holonomy reduced to \( \text{Sp}(2g) \subset \text{SO}(4g) \).

For a detailed exposition of the theory of rank one Higgs bundles on Riemann surfaces, compare Goldman-Xia [43].

3. Stable vector bundles and Higgs bundles

Narasimhan and Seshadri [67] generalized the Abel-Jacobi theory above to identify \( \text{Hom}(\pi, G)/G \) with a moduli space of holomorphic objects over a Riemann
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surface $X$, when $G = \text{U}(n)$. (This was later extended by Ramanathan [69] to general compact Lie groups $G$.)

A notable new feature is that, unlike line bundles, not every topologically trivial holomorphic rank $n$ vector bundle arises from a representation $\pi \to \text{U}(n)$. Furthermore, equivalence classes of all holomorphic $\mathbb{C}^n$-bundles do not form an algebraic set.

Narasimhan and Seshadri define a degree zero holomorphic $\mathbb{C}^n$-bundle $V$ over $X$ to be stable (respectively semistable) if and only if every holomorphic vector subbundle of $V$ has negative (respectively nonpositive) degree. Then a holomorphic vector bundle arising from a unitary representation $\rho$ is semistable, and the bundle is stable if and only if the representation is irreducible. Furthermore, every such semistable bundle arises from a unitary representation. Narasimhan and Seshadri show the moduli space $\mathcal{M}_{n,0}(X)$ of semistable bundles of degree 0 and rank $n$ over $X$ is naturally a projective variety, thus defining such a structure on $\text{Hom}(\pi, G)/G$.

The Kähler structure depends heavily on the Riemann surface $X$, although the symplectic structure depends only on the topology $\Sigma$.

It is useful to extend the notions of stability to bundles which may not have degree 0. In particular, we would like stability to be preserved by tensor product with holomorphic line bundles. Define a holomorphic vector bundle $V$ to be stable if every holomorphic subbundle $W \subset V$ satisfies the inequality

$$\frac{\text{deg}(W)}{\text{rank}(W)} < \frac{\text{deg}(V)}{\text{rank}(V)}.$$  

Semistability is defined similarly by replacing the strict inequality by a weak inequality.

In trying to extend this correspondence to the complexification $G = \text{GL}(n, \mathbb{C})$ of $\text{U}(n)$, one might consider the cotangent bundle $T^*\mathcal{M}_{n,0}(X)$ of the Narasimhan-Seshadri moduli space, and relate it to representations $\pi \to \text{GL}(n, \mathbb{C})$. In particular, since cotangent bundles of Kähler manifolds tend to be hyper-Kähler, relating $\text{Hom}(\pi, G)/G$ to $T^*\mathcal{M}_{n,0}(X)$ might lead to a hyper-Kähler geometry on $\text{Hom}(\pi, G)/G$.

Thus a neighborhood of the $\text{U}(n)$-representations in the space of $\text{GL}(n, \mathbb{C})$ corresponds to a neighborhood of the zero-section of $T^*\mathcal{M}_{n,0}(X)$. In turn, elements in this neighborhood identify with pairs $(V, \Phi)$ where $V$ is a semistable holomorphic vector bundle and $\Phi$ is a tangent covector to $V$ in the space of holomorphic vector bundles. Such a tangent covector is with a Higgs field, by definition, an $\text{End}(V)$-valued holomorphic 1-form on $X$.

Although one can define a hyper-Kähler structure on the moduli space of such pairs, the hyper-Kähler metric is incomplete and not all irreducible linear representations arise. To rectify this problem, one must consider Higgs fields on possibly unstable vector bundles.

Following Hitchin [51] and Simpson [71], define a Higgs pair to be a pair $(V, \Phi)$ where $V$ is a (not necessarily semistable) holomorphic vector bundle and the Higgs field $\Phi$ a $\text{End}(V)$-valued holomorphic 1-form. Define $(V, \Phi)$ to be stable if and only if for all $\Phi$-invariant holomorphic subbundles $W \subset V$, 

$$\frac{\text{deg}(W)}{\text{rank}(W)} < \frac{\text{deg}(V)}{\text{rank}(V)}.$$  

[Note: The text is formatted as a series of paragraphs and does not include any additional notation or figures.]
The Higgs bundle \((V, \Phi)\) is \textit{polystable} if and only if
\[
(V, \Phi) = \bigoplus_{i=1}^{l} (V_i, \Phi_i)
\]
where each summand \((V_i, \Phi_i)\) is stable and
\[
\frac{\deg(V_i)}{\rank(V_i)} = \frac{\deg(V)}{\rank(V)}
\]
for \(i = 1, \ldots, l\).

The following basic result follows from Hitchin [51], Simpson [71], with a key ingredient (the \textit{harmonic metric}) supplied by Corlette [24] and Donaldson [26]:

**Theorem.** The following natural bijections exist between equivalences classes:

\[
\begin{align*}
\{\text{Stable Higgs pairs } (V, \Phi) \text{ over } \Sigma\} & \quad \longleftrightarrow \quad \{\text{Irreducible representations } \pi_1(\Sigma) \rightarrow \GL(n, \mathbb{C})\} \\
\{\text{Polystable Higgs pairs } (V, \Phi) \text{ over } \Sigma\} & \quad \longleftrightarrow \quad \{\text{Reductive representations } \pi_1(\Sigma) \rightarrow \GL(n, \mathbb{C})\}
\end{align*}
\]

When the Higgs field \(\Phi = 0\), this is just the Narasimhan-Seshadri theorem, identifying stable holomorphic vector bundles with irreducible \(U(n)\)-representations. Allowing the Higgs field \(\Phi\) to be nonzero, even when \(V\) is unstable, leads to a rich new class of examples, which can now be treated using the techniques of Geometric Invariant Theory.

4. **Hyperbolic geometry:** \(G = \PSL(2, \mathbb{R})\)

Another important class of surface group representations are \textit{Fuchsian representations}, which arise by endowing \(\Sigma\) with the local geometry of \textit{hyperbolic space} \(H^2\). Here \(G\) is the group of orientation-preserving isometries \(\text{Isom}^+(H^2)\), which, using Poincaré’s \textit{upper half-space model}, identifies with \(\PSL(2, \mathbb{R})\). Fuchsian representations are characterized in many different equivalent ways; in particular a representation \(\pi \rightarrow G = \PSL(2, \mathbb{R})\) is Fuchsian if and only if it is a \textit{discrete embedding}, that is, \(\rho\) embeds \(\pi\) isomorphically onto a discrete subgroup of \(G\).

4.1. **Geometric structures.** Let \(H^2\) be the hyperbolic plane with a fixed orientation and \(G \cong \text{Isom}^+(H^2) \cong \PSL(2, \mathbb{R})\) its group of orientation-preserving isometries. A \textit{hyperbolic structure} on a topological surface \(\Sigma\) is defined by a coordinate atlas \(\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}\) where

- The collection \(\{U_\alpha\}_{\alpha \in A}\) of coordinate patches \textit{covers} \(\Sigma\) (for some index set \(A\));
- Each coordinate chart \(\psi_\alpha\) is an orientation-preserving homeomorphism of the coordinate patch \(U_\alpha\) onto an open subset \(\psi_\alpha(U_\alpha) \subset H^2\).
- For each connected component \(C \subset U_\alpha \cap U_\beta\), there is (necessarily unique) \(g_{C, \alpha, \beta} \in G\) such that

\[
\psi_\alpha|_C = g_{C, \alpha, \beta} \circ \psi_\beta|_C.
\]

The resulting \textit{local} hyperbolic geometry defined on the patches by the coordinate charts is independent of the charts, and extends to a global structure on \(\Sigma\). The surface \(\Sigma\) with this refined structure of local hyperbolic geometry, will be called a \textit{hyperbolic surface} and denoted by \(M\). Such a structure is equivalent to a Riemannian metric of constant curvature \(-1\). The equivalence follows from two basic facts:
• Any two Riemannian manifolds of curvature $-1$ are locally isometric;
• A local isometry from a connected subdomain of $\mathbb{H}^2$ extends globally to an isometry of $\mathbb{H}^2$.

Suppose $M_1, M_2$ are two hyperbolic surfaces. Define a morphism $M_1 \xrightarrow{\phi} M_2$ as a map $\phi$, which, in the preferred local coordinates of $M_1$ and $M_2$, is defined by isometries in $G$. Necessarily a morphism is a local isometry of Riemannian manifolds.

Furthermore, if $M$ is a hyperbolic surface and $\Sigma \xrightarrow{f} M$ is a local homeomorphism, there exists a hyperbolic structure on $\Sigma$ for which $f$ is a morphism. In particular every covering space of a hyperbolic surface is a hyperbolic surface.

In more traditional terms, a morphism of hyperbolic surfaces is just a local isometry.

4.2. Relation to the fundamental group. While the definitions involving coordinate atlases or Riemannian metrics have certain advantages, another point of view underscores the role of the fundamental group.

Let $M$ be a hyperbolic surface. Choose a universal covering space $\tilde{M} \rightarrow M$ and give $\tilde{M}$ the unique hyperbolic structure for which $\tilde{M} \rightarrow M$ is a local isometry. Then there exists a developing map $\tilde{M} \xrightarrow{\text{dev}_M} \mathbb{H}^2$, a local isometry, which induces the hyperbolic structure on $\tilde{M}$ from that of $\mathbb{H}^2$. The group $\pi_1(M)$ of deck transformations of $\tilde{M} \rightarrow M$ acts on $\tilde{M}$ by isometries and $\text{dev}$ is equivariant respecting this action: for all $\gamma \in \pi_1(M)$, the diagram

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\text{dev}_M} & \mathbb{H}^2 \\
| \downarrow \gamma |
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\text{dev}_M} & \mathbb{H}^2 \\
| \downarrow \rho(\gamma) |
\end{array}
\]

commutes. The correspondence $\gamma \mapsto \rho(\gamma)$ is a homomorphism,

$\pi_1(M) \xrightarrow{\text{hol}_M} \text{Isom}^+(\mathbb{H}^2),$

the holonomy representation of the hyperbolic surface $M$. The pair $(\text{dev}_M, \text{hol}_M)$ is unique up to the $G$-action defined by

$$(\text{dev}_M, \text{hol}_M) \xrightarrow{\text{g}} (\text{dev}_M \circ \text{g}, \text{Inn}(\text{g}) \circ \text{hol}_M)$$

for $g \in \text{Isom}^+(\mathbb{H}^2)$.

If the hyperbolic structure is complete, that is, the Riemannian metric is geodesically complete, then the developing map is a global isometry $\tilde{M} \approx \mathbb{H}^2$. In that case the $\pi$-action on $\mathbb{H}^2$ defined by the holonomy representation $\rho$ is equivalent to the action by deck transformations. Thus $\rho$ defines a proper free action of $\pi$ on $\mathbb{H}^2$ by isometries. Conversely if $\rho$ defines a proper free isometric $\pi$-action, then the quotient

$M := \mathbb{H}^2/\rho(\pi)$

is a complete hyperbolic manifold with a preferred isomorphism

$\pi_1(\Sigma) \xrightarrow{\rho} \rho(\pi) \subset G.$

This isomorphism (called a marking) determines a preferred homotopy class of homotopy equivalences

$\Sigma \xrightarrow{\rho} M.$
4.3. Examples of hyperbolic structures. We now give three examples of surface group representations in $\text{PSL}(2, \mathbb{R})$. The first example is Fuchsian and corresponds to a hyperbolic structure on a surface of genus two. The second example is not Fuchsian, but corresponds to a hyperbolic structure with a single branch point, that is a point with local coordinate given by a branched conformal mapping $z \mapsto z^k$ where $k \geq 1$. (The nonsingular case corresponds to $k = 1$.) In our example $k = 2$ and the singular point has a neighborhood isometric to a hyperbolic cone of cone angle $4\pi$.

4.3.1. A Fuchsian example. Here is a simple example of a hyperbolic surface of genus two. Figure 1 depicts a topological construction for a genus two surface $\Sigma$. Realizing this topological construction in hyperbolic geometry gives $\Sigma$ a local hyperbolic geometry as follows. Take a regular octagon $P$ with angles $\pi/4$. Label the sides as $A_1^{-}, B_1^{-}, A_1^{+}, B_1^{+}, A_2^{-}, B_2^{-}, A_2^{+}, B_2^{+}$.

Pair the sides by $a_1, b_1, a_2, b_2 \in \text{PSL}(2, \mathbb{R})$ according to the pattern described in Figure 1. Given any two oriented geodesic segments in $H^2$ of equal length, a unique orientation-preserving isometry maps one to the other. Since the polygon is regular, one can realize all four identifications in $\text{Isom}^+(H^2)$.

The quotient (compare Figure 2) contains three types of points:

- A point in the open 2-cell has a coordinate chart which is the embedding $P \hookrightarrow H^2$.
- A point on the interior of an edge has a half-disc neighborhood, which together with the half-disc neighborhood of its part, gives a coordinate chart for the corresponding point in the quotient.
- Around the single vertex in the quotient is a cone of angle $8(\pi/4) = 2\pi$.

The resulting identification space is a hyperbolic surface of genus $g = 2$. The above isometries satisfying the defining relation for $\pi_1(\Sigma)$:

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} = 1$$

and define a Fuchsian representation

$$\pi_1(\Sigma) \twoheadrightarrow \text{PSL}(2, \mathbb{R}).$$

4.3.2. A branched hyperbolic structure. We can modify the preceding example to include a singular structure, again on a surface of genus two. Take a regular right-angled octagon. Again, labeling the sides as before, side pairings $a_1, b_1, a_2, b_2$ exist. Now 8 right angles compose a neighborhood of the vertex in the quotient space. The quotient space is a hyperbolic structure with one singularity of cone angle $4\pi = 8(\pi/2)$. Since the product of the identification mappings

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$$
**Figure 3.** A regular octagon with vertex angles $\pi/4$ can be realized in the tiling of $\mathbb{H}^2$ by triangles with angles $\pi/2, \pi/4, \pi/8$. The identifications depicted in Figure [1] are realized by orientation-preserving isometries. The eight angles of $\pi/4$ fit together to form a cone of angle $2\pi$, forming a coordinate chart for a hyperbolic structure around that point.

is rotation through $4\pi$ (the identity), the holonomy representation $\hat{\rho}$ of the nonsingular hyperbolic surface $\Sigma \setminus \{p\}$ extends:

$$
\begin{array}{c}
\pi_1(S \setminus \{p\}) \\
\downarrow \\
\pi_1(\Sigma)
\end{array}
\xrightarrow{\hat{\rho}}
\begin{array}{c}
\pi_1(\Sigma) \\
\xrightarrow{\rho} \text{PSL}(2, \mathbb{R})
\end{array}
$$

Although $\rho(\pi)$ is discrete, $\rho$ is not injective.

4.3.3. **A representation with no branched structures.** Consider a degree-one map $f$ from a genus three surface $\Sigma$ to a genus two hyperbolic surface $M$, depicted in Figure [3]. Let $\pi_1(M) \xrightarrow{\mu} G$ denote the holonomy representation of $M$ and consider the composition

$$
\pi = \pi_1(\Sigma) \xrightarrow{f_*} \pi_1(M) \xrightarrow{\mu} G.
$$

Then a branched hyperbolic structure with holonomy $\mu \circ f_*$ corresponds to a mapping with branch singularities

$$
\Sigma \xrightarrow{f_*} \mathbb{H}^2 / \text{Image}(\mu \circ f_*) = M.
$$

inducing the homomorphism

$$
\pi = \pi_1(\Sigma) \xrightarrow{f_*} \pi_1(M).
$$
Figure 4. A regular right-angled octagon can also be realized in the tiling of $H^2$ by triangles with angles $\pi/2, \pi/4, \pi/8$. The identifications depicted in Figure 1 are realized by orientation-preserving isometries. The eight angles of $\pi/2$ fit together to form a cone of angle $4\pi$, forming a coordinate chart for a singular hyperbolic structure, branched at one point.

Figure 5. A degree one map from a genus 3 surface to a genus 2 surface which collapses a handle. Such a map is not homotopic to a smooth map with branch point singularities (such as a holomorphic map).

In particular $F \simeq f$. However, since $\deg(f) = 1$, any mapping with only branch point singularities of degree one must be a homeomorphism, a contradiction.
5. Moduli of hyperbolic structures and representations

To understand “different” geometric structures on the “same” surface, one introduces markings. Fix a topological type \( \Sigma \) and let the geometry \( M \) vary. The fundamental group \( \pi = \pi_1(\Sigma) \) is also fixed, and each marked structure determines a well-defined equivalence class in \( \text{Hom}(\pi, G)/G \). Changing the marking corresponds to the action of the mapping class group \( \text{Mod}(\Sigma) = \text{Out}(\pi) \) on \( \text{Hom}(\pi, G)/G \). Unmarked structures correspond to the orbits of the \( \text{Mod}(\Sigma) \)-action.

5.1. Deformation spaces of geometric structures. A marked hyperbolic structure on \( \Sigma \) is defined as a pair \((M, f)\) where \( M \) is a hyperbolic surface and \( f \) is a homotopy equivalence \( \Sigma \rightarrow M \). Two marked hyperbolic structures \( \Sigma \xrightarrow{f} M, \Sigma \xrightarrow{f'} M' \) are equivalent if and only if there exists an isometry \( M \xrightarrow{\phi} M' \) such that \( f \circ \phi \simeq f' \). The Fricke space \( \mathcal{F}(\Sigma) \) of \( \Sigma \) is the space of all such equivalence classes of marked hyperbolic structures on \( \Sigma \). (Bers-Gardiner [9].) The Fricke space is diffeomorphic to \( \mathbb{R}^{6g-6} \). The theory of moduli of hyperbolic structures on surfaces goes back at least to Fricke and Klein [32].

The Teichmüller space \( \mathcal{T}(\Sigma) \) of \( \Sigma \) is defined similarly, as the space of equivalence classes of marked conformal structures on \( \Sigma \), that is, pairs \((X, f)\) where \( X \) is a Riemann surface and \( \Sigma \xrightarrow{f} X \) is a homotopy equivalence. Teichmüller used quasiconformal mappings to parametrize \( \mathcal{T}(\Sigma) \) by elements of a vector space, define a metric on \( \mathcal{T}(\Sigma) \) and prove analytically that \( \mathcal{T}(\Sigma) \) is a cell. Using these ideas, Ahlfors [1] proved \( \mathcal{T}(\Sigma) \) is naturally a complex manifold.

Since a hyperbolic structure is a Riemannian metric, every hyperbolic structure has an underlying conformal structure. The uniformization theorem asserts that if \( \chi(\Sigma) < 0 \), then every conformal structure on \( \Sigma \) underlies a unique hyperbolic structure. The resulting identification of conformal and hyperbolic structures identifies \( \mathcal{T}(\Sigma) \) with \( \mathcal{F}(\Sigma) \). As discussed below, \( \mathcal{F}(\Sigma) \) identifies with an open subset of \( \text{Hom}(\pi, \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R}) \) which has no apparent complex structure. Thus the complex structure on \( \mathcal{T}(\Sigma) \) is more mysterious when \( \mathcal{T}(\Sigma) \) is viewed as a space of hyperbolic structures. For a readable survey of classical Teichmüller theory see Bers [8].

5.2. Fuchsian components of \( \text{Hom}(\pi, G)/G \). To every equivalence class of marked hyperbolic structures is associated a well-defined element \( [\rho] \in \text{Hom}(\pi, G)/G \).

A representation \( \pi \xrightarrow{\rho} G \) is Fuchsian if and only if it arises as the holonomy of a hyperbolic structure on \( \Sigma \). Equivalently, it satisfies the three conditions:

- \( \rho \) is injective;
- Its image \( \rho(\pi) \) is a discrete subgroup of \( G \);
- The quotient \( G/\rho(\pi) \) is compact.
The first condition asserts that $\rho$ is an embedding, and the second two conditions assert that $\rho(\pi)$ is a cocompact lattice. Under our assumption $\partial \Sigma = \emptyset$, the third condition (compactness of $G/\rho(\pi)$) follows from the first two. In general, we say that $\rho$ is a discrete embedding (or discrete and faithful) if $\rho$ is an embedding with discrete image (the first two conditions).

**Theorem 5.2.1.** Let $G = \text{Isom}(H^2) = \text{PGL}(2, \mathbb{R})$ and $\Sigma$ a closed connected surface with $\chi(\Sigma) < 0$. Fricke space, the subset of $\text{Hom}(\pi, G)/G$ consisting of $G$-equivalence classes of Fuchsian representations, is a connected component of $\text{Hom}(\pi, G)/G$.

This result follows from three facts:

- Openness of Fricke space (Weil [78]),
- Closedness of Fricke space (Chuckrow [21]),
- Connectedness of Fricke space

Chuckrow’s theorem is a special case of a consequence of the Kazhdan-Margulis uniform discreteness (compare Raghunathan [68] and Goldman-Millson [40]). These ideas go back to Bieberbach and Zassenhaus in connection with the classification of Euclidian crystallographic groups. Uniform discreteness applies under very general hypotheses, to show that discrete embeddings form a closed subset of the representation variety. For the proof of connectedness, see, for example, Jost [54], §4.3, Buser [20], Thurston [74] or Ratcliffe [70] for elementary proofs using Fenchel-Nielsen coordinates. Connectedness also follows from the uniformization theorem, together with the identification of Teichmüller space $\mathfrak{T}(\Sigma)$ as a cell.

When $G = \text{Isom}^+(H^2) = \text{PSL}(2, \mathbb{R})$, the situation slightly complicates, due to the choice of orientation. Assume $\Sigma$ is orientable, and orient it. Orient $H^2$ as well. Let $\Sigma \to M$ be a marked hyperbolic structure on $\Sigma$. The orientation of $M$ induces an orientation of $\tilde{M}$ which is invariant under $\pi_1(M)$. However, the developing map $\text{dev}_M$ may or not preserve the (arbitrarily chosen) orientations of $\tilde{M}$ and $H^2$. Accordingly $\text{Isom}^+(H^2)$-equivalence classes of Fuchsian representations in $G$ fall into two classes, which we call orientation-preserving and orientation-reversing respectively. These two classes are interchanged by inner automorphisms of orientation-reversing isometries of $H^2$.

**Theorem 5.2.2.** Let $G = \text{Isom}^+(H^2) = \text{PSL}(2, \mathbb{R})$ and $\Sigma$ a closed connected oriented surface with $\chi(\Sigma) < 0$. The set of $G$-equivalence classes of Fuchsian representations forms two connected components of $\text{Hom}(\pi, G)/G$. One component corresponds to orientation-preserving Fuchsian representations and the other to orientation-reversing Fuchsian representations.

### 5.3. Characteristic classes and maximal representations

Characteristic classes of flat bundles determine invariants of representations. In the simplest cases (when $G$ is compact or reductive complex), these determine the connected components of $\text{Hom}(\pi, G)$.

**5.3.1. The Euler class and components of $\text{Hom}(\pi, \text{PSL}(2, \mathbb{R}))$.** The components of $\text{Hom}(\pi, G)$ were determined in [37] using an invariant derived from the Euler class of the oriented $H^2$-bundle

$$
\begin{array}{c}
H^2 \longrightarrow (H^2)_\rho \\
\downarrow \\
\Sigma
\end{array}
$$
associated to a representation \( \pi \rightarrow \text{PSL}(2, \mathbb{R}) \) as follows. The total space is the quotient

\[
(H^2)^\rho := (\tilde{\Sigma} \times H^2)/\pi
\]

where \( \pi \) acts diagonally on \( \tilde{\Sigma} \times H^2 \) by deck transformations on \( \tilde{\Sigma} \) and via \( \rho \) on \( H^2 \). Isomorphism classes of oriented \( H^2 \)-bundles over \( \Sigma \) are determined by the Euler class, which lives in \( H^2(\Sigma, \mathbb{Z}) \). The orientation of \( \Sigma \) identifies this cohomology group with \( \mathbb{Z} \). The resulting map

\[
\text{Hom}(\pi, G) \xrightarrow{\text{Euler}} H^2(S; \mathbb{Z}) \cong \mathbb{Z}
\]

satisfies

\[
|\text{Euler}(\rho)| \leq |\chi(S)| = 2g - 2.
\]

(Milnor [66] and Wood [81]). Call a representation maximal if equality holds in in (5.3.1), that is, \( \text{Euler}(\rho) = \pm \chi(\Sigma) \):

The following converse was proved in Goldman [35] (compare also [37] and [51]).

**Theorem 5.3.1.** \( \rho \) is maximal if and only if \( \rho \) is Fuchsian.

Suppose \( M \) is a branched hyperbolic surface with branch points \( p_1, \ldots, p_l \) where \( p_i \) is branched of order \( k_i \), where each \( k_i \) is a positive integer. In other words, each \( p_i \) has a neighborhood which is a hyperbolic cone of cone angle \( 2\pi k_i \). Consider a marking \( \Sigma \rightarrow M \), determining a holonomy representation \( \rho \). Then

\[
\text{Euler}(\rho) = \chi(\Sigma) + \sum_{i=1}^{l} k_i.
\]

Consider the two examples for genus two surfaces.

- The first (Fuchsian) example (§4.3.1) arising from a regular octagon with \( \pi/4 \) angles, has Euler class \(-2 = \chi(\sigma)\).
- In the second example (§4.3.2), the structure is branched at one point, so that \( l = k_1 = 1 \) and the Euler class equals \(-1 = \chi(\Sigma) + 1\).

### 5.4. Quasi-Fuchsian representations: \( G = \text{PSL}(2, \mathbb{C}) \)

When the representation

\[
\pi \longrightarrow \text{PSL}(2, \mathbb{R}) \hookrightarrow \text{PSL}(2, \mathbb{C})
\]

is deformed inside \( \text{PSL}(2, \mathbb{C}) \), the action on \( \mathbb{CP}^1 \) is topologically conjugate to the original Fuchsian action. Furthermore there exists a Hölder \( \rho \)-equivariant embedding \( S^1 \hookrightarrow \mathbb{CP}^1 \), whose image \( \Lambda \) has Hausdorff dimension \( > 1 \), — unless the deformation is still Fuchsian. The space of such representations is the quasi-Fuchsian space \( QF(\Sigma) \). By Bers [7], \( QF(\Sigma) \) naturally identifies with

\[
\mathfrak{T}(\Sigma) \times \mathfrak{T}(\Sigma)^* \cong \mathbb{R}^{12g-12}.
\]

Bers’s correspondence is the following. The action of \( \rho \) on the complement \( \mathbb{CP}^1 \setminus \Lambda \) is properly discontinuous, and the quotient

\[
(\mathbb{CP}^1 \setminus \Lambda)/\rho(\pi)
\]

consists of two Riemann surfaces, each with a canonical marking determined by \( \rho \). Furthermore these surfaces possess opposite orientations, so the pair of marked conformal structures determine a point in \( \mathfrak{T}(\Sigma) \times \mathfrak{T}(\Sigma) \). Bonahon [10] and Thurston proved that the closure of \( QF(\Sigma) \) in \( \text{Hom}(\pi, G)/G \) equals the set of equivalence
classes of discrete embeddings. The frontier $\partial QF(\Sigma) \subset \text{Hom}(\pi, G)/G$ is nonrectifiable, and is near non-discrete representations.

However, the two connected components of $\text{Hom}(\pi, G)/G$ are distinguished by the characteristic class (related to the second Stiefel-Whitney class $w_2$) which detects whether a representation in $\text{PSL}(2, \mathbb{C})$ lifts to the double covering $\text{SL}(2, \mathbb{C}) \to \text{PSL}(2, \mathbb{C})$ (Goldman [37]). Contrast this situation with $\text{PSL}(2, \mathbb{R})$ where the discrete embeddings form connected components, characterized by maximality.

Figure 6. A quasi-Fuchsian subgroup of $\text{PSL}(2, \mathbb{C})$ obtained by deforming the genus two surface with a fundamental domain the regular octagon with $\pi/4$ angles in $\mathbb{C}P^1$. The limit set is a nonrectifiable Jordan curve, but the new action of $\pi_1(\Sigma)$ is topologically conjugate to the original Fuchsian action.

5.4.1. Higher rank Hermitian spaces: the Toledo invariant. Domingo Toledo [73] generalized the Euler class of flat $\text{PSL}(2, \mathbb{R})$-bundles to flat $G$-bundles, where $G$ is the automorphism group of a Hermitian symmetric space $X$ of noncompact type.

Let $\pi \rightarrow G$ be a representation and let $X \rightarrow (X)_\rho \rightarrow \Sigma$ be the corresponding flat $(G, X)$-bundle over $\Sigma$. Then the $G$-invariant Kähler form $\omega$ on $X$ defines a closed exterior 2-form $\omega_\rho$ on the total space $(X)_\rho$. Let $\Sigma \rightarrow \Sigma \times (X)_\rho$ be a smooth section. Then the integral

$$\int_{\Sigma} s^* \omega_\rho$$
is independent of $s$, depends continuously on $\rho$ and, after suitable normalization, assumes integer values. The resulting Toledo invariant

$$\text{Hom}(\pi, G) \xrightarrow{\tau} \mathbb{Z}$$

satisfies

$$|\tau(\rho)| \leq (2g - 2) \text{rank}_\mathbb{R}(G).$$

(Domic-Toledo [25], Clerc-Ørsted [22]). Define $\rho$ to be maximal if and only if

$$|\tau(\rho)| = (2g - 2) \text{rank}_\mathbb{R}(G).$$

**Theorem 5.4.1** (Toledo [75]). $\pi \xrightarrow{\rho} \text{U}(n, 1)$ is maximal if and only if $\rho$ is a discrete embedding preserving a complex geodesic, that is, $\rho$ is conjugate to a representation with

$$\rho(\pi) \subset \text{U}(1, 1) \times \text{U}(n - 1).$$

This rigidity has a curious consequence for the local geometry of the deformation space. Let $G := \text{U}(n, 1)$ and

$$G_0 = \text{U}(1, 1) \times \text{U}(n - 1) \subset G.$$

Then, in an appropriate sense,

$$\dim \text{Hom}(\pi, G)/G = 2g + (2g - 2)((n + 1)^2 - 1) = (2g - 2)(n + 1)^2 + 2$$

but Toledo’s rigidity result implies that the component of maximal representations has strictly lower dimension:

$$\dim \text{Hom}(\pi, G_0)/G_0 = 4g + (2g - 2)3 + (2g - 2)((n - 1)^2 - 1)$$

with codimension

$$8(n - 1)(g - 1) - 2.$$

### 5.5. Teichmüller space: marked conformal structures.

The Teichmüller space $\mathcal{T}(\Sigma)$ of $\Sigma$ is the deformation space of marked conformal structures on $\Sigma$.

A marked conformal structure on $\Sigma$ is a pair $(X, f)$ where $X$ is a Riemann surface and $f$ is a homotopy equivalence $\Sigma \to X$. Marked conformal structures

$$\Sigma \xrightarrow{f} X, \quad \Sigma \xrightarrow{f'} X'.$$

are equivalent if and only if there exists a biholomorphism $X \xrightarrow{\phi} X'$ such that

$$\begin{array}{ccc}
\Sigma & \xrightarrow{f'} & X' \\
\downarrow \phi & & \downarrow \\
X & \xrightarrow{f} & \Sigma
\end{array}$$

homotopy-commutes.

**Theorem 5.5.1** (Uniformization). Let $X$ be a Riemann surface with $\chi(X) < 0$. Then there exists a unique hyperbolic metric whose underlying conformal structure agrees with $X$. 
Since every hyperbolic structure possesses an underlying conformal structure, Fricke space $F(\Sigma)$ maps to Teichmüller space $T(\Sigma)$. By the uniformization theorem, $F(\Sigma) \to T(\Sigma)$ is an isomorphism. It is both common and tempting to confuse these two deformation spaces. In the present context, however, it seems best to distinguish between the representation/hyperbolic structure and the conformal structure.

For example, each Fuchsian representation determines a marked hyperbolic structure, and hence an underlying marked conformal structure. An equivalence class of Fuchsian representations thus determines a special point in Teichmüller space. This preferred point can be characterized as the unique minimum of an energy function on Teichmüller space.

The construction, due to Tromba [76], is as follows. Given a hyperbolic surface $M$ and a homotopy equivalence $X \xrightarrow{f} M$, then by Eels-Sampson [27] a unique harmonic map $X \xrightarrow{F} M$ exists homotopic to $f$. The harmonic map is conformal if and only if $M$ is the uniformization of $X$. In general the nonconformality is detected by the Hopf differential $\text{Hopf}(F) \in H^0(X, K^2_X)$, defined as the $(2, 0)$ part of the pullback by $F$ of the complexified Riemannian metric on $M$. The resulting mapping

$$\mathcal{F}(X) \to H^0(X, K^2_X)$$

$$(f, M) \mapsto \text{Hopf}(F)$$

is a diffeomorphism.

Fixing $M$ and letting the marked complex structure $(f, X)$ vary over $T(\Sigma)$ yields an interesting invariant discussed in Tromba [76], and extended in Goldman-Wentworth [42] and Labourie [60]. The energy of the harmonic map $F = F(f, X, M)$ is a real-valued function on $T(\Sigma)$. In the present context it is the square of the $L^2$-norm of $\text{Hopf}(F)$.

**Theorem 5.5.2** (Tromba). The resulting function $T(\Sigma) \to \mathbb{R}$ is proper, convex, and possesses a unique minimum at the uniformization structure $X$.

For more applications of this energy function to surface group representations, compare Goldman-Wentworth [42] where properness is proved for convex cocompact discrete embeddings, and Labourie [60], where the above result is extended to quasi-isometric embeddings $\pi \hookrightarrow G$.

### 5.6. Holomorphic vector bundles and uniformization.

Let $\pi : \Gamma \to \text{PSL}(2, \mathbb{R})$ be a Fuchsian representation corresponding to a marked hyperbolic structure $\Sigma \xrightarrow{f} M$. A spin structure on $\Sigma$ determines a lifting of $\rho$ to

$$\pi : \Gamma \xrightarrow{\tilde{\rho}} \text{SL}(2, \mathbb{R}) \subset \text{SL}(2, \mathbb{C})$$

and hence a flat $\mathbb{C}^2$-bundle $(\mathbb{C}^2)_\rho$ over $\Sigma$.

Choose a marked Riemann surface $X$ corresponding to a point in Teichmüller space $T(\Sigma)$. Since locally constant maps are holomorphic for any complex structure on $\Sigma$, the flat bundle $(\mathbb{C}^2)_\rho$ has a natural holomorphic structure; denote the corresponding holomorphic rank two vector bundle over $X$ by $E_\rho \to X$.

In trying to fit such a structure into a moduli problem over $X$, the first problem is that this holomorphic vector bundle is unstable and does not seem susceptible to...
Geometric Invariant Theory techniques. Indeed, its instability intimately relates to its role in uniformization. Namely, the developing map

\[ \tilde{M} \overset{\text{dev}}{\longrightarrow} \mathbb{CP}^1 \]

determines a holomorphic line bundle \( L \subset E_{\tilde{\rho}} \). Since \( \deg(E_{\tilde{\rho}}) = 0 \), and dev is nonsingular, the well-known isomorphism

\[ T(\mathbb{CP}^1) \cong \text{Hom}(\gamma, \gamma^{-1}) \]

where \( \gamma \to \mathbb{CP}^1 \) is the tautological line bundle implies that

\[ L^2 \cong K_X \]

and \( \deg(L) = g - 1 > 0 \). Therefore \( E_{\tilde{\rho}} \) is unstable. In fact, \( E_{\tilde{\rho}} \) is a nontrivial extension

\[ L \longrightarrow E_{\tilde{\rho}} 
\longrightarrow E_{\tilde{\rho}}/L \cong L^{-1} \]

determined by the fundamental cohomology class \( \varepsilon \) in

\[ H^1(X, \text{Hom}(L^{-1}, L)) \cong H^1(X, K) \cong \mathbb{C} \]

defining Serre duality. (Compare Gunning [49].)

One resolves this difficulty by changing the question. Replace the extension class \( \varepsilon \) by an auxiliary holomorphic object — a Higgs field

\[ \Phi \in H^0(X; K_X \otimes \text{End}(E)) \]

for the vector bundle \( E := L \oplus L^{-1} \) so that the Higgs pair \((E, \Phi)\) is stable in the appropriate sense. In our setting the Higgs field corresponds to the everywhere nonzero holomorphic section of the trivial holomorphic line bundle

\[ \mathbb{C} \cong K_X \otimes \text{Hom}(L, L^{-1}) \subset K_X \otimes \text{End}(E). \]

Now the only \( \Phi \)-invariant holomorphic subbundle of \( E \) is \( L^{-1} \) which is negative, and the pair \((E, \Phi)\) is stable.

6. Rank two Higgs bundles

Now we impose a conformal structure on the surface to obtain extra structure on the deformation space \( \text{Hom}(\pi, G)/G \). As before \( \Sigma \) denotes a fixed oriented smooth surface, and \( X \) a Riemann surface with a fixed marking \( \Sigma \to X \).

6.1. Harmonic metrics. Going from \( \rho \) to \((V, \Phi)\) involves finding a harmonic metric, which may be regarded as a \( \rho \)-equivariant harmonic map

\[ \tilde{M} \overset{h}{\longrightarrow} \text{GL}(n, \mathbb{C})/U(n) \]

into the symmetric space \( \text{GL}(n, \mathbb{C})/U(n) \). The metric \( h \) determines a reduction of structure group of \( E_{\tilde{\rho}} \) from \( \text{GL}(n, \mathbb{C}) \) to \( U(n) \), giving \( E_{\tilde{\rho}} \) a Hermitian structure. Let \( A \) denote the unique connection on \( E_{\tilde{\rho}} \) which is unitary with respect to \( h \). The harmonic metric determines the Higgs pair \((V, \partial_V, \Phi)\) as follows.

- The Higgs field \( \Phi \) is the holomorphic \((1, 0)\)-form \( \partial h \in \Omega^1(\text{End}(V)) \), where the tangent space to \( \text{GL}(n, \mathbb{C})/U(n) \) is identified with a subspace of \( h^*\text{End}(V) \);
- The holomorphic structure \( d''_A \) on \( V \) arises from conformal structure \( \Sigma \) and the Hermitian connection \( A \).
The Higgs pair satisfies the self-duality equations with respect to the Hermitian metric $h$:

\[(d_A)''(\Phi) = 0\]

\[F(A) + [\Phi, \Phi^*] = 0\]  \hspace{1cm} (6.1.1)

Here $F(A)$ denotes the curvature of $A$, and $\Phi^*$ denotes the adjoint of $\Phi$ with respect to $h$. Conversely, Hitchin and Simpson show that every stable Higgs pair determines a Hermitian metric satisfying (6.1.1).

6.2. **Higgs pairs and branched hyperbolic structures.** Choose an integer $d$ satisfying

\[0 \leq d < 2g - 2\]

Hitchin identifies the component $\text{Euler}^{-1}(2 - 2g + d)$ with Higgs pairs $(V, \Phi)$ where

\[V = L_1 \oplus L_2\]

is a direct sum of line bundles $L_1$ and $L_2$ defined as follows. Choose a square-root $K_X^{1/2}$ of the canonical bundle $K_X$ and let $K_X^{-1/2}$ be its inverse. Let $D \geq 0$ be an effective divisor of degree $d$. Define line bundles

\[L_1 := K_X^{-1/2} \otimes D\]

\[L_2 := K_X^{1/2}\]

Define a Higgs field

\[\Phi = \begin{bmatrix} 0 & s_D \\ Q & 0 \end{bmatrix}\]

where:

- $s_D$ is a holomorphic section of the line bundle corresponding to $D$, which determines the component of $\Phi$ in

\[K_X \otimes \text{Hom}(L_2, L_1) \cong D \subset \Omega^1(\Sigma, \text{End}(V))\];

- $Q \in H^0(\Sigma, K_X^2)$ is a holomorphic quadratic differential with $\text{div}(Q) \geq D$, which determines the component of $\Phi$ in

\[K_X \otimes \text{Hom}(L_1, L_2) \cong K_X^2 \subset \Omega^1(\Sigma, \text{End}(V))\].

Then $(V, \Phi)$ is a stable Higgs pair.

When $Q = 0$, this Higgs bundle corresponds to the uniformization representation. In general, when $d = 0$, the harmonic metric is a diffeomorphism (Schoen-Yau [65]).

$Q$ is its Hopf differential.

The Euler class of the corresponding representation equals

\[\deg(L_2) - \deg(L_1) = 2g + d\]

**Theorem 6.2.1** (Hitchin [51]). The component $\text{Euler}^{-1}(2 - 2g + d)$ identifies with a holomorphic vector bundle over the symmetric power $\text{Sym}^d(X)$. The fiber over $D \in \text{Sym}^d(X)$ is the vector space

\[\{Q \in H^0(X, K_X^2) \mid \text{div}(Q) \geq D\} \cong \mathbb{C}^{3(g-1) - d}\]
The quadratic differential $Q$ corresponds to the Hopf differential of the harmonic metric $h$. When $Q = 0$, the harmonic metric is holomorphic, and defines a developing map for a branched conformal structure, with branching defined by $D$.

When $e = 2 - 2g$, then $d = 0$ and the space $\mathcal{F}(X)$ of Fuchsian representations identifies with the vector space $H^0(X, K^2_X) \cong \mathbb{C}^{3(g-1)}$.

### 6.3. Uniformization with singularities

McOwen [63] and Troyanov [77] proved a general uniformization theorem for hyperbolic structures with conical singularities. Specifically, let $D = (p_1) + \cdots + (p_k)$ be an effective divisor, with $p_i \in X$. Choose real numbers $\theta_i > 0$ and introduce singularities in the conformal structure on $X$ by replacing a coordinate chart at $p_i$ with a chart mapping to a cone with cone angle $\theta_i$. The following uniformization theorem describes when there is a singular hyperbolic metric in this singular conformal structure.

**Theorem 6.3.1** (McOwen [63], Troyanov [77]). If

$$2 - 2g + \sum_{i=1}^k (\theta_i - 2\pi) > 0,$$

there exists a unique singular hyperbolic surface conformal to $X$ with cone angle $\theta_i$ at $p_i$.

When the $\theta_i$ are multiples of $2\pi$, then this structure is a branched structure (and the above theorem follows from Hitchin [51]). The moduli space of such branched conformal structures forms a bundle $\mathcal{S}^d$ over $\mathcal{F}(\Sigma)$ where the fiber over a marked Riemann surface $\Sigma \to X$ is the symmetric power $\text{Sym}^d(X)$ where

$$d = \frac{1}{2\pi} \sum_{i=1}^k (\theta_i - 2\pi).$$

The resulting uniformization map

$$\mathcal{S}^d \overset{\Delta}{\to} \text{Euler}^{-1}(2 - 2g + d) \subset \text{Hom}(\pi, G)/G$$

is homotopy equivalence, which is not surjective, by the example in §4.3.3.

**Conjecture 6.3.2.** Every representation with non-discrete image lies in the image of $\mathfrak{M}$.

### 7. Split $\mathbb{R}$-forms and Hitchin's Teichmüller component

When $G$ is a split real form of a semisimple Lie group, Hitchin [52] used Higgs bundle techniques to determine an interesting connected component of $\text{Hom}(\pi, G)/G$, which is not detected by characteristic classes. A Hitchin component of $\text{Hom}(\pi, G)$ is the connected component containing a composition

$$\pi \overset{\rho_0}{\to} \text{SL}(2, \mathbb{R}) \overset{K}{\to} G$$

where $\rho_0$ is Fuchsian and $K$ is the representation corresponding to the 3-dimensional principal subgroup discovered by Kostant [53]. When $G = \text{SL}(n, \mathbb{R})$, then Kostant’s representation $K$ is the irreducible $n$-dimensional representation corresponding to the symmetric power $\text{Sym}^{n-1}(\mathbb{R}^2)$.

The compositions $K \circ \rho_0$ above determine a subset of $\text{Hom}(\pi, G)/G$ which identifies with the Fricke-Teichmüller space, and Hitchin’s main result is that each Hitchin component is a cell of (the expected) dimension $\text{dim}(G)(2g - 2)$. 
For example, if \( G = \text{SL}(n, \mathbb{R}) \), then Hitchin identifies this component with with the \( 2(g - 1)(n^2 - 1) \)-cell
\[
H^0(X; K^2_X) \oplus H^0(X; K^3_X) \oplus \cdots \oplus H^0(X; K^n_X) \\
\cong \mathbb{C}^{3(g-1)} \oplus \mathbb{C}^{5(g-1)} \oplus \cdots \oplus \mathbb{C}^{2n-1}(g-1).
\]
When \( n \) is odd, Hitchin proves there are exactly 3 components. The second Stiefel-Whitney characteristic class is nonzero on exactly one component; it is zero on two components, one of which is the Hitchin-Teichmüller component.

7.1. Convex \( \mathbb{R}P^2 \)-structures: \( G = \text{SL}(3, \mathbb{R}) \). When \( G \cong \text{PGL}(3, \mathbb{R}) \cong \text{SL}(3, \mathbb{R}) \), Hitchin [52] conjectured that his component corresponded to the deformation space \( \mathcal{C}(\Sigma) \) of marked convex \( \mathbb{R}P^2 \)-structures, proved in [38] to be a cell of dimension \( 16(g - 1) \). In [24] Suhyoung Choi and the author proved this conjecture. A convex \( \mathbb{R}P^2 \)-manifold is a quotient \( \Omega/\Gamma \) where \( \Omega \subset \mathbb{R}P^2 \) is a convex domain and \( \Gamma \) a discrete group of collineations acting properly and freely on \( \Omega \). If \( \chi(M) < 0 \), then necessarily \( \Omega \) is properly convex (contains no complete affine line), and its boundary \( \partial\Omega \) is a \( C^{1+\alpha} \) strictly convex curve, for some \( 0 < \alpha \leq 1 \). Furthermore \( \alpha = 1 \) if and only if \( \partial\Omega \) is a conic and the \( \mathbb{R}P^2 \)-structure arises from a hyperbolic structure. These facts are due to Kuiper [56] and Benzécri [6] and have recently been extended and amplified to compact quotients of convex domains in \( \mathbb{R}P^{n-1} \) by Benoist [4, 5].

7.2. Higgs bundles and affine spheres. The Higgs bundle theory of Hitchin [52] identifies, for an arbitrary Riemann surface \( X \), the Hitchin component \( \mathcal{C}(\Sigma) \) with the complex vector space
\[
H^0(X, K^2_X) \oplus H^0(X, K^3_X) \cong \mathbb{C}^{8g-8}
\]
and the component in \( H^0(X, K^2_X) \) of the Higgs field corresponds to the Hopf differential of the harmonic metric. Using the theory of hyperbolic affine spheres developed by Calabi, Loewner-Nirenberg, Cheng-Yau, Gigena, Sasaki, Li, and Wang, Labourie [38, 61] and Loftin [62] proved:

Figure 7. A triangle tesselation in the hyperbolic plane, drawn in the Beltrami-Klein projective model. Its holonomy representation is obtained by composing a Fuchsian representation in \( \text{SL}(2, \mathbb{R}) \) with the irreducible representation \( \text{SL}(2, \mathbb{R}) \longrightarrow \text{SL}(3, \mathbb{R}) \).
Theorem 7.2.1. The deformation space $\mathcal{C}(\Sigma)$ naturally identifies with the holomorphic vector bundle over $\mathcal{F}(\Sigma)$ whose fiber over a marked Riemann surface $\Sigma \to X$ is $H^0(X, K_X^3)$.

For every such representation, there exists a unique conformal structure so that

$$\tilde{\Sigma} \xrightarrow{\tilde{\rho}} SL(3, \mathbb{R})/SO(3)$$

is a conformal map, that is the component of the Higgs field in $H^0(\Sigma, K_X^3)$ — the Hopf differential $\text{Hopf}(h)$ — vanishes. This defines the projection $\mathcal{C}(\Sigma) \to \mathcal{F}(\Sigma)$. The zero-section corresponds to the Fuchsian $\mathbb{RP}^2$-structures, that is, the $\mathbb{RP}^2$-structures arising from hyperbolic structures on $\Sigma$.

It is natural to attempt to generalize this as follows. For any split real form $G$, and Riemann surface $X$ with $\pi_1(X) \cong \pi$, Hitchin [52] identifies a certain direct sum of holomorphic line bundles $\mathcal{U}_X$ naturally associated to $X$ so that a Hitchin component of $\text{Hom}(\pi, G)/G$ identifies with the complex vector space

$$H^0(X, K_X^3) \oplus H^0(X, \mathcal{U}_X).$$

However, this identification depends crucially on the Riemann surface $X$ and fails to be $\text{Mod}(\Sigma)$-invariant. Generalizing the Labourie-Loftin Theorem 7.2.1, we conjecture that each Hitchin component of $\text{Hom}(\pi, G)/G$ identifies naturally with the total space of a holomorphic vector bundle $\mathcal{C}(\Sigma)$ over $\mathcal{F}(\Sigma)$, whose fiber over a marked Riemann surface $X$ equals $H^0(X, \mathcal{U}_X)$.

7.3. Hyperconvex curves. In 2002, Labourie [59] discovered an important property of the Hitchin component:

Theorem 7.3.1 (Labourie). A representation in the Hitchin component for $G = SL(n, \mathbb{R})$ is a discrete quasi-isometric embedding

$$\pi \xrightarrow{\rho} SL(n, \mathbb{R})$$

with reductive image.
A crucial ingredient in his proof is the following notion. A curve $S^1 \to \mathbb{R}P^{n-1}$ is hyperconvex if and only if for all $x_1, \ldots, x_n \in S^1$ distinct,

$$f(x_1) + \cdots + f(x_n) = \mathbb{R}^n.$$  

**Theorem 7.3.2** (Guichard [45, 46], Labourie [59]). ρ is Hitchin if and only if ρ preserves hyperconvex curve.

Recently Fock and Goncharov [28, 29] have studied this component of representations, using global coordinates generalizing Thurston and Penner’s shearing coordinates. In these coordinates the Poisson structure admits a particularly simple expression, leading to a quantization. Furthermore they find a positive structure which leads to an intrinsic characterization of these semi-algebraic subsets of $\text{Hom}(\pi, G)/G$. Their work has close and suggestive connections with cluster algebras and $K$-theory.

8. **HERMITIAN SYMMETRIC SPACES: MAXIMAL REPRESENTATIONS**

We return now to the maximal representations into groups of Hermitian type, concentrating on the unitary groups $U(p,q)$ and the symplectic groups $\text{Sp}(n, \mathbb{R})$.

8.1. **The unitary groups** $U(p,q)$. The Milnor-Wood inequality (5.3.1) may be the first example of the boundedness of a cohomology class. In a series of papers [19, 14, 15, 53, 17, 18], Burger, Monod, Iozzi and Wienhard place the local and global rigidity in the context of the Toledo invariant being a bounded cohomology class. A consequence of these powerful methods for surface groups is the following, announced in [17]:

**Theorem 8.1.1** (Burger–Iozzi–Wienhard [17]). Let $X$ be a Hermitian symmetric space, and maximal representation

$$\pi \: \rho \to G.$$

- The Zariski closure $L$ of $\rho(\pi)$ is reductive;
- The symmetric space associated to $L$ is a Hermitian symmetric tube domain, totally geodesically embedded in the symmetric space of $G$;
- $\rho$ is a discrete embedding.

Conversely, if $X$ is a tube domain, then there exists a maximal $\rho$ with $\rho(\pi)$ Zariski-dense.

For example, if $G = U(p,q)$, where $p \leq q$, then $\rho$ is conjugate to the normalizer $U(p,p) \times U(q-p)$ of $U(p,p)$ in $U(p,q)$. As in the rank one case (compare §5.4.1), the components of maximal representations have strictly smaller dimension. (In earlier work Hernandez [50] considered the case of $U(2,q)$.)

Furthermore every maximal representation deforms into the composition of a Fuchsian representation $\pi \: \rho \to \text{SU}(1,1)$ with the diagonal embedding

$$\text{SU}(1,1) \subset U(1,1) \xrightarrow{\Delta} U(1,1) \times \cdots \times U(1,1) \subset U(p,p) \subset U(p,q)$$

At roughly the same time, Bradlow, García-Prada and Gothen [11] investigated the space of Higgs bundles using infinite-dimensional Morse theory, in a similar
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way to Hitchin [51]. Their critical point analysis also showed that maximal representations formed components of strictly smaller dimension. They found that the number of connected components of \( \text{Hom}(\pi, U(p, q)) \) equals:

\[
2(p + q) \min(p, q) (g - 1) + \gcd(p, q).
\]

(For a survey of these techniques and other results, compare [12] as well as their recent column [13].)

8.2. The symplectic groups \( \text{Sp}(n, \mathbb{R}) \). The case \( G = \text{Sp}(2n, \mathbb{R}) \) is particularly interesting, since \( G \) is both \( \mathbb{R} \)-split and of Hermitian type. Gothen [44] showed there are \( 3 \cdot 2^{2g} + 2g - 4 \) components of maximal representations when \( n = 2 \). For \( n > 2 \), there are \( 3 \cdot 2^{2g} \) components of maximal representations García-Prada, Gothen, and Mundet i Riera [34]). For \( n = 2 \), the components the nonmaximal representations are just the preimages of the Toledo invariant, comprising \( 1 + 2(2g - 3) = 4g - 5 \) components. Thus the total number of connected components of \( \text{Hom}(\pi, \text{Sp}(4, \mathbb{R})) \) equals

\[
2(3 \cdot 2^{2g} + 2g - 4) + 4g - 5 = 6 \cdot 4^g + 10g - 13.
\]

The Hitchin representations are maximal and comprise \( 2^{2g+1} \) of these components. They correspond to deformations of compositions of Fuchsian representations \( \pi \xrightarrow{\rho_0} \text{SL}(2, \mathbb{R}) \) with the irreducible representation

\[
\text{SL}(2, \mathbb{R}) \longrightarrow \text{Aut} \left( \text{Sym}^{2n-1}(\mathbb{R}^2) \right) \hookrightarrow \text{Sp}(2n, \mathbb{R})
\]

where \( \mathbb{R}^{2n} \cong \text{Sym}^{2n-1}(\mathbb{R}^2) \) with the symplectic structure induced from \( \mathbb{R}^2 \).

Another class of maximal representations arises from deformations of compositions of a Fuchsian representation \( \pi \xrightarrow{\rho_0} \text{SL}(2, \mathbb{R}) \) with the diagonal embedding

\[
\text{SL}(2, \mathbb{R}) \xrightarrow{\Delta} \underbrace{\text{SL}(2, \mathbb{R}) \times \cdots \times \text{SL}(2, \mathbb{R})}_{n} \hookrightarrow \text{Sp}(2n, \mathbb{R}).
\]

More generally, the diagonal embedding extends to a representation

\[
\text{SL}(2, \mathbb{R}) \times O(n) \xrightarrow{\Delta} \text{Sp}(2n, \mathbb{R})
\]

corresponding to the \( \text{SL}(2, \mathbb{R}) \times O(n) \)-equivariant decomposition of the symplectic vector space

\[
\mathbb{R}^{2n} = \mathbb{R}^2 \otimes \mathbb{R}^n
\]

as a tensor product of the symplectic vector space \( \mathbb{R}^2 \) and the Euclidean inner product space \( \mathbb{R}^n \). Deformations of compositions of Fuchsian representations into \( \text{SL}(2, \mathbb{R}) \times O(2) \) with \( \Delta \) provide \( 2^{2g} \) more components of maximal representations.

For \( n > 2 \), these account for all the maximal components. This situation is more complicated when \( n = 2 \). In that case, \( 4g - 5 \) components of maximal representations into \( \text{Sp}(4, \mathbb{R}) \) do not contain representations into smaller compact extensions of embedded subgroups isomorphic to \( \text{SL}(2, \mathbb{R}) \). In particular the image of every representation in such a maximal component is Zariski dense in \( \text{Sp}(4, \mathbb{R}) \), in contrast to the situation for \( U(p, q) \) and \( \text{Sp}(2n, \mathbb{R}) \) for \( n > 2 \). See Guichard-Wienhard [48] for more details.
8.3. Geometric structures associated to Hitchin representations. Fuchsian representations into $\text{SL}(2, \mathbb{R})$ correspond to hyperbolic structures on $\Sigma$, and Hitchin representations into $\text{SL}(3, \mathbb{R})$ correspond to convex $\mathbb{RP}^2$-structures on $\Sigma$. What geometric structures correspond to other classes of surface group representations?

Guichard and Wienhard [47] associate to a Hitchin representation in $\text{SL}(4, \mathbb{R})$ an $\mathbb{RP}^3$-structure on the unit tangent bundle $T_1(\Sigma)$ of a rather special type. The trajectories of the geodesic flow on $T_1(\Sigma)$ (for any hyperbolic metric on $\Sigma$), develop to projective lines. The leaves of the weak-stable foliations of this structure develop into convex subdomains of projective planes in $\mathbb{RP}^3$. The construction of this structure uses the hyperconvex curve in $\mathbb{RP}^3$. This convex-foliated structure is a geometric structure corresponding to Hitchin representations in $\text{SL}(4, \mathbb{R})$.

For the special case of Hitchin representations into $\text{Sp}(4, \mathbb{R})$ (which are readily Hitchin representations into $\text{SL}(4, \mathbb{R})$), the convex-foliated structures are characterized by a duality. Furthermore the symplectic structure on $\mathbb{R}^4$ induces a contact structure on $T^1(\Sigma)$ which is compatible with the convex-foliated $\mathbb{RP}^3$-structure. In addition, another geometric structure on another circle bundle over $\Sigma$ arises naturally, related to the local isomorphism $\text{Sp}(4, \mathbb{R}) \rightarrow O(3, 2)$ and the identification of the Grassmannian of Lagrangian subspaces of the symplectic vector space $\mathbb{R}^4$ with the conformal compactification of Minkowski $(2+1)$-space (the $2+1$-Einstein universe. (Compare [3] for an exposition of this geometry.) The interplay between the contact $\mathbb{RP}^3$-geometry, flat conformal Lorentzian structures, the dynamics of geodesics on hyperbolic surfaces, and the resulting deformation theory of promises to be a fascinating extension of ideas rooted in the work of Nigel Hitchin.

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