de Sitter Brane Gravity: from Close-Up to Panorama

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Abstract

We find explicitly the induced graviton propagator on de Sitter branes embedded in various five-dimensional spacetimes; de Sitter branes in AdS and Minkowski space are particular cases. By studying the structure of the momentum-space propagator, we are able to extract interesting physics, much of which is qualitatively different from that of flat branes. We find that 1) there can be a set of graviton-like particles which mediate brane gravity at different scales; 2) localized gravity can exist even on de Sitter branes in Minkowski space; 3) Kaluza-Klein modes also contribute to conventional 4-D gravity for de Sitter branes in AdS; and 4) Newton’s constant can vary considerably with scale. We comment on the implications for the effective cosmological constant.
1 Introduction

The idea that our familiar four-dimensional world might be a slice of a higher-dimensional spacetime in which the extra dimensions are large [1, 4], or even infinite [3], has generated tremendous interest and activity. While gauge fields can be trapped by branes or domain walls, the key finding of Randall and Sundrum (RS) [3] was that gravity, too, could be localized. Later, a very intriguing model was proposed by Gregory, Rubakov, and Sibiryakov (GRS) [4] in which gravity is localized only on intermediate scales and is again five-dimensional at ultra-large scales. Intermediate scale gravity on the brane is mediated not by a normalizable graviton state, as happens in the RS model, but by a resonant state [5, 6]. Several variants of such “quasi-localized gravity” scenarios are now known [7]-[9].

Most brane-world studies have dealt with Ricci-flat branes. One might expect new phenomena in such theories when this restriction is dropped; indeed, it has even been suggested [7, 11, 12] that the cosmological constant problem could be understood within a quasi-localized gravity setting. Here we consider de Sitter branes. (Earlier articles on de Sitter branes include [12]-[16].) For analyzing the effective brane gravity, it is often more convenient, in curved space, to study the graviton propagator rather than the Newton potential. By examining the pole and resonance structure of the propagator in its momentum-space representation, we are able to understand some of the scaling behavior of gravity in (quasi-)localized theories.

In this paper, we shall consider the de Sitter brane extensions of the RS and GRS scenarios, as follows. First we set up the geometry. Then we review the technique for calculating the momentum-space effective propagator on the brane, with a brief discussion on how interesting physics can be read off the propagator. Then, after considering the flat-brane GRS model, we determine the de Sitter brane propagator. The expectation of new physics is indeed borne out. Among the results, we find that Kaluza-Klein modes also contribute to standard 4-D gravity on de Sitter branes in AdS (Eq. (34)), and that gravity is localized even on a de Sitter brane in flat space (Eq. (38)). Most intriguingly, we find multiple poles and resonances (Eqs. (43,44)) for more general set-ups. Since each of these graviton-like excitations dominates at a different energy, we have a picture of gravity at different scales behaving in drastically different ways, with different Newton couplings that are sensitive to the parameters of the configuration. We end with some comments on the effective cosmological constant.

Throughout, we focus on the transverse-traceless modes of the graviton, leaving aside the tensor structure, the radion, questions of stability and phenomenological viability, and other considerations. Indeed, the GRS model has some drawbacks pertaining to some of these issues; we work with it here primarily as a tractable toy model for quasi-localized gravity.

2 The Set-Up

Consider a visible brane at $y = y_0$ with a hidden brane at $y = y_1 > y_0$, with AdS in between, and Minkowski space for $y > y_1$. As usual, the whole configuration is $Z_2$-symmetric about the visible brane and we consider only the half of the spacetime to the right of the visible brane. The metric takes the form

$$ds^2 = dy^2 + e^{-2A(y)} \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu .$$ (1)
Both AdS and Minkowski space admit constant-curvature foliations,
\[ \tilde{R}_{\mu\nu} = 3 \kappa^2 \tilde{g}_{\mu\nu}, \]
where \( \kappa = 1 \) for de Sitter foliations and \( \kappa = 0 \) for flat space foliations, with \( \lambda \) an arbitrary dimensionful constant. All quantities with a tilde are derived from \( \tilde{g}_{\mu\nu} \). The five-dimensional Ricci tensor is zero for Minkowski space and proportional to the metric for AdS, with a proportionality constant of \(-4k^2\). We choose \( \lambda = k > 0 \) without loss of generality. It is convenient to introduce a new coordinate \( z \):
\[ \frac{dz}{dy} = ke^{A(y)}. \]
For a brane placed at a point \( y(z) \), the Israel junction condition relates the tension to the jump in the derivative of the warp factor:
\[ \tau = [\partial_y A(y)] = ke^{A(z)} [\partial_z A(z)]. \]
We shall consider perturbations along the brane. The perturbed bulk metric reads
\[ ds^2 = dy^2 + \left( e^{-2A(y)} \tilde{g}_{\mu\nu}(x) + h_{\mu\nu}(x, y) \right) dx^\mu dx^\nu, \]
where the graviton \( h_{\mu\nu}(x, y) \) is taken to be traceless and to satisfy the gauge choice \( \nabla_\mu h^{\mu\nu} = 0 \).

### 3 The Propagator

In this paper we are interested in the graviton propagator for the perturbations in Eq. (5). The five-dimensional propagator for the transverse-traceless graviton is the solution to the equation
\[ \left( \nabla_5^2 + 2k^2 \right) \Delta_{\mu'\nu'}(x, y; x', y') = G_5 \delta(y - y') \delta^4(x - x') \delta_{\mu'} \delta_{\nu'}, \]
where \( G_5 \) is the five-dimensional Newton constant. The coordinate-space propagator can be computed directly via an integral over the appropriately-normalized Kaluza-Klein and zero modes of the brane. However, this results in a formidable integral of a hypergeometric function. Instead, we expand the graviton propagator as an integral in the Fourier modes of the brane:
\[ \Delta_{\mu'\nu'}(x, y; x', y') = \int \frac{d^4 \tilde{p}}{(2\pi)^4} f_{\mu'\nu'}(\tilde{p}, x, x') \Delta_{\mu}(y, y') e^{4A(y_0)}, \]
where the Fourier modes are defined to satisfy
\[ \nabla_5^2 f_{\mu'\nu'}(\tilde{p}, x, x') = -\tilde{p}^2 f_{\mu'\nu'}(\tilde{p}, x, x'). \]
The factor \( \exp(4A(y_0)) \) in Eq. (7) has been included because Eq. (7) is an expansion over the \( \tilde{p} \)-momentum modes. We are ultimately interested, however, in the physical momentum modes, \( p \), of the visible brane. The two sets of modes are related via \( p^2 = \)
\( p^2 \exp^{+2A(z_0)}. \) By accounting for this factor, we ensure that \( \Delta_p(y, y') \) is in terms of physical momentum.

The analysis for the effective graviton propagator on a brane embedded in a warped compactification scenario was carried out by Giddings, Katz, and Randall (GKR) \[17\]. Here we review the aspects of that work that will be useful later. Defining the momentum-space propagator as

\[
\Delta_p(y, y') \equiv e^{-A(y)/2} \tilde{\Delta}_p(z, z') e^{-A(y')/2} e^{-4A(y_0)},
\]

we find that

\[
\left( \partial_z^2 - V(z) - \left( \frac{\tilde{p}}{k} \right)^2 \right) \tilde{\Delta}_p(z, z') = \frac{G_5}{k} e^{+4A(z)} \delta(z - z'),
\]

where the potential \( V(z) \) is

\[
V(z) = \frac{17}{4} \left( \partial_z A \right)^2 + \frac{1}{2} \partial_z^2 A.
\]

For \( z \neq z' \), \( \tilde{\Delta}_p(z, z') \) satisfies an analog Schrödinger equation, Eq. (10). We now define two functions: \( \tilde{\Delta}_<(z, z') \) if \( z \leq z' \) and \( \tilde{\Delta}_>(z, z') \) if \( z \geq z' \). The general solution thus has four constants of integration in each region of spacetime.

These constants are fixed by an equal number of conditions. At \( z = z' \), Eq. (10) and a matching condition imply that

\[
\tilde{\Delta}_{<|z=z'} = \tilde{\Delta}_{>|z=z'}, \quad \partial_z (\tilde{\Delta}_{> - \tilde{\Delta}_{<}})|_{z=z'} = \frac{G_5}{k} e^{+4A(z)}.
\]

Additionally, the junction conditions at each brane give

\[
[\tilde{\Delta}]|_{\text{brane}} = 0, \quad [\partial_z \tilde{\Delta}]|_{\text{brane}} = -\frac{3}{2} [\partial_z A(z)] \tilde{\Delta}|_{\text{brane}}.
\]

Finally, we have to specify the boundary condition at infinity. We require that there be only outgoing modes there; nothing comes in from infinity.

In the remainder of this paper, we will solve for the effective propagator on the visible brane by setting \( z = z' = z_0 \). From Eq. (10), the propagator as a function of physical momentum on the brane, is

\[
\Delta(p) = \tilde{\Delta}_p(z_0, z_0) e^{-5A(z_0)}.
\]

Much can be understood from the form of the momentum-space graviton propagator. The propagator for the transverse-traceless modes of a graviton on de Sitter space of radius \( R_0 \), when written in terms of \( q^2 \equiv -p^2 \), takes the form

\[
\Delta(q) = \frac{G_N}{q^2 - 2R_0^2}.
\]

Hence, for our effective brane propagator, a pole in \( q^2 \) indicates the presence of a four-dimensional graviton on the visible brane; the residue gives the effective four-dimensional Newton constant. A pole at negative \( q^2 \) indicates that Kaluza-Klein modes conspire to give a graviton-like resonance. On the other hand, an effective propagator that goes as \( 1/q \) indicates that gravity on the brane is still five-dimensional; Gauss’ law is not obeyed on the brane. Also, since there are always KK modes, the propagator has in general an imaginary part which is related to the flux of gravitational radiation into the bulk. And finally, the effective cosmological constant can be read off from the inverse of the momentum-space propagator in the limit of zero momentum.
4 Flat Branes

Although we are mainly interested in de Sitter branes, we pause here to consider the original flat-brane GRS model. The function $A(y)$ is $ky$ for AdS region and is a constant, $ky_1$, in the Minkowski region. The visible brane is at $y = 0$ and has tension $\tau_0 = 2k$ while the hidden brane is at $y = y_1$ and its tension $\tau_1 = -k$ is negative. The basis of solutions of the analog Schrödinger equation, Eq. (10), for this case was analysed in GRS [4] and consists of Bessel functions. One finds that the propagator along the brane ($y = y' = 0$), when written in terms of $x = q/k$, is

$$\Delta(x) = \frac{G_5}{kx} \left( \frac{\alpha J_2(x) - \beta N_2(x)}{\alpha J_1(x) - \beta N_1(x)} \right), \quad (16)$$

where

$$\alpha = iN_2(xz_1) - N_1(xz_1),$$
$$\beta = iJ_2(xz_1) - J_1(xz_1), \quad (17)$$

with $z_1 = e^{k y_1}$. It is instructive to analyze Eq. (16) for small values of $x$. Expanding the numerator and denominator in powers of $x$, we get

$$\text{Re} \, \Delta(x) = \frac{2G_5}{kz_1^2} \left( \frac{1}{x^2} + \frac{1}{(z_1^2 - 1)^2} \right). \quad (18)$$
$$\text{Im} \, \Delta(x) = -\frac{4G_5}{k} \frac{z_1}{(z_1^2 - 1)^2} \frac{1}{x} + \frac{1}{x^2} + \frac{1}{(z_1^2 - 1)^2}. \quad (19)$$

The imaginary part of $\Delta(q)$ has a pole at $q = 0$ near which the propagator is

$$\Delta(q) \approx \frac{G_5}{iq^3} z_1^3. \quad (20)$$

This is a reflection of the fact that gravity on a flat brane becomes effectively five-dimensional at large distances. On the other hand, the structure of $\text{Re} \, \Delta(q)$ establishes the presence of a resonance state in the spectrum. The real part of the propagator is thus effectively four-dimensional with an effective Newton constant of

$$G_N = 2G_5k \left( 1 + \frac{1}{2z_1^2} \right)^{z_1 \rightarrow \infty} 2G_5k. \quad (21)$$

Thus, for flat branes, the effective brane gravity is four-dimensional at intermediate scales but five-dimensional at very small or very large scales. As we shall see, on de Sitter branes things are rather different.

5 de Sitter Branes

Now consider de Sitter branes. The warp factor for de Sitter-foliated AdS and flat space is

$$e^{-2A}|_{\text{AdS}} = \sinh^2(-ky) = \frac{1}{\sinh^2 z},$$
$$e^{-2A}|_{\text{FS}} = k^2(y - b)^2 = e^{2(z + c)}, \quad (22)$$
where $b (c)$ is a constant determined by joining the two spaces at $y = y_1 (z = z_1)$. For AdS, $y = 0$ corresponds to the AdS Cauchy horizon; regions on different sides of the horizon should be considered separately. We see from Eq. (22) that in AdS and flat spaces the warp factor has both decaying and increasing branches. There are therefore four different ways to configure the two spaces, namely: A) AdS and FS both have decreasing warp factors; B) the warp factor increases in AdS but decreases in FS; C) the warp factor decreases in AdS and increases in FS; D) the warp factor increases everywhere. Here we will consider mostly configuration A. Specifically,

$$e^{A(z)} = \begin{cases} 
\sinh z & z_0 \leq z \leq z_1 \\
\sinh z_1 e^{(z-z_1)} & z \geq z_1 .
\end{cases}$$

(23)

The tension of the visible and the hidden brane is, respectively,

$$\tau_0 = 2k \cosh z_0, \quad \tau_1 = -k (\cosh z_1 - \sinh z_1) .$$

(24)

The potential $V(z)$ corresponding to Eq. (23) is

$$V(z) = \begin{cases} 
\frac{17}{4} + \frac{15}{4} \frac{1}{\sinh^2 z} & z_0 \leq z \leq z_1 \\
\frac{17}{4} & z \geq z_1 .
\end{cases}$$

(25)

By switching to a new variable,

$$\xi \equiv + \coth z ,$$

(26)

the solution to the analog Schrödinger equation, Eq. (10), in the AdS region is readily obtained in terms of associated Legendre functions, $P_{3/2}^{iM}(\xi)$ and $Q_{3/2}^{iM}(\xi)$. (Some useful formulas for manipulating these functions are listed in the appendix.) The parameter $M$ that appears in the superscript is

$$M^2 = q^2 R_0^2 - \frac{17}{4} ,$$

(27)

where $q$ is the physical momentum on the visible brane, $q = \bar{q} \exp(A(z_0))$, $\bar{q}^2 = -\bar{p}^2$, and $R_0$ is the physical radius of the brane:

$$R_0^{-1} = k \exp(A(z_0)) .$$

(28)

In the flat space region, the solution is given by a combination of ingoing and outgoing plane waves, $e^{\pm \imath Mz}$. Together with the associated Legendre functions in the AdS region, these form a continuous spectrum with $M^2 \geq 0$. Analyzing the spectrum with $M^2 < 0$ one finds that the only such mode which is both normalizable and satisfies all boundary conditions has the form $e^{-\frac{3}{2}z}$ in the flat space region and $P_{3/2}^{-3/2} (\coth z)$ in Anti-de Sitter region that corresponds to $iM = -\frac{3}{2}$. There is thus a gap between the bound state and the continuous portion of the spectrum [12]. On the visible brane the bound state corresponds to the four-dimensional graviton, which has $q^2 = 2R_0^{-2}$.

Next, we define two functions, $\hat{\Delta}_<(z, z')$ and $\hat{\Delta}_>(z, z')$, valid respectively when $z \leq z'$ ($\xi \geq \xi'$) and $z \geq z'$ ($\xi \leq \xi'$). Since we will ultimately take $z$ and $z'$ both to be on the visible brane, we simplify the algebra by choosing $z'$ to be in the AdS region from the
outset; $z$ may be located in either region. After imposing the junction condition at the visible brane (using Eq. (56) from the appendix), the propagator takes the form

$$\hat{\Delta}_<(z, z') = C_M(\xi') \left( Q_{1/2}(\xi_0) P_{3/2}^M(\xi) - P_{1/2}(\xi_0) Q_{3/2}^M(\xi) \right), \quad (29)$$

and

$$\hat{\Delta}_>(z, z') = \begin{cases} 
    A_M(\xi') P_{3/2}^M(\xi) + B_M(\xi') Q_{3/2}^M(\xi) & z \leq z_1 \\
    D_M(z') e^{iM(z-z_1)} & z \geq z_1 
\end{cases} \quad (30)$$

where, in accordance with our boundary condition at infinity, only outgoing modes have been retained in the flat space region.

The other conditions, Eq. (12) and Eq. (13) at the hidden brane, fix the remaining coefficients $A(\xi'), B(\xi'), C(\xi'),$ and $D(z')$. The expression for the propagator for arbitrary $\xi$ and $\xi'$ is quite lengthy. However, when both points are on the brane ($\xi = \xi' = \xi_0$), the expression takes a nice form. Using Eq. (59) and not forgetting Eq. (14), a little bit of work gives

$$\Delta(q) = G_5 R_0 \frac{1}{2 + iM} \left( \frac{\alpha(M) Q_{3/2}^M(\xi_0) - \beta(M) P_{3/2}^M(\xi_0)}{\alpha(M) Q_{1/2}^M(\xi_0) - \beta(M) P_{1/2}^M(\xi_0)} \right), \quad (31)$$

where all the $q$-dependence is contained in $M$. Here

$$\alpha(M) \equiv P_{3/2}^M(\xi_1) - P_{1/2}^M(\xi_1),$$

$$\beta(M) \equiv Q_{3/2}^M(\xi_1) - Q_{1/2}^M(\xi_1). \quad (32)$$

Eq. (31) is the desired formula; it required solving some linear algebraic equations for the coefficients, but no integration. As we shall see, the detailed structure of our propagator contains a wealth of information.

There are some limiting regimes which are of particular interest.

### 5.1 Curved Randall-Sundrum Limit

Take the limit when $z_1 \to \infty (\xi_1 \to 1)$. Then the radius of the second brane shrinks to zero, and the Minkowski region disappears. We are thus in the de Sitter-brane Randall-Sundrum limit. Now, from properties of the Legendre functions, we know that when $\xi_1$ goes to 1, $\alpha(M)$ goes to 0. Eq. (31) then reads

$$\Delta(q) = G_5 R_0 \frac{P_{3/2}^M(\coth z_0)}{2 + iM P_{1/2}^M(\coth z_0)}. \quad (33)$$

In Fig. 1, we plot the imaginary part of this function. We see that the imaginary part of the propagator is non-zero only for $M^2 > 0$ and is entirely due to the continuous (Kaluza-Klein) modes. This is how the gap in the spectrum shows up in the structure of the propagator. Another manifestation of the gap is a jump in the derivative of the real part of the propagator at $M = 0$.

Moreover, using a Legendre function identity, Eq. (57), we can rewrite Eq. (33) as

$$\Delta(q) = \frac{2G_5 k \cosh z_0}{q^2 - 2R_0^2} - G_5 k \sinh z_0 \frac{1}{2 + i\sqrt{q^2 R_0^2 - 17/4} P_{1/2}^M(\coth z_0)}.$$
The first term above is the contribution of the zero mode while the second term has a Kaluza-Klein origin. Eq. (34) is similar to the corresponding expression obtained in [17] for flat branes. We know from the analysis of [3] and [17] that only the zero mode leads to localized 4-D gravity for flat branes; the Kaluza-Klein modes give a correction. But for de Sitter branes the story is quite different: both the zero mode and the KK modes contribute to the four-dimensional propagator on the brane. This can be seen from the fact that both terms in Eq. (34) have a pole at $q^2 = 2R_0^{-2}$. To simplify the analysis we consider the case when $\xi_0 \gg 1$ ($R_0 k \gg 1$). In this regime, using the asymptotic expression Eq. (63) in the appendix, we find that

$$\Delta(q) \simeq \frac{2G_N k}{(q^2 - 2R_0^{-2})} \left( 1 - \frac{1}{2k^2} (q^2 - 4R_0^{-2}) \ln(R_0 k) \right).$$

(35)

According to our prescription, the induced Newton constant is identified as the residue of the graviton propagator at the pole $q^2 = 2R_0^{-2}$. Thus, for $R_0 k \gg 1$, we find

$$G_N \simeq 2G_N k \left( 1 + (R_0 k)^{-2} \ln(R_0 k) \right).$$

(36)

The value of the induced cosmological constant (or better to say the vacuum energy) $\Lambda$ can be read off as well, using $\Lambda \equiv \frac{G_N}{-2\Delta(q=0)}$. Taking the limit $q \to 0$ in Eq. (35) we find for $R_0 k \gg 1$, that

$$\Lambda \simeq \frac{1}{R_0^2} \left( 1 - (R_0 k)^{-2} \ln(R_0 k) \right).$$

(37)

We mention in passing that since cutoff-AdS/CFT should be applicable to the curved Randall-Sundrum limit, sub-leading terms in the propagator could be interpreted as originating in interactions with conformal matter on de Sitter space. In particular, the logarithmic term in Eqs. (36,37) may be related to the conformal anomaly.

5.2 Coincident Branes: de Sitter Brane in Flat Spacetime

Now consider the opposite limit, in which the two branes are coincident: $z_0 = z_1$. In this limit the anti-de Sitter region between the branes disappears, and it is as if we had a single de Sitter brane in Minkowski space. This brane has positive tension provided we start
with configuration A. Consulting our master formula, Eq. (31), we see that when \( z_0 = z_1 \) the propagator reduces to a simple form:

\[
\Delta(q) = \frac{G_5 R_0}{q^2 + iM}.
\]

(38)

Evidently, Eq. (38) has a pole at \( iM = -\frac{3}{2} \) where \( q^2 = 2R_0^{-2} \). There is thus four-dimensional gravity on the brane even in Minkowski space. This is a consequence of the normalizability of the bound state. (Had we considered a negative tension brane, for which the factor \( e^{-2A} \) grows with \( z \), the pole (here at \( iM = -\frac{3}{2} \)) would not have corresponded to a physical particle since there are now no normalizable bound states.) We also obtain the Newton constant:

\[
\Delta(q) \overset{M \to 3i/2}{\approx} \frac{3G_5}{R_0} \frac{1}{q^2 - 2R_0^{-2}} \Rightarrow G_N = \frac{3G_5}{R_0}.
\]

(39)

This depends on the radius of the brane or, equivalently, on the location of the brane in the five-dimensional flat spacetime. That localized gravity can arise on a curved brane in asymptotically flat space with a position-dependent Newton constant is a by-product of the recent study in [18]. Note that the effect is absent for flat branes. Indeed, the propagator along the brane is then

\[
\lim_{R_0 \to \infty} \Delta(q) = \frac{G_5}{iq},
\]

(40)

which is a five-dimensional propagator. We note as a check that this is just Eq. (20) in the limit \( z_1 \to 1 \).

Turning now to the continuous part of the spectrum, \( M^2 \equiv m^2R_0^2 \geq 0 \), we find that

\[
\text{Re } \Delta(m) = \frac{3}{2} \frac{G_5}{R_0} \frac{1}{m^2 + \frac{9}{4R_0^2}}, \quad \text{Im } \Delta(m) = -\frac{G_5m}{m^2 + \frac{9}{4R_0^2}}.
\]

(41)

This has the characteristic Breit-Wigner form of a resonance. The resonance is concentrated at \( m = 0 \) and has a decay width of \( \frac{3}{2R_0} \). In other words, brane gravity is mediated by both a graviton and a graviton-like resonance. Their Newton couplings to matter differ by a factor of two.

### 5.3 Branes at Finite Separation

A general feature of the graviton propagator for two de Sitter branes in configuration A is that the flat-brane pole \( \frac{1}{iq} \), which signals the re-emergence of 5-D gravity at ultra-large scales, now disappears. It is replaced by a pole at \( iM = -\frac{3}{2} \) or, equivalently, at \( q^2 = 2R_0^{-2} \). This corresponds to the four-dimensional graviton which now mediates the large-scale gravitational interaction on the brane. The strength of this interaction, Newton’s constant, is computed by taking the residue of Eq. (31) at the pole \( iM = -\frac{3}{2} \). This can be done explicitly by computing the residue in Eq. (31). The physics simplifies in the approximation of large radius branes: \( R_0 k, R_1 k \gg 1 \). In this regime, both \( \xi_0 \) and \( \xi_1 \) are large, and the propagator displays not only the pole but also the rich resonance

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4 This requires some care since the Legendre function \( Q_{-3/2}^{-3/2} \) is not defined: one has to first transform \( Q_{3/2}^{3/2} \) into \( Q_{1/2}^{-iM} \) and then take the limit \( iM \to -3/2 \).
structure. Using Eqs. (61,62) we expand the Legendre functions in Eq. (31) for large \( \xi_0 \) and \( \xi_1 \). We find that

\[
\Delta(q) \simeq \frac{G_5 R_0}{\frac{3}{2} + iM} \frac{1}{\alpha} \left( \frac{4kR_0\alpha - (1 - \alpha^4)\left(\frac{3}{2} + iM\right)}{4kR_0\alpha^3 + 2(1 - \alpha^2)\left(\frac{3}{2} - iM\right)} \right),
\]

where we have introduced \( \alpha = \left(\frac{R_1}{R_0}\right) \). From the pole at \( iM = -\frac{3}{2} \), we see that

\[
G_{N}^{\text{pole}} \simeq \left(\frac{R_0}{R_1}\right)^{3}.
\]

Up to a factor \( \alpha^{-3} \), this is similar to the expression we have for coincident branes. This is because at large intervals the merger of two branes looks like one de Sitter brane in flat space. The configuration-dependence of the four-dimensional Newton constant is evident.

On the other hand, when the brane separation is relatively large we expect a resonance to appear in the KK portion of the spectrum, similar to the resonance in flat-brane GRS. In fact, Eq. (42) contains not one, but two resonances! To see this, let \( M = mR_0 \) and \( \alpha^3\xi_0 \gg 1 \). Then

\[
\Delta(q) \simeq \Delta_1(q) + \Delta_2(q),
\]

where

\[
\Delta_1(q) = G_5 \frac{1}{\alpha^3} \frac{\frac{3}{2}R_0}{9 + iM} + m^2,
\]

and

\[
\Delta_2(q) = G_5 \frac{(1 - \alpha^2)(2 + \alpha^2)}{2\alpha^3} \left( \frac{\frac{3}{2}R_0}{9 + 2k\frac{\alpha^3}{1-\alpha^2}} + iM \right) \left( \frac{\frac{3}{2}R_0}{9 + 2k\frac{\alpha^3}{1-\alpha^2}} + m^2 \right)\]

Both Eq. (43) and Eq. (44) have the resonance structure familiar from the flat GRS case, cf. Eqs. (13,14). The gravitational coupling is, respectively,

\[
G_N^{\text{res}1} = \frac{3}{2}G_5 \frac{1}{R_0 \alpha^3}, \quad G_N^{\text{res}2} = kG_5(2 + \alpha^2).
\]

We see that the second resonance is analogous to the resonance in flat-brane GRS. The coupling \( G_N^{\text{res}2} \sim 2kG_5 \) (for small \( \alpha \)) is typical for a single brane embedded in anti-de Sitter space. The first resonance is what one could call a \textit{shadow}, since it always accompanies the pole at \( iM = -\frac{3}{2} \). We have already seen it at the end of the previous section, in Eq. (11). In the flat brane limit when \( R_0 \) goes to infinity (with \( \alpha = \frac{R_1}{R_0} \) fixed) the second resonance becomes the GRS resonance while the “shadow” disappears. But not completely: its imaginary part \( -iG_5\alpha^{-3}/m \) survives. The small \( x \)-divergence of \( \text{Im} \, \Delta(x) \), in Eq. (13) is all that is left of the “shadow.” From the real part of the propagator, Eqs. (45,46), we read off the widths of the resonance:

\[
\Delta m_1 = \frac{3}{2R_0}, \quad \Delta m_2 = \frac{3}{2R_0} + 2k\frac{\alpha^3}{1-\alpha^2} \simeq 2k \left( \frac{R_1}{R_0} \right)^3.
\]

\(^5\)Note that Eq. (12) is not valid for the case of single brane in AdS space since in this case \( \xi_1 = 1 \) (the second brane is taken to infinity).
We see then that we have a “zoo” of graviton-like states – a pole and two resonances – which mediate 4-D gravity on the brane. We will discuss the physical consequences of this in the next section. Here we just mention that the gravitons operate at essentially different scales. The second resonant state is responsible for the gravitational interaction at intervals
\[ \Delta s_2 \sim \frac{1}{\Delta m_2} \approx \frac{1}{2k} \left( \frac{R_0}{R_1} \right)^3, \]

while the pole becomes important at much larger intervals, \( \Delta s \gg \Delta s_2 \). The “shadow” operates somewhere in between. It seems that its role is just to make a smooth transition between two completely different regimes.

6 From Close-Up to Panorama

In this paper, we have calculated the effective momentum-space graviton propagator on a de Sitter brane embedded in various five-dimensional spacetimes. The resulting picture is of gravity with rich scale structure. Not only does the approximate dimensionality of Newton’s law depend on the scale, but the coupling and even the identity of the carrier of the gravitational interaction – whether the graviton or a resonance – depends on the energies at which the physics is probed.

This is because the gravitational interaction on a brane propagates not just along the brane but also through the bulk. The larger the separation between points on the brane, the deeper the part of the bulk that affects the propagation. At very short distances on the brane, gravity is five-dimensional. This is due to large-momentum excitations; it can be seen by taking the limit of infinite \( q \) in any one of our propagators. The brane propagator then behaves as \( \frac{1}{iq} \); that is, as a five-dimensional graviton. The scale at which the physics is five-dimensional is of order \( 1/k \).

When a configuration of two branes is considered, the brane separation sets another scale (see [4]) at which the gravitational interaction on the brane can change dramatically. This scale, \( \Delta s_2 \approx \frac{1}{2k} \left( \frac{R_0}{R_1} \right)^3 \), is determined by the width of the resonant mode which mediates the four-dimensional gravitational force in the range \( 1/k \ll \Delta s \ll \Delta s_2 \). For well-separated branes \( \Delta s_2 \) can be much larger than \( 1/k \). In order to make estimates let us assume that \( R_0 k, R_1 k \gg 1 \). Then the radius of a brane can be approximated as \( R = \frac{1}{k} e^{ky} \), where \( y \) is the geodesic distance between the brane and the AdS horizon. The strength of the gravitational interaction for the above range of intervals is given by \( G_{N}^{res2} \), Eq. (47), which is approximately \( 2kG_5 \). This is because for the intervals \( 1/k \ll \Delta s \ll \Delta s_2 \), the second brane has no effect.

For intervals larger than \( \Delta s_2 \) the propagation through the bulk is affected by the second brane and Minkowski space; for \( \Delta s \gg \Delta s_2 \) propagation is mostly through flat space. Gravity on the brane in this regime is mediated by the pole with Newton constant \( G_{N}^{pole} \), Eq. (43). The interaction then resembles that of a single brane embedded in flat space. Comparing Eq. (43) and Eq. (47) we see that gravity at larger scales is much weaker than at short scales:
\[ G_{N}^{pole} / G_{N}^{res2} \approx \frac{1}{kR_0} \left( \frac{R_0}{R_1} \right)^3. \]

For \( R_0 \sim R_1 \) this is exponentially small. The brane cosmological constant (or vacuum

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energy) is determined, to leading order, by its geometric value: \( \Lambda \approx \frac{1}{R_0^2} \). For the dimensionless quantity \( \Lambda N \) we find

\[
\Lambda G_N^{\text{res}2} \simeq \frac{1}{k^2 R_0^2} \sim e^{-2k y_0} ,
\]

provided we choose \( G_5 \sim k^{-3} \). But measuring the cosmological constant in terms of \( G_N^{\text{pole}} \) we obtain

\[
\Lambda G_N^{\text{pole}} \simeq \frac{1}{(k R_0)^3} \left( \frac{R_0}{R_1} \right)^3 ,
\]

which is smaller than that of Eq. (51) by a factor \( 1/(k R_0) \sim e^{-k y_0} \), assuming that \( y_0 \sim y_1 \).

These estimates are even stronger if we consider configuration B. In this case, the radius of the visible brane is smaller than that of the hidden brane, \( R_0 < R_1 \), so that the visible brane has negative tension. The total tension of the stack of three branes is still positive, though. So at larger scales, gravity on the visible brane is still due to the pole, with the Newton constant given by Eq. (43). At shorter scales one could expect some peculiarities due to the negative tension. Indeed, the overall factor \((1 - \alpha^2)\) in Eq. (46) flips sign when \( R_1 \) becomes greater than \( R_0 \). However, closer inspection of Eq. (46) shows that for \( R_0k \gg 1 \), the Newton constant \( G_N^{\text{res}2} \) is given by Eq. (47) and is still positive. Assuming that \( R_1 \) is much larger than \( R_0 \), we find from Eq. (47) that

\[
G_N^{\text{res}2} \simeq 2k G_5 \left( \frac{R_1}{R_0} \right) .
\]

So for \( R_1 \gg R_0 \) the Newton constant of the resonance in configuration B is much smaller than it is for configuration A. Comparing Eq. (53) with \( G_N^{\text{pole}} \) we see that

\[
G_N^{\text{pole}} / G_N^{\text{res}2} \simeq \frac{1}{k R_0} \left( \frac{R_0}{R_1} \right)^4 \sim e^{-k y_0} e^{-4k|y_0-y_1|} .
\]

The large-scale gravitational interaction is much weaker than at shorter scales mostly due to the difference between \( R_1 \) and \( R_0 \). Consequently, the cosmological constant measured in units of \( G_N \) is exponentially smaller at larger scales.

If the present picture is correct and we live on the visible brane in a stack of three branes arranged in configuration A or B, then there is a relation between the observable Newton constant, \( G_N \), the de Sitter radius of the brane, \( R_0 \), and the five-dimensional Newton constant, \( G_5 \). Inverting Eq. (43) for \( G_5 \) we can estimate the fundamental energy scale in the bulk, \( M_{\text{pl}}(5) = G_5^{-1/3} \). Inserting the measured values of \( G_N \) and the Hubble radius, we find

\[
M_{\text{pl}}(5) \sim 100^{-1} \, \text{GeV} ,
\]

which for \( \alpha \simeq \frac{R_1}{R_0} \sim 10^{-2} \) is of the order of the electroweak energy scale.

Many interesting questions remain. It would be worthwhile to consider the implications for a corresponding holographic theory for a de Sitter brane embedded in AdS. It would also be interesting to study the detailed tensor structure [19] of the propagator to see how robust the results are. That there is no vDVZ discontinuity in de Sitter space [20]-[22] is perhaps a good omen.
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Appendix

In this appendix, we list a few useful formulas concerning associated Legendre functions. The Legendre functions, \( P_{\mu}^\nu(z) \) and \( Q_{\mu}^\nu(z) \), that we use are single-valued and regular for \( \text{Re} z > 1 \), with a branch cut from \(-\infty\) to \(+1\). They satisfy

\[
(z^2 - 1) \frac{d}{dz} P_{\nu}^\mu(z) = \nu z P_{\nu}^\mu(z) - (\nu + \mu) P_{\nu - 1}^\mu(z),
\]

(56)

\[
(\nu - \mu + 1) P_{\nu + 1}^\mu(z) = (2\nu + 1) z P_{\nu}^\mu(z) - (\nu + \mu) P_{\nu - 1}^\mu(z),
\]

(57)

and identical equations for the \( Q \)'s. Some closed-form expressions that are relevant are

\[
P_{-3/2}^{3/2}(\xi) = \frac{2}{3\sqrt{2\pi}} (\xi^2 - 1)^{3/4},
\]

(58)

\[
P_{1/2}^{-3/2}(\xi) = \frac{1}{\sqrt{2\pi}} \left[ \frac{\xi}{(\xi^2 - 1)^{1/4}} - \frac{1}{2(\xi^2 - 1)^{3/4}} \ln \left( \frac{\xi + \sqrt{\xi^2 - 1}}{\xi - \sqrt{\xi^2 - 1}} \right) \right],
\]

(59)

\[
Q_{1/2}^{+3/2}(\xi) = -i \frac{\sqrt{2\pi}}{4} \frac{1}{(\xi^2 - 1)^{3/4}}.
\]

(60)

We also need the asymptotics for large \( \xi \). For \( \nu > 0 \), in the leading order we have

\[
P_{\nu}^\mu(\xi) = \frac{(2\xi)\nu}{\sqrt{\pi}} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu - \mu + 1)} \left( 1 + \mathcal{O}\left(\frac{\ln \xi}{\xi^2}\right) \right),
\]

(61)

\[
Q_{\nu}^\mu(\xi) = \frac{\sqrt{\pi}}{(2\xi)^{\nu+1}} e^{i\mu \pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \left( 1 + \mathcal{O}\left(\frac{\ln \xi}{\xi^2}\right) \right).
\]

(62)

The logarithmic subleading term appears in the ratio

\[
\frac{P_{3/2}^{iM}(\xi)}{P_{1/2}^{iM}(\xi)} = \frac{2\xi}{\frac{3}{2} - iM} \left( 1 - \frac{(1 + 4M^2)}{8\xi^2} \ln \xi + \mathcal{O}\left(\frac{1}{\xi^2}\right) \right).
\]

(63)

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