Notes on coherent backscattering from a random potential

Herbert Spohn

Zentrum Mathematik and Physik Department, TU München,
D - 85747 Garching, Boltzmannstr. 3, Germany

Abstract. We consider the quantum scattering from a random potential of strength $\lambda^{1/2}$ and with a support on the scale of the mean free path, which is of order $\lambda^{-1}$. On the basis of maximally crossed diagrams we provide a concise formula for the backscattering rate in terms of the Green’s function for the kinetic Boltzmann equation. We briefly discuss the extension to wave scattering.
1 Introduction

The motion of a single quantum particle in a weak random potential of strength $\lambda^{1/2}$ is well approximated by a kinetic equation. More precisely, in the limit of small $\lambda$, the disorder averaged Wigner function $W$ is governed by the transport equation:

$$\frac{\partial}{\partial t} W(q,p,t) = -\nabla_p \omega(p) \cdot \nabla_q W(q,p,t) + \rho(q) \int dp' 2\pi \delta(\omega(p) - \omega(p')) (2\pi)^{-3/2} \tilde{\theta}(p - p') (W(q,p',t) - W(q,p,t)).$$

Here $\omega(p)$ is the kinetic energy (= dispersion relation), $\omega(p) = p^2/2m$ for a non-relativistic particle. $\tilde{\theta}$ is the Fourier transform of the translation invariant part of the covariance of the random potential and $\rho(q)^{1/2}$ is the spatially varying strength of the potential. The mean free time and the mean free path are both of order $\lambda^{-1}$ and Eq. (1.1) is written on that scale. As a side remark, the Wigner function integrated against a smooth test function is self-averaging [1]. Thus (1.1) holds even for a typical realization of the random potential.

(1.1) can be thought of as arising from the stochastic motion of a fictitious classical particle: the particle moves along a straight line according to its velocity and randomly changes its momentum, respecting energy conservation, with a rate given through the covariance of the potential. In this sense the wave nature of the quantum motion is completely lost as $\lambda \to 0$. However, as discovered by Langer and Neal [2], even for small $\lambda$ the wave character is still visible in the fine structure of the average scattering rate. We imagine an incoming plane wave with wave number $k$ scattering off the random potential and denote by $\sigma_B(k,k')$ the scattering rate from the incoming wave number $k$ to the outgoing wave number $k'$ according to the transport equation (1.1) and by $\langle \sigma_B(k,k') \rangle_V$ the scattering rate for the Schrödinger equation averaged over the random potential. Then

$$\lim_{\lambda \to 0} \lambda^2 \langle \sigma_B(k,k') \rangle_V = \sigma_B(k,k') \text{ for } k' \neq \pm k.$$  

(1.2)

Note that the total cross section of the scattering potential is of order $\lambda^{-2}$. Therefore in (1.2) one needs to balance by the prefactor $\lambda^2$. For exact backscattering one finds that

$$\lim_{\lambda \to 0} \lambda^2 \langle \sigma(k,-k) \rangle_V = 2\tilde{\sigma}_B(k,-k),$$

(1.3)

where the tilde indicates that from $\sigma_B$ the paths with only a single scattering event have to be omitted. In fact, the enhanced backscattering consists of a narrow peak of width $\lambda$ centered at $k' = -k$. As to be discussed, one finds

$$\lim_{\lambda \to 0} \lambda^2 \langle \sigma_B(k,-k + \lambda \kappa) \rangle_V = \sigma_{\text{back},k}(\kappa).$$

(1.4)

Properties (1.2) and (1.4) can be seen from an expansion of the scattering amplitude in powers of the potential $V$. Summing all ladder diagrams yields (1.2), while
results from the maximally crossed diagrams. Of course, to actually prove the limits (1.2), (1.4) one would have to establish that the contribution of the very many remaining diagrams vanishes as \( \lambda \to 0 \). This task will not be addressed in my notes. But I point out to the reader that very complete and precise estimates have been achieved in the recent work of Erdös, Salmhofer, and Yau [3, 4]. These notes are written, at least in part, with the aim to encourage a similar kind of analysis for the scattering rate.

Coherent backscattering according to (1.4) has been confirmed experimentally including the shape of the peak [5, 6]. The experiment is done for light scattering, since light is so much easier to manipulate than electrons. In (1.4) averaging over disorder is required. In a single scan no peak can be disentangled. Thus one either repeats the experiment many times using samples of disordered glass or, more elegantly, scatters from a turbid solution. Then the Brownian motion of the suspended particles provides the averaging for free. Excellent theoretical texts are available [7, 8, 9]. They are striving for even more refined information, as e.g. the variance of the conductance fluctuations and the statistics of speckle patterns [7]. Does there remain then anything to be done on the level of (1.4)?

In fact, I was looking for a concise expression of \( \sigma_{\text{back},k} \) in terms of the transition probability resulting from Eq. (1.1) and could not find it in the literature. My result is given at the beginning of Section 5. The required computation I find sufficiently illuminating to be put into written form.

In the literature (as far as I checked) the conventional approach is to start from the average of the square of the Green’s function and then to extract from it the scattering. We proceed in a way which looks more systematic to us. For every realization of the disordered medium the scattering rate is given through stationary scattering theory, at least in principle. The average scattering rate is then expanded in the disorder strength.

The standard Born expansion for the scattering amplitude is covered in Section 2 including the average over the random potential. The scattering theory for (1.1) is explained in Section 3. The most lengthy part of our contribution is the summation of the ladder diagrams, while the maximally crossed diagrams then easily follow, see Sections 4 and 5. We add the modifications required when the Schrödinger equation is replaced by a wave equation and close with the diffusion approximation to (1.4).

2 Average scattering rate for electrons

Because it is slightly simpler and more familiar, we first discuss the stationary scattering theory for an electron moving in a random potential. The physically more relevant wave scattering will require only minor modifications, see Section 6.

We consider the hamiltonian

\[
H = -\frac{1}{2}\Delta + \lambda^{1/2}V(x), \quad \lambda > 0,
\]

(2.1)
acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, dx)$. In Fourier space the Schrödinger equation then reads

$$i\frac{\partial}{\partial t} \hat{\psi}(k, t) = \omega(k) \hat{\psi}(k, t) + (2\pi)^{-3/2}\lambda^{1/2} \int dk_1 \hat{V}(k - k_1) \hat{\psi}(k_1, t),$$

(2.2)

where we introduced the dispersion relation $\omega(k)$,

$$\omega(k) = \frac{1}{2} k^2$$

(2.3)

for a nonrelativistic particle of mass 1. The formalism is such that general dispersion relations are allowed. $V(x)$ is a Gaussian random potential with zero mean $\langle V(x) \rangle_V = 0$, $\langle \cdot \rangle_V$ denoting the average over $V$. The statistics of $V(x)$ is translation invariant locally but $V(x)$ vanishes outside some bounded region. To achieve it we introduce the smooth shape function $\rho : \mathbb{R}^3 \to \mathbb{R}$, such that the support of $\rho$, supp $\rho$, is a bounded set. Then the covariance for $\hat{V}(x)$ is given by

$$\langle V(x) V(y) \rangle_V = \rho(\lambda x)^{1/2} \theta(x - y) \rho(\lambda y)^{1/2}.$$  

(2.4)

$\theta(x) = \theta(-x)$ and $\theta$ is assumed to have a rapid decay. Its Fourier transform satisfies $\hat{\theta}(k) \geq 0$, so to have a positive definite covariance.

In Fourier space the covariance (2.4) is peaked near $k' = -k$ and we write

$$\langle \hat{V}(k) \hat{V}(-k + \lambda k_1) \rangle_V$$

$$= (2\pi)^{-3/2} \int dg \hat{\theta}(g + k_1) \lambda^{-3} \hat{\rho}^{1/2}(\lambda^{-1} g)^* \lambda^{-3} \hat{\rho}^{1/2}(\lambda^{-1} g + k_1)$$

$$\cong \hat{\theta}(k) \lambda^{-3} (2\pi)^{-3/2} \int dg \hat{\rho}^{1/2}(g)^* \hat{\rho}^{1/2}(g + k_1)$$

$$= \hat{\theta}(k) \lambda^{-3} (2\pi)^{-3/2} \int dx \rho(x) e^{-i k_1 x} = \hat{\theta}(k) \lambda^{-3} \hat{\rho}(k_1).$$

(2.5)

Therefore, in approximation, we set

$$\langle \hat{V}(k) \hat{V}(k') \rangle_V = \hat{\theta}(k) \hat{\rho}_\lambda(k + k'),$$

(2.6)

where $\hat{\rho}_\lambda(k) = \lambda^{-3} \hat{\rho}(\lambda^{-1} k)$. Below we will always use (2.6), with symmetry and positivity restored through the $O(\lambda)$ corrections.

We consider a weak potential, $\lambda \ll 1$. Then the mean free path of the electron is $O(\lambda^{-1})$. The shape function $\rho$ enforces $V$ to vanish outside a bounded region on the scale $\lambda^{-1}$. Thus the total cross section is of order $\lambda^{-2}$.

For almost every realization of $V$ the Hamiltonian $H$ has a well-defined unitary scattering matrix $S_\lambda$, see [10] Section XI.6 for a discussion of stationary scattering theory. In momentum space $S_\lambda$ has the kernel

$$S_\lambda(k, k') = \delta(k - k') - 2\pi i T_\lambda(k, k') \delta(\omega(k) - \omega(k')).$$

(2.7)
Let us set $\varphi_k(x) = (2\pi)^{-3/2}e^{ik \cdot x}$. Then the $T$-matrix is defined through

$$T_\lambda(k, k') = \lim_{\varepsilon \to 0} \langle \varphi_k, (\lambda^{1/2} V - \lambda^{1/2} V (H - (E + i\varepsilon))^{-1} \lambda^{1/2} V) \varphi_{k'} \rangle_{\mathcal{H}},$$

$$E = \omega(k) = \omega(k'),$$

with $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denoting the scalar product of $\mathcal{H}$. The scattering rate $\sigma_\lambda(k, k')$ from $k$ to $k'$ is the square of the $T$-matrix. Since in (2.7) $k$ refers to the far future and $k'$ to the far past, one obtains

$$\sigma_\lambda(k, k') = (2\pi)^3 |T_\lambda(k', k)|^2 2\pi \delta(\omega(k) - \omega(k'))$$

for $k \neq k'$. Recall that $\sigma_\lambda(k, k')$ is random through the dependence on $V$.

In these notes we deal only with the average scattering rate, $\langle \sigma_\lambda(k, k') \rangle_V$, for small $\lambda$. The variance of $\sigma_\lambda$ is also of great interest and has been studied both through experiments and theoretically [7].

To be able to perform the Gaussian average we expand the resolvent of (2.8) into the Born series as

$$T_\lambda(k', k) = \sum_{n=0}^\infty (-1)^n \langle \varphi_{k'}, \lambda^{1/2} V (G_{E+} \lambda^{1/2} V)^n \varphi_k \rangle_{\mathcal{H}}. \quad (2.10)$$

Here $G_{E+}$ is the free Green’s function,

$$G_{E+} = \lim_{\varepsilon \to 0} (H_0 - (E + i\varepsilon))^{-1}, \quad H_0 = -\frac{1}{2} \Delta. \quad (2.11)$$

Inserting (2.10) in (2.9) one obtains

$$\langle \sigma_\lambda(k, k') \rangle_V = (2\pi)^3 \left| \sum_{n=0}^\infty (-1)^n \langle \varphi_{k'}, \lambda^{1/2} V (G_{E+} \lambda^{1/2} V)^n \varphi_k \rangle_{\mathcal{H}} \right|^2 V 2\pi \delta(\omega(k) - \omega(k')). \quad (2.12)$$

At this stage the average over $V$ can be carried out explicitly. The result is most concisely expressed diagrammatically.

![Figure 1: A diagram and its basic building blocks.](image)

We draw two horizontal lines, see Fig. 1, the top one is directed to the right and the bottom one towards the left. They have external momenta $k', k$. The
top line carries \( n_+ \) vertices, \( n_+ = 1, 2, \ldots \), and corresponding internal momenta \( k_1, \ldots, k_{n_+ - 1} \). Each bond, see (a), with momentum \( k \) carries the Green’s function \( G_{E_+}(k) = (\omega(k) - (E + i\varepsilon))^{-1} \). Each vertex, see (b), with momenta \( k_1, k_2 \) carries the potential \( \lambda^{1/2}(2\pi)^{-3/2}\hat{V}(k_1 - k_2) \). Correspondingly, the bottom line carries \( n_- \) vertices, \( n_- = 1, 2, \ldots \), and internal momentum \( g_1, \ldots, g_{n_- - 1} \). Each bond with momenta \( g \) carries the Green’s function \( G_{E_-}(g) = G_{E_+}(g)^* \) and each vertex with momenta \( g_1, g_2 \) carries the potential \( \lambda^{1/2}(2\pi)^{-3/2}\hat{V}(g_1 - g_2) \). The resulting expression is integrated over all internal momenta \( k_1, \ldots, k_{n_+ - 1}, g_1, \ldots, g_{n_- - 1} \). The Gaussian average generates a sum over all pairings, provided \( n_+ + n_- \) is even. Otherwise the average vanishes. A pairing is indicated by a wavy line, see (c).

For obvious reasons the building block (d) is called a gate. The sum over all gates can be performed thereby modifying the free Green’s function (or propagator) to the effective medium Green’s function as

\[
\sum_{n=0}^{\infty} \left( \begin{array}{c} \text{\includegraphics{d.png}} \end{array} \right)^n = \begin{array}{c} \text{\includegraphics{d.png}} \end{array} .
\]

Thus the diagrams remain as in Fig. 1 only the light lines are replaced by thick lines. Because of the spatial cut-off in the potential the effective medium Green’s function is not as explicit as in the translation invariant case.

In the following we study two particular classes of diagrams, namely the ladder diagrams and the maximally crossed diagrams, schematically represented in Fig. 2.

![Ladder Diagram](image.png)

**Figure 2:** A ladder diagram and a maximally crossed diagram with effective medium propagators.

We denote by \( I_{\text{ladv}}(k', k; \lambda) \) the sum over all ladder diagrams with external momenta \( k' \) and \( k \) and by \( I_{\text{max}}(k', k; \lambda) \) the sum over all maximally crossed diagrams with external momenta \( k' \) and \( k \).

### 3 Scattering rate from the Boltzmann equation

For a weak random potential one can approximate the true wave dynamics by the stochastic motion of a fictitious classical particle. It has position \( q \in \mathbb{R}^3 \), momentum \( p \in \mathbb{R}^3 \), and kinetic energy \( \omega(p) \). It moves with constant momentum and changes its momentum at random times subject to the constraint of constant energy. More precisely, given the current location \( q \) and momentum \( p \), in the short time interval
the probability for the momentum to be scattered into the volume element \( dp' \) is given by

\[
\rho(q)(2\pi)^{-3/2}\hat{\theta}(p - p')2\pi\delta(\omega(p) - \omega(p'))
\]

(3.1)

independently for each small time interval. The corresponding stochastic process is determined by the backward generator \( L \), which is defined through

\[
Lf(q,p) = \nabla_p \omega(p) \cdot \nabla_q f(q,p) + \rho(q) \int dp'2\pi\delta(\omega(p) - \omega(p'))(2\pi)^{-3/2}\hat{\theta}(p - p')(f(q,p') - f(q,p))
\]

(3.2)

as a linear operator acting on functions \( f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C} \). The transition probability from \((q,p)\) to \((dq',dp')\) in time \( t, \ t \geq 0 \), is then given by the integral kernel of \( e^{Lt} \), denoted by

\[
e^{Lt}(q,p|dq' dp'), \ t \geq 0.
\]

(3.3)

Clearly, outside of \( \text{supp} \rho \) the collision operator vanishes and the particle moves freely.

The stationary scattering situation is easily modelled. We choose \( w \in \mathbb{R}^3 \) away from \( \text{supp} \rho \) and a plane \( \Lambda_{w,k} \) through \( w \) and orthogonal to the incoming unit wave vector \( \hat{k} = k/|k| \) and not intersecting \( \text{supp} \rho \). The initial position of the particle is uniformly distributed on \( \Lambda_{w,k} \) and the initial velocity equals \( \nabla \omega(k) \). The corresponding initial measure \( \mu_k \) imposes a uniform flux at momentum \( k \) through the plane \( \Lambda_{w,k} \), i.e.

\[
\mu_k(dq dp) = \delta(p - k)\delta((x - w) \cdot \hat{k})|\hat{k} \cdot \nabla \omega(p)|dq dp.
\]

(3.4)

The Boltzmann scattering rate from \( k \) to \( k' \) is then given by

\[
2\pi\delta(\omega(k) - \omega(k'))\sigma_B(k,k') = \lim_{t \to \infty} \int \int \mu_k(dq dp)e^{Lt}(q,p|dq' dp')\delta(p' - k')
\]

(3.5)

for \( k \neq k' \). For \( k = k' \) there is a \( \delta \)-contribution coming from those paths which pass the potential without any scattering. In our set-up this \( \delta \)-function has infinite weight. More meaningful would be to restrict \( \mu_k(dq dp) \) to those initial conditions for which the particle actually hits \( \text{supp} \rho \). But the strict forward scattering \( k' = k \) is of no concern to us here, anyhow. The right side of (3.5) is proportional to \( \delta(\omega(k) - \omega(k')) \). It is convenient to remove this overall factor. On the energy shell, \( \sigma_B(k,k') \) is uniquely defined and depends smoothly on its arguments.

In (3.5) only the momentum of the scattered particle is resolved. The limit \( t \to \infty \) is needed, so that for any initial condition on the plane \( \Lambda_{w,k} \) the particle has escaped from the scattering region.

We rewrite (3.5) in a slightly more convenient form by splitting the generator as

\[
L = L_0 + L_1
\]

(3.6)
\[ L_1 f(q, p) = \rho(q) \int dp' 2\pi \delta(\omega(p) - \omega(p'))(2\pi)^{-3/2} \hat{\theta}(p - p') f(q, p') . \tag{3.7} \]

Then
\[
e^{Lt} = e^{L_0 t} + \int_0^t ds e^{L_0 s} L_1 e^{L_0 (t-s)} + \int_0^t ds \int_0^s ds' e^{L_0 s'} L_1 e^{L(s-s')} L_1 e^{L_0 (t-s)} . \tag{3.8} \]

Let
\[
\nu(p) = \int dp' 2\pi \delta(\omega(p) - \omega(p'))(2\pi)^{-3/2} \hat{\theta}(p - p') \tag{3.9} \]
be the total cross section at momentum \( p \) and set
\[
f^+_{k'}(q, p) = \delta(p - k) \exp \left[ -\nu(p) \int_0^\infty dt \rho(q + \nabla \omega(p)t) \right] , \]
\[
f^-_{k}(q, p) = \delta(p - k) \exp \left[ -\nu(p) \int_{-\infty}^0 dt \rho(q + \nabla \omega(p)t) \right] . \tag{3.10} \]

Inserting (3.8) in (3.5), the first term does not contribute, since \( k \neq k' \), and the second and third term have limits, since \( \text{supp} \rho \) is bounded. Then
\[
2\pi \delta(\omega(k) - \omega(k')) \sigma_B(k, k') = \langle f^-_{k}, L_1 f^+_{k'} \rangle + \int_0^\infty dt \langle f^-_{k}, L_1 e^{Lt} L_1 f^+_{k'} \rangle \tag{3.11} \]
with \( \langle \cdot, \cdot \rangle \) denoting the scalar product in \( L^2(\mathbb{R}^3 \times \mathbb{R}^3, dq dp) \). The time integral converges, since the probability to stay inside \( \text{supp} \rho \) decays exponentially.

### 4 Summation over ladder diagrams

The goal of this section is to establish that for \( \omega(k) = \omega(k') \) it holds that
\[
\lim_{\lambda \to 0} \lambda^2 (2\pi)^3 I_{\text{ladder}}(k', k; \lambda) = \sigma_B(k, k'). \tag{4.1} \]

The prefactor \( \lambda^2 \) balances the cross section of the scattering region which is of order \( \lambda^{-2} \).

We set
\[
H_\pm(k_1) = \mp i \int dk_2 (2\pi)^{-3/2} \hat{\theta}(k_1 - k_2) G_{\pm}(k_2) \tag{4.2} \]
and note that
\[
H_+(k) + H_-(k) = \nu(k) . \tag{4.3} \]
The gate \( k_1 \quad k_2 \quad k_3 \) corresponds to the integral operator

\[
\int dk_2 G_{E^+}(k_1)G_{E^+}(k_2)\lambda(2\pi)^{-3}\langle \hat{V}(k_1-k_2)\hat{V}(k_2-k_3) \rangle_V
= \lambda(2\pi)^{-3}G_{E^+}(k_1) \int dk_2 \hat{\theta}(k_1-k_2)G_{E^+}(k_2)\hat{\rho}_\lambda(k_1-k_3)
= i\lambda(2\pi)^{-3/2}G_{E^+}(k_1)H_+(k_1)\hat{\rho}_\lambda(k_1-k_3). \tag{4.4}
\]

Therefore the effective medium propagator, \( k_1 \quad \rightarrow \quad k \), is represented by the kernel

\[
F_-(k_1, k; \lambda) = \delta(k_1-k) + \sum_{m=2}^{\infty} \int dk_2 \ldots dk_m \prod_{j=1}^{m} \{ G_{E^+}(k_j)i(2\pi)^{-3/2}H_+(k_j)\lambda\hat{\rho}_\lambda(k_j-k_{j+1}) \}, \tag{4.5}
\]

where in (4.5) we set \( k_{m+1} = k \). We define

\[
I_0(x; \lambda) = \int dk_1 \exp[i\lambda^{-1}(k_1-k) \cdot x]F_-(k_1, k; \lambda)
= \int dk_1 e^{ik_1 \cdot x}\lambda^3F_-(k+k_1, k; \lambda). \tag{4.6}
\]

Shifting the other internal momenta by \( k \) and rescaling by \( \lambda \) yields

\[
I_0(x; \lambda) = 1 + \sum_{m=1}^{\infty} \int dk_1 \ldots dk_m e^{ik_1 \cdot x} \prod_{j=1}^{m} \{ \lambda G_{E^+}(k + \lambda k_j)i(2\pi)^{-3/2}H_+(k + \lambda k_j)\hat{\rho}(k_j-k_{j+1}) \}, \tag{4.7}
\]

where in (4.7) we set \( k_{m+1} = 0 \). \( H_+ \) is a smooth function. Thus \( H_+(k + \lambda k_j) \) may be replaced by \( H_+(k) \). The Green’s function \( G_{E^+} \) has the integral representation

\[
\lambda G_{E^+}(k + \lambda k_1) = i\lambda \int_0^\infty dt e^{-it(\omega(k+\lambda k_1)-\omega(k)-i\epsilon)}
\approx i \int_0^\infty dt e^{-it(|\omega(k)|k_1-i\epsilon)}. \tag{4.8}
\]
Inserting in (4.7) yields

\[
\lim_{\lambda \to 0} I_0(x; \lambda) = 1 + \sum_{m=1}^{\infty} (-H_+(k))^m \int dk_1 \ldots dk_m (2\pi)^{-3m} \int dx_1 \ldots dx_m \rho(x_1) \ldots \rho(x_m) \\
\times \int_0^{\infty} dt_1 \ldots \int_0^{\infty} dt_m \exp \left[-i \sum_{j=1}^{m} (t_j \nabla \omega(k) \cdot k_j + (k_j - k_{j+1}) \cdot x_j) + ik_1 \cdot x\right]
\]

\[
= 1 + \sum_{m=1}^{\infty} (-H_+(k))^m \int_0^{\infty} dt_1 \ldots \int_0^{\infty} dt_m \\
\times \rho(x - \nabla \omega(k)t_1) \ldots \rho(x - \nabla \omega(k)(t_1 + \ldots + t_m))
\]

\[
= \exp \left[-H_+(k) \int_{-\infty}^{0} dt \rho(x + \nabla \omega(k)t)\right]. \tag{4.9}
\]

With this input we compute the single collision diagram with effective medium propagators,

\[
\begin{align*}
\begin{array}{c}
\downarrow \quad k' \\
& \quad \quad \quad \quad \quad k
\end{array}
\end{align*}
\quad \begin{align*}
\begin{array}{c}
k' \\
\downarrow \quad k
\end{array}
\end{align*}
\]

It is given by

\[
I_1(\lambda) = \int dk_1 dk_2 dg_1 dg_2 \lambda (2\pi)^{-3} \langle \hat{\mathcal{V}}(k_1 - k_2) \hat{\mathcal{V}}(g_1 - g_2)^* \rangle_V \\
\times F_+(k_1, k'; \lambda) F_-(k_2, k; \lambda) F_+(g_1, k'; \lambda)^* F_-(g_2, k; \lambda)^*
\]

\[
= (2\pi)^{-3} \lambda^{-3} \int dk_1 dk_2 dg_1 dg_2 (2\pi)^{-3/2} \int dx \rho(x) \hat{\theta}(k' - k + \lambda k_1 - \lambda k_2)
\]

\[
\times e^{-ik_1 \cdot x} \lambda^3 F_-(k' + \lambda k_1, k'; \lambda) \left( e^{-i\theta_1 x} \lambda^3 F_-(k' + \lambda g_1, k'; \lambda) \right)^*
\]

\[
\times e^{ik_2 \cdot x} \lambda^3 F_+(k + \lambda k_2, k; \lambda) \left( e^{-i\theta_2 x} \lambda^3 F_+(k + \lambda g_2, k; \lambda) \right)^*. \tag{4.10}
\]

Here \(F_+\) is obtained from \(F_-\) by substituting \(\hat{\rho}_\lambda(-k_j + k_{j+1})\) for \(\hat{\rho}_\lambda(k_j - k_{j+1})\) in (4.5). Inserting (4.9) one obtains

\[
I_1(\lambda) = (2\pi)^{-3} \lambda^{-2} \int dx \rho(x) (2\pi)^{-3/2} \hat{\theta}(k - k')
\]

\[
\exp \left[-(H_+(k) + H_-(k)) \int_{-\infty}^{0} dt \rho(x + \nabla \omega(k)t) \right.
\]

\[
- (H_+(k') + H_-(k')) \int_{0}^{\infty} dt \rho(x + \nabla \omega(k')t) \right] \left( 1 + \mathcal{O}(\lambda) \right). \tag{4.11}
\]
Using (4.3) yields indeed

$$\lim_{\lambda \to 0} 2\pi \delta(\omega(k) - \omega(k')) \lambda^2 (2\pi)^3 I_1(\lambda) = \langle f^-_k, L_1 f^+_k \rangle. \quad (4.12)$$

As next item we consider the ladder diagram with two collisions,

![Ladder Diagram](image)

The external legs carry only the free propagator. Their gates can be summed to the effective medium propagator as in the first part of this section. The top line has \( m + 2 \) vertices and internal momenta \( k_1, \ldots, k_{m+1}, \) \( m = 0, 1, 2, \ldots \) Correspondingly, the bottom line has \( n + 2 \) vertices and internal momenta \( g_1, \ldots, g_{n+1}, \) \( n = 0, 1, 2, \ldots \) The sum over all gates reads

$$I_2(\lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int dk_1 \ldots dk_{m+1} \int dg_1 \ldots dg_{n+1} \tau(k' - k_1) \tau(k_{m+1} - k) \times (2\pi)^{-6} \lambda \tilde{\rho}_\lambda(-k_1 + g_1) \lambda \tilde{\rho}_\lambda(-k_{m+1} + g_{n+1})^* \times \prod_{j=1}^{m} \left\{ \lambda(2\pi)^{-3/2} G_{E+}(k_j) iH_+(k_j) \tilde{\rho}_\lambda(k_j - k_{j+1}) \right\} \times \prod_{j=1}^{n} \left\{ \lambda(2\pi)^{-3/2} G_{E-}(g_j)(-i)H_- (g_j) \tilde{\rho}_\lambda(g_j - g_{j+1})^* \right\} G_{E+}(k_{m+1}) G_{E-}(g_{n+1}). \quad (4.13)$$

We shift \( k_j \) by \( k_{m+1} \) and \( g_j \) by \( g_{n+1} \) and then substitute \( k_{m+1} = u + \frac{1}{2} \lambda v \), \( g_{n+1} = u - \frac{1}{2} \lambda v \). Finally \( v \) is rescaled to \( \lambda v \), \( k_j \) to \( \lambda k_j \), and \( g_j \) to \( \lambda g_j \). Then (4.13) becomes

$$I_2(\lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int dk_1 \ldots dk_m \int dg_1 \ldots dg_n \int du \int dv \tau(k' - k_1 - u - \frac{1}{2} \lambda v) \times \tau(-k + u + \frac{1}{2} \lambda v)(2\pi)^{-6} \lambda^2 \lambda^{-3} \tilde{\rho}(-k_1 + g_1 - v) \tilde{\rho}(-v)^* \times \prod_{j=1}^{m} \left\{ \lambda(2\pi)^{-3/2} G_{E+}(u + \frac{1}{2} \lambda v + \lambda k_j) iH_+(u + \frac{1}{2} \lambda v + \lambda k_j) \tilde{\rho}(k_j - k_{j+1}) \right\} \times \prod_{j=1}^{n} \left\{ \lambda(2\pi)^{-3/2} G_{E-}(u - \frac{1}{2} \lambda v + \lambda g_j)(-i)H_- (u - \frac{1}{2} \lambda v + \lambda g_j) \tilde{\rho}(g_j - g_{j+1})^* \right\} \times G_{E+}(u + \frac{1}{2} \lambda v) G_{E-}(u - \frac{1}{2} \lambda v), \quad (4.14)$$

where in (4.14) we set \( k_{m+1} = 0 \), \( g_{n+1} = 0 \).
We insert the approximation (4.18) and switch to position space,

\[ I_2(\lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int dk_1 \ldots dk_m \int dg_1 \ldots dg_n \int du dv \hat{\theta}(k' - u) \hat{\theta}(u - k) \times \left( -H_+(u) \right)^m \left( -H_-(u) \right)^n \int_0^\infty dt_1 \ldots dt_m \int_0^\infty ds_1 \ldots ds_n \times \exp \left[ -i \sum_{j=1}^n t_j \left( \lambda^{-1}(\omega(u) - E - i\varepsilon) + \nabla \omega(u) \cdot k_j + \frac{1}{2} \nabla \omega(u) \cdot v \right) \right] + i \sum_{j=1}^n s_j \left( \lambda^{-1}(\omega(u) - E + i\varepsilon) + \nabla \omega(u) \cdot g_j - \frac{1}{2} \nabla \omega(u) \cdot v \right) \times \int dx_1 \ldots dx_m (2\pi)^{-3m} \rho(x_1) \ldots \rho(x_m) \int dy_1 \ldots dy_n (2\pi)^{-3n} \rho(y_1) \ldots \rho(y_n) \times (2\pi)^{-6} \lambda^2 \lambda^{-3} (2\pi)^{-3} \int dx dy \rho(x) \rho(y) \exp \left[ -i \sum_{j=1}^m x_j \cdot (k_j - k_{j+1}) \right] + i \sum_{j=1}^n y_j \cdot (g_j - g_{j+1}) - ix \cdot (-k_1 + g_1 - v) - iy \cdot v \times G_{E+}(u + \frac{1}{2} \lambda v) G_{E-}(u - \frac{1}{2} \lambda v) (1 + \mathcal{O}(\lambda)), \] (4.15)

where in (4.15) we set \( k_{m+1} = 0, \ g_{n+1} = 0 \). We integrate over \( k_1, \ldots, k_m, g_1, \ldots, g_n \) and substitute

\[ \sigma_j = \sum_{i=1}^j s_i, \quad \tau_j = \sum_{i=1}^j t_i. \] (4.16)

Then

\[ I_2(\lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int du dv (2\pi)^{-3} \hat{\theta}(k' - u) \hat{\theta}(u - k) \times \left( -H_+(u) \right)^m \left( -H_-(u) \right)^n \times \int_{0 \leq \tau_1 \leq \ldots \leq \tau_m} d\tau_1 \ldots d\tau_m \int_{0 \leq \sigma_1 \leq \ldots \leq \sigma_n} d\sigma_1 \ldots d\sigma_n (2\pi)^{-6} \lambda^2 \lambda^{-3} \times \int dx dy \rho(x) \rho(y) \prod_{j=1}^m \rho(x - \nabla \omega(u) \tau_j) \prod_{j=1}^n \rho(y - \nabla \omega(u) \sigma_j) \times \exp \left[ i(E - \omega(u)) \lambda^{-1}(\tau_m - \sigma_n) - i\frac{1}{2} \nabla \omega(u) \cdot v(\tau_m + \sigma_n) + i(x - y) \cdot v \right] \times G_{E+}(u + \frac{1}{2} \lambda v) G_{E-}(u - \frac{1}{2} \lambda v) (1 + \mathcal{O}(\lambda)). \] (4.17)

We still have to take the limit \( \lambda \to 0 \) for the two remaining Green’s functions.
Collecting all the \(u,v\) dependence into the smooth function \(f\), this leads to

\[
I_3(\lambda) = \lambda \int dudu \rho(u,v) \exp \left[ i(E - \omega(u))\lambda^{-1}(\tau_m - \sigma_n) - \frac{i}{2} \nabla \omega(u) \cdot v(\tau_m + \sigma_n) \right] \\
\times G_{E+(u + \frac{1}{2} \lambda v)} G_{E-(u - \frac{1}{2} \lambda v)} \\
= \lambda \int dudu \rho(u,v) \int_0^\infty dt \int_0^\infty ds \ 
\times \exp \left[ i(E - \omega(u))\lambda^{-1}(\tau_m - \sigma_n) - \frac{i}{2} \nabla \omega(u) \cdot v(\tau_m + \sigma_n) \right] \\
\times i(t-s)(E-\omega(u)) - \epsilon(t+s) - \frac{i}{2} \nabla \omega(u) \cdot v\lambda(t+s) \right] (1 + O(\lambda)) \\
= \lambda \int dudu \rho(u,v) \int_{\lambda^{-1}\tau_m}^{\lambda^{-1}\sigma_n} dt \int_{\lambda^{-1}\sigma_n}^{\lambda^{-1}\sigma_n} ds \exp \left[ i(E - \omega(u))(t-s) \right] \\
- \frac{1}{2} \nabla \omega(u) \cdot v(t+s) \right] (1 + O(\lambda)) .
\] (4.18)

We rotate \(s,t\) by \(\pi/4\) and rescale \(t+s\) by \(\lambda\). This yields

\[
\lim_{\lambda \to 0} I_3(\lambda) = \int dudu \rho(u,v) \int_{\max(\tau_m,\sigma_n)}^{\infty} dte^{-i\nabla \omega(u) \cdot v t} 2\pi \delta (E - \omega(u)) .
\] (4.19)

Inserting (4.19) in (4.17) one extends the \(t\)-integration from 0 to \(\infty\), while \(\tau_m \leq t\), \(\sigma_n \leq t\). Then

\[
I_2(\lambda) = (2\pi)^{-3} \lambda^{-2} \int_0^\infty dt \int dudu (2\pi)^{-3} \tilde{\theta}(k' - u) \tilde{\theta}(u - k)(2\pi)^{-3} \\
\times \int dxdy \rho(x)\rho(y) 2\pi \delta (E - \omega(u)) \exp \left[-i\nabla \omega(u) \cdot vt + i(x-y) \cdot v\right] \\
\times \exp \left[-H_+(u) \int_0^t ds \rho(x - \nabla \omega(u)s) - H_-(u) \int_0^t ds \rho(y - \nabla \omega(u)s) \right] \\
\times (1 + O(\lambda)) .
\] (4.20)

Let us set \(f^0_k(q,p) = \delta(p - k)\). Then (4.20) amounts to

\[
\lim_{\lambda \to 0} 2\pi \delta(\omega(k) - \omega(k')) \lambda^2 (2\pi)^3 I_2(\lambda) = \int_0^\infty dt \langle f^0_k, L_1 e^{L_0 t} L_1 f^0_{k'} \rangle .
\] (4.21)

Summing over the gates in the external legs amounts to replacing \(f^0_{k'}\) by \(f^+_k\) and \(f^0_k\) by \(f^-_k\). The ladder diagrams with increasing number of rungs generate the time-dependent perturbation theory for \(e^{Lt}\). Thus we have verified the limit (4.1).

5 Backscattering and maximally crossed diagrams

The first subleading contribution to \(\langle \sigma(k,k') \rangle_V\) comes from the maximally crossed diagrams. They are responsible for a narrow peak of width \(\lambda\) centered at \(k' = -k\)
as a correction to $\sigma_B(k, k')$. Recall that $\varphi_k(q) = (2\pi)^{-3/2} e^{ik\cdot q}$. We define

$$2\pi \delta(\omega(k) - \omega(k'))\sigma_B(k, k'; \kappa) = (2\pi)^3 \int_0^\infty dt \langle f_k^- \varphi_\kappa, L_1 e^{Lt} L_1 \varphi_\kappa f_k'^+ \rangle \quad (5.1)$$

and

$$\sigma_{\text{back}, k}(\kappa) = \sigma_B(k, -k; \kappa) . \quad (5.2)$$

Note that the first and last scattering event carries now an extra phase factor. If one chooses $\omega(k) = \omega(-k + \lambda \kappa)$, then the backscattering rate is given by

$$\lim_{\lambda \to 0} (2\pi)^3 \lambda^2 I^\text{max}(k, -k + \lambda \kappa; \lambda) = \sigma_{\text{back}, k}(\kappa) . \quad (5.3)$$

(5.3) is our main result.

To verify our claim does not require a long computation. Let $A^\text{max}$ be a particular maximally crossed diagram. If its internal momenta at the lower line are $g_1, \ldots, g_m$, then we substitute $-g_{m-j}$ for $g_m$. Then $A^\text{max}(k', k) = A^\text{lad}(k', k)$ where $A^\text{lad}(k', k)$ is the corresponding ladder diagram with external momenta $k'$ and $k$ for the upper line and $-k$ and $-k'$ for the lower line. In particular $A^\text{max}(-k, k) = A^\text{lad}(-k, k)$, which implies (1.3). I.e. for precise backscattering at small $\lambda$ the scattering rate is twice the one predicted by the Boltzmann equation, upon omitting the single scattering contribution. This result is independent of the shape function $\rho$.

For the fine structure we look $\lambda \kappa$ away from backscattering and have to sum over all ladder diagrams, as explained in Section 4, with the correspondingly modified external momenta.

Let $k_1, k_2$, resp. $g_1, g_2$ be the internal momenta of the last rung of the ladder diagram. Then the diagram beyond the last rung equals

$$I_r(k, -k + \lambda \kappa; \lambda) = \int dk_2 dg_2 \lambda \langle \tilde{V}(k_1 - k_2) \tilde{V}(g_1 - g_2)^* \rangle V_F^-(k_2, -k + \lambda \kappa; \lambda) F^-(g_2, k; \lambda)^* . \quad (5.4)$$

We shift $k_2$ to $k - \lambda \kappa + k_2$ and $g_2$ to $k + g_2$ and rescale,

$$I_r(k, -k + \lambda \kappa; \lambda) = \lambda \lambda^{-3} \int dk_2 dg_2 \lambda \langle k_1 - k - \lambda k_2 + \lambda \kappa \rangle \rho(\lambda^{-1}(k_1 - g_1) - k_2 + g_2 + \kappa) \times \lambda^3 F^-(k - \lambda \kappa + \lambda k_2, k - \lambda \kappa; \lambda) \lambda^3 F^-(k + \lambda g_2, k; \lambda)^* . \quad (5.5)$$

Thus, compared to the ladder diagram, in position space the diagram picks up the phase factor $e^{-i\kappa \cdot x}$. Repeating the argument for the leftmost rung one concludes the validity of (5.3).

### 6 Backscattering in the diffusive approximation

Let us consider the conventional slab geometry for which $\rho(r) = 1$ if $r_3 > 0$ and $\rho(r) = 0$ otherwise, $r = (r_1, r_2, r_3)$, and, for the sake of illustration, choose $k =$
(0, 0, k_3), k_3 > 0. The scattering rate is proportional to the cross sectional area, through which we divide by imposing that the first scattering is at r = (0, 0, r_3) with r_3 > 0. Even with these simplifications the inverse of L, as needed in (5.1), is not so easily computed. Therefore we approximate the motion between the first and last scattering through a Brownian motion, i.e. L (ω) through D(ω) ∆ r, where D(ω) = D(ω(k)) is the diffusion coefficient obtained from the transport equation (1.1). According to this equation the particle simply leaves the scattering region upon hitting its boundary. Thus ∆ r is taken with Dirichlet boundary conditions at \{r_3 = 0\}. Setting G_D = (∆ r)^{-1} one has

\[ G_D(r; r') = (4\pi| r - r' |)^{-1} - (4\pi| r - \tilde{r}' |)^{-1}, \quad r_3, r'_3 > 0, \quad (6.1) \]

where \( \tilde{r} = (r_1, r_2, -r_3) \). The first and last scattering are approximately a mean free path away from the boundary \{r_3 = 0\}. Therefore the diffusion approximation is taken between the points \( r = (0, 0, \ell^*) \) and \( r' = (r_\parallel, \ell^*) \) with the mean free path \( \ell^* = |\nabla_k \omega(k)|/\nu(k) \).

The backscattering in the diffusive approximation is thus given by

\[ \sigma_{\text{back}, D}(k + \lambda \kappa) = \left( \nu(\omega)^2 / D(\omega) \right) \int_{\mathbb{R}^2} dr_\parallel G_D(0, 0, \ell^*; r_\parallel, \ell^*) \left( 1 + \cos(\kappa \cdot r_\parallel) \right) \quad (6.2) \]

with \( \nu(\omega) = \nu(\omega(k)) \) and \( \kappa = (\kappa_1, \kappa_2, 0) \). In the round bracket to the right the “1” accounts for the incoherent scattering while the “cos” expresses the coherent backscattering. Working out the integral one obtains

\[ \sigma_{\text{back}, D}(k + \lambda \kappa) = \left( \nu(\omega)^2 / D(\omega) \right) \ell^* \left( 1 + (2\ell^*|\kappa|)^{-1} (1 - \exp[-2\ell^*|\kappa|]) \right). \quad (6.3) \]

We refer to [6] for a confirmation of (6.3) in a light scattering experiment.

7 Light scattering

As an illustration we consider the wave equation with a random index of refraction. Other wave equations can be handled in a similar fashion. The wave field \( \phi : \mathbb{R}^3 \rightarrow \mathbb{R} \) is governed by

\[ \frac{\partial^2}{\partial t^2} \phi = c(x)^2 \Delta \phi \quad (7.1) \]

and we set

\[ c(x) = 1 + \sqrt{\lambda} V(x) \quad (7.2) \]

with \( V \) from (2.1). The Gaussian statistics looks unphysical because \( c(x) \) will have regions where it is negative. However for a, say, bounded random potential only the Gaussian part persists for small \( \lambda \) [3, 11].

The general strategy is to rewrite (7.1) in the form of a Schrödinger equation and then to apply the results from before. Such a procedure is not unique and we adopt the one employed in [11], see also [12].
The dispersion relation for (7.1) at $\lambda = 0$ is
\[ \omega(k) = |k|. \] (7.3)
Again, more general dispersions can be handled by our method. If we consider $\omega$ as multiplication in Fourier space, then the corresponding operator in position space is denoted by $\Omega$. We introduce the two-component “wave function”
\[ \psi = (\psi^+, \psi^-), \] (7.4)
where
\[ \psi^\pm(x) = \frac{1}{\sqrt{2}} \left( \Omega \phi(x) \pm i (1 + \sqrt{\lambda} V(x))^{-1} \dot{\phi}(x) \right). \] (7.5)
Then (7.1) is rewritten as
\[ i \frac{\partial}{\partial t} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} + \sqrt{\lambda} \begin{pmatrix} V\Omega + \Omega V & V\Omega - \Omega V \\ -V\Omega + \Omega V & -V\Omega - \Omega V \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \] (7.6)
Note that $\|\psi^+\|^2 = \|\psi^-\|^2$ is the energy of the wave field, hence conserved in time. We regard (7.6) as an evolution equation in $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. The physical solutions are then the subspace defined by (7.5). The right-hand side of (7.6) defines a self-adjoint operator, thus the solution conserves the norm.

The only still missing input are the plane wave solution in case $\lambda = 0$. They are of the form $\phi_k(x) = (2\pi)^{-3/2} \omega(k) \exp[i(x \cdot k - \omega(k)t)]$, where the prefactor is chosen such that the $\phi_k$’s are orthogonal in the energy norm. Thus
\[ \psi^+(x) = (2\pi)^{-3/2} e^{ik \cdot x}, \quad \psi^-(x) = 0. \] (7.7)
We have
\[ (V\Omega \psi)(k) = (2\pi)^{-3/2} \int dk_1 \hat{V}(k - k_1) \omega(k_1) \hat{\psi}(k_1), \]
\[ (\Omega V \psi)(k) = (2\pi)^{-3/2} \omega(k) \int dk_1 \hat{V}(k - k_1) \hat{\psi}(k_1). \] (7.8)
In the perturbative expansion, compared to the Schrödinger case, a vertex carries an extra factor of $\omega$. This amounts to an extra factor of $\omega^2$ in the collision rate. Thus the transport equation for wave propagation has the backward generator
\[ L_w f(q, p) = \nabla \omega(p) \cdot \nabla_q f(q, p) + \rho(q) \omega(p)^2 \int dp' 2\pi \delta(\omega(p) - \omega(p')) \times (2\pi)^{-3/2} \hat{\theta}(p - p') (f(q, p') - f(q, p)), \] (7.9)
compare with (5.2).

For the coherent backscattering of light one simply replaces in (5.1) $L$ by $L_w$. In other words the collision rate $\nu(p)$ is replaced by $\omega(p)^2 \nu(p)$ and the propagation is $\nabla \omega(k) = k/|k|$ rather than $k$. In the diffusion approximation such fine details are no longer visible. Provided $\ell^*$ is taken as the mean free path for light, the shape of the coherent backscattering peak remains unaltered.
References

[1] T. Chen, Convergence in higher mean of a random Schrödinger to a linear Boltzmann evolution, Comm. Math. Phys. 267, 355–392 (2006).

[2] J.L. Langer and T. Neal, Breakdown of the concentration expansion for the impurity resistivity of metals, Phys. Rev. Lett. 16, 984–986 (1966).

[3] L. Erdős, M. Salmhofer, and H.-T. Yau, Quantum diffusion of the random Schrödinger evolution in the scaling limit, arXiv:math-ph/0512014.

[4] L. Erdős, M. Salmhofer, and H.-T. Yau, Quantum diffusion of the random Schrödinger evolution in the scaling limit II. The recollision diagrams, Comm. Math. Phys. 271, 1–53 (2007).

[5] E. Wolf and G. Maret, Weak localization and coherent backscattering of photons in disordered media, Phys. Rev. Lett. 55, 2696–2699 (1985).

[6] E. Akkermans, P.E. Wolf, and R. Maynard, Coherent backscattering of light by disordered media: analysis of the peak line shape, Phys. Rev. Lett. 56, 1471–1474 (1986).

[7] R. Berkovits and S. Feng, Correlations in coherent multiple scattering, Physics Reports 238, 135–172 (1994).

[8] P. Sheng, Introduction to Wave Scattering, Localization, and Mesoscopic Phenomena. Academic Press, San Diego, 1995.

[9] E. Akkermans, P.E. Wolf, R. Maynard, and G. Maret, Theoretical study of the coherent backscattering of light by disordered media, J. Phys. France 48, 77–98 (1988).

[10] M. Reed and B. Simon, Methods of Modern Mathematical Physics III: Scattering Theory. Academic Press, New York, 1979.

[11] J. Lukkarinen and H. Spohn, Kinetic limit for wave propagation in a random medium, Arch. Rat. Anal. 183, 93–162 (2007).

[12] H. Spohn, The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics, J. Stat. Phys. 124, 1041–1104 (2006).