Barankin Vector Locally Best Unbiased Estimates*

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Abstract

The Barankin bound is generalized to the vector case in the mean square error sense. Necessary and sufficient conditions are obtained to achieve the lower bound. To obtain the result, a simple finite dimensional real vector valued generalization of the Riesz representation theorem for Hilbert spaces is given. The bound has the form of a linear matrix inequality where the covariances of any unbiased estimator, if these exist, are lower bounded by matrices depending only on the parametrized probability distributions.

Keywords: Parameter estimation, unbiased estimation, optimal estimator, Barankin bound, performance bounds, linear matrix inequalities, minimal covariance matrix, Cramer-Rao bound.

1 Introduction

The problem considered, following Barankin, [2], and results in Banach, [1], is the optimal unbiased estimation of a deterministic vector of parameters $\nu$ of a family of probability measures $\mathcal{P}_\nu$, or more generally a known real vector function of these parameters $g(\nu)$, using a realization of a vector random variable $X$ drawn from $\mathcal{P}_\nu$. The first issue is to find a function $\psi$ such that

$$\int \psi(X) \, d\mathcal{P}_\nu = g(\nu),$$

for all $\nu$ in some admissibe set. This problem is a vector integral equation and may or may not have a solution, [6, 20]. Furthermore, even if it has solution, it may not have a solution with finite covariance matrix for $\nu_T$.

Barankin, under very simple hypothesis, [2], gives an if and only if condition for the existence of a minimal $s$-th variance unbiased estimator for the scalar case, which is tighter than the classical Cramer-Rao or Bhattacharyya bounds if they exist. In recent years the Barankin bound has attracted attention, since there are problems for which the Cramer-Rao or Bhattacharyya bounds give no satisfactory solution, see e.g. [22], and references there. Following [2], the problem studied here is under what conditions there exists a finite covariance

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vector unbiased estimator of the true vector parameter $\nu_T$, and in that case if a minimal covariance vector unbiased estimator exists.

In Section 2 an overview is presented of the relevant results of measure theory and the Lebesgue integral related to the Barankin formulation. In Section 3 the vector Barankin bound generalization is presented as a linear matrix inequality (LMI). In Section 4 the Barankin functional analysis formalization is generalized to handle the vector case. In Section 5 a finite dimensional real vector valued generalization of the Riesz representation theorem for Hilbert spaces is presented. In Section 6 necessary and sufficient conditions are given for the existence of an optimal vector estimator attaining the bound given by the LMI obtained in Section 3. In Section 7 other alternative LMI formulations for the existence of an optimal vector estimator are given.

2 Formalization of the vector estimation problem

2.1 Measure theoretic setup

Let $(\Omega, \mathcal{F})$ be a measurable space, where $\Omega$ is a well defined abstract set, and $\mathcal{F}$ is a sigma-algebra of subsets of $\Omega$. Let $\Theta$ be an abstract arbitrary set of sub-indexes with no conditions on its structure as in [2], p. 477. Let $\mathcal{B}_\theta$ be a collection of probability measures $\mathcal{P}_\theta$ for the measurable space $(\Omega, \mathcal{F})$, indexed by the sub-indexes $\theta \in \Theta$, i.e. $\mathcal{B} = \{\mathcal{P}_\theta : \theta \in \Theta\}$, as in [2] p. 477. Hence for each $\theta \in \Theta$, the triple $(\Omega, \mathcal{F}, \mathcal{P}_\theta)$ is a probability space. Let $\mathcal{X}$ be a vector random variable, i.e. a measurable function from the measurable space $(\Omega, \mathcal{F})$ to the measurable space $(\mathbb{R}^d, \mathcal{B}_d)$, where $\mathbb{R}^d$ is the vector $d$-dimensional real space, and $\mathcal{B}_d$ is the Borel sigma-algebra for $\mathbb{R}^d$, that is the minimal sigma-algebra generated, e.g., by the open sets of $\mathbb{R}^d$. Then $\mathcal{X}$ is a real vector random variable iff $\forall B \in \mathcal{B}_d$ we have $\mathcal{X}^{-1}(B) \in \mathcal{F}$, if and only if each component of the vector is a real random variable. Define for each $\theta \in \Theta$ the measure $\mathcal{P}_\theta$, for the measurable space $(\mathbb{R}^d, \mathcal{B}_d)$, induced by the random variable $\mathcal{X}$, i.e. for each $B \in \mathcal{B}_d$ define $\mathcal{P}_\theta(B) = \mathcal{P}_\theta(\mathcal{X}^{-1}(B))$. Hence for each $\theta \in \Theta$ the random variable $\mathcal{X}$ induces the probability space $(\mathbb{R}^d, \mathcal{B}_d, \mathcal{P}_\theta)$.

Let $\psi$ be a real measurable vector function from $(\mathbb{R}^d, \mathcal{B}_d)$ to $(\mathbb{R}^p, \mathcal{B}_p)$, that is $\psi : \mathbb{R}^d \to \mathbb{R}^p$, and for each $B \in \mathcal{B}_p$ we have $\psi^{-1}(B) \in \mathcal{B}_d$. For the measurable vector function $\psi$ from $(\mathbb{R}^d, \mathcal{B}_d)$ to $(\mathbb{R}^p, \mathcal{B}_p)$, define the i-th component of the vector $\psi$ as $[\psi]_i$, which is a measurable function from $(\mathbb{R}^d, \mathcal{B}_d)$ to $(\mathbb{R}, \mathcal{B})$ for each $1 \leq i \leq d_p$, equivalently, $[\psi]_i \in L_1(\mathbb{R}^d, \mathcal{B}_d, \mathcal{P}_\theta)$, for all $1 \leq i \leq d_p$, so that $\mathcal{L}_1(\mathbb{R}^d, \mathcal{B}_d, \mathcal{P}_\theta) = (L_1(\mathbb{R}^d, \mathcal{B}_d, \mathcal{P}_\theta))^{d_p}$.

Hence $\psi(\mathcal{X})$ is a random variable from $(\Omega, \mathcal{F})$ to $(\mathbb{R}^p, \mathcal{B}_p)$, since for
$B \in \mathcal{B}_{d_p}$ we have $[\psi(X)]^{-1}(B) = X^{-1}(\psi^{-1}(B))$, but $B \in \mathcal{B}_{d_p}$ so that $\psi^{-1}(B) \in \mathcal{B}_{d_p}$ and then $X^{-1}(\psi^{-1}(B)) \in \mathcal{F}$.

Define the integral of a vector of functions as a vector whose elements are the integrals of each function. Then, [10] p. 45:

$$\int \psi(X) \, d\mathcal{P}_\theta = \int \psi \, d\mathcal{P}_\theta$$

Note that the integral on the left is with respect to the probability space $(\Omega, \mathcal{F}, \mathcal{P}_\theta)$, while the integral on the right is with respect to the probability space $(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$. We will refer indistinctly to $\psi(X)$ and $\psi$ as an estimator, with the understanding that they refer to different probability spaces linked by the previous equality of integrals.

For $\psi \in \mathcal{L}_1(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$ define the expectation of $\psi$ as $E_\theta[\psi] = \int \psi(X) \, d\mathcal{P}_\theta = \int \psi \, d\mathcal{P}_\theta$.

We assume that the random variable $X$ is drawn from some specific probability measure (p. m.) $\mathcal{P}_\theta$, with $\theta_T \in \Theta$, i.e. we will use the realization of this random variable to obtain the estimator for $g(\theta_T)$. The random vector $\psi(X)$ is an unbiased estimator for $g(\theta)$, $\forall \theta \in \Theta$, $g : \Theta \rightarrow \mathbb{R}^{d_p}$, if the integral $\int \psi(X) \, d\mathcal{P}_\theta$ is well defined, $\forall \theta \in \Theta$, and we have $\int \psi(X) \, d\mathcal{P}_\theta = \int \psi \, d\mathcal{P}_\theta = g(\theta)$, $\forall \theta \in \Theta$.

Then the first issue posed in the introduction may be formally stated as:

**Problem 2.1 (Basic Problem).** Given a function $g : \theta \rightarrow \mathbb{R}^{d_p}$, defined for each $\theta \in \Theta$, and a family of p.m.'s indexed by $\theta \in \Theta$, find an unbiased estimator, i.e. find a function $\psi \in \mathcal{L}_1(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$, $\forall \theta \in \Theta$, such that $\int \psi(X) \, d\mathcal{P}_\theta = \int \psi \, d\mathcal{P}_\theta = g(\theta)$, for all $\theta \in \Theta$.

Define the integral of a matrix $\Psi$ of dimensions $N \times M$, $N, M \in \mathbb{N}$, whose elements belong to $L_1(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$, as a matrix whose elements are the integrals of the elements of $\Psi$, so that $E_\theta[\Psi(X)] = \int \Psi(X) \, d\mathcal{P}_\theta = \int \Psi \, d\mathcal{P}_\theta$. For a measurable square integrable function $f : (\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta) \rightarrow (\mathbb{R}, \mathcal{F})$, i.e. $f \in L_2(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$, define $L_2(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$ as the collection of all the measurable functions $u$ from $(\mathbb{R}^{d_x}, \mathcal{B}_{d_x})$ to $(\mathbb{R}, \mathcal{F})$, such that $\int |u|^2 \, d\mathcal{P}_\theta < +\infty$, for $1 \leq i \leq d_p$, equivalently, $|u| \in L_2(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$, for all $1 \leq i \leq d_p$. If the non-centered second order moments of the components of the estimator $\psi$ exist for $\theta_T$, i.e. $\psi| \in L_2(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$, for $1 \leq i \leq d_p$, so that $\psi \in L_2(\mathbb{R}^{d_x}, \mathcal{B}_{d_x}, \mathcal{P}_\theta)$, then, the first order moments of the components of the estimator exist for $\theta_T$. Also, the correlations $\int \psi| \psi_j \, d\mathcal{P}_\theta$, are well defined and are finite for all $i \neq j$, $1 \leq i, j \leq d_p$, and using the Cauchy-Schwarz inequality, we obtain $\int |\psi| |\psi_j| \, d\mathcal{P}_\theta \leq \int \|\psi| \|^2_{L_2} \|\psi_j \|^2_{L_2}$. Additionally assume $\psi(X)$ is unbiased $\forall \theta \in \Theta$, then the covariance matrix of $\psi(X)$ exists for $\theta_T$, and we have $\text{Cov}_{\theta_T}(\psi) = \int [(\psi(X) - g(\theta_T)) (\psi(X) - g(\theta_T))^T] \, d\mathcal{P}_\theta = \mathbb{E}_{\theta_T}[\psi - g(\theta_T)] (\psi - g(\theta_T))^T = \mathbb{E}_{\theta_T}[\psi \psi^T] - g(\theta_T) g(\theta_T)^T$. 
In the same direction of [2], with \( s = r = 2 \), instead of the general Problem 2.1 we pose the problem in terms of estimators with finite covariance matrix at \( \theta_T \):

**Problem 2.2** (Finite Covariance Problem). Given a function \( g : \theta \rightarrow \mathbb{R}^d \), defined for each \( \theta \in \Theta \), and a family of p.m.’s indexed by \( \theta \in \Theta \), find a function \( \psi \in \mathcal{L}_2(\mathbb{R}^d, \mathcal{B}_{d^2}, \mathbb{P}_\theta) \), with \( \psi \in \mathcal{L}_1(\mathbb{R}^d, \mathcal{B}_{d^2}, \mathbb{P}_\theta) \), \( \forall \theta \in \Theta \), such that \( \int \psi \, d\mathbb{P}_\theta = g(\theta) \), for all \( \theta \in \Theta \). If there are several solutions find, if possible, a solution with minimal covariance matrix at \( \theta_T \).

### 2.2 Centered definitions

Define \( \varphi = \psi - g(\theta_T) \), and \( h(\theta) = g(\theta) - g(\theta_T) \) so that \( h(\theta_T) = 0 \). If \( \psi \) is unbiased, then, since \( \int \psi \, d\mathbb{P}_\theta = g(\theta) \) and \( \int g(\theta_T) \, d\mathbb{P}_\theta = g(\theta_T) \), for all \( \theta \in \Theta \), then, \( \int (\psi - g(\theta_T)) \, d\mathbb{P}_\theta = g(\theta) - g(\theta_T) \), for all \( \theta \in \Theta \), so that \( \int \varphi \, d\mathbb{P}_\theta = h(\theta) \), \( \forall \theta \in \Theta \), and \( \text{Cov}_{\theta_T}(\psi) = \mathbb{E}_{\theta_T}[\varphi \varphi^T] \). Also, if \( \psi \) is unbiased, then, since \( h(\theta_T) = 0 \), then \( \mathbb{E}_{\theta_T}[\varphi] = 0 \).

### 2.3 Barankin formulation: basic hypothesis

Following Barankin we will introduce some simple additional hypothesis resumed in Barankin’s Postulate in [2] p. 481.

**Hypothesis 2.1.** The set \( \Theta \) is an arbitrary index set with no conditions on its structure, [2] p. 477, and \( \mathcal{B} \) is a collection of probability measures \( \mathcal{P}_\theta \) for the measurable space \((\Omega, \mathcal{F})\), i.e. \( \mathcal{B} = \{ \mathcal{P}_\theta : \theta \in \Theta \} \) as in [2], p. 477. The random variable \( \mathcal{X} : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^d, \mathcal{B}_{d^2}) \) is drawn from the probability measure (p. m.) \( \mathcal{P}_{\theta_T} \), with \( \theta_T \in \Theta \). Assume that for each \( \theta \in \Theta \) the p.m. \( \mathcal{P}_\theta \) is absolutely continuous with respect to \( \mathcal{P}_{\theta_T} \), i.e. \( \mathcal{P}_\theta << \mathcal{P}_{\theta_T} \), with \( \theta_T \in \Theta \).

**Lemma 2.1.** If Hypothesis 2.1 is true then for each \( \theta \in \Theta \) the p.m. \( \mathcal{P}_\theta \) is absolutely continuous with respect to \( \mathcal{P}_{\theta_T} \), i.e. \( \mathcal{P}_\theta << \mathcal{P}_{\theta_T} \).

**Proof.** Assume \( B \in \mathcal{B}_{d^2} \) is such that \( \mathcal{P}_{\theta_T}(B) = 0 \), then since \( \mathcal{P}_\theta(B) = \mathcal{P}_{\theta_T}(\mathcal{X}^{-1}(B)) \), we obtain \( \mathcal{P}_\theta(\mathcal{X}^{-1}(B)) = 0 \). But \( \mathcal{P}_\theta << \mathcal{P}_{\theta_T} \), hence \( \mathcal{P}_\theta(\mathcal{X}^{-1}(B)) = 0 \). Since \( \mathcal{P}_\theta(B) = \mathcal{P}_{\theta_T}(\mathcal{X}^{-1}(B)) \), then \( \mathcal{P}_\theta(B) = 0 \).

**Observation 2.1.** In the case in which every index \( \theta \in \Theta \) is a possible candidate for \( \theta_T \), then, Hypothesis 2.1 should require that for each \( \theta_1 \in \Theta \) the p.m. \( \mathcal{P}_{\theta_1} \) should be absolutely continuous with respect to each other p.m. \( \mathcal{P}_{\theta_2} \) with \( \theta_2 \in \Theta \), and then \( \mathcal{P}_{\theta_1} << \mathcal{P}_{\theta_2} \) for all \( \theta_1, \theta_2 \in \Theta \).

As a consequence of the previous hypothesis and lemma, the Radon-Nykodim derivatives \( d\mathcal{P}_\theta/d\mathcal{P}_{\theta_T} \) and \( d\mathcal{P}_\theta/d\mathcal{P}_{\theta_T} \) exist for all \( \theta \in \Theta \), [15] p. 315.

**Definition 2.1.** Define \( \pi(\theta) = d\mathcal{P}_\theta/d\mathcal{P}_{\theta_T} \), with \( \pi(\theta) \equiv \pi_\theta(x, \theta_T), x \in \mathbb{R}^{d^2} \), so that \( d\mathcal{P}_\theta/d\mathcal{P}_{\theta_T} = \pi_\theta(\mathcal{X}, \theta_T) \). Define \( \mathfrak{B}_0 = \{ \pi(\theta) : \theta \in \Theta \} \), see [2], p. 481.

We have \( \pi(\theta) \geq 0 \) w.p. 1, for all \( \theta \in \Theta \), [15] p. 315, \( \pi(\theta_T) = 1 \) w.p. 1, and \( \int \pi(\mathcal{X}, \theta_T)d\mathcal{P}_{\theta_T} = \int (d\mathcal{P}_\theta/d\mathcal{P}_{\theta_T})d\mathcal{P}_{\theta_T} = \int d\mathcal{P}_\theta = 1 \), for all \( \theta \in \Theta \).
Hypothesis 2.2. 1. Assume that for each \( \theta \) there is one and only one \( \pi(\theta) \in \mathfrak{B}_0 \), i.e. the correspondence \( \pi : \Theta \to \mathfrak{B}_0 \) is one-to-one.

2. There are at least two values \( \theta_1, \theta_2 \in \Theta \), such that \( g(\theta_1) \neq g(\theta_2) \).

Observation 2.2. Item 1 avoids the identifiability problem. \[12\], pp. 58 and 191. Item 2 implies that we do not consider estimators which are constant with probability 1: if it was \( \psi = \alpha_0 \) w.p. 1 for some \( \alpha_0 \in \mathbb{R}^d \), then \( \int \psi(\mathbf{x}) \, d\mathcal{P}_{\theta_1} = \int \alpha_0 \, d\mathcal{P}_{\theta_1} = \alpha_0 \), similarly \( \int \psi(\mathbf{x}) \, d\mathcal{P}_{\theta_2} = \alpha_0 \), but since we assume that \( \psi \) is unbiased, it should be \( \int \psi(\mathbf{x}) \, d\mathcal{P}_{\theta_1} = g(\theta_1) \) and \( \int \psi(\mathbf{x}) \, d\mathcal{P}_{\theta_2} = g(\theta_2) \), and then it should be \( g(\theta_1) = g(\theta_2) \), which is a contradiction. Additionally, Hypothesis 2.2 implies that there exists at least a \( \theta_0 \in \Theta \) such that \( g(\theta_0) \neq 0 \). Nonetheless, see e.g. \[2\] p. 482 and \[7\] p. 2440, for some comments regarding constant estimators.

2.4 Barankin postulate

The following hypothesis is Barankin’s Postulate in \[2\], p. 481, for \( s = r = 2 \).

Hypothesis 2.3 (Barankin, \[2\], Postulate p. 481). Assume that

\[ \pi(\theta) \in L_2(\mathbb{R}^{ds}, \mathcal{B}_{ds}, \mathbb{P}_{\theta_T}) \quad \forall \theta \in \Theta \]

equivalently \( \mathfrak{B}_0 \subseteq L_2(\mathbb{R}^{ds}, \mathcal{B}_{ds}, \mathbb{P}_{\theta_T}) \).

Observation 2.3. Since \( \int \pi(\theta) \, d\mathbb{P}_{\theta_T} = \int (d\mathbb{P}_{\theta_T} / d\mathbb{P}_{\theta_T}) \, d\mathbb{P}_{\theta_T} = \int d\mathbb{P}_{\theta} = 1 \), for all \( \theta \in \Theta \), then \( \| \pi(\theta) \|_{L_2} \neq 0 \), for all \( \theta \in \Theta \), equivalently \( \| u \|_{L_2} \neq 0 \), for all \( u \in \mathfrak{B}_0 \).

If not, \( \| \pi(\theta) \|_{L_2} = 0 \) implies \( \pi(\theta) = 0 \) w.p. 1, and then \( \int \pi(\theta) \, d\mathbb{P}_{\theta_T} = 0 \), contradiction. Additionally note, taking in account Hypothesis 2.2, that \( \mathfrak{B}_0 \) has at least two elements.

Suppose \( \psi \in L_2(\mathbb{R}^{ds}, \mathcal{B}_{ds}, \mathbb{P}_{\theta_T}) \), since \( \pi(\theta) \in L_2(\mathbb{R}^{ds}, \mathcal{B}_{ds}, \mathbb{P}_{\theta_T}) \) for all \( \theta \in \Theta \), then the integrals \( \int \psi \pi(\theta) \, d\mathbb{P}_{\theta_T} \) are well defined for all \( \theta \in \Theta \), and we have all the equivalent forms:

\[
\int \psi \pi(\theta) \, d\mathbb{P}_{\theta_T} = \int \psi \, d\mathbb{P}_{\theta_T} \int \pi(\theta) = \int \psi \, d\mathbb{P}_{\theta_T} \int \pi(\theta) \, d\mathcal{P}_{\theta_T} = \mathbb{E}_{\theta_T} [\psi \pi(\theta)]
\]

If \( \psi \in L_2(\mathbb{R}^{ds}, \mathcal{B}_{ds}, \mathbb{P}_{\theta_T}) \) is unbiased, then for \( \varphi = \psi - g(\theta_T) \) we have

\[
\mathbb{E}_{\theta_T}[\varphi \pi(\theta)] = \int \varphi \pi(\theta) \, d\mathbb{P}_{\theta_T} = h(\theta) \quad \forall \theta \in \Theta
\]  \hspace{1cm} (2.1)

The introduction of the functions \( \pi \) reduces the consideration of the multiple probability spaces \( L_2(\mathbb{R}^{ds}, \mathcal{B}_{ds}, \mathbb{P}_{\theta}), \forall \theta \in \Theta \), to a single probability space \( L_2(\mathbb{R}^{ds}, \mathcal{B}_{ds}, \mathbb{P}_{\theta_T}) \).
2.5 Probability density function form

Call $\lambda$ the Lebesgue measure for the measurable space $(\mathbb{R}^d, \mathcal{B}_d)$, i.e. the measure that assigns to parallelepipeds in $\mathbb{R}^d$ the value given by the product of the lengths of the edges of the parallelepiped in each direction. Alternatively call $d\lambda = dx$, with $x \in \mathbb{R}^d$. If in turn we have $P_\theta << \lambda$, i.e. the p.m. $P_\theta$ is absolutely continuous with respect to the Lebesgue measure, then, $P_\theta << P_{\theta_T} << \lambda$, so that $P_\theta << \lambda$, and then the Radon-Nykodim derivatives $dP_\theta/d\lambda$ exist, for all $\theta \in \Theta$. These derivatives are the probability density functions (pdf) $p_\theta = dP_\theta/d\lambda$ with $x \in \mathbb{R}^d$. Since, [15] p. 328,

$\int \psi dP_\theta = \int \psi p_\theta d\lambda = \int \psi \pi(\theta) p_{\theta_T} d\lambda$

then, if $\psi$ is unbiased

$g(\theta) = \int \psi dP_\theta = \int \psi p_\theta d\lambda = \int \psi \pi(\theta) p_{\theta_T} d\lambda$

2.6 The Main Problem

With all the previous considerations we may formalize the generalization to the vector case of the Barankin formulation as:

**Problem 2.3 (Main Problem).** Given a function $g : \theta \to \mathbb{R}^d$, defined for each $\theta \in \Theta$, and a family of p.m.‘s indexed by $\theta \in \Theta$, that satisfy the Hypothesis 2.1, 2.2, and 2.3 find a function $\psi \in L^2(\mathbb{R}^d, \mathcal{B}_d, P_{\theta_T})$, such that $\int \psi \pi(\theta) dP_{\theta_T} = g(\theta)$, $\forall \theta \in \Theta$. If there are several solutions find, if possible, a solution with minimal covariance matrix at $\theta_T$.

The solution to this problem is given below in Theorem 7.3.

3 Matrix bound

For a vector $a$ in a finite vector space denote $[a]_i$ the i-th component of the vector. For a matrix $A$, define $[A]_{i,j}$ as the i-th column of the matrix and $[A]_{i,j}$ as i-th, j-th element of the matrix. We have $[A]_{i,j} = \left([A]_{j,i}\right)^T$. Denote $A^T$ the transpose of the matrix $A$, $\det(A)$ the determinant of $A$, and $\text{Tr} [A]$ the trace of $A$. A square symmetric real matrix $A \in \mathbb{R}^{N \times N}$ is a symmetric non-negative definite (s.n.n.d.) matrix iff, $x^T Ax \geq 0$, for all $x \in \mathbb{R}^N$. A real s.n.n.d. matrix $A \in \mathbb{R}^{N \times N}$ is a symmetric positive definite (s.p.d.) matrix if $\det(A) > 0$, iff $x^T Ax > 0$, for all $x \neq 0$. Two s.n.n.d. matrices $A \in \mathbb{R}^{N \times N}$ and $B \in \mathbb{R}^{N \times N}$ are comparable in the Löwner partial order, [24] p. 166, if either $A - B$ is s.n.n.d. and then $A \geq B$, or if $B - A$ is s.n.n.d. and then $B \geq A$, else, they are not comparable. For $A, B$ and $C$ s.n.n.d of dimensions $N \times N$, then if $A \geq B$ and $B \geq C$ then $A \geq C$, and if $A \geq B$ and $B \geq A$, then $A = B$, see e.g. [7] Lemma 3 p. 2444. If $A \in \mathbb{R}^{N \times N}$ is s.p.d., and $S \in \mathbb{R}^{N \times M}$ is arbitrary, such
that $S^{T}AS = 0$, then $S = 0$, see e.g. [7] Lemma 2 p. 2444. For $A \in \mathbb{R}^{N \times N}$, denote the Frobenius norm as $\|A\|_{F} = (\text{Tr}(AA^{T}))^{1/2}$.

The following lemma is a variant of the information inequality [25] p. 172, [13] Lemma 1 p. 1288, [19] pp. 326–328.

**Lemma 3.1.** Let $(X, X, \mu)$ be an arbitrary measure space. Let $L_{2}(X, X, \mu)$ be the collection of all the measurable square integrable real valued functions from $X$ to $\mathbb{R}$. Let $d_{\gamma}, d_{\rho}, d_{A} \in \mathbb{N}$, $\gamma \in (L_{2}(X, X, \mu))^{d_{\gamma}}$, $\rho \in (L_{2}(X, X, \mu))^{d_{\rho}}$, and $A \in \mathbb{R}^{d_{A} \times d_{\rho}}$. Call $F = \int \gamma \rho^{T} d\mu$, $F \in \mathbb{R}^{d_{\gamma} \times d_{\rho}}$, and $B = \int \rho \rho^{T} d\mu$, $B \in \mathbb{R}^{d_{\rho} \times d_{\rho}}$. If $\det (AB A^{T}) \neq 0$, then $\int \gamma \gamma^{T} d\mu \geq F A^{T} (AB A^{T})^{-1} A F^{T}$, with equality if and only if there exists a matrix $\Lambda_{0} \in \mathbb{R}^{d_{\gamma} \times d_{A}}$ such that $\gamma = \Lambda_{0} A \rho \mu$-almost-everywhere ($\mu$-ae), and in that case, it is $\Lambda_{0} = F A^{T} (AB A^{T})^{-1}$.

**Proof.** For each $\Lambda \in \mathbb{R}^{d_{\gamma} \times d_{A}}$, let $M(\Lambda) = \int (\gamma - \Lambda A \rho)(\gamma - \Lambda A \rho)^{T} d\mu$, $M(\Lambda) \in \mathbb{R}^{d_{\gamma} \times d_{\gamma}}$. Then $M(\Lambda)$ is s.n.n.d. for all $\Lambda \in \mathbb{R}^{d_{\gamma} \times d_{A}}$. We have $M(\Lambda) = \int \gamma \gamma^{T} d\mu - F A^{T} \Lambda^{T} - \Lambda A F^{T} + \Lambda A B A^{T} \Lambda^{T}$. By assumption $\det (AB A^{T}) \neq 0$, so that the matrix $A B A^{T}$ is invertible. Define $\Lambda_{0} = F A^{T} (AB A^{T})^{-1}$; then, $M(\Lambda_{0}) = \int \gamma \gamma^{T} d\mu - F A^{T} (AB A^{T})^{-1} A F^{T} \geq 0$ so that $\int \gamma \gamma^{T} d\mu \geq F A^{T} (AB A^{T})^{-1} A F^{T}$, and there is equality iff $M(\Lambda_{0}) = 0$. From the definition of $M(\Lambda)$, if there exists $\Lambda^{*} \in \mathbb{R}^{d_{\gamma} \times d_{A}}$ such that $\gamma = \Lambda^{*} A \rho \mu$-ae, then $M(\Lambda^{*}) = 0$. In that case it will be $\int \gamma \rho^{T} d\mu = \Lambda^{*} A \int \rho \rho^{T} d\mu$, so that $F = \Lambda^{*} A B$, and then $F A^{T} = \Lambda^{*} A B A^{T}$. Since by hypothesis $A B A^{T}$ is invertible, then $\Lambda^{*} = F A^{T} (AB A^{T})^{-1} = \Lambda_{0}$ so that $M(\Lambda_{0}) = M(\Lambda^{*}) = 0$, and then we obtain the equality. Conversely, if $\int \gamma \gamma^{T} d\mu = F A^{T} (AB A^{T})^{-1} A F^{T}$, take $\Lambda_{0} \in \mathbb{R}^{d_{\gamma} \times d_{A}}$, as $\Lambda_{0} = F A^{T} (AB A^{T})^{-1}$, so that by the definition of $M(\Lambda)$ it results $M(\Lambda_{0}) = \int (\gamma - \Lambda_{0} A \rho)(\gamma - \Lambda_{0} A \rho)^{T} d\mu = \int \gamma \gamma^{T} d\mu - F A^{T} (AB A^{T})^{-1} A F^{T} = 0$. Hence $\text{Tr} (M(\Lambda_{0})) = \text{Tr} \left( \int (\gamma - \Lambda_{0} A \rho)(\gamma - \Lambda_{0} A \rho)^{T} d\mu \right) = \sum_{i=1}^{d_{\gamma}} \int (\gamma - \Lambda_{0} A \rho)_{i}^{2} d\mu = 0$, and then $\gamma = \Lambda_{0} A \rho \mu$-ae. \hfill \Box

The following definition specifies all the elements required in the proposed linear matrix inequality (LMI) generalized Barankin bound.

**Definition 3.1.** Given arbitrary $d_{M} \in \mathbb{N}$ and $d_{A} \in \mathbb{N}$, an arbitrary real matrix $A$ of dimensions $d_{A} \times d_{M}$, $A \in \mathbb{R}^{d_{A} \times d_{M}}$, and arbitrary indexes $\theta_{i} \in \Theta$, for $1 \leq i \leq d_{M}$, define $\tau^{T} = (\theta_{1}, \theta_{2}, \ldots, \theta_{d_{M}})$, $\tau \in \Theta^{d_{M}}$, and define the quad-tuple $\mathbf{q}$ as $\mathbf{q} = (d_{M}, d_{A}, A, \tau)$. Define $h(\theta) = g(\theta) - g(\theta_{T})$.

Define $\beta^{T}(\tau) = (\pi(\theta_{1}), \pi(\theta_{2}), \ldots, \pi(\theta_{d_{M}}))$, i.e. $\beta(\tau) \in \mathbb{R}^{d_{M}}$, define the $d_{P} \times d_{M}$ real matrix $G(\tau)$ as $G(\tau) = \left( g(\theta_{1}) - g(\theta_{T}), g(\theta_{2}) - g(\theta_{T}), \ldots, g(\theta_{d_{M}}) - g(\theta_{T}) \right) = \left( h(\theta_{1}), h(\theta_{2}), \ldots, h(\theta_{d_{M}}) \right)$, and define the $d_{M} \times d_{M}$ real matrix $B(\tau)$ as $B(\tau) = \mathbb{E} \left[ \beta(\tau) \beta^{T}(\tau) \right]$. 
Define $\mathcal{C}_A$ as the collection of all the quad-tuples $q$ with $\det(A B(\tau) A^T) \neq 0$, i.e.

$$\mathcal{C}_A = \{q : \forall d_M \in \mathbb{N}, \forall d_A \in \mathbb{N}, \forall A \in \mathbb{R}^{d_A \times d_M}, \forall \tau \in \Theta^{d_M}, \text{ with } \det(A B(\tau) A^T) \neq 0\}$$

**Definition 3.2.** Call $\mathcal{W}_A$ the family of all the finite covariance at $\theta_T$ unbiased estimators of $g(\theta)$, for all $\theta \in \Theta$, for Problem 2.3. Define $\mathcal{W}_A$ as the collection of matrices of the form:

$$W(q) = G(\tau) A^T (A B(\tau) A^T)^{-1} A G^T(\tau) \quad \forall q \in \mathcal{C}_A$$

i.e. $\forall d_M \in \mathbb{N}, \forall d_A \in \mathbb{N}, \forall A \in \mathbb{R}^{d_A \times d_M}, \forall \tau \in \Theta^{d_M}, \text{ with } \det(A B(\tau) A^T) \neq 0$, with $G(\tau)$ and $B(\tau)$ as in Definition 3.1. Hence $\mathcal{W}_A = \{W(q) : q \in \mathcal{C}_A\}$. The matrices $W(q)$ will be called the Barankin covariance lower bound matrices for Problem 2.3.

Let $S(\mathcal{B}_0)$ be the linear span of $\mathcal{B}_0$, i.e. $S(\mathcal{B}_0) = \{u \in L_2(\mathbb{R}^{d_M}, \mathbb{R}_{d_M}, \mathbb{P}) : u = \sum_{i=1}^{d_M} q_i \pi_i \ \text{ w.p.} \ 1, \ \forall d_M \in \mathbb{N}, \forall a_i \in \mathbb{R} \text{ for } 1 \leq i \leq d_M, \ \forall \pi_i \in \mathcal{B}_0 \text{ for } 1 \leq i \leq d_M\}$.

The following theorem gives the first half of the Barankin vector bound.

**Theorem 3.1.** If for Problem 2.3 there exists a finite covariance at $\theta_T$ unbiased estimator $\psi(\mathcal{X}) \in \mathcal{W}_g$ for $g(\theta)$, for all $\theta \in \Theta$, then, see Definition 3.1

$$\text{Cov}_{\theta_T}(\psi) \geq G(\tau) A^T (A B(\tau) A^T)^{-1} A G^T(\tau) \quad \forall q \in \mathcal{C}_A$$

(3.1)

i.e. (3.1) is true for the set of conditions $\mathcal{C}_A$: $\forall d_M \in \mathbb{N}, \forall d_A \in \mathbb{N}, \forall A \in \mathbb{R}^{d_A \times d_M}, \forall \tau \in \Theta^{d_M}, \text{ with } \det(A B(\tau) A^T) \neq 0$. There is equality in (3.1) for some $\psi^* \in \mathcal{W}_g$ and some $q^* = (d_M, d_A, A^*, \tau^*)$, $q^* \in \mathcal{C}_A$, if and only if there exists a matrix $\Lambda^* \in \mathbb{R}^{d_M \times d_A}$ such that $\psi^* = \psi^* - g(\theta_T) = \Lambda^* A^* \beta(\tau^*)$ w.p. 1, if and only if each component $[\phi^*]_i$ is a linear combination of elements in $\mathcal{B}_0$ w.p. 1 for $1 \leq i \leq d_p$, i.e. $\phi^* = \psi^* - g(\theta_T) \in (S(\mathcal{B}_0))^{d_p}$, see Definition 3.2.

**Proof.** The proof will follow from Lemma 3.1. Let $\psi \in \mathcal{W}_g$ be an arbitrary finite covariance at $\theta_T$ unbiased estimator for Problem 2.3. Take an arbitrary $d_M \in \mathbb{N}$, and an arbitrary $\tau \in \Theta^{d_M}$, see Definition 3.1. Since $\psi$ is unbiased, see (2.1),

$$E_{\theta_T}[\phi^T(\tau)] = \left(E_{\theta_T}[\phi \pi(\Theta_1)], \ldots, E_{\theta_T}[\phi \pi(\Theta_{d_M})]\right)$$

$$= \left(\int \phi \pi(\Theta_1) \ d\mathbb{P}_{\theta_T}, \ldots, \int \phi \pi(\Theta_{d_M}) \ d\mathbb{P}_{\theta_T}\right)$$

$$= \left(\int \phi \ d\mathbb{P}_{\theta_1}, \int \phi \ d\mathbb{P}_{\theta_2}, \ldots, \int \phi \ d\mathbb{P}_{\theta_{d_M}}\right)$$

$$= (h(\theta_1), h(\theta_2), \ldots, h(\theta_{d_M})) = G(\tau)$$

Define $\mathcal{C}_A$ as the collection of all the quad-tuples $q$ with $\det(A B(\tau) A^T) \neq 0$, i.e.
then, \( G(\tau) = \mathbb{E}_{\theta_T}[\varphi \beta^T(\tau)] = \mathbb{E}_{\theta_T}[(\psi - g(\theta_T)) \beta^T(\tau)] \), see Definition 3.1, and this is true for any unbiased estimator \( \psi \in \mathcal{A} \). Additionally, we have, \( \int A \beta(\tau) \beta^T(\tau) \, d\mathbb{P}_0 = A B(\tau) A^T \), see Definition 3.1. Take an arbitrary \( d_A \in \mathbb{N} \) and a matrix \( A \in \mathbb{R}^{d_A \times d_A} \) such that \( \det(A B(\tau) A^T) \neq 0 \) otherwise arbitrary. Then the result follows from Lemma 3.1 with \( \gamma = \varphi, \rho = \beta(\tau), F = G(\tau), \) and \( B = B(\tau) \). The first if and only if equality condition follows directly from Lemma 3.1. As for the second equality condition, if there is equality in (3.1) for some \( \psi^* \in \mathcal{A} \) and some \( \varphi^* = (d_M, d_A, A^*, \tau^*) \in \mathcal{C}_A \), then from Lemma 3.1, there exists \( \lambda^* \in \mathbb{R}^{d_A \times d_A} \) such that \( \varphi^* = \psi^* - g(\theta_T) = \Lambda^* A^* \beta(\tau^*) \) w.p. 1.

Since \( \beta(\tau^*) \in \mathcal{B}_0^M \), then each component \( [\varphi^*]_i \) is a linear combination w.p. 1 of elements in \( \mathcal{B}_0 \), for \( 1 \leq i \leq d_p \), i.e. \( \varphi^* = \psi^* - g(\theta_T) \in (S(\mathcal{B}_0))^{d_p} \). Conversely, suppose that \( \psi^* \in \mathcal{A} \), with \( \varphi^* = \psi^* - g(\theta_T) \), and that each component \( [\varphi^*] \), is a linear combination w.p. 1 of elements in \( \mathcal{B}_0 \), i.e. \( \varphi^* = \psi^* - g(\theta_T) \in (S(\mathcal{B}_0))^{d_p} \). Since each \( [\varphi^*] \), \( \in S(\mathcal{B}_0) \) w.p. 1, then, there exist \( M_i \in \mathbb{N} \), \( a_i \in \mathbb{R}^{M_i} \), and \( \tau_i \in \Theta_{M_i} \), such that \( [\varphi^*]_i = a_i \beta(\tau_i) \) w.p. 1 for \( 1 \leq i \leq d_p \). Define \( M_0 = \sum_{i=1}^{d_p} M_i \), and \( \tau^T = (\tau_1^T, \tau_2^T, \ldots, \tau_{d_p}^T) \), \( \tau_i \in \Theta_{M_i} \). Call \( \beta_0 = \beta(\tau^T) \), \( \beta_0 \in \mathcal{B}_0^M \). Define the real matrix \( A_0 \in \mathbb{R}^{d_p \times M_0} \), as the block-diagonal matrix \( A_0 = \text{Diag}(a_1^T, a_2^T, \ldots, a_{d_p}^T) \), where each block \( a_i^T \) is of dimension \( 1 \times M_i \), for \( 1 \leq i \leq M_0 \), so that \( \varphi^* = A_0 \beta_0 \). Starting with the second component of \( \beta_0 \), see Observation 2.3, delete the i-th component if it is a linear combination w.p. 1 of the previous components. There will remain \( M_i \in \mathbb{N} \) elements, with \( 1 \leq M_i \leq M_0 \), see Observation 2.3.

Call \( \tau^T \in \Theta_{M_0} \), the components of \( \beta_0 \), are linearly independent w.p. 1. Then, there exists a real matrix \( A_0 \in \mathbb{R}^{M_0 \times M_0} \), such that \( \beta_0 = A_0 \beta_0 \). If \( \beta_0 \neq 0 \), such that \( A_0 \beta_0 = 0 \), but \( \varphi^* = A_0 \beta_0 \). Define the quad-tuple \( q_0 = (M_i, M_0, \tau_i, \gamma) \), where \( \tau_i \) is the identity matrix of dimensions \( M_i \times M_i \). Call \( B_0 = B(\tau^T) = \mathbb{E}_{\theta_T}[\beta_0^T] \), \( B_0 \in \mathbb{R}^{M \times M} \), so that \( \det(B_0) \neq 0 \). If not, there would exist \( a \in \mathbb{R}^{M_0} \), with \( a \neq 0 \), such that \( a^T B_0 a = 0 \), but \( a^T B_0 a = a^T \mathbb{E}_{\theta_T}[\beta_0^T] a = a^T \mathbb{E}_{\theta_T}[-\beta_0^T] a = a^T \mathbb{E}_T[\alpha^T \beta_0^T] a = a^T \mathbb{E}_T[\alpha^T a^T] \), and then it would be \( \mathbb{E}_T[a^T a] \), which is a contradiction since the components of \( \beta_0 \), are linearly independent w.p. 1. Hence, \( \det(I_0 B_0 I_0^T) = \det(B_0) \neq 0 \), so that \( q_0 \in \mathcal{C}_A \). Since \( G(\tau^T) = \mathbb{E}_{\theta_T}[\varphi^* \beta_0^T] = \mathbb{E}_{\theta_T}[-\beta_0^T] = A_0 \mathbb{E}_{\theta_T}[\beta_0^T] = A_0 \mathbb{E}_{\theta_T}[\beta_0^T] = A_0 \mathbb{E}_{\theta_T}[\beta_0^T] = A_0 \mathbb{E}_{\theta_T}[\beta_0^T] \), then \( W(q_0) = G(\tau^T) B_0^{-1} G(\tau^T) \).

But \( \text{Cov}_{\theta_T}(\psi^*) = \mathbb{E}_{\theta_T}[(\psi^*)^T] = \mathbb{E}_{\theta_T}[\alpha_a \alpha_a^T] = A_0 \mathbb{E}_{\theta_T}[\beta_0^T \beta_0^T] \).

The converse of this theorem, is given in Theorem 7.3, see Section 7.3.

Observation 3.1. The previous proof shows that if \( \psi \in \mathcal{A} \) is a finite covariance unbiased estimator for Problem 2.3, then \( G(\tau) = (h(\theta_1), h(\theta_2), \ldots, h(\theta_{d_M})) = \)
Hence, equivalently the value of \( L_{\theta} (\varphi \beta^T (\tau)) \) is independent of the estimator \( \psi \in \mathcal{H}_\beta \) as a consequence of the unbiasedness of \( \psi \), see (2.1).

**Observation 3.2.** Theorem 3.1 shows that any other finite covariance at \( \theta_T \) unbiased estimator will satisfy (3.1). Then, the covariance matrix of any unbiased estimator in \( \mathcal{H}_\beta \) is comparable, in the Löwner partial order, with any of the matrices in \( \mathcal{W}_A \). Hence:

\[
\text{Cov}_{\theta_T} (\psi) \geq W \quad \forall \psi \in \mathcal{H}_\beta \text{ and } \forall W \in \mathcal{W}_A
\]

with equality if and only if \( \varphi = \psi - g(\theta_T) \in (S(\mathcal{B}_0))_{d_P} \). The covariance matrices of estimators in \( \mathcal{H}_\beta \) need not be comparable between them, as well as, Barankin bound matrices in \( \mathcal{W}_A \) need not be comparable between them.

4 **Functional analysis setup**

4.1 **Definition of the operator** \( L_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathbb{R}^{d_P} \)

From Hypothesis 2.3, we have \( \mathcal{B}_0 \subseteq L_2(\mathbb{R}^{d_\mathcal{B}}, \mathcal{B}_d, \mathbb{P}_{\theta_T}) \). The subset \( \mathcal{B}_0 \) is not a linear subspace, since any \( \pi \in \mathcal{B}_0 \), is a Radon-Nykodim derivative of a p.m. with respect to the p.m. \( \mathbb{P}_{\theta_T} \), then \( \pi \geq 0 \) w.p. 1, [15], p. 315, with \( \| \pi \|_{L_2} \neq 0 \), see Observation 2.3, so that \( -\pi \) cannot belong to \( \mathcal{B}_0 \).

Let \( u_0 \) be an arbitrary element in \( \mathcal{B}_0 \). To this particular element \( u_0 \in \mathcal{B}_0 \) corresponds a unique \( \theta_0 \in \Theta \), such that \( u_0 \equiv \pi(\theta_0) \), see Hypothesis 2.2, so that \( \theta_0 = \pi^{-1}(u_0) \), and, to this index \( \theta_0 \) corresponds a unique well defined value \( h(\theta_0) = g(\theta_0) - g(\theta_T) \in \mathbb{R}^{d_P} \). Hence, to \( u_0 \in \mathcal{B}_0 \) corresponds a unique element \( h(\pi^{-1}(u_0)) \in \mathbb{R}^{d_P} \) which we define as \( L_{\mathcal{B}_0}(u_0) \), so that \( L_{\mathcal{B}_0}(u_0) = h(\pi^{-1}(u_0)) \).

Hence,

\[
L_{\mathcal{B}_0}(\pi(\theta)) = h(\theta) \quad \forall \theta \in \Theta \quad (4.1)
\]

equivalently \( L_{\mathcal{B}_0}(u) = h(\pi^{-1}(u)) \), for all \( u \in \mathcal{B}_0 \). Note that \( L_{\mathcal{B}_0}(\pi(\theta_T)) = h(\theta_T) = 0 \). Then, we may establish a direct relation from \( \mathcal{B}_0 \) to \( \mathbb{R}^{d_P} \), as an operator \( L_{\mathcal{B}_0} \) from \( \mathcal{B}_0 \) to \( \mathbb{R}^{d_P} \), i.e. \( L_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathbb{R}^{d_P} \). This operator is not (without additional conditions) necessarily linear nor bounded. The operator \( L_{\mathcal{B}_0} \) is completely defined by the collection of Radon-Nykodim derivatives in \( \mathcal{B}_0 \), i.e. the elements \( \pi(\theta) \in \mathcal{B}_0 \), for all \( \theta \in \Theta \), and the vectors \( g(\theta) \in \mathbb{R}^{d_P} \), for all \( \theta \in \Theta \), and does not depend on the existence or not of any unbiased estimator, and if it exists, on whether it has finite covariance at \( \theta_T \) or not.

4.2 **Barankin formulation**

The key observation made by Barankin, [2], for \( d_P = 1 \), where he considers \( \mathcal{B}_0 \subseteq L_r(\mathbb{R}^{d_\mathcal{B}}, \mathcal{B}_d, \mathbb{P}_{\theta_T}) \), \( r \geq 1 \), is that if we are able to find an integral representation of the operator \( L_{\mathcal{B}_0} \), then the problem is solved.

In Barankin, [2], the answer is given by the Riesz Representation Theorem which finds an element in the conjugate space \( \phi_0 \in L_*(\mathbb{R}^{d_\mathcal{B}}, \mathcal{B}_d, \mathbb{P}_{\theta_T}) \), with
1/s + 1/r = 1, such that $L_{\mathcal{B}_0}(u) = \int \phi_0 u \, d\mathbf{P}_{\theta_T}$, $\forall u \in \mathcal{B}_0$, with minimum s-norm, i.e. minimum s-th variance. In our case, we generalize to vector estimates, i.e. $d_P > 1$, but we will only consider the case $s = r = 2$ which is the traditional variance and covariance matrices case, which is the most important in applications. To solve the problem the idea is to generalize the Riesz representation theorem to the vector case. The Riesz representation theorem requires that the represented functional be defined from a linear space to the reals. Since $\mathcal{B}_0$ is not a linear subspace, Barankin, see [1], pp. 479-480, extends the operator $L_{\mathcal{B}_0}$ to a linear operator over the whole space, using indirectly the Hahn-Banach theorem, invoking a condition first used by Riesz and generalized by Helly as exposed in [1] footnote in p. 56, see also [18]. In the next sub-section we generalize the Helly-Riesz-Banach condition to handle the vector case. In Section 5 we generalize the Riesz representation theorem to the vector case without requiring the Hahn-Banach theorem, and in Section 6 we apply these results to solve Problem 2.3.

4.3 Vector generalized Barankin hypothesis: Helly, Riesz, Banach, Barankin (HRBB)

The following is the generalization of the hypothesis in [2], pp. 480 and 483–484, see also [1]. Theorems 4 and 5 pp. 55–57. This condition will be called here the HRBB condition for Helly, Riesz, Banach, Barankin. For $u \in L_2(\mathbb{R}^d, \mathcal{P}_{\theta_T})$, define the semi-norm $\|u\|_{L_2} = (\int u^2 \, d\mathbf{P}_{\theta_T})^{1/2}$, and call $\|x\|_{\mathbb{R}^d}$ the standard Euclidean norm for $x \in \mathbb{R}^d$.

**Definition 4.1.** (HRBB condition) The functions $h(\theta) = g(\theta) - g(\theta_T)$, $h(\theta) \in \mathbb{R}^d$, and $\pi(\theta) \in \mathcal{B}_0$, $\forall \theta \in \Theta$, satisfying the Hypothesis 2.1, 2.2, and 2.3, for Problem 2.3 satisfy the HRBB condition iff: $\exists K_H \in \mathbb{R}^+$, i.e. $K_H \geq 0$, such that:

$$\left\| \sum_{i=1}^{d_M} a_i h(\theta_i) \right\|_{\mathbb{R}^d} \leq K_H \left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L_2}$$

(4.2)

for all $d_M \in \mathbb{N}$, for all $a_i \in \mathbb{R}$, $i = 1, 2, \cdots, d_M$, for all $\theta_i \in \Theta$, $i = 1, 2, \cdots, d_M$.

5 Generalized Riesz representation theorem

Here a generalization is given of the Riesz Representation Theorem for Hilbert spaces real functionals, see e.g. [3] p. 112, to operators from an arbitrary Hilbert space $\mathcal{H}$, separable or not, to the real finite dimensional vector space $\mathbb{R}^d$, with $d_P \geq 1$. The proof given here does not require the Hahn-Banach extension theorem, and then, the non-denumerable Axiom of Choice is not required, or some less stringent variant, [18]. The bound proposed in Helly’s theorem, [1] pp. 55–56, is generalized, and will be called the operator OP-HRBB (Helly, Riesz, Banach, Barankin) condition.
5.1 The Theorem.

Let $\mathcal{H}$ denote an arbitrary Hilbert space with semi-inner product $\langle u, v \rangle_\mathcal{H}$, $\forall u, v \in \mathcal{H}$, and semi-norm $\|u\|_\mathcal{H} = (\langle u, u \rangle_\mathcal{H})^{1/2}$. If $\|u\|_\mathcal{H} = 0$, then we say that $u = 0$ in semi-norm, (i.s.n.). Equivalently $u = 0$ i.s.n. iff $\|u\|_\mathcal{H} = 0$. Define $u = v$ i.s.n., iff $\|u - v\|_\mathcal{H} = 0$.

Theorem 5.1. Let $\mathcal{H}$ be an arbitrary Hilbert space. Let $\mathcal{B}_0$ be a non-empty arbitrary subset of $\mathcal{H}$, $\mathcal{B}_0 \neq \emptyset$, $\mathcal{B}_0 \subseteq \mathcal{H}$. Let $L_{\mathcal{B}_0}$ be an operator from $\mathcal{B}_0$ to $\mathbb{R}^d$, $L_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathbb{R}^d$, such that there exists at least one $u_0 \in \mathcal{B}_0$ for which $L_{\mathcal{B}_0}(u_0) \neq 0$. Assume that the operator $L_{\mathcal{B}_0}$ satisfies the following condition, that will be called the operator HRBB condition (OP-HRBB): $\exists K_H \in \mathbb{R}^+$, i.e. $K_H \geq 0$, such that:

$$\| \sum_{i=1}^{d_M} a_i L_{\mathcal{B}_0}(u_i) \|_{\mathbb{R}^d} \leq K_H \| \sum_{i=1}^{d_M} a_i u_i \|_\mathcal{H}$$

(5.1)

for all $d_M \in \mathbb{N}$, for all $a_i \in \mathbb{R}$, $i = 1, 2, \cdots, d_M$, for all $u_i \in \mathcal{B}_0$, $i = 1, 2, \cdots, d_M$.

Call $C(\mathcal{B}_0) \subseteq \mathcal{H}$ the minimal closed linear space containing $\mathcal{B}_0$, i.e. the closed linear span of $\mathcal{B}_0$, \cite{1} p. 11. Then:

1. The operator $L_{\mathcal{B}_0}$ may be extended to a bounded linear operator $L_C$ from $C(\mathcal{B}_0)$ to $\mathbb{R}^d$, $L_C : C(\mathcal{B}_0) \to \mathbb{R}^d$, with $L_C(u) = L_{\mathcal{B}_0}(u)$, for all $u \in \mathcal{B}_0$.

2. The operator $L_C$ has the following representation: There exists $d_L \in \mathbb{N}$, $1 \leq d_L \leq d_P$, and there exist orthonormal $\hat{u}_i$’s, $\hat{u}_i \in C(\mathcal{B}_0)$, for $1 \leq i \leq d_L$, such that:

$$L_C(u) = \sum_{i=1}^{d_L} \langle u, \hat{u}_i \rangle_\mathcal{H} L_C(\hat{u}_i) \quad \forall u \in C(\mathcal{B}_0)$$

(5.2)

Observation 5.1. The standard Riesz representation theorem, corresponds to $\mathcal{B}_0 = S(\mathcal{B}_0) = C(\mathcal{B}_0) = \mathcal{H}$, and $d_P = 1$. In that case the operator $L_{\mathcal{B}_0}$ is taken as a bounded linear operator, so that the OP-HRBB condition is satisfied, and then the conclusion is given by \cite{2} with $d_L = 1$.

5.2 Proof of the generalized Riesz representation theorem

5.2.1 Extension of the operator $L_{\mathcal{B}_0}$ to the span of $\mathcal{B}_0$, $L_S : S(\mathcal{B}_0) \to \mathbb{R}^d$

This extension follows the exposition of Banach in \cite{1} pp. 55–56. Assume the OP-HRBB condition is true. Call $S(\mathcal{B}_0)$ the linear span i.s.n. of $\mathcal{B}_0$, i.e. $S(\mathcal{B}_0) = \{ u \in \mathcal{H} : u = \sum_{i=1}^{d_M} a_i \pi_i \text{ i.s.n.}, \forall d_M \in \mathbb{N}, \forall a_i \in \mathbb{R}, \text{ for } 1 \leq i \leq d_M, \forall \pi_i \in \mathcal{B}_0, \text{ for } 1 \leq i \leq d_M \}$. The span $S(\mathcal{B}_0)$ is called $|\mathcal{B}_0|$ in \cite{2} p. 495. Clearly, $S(\mathcal{B}_0)$ is a linear space. With the help of the OP-HRBB condition
extend the operator \( L_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathbb{R}^{d_p} \) to an operator \( L_S : S(\mathcal{B}_0) \to \mathbb{R}^{d_p} \) by the following procedure: for each \( u \in S(\mathcal{B}_0) \) there exist, dependent on each \( u, d_M \in \mathbb{N}, a_i \)'s \( \in \mathbb{R}, 1 \leq i \leq d_M, \pi_i \)'s \( \in \mathcal{B}_0, 1 \leq i \leq d_M \), such that \( u = \sum_{i=1}^{d_M} a_i \pi_i \) i.n. Define \( L_S(u) \) for \( u \in S(\mathcal{B}_0) \) as \( L_S(u) = \sum_{i=1}^{d_M} a_i L_{\mathcal{B}_0}(\pi_i) \). This procedure gives a well defined value for \( L_S(u) \), since for any other decomposition of \( u = \sum_{j=1}^{d_M} a_j' \pi_j' \) i.n., resulting in \( L_S(u) = \sum_{j=1}^{d_M} a_j' L_{\mathcal{B}_0}(\pi_j') \), because of the OP-HRBB condition we will have:

\[
\| L_S(u) - L'_S(u) \|_{\mathbb{R}^{d_p}} \leq K_H \left( \sum_{i=1}^{d_M} a_i \pi_i - \sum_{j=1}^{d_M} a_j' \pi_j' \right) \|_{\mathcal{F}} = 0
\]

so that \( \sum_{i=1}^{d_M} a_i L_{\mathcal{B}_0}(\pi_i) = \sum_{j=1}^{d_M} a_j' L_{\mathcal{B}_0}(\pi_j') \). The important result here is that now \( S(\mathcal{B}_0) \), unlike \( \mathcal{B}_0 \), is a linear space, and that \( L_S : S(\mathcal{B}_0) \to \mathbb{R}^{d_p} \) is a bounded linear operator with bound \( K_H \), i.e. \( \| L_S(u) \|_{\mathbb{R}^{d_p}} \leq K_H \| u \|_{\mathcal{F}}, \forall u \in S(\mathcal{B}_0) \), and \( L_S(u) = L_{\mathcal{B}_0}(u), \forall u \in \mathcal{B}_0 \).

**Observation 5.2.** Barankin, [2] pp. 480 and 483–484, following [1] Theorems 2 and 4, p. 55, invokes the Hahn–Banach theorem, see e.g. [11] p. 78 or [1] pp. 78 or [1] p. 27, to extend the operator \( L_S \) to the whole space. The Hahn–Banach theorem requires the Axiom of Choice or some slightly less stringent condition, see e.g. [13]. In [1] arbitrary Banach spaces are considered. The fact that here we work with Hilbert spaces, permits us to avoid the use of the Hahn–Banach theorem, and then, the non-denumerable Axiom of Choice is not required.

### 5.2.2 Extension of the operator \( L_S \) to the closure of the span of \( \mathcal{B}_0 \),

\( L_C : C(\mathcal{B}_0) \to \mathbb{R}^{d_p} \)

Define the closure of the span of \( \mathcal{B}_0 \) as \( C(\mathcal{B}_0) = \text{Closure}(S(\mathcal{B}_0)) \), i.e. \( C(\mathcal{B}_0) = \{ u \in \mathcal{F} : \exists (u_n)_{n \in \mathbb{N}} \text{ with } u_n \in S(\mathcal{B}_0) \text{ } \forall n \in \mathbb{N}, \text{ such that } \| u_n - u \|_{\mathcal{F}} \to 0 \} \). It is readily checked that \( C(\mathcal{B}_0) \) is a closed linear subspace of \( \mathcal{F} \). The set \( C(\mathcal{B}_0) \) is called \( \{ \mathcal{B}_0 \} \) in [2], p. 494. Extend the operator \( L_S : S(\mathcal{B}_0) \to \mathbb{R}^{d_p} \) to an operator \( L_C : C(\mathcal{B}_0) \to \mathbb{R}^{d_p} \) by continuity: Let \( u \in C(\mathcal{B}_0) \), then there exists a sequence \( (u_n)_{n \in \mathbb{N}} \) of elements \( u_n \in S(\mathcal{B}_0) \) such that \( \| u_n - u \|_{\mathcal{F}} \to 0 \). Hence this sequence is a Cauchy fundamental sequence, i.e. for each \( \epsilon > 0 \) there exists \( N(\epsilon) \) such that \( \forall \ n, m \geq N(\epsilon) \) we have \( \| u_n - u_m \|_{\mathcal{F}} < \epsilon \). But since \( L_S \) is a bounded linear operator, then \( \| L_S(u_n) - L_S(u_m) \|_{\mathbb{R}^{d_p}} \leq K_H \| u_n - u_m \|_{\mathcal{F}} < K_H \epsilon \). Then, \( (L_S(u_n))_{n \in \mathbb{N}} \) is a Cauchy fundamental sequence in the complete finite dimensional vector space \( \mathbb{R}^{d_p} \). Hence, there exists a limit in \( \mathbb{R}^{d_p} \). Call that limit \( L_C(u) \), so that \( \| L_S(u_n) - L_C(u) \|_{\mathbb{R}^{d_p}} \to 0 \), and then \( L_S(u_n) - L_C(u) \to 0 \) component by component (c.b.c.), i.e. \( L_S(u_n))_i - [L_C(u)]_i \to 0 \), for \( 1 \leq i \leq d_p \). The value \( L_C(u) \) is well defined: assume that for some other sequence \( (u'_j)_{j \in \mathbb{N}} \) of elements \( u'_j \in S(\mathcal{B}_0) \) with \( \| u'_j - u \|_{\mathcal{F}} \to 0 \), we obtain using the previous procedure a limit \( L'_C(u) \) for the sequence \( (L_S(u'_j))_{j \in \mathbb{N}} \), i.e. \( \| L_S(u'_j) - L_C(u) \|_{\mathbb{R}^{d_p}} \to 0 \). We have: \( \| L_S(u'_j) - L_S(u_n) \|_{\mathbb{R}^{d_p}} = \| L_S(u'_j - u_n) \|_{\mathbb{R}^{d_p}} \leq K_H \| u'_j - u_n \|_{\mathcal{F}} = K_H \| u'_j - u - (u_n - u) \|_{\mathcal{F}} \to 0 \), as \( n \to \infty \).
\[ K_H (\| u'_j - u \| \mathcal{H} + \| u_n - u \| \mathcal{H}) \]. Then, \( \| L_C(u) - L_C(u) \|_{\mathbb{R}^d} = \| L_C(u) - L_S(u_n) - (L_C(u) - L_S(u'_j)) - (L_S(u'_j) - L_S(u_n)) \|_{\mathbb{R}^d} \leq \| L_C(u) - L_S(u_n) \|_{\mathbb{R}^d} + \| L_C(u) - L_S(u'_j) \|_{\mathbb{R}^d} + K_H \left( \| u'_j - u \| \mathcal{H} + \| u_n - u \| \mathcal{H} \right), \]

so that taking the limits \( n \to \infty \), and \( j \to \infty \), we obtain \( L_C(u) = L_C(u) \), so that the value \( L_C(u) \in \mathbb{R}^d \) is independent of the chosen sequence. Hence \( L_C(u) \) is a well defined operator from the closed linear subspace \( C(\mathcal{B}_0) \subseteq \mathcal{H} \) to \( \mathbb{R}^d \). It is immediate to show that this operator is linear and that \( L_C(u) = L_S(u), \forall u \in S(\mathcal{B}_0), \) and then \( L_C(u) = L_S(u) = L_{\mathcal{B}_0}(u), \forall u \in \mathcal{B}_0 \). Finally, let’s show that the operator \( L_C \) is bounded with bound \( K_H \). Let \( u \in C(\mathcal{B}_0) \), and \( (u_n)_{n \in \mathbb{N}} \) a sequence of elements \( u_n \in S(\mathcal{B}_0) \) such that \( \| u - u_n \| \mathcal{H} \to 0 \), and then \( \| L_C(u) - L_S(u_n) \|_{\mathbb{R}^d} \to 0 \). Since \( \| u \| \mathcal{H} - \| u_n \| \mathcal{H} \leq \| u - u_n \| \mathcal{H} \), then \( \| u_n \| \mathcal{H} \to \| u \| \mathcal{H} \). Hence, \( \| L_C(u) \|_{\mathbb{R}^d} = \| L_C(u) - L_S(u_n) + L_S(u_n) \|_{\mathbb{R}^d} \leq \| L_C(u) - L_S(u_n) \|_{\mathbb{R}^d} + \| L_S(u_n) \|_{\mathbb{R}^d} \leq \| L_C(u) - L_S(u_n) \|_{\mathbb{R}^d} + K_H \| u_n \| \mathcal{H} \). Taking the limit \( n \to \infty \), we obtain \( \| L_C(u) \|_{\mathbb{R}^d} \leq K_H \| u \| \mathcal{H} \). Hence \( L_C(u) \) is a bounded linear operator from \( C(\mathcal{B}_0) \) to \( \mathbb{R}^d \), such that \( L_C(u) = L_S(u), \forall u \in S(\mathcal{B}_0) \), and \( L_C(u) = L_S(u) = L_{\mathcal{B}_0}(u), \forall u \in \mathcal{B}_0 \).

5.2.3 Null space \( \mathcal{N}_L \) and topological complement \( \mathcal{N}_L^\perp \) of the operator \( L_C : C(\mathcal{B}_0) \to \mathbb{R}^d \)

Define the kernel or null space of the operator \( L_C \) as \( \mathcal{N}_L = \{ u \in C(\mathcal{B}_0) : L_C(u) = 0 \} \). It is readily seen that \( \mathcal{N}_L \) is a closed linear subspace of \( C(\mathcal{B}_0) \), \( \mathcal{N}_L \subseteq C(\mathcal{B}_0) \subseteq \mathcal{H} \). The orthogonal complement of \( \mathcal{N}_L \) with respect to \( C(\mathcal{B}_0) \) is \( \mathcal{N}_L^\perp = \{ u \in C(\mathcal{B}_0) : \langle u, w \rangle_{\mathcal{H}} = 0 \ \forall w \in \mathcal{N}_L \} \). Note that the orthogonal complement of \( \mathcal{N}_L^\perp \) with respect to \( C(\mathcal{B}_0) \) is \( \mathcal{N}_L^\perp \). It is readily shown that \( \mathcal{N}_L^\perp \) is a closed linear subspace of \( C(\mathcal{B}_0) \), \( \mathcal{N}_L^\perp \subseteq C(\mathcal{B}_0) \subseteq \mathcal{H} \). Next, let’s show that \( C(\mathcal{B}_0) = \mathcal{N}_L \oplus \mathcal{N}_L^\perp \), i.e. for each \( u \in C(\mathcal{B}_0) \) there exist unique elements i.s.n. \( v \in \mathcal{N}_L^\perp \) and \( w \in \mathcal{N}_L \), such that \( u = v + w \) i.s.n. We have:

**Fact 1.** (Minimum Distance to a Convex Set, [11] p. 8) Let \( u \in C(\mathcal{B}_0) \), since \( \mathcal{N}_L \) is a closed convex subset of the complete Hilbert vector space \( \mathcal{H} \), there exists a \( w(u) \in \mathcal{N}_L \), such that \( \| u - w(u) \| \mathcal{H} \leq \| u - z \| \mathcal{H}, \forall z \in \mathcal{N}_L \), and that element is unique i.s.n., i.e. if there exists another \( w'(u) \in \mathcal{N}_L \) such that \( \| u - w'(u) \| \mathcal{H} \leq \| u - z \| \mathcal{H}, \forall z \in \mathcal{N}_L \), then \( \| w(u) - w'(u) \| \mathcal{H} = 0 \).

**Fact 2.** (Principle of Orthogonality, [11] p. 9) Define \( v(u) = u - w(u) \), then \( v(u) \) is orthogonal to each of the elements in \( \mathcal{N}_L \), so that \( v(u) \in \mathcal{N}_L^\perp \). Additionally, if Fact 2 is true then Fact 1 is true. The element \( w(u) \) is defined as the orthogonal projection of \( u \) on the closed subspace \( \mathcal{N}_L \) denoted as \( w(u) = \text{Proj}(u | \mathcal{N}_L) \), similarly \( v(u) = \text{Proj}(u | \mathcal{N}_L^\perp) \).

Hence \( u \in C(\mathcal{B}_0) \) may be decomposed as \( u = v(u) + w(u) \) i.s.n. with \( v(u) \in \mathcal{N}_L^\perp \) and \( w(u) \in \mathcal{N}_L \). This decomposition is unique i.s.n.: if we also may write \( u = v'(u) + w'(u) \) i.s.n., with \( v'(u) \in \mathcal{N}_L^\perp \) and \( w'(u) \in \mathcal{N}_L \), then \( v(u) - v'(u) = w(u) - w(u) \) i.s.n. with \( v(u) - v'(u) \in \mathcal{N}_L^\perp \) and \( w'(u) - w(u) \in \mathcal{N}_L \) by linearity. Then, \( \| v(u) - v'(u) \|_{\mathcal{H}} = \| (u - v'(u), v(u) - v'(u))_{\mathcal{H}} \|_{\mathcal{H}} = \langle v(u) - v'(u), w'(u) - w(u) \rangle_{\mathcal{H}} = 0 \). Similarly \( \| w(u) - w'(u) \|_{\mathcal{H}}^2 = 0 \). Hence
$\mathcal{M}_L$ and $\mathcal{M}^\perp_L$ are topological complements, \textit{p. 93}, i.e. $C(\mathcal{B}_0) = \mathcal{M}^\perp_L \oplus \mathcal{M}_L$.

5.2.4 Images of $\mathcal{B}_0$, $S(\mathcal{B}_0)$, $C(\mathcal{B}_0)$ and $\mathcal{M}^\perp_L$

The previous properties are valid if we replace the space $\mathbb{R}^{d_P}$ with an arbitrary Banach space. The following properties depend strongly on the finite dimensional character of $\mathbb{R}^{d_P}$. The main property is that $\mathcal{M}^\perp_L$ is a finite dimensional sub-space of $\mathcal{M}$ as shown below.

Call $[\mathcal{B}_0]$ the image of the operator $L_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathbb{R}^{d_P}$, then, $[\mathcal{B}_0] \subseteq \mathbb{R}^{d_P}$. Since $\mathbb{R}^{d_P}$ has dimension $d_P$ then any $d_P + 1$ vectors in $\mathbb{R}^{d_P}$ are linearly dependent, and there are $d_P$ linearly independent vectors that constitute a basis for $\mathbb{R}^{d_P}$, see e.g. \textit{p. 178-179}. Since $[\mathcal{B}_0] \subseteq \mathbb{R}^{d_P}$ then any $d_P + 1$ vectors in $[\mathcal{B}_0]$ are linearly dependent. Since by hypothesis there exists at least one $u_0 \in \mathcal{B}_0$ such that $L_{\mathcal{B}_0}(u_0) \neq 0$, then there exists $d_L \in \mathbb{N}$ with $1 \leq d_L \leq d_P$, such that any $d_L + 1$ vectors in $[\mathcal{B}_0]$ are linearly dependent, and there are $d_L$ linearly independent vectors $L_{\mathcal{B}_0}(\tilde{\pi}_1), L_{\mathcal{B}_0}(\tilde{\pi}_2), \ldots, L_{\mathcal{B}_0}(\tilde{\pi}_{d_L})$ that belong to $[\mathcal{B}_0]$ with $\tilde{\pi}_i \in \mathcal{B}_0$, for $1 \leq i \leq d_L$. Note that $[\mathcal{B}_0]$ is not necessarily a linear subspace.

The elements $\tilde{\pi}_i \in \mathcal{B}_0$ for $1 \leq i \leq d_L$, are linearly independent i.s.n., i.e. whenever there are real coefficients $a_i \in \mathbb{R}$ for $1 \leq i \leq d_L$, for which we have $\| \sum_{i=1}^{d_L} a_i \tilde{\pi}_i \|_{\mathcal{M}} = 0$, then $a_i = 0$ for $1 \leq i \leq d_L$. If not, there would exist $a_i$, $a_i \in \mathbb{R}$ for $1 \leq i \leq d_L$, not all null, such that $\| \sum_{i=1}^{d_L} a_i \tilde{\pi}_i \|_{\mathcal{M}} = 0$, but then, because of the OP-HRBB condition $\| \sum_{i=1}^{d_L} a_i L_{\mathcal{B}_0}(\tilde{\pi}_i) \|_{\mathbb{R}^{d_P}} \leq K_H\| \sum_{i=1}^{d_L} a_i \tilde{\pi}_i \|_{\mathcal{M}}$, see \textit{p. 5.1}, it would be $\| \sum_{i=1}^{d_L} a_i L_{\mathcal{B}_0}(\tilde{\pi}_i) \|_{\mathbb{R}^{d_P}} = 0$, if $\sum_{i=1}^{d_L} a_i L_{\mathcal{B}_0}(\tilde{\pi}_i) = 0$, but the $L_{\mathcal{B}_0}(\tilde{\pi}_i)$’s are i.i., so that it should be $a_i = 0$, $1 \leq i \leq d_L$, which is a contradiction.

Next, decompose each $\tilde{\pi}_i$ as in the previous item i.e. for $1 \leq i \leq d_L$, $\tilde{\pi}_i = \tilde{v}_i + \tilde{w}_i$ i.s.n., where $\tilde{v}_i = \text{Proj}(\tilde{\pi}_i | \mathcal{M}^\perp_L)$ and $\tilde{w}_i = \text{Proj}(\tilde{\pi}_i | \mathcal{M}_L)$, so that $\tilde{v}_i \in \mathcal{M}^\perp_L \subseteq C(\mathcal{B}_0)$ and $\tilde{w}_i \in \mathcal{M}_L \subseteq C(\mathcal{B}_0)$. Note that, even though $\tilde{\pi}_i \in \mathcal{B}_0$, and then $\tilde{\pi}_i \in S(\mathcal{B}_0)$, in general it may happen that $\tilde{v}_i \notin S(\mathcal{B}_0)$ and $\tilde{w}_i \notin S(\mathcal{B}_0)$. Since, see item \textit{5.2.2} $L_{\mathcal{B}_0}(\tilde{\pi}_i) = L_C(\tilde{v}_i)$, then the vectors $L_C(\tilde{v}_i)$’s are linearly independent. Hence, the elements $\tilde{v}_i$’s are linearly independent i.s.n.: if not, there would exist $a_i$’s, $a_i \in \mathbb{R}$ for $1 \leq i \leq d_L$, not all null, such that $\| \sum_{i=1}^{d_L} a_i \tilde{v}_i \|_{\mathcal{M}} = 0$. Then, since the extension $L_C$ is a bounded linear operator, see item \textit{5.2.2}, then $\| \sum_{i=1}^{d_L} a_i L_C(\tilde{v}_i) \|_{\mathbb{R}^{d_P}} \leq K_H\| \sum_{i=1}^{d_L} a_i \tilde{v}_i \|_{\mathcal{M}}$, so that it would be $\| \sum_{i=1}^{d_L} a_i L_C(\tilde{v}_i) \|_{\mathbb{R}^{d_P}} = 0$, if $\sum_{i=1}^{d_L} a_i L_C(\tilde{v}_i) = 0$. But the $L_C(\tilde{v}_i)$’s are i.i., so that it should be $a_i = 0$, $1 \leq i \leq d_L$, which is a contradiction.

Since the $\tilde{v}_i$’s are linearly independent i.s.n., and they all belong to $\mathcal{M}^\perp_L$, use the Gram-Schmidt procedure, see e.g. \textit{p. 204}, to obtain $d_L$ orthonormal elements $\tilde{u}_i \in \mathcal{M}^\perp_L \subseteq C(\mathcal{B}_0) \subseteq \mathcal{M}$, that span the same space than the $\tilde{v}_i$’s, so that $\| \tilde{u}_i \|_{\mathcal{M}} = 1$, $\langle \tilde{u}_i, \tilde{u}_j \rangle_{\mathcal{M}} = 0$ for $i \neq j$, and $\langle \tilde{u}_i, w \rangle_{\mathcal{M}} = 0$, for $1 \leq i \leq d_L$ and $\forall w \in \mathcal{M}_L$. Hence, each $\tilde{u}_i$ is a linear combination i.s.n. of the $\tilde{v}_i$’s, and since this transformation is invertible, then each $\tilde{v}_i$ is a linear transformation i.s.n. of the $\tilde{u}_i$’s. The vectors $L_C(\tilde{u}_i)$ are linearly independent: if not, there would exist
Observation 5.3. Corresponding finite number of vectors \( \sum_{i=1}^{d_L} a_i \) \( L(C(\tilde{u}_i)) = 0 \), but then, \( L(C(\sum_{i=1}^{d_L} a_i \tilde{u}_i)) = 0 \), so that \( \sum_{i=1}^{d_L} a_i \tilde{u}_i \in \mathscr{N}_L \). Then, for \( 1 \leq k \leq d_L \), we have \( \langle \tilde{u}_k, \sum_{i=1}^{d_L} a_i \tilde{u}_i \rangle_{\mathscr{H}} = \sum_{i=1}^{d_L} a_i \langle \tilde{u}_k, \tilde{u}_i \rangle_{\mathscr{H}} = a_k \| \tilde{u}_k \|_{\mathscr{H}}^2 = a_k \), so that \( a_k = 0 \) for \( 1 \leq k \leq d_L \), which is a contradiction.

Call \( I_L \) the span of the linearly independent vectors \( L(C(\tilde{u}_i)) \), for \( 1 \leq i \leq d_L \), so that \( I(\mathcal{B}_0) \subseteq I_L \). Since \( L(C(\tilde{u}_i)) = L(C(\tilde{u}_i)) \), then \( I_L \) is the span of the linearly independent vectors \( L(C(\tilde{u}_i)) \), for \( 1 \leq i \leq d_L \). Since each \( \tilde{u}_i \) is a linear combination of the linearly independent vectors \( L(C(\tilde{u}_i)) \) and vice-versa, and then \( I_L \) is the span of the linearly independent vectors \( L(C(\tilde{u}_i)) \), for \( 1 \leq i \leq d_L \).

Call \( I(S(\mathcal{B}_0)) \) the image of the operator \( L_S : S(\mathcal{B}_0) \rightarrow \mathbb{R}^{d_L} \). Recall that if \( u \in S(\mathcal{B}_0) \) then \( L_S(u) = L_C(u) \). Clearly, \( I_L \subseteq I(S(\mathcal{B}_0)) \). If \( u \in S(\mathcal{B}_0) \) then \( u \) is a linear combination of a finite number of elements in \( \mathcal{B}_0 \), and the vector \( L(C(\tilde{u}_i)) \in I(S(\mathcal{B}_0)) \), is the same linear combination of the corresponding finite number of vectors in \( I(\mathcal{B}_0) \). But since each vector in \( I(\mathcal{B}_0) \) is a linear combination of the vectors \( L(C(\tilde{u}_i)) \), for \( 1 \leq i \leq d_L \), then \( L(C(\tilde{u}_i)) \) is a linear combination of the independent vectors \( L(C(\tilde{u}_i)) \), for \( 1 \leq i \leq d_L \), hence \( L(C(\tilde{u}_i)) \subseteq I_L \), so that \( I(S(\mathcal{B}_0)) \subseteq I_L \). If \( u \in S(\mathcal{B}_0) \), then since \( I(S(\mathcal{B}_0)) \subseteq I_L \), there exist \( \alpha_i(u) \in \mathbb{R} \), \( 1 \leq i \leq d_L \), such that \( L(C(u)) = \sum_{i=1}^{d_L} \alpha_i(u) \tilde{u}_i \in \mathcal{M}_L \). Define \( w(u) = u - \sum_{i=1}^{d_L} \alpha_i(u) \tilde{u}_i \), then \( L(C(w(u))) = 0 \), so that \( w(u) \) is \( \mathcal{N}_L \) and \( \sum_{i=1}^{d_L} \alpha_i(u) \tilde{u}_i \in \mathcal{N}_L^\perp \). Hence \( u \in S(\mathcal{B}_0) \) may be written as \( u = \sum_{i=1}^{d_L} \alpha_i(u) \tilde{u}_i + w(u) \) i.s.n.

Observation 5.3. Note that, whenever \( u \in C(\mathcal{B}_0) \) may be written as \( u = \sum_{i=1}^{d_L} \alpha_i(u) \tilde{u}_i + w(u) \), where \( w(u) \in \mathcal{M}_L \) and the \( \tilde{u}_i \)'s, \( 1 \leq i \leq d_L \), are orthonormal elements in \( \mathcal{M}_L^\perp \), then we have

\[
\| u \|_{\mathscr{H}}^2 = \sum_{i=1}^{d_L} |\alpha_i(u)|^2 + \| w(u) \|_{\mathscr{H}}^2
\]  

(5.3)

Call \( I(C(\mathcal{B}_0)) \) the image of the operator \( L_C : C(\mathcal{B}_0) \rightarrow \mathbb{R}^{d_L} \). Since \( I(S(\mathcal{B}_0)) \subseteq I(C(\mathcal{B}_0)) \), then \( I_L \subseteq I(C(\mathcal{B}_0)) \). Fix an arbitrary \( u \in C(\mathcal{B}_0) \), then there exists a sequence \( (u_n)_{n\in\mathbb{N}} \) of elements \( u_n \in S(\mathcal{B}_0) \) such that \( \| u_n - u \|_{\mathscr{H}} \rightarrow 0 \), and \( \| L_C(u_n) - L_C(u) \|_{\mathbb{R}^{d_L}} \rightarrow 0 \). Since \( u_n \in S(\mathcal{B}_0) \), then there exist sequences \( (\alpha_i(u_n))_{n\in\mathbb{N}} \), with \( \alpha_i(u_n) \in \mathbb{R} \) for \( 1 \leq i \leq d_L \), \( \forall n \in \mathbb{N} \), and a sequence \( (w(u_n))_{n\in\mathbb{N}} \), with \( w(u_n) \in \mathcal{M}_L \), \( \forall n \in \mathbb{N} \), such that \( u_n = \sum_{i=1}^{d_L} \alpha_i(u_n) \tilde{u}_i + w(u_n) \). Since \( (u_n)_{n\in\mathbb{N}} \) is a Cauchy fundamental sequence in \( \mathcal{H} \), with \( u_n \in S(\mathcal{B}_0) \subseteq C(\mathcal{B}_0) \), then \( (\alpha_i(u_n))_{n\in\mathbb{N}} \) is a Cauchy fundamental sequence of real numbers, and the sequence \( (w(u_n))_{n\in\mathbb{N}} \) is a Cauchy fundamental sequence of elements in \( \mathcal{M}_L \subseteq C(\mathcal{B}_0) \subseteq \mathcal{H} \). Since the reals are complete, there exist real numbers \( a_i \in \mathbb{R} \) for which \( \alpha_i(u_n) \rightarrow a_i \), for \( 1 \leq i \leq d_L \), and, since \( \mathcal{H} \) is complete and \( \mathcal{M}_L \) is closed, there exists an element \( \eta \in \mathcal{M}_L \) such that \( \| w(u_n) - \eta \|_{\mathscr{H}} \rightarrow 0 \). Define
\[ u' = \sum_{i=1}^{d_L} \alpha_i \hat{u}_i + \eta, \text{ then } u' \in \mathcal{C}(\mathfrak{B}_0). \text{ Then, } (5.3) \text{ shows that } \|u_n - u'\|_{\mathfrak{F}} \to 0. \]

Since \[ \|u - u'\|_{\mathfrak{F}} \leq \|u - u_n\|_{\mathfrak{F}} + \|u_n - u'\|_{\mathfrak{F}}, \] taking the limit, we obtain \[ u = u' \] i.s.n. Hence for each \( u \in \mathcal{C}(\mathfrak{B}_0) \) we have found real numbers \( \alpha_i(u) \in \mathbb{R} \), for \( 1 \leq i \leq d_L \), and an element \( w(u) \in \mathcal{N}_L \) such that

\[ u = \sum_{i=1}^{d_L} \alpha_i(u) \hat{u}_i + w(u) \quad \text{i.s.n.} \quad \forall u \in \mathcal{C}(\mathfrak{B}_0) \]  

(5.4)

Then, \( L_C(u) = \sum_{i=1}^{d_L} \alpha_i(u) L_C(\hat{u}_i) \), so that \( L_C(u) \in I_L \), and then \( \mathbb{I}(\mathcal{C}(\mathfrak{B}_0)) = I_L \). Additionally, since for arbitrary \( u \in \mathcal{C}(\mathfrak{B}_0) \), from (5.4), we have

\[ \text{Proj}(u \mid \mathcal{N}_L^\perp) = \sum_{i=1}^{d_L} \alpha_i(u) \hat{u}_i \quad \text{i.s.n., then, } \quad u \in \mathcal{N}_L^\perp \iff u = \sum_{i=1}^{d_L} \alpha_i(u) \hat{u}_i \quad \text{i.s.n., and then } \mathcal{N}_L^\perp \text{ is a finite dimension subspace, } \mathcal{N}_L^\perp \subseteq \mathcal{C}(\mathfrak{B}_0) \subseteq \mathcal{H}, \text{ with dimension } d_L, \text{ even though } \mathcal{H} \text{ might be a non-separable space.} \]

Hence, we have \( \mathbb{I}[\mathcal{N}_L] = \{0\} \), and \( \mathbb{I}[\mathfrak{B}_0] \subseteq I_L = \mathbb{I}[S(\mathfrak{B}_0)] = \mathbb{I}[\mathcal{C}(\mathfrak{B}_0)] = \mathbb{I}[\mathcal{N}_L^\perp] \).

### 5.2.5 Generalized Riesz representation of the operator \( L_C : \mathcal{C}(\mathfrak{B}_0) \to \mathbb{R}^{d_P} \)

From (5.4), it is \( L_C(u) = \sum_{i=1}^{d_L} \alpha_i(u) L_C(\hat{u}_i) \), for all \( u \in \mathcal{C}(\mathfrak{B}_0) \). Since the \( \hat{u}_i \)'s are orthonormal and perpendicular to \( w(u) \), then,

\[ \langle u, \hat{u}_k \rangle_{\mathfrak{F}} = \langle \sum_{i=1}^{d_L} \alpha_i(u) \hat{u}_i + w(u), \hat{u}_k \rangle_{\mathfrak{F}} = \alpha_k(u). \]

Hence

\[ L_C(u) = \sum_{i=1}^{d_L} \langle u, \hat{u}_i \rangle_{\mathfrak{F}} L_C(\hat{u}_i) \quad \forall u \in \mathcal{C}(\mathfrak{B}_0) \]

see (5.2), which is the vector generalized Riesz representation for the extension \( L_C \) of an operator \( L_{\mathfrak{B}_0} : \mathfrak{B}_0 \to \mathbb{R}^{d_P}, \mathfrak{B}_0 \subseteq \mathcal{H} \), satisfying the OP-HRBB condition.

### 6 Optimal estimator under the HRBB condition

The space \( L_2(\mathbb{R}^{d_S}, \mathscr{B}_{d_S}, \mathbb{P}_{\theta_T}) \) is a Hilbert space, \[ [15] \text{ p. 194, with semi-inner product } \langle u_1, u_2 \rangle_{\mathfrak{F}} = \langle u_1, u_2 \rangle_{L_2} = \int u_1 u_2 \, d\mathbb{P}_{\theta_T}, \forall u_1, u_2 \in L_2(\mathbb{R}^{d_S}, \mathscr{B}_{d_S}, \mathbb{P}_{\theta_T}), \]

semi-norm \( \|u\|_{\mathfrak{F}} = \|u\|_{L_2} = \left( \int u^2 \, d\mathbb{P}_{\theta_T} \right)^{1/2} \), and equality in semi-norm (i.s.n.) given by equality with probability 1 (w.p. 1).

**Lemma 6.1.** If the HRBB condition holds for Problem \([2.3]\) see Definition \([4.4]\) then there exists a finite covariance unbiased estimator \( \psi_c \in \mathcal{U}_g \).

**Proof.** Since \( L_2(\mathbb{R}^{d_S}, \mathscr{B}_{d_S}, \mathbb{P}_{\theta_T}) \) is a Hilbert space, then we take the elements of \( \mathcal{H} \) as the functions in \( L_2(\mathbb{R}^{d_S}, \mathscr{B}_{d_S}, \mathbb{P}_{\theta_T}) \). Define the operator \( L_{\mathfrak{B}_0}(u) = h(\pi^{-1}(u)), \text{ for all } u \in \mathfrak{B}_0, \text{ see Section } [4.1] \). Since the HRBB condition holds for
Problem 2.3 see [4.2], then the OP-HRBB condition holds, see [5.1], and then we may apply Theorem 5.1. From [5.2] we obtain

\[ L_C(u) = \sum_{i=1}^{d_L} \langle u, \hat{u}_i \rangle \circ \sigma L_C(\hat{u}_i) = \sum_{i=1}^{d_L} L_C(\hat{u}_i) \int u \, \hat{u}_i \, dP_{\theta_T} \quad (6.1) \]

\[ = \int u \left[ \sum_{i=1}^{d_L} \hat{u}_i \, L_C(\hat{u}_i) \right] dP_{\theta_T} \quad \forall u \in C(\mathcal{B}_0) \]

where the \( \hat{u}_i \)'s are orthonormal, with \( \hat{u}_i \in \mathcal{H}^0_\mathcal{B}_0 \), for \( 1 \leq i \leq d_L \). Note the importance of working with finite dimensions \( d_P \) and \( d_L \), with \( 1 \leq d_L \leq d_P \), since this permits exchanging sums and integrals invoking elementary properties of Lebesgue integrals. Define

\[ \hat{\varphi}_c = \sum_{i=1}^{d_L} \hat{u}_i \, L_C(\hat{u}_i) \quad (6.2) \]

so that \( L_C(u) = \int \hat{\varphi}_c \, u \, dP_{\theta_T}, \forall u \in C(\mathcal{B}_0) \).

Since each \( L_C(\hat{u}_i) \) is some constant real vector, i.e. \( L_C(\hat{u}_i) \in \mathbb{R}^{d_P} \), for \( 1 \leq i \leq d_L \), and each \( \hat{u}_i \in L_2(\mathbb{R}^{d_s}, \mathcal{B}_{d_s}, P_{\theta_T}) \), then each component of the vector \( \hat{\varphi}_c \) is square integrable, i.e. \( [\hat{\varphi}_c]_i \in L_2(\mathbb{R}^{d_s}, \mathcal{B}_{d_s}, P_{\theta_T}) \), equivalently \( \hat{\varphi}_c \in L_2(\mathbb{R}^{d_s}, \mathcal{B}_{d_s}, P_{\theta_T}) \). Then \( \hat{\varphi}_c \) is a measurable function from \( L_2(\mathbb{R}^{d_s}, \mathcal{B}_{d_s}, P_{\theta_T}) \) to \( \mathbb{R}^{d_P} \), so that \( \hat{\varphi}_c(\mathbf{x}) = \sum_{i=1}^{d_L} \hat{u}_i(\mathbf{x}) \, L_C(\hat{u}_i) \) is a random vector, \( \hat{\varphi}_c(\mathbf{x}) : \Omega \to \mathbb{R}^{d_P} \), that does not depend on the sub-indices \( \theta \in \Theta \). Additionally since \( \hat{\varphi}_c \in L_2(\mathbb{R}^{d_s}, \mathcal{B}_{d_s}, P_{\theta_T}) \), then \( \hat{\psi}_c = \hat{\varphi}_c + g(\theta_T) \) has finite covariance as previously discussed in Section 2.1. Since \( L_C(u) = L_{2_{\mathcal{B}_0}}(u) \) if \( u \in \mathcal{B}_0 \), see Section 5.2.2 and, for each \( u \in \mathcal{B}_0 \) there exists \( \theta \in \Theta \) such that \( u = \pi(\theta) \), see Hypothesis 2.2 and \( L_{2_{\mathcal{B}_0}}(\pi(\theta)) = h(\theta), \forall \theta \in \Theta \), see [4.1], then, using [6.1] and [6.2], \( h(\theta) = L_{2_{\mathcal{B}_0}}(\pi(\theta)) = L_C(\pi(\theta)) = \int \hat{\varphi}_c(\pi(\theta)) \, dP_{\theta_T} = \int \hat{\varphi}_c \, dP_{\theta_T} = \int \hat{\varphi}_c(\mathbf{x}) \, dP_{\theta_T} = \mathcal{E}_{\theta} \left[ \hat{\varphi}_c \right] \), \( \forall \theta \in \Theta \), see [2.1]. Then, \( \mathcal{E}_{\theta} \left[ \hat{\psi}_c \right] = \int \hat{\psi}_c \, dP_{\theta_T} = \int \hat{\varphi}_c(\mathbf{x}) \, dP_{\theta_T} = g(\theta), \forall \theta \in \Theta \). Hence, \( \hat{\psi}_c(\mathbf{x}) \) is unbiased for all \( \theta \in \Theta \), and then \( \hat{\psi}_c \in \mathcal{W}_g \).

**Definition 6.1.** Define the HRBB estimator as \( \hat{\psi}_c = \hat{\varphi}_c + g(\theta_T) \), where \( \hat{\varphi}_c \) is given by (6.2) as discussed in Lemma 6.1 so that \( \hat{\psi}_c \in \mathcal{W}_g \).

**Definition 6.2.** (Barankin-efficient estimator). A finite covariance unbiased estimator \( \hat{\psi} \in \mathcal{W}_g \) for Problem 2.3 will be called Barankin-efficient, if \( \text{Cov}_{\theta_T}(\hat{\psi}) \geq \text{Cov}_{\theta_T}(\hat{\psi}_c) \) for all \( \psi \in \mathcal{W}_g \). Equivalently, \( \hat{\psi} \) is a minimum-covariance unbiased estimator for Problem 2.3.

**Definition 6.3.** Let \( \mathcal{W} \) be a collection of real s.n.n.d. matrices of dimensions \( N \times N \). A s.n.n.d. matrix \( A \in \mathbb{R}^{N \times N} \) is an upper (lower) bound for \( \mathcal{W} \) if \( A \geq W \) (\( A \leq W \)), \( \forall W \in \mathcal{W} \). Define, if it exists, the matrix-supreme (msup) of the matrices in \( \mathcal{W} \), as a real s.n.n.d. matrix \( A \) of dimensions \( N \times N \), such that
\( A \geq W, \forall W \in \mathcal{W}, \) and such that for each \( \epsilon \in \mathbb{R}^+, \epsilon > 0, \) there exists \( W(\epsilon) \in \mathcal{W} \) such that \( \|A - W(\epsilon)\|_F < \epsilon. \) The notation will be \( A = \mathop{\text{msup}}_{W \in \mathcal{W}} W. \) If \( A \in \mathcal{W}, \) then \( A \) will be called the matrix-maximum of \( \mathcal{W}. \)

Define \( L \) as the real matrix, \( L \in \mathbb{R}^{dp \times d_L}, \) with columns \( [L]_i = L_C(\hat{u}_i), \) for \( 1 \leq i \leq d_L, \) and define \( \hat{u}^T = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{d_L}), \) so that \( \hat{\varphi}_c = L \hat{u}, \) see (6.2). Since the \( \hat{u}_i \)'s are orthonormal in \( L_2(\mathbb{R}^{d_S}, \mathcal{B}_{d_S}, \mathcal{P}_{\vartheta_T}), \) then \( \mathbb{E}_{\vartheta_T}[\hat{u} \hat{u}^T] = I_{d_L}, \) where \( I_{d_L} \) is the identity matrix of dimensions \( d_L \times d_L. \) From (6.2), we have: \( \mathbb{Cov}_{\vartheta_T}(\hat{\varphi}_c) = \mathbb{E}_{\vartheta_T}[\hat{\varphi}_c \hat{\varphi}_c^T] = \mathbb{E}_{\vartheta_T}[L \hat{u} (L \hat{u})^T] = L \mathbb{E}_{\vartheta_T}[\hat{u} \hat{u}^T]L^T = L I_{d_L}L^T = L L^T, \) so that

\[
\mathbb{Cov}_{\vartheta_T}(\hat{\varphi}_c) = L L^T = \sum_{i=1}^{d_L} L_C(\hat{u}_i)L_C(\hat{u}_i)^T \tag{6.3}
\]

**Theorem 6.1.** If the HRBB condition holds for Problem 2.3, see Definition 4.1, then the HRBB estimator \( \hat{\varphi}_c, \) see Definition 6.1, is an unbiased Barankin-efficient estimator, and \( \mathbb{Cov}_{\vartheta_T}(\hat{\varphi}_c) = \mathop{\text{msup}}_{W \in \mathcal{W}_A} W. \)

**Proof.** The HRBB estimator \( \hat{\varphi}_c \) is unbiased and has finite covariance as a consequence of Lemma 6.1. To show that it is Barankin-efficient let’s consider the following two cases.

1) All the \( \hat{u}_i \)'s belong to \( S(\mathcal{B}_0). \) Then, \( \hat{u} \in (S(\mathcal{B}_0))^{d_L}. \) Since \( \hat{\varphi}_c = L \hat{u}, \) then \( \hat{\varphi}_c \in (S(\mathcal{B}_0))^{dp}, \) and then, see Theorem 3.1 and Observation 3.2, we have equality in (3.1). More precisely, since each \( \hat{u}_i \in S(\mathcal{B}_0), \) then, there exist \( M_i \in \mathbb{N}, a_i \in \mathbb{R}^{M_i}, \) and \( \tau_i \in \mathcal{C}^{M_i}, \) such that \( \hat{u}_i = a_i^T \beta(\tau_i) \) w.p. 1, for \( 1 \leq i \leq d_L. \) Define \( \hat{M} = \sum_{i=1}^{d_L} M_i, \) and \( \hat{\tau}^T = (\tau_1^T, \ldots, \tau_{d_L}^T), \) \( \hat{\beta} \in \mathcal{C}^{\hat{M}} \). Call \( \hat{\beta} = \beta(\hat{\tau}), \) \( \hat{\beta} \in \mathcal{B}_0^{\hat{M}}, \) so that \( \hat{\beta}^T = \beta^T(\hat{\tau}) = (\beta^T(\tau_1), \ldots, \beta^T(\tau_{d_L})), \) and \( \hat{B} = \mathbb{E}_{\vartheta_T}[^T \hat{\beta} \hat{\beta}], \hat{B} \in \mathbb{R}^{\hat{M} \times \hat{M}}. \) Define the real matrix \( \hat{A} \in \mathbb{R}^{\hat{d}_L \times \hat{M}} \), as the block-diagonal matrix \( \hat{A} = \text{diag}(a_1^T, a_2^T, \ldots, a_{d_L}^T), \) where each block \( a_i^T \) is of dimension \( 1 \times M_i, \) for \( 1 \leq i \leq M_i, \) so that \( \hat{u} = \hat{A} \hat{\beta} \) w.p. 1. Since the \( \hat{u}_i \)'s are orthonormal in \( L_2(\mathbb{R}^{d_S}, \mathcal{B}_{d_S}, \mathcal{P}_{\vartheta_T}), \) we have \( \mathbb{E}_{\vartheta_T}[\hat{u} \hat{u}^T] = I_{d_L}. \) Since \( \mathbb{E}_{\vartheta_T}[\hat{u} \hat{u}^T] = \mathbb{E}_{\vartheta_T}[\hat{A} \hat{\beta} \hat{\beta}^T] = \hat{A} \mathbb{E}_{\vartheta_T}[\hat{\beta} \hat{\beta}^T] \hat{A}^T = \hat{A} \hat{B} \hat{A}^T, \) then \( \hat{A} \hat{B} \hat{A}^T = I_{d_L}, \) so that \( \text{Det}(\hat{A} \hat{B} \hat{A}^T) = 1. \) Since \( \hat{\varphi}_c \in \mathcal{W}_A \) is unbiased, then, see Observation 3.1, \( G(\hat{\tau}) = \mathbb{E}_{\vartheta_T}[\hat{\varphi}_c \beta^T(\hat{\tau})] = \mathbb{E}_{\vartheta_T}[L \hat{u} \beta^T(\hat{\tau})] = \mathbb{L} \mathbb{E}_{\vartheta_T}[\hat{u} \beta^T(\hat{\tau})] = L \hat{A} \mathbb{E}_{\vartheta_T}[\hat{\beta} \hat{\beta}^T] = L \hat{A} \hat{B} \hat{\tau}, \) see Definition 3.1 so that \( \hat{q} \in \mathcal{W}_A \) since \( \text{Det}(\hat{A} \hat{B} \hat{A}^T) = 1, \) and then \( W(\hat{q}) \in \mathcal{W}_A. \) Hence \( W(\hat{q}) = G(\hat{\tau}) \hat{A}^T (\hat{A} \hat{B} \hat{A}^T)_{\hat{\tau}} = G(\hat{\tau}) \hat{A}^T \hat{A} G(\hat{\tau}) = L (\hat{A} \hat{B} \hat{A}^T)(\hat{A} \hat{B} \hat{A}^T)_{\hat{\tau}} = L L^T = \mathbb{Cov}_{\vartheta_T}(\hat{\varphi}_c), \) see (6.3). Then \( \mathbb{Cov}_{\vartheta_T}(\hat{\varphi}_c) \)
W(\hat{q}(m)) = G(\hat{\tau}(m)) \hat{A}^T(m) \\
\left(\hat{A}(m) \hat{B}(m) \hat{A}^T(m)\right)^{-1} \hat{A}(m) G(\hat{\tau}(m)) \\
= L S(m) L^T + L E_{\theta_T}[(\hat{u} - \hat{s}(m)) S(m)] L^T \\
+ L E_{\theta_T}[(\hat{u} - \hat{s}(m)) S(m)] L^T \\
+ L E_{\theta_T}[(\hat{u} - \hat{s}(m)) S(m)] L^T \cdot (S(m))^{-1} \\
E_{\theta_T}[(\hat{u} - \hat{s}(m)) S(m)] L^T \\
so that, see Appendix Lemma A.1, \( W(\hat{q}(m)) \rightarrow L L^T = \text{cov}_{\theta_T}(\hat{\psi}_c) \), see (6.3), component by component and then in Frobenius norm.
Unlike the previous case, if for some \(i^*\), \(1 \leq i^* \leq d_L\), we have that \(\hat{u}_{i^*}\) belongs to \(C(\mathcal{B}_0)\) and \(\hat{u}_{i^*} \notin S(\mathcal{B}_0)\), then \(\text{Cov}_{\theta_T}(\hat{\varphi}_c) = \mathbb{L}_L\). If not, \(\text{Cov}_{\theta_T}(\hat{\varphi}_c) \in \mathcal{W}_A\), and then we have equality in (3.1), so that, see Theorem 3.1, \(\mathcal{G}_c \in S(\mathcal{B}_0)\), for each \(1 \leq i \leq d_L\). But \(\mathcal{G}_c = \mathbb{L}_u\), so that \(\mathbb{L}_L^{T} \mathcal{G}_c = \mathbb{L}_L^{T} \mathbf{u}\), with \(\mathbb{L} = (L_C(\hat{u}_1) \cdots L_C(\hat{u}_{d_L}))\). Since the \(d_L\) columns of \(\mathbb{L}\) are linearly independent then \(\det(\mathbb{L}_L^{T} \mathbb{L}) \neq 0\), if not there exists \(\mathbf{a} \in \mathbb{R}^{d_p}\), \(\mathbf{a} \neq 0\), such that \(\mathbb{L}_L^{T} \mathbf{a} = 0\), so that \(\mathbf{a}^{T} \mathbb{L}_L^{T} \mathbf{a} = \|\mathbf{a}\|^{2}_{\mathbb{R}^{d_p}} = 0\), and then \(\mathbf{L}_L \mathbf{a} = 0\), contradiction. Hence \(\hat{\mathbf{u}} = (\mathbb{L}_L^{T})^{-1} \mathbb{L}_L^{T} \mathcal{G}_c\), and then \(|\hat{\mathbf{u}}_i| \in S(\mathcal{B}_0)\), for each \(1 \leq i \leq d_L\), contradiction.

Since \(\varphi \in \mathcal{G}_g\), see Theorem 3.1, it is \(\text{Cov}_{\theta_T}(\varphi) \geq W(\mathcal{G}(m))\), then \(\text{Cov}_{\theta_T}(\varphi) - W(\mathcal{G}(m)) \geq 0\), so that, taking the limit, see Appendix Lemma A.2, we have \(\text{Cov}_{\theta_T}(\varphi) - \text{Cov}_{\theta_T}(\mathcal{G}_c) \geq 0\), and then \(\text{Cov}_{\theta_T}(\varphi) \geq \text{Cov}_{\theta_T}(\mathcal{G}_c), \forall \varphi \in \mathcal{G}_g\). Then, even though \(\text{Cov}_{\theta_T}(\mathcal{G}_c) \notin \mathcal{W}_A\), we have \(\text{Cov}_{\theta_T}(\varphi) \geq \text{Cov}_{\theta_T}(\mathcal{G}_c), \forall \varphi \in \mathcal{G}_g\), and, see Theorem 3.1, \(\text{Cov}_{\theta_T}(\mathcal{G}_c) \geq W, \forall W \in \mathcal{W}_A\). Furthermore, as previously shown, there exists a sequence \(W_m \in \mathcal{W}_A, W_m \equiv W(\mathcal{G}(m))\), such that \(\|\text{Cov}_{\theta_T}(\mathcal{G}_c) - W_m\|_{\mathcal{F}} \to 0\). Hence, though \(\text{Cov}_{\theta_T}(\mathcal{G}_c)\) is not a matrix-maximum for \(\mathcal{W}_A\), it is a matrix-supreme for \(\mathcal{W}_A\), and \(\text{Cov}_{\theta_T}(\mathcal{G}_c)\) is a matrix-minimum for all the covariances of the estimators in \(\mathcal{G}_g\), so that \(\mathcal{G}_c\) is Barankin-efficient. □

**Observation 6.1.** A key point in Theorem 6.1 is that if \(\mathcal{W}_A\) is bounded above, then, the optimal covariance \(\text{Cov}_{\theta_T}(\mathcal{G}_c)\) may be obtained as the matrix-supreme, see Definition 6.3 of the matrices \(W \in \mathcal{W}_A\), see (3.1).

\[
\text{Cov}_{\theta_T}(\mathcal{G}_c) = \max_{\mathcal{W}_A} \text{Cov}(\theta_T) \ A^{T} \ (A \ B(\theta_T))^{-1} \ A \ G^{T}(\tau)
\]

and the matrix-supreme will be a matrix-maximum if and only if \(\mathcal{G}_c - g(\theta_T) = \mathcal{G}_c \in \left(S(\mathcal{B}_0)\right)^{d_p}\).

### 7 LMI equivalent formulation

#### 7.1 Equivalence of the LMI bound and the HRBB condition

The statement that the Barankin covariance lower bounds \(\mathcal{W}_A\) are bounded above, is a disguised form of the HRBB condition, as a matter of fact the converse is also true, see Lemma 7.1 and Theorem 7.1 below.

**Lemma 7.1.** If the Barankin covariance lower bounds \(\mathcal{W}_A\) are bounded above, i.e. the collection \(\mathcal{W}_A\) is bounded, see Definition 3.2, then the HRBB condition holds, see Definition 4.1.

**Proof.** Call \(B_{\mathcal{W}}\) the bound for \(\mathcal{W}_A\), i.e.

\[
B_{\mathcal{W}} \geq W(\mathcal{q}) = G(\tau) \ A^{T} \ (A \ B(\tau))^{-1} \ A \ G^{T}(\tau)
\]  

(7.1)
for all \( q \in \mathcal{G} \). Since this is true for matrices \( A \) of all sizes \( d_A \in \mathbb{N} \) for a given \( d_M \in \mathbb{N} \), \( A \in \mathbb{R}^{d_A \times d_M} \), in particular is true when \( d_A = 1 \), i.e. when \( A \) has one single row. Call \( A^T = (a_1, a_2, \ldots, a_{d_M}) \) the single row, so that \( A = a^T \), with \( a \in \mathbb{R}^{d_M} \). Then, \( A B(\tau) A^T = a^T B(\tau) a = a^T \mathbb{E}_{\theta_T} [\beta(\tau) \beta^T(\tau)] a = E_{\theta_T} [a^T (\beta(\tau) \beta^T(\tau)] a = E_{\theta_T} [\sum_{i=1}^{d_M} a_i \pi(\theta_i)]^2 = \left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2}^2. \)

Observe that \( a^T B(\tau) a \) is a non-negative scalar, i.e. \( a^T B(\tau) a \in \mathbb{R}^+ \), and since we assumed that \( q \in \mathcal{G} \) then \( a^T B(\tau) a \neq 0 \), as a matter of fact \( a^T B(\tau) a = \left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2}^2 > 0 \). On the other hand, \( G(\tau) A^T = G(\tau) a = \sum_{i=1}^{d_M} a_i h(\theta_i) \). Then, (7.1) takes the form

\[
B_{\varphi} \geq \frac{\left( \sum_{i=1}^{d_M} a_i h(\theta_i) \right) \left( \sum_{i=1}^{d_M} a_i h(\theta_i) \right)^T}{\left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2}^2}
\]

Hence,

\[
\text{Tr} \left[ B_{\varphi} \right] \geq \frac{\text{Tr} \left[ \left( \sum_{i=1}^{d_M} a_i h(\theta_i) \right) \left( \sum_{i=1}^{d_M} a_i h(\theta_i) \right)^T \right]}{\left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2}^2}
= \frac{\left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2}^2}{\left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2}^2}
\]

Call \( K_H = (\text{Tr} [B_{\varphi}])^{1/2} \). Since, \( \text{Tr} \left[ \left( \sum_{i=1}^{d_M} a_i h(\theta_i) \right) \left( \sum_{i=1}^{d_M} a_i h(\theta_i) \right)^T \right] = \left\| \sum_{i=1}^{d_M} a_i h(\theta_i) \right\|^2_{R_{d_P} \mathcal{H}} \), then, \( K_H \geq \left\| \sum_{i=1}^{d_M} a_i h(\theta_i) \right\|_{R_{d_P} \mathcal{H}} \). Hence \( \left\| \sum_{i=1}^{d_M} a_i h(\theta_i) \right\|_{R_{d_P} \mathcal{H}} \leq K_H \left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2} \), \( \forall d_M \in \mathbb{N}, \forall a_i \in \mathbb{R}, 1 \leq i \leq d_M, \ \forall \theta_i \in \Theta, 1 \leq i \leq d_M \), such that \( \left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2} \neq 0 \).

If \( \left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) \right\|_{L^2} = 0 \), then \( \sum_{i=1}^{d_M} a_i \pi(\theta_i) = 0 \) w.p. 1. Take an arbitrary \( u^* \in \mathcal{B}_0 \), then, see Observation 2.3 \( \|u^*\|_{L^2} \neq 0 \). Call \( \theta^* = \pi^{-1}(u^*) \). Then \( \sum_{i=1}^{d_M} a_i \pi(\theta_i) + (1/n) u^* = (1/n) u^* \) w.p. 1, for all \( n \in \mathbb{N} \), so that

\[
\left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) + (1/n) u^* \right\|_{L^2} = \| (1/n) u^* \|_{L^2} \neq 0, \forall n \in \mathbb{N}.
\]

Hence the previously obtained inequality applies

\[
\left\| \sum_{i=1}^{d_M} a_i h(\theta_i) + (1/n) h(\theta^*) \right\|_{R_{d_P} \mathcal{H}} \leq K_H \left\| \sum_{i=1}^{d_M} a_i \pi(\theta_i) + (1/n) u^* \right\|_{L^2} = K_H \left\| (1/n) u^* \right\|_{L^2},
\]

so that

\[
\left\| \sum_{i=1}^{d_M} a_i h(\theta_i) + (1/n) h(\theta^*) \right\|_{R_{d_P} \mathcal{H}} \to 0, \ \text{as} \ n \to +\infty.
\]

Since

\[
\left\| \sum_{i=1}^{d_M} a_i h(\theta_i) \right\|_{R_{d_P} \mathcal{H}} = \left\| \sum_{i=1}^{d_M} a_i h(\theta_i) + (1/n) h(\theta^*) - (1/n) h(\theta^*) \right\|_{R_{d_P} \mathcal{H}} \leq \left\| \sum_{i=1}^{d_M} a_i h(\theta_i) + (1/n) h(\theta^*) \right\|_{R_{d_P} \mathcal{H}} + (1/n) \left\| h(\theta^*) \right\|_{R_{d_P} \mathcal{H}}
\]

then, taking the limit \( n \to +\infty \), it results \( \left\| \sum_{i=1}^{d_M} a_i h(\theta_i) \right\|_{R_{d_P} \mathcal{H}} = 0. \)
Hence \( \left\| \sum_{i=1}^{d_M} a_i \, h(\theta_i) \right\|_{\mathbb{R}^p} \leq K_H \left\| \sum_{i=1}^{d_M} a_i \, \pi(\theta_i) \right\|_{L_2}, \forall d_M \in \mathbb{N}, \forall a_i \in \mathbb{R}, \) 
\( 1 \leq i \leq d_M, \forall \theta_i \in \Theta, \) \( 1 \leq i \leq d_M, \) \( \) \( \) such that \( \left\| \sum_{i=1}^{d_M} a_i \, \pi(\theta_i) \right\|_{L_2} \geq 0, \) \( \) as required by Definition 4.1 so that the HRBB condition holds.

**Theorem 7.1.** The HRBB condition holds, see Definition 4.1 if and only if the collection \( \mathcal{W}_A \) is bounded, see Definition 3.2.

**Proof.** If HRBB holds, see Theorem 6.1 then there exists \( \hat{\psi}_c \in \mathcal{U}_g \) such that \( \hat{\psi}_c \) is Barankin-efficient, and then \( \text{Cov}_{\theta^c} (\hat{\psi}_c) \geq W, \forall W \in \mathcal{W}_A \) so that \( \mathcal{W}_A \) is bounded. The converse follows as a consequence of Lemma 7.1.

**Lemma 7.2.** If there exists a finite covariance unbiased estimator \( \psi(\mathcal{X}) \) for \( g(\theta), \forall \theta \in \Theta \) for Problem 2.3, then \( \mathcal{W}_A \) is bounded above, see Definition 3.2.

**Proof.** Since a finite covariance unbiased estimator \( \psi \) exists, then (3.1) shows that \( \mathcal{W}_A \) is bounded.

### 7.2 Other equivalent LMI bounds

One of the key ideas in Barankin’s paper is the use of the free coefficients \( a_i \)'s, see [2] p. 480, that here take the form of the matrices \( A \)'s. As discussed in [2], and here below, the matrices \( A \) are not required for the determination of the optimal matrix bound, but, they are most useful when one needs to compare the Barankin bound with other bounds, such as Cramer-Rao, Bhattacharyya, etc. For the scalar case see [2] Corollaries 5–1 p. 487 and 6–1 p. 488. For the vector Cramer-Rao bound, compare the results here with e.g. [21] and references there.

**Definition 7.1.** Define the pair \( \mathbf{d} = (d_M, \tau) \), where \( d_M \in \mathbb{N} \), and \( \tau \in \Theta^{d_M} \). Define \( \mathcal{C}_B \) as the collection of all the pairs \( \mathbf{d} \) with \( \text{Det}(B(\tau)) \neq 0 \), with \( B(\tau) \) as in Definition 3.1 so that

\[ \mathcal{C}_B = \{ \mathbf{d} : \forall d_M \in \mathbb{N}, \forall \tau \in \Theta^{d_M}, \text{with Det}(B(\tau)) \neq 0 \} \]

Define \( \mathcal{W}_B \) as the collection of matrices

\[ V(\mathbf{d}) = G(\tau)(B(\tau))^{-1} G^T(\tau) \quad \forall \mathbf{d} \in \mathcal{C}_B \]

with \( G(\tau) \) as in Definition 3.1. Equivalently, \( \mathcal{W}_B = \{ V(\mathbf{d}) : \mathbf{d} \in \mathcal{C}_B \} \).

Define the function \( g(\theta) \) as \( \mathfrak{B}_0 \)-compatible if whenever \( \sum_{i=1}^{d_M} a_i \, \pi(\theta_i) = 0 \) w.p. 1, we have \( \sum_{i=1}^{d_M} a_i \, h(\theta_i) = 0, \) with \( d_M \in \mathbb{N}, a_i \in \mathbb{R}, \theta_i \in \Theta, \) for \( 1 \leq i \leq d_M. \)

Note that if \( g(\theta) \) is not \( \mathfrak{B}_0 \)-compatible then no unbiased estimator exists for \( g(\theta), \forall \theta \in \Theta, \) for Problems 2.1 2.2 or 2.3 If \( g(\theta) \) is \( \mathfrak{B}_0 \)-compatible, then for \( \tau \in \Theta^{d_M} \) and \( \mathbf{a} \in \mathbb{R}^{d_M}, \) if \( \mathbf{a}^T \beta(\tau) = 0, \) then \( G(\tau) \mathbf{a} = 0, \) and for \( \mathbf{A} \in \mathbb{R}^{d_A \times d_M}, \) if \( \mathbf{A} \beta(\tau) = 0, \) then \( G(\tau) \mathbf{A}^T = 0. \) Hence, if \( \tau_1 \in \Theta^{d_A} \) and we have \( \mathbf{A} \beta(\tau) = \beta(\tau_1), \) then \( G(\tau) \mathbf{A}^T = G(\tau_1). \)
Theorem 7.2. The collection $\mathcal{W}_A$, see Definition 32, is bounded above, if and only if the collection $\mathcal{W}_B$ is bounded above and $g(θ)$ is $\mathcal{B}_0$-compatible.

Proof. Assume $\mathcal{W}_A$ is bounded. Then, there exists a s.n.n.d. matrix $B_1 \in \mathbb{R}^{d_p \times d_p}$, such that $B_1 \geq W(q)$, $\forall q \in \mathcal{C}_A$. Take an arbitrary $d' \in \mathcal{C}_B$, with $d' = (d'_M, \tau')$, so that $\det(B(\tau')) \neq 0$. Define $q' = (d'_M, d'_M, I_{d'_M}, \tau')$, where $I_{d'_M}$ is the identity matrix of dimensions $d'_M \times d'_M$. Since $\det(I_{d'_M}B(\tau')I_{d'_M}^T) = \det(B(\tau')) \neq 0$, then $q' \in \mathcal{C}_A$, and we have $W(q') = V(d')$, so that $B_1 \geq V(d')$, $\forall d' \in \mathcal{C}_B$, and then $\mathcal{W}_B$ is bounded. Since $\mathcal{W}_A$ is bounded, then the HRBB condition holds, see Lemma 7.1, and then (4.2) shows that $g(θ)$ is $\mathcal{B}_0$-compatible. Conversely, assume $\mathcal{W}_B$ is bounded. Then, there exists a s.n.n.d. matrix $B_2 \in \mathbb{R}^{d_p \times d_p}$, such that $B_2 \geq V(d)$, $\forall d \in \mathcal{C}_B$. Take an arbitrary $q' \in \mathcal{C}_A$, with $q' = (d'_M, d'_A, A', \tau')$, with $A' \in \mathbb{R}^{d'_A \times d'_M}$, so that $\det(A' B(\tau') (A')^T) \neq 0$. As in the proof of the last part of Theorem 3.1, obtain $d'_{\tau} \in \mathbb{N}$, $1 \leq d'_{\tau} \leq d'_M$, $A' \in \mathbb{R}^{d'_A \times d'_{\tau}}$, and $\tau' \in \Theta^{d'_{\tau}}$, by elimination of the components of the vector $\beta(\tau')$ which are linear combinations w.p. 1 of previous components, so that $\beta(\tau') = A^* \beta(\tau^*)$, with $\det(B(\tau^*)) = \det(\mathbb{E}_{θ_1}[\beta(\tau') \beta^T(\tau')]) \neq 0$.

Then, $B(\tau') = \mathbb{E}_{θ_1}[\beta(\tau') \beta^T(\tau')] = A^* \mathbb{E}_{θ_1}[\beta(\tau') \beta^T(\tau^*)] (A^*)^T = A^* B(\tau^*) (A^*)^T$. Since $g(θ)$ is $\mathcal{B}_0$-compatible, then $G(\tau') = G(\tau^*) (A^*)^T$. Hence,

$$W(q') = G(\tau') (A^*)^T (A' B(\tau') (A')^T)^{-1} A' G(\tau') = G(\tau^*) (A^*)^T (A' A^* B(\tau^*) (A^*)^T (A')^T)^{-1} A' A^* G(\tau^*)$$

Define $d^* = (d_{\tau^*}, \tau^*)$ so that $V(d^*) = G(\tau^*)B^{-1}(\tau^*)G(\tau^*)$. Then, the Appendix Lemma A.4 shows that $V(d^*) \geq W(q')$, so that $B_2 \geq V(d^*) \geq W(q')$. Hence $B_2 \geq W(q')$, for all $q' \in \mathcal{C}_A$, so that $\mathcal{W}_A$ is bounded. □

For an arbitrary symmetric matrix $W$ call $\lambda_M(W) \in \mathbb{R}$ its greatest eigenvalue. The operator norm $\|A\|_{op}$ of a matrix $A \in \mathbb{R}^{N \times M}$ is its greatest singular value, 21 p. 12, i.e. the non-negative square root of the greatest eigenvalue of the matrix $A^T A$, so that $\|A\|_{op} = (\lambda_M(A^T A))^{1/2}$. For s.n.n.d. matrices singular values and eigenvalues coincide, 23 p. 19, so that if $W \in \mathcal{W}'$, then $\|W\|_{op} = \lambda_M(W)$. Define a k-identity matrix as a matrix of the form $K M$, where $K \in \mathbb{R}$ and $I_M$ is the identity matrix of dimensions $M \times M$. Then, we have

Lemma 7.3. If $X$ is a s.n.n.d. matrix, $X \in \mathbb{R}^{M \times M}$, then, for $K \in \mathbb{R}$, we have $K I_M \geq X$ if and only if $K \geq \lambda_M(X)$.

Proof. Since $X$ is symmetric, then it is diagonalizable, so that there exist an orthogonal matrix $Q \in \mathbb{R}^{M \times M}$, and a diagonal matrix $Λ \in \mathbb{R}^{M \times M}$, such that $X = Q Λ Q^T$. Then, since $\lambda_M(X) I_M \geq Λ$, we have $\lambda_M(X) I_M = Q \lambda_M(X) I_M Q^T \geq Q Λ Q^T = X$. Hence, if $K \geq \lambda_M(X)$, then $K I_M \geq \lambda_M(X) I_M \geq X$. Conversely if $K I_M \geq X$, then $K I_M \geq Q Λ Q^T$, so that
Lemma 7.4. A non-empty collection $\mathcal{W}$ of s.n.n.d. matrices $W \in \mathbb{R}^{M \times M}$ is upper bounded if and only if there exists $K_{\mathcal{W}} \in \mathbb{R}$, such that $K_{\mathcal{W}} I_M \geq W$, $\forall W \in \mathcal{W}$.

Proof. If there exists $K_{\mathcal{W}} \in \mathbb{R}$, such that $K_{\mathcal{W}} I_M \geq W$, $\forall W \in \mathcal{W}$, then by definition $K_{\mathcal{W}} I_M$ is a matrix bound for $\mathcal{W}$, and then $\mathcal{W}$ is bounded. Conversely, assume $\mathcal{W}$ is bounded. Then there exists $B_{\mathcal{W}} \in \mathbb{R}^{M \times M}$ s.n.n.d., such that $B_{\mathcal{W}} \geq W$, $\forall W \in \mathcal{W}$. Since $B_{\mathcal{W}}$ is symmetric, then there exists the real maximum eigenvalue $\lambda_M(B_{\mathcal{W}}) \in \mathbb{R}$. Take $K_{\mathcal{W}} \in \mathbb{R}$ such that $K_{\mathcal{W}} \geq \lambda_M(B_{\mathcal{W}})$. Then, from Lemma 7.3, $K_{\mathcal{W}} I_M \geq B_{\mathcal{W}} \geq W$, $\forall W \in \mathcal{W}$.

7.3 Main Theorem

Collecting all the previous results, we have

Theorem 7.3 (Main Theorem). The following statements for Problem 2.3 are equivalent, meaning that if any one of them is true, then they are all true:

1. A finite covariance vector unbiased estimator exists, i.e. $\mathcal{U}_g$ is not empty.
2. A Barankin-efficient vector unbiased estimator exists.
3. The HRBB condition holds.
4. The collection $\mathcal{W}_A$ is bounded.
5. There exists $K \in \mathbb{R}^+$ such that $K I_{d_p} \geq W$, $\forall W \in \mathcal{W}_A$.
6. The collection $\mathcal{W}_B$ is bounded, and $g(\theta)$ is $\mathcal{B}_0$-compatible.
7. There exists $K \in \mathbb{R}^+$ such that $K I_{d_p} \geq W$, $\forall W \in \mathcal{W}_B$, and $g(\theta)$ is $\mathcal{B}_0$-compatible.

Proof. $3 \Rightarrow 2 \Rightarrow 1$ follows from Theorem 6.1 $3 \Leftrightarrow 4$ from Theorem 7.1 $1 \Rightarrow 4$ from Lemma 7.2 $4 \Leftrightarrow 5$ and $6 \Leftrightarrow 7$ from Lemma 7.4 finally $4 \Leftrightarrow 6$ follows from Theorem 7.2.

Corollary 7.3.1. As a corollary, the collection $\mathcal{U}_g$ is empty iff $\mathcal{W}_A$ is not bounded, i.e. for each $k \in \mathbb{N}$ there exists $W^*_k \in \mathcal{W}_A$ such that $\|W^*_k\|_F \geq \|W^*_k\|_{op} = \lambda_M(W^*_k) > k$, so that $\lim_{k \to +\infty} \|W^*_k\|_F = +\infty$. Note that $\mathcal{U}_g$ is empty either because there are no unbiased estimators for $g(\theta)$, $\forall \theta \in \Theta$, or if there exist, they don’t have finite covariance matrix at $\theta_T$, see Definition 3.2.
A APPENDIX

Lemma A.1. Let \( \mathcal{H} \) be an arbitrary Hilbert space. Let the elements \( u_i \in \mathcal{H}, \) for \( 1 \leq i \leq d_L < +\infty, \) be orthonormal so that \( \|u_i\| = 1, \) for \( 1 \leq i \leq d_L, \) and \( \langle u_i, u_j \rangle = 0, \) for \( i \neq j, 1 \leq i, j \leq d_L, \) and then \( \langle u_i, u_j \rangle = \delta_{i,j}. \) Assume that for each \( u_i, \) for \( 1 \leq i \leq d_L, \) there exist sequences \( s_i(m) \) with \( s_i(m) \in \mathcal{H}, \) \( \forall m \in \mathbb{N}, \) for \( 1 \leq i \leq d_L, \) such that \( \lim_{m \to \infty} \|u_i - s_i(m)\| = 0, \) for \( 1 \leq i \leq d_L. \)

Define \( S(m) \in \mathbb{R}^{d_L \times d_L}, \) as a matrix with \( i \)-th, \( j \)-th element \( [S(m)]_{ij} = \langle s_i(m), s_j(m) \rangle \) for \( 1 \leq i, j \leq d_L. \)

1. \( \|S(m) - I_{d_L}\|_F \to 0, \) as \( m \to \infty, \) where \( I_{d_L} \) is the identity matrix of dimensions \( d_L \times d_L. \)

2. \( \det(S(m)) \to 1, \) and \( \|S^{-1}(m) - I_{d_L}\|_F \to 0, \) as \( m \to \infty. \)

3. \( \lim_{m \to \infty} \langle u_i - s_i(m), s_j(m) \rangle = 0, \) for all \( 1 \leq i, j \leq d_L. \)

Proof. a) From the Cauchy-Schwarz inequality we obtain \( \|u_i - s_i(m)\|_{\mathcal{H}} \leq \|u_i - s_i(m)\|_{\mathcal{H}}, \) so that \( \lim_{m \to \infty} \langle u_i - s_i(m), u_j \rangle = 0, \) for all \( 1 \leq i, j \leq d_L. \)

b) Also \( \|u_i - s_i(m), u_j - s_j(m)\|_{\mathcal{H}} \leq \|u_i - s_i(m)\|_{\mathcal{H}} \|u_j - s_j(m)\|_{\mathcal{H}}, \) and then \( \lim_{m \to \infty} \langle u_i - s_i(m), u_j - s_j(m)\rangle = 0, \) for all \( 1 \leq i, j \leq d_L. \)

c) We have \( \langle s_i(m), s_j(m) \rangle = \langle s_i(m) - u_i + u_i, s_j(m) - u_j + u_j \rangle = \langle s_i(m) - u_i, s_j(m) - u_j \rangle + \langle u_i, s_j(m) - u_j \rangle + \langle s_i(m) - u_i, u_j \rangle + \langle u_i, u_j \rangle. \) Taking the limit, and using a) and b), \( \lim_{m \to \infty} \langle s_i(m), s_j(m) \rangle = \delta_{i,j}, \) for all \( 1 \leq i, j \leq d_L. \) Then \( S(m) \to I_{d_L} \) component by component, and then in Frobenius norm. Since the determinant of a matrix is an algebraic sum of a finite number of products of a finite number of elements of the matrix, see [5] p. 319, then \( \det(S(m)) \to \det(I_{d_L}) = 1, \) so that \( \exists M_0 \in \mathbb{N}, \) such that \( \forall m \geq M_0, \) it will be \( \det(S(m)) \geq 1/2, \) and then \( \det(S(m)) \neq 0. \) Similarly, since the elements of the inverse of a matrix are the quotients of algebraic sums of a finite number of products of a finite number of elements of the matrix divided the determinant, see [5] p. 325, then \( S^{-1}(m) \) has a limit \( A_0 \) component by component, so that \( S(m) S^{-1}(m) \to I_{d_L}, \) \( \forall m \in \mathbb{N}, \) then \( A_0 = I_{d_L}, \) and then \( S^{-1}(m) \to I_{d_L} \) component by component as \( m \to \infty, \) and then in Frobenius norm, or any other matrix norm, so that we have shown items [1] and [2].

d) We have \( \langle u_i - s_i(m), s_j(m) \rangle = \langle u_i - s_i(m), s_j(m) - u_j + u_j \rangle = \langle u_i - s_i(m), s_j(m) - u_j \rangle + \langle u_i, s_j(m) - u_j \rangle + \langle u_i - s_i(m), u_j \rangle. \) Applying a) and b) we obtain \( \lim_{m \to \infty} \langle u_i - s_i(m), s_j(m) \rangle = 0, \) for all \( 1 \leq i, j \leq d_L, \) so that we have shown item [3].

\[ \square \]

Lemma A.2. Let \( \{A_n\}_{n \in \mathbb{N}} \) be a sequence of s.n.d. matrices \( A_n \in \mathbb{R}^{N \times N}, \) \( A_n \geq 0, \forall n \in \mathbb{N}, \) such that there exists \( A \in \mathbb{R}^{N \times N} \) for which \( A_n \to A \) c.b.c and then in Frobenius norm. Then the matrix \( A \) is s.n.d.
Proof. Take $\alpha \in \mathbb{R}^N$ arbitrary, since $N$ is finite, then $\lim_{n \to +\infty} \alpha^T A_n \alpha = \alpha^T \alpha$. But $\alpha^T A_n \alpha \geq 0$, $\forall n \in \mathbb{N}$, so that $\lim_{n \to +\infty} \alpha^T A_n \alpha \geq 0$, and then $\alpha^T \alpha \geq 0$, $\forall \alpha \in \mathbb{R}^N$.

The following lemma is a LMI weighted form of the Cauchy-Schwarz inequality for matrices, [8] p. 1093. For convenience, a proof is given here.

**Lemma A.3.** Let $M, N, P \in \mathbb{N}$. Let $H \in \mathbb{R}^{M \times M}$ be an arbitrary real s.p.d. matrix, and let $X \in \mathbb{R}^{N \times M}$ and $Y \in \mathbb{R}^{P \times M}$, be otherwise arbitrary real matrices such that $\text{det}(Y H^T Y^T) \neq 0$. Then

$$X H X^T \geq X H Y^T (Y H Y^T)^{-1} Y H X^T$$

with equality if and only if there exists $\Lambda \in \mathbb{R}^{N \times P}$, such that $X = \Lambda Y$, if and only if

$$X = X H Y^T (Y H Y^T)^{-1} Y$$

Proof. Let $\Lambda \in \mathbb{R}^{N \times P}$. Define $T(\Lambda) = (X - \Lambda Y) H (X - \Lambda Y)^T$, so that $T(\Lambda)$ is s.n.n.d., $\forall \Lambda \in \mathbb{R}^{N \times P}$. Define

$$R(\Lambda) = \left( \Lambda - X H Y^T (Y H Y^T)^{-1} \right) Y H Y^T \left( \Lambda - X H Y^T (Y H Y^T)^{-1} \right)^T$$

and $D = X H X^T - X H Y^T (Y H Y^T)^{-1} Y H X^T$. Note that $R(\Lambda) \geq 0$, $\forall \Lambda \in \mathbb{R}^{N \times P}$. Then $T(\Lambda) = R(\Lambda) + D \geq 0$, $\forall \Lambda \in \mathbb{R}^{N \times P}$. For $\Lambda_0 = X H Y^T (Y H Y^T)^{-1}$, we have $R(\Lambda_0) = 0$, so that $T(\Lambda_0) = D \geq 0$, and then the LMI is obtained. If there is equality then $D = 0$, and then $T(\Lambda) = R(\Lambda)$, $\forall \Lambda \in \mathbb{R}^{N \times P}$. In particular for $\Lambda_0$ we have $R(\Lambda_0) = 0$, and then $T(\Lambda_0) = 0$.

But, since $H$ is s.p.d. then $X = \Lambda_0 Y = X H Y^T (Y H Y^T)^{-1} Y$. As for the converse, if there exists $\Lambda_1$ such that $X = \Lambda_1 Y$, then $T(\Lambda_1) = 0$, since $R(\Lambda_1) \geq 0$ by definition, and $D \geq 0$ as previously shown, then, $R(\Lambda_1) = 0$ and $D = 0$, because $T(\Lambda_1) = R(\Lambda_1) + D$. From $D = 0$ we obtain the equality in the LMI inequality, and from $R(\Lambda_1) = 0$, we obtain that $\Lambda_1 = \Lambda_0$, because $\text{det} (Y H Y^T) \neq 0$ and then $Y H Y^T$ is s.p.d. If $X = X H Y^T (Y H Y^T)^{-1} Y$, multiply both sides on the right by $H X^T$, and then the equality for the LMI is obtained.

The following lemma, cf. [9] Lemma 2.4.1, may be interpreted as a LMI generalization of the Rayleigh quotient, [12] p. 117.

**Lemma A.4.** Let $M, N, P \in \mathbb{N}$. Let $B \in \mathbb{R}^{M \times M}$ be an arbitrary real s.p.d. matrix, and let $G \in \mathbb{R}^{N \times M}$ and $A \in \mathbb{R}^{P \times M}$, be otherwise arbitrary real matrices such that $\text{det}(ABA^T) \neq 0$. Then

$$G B^{-1} G^T \geq G A^T (A B A^T)^{-1} A G^T$$

with equality if and only if there exists $\Lambda_0 \in \mathbb{R}^{N \times P}$, such that $G = \Lambda_0 A B$, if and only if

$$G = G A^T (A B A^T)^{-1} A B$$
Proof. Since $B$ is s.p.d. then it has a unique s.p.d. square root $B^{1/2}$, p. 405, and we have $\det(B) \neq 0$ and $\det(B^{1/2}) \neq 0$. The result follows from the previous Lemma [A.3] taking, $X = G B^{-1/2}$, $Y = A B^{1/2}$, and $H$ as the identity matrix of dimensions $M \times M$. \qed

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