Non-stationary Noise Sources in an Optoacoustical Gravitational Antenna

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Abstract. The optoelectronic Pound-Drever-Hall scheme is considered as a registration system of small variations of length and eigenfrequency of reference Fabry-Perot resonator with base of 2 m fixed on the aluminum test body of the GW antenna. In uniform sensitivity analysis shot noise of photoelectrons, intrinsic technical LF laser stochastic frequency deviations and power fluctuations (PF) are considered. Mechanism of PF penetration into the precise feedback chain is pointed out. It is shown that the carrier of PF noise is the quasistatic error signal of the laser frequency tracking system following the drifting frequency of the resonator, since this voltage is formed by synchronous detector from the photodiode current. The frequency discriminator channel of the installation OGRAN can also introduce PF noise, since the output of it’s synchronous detector contains significant LF noise (10÷300 Hz) voltage created by industrial seismic and vibration disturbances.

1. Introduction

The basis of a bar gravitational wave (GW) antenna [1] is a cylindrical aluminum acoustic resonator as a test body. Gravitational field metric perturbations \( h(t) \) are assumed in form \( h = h_m \cos \omega t \) with limited exposure time \( \tau \) [2-4]. Frequency \( \omega \) should be fairly close to bar eigen frequency \( \omega_n \). Effect of GW is reduced to equivalent harmonic signal force with amplitude [2]

\[
F_{sm} = h_m M \omega_n^2 \frac{L_g}{2},
\]

(1)

where \( M \) is an equivalent oscillator mass and \( L_g \) is a bar length.

Small variations in bar length are registered by a sensitive measuring system (measurer). During the development of the project OGRAN, progress is achieved by reducing the threshold signal (resolution) \( \chi_{min} \) of the measurer \((S/N = 1)\) as it’s single relevant parameter.

In absence of a signal, the measurer registers thermal vibrations of the test body. At the antenna’s output they look as a harmonic process with stochastically slowly varying amplitude. The corresponding spectrogram is presented in article [1] in Figure 7. Registration of this process allows us to control achieved measurer resolution as a result of modifications and adjustments. Spectrogram shows noises of the test body and the measurer in common. And anyone can see noise “pedestal” at level
\[ 7 \times 10^{-15} \text{ cm/Hz}^{1/2} (\sim 4 \cdot 10^{-5} \text{ Hz/(Hz)}^{1/2}) \]. The aim is to show that this noise level determines uniquely the numerical value of the new parameter introduced for the first time in the article [1]: bands of 4 Hz for \( h \approx 10^{-19} \text{ Hz}^{1/2} \). Accordingly, this value should be considered as the achieved resolution of the measurer. An explanation of noise origin is offered.

2. Resolution of the measurer; phenomenology

The main result of article [1] is presented in the form of a graph of the spectral density (SD) of the metric in Figure 8. Certain numerical characteristic of this dependence is represented.

To compose the theoretical forecast of this spectrogram we should to describe noise and dynamic parameters of the test body and to write out the equation of an equivalent oscillator

\[ M \ddot{x} + H \mu \dot{x}^2 + k \mu x = F_s + F_T. \]  

(2)

The Nyquist force \( F_N \) has spectral density [1,3]

\[ G_T(f) = 2 \pi G_T(\omega) = 4 k T M \omega^2 = \frac{4 k T M \omega_\mu^2}{Q}. \]  

(3)

Near the resonance there is a dynamic transfer characteristic

\[ \frac{x_n}{F_n} = K_{nx}(\omega) = \frac{Q / M \omega_\mu^2}{1 + (\omega - \omega_\mu)^2 / \delta_\mu}. \]  

(4)

Spectral densities of force and displacements are related: \( G_{nx}(f) = F_n^2 G_T(f) \).

The expression of bar thermal noise follows from expressions (3), (4). With the addition of the phenomenological noise of measurer (sensor) \( G_0(f) \) (as a “pedestal”), we obtain forecast

\[ G_0(f) = G_{nx}(f) + G_0(f) = G_{00} \left[ 1 + \left( \frac{\Delta \omega}{\delta_\mu} \right)^2 \right]^{-1} + G_0(f), \]  

(5)

\[ G_{00} = \frac{4 k T Q}{M \omega_\mu}. \]  

(6)

Expression (5) describes the experimental spectrogram in Figure 7 in ref. [1].

We assert further that spectral density in the Figure 8 has been obtained completely by recalculating of the experimental dependence in the Figure 7:

\[ G_h(f_0 + \Delta f) = G_x(f_0 + \Delta f) / K_{nx}(f_0 + \Delta f). \]  

(7)

Here the transfer function \( K_{nx} \) is defined by expressions (1) and (4):

\[ K_{nx}(f_\mu + \Delta f) = x_n(f_\mu + \Delta f) / l_n = Q L_0 / 2 k(\Delta f) = K_{nx0} k(\Delta f). \]

Here we had considered \( \Delta \omega = \omega - \omega_\mu \), \( |\Delta \omega| \leq 3 \delta_\mu \), \( \Delta f = f - f_\mu \), \( k(\Delta f) = \frac{\Delta \omega / \delta_\mu}{|\Delta f|} \).

The meaning of the new entered band of 4 Hz [1] is a difference between two frequency values as points on the abscissa corresponding to the ordinate value \( (G_h)^{1/2} = 10^{-19} \text{ Hz}^{1/2} \).

Let's prepare relevant numerical parameters. For the values of \( Q = 10^4 \), \( L_0 = 100 \text{ cm}, \ f_0 = 1.3 \cdot 10^3 \text{ Hz} \) [1] we have \( K_{nx0} = 10^7 \text{ cm}, \ \Delta f_0 = \frac{l_N}{Q} = 1.3 \cdot 10^{-2} \text{ Hz} \). Then for the value \( 2 \Delta f^* = 4 \text{Hz} \) of frequency deviation in

\[ \sim 3 \cdot 10^{-15} \text{ cm/Hz}^{1/2} (\sim 4 \cdot 10^{-5} \text{ Hz/(Hz)}^{1/2}) \]. The aim is to show that this noise level determines uniquely the numerical value of the new parameter introduced for the first time in the article [1]: bands of 4 Hz for \( h \approx 10^{-19} \text{ Hz}^{1/2} \). Accordingly, this value should be considered as the achieved resolution of the measurer. An explanation of noise origin is offered.

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each branch, we find \( k(\Delta f) = 3.25 \cdot 10^{-3} \). From formula (7), substituting also \( (G_h)^{1/2} = 10^{-19} \text{ Hz}^{1/2} \), we find \( (G_x)^{1/2} \approx 3 \cdot 10^{-15} \text{ cm/Hz}^{1/2} \). This is just the value that can be seen as the “pedestal” \( (G_s)^{1/2} \) of the spectrogram in Figure 7 and is the main real registered resolution value of the antenna’s laser measurer.

To confirm this approval, we ascertain the fact that in Figure 7 [1] one may discover \( (G_{r0})^{1/2} \approx 10^{-13} \text{ cm/Hz}^{1/2} \). For \( \Delta f^* = 2 \text{ Hz} \) we find forecast estimate:

\[
(G_{xt})^{1/2} = (G_{r0})^{1/2} k(\Delta f) \approx 3 \cdot 10^{-16} \text{ cm/Hz}^{1/2} \ll (G_s)^{1/2}.
\]

It means \( G_x = G_s \). Those two points corresponds to area of the “pedestal”.

The condition \( G_{xt} = G_s \) is satisfied with the value \( \Delta f^* = 0.2 \text{ Hz} \). Within this band \( (|\Delta f| < 0.2 \text{ Hz}) \) bar thermal noise dominates. Here the relation (7) gives the analytical expression

\[
G_x(f) = \left( \frac{4}{L} \right)^2 \frac{kT}{MQ\omega^2}.
\]

This coincides with the resultant expression of the antenna’s theory [1, 3]:

\[
h_{\text{min}} \approx \left( 4/L_w \right) \left( \frac{kT}{M\omega^2 Q} \right)^{1/2} \approx 10^{-20} \sqrt{F\Delta f} \text{ Hz}^{-1/2}.
\]

In addition to confirmation, for the peak of the bar thermal spectrum, at the frequency \( f_0 \), where \( k(\Delta f) = 1 \), we find \( (G_{r0})^{1/2} = (G_{r0})^{1/2} = 10^{-20} \text{ Hz}^{1/2} \); it exactly coincides to the numerical estimate in expression (8).

3. Classic basics of antenna sensitivity analysis

In the basic formula (8), the contribution of the optoelectronic measurer is introduced by the noise factor

\[
F = \frac{2M}{\tau} \left( \frac{G_e}{G_r} \right)^{1/2}.
\]

Parameters of a conceptual schema are presented [1,3]:

\[
G_e = B\omega^2 \left[ 2h\nu / (\eta P) \right] \left[ \lambda / (2\pi N) \right],
\]

where \( \lambda \) is wavelength of laser radiation, \( \eta \sim 0.7 \) is the quantum yield of the photodiode, \( N \) is an “effective number of reflections” in the FP resonator, \( P \) is the optical power and “\( B = 1 - 10^3 \)” is the phenomenological factor that indicates by how many times exceeds the Poisson level” [1].

To highlight the characteristics of this meter, consider the formula in ref. [3]:

\[
(S / N)_{\text{tot}} = \pi \int \frac{1}{G_r + \left( G_{r0} \right)^2} \frac{\omega^2}{G_e} d\omega.
\]

It contains expression for SD of equivalent total stochastic force \( F_N \) in the equation (2):

\[
G_x = G_r + \left| Z_{r0} \right|^2 G_e.
\]

Applying it, as above, to the dynamic system (4), we obtain

\[
G_x = \frac{G_r}{\omega^2} \left| Z_{r0} \right|^2 + \omega^2 G_e = G_{xt} (\omega) + G_s (\omega).
\]

And analytical expression for meter resolution in expression (5) is extracted: \( G^* = \omega^2 G_e \),

\[
G^* (\omega) = \frac{\lambda}{\pi N} \sqrt{2h\nu / 2\eta P} B = \frac{\lambda}{\sqrt{2F}} \sqrt{\frac{h\nu}{\eta P} B}.
\]
In the last expression are taken into account relations: \( N = \frac{1}{1 - R_m} \) and \( F = \frac{\pi}{1 - R_m} \) as finesse [5], where 

\( R_m \) is an energy reflection coefficient in mirrors of Fabry-Perot (FP) resonator.

On the other hand, there is an article [4], in which a meter on an FP interferometer had been presented and described briefly; it’s threshold signal for \( \eta = 1 \) is represented by the formula:

\[
x_{\text{min}} = \frac{(1 - R_m)c}{2\pi} \sqrt{\frac{2h}{N_\eta \nu_0 \tau}} A = \frac{\lambda}{2F} \sqrt{\frac{2hv}{P}} \Delta f .
\]

(10)

Here laser noise is represented by a numerical factor \( A \) too.

In this article the FP resonator mirrors are attached to ends a bar; it’s length variation \( \delta x \) cause locked mode natural frequency variations of an interferometer \( \delta \nu_R \). For enough hard mount mirrors the relation is:

\[
\frac{\delta x}{L_G} = \frac{\delta \nu_R}{\nu_R}.
\]

Registered variations of frequency allow us to determine variations in length. In article [1] this relationship is expressed in the left and right scales of noise spectrograms. For \( L_G = 200 \) cm and \( \nu_R = 2.8 \cdot 10^{14} \) Hz we have for the scales the conversion factor value \( \frac{\delta x}{\delta \nu_R} = 0.7 \cdot 10^{-12} \) cm/Hz.

Condition \( A = B = 1 \) in formulas (9) and (10) establishes the resolution limit, determined by shot noise of photoelectrons in a photodiode.

4. Laser noises

To remove some uncertainty in resolution sensitivity forecast \( (B = 1 + 10^{3}) \) an additional study of laser noise characteristics is actual. So, this noise also requires consideration in heterodyne (homodyne) receiving of optical signals. In articles [6,7] and in the technical essay [8] there had been studied fluctuations as “amplitude chaotic modulation of the laser radiation: \( P(t) = P_0 [1 + \xi(t)] \), where \( \xi(t) \) is a dimensionless modulating process” having the spectral density \( m_\xi(f) \). LF power fluctuation had been measured for the laser used in OGRAN pilot model [9], represented and applied for analysis [10].

Pound-Drever-Hall (PDH) technique using modulation-demodulation is applied to measure displacements in GW detector AURIGA [11]. Low frequency (LF) laser power fluctuations (LPF) in the noise budget are replaced by negligible RF LPF.

In reports [12,13] there are introduced into consideration phenomenological LF stochastic laser frequency deviations (SLFD). In AURIGA project their influence is negligible in consequence of smallness of the interferometer base – \( L_{\nu_\nu} = 0.1 \) mm. On the other hand, the detector AURIGA is sensitive to LPF. They are under control because they create fluctuations in the light pressure force in the resonant displacement transformer.

Spectral density of SLFD had been measured approximately: \( (S_\nu)_{1/2} \approx 10 \) Hz/(Hz)\(^{1/2}\) at the antenna operating frequency of 1.3 kHz [13]. In ref. [9] on the graph of noise spectra one can see value \( (S_\nu)_{1/2} \approx 4 \) Hz/(Hz)\(^{1/2}\).

This noise influence is suppressed by stabilizing laser frequency applying a reference FP interferometer; the GWA resonator just performs this function. For this reason the enlarged gain factor \( K_{05} \) has required in locked feedback circuit [13]; the relevant theory is represented below. The value \( K_{05} = 10^3 \) at 1.3 kHz is implemented in the OGRAN installation [1,9,13], and calculated noise component in laser frequency
deviations is \( \left( \frac{S_{o}}{K_{o}} \right)^{1/2} = \frac{1}{2} \left( \frac{S_{o}}{K_{o}} \right)^{1/2} \approx 4 \cdot 10^{-15} \text{ Hz/(Hz)}^{1/2} \). The conversion factor \( 0.7 \cdot 10^{-12} \text{ cm/Hz} \) gives the resolution forecast for the OGRAN measurer: \( x_{\text{min}} = 3 \cdot 10^{-15} \text{ cm/(Hz)}^{1/2} \); this satisfactory corresponds to the test result, which had been published implicitly, without interpretation \([1, \text{ Figure 7}]\). This result has been revealed above.

Further efforts on suppression of SLFD had not succeeded yet.

To discover an additional noise source there was revealed an effect of residual amplitude modulation (RAM) arising in an imperfect RF modulator \([10, 14]\). Certain efforts had been made to suppress it \([14]\). As a result (see below), RAM noise is assessed as not primary.

Identifying a second powerful source of technical noise is required.

5. PDH scheme sensitivity analysis

In ref.1 RAM noise is pointed out as the primary technical noise. Meanwhile, it’s theory and calculation has not presented. The theory is relevant. Also, introduced non-stationary sources of noise are similar in nature to RAM noise. Therefore, it is relevant to present and apply a universal instrument of sensitivity analysis. It includes conjoint consideration of photoelectron shot noise \( (B=1) \) and mechanisms of penetration of the laser noise into the precise chain.

This is an improved version of the analysis used in the report \([12]\).

The Pound-Drever-Hall (PDH) scheme \([5]\) is presented at Figure 1. It is a laser frequency retuning and stabilization system.

**Figure 1.** 1 – Electro-optical modulator, 2 – photodiode, 3 – current preamplifier, 4 – RF selective filter-amplifier, 5 – servo DC amplifier, OSC – bearing RF generator, SD – synchronous detector.

It uses RF phase modulation of laser radiation and forms a discrimination feature at the output of a synchronous detector (SD); laser frequency \( v_{L}(t) \) follows promptly FP resonator intrinsic frequency \( v_{o}(t) \). Frequency variations \( \delta v_{L} \) containing signal and noise are formed.

Further the electro-optical discriminatory channel converts laser frequency variations into output voltage variations of antennae’s measurer on the slope of it’s discrimination feature.

A beam with power \( P_{IN} \) falls on the interferometer. After electro-optical RF phase modulator 1 we have three spectral components \([5]\):

\[
E_{me} \approx E_{0} \left[ J_{0}(\beta) e^{i\omega t} + J_{1}(\beta) e^{i(\omega + \Omega_{o})t} - J_{1}(\beta) e^{i(\omega - \Omega_{o})t} \right].
\]
Their powers are related: $P_+ + 2P_\mp \equiv P_{iv}$. In a simplified treatment [5], the condition is $R_n + A_n = 1$, where $A_n$ is an energy penetration coefficient in mirrors.

The power in the beam reflected from the interferometer and fallen on photodiode (PD) 2 is given by the expression [5]:

$$P_{ref} = P_+ \left( \frac{\Delta v_{rl}}{\Delta v_{r0}} \right)^2 + 2P_\mp + D_{pg} \Delta v_{rl} \cos \Omega_0 t. \quad (11)$$

Here $D_{pg} = 8 \left( \frac{P_+ P_\mp}{\Delta v_{r0}} \right)^{1/2}$ – power decrement, $\Delta v_{rl} = v_k - v_L$, $\Delta v_{r0}$ is bandwidth of the resonator at the level $-3$ dB; $\Delta v_{r0} = \frac{\Delta v_{fsr}}{F}$, $\Delta v_{fsr} = c/2L$ - free spectral range. Sidebands $P_s$ are reflected in 100%.

Assuming the first term in (11) to be small, for the persistent component of the power on the photodiode we have the relation: $P_+ = 2P_0$.

In reality we have laser power fluctuation and residual amplitude modulation:

$$P_{in} = P_{in0} \left[ 1 + \xi(t) \right] \left( 1 + m_{rc} \cos \Omega_0 t + m_{rs} \sin \Omega_0 t \right). \quad (12)$$

The typical value of the modulation depth is [10]:

$$\left( m_{rc}^2 + m_{rs}^2 \right)^{1/2} \approx (1/2)\%.$$

On condition $\Delta v_{rl} \ll \Delta v_{r0}$ from (11) we find

$$P_{ref} = P_{ph} = 2P_{in0} \left[ 1 + \xi(t) \right] \left( 1 + m_{rc} \cos \Omega_0 t + m_{rs} \sin \Omega_0 t \right) + D_{pg} \Delta v_{rl} \cos \Omega_0 t.$$

At PD variation components of power $P_\pm$ and current $I_\pm$ are

$$P_\pm = D_{pg} \Delta v_{rl} \cos \Omega_0 t \pm 2P_{in0} \left[ 1 + \xi(t) \right] \left( m_{rc} \cos \Omega_0 t + m_{rs} \sin \Omega_0 t \right),$$

$$I_\pm = \frac{\eta e}{h v_L} P_\pm + I_{sh},$$

Expression for persistent current components of PD has the form: $I_0 = 2\frac{\eta e}{h v_L} P_{in0}$, where $h v_L$ is quantum energy, $e$ is an electron charge.

After low noise current amplifier 3 with gain $R_A$ broadband shot noise $I_{sh}$ arrives at a narrowband filter-amplifier. At its output, the noise voltage becomes a narrow-band stochastic process: $U_{sh}(t) = K_F R_s I_{sh} = U_{ac}(t) \cos \Omega_0 t + U_{as}(t) \sin \Omega_0 t$, where quadrature components of the complex amplitude are contained.

Input voltage of the SD 4 is given by

$$U_{sd,in} = K_F R_A \left[ \frac{\eta e}{h v_L} D_{pg} \Delta v_{rl} \cos \Omega_0 t + I_0 \left[ 1 + \xi(t) \right] \left( m_{rc} \cos \Omega_0 t + m_{rs} \sin \Omega_0 t \right) \right] + U_{ac} \cos \Omega_0 t + U_{as} \sin \Omega_0 t.$$

The synchronous detector converts the harmonic voltage amplitude into persistent and LF voltage: $U_{ac} \cos \Omega_0 t \to K_{sd} U_{ac}$; $U_{as} \sin \Omega_0 t \to 0$.

Then at the SD output we have

$$U_{sd,out} = K_{sd} K_F R_A \left[ \frac{\eta e}{h v_L} D_{pg} \Delta v_{rl} + I_0 m_{rc} \left( 1 + \xi(t) \right) + I_{ac} \right] = S + N. \quad (13)$$
Here \( I_{ac} = U_{ac}/K_F R_A \) is reduced quadrature component of noise current. Expression (13) is an error signal [5] containing noises. The theory of narrow-band stochastic processes gives expression for the spectral density of a given quadrature: 

\[
S_{nh}(\omega) = 4eI_0.
\]

First term in (13) determines the slope of discrimination feature; it is a decrement too. In ref. [9] the measured numerical expression of the slope is given; it is 5.8 kHz/V.

Second term in (13) shows how the persistent voltage carries RAM noise.

To start feedback sensitivity analysis, we separate voltage variational components. In general \( \Delta v_R = \Delta v_{RT} + \delta v_R \); here \( \Delta v_{RT} \) is slow thermal drift.

Put at first \( \delta v_{RT} = 0 \); In (13) it means \( \Delta v_{RT} = \delta v_R = V_L/\xi \).

We introduce denotation: \( K_1 = K_{sd} K_{r} R_A \), \( D_{IG} = \frac{\eta e}{h\nu_L} D_{rg} \),

\[
I_N = I_0 m_{RC} \xi + I_{ac}, \quad U_N = K_1 I_N.
\]

When the feedback is locked, a variational component of laser frequency \( \delta v_L \) arises. The expression (13) takes the relevant form:

\[
U_{SD} = K_1 \left[ D_{IG} \left( \delta v_R - \delta v_L \right) + I_{N} \right]. \tag{14a}
\]

Here the term from the persistent component of the SD voltage \( K_1 I_0 m_{RC} \) is eliminated. It introduces the imbalance of the equation for quasi-static. The special adjustment is used in the scheme to compensate it.

The second equation locks the PDH feedback circuit:

\[
\delta v_L = \delta v_N + \beta L K_2 U_{SD}. \tag{14b}
\]

Here \( K_2 \) is a gain of servo amplifier 5, \( \beta L \) is parameter of laser retuning driver, \( \delta v_N(t) \) is chaotic process, introduced phenomenologically into analytical consideration to characterize stochastic laser frequency deviations [12,13].

The solution of the equation system (14) has the form

\[
\delta v_L = \frac{K_{0 S}}{K_{0 S} + 1} \delta v_R + \frac{K_1 K_2 \beta L}{K_{0 S} + 1} I_N = S + N.
\]

Here \( K_{0S} = K_2 I_{OS} D_{IG} \) is entire gain of the feedback loop.

An expression for the minimum detectable frequency variation \( (S/N = 1) \) is

\[
\left( \delta v_L \right)_{min} = \frac{\delta v_N}{K_{0 S}} + \frac{I_N}{D_{IG}}. \tag{15}
\]

The first term determines the suppression of laser frequency fluctuations. This is stabilization of the frequency by means of a reference resonator. The second term shows that for the noise introduced standardly, the presence and depth of feedback does not affect the threshold signal (sensitivity). I.e., one can determine resolution by unlocking feedback speculatively.

Eliminating SLFD and taking into account the relation \( \delta v_R = V_L/\xi \) in (15), we find general expressions for small displacement resolution:

\[
x_{min} = \left( \frac{V_L}{I_0 m_{RC} \xi + I_{ac}} \right) D_{IG} \left( \frac{I_{OS}}{I_0 m_{RC} \xi + I_{ac}} \right), \tag{16}
\]
\[ G_s (f) = \frac{m_i I_0^2 m_R^2 + 4eI_0}{\alpha^2} = S_{sh} + S_{sh}, \]

where \( \alpha_{sl} \) is a parameter of “displacement-photocurrent” conversion, \( \alpha_{sl} = \frac{\delta I_s}{\chi} = \frac{v}{L} D_{Rl}. \)

To carry out actual numerical estimates, we take into account the additional phenomenological parameter \( K_G \) of a real interferometer with energy loss in reflections (\( R_m + A_m < 1 \)). This parameter was introduced in [9], and article [10] shows that it reduces the decrement as \( K_G / 2 \), if \( K_G \leq \frac{1}{3} \). Then we obtain the actual calculation expression

\[ \alpha \approx 8 \frac{K_G F_G}{\lambda} \left( \frac{\eta e}{h v} \right) \sqrt{P_c P_s}. \]

To present numerical estimates we employ the values: \( F_G = 3000, K_G = 0.2, \lambda = 1.06 \mu \text{m}, \eta = 0.7, P_{ph} = 50 \text{mW} \) [1]. The optimal value of the phase modulation index \( \beta = 1.15 \text{ rad} \) [5] corresponds to the ratio \( \frac{P_s}{P_c} \approx 0.4 \). The persistent component of the radiation power on the PD is given by the expression:

\[ P_{ph} = 2P_s + (1 - K_G) P_c. \]

Using it we find \( P_c = 30 \text{ mW}, P_s = 12 \text{ mW} \) and \( \alpha \approx 5 \cdot 10^7 \text{ A/m}. \)

Spectral density of RAM noise current in (16) for values \( I_0 = 30 \text{ mA} \) and \( m_R = 10^{-12} \text{ Hz}^{-1} \) [10] has the value \( (S_{sh})^{1/2} = 3 \cdot 10^{-10} \text{ A/Hz}^{1/2}. \) Value \( (G_s)^{1/2} \approx 3 \cdot 10^{-16} \text{ cm/Hz}^{1/2} \) [1] as the reported RAM noise contribution estimate defines value \( m_{rc} = 0.5\%. \) And a real value \( m_{rc} = 1.5\% \) defines contribution estimate of \( 10^{-15} \text{ cm/Hz}^{1/2}. \) Thus, there is shown that in installation [1] suppression of RAM is not important.

The uniform noise analysis makes it possible to estimate the shot current noise: \( (S_{sh})^{1/2} = 1.4 \cdot 10^{-10} \text{ A/Hz}^{1/2} \) and the forecast of the OGRAN measurer resolution limit as \( 3 \cdot 10^{-16} \text{ cm/Hz}^{1/2}. \)

The initial formula for the limiting sensitivity of the circuit is derived in article [5] and report [12]. Here it is obtained by substituting the relation \( 4eI_0 = \frac{8\eta e^2 P_s}{hv} \) into the expression (16). The expression for the limit resolution of the PDH measurer takes the form:

\[ x_{\min} = \frac{\lambda}{8F} \sqrt{\frac{2h v}{\eta P_c} \Delta f}. \]

There is a small numerical difference from the known formulas [5,12], determined by the conversion of RF noise into LF noise when implementing synchronous detection.

6. Noise of a quasi-static error signal

A different path of low frequency laser power fluctuation penetration into the precision measuring chain may be pointed out. There is significant temperature drift in the aluminum acoustic resonator length (2 m) causing frequency changes of interferometer \( \Delta v_{GR}. \) The laser frequency variation \( \Delta v_{LJR} \) follows it; voltage on the thick piezoelectric block of slow laser tuning is varied. At the moment of tracking capture this voltage is close to zero. Then the voltage rapidly increases as temperature in the MSU boathouse is very unstable. When the voltage reaches a value of 1000 volts the failure of tracking happens. This
temperature drift significantly limits time of data accumulation for high resolution spectrogram (Figure 7 [1]). While tracking, quasi-static error signal is changed. Feedback equation for quasi-static variations gives the solution:

\[ \Delta V_{RL} = \Delta V_{RT} \frac{K_{oo}}{K_{oo} + 1} \].

Here \( K_{oo} >> K_{os} >> 1 \) is a gain in the feedback loop at zero frequency.

Operating point at the FP resonant transmittance curve drifts from it’s maximum:

\[ \Delta V_{RG0} = \Delta V_{FSRG} / F_G \], \( \Delta V_{FSRG} \) - free spectral range, \( \Delta V_{RG0} = c / 2L_G = 75 \text{ MHz} \) for \( L_G = 2 \text{ m} \) and \( \Delta V_{RG0} = 25 \text{ kHz} \) for \( F_G = 3000 \) [1]. Put ratio \( P_S / P_c = 0.25 \); it is close to the optimum value of 0.42.

Then decrement is \( D_{PG} = 16 \frac{P_S}{\Delta V_{RG0}} \). For ideal optical resonator [5] \( D_{PG} = 8 \). For very arbitrary example values \( \Delta V_R = 2.5 \cdot 10^9 \text{ Hz} \) and \( K_c = 10^7 \) we find \( \Delta V_{RL} = 250 \text{ Hz} \) and \( m_r = 8\% \). This estimate is approximate also because the actual contrast value \( K_G = 0.2 \) has not taken into account; it substantially reduces the estimate to 0.5%.

In formalized analyzing this effect, while feedback for variable components is unlocked speculatively, the formula (13) is transformed to the form

\[ U_{SD,OUT} = K_{SD} K_F R_A \left( \frac{h \nu}{\eta e} \right)^{1/2} \frac{P}{\Delta V_{RG0}} \frac{\Delta V_{RL}}{\Delta V_{RG0}} \left[ 1 + \xi(t) \right] + I_{mc}(t) \].

Here the first new term is independent of the decrement expression. As a result, we obtain the formula for additional noise contribution

\[ m_r = 8 \frac{\Delta V_{RL}}{\Delta V_{RG0}} \].

For the above estimate \( \Delta V_{RL} = 250 \text{ Hz} \), we obtain \( (S_{NL})^{1/2} \approx 2.5 \cdot 10^{-4} \text{ Hz/Hz}^{1/2} \) and \( (G_{SNL})^{1/2} \approx 2 \cdot 10^{-16} \text{ cm/Hz}^{1/2} \). Thus, the expressions (17) and (18) make it possible to calculate numerically the threshold signal contribution of noise conditioned by ambient temperature drift.

7. Discriminator vibration noise

Another way of LF laser power fluctuations penetration may be proposed to attention. It occurs in the discriminator channel [1,9,11,13].
Discriminator channel contains an additional FP resonator and second PDH scheme. It converts laser frequency deviations into antenna’s output voltage variations: $U_{SDM} = D_{det} \Delta \nu_L$. Here $D_{det}$ is discriminator decrement or a slope of the PDH discriminatory curve close to it’s zero. The natural (eigen) frequency of the discriminator resonator is tuned by the piezoelectric element, to which one mirror is attached. By means of feedback it’s eigen frequency follows the frequencies of the laser $\Delta \nu_{LF}$ and base sensor resonator $\Delta \nu_{BF}$. The feedback in the second channel is realized at low frequencies (<100 Hz) [1]. This provides its function.

In general, this curve has a well-known nonlinear form [5,11]; when the amplitude of the laser frequency deviations increases (in the range $f > 10$ Hz ), the transformation ceases to be linear and further reaches full limitation.

There is an observable phenomenon: industrial vibration and seismic noise with frequency range (10÷300) Hz is available at the output of the OGRAN discriminator channel. It’s maximum scope is commensurate with the scope of discriminatory curve. Last value can be estimated. So, the standard bandwidth of the resonator is determined by the relation: $\Delta \nu_{RDO} = \frac{c^2}{2L_d}$. For values $L_d = 0.45$ m and $F_d = 1.2 \cdot 10^4$ [1] one can find $\Delta \nu_{RDO} = 28$ kHz, and then the scope can be estimated as (40÷50) kHz.

The fact of influence of significant Moscow industrial seismic background has been noticed. While amplification is nonlinear and significant relatively low-frequency interference is present, the effective gain of a weak signal decreases. For this reason it had been proposed a modified version of a GW detector displacement measurer with the second laser and heterodyning to extend the linearity of frequency demodulation range by an order of magnitude [PIRT’2013]. In OGRAN installation discriminator channel [1] has simple vibration isolation. This second PDH scheme of antenna with slow frequency tracking (10 Hz – 100 Hz) weakly suppresses LF interference. Usually such precise measurements are made at night.

To estimate the effect we impede the interference: $U_{SDM} = D_{det} \Delta \nu_m \sin \Omega_m t$, where $\Delta \nu_m$ is amplitude of interference deviation. It is very likely that vibrational interference arrives at the synchronous detector from the photodiode. Then one can refine analytically

$$U_{SDM} = D_{det} \Delta \nu_m \left[1 + \xi(t)\right] \sin \Omega_m t.$$  

A noise component $U_{OUT, N} = \xi D_{det} \Delta \nu_m \sin \Omega_m t$ has occurred. In this way voltage carries laser power fluctuations; selected narrowband antenna noise is modulated by function $[\sin \Omega_m t]$. Envelope averaging gives the equivalent constant voltage $U_{SDN} = \frac{2}{\pi} D_{det} \Delta \nu_m$. We put $\Delta \nu_m = \frac{\Delta \nu_{RDO}}{4}$. Then

$$S_v = \frac{2}{\pi} \Delta \nu_m \left(n_v\right)^{1/2} = 4.5 \cdot 10^{-3} \text{ Hz/(Hz)}^{1/2}.$$

It determines measurer resolution estimation $3 \cdot 10^{-15} \text{ cm/(Hz)}^{1/2}$ coinciding in level with above main source of noise.

Here, as above, the contribution of the noise source is determined quantitatively by the product $\Delta \nu_m \xi$. So through the voltage at the output of the detector laser noise penetrate into the precise circuit. Thus, there has been presented an explanation version of excessive technical noise level had been registered in the installation OGRAN in conditions of increased Moscow seismic background. In conditions of Baksan Neutrino Observatory of INR RAS one should expect an essential decrease in vibration background.

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