MONOMIZATION OF POWER IDEALS AND PARKING FUNCTIONS

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Abstract. In this note, we find a monomization of a certain power ideal associated to a directed graph. This power ideal has been studied in several settings. The combinatorial method described here extends earlier work of other, and will work on several other types of power ideals, as will appear in later work.

1. Introduction

Ideals generated by powers of linear forms have arisen in several areas recently. They appear in work on linear diophantine equations and discrete splines [4], zonotope algebra [7][5], zonotopal Cox rings [11], ideals of fat points [6], and other areas. Of particular interest is the computation of the dimensions of quotients by these ideals, as well as their hilbert series. In many cases these computations have been connected to the computation of other statistics which are more germane to the problem in which they appear (eg.,[1],[11]).

In this note we demonstrate a fast algorithm for computing the hilbert series in an important special case. In particular, the computation is reduced to the far simpler problem of determining the hilbert series of certain monomial ideal. In this sense, we have “monomialized” the original ideal. This process also introduces a new notion of parking function, extending the phenomena which have appeared earlier in the literature.

2. Parking Functions and Monomization

We first recall some definitions and facts about parking functions and $G$-parking functions. A parking function of length $n$ is a sequence of non-negative integers $(a_1, \ldots, a_n)$ such that for $i$

$$\# \{ j|a_j < i \} \geq i$$

Note that we allow $a_i = 0$; the definition is often written for positive integers instead of non-negative, in which case the ‘$<$’ above is replaced with ‘$\leq$’. The definition is equivalent to requiring that the increasing rearrangement $b_1 \leq \ldots \leq b_n$ has the property $b_i < i$. The origin of the term parking function comes from the following interpretation. Suppose $n$ cars arrive at a linear parking lot with parking spaces labelled between 0 and $n-1$. Each car, $i$, has a preferred parking space, $f(i)$. The cars arrive in order and drive to their preferred spot. If it is already taken, then they drive until they reach the next empty spot and take that one. A preference function $f$ is a parking function if (and only if) every car gets a parking spot without having to turn around.

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A $G$-parking function is a generalization of parking functions, appearing first in \cite{S}, and later in \cite{2,3}. Let $G$ be a digraph on vertices labelled $0,1,\ldots,n$. We will call $0$ the root of $G$. For every non-empty subset $I \subset [n]$ and $i \in I$ define $d_I(i)$ to be the number of edges originating at $i$ and terminating at a vertex not in $I$. A $G$-parking function is defined to be a function $f$ assigning a non-negative integer to the vertices $1,\ldots,n$ such that for every non-empty subset $I$ there is an $i \in I$ such that $f(i) < d_I(i)$. When $G = K_{n+1}$ the complete graph on $n+1$ vertices, the $G$-parking functions are the same as the parking functions defined above.

We now give a reinterpretation of the $G$-parking functions which was in fact the original motivation for their definition. For any $I \subset [n]$, we define $D_I$ to be the total number of edges of $G$ which originate at a vertex in $I$ and terminate at a vertex outside of $I$. Explicitly,

$$N_I = \sum_{i \in I} d_I(i)$$

Now let $k$ be any integer. If $D_I + k > 0$ for every $I$ as above, we define a polynomial $p_I$ in the ring $\mathbb{C}[x_1,\ldots,x_n]$ given by

$$p_I = \left( \sum_{i \in I} x_i \right)^{D_I + k}$$

Note that this will always be the case when $k$ is positive. We then define $\mathcal{I}_{G,k}$ to be the ideal generated by all such $p_I$, and $\mathcal{A}_{G,k}$ to be the quotient $\mathbb{C}[x_1,\ldots,x_n]/\mathcal{I}_{G,k}$. Since $\mathcal{I}_{G,k}$ is a homogeneous ideal, $\mathcal{A}_{G,k}$ will have a basis of monomials. Given a monomial basis $B$ of $\mathcal{A}_{G,k}$, the set of monomials $\mathcal{M} = \mathbb{C}[x_1,\ldots,x_n] \setminus B$ is an ideal, and $B$ is the basis of standard monomials for $\mathbb{C}[x_1,\ldots,x_n]/\mathcal{M}$. We call any such $\mathcal{M}$ a monomization of the ideal $\mathcal{I}_{G,k}$. Our program is to find a monomization for the ideals $\mathcal{I}_{G,k}$ which is natural in some way and easy to compute. Such a theory would greatly simplify the study of the linear structure of the rings $\mathcal{A}_{G,k}$.

In the case $k = 0$, the picture is especially beautiful. In \cite{S}, it is shown that the monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}$ where $(a_1,\ldots,a_n)$ is a $G$-parking function give a monomial basis for $\mathcal{A}_{G,0}$. Translating this to the ideal $\mathcal{J}_{G,0}$ of monomials $x^a$ where $a$ is not a $G$-parking function, we find the following description. For every non-empty $I \subset [n]$, let

$$m_I = x_1^{d_I(i_1)} \cdots x_{t_r}^{d_I(i_r)}$$

Then, $\mathcal{J}_{G,0} = \langle m_I \rangle_{I \subset [n]}$.

There are several features of this monomization we would like to emulate. Firstly, a set of generators for $\mathcal{J}_{G,0}$ is relatively easy to compute from $G$. Secondly, both the generators of $\mathcal{I}_{G,0}$ and $\mathcal{J}_{G,0}$ are indexed by the same set, and in particular are the same size. There is a third very useful feature of $\mathcal{J}_{G,0}$, which we would like to emulate, but won’t be able to. Clearly the group $H = Aut \ast (G)$ of basepoint preserving automorphisms of $G$ acts on $\mathbb{C}[x_1,\ldots,x_n]$, and since it preserves the $p_I$, $H$ also acts on $\mathcal{A}_{G,k}$. Since the definition of $G$-parking function is invariant under the action of $H$, we see that this basis is $H$-invariant. We can thus use the combinatorial structure of the $G$-parking functions to understand not just the linear structure of $\mathcal{A}_{G,0}$, but also its structure as an $H$-representation. For example, this shows that the multiplicity of the trivial representation of $\mathfrak{S}_n$ on $\mathcal{A}_{K_{n+1},0}$ is equal to $rac{1}{n+1} \binom{2n}{n}$, the $n^{th}$ Catalan number.
For \( k \neq 0 \), we cannot in general find an \( \text{Aut}_1(G) \) invariant basis of monomials, even when \( G \) is a complete graph.

**Example 1.** Let \( G = K_3 \), the triangle. Then \( I_{G,0} = (x^2, y^2, (x+y)^2) \). The parking functions are \{((0, 1), (1, 0), (0, 0))\}, and indeed

\[
A_{G,0} = \mathbb{C} \oplus \mathbb{C} x \oplus \mathbb{C} y
\]

Now for \( k = 1 \), \( I_{G,1} = (x^3, y^3, (x+y)^3) \). The monomials which are non-zero in \( A_{G,1} \) are \( 1, x, y, x^2, xy, y^2, x^2 y \), and \( xy^2 \). As we can easily verify, however, the Hilbert series of \( A_{G,1} \) is \( 1 + 2t + 3t^2 + t^3 \). In particular, any monomial basis must contain \( 1, x, y, x^2, xy, y^2 \), and must contain exactly one of \( x^2 y \) or \( xy^2 \). Thus there is no way to choose an \( S_3 \)-invariant basis of monomials.

We now address the case \( k = 1 \). Similar to the \( k = 0 \) case, we define a monomial \( m_I \) for any non-empty subset \( I \subset [n] \). Namely, let

\[
\nu_I(i) = \begin{cases} 
    d_I(i) + 1 & \text{if } i \in I \text{ is minimal} \\
    d_I(i) & \text{if } i \in I \text{ is not minimal} \\
    0 & \text{if } i \notin I
\end{cases}
\]

and define

\[
m_I = \prod_{i=1}^{n} x_i^{\nu_I(i)}
\]

Thus these monomials, as in the case \( k = 0 \), are as close to the center of the Newton polytope of \( p_I \) as possible. Let \( J_{G,1} = (m_I) \). The main result of this note is that \( J_{G,1} \) is a monomization of \( I_{G,1} \).

**Theorem 2.** The standard monomial basis of \( J_{G,1} \) give a monomial basis for \( I_{G,1} \)

We can use this to give a more combinatorial definition of what we will call a \((G, 1)\)-parking function. In the case of the complete graph, we believe this definition appeared first in [7].

**Definition 3.** For a graph \( G \) on the vertex set \( \{0, 1, \ldots, n\} \), a \((G, 1)\)-parking function is a function \( f : [n] \to \mathbb{N} \) such that for any \( I \subset [n] \),

\[
f(i) < \begin{cases} 
    \# \{\text{edges from } i \text{ out of } I\} + 1 & \text{if } i \text{ is minimal} \\
    \# \{\text{edges from } i \text{ out of } I\} & \text{otherwise}
\end{cases}
\]

**Example 4.** Let’s take the following example.
From the graph above we obtain the power ideal
\[ I_{G,1} = (x_1^3, x_2^3, x_3^4, x_4^4, (x_1 + x_2)^5, (x_1 + x_3)^5, (x_1 + x_4)^5, (x_2 + x_3)^6, (x_2 + x_4)^6, (x_1 + x_2 + x_3)^5, (x_1 + x_2 + x_4)^6, (x_1 + x_3 + x_4)^5, (x_1 + x_2 + x_3 + x_4)^5) \]
and the monomization
\[ J_{G,1} = (x_1^3, x_2^3, x_3^4, x_4^4, x_1 x_2^2, x_1 x_3^2, x_1 x_4, x_2 x_3^2, x_2 x_4, x_3 x_4, x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_3 x_4, x_2 x_3 x_4) \]
From this, it is fairly easy to compute that the dimension of \( B_{G,1} \), and therefore \( A_{G,1} \), is equal to 82. Notice, it is fairly easy to reduce the number of generators in \( J_{G,1} \). For example, clearly since we have \( x_1^3 \), we don’t need \( x_1 x_2^2, x_1 x_4, x_2 x_3^2, x_2 x_4, x_3 x_4, x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_3 x_4, x_2 x_3 x_4 \).
Continuing in this way we can reduce to a minimal set of 10 generators.

It is worth mentioning the initial ideal of \( I_{G,k} \) and its differences from the ideals \( J_{G,0} \) and \( J_{G,1} \). Given a term order, i.e. a linear ordering on the monomials, we can form the initial ideal in \( I_{G,k} \), the set of leading terms of every element of \( I_{G,k} \). A Gröbner basis for \( I_{G,k} \) is a set of generators \( \{f_s\}_{s \in S} \) such that the leading terms of the \( f_s \) generate in \( I_{G,k} \). The theory of Gröbner bases is very rich and general, and provides algorithms for studying a surprising amount of structure of any ideal in a polynomial ring. However, determining a Gröbner basis for a general ideal is a potentially time intensive procedure. Additionally, Gröbner bases can be much larger than a given set of generators for an ideal.

The ideals \( J_{G,0} \) and \( J_{G,1} \) are almost never initial ideals of \( I_{G,0} \) and \( I_{G,1} \). Indeed, initial ideals are generated by vertices of the newton polytopes of a Gröbner basis for the ideal whereas our ideals are generated by choosing monomials near the center of the newton polytope. The tradeoff is that the ideals \( J_{G,k} \) are much easier to compute, relying only on valence data of the associated graph. Furthermore, the size of their sets of generators is strictly controlled \((2^n - 1)\), and typically smaller than that of a Gröbner basis.

**Example 5.** If we let \( G = K_5 \), the 15 monomial generators of \( J_{G,1} \) are all non-redundant. However, this is still an improvement over the 26 elements of a Gröbner basis for \( I_{G,1} \), and are much more difficult to compute.
3. Monotone Monomial Ideals

For the remainder of the paper, let $B_{G,1} = \mathbb{C}[x_1, \ldots, x_n]/J_{G,1}$. Our proof will proceed as follows. We first demonstrate the standard monomial basis of $B_{G,1}$ spans $A_{G,1}$. We then show that $\dim(B_{G,1}) = \dim(A_{G,1})$, from which we can conclude the result. In order to show the dimensions are equal, we show that the dimension of $B_{G,1}$ is equal to the number of forests on $G$, and use the equivalent result for $A_{G,1}$, due to Ardila and Postnikov.

**Theorem 6.** [8] The dimension of the algebra $A_{G,1}$ is equal to the number of forests on $G$.

Both of the results will follow from the fact that the set $\{m_I\}$ is a monotone monomial family, in the language of [8]. We will only need the simplest part of this machinery, which we recall now. Let $\{m_I\}$ be any collection of monomials in $\mathbb{C}[x_1, \ldots, x_n]$. Let $m_{I\setminus i}$ be the monomial formed from $m_I$ by removing all $x_i$ with $i \in I$, and let $\bar{I} = [n] \setminus I$. Then the collection $\{m_I\}$ is a monotone monomial family if $m_{I\setminus i} = 1$ and if $I \subset J$, then $m_{I\setminus J}$ divides $m_I$.

It is routine to check that the monomials defined above for $k = 1$ are a monotone monomial family. Indeed, the condition $m_{I\setminus i} = 1$, which simply states that $m_I$ contains $x_i$ only if $i \in I$, is satisfied by definition. To check the second condition, we examine the degree of $x_i$ for $i \in I$ in $m_I$ and $m_{I\setminus i}$. Since $J \supset I$, the number of edges originating at vertex $i$ and terminating outside $J$ is smaller than those terminating outside $I$, ie. $d_J(i) \leq d_I(i)$. So there are two cases. If $i$ is the smallest element of $I$, then either its degree goes from $d_I(i) + 1$ to $d_J(i) + 1$ (in the case that $i$ is still the smallest element of $J$), or it goes from $d_I(i) + 1$ to $d_J(i)$. In both cases the degree drops. If $i$ is not the smallest element of $I$, then it can’t be the smallest element of $J$, so the degree goes from $d_I(i)$ to $d_J(i)$. Therefore, in any case $\deg_{x_i}(m_I) \geq \deg_{x_i}(m_{I\setminus i})$.

Using this we can conclude from [8] Theorem 3.1] our first claim that the standard monomial basis of $B_{G,1}$ spans $A_{G,1}$. To investigate the dimension of the of space $B_{G,1}$, we first use [8] Prop. 8.4] to find an expression for the dimension as an alternating sum. Let $\nu_I(i) = \deg_{x_i}(m_I)$.

**Proposition 7.** The dimension of $B_{G,1}$ is equal to the alternating sum

$$
\sum_{I_1 \subset \ldots \subset I_k} (-1)^k \prod_{i \in I_1} (\nu(i) - \nu_{I_1}(i)) \times \ldots \times \prod_{i \in I_k \setminus I_{k-1}} (\nu(i) - \nu_{I_k}(i)) \times \prod_{i \notin I_k} \nu(i)
$$

where we include the empty chain of subsets where $k = 0$.

We give the following interpretation to the alternating sum. For a given chain $I_1 \subset \ldots \subset I_k$ the product counts the number of directed subgraphs $H$ of $G$ with the following properties

1. There is at most one edge originating at each $i \in [n]$, and there is no edge originating at 0.
2. If $i \in I_j$ for some $j$, then the edge originating at $i$ must end in $I_j$ as well.
3. If $i \in I_j$ is the minimal element of $I_j$, then $i$ has an edge originating at it.

Let us note some properties of these subgraphs. Firstly, any subgraph of $G$ satisfying the first condition appears in the sum with $k = 0$. Secondly, we can embed the set of forests canonically in this collection as follows. For any forest $F$ of $G$, orient each edge of $F$ so that each connected component has a unique sink at the minimal
element of that component. (Insert example). Note that each such graph appears as above with \( k = 0 \), and only with \( k = 0 \); case analysis here. Thus the alternating sum counts each forest exactly once.

We claim that every other subgraph is cancelled out in the sum, so that the alternating sum is precisely equal to the number of subforests of \( G \). To show this, we now construct an involution on the set of pairs \( (H, I_1 \subseteq \ldots \subseteq I_k) \). The involution will only act on the chain of subsets, that is, it will leave \( H \) fixed. A pair will be fixed by the involution if and only if it corresponds to \( (H, \emptyset) \) with \( H \) a canonically oriented forest. and it will take a chain of length \( k \) either to chain of length \( k - 1 \) or length \( k + 1 \). Since there are no fixed points, this will show that any nonforest \( H \) is cancelled out in the alternating sum.

The involution will only use some subset of the vertices of \( H \) which we call special. We use the following algorithm to label the vertices of \( H \) special and non-special.

- Let \( v \) be the smallest unlabelled vertex.
- If \( v \) has an edge originating at it, label it and all remaining unlabelled vertices special and stop. Otherwise, label \( v \) non-special as well as any vertex such that the chain of edges originating from it terminates at \( v \).
- Return to the first step.

Notice that \( 0 \) will always be chosen first, when none of the vertices are labelled. We also have the following claim.

Claim 8. Each connected component is either composed entirely of non-special vertices or special vertices. Those labelled non-special are trees rooted at their minimal element.

Proof. Suppose that \( i \) is non-special. Then \( i \) must lie on a directed path towards a terminal vertex, and in particular the path originating at \( i \) does not contain a circuit. This follows because the only way a vertex can be labelled non-special is in step 2 of the algorithm, and only as part of a path which terminates. Therefore, if \( i \) is non-special, then its connected component must be a tree.

If \( i \) is part of a tree \( T \) and non-special, we claim the tree is rooted, ie. it has a unique sink. More specifically, it is rooted at the end of the path originating from \( i \). If this is the case, then every vertex of the tree was labelled non-special in the same step that \( i \) was. To see that it has a unique sink, let \( w \) be a sink in \( T \). There is a unique undirected path \( (w, v_1, v_2, \ldots, v_k, i) \) from \( w \) to \( i \). The edge from \( w \) to \( v_1 \) must be oriented towards \( w \), because \( w \) is a sink. Because each vertex can have at most one out-edge, this is the unique out-edge from \( v_1 \). Therefore the edge from \( v_2 \) to \( v_1 \) must be oriented towards \( v_1 \).

Continuing in this way, we conclude that the entire path is oriented from \( i \) to \( w \). Therefore \( w \) is the vertex at the end of the path originating from \( i \), and consequently is unique. The only thing left to see is that \( w \) is the minimal vertex of the tree. This is easy, though, since otherwise \( w \) would not have been chosen in step 1 of the algorithm. \( \square \)

Note that the converse of the second part of the claim is false. It is perfectly feasible for a subtree of \( H \) to be oriented towards its minimal vertex and still be labelled special. We do however get the following corollary.

Corollary 9. \( H \) is a canonically oriented forest if and only if the algorithm labels every vertex non-special.
Proof. One direction is clear from the claim: If all the vertices are labelled non-special, then every connected component is a rooted tree oriented toward the minimal vertex, which is a canonically oriented forest. For the converse, suppose $H$ is a canonically oriented forest, but some vertex is marked special. Then at some point in the algorithm, the least unlabelled vertex $v$ has an out-edge. The path coming out of $v$ must terminate at a vertex smaller than $v$ since the forest is canonically oriented, but then this vertex must have been labelled non-special. This, in turn, would imply that $v$ is labelled non-special, and we get a contradiction. □

Corollary 10. If $i \in I_j$, then $i$ is special.

Proof. Let $w_j$ be the minimal vertex in $I_j$. Clearly $w_j$ must be special, because if it were non-special then by claim blah blah it would lie on a tree oriented towards its minimal vertex. However this isn’t possible since there is an edge originating at $w_j$, and the entire path from $w_j$ must lie within $I_j$ by definition. However, if $i$ is non-special, then the path originating at $i$ must terminate at a non-special vertex $v$, and that vertex must lie in $I_j$. Because $v$ must also lie in $I_j$ it must be greater than $w_j$, but then $w_j$ would have been chosen in step 1 of the above algorithm before $v$ was, and $v$ wouldn’t have been marked non-special. This is a contradiction. □

We now define the involution $\kappa$. Let $S$ be the set of special vertices in $H$. Then

$$\kappa((H, I_1 \subset I_2 \subset \ldots \subset I_k)) = \begin{cases} (H, I_1 \subset I_2 \subset \ldots \subset I_k \subset S) & \text{if } S \neq I_k \\ (H, I_1 \subset I_2 \subset \ldots \subset I_{k-1}) & \text{if } S = I_k \end{cases}$$

If the chain of subsets is empty, and there are no special vertices, then $\kappa$ does nothing. By Corollary 9 this means that $H$ is a canonically oriented tree. Otherwise, the length of the chain of subsets is changed by $\kappa$. Therefore, the only fixed points are $(H, \emptyset)$ where $H$ is a canonically oriented tree. Applying $\kappa$ to formula 1 we get

Theorem 11. The dimension of $A$ is equal to the number of forests on $G$.

This completes the proof of 2. The proof actually gives a little more. Since the standard monomials of $B_{G,1}$ span $A_{G,1}$, we get the inequality

$$\text{Hilb}(B, t) \leq \text{Hilb}(A, t)$$

We have just seen that $\text{Hilb}(A, 1) = \dim_k (A)$ is equal to the number of forests in $G$, which implies the corollary.

Corollary 12. $\text{Hilb}(B, t) = \text{Hilb}(A, t)$

From 1 we also have a combinatorial interpretation for the coefficients in $\text{Hilb}(A, t)$. If

$$\text{Hilb}(A, t) = \sum c_n t^n$$

then $c_n$ is equal to the number of forests $F$ on $G$ with external activity equal to $|G| - |F| - n$. This generalizes the results of 9, 10. It would be nice to find a bijection between the monomials of fixed degree and the forests of fixed external activity.
4. Further Work

Some interesting questions remain about these ideals. As noted above, the monomial ideals $J_{G,1}$ are not in general the initial ideals of the ideals $I$. However, since their Hilbert Series are equal, they each correspond to a point on the same Hilbert Scheme. We can then ask whether or not they lie on the same irreducible component of the Hilbert Scheme. If not, we can ask how many irreducible components away they are from each other. Note that this number exists, since the Hilbert scheme is connected.

The original motivation for the study of these rings comes from differential geometry. Let $G = SL_n(\mathbb{C})$ and $B$ the subgroup of matrices fixing a given flag in $\mathbb{C}^n$, then $G/B$ is the complex flag variety, parametrizing complete flags in $\mathbb{C}^n$. There are $n$ tautological line bundles over $G/B$, assigning to each point $p \in G/B$ the quotient of the $k$-plane by the $(k-1)$-plane of the flag corresponding to $p$. Fixing a hermitian structure on $G/B$ gives us a unique connection on each of these line bundles, and therefore curvature forms $\omega_1, \ldots, \omega_n$. Note that the deRham cohomology classes of these 2-forms (after normalizing) are precisely the Chern classes of the corresponding line bundles. For this reason, we can think of the ring $\mathbb{C}[\omega_1, \ldots, \omega_n]$, sitting inside the ring of $C^\infty$ invariant forms on $G/B$, as an extension of the intersection theory obtained from $H^*(G/B, \mathbb{C})$. In [10] it is shown that this ring is isomorphic to the ring $A_{K_{n+1}}$, and therefore has a basis given by the monomials given above. It would be interesting to determine if other rings of Chern forms on homogeneous spaces have similar presentations, and to determine monomizations of their ideals.

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