Determinants of regular singular Sturm–Liouville operators

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Abstract. We consider a regular singular Sturm–Liouville operator

\[ L := -\frac{d^2}{dx^2} + \frac{q(x)}{x^2(1-x)^2} \]

on the line segment \([0,1]\). We impose certain boundary conditions such that we obtain a semi–bounded self–adjoint operator. It is known (cf. Theorem 1.1 below) that the \(\zeta\)–function of this operator

\[ \zeta_L(s) = \sum_{\lambda \in \text{spec}(L) \setminus \{0\}} \lambda^{-s} \]

has a meromorphic continuation to the whole complex plane with 0 being a regular point. Then, according to [RS] the \(\zeta\)–regularized determinant of \(L\) is defined by

\[ \det_\zeta(L) := \exp(-\zeta'_L(0)). \]

In this paper we are going to express this determinant in terms of the solutions of the homogeneous differential equation \(Ly = 0\) generalizing earlier work of S. Levit and U. Smilansky [LS], T. Dreyfus and H. Dym [DD], and D. Burghelea, L. Friedlander and T. Kappeler [BFK1, BFK2]. More precisely we prove the formula

\[ \det_\zeta(L) = \frac{\pi W(\psi, \varphi)}{2^{\nu_0+\nu_1} \Gamma(\nu_0+1)\Gamma(\nu_1+1)} \]

Here \(\varphi, \psi\) is a certain fundamental system of solutions for the homogeneous equation \(Ly = 0\), \(W(\varphi, \psi)\) denotes their Wronski determinant, and \(\nu_0, \nu_1\) are numbers related to the characteristic roots of the regular singular points 0,1.

1. Introduction and statement of the main result

We begin with some elementary remarks on \(\zeta\)–regularized determinants. Let \(L \geq -c + 1\) be a semi–bounded self–adjoint operator in the Hilbert space \(\mathcal{H}\). We assume that

\[ (L+c)^{-1} \in C_1(\mathcal{H}) \]

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is trace class. Usually one deals with the more general assumption that only some power of \((L + c)\)^{-1} is trace class. But since in this paper we deal exclusively with one-dimensional Sturm–Liouville operators, we may content ourselves with the more convenient case (1.1). Moreover, we assume that we have an asymptotic expansion

\begin{equation}
\text{Tr}(L + z^2)^{-1} \sim_{z \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{k_j} A_{j,k} z^{\alpha_j - 2} \log^k z,
\end{equation}

as \(z \to \infty\), \(z \in \mathcal{C} := \{z \in \mathbb{C} \mid \arg z < \delta\}\), \(0 < \delta < \pi/2\) fixed. Here \(k(j) \in \mathbb{Z}_+\) for all \(j\), \((\alpha_j)_{j \in \mathbb{Z}_+}\) is a sequence of complex numbers with Re \(\alpha_j \to -\infty\) (cf. [BL2, Sec. 2]).

We assume furthermore, that the terms \(z^{-2} \log^k z, k \geq 1\), do not occur, i.e. \(k(j) = 0\) if \(\alpha_j = 0\). These assumptions guarantee, that the \(\zeta\)-function of \(L\),

\begin{equation}
\zeta_L(s) = \sum_{\lambda \in \text{spec } L \setminus \{0\}} \lambda^{-s}, \quad \text{Re } s > 0,
\end{equation}

has a meromorphic continuation to \(\mathcal{C}\) with no pole at 0 (we put \(\lambda^{-s} = e^{-i\pi s} |\lambda|^{-s}\) if \(\lambda < 0\)).

Now we can define the \(\zeta\)-regularized determinant: put \(\text{det}_\zeta L = 0\) if 0 \(\in\) spec \(L\), otherwise put

\begin{equation}
\text{det}_\zeta L := \exp(-\zeta_L'(0)).
\end{equation}

This notion was introduced by Ray–Singer [RS].

Since we mostly deal with \(\log \text{det}_\zeta L\) instead of \(\text{det}_\zeta L\) we abbreviate

\begin{equation}
T(L) := -\zeta_L'(0)
\end{equation}

which is \(\log \text{det}_\zeta L\) for invertible \(L\).

If \(L \geq 0\), our assumption (1.1) implies

\begin{equation}
\zeta_L(s) = 2 \frac{\sin \pi s}{\pi} \int_0^\infty z^{1-2s} \text{Tr}(L + z^2)^{-1} dz
\end{equation}

from which one easily derives the formula

\begin{equation}
T(L) = -2 \int_0^\infty z \text{Tr}(L + z^2)^{-1} dz.
\end{equation}

The symbol \(\mathcal{f}\) indicates that the integral has to be regularized. For convenience of the reader we briefly recall the definition of \(\mathcal{f}\) as we will make extensive use of this notion: if \(f : \mathbb{R}_+ \to \mathcal{C}\) is a function having an asymptotic expansion

\begin{equation}
f(x) \sim_{x \to 0} \sum_{j=1}^{N} \sum_{k=0}^{k_j} a_{j,k} x^{\alpha_j} \log^k x + \sum_{k=0}^{k_0} a_{k0} \log^k x + o(1)
\end{equation}

with Re \(\alpha_j \neq 0\), then we define a "regularized limit" by

\begin{equation}
\text{lim}_{x \to 0} f(x) := a_{00}.
\end{equation}
If \( f(x) \) also has an asymptotic expansion of the form (1.8) as \( x \to \infty \), then \( \lim_{x \to \infty} f(x) \) is defined likewise.

Next let \( f : \mathbb{R}_+ \to \mathbb{C} \), such that

\[
\begin{align*}
  f(x) &= \sum_{j=1}^{N} \sum_{k=0}^{k_j} a_{jk} x^{\alpha_j} \log^k x + x^\varepsilon f_1(x), \\
  &= \sum_{j=1}^{M} \sum_{l=0}^{l_j} b_{jl} x^{\beta_j} \log^k x + x^{-\varepsilon} f_2(x),
\end{align*}
\]

with \( f_1 \in L^1_{\text{loc}}([0, \infty)), f_2 \in L^1([1, \infty)), \varepsilon > 0 \).

Then we put

\[
\int_0^\infty f(x) \, dx = \lim_{b \to \infty} \lim_{a \to 0} \int_a^b f(x) \, dx.
\]

This is the partie–fini definition of Hadamard. \( \int_0^\infty f(x) \, dx \) can also be expressed in terms of the Mellin transform (cf. [BS1, L]). Namely,

\[
F_1(s) = \int_0^1 x^s f(x) \, dx, \quad F_2(s) = \int_1^\infty x^s f(x) \, dx
\]

extend to meromorphic functions in a half–plane containing 0. Then

\[
\int_0^\infty f(x) \, dx = \text{constant term in the Laurent expansion of } F_1(s) + F_2(s) \text{ at } s = 0.
\]

One easily calculates

\[
\begin{align*}
  \int_0^1 x^\alpha \log^k x \, dx &= \begin{cases} 0, & \alpha = -1, \\
  \frac{(-1)^k k!}{(\alpha+1)^{k+1}}, & \alpha \neq -1,
\end{cases} \\
  \int_1^\infty x^\alpha \log^k x \, dx &= \begin{cases} 0, & \alpha = -1, \\
  \frac{(-1)^{k+1} k!}{(\alpha+1)^{k+2}}, & \alpha \neq -1,
\end{cases}
\end{align*}
\]

(1.13a) (1.13b)

in particular

\[
\int_0^\infty x^\alpha \log^k x \, dx = 0.
\]

(1.13c)

For an elaborate discussion of \( f \) see [4, Sec. 2.1].

Next we consider a Sturm–Liouville operator

\[
l = -\frac{d^2}{dx^2} + q(x)
\]

(1.14)

on the interval \((0, 1)\), where \( q \in C^\infty(0, 1) \) is a real function.

Assume for the moment that \( q \in C^\infty([0, 1]) \) and impose, for simplicity, Dirichlet boundary conditions. More precisely, we consider the self–adjoint extension, \( L \), of \( l \) with domain

\[
\mathcal{D}(L) = \{ f \in H^2[0, 1] \mid f(0) = f(1) = 0 \} \subset L^2[0, 1].
\]
Then $L$ satisfies (1.1), hence its $\zeta$–regularized determinant is well–defined. According to [BFK2], $\det_\zeta(L)$ can be computed as follows: let $\varphi$ be the unique function with
\[ l\varphi = 0, \quad \varphi(0) = 0, \quad \varphi'(0) = 1. \]
Then
\[ (1.15) \quad \det_\zeta(L) = 2\varphi(1). \]

In this paper we want to generalize this result to Sturm–Liouville operators having regular singularities at 0 and 1.

From now on let $l$ be regular singular, i.e.
\[ (1.16) \quad q(x) = x^{-2}a_0(x^{1/N}) = (1 - x)^{-2}a_1((1 - x)^{1/N}) \]
with $a_0 \in C^\infty([0,1]), a_1 \in C^\infty((0,1])$ and
\[ (1.17) \quad a_0(0), a_1(1) \geq -1/4 \]
(cf. [BL2, Sec. 4] for examples where this kind of potential occurs naturally). For later purposes we write
\[ (1.18) \quad a_0(x) =: a_0(0) + xb_0(x); \quad a_1(x) =: a_1(1) + (1 - x)b_1(x) \]
with $b_0 \in C^\infty([0,1]), b_1 \in C^\infty((0,1])$.

The assumptions (1.17) imply that $l$ is bounded below on $C^\infty_0(0,1) \subset L^2[0,1]$. In the sequel, the domain of an operator is denoted by $\mathcal{D}$ and we put
\[ (1.19) \quad \begin{align*}
    l_{\min} &:= \overline{l} = \text{closure of } l \text{ in } L^2[0,1], \\
    l_{\max} &:= l^*. 
\end{align*} \]

If $a_0(0) \geq 3/4$ (resp. $a_1(1) \geq 3/4$) then 0 (resp. 1) is in the limit point case and no boundary condition is necessary. Otherwise we have to impose boundary conditions to obtain a self–adjoint operator. Since we will be dealing exclusively with separated boundary conditions, it is enough to discuss one boundary point, e.g. 0. We distinguish between two cases:

1. \textit{$q$ is continuous at 0:} we impose a boundary condition at 0 of the form
\[ (1.20) \quad R_0f = 0 \quad \text{for } f \in \mathcal{D}(l_{\max}), \]
where either
\[ (1.21) \quad R_0f = f(0) \quad \text{(Dirichlet)} \]
or
\[ (1.22) \quad R_0f = f'(0) + Af(0) \quad \text{(generalized Neumann)} \]
with some $A \in \mathbb{R}$.

Furthermore we define the order of the boundary operator $R_0$ by
\[ (1.23) \quad \sigma(R_0) := \begin{cases} 
    0, & \text{in case } (1.21), \\
    1, & \text{in case } (1.22). 
\end{cases} \]
2. \textbf{q is not continuous at 0: } in this situation we content ourselves with the 'Dirichlet' condition at 0. Since \( l \) is bounded below, we can form its Friedrichs extension, \( l^F \).

Now for \( f \in D(l_{\text{max}}) \) we require

\begin{equation}
\varphi f \in \mathcal{D}(l^F)
\end{equation}

for any cut-off function \( \varphi \in C_0^\infty([0,1)) \) with \( \varphi = 1 \) in a neighborhood of 0. In other words we consider 'the Friedrichs extension near zero'. It can be checked that for \( f \in \mathcal{D}(l_{\text{max}}) \) the condition \[ (1.24) \]

\[ |f(x)| = O(x^{1/2}), \quad x \to 0. \]

However, we would like to express this boundary condition in terms of a boundary operator. For doing this we put

\begin{equation}
\nu_0 := \sqrt{a_0(0) + 1/4}.
\end{equation}

\( 1/2 \pm \nu_0 \) are just the roots of the indicial equation

\begin{equation}
\lambda(\lambda - 1) - a_0(0) = 0
\end{equation}

of the regular singular point 0. It is well-known that there is a fundamental system \( \varphi, \psi \) of solutions of the homogeneous equation \( lf = 0 \), where

\begin{align}
\varphi(x) &= x^{\nu_0 + 1/2} \varphi_1(x^{1/N}), \\
\psi(x) &= x^{-\nu_0 + 1/2} \psi_1(x^{1/N}) + k \varphi(x) \log(x)
\end{align}

with \( \varphi_1, \psi_1 \in C^\infty([0,1]), \varphi_1(0) = 1, \) and

\begin{equation}
\psi_1(0) = \begin{cases} \\
1/2\nu_0, & \nu_0 > 0, \\
0, & \nu_0 = 0.
\end{cases}
\end{equation}

Note that \( k = 0 \) if \( 2N\nu_0 \not\in \mathbb{Z} \). If \( \nu_0 = 0 \) then 0 is a repeated root of the indicial equation and hence we can choose \( k = -1 \). We call such a fundamental system \textit{normalized at 0}. Obviously, the Wronskian of \( \varphi, \psi \), \( W(\varphi, \psi) = -1 \). We also introduce

\begin{align}
\varphi_{\nu_0}(x) &= x^{\nu_0 + 1/2}, \\
\psi_{\nu_0}(x) &= \begin{cases} \\
x^{-\nu_0 + 1/2}/2\nu_0, & \nu_0 > 0, \\
x^{-1/2} \log(x), & \nu_0 = 0,
\end{cases}
\end{align}

which is a fundamental system of solutions of the differential equation

\begin{equation}
l_{\nu_0}f := - \frac{d^2}{dx^2} f + x^{-2}(\nu_0^2 - 1/4)f = 0.
\end{equation}

Now we put for \( f \in \mathcal{D}(l_{\text{max}}) \)

\begin{equation}
R_0 f := \lim_{x \to 0} \psi_{\nu_0}(x)^{-1} f(x).
\end{equation}

Note that \( R_0 \) is even well–defined on the larger space

\begin{equation}
\mathcal{D}(l) := \{ f + c_1 \varphi + c_2 \psi \mid f \in \mathcal{D}(l_{\text{max}}), c_1, c_2 \in \mathbb{C} \}.
\end{equation}
Moreover if 
\[ f = a\varphi + b\psi \]
then 
\[ R_0 f = \lim_{x \to 0} \psi_{\nu_0}(x)^{-1} f(x) = b. \]  
Finally we define the ‘order’ of this boundary operator to be 
\[ \sigma(R_0) := 1/2 - \nu_0. \]
In order to treat the various boundary conditions in a unified way, we extend the definition of \( \nu_0 \) to continuous \( q \) and Neumann boundary conditions. We put 
\[ \nu_0 = \frac{1}{2} - \sigma(R_0). \]
Summing up we have 
\[ \nu_0 = \begin{cases} 
\frac{1}{2}, & \text{if } q \text{ is continuous at } 0 \text{ and the boundary condition is of type (1.22)}, \\
\frac{1}{2}, & \text{if } q \text{ is continuous at } 0 \text{ and the boundary condition is of type (1.21)}, \\
\sqrt{a_0(0) + 1/4}, & \text{if } q \text{ is of type (1.16) and not continuous at } 0,
\end{cases} \]
and \( \sigma(R_0) = \frac{1}{2} - \nu_0 \). \( \nu_1 \) and \( R_1 \) are defined analogously.

Having chosen boundary operators \( R_0, R_1 \) of the above types we obtain a self–adjoint extension, \( L := (l, R_0, R_1) \), of \( l \) with domain 
\[ \mathcal{D}(L) = \{ f \in \mathcal{D}(l_{\text{max}}) \mid R_0 f = 0, R_1 f = 0 \}. \]
Our aim is to compute the \( \zeta \)–regularized determinant of \( L \). The existence of \( \det \zeta L \) is a consequence of the following result:

**Theorem 1.1.** \((\text{BS1, 3}, \text{BL2})\) \( L \) is a discrete operator, \( (L + z^2)^{-1} \) is trace class and we have an asymptotic expansion

\[ \text{Tr}((L + z^2)^{-1}) \sim_{z \to \infty} \sum_{j=0}^{\infty} a_j z^{-1-2j} + \sum_{j=1}^{\infty} b_j z^{-1-2j} \log z + \sum_{j=0}^{\infty} c_j z^{-2-j/N} \]
\[ \sim_{z \to \infty} a_0 z^{-1} + c_0 z^{-2} + O(z^{-2-1/N} \log z). \]
Moreover
\[ a_0 = 1/2 \]
and
\[ c_0 = (\sigma(R_0) + \sigma(R_1) - 1)/2 = -\frac{1}{2}(\nu_0 + \nu_1). \]
For smooth potentials this result is classical (cf. [3, Sec. 1.7]). For regular singular operators it is due to [BS1] in case of $N = 1$. For the extension to arbitrary $N$ see [BL2].

If $L \geq 0$ then Theorem 1.1 implies in view of (1.7)

$$T(L) = -2 \int_0^\infty z \left[ \text{Tr} (L + z^2)^{-1} - a_0 z^{-1} - c_0 z^{-2} \right] \, dz$$

(1.37)

$$= -2 \int_0^1 z \text{Tr}(L + z^2)^{-1} - a_0 \, dz - 2 \int_1^\infty z \text{Tr}(L + z^2)^{-1} - a_0 - c_0 z^{-1} \, dz$$

Here, we have used $\int_0^1 z^{-1} \, dz = 0$, $\int_0^\infty z^\alpha \, dz = 0, \alpha \in \mathbb{C}$ (cf. (1.13a,b,c)). Finally, we introduce a special solution of the homogeneous equation $lf = 0$.

A function $\varphi : (0, 1) \to \mathbb{R}$ is called a normalized solution of $lf = 0$ at $0$ (resp. $1$) if

$$(1.38a) \quad l \varphi = 0, \quad R_0 \varphi = 0 \quad \text{(resp. } R_1 \varphi = 0 \text{)}$$

and

$$(1.38b) \quad \varphi(x) = x^{\nu_0 + 1/2} \varphi_1(x), \quad \varphi_1(0) = 1$$

(resp. $\varphi(x) = (1 - x)^{\nu_1 + 1/2} \varphi_1(x), \quad \varphi_1(1) = 1$).

It is clear that a normalized solution at $0$ (resp. $1$) exists and is uniquely determined.

Now we can state our main result:

**Theorem 1.2.** Let $l$ be a regular singular Sturm–Liouville operator as defined in (1.14), (1.16), (1.17). Let $R_0, R_1$ be boundary conditions as defined before. Then we have

$$\det \zeta(L) = \frac{\pi W(\psi, \varphi)}{2^{\nu_0 + \nu_1} \Gamma(\nu_0 + 1) \Gamma(\nu_1 + 1)},$$

where $\varphi$ is a normalized solution of $lf = 0$ at $0$ and $\psi$ is a normalized solution of $lf = 0$ at $1$. $W(\psi, \varphi) = \psi \varphi' - \psi' \varphi$ denotes the Wronskian of $\psi, \varphi$.

Some historical remarks are appropriate here:

For smooth potentials, S. Levit and U. Smilansky [LS] showed, that

$$\det \zeta(L) = C W(\psi, \varphi),$$

where $C$ is a constant depending only on the boundary condition. This is basically the variation result Proposition 3.4 below. T. Dreyfus and H. Dym [DD] generalized this result to operators of arbitrary order. The first who were able to calculate the constant were Burghelea, Friedlander and Kappeler, who calculated the determinant for smooth operators of arbitrary order. They considered periodic [BFK1] and separated boundary conditions [BFK2].
Our method of proof is similar to [BFK1, BFK2]. However, we do not use the asymptotic expansion of \( \det \zeta(L + z) \) for large \( z \), nor do we use the theory of complex functions of a certain order. Instead, the problem is reduced to the explicit calculation of the determinant of a single operator. Moreover we use the well-known values \( \zeta_R(0) = -1/2, \zeta_R'(0) = -\frac{1}{2} \log(2\pi) \) of the Riemann \( \zeta \)-function.

2. The determinant of the regular singular model operator

In this section we calculate the determinant of the Friedrichs extension of the model operator \( l_{\nu} [3.30] \), \( \nu \geq 0 \). Let \( R_1 f = f(1) \) and put

\[
L_{\nu} := (l_{\nu}, R_0, R_1) = l_{\nu}^F.
\]

The kernel, \( k(x, y; z) \), of the resolvent \( (L_{\nu} + z^2)^{-1} \) is given in terms of the modified Bessel functions \( I_\nu, K_\nu \) (cf. [BS1]) by

\[
k(x, y; z) = (xy)^{1/2}I_\nu(xz)(K_\nu(yz) - \frac{K_\nu(z)}{I_\nu(z)}I_\nu(yz)), \quad x \leq y.
\]

Moreover, \( L_{\nu} \) is invertible and the kernel of \( L_{\nu}^{-1} \) is

\[
k_{\nu}(x, y) := k_{\nu}(x, y; 0) = \begin{cases} x^{\nu+1/2}(y^{-\nu+1/2} - y^{\nu+1/2})/2\nu, & \nu > 0, \ x \leq y, \\ -x^{1/2}y^{1/2} \log(y), & \nu = 0, \ x \leq y. \end{cases}
\]

We adopt the following notation: multiplication operators by functions are denoted by the corresponding capital letters. For example the multiplication operator by \( x \) is denoted by \( X \).

**Lemma 2.1.** Let \( \nu > 0 \). Then \( X^{-1}L_{\nu}^{-1} \in C_2(L^2[0,1]) \) is a Hilbert–Schmidt operator. Moreover \( \nu \mapsto X^{-1}L_{\nu}^{-1}, \nu > 0, \) is a continuous map into \( C_2(L^2[0,1]) \).

**Proof.** This follows immediately from the kernel representation. For instance, we have for \( x \leq y \)

\[
2\nu x^{-1}|k_{\nu}(x, y)| \leq x^{\nu-1/4}y^{-\nu+1/2}/2\nu + x^{\nu-1/4}y^{\nu+1/2} \leq 2x^{\nu-1/2}y^{\nu+1/2},
\]

thus

\[
\nu^2 \int_0^1 \int_0^y x^{-2}|k_{\nu}(x, y)|^2dx dy \leq \int_0^1 y^{1-2\nu} \int_0^y x^{2\nu-1}dx dy = 1/4\nu < \infty.
\]

The estimate for \( x \geq y \) is similar and the continuity statement is obvious. \( \square \)

From this lemma we infer that \( \text{Tr}((L_{\nu} + z^2)^{-1}), \nu > 0, \) is differentiable with respect to \( \nu \) and

\[
\frac{d}{d\nu} \text{Tr}((L_{\nu} + z^2)^{-1}) = -2\nu \text{ Tr}(X^{-1}(L_{\nu} + z^2)^{-2}X^{-1}).
\]
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Namely, for \( \nu_1, \nu_2 \) we find

\[
(L_{\nu_1} + z^2)^{-1} - (L_{\nu_2} + z^2)^{-1} = (\nu_2^2 - \nu_1^2)(L_{\nu_1} + z^2)^{-1}X^{-1}(L_{\nu_2} + z^2)^{-1}.
\]

Hence by Lemma 2.1, \( \text{Tr}((L_{\nu} + z^2)^{-1}) \) is differentiable as a map from \((0, \infty)\) into the space of trace class operators and we obtain the formula.

Now we come to the main tool for calculating the determinant of \( L_{\nu} \).

**Proposition 2.2.** \( \nu \mapsto T(L_{\nu}), \nu > 0, \) is differentiable and

\[
\frac{d}{d\nu} T(L_{\nu}) = -\log 2 - \frac{\Gamma'(\nu + 1)}{\Gamma(\nu + 1)}.
\]

**Proof.** First we show that we can differentiate under the integral in (1.7). From the preceding considerations, we conclude that

\[
(0, \infty) \times [0, \infty) \ni (\nu, z) \mapsto \text{Tr}((L_{\nu} + z^2)^{-1})
\]

is continuously differentiable, hence we have

\[
(2.5a) \quad \frac{d}{d\nu} \int_0^1 z \text{ Tr}((L_{\nu} + z^2)^{-1}) dz = -2\nu \int_0^1 z \text{ Tr}(X^{-1}(L_{\nu} + z^2)^{-2}X^{-1}) dz.
\]

To show that

\[
(2.5b) \quad \frac{d}{d\nu} \int_1^\infty z \text{ Tr}((L_{\nu} + z^2)^{-1}) dz = -2\nu z \text{ Tr}(X^{-1}(L_{\nu} + z^2)^{-2}X^{-1}) + 1/2 z^{-1} dz,
\]

it is enough to prove the estimate

\[
| -2z \text{ Tr}(X^{-1}(L_{\nu} + z^2)^{-2}X^{-1}) + \frac{1}{2\nu} z^{-1} | \leq c|z|^{-2}
\]

with \( c \) locally independent of \( \nu \). Then the differentiability is a consequence of the dominated convergence theorem.

Since

\[
-2z (L_{\nu} + z^2)^{-2} = \frac{d}{dz}(L_{\nu} + z^2)^{-1},
\]

the kernel of \(-2z X^{-1}(L_{\nu} + z^2)^{-2}X^{-1}\) is given by

\[
\frac{d}{dz} x^{-1} k_{\nu}(x, y; z) y^{-1},
\]

hence

\[
-2z \text{ Tr}(X^{-1}(L_{\nu} + z^2)^{-2}X^{-1}) = \int_0^1 x^{-1} \frac{d}{dz} I_{\nu}(xz) \left( K_{\nu}(xz) - \frac{K_{\nu}(z)}{I_{\nu}(z)} I_{\nu}(xz) \right) dx.
\]
We have
\[ \int_0^1 x^{-1} \frac{d}{dz} I_\nu(xz) K_\nu(xz) \, dx = z^{-1} \int_0^z \frac{d}{dx} I_\nu(x) K_\nu(x) \, dx = z^{-1} I_\nu K_\nu(z) - \frac{1}{2 \nu} z^{-1}. \]

Furthermore
\[ \int_0^1 x^{-1} \frac{d}{dz} I_\nu(xz) K_\nu(xz) \, dx \]
\[ = K_\nu(z) \int_0^1 x^{-1} \frac{d}{dz} I_\nu(xz) \, dx + \int_0^1 x^{-1} I_\nu(xz) \, dx \frac{d}{dz} \frac{K_\nu(z)}{I_\nu(z)} \]
\[ = z^{-1} I_\nu K_\nu(z) + \int_0^z x^{-1} I_\nu(x) \, dx \frac{d}{dz} \frac{K_\nu(z)}{I_\nu(z)}, \]
hence we find
\[ -2z \text{Tr}(X^{-1}(L_\nu + z^2)^{-2} X^{-1}) + \frac{1}{2 \nu} z^{-1} = -\int_0^z x^{-1} I_\nu(x)^2 \, dx \frac{d}{dz} \frac{K_\nu(z)}{I_\nu(z)}. \]

Now the estimate (2.6) follows from the asymptotics of the modified Bessel functions [Wat, 7.23]

(2.7a): \( I_\nu(x) = \frac{1}{\sqrt{2 \pi x}} e^x (1 + O(x^{-2})), \quad x \to \infty, \)

(2.7b): \( K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} (1 + O(x^{-2})), \quad x \to \infty, \)

which can be differentiated with respect to \( x \) and are locally uniform in \( \nu \).

In view of Theorem 1.1, (1.37), (2.5a,b) we have proved that \( T(L_\nu) \) is differentiable and

\[ \frac{d}{d\nu} T(L_\nu) = 4\nu \int_0^1 z \text{Tr}(X^{-1}(L_\nu + z^2)^{-2} X^{-1}) \, dz \]
\[ + 2 \int_1^\infty 2z \nu z \text{Tr}(X^{-1}(L_\nu + z^2)^{-2} X^{-1}) - 1/2 z^{-1} \, dz \]

(2.8)
\[ = 4\nu \int_0^\infty z \text{Tr}(X^{-1}(L_\nu + z^2)^{-2} X^{-1}) \, dz. \]

Furthermore, using (2.2), we find

\[ 4\nu \int_0^\infty z \text{Tr}(X^{-1}(L_\nu + z^2)^{-2} X^{-1}) \, dz \]
\[ = -2\nu \int_0^\infty \int_0^1 x^{-1} \frac{d}{dz} I_\nu(xz) \left( K_\nu(xz) - \frac{K_\nu(z)}{I_\nu(z)} I_\nu(xz) \right) \, dx \]
\[ = -2\nu \int_0^\infty z^{-1} I_\nu K_\nu(z) - \frac{1}{2\nu} z^{-1} - \frac{d}{dz} \left( \frac{K_\nu(z)}{I_\nu(z)} \int_0^1 x^{-1} I_\nu(xz)^2 \, dx \right) \, dz \]
\[ -2\nu \int_0^\infty z^{-1} I\nu K\nu(z)dz + 2\nu \int_0^\infty \frac{d}{dz} \left( \frac{K\nu(z)}{I\nu(z)} \int_0^z x^{-1} I\nu(x)^2 dx \right) dz =: -2\nu I_1 + 2\nu I_2. \]

The first integral is well-known (cf. e.g. [BS2, p. 418]). One has, more generally,

\[ \int_0^\infty x^s I\nu K\nu(x)dx = \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(-s/2\right)\Gamma\left(\nu + \frac{s+1}{2}\right)}{4\sqrt{\pi}\Gamma\left(\nu - \frac{s+1}{2}\right)}. \]

Since the right hand side has a simple pole at \( s = -1 \), we find using the Mellin-transform definition of \( \int \frac{d}{ds} \bigg|_{s=-1} \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(-s/2\right)\Gamma\left(\nu + \frac{s+1}{2}\right) \)

\[ I_1 = \frac{\log 2}{2\nu} + \frac{1}{4\nu} \left( \frac{\Gamma'}{\Gamma}(\nu) + \frac{\Gamma'}{\Gamma}(\nu + 1) \right). \]

Since

\[ K\nu\left(\frac{z}{I\nu}\right) \int_0^z x^{-1} I\nu(x)^2 dx = O(z^{-2}), \quad z \to \infty, \]

\( I_2 \) is actually a regular integral and we find

\[ I_2 = -\lim_{z \to 0} \frac{K\nu}{I\nu}(z) \int_0^z x^{-1} I\nu(x)^2 dx = -(1/2\nu)^2. \]

Thus we end up with

\[ \frac{d}{d\nu} T(L\nu) = -\log 2 - \frac{1}{2} \left( \frac{\Gamma'}{\Gamma}(\nu) + \frac{\Gamma'}{\Gamma}(\nu + 1) \right) - \frac{1}{2\nu} \]

\[ = -\log 2 - \frac{\Gamma'}{\Gamma}(\nu + 1). \]

\[ \Box \]

An immediate consequence is the

**Theorem 2.3.** Let \( L\nu, \nu \geq 0 \), be the operator defined in (2.1). Then the \( \zeta \)-regularized determinant of \( L\nu \) is given by

\[ \det_\zeta L\nu = \frac{\sqrt{2\pi}}{2^\nu \Gamma(\nu + 1)}. \]

**Proof.** From the preceding proposition we infer for \( \nu > 0 \)

\[ \frac{d}{d\nu} \log \det_\zeta L\nu = -\frac{d}{d\nu} \left( \nu \log 2 + \log \Gamma(\nu + 1) \right) = \frac{d}{d\nu} \log \sqrt{2\pi} \frac{1}{2^\nu \Gamma(\nu + 1)}, \]

hence it suffices to check the formula for \( \nu = 1/2 \) and \( \nu = 0 \). \( L_{1/2} \) is just \( -\frac{d}{dx^2} \) with Dirichlet boundary conditions,

\[ \text{spec} \left( L_{1/2} \right) = \left\{ n^2 \pi^2 \mid n \in \mathbb{N} \right\}, \]
thus
\[ \zeta_{L_{1/2}}(s) = \pi^{-2s} \zeta_R(2s), \]
where \( \zeta_R \) denotes the Riemann zeta–function. In view of the well–known formulas
\[ \zeta_R(0) = -\frac{1}{2}, \quad \zeta_R'(0) = -\frac{1}{2} \log (2\pi) \]
we find
\[ \log \det \zeta(L_{1/2}) = -\zeta_{L_{1/2}}'(0) = 2 \log \pi \zeta_R(0) - 2 \zeta_R'(0) = \log 2 = \log \sqrt{2\Gamma(3/2)}. \]

To prove the result for \( \nu = 0 \) it is enough to show that \( \nu \mapsto \det \zeta(L_{\nu}) \) is continuous
at \( \nu = 0 \). Similar to the argument (2.6) in the proof of Proposition 2.2, it suffices to
prove the estimate
\[ |z \text{Tr}((L_{\nu} + z^2)^{-1}) - 1/2 + \frac{1}{2} (\nu + 1/2)z^{-1}| \leq c|z|^{-2} \tag{2.10} \]
with \( c \) locally independent of \( \nu \). Then continuity is a consequence of the dominated
convergence theorem. The estimate (2.10) is a consequence of Theorem 1.1. That
the constant \( c \) is indeed locally independent of \( \nu \) follows easily from the asymptotic
relations (2.7a,b). \( \square \)

3. Variation formulas

Lemma 3.1. Let \( L_{\nu}, \nu \geq 0, \) be the operator defined in \([3.1]\). Then for \( \delta > 0, z \geq 0 \) the operator \( X^{\delta-1}(L_{\nu} + z^2)^{-1/2} \) is Hilbert–Schmidt and we have the estimate
\[ \|X^{\delta-1}(L + z^2)^{-1/2}\|_{C_2} = \begin{cases} O(z^{-\delta}), & 0 < \delta < 1/2, \\ O(z^{-\delta}\log z), & \delta = 1/2, \\ O(z^{-1/2}), & \delta > 1/2, \end{cases} \]
as \( z \to \infty \).

Proof. Introducing the first order operator
\[ D_\nu := \frac{d}{dx} + (\nu - 1/2)X^{-1} \tag{3.1} \]
one checks that \( L_{\nu} = D_{\nu,\text{max}} D_{\nu,\text{min}} \) (cf. \([3.1]\) Lemma 3.1]). Moreover, since \( L_{\nu} \)
is the Friedrichs extension of \( l \), the domain of \( L_{1/2} \) is the completion of \( C_0^\infty(0,1) \) with
respect to the norm
\[ \|f\|_{L_{1/2}}^2 = (f,f) + (L_{1/2}f, L_{1/2}f) = (f,f) + (L_{\nu}f, f) \]
\[ = (f,f) + (D_{\nu}^2 f, f) = (f,f) + (D_{\nu}f, D_{\nu}f). \]
But since the latter is the square of the graph norm of \( D_{\nu} \) we find \( D(L_{1/2}^2) = D(D_{\nu,\text{min}}) \).
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From \[\text{[BS3, Lemma 2.1]}\] we have for \(f \in \mathcal{D}(D_{\nu, \min}) = \mathcal{D}(L_{\nu}^{1/2})\)

\[
|f(x)| \leq c|x \log x|^{1/2} \|D_{\nu, \min}f\| = c|x \log x|^{1/2} \|L_{\nu}^{1/2}f\|.
\]

(3.2)

Now let \(\tilde{k}(x, y; z)\) be the kernel of \((L + z^2)^{-1/2}\). Then \((3.2)\) implies in view of the Theorem of Riesz

\[
\int_0^1 |\tilde{k}(x, y; z)|^2 dy \leq c \ |x| \log x
\]

with \(c\) independent of \(z \geq 0\). Thus \(X^{\delta-1}(L_{\nu} + z^2)^{-1/2}\) is a Hilbert–Schmidt operator with Hilbert–Schmidt norm uniformly bounded for \(z \geq 0\).

This proves that \(X^{\delta-1}(L_{\nu} + z^2)^{-1/2}\) is trace class for \(z \geq 0\). Moreover, since \(X^{\delta-1}(L_{\nu} + z^2)^{-1}X^{\delta-1} \geq 0\) we find

\[
\|X^{\delta-1}(L_{\nu} + z^2)^{-1/2}\|_{\mathcal{L}_2}^2 = \text{Tr} \left( X^{\delta-1}(L_{\nu} + z^2)^{-1}X^{\delta-1} \right)
\]

\[
= \int_0^1 x^{\delta-1} I_{\nu} K_{\nu}(xz) dx - \frac{K_{\nu}}{I_{\nu}}(z) \int_0^1 x^{\delta-1} I_{\nu}(xz)^2 dx
\]

\[
= z^{-\delta} \left( \int_0^z u^{\delta-1} I_{\nu} K_{\nu}(u) du - \frac{K_{\nu}}{I_{\nu}}(z) \int_0^z u^{\delta-1} I_{\nu}(u)^2 du \right).
\]

Now the assertion follows easily from the well–known asymptotics of \(I_{\nu}(x), K_{\nu}(x)\) as \(x \to 0\)

\[
I_{\nu}(x) \sim cx^\nu, \quad K_{\nu}(x) \sim \begin{cases} 
  c x^{-\nu}, & \nu > 0, \\
  c \log x, & \nu = 0,
\end{cases}
\]

and the asymptotics (2.74b) as \(x \to \infty\).

Next we sketch the construction of the resolvent of general \(L = (l, R_0, R_1)\) (cf. \[\text{[BS1, Sec. 4]}\]). Let

\[
l = -\frac{d^2}{dx^2} + q(x)
\]

be a regular singular Sturm–Liouville operator as defined in (1.14), (1.16), (1.17). Let \(L = (l, R_0, R_1)\) be a self–adjoint extension. We consider the case that \(q\) is not continuous at both ends. The other cases are easier. We choose cut–off functions \(\varphi, \tilde{\varphi}, \psi, \tilde{\psi} \in C^\infty([0, 1])\) as follows:

\[
supp \varphi \subset [0, 1/3], \quad supp \psi \subset [2/3, 1],
\]

\[
\varphi[0, 1/6] = 1, \quad \psi[5/6, 1] = 1,
\]

\[
\tilde{\varphi}[0, 1/3] = 1, \quad \tilde{\psi}[2/3, 1] = 1,
\]

\[
\tilde{\varphi} + \tilde{\psi} = 1.
\]

Then we can write

\[
l = -\frac{d^2}{dx^2} + \varphi(v_0^2 - 1/4)X^{-2} + \psi(v_1^2 - 1/4)(1 - X)^{-2} + \tilde{q}(x)
\]

(3.6)

and

\[
\tilde{q}(x) = O(x^{1/N-2}), x \to 0, \quad \tilde{q}(x) = O((1 - x)^{1/N-2}), x \to 1.
\]

(3.7)
For $\nu \geq 0$ we denote by $\tilde{L}_\nu$ the Friedrichs extension of the operator

$$-\frac{d^2}{dx^2} + (1-x)^{-2}(\nu^2 - 1/4)$$

on $C^\infty_0(0,1) \subset L^2[0,1]$ (cf. [2.7]).

Now we put

$$G(z) := \tilde{\varphi}(L_{\nu_0} + z)^{-1} + \tilde{\psi}(\tilde{L}_{\nu_1} + z)^{-1}.$$

In view of (1.31), (1.36), (3.8) we only have to prove the estimate

$$\|L\|_{C(L^2)} \leq C(\chi, \nu, z_0).$$

Furthermore, for $\nu \geq 0$ we denote by $\tilde{L}_\nu$ the Friedrichs extension of the operator

$$-\frac{d^2}{dx^2} + (1-x)^{-2}(\nu^2 - 1/4)$$

on $C^\infty_0(0,1) \subset L^2[0,1]$ (cf. [2.7]).

Now we put

$$G(z) := \tilde{\varphi}(L_{\nu_0} + z)^{-1} + \tilde{\psi}(\tilde{L}_{\nu_1} + z)^{-1}.$$

In view of (1.31), (1.36), $G(z)$ maps $L^2[0,1]$ into the domain of $L = (l, R_0, R_1)$.

Furthermore,

$$(L + z)G(z) = I + [L, \tilde{\varphi}](L_{\nu_0} + z)^{-1} + [L, \tilde{\psi}](\tilde{L}_{\nu_1} + z)^{-1} + \tilde{\varphi}(\tilde{q} + (\varphi - 1)(\nu_0^2 - 1/4)X^{-2})(L_{\nu_0} + z)^{-1} + \tilde{\psi}(\tilde{q} + (\psi - 1)(\nu_1^2 - 1/4)(1 - X)^{-2})(\tilde{L}_{\nu_2} + z)^{-1} =: I + R(z),$$

where $[L, \tilde{\varphi}] = -\tilde{\varphi}'' - 2\tilde{\varphi}' \frac{d}{dx}$.

**Lemma 3.2.** We use the notation introduced before.

1. For any function $\chi \in C^\infty_0(0,1)$ we have for $|z| \geq z_0$

$$\|\chi \frac{d}{dx} (L_{\nu_0} + z)^{-1/2}\|_{C_2} \leq C(\chi, \nu, z_0).$$

2. For $|z| \geq z_0$ we have $\|R(z)\| \leq C(z_0)|z|^{-1/2}$, where the constant $C(z_0)$ depends only on $\varphi, \tilde{\varphi}, \psi, \tilde{\psi}$ and

$$\sup_{0 < x < 1} X^{2-1/N}(1 - X)^{2-1/N} \tilde{q}(x).$$

Furthermore, for $|z|$ large,

$$(L + z)^{-1} = G(z) \sum_{n=0}^{\infty} (-1)^n R(z)^n.$$

**Proof.** 1. Since $l$ is elliptic of order 2, we have $H^1_{loc}(0,1) \subset \mathcal{D}(L_{1/2})$ from which we reach the conclusion immediately.

2. In view of (3.9) we only have to prove the estimate $\|R(z)\| \leq C(z_0)|z|^{-1/2}$. But this is an easy consequence of Lemma 3.1 (3.9) and the proven first part of this lemma.

**Lemma 3.3.** We use the notation of page 13. For $\delta > 0, z > 0$ the operator $X^{\delta-1}(L + z^2)^{-1/2}$ is Hilbert–Schmidt and

$$\|X^{\delta-1}(L + z^2)^{-1/2}\|_{C_2} = \begin{cases} O(z^{-\delta}), & 0 < \delta < 1/2, \\ O(z^{-1/2} \log z), & \delta = 1/2, \\ O(z^{-1/2}), & \delta > 1/2. \end{cases}$$
as \( z \to \infty \). Again, the \( O \)-constant depends only on \( \varphi, \tilde{\varphi}, \psi, \tilde{\psi} \) and
\[
\sup_{0 < x < 1} X^{2-1/N} (1 - X)^{2-1/N} \tilde{q}(x).
\]
The same estimate holds for \( \|(1 - X)^{\delta-1}(L + z^2)^{-1/2}\|_{C_2} \).
If \( L \) is invertible, then \( X^{\delta-1}L^{-1} \) is Hilbert–Schmidt, too.

Proof. Lemma 3.1, Lemma 3.2 and the formula
\[
(L + z^2)^{-1/2} = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (L + z^2 + \lambda)^{-1/2} d\lambda
\]
(3.10)
which holds for \( z > 0 \) large enough, imply the assertion for \( z \geq z_0 \). If \( z \in \mathbb{C} \) with \( L + z^2 \) invertible, then we conclude from
\[
X^{\delta-1}(L + z^2)^{-1/2} = X^{\delta-1}(L + z_0^2)^{-1/2} [(L + z_0^2)^{1/2}(L + z^2)^{-1/2}]
\]
that the operator \( X^{\delta-1}(L + z^2)^{-1/2} \) is Hilbert–Schmidt, too. \( \square \)

Now we introduce smooth families of operators.
\[
l_t = -\frac{d^2}{dx^2} + q_t(x)
\]
is said to be a smooth family of operators if
\[
q_t(x) = a_0(t, x^{1/N}) x^{-2} = (1 - x)^{-2} a_1(t, (1 - x)^{1/N})
\]
with smooth functions \( a_0 \in C^\infty(I \times [0, 1]), a_1 \in C^\infty(I \times (0, 1]) \), \( I \) some interval.

**Proposition 3.4.** Let \( l_t \) be a smooth family of operators with \( \nu_0, \nu_1 \) independent of \( t \). Let \( R_0, R_1 \) be fixed boundary conditions independent of \( t \). Moreover let \( \varphi_t, \psi_t \) be normalized solutions of \( l_t f = 0 \) at 0 resp. 1. If \( L_t = (l_t, R_0, R_1) \) is invertible, then \( T(L_t) \) is smooth and we have the variation formula
\[
d\frac{d}{dt} T(L_t) = \frac{d}{dt} \log W(\psi_t, \varphi_t).
\]
Here, \( W(\psi_t, \varphi_t) = \psi_t \varphi_t' - \psi_t' \varphi_t \) denotes the Wronskian of \( \psi_t, \varphi_t \).

**Remark 3.5.** This Proposition is essentially the result of [LS] and our proof is an adaption of their proof to our more general setting.

Proof. Since \( \nu_0, \nu_1 \) are independent of \( t \), we have the estimate
\[
|\partial_t q_t(x)| \leq c x^{1/N-2} (1 - x)^{1/N-2}
\]
(3.11)
with \( c \) locally independent of \( t \).
We would like to apply the formula \( \frac{d}{dt} T(L_t) = \text{Tr}((\partial_t q_t) L_t^{-1}) \). However, as the referee pointed out to the author, the operator \( (\partial_t q_t) L_t^{-1} \) need not be of trace class. But, in view of Lemma 3.3 the operator

\[
X^{1-1/2N}(1 - X)^{-1/2N}(\partial_t q_t) L_t^{-1} X^{1/2N-1}(1 - X)^{1/2N-1}
\]

is trace class and the kernels of this operator and \( (\partial_t q_t) L_t^{-1} \) coincide on the diagonal.

We introduce the abbreviation \( \omega(x) := x(1 - x) \) and denote by \( \Omega \) the operator of multiplication by \( \omega \).

To make the preceding consideration rigorous we recall from (1.37) (note that \( L_t \) is assumed to be invertible)

\[
T(L_t) = -2 \int_0^\infty z \text{Tr}(L_t + z^2)^{-1} dz
\]

\[
= -2 \int_0^1 z \text{Tr}(L_t + z^2)^{-1} dz
\]

\[
-2 \int_1^\infty z [\text{Tr}(L_t + z^2)^{-1} - a_0 z^{-1} - c_0 z^{-2}] dz.
\]

\( a_0, c_0 \) are independent of \( t \) in view of Theorem 1.1. Formal differentiation under the integral gives

\[
(3.12) \quad \frac{d}{dt} T(L_t) = 2 \int_0^\infty z \text{Tr}((L_t + z^2)^{-1} (\partial_t q_t)(L_t + z^2)^{-1}) dz
\]

To justify this formula we estimate the integrand using Lemma 3.3

\[
|\text{Tr}((L_t + z^2)^{-1} (\partial_t q_t)(L_t + z^2)^{-1})| \\
\leq ||(L_t + z^2)^{-1} (\partial_t q_t)\Omega^{1-1/2N}||_{C_2} ||\Omega^{1/2N-1}(L_t + z^2)^{-1}||_{C_2} \\
\leq ||(L_t + z^2)^{-1/2}||^2 ||(L_t + z^2)^{-1/2}(\partial_t q_t)\Omega^{1-1/2N}||_{C_2} ||\Omega^{1/2N-1}(L_t + z^2)^{-1/2}||_{C_2} \\
= O(z^{-2-1/N})
\]

where the \( O \)–constant is locally independent of \( t \). Now (3.12) follows from the dominated convergence theorem. We continue starting from (3.12) and find

\[
\frac{d}{dt} T(L_t) = 2 \int_0^\infty z \text{Tr}((L_t + z^2)^{-1} \Omega^{1/2N-1} \Omega^{1-1/2N} (\partial_t q_t)(L_t + z^2)^{-1}) dz
\]

\[
= 2 \int_0^\infty z \text{Tr}(\Omega^{1-1/2N} (\partial_t q_t)(L_t + z^2)^{-2} \Omega^{1/2N-1}) dz
\]

\[
= - \int_0^\infty \frac{d}{dz} \text{Tr}(\Omega^{1-1/2N} (\partial_t q_t)(L_t + z^2)^{-2} \Omega^{1/2N-1}) dz
\]

\[
= \text{Tr}(\Omega^{1-1/2N} (\partial_t q_t)L_t^{-1} \Omega^{1/2N-1})
\]

\[
= \int_0^1 ((\partial_t q_t)L_t^{-1})(x, x) dx,
\]

which morally is \( \text{Tr}((\partial_t q_t)L_t^{-1}) \) although the latter in general does not exist.
The kernel of $L_t^{-1}$ is given by

\[ k(x, y) = W(\psi, \varphi)^{-1} \varphi(x)\psi(y), \quad x \leq y. \]  

Note that $W(\psi, \varphi) \neq 0$ since $L_t$ is assumed to be invertible. Differentiating the formula $\varphi'' = q_t \varphi$ with respect to $t$ gives

\[ \partial_t \varphi'' = (\partial_t q_t) \varphi + q_t \partial_t \varphi \]

and hence

\[ (\partial_t q_t) \varphi \psi = (\partial_t \varphi)'\psi - (\partial_t \varphi)\psi = \frac{d}{dx}[(\partial_t \varphi)'\psi - (\partial_t \varphi)\psi'] = \frac{d}{dx}W(\psi, \partial_t \varphi). \]

Thus we find

\[ \frac{d}{dt}T(L_t) = \int_0^1 ((\partial_t q_t) L_t^{-1})(x, x)dx \]

\[ = W(\psi, \varphi)^{-1} \int_0^1 \frac{d}{dx}W(\psi, \partial_t \varphi)(x)dx \]

\[ = W(\psi, \varphi)^{-1} [W(\psi, \partial_t \varphi)(1) - W(\psi, \partial_t \varphi)(0)]. \]

Since $\varphi$ is normalized at 0 and $\nu_0$ is constant, we have

\[ \partial_t \varphi(x) = O(x^{\nu_0+1/2+1/N}), \quad x \to 0 \]

and

\[ \partial_t \varphi'(x) = O(x^{\nu_0-1/2+1/N}), \quad x \to 0, \]

which implies immediately

\[ W(\psi, \partial_t \varphi)(x) = O(x^{1/N} \log(x)), \quad x \to 0 \]

and thus $W(\psi, \partial_t \varphi)(0) = 0$.

Reversing the roles of $\varphi, \psi$ we find $W(\partial_t \psi, \varphi)(1) = 0$. Summing up we have

\[ \frac{d}{dt}T(L_t) = W(\psi, \varphi)^{-1} [W(\psi, \partial_t \varphi)(1) + W(\partial_t \psi, \varphi)(1)] \]

\[ = W(\psi, \varphi)^{-1} \frac{d}{dt}W(\psi, \varphi) = \frac{d}{dt} \log W(\psi, \varphi). \]

\[ \square \]

The next Proposition is basically [BFK2, Proposition 3.2]. The fact that $q$ may be singular at 0 causes no essential new difficulty. To make the exposition self-contained we include a proof.
Proposition 3.6. Assume that $q$ is continuous at $1$ and let $R_{1,t}f = f'(1) + a(t)f(1)$ be a smooth family of boundary operators of order 1. Assume that $L_t = (l, R_0, R_{1,t})$ is invertible. Then we have

$$\frac{d}{dt}T(L_t) = \frac{d}{dt}\log W(\psi_t, \varphi_t),$$

where $\varphi_t, \psi_t$ are as in Proposition 3.4.

Proof. For simplicity, throughout this proof we are going to write $\varphi, \psi$ instead of $\varphi_t, \psi_t$.

Let $g \in L^2[0,1]$ and consider $f_t := L_t^{-1}g$. Differentiation of $lf_t = g$ and $R_{1,t}f_t = 0$ gives

$$l\partial_t f_t = 0, \quad R_{1,t}\partial_t f = -a'(t)f_t(1).$$

Since $L_t$ is invertible, we have $R_{1,t}\varphi \neq 0$. Now note that $\varphi$ is independent of $t$ and

$$W(\psi, \varphi)(1) = \psi(1)\varphi'(1) - \psi'(1)\varphi(1) = \varphi(1)a(t) + \varphi'(1) = R_{1,t}\varphi.$$

(3.14)

Now consider

$$\partial_t f_t + a'(t)W(\psi, \varphi)^{-1}f_t(1)\varphi = u.$$ We find

$$lu = 0, \quad R_0u = R_1u = 0$$

and again since $L_t$ is invertible,

$$\partial_t f_t = -a'(t)W(\psi, \varphi)^{-1}f_t(1)\varphi.$$ Thus $\partial_t L_t^{-1}$ is actually a rank one operator (see (3.13)):

$$(\partial_t L_t^{-1}g)(x) = -a'(t)W(\psi, \varphi)^{-2}\varphi_t(x)\int_0^1 \varphi_t(y)g(y)dy.$$ Now let $\varphi_t(x, z), \psi_t(x, z)$ be the corresponding solutions for $L_t + z^2$. Then we find

$$\partial_t \text{Tr}((L_t + z^2)^{-1}) = -a'(t)W(\psi, \varphi)^{-2}\int_0^1 \varphi_t(y, z)^2dy$$

$$= -a'(t)\int_0^1 (L_t + z^2)^{-1}(1, y)^2dy$$

$$= -a'(t)(L_t + z^2)^{-2}(1, 1)$$

$$= a'(t)\frac{1}{2z}d(L_t + z^2)^{-1}(1, 1)$$

and we reach the conclusion using (3.14) and (1.37)

$$\frac{d}{dt}T(L_t) = -a'(t)\int_0^\infty \frac{d}{dz}(L_t + z^2)^{-1}(1, 1)dz$$

$$= a'(t)(L_t + z^2)^{-1}(1, 1)$$

$$= a'(t)W(\psi, \varphi)^{-1}\varphi(1).$$
and we are done. □

**Proposition 3.7.** Let \( q_\nu(x) := (\nu^2 - 1/4)/x^2 + q_1(x), \nu > 0, \) with \( \text{supp}(q_1) \subset [\varepsilon, 1] \) for some \( \varepsilon > 0 \) and put \( \gamma_\nu := -\frac{\partial}{\partial x} + q_\nu. \) Let \( R_0 \) be as in \([1.31]\) and choose \( R_1 \) fixed. Moreover let \( \varphi_\nu, \psi_\nu \) be normalized solutions of \( \tilde{L}_\nu f = 0 \) at 0 resp. 1. If \( \tilde{L}_\nu := (l_\nu, R_0, R_1) \) is invertible then \( T(\tilde{L}_\nu) \) is smooth and we have the variation formula

\[
\frac{d}{d\nu} T(\tilde{L}_\nu) = \frac{d}{d\nu} \log W(\psi_\nu, \varphi_\nu) + \frac{d}{d\nu} T(L_\nu) = \frac{d}{d\nu} \log \frac{W(\psi_\nu, \varphi_\nu)}{2^\nu \Gamma(\nu + 1)}.
\]

**Remark 3.8.** Note that \( q_1(x) \) may be singular at 1 (cf. \([1.16]\)). This is the reason why this proposition is needed. If \( q_1 \) is smooth on \([0, 1]\) we can just apply Proposition 3.4 and deform \( q_1 \) to 0.

**Proof.** The resolvent expansion Lemma 3.3 shows that the estimate \((2.6)\) holds for \( \tilde{L}_\nu \), too. Then as in the first part of the proof of Proposition 2.2 one infers that \( T(\tilde{L}_\nu) \) is smooth and

\[
(3.15) \quad \frac{d}{d\nu} T(\tilde{L}_\nu) = 4\nu \int_0^\infty z \text{Tr}(X^{-1}(\tilde{L}_\nu + z^2)^{-2})dz.
\]

Now, since \( \text{supp} q_1 \subset [\varepsilon, 1] \) we have

\[
x^{-2}(\tilde{L}_\nu + z^2)^{-1}(x, x) - x^{-2}k_\nu(x, x; z) \in L^1[0, 1].
\]

To see this we consider \( \varphi_\nu, \psi_\nu \) defined in \((1.29a,b)\). We have

\[
k_\nu(x, y) = \varphi_\nu(x)\psi_\nu(y), \quad x \leq y = x^{\nu+1/2}(y^{\nu+1/2} - y^{-\nu+1/2}).
\]

Since \( q_1[0, \varepsilon] = 0, \varphi_\nu, \psi_\nu[0, \varepsilon] \) is also a fundamental system of solutions of the homogeneous equation \( \tilde{L}_\nu f = 0 \) in the interval \([0, \varepsilon]\). Let \( \tilde{\psi} \) be the unique function with

\[
\tilde{L}_\nu \tilde{\psi} = 0, \quad R_1 \tilde{\psi} = 0, W(\tilde{\psi}|[0, \varepsilon], \varphi_\nu|[0, \varepsilon]) = 1.
\]

Then for \( 0 \leq x \leq y \leq \varepsilon \) the kernel of \( \tilde{L}_\nu^{-1} \) is given by

\[
\tilde{L}_\nu^{-1}(x, y) = \varphi_\nu(x)\tilde{\psi}(y).
\]

Moreover, since \( (L_\nu \tilde{\psi})(x) = 0, x \leq \varepsilon \) there exist constants \( a, b \) such that

\[
\tilde{\psi}[0, \varepsilon] = a\varphi_\nu[0, \varepsilon] + b\psi_\nu[0, \varepsilon].
\]

Furthermore

\[
1 = W(\tilde{\psi}|[0, \varepsilon], \varphi_\nu|[0, \varepsilon]) = bW(\psi_\nu, \varphi_\nu) = b
\]

and hence we find for \( x \leq \varepsilon \)

\[
(\tilde{L}_\nu^{-1} - k_\nu)(x, x) = \varphi_\nu(x)(a\varphi_\nu(x) + \psi_\nu(x)) - \varphi_\nu(x)\psi_\nu(x)
\]

\[
= (a - 1)\varphi_\nu(x)^2 = O(x^{2\nu+1}), \quad x \to 0
\]
which shows (3.15) for \( z = 0 \). For arbitrary \( z \) the proof is similar.

Thus it makes sense to abbreviate

\[
\int_{0}^{1} x^{-2} \left[ (\tilde{L}_\nu + z^2)^{-1} - k_\nu(x, x; z) \right] dx
\]

although we do not claim that \( X^{-1} \left[ (\tilde{L}_\nu + z^2)^{-1} - (L_\nu + z^2)^{-1} \right] X^{-1} \) is really trace class.

Now we have

\[
\frac{d}{d\nu} T(\tilde{L}_\nu) = -2\nu \int_{0}^{\infty} \frac{d}{dz} \text{Tr}(X^{-1} \left[ (\tilde{L}_\nu + z^2)^{-1} - (L_\nu + z^2)^{-1} \right] X^{-1})dz + \frac{d}{d\nu} T(L_\nu)
\]

Using (2.3) we find

\[
-2\nu \int_{0}^{1} x^{-2} \tilde{L}_\nu^{-1}(x, x)dx = -\int_{0}^{1} x^{-1} - x^{2\nu-1} dx = \frac{1}{2\nu},
\]

and as in the proof of Proposition 3.4 we infer

\[
2\nu \int_{0}^{1} x^{-2} \tilde{L}_\nu^{-1}(x, x)dx = W(\psi_\nu, \varphi_\nu)^{-1} \left[ W(\psi_\nu, \frac{d}{d\nu} \varphi_\nu)(1) - \text{LIM}_{x \to 0} W(\psi_\nu, \frac{d}{d\nu} \varphi_\nu)(x) \right]
\]

A direct calculation shows that

\[
W(\psi_\nu, \varphi_\nu)^{-1} W(\psi_\nu, \frac{d}{d\nu} \varphi_\nu)(x) = \log x + \frac{1}{2\nu} + O(x^{2\nu}), \quad x \to 0,
\]

hence

\[
\text{LIM}_{x \to 0} W(\psi_\nu, \varphi_\nu)^{-1} W(\psi_\nu, \frac{d}{d\nu} \varphi_\nu)(x) = \frac{1}{2\nu}
\]

and the result is proved.

\[\square\]

4. Proof of the main result and examples

**Proof of Theorem 1.2** If \( L \) is not invertible, then \( \varphi \) satisfies both boundary conditions, hence \( W(\psi, \varphi) = 0 \). So we may assume that \( L \) is invertible. For \( z \in \mathbb{C} \) consider \( L + z \) and let \( \varphi(\cdot, z), \psi(\cdot, z) \) be the corresponding normalized solutions. Then
det_{\zeta}(L + z) and \( W(\psi(\cdot, z), \varphi(\cdot, z)) \) are holomorphic functions in \( \mathbb{C} \) and in view of Proposition 3.4 these functions have the same logarithmic derivative. Hence it suffices to prove the formula for \( L + z \) and Re \( z \) large.

We can deform the potential \( q(x) \), such that

\[
q(x) = \begin{cases} 
(\nu_0^2 - 1/4)x^{-2}, & x \leq \varepsilon, \\
(\nu_1^2 - 1/4)(1 - x)^{-2}, & x \geq 1 - \varepsilon,
\end{cases}
\]

and again by Proposition 3.4 it suffices to prove the result for these potentials and Re \( z \) large. If Re \( z \) is large enough we apply Proposition 3.7 and deform \( \nu_0 \) and \( \nu_1 \) to \( \pm 1/2 \) leaving a potential \( q \in C_0^\infty(0, 1) \) with compact support. Again using Proposition 3.4 we deform \( q \) to 0.

Thus it remains to prove the assertion for the operator \( \frac{d^2}{dx^2} + z \) and \( \nu_0, \nu_1 \in \{\pm 1/2\} \).

1. \( D_1 = -\frac{d}{dx} \) on \( f \in H^2[0, 1] \mid f(0) = f(1) = 0 \),
2. \( D_2 = -\frac{d}{dx} \) on \( f \in H^2[0, 1] \mid f(0) = 0, f'(1) = 0 \),
3. \( D_3 = -\frac{d}{dx} + z \) on \( f \in H^2[0, 1] \mid f'(0) = f'(1) = 0 \), \( z > 0 \).

1. \( D_1 = L_{1/2} \) and by Theorem 2.3 we have \( \det_{\zeta}(D_1) = 2 \). Moreover, \( \varphi(x) = x, \psi(x) = 1 - x \), hence

\[
\frac{\pi}{2\Gamma(3/2)^2} W(\psi, \varphi) = 2.
\]

2. We have \( \text{spec } (D_2) = \{(n + 1/2)^2\pi^2 \mid n \geq 0\} \), thus

\[
\zeta_{D_2}(s) = \pi^{-2s} \sum_{n=1}^{\infty} (n + 1/2)^{-2s} = \pi^{-2s}(2^{2s} - 1)\zeta_R(2s),
\]

and

\[
\zeta_{D_2}'(0) = 2 \log 2 \zeta_R(0) = -\log 2,
\]

hence \( \det_{\zeta}(D_2) = 2 \). Furthermore, \( \varphi(x) = x, \psi(x) = 1 \), thus

\[
\frac{\pi W(\psi, \varphi)}{2^9\Gamma(1/2)\Gamma(3/2)} = 2.
\]

3. Since the result is already proved for \( D_1 \) one finds

\[
\det_{\zeta}(D_1 + z) = 2 \frac{\sinh(\sqrt{z})}{\sqrt{z}}.
\]

Furthermore, since \( \text{spec } (D_3) = \text{spec } (D_1) \cup \{0\} \) we have

\[
\det_{\zeta}(D_3 + z) = 2\sqrt{z} \sinh(\sqrt{z}).
\]
On the other hand, we have \( \varphi(x) = \cosh(\sqrt{zx}) \), \( \psi(x) = \cosh(\sqrt{z}(x-1)) \) and
\[
W(\psi, \varphi) = \sqrt{z} \sinh(\sqrt{z})
\]
and we are done. \( \square \)

We single out the special case in which the Sturm–Liouville operator can be factorized: let
\[
d := \frac{d}{dx} + S(x),
\]
where \( S \in C^\infty((0, 1)) \) such that
\[
S(x) = \frac{s_0}{x} + S_1(x) = \frac{s_1}{1-x} + S_2(x)
\]
with \( S_1 \in C^\infty((0, 1)), S_2 \in C^\infty((0, 1)) \). Put
\[
l := d^2 - \frac{d}{dx^2} + S^2 - S'.
\]
Note that
\[
\nu_0 = |s_0 + 1/2|, \quad \nu_1 = |s_1 - 1/2|.
\]
Then the Friedrichs extension of \( l, L := l^F \), equals \( \min d_{\min} \) (cf. [BL1, Lemma 3.1]).

**Proposition 4.1.** The \( \zeta \)-regularized determinant of \( L \) is given by the following formulas:

- \( s_0 \leq -1/2, s_1 \geq 1/2 \):
  \[
  \det\zeta(L) = 0,
  \]
- \( s_0 > -1/2, s_1 < 1/2 \):
  \[
  \det\zeta(L) = \frac{\pi}{2^{s_0+s_1-2} \Gamma(\nu_0) \Gamma(\nu_1)} \exp \left( - \int_0^1 S(t)dt \right) \int_0^1 \exp \left( 2 \int_0^x S(t)dt \right) dx,
  \]
- \( s_0 > -1/2, s_1 \geq 1/2 \):
  \[
  \det\zeta(L) = \frac{\pi}{2^{s_0+s_1-1} \Gamma(\nu_0-1) \Gamma(\nu_1)} \exp \left( \int_0^1 S(t)dt \right),
  \]
- \( s_0 \leq -1/2, s_1 < 1/2 \):
  \[
  \det\zeta(L) = \frac{\pi}{2^{s_0+s_1-1} \Gamma(\nu_0+1) \Gamma(\nu_1)} \exp \left( - \int_0^1 S(t)dt \right).
  \]

**Proof.** We put
\[
h(x) := \exp \left( - \int_0^x S(t)dt \right) = x^{-s_0} \exp \left( - \int_0^1 S_1(t)dt \right).
\]
\[
= (1-x)^{s_1} \exp \left( - \int_0^1 S(t)dt \right) \exp \left( \int_1^x S_2(t)dt \right).
\]
We have $dh = 0$. Since $L = d_{\max}^* d_{\min} = d_{\min}^* d_{\min}$, we have $\ker L = \ker d_{\min}$ and $\ker d_{\min}$ is non-trivial iff

\[
\begin{align*}
  h(x) &= o(x^{1/2}|\log x|^{1/2}), \\  h(x) &= o((1-x)^{1/2}|\log (1-x)|^{1/2}), 
\end{align*}
\]

$\implies$ $\ker L$ is non-trivial iff $h(x) = o((x^{1/2}|\log x|^{1/2}), x \to 0$, $h(x) = o((1-x)^{1/2}|\log (1-x)|^{1/2}), x \to 1$, (cf. [BS3, Lemma 3.2]), thus $\ker L \neq 0$ iff $s_0 \leq -1/2$ and $s_1 \geq 1/2$.

$s_0 > -1/2, s_1 < 1/2$: We put

\[
\begin{align*}
  \varphi(x) &= (2s_0 + 1)h(x) \int_0^x h(y)^{-2}dy, \\  \psi(x) &= (1 - 2s_1) \exp \left(- \int_0^1 S(t)dt \right) h(x) \int_x^1 h(y)^{-2}dy.
\end{align*}
\]

It is easy to check that $\varphi$ is normalized at 0 and $\psi$ is normalized at 1 and $W(\psi, \varphi) = (2s_0 + 1)(2s_1 + 1) \exp \left(- \int_0^1 S(t)dt \right) \int_0^1 h(y)^{-2}dy$.

Using Theorem 1.2 we reach the conclusion.

$s_0 > -1/2, s_1 \geq 1/2$: We put

\[
\begin{align*}
  \varphi(x) &= (2s_0 + 1)h(x) \int_0^x h(y)^{-2}dy, \\  \psi(x) &= \exp \left( \int_0^1 S(t)dt \right) h(x).
\end{align*}
\]

Then $\varphi$ is normalized at 0 and $\psi$ is normalized at 1 and $W(\psi, \varphi) = (2s_0 + 1) \exp \left( \int_0^1 S(t)dt \right)$

and again we reach the conclusion using Theorem 1.2

$s_0 \leq -1/2, s_1 < 1/2$:

This is proved analogously to the case $s_0 > -1/2, s_1 \geq 1/2$.

As a classical example we discuss

4.1. The Jacobi differential operator

For $\alpha, \beta > -1$, the Jacobi polynomials, $P_n^{(\alpha, \beta)}$, $n \geq 0$, form a complete orthogonal set in the Hilbert space

\[
(4.6) \quad \mathcal{H} := L^2([-1,1], (1-x)^\alpha (1+x)^\beta dx).
\]

We put

\[
(4.7) \quad \varrho(x) := (1-x)^\alpha (1+x)^\beta, \quad p(x) := 1 - x^2.
\]
\( P_n^{(\alpha, \beta)} \) satisfies the differential equation [3] p. 258]

\[
- \frac{1}{g(x)} \frac{d}{dx} g(x)p(x) \frac{d}{dx} P_n^{(\alpha, \beta)}(x) = n(n + \alpha + \beta + 1) P_n^{(\alpha, \beta)},
\]

thus \( P_n^{(\alpha, \beta)} \) are eigenfunctions of the operator

\[
j := -\frac{1}{g(x)} \frac{d}{dx} g(x)p(x) \frac{d}{dx}
\]

and it is not difficult to see that the \( P_n^{(\alpha, \beta)} \) are in the domain of the self–adjoint extension \( J = d_{\text{max}}^* d_{\text{max}} \), where \( d = \sqrt{g(x)} \). Hence we have

\[
\text{spec}(J) = \{ n(n + \alpha + \beta + 1) \mid n = 0, 1, 2, \ldots \}.
\]

Next we transform \( j \) into an operator in \( L^2[0, 1] \). We put

\[
\kappa(x) := \frac{1}{\pi} \arcsin(x) + 1/2, \quad -1 \leq x \leq 1.
\]

Now a straightforward calculation shows:

**Lemma 4.2.** The map

\[
\Phi : L^2[0, 1] \longrightarrow L^2([-1, 1], g(x)dx), \quad f \mapsto \sqrt{\kappa} g^{-1/2} f \circ \kappa
\]

is unitary and we have

\[
\Phi^* j \Phi = \frac{1}{\pi^2} d^* d
\]

with

\[
d = \frac{d}{dx} - \frac{\pi}{2} (1 + \alpha + \beta) \cot(\pi x) + \frac{\pi}{2} \frac{\alpha - \beta}{\sin(\pi x)}.
\]

Thus we are almost in the situation of Proposition 4.1, except that 0 is an eigenvalue of \( J \) and hence of \( \Phi^* J \Phi \). Note that

\[
\Phi^* J \Phi = \frac{1}{\pi^2} d_{\text{max}}^* d_{\text{max}},
\]

hence

\[
\text{spec}(\Phi^* J \Phi) = \text{spec}(\frac{1}{\pi^2} d_{\text{max}}^* d_{\text{max}}) \cup \{0\}
\]

and \( d_{\text{max}}^* d_{\text{max}} = (d_{\text{min}}^*)^* d_{\text{min}} = (dd^*)^F \). Now we have

\[
-d^* = \frac{d}{dx} + \frac{\pi}{2} (1 + \alpha + \beta) \cot(\pi x) - \frac{\pi}{2} \frac{\alpha - \beta}{\sin(\pi x)}
\]

\[
= \frac{d}{dx} + S(x) =: d_{\alpha, \beta},
\]

and

\[
S(x) \sim -\frac{1/2 + \beta}{1 - x}, \quad x \to 0, \quad S(x) \sim \frac{-1/2 - \alpha}{1 - x}, \quad x \to 1.
\]
We calculate \( d_{\alpha,\beta}^t d_{\alpha,\beta} \) explicitly:

\[
\begin{align*}
  d_{\alpha,\beta}^t d_{\alpha,\beta} &= \frac{-d^2}{dx^2} + \frac{\pi^2}{4} \left( ((\alpha + \beta + 2)^2 - 1) \cot^2 \pi x + \frac{(\alpha - \beta)^2}{\sin^2 \pi x} ight) \\
  &= l_{\alpha,\beta}.
\end{align*}
\]

(4.15)

(4.16)

Let \( L_{\alpha,\beta} := l_{\alpha,\beta}^F \) be the Friedrichs extension of \( l_{\alpha,\beta} \). \( L_{\alpha,\beta} \) obviously makes sense even for \( \alpha = -1 \) or \( \beta = -1 \).

Lemma 4.3. We have

\[
\text{spec } (L_{\alpha,\beta}) = \{ \pi^2 n(n + \alpha + \beta + 1) \mid n = 1, 2, \ldots \}, \quad \alpha, \beta \geq -1.
\]

(4.17)

Proof. By Proposition 4.1 \( L_{\alpha,\beta} \) is invertible for \( (\alpha, \beta) \neq (-1, -1) \). Hence, for \( \alpha, \beta > -1 \), the assertion follows from (4.10) and (4.13).

Now, a straightforward calculation shows:

\[
\begin{align*}
  d_{\alpha,\beta}^t d_{\alpha,\beta} &= \frac{-d^2}{dx^2} + \frac{\pi^2}{4} \left( ((\alpha + \beta + 2)^2 - 1) \cot^2 \pi x + \frac{(\alpha - \beta)^2}{\sin^2 \pi x} ight) \\
  &= l_{\alpha-1,\beta-1} - \pi^2 (\alpha + \beta).
\end{align*}
\]

(4.18)

For \( \alpha \geq 0, \beta \geq 0 \) we infer from [BS3, Lemma 3.2] and (4.14) that \( d_{\alpha,\beta,\max} = d_{\alpha,\beta,\min} \) and hence we find

\[
\begin{align*}
  L_{\alpha-1,\beta-1} - \pi^2 (\alpha + \beta) &= (d_{\alpha,\beta}^t d_{\alpha,\beta})^F = d_{\alpha,\beta,\max}(d_{\alpha,\beta}^t)_{\min} \\
  &= d_{\alpha,\beta,\min}(d_{\alpha,\beta,\min})^*.
\end{align*}
\]

(4.19)

Moreover, from Proposition 4.1 we infer \( \ker (d_{\alpha,\beta,\min})^* \neq 0 \), hence \( 0 \in \text{spec } (L_{\alpha-1,\beta-1} - \pi^2 (\alpha + \beta)) \) and (4.19) implies

\[
\begin{align*}
  \text{spec } (L_{\alpha-1,\beta-1} - \pi^2 (\alpha + \beta)) &= \text{spec } ((d_{\alpha,\beta,\min})^* d_{\alpha,\beta,\min}) \cup \{0\} \\
  &= \text{spec } (L_{\alpha,\beta}) \cup \{0\},
\end{align*}
\]

thus

\[
\begin{align*}
  \text{spec } (L_{\alpha-1,\beta-1}) &= \{ \pi^2 n(n + \alpha + \beta + 1) + \pi^2 (\alpha + \beta) \mid n = 0, 1, 2, \ldots \} \\
  &= \{ \pi^2 n(n + (\alpha - 1) + (\beta - 1) + 1) \mid n = 1, 2, \ldots \}.
\end{align*}
\]
We calculate $\det_\zeta(L)$ using Proposition 4.1:

$$-\log h(x) = \int_0^x S(t) dt = \lim_{\varepsilon \to 0} \int_\varepsilon^x S(t) dt$$

$$= \lim_{\varepsilon \to 0} \left\{ \frac{1 + \alpha + \beta}{2} \log \sin(\pi t) \bigg|_{t=\varepsilon}^{t=x} + \frac{\beta - \alpha}{2} \log \left( \frac{1 - \cos(\pi t)}{\sin(\pi t)} \right) \right\}_{t=\varepsilon}^{t=x}$$

$$= \frac{1 + \alpha + \beta}{2} \log \sin(x) - \frac{1 + \alpha + \beta}{2} \log \pi + \frac{\beta - \alpha}{2} \log \left( \frac{1 - \cos(\pi x)}{\sin(\pi x)} \right) + \frac{\alpha - \beta}{2} \log \frac{\pi}{2},$$

and

$$\int_0^1 S(t) dt = \lim_{x \to 1} \int_0^x S(t) dt = (\alpha - \beta) \log \frac{\pi}{2}.$$ 

Moreover

$$\int_0^1 h(x)^{-2} dx = \pi^{-1-2\beta} 2^{1+2\beta} \int_0^1 \left( \sin \left( \frac{\pi}{2} x \right) \right)^{1+2\beta} \left( \cos \left( \frac{\pi}{2} x \right) \right)^{1+2\alpha} dx$$

$$= \pi^{-2-2\beta} 2^{1+2\beta} \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(2 + \alpha + \beta)}.$$ 

Using Proposition 4.1 we have proved:

**Proposition 4.4.** For $\alpha, \beta \geq -1$, the determinant of the Jacobi operator $L_{\alpha, \beta}$ is

$$\det_\zeta(L_{\alpha, \beta}) = 2^{\pi-1-\alpha-\beta} \frac{\Gamma(2 + \alpha + \beta)}{\Gamma(2 + \alpha + \beta)}.$$ 

Note that if $\alpha = \beta = -1$ then $\det_\zeta(L_{\alpha, \beta}) = 0$. Since $\Gamma$ has a pole at 0, the formula also covers this case.

Since we know $\text{spec}(L)$ explicitly, this result can also be proved directly. This in fact leads to an alternative proof of Theorem 1.2, that does not make use of section 2.

For doing this, we introduce the function

$$\zeta_\lambda(s) := \sum_{n=1}^{\infty} n^{-s} (n + \lambda)^{-s}, \quad \lambda > -1, \text{ Re } s > 1/2. \quad (4.20)$$

**Lemma 4.5.** $\zeta_\lambda$ has a meromorphic continuation to $\mathbb{C}$. $\zeta_\lambda$ is regular at $s = 0$ and we have

$$\zeta_\lambda(0) = -\frac{1 + \lambda}{2}, \quad \zeta_\lambda'(0) = -\log 2\pi + \log \Gamma(\lambda + 1).$$
Proof. That $\zeta_\lambda$ has a meromorphic continuation is well-known. A simple way of seeing this is

$$\zeta_\lambda(s) = \sum_{1 \leq n \leq \lambda+1} n^{-s}(n + \lambda)^{-s} + \sum_{n > \lambda+1} n^{-2s}(1 + \frac{\lambda}{n})^{-s}$$

and the right hand side is a meromorphic function in the whole plane. Moreover we have

$$\zeta_\lambda(s) = \zeta_R(2s) - \lambda s \zeta_R(2s + 1) + \sum_{n=1}^{\infty} \left[ n^{-s}(n + \lambda)^{-s} - n^{-2s} + \lambda sn^{-2s-1} \right]$$

Since

$$|n^{-s}(n + \lambda)^{-s} - n^{-2s} + \lambda sn^{-2s-1}| = O(s\lambda n^{-2\text{Re } s-2})$$

this shows that $\zeta_\lambda$ is regular at $s = 0$. We find

$$\zeta_\lambda(0) = \zeta_R(0) - \frac{\lambda}{2} \text{Res}_{s=1} \zeta_R(s) = -\frac{1 + \lambda}{2}$$

and

$$\zeta'_\lambda(0) = 2\zeta'_R(0) - \lambda \frac{d}{ds} \bigg|_{s=0} \left( s\zeta_R(2s + 1) \right) - \sum_{n=1}^{\infty} \left[ \log (n + \lambda) - \log n - \frac{\lambda}{n} \right]$$

In view of [4.17] we have

$$\zeta_L(s) = \pi^{-2s} \zeta_{1+\alpha+\beta}(s)$$

thus

$$\log \det_\zeta(L) = 2 \log \pi \zeta_{1+\alpha+\beta}(0) - \zeta'_{1+\alpha+\beta}(0) = \log \frac{2\pi^{-1-\alpha-\beta}}{\Gamma(2 + \alpha + \beta)}$$

As promised, we sketch a

**Second proof of Theorem 1.2.** As in the first proof, use Propositions 3.4 and 3.6 to show that

$$\det_\zeta(L) = c(\nu_0, \nu_1) W(\psi, \varphi)$$
with some constant \( c(\nu_0, \nu_1) \) depending only on \( \nu_0, \nu_1 \). Since Proposition 4.4 can be proved directly, we may use it to show that

\[
c(\nu_0, \nu_1) = \frac{\pi}{2^{\nu_0 + \nu_1} \Gamma(\nu_0 + 1) \Gamma(\nu_1 + 1)}, \quad \text{if} \quad \nu_0, \nu_1 \geq 0, \, \nu_0 + \nu_1 > 0.
\]

For fixed \( \nu_0 \) choose a symmetric potential \( q(x) = q(1-x) \) with \( \nu_0 = \nu_0(q) \). Then an easy calculation shows that the eigenvalues of \((-\frac{d^2}{dx^2} + q) F\) consist of the union of the eigenvalues of \((-\frac{d^2}{dx^2} + q)\) on \([0, 1/2]\) with Dirichlet and Neumann boundary conditions at 1/2 and the original boundary condition at 0. A direct calculation shows that this implies

\[
c(\nu_0, \nu_0) = \frac{1}{2} c(\nu_0, 1/2) c(\nu_0, -1/2),
\]

proving

\[
c(\nu_0, -1/2) = 2 \frac{c(\nu_0, \nu_0)}{c(\nu_0, 1/2)} \pi \frac{1}{2^{\nu_0 - 1/2} \Gamma(\nu_0 + 1) \Gamma(1/2)}
\]

and

\[
c(1/2, -1/2) = 2.
\]

\( c(-1/2, -1/2) \) is now calculated as in the first proof. The case \( \nu_0 = \nu_1 = 0 \) has to be treated separately. We leave the details to the reader. \( \square \)

4.2. \( \det_\zeta(L + z) \) as an infinite product

**Proposition 4.6.** Let \( L \) be a semibounded invertible self-adjoint operator in the Hilbert space \( \mathcal{H} \) satisfying (1.1) and (1.2). Let \( (\lambda_n)_{n \in \mathbb{N}} \) be the eigenvalues of \( L \). Then we have for \( z \in \mathbb{C} \)

\[
\det_\zeta(L + z) = \det_\zeta(L) \prod_{n=1}^{\infty} \left( 1 + \frac{z}{\lambda_n} \right).
\]

**Proof.** In view of (1.1) left and right hand side of the equation are entire holomorphic functions and we find

\[
\frac{d}{dz} \log \det_\zeta(L + z) = \text{Tr}((L + z)^{-1}) = \sum_{n=1}^{\infty} (\lambda_n + z)^{-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{d}{dz} \log (1 + \frac{z}{\lambda_n}) = \frac{d}{dz} \log \prod_{n=1}^{\infty} (1 + \frac{z}{\lambda_n}).
\]

Since the assertion is obviously true for \( z = 0 \) we reach the conclusion. \( \square \)

We apply this formula to a regular singular Sturm–Liouville operator \( L = (l, R_0, R_1) \). Let \( \varphi(\cdot, z), \psi(\cdot, z) \) be the normalized solutions for \( L + z^2 \). Then applying Theorem 1.2
and the preceding proposition we find
\[
\prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{\lambda_n} \right) = \frac{\det \zeta(L + z^2)}{\det \zeta(L)} = \frac{W(\psi(\cdot, z), \varphi(\cdot, z))}{W(\psi(\cdot, 0), \varphi(\cdot, 0))}.
\]

In the case of the operator \(-\frac{d^2}{dz^2}\) with Dirichlet boundary conditions, this is essentially the product expansion of \(\sinh\), namely we have \(\varphi(x, z) = \sinh(xz)/z\), thus
\[
\prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{n^2\pi^2} \right) = \frac{\varphi(1, z)}{\varphi(1, 0)} = \frac{\sinh(z)}{z}.
\]

More generally, let \((\lambda_{n,\nu})_{n \in \mathbb{N}}\) be the zeros of the Bessel function \(J_\nu\). Then we have
\[
\text{spec } (L_\nu) = \{ \lambda_{n,\nu}^2 | n \in \mathbb{N} \}.
\]
Furthermore
\[
\varphi(x, z) = 2^\nu \Gamma(\nu + 1)z^{-\nu}I_\nu(xz), \quad \varphi(x, 0) = x^{\nu+1/2},
\]
thus
\[
\prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{\lambda_{n,\nu}^2} \right) = \frac{\varphi(1, z)}{\varphi(1, 0)} = 2^\nu z^{-\nu} \Gamma(\nu + 1)I_\nu(z)
\]
or
\[
I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} \prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{\lambda_{n,\nu}^2} \right).
\]

Of course, this formula is classical \([Wat, \text{Sec. 15.41 (3)}]\).

5. An open problem

We briefly outline our initial motivation for proving Theorem 1.2. Let \(M^m\) be a compact Riemannian manifold. Then the analytic torsion \([RS]\) is defined by
\[
\log T(M) = \frac{1}{2} \sum_{i=0}^{m} (-1)^i \zeta_i'(0),
\]
where \(\zeta(s)\) denotes the \(\zeta\)-function of the Laplacian on \(i\)-forms.

The celebrated Cheeger–Müller theorem \([C, M]\) identifies \(T(M)\) with a purely combinatorial object, the combinatorial torsion of \(M\).

**Problem 5.1.** Is there an analogue of the Cheeger–Müller theorem for a suitable class of pseudomanifolds?

Few attempts have been made in this direction. A. Dar \([D1, D2]\) defined and investigated Reidemeister torsion for intersection cohomology and one might expect that the intersection cohomology should show up on the combinatorial side. On the analytic
side, only manifolds with cone–like singularities have been considered. Using work of Cheeger, A. Dar proved

**Proposition 5.2.** [D1] Let $M$ be a compact manifold with cone–like singularities. Then $T(M)$ exists.

The only thing one has to show is that the meromorphic function

\[ \sum_{i=0}^{m} (-1)^i \zeta_i(s) \]  

has no pole at 0. A priori this function has a simple pole at 0 due to a log–term in the heat asymptotics. However, the sum

\[ \sum_{i=0}^{m} (-1)^i \text{Res}_{s=0} \zeta_i(s) \]  

turns out to be 0.

An interesting approach to the Cheeger–Müller theorem is the recent work of Vishik [V], who proves a gluing formula for analytic torsion norms. Adopting this approach, for proving an analogue of the Cheeger–Müller theorem for manifolds with cone–like singularities it would be enough to compare analytic and the (hypothetical) $R$–torsion for the model cone $C(N)$ over a compact manifold $N$. At least this would indicate what a result could look like.

More precisely, let

\[ C(N) = (0, 1) \times N \]  

be the model cone over $N$ with metric

\[ g = dx^2 \oplus x^2 g_N. \]

On the face $\{1\} \times N$ we impose relative boundary conditions. Then separation of variables shows that the Laplacian on $i$–forms is an infinite sum of operators

\[ L_\lambda = -\frac{d^2}{dx^2} + \frac{\lambda}{x^2}, \]

where the $\lambda$'s are essentially the eigenvalues of the Laplacian on $N$. This is the reason why the author considered Theorem 1.2. However, for calculating $T(C(N))$ it is not enough to know $\det \zeta(L_\lambda)$, since one has to deal with an infinite sum of operators. We leave this as a problem

**Problem 5.3.** Calculate $T(C(N))$ for relative/absolute boundary conditions.

Together with Vishik’s result, the solution to this problem should lead to a Cheeger–Müller type result for manifolds with cone–like singularities.

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