We study chaotic regions in the phase space of classical non-Abelian gauge theory, focusing particularly on those which determine the low-energy interactions between BPS monopoles, and comment on the relevance of the obtained results for long-standing speculations which relate classical Yang-Mills chaos to the disordered quantum vacuum and quark confinement.

**Keywords**: Gauge theory; chaos; magnetic monopoles.

1. Why study chaos in (semi-) classical gauge theories?

Deterministic chaos in the time evolution of classical non-Abelian (and hence non-linear) gauge theories has been investigated for about thirty years\(^1\). This ongoing endeavour has several motivations. On the more conceptual side, the chaotic behavior of gauge theories reveals typical signatures of quantum chaos, visible e.g. in the distribution of nearest-neighbor level spacings of lattice Dirac spectra according to Gaussian matrix ensembles\(^2\), shows a continuous cascading of the dynamical degrees of freedom (and their energy) towards the ultraviolet during time evolution\(^3\), and has potential implications for the continuum limit of lattice gauge theories\(^4\). One of the oldest and perhaps most important fundamental motivations for investigating chaos in non-Abelian gauge theories, furthermore, was to shed light on its potential role in the QCD confinement mechanism\(^5\). Much of the work with more phenomenological focus, on the other hand, attempts to gain insight into otherwise hardly accessible non-equilibrium processes by exploiting the increasingly classical behavior of long-wavelength fields with growing temperature. Such processes are of great interest in the context of current experimental heavy-ion programs at RHIC and LHC which create and analyze matter under extreme conditions. The so far best studied examples are particle production from collective fields\(^6\) and relations between the maximal Lyapunov exponents of the classical dynamics and the damping or thermalization rates of hot gauge systems\(^6,7\).

Astrophysical and cosmological applications, motivated by the classical thermodynamics of hot and dense gauge theories as well, range from the description of stellar interiors to the role of chaos during semi-classical evolution phases of the early universe. Interesting examples are topological structure formation and its im-
impact on baryon number violating production processes in the Standard Model at temperatures around the electroweak phase transition.\(^8\) The chaoticity properties of particle motion in various curved spacetimes have also been investigated (see e.g. Ref.\(^9\)).

A significant part of the results on gauge-theory chaos was obtained in Yang-Mills-Higgs (YMH) theories. These are well suited for the study of transitions from quantum to classical chaotic behavior since their weak-coupling and semiclassical limits are controllable at all length scales (in contrast to those of pure YM theories and QCD). The chaotic behavior of the YMH theory with Higgs fields in the fundamental representation of the gauge group (which forms a part of the electroweak sector of the standard model) is e.g. of relevance for the mentioned baryon number violating processes. YMH theories with Higgs fields in the adjoint representation, on the other hand, appear in grand-unified theories and are of special interest because they sustain stable, finite-action monopole solutions.\(^10\)

The latter will be a main focus of this article. More specifically, we will study the regular and chaotic regimes in the low-energy dynamics of the two-monopole system, and the transitions between them. This dynamics is amenable to an enormous but controlled dimensional reduction of the relevant phase space – the geodesic approximation – which we will outline in the following section. In the subsequent Sec.\(^3\) we quantify the chaoticity of the monopole-monopole interactions in various phase space regions, and in Sec.\(^4\) we will address potential implications for vacuum disorder and the quark confinement mechanism in *quantum* Yang-Mills theory and QCD. Section\(^5\) finally, contains a summary and some conclusions.

### 2. Geodesic two-monopole dynamics

In the following two sections we are going to report on a recent investigation of regular and chaotic low-energy interactions among monopoles in the simplest possible setting, i.e. between two electrically charged magnetic Bogomol’nyi-Prasad-Sommerfeld (BPS) monopoles or dyons which solve the SU(2) YMH Bogomol’nyi equation:\(^12\)

\[
B_i^a = \frac{1}{2} \varepsilon_{ijk} F_{jk}^a = \pm \left( \delta^{ac} \partial_i + g \varepsilon^{abc} A_i^b \right) \Phi^c
\]  

(1)

(where \(F_{\mu\nu}^a\) is the field strength tensor of the gauge field \(A_\mu^a\) and \(\Phi^a\) is the (adjoint) Higgs field).

The two-BPS monopole system is prototypical for physically interesting subsystems of the gauge dynamics whose spatially varying fields – here the solitonic monopoles with topologically induced magnetic charge – and time evolution can be studied without invoking either uncontrolled approximations (as e.g. the drastic “homogeneous approximation” used in the pioneering studies) or requiring

\(^a\)These solutions bear interesting similarities to the BPS dyon constituents of caloron solutions with nontrivial holonomy which consist of a BPS monopole-antimonopole pair (for \(N_c = 2\)).
elaborate lattice solution of the full, hyperbolic Yang-Mills-Higgs equations.\cite{1,3,6,7}

This is because the low-energy time evolution of the dyon pair is accurately described by the geodesic motion of a point on the manifold which the few collective degrees of freedom of the two-monopole solution span,\cite{13} i.e. it is governed by ordinary differential equations.

The basis for this geodesic approximation is that all two-monopole solutions form a family whose members are parametrized by continuous collective coordinates or “moduli” $x^\alpha$. (For the one-monopole solution, e.g., these are the three position coordinates of the center and an overall phase angle.) The corresponding moduli space $M_2$ is a manifold whose metric is induced by the metric on the space of all finite-energy field configurations, i.e. by the kinetic terms of the YM-H Lagrangian, and known explicitly. Owing to energy conservation and the degeneracy of all static two-monopole solutions, the low-energy dynamics of two BPS dyons then describes geodesic motion of the associated point on $M_2$. After separating the center-of-mass motion and an overall phase (whose time dependence is associated with the total electric charge), the remaining four-dimensional internal part $M_2^{(0)}$ of the moduli space can be parametrized by three Euler angles $\theta, \varphi$ and $\psi$, which determine the orientation of the two-monopole system, and the distance variable $\rho$ which measures (at large $\rho$) the separation between the two centers. The metric $g_{\alpha\beta}^{(AH)}$ on $M_2^{(0)}$ has been constructed explicitly by Atiyah and Hitchin,\cite{15} and the resulting internal Lagrangian is that of a non-rigid body with distance-dependent “moments of inertia” around the body-fixed axes,\cite{16}

$$L_{AH} (x, \dot{x}) = \frac{m}{2} g^{(AH)}_{\alpha\beta} (x) \dot{x}^\alpha \dot{x}^\beta = L_{AH} (\rho, \theta, \psi, \dot{\theta}, \dot{\psi}, \dot{\psi}),$$  \hspace{1cm} (2)

where $m$ is the reduced mass of the monopoles. Physically, the validity of the geodesic approximation implies that at small velocities (compared to the velocity of light) internal excitations (vibrations) and deexcitations (radiation) of the dyons can be neglected, i.e. the monopole pair adapts adiabatically to its interactions by deforming reversibly and scattering elastically.

### 3. Regularity and chaos in monopole interactions

The possibility for chaotic behavior depends on the number of integrals of the motion of the underlying dynamics, which must be less than the number of degrees of freedom. For the geodesic dynamics (2) of the two-monopole system with its four degrees of freedom, three independently conserved quantities are known explicitly. Those are the total angular momentum $M^2$, the energy $E_{AH}$ and the generalized momentum $p_\varphi$ conjugate to the coordinate $\varphi$ which is cyclic, i.e. does not appear explicitly in the Lagrangian (2). For the two-monopole dynamics to be (Liouville) integrable would therefore require the existence of minimally one additional constant of the motion. Such a fourth conserved quantity indeed exists at least if the two monopoles remain infinitely separated, since then their individual electric charge is conserved. This situation changes when the two dyons begin to approach each
other and only their total charge remains conserved during Higgs-induced charge exchange. Then chaotic motion becomes possible and first numerical evidence for its existence in the two-dyon phase space was gathered in Refs. 17.

In Ref. 11 we have then systematically analyzed regular and chaotic two-monopole motions on the basis of thirteen long-time phase space trajectories for which four-dimensional time series were generated by numerically integrating the equations of motion with high accuracy over typically $2^{25}$ time steps. The initial data sets were chosen to cover a representative range of motion patterns and to explore the low-energy dyon interactions at different strengths. The resulting set of orbits includes sequences of trajectories whose decreasing minimal dyon separations interpolate between asymptotic dyon distances, where charge exchange becomes ineffective and the geodesic dynamics integrable, and relatively small minimal separations for which the interactions are expected to become non-integrable. Hence these orbits allow to map out the order-chaos transition in the two-monopole system.

The first part of our analysis consisted in constructing the Poincaré sections of this orbit set from the Hamiltonian on the reduced four-dimensional phase space in which $p_\phi$ and $M^2$ are conserved and act as fixed “external” parameters. Energy conservation then constrains all orbits to three-dimensional hypersurfaces. (Numerically, $E$, $M^2$ and $p_\phi$ were conserved up to deviations of order $10^{-12}$.) Orbits with weak initial Coulomb attraction between the dyons cover a rather large range of $\rho$ values with relatively moderate variations of the radial velocities which stay well inside the asymptotic region of approximately (or KAM-) integrable motion. Hence their Poincaré sections (in the $(\rho, p_\rho)$ plane at $M_1 = 0$) consist of one-dimensional, continuous closed curves corresponding to quasiperiodic motions. When increasing the initial Coulomb attraction, the variations in dyon distance become smaller (tighter orbits) while their relative momenta vary more strongly. The increased attraction also brings the dyons closer together. From a certain minimal inter-dyon distance $\rho_{\text{min}} \sim 2\pi$ onward the section visibly spreads out into a broadly distributed scatter of points which eventually fills the $(\rho, p_\rho)$ plane. This is a typical signature for the corresponding aperiodic orbit to have become chaotic.

In order to investigate this chaoticity further, we have calculated high-resolution power spectra of the momentum conjugate to the dyon separation for selected orbits. This allows for a more accurate distinction between quasiperiodic and chaotic motion patterns and yields quantitative information about the underlying scales. The resulting spectra indeed clearly separate quasiperiodic from irregular behavior. Two of the orbits, as expected those with the maximal initial Coulomb force between the dyons, were identified as chaotic. This substantially increases previous evidence that the relative low-energy motion of two BPS dyons admits, apart from the asymptotic $\rho \to \infty$ region, only three independent conserved quantities and turns out to be genuinely non-integrable.

In addition, the power spectra characterize quasiperiodic dyon-pair orbits (i.e. those which remain close enough to the asymptotic region) quantitatively by establishing the number of fundamental modes (two), determining their frequencies
and yielding the strength distribution over their various harmonics. The restriction to the minimal number of quasiperiodic modes is rather widespread among nonlinear dynamical systems if they are sufficiently strongly coupled. The common expectation that nonlinear couplings between more than two fundamental modes increasingly turn quasiperiodicity into chaos may therefore apply to the two-dyon system as well and explain why we have encountered only two-mode-quasiperiodic and chaotic trajectories.

In contrast to their rather complete specification of quasiperiodic motion patterns, power spectra do extract relatively little pertinent quantitative information from aperiodic orbits. Hence we have additionally calculated those quantities which perhaps most directly quantify the chaoticity of irregular motion patterns, i.e. the maximal Lyapunov exponents, for a suitable set of orbits. As expected, the Lyapunov exponents of orbits previously identified as quasiperiodic were found to vanish. The two orbits with an irregular broadband power spectrum, on the other hand, turned out to have finite and positive maximal Lyapunov exponents whose values were approximately determined as $L_{\text{max},1} \sim 0.02$ and $L_{\text{max},2} \sim 0.008$. These exponents provide our most unequivocal and quantitative evidence for the chaoticity of the dyon-dyon interactions. The orbit with the smaller Lyapunov exponent, furthermore, shows signs of intermittent behavior.

4. Chaotic monopole systems and confinement

In the following section we will summarize several ideas and speculations on the potential relevance of the discovered two-monopole chaos for vacuum disorder and confinement in quantum YM and YMH theories.

We start by recalling the long-standing conjecture that the vacuum of non-Abelian gauge theories, when undergoing a transition from weakly to strongly coupled fields, also undergoes an order-disorder transition, and that the strongly coupled QCD vacuum is populated by highly irregular color field configurations. In the limit of a large number of colors, in particular, a vacuum made of random Yang-Mills fields is known to be a necessary and sufficient condition for quark confinement. More recently, a numerical investigation of the classical time evolution of Yang-Mills field configurations generated by finite-temperature (quantum) lattice simulations has provided evidence for the confining strong-coupling phase to be indeed substantially more chaotic than its weakly coupled counterpart: as a function of increasing coupling the maximal Lyapunov exponent undergoes a sharp transition to larger values around the confinement transition.

Moreover, the instability of constant color-magnetic vacuum fields made it natural to assume that both gauge invariance and stability of the physical vacuum may be restored by disordering color-magnetic background fields. Under the gluonic infrared degrees of freedom envisioned to exhibit (and maybe cause) this disorder
are random domains as well as vacuum populations of quasi-randomly distributed, percolating center vortices or monopoles.\textsuperscript{19}

The pivotal role which non-Abelian monopoles and their chaotic interactions may play in the context of quark confinement in QCD is further presaged in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in 3+1 dimensions\textsuperscript{24} where BPS monopoles indeed realize a non-Abelian version of the classic ’t Hooft-Mandelstam dual superconductor confinement mechanism\textsuperscript{25}. In this scenario the condensation of magnetic BPS monopole charges screens color magnetic charges and confines color electric charges by the dual Meißner effect. Similar scenarios, in which the condensation of monopole-like objects plays a key role, are expected to unfold in more physical gauge theories as well. As a case in point, in the 2+1 dimensional Yang-Mill-Higgs model ’t Hooft-Polyakov monopoles (a generalization of BPS monopoles which play the role of instantons in this case) generate “weak confinement” by forming a monopole antimonopole plasma, as shown by Polyakov\textsuperscript{26}. In 3+1 dimensional Yang-Mills theories, furthermore, there is lattice evidence for the condensation of Abelian-projected monopoles to generate the bulk of the string tension\textsuperscript{27}. (According to an interesting recent suggestion, the “active” monopoles might actually be BPS dyon constituents of caloron solutions with nontrivial holonomy\textsuperscript{28}.)

In light of the above arguments it is tempting to speculate that a disordered ensemble of monopoles (and anti-monopoles) in a semiclassical vacuum may be generated by the classically chaotic low-energy interactions among individual monopole pairs which we have investigated above. Below we will suggest two approaches towards pursuing and testing such ideas in a more quantitative fashion.

5. Summary and conclusions

We have discussed chaotic regions in the classical phase space of non-Abelian gauge theories and studied, in particular, regular and chaotic motion patterns of two interacting BPS monopoles at low energies. Our analysis is based on a representative set of long-time phase-space trajectories in the geodesic approximation and intended to survey the order-chaos transition and to characterize the quasiperiodic and chaotic behavior quantitatively.

The observed changes in the dimension of the trajectories’ Poincaré sections (from one to two) provide clear evidence for transitions from quasiperiodic to chaotic motion when the monopoles come close enough to each other. These results were confirmed and quantified by the analysis of high-resolution power spectra and Lyapunov exponents for selected orbits. The obtained values for the maximal Lyapunov exponents contain information on the relaxation time and thermalization (damping) rate of a non-equilibrium dyon system at sufficiently high temperatures. Taken together, our results provide convincing evidence for and a quantitative description of both quasiperiodic and chaotic regions in the low-energy phase space of two BPS dyons. They also show that no more than the three explicitly known integrals of the motion are conserved by the geodesic forces between the dyons.
Since the motion of free dyons is integrable, the chaotic behavior analyzed above can be uniquely traced to the interactions between the dyons. This raises hopes that our quantitative understanding of the chaotic dyon-dyon interactions may also generate new insights into the disorder of interacting monopole ensembles of the type which are expected to populate the vacuum of the strong interactions. For sufficiently dilute systems, expansions in the monopole density and more sophisticated many-body techniques may e.g. provide a quantitative treatment of chaotic multi-monopole systems. Another option would be the technically challenging extension of the geodesic approximation to approximate multi-monopole-antimonopole solutions with their more complex interactions.

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