Proper-time methods in the presence of non-constant background fields

Alan Chodos
Center for Theoretical Physics
Yale University
P.O. Box 208120
New Haven, CT 06520-8120

June 19, 2021

Abstract

A formalism is developed to enable the construction of the effective action and related quantities in QED for the case of time-varying background electric fields. Some examples are studied and evidence is sought for a possible transition to a phase in which chiral symmetry is spontaneously broken.

Submitted to the Proceedings of the first Gürsey Memorial Conference (Istanbul, June 6-10, 1994).

The proper-time formalism was used a long time ago by Schwinger [1] to compute the Green function for an electron propagating in the presence of a background electromagnetic field. Although the formalism is general, explicit evaluation of the propagator, and of the associated effective action, was possible only for the case of fields uniform in space and constant in time.

Over the intervening decades, attempts have been made [2] to compute the corrections to Schwinger’s results for the case of varying fields. These take the form
of a derivative expansion in the fields, but even the first non-trivial corrections turn out to be quite unwieldy, and are, moreover, restricted to fields that do not vary too rapidly (else the higher terms in the expansion must be included).

More recently, there has been cause for a new look at this problem. The motivation is the strange results of the GSI heavy-ion scattering experiments [3], in which mysterious narrow peaks are seen in the energy spectra of emitted e+e- pairs. Among the many theoretical ideas that have been advanced, I wish to concentrate on one proposed explanation [4, 5]: that the heavy ions create a very strong and rapidly varying electromagnetic field, which then induces a phase transition in QED to a vacuum in which chiral symmetry is spontaneously broken. The observed e+e- peaks are due to the decay of positronium-like states in the new phase of QED.

To study this possibility, we employ a proper-time representation for the vacuum expectation value of \( \bar{\psi} \psi \), which is an order parameter for this transition. This representation is [5]

\[
\langle \bar{\psi} \psi \rangle = m \int_0^\infty dt e^{-m^2 \tau} U(x, \tau) \tag{1}
\]

where

\[
U(x, \tau) = tr \left( x | e^{iH\tau} | x \right). \tag{2}
\]

Here \( \tau \) is the proper-time (continued to imaginary values) and \( m \) is the electron mass. \( U \) is the trace of a quantum-mechanical matrix element for which the Hamiltonian is \((\gamma \cdot \pi)^2\), with
\[ \pi_\mu = p_\mu - eA_\mu(x). \] (3)

The dynamical degrees of freedom are the four coordinates \( x_\mu(\tau) \), and \( p_\mu \) are the associated canonical momenta. \( A_\mu(x) \) is the potential that encodes the background field. We are working in Euclidean space, so the \( \gamma_\mu \) are Hermitian and \( H \) is positive. The trace in eq. (2) is over the indices carried by the \( \gamma \) matrices. For later reference, note that \( H \) possesses a quantum-mechanical supersymmetry \( \{ H \} \), generated by the charges \( Q_\pm = \frac{1}{2}(1\pm \gamma_5)(\gamma \cdot \pi) \) with

\[ \{ Q_+ , Q_- \} = H. \] (4)

The proper-time expression for \( \langle \bar{\psi} \psi \rangle \) incorporates correctly all the effects of the background field, but completely neglects the role of the dynamical photons. In using eq. (1), one hopes that these photons will not affect the presence or absence of a phase transition induced by the background field.

The signal for the spontaneous breakdown of chiral symmetry is that the limit \( m \to 0 \) of \( \langle \bar{\psi} \psi \rangle \) should not vanish. Because of the explicit factor of \( m \) in eq. (1), one requires the integral to diverge as \( m \to 0 \). In fact, one easily sees \( \{ H \} \) that if the large-\( \tau \) behavior of \( U(x, \tau) \) is \( \tau^{-\frac{1}{2}} \), \( \langle \bar{\psi} \psi \rangle \) will remain finite and non-zero. If the falloff is more rapid, \( \langle \bar{\psi} \psi \rangle \) will vanish, indicating that there is no spontaneous chiral symmetry breaking.

For a free fermion, \( U \sim 1/\tau^2 \) so that \( \langle \bar{\psi} \psi \rangle \to 0 \) as expected. For constant \( F_{\mu\nu} \), one finds from the analytic continuation of Schwinger’s results that \( U(x, \tau) \) is a function of the two Euclidean invariants \( F = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \) and \( G = \vec{E} \cdot \vec{B} \). If
\( G = 0, \quad F \neq 0, \quad U \sim 1/\tau \) and \( \langle \bar{\psi}\psi \rangle \to 0 \). If \( G \neq 0 \), then \( U \sim \text{const.} \), which indicates that \( \langle \bar{\psi}\psi \rangle \to \infty \) as \( m \to 0 \). This behavior is not to be interpreted as spontaneous chiral symmetry breaking, however, since there is an anomaly when \( G \neq 0 \) that explicitly breaks chiral symmetry. In what follows, we shall look at cases where \( \vec{E} \neq 0, \vec{B} = 0 \), so the anomaly is absent.

In a recent paper, Caldi and Vafaeisefat \[7\] have computed \( U(x, \tau) \) numerically using Monte Carlo simulation techniques. For this purpose, it is convenient to recast \( U(x, \tau) \) as a path integral:

\[
U(x, \tau) = \text{tr} \int Dx e^{-S}, \tag{5}
\]

\[
S = \int_0^\tau dt' L(x, \dot{x}), \tag{6}
\]

\[
L(x, \dot{x}) = \frac{1}{4} \dot{x}_\mu \dot{x}_\mu + ie \dot{x}_\mu A_\mu(x) - \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}(x). \tag{7}
\]

Note the following peculiarities: (i) \( L \) is complex (this is a consequence of having continued to imaginary proper-time); and (ii) \( L \) is matrix valued. The symbol \( T \) in eq. (5) denotes \( \tau \)-ordering.

Caldi and Vafaeisefat look initially at background electric field configurations pointing in one direction only, for which the magnitude varies in time and in the one spatial variable. In particular, they consider

\[
\vec{E} = (f(x, t), 0, 0) \tag{7}
\]

with
\[ f(x, t) = eE[cosh^2(x/W_s)]^{-1}exp(-t^2/2W_t^2). \]  

They find, for \( W_s = W_t^{-1} = 3 \) (in units where \( eE = 1 \)) that at suitably chosen values for \( x, U(x, \tau) \) exhibits the desired \( \tau^{-1/2} \) falloff. Although the computations are complicated, their method gains credence from the following observations: they obtain agreement with Schwinger’s analytic results for the case \( \vec{E} = const. \), and, when \( W_s \) and \( W_t \) are taken much larger or much smaller than the above values, the chiral symmetry breaking goes away. This is reasonable because, for large \( W_s \) and \( W_t \) the configuration approximates a constant field, for which chiral symmetry is unbroken, whereas for small \( W_s \) and \( W_t \) the field is varying so rapidly that the vacuum does not have time to realign (i.e., the ”sudden approximation” is valid). In later work \([8]\), Caldi and Vafaeisefat have studied more realistic configurations involving all the spatial variables, and they continue to see chiral symmetry breaking for suitable values of the parameters.

Even assuming the utter reliability of these results, one is still left with virtually no intuition or insight concerning the mechanism whereby chiral symmetry is broken. It is therefore of interest to explore these questions in a more analytic fashion. To this end, we look at a configuration even simpler than that chosen by Caldi and Vafaeisefat, to wit one in which the electric field depends only on time (which is, of course, Euclidean time, and which we call \( x_0 \)):

\[ \vec{E} = (f(x_0), 0, 0). \]

We take the associated vector potential to be
\[ A_0 = -x_1 f(x_0) \] (10)

and

\[ \vec{A} = 0. \] (11)

The Lagrangian then reduces to:

\[ L = \frac{1}{4}(\dot{x}_0^2 + \dot{x}_1^2) + ie\dot{x}_0 x_1 f(x_0) + e\sigma_{01} f(x_0) + \frac{1}{4}(\dot{x}_2^2 + \dot{x}_3^2). \] (12)

Note the following:

(a) For proper normalization (i.e. to obtain the known result when \( f(x_0) = 0 \)) we must take
\[
\int Dye^{-\frac{1}{4}\int_0^\tau \dot{y}^2 dt} = \frac{1}{\sqrt{4\pi}}; \] (13)

(b) The \( x_2 \) and \( x_3 \) integrals are then trivial, yielding a factor of \( \frac{1}{4\pi} \);

(c) The \( x_1 \) integral is almost trivial, since it can be reduced to a Gaussian by a shift in \( x_1 \);

(d) the \( \tau \)-ordering is superfluous because there is only one non-trivial matrix, \( \sigma_{01} \). Furthermore, the trace is reduced to a summation over the eigenvalues of \( \sigma_{01} \), i.e. to the operation \( 2\sum_{\sigma=\pm 1} \) (where we have abbreviated \( \sigma_{01} \) by \( \sigma \)).

(e) The path integral must be evaluated subject to the boundary condition \( x_\mu(0) = x_\mu(\tau) = x_\mu \), where \( x_\mu \) is the argument of \( \bar{\psi}\psi \).
Making use of standard manipulations, one obtains

\[ U(x, \tau) = \frac{1}{4(\pi \tau)^3/2} \sum_{\sigma = \pm 1} \int D\dot{x}_0 e^{-\frac{1}{4} \int_0^\tau dt' \dot{x}_0^2} e^{-e^2 \tau |\langle F^2 \rangle - \langle F \rangle^2|} e^{-e \tau \int_0^\tau d\tau' f}. \] (14)

Here \( F \) is defined by \( \frac{dF}{dx_0} = f(x_0) \), and for any function \( \Phi(\tau) \), we have defined \( \langle \Phi \rangle = \frac{1}{\tau} \int_0^\tau d\tau \Phi(\tau') \). Note that \( U \) is invariant under \( F(x_0) \rightarrow F(x_0) + C \), as it must be. Also note that the \( \langle F \rangle^2 \) term is a non-local interaction.

We can re-express \( U(x, \tau) \) in terms of a local action at the cost of introducing an ordinary integral over a parameter \( \lambda \). Using

\[ e^{\beta^2/4\alpha} = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} d\lambda e^{-\alpha \lambda^2 + \beta \lambda} \] (15)

with \( \alpha = \tau \) and \( \beta = 2e\tau \langle F \rangle \), we have

\[ U(x, \tau) = \frac{1}{4\pi^2 \tau} \sum_{\sigma = \pm 1} \int D\lambda \int_{-\infty}^{\infty} D\dot{x}_0 e^{-\int_0^\tau d\tau' L_\sigma}. \] (16)

where

\[ L_\sigma = \frac{1}{4} \dot{x}_0^2 + (eF - \lambda)^2 + e\sigma f. \] (17)

Thus we are summing and integrating over a family of one-dimensional quantum-mechanical models defined by the Hamiltonians

\[ H_\sigma = p^2 + V_\sigma(x_0), \] (18)
\[ V_\sigma = [eF(x_0) - \lambda]^2 + e\sigma f(x_0). \tag{19} \]

Here we see that \( H_\pm \) have the standard form characteristic of quantum-mechanical supersymmetry,

\[ H_\pm = p^2 + W^2 \pm dW/dx_0, \tag{20} \]

with

\[ W = eF - \lambda. \tag{21} \]

For the purpose of quantitative analysis, it is convenient to re-express \( U(x, \tau) \) as

\[ U(x, \tau) = \frac{1}{4\pi^2 \tau} \sum_{\sigma = \pm} \int_{-\infty}^{\infty} d\lambda < x | e^{-H_\sigma \tau} | x >, \tag{22} \]

and then to insert this in eq. (1), and perform the \( \tau \) integral after division by \( m \) and differentiation with respect to \( m^2 \). One obtains

\[ I(m) = -\frac{\partial}{\partial m^2} \left[ \frac{\langle \bar{\psi}\psi(x) \rangle}{m} \right] = \frac{1}{4\pi^2} \sum_{\sigma} \int d\lambda < x | \frac{1}{H_\sigma + m^2} | x >. \tag{23} \]

Therefore, we wish to compute the Green function \( G_\sigma(x, x') = < x | \frac{1}{H_\sigma + m^2} | x' >, \) which obeys the equation

\[ \left[-\frac{\partial^2}{\partial x'^2} + V_\sigma(x) + m^2\right]G(x, x') = \delta(x - x'). \tag{24} \]
A standard expression for $G$ is

$$G(x, x') = \psi_>(x)\psi_<(x')\theta(x - x') + \psi_>(x')\psi_<(x)\theta(x' - x) \tag{25}$$

where each $\psi$ obeys the homogeneous equation

$$[-\frac{\partial^2}{\partial x^2} + V_\sigma + m^2]\psi = 0, \tag{26}$$

subject to the boundary condition that $\psi_>(\psi_<)$ is well-behaved as $x \to \infty (x \to -\infty)$, and where the Wronskian condition

$$\psi_> \frac{\partial \psi_<}{\partial x} - \psi_< \frac{\partial \psi_>}{\partial x} = 1 \tag{27}$$

is imposed. We then have

$$I(m) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\lambda \sum_{\sigma = \pm 1} \psi_\sigma(x)\psi_\sigma(x). \tag{28}$$

Our computational strategy is to choose $f(x)$ so that $\psi_>$ and $\psi_<$ can be computed explicitly \[10, 11\], to insert them in eq. (28), and to determine therefrom the behavior of $I(m)$ as $m \to 0$. If $\langle \bar{\psi}\psi \rangle$ indeed tends to a finite, non-zero value, we expect $I(m) \sim 1/m^3$. Any less singular behavior will be evidence that $\langle \bar{\psi}\psi \rangle$ is tending to zero.

Actually, without any further computation, we can infer from eq. (23) that $I(m)$ will probably behave as $1/m^2$, provided there is some range of $\lambda$ for which supersymmetry is unbroken. Under these circumstances, one of $H_\pm$ will have zero
as its lowest eigenvalue, and hence the matrix element expanded in energy eigenstates will have a term that goes as $1/m^2$. For the range of $\lambda$ (if any) for which supersymmetry is broken, the situation is worse: Both $H_+$ and $H_-$ will have strictly positive ground state energies, so the contribution to $I(m)$ will be non-singular. It is hard to imagine a system for which the desired $1/m^3$ singularity might appear.

As an illustrative example, we can choose

$$eF(x) = \gamma \text{tanh} \beta x.$$  \hspace{1cm} (29)

This yields a model that is exactly solvable quantum mechanically \cite{11}. One finds that supersymmetry is unbroken for $|\lambda/\gamma| < 1$, and broken when $|\lambda/\gamma| > 1$. For $\lambda = \pm \gamma$, the zero-energy eigenstate is not isolated but sits at the bottom of a continuous spectrum. Some of the energy levels of this model as a function of $\lambda$ are illustrated in Figure 1. For any values of $\gamma$ and $\lambda$ one can solve for $\psi_>$ and $\psi_<$ explicitly in terms of hypergeometric functions $2F_1$. We do not reproduce the formulas here, since they are complicated and not particularly illuminating. When inserted into eq. (28), they yield the expected $1/m^2$ behavior, i.e. no evidence for chiral symmetry breaking.

It is possible to study other exactly solvable quantum mechanical models as well. Examples are available for which supersymmetry is unbroken for all $\lambda$ and there are others for which supersymmetry is broken for all $\lambda$ except $\lambda = 0$. In all the cases we have examined, the singularity at small $m$ of $I(m)$ is no worse than $1/m^2$.

This result is not in conflict, of course, with the numerical work of Caldi
and Vafaeisefat, since their field configurations depend on at least two variables. In deciding how to proceed, one can think of a number of possibilities: (i) extend the search among the one-dimensional models in the hope that an as yet undiscovered class will yield the sought-for $1/m^3$ behavior; (ii) introduce new analytic techniques that will enable one to study the two-variable case. This will permit direct comparison with the Monte Carlo results; (iii) find a way to extract the small $m$ behavior of $I(m)$ (or equivalently the large $t$ behavior of $U(x,\tau)$) without first having to compute the full functional forms of $I(m)$ or $U(x,\tau)$. This would lead to enormous simplifications not only of the analytic work but also of the Monte Carlo calculations, where the large $\tau$ behavior is extracted by computing $U(x,\tau)$ for several values of $\tau$ and finding the slope of the best-fitting straight line on a log-log plot.

As new data from GSI and Argonne are reported, one expects the relevance of the ideas upon which the present work is based either to wax or to wane. If the former, it will be interesting to see whether new insight can in fact be gained about the mechanism whereby time- and space-varying background fields induce a chiral phase transition in QED.

Acknowledgments

I am especially grateful to Daniel Caldi, Andras Kaiser, David Owen and Saeid Vafaeisefat, each of whom contributed substantially to various parts of the work described in this paper. I am also grateful for illuminating conversations with Charles Sommerfield. I am pleased that this work touches on the subject of exactly solvable quantum mechanical models, because this is a subject in which Feza was
interested and to which he made significant contributions in collaboration with
Franco Iachello and Yoram Alhassid. Finally I wish to express my deepest thanks
to Meral Serdaroglu and the rest of the organizers at Bogazici University, to whom
the success of the first Gürsey Memorial Conference is in large measure due. The
research discussed in this paper was supported in part by the U.S. Department of
Energy grant DEFG0292ER40704.

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Figure Caption

The first two energy levels of the $F = \gamma \tanh \beta x$ model discussed in the text. $E_0 = 0$ for $\lambda < \lambda_0 = \gamma$, and $E_0 = E_c = (\gamma - \lambda)^2$ for $\lambda > \gamma$. $E_c$ is the energy at which the continuous spectrum begins. The first excited bound state exists for $\lambda < \lambda_1 = (\gamma - \beta)^2/\gamma$ (provided $\beta < \gamma$), and is given explicitly by $E_1 = \beta(2\gamma - \beta)(1 - \lambda^2/(\gamma - \beta)^2)$. The diagram is symmetric for $\lambda \to -\lambda$.

Figure available by request (chodos@yalph2.physics.yale.edu).
This figure "fig1-2.png" is available in "png" format from:

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