Clustering in random line graphs

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Abstract

We investigate the degree distribution $P(k)$ and the clustering coefficient $C$ of the line graphs constructed on the Erdős-Rényi networks, the exponential and the scale-free growing networks. We show that the character of the degree distribution in these graphs remains Poissonian, exponential and power law, respectively, i.e. the same as in the original networks. When the mean degree $<k>$ increases, the obtained clustering coefficient $C$ tends to 0.50 for the transformed Erdős-Rényi networks, to 0.53 for the transformed exponential networks and to 0.61 for the transformed scale-free networks. These results are close to theoretical values, obtained with the model assumption that the degree-degree correlations in the initial networks are negligible.

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1 Introduction

The science of networks is indeed a new kind of science [1] for its interdisciplinary character and its explosive development; for some recent monographs we refer to [2, 3, 4, 5, 6, 7, 8, 9]. The list of applications of networks contains examples from physics, informatics, biology and social sciences. Basic characteristics of networks are the degree distribution $P(k)$ and the clustering coefficient $C$. The degree of a given node is the number of other nodes connected to that node; the clustering coefficient measures the probability that two neighbours of a given node are connected to each other. As it was indicated only recently by Mark Newman [10], many real networks show a high clustering coefficient, usually some tens of percent. On the contrary to this fact, model random networks show rather low $C$, unless a special procedure is applied to enhance it. Examples of such procedures are described in [11, 12, 13, 14, 15]; we owe this list again to [10]. The idea is to enhance $C$ gradually by linking nodes which are neighbours of the same node. When a node has three neighbours, it is convenient to replace it by a triangle [10]; the trick is similar to the star-triangle or star-delta transformation. The latter has been used also to construct Apollonian networks with a high values of the clustering coefficient [17], and to prove some theorems in the theory of percolation [18].
The transformation from a graph $G$ to its line graph $L(G)$ [19] used here can be seen as a simple reformulation of the star-triangle transformation. The latter converts a subgraph $Y$, i.e. a node with three neighbours, into a subgraph $\Delta$, where these three neighbours are linked to each other and the central node is deleted. In the transformation $G \rightarrow L(G)$ the same original subgraph $Y$ is converted also into a graph of three nodes linked to each other. The idea of the line graph, known also as edge graph [20], is to convert all links into nodes [19]. In this way a new network appears, where the number of nodes is equal to the number of links in the original network. In the new network, two nodes are connected if they are formed from links which shared the same node. In particular, a node of degree $k$ is converted to a fully connected subgraph of $k$ nodes. The definition of distance ensures in particular, that the small world effect in the original network persists also in the transformed network.

Recently the line graph constructed from the scale-free network was discussed in [21] analytically and numerically. The results for dense networks ($<k>=10, 20$ and $30$) supported the rule that the transformed network is also scale-free. The exponent $\gamma'$ defined by the degree distribution $P(k) \propto k^{-\gamma'}$ of the transformed graph was shown to fulfil the relation $\gamma' = \gamma - 1$, where $\gamma$ was the same exponent for the initial network. The line graph was also found to be useful in the problem of identification of communities in networks [22].

The aim of this work is to investigate the clustering coefficient $C$ in the line graphs transformed from random networks. Our motivation is twofold. First, we share the point of view expressed by Mark Newman, as reported in our first paragraph. Our former simulations [16, 23] can actually be seen as the same transformation limited to local subgraphs and to nodes of degree three. This technique allowed us to enhance the clustering coefficient in networks. Second issue can be briefly expressed as follows. In social networks, an active contact between two linked actors excludes at least to some extent the contact between each of these two actors and each of their other neighbours. If we introduce two states of a link, termed for example 'open' and 'closed', then it is clear that there is an anticorrelation between links which share the same node; two such links cannot be open simultaneously. It can be convenient, then, to work on the network of links instead of the network of nodes. However, mathematical formulation of many problems is expressed in terms of networks of nodes. Here is the area where the transformation can be useful.

In the next section we describe some examples of the transformed networks. Section 3 is devoted to our numerical results. Short discussion closes the text.

2 Examples

A chain of $N$ nodes is equivalent to a chain of $N$ links, then under action of the transformation $G \rightarrow L(G)$ it is converted into itself. In a fully connected graph of $N$ nodes each pair is connected; there is $N(N - 1)/2$ links. After the transformation from $G$ to $L(G)$, this is the number of nodes. The transformed graph is not fully connected; this can be shown easily for $N = 4$. There, link 12 is connected to 13 and 14, but not to 24, etc. For arbitrary $N$, each node has
Figure 1: The degree distribution in the network transformed from the Erdős-Rényi network. The stars are numerical results for the initial Erdős-Rényi network with \(< k > = 10\) and the line comes from the Poisson distribution with mean \(\lambda = 20\).

For a network with \(N - 1\) neighbours, then each link shares its each node with \(N - 2\) other links. In the transformed network, each node has therefore \(2(N - 2)\) neighbours. The number of links in the transformed network is then \(N(N - 1)(N - 2)/2\). In particular, for \(N = 4\) in the original graph we have 12 links in the transformed graph - an octahedron, if links are equally long.

Similar arguments apply to a regular graph, where each node is of the same degree, say \(k\). Each link joins two nodes with \(k - 1\) other neighbours. Under the transformation \(G \rightarrow L(G)\), each node is converted into a \(k\)-clique. Obviously, each link in the original network joins two nodes; then it contributes - as a node in the transformed network - to two cliques. Its degree in the transformed network is then \(2(k - 1)\). The number of links in the transformed network is then \((Nk/2) \times (k - 1)\). In the case of \(N = 4\) and \(k = 3\) again a tetrahedron is transformed into an octahedron.

Let us consider a network with the degree distribution \(P(k)\), where is no correlation between degrees of neighboring nodes. The above arguments are now as follows. A link in the original network joins two nodes with degrees \(k_1 - 1\) and \(k_2 - 1\); the considered link is not counted. In the transformed network, this link is a node of degree \(k_1 + k_2 - 2\). The degree distribution \(P_t(k)\) in the transformed network is then [24].
Figure 2: The degree distribution in the networks transformed from the exponential networks. The growing parameter $M=3$, 5 and 8 for the curves from left to right. Numerical results (pluses, X’s, stars) are fitted with (Eq. 2). The obtained values of $c$ are close to $M/(M+1)$.

$$P_t(k) = \sum_{k_1,k_2} k_1 k_2 P(k_1) P(k_2) \delta_{k,k_1+k_2 -2} = \sum_{k_1=1}^{k+1} k_1 (k-k_1+2) P(k_1) P(k-k_1+2)$$

In the case when $P(k)$ is the Poisson distribution with $< k > = \lambda$ the degree distribution $P_t(k)$ for the transformed network is a new Poisson distribution with $< k > = 2\lambda$. For the geometrical distribution $P(k) = (1-c)c^k$ we get

$$P_t(k) = \frac{(1-c)^4}{6}(k+1)(k+2)(k+3)c^k$$

This distribution, when presented as $\log P(k)$ against $k$, gives only logarithmic deviation from the degree distribution $P(k)$ of the original network. For the power function $P(k) \propto k^{-\gamma}$ we have no simple result. Some analytical considerations in terms of Polygamma functions can be found in [21].

Now we consider the clustering coefficient $C$. In a fully connected graph of $N$ nodes each link $12$ is converted under $G \rightarrow L(G)$ to a node of degree $2(N-2)$. Maximal number of links between these $2N-4$ neighbours is $(N-2)(2N-5)$. The actual number of links within each clique of $(N-2)(N-3)/2$. There is also $N-2$ links between nodes converted from links which met at the $N-2$ nodes
Figure 3: The degree distribution in the networks transformed from the scale-free networks for $M = 3, 8$. The lines come from Eq. 1, with $P(k) \propto k^{-3}$.

different from nodes 1 and 2. Then in total we have $(N - 2)(N - 3) + N - 2 = (N - 2)^2$ links, what gives

$$C = \frac{(N - 2)^2}{(N - 2)(2N - 5)} = \frac{N - 2}{2N - 5},$$

(3)

the same for each node. For regular graphs of degree $k$ each link is converted to a node of degree $2(k - 1)$, with $(k - 1)(2k - 3)$ possible links between its neighbours. Neglecting triangles built on the considered link, we have only $2(k - 1)(k - 2)/2$ links within cliques. Then the clustering coefficient is

$$C = \frac{k - 2}{2k - 3},$$

(4)

For a network with degree distribution $P(k)$ we have to find an average

$$C = \sum_{k_1, k_2} k_1 k_2 P(k_1) P(k_2) \frac{(k_1 - 1)(k_1 - 2) + (k_2 - 1)(k_2 - 2)}{(k_1 + k_2 - 2)(k_1 + k_2 - 3)}.$$  

(5)

There, the contribution of pairs of nodes where $k_1 + k_2 < 4$ is zero.

3 Numerical calculations and results

The connectivity matrix for the transformed network is constructed here as follows. In the connectivity matrix $C(i, j)$ of the original network each unit above the main diagonal means a link. We substitute these units by their consecutive
Figure 4: The clusterization coefficient $C$ for the Erdös-Rényi networks against the mean degree $<k>$ (squares) and $C$ against the growing parameter $M$ for the exponential networks (rhombs) and the scale-free networks (circles). For the Erdös-Rényi network, the continuous line is obtained directly from Eq. 5. For the scale-free network we used $P(k) \propto k^{-3}$. For the exponential network, the analytical values of $C$ for large $<k>$ are about 0.545, while the simulation gives $C = 0.53$.

numbers: $r = 1, 2$ and so on. Let us call the obtained matrix $R(i, j)$. The last number $r_m$ is equal to the number of links in the original network; it is then equal also to the size of the transformed network. In the connectivity matrix $C_t(i, j)$ of this network, elements $i$ and $j$ are connected if their numbers $i$ and $j$ appear in the matrix $R$ in the same row or in the same column.

The original Erdös-Rényi network is generated from $N = 10^4$ nodes. Then, the number of nodes in the transformed network is about $pN^2/2$, where $p$ is the density of links in the original network. With $p = 10^{-3}$, as in Fig. 1, we expect $<k>$ close to $Np = 10$ in the original network. The number of nodes in the transformed network should be about $N^2p/2 = 5 \times 10^4$. In the example presented in Fig. 1 we have 50147 nodes. The result of the analytical calculation made above indicates that the mean degree of the transformed network should be equal to 20, what confirms the simulation.

The original exponential network is grown from a fully connected cluster of $M$ nodes. Each next node is attached to randomly selected $M$ different nodes. No preference of attachment is applied for the exponential networks. The original network has $N = 10^4$ nodes, and the transformed network has about $NM$
nodes. The original network is known to have the exponential distribution of node degree; $\log(P(k))$ plotted against $k$ is a straight line. The degree distributions of the transformed network, shown in Fig. 2, seem also to be close to the exponential function, as in the original network. The curves shown are obtained from Eq. 2. The size of the transformed network is 29994, 49985 and 79964 for $M = 3, 5$ and 8, respectively.

To generate the scale-free networks numerically we have only to add the preferential attachment; nodes are selected with the probability proportional to their degree. The original scale-free network is again $10^4$ nodes. In Fig. 3 we show the degree distributions for the growing parameter $M = 3$ and $M = 8$. The plots are not far from straight lines in the log-log scale. On the contrary to the exponential network, the slope of the obtained curves does not depend on the mean degree $<k> = 2M$. The overall results agree with those of [21].

In Fig. 4 we show the comparison of the data on the clustering coefficient $C$, as calculated from direct numerical simulations (points) and from Eq. 5 (lines). It appears that the respective plots met when the mean degree $<k>$ is large enough. This accordance indicates that our model assumption on the lack of correlations works well for dense networks. As it can be seen in Fig. 4, the largest departure of the clustering coefficient $C$ calculated numerically from the analytical values are found for the exponential networks. This suggests that for these networks, the degree-degree correlations are the largest.

4 Conclusions

The rule that a given degree distribution is transformed under $G \rightarrow L(G)$ into the degree distribution from the same family has some justification in the transformation itself. Namely, a node of degree $k$ is transformed into a set of nodes of at least the same degree $k$. This means in particular that hubs are transformed into cliques of hubs, and chains are transformed into chains. For the scale-free networks, our results on the degree distribution $P(k)$ agree with those of [21].

Accordance of the results calculated numerically with the analytical formulas means in our case that the corrections introduced by degree-degree correlations [25] are relatively irrelevant. We demonstrated that the transformation $G \rightarrow L(G)$ leads to clustered networks, where the clustering coefficient $C$ is not smaller than 0.5. This limit value can be lower, if $G \rightarrow L(G)$ is applied to selected local subnetworks and not to the whole system. The density of the transformed nodes can be used to tune $C$, similarly to [11, 12, 13, 14, 15, 16].

An application of the transformation $G \rightarrow L(G)$ to the communication networks needs now a specification of interaction between links, i.e. between nodes of the transformed network. The task is out of frames of this paper. We hope that the idea of interaction between links can find applications in networks, where a relation between two nodes excludes at least partially the relations between one of these nodes and its neighbours. We have in mind trade networks, sexual networks, decision trees and transport networks, where parallel
links are activated along the principle 'this or this'. Links can also activate each other, according to 'this, then this'. Examples could be found in genetic networks, chains of catalytic reactions and social systems, where a process is simultaneously an active agent.

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