COMPUTING SKEW LEFT BRACES OF SMALL ORDERS

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Abstract. By improving Algorithm 5.1 of [Math. Comp. 86 (2017), 2519-2534], we enumerate left braces and skew left braces of orders upto 511 with some exceptions.

1. Introduction

A multiplicatively written group \( G \), with multiplicative structure on \( G \) given by \((g_1, g_2) \mapsto g_1.g_2\), is said to be a skew left brace if it admits an additional group structure given by \((g_1, g_2) \mapsto g_1 \circ g_2\) satisfying

\[
(1.1) \quad g_1 \circ (g_2.g_3) = (g_1 \circ g_2).g_1^{-1}(g_1 \circ g_3)
\]

for all \( g_1, g_2, g_3 \in G \), where \( g_1^{-1} \) denotes the multiplicative inverse of \( g_1 \). We call \((G, \cdot)\) the primary group and \((G, \circ)\) the secondary group of the skew left brace \( G \). A skew left brace \( G \) is said to be a left brace if \( G \) is an abelian groups under multiplicative structure. The concept of left braces was introduced by Rump [16] in 2007 in connection with non-degenerate involutive set theoretic solutions of the quantum Yang-Baxter equations. Thereafter the subject received a tremendous attention of the mathematical community; see [2, 4, 17, 18] and the references therein. Interest in the study of set theoretic solutions of the quantum Yang-Baxter equations was intrigued by the paper [9] of Drinfeld, published in 1992.

Let \( X \) be an arbitrary set and \( R : X \times X \to X \times X \) a bijective map. Recall that the pair \((X, R)\) is said to be a set theoretic solution of the Yang-Baxter equation if

\[
R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}
\]

holds in the set of all maps from \( X \times X \times X \) to itself, where \( R^{ij} \) is just \( R \) acting on the \( i \)th and \( j \)th components of \( X \times X \times X \) and identity on the remaining one. Let us write

\[
R(x, y) = (\sigma_x(y), \tau_y(x)), \quad x, y \in X
\]

with \( \sigma_x \) and \( \tau_y \) component maps from \( X \) to itself.

A solution \((X, R)\) is said to be non-degenerate if the component maps \( \sigma_x \) and \( \tau_y \) are bijections on \( X \) for all \( x, y \in X \). It is said to be involutive if \((\tau' \circ R)^2\) is the identity map, where \( \tau' : X \times X \to X \times X \) is the permutation map given by \( \tau'(x, y) = (y, x) \) for all \( x, y \in X \). The study of non-degenerate set theoretic solutions of the the quantum Yang-Baxter equations has been extensively taken up, e. g., [5, 8, 11, 14, 19] to mention a few.

The concept of skew left brace was introduced by Guarnieri and Vendramin [13] in 2017 in connection with non-involutive non-degenerate set theoretic solutions of the quantum Yang-Baxter equations. They invented an algorithm, by generalising a result of Bachiller [1] for computing all skew left braces of a given order. They themselves computed left braces and skew left braces of lot of groups upto order 120. Vendramin [20] extended the number upto 168 with some exceptions. All these computations are done using computer algebra systems MAGMA [3] and GAP [12] using the algorithm invented in [13]. For more work on skew braces see [6, 7, 15].

This article aims at filling up the gaps in the table produced in [20] to some extent and making further computations for larger orders. An ingenious observation on regular

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subgroups of the holomorph of a given finite group allows us to improve the algorithm obtained in [13], which substantially enhances the performance of MAGMA computation. The improved algorithm, actually, avoids an expensive calculation in the existing algorithm. We compute the number of non-isomorphic left braces and skew left braces of orders upto 511 except certain cases (mainly when the order is a multiple of 32). These results settle [20, Problem 13] and [13, Problem 6.1]. The computations will help in building a database of left braces and skew left braces, which in turn will greatly enrich the library of solutions of the quantum Yang-Baxter equation.

It is striking that there are more than a million skew brace structures of order 2 and more than 20 millions skew brace structures of order 3. The reader will encounter many more surprises while going through the tables. We have used MAGMA on a computer with 3.5 GHz 6-Core Intel Xeon E5 processor and 64 GB memory for these computations.

2. Regular subgroups

Let \( G \) be a group, which acts on a set \( X \). The action of an element \( g \in G \) on an element \( x \in X \) is denoted by \( x^g \). A subgroup \( H \) of \( G \) is said to be action-closed if for each pair \((g, x) \in G \times X \), there exists an element \( h \in H \) such that \( x^g = x^h \).

Let \( G \) be a group and \( \text{Symm}(G) \) be the symmetric group on the set \( G \). Recall that a subgroup \( \mathcal{G} \) of \( \text{Symm}(G) \) is said to be regular if \( \mathcal{G} \)-action on \( G \) is free and transitive. By a free action we here mean that for any element \( g \in G \), its stabilizer in \( \mathcal{G} \) is the trivial subgroup. Observe that when \( G \) is finite, any regular subgroup of \( \text{Symm}(G) \) is of order \(|G|\).

For a group \( G \), \( \text{Hol}(G) \) denotes the holomorph of \( G \), which is defined as the semidirect product of \( G \) with \( \text{Aut}(G) \), the automorphism of \( G \). So

\[ \text{Hol}(G) := \text{Aut}(G) \rtimes G, \]

where the product in \( \text{Hol}(G) \) is given by

\[ (\alpha, g)(\beta, h) = (\alpha\beta, g\alpha(h)). \]

Notice that \( \text{Hol}(G) \) acts on \( G \) transitively under the following action:

\[ g^{(\alpha, h)} = \pi_2([(\alpha, h)(1, g)]) = h\alpha(g) \]

for all \( \alpha \in \text{Aut}(G) \) and \( g, h \in G \), where \( \pi_2 : \text{Hol}(G) \to G \) is the projection map given by \( \pi_2((\alpha, g)) = g \). It follows that the stabilizer of any element of \( G \) in \( \text{Hol}(G) \) is isomorphic to \( \text{Aut}(G) \).

Let \( \mathcal{G} \) be a regular subgroup of \( \text{Hol}(G) \). Then it is not difficult to see that for each \( g \in G \), there exists a unique element \((\alpha, h) \in \mathcal{G} \) such that \( g^{(\alpha, h)} = h\alpha(g) = 1 \). Let \( \text{Reg}(G) \) denote the set of all regular subgroups of \( \text{Hol}(G) \). Then \( \text{Hol}(G) \) acts on \( \text{Reg}(G) \) by conjugation. With this setting, we have the following easy observation, which plays a key role in what follows.

**Lemma 2.1.** \( \text{Aut}(G) \), as a subgroup of \( \text{Hol}(G) \), is action-closed with respect to the conjugation action of \( \text{Hol}(G) \) on \( \text{Reg}(G) \).

**Proof.** Let \( \mathcal{G} \in \text{Reg}(G) \) and \((\alpha, h) \in \text{Hol}(G) \). Then there exists an element \((\alpha_1, h_1) \in \mathcal{G} \) such that \( h^{(\alpha_1, h_1)} = h_1\alpha_1(h) = 1 \). Notice that

\[ (\alpha_1, h_1)(\alpha, h) = (\alpha_1\alpha, h_1\alpha_1(h)) = (\alpha_1\alpha, 1). \]

Let \( \beta := \alpha_1\alpha \), which lies in \( \text{Aut}(G) \). Thus,

\[ \mathcal{G}^{(\beta, 1)} = (\mathcal{G}^{(\alpha_1, h_1)})(\alpha, h) = (\mathcal{G}^{(\alpha_1, h_1)})^{(\alpha, h)} = \mathcal{G}^{(\alpha, h)}. \]

Proof is now complete. \( \square \)

The preceding lemma enables us to get the following generalization of [13, Proposition 4.3].
Theorem 2.2. Let $G$ be a group. Then non-isomorphic skew left brace structures over $G$ are in bijective correspondence with conjugacy classes of regular subgroups in $\text{Hol}(G)$.

Proof. The result follows from [13, Proposition 4.3] along with Lemma 2.1. □

As a result, we get the following algorithm which improves [13, Algorithm 5.1].

Algorithm 2.3. For a finite group $G$, the following sequence of computations constructs all skew left brace structures over $G$:

1. Compute the holomorph $\text{Hol}(G)$ of $G$.
2. Compute the list of regular subgroups of $\text{Hol}(G)$ of order $|G|$ up to conjugation.
3. For each representative $G$ of regular subgroups of $\text{Hol}(G)$, construct the map $\chi : G \to G$ given by $g \mapsto (f, f(g)^{-1})$, where $(f, f(g)^{-1}) \in G$. The triple $(G, G, \chi)$ yields a skew left brace structure over $G$ with multiplication given by $g_1 \circ g_2 = \chi^{-1}(\chi(g_1)\chi(g_2))$ for all $g_1, g_2 \in G$.

As remarked in [13] too, for enumerating skew left brace structures over $G$ we only need first two step of this algorithm.

3. Computations

Throughout this section, for a given positive integer $n$, $b(n)$ and $s(n)$, respectively, denote the total number of left braces and skew left braces of order $n$. For each such $n$, $pf(n)$ stands for the prime factorization of $n$. The following table remedy some gaps in the list obtained in [20].

| $n$    | $32$ | $54$ | $64$ | $72$ | $80$ | $81$ | $96$ | $108$ |
|--------|------|------|------|------|------|------|------|-------|
| $pf(n)$ | $2^5$ | $2^3.3^2$ | $2^6$ | $2^3.3^2$ | $2^4.5$ | $3^4$ | $2^3.3$ | $2^3.3^3$ |
| $b(n)$  | 25281 | 80 | ?  | 489 | 1985 | 804 | 195971 | 494 |
| $s(n)$  | 1223061 | 1028 | 17790 | 74120 | 8436 | ?  | 11223 |

| $n$    | $112$ | $120$ | $126$ | $128$ | $136$ | $144$ | $147$ | $150$ |
|--------|------|------|------|------|------|------|------|-------|
| $pf(n)$ | $2^3.7$ | $2^3.5$ | $2.3.7$ | $2^2$ | $2^3.17$ | $2^3.3^2$ | $3.7^2$ | $2.3.5^2$ |
| $b(n)$  | 1671 | 395 | 36 | 108 | 10215 | 9 | 19 |
| $s(n)$  | 65485 | 22711 | 990 | 986 | 3013486 | 123 | 401 |

| $n$    | $152$ | $158$ | $160$ | $162$ | $164$ | $165$ | $166$ | $168$ |
|--------|------|------|------|------|------|------|------|-------|
| $pf(n)$ | $2^3.19$ | $2.7.9$ | $2^3.5$ | $2.3^4$ | $2^2.41$ | $3.5.11$ | $3.7^2$ | $2.3.5.7$ |
| $b(n)$  | 90 | 2 | 209513 | 1374 | 11 | 2 | 2 | 443 |
| $s(n)$  | 800 | 6 | ? | 45472 | 43 | 12 | 6 | 28505 |

Table 1. Some missing values from [20]

We now enumerate $b(n)$ and $s(n)$ for $n \leq 511$ except some cases for which computations are too big to be handled by our computer. We have given a lower bound on the number of skew left braces of order $3^5$, by taking into account the primary groups with Group Id’s \[243, m\], where $m = 1, \ldots, 31, 33, 37, 38, 48, 61, 67$. By the Group Id we mean the group identification of a group of given order in The Small Groups Library [10] implemented in GAP and MAGMA.
| n   | \( pf(n) \) | \( b(n) \) | \( s(n) \) |
|-----|-------------|-----------|----------|
| 169 | 13^2 2.5.17 3^2.19 2^2.43 173 | 4 14 4 9 1 | 4 36 80 29 1 |
| 170 | 2^2.3.7.5 181 2.7.13 3.61 2.23 5.37 2.3.31 11.17 2^4.47 | 1 129 1 4 2 | 90 1 6 1 9 |
| 171 | 17/ 2 36 29 1 | 36 4 65466 1 6 |
| 172 | 179 180 181 182 | 183 184 185 186 187 188 |
| 173 | 2^2.3.5.181 2.7.13 3.61 2.23 5.37 2.3.31 11.17 2^4.47 | 165 4 1 1 2 | 22 2 4 1 16 |
| 174 | 5849 1 36 8 800 1 78 1 29 |
| 175 | 199 190 191 192 | 193 194 195 196 197 198 |
| 176 | 3^2.7 2.5.19 191 2^6.3 193 2.97 3.5.13 2^3.7^2 197 2.3^3.11 | 165 4 1 1 | 2 2 41 1 16 |
| 177 | 4569 36 1 1 8 389 1 | 294 1 294 |
| 178 | 199 200 201 202 203 204 205 206 207 208 |
| 179 | 2.5.17 3^2.19 2^2.43 173 | 1 129 1 4 2 | 90 1 6 1 9 |
| 180 | 2^2.3.7.5 181 2.7.13 3.61 2.23 5.37 2.3.31 11.17 2^4.47 | 165 4 1 1 2 | 22 2 4 1 16 |
| 181 | 5849 1 36 8 800 1 78 1 29 |
| 182 | 219 220 221 222 223 224 225 226 227 228 |
| 183 | 3^2.7 2.5.19 191 2^6.3 193 2.97 3.5.13 2^3.7^2 197 2.3^3.11 | 165 4 1 1 | 2 2 41 1 16 |
| 184 | 4569 36 1 1 8 389 1 | 294 1 294 |
| 185 | 199 200 201 202 203 204 205 206 207 208 |
| 186 | 2^2.3.7.5 181 2.7.13 3.61 2.23 5.37 2.3.31 11.17 2^4.47 | 165 4 1 1 | 2 2 41 1 16 |
| 187 | 4569 36 1 1 8 389 1 | 294 1 294 |
| 188 | 219 220 221 222 223 224 225 226 227 228 |

Further Computations

Table 2. Further Computations
| \( n \) | \( pf(n) \) | \( b(n) \) | \( s(n) \) |
|---|---|---|---|
| 309 | 320 | 311 | 312 | 313 | 314 | 315 | 316 | 317 | 318 |
| 2 | 6 | 1 | 507 | 1 | 2 | 11 | 9 | 1 | 4 |
| 8 | 94 | 1 | 32075 | 1 | 6 | 47 | 29 | 1 | 36 |
| 319 | 320 | 311 | 322 | 323 | 324 | 325 | 326 | 327 | 328 |
| 4 | 6 | 1 | 507 | 1 | 2 | 11 | 9 | 1 | 4 |
| 1 | ? | 1 | 4 | 1 | 10225 | 4 | 2 | 2 | 108 |
| 1 | ? | 1 | 36 | 1 | ? | 1 | 6 | 8 | 986 |
| 329 | 330 | 331 | 332 | 333 | 334 | 335 | 336 | 337 | 338 |
| 7.47 | 2.3.5.11 | 331 | 2.5.83 | 3.37 | 2.167 | 5.67 | 2.3.7 | 337 | 2.13 |
| 1 | 12 | 1 | 9 | 14 | 2 | 1 | 10990 | 1 | 8 |
| 1 | 564 | 1 | 29 | 80 | 6 | 1 | 5247711 | 1 | 59 |
| 339 | 340 | 341 | 342 | 343 | 344 | 345 | 346 | 347 | 348 |
| 3.113 | 2.5.7.19 | 1131 | 2.3.19 | 7 | 3 | 2.3.43 | 3.5.23 | 2.173 | 347 | 2.3.29 |
| 1 | 35 | 1 | 42 | 61 | 90 | 1 | 2 | 1 | 28 |
| 1 | 739 | 1 | 1164 | 373 | 800 | 1 | 6 | 1 | 410 |
| 349 | 350 | 351 | 352 | 353 | 354 | 355 | 356 | 357 | 358 |
| 2.5.7.11 | 3.3.13 | 2.3.11 | 353 | 2.3.59 | 5.71 | 2.3.89 | 3.7.17 | 2.179 |
| 1 | 16 | 166 | 195479 | 1 | 4 | 2 | 11 | 2 | 2 |
| 1 | 306 | 491 | ? | 1 | 36 | 12 | 43 | 8 | 6 |
| 359 | 360 | 361 | 362 | 363 | 364 | 365 | 366 | 367 | 368 |
| 2.5.3.5 | 192 | 2.181 | 3.112 | 2.2.13 | 5.73 | 2.3.61 | 367 | 2.23 |
| 1 | 2035 | 4 | 5 | 27 | 1 | 6 | 1 | 1670 |
| 1 | 535713 | 4 | 6 | 20 | 395 | 1 | 78 | 1 | 65466 |
| 369 | 370 | 371 | 372 | 373 | 374 | 375 | 376 | 377 | 378 |
| 129 | 4.1 | 2.5.7.19 | 393 | 394 | 395 | 396 | 397 | 398 |
| 4 | 4 | 1 | 34 | 1 | 4 | 54 | 90 | 1 | 548 |
| 4 | 36 | 1 | 606 | 1 | 36 | 253 | 800 | 1 | 47244 |
| 379 | 380 | 381 | 382 | 383 | 384 | 385 | 386 | 387 | 388 |
| 2.3.19 | 1327 | 2.191 | 383 | 2.7.3 | 5.71 | 1.2.193 | 3.2.43 | 2.97 |
| 1 | 27 | 2 | 2 | 1 | ? | 2 | 2 | 11 | 11 |
| 1 | 395 | 8 | 6 | 1 | ? | 12 | 6 | 47 | 43 |
| 389 | 390 | 391 | 392 | 393 | 394 | 395 | 396 | 397 | 398 |
| 3.5.17 | 1723 | 2.3.7 | 3.13 | 2.197 | 5.79 | 2.3.2.11 | 397 | 2.199 |
| 1 | 12 | 1 | 463 | 1 | 2 | 111 | 1 | 2 |
| 1 | 468 | 1 | 18078 | 1 | 6 | 1 | 4985 | 1 | 6 |
| 399 | 400 | 401 | 402 | 403 | 404 | 405 | 406 | 407 | 408 |
| 2.3.7.19 | 2.5.2 | 401 | 2.3.67 | 13.31 | 2.2.101 | 3.5 | 2.7.29 | 11.37 | 2.3.17 |
| 5 | 12744 | 1 | 6 | 1 | 11 | 805 | 6 | 1 | 399 |
| 113 | 3618636 | 1 | 78 | 1 | 43 | 8453 | 110 | 1 | 22923 |
| 409 | 410 | 411 | 412 | 413 | 414 | 415 | 416 | 417 | 418 |
| 3.1.7.19 | 2.5.41 | 3.13 | 2.7.103 | 7.59 | 2.3.23 | 5.83 | 2.3.13 | 3.139 | 2.11.19 |
| 1 | 6 | 1 | 9 | 1 | 16 | 1 | 209507 | 2 | 4 |
| 1 | 94 | 1 | 29 | 1 | 294 | 1 | ? | 8 | 36 |
| 419 | 420 | 421 | 422 | 423 | 424 | 425 | 426 | 427 | 428 |
| 3.7.1.5.7 | 2.3.5.7 | 2.1.211 | 3.4.7 | 2.5.5 | 5.17 | 2.3.7.1 | 7.61 | 2.3.107 |
| 1 | 104 | 1 | 2 | 4 | 106 | 4 | 4 | 1 | 9 |
| 1 | 9052 | 1 | 6 | 4 | 944 | 4 | 36 | 1 | 29 |
| 429 | 430 | 431 | 432 | 433 | 434 | 435 | 436 | 437 | 438 |
| 3.11.13 | 2.5.43 | 3.1.2.3 | 433 | 3.7.31 | 3.5.29 | 2.3.109 | 19.23 | 2.3.73 |
| 2 | 4 | 1 | 115708 | 1 | 4 | 1 | 11 | 1 | 6 |
| 8 | 36 | 1 | ? | 1 | 36 | 1 | 43 | 1 | 78 |
| 439 | 440 | 441 | 442 | 443 | 444 | 445 | 446 | 447 | 448 |
| 3.7.1.5.11 | 2.3.5.11 | 3.2.7 | 2.13.17 | 443 | 2.3.3.7 | 5.89 | 2.23 | 3.149 | 2.6 |
| 1 | 474 | 55 | 4 | 1 | 40 | 1 | 2 | 1 | ? |
| 1 | 31970 | 1110 | 36 | 1 | 782 | 1 | 6 | 1 | ? |

Table 3. Further Computations
We now record some partial computations considering specific primary groups of given orders.

| Group Id | Number | 64, 1 | 64, 2 | 64, 26 | 64, 50 | 64, 55 | 64, 83 |
|----------|--------|------|------|-------|-------|-------|-------|
| 64, 10   | 1      | 11354| 2742 | 142   | ?     | 734410|
| 64, 183  | 3124   | ?    | 253350| 2189661| 585558|

Table 5. Enumerations of left braces of order 64

| Group Id | Number | 480, 4 | 480, 199 | 480, 212 | 480, 919 | 480, 934 | 480, 1180 | 480, 1213 |
|----------|--------|------|--------|---------|--------|--------|----------|----------|
| 480, 128 | 4928   | ?    | 958965 | 99970   | ?      | 39650 |

Table 6. Enumerations of left braces of order 480

We conclude by presenting a comparison on the time taken (in seconds) by [13, Algorithm 5.1] and Algorithm 2.3 for enumerating skew left braces of order 32 for select primary groups which took considerable amount of time on MAGMA.
| Group Id of the primary group | 32, 23 | 32, 24 | 32, 25 | 32, 28 | 32, 29 | 32, 30 |
|-------------------------------|--------|--------|--------|--------|--------|--------|
| Number of skew brace structures | 39488  | 70400  | 138336 | 138336 | 138336 | 137526 |
| Time on Algorithm 5.1 [13] | 11238  | 9808   | 18720  | 10193  | 10083  | 34005  |
| Time on Algorithm 2.3 | 539    | 709    | 1905   | 4308   | 3135   | 34005  |
| Group Id of the primary group | 32, 31 | 32, 32 | 32, 33 | 32, 45 | 32, 47 | 32, 51 |
| Number of skew brace structures | 70944  | 69236  | 91008  | 8015   | 7870   | 744    |
| Time on Algorithm 5.1 [13] | 14568  | 18342  | 17222  | 30102  | 28848  | #      |
| Time on Algorithm 2.3 | 4797   | 7302   | 8869   | 30     | 68     | 8      |

Table 7. Time comparison on skew left braces of order 32

# Program was stopped after running more than a month without a result.

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