Measurement of spatial qubits

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Abstract

In this paper we study the state determination for composite systems of two spatial qubits. We show, theoretically, that one can use the technique of quantum tomography to reconstruct the density matrices of these systems. This tomographic reconstruction is based on the free evolution of the spatial qubits and a post-election process.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The concept of the quantum state plays a central role in quantum theory. The statistical distributions for the results of experiments carried out on a physical system can be completely predicted from its initial state. This has led to the development of techniques for the state determination. In the field of quantum optics, quantum endoscopy was used to determine the state of ions and atoms [1–3]. In quantum tomography of multimode fields could be measured using homodyne detection [4–6], and quantum tomographic reconstruction (QTR) was used for measuring the polarization state of the parametric down-converted photons [7–11]. Beside this, the raising of new technologic fields has brought more motivations for the study of these techniques of reconstruction. For example, in the field of quantum information, protocols such as teleportation [12] and superdense coding [13] require initially knowing the quality of the quantum channel that is being used for implementing them.

In spontaneous parametric down-conversion (SPDC) one photon from a pump beam incident onto a nonlinear crystal originates, with a small probability, two other photons usually called signal (s) and idler (i) or also twin photons [14]. Recently, we have demonstrated that by exploring the transverse momentum correlations of the twin photons it is possible to generate entangled states of two effective D-dimensional quantum systems [15, 16]. Because the dimension of the Hilbert space of these photons is defined by the number of different ways available for their transmission through apertures where they are sent, we call them spatial qudits.

In this work we investigate the state determination of these quantum systems. We show, theoretically, that one can implement the process of QTR to obtain the density operators of two spatial qubits (D = 2) states. The theoretical description of our protocol is closely related to the description given in [8] for the quantum tomography of polarized two-qubit states. Even though we have considered just the case of qubits, the description used here can be generalized to spatial qudit states.

2. The state of two spatial qubits

The angular spectrum of the pump beam is transferred to the two-photon state generated by SPDC, when the monochromatic, paraxial and thin crystal approximations are assumed [17–21]. The spectral matching is implemented by inserting narrow-bandwidth interference filters in front of the detectors [17, 21]. The state of twin photons when they are transmitted through generic apertures, under above approximations, can be written in one dimension as [15, 17]

$$|\Psi\rangle = \int dq_s \int dq_i \mathcal{F}(q_s, q_i) |1_{q_s}\rangle |1_{q_i}\rangle,$$

where $|1_{q_j}\rangle$ is the Fock state for one photon in mode $j = s, i$ with transverse wave vector $q_j$. $\mathcal{F}(q_s, q_i)$ is the two-photon...
amplitude and is given by

$$ F(q_s, q_j) = \int dx_j \int dx_i A_j(x_j) A_i(x_i) \times \exp(i k_p (x_j - x_i)^2/8 z_A) \times W \left( \frac{1}{2} (x_i + x_j); z_A \right) \exp(-i(q_s x_j + q_j x_i)). $$

The longitudinal distance $z_A$ defines the plane where the apertures are placed and $k_p$ is the number of the pump beam used to generate the twin photons. The function $W(x; z_A)$ is the pump field distribution at the plane of the apertures ($z = z_A$) and at the transverse position $x$. $A_j(x_j)$ is the transmission function of the aperture in mode $j$.

In the case where these apertures are two identical double slits, the two-photon state in equation (1) simplifies to [15, 16]

$$ |\Psi\rangle = W_{+,+}|+\rangle|+\rangle + W_{-,-}|+\rangle|-\rangle + W_{-,+}|+\rangle|+\rangle + W_{+-}|+\rangle|-\rangle, $$

where $W_{+,-} = W_{+,+} = \gamma W(0; z_A) e^{ik_p d^2/8 z_A}$ and $W_{-,+} = \gamma W(\pm 2; z_A)$. $d$ is the separation between two consecutive slits and $a$ is each slit’s half width. $\gamma$ is a normalization constant. The $|+\rangle$ (or $|-\rangle$) state is a single-photon state defined, up to a global phase factor, by

$$ |\pm\rangle = \sqrt{\alpha / \pi} \int dq_j e^{iq_j d/2} \text{sinc}(q_j a) |1q_j\rangle, $$

and represents the photon in mode $j$ transmitted by the upper (lower) slit of its respective double slit. These states form an orthonormal basis in the two-dimensional Hilbert space of each photon and are used to define the twin photons’ logical spatial qubit states. Therefore, it is clear that the state $|\Psi\rangle$ represents a system composed of two spatial qubits.

A typical setup used to generate the spatial qubits is outlined in figure 1. A pump beam incident upon a nonlinear crystal generates SPDC, the twin photons which are correlated in their transverse momenta. These photons are then sent through identical double slits ($A_i$ and $A_j$) placed at the $z_A$-plane and, after being transmitted by these apertures, the twin photons will be in the state of equation (3).

3. Detecting the spatial qubits in distinct bases

For doing the quantum tomographic reconstruction of two-qubit states codified in polarization it is necessary to consider the detection of these qubits in some polarization states. The measurement bases that are usually considered in the experiments are [8]: $|\{H\}, |\{V\}\rangle$ (horizontal and vertical polarizations), $|\{D\}, |\{A\}\rangle$ (linearly polarized light at 45° and 135° with respect to the horizontal polarization, respectively) and $|\{L\}, |\{R\}\rangle$ (left and right circularly polarized lights, respectively). The measurement (the projection) in one of the states of these bases is represented by a projector operator, and an adequate choice among the available projections allows the reconstruction of the composite state.

Now we show how the spatial qubits can be measured in distinct bases of $|\{\pm\}⟩, |\{\pm\}⟩$. After being transmitted by the double slit, the twin photons freely evolve in space. Because of diffraction, it spreads out faster in the $x$-direction so that it passes from the discrete states $|\pm\rangle$, to states continuously distributed along this transverse direction. In a certain $z$-plane situated far behind the double slits’ plane ($z = z_A$), each particular superposition of the basis states $|\pm\rangle$ will have a specific transverse probability distribution [22]. To recover the discrete nature of the logical states in this $z$-plane, we need to implement an adequate post-election process for the twin photons. As we shall show in the following, this can be properly done by allocating in this plane one double slit for the transmission of each mode $j$. We refer to these new apertures as detection double slits. The transverse position of their slits is $x_{j,k}^\mu$, where the subscript $j$ denotes the mode at which the double slit was placed (again $j = s, i$) and $k = 0, 1$ stand for the lower and upper slits of this aperture, respectively. The slits’ width is $2b$. The superscript $\mu$ is used to define a set of measurements. For a certain value of $\mu$ we have two transverse positions available, $x_{j,0}^\mu$ and $x_{j,1}^\mu$, for the slits of the double slit. It can be seen as the mean position of the double slit’s centre and, as we shall see, is the index which defines the spatial measurement basis which is being considered. The state of the twin photons that crossed these two detection double slits can be written as

$$ |\Phi\rangle \propto \sum_{k,k'=0,1} B_{k,k'} |f(x_{j,k}^\mu)\rangle |f(x_{j,k'}^\mu)\rangle, $$

where the states $|f(x_{j,k}^\mu)\rangle$ are the states of the post-selected photons which crossed the slit in the transverse position $x_{j,k}^\mu$ of the double slit placed in mode $j$. These states are given by

$$ |f(x_{j,k}^\mu)\rangle = \frac{1}{\sqrt{b}} \int dq_j \exp(-iq_j x_{j,k}^\mu) \times \text{sinc} \left( \frac{q_j a b}{2\alpha} + q_j b \right) |1q_j\rangle, $$

where $\alpha = (z - z_A)/k_p$ and one can see that the logical states used to describe the twin photons are once again discrete. The calculation of the state $|\Phi\rangle$ is shown in appendix A. However, what is really interesting to note now is that there is a relation between the coefficients of this state and the coefficients of the...
original state of the twin photons given by equation (3). That is
\[ B_{k,k'} = \sum_{u,v=\pm} r_u \left( x_{+k}^u \right) r_v \left( x_{+k'}^v \right) W_{u,v}, \]
with
\[ r_{\pm} \left( x_{+k}^u \right) \equiv \text{sinc} \left( \frac{(x_{+k}^u + d) a}{2a} \right) \exp \left( \frac{(x_{+k}^u + d)^2}{4a} \right). \]

Therefore, one can think of the state of the twin photons transmitted through the detection double slits (equation (5)) as the original state of these photons (equation (3)) rewritten in the new basis \( \{|f(x_{+k}^u)|, |f(x_{+k}^v)|\} \) (again \( k, k' = 0 \) and 1). As we mentioned before, the spatial bases are designed in terms of \( \mu \). For distinct values of \( \mu \) one has distinct transverse positions \( x_{+0}^\mu \) and \( x_{+1}^\mu \) for the slits of the detection double slit in mode \( j \). The states \( |f(x_{+0}^\mu)| \) and \( |f(x_{+1}^\mu)| \) are orthonormal, i.e., they form an orthonormal basis for the Hilbert space of the single photon after the detection double slit. We use the superscripts \( \mu, \mu', \mu'', \ldots \) to denote distinct bases. The post-selection of the twin photons in one of the basis states of \( \{|f(x_{+k}^\mu)|, |f(x_{+k}^{\mu'})|\} \) is then represented by the following projector:
\[ P_{\mu,\mu'} \equiv |f(x_{+k}^\mu)|, f(x_{+k}^{\mu'})\rangle \langle f(x_{+k}^{\mu'}), f(x_{+k}^\mu)|. \]

4. Reconstructing the qubit state

As is discussed in [8], the density operator of two qubits is specified by 16 real parameters. Thus, for determining their state, one needs to construct 16 equations that are linearly specified by 16 real parameters. Thus, for determining their state, one needs to construct 16 equations that are linearly specified by 16 real parameters. Thus, for determining their state, one needs to construct 16 equations that are linearly specified by 16 real parameters.

4.1. The protocol

We can now explain the protocol for doing the reconstruction of the two spatial qubits state given by equation (3). To start with this, let us first assume the following general formula for its density operator \( \rho \):
\[ \rho = \sum_{l,m,l',m'} \rho_{l,m,l',m'} |l_s; m_i\rangle \langle l'_s; m'_i|. \]
where the indices \( l, m, l', m' \) assume the values + (or −) when denoting the upper (lower) slits of the double slits \( A_j \)'s. For example, the matrix element, \( \rho_{l_1 \ldots l_N} \), is the probability of detecting the twin photon signal at the slit ‘+’ and the idler photon at the slit ‘−’.

To construct the first set of MLI equations, one should first measure the coincidence rates with two detectors \( D_l \) and \( D_m \) placed just behind the slits of the apertures \( A_l \) and \( A_m \) as shown in figure 2(a). Four MLI equations can be constructed by permuting the detector positions behind the slits of these two double slits. The coincidence rates behind these slits, according to equation (11), can be written as
\[ C_{l,m} = \eta N \text{Tr}(\rho P_{l,m}), \]
where \( N \) is the number of the photon pairs transmitted through the double slits and \( \eta \) is the coincidence counts efficiency for the detectors used. \( P_{l,m} \) is the projector operator given by \( P_{l,m} = |l_s; m_i\rangle \langle l_s, m_i|. \) If we work out this expression, we will see that
\[ C_{l,m} = \eta N \text{Tr}(\rho P_{l,m}), \]
and, therefore it is clear that, after doing the normalization of the coincidence counts seen in this part of the experiment, one will be able to determine the diagonal elements of the spatial qubits density operator \( \rho \).

To obtain four more MLI equations one should now introduce the detection double slits in the \( z \)-plane, with their slits at the transverse positions \( x_{+0}^0 = 0 \) and \( x_{+1}^1 = \frac{b}{a} \) (again, \( a = (z - z_a)/b \)), and record the coincidence rates with the detectors \( D_j \)'s behind these slits (see figure 2(b)). The new coincidence rates are now written in terms of the projectors of equation (9) and are given by
\[ C_{x_{+i}^0,x_{+i}^0} = \left( \frac{2b}{L} \right)^2 \eta N \text{Tr}(\rho \Phi_{x_{+i}^0,x_{+i}^0}), \]
\[ C_{x_{+i}^1,x_{+i}^1} = \left( \frac{2b}{L} \right)^2 \eta N \text{Tr}(\rho \Phi_{x_{+i}^1,x_{+i}^1}), \]
\[ C_{x_{+i}^0,x_{+i}^1} = \left( \frac{2b}{L} \right)^2 \eta N \text{Tr}(\rho P_{x_{+i}^0,x_{+i}^1}), \]
\[ C_{x_{+i}^1,x_{+i}^0} = \left( \frac{2b}{L} \right)^2 \eta N \text{Tr}(\rho P_{x_{+i}^1,x_{+i}^0}), \]
where \( L \) is the transverse size of the idler and signal diffraction patterns (caused by the apertures \( A_j \)'s) at the

3. For practical purposes, it is worth mentioning that this measurement can also be done in the plane of image formation of these apertures [23].

4. Here we consider the usually accepted fair sampling assumption where the ensemble observed in the experiment \( (g \times N \) twin photons) is considered to be a fair sample of the total ensemble of \( N \) twin photons transmitted by the double slits.
possible coincidence rates obtained, again by permuting the detectors’ positions behind the slits of these double slits, are now written as

\[
C_{x_{i,m}^{a_{m}}} = \frac{2b}{L} \eta N \text{Tr}(\rho P_{x_{i,m}^{a_{m}}}),
\]

\[
C_{x_{i,m}^{a_{m}}} = \frac{2b}{L} \eta N \text{Tr}(\rho P_{x_{i,m}^{a_{m}}}),
\]

with again \( m = + (m = -) \) for the detection of the idler photon at the upper (lower) slit of \( A_i \). The projector \( P_{x_{i,m}^{a_{m}}} \) is defined by

\[
P_{x_{i,m}^{a_{m}}} = |f(x_{i,m}^{a_{m}}), m|f(x_{i,m}^{a_{m}}), m\rangle\langle f(x_{i,m}^{a_{m}}), m|f(x_{i,m}^{a_{m}}), m|f(x_{i,m}^{a_{m}}), m\rangle.
\]

These other four equations are MLI equations\(^{6} \) and they allow the determination of new elements of \( \rho \) as shown in table 1.

The last four MLI equations are obtained when the signal photons are detected behind the slits of \( A_s \) and the idler photons behind the slits \( x_{i,0} = 0 \) and \( x_{i,1} = \frac{2\pi}{N} \) at the transverse \( z \)-plane. The projectors which represent these four coincidence measurements are given by

\[
P_{l,s_{i}^{a_{m}}} = |l, f(x_{i,s}^{a_{m}})|l, f(x_{i,s}^{a_{m}})\rangle\langle f(x_{i,s}^{a_{m}}), m|f(x_{i,s}^{a_{m}}), m|f(x_{i,s}^{a_{m}}), m\rangle.
\]

and the coincidence rates are

\[
C_{l,s_{i}^{a_{m}}} = \frac{2b}{L} \eta N \text{Tr}(\rho P_{l,s_{i}^{a_{m}}}),
\]

\[
C_{l,s_{i}^{a_{m}}} = \frac{2b}{L} \eta N \text{Tr}(\rho P_{l,s_{i}^{a_{m}}}),
\]

where \( l = + (l = -) \) holds for the detection of the signal photon at the upper (lower) slit of \( A_s \). The elements of \( \rho \) that can be determined from these last measurements are also shown in table 1.

So, after performing the 16 measurements described above, one will have reconstructed the density operator \( \rho \) of the two spatial qubits state generated in the experiment considered. It is worth mentioning that the protocol presented here can also be applied to any type of two spatial qubit state, including the mixed states recently reported in the literature [24].

Here, we give a brief description of an experimental setup that can be used for implementing this reconstruction scheme. This description is based on already known parameters from our experiments. First, to generate spatial qubits (qudits) we use two identical double (multi) slits aligned in the direction of the signal and idler beams, which are positioned at a distance \( z_a = 200 \text{ mm} \) from the Type II BBO crystal. The slits width is 0.1 mm with a separation of 0.25 mm. The photons transmitted through the double slits are detected in coincidence. In front of the detectors there are interference filters, with very small bandwidths (with a typical FWHM of 1 nm) and centred at twice the pump beam wavelength, which will select the frequency of the generated signal and idler photons [17, 21]. For reconstruction purposes identical single slits at the detection plane can be used at position \( z - z_a = \)

\footnote{This is, of course, an approximation where we assume that the total length of the double slit diffraction pattern, \( L \), is defined by the transverse length of the three principal peaks of diffraction. However, we can say that equation (15) is a good approximation for the values of the coincidence rates considered since most of the intensity of the light scattered by a double slit lies in this \( L \) region.}

\footnote{Here it should be clear that we are always considering that the MLI equations constructed are linearly independent in the system of 16 equations that we want to construct. The reader can check this by calculating equation (7), explicitly, for the values \( x_{i,0}^{a} = 0 \) and \( x_{i,1}^{a} = \frac{2\pi}{N} \) used.}

Figure 2. Schematic diagram of the spatial measurements needed for doing the quantum tomography of two spatial qubits. In (a) the coincidence counts are recorded with the detectors \((D_{i} \text{ and } D_{j})\) fixed behind the slits of the double slits \( A_{s} \) and \( A_{i} \), respectively. In (b) they are fixed behind the detection double slits whose slit transverse positions are: \( x_{i,0}^{a} = 0 \) and \( x_{i,1}^{a} = \frac{2\pi}{N} \). In (c) one detector records the idler photons transmitted by the slits of \( A_{s} \) and the other, the photons transmitted by the signal detection double slit. z-plane\(^{5} \). Therefore, it is clear that the factor \( \frac{2b}{L} \) is a normalization constant which accounts for the effect of one photon diffraction, which causes a reduction of \( \frac{2\pi}{N} \) in the detector’s efficiency that is being used to record this photon at the z-plane. From these measurements one will be able to determine the imaginary and real parts of some of the matrix elements of \( \rho \). In table 1, we give a summary showing which are the components of \( \rho \) that are determined from the measurements considered in this protocol. There is also a comparison with the measurements done in the QTR of two polarized qubits [8]. In appendix B we give the expressions which relate the elements of \( \rho \) and all the coincidence rates considered in this protocol.

Let us now consider the detection of the signal photons behind its detection double slit at the z-plane and the detection of the idler photons behind the slits of the double slit \( A_{s} \) (at the \( z_{s}-\text{plane} \)) as schematically shown in figure 2(c). The four
Table 1. Summary of the spatial measurements needed to do the quantum tomography of two spatial qubits. There is also a comparison with the measurements used in the QTR of polarized qubits. $|H\rangle$, $|V\rangle$, $|D\rangle$ and $|L\rangle$ are the kets representing polarized qubits with horizontal, vertical, diagonal and left-circular polarization, respectively. The indices $\Re$ and $\Im$ stand for the real and imaginary parts, respectively.

| Measurements | Pol. proj. | Coefficients | Spatial proj. | Coefficients |
|--------------|------------|--------------|---------------|--------------|
| $|HH\rangle$ | $|HH\rangle$ | $\rho_{HH,HH}$ | $P_{++}$ | $\rho_{++}$ |
| $|HV\rangle$ | $|HV\rangle$ | $\rho_{HV,HH}$ | $P_{+-}$ | $\rho_{-+}$ |
| $|VV\rangle$ | $|VV\rangle$ | $\rho_{VV,HH}$ | $P_{-+}$ | $\rho_{+-}$ |
| $|DL\rangle$ | $|DL\rangle$ | $\rho_{DL,HH}$ | $P_{+-}$ | $\rho_{-+}$ |
| $|VD\rangle$ | $|VD\rangle$ | $\rho_{VD,HH}$ | $P_{-+}$ | $\rho_{+-}$ |
| $|VL\rangle$ | $|VL\rangle$ | $\rho_{VL,HH}$ | $P_{+-}$ | $\rho_{-+}$ |

600 mm. At this distance, the fourth-order interference pattern has a width of the order of 8 mm. The detection slit has a width of 0.1 mm. This slit can be easily installed in front of single photon detector modules. If a krypton laser, with an average power of 200 mW is used as a pump beam, a maximum number of 200 coincidence counts in 500 s are expected for (b) configuration in figure 2. For this experimental setup we have $x_0^{I}=0$ and $x_0^{I} = 0.496$ mm. These measurement positions are within the fourth-order interference pattern region and they can be well resolved. In case of the measurements of type (a) and (c) 200 coincidence counts are expected for measurement times of 20 s and 100 s, respectively. Furthermore, there is no requirement for the numerical aperture of the detectors.

4.2. Maximum likelihood estimation

The technique of QTR is based on a linear inversion of the measured data as was shown above for the two spatial qubits states. Thus, it is dependent on any experimental errors that may occur while recording the data. They can appear as a consequence of experimental noise or misalignment and the reconstructed state is, in general, only a reasonable approximation of the real quantum state. The density matrices obtained may have properties that are not fully compatible with a quantum state.

To generate only possible density matrices there is an alternative that has been used during the state determination. It is a numerical technique called the maximum likelihood estimation [25]. However, even though it generates only possible density matrices, it has the drawback of enhancing the uncertainty in the state estimation.

Once the two-qubit density operator is reconstructed with the technique proposed here, one can apply the matrix obtained to this numerical optimization. In [8] a practical approach for doing this is extensively discussed and it can also be applied for two spatial qubit systems.

5. Conclusion

In conclusion, we have theoretically shown that one can use the technique of quantum tomography to reconstruct states composed of two spatial qubits. Even though we have considered the state determination only for this special case, we believe that it can be generalized and performed in a similar way for spatial states of more dimensions. The importance of this work comes from the possibility of using the spatial qubits for quantum communication protocols where the ability to characterize may be necessary. The main advantage that we can envisage at this stage is the possibility of using transverse correlations of the down-converted photons for encoding quantum information in qudits instead of qubits. Besides, even in the case of qudits we see some other important advantages. For instance, when entangled polarization qubits are propagated through optical fibres they suffer decoherence because of the depolarization effect of the fibres. The birefringence of the fibre gradually destroys the entanglement of the entangled polarized qubits. The entanglement of the time-bin-entangled qubits also suffers severe decoherence in optical fibre due to the chromatic dispersion effect. However, when the spatial qubits are used for sending information through optical fibres, both depolarization and chromatic dispersion does not affect, directly, their quantum correlations and thus the entanglement of the spatial qubits seems to be more robust against the decoherence effects of optical fibres. We have already started this line of research in our group to demonstrate this idea.
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Appendix A. Calculating the state \(|\Psi\rangle\)

As we showed in section 2, the state of the down-converted photons when they are transmitted through double slits, placed at the z\(_\mu\)-plane, is given by equation (3) (Now, if we consider the free space propagation of the twin photons from these apertures to another transverse z-plane, we obtain a new form for their state [23]):

\[
|\Psi\rangle \propto \int dq_j \int dq_i G(q_i, q_j)|1q_i\rangle|1q_j\rangle, \tag{A.1}
\]

where the new two-photon amplitude \(G(q_i, q_j)\) is

\[
G(q_i, q_j) = \frac{\alpha}{\pi} e^{-i\omega q_i\tau} e^{-i\omega q_j\tau} \sin(q_i a)\sin(q_j a)
\]

\[
\times \left( W_{+} e^{-i(q_i+q_j)d} + W_{-} e^{i(q_i+q_j)d} \right)
\]

\[
+ \frac{\alpha}{\pi} e^{-i\omega q_i\tau} e^{-i\omega q_j\tau} \sin(q_i a)\sin(q_j a)
\]

\[
\times \left( W_{+} e^{-i(q_j-q_i)d} + W_{-} e^{i(q_j-q_i)d} \right), \tag{A.2}
\]

again, \(\alpha = \frac{\omega - z\mu}{\sqrt{\gamma}}\).

Now we consider the determination of the twin-photon state after they are transmitted through the detection double slits considered in our protocol and which are now placed at this transverse z-plane. The transverse positions for the slits of these double slits are \(x_{\mu 0}\) and \(x_{\mu 1}\). To calculate this state we first assume a general form for it

\[
|\Phi\rangle \propto \int dq_j \int dq_i D(q_i, q_j)|1q_i\rangle|1q_j\rangle. \tag{A.3}
\]

The two-photon amplitude \(D(q_i, q_j)\) can be written in terms of the convolution

\[
D(q_i, q_j) = \int dq_j' \int dq_i' G(q_i', q_j') T_i(q_i' - q_i) T_j(q_j' - q_j), \tag{A.4}
\]

where \(T_i\)’s are the Fourier transform of the transission function of the detection double slits and are given by

\[
T(q_j) = 2\eta e^{iq_j\tau/\sqrt{\eta}} \sin(q_j b) + e^{iq_j\tau/\sqrt{\eta}} \sin(q_j b), \tag{A.5}
\]

and by replacing equation (A.5) into equation (A.4) we obtain

\[
D(q_i, q_j) = \int dq_j' \int dq_i' e^{-i(q_i'+q_j')d/\gamma}
\times \sin(q_i'a)\sin(q_j'a)\sin((q_j' - q_j)b)
\times \sin((q_j' - q_j)b) \sum_{u,v=k} W_{u,v} e^{-i(q_j'+uq_j'+vq_j)}
\times \sum_{k,k'} e^{i(q_j'-q_j)v_{k,k'}} e^{i(q_j'-q_j)v_{k,k'}}. \tag{A.6}
\]

By using the following solution

\[
I_{\pm}(x_{\mu,i,j}, q_j) = \int e^{-iux_{\mu,i,j}} e^{iux_{\mu,i,j}} e^{i(q_j'-q_j)x_{\mu,i,j}} \times \sin((q_j' - q_j)b) dq_j
\approx e^{-iux_{\mu,i,j}} e^{iux_{\mu,i,j}} \sin\left(\frac{(x_{\mu,i,j} + l)d}{2\alpha}\right)
\times \sin\left(\frac{x_{\mu,i,j}b}{2\alpha} + q_j b\right), \tag{A.7}
\]

and by defining the coefficients

\[
r_{\pm}(x_{\mu,i,j}) = \sin\left(\frac{(x_{\mu,i,j} + l)d}{2\alpha}\right) \exp\left(i\frac{(x_{\mu,i,j} + l)^2}{4\alpha}\right), \tag{A.8}
\]

one can write the function \(D(q_i, q_j)\) as

\[
D(q_i, q_j) = \sum_{k,k'=0,1} \sum_{u,v=k} W_{u,v} r_k(x_{\mu,i,k}) r_k(x_{\mu,i,k'})
\times e^{-iux_{\mu,i,k}} \sin\left(\frac{x_{\mu,i,k}b}{2\alpha} + q_j b\right)
\times e^{-iux_{\mu,i,k'}} \sin\left(\frac{x_{\mu,i,k'}b}{2\alpha} + q_j b\right). \tag{A.9}
\]

Now if we replace equation (A.9) into equation (A.3) and define the new slit states

\[
|f(x_{\mu,i,k})\rangle = \int dq_j e^{-iux_{\mu,i,k}} \sin\left(\frac{x_{\mu,i,k}b}{2\alpha} + q_j b\right)|1q_j\rangle, \tag{A.10}
\]

we get that the state of the twin photons which were transmitted by the detection double slits is

\[
|\Phi\rangle \propto \sum_{k,k'=0,1} \sum_{u,v=k} W_{u,v} r_k(x_{\mu,i,k}) f(x_{\mu,i,k})|f(x_{\mu,i,k'})\rangle, \tag{A.11}
\]

with

\[
B_{k,k'} = \sum_{u,v=k} r_k(x_{\mu,i,k}) r_k(x_{\mu,i,k'}) W_{u,v}. \tag{A.12}
\]

Appendix B. The coefficients of \(\rho\)

Here we show explicitly the expressions for determining all the coefficients of the density operator, \(\rho\), by following the measurement scheme given in section 4.1 (see figure 2). The first type of measurement is performed with the detectors \(D_i\) and \(D_j\) right behind the slits of the double slits \(A_i\) and \(A_j\), respectively (see figure 2(a)). As discussed above, the diagonal coefficients are given by

\[
\rho_{i,m,j,m} = \frac{1}{\eta N} C_{i,m}, \tag{B.1}
\]

with the normalization that \(\eta N = C_{++} + C_{+-} + C_{-+} + C_{--}\). We remind that \(C_{i,m}\) is the coincidence rate recorded with the
diagonal elements are given in terms of these coincidences $R_{Ds}$ detector behind the slit $l$ of its double slit and with the detector $D_m$ behind the slit $m$ of $A_i$.

Now, we consider the detection of the signal photon behind its detection double slit at the $z$-plane (see figure 2(c)), while the other detector remains behind the slits of the aperture $A_s$ (see section 4.1 for more details). In this case, the coincidence rates are

$$C_{\chi_{s,m}} = \frac{2b}{L} \eta N \text{Tr}(\rho P_{\chi_{s,m}}),$$

$$C_{\chi_{l,m}} = \frac{2b}{L} \eta N \text{Tr}(\rho P_{\chi_{l,m}}),$$

and by considering the values of $\chi_{s,0} = 0$ and $\chi_{s,1} = \frac{2\pi}{3N}$ we get

$$\rho_{+:+,=} = \frac{1}{\eta N} \left( \chi(C_{\chi_{s,0}} - iC_{\chi_{s,1}}) + \frac{C_{++} + C_{+-}}{2} (-1 + i) \right) = \rho_{+:+,=}^*,$$

$$\rho_{-,:-,=} = \frac{1}{\eta N} \left( \chi(C_{\chi_{s,0}} - iC_{\chi_{s,1}}) + \frac{C_{--} + C_{+-}}{2} (-1 + i) \right) = \rho_{-,:-,=}^*,$$

where $\chi = L/(2b)$.

Reversing the role of signal and idler detectors, by considering the coincidence rates of equation (19) in our protocol, we obtain

$$\rho_{+:+,=} = \frac{1}{\eta N} \left( \chi(C_{\chi_{s,0}} - iC_{\chi_{s,1}}) + \frac{C_{++} + C_{+-}}{2} (-1 + i) \right) = \rho_{+:+,=}^*,$$

$$\rho_{-,:-,=} = \frac{1}{\eta N} \left( \chi(C_{\chi_{s,0}} - iC_{\chi_{s,1}}) + \frac{C_{--} + C_{+-}}{2} (-1 + i) \right) = \rho_{-,:-,=}^*.$$  

The last elements of $\rho$ to be calculated are the anti-diagonal coefficients. To determine them we need to consider the measurements which involve the propagation of both photons, i.e., the coincidence rates of equation (15). The anti-diagonal elements are given in terms of these coincidences by

$$\mathbb{R}(\rho_{+:+,-}) = \frac{1}{\eta N} \left[ \chi^2 \left( C_{\chi_{s,0} - \chi_{s,0}} - C_{\chi_{s,1} - \chi_{s,1}} \right) \right] \frac{1}{\eta N} \times \left[ - \chi(C_{\chi_{s,0} - \chi_{s,0}} + C_{\chi_{s,0} - \chi_{s,1}} + C_{\chi_{s,1} - \chi_{s,0}} + C_{\chi_{s,1} - \chi_{s,1}}) \right],$$

$$\mathbb{R}(\rho_{-,:-,+}) = \frac{1}{\eta N} \left[ \chi^2 \left( C_{\chi_{s,0} - \chi_{s,1}} + C_{\chi_{s,1} - \chi_{s,0}} \right) \right] \frac{1}{\eta N} \times \left[ - \chi(C_{\chi_{s,0} - \chi_{s,0}} + C_{\chi_{s,0} - \chi_{s,1}} + C_{\chi_{s,1} - \chi_{s,0}} + C_{\chi_{s,1} - \chi_{s,1}}) \right] + \frac{1}{2},$$

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