Non-linear conductivity and quantum interference in disordered metals

M. Leadbeater(1), R. Raimondi(1), P. Schwab(2), and C. Castellani(2)

(1) Istituto per la Fisica della Materia e Dipartimento di Fisica "E. Amaldi", Università di Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy
(2) Istituto per la Fisica della Materia e Dipartimento di Fisica, Università “La Sapienza”, piazzale A. Moro 2, 00185 Roma - Italy
(March 24, 1999)

We report on a novel non-linear electric field effect in the conductivity of disordered conductors. We find that an electric field gives rise to dephasing in the particle-hole channel, which depresses the interference effects due to disorder and interaction and leads to a non-linear conductivity. This non-linear effect introduces a field dependent temperature scale $T_E$ and provides a microscopic mechanism for electric field scaling at the metal-insulator transition. We also study the magnetic field dependence of the non-linear conductivity and suggest possible ways to experimentally verify our predictions. These effects offer a new probe to test the role of quantum interference at the metal-insulator transition in disordered conductors.

PACS numbers: 72.10-d, 72.15.Rn

Disordered conductors have been the subject of theoretical and experimental study for almost twenty years. Recently there has been a strong resurgence of interest in the field due to the unexpected discovery of a metal-insulator transition in two-dimensional systems. Various suggestions have been made concerning the origin of the temperature dependence of the resistivity in the metallic phase and the nature of the metal-insulator transition. One main question is whether the transition is of a classical origin or if it is a real quantum phase transition. In the first case, if a standard Landau quasiparticle picture applies the observed resistivity could be attributed to a temperature dependent scattering time in the context of the semi-classical Boltzmann-Landau kinetic equation. In the second case, it has been pointed out that the occurrence of a metallic phase and a metal-insulator transition in two dimensional systems is indeed possible within the standard theory of disordered-interacting electrons.

To discriminate between these possibilities one needs specific probes for quantum interference effects. Weak localization (WL) is probed effectively by the application of a magnetic field. This magnetic field also affects the localization (WL) is probed effectively by the application of a magnetic field. This magnetic field also affects the localized states due to disorder and electron-electron interaction (EEI) but only in the particle-hole triplet channel due to the Zeeman coupling. In this paper we propose a new probe for the EEI contribution (both the singlet and triplet) based on the non-linear conductivity in the presence of a static electric field. We recall that WL is not affected by such a field.

To be more specific we have found that a static (or low frequency) electric field: (1) acts as a source of dephasing in the particle-hole channel and introduces a characteristic temperature $T_E = (D_{qp} e^2 E^2)^{1/3}$ below which interference effects are suppressed. Here $D_{qp}$ is the quasiparticle diffusion coefficient defined below. (2) For temperatures above $T_E$ non-linear effects in the conductivity appear as $T_E^2 / T^3$ corrections. (3) For large electric field the scale of the magnetic field in the magnetoresistance is set by $T_E$. Clearly one expects that besides quantum interference, heating effects will also be important in the non-linear conductivity and we shall suggest how to distinguish the two effects. Our results, besides providing a new probe for EEI corrections, offer a microscopic mechanism for the electric field scaling which is observed in two-dimensional systems.

Before giving details of the mathematical derivation a qualitative understanding of the effect may be obtained by simple physical arguments along the lines of Ref. [1]. In a generalised Hartree-Fock picture one electron is scattered by the potential created by all the other electrons. Due to disorder, the electron density is not uniform and hence this potential is random. From a semi-classical point-of-view, a local, single-particle quantity, like current, only involves closed paths. Futhermore, the corrections are dominated by all the other electrons retracing backwards-in-time (as holes) almost the same closed paths. According to Ref. [1] only trajectories which are traversed in a time $\eta < 1 / T$ contribute to quantum corrections. Although the two electrons go around the same closed path they have different starting positions. The first electron starts at the observation point $x_1$ at time zero, while the second electron will only start to retrace the path at the point of interaction $x_2$ at time $t_1$. This means that the second electron will traverse part of the closed path at a different time. In the presence of a vector potential the accumulated phase difference is then

\[
\phi_1 - \phi_2 = e \int_{t_1}^{t_0} dt' \mathbf{x}_1(t') \cdot \mathbf{A}(\mathbf{x}_1(t'), t') - e \int_{t_1}^{t_0} dt' \mathbf{x}_2(t') \cdot \mathbf{A}(\mathbf{x}_2(t'), t').
\]

If the vector potential is time independent (eg. a magnetic field) these phases completely cancel. However, if the vector potential is time dependent (as for a static electric field) the time delay leads to a finite phase difference $\phi_1 - \phi_2 = e (\mathbf{x}_2 - \mathbf{x}_1) \cdot \mathbf{E} \eta$, which suggests
that the EEI correction should be sensitive to a static electric field, in contrast to WL. Such a phase-sensitivity leads to non-linear conductivity. It is possible to estimate the typical electric field scale where dephasing and non-linear effects in a weakly disordered metal become important. The typical time scale is the inverse temperature and the typical length scale is the thermal length \( L_T = (D_{qp}/T)^{1/2} \). The non-linear effects become important when the phase difference induced by the electric field is of order one, which leads to the condition that the voltage drop over a thermal length becomes of the order of the temperature, i.e., when \( eEL_T \sim T \). This condition defines the temperature scale \( T_E \) given above.

We now present a quantitative theory of our results. We start with the expression for the EEI quantum correction to the current due to the interplay between disorder and interaction. Within the real-time Keldysh formalism we obtain:

\[
\delta j(t) = -\frac{4\tau^2_e}{\pi} \int dp d\tau_1 d\tau_2 \left( \frac{\pi T}{\sinh(\pi T \eta)} \right)^2 \sum_q D_{qp} q \\
\times \sum_{J,M} V_{J,M}(q,t_1-t_2) D_{J,M}^{\eta=0} \left( t_2, t-\eta; q \right) \\
\times D_{J,M}^{\eta} \left( t-\eta/2, t_1-\eta/2; q \right),
\]

(1)

A pictorial representation of this equation is shown in Fig. 1. The details of its derivation may be found in [13]. The sum \( \sum_{J,M} \) is over one singlet (\( J = 0, M = 0 \)) and three triplet channels with \( J = 1, M = 0, \pm 1 \). In eq. (1) \( \tau \) is the elastic scattering time which is the short-time cut-off in the problem. \( V_{J,M} \) and \( D_{J,M}^{\eta} \) are the interaction and the diffusion propagator in the spin channel \((J,M)\). Here the time arguments \( t, t' \) refer to the incoming and outgoing centre-of-mass time of the particle-hole pair and \( \eta \) to the relative time which is constant during the propagation. Notice that both \( V_{J,M} \) and \( D_{J,M} \) are retarded functions. The factor containing the hyperbolic sine comes from the Fourier transform of Fermi functions and limits us to trajectories with traverse time \( \eta < 1/T \). The interaction is found by summing ladder diagrams and is given by

\[
V_{J,M}(q, \omega) = \gamma_J \frac{-i\omega + D_{qp} q^2 + iM \tilde{\Omega}_s}{-i(1-2\gamma_J)\omega + D_{qp} q^2 + iM \Omega_s},
\]

(2)

where \( \gamma_J \) is the static amplitude for scattering between particles and holes. The quasi-particle diffusion constant can be expressed in terms of the particle diffusion constant \( D_{qp} = D/Z \). The parameter \( Z \), which arises in the context of the Fermi liquid theory of disordered systems [7] as the energy renormalization, plays the role of mass renormalization, \( m^* / m \), in the effective Fermi liquid theory of disordered systems [3]. The interaction amplitude in the spin singlet channel is given by \( \gamma_{J=0} = 1/2 \) for long range Coulomb forces. The triplet amplitude, for which we adopt in the following the standard notation \( \gamma_J = -\gamma_2/2 \), is related to the Landau parameter \( F_0 \) via \( \gamma_2 = -A_0 - F_0/1 + F_0^2 \). The diffusion propagator \( D_{J,M} \) is given by the solution of the differential equation

\[
\left\{ \frac{\partial}{\partial t} + D_{qp} \left[-i\nabla + eA_0(q, t)\right]^2 + iM \tilde{\Omega}_s \right\} D_{J,M}(t, t') \\
= \frac{1}{T} \delta(t - t') \delta(q - q'),
\]

(3)

where \( A_0(q, t) = A_0(q - \omega t/2 - \omega t/2) \). The term \( iM \tilde{\Omega}_s \) is due to the Zeeman coupling, where \( \tilde{\Omega}_s = (1 + \gamma_2)\Omega_s \) with \( \Omega_s = g\mu_B H \).

For a better understanding of eq. (3) we make contact with the physical arguments given in the introduction. By considering Fig. 1 one observes that for a piece of the path the particle and hole are delayed by a time \( \eta \), the traverse time of the closed path. This corresponds to the second of the diffusons in eq. (1). In the other piece there is no delay between the particle and hole. This corresponds to the first diffuson in eq. (1).

We now evaluate the current explicitly. According to eq. (3) the interaction \( V_{J,M} \) and the first of the two diffusons in (1) do not depend on the vector potential. An electric field, however, affects the second diffuson in (1) due to the non-zero time delay \( \eta \) between the particle and hole. For a static field the vector potential is \( A(t) = -Et \) and the solution of (3) is \( D_{J,M}^{\eta}(t - \frac{r}{2}, t_1 - \frac{r}{2}; q) \)

\[
= \frac{1}{T} \exp \left\{ - \int_{t_1}^{t} \left[ D_{qp}(q - eEt)^2 - iM \tilde{\Omega}_s \right] (t - t_1) \right\}
\]

The equation for the current after integrating over the momentum then becomes

\[
\delta j_{J,M} = -E \frac{4e^2 D_{qp} \gamma_J}{\pi} \frac{1 - 2\gamma_J}{4\pi D_{qp}} \int_{\tau}^\infty \frac{d\eta}{\sinh(\pi T \eta)} \left( \frac{\pi T}{\sinh(\pi T \eta)} \right)^2 \int_{0}^{\eta} \frac{dt_1}{\sinh(\pi T t_1)} \frac{t_1}{(\eta - 2\gamma_J t_1)^{1+\eta}} \\
\times \cos [M \tilde{\Omega}_s (\eta - 2\gamma_J t_1)] \\
\times \exp \left[ -T_E^3 \eta^2 t_1 (\eta - t_1)/(\eta - 2\gamma_J t_1) \right]
\]

(4)

where we have introduced \( T_E^3 = D_{qp}(eE)^2 \) and \( d \) is the dimension. It is clear from this equation that the electric field provides a dephasing time \( \sim T_E^{-1} \), since in the low temperature limit \( T \ll T_E \) the exponential now cuts off all times larger than \( T_E^{-1} \).

We first consider the current in the weak electric field regime and derive the leading non-linear terms. In the absence of magnetic field we find

\[
\delta j = \frac{E e^2}{2d - i\pi} \left( \frac{D_{qp}}{T} \right) \frac{d}{d\eta} \int_{\pi T \tau}^{\infty} dx \frac{\eta^2 - 4}{\sinh^2(x)} \\
\times \left( f^0_2(\gamma_2) + f^3_2(\gamma_2) \right) \frac{x^3/3}{\pi^3 T^3}
\]

(5)

where the functions \( f^0_2, f^3_2 \) are shown in the table. For the sake of completeness we have also included the term linear in the electric field which reproduces the well-known
interaction correction to the conductivity. Notice that the functions $f_{1,3}^2(\gamma_2)$ are the sum of the singlet and the triplet contributions. The remaining integrals are of a standard form. We now have the results for \( \delta \sigma = \delta |j|/|E| \)

\[
\begin{align*}
\delta \sigma_1 &= \frac{e^2}{\pi} L_T \left[-2.46 f_1^1(\gamma_2) + \frac{4.88}{\pi^3} f_1^2(\gamma_2) \frac{T_E^3}{T^3} \right] \\
\delta \sigma_2 &= \frac{e^2}{2\pi^2} \left[ f_2^2(\gamma_2) \ln \left( \frac{e}{2\pi T} \right) + \frac{\pi}{30} f_3^2(\gamma_2) \frac{T_E^3}{T^3} \right] \\
\delta \sigma_3 &= \frac{e^2}{4\pi^2} L_T^{-1} \left[ \frac{1.83}{\pi^2} f_3^1(\gamma_2) + \frac{2.32}{\pi^3} f_3^2(\gamma_2) \frac{T_E^3}{T^3} \right]
\end{align*}
\]

where \( L_T \) is the thermal length defined previously and we have left out temperature independent terms. We recall that, in the case of spin-singlet interactions only (\( \gamma_2 = 0 \)) the conductivity decreases with decreasing temperature (i.e. \( f_1^1(0) > 0 \)), whereas for sufficiently large triplet amplitude \( \gamma_2 \) the latter dominates and leads to an increase of conductivity with decreasing temperature (i.e. \( f_1^1(\gamma_2) < 0 \)). The non-linear coefficient \( f_3^2(\gamma_2) \), however, is generically positive and only changes sign for large \( \gamma_2 \) in \( d = 3 \).

We study the cross-over behaviour from small to large electric fields by numerically integrating eq. (6). The conductivity as a function of the electric field is plotted in fig.3 for two values of \( \gamma_2 \) for \( d = 2 \). At zero field, for \( \gamma_2 = 0 \) (\( \gamma_2 = 5 \)) the correction is localising (anti-localising) with \( \delta \sigma_2 < 0 \) (\( \delta \sigma_2 > 0 \)). The quadratic increase at small fields has a positive curvature irrespective of the value of \( \gamma_2 \). At large field, the temperature scale disappears and the correction \( \delta \sigma \) has the same form as the linear conductivity with \( T_E \) replacing the temperature.

Non-linear effects also appear in the magnetoconductance which originate from the magnetic field depression of the \( M = \pm 1 \) triplet contributions to the current. In particular we find for small \( T_E \) and small Zeeman energy \( \Omega_s \)

\[
\begin{align*}
\Delta \sigma_1 &= \frac{e^2}{\pi^2} L_T \frac{\Omega_s^2}{T^2} \left[ \frac{2.32}{\pi^2} g_1^1(\gamma_2) + \frac{41.85}{\pi^5} g_1^2(\gamma_2) \frac{T_E^3}{T^3} \right] \\
\Delta \sigma_2 &= \frac{e^2}{2\pi^2} \Omega_s^2 \left[ \frac{3\zeta(3)}{2\pi^2} g_2^1(\gamma_2) + \frac{\pi}{42} g_2^2(\gamma_2) \frac{T_E^3}{T^3} \right] \\
\Delta \sigma_3 &= \frac{e^2}{4\pi^2} L_T^{-1} \frac{\Omega_s^2}{T^2} \left[ \frac{1.58}{\pi^2} g_3^1(\gamma_2) + \frac{13.04}{\pi^5} g_3^2(\gamma_2) \frac{T_E^3}{T^3} \right]
\end{align*}
\]

where \( \Delta \sigma = \sigma(\Omega_s) - \sigma(0) \) and the functions \( g_{d}^{1,3} \) are also shown in the table (6). To illustrate the behaviour at large \( \Omega_s \) we again resort to numerical integration of eq. (6). In fig.3 we show the magnetic field dependence of the current for two choices of \( T_E \) for \( \gamma_2 = 2.5 \). Notice that for such a value of \( \gamma_2 \) the zero magnetic field conductivity interference correction has an anti-localizing character. This explains the suppression of conductivity with increasing \( T_E \). For small \( T_E \) we obtained the standard behaviour of a initial quadratic decrease on the scale of the temperature followed by a logarithmic suppression of the corrections. For large \( T_E \) however, the temperature disappears as an energy scale and, although the curve appears similar, the scale of the magnetic field is now set by \( T_E \). Expanding eq. (6) to leading order in the magnetic field for \( T_E \gg T \) one obtains \( \Delta \sigma \propto \Omega_s^2/T_E^2 \).

The effects described in this paper may be detected by measuring the current-voltage characteristics. In such a measurement however the electron temperature changes with the applied voltage and one has to discriminate heating from non-heating non-linear effects. A direct way to isolate the non-linear contribution due to the dephasing effect of \( E \) would be to measure the electron temperature \( T_{el} \) for a given \( E \) (for instance by noise measurements as in (6)). Then \( \sigma(T_{el}, 0) - \sigma(T, E) \) yields the effect of the electric field on the EEI contribution and provides a direct probe of the relevance of quantum interference in the p-h channels. Alternatively, at low temperature, where \( T_{el} \tau_{el-ph} \gg 1 \) (\( \tau_{el-ph} \) is the heat electron-phonon relaxation time) non-heating effects could be detected by exploiting the different time scales \( \tau_{el-ph} \) and \( T_{el}^{-1} \) which control the frequency dependence of heating and non-heating effects respectively. In a time-dependent electric field, \( E(t) = E \cos(\Omega t) \), the electron temperature becomes time dependent. For frequencies \( \Omega > 1/\tau_{el-ph} \) however the temperature cannot follow the electric field, i.e. heating becomes time independent. Non-linearities due to quantum interference on the other hand follow the electric field instantly as long as the frequency remains smaller than the temperature. Thus measuring non-linear response in the presence of a microwave with a frequency of the order \( \Omega \geq 1/\tau_{el-ph} \) offers a possibility to detect the predicted effects (7).

Possible non-heating effects have already been observed in different materials like 2D Si-MOSFETs (10,11,13, GaAs (19), SiGe (20) and in gold films (23,24). In particular in Ref. (10) a remarkable electric field scaling was observed near the metal-insulator transition. Our theory of the non-linear effects provides an explicit mechanism for the electric field scaling via the temperature scale \( T_E \): on the basis of general scaling arguments close to a quantum critical point, the temperature scales as \( T \sim \xi^{-z} \) where \( \xi \) is the correlation length and \( z \) is the dynamical critical exponent (24). In a diffusive system temperature and length scales are related via the diffusion constant with \( T \sim D_{qp}(\xi)/\xi^2 \) implying a scaling of \( D/Z = D_{qp} \) near the critical point as \( D_{qp} \sim \xi^{2-z} \). From our relation \( T_E^3 = D_{qp}e^2E^2 \) one then obtains \( E \sim \xi^{-(1+z)} \). In the experiments \( z \) is near one, which corresponds to a growing quasi-particle diffusion constant and a vanishing quasi-particle density of states near the transition. This small value of \( z \) means that the electronic specific heat would vanish as \( c_v \sim T_E^{z-2} \sim T^{2/z} \). From these considerations one expects large non-linear effects near the critical point. Such large effects have been observed in Ref. (19) for a GaAs metallic sample near the metal-
insulator transition. In this experiment the differential conductivity was found to decrease with increasing voltage. In particular at low temperature the conductivity shows a non-monotonic behaviour of the type shown in fig.2 for large $\gamma_2$. By comparing with experiment it appears that the scale over which the effect is observed is larger than what we would predict based upon a naive estimate of the diffusion constant from the conductivity $\sigma = 2e^2DN_0$ and the free particle density of states $N_0$ (i.e. assuming $D_{qp} \sim D$). However, by allowing for quasi-particle diffusion constant renormalization as implied by scaling one can obtain larger effects.

Finally, we point out that the effects discussed in this paper could have an enhanced relevance in the presence of strong local electric fields such as in percolative metallic systems.

This work was partially supported by MURST under contract no. 970265437 (R.R. and C.C.), by INFM under the PRA-project QTMD (R.R.) and the European Union TMR program (M.L. and P.S.). C.C. and R.R. acknowledge useful discussions with M. Sarachik and S. Vitkalov.

[1] B.L. Altshuler, A.G. Aronov, D.E. Khmelnitskii, and A.G. Larkin, in Quantum Theory of Solids, edited by L.M. Lifshitz (MIR Publishers, Moscow, 1982); B.L. Altshuler and A.G. Aronov, in Electron-Electron Interactions in Disordered Systems, edited by M. Pollak and A.L. Efros (North-Holland, Amsterdam, 1985), p. 1.
[2] P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985); D. Belitz and T.R. Kirkpatrick, Rev. Mod. Phys. 66, 261 (1994).
[3] S.V. Kravchenko, W.E. Mason, G.E. Bower, J.E. Furneaux, V.M. Pudalov, and M. D’Iorio Phys. Rev. B 51, 7038 (1995).
[4] V. Dobrosavljevic, E. Abrahams, E. Miranda, and S. Chakravarty Phys. Rev. Lett. 79, (1997) 455; P. Phillips et al, Nature 395, 253, 1998; S.Chakravarty et al cond-mat/9805383; V.M. Pudalov JETP Lett., 66, 175 (1997); D. Belitz and T.R. Kirkpatrick, Phys. Rev. B58, 8214-8217 (1998); Q. Si and C. M. Varma, Phys. Rev. Lett. 81, 4951 (1998).
[5] B.L. Altshuler and D.L. Maslov, Phys. Rev. Lett. 82, 145 (1999); T.M. Klapwijk and S. Das Sarma cond-mat/9810349; S. Das Sarma and E.H. Hwang cond-mat/9812216.
[6] C. Castellani, C. Di Castro, and P.A. Lee, Phys. Rev. B 57, R9831 (1998).
[7] A.M. Finkel’stein JETP 57, 97 (1983); C. Castellani, C. Di Castro, P.A. Lee, and M. Ma, Phys. Rev. B 30, 527 (1984).
[8] B.L. Altshuler, A.G. Aronov, and P.A. Lee, Phys. Rev. Lett. 44, 1288 (1980); B.L. Altshuler, D. Khmel’nit’zkii, A.I. Larkin, and P.A. Lee, Phys. Rev. B 22, 5141 (1980).
[9] G. Bergmann, Z. Phys. B 49, 133 (1982).
[10] S.V. Kravchenko, D. Simonian, M.P. Sarachik, W. Me- son, and G.E. Fournear, Phys. Rev. Lett. 77, 4938 (1996), R. Heemskerk and T.M. Klapwijk, Phys. Rev. B 58, R1754 (1998).
[11] L.L. Aleiner, B.L. Altshuler and E. Gersherson, cond-mat/9808053.
[12] G. Bergmann, Phys. Rev. B 35, 4205 (1987).
[13] R. Raimondi, P. Schwab, and C. Castellani, cond-mat/9903146.
[14] C. Castellani and C. Di Castro, Phys. Rev. B34, 5935 (1986); C. Castellani, G. Kotliar, and P.A. Lee Phys. Rev. Lett. 59, 323 (1987).
[15] Note that the numerical factor we obtain in front of the $g_2^2(\gamma_2)$ term is slightly larger (about 10%) than the factor given in the literature.
[16] R. de-Picciotto et al, Nature 389, 162 (1997).
[17] S.A. Vitkalov (Private communication).
[18] S.A. Vitkalov, G.M. Gusev, Z.D. Kvon, G.I. Leviev, and V.I. Fal’ko, Sov. Phys. JETP 67, 1080 (1988).
[19] J. Yoon, C.C. Li, D. Shahar, D.C. Tsui, and M. Shayegan, Phys. Rev. Lett. 82, 1744 (1999).
[20] V. Senz, et al, preprint, cond-mat/9903367.
[21] G. Bergmann, Wei Wei, Yao Zou, and R.M. Mueller, Phys. Rev. B 41, 7386 (1990).
[22] J. Liu and N. Giordano, Phys. Rev. B 43, 1385 (1991).
[23] For a recent review see S.L. Sondhi, S.M. Girvin, J.P. Carini, and D. Shahar, Rev. Mod. Phys. 69, 315 (1997).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Pictorial representation of the current formula. Four dashed lines represent a diffusion.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{The electric field dependence of the interaction correction to the conductivity in two dimensions in units of $e^2/\hbar$ for $\gamma_2 = 0$ (dashed line) and $\gamma_2 = 5$ (solid line). Note the different scales used for the two values of $\gamma_2$.}
\end{figure}
FIG. 3. The magnetic field dependence of the conductivity in two dimensions in units of \( e^2/h \) for two different values of \( T_E \) and for \( \gamma_2 = 2.5 \). The inset shows the difference \( \Delta_{TE}\sigma_2(\Omega_s) = \delta\sigma_2(5T,\Omega_s) - \delta\sigma_2(0.1T,\Omega_s) \)of the two curves.

\[
\begin{array}{c|c}
\| f_2^1(\gamma_2) \| & \frac{2}{\pi} - 3(1+\gamma_2)^\frac{3}{2} - 4 - 2d\gamma_2 \\
\| f_2^3(\gamma_2) \| & \frac{4}{d(d+2)} \left( 1 + 3 \frac{24 + (16 - 4d)\gamma_2}{(d+2)(d-2)\gamma_2^2} - 24 - (2 + d)\gamma_2(8 + d\gamma_2) \right) \\
\| f_3^1(\gamma_2) \| & 1 + 3 \left( 1 - \frac{1}{\gamma_2^2} \ln(1 + \gamma_2) \right) \\
\| f_3^2(\gamma_2) \| & 1 + \frac{3}{2} \left( \frac{6 + 5\gamma_2}{\gamma_2^2} - \frac{(6 + 2\gamma_2)(1 + \gamma_2)}{\gamma_2^2} \ln(1 + \gamma_2) \right) \\
\| f_3^3(\gamma_2) \| & \frac{2}{d} \left( 1 + \gamma_2 \right)^\frac{3}{2} - \left( 1 + \gamma_2 \right)^2 (2 + (d-4)\gamma_2) \\
\| g_2^1(\gamma_2) \| & \frac{4}{(d-6)(d-4)\gamma_2} \left( 24 + 4(8 - d)\gamma_2(1 + \gamma_2) \frac{3}{2} - \left( 24 + (d + 2)\gamma_2(8 + (d - d)\gamma_2) \right) (1 + \gamma_2)^2 \right) \\
\| g_2^2(\gamma_2) \| & \frac{1}{(d-8)(d-6)(d-4)(d-2)\gamma_2} \\
\| g_2^3(\gamma_2) \| & \frac{1 + \gamma_2}{\gamma_2^2} \left( 2\gamma_2 + 6\gamma_2^2 - \frac{11}{2} \gamma_2 \ln(1 + \gamma_2) \right) \\
\end{array}
\]

TABLE I. Table of coefficients which appear in the expressions for the current. For small \( \gamma_2 \) these reduce to \( f_2^1(\gamma_2) = 2/d - 3\gamma_2/2, f_2^3(\gamma_2) = 4/(d(2 + d)) - \gamma_2/4, g_2^1(\gamma_2) = \gamma_2/2 \) and \( g_2^3(\gamma_2) = -\gamma_2/12 \).