Relativistic Kramers–Pasternack recurrence relations

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Abstract
Recently we have evaluated the matrix elements \( \langle O_{r} \rangle \), where \( O = \{1, \beta, i\alpha \} \) are the standard Dirac matrix operators and the angular brackets denote the quantum-mechanical average for the relativistic Coulomb problem in terms of generalized hypergeometric functions \( 3F_2(1) \) for all suitable powers and established two sets of Pasternack-type matrix identities for these integrals. The corresponding Kramers–Pasternack-type three-term vector recurrence relations are derived.

1. Introduction

Recent experimental and theoretical progress has renewed interest in quantum electrodynamics of atomic hydrogenlike systems (see, for example, [22, 23, 27, 28, 36, 53, 55] and references therein). In the last decade, the two-time Green’s function method of deriving formal expressions for the energy shift of a bound-state level of high-Z few-electron systems was developed [53] and numerical calculations of QED effects in heavy ions were performed with excellent agreement to current experimental data [22, 23, 55]. These advances motivate, among other technical things, investigation of the expectation values of the Dirac matrix operators between the bound-state relativistic Coulomb wavefunctions. Special cases appear in calculations of the magnetic dipole hyperfine splitting, the electric quadrupole hyperfine splitting, the anomalous Zeeman effect and the relativistic recoil corrections in hydrogenlike ions (see, for example, [1, 52, 54, 57] and references therein).

In the previous paper [57], the matrix elements \( \langle O_{r} \rangle \), where \( O = \{1, \beta, i\alpha \} \) are the standard Dirac matrix operators and the angular brackets denote the quantum-mechanical average for the relativistic Coulomb problem, have been evaluated as sums of three special generalized hypergeometric functions \( 3F_2(1) \) (or Chebyshev polynomials of a discrete variable) for all suitable powers. As a result, two sets of Pasternack-type matrix symmetry relations for these integrals, when \( p \rightarrow -p - 1 \) and \( p \rightarrow -p - 3 \), have been derived. We concentrate on what are essentially radial integrals since, for problems involving spherical symmetry, one can reduce all expectation values to radial integrals by use of the properties of angular momentum.

Our next goal is to establish relativistic analogues of the Kramers–Pasternack recurrence relation [31, 44, 45] (some progress in this direction had been made in [2]). Here several three-term vector recurrence relations are obtained, which follow in a natural way from the well-known theory of classical orthogonal polynomials of a discrete variable [39] and basic matrix algebra. This paper is organized as follows. We review the nonrelativistic case in the next section and then present relativistic extensions, that seem to be new, in sections 3 and 4. The last section contains a new interpretation of the known two-term recurrence relations for these relativistic expectation values [50, 51]. The new recurrence relations are summarized in an appendix in matrix form for the benefit of the reader.

The author believes that these mathematical results are not only natural and elegant but also will be useful in the current theory of hydrogenlike heavy ions and other exotic relativistic Coulomb systems. At the same time, the mathematical structure behind these expectation values remains not fully understood [2]. For example, to the best of my knowledge, numerous recurrence relations for the radial integrals, obtained with the help of a hypervirial theorem (see [1, 50, 51, 54] and references therein), do not follow in an obvious way from the advanced theory of generalized hypergeometric functions [6]. After more than 80 years of thorough investigation, the relativistic Coulomb problem keeps generating some mathematical challenges.
2. Expectation values $\langle r^p \rangle$ for the nonrelativistic Coulomb problem

Evaluation of the matrix elements $\langle r^p \rangle$ between nonrelativistic bound-state hydrogenlike wavefunctions has a long history in quantum mechanics. An incomplete list of references includes [2, 3, 7–13, 15, 18, 19, 25, 31–35, 37, 42, 44, 45, 47, 48, 51, 56, 58, 59, 63, 65]. Different methods were used in order to derive these matrix elements. For example, in [58] (see also [44, 45]) the mean values for states of definite energy have been derived in terms of the Chebyshev polynomials of a discrete variable $t_n(x, N) = h_{0, 0}^{N+1}(x, N)$ originally introduced in [60, 61]. The so-called Hahn polynomials, introduced also by P L Chebyshev [62] and given by

\[ h_{m}^{(\alpha, \beta)}(x, N) = \frac{(1 - N)_{m}(\beta + 1)n}{m!} \times F_2 \left( \begin{array}{c} -m, \alpha + \beta + m + 1, -x \\ \beta + 1, 1 - N \end{array} ; 1 \right), \tag{2.1} \]

were rediscovered and generalized in the late 1940s by W Hahn (we use the standard definition of the generalized hypergeometric series throughout the paper [6, 21]).

The final results have the following closed forms:

\[ \langle k^k \rangle = \frac{1}{2n} \left( \frac{n a_0}{Z} \right)^{k-1} t_k(n-l-1, -2l-1), \tag{2.2} \]

when $k = 0, 1, 2, \ldots$ and

\[ \left( \frac{1}{r^{k+1}} \right) = \frac{1}{2n} \left( \frac{2Z}{n a_0} \right)^{k+1} t_k(n-l-1, -2l-1), \tag{2.3} \]

when $k = 0, 1, 2, \ldots, 2l$. Here $a_0 = \hbar^2/mc^2$ is the Bohr radius (more details can be found in [14, 58]).

The ease of handling of these matrix elements for the discrete levels is greatly increased if use is made of the known properties of these classical polynomials of Chebyshev [21, 26, 29, 39, 40, 60–62]. The direct consequences of (2.2) and (2.3) are an inversion relation

\[ \left( \frac{1}{r^{k+1}} \right) = \frac{2Z}{n a_0} \left( \frac{2Z}{n a_0} \right)^{k+1} \left( \frac{2l-k}{2l+k+1} \right) \langle r^{k-1} \rangle \tag{2.4} \]

with $0 \leq k \leq 2l$ and the three-term recurrence relation

\[ \langle r^k \rangle = \frac{2n(2k+1)}{k+1} \left( \frac{n a_0}{2Z} \right)^{k-1} \langle r^{k-1} \rangle - k \left( \frac{2l+1}{2l+1} + k^2 \right) \frac{n a_0}{2Z} \left( \frac{n a_0}{2Z} \right)^{k-1} \langle r^{k-2} \rangle \tag{2.5} \]

with the initial conditions

\[ \left( \frac{1}{r} \right) = \frac{Z}{a_0 \hbar^2}, \quad \langle 1 \rangle = 1, \tag{2.6} \]

which is convenient for evaluation of the mean values $\langle r^k \rangle$ for $k \geq 1$.

In our approach, the recurrence relation (2.5) is a special case of the three-term recurrence relation for the Hahn polynomials (3.8). It was originally found by Kramers and Pasternack in the late 1930s [31, 44, 45]. The inversion symmetry (2.4), which is also due to Pasternack, was rediscovered many years later [12, 37] (see also [24, 25] for historical comments). Generalizations of (2.4) and (2.5) for off-diagonal matrix elements are discussed in [11, 17, 20, 25, 38, 41, 42, 46, 49–51, 56, 59]. The properties of the hydrogenlike radial matrix elements are considered from a group-theoretical viewpoint in [3, 13, 42]. Extensions to the relativistic case are given in [1, 2, 16, 19, 43, 57, 58] (see also references therein and the following sections of this paper).

In a retrospect, Pasternack’s papers [44, 45] had paved the way to the discovery of the continuous Hahn polynomials in the mid-1980s (see [4, 5, 30, 58] and references therein).

3. Three-term recurrence relations for relativistic matrix elements

In this paper we establish relativistic analogues of the Kramers–Pasternack recurrence relations (2.5) for the following set of integrals of the relativistic radial functions:

\[ A_p = \int_0^{\infty} r^{p+2} (F^2(r) + G^2(r)) \, dr, \tag{3.1} \]

\[ B_p = \int_0^{\infty} r^{p+2} (F^2(r) - G^2(r)) \, dr, \tag{3.2} \]

\[ C_p = \int_0^{\infty} r^{p+2} F(r) G(r) \, dr. \tag{3.3} \]

With the notations from [57, 58], one can evaluate these integrals in terms of the Chebyshev and Hahn polynomials of a discrete variable and present the answer, say, when $p \geq 0$, in the following closed form:

\[ 4\mu \nu^2 (2a \beta)^p A_p = a \kappa (\kappa \nu + \nu) h_{p+1}^{(0, 0)}(n-1, -1-2v) \]

\[ -2 \left( \frac{p+2}{p+1} \right) \mu (\nu^2 - v^2) h_{p+1}^{(0, 1)}(n-1, -1-2v) \]

\[ + a \kappa (\kappa - \nu) h_{p+1}^{(0, 0)}(n, 1-2v), \tag{3.4} \]

\[ 4\mu \nu^2 (2a \beta)^p B_p = a (\kappa + \nu) h_{p+1}^{(0, 0)}(n-1, -1-2v) \]

\[ - a (\kappa - \nu) h_{p+1}^{(0, 0)}(n, 1-2v), \tag{3.5} \]

\[ 8\mu \nu^2 (2a \beta)^p C_p = a \mu (\kappa + \nu) h_{p+1}^{(0, 0)}(n-1, -1-2v) \]

\[ - 2 \left( \frac{p+2}{p+1} \right) \kappa (\nu^2 - v^2) h_{p+1}^{(0, 1)}(n-1, -1-2v) \]

\[ + a \mu (\kappa - \nu) h_{p+1}^{(0, 0)}(n, 1-2v). \tag{3.6} \]

Here,

\[ \kappa = \pm (j + 1/2), \quad \nu = \sqrt{\kappa^2 - \mu^2}, \quad \mu = a Z = Z e^2 / \hbar c, \quad a = \sqrt{1 - \alpha^2}, \quad \alpha = E / mc^2, \quad \beta = mc / \hbar \]

with the total angular momentum $j = 1/2, 3/2, 5/2, \ldots$ (see [57, 58] for more details).

Although a set of useful recurrence relations between the relativistic matrix elements was derived by Shahbaev [50, 51] (see also [1, 19, 54, 57, 64], and the last section of this paper) on the basis of a hypervirial theorem, the corresponding relativistic Kramers–Pasternack-type relations
seem to be missing in the available literature. Our equations (3.4)–(3.6) reveal that they should have a vector form.

Here, one can apply a familiar three-term recurrence relation for the Hahn polynomials [39, 40]:

\[ x_h^{(\alpha, \beta)}(x, N) = \alpha_h h_{m+1}^{(\alpha, \beta)}(x, N) + \beta_h h_m^{(\alpha, \beta)}(x, N) + \gamma_m h_{m-1}^{(\alpha, \beta)}(x, N) \]  

(3.8)

with

\[ \alpha_m = \frac{(m+1)(\alpha + \beta + m + 1)}{(\alpha + \beta + 2m + 1)(\alpha + \beta + 2m + 2)}, \]

\[ \beta_m = \frac{(\alpha - \beta + 2N - 2)(\beta^2 - \alpha^2)(\alpha + \beta + 2N)}{4(4\alpha + \beta + 2m)(\alpha + \beta + 2m + 2)}, \]

\[ \gamma_m = \frac{(\alpha + m)(\beta + m)(\alpha + \beta + N + m)(N - m)}{(\alpha + \beta + 2m)(\alpha + \beta + 2m + 1)} \]  

(3.9)

(3.10)

(3.11)

three for the corresponding special cases in (3.4)–(3.6). Introducing two sets of vectors

\[ \mathbf{A}_p = (2a\beta)^p \begin{pmatrix} a_p \\ b_p \\ c_p \end{pmatrix}, \quad \mathbf{X}_p = \begin{pmatrix} h_{p+1}^{(0,0)}(n - 1, -1 - 2v) \\ \frac{p+1}{2} h_{p+1}^{(1,1)}(n - 1, -1 - 2v) \\ h_{p+1}^{(0,0)}(n, 1 + 2v) \end{pmatrix}, \]  

(3.12)

we conclude that the recurrence relations in question should have the following matrix structure:

\[ \mathbf{A}_p = T \mathbf{X}_p, \quad \mathbf{X}_p = T^{-1} \mathbf{A}_p \]  

(3.13)

with

\[ \mathbf{X}_{p+1} = U_p \mathbf{X}_p + V_p \mathbf{X}_{p-1}, \]  

(3.14)

where \( U_p \) and \( W_p \) are known diagonal matrices. Then

\[ \mathbf{A}_{p+1} = D_p \mathbf{A}_p + E_p \mathbf{A}_{p-1}, \]  

(3.15)

where

\[ D_p = T U_p T^{-1}, \quad E_p = T V_p T^{-1} \]  

(3.16)

are similar matrices.

The elementary linear algebra argument outlined above allows us to obtain a relativistic generalization of the Kramers–Pasternack recurrence relation (2.5) as follows:

\[ (2a\beta)^2 A_{p+1} = 4\beta^2 \mu \frac{2p + 3}{p + 2} A_p \]

\[ - \frac{p(p+2)(4v^2 - (p+1)^2)}{(p+1)(p+2)} A_{p-1} \]

\[ - 4\kappa \frac{p+1}{p+2} B_{p-1} + 8 \mu \frac{p+1}{p+2} C_{p-1}. \]  

(3.17)

\[ (2a\beta)^2 B_{p+1} = 4\beta^2 \mu \frac{2p + 3}{p + 2} B_p \]

\[ - (4v^2 - p(p+2)) \frac{p+1}{p+2} B_{p-1} \]

\[ - 4\kappa \frac{p+1}{p+2} A_{p-1} + 8 \mu \frac{p+1}{p+2} C_{p-1}. \]  

(3.18)

These recurrence relations are summarized for the benefit of the reader in matrix form in the appendix. One should
take special values (A.4)–(A.5) as the initial data. A direct derivation of the relativistic three-term vector recurrence relations (3.17)–(3.19) and (3.23)–(3.25) on the basis of a hypervirial theorem needs to be found.

4. Independent integrals

The integrals (3.1)–(3.3) are linearly dependent:

\[(2\kappa + \varepsilon(p + 1))A_p - (2\kappa\kappa + p + 1)B_p = 4\mu C_p \tag{4.1}\]

(see, for example, [1, 50, 51, 57] for more details). Thus, eliminating \(C_p\), say from (3.23)–(3.24), we obtain the following three-term vector recurrence relation between the integrals \(A_p\) and \(B_p\) only:

\[(2\alpha \beta)^2 A_{p+1} = (2p + 1) \beta
\times \frac{4\varepsilon \mu_2 (a^2(p + 1)^2 - 1) - \alpha^2 p(p + 2)(2\kappa \varepsilon(p + 1))}{a^2 \mu_2 p(p + 1)(p + 2)} A_p
\]

\[+ (2p + 1) \beta \frac{4\varepsilon \mu_2 + a^2 p(p + 2)(2\kappa \kappa + p + 1)}{a^2 \mu_2 p(p + 1)(p + 2)} B_p
\]

\[= \frac{(a^2(p + 1)^2 - 1)(4\varepsilon^2 - p^2)}{a^2(p + 1)(p + 2)} B_{p-1} \tag{4.2} \]

and

\[(2\alpha \beta)^2 B_{p+1} = -(2p + 1)
\times \frac{4\varepsilon \mu_2 + a^2 p(p + 2)(2\kappa \varepsilon(p + 1))}{a^2 \mu_2 p(p + 1)(p + 2)} A_p
\]

\[+ (2p + 1) \beta \frac{4\varepsilon \mu_2 (a^2(p + 1)^2 + \varepsilon^2) + a^2 p(p + 2)(2\kappa \kappa + p + 1)}{a^2 \mu_2 p(p + 1)(p + 2)} B_p
\]

\[= \frac{\varepsilon(4\varepsilon^2 - p^2)}{a^2(p + 1)(p + 2)} A_{p-1}
\]

\[= \frac{(4\varepsilon^2 - p^2)(a^2(p + 1)^2 + \varepsilon^2)}{a^2(p + 1)(p + 2)} B_{p-1}. \tag{4.3} \]

In a similar fashion, from (3.17)–(3.18):

\[(2a \beta)^2 A_{p+1} = 4\beta \varepsilon \mu \frac{2p^2 + 3}{p + 2} \frac{A_p}{A_{p-1}}
\]

\[+ \frac{2\kappa \kappa - (p + 2)(4\varepsilon^2 - (p + 1)^2)}{(p + 1)(p + 2)} A_{p-1}
\]

\[= 2\kappa \varepsilon \mu_2 - 1 (p + 1)(2p + 3) \frac{B_{p-1}}{(p + 1)(p + 2)} \tag{4.4} \]

and

\[(2a \beta)^2 B_{p+1} = 4\beta \varepsilon \mu \frac{2p^2 + 3}{p + 2} \frac{B_p}{B_{p-1}}
\]

\[+ \frac{2\kappa \epsilon + (p + 1)(2p + 3)}{p + 1}(p + 2) \frac{A_{p-1}}{(p + 1)(p + 2)} \tag{4.5} \]

and

\[= (4\kappa \kappa + \varepsilon^2) \frac{p^2 + 1}{p + 2} B_{p-1}. \]

Matrix forms of these identities and the corresponding initial values are given in the appendix.

5. Two-term recurrence relations

The three-term recurrence relations for the relativistic radial integrals, examples of which have been found in the previous sections, are obviously not unique. Moreover, if

\[A_{p+1} = D_p^{(1)} A_p + E^{(p)} A_{p-1}, \tag{5.1} \]

then

\[(\alpha_p + \beta_p)A_{p+1} = (\alpha_p D^{(1)} + \beta_p B^{(2)}) A_p
\]

\[+ (\alpha_p E^{(1)} + \beta_p E^{(2)}) A_{p-1} \tag{5.2} \]

for two arbitrary sequences of scalars \(\alpha_p\) and \(\beta_p\). One special case is of a particular interest.

Subtracting (4.2) and (4.3) from (4.4) and (4.5) one gets the following matrix equation:

\[P_p \begin{pmatrix} A_p \\ B_p \end{pmatrix} = Q_p \begin{pmatrix} A_{p-1} \\ B_{p-1} \end{pmatrix} \tag{5.3} \]

with

\[\det P_p = -8\beta^2 \varepsilon \kappa a^2(p + 2)(p + 1) + 2\varepsilon \mu^2 \tag{5.4} \]

\[\det Q_p = -2\varepsilon(4\varepsilon^2 - p^2) \kappa a^2(p + 2)(p + 1) + 2\varepsilon \mu^2 \tag{5.5} \]

by a computer algebra system. We have omitted the explicit forms of the \(P\) and \(Q\) matrices and their inverses. This should be easily done by the reader.

The two-term recurrence solutions of the form

\[\begin{pmatrix} A_p \\ B_p \end{pmatrix} = S_p \begin{pmatrix} A_{p-1} \\ B_{p-1} \end{pmatrix} \]

were found by Shabaev [50, 51] by a different method. In our notations

\[A_{p+1} = -(p + 1) \frac{4\varepsilon^2 \kappa + 2\kappa (p + 2) + \varepsilon(p + 1)(2\kappa \kappa + p + 2)}{4(1 - \varepsilon^2)(p + 2) \beta \mu} A_p + \frac{4\mu^2(p + 2) + (p + 1)(2\kappa \kappa + p + 1)(2\kappa \kappa + p + 2)}{4(1 - \varepsilon^2)(p + 2) \beta \mu} \tag{5.6} \]

and

\[B_{p+1} = -(p + 1) \frac{4\varepsilon^2 \kappa + 2\kappa (p + 3) + \varepsilon^2(p + 1)(p + 2)}{4(1 - \varepsilon^2)(p + 2) \beta \mu} A_p + \frac{4\mu^2(p + 2) + (p + 1)(2\kappa \kappa + p + 1)(2\kappa \kappa + p + 2)}{4(1 - \varepsilon^2)(p + 2) \beta \mu} \tag{5.7} \]

and

\[A_{p+1} = \beta \frac{4\varepsilon^2 \kappa(p + 1) + p(2\kappa \kappa + p)(2\kappa \kappa + \varepsilon(p + 1))}{\mu(4\varepsilon - p^2)p} A_p
\]

\[+ \frac{4\mu^2(p + 1) + p(2\kappa \kappa + p)(2\kappa \kappa + p + 1)}{\mu(4\varepsilon - p^2)p} B_p. \tag{5.8} \]
\[ B_{p-1} = \beta \frac{4v^2 + 2\varepsilon(2p + 1) + s^2 p(p + 1)}{\mu(4v^2 - p^2)} A_p \]
\[ - \beta \frac{4v^2\varepsilon + 2\varepsilon(p + 1) + \varepsilon p(2\varepsilon + p + 1)}{\mu(4v^2 - p^2)} B_p, \]  
\[ \text{(5.10)} \]

respectively.

We have shown that these solutions can be derived from the three-term vector recurrence relations found in this paper. As a by-product, the factorization of Shabaev’s matrices, namely
\[ S_p = P_p^{-1} Q_p, \quad S_p^{-1} = Q_p^{-1} P_p, \]  
\[ \text{(5.11)} \]
is given. This can be directly verified with the help of a computer algebra system. Then
\[ \det S_p = \frac{\det Q_p}{\det P_p} = \frac{(4\nu^2 - p^2)p}{4(a\beta^2)(p + 1)}. \]  
\[ \text{(5.12)} \]

Further details are left to the reader.

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**Appendix. Matrix form of the three-term recurrence relations**

The matrix structures of our recurrence relations (3.17)–(3.19) and (3.23)–(3.25) are given by

\[ \begin{pmatrix} A_{p+1} \\ B_{p+1} \\ C_{p+1} \end{pmatrix} = (2a\beta^2) \begin{pmatrix} A_p \\ B_p \\ C_p \end{pmatrix} - \frac{4\varepsilon\mu}{\mu(a^2(p + 1)^2 - 1)} \begin{pmatrix} 4\varepsilon\mu(a^2(p + 1)^2 - 1) \\ 4\varepsilon\mu \end{pmatrix} \]
\[ - \frac{4\varepsilon\mu}{a^2(p + 2)} \begin{pmatrix} 4\varepsilon\mu(a^2(p + 1)^2 - 1) \\ 4\varepsilon\mu \end{pmatrix} \begin{pmatrix} 4\varepsilon\mu(a^2(p + 1)^2 - 1) \\ 4\varepsilon\mu \end{pmatrix} \]
\[ \text{(A.1)} \]

and

\[ \begin{pmatrix} A_{p+1} \\ B_{p+1} \\ C_{p+1} \end{pmatrix} = \beta \begin{pmatrix} 2p + 1 \\ p + 1 \end{pmatrix} \begin{pmatrix} a^2 p(p + 2) \\ a^2(p + 2)^2 \end{pmatrix} \begin{pmatrix} -4 \varepsilon \mu \\ -4 \varepsilon \mu \end{pmatrix} \begin{pmatrix} A_p \\ B_p \\ C_p \end{pmatrix} \]
\[ \text{(A.2)} \]

respectively. The initial vectors are

\[ \begin{pmatrix} A_{-1} \\ B_{-1} \\ C_{-1} \end{pmatrix} = \frac{\beta}{\mu \nu} (1 - \varepsilon^2)(\varepsilon \nu + \sqrt{1 - \varepsilon^2}) \begin{pmatrix} \beta a^2 \\ \mu \end{pmatrix} \]
\[ \frac{\kappa}{2\mu \nu} a^2 \beta \]
\[ \text{(A.3)} \]

\[ \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \mu \end{pmatrix} \]
\[ \frac{\kappa}{2\mu} (1 - \varepsilon^2) \]
\[ \text{(A.4)} \]
(2aβ)^2 \begin{pmatrix} A_{p+1} \\ B_{p+1} \end{pmatrix} = \begin{pmatrix} \beta^2p + 1 \\ p + 1 \end{pmatrix} \begin{pmatrix} 2aμ^2(p + 2)(2κ + ε(p + 1)) \\ a^2μp(p + 2) \end{pmatrix} \begin{pmatrix} 4εμ^2 + a^2p(p + 2)(2κ + ε(p + 1)) \\ -εa^2μ^2 + a^2p(p + 2)(2κ + ε(p + 1)) \end{pmatrix} \begin{pmatrix} A_p \\ B_p \end{pmatrix} - \frac{ε(4ε^2 - p^2)}{a^2(p + 1)(p + 2)} \begin{pmatrix} a^2(p + 1)^2 - 1 \\ -εa^2(p + 1)^2 + ε^2 \end{pmatrix} \begin{pmatrix} A_{p-1} \\ B_{p-1} \end{pmatrix}.

\text{and}

(2aβ)^2 \begin{pmatrix} A_{p+1} \\ B_{p+1} \end{pmatrix} = 4βεμ \begin{pmatrix} 2p + 3 \\ p + 2 \end{pmatrix} \begin{pmatrix} A_p \\ B_p \end{pmatrix} - \begin{pmatrix} 2κ(p + 2)(4ε^2 - p^2) \\ (p + 1)(p + 2) \end{pmatrix} \begin{pmatrix} 2κε - 1 + (p + 1)(2p + 3) \\ -2κε + (p + 1)(p + 2) \end{pmatrix} \begin{pmatrix} A_{p-1} \\ B_{p-1} \end{pmatrix}.

One should use the initial data (A.3) and (A.4) in the case of the recurrence relation (A.1) and (A.4)–(A.5) for (A.2).

Our relations (4.2)–(4.3) and (4.4)–(4.5) take the following matrix forms:

\begin{align*}
\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} &= \left( \begin{array}{c}
\frac{3εμ^2 - κ(1 - ε^2)(1 + εκ)}{2βμ(1 - ε^2)} \\
\frac{3ε^2μ^2 - (1 - ε^2)(εκ + ν^2)}{2βμ(1 - ε^2)} \\
\frac{2βμ(1 - ε^2)}{2εκ - 1} \\
\end{array} \right). \\
\text{(A.5)}
\end{align*}

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