Abstract. A survey is given of some Chernoff type bounds for the tail probabilities $P(X - EX \geq a)$ and $P(X - EX \leq a)$ when $X$ is a random variable that can be written as a sum of indicator variables that are either independent or negatively related. Most bounds are previously known and some comparisons are made.

1. Introduction and conclusions

The purpose of this paper is to give a survey of some simple upper bounds for the probabilities $P(X - EX \leq -a)$ and $P(X - EX \geq a)$, where $X$ is a random variable that can be written as a sum $I_1 + \cdots + I_n$ of 0–1 (indicator) random variables. We consider both independent and dependent variables $I_i$ (with strong restrictions in the dependent case). Many of the inequalities extend to sums of more general bounded variables, but we consider for simplicity only the indicator case.

Most of the bounds are known, see in particular Bennett (1962) and Hoeffding (1963), but are included for comparison and (partial) completeness. A few versions seem to be new. Many of the inequalities appear in various places, for example Janson, Łuczak and Ruciński (2000), Chapter 2. See also the book Boucheron, Lugosi and Massart (2013) which presents several of these bounds and many extensions to other situations.

Independent identically distributed summands. The simplest case is when the indicator variables $I_i$ are independent and identically distributed, $I_i \sim \text{Be}(p)$, with $0 < p < 1$ (avoiding trivial cases); then $X$ has the binomial distribution $\text{Bi}(n, p)$. This case has been studied by many authors, giving bounds or asymptotic results (sometimes in greater generality); see for example Khintchine (1929), Cramér (1938), Feller (1943), Chernoff (1952), Bahadur and Rao (1960), Bennett (1962), Hoeffding (1963), Littlewood (1969), and the further references given in these papers.

We are here interested in explicit bounds for finite $n$ rather than asymptotic results. One simple but powerful such bound was given by Chernoff (1952); since for every $t \geq 0$,

$$P(X \geq EX + a) \leq e^{-t(EX + a)} E e^{tX} \quad (1.1)$$

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and $Ee^{tX} = (1+p(e^t-1))^n$, we obtain by simple calculus, letting $\lambda = EX$ and assuming $0 \leq a \leq n - \lambda$,

$$P(X \geq EX + a) \leq \inf_{t \geq 0} \exp\left(-at - npt + n \ln(1 + p(e^t - 1))\right)$$

$$= \exp\left(-(\lambda + a) \ln \frac{\lambda + a}{\lambda} - (n - \lambda - a) \ln \frac{n - \lambda - a}{n - \lambda}\right). \quad (1.2)$$

Similarly, for $0 \leq a \leq \lambda$,

$$P(X \leq EX - a) \leq \inf_{t \geq 0} \exp\left(-at + npt + n \ln(1 + p(e^{-t} - 1))\right)$$

$$= \exp\left(-(\lambda - a) \ln \frac{\lambda - a}{\lambda} - (n - \lambda + a) \ln \frac{n - \lambda + a}{n - \lambda}\right). \quad (1.3)$$

Chernoff (1952) proved also that the estimates (1.2) and (1.3) are asymptotically sharp in the sense that if $n \to \infty$ with $a/n$ and $p$ fixed, then equality holds within factors $1 + o(1)$ in the exponent.

Simpler but (slightly) cruder bounds are easily obtained by finding suitable upper bounds for the right hand sides of (1.2) and (1.3), see for example Alon and Spencer (1992), Appendix A. We state here some, more or less well-known, such bounds. Proofs are given in Section 2.

**Theorem 1.** Suppose that $X \sim Bi(n,p)$ and let $q = 1-p$. Then, for every $a \geq 0$,

$$P(X \geq EX + a) \leq \exp\left(-\frac{a^2}{2(npq + a(q-p)/3)}\right) \leq \exp\left(-\frac{a^2}{2(npq + a/3)}\right) \leq \exp\left(-\frac{a^2}{2npq (1 - a/3npq)}\right), \quad (1.4)$$

$$P(X \geq EX + a) \leq \exp\left(-np\left(1 + \frac{a}{np}\right) \ln\left(1 + \frac{a}{np}\right) - \frac{a}{np}\right), \quad (1.5)$$

$$P(X \geq EX + a) \leq \exp\left(-\frac{a^2}{2np(1 + a/3np)}\right) \leq \exp\left(-\frac{a^2}{2np} \left(1 - \frac{a}{3np}\right)\right), \quad (1.6)$$

$$P(X \leq EX - a) \leq \exp\left(-\frac{a^2}{2(npq - a(q-p)/3)}\right) \leq \exp\left(-\frac{a^2}{2(npq + a/3)}\right) \leq \exp\left(-\frac{a^2}{2npq (1 - a/3npq)}\right), \quad (1.7)$$

$$P(X \leq EX - a) \leq \exp\left(-np\left(1 + \frac{a}{np}\right) \ln\left(1 + \frac{a}{np}\right) + \frac{a}{np}\right), \quad (1.8)$$

$$P(X \leq EX - a) \leq \exp\left(-\frac{a^2}{2np}\right). \quad (1.9)$$

Moreover, if $0 \leq p \leq 1/2$, then

$$P(X \leq EX - a) \leq \exp\left(-\frac{a^2}{2npq}\right). \quad (1.10)$$

**Remark 1.** The estimates (1.4) and (1.7) (as well as (1.2) and (1.3)) are obvious “mirror images” of each other, and are equivalent by the substitution $X \to n - X$. On the other hand, the remaining estimates in Theorem 1 are asymmetric, and are useful mainly when $p$ is small.
Remark 2. Note that (1.5), (1.6), (1.8) and (1.9) use \( n \) and \( p \) only in the combination \( np = \lambda = E X \).

Remark 3. It may also be observed that if we replace \( np \) by \( \lambda \), then (1.5), (1.6), (1.8) and (1.9) are (e.g. by continuity) valid also when \( X \sim Po(\lambda) \); in fact, (1.5) and (1.8) then become the Chernoff bounds for the Poisson distribution. These Chernoff bounds too are asymptotically sharp in the sense, considering for example (1.5), that if \( c > 0 \) is fixed and \( c_1 > (1 + c) \ln (1 + c) - c \), then \( P(X \geq E X + c\lambda) > \exp(-c_1\lambda) \) for \( X \sim Po(\lambda) \) with \( \lambda \) large, and thus also for some \( X \sim Bi(n, p) \) with \( np \) large and \( p \) small. In particular this implies that the simple bound (1.9) is not valid for \( P(X \geq E X + a) \). This implies further, by considering \( n - X \), that (1.10) cannot hold without some restriction on \( p \).

Independent summands with different distributions. The Chernoff bounds given above for the binomial distribution are easily extended to the case when the 0–1 variables \( I_i \) are independent with different distributions, \( I_i \sim Be(p_i) \). In fact, as is well-known (see for example Alon and Spencer (1992), Appendix A), if \( \lambda = E X = \sum p_i \), \( p = \lambda/n \) (the average of \( p_1, \ldots, p_n \)), and we let \( X_0 \sim Bi(n, p) \) be a binomially distributed random variable with the same \( n \) and expectation as \( X \), then by Jensen’s inequality for the convex function \( x \mapsto -\ln(1 + x(e^t - 1)) \),

\[
E e^{tX} = \prod_{i=1}^{n} (1 + p_i(e^t - 1)) \leq (1 + p(e^t - 1))^n = E e^{tX_0}, \quad -\infty < t < \infty. \tag{1.11}
\]

Consequently,

\[
P(X \geq E X + a) \leq e^{-ta - tEX} E e^{tX} \leq e^{-ta - tEX_0} E e^{tX_0}, \tag{1.12}
\]

and thus every Chernoff type bound for the binomial variable \( X_0 \) derived from (1.1), applies also to \( X \).

Theorem 2. The bounds (1.2)–(1.10) hold also when \( X = \sum I_i \) where \( I_i \sim Be(p_i) \) are independent indicator variables and \( p = E X/n \), \( q = 1 - p \). \( \square \)

Remark 4. We do not claim that the actual tail probability \( P(X_0 \geq E X + a) \) is larger than \( P(X \geq E X + a) \), and indeed this is in general false as is shown by the example \( n = 2, p_1 = 1/5, p_2 = 3/5 \), where \( P(X \geq 1) = 17/25 \) while \( X_0 \sim Bi(2, 2/5) \) and \( P(X_0 \geq 1) = 16/25 \).

As mentioned above, the bounds (1.2) and (1.3) are asymptotically sharp for the binomial distribution, but that is no longer generally true when the 0–1 variables have different distributions. In fact, a Taylor expansion shows that the exponent in (1.2) or (1.3) is \( -\frac{a^2}{2\sigma^2}(1 + o(1)) \) provided \( a = o(np(1 - p)) \), cf. (1.4) and (1.7). In the binomial case, this equals \( -\frac{a^2}{2\sigma^2}(1 + o(1)) \), with \( \sigma^2 = Var X \), which is what one would expect from normal approximation heuristics; in general, however, \( \sigma^2 = Var X \) may be much smaller than \( np(1 - p) \), and it would be advantageous to have better bounds with exponents \( -\frac{a^2}{2\sigma^2}(1 + o(1)) \) for moderately large \( a \). This is achieved by Bennett’s inequality, see Bennett (1962) and Hoeffding (1963), which we state as (1.13) in the next theorem; the simple consequence (1.14) is known as Bernstein’s inequality, see Boucheron, Lugosi and Massart (2013). Note that these inequalities give bounds depending on \( a \) and \( \sigma^2 \) only, with an exponent of the expected order for \( a = o(\sigma^2) \). We give a proof in Section 3 using (1.1) as above, but doing a more careful estimation of \( E e^{tX} \) than (1.11).
Theorem 3. Let $X$ be a random variable and suppose that there exist independent 0–1 variables $I_i \sim \text{Be}(p_i)$, $i = 1,\ldots,n$, such that $X \overset{d}{=} \sum_1^n I_i$. Let $\lambda = \text{E}X$ and $\sigma^2 = \text{Var}X$. Then

(i) If $a \geq 0$, then

$$
P(X \geq \text{E}X + a) \leq \exp\left(-\sigma^2\left((1 + \frac{a}{\sigma^2}) \ln\left(1 + \frac{a}{\sigma^2}\right) - \frac{a}{\sigma^2}\right)\right)
$$

(ii) If $a \geq 0$, then

$$
P(X \geq \text{E}X + a) \leq \exp\left(-\frac{a^2}{2\sigma^2}\left(1 + \frac{a}{3\sigma^2}\right)\right) \leq \exp\left(-\frac{a^2}{2\sigma^2}\left(1 - \frac{a}{3\sigma^2}\right)\right).
$$

(iii) If $a \geq c\sigma^2$, with $c > 0$, then

$$
P(X \geq \text{E}X + a) \leq \exp(-(1 + c^{-1}) \ln(1 + c) - 1)a).
$$

The same estimates hold for $P(X \leq \text{E}X - a)$, and thus $P(|X - \text{E}X| \geq a)$ may be estimated by twice the right hand sides in (1.13)–(1.15).

(iv) Moreover, if $a \geq 0$ and $\sigma^2 \geq \lambda/2$, then

$$
P(X \leq \text{E}X - a) \leq \exp\left(-\frac{a^2}{2\sigma^2}\right).
$$

Remark 5. The estimates (1.13) and (1.14) are very similar to (1.5) and (1.6); the only difference is that $np = \lambda$ is replaced by $\sigma^2 < \lambda$. It is easily seen that this always improves the bound in (1.5) and the first bound in (1.6). On the other hand, the bounds for $P(X \leq \text{E}X - a)$ (except (1.16)) are somewhat different from the corresponding bounds in Theorem 1, because of the symmetry of the bounds in Theorem 3.

Remark 6. It is easily seen (by approximating a Poisson distribution) that the constant in (iii) is best possible. In particular, it follows that (1.16) cannot hold without restriction.

Much more precise estimates of the tail probabilities for sums of independent, but not necessarily identically distributed, random variables were obtained by Feller (1943) using different methods (conjugated distributions as in Cramér (1938) together with a Berry–Esseen estimate), and it is interesting to compare our result with Feller’s. Feller’s result (for our case, using $\lambda_n = 1/\sigma$ in Feller (1943)) is, for $0 < a < \sigma^2/12$,

$$
P(X \geq \text{E}X + a) = e^{-\frac{a^2}{2\sigma^2}}Q(x)\left(1 - \Phi(x) + \frac{\theta(x)}{\sigma}e^{-x^2/2}\right),
$$

where $x = a/\sigma$, $|\theta(x)| < 9$, $\Phi$ is the normal distribution function and

$$
Q(x) = \sum_{\nu=1}^{\infty} q_{\nu} x^\nu,
$$

where $q_{\nu}$ depends on the first $\nu + 2$ moments of $X$ and

$$
|q_{\nu}| < \frac{1}{7} \left(\frac{12}{\sigma}\right)^\nu.
$$
If, say, \( \sigma \leq a \leq \sigma^2/24 \), this yields, using \( 1 - \Phi(x) \leq (2\pi)^{-1/2} x^{-1/2} e^{-x^2/2} \) for \( x > 0 \),

\[
\begin{align*}
P(X \geq E X + a) &< \exp \left( -\frac{x^2}{2} (1 + Q(x)) \right) = \exp \left( -\frac{a^2}{2\sigma^2} \left( 1 + Q \left( \frac{a}{\sigma} \right) \right) \right) \\
&< \exp \left( -\frac{a^2}{2\sigma^2} \left( 1 - \frac{24}{7} \frac{a}{\sigma^2} \right) \right). \quad (1.20)
\end{align*}
\]

For \( P(X \leq E X - a) \) we have the same estimates if we replace \( Q(x) \) by \( Q(-x) \) (and \( \theta(x) \) by some \( \theta'(x) \)); this follows by considering \( n - X \).

The bound (1.20) is similar to the ones given in Theorem 3, in particular (1.14). It is somewhat inferior to (1.14) since the constant in the second order term in the exponent is worse, and the range of \( a \) is restricted, but for applications they are essentially equivalent.

Note also that Feller’s result has other advantages. First, (1.17) is an equality (although the exact value of \( \theta(x) \) is unspecified), and it leads also to a lower bound similar to (1.20) and to asymptotic results. In particular, simple asymptotic results follow when \( a/\sigma^2 \to 0 \) and thus \( Q(x) \to 0 \). Secondly, Feller (1943) describes how the coefficients \( q_\nu \) may be explicitly expressed in terms of the semi-invariants \( \kappa_j \) of \( X \) (and thus in terms of the moments); for example (the sign seems to be wrong in Feller (1943), (2.18)–(2.19)),

\[
\begin{align*}
q_1 &= -\frac{\kappa_3}{3\sigma^3}; \quad (1.21) \\
q_2 &= -\frac{\kappa_4}{12\sigma^4} + \frac{1}{4\sigma^6} \kappa_3^2. \quad (1.22)
\end{align*}
\]

(Thus, \( q_1 = -\frac{1}{3} \gamma_1 \) and \( q_2 = -\frac{1}{12} \gamma_2 + \frac{1}{3} \gamma_1^2 \), where \( \gamma_1 \) and \( \gamma_2 \) are the skewness and excess of \( X \), respectively.) For example, using (1.21) for \( q_1 \) and (1.19) for \( q_\nu \), \( \nu \geq 2 \), we obtain for \( \sigma \leq a \leq \sigma^2/24 \), instead of (1.20),

\[
P(X \geq E X + a) \leq \exp \left( -\frac{a^2}{2\sigma^2} \left( 1 - \frac{288}{7} \frac{a^2}{\sigma^4} - \frac{3\kappa_3}{3\sigma^4} \right) \right), \quad (1.23)
\]

which yields an improvement in cases when \( \kappa_3 = E(X-EX)^3 \) is known and either negative or not to large positive.

**Dependent summands.**

Let us now consider the case of dependent 0–1 variables \( I_i \). Of course any bounded non-negative integer valued random variable \( X \) can be written as a sum of dependent 0–1 variables, so nothing can be said in general. We will here consider only 0–1 variables that are negatively related in the following sense, cf. Barbour, Holst and Janson (1992).

(Note that large deviation bounds for a class of sums of positively related indicators are given in Janson (1990) and Barbour, Holst and Janson (1992), Theorem 2.S. In this case only the lower tail probabilities \( P(X \leq E X - a) \) have nice upper bounds.)

**Definition.** The indicator random variables \( (I_i)_{i=1}^n \) (defined on the same probability space) are negatively related if for each \( j \leq n \) there exist further random variables \( (J_{ij})_{i=1}^n \), defined on the same probability space (or an extension of it), such that the distribution of the random vector \( (J_{ij})_{i=1}^n \) equals the conditional distribution of \( (I_i)_{i=1}^n \) given \( I_j = 1 \), and, moreover, for every \( i \) with \( i \neq j \), \( J_{ij} \leq I_i \).

**Example 1.** (Hypergeometric distribution.) Let \( m \), \( n \) and \( N \) be given positive integers with \( \max(m,n) \leq N \). Given \( N \) urns, labelled \( 1, \ldots, N \), and \( m \) balls, put the balls at random into \( m \) different urns (drawing without replacement), and let \( X \) be the total number of balls in urns \( 1, \ldots, n \). Clearly \( X = \sum_1^n I_i \), where \( I_i \) equals 1 if urn \( i \) contains
a ball. In this case it is easy to show that the indicators $I_i$ are negatively related by explicitly construction $J_i$, as follows. After randomly distributing the balls as above, and recording $I_i$, we ensure that there is a ball in urn $j$ by “cheating”: if urn $j$ is empty we select one of the balls at random and move it to urn $j$. Let $J_{ij} = 1$ if urn $i$ now contains a ball. It is clear that $(J_{ij})$ has the right distribution, and that $J_{ij} \leq I_i$ for $i \neq j$.

**Example 2.** Distribute $m$ balls into $n$ urns, but this time put the balls one by one at random, independently of the other choices of urn (drawing with replacement). Let $X$ be the number of empty urns. Clearly $X = \sum^n I_i$, where $I_i = 1$ if urn $i$ is empty. These indicators are negatively related; this follow by a construction very similar to the one in Example 1, removing all balls (if any) in urn $i$ and redistributing them (repeating if necessary).

Further examples of negatively related variables are given in Barbour, Holst and Janson (1992), where also some general results are established. In particular, it is proven (a special case of Corollary 2.D.1) that the variables $I_i$, $i = 1, \ldots, n$, are negatively related if and only if $I_i$ and $\phi(I_1, \ldots, I_{j-1}, I_{j+1}, \ldots, I_n)$ are negatively correlated for every $j$ and every indicator function $\phi$ that is increasing in each variable. (Pairwise negative correlation of the $I_i$ is not enough.) It follows immediately that the variables $(1 - I_i)_{i=1}^n$ are negatively related if $(I_i)_{i=1}^n$ are. It follows also that variables are negatively related if they are negatively associated in the sense of Joag-Dev and Proschan (1983).

**Theorem 4.** Suppose that $X \overset{d}{=} \sum^n I_i$, where $I_i \sim \text{Be}(p_i)$ are negatively related indicator variables. Let $\tilde{I}_i$, $i = 1, \ldots, n$, be independent indicator variables with $\tilde{I}_i \sim \text{Be}(p_i)$, and put $\tilde{X} = \sum^n \tilde{I}_i$. Then, for every real $t$,

$$e^{tX} \leq E e^{t\tilde{X}}.$$

Consequently, any Chernoff type bound for $\tilde{X}$ applies also to $X$. In particular, (1.2)–(1.10) hold with $p = E X/n$ and $q = 1 - p$; for example, with $\lambda = E X$,

$$P(X \geq E X + a) \leq \exp \left( -\frac{a^2}{2\lambda(1 + a/3\lambda)} \right) \leq \exp \left( -\frac{a^2}{2\lambda} \left( 1 - \frac{a}{3\lambda} \right) \right),$$

$$P(X \leq E X - a) \leq \exp \left( -\frac{a^2}{2\lambda} \right).$$

Of course, also the bounds in Theorem 3 (applied to $\tilde{X}$) apply to $X$. The problem is that we have to use $\sigma^2 = \text{Var}(\tilde{X})$ instead of $\text{Var}(X)$, which may be much smaller. In fact, the bounds in Theorem 3 are in general false with $\sigma^2 = \text{Var}X$ in the dependent case; the following theorem implies that it is impossible to have a general bound that is, say, $\exp(-a^2/3\sigma^2)$ when $a = 4\sigma$.

**Theorem 5.** Let $\alpha > 0$, $0 < c < 1/e$ and $A < \infty$. There exists a random variable $X$ which is a finite sum of negatively related indicators such that $\sigma^2 = \text{Var}X > A$ and, with $a = \alpha\sigma$,

$$P(X > E X + a) > ce^{-a/\sigma}.$$  \hspace{1cm} (1.24)

Nevertheless, there are cases where it is possible to do better. A striking example is based on the result by Vatutin and Mikhailov (1982) that certain random variables that occur in some occupancy problems, and have natural representations as sums of negatively related indicators (with the same expectation), also can be represented as sums of
independent indicators with different expectations. (The proof is algebraic, and based on showing that the probability generating function has only real roots; there is no (known) probabilistic interpretation of these indicators, which in general have irrational expectations.) Their result includes the variables in Examples 1 and 2 (using in their notation \(s_1 = N - n, s_2 = N - m\) for Example 1 and \(s_1 = \cdots = s_m = 1\) for Example 2).

Consequently, the variables in Examples 1 and 2 actually satisfy the hypothesis of Theorem 3 (although we do not know the \(p_i\) explicitly). Hence we can apply Theorem 3; note that the bounds in Theorem 3 involve only \(\sigma^2\) and possibly \(\lambda\), and not the unknown \(p_i\). (In fact, this application was one of the motivations for finding bounds of the form given in Theorem 3.)

**Theorem 6.** Let \(X\) be either hypergeometric as in Example 1, or as in Example 2. Then the conclusions of Theorem 3 hold, with \(\lambda = E X\) and \(\sigma^2 = \text{Var} X\).

**Example 2, cont.** For the occupancy problem described above,

\[
\lambda = E X = n \left(1 - \frac{1}{n}\right)^m, \\
\sigma^2 = \text{Var} X = n \left(1 - \frac{1}{n}\right)^m + n(n - 1) \left(1 - \frac{2}{n}\right)^m - n^2 \left(1 - \frac{1}{n}\right)^{2m};
\]

estimates of the tail probabilities are obtained by using these values in any of the formulas \((1.2)-(1.16)\), letting \(p = E X/n\) and \(q = 1 - n\).

For asymptotical results in the case \(m/n \to r > 0\), we easily find

\[
\lambda = E X \sim ne^{-r}, \\
\sigma^2 = \text{Var} X \sim n(e^{-r} - (1 + r)e^{-2r}) = ne^{-2r}(e^r - 1 - r).
\]

The asymptotics for the tail probabilities in this case have been studied in detail by Kambath, Motwani, Palem and Spirakis (1994).

**Remark 7.** A comparison of Theorems 3 and 5 shows that not every random variable that is a sum of negatively related indicators can be represented as a sum of independent indicators; the Vatutin–Mikhailov result depends on some further structure. The first example of such a variable was found by Andrew Barbour (personal communication): Let \(P(X = 3) = 4/13, P(X = 4) = 5/13, P(X = 5) = 4/13\). Then \(X = \sum_{i=1}^5 I_i\), with \(I_i\) indicators and the distribution of \((I_1, \ldots, I_5)\) uniform given \(X\); and these \(I_i\) are easily verified to be negatively related. On the other hand, it is easily seen that \(X\) is not the sum of any number of independent indicators, since the probability generating function has non-real roots.

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## 2. Proof of Theorem 1

The first inequality in \((1.4)\) is trivial for \(a > nq\). For \(0 \leq a \leq nq\), let \(x = a/n \in [0, q]\). Then the bound \((1.2)\) may be written

\[
P(X \geq E X + a) \leq \exp \left(-np \left(1 + \frac{x}{p}\right) \ln \left(1 + \frac{x}{p}\right) - nq \left(1 - \frac{x}{q}\right) \ln \left(1 - \frac{x}{q}\right)\right). \quad (2.1)
\]
Let, for $0 \leq x \leq q$,
\[
f(x) = p \left(1 + \frac{x}{p}\right) \ln \left(1 + \frac{x}{p}\right) + q \left(1 - \frac{x}{q}\right) \ln \left(1 - \frac{x}{q}\right) - \frac{x^2}{2(pq + x(q - p)/3)}.
\]

Then $f(0) = f'(0) = 0$, and an elementary calculation yields
\[
f''(x) = \frac{1}{x + p} + \frac{1}{q - x} - \frac{p^2q^2}{(pq + x(q - p)/3)^3}
\]
\[
= \frac{1}{3} pq(q - p)^2 x^2 + \frac{1}{27} (q - p)^3 x^3 + \frac{p^2q^2x^2}{(x + p)(q - x)(pq + x(q - p)/3)^3} \geq 0
\]
for $0 \leq x \leq q$. Hence $f(x) \geq 0$ in this interval, and thus
\[
P(X \geq E X + a) \leq \exp \left(-n \frac{x^2}{2(pq + x(q - p)/3)} \right).
\]

This proves the first inequality in (1.4). The second follows from $q - p \leq 1$ and the third from
\[
\frac{1}{1 + a/3npq} \geq 1 - \frac{a}{3npq}.
\]

Inequality (1.5) follows directly from (2.1) and
\[
-nq \left(1 - \frac{x}{q}\right) \ln \left(1 - \frac{x}{q}\right) = n(q - x) \ln \left(1 + \frac{x}{q - x}\right) \leq nx.
\]

(2.2)

The inequalities (1.6) follow from (1.4) and
\[
- \frac{1}{npq + a/3} \leq - \frac{1}{np(1 + a/3np)} \leq - \frac{1}{np} \left(1 - \frac{a}{3np}\right);
\]
alternatively, they follow easily from (1.5), cf. (3.10).

The inequalities (1.7) follow from (1.3) by an argument similar to the one given above for (1.4), or (simpler) by applying (1.4) to $n - X \sim \text{Bi}(n, q)$; (1.8) follows from (1.3), using
\[
(2.2) \text{ with } x = -a/n \leq 0; \text{ (1.9) follows from (1.7) and, assuming (as we may) } a \leq np,
\]
\[npq - a(q - p)/3 \leq npq + ap/3 \leq npq + np^2/3 \leq np.\]\nFinally, also (1.10) follows from (1.7) since we now assume $a(q - p)/3 \geq 0$. \hfill $\square$

3. Proof of Theorem 3

We may assume that $X = \sum_1^n I_i$ where $I_i \sim \text{Be}(p_i)$ are independent. Note that
\[
\lambda = E X = \sum_1^n p_i
\]
\[
\sigma^2 = \text{Var} X = \sum_1^n p_i(1 - p_i) = \lambda - \sum_1^n p_i^2
\]
and thus
\[
\sum_1^n p_i^2 = \lambda - \sigma^2.
\]

We assume, to avoid trivialities, that at least one $p_i \neq 0, 1$. Thus $0 < \lambda < n$ and $0 < \sigma^2 < \lambda$.

We begin with a real analysis lemma. It is an analogue of Jensen’s inequality but with a condition on the sign of the third derivative instead of the second.
Lemma 1. Suppose that $\mu$ is a finite positive measure on $[0, 1]$, and define

$$m = \mu([0, 1]),$$

$$x_0 = \int_0^1 x^2 d\mu / \int_0^1 x d\mu,$$

$$\alpha_0 = \left( \int_0^1 x d\mu \right)^2 / \int_0^1 x^2 d\mu,$$

$$x_1 = 1 - \int_0^1 (1 - x)^2 d\mu / \int_0^1 (1 - x)d\mu,$$

$$\alpha_1 = \left( \int_0^1 (1 - x)d\mu \right)^2 / \int_0^1 (1 - x)^2 d\mu.$$

(We here let $0/0 = 0$; this occurs in the degenerate cases where $\mu$ is a point mass at 0 or 1.) If $f$ is a three times continuously differentiable real function on $[0, 1]$ with $f''' \geq 0$, then

$$(m - \alpha_0)f(0) + \alpha_0 f(x_0) \leq \int_0^1 f d\mu \leq (m - \alpha_1)f(1) + \alpha_1 f(x_1). \quad (3.1)$$

If instead $f''' \leq 0$ on $[0, 1]$, then these inequalities are reversed.

Proof. We will show the left inequality of (3.1); the right inequality then follows by symmetry, considering the function $\tilde{f}(x) = -f(1-x)$, which satisfies $\tilde{f}'''(x) = f'''(1-x) \geq 0$, and the similarly reflected measure $\tilde{\mu}(A) = \mu(\{1 - x : x \in A\})$. Similarly, the statement for $f''' \leq 0$ follows by considering $-f$.

Let $\nu$ be the measure $(m - \alpha_0)\delta_0 + \alpha_0 \delta_{x_0}$; thus the sought inequality is $\int f d\nu \leq \int f d\mu$, while the choice of $\alpha_0$ and $x_0$ yields $\int 1 d\nu = m = \int 1 d\mu$, $\int x d\nu = \alpha_0 x_0 = \int x d\mu$, and $\int x^2 d\mu = \alpha_0 x_0^2 = \int x^2 d\mu$. (In fact, it is easily seen that $\nu$ is the unique measure concentrated on a two-point set $\{0, x\}$ for some $x \in [0, 1]$, such that $\int x^k d\nu = \int x^k d\mu$ for $k = 0, 1, 2$.)

We now use Taylor’s formula

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{2} \int_0^x (x-t)^2 f'''(t) dt \quad (3.2)$$

and integrate against the signed measure $\mu - \nu$. Since as we just have shown,

$$\int_0^1 x^k d(\mu - \nu) = 0, \quad k = 0, 1, 2, \quad (3.3)$$

we obtain from (3.2) and Fubini’s theorem,

$$\int_0^1 f(x) d(\mu - \nu) = \int_0^1 \int_0^x \frac{1}{2} (x-t)^2 f'''(t) dt d(\mu - \nu)(x)$$

$$= \int_0^1 \int_t^1 \frac{1}{2} (x-t)^2 d(\mu - \nu)(x) f'''(t) dt$$

$$= \int_0^1 \varphi(t) f'''(t) dt, \quad (3.4)$$

where

$$\varphi(t) = \frac{1}{2} \int_t^1 (x-t)^2 d(\mu - \nu)(x).$$
Lemma 2. Let \( \varphi \) which completes the proof. \( \square \)

On the interval \([0,1]\), finally, on the interval \([0,1]\], and thus \( \varphi \) is convex on \([0,1]\), while \( \varphi \) is concave on \([x_2,x_0]\). Since \( \varphi(0) = \varphi'(0) = 0 \), this implies that \( \varphi \geq 0 \) on \([0,1]\). Similarly, on the interval \([x_0,1]\], we have \( \nu([0,x]) = m \) and thus

\[
F(x) = \mu([0,x]) - \nu([0,x]) = \mu([0,x]) - m \leq 0,
\]

which implies that \( \varphi \) is convex on \([x_0,1]\). Moreover, \( \varphi(1) = \varphi'(1) = 0 \) and thus \( \varphi \geq 0 \) on \([x_0,1]\). Finally, on the interval \([x_2,x_0]\), \( \varphi \) is concave so it attains its minimum at one of the endpoints, but we have already shown \( \varphi(x_2), \varphi(x_0) \geq 0 \) and thus \( \varphi \geq 0 \) also on \([x_2,x_0]\), which completes the proof. \( \square \)

We apply this lemma to estimate the moment generating function of \( X \).

**Lemma 2.** Let \( X \) be as above. If \( 0 \leq t \leq 1 \), then

\[
E(1-t)^X \leq \left( 1 - t \left( 1 - \frac{\sigma^2}{\lambda} \right) \right)^{\lambda^2/(\lambda - \sigma^2)}
\]

or

\[
\ln E(1-t)^X \leq \frac{\lambda^2}{\lambda - \sigma^2} \ln \left( 1 - t \left( 1 - \frac{\sigma^2}{\lambda} \right) \right).
\]

**Proof.** Since the \( I_i \) are independent, and \( E(1-t)^{I_i} = 1 - p_i + p_i(1-t) = 1 - p_it \),

\[
E(1-t)^X = E \prod_{i=1}^n (1-t)^{I_i} = \prod_{i=1}^n E(1-t)^{I_i} = \prod_{i=1}^n (1 - p_i t),
\]
and thus
\[
\ln E(1 - t)^X = \sum_{i=1}^{n} \ln(1 - p_i t) = \int_{0}^{1} \ln(1 - tx) \, d\mu(x),
\]
where \(\mu\) is the measure \(\sum_{i=1}^{n} \delta_{p_i}\) consisting of \(n\) point masses at the (possibly coinciding)
points \(p_i\). Note that
\[
\int_{0}^{1} x \, d\mu = \sum_{i=1}^{n} p_i = \lambda
\]
and
\[
\int_{0}^{1} x^2 \, d\mu = \sum_{i=1}^{n} p_i^2 = \lambda - \sigma^2.
\]

We may assume that \(t < 1\) (the case \(t = 1\) follows then by continuity); then the function
\(f(x) = \ln(1 - tx)\) is infinitely differentiable on \([0, 1]\) with \(f'''(x) = -2t^3/(1 - tx)^3 \leq 0\).
Hence Lemma 1 yields
\[
\ln E(1 - t)^X = \int_{0}^{1} \ln(1 - tx) \, d\mu(x) \leq (m - \alpha_0)f(0) + \alpha_0 f(x_0) = \alpha_0 \ln(1 - tx_0),
\]
where \(m = n, \alpha_0 = \lambda^2/(\lambda - \sigma^2)\) and \(x_0 = (\lambda - \sigma^2)/\lambda\), which is the required estimate. \(\square\)

**Remark 8.** For \(t \geq 0\), a similar argument yields
\[
\ln E(1 + t)^X \leq (n - \alpha_1) \ln(1 + t) + \alpha_1 \ln(1 + tx_1)
\]
\[
= \frac{n\lambda - \lambda^2 - n\sigma^2}{n - \lambda - \sigma^2} \ln(1 + t) + \frac{(n - \lambda)^2}{n - \lambda - \sigma^2} \ln\left(1 + t \frac{\sigma^2}{n - \lambda}\right).
\]  
(3.7)

This inequality could be used instead of (3.6) below, giving the same results. We prefer to use (3.6), which does not involve \(n\) explicitly.

**Remark 9.** Estimates of \(E e^{\lambda X}\) are, of course, obtained by substituting \(t = 1 - e^s\) in (3.6) for \(s \leq 0\) and \(t = e^s - 1\) in (3.7) for \(s \geq 0\).

We can now obtain our basic estimate.

**Lemma 3.** Let \(X\) be as above. If \(0 \leq a \leq \lambda\), then
\[
\ln P(X \leq \lambda - a) \leq -\frac{\lambda}{\lambda - \sigma^2} (a + \sigma^2 - a\sigma^2/\lambda) \ln\left(1 + \frac{a}{\sigma^2} - \frac{a}{\lambda}\right) - (\lambda - a) \ln\left(1 - \frac{a}{\lambda}\right).
\]  
(3.8)

(When \(a = \lambda\), we define \((\lambda - a) \ln(1 - a/\lambda) = 0\).)

**Proof.** For any \(t\) with \(0 \leq t \leq 1\),
\[
E(1 - t)^X \geq (1 - t)^{\lambda - a} P(X \leq \lambda - a),
\]
and thus, using Lemma 2,
\[
\ln P(X \leq \lambda - a) \leq \ln E(1 - t)^X - (\lambda - a) \ln(1 - t)
\]
\[
\leq \frac{\lambda^2}{\lambda - \sigma^2} \ln\left(1 - t \left(1 - \frac{\sigma^2}{\lambda}\right)\right) - (\lambda - a) \ln(1 - t).
\]
Choosing
\[ t = \frac{a}{a + \sigma^2 - a \sigma^2 / \lambda} \]
(which minimizes the right hand side), this yields
\[
\ln P(X \leq \lambda - a) \leq \frac{\lambda^2}{\lambda - \sigma^2} \ln \frac{\sigma^2}{a + \sigma^2 - a \sigma^2 / \lambda} - (\lambda - a) \ln \frac{\sigma^2 (1 - a / \lambda)}{a + \sigma^2 - a \sigma^2 / \lambda}
\]
\[
= \left( \frac{\lambda^2}{\lambda - \sigma^2} - \lambda + a \right) \ln \frac{\sigma^2}{a + \sigma^2 - a \sigma^2 / \lambda} - (\lambda - a) \ln \left(1 - \frac{a}{\lambda}\right)
\]
\[
= -\frac{\lambda \sigma^2 + \lambda a - a \sigma^2}{\lambda - \sigma^2} \ln \frac{\sigma^2 + a - a \sigma^2 / \lambda}{\sigma^2} - (\lambda - a) \ln \left(1 - \frac{a}{\lambda}\right),
\]
which yields the sought result.

While the estimate in Lemma 3 may be useful for numerical evaluation in applications, it is too complicated to be of much other direct use. Hence we will use it to derive the simpler (but slightly weaker) estimates in Theorem 3.

For notational convenience, let \( \sigma^2 = x \lambda \) and \( a = y \sigma^2 = xy \lambda \), where \( 0 < x < 1 \) and \( 0 \leq y \leq 1 / x \). Then (3.8) may be written
\[
\ln P(X \leq \lambda - a) \leq -\sigma^2 g(x, y),
\]
where
\[
g(x, y) = \frac{1}{1 - x} (1 + y - xy) \ln(1 + y - xy) + \frac{1}{x} (1 - xy) \ln(1 - xy).
\]

**Lemma 4.** \( g(x, y) \) is an increasing function of \( x \) in the region
\[
U = \{(x, y) : 0 < x < 1, \ 0 \leq y \leq 1 / x\}.
\]

**Proof.** We want to show that \( \partial g / \partial x \geq 0 \) in the region \( \{(x, y) : 0 < x < 1, \ 0 \leq y \leq 1 / x\} \); note that \( g \) is well-defined and infinitely differentiable in the larger region \( V = \{(x, y) : \ 0 < x < 1, \ -1 / (1 - x) < y < 1 / x\} \), and continuous on \( \overline{V} \cap \{(x, y) : 0 < x < 1\} \supset U \).

Instead of estimating \( \partial g / \partial x \) directly, we first compute
\[
\frac{\partial g}{\partial y} = \ln(1 + y - xy) - \ln(1 - xy)
\]
and
\[
\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x} = -\frac{y}{1 + y - xy} - \frac{y}{1 - xy} = \frac{y^2}{(1 + y - xy)(1 - xy)} \geq 0.
\]
Hence \( \partial g / \partial x \) is an increasing function of \( y \). Moreover, taking \( y = 0 \), we find \( g(x, 0) = 0 \) and thus
\[
\frac{\partial g}{\partial x}(x, 0) = 0, \quad 0 < x < 1.
\]
Hence \( \frac{\partial g}{\partial x}(x, y) \geq 0 \) for all \( (x, y) \in V \) with \( y \geq 0 \).

**Proof of Theorem 3.** If \( x \searrow 0 \) and \( y \geq 0 \) is fixed, then, as is easily seen,
\[
g(x, y) \rightarrow (1 + y) \ln(1 + y) - y.
\]
Lemma 4 thus yields, for $0 < x < 1$ and $0 \leq y \leq 1/x$,

$$g(x, y) \geq (1 + y) \ln(1 + y) - y,$$

and thus by (3.9), for $0 \leq a \leq \lambda$,

$$\ln P(X \leq \lambda - a) \leq -\sigma^2((1 + y) \ln(1 + y) - y),$$

with $y = a/\sigma^2$, which is the analogue of (1.13) for $P(X \leq EX - a)$; note that this estimate trivially holds for $a > \lambda$. In order to obtain (1.13), we consider the variable $X^* = n - X \overset{d}{=} \sum_{i=1}^{n}(1 - I_i)$, and observe that $P(X \geq EX + a) = P(X^* \leq EX^* - a)$ and Var$(X^*) = \text{Var } X$. The estimates (1.14) and (1.15) and their analogues for $P(X \leq EX - a)$ now follow from the elementary estimates, defining $h(y) = (1 + y) \ln(1 + y) - y$,

$$h(y) \geq \frac{y^2}{2(1 + y/3)} \geq \frac{y^2}{2}(1 - y/3), \quad y \geq 0,$$

(3.10)

and

$$h(y) \geq yh(c)/c, \quad y \geq c.$$  

(3.11)

The estimate (3.10) may be verified by observing that $h(y) - \frac{y^2}{2(1 + y/3)}$ vanishes together with its first derivative at 0, while the second derivative equals $(9y^2 + y^3)/(1 + y)(3 + y)^3 \geq 0$. Similarly, (3.11) follows by the convexity of $h$. We omit the details.

Finally, if $\sigma^2 \geq \lambda/2$, then $x \geq 1/2$ and thus by Lemma 4 (assuming as we may that $a \leq \lambda$),

$$g(x, y) \geq g(\frac{1}{2}, y) = (2 + y) \ln(1 + \frac{y}{2}) + (2 - y) \ln(1 - \frac{y}{2}) \geq \frac{1}{2}y^2,$$

which together with (3.9) yields (1.16). \qed

4. PROOF OF THEOREM 4.

Let $Y = \sum_{i=2}^{n} I_i$ and $Z = \sum_{i=1}^{n} J_{i1}$, where $(J_{i1})_{i=1}^{n}$ are as in the definition of negatively related variables. Then $X = I_1 + Y$ and $(Y \mid I_1 = 1) \overset{d}{=} Z$. Moreover, since $J_{i1} \leq I_i$ for $i \geq 2$, we have $Z \leq Y$. Consequently, for any real $t$,

$$E e^{tX} = E e^{tY} = E(e^{tI_1} - 1)e^{tY} = E(e^{tI_1} - 1)I_1 e^{tY}$$

$$= (e^t - 1)p_1 E(e^{tY} \mid I_1 = 1)$$

$$= (e^t - 1)p_1 E e^{tZ}$$

$$\leq (e^t - 1)p_1 E e^{tY}$$

$$= E(e^{tI_1} - 1)E e^{tY}$$

and thus

$$E e^{tX} \leq E e^{tI_1} E e^{tY}.$$

Induction yields

$$E e^{tX} \leq \prod_{i=1}^{n} e^{tI_i} = E e^{tX}.$$

\qed
5. Proof of Theorem 5.

Given $n$, $p$, $k$, with $1 \leq k \leq n$ and $0 < p < 1$, let $U \sim \text{Bi}(n, p)$ and let $X_{npk}$ be the random variable $U$ conditioned on $U \geq k$. Since $U = \sum_{i=1}^{n} I_i$, with $I_i \sim \text{Be}(p)$ independent, $X_{npk} = \sum_{i=1}^{n} (I_i \mid \sum_{i=1}^{n} I_i \geq k)$, and it follows from Barbour, Holst and Janson (1992) Proposition 2.2.10 and Theorem 2.1 that $X_{npk}$ is a sum of negatively related indicators. We claim that, for any $\alpha$, $c$, $A$ as in Theorem 5, some variable $X_{npk}$ satisfies (1.24).

Suppose not. Then for each $n$, $p$, $k$ either $\text{Var}(X_{npk}) \leq A$ or

$$P(X_{npk} - E X_{npk} > \alpha \sigma) \leq ce^{-\alpha}. \tag{5.1}$$

Fix $p \in (0, 1)$, let $q = 1 - p$, choose $\varepsilon$ with $0 < \varepsilon < q$, take $k = \lfloor n(p + \varepsilon) \rfloor + 1$, and let $n \to \infty$. Then, for fixed $i \geq 0$, with $r = p(q - \varepsilon)/(p + \varepsilon)q < 1$,

$$P(X_{npk} = k + i + 1) = \frac{p}{q} \cdot \frac{n - k - i}{k + i + 1} \to \frac{p}{q} = \frac{q - \varepsilon}{p + \varepsilon} = r \quad \text{as } n \to \infty, \tag{5.2}$$

and

$$P(X_{npk} = k + i) \leq r^i P(X_{npk} = k) \leq r^i. \tag{5.3}$$

It follows that $X_{npk} - k$ converges in distribution to a random variable $Y_r$ with geometric distribution Ge$(1 - r)$: $P(Y_r = i) = (1 - r)r^i$, $i \geq 0$. Moreover, by (5.3), every moment $E(X_{npk} - k)^m$ stays bounded, which implies that the moments converge to the corresponding moments of $Y_r$.

The variance of $Y_r$ equals $r/(1 - r)^2$. If $\text{Var}(Y_r) > A$, then also $\text{Var}(X_{npk}) > A$ for large $n$, so by our assumption (5.1) holds and taking the limit as $n \to \infty$ we obtain

$$P(Y_r - E Y_r > \alpha \sqrt{\text{Var} Y_r}) \leq ce^{-\alpha}. \tag{5.4}$$

Now, let $\varepsilon \to 0$ (keeping $p$ fixed). Then $r \to 1$ and $\text{Var}(Y_r) = r/(1 - r)^2 \to \infty$, so (5.4) holds when $r$ is close to 1. Moreover, it is easily seen that as $r \to 1$, $(1 - r)Y_r$ converges in distribution to an exponential variable $Z \sim \text{Exp}(1)$, again with convergence of all moments. Consequently we may take the limit again and obtain from (5.4)

$$P(Z - E Z > \alpha \sqrt{\text{Var} Z}) \leq ce^{-\alpha}. \tag{5.5}$$

But $E Z = \text{Var} Z = 1$, so the left hand side of (5.5) equals $P(Z > 1 + \alpha) = e^{-1-\alpha} > ce^{-\alpha}$, and we have obtained a contradiction. \qed

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