Combinatorial Semi-Bandits with Knapsacks

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Abstract

We unify two prominent lines of work on multi-armed bandits: bandits with knapsacks (BwK) and combinatorial semi-bandits. The former concerns limited “resources” consumed by the algorithm, e.g., limited supply in dynamic pricing. The latter allows a huge number of actions but assumes combinatorial structure and additional feedback to make the problem tractable. We define a common generalization, support it with several motivating examples, and design an algorithm for it. Our regret bounds are comparable with those for BwK and combinatorial semi-bandits.

1 Introduction

Multi-armed bandits (MAB) is an elegant model for studying the tradeoff between acquisition and usage of information, a.k.a. explore-exploit tradeoff [Robbins 1952; Thompson 1933]. In each round an algorithm sequentially chooses from a fixed set of alternatives (sometimes known as actions or arms), and receives reward for the chosen action. Crucially, the algorithm does not have enough information to answer all “counterfactual” questions about what would have happened if a different action were chosen in this round. MAB problems have been studied steadily since 1930-ies, with a huge surge of interest in the last decade.

This paper combines two lines of work related to bandits: on bandits with knapsacks (BwK) [Badanidiyuru et al. 2013] and on combinatorial semi-bandits [György et al. 2007]. BwK concern scenarios with limited “resources” consumed by the algorithm, e.g., limited inventory in a dynamic pricing problem. In combinatorial semi-bandits, actions correspond to subsets of some “ground set”, rewards are additive across the elements of this ground set (atoms), and the reward for each chosen atom is revealed (semi-bandit feedback). A paradigmatic example is an online routing problem, where atoms are edges in a graph, and actions are paths. Both lines of work has received much recent attention, and are supported by numerous examples.

Our contributions. We define a common generalization of combinatorial semi-bandits and BwK, termed Combinatorial Semi-Bandits with Knapsacks (SemiBwK). Following all prior work on BwK, we focus on an i.i.d. environment: in each round, the “outcome” is drawn independently from a fixed distribution over the possible outcomes. Here the “outcome” of a round is the matrix of reward and resource consumption for all atoms. We design an algorithm for SemiBwK, achieving regret rates that are comparable with those for BwK and combinatorial semi-bandits.

Specifics are as follows. As usual, we assume “bounded outcomes”: for each atom and each round, rewards and consumption of each resource is at most 1. Regret is relative to the expected total reward of the best all-knowing policy, denoted OPT. For BwK problems, this is known to be a much stronger benchmark than the traditional best-fixed-arm benchmark. We upper-bound regret in terms of the relevant parameters: time horizon $T$, (smallest) budget $B$, number of atoms $n$, and OPT itself. We obtain

\[ \text{Regret} \leq \tilde{O}(\sqrt{n})(\text{OPT} / \sqrt{B} + \sqrt{T} + \text{OPT}). \]  (1.1)

The “shape” of the regret bound is consistent with prior work: the $\text{OPT} / \sqrt{B}$ additive term appears in the optimal regret bound for BwK, and the $\sqrt{T}$ and $\sqrt{\text{OPT}}$ additive terms are very common in regret bounds for MAB. The per-round running time is polynomial in $n$, and near-linear in $n$ for some important special cases.

Our regret bound is optimal up to polylog factors for paradigmatic special cases. BwK is a special case when actions are paths. For $\text{OPT} > \Omega(T)$, the regret bound becomes $\tilde{O}(T^{1/2}/B + \sqrt{nT})$, where $n$ is the number of actions.

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1 Our model allows arbitrary correlations within a given round, both across rewards and consumption for the same atom and across multiple atoms. Such correlations are essential in applications such as dynamic pricing and dynamic assortment. E.g., customers’ valuations can be correlated across products, and algorithm earns only if it sells; see Section 4 for details.
Combinatorial Semi-Bandits with Knapsacks

Our model captures several application scenarios, incl. dynamic pricing, dynamic assortment, repeated auctions, and repeated bidding. We work out these applications, and explain how our regret bounds improve over prior work.

**Challenges and techniques.** BwK problems are challenging compared to traditional MAB problems with i.i.d. rewards because it no longer suffices to look for the best action and/or optimize expected per-round rewards; instead, one essentially needs to look for a distribution over actions with optimal expected total reward across all rounds. Generic challenges in combinatorial semi-bandits concern handling exponentially many actions (both in terms of regret and in terms of the running time), and taking advantage of the additional feedback. And in BwK, one needs to deal with distributions over subsets of atoms, rather than “just” with distributions over actions.

Our algorithm connects a technique from BwK and a randomized rounding technique from combinatorial optimization. (With five existing BwK algorithms and a wealth of approaches for combinatorial optimization, choosing the techniques is a part of the challenge.)

We build on a BwK algorithm from Agrawal & Devanur (2014a), which combines linear relaxations and a well-known “optimism-under-uncertainty” paradigm. A generalization of this algorithm to BwK results in a fractional solution $x$, a vector over atoms. Randomized rounding converts $x$ into a distribution over feasible subsets of atoms that equals $x$ in expectation. It is crucial (and challenging) to ensure that this distribution contains enough randomness so as to admit concentration bounds not only across rounds, but also across atoms. Our analysis “opens up” a fairly technical proof from prior work and intertwines it with a new argument based on negative correlation.

We present our algorithm and analysis so as to “plug in” any suitable randomized rounding technique. This makes our presentation more lucid, and also leads to faster running times for some important special cases.

**Solving SemiBwK using prior work.** Solving SemiBwK using an algorithm for BwK would result in a regret bound like $\tilde{O}(nT)$ with $n$ replaced with the number of actions. The latter could be on the order of $n^k$ if each action can consist of at most $k$ atoms, or perhaps even exponential in $n$.

SemiBwK can be solved as a special case of a much more general linear-contextual extension of BwK from Agrawal & Devanur (2014a, 2016). In their model, an algorithm takes advantage of the combinatorial structure of actions, yet it ignores the additional feedback from the atoms. Their regret bounds have a worse dependence on the parameters, and apply for a much more limited range of parameters. Further, their per-round running time is linear in the number of actions, which is often prohibitively large.

To compare the regret bounds, let us focus on instances of SemiBwK in which at most one unit of each resource is consumed in each round. (This is the case all our motivating applications, except repeated bidding.) Then Agrawal & Devanur (2014a, 2016) assume $B > \tilde{O}(n^{3/4})$, and achieve regret $\tilde{O}(n^{1/2}T^{3/4} + n^2 \sqrt{T})$. It is easy to see that we improve upon the range and upon both summands. In particular, we improve both summands by the factor of $n^{3/2}$ in a lucid special case when $B > \tilde{O}(T)$ and $\text{OPT} < O(T)^{3/4}$.

We run simulations to compare our algorithm against prior work on BwK and combinatorial semi-bandits.

**Related work.** Multi-armed bandits have been studied since Thompson (1933) in Operations Research, Economics, and several branches of Computer Science, see Gittins et al. (2011) Bubeck & Cesa-Bianchi (2012) for background. Among broad directions in MAB, most relevant is MAB with i.i.d. rewards, starting from Lai & Robbins (1985). Auer et al. (2002).

Bandits with Knapsacks (BwK) were first introduced by Badanidiyuru et al. (2013) as a common generalization of several models from prior work and many other motivating examples. Subsequent papers extended BwK to “smoother” resource constraints and introduced several new algorithms (Agrawal & Devanur 2014a), and generalized BwK to contextual bandits (Badanidiyuru et al. 2014, Agrawal et al. 2016) Agrawal & Devanur 2016. All prior work on BwK and special cases thereof assumed i.i.d. outcomes.

Special cases of BwK include dynamic pricing with li-
Our model and preliminaries

Our model, called Semi-Bandits with Knapsacks (SemiBwK) is a generalization of multi-armed bandits (henceforth, MAB) with i.i.d. rewards. As such, in each round $t = 1, \ldots , T$, an algorithm chooses an action $S_t$ from a fixed set of actions $F$, and receives a reward $\mu_t(S_t)$ for this action which is drawn independently from a fixed distribution that depends only on the chosen action. The number of rounds $T$, a.k.a. the time horizon, is known.

There are $d$ resources being consumed by the algorithm. The algorithm starts out with budget $B_j \geq 0$ of each resource $j$. All budgets are known to the algorithm. If in round $t$ action $S \in F$ is chosen, the outcome of this round is not only the reward $\mu(S)$ but the consumption $C_t(S,j)$ of each resource $j \in [d]$. We refer to $C_t(S) := (C_t(S,j) : j \in [d])$ as the consumption vector.

Following prior work on BwK, we assume that all budgets are the same: $B_j = B$ for all resources $j$. Algorithm stops as soon as any one of the resources goes strictly below 0. The round in which this happens is called the stopping time and denoted $\tau_{\text{stop}}$. The reward in collected in this last round does not count; so the total reward of the algorithm is $\text{rew} = \sum_{t<\tau_{\text{stop}}} \mu_t(S_t)$.

Actions correspond to subsets of a finite ground set $A$, with $n = |A|$; we refer to elements of $A$ as atoms. Thus, the set $F$ of actions corresponds to the family of “feasible subsets” of $A$. The rewards and resource consumption is additive over the atoms: for each round $t$ and each atom $a$ there is a reward $\mu_t(a) \in [0,1]$ and consumption vector $C_t(a) \in [0,1]^d$ such that for each action $S \in F$ it holds that $\mu_t(S) = \sum_{a \in S} \mu_t(a)$ and $C_t(S) = \sum_{a \in S} C_t(a)$.

We assume the i.i.d. property across rounds, but allow arbitrary correlations within the same round. Formally, for a given round $t$ we consider the $n \times (d+1)$ “outcome matrix” $(\mu_t(a), C_t(a) : a \in A)$, which specifies rewards and resource consumption for all resources and all atoms. We assume that the outcome matrix is chosen independently from a fixed distribution $D_h$ over such matrices. The distribution $D_h$ is not revealed to the algorithm. The mean rewards and mean consumption is denoted $\mu(a) := \mathbb{E}[\mu_t(a)]$ and $C(a) := \mathbb{E}[C_t(a)]$. We extend the notation to actions, i.e., to subsets of atoms: $\mu(S) := \sum_{a \in S} \mu(a)$ and $C(S) := \sum_{a \in S} C(a)$.

An instance of SemiBwK consists of the action set $F \subset 2^n$, the budgets $B = (B_j : j \in [d])$, and the distribution $D_h$. The $F$ and and $B$ are known to the algorithm, and $D_h$ is not. As explained in the introduction, SemiBwK subsumes Bandits with Knapsacks (BwK) and semi-bandits. BwK is

\footnote{We use bold font to indicate vectors and matrices.}
the special case when $$\mathcal{F}$$ consists of singletons, and semi-bandits is the special case when all budgets are equal to $$B_j = nT$$ (so that the resource consumption is irrelevant).

Following the prior work on BwK, we compete against the "optimal all-knowing algorithm": an algorithm that optimizes the expected total reward for a given problem instance; its expected total reward is denoted OPT. As observed in Badanidiyuru et al. [2013], OPT can be much larger (e.g., factor of 2 larger) than the expected cumulative reward of the best action, for a variety of important special cases of BwK. Our goal is to minimize regret, defined as OPT minus algorithm’s total reward.

**Combinatorial constraints.** Action set $$\mathcal{F}$$ is given by a combinatorial constraint, i.e., a family of subsets. Treating subsets of atoms as $$n$$-dimensional binary vectors, $$\mathcal{F}$$ corresponds to a finite set of points in $$\mathbb{R}^n$$. We assume that the convex hull of $$\mathcal{F}$$ forms a polytope in $$\mathbb{R}^n$$. In other words, there exists a set of linear constraints over $$\mathbb{R}^n$$ whose set of feasible integral solutions is $$\mathcal{F}$$. We call such $$\mathcal{F}$$ linearizable; the convex hull is called the polytope induced by $$\mathcal{F}$$.

Our main result is for matroid constraints, a family of linearizable combinatorial constraints which subsumes several important special cases such as cardinality constraints, partition matroid constraints, spanning tree constraints and transversal constraints. Formally, $$\mathcal{F}$$ is a matroid if it contains the empty set, and satisfies two properties: (i) if $$\mathcal{F}$$ contains a subset $$S$$, then it also contains every subset of $$S$$, and (ii) for any two subsets $$S, S' \in \mathcal{F}$$ with $$|S| > |S'|$$ it holds that $$S' \cup \{a\}$$ for each atom $$a \in S \setminus S'$$. See Appendix A for more background and examples.

We incorporate prior work on randomized rounding for linear programs. Consider a linearizable action set $$\mathcal{F}$$ with induced polytope $$P \subset [0,1]^n$$. The randomized rounding scheme (henceforth, RRS) for $$\mathcal{F}$$ is an algorithm which inputs a feasible fractional solution $$x \in P$$ and the linear equations describing $$P$$, and produces a random vector $$Y$$ over $$\mathcal{F}$$. For our main result, we consider RRS’s such that $$E[Y] = x$$ and $$Y$$ is negatively correlated; we call such RRS’s negatively correlated. Several such RRS are known: e.g., for cardinality constraints and bipartite matching (Gandhi et al. 2006), for spanning trees (Asadpour et al. 2010), and for matroids (Chekuri et al. 2010).

**Negative correlation.** Let $$\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$$ denote a family of random variables which take values in [0,1]. Let $$X := \frac{1}{m} \sum_{i=1}^{m} X_i$$ be the average, and $$\mu := E[X]$$.

Family $$\mathcal{X}$$ is called negatively correlated if

$$E \left[ \prod_{i \in S} X_i \right] \leq \prod_{i \in S} E[X_i] \quad \forall S \subseteq [m] \quad (2.1)$$

$$E \left[ \prod_{i \in S} (1 - X_i) \right] \leq \prod_{i \in S} E[1 - X_i] \quad \forall S \subseteq [m] \quad (2.2)$$

Independent random variables satisfy both properties with equality. For intuition: if $$X_1, X_2$$ are Bernoulli and (2.1) is strict, then $$X_1$$ is more likely to be 0 if $$X_2 = 1$$.

Negative correlation is a generalization of independence that allows for similar concentration bounds, i.e., high-probability upper bounds on $$|X - \mu|$$. However, our analysis does not invoke them directly. Instead, we use a concentration bound given a closely related property:

$$E \left[ \prod_{i \in S} X_i \right] \leq \left( \frac{1}{2} \right)^{|S|} \quad \forall S \subseteq [m].$$

(2.3)

**Theorem 2.1.** If (2.3), then for some absolute constant $$c$$,

$$\Pr[X \geq \frac{1}{2} + \eta] \leq c \cdot e^{-2m\eta^2} \quad (\forall \eta > 0) \quad (2.4)$$

This theorem easily follows from Impagliazzo & Kabanets [2010], see Appendix A in the supplement.

**Confidence radius.** We bound deviations $$|X - \mu|$$ in a way that gets sharper when the $$\mu$$ is small, without knowing the $$\mu$$ in advance. (We use the notation $$X, \mu$$ as above.) To this end, we use the notion of confidence radius from Kleinberg et al. [2015], Babai et al. [2015], Badanidiyuru et al. [2013] Agrawal & Devanur [2014b].

$$\text{Rad}_a(x, m) = \sqrt{ax/m + \alpha/m}. \quad (2.5)$$

If random variables $$X$$ are independent, then event

$$|X - \mu| < \text{Rad}_a(X, m) < 3 \text{Rad}_a(\mu, m) \quad (2.6)$$

happens with probability at least $$1 - O(e^{-\Omega(a)})$$, for any given $$a > 0$$. We use this notion to define upper/lower confidence bounds on the mean rewards and mean resource consumption. Fix round $$t$$, atom $$a$$, and resource $$j$$. Let $$\hat{\mu}_t(a)$$ and $$\hat{C}_t(a,j)$$ denote the empirical average of the rewards and resource-$$j$$ consumption, resp., between rounds $$1$$ and $$t - 1$$. Let $$N_t(a)$$ be the number of times atom $$a$$ has been chosen in these rounds (i.e., included in the chosen action). The confidence bounds are defined as

$$C^+_t(a,j) = \text{proj} \left( \hat{C}_t(a,j) \pm \text{Rad}_a(\hat{C}_t(a,j), N_t(a)) \right)$$

$$\mu^+_t(a) = \text{proj} \left( \hat{\mu}_t(a) \pm \text{Rad}_a(\hat{\mu}_t(a), N_t(a)) \right) \quad (2.7)$$

where $$\text{proj}(x) := \arg\min_{y \in [0,1]} |y - x|$$ denotes the projection into $$[0,1]$$. We always use the same parameter $$\alpha = c_{\text{conf}} \log(nt)$$, for an appropriately chosen absolute constant $$c_{\text{conf}}$$. We suppress $$\alpha$$ and $$c_0$$ from the notation. We use a vector notation $$\mu^+_t(a)$$ and $$C^+_t(a,j)$$ to denote the corresponding $$n$$-dimensional vectors over all atoms $$a$$.

By (2.6), with probability $$1 - O(e^{-\Omega(a)})$$ it holds that

$$\mu(a) \in [\mu^-_t(a), \mu^+_t(a)] \quad \text{and} \quad C(a,j) \in [C^-(a,j), C(a,j)^+].$$

**3 Main algorithm**

Let us define our main algorithm, called SemiBwK-RRS. The algorithm builds on an arbitrary RRS for the action set $$\mathcal{F}$$. It is parameterized by this RRS, the polytope $$\mathcal{P}$$ induced
feasible action. The pseudocode is given as Algorithm 1.

Algorithm uses the RRS seen as a probability vector over the atoms. Finally, the bounds for consumption. The linear relaxation is also “con-

This linear program defines a linear relaxation of the orig-

earizable action set represented as a collection of linear constraints), and

Subject to $$C_t^{-1}(j) \cdot x \leq \frac{B(1-\epsilon)}{T}, \quad j \in [d]$$ (LP_{ALG})

$${x \in P}$$

$${\text{maximize } \mu_t^+ \cdot x}$$

$${\text{subject to } C_t^{-1}(j) \cdot x \leq \frac{B(1-\epsilon)}{T}, \quad j \in [d]}$$ (LP_{ALG})

Theorem 3.1. Consider the SemiBwK-RSS problem with a lin-

earizable action set F that admits a negatively correlated RRS. Then algorithm SemiBwK-RSS with this RRS achieves expected regret at most

$$O(\log(ndT)) \sqrt{n} \left( \OPT / \sqrt{B} + \sqrt{T + \OPT} \right) \quad (3.1)$$

Here T is the time horizon, n is the number of atoms, and B is the budget. We require $$B > \alpha n + \sqrt{\alpha n T}$$, where $$\alpha = \Theta(\log(nT))$$ is the parameter in confidence radius. Parameter $$\epsilon$$ in the algorithm is set to $$\sqrt{\frac{\alpha n T}{B}} + \frac{\alpha n}{B} + \sqrt{\frac{\alpha n T}{B}}$$.

Corollary 3.2. Consider the setting in Theorem 3.1 and assume that the action set F is defined by a matroid on the set of atoms. Then, using the negatively correlated RRS from (Chekuri et al. 2010), we obtain regret bound (3.1).

The proof of the theorem is very technical. We provide an overview below, and defer the full proof to the supplement. We actually prove a slightly stronger statement involving high-probability regret rather than expected regret.

3.1 Proof overview of Theorem 3.1

First, we argue that [LP_{ALG}] provides a good benchmark that we can use instead of OPT. Specifically, at any given round, the optimal value for [LP_{ALG}] in each round is at least $$\frac{1}{2}(1-\epsilon)$$ OPT with high probability. We prove this by constructing a series of LPs, starting with a generic linear relaxation for BwK and ending with [LP_{ALG}] and showing that the optimal value does not decrease along the series.

Next we define an event that occur with high probability, henceforth called clean event. This event concerns total rewards, and compares our algorithm against [LP_{ALG}]

$$\sum_{t \in [T]} r_t - \sum_{t \in [T]} \sum_{i \in [n]} \mu_{t}^+ \cdot x_i \leq O \left( \sqrt{\alpha n \sum_{t \in [T]} r_t + \sqrt{T + \alpha n} \right) \quad (3.2)$$

We prove that it is indeed a high-probability event in three steps. First, we connect the algorithm’s reward $$\sum_{t} r_t$$ to its expected reward $$\sum_{t} \mu \cdot S_t$$, where we interpret the chosen action $$S_t$$, a subset of atoms, as a binary vector over the atoms. Then we connect $$\sum_{t} \mu \cdot S_t$$ to $$\sum_{t} \mu^+ \cdot S_t$$, replacing expected rewards with the upper confidence bounds. Finally, we connect $$\sum_{t} \mu^+ \cdot S_t$$ to $$\sum_{t} \mu^+ \cdot x_i$$, replacing the output of the RRS with the corresponding expectations. Putting it together, we connect algorithm’s reward to $$\sum_{t} \mu^+ \cdot x_i$$, as needed. It is essential to bound the deviations in the sharpest way possible; in particular, the naive $$O(\sqrt{T})$$ bounds are not good enough. To this end, we use several tools: the confidence radius from (2.5), the negative correlation property of the RRS, and another concentration bound from prior work.

A similar “clean event” (with a similar proof) concerns the total resource consumption of the algorithm. We condition on both clean events, and perform the rest of the analysis via a “deterministic” argument not involving probabilities. In particular, we use the second “clean event” to guarantee that the algorithm never runs out of resources.

We use negative correlation via a rather delicate argument. We extend the concentration bound in Theorem 3.1 to a random process that evolves over time, and only assumes that property (2.3) holds within each round conditional on the history. For a given round, we start with a negative correlation property of $$S_t$$ and construct another family of random variables that conditionally satisfies (2.3). The ex-
tended concentration bound is then applied to this family. The net result is a concentration bound for $\sum_i \mu_i^T \cdot S_i$, as if we had $n \times T$ independent random variables there.

### 3.2 Running time of the algorithm

The algorithm does two computationally intensive steps in each round: solves the linear program $\text{LP}_{\text{ALG}}$ and runs the RRS. For matroid constraints, the RRS from Chekuri et al. (2010) has $O(n^2)$ running time. Hence, in the general case the computational bottleneck is solving the LP, which has $n$ variables and $O(2^n)$ constraints. Matroids are known to admit a polynomial-time separation oracle (e.g., see Schrijver, 2003). It follows that the entire set of constraints in $\text{LP}_{\text{ALG}}$ admits a polynomial-time separation oracle, and therefore we can use the Ellipsoid algorithm to solve $\text{LP}_{\text{ALG}}$ in polynomial time. For some classes of matroid constraints the LP is much smaller: e.g., for cardinality constraints (just $d + 1$ constraints) and for traversal matroids on bipartite graphs (just $2n + d$ constraints). Then near-linear-time algorithms can be used.

Our algorithm works under any negatively correlated RRS. We can use this flexibility to improve the per-round running time for some special cases. (Making decisions extremely fast is often critical in practical applications of bandits (e.g., see Agrawal et al., 2016).) We obtain near-linear per-round running times for cardinality constraints and partition matroid constraints. Indeed, $\text{LP}_{\text{ALG}}$ can be solved in near-linear time, as mentioned above, and we can use a negatively correlated RRS from (Gandhi et al., 2006) which runs in linear time. These classes of matroid constraints are important in our applications (see Section 4).

### 4 Applications and special cases

Let us discuss some notable examples of $\text{SemiBwK}$ (which generalize some of the numerous applications listed Badanidiyuru et al., 2013). Our results for these examples improve exponentially over a naive application of the $BwK$ framework. Compared to what can be derived from Agrawal & Devanur (2014, 2016), our results feature a substantially better dependence on parameters, a much better per-round running time, and apply to a wider range of parameters. However, we leave open the possibility that regret bounds can be improved for some special cases.

**Dynamic pricing.** The dynamic pricing application is as follows. The algorithm has $d$ products on sale with limited supply: for simplicity, $B$ units of each. Following Besbes & Zeevi (2012), we allow supply constraints across products, e.g., a "gadget" that goes into multiple products. In each round $t$, an agent arrives, the algorithm chooses a vector of prices $p_t \in [0,1]^d$ to offer the agent, and the agent chooses what to buy at these prices. For simplicity, the agent is interested in buying (or is only allowed to buy) at most one item of each product. The agent has a valuation vector over products, so that the agent buys a given product if and only if her valuation for this product is at least as high as the offered price. The entire valuation vector is drawn as an independent sample from a fixed and unknown distribution (but valuations may be correlated across products). The algorithm maximizes the total revenue from sales.

To side-step discretization issues, we assume that prices are restricted to a known finite subset $S \subset [0,1]$. Achieving general regret bounds without such restriction appears beyond reach of the current techniques for $BwK$.

To model it as a $\text{SemiBwK}$ problem, the set of atoms is all price-product pairs. The combinatorial constraint is that at most one price is chosen for each product. (If an action does not specify a price for some product, the default price is used.) This is a "partition matroid" constraint, see Appendix B. Rewards correspond to revenue from sales, and resources correspond to inventory constraints.

We obtain regret $\tilde{O}(d\sqrt{dB|S|} + \sqrt{T|S|})$ using Corollary 3.2 whenever $B > \tilde{O}(n + \sqrt{nT})$. This is because $OPT \leq dB$, since that is the maximum number of products available, and the number of atoms is $n = d|S|$.

For comparison, results of Agrawal & Devanur (2014a, 2016) apply only when $B > \sqrt{nT^{3/4}}$, and yield regret bound of $O(d^3|S|^2\sqrt{T})$. Thus, our regret bounds feature a better dependence on the number of allowed prices $|S|$ (which can be very large) and the number of products $d$. Further, our regret bounds hold in a meaningful way for the much larger range of values for budget $B$.

For a naive application of the $BwK$ framework, arms correspond to every possible realization of prices for the $d$ products. Thus, we have $|S|^d$ arms, with a corresponding exponential blow-up in regret.

**Dynamic assortment.** The dynamic assortment problem is similar to dynamic pricing in that the algorithm is selling $d$ products to an agent, with a limited inventory $B$ of each product, and is interested in maximizing the total revenue from sales. As before, agents can have arbitrary valuation vectors, drawn from a fixed but unknown distribution. However, the algorithm chooses which products to offer, whereas all prices are fixed externally. There is a large number of products to choose from, and only $k \ll d$ of them can be offered in any given round.

To model this as $\text{SemiBwK}$, atoms correspond to products,
and actions correspond to subsets of at most $k$ atoms. The combinatorial constraint forms a matroid (see Appendix B). Rewards correspond to sales, and resources correspond to products, as in dynamic pricing. Since $OPT \leq \min(dB, kT)$, Corollary 3.2 yields regret $\tilde{O}(k\sqrt{dT})$ when $B > \Omega(T)$, and regret $\tilde{O}(d\sqrt{dB} + \sqrt{dT})$ in general.

In a naive application of BwK, arms are subsets of $k$ products. Hence, we have $O(d^k)$ arms. The other parameters of the problem remain the same. This leads to regret bound $\tilde{O}(d\sqrt{Bd^k})$, with an exponential dependence on $k$.

**Repeated auctions.** Consider a repeated auction with adjustable parameters, e.g., repeated second-price auction with reserve price that can be adjusted from one round to another. While prior work (Cesa-Bianchi et al. 2013; Badanidiyuru et al. 2013) concerned running one repeated auction, we generalize this scenario to multiple repeated auctions with shared inventory. E.g., the same inventory may be sold via multiple channels to different audiences.

More formally, the auctioneer is running $r$ simultaneous repeated auctions to sell a shared inventory of $d$ products, with limited supply $B$ of each product. (E.g., different auctions can cater to different audiences.) Each auction has a parameter which the algorithm can adjust over time. We assume that this parameter comes from a finite domain $S \subset [0, 1]$. For simplicity, assume the auctions are synchronized with one another. As in prior work, we assume that in every round of each auction a fresh set of participants arrives, sampled independently from a fixed joint distribution, and only a minimal feedback is observed: the products sold and the combined revenue.

Following prior work (Cesa-Bianchi et al. 2013; Badanidiyuru et al. 2013), we only assume minimal feedback: for each auction, what where the products sold and what was the combined revenue from this auction. In particular, we do not assume that the algorithm has access to participants’ bids. Not using participants’ bids is desirable for privacy considerations, and in order to reduce the participants’ incentives to game the learning algorithm.

To model this problem as SemiBwK, atoms correspond to the auction-bid pairs. The combinatorial constraint is that each action must specify at most one bid for each auction. There is a “default bid” for each auction in case an action does not specify the bid for this auction.) There is exactly one resource, which is money and the total budget is $B$. Note that the number of atoms is $n = r|S|$. Hence, our main result yields regret $\tilde{O}(OPT \sqrt{r|S|/B} + \sqrt{r|S|T})$.

A naive application of BwK would have arms that correspond to all possible combinations of bids, for the total of $O(|S|^r)$ arms; so we have an exponential blow-up in regret.

## 5 Numerical Simulations

We ran some experiments on simulated datasets in order to compare our algorithm, SemiBwK-RRS, with some prior work that can be used to solve SemiBwK:

- the primal-dual algorithm for BwK from Badanidiyuru et al. (2013), denoted pdBwK.
- an algorithm for combinatorial semi-bandits with a matroid constraint: “Optimistic Matroid Maximization” from Kveton et al. (2014), denoted OMM.
- the linear-contextual BwK algorithm from Agrawal & Devanur (2016), discussed in the Introduction, denoted linCBwK.

To speed up the computation in linCBwK, we used a heuristic modification suggested by the authors in a private communication. This modification did not substantially affect average rewards in our preliminary experiments. We also made a heuristic improvement to our algorithm, setting $\epsilon = 0$ and $\alpha = 5$. We use the same value of $\alpha$ for the pdBwK algorithm as well.

**Problem instances.** We did not attempt to comprehensively cover the huge variety of problem instances in SemiBwK. Instead, we focus on two families of problem instances which seemed representative.
Combinatorial Semi-Bandits with Knapsacks

The first family is inspired by the dynamic assortment application. As in dynamic assortment, we have \( n \) products, and for each product \( i \) is there an atom \( i \) and a resource \( i \). The (fixed) price for each product is generated as an independent sample from \( U_{[0,1]} \), a uniform distribution on \([0,1]\). At each round, we sample the buyer’s valuation from \( U_{[0,1]} \), independently for each product. If the valuation for a given product is greater than the price, one item of this product is sold (and then reward for atom \( i \) is the price, and consumption of resource \( i \) is 1). Else, we do something different from dynamic assortment: we set reward for atom \( i \) and consumption for resource \( i \) to be the buyer’s valuation.

The second family is inspired by the dynamic pricing application with two products. We have \( n/2 \) allowed prices, uniformly spaced in the \([0,1]\) interval. Recall that atoms correspond to price-product pairs, for the total of \( n \) atoms. In each round \( t \), the valuation \( v_{t,i} \) for each product \( i \) is chosen independently from a normal distribution \( \mathcal{N}(v_0^i, 1) \) truncated on \([0,1]\). The mean valuation \( v_0^i \) is drawn (once for all rounds) from \( U_{[0,1]} \). If the valuation for a given product \( i \) is greater than the offered price \( p \), one item of this product is sold (and then reward for atom \( i \) is the price, and consumption of product \( i \) is 1). If there is no sale for this product, we do something different from dynamic pricing. For each atom \( (p, i) \), if \( p < v_{t,i} \), then the reward for atom \( (p, i) \) is drawn independently from \( U_{[0,1]} \) and resource consumption is 1; else, reward is 0 and consumption is \( 3 \). While dynamic assortment is modeled with a uniform matroid, and dynamic pricing is modeled with a partition matroid, we tried both matroids on each family.

**Experimental setup and results.** We choose various values of \( n, B \) and \( T \) and run our algorithms on the above two datasets assuming both a uniform matroid constraint and a partition matroid constraint. We choose \( n \in \{6, 26\} \), \( T \in \{1000, 2000, 3000, 4000, 5000, 6000\} \) and \( B = T/2 \). The maximum number of atoms in any action is set to \( K = 2 \). For a given algorithm, dataset and configuration of \( n \) and \( T \), we simulate each algorithm for 20 independent runs and take the average. We calculate the total reward obtained by the algorithm at the end of \( T \) time-steps.

Figure [1](#) shows results for \( n = 26 \) (see supplement for \( n = 6 \)). It is clear that our algorithm achieves the best regret among the competitors, as expected.

**Additional experiment.** \( \text{linCBwK} \) and \( \text{pdBwK} \) have running times proportional to the number of actions. We ran an additional experiment which compared per-step running times. We first calculate the average running time for every 10 steps and take the median of 50 such runs. Additionally, since the RRS is randomized, we run it for 30 independent runs and take the majority action. For both Uniform matroid and Partition matroid, we run the faster RRS due to Gandhi et al. [2006]. See Figure [2](#) for results.

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**Figure 1:** Experimental Results for Uniform matroid (left plots) and Partition matroid (right plots) on independent (upper) and correlated (lower) instances for \( n = 26 \).
Figure 2: Variation of per-step running times as $n$ increases for the various algorithms.

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Supplementary materials for “Combinatorial Semi-Bandits with Knapsacks”

This supplement is structured as follows. Section 6 provides full proof of the main result. Section 7 gives the details of the simulations. Finally, we provide two “appendices” for the sake of making this paper more self-contained: we derive Theorem 3.1 in Appendix A and list definitions and special cases of matroids in Appendix B.

All references to Sections 1-5 refer to the main paper; references to subsequent sections refer to this supplement. All citations refer to the bibliography in the main paper.

6 Proof of the main result (Theorem 3.1)

This section presents a detailed and self-contained proof of the main result: Theorem 3.1. We actually prove a slightly stronger statement involving high-probability regret rather than expected regret:

**Theorem 6.1** (main result). **Consider the SemiBwK problem with a linearizable action set \( \mathcal{F} \) that admits a negatively correlated RRS. Then algorithm SemiBwK-RRS with this RRS achieves**

\[
\text{Regret} \leq O(\log(ndT/\delta)) \sqrt{n \left( \text{OPT} / \sqrt{B} + \sqrt{T} + \text{OPT} \right)}
\]

**with probability at least** \( 1 - \delta \). **Here** \( T \) **is the time horizon,** \( n \) **is the number of atoms,** \( B \) **is the budget, and** \( \delta > 0 \) **is a given parameter. Parameter** \( \alpha \) **in the confidence radius is set to** \( \alpha = c_{\text{conf}} \log(ndT/\delta) \), **for a large enough absolute constant** \( c_{\text{conf}} > 0 \). **Parameter** \( \epsilon \) **in the algorithm is set to** \( \epsilon = \sqrt{\frac{\alpha n}{B}} + \frac{\alpha n}{B} + \frac{\sqrt{\alpha n T}}{B} \). **The result holds as long as** \( B > \alpha n + \sqrt{\alpha n T} \).

### 6.1 Linear programs

We argue that \( \text{LP}_{\text{ALG}} \) provides a good benchmark that we can use instead of OPT. Fix round \( t \) and let \( \text{OPT}_{\text{ALG},t} \) denote the optimal value for \( \text{LP}_{\text{ALG}} \) in this round. Then:

**Lemma 6.2.** \( \text{OPT}_{\text{ALG},t} \geq \frac{1}{2}(1 - \epsilon) \text{OPT} \) **with probability at least** \( 1 - \delta \).

We will prove this by constructing a series of LP’s, starting with a generic linear relaxation for BwK and ending with \( \text{LP}_{\text{ALG}} \).

We show that along the series the optimal value does not decrease with high probability.

The first LP, adapted from Badanidiyuru et al. (2013), has one decision variable for each action, and applies generically to any BwK problem.

\[
\begin{align*}
\text{maximize} & \quad \sum_{S \in \mathcal{F}} \mu(S) x(S) \\
\text{subject to} & \quad \sum_{S \in \mathcal{F}} C(S,j) x(S) \leq B/T \quad j = 1, ..., d \\
& \quad 0 \leq \sum_{S \in \mathcal{F}} x(S) \leq 1.
\end{align*}
\]

(LP\_BwK)

Let \( \text{OPT}_{\text{BwK}}(B) \) denote the optimal value of this LP with a given budget \( B \). Then:

**Claim 6.3.** \( \text{OPT}_{\text{BwK}}(B) \geq (1 - \epsilon) \text{OPT}_{\text{BwK}}(B) \geq \frac{1}{2}(1 - \epsilon) \text{OPT} \).

**Proof.** The second inequality in Claim 6.3 follows from (Lemma 3.1 in Badanidiyuru et al., 2013). We will prove the first inequality as follows. Let \( x^* \) denote an optimal solution to \( \text{LP}_{\text{BwK}}(B) \). Consider \( (1 - \epsilon)x^* \); this is feasible to \( \text{LP}_{\text{BwK}}(B) \), since for every \( S \), \( (1 - \epsilon)x^*(S) \leq 1 \) and \( \sum_{S \subseteq A : S \in S} C(S,j)(1 - \epsilon)x(S) \leq B_c/T \). Hence, this is a feasible solution. Now, consider the objective function. Let \( y \) denote an optimal solution to \( \text{LP}_{\text{BwK}}(B) \). We have that

\[
\text{OPT}_{\text{BwK}}(B) = \sum_{S \subseteq A : S \in S} \mu(S) y^*(S) \geq \sum_{S \subseteq A : S \in S} \mu(S)(1 - \epsilon)x^*(S) = (1 - \epsilon) \text{OPT}_{\text{BwK}}(B). \]

Now consider a simpler LP where the decision variables correspond to atoms. As before, \( \mathcal{P} \) denotes the polytope induced by action set \( \mathcal{F} \).

\[
\begin{align*}
\text{maximize} & \quad \mu \cdot x \\
\text{subject to} & \quad C^\top x \preceq B_c/T \quad x \in \mathcal{P} \quad x \in [0,1]^n.
\end{align*}
\]

(LP\_ATOMS)
Here \( C = (C(a, j) : a \in A, j \in d) \) is the \( n \times d \) matrix of expected consumption, and \( C^\top \) denotes its transpose. The notation \( \preceq \) means that the inequality \( \leq \) holds for for each coordinate.

Letting \( \text{OPT}_{\text{atoms}} \) denote the optimal value for LP_{ATOMS}, we have:

**Claim 6.4.** With probability at least \( 1 - \delta \) we have, \( \text{OPT}_{\text{ALG}, t} \geq \text{OPT}_{\text{atoms}} \geq \text{OPT}_{BwK}(B_c) \).

**Proof.** We will first prove the second inequality.

Consider the optimal solution vector \( x \) to LP_{ATOMS}. Define \( S^* := \{ S : x(S) \neq 0 \} \).

We will now map this to a feasible solution of \( x \) to \( \text{LP}_{\text{ATOMS}} \) and show that the objective value does not decrease. This will then complete the claim. Consider the following solution \( y \) defined as follows.

\[
y(a) = \sum_{S \in S^*: a \in S} x(S).
\]

We will now show that \( y \) is a feasible solution to the polytope \( \mathcal{P} \). From the definition of \( y \), we can write it as \( y = \sum_{S \in S^*} x(S) \times \mathbb{I}[S] \). Here, \( \mathbb{I}[S] \) is a binary vector, such that it has 1 at position \( a \) if and only if atom \( a \) is present in set \( S \). Hence, \( y \) is a point in the polytope since it can be written as convex combination of its vertices.

Now, we will show that \( y \) also satisfies the resource consumption constraint.

\[
C(j) \cdot y = \sum_{a \in A} C(a, j) \sum_{S \in S^*: a \in S} x(S) \\
= \sum_{S \in S^*} \sum_{a \in S} C(a, j) x(S) \\
= \sum_{S \in S^*} C(S, j) x(S) \leq B_c / T.
\]

The last inequality is because in the optimal solution, the \( x \) value corresponding to subset \( S^* \) is 1 while rest all are 0. We will now show that \( y \) produces an objective value at least as large as \( x \).

\[
\text{OPT}_{\text{atoms}} = \mu \cdot y^* \geq \mu \cdot y = \sum_{a=1}^{n} \mu(a) \sum_{S \in S^*: a \in S} x(S) \\
= \sum_{S \in S^*} \sum_{a \in S} \mu(a) x(S) = \sum_{S \in S^*} \mu(S) x(S) \\
= \text{OPT}_{\text{subsets}}(B_c).
\]

Now we will prove the first inequality. We will assume the “clean event” that \( \mu^+_t \geq \mu \) and \( C^-_t \leq C_t \) for all \( t \). Hence, the inequality holds with probability at least \( 1 - \delta \).

Consider a time \( t \). Given an optimal solution \( x^* \) to LP_{ATOMS} we will show that this is feasible to LP_{ALG,t}. Note that, \( x^* \) satisfies the constraint set \( x \in \mathcal{P} \) since that is same for both LP_{ALG,t} and LP_{ATOMS}. Now consider the constraint \( C^-_t (j) \cdot x \leq B_t / T \). Note that \( C^-_t (a, j) \leq C(a, j) \). Hence, we have that \( C^-_t (j) \cdot x^* \leq C(j) \cdot x^* \leq B_t / T \). The last inequality is because \( x^* \) is a feasible solution to LP_{ATOMS}.

Now consider the objective function. Let \( y^* \) denote the optimal solution to LP_{ALG,t}.

\[
\text{OPT}_{\text{ALG}, t} = \mu^+_t \cdot y^* \geq \mu^+_t \cdot x^* \geq \mu \cdot y^* = \text{OPT}_{\text{atoms}}. \tag*{\Box}
\]

Hence, combining Claim 6.3 and Claim 6.4 we obtain Lemma 6.2.

### 6.2 Negative correlation and concentration bounds

Our analysis relies on several facts about negative correlation and concentration bounds. First, we argue that property (2.1) in the definition of negative correlation is preserved under a specific linear transformation:
Claim 6.5. Suppose \((X_1, X_2, \ldots, X_m)\) is a family of negatively correlated random variables with support \([0, 1]\). Fix numbers \(\lambda_1, \lambda_2, \ldots, \lambda_m \in [0, 1]\). Consider two families of random variables:

\[
\mathcal{F}^+ = \left( \frac{1 + \lambda_i(X_i - \mathbb{E}[X_i])}{2} : i \in [m] \right) \quad \text{and} \quad \mathcal{F}^- = \left( \frac{1 - \lambda_i(X_i - \mathbb{E}[X_i])}{2} : i \in [m] \right).
\]

Then both families satisfy property (2.1).

Proof. Let us focus on family \(\mathcal{F}^+\); the proof for family \(\mathcal{F}^-\) is very similar.

Denote \(\mu_i = \mathbb{E}[X_i]\) and \(Y_i := (1 + \lambda_i(X_i - \mu_i))/2\) and \(z_i := (1 - \lambda_i\mu_i)/2\) for all \(i \in [m]\). Note that \(Y_i = \lambda_i X_i/2 + z_i\) and \(z_i \geq 0, X_i \geq 0\). Fix a subset \(S \subseteq [m]\). We have,

\[
\mathbb{E} \left[ \prod_{i \in S} Y_i \right] = \mathbb{E} \left[ \prod_{T \subseteq S} \left( \prod_{i \in T} \frac{\lambda_i X_i/2}{2} \right) \prod_{j \in S \setminus T} z_j \right]
\]

by Binomial Theorem

\[
\leq \sum_{T \subseteq S} \mathbb{E} \left[ \prod_{i \in T} \frac{\lambda_i \mu_i/2}{2} \right] \prod_{j \in S \setminus T} z_j \quad \text{(2.1) invariant under non-negative scaling, } X_i \text{ neg. correlated}
\]

\[
= \prod_{i \in S} ((1 - \lambda_i \mu_i)/2 + \lambda_i \mu_i/2)
\]

by Binomial Theorem

\[
= \left( \frac{1}{2} \right)^{|S|} = \prod_{i \in S} \mathbb{E}[Y_i] \quad \square
\]

Second, we extend Theorem 2.1 to a random process that evolves over time, and only assumes that property (2.3) holds within each round conditional on the history.

Theorem 6.6. Let \(Z_T = \{\zeta_{t,a} : a \in A, t \in [T]\}\) be a family of random variables taking values in \([0, 1]\). Assume random variables \(\{\zeta_{t,a} : a \in A\}\) satisfy property (2.1) given \(Z_{t-1}\) and have expectation \(\frac{1}{2}\) given \(Z_{t-1}\), for each round \(t\). Let \(Z = \frac{1}{nT} \sum_{a \in A, t \in [T]} \zeta_{t,a}\) be the average. Then for some absolute constant \(c\),

\[
\Pr[Z \geq \frac{1}{2} + \eta] \leq c \cdot e^{-2m\eta^2} \quad (\forall \eta > 0).
\]

Proof. We prove that family \(Z_t\) satisfies property (2.3), and then invoke Theorem 2.1. Let us restate property (2.3) for the sake of completeness:

\[
\mathbb{E} \left[ \prod_{\{t,a\} \in S} \zeta_{t,a} \right] \leq 2^{-|S|} \quad \text{for any subset } S \subseteq Z_T.
\]

Fix subset \(S \subset Z_T\). Partition \(S\) into subsets \(S_t = \{\zeta_{t,a} \in Z_T \cap S\}\), for each round \(t\). Fix round \(\tau\) and denote

\[
G_{\tau} = \prod_{t \in [\tau]} H_t, \quad \text{where } H_t = \prod_{a \in S_t} \zeta_{t,a}.
\]

We will now prove the following statement by induction on \(\tau\):

\[
\mathbb{E}[G_{\tau}] \leq 2^{-k_{\tau}}, \quad \text{where } k_{\tau} = \sum_{t \in [\tau]} |S_t|.
\]

The base case is when \(\tau = 1\). Note that \(G_{\tau}\) is just the product of elements in set \(\zeta_1\) and they are negatively correlated from the premise. Therefore we are done. Now for the inductive case of \(\tau \geq 2\),

\[
\mathbb{E}[H_{\tau} | Z_{\tau-1}] \leq \prod_{a \in S_{\tau}} \mathbb{E}[\zeta_{\tau,a} | Z_{\tau-1}] \quad \text{From property (2.1) in the conditional space}
\]

\[
\leq 2^{-|S_{\tau}|} \quad \text{From assumption in Lemma 6.6}
\]
Therefore, we have
\[ E[G_{\tau}] = E[E[G_{\tau-1}H_{\tau}|Z_{\tau-1}]] \]
\[ = E[G_{\tau-1}E[H_{\tau}|Z_{\tau-1}]] \]
\[ \leq 2^{-|S_{\tau}|} E[G_{\tau-1}] \]
\[ \leq 2^{-k_{\tau}} \]

This completes the proof of Eq. 6.4. We obtain Eq. 6.3 for \( \tau = T \).

Third, we invoke Eq. 2.6 for rewards and resource consumptions:

**Lemma 6.7.** With probability at least \( 1 - e^{-\Omega(\alpha)} \), we have the following:

\[ |\hat{\mu}(a) - \mu(a)| \leq 2 \text{Rad}(\hat{\mu}(a), \mathcal{N}(a)) + 1 \]
\[ \forall a \in [d] \quad |\hat{C}(a, j) - C(a, j)| \leq 2 \text{Rad}(\hat{C}(a, j), \mathcal{N}(a)) + 1. \]  

(6.7)

Fourth, we use a concentration bound from prior work which gets sharper when the expected sum is very small, and does not rely on independent random variables:

**Theorem 6.8 (Babaioff et al., 2015).** Let \( X_1, X_2, \ldots, X_m \) denote a set of \( \{0, 1\} \) random variables. For each \( t \), let \( \alpha_t \) denote the multiplier determined by random variables \( X_1, X_2, \ldots, X_{t-1} \). Let \( M = \sum_{t=1}^{m} M_t \) where \( M_t = E[X_t|X_1, X_2, \ldots, X_{t-1}] \). Then for any \( b \geq 1 \), we have the following with probability at least \( 1 - m^{-\Omega(b)} \):

\[ |\sum_{t=1}^{m} \alpha_t(X_t - M_t)| \leq b(\sqrt{M \log m} + \log m) \]

6.3 Analysis of the “clean event”

Let us set up several events, henceforth called clean events, and prove that they hold with high probability. Then the remainder of the analysis can proceed conditional on the intersection of these events. The clean events are similar to the ones in [Agrawal & Devanur, 2014b], but are somewhat “stronger”, essentially because our algorithm has access to per-atom feedback and our analysis can use the negative correlation property of the RRS.

In what follows, it is convenient to consider a version of SemiBwK in which the algorithm does not stop, so that we can argue about what happens w.h.p. if our algorithm runs for the full \( T \) rounds. Then we show that our algorithm does indeed run for the full \( T \) rounds w.h.p.

Recall that \( x_t \) be the optimal fractional solution obtained by solving the LP in round \( t \). Let \( Y_t \in \{0, 1\}^n \) be the random binary vector obtained by invoking the RRS (so that the chosen action \( S_t \in \mathcal{F} \) corresponds to a particular realization of \( Y_t \), interpreted as a subset). Let \( \mathcal{G}_t := \{ Y_{t'} : \forall t' \leq t \} \) denote the family of RRS realizations up to round \( t \).

6.3.1 “Clean event” for rewards

For brevity, for each round \( t \) let \( \mu_t = (\mu_t(a) : a \in A) \) be the vector of realized rewards, and let \( r_t := \mu_t(S_t) = \mu_t \cdot Y_t \) be the algorithm’s reward at this round.

**Lemma 6.9.** Consider SemiBwK without stopping. Then with probability at least \( 1 - nT e^{-\Omega(\alpha)} \):

\[ |\sum_{t \in [T]} r_t - \sum_{t \in [T]} \mu_t^{+} \cdot x_t| \leq O\left(\sqrt{\alpha n \sum_{t \in [T]} r_t + \sqrt{\alpha n T + \alpha n}}\right). \]

*Proof.* We prove the Lemma by proving the following three high-probability inequalities.

With probability at least \( 1 - nT e^{-\Omega(\alpha)} \), the following holds:

\[ |\sum_{t \in [T]} r_t - \sum_{t \in [T]} \mu_t^{+} \cdot Y_t| \leq 3nT \text{Rad} \left( \frac{1}{nT} \sum_{t \in [T]} \mu_t^{+} \cdot x_t, nT \right) \]
\[ |\sum_{t \in [T]} \mu_t^{+} \cdot Y_t - \mu_t^{+} \cdot Y_{t'}| \leq 12 \sqrt{\alpha n \sum_{t \in [T]} \mu_t^{+} \cdot x_t} + 12 \sqrt{\alpha n} + 12 \alpha n \]
\[ |\sum_{t \in [T]} \mu_t^{+} \cdot Y_t - \mu_t^{+} \cdot x_t| \leq \sqrt{\alpha n T}. \]

(6.8)
(6.9)
(6.10)
We will use the properties of RRS to prove Eq. 6.10. Proof of Eq. 6.9 is similar to Agrawal & Devanur (2014b), while proof of Eq. 6.8 follows immediately from the setup of the model. Using the parts (6.8) and (6.10) we can now find an appropriate upper bound on \( \sqrt{\sum_{t \in [T]} \mu_t^+ \cdot x_t} \) and using this upper bound, we prove Lemma 6.9.

**Proof of Eq. 6.8** Recall that \( r_t = \mu_t Y_t \). Note that, \( \mathbb{E}[\mu_t Y_t] = \mu Y_t \) when the expectation is taken over just the independent samples of \( \mu \). By Theorem 6.8, with probability \( 1 - e^{-\Omega(a)} \) we have:

\[
| \sum_{t \leq T} r_t - \sum_{t \leq T} \mu \cdot Y_t | \leq 3nT \text{Rad} \left( \frac{1}{nT} \sum_{t \leq T} \mu \cdot Y_t , nT \right)
\]

\[
\leq 3nT \text{Rad} \left( \frac{1}{nT} \sum_{t \leq T} \mu_t^+ \cdot Y_t , nT \right)
\]

\[
\leq 3nT \text{Rad} \left( \frac{1}{nT} \sum_{t \leq T} \mu_t^+ \cdot x_t , nT \right).
\]

The last inequality is because \( Y_t \) is a feasible solution to \( \text{LP}_{\text{ALG}} \).

**Proof of Eq. 6.9** For this part, the arguments similar to Agrawal & Devanur (2014b) follow with some minor adaptations.

For sake of completeness we describe the full proof. Note that we have,

\[
| \sum_{t \leq T} \mu \cdot Y_t - \mu_t^+ \cdot Y_t | \leq \sum_{a=1}^n | \sum_{t \leq T} \mu(a) Y_t(a) - \mu_t^+(a) Y_t(a) |.
\]

Now, using Lemma 6.7 in Appendix, we have that with probability \( 1 - nT e^{-\Omega(a)} \)

\[
| \sum_{t \leq T} \mu(a) Y_t(a) - \mu_t^+(a) Y_t(a) | \leq 12 \sum_{t \leq T} \text{Rad}(\mu(a), N_t(a) + 1).
\]

Hence, we have

\[
\sum_{a=1}^n | \sum_{t \leq T} \mu(a) Y_t(a) - \mu_t^+(a) Y_t(a) | = 12 \sum_{a \in A} \sum_{r=1}^{N_T(a)+1} \text{Rad}(\mu(a), r)
\]

\[
\leq 12 \sum_{a \in A} (N_T(a) + 1) \text{Rad}(\mu(a), N_T(a) + 1)
\]

\[
\leq 12 \sqrt{n} \alpha (\mu \cdot (N_T + 1)) + 12 \alpha n.
\]

The last inequality is from the definition of \text{Rad} function and using the Cauchy-Schwartz inequality. Note that \( \mu N_T = \sum_{t \leq T} \mu \cdot Y_t \). Also, since we have with probability \( 1 - e^{-\Omega(a)} \), \( \mu(a) \leq \mu_t^+(a) \), we have,

\[
12 \sqrt{n} \alpha \mu \cdot (N_T + 1) + 12 \alpha n \leq 12 \sqrt{n} \alpha (\sum_{t \leq T} \mu_t^+ \cdot Y_t) + 12 \sqrt{n} + 12 \alpha n.
\]

Finally note that \( Y_t \) is a feasible solution to the semi-bandit polytope \( \mathcal{P} \). Hence, we have that

\[
\mu_t^+ \cdot Y_t \leq \mu_t^+ \cdot x_t.
\]

Hence,

\[
12 \sqrt{n} \alpha (\sum_{t \leq T} \mu_t^+ \cdot Y_t) + 12 \sqrt{n} + 12 \alpha n \leq 12 \sqrt{n} \alpha (\sum_{t \leq T} \mu_t^+ \cdot x_t) + 12 \sqrt{n} + 12 \alpha n.
\]

**Proof of Eq. 6.10** Recall that for each round \( t \), the UCB vector \( \mu_t^+ \) is determined by the random variables \( G_{t-1} = \{ Y_t : \forall t' < t \} \). Further, conditional on a realization of \( G_{t-1} \), the random variables \( \{ Y_t(a) : a \in A \} \) are negatively correlated from the property of RRS. Let \( \zeta_t(a) := \mu_t^+(a) Y_t(a), a \in A \). Note that we have \( \mathbb{E}[\zeta_t(a)|G_{t-1}] = \mu_t^+(a) x_t(a) \). Define

\[
\zeta_t(a) := \frac{1 + \mu_t^+(a) Y_t(a) - \mu_t^+(a) x_t(a)}{2}.
\]

From Claim 6.5 we have that \( \{ \zeta_t(a) : a \in A \} \) conditioned on \( G_{t-1} \) satisfy (2.1). Further, \( \mathbb{E}[\zeta_t(a)|G_{t-1}] = \frac{1}{2} \). Therefore, the family \( \{ \zeta_t(a) : t \in [T], a \in A \} \) satisfies the assumptions in Theorem 6.6 and hence satisfies Eq. 6.2 for some absolute constant \( c \). Plugging back the \( \zeta_t(a) \)'s, we obtain an upper-tail concentration bound:

\[
\Pr \left[ \frac{1}{nT} \sum_{t=1}^T \sum_{a \in A} \zeta_t(a) - \mu_t^+(a) x_t(a) \geq \eta \right] \leq c \cdot e^{-2 \alpha n T \eta^2}.
\]
To obtain a corresponding concentration bound for the lower tail, we apply a similar argument to
\[ \zeta'_t(a) = \frac{1 + \mu^+_t(a)x_t(a) - \tilde{\zeta}_t(a)}{2}. \]

Once again from Claim 6.5 we have that \( \{\zeta'_t(a) : a \in A\} \) conditioned on \( \mathcal{F}_t \) satisfy (2.1). The family \( \{\zeta'_t(a) : t \in [T], a \in A\} \) satisfies the assumptions in Theorem 6.6 and hence satisfies Eq. 6.2. Plugging back the \( \zeta'_t(a) \)'s, we obtain a lower-tail concentration bound:
\[ \Pr \left[ \frac{1}{nT} \left( \sum_{t=1}^T \sum_{a \in A} \mu^+_t(a)x_t(a) - \tilde{\zeta}_t(a) \right) \geq \eta \right] \leq c \cdot e^{-2nT\eta^2}. \]

Combining these two we have,
\[ \Pr \left[ \frac{1}{nT} \left| \sum_{t=1}^T \sum_{a \in A} \mu^+_t(a)Y_t(a) - \mu^+_t(a)x_t(a) \right| \geq \eta \right] \leq 2c \cdot e^{-2nT\eta^2}. \]

Hence setting \( \eta = \sqrt{\frac{n}{nT}} \), we obtain Eq. 6.10 with probability at least \( 1 - e^{-\Omega(\alpha)} \).

**Combining Eq. (6.8), (6.9) and (6.10)** Let us denote \( H := \sqrt{\sum_{t \in [T]} \mu^+_t \cdot x_t} \). Adding the three equations we get
\[ |\sum_{t \in [T]} r_t - H^2| \leq \sqrt{\alpha H + \alpha + \sqrt{\alpha n}H + O(\alpha n) + \sqrt{\alpha nT}}. \]  

Rearranging and solving for \( H \), we have
\[ H \leq \sqrt{\sum_{t \in [T]} r_t} + O(\sqrt{\alpha n}) + (\alpha nT)^{1/4}. \]

Plugging this back into Eq. 6.12, we get Lemma 6.9.

### 6.3.2 “Clean event” for resource consumption

We define a similar “clean event” for consumption of each resource \( j \). By a slight abuse of notation, for each round \( t \) let \( C_t(j) = (C_t(a, j) : a \in A) \) be the vector of realized consumption of resource \( j \). Let \( \chi_t(j) \) denote algorithm’s consumption for resource \( j \) at round \( t \) (i.e., \( \chi_t(j) = C_t(j) \cdot Y_t \)).

**Lemma 6.10.** Consider SemiBwK without stopping. Then with probability at least \( 1 - nT e^{-\Omega(\alpha)} \):
\[ \forall j \in [d] \quad |\sum_{t \in [T]} \chi_t(j) - \sum_{t \in [T]} C_t^+(j) \cdot x_t| \leq \sqrt{\alpha n}B_n + \alpha n + \sqrt{\alpha n T}. \]

**Proof.** The proof is similar to Lemma 6.9. We will split the proof into following three equations. Fix an arbitrary resource \( j \in [d] \). With probability at least \( 1 - nT e^{-\Omega(\alpha)} \) the following holds:
\[ |\sum_{t \leq T} \chi_t(j) - \sum_{t \leq T} C_t(j) \cdot Y_t| \leq 3nT \text{ Rad} \left( \frac{1}{nT} \sum_{t \leq T} C_t(j) \cdot Y_t , nT \right). \]  

\[ |\sum_{t \leq T} C_t(j) \cdot Y_t - C_t^+(j) \cdot x_t| \leq 12 \sqrt{\alpha n \left( \sum_{t \leq T} C_t(j) \cdot Y_t \right)} + 12 \sqrt{\alpha n} + 12 \alpha n. \]  

\[ |\sum_{t \leq T} C_t^-(j) \cdot Y_t - C_t^-(j) \cdot x_t| \leq \sqrt{\alpha n T}. \]

Using the parts 6.13, 6.14 and 6.15 we can find an upper bound on \( \sqrt{\sum_{t \leq T} C_t(j) \cdot Y_t} \). Hence, combining Lemmas 6.13, 6.14 and 6.15 with this bound and taking an Union Bound over all the resources, we get Lemma 6.10.

**Proof of Eq. 6.13** We have that \( \{C_t(a, j) : a \in A\} \) is a set of independent random variables over a probability space \( C_t \). Note that, \( \mathbb{E}_{C_t} C_t(a, j)Y_t(a) = C_t(a, j)Y_t(a) \). Hence, we can invoke Theorem 6.8 on independent random variables to get with probability \( 1 - nT e^{-\Omega(\alpha)} \).
\[ |\sum_{t\leq T} \chi_t(j) - \sum_{t\leq T} C(j) \cdot Y_t| \leq 3nT \text{Rad} \left( \frac{1}{nT} \sum_{t\leq T} C(j) \cdot Y_t, nT \right). \]

**Proof of Eq. 6.14** This is very similar to proof of 6.9 and we will skip the repetitive parts. Hence, we have with probability \(1 - nT e^{-\Omega(n)}\)

\[ |\sum_{t\leq T} C(j) \cdot Y_t - C_1(j) \cdot Y_t| \leq 12\sqrt{\alpha n (C(j) \cdot (N_T + 1) + 12an} \]

\[ \leq 12\sqrt{\alpha n (C(j) \cdot Y_t)} + 12\sqrt{\alpha n} + 12an. \]

**Proof of Eq. 6.15** Recall that for each round \(t\) and each resource \(j\), the LCB vector \(C_t(j)\) is determined by the random variables \(G_{t-1} = \{Y_t' : \forall t' < t\}\). Similar to the proof of Eq. 6.10, random variables \(\{Y_t(a) : a \in A\}\) obtained from the RRS are negatively correlated given \(G_{t-1}\). As before define \(\zeta_t(a) = C_t(a) Y_t(a), a \in A\). We have that \(\mathbb{E}[\zeta_t(a) | G_{t-1}] = C_t(a) x_t(a)\).

By Claim 6.5 random variables

\[ \zeta_t(a) = \frac{1 + \tilde{\zeta}_t(a) - C_t^{-1}(a) x_t(a)}{2}, a \in A \]

satisfy (2.1), given \(G_{t-1}\). We conclude that family \(\{\zeta_t(a) : t \in [T], a \in A\}\) satisfies the assumptions in Theorem 6.6 and therefore satisfies Eq. 6.2 for some absolute constant \(c\). Therefore, we obtain an upper-tail concentration bound for \(\zeta_t(a)\)’s:

\[ \text{Pr} \left[ \frac{1}{nT} (\sum_{t=1}^{T} \sum_{a \in A} \tilde{\zeta}_t(a) - C_t^{-1}(a) x_t(a)) \geq \sqrt{\alpha n} \right] \leq c \cdot e^{-2nT\eta^2}. \]

To obtain a corresponding concentration bound for the lower tail, we apply a similar argument to

\[ \zeta'_t(a) = \frac{1 + C_t^{-1}(a) x_t(a) - \tilde{\zeta}_t(a)}{2}. \]

Once again, invoking Claim 6.5 we have that \(\{\zeta'_t(a) : a \in A\}\) conditioned on \(G_{t-1}\) satisfy (2.1). Thus, family \(\{\zeta_t(a) : t \in [T], a \in A\}\) satisfies the assumptions in Theorem 6.6 and therefore satisfies Eq. 6.2. We obtain:

\[ \text{Pr} \left[ \frac{1}{nT} (\sum_{t=1}^{T} \sum_{a \in A} C_t^{-1}(a) x_t(a) - \tilde{\zeta}_t(a)) \geq \sqrt{\alpha n} \right] \leq c \cdot e^{-2nT\eta^2}. \]

Combing the two tails we have,

\[ \text{Pr} \left[ \frac{1}{nT} \sum_{t=1}^{T} \sum_{a \in A} C_t^{-1}(a) Y_t(a) - C_t^{-1}(a) x_t(a) \right) \geq \sqrt{\alpha n} \right] \leq 2c \cdot e^{-2nT\eta^2}. \quad (6.16) \]

Once again, setting \(\eta = \sqrt{\frac{\alpha n}{T}}\), we obtain Eq. 6.15 with probability at least \(1 - e^{-\Omega(n)}\).

**Proof of Lemma 6.10** Denote \(G = \sqrt{\sum_{t \leq T} C(j) \cdot Y_t}\). From Equation 6.13, 6.14 and 6.15 we have that \(G^2 - 2\Omega(\sqrt{\alpha n}) \leq \sum_{t \leq T} C_t^{-1}(j) \cdot x_t + O(\alpha n) + \sqrt{\alpha nT}\). Note that \(\sum_{t \leq T} C_t^{-1}(j) \cdot x_t \leq B_e\). Hence, \(G^2 - 2\Omega(\sqrt{\alpha n}) \leq B_e + O(\alpha n) + \sqrt{\alpha nT}\). Hence, re-arranging this gives us \(G \leq \sqrt{B_e} + O(\sqrt{\alpha n}) + (\alpha nT)^{1/4}\). Plugging this back in Equations 6.13, 6.14 and 6.15 we get Lemma 6.10.

\[ \square \]
6.4 Putting it all together

Similar to Agrawal & Devanur (2014b), we will handle the hard constraint on budget, by choosing an appropriate value of $\epsilon$. We then combine the above Lemma on "rewards" clean event to compare the reward of the algorithm with that of the optimal value of LP to obtain the regret bound in Theorem 6.1. Additionally, we use the Lemma on "consumption" clean event to argue that the algorithm doesn’t exhaust the resource budget before round $T$. Formally, consider the following.

Recall that from Lemma 6.2, we have $\text{OPT} \geq \frac{1}{T} (1 - \epsilon) \text{OPT}$. Let us define the performance of the algorithm as $\text{ALG} = \sum_{t \leq T} r_t$. From Lemma 6.9, that with probability at least $1 - nd T e^{-\Omega(\alpha)}$

$$\text{ALG} \geq (1 - \epsilon) \text{OPT} - O(\alpha n \text{OPT}) - \sqrt{\alpha n T}$$

(since $\text{ALG} \leq \text{OPT}$).

Choosing $\epsilon = \sqrt{\frac{\alpha n}{B}} + \frac{\alpha n}{B} + \sqrt{\frac{\alpha n T}{B}}$ and using the assumption that $B > \alpha n + \sqrt{\alpha n T}$, we derive Eq. 6.1 For any given $\delta$, we set $\alpha = \Omega(\log(\frac{nd T}{\delta}))$ to obtain a success probability of at least $1 - \delta$.

Now we will argue that the algorithm does not exhaust the resource budget before round $T$ with probability at least $1 - nd T e^{-\Omega(\alpha)}$. Note that for every resource $j \in [d]$,

$$\sum_{t \leq T} C_t^-(j) \cdot x_t \leq (1 - \epsilon) B.$$

Hence, combining this with Lemma 6.10 we have $\sum_{t \leq T} C_t(j) Y_t \leq (1 - \epsilon) B + \epsilon B \leq B$.

7 Details for Numerical Simulations

Details of heuristic implementation of linCBwK. We now briefly describe the heuristic we use to simulate the linCBwK algorithm. Note that even though the per-time-step running time of linCBwK is reasonable, it takes a significant time when we want to perform simulations for many time-steps. The time-consuming step in the linCBwK algorithm is the solution to a convex program for computing the optimistic estimates (namely $\tilde{\mu}_t$ and $\tilde{W}_t$). Hence, the heuristic gives a faster way to obtain this estimate. We sample multiple times from a multi-variate Gaussian with mean $\hat{\mu}$ and covariance $M_t$ (to obtain estimate $\tilde{\mu}$) and with mean $\hat{w}_t$ and covariance $M_t$ (to obtain estimate $\tilde{w}_t$ for each resource $j$). We use these samples to compute the objective to choose the action at time-step $t$. For each sample, we compute the best action based on the objective in linCBwK. We finally choose the action that occurs majority number of times in these samples. The number of samples we choose is set to 30.

Language Details of algorithms. All algorithms except linCBwK were implemented in Python. The linCBwK algorithm was implemented in MATLAB. This difference is crucial when we compare running times since language construct can speed-up or slow down algorithms in practice. However, it is known that 9 for matrix operations commonly encountered in engineering and statistics, MATLAB implementations run several orders of magnitude faster than the corresponding python implementation. Since linCBwK is the slowest of the four algorithms, our comparison of running times across languages is justified.

Further results. We now show additional plots omitted in the main section in Figure 3. In particular, we show the variation of rewards of various algorithms when $n = 6$. As before, our algorithm has the best performance across the algorithms in all settings.

9https://www.mathworks.com/products/matlab/matlab-vs-python.html
Figure 3: Experimental Results for Uniform matroid (left plots) and partition matroid (right plots) on independent (upper) and correlated (lower) instances for $n = 6$. 
A Proof of Theorem 2.1

Theorem 2.1 follows easily from Theorem 3.3 in [Impagliazzo & Kabanets, 2010].

Theorem (Theorem 2.1). Let $\mathcal{X} = (X_1, X_2, \ldots, X_m)$ denote a collection of random variables which take values in $[0, 1]$, and let $X := \frac{1}{m} \sum_{i=1}^m X_i$ be their average. Suppose $\mathcal{X}$ satisfies (2.3), i.e., $E[\prod_{i \in S} X_i] \leq (\frac{1}{2})^{|S|}$ for every $S \subseteq [m]$. Then for some absolute constant $c$,

$$\Pr[X \geq \frac{1}{2} + \eta] \leq c \cdot e^{-2m\eta^2} \quad (\forall \eta > 0).$$

(A.1)

Proof. Fix $\eta > 0$. From Theorem 3.3 in [Impagliazzo & Kabanets, 2010], we have that

$$\Pr[X \geq \frac{1}{2} + \eta] \leq c \cdot e^{-mD_{KL}(1/2+\eta \| 1/2)},$$

where $D_{KL}(\cdot \| \cdot)$ denotes KL-divergence, so that

$$D_{KL}(\frac{1}{2} + \eta \| \frac{1}{2}) = (\frac{1}{2} + \eta) \log(1 + 2\eta) + (\frac{1}{2} - \eta) \log(1 - 2\eta).$$

(A.2)

From Taylor series expansion we have,

$$\log(1 + x) = x - x^2/2 + x^3/3 + \ldots$$

$$\log(1 - x) = -x - x^2/2 - x^3/3 \ldots \ .$$

Plugging this into (A.2), we derive $D_{KL}(1/2 + \eta \| 1/2) \geq 2\eta^2$, which implies (A.1). \qed

B Matroid constraints

To make this paper more self-contained, we provide more background on matroid constraints and special cases thereof.

Recall that in $\SemiBreak$, we have a finite ground set whose elements are called “atoms”, and a family $\mathcal{F}$ of “feasible subsets” of the ground set which are the actions. To be consistent with the literature on matroids, the ground set will be denoted $E$. Family $\mathcal{F}$ of subsets of $E$ is called a matroid if it satisfies the following properties:

- **Empty set**: The empty set $\phi$ is present in $\mathcal{F}$
- **Hereditary property**: For two subsets $X, Y \subseteq E$ such that $X \subseteq Y$, we have that
  $$Y \in \mathcal{F} \implies X \in \mathcal{F}$$
- **Exchange property**: For $X, Y \in \mathcal{F}$ and $|X| > |Y|$, we have that
  $$\exists e \in X \setminus Y : Y \cup \{e\} \in \mathcal{F}$$

Matroids are **linearizable**, i.e., the convex hull of $\mathcal{F}$ forms a polytope in $\mathbb{R}^E$. (Here subsets of $\mathcal{F}$ are interpreted as binary vectors in $\mathbb{R}^E$.) In other words, there exists a set of linear constraints whose set of feasible integral solutions is $\mathcal{F}$. In fact, the convex hull of $\mathcal{F}$, a.k.a. the **matroid polytope**, can be represented via the following linear system:

$$\begin{align*}
x(S) \leq \text{rank}(S) & \quad \forall S \subseteq E \\
x_e \in [0, 1]^E & \quad \forall e \in E.
\end{align*}$$

(LP-Matroid)

Here $x(S) := \sum_{e \in S} x_e$, and $\text{rank}(S) = \max\{|Y| : Y \subseteq S, Y \in \mathcal{F}\}$ is the “rank function” for $\mathcal{F}$.

$\mathcal{F}$ is indeed the set of all feasible integral solutions of the above system. This is a standard fact in combinatorial optimization, e.g., see Theorem 40.2 and its corollaries in [Schrijver, 2002].

We will now describe some well-studied special cases of matroids. That they indeed are special cases of matroids is well-known, we will not present the corresponding proofs here.

In all LPs presented below, we have variables $x_e$ for each atom $e \in E$, and we use shorthand $x(S) := \sum_{e \in S} x_e$ for $S \subseteq E$. 
Cardinality constraints. Cardinality constraint is defined as follows: a subset $S$ of atoms belongs to $\mathcal{F}$ if and only if $|S| \leq K$ for some fixed $K$. This is perhaps the simplest constraint that our results are applicable to. In the context of SemiBwK, each action selects at most $K$ atoms.

The corresponding induced polytope is as follows:

$$
\begin{align*}
    x(E) & \leq K \\
    x_e & \in [0, 1] \quad \forall e \in E.
\end{align*}
$$

Partition matroid constraints. A generalization of cardinality constraints, called partition matroid constraints, is defined as follows. Suppose we have a collection $B_1, \ldots, B_k$ of disjoint subsets of $E$, and numbers $d_1, \ldots, d_k$. A subset $S$ of atoms belongs to $\mathcal{F}$ if and only if $|S \cap B_i| \leq d_i$ for every $i$. Partition matroid constraints appear in several applications of SemiBwK such as dynamic pricing, adjusting repeated auctions, and repeated bidding. In these applications, each action selects one price/bid for each offered product. Also, partition matroid constraints can model clusters of mutually exclusive products in dynamic assortment application.

The induced polytope is as follows:

$$
\begin{align*}
    x(B_i) & \leq d_i \quad \forall i \in [k] \\
    x_e & \in [0, 1] \quad \forall e \in E.
\end{align*}
$$

Spanning tree constraints. Spanning tree constraints describe spanning trees in a given undirected graph $G = (V, E)$, where the atoms correspond to edges in the graph. A spanning tree in $G$ is a subset $E' \subset E$ of edges such that $(V, E')$ is a tree. Action set $\mathcal{F}$ consists of all spanning trees of $G$.

The induced polytope is as follows:

$$
\begin{align*}
    x(E_S) & \leq |S| - 1 \quad \forall S \subseteq V \\
    x(E_V) & = |V| - 1 \\
    x_e & \in [0, 1] \quad \forall e \in E.
\end{align*}
$$

Here, $E_S$ denotes the edge set in subgraph induced by node set $S \subseteq V$. 

\textbf{Combinatorial Semi-Bandits with Knapsacks}