Jordan Blocks of Richardson Classes in the Classical Groups and the Bala-Carter Theorem

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Abstract

This paper provides new, relatively simple proofs of some important results about unipotent classes in simple linear algebraic groups. We derive the formula for the Jordan blocks of the Richardson class of a parabolic subgroup of a classical group. This result was originally due to Spaltenstein. Secondly, we derive, for good characteristic, the description of the natural partial order of unipotent classes of a classical group in terms of their Jordan blocks. This result was originally due to Gerstenhaber and Hesselink. As a consequence we obtain a proof of the Bala-Carter Theorem which holds even in certain bad characteristics (this proof requires the prior classification of unipotent classes, unlike the original proofs due to Bala, Carter and Pommerening).

Keywords: classical groups, unipotent classes, Richardson classes, partial order of unipotent classes, Jordan blocks, Bala-Carter Theorem.

1 Introduction

Let $G$ be a connected reductive linear algebraic group. Richardson \cite{Richardson} made a vital contribution to the study of unipotent classes in algebraic groups by associating to each parabolic subgroup of $G$ a unipotent class of $G$. This result has had surprisingly powerful implications, some of which we will discuss below. The following theorem is one version of Richardson’s result (see also \cite{Bala} or \cite{Carter} for other proofs).
Richardson’s Theorem. Let $G$ be a connected reductive group, $P$ a parabolic subgroup with unipotent radical $Q$ and Levi factor $L$. The following hold:

(i) There exists a unique unipotent $G$-class $C$ such that $C \cap Q$ is open and dense in $Q$.

(ii) $C \cap Q$ forms a single $P$-class.

(iii) If $u \in C \cap Q$ then $C_G(u)^0 = C_P(u)^0$, whence these centralizers have dimension $\dim L$.

(iv) Let $Z$ be the center of $G$, let $Q'$ be the derived subgroup of $Q$. Then $\dim L/Z \geq \dim Q/Q'$.

Spaltenstein [17] studies generalizations of this result to the case where $G$ is non-connected. We will always have $G$ connected except when $G = O_n$.

We call $C$ the Richardson class of $P$ and we call $C \cap Q$ the Richardson orbit in $Q$.

For many questions it is of fundamental importance to be able to find the Jordan blocks of a unipotent class. The next result indicates how to do this for Richardson classes, but first we introduce some standard notation.

A partition of $n$ is a sequence of natural numbers which add to $n$; we assume that the sequence is weakly decreasing unless indicated otherwise. We write a partition $\lambda$ as $(\lambda_1, \lambda_2, \ldots)$ or $(\lambda_1 \geq \lambda_2 \geq \ldots)$ or $(1^{c(1)}, 2^{c(2)}, 3^{c(3)}, \ldots)$ where $c(x)$ is the multiplicity of $x$ in $\lambda$. We call $\lambda_i$ a part of $\lambda$. The dual of $\lambda$ is a partition of $n$ which we write as $\lambda^*$ and which has parts defined as follows: $\lambda^*_i$ equals the number of parts of $\lambda$ which are greater than or equal to $i$ (i.e. $\lambda^*_i$ equals the number of indices $j$ such that $\lambda_j \geq i$).

Let $G \in \{SO_{2n}, SO_{2n+1}, Sp_{2n}\}$. We fix a root base $\Delta$ for $G$ and label the nodes $\{\alpha_1, \ldots, \alpha_n\}$ as in [3]. Given a parabolic subgroup $P$ let $J \subseteq \Delta$ such that $P$ is conjugate to the standard parabolic associated with $J$. If $\alpha_n \in J$ then let $m$ be the largest integer such that such that the last $m$ nodes $\alpha_{n-m+1}, \ldots, \alpha_n$ are contained in $J$. Then the Levi factor $L$ of $P$ can be written as $L = GL_{n_1} \cdots GL_{n_s} Cl_m$ where $Cl_m \in \{SO_{2m}, SO_{2m+1}, Sp_{2m}\}$. We extend this notation also to the cases $\alpha_n \notin J$ and to $G = GL_n$ by taking $Cl_m = 1$ and $m = 0$. In this way each parabolic subgroup of $GL_n$, $SO_{2n}$, $SO_{2n+1}$, $Sp_{2n}$ determines a partition $n = n_1 + \cdots + n_s + m$. We call this the Levi partition and write it either as $\Lambda = (n_1, \ldots, n_s) \oplus m$ or $\Lambda = (1^{c(1)}, 2^{c(2)}, \ldots) \oplus m$ where $c(x)$ is the multiplicity of the part $x$ in the $n_i$ and the notation “$\oplus$” indicates that we have an ordered pair consisting of the partition $(n_1, \ldots, n_s)$ and the number $m$. 

Table 1: Jordan blocks of a Richardson class

| $G = \text{GL}_n$, $\psi$ is the identity | $G = \text{SO}_{2n}$, $p \neq 2$ | $G = \text{SO}_{2n+1}$, $p \neq 2$ | $G = \text{Sp}_{2n}$ |
|-------------------------------------------|----------------------------------|---------------------------------|---------------------|
| $\psi(m) = 2m$                            | $\psi(j^{(i)}) = j^{2c(j)}$      | $\psi(m) = 2m + 1$             | $\psi(m) = 2m$     |
| $\psi(j^{(i)}) = j^{2c(j)}$               | if $j$ is even or $j \leq 2m$    | $\psi(j^{(i)}) = j^{2c(j)}$    | $\psi(j^{(i)}) = j^{2c(j)}$ |
| $\psi(j^{(i)}) = j + 1$, $j^{2c(j) - 2}$, $j - 1$ | if $j$ is odd and $j > 2m$       | if $j$ is even and $j > 2m + 1$ | if $j$ is even or $j \geq 2m$ |
| $\psi(j^{(i)}) = (j + 1)^2$, $j^{2c(j) - 4}$, $(j - 1)^2$ | if $j$ is odd, $j > 2m$ and $c(j) = 1$ | if $j$ is even and $j > 2m + 1$ | $\psi(j^{(i)}) = j + 1$, $j^{2c(j) - 2}$, $j - 1$ |
| $\psi(j^{(i)}) = j^{2c(j)}$               | if $j$ is odd, $j > 2m$ and $c(j) \geq 2$ |                                 | if $j$ is odd and $j < 2m$ |
Theorem 1 ([17, II.7.4]). Let $G$ be one of $\text{GL}_n$, $\text{SO}_{2n}$, $\text{SO}_{2n+1}$, $\text{Sp}_{2n}$ and exclude the case $\text{SO}_{2n+1}$ if $p = 2$. Let $P$ be a parabolic subgroup of $G$, let $\Lambda$ be the Levi partition of $P$, define the map $\psi$ as in table 1 and let $\lambda$ be the partition of Jordan block sizes of the Richardson class of $P$. Then $\lambda$ equals $\psi(\Lambda)^*$, the dual of $\psi(\Lambda)$.

Remarks 1.1. (a) It is easy to extend this result to the case of $G = \text{SO}_{2n+1}$ and $p = 2$. One applies the formula for $\text{Sp}_{2n}$ using the same Levi partition and adds one block of size 1 to the result. (b) Spaltenstein’s formulas appear rather different from those presented here (and have some minor mistakes). (Spaltenstein also determines the index $\varepsilon$ in his notation or, equivalently, the singularity of the parts of $\lambda$. See [5] for a discussion of this notation.)

Let $C_1$ and $C_2$ be two unipotent classes of $G$. We define $C_1 \leq C_2$ if and only if $C_1 \subseteq \overline{C_2}$ (where $\overline{C_2}$ is the closure of $C_2$). This is the natural partial order on unipotent classes.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be two partitions. We define $\lambda \leq \mu$ if and only if for all $j \geq 1$ we have $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$. This is the dominance partial order on partitions.

Theorem 2 ([7], [6], [17, I.2.10]). Let $G \in \{\text{GL}_n, \text{O}_n, \text{SO}_n, \text{Sp}_n\}$. Let $C_\lambda$ and $C_\mu$ be two unipotent classes in $G$ with $\lambda$ and $\mu$ the partitions consisting of the Jordan blocks of $C_\lambda$ and $C_\mu$ respectively. Assume either that $p \neq 2$ if $G \neq \text{GL}_n$ or that $\mu$ has no even parts with even multiplicity. Then $\lambda < \mu$ if and only if $C_\lambda < C_\mu$.

Spaltenstein [17, I.2.10] generalizes this result to all unipotent classes in bad characteristics, but we will not discuss his generalization here.

We now introduce the necessary terminology to state the Bala-Carter Theorem which gives a parameterization of unipotent classes in simple algebraic groups.

Let $G$ be a connected reductive algebraic group with root system $\Phi$ and root base $\Delta$. Fix $J \subseteq \Delta$ and let $P$ be the standard parabolic subgroup corresponding to $J$. Let $\beta \in \Phi$ and write $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$. The $P$-height is defined to be $h_P(\beta) = \sum_{\alpha \in \Delta - J} n_\alpha$.

Let $L$ be a Levi factor for $P$, $Q$ the unipotent radical of $P$, and $\Phi(Q)$ the roots of $Q$. We say $P$ is distinguished if $\dim L/Z(G)$ equals the number of roots in $\Phi(Q)$ with $P$-height equal to 1.

If $Q'$ is the derived subgroup of $Q$ then Richardson’s Theorem (iv) implies that $\dim L/Z(G) \geq \dim Q/Q'$ for all $P$. If $P$ is distinguished then

$$\dim L/Z(G) = \dim Q/Q'. \quad (*)$$
The converse holds provided $p \neq 2$ if the Dynkin diagram of $G$ contains double bonds, and $p \neq 3$ if the Dynkin diagram of $G$ contains triple bonds (see [2] or [1]).

The work in [2], [4] etc. takes condition (*) as the definition of distinguished, but then applies this definition only with the restrictions on $p$ just described. Thus, the definition we have given here takes the usual list of distinguished parabolics and uses this same list even when $p$ equals 2 or 3. We refer the reader to [4] for a list of the distinguished parabolics (note however that there is a mistake in the second formula for $D_n$).

Throughout this paper, a **Levi subgroup** means a Levi factor of a parabolic subgroup. Let $L$ be a Levi subgroup of $G$ and $u \in L$ a unipotent element. Then $u$ is **distinguished** in $L$ if $u$ is not contained in any proper Levi subgroup of $L$. If $L = G$ has trivial center this is equivalent to having $C_G(u)$ contain no non-trivial torus (see Lemma 5.1 below).

For a reductive group $G$ let $\text{BC-pairs}(G)$ denote the pairs $(L,P)$ where $L$ is a Levi subgroup and $P$ is a distinguished parabolic subgroup of $L$. Let $\psi$ (or $\psi_G$) denote the map from such pairs to unipotent classes in $G$ obtained by extending the Richardson class of $P$ (in $L$) to a $G$-class.

**Theorem 3** (Bala-Carter [2], Pommerening [13]). Let $G$ be a simple algebraic group and let $\psi = \psi_G$ be as just defined. The following hold:

(i) If $X$ is a Levi subgroup the following diagram commutes:

\[
\begin{array}{ccc}
X\text{-classes in } \text{BC-pairs}(X) & \overset{\psi_X}{\longrightarrow} & \text{unipotent classes in } X \\
\downarrow & \circ & \downarrow \\
G\text{-classes in } \text{BC-pairs}(G) & \overset{\psi}{\longrightarrow} & \text{unipotent classes in } G
\end{array}
\]

where the vertical maps extend an $X$-class to the corresponding $G$-class.

(ii) Let $\psi(L,P) = C$ and $u \in C \cap L$. Then $u$ is distinguished in $L$.

(iii) The map $\psi$ is injective. It is a bijection except in the following cases: $G \in \{B_n, C_n, D_n\}$ and $p = 2$; $(G,p)$ is one of $(E_7, 2)$, $(E_8, 2)$, $(E_8, 3)$, $(F_4, 2)$ or $(G_2, 3)$ in which cases there are 1, 4, 1, 4 and 1 extra classes respectively.

**Remarks 1.2.** Although part (i) is obvious, we state it here to bring attention to some of the following points. (a) Part (i) makes the Bala-Carter Theorem more useful than Jordan blocks for comparing unipotent classes in $X$ and unipotent classes in $G$. For example, let $G = E_6$, $X = D_5T_1$ and...
let $C$ be the unipotent class of $X$ which has two Jordan blocks of size 5 in the natural module for $SO_{10}$. The Jordan blocks do not make it clear which class $C$ corresponds to in $G$. However, the Bala-Carter label for $C$ is $A_4$ (i.e. $C$ is represented by a regular element of a Levi subgroup of type $A_4$) both as a class of $X$ and when it is extended to a class of $G$. (b) If $X$ is a maximal rank reductive subgroup which is not a Levi subgroup, one may often still obtain a commutative diagram similar to that in part (i). For instance let $G = E_6$, $X = A_2 A_2 A_2$ and $(L, P) \in BC\text{-pairs}(X)$ where $L$ is a proper Levi subgroup of $X$. Then $(L, P) \in BC\text{-pairs}(G)$ and the same result is obtained if one first extends $(L, P)$ to a $G$-class and then takes the unipotent $G$-class, or if one first takes the unipotent $X$-class and then extends this to a $G$-class. (c) Parts (i) and (iii) show that in most cases the intersection of a unipotent $G$-class with a Levi subgroup forms a single unipotent class for the Levi subgroup. If this is not the case then $G = E_r$, $L$ is of type $D_n$ and the unipotent class is of type $A_{n-1}$. (d) Part (iii) is a stronger version of the Bala-Carter-Pommerening Theorem than usually appears (as in the references above or [4], [8]), although this version seems to be known or assumed by specialists in the field (see, for example, [10]). In addition, the proof given in this paper (see Proof 5.4) uses the classification of unipotent classes for each simple algebraic group whereas the standard proof (as in [4]) constructs the inverse of $\psi$ (at the level of the Lie algebra) and is independent of these classifications.

2 Recollections and Conventions

All algebraic groups in this paper are affine and defined over a fixed algebraically closed field of characteristic $p \geq 0$.

The groups $SO_{2n}$, $SO_{2n+1}$ and $Sp_{2n}$ are defined in terms of a bilinear form and a quadratic form which we will usually denote by $\beta$ and $\varphi$ respectively. Let $V$ be the natural module for one of these groups. A subspace $W$ is \textbf{totally singular} if $\varphi|_W$ is identically zero (which implies that $\beta|_{W \times W}$ also equals zero); it is \textbf{nonsingular} if $\beta|_W$ has trivial radical. If $G = GL_n$ we consider each subspace of its natural module to be totally singular. If $W$ is a nonsingular subspace then $\text{Cl}(W)$ denotes the classical group of the same type as $G$ defined on $W$.

Let $G$ be a classical group with natural module $V$. A \textbf{flag} is a sequence of nested subspaces. Let $f$ be the flag $W_0 \leq W_1 \leq \cdots \leq W_\ell = V$. Then $f$ has length $\ell$ and is \textbf{totally singular} if for each $i$ either $W_i$ is totally singular or $W_i = W^\perp$ for some totally singular subspace $W \leq V$ (if $V$ is
nonsingular this is equivalent to requiring that either $W_i$ or $W_i^\perp$ be totally singular). A subgroup of $G$ is parabolic if and only if it is the stabilizer of a totally singular flag. Let $L = \GL_{n_1} \cdots \GL_{n_s} \Cl_m$ be the Levi factor of a parabolic subgroup $P$. We say a flag $f$ of length $\ell$ is a natural flag for $P$ if the following hold: $f$ is totally singular, $P$ is the stabilizer of $f$, $\ell = s$ if $G = \GL_n$, $\ell = 2s$ if $m = 0$ and $G \in \{\SO_{2n}, \Sp_{2n}\}$, and $\ell = 2s + 1$ if $m \geq 1$ or $G = \SO_{2n+1}$. The unipotent radical of a parabolic equals the set of elements which act trivially upon each factor in a natural flag.

In the classical groups the unipotent classes are described using partitions. We will mention only a few facts here and refer the reader to [4] or [5] for more complete information. Let $G$ be one of $\GL_n$, $\OO_n$, $\SO_n$, $\Sp_n$, let $C$ be a unipotent class of $G$ and let $\lambda$ be the partition of $n$ consisting of the Jordan block sizes of $C$.

The parity conditions on $\lambda$ refer to the following requirements: if $G \in \{\OO_n, \SO_n\}$ and $p \neq 2$ then each even part of $\lambda$ must have even multiplicity; if $G = \Sp_n$ or $G \in \{\OO_n, \SO_n\}$ and $p = 2$ then each odd part of $\lambda$ must have even multiplicity; if $G = \SO_n$ with $n$ even then $\lambda$ must have an even number of parts.

If $G \in \{\OO_n, \Sp_n\}$ and $\lambda$ has no even parts with even multiplicity then all unipotent elements with Jordan blocks equal to $\lambda$ form a single $G$-class. This is generally not the case if $G \neq \GL_n$ and $p = 2$.

If $u \in C$ we say a part $x$ of $\lambda$ is nonsingular if there exists a Jordan chain of $u$ (i.e. a sequence of vectors $(v_i)_{i=0}^x$ such that $v_0 = 0$ and $(u-1)v_i = v_{i-1}$ for all $i \geq 1$) which generates an $x$-dimensional nonsingular subspace.

**Lemma 2.1 ([5]).** Let $G$ be $\OO_n$, $\SO_n$ or $\Sp_n$, with natural module $V$, bilinear form $\beta$, $u \in G$ a unipotent element, $\lambda$ the Jordan blocks of $u$, $x$ a part of $\lambda$ and $v_1, \ldots, v_x$ a Jordan chain of $u$.

(i) The subspace $\langle v_1, \ldots, v_x \rangle$ is nonsingular if and only if $\beta(v_i, v_j) \neq 0$ for some $i, j$ or, equivalently, for all $i, j > 0$ with $i + j = x + 1$. If $V$ is nonsingular and $x$ has multiplicity $1$ then the subspace $\langle v_1, \ldots, v_x \rangle$ is nonsingular.

(ii) If $G$ equals $\OO_n$ or $\SO_n$ and $p \neq 2$ then $x$ is nonsingular if and only if $x$ is odd. If $G = \Sp_n$ and $p \neq 2$ then $x$ is nonsingular if and only if $x$ is even. In any case, if $x \neq 1$ and the multiplicity of $x$ is odd then $x$ is nonsingular.

**Remarks 2.2.** (a) A stronger statement than given here is possible. In particular, keeping track of information about singularity of partitions is enough to parameterize the unipotent classes of $\OO_n$ and $\Sp_n$ in characteristic
2. (b) Spaltenstein [17, I.2.8] gives the following expression (using different terminology) for the dimension of the centralizer of a unipotent element. Suppose \( p = 2 \) and \( G \) equals \( \text{Sp}_n \) or \( \text{SO}_n \) with \( n \) even. Let \( u \in G \) be a unipotent element and let \( u_C \in \text{Sp}_n(\mathbb{C}) \) be a unipotent element with the same Jordan blocks as \( u \). Then \( \dim C_G(u) \) equals \( \dim C_{\text{Sp}_n(\mathbb{C})}(u_C) \) plus the number of even, singular parts of \( \lambda \).

If \( G \) is a reductive (not necessarily connected) group, a unipotent element is regular if the dimension of its centralizer equals the rank of \( G \). If \( G \) is connected then the regular elements form a single unipotent class (see \[18\] or \[4\]), which is the Richardson class of the Borel subgroups. If \( G \) is not connected then the number of regular classes is at most the number of connected components (see \[17\] for more on this and the connection with Richardson classes).

**Lemma 2.3.** Let \( G \) be one of \( GL_n \), \( SO_{2n+1} \), \( SO_{2n} \), \( \text{Sp}_{2n} \) or \( \mathcal{O}_{2n} \) and exclude the case \( SO_{2n+1} \) with \( p = 2 \). Let \( \lambda \) be the Jordan blocks of a regular unipotent class. If \( G = GL_n \) then \( \lambda = n \). If \( G = SO_{2n+1} \) then \( \lambda = 2n+1 \). If \( G = SO_{2n} \) then \( \lambda \) equals \( (2n-1,1) \) or \( (2n-2,2) \) according as \( p \neq 2 \) or \( p = 2 \) respectively. If \( G = \text{Sp}_{2n} \) then \( \lambda = 2n \). If \( G = \mathcal{O}_{2n} \) and \( p = 2 \) then there are two regular unipotent classes and these have Jordan blocks of sizes \( 2n \) and \( (2n−2,2) \). In all cases all parts of \( \lambda \) are nonsingular.

**Proof.** This follows from an easy dimension calculation. \( \square \)

### 3 Proof of Theorems 1 and 2

For this section we use the following notation and assumptions (with three explicit exceptions marked by the phrase “Contrary to our usual assumptions ...”). We assume throughout that \( G \) is one of \( GL_n \), \( SO_{2n+1} \), \( SO_{2n} \), \( \text{Sp}_{2n} \) and exclude the case \( SO_{2n+1} \) when \( p = 2 \). Let \( V \) be the natural module for \( G \) and \( \beta \) the bilinear form on \( V \) if \( G \neq GL_n \). Let \( P \) be a proper parabolic subgroup (we allow \( P = G \) in the statement of Theorem 1 but if this holds there is nothing to prove). Let \( Q \) be the unipotent radical of \( P \) and \( f = (0 = W_0 < \cdots < W_\ell = V) \) a natural flag. Let \( \Lambda = (n_1, \ldots, n_s) \oplus m = (1^{c(1)}, 2^{c(2)}, \ldots) \oplus m \) be the Levi partition of \( P \).

For any \( g \in Q \) we let \( \lambda(g) = (\lambda_1(g), \lambda_2(g), \ldots) \) be the partition of Jordan blocks of \( g \). We fix \( u \in Q \) which represents the Richardson orbit in \( Q \). We fix \( \lambda = (\lambda_1, \lambda_2, \ldots) = \lambda(u) \) and \( \mu = \psi(\Lambda)^* \). We wish to prove that \( \lambda = \mu \).

Essentially the proof of Theorem 1 is inductive. We will produce the largest one or two Jordan blocks of \( \lambda \) and show that they equal the largest
Lemma 3.1. Let the notation be as described above.

(i) Contrary to our usual assumptions, let \( G \leq \text{GL}(V) \) be any algebraic group, let \( C_\lambda \) and \( C_\mu \) be any two unipotent classes of \( G \) with Jordan blocks given by the partitions \( \lambda \) and \( \mu \) respectively. If \( C_\lambda \leq C_\mu \) then \( \lambda \leq \mu \).

(ii) Let \( g \in Q \) and let \( V_r \) be a subspace formed by \( r \) Jordan blocks of \( g \). Then \( \dim V_r \leq \sum_{i=1}^{r} \min\{r, \dim W_i/W_{i-1}\} \) with equality holding if and only if \( \dim V_r \cap W_j = \sum_{i=1}^{j} \min\{r, \dim W_i/W_{i-1}\} \) for all \( j \geq 1 \). In particular, for all \( r \geq 1 \) we have \( \sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{r} \min\{r, \dim W_i/W_{i-1}\} \).

(iii) Let \( G \in \{\text{SO}_{2n+1}, \text{SO}_{2n}, \text{Sp}_{2n}\} \). If \( G = \text{SO}_{2n+1} \) let \( r = 1 \) and otherwise let \( r = 2 \). If there exists \( g \in Q \) with \( (\lambda_1(g), \ldots, \lambda_r(g)) = (\lambda_1, \ldots, \lambda_r) \) such that \( \lambda_1(g), \ldots, \lambda_r(g) \) are nonsingular as Jordan blocks of \( g \) then \( \lambda_1, \ldots, \lambda_r \) are nonsingular as Jordan blocks of \( u \).

Sketch of proof. Part (i). Let \( U \) be the variety of all unipotent elements in \( G \). It is easy to show that for each \( j, b \geq 0 \) the subset \( \{g \in U \mid \dim \ker((g-1)^j) \geq b\} \) is closed in \( U \). (One way to prove this is to use elementary characterizations of rank in terms of determinants of minors of a matrix. Another way is to use the upper semi-continuity of dimension applied to the endomorphism of \( U \times V \) given by \( (g, v) \mapsto (g, (g-1)^j v) \), see [17, III.8.1].) Let \( u_\lambda \in C_\lambda \) and \( u_\mu \in C_\mu \). Since \( C_\lambda \subseteq C_\mu \) we have that \( u_\lambda \) is contained in any \( G \)-invariant, closed subset of \( U \) that contains \( u_\mu \). Thus, for each \( j \), one has \( u_\lambda \in \{g \in U \mid \dim \ker((g-1)^j) \geq \dim \ker(u_\mu - 1)^j\} \). Finally, note that \( \dim \ker((u_\lambda - 1)^j) = \sum_{i=1}^{j} \lambda_i^j \) and \( \dim \ker(u_\mu - 1)^j = \sum_{i=1}^{j} \mu_i^j \).

Part (ii) is elementary linear algebra and induction together with the fact that \( g \) acts trivially upon each factor \( W_i/W_{i-1} \).

Part (iii) uses the following facts. Every \( P \)-invariant, nonempty, open subset of \( Q \) contains \( u \). Let \( X \) be any subset of \( V \) and for \( g \in Q \) define a subspace \( V_g := \langle (g-1)^j v \mid i \geq 1, v \in X \rangle \leq V \). Then the set of \( g \in Q \) such that \( V_g \) is a nonsingular subspace is open set. (Note that one can express the fact that \( V_g \) is nonsingular via a determinant being nonzero.)

Lemma 3.2. If \( G \in \{\text{GL}_n, \text{SO}_{2n+1}\} \) let \( r = 1 \), otherwise let \( r = 2 \). If the following hypotheses hold then \( \lambda = \mu \).

(i) \((\lambda_1, \ldots, \lambda_r) \leq (\mu_1, \ldots, \mu_r)\),
(ii) \(\sum_{i=1}^{\ell}\min\{r, \dim W_i/W_{i-1}\} = \mu_1 + \cdots + \mu_r\),
(iii) there exists \(g \in Q\) with \((\lambda_1(g), \ldots, \lambda_r(g)) = (\mu_1, \ldots, \mu_r)\) and if \(G\) is orthogonal or symplectic \(\lambda_1(g), \ldots, \lambda_r(g)\) are nonsingular as Jordan block sizes of \(g\).

**Remark 3.3.** The previous lemma abstracts the inductive step in showing \(\lambda = \mu\). Essentially one can view Lemma 2.3 and Lemma 3.9 as the base cases. Lemma 3.4 finishes the proof for the case \(G = \text{GL}_n\). For the remaining groups Lemma 3.8 establishes part (i) and (ii) and Lemma 3.9 establishes part (iii).

**Proof.** If \(G\) has rank 1 then \(P\) is a Borel subgroup and we are done by Lemma 2.3. We assume now that Theorem 1 is true for classical groups with natural module \(V'\) where \(\dim V' < \dim V\).

Combining hypothesis (i) and Lemma 3.1(i) gives that \((\lambda_1(g), \ldots, \lambda_r(g)) \leq (\lambda_1, \ldots, \lambda_r) \leq (\mu_1, \ldots, \mu_r),\) whence we have equality by hypothesis (iii). Let \(V_r\) be the space generated by \(r\) Jordan chains of \(u\) of lengths \((\lambda_1, \ldots, \lambda_r) = (\mu_1, \ldots, \mu_r).\) If \(G\) is symplectic or orthogonal we apply Lemma 3.1(iii) and assume that \(V_r\) is nonsingular. By hypothesis (ii) we have \(\dim V_r = \lambda_1 + \cdots + \lambda_r = \mu_1 + \cdots + \mu_r = \sum_{i=1}^{\ell}\min\{r, \dim W_i/W_{i-1}\}\).

The inductive step will proceed as follows. Let \(X = V_r.\) We will produce a \(u\)-stable decomposition \(V = X \oplus Y.\) We will show that \(f\) induces flags in \(X\) and \(Y\) which we will denote by \(f \cap X\) and \(f \cap Y\) such that \(W_i\) is the direct sum of corresponding terms in \(f \cap X\) and \(f \cap Y.\) We will then calculate the Jordan blocks of the Richardson classes in \(\text{Cl}(X)\) and \(\text{Cl}(Y)\) (these are the classical groups defined on \(X\) and \(Y\)) associated with \(f \cap X\) and \(f \cap Y\) and show that these equal \(\lambda(u\vert_X)\) and \(\lambda(u\vert_Y).\) The Jordan blocks of \(u\vert_X\) are \((\lambda_1, \ldots, \lambda_r) = (\mu_1, \ldots, \mu_r)\) by construction (since \(X = V_r\)), and the blocks of \(u\vert_Y\) will be found by induction.

Let \(f \cap X\) denote the flag in \(X = V_r\) with terms given by \(X_i := X \cap W_i\) for \(1 \leq i \leq \ell.\) We will construct below a space \(Y\) and a flag \(f \cap Y\) with terms \(Y_i\) such that \(W_i = X_i \oplus Y_i\) for \(1 \leq i \leq \ell.\) When \(G\) equals \(\text{Sp}_n\) or \(\text{SO}_n\) the flags will be totally singular and in all cases \(u\) will act trivially upon the factors in each flag.

Let \(G = \text{GL}_n.\) Let \(Y_1\) be any direct complement of \(X_1\) in \(W_1;\) this is \(u\)-stable since \(u\) of 1 on \(W_1.\) Let \(i \geq 2\) and suppose \(Y_{i-1}\) has been constructed such that \(W_{i-1} = X_{i-1} \oplus Y_{i-1}.\) Using the fact that \(X_{i-1} \cap Y_{i-1} = \{0\}\) it is easy to show that \(\ker(u - 1)\vert_{X_i} \cap \ker(u - 1)\vert_{Y_{i-1}} = \{0\}.\) Then we may choose a direct complement \(Z\) and a basis \(v_1, v_2, \ldots,\) of another direct
calculate the parts of the Levi partition of $Y$ in the flag $u$. To check that this sum is direct, write $0$ as the sum of an element in each term, then apply $u - 1$ and use the definitions. It is now relatively easy to show that $W_i = X_i \oplus Y_i$. Finally, we take $Y = Y_\ell$.

If $G \neq \GL_n$ let $Y = V_r^\perp$ and $Y_i = Y \cap W_i$ for $1 \leq i \leq \ell$. The dimension of $X_i$ can be calculated using Lemma 3.1(ii) and $\dim Y_i$ is given by $\dim W_i + \dim(W_i^\perp \cap X) - \dim X$. Using dimension calculations one may show that $W_i = X_i \oplus Y_i$ for $1 \leq i \leq \ell$ and that the flags are totally singular.

For $J \in \{X, Y\}$ let $\Cl(J)$ be the classical group on $J$, let $P_J$ be the parabolic in $\Cl(J)$ corresponding to the flag $f \cap J$ and let $Q_J$ be the unipotent radical of $P_J$. We may identify $Q_XQ_Y$ as a subgroup of $Q$. Let $C$ denote the Richardson orbit in $Q$ and note that $C \cap (Q_XQ_Y)$ is an open subset of $Q_XQ_Y$ which is also dense as it contains $u$. Then $C$ contains the Richardson orbits in $Q_X$ and $Q_Y$. Let $u^i \in C \cap (Q_XQ_Y)$ such that $u^i|_X$ and $u^i|_Y$ represent the Richardson orbits in $Q_X$ and $Q_Y$ respectively.

We have that $u$ and $u'$ are conjugate whence $\lambda(u) = \lambda(u')$. We also have that $\lambda(u|_X) = (\lambda_1, \ldots, \lambda_r) = (\mu_1, \ldots, \mu_r)$ by construction. We have $\lambda(u'|_X) \geq \lambda(u|_X)$ since $u'|_X$ represents the Richardson orbit (and using Lemma 3.1(i)). This, together with the fact that $\lambda(u') = \lambda(u)$ implies that $\lambda(u'|_X) = \lambda(u|_X)$. Thus $\lambda(u'|_X) = \lambda(u|_X)$ and we may assume, for our purposes, that $u = u'$.

We have $\lambda(u|_X) = (\lambda_1, \ldots, \lambda_r)$ and $\lambda(u|_Y) = (\lambda_{r+1}, \ldots)$. Since $(\lambda_1, \ldots, \lambda_r) = (\mu_1, \ldots, \mu_r)$ it suffices to show that $(\lambda_{r+1}, \ldots) = (\mu_{r+1}, \ldots)$. One may calculate the parts of the Levi partition of $Y$ using the dimensions of factors in the flag $f \cap Y$. One may verify that $\Lambda(Y) = (\max\{n_i - r, 0\} | 1 \leq i \leq s)$ if $m = 0$ and $\Lambda(Y) = (\max\{n_i - r, 0\} | 1 \leq i \leq s) \oplus (m - 1)$ if $m \geq 1$ and $G \neq \SO_{2n+1}$ and $\Lambda(Y) = (\max\{n_i - 1, 0\} | 1 \leq i \leq s) \oplus m$ if $G = \SO_{2n+1}$ (when $G = \SO_{2n+1}$ then $m$ in $\Lambda$ corresponds to $\SO_{2m+1}$, but $Y$ is even dimensional and $m - 1$ or $m$ in $\Lambda(Y)$ corresponds to $\SO_{2m-2}$ or $\SO_{2m}$).

By induction we may apply Theorem 3.4 to determine the Jordan blocks of this Levi partition $\Lambda(Y)$. By analyzing the cases in Theorem 3.4 one finds that they equal $\mu$ with the first $r$ rows removed.

**Proof 3.4 (Proof of Theorem 3.4 when $G = \GL_n$).** Note that $\mu_1 = \ell = s$. Using Lemma 3.1(ii) it is easy to verify hypotheses (i) and (ii) of Lemma 3.2.
It remains to prove the existence of \( g \in Q \) with \( \lambda_1(g) = \mu_1 \). Let \( X_1 \) be a one dimensional subspace of \( W_1 \) and \( Y_1 \leq W_1 \) such that \( W_1 = X_1 \oplus Y_1 \). For \( i \in \{2, \ldots, \ell\} \) let \( X_i \) be an \( i \) dimensional subspace of \( W_i \) such that \( X_{i-1} \leq X_i \) and \( Y_i \leq W_i \) such that \( Y_{i-1} \leq Y_i \) and \( W_i = X_i \oplus Y_i \). Define \( f \cap X \) to be the flag in \( X = X_\ell \) with terms given by the \( X_i \) and \( f \cap Y \) to be the flag in \( Y = Y_\ell \) with terms given by the \( Y_i \). Let \( P_X \) be the parabolic in \( GL(X) \) of \( f \cap X \), let \( Q_X \) be the unipotent radical of \( P_X \) and identify \( Q_X \) as a subgroup of \( Q \). Then \( P_X \) is a Borel subgroup of \( GL(X) \), whence there exists an element \( g \) in \( Q_X \) which has one block of size \( \ell = \mu_1 \) by Lemma 2.3.

**Corollary 3.5.** Every unipotent class in \( GL_n \) is a Richardson class.

This is also proven in [17, II.5.14] and in [8, 5.5].

**Proof.** If a unipotent class has Jordan blocks given by the partition \( \nu \) then it is the Richardson class of any parabolic with Levi partition equal to \( \nu^* \).

For the next result we introduce some notation. Let \( H \leq J \) be algebraic groups and let \( H \) act upon \( J \) via conjugation. Given a subset \( O \subseteq J \) we denote by \( \overline{O} \) the closure taken within \( J \) and by \( O_J \) the subset \( \bigcup_{g \in J} O^g = \{ gxg^{-1} \mid g \in J, x \in O \} \).

**Lemma 3.6.** Let \( H \leq J \) be algebraic groups and use the notation described above. Let \( O_1 \) and \( O_2 \) be two \( H \)-classes in \( H \).

(i) If \( O_1 \subseteq \overline{O_2} \) then \( O_1^J \subseteq \overline{O_2}^J \).

(ii) If \( H \) has a dense orbit in \( H \cap \overline{O_2} \), has a single orbit in \( H \cap O_2^J \) (i.e. \( H \cap O_2^J = O_2 \)), and \( O_1^J \) is a subset of \( \overline{O_2} \) then \( O_1 \subseteq \overline{O_2} \).

(iii) If \( H \) has a single orbit in \( O_1^J \) (i.e. \( O_1^J = O_1 \)), has finitely many orbits in \( O_2^J \), and \( O_1^J \) is a subset of \( \overline{O_2} \) then \( O_1 \subseteq \overline{O_2} \).

We refer to conditions (ii) and (iii) as “descending from \( J \) to \( H \”).

**Proof.** Part (i) We have:

\[
O_1^J \subseteq \overline{(O_2)^J} = \bigcup_{g \in J} (O_2)^g = \bigcup_{g \in J} \overline{O_2^g} \subseteq \bigcup_{g \in J} \overline{O_2^g} = \overline{O_2^J}.
\]

Part (ii) We claim that:

\[
O_1 \subseteq H \cap O_1^J \subseteq H \cap \overline{O_2^J} = \overline{O_2}.
\]

The final equality is the one to be proved. Let \( C \) denote a dense orbit of \( H \) in \( H \cap \overline{O_2^J} \). Then \( C \subseteq \overline{O_2^J} \) whence \( C^J \subseteq \overline{O_2^J} \) by part (i). On the
other hand, \( O_2 \subseteq \overline{C} \) whence \( O_2^J \subseteq \overline{C^J} \) by (i). Thus \( O_2^J = C^J \) whence 
\( C \subseteq H \cap C^J = H \cap O_2^J = O_2 \) and \( C = O_2 \).

Part (iii). Denote the \( H \)-orbits in \( O_2^J \) by \( O_2 = O_2^{g_1}, O_2^{g_2}, \ldots, O_2^{g_h} \) where 
\( g_1 = 1 \) and \( g_i \in J - H \) for \( i > 1 \). We have: \( O_1 \subseteq O_2^J = O_2^{g_1} \cup \cdots \cup O_2^{g_h} \) which 
implies that \( O_1 \subseteq O_2^{g_j} \) for some \( j \). Then \( O_1 = O_2^{g_j} \subseteq \left( O_2^{g_j} \right)^{g_j^{-1}} = O_2 \). \(\square\)

**Corollary 3.7.** Contrary to our usual assumptions, let \( G \leq \text{GL}_n \) be an 
algebraic group, let \( C_\lambda \) and \( C_\mu \) be two unipotent classes with Jordan blocks 
given by the partitions \( \lambda \) and \( \mu \).

(i) Suppose that all the unipotent elements in \( G \) with Jordan blocks equal 
to \( \mu \) form a single conjugacy class. Then \( \lambda < \mu \) if and only if \( C_\lambda \leq C_\mu \).

(ii) If \( G = \text{GL}_n \), or \( G \in \{ \text{O}_n, \text{Sp}_n \} \) and \( p \neq 2 \), or \( G \in \{ \text{O}_n, \text{Sp}_n \} \) and \( \mu \) 
has no even parts with even multiplicity, or \( G = \text{SO}_n \) with \( n \) even and 
\( p \neq 2 \), then \( \lambda < \mu \) if and only if \( C_\lambda < C_\mu \).

**Proof.** Part (i). By Lemma 5.3(i) we have that \( C_\lambda < C_\mu \) implies \( \lambda < \mu \). We 
prove the converse first for \( G = \text{GL}_n \).

Step 1: Since \( \lambda \leq \mu \) we may fix a sequence of partitions \( \lambda = \lambda^{(0)} < \lambda^{(1)} < \cdots < \lambda^{(r)} = \mu \) such that for each \( i \) we have that \( \lambda^{(i)} \) and \( \lambda^{(i+1)} \) 
differ in exactly two places, i.e. there exist exactly two indices \( j \) such that 
\( \lambda^{(i)}_j \neq \lambda^{(i+1)}_j \) (see [3] p23)). Then by transitivity it suffices to prove that 
\( \lambda < \mu \Rightarrow C_\lambda < C_\mu \) when \( \lambda \) and \( \mu \) differ in exactly two places, which we now 
assume.

Step 2: Since \( \lambda \) and \( \mu \) differ in exactly two places, we may find a subgroup 
\( \text{GL}_{n_1} \times \text{GL}_{n_2} \) of \( G \), a unipotent \( \text{GL}_{n_1} \)-class \( C \), two unipotent \( \text{GL}_{n_2} \)-classes \( C_1 \) 
and \( C_2 \) with \( C_\lambda \) and \( C_\mu \) the extensions to \( G \) of \( CC_1 \) and \( CC_2 \) respectively 
(i.e. the classes \( C_1 \) and \( C_2 \) correspond to the two parts where \( \lambda \) and \( \mu \) differ). 
By Lemma 5.3(i) it suffices to show that \( C_1 \leq C_2 \) (for then \( CC_1 \leq CC_2 \) and 
\( C_\lambda = (CC_1)^G \leq (CC_2)^G = C_\mu \)).

Step 3. It suffices now to prove the result under the assumption that \( \lambda \) 
is a two part partition (whence \( \mu \) has one or two parts). Then the difference 
between \( \mu^* \) and \( \lambda^* \) is that \( \lambda^* \) has one extra 2 and two fewer 1’s. Let \( g \in C_\lambda \). By Corollary 3.3, 
we may find flags \( f_\lambda : 0 \leq W_2 < W_3 < \cdots \) and 
\( f_\mu : 0 \leq W_1 < W_2 < W_3 < \cdots \) such that \( f_\lambda \) and \( f_\mu \) have corresponding 
Levi partitions of \( \lambda^* \) and \( \mu^* \); \( f_\lambda \) and \( f_\mu \) are identical to the right of \( W_3 \), 
and \( g \) represents the Richardson orbit corresponding to \( f_\lambda \) (in particular \( g \) 
acts trivially upon each factor in \( f_\lambda \)). Then \( g \) is in the unipotent radical 
associated with \( f_\mu \), which in turn is contained in \( \overline{C_\mu} \). Whence, \( C_\lambda \subseteq \overline{C_\mu} \).
Now (i) is proven for $G = GL_n$. If $G < GL_n$ we may descend to $G$ via part (ii) of the previous lemma; i.e., apply Lemma 3.6(ii) with $H = G$ and $J = GL_n$ to get $\lambda < \mu$ implies $C_\lambda < C_\mu$.

Part (ii). This is immediate from part (i) (and the comments in Section 2), unless $G = SO_n$ with $n$ even, $p \neq 2$. However, part (i) holds for $O_n$ and one may descend to $SO_n$ by applying Lemma 3.6(iii) with $H = SO_n$ and $J = O_n$.

The following lemma establishes Lemma 3.2(ii) for the cases where $G \neq GL_n$. It will also be used in Lemma 3.9 to establish Lemma 3.2(iii).

Lemma 3.8. If $G = SO_{2n+1}$ let $r = 1$ and if $G \in \{Sp_{2n}, SO_{2n}\}$ let $r = 2$.

Recall that $\ell$ is the number of terms in the natural flag and that the Levi partition is $\Lambda = (1^{c(1)}, 2^{c(2)}, \ldots) \oplus m$. Then $(\mu_1, \ldots, \mu_r)$ are listed below. Furthermore, $(\lambda_1, \ldots, \lambda_r) \leq (\mu_1, \ldots, \mu_r)$ and $\sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\} = \mu_1 + \cdots + \mu_r$.

(i) $G = SO_{2n+1}$ and $p \neq 2$. We have $\mu_1 = \ell$.

(ii) $G = SO_{2n}$ and $p \neq 2$. If $m = 0$ and $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. Otherwise we have $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.

(iii) $G = SO_{2n}$ and $p = 2$. If $m = 0$ and $c(1) \geq 2$ then $(\mu_1, \mu_2) = (\ell - 2, \ell - 2c(1) + 2)$. If $m = 0$, $c(1) = 1$, or $m \geq 1$, $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. If $c(1) = 0$ then $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.

(iv) $G = Sp_{2n}$. If $m \geq 1$ and $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. Otherwise $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.

Proof. Recall that $\psi(\Lambda)$ is defined in Table 1 and that $\mu$ equals $\psi(\Lambda)^*$, the dual of $\psi(\Lambda)$. Thus $\mu_1$ equals the number of parts in $\psi(\Lambda)$ and $\mu_2$ equals the number of parts in $\psi(\Lambda)$ which are greater than or equal to 2. It is easy to verify the stated formulas for $\mu_1$ and $\mu_2$.

Since $\sum_{i=1}^{\ell} \min\{1, \dim W_i/W_{i-1}\} = \ell$ and $\sum_{i=1}^{\ell} \min\{2, \dim W_i/W_{i-1}\} = 2\ell - 2c(1)$ we conclude that $\sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\} = \mu_1 + \cdots + \mu_r$ and that $(\lambda_1, \lambda_2) \leq (\ell, \ell - 2c(1))$ (by Lemma 3.1(ii)). This gives the desired upper bounds on $\lambda$ in case (i) or when $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.

For all the remaining cases we start by proving that one cannot have $\lambda_1 = \ell$, whence $(\lambda_1, \lambda_2) \leq (\ell - 1, \ell - 2c(1) + 1)$. First we assume $G = SO_{2n}$, $m = 0$ (whence $\ell = 2s$) and $c(1) \geq 1$. If $\lambda_1 = \ell$ there is a Jordan chain $v_1, \ldots, v_{2s}$ of length $2s = \ell$. Since $u$ acts trivially upon each factor in the flag, and since the Jordan chain has as many elements as there are terms in the flag, we see that $v_s \in W_s - W_{s-1}$ and $v_{s+1} \in W_{s+1} - W_s$. By Lemma 2.1(i) we have $\beta(v_s, v_{s+1}) \neq 0$. Let $\tilde{W} = \langle v_{s+1} \rangle^\perp \cap W_s$. Then $\tilde{W}$ is a totally
singular \((n-1)\)-space, whence \(\tilde{W}^\perp/\tilde{W}\) is a nonsingular 2-space. But \(v_s\) and \(v_{s+1}\) project to distinct, nontrivial elements in \(\tilde{W}^\perp/\tilde{W}\) whence \(u|_{\tilde{W}^\perp/\tilde{W}}\) is a nontrivial unipotent element of \(SO(\tilde{W}^\perp/\tilde{W}) = SO(2)\), a contradiction. In all other cases where \((\mu_1, \mu_2) < (\ell, \ell - 2c(1))\) the parity conditions upon \(\lambda\) and the fact that \(\lambda_1 + \lambda_2 \leq 2\ell - 2c(1)\) imply \(\lambda_1 \neq \ell\).

It remains to show that if \(G = SO_{2n}, m = 0\) and \(c(1) \geq 2\) then \((\lambda_1, \lambda_2) \leq (\ell - 2, \ell - 2c(1) + 2)\). However, we have shown already that \((\lambda_1, \lambda_2) \leq (\ell - 1, \ell - 2c(1) + 1)\) and if \(\lambda_1 = \ell - 1\) this contradicts the parity conditions upon \(\lambda\).

**Lemma 3.9.** With the usual notation, the following hold and, in particular, Lemma 3.2(ii) holds, whence Theorem 1 is proven.

(i) Let \(G = SO_{2n} \) and \(p = 2\). (i)(a) If \(\Lambda = (2^a)\) then \(\lambda = (2a, 2a)\). (i)(b) If \(\Lambda = (2^a, 1)\) then \(\lambda = (2a + 1, 2a + 1)\). (i)(c) If \(\Lambda = (2^a, 1^b)\) with \(b \geq 2\) then \(\lambda = (2a + 2b - 2, 2a + 2)\).

(ii) Let \(G = SO_{2n} \) and \(p \neq 2\). (ii)(a) If \(\Lambda = (2^a)\) then \(\lambda = (2a, 2a)\). (ii)(b) If \(\Lambda = (2^a, 1^b)\) with \(b \geq 1\) then \(\lambda = (2a + 2b - 1, 2a + 1)\).

(iii) Let \(G = Sp_{2n}\). (iii)(a) If \(\Lambda = (2^a)\) then \(\lambda = (2a, 2a)\). (iii)(b) If \(\Lambda = (2^a, 1^b)\) then \(\lambda = (2a + 2b, 2a)\). (iii)(c) If \(\Lambda = (2^a) \oplus 1\) then \(\lambda = (2a + 1, 2a + 1)\). (iii)(d) If \(\Lambda = (2^a, 1^b) \oplus 1\) with \(b \geq 1\) then \(\lambda = (2a + 2b, 2a + 2)\).

We sketch two proofs of parts (i)-(iii). Neither proof seems entirely satisfactory as each contains a tedious verification of a rather simple fact.

**Proof.** Sketch of first proof of (i)-(iii). For the following statement, \(\lambda\) and \(\mu\) need not have their usual definitions. Let \(\lambda\) and \(\mu\) be two partitions and \(u_\lambda, u_\mu\) two unipotent elements with Jordan blocks given by \(\lambda\) and \(\mu\) respectively. Suppose \(\mu\) has two parts. Then we claim that \(\lambda < \mu\) implies \(\dim C_G(u_\lambda) \geq \dim C_G(u_\mu)\) with the inequality strict provided the parts of \(\mu\) are nonsingular.

Given the claim, we again let \(\lambda\) and \(\mu\) have their usual definitions, whence \(\lambda \leq \mu\) by Lemma 3.8. Let \(L\) be the Levi factor of the parabolic \(P\) under discussion. Then Richardson’s Theorem (iii) and the claim show \(\dim L = \dim C_G(u_\lambda) \geq \dim C_G(u_\mu)\). Using the expression for \(\mu\) in the previous Lemma, one may check that \(\dim C_G(u_\mu)\) equals \(\dim L\) or \(\dim L + 2\) with the latter only if both parts of \(\mu\) are singular. We conclude that both parts of \(\mu\) are nonsingular, and that we must have \(\lambda = \mu\).
decomposing a flag

\[ \text{Lemma 3.7 we have} \]

\[ \text{Lemma 3.8} \]

\[ \text{We note that it would be circular to prove the claim by applying Spaltenstein’s version of Theorem 2 as this is proven in [17] using the version of Theorem 1 which is found there.) To prove the claim directly one may manipulate the formulas for dimensions of centralizers, though this is somewhat tedious. In particular, let \( c \) be the multiplicity of \( \lambda_i^* \) in \( \lambda^* \). If \( \lambda_1^* = 5 \), or \( \lambda_1^* = 4 \), \( c \geq 2 \), or \( \lambda_1^* = 3 \), \( c \geq 4 \) then one may show that \( \sum_{i \geq 1} ((\lambda_i^*)^2 - (\mu_i^*)^2) > 2\lambda_i^* \) which proves the claim (by examining the formulas for dimensions of centralizers). The remaining cases amount to direct calculations.

**Sketch of second proof of (i)-(iii).** There are two cases.

Case 1: \( \mu \) has two equal parts. We claim that there exists \( g \in Q \) with \( \mu = \lambda(g) \). Given the claim, and using Lemma 3.1(i) and Lemma 3.8 we have \( \mu = \lambda(g) \leq \lambda \leq \mu \).

To prove the claim let \( \mu = (n, n) \) where \( n \) is the rank of \( G \). The tedious part of the argument is verifying, inductively, that one may construct, using roots in \( \Phi(Q) \), a root base of an \( A_{n-1} \) root system. Given this root base, the group generated by the maximal torus and the root groups corresponding to \( \mathbb{Z} \)-linear combinations of this base is isomorphic to \( \text{GL}_n \). Let \( g \in \text{GL}_n \) be a regular unipotent element written as the product of a nontrivial element in each root group corresponding to a root in this root base (see [18]). Then \( g \) is in \( Q \) and \( g \) has two blocks of size \( n \) in the natural embedding of \( \text{GL}_n \) in \( G \).

Case 2: \( \mu \) has two distinct parts. As stated in Section 2 one sees that \( G \) has a unique unipotent class \( C_\mu \) with Jordan blocks given by \( \mu \). Let \( C_\lambda \) be the Richardson class of \( P \). By Lemma 3.8 we have \( \lambda \leq \mu \), and by Lemma 3.7 we have \( C_\lambda \leq C_\mu \). One may easily show that \( \dim C_\mu = \dim C_\lambda \) whence \( C_\lambda = C_\mu \) and \( \lambda = \mu \).

**Sketch of proof of (ii).** This proof parallels that given for the case \( G = \text{GL}_n \) (see Proof 3.4 so we will be brief. Recall that the natural flag \( f \) has terms \( W_i \). We will produce a decomposition of \( f \) by constructing and decomposing a flag \( \tilde{f} \) which is isomorphic to (whence conjugate to) \( f \).

Suppose \( G = \text{SO}_{2n+1} \). Let \( X \leq V \) be a nonsingular subspace of dimension \( 2s+1 = \ell \). Choose a totally singular flag \( 0 < X_1 < \cdots < X_\ell = X \) where \( \dim X_i = i \). This flag corresponds to a Borel subgroup of \( \text{Cl}(X) \) which has Levi partition \( \Lambda(X) = (1^s) \). Suppose \( G \in \{ \text{Sp}_{2n}, \text{SO}_{2n} \} \). Let \( X \leq V \) be a nonsingular subspace of dimension \( 2\ell - 2c(1) \). Choose a totally singular flag \( 0 < X_1 < \cdots < X_\ell = X \) where \( \dim X_j = \sum_{i=1}^j \min \{ 2, \dim W_i/W_{i-1} \} \). This flag has Levi partition \( \Lambda(X) \) as follows: If \( m = 0 \) then \( \Lambda(X) = (2^{s-c(1)}, 1^{c(1)}) \); If \( m \geq 1 \) and \( G = \text{Sp}_{2n} \) then \( \Lambda(X) = (2^{s-c(1)}, 1^{c(1)}) \oplus 1 \); If \( m \geq 1 \) and
Let $G = \text{SO}_{2n}$ then $(2^{s-c(1)}, 1^{(1)+1})$.

In each case we define $\tilde{Y} = \tilde{X}^\perp$ and choose a totally singular flag $0 \leq \tilde{Y}_1 \leq \cdots \leq \tilde{Y}_t = \tilde{Y}$ where $\dim Y_i = \dim W_i - \dim X_i$. Define the flag $\tilde{f}$ to have terms $0 < \tilde{W}_1 < \cdots < \tilde{W}_t = V$ where $\tilde{W}_i = \tilde{X}_i \oplus \tilde{Y}_i$. Since $\tilde{f}$ is conjugate to $f$ we see that a similar decomposition holds for $f$ which we express as $V = X \oplus Y$, $f = (f \cap X) \oplus (f \cap Y)$. The Levi partitions for the flag $f \cap X$ are the Levi partitions listed for $\Lambda(\tilde{X})$ above.

Now that $f$ has been decomposed, let $P_X$ be the parabolic in $\text{Cl}(X)$ and let $Q_X$ be its unipotent radical. Then we identify $Q_X$ as a subgroup of $Q$. Let $g \in Q_X \leq Q$ which represents the Richardson orbit in $Q_X$. Then we may apply parts (i)-(iii) to calculate the Jordan blocks of $g$. We find that $\lambda(g) = \mu_1$ if $G = \text{SO}_{2n}+1$ and $\lambda(g) = (\mu_1, \mu_2)$ if $G \in \{\text{SO}_{2n}, \text{Sp}_{2n}\}$ where $\mu_1$ and $\mu_2$ are as in Lemma 5.8.

\section{Richardson Classes of Distinguished Parabolics}

Lemma 4.1. Let $G$ be one of $\text{GL}_n$, $\text{SO}_{2n}+1$, $\text{SO}_{2n}$, and $\text{Sp}_{2n}$. Let $\Psi$ denote the map taking each distinguished parabolic class to the Jordan blocks of its Richardson class. Then $\Psi$ gives a bijection with the set of partitions described in table 2.

For $p \neq 2$, the descriptions in table 2 of the image of $\Psi$ are stated in 2, but it is not stated there that these partitions equal the Jordan blocks of the Richardson class.

Proof. If $G = \text{GL}_n$, then the only distinguished parabolic is the Borel subgroup, which corresponds to the regular class.

We give the proof for $\text{SO}_{2n}$ and leave the other cases (which are simpler) to the reader.

Let $\Lambda$ be the Levi partition of a distinguished parabolic $P$. Using the description of distinguished parabolics given in 4, we may write $\Lambda = (n_1, \ldots, n_s) \oplus m = (1^{c(1)}, \ldots, (2m)^{c(2m)}, (2m+1)^{c(2m+1)}) \oplus m$ where we index the $n_i$ such that $n_s$ is the largest $n_i$. If $m = 0$ then $n_s \in \{1, 2\}$; if $m \geq 1$ then $n_s \in \{2m - 1, 2m\}$ and in all cases $c(i) \geq 1$ if and only if $1 \leq i \leq n_s$.

Let $\psi$ be the map defined in Theorem 4. If $p \neq 2$ and $m = 0$ then $\psi(\Lambda) = (1^{2c(1)-2}, 2^{2c(2)+1})$. If $p \neq 2$ and $m \geq 2$ then $\psi(\Lambda) = (1^{2c(1)}, \ldots, (2m-1)^{2c(2m-1)}, (2m)^{c(2m)+1})$. If $p = 2$ and $m = 0$ then $\psi(\Lambda) = (1^{2c(1)-4}, 2^{2c(2)+2})$. If $p = 2$ and $m \geq 2$ then $\psi(\Lambda) = (1^{2c(1)-2}, 2^{2c(2)+2}, \ldots, (2m-1)^{2c(2m-1)-2}, (2m)^{2c(2m)+2})$. 


Table 2: Jordan blocks of distinguished Richardson classes

| Image of $\Psi$                                                                 |                                                                 |
|--------------------------------------------------------------------------------|-------------------------------------------------------------------|
| $\text{GL}_n$                                                                 | The partition of $n$ consisting of a single block                  |
| $G = \text{SO}_{2n+1}$, $p \neq 2$                                          | Partitions of $2n + 1$ consisting of distinct odd parts            |
| $G = \text{SO}_{2n+1}$, $p = 2$                                              | Partitions of $2n + 1$ of the form $1 \oplus \lambda$ such that:  |
|                                                                              | $\lambda$ is even; the multiplicity of each part of $\lambda$ is |
|                                                                              | at most 2; if $i$ is even then $\lambda_i - \lambda_{i+1} \geq 4$.|
| $G = \text{Sp}_{2n}$                                                         | Partitions of $2n$ consisting of distinct even parts              |
| $G = \text{SO}_{2n}$, $p \neq 2$                                            | Partitions of $2n$ consisting of distinct odd parts               |
| $G = \text{SO}_{2n}$, $p = 2$                                               | Partitions $\lambda$ of $2n$ such that: $\lambda$ has an even   |
|                                                                              | number of parts; each part of $\lambda$ is even; the multiplicity |
|                                                                              | of each part is at most 2; if $i$ is even with $\lambda_{i+1} \neq 0$ |
|                                                                              | then $\lambda_i - \lambda_{i+1} \geq 4$.                         |
We have that $\Psi(P)$ equals $\psi(\Lambda)^*$, the dual of $\psi(\Lambda)$. The formulas for $\psi(\Lambda)$ make it clear that $\Psi$ is injective.

Let $\mu$ be any partition and let $m(i)$ be the multiplicity of $i$ in the dual partition $\mu^*$. Then $\mu$ consists of distinct parts if and only if $\mu^*$ contains each integer between 1 and its maximal part. Also, each part of $\mu$ is odd if and only if for each $j$ we have $\sum_{i\geq j} m(i)$ is odd.

If $P$ is given and $p \neq 2$, the comments just made about $\mu$ show that $\Psi(P) = \psi(\Lambda)^*$ satisfies the properties described in the statement of the Lemma. In other words, the image of $\Psi$ is in the desired set.

Conversely, let $p \neq 2$ and let $\lambda$ be given which satisfies the properties described in the statement of the Lemma. The comments just made about $\mu$ show that $\lambda^*$ can be set equal to an expression of the form given for $\psi(\Lambda)$, and then one may solve for $c(1), c(2)$, etc. In other words, $\Psi$ is surjective.

The case for $p = 2$ may be verified similarly, however the following alternative description may make the proof easier. Let $\lambda(2)$ and $\lambda(\neq 2)$ be the Jordan blocks of the Richardson class of a parabolic associated with $\Lambda$ when $p = 2$ and when $p \neq 2$ respectively. Let $\lambda(\neq 2) = (\lambda_1, \lambda_2, \ldots, \lambda_{2\ell-1}, \lambda_{2\ell})$ where $\lambda_{2\ell-1}$ or $\lambda_{2\ell}$ is the last nonzero part of $\lambda(\neq 2)$. Then $\lambda = (\lambda_1 - 1, \lambda_2 + 1, \ldots, \lambda_{2\ell-1} - 1, \lambda_{2\ell} + 1)$. This description may be verified directly from the formulas for $\psi(\Lambda)$ (Spaltenstein [17, III.7.2, III.8.2] defines a similar map for the Jordan blocks of all unipotent classes; note there is a typographical mistake in the formula for $SO_{2n+1}$).

Given $\lambda = \psi(\Lambda)^*$, it remains to prove that $\lambda$ is nonsingular. For those cases where $\lambda$ has distinct parts this follows from Lemma 2.1. In the remaining cases we have that $G$ is orthogonal and $p = 2$. By Richardson’s Theorem (iii) we know $\dim L = \dim C_G(u)$ where $L$ is the Levi subgroup determined by $\Lambda$ and $u$ is an element of the Richardson class in $G$. It is now easy to finish the proof by using Spaltenstein’s expression for $\dim C_G(u)$ described in Remarks 2.2.

Corollary 4.2. Let $G$ be a simple algebraic group and consider the map which takes each distinguished parabolic class to its Richardson class. This map is injective.

Proof. For the classical groups this follows from the previous lemma. For the exceptional groups, we observe that no two distinct distinguished parabolics have the same dimension of Levi factor. By Theorem I (iii) the dimension of the Levi factor equals the dimension of the centralizer of an element in the unipotent class, whence the result follows by dimension.
5 Proof of the Bala-Carter-Pommerening Theorem

Throughout this section, \( G \) denotes a connected reductive group, unless indicated otherwise.

**Lemma 5.1.** (i) Let \( S \) be a torus in \( G \). Then \( L = C_G(S) \) is a Levi subgroup.

(ii) If \( u \) is a unipotent element and \( S \) a maximal torus of \( C_G(u) \) then \( u \) is distinguished in \( L = C_G(S) \). Furthermore, any Levi subgroup in which \( u \) is distinguished is conjugate to \( L \) via an element of \( C_G(u) \).

**Proof.** For part (i) one may adapt [4, 5.9.2]. For part (ii) one may adapt [4, 5.9.3].

**Corollary 5.2.** Define a map from \( G \)-classes of pairs \((L,C)\) consisting of a Levi subgroup \( L \) of \( G \) and a distinguished unipotent \( L \)-class \( C \) to unipotent \( G \)-classes by extending \( C \). This map gives a bijection.

**Lemma 5.3.** Let \( P \) be a distinguished parabolic of \( G \). Let \( \overline{G} = G/Z(G) \), \( \overline{P} = P/Z(G) \), let \( \overline{Q} \) be the unipotent radical of \( \overline{P} \) and let \( u \) represent the dense orbit of \( \overline{P} \) upon its unipotent radical \( \overline{Q} \). Then \( C_{\overline{G}}(u) = C_{\overline{P}}(u) = C_{\overline{Q}}(u) \). In particular the Richardson class of \( P \) is distinguished in \( G \).

**Proof.** It is easy to reduce to the case \( Z(G) = 1 \) and adapt the proof given in [4, 5.8.7].

**Proof 5.4 (Proof of Theorem 3).** Part (i). This is by definition of the map \( \psi \).

Part (ii). We have \( \psi(L,P) = C \) and \( u \in C \cap L \). Let \( M \leq L \) be a minimal Levi subgroup containing \( u \). We wish to show that \( L = M \). By definition, \( C \) is obtained by extending to \( G \) the Richardson class in \( L \) of \( P \). If \( v \in L \) represents this Richardson class in \( L \) then \( v \) is distinguished in \( L \) by Lemma 5.3. Since \( u \) is conjugate to \( v \) (in \( G \)) we have \( \text{rank } C_G(u) = \text{rank } C_G(v) \). By Lemma 5.3 we have \( \dim Z(M) = \text{rank } C_G(u) = \text{rank } C_G(v) = \dim Z(L) \) whence \( L = M \).

Part (iii). Corollary 4.2 shows that \( \psi \), restricted to those pairs where \( L = G \), is injective and part (ii) shows that the image of this restriction is a subset of the distinguished classes of \( G \). Then Corollary 5.2 shows that \( \psi \) defined on all of BC-pairs\((G)\) is injective.

For surjectivity, we have two cases. If \( G \) is a classical group, we use the description of distinguished unipotent classes in [17, II.7.10] or [5] and apply
Lemma 4.1 to see that $\psi$, applied to those pairs $(L, P)$ where $L = G$, has image equal to all the distinguished classes of $G$. Then Corollary 5.2 shows that $\psi$ is surjective. If $G$ is exceptional it is simpler to count all pairs $(L, P)$ and compare this to the number of unipotent classes in $G$ as found in [10], which draws on [11], [12], [15], [16], [21].

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