Higher order evolution inequalities with nonlinear convolution terms

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Abstract

We are concerned with the study of existence and nonexistence of weak solutions to

\[
\begin{aligned}
\frac{\partial^k u}{\partial t^k} + (-\Delta)^m u &\geq (K * |u|^p)|u|^q &\text{in } \mathbb{R}^N \times \mathbb{R}_+, \\
\frac{\partial^i u}{\partial t^i}(x,0) &= u_i(x) &\text{in } \mathbb{R}^N, 0 \leq i \leq k-1,
\end{aligned}
\]

where $N, k, m \geq 1$ are positive integers, $p, q > 0$ and $u_i \in L^1_{\text{loc}}(\mathbb{R}^N)$ for $0 \leq i \leq k - 1$. We assume that $K$ is a radial positive and continuous function which decreases in a neighbourhood of infinity. In the above problem, $K * |u|^p$ denotes the standard convolution operation between $K(|x|)$ and $|u|^p$. We obtain necessary conditions on $N, m, k, p$ and $q$ such that the above problem has solutions. Our analysis emphasizes the role played by the sign of $\frac{\partial^{k-1} u}{\partial t^{k-1}}$.

Keywords: Higher-order evolution inequalities; nonlinear convolution terms; nonlinear capacity estimates.

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1 Introduction and the main results

Let $N, k, m \geq 1$ be positive integers. In this work we are concerned with the problem

$$
\begin{align*}
\frac{\partial^k u}{\partial t^k} + (-\Delta)^m u &\geq (K \ast |u|^p)|u|^q \quad \text{in } \mathbb{R}^{N+1}_+ := \mathbb{R}^N \times \mathbb{R}_+, \\
\frac{\partial^i u}{\partial t^i}(x,0) &= u_i(x) \quad \text{in } \mathbb{R}^N, \ 0 \leq i \leq k-1,
\end{align*}
$$

where $\mathbb{R}_+ = (0, \infty)$, $N \geq 1$, $p, q > 0$ and $u_i \in L^1_{\text{loc}}(\mathbb{R}^N)$ for $0 \leq i \leq k-1$. We assume that $K \in C(\mathbb{R}_+; \mathbb{R}_+)$ satisfies:

$$(A) \quad \text{there exists } R_0 > 1 \text{ such that } \inf_{r \in (0,R)} K(r) = K(R) \text{ for all } R > R_0.$$ 

In particular, condition $(A)$ above implies that $K$ is decreasing on the interval $(R_0, \infty)$. Typical examples of potentials $K$ satisfying the above conditions are the constant functions as well as

$$K(r) = r^{-\alpha}, \ \alpha \in (0, N) \quad \text{or} \quad K(r) = r^{-\alpha} \log^\beta(1+r), \ \alpha \in (0, N), \beta \in \mathbb{R}, \beta > \alpha - N.$$ 

The picture below illustrates some other functions $K(r)$ satisfying the hypothesis $(A)$.

By $K \ast |u|^p$ we denote the standard convolution operator defined by

$$
(K \ast |u|^p)(x,t) = \int_{\mathbb{R}^N} K(|x-y|)|u(y,t)|^p dy \quad \text{for all } (x,t) \in \mathbb{R}^{N+1}_+
$$

and we shall require the above integral to be finite for almost all $(x,t) \in \mathbb{R}^{N+1}_+$.

We are interested in weak solutions of $(1)$, that is, functions $u \in L^p_{\text{loc}}(\mathbb{R}^{N+1}_+)$ such that

(i) $(K \ast |u|^p)|u|^q \in L^1_{\text{loc}}(\mathbb{R}^{N+1}_+)$;

(ii) for any nonnegative test function $\varphi \in C_c^\infty(\mathbb{R}^{N+1}_+)$, we have
\[
\sum_{i=1}^{k} (-1)^i \int_{\mathbb{R}^N} u_{k-i}(x) \frac{\partial^{i-1}\varphi}{\partial t^{i-1}}(x,0) dx + \int_{\mathbb{R}^{N+1}_+} u \left[ (-1)^k \frac{\partial^k \varphi}{\partial t^k} + (-\Delta)^m \varphi \right] dx dt \\
\geq \int_{\mathbb{R}^{N+1}_+} (K \cdot |u|^p)|u|^q \varphi \ dx dt. \quad (3)
\]

Using a standard integration by parts it is easily seen that any classical solution of (1) is also a weak solution. Let us point out that condition (i) above implies that the convolution integral in (2) is finite for a.a. \((x,t) \in \mathbb{R}^{N+1}_+\). This further yields \(u \in L^p_{\text{loc}}(\mathbb{R}^{N+1}_+)\). Indeed for \(R > R_0\) and \(x, y \in B_R\) we have \(|x - y| \leq 2R\), so that by the definition of \(K\) and its monotonicity we deduce

\[
\infty > (K \cdot |u|^p)(x,t) \geq K(2R) \int_{B_R} |u(y,t)|^p dy. \quad (4)
\]

Since early 1980s many research works have been devoted to the study of the prototype evolution inequalities

\[
\frac{\partial u}{\partial t} - \Delta u \geq |u|^q \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} - \Delta u \geq |u|^q \quad \text{in} \ \mathbb{R}^{N+1}_+.
\]

To the best of our knowledge, the first study of nonexistence of solutions to higher order hyperbolic inequalities is due to L. Vérón and S.I. Pohozaev [21] related to

\[
\frac{\partial^2 u}{\partial t^2} - \mathcal{L}_m(\phi(u)) \geq |u|^q \quad \text{in} \ \mathbb{R}^{N+1}_+,
\]

where

\[
\mathcal{L}_m v = \sum_{|\alpha|=m} D^\alpha(a_\alpha(x,t)v) \quad \text{for some integer} \ m \geq 1
\]

and \(\phi\) is a locally bounded real-valued function such that \(|\phi(u)| \leq c|u|^p\), for some \(c,p > 0\). It is obtained in [21] that if \(q > \max\{p,1\}\) and one of the following conditions hold

either \(m \geq 2N\) or \(m < 2N \leq \frac{m(q+1)}{q-p}\),

then [5] has no solutions in \(\mathbb{R}^{N+1}_+\) satisfying

\[
\int_{\mathbb{R}^N} \frac{\partial u}{\partial t}(x,0) dx \geq 0. \quad (7)
\]

Condition (7) is essential in the study of nonexistence of solutions to [5] and will also play an important role in the study of [1] (see Theorem 1.1 below). Further, if \(m = 2\), it is obtained in [21] that if [6] fails to hold then [5] has a positive solution. Among the results for \(m = 2\), we quote also [10], where a weighted nonlinearity is considered.

Another set of results that motivate our present work are due to G.G. Laptev [16, 17] where the following problem is studied

\[
\begin{cases}
\frac{\partial^k u}{\partial t^k} - \Delta u \geq |x|^{-\sigma}|u|^q, u \geq 0 \quad \text{in} \ \Omega \times \mathbb{R}_+,
\frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0 \quad \text{in} \ \Omega.
\end{cases}
\]

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In the above, $\Omega \subset \mathbb{R}^N$ is either the exterior of a ball or an unbounded cone-like domain; for other results on hyperbolic inequalities in exterior domains see [14, 15, 18]. We observe that solutions of (8) are also required to satisfy (7). It is obtained in [17] that if $\sigma > -2$ then the above problem has no solutions provided that

$$1 < q < q^*_k := 1 + \frac{2N + 2 + \sigma}{N - 2 + 2/k}.$$ 

The above exponent $q^*_k$ coincides with the Fujita-Hayakawa critical exponent if $k = 1$, that is, $q^*_1 = 1 + \frac{2 + \sigma}{N}$ and with the Kato critical exponent if $k = 2$, that is, $q^*_2 = 1 + \frac{2 + \sigma}{N - 1}$.

The study of convolution terms in time-dependent partial differential equations goes back to near a century ago. Indeed, the equation

$$i\psi_t - \Delta \psi = (|x|^\alpha - N \ast \psi^2)\psi \quad \text{in } \mathbb{R}^N, \alpha \in (0, N), N \geq 1,$$ 

(9)

was introduced by D.H. Hartree [11, 12, 13] in 1928 for $N = 3$ and $\alpha = 2$ in relation to the Schrödinger equation in quantum physics. Nowadays, (9) bears the name of Choquard equation. Stationary solutions to (9) and its related equations have been extensively investigated from various perspectives: ground states, isolated singularities, symmetry of solutions; see e.g., [3, 6, 7, 8, 9, 19, 20]. In a different direction, G. Whithman [22, Section 6] considered the nonlocal equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \int_{-\infty}^\infty K(x - y) \frac{\partial u}{\partial x}(y, t)dy = 0$$

to study general dispersion in water waves; for recent results on nonlocal evolution equations we refer the reader to [1, 2]. In [4] it is studied a class of quasilinear parabolic inequalities featuring nonlocal convolution terms in the form

$$\frac{\partial u}{\partial t} - Lu \geq (K \ast u^p)u^q \quad \text{in } \mathbb{R}^{N+1},$$

where $Lu$ is a quasilinear operators that contains as prototype model the $m$-Laplacian and the generalized mean curvature operator.

In the present work we investigate the existence and nonexistence of weak solutions to (1). The stationary solutions to (1) were discussed recently in [6]. Our main result concerning the inequality (1) reads as follows.

**Theorem 1.1.** Assume $N, m, k \geq 1$ and $p, q > 0$.

(i) If $k \geq 1$ is a even integer and $q \geq 1$, then (1) admits some positive solutions $u \in C^\infty(\mathbb{R}^{N+1})$ which verify

$$u_{k-1} = \frac{\partial^{k-1}u}{\partial t^{k-1}}(\cdot, 0) < 0 \quad \text{in } \mathbb{R}^N.$$ 

(ii) If $p + q > 2$ and

$$\limsup_{R \to \infty} K(R)R^{\frac{2N+2m/k - N + 2m(1-1/k)}{p+q}} > 0,$$ 

(10)

then (1) has no nontrivial solutions such that

$$u_{k-1} \geq 0 \quad \text{or} \quad u_{k-1} \in L^1(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} u_{k-1}(x)dx > 0.$$ 

(11)
Let us note that condition (10) and the fact that $K$ is decreasing in a neighbourhood of infinity imply that
\[
\frac{2N + 2m/k}{p + q} \geq N - 2m \left(1 - \frac{1}{k}\right).
\]
Also, under extra assumptions on $u_{k-1}$ and strengthening (10), we can handle the case $\int u_{k-1}(x) = 0$ in (11) for which the same conclusion as in Theorem 1.1(ii) holds (see Proposition 2.3).

The proof of Theorem 1.1 relies on nonlinear capacity estimates specifically adapted to the nonlocal setting of our problem. More precisely, we derive integral estimates in time for the new quantity
\[
J(t) = \int_{\mathbb{R}^N} u^{\frac{p+q}{2}}(x,t) \varphi(x,t) \, dx, \quad t \geq 0,
\]
where $\varphi$ is a specially constructed test function with compact support (see (26)).

Theorem 1.1 shows a sharp contrast in the existence/nonexistence diagram according to whether $\partial_{x_{k-1}}u_{k-1}(x,0)$ has constant sign (positive or negative) on $\mathbb{R}^N$. To better illustrate this fact, let us discuss the case of pure powers in the potential $K(r) = r^{-\alpha}$, $\alpha \in (0,N)$ and $k = 1,2$.

Let us first consider the parabolic problem
\[
\begin{cases}
\frac{\partial u}{\partial t} + (-\Delta)^m u \geq (|x|^{-\alpha} * |u|^p)|u|^q & \text{in } \mathbb{R}^{N+1}_+ := \mathbb{R}^N \times \mathbb{R}_+, \\
u(x,0) = u_0(x) & \text{in } \mathbb{R}^N.
\end{cases}
\]
(12)

Note that since $k = 1$ in (1), condition (11) is satisfied for all nonnegative solutions of (12).

**Corollary 1.2.** Let $N, m \geq 1$, $p, q > 0$ and $\alpha \in (0,N)$.

(i) If $0 < \alpha < m$ and
\[
2 < p + q \leq \frac{2N + 2m}{N + \alpha},
\]
then (11) has no nontrivial nonnegative solutions;

(ii) If $N > 2m$ and
\[
\min\{p,q\} > \frac{N - \alpha}{N - 2m} \quad \text{and} \quad p + q > \frac{2N - \alpha}{N - 2m},
\]
then (12) has positive solutions.

Part (i) in the above result follows from Theorem 1.1 while part (ii) follows from [6, Theorem 1.4] where the stationary case of (12) was discussed. Let us note that for $m = 1$, the nonexistence of a nonnegative solution in Corollary 1.2(i) was already observed in [4].

We next take $K(r) = r^{-\alpha}$, $k = 2$ and $m = 1$ in Theorem 1.1. We thus consider
\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} - \Delta u \geq (|x|^{\alpha} * |u|^p)|u|^q & \text{in } \mathbb{R}^{N+1}_+ := \mathbb{R}^N \times \mathbb{R}_+, \\
u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \\
u_t(x,0) = u_1(x) & \text{in } \mathbb{R}^N.
\end{cases}
\]
(13)

Our result on problem (13) is stated below.
Corollary 1.3. Let $N, m \geq 1$, $p, q > 0$ and $\alpha \in (0, N)$.

(i) If $q \geq 1$, then (1) admits some positive solutions $u \in C^2(\mathbb{R}^{N+1}_+)$ which verify

$$\frac{\partial u}{\partial t}(\cdot, 0) < 0 \quad \text{in } \mathbb{R}^N.$$ 

(ii) If $0 < \alpha < \min\{N, 3/2\}$ and

$$2 < p + q \leq \frac{2N + 1}{N + \alpha - 1},$$

then (13) has no nontrivial solutions satisfying

$$\frac{\partial u}{\partial t}(x, \cdot) \geq 0 \quad \text{in } \mathbb{R}^N \quad \text{or} \quad \int_{\mathbb{R}^N} \frac{\partial u}{\partial t}(x, \cdot)dx > 0. \quad (14)$$

(iii) If $N > 2$ and

$$\min\{p, q\} > \frac{N - \alpha}{N - 2} \quad \text{and} \quad p + q > \frac{2N - \alpha}{N - 2}, \quad (15)$$

then (13) admits some positive solutions which verify $\frac{\partial u}{\partial t}(x, \cdot) > 0$ in $\mathbb{R}^N$.

The diagram of existence/nonexistence of a weak solution to (13) satisfying (14) in the $pq$-plane is given below. The light shaded region represents the region for which (13) admits solutions satisfying (14), while the dark shaded region describes the pairs $(p, q)$ for which no such solutions exist. Corollary 1.3 leaves open the issue of existence and nonexistence in the white regions of the $(p > 0, q > 0)$ quadrant.

![Diagram of existence/nonexistence in the pq-plane](image-url)
Theorem 1.1 also applies to the case where \( K \equiv 1 \) for which (1) reads
\[
\begin{align*}
\frac{\partial^k u}{\partial t^k} - \Delta^m u & \geq \left( \int_{\mathbb{R}^N} |u(y)|^p dy \right) |u|^q \quad \text{in } \mathbb{R}^{N+1}_+, \\
\frac{\partial^k u}{\partial t^k}(x, 0) & = u_i(x) \quad \text{in } \mathbb{R}^N, \quad 0 \leq i \leq k - 1.
\end{align*}
\] (16)

A similar conclusion to Corollary 1.2 and Corollary 1.3 (in which we let \( \alpha = 0 \)) hold for (16). We can further employ the ideas in the study of (1) to the case of systems of type
\[
\begin{align*}
\frac{\partial^k u}{\partial t^k} + (-\Delta)^m u & \geq (K * |v|^p)|v|^q \quad \text{in } \mathbb{R}^{N+1} = \mathbb{R}^N \times (0, \infty), \\
\frac{\partial^k v}{\partial t^k} + (-\Delta)^m v & \geq (L * |u|^n)|u|^s \quad \text{in } \mathbb{R}^{N+1}, \\
\frac{\partial^k u}{\partial t^k}(x, 0) & = u_i(x) \quad \text{in } \mathbb{R}^N, \quad 0 \leq i \leq k - 1, \\
\frac{\partial^k v}{\partial t^k}(x, 0) & = v_i(x) \quad \text{in } \mathbb{R}^N, \quad 0 \leq i \leq k - 1,
\end{align*}
\] (17)
where \( N, m, k \geq 1, p, q, n, s > 0 \). We assume \( u_i, v_i \in L^1_{loc}(\mathbb{R}^N), 0 \leq i \leq k - 1 \) and that \( K, L \) satisfy condition (A) for some \( R_0 > 1 \).

A pair \((u, v) \in L^1_{loc}(\mathbb{R}^{N+1}) \times L^p_{loc}(\mathbb{R}^{N+1})\) is a weak solution of (17) if:

(i) \((L * u^n)u^s, (K * v^p)v^q \in L^1_{loc}(\mathbb{R}^{N+1})\);

(ii) for any nonnegative test function \( \varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}) \), we have
\[
\begin{align*}
\sum_{i=1}^{k}(-1)^i \int_{\mathbb{R}^N} u_{k-i}(x) \frac{\partial^{i-1} \varphi}{\partial t^{i-1}}(x, 0) dx + \int_{\mathbb{R}^{N+1}} u \left[ (-1)^k \frac{\partial^k \varphi}{\partial t^k} + (-\Delta)^m \varphi \right] dx dt & \\
& \geq \int_{\mathbb{R}^{N+1}} (K * |v|^p)|v|^q \varphi dx dt,
\end{align*}
\] (18)
\[
\sum_{i=1}^{k}(-1)^i \int_{\mathbb{R}^N} v_{k-i}(x) \frac{\partial^{i-1} \varphi}{\partial t^{i-1}}(x, 0) dx + \int_{\mathbb{R}^{N+1}} v \left[ (-1)^k \frac{\partial^k \varphi}{\partial t^k} + (-\Delta)^m \varphi \right] dx dt & \\
& \geq \int_{\mathbb{R}^{N+1}} (L * |u|^n)|u|^s \varphi dx dt.
\]

Our main result concerning (18) is stated below.

**Theorem 1.4.** Assume \( \min\{p + q, n + s\} > 2 \). If either
\[
\limsup_{R \to \infty} K(R)L(R) \frac{2^{N+2m/k}}{R^{(n+s)(p+q)}} \frac{N+2m}{n+s} - N+2m \left( \frac{1}{p} \right) > 0,
\] (19)

or
\[
\limsup_{R \to \infty} K(R)^{p/q} L(R) R^{\frac{2^{N+2m/k}}{p+q}} \frac{N+2m}{p+q} - N+2m \left( \frac{1}{q} \right) > 0,
\] (20)
then (18) has no nontrivial solutions such that
\[ u_{k-1}, v_{k-1} \geq 0 \quad \text{or} \quad u_{k-1}, v_{k-1} \in L^1(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} u_{k-1}(x) dx, \int_{\mathbb{R}^N} v_{k-1}(x) dx > 0. \quad (21) \]

The next sections contain the proofs of the above results.

2 Proof of Theorem 1.1

Proof of Theorem 1 (i) Let \( \gamma > 0 \) be such that \( p\gamma > N \) and define
\[ u(x, t) = e^{-Mt} v(x)^{-\gamma/2} \quad \text{with} \quad v(x) = 1 + |x|^2, \quad (22) \]
where \( M > 1 \) will be specified later. Since \( K > 0 \) is continuous in \( \mathbb{R}^+ \) and decreasing in a neighbourhood of infinity (by condition (A)) it follows that
\[ \sup_{[1, \infty)} K = \max_{[1, \infty)} K < \infty. \]

Furthermore, we have
\[
(K * v^{-\gamma/2})(x) = \int_{\mathbb{R}^N} K(|z|)(1 + |z - x|^2)^{-p\gamma/2} dz
\]
\[
\leq \int_{B_1} K(|z|) dz + \left( \max_{[1, \infty)} K \right) \int_{\mathbb{R}^N \setminus B_1} (1 + |z - x|^2)^{-p\gamma/2} dz
\]
\[
\leq \int_{B_1} K(|z|) dz + \left( \max_{[1, \infty)} K \right) \int_{\mathbb{R}^N} (1 + |z - x|^2)^{-p\gamma/2} dz
\]
\[
= \int_{B_1} K(|z|) dz + \left( \max_{[1, \infty)} K \right) \int_{\mathbb{R}^N} (1 + |y|^2)^{-p\gamma/2} dy
\]
\[
\leq C(K, p, \gamma), \quad (23)
\]
since \( p\gamma > N \).

Further, a direct calculation shows that \( -\Delta (v^{-\gamma/2}) = c_1 v^{-\gamma/2 - 1} + c_2 v^{-\gamma/2 - 2} \) in \( \mathbb{R}^N \), where \( c_1, c_2 \) are real coefficients depending on \( N \) and \( \gamma \). Hence, an induction argument yields
\[ (-\Delta)^m (v^{-\gamma/2}) = v^{-2m-\gamma/2} \sum_{j=0}^m c_j v^j \quad \text{in} \quad \mathbb{R}^N, \]
where \( c_j = c_j(\gamma, N, m) \in \mathbb{R} \). Thus, the function \( u \) given by (22) satisfies
\[ \frac{\partial^k u}{\partial t^k} + (-\Delta)^m u = e^{-Mt} \left( (-1)^k M^k v^{2m} + \sum_{j=0}^m c_j v^j \right) v^{-2m-\gamma/2} \quad \text{in} \quad \mathbb{R}^{N+1}, \quad (24) \]
where \( c_j \in \mathbb{R} \) are independent of \( M \). Using the fact that \( k \) is an even integer, by taking \( M > 1 \) large, we may ensure that
\[ M^k v^{2m} + \sum_{j=0}^m c_j v^j \geq C(K, p, \gamma) v^{2m} \quad \text{in} \quad \mathbb{R}^N, \quad (25) \]
where \(C(K, p, \gamma) > 0\) is the constant from (23). Now, combining (23)-(25) and using that 
\[u^p e^{M pt} = v^{-\gamma^2/2},\]
we deduce
\[
\frac{\partial^k u}{\partial t^k} - \Delta^m u \geq C(K, p, \gamma) e^{-M t v^{-\gamma^2/2}} \\
\geq C(K, p, \gamma) e^{-(p+q) M t v^{-\gamma^2/2}} \\
\geq (K * u^p) e^{-q M t v^{-\gamma^2/2}} \\
\geq (K * u^p) (e^{-M t v^{-\gamma^2/2}})^q \\
\geq (K * u^p) u^q \quad \text{in } \mathbb{R}^{N+1},
\]

being \(q \geq 1\), which concludes the proof of part (i). Since \(k\) is even, one has
\[u_{k-1}(x) = \frac{\partial^{k-1} u}{\partial t^{k-1}}(x, 0) = (-1)^{k-1} M^k u(x) < 0.\]

(ii) Assume that \(p + q > 2\), (10) and (11) hold and that (1) admits a weak solution \(u\). We shall first construct a suitable test function \(\varphi\) for (3). To do so, take a standard cut off function \(\varrho \in C_{\infty}^c(\mathbb{R})\) such that:

- \(\varrho = 1\) in \((0, 1)\), \(\varrho = 0\) in \((2, \infty)\);
- \(0 \leq \varrho \leq 1\), \(\text{supp } \varrho \subseteq [0, 2]\).

Now take \(R > 0, \gamma > 0\) to be precised later, \(\kappa \geq 1\) sufficiently large and consider the function
\[
\varphi(x) = \varrho^p \left(\frac{|x|}{R}\right) \varrho^q \left(\frac{t}{R^\gamma}\right) \quad \text{in } \mathbb{R}^{N+1}. \tag{26}
\]

Clearly
\[\text{supp } \varphi \subset B_{2R} \times [0, 2R^\gamma) \subset \mathbb{R}^{N+1}, \quad \tag{27}\]

and
\[
\text{supp } \frac{\partial^k \varphi}{\partial t^k} \subset B_{2R} \times [R^\gamma, 2R^\gamma) \subset \text{supp } \varphi, \quad \text{supp } \Delta \varphi \subset (B_{2R} \setminus B_R) \times [0, 2R^\gamma) \subset \text{supp } \varphi. \tag{28}\]

As in [5, Lemma 3.1], we have the following estimates:

**Proposition 2.1.** Let \(\varphi\) be defined in (26). Then for \(\varsigma > 1\) and \(\kappa \geq 2m\varsigma\) one has
\[
\int_{\text{supp } \varphi} \frac{1}{\varphi^{\varsigma-1}} \left| \frac{\partial^i \varphi}{\partial t^i} \right| dx dt \leq CR^{N+\gamma - i\varsigma}, \quad i \geq 1, \tag{29}\]
\[
\int_{\text{supp } \varphi} \frac{|\Delta^m \varphi|^{\varsigma}}{\varphi^{\varsigma-1}} dx dt \leq CR^{N+\gamma - 2m\varsigma}, \tag{30}\]

where \(C\) is a positive constant changing from line to line.
Since \( \frac{\partial \varphi}{\partial t} (x, 0) = 0 \) in \( \mathbb{R}^N \) for all \( i \geq 1 \), from (3) we deduce

\[
\int_{\mathbb{R}^N} u_{k-1}(x) \varphi(x, 0) \, dx + \int_{\mathbb{R}_+^N} (K * |u|^p) |u|^q \varphi \, dxdt \\
\leq \int_{\mathbb{R}_+^N} |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| \, dxdt + \int_{\mathbb{R}_+^N} |u| |\Delta^m \varphi| \, dxdt.
\] (31)

Observe that by (11) we have \( u_{k-1} \geq 0 \) or \( u_{k-1} \in L^1(\mathbb{R}^N) \) and \( \int_{\mathbb{R}^N} u_{k-1}(x) \, dx > 0 \). In the latter case, from \( u_{k-1} \in L^1_{loc}(\mathbb{R}^N) \), we deduce, using (26),

\[
\lim_{R \to \infty} \int_{\mathbb{R}^N} u_{k-1}(x) \varphi(x, 0) \, dx = \lim_{R \to \infty} \int_{\mathbb{R}^N} u_{k-1}(x) \varphi \left( \frac{|x|}{R} \right) \, dx = \int_{\mathbb{R}^N} u_{k-1}(x) \, dx.
\] (32)

Thus, from (11) we deduce that for large \( R > 0 \) we have

\[
\int_{\mathbb{R}^N} u_{k-1}(x) \varphi(x, 0) \, dx \geq 0,
\] (33)

case in which (31) yields

\[
\int_{\mathbb{R}_+^N} (K * |u|^p) |u|^q \varphi \, dxdt \leq \int_{\mathbb{R}_+^N} |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| \, dxdt + \int_{\mathbb{R}_+^N} |u| |\Delta^m \varphi| \, dxdt,
\] (34)

provided \( R > 0 \) in the definition of \( \varphi \) (see (26)) is large enough.

An important tool of our approach is the following result.

**Lemma 2.2.** For almost all \( t \geq 0 \) we have \( u(\cdot, t) \in L^{p+q}_{loc}(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} (K * |u|^p) |u(x, t)|^q \varphi(x, t) \, dx \geq K(4R) J^2(t) \quad \text{for all} \quad R \geq R_0,
\] (35)

where

\[
J(t) = \int_{\mathbb{R}^N} |u(x, t)|^{\frac{p+q}{2}} \varphi(x, t) \, dx.
\] (36)

**Proof.** First note that for \( x, y \in B_{2R} \) one has \( |x - y| \leq 4R \), so that, thanks to the monotonicity of \( K \),

\[
\int_{\mathbb{R}^N} K(|x - y|) |u(y)|^p \, dy \geq \int_{B_{2R}} K(|x - y|) |u(y)|^p \, dy \geq K(4R) \int_{B_{2R}} |u(y)|^p \, dy.
\]

Hence

\[
\int_{\mathbb{R}^N} (K * |u|^p) |u(x, t)|^q \varphi(x, t) \, dx \geq K(4R) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(y, t)|^p \varphi(y, t) |u(x, t)|^q \varphi(x, t) \, dx \, dy,
\] (37)

where we have used that \( \varphi \leq 1 \) and that \( \varphi(\cdot, t) \equiv 0 \) outside of \( B_{2R} \) for all \( t \).
Furthermore, by Hölder’s inequality we have
\[
\left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(y, t)|^p \varphi(y, t)|u(x, t)|^q \varphi(x, t) \, dx \, dy \right)^2
\]
\[
= \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(y, t)|^p \varphi(y, t)|u(x, t)|^q \varphi(x, t) \, dx \, dy \right) \cdot \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x, t)|^p \varphi(x, t)|u(y, t)|^q \varphi(y, t) \, dx \, dy \right)
\]
\[
\geq \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x, t)|^{\frac{p+q}{2}} |u(y, t)|^{\frac{p+q}{2}} \varphi(x, t) \varphi(y, t) \, dx \, dy \right)^2
\]
\[
= \left( \int_{\mathbb{R}^N} |u(x, t)|^{\frac{p+q}{2}} \varphi(x, t) \, dx \right)^4 = J(t)^4
\]
which, by (37) and part (i) in the definition of a solution, yields \( u(\cdot, t) \in L^\frac{p+q}{p+q-2}(\mathbb{R}^N) \) and (35).

Inserting (35) into (34) we find
\[
K(4R) \int_0^{2R^\gamma} J^2(t) \, dt \leq \int_{R^N_+} |u| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx \, dt.
\]
(38)

We next estimate the integral term on the right hand side of (38). Using Hölder’s inequality, we have
\[
\int_{R^N_+} |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| \, dx \, dt \leq \left( \int_{\text{supp}(\frac{\partial^k \varphi}{\partial t^k})} |u|^{\frac{p+q}{2}} \varphi \, dx \, dt \right)^2 \cdot \left( \int_{\text{supp}(\frac{\partial^k \varphi}{\partial t^k})} \varphi^{-\frac{2}{p+q-2}} \left| \frac{\partial^k \varphi}{\partial t^k} \right|^{\frac{p+q-2}{p+q}} \right)^{\frac{2}{p+q}}
\]
\[
\leq C \left( \int_{\text{supp}(\frac{\partial^k \varphi}{\partial t^k})} |u|^{\frac{p+q}{2}} \varphi \, dx \, dt \right)^{\frac{2}{p+q}} R^{-k \gamma+(N+\gamma)\frac{p+q-2}{p+q}},
\]
(39)
where in the last inequality we have used (29) with \( \zeta = \frac{p+q}{p+q-2}, i = k \) and thanks to (28). Note that
\[
\text{supp}(\frac{\partial^k \varphi}{\partial t^k}) \subset B_{2R} \times [R^\gamma, 2R^\gamma].
\]
(40)

By Hölder’s inequality we find
\[
\int_{R^N_+} |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| \, dx \, dt \leq CR^{-k \gamma+(N+\gamma)\frac{p+q-2}{p+q}} \left( \int_{R^\gamma} \left( \int_{\mathbb{R}^N} |u|^{\frac{p+q}{2}} \varphi \, dx \right) \, dt \right)^{\frac{2}{p+q}}
\]
\[
\leq CR^{-k \gamma+(N+\gamma)\frac{p+q-2}{p+q}} + \frac{\gamma}{p+q} \left( \int_{R^\gamma} \left( \int_{\mathbb{R}^N} |u|^{\frac{p+q}{2}} \varphi \, dx \right)^2 \, dt \right)^{\frac{1}{p+q}}.
\]
(41)

Similarly, by Hölder’s inequality, (30) with \( \zeta = \frac{p+q}{p+q-2} \) and (28), we find
\[
\int_{R^N_+} |u| |\Delta^m \varphi| \, dx \, dt \leq \left( \int_{\text{supp}(\Delta^m \varphi)} |u|^{\frac{p+q}{2}} \varphi \, dx \, dt \right)^{\frac{2}{p+q}} \cdot \left( \int_{\text{supp}(\Delta^m \varphi)} \varphi^{-\frac{2}{p+q-2}} |\Delta^m \varphi|^{\frac{p+q-2}{p+q}} \right)^{\frac{p+q-2}{p+q}}
\]
\[
\leq C \left( \int_{\text{supp}(\Delta^m \varphi)} |u|^{\frac{p+q}{2}} \varphi \, dx \, dt \right)^{\frac{2}{p+q}} R^{-2m+(N+\gamma)\frac{p+q-2}{p+q}}.
\]
(42)
Since

$$\text{supp}(\Delta^m \varphi) \subset (B_{2R} \setminus B_R) \times [0, 2R^\gamma],$$

(43)
a new application of Hölder’s inequality in the above estimate yields

$$\int_{\mathbb{R}^N} |u| |\Delta^m \varphi| dx dt \leq CR^{-2m + (N + \gamma) \frac{p+q}{p+q-2}} \left( \int_0^{2R^\gamma} \left( \int_{B_{2R} \setminus B_R} |u|^{\frac{p+q}{2}} \varphi \, dx \right) \, dt \right)^{\frac{2}{p+q}}$$

(44)

\[ \leq CR^{-2m + (N + \gamma) \frac{p+q}{p+q-2} + \frac{\gamma}{p+q}} \left( \int_0^{2R^\gamma} \left( \int_{B_{2R} \setminus B_R} |u|^{\frac{p+q}{2}} \varphi \, dx \right)^2 \, dt \right)^{\frac{1}{p+q}}. \]

Comparing the powers of $R$ in (41) and (44) we are led to choose $\gamma > 0$ so that $k\gamma = 2m$, that is $\gamma = 2m/k$. Also,

$$-2m + (N + \gamma) \frac{p + q - 2}{p + q} + \frac{\gamma}{p + q} = N - 2m \left( 1 - \frac{1}{k} \right) - \frac{2N + 2m}{p + q}. \]

Thus, (38) together with (41) and (44) yield

$$K(4R) \int_0^{2R^\gamma} J^2(t) \, dt \leq CR^{-2m \left( 1 - \frac{1}{k} \right)} - \frac{2N + 2m/k}{p + q} \times \left[ \left( \int_{B_{2R} \setminus B_R} |u|^{\frac{p+q}{2}} \varphi \, dx \right)^2 \right]^{\frac{1}{p+q}} \]$$

(45)

Using (36), the above estimate implies

$$K(4R) \int_0^{2R^\gamma} J^2(t) \, dt \leq CR^{-2m \left( 1 - \frac{1}{k} \right)} - \frac{2N + 2m/k}{p + q} \left( \int_0^{2R^\gamma} J^2(t) \, dt \right)^{\frac{1}{p+q}}$$

(46)

which further yields

$$\left( \int_0^{2R^\gamma} J^2(t) \, dt \right)^{\frac{p+q-1}{p+q}} \leq C \frac{1}{K(4R) R^{\frac{2N + 2m/k}{p + q} - N + 2m(1-1/k)}}. \]

(47)

Let $\{R_j\}_j$ be a divergent sequence that achieves the limsup in (10), namely

$$K(4R_j) R_j^{\frac{2N + 2m/k}{p + q} - N + 2m(1-1/k)} \to \ell > 0 \quad \text{as} \quad j \to \infty. \]

Passing to a subsequence we may assume $R_j > 2R_{j-1}$ for all $j > 1$.

If $\ell = \infty$, we can pass to the limit in (47), by replacing $R$ with $R_j$, to raise $\int_0^\infty J^2(t) \, dt = 0$ from where $J \equiv 0 \text{ in } \mathbb{R}^+$ and then, by the definition of $J$,

$$\int_{B_R \times (0, \infty)} |u|^{(p+q)/2} \, dx \, dt = 0 \quad \text{for all } \quad R > 0,$$

namely $u \equiv 0 \text{ in } \mathbb{R}^N_{+1}$ as required.

If $\ell \in (0, \infty)$, then (47) shows that $J \in L^2(0, \infty)$. Using this fact we infer that

$$\int_{R_j}^{2R_j} \left( \int_{\mathbb{R}^N} |u(x, t)|^{\frac{p+q}{2}} \varphi(x, t) \, dx \right)^2 \, dt \to 0, \quad \int_0^{2R^\gamma} \left( \int_{B_{2R} \setminus B_{R_j}} |u(x, t)|^{\frac{p+q}{2}} \varphi(x, t) \, dx \right)^2 \, dt \to 0, \quad (48)$$

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as $j \to \infty$. Indeed, the first limit in (48) follows immediately by the fact that $J \in L^2(0, \infty)$. To check the second limit in (48) we observe that

$$\infty > \int_0^\infty \left( \int_{\mathbb{R}^N} |u(x, t)|^\frac{p+q}{p-q} dx \right)^2 dt \geq \int_0^\infty \left( \sum_{j \geq 1} \int_{B_{2R_j} \setminus B_{R_j}} |u(x, t)|^\frac{p+q}{p-q} dx \right)^2 dt$$

$$\geq \int_0^\infty \sum_{j \geq 1} \left( \int_{B_{2R_j} \setminus B_{R_j}} |u(x, t)|^\frac{p+q}{p-q} dx \right)^2 dt$$

$$= \sum_{j \geq 1} \int_0^\infty \left( \int_{B_{2R_j} \setminus B_{R_j}} |u(x, t)|^\frac{p+q}{p-q} dx \right)^2 dt.$$ 

The convergence of the last series in the above estimate implies the second part of (48). Using this fact in (45) we find

$$\left( \int_{0}^{2R_j} J^2(t) dt \right)^\frac{p+q-2}{p-q} \leq C \frac{1}{K(4R_j)R_j^{\frac{2N+2m}{p+q}-N+2m(1-1/k)}} \rho(1) \quad \text{as } j \to \infty.$$ 

Again, by letting $R_j \to \infty$, we can conclude, also when $\ell \in (0, \infty)$, that $\int_0^\infty J^2(t) dt = 0$, namely $u \equiv 0$ in $\mathbb{R}^{N+1}_{+}$.

A similar argument allows us to treat the case $\int_{\mathbb{R}^N} u_{k-1}(x) dx = 0$. In this case we need to be more precise on the behavior of $u_{k-1}$.

**Proposition 2.3.** Let $\varrho \in C^\infty_c(\mathbb{R})$ satisfy supp $\varrho \subseteq [0, 2]$, $0 \leq \varrho \leq 1$ and $\varrho = 1$ in $(0, 1)$. Assume that for some $\kappa \geq 2m$ we have

$$\int_{\mathbb{R}^N} u_{k-1}(x) \varrho^k \left( \frac{x}{R} \right) dx = O(R^{-\beta}) \quad \text{as } R \to \infty, \quad \text{for some } \beta > 0. \quad (49)$$

If

$$\limsup_{R \to \infty} K(R) R^{\min(\beta, \frac{2N+2m}{p+q}-N+2m(1-1/k))} > 0, \quad (50)$$

then (1) has no nontrivial solutions.

**Proof.** With the help of the above $\varrho$ we construct the test function $\varphi$ as defined in (26). Thus, (33) is no more in force and (29) changes to

$$\int_{\mathbb{R}^{N+1}_+} (K * |u|^p)|u|^q \varphi dx dt \leq CR^{-\beta} + \int_{\mathbb{R}^{N+1}_+} |u|^q \varphi dx dt + \int_{\mathbb{R}^{N+1}_+} |u|\Delta^m \varphi dx dt.$$ 

Consequently using (41) and (44), with $\gamma = 2k/m$, and (35), then the above inequality gives

$$K(4R) \int_{0}^{2R^\gamma} J^2(t) dt \leq CR^{-\beta} + CR^{N-2m(1-1/k)} \frac{2N+2m}{p+q} \left( \int_{0}^{2R^\gamma} J^2(t) dt \right)^{\frac{1}{p+q}}. \quad (51)$$

Let $\sigma = \frac{2N+2m}{p+q} - N + 2m(1-1/k)$. By Young’s inequality we have

$$CR^{N-2m(1-1/k)} \frac{2N+2m}{p+q} \left( \int_{0}^{2R^\gamma} J^2(t) dt \right)^{\frac{1}{p+q}} = CR^{-\sigma} \left( \int_{0}^{2R^\gamma} J^2(t) dt \right)^{\frac{1}{p+q}} = CR^{-\sigma} K(4R)^{-\frac{1}{p+q}} (K(4R) \int_{0}^{2R^\gamma} J^2(t) dt)^{\frac{1}{p+q}}$$

$$\leq \frac{K(4R)}{2} \int_{0}^{2R^\gamma} J^2(t) dt + CR^{-\sigma(p+q)} K(4R)^{-\frac{1}{p+q}}.$$
Using this last inequality into (51) we find
\[
\frac{K(4R)}{2} \int_0^{2R^2} J^2(t) dt \leq CR^{-\beta} + CR^{-\sigma(p+q)/(p+q-1)} K(4R)^{-\frac{1}{p+q-1}},
\]
that is,
\[
\int_0^{2R^2} J^2(t) dt \leq \frac{C}{R^\beta K(4R)} + \frac{C}{(R^\sigma K(4R))^{\frac{p+q-1}{p+q-1}}}.
\]
Now, in virtue of (50) we may let \( R \to \infty \) in the above estimate to deduce \( J = 0 \) and then \( u \equiv 0 \).

### 3 Proof of Corollary 1.3

Part (i) and (ii) in Corollary 1.3 follow directly from Theorem 1.1.

(iii) Let \( p, q, \alpha \) satisfy (15) which we may write as
\[
(N - 2) \min\{p, q\} > N - \alpha \quad \text{and} \quad (N - 2)(p + q - 1) > N + 2 - \alpha.
\]
Thus, we may choose \( \beta \in (0, N - 2) \) such that
\[
\beta \min\{p, q\} > N - \alpha, \quad \beta p \neq N \quad \text{and} \quad \beta(p + q - 1) > N + 2 - \alpha \quad (52)
\]
Set \( w(x, t) = (1 + t)^{-a} + 1 + |x|^2 \) where \( a > 0 \) will be precised later and \( u(x, t) = w^{-\beta/2}(x, y) \).
Then
\[
-\Delta u = \beta N w^{-\beta/2-1} - \beta(\beta + 2)w^{-\beta/2-2}|x|^2
= \beta \left[ N - (\beta + 2) \frac{|x|^2}{w} \right] w^{-\beta/2-1} \geq \beta(N - \beta - 2)w^{-(\beta+2)/2} \quad \text{in} \ \mathbb{R}^{N+1}_+.
\]
(53)
Also,
\[
\frac{\partial^2 u}{\partial t^2} = -\frac{a(a + 1)\beta}{2} (1 + t)^{-a-2} w^{-(\beta+2)/2} + \frac{a^2 \beta(\beta + 2)}{4} (1 + t)^{-2a-2} w^{-(\beta+2)/2-1}
= \frac{a\beta}{2} w^{-(\beta+2)/2}(1 + t)^{-a-2} \left[ -a - 1 + \frac{a}{2}(\beta + 2) \frac{(1 + t)^a}{w} \right]
\geq -\frac{a(a + 1)\beta}{2} w^{-(\beta+2)/2} \quad \text{in} \ \mathbb{R}^{N+1}_+.
\]
(54)
Combining now (53) and (54) we have
\[
\frac{\partial^2 u}{\partial t^2} - \Delta u \geq \beta \left( N - 2 - \beta - \frac{a(a + 1)}{2} \right) w^{-(\beta+2)/2} \quad \text{in} \ \mathbb{R}^{N+1}_+.
\]
Thus, by letting \( a > 0 \) small enough, the above estimate yields
\[
\frac{\partial^2 u}{\partial t^2} - \Delta u \geq Cw^{-(\beta+2)/2} \quad \text{in} \ \mathbb{R}^{N+1}_+,
\]
for some constant \( C > 0 \). To proceed further, we need the following result.
Lemma 3.1. Let $\alpha \in (0, N)$, $\sigma > N - \alpha$ and $1 \leq \lambda \leq 2$. Then, there exists a constant $C = C(N, \alpha, \sigma) > 0$ (note that $C$ is independent of $\lambda$) such that

$$
\int_{\mathbb{R}^N} \frac{dy}{|x-y|^\alpha(\lambda + |y|^2)^{\sigma/2}} \leq C \begin{cases} 
(\lambda + |x|^2)^{(N-\alpha-\sigma)/2} & \text{if } \sigma < N, \\
(\lambda + |x|^2)^{-\alpha/2} & \text{if } \sigma > N, 
\end{cases} \quad \text{for all } |x| \geq 1.
$$

For $\lambda = 0$, similar estimates are available in [19] Lemma A.1 as well as in [6] [7] [9].

Proof. Let $|x| \geq 1$. We split the integral into

$$
\int_{\mathbb{R}^N} \frac{dy}{|x-y|^\alpha(\lambda + |y|^2)^{\sigma/2}} = \left\{ \int_{|y| \geq 2|x|} + \int_{\frac{1}{2}|x| \leq |y| \leq 2|x|} + \int_{|y| \leq |x|/2} \right\} \frac{dy}{|x-y|^\alpha(\lambda + |y|^2)^{\sigma/2}}.
$$

If $|y| \geq 2|x|$, then $|x-y| \geq |y| - |x| \geq |y|/2$ and we have

$$
\int_{|y| \geq 2|x|} \frac{dy}{|x-y|^\alpha(\lambda + |y|^2)^{\sigma/2}} \leq C \int_{|y| \geq 2|x|} \frac{dy}{|y|^{\alpha+\sigma}} \leq C|x|^{N-\alpha-\sigma}
$$

$$
\leq C\left(\frac{\lambda + |x|}{3}\right)^{N-\alpha-\sigma}
$$

$$
\leq C(\lambda + |x|)^{N-\alpha-\sigma}
$$

$$
\leq C(\lambda + |x|^2)^{(N-\alpha-\sigma)/2},
$$

where we have used that

$$
|x| \geq \frac{\lambda + |x|}{3} \quad \text{if } |x| \geq 1 \text{ and } \lambda \leq 2. \quad (56)
$$

Next we have

$$
\int_{\frac{1}{2}|x| \leq |y| \leq 2|x|} \frac{dy}{|x-y|^\alpha(\lambda + |y|^2)^{\sigma/2}} \leq C(\lambda + |x|^2)^{-\sigma/2} \int_{\frac{1}{2}|x| \leq |y| \leq 2|x|} \frac{dy}{|x-y|^\alpha}
$$

$$
\leq C(\lambda + |x|^2)^{-\sigma/2} \int_{|y-x| \leq 3|x|} \frac{dy}{|x-y|^\alpha}
$$

$$
\leq C(\lambda + |x|^2)^{-\sigma/2}|x|^{N-\alpha}
$$

$$
\leq C(\lambda + |x|^2)^{(N-\alpha-\sigma)/2},
$$

where we have used that $|y| \leq 2|x|$ implies $|y-x| \leq 3|x|$.

Finally, if $|y| \leq |x|/2$ then $|x-y| \geq |x| - |y| \geq |x|/2$. We have

$$
\int_{|y| \leq |x|/2} \frac{dy}{|x-y|^\alpha(\lambda + |y|^2)^{\sigma/2}} \leq C|x|^{-\alpha} \int_{|y| \leq |x|/2} \frac{dy}{(\lambda + |y|^2)^{\sigma/2}}. \quad (57)
$$

If $\sigma < N$ then

$$
\int_{|y| \leq |x|/2} \frac{dy}{(\lambda + |y|^2)^{\sigma/2}} \leq \int_{|y| \leq |x|/2} \frac{dy}{|y|^{\sigma}} = C|x|^{N-\sigma},
$$
so that by (56) and being \( N - \alpha - \sigma < 0 \)

\[
\int_{|y| \leq |x|/2} \frac{dy}{|x - y|^{\alpha} (\lambda + |y|^2)^{\sigma/2}} \leq C|x|^{N - \alpha - \sigma} \leq C\left(\frac{\lambda + |x|}{3}\right)^{N - \alpha - \sigma} \leq C(\lambda + |x|^2)^{(N - \alpha - \sigma)/2}.
\]

If \( \sigma > N \) then

\[
\int_{|y| \leq |x|/2} \frac{dy}{(\lambda + |y|^2)^{\sigma/2}} \leq \int_{\mathbb{R}^N} \frac{dy}{(1 + |y|^2)^{\sigma/2}} < C < \infty,
\]

and from (57) and (56) one has

\[
\int_{|y| \leq |x|/2} \frac{dy}{|x - y|^{\alpha} (\lambda + |y|^2)^{\sigma/2}} \leq C|x|^{-\alpha} \leq C\left(\frac{\lambda + |x|}{3}\right)^{-\alpha} \leq C(\lambda + |x|^2)^{-\alpha/2},
\]

which concludes our proof. \( \square \)

Let us return to the proof of Corollary 1.3 and observe that

\[
(|x|^{-\alpha} * w^p) u^q \leq (1 + |x|^2)^{-\beta q/2} \int_{\mathbb{R}^N} \frac{dy}{|x - y|^{\alpha} (1 + |y|^2)^{\beta p/2}} \quad \text{for all} \ (x, t) \in \mathbb{R}^{N+1}_+,
\]

since \( w(x, t) \geq 1 + |x|^2 \). Since from (52) we have \( \beta p > 0 \) and \( \alpha < N \), the above integral is finite and thus \( (|x|^{-\alpha} * w^p) u^q \) is uniformly bounded from above for \( (x, t) \in B_1 \times [0, \infty) \). In addition, since \( 1 \leq w \leq 3 \) in \( (x, t) \in B_1 \times [0, \infty) \), then by (55) \( \frac{\partial^2 u}{\partial x^2} - \Delta u \) and \( (|x|^{-\alpha} * w^p) u^q \) are positive, continuous and uniformly bounded functions from below and from above respectively, on \( (x, t) \in B_1 \times [0, \infty) \). We may thus find \( C_1 > 0 \) such that

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u \geq C_1 (|x|^{-\alpha} * w^p) u^q \quad \text{in} \ B_1 \times [0, \infty).
\]

On the other hand, by Lemma 3.1 for \( \lambda = 1 + (1 + t)^{-\alpha} \) and \( \sigma = \beta p \), where \( \sigma > N - \alpha \) by (52), we have

\[
(|x|^{-\alpha} * w^p) u^q \leq C \begin{cases} w^{N - \alpha - \beta q/2} & \text{if } \sigma = \beta p < N, \\ w^{-\beta q - \alpha/2} & \text{if } \sigma = \beta p > N, \end{cases} \quad \text{for all} \ (x, t) \in (\mathbb{R} \setminus B_1) \times [0, \infty).
\]

Using the above estimate together with (52) and (55) we find

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u \geq C w^{-(\beta + 2)/2} \geq C w^{\max\{N - \alpha - \beta q/2, -\beta q - \alpha/2\}} \geq C_2 (|x|^{-\alpha} * w^p) u^q,
\]

for all \( (x, t) \in (\mathbb{R} \setminus B_1) \times [0, \infty) \). Letting now \( M = \left( \max\{C_1, C_2\} \right)^{1/(p + q) - 1} \), it follows from (58) and (59) that \( U = Mu \) is a \( C^\infty(\mathbb{R}^{N+1}_+) \) solution of (13) such that

\[
\frac{\partial U}{\partial t} (x, 0) = \frac{a \beta M}{2} (2 + |x|^2)^{-(\beta + 2)/2} > 0.
\]
4 Proof of Theorem 1.4

Let \((u, v)\) be a weak solution of (17) which satisfies (21). Let \(\varphi\) be the test function given by (26) with \(\gamma = 2m/k\). Arguing as in (31) we have

\[
\begin{aligned}
\int_{\mathbb{R}^N} u_{k-1}(x) \varphi(x, 0) + \int_{\mathbb{R}^N} (K \ast |v|^p) |v|^q \phi \, dx \, dt &\leq \int_{\mathbb{R}^N} u \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx \, dt, \\
\int_{\mathbb{R}^N} v_{k-1}(x) \varphi(x, 0) + \int_{\mathbb{R}^N} (L \ast |u|^n) |u|^s \phi \, dx \, dt &\leq \int_{\mathbb{R}^N} v \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| dx dt + |\Delta^m \varphi| \right) \, dx dt. 
\end{aligned}
\]

Setting

\[
I(t) = \int_{\mathbb{R}^N} |u(x, t)| \frac{n+k}{2} \varphi(x, t) \, dx,
\]

\[
J(t) = \int_{\mathbb{R}^N} |v(x, t)| \frac{p+q}{2} \varphi(x, t) \, dx,
\]

we have \(\text{supp } I \subset [0, 2R^\gamma)\) and \(\text{supp } J \subset [0, 2R^\gamma)\). With the same approach as in Lemma 2.2 for \(R > R_0\) we have

\[
\begin{aligned}
\int_{\mathbb{R}^N} (K \ast |v|^p) |v|^q \phi(x, t) \, dx &\geq K(4R) J^2(t), \\
\int_{\mathbb{R}^N} (L \ast |u|^n) |u|^s \phi(x, t) \, dx &\geq L(4R) I^2(t),
\end{aligned}
\]

for all \(t \geq 0\). Thus, (60), (32) and (21) yield

\[
\begin{aligned}
K(4R) \int_0^{2R^\gamma} J^2(t) \, dt &\leq \int_{\mathbb{R}^N} |u| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx \, dt, \\
L(4R) \int_0^{2R^\gamma} I^2(t) \, dt &\leq \int_{\mathbb{R}^N} |v| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx dt,
\end{aligned}
\]

for \(R > 0\) large. We next employ the estimates (39) and (42) with \(\gamma = 2m/k\). We find

\[
\int_{\mathbb{R}^N} |u| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx dt 
\leq CR^{-2m+\left(N+2m/k\right)\frac{n+2}{n+s}} \left[ \left( \int_{\text{supp}(\rho)} |u|^{\frac{n+s}{2}} \varphi \, dx \, dt \right)^{\frac{2}{n+s}} + \left( \int_{\text{supp}(\Delta^m \rho)} |u|^{\frac{n+s}{2}} \varphi \, dx \, dt \right)^{\frac{2}{n+s}} \right] .
\]

Since \(\text{supp}(\frac{\partial^k \varphi}{\partial t^k}) \subset [R^\gamma, 2R^\gamma] \times \mathbb{R}^N\) and \(\text{supp}(\Delta^m \varphi) \subset [0, 2R^\gamma] \times (B_{2R} \setminus B_R)\), applying Hölder’s inequality we derive

\[
\int_{\mathbb{R}^N} |u| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx dt \leq CR^{-2m+\left(N+2m/k\right)\frac{n+2}{n+s}} \times 
\times \left[ \left( \int_{R^\gamma} \left( \int_{\mathbb{R}^N} |u|^{\frac{n+s}{2}} \varphi \, dx \right)^2 \, dt \right)^{\frac{1}{n+s}} + \left( \int_{B_{2R} \setminus B_R} \left( \int \mathbb{R}^N |u|^{\frac{n+s}{2}} \varphi \, dx \right)^2 \, dt \right)^{\frac{1}{n+s}} \right].
\]

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In particular,

\[
\int_{\mathbb{R}_{+}^{N+1}} |u| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx \, dt \leq C R^{N-2m(1-\frac{k}{n})} - \frac{2N+2m/k}{n+s} \left( \int_0^{2R} I^2(t) \, dt \right)^{\frac{1}{n+s}}. \tag{64}
\]

Similarly

\[
\int_{\mathbb{R}_{+}^{N+1}} |v| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx \, dt \leq C R^{N-2m(1-\frac{k}{n})} - \frac{2N+2m/k}{p+q} \times \left( \int_0^{2R} J^2(t) \, dt \right)^{\frac{1}{p+q}}. \tag{65}
\]

and in particular,

\[
\int_{\mathbb{R}_{+}^{N+1}} |v| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx \, dt \leq C R^{N-2m(1-\frac{k}{n})} - \frac{2N+2m/k}{p+q} \left( \int_0^{2R} J^2(t) \, dt \right)^{\frac{1}{p+q}}. \tag{66}
\]

From (62), (64) and (66) we derive

\[
\begin{cases}
K(4R) \left( \int_0^{2R} J^2(t) \, dt \right)^{\frac{1}{n+s}} \leq C R^{N-2m(1-\frac{k}{n})} - \frac{2N+2m/k}{n+s} \left( \int_0^{2R} I^2(t) \, dt \right)^{\frac{1}{n+s}}, \\
L(4R) \left( \int_0^{2R} I^2(t) \, dt \right)^{\frac{1}{p+q}} \leq C R^{N-2m(1-\frac{k}{n})} - \frac{2N+2m/k}{p+q} \left( \int_0^{2R} J^2(t) \, dt \right)^{\frac{1}{p+q}}.
\end{cases} \tag{67}
\]

Assume now that (19) holds and let \( \{R_j\} \) be an increasing sequence of positive real numbers such that \( R_j > 2R_{j-1} \) for all \( j \geq 2 \) and

\[
K(R_j)L(R_j) \frac{1}{n+s} R_j^{N+2m/k} + N+2m \rightarrow \ell \in (0, \infty) \quad \text{as} \quad j \rightarrow \infty. \tag{68}
\]

Assume first \( \ell = \infty \). Then, using the second estimate of (67) into the first we find

\[
\left( \int_0^{2R} J^2(t) \, dt \right)^{\frac{1}{\frac{1}{n+s}}(p+q)} \leq \frac{C}{K(R_j)L(R_j) \frac{1}{n+s} R_j^{N+2m/k} + N+2m \left( 1-\frac{k}{n} \right)} \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty. \tag{69}
\]

This implies that \( J(t) \equiv 0 \) so that \( v \equiv 0 \). From the second estimate of (67) it follows now that \( I(t) \equiv 0 \), hence \( u \equiv 0 \).

Assume now \( \ell \in (0, \infty) \). From (69) it follows that \( J^2 \in L^1(0, \infty) \) and arguing as in (48), this further implies

\[
\int_{R_j}^{2R} \left( \int_{\mathbb{R}^N} |v(x,t)|^{\frac{p+q}{2}} \varphi(x,t) \, dx \right)^2 \, dt \rightarrow 0, \quad \int_0^{2R} \left( \int_{B_{2R} \setminus B_{R_j}} |v(x,t)|^{\frac{p+q}{2}} \varphi(x,t) \, dx \right)^2 \, dt \rightarrow 0, \tag{70}
\]

as \( j \rightarrow \infty \). Using (70) into (65) (with \( R_j \) instead of \( R \)) it follows that

\[
\int_{\mathbb{R}_{+}^{N+1}} |v| \left( \left| \frac{\partial^k \varphi}{\partial t^k} \right| + |\Delta^m \varphi| \right) \, dx \, dt \leq C R^{N-2m(1-\frac{k}{n})} - \frac{2N+2m/k}{p+q} o(1) \quad \text{as} \quad j \rightarrow \infty.
\]
Using this fact in (62) along with (64) we deduce

\[
\begin{cases}
K(4R) \int_0^{2R^\gamma} J^2(t) \, dt \leq CR^{N-2m\left(1-\frac{1}{k}\right)} \left( \int_0^{2R^\gamma} I^2(t) \, dt \right)^{\frac{1}{n+s}} \\
L(4R) \int_0^{2R^\gamma} I^2(t) \, dt \leq CR^{N-2m\left(1-\frac{1}{k}\right)} \left( \int_0^{2R^\gamma} I^2(t) \, dt \right)^{\frac{1}{n+s}} - \frac{2N+2m}{p+q} o(1)
\end{cases}
\]

as \( j \to \infty \). \hspace{1cm} (71)

Using the second estimate of (71) into the first one obtains

\[
\left( \int_0^{2R^\gamma} J^2(t) \, dt \right)^{1-\frac{1}{(n+s)(p+q)}} \leq \frac{C}{K(R_j)L(R_j) \left( \frac{2N+2m}{n+s}(p+q) \right)^{\frac{N+2m}{n+s} - N+2m\left(1-\frac{1}{k}\right)}} o(1) \to 0,
\]

as \( j \to \infty \). As before we deduce again \( u \equiv v \equiv 0 \).

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