Theory of spin response in underdoped cuprates as strongly fluctuating d-wave superconductors

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We study the spin dynamics in underdoped cuprates at low temperatures by considering them as quasi two dimensional d-wave superconductors with strong quantum phase fluctuations. An effective theory of spin degrees of freedom of nodal quasiparticles coupled to vortex defects in the phase of the superconducting order parameter is formulated. It represents the minimal extension of the $QED_3$ theory of the pseudogap phase into the superconducting region. The theory predicts a single superconductor-spin density wave (SC-SDW) phase transition, without coexistence between the two phases. At the transition, which may be fluctuation induced first-order, vortices condense (and SC is lost) and the chiral symmetry for fermions simultaneously breaks (and SDW is formed). We compute the spin-spin correlation function in the fluctuating superconducting state, and explain the evolution of the spin response with energy in underdoped YBa$_2$Cu$_3$O$_{6+x}$ observed in neutron scattering experiments. In particular, we predict that at very low frequencies $\omega \sim (1/10) \omega_{\text{res}}$, with $\omega_{\text{res}}$ being the energy of the ‘resonance’ at $Q = (\pi, \pi)$, (weak) spin response should become narrowly peaked at four diagonally incommensurate wave vectors that span between the nodes of the superconducting order parameter. These peaks represent the inherent collective mode of the phase fluctuating d-wave superconductor, the condensation of which would bring about the SDW order in the pseudogap phase. Our interpretation of the resonance suggests that it should become more elusive in the superconductors with lower $T_c$, in accord with its’ conspicuous absence in La$_{2-x}$Sr$_x$CuO$_4$.

I. INTRODUCTION

Underdoped high temperature superconductors are highly anisotropic, quasi two-dimensional materials, in which thermal and quantum fluctuations of the phase of the superconducting order parameter (OP) should be important well below the pseudogap temperature. It has recently been shown that a phase fluctuating d-wave superconductor (dSC) at zero temperature ($T = 0$) is inherently unstable towards the formation of the very weak spin density wave (SDW) state with loss of phase coherence [1], [2]. This follows from the observation that the standard Fermi liquid theory for the BCS d-wave quasiparticles, besides the usual spatial symmetries, also acquires an additional continuous (‘chiral’ [1]) symmetry at low energies. This symmetry becomes dynamically broken when the superconducting phase coherence is lost via proliferation of vortex defects. The theory of the phase incoherent (pseudogap) state in this formulation becomes closely related to the three dimensional quantum electrodynamics ($QED_3$) [3], in which the coupling constant (‘charge’) is proportional to the vortex condensate. In the crudest approximation, the fluctuations of the gauge-field may then be neglected in the superconducting state, which leaves quasiparticles as sharp low energy excitations. When the vortices condense and the charge in the $QED_3$ is finite, the chiral symmetry of the dSC becomes dynamically broken, in analogy with the well known phenomenon in particle physics [4]. In the present context this translates into weak SDW ordering, with confined spin-1/2 (spinon) excitations [1].

Several conceptual and practical issues of direct relevance to cuprates arise naturally in such an approach to underdoped high temperature superconductors. Can the SC and the SDW long-range orders coexist? Such a coexistence appears rather generically in various mean-field treatments of microscopic models of cuprates [5], [6]. Several intriguing recent experiments [7], [8], [9], also suggest the coexistence of the two inimical orderings. If there is a single dSC-SDW quantum phase transition, on the other hand, what should be its universality class? What are the effects of the virtual vortex fluctuations inside the superconducting state? These questions all require a better understanding of the quasiparticle-vortex interaction, particularly in the superconducting state.

A theory of low-energy d-wave quasiparticles coupled to phase fluctuations of the superconducting OP which puts the earlier physical ideas of [1], [3] into a consistent field-theoretical framework has recently been proposed in [10], [11]. One result of that study is that, at least within a certain approximation [11], in the superconducting state spin of quasiparticles may be considered asymptotically decoupled from their charge at low energies. Physically this is just what is expected, and may be understood as being a consequence of the long range phase coherence in a singlet superconductor [12], [13]: the condensate screens perfectly the charge of quasiparticles, effectively transforming them into neutral, spin-1/2 excitations.

In this paper we wish to take advantage of this state of affairs and consider a simpler effective $T = 0$ theory (Eq. (1)) for the spin sector of the fluctuating dSC only. We assume that in underdoped high temperature superconductors the amplitude of the superconducting OP may be considered frozen well below the pseudogap temperature $T^*$, but that the quantum fluctuations of its phase will inevitably arise sufficiently near half filling when the interaction between electrons becomes strong. In the phase incoherent (pseudogap) state our effective theory reduces to the previously studied $QED_3$ [1], [3],...
with its concomitant SDW instability. Deep in the dSC, on the other hand, the theory becomes equivalent to the three dimensional Thirring model, with an (irrelevant) short range interaction between the electrically neutral spin-$1/2$ excitations, which we will, as traditionally, call spinons. The effective theory in Eq. (1) connects, and describes the region in between, these two limits, and establishes a physical link between the phase fluctuations and the magnetic excitations in underdoped cuprates.

Having the low energy theory for the spin degrees of freedom we proceed to derive several interesting consequences from it. First, we show that the condensation of vortices (i.e. the loss of superconductivity) and the breaking of the chiral symmetry for fermions (the SDW instability) in the theory coincide. The region of coexistence between the SC and the SDW long-range orders is not found. Second, we find that the quantum dSC-SDW phase transition could be fluctuation-induced first-order, in agreement with our earlier study of the full theory that included charge [10]. Finally, and maybe most importantly, we calculate the spin dynamics induced by vortex fluctuations inside the superconducting state, in the leading approximation. The detailed evolution of the spin response with energy observed in the neutron scattering experiments on $YBa_2Cu_3O_{6+x}$ (YBCO) [14], [15], [16] is explained in terms of four diagonally incommensurate ‘mother’ peaks, that should become discernible at lowest frequencies. These weak and narrow peaks are centered on the wave vectors that span between the diagonally opposite nodes of the superconducting OP, and represent the collective particle-hole mode of the phase fluctuating dSC. The SDW instability obtained previously with the QED$_3$ theory can be understood qualitatively as the condensation of this collective mode. The famous ‘resonance’ at the commensurate wave vector $Q = (\pi, \pi)$ is here obtained as an overlap between the all four mother peaks, which dominates in the spin response within certain window of energies. This energy window shrinks with the superconducting transition temperature $T_c$, which may explain why the resonance has not been observed in $La_{2-x}Sr_xCuO_4$ (LSCO) [17].

The paper is organized as follows. In the following section we introduce the effective theory of spin of quasiparticles, and discuss its physical motivation and the formal justification. In sec. III we show how our theory avoids the coexistence of the SC and the SDW orders. In sec. IV we briefly discuss the nature of the dSC-SDW quantum phase transition. Spin response is discussed in sec. V. The discussion of our results and the relation to other approaches is presented in the concluding section.

II. THEORY OF SPINONS AND VORTICES

Consider the quantum mechanical ($T = 0$) action for the low energy quasiparticles in the two-dimensional phase fluctuating dSC, $S = \int d^2x \mathcal{L}$, $x = (\tau, \vec{r})$, $\tau$ is the imaginary time, and

$$
\mathcal{L} = \sum_{i=1}^{N} \bar{\Psi}_i \gamma_\mu (\partial_\mu - ia_\mu) \Psi_i + \frac{i}{\pi} \vec{A} \cdot (\nabla \times \vec{A}) + \frac{1}{(\nabla - i\vec{A})\Phi}^2 + \mu^2 |\Phi|^2 + \frac{b}{2} |\Phi|^4.
$$

The fluctuating complex field $\Phi$ describes the vortex loops, and $\langle \Phi \rangle = 0$ implies that all such loops are of finite size, and thus the superconducting phase coherence. $\phi$ should be understood as being dual to the standard superconducting OP, and $\langle \Phi \rangle \neq 0$ means that vortex loops proliferated and that superconductivity is lost [18]. Two ($N = 2$) four-component Dirac fermions describe the gapless, neutral, spin-$1/2$ (spinon) excitations near the four nodes of the superconducting order parameter at $\pm K_{1,2}$, one Dirac field for each pair of diagonally opposed nodes, as defined previously in [1]. $\vec{a}$ is the gauge-field that results from absorbing the singular part of the superconducting phase due to vortices into the spinon fields [3], [1]. $\gamma_\mu, \mu = 0, 1, 2$ are the Dirac gamma matrices [1], and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Finally, $\vec{A}$ is an auxiliary (Chern-Simons) field which facilitates the statistical spinon-vortex coupling, as will be explained shortly. Tuning parameter $\mu^2$ controls the magnitude of vortex fluctuations, and we assume it to be related to doping, $\mu^2 \propto (x - x_c)$, with $x_c$ being the critical doping in the underdoped regime. The coupling $b > 0$ describes the short-range repulsion between the vortex loops. We work in units in which $\hbar = k_B = 1$, and we have set the two characteristic velocities of the nodal quasiparticles $v_F = v_\Delta = 1$ in (1), for simplicity. The amplitude of the superconducting order parameter is assumed frozen, and Eq. (1) is supposed to describe only the spin and the vortex degrees of freedom well below the pseudogap temperature $T^*$, as in the previous work [1].

One may physically understand the form of the theory (1) as follows [19]: the exact integration over the gauge-field $\vec{a}$, which appears only linearly in (1), leaves the spinon-vortex coupling $i\vec{A} \cdot \vec{J}_b$, where $\nabla \times \vec{A} = \pi \vec{J}_b$, and $\vec{J}_b$ and $\vec{J}_\phi$ are the spin and vortex current densities, respectively. Vortices and spinons therefore see each other as sources of a fictitious magnetic flux, and circling around a vortex with a spinon (or vice versa) leads to a phase change of $\pi$. This is precisely the statistical (Aharanov-Bohm) part of the interaction between vortices and quasiparticles [3], [1], which couples only to spin. The regular part of the phase, which provides the standard Doppler shift of quasiparticle energies and couples only to charge, can be ‘gauged away’ [13] at the price of introducing an additional, ultimately irrelevant interaction term. Absorbing this regular part of the phase into quasiparticles converts them into spinons [13], represented by $\Psi$ in Eq. (1) [1]. The final point is that at low energies and in the superconducting state the spin of quasiparticles may be considered decoupled from its charge [11]. Essentially, this is because one can consider
the superconducting condensate to carry all the charge in the phase coherent state. Similar conclusion may also be reached within the gauge theory formulation of the t-J model [20].

The separation between spin and charge allows one to study spin excitations separately from the charge in the fluctuating dSC at low energies. Eq. (1) may therefore be understood as an effective theory for the spin of quasiparticles. One can also derive it from the full theory which includes charge [10], [11] by simply omitting the additional Chern-Simons field for the charge degrees of freedom.

Finally, in writing (1) the anisotropy between the Fermi and the OP related velocities, \( v_F \gg v_s \), and all the local interactions between quasiparticles [2] have been neglected, as irrelevant at low energies [21], [22]. We will, however, find it necessary to restore the velocity anisotropy later when we discuss the spin response, where this feature will turn out to play an important role.

Although in this work we are concerned with the vortex fluctuations in the superconducting state, which is a complementary problem to the one studied in the previous papers on the QED3 theory of the pseudogap phase [1], [3], let us nevertheless mention how the theory (1) is related to the QED3 in the mean-field approximation for the vortex field. Entirely neglecting the fluctuations in \( \Phi \), in the superconducting phase (\( \langle \Phi \rangle = 0 \)), the integration over \( A \) would constrain \( \nabla \times A = 0 \), and spinons become free. When vortex condensate (\( \langle \Phi \rangle \neq 0 \)), on the other hand, the gauge-field \( A \) acquires a mass via Higgs mechanism, \( \langle \Phi \rangle^2 A^2 \). The Gaussian integration over \( A \) produces then the Maxwell term for \( A \sim (\nabla \times A)^2/\langle \Phi \rangle^2 \).

Together with the Dirac Lagrangian this constitutes the QED3 for spinons, in which the fermions’ chiral symmetry generated by \( \gamma_3 \) and \( \gamma_5 \) is dynamically broken by the generation of the small mass term \( \sim M \bar{\psi}_i \psi_i \) [4]. In the present context such a mass \( M \sim (\bar{\psi}_i \psi_i) \sim \langle \Phi \rangle^2 \) is proportional to the weak SDW OP, with the (incommensurate) ordering wave vectors \( \pm 2K_{1,2} \) that connect the diagonally opposed nodes of the superconducting OP [1].

Our aim will therefore be to understand how the theory (1) interpolates between the limits of QED3 in the pseudogap, and the non-interacting theory of spinons in the superconducting phases. In particular, we want to understand the evolution of spin dynamics with progressive quantum disordering of the dSC, which is assumed to correspond to underdoping. We begin with the issue of possible coexistence of the SDW and the dSC orderings.

### III. COINCIDENCE OF THE SC AND THE SDW TRANSITIONS

In the mean-field approximation, the vortex condensation and the SDW transition in the theory (1) coincide. It is not obvious that this feature survives the inclusion of the fluctuations, and it is conceivable that the SDW transition may occur within the superconducting phase [1], [5], [6], [23]. To examine this issue we first ask what would be the exact propagator for the gauge field \( A \) if there were no fermions, i.e. when \( N = 0 \) in (1). We then approximate the interacting action for \( A \) that would result from the integrations over \( \Phi \) and \( A \) with the effective Gaussian term that reproduces that exact propagator. In the Landau gauge such a propagator is

\[
G^0_{aa,\mu\nu}(p, m) = \frac{\pi^2 \Pi_{AA}(p, m)}{p^2} (\delta_{\mu\nu} - \hat{p}_\mu \hat{p}_\nu),
\]

where \( \Pi_{AA}(p, m) (\delta_{\mu\nu} - \hat{p}_\mu \hat{p}_\nu) \) is the transverse current-current correlation function in the \( |\Phi|^4 \) theory. Here \( m^2 = \mu^2 + O(b) \) is the fully renormalized “mass” of the vortex field. In general, \( \Pi_{AA}(p, m) = np F_+(m/p) \), for \( m^2 > 0 \) (superconductor). To the lowest order in \( b \)

\[
F_+(z) = \frac{1}{8\pi} \left( 4z^2 + 1 \right) \arctan \left( \frac{1}{2z} \right) - 2z + O(b).
\]

Here we generalized our model to the one with \( n \) complex vortex fields, \( n = 1 \) being the case of physical interest.

We are now in position to study the chiral symmetry breaking for fermions with (2) serving as the bare (without fermion polarization) gauge-field propagator in the QED3. Consider the standard large-N Dyson equation [4] for the fermion self-energy

\[
\Sigma(q) = \frac{1}{4} \text{Tr} \int \frac{d^3 p}{(2\pi)^3} \gamma_\mu \frac{G_{aa,\mu\nu}(p - q, m) \Sigma(p)}{p^2 + \Sigma(p)^2} \gamma_\nu
\]

Here \( [G_{aa,\mu\nu}(p, m)]^{-1} = \Pi_{aa,\mu\nu}(p) + [G^0_{aa,\mu\nu}(p, m)]^{-1} \), with the one-loop fermion polarization \( \Pi_{aa,\mu\nu}(p) = (NP/8) (\delta_{\mu\nu} - \hat{p}_\mu \hat{p}_\nu) \). It is useful to consider first the point of the superconducting phase transition \( m = \phi \), where Eq. (3) implies \( \Pi_{AA}(p, 0) = np(1/16 + O(b_c)) \)

![Fig. 1. Schematic behavior of the inverse gauge-field propagator 1/G^0_{aa,\mu\nu}(p, m) as a function of the momentum in the SC (m^2 > 0), critical (m = 0), and the pseudogap (m^2 < 0) phases of the theory (1), from top to bottom. At large momenta the behavior is always linear. At low momentum, however, the statistical gauge field is massive (\sim m) in the SC phase, and has a Maxwell term (\sim p^2) in the pseudogap (incoherent) phase.](image-url)
where \( b_c \) is the fixed point value of \( b \) in \( d = 3 \) [24]. Linear dependence of \( \Pi_{AA}(p, 0) \) on momentum is an exact result [25]. Inserting this into the Dyson equation we find that at \( m = 0 \) the effect of vortices is only to increase the coefficient \( N - N_{eff} = N + 128/(\pi^2 n) + O(b_c) \). Recalling that the non-trivial solution of the Dyson equation (4) exists only for \( N_{eff} < N_c = 32/\pi^2 [4] \), for \( N = 2 \) we find that chiral symmetry at \( m = 0 \) would be already broken only for \( n > n_c = 10.44(1 + 32/(9\pi^2 n)) \), where we have also included the known \( O(b_c) \) correction to \( \Pi_{AA}(p, 0) \) in the large-\( n \) approximation [24]. Since \( n = 1 \) in the physical case we conclude that right at the superconducting critical point SDW order is most likely absent.

Little further thought shows that, if correct, the above result implies that the SDW order is absent for all \( m^2 > 0 \). Indeed, this is to be expected, since \( G^{0}_{aa}(p, m) < G^{0}_{aa}(p, 0) \), and \( \tilde{\sigma} \) is only stiffer in the superconducting phase. To prove this, assume that for \( m^2 > 0 \) a non-trivial solution of the Eq. (4), \( \tilde{\Sigma}(q) \), does exist. Such a solution would then satisfy

\[
\tilde{\Sigma}(q) < \frac{1}{4} Tr \int \frac{d^3 p}{(2\pi)^3} \frac{G_{aa, \mu \nu}(p-q, 0)\tilde{\Sigma}(p)}{p^2 + \Sigma(p)^2} \gamma_{\mu \nu}. \tag{5}
\]

On the other hand, we already established that there is only a trivial solution of the Dyson equation at \( m = 0 \). This means that assuming a candidate function \( \Sigma(p) \) and inserting it under the integral in the Dyson equation (4) for \( m = 0 \) will produce only a smaller new \( \Sigma(q) \), since under iterations the physically acceptable self-energy must approach the trivial solution. This further implies that any \( \Sigma(q) \) from the domain of attraction of the trivial solution at \( m = 0 \) has to satisfy the inequality opposite to (5). A non-trivial \( \Sigma(q) \) at \( m^2 > 0 \) therefore can not exist if it did not exist at \( m = 0 \).

For \( m^2 < 0 \), superconductivity is lost, and \( \Pi_{AA}(p, m) = n p F_{-}(\langle |\Phi|^2 \rangle / p) \), with \( F_{-}(z) \rightarrow 1/16 \) for \( z \ll 1 \), and \( F_{-}(z) \rightarrow 2z \), for \( z \gg 1 \) (Higgs mechanism), to the leading order. The behavior of the gauge field propagator is depicted in Fig. 1. Since \( G^{0}_{aa}(p, m) > G^{0}_{aa}(p, 0) \) for \( m^2 < 0 \), the above argument no longer applies. In fact, it is easy to see that there is immediately a non-trivial solution of the Eq. (4) when \( m^2 < 0 \). Since \( \Sigma(q) > 0 \) only for \( q \approx |\langle |\Phi|^2 \rangle | \) and smaller, \( |\langle |\Phi|^2 \rangle | \) serves as the effective ultraviolet cutoff in the Eq. (4). The Dyson equation then reduces to the standard one in the \( QED_3 \) [4]. For \( N = 2 < N_c \), \( \Sigma(q = 0) \sim |\langle |\Phi|^2 \rangle |^2 \), and the chiral symmetry is dynamically broken. There is no intermediate (quantum disordered) phase in between the SDW and the dSC in the theory (1), unless the exact value of \( N_c \) in the \( QED_3 \) is actually less than two [26], [27].

IV. NATURE OF THE TRANSITION

The above argument strongly suggests that there is a single dSC-SDW transition in the theory (1), but does not say anything about its nature. This issue in the full theory with both spin and charge was addressed in [10], and here we will provide a brief alternative calculation in our effective theory that supports the earlier results. To study the phase transition it is better to proceed in the opposite direction, and integrate over the fermions first. This gives dynamics to \( \tilde{\sigma} \) via the fermion polarization bubble. If we neglect the quartic and the higher order terms and perform the Gaussian integral over \( \tilde{\sigma} \), the result is the Maxwell-like term for \( \tilde{A} : (A_{\mu}(p)A_{\nu}(-p)) \sim (N^2/8|p|)(\delta_{\mu \nu} - \rho_{\mu} \rho_{\nu}) \). Note that since the inverse of the above average is non-analytic at \( p = 0 \), it can not renormalize [28], and therefore the number of fermions \( N \) represents an exactly marginal coupling. Assuming a constant \( \Phi \) in (1) we may further integrate \( \tilde{A} \) to find the energy per unit volume to be

\[
S[\Phi] - S[0] = \frac{\mu^2}{\Lambda^2} + \frac{N}{48} |\Phi|^2 + \frac{1}{2} b_2 \left( \frac{b}{2\Lambda} - \frac{\pi^2 N^2}{48} \right) |\Phi|^4 + \frac{1}{6\pi^2} \ln(1 + \frac{\pi^2 N^2}{4} |\Phi|^2) + \frac{\pi^4 N^3}{384} |\Phi|^6 \ln(1 + \frac{4}{\pi^2} N^2 |\Phi|^2), \tag{6}
\]

where we have rescaled \( \Phi^2 \), with the ultraviolet cutoff \( \Lambda \) implicit in (1). Standard analysis of the Eq. (6) shows that there is a discontinuous transition for \( N > (4/\pi)\sqrt{2b/\Lambda} \). Using the lowest order fixed point value \( b/\Lambda = (2\pi^2/5)(4 - d) + O((4 - d)^2) \) in \( d = 3 \) as a crude estimate of this bound, we find the first-order transition for \( N > 3.58 \), in rough agreement with [10]. Of course, this conclusion is to be trusted only for \( N \gg 1 \), when the first-order transition occurs at a large \( \mu^2 \), at which our neglect of fluctuations in \( \Phi \) in arriving at the Eq. (6) becomes justified. For \( N \ll 1 \), on the other hand, the phase transition in the theory (1) should be continuous, with weakly modified XY exponents: \( \nu = \nu_{xy} + O(N) \), \( \eta = \eta_{xy} + O(N) \) [10], [19]. The situation is reminiscent of the Ginzburg-Landau superconductor [25], with \( N \gg 1 \) analogous to the strongly type-I, and \( N \ll 1 \) to the extreme type-II case. Of course, in dSC \( N = 2 \) and fixed, so the issue is whether this falls into the type-I or into the type-II regime. In this context it may be interesting to note that \( \mathcal{T}_{c} \) does appear to drop discontinuously to zero at the critical doping in LSCO [17], and very steeply, if not even discontinuously, in YBCO [29].

V. SPIN RESPONSE IN THE SUPERCONDUCTOR

A. Derivation of the response function

Finally, we turn to spin dynamics deep inside the dSC, when \( m/b \gg 1 \). For small momenta, \( p < \Lambda_{T}, \approx 5m \), we find that \( G^{0}_{aa, \mu \nu}(p, m) = (\pi/24m) + O(p^2) \). To keep the algebra simple, in the following we will approximate the gauge-field propagator with the constant mass term, corresponding to the leading contribution in the above. The retention of the full form of \( G^{0}_{aa}(p, m) \)
should not qualitatively change the results, although it is more than likely that there will be some quantitative differences.

Integrating out such a massive $\bar{a}$ in the superconducting phase leads to the effective 2 + 1 dimensional Thirring model [30] for spinons

$$\mathcal{L} = \bar{\Psi}_i \gamma_\mu \partial_\mu \Psi_i + \frac{\pi}{72m} \bar{\Psi}_i \gamma_\mu \gamma_5 \Psi_j \gamma_\mu \Psi_j,$$  \hspace{1cm} (7)

with the summation over repeated indices and the ultraviolet cutoff $\Lambda_{Th}$ assumed. Using the Hubbard-Stratonovich transformation [31] we can rewrite this exactly as

$$\mathcal{L} = \bar{\Psi}_i \gamma_\mu \partial_\mu \Psi_i + \frac{18m}{\pi} Tr[M_\mu^{ij} M_\nu^{ji}] + Tr[M_\mu^{ij} \gamma_\mu \Psi_j \Psi_i].$$  \hspace{1cm} (8)

Making an ansatz $M_\mu^{ij}(x) = (M(x)/3) \delta_{ij} \gamma_\mu$ at the saddle point, and then integrating out the fermions yields

$$S = N \int d^3x [\frac{24m}{\pi} M^2(x) - Tr[\ln(\gamma_\mu \partial_\mu - M(x))]].$$  \hspace{1cm} (9)

Expanding further in powers of $M(x)$ we finally write

$$\frac{S}{N} = \int \frac{d^3q}{(2\pi)^3} \left[ \frac{24m}{\pi} - \frac{\Lambda_{Th}}{\pi^2} \right] + \left| \frac{q}{8} \right| M^2(q) + O(M^4),$$  \hspace{1cm} (10)

where $M^2(q) = M(q)M(-q)$.

Using the definition of the Dirac fields in terms of the electron creation and annihilation operators [1] it readily follows that

$$\langle M(x) \rangle = \frac{\pi}{12Nm} \langle \hat{S}_z(\vec{r}, \tau) \rangle \sum_{i=1}^{2} \cos(2\vec{K}_i \cdot \vec{r}),$$  \hspace{1cm} (11)

where the vectors $\pm \vec{K}_i$, $i = 1, 2$ denote the positions of the four nodes of the d-wave order parameter, as before. $\langle M(q = 0) \rangle$ is therefore the static SDW OP [1], which, of course, was the motivation behind our saddle-point ansatz. Similarly,

$$\langle M^2(q, \omega) \rangle - \frac{\pi}{48Nm} =$$  \hspace{1cm} (12)

$$\left( \frac{\pi}{24Nm} \right)^2 \sum_{i=1}^{2} \left( \langle \hat{S}_z(\vec{q} + 2\vec{K}_i, \omega) \hat{S}_z(-\vec{q} - 2\vec{K}_i, -\omega) \rangle + \langle \hat{S}_z(\vec{q} - 2\vec{K}_i, \omega) \hat{S}_z(\vec{q} + 2\vec{K}_i, -\omega) \rangle \right).$$

Both Eqs. (11) and (12) represent exact equalities.

Since the dSC is rotationally invariant, the spin-spin correlation function is diagonal in indices $\alpha, \beta$, $\langle \hat{S}_\alpha(\vec{k}, \omega) \hat{S}_\beta(-\vec{k}, -\omega) \rangle = \chi(\vec{k}, \omega) \delta_{\alpha\beta}$. We may therefore finally deduce that the imaginary part of the spin response function $\chi''(\vec{k}, \omega) = \chi''(\vec{k}, \omega) + i\chi''(\vec{k}, \omega)$ in the phase fluctuating dSC is

$$\chi''(2\vec{K}_i \pm \vec{q}, \omega) = \left( \frac{12Nm}{\pi} \right)^2 Im(M^2(\vec{q}, \omega)),$$  \hspace{1cm} (13)

Neglecting the quartic and higher order terms in Eq. (10) (which may be formally justified by assuming $N \gg 1$), and analytically continuing to real frequencies $i\omega \rightarrow \omega$, in the Gaussian approximation we finally obtain

$$\chi''(2\vec{K}_i \pm \vec{q}, \omega) = \left( \frac{8m}{\pi} \right)^2 \frac{2N\Theta(\omega^2 - q^2)\sqrt{\omega^2 - q^2}}{(64m/\pi)^2 + \omega^2 - q^2},$$  \hspace{1cm} (14)

where we have written the cutoff $\Lambda_{Th} \approx 5m$ as $\Lambda_{Th} = (16/\pi)m$, for convenience. Eq. (14) is our central result. Its' most important feature is that the relativistic form of the quasiparticle dispersion near the nodes forces $\chi''$ to vanish below a certain momentum-dependent frequency.

First, let us check that deep in the superconducting phase we recover the result for the free quasiparticles. Taking $m \rightarrow \infty$ in the Eq. (14)

$$\lim_{m \rightarrow \infty} \chi''(2\vec{K}_i \pm \vec{q}, \omega) = \frac{N}{64} \Theta(\omega^2 - q^2) \sqrt{\omega^2 - q^2},$$  \hspace{1cm} (15)

in agreement with [32] and [33] for $N = 2$. For $q = 0$, for example, $\chi'' \sim |\omega|$, and the spin response at $\pm 2\vec{K}_i$ is completely suppressed, as expected.

Next, we connect the uniform vortex susceptibility $m$ to the measurable superconducting $T_c$. Assuming a continuous dSC-SDW transition, $m^2 \propto (\mu^2 - \mu_c'^2)^\gamma$, with $\gamma = \nu(2 - \eta)$ characterizing the quantum critical point; also, $T_c \propto (\mu^2 - \mu_c^2)^{z\nu}$. Since the dynamical critical exponent in the theory (1) is $z = 1$, $m \propto T_c^{1-(\eta/2)}$. Sufficiently away from the critical point we may neglect the anomalous dimension $\eta$ and find the result dictated by the (engineering) dimensional analysis, $m = c(\pi T_c/64), $
where \( c \) is a number. This would remain true even if the transition is weakly first order.

On the other hand, retaining the charge degrees of freedom one finds that the \( T = 0 \) superfluid density is \( \rho_{sf} \approx 24m/\pi \), in the underdoped regime [11]. This allows us to estimate the number \( c \) by utilizing the Uemura scaling [34], by which in our units \( \rho_{sf} \approx 2.1T_c \) in underdoped cuprates [35]. This number is known not to be truly universal, but due to structural similarities between the cuprates the variations from one material to another are reasonably small [35]. The results that follow will at any rate turn out to be quite insensitive to its exact value. This finally yields \( c \approx 5.6, \) i. e.

\[
\frac{64n}{\pi} \approx 5.6T_c. \tag{16}
\]

This should be understood as only a crude estimate; changing the cutoff \( \Lambda_{Th} \), for example, will alter the numerical value of the constant \( c \). This inherent limitation of the present calculation underlines the need for a better treatment of the gauge field term in the superconducting phase, that would improve upon our constant mass approximation. We emphasize that this type of incommensurate wave vectors \( \vec{K} \), such four well separated peaks at \( 2\vec{K}_1, -2\vec{K}_1 + (2\pi, 2\pi), 2\vec{K}_2 + (2\pi, 0), \) and \( -2\vec{K}_2 + (0, 2\pi) \), where we have used the periodicity provided by the underlying lattice to shift all the peaks into the vicinity of the commensurate wave vector \( \vec{Q} = (\pi, \pi) \). For illustration, we assumed \( T_c = 67K \) [16], \( v_F/v_\Delta = 7 \) [36], \( v_F = 1.2eV\AA \) [37], and estimated \( 2\vec{K}_1 = 0.465(2\pi, 2\pi) \) from [14]. These ‘mother’ peaks are narrow and of low intensity, since both their width and the magnitude are controlled by the frequency.

2) As the frequency increases both the thermal intensity and the width of the mother peaks in the momentum space grow. Since \( v_F \gg v_\Delta \) they first start to overlap at four ‘parallel’ positions (Fig. 3). This way the superposition of the mother peaks leads to appearance of the maximum response at four incommensurate ‘parallel’ wave vectors. We emphasize that this type of incommensuration has nothing to do here with any sort of one dimensional (‘stripe’) ordering, as often assumed in literature.

B. Evolution with frequency

We look for the maxima of the \( \chi''(\vec{q}, \omega) \) in the momentum space at a fixed \( \omega \). There are four regimes to discern:

1) For low frequencies \( \omega \), the maximum values of \( \chi'' \) are located at four "diagonally" incommensurate wave vectors \( \pm 2\vec{K}_{1,2} \) (i.e. at \( \vec{q} = 0 \)). In Fig. 2 we plot such four well separated peaks at \( 2\vec{K}_1, -2\vec{K}_1 + (2\pi, 2\pi), 2\vec{K}_2 + (2\pi, 0), \) and \( -2\vec{K}_2 + (0, 2\pi) \), where we have used the periodicity provided by the underlying lattice to shift all the peaks into the vicinity of the commensurate wave vector \( \vec{Q} = (\pi, \pi) \). For illustration, we assumed \( T_c = 67K \) [16], \( v_F/v_\Delta = 7 \) [36], \( v_F = 1.2eV\AA \) [37], and estimated \( 2\vec{K}_1 = 0.465(2\pi, 2\pi) \) from [14]. These ‘mother’ peaks are narrow and of low intensity, since both their width and the magnitude are controlled by the frequency.

2) As the frequency increases both the thermal intensity and the width of the mother peaks in the momentum space grow. Since \( v_F \gg v_\Delta \) they first start to overlap at four ‘parallel’ positions (Fig. 3). This way the superposition of the mother peaks leads to appearance of the maximum response at four incommensurate ‘parallel’ wave vectors. We emphasize that this type of incommensuration has nothing to do here with any sort of one dimensional (‘stripe’) ordering, as often assumed in literature.

3) With a further increase of frequency the four initial peaks start overlapping at \( \vec{Q} = (\pi, \pi) \), and for a while the ‘commensurate’ response dominates (Fig. 4). With the above parameters we find the response at \( (\pi, \pi) \) starting developing at \( \sim 40meV \) (see Fig. 6). The energy of the ‘resonance’ at which the response at \( \vec{Q} = (\pi, \pi) \) is maximal is \( \omega_{res} \approx 68meV \). The overlap of all four mother peaks makes the response large in the whole interior of the square in Fig. 2.

4) Finally, at the largest frequencies the maximum in Eq. (14) shifts to a \( |\vec{q}| \neq 0 \), which implies a weak redistribution of the commensurate peak to four ‘parallel’ incommensurate positions again (Fig. 5). However, whereas the position of the ‘parallel’ incommensurate peaks at Fig. 3 was independent of frequency at low frequencies, now it increases with frequency.

C. Comparison with experiment

Let us review some of the features of neutron scattering experiments on underdoped cuprates pertaining to our discussion. We will concentrate on the underdoped YBCO at low temperatures, where the most detailed study of spin response as a function of frequency and wave vector exists [16].

1) At low frequencies \( (\omega < 40meV) \) the maximum response is at four ‘parallel’ incommensurate wave vectors [8], [16]. The location of these peaks is very weakly dependent, or almost independent, on frequency [16].

2) At a frequency \( \omega \sim 40meV \) the maximum shifts to the commensurate position at \( (\pi, \pi) \), and peaks there at a slightly larger energy. This is the so-called ‘resonance’.

3) At a higher frequency \( \omega \sim 50meV \) the resonance becomes flat, and two weak ‘parallel’ incommensurate
peaks can be discerned again. This time, however, their incommensuration increases with frequency.

4) The energy of the resonance decreases with the superconducting $T_c$. [14, 15].

5) At $\vec{Q} = (\pi, \pi)$ there is a ‘spin gap’: the commensurate response is strongly suppressed below a certain energy $\omega_{sg}$. At high frequencies the response also continuously decreases, leaving a maximum at the ‘energy of the resonance’.

6) Overall spin response increases with frequency at low frequencies, peaks around the energy of the resonance, and then decreases with further increase in frequency.

We see that the features 1)-3) are at least qualitatively reproduced by our Eq. (14), once the lattice periodicity and the velocity anisotropy are taken into account. Also, if we assume that $v_F$ and $v_\parallel$ are only weakly doping dependent, the only energy scale left is the gauge field mass $m$, which is connected to the doping dependent superconducting $T_c$. The observed scaling of resonance energy with $T_c$ in underdoped samples therefore automatically follows from our theory. Note that this feature otherwise may appear quite counterintuitive: if one believes that in underdoped cuprates $T_c$ is the temperature where the phase coherence sets in, while the higher and increasing pseudogap temperature $T^*$ is related to the amplitude of the OP, the observed connection between the spin response and the superconducting phase coherence would seem rather mysterious.

The resonance at $\vec{Q} = (\pi, \pi)$, in our notation corresponds to some finite $\vec{q}_0 = \vec{Q} - 2\vec{K}_1$. This implies that as a function of frequency, at $T = 0$ $\chi''$ vanishes below the spin gap $\omega_{sg} \approx v_F q_0 \approx 60 meV$, which is in our theory essentially determined by the incommensuration of the mother peaks and the magnitude of $v_F$ (Fig. 6). While our estimate appears to be too large, our result naturally accounts for the existence of the spin gap. Also, the maximum response in Eq. (14) is at the energy $\omega_{res} = \sqrt{\omega_{sg}^2 + (cT_c)^2}$, which decreases with decreasing $T_c$, as observed [14, 15]. The experimental data of Dai et al. [14] are well reproduced by this simple formula with a just slightly smaller constant $c$ than in Eq. (16), and $\omega_{sg} = 22 meV$ (Fig. 7). As $T_c \to 0$, therefore, the commensurate peak energy $\omega_{res} \to \omega_{sg} \neq 0$. We thus predict that the spin gap, and the $T_c \to 0$ extrapolated energy of the resonance are essentially the same. Remarkably, the spin gap that can be estimated from a different set of data on YBCO (Fig. 17 in [15]) is quite similar, $\omega_{sg} \approx 20 meV$, consistent with our interpretation. At large energies, $\chi''$ should behave as $\sim 1/\omega$ (see Fig. 6), which also appears to be in a qualitative, but not quite a quantitative, accord with the data.

Finally, we predict a new feature of the spin response at the lowest frequencies: four weak and narrow 'diagonally' incommensurate mother peaks, the energy of which vanishes with $T_c$. Their low intensity and sharpness make them rather elusive, and indeed they have yet to be observed. Nevertheless, the fact that the evolution of the spin response with frequency as observed in underdoped YBCO can be straightforwardly understood in terms of these makes one optimistic about their existence. We find that the mother peaks should resolve as in Fig. 2 at a frequency $\sim \omega_{res}/10$. Our optimism is further supported by the fine details of the measured spin response, which appear so far to be in accord with our picture. For example, Eq. (14) predicts that the response function at the 'parallel' peaks must always be less than twice larger than that at $\pm 2\vec{K}_1,\parallel$ at the same frequency. Closer inspection of the data in [14] or [16], for example, conforms to this expectation.
superconductor. The theory extends the energy spin excitations in the strongly fluctuating d-wave gauge field cutoff $\Lambda$ constant in the Thirring model this result by considering the dimensionless coupling con-
der and the phase coherence. One can readily understand
metry for fermions becomes broken at a
Eq. (7). This leads to the conclusion that the chiral sym-
phase, so that after integration over the gauge field the
 coupled to the massive gauge field in the superconducting
a closely related theory to (1). In their model spinons are
region between the dSC and the SDW is also in contrast
mean-field theories of the high temperature superconduc-
YBCO.

VI. CONCLUSION AND DISCUSSION

We have proposed an effective theory for the low-
energy spin excitations in the strongly fluctuating d-wave
superconductor. The theory extends the $QED_3$ of the
pseudogap phase into the superconducting phase. It sug-
jects that there is a single dSC-SDW transition, which
may be fluctuation induced first order. We derived ap-
proximately the spin response function of the fluctuating
d-wave superconductor, and used it to qualitatively ex-
plain the neutron scattering experiments in underdoped
YBCO.

Besides standing in clear opposition to most of the
mean-field theories of the high temperature superconduct-
vivity [5], [6], our conclusion that there is no coexistence
region between the dSC and the SDW is also in contrast
to the result of Perez-Barnea and Franz [23], who studied
a closely related theory to (1). In their model spinons are
coupled to the massive gauge field in the superconducting
phase, so that after integration over the gauge field the
model would become similar to the Thirring model in our
Eq. (7). This leads to the conclusion that the chiral sym-
metry for fermions becomes broken at a finite mass of the
gauge field $m$, which implies coexistence of the SDW or-
der and the phase coherence. One can readily understand
this result by considering the dimensionless coupling con-
stant in the Thirring model $g = \Lambda_{Th}/m$. Indeed, if the
cutoff $\Lambda_{Th}$ is kept constant, $g \rightarrow g_c \sim 1$ at a finite $m$,
and the model exhibits its well-known chiral instability.
This result, however, in the more complete theory (1)
 is avoided by remembering that in the Thirring model,
or in an equivalent description of [23], the cuttof needs to
be $\Lambda_{Th} \sim m$ as well, because for momenta $p \gg m$
the inverse gauge field propagator becomes a linear func-
tion of $p$ (Fig. 1). With this modification, as $m \rightarrow 0$,
$g \rightarrow constant$, with the constant apparently too small
for the chiral instability to occur before the critical point
is reached.

The result that superconducting and antiferromag-
netic orders are not comfortably coexisting in under-
doped cuprates is also in agreement with recent exper-
iments on heat transport and c-axis penetration depth
in clean underdoped YBCO [38], [39]. Both measure-
ments find that d-wave nodes survive in the underdoped
regime, without the quasiparticles becoming gaped. If
there would be a sufficiently strong coexisting antiferro-
magnetic ($\vec{Q} = (\pi, \pi)$) or SDW ($\vec{Q} \neq (\pi, \pi)$) order, on the
other hand, quasiparticles would acquire precisely such a
gap at the nodes. It should be said, however, that the
coeistence of SDW and SC orders is not in principle
impossible, and can be induced by a strong favourable
interaction term, for example, omitted in (1). Our point
is only that vortex condensation as the dynamical mech-
anism behind the appearance of the SDW order in the
theory (1) does not lead to such coexistence.

Our picture of spin response in underdoped cuprates is
similar in spirit to that of Rantner and Wen [32]. There
are important conceptual and formal differences, how-
ever. Rantner and Wen considered the pseudogap regime,
above the superconducting $T_c$, and proposed that spinons
may there effectively be described as coupled to a mass-
less gauge field. Neglecting the chiral instability such a
theory is almost universally believed to have, they pro-
posed the algebraic spin liquid as the pseudogap phase,
in which in the large-N approximation the spin response
is in fact suppressed at low frequencies. Only if the num-
ber of fermion components $N$ is taken sufficiently small
(smaller in fact than the $N_c$ below which the chiral in-
stability occurs in the same calculation) does one recover
the enhancement of the spin response. To account phe-
nomenologically for the phase coherence in the supercon-
ducting state Rantner and Wen added a mass to their
gauge field, which then further suppresses the spin re-
sponse at energies below the mass. We see that a qual-
itatively similar mechanism operates in our theory, but
at $T = 0$: the gauge field coupled to spinons is mass-
less outside the SC, and massive inside. The form of our
spin response function $\chi''$, however, is quite different,
which, for example, leads to reappearance of the incom-

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig6}
\caption{Spin response function $\chi''$ (in arbitrary units) at the commensurate wave vector $\vec{Q} = (\pi, \pi)$ as a function of energy (in units of $\nu_F = 1.2 eV A$). Note that the spin response vanishes below certain frequency (`spin gap'), sharply peaks just above it, and behaves like $\sim 1/\omega$ at large frequencies.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig7}
\caption{The energy of the resonance (in meV) vs. the superconducting $T_c$ (in K). Data are from the Fig. 29 in second ref. 14, and the fit is to $\omega_{res} = \sqrt{(cT_c)^2 + \omega_{sg}^2}$ (see the text), with $c = 4.75$, and $\omega_{sg} = 22 meV$.}
\end{figure}
mensurate peaks at high energies (Fig. 5), among other things. Most importantly, the mass of the gauge field $m$ in the superconducting phase that was postulated in [32] on phenomenological grounds here becomes a result within the effective theory (1).

With the parameters we used the energy of the resonance $\omega_{res} \approx 70\text{meV}$, not far from the generic $\sim 40\text{meV}$ see in YBCO, and more recently in $Bi_2Sr_2CaCu_2O_{8+\delta}$ (BSCCO) [40], [41]. Given the crudeness of our approximation this is an encouraging agreement. However, there is a considerable uncertainty in the parameters we used which may easily make this agreement fortuitous. For example, assuming $v_F = 1.8eV\text{A}$ [42] would push the resonance energy in our calculation to $\sim 100\text{meV}$. Also, we do not find the $\delta$-function response at the commensurate wave vector, but only a $1/\omega$ decay on the high energy side (Fig. 6). This is contrast to the mean-field calculation within the $t - J$ model of Brinckmann and Lee [6], who found that the next-nearest-neighbor hopping may lead to a $\delta$-function term in $\chi^{\nu}$. Our result, however, seems to agree with the measurements of Fong et al. [15] on underdoped YBCO, where the width of the resonance of $\sim 10\text{meV}$ was observed (see Fig. 7 in [15]). It is interesting that Brinckmann and Lee, just like us, also tend to overestimate the energy of the resonance, and found it necessary to reduce the coupling $J$ by hand by a factor of three in their calculation to match the experiment.

We find the commensurate response to dominate only within a certain window of energies around the resonant energy, while both at lower ($\omega < \omega_{sg}$) and higher ($\omega > 1.5\omega_{sg}$) energies the peaks are located at the parallel incommensurate wave vectors. In our calculation the resonance exists for $60\text{meV} < \omega < 90\text{meV}$, with the parameters we used. For a lower $T_c$ the window of the dominant response at $\vec{Q} = (\pi, \pi)$ shrinks. This may provide an explanation why the resonance was never found in LSCO, but has been seen in both YBCO and BSCCO which both have a higher $T_c$, without other parameters being too different. Alternatively, the physics of underdoped LSCO may be qualitatively different, as some measurements would suggest [17].

We end by repeating the central prediction of this work: the appearance of four weak and narrow ‘diagonal’ incommensurate peaks, the energy of which should extrapolate to zero with vanishing superconducting $T_c$. It is the interplay of these fundamental peaks that leads to evolution of spin response with frequency in our calculation in qualitative agreement with experiment. While some other theories [6], [32] would also lead to such ‘diagonal’ incommensuration at low frequencies, we also predict that the energy of these peaks should vanish with $T_c$, since they represent the intrinsic soft mode of the phase fluctuating d-wave superconductor. Detection of this feature in future neutron scattering experiments would provide strong support in favor of the theory of the underdoped cuprates advocated here.

VII. ACKNOWLEDGEMENT

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