Confined Klein-Gordon oscillators in a spacetime and a pseudo-spacetime with a space-like dislocation: PDM KG-oscillators, isospectrality and invariance

Omar Mustafa

Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus, Mersin 10 - Turkey.

Abstract: We revisit the a confined (in a Cornell-type Lorentz scalar potential) KG-oscillator in a spacetime with space-like dislocation background. We show that the effect of space-like dislocation is to shift the energy levels along the dislocation parameter axis, and consequently energy levels crossings are unavoidable. We report some KG-particles in a transformed pseudo-spacetime with space-like dislocation that admit isospectrality and invariance with the confined KG-oscillator in a spacetime with space-like dislocation. An alternative metaphorically PDM settings for the KG-particles (relativistic particles in general) is introduced. We discuss the effects of space-like dislocation and PDM settings on the confined KG-oscillators in a spacetime with space-like dislocation. Three confined PDM KG-oscillators are discussed as illustrative examples, (i) a PDM KG-oscillator from a dimensionless scalar multiplier $m(r) = \exp(2\alpha r^2) \geq 0$, $\alpha \geq 0$, (ii) a PDM KG-oscillator from a power law type dimensionless scalar multiplier $m(r) = Ar^\sigma \geq 0$, and (iii) a PDM KG-oscillator in a Cornell-type confinement with a dimensionless scalar multiplier $m(r) = \exp(\xi r) \geq 0$.

PACS numbers: 03.65.Ge,03.65.Pm,02.40.Gh

Keywords: Klein-Gordon (KG) oscillator, spacetime with space-like screw dislocation, position-dependent mass KG-particles.

I. INTRODUCTION

The grand unified theories have predicted possible topological defects in spacetime [1-4] that have been investigated in many areas of physics. For example, in condensed matter physics [5], in gravitation [6-8] (where the linear defects are due dislocation (torsion) and curvature (disclinations)), in domain wall [2, 9], in cosmic string [9, 10], in global monopole [11], etc. However, in their work on Volterra distortions and cosmic defects, Puntigam and Soleng [6] have generalized the Volterra distortion to (3+1)-dimensions, using differential geometric and gauge theoretical methods, and introduced the concept of Volterra distorted spacetime. Where, distortions are line-like defects characterized by a delta-function-valued curvature (classified as disclination) and torsion (classified as dislocation) distributions that result in rotational and translational holonomy. Dislocation may be in the form of a spiral-type [7] or a screw-type [8, 12]. The latter is in point of the current study.

Such topological defects in spacetime have their figure prints on the spectroscopic structure of relativistic and non-relativistic quantum systems. The Dirac oscillator [13], for example, is investigated by Hassanabadi and co-worker [14, 15] in a Gödel-type cosmic string Som-Raychaudhuri spacetime. The Klein-Gordon (KG) oscillator is studied in the Gödel-type spacetime (e.g., [12, 13, 21]), in cosmic string spacetime and Kaluza-Klein theory (e.g., [12, 22, 20]).

*Electronic address: omar.mustafa@emu.edu.tr
in Som-Raychaudhuri spacetime \cite{27}, in the (2+1)-dimensional Gürses spacetime (e.g., \cite{28–32}).

On the other hand, the concept of position-dependent effective mass (PDM) (initiated by Mathews-Lakshmanan oscillator \cite{33}) has sparked research interest on PDM in both classical and quantum mechanics \cite{33–55}. Such a PDM concept is, in fact, a metaphoric manifestation of coordinate transformation \cite{37–39, 43}. The coordinate transformation, in effect, changes the form of the canonical momentum in classical and the momentum operator in quantum mechanics (e.g., \cite{37, 38, 41, 46} and related references therein). In classical mechanics, for example, negative the gradient of the potential force field is no longer the time derivative of the canonical momentum

$$ p = m \dot{x} $$

but it is rather related to the time derivative of the pseudo-momentum (also called Noether momentum) $$ \pi(x) = \sqrt{m^2 \dot{x}^2} \quad \text{\cite{38}}, $$

where $$ m $$ denotes the mass of the particle at hand and $$ m(x) $$ is a dimensionless positive valued scalar multiplier. In quantum mechanics, however, the PDM momentum operator is constructed by Mustafa and Algadhi \cite{41} to read

$$ \hat{p}(r) = -i \left( \nabla - \frac{\nabla m(r)}{4m(r)} \right) \iff \hat{p}_j(r) = -i \left( \partial_j - \frac{\partial_j m(r)}{4m(r)} \right) ; j = 1, 2, 3. \quad (1) $$

Which, in its most simplistic one-dimensional form, suggests (e.g., \cite{37, 40} for more details) that the von Roos \cite{34} PDM kinetic energy operator

$$ \hat{T}(x) \Phi(x) = \left( \frac{\hat{p}_x(x)}{\sqrt{m(x)}} \right)^2 \Phi(x) = -m(x)^{-1/4} \partial_x m(x)^{-1/2} \partial_x m(x)^{-1/4} \Phi(x), \quad (2) $$

This result clearly indicates that the momentum operator of an effective PDM quantum particle is given by \cite{11}. It has also been reported that such effective PDM quantum particles may very well be trapped in their own byproducted force fields (i.e., quasi-free PDM particles is used to describe such a system \cite{42}). Yet, it has been used to find the PDM creation and annihilation operators for the PDM-Schrödinger oscillator \cite{37}.

Nevertheless, attempts were made to include PDM settings in the Dirac and KG relativistic equations through the assumption that $ m \rightarrow m + S(r) = m(r) $, where $ m $ denotes the rest mass energy, $ S(r) $ is the Lorentz scalar potential (commonly used in heavy quarkonium spectroscopy \cite{57}), and $ m(r) $ denotes PDM (e.g., \cite{56, 58, 59}). In the current study, however, we shall rather argue that analogous to textbook procedure where the momentum operator $ p_j = -i \partial_j $ for constant mass is used in the relativistic wave equations, so should be the case with the PDM-momentum operator \cite{11} to describe effective PDM-relativistic quantum particles (e.g., \cite{37, 41, 42, 46}). That is, for effectively PDM particles (non-relativistic/relativistic), the PDM-momentum operator \cite{11} should replace the constant mass textbook momentum operator $ p_j = -i \partial_j $. In the current methodical proposal, we use such a PDM assumption and investigate the effects of the gravitational field generated by a spacetime with space-like screw dislocation \cite{39} below on some confined PDM KG-oscillators.

The organization of this paper is in order. We start, in section 2, with a confined (in a Cornell-type potential \cite{57, 60}) KG-oscillator in a spacetime with space-like dislocation background. We show that the effect of space-like dislocation is to shift the energy levels along the dislocation parameter axis, and consequently energy levels crossings (i.e., occasional degeneracies) are unavoidable. Energy levels crossings, nevertheless, is a phenomenon responsible for electron transfer in protein, it underlies stability analysis in mechanical engineering, and appears in algebraic geometry (e.g., \cite{61} and references cited therein). Moreover, clusterings of energy levels are found feasible for $ |\delta| >> 1 $, where $ \delta $ denotes space-like dislocation parameter. In section 3, we report some KG-particles in a transformed pseudo-spacetime with space-like dislocation \cite{20}, below, that admit isospectrality and invariance with the KG-oscillator in a spacetime
with space-like dislocation \( \delta \), below. Moreover, we suggest (in section 4) an alternative PDM setting for the KG-particles (relativistic particles in general). Therein, we use the PDM-momentum operator \( \mathbf{P} \), constructed by Mustafa and Algadhi \[41\], and discuss the effects of space-like dislocation and the metaphorically PDM settings on the confined KG-oscillators in a spacetime with space-like dislocation. Three confined PDM KG-oscillators are used/discussed as illustrative examples, (i) a PDM KG-oscillator from a dimensionless scalar multiplier \( m(r) = \exp(2\alpha r^2) \geq 0, \alpha \geq 0 \),
(ii) a PDM KG-oscillator from a power law type dimensionless scalar multiplier \( m(r) = Ar^\sigma \geq 0 \), and (iii) a PDM KG-oscillator in a Cornell-type confinement with a dimensionless scalar multiplier \( m(r) = \exp(\xi r) \geq 0 \). Our concluding remarks are given in section 5.

II. CONFINED KG-OSCILLATOR IN A SPACETIME WITH SPACE-LIKE SCREW DISLOCATION: REVISITED

In this section, we consider a spacetime with space-like screw dislocation (i.e., a Volterra-type spacetime with \( \hbar = c = 1 \) units) described by the line element

\[
ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 + (dz + \delta d\varphi)^2,
\]

where \( \delta \) denotes space-like dislocation parameter (i.e., torsion parameter). The covariant and contravariant metric tensors in this case, respectively, read

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r^2 + \delta^2 & \delta \\
0 & 0 & \delta & 1
\end{pmatrix} \quad \iff \quad \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{1 + \delta^2} & -\frac{\delta}{1 + \delta^2} \\
0 & 0 & -\frac{\delta}{1 + \delta^2} & 1 + \frac{\delta^2}{1 + \delta^2}
\end{pmatrix}; \quad \det(g) = -r^2.
\]

On the other hand, the KG-equation, with a Lorentz scalar potential \( S(r) \) (i.e., \( m \rightarrow m + S(r) \)) \[56, 58\], is given by

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = (m + S(r))^2 \Psi.
\]

Moreover, we may now use similar recipe to those used in \[13, 22, 60, 64\] and consider

\[
p_\mu \rightarrow p_\mu + i\eta \chi_\mu,
\]

where \( \chi_\mu = (0, r, 0, 0) \). This would, in effect, transform KG-equation (5) into

\[
\frac{1}{\sqrt{-g}} (\partial_\mu + \eta \chi_\mu) [\sqrt{-g} g^{\mu\nu} (\partial_\nu - \eta \chi_\nu) \Psi] = (m + S(r))^2 \Psi,
\]

Hereby, one may use \( \eta = m\omega \geq 0 \) (with \( m \) denoting rest mass of the KG-particle) to recover the traditionally used values as in (e.g., \[12\] and other related references cited therein). However, we shall use a more general parameter \( \eta \geq 0 \). In this case, we avoid eminent confusion and inconsistency between \( m^2 \) (denoting \( m^2c^2 = mmc^2 \), the rest mass multiplied by the rest mass energy, should the KG-equation (7) be divided by \( c^2 \)) on the R.H.S. and the rest mass of the particle on the L.H.S. of the KG-equation (7) for \( \eta = m\omega \) case. This point is made implicitly clear by Moshinsky and Szczepaniak \[13\] and Mirza and Mohadesi \[64\] while dealing with the Dirac and KG oscillators, who
kept the speed of light $c$ as is. We therefore stick with our assumption and use the spacetime metric tensor elements in (4), to recast (7) as

$$\left\{ -\partial_t^2 + \left( \partial_r^2 + \frac{1}{r}\partial_r \right) + \frac{1}{r^2}\partial^2_\varphi + \left( 1 + \frac{\delta^2}{r^2}\right) \partial_z^2 - \frac{2\delta}{r^2}\partial_r\partial_z - \eta^2 r^2 - 2\eta - (m + S(r))^2 \right\} \Psi = 0. \quad (8)$$

A substitution in the form of

$$\Psi(t, r, \varphi, z) = \exp\left( i \left[ \ell \varphi + k z z - Et \right] \right) \psi(r) = \exp\left( i \left[ \ell \varphi + k z z - Et \right] \right) \frac{R(r)}{\sqrt{r}} \quad (9)$$

would result in

$$R''(r) + \left[ \lambda - \left( \frac{\ell^2 - 1/4}{r^2} \right) - \eta^2 r^2 - 2mS(r) - S(r)^2 \right] R(r) = 0, \quad (10)$$

where

$$\lambda = E^2 - k_z^2 - 2\eta - m^2; \quad \ell^2 = (\ell - k_z \delta)^2. \quad (11)$$

Notably, the effect of the space-like dislocation is to introduce a shift in the irrational magnetic quantum number $\tilde{\ell} = \pm |\ell - k_z \delta|$ of (11), where $\ell = 0, \pm 1, \pm 2, \cdots$, is the magnetic quantum number. Moreover, equation (10) resembles, with $S(r) = 0$, the two-dimensional radial Schrödinger oscillator (in the units $2m = \hbar = 1$) with an effective oscillation frequency $\eta \geq 0$. Consequently and mathematically inherits its textbook eigenvalues

$$\lambda = 2\eta \left( 2n_r + |\tilde{\ell}| + 1 \right) \iff E^2 = 2\eta (2n_r + |\ell - k_z \delta| + 2) + k_z^2 + m^2 \quad (12)$$

and radial eigenfunctions

$$\psi(r) \sim r^{|\ell - k_z \delta|} \exp\left( -\frac{\eta r^2}{2} \right) L_{n_r}^{|\ell - k_z \delta|}\left( \eta r^2 \right) \iff \psi(r) \sim r^{|\ell - k_z \delta|} \exp\left( -\frac{\eta r^2}{2} \right) L_{n_r}^{|\ell - k_z \delta|}\left( \eta r^2 \right), \quad (13)$$

where $L_{n_r}^{|\ell - k_z \delta|}\left( \eta r^2 \right)$ are the the associated Laguerre polynomial.

![FIG. 1](image-url)  
We plot the energy levels of (19) versus the torsion parameter $\delta$, for $m = k_z = \eta = 1, a = b = 2$ and for (a) $\ell = 0$, $n_r = 0, 1, 2, 3$, (b) $n_r = 1, \ell = 0, \pm 1, \pm 2$, and (c) $n_r = 3, \ell = 0, \pm 3, \pm 5$.

Let us now consider the KG-oscillator above be confined in a Cornell type potential

$$S(r) = ar + \frac{b}{r}. \quad (14)$$
In this case, equation (10) reads
\[ R''(r) + \left[ \tilde{\lambda} - \frac{(\tilde{\gamma}^2 - 1/4)}{r^2} - \tilde{\omega}^2 r^2 - 2mar - \frac{2mb}{r} \right] R(r) = 0, \] (15)
where
\[ \tilde{\lambda} = E^2 - k_z^2 - 2\eta - m^2 - 2ab; \quad \tilde{\gamma}^2 = (\ell - k_z\delta)^2 + b^2; \quad \tilde{\omega}^2 = \eta^2 + a^2. \] (16)

Now, \( \tilde{\gamma} = \pm \sqrt{(\ell - k_z\delta)^2 + b^2} \) is the new irrational magnetic quantum number and \( \tilde{\omega} = \sqrt{\eta^2 + a^2} \geq 0 \) is our new effective oscillation frequency. Equation (15) admits a solution in the form of
\[ \psi(r) = R(r) \sqrt{r^2} \exp \left( -\frac{\tilde{\omega}^2 r^2 + 2amr}{2\tilde{\omega}} \right) H_B \left( 2|\tilde{\gamma}|, \frac{2ma}{\tilde{\omega}}, \frac{a^2 m^2 + \tilde{\lambda}}{\tilde{\omega}^3}, \frac{4mb}{\tilde{\omega}} \right), \] (17)
where \( H_B(\alpha, \beta, \gamma, \delta, r) \) is the biconfluent Heun function that is truncated into polynomial of degree \( n \geq 0 \) by the condition that \( \gamma = 2(n + 1) + \alpha \) to secure finiteness and square integrability of the solution. However, the truncation order of the power series Heun polynomial condition would provide a quantization recipe but does not make \( n \) a valid quantum number. In this case, if we set \( n = 2n_r \geq 0 \), where \( n_r = 0, 1, 2, \cdots \) is the radial quantum number then the condition \( \gamma = 2(2n_r + 1) + \alpha \) would satisfy Ronveaux’s condition \([65, 66]\) and implies
\[ \frac{a^2 m^2 + \tilde{\lambda} \tilde{\omega}^2}{\tilde{\omega}^3} = 2(2n_r + |\tilde{\gamma}| + 1) \iff \tilde{\lambda} = 2\tilde{\omega}(2n_r + |\tilde{\gamma}| + 1) - \frac{m^2 a^2}{\tilde{\omega}^2} \] (18)
In this case, we get the relation for the energy eigenvalues as
\[ E^2 = 2 \left( \sqrt{\eta^2 + a^2} \right) \left( 2n_r + \sqrt{(\ell - k_z\delta)^2 + b^2} + 1 \right) - \frac{m^2 a^2}{\eta^2 + a^2} + 2\eta + k_z^2 + m^2 + 2ab. \] (19)

The choice of \( \gamma = 2(2n_r + 1) + \alpha \) is not a random one but rather manifested by the fact that when \( a = b = 0 \) the

![FIG. 2: We plot the energy levels of (19) versus the torsion parameter \( \delta \), without the Cornell confinement, for \( m = k_z = \eta = 1 \), \( a = b = 0 \) and for (a) \( \ell = 0 \), \( n_r = 0, 1, 2, 3 \), (b) \( \ell = 10 \), \( n_r = 0, 1, 2, 3 \), and (c) \( \ell = -10 \), \( n_r = 0, 1, 2, 3 \).](image-url)

energies in (12) should naturally be recovered (this issue is emphasised in e.g., [32, 65, 66]).

At this point, one should be aware that this result (19), along with that in (17), belong to the set of the so called conditionally exactly solvable quantum mechanical problems. Moreover, it is obvious that for the case when \( \tilde{\omega} = 0 = a \), the biconfluent Heun polynomial energies in (19) tragically fails to provide any information on the spectrum and/or the
radial wave functions of a KG-Coulombic problem. Yet, instead of collapsing into the spectrum of the KG-Coulombic problem, the reported spectrum \([19]\) collapses into the free relativistic particle energies \(E^2 = m^2 + k_z^2\). Nevertheless, we continue with such \textit{conditionally} exact solution \([17]\) and do our analysis.

In Figures 1 and 2, we show the effect of dislocation related parameter \(\delta\) on the energy levels of a confined KG-oscillator in a spacetime with space-like dislocation. We clearly observe that the first term under the square root of \([19]\) determines the shifts in the energy levels at \(\delta = \ell/k_z\), on the \(\delta\)-axis. That is, for negative \(\ell\) values the shifts will be in the negative \(\delta\) region, whereas for positive \(\ell\) values the shifts will be in the positive \(\delta\) region. This would, in effect, manifestly yield energy levels crossings (i.e., occasional degeneracies, as shown in figures 1(a), 1(b), and 1(c), with the Cornell confinement). Moreover, in Figures 2(a), 2(b), and 2(c), we observe eminent energy levels clusterings when \(|\delta| \gg 1\), for each value of the magnetic quantum number \(\ell = 0, \pm 1, \pm 2, \cdots\). The effects of the dislocation parameter on the energy levels of the confined KG-oscillator in a spacetime with space-like dislocation are clear, therefore.

### III. KG-PARTICLES IN A PSEUDO-SPACETIME WITH SPACE-LIKE DISLOCATION ADMITTING ISOSPECTRALITY AND INVARIANCE WITH THE KG-OSCILLATOR OF \([3]\)

Let metric \([3]\) that describes a spacetime with space-like dislocation be transformed in such a way that

\[
\begin{align*}
\tilde{g}_{\mu\nu} &= \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & g(r) & 0 & 0 \\
0 & 0 & (f(r) + \delta^2) & \delta \\
0 & 0 & \delta & 1
\end{pmatrix} \iff \tilde{g}^{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{1}{g(r)} & 0 & 0 \\
0 & 0 & \frac{1}{f(r)} & -\frac{\delta}{f(r)} \\
0 & 0 & -\frac{\delta}{f(r)} & \left(1 + \frac{\delta^2}{f(r)}\right)
\end{pmatrix}; \quad \det (\tilde{g}_{\mu\nu}) = -g(r) f(r).
\end{align*}
\]

We now include the PDM KG-oscillator using the momentum operator \([3]\) of Mirza et al.’s recipe \([64]\) and suggest that \(\chi_\mu = \left(0, \sqrt{g(r)} f(r), 0\right)\) to accommodate a new set of KG-oscillators in the pseudo-spacetime with space-like dislocation settings. This would, in effect, transform the KG-oscillator equation \([7]\) into

\[
\left[ \frac{\partial_r}{\sqrt{g(r) f(r)}} \left( \sqrt{\frac{f(r)}{g(r)}} \partial_r \right) - \partial_t^2 + \frac{\partial^2}{f(r)} + \left[1 + \frac{\delta^2}{f(r)}\right] \partial_z^2 - \frac{2\delta \partial_t \partial_z}{f(r)} - 2\eta - \eta^2 f(r) - (m + S(r))^2 \right] \Psi = 0.
\]

Which upon the substitution

\[
\Psi(t, r, \varphi, z) = \exp \left(i \left[\ell \varphi + k_z z - Et\right]\right) U(r),
\]

(25)
and
\[ S(\tilde{r}) = a \tilde{r} + \frac{b}{\tilde{r}} = a \sqrt{f(r)} + \frac{b}{\sqrt{f(r)}}, \]

yields
\[ \frac{\partial}{\sqrt{g(r)f(r)}} \left( \sqrt{\frac{f(r)}{g(r)}} \partial_r \right) U(r) + \left[ \lambda - \frac{\tilde{r}^2}{f(r)} - \tilde{\omega}^2 f(r) - 2m \left( a \sqrt{f(r)} + \frac{b}{\sqrt{f(r)}} \right) \right] U(r) = 0. \]  

(27)

Where \( \lambda, \tilde{\gamma}^2, \) and \( \tilde{\omega}^2 \) are defined in \( (16) \). Yet, the first term of \( (27) \) can be rewritten, with \( f(r) = \tilde{r}^2 \) and \( \partial_r = \frac{1}{\sqrt{g(r)}} \partial_r, \) as
\[ \frac{1}{\sqrt{f(r)}} \frac{1}{\sqrt{g(r)}} \partial_r \left( \sqrt{\frac{f(r)}{g(r)}} \frac{1}{\sqrt{g(r)}} \partial_r \right) U(r) = \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \frac{\tilde{r}}{\partial \tilde{r}} \right) U(\tilde{r}(r)) = \left( \frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \right) U(\tilde{r}(r)). \]

(28)

To remove the first derivative we may define \( U(\tilde{r}) = R(\tilde{r}) / \sqrt{\tilde{r}} \) to eventually imply
\[ \frac{d^2}{d\tilde{r}^2} R(\tilde{r}) + \left[ \lambda - \frac{(\tilde{\gamma}^2 - 1/4)}{\tilde{r}^2} - \tilde{\omega}^2 \tilde{r}^2 - 2m \tilde{r} - 2m b \frac{\tilde{r}}{\tilde{r}} \right] R(\tilde{r}) = 0. \]

(29)

This equation is in the same form as that in \( (15) \) and they are, therefore, isospectral and invariant. Hence, \( (29) \) inherits the energies reported in \( (19) \) and the radial eigenfunctions in \( (17) \) but with \( \tilde{r} \) replacing \( r \). That is, in terms of the biconfluent Heun polynomials, the eigenfunctions would read
\[ \psi(\tilde{r}) = \frac{R(\tilde{r})}{\sqrt{\tilde{r}}} \sim \tilde{r}^{\tilde{\gamma}^2} \exp \left( -\frac{\tilde{\omega}^2 \tilde{r}^2 + 2am\tilde{r}}{2\tilde{\omega}} \right) H_2 \left( 2|\tilde{\gamma}|, \frac{2ma}{\tilde{\omega}^{3/2}}, a^2 m^2 + \tilde{\lambda} \tilde{\omega}^2 \frac{4mb}{\sqrt{\tilde{\omega}^2}} \right). \]

(30)

As long as \( g(r) \) and \( Q(r) \) (of \( f(r) = \tilde{r}^2 = Q(r) \tilde{r}^2 \)) are correlated through \( (22) \) and they are positive valued functions, then all KG-particles in the transformed pseudo-spacetime with space-like dislocation metric \( (20) \) have identical energy spectra as the spectrum of the KG-oscillator in spacetime with space-like dislocation metric \( (19) \), confined \( S(r) \neq 0 \) or unconfined \( S(r) = 0 \).

IV. PDM KG-Oscillators in a Spacetime with a Space-like Dislocation and Confined/Unconfined KG-Oscillators

We have mentioned that Mustafa and Algadhi \[41\] have shown that an effective PDM-momentum operator is given by \( (11) \). In this section, we shall use such PDM-momentum operator to describe metaphorically PDM KG-particles in a spacetime with a space-like dislocation \( (3) \) and subject them to a Lorentz scalar potential \( S(r) \). The corresponding inverse metric tensor \( g^{\mu\nu} \) is readily given in \( (11) \). Moreover, we shall use the assumption that \( m(r) = m(r) \) (i.e., only radially dependent). Under such settings, the momentum operator in \( (10) \) would take the PDM form so that
\[ \hat{p}_\mu \rightarrow -i \partial_\mu + i \mathcal{F}_\mu; \quad \mathcal{F}_\mu = (0, \mathcal{F}_r, 0, 0), \quad \mathcal{F}_r = \eta r + \frac{m'(r)}{4m(r)}, \]

(31)

is used to construct the KG-oscillators with PDM in a spacetime with a space-like dislocation through
\[ \frac{1}{\sqrt{-g}} \left( \partial_\mu + \mathcal{F}_\mu \right) \left[ \sqrt{-gg^{\mu\nu}} (\partial_\nu - \mathcal{F}_\nu) \Psi \right] = (m + S(r))^2 \Psi. \]  

(32)
This equation (32), with the contravariant metric tensors in (4), would yield
\[
\left[ \frac{1}{r} \partial_r r \partial_r - \partial_t^2 + \frac{1}{r^2} \partial_\varphi^2 + \left( 1 + \frac{\delta^2}{r^2} \right) \partial_z^2 - \frac{2\delta}{r^2} \partial_z \partial_\varphi - \frac{\mathcal{F}_r}{r} - \mathcal{F}_z^2 - (m + S(r))^2 \right] \Psi = 0. \quad (33)
\]

We may now use \( \Psi (t, r, \varphi, z) \) of (9) to obtain
\[
R''(r) + \left[ \lambda - \frac{(\tilde{\ell}^2 - 1/4)}{r^2} - 2mS(r) - S(r)^2 + M(r) \right] R(r) = 0 \quad (34)
\]
where \( \lambda = E^2 - k_z^2 - m^2, \tilde{\ell}^2 = (\ell - k_z \delta)^2 \), and
\[
M(r) = \frac{3}{16} \left( \frac{m'(r)}{m(r)} \right)^2 - \frac{1}{4} \frac{m''(r)}{m(r)} - \frac{m'(r)}{4rm(r)} - \frac{m'(r)}{2m(r)}r. \quad (35)
\]

Obviously, for constant mass settings the dimensionless scalar multiplier is set equal 1, i.e., \( m(r) = 1 \), and equation (34) collapses into that of (10) as should be. Yet, one should notice that when the KG-oscillator’s irrational frequency is off, i.e., \( \eta = 0 \), equation (34) would describe KG-particles in a spacetime with a space-like dislocation, in general.

To study the space-like dislocation effect on such PDM KG-particles, we choose three illustrative examples.

**A. Example 1: A PDM KG-oscillator from a dimensionless scalar multiplier** \( m(r) = \exp(2\alpha r^2) \geq 0, \alpha \geq 0 \).

Let us start with a dimensionless scalar multiplier \( m(r) = \exp(2\alpha r^2) \geq 0, \alpha \geq 0 \) to imply that
\[
M(r) = -\Omega^2 r^2 - 2\Omega; \; \Omega = \alpha + \eta \geq 0. \quad (36)
\]
This would, in turn, implies that equation (34) now reads
\[
R''(r) + \left[ \lambda - \frac{(\tilde{\ell}^2 - 1/4)}{r^2} - \Omega^2 r^2 - 2mS(r) - S(r)^2 \right] R(r) = 0, \quad (37)
\]
where
\[
\lambda = E^2 - k_z^2 - 2\Omega - m^2; \tilde{\ell}^2 = (\ell - k_z \delta)^2. \quad (38)
\]

Obviously, the angular frequency \( \Omega = \alpha + \eta \geq 0 \) of this model suggests that a KG-oscillator could also be a manifestation of a dimensionless scalar multiplier \( m(r) = \exp(2\alpha r^2), \alpha \geq 0 \), when \( \eta = 0 \). This is yet another way to come out with a KG-oscillator like model. Moreover, for a Cornell-type confining potential \( S(r) \) (14) we obtain
\[
R''(r) + \left[ \tilde{\lambda} - \frac{(\tilde{\gamma}^2 - 1/4)}{r^2} - \tilde{\Omega}^2 r^2 - 2mar - \frac{2mb}{r} \right] R(r) = 0, \quad (39)
\]
where
\[
\tilde{\lambda} = E^2 - k_z^2 - 2\Omega - m^2 - 2ab; \quad \tilde{\gamma}^2 = (\ell - k_z \delta)^2 + b^2; \quad \tilde{\Omega}^2 = (\alpha + \eta)^2 + a^2. \quad (40)
\]
Now, \( \tilde{\gamma} = \pm \sqrt{(\ell - k_z \delta)^2 + b^2} \) is the new irrational magnetic quantum number and \( \tilde{\Omega} = \sqrt{\Omega^2 + a^2} \geq 0 \) is our new effective oscillation frequency. This equation is in the same form as that in (14), and hence it admits similar forms of the eigenfunctions (17) and energies (19) with \( \tilde{\Omega} \) replaces \( \tilde{\omega} \). That is,
\[
\psi(r) = \frac{R(r)}{\sqrt{r}} \sim r^{\kappa + \frac{1}{2}} exp \left( -\frac{\tilde{\Omega}^2 r^2 + 2mar}{2 \tilde{\Omega}} \right) H_B \left( 2 |\tilde{\gamma}|, \frac{2ma}{\tilde{\Omega}^{3/2}}; \frac{\alpha^2 m^2 + \tilde{\lambda} \tilde{\Omega}^2}{\tilde{\Omega}^3}, \frac{4mb}{\sqrt{\tilde{\Omega}}} \right), \quad (41)
\]
and
\[ \frac{a^2m^2 + \lambda \Omega^2}{\Omega^3} = 2(2n_r + |\gamma| + 1) \iff \lambda = 2\Omega(2n_r + |\gamma| + 1) - \frac{m^2a^2}{\Omega^2} \] (42)

In this case, we get the relation for the energy eigenvalues as
\[ E^2 = 2\left( \sqrt{(\alpha + \eta)^2 + a^2} \right)(2n_r + \sqrt{(\ell - k_z\delta)^2 + b^2} + 1) - \frac{m^2a^2}{(\alpha + \eta)^2 + a^2} + 2(\alpha + \eta) + k_z^2 + m^2 + 2ab. \] (43)

Which for \( \alpha = 0 \) retrieves the result in (17) and (19).

**B. Example 2: A PDM KG-oscillator from a power law type dimensionless scalar multiplier \( m(r) = Ar^\sigma \geq 0. \)**

A power-law type dimensionless scalar multiplier \( m(r) = Ar^\sigma \geq 0 \) would, through (35), imply that
\[ M(r) = -\frac{\sigma^2}{16r^2} - \frac{\sigma}{2}\eta. \] (44)

Which, in turn, yields
\[ R''(r) + \left[ \mathcal{E} - \frac{(\zeta^2 - 1/4)}{r^2} - \eta^2r^2 - 2mS(r) - S(r)^2 \right] R(r) = 0, \] (45)

where
\[ \mathcal{E} = E^2 - k_z^2 - 2\eta - m^2 - \frac{\sigma^2}{2}\eta; \quad \zeta^2 = (\ell - k_z\delta)^2 + \frac{\sigma^2}{16}. \] (46)

With \( S(r) = 0 \), this equation resembles that of the two-dimensional radial Schrödinger oscillator discussed in section 2 (namely, equations (10), (12), and (13)) and admits eigenvalues
\[ \mathcal{E} = 2\eta(2n_r + |\zeta| + 1) \iff E^2 = 2\eta\left( 2n_r + \sqrt{(\ell - k_z\delta)^2 + \frac{\sigma^2}{16}} + 2 \right) + k_z^2 + m^2 + \frac{\sigma}{2}\eta, \] (47)

and radial eigenfunctions
\[ R(r) \sim r^{|\zeta|+1/2} \exp\left( -\frac{\eta^2}{2} \right) L_n^{(|\zeta|)}(\eta r^2) \iff \psi(r) \sim r^{|\zeta|}\exp\left( -\frac{\eta^2}{2} \right) L_n^{(|\zeta|)}(\eta r^2). \] (48)

Let us now consider the PDM KG-oscillators confined in the Cornell-type potential of (14). This would, in effect, imply that equation (45) be rewritten as
\[ R''(r) + \left[ \tilde{\mathcal{E}} - \frac{(\tilde{\zeta}^2 - 1/4)}{r^2} - \tilde{\omega}^2r^2 - 2mar - \frac{2mb}{r} \right] R(r) = 0, \] (49)

where
\[ \tilde{\mathcal{E}} = \mathcal{E} - 2ab; \quad \tilde{\zeta}^2 = \zeta^2 + b^2; \quad \tilde{\omega}^2 = \eta^2 + a^2. \] (50)

Again, this equation is in the form of (15), and hence it admits similar forms of the eigenfunctions (17) and energies (19), with \( \tilde{\zeta} \) replacing \( \ell \), respectively,
\[ \psi(r) \sim r^{|\tilde{\zeta}|}\exp\left( -\frac{\tilde{\omega}^2r^2 + 2mar}{2\tilde{\omega}} \right) H_B\left( 2|\tilde{\zeta}|, \frac{2ma^2 + \tilde{\mathcal{E}}}{\tilde{\omega}^3} \frac{\tilde{\omega}^2}{\sqrt{\tilde{\omega}}}, \frac{4mb}{\sqrt{\tilde{\omega}}} \sqrt{\tilde{\omega}r} \right), \] (51)
FIG. 3: We plot the energy levels of the exponentially growing PDM for $m = k_z = \eta = 1$. We show in (a) the effect of the PDM parameter $\xi$ for $n_r = \ell = 0$, (b) the effect of the torsion parameter $\delta$, for $n_r = 2, \xi = 0, \ell = 0, \pm 1, \pm 2$, and (c) the effect of the torsion parameter $\delta$, for $\ell = 2, \xi = 4, n_r = 0, 1, 2, 3, 4$.

and

$$\frac{a^2 \sigma^2 + \hat{E} \omega^2}{\hat{\omega}^2} = 2 \left( 2n_r + \left| \xi \right| + 1 \right) \iff \hat{E} = 2\hat{\omega} \left( 2n_r + \left| \xi \right| + 1 \right) - \frac{m^2 a^2}{\omega^2}$$

In this case, we get the relation for the energy eigenvalues as

$$E^2 = 2 \left( \sqrt{\eta^2 + a^2} \right) \left( 2n_r + \sqrt{ \left( \ell - k_z \delta \right)^2 / 16 + b^2 } + 1 \right) - \frac{m^2 a^2}{\eta^2} + 2\eta + k_z^2 + m^2 + 2ab + \frac{\sigma^2}{2} \eta.$$  \hfill (53)

Obviously, such energy levels inherit the behavior of those of (19) discussed in section 2. Namely, Figures 1 and 2. That is, one may rewrite this energy equation as

$$E^2 = 2\tilde{\eta} \left( 2n_r + \sqrt{ \left( \ell - k_z \delta \right)^2 + b^2 } + 1 \right) - \frac{m^2 a^2}{\tilde{\eta}^2} + \left( 2 + \frac{\sigma^2}{2} \right) \eta + k_z^2 + m^2 + 2ab.$$  \hfill (54)

where $\tilde{\eta} = \sqrt{\eta^2 + a^2}$ and $\tilde{b} = b + \sigma^2 / 16$, to observe that similar trends of behavior.

C. Example 3: A PDM KG-oscillator in a Cornell-type confinement with a dimensionless scalar multiplier $m (r) = \exp (\xi r) \geq 0$

An exponentially growing dimensionless scalar multiplier $m (r) = \exp (\xi r) \geq 0$ would yield

$$M \left( r \right) = - \frac{\xi^2}{16} - \frac{1}{4} \frac{\xi}{r} - \frac{1}{2} \xi \eta r.$$  \hfill (55)

Consequently, the PDM KG-oscillator’s equation, with $S(r) = 0$, reads

$$R'' \left( r \right) + \left[ \Sigma - \frac{\left( \tilde{b}^2 - 1/4 \right)}{r^2} - \eta^2 r^2 - \frac{1}{4} \frac{\xi}{r} + \frac{1}{2} \xi \eta r - 2mS \left( r \right) - S \left( r \right)^2 \right] R \left( r \right) = 0,$$  \hfill (56)

where $\Sigma = E^2 - k_z^2 - 2\eta - m^2 - \xi^2 / 16$, and $\tilde{b}^2 = (\ell - k_z \delta)^2$. It is clear that a Cornell-type confinement (i.e., $\xi / 4r + \xi \eta r / 2$) is introduced as a byproduct of the dimensionless scalar multiplier settings at hand. We, therefore, continue with
\[ S(r) = 0. \] This equation \((56)\), with \(S(r) = 0\) and following the same procedure as that in the above examples, admits a solution in the form of biconfluent Heun polynomials

\[ \psi (r) \sim r^{|\ell|} \exp \left( -\frac{1}{2} \eta r^2 - \frac{1}{4} \xi r \right) H_B \left( 2 |\ell|, \frac{\xi |\ell| + 16 \Sigma}{16 \eta}, \frac{\xi r}{2 \sqrt{\eta r}} \right). \]  

Hence, the corresponding energy levels are given by

\[ \frac{\xi^2 + 16 \Sigma}{16 \eta} = 2 \left( 2 n_r + |\ell| + 1 \right) \Rightarrow E^2 = 2 \eta (2 n_r + |\ell - k_z \delta| + 2) + + k_z^2 + m^2 + \frac{1}{4} \eta \xi^2. \]  

The energy levels are shown in Figure 3. In Figure 3(a), we show the energy levels against the PDM parameter \(\xi \geq 0\) and observe eminent clustering of the energy levels as \(\xi\) grows up, but no energy levels crossing are found feasible. On the other hand, the space-like dislocation parameter’s effect on the energy levels, for some fixed values of \(\xi\), maintains the same trend of behavior as that associated with \([19]\) and discussed in section 2.

\[ \text{V. CONCLUDING REMARKS} \]

In this work, we have studied the KG-oscillator in a spacetime with a space-like dislocation. We have started with KG-oscillators confined in a Cornell-type Lorentz scalar potential and discussed the dislocation effect on their conditionally exact energy levels. We observed that the space-like dislocation shifts the energy levels along the dislocation parameter \(\delta\)-axis by \(\delta = \ell/k_z; \ell = 0, \pm 1, \pm 2, \cdots\) (documented in Figures 1(b), 1(c), 2(b), 2(c), 3(b), and 3(c)). That is, for \(\ell = -|\ell|\) values, the shifts are in the direction of negative \(\delta\)-region, whereas for \(\ell = +|\ell|\) values the shifts are in the direction of positive \(\delta\) region. This in turn manifestly resulted in energy levels crossings (as shown in figures 1(b), 1(c), and 3(b)). Moreover, in Figures 2(a), 2(b), and 2(c), we have observed eminent energy levels clusterings when \(|\delta| > > 1\), for each value of the magnetic quantum number \(\ell = 0, \pm 1, \pm 2, \cdots\). We have reported, in section 3, a set of KG-particles in a transformed pseudo-spacetime with space-like dislocation admitting isospectrality and invariance with the confined KG-oscillators in a spacetime with a space-like dislocation. Such KG-particles are found to inherit the same effects discussed above.

We have used, in section 4, the argument that the momentum operator for PDM-particles (metaphorically speaking) is given by \((1)\) \((41)\) and yields a von Roos \((??)\) kinetic energy operator \((2)\) with the so called MM-ordering (e.g., Mustafa and Mazharimousavi’s ordering \((3)\) \((40)\). Such PDM-particles are studied in the context of KG-equation in a spacetime with a space-like dislocation background. Hence, the metaphoric notion \(PDM \, KG\text{-particles}\) is adopted in the process. The effect of space-like dislocation on energy levels of such KG-particles is reported through three illustrative examples, (i) a PDM KG-oscillator from a dimensionless scalar multiplier \(m(r) = \exp(2ar^2) \geq 0, \alpha \geq 0\), (ii) a PDM KG-oscillator from a power law type dimensionless scalar multiplier \(m(r) = Ar^n \geq 0\), and (iii) a PDM KG-oscillator in a Cornell-type confinement with a dimensionless scalar multiplier \(m(r) = \exp(\xi r) \geq 0\). For the PDM KG-oscillators of (i) and (ii), the energy levels are shown to have similar trends of behavior as those of \([19]\) discussed in section 2. Whereas, for the PDM KG-oscillator in (iii), we found that such PDM setting introduces a Cornell-like confinement as its own byproduct. Hereby, obvious clustering of the energy levels are observed, as the PDM parameter \(\xi\) grows up, but no energy levels crossing are found feasible for a fixed space-like dislocation parameter \(\delta\) value (documented in figure 3(a)). Moreover, the effect of the space-like dislocation parameter \(\delta\) on the energy levels, for a fixed PDM parameter \(\xi\), is found to maintain the same trend of behavior as that associated with \([19]\) and discussed in section 2.
Finally, the current methodical proposal may very well be extended to cover a more general case of PDM KG-particles and PDM Dirac-particles in different spacetime backgrounds with topological defects. In our opinion, the metaphoric PDM concept for relativistic particles should follow the procedure described in the above methodical proposal, and not through the assumption that \( m \rightarrow m + S(r) = m(r) \) (as in, e.g., \[56, 58, 59\]). To the best of our knowledge, such a PDM KG-oscillator in spacetime with space-like dislocation methodical proposal has never been reported elsewhere.

**Data Availability Statement** Authors can confirm that all relevant data are included in the article and/or its supplementary information files.
