Matrix Theory Description of Schwarzschild Black Holes in the Regime $N \gg S$

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ABSTRACT. We study the description of Schwarzschild black holes, of entropy $S$, within matrix theory in the regime $N \gg S \gg 1$. We obtain the most general matrix theory equation of state by requiring that black holes admit a description within this theory. It has a recognisable form in various cases. In some cases a $D$ dimensional black hole can plausibly be thought of as a $\bar{D} = D + 1$ dimensional black hole, described by another auxiliary matrix theory, but in its $\bar{N} \sim S$ regime. We find what appears to be a matrix theory generalisation to higher dynamical branes of the normalisation of dynamical string tension, seen in other contexts. We discuss a further possible generalisation of the matrix theory equation of state. In a special case, it is governed by $N^3$ dynamical degrees of freedom.

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1. In the past few years a lot of progress has been made in understanding
the properties of charged black holes using the physics of Dirichlet branes.
The latest advance in this progress is in the study of Schwarzschild black
holes in the context of matrix theory [1]. In [2, 3] Banks, Fischler, Klebanov,
and Susskind (BFKS) have shown that the matrix theory compactified on
$p$ dimensional torus reproduces, upto numerical factors, the correct mass vs
entropy relation for the $D = 11 - p$ dimensional Schwarzschild black holes.
Many other properties such as Hawking radiation, long range interaction,
etc. have also been obtained [1, 3]. The Schwarzschild black holes, thought
of as a collection of matrix partons, have also been studied in the context of
SYM theory and Dirichlet 0-branes [4, 5, 6, 7]. For some of the other related
studies, see [9].

Matrix theory can be thought of as the discretised light cone quantisation
(DLCQ) of M theory [1], with $N$ units of quantised longitudinal momentum.
To describe adequately the properties of a black hole with entropy $S$, $N$ must
be atleast of order $S$. Since the matrix theory computations are difficult
for $N \gg S$, BFKS compute the matrix theory equation of state (e.o.s) for
$N \sim S$ and show that it reproduces, upto numerical factors, the correct mass
vs entropy relation for the Schwarzschild black holes.

The BFKS regime $N \sim S$ turns out to be where a black hole to black
string transition [1] takes place [1, 11]. See also [12]. Essentially, for $N < S$
the black hole does not fit into the longitudinal space. This is a black string
configuration. For $N > S$ the black hole Lorentz contracts and fits into the
longitudinal space easily. This is a black hole configuration. Thus, $N \sim S$
is the regime where the transition takes place. When $N \gg S$, the black hole
size becomes much smaller. For a precise explanation, see [6].

The study of $N \gg S$ is of obvious interest and it is likely to yield valu-
able insights. However, in this regime, the matrix theory computations are
difficult, and not much is known from the SYM theory side either. In this
regime, BFKS take the entropy $S$ to be independent of $N$, but now equal
to the black hole entropy and thus predict the matrix theory e.o.s. They
also show that it is consistent with the physics of the SYM theory [2, 3]. In
a different analysis, Li and Martinec model the black hole as a collection of
matrix partons [4, 6, 7], and study the regime $N \gg S$ by treating the Lorentz
contracted black hole as a collection of $S$ interacting clusters of $\frac{N}{S}$ units each
[8].

In section 2, we give a brief description of the results of BFKS.
In this paper, we study the regime $N \gg S$ by another method. We mainly study if there are regimes $N^c \sim S$, $0 < c \leq 1$ so that $N \gtrsim S \gg 1$, where the resulting e.o.s is of a recognisable form - particularly, of that of a $\gamma + 1$ dimensional gas, with $\gamma$ a positive integer. Such regimes are very likely to have a field theoretic description.

For $N \gg S$, the size of the $D$ dimensional black hole becomes much smaller than the size of the longitudinal space. Hence, plausibly, we can think of such black holes as a $\tilde{D} = D + 1$ dimensional black holes and expect it to be described by another auxiliary matrix theory, in its BFKS regime. We find that this is indeed the case. In section 3, we give explicit illustration of various cases. However, we are unable to make more precise the physical sense in which a $D$ dimensional black hole, in the regime $N^c \sim S$, $0 < c < 1$, is equivalent to $\tilde{D} = D + 1$ dimensional black hole of an auxiliary theory, but in its BFKS regime.

It turns out that the BFKS results as well those in section 3 can all be derived systematically by imposing two requirements: (I) we work within the matrix theory framework, and (II) the $D$ dimensional Schwarzschild black holes admit a description within the matrix theory compactified on a $p$ ($= 11 - D$) dimensional torus. These two requirements lead to the most general expression for the matrix theory e.o.s. In section 4, we derive this expression and study various new regimes where the matrix theory e.o.s is of a recognisable form.

In the course of this study, we find that the tension of the dynamical $\delta$-branes ($\delta$ is a positive integer), defined in a particular natural sense, gets normalised as $N^{-\frac{1}{2}}$. This appears to be a matrix theory generalisation to higher dynamical branes of a phenomenon seen in other contexts where dynamical string tension is normalised as $\frac{1}{N^2}$ [13, 14].

In section 5, we discuss a further possible generalisation of the matrix theory e.o.s derived in the earlier section. This generalisation goes beyond the realm of DLCQ framework. We find one case which appears to be special and where the e.o.s in the regime $N \gg S$ is governed by $N^3$ dynamical degrees of freedom.

Although we believe that the present results are interesting, we also admit to one crucial shortcoming: we are unable to understand these results from the SYM theory side and, hence, are unable to present a simple physical picture, if exists, that underlies these results. In section 6 we conclude with
In this section we present the relevant details of the matrix theory description of $D$ dimensional Schwarzschild black holes, as given in [3, 4, 5]. Thereby, we will set our notation also.

Matrix theory can be thought of as the discretised light cone quantisation (DLCQ) of M theory [1]. Let $R$ be the radius of the light like circle. The quantised longitudinal momentum is then given by

$$P_\perp = \frac{N}{R}$$  

(1)

where $N$ is an integer. The matrix theory Hamiltonian is identified with the DLCQ energy according to

$$E = \frac{M^2}{P_\perp} = \frac{M^2 R}{N}.$$  

(2)

To describe the $D$ dimensional Schwarzschild black holes, the matrix theory is compactified on a $p (= 11 - D)$ dimensional torus. For simplicity, the torus is taken to be “square” with all circumferences equal to $L$. According to matrix theory conjecture, the sector of the theory with given $N$ is described exactly by the $p + 1$ dimensional U(N) SYM theory with 16 real supercharges. The SYM theory lives on a dual square torus of volume $V$ given by

$$V = \left(\frac{l_{11}^3}{RL}\right)^p$$  

(3)

where $l_{11}$ is the eleven dimensional Planck length. The SYM coupling constant is given by

$$g^2 = \frac{VR^3}{l_{11}^6}.$$  

(4)

The effective SYM coupling constant for large $N$ is given by

$$e^2 = g^2 N.$$  

(5)

In the $D$ dimensional non compact space time, the Newton’s constant $G_D$ and the Planck length $l_D$ are given by

$$G_D = l_D^{p_D - p} = \frac{l_{11}^p}{L^p}.$$  

(6)
where we have used $D + p = 11$. The entropy $S$ of the $D$ dimensional Schwarzschild black hole of mass $M$ is then given, up to numerical factors, by

$$S \sim (l_DM)_{\frac{9-p}{5-p}}. \quad (7)$$

To describe adequately the properties of a black hole with entropy $S$, $N$ must be at least of order $S$ as explained in detail in [2]. However, since the matrix theory computations are difficult for $N \gg S$, Banks et al (BFKS) take $N \sim S$ [2, 3, 4]. Their strategy is to first obtain the matrix model equation of state (e.o.s) $E = E(N, S)$. For $N \sim S$, the matrix model Hamiltonian (2) gives the relation

$$M^2 \sim \frac{S}{R} E(S, S) \quad (8)$$

between mass and entropy. This is then to be compared with the mass vs entropy relation of the black hole, as explained in [2].

The matrix model e.o.s is obtained by its conjectured relation to SYM theory. For $p = 3$, the conformal invariance of the theory determines the e.o.s. For other values of $p$, its description as $N$ coincident Dirichlet $p$-branes determines the e.o.s [15]. The e.o.s thus obtained can be written as

$$S = \left( (N^2 V)^{\frac{5-p}{2(p-3)}} e^{2(p-3)} E^{9-p} \right)^{\frac{1}{2(p-3)}}. \quad (9)$$

Applying BFKS strategy now, namely taking $N \sim S$, equation (9) can be seen to reproduce, up to numerical factors of order unity, the correct mass vs entropy relation for the $D$ dimensional Schwarzschild black hole, $4 \leq D \leq 11$.

Note that (9) can also be written as

$$S = \left( N^2 (\sigma E)^\gamma \right)^{\frac{1}{1+\gamma}} \quad (10)$$

where $\gamma = \frac{9-p}{5-p}$ and the parameter $\sigma$ has dimensions of length and is given by a specific combination of the two dimensionful parameters $V$ and $e^2$

$$\sigma = V^{\frac{5-p}{2(p-3)}} e^{\frac{2(p-3)}{2(p-3)}}. \quad (11)$$

The significance of $\sigma$ will become clear later. Thus, for the $D = 7, 8, 10$ dimensional black hole, i.e. for $p = 4, 3, 1$, the e.o.s (9) is that of a $\gamma + 1$ dimensional gas, $\gamma = 5, 3, 2$ respectively, with $N^2$ degrees of freedom. See [2, 3] for details.
The choice of the regime $N \sim S$ means the following \[2\]. Let $R_s$ be the Schwarzschild radius of the black hole in its rest frame. For $R_s > R$, the black hole does not fit into the longitudinal space. When it is boosted longitudinally such that its momentum is $P_− = \frac{N}{R}$, its radius Lorentz contracts to

$$r_s = \frac{M}{P_−} R_s = \frac{M R}{N} R_s. \quad (12)$$

In the regime $N \sim S \sim MR_s$, the Lorentz contracted radius $r_s \sim R$ and thus the black hole just fits inside the longitudinal space. A precise explanation is given in \[3\]. The regime $N \sim S$ also turns out to be where the black hole to black string transition \[10\] takes place \[3, 11\]. See also \[12\].

When $N \gg S$, the entropy must be a constant, independent of $N$. Taking the constant $S$ to be the black hole entropy $\sim MR_s$, and using equation (2) and the thermodynamic relation $dE = T dS$, one obtains the e.o.s:

$$S = \left(N^{2(6-p)} V^{5-p} e^{2(p-3) E^{9-p}}\right)^{\frac{1}{2(8-p)}}. \quad (13)$$

At $N \sim S$ this e.o.s agrees with that given in \[9\]. As this equation is derived using black hole physics alone, it can also be thought of as an implication for the SYM e.o.s in the regime $N \gg S$, and is, indeed, consistent with the physics of SYM theory \[2, 3\]. Note that the length scale appearing in (13) is also given by (11).

3. In this paper, we study the regime $N \gg S$ by another method. We mainly study if there are regimes $N^c \sim S$, $0 < c \leq 1$ so that $N \gtrsim S \gg 1$, where the resulting e.o.s is of a recognisable form - particularly, of that of a $\gamma + 1$ dimensional gas, with $\gamma$ a positive integer. Such regimes are very likely to have a field theoretic description.

In the regime $N \gg S$, the Lorentz contracted Schwarzschild radius $r_s$ given by (12) becomes much smaller than $R$, the size of the longitudinal space. Hence, the original $D$ dimensional black hole can plausibly be thought of as a $\tilde{D} = D + 1$ dimensional black hole. Such a black hole may be expected to be described by another auxiliary matrix theory, but now in its BFKS regime, say $\tilde{N} \sim S$ for a suitable $\tilde{N}$.

Thus, specifically, in the regime

$$N^c \sim S, \quad 0 < c < 1 \quad \text{so that} \quad N \gg S \gg 1, \quad (14)$$
we may expect the e.o.s of a $D$ dimensional black hole, for suitable values of $D$, to be that of a $\gamma + 1$ dimensional gas, where the value of $\gamma$ is the one appropriate for $\tilde{D} = D + 1$ dimensional black hole, but in the BFKS regime of the auxiliary matrix theory. We find that this is indeed the case and now give explicit illustration. Since the gas-like description is possible in the BFKS regime $\tilde{N} \sim S$ of the auxiliary theory for $\tilde{D} = D + 1 = 7, 8, 10$ only, we study below the regime $N^c \sim S$, $0 < c < 1$ for the cases $D = 6, 7, 9$ only.

3 a. Let $D = 6$. The e.o.s in the regime (14) must be of the form

$$S \sim E^\frac{\tilde{\gamma}}{2}$$

which follows because $\gamma = 5$ is the appropriate value for the $\tilde{D} = 7$ dimensional blackhole in the BFKS regime of the auxiliary theory. Throughout this section, we write explicitly the $N$ and $E$, equivalently $M$, dependence only. Since the only relevant length scale in the e.o.s appears to be the one given by (11), we take the e.o.s to be of the form given in (10). Thus, we get

$$S \sim \left( N^{2+\frac{5}{2}} E^5 \right)^\frac{1}{6}.$$  \hspace{1cm} (15)

The fraction $\frac{5}{2}$ arises from the $N$ dependence in $\sigma$ in (10) where $p = 11 - 6 = 5$ and $\gamma = 5$. Upon using (2) to write $E$ in terms of $M$ and $N$ and then using (14) to write $N$ in terms of $S$, we get

$$S^{6+\frac{1}{3}} \sim M^{10}.$$  

For $c = \frac{1}{3}$, this leads to

$$S \sim M^\frac{4}{3}$$

which is the correct mass vs entropy relation for the 6 dimensional Schwarzschild black hole, but now in the regime $N \sim S^3$. Moreover, in the regime $N \sim S^3$, the e.o.s (13), which was obtained for $N \gg S$ with $S$ taken to be constant, agrees with the above e.o.s (15).

The Hawking temperature is also correctly reproduced after deboosting the temperature $T_c$, corresponding to the regime $N \sim S^3$, to the rest frame of the black hole. In the regime $N \sim S^3$ the temperature $T_c$, obtained from (15) along with the thermodynamic relation $dE = T dS$, is given by

$$S \sim N^{2+\frac{5}{2}} T_c^5 \quad \longrightarrow \quad T_c \sim N^{-\frac{5}{6}}.$$
By deboosting to the black hole rest frame we get the Hawking temperature to be

\[ T_H = \frac{N}{M} T_c \sim M^{-\frac{1}{4}} \]

where we have used \( N \sim S^3 \sim M^4 \), valid in the regime under consideration. The Schwarzschild radius in this regime is given by

\[ S \sim M R_s \rightarrow R_s \sim M^{\frac{1}{3}} \]

and, hence, we have \( T_H \sim \frac{1}{R_s} \), the correct scaling relation for Schwarzschild black hole.

3 b. Let \( D = 7 \). The e.o.s in the regime (14) must be of the form

\[ S \sim E^{\frac{3}{4}} \]

which follows because \( \gamma = 3 \) is the appropriate value for the \( \tilde{D} = 8 \) dimensional blackhole in the BFKS regime of the auxiliary theory. As above, taking the e.o.s to be of the form given in (14), we get

\[ S \sim \left( N^{2 + \frac{3}{5}} E^3 \right)^{\frac{1}{4}} \]. \hspace{1cm} (16) \]

The fraction \( \frac{3}{5} \) arises from the \( N \) dependence in \( \sigma \) in (14) where \( p = 11 - 7 = 4 \) and \( \gamma = 3 \). Upon using (2) to write \( E \) in terms of \( M \) and \( N \) and then using (14) to write \( N \) in terms of \( S \), we get

\[ S^{1 + \frac{2}{5}} \sim M^6 \].

For \( c = \frac{1}{2} \), this leads to

\[ S \sim M^{\frac{5}{4}} \]

which is the correct mass vs entropy relation for the 7 dimensional Schwarzschild black hole, but now in the regime \( N \sim S^2 \). Moreover, in the regime \( N \sim S^2 \), the e.o.s (13), which was obtained for \( N \gg S \) with \( S \) taken to be constant, agrees with the above e.o.s (16).

The Hawking temperature is also correctly reproduced after deboosting the temperature \( T_c \), corresponding to the regime \( N \sim S^2 \), to the rest frame of the black hole. In the regime \( N \sim S^2 \) the temperature \( T_c \), obtained from (16) along with the thermodynamic relation \( dE = T dS \), is given by

\[ S \sim N^{2 + \frac{3}{5}} T_c^{\frac{3}{5}} \rightarrow T_c \sim N^{-\frac{5}{16}} \].
By deboosting to the black hole rest frame we get the Hawking temperature to be

\[ T_H = \frac{N}{M} T_c \sim M^{-\frac{1}{4}} \]

where we have used \( N \sim S^2 \sim M^\frac{2}{5} \), valid in the regime under consideration.

The Schwarzschild radius in this regime is given by

\[ S \sim M R_s \quad \rightarrow \quad R_s \sim M^{\frac{1}{2}} \]

and, hence, we have \( T_H \sim \frac{1}{R_s} \), the correct scaling relation for Schwarzschild black hole.

3 c. Let \( D = 9 \). The e.o.s in the regime (14) must be of the form

\[ S \sim E^{\frac{3}{2}} \]

which follows because \( \gamma = 2 \) is the appropriate value for the \( \tilde{D} = 10 \) dimensional blackhole in the BFKS regime of the auxiliary theory. As above, taking the e.o.s to be of the form given in (10), we get

\[ S \sim \left( N^{2 - \frac{2}{5}} E^2 \right)^\frac{1}{7} \quad (17) \]

The fraction \(-\frac{2}{5}\) arises from the \( N \) dependence in \( \sigma \) in (10) where \( p = 11 - 9 = 2 \) and \( \gamma = 2 \). Upon using (2) to write \( E \) in terms of \( M \) and \( N \) and then using (14) to write \( N \) in terms of \( S \), we get

\[ S^{3 + \frac{2}{7}} \sim M^4 \]

For \( c = \frac{2}{3} \), this leads to

\[ S \sim M^{\frac{1}{2}} \]

which is the correct mass vs entropy relation for the 9 dimensional Schwarzschild black hole, but now in the regime \( N \sim S^\frac{2}{5} \). Moreover, in the regime \( N \sim S^\frac{2}{5} \), the e.o.s (13), which was obtained for \( N \gg S \) with \( S \) taken to be constant, agrees with the above e.o.s (17).

The Hawking temperature is also correctly reproduced after deboosting the temperature \( T_c \), corresponding to the regime \( N \sim S^\frac{2}{5} \), to the rest frame of the black hole. In the regime \( N \sim S^\frac{2}{5} \) the temperature \( T_c \), obtained from (17) along with the thermodynamic relation \( dE = TdS \), is given by

\[ S \sim N^{2 - \frac{2}{5}} T_c^2 \quad \rightarrow \quad T_c \sim N^{-\frac{11}{14}} \quad . \]
By deboosting to the black hole rest frame we get the Hawking temperature to be

$$T_H = \frac{N}{M} T_c \sim M^{-\frac{1}{6}}$$

where we have used $N \sim S^{\frac{2}{3}} \sim M^{\frac{7}{6}}$, valid in the regime under consideration. The Schwarzschild radius in this regime is given by

$$S \sim M R_s \implies R_s \sim M^{\frac{1}{6}}$$

and, hence, we have $T_H \sim \frac{1}{R_s}$, the correct scaling relation for Schwarzschild black hole.

Thus, we have seen in the above cases that the $D$ dimensional black hole in the regime $N^c \sim S$, $0 < c < 1$ can plausibly be thought of as a $\tilde{D} = D + 1$ dimensional black hole described by an auxiliary matrix theory, but in its BFKS regime $\tilde{N} \sim S$.

Clearly, the auxiliary matrix theory is to be compactified on a $\tilde{\rho} = p - 1$ dimensional torus. The parameters of the auxiliary matrix theory - the quanta, $\tilde{N}$, of the longitudinal momentum, the radius, $\tilde{R}$, of the light-like circle, the volume, $\tilde{V}$, of the dual $\tilde{p}$ dimensional torus, and the effective coupling constant, $\tilde{e}^2$, of the $\tilde{p} + 1$ dimensional SYM theory - can all be determined explicitly by noting that the physical parameters must be the same. Namely, the entropy $S$, the mass $M$, and the longitudinal momentum $P_-$ of the black hole must be the same in both the original and the auxiliary matrix theories. We thus get

$$\tilde{N} = N^c$$
$$\tilde{R} = N^{c-1} R$$
$$\tilde{V} = N^{\frac{2(p-2)}{8-p}} V^{\frac{7-p}{9-p}} e^{\frac{5}{9-p}}$$
$$\tilde{e}^2 = N^{\frac{2(p-6)}{8-p}} V^{\frac{2}{9-p}} e^{\frac{2(12-p)}{9-p}}.$$  \hspace{1cm} (18)

However, we are unable to make more precise the physical sense in which a $D$ dimensional black hole, in the regime $N^c \sim S$, $0 < c < 1$, is equivalent to $\tilde{D} = D + 1$ dimensional black hole of an auxiliary theory, but in its BFKS regime; equivalently, the physical sense in which the $p + 1$ dimensional SYM theory in the regime $N^c \sim S$, $0 < c < 1$, is equivalent to an auxiliary $\tilde{p} + 1$ ($= (p - 1) + 1$) dimensional SYM theory, but in its BFKS regime. The main problem is that we lack a clear understanding of the e.o.s from the SYM
theory side in the regime $N^c \sim S$, $0 < c < 1$. The relations in (18) may, perhaps, be useful in this regard.

4. The BFKS results as well as those in section 3 can all be derived systematically by imposing two requirements: (I) we work within the matrix theory framework, and (II) the $D$ dimensional Schwarzschild black holes admit a description within the matrix theory compactified on a $p$ ($= 11 - D$) dimensional torus (taken, for simplicity, to be “square” with all circumferences equal to $L$). These two requirements lead to the most general expression for the matrix theory e.o.s. One may then study the description of black holes in the BFKS regime $N \sim S$ or in the generalised regime $N^c \sim S$, $0 < c \leq 1$. That is, in the regime $N \gtrsim S \gg 1$, where the matrix theory has sufficient degrees of freedom to describe the black hole with entropy $S$ [3].

Through the matrix theory conjecture, the requirement II is equivalent to the requirement that the $D$ dimensional Schwarzschild black holes admit a description within the $p + 1$ dimensional $U(N)$ SYM theory with 16 real supercharges. Consequently, the resulting most general e.o.s can be viewed as the most general e.o.s for the SYM theory, following from the requirement that the $D$ dimensional Schwarzschild black holes admit a description within this theory.

The matrix theory Hamiltonian is given by (2), reproduced below:

$$E = \frac{M^2 R}{N}.$$ 

The matrix theory conjecture that the sector of the theory with given $N$ is described exactly by the $p + 1$ dimensional $U(N)$ SYM theory with 16 real supercharges implies that the entropy $S$ must be expressible in terms of the SYM parameters only. Namely, in terms of $N$, $V$ the volume of the dual torus given in (3), and $g^2$ (or equivalently $e^2$) given in (4) (or (5)), only. Thus, most generally the entropy $S$ can be written as

$$S = \left( N^\alpha V^a e^{2b E \gamma} \right)^{1/(1+\gamma)}$$

for some constants $\alpha, a, b,$ and $\gamma$. Within matrix theory, the $N$ dependence can not be determined. But the value $\alpha = 2$, which implies that the dynamics is governed by $N^2$ degrees of freedom, is natural in the context of $U(N)$ SYM theory. The parameter $\gamma$ is not necessarily an integer, nor even positive. Since $S$ is dimensionless, we have

$$ap + b(p - 3) = \gamma$$

(20)
since $V$ and $e^2$ have length dimensions $p$ and $(p - 3)$ respectively. Equations (19) and (20) are thus the consequences of the requirement (I).

Now, the requirement (II) that the $D$ dimensional Schwarzschild black hole admit a description in matrix theory implies that the entropy $S$ must be expressible in terms of $N$, the $D$ dimensional Planck length $l_D$ (6), and the mass $M$ (or equivalently $E_R$) only. Thus, most generally the entropy $S$ can be written, noting that it is dimensionless, as

$$S = N A_1 \left( \frac{l_D^2 E_R}{R} \right)^{A_2}$$

for some constants $A_1$ and $A_2$.

It follows, upon using equations (3)-(6), that the above expression is of the form (19) iff $a$ and $b$ satisfy the relation

$$\frac{b}{a} = \frac{p - 3}{5 - p}.$$  

Solving equations (24) and (22) for $a$ and $b$ gives

$$\frac{a}{\gamma} = \frac{5 - p}{9 - p}, \quad \frac{b}{\gamma} = \frac{p - 3}{9 - p}.$$  

This implies that the scale which dictates the dynamics of matrix theory is a length scale $\sigma$, or equivalently $\sigma_0$, which is a specific combination of the dimensionful SYM parameters $V$ and $e^2$, or equivalently $g^2$, and is given by

$$\sigma = V^{\frac{5 - p}{9 - p}} e^{\frac{2(p-3)}{9-p}},$$

or equivalently by

$$\sigma_0 = V^{\frac{5 - p}{9 - p}} g^{\frac{2(p-3)}{9-p}}.$$  

The relevance of this scale to black hole physics can also be glimpsed by noticing that it is related to the $D$ dimensional Planck length $l_D$ by

$$\sigma_0 = \frac{l_D^2}{R}.$$  

Indeed, precisely this length scale enters in both the e.o.s (9) and (13) in the BFKS analysis in the regimes $N \sim S$ and $N \gg S$ respectively.
Equating the two expressions (19) and (21) now gives

\[ A_1 = \frac{\alpha + b}{1 + \gamma}, \quad A_2 = \frac{\gamma}{1 + \gamma}. \] (27)

Thus, the matrix theory e.o.s (19) can be expressed in any of the following forms:

\[
S = (N^\alpha (\sigma E)^{\gamma})^{\frac{1}{1+\gamma}} \\
= (N^{\alpha+b} (\sigma_0 E)^{\gamma})^{\frac{1}{1+\gamma}} \\
= N^{\frac{\alpha+b}{1+\gamma}} \left( \frac{l_D E}{R} \right)^{\frac{\gamma}{1+\gamma}} \\
= N^{\frac{\alpha+b-\gamma}{1+\gamma}} (l_D M)^{\frac{2p}{1+\gamma}} \] (28) (29) (30) (31)

where \(\alpha\) and \(\gamma\) are constants, \(b\) is given in (23) and, in obtaining (31), we have used equation (2) which relates the matrix theory Hamiltonian and the DLCQ energy. This is the most general form of the matrix theory e.o.s subject only to the matrix theory conjecture and to the requirement that Schwarzschild black holes admit a description within this theory. This can also be viewed, through the matrix theory conjecture, as the most general e.o.s for the \(p+1\) dimensional U(N) SYM theory with 16 real supercharges, subject only to the requirement that Schwarzschild black holes admit a description within this theory.

We now adapt the BFKS strategy and require that the matrix theory e.o.s reproduce the black hole entropy in the regime \(N^c \sim S, 0 < c \leq 1\). The entropy of the \(D\) (= \(11-p\)) dimensional Schwarzschild black hole is given, up to numerical factors, by

\[ S \sim (l_D M)^{\frac{9-p}{5-p}}. \] (32)

Hence, it follows after a simple algebra that the matrix theory e.o.s reproduces black hole entropy in the regime \(N^c \sim S\) iff \(c\) and \(\gamma\) satisfy the relation

\[ c = \frac{2\gamma(6-p) - \alpha(9-p)}{\gamma(7-p) - (9-p)}. \] (33)

For \(c = 1\), equation (33) indeed yields the BFKS result, namely \(\gamma = \frac{9-p}{5-p}\).
We first describe two general results, which we have seen in various cases previously. If
\[ \gamma = \frac{9 - p}{7 - p} \equiv \gamma_* \tag{34} \]
and, furthermore, if
\[ \alpha = \gamma - b = \frac{2(6 - p)}{7 - p} \tag{35} \]
where we have used (23) and (34), then the matrix theory e.o.s (31) is independent of \( N \) and always reproduces the black hole e.o.s. Expressed in terms of \( E \), for example in the form given in (29), the matrix theory e.o.s becomes
\[ S = (N\sigma_0E)^{\frac{\gamma}{\gamma - b}}. \tag{36} \]
As can be checked easily, this equation is same as equation (13) derived a la BFKS [2, 3].

We have seen in various cases previously that the e.o.s (36), equivalently (13), agrees with the e.o.s obtained in the regimes \( N_c \sim S \), \( c \leq 1 \). From the above derivation, it is now clear that this result is valid in general. That is, valid for any \( c \) and any \( D \). This can be checked explicitly in other ways also.

Another result, which we have seen in various cases previously, is also valid in general i.e. valid for any \( c \) and any \( D \): That deboosting the critical temperature \( T_c \) in the regime \( N_c \sim S \) to the black hole rest frame gives the Hawking temperature \( T_H \) with the correct scaling. This can be seen as follows. The e.o.s can be written generally as
\[ S = (\mathcal{A}E^\gamma)^{\frac{1}{1+\gamma}} \tag{37} \]
where \( \mathcal{A} \) is a constant independent of \( E \). Together with the thermodynamic relation \( dE = TdS \), this gives
\[ S = \mathcal{A}T^\gamma. \tag{38} \]
In the regime \( N_c \sim S \equiv S_c \), the critical temperature \( T_c \) is given by
\[ T_c \sim \left( \frac{S_c}{\mathcal{A}} \right)^\frac{1}{\gamma} \sim \frac{E}{S_c} \]
where the second step follows from (37). By deboosting to the black hole rest frame we get the Hawking temperature to be
\[ T_H = \frac{N}{R M} T_c \sim \frac{N}{R M} \frac{E}{S_c}. \]
Upon using (2) to write \( E \) in terms of \( M \), and noting that \( S_c \sim M R_s \) we get the Hawking temperature to be
\[ T_H \sim \frac{M}{S_c} \sim \frac{1}{R_s} \]
which is the correct scaling relation for Schwarzschild black hole. Thus, this result is also true for any \( c \) and any \( D \).

Now we return to equation (33), which ensures that the matrix theory e.o.s reproduces the black hole entropy in the regime \( N_c \sim S, 0 < c \leq 1 \). We study the regimes where the e.o.s is that of a \( \gamma + 1 \) dimensional gas, with \( \gamma \) an integer. We also take \( \alpha = 2 \), the value which is natural within the context of U(N) SYM theory.

(i) In the BFKS regime, \( c = 1 \) and (33) gives \( \gamma = \frac{9 - p}{6 - p} = \frac{D - 2}{D - 6} \), the result obtained in [2, 3].

(ii) In the regime \( N_c \sim S, 0 < c < 1 \), let \( \gamma = \frac{10 - p}{6 - p} \), as appropriate for the \( \tilde{D} = D + 1 \) dimensional black hole in the BFKS regime of the auxiliary theory. Equation (33) then gives \( c = \frac{6 - p}{8 - p} \). For \( D = 6, 7, 9 \), equivalently \( p = 5, 4, 2 \), we get \( c = \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \) respectively, which is the result obtained in section 3.

(iii) One can now search for other regimes, i.e. for values of \( c \), if any, where the e.o.s is that of a \( \gamma + 1 \) dimensional gas, with \( \gamma \) an integer.

(a) For \( p \leq 3 \), there are no new regimes other than the ones given in [2, 3] and in section 3.

(b) For \( p = 4 \), besides the regimes given in [2, 3] and in section 3, there is another regime with \( c = \frac{2}{7} \) and \( \gamma = 4 \). That is, the e.o.s of the 7 dimensional black hole in the regime \( N \sim S_7^2 \) is that of a \( 4 + 1 \) dimensional gas.

(c) For \( p = 5 \), besides the regimes given in [2, 3] and in section 3, there is a series of regimes for every integer \( \gamma \geq 6 \) with \( c = \frac{\gamma - 4}{\gamma - 2} \leq 1 \). That is, the
e.o.s of the 6 dimensional black hole in the regime $N \sim S^{\frac{1}{c}}$ is that of a $\gamma + 1$ dimensional gas, for a series of values of $c$ and $\gamma$ given as above.

In the BFKS regime, $c = 1$, $\gamma = \infty$ formally, indicating infinite specific heat. The e.o.s (28) becomes

$$S \sim eE$$

which is the e.o.s of a dynamical string with tension $\sim \frac{1}{N}$.  

(d) We will consider $p = 7$, before proceeding to $p = 6$. There are no new regimes except the $N \gg S$ regime considered in [4]. In this regime, $\gamma = \gamma_* = \infty$ formally, indicating infinite specific heat, and $\alpha + b = \gamma$. See equations (34) and (35). The e.o.s (29) becomes

$$S \sim N\sigma_0 E$$

which is the e.o.s of a dynamical string with tension $\sim \frac{1}{N^2\sigma_0^2}$. The string length scale, $\lambda_1$, of this dynamical string is given by

$$\lambda_1 \sim N\sigma_0.$$  

Note that the string length scale $\lambda_1$ scales linearly with $N$.

(e) For $p = 6$ in the BFKS regime, $c = 1$ and equation (33) gives $\gamma = -3$ formally, indicating negative specific heat. The e.o.s (29) becomes

$$S = \left(N^{\frac{3}{2}}\sigma_0 E\right)^{\frac{3}{2}}.$$  

The e.o.s of a dynamical $\delta$-brane ($\delta$ is a positive integer) is of the form [16]

$$S \sim E^{\frac{2}{1+\delta}} \equiv (\lambda_\delta E)^{\frac{2}{1+\delta}}$$

where $\lambda_\delta$ is a constant, of length dimension 1. Hence, we see that e.o.s (11) has the form characteristic of that of dynamical 3-branes [3]. That the dynamics here involves some 3 (spatial) dimensional object is not surprising in the light of the indications that the matrix theory compactified on 6 dimensional torus, i.e. $p = 6$, seems to involve gravitational degrees of freedom of 3 + 1 dimensional nature [17].

For dynamical $\delta$-branes, $\delta \neq 1$, there does not appear to be a canonical way of reading out the brane tension, $T_\delta$, from the e.o.s (12) alone. However,
a natural choice would be to identify \( \lambda_\delta \), the coefficient of \( E \) in (42), as the dynamical brane length scale, up to numerical factors. The tension \( T_\delta \) is then given by

\[
T_\delta = (\lambda_\delta)^{-/(1+\delta)} .
\]  

(43)

For \( \delta \)-branes, we can also define a brane volume scale:

\[
v_\delta = (\lambda_\delta)^\delta ,
\]

(44)

which can be thought of as a brane analog of the string length scale.

Thus, with the natural choice of the brane parameters given above, the coefficient of \( E \) in (11) can be interpreted as \( \lambda_3 \), the dynamical 3-brane length scale. The tension and the brane volume scale of the dynamical 3-brane are then given by

\[
T_3 \sim \frac{1}{N^{\frac{1}{3}}\sigma_0^4} , \quad v_3 \sim N\sigma_0^3 .
\]

(45)

Note that the brane volume scale \( v_3 \) scales linearly with \( N \).

There is also another regime \( N^c \sim S, \ c < 1 \), where the e.o.s has a recognisable form. This regime is given by \( c = \frac{6}{7} \), i.e. the regime where \( N \sim S^{\frac{2}{7}} \). In this regime \( \gamma = -4 \) formally, indicating negative specific heat. The e.o.s (29) becomes

\[
S = \left(N^{\frac{4}{7}}\sigma_0E\right)^{\frac{4}{7}}
\]

(46)

which has the form characteristic of that of dynamical 2-branes. Note that the dynamical region (2-brane) now has one less dimension than that (3-brane) in the BFKS regime. This is reminiscent of the phenomenon we have seen in section 3 where \( D \) dimensional black hole in the regime \( N^c \sim S, \ 0 < c < 1 \) can plausibly be thought of as described by a \( \tilde{D} = D + 1 \) dimensional black hole in the BFKS regime of the auxiliary theory. The auxiliary theory is to be compactified on a \( \tilde{p} = p - 1 \) dimensional torus. The corresponding auxiliary SYM theory lives on a \( \tilde{p} + 1 \) dimensional space time, which indeed has one less dimension than that (\( p + 1 \) dimensional SYM theory) in the BFKS regime.

With the natural choice of the brane parameters given above, the coefficient of \( E \) in (11) can be interpreted as \( \lambda_2 \), the dynamical 2-brane length scale. The tension and the brane volume scale of the dynamical 2-brane are then given by

\[
T_2 \sim \frac{1}{N^{\frac{1}{2}}\sigma_0^3} , \quad v_2 \sim N\sigma_0^2 .
\]

(47)
Note that the brane volume scale $v_2$ scales linearly with $N$.

In all the above cases where a dynamical $\delta$-brane ($\delta = 1, 2, 3$, see equations (39), (46), and (41)) appears, the brane tension $T_\delta$ is normalised w.r.t. $N$ according to

$$T_\delta \sim N^{-\frac{1+\delta}{2}}.$$  

(48)

In this case of dynamical strings, $\delta = 1$ and the normalisation becomes $T_1 \sim \frac{1}{N^2}$. For this case, such normalisation has appeared in other contexts also. For example in [13] - where the dynamical open strings attached to the $(N,1)$ strings have an effective tension $\sim \frac{1}{N^2}$. See [14] for another example. Equation (48) appears to be a matrix theory generalisation of this normalisation to higher dynamical branes. Also, very interestingly, the brane-volume scale $v_\delta = (\lambda_\delta)^\delta$, defined in analogy with the string length scale, has a simple scaling in all the above cases:

$$v_\delta = (\lambda_\delta)^\delta \sim N\sigma_\delta^\delta.$$  

(49)

Namely, the brane volume scale $v_\delta$ scales linearly with $N$.

This is an intriguing result. It has a striking similarity to the phenomenon seen in the case of Dirichlet branes, where multiply wound single Dirichlet brane (with winding number $N$) dominate over singly wound multiple branes ($N$ in number). The effective volume of the multiply wound brane is $N$ times the volume of the single brane. That is, the effective volume scales linearly with $N$ [15]. However, here, it is the brane volume scale $v_\delta$ - defined in (44) in analogy with the string length scale - which scales linearly with $N$. Clearly, $v_\delta$ is not the same as the volume of any brane, at least not in any obvious sense. We have no simple explanation for this intriguing result.

5. The most general matrix theory e.o.s, given in equivalent forms in (28)-(31), is obtained by requiring that the $D$ dimensional Schwarzschild black holes admit a description within matrix theory compactified on a $p$ ($= 11 - D$) torus. As such, it can be viewed, perhaps more properly, as the most general e.o.s for the $p+1$ dimensional U($N$) SYM theory with 16 real supercharges, subject only to the requirement that the Schwarzschild black holes admit a description within this theory. Equivalently, it can also be thought of as the e.o.s for the dynamics of $N$ Dirichlet $p$-branes, related to $N$ Dirichlet 0-branes by T-dualities.

From this point of view, the general e.o.s is that given in (13). Requiring that black holes admit a description in this theory implies that (13) is
expressible as in (21) with various constants related by (23) and (27). The resulting e.o.s can be written in any of the equivalent forms (28)-(30). Note, however, that $R$ appears in the combination $\frac{E}{R}$ only.

Now, one further generalisation is possible. Equation (2) relating $E$ to $M$ and $N$ can be generalised to

$$E = \frac{M^2R}{NB}$$

(50)

where $B$ is a constant. That this is the only possible generalisation follows from dimensional analysis and the fact that $R$ appears in the combination $\frac{E}{R}$ only. However, we do not know the physical significance, if any, of the parameter $B$ when $\neq 1$. It may be that there is a Seiberg-Sen type scaling [17] at work here, but which involves $N$ also.

The e.o.s now becomes

$$S = N^{\frac{\gamma + p - \gamma B}{\gamma + p - 1}} (t DM)^{\frac{2\gamma}{7 - p}}.$$  

(51)

Requiring that it reproduce the black hole entropy in the regime $N^c \sim S$, $0 < c \leq 1$ then implies the relation:

$$c = \frac{\gamma(3(1 + 3B) - p(1 + B)) - \alpha(9 - p)}{\gamma(7 - p) - (9 - p)}.$$  

(52)

When $B = 1$ these two equations give (31) and (33) respectively.

As before, if $\gamma = \gamma_*$ given in (34) and, furthermore, if

$$\alpha = \gamma B - b = \frac{3(1 + 3B) - p(1 + B)}{7 - p}.$$  

(53)

where we have used (23) and (34), then the matrix theory e.o.s (51) is independent of $N$ and always reproduces the black hole e.o.s. The e.o.s is applicable in the regime $N \gg S$, for reasons similar to those given in [2, 3].

We now look for special value(s) of $B$, if any, other than 1. Note that $\alpha$, given in equation (53), depends on $p$ in general. Equivalently, equation (53) can be thought of as straight lines, drawn in the $(B, \alpha)$ plane, one for each value of $p$ with slope $\frac{\alpha}{7 - p}$. It turns out that these lines all intersect at a common point in the $(B, \alpha)$ plane. That more than two lines intersect at
one point is evidently non-trivial, and the common intersection point, given by
\[(B, \alpha) = (2, 3),\] (54)
may perhaps signify a special value for \(B\) and \(\alpha\). For these values of \((B, \alpha)\), the e.o.s (28) becomes
\[S = \left(N^3(\sigma E)^\gamma\right)^{1/\gamma},\] (55)
which implies that the dynamics is governed by \(N^3\) degrees of freedom. Such \(N^3\) degrees of freedom also appear in the entropy of \(N\) coincident 5-branes [13] where they are expected to be related to the appearance of tensionless strings [19]. However, it is far from clear whether this is related to the \(N^3\) degrees of freedom appearing in the above e.o.s, applicable in the \(N \gg S\) regime, when \((B, \alpha)\) take the special value \((2, 3)\).

6. To summarise: In this paper, we have studied the description of Schwarzschild black holes within matrix theory in the regime \(N \gg S\), more specifically in the regime \(N^c \sim S, 0 < c \leq 1\). We mainly studied the regimes where the e.o.s is of a recognisable form. We found that in some cases a \(D\) dimensional black hole can plausibly be thought of as a \(D + 1\) dimensional black hole, described by another auxiliary matrix theory, but in its BFKS regime.

We then derived systematically these and the BFKS results by imposing two requirements: (I) we work within the matrix theory framework, and (II) the \(D\) dimensional Schwarzschild black holes admit a description within the matrix theory. Consequently, we obtained the most general expression for the matrix theory e.o.s and studied various new regimes where the matrix theory e.o.s is of a recognisable form. In the course of this study, we found what appears to be a matrix theory generalisation to higher dynamical branes of the normalisation of dynamical string tension, seen in other contexts.

We also discussed a further possible generalisation of the matrix theory e.o.s mentioned above and found, in a special case, that the e.o.s is governed by \(N^3\) dynamical degrees of freedom.

Although we believe that the present results are interesting, there is a crucial shortcoming. We are unable to explain these results from the SYM theory side. The main problem is that we lack a clear understanding of the e.o.s from the SYM theory side in the regime \(N^c \sim S, 0 < c < 1\). Therefore, we are unable to present a simple physical picture, if exists, that underlies
these results. It will indeed be highly satisfactory to derive the present results from the SYM theory side or, conversely, understand their implication to the physics of SYM theory. We are presently studying these issues.

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