Fluctuation-induced first order transition due to Griffiths anomalies of the Cluster glass phase

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ABSTRACT

In itinerant magnetic systems with disorder, the quantum Griffiths phase at $T = 0$ is unstable to formation of a cluster glass (CG) of frozen droplet degrees of freedom. In the absence of the fluctuations associated with these degrees of freedom, the transition from the paramagnetic Fermi liquid (PMFL) to the ordered phase proceeds via a conventional second-order quantum phase transition. However, when the Griffiths anomalies due to the broad distribution of local energy scales are included, the transition is driven first-order via a novel mechanism for a fluctuation induced first-order transition. At higher temperatures, thermal effects restore the transition to second-order. Implications of the enhanced non-Ohmic dissipation in the CG phase are briefly discussed.

Keywords: non-Fermi liquid behavior; rare events; Griffiths phases

1. INTRODUCTION

When quenched disorder is introduced into a magnetic system in its ordered phase, exponentially rare defect-free regions or droplets can form which are nearly magnetically ordered, even if the random magnet itself is not. The system is correlated over the length scale of the individual droplets and long time scale (compared to that of a single spin) fluctuations are induced due to coherent flipping of these large volumes of spins. Such slow fluctuations lead to singularities of the free energy in what is known as the Griffiths phase.\textsuperscript{1} Classically, these Griffiths singularities are weak, essential singularities\textsuperscript{2} and can only make exponentially small corrections to thermodynamic quantities. However, near quantum phase transitions, disorder is perfectly correlated in imaginary time making its effect much stronger than in the corresponding classical problem. Recently, much progress has been made in understanding the effects of disorder on quantum critical points by focusing on the dynamics in the quantum Griffiths phase (QGP).\textsuperscript{3–9}

One compelling reason for much the focus in this area is in understanding the role QGP anomalies play in the non-Fermi liquid behavior of disordered strongly correlated electron systems.\textsuperscript{10} For example, in the random transverse field Ising model in dimensions 1,\textsuperscript{3} 2 and 3,\textsuperscript{5} it has been shown that a quantum Griffiths phase characterized by a broad distribution of local energy scales precedes a $T = 0$ transition in the universality class of the infinite-randomness fixed point (IRFP). The distribution of energies follows a power-law form with a continuously varying exponent $\alpha = d/z'$ where $d$ is the dimensionality and $z'$ is a dynamical critical exponent which diverges at the IRFP. This power-law distribution function leads to strong corrections to Fermi liquid scaling in the QGP. In particular, thermodynamic quantities become divergent at $T = 0$ showing that quantum Griffiths anomalies can have significant impact on the behavior of random systems near their critical points.

Significant progress has been made recently by considering the rare regions in the Griffiths phase as independent droplet degrees of freedom, and an elegant classification scheme was proposed based only on general symmetry arguments (for a review see Ref. 11). It was shown that the effects of quantum Griffiths anomalies range from weak, classical corrections, to complete destruction (rounding) of the phase transition. However, these arguments apply only to insulating magnets for which the droplets can truly be considered as independent. By considering interactions induced between droplets in a metallic host, it was recently shown\textsuperscript{9} that the QGP is

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unstable to formation of a cluster glass phase (CGP), and the conventional quantum critical point is destroyed.
In the present work, we will examine the same model as in Ref. 9 and shown that fluctuations due to Griffiths
phase anomalies have the further effect to drive the transition first-order. At higher temperatures, thermal effects
act to suppress these fluctuations and a second-order transition is restored.

The paper is organized as follows: Sections 2 and 3 introduce the model and formulate the saddle-point
approximation to it; its solution in the limit of uniform droplet sizes is presented in Section 4; Sections 5, 6 and
7 are devoted to solution of the model incorporating the broad distribution of droplet sizes; and Section 8 is for
summary and discussion.

2. THE MODEL OF INTERACTING DROPLETS

By focusing on locally ordered droplets as the relevant degrees of freedom,7,8 a coarse-grained action in terms
of a fluctuating, local order parameter can be devised9 to study the effect of disorder on phase transitions in
random magnets. This coarse-graining procedure is shown schematically in Figure 1. In this framework, a single,
isolated droplet acts as a classical moment with dynamics in imaginary time and can be mapped onto a classical,
one-dimensional spin chain with a local coupling constant determined by the size of the droplet. The local action
for an isolated droplet is then given by

$$ S_{L_i} = \int_0^\beta d\tau d\tau' \phi_i(\tau)\Gamma_i(\tau-\tau')\phi_i(\tau') + \frac{u}{2N} \int_0^\beta d\tau \phi_i^4(\tau) $$

where $\phi_i$ is an N-component order parameter field for the ith droplet and $\Gamma_i(\tau)$ is a generic coarse-grained
two-point vertex. For definiteness, we will focus on the itinerant Heisenberg antiferromagnet12 ($N > 1$) so that

$$ \Gamma_i(\omega_n) = (r_i + |\omega_n|); $$

the $\omega_n$ are bosonic Matsubara frequencies $\omega_n = 2n\pi T$ and the local coupling constant is given by $r_i \approx \delta(L_i)^d$
where $L_i$ is the linear dimension of the droplet, $d$ is the spatial dimension, and $\delta < 0$ is the coupling constant
of the clean magnet. Droplet sizes follow a Poisson distribution $P(L_i) \sim \exp[-\rho(L_i)^d]$ so that, correspondingly,
$P(r_i) \sim \exp[-\rho r_i/\delta]$, where $\rho$ is the volume fraction of droplets.
In the absence of interactions between droplets, the action described by (1) and (2) is well understood.\(^8\) Ohmic dissipation given by the linear \(\omega_n\) term corresponds to long-ranged interactions in imaginary time so that the corresponding Heisenberg spin chain is exactly at its lower critical dimension\(^13\) and the resulting energy gap is found to be\(^7,8\) \(\epsilon_1 \sim \exp[\pi r_i / u].\) This leads to a power-law distribution of local energies scales \(P(\epsilon_i) \sim \epsilon_i^{\alpha-1}\) where the Griffiths exponent \(\alpha = d / z'\) is a non-universal function of parameters and is expected to decrease as the magnetically ordered phase is approached. Standard quantum Griffiths phase (QGP) phenomenology follows from this distribution; for example, the average susceptibility \(\chi \sim T^\alpha\) diverges as \(T \to 0\) for \(\alpha < 1\).

The above scenario applies in the absence of interactions between droplets and, in particular, is valid for insulating magnets. However, in itinerant systems, electrons in the bulk mediate long-ranged interactions which must be included.\(^9\) Formally, the coarse-graining procedure requires that the order parameter fluctuations in the metallic bulk be integrated out, leading to an effective RKKY interaction between droplets which has the asymptotic form

\[
S_{1,ij} = \frac{J_{ij}}{(R_{ij})^d} \int_0^\beta d\tau \phi_i(\tau) \phi_j(\tau)
\]

where \(R_{ij}\) is the distance between droplets \(i\) and \(j\), and \(J_{ij}\) can be taken to be a random amplitude with zero mean and variance \(\langle J_{ij}^2 \rangle = J^2\). The effective description of a random itinerant magnet near a phase transition is then given by the full action

\[
S = \sum_i S_{L,i} + \sum_{i,j} S_{1,ij}.
\]

Recently,\(^9\) action (4) was studied for the Heisenberg antiferromagnet and it was found that infinitesimal interactions can destabilize the QGP found in the non-interacting theory, leading to the formation of the cluster glass phase (CGP). These results showed that the interactions can have a non-trivial effect on the droplet dynamics, even at the saddle-point level which is formally justified in the large-\(N\) limit. In the following Section, we review the saddle-point theory for the action (4).

### 3. SADDLE POINT THEORY

By including RKKY interactions between separated droplets, individual droplets no longer have only their own dynamics given by the kernel (2), but also acquire dynamics through their interactions with an effective bath given by the collective fluctuations of all other droplet degrees of freedom. To obtain a description for this process, we average over the random variables \(J_{ij}\) using the standard method of introducing \(n\) copies or replicas of the system\(^14\) described by (4) and applying the identity

\[
\langle Z \rangle_{\text{dis}} = \lim_{n \to 0} \{Z^n\}_{\text{dis}} - \frac{1}{n}
\]

where the partition function \(Z = \text{Tr}\{\exp[-S]\}\) and \(\langle \cdots \rangle_{\text{dis}}\) represents averaging over random \(J_{ij}\). The interaction term of (4) then becomes

\[
S_{1,ij} = -\frac{1}{2} \frac{J^2}{(R_{ij})^d} \sum_{a,b=1}^n \int_0^\beta d\tau d\tau' \phi_i^a(\tau) \phi_i^b(\tau') \phi_j^a(\tau) \phi_j^b(\tau')
\]

which can be decoupled by introducing a Hubbard-Stratonovich field\(^15\) \(Q_{ij}^{ab}(\tau - \tau')\). The replicated partition function can then be written as

\[
\langle Z^n \rangle = \int D\phi DQ \exp\{-S_{\text{eff}}[\phi, Q]\}
\]

with effective action

\[
S_{\text{eff}}[\phi, Q] = \sum_{a=1}^n \sum_{i,\omega_n} S_{L,i}[\phi_i^a] + \frac{1}{2} \sum_{a,b=1}^n \sum_{i,\omega_n} [\sum_j \frac{J^2}{(R_{ij})^{2d}} Q_j^{ab}(\omega_n)] \phi_i^a(\omega_n)
\]

\[
+ \frac{1}{2} \sum_{a,b=1}^n \sum_{i,j,\omega_n} Q_{ij}^{ab}(\omega_n) \left( \frac{J^2}{(R_{ij})^{2d}} \right) Q_j^{ab}(\omega_n).
\]
Defining the “cavity field”

$$\Delta_i^{ab}(\omega_n) = \sum_j \frac{J^2}{(R_{ij})^{2d}} Q_j^{ab}(\omega_n),$$

we see that the problem of interacting droplets has been reduced to a new single-site problem, given by

$$S_{L,\text{eff}}[\phi, Q] = \sum_{a=1}^n \sum_{i,\omega_n} S_{L,i}[\phi_i^a] + \frac{1}{2} \sum_{a,b=1}^n \sum_{i,\omega_n} \phi_i^a(\omega_n) \Delta_i^{ab}(\omega_n) \phi_i^b(\omega_n),$$

so that the effect of the other degrees of freedom is simply to introduce additional dissipation through the frequency dependence of $\Delta(\omega_n)$. At the saddle-point level, the cavity field is determined self-consistently from the relation

$$\Delta_i^{ab}(\omega_n) = \sum_j \frac{J^2}{(R_{ij})^{2d}} \langle \phi_j^a(\omega_n) \phi_j^b(\omega_n) \rangle_L$$

where the expectation value is taken with respect to $S_{L,\text{eff}}$. In the paramagnetic phase, $\langle \phi_i^a(\omega_n) \phi_i^b(\omega_n) \rangle = \delta_{ab}\chi_L(r,\omega_n)$ is diagonal in replica indices. Noting also that the $r_i$ are independent of droplet position, (11) is equivalent to

$$\Delta_i^{ab}(\omega_n) = \tilde{g} \delta_{ab} \chi_L(\omega_n) = \tilde{g} \delta_{ab} \int dr_i P(r_i) \chi_L(r_i,\omega_n)$$

where $\tilde{g} \equiv J^2 \sum_{ij} (R_{ij})^{-2d}$.

To finalize the formulation of the saddle-point theory, the correlator $\chi_L$ can be calculated in the large-$N$ limit by decoupling the quartic term in (11) with an auxiliary Hubbard Stratonovich field $\lambda_i$. The equations governing the dynamics of the droplets in the interacting theory then become

$$\chi_L(\omega_n) = \frac{1}{2} \int dr_i \frac{P(r_i)}{r_i + \lambda_i + |\omega_n| - \tilde{g} \chi_L(\omega_n)}$$

$$\lambda_i = \frac{uT}{2} \sum_{\omega_n} \frac{1}{r_i + \lambda_i + |\omega_n| - \tilde{g} \chi_L(\omega_n)}$$

which must be evaluated self-consistently for $\lambda_i$ and $\chi_L(\omega_n)$.

### 4. QUANTUM CRITICALITY FOR UNIFORM DROPLETS

To understand the transition in the interacting droplet model, it is instructive to first study equations (13) and (14) in the limit of uniform droplet size, $P(r_i) = \delta(r_i - \bar{r})$. In this limit, the problem becomes equivalent to that of a quantum spin glass which is known to undergo a conventional second-order transition from a paramagnetic Fermi liquid (PMFL) to a spin glass at large enough $\tilde{g}$.

Self-consistency equation (13) reduces to an algebraic equation for $\chi(\omega_n)$ and is easily solved:

$$\chi_U(\omega_n) = \frac{1}{2\tilde{g}} \left( \bar{r} + \lambda + |\omega_n| \pm \sqrt{(\bar{r} + \lambda + |\omega_n|)^2 - 2\tilde{g}} \right)$$

where $\lambda \equiv \lambda(\bar{r})$. Clearly, this solution becomes unstable when the energy scale defined by $\Delta \equiv (\bar{r} + \lambda - \sqrt{2\tilde{g}})$ becomes zero. The phase boundary can then be determined by setting $\Delta = 0$ and solving (14) self-consistently for $\tilde{r}_c(T)$ (or, equivalently, $\lambda_c(T)$). In particular, at $T = 0$, this gives

$$\tilde{r}_c(T = 0) = \sqrt{2\tilde{g}} + \frac{u}{2\pi} \left( \frac{1}{2} + \ln \left( \frac{\sqrt{2\tilde{g}}}{2\Lambda} \right) \right)$$

where $\Lambda$ is an ultra-violet cutoff. In the limit $\tilde{g} \to 0$, $\tilde{r}_c(0) \to -\infty$ showing that frustration induced by the interactions is necessary to stabilize the glass phase.
The phase for $\Delta > 0$ ($\hat{r} > \hat{r}_c$) is characterized by the frequency dependence of $\chi_U$. From the solution (15), we find three separate regimes:

$$\chi_U(0) - \chi_U(\omega) \sim \begin{cases} \omega, & \omega < \Delta \ll \sqrt{2g} \\
\sqrt{\omega}, & \Delta < \omega \ll \sqrt{2g} \\
\chi_U(0) - \omega^{-1}, & \omega \gg \sqrt{2g}. \end{cases}$$ (17)

The linear, low-$\omega$ dependence of $\chi_U$ for $\Delta > 0$ is characteristic of a Fermi liquid, while, right at the transition, $\chi_U$ behaves as $\sqrt{\omega}$. This sub-linear form at $\Delta = 0$ leads to non-Fermi liquid behavior of all thermodynamic quantities, though these violations are fairly mild since the corrections remain finite at $T = 0$. Thus, there is no quantum Griffiths behavior in this model since we have neglected the fluctuations arising from a distribution of droplet sizes.

The transition at $T = 0$ can also be approached from the magnetically ordered side. In this case, the quantum critical point can be determined when the mean-field stability criterion for the glass phase vanishes,$^{17}$ i.e.:

$$\lambda_{SG} \equiv 1 - \sqrt{2g} \chi_U(0) = 0.$$ (18)

Using the solution (15) for $\chi_U$, we find $\lambda_{SG} \sim \sqrt{\Delta}$ which vanishes at $\Delta = 0$.

Thus, examining the transition through the instability of either the paramagnetic Fermi liquid or the spin-glass phase yields a quantum critical point at $\Delta = 0$, consistent with a conventional second-order transition in this problem. In the next Sections, we will consider the model with the full distribution of droplet sizes and show that the two instability criteria do not coincide, revealing the singular effect of droplet fluctuations.

5. DISTRIBUTED DROPLETS AND THE CLUSTER GLASS PHASE

For dilute impurities, the defect-free regions assume a Poisson distribution which can be written in terms of the local coupling constant as

$$P(r_i) = \frac{2\pi \kappa}{u} e^{2\pi \kappa (r_i - \hat{r})/u}, \quad r_i \leq \hat{r} < 0.$$ (19)

The offset $\hat{r}$ is used to tune through the transition and $\kappa = pu/2\pi \delta$. The uniform limit of the previous Section is recovered for $\kappa \to \infty$; however, by accounting for droplets of all sizes, we are also including their fluctuations which are associated with Griffiths phase behavior.

Using the uniform solution (15) as a zeroth order approximation, we can iterate equations (13) and (14) until self-consistency is achieved. Already at one iteration loop (1IL), the instability of the Griffiths phase at $T = 0$ is apparent. At this level of approximation, equation (14) can be integrated exactly. Identifying the local droplet energy $\epsilon_i = \hat{r} + r_i + \lambda_i - \hat{g}_U(0)$, we find the relation between energy scale and local coupling strength, $\epsilon_i \propto \exp[2\pi f^{-1}r_i/u]$ where

$$f \equiv \frac{2\sqrt{\Delta^2 + 2\Delta \sqrt{2g}}}{\Delta + \sqrt{2g} + \sqrt{\Delta^2 + 2\Delta \sqrt{2g}}}.$$ (20)

This allows us to switch from integration over $r_i$ in equation (15), to integration over $\epsilon_i$ via the replacement $P(r_i) dr_i \to \tilde{P}(\epsilon_i) d\epsilon_i$ where

$$\tilde{P}(\epsilon_i) = \left(\kappa f e^{-2\pi \kappa \hat{r}/u}\right)^{\epsilon_{f^{-1}}}.\quad (21)$$

Defining $\alpha \equiv \kappa f$, $^{21}$ takes the power-law form characteristic of droplets in the Griffiths phase with Griffiths exponent $\alpha$. The novelty arises when this distribution is used to calculate the frequency dependent cavity field,

$$\lambda_{GL}(\omega_n) = \frac{\alpha}{2} e^{-2\pi \kappa \hat{r}/u} \int_0^{\Lambda_c} d\epsilon_i \left(\frac{\epsilon_{i}^{\alpha - 1}}{\epsilon_{i} + |\omega| - \hat{g}(\chi_U(\omega) - \chi_U(0))}\right),$$ (22)

where $\Lambda_c$ is an upper cutoff in energy. This gives

$$\lambda_{GL}(\omega) - \chi_U(0) = -\gamma |\omega|^\alpha + \mathcal{O}(|\omega|).$$ (23)
Figure 2. Schematic phase diagram of random itinerant magnets. Interactions destabilize the putative quantum Griffiths phase at $\alpha = 1$, favoring the formation of the cluster glass phase for $\alpha < 2$. The phase boundary has an exponential tail due to the exponentially rare nature of the large frozen droplets.

where $\gamma$ is a constant. Thus, droplets for which $\alpha < 2$ acquire non-Ohmic dissipation and the corresponding Heisenberg spin chains find themselves above their lower critical dimension$^{13}$ and can order in the imaginary time direction. Due to the frustration caused by the effectively random RKKY interaction between these ordered droplets, they form a glassy state dubbed the "cluster glass phase" (CGP).$^9$ Importantly, the CGP is present over the entire range where one would predict the QGP from the isolated droplet model.$^8$

The finite temperature phase boundary can also be estimated in this formalism. The critical coupling constant is determined from (14) at $\epsilon_i = 0$:

$$
\lim_{\epsilon_i \to 0} r_{i,c} = -\frac{u}{\pi} \int_0^\Lambda \frac{d\omega}{\omega + \gamma g \omega^{\alpha - 1}} \approx -\frac{u}{\pi (2 - \alpha)} \ln(1/\gamma g)
$$

and the number of frozen droplets is $n_{fr} \sim \exp(-\rho r_{i,c}/\delta)$. The critical temperature is then estimated as$^7$

$$
T_c \sim n_{fr} \sim \exp \left[ -\frac{\rho u}{\pi \delta (2 - \alpha)} \log(1/\gamma g) \right]
$$

which has an exponential tail as $\alpha \to 2$. The phase diagram is shown schematically in Figure 2.

6. IRFP OF ISOLATED DROPLETS AT QUANTUM CRITICALITY

It is interesting to note that our 1IL expression for the Griffiths exponent hints at the close analogy between the present work and the physics of infinite-randomness fixed points (IRFP).$^3$ From the expression $\alpha = f \kappa$ with $f$ given in equation (20), we would conclude at this level of approximation that the Griffiths exponent vanishes at the uniform droplet quantum critical point as

$$
\alpha \sim \sqrt{\Delta}.
$$

This is significant in that it gives a simple access to the IRFP that has been lacking in the past, while capturing the physics of the quantum Griffiths phase. Of course, this expression for $\alpha$ is only valid at 1IL, and, in particular,
does not hold at the lowest temperatures where the system freezes into the CGP. However, QGP and IRFP behavior can still be expected at temperatures above the CGP transition temperature. We will address this issue further in the final Section.

7. FLUCTUATION INDUCED FIRST ORDER TRANSITION

In the paramagnetic phase with $\alpha > 2$, the leading behavior of $\chi_L(\omega)$ is linear in $\omega$. Writing $\chi_L(\omega) - \chi_L(0) = -m_\chi \omega$ and inserting this into the self-consistency condition (14) gives

$$\epsilon_i - r_i - \hat{r} + g\chi_L(0) = \frac{u}{2\pi} \int_0^\Lambda d\omega \frac{1}{\epsilon_i + (1 - m_\chi)|\omega|}$$

$$= \frac{u}{2\pi(1 - \tilde{g}m_\chi)} \ln \left( \frac{\epsilon_i + (1 - m_\chi)\Lambda}{\epsilon_i} \right).$$

Analyzing this for $r_i \ll 0$ gives

$$\epsilon_i \sim \exp \left( \frac{2\pi(1 - \tilde{g}m_\chi)}{u} \cdot r_i \right)$$

and, comparing to the 1IL results, defines

$$f \equiv (1 - \tilde{g}m_\chi)^{-1}; \quad (29)$$

$$\alpha = f \kappa; \quad (30)$$

which can be checked against the previous results. Note that expression (30) is exact in the paramagnetic phase provided that $m_\chi = \partial \chi_L(\omega)/\partial \omega |_{\omega=0}$ is determined self-consistently, so, following the arguments of Section 5, the instability of the PMFL to CGP formation still corresponds to $\alpha = 2$, to all orders.

This relation can be further illuminated by calculating the slope exactly from (13):

$$m_\chi = -\frac{1}{2} \int dr_i \frac{(1 - \tilde{g}m_\chi)P(r_i)}{|r_i + \lambda_i - g\chi_L(0)|^2}$$

$$= -2(1 - \tilde{g}m_\chi)\chi_L^2(0).$$

Solving this for $m_\chi$, we get

$$1 - \tilde{g}m_\chi = (1 - 2\tilde{g}\chi_L^2(0))^{-1}. \quad (32)$$

What is striking about this result is that the term in parentheses on the right hand side of (32) is the exact analogue of the stability criterion (18) for droplets with distributed site energies; in this case, $\lambda_{SG} = 1 - 2\tilde{g}\chi_L^2(0)$. Relating this to the definition (30) for $\alpha$, we can write

$$\alpha = \lambda_{SG} \kappa \quad (33)$$

so that $\lambda_{SG} > 0$ at the point where the PMFL becomes unstable ($\alpha = 2$). This should be contrasted with the uniform droplet limit of Section 4 where the instability of the PMFL and the instability of the spin-glass phase coincided at a conventional second-order transition. In the present case, the stability criteria signify spinodal lines and the CGP and PMFL coexist in a region around a first-order phase transition.

To gain more insight into the nature of the transition, we recast the problem as an eigenvalue analysis of the constitutive free energy for which the solution to (13) is a minimum. The relevant contribution to the free energy can be expressed near the minimum as

$$F = \int d\omega d\omega' (\chi_L(\omega) - \chi_0(\omega))\Gamma_X(\omega, \omega')(\chi_L(\omega') - \chi_0(\omega'))$$

so that the eigenvalue determining the stability of the PMFL solution $\chi_0$ satisfies

$$\lambda_X \leq \frac{\int d\omega d\omega' (\chi_L(\omega) - \chi_0(\omega))\Gamma_X(\omega, \omega')(\chi_L(\omega') - \chi_0(\omega'))}{\int d\omega (\chi_L(\omega) - \chi_0(\omega))^2}. \quad (35)$$
The self-consistency condition \( \frac{\delta F}{\delta \chi} = 0 \) so that

\[
\int d\omega' \Gamma_\chi(\omega, \omega') \chi_L(\omega') = \chi_L(\omega) - \int dr_i r_i + \lambda_i + |\omega| - \tilde{g}_\chi L(\omega).
\]  

and

\[
\int d\omega' \Gamma_\chi(\omega, \omega') \chi_0(\omega') = 0.
\]

Now, define an eigenvalue \( \lambda^{(n)} \) valid at each step in an iteration scheme

\[
\lambda^{(n)} \chi = \frac{\int d\omega d\omega' (\chi_L^{(n-1)}(\omega) - \chi_L^{(n)}(\omega)) \Gamma_\chi(\omega, \omega') (\chi_L^{(n-1)}(\omega') - \chi_L^{(n)}(\omega'))}{\int d\omega (\chi_L^{(n-1)}(\omega) - \chi_L^{(n)}(\omega))^2}
\]

and proceed numerically. From this, the spinodal line defined by \( \lambda_\chi = 0 \) can be determined at zero and finite temperatures, and compared with the condition \( \lambda_{SG} = 0 \).

The results of this procedure at \( T = 0 \) are shown in Figure (3). As the transition is approached from the PMFL, \( \lambda_\chi \) appears to approach a positive value as \( n \to \infty \). We identify the critical value of \( \Delta_c \) at the transition as the last point where the iterative method converges; the associated curve \( \lambda^{(n)} \chi, c \) vs. \( n^{-1} \) is indicated by the dashed line in the figure and extrapolates to zero at \( n^{-1} = 0 \). Below the transition, \( \lambda_\chi \) eventually becomes negative at some finite iteration step and the procedure breaks down. The inset to Figure (3) shows a scaling analysis for the same data as in the main panel. Using the \textit{ansatz} \( \lambda_\chi/\lambda_{\chi, c} = \Lambda_{\pm}(|\Delta - \Delta_c| n^x) \), we find excellent scaling with exponent \( x = 2.35 \). While the meaning of the exponent is unclear at this point, the scaling shown in Figure (3) demonstrates that the iterative method is well controlled.

At finite temperatures, we find that the two stability criteria continue to disagree over a portion of the phase boundary away from \( T = 0 \), as shown in Figure (4). The upper panel shows the boundary line defined by

![Figure 3](image-url)
Figure 4. Upper panel: The phase boundary at finite temperatures is shown by the symbols and corresponds to \( \lambda = 0 \). The line is a fit to the data as discussed in the text. Lower panel: The static stability criterion calculated at \( T_c \). The line is a guide to the eye. Empty squares correspond to first-order transitions, full circles to second-order transitions.

\( \lambda = 0 \). The solid line is a fit to the data using the form \( \log(T) \sim (\hat{r} - \hat{r}_c)^{-1} \), reflecting the fact that the critical temperature is proportional to the number of frozen droplets, as discussed in Section 5. The value of \( \lambda_{SG} \) at the boundary defined by \( \lambda = 0 \) is shown in the lower panel. We can see that \( \lambda_{SG} > \lambda = 0 \) for \( -0.65 \leq \hat{r} \leq \hat{r}_c(0) \). For \( \hat{r} < -0.65 \), the two criteria merge, indicating that the standard continuous transition is restored. In both panels, empty squares represent the first-order phase boundary and full circles represent the second-order phase boundary.

This scenario is consistent with a first-order transition induced by the fluctuations associated with the Griffiths phase. In the uniform limit, these fluctuations were suppressed since all droplets were of equal size, and the transition was of a conventional, second-order type. However, with the full distribution of droplet sizes included, the transition at \( T = 0 \) was driven first-order. At finite temperature, \( 1/T \) acts an effective cutoff in the imaginary time direction, suppressing fluctuations of the largest droplets. Thus, the restoration of the second-order transition at high enough temperatures is a natural consequence of eliminating fluctuations associated with the longest time-scales in the exponential tail of the distribution, and we conclude that these fluctuations are responsible for the discontinuous nature of the transition at \( T = 0 \) and above.

Finally, in Figure 4, we show the phase boundary on a linear temperature scale and for higher temperatures. At low temperatures, there is an exponential tail reflecting the fact that the largest droplets which freeze to form the CGP are exponentially rare in this regime, as was shown schematically in Figure 2.

8. CONCLUSIONS AND DISCUSSION

The fluctuations of rare, defect-free regions in random magnets can lead to a variety of interesting phenomena near quantum critical points. In the present work, we focused on itinerant antiferromagnets with a \( T = 0 \) transition between the PMFL and CGP, and saw that the Griffiths anomalies provide a novel mechanism responsible for driving the transition first-order. This work could be of relevance to experiments on random systems in metallic environments. For example, it has recently been shown\(^{18}\) that singular corrections due to rare regions are necessary in understanding the transition in a magnetic field of superconducting thin films in the Little-Parks experiment at \( T = 0 \).\(^{19}\) The non-Fermi
liquid behavior arising due to quantum Griffiths anomalies may also be relevant for certain strongly correlated materials. Recent studies on such varied systems as the rare-earth intermetallic Tb$_5$Si$_2$Ge$_2$\textsuperscript{20} and the colossal magnetoresistive La$_{0.7}$Ca$_{0.3}$MnO$_{3.2}$\textsuperscript{21} show power-law diverging susceptibilities reminiscent of quantum Griffiths physics. We conclude with a brief discussion of the temperature scales relevant to experimental observations of cluster glass physics.

From equation (24), we notice that the effects of non-Ohmic dissipation are only apparent below a crossover frequency
\[ \omega^* = \exp[-\ln(1/\gamma\tilde{g})/(2 - \alpha)]. \] (39)
Thus, for temperatures below this energy scale, thermodynamic quantities should be dominated by the contribution due to frozen droplets, giving, for example, $\chi \sim T^{-\alpha}$. Above this temperature scale, Ohmic dissipation dominates and the results from the non-interacting droplet model will apply. In particular, thermodynamics quantities will diverge as a power-law with exponent $\alpha - 1 < 0$ as in the quantum Griffiths phase.

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