More concerning the anelastic and subseismic approximations for low-frequency modes in stars

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ABSTRACT
Two approximations, namely the subseismic approximation and the anelastic approximation, are presently used to filter out the acoustic modes when computing low frequency modes of a star (gravity modes or inertial modes). In a precedent paper (Dintrans & Rieutord 2001), we observed that the anelastic approximation gave eigenfrequencies much closer to the exact ones than the subseismic approximation. Here, we try to clarify this behaviour and show that it is due to the different physical approach taken by each approximation: On the one hand, the subseismic approximation considers the low frequency part of the spectrum of (say) gravity modes and turns out to be valid only in the central region of a star; on the other hand, the anelastic approximation considers the Brunt-Väisälä frequency as asymptotically small and makes no assumption on the order of the modes. Both approximations fail to describe the modes in the surface layers but eigenmodes issued from the anelastic approximation are closer to those including acoustic effects than their subseismic equivalent.

We conclude that, as far as stellar eigenvalue problems are concerned, the anelastic approximation is better suited for simplifying the eigenvalue problem when low-frequency modes of a star are considered, while the subseismic approximation is a useful concept when analytic solutions of high order low-frequency modes are needed in the central region of a star.

Key words: stars: oscillations - subseismic and anelastic approximations - low-frequency g-modes

1 INTRODUCTION
When considering the low-frequency modes of a star, namely gravity modes or inertial modes, the compressibility of the fluid is often a side effect in the determination of eigenfrequencies and eigenmodes; in other words, the dynamics of these modes may be simplified by neglecting the elasticity of the fluid or, equivalently, by filtering out acoustic modes. This is the aim of the subseismic and anelastic approximations; the resulting equations for eigenmodes are much simpler than the original ones and very useful when dealing with the low frequency oscillations of rotating stars (e.g. Dintrans & Rieutord 2000).

Recently, we compared these two approximations (Dintrans & Rieutord (2001) referred to as paper I hereafter). We found that in the two cases which we analysed, namely two polytropes, the anelastic approximation performed much better than the subseismic approximation. We attributed this behaviour to an inconsistency of the subseismic approximation but our argument turns out to be not general and Smeyers (2001) showed that, for low-frequency high order modes, the subseismic approximation gives the first order equations in regions not close to the surface of the star. These results prompted us to re-examine this question in order to clarify the origin of the different behaviour of the two approximations. For this purpose we will focus, in section 2, on two asymptotic developments: a first one where we use, as Smeyers (2001), the frequency as a small parameter and a second one where we use the Brunt-Väisälä frequency as the small parameter. These asymptotic developments will prove to be at the origin of each of these approximations and will allow us to clarify the physics attached to each of them. In section 3, using the same examples as in paper I, we will compare the approximate eigenfunctions to their exact counterparts and show the better behaviour of the anelastic approximation. Our conclusions are drawn in section 4.

2 THE ASYMPTOTIC EQUATIONS
As was shown in I, both approximations imply Cowling’s approximation; we shall therefore neglect the perturbation of
the gravitational potential and will start from the following equations:

\[ \rho' + \text{div}(\rho \vec{e}) = 0, \]
\[ \omega^2 \xi = \vec{\nabla} \left( \frac{P'}{\rho} \right) - \frac{N^2}{\rho g} \delta P \xi, \]
\[ \delta P = \epsilon^2 \delta \rho, \]

where we assumed a time-dependence of the form \( \exp(i\omega t) \) and considered adiabatic oscillations. \( \xi \) is the displacement; \( P' \) and \( \rho' \) respectively denote the Eulerian fluctuations of pressure and density whereas \( \delta P, \delta \rho \) are their Lagrangian counterparts; thus we have

\[ \delta P = P' + \frac{dP}{dr} \xi_r, \quad \delta \rho = \rho' + \frac{d\rho}{dr} \xi_r, \]

with a pressure gradient satisfying the hydrostatic equilibrium \( dP/dr = -\rho g \). Also, \( \rho \) is the equilibrium density, \( \vec{g} = -\vec{g}e_r \) the gravity and \( \gamma = (\partial \ln P/\partial \ln \rho)_{s} \) the first adiabatic exponent. Finally, \( c^2 \) and \( N^2 \) respectively denote the squares of sound speed velocity and Brunt-Väisälä frequency such as

\[ c^2 = \gamma \frac{P}{\rho}, \quad N^2 = g \left( \frac{d \ln P}{\gamma d r} - \frac{d \ln \rho}{d r} \right). \]

### 2.1 The subseismic view

As a first exercise we derive the equations verified by the low frequency gravity modes.

We therefore assume that the frequency reads \( \omega = \varepsilon \omega_1 \) and that \( \frac{d}{dr} \) scales as \( \varepsilon^{-1} \) with \( \varepsilon \ll 1 \) since we focus on high radial order modes. Developing the dependent variables generically as

\[ f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots, \]

and using the classical expansion of the variables on the spherical harmonics,

\[ \xi(r, \theta, \phi) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} \xi_{m}^{\ell}(r) Y_{m}^{\ell}(\theta, \phi) \vec{e}_r + \chi_{m}^{\ell}(r) \vec{\nabla} Y_{m}^{\ell}, \]

and dropping \((\ell, m)\)-indices, we find that \( P'_0 = P'_1 = 0, \rho'_0 = 0, \xi_0 = 0 \) and that

\[ \frac{d}{dr} (r \chi_0) = -\frac{N^2}{\omega_1^2} \xi_1, \]
\[ \frac{d}{dr} (r^2 \xi_1) = \ell (\ell + 1) r \chi_0. \]

A system which is slightly different from the one obtained when setting \( P' = 0 \) which yields the subseismic equations:

\[ \frac{d}{dr} (r \chi) = \left( 1 - \frac{N^2}{\omega_1^2} \right) \xi, \]
\[ \frac{d}{dr} (r^2 \xi) = \ell (\ell + 1) r \chi + \frac{\rho}{c^2} r^2 \xi. \]

However, if we use the expansion (5) into (7-8), we recover (6); therefore, (7-8) contain terms of higher order than (6).

In fact, our expansion (5) breaks down near the origin \( r = 0 \) where the regularity of the solutions (i.e. that \( \xi \propto r^{-\ell-1} \)) is not insured. This comes from the fact that terms like \( f/r \) are no longer negligible compared to derivatives \( d f/dr \).

This difficulty is avoided by Smeyers (2001) with the use of the variable \( \tau = \sqrt{\ell (\ell + 1)} \int_{0}^{r} N(r')dr' \) instead of the radial variable \( r \). \( \tau \) serves as a fast variable while \( r \), the slow variable, is assumed small compared to the scale of variation of the Brunt-Väisälä or the background density. Using this transformation, Smeyers (2001) shows that (8) is verified by the solution at leading order while (7) is approximately verified.

Near the surface layers, Smeyers (2001) has shown that within this development, the Eulerian pressure perturbation is no longer negligible and that the subseismic approximation does not apply.

Thus the subseismic equations govern the oscillations of high radial order gravity modes in the central parts of the star. No constraint is imposed to the Brunt-Väisälä frequency and the equations to be solved are:

\[ \text{div} \vec{\xi} = \frac{g}{c^2} \xi_r, \quad \omega^2 \vec{\xi} = \vec{\nabla} \left( \frac{P'}{\rho} \right) + N^2 \xi_r \vec{e}_r. \]

### 2.2 The anelastic view

Let us now turn to the anelastic approximation. In this case, it is more convenient to write the equations of motion for the velocity field \( \vec{v} \) rather than the displacement \( \xi \), that is,

\[ \begin{aligned}
\omega \rho' + \frac{1}{r^2} \frac{d}{dr} (r^2 \rho u) - \ell (\ell + 1) \frac{\rho u}{r} &= 0, \\
\omega \rho u &= -\frac{d}{dr} \left( \frac{P'}{\rho} \right) + N^2 \frac{\rho g}{\omega} (\omega P' - \rho g u), \\
\omega \nu &= -\frac{\rho}{P'} & & & , \\
\omega (P' - c^2 \dot{\rho}) &= -\frac{\rho c^2 N^2}{g} - u,
\end{aligned} \]

where we used the following spherical harmonics decomposition for \( \vec{v} \) (for clarity, we still dropped in the previous system the \( \ell, m \) indices)

\[ \vec{v}(r, \theta, \phi) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} v_{m}^{\ell}(r) Y_{m}^{\ell}(\theta, \phi) \vec{e}_r + u_{m}^{\ell}(r) \vec{\nabla} Y_{m}^{\ell}. \]

We now assume that the Brunt-Väisälä frequency is vanishingly small; note that as this quantity often diverges at the surface of the star models, it is more appropriate to assume that \( \omega_N, \) the frequency of the lowest order gravity mode, is vanishingly small. Thus we write

\[ N(r) = \varepsilon n(r), \quad \omega = \varepsilon \omega_1, \quad P' = P'_0 + \varepsilon P'_1 + \cdots, \]

Note that we make no assumption concerning the scale of the perturbations which may be of order unity. First orders yield the equations

\[ P'_0 = \rho'_0 = 0, \]

(10)
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Figure 1. Normalized eigenfunctions $\xi$ with $\ell = 2$ and $k = 5$ for the homogeneous polytrope. The solid line shows the exact solution ($\omega^2 \simeq -3.61 \times 10^{-2}$) while the dashed and dotted lines correspond to its subseismic and anelastic approximations, respectively (with $\omega^2_{\text{subs}} \simeq -3.26 \times 10^{-2}$ and $\omega^2_{\text{anel}} \simeq -3.53 \times 10^{-2}$). On right, the surface layers have been magnified.

\begin{align*}
\vec{\nabla} \cdot (\rho \vec{u}_0) &= 0, \\
i\omega_1 u_0 &= -\frac{d(P'/\rho)}{dr} - \frac{n^2}{i\omega_1} u_0, \\
i\omega_1 r v_0 &= -P'/\rho, \\
i\omega_1 \left(P' - c^2 \rho_1\right) &= -\frac{\rho n^2 c^2}{g} u_0.
\end{align*}

from which we write the anelastic system

$$\text{div}(\rho \vec{\xi}) = 0, \quad \omega^2 \vec{\xi} = \vec{\nabla}(P'/\rho) + N^2 \xi \vec{e}_r$$

As for the subseismic approximation the perturbed equation of state is eliminated; but from (14), we note that the Eulerian pressure perturbation is of the same order as $c^2 \rho'$. The subseismic equations can be obtained by just dropping out the Eulerian fluctuation $P'$ in $\delta P$, in the original equations (1-3); from (14), we see that this is not the case in the anelastic approximation. On the other hand, the fluctuation of density in the mass conservation equation can be neglected.

We therefore see that the anelastic approximation applies when the Brunt-Väisälä frequency is small compared to the acoustic frequencies but does not impose any constraint on the scale of the solutions. Near the surface the anelastic solution differs from the exact solution because of the different boundary condition: Exact solutions verify $\delta P = 0$, a condition which transforms into $\xi = 0$ or $u = 0$ for the approximate solution.

2.3 Comments

The foregoing developments show that the anelastic approximation applies under rather more general conditions than the subseismic approximation; indeed, the only requirement is the smallness of the Brunt-Väisälä frequency or, in other words, a large separation between the acoustic spectrum and the gravity spectrum. As this latter condition is often met in stars we can expect that the anelastic approximation performs better when applied to star models.

Concerning the subseismic approximation, it is clear that it can be applied in the central region of a star but that surface layers should be avoided. Smeyers introduces the notion of boundary layer to describe the regions where his asymptotic solutions are valid. However, these boundary layers are somehow special since their thickness can be comparable to the radius of the star (in polytropes for instance)\(^1\).

Broadly speaking, it turns out that the subseismic approximation has a rather local character while the anelastic approximation has a global one. As eigenvalue problem are global problems in nature, the anelastic approximation should be better suited for these problems.

3 EXAMPLES

As in paper I we consider two polytropes: one of constant density and one of index $n = 3$.

3.1 The homogeneous star model

In this case analytic solutions exist either for the exact or the approximate equations (see paper I).

In the asymptotic case of large wavenumbers ($k \to \infty$), one finds that

$$\omega^2 = \Delta - \sqrt{\Delta^2 + \ell(\ell + 1)} = -\frac{\ell(\ell + 1)}{2\Delta} + O\left(\frac{l^2(\ell+1)^2}{\Delta^2}\right),$$

\(^1\) Classical boundary layers have a thickness very small compared to the size of the domain and which tends to zero as the small parameter is decreased.
with $\Delta = \gamma \left[ k \left( \ell + k + \frac{3}{2} \right) + \ell + \frac{3}{2} \right] - 2$. Therefore

$$\omega^2 \simeq \frac{2\ell(\ell+1)}{\gamma} \frac{1}{\frac{1}{2k(2k+2\ell+5)+4\ell+6}-4}/\gamma.$$ 

Now using (22) of paper I we find that for the anelastic approximation

$$\omega^2_{\text{anel}} = -\frac{2\ell(\ell+1)}{\gamma} \frac{1}{\frac{1}{2k(2k+2\ell+5)+4\ell+6}},$$

while the subseismic expression (21) can be rewritten as:

$$\omega^2_{\text{subs}} = -\frac{2\ell(\ell+1)}{\gamma} \frac{1}{\frac{1}{2k(2k+3\ell+4+2\gamma)+6\ell+(2\ell+4)}},$$

It is clear from these three expressions that, for high order modes, the anelastic approximation is very close to the exact expression. In fact, one finds that $(\omega^2 - \omega^2_{\text{anel}})/\omega^2 \sim k^{-2}$ while $(\omega^2 - \omega^2_{\text{subs}})/\omega^2 \sim k^{-1}$, that is, the anelastic expression converges quadratically while the subseismic one only linearly.

This better behaviour of the anelastic approximation is confirmed by the shape of the eigenfunctions as shown in figure 1. There, we clearly see that the subseismic solution is good only in the central regions ($r < 0.2$) while the anelastic approximation remains close to the exact solution almost to the surface.

### 3.2 The polytrope $n = 3$

For a polytrope $n = 3$ a similar behaviour exists although the difference between the two approximations is less pronounced. In I we observed that the eigenfrequencies converged at different rates, the anelastic approximation one converging faster. Here we plot two eigenmodes of high order ($k = 10$ and $k = 20$) computed with the two approximations and with the complete equations (see figure 2). As expected, it is clear from these three expressions that, for high order modes, the anelastic approximation is very close to the exact expression. In fact, one finds that $(\omega^2 - \omega^2_{\text{anel}})/\omega^2 \sim k^{-2}$ while $(\omega^2 - \omega^2_{\text{subs}})/\omega^2 \sim k^{-1}$, that is, the anelastic expression converges quadratically while the subseismic one only linearly.

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![Figure 2](image-url) Normalized eigenfunctions $\xi$ with $\ell = 2$ and $k = 10$ (left) and $k = 20$ (right) for the polytrope $n = 3$. As in Fig. 1, exact solutions are denoted by solid lines (with $\omega^2_{\text{subs}} \simeq 1.98 \times 10^{-3}$ and $\omega^2_{\text{anel}} \simeq 5.99 \times 10^{-4}$) while the dashed and dotted lines correspond to their subseismic and anelastic approximations, respectively (with $\omega^2_{\text{subs}} \simeq 2.08 \times 10^{-3}$, $\omega^2_{\text{anel}} \simeq 2.02 \times 10^{-3}$ for $k = 10$ and $\omega^2_{\text{subs}} \simeq 6.15 \times 10^{-4}$, $\omega^2_{\text{anel}} \simeq 6.04 \times 10^{-4}$ for $k = 20$).

### Figure 3. A schematic picture of the modes of a star viewed from the anelastic viewpoint (above) and subseismic viewpoint (below). $\omega_N$ and $\omega_c$ are respectively the frequency of the lowest order gravity and acoustic modes.

![Figure 3](image-url) 

while both approximations describe the central regions very accurately, the anelastic one remains closer to the ‘exact’ solutions on a larger volume. For the $k = 20$-mode, it departs noticeably from the exact solution close to the surface ($r \sim 0.83$).

### 4 CONCLUSION

In this paper we tried to clarify the differences between the subseismic and anelastic approximations which both aim at describing the low frequency spectrum. The subseismic approximation appears when one concentrates on the low frequency high radial order modes in the central region of a star; no constraint is imposed to the Brunt-Väisälä frequency.

On the other hand the anelastic approximation assumes a weak stratification but imposes no constraint on the degree of the mode.

Hence, while the anelastic approximation makes the
Brunt-Väisälä frequency, and thus the frequency of all gravity modes, vanishingly small compared to acoustic frequencies, the subseismic approximation focuses on gravity modes whose radial order is very large and hence have small frequencies compared to acoustic ones.

In other words, the anelastic approximation removes the elasticity of the fluid by rejecting acoustic frequencies to infinity and therefore allows for a description of the full spectrum of gravity modes while the subseismic approximation, keeping $\omega_c$ and $\omega_N$ in a finite ratio, concentrates on one part of the spectrum, namely that containing high radial order modes which are the least sensitive to the elasticity of the fluid. This situation is summarized in figure 3.

Since in stars the situation is often that $\omega_N \ll \omega_c$, the use of the anelastic approximation is recommended as it is likely closer to the solutions of the complete equations; on the other hand, the subseismic approximation may be useful when one needs an analytic expression of gravity modes in the central regions of a star.

Finally, it is worth mentioning the work of Durran (1989) who discussed these two approximations in the context of atmospheric sciences. In this field, where the subseismic approximation is called the “pseudo-incompressible approximation” and the anelastic approximation the “modified anelastic approximation”, the subseismic approximation appears to be superior to the anelastic approximation as it conserves the energy, a property which is important for nonlinear problems. This result shows that the best choice for filtering out acoustic modes is dependent on the problem at hands. Therefore, our results which favour the anelastic approximation when searching for low-frequency modes of stars, may be specific to eigenvalue problems.

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