THE WEIL REPRESENTATION OF A UNITARY GROUP ASSOCIATED TO A RAMIFIED QUADRATIC EXTENSION OF A FINITE LOCAL RING

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Abstract. We find all irreducible constituents of the Weil representation of a unitary group $U_m(A)$ of rank $m$ associated to a ramified quadratic extension $A$ of a finite, commutative, local and principal ring $R$ of odd characteristic. We show that this Weil representation is multiplicity free with monomial irreducible constituents. We also find the number of these constituents and describe them in terms of Clifford theory with respect to a congruence subgroup. We find all character degrees in the special case when $R$ is a field.

1. Introduction

Let $K$ be a non-archimedean local field with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{p}$ and residue field $F_q = \mathcal{O}/\mathfrak{p}$ of odd characteristic $p$. Let $m \geq 1$ and consider the Weil representation $W : \text{Sp}_{2m}(K) \to \text{GL}(V)$ of the symplectic group $\text{Sp}_{2m}(K)$. This is a projective representation defined over an infinite dimensional complex vector space $V$. The restriction of $W$ to the maximal compact subgroup $\text{Sp}_{2m}(K)$ is an ordinary representation whose decomposition into irreducible constituents was determined by Prasad [12]. These constituents are the trivial module together with two irreducible modules for each factor group $\text{Sp}_{2m}(O/\mathfrak{p}^{2\ell})$, as $\ell$ runs over all positive integers, which are precisely the two irreducible constituents of the top layer of the Weil representation of $\text{Sp}_{2m}(O/\mathfrak{p}^{2\ell})$ constructed in [2] using different methods. While Prasad’s decomposition does not involve odd powers of $\mathfrak{p}$, Cliff and McNeilly [3] obtained an analogous decomposition that only involves odd powers of $\mathfrak{p}$ by restricting $W$ to another maximal compact subgroup of $\text{Sp}_{2m}(K)$ and, again, the resulting constituents are those appearing in [2].

Let $F$ be a quadratic extension of $K$, with ring of integers $\mathcal{R}$ and maximal ideal $\mathfrak{P}$. Let $U_m(F)$ be a unitary group of rank $m$ associated to $F/K$. We may imbed $U_m(F)$ in $\text{Sp}_{2m}(K)$ and consider the restriction of $W$ to $U_m(F)$. This raises the question as to what is the decomposition of the Weil representation of $U_m(F)$ when restricted to a maximal compact subgroup. This question is what motivates us to consider the decomposition problem for the Weil representation of unitary groups $U_m(\mathcal{R}/\mathfrak{P}^\ell)$ for $\ell \geq 1$. Our problem depends in an essential manner on whether the extension $F/K$ is ramified or unramified.
In the unramified case let \( A = \mathcal{R}/\mathcal{P}^\ell \) and \( R = \mathcal{O}/p^\ell \). Then \( A = R[\sqrt{x}] \), where \( x \) is a non-square unit of \( R \). When \( \ell = 1 \) then \( A = F_{q^2} \), which is the classical case solved by Gérardin [6]. The case of arbitrary \( \ell \) was solved by Szechtman [13].

The ramified case remains open and constitutes the main goal of this paper. In this case let \( A = \mathcal{R}/\mathcal{P}^{2\ell} \) and \( R = \mathcal{O}/p^\ell \). Then \( A = R[\sqrt{x}] \), where \( Rx \) is the maximal ideal of \( R \). We find all irreducible constituents of the Weil representation of \( U_m(A) \). The Weil representation is multiplicity free and all its irreducible constituents are monomial. A description of each constituent in terms of Clifford theory with respect to a suitable normal abelian subgroup is given. We also find the total number of irreducible constituents of the Weil module, while their degrees are given only in the case \( \ell = 1 \), i.e. when \( A = F_q[t]/(t^2) \). The calculation of the degrees in the general case is equivalent to finding the index in a unitary group \( U_m(B) \), where \( B \) is quotient of \( A \), of the stabilizer of a basis vector of arbitrary length. This is a non-trivial problem, specially for non-unit lengths, whose solution appears in [4].

What makes the ramified case interesting and more complicated is that the norm map \( A^* \to R^* \) is not surjective, unlike what happens in the unramified case. To exemplify one of the consequences of a defective norm map we mention this: while in the unramified case there is only one non-degenerate hermitian form of rank \( m \) over \( A \) up to equivalence, in the ramified case there are two of these and, more importantly, the number of irreducible constituents of the Weil module as well as their degrees depend on the choice of the actual form; these character degrees depend on the underlying form even when \( m \) is odd, when the isomorphism type of \( U_m(A) \) is uniquely determined. Nothing like this happens in the symplectic or unramified unitary cases.

Interest in the Weil representation shows no sign of slowing down. Since Weil’s original paper [16] dealing with classical groups over local fields, several authors have directed attention to the subject. Howe [9] looked at the context in which the Weil representation appears, namely through the action of the symplectic group on the Heisenberg group, with the aim of generalizing it to other groups. Shortly afterwards Gérardin [6] defined Weil representations for general linear, symplectic and unitary groups over finite fields and found their irreducible constituents. He also gave an explicit character formula in the unitary case. Gow [7] found the character fields and Schur indices of the irreducible constituents of the Weil representation of the symplectic group in the finite field case. Tiep and Zalesskii [15] gave various characterizations of the Weil representation of symplectic and unitary groups in terms of its restriction to standard subgroups. Recently, Thomas [14] gave a fairly explicit formula for the character of the Weil representation of the symplectic group in the finite field case.

The Weil representation of the special unitary group \( SU_2(T) \), where \( T \) is a fairly general finite ring with involution, is considered in a series of papers by Gutiérrez, Pantoja, and Soto-Andrade. Much of their work is centered around finding a Bruhat presentation for \( SU_2(T) \). Once this is achieved they construct a generalized Weil representation [8] by assigning to each Bruhat generator a linear operator, analogous to the Weil operators arising in the classical case, and then verify that the defining relations for \( SU_2(T) \) hold.

Our paper is organized as follows. §2 recalls the definition of Weil representation in a context that is sufficiently general to apply to symplectic, unitary and other groups. §3 describes a general method to imbed a unitary group into a symplectic
group, provides numerous examples of this phenomenon, and shows that the Weil representation is compatible with this type of imbedding. More precisely, one may restrict the Weil representation of a symplectic group to a unitary group or consider a Weil representation of the unitary group directly: they will agree up to a linear character. §4 constructs a Weil module $X$ for $U_m(A)$ that takes advantage of the special features of the ring $A$. §5 lays the foundation for an analysis of the Clifford theory of $X$ with respect to a specific congruence subgroup of $U_m(A)$. §6 determines and describes in terms of Clifford theory all irreducible constituents of the top layer of $X$, while §7 finds the exact number of them. §8 shows that the bottom layer of $X$ is trivial if $\ell = 1$ or a Weil module for a unitary group $U_m(B)$ over a quotient ring $B$ of $A$; this allows us, in §9, to obtain the full decomposition of $X$ by repeatedly peeling off one top layer after the other. §10 refers to the degrees of the components of the top layer of $X$. Finally §11 considers the special case $\ell = 1$, revisits the decomposition of $X$, and explicitly displays all character degrees.

2. **The Schrödinger and Weil Representations**

Before attempting to decompose the restriction to a unitary group of the Weil representation of a symplectic group it will be well to recall the definition of Weil representation, as well as the auxiliary Schrödinger representation. We will do this in a fairly general setting, to take advantage of this flexibility in §4.

Rather than beginning, as in [2], with a non-degenerate, alternating bilinear form $f : V \times V \to R$, where $R$ is a finite ring of odd characteristic and $V$ is a finitely generated $R$-module, we will merely assume here that $R$ and $V$ are finite additive abelian groups of odd order and that $f : V \times V \to R$ is a bi-additive function, in the sense that

$$f(u + v, w) = f(u, w) + f(v, w), \quad f(u, v + w) = f(u, v) + f(u, w), \quad u, v, w \in V.$$ 

Associated to $f$ we have the isometry group $G = G(f)$ and Heisenberg group $H = H(f)$. The group $G$ consists of all group automorphisms of $V$ that preserve $f$, i.e.

$$G = \{ g \in \text{Aut}(V) \mid f(gu, gv) = f(u, v) \text{ for all } u, v \in V \}.$$ 

The group $H$ is comprised of all pairs $(r, v)$, $r \in R$ and $v \in V$, with multiplication

$$(r, v)(s, w) = (r + s + f(v, w), v + w).$$

The associative law follows from the fact that $f$ is bi-additive. Moreover, the neutral element of $H$ is $(0, 0)$, and the inverse of $(r, v)$ is

$$(r, v)^{-1} = (-r + f(v, v), -v).$$

Thus $H$ is indeed a group. Conjugation in $H$ is given by

$$(1) \quad (r, v)(s, w)(r, v)^{-1} = (s, w)(f(v, w) - f(w, v), 0).$$

In particular, $(R, 0)$ is a central subgroup of $H$, and it will be identified with $R^+$. Clearly $G$ acts by automorphisms on $H$ as follows:

$$\sigma(r, v) = (r, gv).$$

Since $R$ is a finite additive abelian group of odd order, multiplication by 2 is an automorphism of $R$, whose inverse will be denoted as multiplication by $1/2$. We thus define the alternating and symmetric parts of $f$ as follows:

$$f^-(u, v) = (f(u, v) - f(v, u))/2, \quad f^+(u, v) = (f(u, v) + f(v, u))/2, \quad u, v \in V.$$
Thus
\[ f = f^- + f^+ \text{ and } G(f) = G(f^-) \cap G(f^+). \]

For \( g \in G \) and a representation \( T : H \to \text{GL}(X) \), the conjugate representation \( T^g : H \to \text{GL}(X) \) is defined by \( T^g(h) = T(gh) \) for all \( h \in H \). We say that \( T \) is \( G \)-invariant if \( T \) is equivalent to \( T^g \) for all \( g \in G \).

We fix a linear character \( \lambda : R^+ \to \mathbb{C}^\times \) for the remainder of this section. By a Schrödinger character of type \( \lambda \) we mean a \( G \)-invariant irreducible character of \( H \) whose restriction to \( R^+ \) is a multiple of \( \lambda \).

**Theorem 2.1.** There is one and only one irreducible character of \( H(f) \) lying over \( \lambda \) and invariant under the central involution, say \( \iota \), of \( G(f) \), given by \( v \mapsto -v \). Moreover, this character is actually \( G(f^-) \)-invariant, that is, it is a Schrödinger character of \( H(f) \) of type \( \lambda \). Its degree is equal to \( \sqrt{|V|}/\sqrt{|V(\lambda)|} \), where \( V(\lambda) = \{ v \in V | \lambda(f(v,u)-f(u,v)) = 1 \text{ for all } u \in V \} = \{ v \in V | \lambda(f^- (v,V)) = 1 \} \).

**Proof.** We first consider the case \( f = f^- \). Let \( V_1 \) be a subgroup of \( V \) maximal subject to
\[ \lambda(f(v,u)) = 1, \quad v, u \in V_1. \]
We extend \( \lambda \) to a linear character \( \rho \) of the normal subgroup \( (R, V_1) \) of \( H \) by setting \( \rho(r,u) = \lambda(r) \). Let \((s,v)\) belong to the stabilizer of \( \rho \) in \( H \). Since \( f = f^- \) and \( R \) has odd order, it follows from (1) that \( \lambda(f(v,V_1)) = 1 \). From the maximality of \( U \) and the fact that \( f(v,v) = f^-(v,v) = 0 \) we see that \( v \in V_1 \), so \((s,v) \in (R,V_1) \). By Clifford theory, \( \chi = \text{ind}_{R,V_1}^H \rho \) is an irreducible character of \( H \) lying over \( \lambda \). The fact that \( \chi \) is \( G(f^-) \)-invariant and that no other \( \iota \)-invariant irreducible character of \( H \) lies over \( \lambda \) follows as in the proof of Proposition 2.1 of [2].

We move to the general case. Consider the map \( \beta : H(f^+) \to H(f^-) \) given by
\[ \beta(r,v) = (r-f(v,v))/2, v. \]
We claim that \( \beta \) is a group isomorphism. Indeed, if \((r,v), (s,w) \in H(f) \) then
\[
\beta[(r,v)(s,w)] = \beta[(r+s+f(v,w), v+w)]
= (r+s+f(v,w) - f(v+w, v+w))/2, v+w
= (r+s - f(v,v)/2 - f(w,w)/2 + f^- (w,v), v+w)
= (r-f(v,v)/2, v)(s-f(w,w)/2, w)
= (r,v)\beta(s,w),
\]
so \( \beta \) is a group homomorphism. Since \( \beta(r,v) = (0,0) \) forces \((r,v) = (0,0) \) and \( \beta(r+f(v,v)/2, v) = (r-f(v,v)/2, v) \), our claim is established. This isomorphism is compatible with the actions of \( G(f) \) on \( H(f) \) and \( H(f^-) \), in the sense that
\[ g\beta(k) = \beta(gk), \quad k \in H(f). \]
Let \( T : H(f^-) \to \text{GL}(X) \) be a Schrödinger representation of \( H(f^-) \) of type \( \lambda \). It then follows from (3) and the fact that \( G(f) \subseteq G(f^-) \) that \( S = T \circ \beta : H(f) \to \text{GL}(X) \) is a Schrödinger representation of \( H(f) \) of type \( \lambda \). Moreover, let \( S_1 : H(f) \to \text{GL}(X_1) \) and \( S_2 : H(f) \to \text{GL}(X_2) \) be \( \iota \)-invariant irreducible representations lying over \( \lambda \). Then \( T_1 = S_1 \circ \beta^{-1} \) and \( T_2 = S_2 \circ \beta^{-1} \) are \( \iota \)-invariant irreducible representations of \( H(f^-) \) lying over \( \lambda \). By the first case, \( T_1 \) and \( T_2 \) are equivalent, whence \( S_1 \) and \( S_2 \) are equivalent. \( \square \)

**Corollary 2.2.** Suppose \( V(\lambda) = 0 \). Then there is one and only one irreducible character of \( H(f) \) lying over \( \lambda \), namely the Schrödinger character of type \( \lambda \).
Lemma 2.5. The degree of $S$ is.

Letting $F$ be a local ring of odd prime characteristic and the proof is complete. □

Proposition 2.4. □

Theorem 2.3. There is one and only one Weil representation of type $\lambda$, up to equivalence and multiplication by a linear character of $G(f)$.

Proof. Let $S : H(f) \to \text{GL}(X)$ be a Schrödinger representation of $H(f)$ of type $\lambda$. By Theorem 2.1, given $g \in G$ there exists $P(g) \in \text{GL}(X)$ such that

$$P(g)S(h)P(g)^{-1} = S(\sigma h), \quad g \in G(f), h \in H(f).$$

Given $g \in G(f)$ we must find $c(g) \in \mathbb{C}^\times$ such that $W(g) = P(g)c(g)$, is a representation. This can be achieved much as in the proof of Theorem 3.1 of [2]. □

In regards to the relationship between the Weil representations of $G(f)$ and $G(f^-)$ we have the following result.

Proposition 2.4. Let $W : G(f^-) \to \text{GL}(X)$ be a Weil representation of of type $\lambda$. Then its restriction to $G(f)$ is a Weil representation of type $\lambda$.

Proof. Let $T : H(f^-) \to \text{GL}(X)$ be a Schrödinger representation of $H(f^-)$ of type $\lambda$ and let $W : G(f^-) \to \text{GL}(X)$ be an associated Weil representation. Thus

$$W(g)T(h)W(g)^{-1} = T(\sigma h), \quad g \in G(f^-), h \in H(f^-).$$

Appealing to the isomorphism (2) and the compatibility condition (3) we may write this as follows:

$$W(g)T(\beta(k))W(g)^{-1} = T(\sigma\beta(k)) = T(\beta(\sigma k)), \quad g \in G(f^-), k \in H(f).$$

Letting $S = T \circ \beta : H(f) \to \text{GL}(X)$ and using $G(f) \subseteq G(f^-)$, we see that

$$W(g)S(k)W(g)^{-1} = S(\sigma k), \quad g \in G(f), k \in H(f).$$

Thus $S$ is Schrödinger of type $\lambda$ and the proof is complete. □

Suppose for the remainder of the paper that $R$ is a finite, principal, commutative local ring of odd prime characteristic $p$. Since 2 is invertible in the subring $\mathbb{Z}/p\mathbb{Z}$ of $R$ we see that $2 \in R^\times$, the unit group of $R$. Let $\mathfrak{m}$ be the maximal ideal of $R$ and let $F_q = R/\mathfrak{m}$ be the residue field of $R$, where $q$ is a power of $p$. The nilpotency degree of $\mathfrak{m}$ will be denoted by $\ell \geq 1$.

For the remainder of this section we let $V$ be a non-zero free $R$-module $V$ and $f : V \times V \to R$ an alternating bilinear form that is non-degenerate, in the sense that the associated linear map $V \to V^*$, given by $v \mapsto f(v, -)$, is an isomorphism.

Lemma 2.5. The $R$-module $V$ has a basis $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ such that

$$f(u_i, u_j) = 0 = f(v_i, v_j), \quad f(u_i, v_j) = \delta_{ij}, \quad 1 \leq i, j \leq m.$$
Proof. Since $f$ is a non-degenerate alternating form, we must have $\text{rank } V > 1$. Let $w_1, w_2, \ldots, w_m$ be a basis of $V$. If $f(w_1, w_i) \in \mathfrak{m}$ for all $i > 1$ then $f(\mathfrak{m}^{\ell-1}w_1, V) = 0$, a contradiction. We may assume without loss of generality that $f(w_1, w_2) = 1$. Set $z_i = w_i - f(w_i, w_2)w_1 + f(w_i, w_1)w_2$ for $i > 2$. Then $w_1, w_2, z_3, \ldots, z_m$ is a basis of $V$ and $f(w_1, z_j) = 0 = f(w_2, z_j)$. Thus the restriction of $f$ to the span of $z_3, \ldots, z_m$ is non-degenerate and the result follows by induction. 

It follows that $\text{rank } V = 2m$ is even, and all non-degenerate alternating bilinear forms on $V$ are equivalent. Thus, the corresponding isometry groups are similar in $\text{GL}(V)$. The symplectic group associated to $f$ is

$$\text{Sp}_{2m}(R) = G(f) \cap \text{GL}(V).$$

A linear character $\lambda : R^+ \to \mathbb{C}^\times$ is said to be primitive if its kernel contains no non-zero ideals of $R$. Such a linear character exists because $R$ has a unique non-zero minimal ideal, namely $\mathfrak{m}^{\ell-1}$.

**Corollary 2.6.** A Weil representation of $\text{Sp}_{2m}(R)$ of primitive type $\lambda$ has degree $|R|^m$.

**Theorem 2.7.** Let $\hat{V}$ be the permutation module associated to $V$. Let $S : H(f) \to \text{GL}(X)$ be a Schrödinger representation of $H(f)$ of primitive type $\lambda$ and let $W : \text{Sp}_{2m}(R) \to \text{GL}(X)$ be an associated Weil representation. Then $\hat{V} \cong \text{End}(X)$ as $\text{Sp}_{2m}(R)$-modules. In particular, if $G$ be a subgroup of $\text{Sp}_{2m}(R)$, $\Omega$ is the Weil character of the $G$-module $X$ and $O_G(V)$ stands for the number of orbits of $G$ acting on $V$, then $[\Omega, \Omega] = O_G(V)$.

**Proof.** The first assertion is proven in Theorem 4.5 of [2]. As for the second, we have $\hat{V} \cong \text{End}(X)$ as $G$-modules, so the trivial $G$-module appears an equal number of times in both of them, that is, $[\Omega, \Omega] = [\Omega \Omega, 1_G] = O_G(V)$.

## 3. Imbedding unitary groups in symplectic groups

In this section we furnish a generic method to imbed a unitary group inside a symplectic group, exhibit various examples of this nature and, more importantly, prove that the Weil representation, as defined in §2, is compatible with this type of imbedding. Thus the restriction to a unitary group of the Weil representation of a symplectic group can be studied by looking directly at the Weil representation of this unitary group, which is what we do from §4 onwards.

We make the following assumptions throughout this section: $A$ is an associative $R$-algebra with identity endowed with an involution $*$ that fixes $R$ elementwise; $A$ is a free $R$-module of finite rank $m$; $h : V \times V \to A$ is hermitian or skew-hermitian form relative to $*$, which means that $h$ is $A$-linear in the second variable and satisfies

$$h(v, u) = \varepsilon h(u, v)^*, \quad u, v \in V,$$

where $\varepsilon = -1$ in the skew-hermitian case and $\varepsilon = 1$ in the hermitian case; $h$ is non-degenerate, i.e., the map $V \to V^*$, given by $v \mapsto h(v, -)$, is an isomorphism of right $A$-modules, where $V^*$ is a right $A$-module via

$$(a\alpha)(v) = a^* \alpha(v), \quad v \in V, a \in A, \alpha \in V^*.$$
Lemma 3.1. Let \( d : A \to R \) be an \( R \)-linear map satisfying:

(C1) If \( a \in A \) and \( d(ab) = 0 \) for all \( b \in A \) then \( a = 0 \).

(C2) \( d(a + \varepsilon a^*) = 0 \) for all \( a \in A \).

Then the map \( f : V \times V \to R \), given by

\[
f(u, v) = d(h(u, v)), \quad u, v \in V
\]

is a non-degenerate alternating \( R \)-bilinear form and \( G(h) \) is a subgroup of \( G(f) \). Consequently the unitary group

\[
U_m(A) = G(h) \cap GL_A(V)
\]

is a subgroup of the symplectic group

\[
SP_{sm}(R) = G(f) \cap GL_R(V).
\]

Proof. It is clear that \( f \) is \( R \)-bilinear and automatic that

\[
G(h) \subseteq G(f).
\]

We next claim that \( f \) is alternating. Indeed, let \( u \in V \). Then

\[
2h(u, u) = h(u, u) + h(u, u) = h(u, u) + \varepsilon h(u, u)^*;
\]

so by (C2)

\[
2f(u, u) = d(2h(u, u)) = 0.
\]

We finally claim that \( f \) is non-degenerate. Indeed, let \( 0 \neq u \in V \). Since \( h \) is non-degenerate there is \( v \in V \) such that \( a = h(u, v) \neq 0 \). By (C1) there is \( b \in A \) such that \( d(ab) \neq 0 \), whence

\[
f(u, vb) = d(h(u, vb)) = d(h(u, v)b) = d(ab) \neq 0.
\]

Thus \( f \) induces a monomorphism \( V \to V^* \), and hence an isomorphism since \( |V| = |V^*| \). \( \square \)

It is natural at this point to ask about the relationship between the Weil representations of \( SP_{sm}(R) \) and \( U_m(A) \).

Theorem 3.2. Let \( \lambda : R^+ \to C^\times \) be a primitive linear character and let \( \mu : A^+ \to C^\times \) be defined by \( \mu(a) = \lambda(d(a)) \) for \( a \in A \). Assume the hypotheses of Lemma 3.1 hold. Then \( \mu \) is primitive, in the sense that the kernel of \( \mu \) contains no non-zero right ideals. Moreover, the restriction to \( U_m(A) \) of a Weil representation of \( SP_{sm}(R) \) of type \( \lambda \) is a Weil representation of \( U_m(A) \) of type \( \mu \).

Proof. Let \( I \) be a right ideal of \( A \) such that \( \mu(I) = 1 \) and let \( a \in I \). Then

\[
\lambda(d(a)r) = \lambda(d(ar)) = \mu(ar) = 1, \quad r \in R,
\]

whence \( d(a) = 0 \) by the primitivity of \( \lambda \). Therefore \( d(I) = 0 \), so \( I = 0 \) by (C1). Thus \( \mu \) is primitive.

We have a group homomorphism \( \beta : H(h) \to H(f) \) given by

\[
\beta(a, v) = (d(a), v), \quad a \in A, v \in V.
\]

Let \( T : H(f) \to GL(X) \) be a Schrödinger representation of \( H(f) \) of type \( \lambda \) with associated Weil representation \( W : SP_{sm}(R) \to GL(X) \). Clearly \( S = T \circ \beta : H(h) \to GL(X) \) is an irreducible representation of \( H(h) \) lying over \( \mu \) and we see, as in the proof of Theorem 2.4, that the restriction of \( W \) to \( U_m(A) \) is a Weil representation. \( \square \)
We proceed to examine various cases where (C1) and (C2) hold. Thus $U_m(A) \subseteq \text{Sp}_m(R)$ in each example, provided a non-degenerate form $h : V \times V \to A$ of the specified type is taken. This is always possible when $\varepsilon = 1$, as well as when $\varepsilon = -1$ and $m$ is even. Such form does not exist when $\varepsilon = -1$ and $s, m$ are odd.

Example 3.3. $A = M_n(R)$, with transposition as an involution, $\varepsilon = -1$ and $d : A \to R$ the trace map.

Example 3.4. $A = H(a, b)$, the generalized quaternion algebra over $R$ associated to $a, b \in R^\times$, with the involution that sends $i, j, k$ to their opposites, and $\varepsilon = -1$. Let $d : A \to R$ pick the coefficient of 1.

Example 3.5. We introduce the notion of quadratic extension of $R$, ramified or unramified. Let $x \in R$. For an unramified extension we take $x$ to be a unit that is not a square. For a ramified extension we take $x$ to be a generator of the maximal ideal of $R$. We consider the ring $A = R[y]$, where $y^2 = x$ (thus $A = R[t]/(t^2 - x)$, a quotient of the polynomial ring $R[t]$). The elements of $A$ are of the form $r + sy$ for unique $r, s \in R$, and multiply in an obvious manner using $y^2 = x$. It is easy to verify that $A$ is a finite, local, principal ring of characteristic $p$. In the unramified case the maximal ideal of $A$ is $m + my$, where $m$ is the maximal ideal of $R$, and the residue field is $F_q$. In the ramified case the maximal ideal of $A$ is $Ay$ and the residue field is $F_q$. We have an involution $*$ on $A$ defined by

$$(r + sy)^* = r - sy, \quad r, s \in R.$$  

In the unramified case we let $\varepsilon = -1$ and take $d$ to be the trace map $A \to R$, given by $a \mapsto a + a^*$. Explicitly, $d(r + sy) = 2r$.

In the ramified case we let $\varepsilon = 1$ and let $d : A \to R$ be the defined by $d(r + sy) = 2s$.

Example 3.6. $R = F_q$ and $A = F_q[t]/(t^s)$. The ring $A$ has an involution sending $t$ to $-t$, where $t^s = 0$. We let $\varepsilon = -1$ if $s$ is odd, and $\varepsilon = 1$ if $s$ is even. Let $d : V \times V \to F_q$ be given by

$$(4) \quad d(a_0 + a_1 t + \cdots + a_{s-1} t^{s-1}) = a_{s-1}, \quad a_i \in F_q.$$  

Example 3.7. $R = F_q$ and $A = F_q[G]$, the group algebra of a finite group $G$ of order $s$ over $F_q$. Consider the involution $*$ on $A$ sending every $x$ in $G$ to $x^{-1}$. In this case $\varepsilon = -1$. Let $d : A \to R$ be the $R$-linear map sending each $a \in A$ to the coefficient of $1_G$.

Example 3.8. This is a family of examples, including Examples 3.3-3.5 above and many more.

Let $M$ be a free $R$-module of finite rank $t > 0$ and let $q : M \times M \to R$ be a symmetric bilinear form, not necessarily non-degenerate. Let $A = C(M, q)$ be the associated Clifford algebra. This is an associative unital algebra, which is free of rank $2^t$ as $R$-module. If $u_1, \ldots, u_t$ form a basis of $M$ then a basis for $A$ is formed by all

$$(5) \quad u_1^{a_1} \cdots u_t^{a_t}, \quad a_i = 0, 1,$$  

where $u_i u_j + u_j u_i = 2q(u_i, u_j)$ for all $1 \leq i, j \leq t$. We may obtain $A$ as the quotient $T(M)/I$, where $T(M)$ is the tensor algebra of $M$ and $I$ is the ideal generated by all $u \otimes u - q(u, u) \cdot 1$, $u \in M$. It is not difficult to see that $M$ admits an orthogonal basis relative to $q$, and we will let $u_1, \ldots, u_t$ denote such a basis.
(a) Here we take \( \varepsilon = -1 \). Let \( d : A \to R \) be the map that picks the coefficient of 1 relative to the basis (5) of \( A \). Then (C1) holds if and only if \( q \) is non-degenerate, i.e. \( q(u_i, u_i) \in R^\times \), \( 1 \leq i \leq t \). To define an involution, let \( A^0 \) be the opposite algebra of \( A \). We have linear map \( M \to A^0 \) given by \( u \mapsto -u \). This can be extended to an algebra anti-homomorphism \( T(M) \to A \), given by \( w_1 \otimes \cdots \otimes w_r \mapsto (-1)^r w_r \cdots w_1 \), for \( w_i \in M \). As the generators \( u \otimes u - q(u, u) \cdot 1 \), \( u \in M \), of \( I \) are sent to 0, we obtain an involution \( A \to A \) such that \( u \mapsto -u \) for \( u \in M \). We see that condition (C2) holds.

As an illustration, when \( q(u_i, u_i) = -1 \) and \( t = 2 \) we obtain Example 3.4, and when \( q(u_1, u_1) = x \) and \( t = 1 \) we obtain the unramified case Example 3.5.

(b) Here we take \( \varepsilon = -1 \) if \( t \equiv 0, 3 \mod 4 \) and \( \varepsilon = 1 \) if \( t \equiv 1, 2 \mod 4 \). Let \( d : A \to R \) be the map that picks the coefficient of \( u_1 \cdots u_t \) relative to the basis (5) of \( A \). Use the same involution as in case (a). Then conditions (C1) and (C2) hold, regardless of the nature of \( q \) (this form can even be 0, in which case \( A \) is the exterior algebra of \( M \)). When \( t = 1 \) and \( q(u_1, u_1) = x \) we obtain the ramified case of Example 3.5.

(c) Here \( \varepsilon = -1 \), \( t = 2 \), \( q(u_i, u_i) = 0 \) and \( q(u_1, u_2) = 1/2 \). Then \( u_1^2 = 0 = u_2^2 \), \( u_1 u_2 + u_2 u_1 = 1 \), where \( u_1, u_2, u_1 u_2, u_2 u_1 \) is a basis of \( A \). We may construct, as in case (a), an involution of \( A \) that interchanges \( u_1 \) and \( u_2 \). We let \( d : A \to R \) pick the sum of the coefficients of \( u_1 u_2 \) and \( u_2 u_1 \). This is Example 3.3 when \( n = 2 \), with \( u_1, u_2, u_1 u_2, u_2 u_1 \) playing the roles of the basic matrices \( e_{12}, e_{21}, e_{11}, e_{22} \).

4. The Weil Module in the Case of a Ramified Quadratic Extension

We wish to study the restriction to the unitary group of the Weil representation of the symplectic group. In view of Theorem 3.2 we may consider the Weil representation of the unitary group directly, as defined in §2, and we shall do so.

The goal of this section is to construct a concrete Weil module for the unitary group that can be advantageous when attempting to decompose it into irreducible constituents.

For the remainder of the paper we will work exclusively within the framework of the ramified case of Example 3.5. Thus, \( R \) is as defined after Proposition 2.4 and \( A = R[y] = R \oplus R y \), where \( y^2 = x \). Here the maximal ideals of \( R \) and \( A \) are \( m = R \ell \) and \( \tau = A y \), with nilpotency degrees \( \ell \) and \( 2 \ell \), respectively, and residue fields \( R/m \cong F_q \cong A/\tau \). We have an involution \( \ast \) on \( A \) defined by \( (r + sy)^\ast = r - sy \). All ideals of \( A \) are powers of \( \tau \) and hence \( \ast \)-invariant. In addition, \( V \) is a free right \( A \)-module of finite rank \( m \geq 1 \). Since \( A \) is commutative, we may view \( V \) as left module in an obvious way and we shall do so. Moreover, \( h : V \times V \to A \) is a non-degenerate \( \ast \)-hermitian form, with unitary group \( U \) and associated non-degenerate alternating \( R \)-bilinear form \( f : V \times V \to R \), given by \( f(u, v) = d(h(u, v)) \), where \( d(r + sy) = 2s \). Note that \( d(R) = 0 \).

We also fix from now on a primitive linear character \( \lambda : R^+ \to C^\times \), the primitive linear character \( \mu : A^+ \to C^\times \), given by \( \mu(a) = \lambda(d(a)) \), and the ideal \( i = \tau^t \) of \( A \). Note that the annihilator, say \( i^\perp \), of \( i \) in \( A \) is \( i \) itself.

Lemma 4.1. There is an orthogonal basis \( v_1, \ldots, v_m \) of \( V \) satisfying \( h(v_i, v_i) \in R^\times \).

Proof. This is [4], Lemma 2.2. \( \square \)
Consider the $U$-invariant submodule $V_0 = iV$ of $V$ and let

$$V_0^\perp = \{ v \in V \mid h(v, V_0) = 0 \}.$$  

It follows from Lemma 4.1 that $(iV)\perp = i^2V$, i.e., $V_0^\perp = V_0$.

We proceed to construct a specific Weil module for the unitary group $U$, as guaranteed in §2.

Let $H = H(h)$ be the Heisenberg group associated to $h$ and consider the subgroup $(A, V_0)$ of $H$. We extend $\mu$ to a linear character $\rho: (A, V_0) \to \mathbb{C}^\times$ by $\rho(a, u) = \mu(a)$.

This works because $h(V_0, V_0) = 0$.

We claim that the stabilizer of $\rho$ in $H$ is $(A, V_0)$. Indeed, for $(b, v)$ in $H$ we have

$$\rho^{(b,v)}(a, u) = \rho(a, u)\mu(h(v, u) - h(u, v)).$$

Suppose $(b, v) \in H$ stabilizes $\rho$. Then $\mu(h(v, u) - h(u, v)) = 1$ for all $u \in V_0$.

From $d(R) = 0$ it follows that $\mu(h(v, u) + h(u, v)) = 1$ for all $u \in V$. Therefore $\mu(h(u, v)) = 1$ for all $u \in V_0$. The primitivity of $\mu$ now implies $v \in V_0 = V_0^\perp = V_0$, so $(b, v) \in (A, V_0)$, as claimed.

Let $Z = \mathbb{C} \zeta$ be a one-dimensional complex vector space. We have a representation of $(A, V_0)$ on $Z$ given by

$$(a, u)\zeta = \rho(a, u)\zeta = \mu(a)\zeta, \quad a \in A, u \in V_0.$$

By Clifford theory the induced module $\text{ind}^H_{(A, V_0)}Z$ is irreducible and lies over $\mu$. Using the $U$-invariance of $V_0$ we may obtain the same module in a more convenient form, as follows. Extend the above action of $(A, V_0)$ on $Z$ to $(A, V_0) \rtimes U$ by means of $(a, u)g\zeta = \mu(a)\zeta$, and consider the induced module $X = \text{ind}^H_{(A, V_0) \rtimes U}Z$. Let $T$ be a transversal for $V_0$ in $V$. Then $(0, v), v \in T$, is a transversal for both $(A, V_0)$ in $H$ and $(A, V_0) \rtimes U$ in $H \rtimes U$. It follows that the restriction of $X$ to $H$ is isomorphic to $\text{ind}^H_{(A, V_0)}Z$. Therefore, by construction, $\text{res}_H X$ is Schrödinger of type $\mu$ and $\text{res}_U X$ is Weil of type $\mu$.

Now

$$X = \mathbb{C}(H \rtimes U) \otimes_{\mathbb{C}((A, V_0) \rtimes U)} Z$$

has $\mathbb{C}$-basis $e_v = (0, v) \otimes \zeta, \quad v \in T$. The action of $U$ on this basis is as follows:

$$ge_v = \mu(h(gv, v'))e_{v'}, \quad g \in U, \quad v, v' \in T, \quad gv \equiv v' \mod V_0.$$  

Indeed, we have

$$ge_v = g(0, v) \otimes \zeta = g(0, v)g^{-1} \otimes \zeta = (0, gv) \otimes g\zeta = (0, v' + (gv - v')) \otimes \zeta$$

$$= (0, v')(0, gv - v')(-h(v', gv - v'), 0) \otimes \zeta = \mu(-h(v', gv - v'))e_{v'}$$

$$= \lambda(-f(v', gv - v'))e_{v'} = \lambda(f(gv - v', v'))e_{v'} = \mu(h(gv, v'))e_{v'}.$$

For $v \in V$ let

$$C(v) = \{ g \in U \mid gv \equiv v \mod V_0 \}.$$  

Thus, for $v \in T$, $C(v)$ is the stabilizer of the subspace $C e_v$ under the action of $U$.

It follows from (6) that for any $v \in V$ then the map

$$\beta_v : C(v) \to \mathbb{C}^\times, \quad \text{given by } g \mapsto \mu(h(gv, v)),$$

is a group homomorphism. This can be verified independently of (6).

Consider the subgroup $U(i)$ of $U$ defined as follows:

$$U(i) = \bigcap_{v \in V} C(v) = \{ g \in U \mid gv \equiv v \mod V_0 \text{ for all } v \in V \}.$$
It is clear that $U(i)$ is a normal subgroup of $U$. Using $i^2 = 0$ we see that $U(i)$ is abelian.

Let $N = \{ z \in A^\times \mid zz^* = 1 \}$ be the norm-1 subgroup of $A$. We identify $z \in N$ with the element of $Z(U)$ such that $v \mapsto z v$ for all $v \in V$.

For $v \in V$ we consider the subgroup $B(v)$ of $U$ given by

$$B(v) = C(v)N.$$  

It is clear that $C(v) \subseteq B(v)$ with $C(v)$ normal in $B(v)$ and $B(v)/C(v)$ abelian.

For $v \in V$ let $\alpha_v : U(i) \to C^\times$ be the function defined by

$$\alpha_v(g) = \mu(h(gv, v)), \quad g \in U(i).$$

It follows from (6) that $\alpha_v$ is a group homomorphism, namely the restriction of $\beta_v$ to $U(i)$.

5. The stabilizer in $U$ of the linear character $\alpha_v$ of $U(i)$

A vector $v \in V$ is said to be primitive if $v \notin iW$. This is equivalent to say that $v$ belongs to a basis of $V$.

The goal of this section is to prove $\text{Stab}_U(\alpha_v) = B(v)$ when $v \in V$ is primitive. This will require a few subsidiary results.

**Lemma 5.1.** Let $a \in i$ and $z_1, z_2 \in V$. Consider the map $\rho_{a, z_1, z_2} : V \to V$ given by

$$\rho_{a, z_1, z_2}(v) = v + ah(z_1, v)z_2 - a^* h(z_2, v)z_1, \quad v \in V.$$  

Then $\rho_{a, z_1, z_2} \in U(i)$.

**Proof.** A direct calculation shows that $\rho_{a, z_1, z_2} \in U$. Since $a \in i$, it follows that $\rho_{a, z_1, z_2} \in U(i)$. \hfill $\square$

**Lemma 5.2.** Let $u_1, \ldots, u_s$ be linearly independent vectors of $V$. Then $s \leq m$. Moreover, if $s = m$ then $u_1, \ldots, u_s$ is a basis of $V$, and if $s < m$ then $u_1, \ldots, u_s$ can be extended to a basis of $V$.

**Proof.** Let $v_1, \ldots, v_m$ be a basis of $V$. Write $u_1$ in terms of this basis. Since $u_1$ is primitive, at least one coefficient must be a unit. Thus, we can replace some basis vector, say $v_1$, by $u_1$ and still have a basis. Repeat the process with $u_2$. Since $u_1, u_2$ are linearly independent and $r$ is nilpotent, some coefficient other than the one from $u_1$ must be a unit, and we get a basis starting with $u_1, u_2$. Repeating the above process we obtain the desired result. \hfill $\square$

**Lemma 5.3.** Let $v_1, \ldots, v_s$ be linearly independent vectors of $V$ and let $a_1, \ldots, a_s \in A$. Then there is a vector $v \in V$ such that $h(v, v_i) = a_i$ for all $1 \leq i \leq s$.

**Proof.** By Lemma 5.2 we may extend the given list to a basis $v_1, \ldots, v_s, \ldots, v_m$ of $V$. Consider the linear functional $\phi : V \to A$ defined by $\phi(v_i) = a_i$ if $i \leq s$ and $\phi(v_i) = 0$ if $s < i \leq m$.

As the map $V \to V^*$ associated to $h$ is an isomorphism there is a $v \in V$ such that $h(v, -) = \phi$. \hfill $\square$

**Proposition 5.4.** Suppose $v, w \in V$ are primitive and satisfy $\alpha_v = \alpha_w$. Then there exist $z \in N$ and a vector $u_0 \in iW$ such that $w = vz + u_0$. 

Proof. By assumption
\[ \mu(h(gv, v)) = \mu(h(gw, w)), \quad g \in U(i). \]

For ease of notation we will write \((u, v)\) instead of \(h(u, v)\) for the remainder of the proof.

For \(a \in i\) and \(z_1, z_2 \in V\) consider the element \(\rho_{a, z_1, z_2}\) of \(U(i)\) defined in Lemma 5.1. Since \(\mu(R) = \lambda(d(R)) = 1\), we have
\[ \mu(v, v) = 1. \]

It follows that
\[ \mu(\rho_{a, z_1, z_2}(v), v) = \mu(v + a(z_1, v)z_2 - a^*(z_2, v)z_1, v) = \mu(b - b^*), \]
where
\[ b = a^*(v, z_1)(z_2, v). \]

Likewise,
\[ \mu(\rho_{a, z_1, z_2}(w), w) = \mu(c - c^*), \]
where
\[ c = a^*(w, z_1)(z_2, w). \]

Again, since \(\mu(R) = 1\), we have
\[ \mu(b + b^*) = 1 = \mu(c + c^*). \]

It follows that
\[ \mu(b + b^*) = 1 = \mu(c + c^*). \]

The primitivity of \(\mu\) and the fact that \(i\) is its own annihilator imply
\[ (v, z_1)(z_2, v) \equiv (w, z_1)(z_2, w) \mod i, \quad z_1, z_2 \in V. \]

Suppose, if possible, that \(v, w\) are linearly independent. Then by Lemma 5.3 there is \(z_1 \in V\) such that \((z_1, v) = 1 = (z_1, w)\). Substituting these values in (7) yields
\[ (z_2, v) \equiv (z_2, w) \mod i, \quad z_2 \in V, \]
which means
\[ (z_2, v - w) \in i, \quad z_2 \in V. \]

We infer from Lemma 4.1 that \(v \in iV\), so \(y^i v - y^i w = 0\), against the linear independence of \(v, w\).

We deduce the existence of \(a, b \in A\), not both zero, such that \(av = bw\). Since \(v, w\) are primitive, we see that neither \(a\) nor \(b\) is zero. Again, by the primitivity of \(v, w\), it follows that \(a = y^j s\) and \(b = y^j r\) for some units \(r, s \in A\) and \(0 \leq j < 2\ell\). Thus
\[ y^j(rw - sv) = 0. \]

Multiplying this by \(r^{-1}\) we infer the existence of a unit \(t \in A\) such that
\[ y^j(w - tv) = 0. \]

The vectors annihilated by \(y^j\) are those in \(y^{2\ell - j} V\). Letting \(i = 2\ell - j\), we see that \(0 < i \leq 2\ell\) and there is \(u \in V\) such that
\[ (8) \quad w = tv + y^iu. \]

Of all such expressions for \(w\) in terms of \(v\), with \(t \in A^\times\), \(0 < i \leq 2\ell\) and \(u \in V\) we choose one with \(i\) as large as possible.
Suppose first $i = 2\ell$. Then $w = tv$. Substituting this in (7) gives
\[(tt^* - 1)(v, z_1)(z_2, v) \in i, \quad z_1, z_2 \in V.\]
By Lemma 5.3 there is $z_1 = z_2 \in V$ such that $(z_1, v) = (z_2, v)$, whence $tt^* - 1 \equiv 1$ \mod $i$.

Suppose next $i < 2\ell$. Then, by the choice of $i$, $u$ must be primitive. We claim that $v, u$ are linearly independent. Otherwise, arguing as above we may find $a \in A^\times$, $0 < k \leq 2\ell$ and $z \in V$ such that
\[u = av + y^kz.\]
But then
\[w = tv + y^i(aw + y^kz) = (t + y^iav + y^{i+k}z,\]
contradicting the choice of $i$. Thus $v, u$ are linearly independent.

Substituting (8) in (7) yields
\[(9) \quad (tt^* - 1)(v, z_1)(z_2, v) + b + c \in i,\]
where
\[b = y^k(t^*(v, z_1))(z_2, u) + (-1)^iy^i(t(u, z_1))(z_2, v)\]
and
\[c = (-1)^iy^{-2i}(u, z_1)(z_2, u).\]
Now there exists a unique $k$ such that $0 \leq k \leq 2\ell$ and $tt^* - 1 = y^k e$ for some unit $e \in A$. Three cases arise, depending on how $k$ compares to $i$.

- $k < i$. By Lemma 5.3 we may choose $z_1 = z_2 \in V$ such that $(z_1, v) = (z_2, v)$. Then (9) implies $\ell \leq k < i$, as required.
- $k > i$. By Lemma 5.3 we may choose $z_1, z_2 \in V$ such that
\[t^*(v, z_1) = 1, t(u, z_1) = (-1)^i, (z_2, u) = 1, (z_2, v) = 1.\]
Then (9) implies $k > i \geq \ell$, as required.
- $k = i$. By Lemma 5.3 we may choose $z_1, z_2 \in V$ such that
\[t^*(v, z_1) = 1, t(u, z_1) = (-1)^{i+1}, (z_2, u) = 1, (z_2, v) = 1.\]
Then (9) implies $k = i \geq \ell$.

We have shown that $w = tv + u_1$, where $tt^* - 1 \equiv 1$ \mod $i$ and $u_1 = y^iu \in iv$. Now $tt^* = 1 + s$, where $s \in i \cap R$. Since $|R|$ has odd order the squaring map of the group $1 + (i \cap R)$ is surjective. Thus, there is $r \in i \cap R$ such that
\[(1 + r)(1 + r)^* = (1 + r)^2 = 1 + s = tt^*.\]
Thus $t(1 + r)^{-1}$ has norm 1, so $t = (1 + r)z$, where $z \in N$. Therefore,
\[w = (1 + r)zv + u_1 = zv + rzv + u_1 = zv + u_0,\]
where $u_0 = rzv + u_1 \in iv$.

**Theorem 5.5.** Let $v \in V$ be primitive vector. Then the stabilizer in $U$ of the linear character $\alpha_v : U(i) \to \mathbb{C}^\times$ is $B(v)$.

**Proof.** As $B(v) = C(v)N$, the group $N$ central in $U$ and $\alpha_v$ is the restriction of $\beta_v : C(v) \to \mathbb{C}^\times$ to $U(i)$, we see that $B(v)$ stabilizes $\alpha_v$. On the other hand, suppose $g_0 \in U$ stabilizes $\alpha_v$ and $w = g_0v$. Then $w$ is primitive and every $g \in U(i)$ satisfies
\[\alpha_v(g) = \alpha_v(g_0^{-1}g) = \mu(h(g_0^{-1}g)v, v) = \mu(h(g_0v, g_0v)) = \mu(h(gw, v)) = \alpha_w(g).\]
It follows from Proposition 5.4 that $g_0 \in B(v)$. \[\square\]
6. Irreducible Constituents of $\mathit{Top}$

We wish to decompose the Weil module $X$ into irreducible constituents. Let $\mathit{Top}$ be the subspace of $X$ spanned by all $e_v$ with $v \in T$ primitive, and $\mathit{Bot}$ the subspace of $X$ spanned by all $e_v$ with $v \in T$ not primitive. Then

$$\dim \mathit{Bot} = q^{(\ell-1)m}, \quad \dim \mathit{Top} = q^m - q^{(\ell-1)m}.$$ 

In view of the action (6) of $U$ on $X$, it is clear that $\mathit{Top}$ and $\mathit{Bot}$ are $U$-submodules of $X$. We clearly have $X = \mathit{Top} \oplus \mathit{Bot}$. We will see in §8 that if $\ell = 1$ then $\mathit{Bot}$ is the trivial $U$-module and if $\ell > 1$ then $\mathit{Bot}$ affords a Weil representation of primitive type for a unitary group of rank $m$ associated to a ramified quadratic extension, where the nilpotency degree of the maximal ideal of the base ring is $\ell - 1$.

Thus, reasoning by induction on $\ell$, it suffices to decompose $\mathit{Top}$ into irreducible constituents. This is the purpose of this section.

Let $\chi$ be an irreducible constituent of $\mathit{Top}$. By (6) the restriction of $\mathit{Top}$ to the abelian normal subgroup $U(i)$ is the sum of all $\alpha_v$, $v \in T$ with $v$ primitive. Thus $\chi$ lies over a given $\alpha_v$, with $v$ primitive. By Theorem 5.5 the stabilizer of $\alpha_v$ is $B(v)$. Now $B(v) = N C(v)$ and $\alpha_v$ extends to the linear character $\beta_v : C(v) \to C^\times$. Since $N$ is abelian, we may extend $\beta_v$ to a linear character $\gamma_v : B(v) \to C^\times$ (see Lemma 6.2). By Gallagher’s theorem (Corollary 6.17 of [10]) the irreducible characters of $B(v)$ lying over $\alpha_v$ are of the form $\gamma_v \tau$, where $\tau$ runs over all irreducible characters of $B(v)$ that are trivial on $U(i)$. By Clifford Theory (Theorem 6.11 of [10]) there is bijection $\tau \mapsto \text{ind}_{B(v)}^{U} \gamma_v \tau$ from the set of all irreducible characters of $B(v)$ lying over $\alpha_v$ and those of $U$ lying over $\alpha_v$. Thus $\chi = \text{ind}_{B(v)}^{U} \gamma_v \tau$ for a unique $\tau$. We will show that the $\tau$ that actually occur in the decomposition of $\mathit{Top}$ are those trivial on $C(v)$. Since $B(v)/C(v)$ is abelian, all these $\tau$ are actually linear characters, whence $\chi$ is a monomial character.

Consider the equivalence relation $\sim$ on $V \setminus yV$, given by

$$v \sim w \text{ if there is } g \in U \text{ such that } gw \equiv w \mod iV.$$ 

Let $S$ be set of representatives of primitive vectors for $\sim$. We may assume that $S$ is contained in $T$, the transversal for $iV$ in $V$ used in §4. For $s \in S$ let

$$\mathit{Top}(s) = U\text{-submodule of } \mathit{Top} \text{ generated by } e_s.$$ 

By means of (6) we see that $\mathit{Top}(s)$ has a basis consisting of all $e_v$ such that $v$ is primitive and $v \sim s$.

**Proposition 6.1.** The $U$-submodules $\mathit{Top}(s)$, $s \in S$, are disjoint, i.e., they have no isomorphic irreducible constituents in common. Moreover, their sum is $\mathit{Top}$. In particular,

$$\mathit{Top} = \bigoplus_{s \in S} \mathit{Top}(s).$$

**Proof.** Let $v \in T$ be primitive. Then $v \sim s$ for some $s \in S$. Thus $gs \equiv v \mod iV$ for some $g \in U$ and by (6) $e_v \in \mathit{Top}(s)$. This shows that $\mathit{Top}$ is contained in the given sum. The linear characters of $U(i)$ appearing in $\mathit{Top}(s)$ are all $\alpha_w$, where $w \in T$ is primitive and $w \sim s$. By Proposition 5.4 the set of these $\alpha_w$ is disjoint from the set corresponding to any other $s' \in S$, $s' \neq s$. \[ \square \]

Let $v \in V$. We have the linear character $\beta_v|_{C(v) \cap N}$ of $C(v) \cap N$. This is an abelian group, so we may extend $\beta_v|_{C(v) \cap N}$ to a linear character $\delta_v : N \to C^\times$. We
\[ \gamma_v(gz) = \beta_v(g)\delta_v(z), \quad g \in C(v), z \in N. \]

**Lemma 6.2.** \( \gamma_v \) is a well-defined linear character of \( B(v) \).

**Proof.** Suppose \( g_1z_1 = g_2z_2 \) for \( g_1, g_2 \in C(v) \) and \( z_1, z_2 \in N \). Then \( g_2^{-1}g_1 = z_2z_1^{-1} \in C(v) \cap N \). Since \( \delta_v \) extends \( \beta_v|_{C(v) \cap N} \), we have
\[
\delta_v(z_2z_1^{-1}) = \beta_v(z_2z_1^{-1}) = \beta_v(g_2^{-1}g_1).
\]
Thus
\[
\delta_v(z_2)\delta_v(z_1^{-1}) = \beta_v(g_2^{-1})\beta_v(g_1),
\]
so
\[
\beta_v(g_2)\delta_v(z_2) = \beta_v(g_1)\delta_v(z_1). \quad \square
\]

Suppose next \( v \in V \) is primitive. We easily see that in this case
\[
C(v) \cap N = (1+i) \cap N,
\]
so
\[
B(v)/C(v) = C(v)N/C(v) \cong N/C(v) \cap N = N/(1+i) \cap N \cong (1+i)N/(1+i).
\]

Thanks to (10) we may identify \( B(v)/C(v) \) with \( N/N \cap (1+i) \). It follows from Lemma 6.2 that the linear characters of \( B(v) \) extending \( \beta_v \) are the form \( \phi \gamma_v \), where \( \phi \) is a linear character of \( N \) trivial on \( N \cap (1+i) \), considered as a linear character of \( B(v) \) trivial on \( C(v) \).

Let \( G \) be the group of linear characters of \( N \) trivial on \( N \cap (1+i) \). For \( v \in T \), still primitive, and \( \phi \in G \) we consider the element \( E_{\phi,v} \) of \( Top \) defined as follows:
\[
E_{\phi,v} = (\sum_{z \in N} \phi(z)^{-1}\gamma_v(z)^{-1})e_v.
\]
Then \( zE_{\phi,v} = \phi(z)\gamma_v(z)E_{\phi,v} \) and \( gE_{\phi,v} = \beta_v(g)E_{\phi,v} \) for \( z \in N \) and \( g \in C(v) \).

**Lemma 6.3.** Let \( v \in T \) be primitive. Then
(a) \( E_{\phi,v} \neq 0 \).
(b) The stabilizer of \( CE_{\phi,v} \) in \( U \) is \( B(v) \).

**Proof.** (a) Let \( D \) be a transversal for \( M = N \cap (1+i) \) in \( N \). Then by (6)
\[
E_{\phi,v} = (\sum_{b \in D} \phi(b^{-1})\gamma_v(b^{-1})b)(\sum_{z \in M} \beta_v(z)^{-1})e_v = (\sum_{b \in D} \phi(b^{-1})\gamma_v(b^{-1})d)M[e_v] \neq 0.
\]
(b) This follows from (6). \( \square \)

**Theorem 6.4.** Fix \( s \in S \). For \( \phi \in G \) let \( Top(\phi, s) \) be the \( U \)-submodule of \( Top(s) \) generated by \( E_{\phi,s} \). Then
(a) The character of \( Top(\phi, s) \) is \( \text{ind}^U_{B(v)}(\phi \gamma_v) \).
(b) The \( U \)-module \( Top(\phi, s) \) is irreducible.
(c) The \( U \)-modules \( Top(\phi, s) \), \( \phi \in G \), are pairwise non-isomorphic.
(d) \( Top(s) = \bigoplus_{\phi \in G} Top(\phi, s) \).

**Proof.** (a) By Lemma 6.3 \( CE_{\phi,s} \) is 1-dimensional with stabilizer \( B(v) \), which acts on \( CE_{\phi,s} \) via \( \phi \gamma_v \). It follows from (6) that \( Top(\phi, s) \) has character \( \text{ind}^U_{B(v)}(\phi \gamma_v) \).
(b),(c) This follows from (a), Theorem 5.5 and Clifford Theory.
(d) As the \( E_{\phi,s}, \phi \in G \), are linearly independent, it follows from (11) that \( e_s \) is
in their span. \( \square \)
Theorem 6.5. (a) Top is multiplicity free, with the following monomial irreducible constituents:

$$\text{Top} = \bigoplus_{s \in S} \bigoplus_{\phi \in G} \text{Top}(\phi, s).$$

(b) Let $\iota$ be the central involution of $U$ defined by $v \mapsto -v$, and let $\text{Top}^+$ and $\text{Top}^-$ be the eigenspaces of $\iota$ acting on $\text{Top}$. Let $G^+$ be the subgroup of $G$ of all $\phi$ such that $\phi(\iota) = 1$ and let $G^-$ be the coset of $G_1$ in $G$ of all $\phi$ such that $\phi(\iota) = -1$. Then $\text{Top}^+$ and $\text{Top}^-$ are $U$-submodules of $\text{Top}$, with

$$\text{Top} = \text{Top}^+ \oplus \text{Top}^-,$$

$$\dim \text{Top}^+ = \dim \text{Top}^- = \frac{q^{\ell m} - q^{(-1)\ell m}}{2}$$

and $\text{Top}^+$, $\text{Top}^-$ have the following decompositions in irreducible constituents:

$$\text{Top}^+ = \bigoplus_{s \in S} \bigoplus_{\phi \in G^+} \text{Top}(\phi, s), \quad \text{Top}^- = \bigoplus_{s \in S} \bigoplus_{\phi \in G^-} \text{Top}(\phi, s).$$

Proof. (a) This follows from Proposition 6.1 and Theorem 6.4.

(b) Only the statement about the dimensions of $\text{Top}^\pm$ requires a proof. To verify these dimensions, select the transversal $T$ for $iV$ in $V$ to be symmetric, in the sense that $v \in T$ if and only if $-v \in T$. Then $\text{Top}^+$ and $\text{Top}^-$ are respectively spanned by $e_v + e_{-v}$ and $e_v - e_{-v}$ as $v$ runs through the primitive vectors of $T$.

7. Counting the Irreducible Constituents of Top

This section finds the exact number of irreducible constituents of $\text{Top}$. It turns out to be equal to the number of $U$-orbits of $V \setminus y^2V$. This fact is independently verified in §8.

Theorem 7.1. Let $v, w \in V$ be primitive vectors satisfying $h(v, v) = h(w, w)$. Then there exists $g \in U$ such that $gv = w$.

Proof. This can be found in [4], Theorem 4.1.

Proposition 7.2. The number of $U$-orbits of vectors in $yV \setminus y^2V$ is equal to the number of $U$-orbits of vectors in $V \setminus yV$.

Proof. Let $E$ be a set of representatives for the $U$-orbits of $V \setminus yV$. It is clear that every vector in $yV \setminus y^2V$ is $U$-conjugate to a vector in $yE$. Thus the map $E \to yE$, given by $e \mapsto ye$, is surjective. We claim that it is also injective and, in fact, that if $e_1, e_2 \in E$ and $ye_1, ye_2$ are $U$-conjugate then $e_1 = e_2$. For this purpose we view $yV$ as a module for $A/y^{2\ell - 1}A$ and consider the map $q : yV \times yV \to A/y^{2\ell - 1}A$ given by

$$q(yv, yw) = h(v, w) + y^{2\ell - 1}A, \quad v, w \in V.$$

Given $g \in U$, for all $v, w \in V$ we have

$$q(gyv, gyw) = q(ygv, ygw) = h(gv, gw) + y^{2\ell - 1}A = h(v, w) + y^{2\ell - 1}A = q(yv, yw),$$

so the restriction of $g$ to $yV$ preserves $q$. Suppose $g \in U$ satisfies $gye_1 = ye_2$. By above,

$$q(ye_2, ye_2) = q(gye_1, gye_1) = q(ye_1, ye_1),$$

which means

$$h(e_1, e_1) - h(e_2, e_2) \in y^{2\ell - 1}A \cap R = x^{\ell - 1}yA \cap R = 0.$$

Thus $e_1, e_2$ are $U$-conjugate by Theorem 7.1. Since $e_1, e_2 \in E$, we infer $e_1 = e_2$. □
Proposition 7.3. Consider the equivalence relation $\simeq$ on $V \setminus yV$, given by
$$v \simeq w \text{ if } h(v, v) \equiv h(w, w) \mod i \cap R.$$ Then $\simeq$ is equal to the equivalence relation $\sim$ defined in §6.

Proof. Suppose first that $v \sim w$. Then $gv = w + u$ for some $g \in U$ and $u \in iV$. Therefore
$$h(v, v) = h(gv, gv) = h(w + u, w + u) = h(w, w) + h(w, u) + h(w, u)^* \equiv h(w, w) \mod i \cap R.$$ Therefore $v \simeq w$.

Suppose conversely that $v \simeq w$. Then $h(v, v) = h(w, w) + r$, where $r \in i \cap R$. Since $w$ is primitive there is a basis $w, w_2, \ldots, w_m$ of $V$. As $h$ is non-degenerate there is $z \in V$ such that $h(w, z) = r/2$ and $h(w_i, z) = 0$. In particular, $h(z, V) \subset i$, so $z \in iV$ by Lemma 4.1. Using $h(z, z) = 0$ and $h(w, z) = r/2$ we deduce
$$h(w + z, w + z) = h(w, w) + r = h(v, v).$$ By Theorem 7.1 there is $g \in V$ such that $gv = w + z$, so $v \sim w$. □

Lemma 7.4. $[R^x : R^{x^2}] = 2$.

Proof. The kernel of the canonical map $R^x \to (R/m)^x$ is $1 + m$, which is a finite group of odd order. It follows that the squaring map $1 + m \to 1 + m$ is an epimorphism. This implies that $r$ is a square in $R^x$ if and only if $r + m$ is a square in $(R/m)^x$. Since $R/m \cong F_q$ and $[F_q^x : F_q^{x^2}] = 2$, the result follows. □

Lemma 7.5. Let $Q: A^x \to R^x$ be the norm map $a \mapsto aa^*$. Then $Q(A^x) = R^{x^2}$.

Proof. Clearly $R^{x^2} \subseteq Q(A^x)$. In view of Lemma 7.4 it suffices to show that $Q$ is not surjective.

The involution that $*$ induces on $A/\tau$ is the identity map, whence the induced norm map $(A/\tau)^x \to (R/m)^x$ is the squaring map of $F_q^x$, so $Q$ is not surjective. □

Theorem 7.6. Up to equivalence, there are exactly two non-degenerate hermitian forms on $V$, depending on whether the determinant of the form, relative to any basis, is a square or not.

Proof. This follows from [4], Corollary 3.6. □

Given $r_1, \ldots, r_m \in R^x$ we say that $h$ is of type $\{r_1, \ldots, r_m\}$ if there is a basis $B$ of $V$ relative to which $h$ has matrix $\text{diag}\{r_1, \ldots, r_m\}$. In that case, $h$ is also of type $\{s_1, \ldots, s_m\}$, for $s_i \in R^x$, if and only if $(r_1 \cdots r_m)(s_1 \cdots s_m)^{-1} \in R^{x^2}$.

We fix an element $\epsilon$ in $R^x$ but not in $R^{x^2}$. Thus $R^x = R^{x^2} \cup \epsilon R^{x^2}$.

Let $\Lambda$ be the set of all values $h(u, u)$ taken on primitive vectors $u \in V$ and let $K$ be the number of $U$-orbits of $V \setminus yV$. It follows from Theorem 7.1 that $K = |\Lambda|$.

Theorem 7.7. (a) Suppose $m = 1$. If $h$ is of type $\{1\}$ then $\Lambda = R^{x^2}$ and if $h$ is of type $\{\epsilon\}$ then $\Lambda = R^x \setminus R^{x^2}$.

(b) Suppose $m = 2$. If $h$ is of type $\{1, -1\}$ then $\Lambda = R$ and if $h$ is of type $\{1, -\epsilon\}$ then $\Lambda = R^x$.

(c) If $m > 2$ then $\Lambda = R$.

Proof. This is [4], Lemma 3.7. □

Theorem 7.8. Let $L$ be the number of irreducible constituents of Top. Then $L$ is equal to the number of $U$-orbits of $V \setminus y^2V$. □
Proof. Let $S$ be set of representatives of primitive vectors for the equivalence relation $\sim$ defined in §6. We know from Proposition 6.1 that $\text{Top}$ is the direct sum of $[S]$ submodules $\text{Top}(s)$. Furthermore, by Theorem 6.4, each $\text{Top}(s)$ is the direct sum of $|N/N \cap (1 + i)| = |N(1 + i)/(1 + i)|$ irreducible constituents. Thus

$$L = |S||N(1 + i)/(1 + i)|$$

On the other hand, by Proposition 7.2, the number of $U$-orbits of $V \setminus y^2V$ is $2K$, where $K$ is the number of $U$-orbits of $V \setminus yV$. Thus, we need to verify that

$$2K = |S||(1 + i)N/(1 + i)|.$$  

(12)

We claim that

$$|S| = K/|R \cap i|.$$  

(13)

Indeed, Proposition 7.3 ensures that $|S|$ is the number of values $h(u, u)$, with $u$ primitive, that are distinct modulo $R \cap i$. Moreover, as indicated above, $K = |A|$ is the number values $h(u, u)$, with $u$ primitive. A full description of $\Lambda$ is given in Theorem 7.7. Three cases arise. $\Lambda = R$, in which case $|S| = |R|/|R \cap i| = K/|R \cap i|$; $\Lambda = R^{\times 2}$ or $\Lambda = R^{\times} \setminus R^{\times 2}$, and in both cases $|S| = |R^{\times 2}/1 + R \cap i| = K/|R \cap i|$; $\Lambda = R^{\times}$, in which case $|S| = |R^{\times}/1 + R \cap i| = K/|R \cap i|$. This proves (13) in all cases.

Substituting (13) in (12) reduces our verification to

$$2|R \cap i| = |(1 + i)N/(1 + i)|.$$  

(14)

Now $A/i$ inherits an involution from $A$ and $R/R \cap i \cong (R + i)/i$ naturally imbeds in $A/i$ as the fixed ring of this involution, yielding a norm map $(A/i)^{\times} \to (R/R \cap i)^{\times}$. The proofs of Lemmas 7.4 and 7.5 apply to show that the image of the norm map $(A/i)^{\times} \to (R/R \cap i)^{\times}$ has index 2 in $(R/R \cap i)^{\times}$ and coincides with the group of squares of $(R/R \cap i)^{\times}$. The kernel of the norm map $(A/i)^{\times} \to (R/R \cap i)^{\times}$ is easily seen to be $N(1 + i)/(1 + i)$. We deduce

$$|N(1 + i)/(1 + i)| \times |(R/R \cap i)^{\times 2}| = |(A/i)^{\times}|,$$

or

$$|N(1 + i)/(1 + i)| \times |(R/R \cap i)^{\times}| = 2|(A/i)^{\times}|.$$  

Since $R/R \cap i$ and $A/i$ are local rings, we see that $R^{\times} \to (R/R \cap i)^{\times}$ and $A^{\times} \to (A/i)^{\times}$ are epimorphisms with kernels $1 + R \cap i$ and $1 + i$, respectively. Therefore

$$|N(1 + i)/(1 + i)| \times |R^{\times}|/|R \cap i| = 2|A^{\times}|/|i|.$$  

(15)

On the other hand, from

$$q^\ell - q^{\ell-1} = (q^{2\ell} - q^{2\ell-1})/q^{\ell},$$

we deduce

$$|R^{\times}| = |A^{\times}|/|i|.$$  

Substituting this in (15) yields (14).

**Corollary 7.9.** The number of irreducible constituents of $\text{Top}$ is:

- $q^\ell - q^{\ell-1}$ if $m = 1$.
- $2(q^\ell - q^{\ell-1})$ if $m = 2$ and $h$ is of type $\{1, -\ell\}$.
- $2q^\ell$ if $m > 2$ or $m = 2$ and $h$ is of type $\{1, -1\}$.
8. Bottom is a Weil module for a unitary group over a quotient ring

This section is similar in spirit to §5 and §6 of [2], with some technical differences.

**Lemma 8.1.** The subgroup \((0, y^{2\ell-1}V)\) of the Heisenberg group \(H\) acts trivially on \(\text{Bot}\).

**Proof.** Let \(v \in yV \cap T\) and \(w \in y^{2\ell-1}V\). We wish to see that \((0, w)e_v = e_v\). Since \(h(v, w) = 0\) and \(y^{2\ell-1}V \subseteq y'V\), we have
\[
(0, w)e_v = (0, w)(0, v) \otimes z = (0, v)(0, w) \otimes z = (0, v) \otimes (0, w)z = (0, v) \otimes z = e_v.
\]

**Proposition 8.2.** We have an isomorphism of \(H\)-modules
\[X \cong \text{ind}^H_{\text{Bot}}(A, yV),\]
where \(\text{Bot}\) is an irreducible \((y^2A, yV)\)-module of dimension \(q^{(\ell-1)m}\).

**Proof.** Let \(X(y^{2\ell-1})\) be the fixed points of \((0, y^{2\ell-1}V)\) in \(X\). Consider the normal subgroup \((A, y^{2\ell-1}V)\) of \(H\) and its linear character \(\rho : \langle A, y^{2\ell-1}V \rangle \to \mathbb{C}^\times\), given by \(\rho(a, v) = \mu(a)\). Using \(d(R) = 0\) and the primitivity of \(\mu\) we see that the stabilizer of \(\rho\) in \(H\) is \((A, yV)\). By definition
\[X(y^{2\ell-1}) = \{ z \in X \mid (a, v)z = \rho(a, v)z = \mu(a)z \text{ for all } a \in A, v \in y^{2\ell-1}V \}.
\]

By Lemma 8.1 we know that \(\text{Bot} \subseteq X(y^{2\ell-1})\). Conclusion: \(\rho\) enters the restriction of \(X\) to \((A, y^{2\ell-1}V)\): the \(\rho\)-homogeneous component for \((A, y^{2\ell-1}V)\) in \(X\) is \(X(y^{2\ell-1})\), which is an irreducible \((A, yV)\)-module, and \(X \cong \text{ind}^H_{\langle A, yV \rangle} X(y^{2\ell-1})\). Since \((A, 0)\) is central in \(H\), it follows that \(X(y^{2\ell-1})\) is irreducible as a module for \((y^2A, yV)\). Counting dimensions
\[q^{\ell m} = \dim X = [H : \langle A, yV \rangle] \times \dim X(y^{2\ell-1}) = q^m \times \dim X(y^{2\ell-1}),\]
so
\[\dim X(y^{2\ell-1}) = q^{(\ell-1)m} = \dim \text{Bot},\]
whence
\[X(y^{2\ell-1}) = \text{Bot}.\]

Suppose that \(\ell > 1\). Let \(\overline{R} = R/x^{\ell-1}R\) and \(\overline{A} = A/y^{2(\ell-1)}A\). Then \(\overline{A}/\overline{R}\) is a ramified quadratic extension and \(\overline{A}\) inherits an involution from \(A\) whose fixed points form \(\overline{R}\). Let \(\overline{V} = V/y^{2(\ell-1)V}\). Then \(h\) gives rise to a non-degenerate hermitian form \(\overline{h} : \overline{V} \times \overline{V} \to \overline{A}\), given by
\[\overline{h}(v + y^{2(\ell-1)V}, w + y^{2(\ell-1)V}) = h(v, w) + y^{2(\ell-1)}A, \quad v, w \in V.\]

Let \(\overline{U}\) and \(\overline{H}\) be the associated unitary and Heisenberg groups.

Consider the map \((y^2A, yV) \to \overline{H}\) given by
\[(y^2a, yv) \mapsto (-a + y^{2(\ell-1)}A, v + y^{2(\ell-1)V}).\]

It is a well-defined group epimorphism with kernel \((0, y^{2\ell-1}V)\). The corresponding isomorphism \((y^2A, yV)/(0, y^{2\ell-1}V) \to \overline{H}\) is compatible with the actions of \(U/U(y^{2(\ell-1)})\) on \((y^2A, yV)/(0, y^{2\ell-1}V)\) and \(\overline{H}\).

As \((0, y^{2\ell-1}V)\) acts trivially on \(\text{Bot}\), we see that \(\text{Bot}\) is an irreducible module for the quotient group \((y^2A, yV)/(0, y^{2\ell-1}V)\), and hence for \(\overline{H}\) via (16). Let \(\overline{\mu} : \overline{A}^+ \to \mathbb{C}^\times\) be the linear character defined by
\[\overline{\mu}(a + y^{2(\ell-1)}A) = \mu(-y^2a), \quad a \in A.\]
Then $\overline{\mu}$ is primitive. It follows from Proposition 8.2 and Corollary 2.2 that the representation of $\overline{\mu}$ afforded by $\text{Bot}$ via (16) is Schrödinger of type $\overline{\mu}$.

We now remove the condition that $\ell > 1$. For $j \geq 0$ we consider the normal subgroup $U(y^j)$ of $U$ defined as follows:

$$U(y^j) = \{g \in U \mid gy^j \equiv v \mod y^jV \text{ for all } v \in V\}.$$ 

**Theorem 8.3.** (a) The congruence subgroup $U(y^{2(\ell-1)})$ acts trivially on $\text{Bot}$. (b) The congruence subgroup $U(y^{2(\ell-1)})$ does not act trivially on any of the irreducible constituents of $\text{Top}$.

**Proof.** (a) If $\ell = 1$ then $\text{Bot} = C_{e_0}$ and (6) shows that $U$ acts trivially on $\text{Bot}$. Suppose $\ell > 1$. Then $2(\ell - 1) \geq \ell$, so $U(y^{2(\ell-1)}) \subseteq U(y^\ell)$. Let $yv \in T \cap yV$. Then $U(y^{2(\ell-1)})$ acts on the basis vector $e_{yv}$ of $\text{Bot}$ via the linear character $g \mapsto \mu(h(gyv, yv))$. By assumption

$$gyv = v + y^{2(\ell-1)}u$$

for some $u \in V$, so

$$\mu(h(gyv, yv)) = \mu(-xh(gv, v)) = \mu(-xh(v, v))\mu(-xh(y^{2(\ell-1)}u, v)) = 1.$$

(b) Suppose first that $\ell > 1$. Then $U(y^{2(\ell-1)}) \subseteq U(y^\ell)$. The irreducible constituents of $\text{Top}$ restricted to $U(y^{2(\ell-1)})$ are the linear characters $g \mapsto \mu(h(gv, v))$ for $v$ primitive. Suppose one of these is trivial. Reasoning as in the proof of Proposition 5.4 (with $a \in y^{2(\ell-1)}A$) we see that

$$h(v, z_1)h(z_2, v) \in y^2A, \ z_1, z_2 \in V.$$ 

Choosing $z_1 = z_2$ so that $h(v, z_1) = 1 = h(z_2, v)$ we reach a contradiction.

Suppose next $\ell = 1$. Then $U(y^{2(\ell-1)}) = U$, which does not act trivially on any irreducible constituent of $\text{Top}$, as $U(y^\ell) = U(y)$ itself does not. Indeed, it acts through a linear character that is not trivial by the same argument just made above.

**Corollary 8.4.** $\text{Bot}$ coincides with the fixed points of $U(y^{2(\ell-1)})$ in $X$.

**Theorem 8.5.** The natural map $U \to \overline{U}$, given by $g \mapsto \overline{g}$, where $\overline{g}(v + y^{2(\ell-1)}V) = g(v) + y^{2(\ell-1)}V$, is a group epimorphism with kernel $U(y^{2(\ell-1)})$.

**Proof.** It is clear that this is a group homomorphism with kernel $U(y^{2(\ell-1)})$. The fact that it is surjective follows from [4], Theorem 5.2. (This is also shown in §4 of [1].)

**Theorem 8.6.** The representation of $\overline{U}$ on $\text{Bot}$ obtained via the isomorphism $U/U(y^{2(\ell-1)}) \to \overline{U}$ is a Weil representation of primitive type $\overline{\mu}$.

**Proof.** Let $S : H \to \text{GL}(X)$ and $W : U \to \text{GL}(X)$ be the Schrödinger and Weil representations of type $\mu$, as constructed in §4. Then

$$W(g)S(k)W(g) = S(gk), \ g \in U, k \in H.$$ 

First restrict $k$ to $(y^{2A}, yV)$ and all above operators to $\text{Bot}$. Then, using the fact that $U(y^{2(\ell-1)})$ and $(0, y^{2(\ell-1)})$ act trivially on $\text{Bot}$, pass to the corresponding representations of $\overline{U}$ and $\overline{\mu}$, to obtain the desired result.
9. Irreducible constituents of the Weil module $X$

Let $\overline{R}, \overline{A}, \overline{V}, \overline{h}$ and $\overline{U}$ be defined as in §8.

**Note 9.1.** The irreducible constituents of the Weil module $X$ are those of $\text{Top}$, as described in Theorem 6.5, plus those of Bot. Moreover, either $\ell = 1$, in which case Bot is the trivial $U$-module, or $\ell > 1$ and Bot is Weil module for $\overline{U}$, in which case its irreducible constituents can be recursively obtained by repeating this process.

**Theorem 9.2.** The Weil module $X$ of $U$ is multiplicity free.

*Proof.* Theorem 8.3 shows that Bot and Top are disjoint, while Top is multiplicity free by Theorem 6.5. As seen in §8, Bot is the trivial $U$-module when $\ell = 1$ and affords a Weil representation for $U$ when $\ell > 1$, so the result follows by induction. □

**Theorem 9.3.** If $\ell > 1$ the number of orbits of $U$ acting on $\overline{V}$ is equal to the number of orbits of $U$ acting on $y^2V$.

*Proof.* Since $U \to \overline{U}$ is an epimorphism the $U$ and $U$ orbits of $\overline{V}$ are the same. On the other hand, the map $y^2V \to \overline{V}$ given by $y^2v \mapsto v + y^{2(\ell - 1)}V$ is a bijection compatible with the actions of $U$, i.e., $y^2V$ and $\overline{V}$ are equivalent $U$-sets. Thus, the number of $U$-orbits of $y^2V$ equals the number of $U$-orbits of $\overline{V}$, which is the number of $\overline{U}$-orbits of $\overline{V}$. □

**Note 9.4.** It follows from Theorems 2.7, 8.3, 8.6, and 9.3 that the number of irreducible constituents of Top is equal to the number of $U$-orbits of $V \setminus y^2V$. This was independently verified in Theorem 7.8.

**Theorem 9.5.** The number irreducible constituents of the Weil module $X$ is equal to the number of orbits of $U$ acting on $V$. This common number is:

- $q^\ell$ if $m = 1$.
- $2q^\ell - 1$ if $m = 2$ and $h$ is of type $\{1, -\ell\}$.
- $2(q^\ell + \cdots + q) + 1$ if $m > 2$ or $m = 2$ and $h$ is of type $\{1, -1\}$.

*Proof.* That the given quantities are the same follows from Theorems 2.7 and 9.2. The remaining assertions follows by induction by means of Corollary 7.9. □

10. Degrees of the irreducible constituents of Top

Let $\overline{R} = R/R \cap i$ and $\overline{A} = A/i$. Then $\overline{A}$ inherits an involution, say $a + i \mapsto \overline{a} + i$, from $A$ whose fixed points form $\overline{R}$. Let $\overline{V} = V/iV$. Then $h$ gives rise to a non-degenerate hermitian form $\overline{h} : \overline{V} \times \overline{V} \to \overline{A}$, given by

$$\overline{h}(v + iV, w + iV) = h(v, w) + i, \quad v, w \in V.$$  

Note also that if $\ell$ is even then $R \cap i = R x^{\ell/2}$ and $\overline{A}$ is a ramified quadratic extension of $\overline{R}$. However, if $\ell$ is odd then $R \cap i = R x^{(\ell+1)/2}$ but $\overline{A}$ is no longer a ramified quadratic extension of $\overline{R}$ since the classes of $y$ and $x^{(\ell-1)/2}$ multiply to 0 in $\overline{A}$.

Let $\overline{U}$ be unitary group associated to the hermitian space $(\overline{V}, \overline{A})$. As shown in [4], Theorem 5.2, as well as in §4 of [1], the canonical map $U \to \overline{U}$ is a group epimorphism.
Let $S$ be set of representatives of primitive vectors for the equivalence relation $\sim$ defined in §6. Let $s \in S$. It follows from (10), (14) and our description of $R \cap i$ that

\[(17) \quad [B(s) : C(s)] = \begin{cases} 2q^{\ell/2} & \text{if } \ell \text{ is even}, \\ 2q^{(\ell-1)/2} & \text{if } \ell \text{ is odd}. \end{cases}\]

Let $D(s)$ be the $U$-pointwise stabilizer of $s + iV$. We see that $C(s)$ maps onto $D(s)$ under the epimorphism $U \to U$, so

$[U : C(s)] = [U : D(s)]$.

Let $G$ be the group of linear characters of $N$ that are trivial on $N \cap (1 + i)$ and let $\phi \in G$. It follows from Theorem 6.4 that the degree of the irreducible constituent $\text{Top}(\phi, s)$ of $\text{Top}$ is

$\deg \text{Top}(\phi, s) = [U : D(s)]/[B(s) : C(s)]$,

where $[B(s) : C(s)]$ is given in (17).

The computation of $[U : D(s)]$ is essentially equivalent to that of $|U|$, a nontrivial problem that will not be considered in this paper. However, if $\ell = 1$ then $U$ is an orthogonal group of rank $m$ over $F_q$, whose order is well-known. This case is considered below.

11. The case $\ell = 1$

We assume here that $\ell = 1$. Thus $R = F_q$ and $A = F_q[y], y^2 = 0$. Moreover, $i = Ay$, so $R \cap i = 0$. Thus the equivalence relation $\sim$ for primitive vectors considered in §6 is given by: $u \sim v$ if $h(u, u) = h(v, v)$. Hence, by Theorem 7.1, a set $S$ of representatives for $\sim$ is a set of representatives for the $U$-orbits of $V \setminus yV$.

Let $i$ be the central involution of $U$ given by $v \mapsto -v$. Let $s \in S$. Then $B(v) = C(v) \times \{1, i\}$ by (17). We may extend the linear character $\beta_v : C(s) \to C^\times$ of §4 to $B(s)$ in 2 ways, by letting $i \mapsto \pm 1$. Let these extensions be denoted by $\beta_v^\pm$.

The Weil module $X = \text{Top} \oplus \text{Bot}$ has dimension $q^m$, with $\dim \text{Top} = q^m - 1$ and $\dim \text{Bot} = 1$. Let our transversal $T$ for $iV$ in $V$ be symmetric, in the sense that $v \in T$ if and only if $-v \in T$. In particular, $0 \in T$. We see at once from (6) that $\text{Bot} = C_0$ is the trivial $U$-module. Let $\text{Top}^\pm$ be the eigenspaces for $i$ acting on $\text{Top}$. Then each of $\text{Top}^+$ and $\text{Top}^-$ has dimension $(q^m - 1)/2$, respectively spanned by $e_v + e_{-v}$ and $e_v - e_{-v}$ as $0 \neq v \in T$ runs through $T$.

We may also assume that $S$ is contained in $T$. For $s \in S$ let $\text{Top}^\pm(s)$ be the $U$-submodule of $\text{Top}^\pm$ generated by $e_s \pm e_{-s}$. Let $\delta \in F_q^\times \setminus F_q^{\times 2}$. We then have the following result.

**Theorem 11.1.** (a) $\text{Top}^+ = \bigoplus_{v \in S} \text{Top}^+(s)$ and $\text{Top}^- = \bigoplus_{v \in S} \text{Top}^-(s)$.

(b) For $s \in S$, $\text{Top}^+(s)$ and $\text{Top}^-(s)$ are irreducible $U$-modules with characters $\text{ind}_{B(v)}^U(\beta_v^+)$ and $\text{ind}_{B(v)}^U(\beta_v^-)$.

(c) The Weil module $X$ is multiplicity free and has $2K + 1$ irreducible constituents, namely the trivial $U$-module and the $K$ constituents given in (a) for each of $\text{Top}^+$ and $\text{Top}^-$. Here $K = q$ if $m > 2$ or $m = 2$ and $h$ is of type $\{1, -1\}; K = q - 1$ if $m = 2$ and $h$ is of type $\{1, -\delta\}; K = (q - 1)/2$ if $m = 1$.

Let $s \in S$. We next turn our attention to computing the degree of $\text{Top}^\pm(s)$.
Let us adopt the notation of §10. Then the degree of $\text{Top}^\pm(s)$ is $[U : D(s)]/2$. We wish to determine $[U : D(s)]$. Here $O = U$ is the orthogonal group of the non-degenerate symmetric bilinear form $b = \overline{h}$ defined on the $m$-dimensional vector space $W = \overline{V}$ over $F_q = \overline{A}$.

The classification of non-degenerate symmetric bilinear forms over $F_q$ is completely analogous to that of hermitian forms stated in Theorem 7.6 (see [11], Theorem 6.9). In particular, $h_1, h_2$ are equivalent if and only if $h_1, \overline{h}_2$ are equivalent. Since the equivalence types of $h$ and $b$ determine each other, we will refer to only that of $b$ from now on. It is well-known (see [4], Theorem 4.1, for example) that two non-zero vectors in $W$ of the same $b$-length are $O$-conjugate. Let $t = h(s, s) \in F_q$. Thus, all we have to do is find the index in $O$ of the stabilizer $S_t$ of a non-zero vector $u$ of length $t$.

If $m = 1$ then $t$ must be a unit, $|S_t| = 1$ and $|O| = 2$. Thus $\deg(\text{Top}^\pm(s)) = 1$, as expected since $U$ is abelian in this case.

Suppose henceforth $m \geq 2$. When $m$ is even the orders of the groups associated with the two inequivalent quadratic forms are different. The group $O_m(q)$ is associated with the type $\{1, -1, \ldots, 1, -1\}$ (type 1) and $O_m(q, \delta)$ is associated with $\{1, -1, \ldots, 1, -1, -\delta\}$ (type $\delta$). When $m$ is odd there is just one orthogonal group $O_m(q)$ up to isomorphism, but there are still two inequivalent quadratic forms: $\{1, -1, \ldots, 1, -1, -1\}$ (type 1), and $\{1, -1, \ldots, 1, -1, -\delta\}$ (type $\delta$). We will freely use the formulas for the orders of these groups as given in [11], Theorem 6.17.

**Theorem 11.1.** Suppose $\ell = 1$ and write $m = 2r$ or $m = 2r + 1$ depending on the parity of $m$, where $r \geq 1$.

Let $s \in S$ and let $t = h(s, s) \in F_q$. The degree of either of the irreducible constituents $\text{Top}^\pm(s)$ of the Weil character of $U_m(A)$ is given by one of the following:

\[
\deg(\text{Top}^\pm(s)) = \frac{q^r(q^r + 1)}{2} \quad \text{in the following cases:}
\]

(a1) $m = 2r + 1$, $b$ of type 1, $-1 \in F^\times_q \setminus F^\times_q$, $t \in F^\times_q \setminus F^\times_q$,

(a2) $m = 2r + 1$, $b$ of type 1, $-1 \in F^\times_q \setminus F^\times_q$, $t \in F^\times_q \setminus F^\times_q$,

(a3) $m = 2r + 1$, $b$ of type $\delta$, $-1 \in F^\times_q \setminus F^\times_q$, $t \in F^\times_q \setminus F^\times_q$, and

(a4) $m = 2r + 1$, $b$ of type $\delta$, $-1 \in F^\times_q \setminus F^\times_q$, $t \in F^\times_q$.

\[
\deg(\text{Top}^\pm(s)) = \frac{q^r(q^r - 1)}{2} \quad \text{in the following cases:}
\]

(b1) $m = 2r + 1$, $b$ of type 1, $-1 \in F^\times_q \setminus F^\times_q$, $t \in F^\times_q \setminus F^\times_q$,

(b2) $m = 2r + 1$, $b$ of type 1, $-1 \in F^\times_q \setminus F^\times_q$, $t \in F^\times_q$,

(b3) $m = 2r + 1$, $b$ of type $\delta$, $-1 \in F^\times_q \setminus F^\times_q$, $t \in F^\times_q$, and

(b4) $m = 2r + 1$, $b$ of type $\delta$, $-1 \in F^\times_q \setminus F^\times_q$, $t \in F^\times_q$.

\[
\deg(\text{Top}^\pm(s)) = \frac{q^r(q^r + 1)}{2} \quad \text{if}
\]

(c) $m = 2r$, $b$ is of type $\delta$, and $t \in F^\times_q$.

\[
\deg(\text{Top}^\pm(s)) = \frac{q^r(q^r - 1)}{2} \quad \text{if}
\]

(d) $m = 2r$, $b$ is of type 1, and $t \in F^\times_q$. 

by a manner that preserves the determinant modulo squares. We may freely replace 

\(-1\) if \(t \in \{W, \ldots, \} \) then \(u \) Weil character depends on the form, not just its isometry group. We can alter 

\(2(b_{\text{orthogonal complement to }} u)\) and take \(u \) \{group of type \(1\) is odd and \(t \) is not a square\}. We leave these details to the reader.

If \(-1\) is a square we may take \(u \) \{to be the last vector in this basis. Then \(S_t \) is isomorphic to an orthogonal group for a quadratic form of type \(\{1, -1, \ldots, 1, -1\}\). Therefore, \(|S_t| = |O_{2r}(q)|\) and we have \(O : S_t| = |O_{2r+1}(q)|/|O_{2r}(q)| = q^{r}(q^{r} + 1).\)

If \(-1\) is not a square then \(-\delta \in F_q^{\times2}.\) Convert the type of \(b\) to \(\{1, -1, \ldots, 1, -\delta, -\delta\}\) and take \(u\) to be the last vector in this basis. Then \(S_t \) is isomorphic to the orthogonal group of type \(\{1, -1, \ldots, 1, -\delta\}\), and we have \(O : S_t| = |O_{2r+1}(q)|/|O_{2r}(q, \delta)| = q^{r}(q^{r} - 1).\)

Assume next \(b\) has type \(\{1, -1, \ldots, 1, -1, -\delta\}\). If \(-1\) is a square, we can take \(u\) to be the second-to-last vector in this basis, so \(S_t \simeq O_{2r}(d, \delta)\). If \(-1\) is not a square, then \(-\delta\) will be a square and we have \(S_t \simeq O_{2r}(q).\)

Using the same techniques as above, we can find the value of \(|O : S_t|\) when \(m\) is odd and \(t\) is not a square in each of the cases of \(b\) of type \(1\) or \(\delta, -1\) a square or non-square. We leave these details to the reader.

Now suppose \(m = 2r\) is even. By making suitable choices for \(u\) and then eliminating it as above, we find that \(S_t \simeq O_{2r-1}(q)\) in all cases. So the calculation of \(|O : S_t|\) only depends on which type of \(b\) we started with.
Suppose next $t = 0$. Then there is a basis $u, v, w_1, \ldots, w_{m-2}$ of $W$ such that
\[ b(u, u) = 0 = b(v, v), b(u, v) = 1, b(u, w_i) = 0 = b(v, w_i) \]
and the stabilizer $S_0$ of $u$ in $O$ has order
\[ |S_0| = q^{m-2}|O'|, \]
where $O'$ is the orthogonal group of rank $m - 2$ associated to the restriction $b'$ of $b$ to the span $W'$ of $w_1, \ldots, w_{m-2}$ (see [17], pg. 72). In particular, since $\{u, v\}$ is a hyperbolic plane (and hence of type $\{1, -1\}$), the form $b'$ has the same type as $b$.

Note that when $m = 2$ this says that $|S_0| = 1$. If $m$ is odd, then
\[ [O : S_0] = \frac{|O_m(q)|}{q^{m-2}|O_m(q)|} = q^{m-1} - 1. \]
If $m$ is even and $b$ has type 1 (this is the only possibility when $m = 2$), then
\[ [O : S_0] = \frac{|O_{2r}(q)|}{q^{2r-2}|O_{2r}(q)|} = (q^r - 1)(q^{r-1} + 1). \]
If $m$ is even and $W$ has type $\delta$, then
\[ [O : S_0] = \frac{|O_{2r}(q, \delta)|}{q^{2r-2}|O_{2r}(q, \delta)|} = (q^r + 1)(q^{r-1} - 1). \]
This completes the proof of the theorem, since the degrees of the corresponding irreducible characters are obtained simply by dividing $[O : S_0]$ by 2. \[ \square \]

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