Regularization by Misclassification in ReLU Neural Networks

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Abstract

We study the implicit bias of ReLU neural networks trained by a variant of SGD where at each step, the label is changed with probability $p$ to a random label (label smoothing being a close variant of this procedure). Our experiments demonstrate that label noise propels the network to a sparse solution in the following sense: for a typical input, a small fraction of neurons are active, and the firing pattern of the hidden layers is sparser. In fact, for some instances, an appropriate amount of label noise does not only sparsify the network but further reduces the test error. We then turn to the theoretical analysis of such sparsification mechanisms, focusing on the extremal case of $p = 1$. We show that in this case, the network withers as anticipated from experiments, but surprisingly, in different ways that depend on the learning rate and the presence of bias, with either weights vanishing or neurons ceasing to fire.

1 Introduction

Neural networks generalize well while being very expressive, with a training process that produces solutions with zero training error. It is thus expected that some form of implicit regularization by the training algorithm makes neural networks extremely useful.

Our focus is on the role played by label noise in such implicit regularization. We study scenarios where at each iteration during training, the true label is changed with probability $p$ to a uniformly random label. With $p = 0$ there is no noise, and with $p = 1$ the label is uniform.

Why would it make sense to add label noise on carefully collected data? The answer is subtle.

Noise is often thought of as a major cause of difficulty in learning. But noise can sometimes improve the accuracy of the model; a canonical example is label smoothing [Szegedy et al., 2016].

Some recent works [Blanc et al., 2019, Müller et al., 2019] explore when and how label noise affects neural networks. Our main finding is that training with label noise results in sparser activation patterns. Namely, neurons fire less often in networks trained with label noise. Here is the measure of sparsity that we consider in this paper.

Definition 1. Let $N(x) = W_2 \cdot \text{ReLU}(W_1 \cdot x + B_1) + B_2$ be a fully connected network with one hidden layer. For $x$ in the dataset, the number of active neurons is $A_N(x) = |\{i \mid w_i \cdot x + b_i > 0\}|$ where $(w_i, b_i)$ correspond to the weights of neuron $i$ in the hidden layer. The typical number of active neurons is $\mathbb{E}_x A_N(x)$, where $x$ is uniformly distributed in the dataset.
To see how label noise sparsifies the firing pattern of a network, we start from a broad question. How does the network change when it is presented with a mislabeled sample? One natural approach for answering this question is to study the evolution of norms in the network during training. In some applications, norms are explicitly regularized, and evidence suggests that regularization helps the network to generalize better (see Chapter 7 in Goodfellow et al. [2016]).

Our Contribution

As a first theoretical observation, in Section 4 we prove that when a general neural network is presented with a mislabeled sample, the Frobenius norm of its weight matrices decreases (see Theorem 1). This is consistent with the fact that sometimes explicit $L_2$ regularization is not required in practice to achieve comparable generalization, as observed in Van Laarhoven [2017].

The decrease of norms suggests the possibility that if the network is given a “too hard” dataset then the weights are nullified and the network effectively dies.

Indeed, in Section 5 we present a theoretical analysis of the extreme setting of completely noisy data ($p = 1$). We identify several different mechanisms of network decay in that case, which are presented in Theorems 2, 3, and 4.

Theorem 2 proves that under pure label noise ($p = 1$) over the normal distribution $D = \mathcal{N}(0, I_d)$, the weights of a single neuron will roughly decay to zero.

We then move to studying networks with multiple neurons arranged in one hidden layer. Empirically, we show that they decay as well and we identify two modes. Without bias neurons, all the weights roughly decay to zero. With bias neurons, the weights do not decay to zero, but all ReLU neurons die during training, so after training, they do not respond to any input.

Theorem 3 proves that after training under pure label noise over the distribution $D = U\{e_1, \ldots, e_d\}$ (uniform over the standard basis), a network without bias neurons roughly outputs zero for all inputs. Theorem 4 shows that in this setting there are two different modes of decay that depend on the size of the learning rate. For a large learning rate, all the ReLU neurons do not respond to any input; this shows that bias neurons are not always the cause of ReLU death. For a small learning rate, the output neuron dies, but a sizable fraction of neurons remains active. We observed this behavior also empirically; larger learning rates induced greater sparsity.

Extreme label noise causes unfruitful results, as one would expect. What about smaller amounts of label noise? In Section 6 we empirically observe that training with label noise can be beneficial. Figure 1 presents an example (see Section 6.3) where the role of label noise is critical for learning. Without label noise, the network easily fits the data but generalizes poorly with a test error of 49%. Modifying $p$, from 0 to 0.2, makes a tremendous difference and the test error drops to 21%.

Over the training set, for a one hidden layer network with 240 neurons, the typical number of active neurons with label noise is roughly 47 (i.e. 20%) whereas without label noise it is roughly 94 (i.e. 40%). These quantities remain the same even when measured over the test set. This may give an insight on why training with label noise yields better test accuracy. The increased sparsity comes at a cost: the training time is longer.

The intermediate value theorem suggests an additional perspective. Consider binary classification with a label noise parameter $p \in [0, 1]$. When $p = 0$ there is no label noise, and when $p = 1$ the label is completely random. For every $p$, consider the value $\alpha(p)$ equal to the typical number of active neurons with noise parameter $p$. The above discussion suggests that $\alpha(0)$ is much larger than $\alpha(1)$. Tuning $p$ from 0 to 1 allows to tune the sparsity from $\alpha(0)$ to $\alpha(1)$ (a rigorous analysis of the continuity of $\alpha(p)$ is in the appendix). And possibly, there exists a value of $p$ where the network still fits the data but with low sparsity. This, in turn, may improve generalization, as we examine further in the experiments in Section 6.

2 Related work

Even though overparametrized neural networks are well known to perfectly fit random examples, there is a line of work arguing that adding independent label noise to every iteration of the SGD effectively results in a problem that cannot be perfectly fit, possibly improving gen-
eralization and resulting in smoother solutions. Some, mostly empirical, works in this vein include Hanson [1990], Clay and Sequin [1992], Murray and Edwards [1994], An [1996], Breiman [2000], Rifai et al. [2011], Sukhbaatar and Fergus [2014], Maennel et al. [2020]. Our contribution is pointing out the connection of this behavior to the regularizing property described above via sparsity.

A good representative is a recent work [Blanc et al., 2019] that investigates label noise of one-dimensional regression. In their setting, they add independent noise to labels in the training set, which results in data that cannot be perfectly fit. They then argue that this steers the gradient descent towards solutions where the gradient of an implicit regularizer vanishes in certain directions. Our empirical results show a similar behavior for classification problems. Interestingly, [Blanc et al. 2019] argue that this “noise regularization” does not occur in networks with only one layer of trainable weights. However, our experiments for MNIST yield similar results even if only one layer of weights is trained.

A theoretical reason to study learning pure label noise is also given by Abbe and Sandon [2020]. This paper studies the “junk flow”, a notion that acts as a surrogate for the number of queries in lower-bound techniques for gradient descent. In particular, understanding the dynamics of learning under noise can explain why a randomly initialized network will fail at learning functions like large parities. Such functions might produce data that appear to the network as random data, and the ReLU neurons will die before the network gets to be trained.

Dying ReLUs are well observed in practice. Lu et al. [2019] study this phenomenon from an initialization perspective and suggest alternative initialization schemes. Arnekvist et al. [2020], Douglas and Yu [2018] treat this problem, as in our case, from the weight dynamics perspective. Mehta et al. [2019] study a related notion, filter level sparsity. They conduct an extensive empirical study on the mechanisms that allow sparsity to emerge.

Implicit regularization of neural networks has been observed and studied in numerous works such as Du et al. [2018], Hanin [2018], Neyshabur et al. [2015], Gunasekar et al. [2018], Soudry et al. [2018, 2017]. For instance, Soudry et al. [2017] shows that for monotonically decreasing loss functions, linear predictors on separable data converge to the max-margin solution.

3 Preliminaries

A neural network consists of a consecutive application of an affine transformation followed by a non-linearity $\sigma$:

$$N(x) = W_k \cdot \sigma(...\sigma(W_2 \cdot \sigma(W_1 \cdot x + B_1) + B_2)... + B_k)$$  \hfill (1)
We typically work with the standard ReLU$(x) = \max\{0, x\}$ non-linearity. More generally, we consider homogeneous non-linearities. A function $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ is homogeneous if it is piecewise differentiable and if for all $x \in \mathbb{R}\setminus\mathcal{K}$ it holds that $f'(x) \cdot x = f(x)$, where $\mathcal{K}$ is a finite set.

We focus our attention on SGD with mini-batch of size one and label noise parameter $p \in [0, 1]$. In each iteration, a single sample $x$ is chosen uniformly at random from the dataset and its label $y$ is changed with probability $p$ to a uniform random label.

We consider samples with $x \in \mathbb{R}^d$ and $y \in \mathcal{Y}$, where $\mathcal{Y}$ is a finite subset of $\mathbb{R}$. The weights are then updated according to the gradient of a loss function $L(W, B, x, y)$, where $W = (W_1, ..., W_k)$ and $B = (B_1, ..., B_k)$. For a learning rate $\eta > 0$, the update rule of the weights is

$$(W^{(t+1)}, B^{(t+1)}) = (W^{(t)}, B^{(t)}) - \eta \nabla_{(W^{(t)}, B^{(t)})} L(W^{(t)}, B^{(t)}, x, y)$$

The update rule above dictates how the weight matrices and the bias vectors of the neural network evolve with time (the weights are possibly shared as in convolutional neural networks).

For the theoretical analysis, we consider binary classification with $\mathcal{Y} = \{\pm 1\}$ and 0-1 surrogate loss functions. That is, loss functions of the form $L(W, B, x, y) = L(-yN(x))$, where $L$ is increasing, convex, piecewise differentiable and its derivative at zero (from the right) is positive. Two examples are the hinge loss with parameter $\beta \geq 0$ defined by $L(\xi) = \text{ReLU}(\beta + \xi)$ and the logistic loss defined by $L(\xi) = \log(1 + \exp(\xi))$.

Finally, we remark that training with label noise is closely related to label smoothing, introduced in [Szegedy et al. 2016]. The purpose of label smoothing is to prevent the network from being overconfident in its predictions. In this framework, the activations in the output layer $\{y = N_\theta(x)\}_{y \in \mathcal{Y}}$ are treated as a probability distribution $p_y = \frac{\exp(z_y)}{\sum \exp(z_y)}$. Then, for example, assuming the correct class is “1”, this distribution is optimized against the distribution $\hat{p} = (1 - p + p/|\mathcal{Y}|, p/|\mathcal{Y}|, ..., p/|\mathcal{Y}|)$, instead of $(1, 0, ..., 0)$, using the cross-entropy loss $\sum_{y \in \mathcal{Y}} \hat{p}_y \log p_y$.

The parameter $p$ of label smoothing is the equivalent quantity for the parameter $p$ of learning with label noise. Label smoothing distributes a probability mass $p$ uniformly across all labels and label noise assigns all of the probability mass to a random label with probability $p$. So in some sense, label smoothing is label noise applied in expectation.

4 Implicit Norm Regularization

The following theorem shows that, when using a small enough learning rate, the Frobenius norm $\|\cdot\|_F$ of the weights decreases when presented with a mislabeled sample. This holds, e.g., for the hinge and logistic loss, as well as for activations such as ReLU or Leaky ReLU.

For simplicity, we consider binary classification $y \in \{\pm 1\}$ and for this theorem to hold, we consider neural networks of the form

$$N(x) = W_k \cdot \sigma(\ldots \sigma(W_2 \cdot \sigma(W_1 \cdot \tilde{x}))\ldots),$$

where $\tilde{x} = (x, 1)$. The bias terms implicitly appear in the matrices $W_i$, so such networks can express the same functions as the networks presented in equation (1). However, the corresponding gradients will be different. Nevertheless, the weight decay (possibly flatlining at a constant level rather than decaying all the way to zero) still holds empirically for networks of type (1) (for example, see Figure 4a).

**Theorem 1.** Let $N^{(t)}: \mathbb{R}^d \rightarrow \mathbb{R}$ be a neural network at time $t$ of training, of any architecture (including weight sharing), with homogeneous activation functions (not necessarily the same for each neuron), a 0-1 surrogate loss function $L$, and let $(x, y)$ be a sample such that $yN^{(t)}(x) < 0$. There exists $h_0 > 0$ such that for all $0 < h < h_0$ and any layer of weights $W$ of $N^{(t)}$ it holds that $\|W^{(t+1)}\|^2 < \|W^{(t)}\|^2$.

**Theorem 1** is derived by applying a 0-1 surrogate loss to Corollary 2.1 and Theorem 2.3 in [Du et al. 2018] (we provide a proof in the appendix, including self-contained simpler versions).

Considering the hinge loss with $\beta = 0$, updates only occur for misclassified data points. In this case, **Theorem 1** immediately implies a special corollary for gradient flow training: $\dot{W} = -\nabla_{W} L$.
Corollary 1. Let the setting be as in Theorem 1 but with hinge loss with \( \beta = 0 \) and gradient flow training. Then, \( \| W^{(t)} \| \) is monotonically decreasing as a function of \( t \).

5 Pure Label Noise Leads to Dead Networks

Corollary 1 summons a natural theoretical question: is it possible for the weights to decay to zero, effectively cutting off all connections in the network?

Or stated more generally, what are the underlying mechanisms that cause a network to die during training?

Theorem 1 suggests an answer. As long as the network is presented with enough misclassified data points, there is a pressure to decrease the weights. An extreme example is learning a random function, or almost equivalently, presenting the network with random labels. Since this case is theoretically tractable, in this section we study it in more detail.

Definition 2. A neural network is trained under pure label noise if at each step \( (x, y) \) is chosen independently according to \( x \sim \mathcal{D} \) and \( y \sim U\{\pm 1\} \) for some distribution \( \mathcal{D} \).

We use the above definition to answer this question in several examples that model some of the dynamics of a network on different scales, from a single neuron to a larger network.

We start with a single neuron. For many typical cases, the following theorem shows that from any initialization, the norm of the weight vector decreases to roughly zero.

Theorem 2. Assume the loss function satisfies \( \sup_x L'(x) = M \) and (a) \( \lim_{x \to 0^+} L'(x) - L'(0) = m > 0 \) (as in hinge loss) or (b) \( L''(0) = m > 0 \) (as in cross entropy) and let \( V^{(0)} \in \mathbb{R}^d \) be an initialization for a single neuron. With high probability, as \( h \) goes to zero with other parameters fixed, after training under pure label noise with \( \mathcal{D} = \mathcal{N}(0, I_d) \), and after at most

(a) \( T = O(\frac{\| V^{(0)} \|}{\text{mad}}) \) steps, \( \| V^{(t)} \| \) decreases to \( O\left(dh \max(\frac{M^2}{m}, 1)\right) \).

(b) \( T = O(\frac{\| V^{(0)} \|}{\text{mad}^2}) \) steps, \( \| V^{(t)} \| \) decreases to \( O\left(d\sqrt{h} \max(\frac{M^2}{m}, 1)\right) \).

Proof in expectation over \( \mathcal{D} \). For ease of presentation, the full version of Theorem 2 does not include a bias term. So here, we prove a weaker statement that the norm decreases in expectation for the hinge loss with \( \beta = 0 \) with a bias term.

Let \( W^{(t)} = (V^{(t)}, b^{(t)}) \) where \( V^{(t)} \in \mathbb{R}^d \) and \( b \in \mathbb{R} \) and \( \tilde{x} = (x, 1) \). At each step, the norm of \( W^{(t)} \) changes according to \( \| W^{(t+1)} \|^2 = (W^{(t)} + hy\tilde{x}) \cdot (W^{(t)} + hy\tilde{x}) = \| W^{(t)} \|^2 + 2yW^{(t)} \cdot \tilde{x}h + \| \tilde{x} \|^2 h^2 \) if \( yW^{(t)} \cdot \tilde{x} < 0 \). Otherwise, there is no change. Notice that when there is a change, we can take a small enough learning rate \( h \), so the norm will decrease as the leading term is \( 2yW^{(t)} \cdot \tilde{x}h \), which is negative by definition.

By the rotational symmetry of \( x \), we have \( W^{(t)} \cdot \tilde{x} \sim \mathcal{N}(b^{(t)}, \| V^{(t)} \|^2) \). We now show that, conditioned on \( W^{(t)} \), the expected decay rate of \( \| W^{(t)} \|^2 \),

\[
r^{(t)} = \mathbb{E}_{x,y}[y(W^{(t)} \cdot \tilde{x})1_{y(W^{(t)} \cdot \tilde{x}) < 0}],
\]

is bounded by \( r^{(t)} < -C \cdot \| W^{(t)} \| \) for some universal constant \( C > 0 \).

To see why this holds, let \( \sigma = \| V^{(t)} \| \) and \( \mu = b^{(t)} \). Assume w.l.o.g. \( \mu \geq 0 \), then, \( r^{(t)} = -\frac{1}{2\sqrt{2\pi}} \int_0^\infty x \exp(-\frac{(x-\mu)^2}{2\sigma^2}) + x \exp(-\frac{(x+\mu)^2}{2\sigma^2}) dx = -\frac{\sigma}{\sqrt{2\pi}} \exp(-\frac{\mu^2}{2\sigma^2}) - \mu \frac{\sigma}{\sqrt{2\pi}} \text{erf}(\frac{\mu}{\sqrt{2\sigma}}).
\]

If \( \mu/\sigma \leq 1 \), \( r^{(t)} < -\frac{\sigma}{\sqrt{2\pi}} \exp(-\frac{\mu^2}{2\sigma^2}) < -\frac{\sqrt{\pi}}{2\sqrt{2\pi}} \exp(-\frac{1}{2}) < -c_1 \cdot \| W^{(t)} \|. \)

If \( \mu/\sigma \geq 1 \), \( r^{(t)} < -\frac{\mu}{\sqrt{2\pi}} \text{erf}(\frac{\mu}{\sqrt{2\sigma}}) < -\frac{\sqrt{\pi}}{2\sqrt{2\pi}} \text{erf}(\frac{\mu}{\sqrt{2\sigma}}) < -c_2 \cdot \| W^{(t)} \|. \)
We pick \( C = \min\{c_1, c_2\} \). This means that as long as \( \|W(t)\| > dh/C \),
\[
\left( \mathbb{E} \|W(t+1)\|^2 - \|W(t)\|^2 \right)^2 \leq \mathbb{E} \|W(t+1)\|^2 - \|W(t)\|^2 = 2dh(t) + h^2 \mathbb{E} \|x\|^2 < -Ch \|W(t)\|
\]
and therefore
\[
\mathbb{E} \|W(t+1)\| - \|W(t)\| = \frac{(\mathbb{E} \|W(t+1)\|^2 - \|W(t)\|^2)}{\mathbb{E} \|W(t+1)\| + \|W(t)\|} < \frac{-Ch \|W(t)\|}{2 \|W(t)\|} = -Ch/2.
\]
Note that this expression predicts that the norm will roughly decrease at a linear rate. This is empirically verified in Figure 2.

**Remark.** Although the expectation in the proof above is bounded away from zero, some updates during SGD will inevitably increase the norm. In the full proof in the appendix, we use a general 0-1 surrogate loss and concentration bounds to show that nevertheless most of the time the norm decreases.

Will the theorem above hold for neural networks with one or more hidden layers? It might come as a surprise that it depends on whether the network has bias neurons or not. For a network with one hidden layer, ReLU activations, and the cross entropy loss, we provide empirical evidence for training under pure label noise over the normal distribution.

**Without bias neurons.** The network weights decay to zero (see Figure 3). In this case, it is important to note that the neurons cannot stop firing for all inputs. In fact, because \( D = \mathcal{N}(0, I_d) \) is symmetric, if a neuron does not fire for \( x \), it will fire for \( -x \).

**With bias neurons.** The network weights do not decay to zero. In this case, the network experiences another kind of decay, where all the ReLU neurons stop firing for almost all inputs from the distribution \( D = \mathcal{N}(0, I_d) \) due to significant negative drift in the bias weights. We call this kind of decay “ReLU death”. Figure 4 provides an explanation. On the one hand, close to initialization, when the bias terms do not dominate other weights, the norm of the weights of the hidden layer decreases by a mechanism similar to Theorem 3; see Figure 4a. This means that a typical preactivation value without the bias term decreases. On the other hand, the bias term of a neuron is monotonically decreasing as in Figure 4c (at initialization, this can be seen from backpropagation formulas). Eventually, the bias becomes so negative that it prevents any input from activating the neuron.

To summarize, a neural network trained under pure label noise over the normal distribution experiences neural “death”: all the weights decay to zero or all the neurons in the hidden layer do not respond to any input. However, the exact mode of this process can be different depending on the architecture, e.g., the presence of bias neurons as well as the data distribution.

In the following two theorems (proofs in the appendix) we consider a simpler data distribution (the uniform distribution over the standard basis) and networks with one hidden layer and no bias neurons. We follow the theoretical setting appearing in Brutzkus et al. [2018] where we train only the hidden layer and only considering the hinge loss. This allows for a clear presentation of the underlying mechanisms behind the behavior of SGD under pure label noise.

**Theorem 3.** Let \( N \) be a network with one hidden layer with \( 2k \) ReLU neurons: \( N(x) = V \cdot \text{ReLU}(Wx) \) and no bias neurons. Assume only \( W \) is trained and that we fix \( V \in \mathbb{R}^{2k} \) to be split equally between 1 and \(-1\): \( V = (1, \ldots, 1, -1, \ldots, -1) \).

Then, for any initialization of \( W \), if \( N \) is trained with the hinge loss (\( \beta = 0 \)) under pure label noise with \( D = U \{e_1, \ldots, e_d\} \) (uniform over the standard basis), after a finite number of steps we will have \( |N(x)| < 2kh \) for every \( x \in D \).

Theorem 3 shows that the output of the network becomes roughly zero for every input \( \{e_1, \ldots, e_d\} \).

The next theorem shows that the type of neural death depends on the learning rate size (even without existence of bias neurons). If the learning rate is small enough, the top neuron dies, its output becoming roughly zero. If the learning rate is large enough, all neurons in the hidden layer die.
Theorem 4. Let $N$ be a network with one hidden layer with $2k$ ReLU neurons: $N(x) = V \cdot \text{ReLU}(Wx)$ and no bias neurons. Assume only $W$ is trained and that its weights are initialized iid $U[-1, 1]$ and that we fix $V$ to be split equally between 1 and $-1$: $V = (1, \ldots, 1, -1, \ldots, -1)$.

As $k$ grows, w.h.p., if $N$ is trained with the hinge loss ($\beta = 0$) under pure label noise with $\mathcal{D} = U\{e_1, \ldots, e_d\}$ (uniform over the standard basis), we will have after training:

1. For a learning rate $h \geq 1$, for all $i$ it holds $\text{ReLU}(We_i) = \Theta$.
2. For a learning rate $h \leq 1/k$, for all $i$ it holds $\text{ReLU}(We_i) \neq \Theta$. Furthermore, at most $k + o(k)$ (i.e., $1/2 + o(1)$ fraction) of the coordinates of the vector $\text{ReLU}(We_i)$ are zero.

The motivation for Theorem 4 is to show an example on a slightly larger scale, where even without bias neurons, the ReLU neurons can stop firing without all the weights decaying to zero. Such behavior can be observed when training over datasets like MNIST and pictures in general, where the data consists of vectors with non-negative entries (with analogy to the standard basis). However, for MNIST, the learning rate in which all the neurons die is much smaller than the rate $h \geq 1$ that appears in Theorem 4. For example, empirically, for MNIST item 1 holds for the rate $h = 0.001$.

6 Noise Induces Sparsity — Experiments

The discussion above suggests that if the data given to a ReLU network is very noisy, it might “destroy” the network during training. We now verify this empirically. At the same time, we show instances where the network benefits from adding some noise to the data.

In this section, we experiment and progress from exactly following our theoretical setting (a single neuron, binary classification, pure label noise, mini-batch size 1) to a real-life neural network (ResNet20, CIFAR-10, label smoothing, mini-batch size 128). Our main finding, of sparse activation patterns, does not depend on the exact noise model. So accordingly, we do not focus our discussion on the differences between them.

For a clear and consistent presentation, in all experiments (unless specified differently), we use input of dimension $d = 30$, ReLU activations, the binary cross entropy loss, SGD with a mini-batch of size one, a fixed learning rate $h = 1/d^2$, and a uniform i.i.d. weight initialization $\sim U[-\sqrt{3}, \sqrt{3}]$ ($\sqrt{3}$ for unit variance). To the best of our knowledge varying any of these parameters does not affect our main findings. For example, increasing the size of SGD batch results in similar outcomes.

6.1 A single neuron under pure label noise

We start with a single neuron. Figure 2 shows that when a neuron is trained under pure label noise, the norm of the weights decays at a linear rate up to an equilibrium point of norm $dh$, as predicted by Theorem 2.

![Figure 2: Norm evolution — learning noise with $x \sim N(0, I_d)$.](image)
6.2 Network decay under pure noise

We trained a network with $4d$ neurons in the hidden layer and observe in Figure 3 that with no bias neurons the weights decay to zero. With cross entropy, longer training time is needed to nullify the weights compared to the hinge loss (with analogy to items (a) and (b) in Theorem 2).

In comparison, Figure 4a shows that the weights do not decay to zero when adding bias neurons. Figure 4b shows that adding bias neurons shuts down the network by making the ReLU neurons stop firing for all inputs. The quantity measured in Figure 4b is a measure of sparsity of representation that we use in the subsequent experiments (see Definition 1).

ReLU neurons dying during training is a well observed phenomenon (for example, see Lu et al. [2019]), which causes difficulties during the training of neural networks. The experiments in the following sections show that label noise and ReLU death is not necessarily bad for neural networks as our theoretical results for pure label noise might suggest.

6.3 Random errors sparsify and improve generalization

In what scenario adding label noise would make a sizable difference? A reasonable guess would be learning a function that the network can represent in a sparse way. Such candidate function is the hypercube boundary function $F : C \rightarrow \{\pm 1\}$, where $C = C_1 \cup C_{-1}$, $C_1 = \{x \in \mathbb{R}^d : \|x\|_\infty = 1\}$, $C_{-1} = \{x \in \mathbb{R}^d : \|x\|_\infty = 1 - \epsilon\}$, and $F(x) = 1_{C_1}(x) - 1_{C_{-1}}(x)$. This function outputs 1 if the input comes from the unit hypercube boundary and $-1$ otherwise.

This function has a simple representation using a one hidden layer neural network $N(x) = W_2 \text{ReLU}(W_1 \cdot x + B) + b$, $W_1$ is a $2d \times d$ matrix where $W_1(i, i) = 1$ and $W_1(i + d, i) = -1$ for $1 \leq i \leq d$ and else the entries are 0. $B$ is a $2d$ vector where $B(i) = -(1 - \epsilon/2)$ for $1 \leq i \leq 2d$, $W_2$ is a $2d$ vector where $W_2(i) = 1$ for $1 \leq i \leq 2d$ and $b = -0.5$. For this representation, and the
input distribution $D = \frac{1}{2} U(C_1) + \frac{1}{2} U(C_{-1})$, the typical number of active neurons will be $de (\epsilon/2$ fraction) for inputs from $C_1$ and no neuron will be active for inputs from $C_{-1}$.

We perform an experiment with a network with $4d$ neurons in the hidden layer. We randomly pick a fixed data set according to $D$ with $\epsilon = 0.3$. We investigate the overparametrized regime, so the size of the dataset is $4d$ (the number of parameters is roughly $4d^2$). We explore two training regimes. First, without label noise, second, with label noise with $p = 0.2$. We train both networks with learning rate $h = 1/d$ till they reach zero classification error and then some more until the test error stabilizes.

The experiment reveals a significant difference between the networks trained with and without label noise. Figure 1b shows that the network trained with label noise has significantly sparser activation patterns. That is, for a typical input most of the neurons in the hidden layer do not fire. This suggests that neural networks trained with label noise may generalize better since not all of the network capacity is used.

Indeed, figure 13 shows a clear advantage in the generalization of the network trained with noisy labels. Although it takes this network longer to reach small training error, it makes up for it with a considerable drop in the test error versus the “vanilla” network that does not perform a lot better than a random guess on unseen data. This behavior is consistent with what was observed by Blanc et al. [2019] in the context of 1D regression (see Figure 2 therein).

6.4 Noisy labels applied on MNIST and CIFAR-10

MNIST. We now examine the effect of noisy labels on MNIST. We trained a network with 600 neurons in the hidden layer (overparametrized network), 10 neurons in the output layer, and a learning rate $h = 0.01$ which we decrease by a factor of 2 every 5 epochs. This is a multiclass classification task, so we work with the loss function

$$\sum_{i=1}^{10} L(-y_i \cdot N(x)_i)$$

where the coordinate in $y$ that corresponds to the correct label is 1, the other entries are $-1$, $N(x)_i$ is the output of neuron $i$ in the output layer, and $L(z) = \log(1 + \exp(z))$. Our loss function is a sum of 10 cross entropy losses, each corresponding to one digit. This way, we can consider the same properties as discussed throughout the rest of the paper. For example, we can apply Theorem 1 to each output neuron separately. Contrastingly, Theorem 1 cannot be applied when using the standard cross-entropy loss because the softmax function is applied over all output neurons. And indeed, working with the loss in (3) produces sparser representations than working with softmax.

We trained the network without and with label noise for $p = 0.1$. Figure 5b shows that training with label noise has better performance in terms of the test error. This may be explained by the sparser activation patterns (25% vs. 6%).
CIFAR-10. To achieve reasonable test accuracy over this dataset (> 90%), one must choose a more complex architecture. For CIFAR-10 we chose a standard ResNet-20 with the hyperparameters appearing in [Delbayev 2020].

To emphasize, in contrast to MNIST, we are working with the standard cross-entropy loss applied over a softmax of the last layer. This setting is more remote from the theoretical setting we previously explored (convolutional layers, skip connections, batch normalization, etc.). Nevertheless, our findings are consistent with previous experiments and our theoretical discussion. Adding noise over the labels sparsifies the firing patterns of the network.

We trained the network with and without label smoothing with $p = 0.1$ (not label noise) to reflect more accurately what is being done in practice. We achieved test accuracy of slightly more than 92% for both networks. The sparsity was most pronounced in the penultimate layer that consists of 4096 neurons. In this layer, the fraction of active neurons for a typical input for a network trained without label smoothing was 46.9% vs. 27.5% with label smoothing. In general, deeper layers corresponded to larger sparsity compared to the network trained without label smoothing.

6.5 Understanding Label Smoothing Better

It was observed in [Szegedy et al. 2016] that a better calibrated network trained with label smoothing actually achieves better accuracy. Figure 1 of [Müller et al. 2019] offers some insight on why this happens. The authors recorded the activations of the penultimate layer of a network trained with and without label smoothing over CIFAR-10. Then, they projected the activations from three classes on a plane. In both cases, three clusters emerged that correspond to the three different labels. With label smoothing the clusters were more tightly packed so the separation between classes is easier and hence the better accuracy. Our work provides some insight to why label smoothing promotes tighter clusters.

We discovered that for MNIST learning with label noise or label smoothing generates sparser representations. In these examples label smoothing actually generates a sparse representation in the penultimate layer of a special kind. Most neurons in the penultimate layer can be associated with a specific label. That is, most neurons mostly fire when they are presented with their associated label. More specifically, for every neuron $i$ in the hidden layer (with the corresponding row of weights $W_i$), we counted the number of times it fired for each label $'0'$, $'1'$, $'2'$, $'3'$, $'4'$, $'5'$, $'6'$, $'7'$, $'8'$, $'9'$. We noticed that these histograms were more concentrated for a network trained with label noise. Examples of such histograms are in the appendix.

This provides an explanation for Figure 1 in [Müller et al. 2019]. In this case, picking a plane that intersects three clusters is easy. For every label pick the feature that corresponds the most with a specific label and then pick a plane such that the projections of the selected features are large. Since label smoothing generates a stronger association between the features and the labels, the clusters will be tighter.

We remark that although all of our experiments in this section were performed with label noise, except for CIFAR-10, we observe the same qualitative sparsification effects for label smoothing. Quantitatively, label smoothing generates denser representations compared to label noise.

7 Conclusion

Throughout this paper we have studied the effects of misclassification during neural network training. Across all the settings we explored, including MNIST and CIFAR-10, label noise or label smoothing induced sparser activation patterns. This even holds over ResNets that are distant from the theoretical setting we explored, as they incorporate convolutional layers, batch normalization, and so on. For MNIST, we note that even without any form of label noise, the activations were sparsified (see Figure 5a). At initialization, the typical fraction of active neurons is 50%, as one would expect, and at the end of training the fraction decreases to 25%. This raises the question, does SGD have some implicit bias towards solutions with sparse activation patterns?
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A Continuity of $a(p)$

Let $A_N(x)$ be the number of active neurons for an input $x$ of a ReLU neural network with one hidden layer $N$ trained in the presence of label noise with noise parameter $p$ over a dataset $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$. Then, $A_N(x)$ depends on several hyperparameters, specifically: the weights initialization $W^{(0)}$, the learning rate $h$, the number of training iterations $T$, the ordering of the training sequence $ord = i_1, \ldots, i_T$ where $1 \leq i_t \leq n$, and the noise parameter $p$. These hyperparameters induce the training sequence $S_p = \{x_{i_1}, c_1 \cdot y_{i_1}, (x_{i_2}, c_2 \cdot y_{i_2}), \ldots, (x_{i_T}, c_T \cdot y_{i_T})\}$ where $c_i = 2b_i - 1$ and $b_i$ are i.i.d. Bernoulli random variables with parameter $1 - p$. Note that the final network $N$ is completely determined by $S_p$.

For a fixed dataset $S$ and fixed $h$ and $T$, we define $a(p)$ for $p \in [0, 1]$ as the typical number of active neurons averaged over all randomness of SGD.

$$a(p) := \mathbb{E}_{S_p} \mathbb{E}_{x \sim U\{x_1, \ldots, x_n\}} A_N(x).$$

For any $p$, we can write

$$a(p) = \mathbb{E}_{S_p} \mathbb{E}_{x \sim U\{x_1, \ldots, x_n\}} A_N(x) = \mathbb{E}_{(b_1, \ldots, b_T)} \mathbb{E}_{ord} \mathbb{E}_{x \sim U\{x_1, \ldots, x_n\}} A_N(x)$$

$$= \mathbb{E}_{(b_1, \ldots, b_T)} \left[ \mathbb{E}_{ord} \mathbb{E}_{x \sim U\{x_1, \ldots, x_n\}} A_N(x) \mid (b_1, \ldots, b_T) \right]$$

$$= \sum_{t=0}^{T} p^{T-t} (1 - p)^t \sum_{\sum_{i=0}^{n} b_i = i} \left[ \mathbb{E}_{ord} \mathbb{E}_{x \sim U\{x_1, \ldots, x_n\}} A_N(x) \mid (b_1, \ldots, b_T) \right]$$

Clearly, $a(p)$ is a polynomial in $p$ and thus continuous in $[0, 1]$.

B Special cases of Theorem 1

Before the general proof, we provide proofs for two special cases, a single neuron and a network with one hidden layer. This provides specific insight for these cases as we explore them in more detail in the paper.

Proof for a single neuron. Let $N(x) = V \cdot x$. Because $y V^{(t)} \cdot x < 0$, the derivative of the loss is $L'(-y V^{(t)} \cdot x) = 1$. So, $V^{(t+1)} = V^{(t)} + h y x$ and

$$\|V^{(t+1)}\|^2 = (V^{(t)} + h y x) \cdot (V^{(t)} + h y x)$$

$$= \|V^{(t)}\|^2 + 2 h y V^{(t)} \cdot x + h^2 \|x\|^2.$$ 

By assumption $y V^{(t)} \cdot x < 0$. Thus, if $h$ is small enough then the norm of $V$ decreases. □

Proof for a network with one hidden layer. Let $N(x) = V \cdot \text{ReLU}(W \cdot x)$. As for a single neuron, $V^{(t+1)} = V^{(t)} + h y \text{ReLU}(W^{(t)} \cdot x)$ and

$$\|V^{(t+1)}\|^2 = \|V^{(t)}\|^2 + 2 h y N^{(t)}(x) + h^2 \|\text{ReLU}(W^{(t)} \cdot x)\|^2.$$ 

The Frobenius norm of $V$ again decreases.

To investigate the norm of $W^{(t)}$, let us consider the $i$-th row $W_i^{(t)}$. By the chain rule/back-propagation, the derivative is non-zero only if $W_i^{(t)} \cdot x > 0$. In words, for the gradient according to $W_i^{(t)}$ to be non-zero, the network must misclassify $x$ and the corresponding neuron $i$ must fire when presented with $x$.

In this case, if $v_i^{(t)}$ is the $i$-th coordinate of $V^{(t)}$, we have $W_i^{(t+1)} = W_i^{(t)} + h y v_i^{(t)} x$ and

$$\|W_i^{(t+1)}\|^2 = (W_i^{(t)} + h y v_i^{(t)} x) \cdot (W_i^{(t)} + h y v_i^{(t)} x)$$

$$= \|W_i^{(t)}\|^2 + 2 h y v_i^{(t)} W_i^{(t)} \cdot x + h^2 (v_i^{(t)})^2 \|x\|^2.$$
We prove the statement for networks without bias terms. We cannot deduce if the norm of \( W_i \) decreases or increases, however, summing over all \( i \) (including non-firing neurons) yields
\[
\| W^{(t+1)} \|^2 = \sum_i \| W_i^{(t+1)} \|^2 
\leq \sum_i \left( \left\| W_i^{(t)} \right\|^2 + 2h y_i \operatorname{ReLU}(W_i^{(t)} \cdot x) + h^2 (v_i^{(t)})^2 \| x \|^2 \right) 
= \| W^{(t)} \|^2 + 2h y_a \operatorname{ReLU}(W^{(t)} \cdot x) + h^2 \| V^{(t)} \|^2 \| x \|^2 
= \| W^{(t)} \|^2 + 2h y N(x) + h^2 \| V^{(t)} \|^2 \| x \|^2 .
\]
The inequality follows since the non-firing neurons do not contribute anything to the middle term in the sum and the last term in the sum is non-negative. We see that although the norm of some \( W_i^{(t)} \) might increase, the norm of \( W^{(t)} \) must decrease for a small enough learning rate.

C Proof of Theorem 1

We prove the statement for networks without bias terms. For the weights in the last layer, the proof is modeled after our discussion of a single neuron case. Let \( V \) be the weights in the last layer and \( a \) denote the activations of the last layer of neurons. Recall that we are considering a sample \((x, y)\) where \( y N(x) = y(a \cdot V) < 0 \) and let \( \ell' := L'(y N(x)) < 0 \). By backpropagation, we have
\[
\frac{\partial L(y N(x))}{\partial V_i} = \ell' y a_i
\]
and accordingly the loss gradient with respect to \( V \) is \( \frac{\partial L(y N(x))}{\partial V} = \ell' y a \). At the same time,
\[
\| V^{(t+1)} \|^2 = \left( V^{(t)} - h \ell' y a \right) \cdot \left( V^{(t)} - h \ell' y a \right) = \| V^{(t)} \|^2 - 2h \ell' y (a \cdot V) + O(h^2) < \| V^{(t)} \|^2,
\]

since \( h \ell' y (a \cdot V) > 0 \).

For the preceding layers, let \( W \) and \( V \) denote two subsequent layers in the network. By the same calculation,
\[
\| W^{(t+1)} \|^2 = \left( W^{(t)} - h \frac{\partial L(y N(x))}{\partial W^{(t)}} \right) \cdot \left( W^{(t)} - h \frac{\partial L(y N(x))}{\partial W^{(t)}} \right) 
= \| W^{(t)} \|^2 - 2h \left( W^{(t)} \cdot \frac{\partial L(y N(x))}{\partial W^{(t)}} \right) + O(h^2) .
\]

On the other hand, consider the gradient flow algorithm with continuous time parameter \( t \) run on our neural network and one sample \((x, y)\). In this gradient flow we have
\[
\frac{d\| W \|^2}{dt} = 2 \left( W \cdot \frac{\partial L(y N(x))}{\partial W} \right) .
\]

However, we can now invoke Corollary 2.1 and Theorem 2.3 of [Du et al., 2018] that state that for all network architectures \( \frac{d\| W \|^2}{dt} = \frac{d\| V \|^2}{dt} \). Applying this together with (4) and (5) for \( V \) and \( W \) and induction we get \( \| W^{(t+1)} \|^2 < \| W^{(t)} \|^2 \) for small enough \( h \).
D Proof of Theorem 2

Proof. Fix an initialization $V(0)$.

Assume (a) $L''(0) = m > 0$. This means that for some neighborhood of $h = 0$ it holds $m_2 := L'(d\sqrt{h}) > L(0) + \frac{2m_2}{d}$ and $m_1 := L'(-d\sqrt{h}) < L(0) - \frac{2m_2}{d}$. We prove the norm of $\|V(t)\|$ will reach below $N = 100 \max\{d\sqrt{M^2/m}, d\sqrt{h}\}$.

Let $(x_1, y_1), (x_2, y_2), \ldots$ be the random instances presented during the optimization. The weights change according to $V(t) = V(t-1) + h't_y V(t-1) \cdot x_t$, where we write $L' = L'(-y_t V(t-1) \cdot x_t)$ for short, and hence the norm $\|V(t)\| = \|V(t-1)\|^2 + 2h'L'y_t V(t-1) \cdot x_t + (h'L' \cdot x_t)^2$.

We denote by $c_t$ and $r_t$ the random variables such that $L'y_t V(t-1) \cdot x_t = c_t \|V(t-1)\|$ and $\sqrt{h}M^2 \|x_t\|^2 = 0.01m r_t \|V(t-1)\|$.

On the one hand,

$$\|V(t)\|^2 - \|V(t-1)\|^2 = 2h'L'y_t (V(t-1) \cdot x_t) + (h'L' \cdot x_t)^2 < 2hc_t \|V(t-1)\| + 0.01h^{3/2}mr_t \|V(t-1)\|^2.$$ 

On the other hand, since $(L - R)^2 \geq 0$ is equivalent to $L^2 - R^2 \geq 2LR - 2R^2 = 2(L - R)R$, we have

$$\|V(t)\|^2 - \|V(t-1)\|^2 \geq 2(\|V(t)\| - \|V(t-1)\|) \|V(t-1)\|.$$ 

Combining the above inequalities yields $\|V(t)\| - \|V(t-1)\| < hc_t + 0.005h^{3/2}mr_t$.

We will now show that, conditioned on $\|V(t-1)\|$, we have $\mathbb{E} c_t < -md\sqrt{h}/4$ and $\mathbb{E} r_t < 1$. In other words, the evolution of the norm is bounded by a negatively biased random walk, with the expectation of the order $-md\sqrt{h}$.

The bound on $\mathbb{E} r_t$ holds because for each $t$ we assume $\|V(t-1)\| > 100d\sqrt{h}M^2/m$ (otherwise we are done), so $\mathbb{E} r_t = \frac{100d\sqrt{h}^2}{m\|V(t-1)\|^2} \mathbb{E} \|x_t\|^2 < \frac{1}{d} \mathbb{E} \|x_t\|^2 = 1$.

For the bound on $\mathbb{E} c_t$, consider that the distribution of $p_t = y_t \frac{V(t-1)}{\|V(t-1)\|} \cdot x_t \sim N(0, 1)$, therefore $P(p_t > 0.01) = P(p_t < -0.01) > 0.49$. Furthermore, recall that $c_t = L'(-\|V(t-1)\|^2) \cdot p_t$.

Since $L'()$ is increasing, by symmetry considerations $\mathbb{E} [c_t \cdot 1(|p_t| \leq 0.01)] < 0$. On the other hand, since we assume $\|V(t-1)\| > 100d\sqrt{h}$ we have

$$\mathbb{E} [c_t \cdot 1(p_t > 0.01)] = \mathbb{E} [L'(-\|V(t-1)\|^2) \cdot p_t \cdot 1(p_t > 0.01)] \leq m_1 \mathbb{E} [p_t \cdot 1(p_t > 0.01)],$$

and similarly $\mathbb{E} [c_t \cdot 1(p_t < -0.01)] \leq m_2 \mathbb{E} [p_t \cdot 1(p_t < -0.01)]$. Putting these inequalities together indeed gives

$$\mathbb{E} [c_t] < -m_2 - m_1 \mathbb{E} [p_t \cdot 1(p_t > 0.01)] < -(m_2 - m_1)/4 \leq -dm\sqrt{h}/4.$$

To complete the proof, we show concentration for our two random variables $r_t$ and $c_t$.

Since $0 \leq r_t < \frac{1}{d} \|x_t\|^2$ and random variables $\|x_t\|^2$ are iid chi-squared with $d$ degrees of freedom, we can apply a standard tail bound for chi-squared distribution (see Lemma 6.1 in Johnstone [2001]):

$$P(\sum_{t=1}^T r_t - T > T) < P(\sum_{t=1}^T \|x_t\|^2 - dT > dT) < \exp(-\frac{dT}{8}).$$

For the concentration of $c_t$, note that $c_t + md\sqrt{h}/4$ is a supermartingale with respect to $V(t)$. Furthermore, $c_t \mid V(t-1)$ is sub-Gaussian (because $p_t \mid V(t-1)$ is a Gaussian). Therefore, we can apply the Azuma concentration inequality for sub-Gaussian differences in Shamir [2011]. Specifically, conditioned on $V(t-1)$, for any $a > 0$ we have

$$\mathbb{P} \left( \left| \sum_{t=1}^T c_t \right| > a \right) \leq \exp\left(-\frac{a^2}{2m^2} \right).$$
\[ \text{Pr}(c_t > a) = \text{Pr}(L'(-\|V^{(t-1)}\|p_t) \cdot p_t > a) < \text{Pr}(p_t > a/M) \leq \exp(-a^2/2M^2) \]

and similarly \( \text{Pr}(c_t < -a) < \exp(-a^2/2M^2) \). Hence, we can apply Theorem 2 from Shamir [2011] and get

\[ P\left( \sum_{i=1}^{T} c_i > -md\sqrt{h}T/4 + eT \right) < \exp\left( -\frac{Tc^2}{64M^2} \right). \]

We choose \( T = \lceil \frac{7\|V^{(0)}\|}{md^2h^3/2} \rceil \) and \( \epsilon = M/T^{0.4} \). Except with probability \( \exp(-\Omega(T^{0.2})) \), for a small enough learning rate, we will have that

\[ \|V^{(T)}\| < \|V^{(0)}\| - \frac{3}{2} \|V^{(0)}\| + M(7\|V^{(0)}\|)^{0.6}h + 0.01 \frac{7\|V^{(0)}\|}{d} < -0.4 \|V^{(0)}\| + O(h^{0.1}), \]

which would be a contradiction, meaning that the norm of \( V^{(t)} \) must have fallen below \( N \) before time \( T \).

For (b), consider \( m_2 := L'(dh) \), \( m_1 := L'(-dh) \), and \( N = 100 \max\{ \frac{dhM^2}{m}, dh \} \), where \( m < m_2 - m_1 \) by definition. The proof now follows in similar lines, only in this case, \( T = \lceil \frac{3\|V^{(0)}\|}{mh} \rceil \).

**Proof.** By orthogonality of \( e_1, \ldots, e_d \), any update in the network when presented with \( e_i \) does not affect the activations corresponding to \( e_j \) for \( j \neq i \). To see this, let \( W^{(t)}_r \) be the \( r \)-th row of \( W \) at time \( t \). Then, \( W^{(t+1)}_r = W^{(t)}_r + \Delta \), so \( W^{(t+1)} \cdot e_j = (W^{(t)} \cdot e_j) + \Delta \cdot e_j = W^{(t)} \cdot e_j \).

This means we can focus only on the first column of \( W = W^{(t)} \) and fix \( x = e_1 \). At time \( t \), let \( P_{os}^{(t)} = \{ 1 \leq i \leq k : W_{i1} > 0 \} \) and \( N_{eg}^{(t)} = \{ k + 1 \leq i \leq 2k : W_{i1} > 0 \} \) be the sets of “live” weights. Note that we have

\[ N^{(t)}(x) = \sum_{i \in P_{os}^{(t)}} W^{(t)}_{i1} - \sum_{i \in N_{eg}^{(t)}} W^{(t)}_{i1}. \]

Observe that \( P_{os}^{(t+1)} \subseteq P_{os}^{(t)} \) and \( N_{eg}^{(t+1)} \subseteq N_{eg}^{(t)} \). Therefore, after a certain amount of time these sets must stabilize, i.e., \( P_{os}^{(t+1)} = P_{os}^{(t)} \) always. Consider a time \( t \) after this point.

Assume that \( N^{(t)} > 0 \), the opposite case being similar. In that case, by the formulas for the gradient, the update happens only when \( y = -1 \) and it decreases \( W_{i1} \) by \( h \) if \( i \in P_{os}^{(t)} \) and increases \( W_{i1} \) by \( h \) if \( i \in N_{eg}^{(t)} \), leaving all other entries of \( W^{(t)} \) unchanged. Let \( \Delta := h \cdot (|P_{os}| + |N_{eg}|) \).

Note that \( \Delta \geq 0 \) and that \( N^{(t+1)} = N^{(t)} - \Delta \).

Now, if \( N^{(t)} \geq \Delta \), then the output of the network decreases exactly by \( \Delta \). On the other hand, if \( N^{(t)} < \Delta \), then also \( |N^{(t+1)}| < \Delta \). Therefore, the value of \( |N^{(t)}| \) must decrease below \( \Delta \) in a finite number of steps and stay there forever.

Since \( |\Delta| \leq 2kh \), the proof is concluded.

**E Proof of Theorem 4**

**Proof.** As in the proof of Theorem 3, we can only focus on \( x = e_1 \).

**Learning rate** \( h \geq 1 \) During training, after encountering \( x \) three times with misclassified labels we will have \( \text{ReLU}(W e_1) = 0 \). To see this, assume w.l.o.g. that at initialization \( N^{(0)}(x) > 0 \). When encountering \( (x, -1) \) for the first time at time \( t \), all the weights associated with elements in \( P_{os}^{(t)} \) will become negative because they lie in \([0, 1]\) and the learning rate \( h \geq 1 \), so \( P_{os}^{(t+1)} = \emptyset \). Also, the weights associated with \( N_{eg}^{(t+1)} \) will now have values in \([1, 2]\). Since \( \{ P_{os}^{(t)} \}_{t=1}^{\infty} \) is a nested sequence, \( P_{os}^{(t)} \) will remain empty for \( i > t \). This means that updates now will occur only when encountering \( (x, 1) \). After two such encounters \( N_{eg}^{(t)} \) will be the empty set and in total \( \text{ReLU}(W x) = 0 \).
Learning rate $h \leq 1/k$ By Hoeffding’s inequality, for $X_1, \ldots, X_n \in [0, 1]$ i.i.d. random variables it holds $P(\sum_{i=1}^n X_i - n\mathbb{E}X_1 > ne) < 2\exp(-2ne^2)$. Then, with high probability (with respect to $k$), all items hold:

1. $k/2 + k^{0.6} > |Pos^{(0)}| > k/2 - k^{0.6}$. Since the probability of a weight to be included in $Pos^{(0)}$ is $1/2$, we have $X_i \sim \text{Bernoulli}(1/2)$, $n = k$, and $\epsilon = \frac{1}{n^{0.6}}$.

2. $k/2 + k^{0.6} > |Neg^{(0)}| > k/2 - k^{0.6}$. Similarly, as above.

3. $3 < |N^{(0)}(x)| < \frac{1}{k}k^{0.6}$. $N^{(0)}(x) = \sum_{i=1}^k X_i - \sum_{i=k+1}^{2k} X_i$ where $X_i$ are i.i.d. random variables such that with probability $1/2$ we have $X_i = 0$ and with probability $1/2$ we have $X_i$ is uniform in $[0, 1]$. For the upper bound, we apply Hoeffding’s inequality for each of the sums with $\epsilon = \frac{1}{k^{0.6}}$. For the lower bound (anti-concentration), we apply Theorem 8 in [Tag2015].

4. $|\{i \mid i \in Pos^{(0)}, W_{i1} < \frac{1}{k^{0.6}}\}| < 2k^{0.6}$. We upper bound the number of small weights $< \frac{1}{k^{0.6}}$ in $Pos^{(0)}$, so we use $X_i \sim \text{Bernoulli}(\frac{1}{k^{0.6}})$ and $\epsilon = \frac{1}{k^{0.6}}$.

5. $|\{i \mid i \in Neg^{(0)}, W_{i1} < \frac{1}{k^{0.6}}\}| < 2k^{0.6}$. Similarly, as above.

Assume $N^{(0)}(x) > 0$, the opposite case being similar. As long as $N^{(t)}(x) > 0$, updates only occur when encountering $(-1, -1)$. In these cases, $N^{(t+1)}(x) = N^{(t)}(x) - h(|Pos^{(t)}| + |Neg^{(t)}|)$. In these updates, $Neg^{(t)}$ does not change. Furthermore, since at each step, each weight in $Pos^{(t)}$ decreases by $h$, then for $t < \frac{1}{kh}$ the total decrease is at most $th < 1/k^{0.4}$, and by item 4 we have $|Pos^{(t)}| > k/2 - 3k^{0.6}$. But then, by item 3, we have $N^{(t)}(x) < k^{0.6}/2 - th(k/2 - 3k^{0.6} + k/2 - k^{0.6})$ and accordingly $N^{(t)} < 0$ for some $t < \frac{1}{kh}$.

From now on, let $t$ be the first time with $N^{(t)} < 0$. By items 1, 2, and 3, it holds that $N^{(t)}(x) > 3 - (k + 2k^{0.6})ht$. Since $h \leq 1/k$, we have $t \geq 2$. By the upper bounds of items 1 and 2, we have $N^{(t)} > -(k + 2k^{0.6})h$.

Now, since $N^{(t+1)} = N^{(t)} + h(|Pos^{(t)}| + |Neg^{(t)}|)$ and by the lower bounds $|Pos^{(t)}|, |Neg^{(t)}| > k/2 - 3k^{0.6}$ we have $N^{(t+1)} = N^{(t)} > 0.9kh$. In this reversed situation, clearly $Pos^{(t+1)} = Pos^{(t)}$. At the same time, also $Neg^{(t+1)} = Neg^{(t)}$ since $t \geq 2$ so each weight from $Neg^{(t)}$ exceeds $2h$. For another update, we will have $Neg^{(t+2)} = Neg^{(t+1)}$. $Pos^{(t+2)} = Pos^{(t+1)}$, so $N^{(t+2)} - N^{(t+1)} > 0.9kh$ and $N^{(t+2)} > 0$. Similarly, from time $t + 2$ on the sets $Pos^{(t)}$ and $Neg^{(t)}$ are stable and the sequence $N^{(t)}$ becomes periodic with a period of 2 and alternating signs.

Finally, since $Pos^{(t)}$ and $Neg^{(t)}$ are stable, we have $|Pos^{(t)}|, |Neg^{(t)}| > k/2 - 3k^{0.6}$ for all $t$ and the vector $\text{ReLU}(Wx)$ indeed has at least $k - o(k)$ non-zero coordinates.

F Label Noise and a Sparse Representation for MNIST

Figure 6 demonstrates how label noise induces a special sparse representation for MNIST when training a one hidden layer neural network with 600 neurons. Figure 6a (without label noise) and Figure 6b (with label noise $p = 0.1$) each show histograms for 30 randomly selected neurons from the hidden layer after training over MNIST. Each histogram shows how many times a neuron fired for each digit ‘0’, ‘1’, ‘9’.

Qualitatively, we can observe more neurons that are associated with a specific digit in Figure 6b compared to Figure 6a. Quantitatively, we say that a neuron is digit associated if it fires at least twice as much for some digit compared to any other digit. For example, in Figure 6a plots 3, 14, 19, 20, 23, 26, and 30 are associated with the digits ‘1’, ‘1’, ‘6’, ‘0’, ‘5’, ‘5’, and ‘1’, respectively; in Figure 6b plots 2, 7, 8, 9, 10, 12, 13, 17, 21, 22, 23, 26, and 29 are associated with the digits ‘5’, ‘6’, ‘1’, ‘6’, ‘7’, ‘7’, ‘9’, ‘1’, ‘3’, ‘6’, ‘3’, ‘0’, and ‘0’, respectively.

The notion of associated digit is reminiscent to the notion of a grandmother cell in neuroscience: a neuron which fires if and only if one thinks of his grandmother. Using our quantification, training
without label noise corresponds to 104 digit associated neurons versus 222 digit associated neurons with label noise $p = 0.1$.

Figure 6: Learning MNIST with and without noisy labels. Every subplot is a histogram for a random neuron which quantifies how many times the neuron fired for each digit in MNIST. When a neuron is digit associated, a red bar highlights the associated digit.