In this paper, it is the first time to construct a complete post-Newtonian (PN) model of a rigid body by means of a new constraint on the mass current density and mass density. In our PN rigid body model most of relations, such as spin vector proportional to the angular velocity, the definition on the moment of inertia tensor, the key relation between the mass quadrupole moment and the moment of inertia tensor, rigid rotating formulae of mass quadrupole moment and the moment of inertia tensor, are just the extension of the main relations in Newtonian rigid body model. When all of $1/c^2$ terms are neglected, the PN rigid body model and the corresponding formulae reduce to Newtonian version. The key relation is obtained in this paper for the first time, which might be very useful in the future application to problems in geodynamics and astronomy.

PACS numbers: 04.25.Nx, 95.10Jk, 91.10Nj
The mass quadrupole moment and the moment of inertia tensor satisfy the key relation

\[ M_{ab} = -I_{ab} + \frac{1}{3} \delta_{ab} I_{cc}. \]  

(2)

By means of the continuity equation \( \partial_T \Sigma + \partial_a \Sigma^a = 0 \), the time derivative of the moment of inertia tensor \( I_{ab} \) is proportional to the angular velocity \( \Omega \)

\[ \dot{I}_{ab} \equiv \frac{d}{dT} I_{ab} = (\epsilon_{apq} I_{qb} + \epsilon_{bpq} I_{aq}) \Omega^p. \]  

(3)

\( \dot{M}_{ab} \) satisfies a similar relation as (3). Therefore, \( I_{ab} \) and \( M_{ab} \) like constant tensors, will rigidly rotate in space with the angular velocity \( \Omega \).

When we discuss PN rigidity, we will use the symbols, notations and conventions following the 1PN theoretical framework presented by Damour, Soffel and Xu (cited below as the DSX scheme [12, 13, 14]) since the DSX scheme is not only rather simple and complete but also describes the 1PN definition of spin in a satisfactory manner. In the DSX scheme a complete 1PN general relativistic celestial mechanics for \( N \) arbitrarily composed and shaped, rotating deformable bodies is described. Here we will briefly summarize the notation and definition in the DSX scheme.

In the post Newtonian expansion we will always abbreviate the order symbol \( O(c^{-n}) \) simply as \( O(n) \). A spatial multi-index containing \( l \) indices is simply denoted by \( L \) (and \( K \) for \( k \) indices, etc), i.e. \( L \equiv i_1 i_2 \cdots i_l \). A multissumation is always understood for repeated multi-indices \( S_l T_L = \sum_i \sum_{i_2} \cdots \sum_i S_{i_1 i_2 \cdots i_l} T_{i_1 i_2 \cdots i_l} \). Given a spatial vector, \( n^i \), its \( l \)th tensorial power is denoted by \( n^i n^j \cdots n^l \). Also, \( \partial_L \equiv \partial_{i_1} \partial_{i_2} \cdots \partial_{i_l} \). Besides angular brackets the symmetric and trace-free (STF) part of a spatial tensor will be denoted by a caret when no ambiguity arises: \( \text{STF}_{i_1 \cdots i_l} (T_{i_1 \cdots i_l}) = T_{i_1 \cdots i_l} - \frac{1}{l!} T_{i_1 \cdots i_l} \). The spatial indices, \( i, j = 1, 2, 3 \) are freely raised or lowered by means of the Cartesian metric \( \delta_{ij} = \delta^{ij} = \text{diag}(+1,+1,+1) \) in Cartesian coordinates. The metric is presented by means of potential \( W \) and vector potential \( W_a \) (see Eq. (4.1) of Ref. [12]). \( W \) and \( W_a \) can be separated into self-part (with a “+”) and external-part (with a bar), i.e. \( W = W^+ + \overline{W} \) and \( W_a = W^+_a + \overline{W}^a \). The self-part \( W^+ \) and \( W^+_a \) will be solved by the gravitational mass density \( \Sigma \) and the mass current density \( \Sigma^a \) as: \( \Sigma \equiv (T^{\mu \nu} + T_{ss})/c^2 \), and \( \Sigma^a \equiv T^{0a}/c \), through the 1PN Einstein field equation and the coordinate conditions (gauge conditions) (see Eq. (4.3) of Ref. [12]), where \( T^{\alpha \beta} \) is the energy-momentum tensor. \( W^+ \) and \( W^+_a \) will be expanded by STF BD mass moments \( M_L \) and STF spin moments \( S_L \) (see Eq. (6.11) of Ref. [12]). The external part \( \overline{W} \) and \( \overline{W}^a \) can be expanded by gravito-electric tidal moments \( G_L \) and gravito-magnetic tidal moments \( H_L \) (see Eq. (4.15) of Ref. [13]). \( G_L \) and \( H_L \) are also STF spatial tensors dependent on time only. BD mass moments \( \tilde{M}_L \) are widely accepted as the best PN mass moments, which have the form

\[ M^A_L(T) \equiv \int_A d^3X \hat{X}^L \Sigma + \frac{1}{2(2l+3)c^2} \int_A d^3X \hat{X}^L \hat{X}^2 \Sigma - \frac{4(2l+1)}{(l+1)(2l+3)c^2} \int_A d^3X \hat{X}^a \hat{X}^a \Sigma^a \ (l \geq 0). \]  

(4)

The PN spin moment has been discussed for a long time [14, 11]. In the DSX scheme, the expression of 1PN spin of body \( A \) is

\[ S^A_{PN} = \epsilon_{abc} \int_A d^3X X^b \left[ \Sigma c^2 \left( 1 + \frac{4}{c^2} W^A \right) - \frac{4}{c^2} \Sigma (W^+_A + \frac{1}{8} \partial_a \partial_T Z^b_A) \right] \\
+ \frac{1}{c^2} \sum_{l \geq 1} \frac{1}{l!} \int_A d^3X \hat{X}^L \hat{X}^2 \left[ H^A_{aL} \hat{N}_L^A - \frac{l}{l+1} M^A_{aL} H^A_L \right] \\
- \epsilon_{abc} \frac{1}{c^2} \sum_{l \geq 1} \frac{1}{l(l+1)(2l+3)} \left[ (l+10) \hat{N}^A_{bL} \hat{G}^A_{cL} \\
+ 8(2l+3) \hat{P}^A_{bL} \hat{G}^A_{cL} - (l+2) \hat{N}^A_{bL} \hat{G}^A_{cL} \right]\right] + O(4), \]  

(5)

where \( Z^+_A \equiv G \int_A d^3X \Sigma^A (T_A, X')/|X - X'| \), \( W^+_A \equiv G \int_A d^3X \Sigma^A (T_A, X')/|X - X'| \), dot means time derivative \( \partial_T \), \( \hat{N}_L \) and \( \hat{P}_L \) are defined as (see Eq. (2.10) of [11]) \( \hat{N}_L \equiv \int_A d^3X \hat{X}^L \Sigma^L \), and \( \hat{P}_L \equiv \int_A d^3X \hat{X}^L \Sigma^a \) respectively. Later we omit to indicate the body label, \( A \), on all quantities. In Ref. [14], \( \hat{N}_L \) and \( \hat{P}_L \) are called “bad moments”. In such a definition of 1PN spin vector, \( S^A_{PN} \) satisfies the rotational equation of motion (see Eq. (3.11) of Ref. [14]). We also
have 1PN continuity equation for $(\Sigma, \Sigma^a)$ (see Eq. (5.6b) of Ref. [12])
\[
\partial_T \Sigma + \partial_a \Sigma^a = \frac{1}{c^2} (\partial_T T^{ab} - \Sigma \partial_T W) + O(4).
\] (6)

Those are the equations which are taken from the DSX scheme and will be used in the following discussion on the PN rigidity.

The definition of the 1PN rigid body has to agree with the Newtonian rigid body when $1/c^2$ terms are neglected. In the 1PN rigid body the angular velocity should be independent of the local coordinate $X^a$ of body $A$. In the DSX scheme we substitute $\Sigma$ and $\Sigma^c$ for the energy-momentum tensor $T^{\alpha\beta}$, therefore the interdependency inside the energy-momentum tensor and gravitational field in Ref. [4, 5, 6] might be replaced by inside $\Sigma$, $\Sigma^c$ and gravitational field. We expect that the interdependency will produce equations similar to (2) and (3) on the PN level. On the PN level it is sufficient to replace $T^{\alpha\beta}$ by $\Sigma$, $\Sigma^c$ and their derivative [17]. Before a further discussion on the rigid body, the rigid BD moments and the rigid 1PN spin vector should be considered. In the rigid body, since $\hat{\Sigma}$ are PN terms we can substitute the Newtonian relations (Eq. 1 and Newtonian continuity equation) for the definitions of $\hat{N}_L$ and $\hat{P}_L$. It is easy to prove
\[
P_{<L>} = -\frac{1}{2l+1} \hat{N}_{<L>}.
\] (7)

**Lemma 1**  The rigid BD mass moments of rigid body $A$ (Eq. 4) can be simplified as
\[
M_L = \int_A d^3X X <L> \left[ \Sigma + \frac{1}{c^2} \left( \frac{l+9}{2(l+1)(2l+3)} \right) X^2 \hat{\Sigma} \right] + O(4).
\] (8)

Proof of Lemma 1.  Starting from Eq. 4 (the definition of BD mass moment) and replacing the third term in the RHS of Eq. 4 by Eq. 7. The prove will be carried out directly. In fact only the relativistic quadrupole moment is interested in, because in the solar system all of the relativistic higher multipole moments are too small to be considered in any modern measurement in the anticipation of the day. Therefore we only take the relativistic quadrupole moments into account:
\[
M_{ab} = \int_A d^3X X <ab> \left( \Sigma + \frac{11}{42} X^2 \hat{\Sigma} \right).
\] (9)

We define the PN part of the mass density $\Sigma_{PN} \equiv \frac{11}{42} X^2 \hat{\Sigma}$ and the total mass density
\[
\Sigma \equiv \Sigma + \Sigma_{PN}.
\] (10)

**Lemma 2**  The rigid PN spin vector of body $A$ (see Eq. 5) can be reduced to
\[
S^c_{aPN} = \epsilon_{abc} \int_A d^3X X^b \left\{ \Sigma^c + \frac{\Sigma}{c^2} \left( \frac{7}{2} \epsilon_{cde} \Omega^d \partial_e Z^+ + \frac{1}{2} \epsilon_{cde} \Omega^d X^f \partial_e Z^+ \right) + \frac{1}{c^2} \sum_{l \geq 0} \frac{l+9}{2(l+1)(2l+3)} \epsilon_{cde} \Omega^d X^{<l>} H_{el} \right\}.
\] (11)

Substituting Eqs. 1 and 7 for the PN part of Eq. 5, integrating by parts, using the surface integration for whole body $A$ to be zero and taking some STF formulae, we obtained Eq. 11.

We define the PN self-part and PN external part of current density as
\[
\Sigma^c_{self} = \Sigma \left( \frac{7}{2} \epsilon_{cde} \Omega^d \partial_e Z^+ + \frac{1}{2} \epsilon_{cde} \Omega^d X^f \partial_e Z^+ \right),
\] (12)
\[
\Sigma^c_{ext} = \sum_{l \geq 0} \frac{1}{l!} \epsilon_{cde} \Omega^d X^c X^{<l>} G_L(T) + \frac{l+10}{2(l+2)(2l+5)} \epsilon_{cde} X^{<d>h} H_{el} \right\}.
\] (13)
Both of $\Sigma_{\text{self}}^c$ and $\Sigma_{\text{ext}}^c$, as well as $\Sigma^c$ itself are spatially compact-supported. When tidal moments $G_L$ and $H_L$ are equal to zero, $\Sigma^c$ and $\Sigma_{\text{self}}^c/c^2$ form the PN self-part of spin vector. We can define

$$\Sigma^c \equiv \Sigma^c + \frac{\Sigma_{\text{self}}^c}{c^2} + \frac{\Sigma_{\text{ext}}^c}{c^2},$$

then Eq. (11) becomes

$$S_{\alpha}^{\text{PN}} = \epsilon_{abc} \int_A d^3 X X^b \Sigma^c.$$  

(15)

By comparing the Newtonian definition of spin with Eq. (15), $\Sigma^c$ is the fully PN quantity.

We add $1/c^2 \left[ \partial_T (\Sigma_{\text{PN}}) + \partial_a (\Sigma_{\text{self}}^a + \Sigma_{\text{ext}}^a) \right]$ to both sides of Eq. (6) and have

$$\partial_T \Sigma + \partial_a \Sigma^a = \frac{1}{c^2} \left[ \partial_T T^{bb} - \Sigma \partial_T W + \partial_T \Sigma_{\text{PN}} + \partial_a (\Sigma_{\text{self}}^a + \Sigma_{\text{ext}}^a) \right].$$

(16)

Now we construct the model of the PN rigid body by constraint on $\Sigma$ and $\Sigma$ to satisfy

$$\Sigma^c + \frac{1}{2c^2} X^a \left[ \partial_T T^{bb} - \Sigma \partial_T W + \partial_T \Sigma_{\text{PN}} + \partial_a (\Sigma_{\text{self}}^a + \Sigma_{\text{ext}}^a) \right] = \epsilon_{abc} \Omega^b X^c \Sigma + O(4).$$

(17)

The relation (Eq.(17)) is our most important assumption for PN rigid body. Considering that $\Sigma^c$ and $\Sigma$ are expressed by $\Sigma^c$ and $\Sigma$, which are related to $T^{\alpha\beta}$ in the DSX scheme, then Eq. (17) is also constraint to $T^{\alpha\beta}$. When $1/c^2$ terms are neglected, Eq. (17) goes to Eq. (1). Later we will see that only in this model PN mass quadrupole moments and the moment of inertia tensors satisfy the similar Newtonian key relations as in Eq. (2). We were not surprised with the appearance of the time derivative of $\Sigma$ in Eq. (17), since in the DSX scheme $T^{\alpha\beta}$ can be fully represented by $\Sigma$ and $\Sigma^a$ and their derivatives without difficulty [17]. In the interdependencies described by Thorne and Gürsel [5] and Klioner [16], they have their own models of the rigid body by another constraint to $T^{\alpha\beta}$. By comparing the constrained equations (to see Eq.(A7) in [5] (or Eq.(7) in [8]) and Eq.(8) in [8]) with Eq. (17), we see that Eq. (17) is more complicated, but still reasonable.

Substituting Eq. (17) for Eq. (15), we obtain the linear relation between the PN spin vector of the rigid body and the angular velocity:

$$S_{\alpha}^{\text{PN}} = I_{ab} \Omega^b + O(4),$$

where the moment of inertia tensor is

$$I_{ab} = I_{ba} = \int_A d^3 X (\delta_{ab} X^2 - X^a X^b) \Sigma + O(4),$$

(19)

in which $\Sigma$ is defined in Eq. (10).

By comparing Eq. (19) with Eq. (9), we have

$$M_{ab} = -I_{ab} + \frac{1}{3} \delta_{ab} I_{cc} + O(4).$$

(20)

Eq. (20) is the key relation between the PN mass quadrupole moment (rigid BD moment) and the PN moment of inertia tensor. It is just this relation that makes the model of the rigid body very useful and applicable on the PN level as shown in the Newtonian case. This is the first time that we have obtained the PN key relation in this paper. Making use of the extended 1PN continuity equation Eq. (10), we immediately have

$$\dot{I}_{ab} = \frac{d}{dT} I_{ab} = (\epsilon_{apq} I_{qb} + \epsilon_{bpq} I_{aq}) \Omega^p + O(4).$$

(21)

PN $\dot{M}_{ab}$ satisfies a similar relation as $\dot{I}_{ab}$. From Eq. (21) the behavior of the PN $I_{ab}$ (and also PN $M_{ab}$) in our rigid model are just like the Newtonian version (Eq. (13)), i.e. $I_{ab}$ and $M_{ab}$ rigidly rotate as a whole.

Eq. (21) means that we always can introduce a rotation matrix $P_{ia}(T)$, which is a time-dependent orthogonal matrix and transform the PN reference system (RS) to the corotating reference system with rigid body (RS+). $P_{ia}(T)$ can be
constructed by the rotational angular velocity $\Omega^a$ of rigid body according to the relation: $\Omega^a(T) = \frac{1}{2} \epsilon_{abc} P_{ib}(T) \dot{P}_{ic}(T)$ (18). In the new corotating coordinates we get

$$\frac{d\tilde{I}_{ij}}{dT} = O(4),$$

(22)

where $\tilde{I}_{ij} = P_{ia} P_{jb} I_{ab}$. $P_{ia}$ satisfies the following relations: $P_{ia} P_{ja} = \delta_{ij}$, $P_{ia} P_{ib} = \delta_{ab}$ and $dP_{ia}/dT = \epsilon_{abc} \Omega^b P_{ic}$ (here we use $\Omega^b$ to substitute for $\omega^j$ in (18)). The proof is easy by means of (21). Eq. (22) shows it is possible to introduce the PN Tisserand reference system.

At last, we should emphasize that the calculation of the PN moment of inertia tensor Eq. (19) is not too difficult, although the constraint relation on $\Sigma^c$ and $\Sigma$ in Eq. (17) in the model of the PN rigid body is complicated. In practical problems, from our PN rigid spin, the PN moment of inertia tensor (Eqs. (18) – (20)) it is possible to define the three principal axes of the body, the spin axis, rotation axis and figure axis as described in the Newtonian theory, which we will discuss in a separate paper in the future.

In conclusion, the rigid BD (PN) mass multipole moments Eq. (8) and the rigid PN spin moment Eq. (15) are discussed in this paper. We successfully have constructed a new PN model of the rigid body in which the constraint on $\Sigma^c$ and $\Sigma$ satisfies Eq. (17). Our PN rigid body model will reduce to the Newtonian one when all of $1/c^2$ terms are neglected. Most of relations in our PN rigid body model, such as the spin vector proportional to the angular velocity $\Omega$ (Eq. (18)), the definition on the moment of inertia tensor (Eq. (19)), the key relation between the mass quadrupole moment and the moment of inertia tensor (Eq. (20)), the rigidly rotating formulae of $I_{ab}$ and $M_{ab}$ (to see Eq. (21)) are similar to the Newtonian rigid body model where the corresponding relations are mentioned at the beginning of this paper. Especially, the PN key relation between $M_{ab}$ and $I_{ab}$ might be applied to the practical problems in geodynamics and astronomy in the future, e.g. the discussion on the relativistic effects of the nutation and precession.

This work was supported by the National Natural Science Foundation of China (Grant Nos. 10273008 and 10233020). We would like to thank Prof. T.-Y. Huang for his helpful discussion.

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