THE ONE-DIMENSIONAL LINE SCHEME OF A CERTAIN FAMILY OF QUANTUM \( \mathbb{P}^3 \)S

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Abstract. A quantum \( \mathbb{P}^3 \) is a noncommutative analogue of a polynomial ring on four variables, and, herein, it is taken to be a regular algebra of global dimension four. It is well known that if a generic quadratic quantum \( \mathbb{P}^3 \) exists, then it has a point scheme consisting of exactly twenty distinct points and a one-dimensional line scheme. In this article, we compute the line scheme of a family of algebras whose generic member is a candidate for a generic quadratic quantum \( \mathbb{P}^3 \). We find that, as a closed subscheme of \( \mathbb{P}^5 \), the line scheme of the generic member is the union of seven curves; namely, a nonplanar elliptic curve in a \( \mathbb{P}^3 \), four planar elliptic curves and two nonsingular conics.

Introduction

A regular algebra of global dimension \( n \) is often viewed as a noncommutative analogue of a polynomial ring on \( n \) variables. Generalizing the language in [1], such an algebra is sometimes called a quantum \( \mathbb{P}^{n-1} \). In [2], quantum \( \mathbb{P}^2 \)'s were classified according to their point schemes, with the point scheme of the most generic quadratic quantum \( \mathbb{P}^2 \) depicted by an elliptic curve in \( \mathbb{P}^2 \).

Consequently, a similar description is desired for quadratic quantum \( \mathbb{P}^3 \)'s using their point schemes or their line schemes, where the definition of line scheme was given in [11]. However, to date, very few line schemes of quadratic quantum \( \mathbb{P}^3 \)'s are known, especially of algebras that are candidates for generic quadratic quantum \( \mathbb{P}^3 \)'s. As explained in [14], if a generic quadratic quantum \( \mathbb{P}^3 \) exists, then it has a point scheme consisting of exactly twenty distinct points and a one-dimensional line scheme. Hence, in this article, we compute the line scheme of a family

\(^*\)This work was supported in part by NSF grants DMS-0900239 and DMS-1302050.
of algebras that appeared in [3, §5], and whose generic member is a candidate for a generic quadratic quantum \( \mathbb{P}^3 \).

The article is outlined as follows. Section 1 begins with some definitions, including the introduction of the family of algebras considered herein. The point schemes of the algebras are computed in Section 2 in Proposition 2.2, whereas Sections 3 and 4 are devoted to the computation of the line scheme and identifying the lines in \( \mathbb{P}^3 \) to which the points of the line scheme correspond. In particular, our main results are Theorems 3.1, 3.3 and 4.1. In the first two, we prove that the line scheme of the generic member is the union of seven curves; namely, a nonplanar elliptic curve in a \( \mathbb{P}^3 \) (a spatial elliptic curve), four planar elliptic curves and two nonsingular conics. In Theorem 4.1, we find that if \( p \) is one of the generic points of the point scheme, then there are exactly six distinct lines of the line scheme that pass through \( p \). An Appendix is provided in Section 5 that lists polynomials that are used throughout the article.

It is hoped that data from the one-dimensional line scheme of any potentially generic quadratic quantum \( \mathbb{P}^3 \) will motivate conjectures and future research in the subject. In fact, the results herein suggest that the line scheme of the most generic quadratic quantum \( \mathbb{P}^3 \) is conceivably the union of two spatial elliptic curves and four planar elliptic curves (see Conjecture 4.2).

1. The Algebras

In this section, we introduce the algebras from [3, §5] that are considered in this article.

Throughout the article, \( k \) denotes an algebraically closed field and \( M(n, k) \) denotes the vector space of \( n \times n \) matrices with entries in \( k \). If \( V \) is a vector space, then \( V^\times \) will denote the nonzero elements in \( V \), and \( V^\ast \) will denote the vector-space dual of \( V \). In this section, we take \( \text{char}(k) \neq 2 \), but, in Sections 3 and 4, we assume \( \text{char}(k) = 0 \) owing to the computations in those sections.

**Definition 1.1.** [3, §5] Let \( \gamma \in k^\times \) and write \( A(\gamma) \) for the \( k \)-algebra on generators \( x_1, \ldots, x_4 \) with defining relations:

\[
x_4x_1 = ix_1x_4, \quad x_3x_1 = x_1x_3 - x_2^2, \quad x_2x_3 = ix_2x_3, \quad x_4x_2 = x_2x_4 - \gamma x_1^2,
\]

where \( i^2 = -1 \).

By construction of \( A(\gamma) \) in [3], \( A(\gamma) \) is a regular noetherian domain of global dimension four with Hilbert series the same as that of the polynomial ring on four variables. As remarked in [3], the special member \( A(1) \) was studied in [10] and, if \( \gamma^2 \neq 4 \), then \( A(\gamma) \) has a finite point scheme consisting of twenty distinct points and a one-dimensional line scheme. Since the computation of the point scheme was omitted from [3], we will outline the computation of it in Section 2.
It should be noted that $\mathcal{A}(\gamma) \cong \mathcal{A}(-\gamma)$, for all $\gamma \in k^\times$, under the map that sends $x_2 \mapsto -x_2$ and $x_k \mapsto x_k$ for all $k \neq 2$. There also exist antiautomorphisms of $\mathcal{A}(\gamma)$ defined by

$$
\psi_1 : x_1 \leftrightarrow x_3 \quad \text{and} \quad x_2 \leftrightarrow x_4,
$$

$$
\psi_2 : x_2 \leftrightarrow \lambda x_3 \quad \text{and} \quad x_4 \leftrightarrow \lambda x_1,
$$

where $\lambda \in k^\times$ with $\lambda^4 = \gamma$. These latter maps will be useful in Sections 3 and 4.

The reader should note that the point scheme given in [10] for $\mathcal{A}(1)$ has some sign errors in the formulae. Moreover, $\mathcal{A}(1)$ was studied in [7] in the context of finding the scheme of lines associated to each point of the point scheme.

For background material on point modules, line modules, point schemes, line schemes, regular algebras and some of the historical development of the subject, the reader is referred to [14].

2. The Point Scheme of $\mathcal{A}(\gamma)$

In this section, we compute the point scheme of the algebras $\mathcal{A}(\gamma)$ given in Definition 1.1. Our method follows that of [2], and we continue to assume that $\text{char}(k) \neq 2$ in this section.

Let $V = \sum_{i=1}^4 \mathbb{K}x_i$. Following [2], we write the relations of $\mathcal{A}(\gamma)$ in the form $Mx = 0$, where $M$ is a $6 \times 4$ matrix and $x$ is the column vector given by $x^T = (x_1, \ldots, x_4)$. Thus, we may take $M$ to be the matrix

$$
M = \begin{bmatrix}
    x_4 & 0 & 0 & -ix_1 \\
    0 & x_3 & -ix_2 & 0 \\
    x_1 & 0 & -ix_3 & 0 \\
    0 & x_2 & 0 & -ix_4 \\
    x_3 & x_2 & -ix_1 & 0 \\
    \gamma x_1 & x_4 & 0 & -x_2
\end{bmatrix},
$$

and, by [2], the point scheme of $\mathcal{A}(\gamma)$ can be identified with the zero locus, $p(\gamma)$, in $\mathbb{P}(V^*)$ of all the $4 \times 4$ minors of $M$. Fifteen polynomials given by these minors are listed in Section 5.1 in the Appendix. We will prove that, if $\gamma^2 \neq 4$, then $p(\gamma)$ is finite with twenty distinct points.

Let $p = (\alpha_1, \ldots, \alpha_4) \in p(\gamma)$. If $\alpha_1 = 0$, then it is straightforward to prove that $p$ is one of the points $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$. Thus, we assume $\alpha_1 = 1$. If, in addition, $\alpha_4 = 0$, then $\text{rank}(M) = 0$ if and only if $\alpha_2 = 0 = \alpha_3$, so we obtain the point $e_1 = (1, 0, 0, 0)$. Hence, we may assume $\alpha_1 = 1$ and $\alpha_4 \neq 0$.

With this assumption, a computer-algebra program such as Wolfram’s Mathematica yields three polynomials that determine the remaining closed points in $p(\gamma)$:

$$
\rho_1 = x_4^8 - 4x_4^4 + \gamma^2, \quad \rho_2 = x_3^2 - ix_3x_4^2 - 1, \quad \rho_3 = \gamma x_2 - 2ix_3^3 + x_3x_4^5.
$$

(In fact, 5.1.1, 5.1.2 and 5.1.5 evaluated at $x_1 = 1$ generate the other polynomials in Section 5.1 evaluated at $x_1 = 1$, and determine $\rho_1$, $\rho_2$, $\rho_3$.) Since $\rho_1 = 0$ if and only if $(x_4^4 - 2)^2 = 4 - \gamma^2$, we find that $\rho_1$ has eight distinct zeros if and only if $\gamma^2 \neq 4$; if $\gamma^2 = 4$, then $\rho_1$ has exactly four
distinct zeros, each of multiplicity two. Given a zero \( x_4 \) to \( \rho_1 \), the equation \( \rho_2 = 0 \) has a unique solution for \( x_3 \) if and only if \( x_4^4 = 4 \), but this implies \( \rho_1 \neq 0 \) as \( \gamma \neq 0 \), which is false; hence \( \rho_2 \) has two distinct zeros for all \( \gamma \in k^\times \).

The following remark will be useful in the proof of Proposition 2.2.

**Remark 2.1.** (cf., [14]) If the zero locus \( Z \) of the defining relations of a quadratic algebra on four generators with six defining relations is finite, then \( Z \) consists of twenty points counted with multiplicity.

**Proposition 2.2.** Let \( A(\gamma) \) and \( p(\gamma) \) be as above and let \( Z_\gamma \) denote the scheme of zeros of \( \rho_1, \rho_2, \rho_3 \) in \( \mathbb{P}(V^*) \).

(a) For every \( \gamma \in k^\times \), \( p(\gamma) = \{e_1, \ldots, e_4\} \cup Z_\gamma \).

(b) If \( \gamma^2 \neq 4 \), then \( p(\gamma) \) has exactly twenty distinct points.

(c) If \( \gamma^2 = 4 \), then \( p(\gamma) \) has exactly twelve distinct points; the eight closed points of \( Z_\gamma \) have multiplicity two in \( p(\gamma) \) and the remaining four points of \( p(\gamma) \) each have multiplicity one.

(d) For every \( \gamma \in k^\times \), the closed points in \( \mathbb{P}(V^*) \times \mathbb{P}(V^*) \) on which the defining relations of \( A(\gamma) \) vanish are given by: \((e_1, e_2), (e_2, e_1), (e_3, e_4), (e_4, e_3)\) and points of the form \( ((1, \alpha_2, \alpha_3, \alpha_4), (1, i\alpha_2\alpha_3^{-2}, \alpha_3^{-1}, -i\alpha_4)) \), where \((1, \alpha_2, \alpha_3, \alpha_4) \in Z_\gamma \) and \( i^2 = -1 \).

**Proof.** The preceding discussion proves that if \( \gamma^2 \neq 4 \), then the number of distinct closed points in \( p(\gamma) \) is twenty, so, by Remark 2.1, (b) follows. On the other hand, if \( \gamma^2 = 4 \), then the zeros of \( \rho_1 \) have multiplicity two, so, counting multiplicity, the eight distinct points in \( Z_\gamma \) have multiplicity two. Thus, each \( e_i \) has multiplicity one, by Remark 2.1. Hence, (c) and (a) follow. Part (d) is easily verified by computation with the matrix \( M \) using polynomials 5.1.1, 5.1.2 and 5.1.10 in the Appendix.

**Corollary 2.3.** For all \( \gamma \in k^\times \), there exists an automorphism \( \sigma : p(\gamma) \to p(\gamma) \) which, on closed points, is defined by:

\[
e_1 \leftrightarrow e_2, \quad e_3 \leftrightarrow e_4,
\]

\[
\sigma((1, \alpha_2, \alpha_3, \alpha_4)) = (1, i\alpha_2\alpha_3^{-2}, \alpha_3^{-1}, -i\alpha_4)
\]

for all \((1, \alpha_2, \alpha_3, \alpha_4) \in Z_\gamma \). Hence, on the closed points of \( p(\gamma) \), \( \sigma \) has two orbits of length two and \( n \) orbits of length four, where \( n = 4 \) if \( |Z_\gamma| = 16 \) and \( n = 2 \) if \( |Z_\gamma| = 8 \).

**Proof.** The fact the map exists on the closed points of \( p(\gamma) \) is a consequence of Proposition 2.2(d); its existence on the scheme follows from [9, Theorem 4.1.3]. The size of the orbits may be verified by computation.
3. The Line Scheme of $A(\gamma)$

In this section, we compute the line scheme $\mathcal{L}(\gamma)$ of the algebras $A(\gamma)$ as a closed subscheme of $\mathbb{P}^5$. Our arguments follow the method given in [12], which is summarized below in Section 3.1. In Section 3.2, we compute the closed points of the line scheme, and, in Section 3.3, we prove that the line scheme is a reduced scheme, and so is given by its closed points. The main results of this section are Theorems 3.1 and 3.3. Henceforth, we assume that $\text{char}(k) = 0$.

3.1. Method.

In [12], a method was given for computing the line scheme of any quadratic algebra on four generators that is a domain and has Hilbert series the same as that of the polynomial ring on four variables. In this subsection, we summarize that method while applying it to $A(\gamma)$; further details may be found in [12].

The first step in the process is to compute the Koszul dual of $A(\gamma)$. This produces a quadratic algebra on four generators with ten defining relations. One then rewrites those ten relations in the form of a matrix equation similar to that used in Section 2; in this case, however, it yields the equation $\hat{M}z = 0$, where $z^T = (z_1, \ldots, z_4)$ (where $\{z_1, \ldots, z_4\}$ is the dual basis in $V^*$ to $\{x_1, \ldots, x_4\}$) and $\hat{M}$ is a $10 \times 4$ matrix whose entries are linear forms in the $z_i$.

One then produces a $10 \times 8$ matrix from $\hat{M}$ by concatenating two $10 \times 4$ matrices, the first of which is obtained from $\hat{M}$ by replacing every $z_i$ in $\hat{M}$ by $u_i \in k$, and the second is obtained from $\hat{M}$ by replacing every $z_i$ in $\hat{M}$ by $v_i \in k$, where $(u_1, \ldots, u_4), (v_1, \ldots, v_4) \in \mathbb{P}^3$. For $A(\gamma)$, this process yields the following $10 \times 8$ matrix:

$$
\mathcal{M}(\gamma) = \begin{bmatrix}
0 & u_1 & 0 & 0 & 0 & v_1 & 0 & 0 \\
u_2 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 \\
0 & 0 & 0 & u_3 & 0 & 0 & 0 & v_3 \\
0 & 0 & u_4 & 0 & 0 & v_4 & 0 & 0 \\
v_3 & 0 & u_1 & 0 & v_3 & 0 & v_1 & 0 \\
0 & u_4 & 0 & u_2 & 0 & v_4 & 0 & v_2 \\
-u_4 & 0 & 0 & iu_1 & -v_4 & 0 & 0 & iv_1 \\
0 & -u_3 & iu_2 & 0 & 0 & -v_3 & iv_2 & 0 \\
u_1 & 0 & u_3 & \gamma u_2 & v_1 & 0 & v_3 & \gamma v_2 \\
u_2 & u_1 & u_4 & 0 & v_2 & v_1 & v_4 & 0
\end{bmatrix}.
$$

Each of the forty-five $8 \times 8$ minors of $\mathcal{M}(\gamma)$ is a bihomogeneous polynomial of bidegree $(4, 4)$ in the $u_i$ and $v_i$, and so each such minor is a linear combination of products of polynomials of the form $N_{ij} = u_i v_j - u_j v_i$, where $1 \leq i < j \leq 4$. Hence, $\mathcal{M}(\gamma)$ yields forty-five quartic polynomials in the six variables $N_{ij}$. Following [12], one then applies the map:

$$
N_{12} \mapsto M_{34}, \quad N_{13} \mapsto -M_{24}, \quad N_{14} \mapsto M_{23}, \\
N_{23} \mapsto M_{14}, \quad N_{24} \mapsto -M_{13}, \quad N_{34} \mapsto M_{12},
$$
to the polynomials, which yields forty-five quartic polynomials in the Plücker coordinates $M_{ij}$ on $\mathbb{P}^5$.

The line scheme $\mathcal{L}(\gamma)$ of $\mathcal{A}(\gamma)$ may be realised in $\mathbb{P}^5$ as the scheme of zeros of these forty-five polynomials in the $M_{ij}$ together with the Plücker polynomial $P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$.

For $\mathcal{A}(\gamma)$, these polynomials were found by using Wolfram’s Mathematica and are listed in Section 5.2 of the Appendix.

In the remainder of this section, we compute and describe $\mathcal{L}(\gamma)$ as a subscheme of $\mathbb{P}^5$. The lines in $\mathbb{P}(V^*)$ that correspond to the points of $\mathcal{L}(\gamma)$ are described in Section 4.

3.2. Computing the Closed Points of the Line Scheme.

Our procedure in this subsection focuses on finding the closed points of the line scheme $\mathcal{L}(\gamma)$ of $\mathcal{A}(\gamma)$; in the next subsection, we will prove that $\mathcal{L}(\gamma)$ is reduced and so is given by its closed points. We denote the variety of closed points of $\mathcal{L}(\gamma)$ by $\mathcal{L}'(\gamma)$ and the zero locus of a set $S$ of polynomials by $\mathcal{V}(S)$.

Subtracting the polynomials 5.2.18 and 5.2.19 produces $M_{14}M_{23}M_{24}$. If $M_{14} = M_{23} = M_{24} = 0$, then $M_{12} = 0 = M_{34}$, so there is a unique solution in this case. This leaves six cases to consider:

\begin{align*}
\text{(I)} & \ M_{14}M_{23} \neq 0, \ M_{24} = 0, & \text{(IV)} & \ M_{23} \neq 0, \ M_{14} = 0 = M_{24}, \\
\text{(II)} & \ M_{23}M_{24} \neq 0, \ M_{14} = 0, & \text{(V)} & \ M_{14} \neq 0, \ M_{23} = 0 = M_{24}, \\
\text{(III)} & \ M_{14}M_{24} \neq 0, \ M_{23} = 0, & \text{(VI)} & \ M_{24} \neq 0, \ M_{14} = 0 = M_{23}.
\end{align*}

We will outline the analysis for (I), (II), (IV) and (VI); the other cases follow from these four cases by using the map $\psi_1$ defined in Section 1. In applying the map $\psi_1$, the reader should recall that $M_{ji} = -M_{ij}$ for all $i \neq j$.

\textbf{Case (I):} $M_{14}M_{23} \neq 0$ and $M_{24} = 0$.

With the assumption that $M_{24} = 0$, a computation of a Gröbner basis yields several polynomials, one of which is $M_{13}^2M_{14}M_{23}$. Hence, $M_{13} = 0$, and another computation of a Gröbner basis yields several polynomials, two of which are:

\[ M_{14}M_{23} + M_{12}M_{34}, \]

\[ M_{34}^4 - M_{14}^2M_{34}^2 - M_{23}^2M_{34}^2 + \gamma M_{14}M_{23}M_{34}^2 + M_{14}^2M_{23}, \]

so that, in particular, $M_{12}M_{34} \neq 0$. Using the first polynomial to substitute for $M_{14}M_{23}$, and using the assumption that $M_{34} \neq 0$, we find that the second polynomial vanishes if and only if $M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2 = 0$. Another computation of a Gröbner basis yields only these polynomials, so that this case provides the component

\[ \mathcal{L}_1 = \mathcal{V}(M_{13}, M_{24}, M_{14}M_{23} + M_{12}M_{34}, M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2). \]
In Theorem 3.1, we will prove that $\mathfrak{L}_1$ is irreducible if and only if $\gamma^2 \neq 16$. Here we show that if $\gamma^2 = 16$, then $\mathfrak{L}_1$ is the union of two nonsingular conics. Since $\mathcal{A}(4) \cong \mathcal{A}(-4)$, it suffices to consider $\gamma = 4$. In fact, let $\alpha \in \mathbb{k}$ and let

$$Q = M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2 + 2\alpha(M_{14}M_{23} + M_{12}M_{34}),$$

and associate to $Q$ the symmetric matrix

$$\begin{bmatrix}
1 & 0 & 0 & \alpha \\
0 & -1 & \alpha + \frac{\gamma}{2} & 0 \\
0 & \alpha + \frac{\gamma}{2} & -1 & 0 \\
\alpha & 0 & 0 & 1
\end{bmatrix},$$

which has rank at most two if and only if $Q$ factors. This happens if and only if $(\gamma, \alpha) = (\pm 4, \mp 1)$. It follows that if $\gamma = 4$, then

$$Q = (M_{12} - M_{34} + M_{14} - M_{23})(M_{12} - M_{34} - M_{14} + M_{23}),$$

and $\mathfrak{L}_1 = \mathfrak{L}_{1a} \cup \mathfrak{L}_{1b}$, where

$$\mathfrak{L}_{1a} = \mathcal{V}(M_{13}, M_{24}, M_{14}M_{23} + M_{12}M_{34}, M_{12} + M_{14} - M_{23} - M_{34}),$$

$$\mathfrak{L}_{1b} = \mathcal{V}(M_{13}, M_{24}, M_{14}M_{23} + M_{12}M_{34}, M_{12} - M_{14} + M_{23} - M_{34}),$$

and each of $\mathfrak{L}_{1a}$ and $\mathfrak{L}_{1b}$ is a nonsingular conic, since using the last polynomial in each case to substitute for $M_{12}$ in $M_{14}M_{23} + M_{12}M_{34}$ yields a rank-3 quadratic form in each case. Moreover, $\mathfrak{L}_{1b}$ is $\psi_1$ applied to $\mathfrak{L}_{1a}$.

**Case (II):** $M_{23}M_{24} \neq 0$ and $M_{14} = 0$.

With the assumption that $M_{14} = 0$, a computation of a Gröbner basis yields several polynomials, two of which are $M_{13}M_{23}M_{34}^2$ and $M_{23}M_{24}M_{34}^2$. Hence, $M_{13} = M_{34} = 0$. With these additional criteria, another computation of a Gröbner basis yields exactly three polynomials: $M_{12}f$, $M_{23}f$, $M_{24}f$, where $f = M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2$. Thus, $f = 0$. It follows that this case yields the irreducible component

$$\mathfrak{L}_2 = \mathcal{V}(M_{13}, M_{14}, M_{34}, M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2)$$

of $\mathfrak{L}'(\gamma)$.

**Case (III):** $M_{14}M_{24} \neq 0$ and $M_{23} = 0$.

This case is computed by applying $\psi_1$ to case (II), giving

$$\mathfrak{L}_3 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{34}^3 - M_{14}^2M_{34} + iM_{14}M_{24}^2).$$

**Case (IV):** $M_{23} \neq 0$ and $M_{14} = 0 = M_{24}$.

If, additionally, $M_{12} \neq 0$, then $M_{13} = 0$ and $M_{i4} = 0$ for all $i = 1, 2, 3$. It follows that
\(M_{12}^2 = M_{23}^2\), and so these assumptions yield a subvariety of \(\mathfrak{L}_2\). Hence, we may assume that \(M_{12} = 0\). It follows that this case yields the irreducible component

\[
\mathfrak{L}_4 = \mathcal{V}( M_{12}, M_{14}, M_{24}, M_{23}^2 M_{34} + i\gamma M_{13}^2 M_{23} - M_{34}^2 )
\]
of \(\mathcal{L}'(\gamma)\), so \(\mathfrak{L}_4\) is \(\psi_2\) applied to \(\mathfrak{L}_2\).

**Case (V):** \(M_{14} \neq 0\) and \(M_{23} = 0 = M_{24}\).

This case is computed by applying \(\psi_1\) to case (IV), giving the irreducible component

\[
\mathfrak{L}_5 = \mathcal{V}( M_{23}, M_{24}, M_{34}, M_{12} M_{14}^2 - i\gamma M_{13}^2 M_{14} - M_{12}^3 )
\]
of \(\mathcal{L}'(\gamma)\), which is also \(\psi_2\) applied to \(\mathfrak{L}_3\).

**Case (VI):** \(M_{24} \neq 0\) and \(M_{14} = 0 = M_{23}\).

Using \(M_{14} = 0 = M_{23}\), a computation of a Gröbner basis yields several polynomials, one of which is \(M_{12} M_{34} - M_{13} M_{24}\) whereas the others are multiples of \(M_{12}^2 + M_{34}^2\). In particular, two of those polynomials are: \(M_{12} M_{24}(M_{12}^2 + M_{34}^2)\) and \(M_{34}^2 (M_{12}^2 + M_{34}^2)\). It follows that \(M_{12}^2 + M_{34}^2 = 0\), so that this case yields the component \(\mathfrak{L}_6 = \mathfrak{L}_{6a} \cup \mathfrak{L}_{6b}\) of \(\mathcal{L}'(\gamma)\), where

\[
\mathfrak{L}_{6a} = \mathcal{V}( M_{14}, M_{23}, M_{12} M_{34} - M_{13} M_{24}, M_{12} + iM_{34} ),
\]

\[
\mathfrak{L}_{6b} = \mathcal{V}( M_{14}, M_{23}, M_{12} M_{34} - M_{13} M_{24}, M_{12} - iM_{34} ),
\]
and each of \(\mathfrak{L}_{6a}\) and \(\mathfrak{L}_{6b}\) is a nonsingular conic, since using \(M_{12} \pm iM_{34}\) to substitute for \(M_{12}\) in \(M_{12} M_{34} - M_{13} M_{24}\) yields a rank-3 quadratic form in each case. Moreover, \(\mathfrak{L}_{6b}\) is \(\psi_1\) applied to \(\mathfrak{L}_{6a}\).

Having completed this analysis, we can see that the point \(\mathcal{V}( M_{12}, M_{14}, M_{23}, M_{24}, M_{34} )\), that was found earlier, is contained in \(\mathfrak{L}_4 \cap \mathfrak{L}_5 \cap \mathfrak{L}_6\). We summarize the above work in the next result.

**Theorem 3.1.** Let \(\mathcal{L}'(\gamma)\) denote the reduced variety of the line scheme \(\mathcal{L}(\gamma)\) of \(\mathcal{A}(\gamma)\). If \(\gamma^2 \neq 16\), then \(\mathcal{L}'(\gamma)\) is the union, in \(\mathbb{P}^5\), of the following seven irreducible components:

(I) \(\mathfrak{L}_1 = \mathcal{V}( M_{12}, M_{24}, M_{14} M_{23} + M_{12} M_{34}, M_{12}^2 + M_{34}^2 + \gamma M_{14} M_{23} - M_{14}^2 - M_{23}^2\), which is a nonplanar elliptic curve in \(\mathbb{P}^3\).

(II) \(\mathfrak{L}_2 = \mathcal{V}( M_{13}, M_{14}, M_{34}, M_{12}^3 - M_{12} M_{23}^2 - iM_{23} M_{24}^2\), which is a planar elliptic curve.

(III) \(\mathfrak{L}_3 = \mathcal{V}( M_{12}, M_{13}, M_{23}, M_{34} - M_{14} M_{34} + iM_{14} M_{24}\), which is a planar elliptic curve.

(IV) \(\mathfrak{L}_4 = \mathcal{V}( M_{12}, M_{14}, M_{24}, M_{23} M_{34} + i\gamma M_{13}^2 M_{23} - M_{34}^2\), which is a planar elliptic curve.

(V) \(\mathfrak{L}_5 = \mathcal{V}( M_{23}, M_{24}, M_{12} M_{14}^2 - i\gamma M_{13} M_{14} - M_{12}^2\), which is a planar elliptic curve.

(VIa) \(\mathfrak{L}_{6a} = \mathcal{V}( M_{14}, M_{23}, M_{12} M_{34} - M_{13} M_{24}, M_{12} + iM_{34}\), which is a nonsingular conic.

(VIb) \(\mathfrak{L}_{6b} = \mathcal{V}( M_{14}, M_{23}, M_{12} M_{34} - M_{13} M_{24}, M_{12} - iM_{34}\), which is a nonsingular conic.
If $\gamma = 4$, then $\mathcal{L}'(\gamma)$ is the union, in $\mathbb{P}^5$, of eight irreducible components, six of which are $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, $\mathcal{L}_5$, $\mathcal{L}_{6a}$, $\mathcal{L}_{6b}$ (as above) and two of which are
\[
\mathcal{L}_{1a} = \mathcal{V}(M_{13}, M_{24}, M_{14}M_{23} + M_{12}M_{34}, M_{12} + M_{14} - M_{23} - M_{34}),
\]
\[
\mathcal{L}_{1b} = \mathcal{V}(M_{13}, M_{24}, M_{14}M_{23} + M_{12}M_{34}, M_{12} - M_{14} + M_{23} - M_{34}),
\]
which are nonsingular conics.

**Proof.** The polynomials were found in the preceding work, as was the geometric description for $\mathcal{L}_{1a}$, $\mathcal{L}_{1b}$, $\mathcal{L}_{6a}$ and $\mathcal{L}_{6b}$, so here we discuss only the geometric description of the other components.

(I) Write $q_1 = M_{14}M_{23} + M_{12}M_{34}$ and $q_2 = M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2$ viewed in $\mathbb{k}[M_{12}, M_{14}, M_{23}, M_{34}]$. Since
\[
q_2 = M_{12}^2 - (\gamma/2)M_{12}M_{34} + M_{34}^2 - (M_{14}^2 - (\gamma/2)M_{14}M_{23} + M_{23}^2)
\]
modulo $q_1$, and since $\text{char}(\mathbb{k}) \neq 2$, we may take the Jacobian matrix of this system of two polynomials to be the $2 \times 4$ matrix
\[
\begin{bmatrix}
M_{34} & M_{23} & M_{14} & M_{12} \\
2M_{12} - (\gamma/2)M_{34} & -(2M_{14} - (\gamma/2)M_{23}) & -(2M_{23} - (\gamma/2)M_{14}) & 2M_{34} - (\gamma/2)M_{12}
\end{bmatrix}.
\]
Assuming that all the $2 \times 2$ minors are zero, we find that $M_{34}^2 = M_{12}^2$ (from columns one and four) and $M_{23}^2 = M_{14}^2$ (from columns two and three). Substituting these relations into the minor obtained from the last two columns yields either $(\gamma \pm 4)M_{12}M_{14} = 0$ or $\gamma M_{12}M_{14} = 0$, so $M_{12}M_{14} = 0$ (since $\gamma(\gamma^2 - 16) \neq 0$). Substitution into $q_1$ implies that there is no solution, and so the Jacobian matrix has rank two at all points of $\mathcal{V}(q_1, q_2)$. It follows that $\mathcal{V}(q_1, q_2)$, viewed as a subvariety of $\mathbb{P}^3 = \mathcal{V}(M_{13}, M_{24})$, is reduced, and so $\mathcal{L}_1$ is reduced. Following the method of the proof of [13, Proposition 2.5], if $\mathcal{V}(q_1, q_2)$ is not irreducible, then there exists a point in the intersection of two of its irreducible components, and so the Jacobian matrix has rank at most one at that point, which is a contradiction. Hence, $\mathcal{V}(q_1, q_2)$ is irreducible, and thus nonsingular since it is reduced.Moreover, its genus is $4 - 2 - 2 + 1 = 1$. It follows that $\mathcal{V}(q_1, q_2)$ is an elliptic curve, and the same is true of $\mathcal{L}_1$.

(II) Viewing $h = M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2$ as a polynomial in $\mathbb{k}[M_{12}, M_{23}, M_{24}]$, the Jacobian matrix of $h$ is a $1 \times 3$ matrix that has rank one at all points of $\mathcal{V}(h)$ (since $\text{char}(\mathbb{k}) \neq 2$), so $\mathcal{V}(h)$ is nonsingular in $\mathbb{P}^2 = \mathcal{V}(M_{13}, M_{14}, M_{34})$.

(III), (IV), (V) These cases follow from (II) by applying $\psi_1$ or $\psi_2$ as appropriate.

3.3. Description of the Line Scheme.

In this subsection, we prove that the line scheme $\mathcal{L}(\gamma)$ of $\mathcal{A}(\gamma)$ is reduced and so is given by $\mathcal{L}'(\gamma)$ described in Theorem 3.1.
Lemma 3.2. For all $\gamma \in k^\times$, the irreducible components of $\mathcal{L}(\gamma)$ have dimension one; in particular, $\mathcal{L}(\gamma)$ has no embedded points.

Proof. By [3], $\mathcal{A}(\gamma)$ is a regular noetherian domain that is Auslander-regular and satisfies the Cohen-Macaulay property and has Hilbert series the same as that of the polynomial ring on four variables. Hence, by [11, Remark 2.10], we may apply [11, Corollary 2.6] to $\mathcal{A}(\gamma)$, which gives us that the irreducible components of $\mathcal{L}(\gamma)$ have dimension at least one. However, by Theorem 3.1, they have dimension at most one, so equality follows. Let $X_1$ denote the 11-dimensional subscheme of $\mathbb{P}(V \otimes V)$ consisting of the elements of rank at most two, and, for all $\gamma \in k^\times$, let $X_2$ denote the 5-dimensional linear subscheme of $\mathbb{P}(V \otimes V)$ given by the span of the defining relations of $\mathcal{A}(\gamma)$. By [11, Lemma 2.5], $\mathcal{L}(\gamma) \cong X_1 \cap X_2$ for all $\gamma \in k^\times$. Since $X_i$ is a Cohen-Macaulay scheme for $i = 1, 2$, and since $\dim(X_1 \cap X_2) = 1$, the proof of [11, Theorem 4.3] (together with Macaulay’s Unmixedness Theorem) rules out the possibility of embedded components.

Theorem 3.3. For all $\gamma \in k^\times$, the line scheme $\mathcal{L}(\gamma)$ is a reduced scheme of degree twenty.

Proof. Let $X_1$ and $X_2$ be as in the proof of Lemma 3.2, and let $X = X_1 \cap X_2$. Since $\deg(X_1) = 20$ by [8, Example 19.10], Bézout’s Theorem for Cohen-Macaulay schemes ([6, Theorem III-78]) implies that $\deg(X) = 20$. However, since $\mathcal{L}(\gamma) \cong X$ by [11, Lemma 2.5], the reduced scheme $X'$ of $X$ is isomorphic to $\mathcal{L}'(\gamma)$. Since the degrees of the irreducible components of $\mathcal{L}'(\gamma)$ in Theorem 3.1 are as small as possible, $\deg(X') \geq 4 + 12 + 4 = 20$; that is, $20 = \deg(X) \geq \deg(X') \geq 20$, giving $\deg(X) = \deg(X')$. As $X$ has no embedded points by Lemma 3.2, it follows that $X = X'$, so $X$ is a reduced scheme. Thus, $\mathcal{L}(\gamma)$ is reduced and has degree twenty since $\deg(\mathcal{L}'(\gamma)) = 20$.

The intersection points of the irreducible components of $\mathcal{L}(\gamma)$ are straightforward to compute and are listed in [4].

4. The Lines in $\mathbb{P}^3$ Parametrized by the Line Scheme

In this section, we describe the lines in $\mathbb{P}(V^\times)$ that are parametrized by the line scheme $\mathcal{L}(\gamma)$ of $\mathcal{A}(\gamma)$. We also describe, in Theorem 4.1, the lines that pass through any given point of the point scheme; in particular, if $p$ is one of the generic points of the point scheme (that is, $p \in \mathcal{Z}_\gamma$), then there are exactly six distinct lines of the line scheme that pass through $p$. Since we will use results from Section 3, we continue to assume that $\text{char}(k) = 0$. 
4.1. The Lines in $\mathbb{P}^3$.

In this subsection, we find the lines in $\mathbb{P}(V^*)$ that are parametrized by the line scheme. We first recall how the Plücker coordinates $M_{12}, \ldots, M_{34}$ relate to lines in $\mathbb{P}^3$; details may be found in [5, §8.6]. Any line $\ell$ in $\mathbb{P}^3$ is uniquely determined by any two distinct points $a = (a_1, \ldots, a_4) \in \ell$ and $b = (b_1, \ldots, b_4) \in \ell$, and may be represented by a $2 \times 4$ matrix

$$
\begin{bmatrix}
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4
\end{bmatrix}
$$

that has rank two; in particular, the points on $\ell$ are represented in homogeneous coordinates by linear combinations of the rows of this matrix. In general, there are infinitely many such matrices that may be associated to any line $\ell$ in $\mathbb{P}^3$, and they are all related to each other by applying row operations.

The Plücker coordinate $M_{ij}$ is evaluated on this matrix as the minor $a_i b_j - a_j b_i$ for all $i \neq j$, and the Plücker polynomial $P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$, given in Section 3.1, vanishes on this matrix. Moreover, $\mathcal{V}(P)$ is the subscheme of $\mathbb{P}^5$ that parametrizes all lines in $\mathbb{P}^3$.

Since dim$(V) = 4$, we identify $\mathbb{P}(V^*)$ with $\mathbb{P}^3$. By Theorem 3.3, $\mathcal{L}(\gamma)$ is given by Theorem 3.1. We continue to use the notation $e_j$ introduced in Section 2.

(I) In this case, $\gamma^2 \neq 16$ and the component is $\mathcal{L}_1$, which is a nonplanar elliptic curve in a $\mathbb{P}^3$ (contained in $\mathbb{P}^5$), where

$$
\mathcal{L}_1 = \mathcal{V}(M_{13}, M_{24}, M_{14}M_{23} + M_{12}M_{34}, M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2).$

It follows that any line $\ell$ in $\mathbb{P}(V^*)$ given by $\mathcal{L}_1$ is represented by a $2 \times 4$ matrix of the form:

$$
\begin{bmatrix}
a_1 & 0 & a_3 & 0 \\
0 & b_2 & 0 & b_4
\end{bmatrix},
$$

where $a_j, b_j \in \mathbb{k}$ for all $j$ and $a_1^2b_2^2 + a_2^2b_3^2 - \gamma a_1b_2a_3b_4 - a_1^2b_4^2 - b_2^2a_3^2 = 0$. In particular, if $p \in \ell$, then $p = (\lambda_1a_1, \lambda_2b_2, \lambda_1a_3, \lambda_2b_4)$, for some $(\lambda_1, \lambda_2) \in \mathbb{P}^1$, such that $a_1^2b_2^2 + a_2^2b_3^2 - \gamma a_1b_2a_3b_4 - a_1^2b_4^2 - b_2^2a_3^2 = 0$. It is easily verified that $p$ lies on the quartic surface

$$
\mathcal{V}(x_1^2x_2^2 + x_3^2x_4^2 - \gamma x_1x_2x_3x_4 - x_1^2x_4^2 - x_2^2x_3^2)
$$

in $\mathbb{P}(V^*)$ for all $(\lambda_1, \lambda_2) \in \mathbb{P}^1$. Hence, the lines parametrized by $\mathcal{L}_1$ all lie on this quartic surface in $\mathbb{P}(V^*)$ and are given by:

$$
\mathcal{V}(x_3, x_2 \pm x_4), \quad \mathcal{V}(x_4, x_1 \pm x_3), \quad \text{and} \quad \mathcal{V}(x_1 - \alpha x_3, x_2 - \beta x_4)
$$

for all $\alpha, \beta \in \mathbb{k}$ such that $(\alpha^2 - 1)(\beta^2 - 1) = \gamma \alpha \beta$. The case $\gamma = 4$ is discussed below.
(II) In this case, the component is \( \mathcal{L}_2 \), which is a planar elliptic curve, where
\[
\mathcal{L}_2 = \mathcal{V}(M_{13}, M_{14}, M_{34}, M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2),
\]
so any line in \( \mathbb{P}(V^*) \) given by \( \mathcal{L}_2 \) is represented by a \( 2 \times 4 \) matrix of the form:
\[
\begin{bmatrix}
a_1 & 0 & a_3 & a_4 \\
0 & 1 & 0 & 0
\end{bmatrix},
\]
such that \( a_1^3 - a_1a_3^2 + ia_3a_1^2 = 0 \). It follows that \( \mathcal{L}_2 \) parametrizes those lines in \( \mathbb{P}(V^*) \) that pass through \( e_2 \) and meet the planar curve \( \mathcal{V}(x_2, x_1^3 - x_1x_3^2 + ix_3x_4^2) \); this planar curve is a (nonsingular) elliptic curve since \( \text{char}(\mathbb{k}) = 0 \).

(III) In this case, the component is \( \mathcal{L}_3 \), which may be obtained as \( \psi_1 \) applied to \( \mathcal{L}_2 \). Hence, \( \mathcal{L}_3 \) parametrizes those lines in \( \mathbb{P}(V^*) \) that pass through \( e_4 \) and meet the planar elliptic curve \( \mathcal{V}(x_4, x_3^2 - x_1^2x_3 + ix_1x_3^2) \).

(IV) In this case, the component is \( \mathcal{L}_4 \), which may be obtained as \( \psi_2 \) applied to \( \mathcal{L}_2 \). Hence, \( \mathcal{L}_4 \) parametrizes those lines in \( \mathbb{P}(V^*) \) that pass through \( e_3 \) and meet the planar elliptic curve \( \mathcal{V}(x_3, x_4^3 - x_2^2x_4 + i\gamma x_2^2x_3) \).

(V) In this case, the component is \( \mathcal{L}_5 \), which may be obtained as \( \psi_1 \) applied to \( \mathcal{L}_4 \). Hence, \( \mathcal{L}_5 \) parametrizes those lines in \( \mathbb{P}(V^*) \) that pass through \( e_1 \) and meet the planar elliptic curve \( \mathcal{V}(x_1, x_2^3 - x_2x_3^2 + i\gamma x_3x_4) \).

(VI) In this case, the component is \( \mathcal{L}_6 = \mathcal{L}_{6a} \cup \mathcal{L}_{6b} \), where
\[
\mathcal{L}_{6a} = \mathcal{V}(M_{14}, M_{23}, M_{12}M_{34} - M_{13}M_{24}, M_{12} + iM_{34}),
\]
\[
\mathcal{L}_{6b} = \mathcal{V}(M_{14}, M_{23}, M_{12}M_{34} - M_{13}M_{24}, M_{12} - iM_{34}),
\]
which are nonsingular conics. Following the argument from case (I), any line in \( \mathbb{P}(V^*) \) given by \( \mathcal{L}_{6a} \) is represented by a \( 2 \times 4 \) matrix of the form:
\[
\begin{bmatrix}
a_1 & a_2 & a_3 & a_4 \\
\alpha a_1 & \beta a_2 & \beta a_3 & \alpha a_4
\end{bmatrix},
\]
such that \( \alpha, \beta, a_j \in \mathbb{k} \) for all \( j \), \( a_1a_2 = ia_3a_4 \) and \( \alpha \neq \beta \). A calculation similar to that used in (I) verifies that every point of the line lies on the quadric \( \mathcal{V}(x_1x_2 - ix_3x_4) \). It follows that \( \mathcal{L}_{6a} \) parametrizes one of the rulings of the nonsingular quadric \( \mathcal{V}(x_1x_2 - ix_3x_4) \); namely, the ruling that consists of the lines \( \mathcal{V}(\delta x_1 - \epsilon x_4, \delta x_3 + i\epsilon x_2) \) for all \( (\delta, \epsilon) \in \mathbb{P}^1 \). Since \( \mathcal{L}_{6b} \) may be obtained by applying \( \psi_1 \) to \( \mathcal{L}_{6a} \), we find \( \mathcal{L}_{6b} \) parametrizes one of the rulings of the nonsingular quadric \( \mathcal{V}(x_3x_4 - ix_1x_2) \); namely, the ruling that consists of the lines \( \mathcal{V}(\delta x_3 - \epsilon x_2, \delta x_1 + i\epsilon x_4) \) for all \( (\delta, \epsilon) \in \mathbb{P}^1 \).
(Ia) and (Ib) In this case, \( \gamma = 4 \) and the component is \( \mathcal{L}_1 = \mathcal{L}_{1a} \cup \mathcal{L}_{1b} \), where
\[
\mathcal{L}_{1a} = \mathcal{V}(M_{13}, M_{24}, M_{14}M_{23} + M_{12}M_{34}, M_{12} + M_{14} - M_{23} - M_{34}), \\
\mathcal{L}_{1b} = \mathcal{V}(M_{13}, M_{24}, M_{14}M_{23} + M_{12}M_{34}, M_{12} - M_{14} + M_{23} - M_{34}),
\]
which are nonsingular conics. Following the argument from case (I), any line in \( \mathbb{P}(V^*) \) given by \( \mathcal{L}_{1a} \) is represented by a \( 2 \times 4 \) matrix of the form \( (\ast) \) such that \( a_1b_2 + a_1b_4 + b_2a_3 = a_3b_4 \). A calculation similar to that used in (I) verifies that every point of the line lies on the nonsingular quadric
\[
Q_a = \mathcal{V}(x_1x_2 + x_1x_4 + x_2x_3 - x_3x_4)
\]
in \( \mathbb{P}(V^*) \). Hence, the lines parametrized by \( \mathcal{L}_{1a} \) all lie on \( Q_a \) and are:
\[
\mathcal{V}(x_3, x_2 + x_4) \quad \text{and} \quad \mathcal{V}(x_1 - \alpha x_3, (\alpha + 1)x_2 + (\alpha - 1)x_4)
\]
for all \( \alpha \in \mathbb{k} \), which yields one of the rulings on the quadric \( Q_a \). Applying \( \psi_1 \) to these lines, it follows that the lines parametrized by \( \mathcal{L}_{1b} \) are:
\[
\mathcal{V}(x_1, x_2 + x_4) \quad \text{and} \quad \mathcal{V}(x_3 - \alpha x_1, (\alpha - 1)x_2 + (\alpha + 1)x_4)
\]
for all \( \alpha \in \mathbb{k} \), which yields one of the rulings on the nonsingular quadric
\[
Q_b = \mathcal{V}(x_3x_4 + x_2x_3 + x_1x_4 - x_1x_2).
\]

4.2. The Lines of the Line Scheme That Contain Points of the Point Scheme.

In this subsection, we compute how many lines in \( \mathbb{P}(V^*) \) that are parametrized by \( \mathcal{L}(\gamma) \) contain a given point of \( p(\gamma) \). By [11, Remark 3.2], if the number of lines is finite, then it is six, counting multiplicity; hence, the generic case is considered to be six distinct lines. The reader should note that a result similar to Theorem 4.1 is given in [7, Theorem IV.2.5] for the algebra \( \mathcal{A}(1) \), but that result is false as stated (perhaps as a consequence of the sign errors in the third relation of (3) on Page 797 of [10]).

**Theorem 4.1.** Suppose \( \gamma \in \mathbb{k}^\times \), and let \( \mathcal{Z}_r \) be as in Proposition 2.2.

(a) For any \( j \in \{1, \ldots, 4\} \), \( e_j \) lies on infinitely many lines that are parametrized by \( \mathcal{L}(\gamma) \).

(b) Each point of \( \mathcal{Z}_r \) lies on exactly six distinct lines of those parametrized by \( \mathcal{L}(\gamma) \).

**Proof.** Since (a) follows from (II)-(V) in Section 4.1, we focus on (b). Let \( p = (1, \alpha_2, \alpha_3, \alpha_4) \in \mathcal{Z}_r \). It follows that \( \alpha_j \neq 0 \) for all \( j \). Suppose that \( \gamma^2 \neq 16 \).

Let \( \alpha = 1/\alpha_3 \) and \( \beta = \alpha_2/\alpha_4 \), so \( (\alpha^2 - 1)(\beta^2 - 1) = \gamma \alpha \beta \), by 5.1.15 in Section 5.1. Hence, \( p \in \mathcal{V}(x_1 - \alpha x_3, x_2 - \beta x_4) \), which is a line that corresponds to an element of \( \mathcal{L}_1 \). Clearly, no other line given by \( \mathcal{L}_1 \) contains \( p \).

Let \( r_2 = (1, 0, \alpha_3, \alpha_4) \) and let \( \ell_2 \) denote the line through \( e_2 \) and \( r_2 \). By 5.1.9, we have
\[
1 - \alpha_3^2 + i\alpha_3 \alpha_4^2 = 0, \quad \text{so} \quad r_2 \in \mathcal{V}(x_2, x_1^3 - x_1 x_3^2 + i x_3 x_4^2).
\]
Thus, \( \ell_2 \) corresponds to an element of \( \mathcal{L}_2 \),
and \( p \in \ell_2 \). Conversely, let \( r'_2 = (b_1, 0, b_3, b_4) \in \mathcal{V}(x_2, x_1^3 - x_1^2x_3 + ix_3x_4^2) \). If \( p \) lies on the line through \( r'_2 \) and \( e_2 \), then there exist \((\lambda_1, \lambda_2) \in \mathbb{P}^1\) such that \( p = (\lambda_1b_1, \lambda_2, \lambda_1b_3, \lambda_1b_4) \). Thus, \( \lambda_1b_i \neq 0 \) and \( \alpha_i = b_i/b_1 \) for \( i = 3, 4 \). Hence, \( r'_2 = (b_1, 0, b_1\alpha_3, b_1\alpha_4) = (1, 0, \alpha_3, \alpha_4) = r_2 \). It follows that no other line given by \( \mathfrak{L}_2 \) contains \( p \).

Let \( r_4 = (1, \alpha_2, \alpha_3, 0) \) and let \( \ell_4 \) denote the line through \( e_4 \) and \( r_4 \). By 5.1.2, we have \( \alpha_4^2 - \alpha_3 + i\alpha_2^2 = 0 \), so \( r_4 \in \mathcal{V}(x_4, x_3^3 - x_2^2x_3 + ix_1x_2^2) \). Thus, \( \ell_4 \) corresponds to an element of \( \mathfrak{L}_3 \), and \( p \in \ell_4 \). An argument similar to that of \( \mathfrak{L}_2 \) proves that no other line given by \( \mathfrak{L}_3 \) contains \( p \).

Let \( r_3 = (1, \alpha_2, 0, \alpha_4) \) and let \( \ell_3 \) denote the line through \( e_3 \) and \( r_3 \). By 5.1.5, we have \( \alpha_4^2 - \alpha_3^2\alpha_4 + i\gamma\alpha_2 = 0 \), so \( r_3 \in \mathcal{V}(x_3, x_3^3 - x_2^2x_4 + i\gamma x_2^2x_3) \). Thus, \( \ell_3 \) corresponds to an element of \( \mathfrak{L}_4 \), and \( p \in \ell_3 \). An argument similar to that of \( \mathfrak{L}_2 \) proves that no other line given by \( \mathfrak{L}_4 \) contains \( p \).

Let \( r_1 = (0, \alpha_2, \alpha_3, \alpha_4) \) and let \( \ell_4 \) denote the line through \( e_1 \) and \( r_1 \). By 5.1.8, we have \( \alpha_2^2 - \alpha_2\alpha_4^2 + i\gamma\alpha_3^2\alpha_4 = 0 \), so \( r_1 \in \mathcal{V}(x_1, x_3^3 - x_2^2x_4 + i\gamma x_2^2x_4) \). Thus, \( \ell_4 \) corresponds to an element of \( \mathfrak{L}_5 \), and \( p \in \ell_4 \). An argument similar to that of \( \mathfrak{L}_2 \) proves that no other line given by \( \mathfrak{L}_5 \) contains \( p \).

By 5.1.1, we have \( \alpha_2 = \pm i\alpha_3\alpha_4 \), so either \( p \in \mathcal{V}(x_1x_2 - ix_3x_4) \) or \( p \in \mathcal{V}(ix_1x_2 - x_3x_4) \) (but not both, since \( \alpha_3\alpha_4 \neq 0 \)). In the first case, \( p \in \mathcal{V}(\alpha_4x_1 - x_4, \alpha_4x_3 + ix_2) \) and, in the second, \( p \in \mathcal{V}(\alpha_4x_1 - x_4, i\alpha_4x_3 + x_2) \). These lines correspond to elements of \( \mathfrak{L}_{6a} \) and \( \mathfrak{L}_{6b} \) respectively. Since each quadric has only two rulings, and since each irreducible component of \( \mathfrak{L}_6 \) parametrizes only one of the rulings in each case, no other line given by \( \mathfrak{L}_6 \) contains \( p \).

If, instead, \( \gamma = 4 \), the only adjustment to the above reasoning is in the case of the lines parametrized by \( \mathfrak{L}_4 \). Since \( \gamma = 4 \), the polynomial 5.1.15 factors, so

\[
\begin{align*}
(\alpha_2 + \alpha_4 + \alpha_2\alpha_3 - \alpha_2\alpha_4)(\alpha_2 - \alpha_4 - \alpha_2\alpha_3 - \alpha_3\alpha_4) &= 0, \\
(1 + \alpha_3)\alpha_2 + (1 - \alpha_3)\alpha_4 & \equiv (1 - \alpha_3)\alpha_2 - (1 + \alpha_3)\alpha_4 = 0,
\end{align*}
\]

that is,

\[
((1 + \alpha_3)\alpha_2 + (1 - \alpha_3)\alpha_4)((1 - \alpha_3)\alpha_2 - (1 + \alpha_3)\alpha_4) = 0,
\]

which provides exactly two lines (of those parametrized by \( \mathfrak{L}_4 \)) that could contain \( p \). These lines are

\[
\begin{align*}
\mathcal{V}(x_1 - (1/\alpha_3)x_3, ((1/\alpha_3) + 1)x_2 + (1/\alpha_3) - 1)x_4
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{V}(x_3 - \alpha_3x_1, (\alpha_3 - 1)x_2 + (\alpha_3 + 1)x_4),
\end{align*}
\]

which correspond to elements of \( \mathfrak{L}_{1a} \) and \( \mathfrak{L}_{1b} \) respectively. If the first factor of (‡) is zero, then \( p \) belongs to the first line, whereas if the second factor of (‡) is zero, then \( p \) belongs to the second line. If both factors of (‡) are zero, then \( \alpha_2 = \alpha_3\alpha_4 \), which forces \( \alpha_3\alpha_4 = 0 \), by 5.1.1, and this contradicts \( p \in \mathcal{Z}_\gamma \). It follows that \( p \) belongs to exactly one line of those parametrized by \( \mathfrak{L}_4 \).
For all $\gamma \in \mathbb{k}^\times$, it is a straightforward calculation to show that the six lines found above are distinct.

Considering Theorems 3.1, 3.3 and 4.1 in the case where $\gamma^2 \neq 16$, we arrive at the following conjecture.

**Conjecture 4.2.** The line scheme of the most generic quadratic quantum $\mathbb{P}^3$ is isomorphic to the union of two spatial (irreducible and nonsingular) elliptic curves and four planar (irreducible and nonsingular) elliptic curves. (Here, spatial elliptic curve means a nonplanar elliptic curve that is contained in a subscheme of $\mathbb{P}^5$ that is isomorphic to $\mathbb{P}^3$.)

This conjecture is motivated by the idea that the “generic” points of the point scheme should have exactly six distinct lines of the line scheme passing through each of them, with each line coming from exactly one component of the line scheme. Moreover, if the component $L_6$ of the line scheme $L(\gamma)$ of $A(\gamma)$ had not split into two smaller components, then it would likely have been a spatial elliptic curve.

5. **Appendix**

In this section, we list the polynomials that define $p(\gamma)$ and $L(\gamma)$.

5.1. **Polynomials Defining the Point Scheme.**

The following are the polynomials that define the point scheme viewed as $p(\gamma) \subset \mathbb{P}(V^*)$ of $A(\gamma)$ that are given by the fifteen $4 \times 4$ minors of the matrix $M$ in Section 2; they are used in Section 2 and in the proof of Theorem 4.1:

5.1.1. $x_1^2x_2^2 + x_3^2x_4^2$,
5.1.2. $x_1(x_3^3 - x_1^2x_3 + ix_1x_2^2)$,
5.1.3. $x_2(x_3^3 - x_1^2x_3 + ix_1x_2^2)$,
5.1.4. $x_4(x_3^3 - x_1^2x_3 + ix_1x_2^2)$,
5.1.5. $x_1(x_4^3 - x_2^2x_4 + i\gamma x_2^2x_2)$,
5.1.6. $x_2(x_4^3 - x_2^2x_4 + i\gamma x_2^2x_2)$,
5.1.7. $x_3(x_4^3 - x_2^2x_4 + i\gamma x_2^2x_2)$,
5.1.8. $x_1(x_3^3 - x_2x_4^2 + i\gamma x_2^2x_4)$,
5.1.9. $x_2(x_3^3 - x_1x_3^2 + ix_3x_4^2)$,
5.1.10. $i\gamma x_1^2x_3^2 - x_1^2x_2x_4 - x_2x_3^2x_4$,
5.1.11. $ix_2x_4^3 - x_1^2x_3x_3 - x_1x_3x_4^2$. 
5.1.12. \(x_1^4 x_4 + \gamma x_1^2 x_2 x_3 - x_1 x_3^2 x_4 + i x_2^2 x_3 x_4,\)
5.1.13. \(x_2^3 x_3 + \gamma x_1 x_2^2 x_4 - x_2 x_3 x_4^2 + i x_1^2 x_3 x_4,\)
5.1.14. \(i \gamma x_1^3 x_3 + \gamma x_1^2 x_2^2 - 2 x_1 x_2 x_3 x_4 + i x_1^2 x_4,\)
5.1.15. \(x_1^2 x_2^2 - x_2^2 x_3 - \gamma x_1 x_2 x_3 x_4 - x_1^2 x_4^2 + x_3^2 x_4,\)

where \(i^2 = -1\) and \(\gamma \in k^\times.\)

5.2. Polynomials Defining the Line Scheme.

The following are the forty-six polynomials in the \(M_{ij}\) coordinates from Section 3 that define the line scheme \(\mathcal{L}(\gamma)\) of \(A(\gamma)\):

5.2.0. \(P = M_{12} M_{34} - M_{13} M_{24} + M_{14} M_{23},\)
5.2.1. \(2 M_{13} M_{14} M_{23} M_{24},\)
5.2.2. \(M_{12}(\gamma M_{13} M_{14} M_{23} + i M_{12} M_{14} M_{24} + i M_{23} M_{24} M_{34}),\)
5.2.3. \(M_{12}(\gamma M_{13} M_{14} M_{23} - i M_{12} M_{14} M_{24} - i M_{23} M_{24} M_{34}),\)
5.2.4. \(M_{13}(\gamma M_{13} M_{14} M_{23} + i M_{12} M_{14} M_{24} + i M_{23} M_{24} M_{34}),\)
5.2.5. \(M_{13}(\gamma M_{13} M_{14} M_{23} - i M_{12} M_{14} M_{24} - i M_{23} M_{24} M_{34}),\)
5.2.6. \(M_{13}(\gamma M_{13} M_{14} M_{23} + i M_{12} M_{14} M_{24} + i M_{23} M_{24} M_{34}),\)
5.2.7. \(M_{14}(\gamma M_{13} M_{14} M_{23} + i M_{12} M_{14} M_{24} + i M_{23} M_{24} M_{34}),\)
5.2.8. \(M_{23}(\gamma M_{13} M_{14} M_{23} + i M_{12} M_{14} M_{24} + i M_{23} M_{24} M_{34}),\)
5.2.9. \(M_{23}(\gamma M_{13} M_{14} M_{23} - i M_{12} M_{14} M_{24} - i M_{23} M_{24} M_{34}),\)
5.2.10. \(M_{24}(\gamma M_{13} M_{14} M_{23} + i M_{12} M_{14} M_{24} + i M_{23} M_{24} M_{34}),\)
5.2.11. \(M_{34}(\gamma M_{13} M_{14} M_{23} + i M_{12} M_{14} M_{24} + i M_{23} M_{24} M_{34}),\)
5.2.12. \(M_{12}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} + i M_{14} M_{23} M_{24}),\)
5.2.13. \(M_{12}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} - i M_{14} M_{23} M_{24}),\)
5.2.14. \(M_{13}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} + i M_{14} M_{23} M_{24}),\)
5.2.15. \(M_{14}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} + i M_{14} M_{23} M_{24}),\)
5.2.16. \(M_{14}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} - i M_{14} M_{23} M_{24}),\)
5.2.17. \(M_{23}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} + i M_{14} M_{23} M_{24}),\)
5.2.18. \(M_{24}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} + i M_{14} M_{23} M_{24}),\)
5.2.19. \(M_{24}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} - i M_{14} M_{23} M_{24}),\)
5.2.20. \(M_{24}(M_{12} M_{13} M_{23} - M_{13} M_{14} M_{34} + i M_{14} M_{23} M_{24}),\)
5.2.21. \(M_{34}(M_{12} M_{13} M_{23} + M_{13} M_{14} M_{34} + i M_{14} M_{23} M_{24}),\)
5.2.22. \( M_{13}^2 M_{23} M_{24} + M_{13} M_{14} M_{23}^2 - M_{13} M_{14} M_{34}^2 + i M_{14} M_{23} M_{24} M_{34} \),
5.2.23. \( M_{12}^2 M_{13} M_{23} + i M_{12} M_{14} M_{23} M_{24} - M_{13} M_{14} M_{24} - M_{13} M_{14} M_{23} \),
5.2.24. \( i \gamma M_{12} M_{13} M_{23}^2 - \gamma M_{14} M_{23} M_{24} - M_{12} M_{14} M_{24} M_{34} - M_{23} M_{24} M_{34}^2 \),
5.2.25. \( i \gamma M_{13} M_{14} M_{23} M_{34} - M_{13} M_{14} M_{24}^2 - M_{14} M_{23} M_{24} + M_{23} M_{24} M_{34}^2 \),
5.2.26. \( i \gamma M_{12} M_{13} M_{14} M_{23} - M_{12}^2 M_{14} M_{24} + M_{13} M_{23} M_{24}^2 + M_{14} M_{23} M_{24} \),
5.2.27. \( \gamma M_{13} M_{14} M_{23} + M_{12} M_{13} M_{23} M_{34} + i M_{12} M_{14} M_{24} + M_{13} M_{14} M_{34}^2 \),
5.2.28. \( \gamma M_{14}^2 M_{23}^2 + i M_{14} M_{14} M_{23} + M_{12} M_{14} M_{24} + M_{12} M_{14} M_{34} + M_{14} M_{23} M_{34}^2 \),
5.2.29. \( -i \gamma M_{12} M_{14} M_{23} + \gamma M_{13} M_{14} M_{23} M_{24} + M_{12} M_{13} M_{24} + i M_{12} M_{14} M_{24} + M_{13} M_{24} M_{34} \),
5.2.30. \( i \gamma M_{12} M_{13} M_{14} + M_{12} M_{13} + i M_{12} M_{14} M_{24} - M_{12} M_{13} M_{24} + M_{12} M_{24} M_{34} \),
5.2.31. \( \gamma M_{12} M_{13} M_{14} M_{23} + M_{12} M_{14} M_{24} - M_{12} M_{13} M_{23}^2 - M_{13} M_{14} M_{23} M_{34} - i M_{13} M_{23} M_{24}^2 \),
5.2.32. \( i \gamma M_{12} M_{13} M_{14} M_{34} + M_{12} M_{13} M_{24} + i M_{12} M_{14} M_{24} - 2 M_{13} M_{14} M_{24} + M_{13} M_{24} M_{34} \),
5.2.33. \( i \gamma M_{12} M_{13} M_{23} - \gamma M_{12} M_{14} M_{23} M_{24} - i \gamma M_{13} M_{14} M_{24} + M_{12} M_{14} M_{24} + M_{14} M_{23} M_{24} M_{34} \),
5.2.34. \( i \gamma M_{12} M_{13} M_{23} - M_{12} M_{13} M_{24} + 2 M_{13} M_{23} M_{24} - M_{13} M_{24} M_{34} + i M_{23} M_{24} M_{34} \),
5.2.35. \( i \gamma M_{12} M_{13} M_{23} - M_{12} M_{24} + M_{12} M_{23} M_{24} - M_{13} M_{24} M_{34} + i M_{23} M_{24} M_{34} \),
5.2.36. \( \gamma M_{14}^2 M_{23} M_{34} - M_{12} M_{14} M_{23} + M_{12} M_{23} M_{34}^2 - M_{14} M_{23} M_{34} + i M_{14} M_{24} M_{34} + M_{14} M_{34} \),
5.2.37. \( i \gamma M_{13} M_{23} - \gamma M_{13} M_{14} M_{23} M_{34} - M_{12} M_{13} M_{24} - i M_{12} M_{14} M_{24} M_{34} + M_{13} M_{23} M_{34} - M_{13} M_{34} \),
5.2.38. \( \gamma M_{12}^2 M_{14} M_{23} + i \gamma M_{13} M_{14} M_{14} + M_{12} M_{14} + M_{12} M_{23} M_{34} - M_{12} M_{14} - M_{14} M_{23} M_{34} \),
5.2.39. \( i \gamma M_{13} M_{14} M_{23}^2 - \gamma M_{14} M_{23} M_{34} + M_{12} M_{14} M_{23}^2 - M_{12} M_{14} M_{34} + M_{23} M_{34} - M_{23} M_{34} \),
5.2.40. \( i \gamma M_{12} M_{14} M_{23}^2 + i M_{12} M_{13} + M_{12} M_{14} M_{34} - M_{12} M_{13} M_{34} - M_{12} M_{23} M_{34} + M_{23} M_{24} \),
5.2.41. \( i \gamma M_{12} M_{13} M_{23} M_{34} - \gamma M_{14} M_{23} M_{24} M_{34} - M_{12} M_{13} M_{24} + M_{14} M_{23} M_{24} M_{34} - i M_{14} M_{24} - M_{24} M_{34} \),
5.2.42. \( i \gamma M_{12} M_{14} M_{23} M_{34} - i M_{12} M_{14} M_{23} - i M_{12} M_{14} M_{34} - M_{12} M_{14} M_{24} - i M_{12} M_{23} M_{34} - i M_{14} M_{23} M_{34} + M_{23} M_{24} M_{34} \),
5.2.43. \( i \gamma M_{12} M_{13} M_{23} - \gamma M_{12} M_{14} M_{23} M_{34} - i \gamma M_{13} M_{14} M_{23} M_{34} + M_{12} M_{14} M_{23} + M_{12} M_{14} M_{34} + M_{12} M_{23} M_{34} + M_{14} M_{24} M_{34} \),
5.2.44. \( \gamma M_{12}^2 M_{14} M_{23} + i \gamma M_{12} M_{13} M_{14} + M_{12}^2 - M_{12} M_{14} M_{23} - M_{12} M_{23} M_{23} - i M_{12} M_{23} M_{24} + M_{13} M_{24}^2 + M_{14} M_{23}^2 \),
5.2.45. \( -i \gamma M_{13} M_{14} M_{23} M_{34} + \gamma M_{14} M_{23} M_{34} + M_{13} M_{24} + M_{14} M_{23} - M_{14} M_{34} + i M_{14} M_{24} M_{34} - M_{23} M_{24} - M_{14} M_{34} \),

where \( i^2 = -1 \) and \( \gamma \in k^x \).
Acknowledgments. The authors gratefully acknowledge support from the NSF under grants DMS-0900239 and DMS-1302050. Moreover, the authors are grateful to B. Shelton for discussions about a potential approach towards computing the line scheme of the algebra defined in [10]; that algebra is a member of the family of algebras investigated herein.

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