PULLBACK ATTRACTORS FOR A CLASS OF NON-AUTONOMOUS THERMOELASTIC PLATE SYSTEMS

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Abstract. In this article we study the asymptotic behavior of solutions, in the sense of pullback attractors, of the evolution system

\[
\begin{align*}
\ddot{u} + \Delta^2 u + a(t)\Delta \theta &= f(t, u), \quad t > \tau, \quad x \in \Omega, \\
\dot{\theta} - \kappa \Delta \theta - a(t)\Delta u_t &= 0, \quad t > \tau, \quad x \in \Omega,
\end{align*}
\]

subject to boundary conditions

\[
u = \Delta u = \theta = 0, \quad t > \tau, \quad x \in \partial \Omega,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with \( N \geq 2 \), which boundary \( \partial \Omega \) is assumed to be a \( C^4 \)-hypersurface. \( \kappa > 0 \) is constant, \( a \) is a Hölder continuous function and \( f \) is a dissipative nonlinearity locally Lipschitz in the second variable. Using the theory of uniform sectorial operators, in the sense of P. Sobolevskiĭ ([23]), we give a partial description of the fractional power spaces scale for the thermoelastic plate operator and we show the local and global well-posedness of this non-autonomous problem. Furthermore we prove existence and uniform boundedness of pullback attractors.

1. Introduction. This paper is concerned with the asymptotic behavior of non-autonomous dynamical systems generated by a non-autonomous thermoelastic plate system, in particular as described by their pullback attractors. More precisely, we consider the following problem associated with description of the small vibrations of a homogeneous, elastic and thermal isotropic Kirchhoff plate. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with \( N \geq 2 \), which boundary \( \partial \Omega \) is assumed to be a \( C^4 \)-hypersurface.

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We consider the following initial-boundary value problem

\[
\begin{align*}
    u_{tt} + \Delta^2 u + a(t) \Delta \theta &= f(t, u), \quad t > \tau, \ x \in \Omega, \\
    \theta_t - \kappa \Delta \theta - a(t) \Delta u_t &= 0, \quad t > \tau, \ x \in \Omega,
\end{align*}
\]

where \( \kappa \) is a positive constant, subject to boundary conditions

\[
\begin{align*}
    u = \Delta u &= 0, \quad t > \tau, \ x \in \partial \Omega, \\
    \theta &= 0, \quad t > \tau, \ x \in \partial \Omega,
\end{align*}
\]

and initial conditions

\[
\begin{align*}
    u(\tau, x) &= u_0(x), \ u_t(\tau, x) = v_0(x) \text{ and } \theta(\tau, x) = \theta_0(x), \ x \in \Omega, \ \tau \in \mathbb{R}.
\end{align*}
\]

Here, we assume that there exist positive constants \( a_0 \) and \( a_1 \) such that

\[
0 < a_0 \leq a(t) \leq a_1, \quad \forall t \in \mathbb{R}.
\]

Furthermore, we assume that the function \( a \) is \((\beta, C)\)-Hölder continuous; that is,

\[
|a(t) - a(s)| \leq C|t - s|^\beta, \quad \forall t, s \in \mathbb{R}.
\]

Below we give conditions under which the non-autonomous problem (1)-(2) is locally and globally well posed in some space that we will specify later. To that end we must assume some growth condition on the nonlinearity \( f \).

To obtain the global existence of solutions and the existence of pullback attractor we assume that \( f : \mathbb{R}^2 \to \mathbb{R} \) is locally Lipschitz in the second variable, and it is a dissipative nonlinearity in the second variable

\[
\limsup_{|s| \to \infty} \frac{f(t, s)}{s} < \lambda_1,
\]

uniformly in \( t \in \mathbb{R} \), where \( \lambda_1 > 0 \) is the first eigenvalue of negative Laplacian operator with zero Dirichlet boundary condition. Due to Sobolev embedding we need to assume that the function \( f \) satisfies subcritical growth condition; that is,

\[
|f_s(t, s)| \leq C(1 + |s|^{\rho-1}), \quad \forall s \in \mathbb{R},
\]

where \( 1 \leq \rho < \frac{N}{N-4} \), with \( N \geq 5 \), and \( C > 0 \) independent of \( t \in \mathbb{R} \). We will justify these restrictions later in the paper. If \( N = 2, 3, 4 \), we suppose the growth condition (6) with \( \rho \geq 1 \).

In the literature the initial boundary-value problem (1)-(2) has been extensively discussed for several authors in different contexts. For instance, Baroun et al. in [5] studied the existence of almost periodic solutions for an evolution system like (1), Liu and Renardy [20] proved that the linear semigroup defined by system (1) with \( f \equiv 0 \) with clamped boundary condition for \( u \) and Dirichlet boundary condition for \( \theta \) is analytic. The typical difficulties in thermoelasticity comes from the boundary condition, which make more complicated the task of getting estimates to show the exponential stability of the solutions or analyticity of the corresponding semigroup. In that direction we have the works of Liu and Zheng [21], Lasiecka and Triggiani [18] to free - clamped boundary condition. In this last work the authors show the exponential stability and analyticity of the semigroup associated with the system (1).

We refer to the book of Liu and Zheng [22] for a general survey on those topics.

The purpose of this paper is to prove, under suitable assumptions, local and global well-posedness (using the theory of uniform sectorial operators, in the sense of [23]), of the non-autonomous problem (1)-(2), the existence and uniform boundedness of pullback attractors, which, to our best knowledge, is still missing.
To formulate the non-autonomous problem (1)-(2) in the nonlinear evolution process setting, we introduce some notations. Here, we denote $X = L^2(\Omega)$ and $\Lambda : D(\Lambda) \subset X \to X$ the biharmonic operator defined by $D(\Lambda) = \{ u \in H^4(\Omega ); u = \Delta u = 0 \text{ on } \partial \Omega \}$ and

$$\Lambda u = (-\Delta)^2 u, \ \forall u \in D(\Lambda),$$

then $\Lambda$ is a positive self-adjoint operator in $X$ with compact resolvent and therefore $-\Lambda$ generates a compact analytic semigroup on $X$ (that is, $\Lambda$ is a sectorial operator, in the sense of Henry [17]). Denote by $X^\alpha$, $\alpha > 0$, the fractional power spaces associated with the operator $\Lambda$; that is, $X^\alpha = D(\Lambda^\alpha)$ endowed with the graph norm. With this notation, we have $X^{-\alpha} = (X^\alpha)'$ for all $\alpha > 0$, see Amann [1] for the characterization of the negative scale. It is of special interest the case $\alpha = \frac{1}{2}$, since $-\Lambda^{\frac{1}{2}}$ is the Laplacian operator with homogeneous Dirichlet boundary conditions.

If we denote $v = u_2$, then we can rewrite the non-autonomous problem (1)-(2) in the matrix form

$$w_t = A_{(a)}(t)w + F(t, w), \ t > \tau, \quad w(\tau) = w_0, \ \tau \in \mathbb{R},$$

where $w = w(t)$ for all $t \in \mathbb{R}$, and $w_0 = w(\tau)$ are given by

$$w = \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \quad \text{and} \quad w_0 = \begin{bmatrix} u_0 \\ v_0 \\ \theta_0 \end{bmatrix},$$

and, for each $t \in \mathbb{R}$, the unbounded linear operator $A_{(a)}(t) : D(A_{(a)}(t)) \subset Y \to Y$ is defined by

$$A_{(a)}(t) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & f & 0 \\ -\Lambda & 0 & -a(t)\Lambda^{\frac{1}{2}} \\ 0 & \frac{v}{a(t)\Lambda^{\frac{1}{2}}} & \kappa^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} -\Lambda u - a(t)\Lambda^{\frac{1}{2}}v \\ -a(t)\Lambda^{\frac{1}{2}}v + \kappa\Lambda^{\frac{1}{2}}\theta \\ a(t)\Lambda^{\frac{1}{2}}v + \kappa\Lambda^{\frac{1}{2}}\theta \end{bmatrix},$$

where

$$Y = (H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$$

is the phase space of problem (1)-(2) and the domain of the operator $A_{(a)}(t)$ is defined by

$$D(A_{(a)}(t)) = X^1 \times X^{\frac{1}{2}} \times X^{\frac{1}{2}},$$

with $X^1 = \{ u \in H^4(\Omega); u = \Delta u = 0 \text{ on } \partial \Omega \}$ and $X^{\frac{1}{2}} = H^2(\Omega) \cap H^1_0(\Omega)$.

We define the nonlinearity $F$ by

$$F(t, w) = \begin{bmatrix} f^c(t, u) \\ 0 \\ 0 \end{bmatrix},$$

where $f^c(t, u)$ is the Nemitskiǐ operator associated with $f(t, u), t \in \mathbb{R}$, that is,

$$f^c(t, u)(x) := f(t, u(x)), \ \forall t \in \mathbb{R}, x \in \Omega.$$

The map $f^c(t, u)$ is Lipschitz continuous in bounded subsets of $X^{\frac{1}{2}}$ uniformly in $t \in \mathbb{R}$.

This paper is organized as: In Section 2 we recall concepts and results about problems singularly non-autonomous, including results on existence of pullback attractors. In Section 3 we deal with the linear problem associated (1)-(2). Section 4 is devoted to study the existence of local and global solutions in some appropriate space. Finally, in Section 5 we present results on dissipativeness of thermoelastic equation and existence of pullback attractors for (1)-(2).
2. Singularly non-autonomous abstract problem. Throughout this paper, \( L(Z) \) will denote the space of linear and bounded operators defined in a Banach space \( Z \). Let \( \mathcal{A}(t), t \in \mathbb{R}, \) be a family of unbounded closed linear operators defined on a fixed dense subspace \( D \) of \( Z \).

2.1. Singularly non-autonomous abstract linear problem. Consider the singularly non-autonomous abstract linear parabolic problem of the form

\[
\begin{cases}
\frac{du}{dt} = -\mathcal{A}(t)u, & t > \tau, \\
u(\tau) = u_0 \in D.
\end{cases}
\]

We assume that

(a) The operator \( \mathcal{A}(t) : D \subset Z \to Z \) is a closed densely defined operator (the domain \( D \) is fixed) and there is a constant \( C > 0 \) (independent of \( t \in \mathbb{R} \)) such that

\[
\|(\lambda I + \mathcal{A}(t))^{-1}\|_{L(Z)} \leq \frac{C}{|\lambda| + 1}; \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \text{Re} \lambda \geq 0.
\]

To express this fact we will say that the family \( \mathcal{A}(t) \) is uniformly sectorial.

(b) There are constants \( C > 0 \) and \( \epsilon_0 > 0 \) such that, for any \( t, \tau, s \in \mathbb{R} \),

\[
\|[(\mathcal{A}(t) - \mathcal{A}(\tau))\mathcal{A}^{-1}(s)](\|L(Z)\) \leq C|t - \tau|^{\epsilon_0}, \quad \epsilon_0 \in (0, 1].
\]

To express this fact we will say that the map \( \mathbb{R} \ni t \mapsto \mathcal{A}(t) \) is uniformly H"{o}lder continuous.

Denote by \( \mathcal{A}_0 \) the operator \( \mathcal{A}(t_0) \) for some \( t_0 \in \mathbb{R} \) fixed. If \( Z^\alpha \) denotes the domain of \( \mathcal{A}_0^{\alpha} \), \( \alpha > 0 \), with the graph norm and \( Z^0 := Z \), denote by \( \{Z^\alpha; \alpha \geq 0\} \) the fractional power scale associated with \( \mathcal{A}_0 \) (see Henry [17]).

From (a), \( -\mathcal{A}(t) \) is the generator of an analytic semigroup \( \{e^{-\tau \mathcal{A}(t)} \in L(Z) : \tau \geq 0\} \). Using this and the fact that \( 0 \in \rho(\mathcal{A}(t)) \), it follows that

\[
\|e^{-\tau \mathcal{A}(t)}\|_{L(Z)} \leq C, \quad \tau \geq 0, \quad t \in \mathbb{R},
\]

and

\[
\|\mathcal{A}(t)e^{-\tau \mathcal{A}(t)}\|_{L(Z)} \leq C\tau^{-1}, \quad \tau > 0, \quad t \in \mathbb{R}.
\]

It follows from (b) that \( \|\mathcal{A}(t)\mathcal{A}^{-1}(\tau)\|_{L(Z)} \leq C, \forall (t, \tau) \in I, \) for some \( I \subset \mathbb{R}^2 \) bounded. Also, the semigroup \( e^{-\tau \mathcal{A}(t)} \) generated by \( -\mathcal{A}(t) \) satisfies the following estimate

\[
\|e^{-\tau \mathcal{A}(t)}\|_{L(Z^\beta, Z^\alpha)} \leq M\tau^{\beta - \alpha}, \quad (11)
\]

where \( 0 \leq \beta \leq \alpha < 1 + \epsilon_0 \).

Next we recall the definition of a linear evolution process associated with a family of operators \( \{\mathcal{A}(t) : t \in \mathbb{R}\} \).

**Definition 2.1.** A family \( \{L(t, \tau) : t, \tau \geq 0 \} \subset L(Z) \) satisfying

1. \( L(\tau, \tau) = I, \)
2. \( L(t, \sigma)L(\sigma, \tau) = L(t, \tau), \) for any \( t \geq \sigma \geq \tau, \)
3. \( \{(t, \tau) \in \mathbb{R}^2 : t \geq \tau\} \times Z \ni ((t, \tau), u_0) \mapsto L(t, \tau)u_0 \in Z \) is continuous,

is called a linear evolution process (process for short) or family of evolution operators.
If the operator $A(t)$ is uniformly sectorial and uniformly Hölder continuous, then there exists a linear evolution process $\{L(t, \tau) : t \geq \tau \in \mathbb{R}\}$ associated with $A(t)$, which is given by

$$L(t, \tau) = e^{-(t-\tau)A(\tau)} + \int_\tau^t L(t, s)[A(\tau) - A(s)]e^{-(s-\tau)A(\tau)}ds.$$ 

The evolution process $\{U(t, \tau) : t \geq \tau \in \mathbb{R}\}$ satisfies the following condition:

$$\|L(t, \tau)\|_{L(X^\beta, X^\alpha)} \leq C(\alpha, \beta)(t-\tau)^{\beta-\alpha},$$

where $0 \leq \beta \leq \alpha < 1 + \epsilon_0$. For more details see [14] and [23].

### 2.2. Existence of pullback attractors.

In this subsection we will present basic definitions and results of the theory of pullback attractors for nonlinear evolution process. For more details we refer to [8], [9], [13] and [15].

We consider the singularly non-autonomous abstract parabolic problem

$$\begin{cases}
\frac{du}{dt} = -A(t)u + g(t, u), & t > \tau, \\
u(\tau) = u_0 \in D,
\end{cases}$$

(13)

where the operator $A(t)$ is uniformly sectorial and uniformly Hölder continuous and the nonlinearity $g$ satisfies conditions which will be specified later. The nonlinear evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ associated with $A(t)$ is given by

$$S(t, \tau) = L(t, \tau) + \int_\tau^t L(t, s)g(s, L(s, \tau))ds, \ \forall t \geq \tau.$$

**Definition 2.2.** Let $g : \mathbb{R} \times X^\alpha \to X^\beta, \ \alpha \in [\beta, \beta + 1)$ be a continuous function. We say that a continuous function $u : [\tau, \tau + t_0] \to X^\alpha$ is a (local) solution of (13) starting in $u_0 \in X^\alpha$, if $u \in C([\tau, \tau + t_0], X^\alpha) \cap C^1([\tau, \tau + t_0], X^\alpha)$, $u(\tau) = u_0$, $u(t) \in D(A(t))$ for all $t \in (\tau, \tau + t_0)$ and (13) is satisfied for all $t \in (\tau, \tau + t_0)$.

We can now state the following result, from [7]. We also refer to [14] for a more general version that includes the critical growth case.

**Theorem 2.3.** Suppose that the family of operators $A(t)$ is uniformly sectorial and uniformly Hölder continuous in $X^\beta$. If $g : \mathbb{R} \times X^\alpha \to X^\beta, \ \alpha \in [\beta, \beta + 1)$, is a Lipschitz continuous map in bounded subsets of $X^\alpha$, then, given $r > 0$, there is a time $t_0 > 0$ such that for all $u_0 \in B_{X^\alpha}(0; r)$ there exists a unique solution of the problem (13) starting in $u_0$ and defined in $[\tau, \tau + t_0]$. Moreover, such solutions are continuous with respect the initial data in $B_{X^\alpha}(0; r)$.

We start remembering the definition of Hausdorff semi-distance between two subsets $A$ and $B$ of a metric space $(X, d)$:

$$\text{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Next we present several definitions about theory of pullback attractors, which can be found in [8, 13, 15].

**Definition 2.4.** Let $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ be an evolution process in a metric space $X$. Given $A$ and $B$ subsets of $X$, we say that $A$ pullback attracts $B$ at time $t$ if

$$\lim_{\tau \to -\infty} \text{dist}_H(S(t, \tau)B, A) = 0,$$

where $S(t, \tau)B := \{S(t, \tau)x \in X : x \in B\}$. 

Definition 2.5. The pullback orbit of a subset $B \subset X$ relatively to the evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in the time $t \in \mathbb{R}$ is defined by $\gamma_p(B, t) := \bigcup_{\tau \leq t} S(t, \tau) B$.

Definition 2.6. An evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in $X$ is pullback strongly bounded if, for each $t \in \mathbb{R}$ and each bounded subset $B$ of $X$, $\gamma_p(B, t)$ is bounded.

Definition 2.7. An evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in $X$ is pullback asymptotically compact if, for each $t \in \mathbb{R}$, each sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$ as $n \to \infty$ and each bounded sequence $\{x_n\}$ in $X$ such that $\{S(t, \tau_n)x_n\} \subset X$ is bounded, the sequence $\{S(t, \tau_n)x_n\}$ is relatively compact in $X$.

Definition 2.8. We say that a family of bounded subsets $\{B(t) : t \in \mathbb{R}\}$ of $X$ is pullback absorbing for the evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$, if for each $t \in \mathbb{R}$ and for any bounded subset $B$ of $X$, there exists $\tau_0(t, B) \leq t$ such that $S(t, \tau)B \subset B(t)$ for all $\tau \leq \tau_0(t, B)$.

Definition 2.9. A family of subsets $\{A(t) : t \in \mathbb{R}\}$ of $X$ is called a pullback attractor for the evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ if it is invariant (that is, $S(t, \tau)A(t) = A(t)$, for any $t \geq \tau$), $A(t)$ is compact for all $t \in \mathbb{R}$, and pullback attracts bounded subsets of $X$ at time $t$, for each $t \in \mathbb{R}$.

In applications, to prove that a process has a pullback attractor we use the Theorem 2.11, proved in [8], which gives a sufficient condition for existence of a compact pullback attractor. For this, we will need the concept of pullback strongly bounded dissipativeness.

Definition 2.10. An evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in $X$ is pullback strongly bounded dissipative if, for each $t \in \mathbb{R}$, there is a bounded subset $B(t)$ of $X$ which pullback absorbs bounded subsets of $X$ at time $s$ for each $s \leq t$; that is, given a bounded subset $B$ of $X$ and $s \leq t$, there exists $\tau_0(s, B)$ such that $S(s, \tau)B \subset B(t)$, for all $\tau \leq \tau_0(s, B)$.

Now we can present the result which guarantees the existence of pullback attractors for non-autonomous problems, see [8].

Theorem 2.11. If an evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in the metric space $X$ is pullback strongly bounded dissipative and pullback asymptotically compact, then $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ has a pullback attractor $\{A(t) : t \in \mathbb{R}\}$ with the property that $\bigcup_{\tau \leq t} A(\tau)$ is bounded for each $t \in \mathbb{R}$.

The next result gives sufficient conditions for pullback asymptotic compactness, and its proof can be found in [8].

Theorem 2.12. Let $\{S(t, s) : t \geq s \in \mathbb{R}\}$ be a pullback strongly bounded evolution process such that $S(t, s) = L(t, s) + U(t, s)$, where there exist a non-increasing function $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, with $k(\sigma, r) \to 0$ when $\sigma \to \infty$, and for all $s \leq t$ and $x \in X$ with $\|x\| \leq r$, $\|L(t, s)x\| \leq k(t - s, r)$, and $U(t, s)$ is compact. Then, the family of evolution process $\{S(t, s) : t \geq s \in \mathbb{R}\}$ is pullback asymptotically compact.

3. Linear analysis. In this section we consider the linear problem associated with (1)-(2), in this case we consider the singularly non-autonomous linear parabolic
problem
\[
\begin{align*}
\begin{cases}
\dot{w}_t = A_{(a)}(t)w, \quad t > \tau, \\
\quad w(\tau) = w_0, \quad \tau \in \mathbb{R},
\end{cases}
\end{align*}
\]
where \( w, w_0 \) are defined in (8) and the linear unbounded operator \( A_{(a)} \) is defined by (9)-(10).

We use the term singularly non-autonomous to express the fact that the unbounded operator \( A_{(a)}(t) \) is time dependent and generates a semigroup that satisfies an estimate as in (11).

It is not difficult to see that \( \det(A_{(a)}(t)) = \kappa \Lambda^{3/2} \) and \( 0 \in \rho(A_{(a)}(t)) \) for any \( t \in \mathbb{R} \).

Moreover we have that the operator \( A_{(a)}^{-1}(t) : D(A_{(a)}^{-1}(t)) \to Y \) is defined by
\[
D(A_{(a)}^{-1}(t)) = L^2(\Omega) \times H^{-2}(\Omega) \times H^{-2}(\Omega),
\]
where \( H^{-2}(\Omega) \) denotes the dual \( X^{-\frac{1}{2}} \) of \( X\frac{1}{2} \) and
\[
A_{(a)}^{-1}(t) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{cases}
\begin{bmatrix} \frac{1}{\kappa}(a(t))^{2}\Lambda^{-\frac{1}{2}} & -\Lambda^{-1} & -\frac{1}{\kappa}a(t)\Lambda^{-1} \\ I & 0 & 0 \\ -\frac{2}{\kappa}a(t)I & 0 & \frac{1}{\kappa}\Lambda^{-\frac{1}{2}} \end{bmatrix} & \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \\
= \begin{bmatrix} \frac{1}{\kappa}(a(t))^{2}\Lambda^{-\frac{1}{2}}u - \Lambda^{-1}v - \frac{1}{\kappa}a(t)\Lambda^{-1}\theta \\ u \\ -\frac{2}{\kappa}a(t)u + \frac{1}{\kappa}\Lambda^{-\frac{1}{2}}\theta \end{bmatrix},
\end{cases}
\]
for all \( t \in \mathbb{R} \).

**Proposition 1.** Denote by \( Y_{-1} \) the extrapolation space of \( Y = X^{\frac{1}{2}} \times X \times X \) generated by operator \( A_{(a)}^{-1}(t) \). The following equality holds
\[
Y_{-1} = X \times X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}.
\]

**Proof.** Recall first that \( Y_{-1} \) is the completion of the normed space \( (Y, \|A_{(a)}^{-1}(t)\cdot\|_Y) \).

Since there are positive constants \( C_1 \) and \( C_2 \) such that for any \( (u, v, \theta) \in Y_{-1} \) we have that
\[
\|A_{(a)}^{-1}(t) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \|_Y \leq C_1 \| \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \|_{Y_{-1}},
\]
and
\[
\| \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \|_{Y_{-1}} \leq C_2 \|A_{(a)}^{-1}(t) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \|_Y.
\]

Below is the proof of this last statement. Let \( [u \ v \ \theta]^T \in Y_{-1} \), and note that
\[
\begin{align*}
\|A_{(a)}^{-1}(t) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \|_Y &= \left\| \begin{bmatrix} \frac{1}{\kappa}(a(t))^{2}\Lambda^{-\frac{1}{2}}u - \Lambda^{-1}v - \frac{1}{\kappa}a(t)\Lambda^{-1}\theta \\ u \\ -\frac{2}{\kappa}a(t)u + \frac{1}{\kappa}\Lambda^{-\frac{1}{2}}\theta \end{bmatrix} \right\|_Y \\
&= \left\| \frac{1}{\kappa}(a(t))^{2}\Lambda^{-\frac{1}{2}}u - \Lambda^{-1}v - \frac{1}{\kappa}a(t)\Lambda^{-1}\theta \right\|_{X^{\frac{1}{2}}} + \|u\|_X + \left\| -\frac{2}{\kappa}a(t)u + \frac{1}{\kappa}\Lambda^{-\frac{1}{2}}\theta \right\|_X \\
&\leq C_1 \left( \left\| \frac{1}{\kappa}(a(t))^{2}\Lambda^{-\frac{1}{2}}u - \Lambda^{-1}v - \frac{1}{\kappa}a(t)\Lambda^{-1}\theta \right\|_{X^{\frac{1}{2}}} + \|u\|_X + \left\| -\frac{2}{\kappa}a(t)u + \frac{1}{\kappa}\Lambda^{-\frac{1}{2}}\theta \right\|_X \right).
\end{align*}
\]
On the other hand, let \([u \ v \ \theta]^T \in Y_{-1}\), and note firstly that
\[
\left\| \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \right\|_{Y_{-1}} = \|u\|_X + \|v\|_{X^{-\frac{1}{2}}} + \|\theta\|_{X^{-\frac{1}{2}}}
\]
(14)

The last two parcels of (14) can be estimated as follows
\[
\|\Lambda^{-\frac{1}{2}}\theta\|_X = \kappa \left\| \frac{1}{\kappa}\Lambda^{-\frac{1}{2}}\theta - \frac{1}{\kappa}a(t)u + \frac{1}{\kappa}a(t)u \right\|_X
\]
(15)

and
\[
\|\Lambda^{-\frac{1}{2}}v\|_X = \left\| -\Lambda^{-\frac{1}{2}}v \right\|_X
\]
(16)

Then, combining (14) with (15) and (16) we obtain that
\[
\left\| \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \right\|_{Y_{-1}} \leq C_2 \left\| A_{a}^{-1}(t) \begin{bmatrix} u \\ a \\ \theta \end{bmatrix} \right\|_Y.
\]

So we conclude that the completion of \((Y, \|A_{a}^{-1}(t) \cdot\|_Y)\) and \((Y, \|\cdot\|_{Y_{-1}})\) coincide.
Remark 1. Following the same ideas from [5] and [19], we obtain that for all \( t, \tau, s \) there exists a positive constant \( M \) (independent of \( t \)), such that

\[
\|(\lambda I + A_{(a)}(t))^{-1}\|_{L(Y)} \leq \frac{M}{1 + |\lambda|}, \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \text{Re}\lambda \geq 0.
\]

From this we can conclude that \( A_{(a)}(t) \) is uniformly sectorial (in \( Y \)).

Note that the operator \( A_a(t) \) can be extended to its closed \( Y_{-1} \)–realization (see Amann [1] p. 262), which we will still denote by the same symbol so that \( A_a(t) \) considered in \( Y_{-1} \) is then sectorial positive operator (see [5]). Our next concern will be to obtain embedding of the spaces from the fractional powers scale \( Y_{\alpha -1} \), \( \alpha \geq 0 \), generated by \( (A_a(t), Y_{-1}) \).

**Theorem 3.1.** The operators \( A_{(a)}(t) \) are uniformly sectorial and the map \( \mathbb{R} \ni t \mapsto A_{(a)}(t) \in \mathcal{L}(Y) \) is uniformly Hölder continuous. Then, for each functional parameter \( a \), there exist a process

\[
\{L_{(a)}(t, \tau) : t \geq \tau \in \mathbb{R}\}
\]

(or simply \( L_{(a)}(t, \tau) \)) associated with the operator \( A_{(a)}(t) \), that is given by

\[
L_{(a)}(t, \tau) = e^{-(t-\tau)A_{(a)}(\tau)} + \int_{\tau}^{t} L_{(a)}(t, s)[A_{(a)}(\tau) - A_{(a)}(s)]e^{-(s-\tau)A_{(a)}(\tau)}ds, \quad \forall t \geq \tau.
\]

The linear evolution operator \( \{L_{(a)}(t, \tau) : t \geq \tau \in \mathbb{R}\} \) satisfies the condition (12).

**Proof.** Following the same ideas from [11] and [19], we can conclude that the operator \( A_{(a)}(t) \) is a sectorial positive operator in \( Y_{-1} \). It is not difficult to see that it is also closed and densely defined. Now, note that for \( [u \ v \ \theta]^T \in Y \), and \( t, s \in \mathbb{R} \), we can estimate the norm \( \|((A_{(a)}(t) - A_{(a)}(s))[u \ v \ \theta]^T\|_{Y_{-1}} \) using (4) in the following way

\[
\left\|(A_{(a)}(t) - A_{(a)}(s)) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \right\|_{Y_{-1}} = \left\| \begin{bmatrix} 0 & 0 & -a(t) \alpha \frac{1}{2} \\ 0 & 0 & -a(s) \alpha \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \right\|_{Y_{-1}}
\]

\[
= \left\| \begin{bmatrix} -a(t-\alpha s) \alpha \frac{1}{2} \theta \\ -a(t-\alpha s) \alpha \frac{1}{2} v \end{bmatrix} \right\|_{Y_{-1}}
\]

\[
= |a(t) - a(s)||(-\Delta)\theta|_{X}^{\frac{1}{2}} + |a(t) - a(s)||(-\Delta)v|_{X}^{\frac{1}{2}}
\]

\[
= |a(t) - a(s)||\theta|_{X} + |a(t) - a(s)||v|_{X}
\]

\[
= |a(t) - a(s)||\theta|_{X} + |a(t) - a(s)|\|v\|_{X}
\]

\[
\leq c|t - s|^\beta \left\| \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \right\|_{X^{\frac{1}{2}} \times X \times X} ,
\]

for any \( t, \tau, s \in \mathbb{R} \), hence the map \( \mathbb{R} \ni t \mapsto A_{(a)}(t) \in \mathcal{L}(Y) \) is uniformly Hölder continuous and

\[
\|A_{(a)}(t) - A_{(a)}(s)\|_{\mathcal{L}(Y, Y_{-1})} \leq c(t - s)^\beta.
\]

Therefore, there exists a linear evolution process \( \{L_{(a)}(t, \tau) : t \geq \tau \in \mathbb{R}\} \) associated with the operator \( A_{(a)}(t) \), that is given by

\[
L_{(a)}(t, \tau) = e^{-(t-\tau)A_{(a)}(\tau)} + \int_{\tau}^{t} L_{(a)}(t, s)[A_{(a)}(\tau) - A_{(a)}(s)]e^{-(s-\tau)A_{(a)}(\tau)}ds, \quad \forall t \geq \tau.
\]
Furthermore, the process \( \{ L(t, \tau) : t \geq \tau \in \mathbb{R} \} \) satisfies the condition (12).

4. Existence of global solutions. In this section we study the existence of global solutions for (7). It is not difficult to prove the following result, see for instance Lemma 2.4 in [10].

**Lemma 4.1.** Let \( f \in C^1(\mathbb{R}^2) \) be a function such that the condition (6) is holds. Then
\[
|f(t, s_1) - f(t, s_2)| \leq c|s_1 - s_2|(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}), \quad \forall \ t, s_1, s_2 \in \mathbb{R}.
\]

**Lemma 4.2** ([10]). Assume that \( 1 < \rho < \frac{N+1}{N-4} \) and let \( f \in C^1(\mathbb{R}) \) be a function such that
\[
|f'(s)| \leq C(1 + |s|^{\rho-1}), \quad \forall s \in \mathbb{R}.
\]
Then there exists \( \alpha \in (0, 1) \) such that the Nemitski˘ı operator \( f^c : X^{1/2} \to X^{-1/2} \) is Lipschitz continuous in bounded subsets of \( X^{1/2} \), uniformly in \( t \in \mathbb{R} \).

**Remark 2.** Since \( L^{2/\alpha}(\Omega) \to X \), it follows from the proof of the [10, Lemma 2.5] that \( f^c : X^{1/2} \to X \) is Lipschitz continuous in bounded subsets; that is,
\[
\|f^c(u) - f^c(v)\|_X \leq \tilde{c}\|f^c(u) - f^c(v)\|_{L^{2/\alpha}(\Omega)} \leq \tilde{c}\|u - v\|_{X^{1/2}},
\]
with \( \tilde{c} = \tilde{c}(\|u\|_{X^{1/2}}, \|v\|_{X^{1/2}}) \). The scheme below describes this situation:
\[
X^{1/2} \hookrightarrow H^2(\Omega) \hookrightarrow L^{2N/(N-4)}(\Omega) \hookrightarrow L^{2N/(N-4)}(\Omega) \hookrightarrow X,
\]
in the last inclusion we use that \( \rho < \frac{N}{N-4} \).

**Proposition 2.** The operator \( A_{(a)}(t) \) given in (9) is maximal accretive.

**Proof.** Note that
\[
\left< A_{(a)}(t) \begin{bmatrix} u \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ \theta \end{bmatrix} \right> = \left< \begin{bmatrix} -\Lambda u - a(t)\Lambda^{1/2}\theta \\ a(t)\Lambda^{1/2}v + \kappa\Lambda^{1/2}\theta \end{bmatrix}, \begin{bmatrix} u \\ \theta \end{bmatrix} \right>_{X^{1/2} \times X} = \langle v, u \rangle_{X^{1/2}} + \langle \Lambda u - a(t)\Lambda^{1/2}\theta, v \rangle_X + \langle a(t)\Lambda^{1/2}v + \kappa\Lambda^{1/2}\theta, \theta \rangle_X
\]
\[
= \langle \Lambda^{1/2}v, \Lambda^{1/2}u \rangle_X - \langle \Lambda^{1/2}u, \Lambda^{1/2}v \rangle_X = \langle \Lambda^{1/2}v, \Lambda^{1/2}u \rangle_X
\]
From this
\[
Re\langle A_{(a)}(t) \begin{bmatrix} u \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ \theta \end{bmatrix} \rangle = \langle \kappa\Lambda^{1/2}\theta, \Lambda^{1/2}\theta \rangle_X \geq 0, \begin{bmatrix} u \\ \theta \end{bmatrix} \in X^1 \times X^{1/2}.
\]
This proves the accretivity of \( A_{(a)}(t) \). Additionally, for each \( \begin{bmatrix} \bar{u} \\ \bar{\theta} \end{bmatrix} \in Y \) the equation
\[
[I + A_{(a)}(t)] \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} \bar{u} \\ \bar{\theta} \end{bmatrix}
\]
possesses a unique solution
\[
\begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} B^{-1}(1 + \kappa\Lambda^{1/2} + a^2(t)\Lambda)\bar{u} - B^{-1}(1 + \kappa\Lambda^{1/2})\bar{v} - B^{-1}a(t)\Lambda^{1/2}\bar{\theta} \\ B^{-1}(1 + \kappa\Lambda^{1/2})\bar{u} + B^{-1}(1 + \kappa\Lambda^{1/2})\bar{v} + B^{-1}a(t)\Lambda^{1/2}\bar{\theta} \\ -B^{-1}a(t)\Lambda^{1/2}\bar{u} - B^{-1}a(t)\Lambda^{1/2}\bar{v} + B^{-1}(1 + \Lambda)\bar{\theta} \end{bmatrix} \in X^1 \times X^{1/2} \times X^{1/2},
\]
where \( B = 1 + \kappa\Lambda^{1/2} + \Lambda + \kappa\Lambda^{1/2} + a^2(t)\Lambda \). This completes the proof. \( \square \)
Remark 3. Below we have a partial description of the fractional power spaces scale for $A_{(\alpha)}(t)$: for convenience we denote $Y$ by $Y_0$, then

$$Y_0 \hookrightarrow Y_{\alpha - 1} \hookrightarrow Y_{-1},$$

for all $0 < \alpha < 1$,

where

$$Y_{\alpha - 1} = [Y_{-1}, Y_0]_{\alpha} = X^{\frac{\alpha}{2}} \times X^{\frac{\alpha - 1}{2}} \times X^{\frac{\alpha - 2}{2}},$$

where $[\cdot, \cdot]_{\alpha}$ denotes the complex interpolation functor (see [24]). The first equality follows from Proposition 2 (since $0 \in \rho(A_{(\alpha)}(t))$) see [1, Example 4.7.3 (b)] and the second equality follows from [11, Proposition 2].

Corollary 1. If $f$ is as in Lemma 4.2, then the function $F : Y \to Y_{\alpha - 1}$ $(\alpha \in (0, 1))$, given by (1), is Lipschitz continuous in bounded subsets of $Y$.

Now, Theorem 2.3 guarantees local well posedness for the problem (7) in the energy space $Y$.

Corollary 2. If $f, F$ are like in the Corollary 1, then given $r > 0$, there is a time $\tau = \tau(r) > 0$, such that for all $w_0 \in BY(0; r)$ there exists a unique solution $w : [t_0, t_0 + \tau] \to Y$ of the problem (7) starting in $w_0$. Moreover, such solutions are continuous with respect the initial data in $BY(0; r)$.

Since $\tau$ can be chosen uniformly in bounded subsets of $Y$, the solutions which do not blow up in $Y$ must exist globally. Alternatively, we obtain a uniform in time estimate of $\|(u(t), \partial_t u(t), \theta(t))\|_Y$, such estimate is needed to justify global solvability of the problem (7) in $Y$.

Consider the original system (1) (or (7) in $Y$). Multiplying the first equation in (1) by $u_t$, and the second equation in (1) by $\theta$, we get the system

$$\begin{cases}
    u_t u_t + \Delta^2 u u_t + a(t) \Delta \theta u_t = f(t, u) u_t, \ t > \tau, \ x \in \Omega, \\
    \theta_t - \kappa \Delta \theta - a(t) \Delta u_t \theta = 0, \ t > \tau, \ x \in \Omega,
\end{cases}$$

and integrating over $\Omega$ we obtain

$$\begin{cases}
    \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t|^2 \ dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta u|^2 \ dx + a(t) \int_{\Omega} \Delta u_t u_t \ dx = \frac{d}{dt} \int_{\Omega} \int_{0}^{u} f(t, s) ds dx, \\
    \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta|^2 \ dx + \kappa \int_{\Omega} |\nabla \theta|^2 \ dx - a(t) \int_{\Omega} \Delta u_t \theta \ dx = 0,
\end{cases} \tag{17}$$

for any $t > \tau$. Note that

$$\int_{\Omega} (\Delta) \theta u_t \ dx = \int_{\Omega} \theta (-\Delta) u_t \ dx. \tag{18}$$

Combining (17) with (18) we have

$$\frac{d}{dt} \frac{1}{2} \left( \int_{\Omega} |u_t|^2 \ dx + \int_{\Omega} |\Delta u|^2 \ dx + \int_{\Omega} |\theta|^2 \ dx - 2 \int_{0}^{u} f(t, s) ds \ dx \right) = -\kappa \int_{\Omega} |\nabla \theta|^2 \ dx, \tag{19}$$

for any $t > \tau$.

The total energy of the system $\mathcal{E}(t)$ associated with the solution $(u(t), \partial_t u(t), \theta(t))$ of (1)-(2) is defined by

$$\mathcal{E}(t) = \frac{1}{2} \|u(t)\|_{X^\alpha}^2 + \frac{1}{2} \|u_t(t)\|_{X}^2 + \frac{1}{2} \|\theta(t)\|_{X}^2 - \int_{0}^{u} f(t, s) ds \ dx. \tag{20}$$
This identity says that the function $t \mapsto \mathcal{E}(t)$ becomes monotone decreasing. We obtain (from (5)) that for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
\[ \int_{\Omega} \int_{0}^{u(\cdot, t)} f(t, s) ds dx \leq \varepsilon \|u(\cdot, t)\|_{X}^{2} + C_\varepsilon, \tag{21} \]
then the property $\mathcal{E}(t) \leq \mathcal{E}(\tau)$ offers an a priori estimate of the solution $(u, \partial_t u, \theta)$ in $Y$. In fact,
\[ \frac{1}{2} \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{Y}^{2} \leq c \mathcal{E}(\tau) + c\varepsilon_0 \|u(\cdot, t)\|_{X}^{2} + C_{\varepsilon_0} \]
\[ \leq c \mathcal{E}(\tau) + c\varepsilon_0 \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{Y}^{2} + C_{\varepsilon_0}, \]
and, if we choose $0 < \varepsilon_0 < \frac{1}{2c}$, we get a boundedness as desired, that is,
\[ \limsup_{t \to +\infty} \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{Y} < +\infty. \]

With this, we ensure that there exists a global solution $w(t)$ for Cauchy problem (7) in $Y$ and it defines an evolution process $\{S_{(a)}(t, \tau) : t \geq \tau \in \mathbb{R}\}$, that is,
\[ S_{(a)}(t, \tau)w_0 = w(t), \ \forall t \geq \tau \in \mathbb{R}. \]

According to [14]
\[ S_{(a)}(t, \tau)w_0 = L_{(a)}(t, \tau)w_0 + \int_{\tau}^{t} L_{(a)}(t, s) F(s, S_{(a)}(s, \tau)w_0) ds, \ \forall t \geq \tau \in \mathbb{R}, \tag{22} \]
where $\{L_{(a)}(t, \tau) : t \geq \tau \in \mathbb{R}\}$ is the linear evolution process associated with the homogeneous problem (7).

5. **Dissipativeness of the thermoelastic equation.** In this section we combine the arguments from [2], [3], [4], [6] and [16] in order to prove the existence of pullback attractors for (1)-(2). To achieve our purpose we consider the functionals
\[ \phi(t) = \int_{\Omega} uu_t dx \tag{23} \]
and
\[ \psi(t) = -\int_{\Omega} u_t (\Delta^{-1} \theta) dx. \tag{24} \]

From this we define an energy functional
\[ \mathcal{L}(t) = M \mathcal{E}(t) + \delta_1 \phi(t) + \delta_2 \psi(t) \tag{25} \]
where
\[ 0 < \delta_1 < \delta_2 < 1 \text{ and } M > 0 \tag{26} \]
will be fixed later. We recall that $\mathcal{E}(t)$ is decreasing since $\mathcal{E}'(t) \leq 0$ from (19).

**Theorem 5.1.** For $M > 0$ sufficiently large, there exist constants $M_1, M_2 > 0$ such that
\[ \mathcal{L}'(t) \leq -M_1 \mathcal{E}(t) + M_2, \tag{27} \]
for all $t \geq 0$. 


Proof. In the following, \( C_0 \) and \( C_1 \) will denote positive constants depending on the embedding constants and initial data, respectively, as far it is necessary.

Note that
\[
\mathcal{L}'(t) = ME'(t) + \delta_1 \phi'(t) + \delta_2 \psi'(t).
\]

Due to (25) and Poincaré’s inequality we have
\[
ME'(t) = -\kappa M \int_\Omega |\nabla \theta|^2 dx \leq -\frac{\kappa M}{2} \int_\Omega |\nabla \theta|^2 dx - \frac{\kappa \lambda_1 M}{2} \int_\Omega |\theta|^2 dx,
\]
where \( \lambda_1 \) is the first eigenvalue of negative Laplacian operator with zero Dirichlet boundary condition. Furthermore,
\[
\delta_1 \phi'(t) = \delta_1 \int_\Omega |u_t|^2 dx + \delta_1 \int_\Omega uu_t dx
\]
\[
= \delta_1 \int_\Omega |u_t|^2 dx - \delta_1 \int_\Omega |\Delta u|^2 dx - a(t) \delta_1 \int_\Omega \Delta \theta u dx + \delta_1 \int_\Omega f(t, u) u dx
\]
\[
= \delta_1 \int_\Omega |u_t|^2 dx - \delta_1 \int_\Omega |\Delta u|^2 dx - a(t) \delta_1 \int_\Omega \theta \Delta u dx + \delta_1 \int_\Omega f(t, u) u dx
\]
and from (3) we get
\[
\delta_1 \phi'(t) \leq \delta_1 \int_\Omega |u_t|^2 dx - \delta_1 \int_\Omega |\Delta u|^2 dx - a_0 \delta_1 \int_\Omega \theta \Delta u dx + \delta_1 \int_\Omega f(t, u) u dx
\]
and by Young’s inequality
\[
\delta_1 \phi'(t) \leq \delta_1 \int_\Omega |u_t|^2 dx - \delta_1 \int_\Omega |\Delta u|^2 dx + \frac{a_0}{2} \int_\Omega |\theta|^2 dx
\]
\[
+ \frac{a_0 \delta_1^2}{2} \int_\Omega |\Delta u|^2 dx + \delta_1 \int_\Omega f(t, u) u dx.
\]

To deal with the integral term, just notice that from dissipativeness condition (5), there exists \( C_\nu > 0 \) such that
\[
\int_\Omega f(t, u) u dx \leq \nu \|u\|_{X}^2 + C_\nu,
\]
and thus,
\[
\delta_1 \phi'(t) \leq \delta_1 \int_\Omega |u_t|^2 dx - \delta_1 \int_\Omega |\Delta u|^2 dx + \frac{a_0}{2} \int_\Omega |\theta|^2 dx
\]
\[
+ \frac{a_0 \delta_1^2}{2} \int_\Omega |\Delta u|^2 dx + \delta_1 \nu \int_\Omega |u|^2 dx + \delta_1 C_\nu
\]
\[
\leq \delta_1 \int_\Omega |u_t|^2 dx - \delta_1 \int_\Omega |\Delta u|^2 dx + \frac{a_0}{2} \int_\Omega |\theta|^2 dx
\]
\[
+ \frac{a_0 \delta_1^2}{2} \int_\Omega |\Delta u|^2 dx + \delta_1 \nu \int_\Omega |\nabla u|^2 dx + \delta_1 C_\nu
\]
\[
\leq \delta_1 \int_\Omega |u_t|^2 dx - \delta_1 \int_\Omega |\Delta u|^2 dx + \frac{a_0}{2} \int_\Omega |\theta|^2 dx
\]
\[
+ \frac{a_0 \delta_1^2}{2} \int_\Omega |\Delta u|^2 dx + \frac{\delta_1 \mu_1 \nu}{\lambda_1} \int_\Omega |\Delta u|^2 dx + \delta_1 C_\nu
\]
Since there exists $\varrho$ and from (3) we get $\nu < \lambda$ 

\[
\delta \varrho^2 \leq \psi^2 \delta + \varrho^2 \delta^2 \leq \psi^2 \delta - \delta \alpha(t) \int |u_t|^2 dx
\]

and from (3) we get

\[
\delta \psi'(t) \leq \frac{\delta^2}{2} \int |\nabla u|^2 dx + \frac{1}{2} \int |\nabla \theta|^2 dx + a(t) \delta^2 \int |\theta|^2 dx - \delta \int f(t, u)(\Delta^{-1} \theta) dx
\]

\[
- \delta \kappa \int u_t \theta dx - \delta \alpha(t) \int |u_t|^2 dx.
\]

Since there exists $c_0 > 0$ such that

\[
\int |\nabla \theta|^2 dx \leq c_0 \int |\nabla \theta|^2 dx,
\]

where $\varrho = \Delta^{-1} \theta$, by Young's inequality and Poincaré's inequality we obtain

\[
\delta \psi'(t) 
\leq \frac{\delta^2}{2} \int |\nabla u|^2 dx + \frac{1}{2} \int |\nabla \theta|^2 dx + a_1 \delta_2 \int |\theta|^2 dx + \frac{\delta^2}{2} \int f(t, u)^2 dx
\]

\[
+ \frac{1}{2} \int |\Delta^{-1} \theta|^2 dx + \frac{\kappa}{2 \delta_1} \int |\theta|^2 dx - \frac{\delta \delta_1 \kappa}{2} \int |u_t|^2 dx
\]

\[
\leq \frac{\delta^2}{2} \int |\nabla u|^2 dx + \frac{1}{2} \int |\nabla \theta|^2 dx + a_1 \delta_2 \int |\theta|^2 dx + \frac{\delta^2}{2} \int f(t, u)^2 dx
\]

\[
+ \frac{1}{2 \delta_1} \int |\nabla \theta|^2 dx + \frac{\kappa}{2 \delta_1} \int |\theta|^2 dx + \left(\frac{\delta \delta_1 \kappa}{2} - \delta_2 a_0\right) \int |u_t|^2 dx.
\]

Hence

\[
\delta \psi'(t) 
\leq \frac{\delta^2}{2} \int |\nabla u|^2 dx + \left(\frac{1}{2} + \frac{c_0}{2 \delta_1}\right) \int |\nabla \theta|^2 dx + a_1 \delta_2 \int |\theta|^2 dx + \frac{\delta^2}{2} \int f(t, u)^2 dx
\]

\[
+ \frac{\kappa}{2 \delta_1} \int |\theta|^2 dx + \left(\frac{\delta \delta_1 \kappa}{2} - \delta_2 a_0\right) \int |u_t|^2 dx
\]

\[
\leq \frac{\delta^2}{2} \int |\nabla u|^2 dx + \left(\frac{1}{2} + \frac{c_0}{2 \delta_1}\right) \int |\nabla \theta|^2 dx + a_1 \delta_2 \int |\theta|^2 dx + \frac{\delta^2}{2} \int f(t, u)^2 dx
\]

\[
+ \frac{\kappa}{2 \delta_1} \int |\theta|^2 dx + \left(\frac{\delta \delta_1 \kappa}{2} - \delta_2 a_0\right) \int |u_t|^2 dx.
\]
From (6), there exists $C > 0$ such that

$$
\int_{\Omega} |f(t,u)|^2 dx \leq C \int_{\Omega} |u|^2 dx + C \int |u|^{2\rho} dx.
$$

Since the condition $1 \leq \rho \leq \frac{N}{N-4}$ implies that $H^2(\Omega) \hookrightarrow L^{2\rho}(\Omega)$, we get

$$
\int_{\Omega} |f(t,u)|^2 dx \leq C \int_{\Omega} |u|^2 dx + C \leq \hat{C}_1 \int_{\Omega} |\Delta u|^2 dx + \hat{C}_2 \tag{32}
$$

whenever $\|u\|_{X^\frac{4}{\lambda}} \leq r$ (as [10] and [12]).

Let $C_{\mu,\lambda} := 1 - \frac{\mu\nu}{\lambda} > 0$, combining (28) together with (29), (30), (31) and (32), we obtain

$$
\mathcal{L}'(t) \leq \left( \frac{1}{2} + \frac{c_0}{2\lambda_1} - \frac{M\kappa}{2} \right) \int_{\Omega} |\nabla \theta|^2 dx + \left[ \delta_1 \left( 1 + \frac{\delta_2\kappa}{2} \right) - \delta_2a_0 \right] \int_{\Omega} |u|^2 dx
$$

$$
+ \left( \frac{\mu_1\delta_2^2}{2} + \frac{\tilde{C}_1\delta_2^2}{2} + \frac{a_0\delta_1^2}{2} - C_{\mu,\lambda}\delta_1 \right) \int_{\Omega} |\Delta u|^2 dx
$$

$$
+ \left( \frac{a_0}{2} + \frac{\kappa}{2\delta_1} + a_1\delta_2 - \frac{\kappa\lambda_1 M}{2} \right) \int_{\Omega} |\theta|^2 dx + \frac{\delta_2\tilde{C}_2}{2} + \delta_1C_{\nu},
$$

and by (26)

$$
\mathcal{L}'(t) \leq \left( \frac{1}{2} + \frac{c_0}{2\lambda_1} - \frac{M\kappa}{2} \right) \int_{\Omega} |\nabla \theta|^2 dx + \left[ \delta_1 \left( 1 + \frac{\delta_2\kappa}{2} \right) - \delta_2a_0 \right] \int_{\Omega} |u|^2 dx
$$

$$
+ \left( \tilde{C}_0\delta_2^2 - C_{\mu,\lambda}\delta_1 \right) \int_{\Omega} |\Delta u|^2 dx + \left( \frac{a_0}{2} + \frac{\kappa}{2\delta_1} + a_1\delta_2 - \frac{\kappa\lambda_1 M}{2} \right) \int_{\Omega} |\theta|^2 dx
$$

$$
+ \frac{\delta_2\tilde{C}_2}{2} + \delta_1C_{\nu},
$$

where $\tilde{C}_0 = \frac{a_0}{2} + \frac{\kappa}{2\delta_1} + \frac{\mu_1\delta_2^2}{2} > 0$.

Let $C_{\kappa} = 1 + \frac{\kappa}{2} > 0$. Now, fixed $0 < \delta_2 < 1$, choose $\delta_1$ such that

$$
0 < \frac{\tilde{C}_0}{C_{\mu,\lambda}} \delta_2^2 < \delta_1 < \frac{a_0}{C_{\kappa}} \delta_2,
$$

and, thus

$$
\tilde{C}_0\delta_2^2 - C_{\mu,\lambda}\delta_1 < 0 \quad \text{and} \quad \left( 1 + \frac{\delta_2\kappa}{2} \right) \delta_1 < C_{\kappa}\delta_1 < \delta_2a_0.
$$

Finally, choose $M > 0$ sufficient large such that

$$
\frac{1}{2} + \frac{c_0}{2\lambda_1} - \frac{M\kappa}{2} < 0 \quad \text{and} \quad \frac{a_0}{2} + \frac{\kappa}{2\delta_1} + a_1\delta_2 - \frac{\kappa\lambda_1 M}{2} < 0.
$$

With these choices, we will have

$$
\mathcal{L}'(t) \leq \left[ \delta_1 \left( 1 + \frac{\delta_2\kappa}{2} \right) - \delta_2a_0 \right] \int_{\Omega} |u|^2 dx + \left( \tilde{C}_0\delta_2^2 - C_{\mu,\lambda}\delta_1 \right) \int_{\Omega} |\Delta u|^2 dx
$$

$$
+ \left( \frac{a_0\delta_2^2}{2} + \frac{\kappa}{2\delta_1} + a_1\delta_2 - \frac{\kappa\lambda_1 M}{2} \right) \int_{\Omega} |\theta|^2 dx + \frac{\delta_2\tilde{C}_2}{2} + \delta_1C_{\nu}.
$$

Thus

$$
\mathcal{L}'(t) \leq -M_1 \left( \frac{1}{2} \|u(t)\|_{X^\frac{4}{\lambda}}^2 + \frac{1}{2} \|u_t(t)\|_{X}^2 + \frac{1}{2} \|\theta(t)\|_{X}^2 \right) + M_2, \tag{33}
$$

where $M_1, M_2 > 0$.
where
\[ M_1 = \min \left\{ 2\delta_1 + \frac{\delta_2 \kappa}{2} - 2\delta_2 \alpha_0, 2\tilde{C}_\delta \delta_1^2 - 2C_\mu \kappa \delta_1, \alpha_0 \delta_1^2 + \frac{\kappa}{\delta_1} + 2a_1 \delta_2 - \kappa \lambda \mu, M \right\} > 0, \]
and \( M_2 = \frac{\delta_2 \tilde{C}_\delta}{2} + \delta_1 C_\nu. \)

Now we observe that if \( u \in H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^{2N/(N-4)}(\Omega), \)
then
\[ |u|^{\rho+1} \in L^{2N/(\rho+1)}(\Omega) \hookrightarrow L^1(\Omega) \]
for all \( 1 < \rho < \frac{N}{N-4}, \)
and our hypothesis on \( f \) implies that \( |f(t, s)| \leq c(1 + |s|^{\rho}), \)
\( s \in \mathbb{R}. \)
Therefore, we can find a constant \( \bar{c} > 1 \) such that for all \( u \in X^\frac{1}{2}, \)
\[ -\int_\Omega \int_0^u f(t, s) ds dx \leq \bar{c} \| u \|_{X^\frac{1}{2}}^2 (1 + \| u \|^{\rho-1}_{X^\frac{1}{2}}), \]
and therefore
\[ -\tilde{d} \int_\Omega \int_0^u f(t, s) ds dx \leq \| u \|_{X^\frac{1}{2}}^2, \quad (34) \]
whenever \( \| u \|_{X^\frac{1}{2}} \leq r \) and \( \tilde{d} = \frac{M_1 \bar{c}}{1 + \rho - 1} < 1. \)

Thanks to (33) and (34) we deduce that (since \( \tilde{d} < 1 \))
\[ L'(t) \leq -M_1 \bar{d} E(t) + M_2, \]
where \( M_1 = \frac{M_1 \bar{d}}{4} > 0, \) for all \( t \geq 0. \) This concludes the proof of the theorem. \( \square \)

**Remark 4.** For every \( t \in \mathbb{R}, \) from (21) we have
\[ E(t) = \frac{1}{2} \| u(t) \|_{X^\frac{1}{2}}^2 + \frac{1}{2} \| u(t) \|_{X^\frac{1}{2}}^2 + \frac{1}{2} \| \theta(t) \|_{X^\frac{1}{2}}^2 - \int_\Omega \int_0^u f(t, s) ds dx \]
\[ \geq \frac{1}{2} \| u(t) \|_{X^\frac{1}{2}}^2 + \frac{1}{2} \| u(t) \|_{X^\frac{1}{2}}^2 + \frac{1}{2} \| \theta(t) \|_{X^\frac{1}{2}}^2 - \varepsilon \| u(t) \|_{X^\frac{1}{2}}^2 - C_\varepsilon \]
\[ \geq \left( \frac{1}{2} - \frac{\varepsilon C_0}{2} \right) \| u(t) \|_{X^\frac{1}{2}}^2 + \frac{1}{2} \| u(t) \|_{X^\frac{1}{2}}^2 + \frac{1}{2} \| \theta(t) \|_{X^\frac{1}{2}}^2 - C_\varepsilon \]
where \( \varepsilon \) is such that \( \varepsilon < \frac{1}{C_0}, \) that is
\[ \| \Delta u(t) \|_{X^\frac{1}{2}}^2 + \| u(t) \|_{X^\frac{1}{2}}^2 + \| \theta(t) \|_{X^\frac{1}{2}}^2 \leq C_1 E(t) + C_\varepsilon, \]
where \( C_1 = \min \left\{ \left( \frac{1}{2} - \frac{\varepsilon C_0}{2} \right), \frac{1}{2} \right\}. \)

**Theorem 5.2.** For \( M > 0 \) sufficiently large, there exist constants \( \beta_1, \beta_2, \beta_3, \beta_4 > 0 \)
such that
\[ \beta_3 E(t) - \beta_4 \leq L(t) \leq \beta_1 E(t) + \beta_2, \quad t \geq 0. \quad (35) \]
Proof. Note that from Remark 4 and (23), there exist \( \tilde{C}_1, \tilde{C}_2 > 0 \) such that
\[
|\phi(t)| \leq \frac{1}{2} \| u(t) \|_X^2 + \frac{\mu_1}{2\lambda_1} \| \Delta u(t) \|_X^2
\]
\[
\leq \max \left\{ \frac{1}{2}, \frac{\mu_1}{2\lambda_1} \right\} \left( \| \Delta u(t) \|_X^2 + \| u(t) \|_X^2 + \| \theta(t) \|_X^2 \right) \quad (36)
\]
\[
\leq \tilde{C}_1 \mathcal{E}(t) + \tilde{C}_2.
\]
Due to Remark 4 and (24) we also have
\[
|\psi(t)| \leq C_{\delta_0} \| u(t) \|_X^2 + \delta_0 \| \theta(t) \|_X^2
\]
\[
\leq \max \{ C_{\delta_0}, \delta_0 \} \left( \| \Delta u(t) \|_X^2 + \| u(t) \|_X^2 + \| \theta(t) \|_X^2 \right) \quad (37)
\]
where \( C_{\delta_0} > 0 \).

Now, observe that the constants \( \delta_1 > 0 \) and \( \delta_2 > 0 \) were fixed in the proof of the Theorem 5.1. Then, combining (25) with (36) and (37) we obtain
\[
\mathcal{E}(t) \leq \beta_1 \mathcal{E}(t) + \beta_2, \quad t \geq 0.
\]
On the other hand, since
\[
M \mathcal{E}(t) = \mathcal{L}(t) - \delta_1 \phi(t) - \delta_2 \psi(t)
\]
from (36) and (37),
\[
(M - \delta_1 \tilde{C}_1 - \delta_2 \tilde{C}_3) \mathcal{E}(t) - \delta_1 \tilde{C}_2 - \delta_2 \tilde{C}_4 \leq \mathcal{L}(t),
\]
and taking \( M > 0 \) sufficiently large such that \( M - \delta_1 \tilde{C}_1 - \delta_2 \tilde{C}_3 > 0 \), we obtain
\[
\beta_3 \mathcal{E}(t) - \beta_4 \leq \mathcal{L}(t).
\]
This concludes the proof of the theorem. \( \square \)

Corollary 3. Under the same conditions of the Theorem 5.1 and Theorem 5.2, if \( B \subset Y \) is a bounded set, and \( (u, v, \theta) : [\tau, \tau + T] \rightarrow Y, \ T > 0, \) is the solution of (1)-(2) starting in \( (u_0, v_0, \theta_0) \in B \), then there exist positive constants \( \bar{\omega}, \gamma_1 = \gamma_1(B) \) and \( \gamma_2 \), such that
\[
\| \Delta u(t) \|_X^2 + \| u(t) \|_X^2 + \| \theta(t) \|_X^2 \leq \gamma_1 e^{-\bar{\omega}(t-\tau)} + \gamma_2, \quad t \in [\tau, \tau + T]. \quad (38)
\]
Proof. From (27) and (35), we obtain
\[
\mathcal{L}'(t) \leq -\sigma_1 \mathcal{L}(t) + \sigma_2,
\]
where \( \sigma_1 = \frac{M_1}{\beta_1} \) and \( \sigma_2 = \frac{M_1 \beta_2}{\beta_1} + M_2 \), and thus,
\[
\mathcal{L}(t) \leq \mathcal{L}(\tau) e^{-\sigma_1 (t-\tau)} + \sigma_2 e^{-\sigma_1 t} \int_{\tau}^{t} e^{\sigma_1 s} ds \leq \mathcal{L}(\tau) e^{-\sigma_1 (t-\tau)} + \frac{\sigma_2}{\sigma_1}. \]

Again, by (35) together with Remark 4, we conclude
\[
\| \Delta u(t) \|_X^2 + \| u(t) \|_X^2 + \| \theta(t) \|_X^2 \leq \gamma_1 e^{-\sigma_1 (t-\tau)} + \gamma_2,
\]
where \( \gamma_1 = \gamma_1(\mathcal{L}(\tau)) > 0 \) and \( \gamma_2 > 0 \). \( \square \)

Theorem 5.3. Under the same conditions of Theorem 5.1 and Theorem 5.2, the problem (1)-(2) has a pullback attractor \( \{ \mathcal{A}(u)(t) : t \in \mathbb{R} \} \) in \( Y \) and
\[
\bigcup_{t \in \mathbb{R}} \mathcal{A}(u)(t) \subset Y.
\]
Proof. From estimate (38) it is easy to check that the evolution process \{S(a)(t, \tau) : t \geq \tau \in \mathbb{R}\} associated with (1)-(2) is pullback strongly bounded.

Hence, applying the same ideas of the proofs of the Theorem 5.1 and Theorem 5.2, we obtain that the family of evolution process \{S(a)(t, \tau) : t \geq \tau \in \mathbb{R}\} is pullback asymptotically compact (see Theorem 2.12). In fact, from (22) we write

\[ S(a)(t, \tau) = L(a)(t, \tau) + U(a)(t, \tau), \]

where

\[ U(a)(t, \tau) := \int_\tau^t L(a)(t, s)F(s, S(a)(t, s))ds. \]

With the same arguments used in the proof of the Theorem 5.1 with \( f \equiv 0 \) in (1) and with the functionals

\[ \mathcal{E}(t) = \frac{1}{2} \|u(t)\|_{X_1}^2 + \frac{1}{2} \|u_t(t)\|_X^2 + \frac{1}{2} \|\theta(t)\|_X^2 \]

and

\[ \mathcal{L}(t) = M\mathcal{E}(t) + \delta_1 \phi(t) + \delta_2 \psi(t) \]

where \( \phi \) is defined in (23) and \( \psi \) is defined in (24), we get from (27) that there exist \( c_1 > 0 \) such that

\[ \mathcal{L}'(t) \leq -c_1 \mathcal{E}(t) \]

and from arguments used in the proof of the Theorem 5.2 with \( f \equiv 0 \) in (1), by (35) we get \( c_2, c_3 > 0 \) such that

\[ c_2 \mathcal{E}(t) \leq \mathcal{L}(t) \leq c_3 \mathcal{E}(t) \]

and hence

\[ \mathcal{E}'(t) \leq -c_0 \mathcal{E}(t) \]

for some \( c_0 > 0 \). This ensures that exist constants \( K, \alpha > 0 \) such that

\[ \|L(a)(t, \tau)\|_{\mathcal{L}(Y)} \leq Ke^{-\alpha(t-\tau)}, \text{ for all } t \geq \tau. \]

The family of evolution process \{U(a)(t, \tau) : t \geq \tau \in \mathbb{R}\} is compact in \( Y \). In fact, the compactness of \( U(a)(t, \tau) \) follows easily from the fact that

\[ X_1^\frac{1}{2} \overset{f(t, \cdot)}{\hookrightarrow} X^{-\frac{1}{2}} \overset{\text{inclusion}}{\hookrightarrow} X^{-\frac{1}{2}}, \]

being the last inclusion compact, since \( \alpha < 1 \) (see Lemma 4.2).

Now, applying Theorem 2.11 we get that the problem (1)-(2) has a pullback attractor \( \{A(a)(t) : t \in \mathbb{R}\} \) in \( Y \) and that \( \bigcup_{t \in \mathbb{R}} A(a)(t) \subset Y \) is bounded. \qed

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