MODIFIED ELLIPTIC GAMMA FUNCTIONS
AND 6d SUPERCONFORMAL INDICES

VYACHESLAV P. SPIRIDONOV

Abstract. We construct a modified double elliptic gamma function which is well defined when one of the base parameters lies on the unit circle. A model consisting of 6d hypermultiplets coupled to a gauge field theory living on a 4d defect is proposed whose superconformal index uses the double elliptic gamma function and obeys $W(E_7)$-group symmetry.

1. Introduction

Six dimensional superconformal field theories currently form an active research field (see, e.g., [1] and references therein). As claimed by Moore [1], these theories should form a gold mine for experts in special functions as a source of amazing identities, which is just one of many important potential mathematical outputs from them. This statement sounds curious and the author agrees with it. Indeed, a principally new class of special functions called elliptic hypergeometric integrals has been discovered in [2]. It came as a big surprise to mathematicians since it was tacitly assumed that $q$-hypergeometric functions form the top level special functions of hypergeometric type with nice exact formulas [3]. Some particular examples of such integrals were interpreted as wave functions or normalizations of wave functions in specific elliptic multiparticle quantum mechanical systems [2]. Recently it was shown by Dolan and Osborn [4] that certain elliptic hypergeometric integrals coincide with superconformal indices of four-dimensional gauge field theories and corresponding identities prove Seiberg dualities (electro-magnetic, strong-weak, or mirror symmetry dualities) in the topological sector. Further detailed investigation of this relationship was performed in many papers among which we mention only a small fraction [5] [6] [7] [8].

The theory of elliptic hypergeometric functions is nowadays a rich mathematical subject with many beautiful new constructions [9]. One of its key ingredients is the elliptic gamma function related to the Barnes multiple gamma function of order three $\Gamma_3(u; \omega_1, \omega_2, \omega_3)$ [10] (see the Appendix for a definition of $\Gamma_m(u; \omega)$). The plain and $q$-hypergeometric functions are directly related to the Barnes gamma functions of order one $\Gamma_1(u; \omega_1)$, which is proportional to the standard Euler’s gamma function $\Gamma(u/\omega_1)$, and order two $\Gamma_2(u; \omega_1, \omega_2)$, respectively. The multiple infinite $q$-products with several bases naturally emerge in the considerations of superconformal indices for higher dimensional theories [11] [12]. In particular, the double elliptic gamma function related to $\Gamma_4(u; \omega)$ describes topological strings partition function [12] [13] [14] and 6d superconformal indices [15] [16] [17] (the latter

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The gamma function has the form
\[ \Gamma(z; \omega) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1}p^{i+1}q^{j+1}}{1 - zp^i q^j}, \quad z \in \mathbb{C}^*, \]
and corresponding particular modular transforms
\[ \tilde{q} = e^{-2\pi i \omega_1}, \quad \tilde{p} = e^{-2\pi i \omega_2}, \quad \tilde{r} = e^{-2\pi i \omega_3}, \quad \tilde{\omega} = e^{-2\pi i \omega_4}. \]

The bases \( p, q, r \) and \( \tilde{p}, \tilde{q}, \tilde{r} \) coincide with those used in [2] [9].

In the increasing order of complexity we define the following infinite products
all of which are well defined only when bases \( q, \ldots, w \) are of modulus less than 1. Denote
\[ (z; q_1, \ldots, q_m) = \prod_{k_1, \ldots, k_m=0}^{\infty} (1 - z q_1^{k_1} \cdots q_m^{k_m}), \quad z \in \mathbb{C}, \]
the standard infinite \( q \)-product and \( \theta(z; p) = (z; p)(p z^{-1}; p) \), a theta function obeying properties \( \theta(pz; p) = \theta(z^{-1}; p) = -z^{-1} \theta(z; p) \). The standard (order one) elliptic
gamma function has the form
\[ \Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1}p^{i+1}q^{j+1}}{1 - zp^i q^j}, \quad z \in \mathbb{C}^*, \]
and the double (i.e., of the order two) elliptic gamma function is
\[
\Gamma(z; p, q, t) = \prod_{i,j,k=0}^{\infty} \left( 1 - z^{-1} p^{i+1} q^{j+1} t^{k+1} \right) \left( 1 - z p^{i} q^{j} t^{k} \right), \quad z \in \mathbb{C}^*.
\]

We use the conventions
\[
\Gamma(a; b; \ldots) := \Gamma(a; \ldots) \Gamma(b; \ldots), \quad \Gamma(az^{-1}; \ldots) := \Gamma(az^{-1}; az^{-1}; \ldots),
\]
\[
\Gamma(az^{-1} y^{-1}; \ldots) := \Gamma(az y^{-1}; \ldots) \Gamma(az^{-1} y^{-1}; \ldots) \Gamma(az^{-1} y^{-1}; \ldots).
\]

Both, \(\Gamma(z; p, q)\) and \(\Gamma(z; p, q, t)\) are symmetric in their bases. For the standard elliptic gamma function one has the difference equations
\[
\Gamma(qz; p, q) = \theta(z; p) \Gamma(z; p, q), \quad \Gamma(pz; p, q) = \theta(z; q) \Gamma(z; p, q).
\]

For the second order function \(\Gamma(z; p, q, t)\) one has
\[
\frac{\Gamma(qz; p, q, t)}{\Gamma(z; p, q, t)} = \Gamma(z; p, t), \quad \frac{\Gamma(pz; p, q, t)}{\Gamma(z; p, q, t)} = \Gamma(z; q, t), \quad \frac{\Gamma(tz; p, q, t)}{\Gamma(z; p, q, t)} = \Gamma(z; p, q).
\]

The inversion relations have the form
\[
\Gamma(z, pqz^{-1}; p, q) = 1, \quad \Gamma(pqtz; p, q, t) = \Gamma(z^{-1}; p, q, t).
\]

In \([2]\) the following modified elliptic gamma function was defined
\[
G(u; \omega_1, \omega_2, \omega_3) := \Gamma(e^{2\pi i \omega_1/\omega_2}; p, q) \Gamma(e^{2\pi i \omega_1/\omega_2}; r, \tilde{q}) = \frac{\Gamma(e^{2\pi i \omega_1/\omega_2}; p, q)}{\Gamma(e^{2\pi i \omega_1/\omega_2}; r, \tilde{q})}.
\]

It satisfies three linear difference equations of the first order
\[
G(u + \omega_1; \omega) = \theta(e^{2\pi i \omega_1/\omega_2}; p) G(u; \omega),
\]
\[
G(u + \omega_2; \omega) = \theta(e^{2\pi i \omega_1/\omega_2}; r) G(u; \omega),
\]
\[
G(u + \omega_3; \omega) = e^{-\pi i B_{2,2}(u; \omega)} G(u; \omega),
\]
where \(B_{2,2}(u; \omega)\) is the diagonal Bernoulli polynomial of order two,
\[
B_{2,2}(u; \omega) = \frac{u^2}{\omega_1 \omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6 \omega_2} + \frac{\omega_2}{6 \omega_1} + \frac{1}{2}.
\]

In \([6]\) the exponential coefficient emerged through the following well-known \(SL(2, \mathbb{Z})\)-modular transformation property of theta functions
\[
\theta \left( e^{-2\pi i \omega_1/\omega_2}; e^{-2\pi i \omega_1/\omega_2} \right) = e^{\pi i B_{2,2}(u; \omega)} \theta \left( e^{2\pi i \omega_1/\omega_2}; e^{2\pi i \omega_1/\omega_2} \right).
\]

One has the reflection equation \(G(a; \omega)G(\omega_1 + \omega_2 + \omega_3 - a; \omega) = 1\). We shall use below the following shorthand notation
\[
G(a \pm b; \omega) := G(a + b, a - b; \omega) := G(a + b; \omega) G(a - b; \omega).
\]

To prove that function \([8]\) is well defined for \(|q| = 1\) we consider another function
\[
G(u; \omega_1, \omega_2, \omega_3) = e^{-\pi i \omega_3} B_{3,3}(u; \omega) \Gamma(e^{-2\pi i \omega_1/\omega_2}; \tilde{r}, \tilde{p}),
\]
where \(B_{3,3}(u; \omega)\) is the diagonal Bernoulli polynomial of order three,
\[
B_{3,3} \left( u + \sum_{n=1}^{3} \frac{\omega_n}{2}; \omega \right) = \frac{u(u^2 - \frac{1}{4} \sum_{k=1}^{3} \omega_k^2)}{\omega_1 \omega_2 \omega_3}.
\]
Obviously, one has the symmetry $\tilde{G}(u; \omega_1, \omega_2, \omega_3) = \tilde{G}(u; \omega_2, \omega_1, \omega_3)$. Using the relation

$$B_{3,3}(u + \omega_3; \omega_1, \omega_2, \omega_3) - B_{3,3}(u; \omega_1, \omega_2, \omega_3) = 3\omega_3 B_{2,2}(u; \omega_1, \omega_2),$$

it is not difficult to check that $\tilde{G}(u; \omega)$ satisfies the same three equations (4) and the normalization condition

$$\tilde{G}(\frac{1}{2} \sum_{k=1}^{3} \omega_k; \omega_1, \omega_2, \omega_3) = \tilde{G}(\frac{1}{2} \sum_{k=1}^{3} \omega_k; \omega_1, \omega_2, \omega_3) = 1.$$

Therefore by the Jacobi theorem one obtains the equality

$$\tilde{G}(u; \omega_1, \omega_2, \omega_3) = G(u; \omega_1, \omega_2, \omega_3)$$

corresponding to one of the $SL(3, \mathbb{Z})$-modular transformation laws for the elliptic gamma function [21].

The crucial property of $G(u; \omega)$ is that it remains a well defined meromorphic function of $u$ even for $\omega_1/\omega_2 > 0$ (i.e., when $|q| = 1$ with the conditions $|p|, |r| < 1$ being obligatory), in difference from $\Gamma(z; p, q)$. This fact is evident from the second form of representation of $G(u; \omega)$ [8].

Take the limit $\omega_3 \to \infty$ in such a way that $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) \to +\infty$ (i.e., $p, r \to 0$). Then,

$$\lim_{p, r \to 0} G(u; \omega_1, \omega_2, \omega_3) = \gamma(u; \omega_1, \omega_2) = \frac{(e^{2\pi i u/\omega_1} \tilde{G}; \tilde{q})}{(e^{2\pi i u/\omega_2}; q)}.$$

This is a modified $q$-gamma function known under many other different names (double sine, hyperbolic gamma function, or quantum dilogarithm, see Appendix A in [18] for interconnections between these functions). For $\text{Re}(\omega_1), \text{Re}(\omega_2) > 0$ and $0 < \text{Re}(u) < \text{Re}(\omega_1 + \omega_2)$ it has the integral representation

$$\gamma(u; \omega_1, \omega_2) = \exp \left( - \int_{\mathbb{R}+i0} \frac{e^{ux}}{(1 - e^{ux}) (1 - e^{ux})} \frac{dx}{x} \right),$$

which shows that $\gamma(u; \omega_1, \omega_2)$ is meromorphic even for $\omega_1/\omega_2 > 0$, when $|q| = 1$ and the infinite product representation [11] is not applicable.

Euler’s gamma function $\Gamma(u)$ can be defined as a special solution of the functional equation $f(u + 1) = u f(u)$. $q$-Gamma functions with $q = e^{2\pi i \omega_1/\omega_2}$ can be defined as special solutions of the equation $f(u + \omega_1) = (1 - e^{2\pi i u/\omega_2}) f(u)$ (in particular the functions [11] and $1/(e^{2\pi i u/\omega_2}; q)$ satisfy this equation). Analogously, the elliptic gamma functions of order one are defined as special solutions of the key equation [4], which does not assume any restriction on the parameter $q$. Its particular solutions $\Gamma(e^{2\pi i x/\omega_2}; p, q)$ and $1/\Gamma(q^{-1} e^{2\pi i x/\omega_2}; p, q^{-1})$ exist only for $|q| < 1$ or $|q| > 1$, respectively. And $G(u; \omega)$ covers the remaining domain $|q| = 1$.

Define now the modified double elliptic gamma function

$$G(u; \omega_1, \ldots, \omega_4) := \frac{\Gamma(e^{2\pi i u/\omega_2}; q, p, t)}{\Gamma(q e^{2\pi i u/\omega_2}; \tilde{q}, \tilde{r}, \tilde{s})}. \quad (12)$$

This is a meromorphic function of $u \in \mathbb{C}$ satisfying the inversion relation

$$G(u + \sum_{k=1}^{4} \omega_k; \omega_1, \ldots, \omega_4) = G(-u; \omega_1, \ldots, \omega_4)$$
and four linear difference equations of the first order

\[ G(u + \omega_1; \omega) = \Gamma(e^{2\pi i \frac{u}{\omega}}; p, t)G(u; \omega), \quad (13) \]
\[ G(u + \omega_2; \omega) = \Gamma(e^{2\pi i \frac{u}{\omega}}; r, s)G(u; \omega), \quad (14) \]
\[ G(u + \omega_3; \omega) = \frac{\Gamma(e^{2\pi i \frac{u}{\omega}}; q, t)}{\gamma e^{2\pi i \frac{u}{\omega}}; \tilde{q}, s)G(u; \omega), \quad (15) \]
\[ G(u + \omega_4; \omega) = \frac{\Gamma(e^{2\pi i \frac{u}{\omega}}; p, q)}{\gamma e^{2\pi i \frac{u}{\omega}}; \tilde{q}, r)G(u; \omega). \quad (16) \]

Note that the latter equation coefficient is simply \( G(u; \omega_1, \omega_2, \omega_3) \). Note also that in the limit \( \omega_4 \to \infty \) taken in such a way that \( s, t \to 0 \), we have

\[ \lim_{s, t \to 0} G(u; \omega_1, \ldots, \omega_4) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i \frac{uj}{\omega_j}} p^j q^k}{1 - e^{2\pi i \frac{uj}{\omega_j}} r^j \tilde{q}^{k+1}}, \]

which is only “a half” of \( 1/G(u; \omega_1, \omega_2, \omega_3) \).

Let us demonstrate that \( G(u; \omega_1, \ldots, \omega_4) \) remains a meromorphic function of \( u \) for \( \omega_1/\omega_2 > 0 \) (when \(|q| = 1\)). First, we find another solution of the above set of equations. Consider the following function

\[ \tilde{G}(u; \omega_1, \ldots, \omega_4) = e^{-\frac{\pi i}{2\omega_4} B_{4,4}(u; \omega)} \frac{\Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{p}, \tilde{r}, \tilde{w})}{\Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{s}, \tilde{t}, \tilde{w})}, \quad (17) \]

where \( B_{4,4}(u; \omega) \) is the diagonal multiple Bernoulli polynomial of order four, whose compact form we have found from (58) as

\[ B_{4,4}(u; \omega_1, \ldots, \omega_4) = \frac{1}{\omega_1^4 \omega_2^4 \omega_3^4 \omega_4^4} \left((u - \frac{1}{2})^4 \sum_{k=1}^{4} \omega_k^2 - \frac{1}{4} \sum_{k=1}^{4} \omega_k^4 \right)^2
\]
\[ - \frac{1}{30} \sum_{k=1}^{4} \omega_k^4 \right) \sum_{1 \leq j < k \leq 4} \omega_j^2 \omega_k^2 \}. \quad (18) \]

This function satisfies four linear difference equations of the first order

\[ \tilde{G}(u + \omega_1; \omega) = e^{-\frac{\pi i}{2\omega_4} B_{3,3}(u; \omega_1, \omega_2, \omega_3, \omega_4)} \frac{\Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{p}, \tilde{t}, \tilde{w})}{\Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{s}, \tilde{t}, \tilde{w})} \tilde{G}(u; \omega), \quad (19) \]
\[ \tilde{G}(u + \omega_2; \omega) = e^{-\frac{\pi i}{2\omega_4} B_{3,3}(u; \omega_1, \omega_2, \omega_3, \omega_4)} \frac{\Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{r}, \tilde{w})}{\Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{s}, \tilde{w})} \tilde{G}(u; \omega), \quad (20) \]
\[ \tilde{G}(u + \omega_3; \omega) = e^{-\frac{\pi i}{2\omega_4} B_{3,3}(u; \omega_1, \omega_2, \omega_3, \omega_4)} \frac{\Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{s}, \tilde{t})G(u; \omega), \quad (21) \]
\[ \tilde{G}(u + \omega_4; \omega) = e^{-\frac{\pi i}{2\omega_4} B_{3,3}(u; \omega_1, \omega_2, \omega_3, \omega_4)} \frac{\Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{p}, \tilde{r})G(u; \omega), \quad (22) \]

following from the previously defined formulas and the relation \( B_{4,4}(u + \omega_4; \omega) - B_{4,4}(u; \omega) = 4\omega_4 B_{3,3}(u; \omega) \). But this is precisely the set of equations \( (13)-(16) \). Indeed, equality of coefficients in \( (16) \) and \( (22) \) is nothing else than the relation \( (9) \). Equality of coefficients in \( (13) \) and \( (19) \) or in \( (15) \) and \( (21) \) follows from \( (9) \) after the replacement \( \omega_1 \to \omega_4 \) or \( \omega_3 \to \omega_4 \), respectively. Equality of coefficients in \( (14) \) and \( (20) \) follows after the replacement in \( (9) \) \( \omega_1 \to \omega_4 \) and subsequent substitution \( \omega_2 \to \omega_1 \). Since \( \omega_j \)'s are incommensurate we conclude that the ratio \( \tilde{G}(u; \omega)/G(u; \omega) \) is a constant independent on \( u \). However, there is no distinguished
value of \( u \) for which the equality of normalizations of \( G \) and \( \tilde{G} \) becomes obvious. The fact that
\[
\tilde{G}(u; \omega_1, \omega_2, \omega_3, \omega_4) = G(u; \omega_1, \omega_2, \omega_3, \omega_4)
\]
follows from an \( SL(4, \mathbb{Z}) \)-modular group transformation law for the double elliptic gamma function established as Corollary 9 in [22].

So, in the same way as in the lower order cases, special solutions of the key equation (13) define double elliptic gamma functions: the functions \( \Gamma(e^{2\pi i/2}; p, q, t) \) and \( 1/\Gamma(q^{-1}e^{2\pi i/2}; p, q^{-1}, t) \) satisfy it for \( |q| < 1 \) and \( |q| > 1 \), respectively, and \( G(u; \omega_1, \ldots, \omega_4) \) covers the domain \( |q| = 1 \). The latter function is defined for \( |p|, |r|, |s|, |t|, |u| < 1 \) and \( |q| \leq 1 \) (more precisely, for the union of the upper half plane \( \text{Im}(\omega_1/\omega_2) > 0 \) and the half line \( \omega_1/\omega_2 > 0 \)), for other admissible domains of values of bases it will take a different form.

3. A 6d/4d theory with \( W(E_7) \)-invariant superconformal index

Superconformal indices are defined as [23] [24]
\[
I(y) = \text{Tr} \left[ (-1)^F \prod_{k=1}^{m} y_k^{G_k} e^{-\beta \mathcal{H}} \right],
\]
where \( F \) is the fermion number, \( G_k \) form the maximal Cartan subalgebra preserving a distinguished supersymmetry relation involving one supercharge and its superconformal partner
\[
\{Q, S\} = 2H, \quad Q^2 = S^2 = 0, \quad [Q, G_k] = [S, G_k] = 0.
\]
The trace is effectively taken over the space of BPS states formed by zero modes of the operator \( \mathcal{H} \) which eliminates dependence on the chemical potential \( \beta \). Computing supersymmetric indices of nonconformal theories on curved backgrounds that flow to certain superconformal field theories one gets superconformal indices of the theories with the same superconformal fixed points [25]. Computation of such indices via the localization techniques was initiated in [26].

We shall not discuss general structure of these indices in 4d field theories since they were described in many previous papers, see, e.g., [6] [7]. Take a particular \( \mathcal{N} = 1 \) 4d theory in the space-time \( S^3 \times S^1 \) with \( SP(2N) \) gauge group and the flavor group \( SU(8) \times U(1) \). In addition to the vector superfield in the adjoint representation of \( SP(2N) \), take 8 chiral matter fields forming the fundamental representation of \( SP(2N) \) with the \( R \)-charge 1/2 and \( U(1) \)-charge \( (1-N)/4 \). Take also one antisymmetric \( SP(2N) \)-tensor field of zero \( R \)- and \( SU(8) \)-charges and unit \( U(1) \)-charge. For \( N = 1 \), the global group \( U(1) \) decouples and the tensor field is absent.

The superconformal index of this theory is described by the following elliptic hypergeometric integral [5]:
\[
I(y_1, \ldots, y_8; t; p, q) = \frac{(p;p)^N(q;q)_N}{2^N N!} \Gamma(t; p, q)^{N-1} \int_{T_N} \prod_{1 \leq j < k \leq N} \frac{\Gamma(tz_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^{N} \frac{\Gamma(t^{1/4}(pq)^{1/2} y_j z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1}; p, q)} \frac{dz_j}{2\pi i z_j}. \tag{24}
\]
Here \( y_i \) are fugacities for \( SU(8) \)-group satisfying the constraint \( \prod_{i=1}^{8} y_i = 1 \), \( t \) is the fugacity for the group \( U(1) \), \( p \) and \( q \) are fugacities for the superconformal group.
generator combinations $R/2 + J_1 \pm J_2$, where $R$ is the $R$-charge and $J_{1,2}$ are Cartan generators of $SO(4)$-rotations. Nontrivial contributions to the index come only from the states with $H = E - 2J_1 - 3R/2 = 0$, where $E$ is the energy. In terms of the variables $t_i = t^{1/N}(pq)^{\frac{1}{2}} y_i$, we have the balancing condition $t^{2N-2} \prod_{i=1}^{8} t_i = (pq)^2$. The constraints $|t_i|, |t_i| < 1$ are needed for the choice of the integration contours as unit circles with positive orientation $T$. For $N = 1$ the integral $I(y_1, \ldots, y_8; p, q)$ is nothing else than an elliptic analogue of the Euler-Gauss hypergeometric function introduced in [2].

In addition to the obvious $S_8$-symmetry in variables $y_i$, function (24) obeys the following hidden symmetry transformation extending $S_8$-group to $W(E_7)$ – the Weyl group of the exceptional root system $E_7$:

$$I(y_1, \ldots, y_8; t; p, q) = \prod_{m=0}^{N-1} \left( \prod_{1 \leq i < j \leq 4} \Gamma(t^{m+\frac{1}{2}}(pq)^{\frac{1}{2}} \sqrt{\frac{y_i y_j}{p, q}}) \right) \cdot \prod_{5 \leq i < j \leq 8} \Gamma(t^{m+\frac{1}{2}N}(pq)^{\frac{1}{2}} y_i y_j; p, q) I(\hat{y}_1, \ldots, \hat{y}_8; t; p, q),$$

where

$$\hat{y}_k = \frac{y_k}{\sqrt{T}}, \quad \hat{y}_{k+4} = \sqrt{T} y_{k+4}, \quad k = 1, \ldots, 4, \quad Y = y_1 y_2 y_3 y_4.$$

Equivalently one can write $Y^{-1} = y_5 y_6 y_7 y_8$. For $N = 1$ this relation was established by the author [2] and it was extended to arbitrary $N$ by Rains [19].

Consider the following ratio involving double elliptic gamma functions

$$I_{6d/4d}(y_1, \ldots, y_8; t; p, q) := \frac{I(y_1, \ldots, y_8; t; p, q)}{\prod_{1 \leq j < k \leq 8} \Gamma(t^{m+\frac{1}{2}}(pq)^{\frac{1}{2}} y_j y_k; p, q, t)}. \tag{25}$$

First, we show that this function is $W(E_7)$-group invariant. Indeed, explicit substitution yields

$$I_{6d/4d}(\hat{y}_1, \ldots, \hat{y}_8; t; p, q) = I_{6d/4d}(y_1, \ldots, y_8; t; p, q), \tag{26}$$

which follows from the relation

$$\prod_{1 \leq j < k \leq 8} \Gamma(t^{m+\frac{1}{2}}(pq)^{\frac{1}{2}} y_j y_k; p, q, t) = \prod_{1 \leq j < k \leq 8} \Gamma(t^{m+\frac{1}{2}N}(pq)^{\frac{1}{2}} y_j y_k; p, q, t) \cdot \prod_{5 \leq j < k \leq 8} \Gamma(t^{m+\frac{1}{2}N}(pq)^{\frac{1}{2}} y_j y_k; p, q, t).$$

In [19] the $W(E_7)$-transformation was also written in the form (26), but for a function different from (25).

Now we would like to interpret equality (26) as a symmetry of the superconformal index of some 6d field theory with a 4d defect (we use the terminology of [20] where similar mixed 4d/2d theories were constructed). The main inspiration for that comes from a beautiful 5d/4d field theory interpretation of the $W(E_7)$-symmetry of the elliptic analogue of Euler-Gauss hypergeometric function given by Dimofte and Gaiotto in [8].
The 6d-index for $\mathcal{N} = (1,0)$ theories on the $S^5 \times S^1$ manifold is

$$I(y; p, q, t) = \text{Tr} \left[ (-1)^F p^{C_1} q^{C_2} t^{C_3} \prod_k y_k^{C_k} \right],$$

where $G_k$ are the flavor group maximal torus generators and $C_{1,2,3}$ are Cartan generators for the space-time symmetry group. In the notations of Imamura [16]

$$p^{C_1} q^{C_2} t^{C_3} = x^{j_1 + 3 R/2} y_3^{j_2} y_8^{j_3},$$

where $R$ is the Cartan of $SU(2)_R$-subalgebra, $j_1$ is the generator of $U(1)_V$ and $j_2, j_3$ are Cartans of $SU(3)_V$ with $U(1)_V \times SU(3)_V$ being a subgroup of the $SO(6)$-isometry group of $S^5$. Perturbative contributions to the index are described by the double elliptic gamma functions [16] [17] with bases

$$p = \frac{x y_3}{y_8}, \quad q = \frac{x}{y_3 y_8}, \quad t = x y_8^2.$$

One can permute $p, q,$ and $t,$ but we stick to this choice leading to

$$C_{1,2} = \frac{1}{3} \left( j_1 - j_3 \right) \pm \frac{j_2}{2} + \frac{R}{2}, \quad C_3 = \frac{1}{3} \left( j_1 + j_3 \right) + \frac{R}{2}. \quad (27)$$

E.g., for a $U(1)$-flavor group hypermultiplet one has the index

$$I_{hyp}(y; p, q, t) = \frac{1}{\Gamma(\sqrt{pq y}; p, q, t)} = \exp \left( \sum_{n=1}^{\infty} \frac{i_{hyp}(y^n; p^n, q^n, t^n)}{n} \right), \quad (28)$$

$$i_{hyp}(y; p, q, t) = \frac{\sqrt{pq t(y + y^{-1})}}{(1 - p)(1 - q)(1 - t)}.$$

For $SU(2)$ gauge group vector superfield one obtains

$$I_{vec}(z; p, q, t) = \kappa \frac{\Gamma(z + 2; p, q, t)}{(1 - z^2)(1 - z^{-2})} = \exp \left( \sum_{n=1}^{\infty} \frac{i_{vec}(z^n; p^n, q^n, t^n)}{n} \right), \quad (29)$$

$$i_{vec}(z; p, q, t) = \left( 1 - \frac{1 + pq t}{(1 - p)(1 - q)(1 - t)} \right) \chi_{adj, SU(2)}(z),$$

$$\kappa = \lim_{x \to 1} \frac{\Gamma(x; p, q, t)}{1 - x} = (p; p)(q; q)(t; t)(pq; p, q)(pt; p, t)(qt; q, t)(pq t; p, q, t)^2.$$

The multiplier $\kappa$ appears naturally from the adjoint representation character $\chi_{adj, SU(2)}(z) = z^2 + z^{-2} + 1.$ One can incorporate into $I_{vec}$ a piece of Haar measure for $SU(2)$ and cancel thus the terms $(1 - z^2)(1 - z^{-2}).$

Take now the 4d interacting gauge theory described above and assume that it lives on a $S^3 \times S^1$ manifold immersed into the taken 6d space-time $S^5 \times S^1$ in the "corner" $x_5 = x_6 = 0$ of $S^5$ defined by the coordinate constraint $\sum_{i=1}^{6} x_i^2 = 1.$ This defect breaks half of 6d supersymmetries, presumably preserving the supercharge for defining the superconformal index. Associate fugacities $p$ and $q$ with the isometries of the space $S^3 \times S^1,$ which connects corresponding 4d/6d Cartan generators as $J_1 \propto j_1 - j_3/2$ and $J_2 \propto j_2.$ The abelian group associated with the generator $C_3$ in (27) is identified from the 4d theory point of view with the $U(1)$-flavor group whose fugacity is $t.$ Take now free 6d gauge invariant hypermultiplets forming the totally antisymmetric tensor of second rank $T_A$ for the mentioned $SU(8)$ flavor group and couple them to the taken 4d defect.

The interaction superpotential that ties together the flavor symmetries and the rotation symmetry of the defect should be of the form $W = \text{Tr} q M_q |_{x_5=x_6=0},$ where
$q$ are the $4d$ quark superfields and $M$ is one of the two chiral fields in the $6d$ hypermultiplet. Here the trace contracts the gauge indices of the quarks with the symplectic form as well as the $SU(8)$ flavor indices of the quarks and $M$. Since $q$ has $U(1)$-charge $(1-N)/4$, this results in the additional $U(1)$-charge of the hypermultiplets equal to $N/2$ after taking into account the rotation symmetries of $M$ and the actual Lagrangian couplings obtained after integration of $W$ over the superspace (the author is indebted to D. Gaiotto for pointing out on such a possibility). This yields a $6d$ model with a $4d$ defect similar to $4d/2d$ systems considered in [20] (in particular, see the toy model considered in Sect. 4.3 of [27]).

As a result, the hypermultiplet index takes the form

$$I_{RA}(\nu; p, q, t) = \prod_{1 \leq j < k \leq 8} \frac{1}{\Gamma(t^{N/2}/pqy_jy_k; p, q, t)} \prod_{k=1}^{8} y_k = 1,$$

which evidently coincides with the multiplier in [25]. The “corner” defect $4d$ theory gives its own contribution to the superconformal index described by the integral $I(y_1, \ldots, y_8; t; p, q)$. The combined index has $W(E_7)$-symmetry indicating that this theory may have the enhanced $E_7$-flavor group, provided there exists an appropriate point in the moduli space. This is a rough potential physical picture behind relation [20] the detailed consideration of which lies beyond the scope of the present note. For $N = 1$ a simplification takes place since the $U(1)$ group decouples from the $4d$-defect. In this case $I_{4d/4d}$ turns into the $W(E_7)$-invariant half-index of [8] in the limit $t \to 0$, but this seems to be a formal coincidence since the $\varphi$ parameter should stay intact in the half-index for $N > 1$.

As proposed in [2], one can build elliptic hypergeometric integrals using the modified elliptic gamma function. This is achieved by mere replacement of $\Gamma(e^{2\pi i u/\omega}; p, q)$-functions by $G(u; \omega_1, \omega_2, \omega_3)$ and appropriate change of the integration contour. Taking the limit $\omega_3 \to \infty$ such that $p, r \to 0$ one obtains hyperbolic hypergeometric integrals expressed in terms of the hyperbolic gamma function $\gamma(2)(u; \omega_1, \omega_2)$ (see the Appendix). Using this procedure the integral [24] can be reduced [28] [24] to the following expression:

$$I_8(x_1, \ldots, x_8; \lambda; \omega_1, \omega_2) = \frac{1}{2^N N!} \gamma(2)(\lambda; \omega_1, \omega_2)^{N-1} \times \prod_{1 \leq j < k \leq N} \gamma(2)(\lambda \pm u_j \pm u_k; \omega_1, \omega_2) \prod_{j=1}^{N} \gamma(2)(\pm u_j; \omega_1, \omega_2) \prod_{j=1}^{N} \frac{du_j}{\sqrt{\omega_1 \omega_2}},$$

where chemical potentials are related to flavor fugacities as $t = e^{2\pi i \lambda/\omega_2}$ and $y_k = e^{2\pi i x_k/\omega_2}$ with

$$\mu_k = x_k + \frac{\omega_1 + \omega_2}{4} - (N-1) \frac{\lambda}{4} - \frac{\lambda}{4}, \quad \sum_{k=1}^{8} x_k = 0.$$

In terms of $\mu_k$ the balancing condition reads

$$2(N-1)\lambda + \sum_{k=1}^{8} \mu_k = 2(\omega_1 + \omega_2).$$

Define now a new function

$$I_{4d/3d}(x_1, \ldots, x_8; \lambda; \omega_1, \omega_2) = \frac{I_8(x_1, \ldots, x_8; \lambda; \omega_1, \omega_2)}{\prod_{1 \leq j < k \leq 8} \gamma(3)(\frac{N+1}{2} \lambda + \mu_j + \mu_k; \omega_1, \omega_2, \lambda)},$$

(31)
where $\gamma^{(3)}(u; \omega_1, \omega_2, \lambda)$ is the hyperbolic gamma function of third order (see the Appendix). Again, it is not difficult to see that this function is $W(E_7)$-invariant as a consequence of known relations for $I_h$-integral:

$$I_{5d/3d}(\hat{x}_1, \ldots, \hat{x}_8; \lambda; \omega_1, \omega_2) = I_{5d/3d}(x_1, \ldots, x_8; \lambda; \omega_1, \omega_2),$$

where

$$\hat{x}_k = x_k - \frac{1}{2} \sum_{l=1}^{4} x_l, \quad \hat{x}_{k+4} = x_{k+4} + \frac{1}{2} \sum_{l=1}^{4} x_l, \quad k = 1, \ldots, 4.$$

Integral (31) may be interpreted as the partition function of some 5d-theory coupled to a 3d defect. Indeed, contribution of 5d hypermultiplets to the partition function is determined by the $1/\gamma^{(3)}$-function which indicates that our 5d theory has the field content similar to the one described earlier with the defect $S^3 \times S^1$ replaced by the squashed three-sphere $S^3_b$ with $b^2 = \omega_1/\omega_2$. The transition from 4d indices to 3d partition functions of theories living on such manifolds was described in [29]. Note that symmetry transformation (32) looks similar to the enhanced $E_8$-global symmetry discussed in [30] asking for an investigation of potential relations between corresponding theories.

4. Relevance of the modular group transformations

Let us discuss physical relevance of modular groups acting on the generalized gamma functions. Quasiperiods $\omega_k$ are usually interpreted as squashing parameters and coupling constants. The generalized gamma functions are defined differently for different domains of these parameters related to each other by modular transformations usually playing the role of $S$-dualities.

The simplest example of the relevance of $SL(2, \mathbb{Z})$-modular group is given by the $q$-gamma function. It can be defined as a solution of the equation $f(u + \omega_1) = (1 - e^{2\pi i u/\omega_1})f(u)$. For $|q| < 1$ its solution $1/(e^{2\pi i u/\omega_1}; q)$ defines the standard $q$-gamma function and serves as a building block of various partition functions. However, to cover the region $|q| = 1$, one needs $SL(2, \mathbb{Z})$-modular transformation and define the modified $q$-gamma function, i.e. to use the ratio of modular transformed elementary partition functions.

Consider now the elliptic gamma function $\Gamma(z; p, q)$ describing the superconformal index for a 4d chiral superfield. In order to define an analogue of this function for the region $|q| = 1$ in [2] the modified elliptic gamma function was proposed as the ratio of this index with a $U(1)$-group fugacity parametrization $z = e^{2\pi i u/\omega_2}$ and superconformal group generator fugacities $q = e^{2\pi i u_2/\omega_2}$ and $p = e^{2\pi i u_3/\omega_2}$ and the index with a different choice of squashing parameters $\Gamma(\tilde{q} e^{2\pi i u_2}; r, \tilde{q})$. Surprisingly, this ratio yields again the chiral field index with yet another parametrization of fugacities $e^{-\frac{1}{4} \tilde{B}_{3,3}(u; \omega)} \Gamma(e^{2\pi i u_3}; \tilde{r}, \tilde{p})$. The exponential cocycle factor spoils this interpretation and requires a physical interpretation. As shown in [6] this $SL(3, \mathbb{Z})$-group action on 4d superconformal indices describes the 't Hooft anomaly matching conditions as the conditions of cancellation of this cocycle contributions described by a curious set of Diophantine equations. Therefore this modular group plays quite important role in the formalism.

A similar picture at the level of free 6d hypermultiplet index was described recently in [14] in relation to the topological strings partition function. Namely,
\(I_{hyp}(y; p, q, t)\) is proportional to the latter function and, as argued in [14], a particular combination of three \(SL(4, \mathbb{Z})\)-transformed versions of it should yield yet another similar partition function. And this expectation is confirmed with the help of an \(SL(4, \mathbb{Z})\)-modular group transformation for the double elliptic gamma function which is written in our case as equality (23).

However, in difference from the \(G(u; \omega_1, \omega_2, \omega_3)\)-function case, the elliptic hypergeometric integrals formed from \(G(u; \omega_1, \omega_2, \omega_3)\) do not reduce to the integrals composed from \(\Gamma(z; p, q, t)\). Now the modular group simply maps them into similar integrals up to the cocycle \(e^{-h_{B_{14}}}\) multiplying the integral kernels. Therefore one should not expect cancellation of these factors from the integrals. Cancellation of even the gauge group chemical potentials is possible only under very strong restrictions, e.g. for \(SU(2)\) gauge group it is possible only for \(N_f = 16\) at the expense of an unusual quadratic restriction on chemical potentials. Such exponentials have the forms resembling the Casimir energy contributions to the indices [17]. Therefore it is necessary to better understand the general structure of full 6d superconformal indices before connecting \(SL(4, \mathbb{Z})\)-modular group transformations to higher dimensional anomalies. Still, we can see an involvement of the \(B_{14}\)-polynomial in the 4d anomaly matching conditions.

Define a modified elliptic hypergeometric integral:

\[
I^{mod}(x_1, \ldots, x_8; \omega_1, \ldots, \omega_4) = \frac{(\hat{p}; \hat{p})^N (\hat{r}; \hat{r})^N}{2^N N!} G(\omega_1; \omega_2, \omega_3)^{N-1}
\times \int_{[-\frac{N}{2}, \frac{N}{2}]^N} \prod_{1 \leq j < k \leq N} G(\omega_1 \pm u_j \pm u_k; \omega_1, \omega_2, \omega_3)
\times \prod_{j=1}^N G(x_j - \frac{N}{4} \omega_4 + \frac{1}{2} \sum_{k=1}^4 \omega_k \pm u_j; \omega_1, \omega_2, \omega_3)
\times G(\pm 2u_j; \omega_1, \omega_2, \omega_3) \frac{du_j}{\omega_3},
\] (33)

which is obtained from (23) simply by the replacement of \(\Gamma\)-functions by \(G\)-functions using exponential representation for fugacities in terms of chemical potentials and passing to the integration over a cube. Note that this integral is well-defined for \(|q| = 1\). Introduce “the modified index”

\[
I^{mod}_{6d/4d}(x_1, \ldots, x_8; \omega_1, \ldots, \omega_4)
= \frac{I^{mod}(x_1, \ldots, x_8; \omega_1, \ldots, \omega_4)}{\prod_{1 \leq j < k \leq 8} G(\omega_4 + \frac{1}{4} \sum_{k=1}^4 \omega_k + x_j + x_k; \omega_1, \ldots, \omega_4)},
\] (34)

containing the modified double elliptic gamma function. It is not difficult to check that this expression is also \(W(E_7)\)-invariant

\[
I^{mod}_{6d/4d}(\hat{x}_1, \ldots, \hat{x}_8; \omega_1, \ldots, \omega_4) = I^{mod}_{6d/4d}(x_1, \ldots, x_8; \omega_1, \ldots, \omega_4).
\] (35)

Now one can replace \(G\)-functions by their modular transformed expressions \(\hat{G}\) containing exponentials of Bernoulli polynomials and check that relation (35) boils down to an \(SL(3, \mathbb{Z})\)-modular transformation of the previous relation (26). At the level of integral (33) with the constraint \(2(x_7 + x_8) = \sum_{k=1}^4 \omega_k + (N - 1)\omega_4\) this was done already in [28]. As mentioned above, the condition of cancellation of Bernoulli polynomial coefficients in the integration variables and external parameters describes the ’t Hooft anomaly matching conditions. Therefore the fourth
order polynomial $B_{4,4}(u; \omega)$ is effectively involved into these anomaly matchings as well.

The residue calculus for elliptic hypergeometric integrals was developed long ago, see [2, 9] and references therein. It has shown that by shrinking the integration contour to zero one can formally represent integrals as sums of bilinear combinations of elliptic hypergeometric series with permuted base variables which describes the factorization of superconformal indices into some more elementary building blocks which in general are not defined in the limit $p \to 0$ or $q \to 0$. This analysis has lead to the discovery of the notion of two-index biorthogonality and the elliptic modular doubling principle [2, 9]. In [7] this residue calculus applied to 4d $\mathcal{N} = 2$ superconformal indices was physically interpreted as a result of insertions of surface defects into the bulk theory.

One can investigate the structure of residues for the modified elliptic hypergeometric integrals/indices and come to similar factorization in terms of different elliptic hypergeometric series. The latter series are related by an $SL(3, \mathbb{Z})$-transformation and remain well defined in the limit $p \to 0$, which leads to hyperbolic integrals. As a result hyperbolic integrals are represented as combinations of products of two $q$-hypergeometric series related by an $SL(2, \mathbb{Z})$-module transformation (their bases are $q$ and $\tilde{q}$) [2, 9]. This factorization was used in [32] for computing partition functions in some 3d $\mathcal{N} = 2$ theories appearing from the reduction of 4d $\mathcal{N} = 4$ SYM theories. The principle difference between 4d (elliptic) and 3d (hyperbolic) cases consists in the fact that in 3d this factorization of sums of residues into modular blocks has rigorous meaning because of the convergence of corresponding infinite series for $|q| < 1$, whereas in 4d such series do not converge for generic values of $p$ and $q$ bases and the factorization of indices has in general a formal meaning. It is not difficult to develop the residue calculus for 6d indices and find triple sums of residues. However, corresponding sums cannot factorize because there are no triply periodic functions. This makes the 4d (elliptic) case rather unique and raises the interest to 6d indices as qualitatively different objects.

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Appendix A. Barnes multiple gamma function

Barnes multiple zeta function $\zeta_m(s, u; \omega)$ [10] is originally defined by an $m$-fold series

$$\zeta_m(s, u; \omega) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{1}{(u + \Omega)^s}, \quad \Omega = n_1 \omega_1 + \ldots + n_m \omega_m,$$

where $s, u \in \mathbb{C}$. It converges for $\text{Re}(s) > m$, provided all $\omega_j$ lie in one half-plane formed by a line passing through zero (then there are no accumulation points of the $\Omega$-lattice in compact domains).

This zeta function satisfies equations

$$\zeta_m(s, u + \omega_j; \omega) - \zeta_m(s, u; \omega) = -\zeta_{m-1}(s, u; \omega(j)), \quad j = 1, \ldots, m, \quad (36)$$
where \( \omega(j) = (\omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_m) \) and \( \zeta_0(s, u; \omega) = u^{-s} \). The multiple gamma function is defined by Barnes as

\[
\Gamma_m(u; \omega) = \exp(\partial \zeta_m(s, u; \omega)/\partial s)|_{s=0}.
\]

As a consequence of \( \text{(36)} \) it satisfies \( m \) finite difference equations

\[
\Gamma_m(u + \omega_j; \omega) = \frac{1}{\Gamma_{m-1}(u; \omega(j))} \Gamma_m(u; \omega), \quad j = 1, \ldots, m,
\]

where \( \Gamma_0(u; \omega) := u^{-1} \).

The multiple sine-function is defined as

\[
S_m(u; \omega) = \frac{\Gamma_m(\sum_{k=1}^m \omega_k - u; \omega)^{(-1)^m}}{\Gamma_m(u; \omega)}
\]

and the hyperbolic gamma function is

\[
\gamma^{(m)}(u; \omega) = S_m(u; \omega)^{(-1)^{m-1}}.
\]

One has equations

\[
\gamma^{(m)}(u + \omega_j; \omega) = \gamma^{(m-1)}(u; \omega(j)) \gamma^{(m)}(u; \omega), \quad j = 1, \ldots, m.
\]

The standard elliptic gamma function can be written as a special ratio of four \( \Gamma_3(u; \omega) \)-functions, and the double elliptic gamma function is given by a product of four \( \Gamma_4(u; \omega) \)-functions \( \text{[9]} \).

One can derive the integral representation \( \text{[22]} \)

\[
\gamma^{(m)}(u; \omega) = \exp \left( -\text{PV} \int \frac{e^{ux}}{\prod_{k=1}^m (e^{\omega_k x} - 1)} \, dx \right)
\]

\[
= \exp \left( -\pi i \frac{1}{m!} B_{m,m}(u; \omega) - \int_{\mathbb{R} + i0} \frac{e^{ux}}{\prod_{k=1}^m (e^{\omega_k x} - 1)} \, dx \right)
\]

\[
= \exp \left( \pi i \frac{1}{m!} B_{m,m}(u; \omega) - \int_{\mathbb{R} - i0} \frac{e^{ux}}{\prod_{k=1}^m (e^{\omega_k x} - 1)} \, dx \right),
\]

where \( \text{Re}(\omega_k) > 0 \) and \( 0 < \text{Re}(u) < \text{Re}(\sum_{k=1}^m \omega_k) \) and \( B_{m,m} \) are multiple Bernoulli polynomials defined by the generating function

\[
\frac{x^m e^{xu}}{\prod_{k=1}^m (e^{\omega_k x} - 1)} = \sum_{n=0}^{\infty} B_{m,n}(u; \omega_1, \ldots, \omega_m) \frac{x^n}{n!}.
\]

In particular, one has the following relation with the modified \( q \)-gamma function \( \gamma(u; \omega_1, \omega_2) \):

\[
\gamma^{(2)}(u; \omega_1, \omega_2) = e^{-\frac{\pi i}{m!} B_{2,2}(u; \omega_1, \omega_2)} \gamma(u; \omega_1, \omega_2).
\]

Collapsing integrals to sums of residues one can derive infinite product representations for \( \gamma^{(m)}(u; \omega) \) \( \text{[22]} \). Particular inversion relations have the form

\[
\gamma^{(2)}(\sum_{k=1}^2 \omega_k + u; \omega_1, \omega_2) \gamma^{(2)}(-u; \omega_1, \omega_2) = 1,
\]

\[
\gamma^{(3)}(\sum_{k=1}^3 \omega_k + u; \omega_1, \omega_2, \omega_3) = \gamma^{(3)}(-u; \omega_1, \omega_2, \omega_3).
\]
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Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow reg. 141980, Russia and Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111, Bonn, Germany