Orbit equivalence and actions of $\mathbb{F}_n$

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Abstract

In this paper we show that there are “$E_0$ many” orbit inequivalent free actions of the free groups $\mathbb{F}_n$, $2 \leq n \leq \infty$ by measure preserving transformations on a standard Borel probability space. In particular, there are uncountably many such actions.

I. Introduction

Let $G_1, G_2$ be countable groups, acting by measure preserving transformations on standard Borel probability spaces $(X_1, \mu_1)$ and $(X_2, \mu_2)$ respectively, giving rise to orbit equivalence relations $E_{G_1}$ and $E_{G_2}$. We say that the actions of $G_1$ and $G_2$ are orbit equivalent if there is a measure preserving bijection $\varphi : X_1 \rightarrow X_2$ such that

$$xE_{G_1}y \iff \varphi(x)E_{G_2}\varphi(y)$$

almost everywhere.

The fundamental theorem in the study of the notion of orbit equivalence is the theorem of H. Dye ([6], [7]), which states that two ergodic measure preserving actions of $\mathbb{Z}$ are orbit equivalent. Ornstein and Weiss ([21], [4]) showed that this theorem extends to all countable amenable groups.

The work of Connes, Weiss and Schmidt ([24], [5]), and more recently Hjorth ([13]), shows that this characterizes amenability: A countable group is amenable if and only if it has, up to orbit equivalence, only one ergodic action by measure preserving transformations on a standard Borel probability space.
The study of orbit equivalence is naturally related to the study of Borel equivalence relations, and the notion of Borel reducibility. (The reader may find a more thorough discussion of Borel reducibility in [16].) For equivalence relations $E$ and $F$ on Polish spaces $X$ and $Y$, we say that $E$ is Borel reducible to $F$, written $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that

$$xEy \iff f(x)Ff(y).$$

In other words, $E$ is Borel reducible to $F$ if we can classify points in $X$ up to $E$ equivalence by a Borel assignment of invariants, that are $F$ equivalence classes.

An equivalence relation $E$ on a Polish space $X$ is said to be smooth or concretely classifiable if $E \leq_B =_{2^\mathbb{N}}$, where $=_{2^\mathbb{N}}$ is the equality relation on the Cantor space $2^\mathbb{N}$. In other words, a smooth equivalence relation admits a Borel assignment of real numbers as complete invariants, classifying elements of $X$ up to $E$ equivalence.

There are equivalence relations which are not smooth. The cardinal example of such an equivalence relation is $E_0$, defined on $2^\mathbb{N}$ by

$$fE_0g \iff (\exists N)(\forall n \geq N)f(n) = g(n).$$

It is not hard to see that $E_0 \not\leq_B =_{2^\mathbb{N}}$, and $=_{2^\mathbb{N}} \leq_B E_0$, i.e. that $=_{2^\mathbb{N}} <_B E_0$. Hence we cannot in a Borel way classify points up to $E_0$ equivalence using real numbers as invariants. This can be understood as saying that, in the sense of $\leq_B$, there are many more $E_0$ classes than there are real numbers.

If $E_0 \leq_B F$ for an equivalence relation $F$, we will say that $F$ has (at least) “$E_0$ many” equivalence classes. In this paper we show:

**Theorem 1.** There are (at least) $E_0$ many orbit inequivalent almost everywhere free actions of $\mathbb{F}_n$, $2 \leq n \leq \infty$, by measure preserving transformations on a standard Borel probability space.

This may be seen as a strengthening of the following result of Gaboriau and Popa:
Theorem (Gaboriau-Popa, [9].) There are continuum many orbit inequivalent a.e. free actions of \( \mathbb{F}_n \), \( 2 \leq n \leq \infty \), by measure preserving transformations on a standard Borel probability space.

It is worth noting explicitly that Theorem 1 is stronger than the result of Gaboriau and Popa since it rules out the possibility of finding a reasonable real-valued complete invariant for orbit equivalence. In the light of Gaboriau’s work on the notion of “cost” [8], one might have hoped otherwise.

Outline and organization. The proof of Theorem 1 relies mostly on elementary methods, and does not involve the operator algebra techniques used by Gaboriau and Popa in their result in [9]. That said, both results rely on similar ideas with origins in Popa’s work on rigidity phenomena and the notion of relative property (T) in [22].

We first consider in section II a particular (a.e.) free action of \( \mathbb{F}_2 \) on a standard Borel probability space \( (X, \mu) \) by measure preserving transformations. This action has the special property, that there is a countable group \( G \subseteq L_0(X, \mathbb{T}) \subseteq L^\infty(X) \), invariant under the action of \( \mathbb{F}_2 \), such that the induced semi-direct product \( G \rtimes \mathbb{F}_2 \) has the relative property (T).

Our strategy is to first obtain \( E_0 \) many actions of \( \mathbb{F}_3 \). Denote by \( \mathcal{M}_\infty(X) \) the (Polish) group of all measure preserving transformations on \( X \). In section III we prove a general lemma which has a consequence that there is a dense \( G_\delta \) set of transformations that extends a given a.e. free m.p. action of \( \mathbb{F}_2 \) to an a.e. free m.p. action of \( \mathbb{F}_3 \).

The main argument is presented in section V. We consider the special action of \( \mathbb{F}_2 \) mentioned above, and introduce an equivalence relation \( \mathcal{R} \) on \( \mathcal{M}_\infty(X) \) by letting \( SRS' \) whenever the equivalence relation induced by the transformation \( S \) and (the action of) \( \mathbb{F}_2 \), and the equivalence relation induced by \( S' \) and \( \mathbb{F}_2 \), are orbit equivalent. Similarly, we let \( SFS' \) whenever \( S \) and \( \mathbb{F}_2 \) induce the same equivalence relation as \( S' \) and \( \mathbb{F}_2 \) (a.e.)

Using the relative Kazhdan property, we show that \( \mathcal{R}/\mathcal{F} \) has countable classes. It then follows easily that \( \mathcal{R} \) is meagre in \( \mathcal{M}_\infty(X) \times \mathcal{M}_\infty(X) \). We then show that a Theorem of Becker and Kechris applies to give us that \( E_0 \leq \beta \mathcal{R} \), which shows Theorem 1 in the case \( \mathbb{F}_3 \).

The case \( \mathbb{F}_n, n > 3 \) is obtained similarly. At the end of section V, we see that the case \( \mathbb{F}_2 \) follows from that of \( \mathbb{F}_3 \) by an expansion argument.
We also get the following surprising corollary:

**Corollary.** Equality a.e. of equivalence relations induced by a.e. free measure preserving actions of $F_n$, $n \geq 2$, is not smooth.

The author does not know if either orbit equivalence or equality in this context is strictly above $E_0$, however it seems natural suspect that this is so.

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**II. An action of $F_2$**

In this section we will describe a particular ergodic action of $F_2$ which has some very interesting properties. The idea behind this is due to Sorin Popa ([22]), and is fundamental to the entire argument of this paper. First we recall the notion of relative property (T) and related concepts.

**Definition ([2], [12]).** If $(\pi, H)$ is unitary representation of a countable group $G$, $Q \subseteq G$ is a subset, and $\varepsilon > 0$, we say that a vector $v \in H$ is $(Q, \varepsilon)$-invariant if

$$\sup_{g \in Q} \|\pi(g)v - v\|_H < \varepsilon \|v\|_H.$$ 

The semidirect product of countable groups $H \rtimes G$ has the relative property (T)\footnote{In the literature one sometimes refers to this situation by saying that $(G, H)$ is a pair with property (T), cf. [2] 1.4.3.}, if there is a finite set $Q \subseteq H \rtimes G$ and $\varepsilon > 0$ such that whenever $(\pi, H)$ is a unitary representation of $H \rtimes G$ with $(Q, \varepsilon)$-invariant vectors, then there is a non-zero $H$-invariant vector.

If $H \rtimes G$ has the relative property (T), it can be seen (as in [2] proposition 1.1.8) that given $\delta > 0$ we may find $\varepsilon > 0$ such that if $v$ is a $(Q, \varepsilon)$-invariant...
vector for $(\pi, \mathcal{H})$, then there is an $H$-invariant vector $v'$ such that $\|v - v'\|_H \leq \delta$. In other words, we can ensure that almost invariant vectors are $\delta$ close to $H$-invariant vectors.

**Examples.** The semidirect product $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$, corresponding to the natural action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$, has the relative property (T). ([17],[3],[25]).

The matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a copy of $\mathbb{F}_2$ inside $SL_2(\mathbb{Z})$. It can be shown that this subgroup has finite index, so that $\mathbb{F}_2$ is a lattice in $SL_2(\mathbb{Z})$. It follows (see [2] Theorem 1.5.1) that the corresponding semidirect product $\mathbb{Z}^2 \rtimes \mathbb{F}_2$ also has the relative property (T).

**The action.** We now describe the particular action of $\mathbb{F}_2$ with which we will be working throughout this paper. We denote by $T$ the 1-torus

$$T = \{ z \in \mathbb{C} : |z| = 1 \}.$$

Note that the map $x \mapsto e^{2\pi ix}$ identifies $\mathbb{R}/\mathbb{Z}$ and with $T$.

We consider the group $\mathbb{Z}^2$ and in particular its dual, $\hat{\mathbb{Z}}^2$. Every character of $\mathbb{Z}^2$ has the form

$$\chi_a \left( \begin{pmatrix} n \\ m \end{pmatrix} \right) = e^{2\pi i (na_1 + ma_2)}$$

where $a = (a_1, a_2) \in \mathbb{R}^2$. Hence we may identify $\hat{\mathbb{Z}}^2$ with the 2-torus $T^2$.

The action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$ induces an action on $\hat{\mathbb{Z}}^2$, as defined by

$$\sigma \cdot \chi \left( \begin{pmatrix} n \\ m \end{pmatrix} \right) = \chi(\sigma^{-1} \begin{pmatrix} n \\ m \end{pmatrix}).$$

We then have the formula

$$\chi_a = \chi(\sigma^{-1})^T a,$$

which shows that the corresponding action of $SL_2(\mathbb{Z})$ on $T^2$ is measure preserving. We have previously noted that $\mathbb{F}_2$ sits inside $SL_2(\mathbb{Z})$ as a lattice.
The restriction of the action of $SL_2(\mathbb{Z})$ on $\mathbb{T}^2$ to this subgroup is the the action of $F_2$ we wish to consider in detail.

The action of $SL_2(\mathbb{Z})$ on $\mathbb{T}^2$ induces an action on $L_0(\mathbb{T}^2, \mathbb{T}) \subseteq L^\infty(\mathbb{T}^2)$, where $L_0(\mathbb{T}^2, \mathbb{T})$ is the group of measurable functions $\mathbb{T}^2 \to \mathbb{T}$ with pointwise multiplication, through

$$\sigma \cdot f(\chi) = f(\sigma^{-1} \cdot \chi).$$

Define $\phi_{(n,m)} \in L_0(\mathbb{T}^2, \mathbb{T})$ by

$$\phi_{(n,m)}(\chi) = \chi \left( \begin{array}{c} n \\ m \end{array} \right).$$

Then the map $\mathbb{Z}^2 \to L^\infty(\mathbb{T}^2)$ defined by

$$\left( \begin{array}{c} n \\ m \end{array} \right) \mapsto \phi_{(n,m)}$$

is an endomorphism of $\mathbb{Z}^2$ into $L_0(\mathbb{T}^2, \mathbb{T})$, and we denote by $G$ the group

$$G = \{ \phi_{(n,m)} : \left( \begin{array}{c} n \\ m \end{array} \right) \in \mathbb{Z}^2 \}.$$ 

It follows easily from the definitions that

$$\sigma \cdot \phi_{(n,m)} = \phi_{\sigma(n,m)}, \ (\sigma \in SL_2(\mathbb{Z})), $$

which shows that $G$ is invariant under the action of $SL_2(\mathbb{Z})$, and the semidirect product $G \rtimes SL_2(\mathbb{Z})$ is isomorphic to $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$. It follows that $G \rtimes SL_2(\mathbb{Z})$ has the relative property (T). Note also that $G$ separates points in $\mathbb{T}^2$.

Finally we observe:

**Claim.** The action of the subgroup $F_2 < SL_2(\mathbb{Z})$ on $\mathbb{T}^2$ is ergodic.

**Proof.** We must show that for any $f \in L_0^2(\mathbb{T}^2) = 1^+$, we have $\sigma \cdot f = f$ for all $\sigma \in F_2$ iff $f = 0$. It is a standard fact of Fourier analysis on locally compact abelian groups that $\{ \phi_{(n,m)} : \left( \begin{array}{c} n \\ m \end{array} \right) \in \mathbb{Z}^2 \}$ forms an orthonormal basis for $L^2(\mathbb{T}^2)$ (see [23], p. 143 ff.). Hence

$$\{ \phi_{(n,m)} : \left( \begin{array}{c} n \\ m \end{array} \right) \neq 0 \}$$
is an orthonormal basis for $L^2_0(\mathbb{T}^2)$.

Suppose first that $f \neq 0$ and 

$$f = a_1 \phi_{(n_1 m_1)} + \cdots + a_l \phi_{(n_l m_l)},$$

$(n_i m_i) \neq (0,0)$ and $a_1, \ldots, a_l \in \mathbb{C}$. Since 

$$A^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}, \quad B^k = \begin{pmatrix} 1 & 0 \\ 2k & 1 \end{pmatrix}$$

we may find $k > 0$ such that for any $m_i \neq 0$ we have $(A^k(n_i m_i))_1 \neq n_j$ for all $j \leq l$. Then find $k' > 0$ such that for all $m_i = 0$ we have $(B^{k'}(n_i m_i))_2 \neq m_j$ for all $j \leq l$. Then for $\sigma = B^{k'} A^k$ we have $\sigma(n_i m_i) \neq (n_j m_j)$ for all $j \leq l$, and 

$$\sigma \cdot f = a_1 \phi_{\sigma(n_1 m_1)} + \cdots + a_l \phi_{\sigma(n_l m_l)},$$

so that $(\sigma \cdot f, f) = 0$. Now for general $0 \neq f \in L^2_0(\mathbb{T}^2)$ it follows that $| (\sigma \cdot f, f) |$ can be made arbitrarily small for appropriate $\sigma \in \mathbb{F}_2$ and hence there is $\sigma \in \mathbb{F}_2$ such that $\sigma \cdot f \neq f$. \hfill \Box

Let us summarize what we have shown:

**Proposition 1.** There is an ergodic action of $\mathbb{F}_2$ on a standard non-atomic probability space $(X, \mu)$, such that

(i) There is a countable group $G \subseteq L_0(X, \mathbb{T}) \subseteq L^\infty(X)$, invariant under the induced action of $\mathbb{F}_2$ on $L^\infty(X)$,

(ii) The group $G$ separates points in $X$, and

(iii) The semidirect product $G \rtimes \mathbb{F}_2$ has the relative property (T).

This is all we will need to know for the argument below.
III. The Category Lemma

Let \((X, \mu)\) be a standard non-atomic probability space, and denote by \(\mathcal{M}_\infty(X)\) the group of all measure preserving transformations on \(X\). This group has two important group topologies (see \[10\] pp. 61 and 69): The first is the weak topology which is defined by the neighborhood basis

\[
N(T; E_1, \ldots, E_k, \varepsilon) = \{S \in \mathcal{M}_\infty(X) : (\forall i \leq k) \mu(T(E_i) \triangle S(E_i)) < \varepsilon\}
\]

for \(T \in \mathcal{M}_\infty(X)\), where \(\varepsilon > 0\) and \(E_1, \ldots, E_k \subseteq X\) are measurable subsets of \(X\). With the weak topology \(\mathcal{M}_\infty(X)\) is a Polish group.

The other topology is the uniform topology, which is induced by the metric

\[
d_U(T, S) = \mu(\{x : T(x) \neq S(x)\}).
\]

The uniform topology is stronger than the weak topology and the metric \(d_U\) is complete. However, the uniform topology is not separable.

The uniform topology will be useful in later sections, but for the considerations of this section we only need the weak topology.

In this section we prove the following category theoretic fact:

The Category Lemma. Let \(G = \{T_n \in \mathcal{M}_\infty(X) : n \in \mathbb{N}\}\) be a countable group of measure preserving transformations, and suppose \(G\) acts freely almost everywhere on \((X, \mu)\). Then

\[
\{S \in \mathcal{M}_\infty(X) : G * \langle S \rangle \text{ acts a.e. freely on } X\}
\]

is a dense \(G_\delta\) subset of \(\mathcal{M}_\infty(X)\) in the weak topology.

Here \(G * \langle S \rangle\) denotes the free product of the group \(G\) and the group generated by \(S\), denoted \(\langle S \rangle\), which may be thought of formally as the set of finite sequences of alternatingly elements from \(G\) and elements of \(\langle S \rangle\), with the obvious concatenate-and-reduce operation as composition.

Before the proof, let us note some useful facts:
**Observation.** If \((A_i)_{i \leq k}\) is a sequence of \(k \in \mathbb{N}\) measurable subsets of \((X, \mu)\), then there is a measure preserving involution \(P \in \mathcal{M}_\infty(X)\) such that \(P(x) \neq x\) a.e. and \(P(A_i) = A_i\) for all \(i \leq k\). The proof is an easy induction on \(k\): For \(k = 1\), it is simply the fact that there always is an involution with almost no fixed points. Now suppose suppose the assertion holds for \(k \geq 1\), and consider \((A_i)_{i \leq k+1}\). Then there is an involution \(T_0 : A_{k+1} \to A_{k+1}\) with almost no fixed points such that \(T_0(A_i \cap A_{k+1}) = A_i \cap A_{k+1}\), and similarly \(T_1 : X \setminus A_{k+1} \to X \setminus A_{k+1}\). Then \(T = T_0 \cup T_1\) is the desired transformation.

We also note the following easy technical lemma:

**Lemma 2.** Let \(T_1, \ldots, T_n : X \to X\) be measurable functions such that

\[
\mu(\{x : (\forall i,j) i \neq j \implies T_i(x) \neq T_j(x)\}) > K > 0.
\]

Then there are finitely many disjoint non-null Borel sets \(E_1, \ldots, E_m\) such that

\[
\mu\left(\bigcup_{l \leq m} E_l\right) > K,
\]

and \(T_i(E_l) \cap T_j(E_l) = \emptyset\) whenever \(i \neq j, l \leq m\).

**Proof.** Assume that \(X\) is equipped with a compatible Polish topology. By Lusin’s Theorem (cf. [18], 17.12), let

\[
F \subseteq \{x : (\forall i,j) i \neq j \implies T_i(x) \neq T_j(x)\}
\]

be a closed set of measure \(\mu(F) > K\), such that all of \(T_1, \ldots, T_n\) are continuous on \(F\). Then for each \(x \in F\), there is a basic open set \(O_x\) such that \(T_i(O_x \cap F) \cap T_j(O_x \cap F) = \emptyset\). Since the collection of sets \(O_x\) is countable, and \(\mu(\bigcup_{x \in F} O_x \cap F) = \mu(F)\), there are finitely many \(O_{x_1}, \ldots, O_{x_k}\), such that

\[
\mu(\bigcup_{l \leq k} O_{x_l} \cap F) > K.
\]

After possibly breaking the collection \((O_{x_l})_{l \leq k}\) into disjoint pieces, we obtain a collection of disjoint Borel sets \(B_1, \ldots, B_m \subseteq X\) with the same properties. Now let \(E_l = B_l \cap F\). \(\square\)
Proof of the Category Lemma. Let \( \langle a \rangle \) be an infinite cyclic group with a single generator \( a \), and consider the free product \( G \ast \langle a \rangle \) of \( G = \{ T_n : n \in \mathbb{N} \} \) and \( \langle a \rangle \). The free product may be thought of as consisting of words in the alphabet \( A = \{ T_n : n \in \mathbb{N} \} \cup \{ a, a^{-1} \} \), reduced according to the rules of the respective groups.

Given such a word \( w \), the evaluation map \( e_w : M_\infty(X) \rightarrow M_\infty(X) \) associated to \( w \) is the map that associates to a transformations \( S \in M_\infty(X) \) the transformation obtained by replacing \( a \) with \( S \) in the word \( w \). Note that since \( M_\infty(X) \) is a topological group, the evaluations map \( e_w \) is continuous.

We will show that for a non-trivial reduced word \( w \) in the alphabet \( A \),

\[
\{ S \in M_\infty(X) : e_w(S)(x) \neq x \text{ a.e.} \}
\]

is a dense \( G_\delta \) set in \( M_\infty(X) \).

The proof goes by induction on the length of the word \( w \). Assume that the above holds for all non-trivial reduced words \( \eta \) with \( \text{lh}(\eta) < n \), and let \( \text{lh}(w) = n \). Let \( \varepsilon > 0 \) be given, and consider the set

\[
G_\varepsilon = \{ S \in M_\infty(X) : e_w(S)(x) \neq x \text{ on a set of measure } > 1 - \varepsilon \}.
\]

It suffices to show this set is (i) open and (ii) dense.

(i) The set \( G_\varepsilon \) is open.

This will follow from:

Claim. If \( P \in M_\infty(X) \) and

\[
\mu(\{ x : P(x) \neq x \}) > K > 0,
\]

then there is a neighborhood \( N \subseteq M_\infty(X) \) of \( P \) such that

\[
\mu(\{ x : S(x) \neq x \}) > K
\]

for all \( S \in N \).

Proof. Let \( \delta > 0 \) be such that

\[
\mu(\{ x : P(x) \neq x \}) > K + \delta.
\]
By lemma 2 applied to the identity transformation $I$ and $P$, there are disjoint Borel sets $E_1, \ldots, E_m$, such that $P(E_i) \cap E_i = \emptyset$ and $\mu(\bigcup E_i) > K + \delta$. Now consider the neighborhood

$$N = N(P; E_1, \ldots, E_m, \frac{\delta}{m}) = \{S : (\forall l \leq m) \mu(P(E_l) \triangle S(E_l)) < \frac{\delta}{m}\}.$$ 

Then for any $S \in N$ we have $\mu(S(E_l) \cap E_l) < \frac{\delta}{m}$, so that

$$\mu(\{x : S(x) \neq x\}) \geq \sum_{l=1}^{m} \mu(E_l) - \frac{\delta}{m} > K,$$

and $N$ is the neighborhood we needed to find.

From the claim it follows easily that $G_\varepsilon$ is open, since

$$G_\varepsilon = e_w^{-1}(\{P : P(x) \neq x \text{ on a set of measure } > 1 - \varepsilon\}),$$

where $e_w$ is the evaluation map.

(iii) The set $G_\varepsilon$ is dense.

Fix a sequence $\eta_0, \ldots, \eta_n = w$ of reduced words in the alphabet $\mathcal{A}$, such that $\text{lh}(\eta_i) = i$, and for all $i < n$ there is a unique $\tau \in \mathcal{A}$, such that $\eta_{i+1} = \tau \eta_i$.

Let $S \in \mathcal{M}_\infty(X)$ and let $N$ be a neighborhood of $S$. We want to show that there is $S' \in N \cap G_\varepsilon$. By our inductive assumption, we can assume that

$$S \in \bigcap_{\text{lh}(\eta) < n} \{S' : e_\eta(S')(x) \neq x \text{ a.e.}\}.$$

Moreover, we can assume the letter $a$ (or $a^{-1}$) occurs at some point in the reduced word $w$, since otherwise there is nothing to show.

Clearly $e_{\eta_i}(S)(x) \neq e_{\eta_j}(S)(x)$ a.e. whenever $i \neq j$, $(i, j < n)$. So by lemma 2, we can find disjoint non-0 measurable sets $E_1, \ldots, E_M \subseteq X$ such that

$$\mu(\bigcup_{l \leq M} E_l) > 1 - \varepsilon$$

and $e_{\eta_i}(S)(E_l) \cap e_{\eta_j}(S)(E_l) = \emptyset$ whenever $i \neq j$. 


Let \((A_i)_{i \in \mathcal{F}_0}\) be a finite family of measurable sets and let \(\delta > 0\) be such that

\[N_0 = N(S; (A_i)_{i \in \mathcal{F}_0}, \delta) \subseteq N,\]

and such that the collection \((A_i)_{i \in \mathcal{F}_0}\) contains all of the sets

\[e_{\eta_i}(S)(E_i), \ (l \leq M, i < n).\]

Define

\[f(P) = \{x : e_w(P)(x) = x\}\]

for \(P \in \mathcal{M}_\infty(X)\). Then either \(\mu(f(S)) < \varepsilon\) (in which case there is nothing to show), or there is some \(E_l\) with \(\mu(f(S) \cap E_l) > 0\). We may assume that \(\mu(f(S) \cap E_1) > 0\). Let \(i_0 < n\) be largest possible such that \(\eta_{i_0 + 1} = a\eta_{i_0}\) or \(\eta_{i_0 + 1} = a^{-1}\eta_{i_0}\). Without loss of generality, assume \(\eta_{i_0 + 1} = a\eta_{i_0}\) and let 

\[B = e_{\eta_{i_0}}(S)(f(S) \cap E_1).\]

As observed before the proof, we may find an involution \(P \in \mathcal{M}_\infty(X)\) such that \(P(x) \neq x\) for almost all \(x \in B\), and \(P(x) = x\) for all \(x \notin B\), such that \(P(A_i) = A_i\) for all \(i \in \mathcal{F}_0\).

Define

\[S_1 = SP.\]

Then \(S_1(A_i) = S(A_i)\) for all \(i \in \mathcal{F}_0\), in particular \(S_1 \in N_0\). Moreover, for almost all \(x \in E_1 \cap f(S)\), we have \(e_w(S_1)(x) \neq e_w(S)(x) = x\), and for \(x \in E_1 \setminus f(S)\) we have \(e_w(S_1)(x) = e_w(S)(x) \neq x\). Thus \(e_w(S_1)(x) \neq x\) for almost all \(x \in E_1\).

By Lemma 2, we may then find disjoint measurable sets \(F_1, \ldots, F_p \subseteq E_1\), such that

\[\mu(\bigcup_{q \leq p} F_q \cup \bigcup_{1 < l \leq M} E_p) > 1 - \varepsilon\]

and \(e_w(S_1)(F_q) \cap F_q = \emptyset\), \(q \leq p\).

Let \((A_i)_{i \in \mathcal{F}_1}\) be the extension of the family \((A_i)_{i \in \mathcal{F}_0}\) obtained by adding all the sets

\[e_{\eta_i}(S_1)(F_q), \ (i < n, q \leq p).\]

If \(\mu(f(S_1)) < \varepsilon\), then we’re done. Otherwise we may find \(l > 1\) for which \(\mu(f(S_1) \cap E_l) > 0\), indeed we may assume \(\mu(f(S_1) \cap E_2) > 0\). Now we can apply the same argument as above, with \((A_i)_{i \in \mathcal{F}_1}\) in place of \((A_i)_{i \in \mathcal{F}_0}\), to get \(S_2 \in N_0\) with \(f(S_2) \cap E_2 = \emptyset\), and a finite collection \((A_i)_{i \in \mathcal{F}_2}\) extending \((A_i)_{i \in \mathcal{F}_1}\). However, as this construction guarantees that \(S_1(A_i) = S_2(A_i)\)
for all $i \in F_1$, we retain that $e_w(S_2)(F_q) \cap F_q = \emptyset$, for $q \leq p$. Hence by repeating the above argument finitely many times, we eventually obtain a transformation $S' = S_l \in N_0$, for some $l \leq M$, such that

$$\mu(\{x : e_w(S')(x) \neq x\}) > 1 - \varepsilon,$$

and since $S' \in N_0 \subseteq N$, this completes the proof.

**Remark.** The author is thankful to Sorin Popa for making the following remarks regarding the Category Lemma in a recent conversation.

The proof of the category lemma does not use the fact that $G = \{T_n \in \mathcal{M}_\infty(X) : n \in \mathbb{N}\}$ is a group. Rather, the proof shows that given a sequence of transformations $\{T_n \in \mathcal{M}_\infty(X) : n \in \mathbb{N}\}$ with almost no fixed points, we can find a dense $G_\delta$ set of $T \in \mathcal{M}_\infty(X)$ that are “independent” of $\{T_n : n \in \mathbb{N}\}$, in the sense that any composition of alternatingly elements $T_n$, $n \in \mathbb{N}$, and $T$ or $T^{-1}$, does not have any fixed points almost everywhere. Popa has pointed out that this observation gives us the following useful corollary:

**Corollary.** Let $(X, \mu)$ be a standard Borel probability space, and suppose $H_1$ and $H_2$ are countable groups of transformations in $\mathcal{M}_\infty(X)$, acting freely a.e. on $X$. Then there is a dense $G_\delta$ set of transformations $T \in \mathcal{M}_\infty(X)$ such that $H_1 * TH_2 T^{-1}$ acts freely a.e. on $X$.

**Proof.** By the remark, we can find a dense $G_\delta$ set of transformations $T \in \mathcal{M}_\infty(X)$ that are independent of the set of transformations $H_1 \cup H_2$, in the above sense. But then $H_1 * TH_2 T^{-1}$ acts freely a.e. on $X$. 

**IV. The group $\mathcal{M}_\infty(X)$**

Before we proceed to prove Theorem 1, we note in this section some standard facts regarding the group $\mathcal{M}_\infty(X)$ and some important subgroups.

If $G$ is a countable group acting by m.p. transformations on the standard Borel probability space $(X, \mu)$, giving rise to the equivalence relation $E_G$,
the full group, or inner group, of \( E_G \) is the group
\[
\text{Inn}(E_G) = \{ S \in \mathcal{M}_\infty(X) : S(x)E_Gx \text{ a.e.}\}.
\]
It is easy to see that in this case \( \text{Inn}(E_G) \) is a Polish group when given the uniform topology. The full group is also denoted \([E_G]\). The set of partial measure preserving functions, i.e. morphisms, whose graph is contained in \( E_G \), is denoted \([[E_G]]\).

**Proposition 2.** Let \( G \) be a countable group acting by m.p. transformations on the standard Borel probability space \((X, \mu)\). Consider \( \mathcal{M}_\infty(X) \) with the weak topology. Then

(i) \( \text{Inn}(E_G) \) is a meagre subgroup of \( \mathcal{M}_\infty(X) \), and

(ii) \( \text{Inn}(E_G) \) is dense if and only if \( E_G \) is ergodic.

**Proof.** (i) Since the uniform topology is stronger than the weak topology, the identity embeds \( \text{Inn}(E_G) \) continuously into \( \mathcal{M}_\infty(X) \), and so \( \text{Inn}(E_G) \) is an analytic (in fact, Borel) subgroup of \( \mathcal{M}_\infty(X) \). In particular, \( \text{Inn}(E_G) \) has the Baire property, and it follows by Pettis Theorem (cf. \[18\]) that either it is meagre, or it contains a neighborhood of the identity. However, the latter cannot be the case, since then \( \text{Inn}(E_G) \) would not be separable in the uniform topology.

(ii) The “only if” direction is obvious. For the “if” direction, let \( A, A' \subseteq X \) be measurable sets with \( \mu(A) = \mu(A') > 0 \). We first show, that there is a morphism \( \varphi : A \rightarrow A', \varphi \in [[E_G]] \). Since
\[
\mu(\bigcup_{g \in G} g \cdot A) = 1,
\]
there is some \( g \in G \) such that \( \mu(g \cdot A \cap A') > 0 \). Let \( B_0 = g^{-1} \cdot A' \cap A \) and define \( \varphi(x) = g \cdot x \) for \( x \in B_0 \). If \( \mu(A \setminus B_0) = 0 \), we’re done. Otherwise, we can repeat the argument with \( A \setminus B_0 \) and \( A' \setminus g \cdot B_0 \). In this way, we eventually exhaust \( A \) (in perhaps transfinitely many steps), and have defined the desired morphism \( \varphi \).

Let \( T \in \mathcal{M}_\infty(X) \), and let \( A_1, \ldots, A_k \) be measurable subsets of \( X \). We claim that there is \( S \in \text{Inn}(E_G) \) such that \( \mu(S(A_i) \triangle T(A_i)) = 0 \) for all
i \leq k. After possibly breaking the sets $A_1, \ldots, A_k$ into smaller pieces, we can assume they are disjoint. Then by the above we can find morphisms $\varphi_0, \ldots, \varphi_k \in [[E_G]]$ such that $\varphi_i : A_i \to T(A_i)$ for $1 \leq i \leq k$ and

$$\varphi_0 : X \setminus \bigcup A_i \to T(X \setminus \bigcup A_i).$$

Define $S = \bigcup_{i \geq 0} \varphi_i$.

Note now that $S \in N(T; A_1, \ldots, A_k, \varepsilon)$ for all $\varepsilon > 0$. Since $T$ and $A_1, \ldots, A_k$ were arbitrary, this shows that $\text{Inn}(E_G)$ is dense in $\mathcal{M}_\infty(X)$.

Let $(O_n)$ be a countable basis for a compatible Polish topology on the standard probability space $(X, \mu)$. Assume the basis $(O_n)$ is closed under finite unions. Then a complete metric for the weak topology on $\mathcal{M}_\infty(X)$ is given by

$$d_w(S, T) = \sum_{m} 2^{-m} \left[ \mu(S(O_m) \triangle T(O_m)) + \mu(S^{-1}(O_m) \triangle T^{-1}(O_m)) \right],$$

(see [18].) We note the following fact about the relation between convergence in $d_w$ and pointwise convergence:

**Proposition 3.** Let $S \in \mathcal{M}_\infty(X)$ and suppose $(S_n)$ is a sequence of measure preserving transformations such that $d_w(S_n, S) < 2^{-n}$. Then $S_n(x) \to S(x)$ a.e.

**Proof.** Let $\varepsilon > 0$ and $\rho > 0$. Let $F \subseteq X$ be a closed set such that $S$ is continuous on $F$ and $\mu(F) > 1 - \frac{\rho}{2}$. We want to show that

$$\mu(\{ x : (\exists N)(\forall n \geq N) \ d(S_n(x), S(x)) < \varepsilon \}) > 1 - \rho.$$

For this, first find finitely many basic open sets $O_{m_1}, \ldots, O_{m_k}$ such that

$$x, y \in O_{m_i} \cap F \implies d(S(x), S(y)) < \varepsilon$$

and $\mu(\bigcup_{i \leq k} O_{m_i} \cap F) > 1 - \frac{\rho}{2}$.

Since $d_w(S_n, S) < 2^{-n}$, we have for each $i$ that $\mu(S_n(O_{m_i}) \triangle S(O_{m_i})) < C_m 2^{-n}$, where $C_m > 0$ is a constant which depends only on $m_i$. Then for $N > 0$ such that $C_m 2^{-N} < \frac{\rho}{2k}$,

$$\mu(S(O_{m_i}) \cap \bigcap_{n > N} S_n(O_{m_i})) \geq \mu(S(O_{m_i})) - \frac{\rho}{2k}.$$
Since if $x, y \in S(O_m \cap F)$ we have $d(x, y) < \varepsilon$, it now follows that

$$
\mu(\{x : (\exists N)(\forall n \geq N) \; d(S_n(x), S(x)) < \varepsilon\}) > 1 - \rho.
$$

\[\Box\]

V. The main argument

In this section we prove Theorem 1. We focus on proving the theorem for $F_3$, the argument for $F_n$, $n > 3$, being similar. The case $F_2$ will eventually follow from that of $F_3$.

Consider an action of $F_2 = \langle a, b \rangle$ on a standard probability space $(X, \mu)$, as described in Proposition 1. Denote by $G \subseteq L_0(X, T) \subseteq L^\infty(X)$ the associated $F_2$-invariant multiplicative subgroup, and let $T_a(x) = a \cdot x$ and $T_b(x) = b \cdot x$ be the m.p. transformations corresponding to the generators $a$ and $b$. Let $Q \subseteq G \rtimes F_2$ be a finite Kazhdan set with Kazhdan constant $\varepsilon > 0$, witnessing the relative property (T) of $G \rtimes F_2$.

For $S \in M_\infty(X)$ denote by

$$
E_S = E_{\langle T_a, T_b, S \rangle}
$$

the equivalence relation generated by the transformations $T_a, T_b$ and $S$. We define two equivalence relations $R$ and $F$ on $M_\infty(X)$ by

$$
SRS' \iff E_S \text{ is orbit equivalent to } E_{S'}
$$

and

$$
SFS' \iff E_S = E_{S'} \text{ a.e.}
$$

Denote by $A \subseteq M_\infty(X)$ the set of transformations $S$ such that $\langle T_a, T_b, S \rangle$ induces an a.e. free action of $F_3$ on $X$. It follows from the Category Lemma that this set is a dense $G_\delta$ set. Then Theorem 1 in the case of $F_3$ can be phrased as
Theorem 1'. \( E_0 \leq_B R \upharpoonright A \).

An outline of the proof is as follows: We will first show that \( F \) has meagre classes. Then we will use the relative property (T) to make an argument modeled on \([22]\) and \([13]\), to show that \( R \) has countable index over \( F \), and deduce that it is a meagre subset of \( \mathcal{M}_\infty(X) \times \mathcal{M}_\infty(X) \). It will then be easy to apply a theorem of Becker and Kechris (\([II]\)) to obtain that \( E_0 \leq_B R \upharpoonright A \).

The proof is presented as a sequence of lemmata. We start by computing the complexity of \( R \) and \( F \):

Lemma 3. The equivalence relations \( R \) and \( F \) are analytic.

Proof. Give the space \( X \) a compatible Polish topology, and let furthermore the metric \( d_w \) on \( \mathcal{M}_\infty(X) \) be as in Proposition 3. Let \((S_n)_{n \in \mathbb{N}} \) be a sequence of Borel measure preserving transformations which is dense in \( \mathcal{M}_\infty(X) \). Define a relation \( \tilde{E}(\phi, x, y) \subseteq \mathbb{N}^\mathbb{N} \times X \times X \) by

\[
\tilde{E}(\phi, x, y) \iff S_{\phi(n)}(x) \rightarrow y.
\]

Then \( \tilde{E} \) is Borel. Define \( \Phi : \mathcal{M}_\infty(X) \rightarrow \mathbb{N}^\mathbb{N} \) by letting \( \Phi(S)(n) \) be the least \( m \in \mathbb{N} \) such that

\[
d_w(S, S_m) < 2^{-n}.
\]

Then clearly \( d_w(S_{\Phi(S)(n)}, S) < 2^{-n} \) for all \( n \), and \( \Phi \) is a Borel map. Hence the set \( E \subseteq \mathcal{M}_\infty(X) \times X \times X \) defined by

\[
E(S, x, y) \iff \tilde{E}(\Phi(S), x, y)
\]

is Borel. By Proposition 3,

\[(\forall^\mu x, y)[E(S, x, y) \iff S(x) = y]\]

and

\[SF S' \iff (\forall^\mu x, y)[xE_S y \iff xE_{S'} y]\]

\[\iff (\forall^\mu x, y)[(\exists \tau) e_\tau(S)(x) = y \iff (\exists \tau') e_{\tau'}(S')(x) = y]\]

\[\iff (\forall^\mu x, y)[(\exists \tau) E(e_\tau(S), x, y) \iff (\exists \tau') E(e_{\tau'}(S'), x, y)]\],

where \( \tau, \tau' \) are words in \\{a, b, c\}, and \( e_\tau \) denotes the evaluation map, as in section III (i.e. \( T_a \) is substituted for \( a \), \( T_b \) is substituted for \( b \), and \( S \) is
substituted for \( c \). Since the measure quantifiers preserve analyticity (see [18], p. 233), we conclude that \( F \) is analytic (in fact, \( F \) is easily seen to be Borel.)

Finally,

\[
SRS' \iff (\exists T \in \mathcal{M}_\infty(X))(\forall x, y)[xE_Sy \iff T(x)E_{S'}T(y)]
\]

So that \( R \) is analytic, since

\[
(\forall x, y)[T(x)E_{S'}T(y) \iff (\forall z, z')E(T, x, z) \land E(T, y, z') \implies zE_{S'}z']
\]

\[
\iff (\exists z, z')E(T, x, z) \land E(T, y, z') \land zE_{S'}z'
\]

\[
\square
\]

**Corollary 1.** The equivalence relation \( F \) is meagre in \( \mathcal{M}_\infty(X) \times \mathcal{M}_\infty(X) \), and has meagre classes.

**Proof.** Since \( F \) is analytic, it has the Baire property. Hence by [18] 8.41, it is enough to show that each \( F \)-class is meagre. But \([S]_R \subseteq \text{Inn}(E_S)\), which is meagre by Proposition 2 (i).

\[
\square
\]

Our next step is to show

**Main Lemma.** The equivalence relation \( R/F \) is countable. That is, each \( R \)-class contains at most countably many \( F \)-classes.

Before the proof, we note the following:

**Observation.** It is an easy observation, that if \((Y, d)\) is a Polish space, and \((y_\alpha)_{\alpha < \omega_1}\), is a sequence, then for every \( \delta > 0 \) there is an unbounded set \( B \subseteq \omega_1 \) such that whenever \( \alpha, \beta \in B \), then \( d(y_\alpha, y_\beta) < \delta \).

Similarly, if \( G \) is a Polish group and \((g_\alpha)_{\alpha < \omega_1}\) is a sequence in \( G \), then for any neighborhood \( N \subseteq G \) of the identity in \( G \), there is an unbounded set \( B \subseteq \omega_1 \) such that whenever \( \alpha, \beta \in B \), then \( g_\alpha g_\beta^{-1} \in N \). To see this, associate to any \( g \in G \) a basic open neighborhood \( N_g \) of \( g \), such that for \((h_1, h_2) \in N_g \times N_g\), we have \( h_1 h_2^{-1} \in N \), using the continuity of the group operations. Since \((N_g)_{g \in G} \) is countable there must be some \( g_0 \) such that \( g_\alpha \in N_{g_0} \) for an unbounded set of \( \alpha \).

We also note:
Lemma 4. For \( g \in L^\infty(X) \) and \( \delta > 0 \), there is a neighborhood \( N \) of the identity \( I \in M^\infty(X) \), such that
\[
\psi \in N \implies \| g - g \circ \psi \|_{L^2(X)} < \delta.
\]

Proof. This is trivial if we note that the weak topology on \( M^\infty(X) \) is precisely the subspace topology inherited from the unitary group on \( L^2(X) \), under the identification \( \psi \mapsto U \psi \) where \( U \psi(g) = g \circ \psi^{-1}, g \in L^2(X) \).

Finally, recall that if \( G \) is a countable group acting by m.p. transformations on \((X, \mu)\), a Borel measure \( M \) on \( E_G \) is defined by
\[
M(A) = \int |A| |d\mu(x)|
\]
for \( A \subseteq E_G \) (see [20], p. 34).

Proof of Main Lemma. Let \((S_\alpha)_{\alpha < \omega_1}\) be a sequence of m.p. transformations such that \( S_\alpha R S_\beta \) for all \( \alpha, \beta \in \omega_1 \). We want to show that \( E_{S_\alpha} = E_{S_\beta} \) for some \( \alpha \neq \beta \).

For each \( \alpha < \omega_1 \), let \( \psi_\alpha : X \to X \) be a m.p. transformation witnessing that \( E_{S_\alpha} \) is orbit equivalent to \( E_{S_\alpha} \). We will write \( \psi_{\alpha, \beta} \) for the transformation \( \psi_\beta \circ \psi_\alpha^{-1} \). A unitary representation \( \pi_{\alpha, \beta} \) of \( G \rtimes F_2 \) is defined on \( L^2(E_{S_\alpha}) \) by
\[
((g, \sigma) \cdot f)(x, y) = g(x)g(\psi_{\alpha, \beta}(y))f(\sigma^{-1} \cdot x, \psi_{\beta, \alpha}^{-1} \psi_{\alpha, \beta}(y))
\]
for each \( \alpha < \omega_1 \). Indeed, we have
\[
((g_1, \sigma_1) \cdot ((g_2, \sigma_2) \cdot f))(x, y)
= g_1(x)g_1(\psi_{\alpha, \beta}(y))(g_2, \sigma_2) \cdot f(\sigma_1^{-1} \cdot x, \psi_{\beta, \alpha}^{-1} \psi_{\alpha, \beta}(y))
= g_1(x)g_1(\psi_{\alpha, \beta}(y))g_2(\sigma_1^{-1} \cdot x)g_2(\sigma_1^{-1} \psi_{\alpha, \beta}(y))
= f(\sigma_2^{-1} \sigma_1^{-1} \cdot x, \psi_{\beta, \alpha}^{-1} \sigma_1^{-1} \psi_{\alpha, \beta}(y))
=((g_1(\sigma_1 \cdot g_2), \sigma_1 \sigma_2) \cdot f)(x, y),
\]
which shows that an action of \( G \rtimes F_2 \) is defined.
Claim. There is $\alpha \neq \beta$ such that $\psi_{\alpha,\beta}(x) = x$ on a non-null set of $x$.

Proof. Recall that $G \rtimes F_2$ has the relative property (T) with Kazhdan pair $(Q, \varepsilon)$. We assume that $Q$ has the form $Q_1 \times Q_2$, for finite sets $Q_1 \subseteq G$ and $Q_2 \subseteq F_2$, and that $\varepsilon$ is chosen so that if $(\pi, \mathcal{H})$ is a unitary representation and $v$ is $(Q, \varepsilon)$-invariant, then there is a $G$-invariant $v'$ such that $\|v - v'\|_\mathcal{H} \leq \frac{1}{2}$.

In order to prove the claim, we will show that there is $\alpha \neq \beta$ such that the function

$$1_{\Delta}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

is $(Q, \varepsilon)$-invariant for the representation $\pi_{\alpha,\beta}$. If we can show this, then there is a $G$-invariant $\xi \in L^2(E_{S_0})$ such that $\|\xi - 1_{\Delta}\|_{L^2(E_{S_0})} \leq \frac{1}{2}$. From this it follows that $\xi(x, x) \neq 0$ on a non-null set and hence

$$g(x)g(\psi_{\alpha,\beta}(x)) = 1$$

for all $g \in G$ on a non-null set. Since $G$ separates points, it follows that $\psi_{\alpha,\beta}(x) = x$ on a non-null set, as we wanted.

Since $Q_1$ is finite, we can use Lemma 4 to find a neighborhood $N \subseteq \mathcal{M}_\infty(X)$ of $I \in \mathcal{M}_\infty(X)$ such that for each $\psi \in N$ and $g \in Q_1$ we have

$$\|g - g \circ \psi\|_{L^2(X)}^2 < \frac{\varepsilon^2}{4}.$$ 

Using the observation preceding the proof, we find an unbounded set $B_0 \subseteq \omega_1$ such that $\psi_{\alpha,\beta} \in N$ for all $\alpha, \beta \in B_0$.

We now consider the transformations $\psi_{\alpha}^{-1}\sigma^{-1}\psi_{\alpha} \in \text{Inn}(E_{S_0})$, $\sigma \in F_2$. Applying the first part of the observation, we can for a given $\sigma \in Q_2$ find an unbounded set $B_1 \subseteq B_0$ such that

$$d_U(\psi_{\alpha}^{-1}\sigma^{-1}\psi_{\alpha}, \psi_{\beta}^{-1}\sigma^{-1}\psi_{\beta}) < \frac{\varepsilon^2}{4}$$

for all $\alpha, \beta \in B_1$, where $d_U$ is the usual complete metric for the uniform topology on $\text{Inn}(E_{S_0})$. Iterating this until the finite set $Q_2$ is exhausted, we get an unbounded set $B \subseteq B_0$ such that (*) holds for all $\sigma \in Q_2$ and all $\alpha, \beta \in B$.

By (*) it holds for $\alpha, \beta \in B$ that the set

$$C_\sigma = \{x \in X : \psi_{\beta,\alpha}^{-1}\psi_{\alpha,\beta}(x) = \sigma^{-1} \cdot x\}$$
has $\mu(C_\sigma) > 1 - \frac{\varepsilon^2}{4}$ for $\sigma \in Q_2$.

Consider then the unitary representation $\pi_{\alpha, \beta}$ for some fixed $\alpha, \beta \in B$. For $(g, \sigma) \in Q = Q_1 \times Q_2$, we have

$$
\|1_\Delta - \pi_{\alpha, \beta}(g, \sigma)1_\Delta\|^2_{L^2(E_{S_\alpha})} = \int \sum_{y \in [x]E_{S_\alpha}} |1_\Delta(x, y) - g(x)\overline{g(\psi_{\alpha, \beta}(y))}1_\Delta(\sigma^{-1} \cdot x, \psi_{\beta, \alpha}\sigma^{-1}\psi_{\alpha, \beta}(y))|^2 \, d\mu(x).
$$

Since for almost all $x$

$$
\sum_{y \in [x]E_{S_\alpha}} |1_\Delta(x, y) - g(x)\overline{g(\psi_{\alpha, \beta}(y))}1_\Delta(\sigma^{-1} \cdot x, \psi_{\beta, \alpha}\sigma^{-1}\psi_{\alpha, \beta}(y))|^2 \leq 1 + \|g\|_\infty^2 \|\overline{g}\|_\infty^2 = 2,
$$

we get

$$
\|1_\Delta - \pi_{\alpha, \beta}(g, \sigma)1_\Delta\|^2_{L^2(E_{S_\alpha})} \leq \int_{C_\sigma} \sum_{y \in [x]E_{S_\alpha}} |1_\Delta(x, y) - g(x)\overline{g(\psi_{\alpha, \beta}(y))}1_\Delta(\sigma^{-1} \cdot x, \psi_{\beta, \alpha}\sigma^{-1}\psi_{\alpha, \beta}(y))|^2 \, d\mu(x) + \frac{\varepsilon^2}{2}
$$

$$
\leq \|1 - g \circ \psi_{\alpha, \beta}\|^2 + \frac{\varepsilon^2}{2}
$$

$$
\leq \|g \circ \psi_{\alpha, \beta}\|^2 \|g \circ \psi_{\alpha, \beta} - g\|^2 + \frac{\varepsilon^2}{2}
$$

$$
\leq \varepsilon^2.
$$

Hence $1_\Delta$ is $(Q, \varepsilon)$-invariant, as claimed. \(\square\)

Let $\alpha \neq \beta$ as in the claim. Since $E_{\mathcal{F}_2}$ is $\mu$-ergodic we may assume, after possibly discarding a set of measure zero, that in each $E_{\mathcal{F}_2}$ class there is $x$ such that

$$
\psi_{\alpha, \beta}(x) = x.
$$

Consider an $E_{S_\alpha}$ class $C = [x]_{E_{S_\alpha}}$. We claim that $C \subseteq [x]_{E_{S_\beta}}$. For this, write

$$
C = \bigcup_{x_1 \in [x]_{E_{\mathcal{F}_2}}} [x_1]_{E_{\mathcal{F}_2}}$$
where each $x_i$ is such that

$$\psi_{\alpha,\beta}(x_i) = x_i.$$ 

Then $x_i E_{\beta} x_j$ for all $i, j$, and so

$$C = \bigcup_{x_i} [x_i] E_{\beta} \subseteq [x] E_{\beta}.$$ 

The opposite inclusion follows by a similar argument, and we conclude that $E_{\alpha} = E_{\beta}$. 

\[\square\]

**Corollary 2.** The relation $R$ is meagre in $\mathcal{M}_\infty(X) \times \mathcal{M}_\infty(X)$. 

**Proof.** By Corollary 1 and the Main Lemma, the $R$-classes are meagre, so we can conclude that $R$ is meagre as in the proof of Corollary 1. 

Recall, that $A$ is the set of transformations $S$, such that $\langle T_a, T_b, S \rangle$ act freely a.e. on $X$. We now prove:

**Theorem 1'.** $E_0 \leq B R|A$. 

**Proof.** We will use the following theorem:

**Theorem (Becker-Kechris, [1] proof of 3.4.5, also [13] p. 32.)** Suppose $E$ is an equivalence relation on the Polish space $X$, which is meagre as a subset of $X \times X$. Suppose further, that there is a group $G$ acting by homeomorphisms on $X$, such that $E_G \subseteq E$, and that there is a dense $G$-orbit. Then $E_0 \leq B E$. 

Let $G = \text{Inn}(E_{\beta})$ act on $\mathcal{M}_\infty(X)$ by conjugation. For $S \in \mathcal{M}_\infty(X)$ and $T \in G$, it is clear that

$$E_{TST^{-1}} \subseteq E_S.$$ 

But for the same reason

$$E_S = E_{T^{-1}TST^{-1}} \subseteq E_{TST^{-1}},$$

so that $E_S = E_{TST^{-1}}$. Hence $E_G \subseteq F \subseteq R$, and $G$ acts by homeomorphisms.
For an aperiodic transformation $S \in \mathcal{M}_\infty(X)$, the conjugacy class of $S$ in $\mathcal{M}_\infty(X)$ is dense (see [10], p. 77). Since by Proposition 2, $G = \text{Inn}E_{F_2}$ is dense in $\mathcal{M}_\infty(X)$, it follows that $[S]_G$ is dense in $\mathcal{M}_\infty$ for any aperiodic $S$. In particular, $E_G$ has a dense orbit.

Since the set $A$ is easily seen to be invariant under the action of $G$ it now follows from Becker-Kechris’ theorem that $E_0 \leq_B R|A$.

Remark. It is worth noting that the above proof gives us as a corollary that $E_0 \leq_B F$. In particular, equality a.e. of equivalence relations induced by actions by m.p. transformations is not concretely classifiable (i.e. smooth).

The case $F_2$. As remarked earlier, a virtually identical argument to the above can be made for the case $F_n$, $3 < n \leq \infty$: We simply add the appropriate number of generating, independent transformations.

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and

\[ T_1(x, i) = \begin{cases} \tau \tilde{S}(x, i) & \text{if } i = 0, \\ \tilde{T}_b \tau(x, i) & \text{if } i = 1. \end{cases} \]

which induce an a.e. free m.p. action of \( F_2 \). This shows that there are \( E_0 \) many orbit inequivalent actions of \( F_2 \), and this completes the proof of Theorem 1.

**Remark.** The author does not know if orbit equivalence for free m.p. actions of \( F_n, n \geq 2 \), is in general strictly more complicated than \( E_0 \). In the case of Kazhdan groups, it turns out that there are at least TFA many orbit inequivalent actions, where TFA denotes the isomorphism relation for countable torsion free abelian groups, cf. [26]. In particular, it follows from a result of Hjorth [14] that orbit equivalence is analytic non-Borel in this case. Hence the author finds it natural to suspect that in the case \( F_n, n \geq 2 \), orbit equivalence is also far more complicated than \( E_0 \).

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