Cubical Categories for Higher-Dimensional Parametricity

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Abstract—Reynolds’ theory of relational parametricity formalizes parametric polymorphism for System F, thus capturing the idea that polymorphically typed System F programs always map related inputs to related results. This paper shows that Reynolds’ theory can be seen as the instantiation at dimension 1 of a theory of relational parametricity for System F that holds at all higher dimensions, including infinite dimension. This theory is formulated in terms of the new notion of a \( p \)-dimensional cubical category, which we use to define a \( p \)-dimensional parametric model of System F for any \( p \in \mathbb{N} \cup \{\infty\} \). We show that every \( p \)-dimensional parametric model of System F yields a split \( \lambda \)-fibration in which types are interpreted as face maps and degeneracy-preserving cubical functors and terms are interpreted as face-map and degeneracy-preserving cubical natural transformations. We demonstrate that our theory is “good” by showing that the PER model of Bainbridge et al. is derivable as another 1-dimensional instance, and that all instances at all dimensions derive higher-dimensional analogues of expected results for parametric models, such as a Graph Lemma and the existence of initial algebras and final coalgebras. Finally, our technical development resolves a number of significant technical issues arising in Ghani et al.’s recent bifibrational treatment of relational parametricity, which allows us to clarify their approach and strengthen their main result. Once clarified, their bifibrational framework, too, can be seen as a 1-dimensional instance of our theory.

I. INTRODUCTION

Strachey [19] distinguished between ad hoc and parametric polymorphic functions in programming languages, defining a polymorphic program to be parametric if it applies the same type-uniform algorithm at each of its type instantiations. Reynolds [16] introduced the notion of relational parametricity to model the extensional behavior of parametric programs in System F [9], the formal calculus at the core of polymorphic functional languages. Relationally parametric models capture a key feature of parametric programs, namely that they preserve all relations between instantiated types. In other words, in relationally parametric models, parametric polymorphic functions always map related arguments to related results.

Implicit in Reynolds’ original formulation of relational parametricity [16] is that a model of System F is relationally parametric if equality in the model is induced by a logical relation. A logical relation assigns to each type of a language not only a basic interpretation as, say, a set or a domain, but simultaneously an interpretation as a relation on that set or domain as well. Logical relations are defined by induction on the language’s type structure, and are constructed in such a way that the relational actions interpreting its type formers propagate relatedness up its type hierarchy. For each logical relation for a language, a parametricity theorem can then be proved. Such a theorem states that (the basic interpretation of) each of the languages’ programs is related to itself by the relational interpretation, via the associated logical relation, of that program’s type. When instantiated judiciously, this seemingly simple result can be used to prove, inter alia, invariance of polymorphic functions under changes of data representation [1], [5], equivalences of programs [11], and so-called “free theorems” via which properties of programs can be inferred solely from their types [20].

The recent bifibrational treatment of relational parametricity in [8] has put forth a more abstract notion of a parametric model of polymorphism. In this treatment every type is still given a interpretation in a sufficiently structured base category, together with a relational interpretation in a category of (now abstractly formulated) relations over that base category, but the two interpretations are defined simultaneously and are required to be connected via a sufficiently structured bifibration. The express aim of [8] is to provide a very general framework for relational parametricity that is directly instantiateable not only to recover well-known relationally parametric models — such as Reynolds’ original model [1] and the PER model of Bainbridge et al. — but also to deliver entirely new models of relational parametricity for System F.

Unfortunately, however, models of relational parametricity often require more careful notions of functor and natural transformation than just the standard categorical ones used in the bifibrational framework of [8]. For example, functors and natural transformations must be internal to the category of types and terms in the Calculus of Inductive Constructions with impredicative Set to recover Reynolds’ original model, and they must be internal to the category of \( \omega \)-sets to recover Bainbridge et al.’s PER model. As a result, neither of these models is a true instance of the bifibrational framework of [8]. Said differently, the bifibrational framework of [8] is not actually an extension of Reynolds’ theory of relational parametricity as claimed. In fact, showing that Reynolds’ original model is parametric in the sense specified by the

1Since there are no set-theoretic models of System F, by the phrase “Reynolds’ original model” we will mean the version of his model that is internal to the Calculus of Inductive Constructions with Impredicative Set (as indicated in [8]).
bifibrational framework requires a complete redevelopment of the framework internally to the Calculus of Inductive Constructions with impredicative Set, and showing that the PER model is parametric requires a redevelopment internal to the category of ω-sets. The need to redevelop the entire framework of [8] internal to a different category for each relationally parametric model of interest makes the bifibrational framework more of a “blueprint” for constructing parametric models than a general theory that actually includes known models properly among its instances. The fact that such redevelopments must also be carried out on an ad hoc basis, without any generally-applicable guidance, only emphasizes the need for a truly instantiable theory of relational parametricity.

But even if uniform guidance for instantiating the framework of [8] were to be given, the framework itself would still be problematic. Unless fibred functors are required to preserve equality on the nose, neither composition nor substitution in (what is intended to be) the base category of the λ2-fibration constructed in the main theorem of [8] can be defined in any standard way. But equality in [8] is only defined — and therefore can only be preserved — up to isomorphism. And even if the original bifibration from which the λ2-fibration in [8] is constructed (U in the terminology there) were assumed to be split, so that the equality functor were defined uniquely rather than only up to isomorphism, Reynolds’ original model still would not be an instance of the bifibrational framework given there: in that case, neither products nor exponentials would preserve equality on the nose, as would be needed to properly interpret arrow types. In the absence of any alternative definitions or discussion of the exact sense in which fibred functors are required to preserve equality, we can only conclude that the standard definitions are the intended ones. As a result, we regard the entire λ2-fibration as being ill-defined, and the beautiful ideas explored in [8] as being in need of careful technical reconsideration.

This paper provides precisely such a reconsideration, as well as a significant extension. We remedy both of the aforementioned difficulties by developing a unifying approach to relational parametricity that turns the bifibrational “blueprint” for constructing parametric models for System F given by the framework of [8] into a single theory whose instantiation actually delivers such models. Our theory combines two key technical ingredients to produce λ2-fibrations that not only are actually well-defined, but do really model relational parametricity. First, we ensure that the categories necessary to our constructions are well-defined by parameterizing our theory over a class of “good” natural isomorphisms, and requiring that fibred functors preserve equality only up to these isomorphisms and (essentially) that fibred functors transformations preserve them. Secondly, we ensure that well-known models of relational parametricity for System F are properly instances of our theory by parameterizing it over a suitably structured ambient category and working internally to that category. But working internally to an appropriate ambient category is more than just a technical device ensuring that all of our constructions are well-defined and that well-known parametric models for System F are instances of our theory. It is also precisely the mechanism by which we restrict the possible interpretations of types and terms in our λ2-fibrations sufficiently to exclude ad hoc polymorphism. This is illustrated concretely in Example [29] below.

In addition to remedying all known problems with the bifibrational framework of [8], the theory we develop in this paper also naturally opens the way to a theory of relational parametricity at higher dimensions. Indeed, our theory deploys the two key ingredients identified above not solely in a bifibrational setting similar to that in [8], but also in combination with ideas inspired by the theory of cubical sets [2], [13], [10]. In this way, it delivers λ2-fibrations that model more than just the single “level”, or “dimension”, of relational parametricity originally identified by Reynolds and considered in [8]. To enforce relational parametricity at higher dimensions we introduce the new notion of a p-dimensional cubical category, in terms of which we define the equally new notion of a p-dimensional parametric model for System F. Here, the dimension p can be any natural number or ∞. Intuitively, cubical categories generalize cubical sets in the obvious way, by considering the codomain Cat instead of Set. Technically, the codomain will be Cat(C), the category of categories internal to some sufficiently structured ambient category C. Cubical categories have (essentially) the same algebraic structure as cubical sets, except that the morphisms in their domain category are restricted to just those generated by face maps and degeneracies, i.e., to just those have natural interpretations as operations on relations. This ensures that morphisms are restricted to those that, intuitively, have interpretations as operations on relations.

Our main technical result (Theorem 25) shows that every p-dimensional parametric model of System F gives rise to a split λ2-fibration. When combined with a suitable variant of Seely’s result that every split λ2-fibration gives rise to a sound model of System F (Theorem 26), this allows us to prove that every p-dimensional parametric model of System F gives a sound model of that calculus in which types are interpreted as p-dimensional face map- and degeneracy-preserving cubical functors and terms are interpreted as p-dimensional face map-preserving and degeneracy-preserving cubical natural transformations. This, our main result, appears as Theorem 27 below. It strengthens the analogous result in [8], which states that natural transformations interpreting terms must be face map-preserving when p = 1, but does not observe that they can also be proved to be degeneracy-preserving even in the 1-dimensional setting. Because they interpret System F terms as face map- and degeneracy-preserving cubical natural transformations, thereby ensuring that these terms cannot exhibit ad hoc polymorphic behavior, we contend that p-dimensional parametric models of System F are deserving of their name. That both Reynolds’ model and that of Bainbridge et al. are both p-dimensional parametric models for System F when p = 1 further shows our definition is both sensible and good. Additional evidence is provided in Section [7] where it is shown that all p-dimensional parametric models of
System F validate higher-dimensional analogues of the “litmus test” properties for “good” parametric models. That is, they validate a higher-dimensional Identity Extension Lemma, a higher-dimensional Graph Lemma, and the existence of initial algebras and final coalgebras for face map- and degeneracy-preserving cubical functors.

Instantiations of our theory for specific choices for \( p \) may already be of particular interest. When \( p = 2 \), our notion of a \( p \)-dimensional parametric model of System F formalizes a notion of proof-relevant relational parametricity that properly generalizes Reynolds’ original theory. When \( p = \infty \), we get a notion of infinite-dimensional relational parametricity for System F that may provide a useful perspective on the homotopy-canonicity conjecture for homotopy type theory, since proof of this conjecture involves constructing an infinitely parametric model of Martin-Löf type theory. Investigation of this matter is, however, beyond the scope of the present paper.

II. Fibrational Preliminaries

We give a brief introduction to fibrations, mainly to set notation. More details can be found in, e.g., [12].

Definition 1. Let \( U : \mathcal{E} \to \mathcal{B} \) be a functor. A morphism \( g : Q \to P \) in \( \mathcal{E} \) is cartesian over \( f : X \to Y \) in \( \mathcal{B} \) if \( Ug = f \) and, for every \( g' : Q' \to P \) in \( \mathcal{E} \) with \( Ug' = f \circ v \) for some \( v : UQ' \to X \), there exists a unique \( h : Q' \to Q \) with \( Uh = v \) and \( g' = g \circ h \). A morphism \( g : P \to Q \) in \( \mathcal{E} \) is opcartesian over \( f : X \to Y \) in \( \mathcal{B} \) if \( Ug = f \) and, for every \( g' : P \to Q' \) in \( \mathcal{E} \) with \( Ug' = v \circ f \) for some \( v : Y \to UQ' \), there exists a unique \( h : Q \to Q' \) with \( Uh = v \) and \( g' = h \circ g \).

We write \( f^b_P \) for the cartesian morphism over \( f \) with codomain \( P \), and \( f^b_P \) for the opcartesian morphism over \( f \) with domain \( P \). Such morphisms are unique up to isomorphism. If \( P \) is an object of \( \mathcal{E} \) then we write \( f^*P \) for the domain of \( f^b_P \) and \( \Sigma_f P \) for the codomain of \( f^b_P \). We omit \( P \) from these notations when it is either unimportant or clear from context.

Definition 2. A functor \( U : \mathcal{E} \to \mathcal{B} \) is a fibration if for every object \( P \) of \( \mathcal{E} \) and every morphism \( f : X \to UP \) of \( \mathcal{B} \), there is a cartesian morphism \( f^b_P : Q \to P \) in \( \mathcal{E} \) over \( f \). Similarly, \( U \) is an opfibration if for every object \( P \) of \( \mathcal{E} \) and every morphism \( f : UP \to Y \) of \( \mathcal{B} \), there is an opcartesian morphism \( f^b_P : P \to Q \) in \( \mathcal{E} \) over \( f \). A functor \( U \) is a bifibration if it is both a fibration and an opfibration.

If \( U : \mathcal{E} \to \mathcal{B} \) is a fibration, opfibration, or bifibration, then \( \mathcal{E} \) is its total category and \( \mathcal{B} \) is its base category. An object \( P \) in \( \mathcal{E} \) is over its image \( UP \) and similarly for morphisms. A morphism is vertical if it is over \( id \). We write \( \mathcal{E}_X \) for the fiber over an object \( X \) in \( \mathcal{B} \), i.e., the subcategory of \( \mathcal{E} \) of objects over \( X \) and morphisms over \( id_X \). For \( f : X \to Y \) in \( \mathcal{B} \), the function mapping each object \( P \) of \( \mathcal{E} \) to \( f^*P \) extends to a functor \( f^* : \mathcal{E}_Y \to \mathcal{E}_X \) mapping each morphism \( k : P \to P' \) in \( \mathcal{E}_Y \) to the morphism \( f^*k \) with \( k f^b_P = f^b_{P'} f^*k \). The universal property of \( f^b_P \) ensures the existence and uniqueness of \( f^*k \). We call \( f^* \) the reindexing functor along \( f \). A similar situation holds for opfibrations; the functor \( \Sigma_f : \mathcal{E}_X \to \mathcal{E}_Y \) extending the function mapping each object \( P \) of \( \mathcal{E} \) to \( \Sigma_f P \) is the opreindexing functor along \( f \).

In this paper we will construct a certain kind of fibration, called a \( \lambda^2 \)-fibration [13], that models higher-dimensional parametricity for System F. Fibrations will also be essential to defining a higher-dimensional graph functor, and bifibrations will be crucial to formulating an alternative characterization of the graph functor that allows us to prove both a higher-dimensional Graph Lemma, and the existence of initial algebras and final coalgebras of face map- and degeneracy-preserving cubical functors.

III. Cubical Categories

Cubical categories, functors, and natural transformations are the key structures from which we will construct our models of higher-dimensional parametricity. To define them we start with the following preliminary category.

Definition 3. The category \( \square \) is given as follows:

- the objects are (finite) sets of natural numbers of the form \( \{0, \ldots, l - 1\} \) for \( l \in \mathbb{N} \)
- the morphisms from \( l_1 \) to \( l_2 \) are functions from \( l_1 \) to \( l_2 + 2 \), where \( 2 \) is the two-element set \( \{\top, \bot\} \)
- the identity morphism on \( l \) is (induced by) the identity function on \( l \), i.e., is the inclusion map \( l \to l + 2 \)
- the composition of two morphisms \( f : l_1 \to l_2 \) and \( g : l_2 \to l_3 \) is the function \( g \circ f : l_1 \to l_3 + 2 \) defined by
  \[
  (g \circ f)(i) = \begin{cases} 
  \ast & \text{if } f(i) = \ast, \ast \in \{\top, \bot\} \\
  g(j) & \text{if } f(i) = j, j \in \mathbb{N} 
  \end{cases}
  \]

We will henceforth denote a set \( \{0, \ldots, l - 1\} \) of natural numbers by \( l \). (Set-theoretically, these are identical.) We call any such set, i.e., any object of \( \square \), a level. The category \( \square \) can also be described as the Kleisli category for the error monad with two distinct error values. The last bullet point then defines composition of morphisms in \( \square \) to be normal composition of functions, except that errors are propagated.

The category \( \square \) contains all functions from \( l_1 \) to \( l_2 + 2 \) for all \( l_1 \) and \( l_2 \). But to model parametricity, we will want to restrict the set of morphisms to those that, intuitively, have interpretations as operations on relations. For this reason, only the face maps and degeneracies defined below, and their compositions, are used to construct our cubical categories. The interpretations of the face maps and degeneracies in the specific setting of [8] are given in Example 6.

For any \( \ast \in \{\top, \bot\} \) and \( l, k \in \mathbb{N} \) with \( k \leq l \), we define the function \( f_\ast : (l, k) : l + 1 \to l + 1 \) by

\[
  f_\ast(l, k)(i) = \begin{cases} 
  i & \text{if } i < k \\
  \ast & \text{if } i = k \\
  i - 1 & \text{if } i > k 
  \end{cases}
  \]

Such a function is called a face map. The terminology comes from regarding each natural number \( l \) as defining an \( l \)-dimensional cube. A face map \( f_\ast(l, k) \) then can be thought of as projecting an \((l + 1)\)-cube onto either the “top” or the
“bottom” $l$-dimensional cube in dimension $k$, according as $*$ is $\top$ or $\bot$. Similarly, for any $l, k \in \mathbb{N}$ with $k \leq l$, we define the function $d(l, k) : l \to l + 1$ by

$$d(l, k)(i) = \begin{cases} i & \text{if } i < k \\ i + 1 & \text{if } i \geq k \end{cases}$$

Such a function is called a degeneracy. A degeneracy $d(l, k)$ can be thought of as constructing an $(l + 1)$-dimensional cube from an $l$-dimensional one by replicating it along dimension $k$.

We also have the following set of categories $\square_p$:

**Definition 4.** Let $p \in \mathbb{N} \cup \{\infty\}$. The category $\square_p$ is the subcategory of $\square$ generated by the following data:

- levels $l \leq p$
- face maps $f_*(l, k)$ for $l < p$
- degeneracies $d(l, k)$ for $l < p$

Alternatively, we can define $\square_p$ as the free category generated by the the data above and the following relations:

- $f_*(l, k) \circ d(l, k) = 1_l$ for $l < p$
- $f_*(l, j) \circ d(l, k) = d(l - 1, k) \circ f_*(l - 1, j - 1)$ for $k < j$ and $l < p$
- $d(l + 1, j) \circ d(l, k) = d(l, k) \circ d(l, j - 1)$ for $k < j$ and $l + 1 < p$
- $f_{*j}(l, k) \circ f_{*k}(l + 1, k) = f_{*j}(l, k) \circ f_{*2}(l + 1, j + 1)$ for $k \leq j$ and $l + 1 < p$

The alternative characterization coincides exactly with Crans’ combinatory treatment of cubes. As proved in [4], any morphism in $\square_p$ can be factored as a composition of face maps followed by a composition of degeneracies. Moreover, the second arguments to the degeneracies in these compositions are non-increasing, and the second arguments to the face maps are strictly decreasing, when read in composition order, i.e., from right to left. Such a factorization gives a representation of each morphism as a surjection followed by an inclusion, as well as a canonical form for each morphism in $\square_p$.

The categories $\square_p$ will serve as the domains of our cubical categories. As such, they are our analogues of the category of names and substitutions, which forms the common domain of all cubical sets in [2]. The differences between our categories $\square_p$ and the category of names and substitutions are that $\square_p$ does not include the “exchange morphisms” of [2] (and [10]), and that membership in $\square_p$ does not explicitly require morphisms to be injective. That all morphisms in each $\square_p$ are, in fact, injective follows from the injectivity of $\square_p$’s generators.

In the remainder of this paper we will always work internally with respect to a finitely complete locally small ambient category $C$, i.e., a category $C$ with pullbacks and a terminal object $1_C$.\footnote{The requirement that $C$ has all pullbacks is actually stronger than necessary. In fact, we need only require $C$ to have all “composable” pullbacks.} Instantiating $C$ appropriately will impose conditions on functors and natural transformations that allow us to produce a theory of higher-dimensional parametricity that subsumes well-known models as (1-dimensional) relational parametricity as instances of our framework. For example, Reynolds’ original model is an instance when $C$ is taken to be the category of types and terms in the Calculus of Inductive Constructions with impredicative Set\footnote{The quite surprising fact that this theory has not yet actually been proved consistent is, however, worth noting.}, and the PER model of Bainbridge et al. is an instance when $C$ is the category of $\omega$-sets; see Examples [28] and [29] below for details.

Let $\text{Cat}(C)$ be the category of categories internal to an ambient category $C$.

**Definition 5.** A $(p$-dimensional) cubical category is a functor $\mathcal{X} : \square_p \to \text{Cat}(C)$.

To ease the notational and conceptual burden, we will henceforth regard a $(p$-dimensional) cubical category as a functor $\mathcal{X} : \square_p \to \text{Cat}$, and similarly identify internal and external constructions when convenient. Under this identification, a $(p$-dimensional) cubical category $\mathcal{X}$ becomes the category-level equivalent of a covariant presheaf on $\square_p$, and gives us a (small) category $\mathcal{X}(l)$ for each level $l \in \square_p$. The category $\mathcal{X}(0)$ can be thought of as an abstract category of “0-relations”, or “types”; $\mathcal{X}(1)$ can be thought of as the category of “1-relations”, or ordinary relations, on types; $\mathcal{X}(2)$ can be thought of as the category of “2-relations”; and so on. Each face map $f : l + 1 \to l$, $\mathcal{X}(f)$ is thus a functor projecting an $l$-relation out of a given $(l + 1)$-relation, and each degeneracy $d : l \to l + 1$, $\mathcal{X}(d)$ is a functor that replicates a given $l$-relation to obtain an $(l + 1)$-relation.

**Example 6.** In the setting of [8], a relations fibration $\text{Rel}(U) : \text{Rel}(\mathcal{E}) \to B \times B$ induces a 1-dimensional cubical category, with the action on objects given by $0 \mapsto B$ and $1 \mapsto \text{Rel}(\mathcal{E})$. The action on morphisms is induced by mapping the two face maps $f_r(0, 0)$ and $f_l(0, 0)$ to the functors $\text{fst} \circ \text{Rel}(U)$ and $\text{snd} \circ \text{Rel}(U)$ respectively, and mapping the degeneracy $d(0, 0)$ to the equality functor $\text{Eq}$ on $B$.

If $\mathcal{X}$ is a cubical category we define the discrete cubical category $[\mathcal{X}]$, and the product cubical category $\mathcal{X}^n$ for $n \in \mathbb{N}$, via the usual constructions for functors. Each construction on cubical categories actually requires an analogous construction on $\text{Cat}(C)$, but these are precisely as expected. For example, for cubical categories $\mathcal{X}, \mathcal{Y} : \square_p \to \text{Cat}(C)$, the cubical category $\mathcal{X} \times \mathcal{Y} : \square_p \to \text{Cat}(C)$ is defined by $(\mathcal{X} \times \mathcal{Y})(l) = \mathcal{X}(l) \times \mathcal{Y}(l)$ for all $l \leq p$. Here, the product on the left-hand side is a product of functors, and the product on the right-hand side is a product of internal categories. The latter exists because $C$ is finitely complete by assumption.

**Definition 7.** Let $\mathcal{X}$ and $\mathcal{Y}$ be $(p$-dimensional) cubical categories. A $(p$-dimensional) cubical functor $\mathcal{F}$ from $\mathcal{X}$ to $\mathcal{Y}$ is a set of functors $\{\mathcal{F}(l) : \mathcal{X}(l) \to \mathcal{Y}(l) | l \leq p\}$. A cubical functor $\mathcal{F}$ is face map-preserving if the following diagram commutes for every face map $h : l_1 \to l_2$ in $\square_p$.\footnote{The requirement that $\mathcal{F}$ has all pullbacks is actually stronger than necessary. In fact, we need only require $\mathcal{F}$ to have all “composable” pullbacks.}
A cubical functor $F$ is degeneracy-preserving if the diagram above commutes up to a chosen natural isomorphism $\varepsilon_F(h)$ for each degeneracy $h : l_1 \to l_2$ in $\Box_p$.

In the setting of \cite{8}, a fibred functor from $|\text{Rel}(U)|^n$ to $\text{Rel}(U)$ is precisely a face map-preserving cubical functor from $|\text{Rel}(U)|^n$ to $\text{Rel}(U)$ (presented as cubical categories). Similarly, an equality-preserving fibred functor from $|\text{Rel}(U)|^n$ to $\text{Rel}(U)$ is a face map-preserving cubical functor that also preserves degeneracies up to “good” natural isomorphisms (for a suitable notion of “good”; see Section IV below).

**Definition 8.** Let $F$ and $G$ be $(p$-dimensional$)$ cubical functors from $X$ to $Y$. A $(p$-dimensional$)$ cubical natural transformation $\eta$ from $F$ to $G$ is a set of natural transformations $\{\eta(l) : F(l) \to G(l) \mid l \leq p\}$. A cubical natural transformation $\eta$ is face map-preserving if $F$ and $G$ are face map-preserving and, for each face map $h : l_1 \to l_2$ in $\Box_p$, the following equality holds for every object $X$ of $X(l_1)$:

$$
\eta(h)(\eta(l_1) X) = \eta(l_2)(\eta(X) X)
$$

A cubical natural transformation $\eta$ is degeneracy-preserving if $F$ and $G$ are degeneracy-preserving and, for each degeneracy $h : l_1 \to l_2$ in $\Box_p$, the following diagram commutes for every object $X$ of $X(l_1)$:

$$
\begin{array}{ccc}
\eta(h)(F(l_1) X) & \xrightarrow{\varepsilon_F(h) X} & F(l_2)(\eta(X) X) \\
\downarrow \varepsilon_G(h) X & & \downarrow \eta(l_2)(\eta(X) X) \\
\eta(h)(G(l_1) X) & \xrightarrow{\varepsilon_G(h) X} & G(l_2)(\eta(X) X)
\end{array}
$$

Here, $\varepsilon_F(h) : Y(h) \circ F(l_1) \to F(l_2) \circ \eta(X) h$ and $\varepsilon_G(h) : Y(h) \circ G(l_1) \to G(l_2) \circ \eta(X) h$ are the natural isomorphisms witnessing that $F$ and $G$ are themselves degeneracy-preserving.

By contrast with the diagram in Definition 7, the one in Definition 8 is required to commute on the nose.

In the setting of \cite{8}, a fibred natural transformation between two fibred functors $F, G : |\text{Rel}(U)|^n \to \text{Rel}(U)$ induces a face map-preserving cubical natural transformation from $F$ to $G$ presented as cubical functors. We note, however, that there is no notion in \cite{8} that induces cubical natural transformations that are both face map- and degeneracy-preserving; indeed, the framework of \cite{8} does not require natural transformations between equality-preserving fibred functors from $|\text{Rel}(U)|^n$ to $\text{Rel}(U)$ to themselves be equality-preserving. Significantly, the results of \cite{8} can still be obtained even if fibred natural transformations are required to be equality-preserving. The analogous requirement for cubical natural transformations — namely, the requirement that cubical natural transformations be both face map- and degeneracy-preserving — thus generalizes structure already present in the 1-dimensional setting of \cite{8}.

We exploit this requirement on cubical natural transformations to construct $\lambda$2-fibrations in which terms of System F are interpreted as face map- and degeneracy-preserving cubical natural transformations. Even when $p = 1$ this gives a stronger result than is obtained in \cite{8}, where natural transformations interpreting terms are not shown to be equality-preserving.

**IV. FIBRATIONAL MODELS OF HIGHER-DIMENSIONAL PARAMETRICITY**

In this section we assume a fixed $(p$-dimensional$)$ cubical category Rel. To ensure that composition (and thus substitution) in the base category of the $\Lambda$2-fibrations we construct is well-defined, as well as that the cubical functors interpreting System F types will be not just face map-preserving but also degeneracy-preserving, we need to consider equality of morphisms up to (certain kinds of) natural isomorphisms. We therefore assume that our fixed cubical category Rel comes equipped with a class $M = \Sigma_{0 \leq n} M(l)$ of “good” isomorphisms of the form $J \to \text{Rel}(l)_1$, where $J$ is some object of $C$ and $\text{Rel}(l)_1$ is the object of morphisms in the internal category $\text{Rel}(l)$. We require that each $M(l)$ contains all identity morphisms in $\text{Rel}(l)$, is closed under composition and inverses in $\text{Rel}(l)$, and is closed under reindexing by morphisms in $C$. The isomorphisms in $M$ will be used below to parameterize various constructions on cubical categories over different notions of equivalence of morphisms. For example, taking $M$ to be the class of identity morphisms will ensure that diagrams commute on the nose — as for Bainbridge et al.’s PER model; see Example 22 below — while taking $M$ to be the class of all isomorphisms will entail that the same diagrams commute up to an arbitrary natural isomorphism. Less extremal choices for $M$ are possible as well: for example, we can define $M$ by induction on $l$, letting $M(l)$ consist of only the identity morphisms, and defining an isomorphism $f$ to belong to $M(l + 1)$ iff $\text{Rel}(h) \circ f$ belongs to $M(l)$ for any face map $h : l + 1 \to l$, where this composition is in $C$. This definition of $M$ is used for Reynolds’ model; see Example 28.

We will use the following cubical categories — one for each $n$ — to interpret System F types:

**Definition 9.** The $(p$-dimensional$)$ cubical category $|\text{Rel}|^n \to \text{Rel}$ is given as follows:

- the objects are triples $(F, \varepsilon_F, \nu_F)$, where
  - $F$ is a face map-preserving $(p$-dimensional$)$ cubical functor from $|\text{Rel}|^n$ to $\text{Rel}$
  - $\varepsilon_F$ is a family of natural isomorphisms witnessing that $F$ is also degeneracy-preserving. Moreover, $\varepsilon_F(h)$ is in $M(l_2)$ for each degeneracy $h : l_1 \to l_2$ in $\Box_p$
  - $\nu_F$ is a function associating to each isomorphism $f : \text{Rel}(l)^m_0 \to \text{Rel}(l)^n_0$ with the property that $\nu_k \circ f$ is in $M(l)$ for each $k \leq n$ an isomorphism $\nu_F(f) : \text{Rel}(l)^m_0 \to \text{Rel}(l)^n_1$ in $M(l)$. Moreover, $\nu$ respects the
source and target operations, as well as identities, composition, and reindexing of isomorphisms

• the morphisms are face map- and degeneracy-preserving (p-dimensional) cubical natural transformations

Generalizing from the 1-dimensional setting of \[\text{[8]}\], in which types with \( n \) free variables are interpreted as equality-preserving functors from \( \text{Rel}(1)^n \) to \( \text{Rel}(1) \), we aim to interpret System F types as face map- and degeneracy-preserving cubical functors from \( \text{Rel}^n \) to \( \text{Rel} \) for various \( n \); the restriction to functors with discrete domains here makes it possible to handle all type expressions in System F, not just the positive ones. This will require that the total categories of the \( \lambda \)2-fibrations we construct have such functors as their objects. But for such functors to form a category, they must support a well-defined notion of composition. To ensure that this is the case even though cubical functors are only required to preserve degeneracies up to isomorphism, and even though those isomorphisms are, importantly, in \( \text{Rel} \) rather than in \( \text{Rel}^n \), we need to arrange that cubical functors from \( \text{Rel}^n \) to \( \text{Rel} \) for various \( n \) preserve enough isomorphisms. The functions \( \nu_F \) accomplish just this: they endow each cubical functor \( F : \text{Rel}^n \to \text{Rel} \) with enough structure to preserve all “good” isomorphisms, and this is what we need to push all of the constructions we require through. Of course, if \( F \) were a cubical functor with domain \( \text{Rel}^n \) rather than \( \text{Rel}^n \) we would get the preservation of (all) isomorphisms in \( \text{Rel}^n \) for free. However, this would make it impossible to handle contravariant type expressions.

When giving a categorical interpretation of System F, a category for interpreting type contexts is required. We therefore associate a category of contexts to each cubical category.

**Definition 10.** The \((p\text{-dimensional})\) category of contexts \( \text{Ctx}(\text{Rel}) \) is given as follows:

• the objects are natural numbers

• the morphisms from \( n \) to \( m \) are \( m \)-tuples of objects in \( \text{Rel}^n \to \text{Rel} \)

Defining the product \( m \times 1 \) in \( \text{Ctx}(\text{Rel}) \) to be the natural number sum \( m + 1 \), we see that \( \text{Ctx}(\text{Rel}) \) enjoys sufficient structure to model the construction of System F type contexts:

**Lemma 11.** The category \( \text{Ctx}(\text{Rel}) \) has a terminal object \( 0 \) and a choice of products \((-) \times 1 \).

To appropriately interpret arrow types we will need to know that each cubical category of the form \( \text{Rel}^n \to \text{Rel} \) is cartesian closed. The next three lemmas show that, under reasonable conditions on \( \text{Rel} \), this is indeed the case. The constructions are variants of familiar ones, except that care must be taken to ensure that the isomorphisms in \( M \) are respected.

**Definition 12.** \( \text{Rel} \) has terminal objects if it comes equipped with a choice of terminal objects \( 1 \) in \( \text{Rel}(l) \) for \( l \leq p \). This choice of terminal objects is stable under face maps if the equality below holds for each face map \( h : l_1 \to l_2 \) in \( \text{Rel}(l) \):

\[
\text{Rel}(h) (1)_{l_1} = 1_{l_2}
\]

It is stable under degeneracies if the equality holds up to an isomorphism in \( M(1) \) for each degeneracy \( h : l_1 \to l_2 \) in \( \text{Rel}(l) \).

We write \( 1 \) rather than \( 1 \) below when \( l \) is clear from context.

**Lemma 13.** If \( \text{Rel} \) has terminal objects that are stable under face maps and degeneracies then we have a choice of terminal objects \( 1 \) in \( \text{Rel}^n \to \text{Rel} \).

**Definition 14.** \( \text{Rel} \) has products if it comes equipped with a choice of products \((x, \text{fst}_l, \text{snd}_l)\) in \( \text{Rel}(l) \) for \( l \leq p \), such that \( M(l) \) for \( l \leq p \) is closed under products. This choice of products is stable under face maps if, for each face map \( h : l_1 \to l_2 \) in \( \text{Rel}(l) \), the equalities below hold for any objects \( A, B \) of \( \text{Rel}(l) \):

\[
\begin{align*}
\text{Rel}(h) (A \times 1_l B) & = (\text{Rel}(h) A) \times 1_l (\text{Rel}(h) B) \\
\text{Rel}(h) (\text{fst}_l [A, B]) & = \text{fst}_l [\text{Rel}(h) A, \text{Rel}(h) B] \\
\text{Rel}(h) (\text{snd}_l [A, B]) & = \text{snd}_l [\text{Rel}(h) A, \text{Rel}(h) B]
\end{align*}
\]

It is stable under degeneracies if, for each degeneracy \( h : l_1 \to l_2 \) in \( \text{Rel}(l) \), the first equality above holds up to an isomorphism \( \varepsilon(h, A, B) \) in \( M(1) \) that makes the following two diagrams commute:

We write \((x, \text{fst}, \text{snd})\) rather than \((x, \text{fst}_l, \text{snd}_l)\) when \( l \) is clear from context.

**Lemma 15.** If \( \text{Rel} \) has products stable under face maps and degeneracies then we have a choice of products \((x_n, \text{fst}_{n_l}, \text{snd}_{n_l})\) in \( \text{Rel}^n \to \text{Rel} \).

If, following the development in \[\text{[8]}\], we did not require cubical natural transformations to preserve degeneracies, then we would not need to require commutativity of the two diagrams above for degeneracies. We would still need Definition \[\text{[14]}\]'s requirement on face maps, however.

**Definition 16.** \( \text{Rel} \) has exponentials if it has products and it comes equipped with a choice of exponentials \((\Rightarrow_l, \text{eval}_l)\) in \( \text{Rel}(l) \) for \( l \leq p \) with respect to the chosen products, such that \( M(l) \) for \( l \leq p \) is closed under exponentials. This choice of
exponentials is stable under face maps if the choice of products is stable under face maps and, for each face map \( h : l_1 \to l_2 \) in \( \square_p \), the equalities below hold for any objects \( A, B \) of \( \text{Rel}(l_1) \):

\[
\text{Rel}(h) (A \Rightarrow_{l_1} B) = (\text{Rel}(h) A) \Rightarrow_{l_2} (\text{Rel}(h) B)
\]

\[
\text{Rel}(h) (\text{eval}_{l_1} [A, B]) = \text{eval}_{l_2} [\text{Rel}(h) A, \text{Rel}(h) B]
\]

It is stable under degeneracies if the choice of products is stable under degeneracies and, for each degeneracy \( h : l_1 \to l_2 \) in \( \square_p \), the first equality above holds up to an isomorphism \( \zeta(h, A, B, A) \) in \( M(l_2) \) that makes the following diagram commute:

\[
\begin{array}{c}
\text{Rel}(h)( (A \Rightarrow_{l_1} B) \times_{l_1} 1) \\
\text{Rel}(h)( (\text{eval}_{l_1} [A, B]) \\
\text{Rel}(h)( (\text{Rel}(h) A) \Rightarrow_{l_1} (\text{Rel}(h) B)) \times_{l_1} (\text{Rel}(h) A)) \\
\text{eval}_{l_1} [\text{Rel}(h) A, \text{Rel}(h) B] \\
\end{array}
\]

Here, \( \zeta(h, A \Rightarrow_{l_1} B, A) \) is the isomorphism in \( M(l_2) \) witnessing the stability of the product in question under \( h \).

We write \( (\Rightarrow, \text{eval}) \) rather than \( (\Rightarrow_{l_1}, \text{eval}_{l_1}) \) when \( l \) is clear from context.

**Lemma 17.** If \( \text{Rel} \) has exponentials stable under face maps and degeneracies then we have a choice of exponentials \( (\Rightarrow_n, \text{eval}_n) \) in \( |\text{Rel}|^n \to \text{Rel} \).

As for products, if we follow the development in [8] and did not require cubical natural transformations to preserve degeneracies, then we would not need to require commutativity of the above diagram for degeneracies. We would still need Definition 16’s requirement on face maps, however.

Putting Lemmas 13, 15, and 17 together gives:

**Proposition 18.** If a cubical category \( \text{Rel} \) has terminal objects, products, and exponentials, all of which are stable under face maps and degeneracies, then \( |\text{Rel}|^n \to \text{Rel} \) is cartesian closed.

In the development above we consider cubical categories to be functors with codomain \( \text{Cat} \), as explained in Section III above. If, however, we more properly view \( p \)-dimensional cubical categories as functors from \( \square_p \) to \( \text{Cat}(C) \), then the construction of terminal objects, products, and exponentials must actually be carried out internally to our ambient category \( C \). This means that in Definition 14 for example, \( A \) and \( B \) are morphisms into \( C \)’s object of objects, and their product is an internal product. The necessary definition of internal products is standard and can be found, for example, in Section 7.2 of [12]. A similar remark applies to Definitions 12 and 16 and at several places below, but we will suppress remarks analogous to this one in the remainder of this paper.

The cubical category \( |\text{Rel}|^n \to \text{Rel} \) will ultimately emerge as the fiber over object \( n \) of \( \text{Ctx}(\text{Rel}) \) in the \( \lambda \text{2} \)-fibration we construct to interpret System F. To interpret \( \forall \)-types we will require a right adjoint to context weakening that moves between such fibers and is appropriate to the cubical setting. To formalize this requirement, we first define the category that will be the total category of our \( \lambda \text{2} \)-fibration.

**Definition 19.** The \((p \text{-dimensional})\) cubical category \( \int_n |\text{Rel}|^n \to \text{Rel} \) is given as follows:

- the objects are pairs \((n, F)\), where \( F \) is an object in \( |\text{Rel}|^n \to \text{Rel} \)
- the morphisms from \((n, F)\) to \((m, G)\) are pairs \((F, \eta)\), where \( F : n \to m \) is a morphism in \( \text{Ctx}(\text{Rel}) \) and \( \eta : F \to G \circ F \) is a morphism in \( |\text{Rel}|^n \to \text{Rel} \)

Since the set of objects of \( |\text{Rel}|^n \to \text{Rel} \) is, by definition, (isomorphic to) the set of morphisms \( \text{Mor}(n, 1) \) in \( \text{Ctx}(\text{Rel}) \), we have not only that \( \int_n |\text{Rel}|^n \to \text{Rel} \) is the total category of a fibration over \( \text{Ctx}(\text{Rel}) \), but that this fibration is actually a split fibration.

**Lemma 20.** The forgetful functor from \( \int_n |\text{Rel}|^n \to \text{Rel} \) to \( \text{Ctx}(\text{Rel}) \) is a split fibration with split generic object 1.

Moreover, cartesian structure from \( \text{Rel} \) lifts to this fibration:

**Lemma 21.** If \( \text{Rel} \) has terminal objects, products, and exponentials, all stable under face maps and degeneracies, then the forgetful functor from \( \int_n |\text{Rel}|^n \to \text{Rel} \) to \( \text{Ctx}(\text{Rel}) \) is a split cartesian closed fibration with split generic object 1.

The split cartesian closed structure identified in Lemma 21 will allow us to interpret of function types. To ensure that we can also interpret \( \forall \)-types we require some additional structure.

**Definition 22.** Let \( U : E \to B \) be a split fibration with a distinguished object \( \Omega \) of \( B \) and a choice of products \((\cdot) \times \Omega \) in \( B \). We say that \( U \) has split simple \( \Omega \)-products if it comes equipped with a choice of right adjoints \( \forall \eta : \forall A : E_{A \times \Omega} \to E_A \) to the weakening functors \( \forall [A, \Omega]^n : E_A \to E_{A \times \Omega} \) for objects \( A \) of \( B \), with the respective unit and counit pairs \((\eta_A, \varepsilon_A)\), satisfying the following conditions for every morphism \( f : A \to B \) in \( B \):

- the following diagram commutes:

\[
\begin{array}{ccc}
E_{B \times \Omega} & \xrightarrow{(f \times 1)^*} & \forall B \times \Omega \\
\downarrow & & \downarrow \forall B & \downarrow \forall B \\
E_A & \xleftarrow{f^*} & \forall A & \xleftarrow{\forall A} & E_A \\
\end{array}
\]

- \( f^*(\eta_B(X)) = \eta_A(f^*(X)) \) for every object \( X \) of \( E(B) \)
- \( (f \times 1)^*(\varepsilon_B(X)) = \varepsilon_A((f \times 1)^*(X)) \) for every object \( X \) of \( E_{B \times \Omega} \)

Fibrations with enough structure to give sound interpretations of System F were dubbed “\( \lambda \text{2} \)-fibrations” by Seely [18]:

**Definition 23.** A split \( \lambda \text{2} \)-fibration is a split cartesian closed fibration \( U : E \to B \), that has a terminal object in \( B \), a split
generic object $\Omega$, chosen products $(-) \times \Omega$ in $\mathcal{B}$, and split simple $\Omega$-products.

**Definition 24.** Rel is a $(p$-dimensional$)$ parametric model of System F if it has terminal objects, products, and exponentials, all stable under face maps and degeneracies, and is such that the forgetful functor from $\int_n |\text{Rel}|^n \rightarrow \text{Rel}$ to $\text{Ctx}$(Rel) has split simple $1$-products.

Our main technical theorem shows that every parametric model of System F naturally gives rise to a split $\lambda 2$-fibration. The construction is also a careful variant of familiar ones.

**Theorem 25.** If Rel is a $(p$-dimensional$)$ parametric model of System F, then the forgetful functor from $\int_n |\text{Rel}|^n \rightarrow \text{Rel}$ to $\text{Ctx}$(Rel) is a split $\lambda 2$-fibration.

We also have the following variant of Seely’s [13] result that every split $\lambda 2$-fibration gives rise to a sound model of System F:

**Theorem 26.** Every split $\lambda 2$-fibration $U : \mathcal{E} \rightarrow \mathcal{B}$ gives a model of System F in which:

- every type context $\Gamma$ is interpreted as an object $[\Gamma]$ in $\mathcal{B}$
- every type $\Gamma \vdash T$ is interpreted as an object $[\Gamma \vdash T]$ in the fiber $\mathcal{E}_{[\Gamma]}$
- every term context $\Gamma; \Delta$ is interpreted as an object $[\Gamma \vdash \Delta]$ in the fiber $\mathcal{E}_{[\Gamma]}$
- every term $\Gamma; \Delta \vdash t : T$ is interpreted as a morphism $[\Gamma; \Delta \vdash t : T]$ from $[\Gamma; \Delta]$ to $[\Gamma \vdash T]$ in the fiber $\mathcal{E}_{[\Gamma]}$

Moreover, if $\Gamma; \Delta \vdash s =_{\beta \eta} t : T$, then $[\Gamma; \Delta \vdash s : T] = [\Gamma; \Delta \vdash t : T]$. Theorems 25 and 26 together imply our main result, namely:

**Theorem 27.** A $(p$-dimensional$)$ parametric model Rel of System F gives a sound model of System F in which:

- every type $\Gamma \vdash T$ is interpreted as a face map-and degeneracy-preserving cubical functor $[\Gamma \vdash T] : |\text{Rel}|^{|\Gamma|} \rightarrow \text{Rel}$
- every term $\Gamma; \Delta \vdash t : T$ is interpreted as a face map-and degeneracy-preserving cubical natural transformation $[\Gamma; \Delta \vdash t : T] : [\Gamma \vdash T] \rightarrow [\Gamma \vdash T]$

Taking $p = 1$ and omitting the requirement that cubical natural transformations be degeneracy-preserving as indicated at several places above shows that Theorem 27 naturally generalizes Theorem 4.6 of [8] to arbitrary (including infinite, when $p = \infty$) higher dimensions. In particular, the fact that our cubical functors interpreting types are degeneracy-preserving gives a higher-dimensional analogue of the fibrational formulation of Reynolds’ Identity Extension Lemma from [8].

V. EXAMPLES

In this section we show how both Reynolds’ original model and the PER model of Bainbridge et al. arise as instances of our theory.

**Example 28.** We consider Reynolds’ original model, which is internal to the Calculus of Inductive Constructions with Impredicative Set. In the interest of clarity, we write $\mathbb{U}$ (rather than Set, as in implementations of Coq) for the impredicative universe $\mathbb{U}$. We then define

- $\text{isProp}(A) := \Pi_{a,b:A} \text{Id}(a, b)$
- $\text{Prop} := \Sigma_{A : \mathbb{U}} \text{isProp}(A)$
- $\text{isSet}(A) := \Pi_{a,b:A} \text{Prop}(|\text{Id}(a, b)|)$
- $\text{Set} := \Sigma_{A : \mathbb{U}} \text{isSet}(A)$

Here, $\Sigma$ forms dependent sums, $\Pi$ forms dependent products, and $\text{Id}$ is the identity type. Intuitively, Set is the type of types in $\mathbb{U}$ that are “discrete”. We therefore treat the terms of Set as if they were types in $\mathbb{U}$. Since $\mathbb{U}$ is impredicative, we have $\text{Set} : \mathbb{U}$.

To capture Reynolds’ construction we take our ambient category $\mathcal{C}$ to be the category whose objects are the types in $\mathbb{U}$, and whose morphisms are equivalence classes of functions. Here, functions $f, g : A \rightarrow B$ are considered equal precisely when the type $\text{eq}(f, g)$ is inhabited. To keep from incorporating any particular computational structure into the categorical structure, it is crucial that we use proof-relevant propositional equality types $\text{eq}(-,-)$, rather than proof-relevant identity types $\text{Id}(-,-)$, here; this ensures, for example, that the uniqueness condition for pullbacks is satisfied. With this definition it is easy to check that $\mathcal{C}$ is finitely complete.

To see the type Set as a category internal to $\mathcal{C}$ we first define the type $\text{Set}(A, B)$ of morphisms from $A$ to $B$ to be $\text{Set}(A, B) := A \rightarrow B$, and then take the object of objects in the internal category to be $\text{Set}$ itself and its object of morphisms to be $\Sigma_{A : \mathbb{U}} A \rightarrow B$. We define a category of relations by

$$R := \Sigma_{A,B : \text{Set}} A \times B \rightarrow \text{Prop}$$

$$\Pi_{a_1:A_1, a_2:A_2} R_A(a_1, a_2) := \Sigma_{f:A_1 \rightarrow B_1} \Sigma_{g:A_2 \rightarrow B_2} \Pi_{a_1:A_1, a_2:A_2} R_A(a_1, a_2) \rightarrow R_B(f(a_1), g(a_2))$$

which we can see as a category internal to $\mathcal{C}$ whose object of objects is $\mathbb{U}$ itself and whose object of morphisms is

$$\Sigma_{A_1, A_2, R_A} \Sigma_{B_1, B_2, R_B} \Pi_{a_1:A_1, a_2:A_2} R_A(a_1, a_2) \rightarrow R_B(f(a_1), g(a_2))$$

We obviously have two internal functors from $R$ to $\text{Set}$ corresponding to the first and second projections, respectively. We also have an equality functor $\text{Eq}$ from $\text{Set}$ to $\mathbb{R}$ defined by

$$\text{Eq} A := (A, A, \text{Id}_A)$$

$$\text{Eq} f := (f, f, \text{ap} f)$$

where $\text{ap} f : \text{Id}_A(a_1, a_2) \rightarrow \text{Id}_B(f(a_1), f(a_2))$ is defined as usual by $\text{Id}$-induction.

We obtain a 1-dimensional cubical category $\text{Rel}$ by defining $\text{Rel}(0) = \text{Set}$ and $\text{Rel}(1) = \mathbb{R}$, and mapping the two face maps to the two projections, and the single degeneracy to $\text{Eq}$. We can define terminal objects, products, and exponentials for $\text{Rel}$ in the obvious ways, relating two pairs iff their first and second components are related, and two functions iff they map related arguments to related values. It is not
hard to check that all these constructs are preserved on the nose by the two face maps (projections), and preserved up to a natural isomorphism whose first and second projections are identities by the single degeneracy (equality functor). All three constructs are therefore stable under both face maps and degeneracies. As noted in the introduction, the difference between fibred functors preserving equality on the nose or only up to natural isomorphism is precisely where the construction in \([13]\) fails. Composition and substitution in (what is intended to be) the base category of the \(\lambda\omega\)-fibration constructed in the main theorem there cannot be defined in any standard way unless equality is preserved on the nose, but equality in \([13]\) is only defined — and therefore can only be preserved — up to isomorphism.

Finally, we define the adjoint \(\forall_n\) by

\[
\forall_n F(0) \equiv \sum_{f: A, R} f(1)(\text{Eq} A, R) (f (\pi_1 R), f (\pi_2 R))
\]

\[
\pi_3 (F(1) \xi R) := (\forall_n F(0) \pi_1 (R), \forall_n F(0) \pi_2 (R), \lambda_{f:F(1) \xi R} \pi_3 F(1)(R, R)) (f (\pi_1 R)) (g (\pi_2 R))
\]

In the above, the term \((f (\pi_1 R)) (f (\pi_2 R))\) stands for the term \(f (\pi_1 R) : F(0)(A, \pi_1 R)\) transported along the equality between the respective types, and similarly for \(\pi_2\) and \(g\). We emphasize again that these terms all exist because \(\text{Rel}\) preserves face maps on the nose.

Example 29. We consider the PER model of Bainbridge et al. internal to the category of \(\omega\)-sets. We follow the development of \([14]\) for concepts related to this category. In particular, this category is defined in Definition 6.3 of \([14]\), and proved in Corollary 8.3 there to be finitely complete. We construct a 1-dimensional cubical functor \(\text{Rel}\) as follows. As our internal category \(\text{Rel}(0)\) of 0-relations we take the category \(\mathcal{M}'\) as in Definition 8.4 of \([14]\). Informally, the objects \(\mathcal{M}'\) are partial equivalence relations, and its morphisms are realizable functions that respect those relations.

To define the internal category \(\text{Rel}(1)\) of 1-relations, we first construct its object of objects. As the carrier of this \(\omega\)-set we take the set of triples \((A, B, R)\), where \(A\) and \(B\) are partial equivalence relations and \(R\) is a saturated predicate on \(A \times B\). Here the product \(A \times B\) of two PERs is constructed in the standard way, using a bijective pairing function \(\langle \cdot, \cdot \rangle\) and relating two pairs iff their respective projections — which we will call \(\text{fst}\) and \(\text{snd}\) below — are related. A saturated predicate on a PER \(A\) is a predicate \(R\) on natural numbers that is closed under \(A\), in the sense that \(n \sim_A n\) and \(R(n)\) imply \(R(n)\). To finish the construction of our object of objects for \(\text{Rel}(1)\) we take any triple \((A, B, R)\) as above to be realized by any natural number.

As the carrier of the object of morphisms for \(\text{Rel}(1)\) we take the set of quadruples of the form

\[
((A_1, B_1, R_1), (A_2, B_2, R_2), \{n\}_{A_1 \rightarrow A_2}, \{m\}_{B_1 \rightarrow B_2})
\]

satisfying the condition that, for any \(k\) such that \(R_1(k)\) holds, we have that \(R_2((n \cdot \text{fst}(k), m \cdot \text{snd}(k)))\) holds as well. The first two components of such a quadruple serve to encode the domain and codomain of the morphism. The third component is a (nonempty) equivalence class under the exponential PER \(A_1 \rightarrow A_2\). Here the exponential \(A \rightarrow B\) of two PERs is constructed in the standard way, using an encoding of partial recursive functions as natural numbers and relating two functions iff they map related arguments to related values. In accordance with \([14]\), we denote the application of the \(n^{th}\) partial recursive function to a natural number \(a\) in its domain by \(n \cdot a\). To finish the construction of our object of morphisms for \(\text{Rel}(1)\), we take a quadruple as above to be realized by a natural number \(k\) if \(\text{fst}(k) \sim_{A_1 \rightarrow A_2} (n \cdot m)\) and \(\text{snd}(k) \sim_{B_1 \rightarrow B_2} m\).

We obviously have two internal functors from \(\text{Rel}(1)\) to \(\text{Rel}(0)\), corresponding to the first and second projections, respectively. We also have an equality functor \(\text{Eq}\) from \(\text{Rel}(0)\) to \(\text{Rel}(1)\) whose action on objects is given by \(\text{Eq} A := (A, A, R_A)\), where \(R_A(k)\) iff \(\text{fst}(k) \sim_A \text{snd}(k)\), and whose action on morphisms is given by \(\text{Eq}(A, B, \{n\}_A \rightarrow B, \{n\}_B) := (\text{Eq} A, \text{Eq} B, \{n\}_A \rightarrow B, \{n\}_B \rightarrow B)\). We therefore have that \(\text{Rel}\) is indeed a 1-dimensional cubical category. We can define terminal objects, products, and exponentials for \(\text{Rel}\) in the obvious ways, inheriting from the corresponding standard constructs on PERs. It is not hard to check that all these constructs are preserved both by the two face maps (projections), and by the single degeneracy (equality functor), on the nose.

Finally, we define the adjoint \(\forall_n\) on objects by

\[
\forall_n F(0) \equiv \{\langle n, k \rangle \mid \text{ for all } A : \mathcal{M}', n \sim_A k, \text{ and for all } R : \text{Rel}(1), \forall_n F(1)(\text{Eq} A, R) (n, k)\}
\]

\[
\forall_n F(1) \equiv \{\forall_n F(0) \pi_1 (R), \forall_n F(0) \pi_2 (R), \forall_n F(1)(\text{Rel}(R)), \forall_n \text{Eq}(R, R) (n)\}
\]

To define \(\forall_n\) on a morphism \(\eta : F \rightarrow G\), we define

\[
\forall_n \eta(0) \equiv \langle \forall_n F(0) \text{Eq}(A, R), \forall_n \text{Eq}(G(0), A) \rangle
\]

Here, \(m\) is any natural number realizing \(\eta(0)\). It is crucial that all natural transformations are “uniformly realized”, in the sense that there is a natural number realizing each such transformation and, because all PERs are defined to be realized by all natural numbers, each is suitably uniform. In particular, if \(\eta\) were not uniformly realized in the above sense, then \(\forall\) would not be well-defined. Using this observation it is possible to show that, in the category-theoretic setting (rather than in the setting of \(\omega\)-sets), the adjoint \(\forall_n\) cannot exist precisely because ad hoc natural transformations — i.e., natural transformations that are not uniformly realizable, even though each of their components may indeed be realizable — are not excluded.
VI. CONSEQUENCES OF PARAMETRICITY

In this section we show that the models constructed in Theorem 27 satisfy the properties that “good” models of parametricity for System F should satisfy. In particular, Lemma 34 below shows that, under reasonable conditions, our models support the definition of a graph for each face map- and degeneracy-preserving cubical functor. Moreover, Theorem 35 and its analogue for final coalgebras show that our higher-dimensional models of relational parametricity for System F also validate the existence of initial algebras and final coalgebras for such functors. These results serve as a sanity check for our theory, and show that it is powerful enough to show that “good” models of relational parametricity for System F can be constructed even at higher dimensions.

A. A Higher-Dimensional Graph Lemma

Every function \( f : A \rightarrow B \) between sets \( A \) and \( B \) defines a graph relation \( (f) = \{(a, b) \mid f(a) = b\} \). This observation can be phrased diagrammatically by stating \( \mathbf{U} : \text{Rel} \rightarrow \mathbf{Set} \) is the standard relations fibration on \( \mathbf{Set} \), and \( \langle f \rangle \) can be obtained by reindexing the equality relation \( \mathbf{Eq} \) on \( B \). In [8], the notion of a graph was extended to more general relations fibrations and a Graph Lemma was proved for their associated models of 1-dimensional parametricity. In this subsection we give a natural generalization of the definition of a graph from [8] to the higher-dimensional setting, and prove a Graph Lemma appropriate to this setting. We begin by introducing the (new) notion of a cubical (bi)fibration.

**Definition 30.** A \((p\text{-dimensional})\) cubical category \( \text{Rel} \) that has products is a \((p\text{-dimensional})\) cubical (bi)fibration if, for each \( l < p \), each functor
\[
\mathbf{f}(l, k) = (\text{Rel}(\mathbf{f}_l(l, k)), \text{Rel}(\mathbf{f}_l(l, k))) : \text{Rel}(l + 1) \rightarrow \text{Rel}(l) \times \text{Rel}(l)
\]
for \( k \leq l \) is a (bi)fibration.

As already noted in Example 5 the (bi)fibrations \( \mathbf{f}(l, k) \) play the role of the relations fibrations in [8], while the \( \mathbf{d}(l, k) \) play the role of equality functors. When \( \mathbf{C} \) is a cubical (bi)fibration, we have that \( \text{Rel}(\mathbf{d}(l, k)) \mathbf{A} \) is indeed over \((\mathbf{A}, \mathbf{A})\) with respect to \( \mathbf{f}(l, k) \) for every object \( \mathbf{A} \in \text{Rel}(l) \), and similarly for every morphism in \( \text{Rel}(l) \).

If \( \mathbf{C} \) is a category, write \( \mathbf{C} \rightarrow \mathbf{C} \) for the **arrow category** of \( \mathbf{C} \), i.e., for the category whose objects are morphisms in \( \mathbf{C} \) and whose morphisms from \( f : A \rightarrow B \) to \( f' : A' \rightarrow B' \) in \( \mathbf{C} \rightarrow \mathbf{C} \) are pairs of morphisms \( g : A \rightarrow A' \) and \( h : B \rightarrow B' \) such that \( f' \circ g = h \circ f \). We define the graph functor for \( \mathbf{C} \) to be the set of functors \( \{(-)_l, k \mid l < p, k \leq l\} \), where each \( (-)_l, k \) is defined as follows:

**Definition 31.** Let \( \text{Rel} \) be a \((p\text{-dimensional})\) cubical category that has terminal objects. For every \( l < p \) and \( k \leq l \), the functor \( (-)_l, k : \text{Rel}(l) \rightarrow \text{Rel}(l + 1) \) is defined by:

- if \( h : A \rightarrow B \) is an object in \( \text{Rel}(l) \), then \( (h)_l, k = (h, \text{id}_B)^* (\text{Rel} \mathbf{d}(l, k) B) \)
- if \( f : A \rightarrow B, f' : A' \rightarrow B' \), and \((g, h) : f \rightarrow f'\) is a morphism in \( \text{Rel}(l) \), then \( (g, h)_l, k \) is the unique morphism from \( (f)_l, k, \text{Rel} \mathbf{d}(l, k) h \circ (f, \text{id}_B) \) via \( (f', \text{id}_B) \).

Intuitively, one of \( f_\bot(l, k) \) and \( f_\top(l, k) \) acts as the \( x \)-axis, and the other acts as a \( y \)-axis, for \( l \)-dimensional graphs projected onto dimension \( k \). Since reindexing preserves identities, we have that \( (\text{id}_A)_l, k = (\text{id}_A, \text{id}_A)^* (\text{Rel} \mathbf{d}(l, k) B) = \text{Rel} \mathbf{d}(l, k) B \). This generalizes the observation that \( \langle \text{id}_A \rangle \) is \( \text{Eq} \) in the 1-dimensional setting of [8].

We also have the following alternative characterization of the graph functor when \( \text{Rel} \) is a bifibration.

**Lemma 32.** If \( \text{Rel} \) is a \((p\text{-dimensional})\) cubical bifibration that has terminal objects, and if \( f : A \rightarrow B \), then \( (f)_l, k = \Sigma (\text{id}_A, \mathbf{f}_r(l, k)) \mathbf{d}(l, k) A \).

By contrast with the analogous characterization in Lemma 5.2 of [8], no Beck-Chevalley condition is required since the bifibrations \( \mathbf{f}(l, k) \) are postulated here, rather than derived from more primitive bifibrations as is done there.

We have the following analogue of Lemma 5.3 of [8]:

**Lemma 33.** \( (-)_l, k \) is full and faithful if \( \text{Rel} \mathbf{d}(l, k) B \) is.

Together, the (fibrical) definition of the graph functor and its opfibrical characterization from Lemma 33 give the following Graph Lemma for our higher-dimensional setting:

**Lemma 34.** (Graph Lemma) Let \( \text{Rel} \) be a \((p\text{-dimensional})\) cubical bifibration that has terminal objects and \( \mathbf{F} : \text{Rel} \rightarrow \text{Rel} \) be a \((p\text{-dimensional})\) face map- and degeneracy-preserving cubical functor. For any \( l < p \), \( f : A \rightarrow B \) in \( \text{Rel}(l) \), and \( k \leq l \), there exist morphisms
\[
\phi_f : \mathbf{F}((l + 1) \mathbf{f})_{l, k} \rightarrow \mathbf{F}(l + 1)(\mathbf{f})_{l, k}
\]
and
\[
\psi_f : \mathbf{F}(l + 1)(\mathbf{f})_{l, k} \rightarrow \mathbf{F}(l) \mathbf{f}_{l, k}
\]
in \( \text{Rel}(l + 1) \) that are vertical with respect to \( \mathbf{f}(l, k) \).

B. Existence of Initial Algebras and Final Coalgebras

In this subsection we use our Graph Lemma to show that the models constructed in Theorem 27 validate the existence of initial algebras and final coalgebras for face map- and degeneracy-preserving cubical functors, and thus for all interpretations of positive type expressions in System F. Our constructions naturally extend those in [8] to the higher-dimensional setting.

If \( \mathbf{C} \) is a \((p\text{-dimensional})\) cubical category and \( \mathbf{C} : \text{Rel} \rightarrow \text{Rel} \) is \((p\text{-dimensional})\) cubical functor, then an \( \mathbf{C}-\text{algebra} \)
\((A, k_A)\) is a set of pairs \(\{(A_l, k_{A_l}) \mid l < p\}\) in which each \(A_l\) is an object of \(\text{Rel}(l)\) and each \(k_{A_l}: F(l)A_l \to A_l\) is a morphism in \(\text{Rel}(l)\). We call the set \(A = \{A_l \mid l < p\}\) the carrier of the \(F\)-algebra and the set \(k = \{k_{A_l} \mid l < p\}\) its structure map. A set of morphisms \(f = \{f_l: A_l \to B_l \mid l < p\}\) with each \(f_l\) in \(\text{Rel}(l)\) is an \(F\)-algebra morphism \(f: (A, k_A) \to (B, k_B)\) if, for each \(f_l\) in \(f\), \(k_{B_l} \circ \langle F(l)f_l \rangle = f_l \circ k_{A_l}\). An \(F\)-algebra \((Z, in)\) is weakly initial if, for any \(F\)-algebra \((A, k_A)\), there exists a mediating \(F\)-algebra morphism \(\langle Z, in \rangle \to (A, k_A)\). It is an initial \(F\)-algebra if \(Z, in\) is unique up to isomorphism.

Now, every \(\lambda_2\)-fibration has an associated internal language. For the \(\lambda_2\)-fibration we construct in Theorem 25 this is a polymorphic lambda calculus for which each type \(\Gamma \vdash A\) is given by a face map- and degeneracy-preserving cubical functor from \([\text{Rel}]^{|\Gamma|}\) to \(\text{Rel}\), and each term \(\Gamma; \Delta \vdash t : A\) is a face map- and degeneracy-preserving cubical natural transformation between such functors. We can use this internal language to reason about our models using System F.

Let \(F: \text{Rel} \to \text{Rel}\) be a \((p\text{-dimensional})\) face map- and degeneracy-preserving cubical functor. A strength for \(F\) is a set \(\sigma = \{\sigma_l \mid l < p\}\) of families of morphisms \((\sigma_l)_{A,B}: A \Rightarrow B \to F(l)A \Rightarrow F(l)B\) such that the mapping of cubical functors to their strengths preserves identities and composition, and, for each \(l < p\) and \(k < \ell\), \(f(l,k)\) \((\sigma_{l+1})_{C,D} = ((\sigma_l)_{A,B}, (\sigma_l)_{A',B'})\) if \(f(l,k)C = (A, B)\) and \(f(l,k)D = (A', B')\). A cubical functor with a strength is said to be strong. Because of the discrete domains, \(\sigma\) is a cubical natural transformation from \(\Rightarrow\) \(\Rightarrow\) to \(\Rightarrow\) \(\Rightarrow\) in \([\text{Rel}]^2 \to \text{Rel}\). The term \(A, B; \vdash \sigma: (A \Rightarrow B) \Rightarrow (F[A] \Rightarrow F[B])\) represents the action of \(F\) on morphisms in the internal language.

To see that every face map- and degeneracy-preserving cubical functor \(F\) has an initial \(F\)-algebra we define \(Z = [\forall X.(F X \Rightarrow X \Rightarrow X)], fold = \Lambda A.\Lambda k : FA \to A.\lambda z : Z.\ z\ k, \ fold\ [A, k] = [fold\ k], where\ A\ and\ k\ are\ the\ internal\ expressions\ corresponding\ to\ the\ components\ of\ another\ \(F\)-algebra \((A, k)\), and \(in = [\lambda x.A.X.\lambda k : FA \to A. k (\text{fold} X k x)]\). Our Graph Lemma can then be used to extend the 1-dimensional construction from [8] to the higher-dimensional setting:

**Theorem 35.** If \(F\) is a \((p\text{-dimensional})\) bifibration that has terminal objects, if \(F: \text{Rel} \to \text{Rel}\) is a \((p\text{-dimensional})\) face map- and degeneracy-preserving cubical functor, if \(d(l,k)\) is full for every \(l < p\) and \(k < \ell\), and if, for every \(l < p\), \(\text{Rel}(l)\) is well-pointed, then \((Z, in)\) is an initial \(F\)-algebra.

We obtain the analogous result for final \(F\)-coalgebras as well.

**VII. Related Work**

The study of parametricity runs both wide and deep. Here, we draw connections with some of the work most closely related to ours.

Ma and Reynolds [13] gave the first categorical formulation of relational parametricity. Generalizing from the evident reflexive graph structure in well-behaved relational models of the simply typed lambda calculus, they reformulated Reynolds’ original notion of relational parametricity for System F in terms of reflexive graphs of Seely’s PL categories [18]; these have sufficient structure to model the type-dependent aspects of System F as well. Jacobs [12] later generalized this reformulation, recasting it in terms of \(\lambda_2\)-fibrations and parameterizing it over a “logic of types” for the polymorphic type theory. His Definition 8.6.2 gives an notion of 1-dimensional relational parametricity that is “external”, in the sense that it describes when an arbitrary \(\lambda_2\)-fibration carries enough structure to formalize that some of the specific models he constructs are “intuitively parametric”. This contrasts with our “internal” approach, which starts with some suitably-structured-but-otherwise-arbitrary components and uses a particular construction to weave them into \(\lambda_2\)-fibrations that are “intuitively parametric” in the same sense as Jacobs’ models, except that our models satisfy this property at higher dimensions, too. Overall, our work can be seen as a first extension to higher dimensions of a formalism capturing the observation that “intuitively parametric” \(\lambda_2\)-fibrations are all generated in essentially the same way.

Ma and Reynolds [13] neither provide models that are relationally parametric in the sense they define, nor give any indication how hard such models might be to construct. This led Robinson and Rosolini [17] to reconsider Ma and Reynolds’ formulation of Reynolds’ relational parametricity from the point of view of internal categories. This supports a narrowing of Ma and Reynolds’ framework that is more promising for model construction. Robinson and Rosolini also use internal categories to clarify the constructions of [15]; our use of internal categories to clarify the constructions of [8] when \(p = 1\) is in the same spirit.

Duñuph and Reddy [6] do not work with internal categories, but they do use reflexive graphs to model relations and functors between reflexive graph categories to model types. The framework they develop is mathematically elegant and powerful enough to derive some expected consequences of relational parametricity, including the existence of initial algebras for strictly positive System F type expressions. The framework of [8] offers an alternative categorical approach to relational parametricity formulated in terms of bifibrations rather than reflexive graphs. It gives a functorial semantics for System F that derives all of the expected consequences of parametricity that Birkedal and Møgelberg prove using Abadi-Plotkin logic [3], including the existence of initial algebras for all positive type expressions, rather than just strictly positive ones. However, the bifibrational framework suffers from the shortcomings already discussed in this paper.

Cubical sets were originally introduced in the context of algebraic topology, but have more recently been shown to model homotopy type theory [4], [10], an extension of Martin-Löf type theory. A key feature of homotopy type theory is that functions are infinitely parametric with respect to propositional equality in a non-trivial way. It is still not fully established whether this theory supports a well-defined notion of computation, even for base types such as natural numbers.
That it does is Voevodsky’s *homotopy-canonicity conjecture*. We are not the first consider parametricity at higher dimensions. In [3], the bifibrational approach to relational parametricity developed in [5] was extended to proof-relevant relations. This was achieved by extending the uniformity condition characterizing parametric functions to proofs by adding a second “dimension” of parametricity on top of Reynolds’ standard one that forces the standard uniformity condition to itself be uniform, in effect requiring that polymorphic programs can be proved to map related arguments to related results via related proofs. The resulting construction delivers a 2-dimensional parametricity theorem appropriate to the proof-relevant setting. We conjecture that this construction can be made an instance of our theory. Note, however, that Definition 22 of [7] actually needs our more general theory in which equality can be required to be preserved only up to natural isomorphism.

**VIII. Conclusion and Future Work**

In this paper we developed a theory of higher-dimensional relational parametricity for System F that not only clarifies and strengthens the results of [3] when $p = 1$, but also naturally generalizes Reynolds’ original notion of relational parametricity for System F to higher dimensions. We have also shown that our theory properly subsumes Reynolds’ original model and the PER model of Bainbridge *et al.* as proper instances of our theory when $p = 1$, and that it formalizes notions of proof-relevant parametricity (when $p = 2$) and infinite-dimensional parametricity when ($p = \infty$) as well. Finally, we have proved that our theory is “good” in the sense that it derives higher-dimensional analogues of expected results for parametric models. In future work we hope to settle our conjecture that our $\lambda 2$-fibrations are relationally parametric in the sense of Jacobs’ “external” notion when $p = 1$, as well as to generalize this “external” notion to relational parametricity to infinitely many dimensions. We also plan to investigate how our theory can be instantiated to give new parametric models for System F at dimension 1. Finally, we plan to investigate connections between our theory and proof-relevant parametricity at dimension 2, and between our theory and the homotopy-canonicity conjecture when $p = \infty$.

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