Relativistic hydrodynamical model in the presence of long-range correlations

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Effects of dynamical long-range correlations over a fluid cell size scale on a relativistic fluid are discussed. It is shown that such correlations among the fluid elements introduced into hydrodynamical model induce some weak dissipation and viscosity into the fluid. The influence of the long-range correlations on the entropy current is also discussed.

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I. INTRODUCTION

Recently, the so called ridge phenomenon has been observed at RHIC [1] (particle yield enhancement in narrow azimuthal angle window $\Delta \phi$ but broad pseudo-rapidity window $\Delta \eta$). It is a jet-related long range correlation in rapidity. For us it is particularly interesting that the so-called soft ridge phenomenon, a similar longitudinal rapidity correlation but without jet trigger, is found in collisions of higher centrality events [2,3]. Although there is no theoretical model that can quantitatively reproduce all experimental data related to this phenomena, several models have been proposed aiming at the explanation of ridge data [4,5].

Actually, correlations over several rapidity units can only occur at the earliest stages of heavy-ion collision, when different rapidity regions are still causally connected [2,6]. It means that the origin of this phenomenon must be placed almost instantaneously after the collision of two nuclei [1,8], namely before or around the initial stage of the hydrodynamical evolution of the fireball created by the relativistic heavy-ion collisions. On the other hand, observation of correlations is related to period after freeze out, what means that the long-range correlations can survive the whole stage of hydrodynamical evolution of the matter. Since little attention has been given to this problem so far it is therefore naturally to ask what are the influence of the long-range correlations on the fluid dynamical evolution because, this will be the subject we shall investigate here. Because such dynamical long-range correlations may cause weak dissipation and viscosity in the fluid we propose a kind of extreme model for the relativistic dissipative hydrodynamics with small dissipative and viscous terms, which originate entirely from the long-range correlations among the fluid elements.

A few words on hydrodynamical models used in particle production processes are in order here. The ideal fluid dynamics [11,15] have successfully reproduced experimental results obtained at Relativistic Heavy-Ion collider (RHIC) (for a review, see for example, [17]). This success brought us a new understanding of the matter created at the RHIC. It turned out that this matter behaves more like an ideal fluid rather than the weakly interacting parton plasma [18]. It is thought that this is because the mean free path of the particles composing it is small comparing to the system size or to the typical scale in the fluid dynamics (strongly interacting quark gluon plasma, sQGP [19,21]). However, there are noticeably differences between predictions from the ideal fluid model and the corresponding data for the flow parameters obtained in [22,24,54]. Hence, some revisions to the ideal fluid picture turned out to be necessary and dissipation and viscosity effects have been introduced to the perfect fluid. This caused some problems with the proper formulation of the relativistic dissipative fluid model [25,33]. In a simple relativistic extension of the Naiver-Stokes equation (so-called first order theories [34,35]), it is known that the causality is violated [36,37] and the solution of the equation is not stable against small perturbations [38,39]. Over the past few years a considerable number of studies have been made aiming at the solution of these problems (see, for example Ref. [40,42]).

The purpose of this paper is to formulate the relativistic imperfect hydrodynamics including the effect of long-range correlations and to find its connections with the usual dissipative hydrodynamics. Note that, in the presence of the long-range correlations over the thermal equilibrium scale, the microscopic state of the particle that composes the fluid does not obey the usual Boltzmann-Gibbs statistics any longer [43]. Also thermodynamic functions such as the energy density and pressure among the separate fluid cells will be affected. The state in the presence of the long-range correlations, which is generally different from the usual equilibrium state (it is a kind of a non-equilibrium state), can be the so-called stationary state near the equilibrium. It approaches to the equilibrium state with a finite (or infinitely long) relaxation time under a non-equilibrium thermodynamical constraint which may make a kind of narrow valley of the free energy generated by dynamical correlations [44]. This relaxation process depends on the initial state selected in the valley of the free energy. Such dependence
on the initial state for the relaxation processes can be then regarded as a kind of memory effect induced by the long-range correlations.

The relaxation process of the stationary state is called pre-thermalization [44] (see also [45]) and leads to Tsallis’s statistics [46]. It is usually considered to be important for describing correlated system and appears when the phase space is reduced by the correlations among constituents of the system. In [47] it is shown that there is a certain correspondence between the fluid dynamics based on the Tsallis statistics and the dissipative fluid dynamics based on the usual Boltzmann-Gibbs statistics. Thus, the dynamical long-range correlations may result in some dissipative contribution in the relativistic fluid.

It should be stressed at this point that situation considered here is rather special. Our model assumes that, because of action of the long-range correlations, some stationary, near equilibrium, state of fluid is formed replacing the usual equilibrium state. The natural question is how real is such assumption. So far there is no hard evidence for creation of such state. On the other hand, there is growing evidence of a power-law-like behavior of the transverse momentum spectra in the mid-pT region (2–6 GeV/c) and this may indicate the existence of such stationary state [48]. The necessary condition for the existence of such pre-thermalization state in the matter created in the relativistic heavy-ion collisions (for example at RHIC) is that its relaxation (or equilibration) time is long enough comparing with the typical hydrodynamical time scale, such as Lhydro/cS, where Lhydro is some typical length scale of fluid and cS is sound velocity. All these calls for some very detail considerations which are, however, outside of the limited scope of the present paper. Here we consider therefore only a kind of thought experiment in which appearance of the stationary state due to the long range correlations is assumed and its consequences investigated in detail.

The long-range correlation influence not only the microscopic state but also the macroscopic state of the fluid such as velocity vectors. Therefore, instead of specifying the type of the long-range correlations, we shall assume that all information on them are included in a flow velocity vector field, u\(\mu\)(x), and in the non-equilibrium thermodynamic functions like local energy density, \(\varepsilon(x)\), and pressure, \(P(x)\) (here \(x\) denotes a space-time four vector).

To quantitatively investigate the effects of the long-range correlations (LRC) on the relativistic fluid, we shall propose a thought experiment assuming that it is possible to switch on and off the LRC for a perfect fluid in such a way as not to break the causality. If one can turn on the switch of the LRC that acts on the perfect fluid at a proper time \(x^0 = \tau_{on}\), then, after \(\tau_{on}\), the fluid starts to expand as a dissipative fluid. We assume that solution of the perfect fluid, which is obtained in the case without LRC, is known for all space-time points (i.e., by solving ideal fluid dynamics after \(\tau = \tau_{on}\) we know the energy density, \(\varepsilon_{eq}(x)\), conserved charge density, \(n_{eq}(x)\), flow velocity, \(U^\mu(x)\), and equation of state \(P_{eq}(x) = P_{eq}(\varepsilon_{eq}, n_{eq})\) for all space-time point \(x\) [49, 50]. It is then assumed that to know evolution of changes (defined as \(\Lambda(x) \equiv \varepsilon - \varepsilon_{eq}\), \(\Pi(x) \equiv P - P_{eq}\), \(\delta n(x) \equiv n - n_{eq}\) and \(\delta u^\mu \equiv u^\mu(x) - U^\mu(x)\)) is equivalent to solving equations of resulting dissipative hydrodynamics. As will be seen below, since \(\Lambda, \Pi\) and \(\delta n\) can be expressed as a function of the enthalpy, \(h_{eq} \equiv \varepsilon_{eq} + P_{eq}, n_{eq}, U^\mu\) and \(\delta u^\mu\), in order to investigate the dissipative effects of the LRC is sufficient to know only the evolution of \(\delta u^\mu(x)\). Hereafter, we shall call the set of solutions of the ideal hydrodynamical equations, \(\varepsilon_{eq}(x), P_{eq}(x), n_{eq}(x)\) and \(U^\mu(x)\), the reference fluid fields (RFF) for the corresponding dissipative fluid field. One may also define RFF by switching-off the LRC for a given fluid with dissipation that originated from LRC.

The paper is organized as follows. In Sec. II, we briefly review the relativistic fluid dynamics. Then, in Sec. III, we introduce the RFF which satisfy ideal hydrodynamics and we express the energy-momentum tensor in terms of \(\delta u^\mu\) and RFF. In Sec. IV equation on \(\delta u^\mu\) is found which agrees with the relativistic dissipative hydrodynamical equations. Sec. V contains the off-equilibrium thermal entropy current corresponding to our model. We close with Sec. VI containing the summary and some further discussion.

II. THE RELATIVISTIC FLUID DYNAMICS

Basic equations of the relativistic hydrodynamics for arbitrary dissipative fluid consist of by local conservation laws for the energy-momentum and conserved charges. In addition to these local conservation laws, the second law of thermodynamics is required. These equations are generally expressed by the covariant derivatives [51] of the energy-momentum tensor, \(T^{\mu\nu}(x)\), the conserved charge current, \(N^\mu(x)\), and the entropy current \(S^\mu(x)\):

\[
\begin{align*}
T^{\mu\nu} & = 0, \quad (1a) \\
N^\mu & = 0, \quad (1b) \\
S^\mu & \geq 0. \quad (1c)
\end{align*}
\]

The general form of the energy-momentum tensor used here is given by

\[
T^{\mu\nu} = \varepsilon u^\mu u^\nu - P \Delta^{\mu\nu}(u) + 2W(r)\varepsilon u^\nu + \pi^{\mu\nu}, \quad (2)
\]

where \(\varepsilon \equiv T^{\mu\nu}u_\mu u_\nu\) is energy density, \(P \equiv -\frac{1}{3}\Delta^{\mu\nu}(u)T^{\mu\nu}\) is pressure, \(W \equiv u_\nu T^{\nu\lambda}\Delta^\lambda_\mu(u)\) is energy flow and \(\pi^{\mu\nu} \equiv T^{(\mu\nu)}\) is shear stress tensor. The fluid is moving with four-velocity \(u^\mu(x)\) such that \(u^\mu u_\mu = 1.\) We
introduce transverse projection tensor $\Delta^{\mu\nu}(u) \equiv g^{\mu\nu} - u^\mu u^\nu$ with general metric $g^{\mu\nu}$ for the space-time considered. This tensor has the property that $u_{\nu}\Delta^{\mu\nu}(u) = u_{\mu}\Delta^{\mu\nu}(u) = 0$. The notation used is: $A^{(\mu}B^{\nu)} \equiv \frac{1}{2} (A^{\mu}B^{\nu} + A^{\nu}B^{\mu})$ denotes symmetric tensor defined by two four-vectors $A^\mu$ and $B^\nu$ and the angular bracket notation $A^{(\mu\nu)} \equiv \frac{1}{2} \left( \Delta^{\mu\nu\gamma}_{\delta} + \Delta^{\mu\gamma\nu}_{\delta} - \frac{1}{2} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) A^{\alpha\beta}$ represents symmetric and trace free part of tensor $A^{\mu\nu}$.

The net charge and entropy currents are generally given by, respectively,

\[ N^\mu = nu^\mu, \quad (3) \]

\[ S^\mu = su^\mu + \Phi^\mu, \quad (4) \]

where $\Phi^\mu \equiv \Delta^\mu_\nu(u)S^\nu$ is entropy flux. In this paper we shall use the so-called Eckart frame \[3\] and fix arbitrary of choice of frame for the four vector flow $u^\mu$. In what follows we assume that baryon number, $N^\mu$, is conserved.

If the local thermal equilibrium is achieved (or, if the LRC is switched off in our thought experiment and after the relaxation process following it has been completed), the entropy flux $\Phi^\mu$ should also disappear, its density reaches the maximum, and it is locally conserved. The energy flow vector $W^\mu$ and viscous shear tensor $\pi^{\mu\nu}$ should also disappear. One gets therefore a perfect fluid, corresponding to Eqs. (2), (3) and (4), for which one has

\[ T_{eq}^{\mu\nu} = \varepsilon_{eq}(T, \mu)U^{\mu}U^{\nu} - P_{eq}(\varepsilon_{eq}, n_{eq})\Delta^{\mu\nu}(U), \quad (5a) \]

\[ N_{eq}^\mu = n_{eq}(T, \mu)U^\mu, \quad (5b) \]

\[ S_{eq}^\mu = s_{eq}(T, \mu)U^\mu, \quad (5c) \]

where $T = T(x)$ and $\mu = \mu(x)$ are the temperature and (baryonic) chemical potential field given by $\varepsilon_{eq}(x)$ and $n_{eq}(x)$. The equation of state $P_{eq}(x) = P_{eq}(\varepsilon_{eq}(x), n_{eq}(x))$ for the perfect fluid are assumed to be known. Here, $U^\mu(x)$ is the flow vector appearing after relaxation has been completed.

### III. RELATIVISTIC HYDRODYNAMICS IN THE PRESENCE OF LONG-RANGE CORRELATIONS

#### A. Reference fluid field (RFF)

Let us introduce reference fluid field (RFF), by which we understand a set of solutions for a given perfect fluid without the LRC (because LRC is assumed to be the origin of dissipation and viscosity, one has perfect fluid if LRC are not present). In our case the perfect fluid dynamics is given by specifying

\[ T_{eq}^{\mu\nu}_{\varepsilon} = 0, \quad (6a) \]

\[ N_{eq,\mu}^{\varepsilon} = 0, \quad (6b) \]

\[ S_{eq,\mu}^{\varepsilon} = 0. \quad (6c) \]

Note that there are 6 unknown variables in the perfect fluid hydrodynamics: energy density $\varepsilon_{eq}(x)$, conserved charge density, $n_{eq}(x)$, pressure $P_{eq}(x)$ and three unknown components of the fluid velocity, $U^\mu(x)$. On the other hand, there are 6 equations: the first two equations of Eq. (3) and equation of state, $P_{eq}(x) = P_{eq}(\varepsilon, n_{eq})$ (equation Eq. (4c), the local entropy conservations, can be derived from the previous two equations.).

#### B. Effects caused by the LRC

After switching on the LRC in our thought experiment (i.e., for a non-equilibrium dissipative situation) we have to solve Eq. (11) with Eqs. (2), (3) and (4). The pressure $P$ is different from the isotropic equilibrium pressure, $P_{eq}(\varepsilon_{eq}, n_{eq})$, which can be obtained by equation of state (EoS) via the equilibrium energy density, $\varepsilon_{eq}$, and the baryon number density, $n_{eq}$ (i.e., $P \neq P_{eq}(\varepsilon_{eq}, n_{eq})$). The difference $P - P_{eq}$ is usually regarded as a bulk pressure, $\Pi \equiv P - P_{eq}$. For energy density it is also natural to introduce bulk energy density, $\Lambda \equiv \varepsilon - \varepsilon_{eq}$, because the microscopic occupation probability does not follow the Boltzmann-Gibbs statistics any longer. Hence, the energy density $\varepsilon$ in Eq. (2) cannot be expressed as a function of the usual temperature $T$ and (baryonic) chemical potential $\mu$ such like $\varepsilon_{eq}(T, \mu)$. Therefore, one can write thermodynamical quantities in the presence of the long-range correlations,

\[ \varepsilon = \varepsilon_{eq}(T, \mu) + \Lambda, \quad (7a) \]

\[ n = n_{eq}(T, \mu) + \delta n, \quad (7b) \]

\[ P = P_{eq}(\varepsilon_{eq}, n_{eq}) + \Pi. \quad (7c) \]

Now, the energy-momentum tensor Eq. (2) can be decomposed by the flow vector $U^\mu(x)$ introduced at the end of Sec. II, which is generally different from $u^\mu(x)$. Denoting by $\delta u^\mu$ the difference between $u^\mu(x)$ and $U^\mu(x)$, one can write

\[ u^\mu(x) = U^\mu(x) + \delta u^\mu(x). \quad (8) \]

Since both $u^\mu$ and $U^\mu$ must satisfy normalization conditions as physical flow vectors, i.e., $u^\mu u_\mu = 1$ and $U^\mu U_\mu = 1$, they are related in the following way:

\[ \frac{1}{2} \delta u^\mu \delta u_\mu = -U^\mu \delta u_\mu = u^\mu \delta u_\mu = \gamma \quad (9) \]

It is desirable to describe the number of unknowns before moving to the main task. In dissipative hydrodynamical model there are 14 unknowns: $\Lambda$, $\Pi$, $\delta n$, 3 components of $W^\mu$ (because condition $W^\mu u_\mu = 0$), 5 components $\pi^{\mu\nu}$ (symmetric and traceless and satisfying conditions $\pi^{\mu\nu} u_\mu = 0$) and 3 components of $\delta u^\mu$. On the other hand, $\varepsilon_{eq}$, $n_{eq}$ and 3 components of the $U^\mu$ (i.e., 5 variables) and EoS, $P_{eq} = P_{eq}(\varepsilon_{eq}, n_{eq})$, are known for the reference fields. As will be shown in Sec. III D and

\[ \text{...} \]
where $\hat{\delta u}^\mu$, total 11 unknowns are function of the 3 unknown of $\delta w^\mu$ and 6 known variables describing the RFF. Note that the number of unknown three variables for $\delta u^\mu$ coincides with the number of spatial dimensions. They depend on the nature of the LRC (for example, if the correlations are totally spatially isotropic for a rest frame of $U^\mu$, the number of unknowns is reduced to unity). Therefore, we need to derive one equation from Eq. (1.1a) and Eq. (1.1b) (besides eqs. (6a), (6b) and EoS) to determine one unknown. This is, as shown later, Eq. (1.10a) or equivalently Eq. (1.11).

C. Tensor decomposition by means of the RFF

Introducing projection operator accompanying $U^\mu$, $\Delta^\mu\nu(U) = g^\mu\nu - U^\mu U^\nu$, one can decompose Eq. (2) as

$$T^\mu\nu = \hat{\epsilon} U^\mu U^\nu - \hat{P} \Delta^\mu\nu(U) + 2 \hat{W}^\mu(U^\nu) + \hat{\pi}^{\mu\nu},$$

where $\hat{\epsilon}$ and $\hat{P}$ are

$$\hat{\epsilon} \equiv T_{\alpha\beta} U^\alpha U^\beta = \epsilon + 3 \Sigma,$$  (11a)

$$\hat{P} \equiv -\frac{1}{3} T_{\alpha\beta} \Delta_{\alpha\beta}(U) = P + \Sigma,$$  (11b)

respectively. The quantity $\Sigma$ that appears both in the Eq. (11a) and (11b) is explicitly given by

$$\Sigma = \frac{1}{3} h(\gamma^2 - 2 \gamma) - \frac{2}{3} E(U) + \frac{1}{3} \gamma^\nu,$$  (12)

where $h = \epsilon + P$ is natural extension of the equilibrium enthalpy $h_{eq} = \epsilon_{eq} + P_{eq}$ and

$$\gamma^\nu = W_\mu \delta u^\mu, \quad \gamma^\nu = \pi_{\mu\nu} \delta u^\mu \delta u^\nu.$$  (13)

For the corresponding energy flow and shear viscosity tensor, we have

$$\hat{\dot{W}}^\mu = \hat{\Delta}^\mu\nu \Delta_{\alpha\beta} U_{\beta}$$

$$= (1 - \gamma)(h X^\mu + Y^\mu) - Z^\mu - \gamma' X^\mu - \gamma'' U^\mu, \quad \quad (14a)$$

$$\hat{\pi}^{\mu\nu} \equiv \left[ \frac{1}{2}(\hat{\Delta}^\alpha\nu \hat{\Delta}_{\alpha\mu} + \hat{\Delta}^\mu\nu \hat{\Delta}_{\alpha\beta}) - \frac{1}{3} \hat{\Delta}^{\mu\nu} \hat{\Delta}_{\alpha\beta} \right] T_{\alpha\beta}$$

$$= \pi^{\mu\nu} + \gamma' U^\mu U^\nu + \Sigma \hat{\Delta}^{\mu\nu}$$

$$+ h X^\mu X^\nu + 2 X^\mu Y^\nu + 2 U^\mu Z^\nu,$$  (14b)

respectively, where

$$X^\mu \equiv \Delta_{\alpha\mu}(U) u^\alpha = \delta u^\mu + \gamma U^\mu,$$  (15a)

$$Y^\mu \equiv \Delta_{\alpha\mu}(U) W^\alpha = \pi_{\mu\nu} \delta u^\nu + \gamma' U^\mu,$$  (15b)

$$Z^\mu \equiv \pi_{\mu\nu} \delta u^\nu,$$  (15c)

which are vectors that can be constructed from $\delta u^\mu$, $W^\mu$ and $\pi^{\mu\nu}$. One can write then that

$$\hat{\epsilon} = \epsilon_{eq} + \Lambda + 3 \Sigma, \quad \hat{P} = \epsilon_{eq} + \Pi + \Sigma.$$  (16a)

Here we approach the essence of our model. In the case when $\hat{\epsilon} = \epsilon_{eq}(x)$ and $\hat{P} = \epsilon_{eq}(x)$ one has that

$$\Lambda = -3 \Sigma,$$  (17a)

$$\Pi = -\Sigma.$$  (17b)

Assuming now that

$$\hat{\dot{W}}^\mu = 0,$$  (17c)

$$\hat{\pi}^{\mu\nu} = 0,$$  (17d)

one can link $T^\mu\nu$ to its equilibrium correspondence $T_{eq}^\mu\nu$. Physically, when the LRC is switched off the flow velocity $U^\mu$ (and also the related thermodynamical quantities) can not change into corresponding $U^\mu$ of the RFF instantaneously. The flow vector field $u^\mu(x)$ approaches its equilibrium limit $U^\mu(x)$ in a finite proper time. One may regard such change as a kind of relaxation process. In the next Section IV we shall derive the corresponding evolution of $\delta u^\mu$.

Conditions Eqs. (17) can be also mathematically expressed by the following transformation:

$$\mathcal{M}\left[ T^\mu\nu(u^\mu) \right] = T_{eq}^\mu\nu(U^\mu).$$  (18)

Assumption (18) is natural because dissipative fluid must correspond to some ideal fluid, which can be obtained by switching off the LRC. It is our basic assumption deciding on the form of the energy flow $W^\mu$ and shear viscosity $\pi^{\mu\nu}$, as will be shown below.

D. The model energy-momentum tensor

The assumption discussed in the previous Section concerning the energy-momentum tensor $T^\mu\nu$ determines also the complete form not only of $\Pi$ but also both of $W^\mu$ and $\pi^{\mu\nu}$. From Eqs. (17a) and (17b), we obtain that

$$\Lambda = 3 \Pi,$$  (19)

i.e., that

$$\epsilon = \epsilon_{eq}(T, \mu) + 3 \Pi,$$  (20a)

$$P = P_{eq}(T, \mu) + \Pi.$$  (20b)

Proceeding further, conditions $\hat{\dot{W}}^\mu \equiv 0$ and $\hat{\pi}^{\mu\nu} \equiv 0$ determine the explicit form of the energy flow vector and the shear tensor in the energy-momentum tensor in Eq. (1.11). To give the $W^\mu$ and $\pi^{\mu\nu}$ correct physical meaning of the, respectively, energy flow vector and shear viscous tensor, we require that $W^\mu u_\mu = 0$ (or $W^\mu \delta u_\mu = \gamma'$) and that $\pi^{\mu\nu} u_\nu = 0$ (or $\pi^{\mu\nu} \delta u_\mu = Z''$). The later condition can fix the form of the four vector $Z'' = -(1 - \gamma) \Sigma + \gamma' X^\mu - \gamma'' U^\mu$. The former condition $W^\mu u_\mu = 0$ gives a relation between $\Sigma$ and $\gamma'$. From
the definition \( \gamma'' \equiv Z^\mu \delta u_\mu \), eliminating \( Z^\mu \) and \( \gamma' \), we can also find relation between \( \gamma'' \) and \( \Sigma \). When these expressions for \( \gamma' \) and \( \gamma'' \) are put into Eq. (22), then one finds that

\[
\Sigma = \frac{\gamma(2 - \gamma)}{(2\gamma - 1)(2\gamma - 3)} h. \tag{21}
\]

Eliminating \( \gamma', \gamma'' \) and \( Z^\mu \) from the equation \( \dot{W}^\mu = 0 \), one can obtain expression for \( W^\mu \) in terms of \( U^\mu \) and \( \delta u^\mu \). This result for \( W^\mu \) put into the definition of \( Y^\mu \) leads to \( Y^\mu = -(h + \Sigma)X^\mu \). Substitution \( X^\mu, Y^\mu, Z^\mu \) and \( \gamma'' \) into the equation \( \dot{\gamma} = \mu \), one gets expression for \( \pi^{\mu\nu} \). Finally, one finds that the energy flow vector and the shear viscous tensor have the following form:

\[
W^\mu = -h_1 \delta u^\mu - [\gamma h_4 - 3\Sigma] U^\mu, \tag{22a}
\]

\[
\pi^{\mu\nu} = \gamma^2 h_4 U^\mu U^\nu - \Sigma \Delta^{\mu\nu}(U) + h_2 \delta u^\mu \delta u^\nu - 2 [\Sigma - \gamma h_4] \delta (u^\mu U^\nu), \tag{22b}
\]

where \( h_n = 1, 2, 4 \equiv h + n\Sigma \). Since \( h = \varepsilon + P \), one obtains that \( h = h_{eq} + 4\Pi \) (see Eq. (Eq. (22a)) and (22b)), where \( h_{eq} \equiv \varepsilon + P_{eq} \). Using this relation and Eq. (21), one can write \( \Pi = -\Sigma \) as function of \( \gamma \) and of equilibrium enthalpy \( h_{eq} \) as \[ \text{[58]} \]

\[
\Pi = -\frac{1}{3} \gamma(2 - \gamma) h_{eq}(T, \mu). \tag{23}
\]

By using Eq. (23), one can rewrite Eqs. (22a) and (22b) in more compact form, such as

\[
W^\mu = h_{eq}(1 - \gamma) \varphi^\mu, \tag{24a}
\]

\[
\pi^{\mu\nu} = h_{eq}\varphi^\mu \varphi^\nu + \Pi \Delta^{\mu\nu}(u) = h_{eq} \delta u^{[\mu} \delta u^{\nu]}, \tag{24b}
\]

where

\[
\varphi^\mu \equiv \gamma U^\mu - (1 - \gamma) \delta u^\mu. \tag{25}
\]

Hence, our model energy-momentum tensor is finally given by

\[
T^{\mu\nu} = [\varepsilon_{eq} + 3I]u^\mu u^\nu - [P_{eq} + \Pi] \Delta^{\mu\nu}(u) + 2h_{eq} [1 - \gamma] \varphi^{[\mu} u^{\nu]} + h_{eq} \delta u^{[\mu} \delta u^{\nu]}. \tag{26}
\]

Since reference fields \( (\varepsilon_{eq}, n_{eq}, P_{eq}, U^\mu) \) are assumed to be known, the number of unknowns in this model energy-momentum tensor, Eq. (26), is only three (let us stress here that this is because of the Eq. (18)).

It should be noted that expressions Eqs. (24a) and (24b), one finds tensor relations

\[
W^\mu W_\mu = -3I (1 - \gamma)^2 h_{eq}, \tag{27a}
\]

\[
\pi^{\mu\nu} W_\mu = -2I P_{eq} W^\mu, \tag{27b}
\]

\[
\pi^{\mu\nu} \pi_{\mu\nu} = 6I^2, \tag{27c}
\]

which are same as found in the Non-extensive/dissipative correspondence \[ \text{[47]} \) providing

\[
h_{eq}(1 - \gamma)^2 \iff w_q(1 + \gamma)^2, \tag{27d}
\]

where \( \gamma_q \) denotes the \( \gamma \) that appear in \[ \text{[47]} \).
Let us estimate $\tau_{\Omega}$ roughly, starting, for simplicity, from $1+1$ dimensional Bjorken scaling solution. The enthalpy evolution is then given by the $h_{\text{eq}}(\tau) \sim (q_{0}/\tau)^{4/3}$, where $q$ is some proper time and $q_0$ is initial proper time needed to equilibrate the fluid without any long-range correlations. In this simple case, one finds that $\tau_{\Omega}$ results in $\tau_{\Omega} = \frac{4}{3} \tau$. It means that, if the fluid at the initial stage is slightly different from the local equilibrium, it is difficult to reach its thermalization as proper time passes.

In the more realistic $1+3$ dimensional case, since the time derivative of the enthalpy $[h_{\text{eq}}]$ in Eq. (32) must be much larger than this simple estimation, the $\tau_{\Omega}$ should be smaller than the previous estimation. However, in this case the freeze out time, $\tau_{\Omega}$, is also shorter than in the $1+1$ dimensional case. Therefore, whether the stationary state (which is assumed to be created at the initial stage) is transformed in the following way: 

$$v_{\parallel} = \frac{\vec{v}_{\parallel} + \vec{U}}{1 + \vec{U} \cdot \vec{v}_{\parallel}}, \quad v_{\perp} = \frac{\vec{v}_{\perp}}{\gamma_{U}[1 + \vec{U} \cdot \vec{v}_{\parallel}]}.$$ (35)

Here, $v_{\parallel} \equiv v_{\parallel} \vec{e}_{\parallel}$ and $v_{\perp} \equiv v_{\perp} \vec{e}_{\perp}$ are, respectively, the parallel and perpendicular components of $v$ with respect to $\vec{U}$ whereas $\vec{e}_{\parallel}$, $\vec{e}_{\perp}$ are their unit vectors:

$$v_{\parallel} = \frac{\gamma_{U}(\vec{v} \cdot \vec{U})}{\sqrt{\gamma_{U}^2 - 1}}, \quad v_{\perp} = |\vec{v}| \sqrt{1 - |\vec{e}_v \cdot \vec{e}_{\parallel}|^2},$$ (36a)

and

$$\vec{e}_{\parallel} = \frac{\gamma_{U}}{\sqrt{\gamma_{U}^2 - 1}} \vec{U}, \quad \vec{e}_{\perp} = \frac{\vec{e}_{v} - |\vec{e}_v \cdot \vec{e}_{\parallel}| \vec{e}_{\parallel}}{\sqrt{1 - |\vec{e}_v \cdot \vec{e}_{\parallel}|^2}}.$$ (36b)

Notice that the velocity field $u^{\mu}(x)$ and, accordingly, also $\delta u^{\mu}(x)$ are determined by the reference velocity field $U^{\mu}(x)$ once we decide, by choosing $v(x)$, on the type of the long-range correlations. As a possible example of the field $v(x)$, let us consider

$$v(x) = I(x) \exp \left[ -\frac{|\vec{x}|^2}{L(x)^2} \right],$$ (37)

where $I(x)$ and $L(x)$ are, respectively, the local correlation intensity and correlation length. In what follows we shall assume that only $I(x)$ varies with the expanding fluid whereas the $L$ is maintained as a global constant over the whole fluid evolution. In other words, we assume the intensity $I(x)$ is determined by the dissipative fluid dynamics whereas the correlation length $L$ is given by the particles composing the fluid. This assumption fits nicely the ridge phenomenon found at RHIC discussed in Section I, which indicates the persistency of the long-range correlations during the fluid evolution.

To summarize this part, in our model its main quantity, $\delta u^{\mu}$, is determined by only one equation Eq. (30a) (or equivalently Eq. (31)) and can be fixed by just one variable, the correlation intensity $I(x)$. It means therefore that correlation intensity itself is controlled by these equations and that it could be identified with a kind of “relaxation” to the equilibrium state taking place in the presence of LRC.

C. Charge conservation

Once $\delta u^{\mu}$ is fixed, $\Pi$ and $\Lambda$ are given by Eq. (19) and (23), respectively. As for $\delta n$, it is determined by the field $u^{\mu}(x)$ can be written as

$$u^{\mu}(x) = [\gamma_{u}, \vec{u}] + \vec{u}_{\perp}$$ (34)

with $\gamma_{u} \equiv u^{0} = \sqrt{1 - |\vec{u}_{\parallel} + \vec{u}_{\perp}|^2}$ and

$$\vec{u}_{\parallel} = \frac{\vec{v}_{\parallel} + \vec{U}}{1 + \vec{U} \cdot \vec{v}_{\parallel}}, \quad \vec{u}_{\perp} = \frac{\vec{v}_{\perp}}{\gamma_{U}[1 + \vec{U} \cdot \vec{v}_{\parallel}]}.$$ (35)
conservation of the (baryonic) charge. For the conserved charge current \( N^\mu \), we can write explicitly that
\[
[(n_{\text{eq}} + \delta n)(U^\mu + \delta u^\mu)]_{\mu} = 0, \quad (38a)
\]
\[
[n_{\text{eq}}U^\mu]_{\mu} = 0. \quad (38b)
\]
From the above two equations it is possible to write
\[
\delta N^\mu_{\mu} = 0, \quad (39)
\]
where \( \delta N^\mu \equiv N^\mu - N_{\text{eq}}^\mu \). Hence, the correction of the (baryonic) charge density in the presence of LRC is given by Eq. (39).

V. OFF-EQUILIBRIUM ENTROPY CURRENT

For the equilibrium thermodynamical quantities one has following fundamental thermodynamic relation,
\[
T s_{\text{eq}} = \varepsilon_{\text{eq}} + P_{\text{eq}} - \mu_{\text{eq}}, \quad (40)
\]
which, using Eq. (7) and (19), one can be rewritten as
\[
s_{\text{eq}}(T, \mu) = \frac{\varepsilon + P - \mu n}{T} - \left[ \frac{4\Pi - \mu\delta n}{T} \right]. \quad (41)
\]
Notice that first term at the r.h.s. of Eq. (41) can be regarded as a natural extension of the equilibrium entropy density. We assume therefore that
\[
s \equiv \frac{\varepsilon + P - \mu n}{T}, \quad (42a)
\]
and, accordingly, that
\[
\delta s = \frac{4\Pi - \mu\delta n}{T}. \quad (42b)
\]
Although there is no principle to determine the explicit form of the entropy flux \( \Phi^\mu \), a possible natural and simple candidate is
\[
\Phi^\mu = \frac{W^\mu}{T} = \frac{h_{\text{eq}}}{T}(1 - \gamma)\varphi^\mu. \quad (43)
\]
Using it the explicit form of the off-equilibrium entropy current is given by
\[
S^\mu = (s_{\text{eq}} + \delta s)\nu^\mu + \Phi^\mu = (1 - \gamma)[s_{\text{eq}}U^\mu] - \alpha[\delta N^\mu] - \beta \mu^\mu, \quad (44)
\]
where \( \alpha \equiv \mu/T \) and \( \beta \equiv \nu^\mu \) with \( \beta \equiv 1/T \). It should be noted here that the first three terms of Eq. (44) are quantities of the first order in \( \gamma \), whereas the last term contains terms \( \gamma^2 \) (see Eq. (24) and recall that \( \gamma = -U^\mu \delta u^\mu \) is the first order infinitesimal quantity correction induced by the LRC). It means that our model, in spite of applying simple form of the entropy flux \( \Phi^\mu \), introduces in a natural way the 2nd order terms, both in the energy-momentum tensor, \( T^\mu{}^\nu \), and in the entropy current, \( S^\mu \). It is interesting to notice that \( \gamma \) can be related to a derivative of the fluid velocity field \( u^\mu \),
\[
\delta u^\mu \sim u^\mu \lambda', \quad (45)
\]
where \( \lambda'(x) \) is a kind of the local space-time scale. It connects \( \delta u^\mu \), which is crucial and most important parameter in our model, with the derivatives of the flow vector \( u^\mu \).

The second law of thermodynamics, \( S^\mu_{\mu} \geq 0 \), can be expressed in our model as;
\[
[\Pi \beta^\mu]_{\mu} \geq (\partial_{\mu} \gamma) h_{\text{eq}} \beta + (\partial_{\mu} \alpha)[\gamma N_{\text{eq}}^\mu + \delta N^\mu], \quad (46)
\]
where current conservations, i.e., the second and third equations of Eqs. (10) and Eq. (39) are used. In the baryon free limit, one can also express the second law of thermodynamics as an inequality of two kinds of time-like derivatives. Denoting \( \frac{\delta \lambda}{\delta \tau} \equiv u^\mu X_{\mu} \) and \( \theta \equiv u^\mu \), one may write
\[
\frac{d\Pi}{d\tau} + \frac{\Pi}{\tau_{\Pi}} \geq \frac{3}{2} \frac{1 - \gamma}{1 - \gamma} \left[ \Pi + \frac{\Pi}{\tau_{\Pi}} \right], \quad (47)
\]
where \( 1/\tau_{\Pi} \equiv \theta - \frac{1}{2} \frac{d\lambda}{d\tau} \). The l.h.s. of Eq. (47) contains comoving time-like derivative \( d/d\tau \) with respect to the flow \( u^\mu \) whereas the r.h.s. contains comoving time-like derivative with respect to the flow \( U^\mu \) (denoted by dot). Note the minus sign in the Eq. (47) which implies that in the case of weak LRC (i.e., when \( |\gamma| < 1 \), the inequality (second law of the thermodynamics) holds providing that
\[
\bar{\Pi} + \frac{\Pi}{\tau_{\Pi}} \geq 0 \quad \text{and} \quad \frac{d\Pi}{d\tau} + \frac{\Pi}{\tau_{\Pi}} \geq 0. \quad (48)
\]
This requirement, which is the consequence of our model presented here, restricts the relaxation process, namely the bulk pressure \( \Pi \) should decrease with proper time \( \tau \) slower than \( \exp[-\tau/\tau_{\Pi}] \) in the rest frame of \( u^\mu \) slower than \( \exp[-\tau/\tau_{\Pi}] \) in the rest frame of \( U^\mu \).

VI. SUMMARY AND CONCLUDING REMARK

We have discussed dissipative effects appearing by introducing the long-range correlations (LRC) between particles composing a perfect fluid. They arise because the perfectness of the fluid is broken in such case, therefore such fluid obeys in general equations of imperfect hydrodynamics. In particular, to clarify the effects caused by the LRC, we have considered a thought experiment in which LRC can be switch ‘on’ and ‘off’. It is assumed that the complete set of solutions for the perfect fluid (called RFF) obtained when LRC are not present are known. Then, the differences from the RFF, in velocity field, \( \delta u^\mu \), in pressure, bulk pressure \( \Pi \), and in energy density, \( \Lambda \), can be regarded as resulting...
entirely from dissipative effects caused by the LRC.

By switching off the LRC, the imperfect fluid should relax into a perfect fluid. Such a relaxation process can be regarded as a transformation of the energy-momentum tensor. It is assumed to be given by Eq. (18) and we have shown that all dissipative terms in Eq. (2), such as $H$, $W^\mu$ and $\pi^\mu$, are scaled by $\delta u^\mu$. Moreover, one of dissipative equations Eq. (15) can be expressed by Eq. (31) (or, equivalently, by Eq. (30a)).

It is noteworthy to observe that tensor relations Eqs. (27) obtained in the presence of LRC, are the same as those found in the dissipative fluid corresponding to the non-extensive perfect fluid ($q$-fluid) discussed in Ref. [17].

When the LRC are spatially isotropic (in the rest frame of $U^\mu$) with a constant correlation length $L$, we have shown that the local correlation intensity $I(x)$ can be determined by Eq. (31) which is equivalent to dissipative hydrodynamical equation Eq. (15a). Interestingly, we found that it is given by the form of constitutive-like equation. Moreover, it can also be presented in our model as expression for the second law of the thermodynamics, Eq. (17).

The check of the causality and stability of solutions obtained in our model as well as numerical solutions are beyond the scope of this brief paper and will be discussed elsewhere.

We close with observation that our model can be applied to a fluid with LRC originated from gravitational interaction. The perfect hydrodynamics applied for the universe in the early stage should be, for example, modified by such effects. However, this problem should be considered using General Relativistic (GR) hydrodynamics, as proposed in Refs. [51–53]. The GR hydrodynamics includes long-range interactions caused by gravity in a natural way. If LRC originated by the gravity also cause dissipative effects in a perfect fluid, then this may drive us to the interesting question whether GR hydrodynamics have certain correspondence in the (special) relativistic dissipative hydrodynamics.

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[54] Ideal hydrodynamical models (without fluctuations in the initial condition) [12–15] seem to work for minimum bias data on elliptic flow parameter $v_2$ [16] but not for data on its centrality dependence observed in Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV [22]. They can also reproduce the mass ordering and magnitude of $v_2$ for the different particles in the region up to 2 GeV/c, but they fail to reproduce then for $p_T > 2$ GeV/c [23, 24].
[55] The covariant derivative is defined by the Christoffel symbol $\Gamma^\nu_{\lambda\mu} \equiv \frac{1}{2}g^{\nu\sigma}(\partial_{\mu}g_{\sigma\lambda} + \partial_{\lambda}g_{\sigma\mu} - \partial_{\sigma}g_{\lambda\mu}).$ Hence, we have $T^\mu_\nu = \partial^\mu T_\nu + \Gamma^\mu_{\nu\sigma} T_{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T_{\sigma\nu} + N^\nu_\mu \equiv \partial^\mu N_\nu + \Gamma^\nu_{\lambda\mu} N^\lambda.$
[56] These limitations does not mean that in general $W^\nu = 0$ and $\pi^{\mu\nu} = 0.$
[57] The factor 3 in the r.h.s. in Eq. (19) comes from the number of spatial dimensions.
[58] Note that we have following relation between $\gamma(x)$ and $\Lambda(x)$: $\gamma(x) \equiv 1 - \sqrt{1 + \frac{\Lambda(x)}{\Lambda_{eq}(x)}}.$ The sign of $\gamma(x)$ depends on the sign of $\Lambda(x).$