EXISTENCE RESULTS FOR SOME NONAUTONOMOUS INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper we make a subtle use of tools from operator theory and the Schauder fixed-point theorem to establish the existence of pseudo-almost automorphic solutions to some classes of nonautonomous integro-differential equations with pseudo-almost automorphic forcing terms. To illustrate our main results, the existence of pseudo-almost automorphic solutions to a parabolic Neumann boundary value problem that models population genetics and nerve pulse propagation will be discussed.

1. Introduction

Integro-differential equations play a crucial role in qualitative theory of differential equations due to their applications to natural phenomena, see, e.g., [26, 27, 42, 41, 44, 45, 46]. Much work has been done in recent years to investigate the existence of periodic, almost periodic, almost automorphic, pseudo-almost periodic, and pseudo-almost automorphic solutions to integro-differential equations, see, e.g., [4, 15, 20, 21, 22, 28, 29, 30, 35, 36, 44, 45, 46]. The existence of solutions to autonomous integro-differential equations in the above-mentioned function spaces is, to some extent, relatively well-understood. The method most widely used to deal with the existence of solutions to those integro-differential equations is the so-called ‘method of resolvents’, see, e.g., [3, 12, 13, 29, 30, 35, 36].

Fix \( \alpha \in (0, 1) \). The purpose of this paper consists of making use of a new approach to study the existence of pseudo-almost automorphic solutions to the class of non-autonomous Volterra integro-differential equations given by

\[
\frac{d\varphi}{dt} = A(t)\varphi + \int_{-\infty}^{t} C(t-s)\varphi(s)ds + f(t, \varphi),
\]

where \( A(t) : D \subset X \mapsto X \) is a family of closed unbounded linear operators on a Banach space \( X \) whose domains \( D(A(t)) = D \) are constant in \( t \in \mathbb{R}, C(t) : D \subset X \mapsto X \) are (possibly unbounded) linear operators upon \( X \), and the function \( f : \mathbb{R} \times X_\alpha \mapsto X \) is pseudo-almost automorphic in the first variable uniformly in the second one with \( X_\alpha \) being the real interpolation space of order \((\alpha, \infty)\) between \( X \) and \( D \).

These types of equations arise very often in the study of natural phenomena in which a certain memory effect is taken into consideration [26, 27, 42]. In [41], for instance, equations of type Eq. (1.1) appeared in the study of heat conduction in materials with memory. The existence, uniqueness, maximal regularity, and asymptotic behavior of solutions to Eq. (1.1) have widely been studied, see, e.g., [41, 6, 8, 9, 25, 34, 38, 39, 40]. However, to the best of our knowledge, the existence of pseudo-almost automorphic solutions to Eq. (1.1) is an untreated original problem with important applications, which constitutes the main motivation of our study.

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In order to investigate the existence of pseudo-almost automorphic solutions to Eq. (1.1), we study the non-autonomous abstract differential equations involving the time-dependent linear operators \( A(t) \) and the function \( B(\cdot) : \mathbb{R} \rightarrow L(C(\mathbb{R}, D), \mathbb{X}) \), that is,

\[
\frac{d\phi}{dt} = A(t)\phi + B(t)\phi + f(t, \phi),
\]

where \( A(t) : D \subset \mathbb{X} \rightarrow \mathbb{X} \) is a family of closed linear operators on \( \mathbb{X} \) with constant domains \( D \) and the linear operators \( B(t) : C(\mathbb{R}, D) \rightarrow \mathbb{X} \) defined by

\[
B(t)\phi := \int_{-\infty}^{t} C(t-s)\phi(s)ds, \quad \phi \in C(\mathbb{R}, D)
\]

with \( C(\mathbb{R}, D) \) being the collection of all continuous functions from \( \mathbb{R} \) into \( D \).

In order to study the existence of solutions to Eq. (1.2), we will make extensive use of exponential dichotomy tools, real interpolation spaces, and suppose that for all \( \phi \in PAA(\mathbb{X}) \), the function \( t \rightarrow B(t)\phi \) belongs to \( PAA(\mathbb{X}) \). Our existence result will then be obtained through the use of the well-known Schauder fixed-point theorem. Obviously, once we establish the sought existence results for Eq. (1.1), then we can easily go back to Eq. (1.2) notably through Eq. (1.3).

The notion of pseudo-almost automorphy is a powerful concept that has been introduced and studied in a series of recent papers by Liang et al. [11, 14, 15]. Such a concept has recently generated several developments and these and related topics have recently been summarized in the new book by Diagana [15].

Some recent contributions on almost periodic and asymptotically almost periodic solutions to integro-differential equations of the form Eq. (1.1) have recently been made in [22, 31] in the case \( A(t) = A \) is constant. Similarly, in [14], the existence of pseudo-almost automorphic solutions to an autonomous version of Eq. (1.1) was studied. The main method used in the above-mentioned papers are resolvents operators. However, to the best of our knowledge, the existence of pseudo-almost automorphic solutions to Eq. (1.1) (and hence to Eq. (1.2)) in the case when \( A(t) \) are sectorial linear operators is an original untreated topic with some interesting applications to the real world problems. Among other things, we will make extensive use of the so-called Acquista-Terreni conditions method associated with sectorial operators \( A(t) \) and the Schauder fixed point to derive some sufficient conditions for the existence of pseudo-almost automorphic (mild) solutions to (1.2) and then to Eq. (1.1). To illustrate our main results, the existence of pseudo-almost automorphic solutions to a parabolic Neumann boundary value problem that models population genetics [45, 47] and nerve pulse propagation [14] will be discussed.

2. Preliminaries

The basic results discussed in this section are mainly taken from the following recent papers by the author [17] and [19].

In this paper, \( (\mathbb{X}, \|\cdot\|) \) denotes a Banach space. If \( A \) is a linear operator upon \( \mathbb{X} \), then the notations \( D(A), \rho(A), \sigma(A), \text{ Adj}(A), \text{ and } R(A) \) stand respectively for the domain, resolvent, spectrum, kernel, and the range of \( A \). Similarly, if \( A : D = D(A) \subset \mathbb{X} \rightarrow \mathbb{X} \) is a closed linear operator, one denotes its graph norm by \( \|\cdot\|_D \) defined by \( \|\phi\|_D := \|\phi\| + \|A\phi\| \) for all \( \phi \in D \). From the closedness of \( A \), one can easily see that \( (D, \|\cdot\|_D) \) is a Banach space. Moreover, one sets \( R(A, L) := (\lambda I - L)^{-1} \) for all \( \lambda \in \rho(A) \). Furthermore, we set \( Q(t) = I - P(t) \) for projections \( P(t) \). If \( \mathbb{Y}, \mathbb{Z} \) are Banach spaces, then the space \( B(\mathbb{Y}, \mathbb{Z}) \) denotes the collection of all bounded linear operators from \( \mathbb{Y} \) into \( \mathbb{Z} \) equipped with its natural uniform operator topology \( \|\cdot\|_{B(\mathbb{Y}, \mathbb{Z})} \). We also set \( B(\mathbb{Y}) = (\mathbb{Y}, \mathbb{Y}) \) whose corresponding norm will be denoted
\[ \parallel \cdot \parallel. \] If \( K \subset \mathbb{K} \) is a subset, we let \( \partial K \) denote the closed convex hull of \( K \). Additionally, \( \mathbb{T} \) will denote the set defined by, \( \mathbb{T} := \{(t, s) \in \mathbb{R} \times \mathbb{R} : t \geq s\} \).

2.1. Evolution Families.

**Definition 2.2.** A family of closed linear operators \( A(t) \) for \( t \in \mathbb{R} \) on \( \mathbb{K} \) with domains \( \mathcal{D}(A(t)) \) (possibly not densely defined) is said to satisfy the so-called Acquistapace–Terreni conditions, if there exist constants \( \omega \in \mathbb{R}, \theta \in (\frac{\pi}{2}, \pi) \), \( K, L \geq 0 \) and \( \mu, \nu \in (0, 1) \) with \( \mu + \nu > 1 \) such that

\[
S_\theta \cup \{0\} \subset \rho(A(t) - \omega I), \quad \|R(\lambda, A(t) - \omega I)\| \leq \frac{K}{1 + |\lambda|}, \quad \text{and}
\]

\[
\|(A(t) - \omega I)R(\lambda, A(t) - \omega I)\| \leq L |t - s|^{\mu} |\lambda|^{-\nu}
\]

for \( t, s \in \mathbb{R}, \lambda \in S_\theta \), where

\[
S_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}.
\]

Among other things, the Acquistapace–Terreni conditions do ensure the existence of a unique evolution family

\[
\mathcal{U} = \{U(t, s) : t, s \in \mathbb{R} \text{ such that } t \geq s\}
\]
on \( \mathbb{K} \) associated with \( A(t) \) such that \( U(t, s)\mathbb{K} \subset \mathcal{D}(A(t)) \) for all \( t, s \in \mathbb{R} \) with \( t > s \), and

(a) \( U(t, s)U(s, r) = U(t, r) \) for \( t, s, r \in \mathbb{R} \) such that \( t \geq s \geq r \);

(b) \( U(t, t) = I \) for \( t \in \mathbb{R} \) where \( I \) is the identity operator of \( \mathbb{K} \); and

(c) for \( t > s \), the mapping \( (t, s) \mapsto U(t, s) \in \mathcal{B}(\mathbb{K}) \) is continuous and continuously differentiable in \( t \) with \( \partial_t U(t, s) = A(t)U(t, s) \). Moreover, there exists a constant \( C' > 0 \) which depends on constants in Eq. (2.1) and Eq. (2.2) such that

\[
\|A^k(t)U(t, s)\| \leq C'(t - s)^{-k}
\]

for \( 0 < t - s \leq 1 \) and \( k = 0, 1 \).

**Definition 2.2.** An evolution family \( \mathcal{U} = \{U(t, s) : (t, s) \in \mathbb{T}\} \) is said to have an exponential dichotomy if there are projections \( P(t) (t \in \mathbb{R}) \) that are uniformly bounded and strongly continuous in \( t \) and constants \( \delta > 0 \) and \( N \geq 1 \) such that

(d) \( U(t, s)P(s) = P(t)U(t, s) \);

(e) the restriction \( U_Q(t, s) : Q(s)\mathbb{K} \rightarrow Q(t)\mathbb{K} \) of \( U(t, s) \) is invertible (we then set \( U_Q(t, s) := U_Q(t, s)^{-1} \) where \( Q(t) = I - P(t) \)); and

(f) \( \|U(t, s)P(s)\| \leq Ne^{-\delta(t-s)} \) and \( \|U_Q(t, s)Q(t)\| \leq Ne^{-\delta(t-s)} \) for \( t \geq s \) and \( t, s \in \mathbb{R} \).

If an evolution family \( \mathcal{U} = \{U(t, s) : (t, s) \in \mathbb{T}\} \) has an exponential dichotomy, we then define

\[
\Gamma(t, s) := \begin{cases} 
U(t, s)P(s), & \text{if } t \geq s, \ t, s \in \mathbb{R}, \\
-U_Q(t, s)Q(s), & \text{if } s > t, \ t, s \in \mathbb{R}.
\end{cases}
\]

2.2. Estimates for \( U(t, s) \). This setting requires some estimates related to \( U(t, s) \). For that, we make extensive use of the real interpolation spaces of order \( (\alpha, \infty) \) between \( \mathbb{K} \) and \( \mathcal{D}(A(t)) \), where \( \alpha \in (0, 1) \). We refer the reader to Amann [3] and Lunardi [37] for proofs and further information on these interpolation spaces.

Let \( A \) be a sectorial operator on \( \mathbb{K} \) (for that, in Definition 2.1 replace \( A(t) \) with \( A \)) and let \( \alpha \in (0, 1) \). Define the real interpolation space

\[
\mathbb{K}_\alpha := \{x \in \mathbb{K} : \|x\|_{\alpha} := \sup_{r>0} \|r^\alpha (A - \omega)R(r, A - \omega)x\| < \infty\},
\]
which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_0^\alpha$. For convenience we further write

$$\mathcal{X}_0^\alpha := \mathcal{X}, \quad \|x\|_0^\alpha := \|x\|, \quad \mathcal{X}_1^\alpha := D(A)$$

and

$$\|x\|_1^\alpha := \|(\omega - A)x\|.$$ 

Moreover, let $\mathcal{X}^\alpha := D(A)$ of $\mathcal{X}$. In particular, we have the following continuous embedding

\begin{equation}
(2.4) \quad D(A) \hookrightarrow \mathcal{X}_0^\alpha \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathcal{X}_1^\alpha \hookrightarrow \mathcal{X},
\end{equation}

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathcal{X}_0^\alpha$ and $\mathcal{X}$. However, we have the following continuous injection

$$\mathcal{X}_0^\alpha \hookrightarrow D(A)$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying Acquistapace–Terreni conditions, we set

$$\mathcal{X}_\alpha^\prime := \mathcal{X}^{A(t)}, \quad \hat{\mathcal{X}}^\prime := \hat{\mathcal{X}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in Eq. (2.4) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{F}_\alpha$ (Definition 1.1.1) and hence there is a constant $c(\alpha)$ such that

$$\|y\|_\alpha^\prime \leq c(\alpha) \|y\|^{1-\alpha} \|A(t)y\|^\alpha, \quad y \in D(A(t)).$$

We have the following estimates for the evolution family $U(t,s)$.

**Proposition 2.3.** [1] Suppose the evolution family $\mathcal{U}$ has exponential dichotomy. For $x \in \mathcal{X}$, $0 \leq \alpha \leq 1$ and $t > s$, the following hold:

(i) There is a constant $c(\alpha)$, such that

\begin{equation}
(2.5) \quad \|U(t,s)P(s)x\|_\alpha^\prime \leq c(\alpha)e^{-\frac{\delta}{6}(t-s)}(t-s)^{-\alpha} \|x\|.
\end{equation}

(ii) There is a constant $m(\alpha)$, such that

\begin{equation}
(2.6) \quad \left\| \tilde{U}_Q(s,t)Q(t)x \right\|_\alpha^\prime \leq m(\alpha)e^{-\delta(t-s)} \|x\|.
\end{equation}

**Remark 2.4.** Note that if an evolution family $\mathcal{U}$ is exponential stable, that is, there exists constants $N, \delta > 0$ such that $\|U(t,s)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$, then its dichotomy projection $P(t) = I$ ($Q(t) = I - P(t) = 0$). In that case, Eq. (2.5) still holds and can be rewritten as follows: for all $x \in \mathcal{X}$,

\begin{equation}
(2.7) \quad \|U(t,s)x\|_\alpha^\prime \leq c(\alpha)e^{-\frac{\delta}{6}(t-s)}(t-s)^{-\alpha} \|x\|.
\end{equation}

**Remark 2.5.** Note that if the evolution family $U(t,s)$ is compact for $t > s$ and is exponential stable, then it can be shown that for each given $t \in \mathbb{R}$ and $\tau > 0$, the family

$$\left\{ U(\cdot, s) : s \in (-\infty, t - \tau) \right\}$$

is equi-continuous in $t$ for the uniform operator topology.
2.3. **Pseudo-Almost Automorphic Functions.** Let $BC(\mathbb{R}, \mathcal{X})$ stand for the Banach space of all bounded continuous functions $\varphi : \mathbb{R} \mapsto \mathcal{X}$, which we equip with the sup-norm defined by $\|\varphi\|_{\infty} := \sup_{t \in \mathbb{R}} \|\varphi(t)\|$ for all $\varphi \in BC(\mathbb{R}, \mathcal{X})$. Similarly, letting $\mathcal{X}_\alpha = (\mathcal{X}, D, \alpha)_{\alpha, \infty}$ for $\alpha \in (0, 1)$, then the space $BC(\mathbb{R}, \mathcal{X}_\alpha)$ will stand for the Banach space of all bounded continuous functions $\varphi : \mathbb{R} \mapsto \mathcal{X}_\alpha$, which we equip with the sup norm defined by $\|\varphi\|_{\alpha, \infty} := \sup_{t \in \mathbb{R}} \|\varphi(t)\|_\alpha$ for all $\varphi \in BC(\mathbb{R}, \mathcal{X}_\alpha)$.

**Definition 2.6.** A function $f \in C(\mathbb{R}, \mathcal{X})$ is said to be almost automorphic if for every sequence of real numbers $(s_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that

$$g(t) := \lim_{n \to \infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \to \infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the convergence above is uniform in $t \in \mathbb{R}$, then $f$ is almost periodic in the classical Bochner’s sense. Denote by $AA(\mathcal{X})$ the collection of all almost automorphic functions $\mathbb{R} \mapsto \mathcal{X}$. Note that $AA(\mathcal{X})$ equipped with the sup-norm turns out to be a Banach space.

Among other things, almost automorphic functions satisfy the following properties.

**Theorem 2.7.** If $f, f_1, f_2 \in AA(\mathcal{X})$, then

(i) $f_1 + f_2 \in AA(\mathcal{X})$,

(ii) $\lambda f \in AA(\mathcal{X})$ for any scalar $\lambda$,

(iii) $f_\alpha \in AA(\mathcal{X})$ where $f_\alpha : \mathbb{R} \mapsto \mathcal{X}$ is defined by $f_\alpha(t) = f(t + \alpha)$,

(iv) the range $R_f := \{f(t) : t \in \mathbb{R}\}$ is relatively compact in $\mathcal{X}$, thus $f$ is bounded in norm,

(v) if $f_\alpha \to f$ uniformly on $\mathbb{R}$ where each $f_\alpha \in AA(\mathcal{X})$, then $f \in AA(\mathcal{X})$ too.

**Definition 2.8.** Let $\mathcal{Y}$ be another Banach space. A jointly continuous function $F : \mathbb{R} \times \mathcal{Y} \mapsto \mathcal{X}$ is said to be almost automorphic in $t \in \mathbb{R}$ if $t \mapsto F(t, x)$ is almost automorphic for all $x \in K$ ($K \subset \mathcal{Y}$ being any bounded subset). Equivalently, for every sequence of real numbers $(s_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that

$$G(t, x) := \lim_{n \to \infty} F(t + s_n, x)$$

is well defined in $t \in \mathbb{R}$ and for each $x \in K$, and

$$\lim_{n \to \infty} G(t - s_n, x) = F(t, x)$$

for all $t \in \mathbb{R}$ and $x \in K$.

The collection of such functions will be denoted by $AA(\mathbb{R} \times \mathcal{X})$.

We now introduce the notion of bi-almost automorphy, which in fact is due to Xiao et al. \[8\].

**Definition 2.9.** A jointly continuous function $F : \mathbb{R} \times \mathcal{T} \mapsto \mathcal{X}$ is called positively bi-almost automorphic if for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, we can extract a subsequence $(s_n)_{n \in \mathbb{N}}$ such that

$$G(t, s) := \lim_{n \to \infty} F(t + s_n, s + s_n)$$

is well defined for $(t, s) \in \mathbb{T}$, and

$$\lim_{n \to \infty} G(t - s_n, s - s_n) = F(t, s)$$
for each \((t, s) \in \mathbb{T}\).

The collection of such functions will be denoted \(bAA(\mathbb{T}, \mathcal{X})\).

For more on almost automorphic functions and their generalizations, we refer the reader to the recent book by Diagana [13].

Define \(PAP_0(\mathbb{R}, \mathcal{X})\) as the collection of all functions \(\varphi \in BC(\mathbb{R}, \mathcal{X})\) satisfying,

\[
\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(s)\| ds = 0.
\]

Similarly, \(PAP_0(\mathbb{R} \times \mathcal{X})\) will denote the collection of all bounded continuous functions \(F : \mathbb{R} \times \mathcal{Y} \mapsto \mathcal{X}\) such that

\[
\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|F(s, x)\| ds = 0
\]

uniformly in \(x \in K\), where \(K \subset \mathcal{Y}\) is any bounded subset.

**Definition 2.10.** (Liang et al. [38] and Xiao et al. [47]) A function \(f \in BC(\mathbb{R}, \mathcal{X})\) is called pseudo almost automorphic if it can be expressed as \(f = g + \phi\), where \(g \in AA(\mathcal{X})\) and \(\phi \in PAP_0(\mathcal{X})\). The collection of such functions will be denoted by \(PAA(\mathcal{X})\).

The functions \(g\) and \(\phi\) appearing in Definition 2.10 are respectively called the almost automorphic and the ergodic perturbation components of \(f\).

**Definition 2.11.** Let \(\mathcal{Y}\) be another Banach space. A bounded continuous function \(F : \mathbb{R} \times \mathcal{Y} \mapsto \mathcal{X}\) belongs to \(AA(\mathbb{R} \times \mathcal{X})\) whenever it can be expressed as \(F = G + \Phi\), where \(G \in AA(\mathbb{R} \times \mathcal{X})\) and \(\Phi \in PAP_0(\mathbb{R} \times \mathcal{X})\). The collection of such functions will be denoted by \(PAA(\mathbb{R} \times \mathcal{X})\).

A substantial result is the next theorem, which is due to Xiao et al. [47].

**Theorem 2.12.** [47] The space \(PAA(\mathcal{X})\) equipped with the sup norm \(\| \cdot \|_\infty\) is a Banach space.

**Theorem 2.13.** [47] If \(\mathcal{Y}\) is another Banach space, \(f : \mathbb{R} \times \mathcal{Y} \mapsto \mathcal{X}\) belongs to \(PAA(\mathbb{R} \times \mathcal{X})\) and if \(x \mapsto f(t, x)\) is uniformly continuous on each bounded subset \(K\) of \(\mathcal{Y}\) uniformly in \(t \in \mathbb{R}\), then the function defined by \(h(t) = f(t, \varphi(t))\) belongs to \(PAA(\mathcal{X})\) provided \(\varphi \in PAA(\mathcal{Y})\).

For more on pseudo-almost automorphic functions and their generalizations, we refer the reader to the recent book by Diagana [13].

3. Main Results

Fix \(\alpha \in (0, 1)\). To study the existence of pseudo-almost automorphic mild solutions to Eq. (1.2), we will need the following assumptions,

(H.1) The linear operators \(|A(t)|_{\infty}\) with constant common domains denoted \(D\), satisfy the Acquistapace–Terreni conditions.

Let \(\mathcal{U} = \{U(t, s) : (t, s) \in \mathbb{T}\}\) denote the evolution family associated with the linear operators \(A(t)\).

(H.2) The evolution family \(U(t, s)\) is not only compact for \(t > s\) but also is exponentially stable, i.e., there exists constants \(N, \delta > 0\) such that

\[
\|U(t, s)\| \leq Ne^{-\delta(t-s)}
\]

for \(t \geq s\).

(H.3) The function \(\mathbb{R} \times \mathbb{R} \mapsto \mathcal{Y}_\alpha, (t, s) \mapsto U(t, s)\varphi\), belongs to \(bAA(\mathbb{T}, \mathcal{Y}_\alpha)\) for all \(\varphi \in \mathcal{Y}_\alpha\).
(H.4) The linear operators $B(t) \in B(BC(\mathbb{R}, \mathbb{X}_a), \mathbb{X})$ for all $t \in \mathbb{R}$. Moreover, the following hold,

a) $C_0 := \sup_{t \in \mathbb{R}} \|B(t)\|_{B(BC(\mathbb{R}, \mathbb{X}_a), \mathbb{X})} \leq \frac{1}{2d(\alpha)}$, where $d(\alpha) := c(\alpha)(2\delta^{-1})^{1-\alpha} \Gamma(1 - \alpha)$.

b) For all $\varphi \in PAA(\mathbb{X}_a)$, the function $t \mapsto B(t) \varphi$ belongs to $PAA(\mathbb{X})$.

(H.5) The function $f : \mathbb{R} \times \mathbb{X}_a \mapsto \mathbb{X}$ is pseudo-almost automorphic in the first variable uniformly in the second one. For each bounded subset $K \subset \mathbb{X}_a$, $f(\mathbb{R}, K)$ is bounded. Moreover, the function $u \mapsto f(t, u)$ is uniformly continuous on any bounded subset $K$ of $\mathbb{X}_a$ for each $t \in \mathbb{R}$. Finally, we suppose that there exists $L > 0$ such that

$$\sup_{t \in \mathbb{R}, \|\varphi\|_{L^1}} \|f(t, \varphi)\| \leq \frac{L}{2d(\alpha)}.$$  

(H.6) If $(u_n)_{n \in \mathbb{N}} \subset PAA(\mathbb{X}_a)$ is uniformly bounded and uniformly convergent upon every compact subset of $\mathbb{R}$, then $f(\cdot, u_n(\cdot))$ is relatively compact in $BC(\mathbb{R}, \mathbb{X})$.

Consider the non-autonomous first-order differential equation,

$$\frac{d\varphi}{dt} = A(t)\varphi + g(t), \quad t \in \mathbb{R},$$  

where $g : \mathbb{R} \mapsto \mathbb{X}$ is a bounded continuous function.

**Definition 3.1.** Under assumption (H.1), a continuous function $\varphi : \mathbb{R} \mapsto \mathbb{X}$ is said to be a mild solution to Eq. (3.1) provided that

$$\varphi(t) = U(t, s)\varphi(s) + \int_s^t U(t, \tau)g(\tau)d\tau, \quad \forall(t, s) \in T.$$  

**Lemma 3.2.** Suppose (H.1) holds and that $\mathcal{U} = \{U(t, s) : (t, s) \in T\}$ has exponential dichotomy with constants $N$ and $\delta$. If $g : \mathbb{R} \mapsto \mathbb{X}$ is a bounded continuous function, then $\varphi$ given by

$$\varphi(t) := \int_{-\infty}^{\infty} \Gamma(t, s)g(s)ds$$  

for all $t \in \mathbb{R}$, is the unique bounded mild solution to Eq. (3.1).

**Proof.** The fact that $\varphi$ given in Eq. (3.3) is a bounded mild solution to Eq. (3.1) is clear, see, e.g., [4, Chap. 4]. Now let $u, v$ be two bounded mild solutions to Eq. (3.1). Setting $w = u - v$, one can easily see that $w$ is bounded and that $w(t) = U(t, s)w(s)$ for all $(t, s) \in T$. Now using property (d) from exponential dichotomy (Definition 2.2) it follows that

$$P(t)w(t) = P(t)U(t, s)w(s) = U(t, s)P(s)w(s),$$  

and hence

$$\|P(t)w(t)\| = \|U(t, s)P(s)w(s)\|$$

$$\leq Ne^{-\delta(t-s)}\|w(s)\|$$

$$\leq Ne^{-\delta(t-s)}\|w\|_{L^1}, \quad \forall(t, s) \in T.$$  

Now, given $t \in \mathbb{R}$ with $t \geq s$, if we let $s \to -\infty$, we then obtain that $P(t)w(t) = 0$, that is, $P(t)u(t) = P(t)v(t)$. Since $t$ is arbitrary it follows that $P(t)w(t) = 0$ for all $t \geq s$.

Similarly, from $w(t) = U(t, s)w(s)$ for all $t \geq s$ and property (d) from exponential dichotomy (Definition 2.2) it follows that

$$Q(t)w(t) = Q(t)U(t, s)w(s) = U(t, s)Q(s)w(s),$$  

for all $t \geq s$.
and hence \( U_Q(s, t)Q(t)w(t) = Q(s)w(s) \) for all \( t \geq s \). Moreover,
\[
\|Q(s)w(s)\| = \|U_Q(s, t)Q(t)w(t)\| \\
\leq Ne^{-\alpha(t-s)}\|w\|_{\infty}, \quad \forall (t, s) \in \mathbb{T}.
\]

Now, given \( s \in \mathbb{R} \) with \( t \geq s \), if we let \( t \to \infty \), we then obtain that \( Q(s)w(s) = 0 \), that is, \( Q(s)u(s) = Q(s)v(s) \). Since \( s \) is arbitrary it follows that \( Q(s)w(s) = 0 \) for all \( t \geq s \). The proof is complete.

**Definition 3.3.** Under assumptions (H.1), (H.2), and (H.4) and if \( f : \mathbb{R} \times \mathbb{X}_a \to \mathbb{X} \) is a bounded continuous function, then a continuous function \( \varphi : \mathbb{R} \to \mathbb{X}_a \) satisfying
\[
\varphi(t) = U(t, s)\varphi(s) + \int_s^t U(t, s)[B(s)\varphi(s) + f(s, \varphi(s))]ds, \quad \forall (t, s) \in \mathbb{T}
\]

is called a mild solution to Eq. (3.4).

Under assumptions (H.1), (H.2), and (H.4) and if \( f : \mathbb{R} \times \mathbb{X}_a \to \mathbb{X} \) is a bounded continuous function, it can be shown that the function \( \varphi : \mathbb{R} \to \mathbb{X}_a \) defined by
\[
\varphi(t) = \int_{-\infty}^t U(t, s)[B(s)\varphi(s) + f(s, \varphi(s))]ds
\]
for all \( t \in \mathbb{R} \), is a mild solution to Eq. (3.5).

Define the following integral operator,
\[
(S\varphi)(t) = \int_{-\infty}^t U(t, s)[B(s)\varphi(s) + f(s, \varphi(s))]ds.
\]

We need the next lemma to establish the main results of the paper.

**Lemma 3.4.** Under assumptions (H.1)–(H.2)–(H.4) and if \( f : \mathbb{R} \times \mathbb{X}_a \to \mathbb{X} \) is a bounded continuous function, then the mapping \( S : BC(\mathbb{R}, \mathbb{X}_a) \to BC(\mathbb{R}, \mathbb{X}_a) \) is well-defined and continuous.

**Proof.** We first show that \( S \) is well-defined and that \( S(BC(\mathbb{R}, \mathbb{X}_a)) \subset BC(\mathbb{R}, \mathbb{X}_a) \). Indeed, letting \( u \in BC(\mathbb{R}, \mathbb{X}_a) \), \( g(t) := f(t, u(t)) \), and using Proposition 2.3, we obtain
\[
\|S u(t)\|_{\alpha} \leq \int_{-\infty}^t \|U(t, s)[B(s)u(t) + g(s)]\|_\alpha ds \\
\leq \int_{-\infty}^t c(\alpha)e^{-\frac{\alpha}{\alpha-1}(t-s)}(t-s)^{-\alpha}\|B(s)u(s)\| + \|g(s)\|ds \\
\leq \int_{-\infty}^t c(\alpha)e^{-\frac{\alpha}{\alpha-1}(t-s)}(t-s)^{-\alpha}\|C_0u(s)\|_{\alpha} + \|g(s)\|ds \\
\leq d(\alpha)(C_0\|u\|_{\alpha, \infty} + \|g\|_{\infty}),
\]
for all \( t \in \mathbb{R} \), where \( d(\alpha) = c(\alpha)(2\delta^{-1})^{1-\alpha}\Gamma(1-\alpha) \), and hence \( Su : \mathbb{R} \to \mathbb{X}_a \) is bounded.

To complete the proof it remains to show that \( S \) is continuous. For that, set
\[
F(s, u(s)) := B(s)u(s) + g(s) = B(s)u(s) + f(s, u(s)), \quad \forall s \in \mathbb{R}.
\]

Consider an arbitrary sequence of functions \( u_n \in BC(\mathbb{R}, \mathbb{X}_a) \) that converges uniformly to some \( u \in BC(\mathbb{R}, \mathbb{X}_a) \), that is, \( \|u_n - u\|_{\alpha, \infty} \to 0 \) as \( n \to \infty \).
Now
\[
\|S u(t) - S u_n(t)\|_{\alpha,\infty} = \left\| \int_{-\infty}^{t} U(t, s)[F(s, u_n(s)) - F(s, u(s))] \, ds \right\|_{\alpha,\infty}
\leq c(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{\frac{s}{2} (t-s)} \left\| F(s, u_n(s)) - F(s, u(s)) \right\| \, ds.
\]
\[
\leq c(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{\frac{s}{2} (t-s)} \left\| f(s, u_n(s)) - f(s, u(s)) \right\| \, ds
+ d(\alpha)C_0 \left\| u_n - u \right\|_{\alpha,\infty}.
\]

Using the continuity of the function \( f : \mathbb{R} \times X_\alpha \mapsto X \) and the Lebesgue Dominated Convergence Theorem we conclude that
\[
\left\| \int_{-\infty}^{t} U(t, s)P(s)[f(s, u_n(s)) - f(s, u(s))] \, ds \right\| \to 0 \quad \text{as} \quad n \to \infty.
\]
Therefore, \( \|S u_n - S u\|_{\alpha,\infty} \to 0 \) as \( n \to \infty \). The proof is complete. \( \square \)

**Lemma 3.5.** Under assumptions (H.1)—(H.5), then \( S(PAA(X_\alpha)) \subset PAA(X_\alpha) \).

**Proof.** Let \( u \in PAA(X_\alpha) \) and define \( h(s) := f(s, u(s)) + B(s)u(s) \) for all \( s \in \mathbb{R} \). Using (H.5) and Theorem 2.13 it follows that the function \( s \mapsto f(s, u(s)) \) belongs to \( PAA(X) \). Similarly, from (H.4), the function \( s \mapsto B(s)u(s) \) belongs to \( PAA(X) \). In view of the above, the function \( s \mapsto h(s) \) belongs to \( PAA(X) \). Now write \( h = h_1 + h_2 \in PAA(X) \) where \( h_1 \in AA(X) \) and \( h_2 \in PAP_0(X) \) and set
\[
Rh_j(t) := \int_{-\infty}^{t} U(t, s)h_j(s) \, ds \quad \text{for all} \quad t \in \mathbb{R}, \quad j = 1, 2.
\]

Our first task consists of showing that \( R(AA(X)) \subset AA(X_\alpha) \). Indeed, using the fact that \( h_1 \in AA(X) \), for every sequence of real numbers \( (\tau_n)_{n \in \mathbb{N}} \) there exist a subsequence \( (\tau_n)_{n \in \mathbb{N}} \) and a function \( f_1 \) such that
\[
f_1(t) := \lim_{n \to \infty} h_1(t + \tau_n)
\]
is well defined for each \( t \in \mathbb{R} \), and
\[
\lim_{n \to \infty} f_1(t - \tau_n) = h_1(t)
\]
for each \( t \in \mathbb{R} \).
Now
\[(Rh_1)(t + \tau_n) - (Rf_1)(t) = \int_{-\infty}^{+\tau_n} U(t + \tau_n, s)h_1(s)ds - \int_{-\infty}^{t} U(t, s)f_1(s)ds\]
\[= \int_{-\infty}^{t} U(t + \tau_n, s + \tau_n)h_1(s + \tau_n)ds - \int_{-\infty}^{t} U(t, s)f_1(s)ds.\]
\[= \int_{-\infty}^{t} U(t + \tau_n, s + \tau_n)(h_1(s + \tau_n) - f_1(s))ds + \int_{-\infty}^{t}(U(t + \tau_n, s + \tau_n) - U(t, s))f_1(s)ds.\]

From Proposition 2.3 and the Lebesgue Dominated Convergence Theorem, it easily follows that
\[\|\int_{-\infty}^{t} U(t + \tau_n, s + \tau_n)(h_1(s + \tau_n) - f_1(s))ds\|_{\alpha} \leq \int_{-\infty}^{t} \|U(t + \tau_n, s + \tau_n)(h_1(s + \tau_n) - f_1(s))\|_{\alpha} ds\]
\[\leq c(\alpha) \int_{-\infty}^{t} (t - s)^{-\alpha}e^{-\frac{\delta}{r}(t-s)}\|h_1(s + \tau_n) - f_1(s)\| ds\]
\[\to 0 \text{ as } n \to \infty.\]

Similarly, from (H.3) and the Lebesgue Dominated Convergence Theorem, it follows
\[\|\int_{-\infty}^{t}(U(t + \tau_n, s + \tau_n) - U(t, s))f_1(s)ds\|_{\alpha} \leq \int_{-\infty}^{t} \|U(t + \tau_n, s + \tau_n) - U(t, s)\|_{\alpha} ds\]
\[\to 0 \text{ as } n \to \infty,\]
and hence
\[(Rf_1)(t) = \lim_{n \to \infty} (Rh_1)(t + \tau_n)\]
for all \(t \in \mathbb{R}.

Using similar arguments as above one obtains that
\[(Rh_1)(t) = \lim_{n \to \infty} (Rf_1)(t - \tau_n)\]
for all \(t \in \mathbb{R},\) which yields, \(t \mapsto (Sh_1)(t)\) belongs to \(AA(\mathcal{X}_\alpha).\)

The next step consists of showing that \(R(PAP_0(\mathcal{X})) \subset PAP_0(\mathcal{X}_\alpha).\) Obviously, \(Rh_2 \in BC(\mathbb{R}, \mathcal{X}_\alpha)\) (see Lemma 2.4). Using the fact that \(h_2 \in PAP_0(\mathcal{X})\) and Proposition 2.3 it can be easily shown that \((Rh_2) \in PAP_0(\mathcal{X}_\alpha).\) Indeed, for \(r > 0,\)
\[\frac{1}{2r} \int_{-r}^{r} \|\int_{-\infty}^{t} U(t, s)h_2(s)ds\|_{\alpha} dt \leq \frac{c(\alpha)}{2r} \int_{-r}^{r} \int_{0}^{\infty} e^{\frac{\delta t}{r} s^{-\alpha}} \|h_2(t-s)\| ds dt\]
\[\leq \int_{0}^{\infty} e^{\frac{\delta t}{r} s^{-\alpha}} \left( \frac{1}{2r} \int_{-r}^{r} \|h_2(t-s)\| dt \right) ds.\]

Using the fact that \(PAP_0(\mathcal{X})\) is translation-invariant it follows that
\[\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \|h_2(t-s)\| dt = 0,\]
as \(t \mapsto h_2(t-s) \in PAP_0(\mathcal{X})\) for every \(s \in \mathbb{R}.

One completes the proof by using the Lebesgue Dominated Convergence Theorem. In summary, \((Rh_2) \in PAP_0(\mathcal{X}_\alpha),\) which completes the proof.

\[\square\]

**Theorem 3.6.** Suppose assumptions (H.1)—(H.6) hold, then Eq. (2.2) has at least one pseudo-almost automorphic mild solution.
Proof. Let \( B_a = \{ u \in \text{PAA}(X_{\alpha}) : \|u\|_{a} \leq L \} \). Using the proof of Lemma 3.4 one can easily show that \( B_a \) is convex and closed. Moreover, from Lemma 3.5, one can see that \( S(B_a) \subset \text{PAA}(X_{\alpha}) \). Now for all \( u \in B_a \),

\[
\|S u(t)\|_a \leq \int_{-\infty}^{t} \|U(t, s) [B(s)u(t) + g(s)]\|_a ds \\
\leq \int_{-\infty}^{t} c(\alpha)e^{-\frac{1}{2}(t-s)^{-\alpha}} \|B(s)u(s)\| + \|f(s, u(s))\| ds \\
\leq \int_{-\infty}^{t} c(\alpha)e^{-\frac{1}{2}(t-s)^{-\alpha}} \left[ C_0 \|u(s)\|_{a} + \|f(s, u(s))\| \right] ds \\
\leq d(\alpha) \left( \frac{L}{2d(\alpha)} + \frac{L}{2d(\alpha)} \right) \\
= L
\]

for all \( t \in \mathbb{R} \) and hence \( Su \in B_a \).

To complete the proof, we have to prove the following:

a) That \( V = \{ Su(t) : u \in B_a \} \) is a relatively compact subset of \( X_{\alpha} \) for each \( t \in \mathbb{R} \);

b) That \( W = \{ Su : u \in B_a \} \subset \text{PAA}(X_{\alpha}) \) is equi-continuous.

To show a), fix \( t \in \mathbb{R} \) and consider an arbitrary \( \varepsilon > 0 \).

Now

\[
(S_{\varepsilon} u)(t) := \int_{-\infty}^{t-\varepsilon} U(t, s) F(s, u(s)) ds, \ u \in B_a
\]

\[
= U(t, t-\varepsilon) \int_{-\infty}^{t-\varepsilon} U(t-\varepsilon, s) F(s, u(s)) ds, \ u \in B_a \\
= U(t, t-\varepsilon) (S u)(t-\varepsilon), \ u \in B_a
\]

and hence \( V_{\varepsilon} := \{ S_{\varepsilon} u(t) : u \in B_a \} \) is relatively compact in \( X_{\alpha} \) as the evolution family \( U(t, t-\varepsilon) \) is compact by assumption.

Now

\[
\|S u(t) - U(t, t-\varepsilon) \int_{-\infty}^{t-\varepsilon} U(t-\varepsilon, s) F(s, u(s)) ds\|_a \\
\leq \int_{t-\varepsilon}^{t} \|U(t, s) F(s, u(s))\|_a ds \\
\leq c(\alpha) \int_{t-\varepsilon}^{t} e^{-\frac{1}{2}(t-s)^{-\alpha}} \|F(s, u(s))\| ds \\
\leq c(\alpha) \int_{t-\varepsilon}^{t} (t-s)^{-\alpha} e^{-\frac{1}{2}(t-s)^{-\alpha}} \|f(s, u(s))\| ds + c(\alpha) \int_{t-\varepsilon}^{t} (t-s)^{-\alpha} e^{-\frac{1}{2}(t-s)^{-\alpha}} \|B(s)u(s)\| ds \\
\leq c(\alpha) \int_{t-\varepsilon}^{t} (t-s)^{-\alpha} e^{-\frac{1}{2}(t-s)^{-\alpha}} \|f(s, u(s))\| ds + c(\alpha) C_0 \|u\|_{a, \infty} \int_{t-\varepsilon}^{t} (t-s)^{-\alpha} e^{-\frac{1}{2}(t-s)^{-\alpha}} ds \\
\leq c(\alpha) L \left( d^{-1}(\alpha) + C_0 \right) \int_{t-\varepsilon}^{t} (t-s)^{-\alpha} ds \\
= c(\alpha) L \left( d^{-1}(\alpha) + C_0 \right) e^{1-\alpha} (1 - \alpha)^{-1},
\]
and hence the set \( V := \{ Su(t) : u \in B_u \} \subset \mathcal{X}_a \) is relatively compact.

The proof for b) follows along the same lines as in Ding et al. [3], Theorem 2.6 and hence is omitted.

Now since \( B_u \) is a closed convex subset of \( \text{PAA}(\mathcal{X}_a) \) and that \( S(B_u) \subset B_u \), it follows that \( \overline{S}S(B_u) \subset B_u \). Consequently, \( S(\overline{S}S(B_u)) \subset S(B_u) \subset \overline{S}S(B_u) \). Further, it is not hard to see that \( \{ u(t) : u \in \overline{S}S(B_u) \} \) is relatively compact in \( \mathcal{X}_a \) for each fixed \( t \in \mathbb{R} \) and that functions in \( \overline{S}S(B_u) \) are equi-continuous on \( \mathbb{R} \). Using Arzelà-Ascoli theorem, we deduce that the restriction of \( \overline{S}S(B_u) \) to any compact subset \( I \) of \( \mathbb{R} \) is relatively compact in \( C(I, \mathcal{X}_a) \). In summary, \( S : \overline{S}S(B_u) \rightarrow \overline{S}S(B_u) \) is continuous and compact. Using the Schauder fixed point it follows that \( S \) has a fixed-point, which obviously is a pseudo-almost automorphic mild solution to Eq. (1.2).

In order to study Eq. (1.1), we need the following additional assumption:

(H.7) There exists a function \( \rho \in L^1(\mathbb{R}, (0, \infty)) \) with \( \| \rho \|_{L^1(\mathbb{R}, (0, \infty))} \leq \frac{1}{2d(\alpha)} \) such that

\[
\left\| C(t) \varphi \right\| \leq \rho(t) \| \varphi \|_a
\]

for all \( \varphi \in \mathcal{X}_a \) and \( t \in \mathbb{R} \).

**Corollary 3.7.** Suppose assumptions (H.1)–(H.2)–(H.3)–(H.5)–(H.6)–(H.7) hold, then Eq. (1.1) has at least one pseudo-almost automorphic mild solution

**Proof.** It suffices to check that (H.7) yields (H.4) in the case when the bounded linear operators \( B(t) \) are defined by

\[
B(t) \varphi := \int_{-\infty}^t C(t-s) \varphi(s) \, ds
\]

for all \( t \in \mathbb{R} \) and \( \varphi \in BC(\mathbb{R}, \mathcal{X}_a) \) with \( \mathcal{X}_a = (\mathcal{X}, D)_{\alpha, \infty} \).

Indeed, since \( \rho \) is integrable, it is clear that the operators \( B(t) \) belong to \( B(BC(\mathbb{R}, \mathcal{X}_a), \mathcal{X}) \) for all \( t \in \mathbb{R} \) with \( \| B(t) \|_{BC(\mathbb{R}, \mathcal{X}_a)} \leq \| \rho \|_{L^1(\mathbb{R}, (0, \infty))} \). In fact, we can take \( C_0 = \| \rho \|_{L^1(\mathbb{R}, (0, \infty))} \).

To complete the proof, we have to show that the function \( \mathbb{R} \rightarrow \mathcal{X}, t \mapsto B(t) \varphi \) is pseudo-almost automorphic for any \( \varphi \in \text{PAA}(\mathcal{X}_a) \). For that, write \( \varphi = \varphi_1 + \varphi_2 \), where \( \varphi_1 \in \text{AA}(\mathcal{X}_a) \) and \( \varphi_2 \in \text{PAP}(\mathcal{X}_a) \). Using the fact that the function \( t \mapsto \varphi_1(t) \) belongs to \( \text{AA}(\mathcal{X}_a) \), for every sequence of real numbers \( (\tau_n)_{n \in \mathbb{N}} \) there exist a subsequence \( (\tau_n)_{n \in \mathbb{N}} \) and a function \( \psi_1 \) such that

\[
\psi_1(t) := \lim_{n \to \infty} \varphi_1(t + \tau_n)
\]

is well defined for each \( t \in \mathbb{R} \), and

\[
\lim_{n \to \infty} \psi_1(t - \tau_n) = \varphi_1(t)
\]

for each \( t \in \mathbb{R} \).

Now

\[
B(t + \tau_n) \varphi_1 - B(t) \psi_1 = \int_{-\infty}^{t+\tau_n} C(t + \tau_n - s) \varphi_1(s) \, ds - \int_{-\infty}^{t} C(t - s) \psi_1(s) \, ds
\]

\[
= \int_{-\infty}^{t} C(t - s) \varphi_1(s + \tau_n) \, ds - \int_{-\infty}^{t} C(t - s) \psi_1(s) \, ds
\]

\[
= \int_{-\infty}^{t} C(t - s) (\varphi_1(s + \tau_n) - \psi_1(s)) \, ds
\]
and hence
\[
\|B(t + \tau_n)\varphi_1 - B(t)\varphi_1\| \leq \int_{-\infty}^t C(t-s)(\varphi_1(s + \tau_n) - \varphi_1(s))ds
\leq \int_{-\infty}^t \rho(t-s)\|\varphi_1(s + \tau_n) - \varphi_1(s)\|_a ds
\]
which by Lebesgue Dominated Convergence Theorem yields
\[
\lim_{n \to \infty} \|B(t + \tau_n)\varphi_1 - B(t)\varphi_1\| = 0.
\]
Using similar arguments, we obtain
\[
\lim_{n \to \infty} \|B(t - \tau_n)\psi_1 - B(t)\psi_1\| = 0.
\]
For \( r > 0 \),
\[
\frac{1}{2r} \int_{-r}^r \int_{-\infty}^t C(t-s)\varphi_2(s)ds\|_a dt \leq \frac{1}{2r} \int_{-r}^r \int_0^\infty \rho(s)\|\varphi_2(t-s)\|_a ds dt
\leq \int_0^\infty \rho(s) \left( \frac{1}{2r} \int_{-r}^r \|\varphi_2(t-s)\|_a dt \right) ds.
\]
Now using the translation invariance of the space \( PAP_0(\mathcal{X}_a) \), it follows that,
\[
\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^r \|\varphi_2(t-s)\|_a dt = 0,
\]
as \( t \mapsto \varphi_2(t-s) \in PAP_0(\mathcal{X}_a) \) for every \( s \in \mathbb{R} \).
Therefore,
\[
\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^r \int_{-\infty}^t C(t-s)\varphi_2(s)ds\|_a dt = 0
\]
by using the Lebesgue Dominated Convergence Theorem.

4. Example

Fix \( \alpha \in (0, 1) \). Let \( \Omega \subset \mathbb{R}^N \) be an open bounded set with smooth boundary \( \partial \Omega \) and let \( \mathcal{X} = L^2(\Omega) \) be equipped with its natural norm \( \| \cdot \|_{L^2(\Omega)} \) defined for all \( \varphi \in L^2(\Omega) \) by
\[
\|\varphi\|_{L^2(\Omega)} = \left( \int_\Omega |\varphi(x)|^2 dx \right)^{1/2}.
\]
Motivated by natural phenomena such as population genetics [15, 26] or nerve pulse propagation [14], in this section we study the existence of pseudo-almost automorphic solutions to the parabolic Neumann boundary value problem given by,
\[
\begin{align*}
\frac{\partial \varphi}{\partial t}(t, x) &= a(t)\Delta \varphi(t, x) + \int_{-\infty}^t b(t-s)\varphi(s, x)ds + f(t, \varphi(t, x)) \\
&\quad + \eta a(t)\varphi(t, x), \quad (t, x) \in \mathbb{R} \times \Omega,
\end{align*}
\]
\[
\frac{\partial \varphi}{\partial n}(t, x) = 0, \quad (t, x) \in \mathbb{R} \times \partial \Omega.
\]
where \( \eta > 0 \) is a constant, \( a : \mathbb{R} \mapsto \mathbb{R} \) and \( b : \mathbb{R} \mapsto (0, \infty) \) are functions, the function \( f : \mathbb{R} \times L^2(\Omega) \mapsto L^2(\Omega) \) is pseudo-almost automorphic in \( t \in \mathbb{R} \) uniformly in \( \varphi \in L^2(\Omega) \), and \( \Delta \) stands for the usual Laplace operator in the space variable \( x \).

Letting

\[
A_\eta(t)\varphi = a(t)(\Delta + \eta)\varphi \quad \text{for all} \quad \varphi \in D(A_\eta(t)) = D(\Delta) = \{ \varphi \in H^2(\Omega) : \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial \Omega \},
\]

\[
C(t)\varphi = b(t)\varphi \quad \text{for all} \quad \varphi \in D(C(t)) = D(\Delta),
\]

and \( f(t, \varphi) = h(t, \varphi) \), one can easily see that Eq. (1.1) is exactly the nonautonomous parabolic Neumann boundary value problem formulated in Eqs. (4.1)-(4.2).

This setting requires the following assumptions,

(H.8) The function \( a : \mathbb{R} \mapsto (0, \infty) \) is almost automorphic with

\[
\inf_{t \in \mathbb{R}} a(t) = a_0 > 0.
\]

(H.9) The function \( b : \mathbb{R} \mapsto (0, \infty) \) belongs to \( L^1(\mathbb{R}, (0, \infty)) \) with

\[
\int_{-\infty}^{\infty} b(s)ds \leq \frac{1}{2\bar{C}d(a)}
\]

where \( \bar{C} \) is the bound of the continuous injection \( L^2(\Omega) := (L^2(\Omega), D(\Delta))_{a,\infty} \hookrightarrow L^2(\Omega) \).

Setting \( A_\eta\varphi = -(\Delta + \eta)\varphi \) for all \( \varphi \in D(A_\eta) = D(\Delta) \), one can easily see that \( A_\eta(t) = -a(t)A_\eta \). Of course, \( -A_\eta \) is a sectorial operator on \( L^2(\Omega) \). Let \( (T(t))_{t \geq 0} \) be the analytic semigroup generated by the operator \( -A_\eta \). It is well-known that the semigroup \( T(t) \) is not only compact for \( t > 0 \) but also is exponentially stable as

\[
\|T(t)\| \leq e^{-\eta t}
\]

for all \( t \geq 0 \).

In view of the above, it is now clear that the evolution family \( U(t, s) \) associated with \( A_\eta(t) \) is given by

\[
U(t, s) = T\left( \int_s^t a(r)dr \right)
\]

for all \( t \geq s, \ t, s \in \mathbb{R} \) as

\[
U(t, s) - U(\tau, s) = T\left( \int_{\tau}^t a(r)dr \right) - I\left( \int_s^\tau a(r)dr \right)
\]

for all \( \tau > t \) and \( \tau, t \in \mathbb{R} \).

Further, under assumptions (H.8), the compactness of the semigroup and the exponential stability of \( T(t) \) it follows that the evolution family \( U(t, s) \) is not only compact for \( t > s \) but also is exponentially stable as

\[
\|U(t, s)\| \leq e^{-\eta_0(t-s)}
\]

for all \( t, s \in \mathbb{R} \) with \( t \geq s \).

Additionally, the functions \( (t, s) \mapsto U(t, s)\varphi, \mathbb{R} \times \mathbb{R} \mapsto L^2_\alpha(\Omega) \) belongs to \( bAA(\mathbb{T}, L^2_\alpha(\Omega)) \) for all \( \varphi \in L^2_\alpha(\Omega) \).

Now
\[ \|C(t)\varphi\|_{L^2(\Omega)} = b(t)\|\varphi\|_{L^2(\Omega)} \leq \tilde{C}b(t)\|\varphi\|_\alpha \]

for all \( \varphi \in L^2(\Omega) \) and \( t \in \mathbb{R} \).

In view of the above, it is clear that assumptions (H.1)-(H.2)-(H.3)-(H.4)-(H.7) are fulfilled. Therefore, using Corollary 3.7, we obtain the following theorem.

**Theorem 4.1.** Under assumptions (H.5)–(H.6)–(H.8)–(H.9), then the system Eqs. (4.1)–(4.2) has at least one pseudo-almost automorphic mild solution.

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EXISTENCE OF PSEUDO-ALMOST AUTOMORPHIC SOLUTIONS

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