Stationary Points of Scalar Fields Coupled to Gravity

H. Kröger\textsuperscript{a}, G. Melkonyan\textsuperscript{a}, F. Paradis\textsuperscript{a}, S.G. Rubin\textsuperscript{b,c}

\textsuperscript{a} Département de Physique, Université Laval, Québec, Québec G1K 7P4, Canada

\textsuperscript{b} Moscow State Engineering Physics Institute,

Kashirskoe sh., 31, Moscow 115409, Russia

\textsuperscript{c} Center for Cosmoparticle Physics ”Cosmion”, Moscow, 125147, Russia

Abstract

We investigate the dynamics of gravity coupled to a scalar field using a non-canonical form of the kinetic term. It is shown that its singular point represents an attractor for classical solutions and the stationary value of the field may occur distant from the minimum of the potential. In this paper properties of universes with such stationary states are considered. We reveal that such state can be responsible for modern dark energy density.

PACS numbers: 98.80.Cq

Keywords: inflationary models, dark energy density, multiverse
I. INTRODUCTION

Scalar fields play an essential role in modern cosmology. A realistic scenario of the origin of our universe is based on the inflationary paradigm and a vast majority of inflationary models use the dynamics of scalar fields. Here we show in a natural way how to produce a class of effective potentials of the scalar field. It is achieved by invoking the simplest form of a potential but non-canonical kinetic terms. The drawback of using scalar fields is the occurrence of potentials with unnatural forms. For example, potentials have to be extremely flat to be consistent with the standard inflationary scenario \[1\].

We consider an action which couples gravity to a scalar field. The latter has a non-trivial kinetic term \(K(\varphi) \neq 1\). By supposition, it contains a singular point of the following form

\[
K(\varphi) = M^n/(\varphi - \varphi_s)^n, n = -1, 1, 2 ,
\]

and investigate their effect on the scalar field dynamics, see also \[6\]. Here \(M\) is some model parameter. The existence of the singular kinetic term opens a rich variety of possibilities for the construction of cosmological models. The well known Brans - Dicke model \[4\] is one of the particular case.

It is known that appropriate change of the field variable, leads to the standard form of kinetic term, i.e. \(K = \pm 1\) what can be done during inflationary stage. The situation becomes much more complex when the field fluctuates around a singular point. The equation of motion for a uniform field distribution has the form

\[
\ddot{\varphi} + 3H\dot{\varphi} - \frac{n}{2(\varphi - \varphi_s)}\dot{\varphi}^2 + V(\varphi_s)'(\varphi - \varphi_s)^n/M^n = 0 .
\]

In the Friedmann-Robertson-Walker universe, \(H\) is the Hubble parameter and expression (1) is taken into account. The field value \(\varphi_s\) is a stationary solution for any smooth potential \(V\) and \(n > 0\) provided that \(\dot{\varphi} = o(\varphi - \varphi_s)\). The cosmological energy density of the vacuum is connected usually with one of its potential minima. Here the situation is different - the vacuum state is connected with the singular point of the kinetic term \(K(\varphi)\). To prove this statement, we consider the simplest form of the potential

\[
V(\varphi) = V_0 + m^2\varphi^2/2 .
\]

In the following we will only consider the class of models characterized by the set of parameters \(m, V_0, M\). The stationary state \(\varphi_s\) is chosen in a way such that it fits the cosmological
The energy density $\sim \Lambda$ in a modern epoch is small compared to any scale during the inflationary stage, which allows us to neglect it whenever this is possible and obtain the relation

$$\varphi_s \approx \sqrt{2|V_0|/m} .$$

To proceed, an auxiliary variable $\chi$ will be taken into account. We suggest the substitution of variables $\varphi \to \chi$ in the form

$$d\chi = \pm \sqrt{K(\varphi)}d\varphi, \quad K(\varphi) > 0 ,$$

which leads to the action in terms of the auxiliary field $\chi$

$$S = \int d^4x\sqrt{-g} \left[ \frac{R}{16\pi G} + sgn(\chi)\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - U(\chi) \right] ,$$

where the potential $U(\chi) \equiv V(\varphi(\chi))$ is a ‘partly smooth’ function. Its form depends on the form of the initial potential $V(\varphi)$, the form of the kinetic term and the position of the singularities at $\varphi = \varphi_s$. Now let us consider some particular cases of $K(\varphi)$.

II. EFFECTIVE POTENTIALS

The case $n = 1$:

In this case formulas (1,4) give the action (5) with the potential

$$U(\chi) \equiv V(\varphi(\chi)) = V_0 + \frac{1}{2}m^2(\varphi_s + sgn(\chi)\frac{\chi^2}{4M})^2 \quad \text{for} \quad \varphi_s > 0 .$$

Here and below we keep the one - to - one correspondence between the physical variable $\varphi$ and auxiliary variable $\chi$ in the intervals:

$$\varphi < \varphi_s \to \chi < 0 ;$$

$$\varphi > \varphi_s \to \chi > 0 .$$

If the auxiliary field starts from $\chi > 0$, it finally approaches the singular point $\chi = 0$ (see Fig. II). If the field obeys $\chi < 0$, than the auxiliary field behaves like a phantom field, which climbs up to the top of the potential and hence tends to the singular point as well. Finally,
FIG. 1: Potential in terms of auxiliary field $\chi$ for the case $n = 1$. If $\chi < 0$ the auxiliary field behaves like a phantom field moving classically to the local extremum at $\chi = 0$. The field settles down in the vicinity of the singular point $\chi = 0$ ($\varphi = \varphi_s$). One concludes that this point is the stationary point and the vacuum energy density equals to $V(\varphi_s)$, (see Eq.(2)) rather than to $V_0$. The value of parameters can be estimated if we interpret the auxiliary field as the inflaton which in addition is responsible for the dark energy. In the course of inflation, a slow roll condition [1] should hold. This happens if the parameters take the values

$$M \sim M_P; \quad |V_0| \sim M_P^4; \quad m \sim 10^{-12} M_P.$$ (7)

The parameter $m$ is small in order to fit data of large scale temperature fluctuations [8].

The problem of smallness of the vacuum energy density, $\Lambda = 10^{-123} M_P^4$, remains topical in this approach although the situation has changed. As mentioned above, the smallness of the vacuum energy density is usually connected with the smallness of a potential minima. In the case considered here the modern energy density is determined by the singular point $\varphi_s$ of the non-canonical kinetic term (see Eq.(2)). The smallness of $\Lambda$ may be realized if the singular point $\varphi_s$ is placed very close to the zero point $\varphi_0$ of the potential ($V(\varphi_0) = 0$). A suitable interval is

$$\varphi_s \in [\varphi_0, \varphi_0 + \Delta \varphi], \quad \Delta \varphi \equiv \sqrt{-2V_0/m^2 + 2\Lambda/m^2} - \sqrt{-2V_0/m^2} \approx \frac{\Lambda}{m \sqrt{2|V_0|}}.$$ (8)

This interval is extremely small, making its explanation still difficult. The next section is devoted to a discussion of this subject matter. We will show that a probabilistic approach may help to obtain a self-consistent picture.
The case $n = 2$:

Now formulas (11, 14) give the action (5) with the potential

$$U(\chi) = \frac{1}{2} m^2 \varphi_s^2 \left[1 + sgn(\varphi_s) \cdot sgn(\chi) \cdot e^{\chi/M} \right]^2 + V_0 .$$  \hspace{1cm} (9)

In the case $\varphi_s < 0; \ \varphi > \varphi_s$ the potential (9) is highly asymmetric, and the behavior of the inflaton is rather different at $\chi < 0$ from that at $\chi > 0$. If we suppose that the inflation starts with $\chi = \chi_{\text{in}} > 0$, the picture is similar to the improved quintessence potential [10]. It is free of problems with the description of the radiation-dominated stage during Big Bang nucleosynthesis which could explain the modern distribution of chemical elements [9]. The chosen parameter values

$$M \sim M_P, \ m \sim M_P, \ |V_0| \sim 10^{-14} M_P^4 .$$  \hspace{1cm} (10)

permit a suitable inflationary stage and they are in agreement with observations of temperature fluctuations [8].

The case $n = -1$:

A nontrivial situation occurs when the kinetic function has not a pole but a root at some point, $K(\varphi) = (\varphi - \varphi_s)/M$. Let the initial field value obey $\varphi = \varphi_{\text{in}} > \varphi_s \sim M_P$, which gives rise to the inflation in early universe. Then the potential of the auxiliary field $\chi$ becomes

$$U(\chi) = \frac{1}{2} m^2 (\varphi_s + sgn(\chi) \cdot \gamma |\chi|^{2/3})^2 + V_0 .$$  \hspace{1cm} (11)

$U(\chi)$ is finite at $\chi = 0$ but its derivative is singular. Classically, the situation looks very strange - the singular point attracts the solution, but forbids it to stay there forever. It looks is similar to quantum mechanics, in particular to the case of an electron in the Coulomb field.

The potential (11) behaves like $\chi^{4/3}$ at large field values. It leads to standard inflation with moderate fine tuning of the parameters. Namely

$$M \sim M_P, \ m \sim 10^{-6} M_P, \ V_0 \sim 10^{-12} M_P^4 .$$  \hspace{1cm} (12)

If $\varphi_s > 0$, the field $\varphi$ will fluctuate around some critical point with energy density [2]. This motion never attenuates completely because classical stationary points are absent in this region.
III. PROBABILISTIC APPROACH TO THE FORM OF ACTION

Here we have investigated several specific forms of effective potentials. Many other potentials and kinetic terms have been discussed in the literature. A substantial number of them does not contradict observational data. In this context the question can be raised and need to be answered: Why is it that particular shape of potential and kinetic term is realized in nature? What are the underlying physical reasons?

Some theoretical hints on the form of the potential have been given in the context of supergravity, which predicts an infinite power series expansion in the scalar field potential \[1\]. Its minima, if they exist, correspond to stationary states of the field. The potential, due to an infinite number of terms in a power series could correspond to a function with an infinite set of potential minima. This assumption with randomly distributed minima appears to be self-consistent \[2\]. In the low energy regime it is reasonable to retain only a few terms (lowest powers in the Taylor expansion) of the scalar field \[12\]. In the vicinity of each of those minima the potential has a particular form. A similar behavior may hold also for the kinetic term. If the scalar field is responsible for the inflation, each local minimum produces an individual universe, different from any other universe. Our own universe is associated with a particular potential minimum, not necessarily located at \(\varphi = 0\).

The observed smallness of the value of the \(\Lambda\)-term is explained usually in terms of a more fundamental theory like supergravity or the anthropic principle. Our point of view is that we have to merge these approaches. The more fundamental theory supplies us with an infinite set of minima of the potential. These minima having an individual shape are responsible for the formation of those universes used in the anthropic consideration.

Practically, it could be performed in the framework of the random potential \[2, 3\] and the kinetic term of the scalar field discussed in sect.1. A part of such potential and the kinetic term in a finite region of the field \(\varphi\) are represented in Fig.2. Fluctuations of the scalar field being generated at high energies in the inflationary stage move classically to stationary points. Those of them who reach stationary points with appropriate energy density could form a universe similar to our Universe. This energy density \((\sim 10^{-123}M_{P}^{4})\) is the result of a small value of the concrete potential minimum or a small value of the difference \(\varphi_{s} - \varphi_{m}\), where \(\varphi_{m}\) is a zero of the potential \((V(\varphi_{m}) = 0)\). The fraction of such universes is relatively small, but nevertheless is infinite because of an infinite number of stationary states.
How could one decide which of the stationary points is most promising? To get an idea we should recall that the main defect of the inflationary scenario is the smallness of some intrinsic parameter compared to unity. It is the value of selfcoupling $\lambda \sim 10^{-13}$ for the potential $V_4 = \lambda \varphi^4$ or the smallness of the mass of the inflaton field in Planck units, $m/M_P \sim 10^{-6}$ for the potential $V_2 = m^2 \varphi^2/2$. Let us consider an infinite set of potential wells corresponding to infinite set of its minima as discussed above. Then we can use the concept of probability to find a potential well with specific properties. To estimate the relative number of specific universes, let us suppose that if there are no observational data on the value of a parameter $g$, the probability density $W$ for any parameter $g$ is distributed by a random uniform distribution in the range $(0, 1)$ in Planck scale. An immediate conclusion is that the probability of a potential $\lambda \varphi^4$ is about $10^{-13}$ while the probability of a potential $m^2 \varphi^2/2$ is about $10^{-6}$. It means that the latter is realized $10^6$ times more frequently.

In fact the probability is much smaller due to smallness of the cosmological $\Lambda$-term. So the probability to find a universe with such small vacuum energy is $P_\Lambda = 10^{-123}$. Recall that the set of potential minima is infinite. It means that the set of universes with an appropriate vacuum energy density is relatively small but still infinite. So the probability to find an appropriate potential $V_4$ is

$$P(V_4) = 10^{-13}P_\Lambda, \quad V_4 \sim \varphi^4,$$

while the same for the potential $V_2$ is

$$P(V_2) = 10^{-6}P_\Lambda, \quad V_2 \sim \varphi^2.$$

The lowest stationary state could be a singular point of the kinetic term, rather than a potential minimum. Thus we could expect that singular point(s) $\varphi_s$ may be found near some minima $\varphi_m$ of the potential. Now the problem is reformulated as follows: “which part of infinite number of minima contains singular points located closely to them? ” This part is very small, but not zero, due to infinite number of the minima. Only this part is important - it represents those vacua where galaxies could be formed due to extremely small value of $\Lambda-$ term. Following the way discussed above we can compare the probability of realization of such potentials. Their common factor is connected with the probability to find the singular point of the kinetic term in a small interval Eq.(8),

$$P_0 = \Delta \varphi_s/M_P \cong \frac{\Lambda}{M_P m \sqrt{2|V_0|}} = P_\Lambda \frac{M_P^3}{m^2 \sqrt{2V_0}}.$$

(15)
For the case $n = 1$ the only additional smallness is dictated by expression (7) and the probability for such universes to occur is

$$P_1 \sim \frac{m}{M}P_0 = P_\Lambda \frac{M_p^3}{M\sqrt{2V_0}} \approx P_\Lambda .$$

(16)

Universes with the properties described in the case $n = 2$ are distributed with probability

$$P_2 \sim \frac{V_0}{M_p^4}P_0 \simeq P_\Lambda \frac{\sqrt{2V_0}}{mM_p} \sim 10^{-7}P_\Lambda ,$$

(17)

if the inflation starts at the right branch of the potential. Here we assumed $m \sim M_p, V_0 \sim M_P^4$. The last case considered, $n = 2$, has a probability by an order of magnitude larger

$$P_{-1} \sim \frac{m}{M} \frac{V_0}{M_p^4}P_0 \simeq P_\Lambda \frac{\sqrt{2V_0}}{M_P^2} \sim 10^{-6}P_\Lambda .$$

(18)

An important conclusion from this consideration is that the model with kinetic term $K \sim (\varphi - \varphi_s)^{-1}$ is much more probable (at least by a factor $10^6$) comparing with other models discussed above, including the models with a standard kinetic term and potentials $\sim \varphi^2$ and $\sim \varphi^4$, see expressions (14), (13). It means that our Universe is likely governed by the model with kinetic term $K \sim (\varphi - \varphi_s)^{-1}$.

In conclusion we have discussed several inflationary models having common features like the occurrence of singular points in non-canonical kinetic terms. We have shown that the
existence of such points where the kinetic term changes its sign or tends to infinity opens new possibilities for scalar field dynamics. It takes place even for the simplest form of the potential. Depending on a position of the singular point of the kinetic term, specific forms of the potential of the auxiliary field have been obtained. One of the main results is that the stationary value of scalar field could occur at singular points of kinetic term rather than at minima of the potential. We estimated the parameter values for three type of new inflationary models. The probabilities to find universes with specific values of parameters have been estimated. It was shown that the probability is much greater for the model with kinetic term $K \sim (\varphi - \varphi_s)^{-1}$ than for the other models including the most promising model of chaotic inflation with the quadratic potential. Another interesting result is that if the singular point is a root of the kinetic term, the final state is intrinsically a quantum state.

Acknowledgment. This work was partially performed in the framework of Russian State contract 40.022.1.1.1106, RFBR grant 02 – 02 – 17490. H.K. has been supported by NSERC Canada.

[1] A.D. Linde, The Large-scale Structure of the Universe. London: Harwood Academic Publishers, 1990.
[2] S. Rubin, Gravitation & Cosmology 9 (2003) 243-248; hep-ph/0309184.
[3] S. Rubin, H. Kröger and G. Melkonian; astro-ph/0310182, 2003.
[4] C. Brans and R. Dicke, Phys. Rev. 124 (1961) 925-935.
[5] V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D9 (2000) 373-444.
[6] K. Bronnikov, J. Math. Phys. 43 (2002) 6096-6115; gr-qc/0204001.
[7] S. Rubin, H. Kröger, G. Melkonian, Gravitation & Cosmology, Supplement 8 (2002) 27-31; hep-ph/0207109.
[8] C.L. Bennett, Astrophys. J. Lett. 464 (1996) L1-L4.
[9] J.P. Kneller and G. Steigman; astro-ph/0210500.
[10] A. Albrecht and C. Skordis, Phys. Rev. Lett. 84 (2000) 2076-2079.
[11] H.P. Nilles, Phys. Repts. 110 (1984) 1-162.
[12] D.H. Lyth and E.D. Stewart, Phys. Rev. D54 (1996) 7186-7190.
[13] S. Weinberg; astro-ph/0005265