Covariant Axial Gauge

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Abstract

We consider abelian gauge theories on a lattice and develop properties of an axial gauge that is covariant under lattice symmetries. Particular attention is paid to a version that behaves nicely under block averaging renormalization group transformations.

1 Introduction

Gauge quantum field theory can be formulated in various gauges. Prominent choices are the axial gauge in which a component of the gauge field is set to zero and covariant gauges like Feynman or Landau. The axial gauge is good for defining the theory and exhibiting the positivity of the action. The covariant gauges are good for ultraviolet regularity and exhibiting the space-time symmetries of the theory.

Balaban in his studies of renormalization group methods for lattice gauge theories found a way to exploit some good properties of both types of gauges [5], [6], [7]. In this formulation the gauge field is a function on bonds in the lattice, and the axial gauge was realized by setting the field to zero on certain trees. However the axial gauge still spoiled the space-time covariance. When he came to applying these methods to pure Yang Mills in $d = 3, 4$ [8], [9] Balaban found that this was a significant obstacle. Instead he developed a covariant axial gauge in which he averaged over various trees to regain the covariance. However details about taking over the results of [5], [6], [7] were absent. Furthermore in the Yang-Mills papers he used an exponential gauge fixing rather than the original delta function gauge fixing.

In this paper we reconsider this covariant axial gauge with delta function gauge fixing, and establish results from [5], [6], [7] for this case. Our purpose is to use them in an analysis of ultraviolet problems for scalar QED in dimension $d = 3$ [13]. Scalar QED was originally studied by Balaban [1], [2], [3], [4]. Some results were extended to the abelian Higgs model by Balaban, Imbrie, and Jaffe [11], [12]. See also [10], [15] for further discussion of these problems.

In view of the intended application we mainly work in dimension $d = 3$, but really the results are not specific to any dimension. In section 2 we develop the covariant axial gauge for the free electromagnetic field on unit lattice cube. In the remainder of the paper, section 3 we extend these results to a toroidal lattice with arbitrarily small lattice spacing. By scaling this is equivalent to a unit lattice with a large volume. The results of section 2 do not give good bounds in this case. This is a case of a massless model in a large volume, and this is just the arena for renormalization group methods. We show how to implement the covariant axial gauge in a way compatible with block averaging renormalization group methods.

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2 Axial gauges

2.1 gauge fixing on a tree

Consider an abelian gauge theory on a finite unit lattice of dimension $d = 2, 3$; specifically for an odd integer $L$ on the square or cube

$$B(0) = [-L/2, L/2]^d \cap \mathbb{Z}^d$$

centered on the origin. The gauge field $A(b) = A(x, x')$ is an $\mathbb{R}^d$ valued function on bonds (=nearest neighbor pairs) in $B(0)$ which satisfies $A(x', x) = -A(x, x')$. The field strength is defined on plaquettes (= squares) and is

$$dA(p) = \prod_{b \in \partial p} A(b)$$

This is invariant under gauge transformation $A^\lambda = A - \partial \lambda$. The action is

$$\frac{1}{2} \|dA\|^2 = \frac{1}{2} \sum_p |dA(p)|^2$$

and we are interested in integrals of the form

$$\int f(A) \exp \left( -\frac{1}{2} \|dA\|^2 \right) DA = \prod_{b \in B(0)} A(b)$$

Here $f(A)$ is assumed bounded on compacts and gauge invariant, but with no particular decay at infinity.

The integral is not convergent since $dA$ has a large null space. The axial gauge is the remedy. We first explain the tree axial gauge. Let $\Gamma_{0x}$ be the rectilinear path in the lattice from 0 to $x$ obtained by successively increasing each coordinate to its final value. Thus in $d = 3$, $\Gamma_{0x}$ is the path

$$\Gamma_{0x} = \left[ (0,0,0), (x_1,0,0), (x_1,x_2,0), (x_1,x_2,x_3) \right]$$

Introduce new variables $\tau^0 A$ defined on lattice sites $x \in B(0), x \neq 0$ by

$$(\tau^0 A)(x) = A(\Gamma_{0x}) \quad A(\Gamma) = \sum_{b \in \Gamma} A(b)$$

Note that under a gauge transformation we have

$$(\tau^0 A^\lambda)(x) = (\tau^0 A)(x) - \sum_{b \in \Gamma_{0x}} \partial \lambda(b) = (\tau^0 A)(x) - (\lambda(x) - \lambda(0))$$

Let $T$ be the oriented tree consisting of all bonds that occur in any $(\tau^0 A)(x)$. See figures 1 and 2. The tree axial gauge means that in the formal integral we set $A(b) = 0$ for all $b \in T$. If $< x, x' >$ is on the tree then $A(x, x') = (\tau^0 A)(x') - (\tau^0 A)(x)$, so it is equivalent to set $(\tau^0 A)(x) = 0$ for all $x$.

Figure 1: The tree $T$ for $d = 2, L = 5$
This mutilation can be motivated by a Fadeev-Popov argument. Let $Q\lambda$ be the average

$$Q\lambda = L^{-d} \sum_{x \in B(0)} \lambda(x)$$

and define

$$\delta(\tau^0 A) = \prod_{x \neq 0} \delta\left( (\tau^0 A)(x) \right) \quad D\lambda = \prod_{x \in B(0)} d(\lambda(x))$$

Start with the identity

$$\int \delta(Q\lambda) \delta(\tau^0 A^\lambda) \, D\lambda = \int \delta(\tau^0 A^\lambda - (\lambda - \lambda(0))) \, D\lambda = \text{const}$$

This can be seen by making the change of variables $\{\lambda(x)\}$ to $\{\lambda(x) - \lambda(0)\}_{x \neq 0}, Q\lambda$ which has a constant Jacobian. Insert this under the integral sign in (3) and change the order of integration to obtain up to a constant multiple

$$\int \left[ \int f(A) \delta(\tau^0 A^\lambda) \exp\left( -\frac{1}{2} \|dA\|^2 \right) \, DA \right] \delta(Q\lambda) \, D\lambda$$

Now in the bracketed expression make the change of variables $A \to A^{-\lambda}$. Since $f(A)$ and $dA$ are gauge invariant we get the same thing with $\lambda = 0$. Take the bracketed expression outside the $\lambda$ integral and then throw away the remaining infinite $\lambda$ integral. We end with the desired expression

$$\int f(A) \delta(\tau^0 A^\lambda) \exp\left( -\frac{1}{2} \|dA\|^2 \right) \, DA$$

2.2 covariant axial gauge

For the covariant axial gauge we average over the ordering of the coordinates in the path from 0 to $x$. Let $\pi$ be a permutation of $(1,2,\ldots,d)$ and let $\Gamma^\pi_{0x}$ be all rectilinear paths from 0 to $x$ in which the coordinates are taken to their final values in the order determined by $\pi$. If $\pi$ is the identity then $\Gamma^0_{0x} = \Gamma_{0x}$. We replace $\tau^0$ by an average over permutations

$$(\tau A)(x) = \frac{1}{d!} \sum_{\pi} A(\Gamma^\pi_{0x}) = \frac{1}{|G(0,x)|} \sum_{\Gamma \in G(0,x)} A(\Gamma)$$

In the second form we let $G(0,x)$ stand for the set of all $\Gamma^\pi_{0x}$ and $|G(0,x)|$ is again $d!$.

\footnote{For a general theory of integrals of the form $\int \delta(\phi(x)) f(x) dx$ see Gelfand and Shilov \[14\]. For us $\phi$ will always be linear or affine.}
This is covariant in the following sense. Let \( r \) be a lattice symmetry leaving the origin fixed and let \( A_r(b) = A(r^{-1}b) \). Then \( A_r(\Gamma) = A(r^{-1}\Gamma) \) and \( rG(0, x) = G(0, rx) \) imply that
\[
(\tau A_r)(x) = (\tau A)_r(x) \equiv (\tau A)(r^{-1}x)
\] (13)
It follows that
\[
\delta(\tau A_r) = \delta(\tau A)
\] (14)
We still have
\[
(\tau A^\lambda)(x) = (\tau A)(x) - (\lambda(x) - \lambda(0))
\] (15)
Hence by exactly the same formal argument that led from (3) to (11) we can go instead from (3) to
\[
\int \delta(\tau A)f(A) \exp \left( -\frac{1}{2}\|dA\|^2 \right) DA
\] (16)
This is our starting point.

The next result shows that the gauge fixing has done its job and the the integrals (11) and (16) are finite.

**Proposition 1.** On the cube \( B(0) \)

1. If \( \tau^0 A = 0 \) and \( dA = 0 \) then \( A = 0 \).
2. If \( \tau A = 0 \) and \( dA = 0 \) then \( A = 0 \).
3. There exists a constant \( C \) (depending on \( L \)) such that if either \( \tau^0 A = 0 \) or \( \tau A = 0 \)
   \[
   \|dA\|^2 \geq C\|A\|^2
   \] (17)
4. If \( f(A) \) is exponentially bounded the integrals (11), (16) exist.

**Proof.** For the first we use the principle that if \( dA(p) = 0 \) and we know that \( A(b) = 0 \) for three of the bonds in \( \partial p \) then \( A(b) = 0 \) for the fourth bond. Hence starting at the origin and working outward we deduce that \( A(b) = 0 \) for bonds \( b \) not on the tree \( T \), and hence for all \( b \). For the second point note that
\[
\tau A - \tau^0 A = \frac{1}{d!} \sum_{\pi} A(\Gamma_0^\pi) - A(\Gamma_{0\pi}) = 0
\] (18)
since \( \Gamma_{0\pi}^\pi = \Gamma_{0\pi} \) is a closed path and \( dA = 0 \) means the integral of \( A \) over closed paths vanishes. Hence \( \tau A = 0 \) implies \( \tau^0 A = 0 \) and again \( A = 0 \). The third follows since a positive definite quadratic form on a finite dimensional vector space is bounded below on the unit sphere. The fourth follows from the third.

### 2.3 Parametrization

We can carry out integrals in the axial gauge by introducing new coordinates which include \( \tau A \). We have
\[
\int \delta(\tau A)f(A)DA = \int_{\ker \tau \times (\ker \tau)'} \delta(\tau A_2)f(A_1 + A_2) \, DA_1DA_2
\]
\[
= \det \left( \tau \ | (\ker \tau)' \right)^{-1} \int_{\ker \tau \times \mathbb{R}^{B(0)-0}} \delta(T)f(A_1 + \tau^{-1}T) \, DA_1DT
\] (19)
\[
= \det \left( \tau \ | (\ker \tau)' \right)^{-1} \int_{\ker \tau} f(A_1) \, DA_1
\]
We have made a change of variables \( T = \tau A_2 \) using that \( \tau : (\ker \tau)' \to \mathbb{R}^{B(0)-0} \) is a bijection. Indeed it is injective since the kernel is zero and it is surjective since \( (\tau(d\lambda))(x) = \lambda(x) - \lambda(0) \). Finally the integral \( \int_{\ker \tau} f(A_1) \, DA_1 \) is evaluated by picking any orthonormal basis for \( \ker \tau \) and reducing it to an integral over \( \mathbb{R}^n \) where \( n = \dim(\ker \tau) \).
2.4 torus

We now discuss how these developments can be extended to a torus. For the torus we again take the cube $B(0)$, but now include bonds joining points on opposite sides, the dotted lines in figure 3. We cannot extend the tree to include these bonds since that would mean closing a loop which cannot be justified. Working with the old tree we gauge fix as before and obtain again

$$\int \delta(\tau^0 A) f(A) \exp \left( -\frac{1}{2} \| dA \|^2 \right) DA$$  \hspace{1cm} (20)

However this integral is still not convergent since there is no decay in the gauge field on the new bonds.

To fix this define a constant vector by

$$(Q \cdot A)_{\mu} = L^{-d} \sum_{x \in B(0)} A(\Gamma_{x,\mu})$$  \hspace{1cm} (21)

where $\Gamma_{x,\mu}$ is the path around the torus through $x$ in the direction $e_\mu$. We insert a delta function enforcing that $(Q \cdot A)_{\mu} = 0$ under the integral and obtain a new starting point

$$\int \delta(Q \cdot A) \delta(\tau^0 A) f(A) \exp \left( -\frac{1}{2} \| dA \|^2 \right) DA$$  \hspace{1cm} (22)

This is not gauge fixing. Rather it is suppressing the contribution of torons (Wilson lines), something which presumably is inconsequential in the infinite volume limit.

The integral is now convergent. To see it we show that $Q \cdot A = 0$ and $\tau^0 A = 0$ and $dA = 0$ imply that $A = 0$. As before $\tau^0 A = 0$ and $dA = 0$ imply that $A = 0$ on all the bonds joining points in $T$. For a new bond $< x, x + e_\mu >$, the dotted lines in figure 3 $dA = 0$ and $A = 0$ on bonds joining points in $T$ imply that $A_\mu(x, x + e_\mu) = c_\mu$ a constant. Since now $(Q \cdot A)_\mu = c_\mu$ the constant must be zero and hence the result.

The reason for working on a torus is to increase group of lattice symmetries. We have made several special choices here to spoil those symmetries. This could be fixed by averaging over the various choices. But this is not necessary for our purposes. In the renormalization group approach one works on a large torus which is broken up into cubes. On each cube we can use the covariant gauge fixing of section 2.1 to preserve the symmetries of the effective interaction. Only in the last step when the volume has shrunk to a single cube do we need the torus version of gauge fixing. Here covariance does not matter since after the final integral all the fields are gone.

We proceed to explain the renormalization group program in more detail.
3 Renormalization group

3.1 orientation

For an abelian gauge theory in dimension \( d = 3 \) we take a fixed large \( L \) and introduce toroidal lattices

\[
\mathbb{T}_M^{-N} = L^{-N} \mathbb{Z}^3 / L^M \mathbb{Z}^3
\]

with spacing \( L^{-N} \) and volume \( L^3 M \).

For ultraviolet problems we start on \( \mathbb{T}_0^{-N} \) with spacing \( L^{-N} \) and unit volume and consider formal integrals like

\[
\int f(A) \exp \left( -\frac{1}{2} \|dA\|^2 \right) DA = \prod_{b \in \mathbb{T}_0^{-N}} d(A(b))
\]

The function \( f(A) \) carries the contribution of any other fields and is assumed gauge invariant.

The general problem is first to make sense of this integral and second take the limit \( N \to \infty \) or at least get bounds uniform in \( N \). Again the solution to the first problem is gauge fixing. But if we gauge fix on a giant tree and use it directly we do not get bounds uniform in \( N \). Instead we gauge fix on a hierarchical tree which we now explain. This is more compatible with renormalization group transformations which provide the solution to the second problem.

First some definitions. Let \( \mathcal{A} \) be a function on bonds on a lattice \( T_s^{-k} \) with spacing \( L^{-k} \) and arbitrary volume. We define an averaged field \( \mathcal{Q} \mathcal{A} \) defined on oriented bonds \( <y, y + L^{-k+1} e_\mu> \) in \( T_s^{-k+1} \) by (for reverse oriented bonds take minus this)

\[
(\mathcal{Q} \mathcal{A})(y, y + L^{-k+1} e_\mu) = \sum_{x \in B(y)} L^{-4} \mathcal{A}(\Gamma_{x,x+L^{-k+1} e_\mu})
\]

Here \( B(y) \) is a block with \( L \) sites in each direction centered on \( y \), and \( \mathcal{A}(\Gamma) = \sum_{b \in \Gamma} \mathcal{A}(b) \) is an unweighted sum over bonds \( b \) in \( T_s^{-k} \). This is scale invariant. We also define the \( n \) fold composition \( Q_n = \mathcal{Q} \circ \cdots \circ \mathcal{Q} \) which takes fields on \( T_s^{-k} \) to fields on \( T_s^{-k+n} \) and is given by

\[
(Q_n \mathcal{A})(y, y + L^{-k+n} e_\mu) = \sum_{x \in B^n(y)} L^{-4n} \mathcal{A}(\Gamma_{x,x+L^{-k+n} e_\mu})
\]

Here \( B^n(y) \) is a block with \( L^n \) sites in each direction centered on \( y \)

Also on any lattice \( T_s^{-k} \) for \( x \in B(y) \) define

\[
(\tau \mathcal{A})(y, x) = \frac{1}{d!} \sum_{\pi} \mathcal{A}(\Gamma_{yx}) = \frac{1}{|G(y,x)|} \sum_{\Gamma \in G(y,x)} \mathcal{A}(\Gamma)
\]

Here \( \Gamma_{yx} \) is the rectilinear paths from \( y \) to \( x \in B(y) \) in which the coordinates are taken from \( y \) to their final values \( x \) in the order determined by \( \pi \), and \( G(y,x) \) is all such paths. Note that since \( \mathcal{A}(\Gamma) \) defined with an unweighted sum we have \( (\partial \lambda)(\Gamma_{yx}) = L^k(\lambda(x) - \lambda(y)) \) and hence \( (\tau \partial \lambda)(y, x) = L^k(\lambda(x) - \lambda(y)) \). Note also that if \( r \) is a lattice symmetry then \( r G(y, x) = G(r y, r x) \) and so \( (\tau \mathcal{A}_r)(y, x) = (\tau \mathcal{A})(r^{-1} y, r^{-1} x) \).

Returning to our problem on \( \mathbb{T}_0^{-N} \) we first define a gauge fixing function for \( \mathcal{A} \) on bonds in \( \mathbb{T}_0^{-N+j} \) for \( j = 0, \ldots, N - 1 \) by

\[
\delta(\tau \mathcal{A}) = \prod_{y} \prod_{x \in B(y), x \neq y} \delta\left((\tau \mathcal{A})(y, x)\right)
\]

where \( y \in \mathbb{T}_0^{-N+j+1} \) and \( x \in \mathbb{T}_0^{-N+j} \). This satisfies \( \delta(\tau \mathcal{A}_r) = \delta(\tau \mathcal{A}) \). Then for a function \( \mathcal{A} \) on bonds in \( \mathbb{T}_0^{-N} \), \( \mathcal{Q} \mathcal{A} \) is a function on bonds in \( \mathbb{T}_0^{-N+1} \), and we define the gauge fixing function

\[
\delta_N^{-1}(\mathcal{A}) = \prod_{j=0}^{N-1} \delta(\tau \mathcal{Q}_j \mathcal{A})
\]
This is also invariant under lattice symmetries.

We would like to insert $\delta_N^X(A)$ in the integral \([24]\). To motivate this we need:

**Proposition 2.** For $A$ on $T_0^{-N}$ the integral

$$
\int \delta(Q_N \lambda) \delta_N^X(A^\lambda) \, D\lambda = \prod_{x \in \mathbb{T}_0^{-N}} d(\lambda(x)) \tag{30}
$$

is constant.

**Remark.** In general $Q_k = Q \circ \cdots \circ Q$ ($k$ times) averages scalars over blocks with $L^{3k}$ sites in each direction, and is given by

$$(Q_k f)(y) = L^{-3k} \sum_{x \in B^k(y)} f(x) \tag{31}$$

So for $\lambda$ on $\mathbb{T}_0^{-N}$ we have that $Q_N \lambda$ is a single number equal to the average of $\lambda$ over the whole lattice.

**Proof.** We have

$$
\int \delta(Q_N \lambda) \delta_N^X(A^\lambda) \, D\lambda = \int \delta(Q_N \lambda) \prod_{j=0}^{N-1} \delta(\tau Q_j A^\lambda) \, D\lambda
= \int \delta(Q_N \lambda) \prod_{j=0}^{N-1} \prod_{y_j, x_j} \delta \left( (\tau Q_j A)(y_j, x_j) - L^{-N-1}(\, Q_j \lambda(x_j) - Q_j \lambda(y_j)) \right) \, D\lambda \tag{32}
$$

The interior product is over

$$y_j \in \mathbb{T}_0^{-N+1+j} \quad x_j \in \mathbb{T}_0^{-N+j} \quad x_j \in B(y_j) \quad x_j \neq y_j \tag{33}$$

and we have used the identities $Q_j \partial \lambda = \partial Q_j \lambda$ and $(\tau \partial Q_j \lambda)(y_j, x_j) = L^{-N-1}(Q_j \lambda(x_j) - Q_j \lambda(y_j))$.

We change variables from $\{\lambda(x)\}$ for $x \in \mathbb{T}_0^{-N}$ to

$$Q_N \lambda \quad \text{and} \quad \{Q_j \lambda(x_j) - Q_j \lambda(y_j)\}_{x_j \neq y_j} \quad j = 0, \ldots, N - 1 \tag{34}$$

We claim this linear transformation is non-singular. The number of variables is the same, namely

$$
(L^{3N} - L^{(3N-1)}) + \cdots + (L^6 - L^3) + (L^3 - 1) + 1 = L^{3N} \tag{35}
$$

so it suffices to show kernel is zero. But $Q_N \lambda = QQ_{N-1} \lambda = 0$ and $Q_{N-1} \lambda(x_{N-1}) - Q_{N-1} \lambda(y_{N-1}) = 0$
for $x_{N-1} \neq y_{N-1}$ imply $Q_{N-1} \lambda(x_{N-1}) = 0$ for all $x_{N-1}$. This is the same as $(QQ_{N-2} \lambda)(y_{N-2}) = 0$
for all $y_{N-2}$. Combine this with $Q_{N-2} \lambda(x_{N-2}) - Q_{N-2} \lambda(y_{N-2}) = 0$ for all $x_{N-2} \neq y_{N-2}$ and conclude that $Q_{N-2} \lambda(x_{N-2}) = 0$ for all $x_{N-2}$. Continue the argument until we get to $Q \lambda(y_0) = 0$ and
$
\lambda(x_0) - \lambda(y_0) = 0 \quad \text{for} \quad x_0 \neq y_0$ to conclude $\lambda(x_0) = 0$ for all $x_0$.

Thus we can write in \([32]\)

$$
D\lambda = \text{const } d(Q_N \lambda) \prod_{j=0}^{N-1} \prod_{y_j, x_j} d \left( (Q_j \lambda)(y_j) - (Q_j \lambda)(x_j) \right) \tag{36}
$$

Carrying out the integrals in \([32]\) with these variables yields a constant. This completes the proof.
Using this result we again make a Fadeev-Popov argument. Insert the integral $\int \delta(QN\lambda)\delta_N^{\lambda}(A^4)D\lambda$ in (24) and the change the order of integration to get

$$\int \left[ \int f(A)\delta_N^{\lambda}(A^4) \exp \left( -\frac{1}{2}\|dA\|^2 \right) DA \right] \delta(QN\lambda)D\lambda$$

(37)

Now gauge away the $\lambda$ dependence in the integral over $A$ and then throw away the infinite integral over $\lambda$ and any other constants. This yields the gauge fixed integral

$$\int f(A)\delta_N^{\lambda}(A) \exp \left( -\frac{1}{2}\|dA\|^2 \right) DA$$

(38)

This is still not well-defined because of the toron problem mentioned in section 2.4. The remedy is the same. Define

$$Q_N^\bullet = Q^\bullet \circ Q_{N-1}$$

(39)

Here $Q^\bullet$ is defined as in (21) but now on an $L^{-1}$ lattice instead of a unit lattice. Insert $\delta(Q_N^\bullet A)$ under the integral sign to obtain

$$\int f(A)\delta(Q_N^\bullet A)\delta_N^{\lambda}(A) \exp \left( -\frac{1}{2}\|dA\|^2 \right) DA$$

(40)

This is our new starting point. We will see in the next section that it is well-defined.

### 3.2 renormalization group transformations

We next explain the renormalization group transformations. For this it is convenient to work with unit lattice variables so we start by scaling up to the torus $T^N_N$ with unit spacing and volume $L^{3N}$. If $A$ is field on $T^0_N$ then

$$A(b) = A_{L^{-N}}(b) \equiv L^{-N}A(L^N b)$$

(41)

is a field on $T^{L^{-N}}_0$. We substitute it into the original density $f(A) \exp \left( -\frac{1}{2}\|dA\|^2 \right)$, use the fact that $\|dA\|^2$ is invariant, and get a new unit lattice density

$$\rho_0(A) = F_0(A) \exp \left( -\frac{1}{2}\|dA\|^2 \right) \quad F_0(A) \equiv f_{L^N}(A) \equiv f(A_{L^{-N}})$$

(42)

Starting with $\rho_0$ on $T^0_N$ we generate the integral (40) in a series of steps. We successively define densities $\rho_k$ on fields in $T^0_{N-k}$ by block averaging. Given $\rho_k$ we first define $\tilde{\rho}_{k+1}$ on fields $A_{k+1}$ on $T^0_{N-k}$ by

$$\tilde{\rho}_{k+1}(A_{k+1}) = \int \delta(A_{k+1} - QA_k)\delta(\tau A_k)\rho_k(A_k) DA_k$$

(43)

Here $\delta(\tau A_k)$ is defined as in (28) except that now $y \in T^1_{N-k}$ and $x \in T^0_{N-k}$. The other delta function is

$$\delta(A_{k+1} - QA_k) = \prod_{<y, y'>} \delta\left((A_{k+1} - QA_k)(y, y')\right)$$

(44)

Then we define densities $\rho_{k+1}$ on fields $A_{k+1}$ on $T^0_{N-k-1}$ by

$$\rho_{k+1}(A_{k+1}) = L^3L^{d_{k+1}}\tilde{\rho}_{k+1}(A_{k+1, k})$$

(45)

Here in general for a $d = 3$ toroidal lattice with $L^M$ sites on side we set $b_M = 3L^{3M}$ as the number of bonds and $s_M = L^{3M}$ as the number of sites. Note that the $\rho_k$ are not gauge invariant due to the gauge fixing.
Proposition 3. For $A_k$ on $T^0_{N-k}$ and $\mathcal{A}$ on $T^{−k}_{N−k}$

$$\rho_k(A_k) = \int \delta(A_k - Q_k \mathcal{A}) \delta_k^X(\mathcal{A}) \rho_{0,L−k}(\mathcal{A}) \, D\mathcal{A} \quad 1 \leq k \leq N−1$$ (46)

In the last step we replace $\delta(A_k - Q_k \mathcal{A})$ by $\delta(Q^*_k \mathcal{A})$ and have

$$\rho_N = \int \delta(Q^*_k \mathcal{A}) \delta_k^X(\mathcal{A}) \rho_{0,L−N}(\mathcal{A}) \, D\mathcal{A}$$ (47)

Remarks.

1. Here $\delta_k^X(\mathcal{A})$ is defined as before

$$\delta_k^X(\mathcal{A}) = \prod_{j=0}^{k-1} \delta(\tau Q_j \mathcal{A}) = \prod_{j=0}^{k-1} \prod_{y_j, x_j} \delta(\tau Q_j \mathcal{A}(y_j, x_j))$$ (48)

with $x_j \in B(y_j), x_j \neq y_j$, except that now $y_j \in T^{−k+j+1}_{N−k}$ and $x_j \in T^{−k+j}_{N−k}$.

2. Since $\rho_{0,L−N}(\mathcal{A}) = f(\mathcal{A}) \exp\left(-\frac{1}{4} \|\delta \mathcal{A}\|^2\right)$ we see that $\rho_N$ is the desired expression $[10]$. The basic idea of the renormalization group is to control the partition function $\rho_N$ by controlling the sequence $\rho_0, \rho_1, \ldots, \rho_N$.

3. Note that $Q_k \mathcal{A}$ can also be written with weighted sums appropriate for the lattice $T^k_{N−k}$. For oriented bonds we have

$$(Q_k \mathcal{A})(y, y + e_\mu) = \int_{|y - y'| < \frac{1}{2}} < \mathcal{A}, \Gamma_{x,x+e_\mu} >$$ (49)

where $\int \ldots dx = \sum_x L^{-3k}\ldots$ and

$$< \mathcal{A}, \Gamma > \equiv L^{-k} \mathcal{A}(\Gamma) = L^{-k} \sum_{b \in \Gamma} B(b) \equiv \int_\Gamma A$$ (50)

Proof. In the next proposition we show that $\rho_k$ is well-defined. Assuming this we show that the representation [16] for $\rho_k$ yield the representation [15] for $\rho_{k+1}$. We have

$$\tilde{\rho}_k(A_{k+1}) = \int \delta(A_{k+1} - Q_k \mathcal{A}) \delta(\tau A_k) \delta(A_k - Q_k \mathcal{A}) \delta_k^X(\mathcal{A}) \rho_{0,L−k}(\mathcal{A}) \, D\mathcal{A} \, D\mathcal{A}_k$$

$$= \int \delta(A_{k+1} - Q_{k+1} \mathcal{A}) \delta(\tau Q_k \mathcal{A}) \delta_k^X(\mathcal{A}) \rho_{0,L−k}(\mathcal{A}) \, D\mathcal{A}$$ (51)

Replace $A_{k+1}$ by $A_{k+1,L}$ now with $A_{k+1}$ on $T^0_{N−k−1}$ and replace $\mathcal{A}$ by $\mathcal{A}_L$ now with $\mathcal{A}$ on $T^{−k−1}_{N−k−1}$. Use the facts that $Q$ is scale invariant, $Q(\mathcal{A}_L) = (Q \mathcal{A})_L$, and that $\tau$ is scale invariant, $\tau(\mathcal{A}_L) = (\tau \mathcal{A})_L$, to obtain

$$\rho_k(A_{k+1}) = L^\frac{4}{b_{k+1}} \int \delta\left((A_{k+1} - Q_{k+1} \mathcal{A})_L\right) \delta\left((\tau Q_k \mathcal{A})_L\right) \delta_k^X(\mathcal{A}_L) \rho_{0,L−k}(\mathcal{A}_L) \, D(\mathcal{A}_L)$$ (52)

However $D(\mathcal{A}_L) = L^{-\frac{2}{b_{k+1}}} D\mathcal{A}$ and

$$\delta\left((A_{k+1} - Q_{k+1} \mathcal{A})_L\right) = L^\frac{2}{b_{N−k−1}} \delta\left(A_{k+1} - Q_{k+1} \mathcal{A}\right)$$ (53)
Further since \( Q_jA \) is a field on \( T_{N-k-1}^{-} \) which has \( L^{N-j} \) sites on a side, \((\tau Q_jA)\) takes values at \( s_{N-j} - s_{N-j-1} \) points and so

\[
\delta\left((\tau Q_kA)L\right)\delta_k^X(A_L) = \prod_{j=0}^{k} \delta\left((\tau Q_jA)L\right) = \prod_{j=0}^{k} L_x^{s_{N-j}-s_{N-j-1}} \delta\left((\tau Q_jA)\right) = L_x^{s_{N-j}s_{N-j-1}} \delta_{k+1}^X(A)
\]

(54)

The powers of \( L \) from under the integral sign collect to form \( L^{-t_{k+1}} \) which cancels the \( L^t_{k+1} \) in front. Thus we have the desired

\[
\rho_{k+1}(A_{k+1}) = \int \delta(A_{k+1} - Q_{k+1}A)\delta_{k+1}^X(A) \rho_{0,L-k-1}(A) \, dA
\]

(55)

This completes the proof.

**Proposition 4.** If \( F_0 \) is exponentially bounded \( \rho \) is well-defined for \( 1 \leq k \leq N \).

**Proof.** (after \[10\], \[11\] for \( k = 1, \tau^0 \).) For \( k < N \) we have

\[
\rho_k(A_k) = \int \delta(A_k - Q_kA)\delta_{k}^X(A) F_{0,L-k}(A) \exp\left(-\frac{1}{2}\|dA\|^2\right) \, dA
\]

(56)

The delta functions restrict the integral to the surface

\[
Q_kA = A_k, \quad \tau Q_{k-1}A = 0, \ldots, \quad \tau Q_A = 0, \quad \tau A = 0
\]

(57)

We first note that this surface is not empty. There are certainly \( A \) satisfying the null conditions since they represent the kernel of a linear operator to a lower dimension space. These conditions do not involve bonds joining neighboring unit cubes \( B^k(y), B^k(y') \) in \( T_{N-k}^{-} \). By adjusting \( A(b) \) for such bonds there is plenty of freedom to meet the final condition \( Q_kA = A_k \).

Let \( A_0 \) satisfy (57). Change variables by \( A = A_0 + Z \) and then the integral is

\[
\rho_k(A_k) = \int \delta(Q_kZ)\delta_{k}^X(Z) F_{0,L-k}(A_0 + Z) \exp\left(-\frac{1}{2}\|dA_0\|^2 - \frac{1}{2}\|dZ\|^2 - <dA_0,dZ>\right) \, dZ
\]

(58)

It suffices to show for \( Z \in T_{N-k}^{-} \) that \( \|dZ\|^2 \) is positive definite on the surface

\[
Q_kZ = 0, \quad \tau Q_{k-1}Z = 0, \ldots, \quad \tau QZ = 0, \quad \tau Z = 0
\]

(59)

Then this integral converges and the original integral converges.

First suppose \( k = 1 \) so we need to show for \( Z \in T_{N-k}^{-} \) that \( QZ = 0 \) and \( \tau Z = 0 \) and \( dZ = 0 \) imply \( Z = 0 \). Arguing as in Proposition 1 but with the \( L \)-cube \( T_0 \) replaced by a unit cube \( T_y \) centered on \( y \in T_{N-k}^{-} \), we find that \( Z \) vanishes on each cube \( B(y) \). Consider \( Z(b) \) for bonds connecting neighboring cubes \( B(y), B(y') \). The condition \( dZ = 0 \) implies that they all have the same value. Then the condition \( (QZ)(y,y') = 0 \) implies that that value is zero.

Back to the general case we note that \( dZ = 0 \) implies \( dQZ = 0 \). This follows since if \( x \in T_{N-k}^{-}, \quad y \in T_{N-k+1}^{-} \) and \( \eta = L^{-k} \):

\[
(dQZ)\left([y, y + L\eta \varepsilon_\mu, y + L\eta \varepsilon_\mu + L\eta \nu \varepsilon_\nu, y + L\eta \varepsilon_\nu, y]\right)
\]

\[
= \sum_{x \in B(y)} \left[\Gamma[x, x + L\eta \varepsilon_\mu, x + L\eta \varepsilon_\mu + L\eta \varepsilon_\nu, x + L\eta \varepsilon_\nu, x]\right]
\]

(60)
where $\Gamma[\cdots]$ in the closed path passing through the indicated points. But $Z(\Gamma[\cdots]) = 0$ follows from $dZ = 0$ and the lattice Stokes theorem. Hence the result. More generally $dZ = 0$ implies $dQ_j Z = 0$ for any $j$.

Now for $k > N$ we argue that $Q_k Z = Q(Q_{k+1} Z) = 0$ and $\tau Q_{k+1} Z = 0$ and $dQ_{k+1} Z = 0$ imply $Q_k Z = 0$ just as for $k = 1$. Then $Q_k Z = Q(Q_{k-1} Z) = 0$ and $\tau Q_{k-1} Z = 0$ and $dQ_{k-1} Z = 0$ imply $Q_k Z = 0$. Continuing we reduce to the case $k = 1$ and the conclusion $Z = 0$.

For $k = N$ we have $A_N = 0$ and are looking at the same expression (65), but with $\delta(Q_k Z)$ replaced with $\delta(Q^*_N Z)$. Then $Q^*_N Z = Q^*(Q_{N-1} Z) = 0$ and $\tau Q_{N-1} Z = 0$ and $dQ_{N-1} Z = 0$ imply the same with $\tau^0 Q_{N-1} Z = 0$ and hence $Q_{N-1} Z = 0$ on all bonds as in section 2.4 The rest of the argument proceeds as before.

Thus in all cases $\|dZ\|^2 \geq \text{const}\|Z\|^2$. The constant is not uniform in $N$, but the bound is sufficient to show the integral is well-defined. This completes the proof.

3.3 minimizers

Let $\mathcal{H}_k^A$ be the minimizer of $\|dA\|^2$ on the subspace (59). One can obtain explicit representation of $\mathcal{H}_k^A$ see [11], [15]. This representation involves a certain Green’s function $G_k^A$ which is essentially the inverse of $d^2 d$ on the surface (59) with $\tau^0$ instead of $\tau$ . The same representation holds for our case if we define $G_k^A$ as the inverse of $d^T d$ on the surface (59). It exists since the quadratic form $\langle A, d^T dA \rangle = \|dA\|^2$ is positive definite as we have just seen in the proof of Proposition 3. In any case we do not need to know much about $\mathcal{H}_k^A$ beyond existence. It is not very regular and will not be used directly.

Expand around the minimizer as above by $A = \mathcal{H}_k^A + Z$. The linear term vanishes we find

$$\rho_k(A_k) = Z_k F_k(\mathcal{H}_k^A) \exp \left(- \frac{1}{2} < A_k, \Delta_k A_k > \right)$$

(61)

where

$$< A_k, \Delta_k A_k > = \|dH_k^A A_k\|^2$$

$$F_k(\mathcal{H}_k^A) = Z_k^{-1} \int \delta(Q_k Z) \delta_k^X(Z) F_{0,k-l+i}(\mathcal{H}_k^A + Z) \exp \left(- \frac{1}{2} \|dZ\|^2 \right) DZ$$

(62)

$$Z_k = \int \delta(Q_k Z) \delta_k^X(Z) \exp \left(- \frac{1}{2} \|dZ\|^2 \right) DZ$$

The $F_k$ are functions of fields defined on bonds in $\mathcal{T}_{N-k}^{-k}$. Controlling the sequence $F_1, F_2, \ldots$ is the main issue for a complete analysis. To study this we consider how to pass from $F_k$ to $F_{k+1}$.

3.4 the next step

Suppose we are starting with the expression (61). In the next step we consider

$$\tilde{\rho}_{k+1}(A_{k+1}) = \int \delta(A_{k+1} - QA_k) \delta(\tau A_k) \rho_k(A_k) DA_k$$

$$= Z_k \int \delta(A_{k+1} - QA_k) \delta(\tau A_k) F_k(\mathcal{H}_k^A) \exp \left(- \frac{1}{2} < A_k, \Delta_k A_k > \right) DA_k$$

(63)

Let $H_k^A_{k+1}$ be the minimizer for $\frac{1}{2} < A_k, \Delta_k A_k >$ subject to the constraints. Again an explicit expression can be found [11], [15]. Expanding around the minimizer with $A_k = H_k^A_{k+1} Z$ we get

$$\tilde{\rho}_{k+1}(A_{k+1}) = Z_k Z_k^T \exp \left(- \frac{1}{2} < H_k^A_{k+1}, \Delta_k H_k^A_{k+1} > \right) F_k^T(\mathcal{H}_k^A H_k^A_{k+1})$$

(64)
where

\[ F_k^e (H_k^e H_k^e A_{k+1}) = (Z_k^e)^{-1} \int \delta(QZ) \delta(\tau Z) F_k \left( H_k^e H_k^e A_{k+1} + H_k^e Z \right) \exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) DZ \]

(65)

This yields the identity

\[ Z_k^e = \int \delta(QZ) \delta(\tau Z) \exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) DZ \]

After scaling then we have

\[ \rho_{k+1}(A_{k+1}) = Z_k^e Z_k^e L^\frac{1}{2} e^{\delta_k+1} \exp \left( -\frac{1}{2} < H_k^e A_{k+1,L}, \Delta_k H_k^e A_{k+1,L} > \right) F_k^e (H_k^e H_k^e A_{k+1,L}) \] (66)

Take the special case in which \( F_0(A_0) = 1 \). Then \( F_k = 1 \) and \( F_k^e = 1 \) for all \( k \). Then taking \( A_{k+1} = 0 \) and comparing (66) with (61) for \( k+1 \) we have the identity

\[ Z_{k+1} = Z_k Z_k^e L^\frac{1}{2} e^{\delta_k+1} \] (67)

We also see that the quadratic form in (66) must be \( < A_{k+1}, \Delta_{k+1}A_{k+1} > \).

Now in the general case (66) says

\[ \rho_{k+1}(A_{k+1}) = Z_{k+1} \exp \left( -\frac{1}{2} < A_{k+1}, \Delta_{k+1}A_{k+1} > \right) F_k^e (H_k^e H_k^e A_{k+1,L}) \] (68)

Comparing this with (61) for \( k+1 \) yields

\[ F_k^e (H_k^e H_k^e A_{k+1,L}) = F_k (H_{k+1}^e A_{k+1}) \] (69)

Next take the special case in which \( F_0(A_0) = < A_0, J > \). It is not gauge invariant, but does not have to be for this argument. From (62) we get \( F_k (H_k^e A_k) = < (H_k^e A_k)_L, J > \) for all \( k \). Then (65) yields \( F_k^e (H_k^e H_k^e A_{k+1}) = < (H_k^e H_k^e A_{k+1,L})_L, J > \) for all \( k \). Hence (69) says

\[ < (H_k^e H_k^e A_{k+1,L,L})_L, J > = < (H_{k+1}^e A_{k+1})_L, J > \] (70)

This yields the identity

\[ H_k^e H_k^e A_{k+1,L} = (H_{k+1}^e A_{k+1})_L \] (71)

Back to the general case (65) with \( A_{k+1} \rightarrow A_{k+1,L} \) becomes

\[ F_{k+1}(H_k^e A_{k+1}) = (Z_k^e)^{-1} \int \delta(QZ) \delta(\tau Z) F_k \left( H_k^e A_{k+1,L} + H_k^e Z \right) \exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) DZ \] (72)

More generally we define fluctuation integrals for any \( A \) on \( T_{N-k}^{-1} \) by

\[ F_{k+1}(A) = (Z_k^e)^{-1} \int \delta(QZ) \delta(\tau Z) F_k \left( A_L + H_k^e Z \right) \exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) DZ \] (73)

Here \( Z \) is on \( T_{N-k}^0 \) while \( H_k^e Z \) and \( A_L \) are on \( T_{N-k}^{-1} \). This generates the sequence \( F, F_1, F_2, \ldots \). If \( F_0 \) is gauge invariant then \( F_k \) is gauge invariant for all \( k \).

### 3.5 Feynman gauges

Consider integrals of the form and

\[ \rho_k(A_k) = \int \delta(A_k - Q_k A) \delta_k^e (A) f(A) \exp \left( -\frac{1}{2} ||dA||^2 \right) DA \] (74)

where \( f(A) \) is a gauge invariant function on fields \( A \) defined on \( T_{N-k}^{-1} \). The expression (16) is of this form. Also let \( d^2 \) which be the adjoint of \( d = \partial \) on scalars \( (d^2 = \text{the divergence}) \), and let \( R_k \) is the projection onto the subspace \( \Delta(\ker Q_k) \).

We introduce the modified Feynman gauge developed in [5], [10]:

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Proposition 5. The integral (74) can be expressed for any $\alpha > 0$ as
\[
\text{const} \int \delta(A_k - Q_k A) \ f(A) \exp \left( -\frac{1}{2} \|dA\|^2 - \frac{1}{2\alpha} < d^T A, R_k d^T A > \right) DA
\] (75)
This includes the Landau gauge at $\alpha = 0$ in which case
\[
\rho_k(A_k) = \text{const} \int \delta(A_k - Q_k A) \delta(R_k d^T A) \ f(A) \exp \left( -\frac{1}{2} \|dA\|^2 \right) DA
\] (76)
where $\delta(R_k)$ is the delta function in the subspace $\text{ran } R_k = \Delta(\ker Q_k)$.

Proof. We sketch the proof. One employs a Fadeev- Popov procedure. Define
\[
Z_k(A) = \int \delta(Q_k \lambda) \exp \left( -\frac{1}{2\alpha} \|d^T \lambda A\|^2 \right) D\lambda
\] (77)
and insert $1 = Z_k(A)/Z_k(A)$ under the integral sign in (74). In the numerator change the order of integration. This yields
\[
\int \left[ \int Z_k(A)^{-1} \delta(A_k - Q_k A) \delta_k X(A) f(A) \exp \left( -\frac{1}{2} \|dA\|^2 - \frac{1}{2\alpha} \|d^T \lambda A\|^2 \right) DA \right] \delta(Q_k \lambda) \ D\lambda
\] (78)
Now let $A \to A^{-\lambda} = A + \partial \lambda$. The delta function $\delta(A_k - Q_k A)$ is invariant since $Q_k \partial \lambda = \partial Q_k \lambda = 0$ and $\|dA\|^2$ is invariant. Furthermore $Z_k(A)$ is invariant since $Q_k \lambda = 0$ implies
\[
Z_k(A^{-\lambda}) = \int \delta(Q_k \lambda') \exp \left( -\frac{1}{2\alpha} \|d^T \lambda A\|^2 \right) D\lambda'
\] (79)
Thus our expression becomes
\[
\int \left[ \int Z_k(A)^{-1} \delta(A_k - Q_k A) \delta_k X(A^{-\lambda}) f(A) \exp \left( -\frac{1}{2} \|dA\|^2 - \frac{1}{2\alpha} \|d^T \lambda A\|^2 \right) DA \right] \delta(Q_k \lambda) D\lambda
\] (80)
Now change the order of integration again. The $\lambda$ integral is $\int \delta_k X(A^{-\lambda}) \delta(Q_k \lambda) D\lambda$ and as in Proposition 2 it is constant. Thus
\[
\rho_k(A_k) = \text{const} \int Z_k(A)^{-1} \delta(A_k - Q_k A) \ f(A) \exp \left( -\frac{1}{2} \|dA\|^2 - \frac{1}{2\alpha} \|d^T \lambda A\|^2 \right) DA
\] (81)
Finally a computation [5] shows that
\[
Z_k(A) = \text{const} \exp \left( -\frac{1}{2\alpha} < d^T A, P_k d^T A > \right)
\] (82)
where
\[
P_k = G_k Q_k^T (Q_k G_k^2 Q_k^T)^{-1} Q_k G_k
\] (83)
and for any $\alpha \geq 0$
\[
G_k = (-\Delta + a Q_k^T Q_k)^{-1}
\] (84)
This gives the Feynman gauge expression (75) with $R_k = I - P_k$. One verifies that $R_k$ is the projection on $\Delta(\ker Q_k)$ as claimed.

The Landau gauge expression (76) follows by taking the limit $\alpha \to 0$. Alternatively one can use the Fadeev- Popov procedure again to pass from from (74) to (76); see the appendix. This completes the proof.

Let $H_k A_k$ be the minimizer of $\frac{1}{2}\|dA\|^2 - (2\alpha)^{-1} < d^T A, R_k d^T A >$ subject to the constraint $Q_k A = A_k$ imposed in (75). Then one can establish the following facts concerning this Feynman gauge minimizer:
1. $H_k$ is independent of $\alpha$ and is also the minimizer for Landau gauge. \[10\]

2. $H_k^x = H_k + dD_k$ for some operator $D_k$. \[11, 15\]

3. $H_k, \partial H_k, \delta_\alpha \partial H_k$ have kernels with exponential decay. ($\delta_\alpha$ is the Holder derivative of order $\alpha$.) \[5, 6\].

Point (2.) says that in a gauge invariant expression we can replace the axial gauge minimizer $H_k^x$ by $H_k$. Point (3.) says this is useful since $H_k$ has good regularity and decay properties. Note in particular that instead of $< A_k, \Delta k A_k >= \|dH_k^x A_k\|^2$ we can take

$$< A_k, \Delta k A_k >= \|dH_k A_k\|^2 \quad (85)$$

3.6 a lower bound on $\Delta k$

For the quadratic form $\Delta k$ we have a lower bound independent of $N, k$.

**Proposition 6.** There is a constant $C$ (depending on $L$) such that for $A$ on $T^0_{N-k}$ and satisfying $QA = 0, \tau A = 0$

$$< A, \Delta k A > \geq C\|A\|^2 \quad (86)$$

**Proof.** The proof follows lemma 2.4 in \[6\], with minor modifications for the covariant axial gauge and specialized to $d = 3$. Start with (85). Using an explicit formula for $H_k$ and working in Fourier transform space one shows (section D in \[5\])

$$< A, \Delta k A > \geq C\|dA\|^2 \quad (87)$$

Thus our claim is reduced to showing $\|dA\|^2 \geq C\|A\|^2$, and since $QA = 0$ this is equivalent to showing that for $\tau A = 0$

$$\|dA\|^2 + \|QA\|^2 \geq C\|A\|^2 \quad (88)$$

The proof divides into three parts

1. We first consider an $L$-cube $B(y)$ centered on $y \in T^1_{N-k}$ and show that (c.f (2.123) in \[6\])

$$\|A\|_{B(y)}^2 \leq 3L^3\|dA\|_{B(y)}^2 \quad (89)$$

To see this note that for any permutations of coordinates $\pi$ and any bond $< x, x + e_\mu >$ in $B(y)$

$$A(\Gamma^\pi_{yx}) + A(x, x + e_\mu) - A(\Gamma^\pi_{y,x+e_\mu}) \quad (90)$$

is a closed curve. Hence it bounds a surface $\Sigma^\pi_{y,x,\mu}$ and by Stokes theorem the expression is equal to $dA(\Sigma^\pi_{y,x,\mu})$. Averaging over permutations gives

$$(\tau A)(y, x) + A(x, x + e_\mu) - (\tau A)(y, x + e_\mu) = \frac{1}{d!}\sum_{\pi} dA(\Sigma^\pi_{y,x,\mu}) \quad (91)$$

But $\tau A = 0$ and so

$$A(x, x + e_\mu) = \frac{1}{d!}\sum_{\pi} dA(\Sigma^\pi_{y,x,\mu}) \quad (92)$$

Now

$$|dA(\Sigma^\pi_{y,x,\mu})| \leq \sum_{p \in \Sigma^\pi_{y,x,\mu}} |dA(p)| \leq \left( \sum_{p \in \Sigma^\pi_{y,x,\mu}} 1 \right)^\frac{1}{2} \left( \sum_{p \in \Sigma^\pi_{y,x,\mu}} |dA(p)|^2 \right)^\frac{1}{2} \leq L^\frac{1}{2}\left( \sum_{p \in \Sigma^\pi_{y,x,\mu}} |dA(p)|^2 \right)^\frac{1}{2} \quad (93)$$
and
\[
\sum_{<x, x+y, \mu> \in B(y)} |dA(\Sigma_{y,x,\mu})|^2 \leq L \sum_{<x, x+y, \mu> \in B(y)} \sum_{p \in \Sigma_{y,x,\mu}} |dA(p)|^2
\]
\[
= L \sum_{y \in B(y)} |dA(p)|^2 \sum_{<x, x+y, : \Sigma_{y,x,\mu} > p} 1 \leq 3L^3 \|dA\|^2_{B(y)}
\] (94)

This yields
\[
\|A\|_{B(y)} \leq \frac{1}{d!} \sum_{\pi} \|dA(\Sigma_{y,\pi})\|_{B(y)} \leq 3^\frac{1}{2} L^\frac{3}{2} \|dA\|_{B(y)}
\] (95)

which proves (89).

2. Now consider bonds or plaquettes that join neighboring unit cubes \(B(y), B(y')\) denoted \(B(y, y')\). One shows that (2.127) in [6]
\[
\sum_{p \in B(y, y')} |dA(p)|^2 + |QA(y, y')|^2
\]
\[
\geq \frac{1}{3L^4} \sum_{b \in B(y, y')} |A(b)|^2 - \frac{1}{L^3} \sum_{b \in B(y)} |A(b)|^2 - \frac{1}{L^3} \sum_{b \in B(y')} |A(b)|^2
\] (96)

Summing over oriented bonds \(<y, y'>\) and using that
\[
\sum_{<y, y'>} \sum_{b \in B(y)} |A(b)|^2 = 3 \sum_{y} \sum_{b \in B(y)} |A(b)|^2
\] (97)

we have
\[
\sum_{<y, y'>} \left( \sum_{p \in B(y, y')} |dA(p)|^2 + |QA(y, y')|^2 \right)
\]
\[
\geq \frac{1}{3L^4} \sum_{<y, y'>} \sum_{b \in B(y, y')} |A(b)|^2 - \frac{6}{L^3} \sum_{y} \sum_{b \in B(y)} |A(b)|^2
\] (98)

3. We combine (89) and (98) to estimate
\[
\|dA\|^2 + \|QA\|^2
\]
\[
= \sum_y \sum_{p \in B(y)} |dA(p)|^2 + \sum_{<y, y'>} \left( \sum_{p \in B(y, y')} |dA(p)|^2 + |QA(y, y')|^2 \right)
\]
\[
\geq \frac{1}{3L^4} \sum_y \sum_{b \in B(y)} |A(b)|^2 + \frac{1}{36} \sum_{<y, y'>} \sum_{b \in B(y, y')} |A(b)|^2 - \frac{6}{L^3} \sum_y \sum_{b \in B(y)} |A(b)|^2
\] (99)
\[
= \frac{1}{6L^3} \sum_y \sum_{b \in B(y)} |A(b)|^2 + \frac{1}{108L^4} \sum_{<y, y'>} \sum_{b \in B(y, y')} |A(b)|^2 \geq \frac{1}{108L^2} \|A\|^2
\]

Thus (88) is established and the proof is complete.
3.7 parametrization of the fluctuation integral

Next we parametrize the fluctuation integrals (73), or more generally integrals of the form

\[ \int \delta(\tau Z) \delta(QZ) f(Z) \exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) DZ / \{ f = 1 \} \]

(100)

where \( \{ f = 1 \} \) is the same integral with \( f = 1 \). The analysis is a variation of [13].

The field \( Z \) is a function on bonds in a unit lattice like \( \mathbb{T}_N^{-k} \). We split the bonds into those in \( L \)-cubes and those joining \( L \)-cubes by \( Z = (Z_1, Z_2) \) where

\[
Z_1 = \{ Z(b) \} \quad b \in \bigcup_y B(y) \\
Z_2 = \{ Z(b) \} \quad b \in \bigcup_{<y,y'>} B(y,y')
\]

(101)

The integral over \( \delta(\tau Z) = \delta(\tau Z_1) \) is just an integral over the subspace \( \ker \tau \) as in section (2.3). Thus with \( \tilde{Z}_1 \in \ker \tau \) we have

\[ \int \left[ \delta(QZ) f(Z) \exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) \right]_{Z=(\tilde{Z}_1,Z_2)} D\tilde{Z}_1 DZ_2 / \{ f = 1 \} \]

(102)

For the remaining delta function we are integrating over the subspace \( QZ = Q(\tilde{Z}_1, Z_2) = 0 \). Let \( b(y,y') \) be the central bond in \( B(y,y') \) and let \( \tilde{Z}_2 \) be the non-central bonds:

\[ \tilde{Z}_2 = \{ Z(b) \} \quad b \in \bigcup_{<y,y'>} \left( B(y,y') - b(y,y') \right) \]

(103)

The condition is now \( Q(\tilde{Z}_1, \tilde{Z}_2, \{ Z(b(y,y')) \} = 0 \) and can be solved for the variables \( Z(b(y,y')) \) and written as

\[ Z(b(y,y')) = \left( S(\tilde{Z}_1, \tilde{Z}_2) \right)(b(y,y')) \]

(104)

for some local linear operator \( S \). For an explicit formula for \( S \) see [13]. Then \( Z_2 = (\tilde{Z}_2, S(\tilde{Z}_1, \tilde{Z}_2)) \) and we can evaluate the delta function by

\[ \int \left[ f(Z) \exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) \right]_{Z=(\tilde{Z}_1,\tilde{Z}_2, S(\tilde{Z}_1, \tilde{Z}_2))} D\tilde{Z}_1 D\tilde{Z}_2 / \{ f = 1 \} \]

(105)

Finally put \( \tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2) \) and define

\[ Z = C\tilde{Z} = (\tilde{Z}, S(\tilde{Z})) \]

(106)

Note that \( C \) is a local operator mapping to the subspace \( QZ = 0, \tau Z = 0 \). Now (105) can be written

\[ \int f(C\tilde{Z}) \exp \left( -\frac{1}{2} < C\tilde{Z}, \Delta_k C\tilde{Z} > \right) D\tilde{Z} / \{ f = 1 \} \]

(107)

Finally define

\[ C_k = (C^T \Delta_k C)^{-1} \]

(108)

and identify the Gaussian measure \( \mu_{C_k} \) with covariance \( C_k \). Then (107) and hence (100) is expressed as

\[ \int f(C\tilde{Z}) d\mu_{C_k}(\tilde{Z}) \]

(109)
3.8 representation for \(C_k\)

To analyze integrals like (109) note that \(C^T \Delta_k C\) is a uniformly bounded strictly positive operator with exponentially decaying kernel by (85), (86). It follows by a lemma of Balaban (4, section 5) that \(C_k = (C^T \Delta_k C)^{-1}\) has the same properties. Then one can employ a cluster expansion to get estimates on the integral. However we eventually want to use a version of the cluster expansion which employs a random walk expansion for \(C_k\). For this Balaban’s lemma is not sufficient. The analysis is not straightforward since \(C_k\) is not the inverse of a local operator.

We develop another representation for \(C_k\). The following is a simpler version of an analysis by Balaban in [7]. There he treats a multi-scale non-abelian problem while here it is single scale and abelian. It is easier to consider \(CC_kC^T\) and the argument has a number of steps.

**Step 1:** Start with the representation for \(J, Z\) on \(T^0_{N-k}\)

\[
\exp \left( \frac{1}{2} < J, CC_k C^T J > \right) = \int e^{<C \tilde{Z}, \Delta_k \tilde{Z}>} \exp \left( -\frac{1}{2} < C \tilde{Z}, \Delta_k \tilde{Z}> \right) D \tilde{Z} \quad \text{if } J = 0
\]

Combining (61) and (76) with \(f = 1\) we have

\[
\exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) = \text{const} \int \delta (Z - Q_k A) \delta (R_k d^T A) \exp \left( -\frac{1}{2} ||dA||^2 \right) DA
\]

and we insert this in (110). It is tempting to now do the integral over \(Z\). But this turns out to lead to difficulties so we postpone it. Instead we further complicate things by inserting for \(\lambda\) on \(T^0_{N-k}\)

\[
1 = \text{const} \int \delta (Q_k+1 \lambda) \delta (R_{k+1} (d^T A - \Delta \lambda)) D \lambda
\]

See Proposition 9 in the appendix for details. This will free up some intermediate gauge fixing. Then (110) becomes

\[
\text{const} \int \delta (Q_k \lambda) \delta (R_k (d^T A)) \delta (R_{k+1} (d^T A - \Delta \lambda)) \delta (Q_k+1 \lambda) \exp \left( -\frac{1}{2} ||dA||^2 + < Z, J > \right) DA \quad DZ \quad DL
\]

**Step 2:** Next let \(A \rightarrow A^\lambda = A - \partial \lambda \) so \(d^T A \rightarrow d^T A + \Delta \lambda\). We also write for \(\mu\) on \(T^0_{N-k}\)

\[
\delta (Q_k+1 \lambda) = \int \delta (Q \mu) \delta (\mu - Q_k \lambda) D \mu
\]

and use \(Q_k \partial = \partial Q_k\) to obtain

\[
\text{const} \int \delta (Q \mu) \delta (\mu - Q_k \lambda) \delta (\tau \lambda) \delta (R_k (d^T A + \Delta \lambda)) \delta (R_{k+1} (d^T A)) \delta (Q_k \lambda) \exp \left( -\frac{1}{2} ||dA||^2 + < Z, J > \right) DA \quad DZ \quad DL \quad D \mu
\]

**Step 3:** Now make the change of variables

\[
\lambda' = \lambda + \lambda_0
\]
where \( \lambda_0 \) is chosen so that \( Q_k \lambda_0 = \mu \) and \( R_k \Delta \lambda_0 = 0 \). These equation have a unique solution which turns out to be [7]

\[
\lambda_0 = G_k^2 Q_k^2 (Q_k G_k Q_k)^{-1} (I - aQ_k G_k Q_k^T) \mu + aG_k Q_k^T \mu
\]

(117)

where \( G_k = (-\Delta + aQ_k^T Q_k)^{-1} \) and \( a > 0 \) is arbitrary. Then \( \delta(\mu - Q_k \lambda) = \delta(Q_k \lambda') \) and since \( Q_k \lambda' = 0 \) we have \( \partial Q_k \lambda = \partial Q_k \lambda' - \partial \mu = -\partial \mu \). Also \( R_k \Delta \lambda = R_k \Delta \lambda' \). Hence we get

\[
\text{const} \int \delta(QZ) \delta(\tau Z) \delta(Z - Q_k A - \partial \mu) \cdot \delta_{R_k} \left( R_k (d^T A + \Delta \lambda') \right) \delta_{R_{k+1}} \left( R_{k+1} d^T A \right)\]

\[
\delta(Q) \delta(Q \lambda') \exp \left( -\frac{1}{2} \|dA\|^2 + <Z, J > \right) D A D Z D \lambda' D \mu
\]

(118)

We now have seven delta functions and the task is to remove all of them.

**step 4:** The integral over \( \lambda' \) is now (Proposition 9 again)

\[
\int \delta(Q \lambda') \delta_{R_k} \left( R_k (d^T A + \Delta \lambda') \right) D \lambda' = \text{const}
\]

(119)

So this leaves us with

\[
\text{const} \int \delta(QZ) \delta(\tau Z) \delta(Z - Q_k A - \partial \mu) \cdot \delta_{R_k} \left( R_k (d^T A + \Delta \lambda) \right) \delta_{R_{k+1}} \left( R_{k+1} d^T A \right) \delta(Q \mu) \]

\[
\exp \left( -\frac{1}{2} \|dA\|^2 + <Z, J > \right) D A D Z D \mu
\]

(120)

**step 5:** Next make the gauge transformation \( Z \rightarrow Z + \partial \mu \). This leaves \( \delta(QZ) \) invariant since \( Q \mu = 0 \) and we have

\[
\text{const} \int \delta(QZ) \delta(\tau Z + \partial \mu) \delta(Z - Q_k A) \cdot \delta_{R_{k+1}} \left( R_{k+1} d^T A \right) \delta(Q \mu) \]

\[
\exp \left( -\frac{1}{2} \|dA\|^2 + <Z + \partial \mu, J > \right) D A D Z D \mu
\]

(121)

**step 6:** In general for \( Z, \mu \) on a unit lattice let \( \mu = MZ \) is the solution of the equations for \( x \in B(y) \)

\[
(\tau(Z + d\mu))(y, x) = 0 \quad x \neq y \quad Q \mu(y) = 0
\]

(122)

This can also be written

\[
(\tau Z)(y, x) + \mu(x) - \mu(y) = 0 \quad x \neq y \quad \mu(y) = -\sum_{x' \neq y} \mu(x')
\]

(123)

The solution is

\[
\mu(x) = MZ(x) = -(\tau Z)(y, x) + L^{-3} \sum_{x' \neq y} (\tau Z)(y, x')
\]

(124)

The integral over \( \mu \) in [121] is

\[
\int \delta(\tau(Z + d\mu)) \delta(Q \mu) \exp \left( <Z + \partial \mu, J > \right) D \mu
\]

(125)

The delta functions select \( \mu = MZ \) and so [122] becomes

\[
\text{const} \int \delta(QZ) \delta(Z - Q_k A) \cdot \delta_{R_{k+1}} \left( R_{k+1} d^T A \right) \exp \left( -\frac{1}{2} \|dA\|^2 + <Z + \partial MZ, J > \right) D A D Z
\]

(126)
Now do the integral over $Z$ and get
\[ \text{const} \int \delta(Q_{k+1}A) \delta_{R_{k+1}}(R_{k+1}d^TA) \exp \left( -\frac{1}{2} \|dA\|^2 + \langle I + \partial M \rangle Q_kA, J > \right) DA \] \hspace{1cm} (127)

**step 7:** Next we change from the delta function gauge fixing $\delta_{R_{k+1}}(R_{k+1}d^TA)$ to exponential gauge fixing given by $\exp(-\frac{1}{2}\|R_{k+1}d^TA\|^2)$. The cost is that we make the gauge transformation
\[ A \rightarrow A - \partial G_{k+1} R_{k+1}d^TA \] \hspace{1cm} (128)

($R_{k+1}, G_{k+1}$ are still on $\mathbb{T}_{N-k}^k$.) This is a Fadeev-Popov argument; see Proposition 10 in the appendix for the details. But this particular gauge transformation changes nothing in (127) as we now explain.

First we claim that $Q_kG_kR_k = 0$. Indeed for any scalar $f$, $R_k f = \Delta \lambda$ for some $\lambda$ satisfying $Q_k \lambda = 0$ and then
\[ Q_kG_kR_k f = Q_kG_k \Delta \lambda = -Q_kG_k(-\Delta + aQ_k^TQ_k) \lambda = -Q_k \lambda = 0 \] \hspace{1cm} (129)

Hence the change in $Q_{k+1}A$ under the gauge transformation is
\[ Q_{k+1} \partial G_{k+1} R_{k+1}d^TA = \partial Q_{k+1} G_{k+1} R_{k+1}d^TA = 0 \] \hspace{1cm} (130)

The change in $(I + \partial M)Q_kA$ under the gauge transformation is
\[ (I + \partial M)Q_k \partial G_{k+1} R_{k+1}d^TA = (I + \partial M) \partial Q_{k+1} G_{k+1} R_{k+1}d^TA = 0 \] \hspace{1cm} (131)

since $Q(Q_kG_{k+1} R_{k+1}d^TA) = Q_{k+1} G_{k+1} R_{k+1}d^TA = 0$ and in general if $Q\mu = 0$ then $M\partial \mu = -\mu$.

Thus our expression has become
\[ \text{const} \int \delta(Q_{k+1}A) \exp \left( -\frac{1}{2} \|dA\|^2 - \frac{1}{2} \|R_{k+1}d^TA\|^2 + \langle I + dM \rangle Q_kA, J > \right) DA \] \hspace{1cm} (132)

**step 8:** The last integral has the form
\[ \int \delta(Q_{k+1}A) \exp \left( -\frac{1}{2} \|dA\|^2 - \frac{1}{2} \|R_{k+1}d^TA\|^2 + \langle A, J > \right) DA \] \hspace{1cm} (133)

Note that for any $a > 0$ we can insert a term $-\frac{1}{2} \|Q_{k+1}A\|^2$ in the exponential. By computing the minimizer in $A$ the exponential subject to $Q_{k+1}A = 0$ one finds that the minimizer is $A = \hat{G}_{k,0}J$

where
\[ \hat{G}_{k,0} = G_{k,0} - G_{k,0}Q_{k+1}^T \left( Q_{k+1} G_{k,0} Q_{k+1}^T \right)^{-1} Q_{k+1} G_{k,0} \] \hspace{1cm} (134)

and where $G_{k,0}$ (also called $G_{k+1}^0$ since it scales to $G_{k+1}$) is the operator on functions on $\mathbb{T}_{N-k}^k$:
\[ G_{k,0} = \left( d^T d + d R_{k+1} d^T + a Q_{k+1}^T Q_{k+1} \right)^{-1} \] \hspace{1cm} (135)

We can also write
\[ \hat{G}_{k,0} = G_{k,0}^+ \left[ I - G_{k,0}^+ Q_{k+1}^T \left( Q_{k+1} G_{k,0} Q_{k+1}^T \right)^{-1} Q_{k+1} G_{k,0}^+ \right] G_{k,0}^+ \] \hspace{1cm} (136)

The bracketed expression is identified as a projection operator, and hence
\[ \hat{G}_{k,0} G_{k,0}^{-1} \hat{G}_{k,0} = \hat{G}_{k,0} \] \hspace{1cm} (137)
The integral \( \int \delta(Q_{k+1}A) \exp \left( -\frac{1}{2} < A, G_{k,0}A > + < A, J > \right) DA \). Expanding around the minimum \( A = \tilde{G}_{k,0}J \) and using the identity (137) we find

\[
\text{(138)}
\]

Therefore (132) can be written

\[
\text{(139)}
\]

This gives the desired representation:

**Proposition 7.**

\[
CC_kC^T = (I + dM)\tilde{Q}_k(I + dM)^T
\]

3.9 **representation for** \( C^\frac{1}{2}_k \)

It will also be useful to have a representation for \( C^\frac{1}{2}_k \) or \( CC^\frac{1}{2}_kC^T \) following [9]. Start with \( C_k = (C^T\Delta_kC)^{-1} \) and represent the square root as

\[
C^\frac{1}{2}_k = \frac{1}{\sqrt{\pi}} \int_0^\infty dx C_{k,x} \quad C_{k,x} = \left( C^T\Delta_kC + x \right)^{-1}
\]

It is sufficient then to find a representation for \( C_{k,x} \) or \( CC_{k,x}C^T \).

Following the proof for \( CC_kC^T \) we start with

\[
\exp \left( \frac{1}{2} < J, CC_{k,x}C^T J > \right) = \text{const} \int e^{<C\tilde{Z},J>} \exp \left( -\frac{1}{2} < C\tilde{Z}, \Delta_kC\tilde{Z} > -\frac{x}{2} ||\tilde{Z}||^2 \right) D\tilde{Z}
\]

But \( C\tilde{Z} = (\tilde{Z}, S(\tilde{Z})) \) where \( S\tilde{Z} \) is defined on the central linking bonds \( b(y,y') \) Thus if \( \chi^* \) is the characteristic function of \( T_{N,k} - \{ b(y,y') \} \) we have \( \tilde{Z} = \chi^*C\tilde{Z} \). Then we can write the integral as

\[
\text{const} \int e^{<Z,J> - \delta(QZ)\delta(\tau Z)} \exp \left( -\frac{1}{2} < Z, \Delta_kZ > -\frac{x}{2} ||\chi^*Z||^2 \right) DZ
\]

The only difference from the previous lemma is the term \(-\frac{x}{2} ||\chi^*Z||^2 \) in the exponential. This has no effect up to step 4 which now reads

\[
\text{const} \int \delta(QZ)\delta(\tau Z) \delta(Z - Q_kA - \partial\mu) \delta_{R_{k+1}} \left( R_{k+1}d^T A \right) \delta(Q\mu)
\]

\[
\exp \left( -\frac{1}{2} ||dA||^2 - \frac{x}{2} ||\chi^*Z||^2 + < Z, J > \right) DA DZ D\mu
\]

In subsequent steps we have \( Z \to Z + \partial\mu \), then \( \mu = MZ \), and then \( Z = Q_kA \). This brings us to

\[
\text{const} \int DA \delta(Q_{k+1}A)
\]

\[
\exp \left( -\frac{1}{2} ||dA||^2 - \frac{1}{2} ||R_{k+1}d^T A||^2 - \frac{x}{2} ||\chi^*(I + \partial M)Q_kA||^2 + < (I + \partial M)Q_kA, J > \right)
\]
Define $\tilde{G}_{k,x}$ by

$$\exp \left( \frac{1}{2} < f, \tilde{G}_{k,x} J > \right)$$

$$= \text{const} \int \delta(Q_{k+1} A) \exp \left( - \frac{1}{2} \| dA \|^2 - \frac{1}{2} \| R_{k+1} dT A \|^2 - \frac{x}{2} \| \chi^*(I + \partial M) Q_k A \|^2 + < A, J > \right) D A$$

(146)

The minimizer of the exponential subject to $Q_{k+1} A = 0$ is $A = \tilde{G}_{k,x} J$ where

$$\tilde{G}_{k,x} = G_{k,x} - G_{k,x} Q_{k+1} \left( Q_{k+1} G_{k,x} Q^T_{k+1} \right)^{-1} Q_{k+1} G_{k,x}$$

(147)

and where for any $a > 0$

$$G_{k,x} = \left( d^T d + dR_{k+1} d^T + a Q^T_{k+1} Q_{k+1} + x Q^T_k (I + dM)^T \chi^*(I + dM) Q_k \right)^{-1}$$

(148)

Expanding around the minimizer (145) becomes

$$\exp \left( \frac{1}{2} < Q^T_{k+1} (I + \partial M)^T J, \tilde{G}_{k,x} Q^T_{k+1} (I + \partial M)^T J > \right)$$

(149)

Thus we have established the representation:

**Proposition 8.**

$$CC_{k,x} C^T = (I + \partial M) Q_k \tilde{G}_{k,x} Q^T_{k+1} (I + \partial M)^T$$

(150)

### 3.10 Summary

Thanks to the parametrization of section 3.7, the fluctuation integral (148) can be written as the Gaussian integral

$$F_{k+1}(A) = \int F_k \left( A_L + H_k^0 \tilde{Z} \right) d\mu_{C_k} (\tilde{Z})$$

(151)

We want to evaluate this at $A = H_{k+1}^0 A_{k+1}$. But since $F_k$ is gauge invariant and since $H_k$ and $H_k^0$ are related by a gauge transformation we can equally evaluate it at the more regular $A = H_{k+1} A_{k+1}$ and with $H_k^0 \tilde{Z}$ replaced by $H_k C \tilde{Z}$. Furthermore we can remove the non-locality from the Gaussian measure by the change of variables $\tilde{Z} = C_k^{1/2} \tilde{W}$ where $\tilde{W}$ is a variable of the same type as $\tilde{Z}$. Then we have

$$F_{k+1}(A) = \int F_k \left( A_L + H_k C C_k^{1/2} \tilde{W} \right) d\mu_L (\tilde{W})$$

(152)

In this last form the fluctuation integral is subject to rigorous analysis. The fields $A = H_{k+1} A_{k+1}$ and $H_k C C_k^{1/2} \tilde{Z}$ have good regularity properties. The operator $C_k^{1/2}$ has a random walk expansion which can be derived from the representation (150). Hence it can be broken into local pieces and the integral can then be treated by the standard technique of a cluster expansion. This means that if $F_k$ has an expansion into local pieces, then one can expand $F_{k+1}$ into local pieces. This is the key issue in studying the mapping $F_k \to F_{k+1}$ and controlling the flow. Variations of this program are carried out in [9], [12], [13].
A change of gauge

We explain how to change between the generalized Landau gauge to a generalized Feynman gauge. First a preliminary result:

Proposition 9. For \( \lambda \) on \( \mathbb{R}^k_{N-k} \) and \( R_k \) the projection onto \( \Delta(\ker Q_k) \) and \( \Delta = -d^T d \)

\[
1 = \text{const} \int \delta(Q_k \lambda) \exp \left( -\frac{1}{2} \| R_k d^T A - \Delta \lambda \|^2 \right) D \lambda
\]

(153)

\[
1 = \text{const} \int \delta(Q_k \lambda) \delta(R_k \Delta(\lambda)) D \lambda
\]

(154)

Remark. Since \( Q_k \lambda = 0 \) we can replace \( \Delta \lambda \) by \( R_k \Delta \lambda \) in these formulas.

Proof. It suffices to show that the mapping \( \lambda \rightarrow (Q_k \lambda, R_k \Delta \lambda) \) from \( \mathbb{R}^k_{N-k} \) to \( \mathbb{R}^0_{N-k} \times \text{ran} R_k \) is a bijection. Then either result follows by making this change of variables. We use that \( \Delta \) is a bijection from \( \ker Q_k \) to \( \text{ran} R_k \). The mapping is injective since if \( Q_k \lambda = 0 \) and \( R_k \Delta \lambda = 0 \) then \( \Delta \lambda = 0 \) and hence \( \lambda = 0 \). The dimensions match since

\[
\dim(\text{ran} Q_k) + \dim(\ker Q_k) = \dim(\mathbb{R}^k_{N-k})
\]

(155)

Hence the result.

Proposition 10. For \( G_k = (-\Delta + aQ_k^T Q_k)^{-1} \) and any \( a \geq 0 \)

\[
\int \delta(Q_k \lambda) \delta(R_k \Delta(\lambda)) f(A) \exp \left( -\frac{1}{2} \| dA \|^2 \right) DA
\]

(157)

\[
\int \delta(Q_k \lambda) \delta(R_k \Delta(\lambda)) f(A) \exp \left( -\frac{1}{2} \| dA \|^2 - \frac{1}{2} \| R_k d^T A \|^2 \right) DA
\]

(158)

Proof. Insert (153) under the integral sign on the left side of (157). Change the order of integration and make the gauge transformation \( A \rightarrow A^\lambda = A - \partial \lambda \). Then \( d^T A \rightarrow d^T A + \Delta \lambda \) and \( Q_k A \) is invariant since \( Q_k \lambda = 0 \). This yields

\[
\text{const} \int \delta(Q_k \lambda) \left[ \int \delta(Q_k A) \delta(R_k \Delta(\lambda)) f(A - \partial \lambda) \exp \left( -\frac{1}{2} \| dA \|^2 - \frac{1}{2} \| R_k d^T A \|^2 \right) DA \right] D \lambda
\]

(158)

Change the order of integration again and do the integral over \( \lambda \). The delta functions in \( \lambda \) select

\[
Q_k \lambda = 0 \quad \Delta \lambda = -R_k d^T A
\]

(159)

or equivalently

\[
Q_k \lambda = 0 \quad (-\Delta + aQ_k^T Q_k) \lambda = R_k d^T A
\]

(160)

The unique solution is (recall \( Q_k G_k R_k = 0 \))

\[
\lambda = G_k R_k d^T A
\]

(161)

Make this replacement in \( f(A - \partial \lambda) \). Then the integral over \( \lambda \) is constant by (154) and we are left with the right side of (157).
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