Radial distribution of Julia sets of derivatives of solutions of complex linear differential equations

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Abstract

In this paper we mainly investigate the radial distribution of Julia set of derivatives of entire solutions of some complex linear differential equations. Under certain conditions, we find the lower bound of it which improve some recent results.

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1 Introduction and main results

In this paper, we assume the reader is familiar with standard notations and basic results of Nevanlinna’s value distribution theory; see [4, 5, 8, 14, 16]. Some basic knowledge of complex dynamics of meromorphic functions is also needed; see [3, 18]. Let $f$ be a meromorphic function in the whole complex plane. We use $\sigma(f)$ and $\mu(f)$ to denote the order and lower order of $f$ respectively; see [16 p.10] for the definitions.

We define $f^n, n \in \mathbb{N}$ denote the $n$th iterate of $f$. The Fatou set $F(f)$ of transcendental meromorphic function $f$ is the subset of the plane $\mathbb{C}$ where the iterates $f^n$ of $f$ form a normal family. The complement of $F(f)$ in $\mathbb{C}$ is called the Julia set $J(f)$ of $f$. It is well known that $F(f)$ is open and completely invariant under $f$, $J(f)$ is closed and non-empty.

We denote $\Omega(\alpha, \beta) = \{z \in \mathbb{C} | \arg z \in (\alpha, \beta)\}$, where $0 < \alpha < \beta < 2\pi$. Given $\theta \in [0, 2\pi)$, if $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap J(f)$ is unbounded for any $\varepsilon > 0$, then we call the ray $\arg z = \theta$ the radial distribution of $J(f)$. Define

$$\Delta(f) = \{\theta \in [0,2\pi)|J(f) \text{ has the radial distribution with respect to } \arg z = \theta\}.$$
Obviously, $\Delta(f)$ is closed and so measurable. We use the $\text{meas}\Delta(f)$ to denote the linear measure of $\Delta(f)$. Many important results of radial distribution of transcendental meromorphic functions have been obtained, for example [2, 10, 11, 12, 13, 19]. Qiao [10] proved that $\text{meas}\Delta(f) = 2\pi$ if $\mu(f) < 1/2$ and $\text{meas}\Delta(f) \geq \pi/\mu(f)$ if $\mu(f) \geq 1/2$, where $f(z)$ is a transcendental entire function of finite lower order. Recently, Huang et al [6, 7] considered radial distribution of Julia set of entire solutions of linear complex differential equations. Their results are stated as follows.

**Theorem A** [6] Let $\{f_1, f_2, \ldots, f_n\}$ be a solution base of

$$f^{(n)} + A(z)f = 0,$$  

(1.1)

where $A(z)$ is a transcendental entire function with finite order, and denote $E = f_1f_2 \cdots f_n$. Then $\text{meas}\Delta(E) \geq \min\{2\pi, \pi/\sigma(A)\}$.

**Theorem B** [7] Let $A_i(z)(i = 0, 1, \ldots, n-1)$ be entire functions of finite lower order such that $A_0$ is transcendental and $m(r, A_i) = o(m(r, A_0)), (i = 1, 2, \ldots, n-1)$ as $r \to \infty$. Then every non-trivial solution $f$ of the equation

$$f^{(n)} + A_{n-1}f^{(n-1)} + \ldots + A_0f = 0$$  

(1.2)

satisfies $\text{meas}\Delta(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$.

For entire functions and their derivatives, the difference between their local properties are astonishing, because a small disturbance of the parameter may cause a gigantic change of the dynamics for some given entire functions. So no one seems to believe that there are some neat relation between them in dynamical properties. However, Qiao [9, 11] proved that the Julia set of a transcendental entire function of finite lower order and its derivative have a large amount of common radial distribution and their distribution densities influence each other. A natural question is that what happens to the radial distribution of Julia set between entire function with infinite lower order and its derivative?

It is easy to know that, by the logarithmic derivative lemma, the non-trivial entire solutions of equations (1.1) and (1.2) have infinite lower order, see details in [6] and [7]. In the present paper, we study the radial distribution of Julia set of the derivatives of entire solutions of equations (1.1) and (1.2) and try to answer that above question partially. Indeed, we obtain the following results.

**Theorem 1.1** Let $A_i(z)(i = 0, 1, \ldots, n-1)$ be entire functions of finite lower order such that $A_0$ is transcendental and $m(r, A_i) = o(m(r, A_0)), (i = 1, 2, \ldots, n-1)$ as $r \to \infty$. Then every non-trivial solution $f$ of the equation (1.2) satisfies $\text{meas}(\Delta(f) \cap \Delta(f^{(k)})) \geq \min\{2\pi, \pi/\mu(A_0)\}$, where $k$ is a positive integer.

**Corollary 1.1** Under the hypothesis of Theorem 1.1 we have $\text{meas}(\Delta(f^{(k)})) \geq \min\{2\pi, \pi/\mu(A_0)\}$, where $k$ is a positive integer.

Obviously, Theorem B is a corollary of Theorem 1.1. For entire solutions of equation (1.1), we have
Corollary 1.2 Assume that $f$ is any non-trivial solution of equation (1.1), we have \[ \text{meas}(\Delta(f^{(k)})) \geq \min\{2\pi, \pi/\mu(A)\} \] where $k$ is a positive integer.

Furthermore, we obtain the following.

Theorem 1.2 Under the hypothesis of Theorem A, we have \[ \text{meas}(\Delta(E^{(k)})) \geq \min\{2\pi, \pi/\sigma(A)\} \] where $k$ is a positive integer.

By Theorem 1.1, we have the next corollary even more.

Corollary 1.3 Suppose that $A_i(z)(i = 0, 1, \ldots, n - 1)$ be entire functions satisfying $\sigma(A_j) < \mu(A_0)(j = 1, 2, \ldots, n - 1)$ and $\mu(A_0)$ is finite. Then every non-trivial solution $f$ of the equation (1.2) satisfies \[ \text{meas}(\Delta(f) \cap \Delta(f^{(k)})) \geq \min\{2\pi, \pi/\mu(A_0)\} \] where $k$ is a positive integer.

2 Preliminary lemmas

At first, we recall the Nevanlinna characteristic in an angle; see [4]. We set

\[ \Omega(\alpha, \beta, r) = \{z : z \in \Omega(\alpha, \beta), |z| < r\} \]
\[ \Omega(r, \alpha, \beta) = \{z : z \in \Omega(\alpha, \beta), |z| \geq r\} \]

and denote by $\overline{\Omega}(\alpha, \beta)$ the closure of $\Omega(\alpha, \beta)$. Let $g(z)$ be meromorphic on the angle $\overline{\Omega}(\alpha, \beta)$, where $\beta - \alpha \in (0, 2\pi]$. Following [4], we define

\[ A_{\alpha, \beta}(r, g) = \frac{w}{\pi} \int_{1}^{r} \left( \frac{1}{t^2} - \frac{t^w}{r^{2w}} \right) \{\log^+ |g(te^{i\alpha})| + \log^+ |g(te^{i\beta})|\} \frac{dt}{t}; \]
\[ B_{\alpha, \beta}(r, g) = \frac{2w}{\pi r^w} \int_{\alpha}^{\beta} \log^+ |g(re^{i\theta})| \sin w(\theta - \alpha) d\theta; \]
\[ C_{\alpha, \beta}(r, g) = 2 \sum_{1 < |b_n| < r} \left( \frac{1}{|b_n|^w} - \frac{|b_n|^w}{r^{2w}} \right) \sin w(\beta_n - \alpha), \]

where $w = \pi/(\beta - \alpha)$, and $b_n = |b_n|e^{i\beta_n}$ are poles of $g(z)$ in $\overline{\Omega}(\alpha, \beta)$ appearing according to their multiplicities. The Nevanlinna angular characteristic is defined as

\[ S_{\alpha, \beta}(r, g) = A_{\alpha, \beta}(r, g) + B_{\alpha, \beta}(r, g) + C_{\alpha, \beta}(r, g). \]

In particular, we denote the order of $S_{\alpha, \beta}(r, g)$ by

\[ \sigma_{\alpha, \beta}(g) = \limsup_{r \to \infty} \frac{\log S_{\alpha, \beta}(r, g)}{\log r}. \]

We call $W$ is a hyperbolic domain if $\overline{\mathbb{C}} \setminus W$ contains three points, where $\overline{\mathbb{C}}$ is the extended complex plane. For an $a \in \mathbb{C} \setminus W$, define

\[ C_W(a) = \inf\{\lambda_W(z)|z - a| : \forall z \in W\}, \]
where \( \lambda_W(z) \) is the hyperbolic density on \( W \). It’s well known that, if every component of \( W \) is simply connected, then \( C_W(a) \geq 1/2 \).

**Lemma 2.1.** ([19] Lemma 2.2) Let \( f(z) \) be an analytic in \( \Omega(r_0, \theta_1, \theta_2) \), \( U \) be a hyperbolic domain, and \( f : \Omega(r_0, \theta_1, \theta_2) \rightarrow U \). If there exists a point \( a \in \partial U \setminus \{\infty\} \) such that \( C_U(a) > 0 \), then there exists a constant \( d > 0 \) such that, for sufficiently small \( \varepsilon > 0 \), we have

\[
|f(z)| = O(|z|^d), \quad z \to \infty, \; z \in \Omega(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon).
\]

The next lemma shows some estimates for the logarithmic derivative of functions being analytic in an angle. Before this, we recall the definition of an R-set; for reference, see [8]. Set \( B(z_n, r_n) = \{z : |z - z_n| < r_n\} \). If \( \sum_{n=1}^{\infty} r_n < \infty \) and \( z_n \to \infty \), then \( \cup_{n=1}^{\infty} B(z_n, r_n) \) is called an R-set. Clearly, the set \( \{z : z \in \cup_{n=1}^{\infty} B(z_n, r_n)\} \) is of finite linear measure.

**Lemma 2.2.** ([7] Lemma 2.2) Let \( z = re^{i\psi}, r_0 + 1 < r \) and \( \alpha \leq \psi \leq \beta \), where \( 0 < \beta - \alpha \leq 2\pi \). Suppose that \( n(\geq 2) \) is an integer, and that \( g(z) \) is analytic in \( \Omega(r_0, \alpha, \beta) \) with \( \sigma_{\alpha, \beta}(g) < \infty \). Choose \( \alpha < \alpha_1 < \beta_1 < \beta \). Then, for every \( \varepsilon_j \in (0, (\beta_j - \alpha_j)/2)(j = 1, 2, \ldots, n - 1) \) outside a set of linear measure zero with

\[
\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \ldots, n - 1.
\]

there exists \( K > 0 \) and \( M > 0 \) only depending on \( g, \varepsilon_1, \ldots, \varepsilon_{n-1} \) and \( \Omega(\alpha_{n-1}, \beta_{n-1}) \), and not depending on \( z \), such that

\[
\left| \frac{g'(z)}{g(z)} \right| \leq Kr^M (\sin k(\psi - \alpha))^{-2}
\]

and

\[
\left| \frac{g^{(n)}(z)}{g(z)} \right| \leq Kr^M \left( \sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_{\varepsilon_j}(\psi - \alpha_j) \right)^{-2}
\]

for all \( z \in \Omega(\alpha_{n-1}, \beta_{n-1}) \) outside an R-set \( D \), where \( k = \pi/(\beta - \alpha) \) and \( k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j)(j = 1, 2, \ldots, n - 1) \).

**Lemma 2.3.** ([15, 18]) Let \( f(z) \) be a transcendental meromorphic function with lower order \( \mu(f) < \infty \) and order \( 0 < \sigma(f) \leq \infty \). Then, for any positive number \( \lambda \) with \( \mu(f) \leq \lambda \leq \sigma(f) \) and any set \( H \) of finite measure, there exists a sequence \( \{r_n\} \) satisfies

1. \( r_n \not\in H, \lim_{n \to \infty} r_n/n = \infty \);
2. \( \liminf_{n \to \infty} \log T(r_n, f)/\log r_n \geq \lambda \);
3. \( T(r, f) < (1 + o(1))(2t/r_n)\lambda T(r_n/2, f), t \in [r_n/n, nr_n]; \)
4. \( t^{-\lambda - \varepsilon_n}T(t, f) \leq 2^{\lambda + 1} r_n^{-\lambda - \varepsilon_n} T(r_n, f), 1 \leq t \leq nr_n, \varepsilon_n = (\log n)^{-2} \).

Such \( \{r_n\} \) is called a sequence of Pólya peaks of order \( \lambda \) outside \( H \). The following lemma, which related to Pólya peaks, is called the spread relation; see [1].

**Lemma 2.4.** ([1]) Let \( f(z) \) be a transcendental meromorphic function with positive order and finite lower order, and has a deficient value \( a \in \mathbb{C} \). Then, for any sequence of Pólya peaks \( \{r_n\} \)
of order $\lambda > 0$, $\mu(f) \leq \lambda \leq \sigma(f)$, and any positive function $\Upsilon(r) \to 0$ as $r_n \to \infty$, we have

$$\liminf_{r_n \to \infty} \text{meas} D_{\Upsilon}(r_n, a) \geq \min \left\{ 2\pi, \frac{4}{\lambda} \arcsin \sqrt{\frac{\delta(a, f)}{2}} \right\},$$

where

$$D_{\Upsilon}(r, a) = \left\{ \theta \in [0, 2\pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \Upsilon(r) T(r, f) \right\},$$

and

$$D_{\Upsilon}(r, \infty) = \left\{ \theta \in [0, 2\pi) : \log^+ |f(re^{i\theta})| > \Upsilon(r) T(r, f) \right\}.$$

3 Proof of Theorems

Proof of theorem 1.1 We know that every non-trivial solution $f$ of the equation is an entire function with infinite lower order. We obtain the assertion by reduction to contradiction. Assume that

$$\text{meas} (\Delta(f) \cap \Delta(f^{(k)})) < \nu = \min \{2\pi, \pi/\mu(A_0)\}$$

and so

$$\xi := \nu - \text{meas} (\Delta(f) \cap \Delta(f^{(k)})) > 0.$$ (3.1)

Applying Lemma 2.3 to $A_0$, we have a Pólya peak $\{r_j\}$ of order $\mu(A_0)$ with all $r_j \notin H$. Since $A_0$ is transcendental entire function, it follows the Nevanlinna deficient $\delta(\infty, A_0) = 1$. By Lemma 2.4, for the Pólya peak $\{r_j\}$, we have

$$\liminf_{r_j \to \infty} \text{meas} (D_{\Upsilon}(r_j, \infty)) \geq \pi/\mu(A_0),$$

where the function $\Upsilon(r)$ is defined by

$$\Upsilon(r) = \max \left\{ \sqrt{\frac{\log r}{m(r, A_0)}}, \sqrt{\frac{m(r, A_0)}{m(r, A_i)}}, i = 1, 2, \ldots, n-1 \right\}$$

and $m(r, A_j)$ is the proximation function of $A_j, j = 0, 1, \ldots, n-1$. Obviously, $\Upsilon(r)$ is positive and $\lim_{r \to \infty} \Upsilon(r) = 0$. For the sake of simplicity, we denote $D_{\Upsilon}(r_j, \infty)$ by $D(r_j)$ in the following. We shall show that there must exist an open interval

$$I = (\alpha, \beta) \subset \Delta(f^{(k)})^c, \quad 0 < \beta - \alpha < \nu$$

such that

$$\lim_{j \to \infty} \text{meas} (\Delta(f) \cap D(r_j) \cap I) > 0,$$

where $\Delta(f^{(k)})^c := [0, 2\pi) \setminus \Delta(f^{(k)})$. In order to achieve this goal, we shall prove the following firstly.

$$\lim_{j \to \infty} \text{meas} (D(r_j) \setminus \Delta(f)) = 0.$$ (3.5)
Otherwise, suppose that there is a subseries \( \{r_{jk}\} \) such that
\[
\lim_{k \to \infty} \text{meas}(D(r_{jk}) \setminus \Delta(f)) > 0, \quad (3.8)
\]
then there exists \( \theta_0 \in \Delta(f)^c \) and \( \eta > 0 \) satisfying
\[
\lim_{k \to \infty} \text{meas}((\theta_0 - \eta, \theta_0 + \eta) \cap (D(r_{jk}) \setminus \Delta(f))) > 0. \quad (3.9)
\]
Since \( \arg z = \theta_0 \) is not a radial distribution of \( J(f) \), there exists \( r_0 > 0 \) such that
\[
\Omega(r_0, \theta_0 - \eta, \theta_0 + \eta) \cap J(f) = \emptyset. \quad (3.10)
\]
This implies that there exists an unbounded component \( U \) of Fatou set \( F(f) \), such that \( \Omega(r_0, \theta_0 - \eta, \theta_0 + \eta) \subset U \). Take a unbounded and connected set \( \Gamma \subset \partial U \), the mapping \( f : \Omega(r_0, \theta_0 - \eta, \theta_0 + \eta) \to \mathbb{C} \setminus \Gamma \) is analytic. Since \( \mathbb{C} \setminus \Gamma \) is simply connected, then for any \( a \in \Gamma \setminus \{\infty\} \), we have \( C_{\mathbb{C} \setminus \Gamma}(a) \geq 1/2 \). Now applying Lemma 2.1 to \( f \) in \( \Omega(r_0, \theta_0 - \eta, \theta_0 + \eta) \), for any \( \zeta > 0, \zeta < \eta \), we have
\[
|f(z)| = O(|z|^{d_1}), \quad z \in \Omega(r_0, \theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta), \quad |z| \to \infty, \quad (3.11)
\]
where \( d_1 \) is a positive constant. Recalling the definition of \( S_{\alpha, \beta}(r, f) \), we immediately get that
\[
S_{\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta}(r, f) = O(1). \quad (3.12)
\]
Therefore, by Lemma 2.2, there exists constants \( M > 0 \) and \( K > 0 \) such that
\[
\left| \frac{f^{(s)}(z)}{f(z)} \right| \leq Kr^M, \quad (s = 1, 2, \ldots, n - 1), \quad (3.13)
\]
for all \( z \in \Omega(r_0, \theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta) \), outside a R-set \( H \).

Since \( \zeta \) can be chosen sufficiently small, from \( (3.9) \) we have
\[
\lim_{k \to \infty} \text{meas}((\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta) \cap D(r_j)) > 0. \quad (3.14)
\]
Thus, we can find an infinite series \( \{r_{jk} e^{i \theta_j} \} \) such that for all sufficiently large \( k \),
\[
\log^+ |A_0(r_{jk} e^{i \theta_j})| > \Upsilon(r_{jk}) T(r_{jk}, A_0) = \Upsilon(r_{jk}) m(r_{jk}, A_0) \quad (3.15)
\]
where \( \theta_j \in F_{jk} := (\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta) \cap D(r_j) \). Then, for sufficiently large \( k \), we have
\[
\int_{F_{jk}} \log^+ |A_0(r_{jk} e^{i \theta_j})| d\theta \geq \text{meas}(F_{jk}) \Upsilon(r_{jk}) m(r_{jk}, A_0). \quad (3.16)
\]
On the other hand, combining \( (3.12) \) and \( (3.9) \) leads to
\[
\int_{F_{jk}} \log^+ |A_0(r_{jk} e^{i \theta_j})| d\theta \leq \int_{F_{jk}} \left( \sum_{s=1}^{n} \log^+ \left| \frac{f^{(s)}(r_{jk} e^{i \theta_j})}{f(r_{jk} e^{i \theta_j})} \right| + \sum_{i=1}^{n-1} \log^+ |A_i(r_{jk} e^{i \theta_j})| \right) d\theta + O(1)
\]
\[
= \int_{F_{jk}} \left( \sum_{i=1}^{n-1} \log^+ |A_i(r_{jk} e^{i \theta_j})| \right) d\theta + O(\log r_{jk})
\]
\[
\leq \sum_{i=1}^{n-1} m(r_{jk}, A_i) + O(\log r_{jk})
\]
\[
\leq c_0 \Upsilon^2(r_{jk}) m(r_{jk}, A_0) \quad (3.17)
\]
where $c_0$ is a positive constant. From (3.16) and (3.17), we have

$$0 < \text{meas}(F_{j_k}) \leq c_0 \Upsilon(r_{j_k})$$

(3.18)

which contradicts to the fact $\Upsilon(r_{j_k}) \to 0$ as $k \to \infty$. This contradiction implies (3.7) is valid.

By Theorem B, we know that

$$\text{meas} \Delta(f) \geq \nu.$$  

(3.19)

From Lemma 2.4, we have, for all sufficiently large $j$ and any positive $\varepsilon$,

$$\text{meas} D(r_j) > \nu - \varepsilon.$$  

(3.20)

Combining (3.7), (3.19) and (3.20) follows that, for all sufficiently large $j$,

$$\text{meas}(\Delta(f) \cap D(r_j)) \geq \nu - \varepsilon/4,$$

(3.21)

where $\xi$ is defined in (3.2). Since $\Delta(f)$ is closed, clearly $\Delta(f)^c$ is open, so it consists of at most countably open intervals. We can choose finitely many open intervals $I_j, (j = 1, 2, \ldots, m)$ satisfying

$$I_j \subset \Delta(f)^c, \; \text{meas}(\Delta(f)^c \cup \bigcup_{i=1}^m I_i) < \xi/4.$$  

(3.22)

Since, for sufficiently large $j$,

$$\text{meas}(\Delta(f) \cap D(r_j) \cap (\bigcup_{i=1}^m I_i)) + \text{meas}(\Delta(f) \cap D(r_j) \cap \Delta(f))$$

$$= \text{meas}(\Delta(f) \cap D(r_j) \cap (\Delta(f)^c \cup (\bigcup_{i=1}^m I_i))) \geq \nu - \varepsilon/2,$$

(3.23)

we have

$$\text{meas}(\Delta(f) \cap D(r_j) \cap (\bigcup_{i=1}^m I_i)) \geq \nu - \varepsilon/2 - \text{meas}(\Delta(f) \cap D(r_j) \cap \Delta(f))$$

$$\geq \nu - \varepsilon/2 - \text{meas}(\Delta(f) \cap \Delta(f)) = \xi/2.$$  

(3.24)

Thus, there exists an open interval $I_{\tilde{r}_0} = (\alpha, \beta) \subset \bigcup_{i=1}^m I_i \subset \Delta(f)^c$ such that, for infinitely many sufficiently large $j$,

$$\text{meas}(\Delta(f) \cap D(r_j) \cap I_{\tilde{r}_0}) \geq \frac{\xi}{2m} > 0.$$  

(3.25)

Then, we prove (3.6) holds.

From (3.6), we know that there are $\tilde{\theta}_0$ and $\tilde{\eta} > 0$ such that

$$(\tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta}) \subset I$$

(3.26)

and

$$\lim_{j \to \infty} \text{meas}(\Delta(f) \cap D(r_j) \cap (\tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta})) > 0.$$  

(3.27)

Then, there exists $\tilde{r}_0$ such that $\Omega(\tilde{r}_0, \tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta}) \cap J(f^{(k)}(z)) = \emptyset$. By the similar argument between (3.10) and (3.11), for any $\zeta > 0, \zeta < \tilde{\eta}$, we have

$$|f^{(k)}(z)| = O(|z|^{d_2}), \; z \in \Omega(\tilde{r}_0, \tilde{\theta}_0 - \tilde{\eta} + \tilde{\zeta}, \tilde{\theta}_0 + \tilde{\eta} - \tilde{\zeta}), \; |z| \to \infty,$$

(3.28)
where $d_2$ is a positive constant. By (3.27) we can choose an unbounded series \( \{ r_j e^{i \theta_j} \} \), for all sufficiently large \( j \) such that

\[
\log^+ |A_0(r_j e^{i \theta_j})| > \Upsilon(r_j) m(r_j, A_0),
\]

where

\[
\theta_j \in \Delta(f) \cap D(r_j) \cap (\tilde{\theta}_0 - \bar{\eta}, \bar{\theta}_0 + \bar{\eta}).
\]

Fixed \( r_j e^{i \theta_j} \), and take a \( r_j e^{i \theta_j} \in \{ r_j e^{i \theta_j} \} \). Take a simple Jordan arc \( \gamma \) in \( \Omega(\tilde{r}_0, \tilde{\theta}_0 - \bar{\eta}, \tilde{\theta}_0 + \bar{\eta}) \) which connecting \( r_j e^{i \theta_j} \) to \( r_j e^{i \theta_j} \) along \( |z| = r_j \), and connecting \( r_j e^{i \theta_j} \) to \( r_j e^{i \theta_j} \) along \( \arg z = \theta_j \). For any \( z \in \gamma \), \( \gamma_z \) denotes a part of \( \gamma \), which connecting \( r_j e^{i \theta_j} \) to \( z \). Let \( L(\gamma) \) be the length of \( \gamma \). Clearly,

\[
L(\gamma) = O(r_j), \quad j \to \infty.
\]

By (3.28), it follows

\[
|f^{(k-1)}(z)| \leq \int_{\gamma_z} |f^{(k)}(z)||dz| + c_k \\
\leq O(|z| d^2 L(\gamma)) + c_k \\
\leq O(r_j^{d^2+1}), \quad j \to \infty.
\]

Similarly, we have

\[
|f^{(k-2)}(z)| \leq \int_{\gamma_z} |f^{(k-1)}(z)||dz| + c_{k-1} \\
\leq O(r_j^{d^2+2}), \quad j \to \infty.
\]

\[
\vdots
\]

\[
|f(z)| \leq \int_{\gamma_z} |f'(z)||dz| + c_1 \\
\leq O(r_j^{d^2+k}), \quad j \to \infty.
\]

where \( c_1, c_2, \ldots, c_k \) are constants, which are independent of \( j \). Therefore,

\[
S_{\tilde{\theta}_0 - \bar{\eta} + \tilde{\zeta}, \bar{\theta}_0 + \bar{\eta} - \tilde{\zeta}}(r, f) = O(1).
\]

By Lemma 2.2, we know (3.13) also holds for all \( z \in \Omega(\tilde{r}_0, \tilde{\theta}_0 - \bar{\eta} + \tilde{\zeta}, \tilde{\theta}_0 + \bar{\eta} - \tilde{\zeta}) \), outside a R-set \( H \). Combining (3.13) and (3.29), and applying the similar argument as (3.16) and (3.17), we can deduce a contradiction. Therefore, it follows

\[
\text{meas}(\Delta(f) \cap \Delta(f^{(k)})) \geq \min\{2\pi, \pi/\mu(A)\}.
\]

(3.32)

The proof is complete.

**Proof of theorem 1.2** The main idea of this proof comes from that of the proof of Theorem 1.1 in [6], but need some changes. We assume that \( \text{meas}(\Delta(E^{(k)})) < \min\{2\pi, \pi/\sigma(A)\} \). By similar argument in [6], there exists an angular domain \( \Omega(\alpha, \beta) \) such that

\[
\Omega(\alpha, \beta) \cap \Delta(E^{(k)}) = \emptyset, \quad \Omega(r_0, \alpha, \beta) \cap J(E^{(k)}) = \emptyset
\]

(3.33)
for sufficiently large $r_0$. Then by the same method between (3.10) and (3.11), we have

$$|E^{(k)}(z)| = O(|z|^d), \quad z \in \Omega(r_0, \alpha, \beta), \quad |z| \to \infty,$$

where $d$ is a positive constant. Take a simple Jordan arc $\gamma$, which connected points $z_0$ and $z$, satisfying $\gamma \subset \Omega(r_0, \alpha, \beta)$. Applying the method which is used in (3.30), we obtain

$$|E(z)| = O(|z|^{d+k}), \quad z \in \Omega(r_0, \alpha, \beta), \quad |z| \to \infty.$$

Therefore, Theorem 1.2 can be proved word by word following the proof of Theorem 1.1 in [6].

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