Perturbation Calculation of the Axial Anomaly of a Ginsparg-Wilson lattice Dirac Operator

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Abstract

A recent proposal suggests that even if a Ginsparg-Wilson lattice Dirac operator does not possess any topological zero modes in topologically-nontrivial gauge backgrounds, it can reproduce correct axial anomaly for sufficiently smooth gauge configurations, provided that it is exponentially-local, doublers-free, and has correct continuum behavior. In this paper, we calculate the axial anomaly of this lattice Dirac operator in weak coupling perturbation theory, and show that it recovers the topological charge density in the continuum limit.

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1 Introduction

Recently, one of us (TWC) has constructed a Ginsparg-Wilson Dirac operator which is $\gamma_5$-hermitian, exponentially-local, doublers-free, and has correct continuum behavior, but it does not possess any topological zero modes for topologically-nontrivial background gauge fields. This suggests that one might have the option to turn off the topological zero modes of a Ginsparg-Wilson lattice Dirac operator, without affecting its physical behaviors (axial anomaly, fermion propagator, etc.) at least for the topologically-trivial gauge sector. Therefore it is interesting to verify explicitly that it indeed reproduces the continuum axial anomaly for smooth gauge backgrounds, in the framework of weak coupling perturbation theory which is amenable to analytic calculations.

The Ginsparg-Wilson lattice Dirac operator proposed in Ref. [1] is

$$D = a^{-1}D_c(1 + rD_c)^{-1}, \quad r > 0,$$

with

$$D_c = \sum_\mu \gamma_\mu T^\mu, \quad T^\mu = ft^\mu f,$$

$$f = \left(\frac{2c}{\sqrt{t^2 + w^2}}\right)^{1/2}, \quad t^2 = -\sum_\mu t^\mu t^\mu.$$  

Here $\gamma^\mu t^\mu$ is the naive lattice fermion operator and $-w$ is the Wilson term with a negative mass $-c$ ($0 < c < 2$)

$$t^\mu(x, y) = \frac{1}{2}\left[U_\mu(x)\delta_{x+\hat{\mu}, y} - U_\mu(y)\delta_{x-\hat{\mu}, y}\right],$$

$$U_\mu(x) = \exp\left[iagA_\mu\left(x + \frac{a}{2}\hat{\mu}\right)\right],$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix},$$

$$\sigma^\mu \sigma^\nu + \sigma^\nu \sigma^\mu = 2\delta^\mu_\nu,$$

$$w(x, y) = c - \frac{1}{2}\sum_\mu \left[2\delta_{x,y} - U_\mu(x)\delta_{x+\hat{\mu}, y} - U_\mu(y)\delta_{x-\hat{\mu}, y}\right], \quad 0 < c < 2,$$

where the Dirac, color and flavor indices have been suppressed. Note that in [1], we do not fix $r = 1/2c$, but keep it as a free parameter, since we intend to show that the axial anomaly is independent of $r$ in the continuum limit.

Evidently, the lattice Dirac operator (1) is $\gamma_5$-hermitian

$$D^\dagger = \gamma_5 D\gamma_5,$$
and satisfies the Ginsparg-Wilson relation \[2\]

\[D\gamma_5 + \gamma_5 D = 2raD\gamma_5 D . \] (10)

In the free fermion limit, \(D\) is exponentially-local, doublers-free, and has correct continuum behavior \[1\]. These properties should be sufficient to guarantee that \(D\) can reproduce correct axial anomaly in a smooth gauge background.

In the next section, we calculate the axial anomaly of (1) in weak coupling perturbation theory, and show that, in the continuum limit, it recovers the topological charge density of the gauge background

\[\frac{g^2}{32\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F_{\mu\nu}F_{\lambda\sigma}) ,\]

where

\[F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ig[A_{\mu}, A_{\nu}] .\]

Before we proceed to the perturbation calculations in the next section, we first clarify that even though a Ginsparg-Wilson lattice Dirac operator can reproduce the continuum axial anomaly for sufficiently smooth gauge configurations, it does not necessarily imply that it could possess topological zero modes for topologically-nontrivial gauge backgrounds. This can be seen as follows.

Consider a gauge configuration with positive definite topological charge \(Q > 0\), and its topological charge density \(\rho(x)\) is very smooth for all \(x\),

\[\int d^4x \; \rho(x) = Q . \] (11)

Then, in principle, \(\rho(x)\) can be decomposed into two parts,

\[\rho(x) = \rho_w(x) + \rho_0(x) , \] (12)

where \(\rho_w(x)\) is the "winding number density" which is positive for all \(x\), and it is the smoothest particular solution of (11),

\[\int d^4x \; \rho_w(x) = Q , \; \rho_w(x) > 0 \; \forall x , \] (13)

and \(\rho_0(x)\) is the solution of the homogeneous equation of (11),

\[\int d^4x \; \rho_0(x) = 0 . \] (14)

If one assumes that the amount of the topological charge \(Q\) is bounded by a constant times the square root of the space-time volume, then in the infinite
volume and continuum limit, $\rho_w(x)$ becomes infinitesimally small at any $x$, thus it becomes unobservable. In other words, even if one calculates the axial anomaly $A(x)$ of a Ginsparg-Wilson lattice Dirac operator, and shows that it agrees with $\rho(x) \simeq \rho_0(x)$ in the infinite volume and continuum limit, however, one still cannot be sure whether the integral $\int d^4x A(x)$ is equal to the topological charge $Q$ or not.

In order to have a definite answer to this question, one needs to perform a nonperturbative calculation of the axial anomaly on a finite lattice. Otherwise, it could hardly detect the ”winding number density” $\rho_w(x)$ at a site. However, to our knowledge, there is no preceding nonperturbative analytic calculations of the axial anomaly for any Ginsparg-Wilson lattice Dirac operator, on a finite lattice. Therefore, in this paper, we do not intend to tackle the difficult problem of performing a nonperturbative analytic calculation of the axial anomaly on a finite lattice.

Nevertheless, from the argument presented in Ref. [1], it is clear that (1) does not possess any topological zero modes. Thus, it follows that, on a finite lattice, the axial anomaly for at least some of the lattice sites could not agree with the topological charge density, since its sum over all sites is equal to the zero index, rather than the nonzero topological charge $Q$ of the gauge background. Therefore, for any topologically-nontrivial gauge background, we have no difficulties to understand why the axial anomaly of (1) can not be exactly equal to the topological charge density for all sites on a finite lattice. However, in the infinite volume and continuum limit, $\rho_w(x) \to 0$, then the axial anomaly $A(x)$ can appear to agree with the topological charge density $\rho(x)$ for sufficiently smooth gauge backgrounds, even though the integral $\int d^4x A(x)$ is always zero. Thus, for a sufficiently smooth gauge background ( except for any possible ”winding” at the infinity ), we can calculate the axial anomaly ( at any point except the infinity ) of the lattice Dirac operator (1) in weak coupling perturbation theory, and show that in the continuum limit, it agrees with the topological charge density of the gauge background, regardless of the global topological charge.

2 Perturbation Calculation of the Axial Anomaly

The axial anomaly of a flavor-singlet of Ginsparg-Wilson lattice fermions can be written as

$$A(x) = \text{tr}[\gamma_5(\mathds{1} - raD)(x,x)]$$

$$= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{i(p-q)\cdot x} \text{tr}[\gamma_5(\mathds{1} - raD)(p,q)] , \quad (15)$$

$^1$It is understood that the axial anomaly at a lattice site has to be normalized by the volume of the unit cell, $a^4$, when it is compared with the topological charge density in the continuum.
where the trace runs over the Dirac, color and flavor space. Now we expand $D$ in power series of the gauge coupling (through the link variables)

$$D = D_0 + D_1 + D_2 + D_3 + D_4 + O(g^5)$$

(16)

where $D_n$ denotes the terms containing the factor $g^n$, the gauge coupling to the $n$-th power. Then we have

$$\mathbb{1} - raD = e_0 - e_0d_1e_0 - e_0(d_2 - d_1e_0d_1)e_0 - e_0(d_3 - d_1e_0d_2 - d_2e_0d_1 + d_1e_0d_1e_0d_1)e_0$$

$$- e_0(d_4 - d_1e_0d_3 - d_3e_0d_1 - d_2e_0d_2 + d_1e_0d_1e_0d_2 + d_1e_0d_2e_0d_1 +$$

$$+ d_2e_0d_1e_0d_1 - d_1e_0d_1e_0d_1e_0d_1)e_0 + O(g^5)$$

(17)

where

$$e_0 = \mathbb{1} - raD_0 \equiv (\mathbb{1} + d_0)^{-1},$$

(18)

$$d_1 = r \sum_\mu \gamma_\mu(f_1t_1^\mu f_0 + f_0t_1^\mu f_1),$$

(19)

$$d_2 = r \sum_\mu \gamma_\mu(f_2t_2^\mu f_0 + f_0t_2^\mu f_2 + f_1t_1^\mu f_2 + f_1t_2^\mu f_1 + f_0t_1^\mu f_1),$$

(20)

$$d_3 = r \sum_\mu \gamma_\mu \sum_{i=0}^{3} \sum_{j=0}^{3-i} (f_it_j^\mu f_{3-i-j}),$$

(21)

$$d_4 = r \sum_\mu \gamma_\mu \sum_{i=0}^{4} \sum_{j=0}^{4-i} (f_it_j^\mu f_{4-i-j}),$$

(22)

$$f = f_0 + f_1 + f_2 + f_3 + f_4 + O(g^5),$$

(23)

$$t^\mu = t_0^\mu + t_1^\mu + t_2^\mu + t_3^\mu + t_4^\mu + O(g^5).$$

(24)

Evidently some terms in $\mathbb{1}$ do not contribute to the axial anomaly, since they only have a factor of a product less than four distinct $\gamma$ matrices, thus they vanish after taking the trace with $\gamma_5$ in (15). Note that each term of $d_n$ only has one factor of $\gamma$ matrix, and $e_0$ contributes at most one factor of $\gamma$ matrix [see Eq. (13)]. Thus, the terms $e_0$, and $e_0d_0e_0$, and for any n) can be dropped from the series, when (17) is substituted into (15).

Then the axial anomaly at the order $g^2$ is due to the fourth term in (17),

$$\mathcal{A}_2(x) = \int_p \int_q \int_k e^{i(p-q)k} \text{tr}[\gamma_5e_0(p)d_1(p,k)e_0(k)d_1(k,q)e_0(q)]$$

(25)

where

$$\int_p \equiv \int \frac{d^4p}{(2\pi)^4}, \text{ etc.}$$

Similarly, substituting higher order terms of (17) into (15) gives the axial anomaly at orders $g^3$ and $g^4$, respectively. Although there seems to be
many terms involving higher order vertices (\(d_2, d_3\), etc.) at higher orders, the nonvanishing contributions at orders \(g^3\) and \(g^4\) only come from terms \(e_0 d_1 e_0 d_1 e_0\) and \(e_0 d_1 e_0 d_1 e_0 e_0 d_1\) respectively, as we shall show later. At this point, it is instructive to identify 
\(e_0 = (\mathbb{1} + d_0)^{-1}\) \(e_0\) with the free fermion propagator, \(d_1\) with the "quark-quark-gluon" vertex, \(d_2\) with the "quark-quark-gluon-gluon" vertex, etc. Then the expression 
\[ \text{tr}(\gamma_5 e_0 d_1 e_0 d_1 e_0) \]
contributing to the axial anomaly at \(O(g^2)\), \(e_0\), corresponds to the triangle diagram coupling the axial current to two external gauge bosons through an internal fermion loop. Similarly, the expressions 
\[ \text{tr}(\gamma_5 e_0 d_1 e_0 d_1 e_0) \] and 
\[ \text{tr}(\gamma_5 e_0 d_1 e_0 d_1 e_0 d_1 e_0) \]
contributing to the axial anomaly at \(O(g^3)\) and \(O(g^4)\), correspond to the quadrilateral (box) and the pentagon diagrams coupling the axial current to three and four external gauge bosons, respectively. These are exactly the Feynman diagrams \(2\) which contribute to the axial anomaly in continuum nonabelian gauge theories \(3\). Note that even though higher order vertices are allowed at finite lattice spacing, the symmetry forbids their contributions to the axial anomaly, thus only the "quark-quark-gluon" vertex \(d_1\) can enter the fermion loop contributing to the axial anomaly.

In the following three subsections, we evaluate the axial anomaly at orders \(g^2\), \(g^3\) and \(g^4\) respectively.

### 2.1 The Axial Anomaly at the Order \(g^2\)

To evaluate \(25\), first, we obtain \(d_1\) by expanding \(f\) and \(t^\mu\) in power series of the gauge coupling, \(23\) and \(24\).

Consider

\[ h = \sqrt{t^2 + w^2} + w = h_0 + h_1 + h_2 + h_3 + h_4 + O(g^5), \]

which satisfies the identity \(f \cdot f \cdot h = 2c\), i.e.,

\[ (f_0 + f_1 + \cdots)(h_0 + f_1 + \cdots)(h_0 + h_1 + \cdots) = 2c. \]

To order \(g\), it gives

\[ f_0 f_0 h_0 = 2c, \]
\[ f_1 f_0 h_0 + f_0 f_1 h_0 + f_0 f_0 h_1 = 0. \]

Solving these two equations, we obtain

\[ f_0(p, k) = \left(\frac{2c}{\sqrt{t_0^2(p) + w_0^2(p) + w_0(p)}}\right)^{1/2} \delta^4(p - k) \equiv f_0(p) \delta^4(p - k), \]
\[ f_1(p, k) = -\frac{1}{2c} \frac{f_0^2(p) f_0^2(k)}{f_0(p) + f_0(k)} h_1(p, k), \]

\(^2\)See, for example, Figure 22.2 in Ref. \(4\)
where

\[ t_0^2(p) = - \sum_\mu t_0^\mu(p) t_0^\mu(p) = \sum_\mu \sin^2(p_\mu a) \quad . \]  

(30)

Similarly, we obtain

\[ h_1(p, k) = \frac{w_1(p, k)[w_0(p) + w_0(k)] - \sum_\mu t_1^\mu(p, k)[t_0^\mu(p) + t_0^\mu(k)]}{\sqrt{t_0^2(p) + w_0^2(p) + \sqrt{t_0^2(k) + w_0^2(k)}}} + w_1(p, k) \]

(31)

Here we recall some well-known basic formulas in weak coupling perturbation theory (see Appendix A),

\[ t_\mu(p, k) = t_\mu^0(p) + O(g^2) \quad , \]

\[ w(p, k) = w_0(p) + O(g^2) \quad , \]

where

\[ t_0^\mu(p) = i \sin(p_\mu a) \quad , \]

\[ w_0(p) = c - \sum_\mu [1 - \cos(p_\mu a)] \quad , \]

\[ t_1^\mu(p, k) = g \tilde{A}_\mu(p - k) \partial_\mu t^0 \left( \frac{p + k}{2} \right) \quad , \]

\[ w_1(p, k) = \sum_\mu g \tilde{A}_\mu(p - k) \partial_\mu w_0 \left( \frac{p + k}{2} \right) \quad , \]

\[ \tilde{A}_\mu(p - k) = \sum_x e^{-i(p - k) \cdot (x + \frac{a_\mu}{2})} A_\mu \left( x + \frac{a}{2} \right) \quad . \]

Therefore, the axial anomaly (25) at the order \( g^2 \) can be written as

\[ \mathcal{A}_2(x) = g^2 \int_p \int_q e^{i(p - q) \cdot x} \sum_{\mu, \nu} \int_k \text{tr}\{ \tilde{A}_\mu(p - k) \tilde{A}_\nu(k - q) \} \times \text{tr}[\gamma_5 e_0(p)d_{1,\mu}(p, k)e_0(k)d_{1,\nu}(k, q)e_0(q)] \quad , \]

(32)

where

\[ e_0(p) = 1 - r a D_0(p) = \frac{1}{1 + d_0(p)} = \frac{1 - r f_0^2(p) \gamma \cdot t_0(p)}{1 + r^2 f_0^4(p) t_0^2(p)} \quad , \]

\[ \equiv b(p) + \sum_\mu \gamma^\mu c^\mu(p) \quad , \]

(33)

\[ d_0(p) = r f_0^2(p) \gamma \cdot t_0(p) \equiv \sum_\mu \gamma^\mu d_0^\mu(p) \quad , \]

(34)

\[ d_1(p, k) = g \sum_\mu \tilde{A}_\mu(p - k)d_{1,\mu}(p, k) \quad , \]

(35)
\[ d_{1,\mu}(p, k) = r [f_{1,\mu}(p, k)\gamma \cdot t_0(k) f_0(k) + f_0(p)\gamma \cdot t_0(p) f_{1,\mu}(p, k) + f_0(p)\gamma^\mu \partial^\mu \left(\frac{p + k}{2}\right) f_0(k)] , \]  
\[ f_{1,\mu}(p, k) = -\frac{1}{2c} \frac{f_0^2(p) f_0^2(k)}{f_0(p) + f_0(k)} \times \]
\[ \left[ \partial_\mu w_0 \left(\frac{p + k}{2}\right) + \frac{\partial_\mu w_0 \left(\frac{p + k}{2}\right) (w_0(p) + w_0(k)) - \partial_\mu t_0^\mu \left(\frac{p + k}{2}\right) (t_0^\mu(p) + t_0^\mu(k))}{\sqrt{t_0^2(p) + w_0^2(p) + \sqrt{t_0^2(k) + w_0^2(k)}}} \right] . \]

(37)

In the limit \( k = p \), (37) and (36) reduce to
\[ f_{1,\mu}(p, p) = \partial_\mu f_0(p) , \]  
\[ d_{1,\mu}(p, p) = \partial_\mu [r f_0(p)\gamma \cdot t_0(p) f_0(p)] = \partial_\mu d_0(p) . \]

(38)  
(39)

To evaluate the integral in (32), we change the variables \( p \to p + k \) and \( q \to q + k \), then (32) becomes
\[ \mathcal{A}_2(x) = g^2 \int_p \int_q e^{i(p-q)\cdot x} \sum_{\mu,\nu} \text{tr} \{ \tilde{A}_\mu(p) \tilde{A}_\nu(-q) \} G_{\mu\nu}(p, q) \]

(40)

where
\[ G_{\mu\nu}(p, q) = \int_k \text{tr} [\gamma_5 e_0(p + k)d_{1,\mu}(p + k, k)e_0(k)d_{1,\nu}(k, q + k)e_0(q + k)] . \]

In the following, we show that
\[ G_{\mu\nu}(p, q) = \sum_{\lambda,\sigma} p_\lambda q_\sigma \frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q)|_{p,q=0} + O(a) \]
\[ = -M(c) \sum_{\lambda,\sigma} 4 \ p_\lambda q_\sigma \epsilon_{\mu\nu\lambda\sigma} + O(a) \]

(41)

which, when substituted into (30), leads to the axial anomaly at the order \( g^2 \),
\[ \mathcal{A}_2(x) = g^2 M(c) \sum_{\mu,\nu,\lambda,\sigma} \epsilon_{\mu\nu,\lambda,\sigma} \text{tr} \{ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda)(x) \} , \]

(42)

where \( M(c) \) is a coefficient which tends to \( \frac{1}{32\pi^2} \) ( for \( 0 < c < 2 \) ) in the continuum limit.

We expand \( G_{\mu\nu}(p, q) \) in power series of \( p \) and \( q \),
\[ G_{\mu\nu}(p, q) = G_{\mu\nu}(0, 0) + \sum_{\lambda} p_\lambda \frac{\partial}{\partial p_\lambda} G_{\mu\nu}(p, q)|_{p,q=0} + \sum_{\sigma} q_\sigma \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q)|_{p,q=0} \]
\[ + \sum_{\lambda,\sigma} p_\lambda q_\sigma \frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q)|_{p,q=0} + O(a) . \]

(43)
It is easy to see that the zeroth order and the first order terms in (43) vanish, by symmetry and the basic properties of $\gamma$ matrices,

$$\text{tr}(\gamma^5) = \text{tr}(\gamma^5\gamma^\alpha) = \text{tr}(\gamma^5\gamma^\alpha\gamma^\beta) = \text{tr}(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 0,$$

(44)

$$\text{tr}(\gamma^5\gamma^\alpha\beta\gamma^\gamma\gamma^\sigma) = \text{tr}(\mathbb{1})\epsilon_{\alpha\beta\gamma\delta}.$$

(45)

Explicitly,

$$G_{\mu\nu}(0, 0) = \int_k \text{tr}\{\gamma_5 e_0(k) \partial_\mu d_0(k) e_0(k) \partial_\nu d_0(k) e_0(k)\}$$

$$= \text{tr}(\mathbb{1}) \int_k \epsilon_{\alpha\beta\gamma\delta}(k) c^\alpha(k) c^\delta(k) b(k) \partial_\mu d_0^\beta(k) \partial_\nu d_0^\gamma(k)$$

(46)

where (33), (34), (39), (44) and (45) have been used. Note that repeated indices are summed over in Eqs. (46), (50), (51), (54) and (55). Evidently the integrand in (46) vanishes due to contraction of the completely antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$ with the symmetric tensor $c^\alpha(k) c^\delta(k), i.e.,$

$$\sum_{\alpha\delta} \epsilon_{\alpha\beta\gamma\delta}(k) c^\alpha(k) c^\delta(k) = 0.$$  

(47)

Hence, we have

$$G_{\mu\nu}(0, 0) = 0.$$  

Next consider the first order term of the power series (43),

$$\frac{\partial}{\partial p^\lambda}G_{\mu\nu}(p, q)|_{p, q=0}$$

$$= \int_k \{ \text{tr}[\gamma_5 e_0(k) \partial_\mu^\lambda d_1_{\mu\nu}(p, k)] e_0(k) d_1_{\mu\nu}(k, k) e_0(k)\}$$

$$+ \text{tr}[\gamma_5 \partial_\lambda e_0(k) d_1_{\mu\nu}(k, k) e_0(k) d_1_{\mu\nu}(k, k) e_0(k)]$$

(48)

From (36), it is easy to see that

$$\partial_\mu^\lambda d_1_{\mu\nu}(p, k)|_{p=k} = \sum_{\sigma} \gamma^\sigma h^\sigma_{\lambda\mu},$$

(49)

where each term has one factor of $\gamma$ matrix, and the explicit expression of $h^\sigma_{\lambda\mu}$ is not required for our purpose.

Then the first term and the second term of the integrand (48) both vanish,

$$\text{tr}[\gamma_5 e_0(k) \gamma \cdot h^\sigma_{\lambda\mu}(k) e_0(k) \partial_\mu d_0(k) e_0(k)]$$

$$= \text{tr}(\mathbb{1}) \epsilon_{\alpha\beta\gamma\delta}(k) c^\alpha(k) c^\beta(k) b(k) h^\gamma_{\lambda\mu}(k) \partial_\mu d_0^\delta(k) = 0,$$

(50)

and

$$\text{tr}[\gamma_5 \partial_\lambda e_0(k) \partial_\mu d_0(k) e_0(k) \partial_\nu d_0(k) e_0(k)]$$

$$= \text{tr}(\mathbb{1}) (\epsilon_{\delta\alpha\beta\gamma} + \epsilon_{\delta\alpha\sigma\beta}) c^\beta(k) b(k) \partial_\lambda c^\delta(k) \partial_\mu d_0^\gamma(k) \partial_\nu d_0^\delta(k)$$

$$+ \text{tr}(\mathbb{1}) \epsilon_{\alpha\beta\gamma\delta}(k) c^\beta(k) c^\delta(k) \partial_\lambda b(k) \partial_\mu d_0^\alpha(k) \partial_\nu d_0^\gamma(k)$$

$$= 0,$$

(51)
where (33), (34), (39), (49), (44), (45) and (47) have been used. Thus

\[ \frac{\partial}{\partial p_\lambda} G_{\mu\nu}(p, q) \big|_{p,q=0} = 0 . \]

Similarly, we have

\[ \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q) \big|_{p,q=0} = 0 . \]  \tag{52}

Now consider the second order term of the power series (43),

\[ \frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q) \big|_{p,q=0} \]

\[ = \int_k \left\{ \text{tr} \left[ \gamma_5 \partial_\lambda e_0(k) \right] d_1,\mu(k, k) e_0(k) \partial_\sigma^q d_1,\mu(k, q) |_{q=k} e_0(k) \right\] 

\[ + \text{tr} \left[ \gamma_5 e_0(k) \partial_\mu^\lambda d_1,\mu(p, k) |_{p=k} e_0(k) d_1,\mu(k, k) \partial_\sigma e_0(k) \right] \]

\[ + \text{tr} \left[ \gamma_5 e_0(k) \partial_\mu^\lambda d_1,\mu(p, k) |_{p=k} e_0(k) d_1,\mu(k, q) |_{q=k} e_0(k) \right] \]

\[ + \text{tr} \left[ \gamma_5 \partial_\lambda e_0(k) d_1,\mu(k, k) e_0(k) d_1,\mu(k, k) \partial_\sigma e_0(k) \right] \} . \]  \tag{53}

Evidently, the first three terms of the integrand are zero, by symmetry. Explicitly, the first term is equal to

\[ \text{tr} \left[ \gamma_5 \partial_\lambda e_0(k) \right] d_1,\mu(k, k) e_0(k) \partial_\sigma^q d_1,\mu(k, q) |_{q=k} e_0(k) \]

\[ = \text{tr} (\mathbb{1}) \left( \epsilon_{\delta\alpha\beta\gamma} + \epsilon_{\delta\alpha\gamma\beta} \right) c^\beta(k) b(k) \partial_\lambda c^\delta(k) \partial_\mu^\lambda d_0^\sigma(k) h_\alpha^\gamma(k) \]

\[ + \text{tr} (\mathbb{1}) \epsilon_{\alpha\beta\delta\gamma} c^\beta(k) c^\delta(k) \partial_\lambda b(k) \partial_\mu^\lambda d_0^\sigma(k) h_\gamma^\alpha(k) \]

\[ = 0 , \]  \tag{54}

the second term is also zero since it has the same form of the first, and the third term is

\[ \text{tr} \left[ \gamma_5 e_0(k) \right] \gamma \cdot h_{\lambda\mu}(k) e_0(k) \gamma \cdot h_{\sigma\nu}(k) e_0(k) \]

\[ = \text{tr} (\mathbb{1}) \epsilon_{\alpha\beta\delta\gamma} c^\alpha(k) c^\delta(k) b(k) h_\lambda^\beta(k) h_\gamma^\alpha(k) h_\sigma^\mu(k) = 0 , \]  \tag{55}

where (33), (34), (39), (40), (41), (42) and (47) have been used. Thus, (53) becomes

\[ \frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q) \big|_{p,q=0} \]

\[ = \int_k \text{tr} \left\{ \partial_\lambda \left[ \gamma_5 e_0(k) \right] \right\} \partial_\mu d_0(k) e_0(k) \partial_\sigma d_0(k) e_0(k) \} . \]  \tag{56}

From (33) and (34), it is easy to see that \( e_0(k) \) also satisfies the Ginsparg-Wilson relation

\[ \gamma_5 + e_0^{-1}(k) \gamma_5 e_0(k) = 2 \gamma_5 e_0(k) , \]  \tag{57}
which, after differentiation with respect to $k_\lambda$, gives

\[
[\partial_\lambda e_0^{-1}(k)]\gamma_5 e_0(k) + e_0^{-1}(k)\gamma_5[\partial_\lambda e_0(k)] = 2\partial_\lambda[\gamma_5 e_0(k)].
\] (58)

Substituting (58) and $d_0(k) = e_0^{-1}(k) - 1$ [from Eq. (58)] into the integrand of (56), then the integrand becomes

\[
\frac{1}{2}\text{tr}\{\partial_\lambda e_0^{-1}(k) \gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu e_0^{-1}(k) \partial_\sigma e_0(k)\}
\]

\[+\frac{1}{2}\text{tr}\{e_0^{-1}(k) \gamma_5 \partial_\lambda e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu e_0^{-1}(k) \partial_\sigma e_0(k)\}
\]

which can be further reduced to

\[
-\frac{1}{2}\text{tr}\{\gamma_5 \partial_\mu e_0(k) \partial_\nu e_0^{-1}(k) \partial_\sigma e_0(k) \partial_\lambda e_0^{-1}(k)\}
\]

\[+\frac{1}{2}\text{tr}\{\gamma_5 \partial_\lambda e_0(k) \partial_\mu e_0^{-1}(k) \partial_\nu e_0(k) \partial_\sigma e_0^{-1}(k)\},
\] (59)

where the identities

\[e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) = -\partial_\mu e_0(k)
\] (60)

\[e_0^{-1}(k) \partial_\mu e_0(k) e_0^{-1}(k) = -\partial_\mu e_0^{-1}(k)
\] (61)

have been used. By symmetry, it is obvious that the contribution of the second expression of (59) is equal to that of the first one, thus (56) becomes

\[I_{\mu\nu\lambda\sigma} \equiv \frac{\partial}{\partial p_{\lambda}} \frac{\partial}{\partial q_{\sigma}} G_{\mu\nu}(p, q)|_{p, q = 0}
\]

\[= \int \frac{d^4 k}{(2\pi)^4} \text{tr}\{\gamma_5 \partial_\mu e_0(k) \partial_\nu e_0^{-1}(k) \partial_\lambda e_0(k) \partial_\sigma e_0^{-1}(k)\}
\]

\[= \int \frac{d^4 k}{(2\pi)^4} \text{tr}\left\{\gamma_5 \partial_\mu \left(\frac{1}{1 + d_0(k)}\right) \partial_\nu d_0(k) \partial_\lambda \left(\frac{1}{1 + d_0(k)}\right) \partial_\sigma d_0(k)\right\}
\] (62)

where the domain of integration is the 4-torus, $\mathcal{T}_4 = \times_{i=1}^4 [-\pi/a, \pi/a]$, in which the endpoints $(\pm \pi/a)$ in each direction are identified to be the same point. In the limit $a \to 0$, $\mathcal{T}_4 \to \times_{i=1}^4 (-\infty, \infty)$, which is invariant under the transformation

\[k_\mu \to -\frac{1}{r^2 k_\mu}, \quad \mu = 1, \cdots, 4,
\] (63)

for any $r \neq 0$.

Now we change the variables according to (64), then (62) becomes

\[I_{\mu\nu\lambda\sigma} = \int \frac{d^4 k}{(2\pi)^4} \text{tr}\left\{\gamma_5 \partial_\mu \left(\frac{1}{1 + d_0(K)}\right) \partial_\nu d_0(K) \partial_\lambda \left(\frac{1}{1 + d_0(K)}\right) \partial_\sigma d_0(K)\right\}
\] (64)
where the arguments of \(d_0\) are \(K_\mu = -(r^2 k_\mu)^{-1}\). Note that \(dk_\mu \partial_\mu\) is invariant under the transformation \((63)\).

Therefore, our ansatz of evaluating \((64)\) in the continuum limit is to substitute \(d_0(K)\) by \(d_0^{-1}(k)\) in the integrand of \((64)\), since \(d_0(k) \to irak\) as \(a \to 0\), and the trace with \(\gamma_5\) picks out four distinct \(\gamma\) matrices, one from each of the four factors with partial derivatives. Then \((64)\) reads as

\[
I_{\mu\nu\lambda\sigma} = \int \frac{d^4k}{(2\pi)^4} \text{tr} \left\{ \gamma_5 \partial_\mu \left( \frac{d_0(k)}{1 + d_0(k)} \right) \partial_\nu d_0^{-1}(k) \partial_\lambda \left( \frac{d_0(k)}{1 + d_0(k)} \right) \partial_\sigma d_0^{-1}(k) \right\}
\]

\[
= \int \frac{d^4k}{(2\pi)^4} \text{tr} \left\{ \gamma_5 \partial_\mu h(k) \partial_\nu h^{-1}(k) \partial_\lambda h(k) \partial_\sigma h^{-1}(k) \right\}, \tag{65}
\]

where

\[
h(k) = \frac{d_0(k)}{1 + d_0(k)}.
\]

The integral \((65)\) can be evaluated by first removing an infinitesimal ball \(B_\epsilon\) of radius \(\epsilon\) from the origin \((k = 0)\) of the 4-torus \(T_4\), then performing the integration, and finally taking \(\epsilon \to 0\), i.e.,

\[
I_{\mu\nu\lambda\sigma} = \frac{1}{16\pi^4} \lim_{\epsilon \to 0} \int_{T_4 \setminus B_\epsilon} d^4k \text{tr} \left\{ \gamma_5 \partial_\mu h(k) \partial_\nu h^{-1}(k) \partial_\lambda h(k) \partial_\sigma h^{-1}(k) \right\} \tag{66}
\]

\[
= \frac{1}{16\pi^4} \lim_{\epsilon \to 0} \int_{T_4 \setminus B_\epsilon} d^4k \partial_\mu \text{tr} \left\{ \gamma_5 h(k) \partial_\nu h^{-1}(k) \partial_\lambda h(k) \partial_\sigma h^{-1}(k) \right\} \tag{67}
\]

where the \(\partial_\mu\) operation in \((67)\) produces \((66)\), plus three terms which are symmetric in \(\mu\nu\), \(\mu\lambda\), and \(\mu\sigma\), respectively, hence neither one of these three terms contributes to the integral.

Then according to the Gauss theorem, the volume integral over \(T_4 \setminus B_\epsilon\) can be expressed as a surface integral on the surface \(S_\epsilon\) of the ball \(B_\epsilon\), provided that the integrand is continuous in \(T_4 \setminus B_\epsilon\). The last condition is satisfied since \(h(k) = r a D_0(k)\) is analytic, and \(h(k) \neq 0\) (free of species doublings, for \(0 < c < 2\)) for any \(k \in T_4 \setminus B_\epsilon\) \(\mathbb{I}\). Thus \((67)\) becomes

\[
I_{\mu\nu\lambda\sigma} = \frac{1}{16\pi^4} \lim_{\epsilon \to 0} \int_{S_\epsilon} d^3s n_\mu \text{tr} \left\{ \gamma_5 h(k) \partial_\nu h^{-1}(k) \partial_\lambda h(k) \partial_\sigma h^{-1}(k) \right\} \tag{68}
\]

where \(n_\mu\) is the \(\mu\)-th component of the outward normal vector on the surface \(S_\epsilon\). Since \(d_0(k) \to irak\) as \(k \to 0\), we can set \(d_0(k) = irak\) on the surface \(S_\epsilon\) and obtain

\[
I_{\mu\nu\lambda\sigma} = \frac{1}{16\pi^4} \lim_{\epsilon \to 0} \int_{S_\epsilon} d^3s n_\mu \text{tr} \left\{ \gamma_5 \left( \frac{\bar{k}}{1 + r^2 a^2 k^2} \right) \frac{\gamma_\nu}{k^2} \left( \frac{\gamma_\lambda}{1 + r^2 a^2 k^2} \right) \frac{\gamma_\sigma}{k^2} \right\}
\]

\[
= \frac{1}{16\pi^4} \text{tr}(\mathbb{I}) \epsilon_{\mu\nu\lambda\sigma} \lim_{\epsilon \to 0} \int_{S_\epsilon} d^3s \frac{n_\mu k_\mu}{k^3(1 + r^2 a^2 k^2)^2} \tag{69}
\]
where we have used (44), (45), and the property
\[ \int_{S} d^{3}s \, n_{\mu}k_{\nu}F(k^{2}) = \delta_{\mu\nu} \int_{S} d^{3}s \, n_{\mu}k_{\mu}F(k^{2}) . \]

Finally, the result of the integral (69) is
\[ I_{\mu\nu\lambda\sigma} = -N_{f} \frac{4\pi}{4\pi^{4}}\epsilon_{\mu\nu\lambda\sigma} \lim_{\epsilon \to 0} \frac{1}{(1 + r^{2}a^{2}\epsilon^{2})^{2}} \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta_{2} \sin \theta_{2} \int_{0}^{\pi} d\theta_{1} \sin^{2} \theta_{1} \cos^{2} \theta_{1} \]
\[ = -N_{f} \frac{\epsilon_{\mu\nu\lambda\sigma}}{8\pi^{2}} , \] (70)

where \( N_{f} \) denotes the number of fermion flavors. Note that (70) is independent of the parameter \( r \) in \( D \). From (70) and (41), we obtain
\[ M(c) = \frac{N_{f}}{32\pi^{2}} , \quad 0 < c < 2 , \] (71)

and the axial anomaly (12) at the order \( g^{2} \),
\[ A_{2}(x) = \frac{g^{2}N_{f}}{32\pi^{2}} \sum_{\mu \nu \lambda \sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr}\{(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial_{\lambda}A_{\sigma} - \partial_{\sigma}A_{\lambda})(x)\} , \] (72)

where
\[ A_{\mu}(x) = \int_{p} e^{ip \cdot x} \tilde{A}_{\mu}(p) , \text{ etc.} \]

### 2.2 The Axial Anomaly at the Order \( g^{3} \)

Inserting the order \( g^{3} \) terms of (14) into (15), we get
\[ A_{3}(x) = g^{3} \int_{q} \int_{r} e^{i(p-q) \cdot x} \{ J_{a}(p, q) + J_{b}(p, q) \} - g^{3} \int_{p} \int_{r} \int_{q} e^{i(p-q) \cdot x} J_{c}(p, r, q) \] (73)

where
\[ J_{a}(p, q) = \int_{k} \text{tr}\{\gamma_{5}e_{0}(p)d_{1}(p, k)e_{0}(k)d_{2}(k, q)e_{0}(q)\} , \]
\[ J_{b}(p, q) = \int_{k} \text{tr}\{\gamma_{5}e_{0}(p)d_{2}(p, k)e_{0}(k)d_{1}(k, q)e_{0}(q)\} , \]
\[ J_{c}(p, r, q) = \int_{k} \text{tr}\{\gamma_{5}e_{0}(p)d_{1}(p, r)e_{0}(r)d_{1}(r, k)e_{0}(k)d_{1}(k, q)e_{0}(q)\} , \]

and \( e_{0} \) and \( d_{1} \) are given in (33) and (35) respectively. It is easy to show that \( J_{a} \) and \( J_{b} \) vanish in the continuum limit, and only \( J_{c} \) has nonzero contribution to the axial anomaly at the order \( g^{3} \). To proceed, we derive \( d_{2} \) (20) by expanding \( f \) and \( t^{\nu} \) in power series of the gauge coupling, (23) and (24), similar to the

\footnote{the term \( e_{0}d_{3}e_{0} \) has been dropped since \( \text{tr}(\gamma_{5}e_{0}d_{3}e_{0}) = 0. \)
A procedure of deriving $d_1$ in Section 2.1. First, we solve for $f_2$ from the higher order equations in (27), then together with the formulas of $t_n^\mu$ (126) and $w_n$ (127) derived in the Appendix A, we obtain

$$d_2(p, k) = g^2 \sum_\mu \tilde{A}_\mu^{(2)}(p - k) d_{2,\mu}(p, k) = g^2 \sum_{\mu, \nu} \tilde{A}_\mu^{(2)}(p - k) \gamma^\nu d_{2,\mu}(p, k) ,$$ (74)

where $\tilde{A}_\mu^{(2)}(p - k)$ is defined in (118), and the explicit expression of $d_{2,\mu}(p, k)$ is not required for our subsequent calculations. In general, to any order of $g$, we have

$$d_n(p, k) = g^n \sum_\mu \tilde{A}_\mu^{(n)}(p - k) d_{n,\mu}(p, k) = g^n \sum_{\mu, \nu} \tilde{A}_\mu^{(n)}(p - k) \gamma^\nu d_{n,\mu}(p, k) ,$$ (75)

which can be used for higher order calculations.

Using (35) and (74), and changing variables $p \rightarrow p + k$ and $q \rightarrow q + k$ in $J_a$ and $J_b$, while $p \rightarrow p + r + k$, $r \rightarrow r + k$ and $q \rightarrow q + k$ in $J_c$, then we can rewrite (73) as

$$A_3(x) = g^3 \int_p \int_q e^{i(p-q)x} \{ I_a(p, q) + I_b(p, q) \} - g^3 \int_p \int_q e^{i(p+q-k)x} I_c(p, q, r)$$ (76)

where

$$I_a(p, q) = \sum_{\mu, \nu} \text{tr}[\tilde{A}_\mu(p) \tilde{A}_\nu^{(2)}(-q)] G^{\mu\nu}_a(p, q) ,$$ (77)

$$I_b(p, q) = \sum_{\mu, \nu} \text{tr}[\tilde{A}_\mu(p) \tilde{A}_\nu(-q)] G^{\mu\nu}_b(p, q) ,$$ (78)

$$I_c(p, r, q) = \sum_{\mu, \nu, \lambda} \text{tr}[\tilde{A}_\mu(p) \tilde{A}_\nu(r) \tilde{A}_\lambda(-q)] G^{\mu\nu\lambda}_c(p, r, q) ,$$ (79)

and

$$G^{\mu\nu}_a(p, q) = \int_k \text{tr}[\gamma_5 e_0(p + k) d_{1,\mu}(p + k, k) e_0(k) d_{2,\nu}(k, q + k) e_0(q + k)]$$ (80)

$$G^{\mu\nu}_b(p, q) = \int_k \text{tr}[\gamma_5 e_0(p + k) d_{2,\mu}(p + k, k) e_0(k) d_{1,\nu}(k, q + k) e_0(q + k)]$$ (81)

$$G^{\mu\nu\lambda}_c(p, r, q) = \int_k \text{tr}[\gamma_5 e_0(p + r + k) d_{1,\mu}(p + r + k, r + k) e_0(r + k) \times d_{1,\nu}(r + k, k) e_0(k) d_{1,\lambda}(k, q + k) e_0(q + k)] .$$ (82)

In the following, we show that $G_a$ and $G_b$ vanish in the continuum limit. First, we expand (80) in powers series of $p$ and $q$,

$$G^{\mu\nu}_a(p, q) = G^{\mu\nu}_a(0, 0) + \sum_\sigma p^\sigma \frac{\partial}{\partial p^\sigma} G^{\mu\nu}_a(p, 0)|_{p=0} + \sum_\sigma q^\sigma \frac{\partial}{\partial q^\sigma} G^{\mu\nu}_a(0, q)|_{q=0} + O(a) ,$$ (83)
where the zeroth order and first order terms can be easily shown to be zero, by symmetry. Explicitly,

\[ G_0^{\mu\nu}(0, 0) = \int d_k [\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{2,\nu}(k, k) e_0(k)] = 0 , \]

since the integrand vanishes identically,

\[ \text{tr}[\gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) d_{2,\nu}(k, k) e_0(k)] = -\text{tr}[\gamma_5 \partial_\mu e_0(k) d_{2,\nu}(k, k) e_0(k)] = 0 , \]

where Eqs. (33), (39), (60) and (44) have been used.

Since its integrand is equal to

\[ \text{tr}[\gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) d_{2,\nu}(k, k) e_0(k)] = -\text{tr}[\gamma_5 \partial_\mu e_0(k) d_{2,\nu}(k, k) e_0(k)] = 0 , \]

where Eqs. (33), (34), (84), (44), (45) and (47) have been used. Similarly, we have

\[ \frac{\partial}{\partial p^\sigma} G_0^{\mu\nu}(p, 0)|_{p=0} \]

\[ = \int k \{ \gamma_5 (\partial_\sigma e_0(k) \partial_\mu d_0(k) + e_0(k) \gamma \cdot h_{\sigma\mu}(k)) e_0(k) d_{2,\nu}(k, k) e_0(k) \} = 0 , \]

since its integrand is equal to

\[ \text{tr}(r) \epsilon_{\alpha\beta\lambda\delta} \left\{ \left[ \partial_\sigma b(k) \partial_\mu d_0^\alpha(k) + b(k) h_{\sigma\mu}^\alpha(k) \right] c^\beta(k) c^\delta(k) d_{2,\nu}^\lambda(k) \right\} \]

\[ + \text{tr}(r) \epsilon_{\alpha\beta\lambda\delta} \left\{ \left[ \partial_\sigma c^\alpha(k) \partial_\mu d_0^\beta(k) + c^\alpha(k) h_{\sigma\mu}^\beta(k) \right] c^\lambda(k) d_{2,\nu}^\delta(k) \right\} = 0 , \]

where (33), (34), (84), (44), (45) and (47) have been used. Similarly, we have

\[ \frac{\partial}{\partial q^\sigma} G_0^{\mu\nu}(0, q)|_{q=0} \]

\[ = \int k \{ \gamma_5 e_0(k) \partial_\mu d_0(k) e_0(k) [\gamma \cdot g_{\sigma\nu}(k) e_0(k) + d_{2,\nu}(k, k) \partial_\sigma e_0(k) \} \} = 0 . \]

Therefore \( G_0^{\mu\nu}(p, q) = O(a) \rightarrow 0 \) in the limit \( a \rightarrow 0 \). By the same token, \( G_6^{\mu\nu}(p, q) = 0 \) in the continuum limit.

Next we expand \( G_c \) in power series of \( p, q \) and \( r \),

\[ G_c^{\mu\nu}(p, r, q) = G_c^{\mu\nu}(0, 0, 0) + \sum_{\sigma} p^\sigma \frac{\partial}{\partial p^\sigma} G_c^{\mu\nu}(p, 0, 0)|_{p=0} \]

\[ + \sum_{\sigma} r^\sigma \frac{\partial}{\partial r^\sigma} G_c^{\mu\nu}(0, r, 0)|_{r=0} + \sum_{\sigma} q^\sigma \frac{\partial}{\partial q^\sigma} G_c^{\mu\nu}(0, 0, q)|_{q=0} + O(a) . \]

\[ \text{Note that each term of } d_1 \text{ (19) and } d_2 \text{ (20) has only one factor of } \gamma \text{ matrix.} \]

\[ 4 \]
It is easy to see that the zeroth order term vanishes,
\[ G_{c}^{\mu\nu\lambda}(0, 0, 0) = \int k \text{tr} \left[ \gamma_{5}e_{0}(k)d_{1,\mu}(k, k)e_{0}(k)d_{1,\nu}(k, k)e_{0}(k)d_{1,\lambda}(k, k)e_{0}(k) \right] = 0 \]  
(87)

since its integrand vanishes,
\[ \text{tr} \left[ \gamma_{5}e_{0}(k)d_{1,\mu}(k, k)e_{0}(k)d_{1,\nu}(k, k)e_{0}(k)d_{1,\lambda}(k, k)e_{0}(k) \right] = \text{tr} \left[ \gamma_{5}\partial_{\mu}e_{0}(k)\partial_{\nu}e_{0}^{-1}(k)\partial_{\lambda}e_{0}(k) \right] = 0 , \]
where (39), (60) and (44) have been used.

Next we evaluate the first order terms
\[ \frac{\partial}{\partial p}\sigma G_{c}^{\mu\nu\lambda}(p, 0, 0)|_{p=0} = -\int k \text{tr} \left\{ \gamma_{5}[\partial_{\sigma}e_{0}(k)\partial_{\mu}e_{0}^{-1}(k) + e_{0}(k)\gamma \cdot h_{\sigma\mu}(k)]e_{0}(k)\partial_{\nu}e_{0}^{-1}(k)\partial_{\lambda}e_{0}(k) \right\} \]
\[ = -\int k \text{tr} \left\{ \gamma_{5}[\partial_{\sigma}e_{0}(k)\partial_{\mu}e_{0}^{-1}(k)]e_{0}(k)\partial_{\nu}e_{0}^{-1}(k)\partial_{\lambda}e_{0}(k) \right\} , \]
(88)

where (39), (84) and (60) have been used, and the last equality is due to the vanishing of the second term (containing the factor \( \gamma \cdot h_{\sigma\mu}(k) \)) in the integrand,
\[ \text{tr} \left\{ \gamma_{5}[\partial_{\sigma}e_{0}(k)\gamma \cdot h_{\sigma\mu}(k)]e_{0}(k)\partial_{\nu}e_{0}^{-1}(k)\partial_{\lambda}e_{0}(k) \right\} \]
\[ = \text{tr} \left( \mathbb{I} \right)e_{\alpha\beta\gamma\delta}\partial_{\lambda}b(k)c^{\alpha}(k)c^{\gamma}h_{\sigma\mu}(k)\partial_{\nu}d_{\delta}^{0}(k) \]
\[ + \text{tr} \left( \mathbb{I} \right)(\epsilon_{\alpha\beta\gamma\delta} + \epsilon_{\beta\alpha\gamma\delta})b(k)c^{\alpha}(k)\partial_{\lambda}c^{\delta}(k)h_{\sigma\mu}(k)\partial_{\nu}d_{\delta}^{0}(k) \]
\[ = 0 . \]

Comparing (88) with (57), one immediately obtains\(^5\)
\[ \frac{\partial}{\partial q}\sigma G_{c}^{\mu\nu\lambda}(0, 0, q)|_{q=0} = I_{\mu\nu\lambda\sigma} = -\frac{N_{f}}{8\pi^{2}}\epsilon_{\mu\nu\lambda\sigma} , \]
(89)

which has been evaluated in (70).

Similarly, we obtain
\[ \frac{\partial}{\partial q}\sigma G_{c}^{\mu\nu\lambda}(0, 0, q)|_{q=0} = -I_{\mu\nu\lambda\sigma} = \frac{N_{f}}{8\pi^{2}}\epsilon_{\mu\nu\lambda\sigma} . \]
(90)

\(^{5}\)Note that \( \partial_{\mu}e_{0}^{-1}(k) = \partial_{\mu}d_{0}(k) \), from (33).
where in the integrand of (91), the first two terms cancel each other, and the last two terms vanish respectively, after the (by now familiar) manipulations using (33), (39), (60), (61), (44), (45) and (47).

Substituting (87), (89), (90) and (92) into (86), we obtain

\[ G_{\mu
\nu\lambda}(p, r, q) = -\frac{N_f}{8\pi^2} \sum_\sigma (p_\sigma - q_\sigma) \epsilon_{\mu
\nu\lambda\sigma} + O(a) . \] (93)

Therefore, in the continuum limit, the axial anomaly (78) at \( O(g^3) \) is

\[ A_3(x) = \frac{g^3 N_f}{8\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \int_p \int_q \int \, e^{i(p+q-r)s} (p_\sigma - q_\sigma) \text{tr}[\tilde{A}_\mu(p)\tilde{A}_\nu(r)\tilde{A}_\lambda(-q)] \]

\[ = -i \frac{g^3 N_f}{32\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr}\{(\partial_\sigma A_\mu(x) - \partial_\mu A_\sigma(x))[A_\nu(x), A_\lambda(x)] + [A_\mu(x), A_\nu(x)](\partial_\sigma A_\lambda(x) - \partial_\lambda A_\sigma(x))\} , \] (94)

where

\[ A_\mu(x) = \int_p e^{i p \cdot x} \tilde{A}_\mu(p) , \text{ etc.} \]

### 2.3 The Axial Anomaly up to Order \( g^4 \)

Inserting the order \( g^4 \) terms of (17) into (15), we have\(^6\)

\[ A_4(x) = g^4 \int_p \int_q e^{i(p-q) \cdot x} \left\{ J_1(p, q) + J_2(p, q) + J_3(p, q) \right\} \]

\[ -g^4 \int_p \int_q \int_r e^{i(p-r) \cdot x} \left\{ J_4(p, r, q) + J_5(p, r, q) + J_6(p, r, q) \right\} \]

\[ +g^4 \int_p \int_q \int_s \int_r e^{i(p-r) \cdot x} J_7(p, r, s, q) \] (95)

\(^6\)the term \( e_0 d_4 e_0 \) has been dropped since \( \text{tr}(\gamma_5 e_0 d_4 e_0) = 0. \)
where

\[
J_1(p,q) = \int_k \text{tr}[\gamma_v e_0(p)d_1(p,k)e_0(k)d_3(k,q)e_0(q)]
\]

\[
J_2(p,q) = \int_k \text{tr}[\gamma_v e_0(p)d_2(p,k)e_0(k)d_2(k,q)e_0(q)]
\]

\[
J_3(p,q) = \int_k \text{tr}[\gamma_v e_0(p)d_3(p,k)e_0(k)d_1(k,q)e_0(q)]
\]

\[
J_4(p,r,q) = \int_k \text{tr}[\gamma_v e_0(p)d_1(p,r)e_0(r)d_1(r,k)e_0(k)d_2(k,q)e_0(q)]
\]

\[
J_5(p,r,q) = \int_k \text{tr}[\gamma_v e_0(p)d_2(p,r)e_0(r)d_2(r,k)e_0(k)d_1(k,q)e_0(q)]
\]

\[
J_6(p,r,q) = \int_k \text{tr}[\gamma_v e_0(p)d_3(p,r)e_0(r)d_3(r,k)e_0(k)d_1(k,q)e_0(q)]
\]

\[
J_7(p,r,s,q) = \int_k \text{tr}[\gamma_v e_0(p)d_1(p,r)e_0(r)d_3(r,s)e_0(s)d_1(s,k)e_0(k)d_1(k,q)e_0(q)]
\]

Now we change the variables as follows:
(i) \( p \to p + k \) and \( q \to q + k \) in \( J_1, J_2, \) and \( J_3; \)
(ii) \( p \to p + r + k, \) \( r \to r + k, \) and \( q \to q + k \) in \( J_4, J_5, \) and \( J_6; \)
(iii) \( p \to p + r + s + k, \) \( r \to r + s + k, \) \( s \to s + k, \) and \( q \to q + k \) in \( J_7. \)

Then (93) becomes

\[
\mathcal{A}_4(x) = g^4 \int_p \int_q e^{i(p-q)\cdot x} \left\{ I_1(p,q) + I_2(p,q) + I_3(p,q) \right\}
\]

\[
- g^4 \int_p \int_r \int_q e^{i(p+r-q)\cdot x} \left\{ I_4(p,r,q) + I_5(p,r,q) + I_6(p,r,q) \right\}
\]

\[
+ g^4 \int_p \int_r \int_s \int_q e^{i(p+r+s-q)\cdot x} I_7(p,r,s,q)
\]

(96)

where

\[
I_1(p,q) = \sum_{\mu,\nu} \text{tr}[\tilde{A}_\mu(p)\tilde{A}_\nu^{(3)}(-q)]G_{\mu\nu}^1(p,q)
\]

\[
I_2(p,q) = \sum_{\mu,\nu} \text{tr}[\tilde{A}_\mu^{(2)}(p)\tilde{A}_\nu^{(2)}(-q)]G_{\mu\nu}^2(p,q)
\]

\[
I_3(p,q) = \sum_{\mu,\nu} \text{tr}[\tilde{A}_\mu^{(2)}(p)\tilde{A}_\nu(-q)]G_{\mu\nu}^3(p,q)
\]

\[
I_4(p,r,q) = \sum_{\mu,\nu,\lambda} \text{tr}[\tilde{A}_\mu(p)\tilde{A}_\nu(r)\tilde{A}_\lambda^{(2)}(-q)]G_{\mu\nu\lambda}^1(p,r,q)
\]

\[
I_5(p,r,q) = \sum_{\mu,\nu,\lambda} \text{tr}[\tilde{A}_\mu(p)\tilde{A}_\nu^{(2)}(r)\tilde{A}_\lambda(-q)]G_{\mu\nu\lambda}^5(p,r,q)
\]

\[
I_6(p,r,q) = \sum_{\mu,\nu,\lambda} \text{tr}[\tilde{A}_\mu^{(2)}(p)\tilde{A}_\nu(r)\tilde{A}_\lambda(-q)]G_{\mu\nu\lambda}^6(p,r,q)
\]

\[
I_7(p,r,s,q) = \sum_{\mu,\nu,\lambda,\sigma} \text{tr}[\tilde{A}_\mu(p)\tilde{A}_\nu(r)\tilde{A}_\lambda(s)\tilde{A}_\sigma(-q)]G_{\mu\nu\lambda\sigma}^7(p,r,s,q)
\]
and

\[ G_1^{\mu\nu}(p, q) = \int k \text{tr}[\gamma_5 e_0(p + k) d_{1,\mu}(p + k, k) e_0(k) d_{3,\nu}(k, q + k) e_0(q + k)] \]

\[ G_2^{\mu\nu}(p, q) = \int k \text{tr}[\gamma_5 e_0(p + k) d_{2,\mu}(p + k, k) e_0(k) d_{2,\nu}(k, q + k) e_0(q + k)] \]

\[ G_3^{\mu\nu}(p, q) = \int k \text{tr}[\gamma_5 e_0(p + k) d_{3,\mu}(p + k, k) e_0(k) d_{1,\nu}(k, q + k) e_0(q + k)] \]

\[ G_4^{\mu\nu}(p, r, q) = \int k \text{tr}[\gamma_5 e_0(p + r + k) d_{1,\mu}(p + r + k, r + k) e_0(r + k) \times d_{1,\nu}(r + k, k) e_0(k) d_{2,\lambda}(k, q + k) e_0(q + k)] \]

\[ G_5^{\mu\nu}(p, r, q) = \int k \text{tr}[\gamma_5 e_0(p + r + k) d_{1,\mu}(p + r + k, r + k) e_0(r + k) \times d_{2,\nu}(r + k, k) e_0(k) d_{1,\lambda}(k, q + k) e_0(q + k)] \]

\[ G_6^{\mu\nu}(p, r, q) = \int k \text{tr}[\gamma_5 e_0(p + r + s + k) d_{2,\mu}(p + r + s + k, r + s + k) \times e_0(r + s + k) d_{1,\nu}(r + s + k, s + k) e_0(s + k) \times d_{1,\lambda}(s + k, k) e_0(k) d_{1,\sigma}(k, q + k) e_0(q + k)] \]

\[ G_7^{\mu\nu}(p, r, q) = \int k \text{tr}[\gamma_5 e_0(p + r + s + k) d_{1,\mu}(p + r + s + k, r + s + k) \times e_0(r + s + k) d_{1,\nu}(r + s + k, s + k) e_0(s + k) \times d_{1,\lambda}(s + k, k) e_0(k) d_{1,\sigma}(k, q + k) e_0(q + k)] \]

where (35), (33), (39), (75), (44), (45) and (47) have been used, and \( \tilde{A}_\mu^{(n)}(p) \), etc. are defined in Eq. (118).

In the following, we show that all \( G \)'s vanish in the continuum limit except \( G_7 \), by expanding each one of them in power series of \( p, q, r \) and \( s \), respectively.

First consider

\[ G_1^{\mu\nu}(p, q) = G_1^{\mu\nu}(0, 0) + O(a) \]

where

\[ G_1^{\mu\nu}(0, 0) = \int k \text{tr}[\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{3,\nu}(k, k) e_0(k)] = 0 , \]

since its integrand vanishes,

\[ \text{tr}[\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{3,\nu}(k, k) e_0(k)] = \text{tr}(1) e_{\alpha\beta\gamma\delta} b(k) c^{\alpha}(k) c^{\beta}(k) \partial_{\mu} d_{0}^{\gamma}(k) d_{3,\nu}^{\delta}(k) = 0 , \]

where (33), (39), (74), (73), (14), (15) and (17) have been used. Similarly, we obtain

\[ G_2^{\mu\nu}(0, 0) = G_3^{\mu\nu}(0, 0) = 0 . \]

Next consider

\[ G_4^{\mu\nu}(p, r, q) = G_4^{\mu\nu}(0, 0, 0) + O(a) , \]

\[ G_5^{\mu\nu}(p, r, q) = G_5^{\mu\nu}(0, 0, 0) + O(a) , \]

\[ G_6^{\mu\nu}(p, r, q) = G_6^{\mu\nu}(0, 0, 0) + O(a) , \]

\[ G_7^{\mu\nu}(p, r, q) = G_7^{\mu\nu}(0, 0, 0) + O(a) , \]

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where

\[ G^\mu_4(0, 0, 0) = \int_k \operatorname{tr}[\gamma_5 e_0(k)d_{1,\mu}(k, k)e_0(k)d_{1,\nu}(k, k)e_0(k)d_{2,\lambda}(k, k)e_0(k)] = 0 , \quad (99) \]

since its integrand vanishes,

\[ \operatorname{tr}[\gamma_5 e_0(k)\partial_\mu e^{-1}_0(k)e_0(k)\partial_\nu d_0(k)e_0(k)d_{2,\lambda}(k, k)e_0(k)] = 0 , \quad (100) \]

where (33), (39), (50), (74), (44), (45) and (47) have been used. By the same token, we have

\[ G^\mu_5(0, 0, 0) = G^\mu_6(0, 0, 0) = 0 . \quad (101) \]

Finally, we consider

\[ G^\mu_7(\rho, \gamma, s, q) = G^\mu_6(0, 0, 0) + O(a) \quad (102) \]

where

\[ G^\mu_7(0, 0, 0, 0) = \int_k \operatorname{tr}[\gamma_5 e_0(k)d_{1,\mu}(k, k)e_0(k)d_{1,\nu}(k, k)e_0(k)d_{1,\lambda}(k, k)e_0(k)d_{1,\sigma}(k, k)e_0(k)] = 0 . \quad (103) \]

Here we have used (39) and (60) to simplify the integrand. Comparing (103) with (50), one immediately obtains

\[ G^\mu_7(0, 0, 0, 0) = I_{\mu, \lambda, \sigma} = -\frac{N_f}{8\pi^2}\epsilon_{\mu, \lambda, \sigma} , \quad (104) \]

which has been evaluated in (70).

Thus, in the continuum limit, only \( G_7 \) has nonzero contribution to the axial anomaly. Substituting (102) into (96), we obtain the axial anomaly at the order \( g^4 \),

\[ \mathcal{A}_4(x) = -\frac{g^4 N_f}{8\pi^2} \sum_{\mu, \nu, \lambda, \sigma} \epsilon_{\mu, \nu, \lambda, \sigma} \int_p \int_r \int_s \int_q e^{i[p+q+\sigma-q]x} \tilde{A}_\mu(p)\tilde{A}_\nu(r)\tilde{A}_\lambda(s)\tilde{A}_\sigma(-q) \]

\[ = -\frac{g^4 N_f}{8\pi^2} \sum_{\mu, \nu, \lambda, \sigma} \epsilon_{\mu, \nu, \lambda, \sigma} \operatorname{tr}\{A_\mu(x)A_\nu(x)A_\lambda(x)A_\sigma(x)\} \]

\[ = -\frac{g^4 N_f}{32\pi^2} \sum_{\mu, \nu, \lambda, \sigma} \epsilon_{\mu, \nu, \lambda, \sigma} \operatorname{tr}\{[A_\mu(x), A_\nu(x)][A_\lambda(x), A_\sigma(x)]\} . \quad (105) \]
Finally, adding (72), (94) and (105) together gives the axial anomaly (15) in the continuum limit,
\[
\mathcal{A}(x) = \text{tr}[\gamma_5(1 - arD)(x, x)] = g^2 N_f 32\pi^2 \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr}\{F_{\mu\nu}(x)F_{\lambda\sigma}(x)\} ,
\]
where
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g [A_\mu, A_\nu] .
\]
Note that the axial anomaly (106) is invariant for any \( r > 0 \), since the anomaly coefficient (70) is independent of \( r \).

This completes our perturbation calculation of the axial anomaly of the Ginsparg-Wilson lattice Dirac operator proposed in Ref. [1].

3 Concluding Remarks

Several remarks are as follows.

It is obvious that (106) also holds for other \( T_\mu \) such that \( D \) is doublers-free, exponentially-local, and has correct continuum behavior. In particular, (106) holds for
\[
f = \left( \frac{2c}{\sqrt{t^2 + w^2 + w}} \right)^\alpha , \quad \alpha \geq \frac{1}{2} , \quad 0 < c < 2 ,
\]
as proposed in Ref. [1].

It is instructive to examine how well the continuum axial anomaly can be recovered on a finite lattice, by evaluating the integral \( I_{1234} \) as a numerical sum over the discrete momenta on a finite lattice. In Table 1, the ratios of \( I_{1234} \) to \( 1/8\pi^2 \) are listed for several lattice sizes (\( L^4 \)), as well as for a range of \( r \), respectively. Here the lattice spacing \( a \) is set to one, and the value of \( c \) [ see Eqs. (3) and (5) ] is fixed at 1.0. Evidently, the integral \( I_{1234} \) tends to the continuum value \( 1/8\pi^2 \) as \( L \to \infty \), independent of the parameter \( r \). This also provides a verification of our ansatz to substitute \( d_0(K) \) by \( d_0^{-1}(k) \) in evaluating \( I_{\mu\nu\lambda\sigma} \) in the continuum limit.

Next we repeat the same calculations of \( I_{1234} \) for another \( f \), namely, replacing (3) by
\[
f = \frac{2c}{\sqrt{t^2 + w^2 + w}} .
\]
The results are listed in Table 2, which show that the integral \( I_{1234} \) (as a function of lattice size \( L^4 \)) approaches the continuum value \( 1/8\pi^2 \) much faster.
Table 1: The ratio of the integral $I_{1234}$ [Eq. (62)] to $1/8\pi^2$, for lattices of sizes $16^4, 32^4, 64^4, 128^4, 256^4$, as well as for $r = 0.5, 1.0, 2.0, 4.0$, respectively. The lattice spacing $a$ is set to one, and the parameter $c$ [Eqs. (3) and (8)] is fixed at 1.0.

| $L$ | $r$ |
|-----|-----|
|     | 0.5 | 1.0 | 2.0 | 4.0 |
| 16  | 0.9204 | 0.9800 | 0.9848 | 0.7617 |
| 32  | 0.9810 | 0.9952 | 0.9988 | 0.9907 |
| 64  | 0.9953 | 0.9988 | 0.9997 | 0.9999 |
| 128 | 0.9988 | 0.9997 | 0.9999 | 1.0000 |
| 256 | 0.9997 | 0.9999 | 1.0000 | 1.0000 |

Table 2: The ratio of the integral $I_{1234}$ [Eq. (62)] to $1/8\pi^2$, for $f = 2c/(\sqrt{t^2+w^2+w})$. Other parameters are the same as those in Table 1.

| $L$ | $r$ |
|-----|-----|
|     | 0.5 | 1.0 | 2.0 | 4.0 |
| 16  | 1.0000 | 1.0000 | 0.9911 | 0.7565 |
| 32  | 1.0000 | 1.0000 | 1.0000 | 0.9914 |
| 64  | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 128 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 256 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

than that in Table 1. This indicates that (109) may be a better choice than (8), especially for small lattices.

In general, $T_\mu$ (2) may not be in the form $T_\mu = f t_\mu f$, then our present perturbation calculation may not go through without modifications. We refer to our former derivation (3) for the general case, though some of our intermediate steps need further clarifications. If one assumes that the coefficient of the axial anomaly, $g^2/32\pi^2$, is the same for $U(1)$ and $SU(n)$ background gauge fields, then it can also be determined (8) by imposing a gauge configuration with constant field tensors to Lüscher’s formula (7) for the axial anomaly of Ginsparg-Wilson lattice Dirac fermions in a $U(1)$ background gauge field, provided that $D$ is topologically-proper. However, for a topologically-trivial Ginsparg-Wilson lattice lattice Dirac operator such as (1), it seems to be necessary to perform an explicit calculation in order to determine its axial anomaly. In passing, we also refer to other axial anomaly calculations (3, 10, 11, 12) for the overlap Dirac operator (13, 14), as well as that in the original Ginsparg-Wilson paper (3). So far, a nonperturbative analytic calculation of the axial anomaly for any Ginsparg-Wilson lattice Dirac operator, on a finite lattice, is still lacking.
In summary, we have shown that the lattice Dirac operator (1) reproduces continuum axial anomaly for smooth gauge configurations, even though it does not possess any topological zero modes in topologically-nontrivial gauge backgrounds. If one insists that the topologically zero modes of a lattice Dirac operator are crucial for lattice QCD to reproduce the low energy hadron phenomenology, then one should assure that a Ginsparg-Wilson lattice Dirac operator is indeed topologically-proper, before it could be employed for lattice QCD computations. However, so far, there does not seem to have compelling experimental evidence that these topological zero modes are physically relevant, unlike the axial anomaly in the trivial gauge sector, which accounts for the decay rate of the neutral pion. So there might be a very slight possibility that lattice QCD with topologically-trivial quarks could reproduce low energy hadron phenomenology. These issues seem to deserve further studies.

A Basic Formulas in Weak Coupling Perturbation Theory

In this appendix, we set up our notations by deriving some basic formulas in weak coupling perturbation theory. These formulas are used in our anomaly calculation in Section 2.

Consider the forward and backward difference operators

\[ \nabla_+^\mu(x, y) \equiv U_\mu(x)\delta_{x+a\hat{\mu}, y} - \delta_{x, y}, \]
\[ \nabla_-^\mu(x, y) \equiv \delta_{x, y} - U_\mu^\dagger(x-a\hat{\mu})\delta_{x-a\hat{\mu}, y}, \]

where the link variables (for SU(n) gauge group) are defined as

\[ U_\mu(x) = \exp[iagA_\mu(x + \frac{a}{2}\hat{\mu})], \]
\[ U_\mu^\dagger(x-a\hat{\mu}) = \exp[-iagA_\mu(x - \frac{a}{2}\hat{\mu})]. \]

Expanding \( \nabla^\pm \) in power series of the gauge coupling \( g \),

\[ \nabla_\mu^\pm = (\nabla_\mu^\pm)_0 + (\nabla_\mu^\pm)_1 + (\nabla_\mu^\pm)_2 + \cdots \]

and performing the Fourier transform

\[ \sum_{x,y} e^{-i(p\cdot x - q\cdot y)}(\nabla_\mu^\pm)_n(x, y) \equiv (\nabla_\mu^\pm)_n(p, q), \]

we obtain

\[ (\nabla_\mu^\pm)_0(p, q) = (e^{ip\mu a} - 1) \delta^4(p - q) \equiv (\nabla_\mu^\pm)_0(p) \delta^4(p - q) \]

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and

\[(\nabla^+_{\mu})_n(p, q) = \sum_{x, y} e^{-i(p-x-q)y} \frac{1}{n!} (iag)^n \left[ A_{\mu} \left( x + \frac{a}{2} \hat{\mu} \right) \right]^n \delta_{x+a\hat{\mu}, y} \]

\[= \sum_x e^{-i(p-q)-(x+a\hat{\mu})/2} \frac{1}{n!} (iag)^n \left[ A_{\mu} \left( x + \frac{a}{2} \hat{\mu} \right) \right]^n e^{i(p_\mu+q_\mu)a/2} \]

\[= \frac{(iag)^n}{n!} A^{(n)}_{\mu}(p-q) e^{i(p_\mu+q_\mu)a/2} \]

\[= \frac{g^n}{n!} A^{(n)}_{\mu}(p-q) \partial^\mu_0 (\nabla^+_{\mu})_0 \left( \frac{p+q}{2} \right) \]

(117)

where

\[A^{(n)}_{\mu}(p-q) = \sum_x e^{-i(p-q)-(x+a\hat{\mu})/2} \left[ A_{\mu} \left( x + \frac{a}{2} \hat{\mu} \right) \right]^n, \]

(118)

\[\left( \nabla^+_{\mu} \right)_0 \left( \frac{p+q}{2} \right) = e^{i(p_\mu+q_\mu)a/2} - 1. \]

Similarly, we have

\[(\nabla^-_{\mu})_0(p, q) = (1 - e^{-ip_\mu a}) \delta^4(p-q) \equiv (\nabla^-_{\mu})_0(p) \delta^4(p-q) , \]

(120)

\[(\nabla^-_{\mu})_n(p, q) = g^n \frac{n!}{n!} A^{(n)}_{\mu}(p-q) \partial^\mu_0 (\nabla^-_{\mu})_0 \left( \frac{p+q}{2} \right). \]

(121)

Then we obtain the weak coupling perturbation series of \(t^\mu\) (11) and \(w\) (8) in the momentum space:

\[t^\mu(p, q) = \frac{1}{2} [\nabla^+_{\mu}(p, q) + \nabla^-_{\mu}(p, q)] \]

\[= t^\mu_0(p)\delta^4(p-q) + t^\mu_1(p, q) + t^\mu_2(p, q) + \cdots \]

(122)

\[w(p, q) = c - \frac{1}{2} \sum_{\mu} [\nabla^+_{\mu}(p, q) + \nabla^-_{\mu}(p, q)] \]

\[= w_0(p)\delta^4(p-q) + w_1(p, q) + w_2(p, q) + \cdots \]

(123)

where

\[t^\mu_0(p) = i \sin(p_\mu a), \]

(124)

\[w_0(p) = c - \sum_{\mu} [1 - \cos(p_\mu a)], \]

(125)

\[t^\mu_n(p, q) = \frac{g^n}{n!} A^{(n)}_{\mu}(p-q) \partial^\mu_0 t^\mu_0 \left( \frac{p+q}{2} \right), \]

(126)

\[w_n(p, q) = \frac{g^n}{n!} \sum_{\mu} A^{(n)}_{\mu}(p-q) \partial^\mu_0 w_0 \left( \frac{p+q}{2} \right). \]

(127)
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