Fine-tuning of the cosmological constant is not needed

Qingdi Wang

Department of Physics and Astronomy, The University of British Columbia, Vancouver, Canada V6T 1Z1

We show that the fluctuations of the quantum vacuum stress-energy tensor would produce a large positive contribution to the averaged macroscopic spatial curvature of the universe. In order to cancel this contribution, the bare cosmological constant in Einstein equations has to take large negative values, and if it is large enough, the spacetime structure would be similar to the cyclic model of the universe in the sense that at small scales every point in space is a “micro-cyclic universe” which is following an eternal series of oscillations between expansions and contractions. Moreover, due to the weak parametric resonance effect caused by the fluctuations of the quantum vacuum stress-energy, the size of each “micro-universe” increases a tiny bit at a slowly accelerating rate during each microcycle of oscillation. Accumulation of this effect over the cosmological scale gives an accelerating universe. More importantly, the extreme fine-tuning of the cosmological constant to an accuracy of $10^{-122}$ is not needed. This resolves the cosmological constant problem and suggests that it is the quantum vacuum fluctuations serve as the dark energy which is accelerating the expansion of our Universe.

I. INTRODUCTION

The cosmological constant problem is widely regarded as one of the major obstacles to further progress in fundamental physics (for example, see [1–5]). It arises when one tries to put quantum mechanics and general relativity together to study the gravitational property of quantum vacuum. Since we do not have a satisfactory quantum theory of gravity yet, the usual assumption is the semi-classical Einstein equations

$$G_{ab} + \lambda_B g_{ab} = 8\pi G \langle T_{ab} \rangle,$$  \hspace{2cm} (1)

where $\lambda_B$ is the bare cosmological constant and the source of gravity is the expectation value of the quantum vacuum stress energy tensor. Lorentz invariance requires that in the vacuum $\langle T_{ab} \rangle$ takes the form

$$\langle T_{ab} \rangle = - \langle \rho \rangle g_{ab},$$  \hspace{2cm} (2)

where $\rho$ is the vacuum energy density. Then the gravitational effect of the vacuum would be equivalent to a cosmological constant that the Einstein equations (1) can be written as

$$G_{ab} + \lambda_{\text{eff}} g_{ab} = 0,$$  \hspace{2cm} (3)

where the effective cosmological constant $\lambda_{\text{eff}}$ is defined by

$$\lambda_{\text{eff}} = \lambda_B + 8\pi G \langle \rho \rangle.$$  \hspace{2cm} (4)

Quantum field theory predicts a huge vacuum energy density from various sources whose magnitude depends on the high energy cutoff $\Lambda$:

$$\langle \rho \rangle \sim \Lambda^4.$$  \hspace{2cm} (5)

If we take $\Lambda$ up to Planck energy, i.e. $\Lambda = 1$ in Planck units, we have

$$\langle \rho \rangle \sim 1.$$  \hspace{2cm} (6)

Unfortunately, the observed value of the effective cosmological constant in Planck units is

$$\lambda_{\text{eff}} = 5.6 \times 10^{-122},$$  \hspace{2cm} (7)

then according to Eq. (4), one has to fine-tune $\lambda_B$ to a precision of 122 decimal places to cancel the huge vacuum energy density to match the observations. This problem of extreme fine-tuning is the so called cosmological constant problem [2].

The above usual formulation of the cosmological constant problem treats the vacuum energy density $\rho$ as a constant. However, since the vacuum is not an eigenstate of the energy density operator $T_{00}$ (although it is an eigenstate of the Hamiltonian $H = \int dx^3 T_{00}$), $\rho$ can not be a constant—it is always fluctuating. In fact, the magnitude of the fluctuation is as large as its expectation value $\langle \rho \rangle$:

$$\Delta \rho \sim \langle \rho \rangle.$$  \hspace{2cm} (8)

Due to this huge fluctuation, the usual formulation of the cosmological constant problem would break down. Even if one has successfully fine-tuned $\lambda_B$ to the needed accuracy of $10^{-122}$, the Universe would still explode since $\lambda_B$ only cancels the expectation value of $\rho$ but not its fluctuations (see Sec.IV of [7] for more detailed discussion).

Moreover, the spacetime sourced by the wildly fluctuating vacuum stress-energy tensor is by no means homogeneous at small scales. Instead, the spacetime would have a foamy, jittery nature and would consist of many small, ever-changing, regions in which spacetime is not definite, but fluctuates [8, 9]. One consequence is that at small scales, the spatial curvature at each point must be large and fluctuating.

However, the observed average spatial curvature of the Universe is very small (flat with only a 0.4 percent margin of error). So one natural question is: can the large curvature at small scales averages to small value macroscopically?

In this paper, we show that the wildly fluctuating “foamy” structure of the spacetime would produce a large
positive contribution to the averaged macroscopic spatial curvature of the Universe. In order to cancel this contribution, the bare cosmological constant $\lambda_B$ has to take large negative values. If $|\lambda_B|$ is large enough, the spacetime exhibits interesting features where at small scales each spatial point oscillates between expansion and contraction. The expansion and the contraction almost cancel except the expansion wins out a little bit due to the weak parametric resonance effect produced by the vacuum fluctuations. This tiny net expansion accumulates on cosmological scale, gives the observed slowly accelerating expansion of the Universe without the need of fine-tuning of the cosmological constant.

II. FLUCTUATION AND THE SPATIAL CURVATURE

Let $\Sigma$ be a Cauchy surface of our fluctuating spacetime $M$. The spacetime metric $g_{ab}$ induces a spatial metric $h_{ab}$ on $\Sigma$ and the “bending” of $\Sigma$ in $M$ is described by the extrinsic curvature $K_{ab}$. The initial data $(h_{ab}, K_{ab})$ on $\Sigma$ are not arbitrary. One constraint equation they must satisfy is

$$R^{(3)} - K_{ab}K^{ab} + K^2 = 16\pi G\rho + 2\lambda_B,$$  \hspace{1cm} (9)

where $R^{(3)}$ is the scalar curvature of $\Sigma$, $K = h^{ab}K_{ab}$ is the mean curvature of $\Sigma$. Expanding the terms $K_{ab}K^{ab} - K^2$ in $\Sigma$ gives

$$K_{ab}K^{ab} - K^2 = \sum_{i \neq j \neq k} M_k K_{ij}^2 + \sum_{\{ij\} \neq \{k,l\}} (h^{ik}h^{jl} - h^{ij}h^{kl}) K_{ij}K_{kl},$$  \hspace{1cm} (10)

where

$$M_k = h^{ii}h^{jj} - (h^{ij})^2, \quad k \neq i \neq j,$$  \hspace{1cm} (11)

is the $k$th principal minor of $h_{ab}$. Since by definition the metric matrix $h_{ab}$ is positive definite, we have $M_k > 0$.

Since general relativity is time reversal invariant, then for every expanding solution there is a corresponding contracting solution, i.e. if $(h_{ab}, K_{ab})$ is allowed initial data, so is $(h_{ab}, -K_{ab})$ \[10\]. Thus the following four pairs of components

$$(K_{ij}, K_{kl}), (K_{ij}, -K_{kl}), (-K_{ij}, K_{kl}), (-K_{ij}, -K_{kl})$$  \hspace{1cm} (12)

are equally likely. Because in general, there is no particular relationship between the components of $K_{ab}$, the above four cases would statistically cancel each other that the macroscopic (spatial) average

$$\langle (h^{ik}h^{jl} - h^{ij}h^{kl}) K_{ij}K_{kl} \rangle = 0, \quad \{i, j\} \neq \{k, l\}$$  \hspace{1cm} (13)

for a very large class of randomly chosen data on $\Sigma$. Note that the macroscopic average does not require a very large volume: a cubic centimeter contains some $10^{109}$ Planck-size regions \[10\].

Then taking the spatial average of $\langle R^{(3)} \rangle$ over $\Sigma$ gives

$$\langle R^{(3)} \rangle = 2 \left( \lambda_B + 8\pi G\rho + \sum_{1 \leq i < j \leq 3} \langle M_k K_{ij}^2 \rangle \right).$$  \hspace{1cm} (14)

For the wildly fluctuating spacetime, we have the last term $\langle M_k K_{ij}^2 \rangle > 0, \{i \neq j\}$ for a large class of randomly chosen data. The vacuum energy density $\langle \rho \rangle$ is usually assumed to be positive, so the bare cosmological constant $\lambda_B$ has to take extra large negative values to cancel the positive contribution from the term $\sum_{1 \leq i < j \leq 3} \langle M_k K_{ij}^2 \rangle$ to make the observed $\langle R^{(3)} \rangle$ given by \[14\] small.

There is not a unique value for $\lambda_B$ to exactly cancel the contribution from the fluctuating spacetime since the magnitude of the fluctuation also depends on the magnitude of $|\lambda_B|$. It turns out that the case $-\lambda_B$ dominates over the vacuum stress-energy fluctuation is most interesting. We are going to investigate this case in the following sections.

III. THE MICRO-CYCLIC “UNIVERSES”

To describe the spacetime fluctuations, we generalize the usual homogeneous FLRW metric

$$ds^2 = -dt^2 + a^2(t)dx^2$$  \hspace{1cm} (15)

to the following inhomogeneous metric

$$ds^2 = -dt^2 + h_{ab}(t, x)dx^adx^b,$$  \hspace{1cm} (16)

with $x = (x, y, z)$. In this coordinate, each time line $x = \text{Constant}$ is a geodesic normal to $\Sigma$. The mean curvature $K$ is then the expansion of the geodesic congruence of these time lines. It is related to the determinant of the spatial metric $h = \det(h_{ab})$ by

$$K = \frac{\dot{h}}{2h} = \frac{\sqrt{h}}{\sqrt{h}}.$$  \hspace{1cm} (17)

Since $\sqrt{h}dx \wedge dy \wedge dz$ is the spatial volume element, $K$ indeed measures the local volume expansion rate of the 3-dimensional hypersurface $\Sigma$. It satisfies the following Raychaudhuri’s equation

$$\dot{K} + \frac{1}{3} K^2 + 2\sigma^2 - \lambda_B + 4\pi G(\rho + trT) = 0,$$  \hspace{1cm} (18)

where $trT = h^{ab}T_{ab}, \sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$ and $\sigma_{ab} = K_{ab} - \frac{1}{3}Kh_{ab}$ is the trace free part of $K_{ab}$.

Define

$$h = a^6,$$  \hspace{1cm} (19)

we then have $K = 3\frac{\dot{a}}{a}$ and \[18\] becomes

$$\ddot{a} + \frac{1}{3} (2\sigma^2 - \lambda_B + 4\pi G(\rho + trT)) a = 0.$$  \hspace{1cm} (20)
The \( a \) defined by (19) is a generalization of the scale factor in the usual homogeneous FLRW metric (15). It locally describes the relative size of each spatial point.

Since \( \sigma^2 \geq 0 \) and \( -\lambda_B \) is large positive and dominant over \( 4\pi G (\rho + 3 T) \), the above equation (20) describes an oscillator with varying frequency. Thus the solution for \( a \) must be oscillating around 0 (see the left graph of Fig. 1). Then the local volume expansion rate \( K = 3 \frac{\dot{a}}{a} \) ranges from \(-\infty\) to \(+\infty\). \( K > 0 \) represents expansion while \( K < 0 \) represents contraction. It jumps discontinuously from \(-\infty\) to \(+\infty\) each time when \( a \) goes across 0 (see the right graph of Fig. 1). In this process, the determinant \( h = a^6 \geq 0 \) decreases continuously to 0 and then increases back to positive values as \( a \) crosses 0 (see the middle graph Fig. 1). Physically, this means, on average, the space locally collapses to zero size and immediately expands back.

Note that the solution to the oscillator equation (20) depends on the values of \( a(0, x) \) and \( \dot{a}(0, x) \) on the initial hypersurface \( t = 0 \). In general, \( a(0, x) \) and \( \dot{a}(0, x) \) would take different values for different \( x \) and the phases of the oscillations of \( a(t, x) \) at neighboring spatial points would be different in this wildly fluctuating spacetime. Due to the phase differences, the local expansion and contraction at small scales would be largely canceled out at macroscopic scales after spatial averaging.

The turning points \( a = 0 \) at which the space switches from contractions to expansions are actually spacetime singularities. They are very similar to the big bang singularity (see Sec.VIII A of [7] for more discussions about the issue of the singularities). The alternatively expanding and contracting picture is similar to the cyclic model (or oscillating model) of the universe in the sense that every point in space is a “micro-cyclic universe” which is following an eternal series of oscillations. Each “micro-universe” begins with a “big bang” and ends with a “big crunch” and then a “big bounce” happens that the cycle starts over again (see FIG. 2).

**IV. THE ACCELERATING EXPANSION**

The local scale factor \( a \) characterizes the relative distances between the geodesic lines \( x = \text{Constants} \). In order to investigate the dynamics of \( a \) in more detail, we need to study the relative motions between these geodesics. To do this, we construct a free falling observer \( \gamma \)’s own local inertial frame \( \{\xi^a, e^a_1, e^a_2, e^a_3\} \) (see FIG. 3) and study in general how a free falling test particle moves around the observer.

The metric components \( g_{\mu\nu} \) along \( \gamma \) in this frame are exactly \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \). The components, \( x^i \), of the deviation vector \( x^\mu \) (see FIG. 3) which describes the displacement to an infinitesimally nearby geodesic from \( \gamma \) satisfy the geodesic deviation equation (see e.g. pages 47, 225 of [14]):

\[
\frac{d^2 x^i}{d\tau^2} = - \sum_{j=1}^{3} R^i_{0j0}(\tau) x^j, \quad i = 1, 2, 3, \tag{21}
\]

where \( \tau \) is \( \gamma \)’s proper time. \( R^i_{0j0} \) are components of the Riemann curvature tensor, they can be expressed in the following form (see Sec.II and Sec.IV A of [7] for more detailed derivation):

\[
R^i_{0j0} = \langle R^i_{0j0} \rangle + \Delta R^i_{0j0}, \quad \tag{22}
\]

\[
R^i_{0j0} = C_{0i0j} - 4\pi G T_{ij}, \quad i \neq j, \tag{23}
\]

with

\[
\langle R^i_{0j0} \rangle = - \frac{\lambda_B}{3} + \frac{4\pi G}{3} \left( \langle \rho \rangle + 3 \langle P \rangle \right), \tag{24}
\]

\[
\Delta R^i_{0j0} = \frac{4\pi G}{3} \left( T_{00} - \langle \rho \rangle + 2 \sum_{k=1}^{3} T_{kk} - 3 T_{ii} - 3 \langle P \rangle \right), \tag{25}
\]
The case when $-\lambda_B$ dominates over the vacuum stress-energy fluctuations ($-\lambda_B \gg \Lambda^2 \geq GA^4$), assuming $\Lambda \leq E_P$, where $E_P$ is the Planck energy) is most interesting. Another important quantity is the time scale of variations $R_{0i0j}(i \neq j)$, which is given by $\Delta t = 1/\Lambda$, i.e., $R_{0i0j}(i \neq j)$ would become appreciably different after a time interval of the order $1/\Lambda$.

If we ignore the small off-diagonal terms $R_{0i0j}(i \neq j)$ which describe the interactions between oscillations in the three spatial directions but keep the small diagonal fluctuation terms $\Delta R_{0i0j}$, (21) becomes

$$\frac{d^2x^i}{d\tau^2} + \Omega_i^2(\tau)x^i = 0,$$

where

$$\Omega_i^2(\tau) = R_{0i0j}^0,$$

$$x^i \sim A^i e^{H \tau} \sqrt{\frac{\Omega_i(0)}{\Omega_i(\tau)}} \cos \left( \int_0^\tau \Omega_i(\tau')d\tau' + \theta^i \right),$$

where the exponent $H$ measures the strength of the parametric resonance, which must be weak since this is an adiabatic process.

When all the fluctuations including the off-diagonal terms are considered, the system is still adiabatic and the weak parametric resonance still occurs. We have a similar quantity $H$ to describe the rate of change of the amplitudes of the oscillation. Its magnitude depends on the parameters $\lambda_B$ and $\Lambda$. The exact dependence of $H$ on them can be obtained by studying the evolution of the adiabatic invariant of the system, which is proportional to the square of the amplitudes. In fact, it has been well-established that the error in adiabatic invariant is exponentially small [12] [13]. Detailed analysis using the technique of contour integral gives (see Sec.VII of [2] for more detailed derivation)

$$H = \alpha \Lambda e^{-\beta \frac{\sqrt{-\lambda_B}}{\Lambda}},$$

where $\alpha, \beta > 0$ are two dimensionless constants which depend on the variation details of the system of oscillations. Larger $\frac{\sqrt{-\lambda_B}}{\Lambda}$ means slower fluctuations compared to the oscillations and thus a smaller rate of change $H$. The extra factor $\Lambda$ in front of $e^{-\beta \frac{\sqrt{-\lambda_B}}{\Lambda}}$ is because faster fluctuations give stronger parametric resonance. The micro “big bounces” at the singularities only reverse the moving directions of the test particle. They do not change the amplitude of the oscillations. Thus the result (33) for $H$ is not affected by the bounces at the singularities.

Therefore every free falling observer $\gamma$ would see that the average distance from him to the nearby free falling test particles increases exponentially as $\sim e^{H \tau}$, with the magnitude of $H$, given by (33), being exponentially suppressed. Since this is true for every free falling observer moving in the fluctuating spacetime which has the same (statistical) properties everywhere, we obtain an accelerating universe with $H$ describing the global Hubble expansion rate. In this picture, the expansion wins out the contraction a tiny bit during each oscillation of $a$ that its amplitude (the size of each “micro-universe”) grows as $\sim e^{H \tau}$ (see the left graph of FIG. 1). The expectation value of the local volume expansion rate $K$ would be (see the right graph of FIG. 1):

$$\langle K \rangle = 3H.$$

Moreover, instead of the usual relation (1) between the observed effective cosmological constant $\Lambda_{\text{eff}}$ and the bare cosmological constant $\lambda_B$, we obtain a new relation:

$$\Lambda_{\text{eff}} = 3H^2 = \alpha^2 \Lambda^2 e^{-2\beta \frac{\sqrt{-\lambda_B}}{\Lambda}}.$$

Because of the exponential suppression, the extreme fine-tuning of $\lambda_B$ to the accuracy of $10^{-122}$ is no longer needed. This resolves the cosmological constant problem.
and suggests that it is the quantum vacuum fluctuations serve as the dark energy which is accelerating the expansion of our Universe.

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