Normal Bundles, Pfaffians and Anomalies

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**ABSTRACT:** We deal with the problem of diffeomorphism anomaly in theories with branes. In particular we thoroughly analyze the problem of the residual chiral anomaly of a five–brane immersed in M–theory, paying attention to its global formulation in the five–brane world–volume. We conclude that the anomaly can be canceled by a *local* counterterm in the five–brane world–volume.

**KEYWORDS:** Locality, Normal bundle, Pfaffian, Reducible connection, M–5–brane anomaly.
1. Introduction

This article deals with the problem of diffeomorphism anomalies in theories with branes. With respect to the traditional anomaly analysis in field or superstring theories without branes, the presence of branes introduces new questions. Macroscopic branes in superstring or M–theory are represented, from a geometrical point of view, by submanifold embedded in the 10 or 11 dimensional ambient manifold. On the brane world–volumes there live fields that represent the dynamical degrees of freedom of the brane. Simultaneously the brane interacts with the ambient theory. Let us suppose that the latter is anomaly free and that the fermionic degrees of freedom on the brane are chiral. Then the overall theory of the brane embedded in the ambient theory might contain chiral anomalies which break the invariance under those diffeomorphisms that map the brane world–volume to itself. Anomaly contributions may be of three types: there may be anomalies of the brane theory in isolation, anomalies due to the embedding, i.e. anomalies due to pulled–back metrics or connections from the ambient manifold, and finally inflow anomalies, i.e. anomalies due to the interaction of the brane with the ambient theory. All these types contain contributions from the tangent bundle of the world–volume $W$ of the embedded brane, but the last two types of anomalies contain also normal bundle contributions. They come in fact from characteristic classes of the ambient tangent bundle $TM$, where $M$ is the ambient space–time; the well–known decomposition $TM|_W = TW \oplus N$ holds, $N$ being the bundle normal to the brane and the characteristic classes of $TM$ will split accordingly. The purely $TW$
part of the anomaly can be thought of as due to diffeomorphisms that map $W \rightarrow W$. On the other hand, the $N$ part of the anomaly can be thought of as due to diffeomorphisms that leave $W$ pointwise fixed and can be interpreted as gauge transformations of $N$. These two cases have an easy geometrical interpretation. They are essentially the two cases considered so far in the literature. However they do not represent the most general situation. Apart from this, in all the examples considered so far, only local expressions of the relevant anomalies have been used. Reality is more complicated than this. We need to refine the analysis in at least two directions.

To start with there are other diffeomorphisms of $M$, which map $W$ to $W$ but are not comprised in the two subgroups mentioned above. We remark that if we apply a diffeomorphism that maps $W$ to $W$, in general the normal bundle is deformed and not mapped to itself. We will argue later on that one must consider only diffeomorphisms which map $N$ (globally) to itself. Even with this simplification the relevant anomalies cannot be thought of and treated as usual gauge anomalies. One has to rely on a more general formalism which involves both diffeomorphisms and gauge transformations on the same footing. This can be done by considering general automorphisms of the relevant principal bundle (rather than vertical automorphisms, i.e. ordinary gauge transformations, alone).

In addition we will need formulas for anomalies expressed as basic forms in the space–time manifold $M$. The only way to achieve this is by introducing a background connection, i.e. a spectator connection which is not transformed under the relevant transformations. Formulas for anomalies with a background connection can be found in the literature, [2, 3]. Introducing a background connection is allowed for the following reason. Let us consider a principal fiber bundle $P(M, G)$ and a given connection $A$. Any automorphism $\psi \in AutP$ maps $P$ to $P$ but ‘rotates’ it, i.e. $P \rightarrow \psi^*P$, and maps any connection $A$ to $\psi^*A$. However $A$ is a connection both in the original bundle and in the transformed one. Therefore it is consistent to keep one connection $A_0$ fixed, while rotating the bundle and all the other connections. The background connection $A_0$ allows us to write down local expressions for anomalies. This is of course not in contradiction with the usual local expressions of the same anomalies, which can be recovered by simply setting (locally) $A_0 = 0$. As we shall see, the framework considered in the paper needs in fact further qualifications with respect to the ordinary gauge theories setting just described.

As far as anomalies are concerned, however, one does not expect any significant complication from this more general treatment whenever anomaly cancellation takes place at the level of characteristic classes (apart, of course, for the necessity to specify an appropriate geometrical setup). For in this case anomalies are canceled at the source, so to speak, and there is nothing left on which an anomalous behaviour can build up. However the situation is different when a residual anomaly is canceled by means of a mechanism ´a la Green–Schwarz. In the latter case there is a physical input (the existence of a suitable local field) that does not follow simply from the automatisms of the descent equations.

In this regard there is a gap in the analysis carried out so far in the literature on the subject. In [8] we have shown how to implement the anomaly cancellation mechanism with background connection in the original Green–Schwarz case. In this paper we wish to
extend the analysis to theories with branes. Actually we will concentrate on the case of the M–5–brane anomaly, which alone contains all the above complications: it is a normal bundle anomaly which can only be canceled via a Green–Schwarz mechanism,[4][5][6][7][10] and[4] (analogous problems arise in other cases,[11][12][13], which will not be discussed here but can be treated along the same lines).

Let us summarize the M–5–brane problem. The geometric setting for this problem,[4], is specified by the 11d manifold \(X\) of M–theory and by the 6d manifold \(W\), representing the world–volume of the 5–brane embedded in it. In addition we have the well–known decomposition \(TX|_W = TX \oplus N\), \(N\) being the bundle normal to the brane world–volume, whose structure group is \(SO(5)\). In isolation, both theories on \(X\) and on \(W\) are anomaly–free. But, due to the embedding of \(W\) in \(X\) and to the physical coupling of the 5–brane (see eq.(1.1) below), one gets both induced and inflow anomalies on \(W\). These contributions to the anomaly do not cancel completely: the residual anomaly is generated via the descent equations by the 8–form \(\frac{1}{24}p_2(N)\), where \(p_2(N)\) represents the second Pontryagin class of the normal bundle. Now, the brane is magnetically coupled to M–theory via

\[
dF_4 = \delta_W
\]

where \(\delta_W\) is a representative of the Poincaré dual of \(W\), and \(F_4\) is the 4–form field strength of 11d supergravity. We showed in[4] that, due to (1.1), \(W = \partial Y\), i.e. \(W\) is the boundary of some 7–manifold \(Y\). This implies that the normal bundle \(N\) is a direct sum, \(L \oplus N'\), of a trivial line bundle \(L\) and a vector bundle \(N'\) whose structure group is reduced to \(SO(4)\). It follows that the second Pontryagin class of \(N\) becomes:

\[
p_2(N) = p_2(N') = e(N')^2 = e^2
\]

where \(e\) is the Euler class of the normal bundle. Therefore the anomaly we have to do with is generated via the descent equations by \(e\). This is, roughly speaking, a summary of the M–5–brane anomaly problem. We will clarify, in the course of the paper, various aspects of this problem. But we would like to emphasize from the start the aspect of locality. Throughout the paper the form \(\delta_W\) in (1.1) is taken to be the Dirac–delta 5–form. This allows one to work within a local field theory framework (with a topological defect). Since the residual anomaly has a local expression on the brane world–volume, one has to require that it be canceled by a local counterterm. From this point of view, however, the existing literature does not offer a satisfactory solution. This is the main problem we want to cope with.

In the present paper we intend to fill in the gaps described above. We propose a local counterterm to cancel the residual M–5–brane anomaly. This mechanism is tailored to take into account eq.(1.1). It is worth insisting that this equation, which expresses the magnetic coupling of the 5–brane to M–theory, requires the normal bundle splitting \(N = L \oplus N'\): this must be reflected both in the form of the anomaly and of the counterterm. We will see that this splitting is essential for the anomaly cancellation.

While meeting the above requirements, we make a point of using basic expressions both for the anomaly and the counterterm and properly take into account the problems connected with diffeomorphisms in the presence of a normal bundle.
The paper is organized as follows. The next section is preliminary to the anomaly analysis. Since a crucial role in our problem is played by the reduction from $SO(5)$ bundles and connections to $SO(4)$ ones, we devote a few pages to deriving a workable formalism to deal with reducible connections. In particular we find for them an explicit formula, (2.3), in terms of a unimodular section $v$ which defines the corresponding reduction. Next we show that for reducible connections the form which represents the second Pontryagin class factorizes into the square of a Pfaffian. In section 3 we set out to calculate the expression of the anomaly corresponding to such a factorized class, and succeed in finding an expression for it which is globally defined on the world-volume of the 5–brane. Finally in section 4 we introduce the local and globally defined counterterm which cancels such an anomaly. We discuss the meaning of the counterterm and of the field variables $v$ introduced to parametrize the various reductions.

2. Reducible connections, Pfaffians and Anomalies

Our purpose in this section is to find a basic expression of the residual anomaly of the M–5–brane, taking into account the two complications mentioned in the introduction.

2.1 The geometric setting

The first task is to specify what subgroup of the diffeomorphisms of $X$ is to be considered. It would seem natural to consider diffeomorphisms of $X$ which map $W$ to itself. However we notice that, while $TW$ is mapped to itself by any diffeomorphism of this kind, the same is not true for the normal bundle: if $\psi$ is any such diffeomorphism, $N$ and $\psi^*N$ are in general different subbundles of $TX$. Now, while a generic connection $A$ in $N$ is mapped to a connection $\psi^*A$ in $\psi^*N$, this is not true anymore for a background connection $A_0$. Recalling the above remarks on background connections we point out that, if $A_0$ is a fixed connection of $N$, it cannot in general be a connection in $\psi^*N$. If we want a basic expression of the anomaly, we have to come to terms with this fact. Therefore we restrict the subgroup of allowed diffeomorphisms to those that leave $N$ globally invariant and denote it as $\text{Diff}(X,N)$. This implies that $N$ has some physical significance. In fact the M–5–brane spectrum contains five scalar fields, which span the normal directions to the 5–brane. A symmetry transformation permitted by physics can only transform each one of five fields into a combination of them, but will not be allowed to switch on new directions. This is exactly what the global invariance of $N$ means.

The subgroup $\text{Diff}(X,N)$ can also be seen more usefully as $\text{Aut}P$, the group of automorphisms of the principal fiber bundle $P$ with structure group $SO(5)$, associated to the normal bundle $N$. In this paper we will consider only the infinitesimal version, its Lie algebra $\text{aut}P$. The anomaly of the M–5–brane is the anomaly with respect to the transformations $Z$, which are the vector fields in $\text{aut}P$. The next thing to consider is the splitting $N = L \oplus N'$. This is an inevitable consequence of (1.1), see [7], and induces the reduction of the structure group from $SO(5)$ to $SO(4)$. There is a manifold of such splittings and one can take two attitudes: either one assumes there is a privileged one and then further limit $\text{Diff}(X,N)$ to those diffeomorphisms that preserve such a splitting, or
else one takes all of them into account. We will resume later on this discussion. For the
time being we treat the problem in complete generality: first of all, we classify all possible
reductions of $SO(5)$ to $SO(4)$; then we notice that a diffeomorphism of $\text{Diff}(X, N)$ maps
a reduced bundle into another reduced bundle, on the other hand the transform of a non–
reducible connection in the first bundle is not a connection in the transformed one; so it
makes sense to consider in $p_2(N)$ only reducible connections. Therefore we write down a
general formula for reducible connections and show that $p_2(N)$ decomposes into the square
of a Pfaffian; finally we derive a basic formula for the anomaly. In the next section, we
show that it is possible to cancel it with a local counterterm in $W$.

2.2 The role of gamma matrices

A relevant role in the following is played by gamma matrices and by the relation between
the fundamental representation of $\mathfrak{sp}(2)$ and the representation $4$ of $\mathfrak{so}(5)$. We devote the
present subsection to these pedagogical topics.

Gamma matrices for $\mathfrak{spin}(5) = \mathfrak{so}(5)$ are defined as follows: take Euclidean gamma
matrices in 4D, $\gamma_1, \ldots, \gamma_4$ and define $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$. Then $\gamma_a$ with $a = 1, \ldots, 5$ satisfy
\[
\{ \gamma_a, \gamma_b \} = 2 \delta_{ab}
\]

They form a $4 \times 4$ matrix representation of the Clifford algebra of $\mathbb{R}^5$. We have $\gamma_a^2 = 1$ and
\[
\text{Tr}(\gamma_a \gamma_b \gamma_c \gamma_d \gamma_e) = 4 \epsilon_{abcde}
\]

with $\epsilon_{12345} = 1$. The quadratic combinations
\[
\Sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]
\]
satisfy
\[
[\Sigma_{ab}, \Sigma_{cd}] = -\delta_{ac} \Sigma_{bd} + \delta_{ad} \Sigma_{bc} - \delta_{bd} \Sigma_{ac} + \delta_{bc} \Sigma_{ad}
\]
\[
[\Sigma_{ab}, \gamma_c] = \delta_{bc} \gamma_a - \delta_{ac} \gamma_b
\]

Therefore they are the generators of the representation $4$ of $\mathfrak{so}(5)$. The latter can be
identified with the fundamental representation of $\mathfrak{sp}(2)$ via the Lie-algebra isomorphism
$\mathfrak{so}(5) \approx \mathfrak{sp}(2)$. Consider now the adjoint representation

$\text{Ad}: \text{Spin}(5) \to SO(5)$

The associated Lie algebra isomorphism

$\text{Ad}_*: \mathfrak{spin}(5) \to \mathfrak{so}(5)$

is given on the basis elements $\Sigma_{ij} = \frac{1}{4} [\gamma_i, \gamma_j]$ ($i < j$) by

$\Sigma_{ij} \to e_i \wedge e_j$
Notice that \( \mathfrak{so}(5) \) acts on vectors of \( \mathbb{R}^5 \), in the usual way
\[
(e_i \wedge e_j) e_k = e_i \delta_{jk} - e_j \delta_{ik}
\]
and so \( e_i \wedge e_j \) can be identified with the \( 5 \times 5 \) matrix \((E_{ij})_{lm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{lj}\). Therefore the corresponding representations can be implemented at the level of \( \text{spin}(5) \) on \( \Delta = \text{span}(\gamma_i) \subset \text{Mat}(4) \) as follows \( \Sigma_{ij} \cdot \gamma_k = [\Sigma_{ij}, \gamma_k] \). Therefore for any vector \( v^a, a = 1, \ldots, 5 \), it is convenient to define \( v = v^a \gamma_a \), i.e. \( v^a = \frac{1}{4} \text{Tr}(\gamma^a v) \).

### 2.3 Reducible connections

Our purpose in this subsection is to justify formula (2.3) for reducible connections. On first reading one can jump directly to that formula and to the next subsection.

Consider the principal bundle \( P \to W \) associated to the normal bundle \( N \) and a reduction \( R \to W \) with structure group \( SO(4) \) with associated bundles \( N' \) and fiber \( \mathbb{R}^4 \). Any reduction \( j : R \to P \) corresponds to a section \( \sigma : W \to E \) of the bundle \( E \) associated to \( P \) with fiber \( \frac{SO(5)}{SO(4)} \). We will identify \( \frac{SO(5)}{SO(4)} = S^4 \). To make this identification precise consider the vector \( e = (0, 0, 0, 0, 1) \in \mathbb{R}^5 \) and let \( j : SO(4) \to SO(5) \) be given by
\[
j(h) = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}
\]
Then \( SO(4) e = e \) and we map \([g] \in \frac{SO(5)}{SO(4)}\) to \( ge \in S^4 \). Once this identification is made it is easy to identify \( E \) with the unit sphere subbundle \( S(N) \) of the normal bundle \( N \). To the section \( \sigma : W \to S(N) \) we can associate as usually a unimodular map \( v : P \to \mathbb{R}^5 \), such that \( v(pg) = g^{-1}v(p) \). In fact any point \( p \in P \) (i.e. a basis \( p = (X_1, \ldots, X_5) \) of \( N_{\pi(p)} \)) defines a map \( \widehat{p} : \mathbb{R}^5 \to N_{\pi(p)} M \) via \( \widehat{p}(w) = \sum_{i=1}^5 X_i w_i \). Now define a function \( v(p) \equiv \widehat{p}^{-1} \sigma(\pi(p)) \).

Notice that \( v \) determines a line bundle \( L_v \) within the normal bundle \( N \) and thus a splitting \( N = L_v \oplus N'_v \), where \( N'_v \) has structure group \( SO(4) \).

Let now \( A \) be a connection in \( P \). By definition \( d_A v \) is a tensorial 1-form on \( P \) with values in \( \mathbb{R}^5 \). Consider the 1-form \( \omega = v \wedge d_A v \) on \( P \). Given the canonical basis \( e_i \) of \( \mathbb{R}^5 \) we can decompose \( v \) as \( v = \sum_{i=1}^5 v_i e_i \), and
\[
v \wedge d_A v = \sum_{i,j=1}^5 v_i (d_A v)_j [e_i \wedge e_j] = \sum_{i,j=1}^5 v_i (d_A v)_j E_{ij}.
\]
We have the following properties

- \( \omega \) takes values in \( \mathfrak{so}(5) \).
- \( \omega \) vanishes on vertical vectors.
- For \( g \in G, p \in P, X \in T_p P \) we have \( \omega((R_g)_*, X)_p = \text{ad}_{g^{-1}} \omega(X)_p \).

Therefore \( \omega \in \Omega^1(W, \text{ad}P) \). We can therefore consider the connection on \( P \)
\[
B = A - \omega = A - v \wedge d_A v
\]
Assume now that $\langle v, v \rangle = 1$. Then

$$d_B v = dv - (A - v \wedge d_A v)v = dv - Av + (v \wedge d_A v)v = dv - Av - d_A v = 0$$

In fact

$$[v_i d_A v_j v_k (E_{ij}) e_k]_m = v_i d_A v_j v_k [E_{ij}]_{ml} (e_k)_l =$$

$$= v_i d_A v_j v_k - v_k d_A v_m v_k = (\langle d_A v, v \rangle v - \langle v, v \rangle d_A v)_m = -d_A v$$

because $\langle d_A v, v \rangle = 0$ if we assume that $\langle v, v \rangle = 1$. The equation $d_B v = 0$, means that $v$ is parallel with respect to $B$, i.e. $B$ is a reducible connection (in $P$), with reduced group $SO(4)$ (see [1], vol.1, Proposition 7.4 of Chapter 2). $B$ is therefore a connection reducible to the subbundle determined by $v$. We will make now the assumption that the bundle $P$ has a double covering $\tilde{P} \rightarrow W$ with structure group $Spin(5)$. In order for this assumption to hold it is enough that both $W$ and $X$ have a spin-structure. Now every connection $A$ on $\tilde{P}$ induces a connection, which we call again $A$ on $P$ via the isomorphism $\tilde{P}$. The above argument generalizes to $\tilde{P}$.

If, however, one works in $\tilde{P}$ with the spin connection and use the identifications described above, one must be aware that many familiar conventions change. In the next subsection we collect a set of useful formulas and results by making use of this formalism.

### 2.4 Some explicit formulae and the Pfaffian

For simplicity, let us speak about a principal fiber bundle $P$ with structure group $SO(5)$, a connection $A$ in it and a map $v$ from $W$ to $\mathbb{R}^5$. We start with

$$d_A v = dv - \frac{1}{2} [A, v]$$

$$F_A = dA - \frac{1}{4} [A, A]$$

$$d_A d_A v = -\frac{1}{2} [F_A, v]$$

The transformation properties under a gauge transformation $\Lambda = \Lambda^{ab} \Sigma_{ab}$ are

$$\delta v = -\frac{1}{2} [v, \Lambda] \longrightarrow \delta v^a = \Lambda^a_b v^b$$

$$\delta A = d_A \Lambda = dA - \frac{1}{2} [A, \Lambda]$$

$$\delta F_A = -\frac{1}{2} [F, \Lambda]$$

and so on. Notice that we have

$$\sum_{a=1}^{5} \text{Tr}(\gamma_a \Sigma_{j_1 j_2} \Sigma_{j_3 j_4}) \text{Tr}(\gamma^a \Sigma^{i_1 i_2} \Sigma^{i_3 i_4}) = 5 \left( \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3} \delta^{i_4}_{j_4} \pm \text{permutations} \right)$$
and so

\[ p_2(N) = \frac{1}{(2\pi)^4} \sum_{a=1}^{5} \text{Tr}(\gamma_a FF)\text{Tr}(\gamma^a FF), \quad F = F^{ab}\Sigma_{ab} \]  \tag{2.2}

Setting

\[ \chi_a = \text{Tr}(\gamma_a FF) = \epsilon_{abcde} F^{bc} F^{de} \]

and \( \Lambda = \Lambda^{ab}\Sigma_{ab} \), we get

\[ \delta \chi_a = -2\Lambda_a^b \chi_b \]

We will often use the following obvious vanishing argument. Write

\[ \epsilon_{abcde} v^a_1 v^b_2 v^c_3 v^d_4 v^e_5 \sim \text{Tr}(v_1 v_2 v_3 v_4 v_5) \]

where \( v_i, i = 1, \ldots, 5 \) are vectors in \( \mathbb{R}^5 \). If they are all orthogonal to the same vector \( v \) in then the above expression vanishes.

The reducible connection \( B \) in \( P \) becomes in \( \tilde{P} \)

\[ B_v = A - \frac{1}{2}[v, d_A v] \]  \tag{2.3}

Very often we will drop the label \( v \) in \( B_v \). Whenever this happens it is understood that we refer to the reduction represented by the \( v \) in question. We will not insist either on the distinction between \( P \) and \( \tilde{P} \). It is easy to directly verify that

\[ d_{B_v} v \equiv d_{B_v} = 0 \]  \tag{2.4}

where use has been made of \( \langle v, d_A v \rangle = 0 \). As a consequence of (2.3) we have:

\[ F_B = F_A + \frac{1}{4}[v, [F_A, v]] - \frac{1}{4}[d_A v, d_A v] \]

From (2.4) we also deduce that \( [F_B, v] = 0 \). This in turn implies that \( F_B \perp v \), i.e. \( F_B^{ab} v_a = 0 = F_B^{ab} v_b \).

Another useful result is the following: if \( Z \in autP \) is vertical, then, from (2.3), \( B(Z) = A(Z) \).

Let us consider next the effect of a generic \( Z \in autP \) on \( v \). We have

\[ L_Z v = \frac{1}{2}[i_Z B, v] \]  \tag{2.5}

since \( [i_Z F_B, v] = 0 \). Any transformation \( \delta v \) of this kind maps a reducible connection \( B \) into a new reducible connection \( B + L_Z B \). In fact

\[ d_{B + L_Z B}(v + L_Z v) = 0 \]

up to infinitesimals of higher order. Notice that \( Z \) maps \( N \) to \( N' \), but deforms \( N'_v \), for it maps it to \( N'_{v + \delta v} \).

Any reduction \( R \) of \( P \) determines a factorization of \( p_2(N) \) into the square of \( Pf(A, v) = \text{Tr}(v F_B F_B) \). In fact, given a reduction determined by a vector \( v \), it makes sense to consider
in $p_2(N)$ only the relevant reducible connections. Therefore in (2.2) we must replace everywhere $A$ with $B \equiv B_v$. But since, as we have noticed, $F_B \perp v$, only the component of $\chi_a$ parallel to $v$ will contribute in the expression of $p_2(N)$ due to the vanishing argument. Therefore we can write

$$\left. \frac{1}{24} p_2(N) \right|_v = \alpha^2 \left( Pf(A, v) \right)^2, \quad Pf(A, v) = \text{Tr}(v F_B F_B), \quad \alpha = (24)^{-1} (2\pi)^{-2} (2.6)$$

with obvious meaning of the subscript $v$. The number of independent degrees of freedom in $v$ is 4, the same as the difference between the $SO(5)$ gauge variables and the $SO(4)$ ones. What happens is clear: when considering reducible connections we trade the gauge parameters lost in the reduction process with the free parameters in $v$.

Notice that in $Pf(A, v)$, the Pfaffian corresponding to $N'_v$, only terms linear in $v$ and with an even number of $d_A v$ survive (compare with [3]).

3. Expression of the anomaly

Our aim is to deduce the anomaly from the usual descent equations, starting from the Pfaffian $Pf(A, v)$ introduced above while introducing a background connection. The derivation is far from straightforward. So before we deal with it, let us work out a simpler well–known example and use it as a guide.

Let us consider a generic connection $A$ in a principal bundle $P$ with base $M$, together with a background connection $A_0$. Let $I$ denote the unit interval over which a parameter $t$ is defined. Then we can think of the interpolating connection $A_t = A_0 + t(A - A_0)$ as a connection on $P \times I$ and denote it $\hat{A}$ (from now on hatted symbols will indicate quantities relevant to $P \times I$). Its curvature will be

$$\hat{F} = F_{\hat{A}} - (A - A_0) dt$$

Correspondingly, if $d$ denotes the exterior derivative in $M$, $\hat{d}$ will be the exterior derivative on $M \times I$. We can easily derive the Chern–Simons formula:

$$d \int_I Q(\hat{F}) = \int_I \hat{d}Q(\hat{F}) - Q(F_t)|_0 = -Q(F) + Q(F_0)$$

where $Q$ is any symmetric ad–invariant polynomial\(^1\). So the Chern–Simons term can be written

$$W_Q(A, A_0) = -\int_I Q(\hat{F})$$

We would like now to express in a similar way the corresponding anomaly. Consider the derivative in the space of connections on $P \times I$ along the vector $t\xi$, with $\xi = L_Z A \in \Omega^1(M, \text{ad}P)$, for any $Z \in \text{aut}P$. Then $\delta \hat{F} = \hat{d}_A t\xi = td_A \xi - \xi dt$, and

$$\delta \int_I Q(\hat{F}) = \int_I n Q(\delta \hat{F}, \hat{F}) = \int_I n Q(\hat{d}_A t\xi, \hat{F}) = \int_I n \hat{d}Q(t\xi, \hat{F}) =$$

\(^1\)Q has $n$ entries, but here and in the following we adopt the convention that whenever several entries coincide, we write down only one of them, say $Q(F, \ldots, F) \equiv Q(F)$.  

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\[ d \int_I nQ(t\xi, \hat{F}) - nQ(\xi, F) = d \left( n(1 - n) \int_I dt \, tQ(\xi, A - A_0, \mathcal{F}_{A_t}) \right) - nQ(\xi, F) \]

The last term drops for dimensional reasons (if \( \text{dim} \, M \leq 2n - 2 \)) and the term in brackets gives the well-known expression of the anomaly.

Let us return now to our original problem on the 5–brane world volume \( W \). For a consistent construction we have to introduce both a reducible reference connection \( A_0 \) and a corresponding reference reduction \( v_0 \). So we can define \( Pf(A_0, v_0) = \text{Tr}(v_0F_{B_0}F_{B_0}) \). Now we introduce the interpolating connection \( A_t = tA + (1 - t)A_0 \) and a path of unit vectors \( v_t \) that interpolates between \( v_0 \) and \( v_1 = v \). There are of course many different paths from \( v_0 \) to \( v \). All we say in this and the following section does not depend on what path we choose. However this issue will become important in the final section, in which we will need the path to depend only on the initial and final vectors. To satisfy this requirement we select one single path from \( v_0 \) to \( v = v_1 \): since \( v_0 \) and \( v \) lie both on a sphere \( S^4 \) embedded in \( \mathbb{R}^5 \), we will choose the geodesic (with respect to the embedded metric) passing through \( v_0 \) and \( v \). This is a canonical choice of a path between \( v_0 \) and \( v \).

Now we set

\[ \mathcal{B}_{v_t} \equiv \mathcal{B}_t = A_t - \frac{1}{2}[v_t, d_{A_t}v_t] \]

We have, of course,

\[ d_{\mathcal{B}_t}v_t = 0 \]

Now let us consider the family \( R_t \) of reductions of \( P \), determined by \( v_t \). Then we have a bundle \( \mathcal{R} \rightarrow W \times I \). The path \( \mathcal{B}_t \), just introduced, can be viewed as a connection \( \mathcal{B} \) on \( \mathcal{R} \). We can then repeat the above procedure. For simplicity, from now on we write our Pfaffian in a more standard way:

\[ Q(A, B, C) \equiv \alpha \text{Tr}(ABC) \]

(the constant \( \alpha \) has been introduced in \([2.10]\)). The polynomial \( Q \) is indeed symmetric and ad–invariant.

Let \( \hat{A} \equiv A_t = A_0 + t(A - A_0) \) on \( P \times I \) and \( \hat{\mathcal{B}} = \mathcal{B}_t - \frac{1}{2}[v_t, \dot{v}_t]dt = A_t - \frac{1}{2}[v_t, \dot{A}_t] \).

Then \( \hat{d}_{\mathcal{B}_t}v_t = d_{\mathcal{B}_t}v_t + (\dot{v}_t + \frac{1}{4}[v_t, \dot{v}_t]) dt = 0 \) (using the gamma matrix algebra) and \( \hat{d}_{\mathcal{B}_t}\hat{F}_{\mathcal{B}_t} = 0 \), where

\[ \hat{F}_{\mathcal{B}_t} = F_{\mathcal{B}_t} - \left( \hat{\mathcal{B}}_t + \frac{1}{2}d_{\mathcal{B}_t}([v_t, \dot{v}_t]) \right) dt \]

Correspondingly we have

\[ d \int_I Q(v_t, \hat{F}_{\mathcal{B}_t}, \hat{F}_{\mathcal{B}_t}) = \int_I dQ(v_t, \hat{F}_{\mathcal{B}_t}, \hat{F}_{\mathcal{B}_t}) - Q(v_t, \hat{F}_{\mathcal{B}_t}, \hat{F}_{\mathcal{B}_t})|_0^1 = -Q(v_1, F_B, F_B) + Q(v_0, F_{B_0}, F_{B_0}) \]

So the Chern–Simons term we were looking for is

\[ W_Q(v, v_0, B, B_0) = - \int_I Q(v_t, \hat{F}_{\mathcal{B}_t}, \hat{F}_{\mathcal{B}_t}) \]

\( \text{\footnotesize\( ^2 \text{This prescription is not unique whenever } v_1 = -v_0. \) However, except when both } v_0 \text{ and } v_1 \text{ are constant sections, a case we can easily exclude, the ambiguity can be resolved by continuity.} \)
We would like now to express in a similar way the anomaly. Consider the derivative in the space of connections on $\mathcal{R}$ along the vector $t\xi$, with $\xi = L_Z A \in \Omega^1(W, \text{ad}P)$ and along $\delta v = L_Z v$. Then

$$\delta \hat{B} \equiv \delta(B_t - \frac{1}{2}[v_t, \dot{v}_t]dt) = \hat{G}(\xi)$$

where

$$\hat{G}(\xi) = t(\xi + \frac{1}{4}[v_t, [\xi, v_t]]) - \frac{1}{2}[\delta v_t, \dot{d}_A v_t] - \frac{1}{2}[v_t, \dot{d}_A \delta v_t] \equiv G_t(\xi) + (\ldots)dt$$

and $\dot{d}_A, v_t = d_A, v_t + v_t dt$, etc. Then

$$\delta \hat{F}_{\hat{B}} = \dot{d}_B \hat{G}(\xi)$$

We have

$$\delta \int_I Q(v_t, \hat{F}_{\hat{B}}, \hat{F}_{B}) = \int_I Q(\delta v_t, \hat{F}_{\hat{B}}, \hat{F}_{B}) + 2 \int_I Q(v_t, \delta \hat{F}_{\hat{B}}, \hat{F}_{B})$$

$$= 2 \int_I Q(v_t, \dot{d}_B \hat{G}(\xi), \hat{F}_{B}) = 2 \int \dot{d}Q(v_t, \hat{G}(\xi), \hat{F}_{B})$$

$$= 2d \left( \int_I Q(v_t, \hat{G}(\xi), \hat{F}_{B}) \right) - 2Q(v_t, G_1(\xi), F_B) \quad (3.2)$$

In fact

$$\int_I Q(\delta v_t, \hat{F}_{\hat{B}}, \hat{F}_{B}) = 2 \int dtQ(\delta v_t, \hat{B}_t + \frac{1}{2}d_{B_t}[v_t, \dot{v}_t], F_{B_t})$$

$$= 2 \int dtQ(\delta v_t, \hat{B}_t + \frac{1}{4}[v_t, [\hat{B}_t, v_t]], F_{B_t}) \quad (3.3)$$

having used $d_{B_t} v_t = 0$ differentiated with respect to $t$. Now it is easy to prove that any quantity

$$\hat{C} = C + \frac{1}{4}[v, [C, v]], \quad C = C^{ab}\Sigma_{ab}$$

is such that $[\hat{C}, v] = 0$, i.e. $\hat{C} \perp v$. Therefore, using $[3.5]$ applied to $v_t$ and using the ad–invariance of the polynomial $Q$, it is easy to see that $[3.3]$ vanishes.

Now we recall that

$$\delta v = L_Z v, \quad \delta v_0 = 0 \quad (3.4)$$

and notice that $G_1(\xi) = L_Z B$. Therefore we can write

$$\delta \int_I Q(v_t, \hat{F}_{\hat{B}}, \hat{F}_{B}) = 2d \left( \int_I Q(v_t, \hat{G}(\xi), \hat{F}_{B}) - Q(v, i_Z B, F_B) \right)$$

$$- i_Z Q(v, F_B, F_B)$$

$$\equiv dA_2^1 - i_Z Q(v, F_B, F_B) \quad (3.5)$$

$A_2^1$ corresponds to the familiar 2d anomaly. The expression of $A_2^1$ is as follows

$$A_2^1 = -2 \left( \int dtQ(v_t, L_Z B_t, \hat{B}_t + \frac{1}{4}[v_t, [\hat{B}_t, v_t]]) + Q(v, i_Z B, F_B) \right)$$

$$- \int dtQ(v_t, [L_Z v_t, \dot{v}_t], F_{B_t}) - \int dtQ(v_t, [v_t, \delta \dot{v}_t], F_{B_t}) \quad (3.6)$$
The last term vanishes since \([F_{B_t}, v_t] = 0\).

Now let us extract the full anomaly. The Chern–Simons term is

\[
W_{\text{tot}}(v, v_0, B, B_0) = W_Q(v, v_0, B, B_0)\left(Q(F_B) + Q(F_{B_0})\right)
\]

where \(Q(F_B) = Q(v, F_B, F_B), Q(F_{B_0}) = Q(v_0, F_{B_0}, F_{B_0}),\) and

\[
W_Q(v, v_0, B, B_0) = -2 \int dt Q(v_t, \dot{B}_t) + \frac{1}{4}[v_t, [\dot{B}_t, v_t]], F_{B_t})
\]

With standard steps we obtain

\[
\delta W_{\text{tot}} = i_Z (Q(F_B)Q(F_B)) - d \left[A_2^1 (Q(F_B) + Q(F_{B_0})) + W_Q(v, v_0, B, B_0)i_Z Q(F_B)\right]
\]

The first term on the RHS vanishes for dimensional reasons. The term in square bracket is the anomaly. Therefore

\[
\text{Anom}_v = A_2^1 (Q(F_B) + Q(F_{B_0})) + W_Q(v, v_0, B, B_0)i_Z Q(F_B)
\]  

(3.7)

The label \(v\) is in order to stress that this expression depends on the particular reduction \(v\) we have chosen.

4. The M–5–brane anomaly cancellation

Before we embark in the discussion of anomaly cancellation, let us summarize what we have done and clarify the role of the background connection \(A_0\) in this context. The anomaly just obtained is the residual M–5–brane anomaly generated via the descent equations from the 8–form \(p_2(N)\), i.e. the second Pontryagin class of the normal bundle. For reducible connections \(p_2(N)\) splits into the square of a Pfaffian. Any such splitting can be thought to correspond to a section \(v\). Therefore we can write

\[
N = L_v \oplus N'_v
\]  

(4.1)

At this point we remark that, if in \(p_2(N)\) we consider a connection reducible to \(SO(4)\) (i.e. the connection \(B_v\) introduced above), the second Pontryagin class of \(N\) becomes:

\[
p_2(N) = p_2(N'_v) = e(N'_v)^2 = 24 Q(v, F_B, F_B)^2
\]  

(4.2)

where \(e\) represents the Euler class. Therefore the anomaly we have to cancel is generated by \(Q(v, F_B, F_B)\) and given by (3.7).

It is thanks to the factorization (4.2) that in the previous section we were able to derive eq. (3.7) for the anomaly, and in this section we are able to cancel it via a suitable counterterm. At this point however, as partially anticipated in the previous section, we can take two different attitudes.

The first attitude is based on the idea that one has to restrict the subgroup of relevant diffeomorphisms of \(X\) to those which, not only map \(W \to W\) and \(N \to N\), but also preserve the splitting \(N = L_v \oplus N'_v\), i.e. in particular preserve \(v\). Let us call this
subgroup $\text{Diff}_v(X, N)$. Now we can safely pick a background connection $A_0$ in $N'_v$: it will remain a connection in any transformed $N'_v$, as long as the diffeomorphisms considered are those of $\text{Diff}_v(X, N)$. They correspond to the automorphisms of a principal fiber bundle $P'$ whose structure group is $SO(4)$. With this understanding the anomaly is given by (3.7) with fixed $v$. This attitude assumes that $v$ has some physical meaning. We recall that $L_v$ represents the direction normal to $W$ which is tangent to $Y$, the manifold which bounds $W$. This means therefore that $Y$, or at least a collar which represents the part of $Y$ nearest to $W$, retains some physical information too. In other words, it would seem that physical information about the 5–brane is not stored only in $W$ but also on $Y$. This sounds curiously similar to what has been proposed in a different context in [14], where two different manifolds terminating on the same space–time manifold carry different physical information. We shall call this scheme the restricted scheme. This is essentially the scheme adopted in [1, 3]. Another context in which this scheme applies is to describe the compactification of M theory on a circle $S^1$ along a direction transverse to the 5–brane. In this case the latter becomes the NS 5–brane of type IIA theory. As pointed out in [4], the normal bundle to the 5–brane then splits into the direct sum of a vector bundle with structure group $SO(4)$ and a trivial line bundle, which actually coincides with the tangent bundle of the compactification circle. Although the construction is not exactly the same as above, the final setup is. Therefore we can choose a section of the trivial line bundle, say $v$, and redo everything without changing a single word. In this case the physical nature of $v$ is immediately visible.

The second attitude, or general scheme, assumes instead that, although the splitting $N = L_v \oplus N'_v$ has of course a physical meaning, since it represents, via (1.1), the magnetic coupling of the 5–brane to M theory, there is no privileged reductions. Therefore we have to consider all possible $v$’s and integrate over them in the relevant path integral (see below for further comments on this point). In this case the gauge transformations are all $Z \in \text{aut}P$, where $P$ is the principal fiber bundle with structure group $SO(5)$ associated to $N$. The intermediate steps in the derivation of the anomaly make sense since we have seen that a transformation by any $Z \in \text{aut}P$ maps a reduction to another reduction and a reducible connection into another reducible connection. There is no problem either with the background connection $A_0$ and the background vector $v_0$, which remain fixed throughout. Finally the anomaly is again given by (3.7).

4.1 The counterterm

The basis for anomaly cancellation is that $e$ be cohomologically trivial, i.e.

$$Q(v, F_B, F_B) = d\eta_v|_W$$

(4.3)

where $\eta_v$ is some 3–form field in the theory. We will discuss in the following subsection how to relate $\eta_v$ to the theory. For the time being let us suppose that it exists and is local. Consistently with (4.3) we set

$$Q(v_0, F_0, F_0) = d\eta_{v_0}$$

(4.4)
where $\eta v_0$ need not be a local field, it may be a purely differential-geometric 3–form. Moreover, consistently with our definitions and with $\delta A_0 = 0$, we set
\begin{equation}
\delta \eta_v = L_Z \eta_v, \quad \delta \eta_{v_0} = 0 \tag{4.5}
\end{equation}

As we said above, the anomaly we have to cancel is given by (3.7). Our proposed counterterm to cancel it is
\begin{equation}
S_v = \int_W \left( (\eta_v + \eta_{v_0}) \wedge W_Q(v, v_0, B, B_0) + \eta_v \wedge \eta_{v_0} \right) \tag{4.6}
\end{equation}

We have
\begin{equation}
\delta S_v = \int_W \left[ L_Z \eta_v \wedge W_Q(v, v_0, B, B_0) + L_Z \eta_{v_0} \wedge \eta_{v_0} \right. \\
- (\eta_v + \eta_{v_0}) \wedge \left( dA_2^1 - i_Z Q(F_B) \right) \\
= \int_W \left[ i_Z Q(F_B) \left( W_Q(v, v_0, B, B_0) + \eta_{v_0} \right) - i_Z \eta_v \wedge \left( W_Q(v, v_0, B, B_0) + \eta_{v_0} \right) \right. \\
+ (\eta_v + \eta_{v_0}) \wedge i_Z Q(F_B) - (Q(F_B) + Q(F_{B_0})) A_2^1 \left. \right] \\
= \int_W \left[ i_Z Q(F_B) W_Q(v, v_0, B, B_0) - i_Z (Q(F_B) \wedge \eta_v) \right. \\
- (Q(F_B) + Q(F_{B_0})) A_2^1 \left. \right] \\
= \int_W \left[ i_Z Q(F_B) W_Q(v, v_0, B, B_0) - (Q(F_B) + Q(F_{B_0})) A_2^1 \right]
\end{equation}

where we have used $i_Z [\eta_v \wedge Q(F_B, F_B)] = 0$ for dimensional reasons. Therefore adding the counterterm $S$ to the action cancels the residual M–5–brane anomaly.

The anomaly cancellation works in both restricted and general schemes.

4.2 The nature of $\eta_v$

The previous cancellation mechanism is based on the existence in the M–theory with a 5–brane of a 3–form field with the transformation property (1.1). In such a theory there are several 3–forms. From 11 dimensional supergravity we have the 3–form $C_3$. On the 5–brane we have a 2–form $B_2$, by means of which in the interacting theory we can form the combination $H_3 = dB_2 - C_3$. But neither $C_3$ nor $H_3$ can be identified with $\eta_v$, even though the transformation property would be the right one (1.5): we know that in the absence of the 5–brane we have $dC_3 = F_4$, while, when the 5–brane is present, $F_4$ is modified in such a way that (1.1) holds; therefore $C_3$ does not contain any information concerning $e$.

From this discussion it is evident that $\eta_v$ must be constructed out of $F_4$. Let us generalize the construction presented in [7]. The section $v$, which determines the decomposition $N = L_v \oplus N_v'$, can be seen as a vector field on the 11 dimensional space $X$: $v = \sum_i v^i \frac{\partial}{\partial x^i}$. In fact $v$ is a section of a line bundle which lies in $N$ and therefore in $TX^3$. In the following we will need the equation
\begin{equation}
i_v \Phi(L)|_W = 1 \tag{4.7}
\end{equation}

3We recall that $v$ is a vector field in $X$, while $v$ represents a set of scalar fields in $W$
where Φ denotes the Thom form on L. To show (4.7), notice that for a generic vector field v with nonzero components only along L, $i_v\Phi(L)|_W$ is a non-vanishing function on W, therefore a suitable rescaling is enough to produce the desired result. Whichever the choice, we remark that what we have achieved so far is the definition of v only on W; outside W we can define it in an arbitrary way.

Now, assuming the triviality of $Q(v, F_B, F_B)$, eq. (1.1), given v, is equivalent to the existence of a local 3–form $\chi_v$ that solves the equation

$$L_vF_4|_W = d\chi_v$$

In fact using $L_v = di_v + i_vd$ we have

$$(di_vF_4 + i_vdF_4)|_W = d\chi \Leftrightarrow i_v\delta W|_W = dW(\chi_v - i_vF_4|_W)$$

We recall from [7] that $\delta W = \Phi(N'_v) \wedge \Phi(L_v)$ and the Thom form $\Phi(N'_v)$ can be interchanged with the Euler form $e$ of $N'_v$. Then we get

$$i_v\delta W|_W = \Phi(N'_v)|_W = e(N'_v) = Q(v, F_B, F_B)$$

So, finally,

$$Q(v, F_B, F_B) = d\eta_v|_W, \quad \eta_v = \chi_v - i_vF_4$$

This $\eta_v$ satisfies all the requirements. The form $\chi_v$ is left undetermined by our analysis.

4.3 The fate of v

It remains for us to discuss the implications that come for a theory from the presence of v. In the general scheme, in fact, we have still to explain how we deal with v from a field–theoretic point of view: what kind of field is v and what is the path integral treatment of it beyond the anomaly problem, where v is a spectator?

We summarize what has been said so far: the possible magnetic couplings of the 5–brane inside the M–theory are spanned by the v sections. Each v represents a reduction from the structure group of the normal bundle of W from $SO(5)$ to $SO(4)$. In view of this physical input (i.e. the magnetic coupling) it only makes sense to consider reducible connections, i.e. connections valued in the Lie algebra $so(5)$, which, when restricted to the reduced bundle, are connections valued in the Lie subalgebra $so(4)$. Given a connection $A$ in the principal fiber bundle $P$ with gauge group $SO(5)$, and a section $v$ of the associated bundle with fiber $SO(5)/SO(4)$, we can construct a reducible connection $B_v$ via $(2.3)$. Therefore the relevant theory is obtained starting from the theory in the eleven dimensional manifold $X$ coupled to the 5–brane with world–volume $W$, considering the splitting of the spin connection of $X$ on $W$ into a tangential and normal part (the latter is exactly $A$) and then replacing $A$ everywhere with $B_v$. Here we have to be careful about possible Jacobian factors. The measure over $A$ in the path integral is provided by the theory. As for the measure over v, it is very natural to adopt the measure of the gauge transformations that map v to a fixed section $v_0$ (we have already remarked that, in the process of reducing the structure group, we trade such gauge transformations for the v’s). Now, surprisingly
enough, the Jacobian for the passage from $A$ to $B_v$ at fixed $v$ is a constant. Therefore we can use, as path integral measure for the theory formulated in terms of $B_v$, the product of the measure of $A$ and the measure of $v$.

After defining the relevant path integral measures, let us turn to the action. Now comes the crucial point: as remarked above, although the action is now expressed in terms of reducible connections, the gauge symmetry group is still $SO(5)$ (not simply $SO(4)$), because an $autP$ transformation (with structure group $SO(5)$) maps a reducible connections into reducible connections while keeping the the background connection unchanged – we stress that this is true in the present case, but is not true in general. This is reflected in the fact that we have considered anomalies of $autP$, not anomalies of the gauge transformations with group $SO(4)$.

Next, while computing anomalies, both $A$ and $v$ are spectators. We have seen that a suitable choice of the counterterm (depending on $A$ and $v$) allows us to free the theory from anomalies. The question is now: what do we do next with $B_v$ and $v$? In particular, what is the fate of $v$?

If we go on with the path integral quantization, after taking care of the fermions determinants, it is necessary to fix the gauge. A simple way (not necessarily the best one) to do it is the following. The infinitesimal gauge parameters $^4$ split into the direct sums of gauge transformations that leave $v$ invariant and the ones that modify $v$. We can fix the gauge by first choosing one fixed $v$, say $v_0$, and then fixing the remaining $SO(4)$ (or, better, $autP$ with structure group $SO(4)$) gauge invariance in the ordinary Faddeev–Popov way$^5$.

In this way we have closed the circle. What was originally a set of gauge degrees of freedom (i.e., $v$) have met their fate, that is they have been gauge-fixed and have disappeared from the game (except for the remnant $v_0$).

To conclude, it is worth emphasizing the roots of the successful cancellation of the M–5–brane anomaly, which originates from the fundamental role played by eq. (1.1). This equation entails the normal bundle splitting, which, in turn, implies that the second Pontryagin class of the normal bundle is factorizable as the square of the Pfaffian. It is only thanks to such occurrence that we can write down the local counterterm (4.6).

### 4.4 One final comment

Our approach in this paper is partially on–shell. In fact we suppose throughout that (1.1) be satisfied. It would be interesting to know whether one can extend it to an off–shell treatment. In a local action for the M–5–brane was proposed that overcomes the traditional difficulty connected with the kinetic term of the self–dual two form of the M–5–brane. This is done by introducing additional fields and gauge symmetries so that the additional degrees of freedom turn out to be pure gauge and the gauge freedom implies the sought for equations of motion. In this scheme was extended by embedding the M–5–brane action into the full action of 11 dimensional supergravity.

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$^4$For global gauge transformations it is perhaps necessary to deal with this problem more carefully.

$^5$It would seem that in this way we return to the restricted scheme. This is not so, because we first make sure that anomaly are canceled so that the full gauge symmetry is restored, and only afterwards do we fix the gauge.
We do not know whether one can deal with the anomaly problem in such more general framework. However it is interesting to point out some similarities. One of the additional fields introduced in [16] is an eleven dimensional vector field $u$, a section of $TX$. In the presence of the 5–brane $u$, restricted to $W$, splits naturally into two components, one in the tangent bundle to $W$ and another in the normal bundle. It is natural to identify the latter components with the section $v$ which is responsible for the reduction of the structure group of the normal bundle. Said otherwise, we can immerse $v$ in the formalism of $[16]$ by assimilating it to the additional $u$ field. The latter is then dealt with as a pure gauge degree of freedom.

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