Three-Qubit Ground State and Thermal Entanglement of XXZ Model With Dzyaloshinskii-Moriya Interaction

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We have studied the symmetric and non-symmetric pairwise ground state and thermal entanglement in three-qubit anisotropic Heisenberg (XXZ) and Ising in a magnetic field models and in the presence of Dzyaloshinskii-Moriya (DM) interaction. We have found that increasing of the DM interaction and magnetic field can enhance and reduce the entanglement of system. We have shown that the non-symmetric pairwise has higher value concurrence and critical temperature (above which the entanglement vanishes) than the symmetric pairwise. For negative anisotropy the non-symmetric entanglement is a monotonic function of DM interaction while for positive anisotropy it has a maximum versus DM parameter and vanishes for larger values of DM interaction. The conditions for the existence of thermal entanglement are discussed in details. The most remarkable result happens at zero temperature where 3-qubit ground state entanglement of the system (in spite of 2-qubit counterpart) shows the fingerprint of quantum phase transition for an infinite size system.

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INTRODUCTION

Entanglement is a property of quantum state which has been studied intensively in recent years as a specific nonlocal quantum mechanical correlation[1,2,3] and it becomes recently a key feature of quantum information theory[4]. When the entanglement is generated, observation or manipulation of it in practice constitutes a major obstacle, because of the fragility of quantum entanglement to the decoherence induced by environment. Therefore, how to generate, maintain and control the entanglement in the presence of dissipative coupling of the system to the environment is of utmost importance in the implementation of quantum information processing.

However, similar to the superposition of two coherent electromagnetic waves which enable us to learn some global information from a localized spatial area, people expect that the entanglement, which roots in the same superposition principle, can enable us to learn some global properties from a small part of the system. This observation may be one of the main motivations in the recent studies[2,4,5] on the role of entanglement between a small part, e.g. a block consisting of one or more sites, and the rest of the system in the quantum phase transition. These results suggested that the local entanglement may be used as a good marker of quantum phase transition.

Then over the past few years there has been an ongoing effort to characterize the entanglement properties of condensed matter systems and apply them in quantum information. The quantum entanglement in solid state systems such as spin chains is an important emerging field[8,9,10,11,12,13,14,15,16]. Spin chain are natural candidates for realization of entanglement, and spin effects have been investigated in many other systems, such as superconductors[17], quantum dots[18] and trapped ions[19]. A most known models in the spin chains is Heisenberg model and Ising model as a special case of Heisenberg model. The Heisenberg model can describe interaction of qubits not only in solid physical systems but also in many other systems such as quantum dots[20], nuclear spin[21], cavity QED[22,23], optical lattice[24], quantum computation[25] and controlled-Not gate[26].

In recent years the two-qubit thermal entanglement which includes spin-spin interactions[10,12,14,20,27,28], and spin-orbit coupling[29,30,31,32,33,34] (Dzyaloshinskii[35]-Moriya[36] interaction) has been studied. Entanglement in two-qubit state has been well studied in the literature along various kind of three-qubit entanglement states[37,38,39,40,41]. The three-qubit entanglement states have been shown to possess advantage over the two-qubit states in quantum teleportation[42], dense coding[43] and quantum cloning[44]. More specifically, both quantitative and qualitative behavior of the entanglement in two-qubit and three-qubit systems are different. The concurrence is always increasing in a two-qubit model versus DM parameter while it is both increasing and decreasing versus DM for a three-qubit model. The position where the concurrence is maximum or zero for a three-qubit model corresponds to the quantum critical point of the infinite size system while it is not the case for a two-qubit model. Moreover, the pairwise entanglement can be defined as symmetric and non-symmetric ones with different properties for three-qubit as will be discussed in this article.

In addition to the above facts, recently some novel magnetic systems with antiferromagnetic (AF) properties, such as $Cu(C_6D_5COO)_23D_2O[33]$, $Yb_{1.83}[40]$, $BaCu_2Si_2O_7[17]$, $\alpha - Fe_2O_3$, $LaMnO_3[48]$ and
were discovered in the category of quasi-one dimensional materials which are known to belong to an antisymmetric interaction of the form $D_i (S_i \times S_j)$ which is known as the Dzyaloshinskii-Moriya (DM) interaction \[33, 32\]. Thus, investigation of the quantum effects in such spin models requires more research in this direction.

In this paper, we have investigated the influence of the anisotropy coupling, magnetic field and the $z$-component DM interaction on the non-symmetric ($\rho_{12} = Tr_3 \rho(T)$) and symmetric pairwise ($\rho_{13} = Tr_2 \rho(T)$) ground state and thermal entanglement of the three-qubit XXZ and Ising models in a magnetic field and in the presence of DM interaction. We will show that the DM interaction, anisotropy and magnetic field parameters are efficient control parameters of entanglement. Increasing the DM coupling and anisotropy have a different effects on the entanglement and can enhance or reduce the entanglement, whereas these parameter just increase the entanglement in two-qubit $^{29, 50}$ counterpart. However we show that at $T = 0$ the 3-qubit quantum phase transition points of this system correspond to those ones at thermodynamic limit $^{13, 50}$.

**THE MODEL INTERACTION**

The Hamiltonian of $N$-qubit $XXZ$ model with $z$-component of DM Interaction is

\[
H(\bar{J}, \Delta) = \frac{\bar{J}}{4} \sum_i^N \left[ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right] + D (\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x),
\]

where $\bar{J}$ is the exchange coupling, $D$ is the strength of $z$ component DM interaction and $\Delta$ defines the easy-axis anisotropy which can be positive or negative. The positive or negative $\bar{J}$ corresponds to the antiferromagnetic (AF) or ferromagnetic (F) cases, respectively. $\sigma_i^\alpha$ refers to the $\alpha$-component of Pauli matrix at site $i$. A $\pi$-rotation around z axis on odd (or even) sites maps the F case ($\bar{J} < 0$) to the AF case with the opposite sign of anisotropy,

\[
H(\bar{J}, \Delta) = \frac{\bar{J}}{4} \sum_i^N \left[ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \Delta \sigma_i^z \sigma_{i+1}^z \right] + D (\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x), \quad J = |\bar{J}| > 0.
\]

So we can restrict ourselves to AF case ($J > 0$) with $D > 0$ and arbitrary anisotropy ($\Delta < 0$ and $\Delta > 0$) without loss of generality.

We can also restore the Ising model with DM interaction (IDM) in the limit $J \rightarrow 0, \Delta \rightarrow \infty$ and $D \rightarrow \infty$ where $J\Delta = \hat{J}$ and $\hat{D} = \hat{D}$. The resulting Hamiltonian is given by

\[
H(\hat{J}, \hat{D}) = \frac{\hat{J}}{4} \sum_i^N \left[ \sigma_i^x \sigma_{i+1}^x + \hat{D} (\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x) \right].
\]

It should be noticed that both AF (corresponds to the positive anisotropy in $XXZ$ with DM) and F cases (corresponds to the negative anisotropy in $XXZ$ with DM) have to be considered separately, since the mentioned symmetry can not be justified for Eq. (1). We will consider the above Hamiltonians (Eqs. [4\hspace{0.33em}13\hspace{0.33em}32\hspace{0.33em}33\hspace{0.33em}34\hspace{0.33em}35\hspace{0.33em}36\hspace{0.33em}37\hspace{0.33em}38\hspace{0.33em}39\hspace{0.33em}40\hspace{0.33em}41\hspace{0.33em}42\hspace{0.33em}43\hspace{0.33em}44\hspace{0.33em}45\hspace{0.33em}46\hspace{0.33em}47\hspace{0.33em}48\hspace{0.33em}49\hspace{0.33em}50]) on a string of 3-qubits and obtain the entanglement of them in terms of the model parameters.

The entanglement of two-qubit can be measured by the concurrence which is defined as $^{51}$

\[
C_{ij} = \max \left\{ 2 \max(\lambda_k) - \sum_{k=1}^4 \lambda_k, 0 \right\}
\]

where $\lambda_k$ (k=1,2,3,4) are the square roots of the eigenvalues of the following operator

\[
R = \rho_{ij} (\sigma_i^x \otimes \sigma_j^x) \rho_{ij}^* (\sigma_i^y \otimes \sigma_j^y),
\]

$\rho_{ij}$ is the density matrix of pair $i$ and $j$ spins and asterisk denotes the complex conjugate. For a system in equilibrium at temperature $T$ the state of a system is determined by the density matrix

\[
\rho(T) = \frac{e^{-\beta H}}{Z},
\]

where $H$ is the system Hamiltonian, $Z = Tr(e^{-\beta H})$ is the partition function and $\beta = \frac{1}{k_B T}$ where $T$ is temperature and $k_B$ is Boltzmann constant. For simplicity we take $k_B = 1$.

To get the concurrence of two qubits in a string of 3-qubits we define two types of reduced density matrix, the symmetric and non-symmetric ones. Let label the 3-qubits as 1,2,3 sequentially. The symmetric reduced density matrix ($\rho_{13}$) is defined as $\rho_{13} = Tr_2 (\rho)$, where $\rho$ is the density matrix of 3-qubits. The non-symmetric reduced density matrix is $\rho_{12} = tr_3 (\rho)$.

**THREE QUBITS XXZ WITH DM INTERACTION**

A straightforward calculation gives the eigenstates and eigenvalues of the 3-qubit XXZ+DM Hamiltonian $^{13}$. The positive square roots of the eigenvalues of the matrix
The concurrence of non-symmetric pairwise is plotted versus $\Delta$ for different values of $T$, ($D = 0.0$).

The concurrence of non-symmetric pairwise against anisotropy ($\Delta$) for different values of $T$, ($D = 2.0$).

The square root of matrix $R$ is invariant under the substitution $D \rightarrow -D$, so the generality of AF case with $D > 0$ and arbitrary anisotropy, remains pristine.

At $T = 0$, for $\Delta > -\sqrt{1 + D^2}$ the concurrence of the doubly-degenerate ground state is

$$C_{12}(T = 0) = \frac{2(q + \Delta - \sqrt{1 + D^2})\sqrt{1 + D^2}}{q(q + \Delta)},$$

while it is a disentangled ferromagnetic states for $\Delta < -\sqrt{1 + 8D^2}$. At the level crossing line ($\Delta = -\sqrt{1 + 8D^2}$) the ground state is 4-fold degenerate and disentangled.

For nonzero temperature ($T \neq 0$) the concurrence is

$$C_{12}^{AF}(T) = \frac{e^{\frac{4}{3}q\Delta}}{Z} \max \left\{ \left( \frac{1 + 8D}{q} \right) \sinh \left( \frac{J}{4} \beta q \right) - \cosh \left( \frac{J}{4} \beta q \right) - e^{-\frac{4}{3}(q+\Delta)}, 0 \right\}.$$  

The model is entangled if $T > T_c(\Delta, D)$, and $T_c(\Delta, D)$ is determined by the following nonlinear equation

$$(1 + 8D/q) \sinh \left( \frac{J}{4} \beta q \right) - \cosh \left( \frac{J}{4} \beta q \right) = e^{-\frac{4}{3}\beta} + 2e^{-\frac{2}{3}\beta}. \quad (9)$$

To determine whether the entanglement exists or not, we have to consider two different cases.

- $\Delta < 0$
  
  The ground state entanglement of non-symmetric pairwise have been shown in Fig. 1 and Fig. 2 as a function of $\Delta$ for $D = 0$ and $D = 2$ which are denoted by $T = 0$ plots. Both figures manifest that the concurrence of ground state decreases

\[ R \text{ for non-symmetric pairwise is obtained} \]

\[
\lambda_1 = \lambda_2 = \frac{1}{Z} \left\{ \frac{1}{2} + e^{-\frac{4}{3}\beta\Delta} \right\},
\]

\[
\lambda_3 = \frac{1}{Z} \left\{ \frac{1}{2} + e^{-\frac{4}{3}\beta\Delta} \right\} \times
\left\{ \frac{3}{2} \cosh \left( \frac{J}{4} \beta q \right) + \frac{\Delta - 8\sqrt{1 + D^2}}{2q} \sinh \left( \frac{J}{4} \beta q \right) \right\},
\]

\[
\lambda_4 = \frac{1}{Z} \left\{ \frac{1}{2} + e^{-\frac{4}{3}\beta\Delta} \right\} \times
\left\{ \frac{3}{2} \cosh \left( \frac{J}{4} \beta q \right) + \frac{\Delta + 8\sqrt{1 + D^2}}{2q} \sinh \left( \frac{J}{4} \beta q \right) \right\}.
\]

\[ (7) \]
by reduction of the anisotropy and suddenly become zero below the critical line $\Delta_c = -\sqrt{1 + D^2}$. In the other words, for each value of $D$, there is a threshold $\Delta_c = -\sqrt{1 + D^2}$ above which the ground state will be entangled. Accordingly, for each value of anisotropy ($\Delta \leq -1$) there is a value of DM coupling $D_c = \sqrt{\Delta^2 - 1}$ under which the ground state looses its entanglement. The critical line $\Delta_c = -\sqrt{1 + D^2}$ is at the position where the level crossing occurs and corresponds to the critical line of this model at thermodynamic limit (i.e. infinite number of qubits). At this critical line the global $U(1) \times Z_2$ symmetry of the Hamiltonian changes to the local $SU(2)$ symmetry.

At finite temperature, thermal entanglement behaves similar to the ground state counterpart except that it becomes disentangled gradually by increasing temperature. For $T \neq 0$, the critical line below which the thermal entanglement vanishes is a function of $D$ and temperature, i.e. $\Delta_c(D, T)$ and is given by Eq.(9). Generally, the decrease in anisotropy parameter and increase in temperature have a reduction influence on the concurrence. The effect of temperature is plotted in Fig.(1) and Fig.(2) for some fixed values of temperature. To scan the influence of DM coupling the concurrence has been plotted in Fig.(3) and Fig.(4) at fixed values of $\Delta = -1.5$ and $\Delta = -0.5$ versus $D$. For $\Delta < -1$, the concurrence jumps suddenly to non-zero value as $D$ crosses the critical value $D_c = \sqrt{\Delta^2 - 1}$ as shown in Fig.(3). Contrary to the previous case, for $-1 \leq \Delta < 0$ the ground state is always entangled while the thermal entanglement becomes zero for $|D| < D_c(\Delta, T)$ which is justified in Fig.(4).

At $\Delta = 0$ the concurrence of ground state is independent of DM coupling and take a constant value,

$$C_{12}(T = 0, \Delta = 0) = \frac{2\sqrt{2} - 1}{4}.$$  

• $\Delta > 0$

In the positive anisotropic region, increasing the anisotropy enhances the ground state and thermal entanglement up to a maximum and then decreases gradually which can be seen in Fig.(5) and Fig.(6). At $T = 0$ the maximum value of concurrence is due to the maximum fluctuations which exist at the critical line $\Delta_c = \sqrt{1 + 8D^2}$. The critical line belongs to the spin-fluid to Néel phase transition for infinite number of qubits [15]. Contrary to what happened in the negative anisotropy region where
the ground state entanglement vanishes immediately below the critical line \( \Delta_c = -\sqrt{1+D^2} \), in the positive anisotropy region the ground state entanglement has non-zero value in the both side of critical line \( \Delta_c = \sqrt{1+D^2} \). However, in the both cases the global \( U(1) \times Z_2 \) symmetry of the Hamiltonian breaks to the local \( SU(2) \) symmetry on the critical line \( \Delta_c = \mp \sqrt{1+D^2} \). The influence of temperature is to increase the concurrence and increase the critical anisotropy parameter \( \Delta_c(D,T) \) beyond which the system being entangled (as given by Eq.\((9)\)).

To survey the effect of DM interaction the ground state and thermal concurrence have been plotted in Fig.\((6)\) and Fig.\((7)\) versus \( D \) for \( \Delta = 0.5, \Delta = 1.5 \) and different values of temperature. As far as \( 0 < \Delta < 1 \) the influence of DM interaction is to reduce the concurrence of ground state (Fig.\((6)\)) while for \( \Delta \geq 1 \) the ground state concurrence gets a maximum at the position of the critical line \( \Delta_c = \sqrt{1+D^2} \). In the other words, for \( \Delta \geq 1 \) increasing of the DM interaction raises the ground state concurrence until its maximum value and then decreases slowly (Fig.\((6)\)). This is a feature which is only observable in 3-qubits system and is related to the fact that the 3-qubits system show the critical behavior of infinite qubits limit correctly. For low temperatures, thermal entanglement decreases similar to the ground state concurrence, but at high temperatures the thermal entanglement is increased on the onset of the DM coupling.

- Symmetric pairwise

We have discussed extensively the non-symmetric pairwise entanglement \( (C_{12}) \) which has been calculated from \( \rho_{12} \). Now, we summarize the main features of the symmetric pairwise entanglement \( (C_{13}) \) in comparison with the non-symmetric one without presenting the details.

In the negative anisotropy region \( \Delta < 0 \), there is a threshold anisotropy and \( D \) parameters beyond that the symmetric and non-symmetric pairwise are entangled. Decreasing the anisotropy increases the ground state entanglement of symmetric pairwise but decreases the ground state entanglement of non-symmetric pairwise. However, reduction of the anisotropy increases the thermal entanglement of symmetric pairwise up to a maximum value and then decreases it gradually to be vanished while it decreases the non-symmetric pairwise thermal concurrence to zero. In symmetric pairwise the increment of DM coupling decreases the ground state entanglement but enhances the thermal entanglement up to the maximum value and then decreases it. For non-symmetric pairwise the influence of DM interaction on ground state and thermal concurrence is incremental. For the positive anisotropy region \( \Delta > 0 \), the symmetric case is disentangled in the presence or absence of temperature.

### THREE QUBITS ISING WITH DM INTERACTION

We have studied both symmetric and non-symmetric pairwise entanglement of the Ising model with DM interaction defined by Hamiltonian in Eq.\((3)\). Its qualitative behavior is similar to the XXZ model with DM interaction while only some quantitative changes observed, for example the critical point for both AF and F cases is \( D_c = 1 \). Hence, we do not present the results of 3-qubits Ising model with DM interaction here; however, we will show the effect of magnetic field on the entanglement in this model. Let rewrite the Hamiltonian of 3-qubit Ising model with DM interaction in the following form

\[
H = \frac{J}{4} \left[ (\sigma_1^x \sigma_2^x + \sigma_2^x \sigma_3^x) + D(\sigma_1^z \sigma_2^z - \sigma_1^y \sigma_2^y) + D(\sigma_2^z \sigma_3^z - \sigma_2^y \sigma_3^y) + h(\sigma_1^z + \sigma_2^z + \sigma_3^z) \right],
\]

where \( h \) is proportional to the strength of magnetic field, \( J \) and \( D \) are the exchange and DM couplings, respectively. The square root of the eigenvalues of matrix \( R \) (Eq.\((5)\)), are invariant under the substitutions \( D \to -D \) and \( h \to -h \), so we will consider only \( D > 0 \) and \( h > 0 \) without loss of generality.

As mentioned before we consider the AF and F cases of Ising model with DM interaction separately. However, we only study the non-symmetric pairwise entanglement \( (C_{13}) \) because the symmetric one \( (C_{12}) \) shows the same results qualitatively.

- **AF case \((J > 0)\)**

For zero temperature \((T = 0)\), the ground state is an entangled one and its concurrence is given by

\[
C_{12}^{AF}(T = 0, h \neq 0) = \frac{2D}{9},
\]

which is plotted in Fig.\((7)\) and denoted by \( T = 0 \). The thermal concurrence is nonzero for \( T < T_c(D,h) \) which is shown for different temperatures and \( h = 2.0 \) versus \( D \) in Fig.\((7)\). We have observed that the critical temperature decreases with increasing of the magnetic field. Moreover, the DM interaction creates entanglement in the system. For zero temperature, the onset of DM interaction leads to nonzero concurrence while for finite temperature \( D \) should be greater than a critical value to have nonzero concurrence. The concurrence is plotted versus \( h \) for different values of temperature and \( D = 2.0 \) in Fig.\((7)\). This figure shows that for low temperature the concurrence increases with
the increase of magnetic field initially to reach a maximum and then decreases gradually for larger magnetic field, whereas for mid-range temperature the magnetic field reduces the concurrence slowly. In other words, for low temperatures field induces entanglement in a 3-qubit system of Ising model with DM interaction.

• F case ($J < 0$)

For $D \leq \sqrt{[(3 + 2 h)^2 - 1]/8}$, the entanglement of the ground state is zero while it is nonzero for $D \geq \sqrt{[(3 + 2 h)^2 - 1]/8}$. In the entangled region the concurrence is

$$C_{12}^F(T = 0, h \neq 0) = \frac{2D}{q}.$$  

For non-zero temperature and $D < D_c(T, h)$ the concurrence is zero as shown in Fig. 9 for $h = 2.0$. The increment of DM interaction and the magnetic field induce entanglement in this system as far as $T < T_c(D, h)$. The critical value of DM coupling ($D_c(T, h)$) is increased with the increment of magnetic field. Moreover, we have observed that the amount of concurrence is enhanced for higher magnetic fields. More importantly, all figures reveal that the system can be entangled in a region which is not entangled at $T = 0$ by the effect of thermal fluctuations.

SUMMARY AND CONCLUSIONS

The ground state and thermal entanglement of symmetric and non-symmetric pairwise in three-qubit $XXZ$ and Ising models in the presence of DM interaction has been investigated. We have studied the influence of DM and anisotropy parameters on the concurrence of these models.

The DM interaction and anisotropy are the efficient control parameters of entanglement. For the positive anisotropy region (AF case of IDM), the symmetric pairwise is disentangled in finite and zero temperature. For non-symmetric pairwise, increasing of anisotropy enhances the ground state and thermal entanglement of the system to its maximum value and then decreases it slowly. In this region the DM interaction has a different...
FIG. 10: (color online) The concurrence of non-symmetric pairwise is plotted versus $h$ for different values of $T$ in F case ($J = -1$), $D = 2.0$. 

effect on the concurrence. For $0 < \Delta < 1$, the increment of DM coupling decreases the ground state entanglement and concurrence at low temperature but at high temperature DM interaction enhances thermal concurrence. For $\Delta \geq 1$, increment of DM coupling enhances the ground state entanglement and low temperature concurrence to a maximum value and then decreases it gradually but it has only an increment effect on the concurrence at hight temperature.

In the negative anisotropy region (F case of IDM) there is a threshold for anisotropy and $D$ parameters above which both symmetric and non-symmetric pairwise are entangled. Decreasing the anisotropy increases the ground state entanglement of symmetric pairwise but decreases the ground state entanglement of non-symmetric pairwise. However, lowering the anisotropy increases the thermal entanglement of symmetric pairwise up to a maximum value and then decreases gradually to be vanished, but it decreases the non-symmetric pairwise thermal concurrence to be eliminated. In symmetric pairwise the increment of DM coupling decreases the ground state entanglement but enhances the thermal entanglement up to the maximum value and then decreases it. For non-symmetric pairwise the influence of DM interaction on the ground state and thermal concurrence is incremental.

The most noteworthy result occurs at $T = 0$ where the 3-qubit ground state entanglement of the systems shows the fingerprint of quantum phase transition for an infinite size system. For negative anisotropy case of both symmetric and non symmetric pairwise, level crossing of the ground state and first excited state occurs at $\Delta = -\sqrt{1 + D^2}$ ($D = 1$ for F case of IDM) under which the concurrence is zero and jumps to a nonzero value for $\Delta > -\sqrt{1 + D^2}$ ($D < 1$). For positive anisotropy case of non-symmetric pairwise, the symmetry breaking of the ground state occurs similar to the negative anisotropy case without level crossing at $\Delta = \sqrt{1 + D^2}$ ($D = 1$ for AF case of IDM) and the concurrence has a maximum value due to the maximum quantum fluctuations. At these points the global $U(1) \times Z_2$ symmetry of the Hamiltonian is changed to local hidden $SU(2)$ symmetry.

We have probed the influence of magnetic field in 3-qubit IDM model. In non-symmetric pairwise and at low temperature the magnetic field induces the entanglement to the system which get a maximum for finite field. Obviously for high magnetic field the model enters a paramagnetic phase which is disentangled. For middle range of temperature the entanglement is reduced by adding the magnetic field, whereas for high temperature, there are many states populated and its reduction is very tiny. It is related to the thermal mixing of states which is a source of entanglement in the system. For the ferromagnetic case, the variations of entanglement is faster than the AF one.

For symmetric pairwise in the AF case the magnetic field can induce entanglement in a non-entangled system ($h = 0$) and the entanglement enhances with increasing of the DM interaction. The entanglement is increased with magnetic field to reach a maximum value while further increment of magnetic field decreases the entanglement which gradually disappears. This happens very quickly for F case and the critical temperature where the entanglement is vanishing is lower than the AF case. The optimal mixing of all eigenstates in the system leads the maximum value.

More generally, the entanglement properties of a finite system depends on the number of qubits as discussed in the text.

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