Some Goodness of Fit Tests based on Centre Outward Spacings

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Abstract

Data depth provides a centre-outward ordering for multivariate data. Recently, some univariate GoF tests based on data depth have been studied by Li (2018). This paper discusses some univariate goodness of fit tests based on centre-outward spacings. These tests have similar asymptotic properties (distribution and efficiency) as those based on usual spacings. A simulation study reveals that for light-tailed symmetric alternatives, the proposed tests perform better than those based on usual spacings.

Keywords: Asymptotic relative efficiency; Centre outward spacings; Goodness of fit test.

1 Introduction

For absolutely continuous distribution functions (dfs), a popular method of univariate goodness of fit (GoF) tests is based on sample spacings. Let \( X_1, \ldots, X_{n-1} \) be independent and identically distributed (i.i.d.) random variables from an absolutely continuous df \( F \). Let \( X_{(1)} \leq \cdots \leq X_{(n-1)} \) denote the corresponding order statistics. Define \( X_{(0)} = -\infty \) and \( X_{(n)} = \infty \). The \( m \)-step spacings are defined as \( D_k^{(m)} := F(X_{(k+m-1)}) - F(X_{(k-1)}) \) for \( k = 1, 2, \ldots, n - m + 1 \). For \( m = 1 \), these are known as simple spacings, usually denoted by \( D_k \)'s. A typical GoF test statistic based on spacings has the form \( W(h) := \frac{1}{n} \sum_{i=1}^{n} h(nD_i) \), where \( h \) is some convex function. Some popular choices of function \( h \) are as follows:

| \( h(x) \) | Statistic |
|---|---|
| \( x^2 \) | Greenwood Statistic (Greenwood 1946) |
| \( -\log(x) \) | Log Spacing Statistic (Moran 1951) |
| \( |x - 1| \) | Rao’s Spacing Statistic (Rao 1976) |
| \( x \log(x) \) | Relative Entropy Spacing Statistic (Misra and van der Meulen 2001) |
Such a statistic is an estimator of a $\phi$-divergence and a natural candidate for a GoF test statistic. Initially, Sethuraman and Rao (1970) and Rao and Sethuraman (1975) discovered that a class of such statistics are asymptotically normal under simple null and a smooth sequence of alternative converging to null at the rate of $n^{1/4}$. They found that the Greenwood test is asymptotically the most efficient in terms of the Pitman asymptotic relative efficiency (ARE) for this sequence of alternatives. Using the same approach, Del Pino (1979) found that the Greenwood type test based on disjoint $m$-step spacings is asymptotically more efficient than the usual Greenwood test. Rao and Kuo (1984) observed that, for the fixed step $m$, tests based on overlapping spacing are asymptotically more efficient than the corresponding tests based on disjoint spacings and the Greenwood type test is asymptotically most efficient among tests based on symmetric functions of overlapping $m$-step spacings.

In the multivariate statistics literature, the data depth of a point is a measure of centrality of the point with respect to the data cloud or the underlying df. There are various notions of data depth (see e.g., Zuo and Serfling 2000). Two popular notions are half-space depth (Tukey 1975) and simplicial depth (Liu 1990). In fact, a data depth induces centre-outward (CO) ordering. This ordering in the univariate case can be utilised for GoF tests. Recently, Li (2018) studied Kolmogorov-Smirnov, Anderson-Darling, Cramer von-Mises tests based on CO ordering. They found that the GoF tests based on CO ordering perform better than their usual counterparts for alternatives with scale differences.

In this paper, we define sample spacings based on CO ordering and study GoF tests based on such spacings. Such tests have not been studied in the literature and are of potential theoretical and practical interest. We also perform a small simulation study. The aim of this simulation study is to compare performances of the proposed tests and GoF tests based on usual spacings.

2 Centre-Outward Spacings

For the univariate case, let $S \equiv 1 - F$ denote the survival function. Then, the half-space depth and the simplicial depth of a point $x \in \mathbb{R}$ with respect to the df $F$ are given by $\min(F(x), S(x))$ and $2F(x)S(x)$, respectively. For the univariate case, half-space depth and simplicial depth achieve maximum at the median of the df and monotonically decrease to zero on either side of median. So, we can use either of them to construct CO ordering of observations. Li (2018) discussed both the univariate half-space and simplicial depths, and found that they provide the same CO ordering.

Denote the depth (half-space, or simplicial) with respect to the df $F$ by $D_F$. Define $R_Y = P_F[D_F(X) \geq D_F(Y) | Y]$ for $X \sim F$. Then, $R_X = |2F(X) - 1|$, and $R_{X_i} \overset{i.i.d.}{\sim} U(0,1)$ for $i = 1, 2, \ldots, n - 1$ (see Li 2018). Note that $R_Y$ is a decreasing function of $D_F(Y)$. Let $R^{(1)}, R^{(2)}, \ldots, R^{(n-1)}$ be the order statistics corresponding to $R_{X_1}, R_{X_2}, \ldots, R_{X_{n-1}}$, $R^{(0)} = 0$
and $R^{(n)} = 0$. Now, we can define sample spacings based on $R_i$.s. We call these spacings the “CO spacings”.

**Definition 1.** Under the above described set-up, we define CO spacings as

$$DS_i = R^{(i)} - R^{(i-1)} \quad \text{for } i = 1, 2, \ldots, n.$$ 

The following result gives the distribution of CO spacings.

**Lemma 1.** For an absolutely continuous df $F$, we have

$$(DS_1, DS_2, \ldots, DS_n) \overset{d}{=} (T_1, T_2, \ldots, T_n),$$

where $(T_1, T_2, \ldots, T_n)$ are simple spacings corresponding to a random sample of size $n - 1$ from the $U(0,1)$ df.

This result is a consequence of the fact that $R_i \overset{i.i.d.}{\sim} U(0,1)$ for $i = 1, 2, \ldots, n - 1$. Thus, the CO spacings have the same distribution as the usual spacings.

### 3 Goodness of Fit Tests based on CO Spacings

The goal is to test $H_0 : F = F_0$ against $H_1 : F \neq F_0$, where $F_0$ is a completely specified df. Using the probability integral transform, this is equivalent to testing uniformity, i.e., $H_0 : F(x) = x \quad \forall x \in [0,1]$ against $H_1 : F(x) \neq x$ for some $x \in [0,1]$, where the support of $F$ is $[0,1]$. Under $H_0$, the CO ordering random variable is $R_X = |2X - 1|$. For $X \sim F$, the df of $R_X$ is as follows:

$$F_{R}(y) = P(R_X \leq y) = \begin{cases} 
0, & \text{if } y < 0, \\
F\left(\frac{1+y}{2}\right) - F\left(\frac{1-y}{2}\right), & \text{if } y \in [0,1], \\
1, & \text{if } y > 1.
\end{cases}$$

Denote the density function of $F$ by $f$. Then the density function of $R_X$ is given by

$$f_R(y) = \begin{cases} 
\frac{1}{2} \left(f\left(\frac{1+y}{2}\right) + f\left(\frac{1-y}{2}\right)\right), & \text{if } y \in [0,1] \\
0, & \text{otherwise}.
\end{cases}$$

Let $F_1$ and $F_2$ be two dfs with corresponding density functions $f_1$ and $f_2$, respectively. Then, the Hellinger distance (HD) between the dfs $F_1$ and $F_2$ is defined as $HD(F_1, F_2) = \sqrt{1 - \int_{\mathbb{R}} \sqrt{f_1(x)f_2(x)} \, dx}$. 

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Lemma 2. Let $F_0$ denote the df of $U(0,1)$, $X \sim F$ and $R_X = |2X - 1|$. Denote the df of $R_X$ by $F_R$. Then, $HD(F_0, F_R) \leq HD(F_0, F)$. Moreover, if $X$ is symmetric about $1/2$, then $HD(F_0, F_R) = HD(F_0, F)$.

Proof. Observe that $\sqrt{2}\sqrt{a+b} \geq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, and equality holds iff $a = b$. Here, $HD(F_0, R_X) = 1 - \int_0^1 \sqrt{f_R(y)}dy$.

$$
\int_0^1 \sqrt{f_R(y)}dy = \frac{1}{\sqrt{2}} \int_0^1 \sqrt{f\left(\frac{1+y}{2}\right) + f\left(\frac{1-y}{2}\right)}dy \\
\geq \frac{1}{2} \left[ \int_0^1 \sqrt{f\left(\frac{1+y}{2}\right)}dy + \int_0^1 \sqrt{f\left(\frac{1-y}{2}\right)}dy \right] \\
= \frac{1}{2} \left[ 2 \int_0^{0.5} \sqrt{f(x)}dx + 2 \int_0^{0.5} \sqrt{f(x)}dx \right] = \int_0^1 \sqrt{f(x)}dx.
$$

Hence, $HD(F_0, F_R) \leq HD(F_0, F)$ and equality holds if $f\left(\frac{1+y}{2}\right) = f\left(\frac{1-y}{2}\right)$ $\forall y \in [0,1]$, i.e., $f$ is symmetric about $1/2$. 

Lemma 2 suggests that, when the underlying distribution is not symmetric, the HD between the df of CO ordering random variable and $U(0,1)$ df is less than the HD between the underlying df and $U(0,1)$ df. This explains why CO ordering based GoF tests have low power in detecting location differences, which was also observed by Li (2018) in simulation studies.

Inspired by GoF tests based on usual spacings, we propose the following class of GoF test statistics based on CO spacings

$$W^*(h) = \frac{1}{n} \sum_{i=1}^n h(nDS_i).$$

Note that these test statistics based on CO spacings are distribution-free and have the same distribution as the corresponding test statistics based on usual spacings. So, a test based on CO spacings has the same critical values as corresponding usual spacings based test.

3.1 Some Asymptotic Results

We consider test statistics based on CO spacings of type: $W^*(h) = \frac{1}{n} \sum_{i=1}^n h(nDS_i)$, where $h$ satisfies assumption (3.3) of Del Pino (1979). Following Sethuraman and Rao (1970), we consider sequence of local alternatives of the type

$$F_n(x) = x + \frac{L_n(x)}{\sqrt{n}} \text{ for } 0 \leq x \leq 1,$$  

(1)
Theorem 2. The asymptotic distribution of
\[ \sqrt{n} \sup_{0 \leq x \leq 1} |L_n(x) - L(x)| = o(1), \]
\[ \sqrt{n} \sup_{0 \leq x \leq 1} |L_n'(x) - L'(x)| = o(1) \]
and
\[ \sqrt{n} \sup_{0 \leq x \leq 1} |L_n''(x) - L''(x)| = o(1). \]

For the above mentioned sequence of local alternatives, the df of \( R_X \) is given by
\[ F_{nR}(y) = y + \frac{L_n(\frac{1+y}{2}) - L_n(\frac{1-y}{2})}{\sqrt{n}} \text{ for } 0 \leq x \leq 1. \]

Denote \( L^*_n(y) := L_n \left( \frac{1+y}{2} \right) - L_n \left( \frac{1-y}{2} \right) \) and \( L^*(y) := \frac{1+y}{2} - \frac{1-y}{2} \). Now, using Theorem 3 of Sethuraman and Rao (1970), we obtain asymptotic distribution of \( W^*(h) \) under the null as well as the local alternatives (1), as detailed in the following theorems.

Theorem 1. The asymptotic distribution of \( W^*(h) \) under null hypothesis is given by
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [h(nDS_i) - \mathbb{E}h(Z)] \xrightarrow{d} N(0, \sigma_h^2) \text{ as } n \to \infty, \]
where \( \sigma_h^2 = \text{Var}(h(Z)) - \text{Cov}^2(h(Z), Z) \) and \( Z \) is a standard exponential random variable.

Theorem 2. The asymptotic distribution of \( W^*(h) \) under the sequence of local alternatives (1) is given by
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [h(nDS_i) - \mathbb{E}h(Z)] \xrightarrow{d} N(\mu_h, \sigma_h^2) \text{ as } n \to \infty, \]
where \( \mu_h = \frac{1}{2} \left( \int_0^1 [L^*(u)]^2 du \right) \text{Cov}[h(Z), (Z - 2)^2] \) and \( Z \) is a standard exponential random variable.

3.2 Asymptotic Relative Efficiency under a sequence of Local Alternatives

Suppose there are two competing tests corresponding to test statistics \( V_n(g_i) := \frac{1}{n} \sum_{k=1}^{n} g_i(DS_k) \) for \( i = 1, 2 \). Let \( V_n(g_i) \)'s have asymptotic means zero and finite variances under null hypothesis. Under the sequence of alternatives stated in (1), let \( V_n(g_i) \) have asymptotic mean and variance \( \mu(g_i) \) and \( \sigma^2(g_i) \), respectively, for \( i = 1, 2 \). Then, the Pitman asymptotic relative efficiency (ARE) of \( V_n(g_1) \) relative to \( V_n(g_2) \) is given by
\[ \text{ARE}(g_1, g_2) = \frac{\sigma^2(g_1)}{\sigma^2(g_2)} = \left( \frac{\mu^2(g_1)}{\mu^2(g_2)} \right)^2. \]
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The quantity $e(g_i) = \mu^2(g_i)/\sigma^2(g_i)$ is called the efficacy of the test based on $V_n(g_i)$ for $i = 1, 2$. Under a sequence of local alternatives converging to the null hypothesis, the test with maximum efficacy is asymptotically locally most powerful in terms of the Pitman ARE. Sethuraman and Rao (1970) obtained efficacy for tests based on usual spacings $W(h) := \frac{1}{n} \sum_{i=1}^{n} h(nD_i)$ as below

$$e(h) = \frac{\int_0^1 t^2(u) Cov[h(Z), (Z-2)^2]}{2[Var(h(Z)) - Cov^2(h(Z), Z)^{1/2}],}$$

where $l(x) := L'(x)$ and $Z$ is a standard exponential random variable. Similarly, we obtain efficacy of tests based on CO spacings, which is given by the following lemma.

**Lemma 3.** For the test statistic $W_n^*(h) := \frac{1}{n} \sum_{k=1}^{n} h(nDS_k)$, the efficacy under the sequence of alternative (1) is given by

$$e^*(h) = \frac{\int_0^1 t^2(u) Cov[h(Z), (Z-2)^2]}{2[Var(h(Z)) - Cov^2(h(Z), Z)^{1/2}],}$$

where $l^*(x) = l \left( \frac{1+x}{2} \right) - l \left( \frac{1-x}{2} \right)$ and $Z$ is a standard exponential random variable.

The following result provides the asymptotically locally most powerful (ALMP) test among tests based on statistics of the type $W_n^*(h)$.

**Theorem 3.** For the sequence of alternatives (1), among tests based on statistics of the type $W_n^*(h) = \frac{1}{n} \sum_{k=1}^{n} h(nDS_k)$, the test corresponding to $h(x) = x^2$ is most efficient in terms of the Pitman ARE.

The above theorem is a consequence of a result of Sethuraman and Rao (1970). As expected, the ALMP test is the Greenwood test based on CO spacings. Similar to tests based on usual spacings, tests based on statistics of the type $W_n^*(h)$ can not detect alternatives converging to the null distribution at a rate faster than $n^{-1/4}$.

**Remark 1.** We can define higher order disjoint and overlapping spacings based on CO ordering. For these higher order CO spacings, results similar to those for usual higher order spacings in the existing literature hold true (see, e.g., Del Pino 1979; Rao and Kuo 1984; Misra and van der Meulen 2001). Also, results similar to those in Tang and Jammalamadaka (2012b,a) hold true for CO spacings.

### 4 Simulation Studies

We now perform some simulation studies to assess the finite sample performance of the proposed tests, and compare their performance with tests based on usual spacings. Suppose
GS, LS, ES and RS denote the test statistics corresponding to the Greenwood, log spacing, relative entropy and Rao spacing, respectively, based on usual spacings. Let GS*, LS*, ES* and RS* denote test statistics corresponding to the Greenwood, log spacing, relative entropy and Rao spacing, respectively, based on CO spacings. For our study, we take the level of significance to be 0.05. The empirical powers of the tests are calculated from 10000 iterates. We consider sample sizes 10, 20, 30, 50, 80, 100, 200 and 300.

4.1 Uniformity Tests

Following Stephens (1974), first we consider alternatives of the following three types (for \( k > 0 \)),

A_k : \( F(x) = 1 - (1 - x)^k, 0 \leq x \leq 1; \)

\[
B_k : \quad F(x) = \begin{cases} 
2^{k-1}x^k, & \text{if } 0 \leq x \leq 0.5, \\
1 - 2^{k-1}(1 - x)^k, & \text{if } 0.5 \leq x \leq 1; 
\end{cases}
\]

\[
C_k : \quad F(x) = \begin{cases} 
0.5 - 2^{k-1}(0.5 - x)^k, & \text{if } 0 \leq x \leq 0.5, \\
0.5 + 2^{k-1}(x - 0.5)^k, & \text{if } 0.5 \leq x \leq 1. 
\end{cases}
\]

These families of distribution give a wide variety of dfs supported on \([0, 1]\). For \( k > 1 \), the family \( A_k \) yields skewed distributions with a cluster near zero, whereas \( B_k \) gives symmetric distributions with cluster near 0.5 and \( C_k \) gives symmetric distributions with two clusters near zero and one. Also, the family \( B_k \) has lighter tail than the \( U(0,1) \) df, whereas the family \( C_k \) has heavier tail than the \( U(0,1) \) df. For the simulation study, we take \( k = 1.5 \).

The empirical powers for various tests are listed in Table 1.

Table 1 suggests that, tests based on usual spacings perform better for alternatives \( A_{1.5} \) and \( C_{1.5} \), but tests based on CO spacings perform better for alternatives \( B_{1.5} \). The alternative \( A_{1.5} \) is not symmetric about 1/2, and so power of tests based on CO spacings are lower (which is explained by Lemma 2). For alternatives \( B_{1.5} \) and \( C_{1.5} \), powers of tests based on usual spacings and those based on CO spacings are comparable, which is also explained by Lemma 2. For the light-tailed alternative \( B_{1.5} \), tests based on CO spacings are superior to those based on usual spacings.

Table 1 also suggests that, tests based on usual spacings perform better than those based on CO spacings.

The empirical powers of the competing tests are reported in Table 2. It is evident that, for the heavy-tailed alternative \( Beta(0.5, 0.5) \), tests based on usual spacings have better powers. For the light-tailed symmetric alternatives \( Beta(1.5, 1.5) \) and \( Beta(2.5, 2.5) \), tests based on CO spacings perform better than those based on usual spacings.

Remark 2. Similar to Li (2018), we can combine tests based on usual spacings and CO spacings, i.e., a test based on \( \max(W(h), W^*(h)) \). Such tests are also distribution-free. Based
Table 1: Empirical powers for $A_{1.5}$, $B_{1.5}$ and $C_{1.5}$ alternatives for the $U(0,1)$ null

| Alternative | $n$ | $G$   | $G^*$ | $L$   | $L^*$ | $E$   | $E^*$ | $R$   | $R^*$ |
|-------------|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| $A_{1.5}$   | 10  | 0.080 | 0.043 | 0.068 | 0.047 | 0.078 | 0.044 | 0.073 | 0.047 |
|             | 20  | 0.112 | 0.050 | 0.084 | 0.049 | 0.105 | 0.047 | 0.090 | 0.047 |
|             | 30  | 0.156 | 0.051 | 0.095 | 0.049 | 0.139 | 0.049 | 0.110 | 0.050 |
|             | 50  | 0.227 | 0.052 | 0.124 | 0.052 | 0.205 | 0.052 | 0.155 | 0.051 |
|             | 80  | 0.338 | 0.053 | 0.172 | 0.053 | 0.306 | 0.055 | 0.220 | 0.057 |
|             | 100 | 0.404 | 0.052 | 0.198 | 0.056 | 0.368 | 0.054 | 0.249 | 0.054 |
|             | 200 | 0.662 | 0.056 | 0.310 | 0.056 | 0.600 | 0.055 | 0.399 | 0.053 |
|             | 300 | 0.821 | 0.059 | 0.414 | 0.054 | 0.753 | 0.058 | 0.527 | 0.059 |
| $B_{1.5}$   | 10  | 0.025 | 0.077 | 0.038 | 0.063 | 0.026 | 0.074 | 0.034 | 0.071 |
|             | 20  | 0.055 | 0.116 | 0.059 | 0.081 | 0.058 | 0.110 | 0.059 | 0.092 |
|             | 30  | 0.086 | 0.154 | 0.070 | 0.100 | 0.083 | 0.145 | 0.081 | 0.117 |
|             | 50  | 0.142 | 0.230 | 0.095 | 0.132 | 0.138 | 0.215 | 0.122 | 0.166 |
|             | 80  | 0.247 | 0.339 | 0.137 | 0.166 | 0.232 | 0.304 | 0.175 | 0.217 |
|             | 100 | 0.300 | 0.411 | 0.164 | 0.194 | 0.277 | 0.372 | 0.204 | 0.255 |
|             | 200 | 0.581 | 0.663 | 0.280 | 0.317 | 0.530 | 0.601 | 0.362 | 0.396 |
|             | 300 | 0.765 | 0.824 | 0.388 | 0.414 | 0.707 | 0.752 | 0.506 | 0.527 |
| $C_{1.5}$   | 10  | 0.166 | 0.076 | 0.099 | 0.063 | 0.152 | 0.075 | 0.133 | 0.074 |
|             | 20  | 0.215 | 0.116 | 0.119 | 0.080 | 0.192 | 0.111 | 0.145 | 0.099 |
|             | 30  | 0.263 | 0.158 | 0.126 | 0.096 | 0.229 | 0.144 | 0.164 | 0.115 |
|             | 50  | 0.356 | 0.229 | 0.161 | 0.129 | 0.312 | 0.210 | 0.208 | 0.162 |
|             | 80  | 0.466 | 0.336 | 0.206 | 0.168 | 0.407 | 0.301 | 0.261 | 0.216 |
|             | 100 | 0.520 | 0.403 | 0.227 | 0.198 | 0.458 | 0.365 | 0.287 | 0.242 |
|             | 200 | 0.760 | 0.663 | 0.347 | 0.311 | 0.685 | 0.602 | 0.437 | 0.404 |
|             | 300 | 0.875 | 0.821 | 0.449 | 0.413 | 0.810 | 0.756 | 0.564 | 0.534 |

on observations of Li (2018), we expect that such tests can perform well for a wide variety of alternatives.

5 Conclusion

In this paper, we have studied several GoF tests based on centre-outward (CO) spacings. New tests are constructed similar to some popular GoF tests based on usual spacings. For a skewed alternative, tests based on CO ordering data have less power compared to those based on the original data. This was also observed by Li (2018) in a simulation study. This is explained by the fact that the Hellinger distance decreases for CO ordering based data in the case of skewed alternatives. When the alternative distribution is symmetric and light-tailed, the proposed GoF tests perform better than those based on usual spacings.

Theoretical results on GoF tests based on higher order spacings extend easily for GoF tests based on higher order CO spacings. There exist studies concerning estimation and parametric tests based on spacings (see, e.g., Ghosh and Jammalamadaka 2001; Ekström 2013).
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Table 2: Empirical powers for $Beta(k, k)$ alternatives for the $U(0, 1)$ null

| Alternative  | $n$ | $G$   | $G^*$  | $L$   | $L^*$ | $E$   | $E^*$  | $R$   | $R^*$  |
|--------------|-----|-------|--------|-------|-------|-------|--------|-------|--------|
| $Beta(0.5, 0.5)$ | 10  | 0.253 | 0.205  | 0.418 | 0.343 | 0.312 | 0.253  | 0.301 | 0.247  |
|               | 20  | 0.300 | 0.255  | 0.521 | 0.436 | 0.398 | 0.333  | 0.384 | 0.320  |
|               | 30  | 0.351 | 0.310  | 0.600 | 0.513 | 0.466 | 0.409  | 0.451 | 0.397  |
|               | 50  | 0.444 | 0.414  | 0.722 | 0.652 | 0.591 | 0.549  | 0.586 | 0.536  |
|               | 80  | 0.562 | 0.525  | 0.832 | 0.783 | 0.726 | 0.687  | 0.716 | 0.684  |
|               | 100 | 0.613 | 0.590  | 0.882 | 0.848 | 0.794 | 0.762  | 0.774 | 0.746  |
|               | 200 | 0.840 | 0.833  | 0.980 | 0.975 | 0.949 | 0.947  | 0.941 | 0.938  |
|               | 300 | 0.944 | 0.934  | 0.996 | 0.995 | 0.990 | 0.987  | 0.989 | 0.984  |
| $Beta(1.5, 1.5)$ | 10  | 0.024 | 0.048  | 0.033 | 0.050 | 0.026 | 0.046  | 0.031 | 0.050  |
|               | 20  | 0.037 | 0.080  | 0.046 | 0.063 | 0.040 | 0.076  | 0.045 | 0.072  |
|               | 30  | 0.057 | 0.101  | 0.051 | 0.072 | 0.055 | 0.093  | 0.056 | 0.080  |
|               | 50  | 0.090 | 0.137  | 0.070 | 0.084 | 0.086 | 0.128  | 0.081 | 0.100  |
|               | 80  | 0.148 | 0.205  | 0.091 | 0.113 | 0.137 | 0.179  | 0.114 | 0.136  |
|               | 100 | 0.174 | 0.252  | 0.111 | 0.130 | 0.166 | 0.220  | 0.128 | 0.151  |
|               | 200 | 0.362 | 0.441  | 0.167 | 0.192 | 0.309 | 0.377  | 0.207 | 0.239  |
|               | 300 | 0.515 | 0.597  | 0.217 | 0.240 | 0.438 | 0.501  | 0.280 | 0.306  |
| $Beta(2.5, 2.5)$ | 10  | 0.034 | 0.168  | 0.057 | 0.104 | 0.043 | 0.155  | 0.056 | 0.132  |
|               | 20  | 0.149 | 0.377  | 0.117 | 0.184 | 0.160 | 0.344  | 0.152 | 0.249  |
|               | 30  | 0.310 | 0.558  | 0.180 | 0.254 | 0.301 | 0.501  | 0.249 | 0.343  |
|               | 50  | 0.614 | 0.804  | 0.325 | 0.409 | 0.590 | 0.753  | 0.449 | 0.536  |
|               | 80  | 0.881 | 0.950  | 0.511 | 0.592 | 0.848 | 0.925  | 0.664 | 0.725  |
|               | 100 | 0.950 | 0.981  | 0.616 | 0.687 | 0.930 | 0.968  | 0.756 | 0.808  |
|               | 200 | 1.000 | 1.000  | 0.913 | 0.931 | 0.999 | 1.000  | 0.971 | 0.977  |
|               | 300 | 1.000 | 1.000  | 0.983 | 0.987 | 1.000 | 1.000  | 0.997 | 0.997  |

Such studies based on CO spacings are some potential future problems in this direction.

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