GAUGE THEORETIC APPROACH TO FLUID DYNAMICS

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Abstract:
The Hamiltonian dynamics of a compressible inviscid fluid is formulated as a gauge theory. The idea of gauge equivalence is exploited to unify the study of apparently distinct physical problems and solutions of new models can be generated from known fluid velocity profiles.

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The notions of gauge symmetry and gauge invariance, or the study of constraint systems (in the Hamiltonian framework) in a more general sense [1], have become indispensable tools in particle physics and to some extent in condensed matter physics. We show in the present paper that they can play an equally important role in fluid dynamics.

Formulating a gauge theory to describe the dynamics of a compressible fluid will be fruitful because a wide class of apparently distinct fluid systems can be shown to be gauge equivalent, meaning that they will yield identical results regarding physical quantities in the gauge invariant sector. New fluid models can be generated and solved from the explicit knowledge of flow patterns of simple systems. The gauge invariant model that we construct, following the Batalin-Tyutin prescription [2], has the advantage of having a completely canonical phase space Poisson Bracket structure and thus is eminently suitable for quantization programme and in the quantum version, powerful results of three or lower spatial dimensional gauge theories can be borrowed from elsewhere. Note that the construction of the gauge theory is not unique and alternative gauge models may not have the above attractive features.

It would be interesting to contrast the nature of (gauge) equivalence in the present theory with electrodynamics on one hand and Canonical Transformation (CT) connecting inertial and non-inertial frames on the other hand. In the former, the Maxwell-Lorentz equations of motion are manifestly gauge invariant and the gauge transformation does not show up anywhere. However, in the latter, the same physical situation, if described in a non-inertial frame, (connected to an inertial one by time dependent CT), will require inertial or pseudo forces involving the transformation parameters. The gauge fluid theory is a generalization of the latter, since here, (unlike in CT), the Poisson Bracket structure can change under Gauge Transformations (GT). This and the appearance of interactions arising due to the GT, (analogous to pseudo interactions in CT), makes the equivalence non-trivial. The compressible fluid equations of motion are

\[
\partial_t \rho + \partial_i (\rho v_i) = X(\rho, v_j, f(x)), \quad \partial_t v_i + (v_j \partial_j)v_i = X_i(\rho, v_j, g(x)),
\]

where \(\rho\) and \(v_i\) denote the density and velocity fields respectively, and \(X\) and \(X_i\) represent some source and force terms originating from phenomenological considerations, with \(f\) and \(g\) being c-number functions. \(X = 0\) signifies mass conservation and \(X_i = 0\) reduces to the "free" Euler equation. The aim is to solve these dynamical equations with appropriate boundary conditions.

We will demonstrate that (1) and another system with the dynamical variables transformed to \(\rho \rightarrow \tilde{\rho} \equiv \rho, \ v_i \rightarrow \tilde{v}_i \equiv v_i + A_i\), are gauge equivalent, provided \(A_i\) (which can depend on the fields \(\rho\) and \(v_i\) as well), obeys certain simple restrictions. Our main result is the explicit form of \(A_i\). Thus the theory enjoys a large amount of arbitrariness that can be exploited to connect different theories via choice of gauge conditions. The fact that different sets of source and interaction terms can be traded will have applications both theoretical and experimental studies.

The Hamiltonian system: Time evolution of fields (1) are

\[
\partial_t \rho(x) = \{H, \rho(x)\}, \quad \partial_t v_i(x) = \{H, v_i(x)\}, \quad H[\rho, v_i] = \int dy H(y),
\]

with an (in general non-canonical) Poisson bracket structure satisfying antisymmetry and Jacobi identity. The free theory corresponding to (1) can be derived from the following Hamiltonian
and algebra \[3\]
\[ H = \frac{1}{2} \rho \dot{v}_i v_i, \quad \{v_i(x), \rho(y)\} = \frac{\partial}{\partial x^i} \delta(x-y), \quad \{v_i(x), v_j(y)\} = -\frac{\partial v_j - \partial v_i}{\rho} \delta(x-y). \quad (3) \]

We consider those systems where additional terms in \(H\) can generate \(X\) and \(X_i\) in \([1]\). For example, a term \(p(x)f(\rho(x))\) in \(H\) can induce a pressure like term in the \(v_i\) equation. This leaves out viscid fluids, at least for the present work. We use the equivalent Clebsch parametrisation of \(v_i\) \([3, 4]\), \(v_i(x) \equiv \partial_i \theta(x) + \alpha(x) \partial_i \beta(x)\), satisfying
\[ \{\theta(x), \alpha(y)\} = -\frac{\alpha}{\rho} \delta(x-y), \quad \{\beta(x), \alpha(y)\} = -\frac{1}{\rho} \delta(x-y), \quad \{\theta(x), \rho(y)\} = \delta(x-y). \quad (4) \]

The explicit form of \(A_i\) is,
\[ \tilde{v}_i \equiv v_i + A_i = \partial_i (\theta + \alpha G) + (\alpha + \frac{F}{\rho}) \partial_i (\beta - G), \quad (5) \]

where different choices of \(F\) and \(G\), (which can depend on the fields also), constitute gauge equivalent theories. The need to restrict \(F\) and \(G\) is intuitively clear. The gauge choices, \(G = \beta\) or \(F = -\rho \alpha\) will render \(v_i\) irrotational, which can not be equivalent to a velocity field having vorticity, (due to Helmholtz’ vorticity theorem). Similarly, if invoked, the condition \(G = -\theta/\alpha\) will remove the longitudinal part completely. It will be seen that the above are not proper GTs.

Lastly we mention that since we are concentrating on Hamiltonian systems, the subsequent gauge equivalent models will also be Hamiltonian, with the following structure, \(\partial_i \tilde{\rho} = \{\tilde{H}, \tilde{v}_i\}_{DB}, \quad \partial_i \tilde{v}_i = \{\tilde{H}, \tilde{v}_i\}_{DB}, \quad \tilde{H} = H(\tilde{\rho}, \tilde{v}_i)\), where \(DB\) stands for (the antisymmetric and Jacobi identity satisfying) Dirac Bracket \([1]\), which is a generalization of the Poisson Bracket, appropriate for theories with constraints. We emphasise that if the systematic procedure followed here is not adopted, obtaining the above algebra and \(\tilde{H}\) is indeed non-trivial.

**Applications:** A crucial feature of our formalism \([2]\) is the generic form
\[ (\tilde{\rho}, \tilde{v}_i) = (\rho, v_i) + extension \]

which ensures that \(\tilde{H} = H + \text{extension}\), (if it is polynomial in nature). Hence the equations of motion in the gauge theory will be of the form,
\[ \dot{\rho} + \partial_i (\rho v_i) = X_0(\rho, \tilde{v}_j, f) - \partial_i (\rho A_i), \quad \dot{v}_i + (v_j \partial_j) v_i = X_i(\rho, \tilde{v}_j, g) - \dot{A}_i - (v_j \partial_j) A_i - (A_j \partial_j) v_i. \quad (6) \]

Indeed the above equations are not obvious and are inherent in the BT scheme \([2]\). Notice that compared to \([1]\), here we have a different set of \(A_i\) dependent source and interaction terms but the two systems are gauge equivalent because of the existence of the gauge invariant set of variables \(\tilde{\rho} \equiv \rho\) and \(\tilde{v}_i\), in terms of which the equations of motion are structurally identical, \([2]\). Experimental or theoretical analysis may become easier in one system than in the other, such as the point charge-magnetic field system simplifies in a suitable rotating frame where the Coriolis force removes the magnetic force. \([5]\) is also a new model which is solved knowing \(v_i\), \(\rho\) and the chosen form of \(A_i\).

**The theory:** It is necessary to embed the system in an enlarged space having independent canonical pairs \((\theta, \Pi_\theta \equiv \rho), (\alpha, \Pi_\alpha)\) and \((\beta, \Pi_\beta)\) with \(\{\theta(x), \Pi_\theta(y)\} = \delta(x-y)\) etc. instead of
Next we introduce non-commuting Second Class Constraints (SCC) $\eta_a$ which induce the previous non-canonical brackets in (4) as Dirac Brackets defined by

$$\{A(x), B(y)\}_{DB} = \{A(x), B(y)\} - \int (d^3z d^3w) \{A(x), \eta_a(z)\} \{\eta_a(z), \eta_b(w)\}^{-1} \{\eta_b(w), B(y)\}. \tag{7}$$

The DB vanishes if either $A$ or $B$ is $\eta_a$. The following SCCs $\eta_1 \equiv \alpha \Pi_\theta - \Pi_\beta$; $\eta_2 \equiv \Pi_\alpha$ (8) reproduce (4) as DBs from the canonical set. The degrees of freedom count remains the same since there are two SCCs for the additional variables $\Pi_\alpha$ and $\Pi_\beta$.

Finally we bring in additional Batalin-Tyutin (BT) auxiliary variables such that in the final BT-extended phase space the theory is converted into a gauge theory, meaning that the final theory has only commuting or First Class Constraints (FCC) $\eta_1$ $\equiv$ $\alpha \Pi_\theta - \Pi_\beta$; $\eta_2$ $\equiv$ $\Pi_\alpha$ $\eta_a \equiv \alpha \Pi_\theta - \Pi_\beta$, $\eta_a \equiv \Pi_\alpha$. (8)

The BT fields obey $\{\phi_1(x), \phi_2(y)\} = \delta(x-y)$. To ensure that there are no further constraints, we need a Hamiltonian that commutes with the FCCs. The following variables, $\tilde{\theta} = \theta + \alpha \phi_2$, $\tilde{\Pi}_\theta = \Pi_\theta$, $\tilde{\alpha} = \alpha + \phi_1 \Pi_\theta$, $\tilde{\Pi}_\alpha = \Pi_\alpha - \Pi_\theta \phi_2$, $\tilde{\beta} = \beta - \phi_2$, $\tilde{\Pi}_\beta = \Pi_\beta$, $\tilde{\phi}_i = 0$. (10) are gauge invariant in the sense that they commute with the FCCs. Hence all quantities written in terms of the redefined variables, e.g. the following (free) Hamiltonian, are gauge invariant in the extended space,

$$\tilde{H} \mid_{free} = \frac{1}{2}(\tilde{\Pi}_\theta \tilde{v}_i \tilde{v}_i) = \frac{1}{2} \Pi_\theta [\partial_i (\theta + \alpha \phi_2) + (\alpha + \phi_1 \Pi_\theta) \partial_i (\beta - \phi_2)]^2. \tag{11}$$

The remaining interaction terms in $H$ will also be extended in a similar way. This Hamiltonian (11) together with the FCCs (8) and the canonical phase space is the gauge invariant system we were looking for.

We note a fortuitous simplification in the extension structures (10). Unlike in other theories with non-linear SCCs $\eta_a$, where some of the extensions turn out to be infinite sequences of higher order terms in $\phi_i$-s, the present theory with non-linear SCCs (8), is free from this pathology.

To make contact with the physical system, the dimension of the BT extended phase space has to be reduced by additional gauge fixing constraints, (two in this case, $\tilde{\eta}_3$ and $\tilde{\eta}_4$, corresponding to two FCCs), with the only restriction that $\tilde{\eta}_a$, $a = 1,..,4$ constitute an SCC system that is $\{\tilde{\eta}_a, \tilde{\eta}_b\} \neq 0$. A consistency check is to see that the original system is recovered in the so called unitary gauge, $\tilde{\eta}_3 = \tilde{\eta}_4 = 0$. (Also it is now clear that some of the unphysical gauge choices mentioned before are not valid ones.) For a particular gauge, one has to construct the corresponding DB and compute the equations of motion using the DBs in reduced phase space, where the SCCs have been used strongly. Once again, the degrees of freedom count agrees with the original one. Consider the special class of gauge transformations: $\phi_1 = 0$; $\phi_2 = constant$. These will not change the $(v_i, \rho)$ algebra. Hence they can be identified as the conventional...
canonical transformations. It might be convenient, (although not necessary), to consider the gauges of the form \( \tilde{\eta}_3 \equiv \phi_1 - F \), \( \tilde{\eta}_4 \equiv \phi_2 - G \), to remove the BT fields directly to obtain (5).

Furthermore, additional constraints, such as incompressibility [1], can be included in this setup in the form \( \rho = \text{constant} \), which under time persistence generates another constraint \( \partial_i (\rho \tilde{v}_i) \). This SCC pair leads to [4].

The constants of motion for the free theory are obviously the energy \( \tilde{H} \), the momenta \( \tilde{P}_i = \int (\rho \partial_i \theta + \Pi_\alpha \partial_i \alpha + \Pi_\beta \partial_i \beta + \phi_2 \partial_i \phi_1) \), the angular momenta \( \tilde{L}^{ij} = \int (r_i \tilde{P}_j - r_j \tilde{P}_i) \) and the boost generator \( \tilde{B}^i = t \tilde{P}_i - \int (r_i \rho) \), effecting the transformation

\[
\{ \tilde{v}_i, u_j \tilde{B}_j \} = -t(u_j \partial_j) \tilde{v}_i + u_i, \quad \{ \rho, u_j \tilde{B}_j \} = -t(u_j \partial_j) \rho.
\]

Obtaining the Lagrangian is indeed straightforward. The first order form is

\[
\mathcal{L} = \Pi_\theta \dot{\theta} + \Pi_\alpha \dot{\alpha} + \Pi_\beta \dot{\beta} + \phi_2 \dot{\phi}_1 - \tilde{H} - \lambda_1 \tilde{\eta}_1 - \lambda_2 \tilde{\eta}_2
\]

\[
\equiv \Pi_\theta \dot{\theta} + \phi_2 \dot{\phi}_1 + \beta (\alpha \Pi_\theta + \phi_1) + \dot{\alpha} \Pi_\theta \phi_2 - \tilde{H},
\]

where \( \lambda_1 \) and \( \lambda_2 \) are multiplier fields and some of the variables have been removed using the equations of motion. At this stage, one can check explicitly that (12) is invariant under the following two independent sets of gauge transformations:

\[
\tilde{\eta}_1 \rightarrow \delta_1 \Pi_\theta = 0, \delta_1 \theta = -\alpha \psi_1, \delta_1 \beta = \psi_1, \delta_1 \alpha = 0, \delta_1 \phi_1 = 0, \delta_1 \phi_2 = \psi_1;
\]

\[
\tilde{\eta}_2 \rightarrow \delta_2 \Pi_\theta = 0, \delta_2 \theta = \phi_2 \psi_2, \delta_2 \beta = 0, \delta_2 \alpha = -\psi_2, \delta_2 \phi_1 = \Pi_\theta \psi_2, \delta_2 \phi_2 = 0,
\]

where \( \psi_1 \) and \( \psi_2 \) are gauge transformation parameter functions. Naively taking the unitary gauge, i.e. \( \phi_1 = \phi_2 = 0 \), we can recover the Lagrangian posited in [4].

We conclude by mentioning some of the future prospects. From the fluid mechanics point of view, the major interest lies in finding explicit forms of the GT leading to realistic situations. It will also be interesting if the BT fields can be identified with other physical variables such as temperature, pressure etc. Notice the coincidence that the fluid gauge theory and the Maxwell electrodynamics have the same number of fields and FCCs. Hence it might be worthwhile to look for some non-linear transformations connecting the two [4]. Finally, studying symmetry properties of the gauge fluid theory along the lines of [5] can be rewarding.

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