ON THE INVARIANCE OF WELSCHINGER INVARIANTS

ERWAN BRUGALLÉ

Abstract. We collect in this note some observations about original Welschinger invariants defined in [Wel05]. None of their proofs is difficult, nevertheless these remarks do not seem to have been made before. Our main result is that when $X_R$ is a real rational algebraic surface, Welschinger invariants only depend on the number of real interpolated points, and some homological data associated to $X_R$. This strengthened the invariance statement initially proved by Welschinger.

This main result follows easily from a formula relating Welschinger invariants of two real symplectic manifolds differing by a surgery along a real Lagrangian sphere. In its turn, once one believes that such formula may hold, its proof is a mild adaptation of the proof of analogous formulas previously obtained by the author on the one hand, and by Itenberg, Kharlamov and Shustin on the other hand.

We apply the two aforementioned results to complete the computation of Welschinger invariants of real rational algebraic surfaces, and to obtain vanishing, sign, and sharpness results for these invariants that generalize previously known statements. We also discuss some hypothetical relations of our work with tropical refined invariants defined in [BG16] and [GS16].

1. Main results

A real symplectic manifold $X_R = (X, \omega_X, \tau_X)$ is a symplectic manifold $(X, \omega_X)$ equipped with an anti-symplectic involution $\tau_X$. The real part of $(X, \omega_X, \tau_X)$, denoted by $R_X$, is by definition the fixed point set of $\tau_X$. An almost complex structure $J$ on $X$ is called $\tau_X$-compatible if it is tamed by $\omega$, and if $\tau_X$ is $J$-anti-holomorphic. In this note, the manifold $X_R$ will always be compact of dimension 4 with a non-empty real part, and we denote by $H^2_{\tau_X}(X; \mathbb{Z})$ the space of $\tau_X$-anti-invariant classes. A non-singular projective real algebraic variety is always implicitly assumed to be equipped with some Kähler form which turns it into a real symplectic manifold. All algebraic surfaces considered here are assumed to be projective and non-singular.

Let $X_R = (X, \omega_X, \tau_X)$ be a real compact symplectic manifold of dimension 4, and denote by $L_1, \cdots, L_k$ the connected components of $R_X$. Choose a class $d \in H^2(X; \mathbb{Z})$, and a vector $\rho = (r_1, \cdots, r_k) \in \mathbb{Z}^k_{\geq 0}$ such that

$$c_1(X) \cdot d - 1 - \sum_{i=1}^k r_i = 2s \in 2\mathbb{Z}_{\geq 0}.$$ 

Choose a configuration $\underline{x}$ made of $r_i$ points in $L_i$ for $i = 1, \cdots, k$, and $s$ pairs of $\tau_X$-conjugated points in $X \setminus R_X$. Given a $\tau_X$-compatible almost complex structure $J$, we denote by $C_{X_R}(d, \underline{x}, J)$ the set of real rational $J$-holomorphic curves in $X$ realizing the class $d$, and passing through $\underline{x}$. Then we define the integer

$$W_{X_R, \rho}(d; s) = \sum_{C \in C_{X_R}(d, \underline{x}, J)} (-1)^{m(C)},$$

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where $m(C)$ is the number of nodes of $C$ in $\mathbb{R}X$ with two $\tau_X$-conjugated branches. For a generic choice of $J$, the set $C_{X_\mathbb{R}}(d, \underline{x}, J)$ is finite, and $W_{X_\mathbb{R}, \rho}(d; s)$ depends neither on $\underline{x}$, $J$, nor on the deformation class of $X_\mathbb{R}$ (see [Wel05, Wel15]). We call these numbers the Welschinger invariants of $X_\mathbb{R}$. The main result of this paper, Theorem 1.3 below, is that when $X_\mathbb{R}$ is a real rational algebraic surface, Welschinger invariants eventually only depend on $s$ and some homological data of $X_\mathbb{R}$.

**Remark 1.1.** The set $C_{X_\mathbb{R}}(d, \underline{x}, J)$ is clearly empty when either $c_1(X) \cdot d \leq 0$, or $d \not\in H_2^{-\tau_X}(X; \mathbb{Z})$, or the partition $\rho$ contains two positive elements. This implies that $W_{X_\mathbb{R}, \rho}(d; s) = 0$ in these cases.

**Remark 1.2.** We restrict ourselves to the original Welschinger invariants as defined in [Wel05], and do not consider the more general modified Welschinger invariants defined in [IKS13, IKS15, IKS17], and also considered in [BP15, Bru18]. Further, because of Remark 1.1, the invariants $W_{X_\mathbb{R}, \rho}(d; s)$ are usually considered with a partition $\rho$ containing a single positive entry corresponding to a connected component $L$ of $\mathbb{R}X$. In this case, the invariant $W_{X_\mathbb{R}, \rho}(d; s)$ in this text corresponds to the invariant $W_{X_\mathbb{R}, L.\mathbb{R}X \setminus L}(d; s)$ in [BP15, Bru18], to the invariant $W_s(X_\mathbb{R}, d, L, 0)$ in [IKS17], and to the invariant $W(X_\mathbb{R}, d, L, 0)$ in [IKS13, IKS15] if $s = 0$. Our motivation to consider arbitrary partitions $\rho$ comes from Theorem 1.3.

Two real rational algebraic surfaces $X_1, \mathbb{R}$ and $X_2, \mathbb{R}$ are said to be *homologically equivalent* if both are obtained, up to deformation, as a real blow-up $\pi_i : X_i, \mathbb{R} \rightarrow X_0, \mathbb{R}$ of a real minimal algebraic surface $X_0, \mathbb{R}$ at $p$ distinct real points and $q$ distinct pairs of $\tau_{X_0}$-conjugated points. We emphasize that the distributions of the $p$ real points among connected components of $\mathbb{R}X_0$ may not coincide for $\pi_1$ and $\pi_2$. Note nevertheless that

$$\chi(\mathbb{R}X_1) = \chi(\mathbb{R}X_2) = \chi(\mathbb{R}X_0) - p.$$  

Denoting by $E_1, \cdots E_p$ (resp. $F_1, \overline{F}_1, \cdots, F_q, \overline{F}_q$) the exceptional divisors coming from the blow-ups at real points (resp. at pairs of $\tau_{X_0}$-conjugated points), the map $\pi_i$ induces the following decomposition

$$H_2(X_i; \mathbb{Z}) = H_2(X_0; \mathbb{Z}) \oplus \bigoplus_{j=1}^p \mathbb{Z}[E_j] \oplus \bigoplus_{j=1}^q \mathbb{Z}[F_j] \oplus \mathbb{Z}[\overline{F}_j]$$

which is orthogonal with respect to the intersection form. Furthermore, the action of $\tau_{X_i}$ is given by

$$\tau_{X_i,*} : H_2(X_0; \mathbb{Z}) = \tau_{X_0,*} \quad \tau_{X_i,*}(E_j) = -[E_j] \quad \text{for} \ j = 1, \cdots, p \quad \tau_{X_i,*}(F_j) = -[\overline{F}_j] \quad \text{for} \ j = 1, \cdots, q.$$  

In particular, the two maps $\pi_1$ and $\pi_2$ provide an identification of the groups $H_2(X_1; \mathbb{Z})$ and $H_2(X_2; \mathbb{Z})$ commuting with both intersection forms and action of the anti-symplectic involutions. We denote by $[X_\mathbb{R}]$ the homological equivalence class of a real rational algebraic surface $X_\mathbb{R}$.

**Theorem 1.3.** If $X_\mathbb{R}$ is a real rational algebraic surface, then $W_{X_\mathbb{R}, \rho}(d; s)$ does not depend on $\rho$, nor on a particular representative of $[X_\mathbb{R}]$.

As a consequence of Theorem 1.3, we simply denote by $W(X_\mathbb{R})(d; s)$ the invariant $W_{X_\mathbb{R}, \rho}(d; s)$. Note that some particular instances of Theorem 1.3 have already been noticed in [Bru15, Corollary 6.11 and Theorem 7.5] and [Bru18, Corollary 4.5]. Further, it follows from the classification of real rational algebraic surfaces up to deformation established in [DK02, Main Theorem and Theorem 2.4.1] that all Welschinger invariants of projective real rational algebraic surfaces are determined by the following ones:

- $W_{\mathbb{CP}^2_{p,q}}(d; s)$, where $\mathbb{CP}^2_{p,q}$ is a real blow-up of $\mathbb{CP}^2$ in $p$ real points and $q$ pairs of complex conjugated points;  

\[1\text{Recall that all real algebraic surfaces considered here are assumed to have a non-empty real part.}\]
Remark 1.4. Loosely speaking, Theorem 1.3 states that \( W_{Y_\mathbb{R}, \rho}(d; s) \) only depends on \( s \) and the lattice \( H_2(X; \mathbb{Z}) \) equipped with the intersection form and the action of \( \tau_{X, s} \). It may be interesting to work this out more rigorously. It may also be interesting to study generalizations of Theorem 1.3 to modified Welschinger invariants introduced in [IKS15], as well as to higher genus Welschinger invariants introduced in [Shu14], or to the higher dimensional invariants recently defined in [Geo16, GZ15].

Theorem 1.3 easily implies next corollary, which generalizes [BPT15, Theorem 1.1(1)] in the case \( F = [\mathbb{R}X_\mathbb{R} \setminus \mathbb{Q}] \).

Corollary 1.5. Let \( X_\mathbb{R} \) be a compact real rational algebraic surface with a disconnected real part. Suppose that \( X_{\mathbb{R}} \) is a real blow-up of another real rational algebraic surface in at least two real points, and denote by \( E_1 \) and \( E_2 \) the corresponding exceptional divisors. Then for any \( d \in H_2(X; \mathbb{Z}) \) such that both \( d \cdot [E_1] \) and \( d \cdot [E_2] \) are odd, one has \( W_{[X_{\mathbb{R}}]}(d; s) = 0 \).

Combining Theorem 1.3 with [Wel07, Theorem 1.1] and Corollary 1.5, we obtain the following.

Theorem 1.6. Let \( X_\mathbb{R} \) be a compact real rational algebraic surface with a disconnected real part, and assume that \( c_1(X) \cdot d - 1 - 2s > 0 \). Then one has

\[
(-1)^{\frac{d^2-c_1(X) \cdot d+2}{2}} \cdot W_{[X_{\mathbb{R}}]}(d; s) \geq 0.
\]

Furthermore, the invariant \( W_{[X_{\mathbb{R}}]}(d; s) \) is sharp in the following sense: there exists a compact real rational algebraic surface \( Y_{\mathbb{R}} \in [X_{\mathbb{R}}] \), a real configuration \( \underline{x} \) of \( c_1(X) \cdot d-1 \) points in \( Y \) with \( |\underline{x} \cap \mathbb{R}Y| = c_1(Y) \cdot d - 2s \), and a generic \( \tau_Y \)-compatible almost complex structure \( J \) on \( Y \) such that

\[
\text{Card}(\mathcal{C}_Y(d, \underline{x}, J)) = |W_{[X_{\mathbb{R}}]}(d; s)|.
\]

Remark 1.7. A configuration \( \underline{x} \) and a \( \tau_Y \)-compatible almost complex structure \( J \) as in Theorem 1.6 may not exist for any representative \( Y_{\mathbb{R}} \) of \([X_{\mathbb{R}}] \), even up to deformation, see [Bru13, Remark 6.13].

One of the main ingredients in our proof of Theorem 1.3 is Theorem 2.1 that relates Welschinger invariants of two real symplectic 4-manifolds differing by a so-called surgery along a real Lagrangian sphere. We refer to Section 2 for more details about this operation, and for a statement of Theorem 2.1 together with its proof. This latter theorem partially generalizes both [IKS15, Corollary 4.2] and [Bru18, Theorem 1.1, Remark 1.3]. We point out that its proof is an easy adaptation of the proof of [IKS15, Corollary 4.2], using [BPT15, Theorem 2.5(1)]. It just required to believe in the correctness of the statement to prove it.

Combining [DK02, Main Theorem and Theorem 2.4.1] together with the classical rigid isotopy classifications of plane real quartics, of real cubic sections of the quadratic cone in \( \mathbb{CP}^3 \), and of real...
quadrics in $\mathbb{CP}^3$ (see for example [DK00]), one easily classifies real rational algebraic surfaces up to deformation, real blow-up, and surgery along a real Lagrangian sphere: any real rational surface is obtained by a finite sequence of these three operations starting from either $\mathbb{CP}^2$ or the quadric hyperboloid $QH$. In particular we have the following result.

**Theorem 1.8.** Let $X_\mathbb{R}$ be a real algebraic rational surface. Then by finitely many successive applications of Theorem 2.1, all Welschinger invariants of $X_\mathbb{R}$ can be computed out of Welschinger invariants of either $\mathbb{CP}^2_{p,0}$ or $QH$.

Since all Welschinger invariants of $QH$ and $\mathbb{CP}^2_{p,0}$ have been computed, see for example [Mik05, BM08, HS12, Che18], Theorem 1.8 completes the computation of Welschinger invariants $W_{[X_\mathbb{R}]}(d; s)$ of real rational algebraic surfaces.

Next statement can be seen as an increasingness property of $W_{[X_\mathbb{R}]}$ with respect to $\chi(\mathbb{R}X_\mathbb{R})$, and goes in a somewhat different direction than [BP13, Theorem 3.4], [BP15, Proposition 2.8], and [Bru15, Corollaries 4.4 and 6.10].

**Theorem 1.9.** Let $X_\mathbb{R}$ and $Y_\mathbb{R}$ be two compact real rational algebraic surfaces with a disconnected real part, differing by a surgery along a real Lagrangian sphere, and such that $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2$. Then for any $d \in H^2_\tau(Y; \mathbb{Z})$ and $s \in \mathbb{Z}_{\geq 0}$ such that $c_1(X) \cdot d - 1 - 2s > 0$, one has

$$|W_{[Y_\mathbb{R}]}(d; s)| \geq |W_{[X_\mathbb{R}]}(d; s)|.$$

Theorem 2.1 is obtained thanks to a real version of a (very simple instance) of the symplectic sum formula from [IP04, LR01, TZ14], see also [Li02, Li04] for an analogous formula in the complex algebraic category. It turns out that the same strategy provides a formula similar to Theorem 2.1 for relative Gromov-Witten invariants of symplectic 4-manifolds. This observation suggests a possible connection of our work to tropical refined invariants defined in [BG16, GS16]. We discuss this aspect in Section 4. In particular, we provide there an alternative explanation for the specializations in $q = \pm 1$ of the tropical refined descendant invariants from [GS16]. We also show that a refined version of a conjecture by Itenberg, Kharlamov and Shustin [IKS04, Conjecture 6] holds, although it was known to be wrong in the non-refined case.

The remaining part of the paper is organised as follows. We introduce surgeries along real Lagrangian spheres in Section 2, then state and prove Theorem 2.1. All statements given in the present section can easily be derived from Theorem 2.1 and previously known results. Proofs are given in Section 3. Finally, we discuss in Section 4 connections of our work to tropical refined invariants of algebraic surfaces and refined Severi degrees.

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2. Surgery along a real Lagrangian sphere

Let $X_\mathbb{R} = (X, \omega_X, \tau_X)$ be a real compact symplectic manifold of dimension 4, and let $S \subset X$ be a Lagrangian sphere globally invariant under $\tau_X$. Locally, a neighborhood $V$ of $S$ is given by a neighborhood in the real affine quadric $(Q, \omega_Q, \tau)$ in $\mathbb{C}^3$ given by the equation

$$(-1)^{\varepsilon_1}x^2 + (-1)^{\varepsilon_2}y^2 + (-1)^{\varepsilon_3}z^2 = 1 \quad \text{with} \quad \varepsilon_i \in \{0, 1\},$$

of the sphere $S_Q$ in $i^{\varepsilon_1}\mathbb{R} \times i^{\varepsilon_2}\mathbb{R} \times i^{\varepsilon_3}\mathbb{R}$ with equation

$$x^2 + y^2 + z^2 = 1.$$
As explained in [Bru18, Section 2.2], one can modify the symplectic and real structure of \(X_\mathbb{R}\) in \(V\) so that \(V\) is now given by a neighborhood of \(S_Q\) in the real affine quadric in \(\mathbb{C}^3\) with equation
\[(-1)^{e_1}x^2 + (-1)^{e_2}y^2 + (-1)^{e_3}z^2 = -1.\]
The resulting real symplectic manifold \(Y_\mathbb{R}\) is called a surgery of \(X_\mathbb{R}\) along \(S\) (see Figure 1 for a local picture). Note that, with the convention that \(\chi(\emptyset) = 0\), we have

\[\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) \pm 2.\]

Furthermore the class \([S]\) in \(H_2(X;\mathbb{Z})\) is \(\tau_X\)-anti-invariant if and only if it is \(\tau_Y\)-invariant (in which case we have \(\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2\)).

Suppose that \(\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2\), and let \(\rho\) be a vector whose entries are indexed by connected components of \(\mathbb{R}Y\). If \(S \subset \mathbb{R}Y\), we further assume that the entry of \(\rho\) corresponding to \(S\) vanishes. We denote by \(S^{\tau_X}\) (resp. \(S^{\tau_Y}\)) the fixed point set of the involution \(\tau_X|_S\) (resp. \(\tau_Y|_S\)). In particular, we have either \(S^{\tau_X} = S^1\) and \(S^{\tau_Y}\) consists in 2 points, or \(S^{\tau_X} = \emptyset\) and \(S^{\tau_Y} = S\). The connected components of \(\mathbb{R}Y\setminus S^{\tau_Y}\) are canonically in one-to-one correspondence with the connected components of \(\mathbb{R}X\setminus S^{\tau_X}\). Hence one can associate to \(\rho\) a vector \(\tilde{\rho}\) whose entries are indexed by the connected components of \(\mathbb{R}X\): the entry corresponding to the connected component \(L\) of \(\mathbb{R}X\) is the sum of the entries corresponding to the connected component of \(\mathbb{R}Y\setminus S^{\tau_Y}\) corresponding to \(L\setminus S^{\tau_X}\).

Next theorem partially generalizes both [IKS15, Corollary 4.2] and [Bru18, Theorem 1.1, Remark 1.3].

**Theorem 2.1.** Let \(X_\mathbb{R}\) be a compact real symplectic manifold of dimension 4. Let \(S\) be a real Lagrangian sphere in \(X_\mathbb{R}\), and \(Y_\mathbb{R}\) a surgery of \(X_\mathbb{R}\) along \(S\). Suppose that \(\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2\). Then for any class \(d \in H_2^{TV}(X;\mathbb{Z})\), the following identity holds
\[W_{Y_\mathbb{R},\rho}(d; s) = \sum_{k \in \mathbb{Z}} (-1)^k W_{X_\mathbb{R},\tilde{\rho}}(d - k[S]; s),\]
whenever the entry of \(\rho\) corresponding to \(S\) vanishes if \(S \subset \mathbb{R}Y\).

**Proof.** Note first that the identity we want to prove is independent on the orientation chosen on \(S\) to define an element \([S]\) \(H_2(X;\mathbb{Z})\). By [Bru18, Remark 1.3], this identity can be rewritten in the following form once such orientation is chosen:
\[W_{Y_\mathbb{R},\rho}(d; s) = W_{X_\mathbb{R},\tilde{\rho}}(d; s) + 2 \sum_{k \geq 1} (-1)^k W_{X_\mathbb{R},\tilde{\rho}}(d - k[S]; s).\]
Recall (see [Bru18 Section 2.2]) that $X_{\mathbb{R}}$ can be deformed to a real symplectic 4-manifold $(Z, \omega_Z, \tau_Z)$ for which $S$, equipped with the chosen orientation, becomes symplectic. Choose a real configuration of points $x$ in $Z$ with $s$ pairs of $\tau_Z$-conjugated points, and $\rho_L$ points in $L$ for each connected component $L$ of $\mathbb{R}Z \setminus \mathbb{R}S$. Choose also a $\tau_Z$-compatible almost complex structure $J$ on $Z$ for which $S$ is $J$-holomorphic. Given an integer $k \geq 0$, we denote by $C^\beta(d - k[S], x, J)$ the set of all reducible rational real $J$-holomorphic curves in $(Z, \omega_Z, \tau_Z)$ passing through all points in $x$, realizing the class $d - k[S]$, and intersecting $S \setminus \mathbb{R}S$ in exactly $\beta$ pairs of $\tau_Z$-conjugated points. For a generic choice of $J$ satisfying the above conditions, the set $C^\beta(d - k[S], x, J)$ is finite and only contains nodal curves by [BP15 Lemma 3.1 and Proposition 3.3]. We define

$$W_{Z_{a, \rho}}^\beta(d - k[S]; s) = \sum_{C \in C^\beta(d - k[S], x, J)} (-1)^{m(C)}.$$

Note that $W_{Z_{a, \rho}}^\beta(d - k[S]; s)$ may depend on the choices of $x$ and $J$, nevertheless we will not record this dependence in our notation in order to lighten the exposition. By [BP15 Theorem 2.5(1)], we have

$$W_{Y_{a, \rho}}(d; s) = \sum_{l \geq 0} (-2)^l W_{Z_{a, \rho}}^l(d - l[S]; s),$$

and

$$W_{X_{a, \rho}}(d; s) = \sum_{j, b, \beta \geq 0} \left( \begin{array}{c} 2j - 2b \\ j - 2\beta \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right) W_{Z_{a, \rho}}^b(d - j[S]; s).$$

Denoting by $A$ the right-hand-side of (1), and using the last identity, we obtain

$$A = \sum_{j, b, \beta \geq 0} \left( \begin{array}{c} 2j - 2b \\ j - 2\beta \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right) W_{Z_{a, \rho}}^b(d - j[S]; s)$$

$$+ 2 \sum_{k \geq 1} (-1)^k \sum_{j, b, \beta \geq 0} \left( \begin{array}{c} 2k + 2j - 2b \\ j - 2\beta \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right) W_{Z_{a, \rho}}^b(d - (j + k)[S]; s).$$

By the change of variable $j = l - k$, we obtain

$$A = \sum_{l, b, \beta \geq 0} \left( \begin{array}{c} 2l - 2b \\ l - 2\beta \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right) W_{Z_{a, \rho}}^b(d - l[S]; s)$$

$$+ 2 \sum_{k, l \geq 1, b, \beta \geq 0} (-1)^k \left( \begin{array}{c} 2l - 2b \\ l - k - 2\beta \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right) W_{Z_{a, \rho}}^b(d - l[S]; s).$$

Hence the coefficient of $W_{Z_{a, \rho}}^b(d - l[S]; s)$ in $A$ is

$$\sum_{\beta \geq 0} \left( \begin{array}{c} 2l - 2b \\ l - 2\beta \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right) + 2 \sum_{k \geq 1} \sum_{\beta \geq 0} (-1)^k \left( \begin{array}{c} 2l - 2b \\ l - k - 2\beta \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right),$$

that is to say

$$\sum_{\beta \geq 0} \left( \begin{array}{c} 2l - 2b \\ l - 2\beta \end{array} \right) + 2 \sum_{k \geq 1} (-1)^k \left( \begin{array}{c} 2l - 2b \\ l - k - 2\beta \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right).$$
We denote by \( u_{l,b,\beta} \) the coefficient of \( \left( \begin{array}{c} b \\ \beta \end{array} \right) \) in the latter sum. We have
\[
 u_{l,b,\beta} = \left( \begin{array}{c} 2l - 2b \\ l - 2\beta \end{array} \right) + 2 \sum_{k \geq 1} (-1)^k \left( \begin{array}{c} 2l - 2b \\ l + k - 2\beta \end{array} \right).
\]
Hence we get
\[
 u_{l,b,\beta} + u_{l,b,\beta} = 2 \times (-1)^l \sum_{p \geq 0} (-1)^p \left( \begin{array}{c} 2l - 2b \\ p \end{array} \right),
\]
and so
\[
 u_{l,b,\beta} + u_{l,b,\beta} = 0
\]
if \( l > b \), and
\[
 u_{l,l,\beta} + u_{l,l,\beta} = 2 \times (-1)^l.
\]
This implies that
\[
 \sum_{\beta \geq 0} u_{l,b,\beta} \left( \begin{array}{c} b \\ \beta \end{array} \right) = 0
\]
if \( l > b \), and
\[
 \sum_{\beta \geq 0} u_{l,l,\beta} \left( \begin{array}{c} l \\ \beta \end{array} \right) = (-2)^l,
\]
which is precisely what we have to prove.

**Remark 2.2.** Theorem 2.1 can clearly be generalised to modified Welschinger invariants introduced in [IKS13], at the cost of much heavier notations. It may be nevertheless interesting to work out such generalisation.

As an example of application of Theorem 2.1 we have the following.

**Proposition 2.3.** Let \( X_{\mathbb{R}} \) be a compact real symplectic manifold of dimension 4. Let \( \tilde{X}_{\mathbb{R}} \) be the blow up of \( X_{\mathbb{R}} \) at two disjoint real balls, and \( \tilde{Y}_{\mathbb{R}} \) be the blow up of \( X_{\mathbb{R}} \) at two \( \tau_X \)-conjugated disjoint balls. In both cases we denote by \( E_1 \) and \( E_2 \) the two exceptional divisors. Then for any class \( d \in H^{-\tau_X}(X;\mathbb{Z}) \) and any \( l \in \mathbb{Z} \geq 0 \), one has
\[
 W_{\tilde{Y}_{\mathbb{R}},\rho}(d-l[E_1]-l[E_2];s) = W_{\tilde{X}_{\mathbb{R}},\rho}(d-l[E_1]-l[E_2];s) + 2 \sum_{\lambda=1}^{l} (-1)^\lambda W_{\tilde{X}_{\mathbb{R}},\rho}(d-(l-\lambda)[E_1]-(l+\lambda)[E_2];s).
\]

The notation \( \rho \) both for \( \tilde{X}_{\mathbb{R}} \) and \( \tilde{Y}_{\mathbb{R}} \) in the theorem makes sense since the connected components of \( \tilde{X} \) and \( \tilde{Y} \) are in a canonical one-to-one correspondence.

**Proof.** This is Theorem 2.1 applied to the class \([E_1] - [E_2]\). \( \square \)

**Proposition 2.3** applied with \( l = 1 \), combined with [DH18] Theorems 1.1 and 1.2, specializes to Welschinger Formula [Wel05, Theorem 0.4].

**Corollary 2.4.** Let \( X_{\mathbb{R}} \) be a compact real symplectic manifold of dimension 4, and let \( \tilde{X}_{\mathbb{R}} \) be the blow up of \( X_{\mathbb{R}} \) at one real ball. We denote by \( E \) the exceptional divisor. Then for any \( d \in H^{-\tau_X}(X;\mathbb{Z}) \) and \( s \in \mathbb{Z} \geq 0 \) such that \( c_1(X) \cdot d - 2s \geq 3 \), one has
\[
 W_{X_{\mathbb{R}},\rho}(d; s+1) = W_{X_{\mathbb{R}},\rho}(d; s) - 2W_{\tilde{X}_{\mathbb{R}},\rho}(d-2E; s).
\]
3. Proofs of the main results

As mentioned in Section 1, a classification of real rational algebraic surfaces up to deformation, real blow-up, and surgery along a real Lagrangian sphere is easily obtained combining [DK02, Main Theorem and Theorem 2.1] together with the classical rigid isotopy classifications of plane real quartics, of real cubic sections of the quadratic cone in $\mathbb{C}P^3$, and of real quadrics in $\mathbb{C}P^4$ (see for example [DK00]). We implicitly use this classification in the following five proofs.

Proof of Theorem 1.3. The surface $X_R$ can be degenerated to a nodal real algebraic rational surface $\overline{X}_R$ with a connected real part and having only real nodes. Blowing up these nodes, we obtain a non-singular real algebraic rational surface $Z_R$ with a connected real part. By construction $X_R$ is obtained, up to deformation, by surgeries along the disjoint union of the exceptional divisors of the desingularization $Z_R \to \overline{X}_R$. Since $\mathbb{R}Z$ is connected, the Welschinger invariants of $Z_R$ depend neither on the choice of $x$ nor on the position of the real blown-up points on a minimal model of $Z$. Now the proposition is an immediate consequence of Theorem 2.1.

Proof of Corollary 1.5. Up to choosing another representative of $[X_R]$, we may assume that $\mathbb{R}E_1$ and $\mathbb{R}E_2$ do not lie on the same connected component of $\mathbb{R}X$. In particular by connectedness of $\mathbb{R}P^1$, the set $C_{X_R}(d, x, J)$ is clearly empty for any configuration $x$.

Proof of Theorem 1.6. Up to choosing another representative of $[X_R]$, we may assume that a connected component $L$ of $\mathbb{R}X$ is homeomorphic to the sphere $S^2$. If $c_1(X) \cdot d - 1 - 2s \geq 2$, the result follows from Remark 1.1. If $c_1(X) \cdot d - 1 - 2s = 1$, then we choose the configuration $x$ such that $x \cap \mathbb{R}X \subset L$. Now the result follows from [Wel07, Theorem 1.1].

Remark 3.1. If $c_1(X) \cdot d - 1 - 2s = 0$, then [Wel07, Theorem 1.1] states that one can find $x$ and $J$ so that there are no curve $C$ in $C_{X_R}(d, x, J)$ with $\mathbb{R}C \subset L$. Nevertheless, there may exist curves $C$ in $C_{X_R}(d, x, J)$ with $\mathbb{R}C \subset \mathbb{R}X \setminus L$.

Proof of Theorem 1.8. Any real rational algebraic surface can be obtained by a finite sequence of deformations, real blow-ups, and surgery along real Lagrangian spheres, starting from either $\mathbb{C}P^2$ or the quadric hyperboloid $QH$ in $\mathbb{C}P^4$.

Let $Y_{1,R}$ and $Y_{2,R}$ be two real algebraic surfaces obtained by blowing up a real rational surface $Y_R$ in respectively two real and one imaginary-conjugated points. Denoting by $E_1$ and $E_2$ the two exceptional divisors, the real surface $Y_{2,R}$ is obtained by a surgery along a real Lagrangian sphere realizing the class $[E_1] - [E_2]$. Since $QH$ blown-up in $l \geq 1$ real points is also $\mathbb{C}P^2$ blown-up in $l + 1$ real points, the result follows.

Proof of Theorem 1.9. Let $S$ be a real Lagrangian sphere allowing to pass from $X_R$ to $Y_R$ by surgery. Since one has $|S|^2 = -2$ and $c_1(X) \cdot |S| = 0$, it follows from Theorem 1.6 that $W_{[X_R]}(d; s)$ and $(-1)^k W_{[X_R]}(d - k|S|; s)$ have the same sign. Now the result follows from Theorem 2.1.

4. Hypothetical connection to tropical refined invariants

We end this note by pointing out a possible connection of Theorem 2.1 to tropical refined invariants of algebraic surfaces, and refined Severi degrees.

4.1. Tropical refined invariants and surgeries along real Lagrangian spheres. A non-degenerate convex polygon $\Delta \subset \mathbb{R}^2$ with vertices in $\mathbb{Z}^2$ defines a complex toric surface $X_\Delta$ together with a complete linear system $d$. Block and Göttsche proposed in [BG16] to enumerate irreducible tropical curves with Newton polygon $\Delta$ and genus $g$ as proposed in [Mik02], but replacing Mikhalkin’s complex multiplicity with its quantum analog. One obtains in this way a Laurent polynomial in the variable $q$, called tropical refined invariant and denoted by $G_{X_\Delta}(d, g)$, which does not depend on the
configuration of points chosen to define it [IM13]. By [Mik05], the value $G_{X_{\Delta}}(d, g)(1)$ recovers the number of complex irreducible algebraic curves of genus $g$ in $X_{\Delta}$ realizing the class $d$, and passing through a generic configuration of $c_1(X_{\Delta}) \cdot d - 1 + g$ points. Any complex toric surface has a standard real structure induced by the standard real structure on $(\mathbb{C}^*)^2$, and we denote by $X_{\Delta, R}$ the corresponding real toric surface. It also follows from [Mik05] that when $X_{\Delta, R}$ is a real unnodal toric del Pezzo surface (i.e. $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2$ blown up in at most 3 real points not contained in a line), one has

$$G_{\Delta}(0)(-1) = W_{[X_{\Delta, R}]}(d; 0).$$

In other words, tropical refined invariants interpolate between Gromov-Witten invariants (for $q = 1$) and Welschinger invariants with $s = 0$ (for $q = -1$) of $X_{\Delta, R}$ when both are defined. Tropical refined invariants are conjectured to agree with the $\chi_y$-refinement of Severi degrees introduced in [GS14].

In the case of a real unnodal toric del Pezzo surface $X_{\Delta, R}$, Göttsche and Schroeter defined in [GS16] some tropical refined descendant invariants, denoted by $G_{X_{\Delta}}(d, 0; s)$, interpolating between some genus 0 tropical descendent invariants and Welschinger invariants $W_{[X_{\Delta, R}]}(d; s)$, and this for arbitrary values of $s$. This work has been then generalized to all genus 0 tropical descendent invariants by Blechman and Shustin in [BS16], leaving open the general interpretation of the value at $q = -1$ of these new tropical refined descendant invariants.

Despite recent progress, e.g. [Mik17, NPS18, Bou17], a general enumerative interpretation of tropical refined invariants seems to behave nicely under degenerations of the ambient algebraic surface (or symplectic 4-manifold). In particular, a connection of our results to tropical refined invariants may be suggested by the following observation: relative Gromov-Witten invariants of symplectic 4-manifolds satisfy a formula similar to the one from Theorem 2.1. To make this precise, we need first to introduce some additional notations.

Let $(X, \omega_X)$ be a compact symplectic manifold of dimension 4, containing a finite union $U = E_1 \cup \ldots \cup E_\kappa$ of pairwise disjoint embedded symplectic spheres with $[E_i]^2 = -2$. Let also $J$ be an almost complex structure on $X$ tamed by $\omega_X$, for which all curves $E_1, \ldots, E_\kappa$ are $J$-holomorphic.

Given $d \in H_2(X; \mathbb{Z})$, let us choose a configuration $\underline{z}$ of $c_1(X) \cdot d - 1$ distinct points in $X \setminus \bigcup_{i=1}^\kappa E_i$.

We define $GW_{(X, \omega)}^U(d)$ as the number of irreducible $J$-holomorphic rational curves $f : C \to X$ with $f_*[C] = d$, passing through all points in $\underline{z}$, and whose image is not contained in $\bigcup_{i=1}^\kappa E_i$. For a generic choice of $J$, this number is finite and does not depend on $\underline{z}$, it is called a Gromov-Witten invariant of $(X, \omega)$ relative to $U$. Given an element $E$ of $U$, we define $\bar{U} = U \setminus \{E\}$. Given two integer numbers $m$ and $k$, we define

$$u_{m, k} = (-1)^k \left( \binom{m + k}{m} + \binom{m + k - 1}{m} \right).$$

**Proposition 4.1.** One has

$$GW_{(X, \omega)}^U(d) = \sum_{k \geq 0} u_{d; [E], k} GW_{(X, \omega)}^{\bar{U}}(d - k[E]).$$

In particular if $d \cdot [E] = 0$, then one has

$$GW_{(X, \omega)}^U(d) = \sum_{k \in \mathbb{Z}} (-1)^k GW_{(X, \omega)}^{\bar{U}}(d - k[E]).$$
Proof. It follows immediately from [BPT15 Corollary 3.8] that

\[ GW^{\tilde{U}}(X,\omega)(d) = \sum_{k \geq 0} \left( \frac{d \cdot [E] + 2k}{k} \right) GW^U(X,\omega)(d - k[E]). \]

So the first formula follows from [Bru18 Proposition 3.12]. If \( d \cdot [E] = 0 \), this formula becomes

\[ GW^{\tilde{U}}(X,\omega)(d) = GW^U(X,\omega)(d) + 2 \sum_{k \geq 0} (-1)^k GW^{\tilde{U}}(X,\omega)(d - k[E]). \]

Thanks to the equality \( GW^U(X,\omega)(d') = GW^U(X,\omega)(d + (d' \cdot [E])|E)) \) for any \( d' \in H_2(X;\mathbb{Z}) \), we obtain the desired result.

The combination of Theorem 2.1 with Proposition 4.1 implies the following “refined flavored” corollary.

**Corollary 4.2.** Let \( X_\mathbb{R} \) be either the quadric hyperboloid \( QH \) in \( \mathbb{C}P^3 \), or \( \mathbb{C}P^2 \) blown-up at finitely many real points. Let also \( U = E_1 \cup \ldots \cup E_\kappa \) be a finite collection of pairwise disjoint real embedded symplectic spheres with \( [E_i]^2 = -2 \). Suppose that we are given a function \( \Gamma_{X_\mathbb{R}} : H_2(X;\mathbb{Z}) \to \mathbb{Z}[q, q^{-1}] \) such that for any \( d \in H_2(X;\mathbb{Z}) \), one has

\[ \Gamma_{X_\mathbb{R}}(d)(1) = GW^U(X,\omega)(d) \quad \text{and} \quad \Gamma_{X_\mathbb{R}}(d)(-1) = W_{[X_\mathbb{R}]}(d;0). \]

Let \( Y_\mathbb{R} \) be the surgery of \( X_\mathbb{R} \) along the real Lagrangian spheres \( E_1, \ldots, E_\kappa \), and define the following function

\[ \Gamma_{Y_\mathbb{R}} : H_2(X;\mathbb{Z}) \to Z[q, q^{-1}] \]

\[ d \mapsto \sum_{k_1, \ldots, k_\kappa \geq 0} \left( \prod_{i=1}^{\kappa} u_{d[E_i], k_i} \right) \Gamma_{X_\mathbb{R}}(d - k_1[E_1] - \cdots - k_\kappa[E_\kappa]). \]

Then \( \Gamma_{Y_\mathbb{R}} \) satisfies the two following properties:

\[ \forall d \in H_2(X;\mathbb{Z}), \quad \Gamma_{Y_\mathbb{R}}(d)(1) = GW^{\tilde{U}}(X,\omega)(d); \]

\[ \forall d \in H_2(X;\mathbb{Z}), \quad \Gamma_{Y_\mathbb{R}}(d)(-1) = W_{Y_\mathbb{R}}(d;0). \]

Following Corollary 4.2, we discuss in the next two subsections a refined version of Corollary 2.4 and of [BPT15 Proposition 2.7].

### 4.2. A refined Corollary 2.4

Refined tropical descendant invariants defined in [GS16] can be computed using floor diagrams from [BGM12], equipped with refined weights as indicated in [BS16]. Next proposition, whose proof we omit, is an easy corollary of this floor diagrammatic computation. Recall that \( QH \) is the real quadric hyperboloid in \( \mathbb{C}P^3 \).

**Proposition 4.3.** Let \( X_\mathbb{R} \) be either \( QH \) or \( \mathbb{C}P^2 \) blown-up in at most two distinct real points, and let \( X_\mathbb{C} \) be the blow-up of \( X_\mathbb{R} \) at one real point. We denote by \( E \) the exceptional divisor. Then for any \( d \in H_2(X;\mathbb{Z}) \) and \( s \in \mathbb{Z}_{\geq 0} \) such that \( c_1(X) \cdot d - 2s \geq 3 \), one has

\[ G_{X_\mathbb{R}}(d, 0; s + 1) = G_{X_\mathbb{R}}(d, 0; s) - 2G_{X_\mathbb{C}}(d - 2E, 0; s). \]

By Corollary 4.2 Proposition 4.3 is clearly a refined version of Corollary 2.4. In particular, an induction on \( s \) provides an alternative proof of [GS16 Corollaries 3.14 and 3.29] (however it does not explain the tropical invariance of the numbers \( G_{X_\mathbb{R}}(d, 0; s) \)).

**Example 4.4.** One computes easily, for example using floor diagrams, that the coefficient of degree \( \frac{(d-1)(d-2)}{2} - 1 \) of \( G_{\mathbb{C}P^2}(d, 0; 0) \) is \( 3d + 1 \). Since this coefficient is 1 for \( G_{\mathbb{C}P^2}(d - 2E, 0; s) \), we deduce from Proposition 4.3 that the coefficient of degree \( \frac{(d-1)(d-2)}{2} - 1 \) of \( G_{\mathbb{C}P^2}(d, 0; s) \) is \( 3d + 1 - 2s \). In particular when \( s \) is maximal, this coefficient is equal to 2 or 3 depending on the parity of \( d \).
Since all coefficients of $G_{X_R}(d, 0; s)$ are positive for an unnodal real del Pezzo toric surface $X_R$, Proposition 4.3 immediately implies the following.

**Corollary 4.5.** Let $X_R$ be either $QH$ or $\mathbb{CP}^2$ blown-up in at most two distinct real points. For $d \in H_2(X; \mathbb{Z})$, we denote by $a_{r,s}$ the coefficient of degree $r$ of $G_{X_R}(d, 0; s)$. Then one has
\[ a_{r,0} \geq a_{r,1} \geq \cdots \geq a_r \left[ \frac{c_2(X).d - 1}{2} \right] > 0. \]

Corollary 4.5 may be seen as a refined version of [IKS04, Conjecture 6]. It is amusing that although this conjecture has been proven to be wrong in [Wel07] and [ABLdM11], its refined version eventually holds.

### 4.3. Refined invariants of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\Sigma_2$

Recall that the quadric hyperboloid $QH$ can be deformed to the second Hirzebruch surface $\Sigma_2$ equipped with its toric real structure. This can be done algebraically by degenerating the quadric hyperboloid $QH$ to a nodal quadric, and by blowing up the node. The surgery of $X_R$ along the exceptional curve in $\Sigma_2$ is, up to deformation, the quadric ellipsoid $QE$ in $\mathbb{CP}^3$. Since both $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\Sigma_2$ are toric, Corollary 4.2 suggests the following conjecture.

**Conjecture 4.6.** We denote by $\Box_{a,b}$ (resp. $\triangle_{a,b}$) the class in $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{Z})$ (resp. $\Sigma_2$) defined by the convex polygon in $\mathbb{R}^2$ with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and $(a, b)$ (resp. $(0, 0)$, $(2a + b, 0)$, $(0, a)$, and $(b, a)$), see Figure 2. Then for any integers $a, b, g \geq 0$, the tropical refined invariants $G_{QH}(\Box, g)$ and $G_{\Sigma_2}(\triangle, g)$ satisfy the following relations:
\[ G_{\Sigma_2}(\triangle_{a,b}, g) = \sum_{k \geq 0} u_{b,k} G_{QH}(\Box_{a+b+k,a-k,b}, g). \]

Note that this can be seen as a refined version of [BPT15, Proposition 2.7]: denoting by $h$ the hyperplane section class of the quadric ellipsoid $QE$, one has
\[ \forall a \geq 0, \ G_{\Sigma_2}(\triangle_{a,0}, 0)(-1) = W_{QE}(ah; 0). \]

In the next examples, we check that Conjecture 4.6 holds in a few cases.

**Example 4.7.** The cases $a = 1$ or $g = (a - 1)(a - 1 + b)$ hold trivially.

The value of all refined invariants needed in the next examples are provided in Appendix A.

**Example 4.8.** In the case $(a, b) = (2, 0)$ and $g = 0$, one has
\[ G_{\Sigma_2}(\triangle_{2,0}, 0; s) = G_{QH}(\Box_{2,2}, 0; s) - 2 G_{QH}(\Box_{3,1}, 0; s). \]

\[ \text{As stated, Corollary 4.2 only suggests the conjecture for } g = 0. \text{ Nevertheless, [Bru18, Corollary 3.17] suggests its extension to any genus.} \]

\[ \text{This conjecture has been now proved by Bousseau in [?].} \]
Example 4.9. The case \((a, b) = (2, 2)\) gives
\[
\begin{align*}
g = 2 & : \quad G_{\Sigma_2}(\triangle_{2,2}, 2) = G_{QH}(\square_{4,2}, 2); \\
g = 1 & : \quad G_{\Sigma_2}(\triangle_{2,2}, 1) = G_{QH}(\square_{4,2}, 1); \\
g = 0 & : \quad G_{\Sigma_2}(\triangle_{2,2}, 0; s) = G_{QH}(\square_{4,2}, 0; s) - 4G_{QH}(\square_{5,1}, 0; s).
\end{align*}
\]

Example 4.10. A more interesting case is given by \((a, b) = (3, 0)\) and \(g \in \{0, 1, 2, 3\}\). We obtain the following
\[
\begin{align*}
g = 3 & : \quad G_{\Sigma_2}(\triangle_{3,0}, 3) = G_{QH}(\square_{3,3}, 3) - 2G_{QH}(\square_{4,2}, 3). \\
g = 2 & : \quad G_{\Sigma_2}(\triangle_{3,0}, 2) = G_{QH}(\square_{3,3}, 2) - 2G_{QH}(\square_{4,2}, 2). \\
g = 1 & : \quad G_{\Sigma_2}(\triangle_{3,0}, 1) = G_{QH}(\square_{3,3}, 1) - 2G_{QH}(\square_{4,2}, 1). \\
g = 0 & : \quad G_{\Sigma_2}(\triangle_{3,0}, 0; s) = G_{QH}(\square_{3,3}, 0; s) - 2G_{QH}(\square_{4,2}, 0; s) + 2G_{QH}(\square_{5,1}, 0; s).
\end{align*}
\]

A refined version of the strategy used in [BM16] may lead to a combinatorial proof of Conjecture 4.6, however it is not clear that such a combinatorial proof will be geometrically meaningful.

APPENDIX A. SOME COMPUTATIONS OF TROPICAL REFINED INVARIANTS

In this appendix we provide a few values of tropical refined invariants of tropical projective plane and Hirzebruch surfaces of small degree. All non-trivial computations of absolute refined invariants have been done using floor diagrams [BM08, BG16, BIMS15]. The computations of tropical refined descendant invariants have been done using floor diagrams from [BGM17] equipped with refined weights as indicated in [BS16].

A.1. \(\mathbb{T}P^1 \times \mathbb{T}P^1\).

Here we give the values of \(G_{QH}(\square_{a,b}, g)\) for a few \(a\) and \(b\). Note that \(G_{QH}(\square_{a,b}) = G_{QH}(\square_{b,a})\)

- \(a = 1:\)
  \[
  G_{QH}(\square_{1,b}, 0) = 1
  \]

- \((a, b) = (2, 0):\)
  \[
  \begin{align*}
  g = 1 & : \quad G_{QH}(\square_{2,2}, 1) = 1 \\
  g = 0 & : \quad G_{QH}(\square_{2,2}, 0; s) = q^{-1} + (10 - 2s) + q
  \end{align*}
  \]

- \((a, b) = (2, 4):\)
  \[
  \begin{align*}
  g = 3 & : \quad G_{QH}(\square_{2,4}, 3) = 1 \\
  g = 2 & : \quad G_{QH}(\square_{2,4}, 2) = 3q^{-1} + 22 + 3q \\
  g = 1 & : \quad G_{QH}(\square_{2,4}, 1) = 3q^{-2} + 36q^{-1} + 162 + 36q + 3q^2 \\
  g = 0 & : \quad G_{QH}(\square_{2,4}, 0; 0) = q^{-3} + 14q^{-2} + 95q^{-1} + 420 + 95q + 14q^2 + q^3 \\
  & \quad G_{QH}(\square_{2,4}, 0; 1) = q^{-3} + 12q^{-2} + 71q^{-1} + 280 + 71q + 12q^2 + q^3 \\
  & \quad G_{QH}(\square_{2,4}, 0; 2) = q^{-3} + 10q^{-2} + 51q^{-1} + 180 + 51q + 10q^2 + q^3 \\
  & \quad G_{QH}(\square_{2,4}, 0; 3) = q^{-3} + 8q^{-2} + 35q^{-1} + 112 + 35q + 8q^2 + q^3 \\
  & \quad G_{QH}(\square_{2,4}, 0; 4) = q^{-3} + 6q^{-2} + 23q^{-1} + 68 + 23q + 6q^2 + q^3 \\
  & \quad G_{QH}(\square_{2,4}, 0; 5) = q^{-3} + 4q^{-2} + 15q^{-1} + 40 + 15q + 4q^2 + q^3
  \end{align*}
  \]
\* \( (a, b) = (3, 3) \):

\[
\begin{align*}
g &= 4 : & G_{QH}(\square_{3,3}, 4) &= 1 \\
g &= 3 : & G_{QH}(\square_{3,3}, 3) &= 4q^{-1} + 26 + 4q \\
g &= 2 : & G_{QH}(\square_{3,3}, 2) &= 6q^{-2} + 64q^{-1} + 256 + 64q + 6q^2 \\
g &= 1 : & G_{QH}(\square_{3,3}, 1) &= 4q^{-3} + 52q^{-2} + 332q^{-1} + 1144 + 332q + 52q^2 + 4q^3 \\
g &= 0 : & G_{QH}(\square_{3,3}, 0; 0) &= q^{-4} + 14q^{-3} + 109q^{-2} + 592q^{-1} + 2078 + 592q + 109q^2 + 14q^3 + q^4 \\
& & G_{QH}(\square_{3,3}, 0; 1) &= q^{-4} + 12q^{-3} + 83q^{-2} + 404q^{-1} + 1270 + 404q + 83q^2 + 12q^3 + q^4 \\
& & G_{QH}(\square_{3,3}, 0; 2) &= q^{-4} + 10^{-3} + 61q^{-2} + 264^{-1} + 742 + 264q + 61q^2 + 10q^3 + q^4 \\
& & G_{QH}(\square_{3,3}, 0; 3) &= q^{-4} + 8q^{-3} + 43q^{-2} + 164q^{-1} + 414 + 164q + 43q^2 + 8q^3 + q^4 \\
& & G_{QH}(\square_{3,3}, 0; 4) &= q^{-4} + 6q^{-3} + 29q^{-2} + 96q^{-1} + 222 + 96q + 29q^2 + 6q^3 + q^4 \\
& & G_{QH}(\square_{3,3}, 0; 5) &= q^{-4} + 4q^{-3} + 19q^{-2} + 52q^{-1} + 118 + 52q + 19q^2 + 4q^3 + q^4 \\
\end{align*}
\]

A.2. \( T \Sigma_2 \).

Here we give the values of \( G_{\Sigma_2}(\Delta_{a,b}, g) \) for a few \( a \) and \( b \).

\* \( a = 1 \):

\[
G_{\Sigma_2}(\Delta_{1,b}, 0) = 1
\]

\* \( (a, b) = (2, 0) \):

\[
\begin{align*}
g &= 1 : & G_{\Sigma_2}(\Delta_{2,0}, 1) &= 1 \\
g &= 0 : & G_{\Sigma_2}(\Delta_{2,0}, 0; s) &= q^{-1} + (8 - 2s) + q
\end{align*}
\]

\* \( (a, b) = (2, 2) \):

\[
\begin{align*}
g &= 3 : & G_{\Sigma_2}(\Delta_{2,2}, 3) &= 1 \\
g &= 2 : & G_{\Sigma_2}(\Delta_{2,2}, 2) &= 3q^{-1} + 22 + 3q \\
g &= 1 : & G_{\Sigma_2}(\Delta_{2,2}, 1) &= 3q^{-2} + 36q^{-1} + 162 + 36q + 3q^2 \\
g &= 0 : & G_{\Sigma_2}(\Delta_{2,2}, 0; 0) &= q^{-3} + 14q^{-2} + 95q^{-1} + 416 + 95q + 14q^2 + q^3 \\
& & G_{\Sigma_2}(\Delta_{2,2}, 0; 1) &= q^{-3} + 12q^{-2} + 71q^{-1} + 276 + 71q + 12q^2 + q^3 \\
& & G_{\Sigma_2}(\Delta_{2,2}, 0; 2) &= q^{-3} + 10q^{-2} + 51q^{-1} + 176 + 51q + 10q^2 + q^3 \\
& & G_{\Sigma_2}(\Delta_{2,2}, 0; 3) &= q^{-3} + 8q^{-2} + 35q^{-1} + 108 + 35q + 8q^2 + q^3 \\
& & G_{\Sigma_2}(\Delta_{2,2}, 0; 4) &= q^{-3} + 6q^{-2} + 23q^{-1} + 64 + 23q + 6q^2 + q^3 \\
& & G_{\Sigma_2}(\Delta_{2,2}, 0; 5) &= q^{-3} + 4q^{-2} + 15q^{-1} + 36 + 15q + 4q^2 + q^3
\end{align*}
\]

\* \( (a, b) = (3, 0) \):
\( g = 4 \) : \( G_{\Sigma_2}(\triangle_{3,0}, 4) = 1 \)

\( g = 3 \) : \( G_{\Sigma_2}(\triangle_{3,0}, 3) = 4q^{-1} + 24 + 4q \)

\( g = 2 \) : \( G_{\Sigma_2}(\triangle_{3,0}, 2) = 6q^{-2} + 58q^{-1} + 212 + 58q + 6q^2 \)

\( g = 1 \) : \( G_{\Sigma_2}(\triangle_{3,0}, 1) = 4q^{-3} + 46q^{-2} + 260q^{-1} + 820 + 260q + 46q^2 + 4q^3 \)

\( g = 0 \) : \( G_{\Sigma_2}(\triangle_{3,0}, 0; 0) = q^{-4} + 12q^{-3} + 81q^{-2} + 402q^{-1} + 1240 + 402q + 81q^2 + 12q^3 + q^4 \)

\( G_{\Sigma_2}(\triangle_{3,0}, 0; 1) = q^{-4} + 10q^{-3} + 59q^{-2} + 262q^{-1} + 712 + 262q + 59q^2 + 10q^3 + q^4 \)

\( G_{\Sigma_2}(\triangle_{3,0}, 0; 2) = q^{-4} + 8q^{-3} + 41q^{-2} + 162q^{-1} + 384 + 162q + 41q^2 + 8q^3 + q^4 \)

\( G_{\Sigma_2}(\triangle_{3,0}, 0; 3) = q^{-4} + 6q^{-3} + 27q^{-2} + 94q^{-1} + 192 + 94q + 27q^2 + 6q^3 + q^4 \)

\( G_{\Sigma_2}(\triangle_{3,0}, 0; 4) = q^{-4} + 4q^{-3} + 17q^{-2} + 50q^{-1} + 88 + 50q + 17q^2 + 4q^3 + q^4 \)

\( G_{\Sigma_2}(\triangle_{3,0}, 0; 5) = q^{-4} + 2q^{-3} + 11q^{-2} + 22q^{-1} + 40 + 22q + 11q^2 + 2q^3 + q^4 \)

REFERENCES

[ABLdM11] A. Arroyo, E. Brugallé, and L. López de Medrano. Recursive formula for Welschinger invariants. Int Math Res Notices, 5:1107–1134, 2011.

[BG16] F. Block and L. Göttsche. Refined curve counting with tropical geometry. Compos. Math., 152(1):115–151, 2016.

[BGM12] F. Block, A. Gathmann, and H. Markwig. Psi-floor diagrams and a Caporaso-Harris type recursion. Israel J. Math., 191(1):405–449, 2012.

[BIMS15] E. Brugallé, I. Itenberg, G. Mikhalkin, and K. Shaw. Brief introduction to tropical geometry. In Proceedings of the Gökova Geometry-Topology Conference 2014, pages 1–75. Gökova Geometry/Topology Conference (GGT), Gökova, 2015.

[BM08] E. Brugallé and G. Mikhalkin. Floor decompositions of tropical curves : the planar case. Proceedings of 15th Gökova Geometry-Topology Conference, pages 64–90, 2008.

[BM16] E. Brugallé and H. Markwig. Deformation and tropical Hirzebruch surfaces and enumerative geometry. J. Algebraic Geom., 25(4):633–702, 2016.

[Bou17] P Bousseau. Tropical refined curve counting from higher genera and lambda classes. arXiv:1706.0776, 2017.

[BP13] E. Brugallé and N. Puignau. Behavior of Welschinger invariants under Morse simplifications. Rend. Semin. Mat. Univ. Padova, 130:147–153, 2013.

[BP15] E. Brugallé and N. Puignau. On Welschinger invariants of symplectic 4-manifolds. Comment. Math. Helv., 90(4):905–938, 2015.

[Bru15] E. Brugallé. Floor diagrams relative to a conic, and GW–W invariants of Del Pezzo surfaces. Adv. Math., 279:438–500, 2015.

[Bru18] E. Brugallé. Surgery of real symplectic fourfolds and welschinger invariants. Journal of Singularities, 17:267–294, 2018.

[BS16] L. Blechman and E. Shustin. Refined descendant invariants of toric surfaces. https://arxiv.org/abs/1602.06471, 2016.

[Che18] X Chen. Solomon’s relations for Welschinger’s invariants: Examples. arXiv:1809.08938, 2018.

[DH18] Y. Ding and J. Hu. Welschinger invariants of blow-ups of symplectic 4-manifolds. Rocky Mountain J. Math., Volume 48(4):1105–1144, 2018.

[DK00] A. I. Degtyarev and V. M. Kharlamov. Topological properties of real algebraic varieties: Rokhlin’s way. Russian Math. Surveys, 55(4):735–814, 2000.

[DK02] A. Degtyarev and V. Kharlamov. Real rational surfaces are quasi-simple. J. Reine Angew. Math., 551:87–99, 2002.

[Geo16] P. Georgieva. Open Gromov-Witten disk invariants in the presence of an anti-symplectic involution. Adv. Math., 301:116–160, 2016.

[GS14] L. Göttsche and V. Shende. Refined curve counting on complex surfaces. Geom. Topol., 18(4):2245–2307, 2014.

[GS16] L. Göttsche and F. Schroeter. Refined brocoli invariants. arXiv:1606.09631, 2016.
[GZ15] P. Georgieva and A. Zinger. Real Gromov-Witten theory in all genera and real enumerative geometry: Construction. arXiv:1504.06617, 2015.

[HS12] A. Horev and J. Solomon. The open Gromov-Witten-Welschinger theory of blowups of the projective plane. arXiv:1210.4034, 2012.

[IKS04] I. Itenberg, V. Kharlamov, and E. Shustin. Logarithmic equivalence of Welschinger and Gromov-Witten invariants. Uspehi Mat. Nauk, 59(6):85–110, 2004. (in Russian). English version: Russian Math. Surveys 59 (2004), no. 6, 1093-1116.

[IKS13] I. Itenberg, V. Kharlamov, and E. Shustin. Welschinger invariants of real del Pezzo surfaces of degree $\geq 3$. Math. Ann., 355(3):849–878, 2013.

[IKS15] I. Itenberg, V. Kharlamov, and E. Shustin. Welschinger invariants of real del Pezzo surfaces of degree $\geq 2$. Internat. J. Math., 26(8):1550060, 63, 2015.

[IKS17] I. Itenberg, V. Kharlamov, and E. Shustin. Welschinger invariants revisited. In Analysis meets geometry, Trends Math., pages 239–260. Birkhäuser/Springer, Cham, 2017.

[IM13] I. Itenberg and G. Mikhalkin. On Block-Göttsche multiplicities for planar tropical curves. Int. Math. Res. Not. IMRN, (23):5289–5320, 2013.

[IP04] E.-N Ionel and T. H. Parker. The symplectic sum formula for Gromov-Witten invariants. Ann. of Math., 159(2):935–1025, 2004.

[Li02] J. Li. A degeneration formula of GW-invariants. J. Differential Geom., 60(2):199–293, 2002.

[Li04] J. Li. Lecture notes on relative GW-invariants. In Intersection theory and moduli, ICTP Lect. Notes, XIX, pages 41–96 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[LR01] A. Li and Y. Ruan. Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds. Invent. Math., 145(1):151–218, 2001.

[Mik05] G. Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. J. Amer. Math. Soc., 18(2):313–377, 2005.

[Mik17] G. Mikhalkin. Quantum indices of real plane curves and refined enumerative geometry. Acta Math., 219(1):135–180, 2017.

[NPS18] J. Nicaise, S. Payne, and F. Schroeter. Tropical refined curve counting via motivic integration. Geom. Topol., 22(6):3175–3234, 2018.

[Shu14] E. Shustin. On higher genus Welschinger invariants of del pezzo surfaces. Intern. Math. Res. Notices, 2014. doi: 10.1093/imrn/rnu148.

[TZ14] M.F. Tehrani and A. Zinger. On symplectic sum formulas in Gromov-Witten theory. arXiv:1404.1898, 2014.

[Wel05] J. Y. Welschinger. Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry. Invent. Math., 162(1):195–234, 2005.

[Wel07] J. Y. Welschinger. Optimalité, congruences et calculs d’invariants des variétés symplectiques réelles de dimension quatre. arXiv:0707.4317, 2007.

[Wel15] J.-Y. Welschinger. Open Gromov-Witten invariants in dimension four. J. Symplectic Geom., 13(4):1075–1100, 2015.