Homogeneous and Non Homogeneous Ordinary Differential Equations with the Second Order

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Since Newton time, Differential Equations have been used to comprehend most of physical, geometrical, and vital science in addition to their participation in the study of arithmetic analysis. Furthermore, they have been used in economical and social aspects. They have been developed and got significant position in various sciences.

This paper presents for the non-homogeneous ordinary differential equations with the second order. This idea starts in chapter one which talks about the notion of those equations, their orders, in addition to the study of the linear differential equation with the first order and its solution. Whereas the second chapter studies the ordinary differential equations (homogeneous and non-homogeneous) and their solution. Each chapter is supported by solved examples that cover most of the aspects of the topic and we hope that this paper will make a good reference for those who would like to study this topic furtherly.

1. Basic Concepts

1.1 Definition (1)

Differential Equation: a relation between the relevant variable and the independent one that includes derivatives or differentials.

Examples: let x be the independent variable, y is the relevant one, so the following relations represent ordinary differential equations:

\[ 1 - \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dx}{dy} + y = 0 \]
Types of Differential Equations

Ordinary Differential Equation: is the differential equation based on one independent variable only.

Partial Differential Equation: is the differential equation based on two or more independent variables.

Example: let $U$ be the relevant variable, $x, y,$ and $z$ are independent ones, so the following relations represent partial differential equations:

1) $\frac{\partial U}{\partial x} + 3 \frac{\partial U}{\partial y} = 0$
2) $\frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial U}{\partial y} = x$
3) $x^2 \frac{\partial^2 U}{\partial x \partial y} + 3y \frac{\partial y}{\partial x} + (x - y^2)U = 0$

1.2 Definition (1)

Differential Equation Order: when the $n$th derivative $y_n$ is a higher derivative which appears in the ordinary differential equation, then it is an equation of $n$ order, i.e the differential equation order is determined by its higher order.

Example: the differential equation: $\left( \frac{d^2 y}{dx^2} \right)^3 - 2 \frac{dy}{dx} + 8 = x^3 + \cos x$ is an ordinary differential equation of the third order.

1.3 Definition (1)

Differential Equation Degree: is the power of a higher derivative that appears in the differential equation. Before determining the equation degree, it needs to be put in a regular and correct form based on its derivatives.

Example: the differential equation: $\left( \frac{d^2 y}{dx^2} \right)^3 - 2 \frac{dy}{dx} + 8 = x^3 + \cos x$ is an ordinary differential equation of a third order and third degree. Before determining the equation degree, it needs to be put in a regular and correct form based on its derivatives.

Before determining any equation degree, it needs to be put in a form free of roots. For instance, the following differential equation:

$$(y')^3 = 3 \sqrt{4y + (y'')^2}$$
so:

\[(y')^9 = 4y + (y')^2\]

This is a second order and degree differential equation.

1.4 **Solving Differential Equation (2):**

any given relation among the differential equation variables which is:

1. free of derivatives
2. defined on a specific period
3. achieve the differential equation

**Example 1**: prove that \(y = x \ln|x| - x\) is one of the following equation solutions:

\[x \frac{dy}{dx} = x + y, x > 0 \quad \ldots \ldots \text{ (1)}\]

This is an equation free of derivatives and defined on a specific period of the differential equation itself \(x > 0\), it also achieves it by applying directly:

\[
LHS = x \frac{dy}{dx} = x\left(\frac{x}{x} + \ln|x|(1) - 1\right) \\
= x \cdot (1 + \ln|x| - 1) = x \ln|x| \\
RHS = x + y = x + x \ln|x| - x = x \ln|x| \\
∴ LHS = RHS
\]

1) **Differential Equations Solutions (2):**

1. General Solution (The relevant): is the solution that contains a number of optional constants equal to the equation order.

2. Special Solution: is the solution resulted from the previous one after applying certain numeric values on the optional constants.

**Example**: solve the following differential equation:

\[2y' = 3y^\frac{1}{2}\]

Solution

\[2y' = 3y^\frac{1}{2}\]

\[\rightarrow 2 \frac{dy}{dx} = 3y^\frac{1}{2}\]
2\,dy = 3y^\frac{1}{3}dx

2\frac{dy}{y^\frac{1}{3}} = 3dx \rightarrow \int 2y^{-\frac{1}{3}} \, dy = \int 3dx

\rightarrow 2\frac{y^{\frac{2}{3}}}{\frac{2}{3}} = 3x + c

\rightarrow 3y^{\frac{2}{3}} = 3x + c

y^{\frac{2}{3}} = x + \frac{c}{3}

y^{\frac{2}{3}} = x + c_1

\therefore \text{ We assume } c_1 = \frac{c}{3}

y^2 = (x + c_1)^3 \text{ general solution}

If we assume that the given point \((1, 2\sqrt{2})\)

\((2\sqrt{2})^2 = (1 + c_1)^3\)

8 = (1 + c_1)^3

2 = 1 + c_1 \rightarrow c_1 = 1

\therefore y^2 = (x + 1)^3 \text{ special solution}

1.6 Linear Differential Equation of the first order (3)

It takes the following form:

\[ \frac{dy}{dx} = p(x)y = Q(x) \]

It can be solve by the use of the following form:

\[ y\mu = \int \mu Q(x)dx + c \]

where \( \mu \) is the integrated factor and is defined as follows:
\[
\mu = e^{\int p(x) dx}
\]

**Example:** solve the following equation:

\[
x \frac{dy}{dx} + 2y = x^2
\]

**Solution:**

It is a linear equation in \( y \). We put it in this form:

\[
\frac{dy}{dx} + P(x)y = Q(x) \quad \ldots \ldots \ldots (1)
\]

\[
\frac{dy}{dx} + \frac{2}{x}y = x^2 \quad \ldots \ldots \ldots (2)
\]

that is:

By comparing (1) and (2) we find that:

\[
P(x) = \frac{2}{x}, \quad Q(x) = x^2
\]

so the integrated factor \( \mu \) of this problem can be found as follows:

\[
\mu = e^{\int P(x) dx} = e^{\ln x^2} = x^2
\]

and the solution is as follows:

\[
yx^2 = \int x^2 x^2 dx + c
\]

\[
yx^2 = \int x^4 dx + c
\]

\[
yx^2 = \frac{1}{5}x^5 + c
\]

\[
y = \frac{1}{5}x^3 + cx^{-2}
\]

2. Homogeneous and Non Homogeneous Ordinary Differential Equations with the Second Order

2.1 Linear Differential Equation of the second order (6)

It is written in its general form as follows:
\[ a_0 y'' + a_1 y' + a_2 y = f(x) \] \quad (1)

Where \(a_0 \neq 0\)

if all coefficients \(a_0, a_1, a_2\) are constant values, the equation then is linear with constant coefficients. But if one of the coefficients a function in \(x\), it becomes an equation of variable coefficients. Thus it will be:

\[ a_0 y'' + a_1 y' + a_2 y = 0 \] \quad (2)

is a linear homogeneous, \(f(x)=0\) in the equation \(1\)

**Note:** if \(f(x)\neq0\), equation \(1\) is a non-homogeneous linear.

**Definition: Differential effect \(D\):** we define \(D = \frac{d}{dx}\) the first derivative according to \(x\). also \(D^2 = \frac{d^2}{dx^2}\) the second derivative according to \(x\).

**Example:**

1) \(D e^{3x} = \frac{d}{dx} e^{3x} = 3e^{3x}\)

2) \(D^2 e^{3x} = \frac{d^2}{dx^2} e^{3x} = 9e^{3x}\)

**Some \(D\) features:**

1) \(D[f_1(x) \pm f_2(x)] = Df_1(x) \pm Df_2(x)\)

2) \(D[kf(x)] = kDf(x)\)

\(F\) is a multi limits in 3) \(F(D)e^{ax} = F(a)e^{ax}\)

Based on what we had before, equation \(1\) is written in a form by the use of \(D\) effect:

\[ a_0 D^2 + a_1 D + a_2 y = f(x) \]

we find that

\(F(D) = a_0 D^2 + a_1 D + a_2 y\)

is a multi limit function in \(D\) and of the second degree

so, equation \(1\) becomes as follows
\[ F(D)y = f(x) \]

and it is a non-homogeneous linear equation, whereas the equation

\[ F(D)y = 0 \]

is a homogeneous linear equation.

Now we tend to present for some of the basic features of the solutions of the homogeneous equation of the second order.

### 2.2 Features of the solutions of the homogenous linear differential equations of the second order (4):

We assume that this type of equation take the following form:

\[ y'' + a_1y' + a_2y = 0 \quad \ldots \ldots \ldots (1) \]

**Theory:** if \( y_1 \) and \( y_2 \) are a solution for the equation (1) each, then \( y = c_1y_1 + c_2y_2 \) is also a solution for the same equation (1) in the sense that \( c_1 \) and \( c_2 \) constants.

**Definition:**

1. Both \( y_1, y_2 \) of equation (1) are linearly independent if \( \frac{y_2}{y_1} \neq c \) (constant):
   \[ y_2 \neq cy_1 \]

2. Both \( y_1, y_2 \) of equation (1) are linearly connected if \( \frac{y_2}{y_1} = c \) (constant):
   \[ y_2 \neq cy_1 \]

**Definition (Wronskian):** if \( y_1(x), y_2(x) \) are derivable functions in their definition level, so their wronskian is defined as follows:

\[ W\{y_1(x), y_2(x)\} = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} \]

**Definition:**

1. \( y_1, y_2 \) of equation (1) are linearly independent if and only if \( W\{y_1, y_2\} \neq 0 \)

2. \( y_1, y_2 \) of equation (1) are linearly connected if and only if \( W\{y_1, y_2\} = 0 \)
Example: search for the connection and independence of each group of the following functions:

1) $e^x, e^{-x}$, 2) $e^x, 2e^x, x$

**Solution:**

1) $W\{e^x, e^{-x}\} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$

∴ Thus, the two functions are linearly independent.

2) $W\{e^x, 2e^x, x\} = \begin{vmatrix} e^x & 2e^x & x \\ e^x & 2e^x & 1 \\ e^x & 2e^x & 0 \end{vmatrix} = 0$

Thus the functions are linearly connected.

**Definition: General Solution:** if $y_1, y_2$ are independent solutions of the equation (1), then $y = c_1y_1 + c_2y_2$ represents the general solution of equation (1) in the sense that $c_1, c_2$ are optionally constant.

2.3 Finding the general solution of the homogeneous linear differential equation of the second order with constant coefficients. (4)

We assume that the equation

$$y'' + a_1y' + a_2y = 0$$

$a_1, a_2$ are constant

To find the general solution of that equation, we tend to find two linear independent solutions

also we try to use $y = e^{\lambda x}$

as a solution of the equation (1) in the sense that $\lambda$ is a constant value.

We put the equation in the form:

$$(D^2 + a_1D + a_2)y = 0$$

then we apply the assumed solution:

$$Dy = De^{\lambda x} = \lambda e^{\lambda x}, D^2y = D^2e^{\lambda x} = \lambda^2 e^{\lambda x}$$
We get the equation

\[(\lambda^2 + a_1\lambda + a_2)e^{\lambda x} = 0\]

in the sense that \(e^{\lambda x} \neq 0\) we find:

\[\lambda^2 + a_1\lambda + a_2 = 0\]

This equation is called the special equation (helping) and we can get it directly from the original differential equation by the effect \(D\) by putting \(\lambda\) instead of \(D\). It is a square equation of the second degree in \(\lambda\), so it has two roots where:

\[\lambda_1, \lambda_2 = -a \pm \sqrt{a_1^2 - 4a_2}\]

These roots have three cases:

1) different and real \(\lambda_1 \neq \lambda_2\)
2) real and equal \(\lambda_1 = \lambda_2\)
3) composite

we tend to study each case independently.

The Special Equation Roots are real and different: \(\lambda_1 \neq \lambda_2\) so \(\lambda_1 = e^{\lambda_1 x}, \lambda_2 = e^{\lambda_2 x}\). Thus, the general solution is: \(y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\) \(c_1, c_2\) are optionally constant.

Example: find the general solution of the equation:

\[y'' + 3y' - 4y = 0\]

Solution

We put the equation in a form: \((D^2 + 3D - 4)y = 0\)

\[D \frac{d}{dx}\]

we assume that \(y = e^{\lambda x}\) is a solution of the equation, so the helping equation is:

\[\lambda^2 + 3\lambda - 4 = 0\]

\[(\lambda + 4)(\lambda - 1) = 0\]

That is \(\lambda = -4, \lambda = 1\)
is the general solution. \( y = c_1 e^{-4x} + c_2 e^x \)

2) **The Special Equation Roots are real and equal:** \( \lambda_1 = \lambda_2 \) so \( y_1 = e^{\lambda_1 x} \) is the first solution and connected with \( y_2 = e^{\lambda_2 x} \). So, we tend to find another solution \( y_2 \) which not connected with \( y_1 = e^{\lambda_1 x} \). It was clear that \( y_2 = e^{\lambda_2 x} \) represents a solution for the equation and it is not connected with the first one \( y_1 \). Thus, the general solution of the equation is:

\[
y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_2 x}
\]

Example:

If find the general solution of the equation \( y'' - 4y' + 4y = 0 \)

Solution:

we put the equation in a form: \( (D^2 - 4D + 4)y = 0 \)

we assume that \( y = e^{\lambda x} \) is

\[
\therefore \text{the helping equation is} \quad \lambda^2 - 4\lambda + 4 = 0
\]

\[
(\lambda - 2)^2 = 0
\]

and it has two roots: \( \lambda = 2, 2 \)

so the general solution is \( y = (c_1 + c_2x)e^{2x} \)

3) **The two roots of the helping equation are composite:**

If one of the equation roots is a composite number \( \lambda_1 = \alpha + i\beta \) in the sense that \( i = \sqrt{-1} \), so the other root \( \lambda_2 \) is in the form \( \lambda_2 = \alpha - i\beta \) (the attached root) \( \beta \neq 0 \). From that, \( \lambda_1 \neq \lambda_2 \) and the general solution is:

\[
y = A_1 e^{(\alpha + i\beta)x} + A_2 e^{(\alpha - i\beta)x} .......... .......... (1)
\]

\( A_1, A_2 \) are optionally constant

This can be proved in the form:
\[ y = e^{ax}[c_1 \cos \beta x + c_2 \sin \beta x] \]

we put (1) in a form:

\[ y = A_1 e^{ax} e^{i\beta x} + e^{ax} e^{ax} e^{-i\beta x} = e^{ax} [A_1 e^{i\beta x} + A_2 e^{-i\beta x}] \]

we know that:

\[ e^{-i\beta x} = \cos \beta x - i \sin \beta x, e^{i\beta x} = \cos \beta x + i \sin \beta x \]

the solution will be:

\[ y = e^{ax} [A_1 \cos \beta x + i \sin \beta x] + A_2 (\cos \beta x - i \sin \beta x) \]

\[ = e^{ax} [(A_1 + A_2) \cos \beta x + i(A_1 - A_2) \sin \beta x] \]

we consider:

\[ (A_1 + A_2) = C_1, \quad i(A_1 - A_2) = C_2 \]

the solution will be:

\[ y = e^{ax} [C_1 \cos \beta x + C_2 \sin \beta x] \]

**Example**: find the general solution of the equation: \( y'' + 2y' + 5y = 0 \)

Solution:

we put the equation in a form: \((D^2 + 2D + 4)y = 0\)

we assume that \( y = e^{\lambda x} \) is a solution of the given equation,

thus the helping equation is:

\[ \lambda^2 + 2\lambda + 5 = 0 \]

\[ \therefore \lambda = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i \]

And the general solution of the equation is:

\[ y = e^{-x}[c_1 \cos 2x + c_2 \sin 2x] \]

### 2.4 Non Homogeneous Linear Differential Equations with the Second Order (5)

The general form of this type of equations is
\[ y'' + a_1 y' + a_2 y = f(x) \]

If \( y_p \) is a special solution of the above equation, so the solution of the above homogeneous equation \((f(x)=0)\) is \( y_c \). Whereas the special solution can be got from several ways. After that we have the general solution of the equation as follows: \( y = y_c + y_p \).

The main ways are:

1 ) non determined coefficients way
2 ) constants change way
3) effect way

2.5 non determined coefficients way (5)

A ) if \( f(x) \) is multi limits: the way to find the special solution of the non-homogeneous linear differential equation is by assuming that this special solution is a multi limits whose degree is equal to that of \( f(x) \). Then we tend to find coefficients of multi limits as it is shown in the following example:

Example:

The differential equation:

\[ 3y'' - 5y' - 2y = 6x^2 - 7 \]

It represents a non-homogeneous linear differential equation, i.e \( f(x) \) is a multi limits and can be solved as follows:

\[ 3y'' - 5y' - 2y = 0 \]

then we write the special equation:

\[ 3\lambda^2 - 5\lambda - 2 = 0 \]

\[ (3 \lambda + 1)(\lambda - 2) = 0 \]

\[ \lambda = \frac{1}{3}, \quad \lambda = 2 \]

the complement function is:

\[ y_c = c_1 e^{2x} + c_2 e^{-\frac{x}{3}} \]
whereas the special solution $f(x) = 6x^2 - 7$

is a multi limits of the second order. We tend to assume a multi limits of the second order to that solution $y = ax^2 + bx + c$ and to find the coefficients $a, b, c$ we need three equations which we can get by applying $y''$, $y'$, $y$ in the original equation. So we tend to find $y''$, $y'$ first from the hypothesis as it is $y' = 2ax + b$          $y'' = 2a$ , and by applying $y''$, $y'$, $y$ in the differential equation, we get:

$$3(2a) - 5(2ax + b) - 2(ax^2 + bx + c) = 6x^2 - 7$$

And it is true for all $x$ values. To find $a, b, c$, we tend to equalize $x$ power coefficients as it follows:

Coefficient $-2a = 6: x^2$ from it we get $a = -3$

Coefficient: $-10a - 2b = 0$ from it we get $b = 15$

The empty limit of $x$: $6a - 5b - 2c = -7$ form it we get $c = -43$

So the special solution is

$$y_p = -3x^2 + 5x - 43$$

whereas the general solution is:

$$y = y_c + y_p$$

$$y = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x} - 3x^2 + 5x - 43$$

**B ) if** $f(X) = be^{ax}$ **to find the special solution we need to find the complement function by finding the special equation of the homogeneous equation.**

$F(D)y = 0$

If $a$ is not a root of the special equation, then we assume that the special solution of the non-homogeneous equation $F(D)y = f(x)$ is represented in the following hypothesis: $y = Ae^{ax}$, $A$ is the unknown element which could be found by applying $y$ and its derivatives in the given differential equation.

- But if $a$ is one of the roots of the special equation and it is not repeated, we assume the following hypothesis to be the solution:
\[ y = AXe^{ax} \]

- If \( a \) is one of the roots of the mentioned special equation and it is repeated by \( n \) times, we assume the solution to be the following hypothesis:

\[ y = Ax^n e^{ax} \]

**Example:** the following differential equation \( e^{2x} y'' - 5y' - 2y = 5 \)

represents a non-homogeneous linear differential equation, \( f(x) = be^{ax} \) and could be solved:

We note that \( a=2 \) is one of the two roots of the special equation of the homogenous one, so we assume the solution in the function:

\[ y = Ax e^{2x} \]

\[ y' = Ae^{2x} + 2Ax e^{2x} \]

\[ y'' = 2Ae^{2x} + 2Ae^{2x} + 4Ae^{2x} \]

By applying \( y, y', y'' \) in the differential equation we find that:

\[ (12 - 10 - 2)Ae^{2x} + (12 - 5)Ae^{2x} = 5e^{2x} \]

\[ 0 + 7Ae^{2x} = e^{2x} \]

\[ A = \frac{5}{7} \]

So the special solution is \( y_p = \frac{5}{7}xe^{2x} \)

And the general solution is \( y = y_c + y_p \)

\[ y = c_1 e^{2x} + c_2 e^{-1x} + \frac{5}{7}xe^{2x} \]

C ) if \( f(x) = b \cos ax \)

Or \( f(x) = b \sin ax \)

to find the special solution of the non-homogeneous equation in such the case we assume:

\[ y = A \cos ax + B \sin ax \]

\( A, B \) are unknown.
If $\lambda = a_i$ is not the two roots of the special equation, we use the differential equation instead of $y$ and its derivatives, to find $A,B$.

If $\lambda = a_i$ is one of the roots of the special equation and it is not repeated, we assume that the special solution is the function:

$$y = x(A \cos ax + B \sin ax)$$

If the roots are repeated by $n$ times, we assume the solution to be the function

$$y = x^n(A \cos ax + B \sin ax)$$

**Example:** The following differential equation:

$$3y'' - 5y' - 2y = 4 \sin 2x$$

is a non-homogeneous linear differential equation $f(x) = b \sin ax$, and it could be solved by:

This special equation does not have loop roots, so we assume the solution to be the function

$$y = A \cos 2x + B \sin 2x$$

then we find:

$$y' = -2A \sin 2x + 2B \cos 2x$$

$$y'' = -4A \cos 2x - 4B \sin 2x$$

then we apply about $y, y', y''$ in the differential equation and find that:

$$(-12A - 10B - 2A) \cos 2x + (-12B + 10A - 2B) \sin 2x$$

$$= 0 \cos 2x + 4 \sin 2x$$

And by equalizing the coefficients of $\sin 2x, \cos 2x$, we get the two equations:

$-14A - 10B = 0$

$10A - 14B = 4$

From which we find $28A = 5, 28B = -7$

$$A = \frac{5}{37}, B = \frac{-7}{37}$$
The special solution is \( \frac{7}{37} \sin 2x \cos 2x - y_p = \frac{5}{37} \)

while the general solution is

\[
y = y_c + y_p \\
y = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x} + \frac{5}{37} \cos 2x - \frac{7}{37} \sin 2x
\]

2.6 Constants Change way (5)

This way can be used to find the special solution for any non-homogeneous linear differential equation. This way is preferable when \( f(x) \) is not one of the three cases mentioned previously. The main feature of this is the change of the optional constants of the complement function into some functions.

To find the special solution of the differential equation of the second order:

\[
\frac{d^2x}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x)
\]

we first find the complement function by writing:

\[(D^2 + a_1 D + a_2)y = 0\]

Let \( y = c_1 y_1 + c_2 y_2 \) is the needed complement function, then we tend to find two functions \( c_1, c_2 \):

\[
y = c_1 y_1 + c_2 y_2
\]

which is the special solution of the differential equation

This needs finding two equations by \( c_1, c_2 \) and by direct applying of \( y, y', y'' \) in the differential equation

we start with the definition and finding the second equation first

When:

\[
y = c_1 y_1 + c_2 y_2 \\
y' = c_1 y'_1 + c_1' y_1 + c_2 y'_2 + c_2' y_2
\]

The second condition is the hypothesis \( c_1 y_1 + c_2 y_2 = 0 \)

\[
c_1' y_1 + c_2' y_2 = 0 \ldots \ldots (1)
\]
To find the second equation, we write

\[ y' = c_1 y_1' + c_2 y_2 \]

\[ y'' = c_1 y_1'' + c_1' y_1' + c_2 y_2'' + c_2' y_2' \]

And by applying \( y, y', y'' \) in the differential equation, we get:

\[ (c_1 y_1'' + c_1' y_1' + c_2 y_2'') + a_1 (c_1 y_1' + c_2 y_2) + a_2 (c_1 y_1 + c_2 y_2) = f(x) \]

\[ c_1 (y_1'' + a_1 y_1' + a_2 y_1) + c_2 (y_2'' + a_1 y_2' + a_2 y_2) + (c_1' y_1' + c_2' y_2') = f(x) \]

\[ 0 + 0 + (c_1' y_1' + c_2' y_2') = f(x) \]

\[ c_1' y_1' + c_2' y_2' = f(x) \quad \ldots \quad (2) \]

Then we find \( c_1, c_2 \) by solving the two equations:

\[ c_1' y_1' + c_2' y_2 = 0 \]

\[ c_1' y_1' + c_2' y_2' = f(x) \]

\[ c_1' = -\frac{y_2}{y_1 y_2' - y_2 y_1} f(x) \quad , \quad c_2' = \frac{y_1}{y_1 y_2' - y_2 y_1} f(x) \]

Then we find \( c_1, c_2 \) by one integration \( 'c_1', c_2' \) as it is shown in the following example:

**Example:**

the following differential equation

\[ (D^2 + 4)y = \sec 2x \]

is a non-homogeneous linear differential equation which can be solved as it follows:

The complement function

\[ y = c_1 \sin 2x + c_2 \cos 2x \]

We assume the special solution of the equation is \( y = c_1 \sin 2x + c_2 \cos 2x \)

The first condition is:

\[ c_1' \sin 2x + c_2' \cos 2x = 0 \]
The second condition is

$$2c_1' \cos 2x - 2c_2' \sin 2x = \sec 2x$$

$$c_1' = \frac{-\cos 2x}{-2 \sin 2x \sin 2x - 2 \cos 2x \cos 2x} \sec x = \frac{1}{2}$$

$$c_1' = \frac{x}{2}$$

$$c_2' = \frac{\sin 2x}{-2 \sin 2x \sin 2x - 2 \cos 2x \cos 2x} \sec 2x = -\frac{1}{2} \tan 2x$$

$$c_2 = \frac{1}{4} \ln \cos 2x$$

The special solution is

$$y_p = \frac{x}{2} \sin 2x + \frac{1}{4} (\ln \cos 2x) \cos 2x$$

The general solution is:

$$y = y_c + y_p$$

$$y = c_1 \sin 2x + c_2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} (\ln \cos 2x) \cos 2x$$

2.7 The Effect Way (7):

We explained in the previous sections that the non-homogeneous linear differential equations of the constant coefficients is written by the effect way as $F(D)y = f(x)$. so the special solution of this equation is:

$$y = \frac{1}{F(D)}f(x)$$

in the sense that

$$F(D) = D^2 + a_1 D + a_2$$

To find the special solution of the non-homogeneous equation, we need to deal with types of functions that is taken by $f(x)$. we have the following cases:

**First Case: if** $f(x) = e^{bx}$
If \( f(b) \neq 0 \) so the special solution of the equation \( F(D)y = e^{bx} \) is:

\[
y = \frac{1}{F(D)} e^{bx} = \frac{1}{f(b)} e^{bx}
\]

Example: The following differential equation:

\[
y'' - 5y' + 6 = e^x
\]

represents a non-homogeneous linear differential equation that can be solved by the following way:

The special function of the homogeneous equation by the effect is:

\[
D^2 - 5D + 6 = 0
\]

\[
(D - 3)(D - 2) = 0
\]

\[
D = 2,3
\]

So the complement function is:

\[
y_c = c_1 e^{2x} + c_2 e^{3x}
\]

And because

\[
F(D) = D^2 - 5D + 6
\]

\[
f(b) = f(1) = 1 - 5 + 6 = 2 \neq 0
\]

So the special solution is:

\[
y_p = \frac{1}{f(1)} e^x = \frac{1}{2}
\]

\[
y = y_c + y_p
\]

\[
y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^x
\]

Note: if \( f(x) = 0 \) which means \( F(D) \) includes a factor of the type \((D - b)^r\) in the sense that \( r \) is a true positive number, so the special solution is:

\[
y = \frac{1}{g(b)} e^{bx} \frac{x^r}{r!}
\]

In the sense that \( g(b) \neq 0 \), \( F(D) = (D - b)^r g(D) \)

Example: the following differential equation
\[ y'' - 5y' + 6 = e^{2x} \]

is a non-homogeneous linear differential equation that can be solved by:

The special function of the homogeneous equation by the effect way is:

\[ D^2 - 5D + 6 = 0 \]

\[ (D - 3)(D - 2) = 0 \]

\[ D = 2, 3 \]

So the complement function is \( y_c = c_1 e^{2x} + c_2 e^{3x} \)

The special solution is

\[ f(D) = D^2 - 5D + 6 \]

\[ f(b) = f(2) = 4 - 10 + 6 = 0 \]

\[ y_p = \frac{1}{f(b)} e^{bx} = \frac{1}{0} e^{2x} \]

\[ F(D) = (D - 6)(D + 1) \]

\[ F(D) = (D - b)^r g(D) \]

in the sense that \( r=2, b=2 \)

\[ g(D) = (D + 1) \]

\[ g(b) = g(2) = (2 + 1) = 3 \]

\[ y_p = \frac{1}{g(b)} \frac{x^r}{r!} e^{bx} \]

\[ = \frac{1}{3} \frac{x}{r!} e^{2x} \]

\[ = \frac{1}{3} x e^{2x} \]

the general solution is:
\[
y = y_c + y_p
\]

\[
y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{3} xe^{2x}
\]

**Second case: if \( f(x) = \cos bx, \ f(x) = \sin bx \)**

To find the special solution in this case we can follow one of two ways. Firstly, we put \( e^{ibx} = \cos bx + i \sin bx \) then we find the special solution of the function \( e^{ibx} \) by the use of the first way, next we find the special solution of the function \( \cos bx \) since it is the main part of that special solution of the power function \( e^{ibx} \) and the special solution of the function \( \sin bx \) since it is the assumed part of the same power function.

By applying

\[
(-b^2) \text{ في } D^2 \text{ و عن } D^3 \text{ في } (-b^2 D) \text{ في } F(D), \ i.e \text{ it could be written as:}
\]

\[
\frac{1}{D^2 + a^2} \cos bx = \frac{1}{-b^2 + a^2} \cos bx
\]

\[
\frac{1}{D^3 + a^3} \cos bx = \frac{1}{a^3 - b^2 D} \cos bx
\]

\[
\frac{a^3 + b^2 D}{a^6 - b^4 D^2} \cos bx
\]

\[
\frac{a^3 + b^2 D}{a^6 + b^6} \cos bx
\]

\[
\frac{1}{a^6 + b^6} (a^3 \cos bx - b^3 \sin bx)
\]

**Example:**

The following differential equation:

\[
y'' + 4y = 15 \cos 3x
\]

represents a non-homogeneous linear differential equation that can be solved as such:
The special equation is:

\[ D^2 + 4 = 0 \]

\[ D = \pm 2i \]

The complement function is:

\[ y_c = c_1 \cos 2x + c_2 \sin 2x \]

The special solution is:

\[ 15 \cos 3x y_p = \frac{1}{D^2 + 4} \]

\[ = \frac{1}{-(3)^2 + 4} 15 \cos 3x \]

\[ = -3 \cos 3x \]

The general solution is:

\[ y = y_c + y_p \]

\[ y = c_1 \cos 2x + c_2 \sin 2x - 3 \cos 3x \]

**Third Case: if f(x) is a multi limit in x:**

To find the special solution of the function \( F(D)y = x^m \), where \( m \) is a true positive number, we write \( \frac{1}{F(D)} \) with an ascending power to \( D \) which include \( D^m \) for instance, if \( F(D) = 1 - D \), \( F(D) = 1 + D \) we write:

\[ \frac{1}{F(D)} = \frac{1}{1 - D} = (1 - D)^{-1} = 1 + D + D^2 + \cdots + D^m \]

\[ \frac{1}{F(D)} = \frac{1}{1 + D} = (1 + D)^{-1} = 1 - D + D^2 - D^3 + \cdots \]

When \( x^m, D^{m+r} \) then the special solution is:

\[ y_p = \frac{1}{F(D)} x^m = (1 + D + D^2 + \cdots + D^m)x^m \]

\[ = x^m + mx^{m-1} + m(m - 1)x^{m-2} + \cdots + m! \]

**Example:** the following differential equation: \( y''4y = 8x^3 \)
represents a non-homogeneous linear differential equation that can be solved as such:

The special solution is:

\[ D^2 + 4 = 0 \]
\[ D = \pm 2i \]

The complement function is: \( y_c = c_1 \cos 2x + c_2 \sin 2x \)

The special solution is:

\[ y_c = \frac{8}{4 + D^2} x^3 \]
\[ = \frac{8}{4(1 + \frac{D^2}{4})} x^3 \]
\[ = 2 \left[ 1 - \frac{D^2}{4} + \left( \frac{D^2}{4} \right)^2 - \cdots \right] x^3 \]
\[ = 2 \left( x^3 - \frac{6x}{4} \right) = 2x^3 - 3x \]

The general solution is:

\[ y = y_c + y_p \]
\[ y = c_1 \cos 2x + c_2 \sin 2x + 2x^3 - 3x \]

**Fourth Case:** if \( f(x) = e^{bx} v(x) \) where \( v(x) \) is one of \( x^m \) or \( \sin bx \) or \( \cos bx \) functions. The special solution is:

\[ y_p = \frac{1}{F(D)} e^{bx} v(x) \]
\[ = e^{bx} \frac{1}{F(D + b)} v(x) \]

to get \( F(D + b) \) we apply \( D \) in \( D + b \) in an effect with multi limits \( F(D) \).

**Example:** the following differential equation:
$$(D^2 + 2D + 5)y = xe^x$$

$x e^x$ represents a non-homogeneous linear differential equation that can be solved as such:

The special function is:

$$D^2 + 2D + 5 = 0$$
$$D = -1 \pm 2i$$

The complement function is:

$$y_c = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

The special solution is:

$$y_c = \frac{1}{D^2 + 2D + 5}xe^x$$

$$= e^x \frac{1}{(D + 1)^2 + 2(D + 1) + 5}x$$

$$= e^x \frac{1}{D^2 + 4D + 8}x$$

$$= e^x \frac{1}{8(1 + \frac{D}{2} + \frac{D^2}{8})}x$$

$$= \frac{1}{8} e^x \left(1 - \frac{D}{2} + \frac{D^2}{8} - \cdots \right)x$$

$$= \frac{1}{8} e^x \left(1 - \frac{1}{2} D + \cdots \right)x$$

$$= \frac{1}{8} e^x \left(x - \frac{1}{2} \right)$$

The general solution is:

$$y = y_c + y_p$$

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{8} e^x \left(x - \frac{1}{2} \right)$$
Fifth Case: if \( f(x) = x^n v(x) \) where \( v(x) \) is one of \( \cos bx \) or \( \cos bx \) functions.

To find the special solution of that case, we show the following example:

Example: the following differential equation:

\[
(D^2 + 1)y = x^2 \sin 2x
\]

represents a non-homogeneous linear differential equation that can be solved as such:

The special function is:

\[
D^2 + 1 = 0
\]

\[
D = \pm i
\]

The complement function is

\[
y_c = (c_1 \cos 2x + c_2 \sin 2x)
\]

The private solution is:

\[
y_p = \frac{1}{D^2 + 1} x^2 \sin 2x
\]

= it equals the assumed part of the special solution \( \frac{1}{D^2 + 1} x^2 e^{2ix} \)

To find it we follow this way:

\[
\frac{1}{D^2 + 1} x^2 e^{2ix} = e^{2ix} \frac{1}{(D + 2i)^2 + 1} x^2
\]

\[
= e^{2ix} \frac{1}{D^2 + 4iD - 3} x^2
\]

\[
= e^{2ix} \frac{1}{-3(1 - \frac{4i}{3} D - \frac{D^2}{3})} x^2
\]

\[
= \frac{1}{3} e^{2ix} \left[ 1 + \frac{4i}{3} D + \frac{D^2}{3} + \left( \frac{4i}{3} D + \frac{D^2}{3} \right)^2 + \cdots \right] x^2
\]

\[
= \frac{1}{3} e^{2ix} \left( 1 + \frac{4i}{3} D + \frac{D^2}{3} + \frac{-16D^2}{9} \right) x^2
\]
\[ = -\frac{1}{3} e^{2ix} \left( x^2 + \frac{8}{3}i - \frac{26}{9} \right) \]
\[ = -\frac{1}{3} \left( \cos 2x + i \sin 2x \right) \left[ \left( x^2 - \frac{26}{9} \right) + \frac{8}{3}i \right] \]

The private solution of this function is:
\[ y_p = -\frac{1}{3} \left[ \left( x^2 - \frac{26}{9} \right) \sin 2x + \frac{8}{3} \cos 2x \right] \]

The general solution is:
\[ y = y_c + y_p \]
\[ y = c_1 \cos x + c_2 \sin x - \frac{1}{3} \left[ \left( x^2 - \frac{26}{9} \right) \sin 2x + \frac{8}{3} \cos 2x \right] \]

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