Elliptic Quantum Groups $U_{q,p}(\hat{\mathfrak{gl}}_N)$ and $E_{q,p}(\hat{\mathfrak{gl}}_N)$

Hitoshi Konno

Department of Mathematics, Tokyo University of Marine Science and Technology, Etchujima, Tokyo 135-8533, Japan

Dedicated to Professor Masatoshi Noumi on his 60th birthday.

Abstract

We reformulate a central extension of Felder’s elliptic quantum group in the FRST formulation as a topological algebra $E_{q,p}(\hat{\mathfrak{gl}}_N)$ over the ring of formal power series in $p$. We then discuss the isomorphism between $E_{q,p}(\hat{\mathfrak{gl}}_N)$ and the elliptic algebra $U_{q,p}(\hat{\mathfrak{gl}}_N)$ of the Drinfeld realization. An evaluation $H$-algebra homomorphism from $U_{q,p}(\hat{\mathfrak{gl}}_N)$ to a dynamical extension of the quantum affine algebra $U_q(\hat{\mathfrak{g}}_N)$ resolves the problem into the one discussed by Ding and Frenkel in the trigonometric case. We also provide some useful formulas for the elliptic quantum determinants.

1 Introduction

An elliptic quantum algebra is an associative algebra related to an elliptic solution to the Yang-Baxter equation (YBE) or the dynamical Yang-Baxter equation (DYBE). Equipped with a co-algebra structure it is called the elliptic quantum group (EQG). Depending on YBE or DYBE, the corresponding EQG is called the vertex type or the face type, respectively [26,37]. Through this paper we use the terminology DYBE [19] as an equation equivalent to the face type YBE or the star triangle equation (see for example [33]).

Let $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ denote a simple Lie algebra and an (untwisted) affine Lie algebra, respectively. In known quantum groups, such as the Yangian $Y(\mathfrak{g})$ (or its double $DY(\mathfrak{g})$) associated to the rational solutions to the YBE and the affine quantum group $U_q(\hat{\mathfrak{g}})$ associated to the trigonometric solutions there are some different formulations depending on the types of the generators. In particular for $U_q(\hat{\mathfrak{g}})$ they are the Drinfeld-Jimbo formulation [10,32] in terms of an analogue of the Chevalley generators, Drinfeld’s new realization [11] whose generators, called the Drinfeld generators, are natural analogues of those in the the loop algebras $\mathfrak{g}[t,t^{-1}]$, and the Faddeev-Reshetikhin-Semenov-Tian-Shansky-Takhtadjan (FRST) formulation [16,59] in terms of the $L$
operators satisfying the $RLL$ relations associated with the $R$ matrix, a solution to the YBE. The isomorphisms among these three have been discussed by several authors [4,9,11,30,31,39].

Correspondingly there are three different formulations of EQGs: $A_{q,p}(\widehat{sl}_N)$ and $B_{q,\lambda}(\hat{g})$ [37] in terms of the Chevalley type generators, $U_{q,p}(\hat{g})$ [36,47] and $E_{\tau,\eta}(\hat{gl}_N)$ [13,14] in terms of the Drinfeld generators and $A_{q,p}(\widehat{sl}_2)$ [21] and $E_{\tau,\eta}(\hat{gl}_N)$ [15,19,21,41] in terms of the $L$ operators. Here only $A_{q,p}(\widehat{sl}_N)$ is the vertex type EQG, which is related to Baxter-Belavin’s elliptic $R$ matrix [3,5]. The others are the face type which are related to the elliptic solutions to the face type YBE, for example [33]. These have their own co-algebra structures: the quasi-Hopf algebra structure [12] for $A_{q,p}(\widehat{sl}_N)$, $B_{q,\lambda}(\hat{g})$ [37] and $E_{\tau,\eta}(\widehat{sl}_2)$ [14], and the Hopf algebroid structure [15,40] for $E_{\tau,\eta}(\hat{gl}_N)$ [21,28,41] and $U_{q,p}(\hat{g})$ [50].

As like the cases in $Y(\hat{g}), DY(\hat{g})$ and $U_q(\hat{g})$, each formulation has both advantages and disadvantages. The quasi-Hopf algebra formulations $A_{q,p}(\widehat{sl}_N)$ and $B_{q,\lambda}(\hat{g})$ [37] are suitable for studying formal algebraic structures such as the universal elliptic dynamical $R$ matrices, the universal form of the dynamical $RLL$ relations etc., but it is hard to derive concrete representations due to the complexity of the quasi-Hopf twist operation.

The Drinfeld realization $U_{q,p}(\hat{g})$ is suitable for studying both finite and infinite dimensional representations [36,43,44,46,49,50,52] due to the nature of the Drinfeld generators. Recent developments include a characterization of the finite dimensional representations in terms of a theta function analogue [50] of the Drinfeld polynomials [8,11] and a clarification of the quantum $Z$-algebra structures of the infinite dimensional representations [17]. An application to the algebraic analysis of the solvable lattice models [35] also have made a great success [7,36,38,46,49]. See also rather older works [1,53,54] whose results, in particular the vertex operators and the screening operators, are able to be reformulated by the representation theory of $U_{q,p}(\widehat{sl}_N)$ [43,46]. In addition there are deep relationships between $U_{q,p}(\hat{g})$ and the deformed $W(\hat{g})$ algebras: the generating functions of the Drinfeld generators (the elliptic currents) $e_j(z)$ and $f_j(z)$ of $U_{q,p}(\hat{g})$ are identified with the screening currents of the deformed $W(\hat{g})$ algebras of the coset type [17,36,45,47].

The FRST formulation is suitable for studying finite dimensional representations by a fusion procedure or by taking a coproduct. In this way finite dimensional representations of $E_{\tau,\eta}(\hat{gl}_N)$ have been studied well [21,41,42] (see also [34]) and applied to the elliptic Ruijsenaars models [22,23], the elliptic hypergeometric series [41,42,57], the partition function of the solvable lattice model [56,58] and the elliptic Gaudin model [60].

On the other hand in order to formulate infinite dimensional representations of $E_{\tau,\eta}(\hat{gl}_N)$ one needs it’s central extension. There are two different proposals provided by [14] and [36].
Accordingly $E_{\tau,\eta}(\mathfrak{sl}_2)$ in [14] and $U_{q,p}(\widehat{\mathfrak{g}})$ in [17,36,47] have been the two proposals for their Drinfeld realizations. However the isomorphism between $E_{\tau,\eta}(\mathfrak{g}_N)$ in the FRST formulation and neither of these two Drinfeld realizations has been discussed precisely.

The aim of this paper is to establish the isomorphism between $U_{q,p}(\widehat{\mathfrak{g}}_N)$ and a central extension of $E_{\tau,\eta}(\mathfrak{gl}_N)$ in the FRST formulation as a Hopf algebroid. For this purpose, we first reformulate $E_{\tau,\eta}(\mathfrak{gl}_N)$ as a topological algebra over the ring of formal power series in $p$ and at the same time we give a central extension of it according to the argument in [36,37]. We denote the resultant algebra by $E_{q,p}(\widehat{\mathfrak{gl}}_N)$, where the generators are clear and their defining relations are well defined in the $p$-adic topology as in $A_{q,p}(\widehat{\mathfrak{sl}}_2)$ [24] and $U_{q,p}(\widehat{\mathfrak{g}})$ [17]. Secondly we discuss dynamical representations of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$. We especially introduce an evaluation $H$-algebra homomorphism from $U_{q,p}(\widehat{\mathfrak{g}}_N)$ to a dynamical extension of the quantum affine algebra $U_q(\widehat{\mathfrak{g}}_N)$. This allows us to obtain the dynamical representations (of both finite and infinite dimensional) from any representations of $U_q(\widehat{\mathfrak{g}}_N)$. As a result the problem resolves itself into the one discussed by Ding and Frenkel in the trigonometric case [9].

A part of the results have been reported in the workshops “Recent advances in quantum integrable systems 2012”, 10-14 September 2012, Angers, France and “Elliptic Integrable Systems and Hypergeometric Functions”, 15-19 July 2013, Lorentz Center, Leiden, the Netherlands.

This paper is organized as follows. In Section 2 preparing notations and conventions we introduce the elliptic dynamical $R$ matrix. In Section 3 we define $U_{q,p}(\widehat{\mathfrak{g}}_N)$ and $E_{q,p}(\widehat{\mathfrak{g}}_N)$ as topological algebras over the ring of formal power series in $p$. We also give the trigonometric ($p = 0$) counter parts of them. In section 4 we show that both $U_{q,p}(\widehat{\mathfrak{g}}_N)$ and $E_{q,p}(\widehat{\mathfrak{g}}_N)$ are $H$-algebras (Proposition [4.3] and [4.4]). Then we introduce an $H$-Hopf algebroid structure to them. In Section 5 we introduce dynamical representations of $U_{q,p}(\widehat{\mathfrak{g}})$ and give a construction of the evaluation dynamical representations from any representations of $U_q(\widehat{\mathfrak{g}}_N)$. In Section 6 we discuss an isomorphism between $U_{q,p}(\widehat{\mathfrak{g}}_N)$ and $E_{q,p}(\widehat{\mathfrak{g}}_N)$ as an $H$-Hopf algebroid. Our arguments mainly follow those by Ding and Frenkel in the trigonometric case [9] with some additional formulas for the lower rank subalgebras of $E_{q,p}(\widehat{\mathfrak{g}}_N)$, which make the induction process more transparent. In particular by making use of the evaluation dynamical representations in Sec. 5 our proof on the injectivity resolves itself into the results in [9]. Appendix A contains a definition of the quantum affine algebra $U_q(\widehat{\mathfrak{g}}_N)$ which we use in Sec.5. In Appendix B we list the formulas necessary for discussing the evaluation dynamical representations. In Appendix C we summarize the formulas which identify a combination of the Gauss components of the $L$ operator with the elliptic currents of $U_{q,p}(\widehat{\mathfrak{g}}_N)$. In Appendix D we summarize some formulas on adding ‘fractional powers in $z$’ which clarify a connection between $U_{q,p}(\widehat{\mathfrak{g}})$ in the current paper
and the previous one in [36, 43, 14, 47, 50]. Appendix E contains some formulas for the elliptic quantum determinants.

2 The $R$-matrices

Let $\{e_j \ (1 \leq j \leq N)\}$ be the orthonormal basis in $\mathbb{R}^N$ with the inner product $\langle e_j, e_k \rangle = \delta_{j,k}$. Setting $\bar{e}_j = e_j - e_\alpha$, $\epsilon = \frac{1}{N} \sum_{j=1}^{N} e_j$, we define the weight lattice $P$ of $A_{N-1}$ type by $P = \sum_{j=1}^{N} \mathbb{Z} \bar{e}_j$. Let $I = \{1, 2, \ldots, N-1\}$. We set $\alpha_j = \bar{e}_j - \bar{e}_{j+1}$, $\Lambda_j = \bar{e}_1 + \cdots + \bar{e}_j$ ($j \in I$) and define $Q = \mathbb{Z} \alpha_1 + \cdots + \mathbb{Z} \alpha_{N-1}$ and $\bar{h}^* = \mathbb{C} \Lambda_1 + \cdots + \mathbb{C} \Lambda_{N-1}$. We also define elements $h_{\bar{e}_j}$ ($1 \leq j \leq N$) in the dual space $\bar{h}$ by $\langle \bar{e}_i, h_{\bar{e}_j} \rangle = \langle \bar{e}_i, \bar{e}_j \rangle = \delta_{j,k} - \frac{1}{N}$. Setting $h_j = h_{\bar{e}_j} - h_{\bar{e}_{j+1}}$ ($j \in I$) we have $\bar{\Lambda}_i, h_j = \delta_{i,j}$ so that $\bar{h} = \mathbb{C} \Lambda_1 + \cdots + \mathbb{C} \Lambda_{N-1}$. For $\alpha = \sum_j a_j \bar{e}_j \in \bar{h}^*$, we define $h_{\alpha} \in \bar{h}$ by $h_{\alpha} = \sum_j a_j h_{\bar{e}_j}$ and $h_0 = 0$. We also need two more elements $c$ and $\Lambda_0$ satisfying $\langle \Lambda_0, c \rangle = 1, \langle \Lambda_0, h_j \rangle = 0 = \langle \Lambda_j, c \rangle$ ($1 \leq j \leq N$). We regard $\bar{h} \oplus \bar{h}^*$ as a Heisenberg algebra by

$$[h_{\bar{e}_j}, \bar{e}_k] = \langle \bar{e}_j, \bar{e}_k \rangle, \quad [h_{\bar{e}_j}, h_{\bar{e}_k}] = 0 = [\bar{e}_j, \bar{e}_k].$$  \ (2.1)

We also introduce the dynamical parameters $P_{\bar{e}_j}$ ($j \in I$) and their duals $Q_{\bar{e}_j}$. They are the Heisenberg algebra defined by

$$[P_{\bar{e}_j}, Q_{\bar{e}_k}] = \langle \bar{e}_j, \bar{e}_k \rangle, \quad [P_{\bar{e}_j}, P_{\bar{e}_k}] = 0 = [Q_{\bar{e}_j}, Q_{\bar{e}_k}].$$  \ (2.2)

We set $P_{\alpha} = \sum_j a_j P_{\bar{e}_j}$ for $\alpha = \sum_j a_j \bar{e}_j$ and $P_0 = 0$ etc. In particular we set $P_j = P_{\alpha_j} = P_{\bar{e}_j} - P_{\bar{e}_{j+1}}$ and $Q_j = Q_{\alpha_j} = Q_{\bar{e}_j} - Q_{\bar{e}_{j+1}}$.

For the abelian group $\mathcal{R}_Q = \sum_i \mathbb{Z} Q_{\bar{e}_i}$, we denote by $\mathbb{C}[\mathcal{R}_Q]$ the group algebra over $\mathbb{C}$ of $\mathcal{R}_Q$. We denote by $e^{Q_\alpha}$ the element of $\mathbb{C}[\mathcal{R}_Q]$ corresponding to $Q_\alpha \in \mathcal{R}_Q$. These $e^{Q_\alpha}$ satisfy $e^{Q_\alpha} e^{Q_\beta} = e^{Q_\alpha + Q_\beta}$ and $(e^{Q_\alpha})^{-1} = e^{-Q_\alpha}$. In particular, $e^0 = 1$ is the identity element.

Now let us introduce a dynamical extension of $\bar{h}$ and $\bar{h}^*$: $H = \bar{h} \oplus \sum_j \mathbb{C} P_{\bar{e}_j} + \mathbb{C} c = \sum_j \mathbb{C} (P + h)_{\bar{e}_j} + \sum_j \mathbb{C} P_{\bar{e}_j} + \mathbb{C} c$ and $H^* = \bar{h}^* \oplus \sum_j \mathbb{C} Q_{\bar{e}_j} + \mathbb{C} \Lambda_0$. Through this paper we often use the abbreviation $(P + h)_{\bar{e}_j}$ for $P_{\bar{e}_j} + h_{\bar{e}_j}$. We have the paring: $\langle Q_\alpha, P_\beta \rangle = \langle \alpha, \beta \rangle = \langle \alpha, \bar{h}_\beta \rangle = \alpha, \beta \in \bar{h}^*$, $\langle \Lambda_0, c \rangle = 1$, and the others vanish. Let $\mathcal{M}_{H^*}$ be the field of meromorphic functions on $H^*$. We denote by $\hat{f} = f(P + h, P)$ an element of $\mathcal{M}_{H^*}$, where $P + h = \sum_j a_j (P + h)_{\bar{e}_j}, P = \sum_j b_j P_{\bar{e}_j} \in H$. The function $\hat{f}$ is evaluated at $\mu \in H^*$ as $\hat{f}(\mu) = f(<\mu, P + h>, <\mu, P>)$ etc.. Hereafter we set $\mathbb{F} = \mathcal{M}_{H^*}$.

Let $h$ and $p$ be indeterminates. We set $q = e^h$. Through this paper we also use $p^* = pq^{-2c}$. 

4
The following notations are often used.

\[ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty, \]

\[(x; q_1, q_2, \cdots, q_k)_\infty = \prod_{n_1, n_2, \cdots, n_k = 0}^\infty (1 - xq_1^{n_1}q_2^{n_2} \cdots q_k^{n_k}), \quad \{z\} = (z;p,q^{2N})_\infty, \]

\[(x_1, x_2, \cdots, x_l; q_1, q_2, \cdots, q_k)_\infty = \prod_{i=1}^l (x_i; q_1, q_2, \cdots, q_k)_\infty. \]

### 2.1 The elliptic dynamical $R$-matrices

Let \( V = \oplus_{i=1}^N \mathbb{C}v_i, \ E_{i,j}v_k = \delta_{j,k}v_i. \) We consider the following elliptic dynamical $R$-matrices \( R^\pm(z,s) \in \text{End}(V \otimes V) \) of type \( A^{(1)}_{N-1}. \) For \( s \in H, \)

\[
R^\pm(z,s) = \rho^\pm(z) \mathcal{R}(z,s), \\
\mathcal{R}(z,s) = \sum_{j=1}^N E_{jj} \otimes E_{jj} + \sum_{1 \leq j < l \leq N} \left( b(z, s_{j,l}) E_{jj} \otimes E_{ll} + \bar{b}(z) E_{ll} \otimes E_{jj} \right),
\]

where \( s_{j,l} = s_{\ell j} - s_{\ell i} \) (1 \( \leq j < l \leq N \)) and

\[
\rho^+(z) = q^{-\frac{N-1}{2}} \left\{ \frac{q^2z}{z} \right\} \left\{ \frac{q^{2N-2}z}{z} \right\} \left\{ \frac{pq^{2N}}{pq^2} \right\} \left\{ \frac{1}{z} \right\} \left\{ \frac{pq^{2N-2}}{pq^2} \right\},
\]

\[
\rho^-(z) = q^{\frac{N-1}{2}} \left\{ \frac{pq^2z}{z} \right\} \left\{ \frac{pq^{2N-2}z}{z} \right\} \left\{ \frac{1}{z} \right\} \left\{ \frac{pq^{2N}}{pq^2} \right\} \left\{ \frac{q^{2N-2}}{q^{2N}} \right\} \left\{ \frac{1}{z} \right\},
\]

\[
b(z, s) = \frac{\Theta_p(q^{2z}) \Theta_p(q^{2s}z) \Theta_p(q^{2s}z)}{\Theta_p(q^{2s})^2 \Theta_p(q^2z)},
\]

\[
\bar{b}(z) = \frac{\Theta_p(z)}{\Theta_p(q^2z)},
\]

\[
c(z, s) = \frac{\Theta_p(q^2z) \Theta_p(q^{2s}z) \Theta_p(q^{2s}z)}{\Theta_p(q^{2s})^2 \Theta_p(q^2z)}, \quad \bar{c}(z, s) = c(z, -s).
\]

We also denote by \( R^{\pm*}(z, s) \) the $R$ matrices obtained from \( R^{\pm}(z, s) \) by replacing \( p \) with \( p^*. \) Note that

\[
\rho^+(zp) = q^{-\frac{N-1}{2}} \rho^-(z).
\]

In particular,

\[
R^-(z,s)^{-1} = PR^+(z^{-1}, s)P.
\]

Furthermore if we set

\[
\rho(z) = \frac{\rho^{++}(z)}{\rho^+(z)}
\]
Then we have
\[ \rho(z)^{-1} = \rho(z^{-1}), \quad \rho(z) = \frac{\rho^+(z)}{\rho^-(z)}. \] (2.13)

**Proposition 2.1.** [19] The \( R^+(z, s) \) satisfies the following dynamical Yang-Baxter equation.

\[
\begin{align*}
R^{+(12)}(z_1/z_2, P + \pi_V(h)(3))R^{+(13)}(z_1, P)R^{+(23)}(z_2, P + \pi_V(h)(1)) \\
= R^{+(23)}(z_2, P)R^{+(13)}(z_1, P + \pi_V(h)(2))R^{+(12)}(z_1/z_2, P),
\end{align*}
\] (2.14)

where \( \pi_V(h)^{(1)} = \pi_V(h) \otimes 1 \otimes 1 \) and \( \pi_V(h)^{(1)}_{j,l} = \pi_V(h_{\xi_j})^{(1)} - \pi_V(h_{\xi_l})^{(1)} \) with \( \pi_V(h_{\xi_j}) = E_{jj} - \frac{1}{N} I \) etc.. Here \( I \) denotes the \( N \times N \) unit matrix.

**Remark 1.** The elliptic dynamical \( R \)-matrix [23] is gauge equivalent to the \( A_N^{(1)} \) type face weight obtained by Jimbo, Miwa and Okado [33]. See Appendix D.

**Remark 2.** The \( R \)-matrix preserves the weights

\[ [R^\pm(z, s), \pi_V(h) \otimes \pi_V(h)] = 0 \quad \forall h \in \mathfrak{h}. \] (2.15)

Now let us set

\[ \rho_0(z) = q^{-1}N^{-1} \left( \frac{(q^2 z; q^2N)_\infty(q^{2N-2} z; q^2N)_\infty}{(z; q^2N)_\infty(q^{2N-2} z; q^2N)_\infty} \right), \] (2.16)

\[ \alpha(z) = \left\{ \frac{pq^2 z}{p q^{2N-2} z} \right\} \left\{ \frac{p z}{p q^{2N-2} z} \right\} \left\{ \frac{p q^{2N-2} z}{p q^{2N} z} \right\} \left\{ \frac{q^{2N-2} z}{q^{2N} z} \right\}, \] (2.17)

\[ \Xi_p(z) = (p z; p)_{\infty} (p z; p)_{\infty}. \] (2.18)

Then we have

\[ \begin{align*}
\rho^+(z) b(z, s) &= q^{\pm 1} \rho_0(z^{\pm 1}) \pm 1 \left( 1 - (q^{2s} q^{-2})^{\pm 1} \right) \left( 1 - (q^{2s} q^{-2})^{\pm 1} \right) \frac{1 - z^{\pm 1}}{1 - (q^2 z)^{\pm 1}} \\
&\quad \times \alpha(z) \Xi_p(q^{2s} q^{-2}) \Xi_p(q^{2s} q^{2}) \Xi_p(z) \Xi_p(q^{2} z), \quad (2.19)
\end{align*} \]

\[ \begin{align*}
\rho^-(z) b(z, s) &= q^{\pm 1} \rho_0(z^{\pm 1}) \pm 1 \frac{1 - z^{\pm 1}}{1 - (q^{2s} z)^{\pm 1}} \alpha(z) \Xi_p(z) \Xi_p(q^{2} z), \quad (2.20)
\end{align*} \]

\[ \begin{align*}
\rho^+(z) c(z, s) &= \rho_0(z^{\pm 1}) \pm 1 \frac{1 - q^{2s} q^{2} - (q^{2s} z)^{\pm 1}}{1 - q^{2s} z^{\pm 1}} \alpha(z) \Xi_p(q^{2} q^{2s}) \Xi_p(q^{2s} z) \Xi_p(q^{2} z), \quad (2.21)
\end{align*} \]

In [24] a similar expression was obtained for Baxter’s elliptic \( R \) matrix. In \( R^\pm(z, s) \), we specify the factors \( \frac{1 - q^{2s} z^{\pm 1}}{1 - (q^{2s} z)^{\pm 1}} \) and \( \frac{1 - (q^{2s} z)^{\pm 1}}{1 - (q^{2s} z)^{\pm 1}} \) in (2.19)-(2.21) to be power series in \( z^{\pm 1} \). We then treat \( R^\pm(z, s) \) as formal Laurent series in \( z \)

\[ R^\pm(z, s) = \sum_{n \in \mathbb{Z}} R^\pm(s)_{n} z^{n} \] (2.22)
whose coefficients are in the ring \( \mathbb{F}[[p]] \) of formal power series in \( p \). Note that \( \alpha(z), \frac{\Xi(z)}{\Xi(q^2z)} \) and \( \Xi(q^{2s+j}z) \) are well defined formal Laurent series in \( z \) with coefficients in \( \mathbb{F}[[p]] \). Then the matrices \( R^\pm(z,s) \) satisfy

\[
R^\pm(s)_n \equiv 0 \mod p^{\max(\pm n,0)}\mathbb{F}[[p]] \quad \forall n \in \mathbb{Z}. \tag{2.23}
\]

In particular, at \( p = 0 \) \( R^+(z,s) \) (reps. \( R^-(z,s) \)) contains only non-negative (reps. non-positive) powers in \( z \). Explicitly we have \( R^\pm_0(z,s) \equiv R^\pm(z,s) \bigg|_{p=0} \),

\[
R^\pm_0(z, s) = \rho_0^\pm(z) \prod_{i,j} \frac{1}{1 - q^2 s^2 i^2 j^2}, \quad \prod_{i,j} \frac{1}{1 - q^2 s^2 i^2 j^2} = 1,
\]

\[
\frac{\prod_{i,j} \frac{1}{1 - q^2 s^2 i^2 j^2}}{\prod_{i,j} \frac{1}{1 - q^2 s^2 i^2 j^2}} = 1.
\]

Hence one can regard the matrix element \( R_0(z,s)^{kl}_{ij} \) as a formal power series in the (multiplicative) dynamical variables \( q^{2s+i,j} \). The 0-th order term in \( q^{2s+i,j} \) coincides with the corresponding component of the standard trigonometric \( R \) matrix

\[
R_0(z) = \rho_0(z)\mathcal{R}_0(z), \tag{2.24}
\]

where

\[
\rho_0^+(z) = \rho_0(z), \quad \rho_0^-(z) = \rho_0(z^{-1})^{-1},
\]

\[
b_0^+(z,s) = \frac{1 - q^2 s^2}{1 - q^2 s^2}, \quad \prod_{i,j} \frac{1}{1 - q^2 s^2 i^2 j^2} = 1,
\]

\[
b_0^-(-z) = \frac{1 - q^2 z}{1 - q^2 z}, \quad \prod_{i,j} \frac{1}{1 - q^2 z i^2 j^2} = 1.
\]

\[
e_0^+(z,s) = \frac{1 - q^2 z}{1 - q^2 z}, \quad \prod_{i,j} \frac{1}{1 - q^2 z i^2 j^2} = 1,
\]

\[
e_0^-(z,s) = \frac{1 - q^2 z^{-1}}{1 - q^2 z^{-1}}, \quad \prod_{i,j} \frac{1}{1 - q^2 z^{-1} i^2 j^2} = 1.
\]

Note that one can parametrize the elliptic dynamical \( R \) matrices associated with the other types of affine Lie algebras, at least \( \mathfrak{g} = B_N^{(1)}, C_N^{(1)}, D_N^{(1)} \), in a similar way to (2.19) - (2.21) so that they have the same property at \( p = 0 \).
3 The Elliptic Quantum Algebras \(U_{q,p}(\hat{\mathfrak{gl}}_N)\) and \(E_{q,p}(\hat{\mathfrak{gl}}_N)\)

In this section we define two elliptic algebras \(U_{q,p}(\hat{\mathfrak{gl}}_N)\) and \(E_{q,p}(\hat{\mathfrak{gl}}_N)\) as topological algebras over \(\mathbb{F}[[p]]\).

3.1 \(U_{q,p}(\hat{\mathfrak{gl}}_N)\)

**Definition 3.1.** The elliptic algebra \(U_{q,p}(\hat{\mathfrak{gl}}_N)\) is a topological algebra over \(\mathbb{F}[[p]]\) generated by \(e_{j,m}, f_{j,m}, k_{l,m}, (1 \leq j \leq N - 1, 1 \leq l \leq N, m \in \mathbb{Z})\), \(\hat{d}\) and the central element \(q^{\pm c/2}\). We set

\[
e_{j}(z) = \sum_{m \in \mathbb{Z}} e_{j,m} z^{-m}, \quad f_{j}(z) = \sum_{m \in \mathbb{Z}} f_{j,m} z^{-m}, \quad (3.1)
\]

\[
k_{l}^{\pm}(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} k_{l,-m} z^{m} + \sum_{m \in \mathbb{Z}_{> 0}} k_{l,m} p^{m} z^{-m}, \quad (3.2)
\]

\[
k_{l}^{-}(z) = q^{2h_{l}} k_{l}^{+}(z) (zp^{*}q^{*}). \quad (3.3)
\]

The defining relations are as follows. For \(g(P), g(P + h) \in \mathbb{F}\),

\[
g(P + h) e_{j}(z) = e_{j}(g(P + h)), \quad g(P) e_{j}(z) = e_{j}(g(P - < Q_{\alpha_{j}}, P >)), \quad (3.4)
\]

\[
g(P + h) f_{j}(z) = f_{j}(g(P + h - < \alpha_{j}, P + h >)), \quad g(P) f_{j}(z) = f_{j}(g(P)), \quad (3.5)
\]

\[
g(P) k_{l}^{+}(z) = k_{l}^{+}(g(P - < Q_{\ell_{i}}, P >)), \quad g(P + h) k_{l}^{+}(z) = k_{l}^{+}(z) g(P + h - < Q_{\ell_{i}}, P >), \quad (3.6)
\]

\[
[\hat{d}, g(P + h)] = 0 = [\hat{d}, g(P)], \quad (3.7)
\]

\[
[\hat{d}, k_{l}^{+}(z)] = -z \frac{\partial}{\partial z} k_{l}^{+}(z), \quad [\hat{d}, e_{j}(z)] = -z \frac{\partial}{\partial z} e_{j}(z), \quad [\hat{d}, f_{j}(z)] = -z \frac{\partial}{\partial z} f_{j}(z), \quad (3.8)
\]

\[
\rho_{l}^{+}(z_{2}/z_{1}) k_{l}^{+}(z_{2}) = \rho_{l}^{+}(z_{1}/z_{2}) k_{l}^{+}(z_{1}), \quad (1 \leq l \leq N), \quad (3.9)
\]

\[
\rho_{l}^{+}(z_{2}/z_{1}) (p^{*}q^{*}z_{2}/z_{1}; p^{*})_{\infty} (pq^{2}z_{2}/z_{1}; p^{*})_{\infty} k_{j}^{+}(z_{2}) k_{l}^{+}(z_{1}) \quad (1 \leq j < l \leq N), \quad (3.10)
\]

\[
\begin{align*}
\frac{(p^{*}q^{*}z_{2}/z_{1}; p^{*})_{\infty}}{(p^{*}q^{-2-j}z_{2}/z_{1}; p^{*})_{\infty}} k_{j}^{+}(z_{1}) e_{j}(z_{2}) &= q^{-1} \frac{(q^{-c+j}z_{1}/z_{2}; p^{*})_{\infty}}{(q^{-c+2+j}z_{1}/z_{2}; p^{*})_{\infty}} e_{j}(z_{2}) k_{j}^{+}(z_{1}), \quad (3.11) \\
\frac{(p^{*}q^{*}z_{2}/z_{1}; p^{*})_{\infty}}{(p^{*}q^{-c+j}z_{2}/z_{1}; p^{*})_{\infty}} k_{j-1}^{+}(z_{1}) e_{j}(z_{2}) &= q^{-(c-j)z_{1}/z_{2}} e_{j}(z_{2}) k_{j-1}^{+}(z_{1}), \quad (3.12) \\
k_{l}^{+}(z_{1}) e_{j}(z_{2}) k_{l}^{+}(z_{2})^{-1} &= e_{j}(z_{2}) \quad (l \neq j, j + 1), \quad (3.13) \\
\frac{(pq^{-1}z_{2}/z_{1}; p^{*})_{\infty}}{(pq^{2}z_{2}/z_{1}; p^{*})_{\infty}} k_{j}^{+}(z_{1}) f_{j}(z_{2}) &= q^{-(2+j)z_{1}/z_{2}} f_{j}(z_{2}) k_{j}^{+}(z_{1}), \quad (3.14) \\
\frac{(pq^{-1}z_{2}/z_{1}; p^{*})_{\infty}}{(pq^{2}z_{2}/z_{1}; p^{*})_{\infty}} k_{j-1}^{+}(z_{1}) f_{j}(z_{2}) &= q^{-1} \frac{(q^{2+j}z_{1}/z_{2}; p^{*})_{\infty}}{(q^{3}z_{1}/z_{2}; p^{*})_{\infty}} f_{j}(z_{2}) k_{j-1}^{+}(z_{1}), \quad (3.15) \\
k_{l}^{+}(z_{1}) f_{j}(z_{2}) k_{l}^{+}(z_{2})^{-1} &= f_{j}(z_{2}) \quad (l \neq j, j + 1), \quad (3.16)
\end{align*}
\]
where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$, $\rho(z)$ is given in (2.12),

$$
\rho^\pm(z) = \frac{\{q^2z\}^* \{ q^{-2}q^{2N}z \}^* \{ z \} \{ q^{2N}z \}}{\{ z \}^* \{ q^{2N}z \}^* \{ q^2z \} \{ q^{-2}q^{2N}z \}},
$$

and $\kappa$ is given by

$$
\kappa = \frac{(p; p)_{\infty} (p^* q^2; p^*)_{\infty}}{(p^*; p^*)_{\infty} (pq^2; p)_{\infty}}.
$$

We call $e_j(z), f_j(z), k^\pm_i(z)$ the elliptic currents. We also denote by $U^\prime_{q,p}(\hat{gl}_N)$ the subalgebra obtained by removing $\hat{d}$. 

We treat these relations as formal Laurent series in $z, w$ and $z_j$'s. In each term of (3.10)-(3.22) and (3.24)-(3.25), the expansion direction of the structure function given by a ratio of infinite
products is chosen according to the order of the accompanied product of the elliptic currents. For example, in the l.h.s of (3.17), \( \frac{(q^2 z_2/z_1 p^*)}{(p^* q^{-1} z_2/z_1 p^*)} \) should be expanded in \( z_2/z_1 \), whereas in the r.h.s \( \frac{(q^2 z_1/z_2 p^*)}{(p^* q^{-1} z_1/z_2 p^*)} \) should be expanded in \( z_1/z_2 \). All the coefficients in \( z_j \)'s are well defined in the \( p \)-adic topology.

For a practical use, we remark that in the sense of analytic continuation (3.9)-(3.15) and (3.17)-(3.21) can be rewritten as follows.

\[
\begin{align*}
    k_i^+(z_1)k_i^+(z_2) &= \rho(z_1/z_2)k_i^+(z_2)k_i^+(z_1), \quad (1 \leq l \leq N), \\
    k_j^+(z_1)k_j^+(z_2) &= \rho(z_1/z_2)\frac{\Theta_{p^*}(q^{-2} z_1/z_2)\Theta_{p}(z_1/z_2)}{\Theta_{p^*}(z_1/z_2)\Theta_{p}(q^{-2} z_1/z_2)}k_j^+(z_2)k_j^+(z_1) \quad (1 \leq j < l \leq N), \\
    k_j^+(z_1)e_j(z_2)k_j^+(z_1)^{-1} &= q^{-1}\frac{\Theta_{p^*}(q^{-c+j} z_1/z_2)}{\Theta_{p^*}(q^{-c-2+j} z_1/z_2)}e_j(z_2), \\
    k_{j+1}^+(z_1)e_j(z_2)k_{j+1}^+(z_1)^{-1} &= q^{-1}\frac{\Theta_{p^*}(q^{-c+j} z_1/z_2)}{\Theta_{p^*}(q^{-c+2+j} z_1/z_2)}e_j(z_2), \\
    k_j^+(z_1)j_f(z_2)k_j^+(z_1)^{-1} &= q^{-1}\frac{\Theta_{p^*}(q^{-2+j} z_1/z_2)}{\Theta_{p}(q^{j} z_1/z_2)}f_j(z_2), \\
    k_{j+1}^+(z_1)j_f(z_2)k_{j+1}^+(z_1)^{-1} &= q^{-1}\frac{\Theta_{p^*}(q^{2+j} z_1/z_2)}{\Theta_{p}(q^{j} z_1/z_2)}f_j(z_2), \\
    e_j(z_1)e_j(z_2) &= -\frac{2}{z_2}\frac{\Theta_{p^*}(q^{2} z_1/z_2)}{\Theta_{p}(q^{2} z_2/z_1)}e_j(z_2)e_j(z_1), \\
    e_j(z_1)e_{j+1}(z_2) &= -\frac{2}{z_2}\frac{\Theta_{p^*}(q^{-1} z_1/z_2)}{\Theta_{p}(q^{-1} z_2/z_1)}e_j(z_2)e_{j+1}(z_1), \\
    j_f(z_1)f_j(z_2) &= -\frac{2}{z_2}\frac{\Theta_{p^*}(q^{-2} z_1/z_2)}{\Theta_{p}(q^{-2} z_2/z_1)}f_j(z_2)f_j(z_1), \\
    j_f(z_1)f_{j+1}(z_2) &= -\frac{2}{z_2}\frac{\Theta_{p^*}(q z_1/z_2)}{\Theta_{p}(q z_2/z_1)}f_{j+1}(z_2)f_j(z_1).
\end{align*}
\]

**Proposition 3.2.** Let us set

\[
K(z) = k_1^+(z)k_2^+(q^{-2} z)\cdots k_N^+(q^{-2(N-1)} z).
\]

Then \( K(z) \) belongs to the center of \( U_{q,p}^\prime(\mathfrak{gl}_N) \).

**Proof.** Direct calculation using (3.6), (3.13), (3.16), (3.28)-(3.33) shows that \( K(z) \) commutes with \( F \) and all of the elliptic currents of \( U_{q,p}^\prime(\mathfrak{gl}_N) \). In particular, \([K(z), k_i^+(w)] = 0 \quad (1 \leq l \leq N)\) follows from the identity

\[
\prod_{j=1}^{N} \rho(q^{-2(j-1)} z) = \frac{\Theta_{p^*}(z)\Theta_{p}(q^{-(N-1)} z)}{\Theta_{p^*}(q^{-(N-1)} z)\Theta_{p}(z)}.
\]

**Remark.** In Appendix [2] we identify \( K(z) \) with the \( q \)-determinant of the \( L \)-operator.
Let us consider the following $E^+_{m^\perp}$ (1 $\leq l \leq N$, $m \in \mathbb{Z}_{\neq 0}$), which we call the elliptic bosons of the orthonormal basis type [17].

$$E^+_{m^\perp} = \frac{q^{lm}}{(q - q^{-1})[m]^2_N[Nm]_q} \left(-q^{-Nm} \sum_{k=1}^{l-1} [km]_q \alpha_{k,m} + \sum_{k=l}^{N-1} [(N - k)m]_q \alpha_{k,m} \right)$$

(1 $\leq l \leq N - 1$),

$$E^+_{m^\perp} = \frac{1}{(q - q^{-1})[m]^2_N[Nm]_q} \sum_{k=1}^{N} [km]_q \alpha_{k,m}.$$ 

They satisfy

$$[E^+_{m^\perp}, E^+_{n^\perp}] = \delta_{m+n,0} \frac{[cm]_q ((N - 1)m)_q}{m(q - q^{-1})^2 [m]^2_N[Nm]_q} \frac{1 - p^m}{1 - p^m q^{-cm}},$$

(3.39)

$$[E^+_{m^\perp}, E^+_{n^\perp}] = -\delta_{m+n,0} q^{(sgn(l-j)N-l+j)m} \frac{[cm]_q}{m(q - q^{-1})^2 [m]^2_N[Nm]_q} \frac{1 - p^m}{1 - p^m q^{-cm}},$$

(3.40)

$$[\alpha_{i,m}, E^+_{n^\perp}] = \delta_{m+n,0} \frac{[cm]_q}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^m q^{-cm}} (q^{-m} \delta_{i,l} - \delta_{i,l-1}).$$

(3.41)

Let $K^\pm_{\ell_j}$ satisfies for $g(P), g(P + h) \in \mathbb{F}$

$$g(P)K^+_{\ell_j} = K^+_{\ell_j} g(P - <Q_{\ell_j}, P >),$$

(3.42)

$$g(P + h)K^+_{\ell_j} = K^+_{\ell_j} g(P + h - <Q_{\ell_j}, P >).$$

(3.43)

Then the following $k^+_l(z)$ satisfy the desired relations.

$$k^+_l(z) = K^+_{\ell_l} \exp \left\{ \sum_{m \neq 0} \frac{(q^m - q^{-m})^2 p^m}{1 - p^m} E^+_{m^\perp} (q^l z)^{-m} \right\}.$$  

(3.44)

Furthermore if we further require that $\alpha_{i,m}$ and $K^\pm_{\ell_l}$ satisfy

$$[g(P), \alpha_{i,m}] = [g(P + h), \alpha_{i,m}] = 0,$$

(3.45)

$$[\hat{\alpha}, \alpha_{j,n}] = n \alpha_{j,n},$$

(3.46)

$$[\alpha_{i,m}, e_j(z)] = \frac{[a_{ij}m]_q}{m} \frac{1 - p^m}{1 - p^m q^{-cm}} z^m e_j(z),$$

(3.47)

$$[\alpha_{i,m}, f_j(z)] = \frac{[a_{ij}m]_q}{m} z^m f_j(z),$$

(3.48)

$$K^+_{\ell_l} e_j(z) = q^{-<\ell_l,h_j>} e_j(z) K^+_{\ell_l}, \quad K^+_{\ell_l} f_j(z) = q^{<\ell_l,h_j>} f_j(z) K^+_{\ell_l},$$

(3.49)
$k^+_j(z)$ satisfy the remaining relations (3.38), (3.11) - (3.16).

Now let us define $\psi^+_j(z)$ (1 ≤ j ≤ N - 1) by

$$
\psi^+_j(q^{-c/2}z) = \kappa k^+_j(z)k^+_{j+1}(z)^{-1},
$$

$$
\psi^-_j(q^{-c/2}z) = \kappa k^-_j(z)k^-_{j+1}(z)^{-1}.
$$

We have

$$
\psi^+_j(q^{-\frac{c}{2}}z) = K^+_j \exp \left( -(q - q^{-1}) \sum_{n>0} \frac{\alpha_{i,n}}{1 - p^n} z^n \right) \exp \left( (q - q^{-1}) \sum_{n>0} \frac{p^n \alpha_{i,n}}{1 - p^n} z^{-n} \right) (3.50)
$$

and $\psi^-_j(z) = q^{2h_j} \psi^+_j(z)$ where we set $K^+_j = K^+_{i_j} K^+_{i_{j+1}}$.

**Proposition 3.3.** The elliptic algebra $U_{q,p}(\hat{sl}_N)$ is characterized by (3.45) - (3.48) and the following relations. For $g(P), g(P + h) \in \mathbb{F}$,

$$
g(P + h)e_j(z) = e_j(z)g(P + h), \quad g(P)e_j(z) = e_j(z)g(P - <Q_{\alpha_j}, P>), \quad (3.51)
$$

$$
g(P + h)f_j(z) = f_j(z)g(P + h - <\alpha_j, P + h>), \quad g(P)f_j(z) = f_j(z)g(P), \quad (3.52)
$$

$$
[\hat{d}, g(P + h, P)] = 0, \quad (3.53)
$$

$$
[z_1 (q^{a_{ij}} - 2/z_1; p^*)_\infty] e_i(z_1) e_j(z_2) = -z_2 (q^{a_{ij}} - 2/z_2; p^*)_\infty e_j(z_1) e_i(z_2), \quad (3.55)
$$

$$
[z_1 (q^{-a_{ij}} - 2/z_1; p^*)_\infty] f_i(z_1) f_j(z_2) = -z_2 (q^{-a_{ij}} - 2/z_2; p^*)_\infty f_j(z_2) f_i(z_1), \quad (3.56)
$$

$$
[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta(q^{-c}z_1/z_2) \psi^-_j(q^{\frac{c}{2}}z_2) - \delta(q^c z_1/z_2) \psi^+_j(q^{-\frac{c}{2}}z_2) \right), \quad (3.57)
$$

$$
\sum_{\sigma \in S_N} \prod_{1 \leq m < k \leq a} \frac{(p^* q^2 z_{\sigma(k)}/z_{\sigma(m)}; p^*)_\infty}{(p^* q^2 z_{\sigma(k)}/z_{\sigma(m)}; p^*)_\infty} \prod_{q^1 \leq m \leq q^s} \frac{(p^* q^{a_{ij}} w/z_{\sigma(m)}; p^*)_\infty}{(p^* q^{a_{ij}} w/z_{\sigma(m)}; p^*)_\infty} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{a_{ij}} z_{\sigma(m)}/w; p^*)_\infty}{(p^* q^{a_{ij}} z_{\sigma(m)}/w; p^*)_\infty}
$$

$$
\sum_{a} \prod_{\sigma \in S_N} \frac{(p^* q^{-2} z_{\sigma(k)}/z_{\sigma(m)}; p^*)_\infty}{(p^* q^{-2} z_{\sigma(k)}/z_{\sigma(m)}; p^*)_\infty} \prod_{q^1 \leq m \leq q^s} \frac{(p^* q^{-a_{ij}} w/z_{\sigma(m)}; p^*)_\infty}{(p^* q^{-a_{ij}} w/z_{\sigma(m)}; p^*)_\infty} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{-a_{ij}} z_{\sigma(m)}/w; p^*)_\infty}{(p^* q^{-a_{ij}} z_{\sigma(m)}/w; p^*)_\infty}
$$

$$
\times e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(a)}) e_j(w) e_i(z_{\sigma(s+1)}) \cdots e_i(z_{\sigma(a)}) = 0, \quad (3.58)
$$

$$
\sum_{\sigma \in S_N} \prod_{1 \leq m < k \leq a} \frac{(p^* q^{-2} z_{\sigma(k)}/z_{\sigma(m)}; p^*)_\infty}{(p^* q^{-2} z_{\sigma(k)}/z_{\sigma(m)}; p^*)_\infty} \prod_{q^1 \leq m \leq q^s} \frac{(p^* q^{a_{ij}} w/z_{\sigma(m)}; p^*)_\infty}{(p^* q^{a_{ij}} w/z_{\sigma(m)}; p^*)_\infty} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{a_{ij}} z_{\sigma(m)}/w; p^*)_\infty}{(p^* q^{a_{ij}} z_{\sigma(m)}/w; p^*)_\infty}
$$

$$
\times f_i(z_{\sigma(1)}) \cdots f_i(z_{\sigma(a)}) f_j(w) f_i(z_{\sigma(s+1)}) \cdots f_i(z_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}), \quad (3.59)
$$

\footnote{Our $\psi^+_j(z)$ are $\hat{\psi}^+_j(z)$ in [30, 33].}
Proposition 3.4. In the sense of analytic continuation, we have

\[ \psi^+_i(z_1)\psi^+_j(z_2) = \frac{\Theta_{p^*}(q^{a_{ij}} z_1/z_2)\Theta_{p}(q^{-a_{ij}} z_1/z_2)}{\Theta_{p^*}(q^{-a_{ij}} z_1/z_2)\Theta_{p}(q^{a_{ij}} z_1/z_2)} \psi^+_j(z_2)\psi^+_i(z_1), \]  

(3.60)

\[ \psi^+_i(z_1)e_j(z_2) = q^{-a_{ij}} \frac{\Theta_{p^*}(q^{a_{ij}-c/2} z_1/z_2)}{\Theta_{p^*}(q^{-a_{ij}-c/2} z_1/z_2)} e_j(z_2)\psi^+_i(z_1), \]  

(3.61)

\[ \psi^+_i(z_1)f_j(z_2) = q^{a_{ij}} \frac{\Theta_{p}(q^{-a_{ij}+c/2} z_1/z_2)}{\Theta_{p}(q^{a_{ij}+c/2} z_1/z_2)} f_j(z_2)\psi^+_i(z_1). \]  

(3.62)

Let \( U_q(\mathfrak{g}) \) be the quantum affine algebra over \( \mathbb{C} \) associated with the untwisted affine Lie algebra \( \hat{\mathfrak{g}} \) in the Drinfeld realization \([11]\) and \( x_j^\pm(z), k_i^\pm(z) \) be the Drinfeld currents. See Appendix \([A]\) for the \( \mathfrak{gl}_N \) case. The other cases can be found, for example in \([17]\). Then \( U_{q,p}(\hat{\mathfrak{g}}) \) is a natural face type (i.e. dynamical) elliptic deformation of \( U_q(\hat{\mathfrak{g}}) \) in the following sense.

Theorem 3.5. \([17]\)

\[ U_{q,p}(\hat{\mathfrak{g}})/pU_{q,p}(\hat{\mathfrak{g}}) \cong (\mathbb{F} \otimes \mathbb{C} U_q(\hat{\mathfrak{g}})) \sharp \mathbb{C}[\mathcal{R}_Q] \]

by the following identification at \( p = 0 \).

\[ e_j(z) = x_j^+(z)e^{-Q_{a_j}}, \quad f_j(z) = x_j^-(z), \quad k_i^+(z) = k_i^-(z)e^{-Q_{c_i}}. \]

Here the smash product \( \sharp \) is defined as follows.

\[
g(P, P + h) a \otimes e^{Q_a} \cdot f(P, P + h) b \otimes e^{Q_b} = g(P, P + h) f(P - <Q_a, P >, P + h - <Q_a + \text{wt}(a), P + h>) ab \otimes e^{Q_a + Q_b}
\]

where \( \text{wt}(a) \in \hat{\mathfrak{h}}^* \) s.t. \( q^h a q^{-h} = q^{<\text{wt}(a), h>} a \) for \( a, b \in U_q(\hat{\mathfrak{g}}), f(P), g(P) \in \mathbb{F}, e^{Q_a}, e^{Q_b} \in \mathbb{C}[\mathcal{R}_Q] \).

Definition 3.6. Let us introduce the multiplicative dynamical parameters \( x = (x_1, \ldots, x_N), x_i = q^{2P_{x_i}} \). We set \( U_{q,x}(\hat{\mathfrak{g}}) = U_{q,p}(\hat{\mathfrak{g}})/pU_{q,p}(\hat{\mathfrak{g}}) \) and call it the dynamical quantum affine algebra in the Drinfeld realization.

3.2 \( E_{q,p}(\hat{\mathfrak{g}}_N) \)

Let \( \mathcal{L}_{i,j,n} (n \in \mathbb{Z}, 1 \leq i, j \leq N) \) be abstract symbols. We define \( L^+(z) = \sum_{1 \leq i, j \leq N} E_{ij} L^+_{ij}(z) \) by

\[ L^+_{ij}(z) = \sum_{n \in \mathbb{Z}} L_{i,j,n} z^{-n}, \quad L_{i,j,n} = p^{\text{max}(n,0)} \mathcal{L}_{i,j,n}, \]

(3.63)

Definition 3.7. Let \( R^+(z, s) \) be the same \( R \) matrix as in Sec\([27]\). The elliptic algebra \( E_{q,p}(\hat{\mathfrak{g}}_N) \) is a topological algebra over \( \mathbb{F}[[p]] \) generated by \( \mathcal{L}_{i,j,n}, \hat{d} \) and the central element \( q^{\pm c/2} \) satisfying
the following relations.

\[
R^{(12)}(z_1/z_2, P + h)L^{(1)}(z_1)L^{(2)}(z_2) = L^{(2)}(z_2)L^{(1)}(z_1)R^{+(12)}(z_1/z_2, P),
\]
\[\text{(3.64)}\]
\[
g(P + h)\tilde{L}_{ij,n} = \tilde{L}_{ij,n} g(P + h - <Q_{i}, P + h>),
\]
\[\text{(3.65)}\]
\[
g(P)\tilde{L}_{ij,n} = \tilde{L}_{ij,n} g(P - <Q_{i}, P>),
\]
\[\text{(3.66)}\]
\[
[\tilde{d}, L^{+}(z)] = -z\frac{\partial}{\partial z} L^{+}(z),
\]
\[\text{(3.67)}\]

where \(g(P + h), g(P) \in \mathbb{F}\) and

\[
L^{(1)}(z) = L^{+}(z) \otimes \text{id}, \quad L^{(2)}(z) = \text{id} \otimes L^{+}(z).
\]

We regard \(L^{+}(z) \in \text{End}\mathcal{V} \otimes E_{q, p}(\hat{\mathfrak{sl}}_2)\). We treat \((3.64)\) as a formal Laurent series in \(z_1\) and 
\(z_2\). Then the coefficients of \(z_1, z_2\) are well defined in the \(p\)-adic topology. See \[24\] for a similar 
formulation for the vertex type elliptic quantum algebra \(A_{q, p}(\hat{\mathfrak{sl}}_2)\). Note also that due to the 
RLL-relation \((3.64)\) the \(L\)-operator \(L^{+}(z)\) is invertible. See Appendix \[23\]

For later convenience we define \(L^{--}(z) = \sum_{1 \leq i, j \leq N} E_{ij} L^{-}_{ij}(z)\) by \[37\]

\[
L^{--}(z) = \left(\text{Ad}(q^{-2\nu(P)} \otimes \text{id}) \left(q^{2T_{V}} L^{+}(z^{p^{*}} q^{c})\right)\right),
\]
\[\text{(3.68)}\]
\[
\theta_{V}(P) = -\sum_{j=1}^{N-1} \left(\frac{1}{2} \nu_{V}(h_{j}) \nu_{V}(h^{j}) + P_{j} \nu_{V}(h^{j})\right),
\]
\[\text{(3.69)}\]
\[
T_{V} = \sum_{j=1}^{N-1} \nu_{V}(h_{j}) \otimes h^{j}.
\]
\[\text{(3.70)}\]

Here \((\text{Ad}X)Y = XYX^{-1}, h^{j} = h_{\lambda_{j}}\) \((j \in I)\), \(\nu_{V}(h_{j}) = E_{jj} - E_{j+1,j+1}\) and \(\nu_{V}(h^{j}) = \sum_{i=1}^{I} \nu_{V}(h_{\epsilon_{i}})\) \((j \in I)\). Then one can verify the following.

Proposition 3.8. The \(L\) operators \(L^{+}(z)\) and \(L^{--}(z)\) satisfy the following relations.

\[
R^{-(12)}(z_1/z_2, P + h)L^{-(1)}(z_1)L^{-(2)}(z_2) = L^{-(2)}(z_2)L^{-(1)}(z_1)R^{-(12)}(z_1/z_2, P),
\]
\[\text{(3.71)}\]
\[
R^{+(12)}(q^{\pm c} z_{1}/z_{2}, P + h)L^{+(1)}(z_{1})L^{+(2)}(z_{2}) = L^{+(2)}(z_{2})L^{+(1)}(z_{1})R^{+(12)}(q^{\mp c} z_{1}/z_{2}, P).
\]
\[\text{(3.72)}\]

proof) Replace \(z_i\) with \(z_{i}p^{i}q^{c}\) \((i = 1, 2)\) in \((3.64)\). Note that \((3.65), (3.66)\) and \((3.68)\) yields

\[
L^{+}(p^{*} q^{c} z) = q^{2^{N-1}} \sum_{i,j} q^{-2(P+h)_{i,j}} q^{2P_{i,j}} E_{ij} L^{-}_{ij}(z).
\]

By a componentwise comparison we obtain

\[
R^{+}(z_1/z_2, P + h)L^{-}(z_1)L^{-}(z_2) = L^{-}(z_2)L^{-}(z_1)R^{+(12)}(z_1/z_2, P).
\]

Then noting \((2.11)\) and \((2.13)\), we obtain \((3.71)\).
Similarly let us replace \( z_1 \) by \( z_1^+q^c \) in (3.64). Noting \( p^*q^c = pq^{-c} \), the components of \( R^+ \) are changed as

\[
p^+(zpq^{-c}) = q^{-N_{x+1}}p^+(zq^{-c}),
\]
\[
b(zpq^{-c}, s) = q^2b(zq^{-c}, s), \quad \bar{b}(zpq^{-c}) = q^2\bar{b}(zq^{-c}), \tag{3.73}
\]
\[
c(zpq^{-c}, \pm s) = q^{\mp 2s+2}c(zq^{-c}, \pm s)
\]

and similarly for \( R^{++} \). Then from (2.3) and (2.11), we obtain the second (lower sign) relation in (3.72). Note that a factor arising from the action of \( \text{Ad}(q^{-20\delta(P)}) \otimes \text{id} \) on the \( L \)-operators cancels the extra factors in (3.73).

To obtain the first relation in (3.72), exchange \( z_1 \) and \( z_2 \) in the second relation of (3.72). Then we have

\[
R^-(q^{-c}z_2/z_1, P + h)^{-1}L^+(z_1)L^-(z_2) = L^-(z_2)L^+(z_1)R^{+*}(q^c z_2/z_1, P)^{-1}.
\]

Using (2.11), we obtain the desired result.

Remark. We can expand (3.64) and (3.71) in both \( z = z_1/z_2 \) and \( z^{-1} = z_2/z_1 \). However (3.72) admits an expansion only in \( z \) (resp. \( z^{-1} \)) for the upper (resp. lower) sign case for the sake of the well-definedness in the \( p \)-adic topology. It is instructive to compare this with the trigonometric case [9].

In the component form (3.64), (3.71) and (3.72) are

\[
\sum_{i', j'} R^\pm(z_1/z_2, P + h)_{i, j'} L_{i', j'}^\pm(z_1)L_{i, j'}^\pm(z_2) = \sum_{i', j'} L_{i', j'}^\pm(z_2)L_{i, j'}^\pm(z_1)R^{\pm*}(z_1/z_2, P)_{i', j'}^{i, j'}, \tag{3.74}
\]
\[
\sum_{i', j'} R^\pm(q^{-c}z_1/z_2, P + h)_{i, j'} L_{i', j'}^\pm(z_1)L_{i, j'}^\pm(z_2) = \sum_{i', j'} L_{i', j'}^\pm(z_2)L_{i, j'}^\pm(z_1)R^{\pm*}(q^c z_1/z_2, P)_{i', j'}^{i, j'}, \tag{3.75}
\]

We call (3.74) the \((i, j), (i'', j'')\) component of (3.64), etc.

Remark. In order to obtain a ‘fully’ dynamical RLL-relations used in [19, 21] with a central extension one may introduce the \( L \)-operators \( L^\pm(z, P) \) related to our \( L^\pm(z) \) by

\[
L^\pm(z, P) = L^\pm(z)e^{\sum_{i=1}^{N} \pi_V(h_{ei}) \otimes Q_{ei}, \tag{3.76}
\]

where \( \pi_V(h_{ei}) = E_{i, i} \). In fact from (3.65) and (3.66) we have

\[
[L_{i, j}^\pm(z, P), f(P)] = 0, \tag{3.77}
\]
\[
g(h)L_{i, j}^\pm(z, P) = L_{i, j}^\pm(z, P)g(h - <\alpha_{ij}, h>), \tag{3.78}
\]
\[
[\hat{a}, L^\pm(z)] = -z \frac{\partial}{\partial z} L^\pm(z). \tag{3.79}
\]
(3.47) indicates that $L^+(z, P)$ is independent of $\mathbb{C}[\mathcal{R}_Q]$. Furthermore from (2.2), (3.64) (3.71) and (3.72), $L^\pm(z, P)$ satisfy the following full dynamical RLL-relations

$$ R^{\pm(12)}(z_1/z_2, P + h)L^{\pm(1)}(z_1, P)L^{\pm(2)}(z_2, P + \pi_V(h)) = L^{\pm(2)}(z_2, P)L^{\pm(1)}(z_1, P + \pi_V(h))R^{\pm(12)}(z_1/z_2, P), \quad (3.80) $$

$$ R^{\pm(12)}(q^\pm c z_1/z_2, P + h)L^{\pm(1)}(z_1, P)L^{\mp(2)}(z_2, P + \pi_V(h)) = L^{\mp(2)}(z_2, P)L^{\pm(1)}(z_1, P + \pi_V(h))R^{\pm(12)}(q^\mp c z_1/z_2, P). \quad (3.81) $$

Here the generators are clear. If we set $L^\pm(z, P) = \sum_{i,j} E_{i,j} L^\pm_{ij}(z, P)$ with $L^\pm_{ij}(z, P) = \sum_{m\in\mathbb{Z}} L^\pm_{ij,m}(P) z^{-n}$, then from (3.63) we have $L^\pm_{ij,m}(P) = L^\pm_{ij,m} e^{-Q_{ij}}$.

Remark. The dynamical RLL relations (3.80)-(3.81) coincides with those derived from the universal DYBE for $\mathcal{B}_{q,\lambda}(\hat{g})$ in [36, 37].

3.3 Reflection equations

Following [59], let us set

$$ \mathcal{L}(z) = L^+(z q^c)L^-(z)^{-1}. $$

Then using (3.64), (3.71)-(3.72), one can show the following relations.

Proposition 3.9.

$$ R^{+(12)}(z_1/z_2, P + h)\mathcal{L}^{(1)}(z_1)R^{+(21)}(q^2 c z_1/z_2, P + h)\mathcal{L}^{(2)}(z_2) = \mathcal{L}^{(2)}(z_2)R^{+(12)}(q^2 c z_1/z_2, P + h)\mathcal{L}^{(1)}(z_1)R^{+(21)}(z_1/z_2, P + h), $$

$$ R^{+(12)}(z_1/z_2, P + h)L^{+(1)}(z_1 q^c)\mathcal{L}^{(2)}(z_2) = \mathcal{L}^{(2)}(z_2)R^{+(12)}(q^2 c z_1/z_2, P + h)L^{+(1)}(z_1 q^c). $$

3.4 The trigonometric limit

Let us consider the trigonometric counterpart of $E_{q,p}(\hat{g})$ according to an idea described in [59]. Set $L^\pm_{0,ij}(z) = L^\pm_{ij}(z)|_{p=0}$. From (3.63) and (3.68), we have

$$ L^\pm_{0,ij}(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} L^\pm_{0,ij,m} z^{\pm m} \quad (1 \leq i, j \leq N) $$

where

$$ L^+_{0,ij,-m} = L_{ij,-m}|_{p=0}, \quad L^-_{0,ij,m} = q^{2(P+h)i} L_{ij,m}|_{p=0} q^{-2P_i} q^{cm} \quad (m \in \mathbb{Z}_{\geq 0}). $$

16
Let \( R_0^\pm(z,s) \) be the trigonometric dynamical \( R \) matrix in \((2.24)\). From \((3.64)\), \((3.71)\) and \((3.72)\), \( L_0^\pm(z) = \sum_{1 \leq i,j \leq N} E_{ij} L_{0;ij}^\pm(z) \) satisfy for \( g(P+h), g(P) \in \mathbb{F} \)

\[
\begin{align*}
P_0^\pm(12) (z_1/z_2, P+h)L_0^\pm(1) (z_1)L_0^\pm(2) (z_2) = & \quad L_0^\pm(2) (z_2)L_0^\pm(1) (z_1)R_0^\pm(12) (z_1/z_2, P), \\
P_0^\pm(12) (q^{+c}z_1/z_2, P+h)L_0^\pm(1) (z_1)L_0^\pm(2) (z_2) = & \quad L_0^\pm(2) (z_2)L_0^\pm(1) (z_1)R_0^\pm(12) (q^{+c}z_1/z_2, P). 
\end{align*}
\]

Let \( 0 \leq i,j \leq m \) be the trigonometric dynamical \( R \) matrix in \((3.73)\). Then \( L_0^\pm(1) (z_1)L_0^\pm(2) (z_2) \) satisfy the same relations as \((3.77)-(3.79)\) as well as the trigonometric limit of the dynamical dependence from \((3.82)\) subject to \((3.84)-(3.86)\). We call \( U_{q,x}^R(\hat{\mathfrak{g}}_N) \) the dynamical quantum affine algebra in the FRST formulation.

Hence we have

**Proposition 3.11.**

\[
E_{q,p}(\hat{\mathfrak{g}}_N)/pE_{q,p}(\hat{\mathfrak{g}}_N) \cong U_{q,x}^R(\hat{\mathfrak{g}}_N).
\]

In order to clarify a relation between the dynamical \( U_{q,x}^R(\hat{\mathfrak{g}}_N) \) and the usual quantum affine algebra \( U_q^R(\hat{\mathfrak{g}}_N) \) in the FRST formulation \([59]\), one needs to further remove the \( \mathbb{C}[R_Q] \) dependence from \( U_{q,x}^R(\hat{\mathfrak{g}}_N) \). This can be done by considering the algebra generated by the trigonometric limit \( L_0^\pm(z, P) \) of \( L^\pm(z, P) \) in \((3.76)\). Then \( L_0^\pm(z, P) \) satisfy the same relations as \((3.77)-(3.79)\) as well as the trigonometric limit of the dynamical RLL-relations \((3.80)-(3.81)\), where \( R^\pm(z, s) \) and \( R_0^\pm(z, s) \) are replaced by \( R_0^\pm(z, s) \) and \( R_0^\pm(z, s) \), respectively. We set

\[
L_{0;ij}^\pm(z, P) \equiv \sum_{m \in \mathbb{Z}_{\geq 0}} L_{0;ij;+m}^\pm(P)z^m 
\]

and denote by \( \tilde{U}_{q,x}^R(\hat{\mathfrak{g}}_N) \) the unital associative algebra over \( \mathbb{F} \) generated by \( L_{0;ij;+m}^\pm(P) \). Then we have

\[
U_{q,x}^R(\hat{\mathfrak{g}}_N) \cong \tilde{U}_{q,x}^R(\hat{\mathfrak{g}}_N)[[\mathbb{C}[R_Q]].
\]

Recall that \( R_0^\pm(z, P)^{kl}_{ij} \) can be expanded to a formal power series in \( x_{i,j} = q^{2P_{i,j}} \) and the 0-th order term gives the trigonometric \( R \) matrix in \((2.25)\). We assume the same property for \( L_0^\pm(z, P) \). Let \( L_0^\pm(z, P) = \sum_{|k|=0}^{\infty} \sum_{k \in \mathbb{N}^N} L_0^\pm(z; k)x^k, \) \( L_0^\pm(z; k) = \sum_{1 \leq i,j \leq N} E_{ij} L_{0;ij}^\pm(z, k) \), where \( k = (k_1, \ldots, k_N), |k| = k_1 + \cdots + k_N \) and \( x^k = x_1^{k_1} \cdots x_N^{k_N} \). Then \( L_0^\pm(z; 0) = \sum_{1 \leq i,j \leq N} E_{ij} L_{0;ij}^\pm(z, 0) \), \( L_{0;ij}(z; 0) = \sum_{m \in \mathbb{Z}_{\geq 0}} L_{0;ij;+m}(0)z^m \) satisfy the same RLL-relations as the quantum affine algebra \( U_q^R(\hat{\mathfrak{g}}_N) \) over \( \mathbb{C} \) in the FRST formulation \([59]\). Hence

\[
\tilde{U}_{q,x}^R(\hat{\mathfrak{g}}_N)/\left( \sum_i x_i \tilde{U}_{q,x}^R(\hat{\mathfrak{g}}_N) \right) \cong U_q^R(\hat{\mathfrak{g}}_N).
\]
4 Hopf Algebroid Structure

In this section, we introduce an $H$-Hopf algebroid structure \[15, 40, 50\] into the elliptic algebras $E_{q,p}(\hat{\mathfrak{g}}_N)$ and $U_{q,p}(\hat{\mathfrak{g}}_N)$, and formulate them as elliptic quantum groups.

4.1 $U_{q,p}(\hat{\mathfrak{g}}_N)$ and $E_{q,p}(\hat{\mathfrak{g}}_N)$ as $H$-Algebras

Let $A$ be an associative algebra, $H$ be a commutative subalgebra of $A$, and $M_{H^*}$ be the field of meromorphic functions on $H^*$ the dual space of $H$.

Definition 4.1. An associative algebra $A$ with 1 is said to be an $H$-algebra, if it is bigraded over $H^*$, $A = \bigoplus_{\alpha, \beta \in H^*} A_{\alpha \beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r: M_{H^*} \to A_{00}$ (the left and right moment maps), such that

$$\mu_l(\hat{f})a = a\mu_l(T_{\alpha}\hat{f}), \quad \mu_r(\hat{f})a = a\mu_r(T_{\beta}\hat{f}), \quad a \in A_{\alpha \beta}, \quad \hat{f} \in M_{H^*},$$

where $T_{\alpha}$ denotes the automorphism $(T_{\alpha}\hat{f})(\lambda) = \hat{f}(\lambda + \alpha)$ of $M_{H^*}$.

Definition 4.2. An $H$-algebra homomorphism is an algebra homomorphism $\pi: A \to B$ between two $H$-algebras $A$ and $B$ preserving the bigrading and the moment maps, i.e. $\pi(A_{\alpha \beta}) \subseteq B_{\alpha \beta}$ for all $\alpha, \beta \in H^*$ and $\pi(\mu_l^A(\hat{f})) = \mu_l^B(\hat{f}), \pi(\mu_r^A(\hat{f})) = \mu_r^B(\hat{f})$.

Let $A$ and $B$ be two $H$-algebras. The tensor product $A \tilde{\otimes} B$ is the $H^*$-bigraded vector space with

$$(A \tilde{\otimes} B)_{\alpha \beta} = \bigoplus_{\gamma \in H^*} (A_{\alpha \gamma} \otimes_{M_{H^*}} B_{\gamma \beta}),$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the following relation.

$$\mu_r^A(\hat{f})a \otimes b = a \otimes \mu_r^B(\hat{f})b, \quad a \in A, b \in B, \hat{f} \in M_{H^*}.$$

The tensor product $A \tilde{\otimes} B$ is again an $H$-algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ and the moment maps

$$\mu_l^{A \tilde{\otimes} B} = \mu_l^A \otimes 1, \quad \mu_r^{A \tilde{\otimes} B} = 1 \otimes \mu_r^B.$$

Let $D$ be the algebra of automorphisms $M_{H^*} \to M_{H^*}$

$$D = \{ \sum_i \hat{j}_T \beta_i | \hat{j}_i \in M_{H^*}, \beta_i \in H^* \}.$$
Equipped with the bigrading $D_{\alpha} = \{ \hat{f}_{\alpha} | \hat{f} \in \mathcal{M}_{\mathcal{H}^*}, \alpha \in \mathcal{H}^* \}$, $D_{\alpha\beta} = 0$ ($\alpha \neq \beta$) and the moment maps $\mu^P, \mu^D : \mathcal{M}_{\mathcal{H}^*} \to D_{00}$ defined by $\mu^P(\hat{f}) = \mu^D(\hat{f}) = \hat{f}T_0$, $D$ is an $\mathcal{H}$-algebra. For any $\mathcal{H}$-algebra $A$, we have the canonical isomorphism as an $\mathcal{H}$-algebra

$$A \cong A \hat{\otimes} D \cong D \hat{\otimes} A$$

(4.2)

by $a \cong a \hat{\otimes} T_{-\beta} \cong T_{-\alpha} \hat{\otimes} a$ for all $a \in A_{\alpha\beta}$.

Now let $H$ be the same as defined in Sec 2 and take $\mathcal{H} = H$.

Proposition 4.3. The $U = U_{q,p}(\hat{\mathfrak{g}}_N)$ is an $H$-algebra by

$$U = \bigoplus_{\alpha, \beta \in H^*} U_{\alpha, \beta}$$

(4.3)

$$U_{\alpha, \beta} = \{ a \in U \mid q^{P+h}a-q^{\alpha, P+h}a, \quad q^{P+h}a-q^{\beta, P}a, \quad \forall P, h, P \in H \}$$

and $\mu_1, \mu_r : \mathbb{F} \to U_{0,0}$ defined by [50]

$$\mu_1(\hat{f}) = f(P + h, p) \in \mathbb{F}[[p]], \quad \mu_r(\hat{f}) = f(P, p^*) \in \mathbb{F}[[p]].$$

Proposition 4.4. The $E = E_{q,p}(\hat{\mathfrak{g}}_N)$ is an $H$-algebra by

$$E = \bigoplus_{\alpha, \beta \in H^*} E_{\alpha, \beta},$$

(4.4)

$$E_{\alpha, \beta} = \{ a \in E \mid q^{P+h}a-q^{\alpha, P+h}a, \quad q^{P+h}a-q^{\beta, P}a, \quad \forall P, h, P \in H \}$$

and $\mu_1, \mu_r : \mathbb{F} \to E_{0,0}$ defined by the same $\mu_1, \mu_r$ as in $U$. Note that $\hat{L}_{ij,n} \in (E_{q,p})_{-Q_{ij}, -Q_{ij}}$.

We regard $T_\alpha = e^{-Q_\alpha} \in \mathbb{C}[\mathcal{R}_Q]$ as the shift operator $\mathbb{F}[[p]] \to \mathbb{F}[[p]]$

$$(T_\alpha \mu_r(\hat{f})) = e^{-Q_\alpha}f(P, p^*)e^{Q_\alpha} = f(P + <Q_\alpha, P >, p^*),$$

$$(T_\alpha \mu_1(\hat{f})) = e^{-Q_\alpha}f(P, h, p)e^{Q_\alpha} = f(P + h + <Q_\alpha, P >, h, p).$$

Then $D = \mathbb{F} \hat{\otimes} \mathbb{C}[\mathcal{R}_Q]$ becomes the $H$-algebra having the property (4.2) for $A = U, E$.

Hereafter we abbreviate $f(P + h, p)$ and $f(P, p^*)$ as $f(P + h)$ and $f^*(P)$, respectively.

### 4.2 H-Hopf algebroids $E_{q,p}(\hat{\mathfrak{g}}_N)$ and $U_{q,p}(\hat{\mathfrak{g}}_N)$

Let us first recall the $\mathcal{H}$-Hopf algebroid following [15, 40].

**Definition 4.5.** An $\mathcal{H}$-bialgebroid is an $\mathcal{H}$-algebra $A$ equipped with two $\mathcal{H}$-algebra homomorphisms $\Delta : A \to A \hat{\otimes} A$ (the comultiplication) and $\varepsilon : A \to D$ (the counit) such that

$$(\Delta \hat{\otimes} \text{id}) \circ \Delta = (\text{id} \hat{\otimes} \Delta) \circ \Delta,$$

$$(\varepsilon \hat{\otimes} \text{id}) \circ \Delta = (\text{id} \hat{\otimes} \varepsilon) \circ \Delta,$$

under the identification (4.2).
Definition 4.6. An $\mathcal{H}$-Hopf algebroid is an $\mathcal{H}$-bialgebroid $A$ equipped with a $\mathbb{C}$-linear map $S : A \to A$ (the antipode), such that

\[ S(\mu_r(\hat{f})a) = S(a)\mu_l(\hat{f}), \quad S(a\mu_l(\hat{f})) = \mu_r(\hat{f})S(a), \quad \forall a \in A, \hat{f} \in \mathcal{M}_{\mathcal{H}^*}, \]
\[ m \circ (\text{id} \otimes S) \circ \Delta(a) = \mu_l(\varepsilon(a)1), \quad \forall a \in A, \]
\[ m \circ (S \otimes \text{id}) \circ \Delta(a) = \mu_r(T_\alpha(\varepsilon(a)1)), \quad \forall a \in A_{\alpha\beta}, \]

where $m : A \otimes A \to A$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in \mathcal{M}_{\mathcal{H}^*}$.

Remark. Definition 4.6 yields that the antipode of an $\mathcal{H}$-Hopf algebroid uniquely exists and gives the algebra antihomomorphism.

The $\mathcal{H}$-algebra $D$ is an $\mathcal{H}$-Hopf algebroid with $\Delta_D : D \to D \otimes D$, $\varepsilon_D : D \to D$, $S_D : D \to D$ defined by

\[ \Delta_D(\hat{f}T_{-\alpha}) = \hat{f}T_{-\alpha} \otimes T_{-\alpha}, \]
\[ \varepsilon_D = \text{id}, \quad S_D(\hat{f}T_{-\alpha}) = T_\alpha \hat{f} = (T_\alpha \hat{f})T_\alpha. \]

Now let us consider the $H$-algebras $E$ and $U$. Let us first consider the $H$-Hopf algebroid structure on $E$. We define two $H$-algebra homomorphisms, the co-unit $\varepsilon : E \to D$ and the co-multiplication $\Delta : E \to E \otimes E$ by

\[ \varepsilon(L_{ij,n}) = \delta_{i,j}\delta_{n,0}T_{k_i} \quad (n \in \mathbb{Z}), \quad \varepsilon(e^Q) = e^Q, \quad (4.5) \]
\[ \varepsilon(\mu_l(\hat{f})) = \varepsilon(\mu_r(\hat{f})) = \hat{f}T_0, \quad (4.6) \]
\[ \Delta(L_{ij}^+(z)) = \sum_k L_{ik}^+(z) \otimes L_{kj}^+(z), \quad (4.7) \]
\[ \Delta(e^Q) = e^Q \otimes e^Q, \quad \Delta(d) = \hat{d} \otimes 1 + 1 \otimes \hat{d}, \quad (4.8) \]
\[ \Delta(\mu_l(\hat{f})) = \mu_l(\hat{f}) \otimes 1, \quad \Delta(\mu_r(\hat{f})) = 1 \otimes \mu_r(\hat{f}). \quad (4.9) \]

One can check that $\Delta$ preserves the relation in Definition 3.7.

Lemma 4.7. The maps $\varepsilon$ and $\Delta$ satisfy

\[ (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \quad (4.10) \]
\[ (\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta. \quad (4.11) \]

Proof. Straightforward. \hfill \Box

We also have the following formulae.
Proposition 4.8. 
\[
\Delta \left( \frac{f(P, p^*)}{f(P + h, p)} \right) = \frac{f(P, p^*)}{f(P + h, p)} \tilde{\otimes} \frac{f(P, p^*)}{f(P + h, p)}. \tag{4.12}
\]

Hence \((\mathcal{E}, \Delta, \mathcal{M}_{H^*}, \mu_l, \mu_r, \varepsilon)\) is a \(H\)-bialgebroid.

We define an algebra antihomomorphism (the antipode) \(S : \mathcal{E} \to \mathcal{E}\) by
\[
S(L_{ij}^+(z)) = (L^+(z)^{-1})_{ij}, \tag{4.13}
\]
\[
S(e^Q) = e^{-Q}, \quad S(\mu_r(\tilde{f})) = \mu_l(\tilde{f}), \quad S(\mu_l(\tilde{f})) = \mu_r(\tilde{f}). \tag{4.14}
\]

The explicit formula for (4.13) in terms of the components of the \(L\)-operator is given in Appendix E. Then \(S\) preserves the \(RLL\) relation (3.64) and satisfies the antipode axioms. We hence obtain

**Theorem 4.9.** The \(H\)-algebra \(\mathcal{E}\) equipped with \((\Delta, \varepsilon, S)\) is an \(H\)-Hopf algebroid.

**Definition 4.10.** We call the \(H\)-Hopf algebroid \((\mathcal{E}, H, \mathcal{M}_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, S)\) the elliptic quantum group \(E_{q,p}(\hat{\mathfrak{g}}_N)\).

Remark. The coproduct for \(L^+(z, P)\) used in [19] is essentially obtained from (4.7) via (3.76):
\[
\Delta(L^+(z, P)) = L^+(z, P) \otimes L^+(z, P + h^{(1)}). \tag{4.17}
\]

By making use of the isomorphism between \(U\) and \(\mathcal{E}\) given in Sec [18] we can define the \(L\)-operators of \(U\) by identifying them with those of \(\mathcal{E}\) in [6.1]. Then the \(H\)-Hopf algebroid structure of \(U\) is defined by using the same \(\Delta, \varepsilon, S\) as \(\mathcal{E}\). See [50] for the \(\hat{\mathfrak{sl}}_2\) case.

**Definition 4.11.** We call the \(H\)-Hopf algebroid \((U, H, \mathcal{M}_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, S)\) the elliptic quantum group \(U_{q,p}(\hat{\mathfrak{g}}_N)\).

Hence the isomorphism obtained in Sec [18] can be extended to as an \(H\)-Hopf algebroid.

Remark. \(U_{q,p}(\hat{\mathfrak{g}})\) admits another co-algebra structure through another coproduct called the Drinfeld coproduct [36, 51].

### 5 Dynamical Representations

#### 5.1 Definition

We summarize some basic facts on the dynamical representation of \(U_{q,p}(\hat{\mathfrak{g}}_N)\). Most of them can be extended to the arbitrary untwisted affine Lie algebra \(\hat{\mathfrak{g}}\) case [17].

Let us consider a vector space \(\hat{V}\) over \(\mathbb{F}[[p]]\), which is \(H\)-diagonalizable, i.e.
\[
\hat{V} = \bigoplus_{\lambda, \mu \in H^*} \hat{V}_{\lambda, \mu}, \quad \hat{V}_{\lambda, \mu} = \{ v \in \hat{V} \mid q^{P + h} \cdot v = q^{<\lambda, P + h> \mu} v, \quad q^P \cdot v = q^{<\mu, P> \lambda} v \forall P + h, P \in H \}.
\]
Let us define the $H$-algebra $\mathcal{D}_{H,\hat{V}}$ of the $\mathbb{C}$-linear operators on $\hat{V}$ by

$$\mathcal{D}_{H,\hat{V}} = \bigoplus_{\alpha, \beta \in H^*} (\mathcal{D}_{H,\hat{V}})_{\alpha \beta},$$

$$(\mathcal{D}_{H,\hat{V}})_{\alpha \beta} = \left\{ X \in \text{End}_{\hat{V}} \left| \begin{array}{c}
  f(P + h)X = X f(P + h + < \alpha, P + h >), \\
  f(P)X = X f(P + < \beta, P >), \forall f(P), f(P + h) \in \mathbb{F}[p], \\
  X \cdot \hat{V}_{\lambda, \mu} \subseteq \hat{V}_{\lambda + \alpha, \mu + \beta}
  \end{array} \right. \right\},$$

$$\mu^D_{H,\hat{V}}(f)v = f(< \lambda, P + h, >, p)v, \quad \mu^D_{H,\hat{V}}(f)v = f(< \mu, P, >, p^*)v, \quad f \in \mathbb{F}[p], \quad v \in \hat{V}_{\lambda, \mu}.$$

**Definition 5.1.** We define a dynamical representation of $U_{q,p}(\hat{\mathfrak{gl}}_N)$ on $\hat{V}$ to be an $H$-algebra homomorphism $\pi : U_{q,p}(\hat{\mathfrak{gl}}_N) \to \mathcal{D}_{H,\hat{V}}$. By the action $\pi$ of $U_{q,p}(\hat{\mathfrak{gl}}_N)$ we regard $\hat{V}$ as a $U_{q,p}(\hat{\mathfrak{gl}}_N)$-module.

**Definition 5.2.** For $k \in \mathbb{C}$, we say that a $U_{q,p}(\hat{\mathfrak{gl}}_N)$-module has level $k$ if $q^{\pm c/2}$ acts as the scalar $q^{\pm k/2}$ on it.

**Definition 5.3.** Let $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$ be the subalgebras of $U_{q,p}(\hat{\mathfrak{gl}}_N)$ generated by $q^{\pm c/2}$, $d$, $k_{i,0}$ ($1 \leq i \leq N$), $k_{i, n}$ ($1 \leq i \leq N, n \in \mathbb{Z}_{>0}$), $e_{j, n}$ ($j \in I, n \in \mathbb{Z}_{>0}$) $f_{j, n}$ ($j \in I, n \in \mathbb{Z}_{>0}$) and $k_{i, -n}$ ($1 \leq i \leq N, n \in \mathbb{Z}_{>0}$), $e_{j, -n}$ ($j \in I, n \in \mathbb{Z}_{>0}$), $f_{j, -n}$ ($i \in I, n \in \mathbb{Z}_{>0}$), respectively.

**Definition 5.4.** For $k \in \mathbb{C}$, $\lambda, \mu \in H^*$, a dynamical $U_{q,p}(\hat{\mathfrak{g}})$-module $\hat{V}(\lambda, \mu)$ is called the level-$k$ highest weight module with the highest weight $(\lambda, \mu)$, if there exists a vector $v \in \hat{V}(\lambda, \mu)$ such that

$$\hat{V}(\lambda, \mu) = U_{q,p}(\hat{\mathfrak{g}}) \cdot v, \quad \mathfrak{n}_+ \cdot v = 0, \quad q^{\pm c/2} \cdot v = q^{\pm k/2}v,$$

$$k_{i,0} \cdot v = q^{-<\lambda - \mu, h_i>} v, \quad f(P) \cdot v = f(<\mu, P>)v, \quad f(P + h) \cdot v = f(<\lambda, P + h>)v.$$

### 5.2 The evaluation $H$-algebra homomorphism

Let $k^\pm_{0,i}(z)$, $x^\pm_j(z)$ be the Drinfeld currents of the quantum affine algebra $U_q(\hat{\mathfrak{gl}}_N)$. See Appendix A. Let us introduce the currents $u^+_{\varepsilon i}(z, p) \in (U_q(\hat{\mathfrak{gl}}_N)[[p]][[z]], u^-_{\varepsilon i}(z, p) \in (U_q(\hat{\mathfrak{gl}}_N)[[p]][[z^{-1}]] (1 \leq i \leq N)$ by

$$u^+_{\varepsilon i}(z, p) = \prod_{n=1}^{\infty} \left( k^-_{i,0} \cdot k^+_{0,i}(p^n q^{c-1}z) \right),$$

$$u^-_{\varepsilon i}(z, p) = \prod_{n=1}^{\infty} \left( k^+_{i,0} \cdot k^-_{0,i}(p^{-n} q^{c-i}z) \right).$$

We also set

$$u^\pm_j(z, p) = u^\pm_{\varepsilon j}(z, p) u^\pm_{\varepsilon j+1}(qz, p)^{-1} \quad (1 \leq j \leq N - 1).$$

22
These are well defined elements in \((U_q(\mathfrak{gl}_N)[[p]])[[z, z^{-1}]]\) in the \(p\)-adic topology.

Now let us define the ‘dressed’ currents \(x^\pm_j(z, p)\) \((1 \leq j \leq N - 1)\), \(k^\pm_i(z, p)\) \((1 \leq i \leq N)\) by

\[
x^+_j(z, p) = u^+_j(z, p)x^+_j(z)e^{-Q_{rj}},
\]
\[
x^-_j(z, p) = x^-_j(z)u^-_j(z, p),
\]
\[
k^+_i(z, p) = u^+_i(q^{-c+j}z, p)k^+_i(z)u^-_i(q^jz, p)e^{-Q_{ri}},
\]
\[
k^-_i(z, p) = u^+_i(q^jz, p)k^-_i(z)u^-_i(q^{-c+j}z, p)e^{-Q_{ri}}.
\]

**Theorem 5.5.** The map \(\phi_p : U_{q,p}(\mathfrak{gl}_N)[[z, z^{-1}]] \to (\mathbb{F}[[p]] \otimes \mathbb{C} U_q(\mathfrak{gl}_N))[[z, z^{-1}]] \otimes \mathbb{C}[\mathcal{R}_Q]\) defined by

\[
e_i(z) \mapsto x^+_i(z, p), \quad f_i(z) \mapsto x^-_i(z, p), \quad k^+_i(z) \mapsto k^-_i(z, p)
\]

is an \(H\)-algebra homomorphism. We call \(\phi_p\) the evaluation \(H\)-algebra homomorphism.

**Proof.** Direct calculations using Lemma B.3.1 \(\square\)

Let \((\varphi_V, V)\) be a representation of \(U_q(\mathfrak{gl}_N)\). We assume \(V\) is an \(\mathfrak{h}\)-diagonalizable vector space over \(\mathbb{C}\). We set \(V_{\mathbb{F}[[p]]} = \mathbb{F}[[p]] \otimes \mathbb{C} V\). Let \(V_Q\) be a vector space over \(\mathbb{C}\), on which an action of \(e^Q\) is defined appropriately. Two important examples of \(V_Q\) are \(V_Q = \mathbb{C} 1\) and \(V_Q = \mathbb{C}[\mathcal{R}_Q]\), where \(1\) denotes the vacuum state satisfying \(e^Q 1 = 1\). Let us consider the vector space \(\widetilde{V}_{\mathbb{F}[[p]]} = V_{\mathbb{F}[[p]]} \otimes \mathbb{C} V_Q\), on which the actions of \(f(P, h, p) \in \mathbb{F}[[p]]\) and \(e^Q\) are defined as follows. For \(v \otimes \xi \in V \otimes V_Q\),

\[
f(P, h, p). (v \otimes \xi) = f(P, \text{wt}(v), p)v \otimes \xi,
\]
\[
e^Q(f(P, h, p)v \otimes \xi) = f(P - <Q, P >, h, p)v \otimes e^Q \xi,
\]

where \(h.v = \text{wt}(v)v\). We extend \(\varphi_V : U_q(\mathfrak{gl}_N) \to \text{End}_\mathbb{C} V\) to a dynamical representation \(\varphi_V : \mathbb{F}[[p]] \otimes \mathbb{C} U_q(\mathfrak{gl}_N)[[z, z^{-1}]] \otimes \mathbb{C}[\mathcal{R}_Q] \to \mathcal{D}_H \widetilde{V}_{\mathbb{F}[[p]]}\) by

\[
\varphi_V(f(P)) = f(P), \quad \varphi_V(e^{Q_\alpha}) = e^{Q_\alpha} \quad \forall e^{Q_\alpha} \in \mathbb{C}[\mathcal{R}_Q].
\]

Note that if we specialize \(p = 0\), \(\widetilde{V}_{\mathbb{F}[[p]]}\) becomes \(V_{\mathbb{F}} \otimes V_Q\), where \(V_{\mathbb{F}} = \mathbb{F} \otimes \mathbb{C} V\).

Then from Theorem 5.5 we obtain the following.

**Corollary 5.6.** A map \(\varphi^p_V = \varphi_V \circ \phi_p : U_{q,p}(\mathfrak{gl}_N) \to \mathcal{D}_H \widetilde{V}_{\mathbb{F}[[p]]}\) gives a dynamical representation of \(U_{q,p}(\mathfrak{gl}_N)\) on \(\widetilde{V}_{\mathbb{F}[[p]]}\). We call \((\varphi^p_V, \widetilde{V}_{\mathbb{F}[[p]]})\) the evaluation dynamical representation.

Due to this corollary, any representation of \(U_q(\mathfrak{gl}_N)\) admits an ‘elliptic and dynamical deformation’ for generic \(p\). One can easily extend this to any untwisted affine Lie algebra case by using the evaluation homomorphism given in Appendix A of [36]. See [50] for \(\mathfrak{sl}_2\) case.
6 Isomorphism Between $U_{q,p} (\hat{\mathfrak{g}}_N)$ and $E_{q,p} (\hat{\mathfrak{g}}_N)$

We introduce the Gauss components of the $L$ operator of $E = E_{q,p} (\hat{\mathfrak{g}}_N)$ and the half currents of $U = U_{q,p} (\hat{\mathfrak{g}}_N)$. Then we show the isomorphism between $U$ and $E$.

6.1 The $L$-operators of $E$

Let us set

$$E^\pm = \{ A(z) \in E[[p]][[z, z^{-1}]] \mid A(z) \in E[[z^\pm]] \text{ mod } pE[[p]][[z, z^{-1}]] \}.$$ 

Then it is easy to show

**Lemma 6.1.** For $A(z), B(z) \in E^\pm$, the product $A(z)B(z)$ is a well-defined element in $E^\pm$ in the $p$-adic topology, respectively. Conversely, if $A(z), B(z) \in E[[p]][[z, z^{-1}]]$ satisfy $A(z)B(z) \in E^\pm$, then $A(z), B(z) \in E^\pm$, respectively.

**Definition 6.1.** We define the Gauss components $E_{l,j}^\pm(z), F_{j,l}^\pm(z), K_m^\pm(z)$ ($1 \leq j < l \leq N, 1 \leq m \leq N$) of the $L$-operator $L^\pm(z)$ of $E$ as follows.

$$L^\pm(z) = \begin{pmatrix} 1 & F_{1,2}^\pm(z) & F_{1,3}^\pm(z) & \cdots & F_{1,N}^\pm(z) \\ 0 & 1 & F_{2,3}^\pm(z) & \cdots & F_{2,N}^\pm(z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & F_{N-1,N}^\pm(z) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K_1^\pm(z) & 0 & \cdots & 0 \\ 0 & K_2^\pm(z) & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & K_N^\pm(z) \end{pmatrix}$$

\[ \times \begin{pmatrix} 1 & 0 & \cdots & 0 \\ E_{2,1}^\pm(z) & 1 & \ddots & \vdots \\ E_{3,1}^\pm(z) & E_{3,2}^\pm(z) & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ E_{N,1}^\pm(z) & E_{N,2}^\pm(z) & \cdots & E_N^\pm(z) \end{pmatrix} \]

In particular we call $E_{l,j}^\pm(z), F_{j,l}^\pm(z), K_m^\pm(z)$ the basic Gauss components.

**Remark.** By definition the matrix elements $L_{l,j}^\pm(z)$ are the elements in $E^\pm$, respectively. Then from Lemma 6.1 the matrix elements $E_{l,j}^\pm(z), F_{j,l}^\pm(z), K_m^\pm(z)$ of the right hand side of (6.1) are elements in $E^\pm$, respectively and their products are well defined formal Laurent series in $z$ in the $p$-adic topology. In addition, since $L^\pm(z)$ are invertible, $K_m^\pm(z)$ ($1 \leq m \leq N$) are invertible. Therefore all the components $E_{l,j}^\pm(z), F_{j,l}^\pm(z)$ and $K_m^\pm(z)$ are determined uniquely by $L_{l,j}^\pm(z)$, respectively.

Hence we define the coefficients of the Gauss components $E_{l,j}^\pm(z), F_{j,l}^\pm(z), K_m^\pm(z)$ as follows.
Definition 6.3.

\[ E^+_{i,j}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} E^+_{i,j,-n} z^n + \sum_{n \in \mathbb{Z}_{\geq 0}} E^+_{i,j,n} p^n z^{-n}, \quad (6.2) \]
\[ F^+_{i,j}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} F^+_{i,j,-n} z^n + \sum_{n \in \mathbb{Z}_{\geq 0}} F^+_{i,j,n} p^n z^{-n}, \quad (6.3) \]
\[ K^+_{j}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} K^+_{j,-n} z^n + \sum_{n \in \mathbb{Z}_{\geq 0}} K^+_{j,n} p^n z^{-n}. \quad (6.4) \]

In addition, from the definition of \( L^-(z) \) \[^{[6.8]}\], we have

\[ E^-_{j,i}(z) = q^{2P_{ij}} E^+_{j,i,n} (zpq^{-c}) q^{-2P_{ii}}, \quad (6.5) \]
\[ F^-_{i,j}(z) = q^{2(P+h) \epsilon_i} F^+_{i,j,n} (zpq^{-c}) q^{-2(P+h) \epsilon_j}, \quad (6.6) \]
\[ K^-_{i}(z) = q^{2(N-1) \epsilon_i} q^{2(P+h) \epsilon_i} K^+_{i,n} q^{-2P_{ii}}. \quad (6.7) \]

Hence we define

**Definition 6.3.**

\[ E^-_{j,i,n} = q^{2P_{ij}} E^+_{j,i,n} q^{-2P_{ii}}, \quad F^-_{i,j,n} = q^{2(P+h) \epsilon_i} F^+_{i,j,n} q^{-2(P+h) \epsilon_j}, \]
\[ K^-_{i,n} = q^{2(N-1) \epsilon_i} q^{2(P+h) \epsilon_i} K^+_{i,n} q^{-2P_{ii}} \]

for \( n \in \mathbb{Z} \).

Then we have

\[ E^-_{j,i}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} E^-_{j,i,-n} p^n (q^{-c} z)^n + \sum_{n \in \mathbb{Z}_{\geq 0}} E^-_{j,i,n} (q^{-c} z)^{-n}, \quad (6.9) \]
\[ F^-_{i,j}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} F^-_{i,j,-n} p^n (q^{-c} z)^n + \sum_{n \in \mathbb{Z}_{\geq 0}} F^-_{i,j,n} (q^{-c} z)^{-n}, \quad (6.10) \]
\[ K^-_{i}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} K^-_{i,-n} p^n (q^{-c} z)^n + \sum_{n \in \mathbb{Z}_{\geq 0}} K^-_{i,n} (q^{-c} z)^{-n}. \quad (6.11) \]

### 6.2 Subalgebras

For \( 1 \leq l < N \), let us define the reduced \( R \)-matrix and \( L \)-operators by

\[ R^\pm_l(z,s) = (R^\pm_{(z,s)} {^i_j})_{l \leq i,j, l' j' \leq N} \]
\[ L^\pm_l(z) = (L^\pm_{(z)})_{l \leq i,j \leq N}. \]

Note that up to overall factors \( R^\pm_l(z,s) \) are the elliptic dynamical \( R \)-matrix of type \( A^{(1)}_{N-l} \). Note also that if \( R^\pm_{(z,s)} {^i_j}^l j' \neq 0 \) for \( 1 \leq i, j \leq l \) (resp. \( l \leq i, j \leq N \)), then \( 1 \leq l', j' \leq l \) (resp. \( l \leq l', j' \leq N \)). Hence we obtain
Proposition 6.4. The reduced $L$-operators $L^\pm_l(z)$ satisfy

\[ R^\pm_{l}^{(12)}(z_1/z_2, P + h)L^\pm_l(1)(z_1)L^\pm_l(2)(z_2) = L^\pm_l(2)(z_2)L^\pm_l(1)(z_1)R^\pm_{l}^{(12)}(z_1/z_2, P), \]  
\[ R^\pm_{l}^{(12)}(q^{\pm c}z_1/z_2, P + h)L^\pm_l(1)(z_1)L^\pm_l(2)(z_2) = L^\pm_l(2)(z_2)L^\pm_l(1)(z_1)R^\pm_{l}^{(12)}(q^{\pm c}z_1/z_2, P). \]  

Therefore $L^\pm_l(z)$, $\tilde{d}$, $q^{+c/2}$ generate a subalgebra of $E_{q,p}(\hat{\mathfrak{g}}_N)$, which is isomorphic to $E_{q,p}(\hat{\mathfrak{g}}_{N-l+1})$.

Note that from (6.1) we have

\[
L^\pm_l(z) = \begin{cases}
1 & F^\pm_{l,l+1}(z) & F^\pm_{l,l+2}(z) & \cdots & F^\pm_{l,N}(z) \\
0 & 1 & F^\pm_{l+1,l+2}(z) & \cdots & F^\pm_{l+1,N}(z) \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & F^\pm_{N-1,N}(z) & \\
0 & \cdots & \cdots & 0 & 1
\end{cases}
\]

\[
\begin{pmatrix}
K^\pm_l(z) & 0 & \cdots & 0 \\
0 & K^\pm_{l+1}(z) & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & K^\pm_N(z)
\end{pmatrix}
\]

\[
\times
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
E^\pm_{l+1,l}(z) & 1 & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
E^\pm_{N,l}(z) & E^\pm_{N,l+1}(z) & \cdots & E^\pm_{N,N-1}(z) & 1
\end{pmatrix}
\]

Hence

\[
L^\pm_l(z)^{-1} = \begin{pmatrix}
K^\pm_l^{-1} & -K^\pm_l^{-1}F^\pm_{l,l+1} & \cdots \\
-K^\pm_l^{-1}F^\pm_{l,l+1} & K^\pm_l^{-1}E^\pm_{l,l+1} + K^\pm_{l+1} & \cdots \\
y_lK^\pm_l^{-1} & -y_lK^\pm_l^{-1}F^\pm_{l,l+1} + E^\pm_{l+2,l+1}K^\pm_{l+1}^{-1} & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix},
\]

where we omitted the argument $z$ and set

\[
x_l(z) = F^\pm_{l,l+1}(z)F^\pm_{l,l+2}(z) - F^\pm_{l,l+2}(z),
\]
\[
y_l(z) = E^\pm_{l+2,l+1}(z)E^\pm_{l+1,l}(z) - E^\pm_{l+2,l}(z).
\]

Furthermore, for $l < m \leq N$ let us define

\[
R^m_l(z,s) := (R^\pm_l(z,s))^{(l')}_{ij}|_{l \leq i, j, l' \leq m},
\]
\[
(L^\pm_l(z)^{-1})^m := ((L^\pm_l(z)^{-1}))^{(l')}_{ij}|_{l \leq i, j \leq m}.
\]
Proposition 6.5. If we set

\[ L_l^{m\pm}(z) = \begin{pmatrix} 1 & F_{l,l+1}^+(z) & F_{l,l+2}^+(z) & \cdots & F_{l,m}^+(z) \\ 0 & 1 & F_{l+1,l+2}^+(z) & \cdots & F_{l+1,m}^+(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & F_{m-1,m}^+(z) \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} K_l^+(z) & 0 & \cdots & 0 \\ 0 & K_{l+1}^+(z) & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & K_m^+(z) \end{pmatrix} \times
\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E_{l+1,l}^+(z) & 1 & \cdots & \vdots & \vdots \\ E_{l+2,l}^+(z) & E_{l+2,l+1}^+(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E_{N,l}^+(z) & E_{m,l+1}^+(z) & \cdots & E_{m,m-1}^+(z) & 1 \end{pmatrix} \]

Then we have

\[ (L_l^+(z))^{-1} = L_l^{m\pm}(z)^{-1}. \]  \hspace{1cm} (6.22)

Note that

\[ L_l^{m\pm}(z) \neq (L_{ij}^\pm(z))_{l \leq i,j \leq m}. \]

Hence we have

Lemma 6.6. The restriction of the relations

\[ L_l^{(1)}(z) L_l^{(2)}(z) - R_l^{(12)}(z, P + h) = R_l^{-(12)}(z, P)L_l^{(2)}(z) - L_l^{(1)}(z), \]  \hspace{1cm} (6.23)

\[ L_l^{(1)}(z) L_l^{(2)}(z) - R_l^{(12)}(zq^{\pm}c, P + h) = R_l^{+(12)}(zq^{\mp}c, P)L_l^{(2)}(z) - L_l^{(1)}(z), \]  \hspace{1cm} (6.24)

where \( z = z_1/z_2 \).

Therefore we obtain the following statement.

Theorem 6.7. \( L_l^{m+}(z), \hat{d}, q^{\pm}c/2 \) generate the subalgebra of \( E_{q,p}(\hat{gl}_N) \), which is isomorphic to \( E_{q,p}(\hat{gl}_{m-l+1}) \).
From (6.25) and (6.26), we have

**Proposition 6.8.**

\[
L_i^{m \pm(2)}(z_2)^{-1} R_i^{m \pm(12)}(z, P + h) L_i^{m \pm(1)}(z_1) = L_i^{m \pm(1)}(z_1) R_i^{m \pm* (12)}(z, P) L_i^{m \pm(2)}(z_2)^{-1},
\]

(6.27)

\[
L_i^{m \pm(1)}(z_1)^{-1} R_i^{m \pm(2)}(z_2)^{-1} R_i^{m \pm(12)}(z, P) L_i^{m \pm(2)}(z_2)^{-1} L_i^{m \pm(1)}(z_1)^{-1},
\]

(6.28)

\[
L_i^{m \pm(2)}(z_2)^{-1} R_i^{m \pm(12)}(zq^{\pm c}, P + h) L_i^{m \pm(1)}(z_1) = L_i^{m \pm(1)}(z_1) R_i^{m \pm* (12)}(zq^{\mp c}, P) L_i^{m \mp(2)}(z_2)^{-1},
\]

(6.29)

\[
L_i^{m \pm(1)}(z_1)^{-1} L_i^{m \mp(2)}(z_2)^{-1} R_i^{m \pm(12)}(zq^{\mp c}, P + h) = R_i^{m \pm* (12)}(zq^{\mp c}, P) L_i^{m \mp(2)}(z_2)^{-1} L_i^{m \pm(1)}(z_1)^{-1}.
\]

(6.30)

**Lemma 6.9.** For \(2 \leq l + 1 < m \leq N\), we have

\[
\rho^\pm(z) \tilde{b}(z) K_l^\pm(z_2) K_m^\pm(z_1) = \rho^\pm(z) \tilde{b}(z)^* K_m^\pm(z_1) K_l^\pm(z_2)^{-1},
\]

(6.31)

\[
\rho^\pm(q^{\pm c} z) \tilde{b}(q^{\mp c} z) K_l^\pm(z_2) K_m^\pm(z_1) = \rho^\pm(q^{\mp c} z)^* \tilde{b}(q^{\pm c} z)^* K_m^\pm(z_1) K_l^\pm(z_2)^{-1},
\]

(6.32)

\[
[E_{l+1, l}^\pm(z_1), K_m^\pm(z_2)] = 0 = [E_{m, l+1}^\pm(z_1), K_l^\mp(z_2)],
\]

(6.33)

\[
[E_{l+1, l}^\pm(z_2), K_m^\pm(z_1)] = 0 = [E_{m, l+1}^\pm(z_1), K_l^\mp(z_2)].
\]

(6.34)

where \(z = z_1/z_2\).

**Proof.** (6.31) and (6.32) follows from the \((m, l), (m, l)\) component of (6.27) and (6.28), respectively. Similarly, (6.33) (resp. (6.34)) follows from (6.31) and the \((m, l + 1), (m, l)\) component of (6.29) (resp. (6.32) and the same component of (6.29)).

Other relations among the basic Gauss components are given in Appendix C.

The following lemma indicates that the whole Gauss components of \(L^\pm(z)\) can be determined recursively by the basic ones.

**Lemma 6.10.** Let \(I_{a, b} = \{ (j, k) \mid a \leq j \leq b - 1, \ j + 1 \leq k \leq b \} \setminus \{(a, b)\} \). For \(2 \leq l + 1 < m \leq N\), \(E_{m, l}^+(z)\) (resp. \(F_{l, m}^+(z)\)) is determined by \(\{E_{m, l}^+(z) (l, m) \in I_{l, m}, K_j^+(z) \ l \leq j \leq m\} \) (resp. \(\{F_{l, m}^+(z) (l, m) \in I_{l, m}, K_j^+(z) \ l \leq j \leq m\}\)).

**Proof.** Let us consider \(E_{m, l}^+(z)\). The \(F_{l, m}^+(z)\) case is similar. From (6.31) and the \((m, l + 1), (l + 1, l)\) component of (6.27) we have

\[
-\tilde{b}^\ast(z) E_{l+1, l}^\pm(z_2^\pm) E_{m, l+1}^\pm(z_1^\pm)
= E_{m, l}^\pm(z_1^\mp) c^\ast(z, P_{l, l+1}) - E_{m, l+1}^\pm(z_2^\pm) E_{l+1, l}^\pm(z_1^\mp) + c^\ast(z, P_{l+1, m}) (L_i^{m \pm}(z_2^\pm)^{-1})_{ml} K_i^\pm(z_2^\pm)
+ \sum_{l+2 \leq k \leq m-1} E_{m, k}^\pm(z_1^\mp) c^\ast(z, P_{l+1, k}) (L_i^{m \pm}(z_2^\pm)^{-1})_{kl} K_i^\pm(z_2^\pm).
\]

(6.35)
Similarly, from (6.32) and the \((m, l + 1), (l + 1, l)\) component of (6.29), we have

\[
-\bar{b}^\ast(z) E_{m,l+1}^\pm (z_1^\pm) E_{m,l+1}^\pm (z_2^\pm) + E_{m,l+1}^\pm (z_1^\pm) c^\ast(z, P_{l+1}) - E_{m,l+1}^\pm (z_2^\pm) + c^\ast(z, P_{l+1,m}) (L_{l}^{m\mp} (z_2^\mp)^{-1})_{m,l} K_{l}^{\mp} (z_2^\mp)
\]

\[+
\sum_{t+2 \leq k \leq m-1} E_{m,k}^\pm (z_1^\pm) c^\ast(z, P_{l+1,k}) (L_{l}^{m\mp} (z_2^\mp)^{-1})_{k,l} K_{l}^{\mp} (z_2^\mp) . \quad (6.36)
\]

Subtracting (6.36) with the upper and lower signs reversed from (6.35), we have

\[
E_{m,l}^\pm (z_1^\pm) - E_{m,l}^\pm (z_2^\pm) = \frac{1}{c^\ast(z, P_{l+1})} \left( (E_{m,l+1}^\pm (z_1^\pm) - E_{m,l+1}^\pm (z_2^\pm)) E_{l+1,l}^\pm (z_2^\pm)
\]

\[-\bar{b}^\ast(z) E_{l+1,l}^\pm (z_2^\pm) \left( E_{m,l+1}^\pm (z_1^\pm) - E_{m,l+1}^\pm (z_2^\pm) \right)
\]

\[-\sum_{t+2 \leq k \leq m-1} \left( E_{m,k}^\pm (z_1^\pm) - E_{m,k}^\pm (z_2^\pm) \right) c^\ast(z, P_{l+1,k}) (L_{l}^{m\mp} (z_2^\mp)^{-1})_{k,l} K_{l}^{\mp} (z_2^\mp) \right) . \quad (6.37)
\]

Then due to (6.2) and (6.9), in each sign case the right hand side of (6.37) determines both

\(E_{m,l}^\pm (z)\) and \(E_{m,l}^\pm (z_1^\pm)\) uniquely as formal Laurent series in \(z\).

**Remark.** The upper and the lower sign cases of (6.37) give two expressions for \(E_{m,l}^\pm (z_1^\pm) - E_{m,l}^\pm (z_1^\pm)\). It is instructive to derive their consistency condition. Equating them, we obtain

\[
\left( E_{m,l+1}^\pm (z_1^\pm) - E_{m,l+1}^\pm (z_1^\pm) \right) E_{l+1,l}^\pm (z_2^\pm)
\]

\[-\bar{b}^\ast(z) E_{l+1,l}^\pm (z_2^\pm) \left( E_{m,l+1}^\pm (z_1^\pm) - E_{m,l+1}^\pm (z_1^\pm) \right)
\]

\[-\sum_{t+2 \leq k \leq m-1} \left( E_{m,k}^\pm (z_1^\pm) - E_{m,k}^\pm (z_1^\pm) \right) c^\ast(z, P_{l+1,k}) (L_{l}^{m\mp} (z_2^\mp)^{-1})_{k,l} K_{l}^{\mp} (z_2^\mp) \right) = \left( E_{m,l+1}^\pm (z_1^\pm) - E_{m,l+1}^\pm (z_1^\pm) \right) E_{l+1,l}^\pm (z_2^\pm)
\]

\[-\bar{b}^\ast(z) E_{l+1,l}^\pm (z_2^\pm) \left( E_{m,l+1}^\pm (z_1^\pm) - E_{m,l+1}^\pm (z_1^\pm) \right)
\]

\[-\sum_{t+2 \leq k \leq m-1} \left( E_{m,k}^\pm (z_1^\pm) - E_{m,k}^\pm (z_1^\pm) \right) c^\ast(z, P_{l+1,k}) (L_{l}^{m\mp} (z_2^\mp)^{-1})_{k,l} K_{l}^{\mp} (z_2^\mp) .
\]

Hence the consistency condition is

\[
\left( E_{m,l+1}^\pm (z_1^\pm) - E_{m,l+1}^\pm (z_1^\pm) \right) \left( E_{l+1,l}^\pm (z_2^\pm) - E_{l+1,l}^\pm (z_2^\pm) \right) - \bar{b}^\ast(z) \left( E_{l+1,l}^\pm (z_2^\pm) - E_{l+1,l}^\pm (z_2^\pm) \right) \left( E_{m,l+1}^\pm (z_1^\pm) - E_{m,l+1}^\pm (z_1^\pm) \right)
\]

\[-\sum_{t+2 \leq k \leq m-1} \left( E_{m,k}^\pm (z_1^\pm) - E_{m,k}^\pm (z_1^\pm) \right) c^\ast(z, P_{l+1,k}) (L_{l}^{m\mp} (z_2^\mp)^{-1})_{k,l} K_{l}^{\mp} (z_2^\mp) \right) .
\]

In particular, for \(m = l + 2\) we have

\[
\left( E_{l+2,l+1}^\pm (z_1^\pm) - E_{l+2,l+1}^\pm (z_1^\pm) \right) \left( E_{l+1,l}^\pm (z_2^\pm) - E_{l+1,l}^\pm (z_2^\pm) \right) - \bar{b}^\ast(z) \left( E_{l+1,l}^\pm (z_2^\pm) - E_{l+1,l}^\pm (z_2^\pm) \right) \left( E_{l+2,l+1}^\pm (z_1^\pm) - E_{l+2,l+1}^\pm (z_1^\pm) \right)
\]

\[= \bar{b}^\ast(z) \left( E_{l+1,l}^\pm (z_2^\pm) - E_{l+1,l}^\pm (z_2^\pm) \right) \left( E_{l+2,l+1}^\pm (z_1^\pm) - E_{l+2,l+1}^\pm (z_1^\pm) \right) . \quad (6.38)
\]
for \(1 \leq l \leq N - 2\). In Appendix C we identify these relations with the commutation relations of the total elliptic currents of \(U_{a,p}(\hat{\mathfrak{gl}}_N)\). This provides an example suggesting the injectivity of the \(H\)-algebra homomorphism \(\Phi : \mathcal{U} \to \mathcal{E}\) given in the next subsection.

### 6.3 The half currents of \(\mathcal{U}\)

Let us define the basic half currents of \(\mathcal{U}\) as follows. For \(1 \leq j \leq N - 1\)

\[
e^+_{j+1,j}(z) = \frac{a^*_{j+1,j} \Theta p^*(q^2)}{(p^*; p^*)^3}_\infty \left( \sum_{m \geq 0} e_{j,-m} \frac{1}{1 - q^{-2(P_{a,j}-1)} p^m} (zq^{j-c})^m \right)
- \sum_{m > 0} e_{j,m} \frac{q^{2(P_{a,j}-1)} p^m}{1 - q^{2(P_{a,j}-1)} p^m} (zq^{j-c})^{-m}, \tag{6.39}
\]

\[
f^+_{j,j+1}(z) = \frac{a_{j,j+1} \Theta p(q^2)}{(p; p)^3}_\infty \left( \sum_{m \geq 0} f_{j,-m} \frac{1}{1 - q^{2((P+h)_{a,j}-1)} p^m} (zq^j)^m \right)
- \sum_{m > 0} f_{j,m} \frac{q^{2((P+h)_{a,j}-1)} p^m}{1 - q^{2((P+h)_{a,j}-1)} p^m} (zq^j)^{-m}, \tag{6.40}
\]

\[
e^-_{j+1,j}(z) = q^{2P_{j+1}} e^+_{j+1,j}(z) q^{-2P_{j+1}}
= \frac{a^*_{j+1,j} \Theta p^*(q^2)}{(p^*; p^*)^3}_\infty \left( \sum_{m \geq 0} e_{j,-m} \frac{q^{2(P_{a,j}-1)} p^m}{1 - q^{2(P_{a,j}-1)} p^m} (zq^{j-c})^m \right)
- \sum_{m > 0} e_{j,m} \frac{1}{1 - q^{2(P_{a,j}-1)} p^m} (zq^{j-c})^{-m}, \tag{6.41}
\]

\[
f^-_{j,j+1}(z) = q^{2(P+h)_{a,j}} f^+_{j,j+1}(z) q^{-2(P+h)_{a,j}}
= \frac{a_{j,j+1} \Theta p(q^2)}{(p; p)^3}_\infty \left( \sum_{m \geq 0} f_{j,-m} \frac{q^{2((P+h)_{a,j}-1)} p^m}{1 - q^{2((P+h)_{a,j}-1)} p^m} (zq^j)^m \right)
- \sum_{m > 0} f_{j,m} \frac{1}{1 - q^{2((P+h)_{a,j}-1)} p^m} (zq^{j-c})^{-m}, \tag{6.42}
\]

where \(a^*_{j+1,j}\) and \(a_{j,j+1}\) are constants given by

\[
a^*_{j+1,j} = q^{-1} \frac{(p^*; p^*)^3}_\infty, \quad a_{j,j+1} = q^{-1} \frac{(p; p)^3}{(pq^{-2}; p)^3}. \tag{6.43}
\]

We then obtain

\[
e^+_{j+1,j}(z^+) - e^-_{j+1,j}(z^-) = \frac{a^*_{j+1,j} \Theta p^*(q^2)}{(p^*; p^*)^3}_\infty e_j(zq^{j-c/2}),
\]

\[
f^+_{j,j+1}(z^-) - f^-_{j,j+1}(z^+) = \frac{a_{j,j+1} \Theta p(q^2)}{(p; p)^3}_\infty f_j(zq^{j-c/2}).
\]

Here we set \(z^\pm = zq^{\pm c/2}\).
Note that at $p = 0$

\[
e^+_{j+1,j}(z) = a^+_{j+1,j}(1 - q^2) \left( e_{j,0} \frac{1}{1 - q^{-2(P_{m_j} - 1)}} + \sum_{m > 0} e_{j,m}(zq^{j-c})^m \right), \quad (6.44)
\]

\[
e^-_{j+1,j}(z) = a^+_{j+1,j}(1 - q^2) \left( e_{j,0} \frac{q^{-2(P_{m_j} - 1)}}{1 - q^{-2(P_{m_j} - 1)}} - \sum_{m \geq 0} e_{j,m}(zq^j)^{-m} \right), \quad (6.45)
\]

\[
f^+_{j,j+1}(z) = a_{j,j+1}(1 - q^2) \left( f_{j,0} \frac{1}{1 - q^{2(P_{m_j} + h)_{a_j} - 1}} + \sum_{m \geq 0} f_{j,m}(zq^j)^m \right), \quad (6.46)
\]

\[
f^-_{j,j+1}(z) = a_{j,j+1}(1 - q^2) \left( f_{j,0} \frac{q^{2(P_{m_j} + h)_{a_j} - 1}}{1 - q^{2(P_{m_j} + h)_{a_j} - 1}} - \sum_{m \geq 0} f_{j,m}(zq^j)^{-m} \right). \quad (6.47)
\]

Noting (3.1) and the expansion formula

\[
\frac{\Theta_p(q^{2s}z)(p; p)_\infty}{\Theta_p(q^{2^2})\Theta_p(z)} = \sum_{n \in \mathbb{Z}} \frac{1}{1 - q^{2s}p^n}z^n = \sum_{l \geq 0} \left( \frac{q^{2sl}}{1 - p^l} - \frac{q^{-2s(l+1)}p^{l+1}/z}{1 - p^{l+1}/z} \right), \quad (6.48)
\]

for $|p| < |z| < 1$, one can express the basic half currents as follows.

**Proposition 6.11.**

\[
e^+_{j+1,j}(z) = a^+_{j+1,j} \int_{C^*} \frac{dz'}{2\pi iz'}e^+_{j}(z') \Theta_p(zq^j - q^{2(1-P_{m_j})}/z')\Theta_p(q^{2^2}/z') \Theta_p(q^{2^2})\Theta_p(zq^j/z'), \quad (6.49)
\]

\[
f^+_{j,j+1}(z) = a_{j,j+1} \int_{C^*} \frac{dz'}{2\pi iz'}f^+_{j}(z') \Theta_p(zq^j q^{2(P_{m_j} + h)_{a_j} - 1}/z')\Theta_p(q^{2^2}/z') \Theta_p(zq^j/z'), \quad (6.50)
\]

where $C^* : |q^j - c| < |z'| < |p^{-1}q^j - c|$, $C : |q^j z| < |z'| < |p^{-1}q^j z|$.

**Proposition 6.12.** The basic half currents $e^+_{j+1,j}(z), f^+_{j,j+1}(z) (j \in I)$ and $k^+_{l}(z) (1 \leq l \leq N)$ satisfy the following relations.

\[
k^+_{l+1}(z) - e^+_{j+1,j}(z_2)k^+_{j+1}(z_1) = e^+_{j+1,j}(z_2) \frac{1}{b^*(z_1/z_2)} - e^+_{j+1,j}(z_1) \frac{c^*(z,P_{m_j} + 1)}{b^*(z)}, \quad (6.51)
\]

\[
k^+_{l+1}(z_1)f^+_{j,j+1}(z_2)k^+_{j+1}(z_1) = \frac{1}{b(z)}f^+_{j,j+1}(z_2) - \frac{c(z,(P + h)_{j,j+1})}{b(z)}f^+_{j,j+1}(z_1), \quad (6.52)
\]

\[
\frac{1}{b^*(1/z)}e^+_{j+1,j}(z_1)e^+_{j+1,j}(z_2) - e^+_{j+1,j}(z_2)2\frac{c^*(1/z,P_{m_j} + 1 - 2)}{b^*(1/z)} = \frac{1}{b^*(z)}e^+_{j+1,j}(z_2)e^+_{j+1,j}(z_1) - e^+_{j+1,j}(z_1)2\frac{c^*(z,P_{m_j} + 1 - 2)}{b^*(z)}, \quad (6.53)
\]
\[
\frac{1}{b(z)} f_{j,j+1}^+(z_1) f_{j,j+1}^+(z_2) - f_{j,l}^+(z_1)^2 \tilde{c}(z, (P + h)_{j,j+1} - 2) = \frac{1}{b(1/z)} f_{j,j+1}^+(z_2) f_{j,j+1}^+(z_1) - f_{j,j}^+(z_2)^2 \tilde{c}(1/z, (P + h)_{j,j+1} - 2), \\
(6.54)
\]
\[
[e_{j+1,j}^+(z_1), f_{j,j+1}^+(z_2)] = k_{j}^+(z_2) k_{j,j+1}^+(z_2) - k_{j,j+1}^+(z_1) - k_{j}^+(z_1) \tilde{c}(z, (P + h)_{j,j+1} - 1), \\
(6.55)
\]

where \( z = z_1/z_2 \).

**Proof.** Direct calculation using Proposition 6.11 and the relations in Definition 3.1 and (3.28).

For \( 1 \leq j \leq N - 1 \), let us consider the subalgebra \( U_{q,p}(\hat{gl}_2) \) of \( U \) generated by \( e_j(z), f_j(z), k_j^+(z), k_{j+1}^+(z), q^{-c/2}, \hat{d} \). Let us define the \( L \)-operator by the associated basic half currents by

\[
L_j^\pm(z) = \begin{pmatrix}
1 & f_{j,j+1}^+(z) & k_j^+(z) & 0 \\
0 & 1 & 0 & k_{j+1}^+(z) \\
k_j^+(z) + f_{j,j+1}^+(z) k_{j+1}^+(z) e_{j+1,j}^+(z) & f_{j,j+1}^+(z) k_{j+1}^+(z) & e_{j+1,j}^+(z) & 1 \\
k_{j+1}^+(z) e_{j+1,j}^+(z) & k_{j+1}^+(z) & 0 & 1
\end{pmatrix}
\]

Then comparing the relations in Proposition 6.12 and those of the basic Gauss components of \( L_j^\pm(z) \) in \( E_{q,p}(\hat{gl}_2) \) in Section 3.1, we obtain the following.

**Theorem 6.13.** For each \( j \), the \( L \)-operators \( L_j^\pm(z) \) satisfy the same RLL-relations (6.25) - (6.26) at \( l = j, m = j + 1 \) replacing \( L_i^\pm(z) \) with \( L_j^\pm(z) \). Hence the following map gives a surjective \( H \)-algebra homomorphism.

\[
\Phi(j) : U_{q,p}(\hat{gl}_2) \to E_{q,p}(\hat{gl}_2), \\
(6.56)
\]

\[
e_{j+1,j}^+(z) \mapsto E_{j+1,j}^+(z), \quad f_{j,j+1}^+(z) \mapsto F_{j,j+1}^+(z), \quad k_j^+(z) \mapsto K_j^+(z), \quad k_{j+1}^+(z) \mapsto K_{j+1}^+(z).
\]

Now let us consider the canonical extension of the map \( \Phi(j) \) to \( \Phi : U \to \mathcal{E} \) by

\[
e_{j+1,j}^+(z) \mapsto E_{j+1,j}^+(z), \quad f_{j,j+1}^+(z) \mapsto F_{j,j+1}^+(z), \quad k_l^+(z) \mapsto K_l^+(z) \quad (j \in I, \ 1 \leq l \leq N).
\]

**Theorem 6.14.** \( \Phi \) gives an isomorphism as a topological \( H \)-algebra over \( \mathbb{F}[[p]] \).

**Proof.**

1) Surjectivity: From Theorem 6.7 with \( l = j, m = j + 1 \) the basic Gauss components \( E_{j+1,j}^+(z), F_{j,j+1}^+(z), K_j^+(z) \) and \( \hat{d} \) generate the subalgebra \( E_{q,p}(\hat{gl}_2) \). From Lemma 6.10 the RLL relations allows us to construct the other Gauss components \( E_{k,j}^+(z), F_{j,k}^+(z) \) (\( 3 \leq j + 2 \leq k \leq N \)) recursively from the basic ones \( E_{j+1,j}^+(z), F_{j,j+1}^+(z), K_j^+(z), K_{j+1}^+(z) \). Then the surjectivity follows from Theorem 6.13.
2) Injectivity: Let \((\varphi^q_{\lambda,k}, V)\) be a highest weight representation of \(U_q = U_q(\widehat{\mathfrak{g}(N)})\) with the highest weight \(\lambda\) and the level \(k\). We extend \(\varphi^q_{\lambda,k}\) to the dynamical representation \((\varphi^{q,p}_{\lambda,k}, \hat{V}_F[[p]]\) of \(U\) as in Corollary 5.6. Then we define \(\hat{\varphi}^{q,p}_{\lambda,k} : \mathcal{E} \to \text{End}_\mathbb{C}\hat{V}_F[[p]]\) by \(\hat{\varphi}^{q,p}_{\lambda,k}(A) = \varphi^q_{\lambda,k}(\phi_p(a))\) for \(A = \Phi(a), a \in \mathcal{U}\). Let \(\pi_p : \mathcal{U} \to \mathcal{U}/p\mathcal{U}\) be the canonical projection. From the remark above Corollary 5.6, we also have the corresponding canonical projection \(\pi_p : \text{End}_\mathbb{C}\hat{V}_F[[p]] \to \text{End}_\mathbb{C}(V_F \otimes V_Q)\). We then consider the following diagram.

\[
\begin{array}{cccc}
\mathcal{U} & \xrightarrow{\Phi} & \mathcal{E} & \xrightarrow{\hat{\varphi}^{q,p}_{\lambda,k}} & \mathcal{D}_{H,\hat{V}_F[[p]]} \\
\pi_p \downarrow & & \phi_p & & \pi_p \downarrow \\
\mathcal{U}/p\mathcal{U} & (\mathbb{F}[[p]] \otimes_{\mathbb{C}} \mathcal{U}_q) \mathbb{C}[\mathcal{R}_Q] & & (\mathbb{F} \otimes_{\mathbb{C}} \mathcal{U}_q) \mathbb{C}[\mathcal{R}_Q] & (\mathbb{F} \otimes_{\mathbb{C}} \mathcal{U}_q) \mathbb{C}[\mathcal{R}_Q] \\
\end{array}
\]

Lemma 6.15.

(i) \(\pi_p \circ \phi_p = \pi_p\)

(ii) \(\pi_p \circ \varphi^q_{\lambda,k} = \varphi^q_{\lambda,k} \circ \pi_p\) on \((\mathbb{F}[[p]] \otimes_{\mathbb{C}} \mathcal{U}_q) \mathbb{C}[\mathcal{R}_Q]\)

Proof. (i) follows from \(u^z_i(z, p) = 1\) at \(p = 0\).

(ii) follows from (i) and \(\hat{V}_F[[p]] = (\mathbb{F}[[p]] \otimes_{\mathbb{C}} V) \otimes V_Q\). \(\square\)

Lemma 6.16. \(\text{Ker } \Phi \subset p\mathcal{U}\).

Proof. Assume \(\text{Ker } \Phi \not\subset p\mathcal{U}\). Then there exists a non zero element \(a \in \text{Ker } \Phi\) such that \(\pi_p(a) \neq 0\). Then from Lemma 6.15 for any level-\(k\) highest weight representation \(\varphi^q_{\lambda,k}\) of \(\mathcal{U}_q\), we have

\[
0 = \pi_p \circ \varphi^{q,p}_{\lambda,k} \circ \Phi(a) = \pi_p \circ \varphi^q_{\lambda,k} \circ \phi_p(a) = \varphi^q_{\lambda,k} \circ \pi_p(a).
\]

This contradicts the fact \(\bigcap_{\lambda,k} \text{Ker } \varphi^q_{\lambda,k} = 0\) given in [9]. \(\square\)

Proof of the injectivity. Let us assume \(\text{Ker } \Phi \neq 0\). Let \(a \neq 0 \in \text{Ker } \Phi\). Then from Lemma 6.16 there exists \(\tilde{a} \in \mathcal{U}\) such that \(a = p^n \tilde{a}\) for some positive integer \(n\) and \(\pi_p(\tilde{a}) \neq 0 \in \mathcal{U}_q\). Then the same argument as (6.57) yields for any \(\varphi^q_{\lambda,k}\)

\[
0 = \pi_p \circ \varphi^{q,p}_{\lambda,k} \circ \Phi(a) = p^n \varphi^q_{\lambda,k} \circ \pi_p(\tilde{a})
\]

This again contradicts \(\bigcap_{\lambda,k} \text{Ker } \varphi^q_{\lambda,k} = 0\). \(\square\)
A Quantum Affine Algebra \( U_q(\hat{\mathfrak{g}}_N) \)

**Definition A.1.** The quantum affine algebra \( U_q(\hat{\mathfrak{g}}_N) \) is a topological algebra over \( \mathbb{C} \) generated by \( k_{i,m}^\pm, x_{j,n}^\pm, d \) \((1 \leq i \leq N, 1 \leq j \leq N - 1, m \in \mathbb{Z}, n \in \mathbb{Z})\) and the central element \( q^{\pm c/2} \). The defining relations are conveniently written in terms of the generating functions called the Drinfeld currents:

\[
\begin{align*}
    k_{0,i}^\pm(z) &= \sum_{m \in \mathbb{Z}} k_{i,m}^\pm z^m, \\
    x_j^\pm(z) &= \sum_{n \in \mathbb{Z}} x_{j,n}^\pm z^{-n}.
\end{align*}
\]

The relations are given by

\[
\begin{align*}
    [d, k_{i}^\pm(z)] &= \pm z \frac{\partial}{\partial z} k_{i}^\pm(z), & [d, x_j^\pm(z)] &= -z \frac{\partial}{\partial z} x_j^\pm(z), \\
    k_{i,0}^+ k_{i,0}^- &= 1 = k_{i,0}^- k_{i,0}^+, \\
    k_{0,j}^+(z_1) k_{0,j}^-(z_2) &= k_{0,l}^+(z_2) k_{0,l}^-(z_1), \\
    k_{0,j}^-(z_1) k_{0,j}^+(z_2) &= \frac{(q^{c+2} z_2 / z_1, q^{2N} q^{-c-2} z_2 / z_1, q^{2N} q^{-c} z_2 / z_1, q^{c} z_2 / z_1; q^{N})_{\infty}}{(q^{c+2} z_2 / z_1, q^{2N} q^{-c-2} z_2 / z_1, q^{2N} q^{-c} z_2 / z_1, q^{c} z_2 / z_1; q^{N})_{\infty}} k_{0,j}^+(z_2) k_{0,j}^-(z_1), \\
    k_{0,j}^+(z_1) k_{0,j}^-(z_2) &= \frac{(q^{c+2} z_2 / z_1, q^{2N} q^{-c-2} z_2 / z_1, q^{2N} q^{-c} z_2 / z_1, q^{c} z_2 / z_1; q^{N})_{\infty}}{(q^{-c+2} z_2 / z_1, q^{c-2} z_2 / z_1, q^{2N} q^{-c} z_2 / z_1, q^{2N} q^{c} z_2 / z_1; q^{N})_{\infty}} k_{0,j}^+(z_2) k_{0,j}^-(z_1) \quad (j < l), \\
    k_{0,j}^-(z_1) k_{0,l}^+(z_2) &= \frac{(q^{c+2} z_2 / z_1, q^{c-2} z_2 / z_1, q^{-c} z_2 / z_1, q^{-c-2} z_2 / z_1; q^{N})_{\infty}}{(q^{-c+2} z_2 / z_1, q^{c-2} z_2 / z_1, q^{2N} q^{-c} z_2 / z_1, q^{2N} q^{c} z_2 / z_1; q^{N})_{\infty}} k_{0,j}^+(z_2) k_{0,j}^-(z_1) \quad (j > l), \\
    k_{0,j}^+(z_1) x_j^+(z_2) k_{0,j}^+(z_1)^{-1} &= q^{-1} \frac{1 - q^{-c+j} z_1 / z_2}{1 - q^{-c-2+j} z_2 / z_1} x_j^+(z_2), \\
    k_{0,j+1}^+(z_1) x_j^+(z_2) k_{0,j+1}^+(z_1)^{-1} &= q \frac{1 - q^{-c+j} z_1 / z_2}{1 - q^{-c+2+j} z_2 / z_1} x_j^+(z_2), \\
    k_{0,j}^-(z_1) x_j^-(z_2) k_{0,j}^-(z_1)^{-1} &= q^{1} \frac{1 - q^{-2+j} z_1 / z_2}{1 - q^{j} z_2 / z_1} x_j^-(z_2), \\
    k_{0,j+1}^-(z_1) x_j^-(z_2) k_{0,j+1}^-(z_1)^{-1} &= q^{-1} \frac{1 - q^{2+j} z_1 / z_2}{1 - q^{j} z_2 / z_1} x_j^-(z_2), \\
    k_{0,l}^+(z_1) x_j^\pm(z_2) k_{0,l}^+(z_1)^{-1} &= x_j^\pm(z_2) \quad (l \neq j, j + 1),
\end{align*}
\]
From the relations in Definition A.1 the following commutation relations hold.

\[
k_{0,j}(z_1)^{-1}x^+_j(z_2)k_{0,j}(z_1) = q^{-1} \frac{1 - q^{-2-j}z_2/z_1}{1 - q^{-j}z_2/z_1} x^+_j(z_2),
\]

\[
k_{0,j+1}(z_1)^{-1}x^+_j(z_2)k_{0,j+1}(z_1) = q^{-1} \frac{1 - q^{-2-j}z_2/z_1}{1 - q^{-j}z_2/z_1} x^+_j(z_2),
\]

\[
k_{0,j}(z_1)^{-1}x^-_j(z_2)k_{0,j}(z_1) = q^{-1} \frac{1 - q^{-j}z_2/z_1}{1 - q^{-j}z_2/z_1} x^-_j(z_2),
\]

\[
k_{0,j+1}(z_1)^{-1}x^-_j(z_2)k_{0,j+1}(z_1) = q^{-1} \frac{1 - q^{-c-j}z_2/z_1}{1 - q^{-c^{-j}}z_2/z_1} x^-_j(z_2),
\]

\[
k_{0,l}(z_1)^{-1}x^+_j(z_2)k_{0,l}(z_1) = x^+_j(z_2) \quad (l \neq j, j + 1),
\]

\[
z_1(1 - q^{\pm 2}z_2/z_1)x^+_i(z_1)x^+_j(z_2) = -z_2(1 - q^{\pm 2}z_1/z_2)x^+_j(z_2)x^+_i(z_1),
\]

\[
z_1(1 - q^{\pm 1}z_2/z_1)x^+_i(z_1)x^+_j(z_2) = -z_2(1 - q^{\pm 1}z_1/z_2)x^+_j(z_2)x^+_i(z_1),
\]

\[
x^+_j(z_1)x^+_i(z_2) = x^+_i(z_2)x^+_j(z_1) \quad (l \neq j, j + 1),
\]

\[
[x^+_i(z_1), x^-_j(z_2)] = \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta(q^{-c}z_1/z_2)k_{0,i}(q^{-c/2}z_1)k_{0,i+1}(q^{-c/2}z_1)^{-1}
\right.

\[
- \delta(q^{-c}z_1/z_2)k_{0,i}(q^{-c/2}z_2)k_{0,i+1}(q^{-c/2}z_2)^{-1}
\left. \right),
\]

\[
\left\{x^+_i(z_1)x^+_j(z_2)x^+_j(w) - (q + q^{-1})x^+_i(z_1)x^+_j(w)x^+_j(z_2) + x^+_j(w)x^+_i(z_1)x^+_j(z_2)\right\}
\]

\[+(z_1 \leftrightarrow z_2) = 0, \quad |i - j| = 1.
\]

**B Relations Among** $u^+_{i\epsilon_1}(z, p)$ **and** $u^+_{j\epsilon}(z, p)$

From the relations in Definition A.1 the following commutation relations hold.

**Lemma B.1.** Let us set $z = z_1/z_2$.

\[
k_{0,j}(z_1)u^+_{i\epsilon_1}(q^j z_2, p) = \left( p^* q^2 z, p^* q^2 N q^{-2} z, p q^2 N z, p q^2 N z, p^* p q^2 N z, p^* p q^2 N z, p q^2 N z \right)_\infty u^+_{i\epsilon_1}(q^j z_2, p) k_{0,j}(z_1),
\]

\[
k_{0,j}(z_1)u^+_{j\epsilon_1}(q^j z_2, p) = \left( p^* q^2 z, p^* q^{-2} z, p q^2 z, p q^2 z, p q^2 z, p q^2 z, p q^2 z \right)_\infty u^+_{j\epsilon_1}(q^j z_2, p) k_{0,j}(z_1) \quad (j < l),
\]

\[
k_{0,j}(z_1)u^+_{j\epsilon}(q^j z_2, p) = \left( p^* q^2 N q^2 z, p^* q^2 N q^2 z, p q^2 N z, p q^2 N z, p q^2 N z, p q^2 N z, p q^2 N z \right)_\infty
\]

\[ \times u^+_{j\epsilon}(q^j z_2, p) k_{0,j}(z_1) \quad (j > l),
\]

\[
k_{0,j}(z_1)u^-_{j\epsilon}(q^j z_2, p) = \left( p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z \right)_\infty
\]

\[ u^-_{j\epsilon}(q^j z_2, p) k_{0,j}(z_1),\]

\[
k_{0,j+1}(z_1)u^-_{j\epsilon}(q^j z_2, p) = \left( p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z \right)_\infty
\]

\[ u^-_{j\epsilon}(q^j z_2, p) k_{0,j+1}(z_1),\]

\[
k_{0,l}(z_1)u^-_{j\epsilon}(q^j z_2, p) = u^-_{j\epsilon}(q^j z_2, p) k_{0,l}(z_1) \quad (l \neq j, j + 1),
\]

\[
k_{0,j}(z_1)u^-_{j\epsilon_1}(q^j z_2, p) = \left( p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z \right)_\infty
\]

\[ u^-_{j\epsilon_1}(q^j z_2, p) k_{0,j}(z_1),\]

\[
k_{0,j+1}(z_1)u^-_{j\epsilon_1}(q^j z_2, p) = \left( p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z, p q^{-2} z \right)_\infty
\]

\[ u^-_{j\epsilon_1}(q^j z_2, p) k_{0,j+1}(z_1),\]

\[
k_{0,l}(z_1)u^-_{j\epsilon_1}(q^j z_2, p) = u^-_{j\epsilon_1}(q^j z_2, p) k_{0,l}(z_1) \quad (l \neq j, j + 1),\]
\[ u_{\varepsilon_j}^{+}(q^{-c+j}z_1,p)x_{j}^{+}(z_2) = \frac{(p^*q^{-c+j}z_1;p^*)_{\infty}}{(p^*q^{-c-2+j}z_1;p^*)_{\infty}}x_{j}^{+}(z_2)u_{\varepsilon_j}^{+}(q^{-c+j}z_1,p), \]
\[ u_{\varepsilon_j}^{-}(q^jz_1,p)x_{j}^{+}(z_2) = \frac{(pq^{-c-j}z_1;p^*)_{\infty}}{(pq^{-c-2-j}z_1;p^*)_{\infty}}x_{j}^{+}(z_2)u_{\varepsilon_j}^{-}(q^jz_1,p), \]
\[ u_{\varepsilon_j+1}^{+}(q^{-c+j+1}z_1,p)x_{j}^{+}(z_2) = \frac{(p^*q^{-c+j}z_1;p^*)_{\infty}}{(p^*q^{-c+2+j}z_1;p^*)_{\infty}}x_{j}^{+}(z_2)u_{\varepsilon_j+1}^{+}(q^{-c+j+1}z_1,p), \]
\[ u_{\varepsilon_j+1}^{-}(q^jz_1,p)x_{j}^{+}(z_2) = \frac{(pq^{-c-j}z_1;p^*)_{\infty}}{(pq^{-c-2-j}z_1;p^*)_{\infty}}x_{j}^{+}(z_2)u_{\varepsilon_j+1}^{-}(q^jz_1,p), \]
\[ u_{\varepsilon_j}^{+}(q^{-c+j}z_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j}^{+}(q^{-c+j}z_1,p) \quad (l \neq j, j + 1), \]
\[ u_{\varepsilon_j}^{-}(q^jz_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j}^{-}(q^jz_1,p), \]
\[ u_{\varepsilon_j+1}^{+}(q^{-c+j+1}z_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j+1}^{+}(q^{-c+j+1}z_1,p), \]
\[ u_{\varepsilon_j+1}^{-}(q^jz_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j+1}^{-}(q^jz_1,p), \]
\[ u_{\varepsilon_j}^{+}(q^{-c+j}z_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j}^{+}(q^{-c+j}z_1,p), \]
\[ u_{\varepsilon_j}^{-}(q^jz_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j}^{-}(q^jz_1,p), \]
\[ u_{\varepsilon_j+1}^{+}(q^{-c+j+1}z_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j+1}^{+}(q^{-c+j+1}z_1,p), \]
\[ u_{\varepsilon_j+1}^{-}(q^jz_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j+1}^{-}(q^jz_1,p), \]
\[ u_{\varepsilon_j}^{+}(q^{-c+j}z_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j}^{+}(q^{-c+j}z_1,p), \]
\[ u_{\varepsilon_j}^{-}(q^jz_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j}^{-}(q^jz_1,p), \]
\[ u_{\varepsilon_j+1}^{+}(q^{-c+j+1}z_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j+1}^{+}(q^{-c+j+1}z_1,p), \]
\[ u_{\varepsilon_j+1}^{-}(q^jz_1,p)x_{j}^{-}(z_2) = x_{j}^{+}(z_2)u_{\varepsilon_j+1}^{-}(q^jz_1,p), \]
\[ L_j^{j+1\pm}(z) = \begin{pmatrix} 1 & F_{j,j+1}(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_j^\pm(z) & 0 \\ 0 & K_j^\pm(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & K_j^\pm(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} K_j^\pm(z) + F_{j,j+1}(z)K_j^\pm(z)E_{j+1,j}^\pm(z) & F_{j,j+1}(z)K_j^\pm(z) \\ K_j^\pm(z)E_{j+1,j}^\pm(z) & K_j^\pm(z) \end{pmatrix} \]

\[ L_j^{j+1\pm}(z)^{-1} = \begin{pmatrix} K_j^\pm(z)^{-1} & -K_j^\pm(z)^{-1}F_{j,j+1}(z) \\ -E_{j+1,j}(z)K_j^\pm(z)^{-1} & E_{j+1,j}(z)K_j^\pm(z)^{-1}F_{j,j+1}(z) + K_j^\pm(z)^{-1} \end{pmatrix}. \tag{C.1} \]

From \((j,j), (j,j)\) of (6.28) and \((j+1,j+1), (j+1,j+2)\) of (6.25)

\[ K_l^\pm(z_1)K_l^\pm(z_2) = \rho(z)K_l^\pm(z_2)K_l^\pm(z_1) \quad (l = j, j+1), \tag{C.2} \]

where \(z = z_1/z_2\). Similarly, from \((j,j), (j,j)\) of (6.30) and \((j+1,j+1), (j+1,j+1)\) of (6.26)

\[ K_l^\pm(z_1)K_l^\mp(z_2) = \frac{\rho_{\pm\ast}(zq^{\mp\ast})}{\rho_{\pm\ast}(zq^{\pm\ast})} K_l^\mp(z_2)K_l^\pm(z_1), \quad (l = j, j+1). \tag{C.3} \]

From \((j+1,j), (j+1,j)\) of (6.27)

\[ K_j^\pm(z_1)K_{j+1}^\pm(z_2) = \rho(z)\frac{\hat{b}(z)}{b^\ast(z)}K_{j+1}^\pm(z_2)K_j^\pm(z_1). \tag{C.4} \]

In particular, in the limit \(z_1 \to z_2\), we have

\[ K_j^\pm(z_2)K_{j+1}^\pm(z_2) = \frac{(p^\ast; p^\ast)_3}{(p^\ast; p^\ast)_3} \Theta_{p^\ast}(q^2)K_{j+1}^\pm(z_2)K_j^\pm(z_2). \tag{C.5} \]

From \((j+1,j), (j+1,j)\) of (6.29)

\[ K_{j+1}^\pm(z_1)K_j^\mp(z_2) = \frac{\rho_{\pm\ast}(zq^{\mp\ast})}{\rho_{\pm\ast}(zq^{\pm\ast})} \frac{\hat{b}(zq^{\pm\ast})}{b(zq^{\mp\ast})} K_j^\mp(z_2)K_{j+1}^\pm(z_1). \tag{C.6} \]

From \((j+1,j+1), (j+1,j)\) of (6.26), we obtain

\[ K_{j+1}^\pm(z_1)E_{j+1,j}^\pm(z_2)K_{j+1}^\pm(z_1) = E_{j+1,j}^\pm(z_2)\frac{1}{b^\ast(z)} - E_{j+1,j}^\pm(z_1)\frac{e^\ast(z, P_{j,j+1})}{b^\ast(z)}. \tag{C.7} \]

From \((j+1,j+1), (j+1,j)\) of (6.28), we obtain

\[ K_{j+1}^\pm(z_1)E_{j+1,j}^\pm(z_2)K_{j+1}^\pm(z_1) = E_{j+1,j}^\pm(z_2)\frac{1}{b^\ast(zq^{\pm\ast})} - E_{j+1,j}^\pm(z_1)\frac{e^\ast(zq^{\mp\ast}, P_{j,j+1})}{b^\ast(zq^{\pm\ast})}. \tag{C.8} \]
Similarly, from $(j + 1, j), (j + 1, j + 1)$ of (6.25),
\[
K_{j+1}^\pm(z_1)F_{j,j+1}^\pm(z_2)K_{j+1}^\pm(z_1)^{-1} = \frac{1}{b(z)} E_{j,j+1}^\pm(z_2) - \frac{\bar{c}(z, (P + h)_{j,j+1})}{b(z)} F_{j,j+1}^\pm(z_1). \tag{C.9}
\]
From $(j + 1, j), (j + 1, j + 1)$ of (6.26),
\[
K_{j+1}^\pm(z_1)F_{j,j+1}^\pm(z_2)K_{j+1}^\pm(z_1)^{-1} = \frac{1}{b(zq^{\pm c})} E_{j,j+1}^\pm(z_2) - \frac{\bar{c}(zq^{\pm c}, (P + h)_{j,j+1})}{b(zq^{\pm c})} F_{j,j+1}^\pm(z_1). \tag{C.10}
\]
From $(j + 1, j), (j, j)$ of (6.28),
\[
K_j^\pm(z_2)^{-1} E_{j+1,j}^\pm(z_1) K_j^\pm(z_2) = \frac{1}{b^*(z)} E_{j+1,j}^\pm(z_1) - \frac{\bar{c}^*(z, P_{j,j+1})}{b^*(z)} E_{j+1,j}^\pm(z_2). \tag{C.11}
\]
From $(j, j), (j + 1, j)$ of (6.28),
\[
K_j^\pm(z_2) F_{j,j+1}^\pm(z_1) K_j^\pm(z_2)^{-1} = F_{j,j+1}^\pm(z_1) \frac{1}{b(z)} - F_{j,j+1}^\pm(z_2) \bar{c}(z, (P + h)_{j,j+1}). \tag{C.12}
\]
From $(j + 1, j + 1), (j, j)$ of (6.25), we obtain
\[
K_{j+1}^\pm(z_1) F_{j,j+1}^\pm(z_2) K_{j+1}^\pm(z_1)^{-1} = F_{j,j+1}^\pm(z_1) \frac{1}{b(z)} - F_{j,j+1}^\pm(z_2) \bar{c}(z, (P + h)_{j,j+1}). \tag{C.13}
\]
From (C.7) and (C.14), we obtain
\[
E_{j+1,j}^\pm(z_1) E_{j+1,j}^\pm(z_2) \frac{1}{b^*(1/z)} - E_{j+1,j}^\pm(z_2) \frac{1}{b^*(1/z)}^2 \frac{c^*(1/\bar{z}, P_{j,j+1} - 2)}{b^*(1/z)}
= E_{j+1,j}^\pm(z_2) E_{j+1,j}^\pm(z_1) \frac{1}{b(z)} - E_{j+1,j}^\pm(z_1) \frac{1}{b(z)}^2 \frac{c^*(z, P_{j,j+1} - 2)}{b(z)}. \tag{C.15}
\]
From $(j + 1, j + 1), (j, j)$ of (6.26), we obtain
\[
K_{j+1}^\pm(z_1) F_{j,j+1}^\pm(z_2) K_{j+1}^\pm(z_1)^{-1} = F_{j,j+1}^\pm(z_1) \frac{1}{b^*(1/(zq^{\pm c}))} - F_{j,j+1}^\pm(z_2) \frac{1}{b^*(1/(zq^{\pm c}))} \frac{c^*(1/(zq^{\pm c}), P_{j,j+1} - 2)}{b^*(1/(zq^{\pm c}))}
= E_{j+1,j}^\pm(z_2) E_{j+1,j}^\pm(z_1) \frac{1}{b^*(zq^{\pm c})} - E_{j+1,j}^\pm(z_1) \frac{1}{b^*(zq^{\pm c})} \frac{c^*(zq^{\pm c}, P_{j,j+1} - 2)}{b^*(zq^{\pm c})}. \tag{C.16}
\]
In a similar way, we obtain
\[
F_{j,j+1}^\pm(z_1) F_{j,j+1}^\pm(z_2) \frac{1}{b(z)} - F_{j,j+1}^\pm(z_1) \frac{1}{b(z)} \frac{\bar{c}(z, (P + h)_{j,j+1} - 2)}{b(z)}
= F_{j,j+1}^\pm(z_2) F_{j,j+1}^\pm(z_1) \frac{1}{b(1/z)} - F_{j,j+1}^\pm(z_2) \frac{1}{b(1/z)} \frac{\bar{c}(1/z, (P + h)_{j,j+1} - 2)}{b(1/z)}, \tag{C.17}
\]
\[
F_{j,j+1}^\pm(z_1) F_{j,j+1}^\pm(z_2) \frac{1}{b(zq^{\pm c})} + F_{j,j+1}^\pm(z_1) \frac{1}{b(zq^{\pm c})} \frac{\bar{c}(zq^{\pm c}, (P + h)_{j,j+1} - 2)}{b(zq^{\pm c})}
= F_{j,j+1}^\pm(z_2) F_{j,j+1}^\pm(z_1) \frac{1}{b(1/(zq^{\pm c}))} - F_{j,j+1}^\pm(z_2) \frac{1}{b(1/(zq^{\pm c}))} \frac{\bar{c}(1/(zq^{\pm c}), (P + h)_{j,j+1} - 2)}{b(1/(zq^{\pm c}))}. \tag{C.18}
\]
From \((j + 1, j), (j, j + 1)\) of (6.25),
\[
\rho^\pm(z) \left\{ b^*(z)K_{j+1}^\pm(z_1)E_{j+1,j}^\pm(z_1)F_{j,j+1}^\pm(z_2)K_{j+1}^\pm(z_2) + c(z,(P + h)_{j,j+1} \left( K_{j}^\pm(z_2) + F_{j,j+1}^\pm(z_2)K_{j+1}^\pm(z_2)E_{j+1,j}^\pm(z_2) \right) K_{j+1}^\pm(z_1)c^*(z,P_{j,j+1}) + F_{j,j+1}^\pm(z_2)K_{j+1}^\pm(z_1)E_{j+1,j}^\pm(z_1)b^*(z,P_{j,j+1}) \right\}.
\] (C.20)

Using \((j + 1, j), (j + 1, j + 1)\) and \((j + j + 1), (j, j + 1)\) of (6.25), we get
\[
[E_{j+1,j}^\pm(z_1), F_{j,j+1}^\pm(z_2)] = K_{j+1}^\pm(z_2)K_{j+1}^\pm(z_2)^{-1} \frac{c^*(z,P_{j,j+1} - 1)}{b^*(z)} - K_{j+1}^\pm(z_1)K_{j+1}^\pm(z_1)^{-1} \frac{c(z,(P + h)_{j,j+1} - 1)}{b(z)}. \] (C.21)

Similarly, from \((j + 1, j), (j, j + 1)\) of (6.26),
\[
\rho^\pm(q^{-c}z) \left\{ b^*(zq^{-c})K_{j+1}^\pm(z_1)E_{j+1,j}^\pm(z_1)F_{j,j+1}^\pm(z_2)K_{j+1}^\pm(z_2) + c(zq^{-c},(P + h)_{j,j+1} \left( K_{j}^\pm(z_2) + F_{j,j+1}^\pm(z_2)K_{j+1}^\pm(z_2)E_{j+1,j}^\pm(z_2) \right) K_{j+1}^\pm(z_1)c^*(zq^{-c},P_{j,j+1}) + F_{j,j+1}^\pm(z_2)K_{j+1}^\pm(z_1)E_{j+1,j}^\pm(z_1)b^*(zq^{-c},P_{j,j+1}) \right\}.
\] (C.22)

Using \((j + 1, j), (j + 1, j + 1)\) and \((j + 1, j + 1), (j, j + 1)\) of (6.26), we get
\[
[E_{j+1,j}^\pm(z_1), F_{j,j+1}^\pm(z_2)] = K_{j+1}^\pm(z_2)K_{j+1}^\pm(z_2)^{-1} \frac{c^*(zq^{-c},P_{j,j+1} - 1)}{b^*(zq^{-c})} - K_{j+1}^\pm(z_1)K_{j+1}^\pm(z_1)^{-1} \frac{c(zq^{-c},(P + h)_{j,j+1} - 1)}{b(zq^{-c})}. \] (C.23)

### C.2 Identification with the elliptic currents

In this section we demonstrate an identification of a combination of the basic Gauss components with the elliptic currents of \(U_{q,p}(\widehat{g}_N)\).

Let us define the total current by \(E_j(zq^{2-c/2}) = \mu^*(E_{j+1,j}^\pm(z^+) - E_{j+1,j}^\pm(z^-))\) with \(\mu^* = \frac{(p^* + p)^3}{a_{j+1,j}^* \Theta_{p^*}(q^{c/2})}\). Then we obtain from (C.7) and (C.8)
\[
K_{j+1,j}^+(z_1)^{-1}E_j(z_1q^{-c/2})K_{j+1,j}^+(z_1) = K_{j+1,j}^+(z_1)^{-1} \frac{1}{b^*(zq^{-c/2})} E_{j+1,j}^+(z_2) + c^*(zq^{-c/2},P_{j,j+1}) \bigg) K_{j+1,j}^+(z_1)
\]
\[
= \mu^* \left( E_{j+1,j}^+(z_2) \frac{1}{b^*(zq^{-c/2})} - E_{j+1,j}^+(z_1) \frac{c^*(zq^{-c/2},P_{j,j+1})}{b^*(zq^{-c/2})} \bigg) + c^*(zq^{-c/2},P_{j,j+1}) \bigg) E_{j+1,j}^+(z_1) \frac{c^*(zq^{-c/2},P_{j,j+1})}{b^*(zq^{-c/2})} \bigg) \bigg)
\]
\[
= E_j(z_1q^{-c/2}) \frac{1}{b^*(zq^{-c/2})}.
\] (C.24)
Comparing this with (3.33), we identify \( E_j(z), K_j^+(z) \) with \( e_j(z), k_j^+(z) \), respectively.

Next, inserting (C.15) and (C.17) into \((E_{j+1,j}^+(z_1^+) - E_{j+1,j}(z_1^-))(E_{j+1,j}^+(z_2^+ - E_{j+1,j}(z_2^-))\), we obtain

\[
E_j(z_1)E_j(z_2) = -\frac{1}{z} \frac{\Theta_{p^*}(zq^2)}{\Theta_{p^*}(q^2/z)} E_j(z_2)E_j(z_1).
\]

This is consistent to (3.34).

To obtain (3.35) we use (6.38), which is derived from (6.29) with \( m = l + 2 \).

Similarly, if we define \( F_j(zq^{j-c/2}) = \mu(F_{j,j+1}^+(z_1) - F_{j,j+1}^-(z_1)) \) with \( \mu = \frac{(p,p)^3}{a_{j,j+1}\Theta_{p}(q^2)} \), we recover the relations (3.36)-(3.37) from (C.9)-(C.10), (C.18)-(C.19) and the \( F^+ \) counterpart of (6.38). Hence we identify \( F_j(z) \) with \( f_j(z) \).

Finally let us check the relation (3.23). From (C.21) and (C.23), we have

\[
(\mu \mu^*)^{-1 \mid E(z_1), F(z_2)} = [E_{j+1,j}^+(z_1^+) - E_{j+1,j}^-(z_1^-); F_{j,j+1}^+(z_2^+) - F_{j,j+1}^-(z_2^-)] \\
= [E_{j+1,j}^+(z_1^+), F_{j,j+1}^+(z_2^+)] + [E_{j+1,j}^-(z_1^-), F_{j,j+1}^-(z_2^-)] \\
- [E_{j+1,j}^+(z_1^+), F_{j,j+1}^-(z_2^-)] - [E_{j+1,j}^-(z_1^-), F_{j,j+1}^+(z_2^+)].
\]

Let us substitute (C.21) and (C.23) into this. Noting the remark below Proposition 3.8, one finds that the terms containing \( K_j^-(z_1^-)K_{j+1}^+(z_2^+) \) from the 2nd and 3rd terms in (C.25) cancel out each other and the same is true for the terms containing \( K_{j+1}^+(z_1^+)K_j^-(z_2^-) \) from the 1st and the 3rd terms in (C.25). We obtain

\[
(\mu \mu^*)^{-1 \mid E_j(z_1), F_j(z_2)} \\
= K_j^+(z_2^-)K_{j+1}^+(z_2^-)^{-1}q^{-1}\Theta_{p^*}(q^2) \left( \frac{\Theta_{p^*}(q^{-2((P+h)_{j,j+1-1})}q^{-c}z)}{\Theta_{p^*}(q^{-2((P+h)_{j,j+1-1})})\Theta_{p^*}(q^{-2(c)})} - \frac{\Theta_{p^*}(q^{-2((P+h)_{j,j+1-1})}q^{-c}z)}{\Theta_{p^*}(q^{-2((P+h)_{j,j+1-1})})\Theta_{p^*}(q^{-2(c)})} \right) \\
- K_j^+(z_1^-)^{-1}K_{j+1}^+(z_1^-)q^{-1}\Theta_{p}(q^2) \left( \frac{\Theta_{p}(q^{-2((P+h)_{j,j+1-1})}q^{-c}z)}{\Theta_{p}(q^{-2((P+h)_{j,j+1-1})})\Theta_{p}(q^{-2(c)})} - \frac{\Theta_{p}(q^{-2((P+h)_{j,j+1-1})}q^{-c}z)}{\Theta_{p}(q^{-2((P+h)_{j,j+1-1})})\Theta_{p}(q^{-2(c)})} \right).
\]

Here

\[
\frac{\Theta_{p}(q^{2s}z)(p;p)^3}{\Theta_{p}(q^{2s})\Theta_{p}(z)} \left|_+ \right. = \frac{\Theta_{p}(q^{2s}z)(p;p)^3}{\Theta_{p}(q^{2s})\Theta_{p}(z)} \\
= \sum_{n\in\mathbb{Z}} \frac{1}{1-q^{2s}p^n z^n} = \sum_{l\in\mathbb{Z}_{\geq 0}} \left( \frac{q^{2s l}}{1-p^l z} - \frac{q^{-2s(l+1)}p^{l+1}/z}{1-p^{l+1}/z} \right). \tag{C.26}
\]
for \(|p| < |z| < 1\) and
\[
\frac{\Theta_p(q^{2s}z)(p;p)^3}{\Theta_p(q^{2s})\Theta_p(z)} \bigg|_\infty = -\frac{\Theta_p(q^{-2s}/z)(p;p)^3}{\Theta_p(q^{-2s})\Theta_p(1/z)}
\]
\[
= -\sum_{n\in\mathbb{Z}} \frac{1}{1 - q^{-2sp^n}z^{-n}}
\]
\[
= -\sum_{l\in\mathbb{Z} \geq 0} \left( \frac{q^{-2slp^l/z}}{1 - p^l/z} - \frac{q^{2s(l+1)}}{1 - p^{l+1}z} \right) \quad (C.27)
\]
for \(1 < |z| < |p^{-1}|\). Then the difference between these two expansions turns out to be the formal delta function \(\delta(z) = \sum_{n\in\mathbb{Z}} z^n\). We thus obtain
\[
[E_j(z_1), F_j(z_2)]
\]
\[
= \mu^{\ast \ast} q^{-1} \left\{ K^{+}(z_2^{-})K^{+}_{j+1}(z_2^{-})^{-1} \frac{\Theta_p(q^2)}{(p^2; p^2)^3}_\infty \delta\left( \frac{z_1}{z_2} p^c \right) - K^{-}(z_1^{-})^{-1} K^{-}_{j+1}(z_1^{-}) \frac{\Theta_p(q^2)}{(p^2; p^2)^3}_\infty \delta\left( \frac{z_1}{z_2} q^{-c} \right) \right\}
\]
\[
= -\frac{\kappa}{q - q^{-1}} \left\{ K^{+}(z_2^{-})K^{+}_{j+1}(z_2^{-})^{-1} \delta\left( \frac{z_1}{z_2} p^c \right) - K^{-}(z_1^{-})^{-1} K^{-}_{j+1}(z_1^{-})^{-1} \delta\left( \frac{z_1}{z_2} q^{-c} \right) \right\}. \quad (C.28)
\]
In the last line we used (C.5) and
\[
q^{-1} \mu^{\ast \ast} \frac{\Theta_p(q^2)}{(p^2; p^2)^3}_\infty = -\frac{\kappa}{q - q^{-1}}.
\]
This is consistent to (3.23).

**D  Relation to Jimbo-Miwa-Okado’s Face Weight**

In this Appendix we give a relationship between our \(R\)-matrix (2.2) and Jimbo-Miwa-Okado’s \(A_{n-1}^{(1)}\) type face weight in [33].

**D.1  Fractional powers in \(z\)**

So far we have defined the elliptic algebras \(U_{q,p}(\tilde{g}_N)\) and \(E_{q,p}(\tilde{g}_N)\) by using the generating functions \(e_j(z), f_j(z), k_j(z)\) and \(L_{ij}(z)\), respectively. The coefficients of their relations are given in terms of the theta function \(\Theta_p(z)\), which enables us to expand every things to power series in \(p\).

However for a practical use it is convenient to introduce operators such as \(z^{\pm \frac{r+h-1}{r^*}}\) and \(z^{\pm \frac{r-1}{r}}\) into the algebras. Here \(r\) and \(r^*\) are introduced by \(p = q^{2r}\) and \(p^* = pq^{-2c} = q^{2r^*}\) with \(r^* = r - c\). The main reason for this can be seen in the following example. Let us consider Jacobi’s theta function
\[
\vartheta_1(u, \tau) = i \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i r(n-1/2)^2} e^{2\pi i u(n-1/2)}.
\]
Identifying \( z = q^{2u}, \ p = e^{-\frac{2\pi i}{\tau}} \) we have

\[
\vartheta_1(u, \tau) = e^{\frac{u_i}{4}} \tau^{-\frac{1}{2}} p^{\frac{i}{2}} q^{-\frac{u_2}{2}} \Theta_p(z).
\]

Then we have

\[
z^{\frac{1}{\tau}} c(z, s) = z^{\frac{1}{\tau}} \frac{\Theta_p(q^2) \Theta_p(q^{2s} z)}{\Theta_p(q^{2s}) \Theta_p(q^2 z)} = \frac{\vartheta_1(\frac{z}{\tau}, \tau)}{\vartheta_1(\frac{s}{\tau}, \tau)} \vartheta_1(\frac{z}{\tau}, \tau).
\]

This is invariant under the shift \( z \mapsto \tau \) i.e. \( u \mapsto u + r \). Compare this with (3.73).

Motivated by this, let us consider the transformation

\[
\hat{R}^\pm(z, s) = \left( \text{Ad} z^{\frac{1}{2} \theta_V(s)} \otimes \text{id} \right) \left( z^{\frac{1}{2} T_{V, V}} R^\pm(z, s) \right),
\]

where \( \theta_V(s) \) is given in (3.69) and

\[
T_{V, V} = \sum_{j=1}^{N-1} \pi_V(h_{j}) \otimes \pi_V(h^j)
\]

Then one finds

\[
\hat{R}^\pm(u, s) = \hat{\rho}^\pm(u) \hat{R}(u, s),
\]

\[
\hat{R}(u, s) = \sum_{j=1}^{N} E_{jj} \otimes E_{jj} + \sum_{1 \leq j < l \leq N} \left( b(z, s, j, l) E_{jj} \otimes E_{ll} + \hat{b}(z) E_{ll} \otimes E_{jj} \right)
\]

\[
+ \sum_{1 \leq j < l \leq N} \left( \hat{c}(z, s, j, l) E_{jl} \otimes E_{lj} + \hat{c}(z, -s, j, l) E_{lj} \otimes E_{jl} \right),
\]

with

\[
\hat{\rho}^+(u) = q^{\frac{N+1}{N} \frac{z^{\frac{1}{2}}}{} \sum_{j=1}^{N-1} \{q^2 z \} \{q^{2N-2} z \} \{p/z \} \{pq^{2N}/z \} \{z \} \{q^{2N} z \} \{pq^2/z \} \{pq^{2N-2}/z \} },
\]

\[
\hat{\rho}^-(u) = q^{\frac{N+1}{N} \frac{z^{\frac{1}{2}}}{} \sum_{j=1}^{N-1} \{pq^2 z \} \{pq^{2N-2} z \} \{1/z \} \{q^{2N}/z \} \{pq^{2N} z \} \{q^2/z \} \{q^{2N-2}/z \} },
\]

\[
\hat{b}(u, s) = \frac{[s + 1][s - 1][u]}{[s][u][u + 1]}; \quad \hat{b}(u) = \frac{[u]}{[u + 1]},
\]

\[
\hat{c}(u, \pm s) = \frac{[1][s + u]}{[s][u + 1]},
\]

where we set

\[
[u] = \vartheta_1 \left( \frac{u}{r}, \tau \right).
\]

We also need for \( p^* = e^{-\frac{2\pi i}{\tau^*}} \)

\[
[u]^* = \vartheta_1 \left( \frac{u}{r^*}, \tau^* \right).
\]

43
We have
\[ \rho^+(u + r) = \rho^-(u), \quad \frac{\rho^+(u)}{\rho^{*-1}(u)} = \rho^-(u), \quad \hat{\rho}^+(u)\hat{\rho}^-(u) = 1, \] (D.9)
\[ \hat{R}^-(u, s)^{-1} = P\hat{R}^+(-u, s)P \] (D.10)
for \( \hat{\rho}^\pm(u) = \hat{\rho}^\pm(u)|_{r \to r^*, p \to p^*}. \) Hence we obtain

**Proposition D.1.**
\[ \hat{R}^+(u + r, s) = \hat{R}^-(u, s). \] (D.11)

Let us further define \( \hat{L}^\pm(u) \) from \( L^+(z) \) in Sec.3.2 by
\[ \hat{L}^+(u) = \left( z^{-\frac{1}{2}V(P)} \otimes \text{id} \right) z^{\frac{1}{2}T_{\nu}(z)} \left( z^{\frac{1}{2}V(P)} \otimes \text{id} \right), \] (D.12)
\[ \hat{L}^-(u) = \hat{L}^+(u + r^* + c/2), \]
where \( z = q^{2u} \), and \( T_{\nu} \) is given in (3.70). Note that we do not need the extra \( \text{Ad} q^{-2\theta V(P)} \) action to define \( \hat{L}^-(u) \) unlike (3.68).

**Proposition D.2.** \( \hat{L}^\pm(u) \) satisfy the same relations as (3.64), (3.71) and (3.72) with replacing \( R^\pm(z, s) \) by \( \hat{R}^\pm(u, s) \). Namely,
\[ \hat{R}^\pm(u, P + h)\hat{L}^\pm(u_1)\hat{L}^\pm(u_2) = \hat{L}^\pm(u_2)\hat{L}^\pm(u_1)\hat{R}^\pm(u, P), \] (D.13)
\[ \hat{R}^\pm(u \pm c/2, P + h)\hat{L}^\pm(u_1)\hat{L}^\pm(u_2) = \hat{L}^\pm(u_2)\hat{L}^\pm(u_1)\hat{R}^\pm(u \mp c/2, P), \] (D.14)
where \( u = u_1 - u_2 \).

**Remark.** The \( R \)-matrices and \( L \)-operators in the previous works [36,43,47,50] are \( \hat{R}^\pm(u, s) \) and \( \hat{L}^\pm(u) \) given in this section except for the prefactor: \( \rho^\pm(u) \) in the previous works are \( \hat{\rho}^\pm(u) \) in (D.6) and (D.5).

**D.2 Gauge transformation from Jimbo-Miwa-Okado’s \( W \)**

For \( 1 \leq j \leq N \) let us define \( F(P, P + \hat{j}) \) by
\[ F(s, s + \hat{j}) = \left( \prod_{m=\hat{j}+1}^N \frac{[s_{j,m} + 1]}{[s_{j,m}]} \right)^{\frac{1}{2}}, \]
where \( s_{j,m} = s_{\epsilon_j} - s_{\epsilon_m} \) as in Sec.2.

Our \( R \)-matrix is related to Jimbo-Miwa-Okado’s \( W \) denoted by \( W_{JMO} \) as follows. Let us set
\[ \hat{R}^+(u, s)^{ij}_{kl} = W \left( \begin{array}{ccc} s & s + \hat{i} & s + \hat{j} \\ s + \hat{l} & a + \hat{i} + \hat{j} & (i + j = k + l) \end{array} \right) u \]
Then we have
\[ W \left( \begin{array}{c|c} s & s + \hat{i} \\ \hline s + \hat{l} & s + \hat{i} + \hat{j} \end{array} \right) u \right) = \hat{\rho}^+(u) \left[ \frac{1}{u + 1} \right] \frac{F(s, s + \hat{i})F(s + \hat{i}, s + \hat{j} + \hat{k})}{F(s + \hat{l}, s + \hat{l} + \hat{k})} \times W_{JMO} \left( \begin{array}{c|c} s & s + \hat{i} \\ \hline s + \hat{l} & s + \hat{i} + \hat{j} \end{array} \right) u \right) ; \]

Here \((s + \hat{i})_{\hat{e}_j} = s_{\hat{e}_i} + \delta_{i,j}\) etc. and
\[ W_{JMO} \left( \begin{array}{c|c} s & s + \hat{i} \\ \hline s + \hat{i} & s + 2\hat{i} \end{array} \right) u \right) = \left[ \frac{1 + u}{u} \right] , \]
\[ W_{JMO} \left( \begin{array}{c|c} s & s + \hat{i} \\ \hline s + \hat{i} & s + \hat{i} + \hat{j} \end{array} \right) u \right) = \left[ \frac{s_{i,j} - u}{s_{i,j}} \right] , \]
\[ W_{JMO} \left( \begin{array}{c|c} s & s + \hat{j} \\ \hline s + \hat{i} & s + \hat{i} + \hat{j} \end{array} \right) u \right) = \left[ \frac{u}{1} \right] \left( \frac{[s_{i,j} + 1][s_{i,j} - 1]}{[s_{i,j}]^2} \right)^{1/2} . \]

E Elliptic Quantum Determinant

E.1 Jimbo-Kuniba-Miwa-Okado’s projection

Quantum determinant of the \(L\)-operator depends on a choice of the gauge for the \(R\)-matrix. Let \(V = \mathbb{F} \otimes \mathbb{C} V, V = \bigoplus_{a=1}^{N} C v_{a}\) and \(E_{i,j} v_{a} = \delta_{j,a} v_{i}\). Let us consider the following \(R\) matrix \(R'(z, s) \in \text{End}(\mathcal{V} \otimes \mathcal{V})\) given by
\[ R'(u, s) = \hat{\rho}^{+'}(u) \left[ \sum_{j=1}^{N} \alpha(u) E_{jj} \otimes E_{jj} + \sum_{j \neq l} \left( \beta(u, s_{l,j}) E_{jj} \otimes E_{ll} + \gamma(u, s_{l,j}) E_{jl} \otimes E_{lj} \right) \right] , \tag{E.1} \]
where
\[ \hat{\rho}^{+'}(u) = \hat{\rho}^+(u) \left[ \frac{1}{u + 1} \right] , \]
\[ \alpha(u) = \frac{u + 1}{1}, \quad \beta(u, s) = \frac{u [s + 1]}{[s]}, \quad \gamma(u, s) = \frac{s - u}{s} . \]

Note that \(R'(u, s)^{ij}_{kl} = W \left( \begin{array}{c|c} s & s + \hat{i} \\ \hline s + \hat{l} & s + \hat{i} + \hat{j} \end{array} \right) \) is the face weight in \([34]\), which is gauge equivalent to \(\hat{R}^+(u, s)\) in \([D.3]\).

Instead of giving the gauge transformation of the \(R\) matrix we give the gauge transformation of the \(L\)-operator \(\hat{L}^+(u)\) in \([D.12]\) to the one satisfying the \(RLL\) relation with the new \(R\) matrix.
\textbf{E.1.} Namely we define \( L(u) = \sum_{1 \leq i,j \leq N} E_{ij} L_{ij}(u) \) by
\[
L_{ij}(u) = \prod_{m=i+1}^{N} \frac{[(P + h)_{jm} + 1]}{[1]} \sum_{n=1}^{j-1} \frac{[1]^*}{P_{nj} + 1}.
\]

Then from (D.13) we obtain

**Proposition E.1.**
\[
R^{(12)}(u_1 - u_2, P + h) L^{(1)}(u_1) L^{(2)}(u_2) = L^{(2)}(u_2) L^{(1)}(u_1) R^{* (12)}(u_1 - u_2, P).
\]

One has
\[
R'(-1, s) = \rho_0 \sum_{j \neq l} \frac{[s_{lj} + 1]}{[s_{lj}]} \left( E_{jj} \otimes E_{ll} - E_{jl} \otimes E_{lj} \right),
\]
where
\[
\rho_0 = - \lim_{u \to -1, (z \to q^-)} \tilde{\rho}^+(u).
\]

Hence
\[
R'(-1, s) v_a \otimes v_b = \rho_0 \left( \frac{[s_{ba} + 1]}{[s_{ba}]} v_a \otimes v_b - \frac{[s_{ab} + 1]}{[s_{ab}]} v_b \otimes v_a \right) \in \mathcal{V} \wedge \mathcal{V}.
\]

In order to generalize this it is convenient to consider the ‘transposition’ of \( R'(u, s) \):
\[
R(u, s) = t_1 t_2 R^{(21)}(u, s), \quad R^*(u, s) = R(u, s)|_{t \to t^*, p \to p^*}.
\]

In fact this yields
\[
R(-1, s) v_a \otimes v_b = \rho_0 \left( \frac{[s_{ab} + 1]}{[s_{ab}]} (v_a \otimes v_b - v_b \otimes v_a) \right) \in \mathcal{V} \wedge \mathcal{V}.
\]

Accordingly taking the transpositions \( t_1 \) and \( t_2 \) of (E.3), flipping the two tensor components and exchanging \( u_1 \) and \( u_2 \), we obtain
\[
R^{* (12)}(u_2 - u_1, P) t L^{(1)}(u_1) t L^{(2)}(u_2) = t L^{(2)}(u_2) t L^{(1)}(u_1) R^{(12)}(u_2 - u_1, P + h).
\]

Let us generalize these formulas as follows \([23, 29, 34]\). For \( 2 \leq k \leq N \), define
\[
R(u_1, \ldots, u_{k-1}; u_k, s)^{1\ldots k-1; k} \text{ and } \Pi_k(u_1, \ldots, u_{k-1}; u_k, s) \in \text{End}_q(V \otimes^k)
\]
by
\[
R(u_1, \ldots, u_{k-1}; u_k, s)^{1\ldots k-1; k} = R^{k-1k}(u_k - u_{k-1}, s) R^{k-2k}(u_k - u_{k-2}, s + h^{k-1}) \cdots R^{1k}(u_k - u_1, s + \sum_{j=2}^{k-1} h^{(j)}),
\]

and
\[
\Pi_k(u_1, \ldots, u_{k-1}; u_k, s) = \frac{1}{k!} R'(u_1, \ldots, u_{k-1}; u_k, s)^{1\ldots k-1; k} R(u_1, \ldots, u_{k-2}; u_{k-1}, s)^{1\ldots k-2; k-1} \cdots R(u_1, u_2; u_3, s)^{12; 3} R(u_1; u_2, s)^{1; 2}.
\]
We also need $\Pi^*_k(s)$ defined by the same formula as \eqref{157} with replacing $R(u, s)$ by $R^*(u, s)$. By using the DYBE \eqref{2,14} repeatedly, one obtains another expression of $\Pi_k(u_1, \cdots, u_{k-1}; u_k, s)$

**Proposition E.2.**

$$\Pi_k(u_1, \cdots, u_{k-1}; u_k, s) = \frac{1}{k!} R(u_1; u_2, \cdots, u_k, s)^1 \cdots R(u_2; u_3, \cdots, u_k, s)^{2k} \cdots R(u_{k-1}; u_k, s)^{k-1}. $$

where for $j < k$

$$R(u_j; u_{j+1}, \cdots, u_k, s)^{jj+1} = R^{jj+1}(u_j+1 - u_j, s + \sum_{i=1}^{k} h(i)) R^{jj+2}(u_{j+2} - u_j, s + \sum_{i=1}^{k} h(i)) \cdots R^k(u_k - u_j, s).$$

**Proposition E.3.** Let $L(z)$ be the $L$ operator defined by \eqref{152}. Then we have

$$\Pi^*_k(u_1, \cdots, u_{k-1}; u_k, P) \cdot L^{(1)}(u_1) L^{(2)}(u_2) \cdots L^{(k)}(u_k) = L^{(k)}(u_k) L^{(k-1)}(u_{k-1}) \cdots L^{(1)}(u_1) \Pi_k(u_1, \cdots, u_{k-1}; u_k, P + h - \sum_{j=1}^{k} h(j)).$$ \hfill (E.8)

Note that

$$R^{ij}(u, s + h(i) + h(j)) = R^{ij}(u, s).$$

Now let us consider the operators $\Pi_k(u_1, \cdots, u_{k-1}; u_k, P + h)$ and $\Pi^*_k(u_1, \cdots, u_{k-1}; u_k, P)$ with the specialization of the spectrum parameters $(u_1, u_2, \cdots, u_k) = (u, u - 1, \cdots, u - (k - 1))$. We denote the resultant operators by $\Pi_k(P + h)$ and $\Pi^*_k(P)$, respectively. Let us set $[1, N] = \{1, 2, \cdots, N\}, I = \{i_1, i_2, \cdots, i_k\} \subseteq [1, N]$ with $i_1 < i_2 < \cdots < i_k$ and define

$$v_I = C_I v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k},$$

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn} \sigma v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(k)}},$$

$$C_I = \prod_{1 \leq a < b \leq k} \sqrt{\frac{\rho^*[a]}{[1]^*} \frac{\rho^*[a]}{[1]^*} \frac{(P + h)_{i_a, i_b} + 1}{[P_{i_a, i_b}]^*}}.$$ 

**Proposition E.4.** For $2 \leq k \leq N$ and $s = P, P + h \in H$,

$$\text{Im } \Pi_k(s) = \wedge^k \mathcal{V}.$$ \hfill (E.9)

In particular, $\text{Im } \Pi_N(s)$ is the one dimensional subspace of $\mathcal{V}^N$ spanned by

$$v_{[1,N]} = C_{[1,N]} v_1 \wedge v_2 \wedge \cdots \wedge v_N.$$
Proof. By induction one has
\[
\Pi_k^e(P) v_{i_1} \tilde{\otimes} v_{i_2} \tilde{\otimes} \cdots \tilde{\otimes} v_{i_k} = \prod_{1 \leq a < b \leq k} \rho_0 \left[ a \right] \left[ P_{i_a,i_b} + 1 \right] v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \in \wedge^k \mathcal{V}.
\]
Then noting the identity
\[
\prod_{1 \leq a < b \leq k} \rho_0 \left[ a \right] \left[ P_{i_a,i_b} + 1 \right] = \mathcal{N}_I \mathcal{C}_I
\]
where
\[
\mathcal{N}_I = \prod_{1 \leq a < b \leq k} \sqrt{\frac{\rho_0 \left[ a \right]}{\rho_0 \left[ a \right] \left[ P_{i_a,i_b} + 1 \right]}}
\]
one obtains
\[
\Pi_k^e(P) v_{i_1} \tilde{\otimes} v_{i_2} \tilde{\otimes} \cdots \tilde{\otimes} v_{i_k} = \mathcal{N}_I v_I.
\]
Similarly using the identity
\[
\prod_{1 \leq a < b \leq k} \rho_0 \left[ a \right] \left[ (P + h)_{i_a,i_b} + 1 \right] = \mathcal{N}'_I \mathcal{C}_I,
\]
with
\[
\mathcal{N}'_I = \prod_{1 \leq a < b \leq k} \sqrt{\frac{\rho_0 \left[ a \right]}{\rho_0 \left[ a \right] \left[ (P + h)_{i_a,i_b} \right]}}
\]
one obtains
\[
\Pi_k(P + h) v_{i_1} \tilde{\otimes} v_{i_2} \tilde{\otimes} \cdots \tilde{\otimes} v_{i_k} = \mathcal{N}'_I v_I.
\]
\[
\square
\]
Consider the projection operator \( A_k : \mathcal{V}^\otimes k \rightarrow \wedge^k \mathcal{V} \)
\[
A_k = \frac{1}{k!} \sum_{1 \leq j_1, \ldots, j_k \leq N} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma E_{j_{\sigma(1)} j_{\sigma(1)} \otimes \cdots \otimes E_{j_{\sigma(k)} j_{\sigma(k)}}}.
\]
Note that \( A_k \Pi_k^e(P) = \Pi_k^e(P) \).

**Definition E.5.** Let \( I = \{i_1, i_2, \ldots, i_k\}, J = \{j_1, j_2, \ldots, j_k\} \subset [1, N] \) with \( i_a < i_b, j_a < j_b \) for \( 1 \leq a < b \leq k \). We define the quantum minor determinant \( l(z)_I \) of \( L(z) \) by
\[
\Pi_k^e(P) \ t L^{(1)}(u) t L^{(2)}(u-1) \cdots t L^{(k)}(u-(k-1)) v_{i_1} \tilde{\otimes} \cdots \tilde{\otimes} v_{i_k} = A_k \ t L^{(k)}(u-(k-1)) t L^{(k-1)}(u-(k-2)) \cdots t L^{(1)}(u) \Pi_k(P + h) v_{i_1} \tilde{\otimes} \cdots \tilde{\otimes} v_{i_k} = \sum_{1 \leq j_1 < \cdots < j_k \leq N} l(z)_I v_J.
\]
For \( \tau \in \mathcal{S}_k \) we set \( \tau(I) = \{ i_{\tau(1)}, \ldots, i_{\tau(k)} \} \) and define \[28\]

\[
\text{sgn}_I(\tau, P + h) = \prod_{1 \leq a < b \leq k \atop \tau(a) > \tau(b)} \left[ \frac{((P + h)_{i_{\tau(a)}, i_{\tau(b)}} + 1)\ast}{((P + h)_{i_{\tau(b)}, i_{\tau(a)}} + 1)} \right]
\]

and define \[28\]

\[
\text{sgn}_I^\ast(\tau, P) = \prod_{1 \leq a < b \leq k \atop \tau(a) > \tau(b)} \left[ \frac{[P_{i_{\tau(a)}, i_{\tau(b)}} + 1]\ast}{[P_{i_{\tau(b)}, i_{\tau(a)}} + 1]} \right].
\]

Then we have

**Proposition E.6.**

\[
l(u)^I = N_J \sum_{\sigma \in \mathcal{E}_k} \text{sgn}_I(\sigma, P)L_{i_{\sigma(1)}}(u)L_{i_{\sigma(2)}}(u-1) \cdots L_{i_{\sigma(k)}}(u-(k-1))
\]

\[
= N_J' \sum_{\sigma \in \mathcal{E}_k} \text{sgn}_I(\sigma, P)L_{i_{\sigma(1)}}(u)L_{i_{\sigma(2)}}(u-1) \cdots L_{i_{\sigma(k)}}(u-(k-1))
\]

where

\[
F_J(P + h) = \prod_{1 \leq a < b \leq k} \left[ \frac{((P + h)_{i_{a}, i_{b}} + 1)\ast}{((P + h)_{i_{a}, i_{b}} + 1)} \right].
\]

In particular, in the case \( I = J = [1, N] \) we obtain the quantum determinant of \( L(z) \):

\[
q \text{-det} L(u) = N_{[1, N]} \sum_{\sigma \in \mathcal{E}_N} \text{sgn}_I(\sigma, P)L_{i_{\sigma(1)}}(u)L_{i_{\sigma(2)}}(u-1) \cdots L_{i_{\sigma(N)}}(u-(N-1))
\]

\[
= N_{[1, N]}' \sum_{\sigma \in \mathcal{E}_N} \text{sgn}_I(\sigma, P)L_{i_{\sigma(1)}}(u)L_{i_{\sigma(2)}}(u-1) \cdots L_{i_{\sigma(N)}}(u-(N-1))
\]

**Proof.** The statement follows from a standard calculation given for example in \[53\] and the formulas

\[
\text{sgn}_I(\sigma, P) = \prod_{1 \leq a < b \leq N \atop \sigma(a) > \sigma(b)} \left[ \frac{[P_{i_{\sigma(a)}, i_{\sigma(b)}} + 1]\ast}{[P_{i_{\sigma(b)}, i_{\sigma(a)}} + 1]} \right] = \prod_{1 \leq a < b \leq N} \left[ \frac{[P_{i_{a}, i_{b}} + 1]\ast}{[P_{i_{a}, i_{b}} + 1]} \right].
\]

Note that the formulas in Proposition \[E.6\] are consistent to the ones obtained by Hartwig using the co-module algebras \[28\].

Now using the identity \( \text{sgn}_I C_\tau(I) = \text{sgn}_I(\tau, P + h) \) we have

**Proposition E.7.**

\[
v_\tau(I) = \text{sgn}_I(\tau, P + h)v_I.
\]

(E.14)

Then we obtain
Proposition E.8. For $\tau \in \mathcal{S}_k$,

$$l(u)_{\tau I}^J = \text{sgn}_I(\tau, P + h) \ l(u)_{I}^J,$$  \hspace{0.5cm} \text{(E.15)}

$$l(u)_{\tau P}^J = \frac{1}{\text{sgn}_J(\tau, P + h)} \ l(u)_{I}^J.$$  \hspace{0.5cm} \text{(E.16)}

Then noting (4.9) and Proposition 4.8 we obtain

Proposition E.9.

$$\Delta(l(u)_{\tau I}^J) = \sum_{1 \leq l_1 < \cdots < l_k \leq N} \frac{N_J}{N_L} \ l(u)_{I}^J \otimes l(u)_{L}^J.$$  \hspace{0.5cm} \text{(E.17)}

In particular,

$$\Delta(q\det L(u)) = q\det L(u) \otimes q\det L(u).$$  \hspace{0.5cm} \text{(E.18)}

Next for $1 \leq l \leq k$ let us set $\hat{i}_l = I\backslash\{i_l\}$ and define

$$N_l^{(l)} = \frac{N_I}{N_{\hat{i}_l}}; \quad F_l^{(l)}(P + h) = \frac{F_I(P + h)}{F_{\hat{i}_l}(P + h)}$$

etc. Note that

$$F_l^{(l)}(P + h) = \prod_{1 \leq a < l} \frac{[(P + h)_{ia,ja} + 1] \times \prod_{l < a \leq k} \frac{[(P + h)_{ia,ja} + 1]}{[(P + h)_{ia,ia}]}}.$$  

Then by a direct calculation using the expressions of $l(u)_{\tau I}^J$ in Proposition E.6 we obtain

Proposition E.10.

$$l(u)_{\tau I}^J = \sum_{l=1}^k N_l^{(l)} \ \prod_{1 \leq a \leq k} \frac{[P_{ja,ja} + 1] \times \prod_{l \leq a < l} \frac{P_{ja,ja} + 1]}{[P_{ja,ja} + 1]} \ l(u)_{i_k j_l}^j \ l_i j_l (u - (k - 1)),$$

$$= \sum_{l=1}^k L_{i_l j_l} (u) \ l(u - 1)_{i_k} \ N_l^{(l)} \ \prod_{1 \leq a < l} \frac{P_{ja,ja} + 1] \times \prod_{l < a \leq k} \frac{P_{ja,ja} + 1]}{[P_{ja,ja} + 1]} \ l_i j_l (u - (k - 1)),$$

$$= \sum_{l=1}^k L_{i_l j_l} (u - (k - 1)) \ l(u)_{i_k} \ N_l^{(l)} \ \prod_{1 \leq a < l} \frac{P_{ja,ja} + 1} \times \prod_{l < a \leq k} \frac{P_{ja,ja} + 1]}{[P_{ja,ja} + 1]} \ l_i j_l (u) - 1) \ N_l^{(l)} \ \prod_{1 \leq a < l} \frac{P_{ja,ja} + 1]}{[P_{ja,ja} + 1]} \ l_i j_l (u - 1) \ N_l^{(l)} \ \prod_{1 \leq a < l} \frac{P_{ja,ja} + 1]}{[P_{ja,ja} + 1]} \ l_i j_l (u)$$

$$= \sum_{l=1}^k N_l^{(l)} \frac{F_l^{(l)}(P + h)}{F_{\hat{i}_l}(P + h)} \ (u - 1) \ l(u - 1)_{i_k} \ l_i j_l (u - 1) \ N_l^{(l)} \ \prod_{1 \leq a < l} \frac{P_{ja,ja} + 1]}{[P_{ja,ja} + 1]} \ l_i j_l (u)$$
Proposition E.11. For $1 \leq i \leq N$,

$$ q\text{-det}L(u) = \sum_{l=1}^{N} N_{[1,N]}^{(l)} \prod_{1 \leq a \leq N} \left( \frac{(P + h)_{i,a} + 1}{(P + h)_{i,i} + 1} \right) \prod_{1 \leq a \leq N} \left( \frac{[P_{a,a} + 1]^*}{[P_{i,a} + 1]^*} \right) l(u)_{i,i}^{\tau} L_{ii}(u - (N - 1)), $$

$$ = \sum_{l=1}^{N} L_{ii}(u) \left( u - (N - 1) \right) l(u)_{i,i}^{\tau} (-1)^{N-l} \prod_{1 \leq a < i} \left( \frac{[P_{a,a} + 1]^*}{[P_{a,a} + 1]^*} \right), $$

$$ = \sum_{l=1}^{N} (-1)^{N-l} N_{[1,N]}^{(i)} F_{[1,N]}^{(l)}(P + h) F_{[1,N]}^{(l)}(P + h) \prod_{1 \leq a \leq N} \left( \frac{[P_{a,a} + 1]^*}{[P_{a,a} + 1]^*} \right) l(u - 1)_{i,i}^{\tau} L_{ii}(u) $$

Proof. Consider the case $I = J = [1, N]$ in Proposition E.10 and use Proposition E.8 for the cyclic permutations $\tau = (i, i + 1, \cdots, N)$ in the 1st and the 4th lines, whereas for $\tau = (i, i - 1, \cdots, 2, 1)$ in the 2nd and the 3rd lines.

E.2 Gauge transformation

Inserting (E.2) into the expressions of $l(u)_{I}^J$ in Proposition E.6 we define the quantum minor determinant $\tilde{l}_+(u)_{I}^J$ of $\tilde{L}_+(u)$ by

$$ l(u)_{I}^J = \left( \prod_{1 \leq a < b \leq k} \left[ 1^* \frac{[P_{a,a} + 1]^*}{[P_{a,a} + 1]^*} \right] \right) \tilde{l}_+(u)_{I}^J. \tag{E.19} $$

For $\sigma \in \Sigma$ we set

$$ \hat{\text{sgn}}_I(\sigma, P + h) = \prod_{1 \leq a < b \leq k} \left[ \frac{[P_{a,a} + 1]^*}{[P_{a,a} + 1]^*} \right] \hat{\text{sgn}}_I(\sigma, P) = \prod_{1 \leq a < b \leq k} \left[ \frac{[P_{a,a} + 1]^*}{[P_{a,a} + 1]^*} \right]. $$

Proposition E.12.

$$ \tilde{l}_+(u)_{I}^J $$

$$ = \mathcal{N}_k \sum_{\sigma \in \Sigma_k} \hat{\text{sgn}}_I^*(\sigma, P) \tilde{L}_{i_{1},j_{1}(1)}^+ (u) \tilde{L}_{i_{2},j_{2}(2)}^+ (u - 1) \cdots \tilde{L}_{i_{k},j_{k}(k)}^+ (u - (k - 1)), $$

$$ = \mathcal{N}_k^{-1} \sum_{\sigma \in \Sigma_k} \hat{\text{sgn}}_I(\sigma, P + h) \tilde{L}_{i_{1},j_{1}(1)}^+ (u - (k - 1)) \tilde{L}_{i_{2},j_{2}(2)}^+ (u - (k - 2)) \cdots \tilde{L}_{i_{k},j_{k}(k)}^+ (u), $$

where

$$ \mathcal{N}_k = \prod_{1 \leq a < b \leq k} \sqrt{\frac{\rho_0^a [a]^*[1]}{\rho_0^a [a]^*[1]^*}}. $$
Proof. Inserting (E.22) and using (3.65) and (3.66), we have

\[
L_{i_2j_1(1)}(u) L_{i_2j_2(2)}(u - 1) \cdots L_{i_kj_k(k)}(u - (k - 1))
\]
\[
= \prod_{1 \leq a < b \leq k} \left( \frac{[P + h]_{i_a, i_b} + 1}{[1]} \right) \hat{L}_{i_2j_1(1)}^+(u) \hat{L}_{i_2j_2(2)}^+(u - 1) \cdots \hat{L}_{i_kj_k(k)}^+(u - (k - 1))
\]
\[
\times \prod_{1 \leq a < b \leq k} \left[ \frac{[P_{j_a j_b} + 1]}{[1]^*} \right] \prod_{1 \leq a < b \leq k} \left[ \frac{[P_{j_a j_b} + 1]^*}{[P_{j_{a(b)} j_{b(a)}} + 1]^*} \right]
\]
\[
L_{i_{a(k)} j_k}(u - (k - 1)) L_{i_{a(k-1)} j_{k-1}}(u - (k - 2)) \cdots L_{i_{a(1)} j_1}(u)
\]
\[
= \prod_{1 \leq a < b \leq k} \left( \frac{[P + h]_{i_a, i_b}}{[1]} \right) \prod_{1 \leq a < b \leq k} \left( \frac{[P + h]_{i_{a(b)}, i_{b(a)}} + 1}{[(P + h)_{i_{a(b)}, i_{b(a)}} + 1]} \right)
\]
\[
\times \hat{L}_{i_{a(k)} j_k}^+(u - (k - 1)) \hat{L}_{i_{a(k-1)} j_{k-1}}^+(u - (k - 2)) \cdots \hat{L}_{i_{a(1)} j_1}^+(u) \prod_{1 \leq a < b \leq k} \left[ \frac{[1]^*}{[P_{j_{a(b)} j_{b(a)}}]^*} \right]
\]

\]

Corollary E.13.

\[
q\text{-det}\hat{L}^+(u)
\]
\[
= N_N \sum_{\sigma \in \mathcal{S}_N} \text{sgn}_N^*[1, N](\sigma, P) \hat{L}_{1\sigma(1)}^+(u) \hat{L}_{2\sigma(2)}^+(u - 1) \cdots \hat{L}_{N\sigma(N)}^+(u - (N - 1)),
\]
\[
= N_N^{-1} \sum_{\sigma \in \mathcal{S}_N} \text{sgn}_N[1, N](\sigma, P + h) \hat{L}_{\sigma(N)N}^+(u - (N - 1)) \hat{L}_{\sigma(N-1)N-1}^+(u - (N - 2)) \cdots \hat{L}_{\sigma(1)1}^+(u).
\]

Proposition E.14.

\[
\hat{L}^+(u)^J\tau_I(1) = \text{sgn}_I(\tau, P + h) \hat{L}^+(u)_I^J,
\]
\[
\hat{L}^+(u)^\tau_J(I) = \text{sgn}_I^*(\tau, P) \hat{L}^+(u)_I^J.
\]

Proposition E.15.

\[
\Delta(\hat{L}^+(u)_I^J) = \sum_{1 \leq l_1 < \cdots < l_k \leq k} \hat{L}^+(u)_{l_1}^J \otimes \hat{L}^+(u)_{l_k}^J.
\]

Proof. Note $\Delta(N_k) = N_k \otimes N_k$ and

\[
\Delta(\text{sgn}_I^*(\sigma, P)) = 1 \otimes \text{sgn}_I^*(\sigma, P), \quad \Delta(\text{sgn}_I(\sigma, P + h)) = \text{sgn}_I(\sigma, P + h) \otimes 1.
\]
Proposition E.16.

\[ \hat{L}^+(u)_{il} = \sum_{l=1}^{k} N_{ik}^* \prod_{1 \leq a < l} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right] \hat{L}^+(u)_{ij} \hat{L}^+(u - (k - 1)), \]

\[ = \sum_{l=1}^{k} \hat{L}_{ijl}(u) \hat{L}^+(u - 1)_{ij} N_k^* \prod_{1 \leq a < l} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right], \]

\[ = \sum_{l=1}^{k} \hat{L}_{ijl}(u - (k - 1)) \hat{L}^+(u - 1)_{ij} N_k^* \prod_{1 \leq a < l} \left[ \frac{(P + h)_{ja,ja} + 1}{(P + h)_{ja,i}} \right], \]

\[ = \sum_{l=1}^{k} N_{ik}^* \prod_{1 \leq a < l} \left[ \frac{(P + h)_{ja,ja} + 1}{(P + h)_{ja,i}} \right] \hat{L}^+(u - 1)_{ij} \hat{L}^+(u), \]

where

\[ N_k^* = \frac{N_k}{N_k - 1} \prod_{1 \leq a < k - 1} \sqrt{\frac{P_0[a]^*}{P_0[a]}}. \]

Proposition E.17. For \(1 \leq i \leq N\),

\[ q\text{-det}\hat{L}^+(u) = \prod_{l=1}^{N} N_{ik}^* \prod_{1 \leq a < N} \left[ \frac{(P + h)_{ja,ja}}{(P + h)_{ja,i}} \right] \prod_{1 \leq a < l} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right] \hat{L}^+(u)_{ij} \hat{L}^+(u - (N - 1)), \]

\[ = \sum_{l=1}^{N} \hat{L}_{il}(u) \hat{L}^+(u - 1)_{il} N_{ik}^* \prod_{1 \leq a < i} \left[ \frac{(P + h)_{ja,ja}}{(P + h)_{ja,i}} \right] \prod_{1 \leq a < l} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right], \]

\[ = \sum_{l=1}^{N} \hat{L}_{il}(u - (N - 1)) \hat{L}^+(u - 1)_{il} N_{ik}^* \prod_{1 \leq a < N} \left[ \frac{(P + h)_{ja,ja} + 1}{(P + h)_{ja,i}} \right] \prod_{1 \leq a < l} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right], \]

\[ = \sum_{l=1}^{N} N_{ik}^* \prod_{1 \leq a < l} \left[ \frac{(P + h)_{ja,ja} + 1}{(P + h)_{ja,i}} \right] \prod_{1 \leq a < i} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right] \hat{L}^+(u - 1)_{il} \hat{L}^+(u). \]

Comparing this and the axiom for the antipode \(S\), we determine the action of \(S\) on \(\hat{L}_{il}^+(u)\) and \(\hat{L}^+(u)_{il}^*\). For each there are four different expressions. For example,

Proposition E.18.

\[ S(\hat{L}_{il}^+(u)) = \hat{L}^+(u - 1)_{il}^* N_{ik}^* \prod_{1 \leq a < l} \left[ \frac{(P + h)_{ja,ja} + 1}{(P + h)_{ja,i}} \right] \prod_{1 \leq a < i} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right] (q\text{-det}\hat{L}^+(u))^{-1}, \]

\[ S(\hat{L}^+(u)_{il}^*) = N_{ik}^* \prod_{1 \leq a < N} \left[ \frac{(P + h)_{ja,ja}}{(P + h)_{ja,i}} \right] \prod_{1 \leq a < l} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right] (q\text{-det}\hat{L}^+(u))^{-1}. \]

Combining these we obtain

Proposition E.19.

\[ S^2(\hat{L}_{il}^+(u)) = \prod_{a \in i} \left[ \frac{(P + h)_{ja,ja} + 1}{(P + h)_{ja,i}} \right] \hat{L}_{il}^+(u - N) \prod_{a \in l} \left[ \frac{P_{ja,ja} + 1}{P_{ja,i}} \right]. \]
Proposition E.16 also yields

**Proposition E.20.**

\[
(\hat{L}^+(u)^{-1})_{ij} = S(\hat{L}^+_{ij}(u)).
\]

### E.3 Formulas for the half currents

In this section we follow the idea in [31].

For \(1 \leq a, b \leq N\) let us define \(\hat{L}^+(u)_{a,a} = (\hat{L}^+_{i,j}(u))_{a \leq i, j \leq N}\) and

\[
\hat{L}^+(u)_{a,b} = \begin{pmatrix}
\hat{L}^+_{ab}(u) & \hat{L}^+_{a1}(u) & \cdots & \hat{L}^+_{aN}(u) \\
\hat{L}^+_{a1b}(u) & \hat{L}^+_{a1+1}(u) & \cdots & \hat{L}^+_{a1N}(u) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{L}^+_{aNb}(u) & \hat{L}^+_{aN+1}(u) & \cdots & \hat{L}^+_{aNN}(u)
\end{pmatrix}
\]

for \(a > b\) \hspace{1cm} (E.23)

\[
= \begin{pmatrix}
\hat{L}^+_{ab}(u) & \hat{L}^+_{ab+1}(u) & \cdots & \hat{L}^+_{aN}(u) \\
\hat{L}^+_{b+1b}(u) & \hat{L}^+_{b+1b+1}(u) & \cdots & \hat{L}^+_{b+1N}(u) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{L}^+_{N-1b}(u) & \hat{L}^+_{N-1b+1}(u) & \cdots & \hat{L}^+_{NN}(u)
\end{pmatrix}
\]

for \(a < b\). \hspace{1cm} (E.24)

Then we have the following Gauss decompositions.

**Lemma E.21.** For \(a > b\)

\[
\hat{L}^+(u)_{a,b} = \begin{pmatrix}
1 & F^+_{a,a+1}(z) & F^+_{a,a+2}(z) & \cdots & F^+_{a,N}(z) \\
0 & 1 & F^+_{a+1,a+2}(z) & \cdots & F^+_{a+1,N}(z) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 1 & F^+_{N-1,N}(z) \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix} \begin{pmatrix}
K^+_{a}(z)E^+_{a,b}(u) & 0 & \cdots & 0 \\
0 & K^+_{a+1}(z) & \cdots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & K^+_{N}(z)
\end{pmatrix}
\]

\[
\times \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
E^+_{a+1,b}(z) & 1 & \cdots & \vdots \\
E^+_{a+2,b}(z) & E^+_{a+2,a+1}(z) & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
E^+_{N,b}(z) & E^+_{N,a+1}(z) & \cdots & E^+_{N,N-1}(z) & 1
\end{pmatrix}
\]

(E.25)
For \( a < b \)
\[
\hat{L}^+(u)_{a,b} = \begin{pmatrix} 1 & F_{a,b+1}^+(z) & F_{a,b+2}^+(z) & \cdots & F_{a,N}^+(z) \\ 0 & 1 & F_{b+1,b+2}^+ & \cdots & F_{b+1,N}^+ \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & F_{N-1,N}^+ \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} F_{a,b}^+(u)K_b^+(z) & 0 & \cdots & 0 \\ 0 & K_{b+1}^+(z) & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & K_{N}^+(z) \end{pmatrix}
\]

\[
E_{b+1,b}^+(1) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E_{b+1,b}^+(z) & 1 & \cdots & \cdots & \vdots \\ E_{b+2,b+1}^+(z) & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E_{N,b}^+(z) & E_{N,b+1}^+(z) & \cdots & E_{N,N-1}^+(z) & 1 \end{pmatrix},
\]

\[\text{(E.26)}\]

The formula for \( \hat{L}^+(u)_{a,a} \) is the same as \( [6,16] \) with \( l=a \).

Let us write the Gauss decomposition of \( \hat{L}^+(u)_{a,b} \) in the above Lemma as
\[
\hat{L}^+(u)_{a,b} = F_{a,b}(u)K_{a,b}(u)E_{a,b}(u).
\]

Then we have
\[
F_{a,b}(u)^{-1} = K_{a,b}(u)E_{a,b}(u)\hat{L}^+(u)_{a,b}^{-1}.
\]

Comparing the \((1,1)\) component of the both sides we obtain the following.

**Lemma E.22.**

\[
K_a^+(u) = \frac{1}{(\hat{L}^+(u)_{a,a}^{-1})_{11}} \quad \text{for} \quad a = b, \quad \text{(E.27)}
\]

\[
E_{a,b}(u) = (\hat{L}^+(u)_{a,a}^{-1})_{11} \frac{1}{(\hat{L}^+(u)_{a,b}^{-1})_{11}} \quad \text{for} \quad a > b, \quad \text{(E.28)}
\]

\[
F_{a,b}(u) = \frac{1}{(\hat{L}^+(u)_{a,b}^{-1})_{11}}(\hat{L}_b^+(u)^{-1})_{11} \quad \text{for} \quad a < b. \quad \text{(E.29)}
\]

Noting
\[
(\hat{L}^+(u)_{a,b}^{-1})_{11} = (\hat{I}^+(u-1)_{a,b})_1^T(q-\det \hat{L}^+(u)_{a,b})^{-1}N_{N-a+1}^\prime,
\]

\[
(\hat{I}^+(u-1)_{a,b})_1^T = q-\det \hat{L}^+(u-1)_{a+1,a+1},
\]

we have
Theorem E.23.

\[
K_a^+(u) = \frac{q-\det \hat{L}^+(u)_{a,a}}{N_{N-a-1}^a} \frac{1}{q-\det \hat{L}^+(u-a)_{a+1,a+1}}, \quad (E.30)
\]

\[
E_{a,b}^+(u) = \frac{1}{q-\det \hat{L}^+(u)_{a,a}} q-\det \hat{L}^+(u)_{a,b} \quad \text{for } a > b, \quad (E.31)
\]

\[
F_{a,b}^+(u) = q-\det \hat{L}^+(u)_{a,b} \frac{1}{q-\det \hat{L}^+(u)_{b,b}} \quad \text{for } a < b. \quad (E.32)
\]

Corollary E.24. Let us define

\[
K(u) = K_1^+(u)K_2^+(u-1) \cdots K_N^+(u-(N-1)). \quad (E.33)
\]

Then

\[
q-\det \hat{L}^+(u) = N_N K(u). \quad (E.34)
\]

Moreover the \( q \)-determinant \( q-\det \hat{L}^+(u) \) belongs to the center of \( E_{q,p}(\hat{gl}_N) \).

Proof. Since \( K_i^+(v), E_i(v) = \text{const.}(E_i^{+0,1}(v+c/4) - E_i^{-0,1}(v-c/4)), F_i(v) = \text{const.}(F_i^{+0,1}(v-c/4) - F_i^{-0,1}(v+c/4)) \) (1 ≤ \( i \) ≤ \( N \), 1 ≤ \( l \) ≤ \( N-1 \)) satisfy the same commutation relations as the elliptic currents of \( U_{q,p}(\hat{gl}_N) \), \( K(u) \) commutes with \( K_i^+(v), E_i(v), F_i(v), F \) due to Proposition [32].

Hence by Definition [6.2] \( K(u) \) commutes with \( K_i^{\pm}(v), E_i^{\pm0,1}(v), F_i^{\pm0,1}(v) \) so that \( K(u) \) commutes with \( \hat{L}^+(v) \).

References

[1] Y.Asai, M.Jimbo, T.Miwa, and Y.Pugai, Bosonization of Vertex Operators for the \( A_{n-1}^{(1)} \) Face Model. J. Phys. A 29 (1996), 6595-6616.

[2] H.Awata, H.Kubo, S.Odake and J.Shiraishi, Quantum \( W_N \) Algebras and Macdonald Polynomials, Comm.Math.Phys. 179 (1996), 401–416.

[3] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press, London (1982)

[4] J.Beck, Braid Group Action and Quantum Affine Algebras. Comm.Math.Phys.165 (1994), 555-568.

[5] A.A. Belavin, Dynamical symmetry of integrable quantum systems. Nucl. Phys. B180 [FS2] (1981), 189-200.

[6] P.Bouwknegt and K.Schoutens, W Symmetry in Conformal Field Theory, Phys.Rep. 223 (1993), 183–276.
[7] J.-S.Caux, H. Konno, M.Sorrell and R. Weston, Tracking the Effects of Interactions on Spinons in Gapless Heisenberg Chains”, Phys. Rev. Lett. 106 (2011), 217203 (4 pages); Exact Form-factor Results for the Longitudinal Structure Factor of the Massless XXZ Model in Zero Field, J. Stat. Mech. (2012), P01007 (40 pages).

[8] V. Chari and A. Pressley, Yangians and R-Matrices, L’Enseignement Math. 36 (1990), 267-302; Quantum Affine Algebras, Comm. Math. Phys. 142 (1991), 261-283

[9] J.Ding and I.B.Frenkel, Isomorphism of Two Realizations of Quantum Affine Algebra $U_q(\widehat{gl(n)})$. Comm.Math.Phys.156 (1993) 277-300.

[10] V.G. Drinfeld, Quantum Groups. Proc.ICM Berkeley 1 (1986), 789-820.

[11] V.G. Drinfeld, A New Realization of Yangians and Quantized Affine Algebras. Soviet Math. Dokl. 36 (1988) 212-216.

[12] V.G.Drinfeld, Quasi-Hopf Algebras, Leningrad Math. J. 1 (1990), 1419-1457.

[13] B. Enriquez and V.N.Rubtsov, Quantum groups in higher genus and Drinfeld's new realizations method (sl2 case) , Ann. Sci. École Norm. Sup., 30, (1997), 821-846; Quasi-Hopf Algebras Associated with $sl_2$ and Complex Curves. Israel J. Math. 112 (1999), 61-108.

[14] B. Enriquez and G. Felder, Elliptic Quantum Groups $E_{\tau,\eta}(sl_2)$ and Quasi-Hopf Algebra , Comm.Math.Phys., 195 (1998), 651–689.

[15] P.Etingof and A.Varchenko, Solutions of the Quantum Dynamical Yang-Baxter Equation and Dynamical Quantum Groups, Comm.Math.Phys. 196 (1998), 591–640 ; Exchange Dynamical Quantum Groups, Comm.Math.Phys. 205 (1999), 19–52.

[16] L.D.Faddeev, N.Yu.Reshetikhin and L.A.Takhtadjan, Quantization of Lie Groups and Lie Algebras, Leningrad Math.J 1 (1989), 178-201.

[17] R.M.Farghly, H.Konno, K.Oshima, Elliptic Algebra $U_{q,p}(\widehat{g})$ and Quantum Z-algebras, Alg. Rep. Theory 18 (2014), 103-135.

[18] B.Feigin and E.Frenkel, Quantum W-Algebras and Elliptic Algebras, Comm.Math.Phys. 178 (1996), 653–678.

[19] G. Felder, Elliptic Quantum Groups, Proc. ICMP Paris-1994 (1995), 211–218.

[20] G.Felder and A.Varchenko, Integral Representation of Solutions of the Elliptic Knizhnik-Zamolodchikov-Bernard Equations. Internat. Math. Res. Notices 5 (1995), 221-233.
[21] G. Felder and A. Varchenko, On Representations of the Elliptic Quantum Group $E_{r,\eta}(sl_2)$. Comm. Math. Phys. 181 (1996), 741-761.

[22] G. Felder and A. Varchenko, Algebraic Bethe ansatz for the Elliptic Quantum Group $E_{r,\eta}(sl_2)$. Nuclear Phys. B 480 (1996), 485-503.

[23] G. Felder and A. Varchenko, Elliptic quantum groups and Ruijsenaars models. J. Statist. Phys. 89 (1997), 963-980.

[24] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa, H. Yan, An Elliptic Quantum Algebra for $sl_2$. Lett. Math. Phys. 32 (1994), 259-268; Notes on Highest Weight Modules of the Elliptic Algebra $A_{q,p}(sl_2)$. Quantum field theory, integrable models and beyond (Kyoto, 1994). Progr. Theoret. Phys. Suppl. 118 (1995), 1-34.

[25] E. Frenkel and N. Reshetikhin, Deformation of W-Algebras Associated to Simple Lie Algebras, q-alg/9708006.

[26] C. Frønsdal, Quasi-Hopf Deformations of Quantum Groups, Lett. Math. Phys. 40 (1997), 117–134.

[27] S. L. Lukyanov and V. A. Fateev, Additional Symmetries and Exactly-Soluble Models in Two-Dimensional Conformal Field Theory, Sov. Sci. Rev. A. Phys. 15 (1990), 1-117.

[28] J. Hartwig, The Elliptic GL(n) Dynamical Quantum Group as an $H$-Hopf Algebroid. Int. J. Math. Math. Sci. (2009), Art. ID 545892, 41 pp.

[29] K. Hasegawa, Ruijsenaars’ Commuting Difference Operators as Commuting Transfer Matrices. Comm. Math. Phys. 187 (1997), 289-325.

[30] N. Hayaishi and K. Miki, L Operators and Drinfeld’s Generators. J. Math. Phys. 39 (1998), 1623-1636.

[31] K. Iohara, Bosonic Representations of Yangian Double $DY_h(g)$ with $g = gl_N, sl_N$. J. Phys. A 29 (1996), 4593-4621.

[32] M. Jimbo, A $q$-difference analogue of $U_q(g)$ and the Yang-Baxter Equation, Lett. Math. Phys., 10 (1985), 63–69.

[33] M. Jimbo and T. Miwa and M. Okado, Solvable Lattice Models Related to the Vector Representation of Classical Simple Lie Algebras, Comm. Math. Phys. 116 (1988), 507–525.
[34] M. Jimbo, A. Kuniba, T. Miwa, M. Okado, The $A_n^{(1)}$ Face Models. *Comm. Math. Phys.* **119** (1988) 543-565.

[35] M. Jimbo, and T. Miwa, Algebraic Analysis of Solvable Lattice Models. *Conference Board of the Math. Sci., Regional Conference Series in Mathematics* **85** (1995) and references therein.

[36] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, Elliptic Algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$: Drinfeld Currents and Vertex Operators, *Comm. Math. Phys.* **199** (1999), 605-647.

[37] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, Quasi-Hopf Twistors for Elliptic Quantum Groups, *Transformation Groups* **4** (1999), 303–327.

[38] M. Jimbo, H. Konno, S. Odake, Y. Pugai and J. Shiraishi, Free Field Construction for the ABF Models in Regime II, *J. Stat. Phys.* **102** (2001), 883–921.

[39] N. Jing, On Drinfeld Realization of Quantum Affine Algebras, in *The Monster and Lie algebras*, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, (1998) 195-206.

[40] E. Koelink and H. Rosengren, Harmonic Analysis on the $SU(2)$ Dynamical Quantum Group, *Acta. Appl. Math.*, **69** (2001), 163–220.

[41] E. Koelink, Y. van Norden and H. Rosengren, Elliptic $U(2)$ Quantum Group and Elliptic Hypergeometric Series, *Comm. Math. Phys.*, **245**, (2004), 519–537.

[42] E. Koelink and Y. van Norden, Pairings and Actions for Dynamical Quantum Group, *Adv. Math.*, **208**, (2007) 1-39.

[43] T. Kojima and H. Konno, The Elliptic Algebra $U_{q,p}(\hat{\mathfrak{sl}}_N)$ and the Drinfeld Realization of the Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(\hat{\mathfrak{sl}}_N)$, *Comm. Math. Phys.* **239** (2003), 405-447.

[44] T. Kojima and H. Konno, The Drinfeld Realization of the Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$, *J. Math. Phys.*, **45** (2004), 3146–3179.

[45] T. Kojima and H. Konno, The Elliptic Algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and the Deformation of $W_N$ Algebra, *J. Phys. A* **37** (2004), 371-383.

[46] T. Kojima and H. Konno and R. Weston, The Vertex-Face Correspondence and Correlation Functions of the Fusion Eight-Vertex Models I: The General Formalism, *Nucl. Phys.* **B720** (2005), 348-398.

[47] H. Konno, An Elliptic Algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and the Fusion RSOS Models, *Comm. Math. Phys.* **195** (1998), 373–403.
[48] H. Konno, Dynamical $R$ Matrices of Elliptic Quantum Groups and Connection Matrices for the $q$-KZ Equations, *SIGMA*, 2 (2006), Paper 091, 25 pages.

[49] H. Konno, Elliptic Quantum Group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and Vertex Operators, *J.Phys.A* 41 (2008) 194012.

[50] H. Konno, Elliptic Quantum Group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, Hopf Algebroid Structure and Elliptic Hypergeometric Series, *J. Geom. Phys.* 59 (2009), 1485-1511.

[51] H. Konno, Elliptic Quantum Group, Drinfeld Coproduct and Deformed $W$-Algebras, *Recent Advances in Quantum Integrable Systems 2014*, Dijon.

[52] H.Konno, K.Oshima, Elliptic Quantum Group $U_{q,p}(B_N^{(1)})$ and Vertex Operators, *RIMS Kokyuroku Bessatsu*, to appear.

[53] M.Lashkevich and Y. Pugai. Free field construction for correlation functions of the eight-vertex model. *Nucl. Phys. B* 516 (1998), 623-651.

[54] S. Lukyanov and Y. Pugai. Multi-point Local Height Probabilities in the Integrable RSOS Model, *Nucl.Phys. B* 473(1996), 631-658.

[55] A. Molev, Yangians and Classical Lie Algebras, *Mathematical Surveys and Monographs* 143 (2007) AMS.

[56] S.Pakuliak, V.Rubtsov and A.Silantyev, The SOS Model Partition Function and the Elliptic Weight Functions. *J. Phys. A* 41 (2008), 295204, 20 pp.

[57] H.Rosengren, Felder’s elliptic quantum group and elliptic hypergeometric series on the root system $A_n$, *Int. Math. Res. Not.* 13, (2011), 2861-2920.

[58] H.Rosengren, An Izergin-Korepin-type identity for the 8VSOS model, with applications to alternating sign matrices, *Adv. in Appl. Math.* 43 (2009), 137-155.

[59] N.Yu.Reshetikhin and M.A.Semenov-Tian-Shansky, Central Extensions of Quantum Current Groups. *Lett.Math.Phys.* 19 (1990), 133-142.

[60] V.Rubtsov, A.Silantyev and D.Talalaev, Manin Matrices, Quantum Elliptic Commutative Families and Characteristic Polynomial of Elliptic Gaudin Model. *SIGMA Symmetry Integrability Geom. Methods Appl.* 5 (2009), Paper 110, 22 pp.