A PRIORI ESTIMATES OF STATIONARY SOLUTIONS OF AN ACTIVATOR-INHIBITOR SYSTEM

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ABSTRACT. We consider positive solutions of the stationary Gierer-Meinhardt system

\[
\begin{align*}
  d_1 \Delta u - u + \frac{u^p}{v^q} + \sigma &= 0 \quad \text{in } \Omega, \\
  d_2 \Delta v - v + \frac{u^r}{v^s} &= 0 \quad \text{in } \Omega, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

where \( \Delta \) is the Laplace operator, \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \), \( n \geq 1 \) and \( \nu \) is the unit outer normal to \( \partial \Omega \). Under suitable conditions on the exponents \( p, q, r \) and \( s \), different types of a priori estimates are obtained, existence and non-existence results of nontrivial solutions are derived, for both \( \sigma > 0 \) and \( \sigma = 0 \) cases.

1. Introduction

In 1972, following an ingenious idea of A. Turing [13], A. Gierer and H. Meinhardt [2] proposed a mathematical model for pattern formations of spatial tissue structures of \textit{hydra} in morphogenesis, a biological phenomenon discovered by A. Trembley in 1744 [12]. It is a system of reaction-diffusion equations of the form

\[
\begin{align*}
  u_t &= d_1 \Delta u - u + \frac{u^p}{v^q} + \sigma \quad \text{in } \Omega \times [0, T), \\
  \tau v_t &= d_2 \Delta v - v + \frac{u^r}{v^s} \quad \text{in } \Omega \times [0, T), \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times [0, T)
\end{align*}
\]

where \( \Delta \) is the Laplace operator, \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \), \( n \geq 1 \) and \( \nu \) is the unit outer normal to \( \partial \Omega \). Here \( u, v \) represent respectively the concentrations of two substances, activator and inhibitor, with diffusion rates \( d_1, d_2 \), and are therefore always assumed to be positive throughout this paper. The source term \( \sigma \) is a nonnegative constant representing the production of the activator, \( \tau > 0 \) is the response rate of \( v \) to the change of \( u \), and the exponents \( p, q, r, s \) are nonnegative numbers satisfying the condition

\[
0 < \frac{p - 1}{r} < \frac{q}{s + 1}.
\]

We remark that the response rate \( \tau \) was introduced mathematically and is an important parameter on the stability of the system.

The idea behind (1.1) is the celebrated \textit{diffusion-driven instability}, originally due to A. Turing [13], which asserts that different diffusion rates could lead to nonhomogeneous distributions of the reactants. Indeed, spike-layer stationary solutions have been proved to exist when \( \Omega \) is axially symmetric [11] or \( n = 1 \) [11]. When
The global existence of (1.1), despite partial progress made in the past 25 years ([4], [5], [9]), is only settled recently by the first author in [3] for any positive initial data if \( \frac{p-1}{r} < 1 \).

Furthermore, if in addition \( \sigma > 0 \), there exists an attracting rectangle bounded away from zero and infinity for (1.1). On the other hand, in a recent paper [6], W. Ni, K. Suzuki and I. Takagi completely classified the dynamics of the corresponding kinetic system

\[
\begin{align*}
    u_t &= -u + \frac{u^p}{v^r} \\
    \tau v_t &= -v + \frac{u^r}{v^s} \\
\end{align*}
\]

(1.3) in \([0, T)\),

In particular, it is shown that when \( \frac{p-1}{r} > 1 \), there exist initial values such that \( u, v \) blows up in finite time. (Also see earlier results in [4].) However, the behavior of solutions to (1.1) is not well understood in general.

Our goal here is to understand the dependence of positive steady states to (1.1) as the diffusion coefficients \( d_1, d_2 \) vary. Especially, we want to study through a priori estimates the existence and nonexistence of nontrivial positive stationary patterns. When the dimension \( n = 1 \), positive lower and upper a priori bounds for positive steady states of (1.1) have been derived by I. Takagi [10], [11] under the general assumption (1.2). The method used in [10], [11] seems difficult to be extended to multi-dimensional case. When \( \sigma > 0 \) and in any space dimension, the trivial positive lower bounds

\[
u > \sigma, \quad v > \sigma^\frac{s}{r},
\]

immediately follow from maximum principle, and a priori upper bounds for Hölder norms have been obtained by W. Ni and I. Takagi [7] using energy method under the assumption

\[
\frac{p}{q} < \frac{r}{s+1} \quad \text{and} \quad r \geq \max \left( \frac{p \cdot n \cdot (p-1)}{2}, \frac{1}{\sigma} \right).
\]

Also when \( \sigma > 0 \), while studying asymptotic behavior of time-dependent solutions to (1.1), a priori upper bounds have been obtained by K. Masuda and K. Takahashi [5] under the assumption

\[
\frac{p-1}{r} < \min \left\{ 1, \frac{2}{n} \right\}
\]

and by the first author [3] under the assumption \( \frac{p-1}{r} < 1 \). When \( \sigma = 0 \), due to the possible singularity caused by \( v \) in the denominators of the nonlinear terms, a priori bounds usually are harder to obtain. Nonetheless, a priori upper bounds of Hölder norms have been obtained in [4] under the assumption

\[
\frac{p}{q} = \frac{r}{s+1} \quad \text{and} \quad s < \frac{2}{n-2};
\]

and positive a priori lower bounds have been obtained by M. del Pino [1] using compactness argument under the assumption

\[
1 < r < \frac{n}{n-2} \quad \text{and} \quad \frac{s}{r-1} < \frac{n}{n-2}.
\]

Throughout this entire paper, we will always assume that (1.2) holds and use \( (u, v) \) to denote a smooth positive steady state of (1.1), unless otherwise explicitly stated, and \( (u^*, v^*) \) to denote its unique constant steady state. We now come to our main results.
Theorem 1.1. Suppose that $\sigma \geq 0$.

(i) If $q < s + 1$, then there exists $k_1 > 0$ depending on $p, q, r, s, \sigma$, such that whenever $d_1 \leq k_1$, we have $u \leq u^*, v \leq v^*$.

(ii) If $r < s + 1$, then there exists $k_2 > 0$ depending on $p, q, r, s, \sigma$, such that whenever $d_1 \leq k_2$, we have $u \geq u^*, v \geq v^*$.

(iii) If $\max\{q, r\} < s + 1$, then whenever $d_1 \leq k = \min\{k_1, k_2\}$, we have $(u, v) \equiv (u^*, v^*)$.

Remark 1.2. The constants $k_1, k_2$ and $k$ can be calculated explicitly. For example, when $\sigma = 0$ and $(p, q, r, s) = (2, 4, 2, 4)$, the "common source" case, we have $k_1 = 1$ and $k_2 = 11 - 4\sqrt{6}$, hence $k = 1$. See Theorem 3.10 for more details.

Theorem 1.1 is new even when $n = 1$. It seems interesting that the above theorem indicates that the ratio of two diffusion rates alone can prevent the existence of nontrivial patterns while all previously known nonexistence results for this system require that at least one of the diffusion rates $d_1, d_2$ be suitably large. Our method also suggests that a priori estimates depending on $d_1$ are quite natural, as the following result shows.

Theorem 1.3.

(i) Let $\sigma = 0$ and $q < s + 1$. Then

$$u \leq c \left(1 + \left(\frac{d_2}{d_1}\right)^\gamma\right), \quad v \leq c \left(1 + \left(\frac{d_2}{d_1}\right)^{\frac{\gamma}{p-\gamma}}\right)$$

where $c, \gamma$ are positive constants independent of $d_1, d_2$.

(ii) Let $\sigma = 0$ and

$$\left(\max\left\{1, \frac{d_2}{d_1}\right\}\right) r < s + 1.$$

Then we have

$$u \geq c, \quad v \geq c^{\frac{1}{\gamma/p}},$$

where $c \to 0$ as $d_1 \to \infty$.

(iii) Let $\sigma > 0$ and $p - 1 < r$. Then we have

$$u \leq c \left(1 + \left(\frac{d_2}{d_1}\right)^\gamma\right), \quad v \leq c \left(1 + \left(\frac{d_2}{d_1}\right)^{\frac{\gamma}{p-\gamma}}\right)$$

where $c, \gamma$ are positive constants independent of $d_1, d_2$.

Remark 1.4. Under the same assumption in part (iii) of Theorem 1.3, similar upper bounds have also been obtained by the first author in [3]. However, our bounds here are more precise.

A common assumption for system (1.1) in modeling biological pattern formation is that the activator diffuses slowly while the inhibitor diffuses rapidly, i.e., $d_1$ is much smaller than $d_2$. If we fix $d_1$ and let $d_2 \to \infty$, formally, $v$ tends to a spatially homogeneous function $\xi = \xi(t)$, and (1.1) is reduced to the shadow system

$$
\begin{align*}
\begin{cases}
u_t &= d_1 \Delta u - u + \frac{\frac{\nu}{p}}{\xi} + \sigma \quad \text{in} \quad \Omega \times [0, T), \quad \\
\tau \xi_t &= -\xi + \frac{\int_{\Omega} u^\gamma(x) dx}{|\Omega|} \xi \quad \text{in} \quad \Omega \times [0, T), \quad \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega \times [0, T].
\end{cases}
\end{align*}
(1.4)
$$

Such formal derivation can be justified if we have a priori estimates independent of $d_2$ as $d_2 \to \infty$. 
Theorem 1.5.  (i) Let $\sigma = 0$ and
\[ \frac{q}{s+1} < \min \left\{ 1, \frac{2}{n} \right\}. \]
Then we have
\[ u \leq c \left( 1 + d_1^{-\gamma} \right), \quad v \leq c \left( 1 + d_1^{-\frac{s+1}{r}} \right) \]
where $c, \gamma$ are positive constants independent of $d_1, d_2$.

(ii) Let $\sigma > 0$, $\frac{p-1}{r} < \min \left\{ 1, \frac{2}{n} \right\}$. Then we have
\[ u \leq c \left( 1 + d_1^{-\gamma} \right), \quad v \leq c \left( 1 + d_1^{-\frac{s+1}{r}} \right) \]
where $c, \gamma$ are positive constants independent of $d_1, d_2$.

The following theorem provides both lower bounds and upper bounds which are independent of $d_1, d_2$ when $d_1, d_2$ are large.

Theorem 1.6. Let $\sigma = 0$ and $d_1, d_2 > \eta$ where $\eta$ is a given positive number.

(i) Assume that $r < \frac{n}{n-2}$ and there exists $\delta \in (0, 1]$ such that
\[ \frac{1-\delta}{r} + \frac{\delta}{p} < 1, \]
and
\[ \frac{(1-\delta) s + \delta q}{r-1+s} - \frac{\delta}{p} < \frac{n}{n-2} \text{ or } \frac{(1-\delta) s + \delta q}{r-1+s} - \frac{\delta}{p} \leq s+1. \]
Then
\[ u \geq c_1, \quad v \geq c_1^{\frac{s+1}{p}} \]
where $c_1 = c_1(n, p, q, r, s, \eta)$.

(ii) Assume in addition that $\frac{p-1}{r} < \min \left\{ 1, \frac{2}{n} \right\}$. Then we have
\[ u \leq c_2, \quad v \leq c_2^{\frac{s+1}{p}} \]
where $c_2 = c_2(n, p, q, r, s, \eta)$.

The assumptions in part (i) of the above theorem seem complicated; however, since $\delta$ is a free parameter, we can choose different $\delta$ to yield a family of estimates. For example, The lower bound by M. del Pino in [1] is contained in part (i) with $\delta = 0$. Also when $n = 2$, the assumptions in part (i) hold automatically as long as we have (1.2).

An important consequence of the above $a \text{ priori}$ estimates is that $(u^*, v^*)$ is the only steady state of (1.1) when $d_1$ is suitably large. (See Theorem 5.1.)

Theorem 1.7. Let $p - 1 < r$.

(i) Assume in addition $\sigma > 0$. Then for any $K > 0$, there exists constant $c > 0$, such that whenever $Kd_1 \geq d_2$ and $d_1 \geq c$, (1.1) has no nonconstant steady states.

(ii) Assume in addition $\sigma > 0$ and $\frac{p-1}{r} < \frac{2}{n}$. Then there exists constant $c > 0$, such that whenever $d_1 \geq d_2 \geq c$, (1.1) has no nonconstant steady states.

(iii) Assume in addition $\sigma = 0$ and $n = 2$. Then for any $d^* > 0$, there exists constant $c > 0$, such that whenever $d_2 \geq d^*$, and $d_1 \geq c$, (1.1) has no nonconstant steady states.
Remark 1.8. In part (iii) of Theorem 1.7, assumption $n = 2$ can be replaced by the more general assumptions in Theorem 1.6.

Another application of our a priori estimates is the existence of nontrivial steady states when $d_1$ is sufficiently small. We refer the readers to Theorems 7.3 and 7.9 below for more details. The main idea is to show that the Leray-Schauder degree of the associated map is nonzero in a region excluding the trivial steady state.

Our techniques work for more general reaction-diffusion systems, but in order to make our ideas clear, we will not pursue such generality here.

The paper is organized in the following way. We first present some basic estimates in Section 2. In Section 3, we use maximum principle to establish a priori bounds depending on $d_2$, especially, Theorems 1.1, 1.3 will be proved. In Sections 4 and 5, we use two different energy methods to establish Theorems 1.5 and 1.6. In Section 6, we will discuss nonexistence results. Finally, in Section 7, we will use topological degree theory to show the existence of nontrivial steady states under certain situations.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded smooth domain. We consider positive stationary solutions of

\[
\begin{aligned}
&d_1 \Delta u - u + \frac{u^p}{v^q} + \sigma = 0 \quad \text{in} \quad \Omega, \\
&d_2 \Delta v - v + \frac{u^r}{v^s} = 0 \quad \text{in} \quad \Omega, \\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

(2.1)

where $d_1, d_2 > 0$ are diffusion constants, the exponents $p, q, r, s$ are nonnegative constants satisfying (1.2) and the source term $\sigma$ is a nonnegative constant.

For any $\sigma \geq 0$, (2.1) has a unique constant solution $(u^*, v^*)$ such that

\[
\begin{aligned}
&-u^* + (u^*)^{p-\frac{qr}{s+1}} + \sigma = 0, \\
&v^* = (u^*)^{\frac{s+1}{r}}.
\end{aligned}
\]

(2.2)

When $\sigma = 0$, we have $(u^*, v^*) \equiv (1, 1)$. Furthermore, we have

Proposition 2.1. For any $\sigma \geq 0$, \( \frac{du^*}{d\sigma} > 0 \) and $\lim_{\sigma \to \infty} u^* = \infty$.

Proof. Differentiating (2.2) with respect to $\sigma$, we have

\[-\frac{du^*}{d\sigma} + \left(p - \frac{qr}{s+1}\right) (u^*)^{p-1-\frac{qr}{s+1}} \frac{du^*}{d\sigma} + 1 = 0,
\]

hence

\[
\frac{du^*}{d\sigma} = \frac{1}{1 + \left(\frac{qr}{s+1} - p\right) (u^*)^{p-1-\frac{qr}{s+1}}}.
\]

(2.3)

If $\frac{qr}{s+1} \geq p$, we have $\frac{du^*}{d\sigma} > 0$. If $\frac{qr}{s+1} < p$, using (2.2) to rewrite (2.3), we have

\[
\frac{du^*}{d\sigma} = \frac{1}{\frac{qr}{s+1} - (p-1) + \sigma \left(p - \frac{qr}{s+1}\right) (u^*)^{-1}} > 0.
\]

\[\Box\]
From now on, we assume that \((u, v)\) is a positive smooth solution of (2.1), i.e., \(u, v \in C^\infty(\Omega)\) and \(u, v > 0\) in \(\Omega\). (Actually, \(u \geq 0\) and \(u \not\equiv 0\) implies \(u, v > 0\).)

With each solution \((u, v)\), we define the following quantities:

\[
\bar{u} = \max_{x \in \Omega} u, \quad \underline{u} = \min_{x \in \Omega} u,
\]

\[
\bar{v} = \max_{x \in \Omega} v, \quad \underline{v} = \min_{x \in \Omega} v.
\]

First, we recall a basic convexity property of a \(C^2\) function at its local extrema.

**Lemma 2.2.** Let \(w \in C^2(\Omega)\) satisfy \(\frac{\partial w}{\partial \nu} = 0\) on \(\partial \Omega\).

(i) If \(w\) has a local maximum at \(x_1 \in \Omega\), then
\[
\nabla w(x_1) = 0, \quad \triangle w(x_1) \leq 0;
\]

(ii) If \(w\) has a local minimum at \(x_2 \in \Omega\), then
\[
\nabla w(x_2) = 0, \quad \triangle w(x_2) \geq 0.
\]

**Remark 2.3.** Neumann boundary condition is needed for the above lemma to hold if the local extremum is located on \(\partial \Omega\).

Applying Lemma 2.2 to \(u, v\), we have

**Proposition 2.4.**

\[
\bar{v} \leq \bar{u}^{\frac{p}{p-1}}, \quad \underline{v} \geq \underline{u}^{\frac{p}{p-1}}, \quad \bar{u} \geq \frac{\underline{u}^p}{\bar{v}^q} + \sigma, \quad \underline{u} \leq \frac{\bar{u}^p}{\bar{v}^q} + \sigma.
\]

**Proof.** Let \(x^* \in \Omega\) be such that \(v(x^*) = \bar{v}\), then at \(x^*\),
\[
\triangle v = \frac{1}{d_2} \left( v - \frac{u^{\frac{p}{p-1}}}{v^q} \right) \leq 0,
\]

hence, \(\bar{v}^{\frac{p}{p-1}} \leq u^{\frac{p}{p-1}}(x^*) \leq \bar{u}^{\frac{p}{p-1}}\). The other three inequalities can be proved in the same manner. \(\square\)

Next, we include basic energy estimates.

**Lemma 2.5.**

\[
\int_{\Omega} \frac{u^p}{v^q} + \sigma |\Omega| = \int_{\Omega} u, \quad \int_{\Omega} \frac{u^{p-1}}{v^q} + \sigma \int_{\Omega} u^{-1} \leq |\Omega|,
\]

\[
\int_{\Omega} \frac{1}{v^q} + \sigma \int_{\Omega} \frac{1}{u^p} \leq \int_{\Omega} \frac{1}{v^q}, \quad \int_{\Omega} \frac{v^q}{u^p} = \int_{\Omega} v,
\]

\[
\int_{\Omega} \frac{u^r}{v^s} \leq |\Omega|, \quad \int_{\Omega} v^{s+1} \leq \int_{\Omega} u^r.
\]

**Proof.** The first identity follows from integrating (2.1) over \(\Omega\). Next, multiplying (2.1) with \(\frac{1}{u}\) then integrating over \(\Omega\), we have
\[
\int_{\Omega} d_1 |\nabla u|^2 \frac{1}{u^2} - |\Omega| + \int_{\Omega} \frac{u^p-1}{v^q} + \sigma \int_{\Omega} u^{-1} = 0,
\]

which establishes the second inequality. The other estimates can be obtained in a similar manner. \(\square\)

The following \(L^1\) estimates come from standard elliptic theory.
Lemma 2.6. Let \((u, v)\) be a solution to (2.1). For any \(0 < \gamma < \frac{n}{n-2}\), we have
\[
\int \Omega u^\gamma \leq c \left(1 + d_1^{-\gamma}\right) \left(\int \Omega u\right)^\gamma, \quad \int \Omega v^\gamma \leq c \left(1 + d_2^{-\gamma}\right) \left(\int \Omega v\right)^\gamma
\]
where \(c\) is a constant independent of \(d_1, d_2\).

Proof. Since
\[
\triangle u = \frac{1}{d_1} \left(u - \frac{u^p}{v^q} - \sigma\right),
\]
for any \(1 \leq \gamma < \frac{n}{n-2}\), we have
\[
\|u\|_{\gamma} \leq c \left(\frac{1}{d_1} \left\|u - \frac{u^p}{v^q} - \sigma\right\| + \|u\|_1\right)
\leq c \left(\frac{1}{d_1} \left(\|u\|_1 + \left\|\frac{u^p}{v^q} + \sigma\right\|_1\right) + \|u\|_1\right)
= c \left(\frac{2}{d_1} + 1\right) \int \Omega u.
\]
And the case \(0 < \gamma < 1\) follows from Hölder’s inequality. The estimate for \(v\) can be proved in the same manner. \(\square\)

We will need the following lemma which was proved in [1] using Green’s function approach.

Lemma 2.7. Let \(\alpha\) be a positive constant and \(w \in C^2(\overline{\Omega})\) be a nonnegative function satisfying
\[
\begin{cases}
-\triangle w + \alpha w \geq 0 & \text{in } \Omega, \\
\frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Then
\[
w(x) \geq c \int \Omega w
\]
holds for any \(x \in \overline{\Omega}\), here \(c\) is a positive constant depending only on \(\alpha, n\) and \(\Omega\).

A direct application of Lemma 2.7 yields the following estimate of solutions to (2.1).

Lemma 2.8.
\[
u \geq c_1 \int \Omega u, \quad v \geq c_2 \int \Omega v
\]
where, for \(i = 1, 2\), the constant \(c_i = c_i(n, d_i)\) and it can be made uniform when \(d_i\) is large.

Proof. Assume \(d_1 \geq \eta\), we rewrite the equation for \(u\) as
\[
-\triangle u + \frac{1}{\eta} u = \frac{1}{d_1} \left(\frac{u^p}{v^q} + \sigma\right) + \left(\frac{1}{\eta} - \frac{1}{d_1}\right) u \geq 0,
\]
hence the estimate follows from Lemma 2.7. \(\square\)

Finally, we will need the following lemma on refined Sobolev inequality.
Lemma 2.9. For any $0 < \varepsilon \leq 1$, $2 \leq k \leq \frac{2n}{n-2}$ if $n \geq 3$ and $k \geq 2$ if $n = 1$ or $2$, we have for any $w \in H^1(\Omega)$,

$$\left( \int_{\Omega} w^k dx \right)^{\frac{1}{k}} \leq C \varepsilon^{\frac{2}{k} - \frac{2}{n}} \left( \int_{\Omega} (\varepsilon^2 |\nabla w|^2 + |w|^2) dx \right)^{\frac{1}{2}},$$

where $C$ is a positive constant independent of $\varepsilon$.

Proof. Let $w_\varepsilon (y) = w(\varepsilon y)$, and $x = \varepsilon y$, then

$$\int_{\Omega} (\varepsilon^2 |\nabla w|^2 + |w|^2) dx = \varepsilon^n \int_{\frac{1}{\varepsilon}\Omega} (|\nabla w_\varepsilon|^2 + |w_\varepsilon|^2) dy \geq C \varepsilon^n \left( \int_{\frac{1}{\varepsilon}\Omega} w_\varepsilon^k dy \right)^{\frac{1}{k}} = C \varepsilon^{n-\frac{2}{k}} \left( \int_{\Omega} w^k dx \right)^{\frac{1}{k}},$$

here $C$ depends on $n, k$, and cone property of $\frac{1}{\varepsilon}\Omega$. Hence, for any $\varepsilon \leq 1$, we have

$$\int_{\Omega} (\varepsilon^2 |\nabla w|^2 + |w|^2) dx \geq C \varepsilon^{n-\frac{2}{k}} \left( \int_{\Omega} w^k dx \right)^{\frac{1}{2}},$$

where $C$ is independent of $\varepsilon$. □

3. Maximum Principle

In this section, we will deduce a priori bounds for positive solutions of (2.1) which depend on $d_2^2 d_1^2$.

First, we apply Lemma 2.2 to $u^\lambda v^\lambda$, where $\lambda$ is any real number.

Lemma 3.1. Let $0 \leq \lambda \leq 1$. Then

$$1 - \frac{u^{p-1}}{v^q} - \sigma u^{-1} - \frac{\lambda d_1}{d_2} \left( 1 - \frac{u^r}{v^{s+1}} \right) \leq 0$$

holds at any point $x^* \in \Omega$ where $\frac{u}{v^\lambda}$ achieves its local maximum.

Proof. If $\frac{u}{v^\lambda}$ has a local maximum at $x^*$, then at $x^*$

$$\nabla \frac{u}{v^\lambda} = \frac{\nabla u}{v^\lambda} - \lambda \frac{u}{v^{\lambda+1}} \nabla v = 0,$$

and

$$\Delta \frac{u}{v^\lambda} = \frac{\Delta u}{v^\lambda} - \lambda \frac{u}{v^{\lambda+1}} \Delta v - 2 \lambda \frac{\nabla u \cdot \nabla v}{v^{\lambda+1}} + \lambda (\lambda + 1) \frac{u}{v^{\lambda+2}} |\nabla v|^2 \leq 0.$$

From (3.2), we have

$$\nabla u = \lambda \frac{u}{v} \nabla v.$$

Hence dividing (3.3) by $\frac{u}{v}$ and using (2.1) together with (3.4), we deduce

$$\frac{1}{d_1} \left( 1 - \frac{u^{p-1}}{v^q} - \sigma u^{-1} \right) - \frac{\lambda}{d_2} \left( 1 - \frac{u^r}{v^{s+1}} \right) - \lambda (\lambda - 1) \frac{|\nabla v|^2}{v^2} \leq 0.$$

Since $\lambda (\lambda - 1) \leq 0$, 3.1 follows. □

Similarly, we have
Lemma 3.2. Let $\lambda \leq 0$ or $\lambda \geq 1$. Then
\begin{equation}
1 - \frac{u^{p-1}}{v^q} - \sigma u^{-1} - \frac{\lambda d_1}{d_2} \left( 1 - \frac{u^r}{v^{q+1}} \right) \geq 0
\end{equation}
holds at any point $x^* \in \Omega$ where $\frac{u}{v^r}$ achieves its local minimum.

Our strategy is to use the bounds of $\frac{u}{v^r}$ to control $u$ and $v$ because we have

Lemma 3.3. Let $0 < \lambda < \frac{s+1}{r}$, then
\[
\inf_{\Omega} \frac{u}{v^\lambda} \leq \frac{u^{1-\frac{\lambda}{s+1}r}}{v^\lambda} \leq \frac{u^{1-\lambda}}{v^\lambda} \leq \frac{u^{1-\lambda}}{v^\lambda} \sup_{\Omega} \frac{u}{v^\lambda}
\]

Proof. Let $x \in \Omega$ be such that $u(x) = \bar{u}$, then we have
\[
\bar{u} = u(x) = \frac{u(x)}{v^\lambda(x)} v^\lambda(x) \leq \left( \sup_{\Omega} \frac{u}{v^\lambda} \right) v^\lambda \leq \left( \sup_{\Omega} \frac{u}{v^\lambda} \right) \frac{u^{1-\lambda}}{v^\lambda},
\]
hence
\[
\bar{u}^{1-\frac{\lambda}{s+1}r} \lambda \leq \sup_{\Omega} \frac{u}{v^\lambda}.
\]
Similarly, we can prove
\[
\frac{u^{1-\lambda}}{v^\lambda} \leq \inf_{\Omega} \frac{u}{v^\lambda}.
\]
The inequalities in between follow from (2.4).

Next, we introduce a family of functions
\begin{equation}
f_\sigma(\lambda) = \frac{\lambda (s+1-\lambda r)}{(q-\lambda(p-1))(u^*)^{\frac{q-\lambda(p-1)(s+1)}{q+1}} + \lambda \sigma (u^*)^{-1}}
\end{equation}
which are continuous and nonnegative in $[0, \frac{s+1}{r}]$. It is easy to check that
\[f_\sigma(0) = f_\sigma\left(\frac{s+1}{r}\right) = 0\]
and $f_\sigma$ has a unique critical point in $[0, \frac{s+1}{r}]$ which is a local maximum. Using (2.2), we have
\[f_\sigma(\lambda) = \frac{\lambda (s+1-\lambda r)}{(q-\lambda(p-1))(u^*)^{\frac{q-\lambda(p-1)(s+1)}{q+1}} + \lambda}.
\]
Proposition 2.4 implies that $f_\sigma(\lambda)$ is monotone in $\sigma$ for any given $\lambda \in (0, \frac{s+1}{r})$, and
\[\lim_{\sigma \to \infty} f_\sigma(\lambda) = s+1 - \lambda r.
\]
When $\sigma = 0$, we have $u^* = 1$, and the function $f_\sigma$ has a simple form
\[f_0(\lambda) = \frac{\lambda (s+1-\lambda r)}{q-\lambda(p-1)}.
\]
The unique critical point of $f_0$ in $(0, \frac{s+1}{r})$ can be calculated
\[
\lambda^* = \frac{q}{p-1} \left( 1 - \sqrt{1 - \frac{(p-1)(s+1)}{qr}} \right) = \frac{\sqrt{\frac{s+1}{r}}}{1 + \sqrt{1 - \frac{(p-1)(s+1)}{qr}}},
\]
We define the quantities
\[ a = \frac{q - \lambda(p-1)}{s+1 - \lambda r}, \quad b = \frac{s+1 - \lambda r}{q - \lambda(p-1)} = \frac{1}{a}, \]
\[ a_0 = \frac{\lambda}{s+1 - \lambda r}, \quad b_0 = \frac{s+1 - \lambda r}{\lambda} = \frac{1}{a_0}, \]
which will appear frequently in our proofs. The quantities \( \frac{\sigma}{\lambda} \) and \( \frac{\sigma r}{v^s} \) are related through \( \frac{\sigma}{\lambda v^s} \) in the following identities
\[
\frac{u^{p-1}}{v^q} = \left( \frac{u^r}{v^{s+1}} \right)^a \left( \frac{u}{v^\lambda} \right)^{1-a} \frac{q - (p-1)(s+1)}{(s+1) - q - \lambda(p-1)} \right)^{1-a},
\]
\[
\frac{u^r}{v^{s+1}} = \left( \frac{u^{p-1}}{v^q} \right)^b \left( \frac{u}{v^\lambda} \right)^{1-b} \frac{q - (p-1)(s+1)}{(s+1) - q - \lambda(p-1)} \right)^{1-b}.
\]
When \( \sigma > 0 \), we need the following identities to connect \( \frac{u^r}{v^{s+1}} \) and \( \frac{1}{u} \)
\[
\frac{1}{u} = \left( \frac{u^r}{v^{s+1}} \right)^{a_0} \left( \frac{u}{v^\lambda} \right)^{1-a_0} \frac{q - (p-1)(s+1)}{(s+1) - q - \lambda(p-1)} \right)^{1-a_0},
\]
\[
\frac{u^r}{v^{s+1}} = \left( \frac{1}{u} \right)^{b_0} \left( \frac{u}{v^\lambda} \right)^{1-b_0} \frac{q - (p-1)(s+1)}{(s+1) - q - \lambda(p-1)} \right)^{1-b_0}.
\]
When \( 0 < \lambda < \frac{s+1}{r} \), using quantities \( a \) and \( a_0 \), we have
\[
f_\sigma(\lambda) = \frac{\lambda}{a (u^*)^{\frac{q - (p-1)(s+1)}{r^{s+1}}} + a_0 (u^*)^{1-a} = \frac{\lambda}{(a - a_0) (u^*)^{\frac{q - (p-1)(s+1)}{r^{s+1}}} + a_0}.
\]
And when \( \sigma = 0 \), we have
\[
f_0(\lambda) = \frac{\lambda}{a}.
\]

Our first result is the upper bounds when \( \sigma = 0 \).

**Theorem 3.4.** Assume \( \sigma = 0 \) and \( q < s + 1 \). Then there exists positive constants \( c \) and \( \gamma \) independent of \( d_1, d_2 \) such that
\[
u \leq c \left( 1 + \frac{d_2}{d_1} \right)^{\gamma} \quad \text{and} \quad \nu \leq c \left( 1 + \frac{d_2}{d_1} \frac{1}{r-1} \right).
\]
Furthermore, if \( \frac{d_2}{d_1} \in f_0(\Lambda_1) \), where
\[
\Lambda_1 = (0, 1] \cap \left( 0, \frac{s+1}{r - (p-1)} \right),
\]
then \( u \leq 1, \nu \leq 1 \).

**Proof.** Let \( \lambda \in \tilde{\Lambda}_1 \cap \left( 0, \frac{d_2}{d_1} \right) \), where
\[
\tilde{\Lambda}_1 = (0, 1] \cap \left( 0, \frac{s+1}{r - (p-1)} \right).
\]
Since \( 0 < \lambda \leq 1 \), we have from Lemma 3.1 at any point \( x^* \in \Omega \) where \( \frac{\nu}{\lambda} \) achieves its maximum,
\[
1 - \frac{\lambda d_1}{d_2} \leq \frac{u^{p-1}}{v^q} - \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}}.
\]
Since
\[ 0 < \frac{p - 1}{r} < \frac{q}{s + 1} < 1, \]
we have
\[ \lambda < \frac{s + 1 - q}{r - (p - 1)} < \frac{s + 1}{r} < \frac{q}{p}, \]
hence
\[ a = \frac{q - \lambda(p - 1)}{s + 1 - \lambda r} \in (0, 1). \]

Applying Young’s inequality, we have
\[
\frac{u^{p-1}}{v^q} = \left( \frac{u^r}{v^s+1} \right)^a \left( \frac{u}{v^\lambda} \right)^{1-a} - \frac{q}{s+1} - \frac{s+1}{q} \left( \frac{u}{v^\lambda} \right)^{1-a} \left( \frac{u}{v^\lambda} \right)^{-\frac{q}{s+1}}.
\]

where we used Lemma 3.3 in the last inequality. Combining this with (3.7), we have
\[
1 - \lambda \frac{d_1}{d_2} \leq \frac{1 - a}{\left( \lambda \frac{d_1}{d_2} \right)^{1-a}} \bar{u} - \frac{q - (p-1)(s+1)}{(s+1)(1-a)}.
\]

Since
\[ 1 - \lambda \frac{d_1}{d_2} > 0, \]
we deduce
\[
\bar{u} \leq \left[ \frac{(1-a)^{1-a}}{1 - \lambda \frac{d_1}{d_2}} \right]^{\frac{q-p}{s+1}}.
\]

Next we prove the optimal bounds, if
\[ \frac{d_2}{d_1} \in f_0(\Lambda_1), \]
then we have for some \( \lambda \in \Lambda_1, \)
\[ \frac{d_2}{d_1} = f_0(\lambda) = \lambda \frac{a}{d_2} - \frac{d_2}{d_1}. \]

We first assume \( \lambda \in \bar{\Lambda}_1, \) then we have \( a \in (0, 1) \) and
\[ \lambda = \frac{d_2}{d_1} \cdot a < \frac{d_2}{d_1}, \]
hence (3.8) holds and the bound becomes \( \bar{u} \leq 1 \). If \( \lambda \notin \tilde{\Lambda}_1 \), then \( \lambda = \frac{s+1-q}{r-(p-1)} \) and \( a = 1 \), hence

\[
\frac{\lambda d_1}{d_2} = 1.
\]

Furthermore, there exists a sequence \( \{\lambda_k\} \subset \tilde{\Lambda}_1 \) such that \( \lambda < \lambda_k \) and

\[
\lambda = \lim_{k \to \infty} \lambda_k.
\]

Let

\[
a_k = \frac{q - \lambda_k (p-1)}{s+1 - \lambda_k r},
\]

then \( a_k \in (0,1) \) and

\[
\lim_{k \to \infty} a_k = 1.
\]

For each \( k \), we have

\[
\bar{u} \leq \left[ \frac{(1-a_k)^{1-a_k} a_k^{a_k}}{(1 - \lambda_k d_1/d_2)^{1-a_k} \left( \frac{\lambda_k d_1}{d_2} \right)^{a_k}} \right]^{\frac{s+1}{s+1-qs+1-(p-1)(s+1)}}.
\]

Now

\[
\lim_{k \to \infty} \left( \frac{1-a_k}{1 - \lambda_k d_1/d_2} \right)^{1-a_k} = \lim_{k \to \infty} \left( \frac{s+1-q}{s+1 - \lambda_k r} \right)^{1-a_k} = 1,
\]

hence \( \bar{u} \leq 1 \). And the optimal bound \( \bar{v} \leq 1 \) follows from (2.4). The set \( \Lambda_1 \) is a nonempty interval with left end point zero, from the property of function \( f_0 \), the set \( f_0(\Lambda_1) \) is a nonempty interval of the form \( (0, k_1] \) for some constant \( k_1 \) depending on \( p, q, r, s \). Inequality

\[
u \leq c \left( 1 + \left( \frac{d_2}{d_1} \right)^{\gamma} \right)
\]

automatically holds if \( \frac{d_2}{d_1} \leq k_1 \). When \( \frac{d_2}{d_1} \geq k_1 \), it can be deduced from (3.8) by taking

\[
\lambda = \frac{1}{2} \min \left\{ 1, k_1, \frac{s+1-q}{r-(p-1)} \right\} \in \tilde{\Lambda}_1 \cap \left( 0, \frac{d_2}{d_1} \right)
\]

Finally

\[
\bar{v} \leq c \left( 1 + \left( \frac{d_2}{d_1} \right)^{\frac{s+1}{s+1-qs+1-(p-1)(s+1)}} \right)
\]

follows from (2.4).

\[\square\]

**Remark 3.5.** It is easy to see that for any \( \tau, t \in (0,1) \), we always have

\[
\tau^t (1-\tau)^{1-t} \leq t^t (1-t)^{1-t},
\]

and the equality holds if and only if \( \tau = t \).
Remark 3.6. $f_0(\Lambda_1)$ is a nonempty interval of the form $(0, k_1]$ for some constant $k_1$ depending only on $p, q, r, s$. If $\frac{s+1-q}{r(p-1)} > 1$, then we have $\Lambda_1 = (0, 1]$ and

$$f_0(\Lambda_1) \supset \left( 0, \frac{s+1-r}{q-(p-1)} \right].$$

And if $\frac{s+1-q}{r(p-1)} \leq 1$, then we have

$$\Lambda_1 = \left( 0, \frac{s+1-q}{r-(p-1)} \right].$$

Especially, if $(p, q, r, s) = (2, 4, 2, 4)$, we have $\Lambda_1 = (0, 1]$.

The optimal bounds $u \leq 1, v \leq 1$ when $\frac{d_2}{d_1}$ is sufficiently small indicate that a priori estimates depending on $d_1, d_2$ in terms of $\frac{d_2}{d_1}$ could be natural. Such estimates are new even in one dimensional case.

Next, we consider lower bounds of stationary solutions when $\sigma = 0$.

**Theorem 3.7.** Assume $\sigma = 0$ and

$$\max \left\{ 1, \frac{d_2}{d_1} \right\} r < s + 1.$$  

Then

$$u \geq c, \quad v \geq c^{\frac{s}{r}}$$

where $c$ is a positive constant depending on $p, q, r, s$ and $\frac{d_2}{d_1}$ and satisfies

$$\lim_{\frac{d_2}{d_1} \to \frac{s+1}{r}} c = 0.$$  

Furthermore, if $\frac{d_2}{d_1} \in f_0(\Lambda_2)$, where

$$\Lambda_2 = \left\{ \lambda \in \left[ 1, \frac{s+1}{r} \right] : \frac{s+1-\lambda r}{q-\lambda(p-1)} \leq 1 \right\},$$

then $u \geq 1, v \geq 1$.

**Proof.** Let $\lambda \in \left( \frac{d_2}{d_1}, \infty \right) \cap \tilde{\Lambda}_2$, where

$$\tilde{\Lambda}_2 = \left\{ \lambda \in \left[ 1, \frac{s+1}{r} \right] : \frac{s+1-\lambda r}{q-\lambda(p-1)} < 1 \right\}.$$  

Since $\lambda \geq 1$, we have from Lemma 3.2

$$\frac{\lambda d_1}{d_2} - 1 \leq \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} - \frac{u^{p-1}}{v^q}.$$  

holds at any point \( x^* \in \overline{\Omega} \) where \( \frac{u}{v^r} \) achieves its minimum. Since
\[
\lambda < \frac{s + 1}{r} < \frac{q}{p - 1},
\]
we have
\[
b = \frac{s + 1 - \lambda r}{q - \lambda (p - 1)} \in (0, 1).
\]
Applying Young’s inequality, we have
\[
\frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} = \left[ \frac{1}{b} \frac{u^{b-1}}{v^q} \left( \frac{\lambda d_1}{\lambda_2} \right)^{1-b} \left( \frac{u}{v^r} \right)^{\frac{q r - (p-1)(s+1)}{q - \lambda (p-1) - (s+1)}} \right]^{1-b}
\]
\[
\leq \frac{u^{b-1}}{v^q} + (1-b) \frac{b^{1-b}}{b^b (1-b)^{1-b}} \left( \frac{\lambda d_1}{\lambda_2} \right)^{1-b} \left( \frac{u}{v^r} \right)^{\frac{q r - (p-1)(s+1)}{q - \lambda (p-1) - (s+1)}}.
\]
where we have used Lemma 3.3 in the last inequality. Combining (3.7), we have
\[
(3.11) \quad u \geq \left[ \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} \right]^{1-b} + \left( \frac{\lambda d_1}{\lambda_2} \right)^{1-b} \left( \frac{u}{v^r} \right)^{\frac{q r - (p-1)(s+1)}{q - \lambda (p-1) - (s+1)}}.
\]
which yields a lower bound for \( u \). Lower bound for \( v \) follows from (2.4).

Next, if \( \frac{d_2}{d_1} \in f_0(\Lambda_2) \), then we have for some \( \lambda \in \Lambda_2 \),
\[
\frac{d_2}{d_1} = f_0(\lambda) = \frac{\lambda (s + 1 - \lambda r)}{q - \lambda (p - 1)} = \lambda b.
\]
If \( \lambda \in \hat{\Lambda}_2 \), then \( b = 0, 1 \) and \( \lambda > \frac{d_2}{d_1} \), hence (3.11) holds and becomes \( u \geq 1 \). If \( \lambda \not\in \hat{\Lambda}_2 \), then
\[
b = \frac{s + 1 - \lambda r}{q - \lambda (p - 1)} = 1
\]
and there exist \( \lambda_k \in \hat{\Lambda}_2 \) such that \( \lambda_k > \lambda \) and
\[
\lim_{k \to \infty} \lambda_k = \lambda.
\]
Now for each \( k \), we have
\[
u \geq \left[ \frac{\lambda d_1}{\lambda_k d_1} \frac{u^r}{v^{s+1}} \right]^{1-b_k} + \left( \frac{\lambda d_1}{\lambda_k d_1} \right)^{1-b_k} \left( \frac{u}{v^r} \right)^{\frac{q r - (p-1)(s+1)}{q - \lambda_k (p-1) - (s+1)}}
\]
where
\[
b_k = \frac{s + 1 - \lambda_k r}{q - \lambda_k (p - 1)} \in (0, 1).
\]
Since
\[ \lim_{k \to \infty} \left[ \frac{\left( \frac{d_2}{\lambda_k d_1} \right)^{b_k} \left( 1 - \frac{d_2}{\lambda_k d_1} \right)^{1-b_k}}{b_k (1-b_k)^{1-b_k}} \right] = \lim_{k \to \infty} \left( 1 - \frac{d_2}{\lambda_k d_1} \right)^{1-b_k} \]
we again have \( u \geq 1 \). Optimal bound for \( v \) follows from (2.4).
\[ \square \]

**Remark 3.8.** The admissible set \( \Lambda_2 \) is a nonempty interval with right end point \( \frac{s+1}{r} \), from the property of function \( f_0 \), the set \( f_0(\Lambda_2) \) is a nonempty interval of the form \( (0, k_2] \) for some constant \( k_2 \) depending on \( p, q, r, s \). If \( \frac{p-1}{r} \geq 1 \), it is easy to check \( \Lambda_2 = \left[ 1, \frac{s+1}{r} \right), f_0(\Lambda_2) \supset (0, \frac{s+1-r}{q-1}) \]

And if \( \frac{p-1}{r} < 1 \),
\[ \Lambda_2 = \left[ \max \left\{ 1, \frac{(s+1)-q}{r-(p-1)} \right\}, \frac{s+1}{r} \right), f_0(\Lambda_2) \supset (0, \frac{s+1-r}{q-1}) \]

so if in addition, \( \frac{(s+1)-q}{r-(p-1)} < 1 \), we have
\[ \Lambda_2 = \left[ 1, \frac{s+1}{r} \right), f_0(\Lambda_2) \supset (0, \frac{s+1-r}{q-1}) \]

and if in addition, \( \frac{(s+1)-q}{r-(p-1)} \geq 1 \), we have
\[ \Lambda_2 = \left[ \frac{(s+1)-q}{r-(p-1)}, \frac{s+1}{r} \right), f_0(\Lambda_2) \supset (0, \frac{s+1-r}{q-1}) \]

When \((p, q, r, s) = (2, 4, 2, 4)\), \( \Lambda_2 = \left( 1, \frac{5}{2} \right) \) and \( f_0(\Lambda_2) = (0, 11 - 4\sqrt{6}) \).

Combining the optimal bounds in Theorems 3.4 and 3.7, we have

**Theorem 3.9.** Let
\[ s + 1 > \max \{q, r\} \]
Then \( u \equiv 1, v \equiv 1 \) is the only solution whenever
\[ \frac{d_2}{d_1} \in f_0(\Lambda_1) \cap f_0(\Lambda_2) \].

**Remark 3.10.** In general, \( f_0(\Lambda_1) = (0, k_1] \) and \( f_0(\Lambda_2) = (0, k_2] \), hence
\[ f_0(\Lambda_1) \cap f_0(\Lambda_2) = (0, k] \]
where \( k = \min \{k_1, k_2\} \). When \((p, q, r, s) = (2, 4, 2, 4)\), we have \( k = 1 \), hence \( u \equiv 1, v \equiv 1 \) is the only solution when \( d_2 \leq d_1 \).

Now we extend our optimal bounds to the case \( \sigma > 0 \).
Theorem 3.11. Assume $\sigma > 0$ and $q < s + 1$. If

$$\frac{d_2}{d_1} \in f_\sigma (\Lambda_3),$$

where

$$\Lambda_3 = (0, 1] \cap \left(0, \frac{s + 1 - q}{r - (p - 1)}\right) \cap \left(0, \frac{s + 1}{r + 1}\right),$$

then $u \leq u^*, v \leq v^*$.

Proof. Let $\lambda \in \tilde{\Lambda}_3 \cap \left(0, \frac{d_a}{d_1}\right)$ where

$$\tilde{\Lambda}_3 = (0, 1] \cap \left(0, \frac{s + 1 - q}{r - (p - 1)}\right) \cap \left(0, \frac{s + 1}{r + 1}\right).$$

Since $0 < \lambda \leq 1$, we have from Lemma 3.11 at any point $x^* \in \Omega$ where $\frac{d_a}{v^*}$ achieves its maximum,

$$1 - \frac{u^{p-1}}{v^q} - \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} - \sigma u^{-1} \leq 0.$$

Since $\lambda \in \left(0, \frac{s + 1 - q}{r - (p - 1)}\right) \cap \left(0, \frac{s + 1}{r + 1}\right)$, we also have

$$a = \frac{q - \lambda (p - 1)}{s + 1 - \lambda r} \in (0, 1) \quad \text{and} \quad a_0 = \frac{\lambda}{s + 1 - \lambda r} \in (0, 1).$$

Let $\delta$ be any given number in $(0, 1)$, we have from Young’s inequality

$$\frac{u^{p-1}}{v^q} = \left(\frac{\delta}{a} \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} \right)^a \left(\frac{\delta}{a} \frac{\lambda d_1}{d_2} \frac{u^{1-a}}{v^{s+1-a}} \right)^{1-a} \leq \frac{\delta \lambda d_1}{d_2} \frac{u^r}{v^{s+1}} + \frac{1-a}{\left(\frac{\delta}{a} \frac{\lambda d_1}{d_2}\right)^a} \left(\frac{u}{v^a}\right)^{1-a},$$

and

$$\sigma u^{-1} = \left(1 - \frac{\delta \lambda d_1}{d_2} \frac{u^r}{v^{s+1}}\right)^{a_0} \left(\frac{\sigma^{1-a_0}}{a_0} \frac{1 - \delta \lambda d_1}{d_2} \frac{u^{1-a}}{v^a}\right)^{1-a_0} \leq \left(1 - \frac{\delta \lambda d_1}{d_2} \frac{u^r}{v^{s+1}}\right)^{a_0} \left(\frac{\sigma^{1-a_0}}{a_0} \frac{1 - \delta \lambda d_1}{d_2} \frac{u^{1-a}}{v^a}\right)^{1-a_0}.$$

Combining the three inequalities above and applying Lemma 3.8, we deduce

$$1 - \frac{\lambda d_1}{d_2} \leq \frac{1-a}{\left(\frac{\delta}{a} \frac{\lambda d_1}{d_2}\right)^a} \left(\frac{u}{v^a}\right)^{1-a} + \frac{\sigma^{1-a_0}}{a_0} \frac{1 - \delta \lambda d_1}{d_2} \frac{u^{1-a}}{v^a} \leq \frac{1-a}{\left(\frac{\delta}{a} \frac{\lambda d_1}{d_2}\right)^a} \left(\frac{u}{v^a}\right)^{1-a} + \frac{\sigma^{1-a_0}}{a_0} \frac{1 - \delta \lambda d_1}{d_2} \frac{u^{1-a}}{v^a},$$

which yields an upper bound for $\tilde{u}$ since

$$1 - \frac{\lambda d_1}{d_2} > 0.$$

Upper bound for $\tilde{v}$ follows from 2.4.
Next, if \( \frac{d^2}{dt^2} \in f_\sigma (\Lambda_3) \), then there exists \( \lambda \in \Lambda_3 \) such that \( \frac{d^2}{dt^2} = f_\sigma (\lambda) \). We first assume \( \lambda \in \Lambda_3 \), then \( \text{(3.12)} \) holds for any \( \delta \in (0, 1) \). Let

\[
\delta = \alpha \frac{d^2}{\lambda d^3} (u^*) - \frac{q - (p - 1)(s + 1)}{s + 1} \equiv 1 - a_0 \frac{d^2}{\lambda d^3} (u^*)^{-1} \in (0, 1),
\]

then \( \text{(3.12)} \) becomes

\[
(3.13) \quad 1 - \frac{\lambda d_1}{d_2} \leq (1 - a) \left( \frac{\bar{u}}{(u^*)^\alpha} \right)^{- \frac{q - (p - 1)(s + 1)}{s + 1} \frac{1}{1 - \alpha}} + \sigma (1 - a_0) \left( \frac{\bar{u}}{(u^*)^\alpha} \right)^{- \frac{1}{\alpha - 1}}.
\]

We observe that if \( \bar{u} = u^* \), then

\[
(1 - a) \left( \frac{u^*}{(u^*)^\alpha} \right)^{- \frac{q - (p - 1)(s + 1)}{s + 1} \frac{1}{1 - \alpha}} + \sigma (1 - a_0) \left( \frac{u^*}{(u^*)^\alpha} \right)^{- \frac{1}{\alpha - 1}}
\]

\[
= (1 - a) (u^*)^{- \frac{q - (p - 1)(s + 1)}{s + 1} \frac{1}{1 - \alpha}} + (1 - a_0) \sigma (u^*)^{-1}
\]

\[
= (u^*)^{- \frac{q - (p - 1)(s + 1)}{s + 1} \frac{1}{1 - \alpha}} + \sigma (u^*)^{-1} - \left( \frac{u^*}{(u^*)^\alpha} \right)^{- \frac{q - (p - 1)(s + 1)}{s + 1} \frac{1}{1 - \alpha}} + a_0 \sigma (u^*)^{-1}
\]

\[
= 1 - \frac{\lambda d_1}{d_2}.
\]

Since the right hand side of \( \text{(3.13)} \) is monotone decreasing in \( \bar{u} \), the equality above implies \( \bar{u} \leq u^* \). And from \( \text{(3.24)} \), we have \( \bar{v} \leq (u^*)^{\frac{r}{s - 1}} = v^* \). If \( \lambda \notin \tilde{\Lambda}_3 \), then \( \lambda = \frac{s + 1 - q}{r} \) or \( \frac{s + 1}{r + 1} \), and there exist \( \lambda_k \in \tilde{\Lambda}_3 \) such that \( \lambda_k < \lambda \) and

\[
\lim_{k \to \infty} \lambda_k = \lambda.
\]

Let

\[
a_k = \frac{q - \lambda_k (p - 1)}{s + 1 - \lambda_k r} \in (0, 1) \quad \text{and} \quad a_{0k} = \frac{\lambda_k}{s + 1 - \lambda_k r} \in (0, 1),
\]

we have

\[
1 - \frac{\lambda_k d_1}{d_2} \leq (1 - a_k) \left( \frac{\bar{u}}{(u^*)^{\alpha_k}} \right)^{- \frac{q - (p - 1)(s + 1)}{s + 1} \frac{1}{1 - a_k}} + \sigma (1 - a_{0k}) \left( \frac{\bar{u}}{(u^*)^{\alpha_{0k}}} \right)^{- \frac{1}{\alpha - 1}}.
\]

For each \( k \), let \( u_k \) be the unique positive number such that

\[
(3.14) \quad 1 - \frac{\lambda_k d_1}{d_2} \leq (1 - a_k) \left( \frac{u_k}{(u^*)^{\alpha_k}} \right)^{- \frac{q - (p - 1)(s + 1)}{s + 1} \frac{1}{1 - a_k}} + \sigma (1 - a_{0k}) \left( \frac{u_k}{(u^*)^{\alpha_{0k}}} \right)^{- \frac{1}{\alpha - 1}},
\]

then we have \( \bar{u} \leq u_k \). We claim

\[
\lim_{k \to \infty} u_k = u^*.
\]

First, since \( (u^*, v^*) \) is a solution, \( \bar{u} \leq u_k \) implies that for each \( k \), \( u_k \geq u^* \). If

\[
\lambda = \frac{s + 1 - q}{r} < \frac{s + 1}{r + 1},
\]

then direct calculation yields

\[
1 - \frac{\lambda_k d_1}{d_2} > 1 - \frac{\lambda d_1}{d_2} = \sigma (u^*)^{-1} \frac{s + 1 - \lambda (r + 1)}{s + 1 - \lambda r} > 0.
\]
It is easy to check
\[ \lim_{k \to \infty} a_k = 1, \quad \lim_{k \to \infty} a_{0k} = a_0 \in (0, 1). \]

Hence
\[
(1 - a_k) \left( \frac{u_k}{(u^*)^{a_k}} \right)^{-\frac{qr - (p-1)(s+1)}{s+1}} \leq (1 - a_k) (u^*)^{-\frac{qr - (p-1)(s+1)}{s+1}}
\]
which converges to zero as \( k \to \infty \). So we have \( \lim_{k \to \infty} u_k \) exists, and
\[
1 - \frac{\lambda d_1}{d_2} = \sigma (1 - a_0) \left( \frac{\lim_{k \to \infty} u_k}{(u^*)^{a_0}} \right)^{-\frac{1}{1-a_0}},
\]
hence \( \lim_{k \to \infty} u_k = u^* \).

If
\[
\lambda = \frac{s + 1}{r + 1} < \frac{s + 1 - q}{r - (p - 1)},
\]
we have
\[
\lim_{k \to \infty} a_k = a \in (0, 1), \quad \lim_{k \to \infty} a_{0k} = 1.
\]

We can obtain \( \lim_{k \to \infty} u_k = u^* \) similarly. Finally, we consider the case
\[
\lambda = \frac{s + 1}{r + 1} = \frac{s + 1 - q}{r - (p - 1)},
\]
then
\[
\lim_{k \to \infty} a_k = 1, \quad \lim_{k \to \infty} a_{0k} = 1
\]
and
\[
\frac{\lambda d_1}{d_2} = 1,
\]
dividing equation (3.14) by \( \lambda - \lambda_k \), we have
\[
\frac{d_1}{d_2} = \frac{1 - a_k}{(\lambda - \lambda_k) \left( \frac{\lambda d_1}{d_2} \right)^{-\frac{qr - (p-1)(s+1)}{s+1}} u_k^{-\frac{1}{1-a_0}}} + \frac{\sigma (1 - a_0)}{(\lambda - \lambda_k) \left( \frac{\lambda d_1}{d_2} \right)^{-\frac{1}{1-a_0}}} u_k^{-\frac{1}{1-a_0}}
\]
\[
= \frac{qr - (p - 1)(s + 1)}{(s + 1 - \lambda r)(s + 1 - \lambda_k r)} \left( \frac{u_k}{(u^*)^{a_k}} \right)^{-\frac{qr - (p-1)(s+1)}{s+1}} u_k^{-\frac{1}{1-a_0}} + \frac{\sigma (s + 1)}{(s + 1 - \lambda r)(s + 1 - \lambda_k r)} \left( \frac{u_k}{(u^*)^{a_k}} \right)^{-\frac{1}{1-a_0}}.
\]

If
\[
\lim_{k \to \infty} u_k > u^*,
\]
then both terms in the right hand side tend to zero, which is a contradiction; If
\[
\lim_{k \to \infty} u_k < u^*,
\]
then both terms in the right hand side tend to infinity, which is a also contradiction. Hence, we have \( \lim_{k \to \infty} u_k = u^* \).

Hence, we have \( \lim_{k \to \infty} u_k = u^* \).

Next, we extend optimal lower bound to the case \( \sigma > 0 \).
Theorem 3.12. If \( s + 1 > r \) and \( \frac{d^2}{d^2 \tau} \in f_r (A_4) \), where

\[
A_4 = \left\{ \lambda \in [1, \infty) : \frac{s + 1}{r + 1} \leq \lambda < \frac{s + 1 - \lambda r}{q - \lambda (p - 1)} \leq 1 \right\}
\]

then we have \( u \geq u^* \), \( v \geq v^* \).

Proof. Let \( \lambda \in \hat{A}_4 \cap \left( \frac{d^2}{d^2 \tau}, \infty \right) \) where

\[
\tilde{A}_4 = \left\{ \lambda \in [1, \infty) : \frac{s + 1}{r + 1} < \lambda < \frac{s + 1 - \lambda r}{q - \lambda (p - 1)} < 1 \right\}.
\]

Since \( \lambda \geq 1 \), we have from Lemma 3.2, at any point \( x^* \in \Omega \) where \( \frac{d^2}{d^2 \tau} \) achieves its minimum,

\[
1 - \frac{u^{p-1}}{v^q} - \sigma u^{-1} - \frac{\lambda d_1}{d_2} \left( 1 - \frac{u^r}{v^{s+1}} \right) \geq 0.
\]

Under our assumptions, we also have

\[
b = \frac{s + 1 - \lambda r}{q - \lambda (p - 1)} \in (0, 1), \quad b_0 = \frac{s + 1 - \lambda r}{\lambda} \in (0, 1).
\]

Applying Young’s inequality, we have

\[
\frac{\lambda d_1}{d_2} - 1 \leq \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} - \frac{u^{p-1}}{v^q} - \sigma u^{-1}
\]

\[
= \left[ \frac{1}{b} \frac{u^{p-1}}{v^q} \right]^{\frac{1}{b}} \left( \frac{\lambda d_1}{d_2} \right)^{\frac{1}{b}} b \frac{\lambda d_1}{d_2} \left( \frac{u}{v^\lambda} \right)^{\frac{q - (p - 1) (s + 1)}{q - \lambda (p - 1) (s + 1 - \lambda r)}} \right)^{1-b} - \left( \frac{1}{b_0} \sigma u^{-1} \right)^{b_0} \left( 1 - \delta \right) \frac{\lambda d_1}{d_2} \left( \frac{u}{v^\lambda} \right)^{\frac{q - (p - 1) (s + 1)}{q - \lambda (p - 1) (s + 1 - \lambda r)}}
\]

\[
\leq (1 - b) \left( \frac{\lambda d_1}{d_2} \right)^{\frac{1}{b}} b \frac{\lambda d_1}{d_2} \left( \frac{u}{v^\lambda} \right)^{\frac{q - (p - 1) (s + 1)}{q - \lambda (p - 1) (s + 1 - \lambda r)}} \right)^{1-b} - \left( \frac{1}{b_0} \sigma u^{-1} \right)^{b_0} \left( 1 - \delta \right) \frac{\lambda d_1}{d_2} \left( \frac{u}{v^\lambda} \right)^{\frac{q - (p - 1) (s + 1)}{q - \lambda (p - 1) (s + 1 - \lambda r)}}
\]

\[
\leq (1 - b) \left( \frac{\lambda d_1}{d_2} \right)^{\frac{1}{b}} b \frac{\lambda d_1}{d_2} \left( \frac{u}{v^\lambda} \right)^{\frac{q - (p - 1) (s + 1)}{q - \lambda (p - 1) (s + 1 - \lambda r)}} \right)^{1-b} - \left( \frac{1}{b_0} \sigma u^{-1} \right)^{b_0} \left( 1 - \delta \right) \frac{\lambda d_1}{d_2} \left( \frac{u}{v^\lambda} \right)^{\frac{q - (p - 1) (s + 1)}{q - \lambda (p - 1) (s + 1 - \lambda r)}}
\]

which yields a lower bound of \( u \) since

\[
\frac{\lambda d_1}{d_2} - 1 > 0.
\]

Lower bound of \( v \) follows from 3.3. When

\[
\frac{d^2}{d^2 \tau} = f_\sigma (\lambda)
\]

for some \( \lambda \in A_4 \), \( u \geq u^* \), \( v \geq v^* \) can be proved similarly as in the proof of Theorem 3.11 with the same choice of \( \delta \).
Combining Theorems 3.11 and 3.12 we have

**Corollary 3.13.** Assume \( \sigma > 0 \) and \( \max\{q, r\} < s + 1 \). Then we have \( u \equiv u^*, v \equiv v^* \) if
\[
\frac{d_2}{d_1^*} \in f_\sigma(\Lambda_3) \cap f_\sigma(\Lambda_4).
\]

**Remark 3.14.** It is easy to see that \( f_\sigma(\Lambda_3) = (0, k_1] \) and \( f_\sigma(\Lambda_4) = (0, k_2] \) for some \( k_1, k_2 > 0 \). Hence, \( f_\sigma(\Lambda_3) \cap f_\sigma(\Lambda_4) = (0, k] \) with \( k = \min\{k_1, k_2\} > 0 \).

When \( \sigma > 0 \), there are simple lower bounds
\[
u > \sigma, \hspace{1cm} \nu > \sigma^{\frac{r}{s+1}}.
\]

In Theorem 3.11, we haven’t made use of such bounds. Actually, these lower bounds will help to relax the conditions in Theorem 3.11. However, we no longer obtain optimal bounds.

**Theorem 3.15.** Assume \( \sigma > 0 \) and \( p - 1 < r \). Then
\[
u \leq c \left(1 + \left(\frac{d_2}{d_1^*}\right)^\gamma\right), \hspace{1cm} \nu \leq c \left(1 + \left(\frac{d_2}{d_1^*}\right)^{\frac{r}{s+1}+\gamma}\right),
\]
where \( c, \gamma \) are positive constants independent of \( d_1, d_2 \).

**Proof.** Let \( \varepsilon \geq 0 \) be such that
\[
0 < \frac{p-1}{r} < \frac{q-\varepsilon}{s+1} < 1.
\]
Let \( \lambda \in \left(0, \frac{d_2}{d_1^*}\right) \cap \Lambda_5 \) where
\[
\Lambda_5 = \left\{ \lambda \in (0, 1) : \lambda < \min\left\{ \frac{s+1 - (q-\varepsilon)}{r-(p-1)}, \frac{s+1}{r+1} \right\} \right\}.
\]
Since \( 0 < \lambda \leq 1 \), we have from Lemma 3.1 at any point \( x^* \in \Omega \) where \( u^* \) achieves its maximum,
\[
1 - \frac{u^{p-1}}{v^q} - \frac{\lambda d_1}{d_2} + \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} - \sigma u^{-1} \leq 0.
\]

Since
\[
a_\varepsilon = \frac{q-\varepsilon - \lambda (p-1)}{s+1 - \lambda r} \in (0, 1) \hspace{1cm} \text{and} \hspace{1cm} a_0 = \frac{\lambda}{s+1 - \lambda r} \in (0, 1),
\]
for any \( \delta \in (0, 1) \), we have from Young’s inequality,
\[
\begin{align*}
\frac{u^{p-1}}{v^q} &< \sigma^{\frac{\gamma}{\gamma+1}} \frac{u^{p-1}}{v^{q-\varepsilon}} \\
&= \left(\frac{\delta}{a_\varepsilon} \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}}\right)^{a_\varepsilon} \left(\sigma^{\frac{\gamma}{\gamma+1}} \frac{\lambda d_1}{a_\varepsilon} \frac{u^r}{d_2 v^{s+1}}\right)^{1-a_\varepsilon} - \frac{u^r}{v^{s+1}} - \frac{(q-\varepsilon)r-(p-1)(s+1)}{s+1 - \lambda r} \leq \left(\frac{\delta}{a_\varepsilon} \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}}\right)^{1-a_\varepsilon} - \frac{(q-\varepsilon)r-(p-1)(s+1)}{s+1 - \lambda r}.
\end{align*}
\]
and
\[
\sigma u^{-1} = \left( \frac{1 - \delta \lambda d_1}{a_0} \right)^{a_0} \left( \frac{1 - \delta \lambda d_1}{a_0} \right)^{1-a_0} \left( \frac{u}{v^{\lambda}} \right)^{1-a_0} \leq (1 - \delta) \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} \lambda \frac{u}{v^{\lambda}} = \frac{(1 - \delta) \lambda d_1}{d_2} \frac{u^r}{v^{s+1}} \lambda \frac{u}{v^{\lambda}}.
\]
Combining the above inequalities and applying Lemma 3.3, we have
\[
1 - \frac{\lambda d_1}{d_2} \leq (1 - a_{\epsilon}) \frac{\delta \lambda d_1}{d_2} \frac{u^r}{v^{s+1}} \lambda \frac{u}{v^{\lambda}} - \frac{1}{1 - a_{\epsilon}} \frac{\delta \lambda d_1}{d_2} \frac{u^r}{v^{s+1}} \lambda \frac{u}{v^{\lambda}}.
\]
which yields an upper bound of \(\bar{u}\) since
\[
1 - \frac{\lambda d_1}{d_2} > 0.
\]
Upper bound for \(\bar{v}\) follows from 2.4.

Theorems 3.1 and Theorem 3.3 are proved by combining all the results we have so far in this section.

In the remaining part of this section, we consider \ref{2.1} with common sources, i.e., we require \(p = r, q = s\). The assumption
\[
\frac{p - 1}{r} < \frac{q}{s + 1}
\]
is then reduced to \(s + 1 > r\). And \(\Lambda_i, i = 1, 2, 3, 4\) have simpler forms
\[
\Lambda_1 = (0, 1], \quad \Lambda_2 = \left[1, \frac{s + 1}{r}\right), \quad \Lambda_3 = (0, 1] \cap \left(0, \frac{s + 1}{r + 1}\right], \quad \Lambda_4 = \left[1, \frac{s + 1}{r + 1}\right] \cap \left[\frac{s + 1}{r + 1}, \frac{s + 1}{r}\right).
\]
For any \(\lambda \in \left[0, \frac{s + 1}{r}\right]\),
\[
f_\sigma(\lambda) = \frac{\lambda (s + 1 - \lambda r)}{(s - \lambda r) (u^*)^{s + 1}} + 1.
\]
Obviously,
\[
f_\sigma\left(\frac{s}{r}\right) \equiv f_0\left(\frac{s}{r}\right) = 1.
\]
Since \(u^*\) is monotone increasing in \(\sigma\), we have for any fixed \(\lambda \in (0, \frac{s}{r})\), \(f_\sigma(\lambda)\) is monotone increasing in \(\sigma\); for any \(\lambda \in \left(\frac{s}{r}, \frac{s + 1}{r}\right), \ f_\sigma(\lambda)\) is monotone decreasing in \(\sigma\). Furthermore, since \(\lim_{\sigma \to \infty} u^* = \infty\), for any \(\lambda \in \left(0, \frac{s + 1}{r}\right), \)
\[
\lim_{\sigma \to \infty} f_\sigma(\lambda) = s + 1 - \lambda r.
\]
Let \(k_\sigma\) be such that for \(\sigma = 0, (0, k_0] = f_0(\Lambda_1) \cap f_0(\Lambda_2)\) and for \(\sigma > 0, (0, k_\sigma] = f_\sigma(\Lambda_3) \cap f_\sigma(\Lambda_4)\). Then for any \(\sigma \geq 0\), the system has no nontrivial solution whenever
\[
\frac{d_2}{d_1} \leq k_\sigma.
\]
The following theorem describes the dependence of \(k_\sigma\) on \(\sigma\).

**Theorem 3.16.** Let \(\sigma \geq 0\).

(i) \(k_0 = 1\).
(ii) If $s = r$, then for any $\sigma > 0$, $k_\sigma = 1$.

(iii) If $s > r$, then $k_\sigma$ is monotone increasing for $\sigma \in (0, \infty)$, and

$$\lim_{\sigma \to 0} k_\sigma = k_0 = 1, \quad \lim_{\sigma \to \infty} k_\sigma = \frac{s + 1}{r + 1}.$$ 

(iv) If $s < r$, then $k_\sigma$ is monotone decreasing for $\sigma \in (0, \infty)$, and

$$\lim_{\sigma \to 0} k_\sigma = k_0 = 1, \quad \lim_{\sigma \to \infty} k_\sigma = s + 1 - r.$$ 

Proof. (i). We have

$$\Lambda_1 = (0, 1], \Lambda_2 = \left[1, \frac{s + 1}{r}\right].$$

Since $f_0$ has only one critical point in $(0, \frac{s + 1}{r})$, we have

$$(0, k_0] = f_0(\Lambda_1) \cap f_0(\Lambda_2) = (0, f_0(1)] = (0, 1],$$

hence $k_0 = 1$.

(ii). If $s = r$, we have

$$\Lambda_3 = (0, 1] \cap \left(0, \frac{s + 1}{r + 1}\right) = (0, 1],$$

and

$$\Lambda_4 = \left[1, \frac{s + 1}{r}\right] \cap \left[\frac{s + 1}{r + 1}, \frac{s + 1}{r}\right) = \left[1, \frac{s + 1}{r}\right].$$

From the property of $f_\sigma$, we have $k_\sigma = f_\sigma(1) = 1$.

(iii). $s > r$ implies

$$0 < 1 < \frac{s + 1}{r + 1} < \frac{s}{r} < \frac{s + 1}{r}.$$ 

Hence

$$\Lambda_3 = (0, 1] \cap \left(0, \frac{s + 1}{r + 1}\right) = (0, 1],$$

and

$$\Lambda_4 = \left[1, \frac{s + 1}{r}\right] \cap \left[\frac{s + 1}{r + 1}, \frac{s + 1}{r}\right] = \left[\frac{s + 1}{r + 1}, \frac{s + 1}{r}\right).$$

For any $\lambda < \frac{s}{r}$, since $f_\sigma$ is monotone increasing in $\sigma$, the set $f_\sigma((0, 1])$ is monotone increasing. On the other hand, it is easy to verify that in $(0, \frac{s + 1}{r})$, $f_0$ has a unique critical point

$$1 < \lambda_0 = \frac{s + 1}{1 + \sqrt{s + 1 - r}} < \frac{s + 1}{r + 1},$$

and since $f_\sigma$ is monotone increasing in $\sigma$ for any $\lambda < \frac{s}{r}$, and monotone decreasing in $\sigma$ for any $\lambda > \frac{s}{r}$, we have

$$\max_{\lambda \in \left[\frac{s + 1}{r + 1}, \frac{s + 1}{r}\right]} f_\sigma(\lambda) = \max_{\lambda \in \left[\frac{s + 1}{r + 1}, \frac{s}{r}\right]} f_\sigma(\lambda)$$

which increases with $\sigma$. Hence, $k_\sigma$ is monotone increasing. Next,

$$\lim_{\sigma \to 0} k_\sigma = k_0 = 1$$

follows from

$$f_0\left(\frac{s}{r}\right) = f_0(1) = 1.$$
And
\[ \lim_{\sigma \to \infty} k_\sigma = \frac{s + 1}{r + 1} \]
follows from
\[ \lim_{\sigma \to \infty} f_\sigma (\lambda) = s + 1 - \lambda r. \]

(iv). When \( s < r \), we have
\[ 0 < \frac{s}{r} < \frac{s + 1}{r + 1} < 1 < \frac{s + 1}{r}, \]
hence
\[ \Lambda_3 = \left( 0, \frac{s + 1}{r + 1} \right], \Lambda_4 = \left[ 1, \frac{s + 1}{r} \right). \]
Now the critical point \( \lambda_0 \) of \( f_0 \) satisfies
\[ \frac{s + 1}{r + 1} < \lambda_0 < 1, \]
and we have
\[ f_0 \left( \frac{s}{r} \right) = f_0 (1) = 1. \]
For any \( \lambda \geq 1 > \frac{s}{r} \),
\[ f_\sigma (\lambda) \leq f_0 (\lambda) \leq f_0 (1) = f_\sigma \left( \frac{s}{r} \right), \]
hence
\[ f_\sigma (\Lambda_3) \cap f_\sigma (\Lambda_4) = (0, f_\sigma (1)], \]
so we have \( k_\sigma = f_\sigma (1) \) which is monotone decreasing. Limits of \( k_\sigma \) follow similarly as the case when \( s > r \). \( \square \)

When \( d_2 = d_1 \), the uniqueness of solutions to (2.1) with common sources can be proved directly.

**Theorem 3.17.** Let \( p = r, q = s \) and (1.2) be satisfied. Then \( u \equiv u^*, v \equiv v^* \) is the only solution when \( d_2 = d_1 \).

**Proof.** Let \( w = u - v \), then we have
\[
\begin{cases}
    d_1 \Delta w - w + \sigma = 0 & \text{in } \Omega, \\
    \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]
hence \( w = \sigma \), and \( u = v + \sigma \), so we have
\[
\begin{cases}
    d_1 \Delta v - v + \frac{(v + \sigma)r}{v} = 0 & \text{in } \Omega, \\
    \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Now maximum principle implies
\[ \bar{v}^\frac{s + 1}{r + 1} \leq \bar{v} + \sigma \text{ and } \underline{v}^\frac{s + 1}{r} \geq \underline{v} + \sigma. \]
Since
\[ \frac{s + 1}{r} > 1, \]
the function
\[ g (t) = t^\frac{s + 1}{r} - t - \sigma \]
satisfies
\[ g (0) = -\sigma, \quad g (v^*) = 0, \quad \lim_{t \to \infty} g (t) = \infty \]
and \( g \) has only one critical point \( t_1 \) in \((0, \infty)\) which is a local minimum with \( g(t_1) < -\sigma \). Now \( g(u) \geq 0 \) implies \( u \geq v^* \) where \( v^* \) is the unique zero of \( g \), and \( g(\bar{v}) \leq 0 \) implies \( \bar{v} \leq v^* \), hence we can conclude \( u = \bar{v} = v^* \).

\[ \square \]

**Remark 3.18.** When \( s < r \) and \( \sigma > 0 \), we have \( k_\sigma < 1 \), hence the above theorem is not covered by Corollary 3.13.

### 4. Energy method I

In this section, we will establish *a priori* bounds independent of \( d_2 \). The main idea is to use the bound
\[
\int_\Omega \frac{u^r}{v^{s+1}} \leq |\Omega|
\]
to control the nonlinear term in the equation for \( u \). This approach can be traced back to [7] by W. Ni and I. Takagi and [5] by K. Masuda and K. Takahashi. The new ingredient here is the Moser iteration technique, which enables us to obtain more general result.

We first consider the case when \( \sigma = 0 \).

**Theorem 4.1.** Assume that \( \sigma = 0 \) and
\[
\frac{q}{s+1} < \min \left\{ 1, \frac{2}{n} \right\}.
\]
Then
\[
u \leq c \left(1 + d_1^{-\gamma}\right), \quad v \leq c \left(1 + d_1^{-\frac{q}{s+1}}\gamma\right),
\]
where \( c, \lambda \) are constants independent of \( d_1, d_2 \).

**Proof.** For any \( l > 1 \), multiplying (2.1) with \( u^{l-1} \) and integrating over \( \Omega \), we obtain
\[
(l-1) \int_\Omega d_1 |\nabla u|^{2} u^{l-2} + \int_\Omega u^l = \int_\Omega \frac{u^{p+1-1}}{v^q}.
\]
Since \( \frac{r}{s+1} < 1 \), applying Hölder’s inequality, we have
\[
\int_\Omega \frac{u^{p+1-1}}{v^q} = \int_\Omega \left( \frac{u^r}{v^{s+1}} \right)^{\frac{q}{s+1}} \left( u^{(l+p-1-\frac{qr}{s+1})\frac{p+1}{s+1}} \right)^{\frac{q}{s+1}} \frac{1}{\frac{q}{s+1}}
\leq \left( \int_\Omega \frac{u^r}{v^{s+1}} \right)^{\frac{q}{s+1}} \left( \int_\Omega u^{(l-\frac{qr}{s+1})\frac{p+1}{s+1} - \frac{qr}{s+1}} \right)^{\frac{1}{\frac{q}{s+1}}}
\leq |\Omega|^{\frac{s}{s+1}} \left( \int_\Omega u^{\frac{p+1}{p+1}} \right)^{\frac{q}{s+1}} \left( \int_\Omega u^{\frac{q}{s+1}} \right)^{\frac{q}{s+1}}.
\]
On the other hand, for any \( l \geq \max \{2, 4d_1\} \), we have
\[
\frac{2d_1}{l} \leq d_1 \frac{4(l-1)}{l^2} \leq 1.
\]
Lemma 2.9 implies that for any $k \in \left(1, \frac{2n}{n-2}\right]$ if $n \geq 3$ and $k \in (1, \infty)$ if $n = 1$ or 2,

$$(l-1) \int_{\Omega} d_1 |\nabla u|^2 u^{l-2} + \int_{\Omega} u^l = \int_{\Omega} d_1 \frac{4(l-1)}{l^2} |\nabla u^l|^2 + \int_{\Omega} |u^l|^2 \geq c \left( d_1 \frac{4(l-1)}{l^2} \right)^{\frac{2}{p-\frac{2}{p}} \left( \int_{\Omega} (u^l)^k \right)^{\frac{2}{p}}}
$$

$$\geq c \left( \frac{d_1}{l} \right)^{\frac{2}{p-\frac{2}{p}} \left( \int_{\Omega} (u^l)^k \right)^{\frac{2}{p}}}$$

here $c$ is a positive constant depending only on $n$ and $\Omega$. Hence,

$$c \left( \frac{d_1}{l} \right)^{\frac{2}{p-\frac{2}{p}} \left( \int_{\Omega} (u^l)^k \right)^{\frac{2}{p}}} \leq |\Omega|^{\frac{2}{p-\frac{2}{p}}} \left( \int_{\Omega} u^{\frac{s+1}{s+1-q} - \frac{qr}{s+1-q} - \frac{(p-1)(s+1)}{2(s+1)}} \right)^{k(s+1-q)}$$

i.e.,

$$\int_{\Omega} u^{\frac{k}{s+1-q}} \leq c \left( \frac{1}{d_1} \right)^{\frac{k}{s+1-q} \left( \int_{\Omega} u^{\frac{k}{s+1-q} - \frac{qr}{s+1-q} - \frac{(p-1)(s+1)}{2(s+1)}} \right)^{k(s+1-q)}}.$$

Setting $l_0 = 2 + 4d_1$, we define $l_i$ recursively by

$$\frac{l_i}{2} = \frac{s+1}{s+1-q} l_{i+1} - \frac{qr - (p-1)(s+1)}{s+1-q},$$

then

$$\int_{\Omega} u^{\frac{k l_{i+1}}{2}} \leq c \left( \frac{l_{i+1}}{d_1} \right)^{\frac{k}{s+1-q} \left( \int_{\Omega} u^{\frac{k}{s+1-q} - \frac{qr}{s+1-q} - \frac{(p-1)(s+1)}{2(s+1)}} \right)^{k(s+1-q)}}.$$

Let

$$a = \frac{k s+1-q}{2}, \quad b = \frac{qr - (p-1)(s+1)}{s+1},$$

then

$$l_{i+1} = a l_i + b.$$

Since $\frac{q}{s+1} < \frac{2}{n}$, we can choose $k$ so that $a > 1$, and direct calculation yields

$$l_i = a^i \left( l_0 + \frac{b}{a-1} \right) - \frac{b}{a-1}.$$
Hence
\[
\int u^{\frac{k}{d_1} + 1} \leq \frac{c}{d_1^{\frac{k}{d_1} - \frac{n}{a-1}}} \left( \int u^{\frac{k}{d_1} + 1} \right)^a \\
\leq \left( \frac{c}{d_1^{\frac{k}{d_1} - \frac{n}{a-1}}} \right)^{a-1} a^{(\frac{k}{d_1} - \frac{n}{a-1})} \left( \int u^{\frac{k}{d_1} + 1} \right)^a \\
\leq \left( \frac{c}{d_1^{\frac{k}{d_1} - \frac{n}{a-1}}} \right)^{a-1} a^{(\frac{k}{d_1} - \frac{n}{a-1})} \left( \int u^{\frac{k}{d_1} + 1} \right)^a \\
= \left( \frac{c}{d_1^{\frac{k}{d_1} - \frac{n}{a-1}}} \right)^{a-1} a^{(\frac{k}{d_1} - \frac{n}{a-1})} \left( \int u^{\frac{k}{d_1} + 1} \right)^a,
\]
and
\[
\left( \int u^{\frac{k}{d_1} + 1} \right)^a \leq \left[ c_1 \left( \int u^{\frac{k}{d_1} + 1} \right)^a \right] \frac{a^{(\alpha + 2) \left( \alpha + 1 \right)}}{(\alpha - 1)^2}.
\]
where
\[
c_1 = \left( \frac{c}{d_1^{\frac{k}{d_1} - \frac{n}{a-1}}} \right)^{a-1} a^{(\frac{k}{d_1} - \frac{n}{a-1})} \left( \int u^{\frac{k}{d_1} + 1} \right)^a.
\]
Letting \( i \to \infty \), we have
\[
\|u\|_{L^\infty(\Omega)} \leq \left[ c \left( \frac{\mathbb{L}_0 + \frac{b}{a-1}}{d_1} \right)^{\frac{k}{d_1} - \frac{n}{a-1}} a^{\left( \frac{k}{d_1} - \frac{n}{a-1} \right)} \left( \int u^{\frac{k}{d_1} + 1} \right)^a \right] \frac{\mathbb{L}_0}{(\mathbb{L}_0 + \frac{b}{a-1})^{\frac{n}{a-1}}}.
\]
Hence,
\[
\|u\|_{L^\infty(\Omega)} \leq \left[ c \left( \frac{\mathbb{L}_0 + \frac{b}{a-1}}{d_1} \right)^{\frac{k}{d_1} - \frac{n}{a-1}} a^{\left( \frac{k}{d_1} - \frac{n}{a-1} \right)} \left( \int u^{\frac{k}{d_1} + 1} \right)^a \right] \frac{\mathbb{L}_0}{(\mathbb{L}_0 + \frac{b}{a-1})^{\frac{n}{a-1}}} \\
\leq c \left( \frac{\mathbb{L}_0 + \frac{b}{a-1}}{d_1} \right)^{\frac{k}{d_1} - \frac{n}{a-1}} \leq c \left( 1 + \frac{\mathbb{L}_0 + \frac{b}{a-1}}{d_1} \right)^{\frac{k}{d_1} - \frac{n}{a-1}}.
\]
The upper bound for \( v \) follows from \[24\].
If $p \geq \frac{qr}{s+1}$, we can actually obtain Hölder estimate for $u$ directly without using the Moser iteration technique.

**Theorem 4.2.** Assume in addition to the assumptions of Theorem 4.1,

$$p \geq \frac{qr}{s+1}.$$

Then for some $\theta \in (0, 1)$,

$$\|u\|_{C^\theta(\Omega)} \leq c \left(1 + d_1^{-\gamma}\right),$$

where $c, \gamma$ are positive constants independent of $d_1, d_2$.

**Proof.** We have from the proof of Theorem 4.1, for any $k \in (1, 2)$ if $n \geq 3$, $k \in (1, \infty)$ if $n = 1$ or 2, and for any $l \geq \max\{2, 4d_1\}$,

$$\int_\Omega u^{\frac{k_1}{q}} \leq c \left(\frac{l}{d_1}\right)^{\frac{k_1}{q} - \frac{s}{2}} \left(\int_\Omega u^{\frac{s+1}{q+1} - \frac{qr-(p-1)(s+1)}{2(s+1)}}\right)^{\frac{k_1-1}{k_1-q}}.$$

Setting

$$k = 2 \frac{s+1}{s+1-q},$$

and applying Hölder’s inequality, we have

$$\int_\Omega u^{\frac{k_1}{q}} \leq c \left(\frac{l}{d_1}\right)^{\frac{k_1}{q} - \frac{s}{2}} \left(\int_\Omega u^{\frac{s+1}{q+1} - \frac{qr-(p-1)(s+1)}{2(s+1)}}\right)^{\frac{k_1-1}{k_1-q}},$$

hence

$$\left(\int_\Omega u^{\frac{s+1}{q+1} - \frac{qr-(p-1)(s+1)}{2(s+1)}}\right)^{\frac{s+1}{2} - \frac{s}{q}} \leq c \left(\frac{l}{d_1}\right)^{\frac{n}{q} - \frac{s}{2}}.$$

Next, for any

$$\frac{n}{2} < \beta < \frac{s+1}{q},$$

applying Hölder’s inequality, we have

$$\int_\Omega \left(\frac{u^p}{v^q}\right)^\beta \leq \left(\int_\Omega \frac{u^r}{v^{q+1}}\right)^{\frac{\beta}{s+1}} \left(\int_\Omega \frac{u^{\beta(p-qr)}}{v^{\beta p+1}}\right)^{1-\frac{\beta}{s+1}} \leq c \left(1 + d_1^{-\gamma}\right)$$

where in the last inequality, we used the fact that $u \in L^{\frac{(s+1)}{s+1-q}}$ for any $l$ large, and

$$\frac{\beta(p-qr)}{1-\beta^2} > 0.$$

Now standard elliptic regularity theory yields

$$\|u\|_{W^{2,\beta}(\Omega)} \leq c \left(\frac{1}{d_1}\right) \left\|u - \frac{u^p}{v^q}\right\|_{L^\beta(\Omega)} + \|u\|_{L^2(\Omega)} \leq c \left(1 + d_1^{-\gamma}\right).$$
Since $\beta > \frac{n}{2}$, bound for Hölder norm of $u$ follows from the Sobolev imbedding. □

**Remark 4.3.** When $n \geq 2$ and $\sigma = 0$, W. Ni and I. Takagi [17] obtained Hölder bound for $u$ when

$$\frac{p}{q} = \frac{r}{s + 1}, \quad \frac{r}{p} > \frac{n}{2}$$

which is a special case of our theorem.

Next, we consider the case $\sigma > 0$. As in Theorem 3.16, we can relax the conditions on $p, q, r, s$ with the help of trivial lower bounds.

**Theorem 4.4.** Let $\sigma > 0$ and

$$\frac{p - 1}{r} < \frac{q - \varepsilon}{s + 1} < \min \left\{1, \frac{2}{n}\right\}.$$  

Then we have

$$u \leq c \left(1 + d_1^{\gamma}\right), \quad v \leq c \left(1 + d_1^{-\frac{n}{r} + \gamma}\right),$$

where $c, \gamma$ are positive constants independent of $d_1, d_2$.

**Proof.** We choose $\varepsilon \geq 0$, such that

$$\frac{p - 1}{r} < q - \varepsilon < \min \left\{1, \frac{2}{n}\right\}.$$  

For any $l > 1$, we have the energy estimate

$$(l - 1) \int_{\Omega} d_1 |\nabla u|^2 u^{l-2} + \int_{\Omega} u^l \leq \int_{\Omega} \frac{u^{p+1}}{v^q} + \sigma \int_{\Omega} u^{l-1}. $$

If

$$(l - 1) \int_{\Omega} d_1 |\nabla u|^2 u^{l-2} + \int_{\Omega} u^l \leq 2\sigma \int_{\Omega} u^{l-1} \leq 2\sigma |\Omega|^{\frac{1}{l-1}},$$

then

$$\int_{\Omega} u^l \leq 2\sigma \int_{\Omega} u^{l-1} \leq 2\sigma |\Omega|^{\frac{1}{l-1}} \left(\int_{\Omega} u^l\right)^{\frac{l-1}{l}},$$

hence

$$\int_{\Omega} u^l \leq (2\sigma)^l |\Omega|. $$

Otherwise,

$$(l - 1) \int_{\Omega} d_1 |\nabla u|^2 u^{l-2} + \int_{\Omega} u^l \leq 2 \int_{\Omega} \frac{u^{p+1}}{v^q} \leq 2\sigma \frac{r}{s + 1} \int_{\Omega} \frac{u^{p+1}}{v^q},$$

the arguments leading to (4.2) implies that for any $l \geq \max \left\{2, 4d_1\right\}$,

$$\left(\int_{\Omega} \frac{u^{(s+1)l}}{v^{(q-s+q-\varepsilon)l}}\right)^{\frac{s+1-\varepsilon}{s+1}} \leq c \left(\frac{l}{d_1}\right)^{\frac{n(s-\varepsilon)}{n(s-\varepsilon) + (s-\varepsilon)(q-s+q-\varepsilon)}}.,$$

here $c$ is a constant independent of $d_1, d_2$. Hence, in any case, we have for $l$ large

$$\left(\int_{\Omega} u^l\right)^{\frac{1}{l}} \leq c \left(1 + d_1^{-\gamma}\right)$$

where $c, \gamma$ are positive constants independent of $d_1, d_2$. Next,

$$\frac{u^p}{v^q} \leq \sigma^{-q} u^p,$$
hence $\frac{u^p}{v^q} \in L^\beta$ for any $\beta$. Choosing $\beta > \frac{n}{2}$, elliptic regularity theory yields $u \in W^{2,\beta} \hookrightarrow C^0(\Omega)$, especially, we have $u \leq c \left(1 + d_1^{-\lambda}\right)$. Upper bound for $v$ follows from $2.4$. □

Proof of Theorem 1.5. Part (i) is exactly Theorem 4.1; Part (ii) follows from Theorem 4.4. □

5. Energy Method II

In this section, we will modify the approach in [1] by M. del Pino to establish a priori bounds which are uniform when $d_1, d_2$ are large. The main idea is to use $L^1$ norm to control upper and lower bounds for $u, v$.

We first look at the case when $\sigma = 0$.

Theorem 5.1. Assume that $\sigma = 0$, $r < \frac{n}{n-2}$ and there exists $\delta \in (0, 1]$ such that

$$0 < \frac{1-\delta}{r} + \frac{\delta}{p} < 1,$$

and

$$\frac{(1-\delta)s + \frac{\delta q}{p}}{r-1+\frac{\delta}{p}} < \frac{n}{n-2} \quad \text{or} \quad \frac{(1-\delta)s + \frac{\delta q}{p}}{r-1+\frac{\delta}{p}} \leq s + 1.$$

Then for any $\eta > 0$, there exists positive constant $c = c(p, q, r, s, \eta)$ such that

$$u \geq c, \quad v > c^{\frac{n+1}{s+1}}$$

whenever $d_1, d_2 \geq \eta$.

Proof. Under the condition $0 < \delta \leq 1$, $0 < \frac{1-\delta}{r} + \frac{\delta}{p} < 1$, applying Hölder’s inequality, we have

$$\int_{\Omega} u \leq \left( \int_{\Omega} \frac{u^r}{v^s} \right)^{\frac{1-\delta}{r}} \left( \int_{\Omega} \frac{u^q}{v^r} \right)^{\frac{\delta}{p}} \left( \int_{\Omega} \frac{v^{s+1}}{r} \right)^{\frac{1}{s+1} - \frac{\delta}{p}}.$$

Here if $\delta = 1$, we simply drop the term on $\int_{\Omega} \frac{u^r}{v^s}$. Using Lemmas 2.3 and 2.6, we have

$$\int_{\Omega} \frac{u^r}{v^s} = \int_{\Omega} v \leq c \left( \int_{\Omega} v^{s+1} \right)^{\frac{1}{s+1}} \leq c \left( \int_{\Omega} u^r \right)^{\frac{1}{r}} \leq c \left( \int_{\Omega} u \right)^{\frac{1}{r}}.$$

where we have used the assumption $r < \frac{n}{n-2}$.

Next, if

$$\frac{(1-\delta)s + \frac{\delta q}{p}}{r-1+\frac{\delta}{p}} > \frac{n}{n-2},$$

we can apply Lemma 2.6 and obtain

$$\int_{\Omega} v \frac{(1-\delta)s + \frac{\delta q}{p}}{r-1+\frac{\delta}{p}} \leq c \left( \int_{\Omega} v \right)^{\frac{(1-\delta)s + \frac{\delta q}{p}}{r-1+\frac{\delta}{p}}} \leq c \left( \int_{\Omega} u \right)^{\frac{n+s+1}{n-1}}.$$
and if
\[
\left(\frac{1-\delta}{r}\right) s + \frac{\delta q}{r-1+\delta} \leq s + 1,
\]
we can apply Hölder’s inequality,
\[
\int_{\Omega} v \left(1 - \delta \right) s + \frac{\delta q}{p} - 1 + \delta p \leq s + 1,
\]
we can apply Hölder’s inequality,
\[
\int_{\Omega} v \left(1 - \delta \right) s + \frac{\delta q}{p} - 1 + \delta p \leq s + 1
\]
we can apply Hölder’s inequality,
\[
\int_{\Omega} u \left(1 - \delta \right) s + \frac{\delta q}{p} - 1 + \delta p \leq c \left( \int_{\Omega} u \right)^{\frac{1}{s+1}}
\]
Finally,
\[
\int_{\Omega} u \leq c \left( \int_{\Omega} u \right)^{1 + \frac{s-1}{p(s+1)(qr-(p-1)(s+1))}}
\]
Since
\[
\frac{\delta}{p(s+1)} (qr-(p-1)(s+1)) > 0,
\]
we have
\[
\int_{\Omega} u \geq c.
\]
Now Lemma 2.3 implies,
\[
\min_{x \in \Omega} u \geq c,
\]
and from (2.4),
\[
v \geq \frac{c}{r} \geq c^{\frac{1}{r}}.
\]
\[\square\]

Remark 5.2. The assumptions in theorem 5.1 seem complicated. However, these assumptions automatically hold when the dimension \(n = 2\).

Since \(\delta\) is a free parameter, a family of a priori lower bounds can be deduced from Theorem 5.1.

Corollary 5.3. Assume \(\sigma = 0\), \(r < \frac{n}{n-2}\) and
\[
\frac{q}{p-1} < \frac{n}{n-2} \quad \text{or} \quad \frac{q}{p-1} \leq s + 1.
\]
Given \(\eta > 0\), there exists a positive constant \(c = c(p, q, r, s, \eta)\) such that
\[
u \geq c, \quad v \geq c^{\frac{s+1}{r}}
\]
whenever \(d_1, d_2 \geq \eta\).

Proof. Let \(\delta = 1\), it is straightforward that our conditions are equivalent to the assumptions in Theorem 5.1. \[\square\]

Corollary 5.4. Assume \(\sigma = 0\), \(1 < r < \frac{n}{n-2}\) and
\[
\frac{s}{r-1} < \frac{n}{n-2} \quad \text{or} \quad \frac{s}{r-1} < s + 1.
\]
Given \(\eta > 0\), there exists a positive constant \(c = c(p, q, r, s, \eta)\) such that
\[
u \geq c, \quad v \geq c^{\frac{s+1}{r}}
\]
whenever \(d_1, d_2 \geq \eta\).
whenever $d_1, d_2 \geq \eta$.

**Proof.** Under our assumptions, it is easy to check that the assumptions in Theorem 5.1 will be satisfied for sufficiently small $\delta > 0$. □

**Remark 5.5.** In [1], it was proved that $1 < r < \frac{n}{n-2}$ and $\frac{s}{r-1} < \frac{n}{n-2}$ imply positive lower a priori bounds which is a special case of the above corollary.

Next, we consider the *a priori* upper bounds of $u, v$. We have two different methods to get such bounds which were discussed respectively in Section 3 and Section 4.

**Theorem 5.6.** Let $\sigma = 0,$

$$\frac{p-1}{r} < \min \left(1, \frac{2}{n}\right)$$

and assume that $v \geq c_1$. Then

$$u \leq c_2, \quad v \leq c_2,$$

where $c_2$ is a positive constant depending on $n, p, q, r, s, d_1$ and $c_1$.

**Proof.** See the proof of Theorem 4.4. □

**Proof of Theorem 1.6.** Part (i) follows from 5.1 Part (ii) follows from 5.6 □

When $n = 2$, our theorems imply

**Corollary 5.7.** Let $n = 2$ and $\sigma = 0$. Given $\eta > 0$, there exists a positive constant $c_1$ depending only on $p, q, r, s, \eta$ and $\Omega$, such that

$$u \geq c_1, \quad v \geq c_1,$$

whenever $d_1, d_2 > \eta$. If in addition

$$\frac{p-1}{r} < 1,$$

then there exists a positive constant $c_2$ depending only on $p, q, r, s, \eta$ and $\Omega$, such that

$$u \leq c_2, \quad v \leq c_2,$$

whenever $d_1, d_2 > \eta$.

6. **Nonexistence of nontrivial solutions via energy estimate**

It is generally expected that uniform bounds imply nonexistence of nontrivial solutions when the diffusion constants are sufficiently large. More precisely, we have the following result.

**Theorem 6.1.** Let $D \subset (0, \infty) \times (0, \infty)$ be nonempty. Assuming that we have positive lower and upper a priori bounds for solutions to (2.4) which are uniform for $(d_1, d_2) \in D$. Then there exists a positive constant $C$ such that for any $(d_1, d_2) \in D$ satisfying

$$d_1 \geq C,$$

$(u, v) \equiv (u^*, v^*)$ is the only solution of (2.4).
Proof. Let 
\[ \bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx, \quad \bar{v} = \frac{1}{|\Omega|} \int_{\Omega} v(x) \, dx. \]
First, multiplying the second equation of (2.1) with \( v - \bar{v} \) and integrating over \( \Omega \), we have
\[
d_2 \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |v - \bar{v}|^2 = \int_{\Omega} \left( \frac{u^r}{v^s} - \frac{\bar{u}^r}{\bar{v}^s} \right) (v - \bar{v})
\]
\[
= \int_{\Omega} \left( \frac{u^r}{v^s} - \frac{\bar{u}^r}{\bar{v}^s} \right) (v - \bar{v}) + \int_{\Omega} \left( \frac{u^r}{v^s} - \frac{\bar{u}^r}{\bar{v}^s} \right) (v - \bar{v})
\]
\[
\leq \int_{\Omega} \left( \frac{u^r}{v^s} - \frac{\bar{u}^r}{\bar{v}^s} \right) (v - \bar{v}) \leq C \int_{\Omega} (u - \bar{u}) (v - \bar{v})
\]
\[
\leq \frac{1}{2} \int_{\Omega} |v - \bar{v}|^2 + C \int_{\Omega} |u - \bar{u}|^2
\]
hence
\[
(6.1) \quad \int_{\Omega} |v - \bar{v}|^2 \leq C \int_{\Omega} |u - \bar{u}|^2.
\]
Next, from the first equation of (2.1), we have
\[
d_1 \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |u - \bar{u}|^2 = \int_{\Omega} \left( \frac{u^p}{v^q} - \frac{\bar{u}^p}{\bar{v}^q} \right) (u - \bar{u})
\]
\[
= \int_{\Omega} \left( \frac{u^p}{v^q} - \frac{\bar{u}^p}{\bar{v}^q} \right) (u - \bar{u}) + \left( \frac{\bar{u}^p}{\bar{v}^q} - \frac{\bar{u}^p}{\bar{v}^q} \right) (u - \bar{u})
\]
\[
\leq C \int_{\Omega} (u - \bar{u})^2 + \int_{\Omega} |u - \bar{u}| \cdot |v - \bar{v}|
\]
\[
\leq C \int_{\Omega} (u - \bar{u})^2 + \int_{\Omega} |v - \bar{v}|^2 \leq C \int_{\Omega} (u - \bar{u})^2.
\]
Hence
\[
d_1 \| \nabla u \|_{L^2(\Omega)}^2 \leq C \int_{\Omega} (u - \bar{u})^2 \leq C \| \nabla u \|_{L^2(\Omega)}^2.
\]
If \( d_1 \geq C \), we have
\[
\| \nabla u \|_{L^2(\Omega)} = 0,
\]
and \( u \) is constant. Now (6.1) implies that \( v \) is also constant, and the only constant solution to (2.1) is \( (u, v) = (u^*, v^*) \).

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. Part (i) follows from Theorem 3.1; (ii) follows from Theorems 4.4; (iii) follows from Corollary 5.7. □

7. Existence of nontrivial solutions

In this section, we will use topological degree theory to show the existence of nontrivial solutions to (2.1) under suitable conditions. This approach has been used by many authors, for example, M. del Pino in \[1\].

Let \( X = C^0(\Omega) \times C^0(\Omega) \) be the Banach space with norm
\[
||(u, v)||_X = \max \big\{ \|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)} \big\}
\]
and $X^+$ be the positive cone in $X$, i.e.,
\[
X^+ = \{(u, v) \in X : u > 0, v > 0 \text{ in } \overline{\Omega}\}.
\]
We define solution operators $S = (I - d_2 \triangle)^{-1}$ and $R = (\varpi I - d_1 \triangle)^{-1}$ under Neumann boundary conditions. Here $\varpi > 0$ is a large constant to be determined later. Let
\[
T(u, v) = (R(f(u, v) + \varpi u), S(g(u, v) + v))
\]
where
\[
f(u, v) = -u + \frac{u^p}{v^q} + \sigma g(u, v) = -v + \frac{u^r}{v^s}.
\]
Then $T$ is an operator defined on $X^+$ and it is easy to check that $(u, v)$ is a positive solution to (2.1) if and only if it is a fixed point of $T$ in $X^+$, i.e.,
\[
T(u, v) = (u, v).
\]

Now we consider the linearization of (2.1) around $(u^*, v^*)$,

\[
\begin{aligned}
\left \{ 
\begin{array}{l}
d_1 \triangle h + f_u (u^*, v^*) h + f_v (u^*, v^*) k = 0 \quad \text{in } \Omega, \\
d_2 \triangle k + g_u (u^*, v^*) h + g_v (u^*, v^*) k = 0 \quad \text{in } \Omega, \\
\frac{\partial h}{\partial \nu} = \frac{2k}{2q} = 0 \\
\end{array}
\right. \\
\end{aligned}
\]  

(7.1)

Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of $-\triangle$ under Neumann boundary conditions in $\Omega$. We also use $m_i$ to denote the multiplicity of eigenvalue $\lambda_i$, $i = 0, 1, 2, \cdots$. In the remaining part of the paper, we will simply use $f_u$ to denote $f_u (u^*, v^*)$, the same applies to $f_v, g_u$ and $g_v$. The linear system (7.1) will possess a nontrivial solution if and only if the matrix
\[
\begin{pmatrix}
f_u - d_1 \lambda_i & f_v \\
g_u & g_v - d_2 \lambda_i
\end{pmatrix}
\]
is singular for some $i$. Hence, given $d_2 > 0$, for each $i \geq 1$, the linear system (7.1) will possess a nontrivial solution if and only if
\[
d_1 = d_{1i} = \frac{1}{\lambda_i} \left[ p (u^*)^{p-1-\frac{q}{2r}} - 1 - \frac{q (u^*)^{p-\frac{(q+1)s}{r+1}}}{s + 1 + d_2 \lambda_i} \right].
\]

We also define for any $d > 0$,
\[A_d = \{i \geq 1 : d < d_{1i}\}, \quad N_d = \sum_{i \in A_d} m_i.\]

**Lemma 7.1.** If $d_1 \neq d_{1i}$, $i = 1, 2, 3, \cdots$, then for every sufficiently small neighborhood $V$ of $(u^*, v^*)$, $T$ has no fixed point on $\partial V$ and
\[
\deg (I - T, V, (0, 0)) = (-1)^{N_d},
\]
provided that $\varpi$ is sufficiently large.

**Proof.** Let $L$ be the Fréchet derivative of $T$ at $(u^*, v^*)$. For any $(h, k) \in X$, we have
\[
L(h, k) = (R(f_u h + f_v k + \varpi h), S(g_u h + g_v k + k))
\]
If $I - L$ is nonsingular, then $(u^*, v^*)$ is an isolated fixed point of $T$ and, for every sufficiently small neighborhood $V$ of $(u^*, v^*)$,
\[
\deg (I - T, V, (0, 0)) = (-1)^{N_d}.
\]
where \( \eta \) is the number of negative eigenvalues counting algebraic multiplicities of \( I - L \). So we only need to show \( I - L \) is nonsingular and \( \eta = N_{d_i} \). Observe that

\[
-\mu \leq 0 \quad \text{is an eigenvalue if and only if the system}
\]

\[
\begin{cases}
- (\mu + 1) d_1 \triangle h + \mu \partial h = f_u h + f_v k, & \text{in } \Omega, \\
- (\mu + 1) d_2 \triangle k + \mu k = g_u h + g_v k, & \text{in } \Omega, \\
\frac{\partial h}{\partial \nu} = \frac{\partial k}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]

(7.2)

has nontrivial solutions, which is equivalent to that the matrix

\[
\begin{pmatrix}
f_u - \omega \mu - (\mu + 1) d_1 \lambda_i & f_v \\
g_u - \mu - (\mu + 1) d_2 \lambda_i & g_v - \mu - (\mu + 1) d_2 \lambda_i
\end{pmatrix}
\]

\[
= \begin{pmatrix}
 p (u^*)^{p-1} \frac{\omega}{\lambda_i} - 1 - \omega \mu - (\mu + 1) d_1 \lambda_i & -q (u^*)^{p-1} \frac{r(q+1)}{s_i^2} - 1 + d_1 \lambda_i \\
q r (u^*)^{p-1} \frac{1}{s_i^2} - 1 - d_1 \lambda_i & -s - 1 - \mu - (\mu + 1) d_2 \lambda_i
\end{pmatrix}
\]

is singular for some \( i \geq 0 \), i.e.,

\[
(\omega + d_1 \lambda_i) (1 + d_2 \lambda_i) \mu^2 + \left[ (s + 1 + d_2 \lambda_i) \omega + d_2 \lambda_i \right] \left( p (u^*)^{p-1} \frac{\omega}{\lambda_i} + 1 + d_1 \lambda_i \right) \mu
\]

(7.3)

\[
+ q r (u^*)^{p-1} \frac{1}{s_i^2} - \left[ p (u^*)^{p-1} \frac{1}{s_i^2} - 1 - d_1 \lambda_i \right] \left[ s + 1 + d_2 \lambda_i \right] = 0.
\]

If we choose \( \omega \) sufficiently large, the left hand side of (7.3) is monotone increasing in \( \mu \geq 0 \). When \( i = 0 \), we have \( \lambda_i = 0 \), using \( u^* \geq 1 \) and (1.1), it is easy to check

\[
q r (u^*)^{p-1} \frac{1}{s_i^2} - \left( p (u^*)^{p-1} \frac{1}{s_i^2} - 1 \right) (s + 1) > 0,
\]

hence the matrix can’t be singular for any \( \mu \geq 0 \). Now for any \( i \geq 1 \), we can solve \( d_1 \) from (7.3).

\[
d_1 = \frac{p (u^*)^{p-1} \frac{\omega}{\lambda_i} - 1 - \omega \mu - q r (u^*)^{p-1} \frac{1}{s_i^2} - 1}{(\mu + 1) \lambda_i (s + 1 + \mu + (\mu + 1) d_2 \lambda_i)} \equiv p_i (\mu).
\]

When \( \omega \) is sufficiently large, one can check that \( p_i (\mu) \) is monotone decreasing for any \( \mu \geq 0 \), and

\[
p_i (0) = d_{11}, \quad \lim_{\mu \to \infty} p_i (\mu) = -\frac{\omega}{\lambda_i}.
\]

Since \( d_1 \neq d_{11} \), we conclude that \( \mu = 0 \) is not an eigenvalue to (7.2), hence \( I - L \) is nonsingular. If \( d_1 < d_{11} \), then there exists a unique \( \mu > 0 \) such that

\[
d_1 = p_i (\mu).
\]

And each eigenfunction of \( \lambda_i \) gives rise to an eigenfunction of (7.2). Hence, we have \( \eta = N_{d_i} \).

Next, we consider a one-parameter family of elliptic systems

\[
\begin{cases}
d_1 \triangle u - u + \tau \left( \frac{u^p}{\lambda_i^p} + \sigma \right) + (1 - \tau) \rho = 0 & \text{in } \Omega, \\
d_2 \triangle v - v + \tau \frac{u^p}{\lambda_i^p} + (1 - \chi_\tau) \rho = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]

(7.4)

with parameter \( \tau \in [0, 1] \). (We have abused the notation here since the parameter \( \tau \) has nothing to do with the response rate in (1.1).) In (7.4), \( \rho \) is a given positive constant and

\[
\chi_\tau = \begin{cases} 
2 \tau & \text{if } \tau \in [0, \frac{1}{2}], \\
1 & \text{if } \tau \in \left[ \frac{1}{2}, 1 \right].
\end{cases}
\]
When $\tau$ changes from 0 to 1, (7.4) serves as a deformation from a trivial system which has a unique solution $(u, v) \equiv (\rho, \rho)$ to (2.1). Our choice of $\chi_\tau$ simplifies the proof of a priori estimates in Proposition 7.7.

For $\eta > 0$, we denote
$$\Lambda_\eta = \left\{ (u, v) \in X : \eta < u, v < \frac{1}{\eta} \text{ in } \Omega \right\}.$$  

**Lemma 7.2.** Assume that positive solutions to (7.4) satisfies a priori bound
(7.5)  
$$0 < \alpha \leq u, v \leq \beta$$
for some positive constants $\alpha, \beta$ independent of $\tau$. Then there exists $\eta > 0$ such that $T$ has no fixed point on $\partial \Lambda_\eta$ and
$$\deg (I - T, \Lambda_\eta, (0, 0)) = 1.$$

**Proof.** Let
$$T_\tau (u, v) = (R_\tau (\tau f (u, v) + (1 - \tau) \rho + \tau \varphi u), S (\tau g (u, v) + v) + (1 - \chi_\tau) \rho))$$
where
$$R_\tau = ((\tau \varphi + 1 - \tau) I - d_1 \Delta)^{-1}.$$  

Then for each $\tau \in [0, 1]$, $T_\tau$ is a compact operator from $\Lambda_\eta$ into $X$. Furthermore, $T_1 = T$ and $T_0 \equiv (\rho, \rho)$. It is easy to check that $(u, v)$ is a fixed point of $T_\tau$ if and only if it is a solution to (7.4). The bounds in (7.5) then implies that for $\eta$ sufficiently small, for each $\tau \in [0, 1]$, $T_\tau$ has no fixed point on $\partial \Lambda_\eta$. Hence,
$$\deg (I - T_1, \Lambda_\eta, (0, 0)) = \deg (I - T_0, \Lambda_\eta, (0, 0)),$$
i.e.,
$$\deg (I - T, \Lambda_\eta, (0, 0)) = \deg (I - (\rho, \rho), \Lambda_\eta, (0, 0)) = \deg (I, \Lambda_\eta, (\rho, \rho)) = 1$$since $(\rho, \rho) \in \Lambda_\eta$. \hfill \Box

**Theorem 7.3.** Under the assumption of Lemma 7.2. If $d_1 \neq d_{i_1}, i = 1, 2, 3, \ldots$, and $N_{d_1}$ is odd, then there exists at least one nontrivial positive solution to (2.1).

**Proof.** From the properties of topological degree, we have
$$\deg (I - T, \Lambda_\eta \setminus \nabla, (0, 0)) = 1 - (-1)^{N_{d_1}} = 2 \neq 0,$$hence $T$ has at least one fixed point in $\Lambda_\eta \setminus \nabla$, which is a nontrivial solution to (2.1). \hfill \Box

**Remark 7.4.** One necessary condition to apply Theorem 7.3 is that $\sigma$ is small. Actually, if
(7.6)  
$$\sigma \geq (p - 1) p^{\frac{s + 1}{\sigma - (p - 1)(s + 1)}}^{-1},$$
we would have
$$p (u^*)^{p - 1 - \frac{s + 1}{\sigma - (p - 1)(s + 1)}} \leq 1,$$hence $d_{i_1} < 0$, $i = 1, 2, 3, \ldots$. So we have for any $d_1 > 0$, $A_{d_1} = \emptyset$ and $N_{d_1} = 0$.

Finally, we give some sufficient conditions for the existence of a priori bounds uniform in $\tau$ of (7.4).

**Lemma 7.5.** Let $(u, v)$ be a positive solution to (7.4). Then we have
(i) \[ u \geq c \int_{\Omega} u, \quad v \geq c \int_{\Omega} v \]

where \( c > 0 \) is a constant independent of \( \tau \).

(ii) For any \( 0 < \gamma < \frac{n}{n-2} \),

\[ \int_{\Omega} u^\gamma \leq c \left( \int_{\Omega} u \right)^\gamma, \quad \int_{\Omega} v^\gamma \leq c \left( \int_{\Omega} v \right)^\gamma \]

where \( c > 0 \) is a constant independent of \( \tau \).

(iii) \[ \tau \int_{\Omega} \frac{u^p}{v^q} \leq \int_{\Omega} u, \quad \tau \int_{\Omega} \frac{u^r}{v^s} \leq \int_{\Omega} v, \quad \tau \int_{\Omega} \frac{u^r}{v^{s+1}} \leq |\Omega| \]

and if \( \tau \in \left[ \frac{1}{2}, 1 \right] \),

\[ \int_{\Omega} v^{s+1} \leq 2 \int_{\Omega} u^r. \]

Proof. We refer the readers to the proofs of Lemmas 2.8, 2.6 and 2.5.

When \( \sigma > 0 \), a priori lower bounds uniformly in \( \tau \) can be obtained using maximum principle.

**Proposition 7.6.** Let \( \sigma > 0 \) and \((u, v)\) be a positive solution to (7.4), then

\[ u \geq c_1, \quad v \geq c_2 \]

where \( c_1, c_2 \) are positive constants depending only on \( \sigma \) and \( \rho \).

Proof. Let \( x^* \in \overline{\Omega} \) be a point such that

\[ u(x^*) = \inf_{x \in \overline{\Omega}} u(x). \]

Then we have at \( x^* \), \( \Delta u \geq 0 \), hence

\[ u(x^*) \geq \tau \left( \frac{u^p}{v^q} + \sigma \right) + (1-\tau) \rho \geq \tau \sigma + (1-\tau) \rho \geq \min \{ \sigma, \rho \}. \]

Next, let \( x^{**} \in \overline{\Omega} \) be a point such that

\[ v(x^{**}) = \inf_{x \in \overline{\Omega}} v(x). \]

Then we have at \( x^{**} \), \( \Delta v \geq 0 \), hence

\[ v(x^{**}) \geq \tau \frac{u^r}{v^s} + (1-\chi_{\tau}) \rho. \]

If \( \tau \in \left[ 0, \frac{1}{4} \right] \), then we have

\[ v(x^{**}) \geq (1-\chi_{\tau}) \rho \geq \frac{1}{4} \rho. \]

And if \( \tau \in \left[ \frac{1}{4}, 1 \right] \), we have

\[ v(x^{**}) \geq \tau \frac{u^r}{v^s} \geq \frac{1}{4} \frac{u^r}{v^s}, \]

hence

\[ v(x^{**}) \geq \left( \frac{1}{4} u^r \right)^{\frac{1}{r}} \geq \left( \frac{1}{4} \left( \min \{ \sigma, \rho \} \right)^r \right)^{\frac{1}{r}}. \]

□
When \( \sigma = 0 \), we will use the energy method in Section 5 to obtain lower bounds.

**Proposition 7.7.** Assume \( \sigma = 0 \), \( r < \frac{n-2}{n} \) and there exists \( \delta \in (0, 1] \) such that
\[
0 < \frac{1 - \delta}{r} + \frac{\delta}{p} < 1,
\]
and
\[
\frac{(1-\delta)s + \delta q}{r} < \frac{n}{n-2} \quad \text{or} \quad \frac{(1-\delta)s + \delta q}{r} < s + 1.
\]
Then (7.4) has a priori lower bounds uniform in \( \tau \).

**Proof.** First we assume \( \tau \in \left[ \frac{1}{2}, 1 \right] \) and we will closely follow the proof of Lemma 5.1. Applying Hölder inequality, we have
\[
\int_\Omega u \leq \left( \int_\Omega u^{r} \right)^{\frac{1-s}{s}} \left( \int_\Omega u^{p} v^{q} \right)^{\frac{s}{p}} \left( \int_\Omega v^{(1-\delta)s + \delta q} \right)^{\frac{1}{1+s}} \leq c
\]
where we have used part (ii) and part (iii) of Lemma 7.5. Hence, we deduce
\[
\int_\Omega u \geq c.
\]
Applying part (i) of Lemma 7.6 we have
\[
u \geq \frac{1}{2} \rho
\]
where \( \rho \) is a positive constant independent of \( \tau \in \left[ \frac{1}{2}, 1 \right] \). Applying maximum principle to the equation for \( v \), we have
\[
v \geq \left( \tau u^{r} \right)^{\frac{1}{r}} \geq \left( \frac{1}{2} \rho \right)^{\frac{1}{r}}.
\]
Next, we assume \( \tau \in [0, \frac{1}{2}] \). Similar to the proof of Proposition 7.6 maximum principle implies
\[
u \geq \frac{1}{2} \rho
\]
and
\[
u \geq \min \left\{ \frac{1}{2} \rho, \left( \frac{1}{4} \left( \frac{1}{2} \rho \right)^{r} \right)^{\frac{1}{r}} \right\}.
\]

Once we have a priori lower bounds, upper bounds can be obtained using the method in Section 3.

**Proposition 7.8.** Assume that we have positive lower bounds for (7.4) which is uniform in \( \tau \). Assume in addition that \( \frac{p-1}{r-1} < 1 \). Then (7.4) has a priori upper bounds uniform in \( \tau \).
Proof. Let $0 < \lambda < \min \left\{ 1, \frac{d_2}{2d_1} \right\}$. At any point $x^* \in \Omega$ where

$$\frac{u}{v^\lambda}(x^*) = \max_{x \in \Omega} \frac{u}{v^\lambda},$$

following the proof of Lemma 3.1, we have

$$1 - \tau \frac{u^{p-1}}{v^q} - (\tau \sigma + (1 - \tau) \rho) u^{-1} - \frac{\lambda d_1}{d_2} \left( 1 - \tau \frac{u^r}{v^{s+1}} - (1 - \chi \tau) \rho v^{-1} \right) \leq 0.$$  

We rewrite the inequality into the form of

$$\left[ \left( 1 - \frac{\lambda d_1}{d_2} - \frac{\tau}{2} \right) - (\tau \sigma + (1 - \tau) \rho) u^{-1} \right] + \tau \left[ \frac{1}{2} + \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} - \frac{u^{p-1}}{v^q} \right]$$

$$\leq - \frac{\lambda d_1}{d_2} (1 - \chi \tau) \rho v^{-1} \leq 0.$$  

If

$$\left( 1 - \frac{\lambda d_1}{d_2} - \frac{\tau}{2} \right) - (\tau \sigma + (1 - \tau) \rho) u^{-1} \leq 0,$$

then we have

$$u(x^*) \leq \frac{\tau \sigma + (1 - \tau) \rho}{1 - \frac{\lambda d_1}{d_2} - \frac{\tau}{2}} \leq \frac{\sigma + \rho}{2 - \frac{\lambda d_1}{d_2}},$$

hence, using the positive lower bound for $v$,

$$\frac{u}{v^\lambda}(x^*) \leq \frac{u(x^*)}{v^\lambda} \leq c$$

where $c$ is a positive constant independent of $\tau$.

Otherwise, we have

$$\tau \left[ \frac{1}{2} + \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} - \frac{u^{p-1}}{v^q} \right] < 0.$$  

Since $\frac{p-1}{r} < 1$ and $\frac{p-1}{r}$ holds, we can choose $\varepsilon > 0$ such that

$$\frac{p-1}{r} < \frac{q-\varepsilon}{s+1} < 1,$$

and we further assume

$$\lambda < \frac{s + 1 - (q - \varepsilon)}{r - (p - 1)}.$$  

Let

$$a_\varepsilon = \frac{(q - \varepsilon) - \lambda (p - 1)}{s + 1 - \lambda r},$$

then it is easy to verify $a_\varepsilon \in (0, 1)$. Using Young’s inequality,

$$\frac{1}{2} + \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} \leq \frac{u^{p-1}}{v^q} \leq \frac{u^{p-1}}{v^q} \leq u^{-\alpha_\varepsilon} \left( \frac{u}{v^\lambda} \right)^{a_\varepsilon} \left( \frac{u}{v^\lambda} \right)^{1-a_\varepsilon}$$

$$\leq \frac{\lambda d_1}{d_2} \frac{u^r}{v^{s+1}} + c \left( \frac{u}{v^\lambda} \right)^{-\frac{(q-\varepsilon)(r-(p-1)(s+1))}{r-(q-\varepsilon)(p-1)}},$$

hence we again have

$$\frac{u}{v^\lambda}(x^*) \leq c.$$
where $c$ is a positive constant independent of $\tau$. Finally, let $x^{**} \in \Omega$ be such that

$$u(x^{**}) = \max_{x \in \Omega} u.$$  

From maximum principle, we have at $x^{**}$,

$$u \leq \tau \left( \frac{u^p}{v^q} + \sigma \right) + (1 - \tau) \rho \leq \frac{u^p}{v^q} + \sigma + \rho \leq c p v^p \lambda - q + \sigma + \rho.$$  

If we choose $\lambda$ so that $\lambda < \frac{q}{p}$, we have

$$\max_{x \in \Omega} u = u(x^{**}) \leq c p v^p \lambda - q + \sigma + \rho$$

which is a bound independent of $\tau$. Finally, at the point where $v$ achieves its maximum, we have

$$v \leq \tau \frac{u^r}{v^s} + (1 - \chi \tau) \rho \leq \frac{u^r}{v^s} + \rho.$$

\[ \blacksquare \]

Combining a priori estimates in Propositions 7.6, 7.7 and 7.8 with Theorem 7.3, we have

**Theorem 7.9.** Assume that $\sigma > 0$ or the assumptions in Proposition 7.7 hold. Assume in addition that $\frac{p-1}{r} < 1$. If $d_i \neq d_{i1}$, $i = 1, 2, 3, \cdots$, and $N_{d_i}$ is odd, then there exists at least one nontrivial solution to (2.1).

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