DIXMIER GROUPS AND BOREL SUBGROUPS

YURI BEREST, ALIMJON ESHMATOV, AND FARKHOD ESHMATOV

Abstract. In this paper, we study a family \( \{G_n\}_{n \geq 0} \) of infinite-dimensional (ind-)algebraic groups associated with algebras Morita equivalent to the Weyl algebra \( A_1(\mathbb{C}) \). We give a geometric presentation of these groups in terms of amalgamated products, generalizing classical theorems of Dixmier and Makar-Limanov. Our main result is a classification of Borel subgroups of \( G_n \) for all \( n \). We show that the conjugacy classes of non-abelian Borel subgroups of \( G_n \) are in bijection with the partitions of \( n \). Furthermore, we prove an infinite-dimensional analogue of the classical theorem of Steinberg [St] that characterizes Borel subgroups in purely group-theoretic terms. Combined together the last two results imply that the \( G_n \) are pairwise non-isomorphic as abstract groups. This settles an old question of Stafford [S].

1. Introduction and statement of results

It is well known (see, e.g., [Sp], [SV], [GP]) that many interesting linear algebraic groups arise as the automorphism groups of finite-dimensional simple algebras: a prototypical example is the family of matrix algebras \( M_n(\mathbb{k}) \), the corresponding groups are \( \text{PGL}_n(\mathbb{k}) \).

In this paper, we will study a class of infinite-dimensional examples related to algebras Morita equivalent to the first complex Weyl algebra \( A_1 := \mathbb{C}(x,y)/(xy - yx - 1) \). Recall that \( A_1 \) is a simple associative \( \mathbb{C} \)-algebra isomorphic to the ring of differential operators on the affine line \( \mathbb{C}^1 \). The algebras Morita equivalent to \( A_1 \) can be divided into two classes: the matrix algebras over \( A_1 \) and the rings \( D(X) \) of differential operators on rational singular curves \( X \) with normalization \( \bar{X} \cong \mathbb{C}^1 \) (see [SS]). The matrix algebras \( M_r(A_1) \) are classified, up to isomorphism, by the integer \( r \geq 1 \) (the dimension of matrices). A remarkable and much less obvious fact\(^1\) is that the algebras \( D(X) \) are also classified, up to isomorphism, by a single non-negative integer \( n = n(X) \), which is called the differential genus of \( X \) (see [BW2]). In the present paper, we will study the automorphism groups of \( D(X) \). For each \( n \geq 0 \), we fix a curve \( X_n \) of differential genus \( n \), with \( X_0 := \mathbb{C}^1 \), and let \( D_n := D(X_n) \) denote the algebra of differential operators on \( X_n \). We define \( G_n \) to be the automorphism group of \( D_n \). The group \( G_0 \) is thus the automorphism group of the Weyl algebra \( A_1 \) originally studied by J. Dixmier [D]. We therefore call \( \{G_n\} \) the Dixmier groups.

The structure of \( G_0 \) as a discrete group is well-known and well-understood: in particular, thanks to [D], we know that \( G_0 \) is generated by two abelian subgroups \( G_x \) and \( G_y \) consisting of transformations \( (x,y) \mapsto (x,y+p(x)) \) and \( (x,y) \mapsto (x+q(y),y) \) with \( p(x) \in \mathbb{C}[x] \) and \( q(y) \in \mathbb{C}[y] \). Thanks to the work of L. Makar-Limanov [ML2], we also know that \( G_0 \) can be described as the amalgamated free product \( A \ast_U B \) of two elementary subgroups (see Section 2.2). Motivated by these results, J. T. Stafford [S] asked about the structure of the automorphism groups \( \text{Aut}_C D(X) \), where \( X \) is a singular curve; more precisely, he raised the following two questions (cf. [S], p. 636):

1. Does there exist a description of \( G_n \) for \( n > 0 \) similar to that of \( G_0 \)?
2. Is it actually true that \( G_n \not\cong G_0 \) when \( n \neq 0 \)?

Part of the problem is that, for singular \( X \), the rings \( D(X) \) have a fairly complicated structure; unlike \( A_1 \), they do not have ‘canonical’ generators satisfying some simple relations, and it is not immediately clear how to describe them in an elementary way.

In this paper, we answer affirmatively both questions of [S]. For (1), we give a presentation for \( G_n \) in terms of (generalized) amalgamated products using the Bass-Serre theory of groups acting

\(^{1}\)This fact was first established in [Ko] following an earlier work of G. Letzter and L. Makar-Limanov (see [LM, Le]). It was rediscovered independently by G. Wilson and the first author in [BW1]. A conceptual proof and explanations can be found in the survey paper [BW2].
on graphs (see Section 3). As for question (2), we actually solve the isomorphism problem for the family \( \{ G_n \} \) proving that \( G_n \cong G_m \) if and only if \( n = m \) (cf. Corollary 1). In a nutshell, the idea of our proof is the following: we put on each \( G_n \) an algebraic structure making it an ind-algebraic group; then, we construct natural invariants of \( G_n \) that distinguish these groups up to isomorphism as ind-algebraic groups; finally, we prove that our invariants are independent of the algebraic structure.

We now proceed with a detailed discussion of results of the paper. A theorem of Makar-Limanov [ML2] asserts that \( G_0 \) is isomorphic to the group \( G \) of symplectic (unimodular) automorphisms of the free associative algebra \( R = \mathbb{C}(x,y) \), the isomorphism \( G \cong G_0 \) being induced by the natural projection \( R \to A_1 \). We will use this isomorphism to identify \( G_0 \) with \( G \); the groups \( G_n \) for \( n \geq 1 \) can then be identified in a natural way with subgroups of \( G \). The embeddings \( G_n \hookrightarrow G \) arise from a geometric interpretation of \( G_n \) in terms of the Calogero-Moser spaces \( \mathcal{C}_n \), which will play a key role in the present paper. Recall that \( \mathcal{C}_n \) \((n \geq 1)\) is an algebraic variety parametrizing the conjugacy classes of pairs of \( n \times n \) matrices \( (X,Y) \) such that \( [X,Y]+I \) has rank 1, i.e.

\[
\mathcal{C}_n := \{ (X,Y) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) : \text{rk}([X,Y]+I) = 1 \}/\text{PGL}_n(\mathbb{C}).
\]

Named after a class of integrable systems in mechanics (see [KKS]) these varieties play an important role in several areas, especially in geometry and representation theory (see, e.g., [N], [EG], [Go]). They were introduced and studied in [W], where it was shown (among other things) that \( \mathcal{C}_n \) is smooth affine irreducible complex variety of dimension \( 2n \). Furthermore, in [BW], it was shown that each \( \mathcal{C}_n \) carries a transitive \( G \)-action, which is obtained, roughly speaking, by thinking of \( \mathcal{C}_n \) as a subvariety of \( n \)-dimensional representations of \( R \) (see Section 2 for a precise definition). It turns out that \( G_n \) is isomorphic to the stabilizer of a point under this transitive action (cf. Theorem 8): thus, fixing a basepoint in \( \mathcal{C}_n \) we can identify \( G_n \) with a specific subgroup of \( G \). Our general strategy will be to study \( G_n \) in terms of the action of \( G \) on \( \mathcal{C}_n \).

Next, we recall that the notion of an ind-algebraic group was introduced by I. Shafarevich (see [Sh, Sh1]). The fundamental example is the group \( \text{Aut}(\mathbb{C}^d) \) of polynomial automorphisms of the affine \( d \)-space. This group (sometimes called the \( d \)-dimensional affine Cremona group) has been extensively studied, especially for \( d = 2 \) (see, e.g., [J, vdK, Da, GD, Wr, K, K1, FuL, FuM]). It is known [Cz, ML1] that as a discrete group, \( \text{Aut}(\mathbb{C}^2) \) is isomorphic to the automorphism group of the free algebra \( \mathbb{C}(x,y) \) and hence contains each \( G_n \) as a discrete subgroup. However, the ind-algebraic structure that we put on \( G_n \) is different (i.e., not induced) from \( \text{Aut}(\mathbb{C}^2) \). This ind-algebraic structure was originally proposed by G. Wilson and the first author in [BW], but the details were not worked out in that paper. It is interesting that the ind-algebraic group \( G \) can be defined in a simpler and somewhat more natural way than \( \text{Aut}(\mathbb{C}^2) \) and \( \text{Aut}(A_1) \). The reason for this is the remarkable fact [Di] that (the analogue of) the Jacobian Conjecture is known to be true for \( \mathbb{C}(x,y) \), while it is still open for the polynomial ring \( \mathbb{C}[x,y] \) and the Weyl algebra \( A_1 \).

Solvable subgroups play an important role in the structure theory of linear algebraic groups (see [Bo]) as well as infinite-dimensional Kac-Moody groups [Ku]. It is natural to expect that they should also play a role in the theory of ind-algebraic groups (cf. [P]). In this paper, we will study the Borel subgroups of \( G_n \); our goal is to describe and classify such subgroups for all \( n \). To begin with, we recall that a Borel subgroup of a topological group is a connected solvable subgroup that is maximal among all connected solvable subgroups. The group \( G \) has an obvious candidate: the subgroup \( B \) of triangular\(^2 \) transformations: \((x,y) \mapsto (ax + q(y), a^{-1}y + b)\), where \( q(y) \in \mathbb{C}[y]\), \( a \in \mathbb{C}^* \) and \( b \in \mathbb{C} \). It is not difficult to prove that \( B \) is indeed a Borel subgroup of \( G \); moreover, as in the finite-dimensional case, we have the following theorem.

**Theorem 1.** Any Borel subgroup of \( G \) is conjugate to \( B \).

For \( n > 0 \), the situation is more interesting. Let \( \mathcal{B}_n \) denote the set of all Borel subgroups of \( G_n \) on which \( G_n \) acts by conjugation. We will show that every Borel subgroup of \( G_n \) is conjugate in \( G \) to a subgroup of \( B \); this defines a \( G_n \)-equivariant map \( \iota : \mathcal{B}_n \to B/G \), where \( G_n \) acts on \( B/G \) by right multiplication. It turns out that, at the quotient level, the map \( \iota \) induces a canonical

\(^2\)also known as de Jonquières transformations in the commutative case
injection

(1) \[ \mathcal{B}_n / \text{Ad } G_n \hookrightarrow C_n / B. \]

Thus, the Borel subgroups of \( G_n \) are classified (up to conjugation) by orbits in \( C_n \) of the Borel subgroup of \( G \). In general (precisely, for \( n \geq 2 \)), the map (1) is not surjective — not every \( B \)-orbit in \( C_n \) corresponds to a Borel subgroup of \( G_n \) — however, the image of (1) has a nice geometric description in terms of the \( C^* \)-action on \( C_n \). To be precise, let \( T := \{(ax, a^{-1}y) : a \in \mathbb{C}^*\} \subset B \) denote the group of scaling automorphisms, which is a maximal torus in \( G \). We will prove

**Theorem 2.** A \( B \)-orbit \( O \) in \( C_n \) corresponds to a conjugacy class of Borel subgroups in \( G_n \) if and only if one of the following conditions holds:

(A) \( T \) acts freely on \( O \).

(B) \( T \) has a fixed point in \( O \).

The orbits of type (A) correspond precisely to the abelian Borel subgroups of \( G_n \), while the orbits of type (B) correspond to the non-abelian ones.

Each of the two possibilities of Theorem 2 actually occurs: the orbits of type (A) exist in \( C_n \) for \( n \geq 3 \), while the orbits of type (B) exist for all \( n \). Thus, in general, \( G_n \) has both abelian and non-abelian Borel subgroups. While the existence of abelian Borel subgroups seems mysterious to us, we have a fairly good understanding of the non-abelian ones. It is known (see [W]) that the \( T \)-fixed points in \( C_n \) are represented by nilpotent matrices \((X,Y)\) and the latter are classified by the partitions of \( n \). We will show that the \( T \)-fixed points actually belong to distinct \( B \)-orbits, which are closed in \( C_n \). Thus Theorem 2 implies

**Theorem 3.** The conjugacy classes of non-abelian Borel subgroups of \( G_n \) are in bijection with the partitions of \( n \). In particular, for each \( n \geq 0 \), there are exactly \( p(n) \) conjugacy classes of non-abelian Borel subgroups in \( G_n \).

The last result that we want to state in the Introduction provides an abstract group-theoretic characterization of non-abelian Borel subgroups of \( G_n \).

**Theorem 4.** A non-abelian subgroup \( H \) of \( G_n \) is Borel if and only if

(B1) \( H \) is a maximal solvable subgroup of \( G \),

(B2) \( H \) contains no proper subgroups of finite index.

Theorem 4 is an infinite-dimensional generalization of a classical theorem of R. Steinberg [St] that characterizes (precisely by properties (B1) and (B2)) the Borel subgroups in reductive affine algebraic groups. However, unlike in the finite-dimensional case, Steinberg’s characterization does not seem to extend to all Borel subgroups of \( G_n \) (in fact, even for \( n = 0 \), there exist abelian subgroups that satisfy (B1) and (B2) but are countable and hence totally disconnected in \( G \)).

Theorem 3 and Theorem 4 combined together imply the following important

**Corollary 1.** The groups \( G_n \) are pairwise non-isomorphic (as abstract groups).

In fact, the groups \( G_n \) are distinguished from each other by the sets of conjugacy classes of their non-abelian Borel subgroups: by Theorem 3, these sets are finite and distinct, while by Theorem 4, they are independent of the algebraic structure.

Although the Borel subgroups of \( G_n \) have geometric origin and their classification is given in geometric terms, our proofs of Theorem 2 and Theorem 4 are not entirely geometric nor algebraic. The crucial ingredient is Friedland-Milnor’s classification of polynomial automorphisms of \( \mathbb{C}^2 \) according to their dynamical properties (see [FM]). This classification was refined by Lamy [L] who extended it to a classification of subgroups of \( \text{Aut}(\mathbb{C}^2) \). We will identify \( G \) as a discrete group with \( \text{Aut}(\mathbb{C}^2) \) and use Lamy’s classification as a main tool to study the subgroups of \( G \).

In the end, we mention that the original goal of the present paper was to prove the result of Corollary 1. Our interest in this result is motivated by the following generalization of the Dixmier Conjecture (for \( A_1 \)) proposed in [BEE].
Conjecture. For all \( n, m \geq 0 \),
\[
\text{Hom}(D_n, D_m) = \begin{cases} 
\emptyset & \text{if } n \neq m \\
G_n & \text{if } n = m 
\end{cases}
\]
where ‘Hom’ is taken in the category of unital associative \( \mathbb{C} \)-algebras.

Corollary 1 implies that the endomorphism monoids \( \text{Hom}(D_n, D_n) \) are pairwise non-isomorphic for different \( n \). Still, we do not know whether the above conjecture is actually stronger than the original Dixmier Conjecture which is formally the special case of (2) corresponding to \( n = m = 0 \).

The paper is organized as follows. In Section 2, we introduce notation, review basic facts about the Calogero-Moser spaces, the Weyl algebra and automorphism groups. This section contains no new results (except, possibly, for the proof of Theorem 8, which has not appeared in the literature).

In Section 3, we describe the structure of \( G_n \) as a discrete group, using the Bass-Serre theory of groups acting on graphs. The main result of this section (Theorem 9) gives an explicit presentation of \( G_n \) in terms of generalized amalgamated products. This result can be viewed as a generalization of the classical theorem of Jung and van der Kulk on the amalgamated structure of \( G \).

In Section 4, we study \( G_n \) as ind-algebraic groups. After a brief review of ind-varieties and ind-groups in Section 4.1, we define the structure of an ind-group on \( G \) in Section 4.2 and on \( G_n \) (for \( n \geq 1 \)) in Section 4.3. We show that \( G \) is connected (Theorem 10) and acts algebraically on \( C_n \) (Theorem 12). The connectedness of \( G_n \) for \( n > 0 \) is a more subtle issue: we prove that \( G_n \) is connected for \( n = 1 \) and \( n = 2 \) (Proposition 3) but leave it as a conjecture in general.

The main results of the paper are proved in Section 5. Specifically, Theorem 1 and Theorem 4 (for \( n = 0 \)) are proved in Section 5.3, where we study the Borel subgroups of \( G \). Theorem 2 is proved in Section 5.4, while Theorems 3 and 4 (for \( n \geq 1 \)) in Section 5.5. Finally, in Section 5.6, we give a geometric construction of Borel subgroups in terms of singular curves and Wilson’s adelic Grassmannian (see Proposition 8 and Corollary 6). We also give a complete list of representatives of the conjugacy classes of non-abelian Borel subgroups of \( G_n \) for \( n = 1, 2, 3, 4 \).

Acknowledgments. We thank P. Etingof, V. L. Popov, J. T. Stafford and G. Wilson for interesting suggestions, questions and comments. We also thank S. Lamy and J. P. Furter for answering our questions and guiding through the literature on polynomial automorphisms. Yu. B. is grateful to Forschungsinstitut für Mathematik (ETH, Zürich) and MSRI (Berkeley) for their hospitality and support during the period when this work was carried out.

F. E. is grateful to the Max-Planck-Institut für Mathematik (Bonn), IHÉS (Bures-sur-Yvette) and the Mathematics Department of Indiana University (Bloomington). This work was partially supported by NSF grant DMS 09-01570.

2. Preliminaries

In this section, we fix notation and review basic facts from the literature needed for this paper.

2.1. The Calogero-Moser spaces. For an integer \( n \geq 1 \), let \( \mathcal{M}_n(\mathbb{C}) \) denote the space of complex \( n \times n \) matrices. Let \( \tilde{C}_n \subseteq \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \) be the subvariety of pairs of matrices \((X, Y)\) satisfying the equation
\[
\text{rank } ([X, Y] + I_n) = 1 ,
\]
where \( I_n \) is the identity matrix in \( \mathcal{M}_n(\mathbb{C}) \). It is easy to see that \( \tilde{C}_n \) is stable under the diagonal action of \( \text{GL}_n(\mathbb{C}) \) on \( \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \) by conjugation of matrices, and the induced action of \( \text{PGL}_n(\mathbb{C}) \) on \( \tilde{C}_n \) is free. Following [W], we define the \( n \)-th Calogero-Moser space to be the quotient variety \( C_n := \tilde{C}_n/\text{PGL}_n(\mathbb{C}) \). It is shown in [W] that \( C_n \) is a smooth irreducible affine variety of dimension \( 2n \).

It is convenient to make sense of \( C_n \) for \( n = 0 \): as in [W], we simply assume that \( C_0 \) is a point, and with this convention, we set
\[
C := \bigsqcup_{n \geq 0} C_n .
\]

Abusing notation, we will write \((X, Y)\) for a pair of matrices in \( \tilde{C}_n \) as well as for the corresponding point (conjugacy class) in \( C_n \).
The Calogero-Moser spaces can be obtained by (complex) Hamiltonian reduction (cf. [KKS]): specifically,

\[ C_n \cong \mu^{-1}(I_n)/\text{GL}_n(\mathbb{C}) , \]

where \( \mu : T^*(\mathfrak{gl}_n \times \mathbb{C}^n) \to \mathfrak{gl}_n \), \((X, Y, v, w) \mapsto -[X, Y] + vw\) is the moment map corresponding to the symplectic action of \( \text{GL}_n \) on the cotangent bundle \( T^*(\mathfrak{gl}_n \times \mathbb{C}^n) \). With natural identification \( T^*(\mathfrak{gl}_n \times \mathbb{C}^n) \cong \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \times \mathbb{C}^n \times (\mathbb{C}^n)^* \), this action is given by

\[ (X, Y, v, w) \mapsto (gXg^{-1}, gYg^{-1}, gv, wg^{-1}), \quad g \in \text{GL}_n(\mathbb{C}) . \]

It is easy to see that the orbit of \((X, Y, v, w) \in \mu^{-1}(I_n)\) under (5) is uniquely determined by the conjugacy class of \((X, Y) \in C_n\); whence the isomorphism (4).

The above construction shows that the Calogero-Moser spaces carry a natural symplectic structure. In fact, it is known that each \( C_n \) is a hyperkähler manifold, and the symplectic structure on \( C_n \) is just part of a hyperkähler structure (see [N, Sect. 3.2] and [W]). In this paper, we will not use the hyperkähler structure and will regard \( C_n \) simply as a complex variety.

2.2. The group \( G \) and its action on \( C_n \). Let \( R = \mathbb{C}[x, y] \) be the free associative algebra on two generators \( x \) and \( y \). Denote by \( \text{Aut}(R) \) the automorphism group of \( R \). Every \( \sigma \in \text{Aut}(R) \) is determined by its action on \( x \) and \( y \): we will write \( \sigma \) as \((\sigma(x), \sigma(y))\), where \( \sigma(x) \) and \( \sigma(y) \) are noncommutative polynomials in \( R \) given by the images of \( x \) and \( y \) under \( \sigma \). A fundamental theorem of Czerniakiewics [C2] and Makar-Limanov [ML1] states that \( \text{Aut}(R) \) is generated by the affine automorphisms:

\[ (ax + by + e, cx + dy + f) , \quad a, b, \ldots, f \in \mathbb{C} , \]

and the triangular (Jonquière) automorphisms:

\[ (ax + q(y), by + h) , \quad a, b \in \mathbb{C}^*, \quad h \in \mathbb{C} , \quad q(y) \in \mathbb{C}[y] . \]

This fact is often stated by saying that every automorphism of \( R \) is tame.

In this paper, we will study a certain family \( \{G_0, G_1, G_2, \ldots\} \) of subgroups of \( \text{Aut}(R) \) associated with Calogero-Moser spaces. The first member in this family, which we will often denote simply by \( G \), is the group of symplectic automorphisms of \( R \):

\[ G = G_0 := \{ \sigma \in \text{Aut}(R) : \sigma([x, y]) = [x, y] \} . \]

The structure of this group is described by the following theorem which is a simple consequence of the Czerniakiewics-Makar-Limanov Theorem.

Theorem 5 ([C2], [ML1]). The group \( G \) is the amalgamated free product

\[ G = A *_U B , \]

where \( A \) is the subgroup of symplectic affine transformations:

\[ (ax + by + e, cx + dy + f) , \quad a, b, \ldots, f \in \mathbb{C} , \quad ad - bc = 1 , \]

\( B \) is the subgroup of symplectic triangular transformations:

\[ (ax + q(y), a^{-1}y + h) , \quad a \in \mathbb{C}^*, \quad h \in \mathbb{C} , \quad q(y) \in \mathbb{C}[y] , \]

and \( U \) is the intersection of \( A \) and \( B \) in \( G \):

\[ (ax + by + e, a^{-1}y + h) , \quad a \in \mathbb{C}^*, \quad b, e, h \in \mathbb{C} . \]

Theorem 5 can be deduced from the well-known result of Jung [J] and van der Kulk [vdK] on the structure of the automorphism group of the polynomial algebra \( \mathbb{C}[x, y] \) in two variables. The key observation of [C2] and [ML1] was the following

**Proposition 1.** The natural projection \( R \to \mathbb{C}[x, y] \) induces an isomorphism of groups \( \text{Aut}(R) \cong \text{Aut} \mathbb{C}[x, y] \). Under this isomorphism, \( G \) corresponds to the subgroup \( \text{Aut}_\omega \mathbb{C}[x, y] \) of Poisson automorphisms (i.e. those with Jacobian 1).
Remark. Proposition 1 implies that the natural action of $G$ on $\mathbb{C}^2$ is faithful; this allows one to identify $G$ with a subgroup of $\text{Aut}(\mathbb{C}^2)$ and view the elements of $G$ as polynomial automorphisms of $\mathbb{C}^2$; we will use this identification in Section 5. For a detailed proof of the Jung-van der Kulk Theorem as well as Proposition 1 we refer to [Co] (see, loc. cit., Theorem 6.8.6 and Theorem 6.9.3, respectively). A direct proof of Theorem 5 can be found in [Co1].

Theorem 5 implies that $G$ is generated by the automorphisms

\begin{equation}
(11) \quad \Phi_p := (x, y + p(x)) , \quad \Psi_q := (x + q(y), y) ,
\end{equation}

where $p(x) \in \mathbb{C}[x]$ and $q(y) \in \mathbb{C}[y]$. We denote the corresponding subgroups of $G$ by $G_x := \langle \Phi_p : p \in \mathbb{C}[x] \rangle$ and $G_y := \langle \Psi_q : q \in \mathbb{C}[y] \rangle$. These are precisely the stabilizers of $x$ and $y$ under the natural action of $G$ on $R$.

Next, following [BW], we define an action of $G$ on the Calogero-Moser spaces $\mathcal{C}_n$. First, thinking of pairs of matrices $(X, Y)$ as points dual to the coordinate functions $(x, y) \in R$, we let $G$ act on $\mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C})$ by

\begin{equation}
(12) \quad (X, Y) \mapsto (\sigma^{-1}(X), \sigma^{-1}(Y)) , \quad \sigma \in G .
\end{equation}

Since $G$ preserves commutators, this action restricts to the subvariety $\hat{\mathcal{C}}_n$ of $\mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C})$ defined by (3) and commutes with the conjugation-action by $\text{PGL}_n(\mathbb{C})$. Hence (12) defines an action of $G$ on $\mathcal{C}_n$. Note that, for $n = 1$, the action of $G$ on $\mathcal{C}_1 = \mathbb{C}^2$ agrees with the natural one coming from Proposition 1.

Knowing the structure of the group $G$ (more precisely, the fact that $G$ is generated by the triangular automorphisms (11)), it is easy to see that $G$ acts on $\mathcal{C}_n$ symplectically and algebraically. Much less obvious is the following fact.

**Theorem 6 ([BW]).** For each $n \geq 0$, the action of $G$ on $\mathcal{C}_n$ is transitive.

Theorem 6 plays a crucial role in the present paper. First, we use this theorem to define the groups $G_n$ for $n \geq 1$: we let $G_n$ be the stabilizer of a point in $\mathcal{C}_n$ under the action of $G$. By transitivity, this determines $G_n$ uniquely up to conjugation in $G$. To do computations it will be convenient for us to choose specific representatives in each conjugacy class $[G_n]$; to this end we fix a basepoint $(X_0, Y_0) \in \mathcal{C}_n$ with

\begin{equation}
(13) \quad X_0 = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}, \quad Y_0 = \begin{pmatrix}
0 & 1 - n & 0 & \ldots & 0 \\
0 & 0 & 2 - n & \ldots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & -1 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix},
\end{equation}

and set

\begin{equation}
(14) \quad G_n := \text{Stab}_G(X_0, Y_0) , \quad n \geq 1 .
\end{equation}

We have recently proved (see [BEE1]) that Theorem 6 holds in a stronger form: namely, the action of $G$ on $\mathcal{C}_n$ is doubly transitive. This means that $G$ acts transitively not only on $\mathcal{C}_n$, but also on the configuration space of (ordered) pairs of points on $\mathcal{C}_n$. An important consequence of this result is that the $G_n$ are maximal subgroups of $G$ for all $n \geq 1$ (see op. cit., Corollary 1).

### 2.3. The Calogero-Moser correspondence.

Next, we recall the connection between the Calogero-Moser spaces and the Weyl algebra $A_1(\mathbb{C}) := R/(xy - yx - 1)$ described in [BW]. In his 1968 paper [D], Dixmier proved that the automorphism group of $A_1$ is generated by the same transformations (11) as the group $G$. This result was refined by Makar-Limanov [ML2] who showed that the analogue of Proposition 1 also holds for $A_1(\mathbb{C})$: namely, the natural projection $R \to A_1$ induces an isomorphism of groups

\begin{equation}
(15) \quad G \cong \text{Aut}(A_1) .
\end{equation}

Identifying $G = \text{Aut}(A_1)$ via (15), we will look at the action of $G$ on the space of ideals of $A_1$. To be precise, let $\mathcal{R} = \mathcal{R}(A_1)$ denote the set of isomorphism classes of nonzero right ideals of
A_1. The automorphism group of A_1 acts naturally on the set of all right ideals (one simply treats an ideal as a subspace of A_1), and this action is compatible with isomorphism. Thus, we get an action: \( G \times \mathcal{R} \to \mathcal{R}, (\sigma, [M]) \mapsto [\sigma(M)]. \) The following result is another main ingredient of the present paper.

**Theorem 7 ([BW]).** There is a bijective map \( \omega : \mathcal{C} \to \mathcal{R} \) which is equivariant under the action of \( G. \)

Note that in combination with Theorem 7, Theorem 6 shows that \( \omega(\mathcal{C}_n) \) are precisely the orbits of \( G \) in \( \mathcal{R}. \) The map \( \omega \) can be described explicitly as follows (see [BC]). Recall that a point of \( \mathcal{C}_n \) is represented by a pair of matrices \( (X, Y) \) satisfying the equation (3). Factoring \( [X, Y] + I_n = vw \) with \( v \in \mathbb{C}^n \) and \( w \in (\mathbb{C}^n)^* \), we define the (fractional) right ideal

\[
M(X,Y) = \det(X - x I_n) A_1 + \chi(X,Y) \cdot \det(Y - y I_n) A_1.
\]

where \( \chi(X,Y) := 1 + w (X - x I_n)^{-1}(Y - y I_n)^{-1}v \) is an element of the quotient field of \( A_1. \) Now, the assignment \( (X,Y) \mapsto M(X,Y) \) induces a map from \( \mathcal{C}_n \) to the set of isomorphism classes of ideals of \( A_1; \) amalgamating such maps for all \( n \) yields the required bijection \( \omega : \mathcal{C} \sim \to \mathcal{R}. \) Substituting the matrices (13) in (16), we find that the basepoint \( (X_0, Y_0) \in \mathcal{C}_n \) corresponds to (the class of) the ideal

\[
M(X_0, Y_0) = x^n A_1 + (y + nx^{-1}) A_1.
\]

We will denote the ideal (17) by \( M_n \) and write \( D_n := \text{End}_{A_1}(M_n) \) for its endomorphism ring. Note that \( D_0 = A_1 \).

2.4. Automorphism groups. Theorem 7 allows one to translate algebraic questions about \( A_1 \) and its module category to geometric questions about the Calogero-Moser spaces and the action of \( G \) on these spaces. One important application of this theorem is a classification of algebras (domains) Morita equivalent to \( A_1 \). Briefly, by Morita theory, every such algebra can be identified with the endomorphism ring of a right ideal in \( A_1; \) by a theorem of Stafford (see [S]), two such endomorphism rings are isomorphic (as algebras) iff the classes of the corresponding ideals lie in the same orbit of \( \text{Aut}(A_1) \) in \( \mathcal{R}. \) Now, using Theorem 7, we can identify the orbits of \( \text{Aut}(A_1) \) in \( \mathcal{R} \) with the Calogero-Moser spaces \( \mathcal{C}_n. \) Thus, the domains Morita equivalent to \( A_1 \) are classified (up to isomorphism) by the single integer \( n \geq 0; \) every such domain is isomorphic to the algebra \( D_n \), and moreover \( D_n \not\sim D_m \) for \( n \neq m. \) This classification was originally established in [Ko] by a direct calculation; it has several interpretations and many interesting implications which the reader may find in [BW2]. We conclude this section by recording a proof of the following fact which is mentioned in passing in [BW2].

**Theorem 8.** Let \( [M] \in \mathcal{R} \) be the ideal class corresponding to a point \( (X,Y) \in \mathcal{C}_n \) under the Calogero-Moser map \( \omega. \) Then, there is a natural isomorphism of groups

\[
\text{Stab}_G(X,Y) \sim \text{Aut}[\text{End}_{A_1}(M)] \,.
\]

In particular, for all \( n \geq 0, \)

\[
G_n = \text{Aut}(D_n) \,.
\]

**Proof.** First, we note that \( \text{Aut}[\text{End}_{A_1}(M)] \) can be naturally identified with a subgroup of \( \text{Aut}(A_1). \) To be precise, let \( \text{Pic}(A) \) denote the Picard group of a \( \mathbb{C} \)-algebra \( A. \) Recall that \( \text{Pic}(A) \) is the group of \( \mathbb{C} \)-linear Morita equivalences of the category of \( A \)-modules; its elements are represented by the isomorphism classes of invertible \( A \)-bimodules \( P. \) There is a natural group homomorphism \( \alpha_A : \text{Aut}(A) \to \text{Pic}(A), \) taking \( \tau \in \text{Aut}(A) \) to the class of the bimodule \([1, A_n], \) and if \( D \) is a ring Morita equivalent to \( A, \) with a progenator \( M, \) then there is a group isomorphism \( \beta_D : \text{Pic}(D) \sim \to \text{Pic}(A) \) given by \( [P] \mapsto [M^* \otimes_D P \otimes_D M] \). Now, for \( A := A_1 \) and \( D := \text{End}_{A_1}(M), \)
we have the following diagram

\[
\begin{array}{ccc}
\text{Aut}(D) & \xrightarrow{\alpha_D} & \text{Pic}(D) \\
\downarrow{i_M} & & \downarrow{\beta_M} \\
\text{Aut}(A) & \xrightarrow{\alpha_A} & \text{Pic}(A)
\end{array}
\]

(19)

where $\beta_M$ is an isomorphism and the two horizontal maps are injective. A theorem of Stafford (see [S], Theorem 4.7) implies that $\alpha_A$ is actually an isomorphism. Inverting this isomorphism, we define the embedding $i_M : \text{Aut}(D) \hookrightarrow \text{Aut}(A)$, which makes (19) a commutative diagram.

Now, writing $H := \text{Stab}_G(X, Y)$, we have group homomorphisms

\[ H \hookrightarrow G \xrightarrow{i_M} \text{Aut}(A) \xrightarrow{i_M} \text{Aut}(D), \]

where the first map is the canonical inclusion and the second is the Makar-Limanov isomorphism (15). We claim that the image of $H$ in $\text{Aut}(A)$ coincides with the image of $i_M$; this gives the required isomorphism $H \xrightarrow{i_M} \text{Aut}(D)$. In view of Theorem 7, it suffices to show that

\[ \text{Im}(i_M) = \{ \tau \in \text{Aut}(A) : \tau(M) \cong M \}. \]

First, we prove the inclusion $\text{Im}(i_M) \subseteq \{ \tau \in \text{Aut}(A) : \tau(M) \cong M \}$. Given $\sigma \in \text{Aut}(D)$, $i_M(\sigma)$ is defined to be the (unique) automorphism $\tau \in \text{Aut}(A)$ such that

\[ 1_A \cong M^* \otimes_D (1_D \sigma) \otimes_D M \quad \text{(as $A$-bimodules)} \]

(20)

The right-hand side of (20) can be identified with the subspace $M^* \sigma(M) \subseteq Q$ in the quotient field of $A$, and we denote by $f$ the corresponding isomorphism

\[ 1_A \cong M^* \otimes_D (1_D \sigma) \otimes_D M \xrightarrow{f} M^* \sigma(M). \]

Then, for any $a \in M \subseteq A$, we have $f(a) = f(a, 1) = a f(1)$. On the other hand, $f(a) = f(1, a) = f(1) \sigma^{-1}(a)$. Thus, writing $b = f(1)$, we see that

\[ \tau(M) = b \sigma(M) b^{-1} = b M M^* \sigma(M) b^{-1} = b M. \]

Conversely, suppose that $\tau(M) = b M$ for some $b \in M^*$. Then $\tau(D) = \tau(M M^*) = b M M^* b^{-1} = b D b^{-1}$ in $Q$. If we let $\sigma := \text{Ad}_b \circ \tau \in \text{Aut}(D)$, where $\text{Ad}_b : a \mapsto b^{-1} a b$, then it is easy to see that $\tau = i_M(\sigma)$.

\[ \square \]

Remark. The above proof shows that for $n = 0$ the isomorphism (18) specializes to (15). Theorem 8 can thus be viewed as an extension of the Dixmier-Makar-Limanov theorem about $\text{Aut}(A_1)$ to algebras Morita equivalent to $A_1$.

3. The Structure of $G_n$ as a Discrete Group

In this section, we will use the Bass-Serre theory of graphs of groups to give an explicit presentation of $G_n$. We associate to each space $C_n$ a graph $\Gamma_n$ consisting of orbits of certain subgroups of $G$ and identify $G_n$ with the fundamental group $\pi_1(\Gamma_n, \ast)$ of a graph of groups $\Gamma_n$ defined by the stabilizers of those orbits in $\Gamma_n$. The Bass-Serre theory will then provide an explicit formula for $\pi_1(\Gamma_n, \ast)$ in terms of generalized amalgamated products. The results of this section were announced in [BEE].

3.1. Graphs of groups. To state our results in precise terms we recall the notion of a graph of groups and its fundamental group (see [Se, Chapter I, §5]).

A graph of groups $\Gamma = (\Gamma, H)$ consists of the following data:

1. a connected graph $\Gamma$ with vertex set $V = V(\Gamma)$, edge set $E = E(\Gamma)$ and incidence maps $i, t : E \to V$,
2. a group $H_a$ assigned to each vertex $a \in V$,
3. a group $H_e$ assigned to each edge $e \in E$,
4. injective group homomorphisms $H_{i(e)} \xrightarrow{\alpha_e} H_e \xrightarrow{\beta_e} H_{t(e)}$ defined for each $e \in E$. 


DIXMIER GROUPS 9

Associated to $\Gamma$ is the path group $\pi(\Gamma)$, which is given by the presentation

$$\pi(\Gamma) := \langle \{a \in V H_a \rangle * \langle E \rangle \rangle / \langle e^{-1} a e(g) e = \beta e(g) : \forall e \in E, \forall g \in H_e \rangle \rangle,$$

where '$*$' stands for the free product (i.e. coproduct in the category of groups) and $\langle E \rangle$ for the free group with basis set $E = E(\Gamma)$. Now, if we fix a maximal tree $T$ in $\Gamma$, the fundamental group $\pi_1(\Gamma, T)$ of $\Gamma$ relative to $T$ is defined as a quotient of $\pi(\Gamma)$ by 'contracting the edges of $T$ to a point': precisely,

$$\pi_1(\Gamma, T) := \pi(\Gamma) / \langle e = 1 : \forall e \in E(T) \rangle.$$

For different maximal trees $T \subseteq \Gamma$, the groups $\pi_1(\Gamma, T)$ are isomorphic. Moreover, if $\Gamma$ is trivial (i.e. $H_a = \{1\}$ for all $a \in V$), then $\pi_1(\Gamma, T)$ is isomorphic to the usual fundamental group $\pi_1(\Gamma, a_0)$ of the graph $\Gamma$ viewed as a CW-complex. In general, $\pi_1(\Gamma, T)$ can be also defined in a topological fashion by introducing an appropriate notion of path and homotopy equivalence of paths in $\Gamma$ (cf. [B], Sect. 1.6).

When the underlying graph of $\Gamma$ is a tree (i.e., $\Gamma = T$), $\Gamma$ can be viewed as a directed system of groups indexed by $T$. In this case, formula (21) shows that $\pi_1(\Gamma, T)$ is just the inductive limit $\lim T \Gamma$, which is called the tree product of groups $\{H_a\}$ amalgamated by $\{H_e\}$ along $T$.

For example, if $T$ is a segment with $V(T) = \{0, 1\}$ and $E(T) = \{e\}$, the tree product is the usual amalgamated free product $H_0 * H_1$. In general, abusing notation, we will denote the tree product by

$$H_{a_1} * H_{a_2} * H_{a_3} * H_{a_4} \cdots.$$

3.2. $G_n$ as a fundamental group. To define the graph $\Gamma_n$, we take the subgroups $A$, $B$ and $U$ of $G$ given by the transformations (8), (9) and (10), respectively. Restricting the action of $G$ on $\mathcal{C}_n$ to these subgroups, we let $\Gamma_n$ be the oriented bipartite graph, with vertex and edge sets

$$V(\Gamma_n) := \{(c_n/A) \bigcup (c_n/B)\}, \quad E(\Gamma_n) := \mathcal{C}_n / U,$$

and the incidence maps $E(\Gamma_n) \to V(\Gamma_n)$ given by the canonical projections $i : \mathcal{C}_n / U \to \mathcal{C}_n / A$ and $t : \mathcal{C}_n / U \to \mathcal{C}_n / B$. Since the elements of $A$ and $B$ generate $G$ and $G$ acts transitively on each $\mathcal{C}_n$, the graph $\Gamma_n$ is connected.

Now, on each orbit in $\mathcal{C}_n / A$ and $\mathcal{C}_n / B$ we choose a basepoint and elements $\sigma_A \in G$ and $\sigma_B \in G$ moving these basepoints to the basepoint $(X_0, Y_0)$ of $\mathcal{C}_n$. Next, on each $U$-orbit $Q_U \in \mathcal{C}_n / U$ we also choose a basepoint and an element $\sigma_U \in G$ moving this basepoint to $(X_0, Y_0)$ and such that $\sigma_U \in \sigma_A A \cap \sigma_B B$, where $\sigma_A$ and $\sigma_B$ correspond to the (unique) $A$- and $B$-orbits containing $Q_U$. Then, we assign to the vertices and edges of $\Gamma_n$, the stabilizers $A_\sigma = G_n \cap \sigma A \sigma^{-1}$, $B_\sigma = G_n \cap \sigma B \sigma^{-1}$, $U_\sigma = G_n \cap \sigma U \sigma^{-1}$ of the corresponding elements $\sigma$ in the graph of right cosets of $G$ under the action of $G_n$. These data together with natural group homomorphisms $\alpha_\sigma : U_\sigma \to A_\sigma$ and $\beta_\sigma : U_\sigma \to B_\sigma$ define a graph of groups $\Gamma_n$ over $\Gamma_n$, and its fundamental group $\pi_1(\Gamma_n, T)$ relative to a maximal tree $T \subseteq \Gamma_n$ has canonical presentation, cf. (21):

$$\pi_1(\Gamma_n, T) = \langle (A_\sigma * U_\sigma \ast B_\sigma * \ldots) \ast \langle E(\Gamma_n \backslash T) \rangle \rangle / \langle e^{-1} a e(g) e = \beta e(g) : \forall e \in E(\Gamma_n \backslash T), \forall g \in U_\sigma \rangle \rangle.$$

In (23), the amalgam $(A_\sigma * U_\sigma \ast B_\sigma * \ldots)$ stands for the tree product taken along the edges of $T$, while $\langle E(\Gamma_n \backslash T) \rangle$ denotes the free group generated by the set of edges of $\Gamma_n$ in the complement of $T$. The main result of this section is the following

**Theorem 9.** For each $n \geq 0$, the group $G_n$ is isomorphic to $\pi_1(\Gamma_n, T)$. In particular, $G_n$ has an explicit presentation of the form (23).

**Proof.** Let $\mathcal{C}_n := \mathcal{C}_n \times G$ denote the (discrete) transformation groupoid corresponding to the action of $G$ on $\mathcal{C}_n$. The canonical projection $p : \mathcal{C}_n \to G$ is then a covering of groupoids3, which maps identically the vertex group of $\mathcal{C}_n$ at $(X_0, Y_0) \in \mathcal{C}_n$ to the subgroup $G_n \subseteq G$. Now, each

3Recall that if $\mathcal{E}$ and $\mathcal{B}$ are (small connected) groupoids, a covering $p : \mathcal{E} \to \mathcal{B}$ is a functor that is surjective on objects and restricts to a bijection $p : x \mathcal{E} \cong p(x) \mathcal{B}$ for all $x \in \text{Ob}(\mathcal{E})$, where $x \mathcal{E}$ is the set of arrows in $\mathcal{E}$ with source at $x$ (see [M], Chap. 3).
of the subgroups $A$, $B$ and $U$ of $G$ can be lifted to $\mathcal{S}_n$: $p^{-1}(A) = \mathcal{S}_n \times_G A$, $p^{-1}(B) = \mathcal{S}_n \times_G B$ and $p^{-1}(U) = \mathcal{S}_n \times_G U$, and these fibre products are naturally isomorphic to the subgroups $A_n := \mathcal{C}_n \times A$, $B_n := \mathcal{C}_n \times B$ and $U_n := \mathcal{C}_n \times U$ of $\mathcal{S}_n$, respectively. Since the coproducts of groups agree with coproducts in the category of groupoids and the latter can be lifted through coverings (see [O, Lemma 3.1.1]), the decomposition (7) implies

$$\mathcal{S}_n = A_n * U_n \cdot B_n, \quad \forall n \geq 0.$$  

Unlike $\mathcal{S}_n$, the groupoids $A_n$, $B_n$ and $U_n$ are not transitive (if $n \geq 1$), so (24) can be viewed as an analogue of the Seifert–Van Kampen Theorem for non-connected spaces. As in topological situation, computing the fundamental (vertex) group from (24) amounts to contracting the connected components (orbits) of $A_n$ and $B_n$ to points (vertices) and $U_n$ to edges (see, e.g., [Ge, Chap. 6, Appendix]). This defines a graph which is exactly $\Gamma_n$. Now, choosing basepoints in each of the contracted components and assigning the fundamental groups at these basepoints to the corresponding vertices and edges defines a graph of groups (cf. [HMM, p. 46]). By [HMM, Theorem 3], this graph of groups is (conjugate) isomorphic to the graph $\Gamma_n$ described above, and our group $G_n$ is isomorphic to $\pi_1(\Gamma_n, T)$. \hfill \Box

### 3.3. Examples

We now look at the graphs $\Gamma_n$ and groups $G_n$ for small $n$.

#### 3.3.1. For $n = 0$, the space $\mathcal{C}_n$ is just a point, and so are all its orbit spaces. The graph $\Gamma_0$ is thus a segment, and the corresponding graph of groups $\Gamma_0$ is given by $[ A \overset{U}{\to} B ]$. Formula (23) then says that $G_0 = A * U B$ which agrees with (7).

#### 3.3.2. For $n = 1$, we have $\mathcal{C}_1 \cong \mathbb{C}^2$, with $(X_0, Y_0) = (0, 0)$. Since each of the groups $A$, $B$ and $U$ contains translations $(x + a, y + b)$, $a, b \in \mathbb{C}$, they act transitively on $\mathcal{C}_1$. So again $\Gamma_1$ is just the segment, and $\Gamma_1$ is given by $[ A_1 \overset{U_1}{\to} B_1 ]$, where $A_1 := G_1 \cap A$, $B_1 := G_1 \cap B$ and $U_1 := G_1 \cap U$. Since, by definition, $G_1$ consists of all $\sigma \in G$ fixing the origin, the groups $A_1$, $B_1$ and $U_1$ are obvious:

- $A_1 : ax + by, cx + dy$, $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$
- $B_1 : ax + q(y), a^{-1}y$, $a \in \mathbb{C}^*$, $q \in \mathbb{C}[y]$, $q(0) = 0$
- $U_1 : ax + by, a^{-1}y$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$

It follows from (23) that $G_1 = A_1 * U_1 B_1$, where $A_1 = \text{SL}_2(\mathbb{C})$. In particular, $G_1$ is generated by the subgroups

$$\begin{align*}
G_{1,x} & := G_1 \cap G_x = \{ \Phi_p \in G : p \in \mathbb{C}[x], p(0) = 0 \}, \\
G_{1,y} & := G_1 \cap G_y = \{ \Psi_q \in G : q \in \mathbb{C}[y], q(0) = 0 \}.
\end{align*}$$

#### 3.3.3. The group $G_2$ has a more interesting structure. To describe the corresponding graph $\Gamma_2$ we decompose

$$C_2 = C_2^{\text{reg}} \bigsqcup C_2^{\text{sing}},$$

where $C_2^{\text{reg}}$ is the subspace of $C_2$ with $Y$ diagonalizable. The following lemma is established by elementary calculations.

**Lemma 1.** The action of $U$ on $C_2$ preserves the decomposition (25). Moreover,

(a) $C_2^{\text{reg}}$ is a single $U$-orbit $O_2^{\text{reg}}$ passing through $(X_1, Y_1) \in C_2$ with

$$X_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  

(b) $C_2^{\text{sing}}$ consists of two orbits $O(2, 1)$ and $O(2, 2)$ passing through $(X(2, 1), Y_2)$ and $(X(2, 2), Y_2)$ with

$$X(2, 1) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad X(2, 2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$  

Note that the orbit $O_2^{\text{reg}}$ is open (and therefore has dimension 4); $O(2, 1)$ and $O(2, 2)$ are closed orbits of dimension 3.
Lemma 2. The $B$-orbits in $C_2$ coincide with the $U$-orbits.

Proof. Note that $B$ belongs to the right coset $U \Psi_q$ generated by $\Psi_q = (x + q(y), y)$ with $q = ay^2 + by^3 + \ldots$. Since $Y_2$ is nilpotent, such $\Psi_q$ acts trivially on $(X(2, r), Y_2)$, so $B (X(2, r), Y_2) = U (X(2, r), Y_2) = O(2, r)$ for $r = 1, 2$. It follows that $O(2, 1)$ and $O(2, 2)$ are distinct $B$-orbits. Since there are only three $U$-orbits in $C_2$, $O_{\text{reg}}$ must be a separate $B$-orbit. \qed

Lemma 3. The group $A$ acts transitively on $C_2$.

Proof. Assume that $A$ has more than one orbit in $C_2$. Since there are only three $U$-orbits, at least one of the $A$-orbits (say, $O_A$) consists of a single $U$-orbit. But then, by Lemma 2, $O_A$ is also a $B$-orbit. Since $A$ and $B$ generate $G$, this means that $O_A$ is a $G$-orbit and hence, by Theorem 7, coincides with $C_2$. Contradiction. \qed

Summing up, we have $C_2/A = \{O_A\}$ and $C_2/B = \{O_{B, \text{reg}}, O_B(2, 1), O_B(2, 2)\}$, $C_2/U = \{O_{U, \text{reg}}, O_U(2, 1), O_U(2, 2)\}$, where $O_B$ and $O_U$ denote the same subspaces in $C_2$ but viewed as $B$- and $U$-orbits respectively. Thus the graph $\Gamma_2$ is a tree which looks as

\[ \begin{array}{c}
\Gamma_2 : \\
O_A \xrightarrow{O_{U, \text{reg}}} O_B \xrightarrow{O_{B, \text{reg}}} O_B(2, 1) \xrightarrow{O_U(2, 1)} O_B(2, 2) \xrightarrow{O_U(2, 1)} O_B(2, 2)
\end{array} \]

Computing the stabilizers of basepoints for each of the orbits, we obtain the graph of groups

\[ \begin{array}{c}
\Gamma_2 : \\
T \xrightarrow{\mathbb{Z}_2} G_{2,y} \times \mathbb{Z}_2 \xrightarrow{T} G_{2,x} \times T
\end{array} \]

where $T \subset G$ is the subgroup of scaling transformations $(tx, t^{-1}y)$, $t \in \mathbb{C}^*$, and the groups $G_{2,x}$, $G_{2,y}$ and $G_{2,y}^{(1)}$ are defined in terms of generators (11) by

\[ G_{2,x} := \{ \Phi_p \in G : p \in \mathbb{C}[x] \ , \ p(0) = p'(0) = 0 \} , \]
\[ G_{2,y} := \{ \Psi_q \in G : q \in \mathbb{C}[y] \ , \ q(0) = q'(0) = 0 \} , \]
\[ G_{2,y}^{(1)} := \{ \Phi \cdot_q \Phi \in G : q \in \mathbb{C}[y] \ , \ q(\pm 1) = 0 \} . \]

Formula (23) yields the presentation

\[ (26) \quad G_2 = (G_{2,x} \times T) *_{T} (G_{2,y} \times T) *_{\mathbb{Z}_2} (G_{2,y}^{(1)} \times \mathbb{Z}_2) . \]

In particular, $G_2$ is generated by its subgroups $G_{2,x}$, $G_{2,y}$, $G_{2,y}^{(1)}$ and $T$.

Using the above explicit presentations, it is easy to show that the groups $G$, $G_1$ and $G_2$ are pairwise non-isomorphic (see [BEE]). In Section 5, we will give a general proof of this fact for
all groups $G_n$. For $n \geq 3$, the amalgamated structure of $G_n$ seems to be more complicated; the corresponding graphs $\Gamma_n$ are no longer trees (in fact, there are infinitely many cycles).

4. $G_n$ AS AN ALGEBRAIC GROUP

In this section, we will equip $G$ with the structure of an ind-algebraic group that is compatible with the action of $G$ on the varieties $C_n$. Each $G_n \subseteq G$ will then become a closed subgroup and hence will acquire an ind-algebraic structure as well. We begin by recalling the definition and basic properties of ind-algebraic varieties. Apart from the original papers of Shafarevich [Sh, Sh1] a good reference for this material is Chapter IV of [Ku].

4.1. Ind-algebraic varieties and groups. An ind-algebraic variety (for short: an ind-variety) is a set $X = \bigcup_{k \geq 0} X^{(k)}$ given together with an increasing filtration

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots$$

such that each $X^{(k)}$ has the structure of a finite-dimensional (quasi-projective) variety over $C$, and each inclusion $X^{(k)} \hookrightarrow X^{(k+1)}$ is a closed embedding of varieties. An ind-variety has a natural topology where a subset $S \subseteq X$ is open (resp, closed) iff $S^{(k)} := S \cap X^{(k)}$ is open (resp, closed) in the Zariski topology of $X^{(k)}$ for all $k$. In this topology, a closed subset $S$ acquires an ind-variety structure defined by putting on $S^{(k)}$ the closed (reduced) subvariety structure from $X^{(k)}$. We call $S$ equipped with this structure a closed ind-subvariety of $X$. More generally, any locally closed subset $S \subseteq X$ acquires from $X$ the structure of an ind-variety since each $S^{(k)}$ is a locally closed subset and hence a subvariety in $X^{(k)}$.

An ind-variety $X$ is said to be affine if each $X^{(k)}$ is affine. For an affine ind-variety $X$, we define its coordinate ring by $\mathbb{C}[X] := \lim_{\leftarrow} \mathbb{C}[X^{(k)}]$, where $\mathbb{C}[X^{(k)}]$ is the coordinate ring of $X^{(k)}$. Naturally, $\mathbb{C}[X]$ is a topological algebra equipped with the inverse limit topology.

If $X$ and $Y$ are two ind-varieties with filtrations $\{X^{(k)}\}$ and $\{Y^{(k)}\}$, a map $f : X \to Y$ defines a morphism of ind-varieties if for each $k \geq 0$, there is $m \geq 0$ (depending on $k$) such that $f(X^{(k)}) \subseteq Y^{(m)}$ and the restriction of $f$ to $X^{(k)}$, $f_k : X^{(k)} \to Y^{(m)}$, is a morphism of varieties. A morphism of ind-varieties $f : X \to Y$ is continuous with respect to ind-topology and, in the case of affine ind-varieties, induces a continuous algebra map $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$.

Example 1. For any ind-varieties $X$ and $Y$, the set $X \times Y$ has the canonical ind-variety structure defined by the filtration $(X \times Y)^{(k)} := X^{(k)} \times Y^{(k)}$, where $(X^{(k)} \times Y^{(k)})$ is a product in the category of varieties (cf. [Ku, Example 4.1.3 (2)]). The two natural projections $X \hookrightarrow X \times Y \twoheadrightarrow Y$ are then morphisms of ind-varieties.

A morphism of ind-varieties $f : X \to Y$ is called an isomorphism if $f$ is bijective and $f^{-1}$ is also a morphism. It is easy to see that a morphism $f : X \to Y$ of affine ind-varieties is an isomorphism iff the induced map $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ is an isomorphism of topological algebras. Two ind-variety structures on the same set $X$ are said to be equivalent if the identity map $\text{Id} : X \to X$ is an isomorphism. It is natural not to distinguish between equivalent structures on $X$.

Example 2. Any vector space $V$ of countable dimension can be given the structure of an (affine) ind-variety by choosing a filtration $V^{(k)}$ by finite-dimensional subspaces. It is easy to see that up to equivalence, this structure is independent of the choice of filtration; hence $V$ has the canonical structure of an ind-variety which is denoted $\mathbb{C}^\infty$ (cf. [Ku, Example 4.1.3 (4)]).

A morphism of ind-varieties $f : X \to Y$ is called a closed embedding if all the morphisms $f_k : X^{(k)} \to Y^{(m)}$ are closed embeddings, $f(X)$ is closed in $Y$ and $f : X \to f(X)$ is a homeomorphism under the subspace topology on $f(X)$. The next lemma gives a useful characterization of morphisms of ind-varieties in terms of closed embeddings (cf. [Ku, Lemma 4.1.2]).
Lemma 4. Let $X$, $Y$, $Z$ be ind-varieties. Let $f : X \rightarrow Y$ be a closed embedding, and let $g : Z \rightarrow X$ be a map of sets with the property that for every $k \geq 0$ there is $m \geq 0$ such that $g(Z^{(k)}) \subseteq X^{(m)}$. Then $f$ is a morphism (resp., closed embedding) iff $f \circ g : Z \rightarrow Y$ is a morphism (resp., closed embedding).

For example, if $Z \subseteq Y$ is a closed ind-subvariety of $Y$, then the inclusion $Z \hookrightarrow Y$ a closed embedding. Lemma 4 shows that the converse is actually also true:

Corollary 2. If $Z \subseteq Y$ is a closed subset of an ind-variety $Y$, there is a unique ind-variety structure on $Z$ making $Z \hookrightarrow Y$ a closed embedding.

Proof. Assume that $Z$ has two ind-variety structures, say $Z'$ and $Z''$, making $Z \hookrightarrow Y$ into closed embeddings: $i' : Z' \rightarrow Y$ and $i'' : Z'' \rightarrow Y$. To apply Lemma 4 we first take $f := i''$ and $g : Z' \rightarrow Z''$ to be the identity map $\text{Id}_Z$. Since $f \circ g = i'$ and $g = \text{Id}$ obviously satisfies the assumption of the lemma, we conclude that $\text{Id} : Z' \rightarrow Z''$ is a morphism of ind-varieties. Reversing the roles of $Z'$ and $Z''$, we similarly conclude that $\text{Id} : Z'' \rightarrow Z'$ is a morphism. Thus $Z' \cong Z''$. □

An ind-algebraic group (for short: an ind-group) is a group $H$ equipped with the structure of an ind-variety such that the map $H \times H \rightarrow H$, $(x, y) \mapsto xy^{-1}$, is a morphism of ind-varieties. A morphism of ind-groups is a group homomorphism which is also a morphism of ind-varieties.

For example, any closed subgroup $K$ of $H$ is again an ind-group under the closed ind-subvariety structure on $H$, and the natural inclusion $K \hookrightarrow H$ is a morphism of ind-groups. Finally, an action of an ind-group $H$ on an ind-variety $W$ is said to be algebraic if the action map $H \times W \rightarrow W$ is a morphism of ind-varieties.

4.2. The ind-algebraic structure on $G$. Recall that $R$ is the free associative algebra on two generators $x$ and $y$. Letting $V$ be the vector space spanned by $x$ and $y$ we identify $R$ with the tensor algebra $T_C(V) := \bigoplus_{n \geq 0} V^\otimes n$. Then, associated to the natural tensor algebra grading is a filtration on $R$ by vector subspaces:

\begin{equation}
R^{(0)} \subseteq R^{(1)} \subseteq \ldots \subseteq R^{(k)} \subseteq R^{(k+1)} \subseteq \ldots ,
\end{equation}

where $R^{(k)} := \bigoplus_{n \leq k} V^\otimes n$. Since each $R^{(k)}$ has finite dimension, this filtration makes $R$ an affine ind-variety, which is isomorphic to $\mathbb{C}^\infty$ (see Example 2). We write $\deg : R \rightarrow \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ for the degree function associated with (27): explicitly, if $p \in R$ is nonzero, $\deg(p) := k$ if $p \in R^{(k)} \setminus R^{(k-1)}$, while $\deg(0) := -\infty$ by convention. Thus $R^{(k)} = \{ p \in R : \deg(p) \leq k \}$.

Now, let $E := \text{End}(R)$ denote the set of all algebra endomorphisms of $R$. Each endomorphism is determined by its values on $x$ and $y$; hence we can identify

\begin{equation}
E = R \times R , \quad \sigma \mapsto \langle \sigma(x), \sigma(y) \rangle .
\end{equation}

This identification allows us to equip $E$ with an ind-variety structure by taking the product on ind-variety structures on $R$ (see Example 1):

\begin{align*}
E^{(k)} := R^{(k)} \times R^{(k)} &= \{ (p, q) \in R \times R : \deg(p) \leq k , \deg(q) \leq k \}
\end{align*}

Clearly, $E$ is an affine ind-variety, which is actually isomorphic to $\mathbb{C}^\infty$. We define the degree function on $E$ by $\deg(\sigma) := \max\{ \deg(p), \deg(q) \}$, where $\sigma = (p, q) \in E$.

Next, recall that we have defined $G$ to be the subset of $E$ consisting of invertible endomorphisms that preserve $w = [x, y] \in R$. On the other hand, a well-known theorem of Dicks, which is an analogue of the Jacobian conjecture for $R$, implies that every endomorphism of $R$ that preserves $w$ is actually invertible (see [Co, Theorem 6.9.4]). We will use this result to put an ind-variety structure on $G$.

Proposition 2. There is a unique ind-variety structure on $G$ making the inclusion $G \hookrightarrow E$ a closed embedding.

Proof. Consider the map

\begin{equation}
c : E \rightarrow R , \quad (p, q) \mapsto [p, q] ,
\end{equation}

The ind-algebraic structure on $G$ is then the free associative algebra $T_C(G)$ on the generators $[x, y]$ and $[y, x]$.

DIXMIER GROUPS 13
where \([p, q] := pq - qp\) is the commutator in \(R\). Since \(c(E^{(k)}) \subseteq R^{(2k)}\) for all \(k \geq 0\) and the restrictions \(c_k : E^{(k)} \to R^{(2k)}\) are given by polynomial equations, (29) is a morphism of ind-varieties (in particular, a continuous map). Under the identification (28), we have \(c(\sigma) = w\) for all elements \(\sigma \in G\). Now, by Dicks’ Theorem, \(G\) actually coincides with the preimage of \(w\). Since \(c\) is a continuous map (and \(w \in R\) is a closed point), \(G = c^{-1}(w)\) is a closed subset in \(E\). Letting \(G^{(k)} = G \cap E^{(k)}\) for \(k \geq 0\) and putting on \(G^{(k)}\) the closed (reduced) subvariety structure from \(E^{(k)} = R^{(k)} \times R^{(k)}\), we make \(G\) a closed ind-subvariety of \(E\). Then \(G \hookrightarrow E\) is a closed embedding, and the uniqueness follows from Corollary 2.

The group \(G\) equipped with the ind-variety structure of Proposition 2 is actually an affine ind-group. Indeed, by construction, \(G\) is an affine ind-variety. We need only to show that \(\mu : G \times G \to G\), \((\sigma, \tau) \mapsto \sigma \circ \tau^{-1}\), is a morphism of ind-varieties. For this, it suffices to show that for each \(k \geq 0\), there is \(m = m(k) \geq 0\) such that \(\mu((G \times G)^{(k)}) \subseteq G^{(m)}\). But since \(G\) admits an amalgamated decomposition, a standard inductive argument (see, e.g, [K, Lemma 4.1]) shows that \(\deg(\tau^{-1}) \leq \deg(\tau)\) for all \(\tau \in G\). This implies that \(\mu((G \times G)^{(k)}) \subseteq G^{(k^2)}\) for all \(k\).

**Remark.** The full automorphism group \(\hat{G} := \text{Aut}(R)\) of the algebra \(R\) has also a natural structure of an ind-group. In fact, by Dicks’ Theorem, \(\hat{G}\) is the preimage of the subset \(\{xw \in R : x \in \mathbb{C}^*\}\) which is locally closed in the ind-topology of \(R\). By [FuM, Lemma 2], \(\hat{G}\) is then locally closed in \(E\) and hence has the structure of an ind-subvariety of \(E\) with induced filtration \(\hat{G}^{(k)} = \hat{G} \cap E^{(k)}\).

We record two basic properties of the ind-group \(G\) which are similar to the properties of the Shafarevich ind-group \(\text{Aut} \mathbb{C}[x, y]\) (see [Sh, Sh1]). First, recall (cf. [Sh1, Bl]) that an ind-variety \(X\) is path connected if for any \(x_0, x_1 \in X\), there is an open set \(U \subseteq \mathbb{A}^1_{\mathbb{C}}\) containing 0 and 1 and a morphism \(f : U \to X\) such that \(f(0) = x_0\) and \(f(1) = x_1\).

**Lemma 5.** A path connected ind-variety is connected.

**Proof.** Indeed, a morphism of ind-varieties \(f : U \to X\) is continuous. Hence, if \(X\) is the disjoint union of two proper closed subsets which intersect with \(\text{Im}(f)\), the preimages of these subsets under \(f\) must be non-empty, closed and disjoint in \(U\). This contradicts the fact that \(U\) is connected in the Zariski topology.

**Theorem 10.** The group \(G\) is connected and hence irreducible.

**Proof.** By Lemma 5, it suffices to show any element of \(G\) can be joined to the identity element \(e \in G\) by a morphism \(f : U \to G\). By Theorem 5, any \(\sigma \in G\) can be written as a composition \(\sigma = \Phi_{p_1} \Psi_{q_1} \ldots \Phi_{p_n} \Psi_{q_n}\) of transformations (11). Rescaling the polynomials \(p_i\) and \(q_i\), we define

\[
(30) \quad f : \mathbb{A}^1_{\mathbb{C}} \to G, \quad t \mapsto \Phi_{tp_1} \Psi_{tq_1} \ldots \Phi_{tp_n} \Psi_{tq_n},
\]

which is obviously a morphism of ind-varieties such that \(f(1) = \sigma\) and \(f(0) = e\). Hence \(G\) is connected. On the other hand, it is known that any connected ind-group is actually irreducible (see [Sh1, Prop. 3] and also [Ku, Lemma 4.2.5]).

The next theorem is the analogue of [Sh, Theorem 8].

**Theorem 11.** Every finite-dimensional algebraic subgroup of \(G\) is conjugate to either a subgroup of \(A\) or a subgroup of \(B\).

**Proof.** The proof is essentially the same as in the classical case; we recall it for reader’s convenience. By definition, an algebraic subgroup \(H\) of \(G\) is a closed subgroup which is again an ind-group with respect to the closed subvariety structure on \(H^{(k)} = H \cap G^{(k)}\). In particular, each \(H^{(k)}\) is closed in \(H\) and hence, when \(H\) is finite-dimensional, there is a \(k \geq 0\) such that \(H^{(k)} = H^{(k+1)} = \ldots = H\). It follows that \(H \subseteq G^{(k)}\) for some \(k\), which means that

\[
(31) \quad \deg(\sigma) \leq k \quad \text{for all} \quad \sigma \in H.
\]

On the other hand, by Theorem 5, every element \(\sigma \in G\) has a reduced decomposition of the form

\[
\sigma = a_1 b_1 \ldots a_l b_l a_{l+1}
\]
where the $b_i \in B \setminus A$ for all $1 \leq i \leq l$ and $a_j \in A \setminus B$ for $2 \leq j \leq l$. The number $l$ is independent of the choice of a decomposition and called the length of $\sigma$. As in the case of polynomial automorphisms (see [Wr, FM]), the length and the degree of $\sigma$ are related by the formula $\deg(\sigma) = \deg(b_1) \deg(b_2) \ldots \deg(b_l)$, which shows that any subset of $G$, bounded in degree, is also bounded in length. Thus (31) implies that $H$ is a subgroup of $G$ bounded in length. A theorem of Serre [Se, Thm. 4.3.8] then implies that $H$ is contained in a conjugate of $A$ or $B$. □

Although we do not need it in the present paper, it is a natural question to ask what the Lie algebra $\text{Lie}(G)$ of the ind-group $G$ is. We close this section with a simple conjecture describing $\text{Lie}(G)$ in terms of derivations of the free algebra $R = \mathbb{C}[x, y]$. Let $\text{Der}(R)$ be the space of linear derivations of $R$, which is naturally a Lie algebra with respect to the commutator bracket. Let $\text{Der}_w(R)$ denote the subalgebra of $\text{Der}(R)$ consisting of derivations that vanish at $w = [x, y]$. Identify $\text{Der}(R) = R^2$ via the evaluation map $\delta \mapsto (\delta(x), \delta(y))$. Then one can check easily that $\text{Der}_w(R) = \{(u, v) \in R^2 : [x, v] + [u, y] = 0\}$. Now, define two (abelian) subalgebras of $\text{Der}_w(R)$:

(32) $\text{Lie}(G_x) := \{(0, p(x)) \in R^2 : p(x) \in \mathbb{C}[x]\},$

(33) $\text{Lie}(G_y) := \{(q(y), 0) \in R^2 : q(y) \in \mathbb{C}[y]\}.$

As the notation suggests, $\text{Lie}(G_x)$ and $\text{Lie}(G_y)$ can be identified with the Lie algebras of the abelian ind-groups $G_x$ and $G_y$.

**Conjecture 1.** $\text{Lie}(G)$ is isomorphic to the Lie subalgebra of $\text{Der}_w(R)$ generated by (32) and (33).

**Remarks.** 1. It is known that $\text{Lie}(G_x)$ and $\text{Lie}(G_y)$ generate a proper Lie subalgebra of $\text{Der}_w(R)$ (cf. [EG, Remark, p. 289]). One can also show that $\text{Lie}(G) \subseteq \text{Der}_w(R)$.

2. There is an explicit description of the Lie algebra $\text{Der}_w(R)$ due to Kontsevich [Kon] (see also [G]). Specifically, $\text{Der}_w(R)$ can be identified with the space $\mathcal{L} := R/([R, R] + \mathbb{C})$ of cyclic words in the letters $x$ and $y$. Under this identification, the Lie bracket on $\text{Der}_w(R)$ corresponds to a ‘noncommutative’ Poisson bracket on $\mathcal{L}$ defined in terms of cyclic derivatives (cf. [Kon, Sect. 6]).

### 4.3. The ind-algebraic structure on $G_n$

Recall that, as shown in [BEE1], each $G_n$ is a maximal subgroup of $G$. It is therefore natural to expect that $G_n$ is an algebraic (closed) subgroup of $G$. This follows formally from the next theorem.

**Theorem 12.** The ind-group $G$ acts algebraically on each space $\mathcal{C}_n$.

**Proof.** Recall that the action of $G$ on $\mathcal{C}_n$ is defined by (12). To see that this action is algebraic we first consider

(34) $G \times \mathcal{M}_n(\mathbb{C})^{\times 2} \xrightarrow{s \times \text{Id}} G \times \mathcal{M}_n(\mathbb{C})^{\times 2} \xrightarrow{i \times \text{Id}} E \times \mathcal{M}_n(\mathbb{C})^{\times 2} \xrightarrow{\text{ev}} \mathcal{M}_n(\mathbb{C})^{\times 2},$

where $s : G \to G$ is the inverse map on $G$, $i : G \hookrightarrow E$ is the natural inclusion and $\text{ev}$ is the evaluation map defined by $[(p, q), (X, Y)] \mapsto (p(X, Y), q(X, Y))$. Each arrow in (34) is a morphism of ind-varieties with respect to the product ind-variety structure on $E \times \mathcal{M}_n(\mathbb{C})^{\times 2}$; hence (34) defines an algebraic action of $G$ on $\mathcal{M}_n(\mathbb{C})^{\times 2}$. This restricts to an action $G \times \mathcal{\hat{C}}_n \to \mathcal{\hat{C}}_n$, which is also algebraic since $\mathcal{\hat{C}}_n$ is a closed subvariety of $\mathcal{M}_n(\mathbb{C})^{\times 2}$. Finally, as $\text{PGL}_n$ acts freely on $\mathcal{\hat{C}}_n$ and $G$ commutes with $\text{PGL}_n$, the quotient map $G \times \mathcal{\hat{C}}_n \to \mathcal{\hat{C}}_n \to \mathcal{C}_n$ is algebraic and it induces an algebraic action $G \times \mathcal{C}_n \to \mathcal{C}_n$, which is precisely (12). □

By definition, $G_n$ is the fibre of the action map $p : G \to \mathcal{C}_n$, $\sigma \mapsto \sigma(X_0, Y_0)$, over the basepoint $(X_0, Y_0) \in \mathcal{C}_n$. By Theorem 12, this map is a morphism of ind-varieties; hence, $G_n$ is a closed subgroup of $G$. We believe that the following is true.

**Conjecture 2.** The groups $G_n$ are connected and hence irreducible for all $n \geq 0$.

In Section 3.3, we have explicitly described the structure of $G_n$ as a discrete group for small $n$. Using this explicit description, we can easily prove

**Proposition 3.** Conjecture 2 is true for $n = 0, 1, 2$. 

Proof. For \( n = 0 \), this is just Theorem 10. The argument of Theorem 10 can be also extended to \( G_1 \) and \( G_2 \), since we know explicit generating sets for these groups. Precisely, as shown in Section 3.3.2, \( G_1 \) is generated by \( \Phi_p \) and \( \Psi_q \) with \( p \in \mathbb{C}[x] \) and \( q \in \mathbb{C}[y] \) satisfying \( p(0) = q(0) = 0 \). Hence, for any \( \sigma \in G_1 \), the morphism (30) constructed in the proof of Theorem 10 has its image in \( G_1 \), which, by Lemma 5, implies that \( G_1 \) is connected. Similarly, by (26), \( G_2 \) is generated by its subgroups \( G_{2,x} \), \( G_{2,y} \), \( G_{2,xy} \) and \( T \), each of which is path connected. For example, every \( \sigma \in G_{2,y} \) has the form \( \Phi_y \Psi_q \Phi_x \), with \( q \in \mathbb{C}[y] \) satisfying \( q(\pm 1) = 0 \). The last condition is preserved under rescaling \( t \mapsto t q \). Hence, we can join \( \sigma \) to \( e \) within \( G_{2,y} \) by the algebraic curve \( f : t \mapsto \Phi_y \Psi_q \Phi_x \). This shows that \( G_2 \) is connected.

Unfortunately, for \( n \geq 3 \), the Bass-Serre decomposition of \( G_n \) is too complicated, and its factors (generating subgroups of \( G_n \)) are much harder to analyze.

5. Borel Subgroups

In this section, we will study the Borel subgroups of \( G_n \) and prove our main classification theorems stated in the Introduction. Recall that the natural action of the group \( G \) on \( \mathbb{C}^2 \) is faithful (see Proposition 1 and remark thereafter). Using this action, we will identify the elements of \( G \) (and hence \( G_n \)) as automorphisms of \( \mathbb{C}^2 \). This will allow us to apply the results of [FM] and [L]. On the other hand, to define Borel subgroups we will regard \( G \) and \( G_n \) as topological groups with (reduced) ind-Zariski topology introduced in Section 4.

5.1. Friedland-Milnor-Lamy classification. By [FM], the elements of \( G \) can be divided into two separate classes according to their dynamical properties as automorphisms of \( \mathbb{C}^2 \): every \( g \in G \) is conjugate to either an element of \( B \) (see (9)) or a composition of generalized Hénon automorphisms:

\[
g = \sigma g_1 g_2 \ldots g_m \sigma^{-1},
\]

where \( g_i = (y, x + q_i(y)) \) with polynomials \( q_i(y) \in \mathbb{C}[y] \) of degree \( \geq 2 \). We say that an element \( g \) is elementary if it is a conjugate of an element of \( B \) and \( g \) is Hénon type if it has the form (35).

A subgroup \( H \subseteq G \) is called elementary if each element of \( H \) is elementary.

It is convenient to reformulate this classification in terms of the action of \( G \) on the standard tree \( T \) associated to the amalgam \( G = A *_U B \). By definition, the vertices \( V(T) \) of \( T \) are the left cosets \( G/A \cup G/B \), while the set of edges is \( E(T) = G/U \). The group \( G \) acts on \( T \) by left translations. Notice that if \( g \in G \) fixes two vertices in \( T \), then it also fixes all the vertices linking these two vertices; thus, for each \( g \in G \), we may define a subtree \( \text{Fix}(g) \subseteq T \) fixed by \( g \). More generally, if \( H \) is a subgroup of \( G \), following [L], we put \( \text{Fix}(H) := \bigcap_{g \in H} \text{Fix}(g) \). It is easy to see that \( \text{Fix}(g) \) is non-empty iff \( g \) is elementary, and \( \text{Fix}(g) = \emptyset \) iff \( g \) is of Hénon type. In the latter case, following [Se], we may define the geodesic of \( g \) to be the set of vertices of \( T \) that realizes the infimum \( \inf_{p \in V(T)} \text{dist}(p, g(p)) \), where \( \text{dist}(p, q) \) is the number of edges of the shortest path joining the vertices \( p \) and \( q \) in \( T \).

The following theorem is a consequence of the main result of S. Lamy.

Theorem 13 ([L]). Let \( H \) be a subgroup of \( G \). Then, one and only one of the following possibilities occurs:

(I) \( H \) is an elementary subgroup conjugate to a subgroup of \( A \) or \( B \).

(II) \( H \) is an elementary subgroup which is not conjugate to a subgroup of \( A \) or \( B \). Then \( H \) is countable and abelian.

(III) \( H \) contains elements of Hénon type, and all such elements in \( H \) share the same geodesic. Then \( H \) is solvable and contains a subgroup of finite index isomorphic to \( \mathbb{Z} \).

(IV) \( H \) contains two elements of Hénon type with distinct geodesics. Then \( H \) contains a free subgroup on two generators.

Remarks. 1. Theorem 13 is essentially Théorème 2.4 of [L], except that this last paper is concerned with subgroups of the full automorphism group \( \text{Aut}(\mathbb{C}^2) \). As a subgroup of \( \text{Aut}(\mathbb{C}^2) \),
$G$ coincides with the kernel of the Jacobian map $\text{Jac} : \text{Aut}(\mathbb{C}^2) \to \mathbb{C}^*$. Using this we can easily deduce Theorem 13 from [L, Théorème 2.4]. Indeed, if $H$ is an elementary subgroup of $G$, then (by definition) it is elementary in $\text{Aut}(\mathbb{C}^2)$ and hence is either of type I or type II in that group. If $H$ is of type II in $\text{Aut}(\mathbb{C}^2)$ then it is automatically of type II in $G$. If $H$ is of type I in $\text{Aut}(\mathbb{C}^2)$, then, by [L, Théorème 2.4], it can be conjugate to a subgroup $\tilde{H}$ of $\text{Aut}(\mathbb{C}^2)$, which is either in $A \times \mathbb{C}^*$ or $B \times \mathbb{C}^*$. But $\text{Jac}(\tilde{H}) = \text{Jac}(H) = 1$, hence $\tilde{H} \subset G$, and since $\text{Aut}(\mathbb{C}^2) \cong G \times \mathbb{C}^*$, we can conjugate $H$ to $\tilde{H}$ within $G$. For types III and IV, the implication Théorème 2.4 $\Rightarrow$ Theorem 13 is automatic.

2. We have added to [L, Théorème 2.4] that any subgroup $H$ of type III contains a finite index subgroup isomorphic to $\mathbb{Z}$. Indeed, by [L, Prop. 4.10], all subgroups of $G$ satisfying property (III) for a fixed geodesic generate a unique largest subgroup, which contains a copy of $\mathbb{Z}$ as a subgroup of finite index. The last condition means that $H \cap \mathbb{Z}$ is a subgroup of finite index in $H$; hence $H$ either contains $H \cap \mathbb{Z} \cong \mathbb{Z}$ as a subgroup of finite index or is finite (if $H \cap \mathbb{Z} = \{1\}$). It remains to note that the last possibility does not occur, since, by [Se, Theorem 8, Sect. I.4.3], any finite subgroup of $G$ is conjugate to a subgroup of $A$ or $B$ and hence is of type I.

5.2. Solvable subgroups of $G$. We begin with the following observation which may be of independent interest.

**Lemma 6.** For $g \in G$, one and only one of the following possibilities occurs:

(a) $g$ is elementary, and $\langle g \rangle \cong \mathbb{Z}$,

(b) $g$ is elementary and $\langle g \rangle \cong \mathbb{Z}_n$ for some $n \geq 1$,

(c) $g$ is Hénon type and $\langle g \rangle \cong \mathbb{Z}$.

Moreover, $\langle g \rangle$ is a closed subgroup of $G$ if and only if it is as in (b) or (c).

**Proof.** By the Friedland-Milnor classification, we know that any element of $G$ is either elementary or of Hénon type. Suppose that $g$ is elementary. Then we may assume that $g$ is contained in $B$. If $g$ has finite order, $\langle g \rangle \cong \mathbb{Z}_n$ for some $n \geq 1$. Since $\langle g \rangle$ is finite, it is a closed subgroup of $G$. If $g$ has infinite order then $\langle g \rangle \cong \mathbb{Z}$. Moreover $\langle g \rangle \subset G^{(k)}$, for some $k$, where $G^{(k)}$ is $k$-th filtration component of $G$. Since $\|(\langle g \rangle)\|$ is countable, it cannot be closed in $G^{(k)}$.

Suppose $g$ is of Hénon type. For a Hénon automorphism, the sequence $\{\deg(g^k)\}_{k=1}^{\infty}$ is strictly increasing, and we have $\lim_{k \to \infty} \deg(g^k) = \infty$. For any $n > 0$, $G^{(n)} \cap \langle g \rangle$ is finite and hence closed. Thus $\langle g \rangle$ is equipped with an increasing filtration of closed sets therefore it is an ind-subgroup of $G$. \hfill \Box

**Proposition 4.** Let $H$ be a subgroup of $G$ with either of the following properties:

(S1) $H$ is a solvable group without a proper subgroup of finite index,

(S2) $H$ is a connected solvable group.

Then $H$ cannot be of type III (in the nomenclature of Theorem 13).

**Proof.** Suppose $H$ is a type III subgroup. Then $H$ is a subgroup of the group $K$ explicitly described in [L, Proposition 4.10]. By the proof of this proposition, $K$ has a finite index subgroup generated by a Hénon type element. We denote this subgroup by $K_1$. Since $H \subseteq K$, we have $H/(H \cap K_1) \subseteq K/K_1$ and $H/(H \cap K_1)$ is finite. This shows that $H$ with property (S1) cannot be of type III, since $H \cap K_1$ is a subgroup of finite index in $H$.

Since $K_1$ is closed in $G$, by Lemma 6(c), $H \cap K_1$ is closed in $H$. Therefore $H = \bigcup_{i=1}^{n} g_i(H \cap K_1)$ is the disjoint union of closed subsets. Since $H$ is connected, we must have $H \cap K_1 = H$, hence $H \subseteq K_1$. It follows that either $H = \langle g \rangle$ for some $g \in K_1$ or $H = 1$. One can easily see that $H = \langle g \rangle$ cannot be connected: $H_1 = \langle g^2 \rangle$ is its closed subgroup of index 2, hence $H = gH_1 \cup H_1$ is the disjoint union of closed subsets. Therefore $H = 1$. This proves (S2). \hfill \Box

5.3. Borel subgroups of $G$. Recall that a subgroup of a topological group is called *Borel* if it is connected, solvable and maximal among all connected solvable subgroups. For basic properties of Borel subgroups we refer to [Bo, § 11]. We only note that any Borel subgroup is necessarily a closed subgroup.

We begin with the following proposition which establishes the main properties of the subgroup of triangular automorphisms in $G$. 

DIXMIER GROUPS 17
Proposition 5. Let $B$ be the subgroup of $G$ defined by (9). Then

(a) $B$ is a solvable group of derived length 3.
(b) $\operatorname{Fix}(B) = \{1 \cdot B\}$ consists of a single vertex.
(c) $N_G(B) = B$.
(d) $B$ is a connected subgroup of $G$.
(e) $B$ is a maximal solvable subgroup of $G$.

In particular, $B$ is a Borel subgroup of $G$.

Proof. (a) One can easily compute the derived series of $B$, which is given by

$$B^{(1)} = \{(x + p(y), y + f) \mid f \in \mathbb{C}, p(y) \in \mathbb{C}[y]\}, \quad B/B^{(1)} \cong \mathbb{C}^*$$

$$B^{(2)} = \{(x + p(y), y) \mid p(y) \in \mathbb{C}[y]\}, \quad B^{(1)}/B^{(2)} \cong \mathbb{C}$$

(b) It is clear that $\operatorname{Fix}(B)$ contains $\{1 \cdot B\}$. Now, by [L, Proposition 3.3], there is an element $f \in B$ such that $\operatorname{Fix}(f) = \{1 \cdot B\}$. Hence $\operatorname{Fix}(B) = \{1 \cdot B\}$.

(c) Let $g \in N_G(B)$, i.e., $g^{-1}Bg \subseteq B$. Then $B \subseteq gBg^{-1}$. Hence $B$ must also fix the vertex $g \cdot B$.

By part (b), $g \cdot B = 1 \cdot B$ and $g \in B$.

(d) By Lemma 5, it suffices to show that $B$ is path connected. Let $b = (tx + p(y), t^{-1}y + f)$ be an arbitrary element in $B$. Consider $b_s = (tx + sp(y), t^{-1}y + sf)$ for $s \in \mathbb{C}$. We have $b_0 = (tx, t^{-1}y)$ and $b_1 = b$. Thus, every element of $B$ is connected to the subgroup $T = \{(tx, t^{-1}y) \mid t \in \mathbb{C}^*\}$. On the other hand, $T$ is path connected, hence $B$ is path connected as well.

(e) Suppose $B$ is contained in a solvable subgroup $H \subset G$. Then, $H$ is a solvable group of length at least 3. Then, by Theorem 13, it is either of type I or type III. By Proposition 4, it can be only of type I: i.e., it is conjugate to a subgroup of either $A$ or $B$. Suppose that there is $g \in G$ such that $g^{-1}Bg \subset g^{-1}Hg \subset A$. Then $B \subset gAg^{-1}$. This implies that $B$ fixes the vertex $g \cdot A$, which contradicts part (b). Suppose that there is $g \in G$ such that $g^{-1}Bg \subset g^{-1}Hg \subset B$. Once again, we can conclude that $B$ fixes $g \cdot B$ and hence $g \in B$ by (b). It follows that $B = H$ and hence $B$ is maximal solvable.

Now, we can prove Theorem 1 from the Introduction.

Proof of Theorem 1. Let $H$ be a Borel subgroup of $G$. Then, by classification of Theorem 13, $H$ can only be a subgroup of type I. Indeed, it is obvious that $H$ cannot be of type IV (since it is solvable); it cannot be of type III (by Proposition 4), and it cannot be of type II, since, by [L, Proposition 3.12], any type II subgroup of $G$ is given by a countable union of finite cyclic groups and hence is totally disconnected in the ind-topology of $G$ (cf. [Ku, 4.1.3(5)]). Thus $H$ is conjugate to either a subgroup of $A$ or a subgroup of $B$. In the first case, it must be a Borel subgroup of $A$. Since $A$ is a connected algebraic subgroup of $G$, by the classical Borel Theorem, all Borel subgroups of $A$ are conjugate to each other. Since $U$ is a Borel in $A$, $H$ must be conjugate to $U$. This obviously contradicts the maximality of $H$ since $U$ is properly contained in $B$. Hence $H$ is conjugate to a subgroup of $B$; by maximality, it must then be conjugate to $B$.

The next lemma is elementary (its proof can be found, for example, in [H]).

Lemma 7. Let $G$ be an abstract group.

(a) If $G$ has a proper subgroup of finite index then $G$ has a proper normal subgroup of finite index.
(b) If $G$ has no proper subgroup of finite index then any homomorphic image of $G$ has no proper subgroup of finite index.
(c) If $G$ is solvable and has no proper finite index subgroup, then it is infinitely generated.

Using Lemma 7, we can now prove

Lemma 8. The group $B$ contains no proper subgroups of finite index.

Proof. Suppose $H$ is a proper finite index subgroup of $B$. By Lemma 7(a), we may assume that $H$ is normal. Consider the quotient map $p_1 : B \to B/B^{(1)} \cong \mathbb{C}^*$. Then, the image of $H$ under $p_1$ is a
finite index subgroup of \(\mathbb{C}^*\). But \(\mathbb{C}^*\) has no proper finite index subgroups. Hence \(p_1(H) = B/B^{(1)}\) and therefore \(B = B^{(1)}H\). Now, let \(H_1 := B^{(1)} \cap H\). Then

\[
B \div H = B^{(1)}H \div H \cong B^{(1)} \div H_1.
\]

This implies that \(H_1\) is a finite index subgroup of \(B^{(1)}\). Next, we consider \(p_2 : B^{(1)} \to B^{(1)}/B^{(2)} \cong \mathbb{C}\). Again, \(p_2(H_1)\) is a finite index subgroup of \(\mathbb{C}\). Since \(\mathbb{C}\) has no proper finite index subgroups, we conclude \(p_2(H_1) = B^{(1)}/B^{(2)}\). Hence \(B^{(1)} = B^{(2)}H_1\), and we have

\[
B^{(1)}/H_1 \cong B^{(2)}/B^{(2)} \cap H_1.
\]

Thus \(B^{(2)} \cap H_1\) is a finite index subgroup of \(B^{(2)}\). On the other hand \(B^{(2)} \cong \mathbb{C}[y]\) which has no proper finite index subgroups. Hence \(B^{(2)} \cap H_1 = B^{(2)}\), which implies that \(B^{(2)}\) is a subgroup of \(H_1\). Next, since \(B^{(1)} = B^{(2)}H_1\), we have \(B^{(1)} = H_1 = B^{(1)} \cap H\). From this last equality we see that \(B^{(1)} \subseteq H\). Finally, from \(B = B^{(1)}H\) we get \(H = B\). This contradicts the properness of \(H\).

Before characterizing the Borel subgroups of \(G\), we recall a classical characterization of solvable subgroups of \(GL_n(\mathbb{C})\) due to A. I. Maltsev. Maltsev’s theorem can be viewed as a generalization of the Lie-Kolchin Theorem (cf. \([Sp, 6.3.1]\)); for its proof we refer to \([LR, \text{Theorem 3.1.6}]\).

**Theorem 14** (Maltsev). Let \(\Gamma\) be any solvable subgroup of \(GL_n(\mathbb{C})\). Then \(\Gamma\) has a finite index normal subgroup which is conjugate to a subgroup of upper triangular matrices.

We are now in position to prove Steinberg’s Theorem for the group \(G\).

**Theorem 15.** Let \(H\) be a non-abelian subgroup of \(G\). Then \(H\) is Borel iff

(B1) \(H\) is a maximal solvable subgroup of \(G\),

(B2) \(H\) contains no proper subgroups of finite index.

**Proof.** (\(\Rightarrow\)) Suppose \(H\) is Borel. Then, by Theorem 1, \(H\) is conjugate to \(B\). Hence, by Proposition 5 and Lemma 8, \(H\) satisfies (B1) and (B2) respectively.

(\(\Leftarrow\)) Let \(H\) be a non-abelian subgroup of \(G\) satisfying (B1) and (B2). Then, by Theorem 13, it is either of type I or type III. By Proposition 4, it cannot be of type III. Therefore, it is conjugate to a subgroup of \(A\) or \(B\). Suppose that \(H\) is conjugate to a subgroup of \(A\). The image of composition \(g^{-1}Hg \to A \to SL_2(\mathbb{C})\) is then a solvable subgroup of \(SL_2(\mathbb{C})\). We denote this group by \(S\). By Theorem 14, \(S\) has a finite index normal subgroup \(T\), which is a subgroup of upper triangular matrices in \(SL_2(\mathbb{C})\). By Lemma 7(b), the group \(S\), being the image of \(H\), contains no proper subgroups of finite index. Thus, \(S = T\) and \(H\) is conjugate to a subgroup of \(U\), which is a proper solvable subgroup of \(B\). This contradicts the assumption that \(H\) is a maximal solvable subgroup of \(G\). Hence, \(H\) can be only conjugate to a subgroup of \(B\). Since \(H\) is maximal solvable, it must be conjugate to \(B\) itself.

Theorem 15 is the special case of Theorem 4 corresponding to \(n = 0\). We now turn to the general case.

5.4. **Borel subgroups of** \(G_n\). We begin with a few technical lemmas. First, recall that, for any element \(g = (P, Q) \in G\), we defined its degree in \(G\) by

\[
\deg(g) := \max\{\deg(P), \deg(Q)\},
\]

where \(\deg(P)\) and \(\deg(Q)\) are the degrees of \(P = P(x, y)\) and \(Q = Q(x, y)\) in the free algebra \(\mathbb{C}(x, y)\) (cf. Section 4.2). It is easy to see that \(\deg(g)\) thus defined coincides with the degree of \(g\) viewed as an automorphism of \(\mathbb{C}^2\).

**Lemma 9.** Let \(H\) be a subgroup of \(B\) with the property that for any \(N > 0\), there is \(h \in H\) such that \(\deg(h) > N\). If \(H \subseteq g^{-1}Bg \cap B\) for some \(g \in G\), then \(g^{-1}Bg \cap B = B\) and \(g \in B\).
Proof. If \( g \in B \), then \( g^{-1}Bg = B \) and therefore \( g^{-1}Bg \cap B = B \). Assume now that \( g \in G \setminus B \). Then we can write \( g = w_0w_1\ldots w_l \), where \( w_0 \in U \) and \( \{w_1, \ldots, w_l\} \) are representatives of some cosets in \( A/U \) or \( B/U \). Without loss of generality, we may assume that \( w_0 = 1 \) and \( w_1 \) is a coset representative from \( A/U \). Then
\[
g^{-1}Bg \cap B = (w_1^{-1}w_2^{-1}w_1^{-1}uw_1w_2\ldots w_l) \cap B,
\]
since \( g^{-1}(B \setminus U)g \) consists of words of length \( 2l + 1 \) and \( g^{-1}(B \setminus U)g \cap B = \emptyset \). Let \( \deg(g) = n \).
Then, by [K, Lemma 4.1], \( \deg(g^{-1}) \leq n \), and the degrees of all elements in (38) are at most \( n^2 \). This contradicts the assumption that (38) contains \( H \) whose elements have arbitrary large degrees. \[\square\]

Now, for \( g \in G \), we define \( B_g := g^{-1}Bg \cap G_n \). Clearly, \( B_g \) is a subgroup of \( G_n \) that depends only on the right coset of \( g \in G \) (mod \( B \)). We write \( V_n(B) := \{B_g\}_{g \in B} \) for the set of all such subgroups of \( G_n \) and note that \( G_n \) acts on \( V_n(B) \) by conjugation.

Lemma 10. The assignment \( g \mapsto B_g \) induces a bijection
\[
\eta : B \setminus G \sim V_n(B),
\]
which is equivariant under the (right) action of \( G_n \).

Proof. It is clear that the map \( \eta \) is well defined and surjective. We need only to prove that \( \eta \) is injective. Suppose that \( \eta^{-1}Bg_1 \cap G_n = \eta^{-1}Bg_2 \cap G_n \) for some \( g_1, g_2 \in G \).
Then
\[
g_2g_1^{-1}Bg_1g_2^{-1} \cap g_2G_ng_2^{-1} = B \cap g_2G_ng_2^{-1},
\]
which implies \( B \cap g_2G_ng_2^{-1} \subseteq g_2g_1^{-1}Bg_1g_2^{-1} \cap B \). Now, observe that \( B \cap g_2G_ng_2^{-1} = \text{Stab}_B[g_2 \cdot (X_0,Y_0)] \). Hence \( H := B \cap g_2G_ng_2^{-1} \) satisfies the assumptions of Lemma 9, and we conclude: \( g_2g_1^{-1}Bg_1g_2^{-1} = B \) and \( g_1g_2^{-1}B \). It follows that \( Bg_1 = Bg_2 \). To see the equivariance of \( \eta \), for \( h \in G_n \), we compute
\[
B_{gh} := (gh)^{-1}B(gh) \cap G_n = h^{-1}(g^{-1}Bg \cap G_n)h = h^{-1}Bgh.
\]
\[\square\]

Dividing the map \( \eta \) of Lemma 10 by the action of \( G_n \), we get
\[
C_n/B \sim V_n(B)/\text{Ad}G_n,
\]
where we have identified \( B \setminus G/G_n = C_n/B \) via \( BgG_n \leftrightarrow Bg(X_0,Y_0) \).

Notice that \( V_n(B) \) is the set of \( B \)-vertex groups of the graph \( \Gamma_n \) constructed in Section 3.2. The next lemma gives a simple description of all vertex groups of \( \Gamma_n \).

Lemma 11. Assume that \( n \geq 1 \). Then, for any \( g \in G \), there is
\[
\begin{array}{ll}
(1) & \tilde{g} \in A_g \text{ such that } g^{-1}Ag \cap G_n = \tilde{g}^{-1}\text{SL}_2(\mathbb{C})\tilde{g} \cap G_n, \\
(2) & \tilde{g} \in B_g \text{ such that } g^{-1}Bg \cap G_n = \tilde{g}^{-1}(T \ltimes G_y)\tilde{g} \cap G_n.
\end{array}
\]
In particular, if \( n \geq 1 \), every \( B \)-vertex group of \( \Gamma_n \) is a solvable subgroup of \( G_n \) of derived length \( \leq 2 \).

Proof. This follows from the fact that each \( A \)- and \( B \)-orbit in \( C_n \) contains a point \((X,Y)\) with \( \text{Tr}(X) = \text{Tr}(Y) = 0 \). Indeed, both \( A \) and \( B \) contain translations, so we can move \((X,Y)\) to \((X - \frac{1}{n}\text{Tr}(X)I, Y - \frac{1}{n}\text{Tr}(Y)I)\) along the orbits. \[\square\]

Proposition 6. Assume that \( n \geq 1 \). Let \( B_g \in V_n(B) \). Then
\[
\begin{array}{ll}
(a) & B_g \text{ is a solvable group of derived length } \leq 2, \\
(b) & N_{G_n}(B_g) = B_g, \\
(c) & B_g \text{ is a maximal solvable subgroup of } G_n.
\end{array}
\]

Proof. (a) By Lemma 11, \( B_g \) is isomorphic to a subgroup of \( T \ltimes G_y \). Since \( T \ltimes G_y \) is solvable of derived length 2, \( B_g \) is solvable of derived length at most 2.
(b) follows from Lemma 10 and the (obvious) fact that \( \text{Stab}_{G_n}(B_g) = B_g \).
(c) Let \( H \) be a solvable subgroup of \( G_n \) containing \( B_g \). Since \( H \) is uncountable, by Theorem 13, it can only be of type I: i.e, conjugate either to a subgroup of \( A \) or a subgroup of \( B \). In the
Corollary 3. Any Borel subgroup of $G_n$ equals $B_g$ for some $g \in G$.

Proof. Suppose $H$ is a Borel subgroup of $G_n$. Then $H$ is a connected solvable subgroup of $G$, and hence, by Theorem 13, it must be of type I or type III. By Proposition 4, it can only be of type I: i.e. conjugate to a subgroup of $A_1$ or $B_1$. In the first case, by Lemma 11(1), it can be conjugated to a subgroup of $SL_2(\mathbb{C})$. In fact, since $H$ is connected and solvable, it can be conjugated to a subgroup of upper triangular matrices: $U_0 = U \cap SL_2(\mathbb{C}) \subseteq U$. In particular, there is $g \in G$ such that $H \subseteq g^{-1}Ug \cap G_n$ which is always a proper subgroup of $g^{-1}Bg \cap G_n$. This contradicts the maximality of $H$. Thus $H$ can only be conjugated to a subgroup of $B$, i.e. there is $g \in G$ such that $H \subseteq g^{-1}Bg \cap G_n$. Since $H$ is a maximal solvable subgroup of $G_n$, we must have $H = g^{-1}Bg \cap G_n = B_g$. □

As a consequence of Theorem 16 and Proposition 6(b), we get the following infinite-dimensional generalization of a well-known theorem of Borel [Bo].

Corollary 4. Any Borel subgroup of $G_n$ equals its normalizer.

Now, let $\mathfrak{B}_n$ denote the set of all Borel subgroups of $G_n$. By Theorem 16, we have a natural inclusion $\iota: \mathfrak{B}_n \hookrightarrow V_n(B)$, which is obviously equivariant with respect to the adjoint action of $G_n$. Taking quotients by this action and combining the induced map of $\iota$ with the inverse of (39), we get

$$\mathfrak{B}_n/Ad G_n \hookrightarrow C_n/B,$$

which is precisely the embedding (1) mentioned in the Introduction. Our aim now is to prove Theorem 2. We begin by recalling the following important fact proved by G. Wilson in [W, Sect. 6].

Theorem 17 ([W]). For each $n$, the variety $C_n$ has exactly $p(n)$ torus-fixed points $(X,Y)$ which are in bijection with the partitions of $n$. These points are characterized by the property that both $X$ and $Y$ are nilpotent matrices.

We will refer to points $(X,Y) \in C_n$, with $X$ and $Y$ being nilpotent matrices, as ‘nilpotent points.’ The next observation is an easy consequence of Theorem 17.

Corollary 4. Let $\Gamma$ be a subgroup of $T$ containing a cyclic group of order $> n$ (possibly infinite). If a point $(X,Y) \in C_n$ is fixed by $\Gamma$ then it is also fixed by $T$.

Proof. If $(X,Y)$ is fixed by $\Gamma$, then $Tr(X^k)$ and $Tr(Y^k)$ vanish for all $k \leq n$. Hence $Tr(X^k) = Tr(Y^k) = 0$ for all $k > 0$. This means that $X$ and $Y$ are both nilpotent matrices, and the claim follows from Theorem 17. □

Next, for each $(X,Y) \in C_n$, we define the following canonical map

$$(41) \quad \chi_{(X,Y)}: Stab_B(X,Y) \hookrightarrow B \twoheadrightarrow B/[B,B].$$

Note that the image of (41) depends only on the $B$-orbit of $(X,Y)$ in $C_n$ (not on the specific representative). The target of (41) plays the role of an ‘abstract’ Cartan subgroup of $G$, which (just as in the finite-dimensional case, cf. [CG, Sect. 3.1]) can be identified with a maximal torus:

$$(42) \quad B/[B,B] \cong T, \quad [(tx + p(y), t^{-1}y + f)] \leftrightarrow (tx, t^{-1}y).$$

In terms of (41), we can give the following useful characterization of $B$-orbits with $T$-fixed points.

Lemma 12. A $B$-orbit of $(X,Y) \in C_n$ contains a $T$-fixed point if and only if the map $\chi_{(X,Y)}$ is surjective. For this, it suffices that the image of $\chi_{(X,Y)}$ contains an element of order $> n$. 
Proof. First, in view of (42), it is obvious that $\chi_{(X,Y)}$ is surjective if $T \subseteq \text{Stab}_B(X,Y)$. For the converse, we will prove the existence of a $T$-fixed point under the assumption that $\chi_{(X,Y)}$ contains an element of order $> n$. By this assumption, there is an element $h = (tx + p(y), t^{-1}y + f) \in \text{Stab}_B(X,Y)$ such that $t$ has order $\geq n + 1$ in $\mathbb{C}^*$. By the Cayley-Hamilton Theorem, we may assume that $\deg p(y) \leq n - 1$. Applying the automorphism $b_1 := (x - \frac{1}{2}\text{Tr}(X), y - \frac{1}{2}\text{Tr}(Y)) \in B$ to $(X,Y)$, we get a point $(X_1, Y_1)$ with $\text{Tr}(X_1) = \text{Tr}(Y_1) = 0$. Hence $b_1 [\text{Stab}_B(X,Y)] b_1^{-1} \subseteq T \ltimes G_y$ and $h_1 = b_1 h b_1^{-1} = (tx + p_1(y), t^{-1}y)$ with $\deg p_1(y) \leq n - 1$. We now show that $h_1$ can be conjugated to $(tx, t^{-1}y)$. Indeed, write $p_1(y) = \sum_{i=0}^{n-1} a_i y^i$ and conjugate
\[
b_2 h_1 b_2^{-1} = (tx + tq(y) - q(t^{-1}y) + p_1(y), t^{-1}y),\]
where $b_2 := (x + q(y), y)$ with $q(y) = \sum_{i=0}^{n-1} c_i y^i \in \mathbb{C}[y]$. Setting
\[
tq(y) - q(t^{-1}y) + p_1(y) = 0,
\]
we get a linear system for the coefficients of $q(y)$ of the form
\[
c_i(t - t^{-1}) = a_i \quad i = 0,\ldots, n - 1.
\]
Hence, if we take $c_i = a_i (t - t^{-1})^{-1}$ for $b_2$ and set $b := b_2 b_1$, then $bh b^{-1} = (tx, t^{-1}y)$. Thus $b [\text{Stab}_B(X,Y)] b^{-1}$ contains $(tx, t^{-1}y)$. By Corollary 4, we now conclude that $b \cdot (X,Y)$ is a nilpotent point, and hence, by Theorem 17, it is $T$-fixed. 

Now, for $(X,Y) \in \mathcal{C}_n$, let $G_y(X,Y)$ denote the stabilizer of $(X,Y)$ in $G_y$. Note that $G_y(X,Y) \subseteq \text{Stab}_B(X,Y)$ for any $(X,Y)$, since $G_y \subseteq B$. The next lemma is a direct consequence of Proposition 8, which is proved in Section 5.6; it shows that all groups $G_y(X,Y)$ are path connected (and hence connected).

Lemma 13. For any $(X,Y) \in \mathcal{C}_n$, if $(x + q(y), y) \in G_y(X,Y)$, then $(x + \lambda q(y), y) \in G_y(X,Y)$ for any $\lambda \in \mathbb{C}$.

We now give a classification of $B$-orbits in $\mathcal{C}_n$ and their isotropy groups.

Proposition 7. For a $B$-orbit $\mathcal{O}_B$ in $\mathcal{C}_n$, one and only one of the following possibilities occurs:

(A) $T$ acts freely on $\mathcal{O}_B$, $\text{Stab}_B(X,Y) = G_y(X,Y)$ and the map $\chi_{(X,Y)}$ is trivial (i.e., its image is 1) for every $(X,Y) \in \mathcal{O}_B$.

(B) $\mathcal{O}_B$ contains a $T$-fixed $(X,Y)$, $\text{Stab}_B(X,Y) = T \ltimes G_y(X,Y)$, and the map of $\chi_{(X,Y)}$ is surjective.

(C) $\mathcal{O}_B$ contains a point $(X,Y)$ such that $\text{Stab}_B(X,Y) = \mathbb{Z}_k \ltimes G_y(X,Y)$ for some $0 < k \leq n$, and the image of $\chi_{(X,Y)}$ is isomorphic to $\mathbb{Z}_k$.

Proof. Let $\mathcal{O}_B$ be a fixed $B$-orbit. For any $(X,Y) \in \mathcal{O}_B$, the character map (41) combined with (42) gives the short exact sequence
\[
1 \rightarrow G_y(X,Y) \rightarrow \text{Stab}_B(X,Y) \rightarrow K \rightarrow 1
\]
where $K$ is the image of $\chi_{(X,Y)}$ in $T$.

If $K = 1$ for some point in $\mathcal{O}_B$, then $K = 1$ for all $(X,Y) \in \mathcal{O}_B$ and hence $\text{Stab}_B(X,Y) = G_y(X,Y)$ for all $(X,Y) \in \mathcal{O}_B$, which means that $T$ acts freely on $\mathcal{O}_B$. This is case (A).

If $K$ contains an element of order $\geq n + 1$ (possibly $\infty$) for some point in $\mathcal{O}_B$, then, by Lemma 12, $\mathcal{O}_B$ contains a $T$-fixed point $(X,Y)$ and $K = T$. Then $\text{Stab}_B(X,Y)$ contains $T$, the above short exact sequence splits, and we have $\text{Stab}_B(X,Y) = \mathbb{Z}_k \ltimes G_y(X,Y)$ for some $0 < k \leq n$. This is case (B).

Finally, assume that neither (A) nor (B) holds. Then, by Lemma 11, there is still a point $(X,Y) \in \mathcal{O}_B$ such that $\text{Stab}_B(X,Y) \subseteq T \ltimes G_y$. By our assumption, the corresponding $K \subseteq T$ must be a cyclic group of order $k$ for $0 < k \leq n$. Let $(\lambda, \lambda^{-1}) \in K$ be the generator of $K$. Write $\phi = (\lambda x + p(y), \lambda^{-1}y)$ for the preimage of $(\lambda, \lambda^{-1})$ in $\text{Stab}_B(X,Y)$. Iterating $\phi$, we get
\[
\phi^k = (x + \sum_{j=1}^{k} \lambda^{k-j}p(\lambda^{1-j}y), y)
\]
Explicitly, if \( p(y) = \sum_{i=0}^{m} a_i y^i \), then the coefficient under \( y^i \) in the first component of \( \phi^k \) is equal to
\[
a_i \sum_{j=1}^{k} \lambda^{-k-j+(1-j)i} = a_i \lambda^i \sum_{j=1}^{k} \lambda^{-j(i+1)}
\]
Since \( \sum_{j=1}^{k} \lambda^{k-j} = 0 \), all these coefficients vanish except those with \( i \equiv -1 \pmod{k} \). Thus \( \phi^k = (x + k p_1(y), y) \), where \( p_1(y) = a_k y^{k-1} + a_{2k-1} y^{2k-1} + \ldots \) is a polynomial obtained from \( p(y) \) by removing all coefficients except those with \( i \equiv -1 \pmod{k} \). Since \( \phi^k \in \text{Stab}_B(X,Y) \), by Lemma 13, \( (x - \lambda^{-1} p_1(y), y) \in \text{Stab}_B(X,Y) \). Hence \( \phi_1 := (lx + p(y) - p_1(y), \lambda^{-1} y) \in \text{Stab}_B(X,Y) \) and \( \phi_1^k = 1 \). Now, the mapping \( (\lambda, \lambda^{-1}) \mapsto \phi_1 \) splits (43). Hence \( \text{Stab}_B(X,Y) = \mathbb{Z}_k \ltimes G_g(X,Y) \), where \( \mathbb{Z}_k \) is generated by \( \phi_1 \). This is case (C).

We are now ready to prove Theorem 2 from the Introduction.

**Proof of Theorem 2.** By Theorem 16, any Borel subgroup of \( G_n \) has the form \( B_g := g^{-1} B g \cap G_n \), while \( B_g = g^{-1} [\text{Stab}_B(X,Y)] g \), where \( (X,Y) = g \cdot (X_0,Y_0) \in \mathcal{C}_n \). Now, by classification of Proposition 7, the group \( \text{Stab}_B(X,Y) \) is connected if and only if the corresponding \( B \)-orbit is of type (A) or type (B). Indeed, in case (A), we have \( \text{Stab}_B(X,Y) = G_g(X,Y) \). Hence, by Lemma 13, \( \text{Stab}_B(X,Y) \) is path connected and therefore connected. Note also that \( \text{Stab}_B(X,Y) \) is abelian, since so is \( G_g(X,Y) \).

In case (B), we may assume that \( \text{Stab}_B(X,Y) = T \ltimes G_g(X,Y) \). Then any element of \( \text{Stab}_B(X,Y) \) can be written in the form \( b = (ax + q(y), a^{-1} y) \), where \( q(y) \in \mathbb{Z}[y] \). By Lemma 13, if \( b \in \text{Stab}_B(X,Y) \) then \( b_1 := (ax + t q(y), a^{-1} y) \in \text{Stab}_B(X,Y) \) for all \( t \in \mathbb{C} \), hence we can join \( b = b_1 \) to \( b_0 = (ax, a^{-1} y) \in T \) within \( \text{Stab}_B(X,Y) \). It follows that \( \text{Stab}_B(X,Y) \) is connected since so is \( T \). Note that in this case, \( \text{Stab}_B(X,Y) \) is a solvable but non-abelian subgroup of \( G \).

In case (C), the group \( \text{Stab}_B(X,Y) \) is obviously disconnected. Hence the corresponding \( B_g \) cannot be a Borel subgroup of \( G_n \).

\[ \square \]

5.5. **Conjugacy classes of non-abelian Borel subgroups.** Following [W], we denote the \( T \)-fixed points of \( \mathcal{C}_n \) by \( (X_\mu,Y_\mu) \), where \( \mu = (n_1, n_2, \ldots, n_k) \) is a partition of \( n \) with \( n_1 \leq n_2 \leq \ldots \leq n_k \). We consider the \( B \)-orbits of these points in \( \mathcal{C}_n \) as vertices of the graph \( \Gamma_n \) defined in Section 3.2. For a fixed collection of elements \( g_\mu \in G \) such that \( g_\mu(X_0,Y_0) = (X_\mu,Y_\mu) \), we define the subgroups \( B_\mu \subset G_n \) by
\[
B_\mu := g_\mu^{-1} B g_\mu \cap G_n.
\]
These are \( B \)-vertex groups attached to the \( B \)-orbits \( B(X_\mu,Y_\mu) \) in \( \Gamma_n \). Geometrically, \( B_\mu \) are the conjugates of subgroups of \( B \) fixing the points \( (X_\mu,Y_\mu) \) in \( \mathcal{C}_n \). More explicitly \( B_\mu = g_\mu^{-1} B(\mu) g_\mu \), where \( B(\mu) := \text{Stab}_B(X_\mu,Y_\mu) \).

As an immediate consequence of Theorem 2, we have

**Corollary 5.** \( B_\mu \) is a Borel subgroup of \( G_n \).

Next, we prove

**Theorem 18.** Any non-abelian Borel subgroup of \( G_n \) is conjugate to some \( B_\mu \).

**Proof.** Suppose \( H \) is a non-abelian Borel subgroup of \( G_n \). By Theorem 16, any Borel group is equal to \( H = B_g \) for some \( g \in G \). Then, by Theorem 2, \( H \) is Borel if either (A) \( T \) acts freely on corresponding \( B \)-orbit or (B) \( T \) has a fixed point on the corresponding \( B \)-orbit. In the first case, \( H \) must be abelian, which contradicts our assumption. In the second case, \( H \) is conjugate to \( T \ltimes G_g(X,Y) \), where \( (X,Y) \) is a nilpotent point. Hence \( H \) is conjugate to \( B_\mu \) for some \( \mu \).

\[ \square \]

**Lemma 14.** \( B_\mu \) contains no proper subgroup of finite index.

**Proof.** Similar to the proof of Lemma 8.

\[ \square \]

Now we are ready to prove Steinberg’s Theorem in full generality.
Proof of Theorem 4. \(\Rightarrow\) Let \(H\) be a Borel subgroup of \(G_n\). Then, by Theorem 18, \(H\) is conjugate to \(B_\mu\). Hence, by Proposition 6(c) and Lemma 14, \(H\) satisfies properties (B1) and (B2) respectively.

\((\Leftarrow)\) Let \(H\) be a subgroup of \(G_n\) satisfying (B1) and (B2). By Theorem 13, it is then either of type I or type III. By Proposition 4, it cannot be of type III. Therefore, it is conjugate to either a subgroup of \(A\) or a subgroup of \(B\). Suppose that it is conjugate to a subgroup of \(A\). The image of \(g^{-1}Hg \to A \to \text{SL}_2(\mathbb{C})\) is then a solvable subgroup of \(\text{SL}_2(\mathbb{C})\). We denote this group by \(S\). By Theorem 14, \(S\) has a finite index normal subgroup \(T\), which is a subgroup of upper triangular matrices in \(\text{SL}_2(\mathbb{C})\). By Lemma 7(b), the group \(S\), being a homomorphic image of \(H\), contains no proper subgroup of finite index. Thus \(S = T\) and \(H\) is conjugate to a subgroup of \(T\) consisting of upper triangular matrices in \(\text{SL}_2(\mathbb{C})\).

\(\Rightarrow\) We will prove that the subgroups \(B_\mu\) are pairwise non-conjugate in \(G_n\). We begin with the following lemma, the result of which is implicit in [W].

Lemma 15. The nilpotent points \((X_\mu, Y_\mu)\) in \(C_n\) belong to distinct \(B\)-orbits.

Proof. Consider the subgroup \(B_0\) consisting of the automorphisms \((x + p(y), y) \in G\) with \(p(0) = 0\). It is easy to see that any two nilpotent points are in the same \(B\)-orbit if they are in the same \(B_0\)-orbit. Indeed, \(T\) fixes each of the nilpotent points, hence does not contribute to the \(B\)-orbit. On the other hand, applying an automorphism with nonzero constant terms to a nilpotent point moves it to a point with a nonzero trace, which is not nilpotent. Therefore we will only consider orbits of \(B_0\). By [W, Proposition 6.11], the points \((X_\mu, Y_\mu)\) are exactly the centers of distinct \(n\)-dimensional cells in \(C_n\) which have pairwise empty intersection. Now, if we show that these cells contain the \(B_0\)-orbits of \((X_\mu, Y_\mu)\), the result will follow. We start by looking at the simplest case the point corresponding with partition: \(\mu = \mu(n, r)\) where \(\mu(n, r) = (1, \ldots, 1, n - r + 1)\). In this case \((X_\mu, Y_\mu)\) is given by

\[
X_\mu = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & a_1 & 0 & \ldots & 0 \\
0 & a_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & a_{n-1} & 0
\end{pmatrix},
Y_\mu = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 1 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

where \((a_1, \ldots, a_{n-1}) = (1, 2, \ldots, r - 1; -(n - r), \ldots, -2, -1)\). Then, the \(B_0\)-orbit of \((X_\mu, Y_\mu)\) consists of the points \((X, Y_\mu)\), where \(X = X_\mu + \sum_{k=1}^{n-1} X^{(k)}\) with matrices \(X^{(k)}\) having nonzero terms only on the \(k\)-th diagonal. Applying a transformation \(Q_t\), which is essentially a scaling transformation followed by conjugation by \(\text{diag}(1, t, \ldots, t^{n-1})\) (see [W, (6.5)]), we obtain

\[
Q_t(X, Y_\mu) = (X_\mu + \sum_{k=1}^{n-1} t^{-k-1} X^{(k)}, Y_\mu)
\]

As \(t \to \infty\), we see that \(Q_t(X, Y_\mu) \to (X_\mu, Y_\mu)\), hence \((X, Y_\mu)\) is still in a cell with the center \((X_\mu, Y_\mu)\).

More generally, consider the partition \(\mu = \mu(n_1, r_1, \ldots, n_k, r_k)\) which corresponds to the Young diagram with one-hook partitions \((1, \ldots, 1, n_1 - r_1 + 1)\) placed inside each other; such that neither the arm nor the leg of any hook is allowed to poke out beyond the preceding one. In this case, \(Y_\mu = \bigoplus_{i=1}^k J(n_i)\) as a sum of several nilpotent Jordan blocks of dimensions \(n_i\); and \(X_\mu\) is a block matrix consisting of the diagonal blocks \(X_{ij} = X_{1,1,1,n_i-r_i+1}\) described as in the previous paragraph and certain (unique) matrices \(X_{ij}\) with non-zero entries only on the \((r_j - r_i - 1)\)-th
diagonal. The $B_0$-orbit of $(X_\mu, Y_\mu)$ then consists of the points $(\tilde{X}, Y_\mu)$, where $\tilde{X}_{ij} = X_{ij}$ for $i \neq j$ and $\tilde{X}_{ii} = X_{ii} + \sum_{k=1}^{n-1} X^{(i,k)}$ is the sum of matrices $X^{(i,k)}$ with only nonzero terms on the $k$th diagonal of the corresponding block matrix. Once again, looking at $Q_t(\tilde{X}, Y_\mu)$ one can easily show that the diagonal blocks $\tilde{X}_{ii}$ flow to $X_{ii}$ as $t \to \infty$. On the other hand, the only non-zero diagonal of $\tilde{X}_{ij}$ is the $(r_j - r_i - 1)$-th diagonal, counting within the $(i,j)$-block; or, if we count diagonals inside the big matrix $\tilde{X}$, it is the one with number $q_j - q_i - 1$, where

$$q_i := n_1 + \ldots + n_{i-1} + r_i .$$

Thus, the map $Q_t$ multiplies the non-zero diagonal of $\tilde{X}_{ij}$ by $t^{q_j-q_i}$. If we now conjugate by the block-scalar matrix $\oplus t^{-q_j} I_{n_j}$, then the $(i,j)$-block gets multiplies by $t^{q_i-q_j}$, so we get $X_{ij}$. Thus, summing up, we obtain that $Q_t(\tilde{X}, Y_\mu) \to (X_\mu, Y_\mu)$ as $t \to \infty$, hence the corresponding $B_0$-orbit is in the cell.

**Theorem 19.** The subgroups $B_\mu$ are pairwise non-conjugate in $G_n$, i.e. there is no $g \in G_n$ such that $g^{-1}B_\mu g = B_\lambda$ unless $\mu = \lambda$.

**Proof.** This is a consequence of Lemma 10 (see (39)) and Lemma 15. \hfill \Box

Now, we can prove Theorem 3 and Corollary 1 stated in the Introduction.

**Proof of Theorem 3.** Combine Theorem 18 and Theorem 19. \hfill \Box

**Proof of Corollary 1.** Suppose that there exists an (abstract) group isomorphism $G_k \cong G_n$ for some $k$ and $n$. Then, by Theorem 4, it must induce a bijection between the sets of conjugacy classes of non-abelian Borel subgroups in $G_k$ and $G_n$. By Theorem 3, these sets are finite sets consisting of $p(k)$ and $p(n)$ elements. Hence $p(k) = p(n)$ and therefore $k = n$. \hfill \Box

### 5.6. Adelic construction of Borel subgroups

We conclude this section by giving an explicit description of the special subgroups $B(\mu)$. To this end we will use an infinite-dimensional *adelic Grassmannian* $\text{Gr}^{ad}$ introduced in [W1]. We recall that $\text{Gr}^{ad}$ is the space parametrizing all primary decomposable subspaces of $\mathbb{C}[z]$ modulo rational equivalence. To be precise, a subspace $W \subseteq \mathbb{C}[z]$ is called primary decomposable if there is a finite collection of points $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \subseteq \mathbb{C}$ such that $W = \bigcap_{i=1}^{N} W_{\lambda_i}$, where $W_{\lambda}$ is a $\lambda$-primary (i.e., containing a power of the maximal ideal $m_\lambda$) subspace of $\mathbb{C}[z]$. Two such subspaces, say $W$ and $W'$, are (rationally) equivalent if $pW = qW'$ for some polynomials $p$ and $q$. Every equivalence class $[W] \in \text{Gr}^{ad}$ contains a unique irreducible subspace, which is characterized by the property that it is not contained in a proper ideal of $\mathbb{C}[z]$. We may therefore identify $\text{Gr}^{ad}$ with the set of irreducible primary decomposable subspaces in $\mathbb{C}[z]$.

Now, by $[W]$ and $[BW]$, there is a natural bijection $\beta : \bigsqcup_{n \geq 0} \mathcal{C}_n \xrightarrow{\sim} \text{Gr}^{ad}$, which is equivariant under $G$. It is not easy to describe the action of the full group $G$ on $\text{Gr}^{ad}$; however, for our purposes, it will suffice to know the action of its subgroup $G_y$, which is not difficult to describe. We will use the construction of the action of $G_y$ on $\text{Gr}^{ad}$ given in [BW] (where $G_y$ is denoted by $\Gamma$).

Let $\mathcal{H}$ denote the space of entire analytic functions on $\mathbb{C}$ equipped with its usual topology (uniform convergence on compact subsets). Given a subspace $W \subseteq \mathbb{C}[z]$ we write $\overline{\mathcal{W}} \subseteq \mathcal{H}$ for its completion in $\mathcal{H}$, and conversely, given a closed subspace $W \subseteq \mathcal{H}$ we set $\mathcal{W}^{alg} := W \cap \mathbb{C}[z]$. Then, for any $q \in \mathbb{C}[z]$, we define

$$e^q : W := (e^q \overline{W})^{alg}$$

The action of $G_y$ under $\beta$ transfers to $\text{Gr}^{ad}$ as follows (see [BW, Sect. 10]): if $W = \beta(X, Y) \in \text{Gr}^{ad}$ then

$$e^q : W = \beta(X + q'(Y), Y), \quad \forall q' \in \mathbb{C}[z] .$$

Now, for any $W \in \text{Gr}^{ad}$, put

$$A_W := \{ q \in \mathbb{C}[z] : qW \subseteq \overline{W} \} .$$

Clearly $A_W$ is a commutative algebra, $W$ being a finite module over $A_W$. Geometrically, $A_W$ is the coordinate ring of a rational curve $X = \text{Spec}(A_W)$, on which $W$ defines a (maximal) rank 1
The inclusion \( A_W \hookrightarrow \mathbb{C}[z] \) gives normalization \( \pi : \mathbb{A}^1 \to X \) (which is set-theoretically a bijective map). In this way, \( \text{Gr}^{ad} \) parametrizes the isomorphism classes of triples \( (\pi, X, \mathcal{L}) \) (see [W1]).

**Proposition 8.** For any \( W \in \text{Gr}^{ad} \), \( \text{Stab}_{G_u}(W) = \{ (x + q'(y), y) \in G : q \in A_W \} \).

**Proof.** By [BW, Lemma 2.1] and the above discussion, the claim is equivalent to

\[
A_W = \{ q \in \mathbb{C}[z] : e^q W = W \}.
\]

The inclusion \( ' \subset ' \) is easy: if \( q \in A_W \) then \( q^n W \subset W \) for all \( n \in \mathbb{N} \), hence \( e^q W \subset W \) and therefore \( e^q W = W \).

To prove the other inclusion it is convenient to use the ‘dual’ description of \( \text{Gr}^{ad} \) in terms of algebraic distributions (see [W1]). To this end assume that \( W \) is supported on \( \{ \lambda_1, \lambda_2, \ldots, \lambda_N \} \subset \mathbb{C} \). Then, for each \( \lambda_i \in \text{supp}(W) \), there is a finite-dimensional subspace \( W^*_{\lambda_i} \) of linear functionals on \( H \) supported at \( \lambda_i \) such that

\[
W = \{ f \in \mathbb{C}[z] : \langle \varphi_i, f \rangle = 0 \text{ for all } \varphi_i \in W^*_{\lambda_i} \text{ and for all } i = 1, 2, \ldots, N \}.
\]

By [BW, Lemma 2.1], we then also have

\[
W = \{ f \in H : \langle \varphi_i, f \rangle = 0 \text{ for all } \varphi_i \in W^*_{\lambda_i} \text{ and for all } i = 1, 2, \ldots, N \}.
\]

Now, suppose that \( e^q W = W \) for some \( q \in \mathbb{C}[z] \). Then \( e^{qW} W = W \) for all \( t \in \mathbb{C} \). Indeed, for fixed \( \varphi_i \in W^*_{\lambda_i} \) and \( f \in W \), the function \( P(t) := \langle \varphi_i, e^{qt} f \rangle \) is obviously a quasi-polynomial in \( t \) of the form \( P(t) = p(t) e^{q \delta(t)} \), where \( p(t) \in \mathbb{C}[t] \). Since \( e^{qW} W = W \) implies \( e^{qW} W = W \) for all \( k \in \mathbb{Z} \), we have \( P(k) = 0 \) and hence \( p(k) = 0 \) for all \( k \in \mathbb{Z} \). This implies \( P(t) \equiv 0 \). In particular, we have \( P'(0) = \langle \varphi_i, q f \rangle = 0 \). Since this equality holds for all \( \varphi \in W^*_{\lambda_i} \), for all \( i \) and for all \( f \in W \), we conclude \( q W \subset W \). Thus \( q \in A_W \).

Now, let \( (X_\mu, Y_\mu) \) be the \( T \)-fixed point of \( C_\mu \) corresponding to a partition \( \mu = \{ n_1 \leq n_2 \leq \cdots \leq n_k \} \). Then, the corresponding (irreducible) primary decomposable subspace of \( \text{Gr}^{ad} \) is given by

\[
W_\mu = \text{span}\{1, x^{r_1}, x^{r_2}, x^{r_3}, \ldots \},
\]

where \( r_i = i + n_k - n_{k-i} \) (with convention \( n_j = 0 \) for \( j < 0 \)). Write \( R_\mu := \{ r_0 = 1, r_1, r_2, \ldots \} \) for the set of exponents of monomials occurring in \( W_\mu \), and denote by \( S_\mu := \{ k \in \mathbb{N} : k + R_\mu \subset R_\mu \} \) the subsemigroup of \( \mathbb{N} \) preserving \( R_\mu \). Then \( A_{W_\mu} = \text{span}\{x^s : s \in S_\mu\} \), and as a consequence of Proposition 8, we get

**Corollary 6.** For any partition \( \mu \), \( B(\mu) = T \ltimes G_{\mu,y} \), where \( G_{\mu,y} \) is the subgroup of \( G_y \) generated by the transformations \( \{ (x + \lambda y a^{-1}, y) : s \in S_\mu, \lambda \in \mathbb{C} \} \).

To illustrate Corollary 6, we list below all special Borel subgroups of \( G_n \) for \( n = 1, 2, 3, 4 \).

5.6.1. **Examples.** For \( n = 1 \), there is only one \( T \)-fixed point \((0,0) \in C_1 \) and the corresponding Borel subgroup is

\[
B(1) = T \ltimes \{ \Psi_{cy^k} \mid c \in \mathbb{C}, k \geq 1 \} = \{(ax + cy^k, a^{-1}y) \mid a \in \mathbb{C}^*, c \in \mathbb{C}, k \geq 1 \}.
\]

For \( n = 2 \), the fixed points are \((X_2, Y_2)\) and \((X_{1,1}, Y_{1,1})\), where

\[
X_2 := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad X_{1,1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y_2 = Y_{1,1} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

The corresponding Borel subgroups are given by

\[
B(2) = T \ltimes \{ \Psi_{cy^k} \mid c \in \mathbb{C}, k \geq 2 \}, \quad B(1,1) = T \ltimes \{ \Phi_{cx^k} \mid c \in \mathbb{C}, k \geq 2 \}
\]

\(^5\text{Note that the elements of } W^*_{\lambda_i} \text{ can be written as } \varphi_i = \sum c_k \delta^{(k)}(z - \lambda_i), \text{ where } \delta^{(k)}(z - \lambda_i) \text{ are the derivatives of the } \delta \text{-function with support at } \lambda_i.\)
For \( n = 3 \), the fixed points are

\[
X(3) = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_{(1,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad X_{(1,2)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}
\]

and

\[
Y_{(3)} = Y_{(1,1)} = Y_{(1,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

The corresponding Borel subgroups are given by

\[
B_{(3)} = T \ltimes \{ \Psi_{c y^k} \mid c \in \mathbb{C}, k \geq 3 \},
\]

\[
B_{(1,1,1)} = T \ltimes \{ \Phi_{c x^k} \mid c \in \mathbb{C}, k \geq 3 \},
\]

\[
B_{(1,2)} = \Psi_{-y^2} \Phi_{-x^2} \Psi_{-2y^2} B(1,2) \Psi_{2y^2} \Phi_{x^2} \Psi_{y^2},
\]

where

\[
B(1,2) := T \ltimes \{ \Psi_{q(y)} \mid q(y) \in \mathbb{C}y + y^3\mathbb{C}[y] \}
\]

For \( n = 4 \), there are five fixed points:

\[
X_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad X_{(1,3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad X_{(1,1,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\]

\[
X_{(2,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -3 & 0 & 0 & 0 \end{pmatrix}, \quad X_{(1,1,1,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}
\]

\[
Y_{(2,2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_\mu = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

where \( \mu = \{(4), (1,3), (1,1,2), (1,1,1,1)\} \). The corresponding Borel subgroups are

\[
B_{(4)} = B(4), \quad B_{(1,3)} = \Psi_{-y^3} \Phi_{x^3} \Psi_{-3y^3} B(1,3) \Psi_{3y^3} \Phi_{-x^3} \Psi_{y^3},
\]

\[
B_{(1,1,2)} = \Psi_{-y^3} \Phi_{x^3} \Psi_{3y^3} B(1,1,2) \Psi_{-3y^3} \Phi_{-x^3} \Psi_{y^3}
\]

\[
B_{(1,1,1,1)} = \Psi_{-y^3} \Phi_{x^3} \Psi_{-y^3} B(1,1,1,1) \Psi_{y^3} \Phi_{-x^3} \Psi_{y^3}
\]

\[
B_{(2,2)} = \Psi_{-y^3} \Phi_{-x^2} \Psi_{-2y^2} B(2,2) \Psi_{2y^2} \Phi_{x^2} \Psi_{y^2},
\]

where

\[
B(4) = T \ltimes \{ \Psi_{q(y)} \mid q(y) \in \mathbb{C}y^4[y] \},
\]

\[
B(1,3) = T \ltimes \{ \Psi_{q(y)} \mid q(y) \in \mathbb{C}y^2 + y^4\mathbb{C}[y] \},
\]

\[
B(1,1,2) = T \ltimes \{ \Psi_{q(y)} \mid q(y) \in \mathbb{C}y^2 + y^4\mathbb{C}[y] \},
\]

\[
B(1,1,1,1) = T \ltimes \{ \Psi_{q(y)} \mid q(y) \in \mathbb{C}y^4[y] \},
\]

\[
B(2,2) = T \ltimes \{ \Psi_{q(y)} \mid q(y) \in \mathbb{C}y^3[y] \}.
\]
[Ko] M. Kouakou, *Isomorphismes entre algèbres d'opérateurs différentiels sur les courbes algébriques affines*, Thèse de Doctorat, Université Claude Bernard-Lyon I, 1994.

[Ku] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Mathematics 204, Birkhäuser, Boston, 2002.

[L] S. Lamy, *L’alternative de Tits pour Aut[C²]*, J. Algebra 239(2) (2001), 413–437.

[Li] S. Lamy, *Dynamique des groupes paraboliques d’automorphismes polynomiaux de C²*, Bol. Soc. Brasil. Mat. (N.S.) 32(2) (2001), 185–212.

[LR] J. C. Lennox and D. Robinson, *The Theory of Infinite Soluble Groups*, Oxford University Press, Oxford, 2004.

[Le] G. Letzter, *Non-isomorphic curves with isomorphic rings of differential operators*, J. London Math. Soc. 45(2) (1992), 17–31.

[Lm] G. Letzter and L. Makar-Limanov, *Rings of differential operators over rational affine curves*, Bull. Soc. Math. France 118 (1990), 193–209.

[ML1] L. Makar-Limanov, *Automorphisms of a free algebra with two generators*, Funct. Anal. Appl. 4 (1971), 262–264.

[ML2] L. Makar-Limanov, *On automorphisms of the Weyl algebra*, Bull. Soc. Math. France 112 (1984), 359–363.

[M] J. P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press, Chicago and London, 1999.

[N] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Rhode Island, 1999.

[N1] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. 76 (1994), 365–416.

[O] E. Ordman, *On subgroups of amalgamated free products*, Proc. Camb. Phil. Soc. 69 (1971), 13–23.

[P] V. L. Popov, *Open problems in: Affine Algebraic Geometry*, Contemp. Math. 369, Amer. Math. Soc., Providence, RI, 2005, pp. 12–16.

[PI] V. L. Popov, *On infinite-dimensional algebraic transformation groups*, Transform. Groups, 19(2) (2014), 549–568.

[Se] J.-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980.

[Sh] I. R. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. e Appl. (5) 25 (1966), 208–212.

[Sh1] I. R. Shafarevich, *On some infinite-dimensional groups II*, Math. USSR Izv. 18 (1982), 214–226.

[SS] S. P. Smith and J. T. Stafford, *Differential operators on an affine curve*, Proc. London Math. Soc. (3) 56 (1988), 229–259.

[SV] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan Algebras and Exceptional Groups*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[Sp] T. A. Springer, *Linear Algebraic Groups*, 2nd Edition, Progr. Math. 9, Birkhäuser, Boston, 1998.

[S] J. T. Stafford, *Endomorphisms of right ideals of the Weyl algebra*, Trans. Amer. Math. Soc. 299 (1987), 623–639.

[St] R. Steinberg, *Abstract homomorphisms of simple algebraic groups (after A. Borel and J. Tits)*, Séminaire Bourbaki, 25ème année (1972/1973), Exp. No. 435. Lecture Notes in Math. 383, Springer, Berlin, 1974, pp. 307–326.

[vdK] W. van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wisk. 1 (3) (1953), 33–41.

[W] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian* (with an Appendix by I. G. Macdonald), Invent. Math. 133 (1998), 1–41.

[WI] G. Wilson, *Bispectral commutative ordinary differential operators*, J. reine angew. Math. 442 (1993), 177–204.

[Wr] D. Wright, *Abelian subgroups of Aut_k(k[X,Y]) and applications to actions on the affine plane*, Ill. J. Math. 23 (1979), 579–634.

DEPARTMENT OF MATHEMATICS, Cornell University, Ithaca, NY 14853-4201, USA
E-mail address: berest@math.cornell.edu

DEPARTMENT OF MATHEMATICS, University of Western Ontario, London, Ontario N6A 5B7, Canada
E-mail address: aeshmato@uwo.ca

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, P.O.Box: 7280 53072, BONN, GERMANY
Current address: School of Mathematics, Sichuan University, Chengdu 610064, China
E-mail address: olimjon55@hotmail.com