Running coupling effects for the singlet structure function $g_1$ at small $x$

B.I. Ermolaev
CFTC, University of Lisbon, Av. Prof. Gama Pinto 2, P-1649-003 Lisbon, Portugal

M. Greco
Dept of Physics and INFN, University Rome III, Rome, Italy

S.I. Troyan
St.Petersburg Institute of Nuclear Physics, 188300 Gatchina, Russia

The running of the QCD coupling is incorporated into the infrared evolution equations for the flavour structure function $g_1$. The explicit expressions for $g_1$ including the total resummation of the double-logarithmic contributions and accounting for the running coupling are obtained. We predict that asymptotically $g_1 \sim x^{-\Delta S}$, with the intercept $\Delta_S = 0.86$, which is more than twice larger than the non-singlet intercept $\Delta_{NS} = 0.4$. The impact of the initial quark ($\delta q$) and gluon ($\delta g$) densities on the sign of $g_1$ at $x \ll 1$ is discussed and explicit expressions relating $\delta q$ and $\delta g$ are obtained.

PACS numbers: 12.38.Cy

I. INTRODUCTION

The flavour singlet structure function $g_1$ has been the object of intensive theoretical investigations. First $g_1$ was calculated in Refs. [1, 2], where the LO DGLAP evolution equations [2, 3] were used. Since that, the DGLAP approach has become the standard instrument for the theoretical description of the singlet and non-singlet components of $g_1$ and provides good agreement between experimental and theoretical results (see e.g. Ref. [4] and the recent review [5]). On the other hand, despite this agreement, it is known that from a theoretical point of view the DGLAP approximation (DLA) and it was shown that the DL intercepts $\Delta$ of $g_1$ are fixed, whereas, as well known, the running coupling effects are relevant. As a matter of fact, these contributions become very important in the small $x$ region and should be accounted for to all orders in $\alpha_s$. Such total resummation was done in Refs. [7, 8] for the non-singlet component, $g_{1NS}$, of $g_1$ and in Ref. [9] for the singlet one. These calculations were done in the double-logarithmic approximation (DLA) and it was shown that $g_1$ has the power-like (Regge) behaviour $\sim x^{-\Delta}$ when $x \rightarrow 0$. However, the QCD coupling in Refs. [5, 6] is kept fixed, whereas, as well known, the running coupling effects are relevant. As the DL intercepts $\Delta$ of $g_1$ obtained in Refs. [6, 8] are proportional to $\sqrt{\alpha_s}$ fixed at an unknown scale, it makes the results of Refs. [6, 8] unclear and not suitable for practical use. A parametrisation for fixed $\alpha_s = \alpha_s(Q^2)$ for both the singlet and non-singlet component was suggested in Refs. [10]-[14]. In the DGLAP framework, in a general ladder rung $\alpha_s$ depends on the transverse momentum $k_\perp$ of the ladder parton (a quark or a gluon), with $\mu^2 < k_\perp^2 < Q^2$, where $\mu^2$ is the starting point of the $Q^2$-evolution. The arguments in favour of such a dependence were given in Ref. [6]. However, in Ref. [15] it was shown that the arguments of Ref. [6] are valid only when $x$ is large ($x \sim 1$), while on the contrary in the small-$x$ region, in a general ladder rung $\alpha_s$ depends on the virtuality of the “horizontal” gluon, $(k_i - k_{i-1})^2$. This dependence was used in Refs. [16] for studying $g_{1NS}$ at small $x$, with running $\alpha_s$ accounted for. Our prediction for the non-singlet intercept was confirmed by phenomenological analyses of the experimental data in Refs. [15].

In the present paper we apply the same arguments of Ref. [15] in order to account for the running $\alpha_s$ effects for the description of the small-$x$ behaviour of the flavour singlet component of $g_1$. The paper is organised as follows: In Sect. 2 we construct and solve the system of the infrared evolution eqs. for $g_1$. In Sect. 3 we obtain the explicit expressions for the anomalous dimensions of $g_1$. These expressions account for all DL contributions and also for running $\alpha_s$. The intercept of $g_1$ is calculated in Sect. 4. In Sect. 5, we specify the general solutions obtained in Sect. 2 through the phenomenological quark and gluon inputs and estimate the sign of $g_1$ at $x \ll 1$. Then we calculate the

*Permanent Address: Ioffe Physico-Technical Institute, 194021 St.Petersburg, Russia
II. IREE FOR THE STRUCTURE FUNCTION $g_1$

In order to obtain $g_1$ with all DL contributions accounted for at small $x$, we cannot use the DGLAP equations. Instead of that, we construct for it a set of infrared evolution equations (IREE), i.e. equations for the evolution with respect to the infrared cut-off $\mu$. Instead of that, we construct for it a set of infrared evolution equations (IREE), i.e. equations for the evolution with respect to both $Q^2$ and $x$. In contrast to the DGLAP equations where only $Q^2$-evolution is studied, the IREE are two-dimensional. In order to derive these IREE, it is convenient to operate with the spin-dependent invariant amplitude $M_q$ which can be extracted from the forward Compton scattering amplitude $M_{\mu\nu}$ by using the projection operator $i\epsilon_{\mu\nu\lambda\sigma}q_{\lambda}p_{\sigma}/pq$. Through this paper we use the standard notations, so that $g_\lambda$ is the momentum of the off-shell photon ($-q^2 = Q^2 \geq \mu^2$) and $p_\mu$ is the momentum of the (nearly) on-shell quark. When $M_q(s,Q^2)$ is obtained, the polarized quark contribution to $g_1,g_q$, can be easily found:

$$g_q(x,Q^2) = \frac{1}{\pi} \Im_s M_q(s,Q^2)$$

(1)

where $s$ is the standard Mandelstam variable, $s \approx 2pq$. The subscript $q$ in the rhs of Eq. (1) means that the off-shell photon is scattered by the quark. It turns out that the IREE for $M_q(x,Q^2)$ involve the spin-dependent invariant amplitude, $M_q(s,Q^2)$ of the forward Compton scattering where the off-shell photon is scattered by a nearly on-shell gluon with momentum $p$. Similarly to Eq. (1), the polarized gluon contribution, $g_g$ is related to $M_g$ as follows:

$$g_g(x,Q^2) = \frac{1}{\pi} \Im_s M_g(s,Q^2).$$

(2)

Therefore, in our notations

$$g_1(x,Q^2) = g_q(x,Q^2) + g_g(x,Q^2).$$

(3)

There is no difference between our approach and DGLAP in this respect. The main difference between the IREE and the DGLAP equations is that due to the $k_\perp$-ordering which is the key point of DGLAP, the ladder partons with minimal transverse momenta should always be in the lowest rung. However, such an assumption is an approximation which is true only for large $x$. When $x$ is small, such partons can be in any rung in the ladder. More generally, each rung consists of two ladder partons which can be either quarks or gluons. Each of these two cases yields DL contributions and therefore have to be accounted for. It is convenient to write down the IREE in the $\omega$-space related to the momentum space through the asymptotic form of the Sommerfeld-Watson (SW) transform for the scattering amplitudes. However, the SW transform is defined for the signature amplitudes $M(\pm)$. In particular for the negative signature amplitudes it reads (we drop the superscript "(-)" as we do not discuss the positive signature amplitudes in the present paper):

$$M_r(s,Q^2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (s/\mu^2)^{\omega} \xi(\omega) F_r(\omega,Q^2)$$

(4)

where $r = q,g$ and $\xi(\omega)$ is the negative signature factor, $\xi(\omega) = [1 - e^{-i\pi\omega}] / 2 \approx i\pi\omega / 2$. It is necessary to note that the transform inverse to Eq. (4) involves the imaginary parts of $M_r$:

$$F_r(\omega,Q^2) = \frac{2}{\pi\omega} \int_{0}^{\infty} d\rho \rho e^{-\rho\omega} \Im M_r(s,Q^2)$$

(5)

where we have used the logarithmic variable $\rho = \ln(s/\mu^2)$. We will also use another logarithmic variable $y = \ln(Q^2/\mu^2)$. In these terms, the system of IREE for $F_r(\omega,Q^2)$ can be written down as follows:

$$(\omega + \frac{\partial}{\partial y}) F_q(\omega,y) = \frac{1}{8\pi^2} \left[ F_{qq}(\omega) F_q(\omega,y) + F_{qg}(\omega) F_g(\omega,y) \right],$$

$$(\omega + \frac{\partial}{\partial y}) F_g(\omega,y) = \frac{1}{8\pi^2} \left[ F_{qq}(\omega) F_q(\omega,y) + F_{gg}(\omega) F_g(\omega,y) \right]$$

(6)
where the anomalous dimensions $F_{ik}$, with $i, k = q, g$, correspond to the forward amplitudes for quark and/or gluon scattering, having used the standard DGLAP notations for the subscripts. It is convenient to absorb factors $1/8\pi^2$ in Eqs. (6) into the definition of new amplitudes $H_{ik}$ related to $F_{ik}$ by

$$H_{ik}(\omega) = (1/8\pi^2)F_{ik}(\omega).$$

(7)

The lhs of Eqs. (6) corresponds to apply the operator $-\mu^2(\partial/\partial\mu^2)$ to $M_k$. The first (second) term in the rhs of each equation (4) corresponds to the case when the ladder rung with minimal transverse momentum is made of a quark (gluon) pair. The Born contribution does not appear in the rhs of Eqs. (6) because the Born amplitude $M_{ik}^{Born} = 0$ and $M_{qg}^{Born} = e_q^2 s/(s - Q^2 + i\epsilon)$ and therefore it vanishes when differentiating with respect to $\mu$. The main difference between the system of the Altarelli-Parisi (AP) eqs. for the singlet and the ladder contributions involving $H_{ik}$ contain NLO terms with two-loop accuracy, whereas $H_{ik}$ in Eqs. (6) account for all NLO terms in DLA. Then, these NLO terms cannot be included within the DGLAP approach whereas in our approach we take into account in Sect. 3, using $\mu^2$-evolution. Solving Eqs. (9) and using Eqs. (11,12), we arrive at the following expressions for $g_q$ and $g_g$:

$$g_q(x, Q^2) = \int_{-1}^{1} \frac{d\omega}{2\pi}\frac{1}{(1/\omega)\omega}\left[C_+(\omega)e^{\Omega_+} + C_-(\omega)e^{\Omega_-}\right].$$

(8)

$$g_g(x, Q^2) = \int_{-1}^{1} \frac{d\omega}{2\pi}\frac{1}{(1/\omega)\omega}\left[C_+(\omega)X + \sqrt{R}e^{\Omega_+} + C_-(\omega)X - \sqrt{R}e^{\Omega_-}\right].$$

(9)

Factors $C_{\pm}(\omega)$ will be specified in Sect. IV. We have denoted here

$$X = H_{gg} - H_{qq},$$

(10)

$$R = (H_{gg} - H_{qq})^2 + 4H_{qg}H_{gg}.$$  

(11)

and

$$\Omega_{\pm} = \frac{1}{2} \left[H_{qq} + H_{gg} \pm \sqrt{(H_{qq} - H_{gg})^2 + 4H_{qg}H_{gg}}\right].$$

(12)

Obviously, $\Omega_+ > \Omega_-$. Eq. (9) includes the coefficient functions $C_\pm(\omega)$ that should be specified. This can be done by different ways. Below we obtain $C_\pm$ in terms of the phenomenological quark and gluon inputs and then calculate the perturbative contributions to these inputs. However before doing so, we first calculate the anomalous dimensions $H_{ik}$, $(i, k = q, g)$.

III. CALCULATING $H_{ik}$

As explicitly shown in eq. (12), $\Omega_\pm$ are expressed in terms of the amplitudes $H_{ik}$. As we are going to calculate $\Omega_\pm$ to all orders, we consequently have to know $H_{ik}$ to all orders in $\alpha_s$. To this aim we write and solve some IREE for $H_{ik}$. The lhs of the new IREE again corresponds to differentiating the amplitudes with respect to $-\mu^2(\partial/\partial\mu^2)$ (cf Eqs. (6)). The rhs include, besides the ladder contributions involving $H_{ik}$, their Born terms

$$H_{ik}^{Born} = a_{ik}/\omega,$$

(13)

and non-ladder contributions $V_{ik}$, which we specify later. The system of eqs. reads

$$\omega H_{qq} = a_{qq} + V_{qq} + H_{qq}^2 + H_{qg}H_{gg},$$

$$\omega H_{gg} = a_{gg} + V_{gg} + H_{gg}^2 + H_{qg}H_{qq},$$

$$\omega H_{qg} = a_{qg} + V_{qg} + H_{qg}(H_{qq} + H_{gg}),$$

$$\omega H_{gq} = a_{gq} + V_{gq} + H_{gq}(H_{qq} + H_{gg}).$$

(14)
The non-ladder DL terms $V_{ik}$ appear in Eqs. (14) when the parton with minimal transverse momentum is a non-ladder gluon. Such a gluon can be factorized, i.e. its propagator is attached to external lines. As the factorized gluon bears a colour quantum number, the remaining DL contribution -defined $H^c$ for the sake of simplicity - gets also a coloured content. When the factorized gluon propagates in the $t$-channel, $H^c$ belongs to the octet $t$-channel representation of $SU(3)$, whereas all $H_{ik}$ belong to the singlet representation. Therefore in order to solve Eqs. (14) one has to calculate the octet amplitudes first. Fortunately, they can be approximated quite well by their Born values as they fall with energy very rapidly. With this approximation one obtains

$$V_{ik} = \frac{m_{ik}}{\pi^2} D(\omega), \quad (15)$$

where

$$m_{qq} = \frac{C_F}{2N}, \quad m_{gg} = -2N^2, \quad m_{qg} = n_f \frac{N}{2}, \quad m_{gq} = -NC_F. \quad (16)$$

where $n_f$ is the number of the flavours. We assume $n_f = 4$. Furthermore $D(\omega)$ in Eq. (15) accounts for the running QCD effects for $V_{ik}$. According to Ref. [15] it is given by

$$D(\omega) = \frac{1}{2b^2} \int_0^\infty d\rho e^{-\omega\rho} \ln \left( \frac{\rho + \eta}{(\rho + \eta)^2 + \pi^2} + \frac{1}{\rho + \eta} \right) \quad (17)$$

where $\eta = \ln(\mu^2/\Lambda^2_{QCD})$ and $b = (33 - 2n_f)/12\pi$.

All $a_{ik}$ in Eqs. (14) are proportional to $\alpha_s$. However, in Ref. [15] it is suggested that in contrast to the DGLAP prescription $\alpha_s = \alpha_s(k_2^\perp)$ ($k_2^\perp$ is the transverse momenta of the ladder partons), which holds universally in every rung of the ladder, the arguments of $\alpha_s$ are different for different kinds of the rungs. In particular, the expressions for the quark-quark and the gluon-gluon rungs include $\alpha_s$ depending on the time-like virtuality of the intermediate gluon, leading to the following expressions for $a_{qq}$ and $a_{gg}$:

$$a_{qq} = A(\omega) C_F \frac{2A(\omega)N}{\pi}, \quad a_{gg} = 2A(\omega)N \quad (18)$$

where

$$A(\omega) = \frac{1}{b} \left[ \frac{\eta}{\eta^2 + \pi^2} - \int_0^\infty \frac{d\rho e^{-\omega\rho}}{(\rho + \eta)^2 + \pi^2} \right]. \quad (19)$$

The $\pi^2$-terms appear in Eq. (19) because $a_{qq}$ and $a_{gg}$ involve $\alpha_s$ with the time-like argument. On the other hand, in the expressions for the rungs mixing quarks and gluons, $\alpha_s$ depends the space-like virtuality of the ladder gluons, which is approximated by an expression similar to (19) without the $\pi^2$ -terms. So, we arrive at

$$a_{gq} = -n_f A'(\omega) \frac{2}{2\pi}, \quad a_{qg} = A'(\omega) C_F \frac{2}{\pi}, \quad (20)$$

with

$$A'(\omega) = \frac{1}{b} \left[ \frac{1}{\eta} - \int_0^\infty \frac{d\rho e^{-\omega\rho}}{(\rho + \eta)^2} \right]. \quad (21)$$

Once $a_{ik}$ and $V_{ik}$ are determined, it is easy to obtain explicit expressions for $H_{ik}$. Indeed, defining $b_{ik}$ as

$$b_{ik} = a_{ik} + V_{ik}, \quad (22)$$

1 We use the Feynman gauge through the paper.
and
\[
Y = -\sqrt{\left(\omega^2 - 2(b_{qq} + b_{gg}) + \sqrt{(\omega^2 - 2(b_{qq} + b_{gg}))^2 - 4(b_{qq} - b_{gg})^2 - 16b_{qq}b_{gg}}\right)/2},
\] (23)
we obtain
\[
H_{gg} = \frac{1}{2}(\omega + Y - \frac{b_{qq} - b_{gg}}{Y}), H_{qq} = \frac{1}{2}(\omega + Y - \frac{b_{qq} - b_{gg}}{Y}),
\] (24)
\[
H_{gg} = -\frac{b_{gg}}{Y}, \quad H_{gg} = -\frac{b_{gg}}{Y}.
\]
As a matter of fact, the expression (24) for \(Y\) appears in solving Eqs. (14) as one of the four solutions of a biquadratic algebraic equation. The signs before the roots in Eq. (24) are chosen so to get a matching between \(H_{ik} = a_{ik}/\omega + O(1/\omega^2)\) when \(\omega \geq 1\) and the Born contributions of Eqs. (20). Now all ingredients for \(\Omega_\pm\) are specified and \(\Omega_\pm\) can be obtained with numerical calculations.

IV. INTERCEPT OF \(g_1\)

Eqs. (8) and (9) give general expressions for \(g_1\). These expressions account for both DL contributions and running \(\alpha_s\) effects. The integrands in Eqs. (8,9) contain the coefficient functions and exponentials. In the present Sect. we consider the exponentials. We will study the coefficient functions in the next Sect. When the limit \(x \to 0\) is considered, one can neglect the exponents with \(\Omega_-\) in these equations and simplify the expression for \(\Omega_+\). Indeed, the behaviour of \(g_1\) in this limit is driven by the leading singularity in Eq. (12). The singularities of \(\Omega_+\) are related to the branching points of the square root. Using Eq. (23) one can see that the leading singularity of \(\Omega_+(\omega)\) is given by the rightmost root, \(\omega_0\) of the equation below:
\[
\omega^4 - 4(b_{qq} + b_{gg})\omega^2 + 16(b_{qq}b_{gg} - b_{gg}b_{gg}) = 0.
\] (25)
Therefore, Eqs. (8,9) predict that
\[
g_1 \sim C(\omega_0)(1/x)^{\omega_0}(Q^2/\mu^2)^{\omega_0/2}
\] (26)
when \(x \to 0\), where \(\omega_0\) is discussed below and the factor \(C(\omega_0)\) in Sect. V. Eq. (25) can be solved numerically. In our approach, \(\omega_0\) depends on \(\eta = \ln(\mu^2/\Lambda_{QCD}^2)\). The result of the numerical calculation for \(\omega_0 = \omega_0(\eta)\) is represented by the curve 1 in Fig. 1. This curve first grows with \(\eta\), achieves a maximum where approximately \(\omega_0 = 0.86\) and smoothly decreases for large \(\eta\). In other words we have obtained that the intercept depends strongly on the infrared cut-off \(\mu\) for small values of \(\mu\) and smoothly thereafter. Quite a similar situation was occurring in Refs. 16 for the intercept of the non-singlet structure function \(g_1^{NS}\). We suggest a possible explanation for this effect. The cut-off \(\mu\) is defined as the starting point in the description of the perturbative evolution. Everything that affects the intercept at scales smaller than \(\mu\) is attributed to non-leading effects and/or non-perturbative contributions. Had they been accounted for, the intercept would have been \(\mu\)-independent. Without those non-leading/non-perturbative effects taken into account, we then observe an important \(\mu\)-dependence, which becomes weaker for large \(\mu\). The maximal perturbative contribution to the intercept \(\Delta_S\) of \(g_1\) is obtained from the maximal value of \(\omega_0\). Therefore we estimate the intercept as
\[
\Delta_S = max(\omega_0(\eta)) \approx 0.86.
\] (27)
Eq. (27) includes the contributions of both virtual quarks and gluons. These contributions have opposite signs and partly cancel each other. It is interesting to note that when only virtual gluon contributions are taken into account, this purely gluonic intercept, \(\Delta_g\) is given by the maximum of the curve 2 in Fig. 1, which is slightly greater than 1. This value obviously exceeds the unitarity limit, similarly to the intercept of the LO BFKL [17], though in a much softer way. Fortunately, by including also the contributions of the virtual quarks the intercept decreases down to \(\Delta_S\) of Eq. (27), without violating unitarity.
We would like also to add the following observation. The total resummation of the DL contributions to \( g_1 \) performed in Refs. [7, 8], where \( \alpha_s \) was suggested to be fixed at the scale \( Q^2 \), as in the framework of DGLAP. In other words, the parametrization \( \alpha_s^{DL} = \alpha_s(Q^2) \) was used in the DL expression for the intercept, \( \Delta^{DL} = 3.5 \sqrt{3} \alpha_s^{DL} / (2 \pi) \) obtained in Ref. [8]. Now, comparing this expression and Eq. (27), one easily concludes that the effective scale of \( \alpha_s^{DL} \) has no relation with \( Q^2 \). This is in a complete accordance with the results of Refs. [12, 16]. Indeed, the parametrization \( \alpha_s = \alpha_s(Q^2) \) is valid only when the factorisation of the longitudinal and transverse momentum space is assumed. This approximation is correct for the kinematic AI region of large \( x \) and does not hold for small \( x \).

V. COEFFICIENT FUNCTIONS \( C_\pm \)

In order to find \( g_q \) and \( g_g \) in Eqs. (30, 31), one has to finally specify the coefficient functions \( C_\pm \). One way to do it is to impose the matching

\[
g_q(x, Q^2)|_{Q^2=\mu^2} = \tilde{\Delta}q(x_0), \quad g_g(x, Q^2)|_{Q^2=\mu^2} = \tilde{\Delta}g(x_0) \tag{28}
\]

where \( x_0 = \mu^2/s; \tilde{\Delta}q(x_0) \) and \( \tilde{\Delta}g(x_0) \) are the initial densities of the polarized quarks and gluons respectively. Using the integral transform of Eqs. (26) at \( Q^2 = \mu^2 \) in the \( \omega \)-space the matching of Eq. (28) can be rewritten as

\[
C_+ + C_- = \Delta q, \quad C_+ \frac{X + \sqrt{R}}{2H_{qg}} + C_- \frac{X - \sqrt{R}}{2H_{qg}} = \Delta g , \tag{29}
\]

with both \( \Delta q \) and \( \Delta g \) depending on \( \omega \). Combining Eqs. (28) and (29), we express \( C_\pm \) through \( \Delta q, \Delta g \) and arrive at

\[
g_q(x, Q^2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} (1/x)^\omega \left[ \left( A^{(-)} \Delta q + B \Delta g \right) e^{\Omega_{+y}} + \left( A^{(+)} \Delta q - B \Delta g \right) e^{\Omega_{-y}} \right] , \tag{30}
\]

\[
g_g(x, Q^2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} (1/x)^\omega \left[ \left( E \Delta q + A^{(+)} \Delta g \right) e^{\Omega_{+y}} + \left( -E \Delta q + A^{(-)} \Delta g \right) e^{\Omega_{-y}} \right] \tag{31}
\]

with

\[
A^{(\pm)} = \left( \frac{1}{2} \pm \frac{X}{2\sqrt{R}} \right), \quad B = \frac{H_{qg}}{\sqrt{R}}, \quad E = \frac{H_{gq}}{\sqrt{R}} . \tag{32}
\]

Eqs. (30, 31) describe the \( Q^2 \) evolution of the polarized quark and gluon densities \( g_q \) and \( g_g \) from \( Q^2 = \mu^2 \), where they are \( \Delta q(\mu^2/s) \) and \( \Delta g(\mu^2/s) \) respectively, to the region \( Q^2 \gg \mu^2 \). In the framework of our approach, \( \mu^2 \) is the starting point for the \( Q^2 \)-evolution. Then one option would be to fix the initial parton densities \( \tilde{\Delta}q \) and \( \tilde{\Delta}g \) from phenomenological considerations.

In this case, one can use Eqs. (30, 31, 32) in order to fix the sign of \( g_1 \) in the small-\( x \) region. The problem of the sign involves an interplay of the gluon and quark contributions to \( g_1 \) (see e.g. Refs. [8, 19]). Let us estimate the sign of \( g_1 \) in the small-\( x \) region, calculating the asymptotics of \( A^{(\pm)} \), \( B \), \( E \). When \( x \to 0 \), the main contributions in Eqs. (30, 31) come from the terms proportional to \( \exp[\Omega_{+y}] \). According to the results of Sect. 4, the small-\( x \) asymptotics of \( g_1 \) is

\[
g_1 \sim [A^{(-)} (S'_S + E_S) \Delta q + (A^{(+)}/B) \Delta g] (1/x)^{\Delta_S (Q^2/\mu^2)^{\Delta_S/2}} \tag{33}
\]

where we have provided the factors \( A^{(\pm)} \), \( B \), \( E \) with the subscript "\( \omega \)" in order to show their explicit dependence on \( \omega = \Delta_S \). Substituting the numerical values \( A^{(-)} = -0.31, A^{(+)} = 1.31, B_S = 0.52, E_S = -0.79 \) into Eq. (33), we arrive at

\[
g_1 \sim [-1.1 \Delta q + 1.8 \Delta g] (1/x)^{\Delta_S (Q^2/\mu^2)^{\Delta_S/2}} \tag{34}
\]
As $\Delta q$ is positive, $g_1$ is negative when

$$\Delta q > 1.7\Delta g .$$  \hfill (35)

Eqs. (34,35) are expressed in terms of the quark and gluon initial densities $\Delta q$, $\Delta g$ defined (see Eq. (28)) at the scale $x \approx \mu^2/s$. On the contrary, the standard DGLAP expressions for $g_1$ involve the initial densities $\delta q$ and $\delta g$ defined at the scale $x \approx 1$. In order to compare our results to DGLAP, we should express $\Delta q$ and $\Delta g$ in terms of $\delta q$ and $\delta g$. In the $\omega$-space, it means expressing $\Delta q(\omega)$ and $\Delta g(\omega)$ through $\delta q(\omega)$ and $\delta g(\omega)$. We can do it within our framework by using the evolution of $\Delta q$ and $\Delta g$ with respect to $s$ at fixed $Q^2 \approx \mu^2$ from the starting point $s \approx \mu^2$ to the region $s \gg \mu^2$. The system of the IREE for $\Delta q$ and $\Delta g$ is similar to Eq. (9) with the exception of the fact that $\Delta q$ and $\Delta g$ do not depend on $Q^2$ and therefore the IREE for them do not include the derivatives $\partial/\partial y$. Therefore, we arrive at the following algebraic IREE:

$$\Delta q(\omega) = (e_q^2/2)\delta q(\omega) + (1/\omega)[H_{qq}(\omega)\Delta q(\omega) + H_{qg}(\omega)\Delta g(\omega)],$$

$$\Delta g(\omega) = (e_g^2/2)\delta g(\omega) + (1/\omega)[H_{gq}(\omega)\Delta g(\omega) + H_{gg}(\omega)\Delta g(\omega)].$$  \hfill (36)

where $e_q^2$ is the sum of the quark electric charges ($e_q^2 = 10/9$ for $n_f = 4$), $\delta q$ is the sum of the initial quark and antiquark densities and $\delta g \equiv -(A'(\omega)/2\pi\omega^2)\delta g$ ($A'(\omega)$ is defined in Eq. (21)) is the starting point of the evolution of the gluon density $\delta g$. Eqs. (36) describe the $s$-evolution of the quark and gluon inputs from $s \sim \mu^2$ to $s \gg \mu^2$. Solving Eqs. (36), we obtain:

$$\Delta q = (e_q^2/2)\omega^{-2}\omega\omega[H_{gg}(\omega)\delta g + \omega H_{qg}(\omega)]\Delta g[\omega - H_{gg}(\omega) + (H_{qg}H_{gg} - H_{gg}H_{qg})] ,$$  \hfill (37)

$$\Delta g = (e_g^2/2)\omega^{-2}\omega\omega[H_{gg}(\omega)\delta q + \omega H_{qg}(\omega)]\Delta q[\omega - H_{gg}(\omega) + (H_{qg}H_{gg} - H_{gg}H_{qg})] .$$  \hfill (38)

Substituting Eqs. (37,38) into Eq. (34) allows to express asymptotics of $g_1$ in terms of $\delta q, \delta g$:

$$g_1 \sim (1/2)\omega^{-1}[1.2\delta q - 0.08\delta g](1/\omega)^{s}(Q^2/\mu^2)^{\Delta s}/2 .$$  \hfill (39)

In contrast to Eq. (24) which involves the parton distributions $\Delta q$ and $\Delta g$ defined at the low-$x$ scale, at $x \approx \mu^2/s$, Eq. (39) is expressed in terms of the densities $\delta q$ and $\delta g$ at $x \sim 1$. These densities can be fixed from phenomenological considerations. Eq. (39) states that $g_1$ can be positive at $x \ll 1$ only when $\delta g$ is negative and large:

$$15\delta q + \delta g < 0 .$$  \hfill (40)

VI. SUMMARY AND OUTLOOK

By constructing and solving the system (61) of infrared evolution equations, we have obtained the explicit expressions (30,31) for the polarized quark and gluon distributions $g_1(x,Q^2)$ and $g_6(x,Q^2)$. These expressions account for DL contributions to all orders and, at the same time, for the running $\alpha_s$ effects. Both $g_q(x,Q^2)$ and $g_6(x,Q^2)$ depend on the low-$x$ quark and gluon distributions, $\Delta q$ and $\Delta g$. Eq. (34) shows that $g_1$ has the negative sign in the small-$x$ region if $\Delta q > 1.7\Delta g$. By the same method, and using the $s$-evolution at $Q^2 \approx \mu^2$, the distributions $\Delta q$ and $\Delta g$ are expressed in Eq. (34) in terms of the initial densities $\delta q$ and $\delta g$ which are supposed to be fixed from phenomenological considerations at $x \sim 1$. Eq. (39) shows that $g_1$ is positive when the initial gluon density is negative and large: $\delta g < -15\delta q$, otherwise $g_1$ is negative.

We obtain that the expressions (30,31) lead to the Regge behaviour (34,38) of $g_1$ when $x \rightarrow 0$. The value of the intercept $\Delta S$ depends on the infrared cut-off $\mu$, but this dependence is quite weak for $\mu$ much larger than $Q_{CD}$. The estimated value of the intercept is given by Eq. (27), and is a factor of 2.2 larger than the non-singlet intercept. The value of $\Delta S = 0.86$ is in a good agreement with the estimate $\Delta S = 0.88$ obtained in Ref. 20 from analysis of the HERMES data. Our results also show that accounting for the gluon contributions only would obtain a value of the intercept exceeding unity and therefore violating unitarity - similarly to the LO BFKL intercept - whereas the inclusion of the quark contributions stabilises the result. We prove that it is unrealistic to combine the resummation of the DL contributions to $g_1$ with the DGLAP-like parametrisation for $\alpha_s$ in the expressions for the intercepts.

Finally, we note that it would be very interesting to implement our results by non-perturbative (lattice) calculations in order to check explicitly the independence of the total intercept on $\mu$. 

VII. ACKNOWLEDGEMENT

We are grateful to S.I. Krivonos and S.M. Oliveira for useful discussions concerning the numerical calculations. The work is supported by grants POCTI/FNU/49523/2002 and RSGSS-1124.2003.2.

VIII. FIGURE CAPTIONS

Fig. 1: Dependence on $\eta$ of the rightmost root of Eq. (25), $\omega_0$. Curve 2 corresponds to the case when gluon contributions only are taken into account; curve 1 is the result of accounting for both gluon and quark contributions.

[1] M.A. Ahmed and G.G. Ross. Nucl. Phys. B 111(1976)441.
[2] G. Altarelli and G. Parisi. Nucl. Phys. B 126(1977)298.
[3] V.N. Gribov and L.N. Lipatov. Sov. J. Nucl. Phys. 15(1978)438 and 675; Yu.L. Dokshitzer. Sov. Phys. JETP 46(1977)641.
[4] G. Altarelli, R. Ball, S. Forte and G. Ridolfi. Acta Phys. Polon. B 29(1998)1201; Nucl. Phys. B 496(1999)337.
[5] J. Blumlein and H. Bottcher. Nucl. Phys. B 636(2002)225.
[6] M.G. Gluck, E. Reya, M. Straumann and W. Vogelsang. Phys. Rev. D 53(1996)4775.
[7] B.I. Ermolaev, S.I. Manaenkov and M.G. Ryskin. Z. Phys. C 69(1996)259; J. Bartels, B.I. Ermolaev and M.G. Ryskin. Z. Phys. C 70(1996)273.
[8] J. Bartels, B.I. Ermolaev and M.G. Ryskin. Z. Phys. C 72(1996)627.
[9] D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini and G. Veneziano. Nucl. Phys. B 173(1980)428.
[10] J. Kwiecinski. Acta Phys. Polon. B 29(2001)1201.
[11] D. Kotlorz and A. Kotlorz. Acta Phys. Polon. B 32(2001)2883.
[12] B. Badalek, J. Kiryluk and J. Kwiecinski. Phys. Rev. D 61(2000)014009.
[13] J. Kwiecinski and B. Ziaja. Phys. Rev. D 60(1999)0802386; J. Kwiecinski and B. Ziaja. hep-ph/9802386.
[14] B. Ziaja. Phys. Rev. D 66(2002)114017; hep-ph/0304208.
[15] B.I. Ermolaev, M. Greco and S.I. Troyan. Phys. Lett. B 522(2001)57.
[16] B.I. Ermolaev, M. Greco and S.I. Troyan. Nucl. Phys. B 594(2001)71; ibid 571(2000)137.
[17] V.S. Fadin, E.A. Kuraev and L.N. Lipatov. Sov. Phys. JETP, 44(1976)443, ibid 45(1977)199.
[18] J. Soffer and O.V. Teryaev. Phys. Rev. D 56(1997)1549; A.L. Kataev, G. Parente, A.V. Sidorov. CERN TH-2001-058; Phys. Part. Nucl. 34(2003)20; Fiz. Elem. Chast. Atom. Yadra 34(2003)43; Nucl. Phys. A 666(2000)184; A.V. Kotikov, A.V. Lipatov, G. Parente, N.P. Zotov. Eur. Phys. J. C 26(2002)51; V.G. Krivokhijine, A.V. Kotikov, hep-ph/0108221; A.V. Kotikov, D.V. Peshekhonov, hep-ph/0110229.
[19] A. De Roeck, A. Deshpande, V.W. Hughes, J. Lichtenstadt, G. Radel. Eur. Phys. J. C 6(1999)121.
[20] N.I. Kochelev, K. Lipka, W.-D. Nowak, V. Vento, A.V. Vinhkov. Phys. Rev. D 67(2003)074014.
