Self-Duality Equations on $S^6$ from $\mathbb{R}^7$ monopole

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Abstract

In this note we identify a correspondence between a seven-dimensional monopole configuration of the Yang-Mills-Higgs system and the generalized self-dual configuration of the Yang-Mills system on a six-dimensional sphere. In particular, the topological charge of the self-duality configurations belongs to the sixth homotopy group of the coset $G/H$ associated with the symmetry breaking $G \to H$ induced by a non-trivial Higgs configuration in seven-dimensions.
In this short note we make an observation about the self-duality equations on the six-dimensional sphere. We make use of the work of [1, 2, 3, 4], the details of which we omit. It is well known [5] that a four-dimensional instanton configuration has second Chern character, which is in turn, related to the third homotopy group \( \pi_3(G) \) of the gauge group \( G \). We show there is a correspondence between seven-dimensional monopoles and self-duality equations on the six-dimensional sphere. There have been numerous efforts to generalize monopoles to higher dimensions, some of which have appeared in [1, 6, 7, 8, 9, 10].

In analogy in six-dimensions, when \( G = SU(N) \), the third Chern character \( \text{Tr} F^3 \) is considered as a topological charge and takes values in \( \pi_5(G) \), with \( \pi_5(SU(N)) = \mathbb{Z} \) for \( N \geq 3 \). In particular, for \( SU(4) \simeq SO(6) \), pure Yang-Mills theory on \( S^6 \), one has a non-trivial gauge configuration [4], which satisfies the generalized self-duality relation

\[
cF \wedge F = \ast \tilde{c} \{ D\phi, F \}.
\]

Here, \( c = 3/(qR_0^2) \) is a covariantly constant scalar given in terms of the gauge coupling \( q \) and radius of \( S^6 R_0 \).

A few examples of other configurations for \( \pi_5(G) \neq 0 \) have appeared in the literature in [3]. In this note, our focus is non-trivial solutions of self-duality equations on \( S^6 \) with gauge group \( G \) with \( \pi_5(G) = 0 \).

In one dimension higher, the above equation takes the form

\[
F \wedge F = \ast \gamma \tilde{c} \{ D\phi, F \},
\]

where \( \tilde{c} \) is a constant. The above equation can be obtained from the Bogomol’nyi equation [9]. Here \( F \) is a gauge field strength two-form and “\( \ast \gamma \)” is the Hodge dual operator with respect to the Euclidean metric on \( \mathbb{R}^7 \). \( \phi^a \) are scalar fields forming a fundamental multiplet of \( SO(7) \), \( \phi := \phi^a \gamma_a \) and finally, \( D \) is the covariant exterior derivative: \( D\phi = d\phi + g[A, \phi] \).

The Hermitian matrices \( \gamma_a \), \( (a = 1, 2, \cdots, 7) \), are Dirac matrices with respect to \( SO(7) \), with \( \gamma_{ab} := (1/2)[\gamma_a, \gamma_b] \) satisfying the commutation relations of \( SO(7) \). \( \phi \) induces symmetry breaking when it acquires an expectation value \( \| \langle \phi^a \rangle \| = H_0 \).

To substantiate this connection, we suppose that the gauge configuration is concentrated around the origin of \( \mathbb{R}^7 \). Solutions of Eq. (2) represent monopole configurations with

\[^1\text{This is easily embedded in } SU(N) \text{ with } N \geq 4.\]
corresponding topological charge,
\[
Q = \int_{B(R_0)} \text{Tr} D\phi F^3 = \int_{S^6_{R_0}} \text{Tr} \phi F^3 ,
\]
where \(B(R_0) = \{ x \in \mathbb{R}^7 \| x \| \leq R_0 \} \). This charge \(Q\) relates to the mapping class degree of \(S^6_{R_0} \to SO(7)/SO(6) = S^6\) for the case where \(R_0 \gg 1\). To see this, we suppose that gauge field \(A\) and scalar field \(\phi\) have the following form,
\[
A = \frac{1 - K(r)}{2q} e \, de , \quad \phi = H_0 U(r) e , \quad e = \frac{x^a}{r} \gamma_a ,
\]
where \(q\) is again the gauge coupling, \(r = \sqrt{x^a x^a}\) and the functions \(U(r)\) and \(K(r)\) satisfy the following boundary conditions: \(U(0) = 1, K(0) = 1, U(\infty) = 1\) and \(K(\infty) = 0\). The corresponding \(F\) and \(D\phi\) are
\[
F = \frac{1 - K^2}{4q} de \wedge de - \frac{K'}{2q} e dr \wedge de , \quad D\phi = H_0 (K U e + U' dr) .
\]
For this particular configuration, Eq. (2) reduces to a first order nonlinear ordinary differential equation [9].

In the asymptotic region, \(F\) and \(D\phi\) become
\[
F \to \frac{1}{4q} de \wedge de , \quad D\phi \to H_0 U' dr ,
\]
where, as may be seen, \(F\) is aligned perpendicular to the radial direction and thus, along the \(S^6\). Hence \(F\) can be regarded as a differential form on \(S^6\). In this asymptotic region, Eq. (2) is tranformed into Eq. (1) with a suitable scalar.

However, the above discussion includes some degree of approximation: the self-duality is not exact. If we now relax the constraint of demanding a finite energy configuration by considering the singular configuration
\[
A = \frac{1}{2q} e de , \quad \phi = -\frac{\kappa}{r} e ,
\]
where \(\kappa\) is a constant, the seven-dimensional equation
\[
F \wedge F = * i \mu \{ D\phi, F \} , \quad \mu = \frac{3}{2q\kappa} ,
\]
reduces to Eq. (1).

Having constructed a concrete example, we now consider other embeddings. In general, we may consider a gauge group \(G\) with non-trivial \(\pi_6(G/H)\) with symmetry breaking \(G \to H\),
from a seven-dimensional monopole solution. For simplicity suppose that $G$ is a simple group and the rank of group $G$ is greater than or equal to 3. From the long exact sequence of homotopy group we obtain

$$\pi_6(G/H) \simeq \text{Ker}\{\pi_5(H) \to \pi_5(G)\}. \quad (9)$$

If $\pi_5(G) = 0$ and $H$ includes Spin(6) or SU($N$) ($N \geq 3$) as a factor group, then $\pi_6(G/H) \neq 0$.

In contrast to the earlier example where the Higgs is in the fundamental 7 of SO(7), it is possible to embed it and the adjoint 21 of SO(7) in the adjoint 28 of SO(8). Here $\pi_6(G/H) \neq 0$, and we can embed the above solution into the larger gauge theory with adjoint Higgs field and it does not come loose as a result of a gauge transformation of the larger group. E$_8$, SU($N$), ($N \geq 8$) and SO($N$) ($N \geq 8$) also permit the same configuration with adjoint Higgs. It would be interesting to explore embeddings of this configuration in string theory or M-theory: the gauge groups SO(16) and E$_8$, both appear in \[1\]. For example, it may be possible to consider symmetry breakings $SO(16) \to SO(6) \times SU(5) \times U(1)$ and $E_8 \to SU(4) \times SU(5) \times U(1)$, inspired by the symmetry breaking of SO(10) GUT: $SO(10) \to SU(5) \times U(1)$.

For these symmetry breakings $\pi_6(G/H) \neq 0$. It may also be of interest to consider coupling this system to gravity in a similar fashion to studies appearing in \[12, 13, 14\], the latter of which addresses the possibility of cosmological models as a result of dynamical compactification on $S^6$.

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