On the convergence of an exotic formal series solution of an ODE

R. R. Gontsov, I. V. Goryuchkina

We consider a non-linear ordinary differential equation (ODE)

\[ F(x, y, \delta y, \ldots, \delta^n y) = 0, \]  

(1)

where \( \delta = x(d/dx) \) and \( F(x, y_0, y_1, \ldots, y_n) \) is a holomorphic function in a small polydisc

\[ \Delta = \{ |x| < \rho, |y_j| < \varepsilon_j, j = 0, \ldots, n \}. \]

Suppose that the equation (1) possesses a formal solution \( y = \varphi \),

\[ \varphi = \sum_{k=0}^{\infty} \alpha_k(x \eta) x^k, \quad i = \sqrt{-1}, \quad \eta \in \mathbb{R} \setminus \{0\}, \]

(2)

where \( \alpha_k(t) \) are meromorphic functions at the origin:

\[ \alpha_k(t) = t^{-r_k} \sum_{\ell=0}^{\infty} \alpha_{k\ell} t^\ell, \quad t^{r_k} \alpha_k(t) \in \mathbb{C}\{t\}, \quad r_k \in \mathbb{Z}, \]

with some common punctured disc \( D = \{ 0 < |t| < \varepsilon \} \) of convergence.

The series (2) will be called exotic, in the terminology of A. D. Bruno [1]. In particular, the Painlevé III, V, VI equations possess formal solutions of such type [2], [3], [8]. Thus, our present goal is to obtain some general condition for the convergence of an exotic formal series solution of the equation (1). In this sense, our work continues a series of articles where similar questions were studied for generalized formal power series [4], [6] and formal Dulac series [7], which were inspired by the original paper of B. Malgrange [9] on the classical formal power series solutions of a non-linear ODE.

Theorem 1. Let (2) be a formal solution of the equation (1):

\[ F(x, \Phi) = 0, \quad \Phi := (\varphi, \delta \varphi, \ldots, \delta^n \varphi), \]

such that \( \frac{\partial F}{\partial y_n}(x, \Phi) \neq 0 \). Furthermore, let each exotic formal series \( \frac{\partial F}{\partial y_i}(x, \Phi) \) be of the form

\[ \frac{\partial F}{\partial y_i}(x, \Phi) = B_i(x \eta) x^N + D_i(x \eta) x^{N+1} + \ldots, \quad i = 0, 1, \ldots, n, \]

with the same \( N \) for all \( i \), \( B_n \neq 0 \) and

\[ \text{ord}_0 B_i \geq \text{ord}_0 B_n, \quad i = 0, 1, \ldots, n. \]

1Institute for Information Transmission Problems of RAS, Bolshoy Karetny per. 19, build.1, Moscow 127051 Russia, gontsovrr@gmail.com.
2Keldysh Institute of Applied Mathematics of RAS, Miusskaya sq. 4, Moscow 125047 Russia, igoryuchkina@gmail.com.
Then the series (2) converges uniformly in some open sector $S \subset \mathbb{C} \setminus \mathbb{R}_+$ with the vertex at the origin and of sufficiently small radius.

We give a sketch of the proof of Theorem 1 in a series of lemmas. The first one is on the reduction of the initial equation to a special form, with the proof in a spirit of Malgrange’s proof [9] of the corresponding lemma for a classical formal power series solution.

**Lemma 1.** Under the assumptions of Theorem 1, there exists $m' \in \mathbb{N}$ such that for any $m \geq m'$, the transformation

$$y = \sum_{k=0}^{m} \alpha_k(t)x^k + x^mu$$

reduces the equation (1) to an equation of the form

$$x^{inr} \sum_{i=0}^{n} A_i(x^{in}) (\delta + m)^i u = x M(x^{in}, x, u, \delta u, \ldots, \delta^n u),$$

where $r \in \mathbb{Z}_+$, $A_i(t)$ are holomorphic near $t = 0$, $A_n(0) \neq 0$, $M(x, t, u_0, \ldots, u_n)$ is holomorphic near $0 \in \mathbb{C}^{n+3}$.

The reduced equation (4) possesses an exotic formal series solution

$$\psi = \sum_{k=1}^{\infty} c_k(x^{inr}) x^k, \quad c_k(t) = \alpha_{k+m}(t) = t^{-\nu_k} \sum_{\ell=0}^{\infty} c_{k\ell} t^\ell,$$

moreover, this is its unique formal solution of such a form, since the integer $m$ can be chosen in such a way that for any $k \geq 1$, a Fuchsian linear differential operator (near $t = 0$)

$$\sum_{i=0}^{n} A_i(t) \left( k + m + i \eta \frac{dt}{dt} \right)^i$$

has no integer exponents. To prove Theorem 1, one should prove the convergence of the series $\psi$.

Let us denote

$$L(t)u := \sum_{i=0}^{n} A_i(0)(\delta + m)^i u,$$

$$x^{in} H(x^{in}, u, \delta u \ldots, \delta^n u) := L(t)u - \sum_{i=0}^{n} A_i(x^{in})(\delta + m)^i u,$$

and write down the equation (4) in the form

$$x^{inr} L(t)u = x^{in(r+1)} H(x^{in}, u, \delta u \ldots, \delta^n u) + x M(x^{in}, x, u, \delta u, \ldots, \delta^n u).$$

First, by the equation (6) we construct an algebraic equation

$$t^r \sigma v = t^{r+1} \widetilde{H}(t, v, v, \ldots, v) + x \widetilde{M}(t, x, v, v, \ldots, v), \quad \sigma > 0,$$
majorant for the equation (11) in the sense that it has a unique formal solution of the form
\[ \tilde{\psi} = \sum_{k=1}^{\infty} C_k(t) x^k, \quad C_k(t) = t^{-\nu_k} \sum_{\ell=0}^{\infty} C_{k\ell} t^\ell, \quad C_{k\ell} \geq 0, \quad (8) \]
and \( |c_{k\ell}| \leq C_{k\ell} \) for all \( k, \ell \). Let us briefly describe this construction.

The functions \( H, M \) of the right hand side in (6) are represented by power series convergent in \( D \times \Delta \):
\[
H(t, u_0, \ldots, u_n) = \sum_{i=0}^{n} \sum_{s=0}^{\infty} h_{s,i} t^s u_i, \quad h_{s,i} \in \mathbb{C},
\]
\[
M(t, x, u_0, \ldots, u_n) = \sum_{(s,p,Q) \in \mathbb{Z}_+^{n+3}} \alpha_{s,p,Q} t^s x^p u_0^{q_0} u_1^{q_1} \ldots u_n^{q_n}, \quad Q = (q_0, q_1, \ldots, q_n), \quad \alpha_{s,p,Q} \in \mathbb{C}.
\]

Then the functions \( \tilde{H}, \tilde{M} \) in the equation (7) are defined as follows:
\[
\tilde{H}(t, v, \ldots, v) = \sum_{i=0}^{n} \sum_{s=0}^{\infty} |h_{s,i}| t^s v,
\]
\[
\tilde{M}(t, x, v, \ldots, v) = \sum_{(s,p,Q) \in \mathbb{Z}_+^{n+3}} |\alpha_{s,p,Q}| t^s x^p v^{q_0} v^{q_1} \ldots v^{q_n}.
\]

The number \( \sigma \) is defined by the formula
\[
\sigma = \inf_{k \in \mathbb{N}, \ell \in \mathbb{Z}_+} |L(k + m + (\ell - kr)i \eta)|.
\]
This value is strictly positive, as \( L(k + m + (\ell - kr)i \eta) \neq 0 \) for \( k \in \mathbb{N}, \ell \in \mathbb{Z}_+ \).

**Lemma 2.** The equation (7) has a uniquely determined formal solution of the form (8), which is majorant for the exotic formal series solution (5) of the equation (6): \( |c_{k\ell}| \leq C_{k\ell} \).

The proof of this lemma is somehow similar to that of Lemma 2 from [5], where a proof of Malgrange’s theorem for a classical power series by the majorant method was proposed.

The second step is to prove the convergence of the series (8). Let us write down the equation (7) in the form
\[
v = t^{r/\sigma} \tilde{H}(t, v, \ldots, v) + \frac{x}{\sigma t^r} \tilde{M}(t, x, v, \ldots, v).
\]
Its right hand side is holomorphic in the domain
\[
\tilde{D} = \{ 0 < \tau < |t| < \varepsilon, |x| < \rho, |v| < \varepsilon_n \},
\]
and we have
\[
\mathcal{M} = \sup_{\tilde{D}} \left| \frac{t}{\sigma} \tilde{H}(t, v, \ldots, v) + \frac{x}{\sigma t^r} \tilde{M}(t, x, v, \ldots, v) \right| < +\infty.
\]
Further we look at the equation (9) as at an equation in two variables $x$, $v$, with a nonzero parameter $t$. For each fixed value $t = t_0$, $\tau < |t_0| < \varepsilon$, such that

$$
\frac{t_0}{\sigma} \sum_{i=0}^{n} \sum_{s=0}^{\infty} |h_{s,i}| t_0^s \neq 1,
$$

we can apply the implicit function theorem to the equation (9). According to this theorem, the equation has a unique holomorphic solution $v = v(x, t_0)$ in some open disc

$$
\mathcal{V}_{t_0} = \left\{ |x| < \rho \left( \frac{\varepsilon_n}{\varepsilon_n + 2M_{t_0}} \right)^2 \right\}, \quad M_{t_0} \leq M,
$$

whose power series should coincide with the formal series (8), for $t = t_0$. Hence, the latter converges uniformly in some open domain

$$
\mathcal{V} = \left\{ 0 < \tau < |t| < \tau', \quad |x| < \rho \left( \frac{\varepsilon_n}{\varepsilon_n + 2M} \right)^2 \right\},
$$

which concludes the proof of Theorem 1.

References

[1] Bruno, A. D., Exotic expansions of solutions to an ordinary differential equation, *Doklady Math.* 76:2 (2007), 729–733.

[2] Bruno, A. D., Goryuchkina, I. V., All asymptotic expansions of solutions to the sixth Painlevé equation, *Doklady Math.* 76:3 (2007), 851–855.

[3] Bruno, A. D., Parusnikova, A. V., Local expansions of solutions to the fifth Painlevé equation, *Doklady Math.* 83:3 (2011), 348–352.

[4] Gontsov, R. R., Goryuchkina, I. V., On the convergence of generalized power series satisfying an algebraic ODE, *Asympt. Anal.* 93:4 (2015), 311–325.

[5] Gontsov, R. R., Goryuchkina, I. V., An analytic proof of the Malgrange theorem on the convergence of formal solutions of an ODE, *J. Dyn. Control Syst.* 22:1 (2016), 91–100.

[6] Gontsov, R. R., Goryuchkina, I. V., The Maillet–Malgrange type theorem for generalized power series, *Manuscripta Math.* 156:1 (2018), 171–185.

[7] Gontsov, R. R., Goryuchkina, I. V., On the convergence of formal Dulac series satisfying an algebraic ODE, *Sbornik Math.*, 2019 (to appear).

[8] Guzzetti, D., Tabulation of Painlevé 6 transcendent, *Nonlinearity* 25 (2012), 3235–3276.

[9] Malgrange, B., Sur le théorème de Maillet, *Asympt. Anal.* 2 (1989), 1–4.