Disentangling Classicality and Kochen-Specker Contextuality

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Abstract

In recent years, much effort has been devoted to showing that classical light can reproduce phenomena that are commonly seen as "intrinsically quantum" and, it has been demonstrated that contextuality is one of them. Despite that, the view that contextuality is a signature of non-classicality is still widespread, and many authors still claim, without further explanation, that contextuality is a non-classical feature of quantum systems. This contrast shows that the relationship between contextuality and classicality needs some clarification, so in this paper we try to shed some light on this issue. We draw a distinction between "ontic" and "epistemic" measurements and argue that contextuality "defies classical understanding" only when measurements are ontic. It means that, unless we can say with certainty that this is the case with all projective measurements in an experimental observation of quantum contextuality, we should not take for granted that this observation is a signature of non-classicality. To illustrate this point, we present a very intuitive "thought experiment" where a classical device based on epistemic measurements generates non-disturbing data violating the famous KCBS inequality. This device is a black box, so we use the same thought experiment to argue that, unlike Bell inequalities, whose violation allegedly implies non-locality, noncontextuality inequalities have no phenomenological significance per se: in principle, no physical phenomenon is witnessed when these inequalities are violated by a measuring device.

1 Introduction

Simon Kochen and Ernst Specker famously showed that, under suitable conditions, one cannot understand selfadjoint operators on a Hilbert space $H$ ($\dim(H) > 2$) as representing properties simultaneously possessed by a physical system [1]. The "suitable conditions" are basically that (1) each bounded selfadjoint operator represents one, and only one, property of the system, (2) the value of a property represented by a selfadjoint operator $A$ lies in its spectrum $\sigma(A)$ and (3) if a selfadjoint operator $B$ is a function of an operator $A$ by means of the Borel functional calculus, i.e., if $B = g(A)$ for some Borel function $g : \sigma(A) \to \mathbb{R}$, then the value $V(B)$ possessed by $B$ has to be equal to $f(V(A))$, where $V(A)$ denotes the value of $A$. A function $V : B(H)_{sa} \to \mathbb{R}$, where $B(H)_{sa}$ denotes the set of all bounded selfadjoint operators on $H$, satisfying conditions (2) and (3) listed above is said to be a valuation on $H$, and the Kochen-Specker theorem consists in the statement "there is no valuation on $H$ if $\dim(H) > 2$" [2, 3, 4]. As Kochen and Specker themselves pointed out [1], a corollary of this result is that, if $\dim(H) > 2$, no noncontextual hidden variable (NCHV) model (see definition 10) for $H$ exists. The reason is simple, as Andreas Döring explains [3]: in such hypothetical model, each selfadjoint operator $A$ would be associated to a random variable $f_A : \Lambda \to \sigma(A)$ on the set $\Lambda$ of "hidden states", whereas the mapping $A \mapsto f_A$ would "commute with the Borel functional calculus", that is to say, if $B = g(A)$, then $f_B = g \circ f_A$. As a consequence, a hidden state $\lambda \in \Lambda$ would define a valuation $V_{\lambda}$ on $B(H)_{sa}$ by $V_{\lambda}(A) = f_A(\lambda)$, contradicting Kochen-Specker theorem. Nowadays, the non-existence of NCHV models for quantum systems is what is usually called "Kochen-Specker contextuality", and the so-called noncontextuality inequalities are inequalities characterizing this condition [5].

Kochen-Specker theorem brings out a tension between quantum theory and realism. In fact, as Chris Isham and Andreas Döring put it [6], although the exactly meaning of "realism" is debatable, when used in classical physics it means, among other things, that (1) "the idea of ‘a property of the system’ is meaningful, and mathematically representable in the theory"; and that (2) there is a space of ontic states (they actually say "microstates") such that specifying an ontic state determines "the way things are"; in particular, an ontic state ascribes a definite value for every property of the system. Note that,
in any realist theory, a property $A$ of the system can be represented as a function $f_A$ associating each ontic state $\lambda$ to the value $A$ assumes in this state, just as in the NCHV model proposed by Kochen and Specker. As Isham and Döring say, Kochen-Specker theorem rules out the “naive realist interpretation” of quantum theory [6], according to which all selfadjoint operators would represent properties that are fully specified by the (hidden) ontic state of the system, while all functional relations between operators would be preserved. This is exactly the view behind NCHV models (definition 10), so we can call them “naive realist models” for quantum systems. Furthermore, the fact that there is no valuation for some “small” subsets of $B(H)_{sa}$ (sets containing only nine operators, for example [7]) indicates that “NCHV-like” models for quantum predictions, i.e., models where projective measurements are random variables and states are probability measures (see definition 5), will, in general, suffer from the same problem. In any case, unless we give up preserving functional relations between operators, Kochen-Specker theorem forces us to choose between one of the two following alternatives.

- We can keep the ontic view on selfadjoint operators, according to which all of them represent properties of the system, but we have to abandon the idea that the (maybe hidden) ontic state of the system determines “the way things are”: whatever it is (and it may be, for example, the wave function), the ontic state of a system cannot ascribe a single value to each property simultaneously. This view is shared by some (non-naive) realist interpretations – or reformulations – of quantum theory, such as the contravariant topos approach [6].

- We can hold an epistemic view on selfadjoint operators, according to which they represent measurement procedures with no necessary connection with properties. In this case, we can – but do not need to – maintain the hypothesis that there may be a (maybe hidden) ontic state determining “the way things are”. This view on selfadjoint operators, shared by Bohr [8], is suggested by the orthodox interpretation, and it is compatible with basically any non-realistic interpretation of quantum theory. It is also compatible with some non-naive realist interpretations – or reformulations – of quantum theory, such as the pilot-wave theory [9].

Together with Kochen-Specker theorem, the ontic view on selfadjoint operators suggests that quantum systems “defy classical understanding”: properties cannot be simultaneously possessed by the system, “the way things are” may depend on the observer, the system may have no properties when no one is looking at it, nature is intrinsically non-deterministic – or it is deterministic but there are “many worlds” –, contextuality is a counter-intuitive feature in the quantum realm, and so on. According to the epistemic view, on the other hand, Kochen-Specker theorem has no immediate phenomenological significance, given that selfadjoint operators do not necessarily represent properties: in principle, an obstruction to the existence of valuations is not an obstruction to the existence of an ontic state ascribing definite values to all properties of the system; moreover, if selfadjoint operators do not necessarily represent properties, there is no reason to expect each one of them to have a definite value at any given time. As a consequence, the existence of a “realist ontological model” for the system is not ruled out by Kochen-Specker theorem, given that only properties must be defined as functions on the hypothetical space of ontic states. Therefore, at least those who share the epistemic view should not take for granted that Kochen-Specker theorem is some kind of signature of non-classicality, i.e., a criterion telling us when, as Michael D. Mazurek et al. put it, “nature fails to respect classical physics” [10], given that such a criterion must have a clear phenomenological meaning. To sum up, Kochen-Specker theorem rules out the naive realist interpretation of quantum theory; if, despite the Kochen-Specker theorem, we keep assuming that all selfadjoint operators represent properties, we are forced to accept that, in some sense, quantum phenomena defy classical understanding; on the other hand, if we abandon the ontic view and assume an epistemic view on selfadjoint operators, Kochen-Specker theorem has no clear phenomenological implication. Therefore, Kochen-Specker theorem is not a criterion of classicality per se: its phenomenological significance depends on how we interpret selfadjoint operators, thus it does not necessarily imply that, in the quantum realm, “nature fails to respect classical physics”. Finally, it is worth emphasizing that Kochen-Specker theorem does not rule out realism, only “naive realism”, and also that the epistemic view on selfadjoint operators is compatible with many realist interpretations of quantum theory.

Following this reasoning, it seems plausible to conclude that, like Kochen-Specker theorem, experimental violations of noncontextuality inequalities [11, 12, 13] have no phenomenological significance per se, and that, in particular, they are not signatures of non-classicality. In fact, these violations show that experimental data obtained from quantum systems cannot be described by “NCHV-like models” (definition 5), i.e., models where projective measurements (those described by selfadjoint operators in quantum theory) are random variables sharing the same sample space $\Lambda$ and states are probability measures on $\Lambda$. In these models, measurements can always have definite values, given that any element $\lambda \in \Lambda$ defines a valuation ascribing values to all measurements,
so this is exactly the kind of model Kochen-Specker theorem rules out. Violations of noncontextuality inequalities, thus, endorse Kochen-Specker theorem by suggesting that some quantum predictions are not compatible with, let’s say, “naive realist formalisms”. If the phenomenological status of Kochen-Specker theorem depends on how we interpret selfadjoint operators, the same must be true for noncontextuality inequalities. All that these violations suggest is that some projective measurements are not revealing pre-existing values of properties, and we only have to admit that it “defies classical understanding” if all projective measurements involved in the experiment are necessarily associated with properties of the system – something that, as far as we can tell, can hardly be done. One who accept the epistemic view believes that, even if there is a “space of ontic states” Lambda, there is no reason to expect that noncontextuality inequalities will not be violated by quantum predictions, given that one of the assumptions behind this inequalities, namely that measurements are random variables, does not need to be satisfied. Therefore, according to the epistemic view, these violations may be no more than expected consequences of the type of measurements we are taking into account.

Thus far we have argued that, if measurements are not necessarily associated with properties, violations of noncontextuality inequalities should be expected, regardless of whether or not the system under study is, let’s say, “non-classical”. This claim has experimental support. In fact, much effort has been made recently to show that classical light can reproduce many phenomena that are commonly seen as “intrinsically quantum” [14, 15, 16], and contextuality is no exception. In 2017, Tao Li and collaborators [17] showed, by implementing “projection measurements” on an optical system, that the famous KCBS inequality [18], named after Klyachko, Can, Binicioğlu, and Shumovsky, can be violated by classical light; more recently, it has been shown that even state-independent contextuality can be simulated by this kind of classical system [19]. As far as we can tell, the reason why contextuality (i.e., the violation of noncontextuality inequalities) is witnessed in these experiments is that, in classical optics, projection measurements are not associated with properties of the system. In any case, these experiments definitely show that, if we cannot say with certainty that measurement procedures in a given experimental setup are associated with properties of a system, violations of noncontextuality inequalities are not signatures of non-classicality: nature does not need to “fail to respect classical physics” [10] for these inequalities to be violated.

The reader who is familiar with recent research on contextuality knows that this peaceful coexistence between contextuality and classicality is not easily recognized; contextuality is commonly presented as a non-classical feature of quantum systems, and it seems that not even the work of Tao Li and his collaborators has been capable of changing it. In effect, it is easy to find very recent papers where contextuality is said to be, for example, a “key signature of quantum non-classicality” [20], a “non-classical behaviour that can be exhibited by quantum systems” [21], a “nonclassical feature, challenging our everyday intuition” [22], or that “Contextuality and nonlocality are nonclassical properties exhibited by quantum statistics” [23]. The fact that, on the one hand, we have experimental evidence that noncontextuality inequalities can be violated by classical systems, whereas, and on the other hand, it is frequently claimed, with no further justification, that contextuality is a signature of non-classicality, shows that the relationship between contextuality and classicality need some clarification; this is the aim of this paper.

The way we see it, this almost unanimous acceptance of contextuality as a signature of non-classicality is due to the fact that the notions of classicality and “naive realism” have been scrambled together and have become synonyms, so that, in many cases, when one call contextuality a non-classical feature one solely means that it is not compatible with “naive realist” descriptions, namely NCHV-like models. This is precisely the case with one of the most famous model-independent frameworks to contextuality, the compatibility-hypergraph approach [5]. In this framework, “non-contextuality” consists, by definition, in the existence of a global section for the data we are analyzing (definition 7), which is equivalent to say that the data do not violate any noncontextuality inequality; “classicality”, in turn, means, by definition, the existence of a NCHV-like model for the data (definition 5). One can easily show that both these definitions are equivalent [5], so, according to this framework, “contextuality” and “non-classicality” are equivalent concepts, corresponding to obstructions to the existence of NCHV-like models for the data. We believe any signature of non-classicality must be an obstruction to phenomenological explanations based on classical physics; violations of noncontextuality inequalities are obstructions only to the existence of naive realist descriptions, and we have argued that such an obstruction does not imply that non-classical phenomena are required for a proper explanation of the data under analysis. Therefore, we believe the notions of contextuality and non-classicality must be disentangled.

In this paper we will try to convince the reader that
an “epistemic” view on measurements might be capable, at least in some situations, of turning contextuality (i.e., the violation of noncontextuality inequalities) into an expected consequence of experimental procedures. In order to do so we will present a very intuitive – and maybe oversimplified – “thought experiment”, that is basically a straightforward generalization of the idea of a coin toss, where a classical device based on epistemic measurements generates data violating the famous KCBS inequality, the same inequality the first classical experimental done by Tao Li and his collaborators [17] has violated. The idea is that, if measurements do not reveal properties possessed by a system to the agent, and if there is a restriction on what can be measured jointly, then we have no ground for expecting noncontextuality in the data, whether or not the system is, in any sense, “non-classical”. This simple example reinforces what we have said about violations of noncontextuality inequalities by quantum systems, namely that it only “defies classical understanding” if all projective measurements involved in the experiment undoubtlydrepresent properties of the system. Besides that, the classical device we consider constitutes a black box, i.e., an input/output device, which is in accordance with many “model-independent” or “operational” frameworks to contextuality. Our device works along the lines set out in Refs. [24, 25], for example. Therefore, regardless of what we say about quantum contextuality, a weaker conclusion is still possible, namely that noncontextuality inequalities have no phenomenological significance per se. The fact that, unlike locality, contextuality cannot be defined in terms of physical principles suggests that contextuality is not a physical phenomenon, let alone a non-classical phenomenon, thus noncontextuality inequalities are not on the same footing as Bell inequalities, as Christian de Ronde points out [4]. Given that many model-independent frameworks intending to make contextuality analysis of experimental data “beyond quantum mechanics” possible have been proposed in recent years [26, 27, 28, 21, 29], and that many authors motivate these constructions by saying that contextuality is a non-classical or counter-intuitive feature of quantum systems [26, 29, 5, 13], we believe this discussion is important for model-independent contextuality either.

2 Black boxes, noncontextuality inequalities and the CH approach

The example we propose should be thought of as a thought experiment or a toy mechanism, and, as we mentioned, it constitutes a “black box” or “input/output device” operating along the lines set out in Refs. [24, 25]. In this section we motivate this model-independent view on experimental setups and introduce the compatibility-hypergraph (CH) approach to contextuality [5], which is the framework we will work with.

Model-independent or operational views on physics have been dominating many contemporary discussions in quantum foundations, and contextuality is no exception. As Sandu Popescu puts it, such views are grounded on the realization that, allegedly, physics can be represented “in a way that is largely independent of the details of the specific underlying theories” [30]. For all practical purposes, it means that experiments can be seen as input/output devices” or “black boxes”. Popescu exemplifies this perspective as follows [30].

“Suppose Alice has a box that accepts inputs $x$ and yields outputs $a$. One can imagine that inside the box there is an automated laboratory, containing particles, measuring devices, and so on. The laboratory is prearranged to perform some specific experiments; the input $x$ simply indicates which experiment is to be performed. Suppose also that for every measurement we know in advance the set of the possible outcomes; the output $a$ is simply a label that indicates which of the results has been obtained. In this framework, the entire physics is encapsulated in $P(a|x)$, the probability that output a occurs given that measurement $x$ was made.”

(emphasis added). Note that an entire laboratory counts as a black box, so, according to this view, basically any experimental setup is a legitimate device.

Popescu emphasizes the model-independent view to illustrate how nonlocality “beyond quantum mechanics” is possible: it does not matter what Alice and Bob have inside their boxes, if the data they obtain violates Bell’s inequalities, we can say that non-locality has been witnessed. As he says, “the entire physics is encapsulated” in the data. Similarly, when contextuality is defined in terms of noncontextuality inequalities or some similar concept like global sections, contextuality “beyond quantum mechanics” becomes mathematically – though not necessarily conceptually – meaningful: if a experimental setup generates contextual data (namely data violating noncontextuality inequalities or, equivalently, having no global section), we can say that contextuality has been witnessed; from this viewpoint, contextuality is a property of experimental data, as those who believe that the entire physics is encapsulated in the data should expect. As we mentioned, lots of model-independent frameworks to contextuality have been proposed in recent years [26, 27, 28, 21, 29]. These frameworks allow us to characterize and quantify contextuality regardless of the
reason why it occurs. In this paper we will work with the so-called compatibility-hypergraph (CH) approach to contextuality, which is closely related to the so-called sheaf-theoretic approach [26]. We will briefly introduce it now. A more precise description of this framework can be found in appendix A.

The starting point of the CH approach is the idea of measurement scenario (definition 2). It consists in a finite set \( A \) representing measurements that can be performed over some physical system, together with a collection \( C \) of maximal contexts. A context is a set \( C \subseteq A \) of compatible measurements, i.e., measurements which can be performed jointly, and we assume that the collection \( C \) satisfies the following conditions: (1) Any \( C \subseteq C \) is maximal, i.e., if \( C', C'' \subseteq C \) satisfy \( C'' \subseteq C' \) then \( C' = C'' \), and (2) each measurement belongs to at least one context, that is, \( \cup C = A \). Denoting by \( O \) the set of possible outcomes for all measurements, we say that the triple of sets \( S \equiv (A, C, O) \) is a measurement scenario [5, 26].

If the system is prepared in a certain way and all measurements of a context \( C \) are performed, a joint outcome, which we represent as a function \( C \to O \), will be obtained. If we prepare the system in exactly the same way – by “exactly the same” we mean that the agent cannot distinguish both preparation procedures – and run the experiment a second time, we will not necessarily obtain the same joint outcome. It means that a preparation procedure does not determine an outcome for each measurement of \( C \) but it actually defines a probability distribution \( p(\cdot|C) \) on the set \( O^C \) of all functions \( C \to O \); this probability distribution is experimentally accessible by means of successive runs of the experiment. This is true for any context \( C \subseteq C \); therefore a preparation procedure defines a mapping \( C \mapsto p(\cdot|C) \) associating to each context \( C \) a probability distribution \( p(\cdot|C) \) on \( O^C \). This mapping \( p \) is said to be a behavior (or empirical model [26]) in the measurement scenario \( S \) – see definition 3.

A behavior \( p \) in a scenario \( S \) is said to be noncontextual (definition 7) if a probability distribution \( p \) on \( O^A \) exists such that, for any context \( C \), its marginalization on \( O^C \) coincides with \( p(\cdot|C) \); otherwise \( p \) is said to be contextual. The set of all noncontextual behaviors in a scenario \( S \) defines a polytope, so it can be characterized by finitely many linear inequalities; a noncontextuality inequality is a linear inequality characterizing the polytope of noncontextual behaviors on \( S \); it is satisfied by all noncontextual behaviors but violated by some contextual ones – we suggest Ref. [5] for an explicit definition. The important point is that a noncontextuality inequality is violated by a behavior \( p \) only if \( p \) is contextual, whereas a contextual behavior necessarily violates some noncontextuality inequality of the scenario.

To conclude this section, let’s see how this definition of contextuality relates with random variables. Let \( p \) be a behavior in a scenario \( S \equiv (A, C, O) \). We say that a probability space \( \Lambda \equiv (\Lambda, \Sigma, \mu) \) is a classical realization for \( p \) if a mapping \( A \to f_A \) associating measurements \( A \subseteq A \) to random variables \( f_A : A \to O \) exists such that, for any context \( C \), \( p(\cdot|C) \) is the joint distribution of the collection \( f_A, A \in C \) (see definition 5). Note that this definition is strongly based on definition 10, i.e., a classical realization is basically a noncontextual hidden variable model for \( p \), and a precise connection between these definitions is established by the idea of quantum behavior (see definition 6). Classical realizations are what we called “NCHV-like models” in the introduction. Finally, a classical realization for \( p \) exists if and only if \( p \) is noncontextual, as one can easily show [5]. Therefore, an experimental test showing that a quantum behavior violates a noncontextuality inequality tells us that we cannot describe this behavior using a “NCHV-like model”, as we said in the introduction.

## 3 Classical observables, coin toss and epistemic measurements

We now motivate the choices we make in our classical device. The measurement procedures we consider are strongly based on coin tossing, which in turn is frequently presented as an example of (or at least as operationally equivalent to) a valid measurement procedure in operational theories – see, for example, Refs. [10, 31]. We also explain what we mean by “epistemic” measurement in classical physics.

For the sake of clarity, let’s recall how to describe a system of \( N \)-particles in classical mechanics. To begin with, we fix a phase space - which we assume to be \( \Lambda \equiv \mathbb{R}^{3N} \times \mathbb{R}^{3N} \) here - encoding the collection of all possible pure states for this system with respect to an inertial reference frame previously chosen. A (pure or ontic) state is a pair \( \lambda \equiv (q, p) \), where \( q \in \mathbb{R}^{3N} \) encodes the positions of all particles and \( p \in \mathbb{R}^{3N} \) encodes their momenta. A mechanical property (we also say mechanical observable or simply observable) of a classical system can always be represented as a (measurable) function \( f : \Lambda \to \mathbb{R} \), and the reason is that, according to the realist foundation of classical mechanics, the ontic state of a system, encoded in the theory by a pure state, determines ‘the way things are’ [6]: if \( A \) is a mechanical property, by knowing the ontic state \( \lambda \) of the system we must be able to know the value of \( A \), given that this state allows us to know everything that, from a mechanical perspective, can be known about the system; thus, the theoretical representation of \( A \) is a function \( f_A \) associating each ontic state \( \lambda \) to the value \( A \) assumes in this state. Note that, by construction, the value of any (mechanical) observable can be inferred from the values
of only two classical properties: position and momentum.

A pure state \( \lambda \) in a classical system ascribes values for all (mechanical) observables, which means that it defines a “valuation” on \( \mathcal{M}(\Lambda) \), the set off all measurable functions on \( \Lambda \), by \( V_\lambda(f) \equiv f(\lambda) \) – note that, if \( g = h \circ f \), then \( V_\lambda(g) = h(f(\lambda)) = h(V_\lambda(f)) \). A standard measurement in classical mechanics (which we call ontic measurement here) is a procedure that reveals to the agent the value of an observable. When such measurement is performed, only one outcome is obtained, namely the value determined by the pure state \( \lambda \) describing the ontic state of the system at that time. If the agent knows this state, she can predict this value with certainty; if she does not have complete knowledge about the system, then all she can do is to determine a probability measure \( \mu \) (or, equivalently, a probability density) on \( \Lambda \), which will allow her to infer, given any Borel set \( \Delta \subset \mathbb{R} \), the probability \( \mu(f^{-1}(\Delta)) \) of obtaining an outcome \( a \in \Delta \) in this measurement, being \( f \) (the theoretical representation of) the observable she is measuring [6].

As we know, a coin toss is a procedure that is completely different from an ontic measurement in classical mechanics, and this distinction is far from being explained by the fact that we usually do not have complete knowledge about the state of a coin when it is tossed. In fact, even though a coin is an object that can be described by classical mechanics – i.e., a coin is a classical system –, the outcomes of a coin toss, namely heads or tails, are not values of an observable of this system. There is no mechanical observable being measured when a coin is tossed; there is nothing like a “flipness” property for such a system. Outcomes of a coin toss only make sense when the measurement event happens: asking someone if the coin I have in my pocket is heads or tails is totally meaningless – not because he does not have complete knowledge about the state of the system but because the measurement has not been performed (or, to put it differently, because there is no such thing as “flipness”). To sum up, a coin toss is a measurement procedure based on a classical system (a coin) that satisfies the following conditions.

- There is no property of the classical system associated with the measurement event.
- Outcomes are intrinsically dependent on the measurement procedure and are meaningless unless the measurement is performed. In particular, they cannot be “possessed” by the system.

From a theoretical perspective, we can say that, if there is no mechanical property associated with a given measurement (like a coin toss), there is no reason to require that the formalism of classical mechanics must be able to represent this measurement as a function on the phase space \( \Lambda \) of the system – in the case of a coin toss, we can represent a coin as a system of 3-particles, for the sake of simplicity. This is true even when pure states allow us to predict with certainty the outcome of the measurement – again, coin tossing is an example. Such representation would be misleading, because it would suggest that, if \( f \) is the function representing the measurement and \( \lambda \) is the ontic state of the system at a given time, then \( f(\lambda) \) is the value of this measurement at that time, contradicting the fact that there is no value if no measurement is performed.

A coin toss is an example of what we call an epistemic measurement in classical mechanics. Following Bohr’s well known view on measurements [8, 32], by an epistemic measurement we mean a reproducible interaction between a classical system and an apparatus (which can range from a photon to something way more complex) satisfying the following conditions.

(a) Every time it happens, the interaction gives rise to a measurement event (which we call the outcome of the measurement), depending on the state of the classical system right before the beginning of the interaction.

(b) Each state (pure or not) of the system determines a probability distribution (possibly deterministic) on the set of outcomes of the epistemic measurement. This distribution is experimentally accessible by means of successive runs of the experiment.

(c) The measurement event does not necessarily reveal a property of the classical system to the agent. Consequently, assertions about outcomes are, in principle, meaningless if no interaction occurs.

Note that the distinction we draw between ontic and more general epistemic measurements is only possible because we have direct access to macroscopic objects and because we have at hand the conceptual framework of classical mechanics. We can only say that a specific measurement “does not reveal a property of the classical system” because we know what a property in classical mechanics is.

4 A contextual but non-disturbing behavior

As we said in section 2, we will use the CH approach to contextuality to analyze data. The measurement scenario (definition 2) we work with is the well known n-cycle scenario. A n-cycle is a scenario \( S_n \) containing \( n \) dichotomic measurements \( A_0, \ldots, A_n \) and \( n \) maximal contexts \( C_i \equiv \{A_i, A_{i+1}\}, i = 0, \ldots, n \) (if \( i = n \), then \( i + 1 \equiv 0 \)). Given a behavior \( p \) on \( S_n \) (definition 3), we will sometimes denote its component \( C_i \) by \( p(\cdot|A_i, A_{i+1}) \)
We have only four possible joint outcomes in context \( C_i \), namely \( (\perp, \perp), (\perp, \top), (\top, \perp) \) and \( (\top, \top) \). We are interested in the following famous example of maximally contextual behavior in the 5-cycle (see Ref. [5]).

**Definition 1 (generalized coin toss)** Let \( S_5 \) be the 5-cycle. A “generalized coin toss” on \( S_{n} \) is the behavior \( p \) defined as follows. For any context \( C_i \equiv \{ A_i, A_{i+1} \} \),

\[
p(\cdot | C_i) \equiv p(\cdot | A_i, A_{i+1}) \quad \text{is given by}
\]

\[
p(\perp, \perp | A_i, A_{i+1}) = 0 = p(\top, \top | A_i, A_{i+1}) \quad (1)
\]

\[
p(\perp, \top | A_i, A_{i+1}) = \frac{1}{2} = p(\top, \perp | A_i, A_{i+1}) \quad (2)
\]

Straightforward calculations show that this behavior is nondisturbing (definition 4). As a consequence, \( p \) associates, for any measurement \( A_i \), a unique (i.e., context-independent) probability distribution \( p(\cdot | A_i) \) on \( \{ \perp, \top \} \) – just take any component of \( p \) containing \( A_i \) and marginalize it. All measurements \( A_i, i = \ldots, 4 \), have the same distribution, given by

\[
p(\perp | A_i) = \frac{1}{2} = p(\top | A_i). \quad (3)
\]

In the CH approach, two measurement events (definition 8) are said to be **exclusive** if they ascribe distinct outcomes to a common measurement (definition 9). In a \( n \)-cycle scenario we can say, without loss of generality, that two events \( (u_i, u_{i+1}| C_i) \), \( (v_j, v_{j+1}| C_j) \) are exclusive if and only if one of the following conditions are satisfied: (1) \( i = j \) and \( (u_i, u_{i+1}) \neq (v_j, v_{j+1}) \) or (2) \( j = i + 1 \) and \( u_{i+1} \neq v_{i+1} \). For any \( i = 0, \ldots, 4 \), denote by \( E_i \) the measurement event \( (\top, \perp | A_i, A_{i+1}) \). The **exclusivity graph** determined by these events consists in a graph \( G \) having \( \{ E_0, \ldots, E_4 \} \) as set of vertices where a pair of vertices is connected by an edge if and only if they are exclusive events. This graph \( G \) is cyclic and can be represented as follows.

By definition, the generalized coin toss ascribe probability \( \frac{1}{2} \) for each of these events (see definition 8), which implies that \( \sum_{i=0}^{4} p(E_i) = \frac{5}{2} \). This number is greater than the Lovász number of this graph, which is known to be \( \sqrt{5} \). As proved in [27], the fact that \( \sum_{i=0}^{4} p(E_i) \) is strictly greater than the Lovász number of \( G \) implies that this probability assignment on \( G \) cannot be reproduced by quantum theory, in the sense that we cannot associate each vertex \( E_i \) to a projection \( P_i \) on a finite-dimensional Hilbert space \( H \) in such a way that the following conditions are satisfied: (1) \( P_i P_j = 0 = P_j P_i \) whenever \( E_i \) and \( E_j \) are exclusive and (2) there is a density operator \( \rho \) on \( H \) satisfying, for any \( i = 0, \ldots, 4 \), \( p(E_i) = \text{tr}(\rho P_i) \).

It proves the well known result [5] that the generalized coin toss has no quantum realization (definition 6); in particular, it is contextual (definition 7) or, equivalently, non-classical (definition 5). Finally, it can be shown that it violates the KCBS inequality [5].

5 **Inside the device: a coin of many sides**

Now let’s show how a classical input/output device can give rise to the behavior we named “generalized coin toss”.

As we have discussed in section 3, a coin toss is a measurement procedure which, by means of a convention, returns one out two previously fixed characteristics (opposite sides) of a coin. Given that, for all practical purposes, a coin has only one pair of opposite sides, we usually understand a coin toss as a measurement procedure where only one dichotomic measurement is performed. In a similar fashion, by conceiving a classical object having \( n \) pairs of opposite sides, we can propose \( n \) distinct epistemic measurements - one for each pair of opposite sides - to be performed with such an object. We represent this object as a regular polygon of \( 2n \) sides, and figure 1 depicts it for \( n = 5 \). That is what we mean by a “coin of many sides”.

![Figure 1: Schematic representation of the object.](image)

A coin toss can be idealized as a rotation around a
fixed axis passing through the center of the coin, and so will be our measurements; the only difference is that, in the case of a “generalized coin toss”, we must take contexts into account. In this device, a context consists of an axis around which the classical object can rotate, together with a detector that is capable of identifying colors and distinguishing between light and dark color. In figure 1 we represent the mechanism determining context $C_2 \equiv \{A_2, A_3\}$; the vertical dashed line represents the axis, and the detector is represented by the dashed segment at the right hand side of the figure. The complete mechanism, including all contexts, is depicted in Figure 2; note that each context (i.e., each pair “axis + detector”) is identified by a type of dashed line.

Figure 2: Schematic representation of the measuring device.

In our thought experiment, this toy mechanism lies inside a black box, which means that the agent only has access to the data this device generates. Following the description of black box for contextuality analysis described in Ref. [25], we can imagine that outside the black box there are buttons, one for each measurement, and that pressing a button is equivalent to performing a measurement. The axes (i.e., contexts) determine which buttons can be pressed at the same time, therefore only pairs of consecutive buttons can be pressed simultaneously. If a single button is pressed (let’s say, $A_0$), one of the two contexts containing this measurement (see figure 3) will be randomly selected by the mechanism – alternatively, we can let the agent choose the context –, and a ‘coin toss’ around the axis composing this context will be performed, which means that that an uncontrolled kick will rotate the object around this axis. If we are measuring, for example, $A_0$, at the end of the movement the detector will return $(\top | A_0)$ in case it detects “dark red”, and $(\bot | A_0)$ in case it detects “light red”; the probability of both these events is $\frac{1}{2}$, because the measurement procedure is an idealized coin toss. The same statistics will be obtained for any single measurement. Similarly, if we press a pair of consecutive buttons simultaneously, i.e., if we perform a joint measurement within a context, a “coin toss” around the axis composing this context will be performed. If we measure $A_2$ and $A_3$ jointly, for example, the detector will return $(\top, \bot | A_2, A_3)$ in case it detects “dark yellow” and “light green”, and $(\bot, \top | A_2, A_3)$ in case it detects “light yellow” and “dark green”; note that, thanks to a mere geometrical property of the object, these are the only possible joint outcomes for this joint measurement (see figure 1), and the analogous result holds for any context. Whenever we perform a joint measurement $A_i, A_{i+1}$ in this device, the probability of obtaining $(\top, \bot | A_i, A_{i+1})$ is half, just as the probability of obtaining $(\bot, \top | A_i, A_{i+1})$; the probability of obtaining $(\top, \top | A_i, A_{i+1})$ or $(\bot, \bot | A_i, A_{i+1})$ is zero. Thus, if all we are able to do is to perform this context-dependent collection of epistemic measurements upon this simple classical system, the data we will obtain will be, in the limit of many trials, the famous example of contextual (equivalently, non-classical) behaviour we named “generalized coin toss” (definition 1). Therefore, the data generated by this black box is, according to standard contextuality analysis, non-classical, and it violates the KCBS inequality.

Figure 3: Schematic representation of the pair of contexts containing measurement $A_0$ (red).

To conclude this section, we would like to discuss the connection between contextuality and context-dependence of outcomes in our thought experiment. Let $A_i$ be any measurement and $C_i, C_{i+1}$ the pair of contexts containing it. It make sense to say that, according to this device, given a final configuration of the object, contexts $C_i$ and
As one measurement only generate the “generalized coin toss” if, for at least one measurement \( A_i \), outcomes are context-dependent, i.e., the detectors associated to \( C_i \) and \( C_{i+1} \) lies in opposite sides of the object.

The interest in experiments where quantum systems violate noncontextuality inequalities comes from the fact that, as we said, they reinforce Kochen-Specker theorem by suggesting that quantum measurements are not revealing pre-existing values of properties. These violations indicate that the values we obtain when we perform measurements are assigned to the system during the act of measuring, and, more than that, that this assignment is context-dependent (otherwise it would define a valuation, conflicting with Kochen-Specker theorem). In our device, outcomes are “created” during the measurement procedure in a context-dependent way, and, as we showed, without this dependence on contexts we would not be able to generate contextual data. Therefore, our device violates noncontextuality inequalities precisely because it is in accordance with what is expected from quantum systems when such inequalities are violated by them. In our device, this dependence on contexts is not “counter-intuitive”, nor does it “challenges our everyday intuition”, illustrating how natural the non-existence of valuations – and, consequently, the non-existence of NCHV models – can be when measurements are epistemic.

\[ \frac{1}{2} = p(\top, \bot, \top, \bot|A_0, A_1, A_2, A_3, A_4) \]
\[ = p(\bot, \bot, \bot, \bot|A_0, A_1, A_2, A_3, A_4) \]

These violations demonstrate that quantum systems do not allow us to access microscopic systems in a manner that is compatible with “naive realist” descriptions, and, as a consequence, it endorses Kochen-Specker theorem by suggesting that quantum theory cannot provide a naive realist description of the microscopic world. Our thought experiment illustrates, in a very intuitive – and maybe oversimplified – way, that contextuality and classicality can harmoniously coexist when measurement

6 Conclusion

We have described an input/output device working along the lines set out in Refs. [24, 25] which is capable of generating contextual – and even non-quantum – data, despite the fact that it is based on a simple classical object. The reason why contextual data is generated by this classical device is twofold. Firstly, we deal with “epistemic” measurements, namely measurements that do not reveal to the agent classical properties possessed by the system. Secondly, the setup is context-dependent, that is to say, not all measurements can be performed simultaneously. According to the epistemic view on self-adjoint operators – a view endorsed by many interpretations of quantum theory, including the orthodox one –, these conditions may be satisfied by measurement setups in quantum systems, thus we believe one should not carelessly assume that quantum contextuality is a signature of non-classicality. In fact, violations of noncontextuality inequalities by quantum systems suggest that projective measurements and context-dependent procedures do not allow us to access microscopic systems in a manner that is compatible with “naive realist” descriptions, and, as a consequence, it endorses Kochen-Specker theorem by suggesting that quantum theory cannot provide a naive realist description of the microscopic world. Our thought experiment illustrates, in a very intuitive – and maybe oversimplified – way, that contextuality and classicality can harmoniously coexist when measurement
procedures are epistemic, so we believe those who agree that quantum theory does not provide a naive realist description of nature should not take for granted that violations of noncontextuality inequalities are signatures of non-classicality. If this is correct, then unless we can say with certainty that all projective measurements involved in an experimental violation of noncontextuality inequalities are “ontic” – something that, to our knowledge, can hardly be done –, we should not say, without further explanation, that this violation “defies classical understanding”. Kochen-Specker theorem is about realist interpretations of quantum theory, it is not about the phenomenological status of microscopic systems, and the fact that quantum theory cannot be understood in a “naive realist” way does not imply that microscopic systems violate the laws of classical physics – we know they do, but for different reasons. Beyond quantum theory, i.e., considering model-independent contextuality, our example suggests that noncontextuality inequalities have no phenomenological significance per se: contextuality analysis of data generated by a black box provides no ground for assertions about the phenomenological status of its underlying structure. In particular, at least in the case of black boxes operating along the lines set out in Refs. [24, 25], contextuality does not require non-classical phenomena. Unlike non-locality, contextuality cannot be defined in terms of physical principles, so it is not a physical phenomenon, let alone a non-classical phenomenon; therefore, as Christian de Ronde points out [4], noncontextuality and Bell inequalities are not on the same footing.

To conclude, it is worth emphasizing that our device is just a thought experiment whose purpose is to intuitively illustrate that, if measurements are not associated with properties of a system, then violations of noncontextuality inequalities should be expected, whether or not the system under study is, in any sense, “non-classical”. As we mentioned, there already exists experimental evidence that classical systems can violate such inequalities [17], so our intention is just to discuss the reasons why such violations occur. We believe this discussion is important because the view that contextuality is some kind of signature of non-classicality is still widespread, even after the classical violations we mentioned. Furthermore, it is important to recall that many model-independent frameworks to contextuality [26, 21, 33, 27] and many operational theories [34] require basically nothing from measuring devices; coin flips, which motivate our example, are frequently presented as valid measurement procedures in operational theories [31, 10]. Also, as we said, our device follows the rules imposed in Refs. [24, 25]. Thus, even if all we have said about quantum theory is misleading, we believe our example at least shed some light on the dangers of approaching contextuality – and physics in general – from a purely operational perspective, specially when we are exceedingly permissive with the notion of measurement procedure. When we ignore the underlying structure of an experimental setup and analyze only the data this setup generates, we may leave aside aspects of the experiment that are crucial for a proper understanding of the phenomena we are studying. Thus, as we see it, the entire physics does not seem to be encapsulated in the data, as the model-independent view on physics suggests. Last but not least, we should not forget that model-independent frameworks to contextuality were originally proposed only as tools for characterizing and quantifying contextuality, not as a means for phenomenological speculation.

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A Compatibility-hypergraph approach to contextuality

The main reference of this appendix is Ref. [5].

Definition 2 (Scenario) A scenario is a triple $S \equiv (A, C, O)$ where $A, O$ are finite sets (whose elements represent measurements and outcomes respectively) and $C$ is a collection of subsets of $A$ (representing maximal sets of compatible measurements) satisfying the following conditions.

(a) $A = \cup C$

(b) For $C, C' \in C$, $C' \subset C'$ implies $C' = C$

The approach is named "hypergraph-approach" because a scenario $(A, C, O)$ can be associated with a hypergraph whose vertices are the elements of $A$ and whose hyperedges are the elements of $C$ [5].

The result of a joint measurement over a context $C$ can be represented as a function $C \rightarrow O$. Therefore, the set $O^C$ of all functions $C \rightarrow O$ can be understood as the set of all possible outcomes of a joint measurement on $C$. Behavior allows us to encode experimental data obtained from joint measurements on a measurement scenario[5].

Definition 3 (behavior) Let $S$ be a scenario. A behavior on $S$ is a function $p$ which associates to each context $C$ a probability distribution $p(\cdot | C)$ on $O^C$, that is to
say, for each context $C$, $p(\cdot|C)$ is a function $O^C \to [0,1]$ satisfying $\sum_{s \in O^C} p(u(C)) = 1$.

A behavior whose components match in intersections of contexts is said to be non-disturbing [5]:

**Definition 4 (Non-disturbance)** A behavior $p$ in a scenario $S$ is said to be non-disturbing if, for any pair of intersecting contexts $C, D$, equality

$$p(\cdot|C \cap D, C) = p(\cdot|C \cap D, D)$$

holds true, where, for any $E \in C$ and $E' \subset E$, $p(\cdot|E', E)$ denotes the marginal of $p(\cdot|E)$ on $O^{E'}$, namely

$$\forall t \in O^{E'}: \quad p(t|E', E)(t) = \sum_{s \in O^E \cap s|E' = t} p(s|E).$$

Classical and quantum behaviors arise when ideal measurements are performed upon classical and quantum systems respectively. They are defined as follows [5]:

**Definition 5 (Classical behavior)** Let $p$ be a behavior in a scenario $S \equiv (A, C, O)$. A probability space $A \equiv (\Lambda, \Sigma, \mu)$ is said to be a classical realization for $p$ if we can associate each measurement $A$ of $S$ to a random variable $f_A : A \to O$ on $A$ is such a way that, for any context $C$, $p(\cdot|C)$ is the joint distribution of $f_A : A \in C$, which means that, for any $s \in O^C$,

$$p(s|C) = \mu \left( \bigcap_{A \in C} f_A^{-1}(\{s_A\}) \right),$$

where $s_A \equiv s(A)$. We say that a behavior $p$ is classical if a classical realization for $p$ exists.

Note that classical realizations are closely related to NCHV models (definition 10). The definitions of quantum realization and quantum behavior goes as follows [5]:

**Definition 6 (Quantum behavior)** Let $p$ be a behavior in a scenario $S \equiv (A, C, O)$. A pair $(H, \rho)$, where $H$ is a finite dimensional Hilbert space and $\rho$ is a density operator on $H$, is said to be a quantum realization for $p$ if we can associate each measurement $A \in A$ to a selfadjoint operator $T_A \in B(H)$ is such a way that the following conditions are satisfied.

(a) for any measurement $A$, $O$ is isomorphic to the spectrum of $T_A$

(b) The image of each context lies inside a commutative algebra, i.e., if $A, B \in C$ for some context $C$, then $[T_A, T_B] = 0$

- For each context $C$, $p(\cdot|C)$ can be reproduced by the Born rule, i.e., for any $s \in O^C$,

$$p(s|C) = \text{tr} \left( \rho \prod_{A \in C} P_{s_A}^{(A)} \right),$$

where $P_{s_A}^{(A)}$ denotes the projection associated with the subspace of $H$ spanned by the eigenvalue $s_A$ of $A$.

We say that a behavior $p$ is quantum if a quantum realization for $p$ exists.

The definition of non-contextuality in the CH approach goes as follows [5]:

**Definition 7 (Non-contextuality)** A behavior $p$ in a scenario $S$ is said to be non-contextual if there is a probability distribution $\overline{p} : O^A \to [0,1]$ satisfying, for any context $C$,

$$p(\cdot|C) = \overline{p}_C,$$

where $\overline{p}_C$ denotes the marginal of $\overline{p}$ in $O^C$, i.e., for any $s \in O^C$,

$$\overline{p}_C(s) = \sum_{t \in O^A \mid t|_C = s} \overline{p}(t).$$

One can easily prove that non-contextuality and classicality (i.e., being classical) are equivalent concepts in the compatibility-hypergraph approach [5].

It is easy to show that any classical behavior has a quantum realization, which means that any classical behavior is quantum [5]. It is also easy to show that any quantum behavior is non-disturbing [5]. Therefore, if we denote by $\mathcal{NC}(S)$, $\mathcal{L}(S)$ and $\mathcal{N\mathcal{D}}(S)$ the sets of noncontextual (or, equivalently, classical), quantum and non-disturbing behaviors, respectively, on a scenario $S$, we obtain the well known chain of inclusions

$$\mathcal{NC}(S) \subset \mathcal{L}(S) \subset \mathcal{N\mathcal{D}}(S).$$

The behavior we are interested in (definition 1) is disturbing but lies outside the quantum set.

To conclude, we introduce measurement events and the exclusivity relation between them.

**Definition 8 (Measurement event)** Let $S$ be a measurement scenario. A measurement event on $S$ is pair $(s|C)$ where $C$ is a context and $s$ is a joint outcome on $C$ (i.e., $s \in O^C$). Given a behavior $p$ on $S$, the probability of $(s|C)$ with respect to $p$ is the number $p(u(C))$ determined according to definition 3.

**Definition 9 (Exclusive events)** Let $S$ be the a scenario and let $E \equiv (u(C), F \equiv (v|D)$ be measurement events on it. Then $E$ and $F$ are said to be exclusive if there is a measurement $A \in C \cap D$ such that $u(A) \neq v(A)$.

Note that, given any context $C$, $(u(C))$ and $(v|C)$ are exclusive if and only if $u \neq v$. 

11
B Hidden variable models for quantum systems

By a noncontextual hidden variable model for a quantum system we mean the following [1, 3].

Definition 10 (NCHV model) Let $H$ be a separable Hilbert space. Let $M = (\Lambda, \Phi, \Omega)$ a triple where $\Lambda \equiv (\Lambda, \Sigma)$ is a measurable space, $\Phi$ is a mapping associating each bounded selfadjoint operator $A \in \mathcal{B}(H)$ to a measurable function $f_A \equiv \Phi(A)$ on $\Lambda$ onto $\sigma(A)$ and, finally, $\Omega$ is a mapping associating each pure state $P_\psi \equiv \langle \psi, \cdot \rangle \psi$ to a probability measure $\Omega(P_\psi) \equiv \mu_\psi$ on $\Lambda$. Then $M$ is said to be a NCHV model for $H$ if the following conditions are satisfied.

(a) $\Phi$ “commutes with the Borel functional calculus”, i.e., for any bounded selfadjoint operator $A$ and any measurable function $g : \sigma(A) \rightarrow \mathbb{R}$ we have $f_g(A) = g \circ f_A$.

(b) For any bounded selfadjoint operator $A \in \mathcal{B}(H)$ and any quantum state $P_\psi \equiv \langle \psi, \cdot \rangle \psi$, $\langle \psi, A\psi \rangle = \int_{\Lambda} f_A \, d\mu_\psi$.

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