EXACT CATEGORIES OF TOPOLOGICAL VECTOR SPACES WITH LINEAR TOPOLOGY

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Abstract. We explain why the naïve definition of a natural exact category structure on complete, separated topological vector spaces with linear topology fails. In particular, contrary to [5], the category of such topological vector spaces is not quasi-abelian. We present a corrected definition of exact category structure which works OK. Then we explain that the corrected definition still has a shortcoming in that a natural tensor product functor is not exact in it, and discuss ways to refine the exact category structure so as to make the tensor product functors exact.

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Introduction

Topological algebra is a treacherous ground. So is general topology, of which topological algebra is a part. Well-behaved classes of topological algebraic structures are few and far between, and hard to come by. The aim of this paper is to warn the reader about some of the dangers.
Topological vector spaces as a subject have a functional analysis flavour. In this paper we consider the most “algebraic” class of topological vector spaces, viz., topological vector spaces with linear topology (VSLTs). Hence the results we discuss tend to be true or false irrespectively of the ground field $k$ (which is discrete).

An optimistic vision of topological vector spaces with linear topology was expressed in Beilinson’s paper [5]. In the present paper we explain that some of most basic assertions in [5] are actually not true. The construction of counterexamples which we operate with is taken from the book [2] by Arnautov et al.; in fact, it goes back to the much earlier book of Roelcke and Dierolf [47].

The present author became aware of the failure of straightforward attempts to prove some assertions from [5] back in May 2008, when working on the book [32]. So I tried to exercise extra caution in [32, Section D.1], in order to avoid the traps. The conceptual and terminological system of the July 2018 preprint [4, Sections 1–10] reflected this understanding.

Still we did not know whether the problematic assertions in [5] were true or not true. It was only in October 2018 that I learned about the book [2] and the counterexamples in [2, Theorem 4.1.48]. Subsequently the reference to [2, Theorem 4.1.48] appeared in the July 2019 version of [36]. In fact, the counterexamples in [23, Problem 20D] are already sufficient for some purposes.

Surprisingly, the categories of incomplete (arbitrary or Hausdorff) topological vector spaces have better exactness properties than the category of complete, separated topological vector spaces. In particular, the categories of incomplete topological vector spaces are quasi-abelian (unlike the category of complete ones).

Weakening the axioms of abelian category in order to develop homological functional analysis is a natural idea, which was tried, e.g., in the long paper [31]. The observation that the categories of incomplete topological vector spaces are quasi-abelian, while the category of complete ones is not, was elaborated upon (in the context of locally convex spaces) in the 2000 paper [44].

The problem is that the quotient space of a complete topological vector space by a closed subspace need not be complete. In the functional analysis context, this observation seems to go back to Köthe’s 1947 paper [25], see also [26, §§19.5 and 31.6]. Moreover, in the same context Dierolf proved that any topological vector space is a quotient of a complete one [12, 13, 15, Section 2.1]. This result was generalized to topological abelian groups of quite general nature in the 1981 book [17, Proposition 11.1]; the same construction is presented in [2, Theorem 4.1.48].

Many a mathematician unfamiliar with the subject would guess nowadays that the embedding of topological vector spaces with linear topology into the abelian category of pro-vector spaces resolves many problems. We discuss the idea in detail, and the conclusion is that this is rather not the case. The category of complete, separated topological vector spaces with linear topology, viewed as a full subcategory in the abelian category of pro-vector spaces, is not closed under extensions, and does not inherit an exact category structure.
In the final sections of the paper we study Beilinson’s constructions of three tensor product operations on topological vector spaces with linear topology. These topological tensor product functors have a number of good properties which we verify, but once again we show that there are subtle issues involved. The problem is that, even when a quotient space $C$ of a complete vector space $U$ is complete, it is, generally speaking, impossible to lift a given zero-convergent sequence or family of elements in $C$ to a zero-convergent family of elements in $U$.

All in all, it is not obvious from our discussion (as well as from the one in [44]) that the language of exact categories provides the most suitable category-theoretic point of view on topological algebra, particularly if one is interested in complete topological abelian groups or vector spaces (which is usually the case). Nevertheless, we hope that the present paper will serve as a useful basic reference source on topological algebraic structures with linear topology and their categorical properties.

Let us say a few words about the linear topology terminology used throughout this paper starting from the title. It may be unfamiliar to some people in functional analysis (even though it appears in Köthe’s book [26, §10]). This terminology, going back to Lefschetz’ classics [28 Definitions II.1.2 and II.25.1], is a standard nomenclature in algebra. See, e.g., Fuchs’ book [16 Section 7 in Chapter I], or the book of Beilinson and Drinfeld [6, Section 3.6.1], or the present author’s book [32, Section D.1.1].

One speaks about linear topologies on abelian groups or vector spaces, meaning topologies with a base of neighborhoods of zero formed by subgroups/subspaces; linear topologies on modules over rings, meaning topologies with a base of neighborhood of zero formed by submodules; left and right linear ring topologies (or left and right linear topological rings), meaning topological rings with a base of neighborhoods of zero consisting of left/right ideals, etc. Topological vector spaces with linear topology form a natural class of topological vector spaces over discrete fields, analogous to the class of locally convex topological vector spaces over the normed fields of real or complex numbers in functional analysis.

Let us describe the content of the paper section by section. An introductory discussion of topological algebraic groups with linear topology is presented in Section 1. We proceed to describe the main construction of counterexamples to completeness of quotients, which will play a key role in the rest of the paper, in Section 2. Topological vector spaces with linear topology are introduced and the basic properties of them discussed in Section 3.

The language of exact categories in Quillen’s sense, which forms the category-theoretic background for the discussion in the rest of the paper, is elaborated upon in Section 4. Two important particular cases, namely the quasi-abelian categories and the maximal exact category structures (on weakly idempotent-complete additive categories) are considered in Sections 5 and 6.

The properties of the categories of topological abelian groups and vector spaces with linear topology, in the context of the general theory of additive categories, are discussed in Sections 7 and 8. The reader can find a brief summary in Conclusion 8.9.
The interpretation of complete, separated topological vector spaces as objects of the abelian category of pro-vector spaces is discussed in Section 9. For incomplete topological vector spaces, what we call supplemented pro-vector spaces are needed, and we consider these in Section 10. The theory which we develop in these sections provides a kind of category-theoretic generalization of topological algebra. See Conclusion 10.10 for a brief summary.

The construction of the abelian group/vector space of zero-convergent families of elements, together with the related notions of strongly surjective maps and the strong exact category structure, are presented in Section 11. These play an important role in the theory of contramodules over topological rings, as developed in our recent works [34, 39, 40, 37] and particularly [36, 41]. Besides, this construction occurs as a particular case of some of Beilinson’s topological tensor products (with a discrete vector space), and serves as a source of our counterexamples in this context.

The uncompleted versions of Beilinson’s topological tensor products (for incomplete topological vector spaces) are studied in Section 12, and the completed topological tensor products (of complete VSLTs) are discussed in Section 13. Some natural questions arising in this context we were unable to answer; they are formulated as open questions in Section 13. The reader can find the final summary in Conclusion 13.11.

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1. Abelian Groups with Linear Topology

A topological abelian group $A$ is a topological space with an abelian group structure such that the summation map $+: A \times A \rightarrow A$ is continuous (as a function of two variables) and the inverse element map $- : A \rightarrow A$ is also continuous. A topological abelian group $A$ is said to have linear topology if open subgroups form a base of neighborhoods of zero in $A$. In this paper, all the “topological abelian groups” will be presumed to have linear topology.

A linear topology on an abelian group $A$ is determined by the collection of all open subgroups in $A$. A collection of subgroups in an abelian group $A$ is the collection of all open subgroups in a (linear) topology if and only if it is a filter, i.e., the following conditions are satisfied:

- $A$ is an open subgroup in itself;
- if $U'' \subset U' \subset A$ are subgroups in $A$ and $U'$ is an open subgroup, then so is $U''$;
- the intersection of any two open subgroups in $A$ is open.
A collection of subgroups $B$ in $A$ is a base of neighborhoods of zero in some linear topology on $A$ if and only if

- $B$ is nonempty; and
- for any $U', U'' \in B$ there exists $U \in B$ such that $U \subset U' \cap U''$.

In this case, a subgroup in $A$ is open if and only if it contains a subgroup belonging to $B$.

The *morphisms* of topological abelian groups $f: A \to B$ are the continuous additive maps, that is, the abelian group homomorphisms such that $f^{-1}(U)$ is an open subgroup in $A$ for every open subgroup $U \subset B$.

The *completion* of a topological abelian group $A$ is defined as the projective limit

$$A = \mathfrak{A} = \lim_{\leftarrow U \subset A} A/U,$$

where $U$ ranges over the poset of all open subgroups in $A$ (or equivalently, any base of open subgroups in $A$). The projective limit is taken in the category of abelian groups. The abelian group $\mathfrak{A}$ is endowed with the topology in which the open subgroups $\mathfrak{U} \subset \mathfrak{A}$ are precisely the kernels of the projection maps $\mathfrak{U} = \ker(\mathfrak{A} \to A/U)$, where $U$ ranges over the open subgroups in $A$. When $U \in B$ ranges over a base of open subgroups in $A$, the related subgroups $\mathfrak{U}$ form a base of open subgroups in $\mathfrak{A}$.

The collection of projection maps $A \to A/U$ defines a natural continuous abelian group homomorphism $\lambda_A: A \to \mathfrak{A}$, which is called the *completion map*. A topological abelian group $A$ is said to be *separated* (or *Hausdorff*) if the map $\lambda_A$ is injective. Equivalently, this means that the intersection of all open subgroups in $A$ is the zero subgroup. A topological abelian group $A$ is said to be *complete* if the map $\lambda_A$ is surjective. A topological abelian group $A$ is separated and complete if and only if $\lambda_A$ is an isomorphism of topological abelian groups.

It is clear from the definitions that the completion $\mathfrak{A} = A^\sim$ of any topological abelian group $A$ is a complete, separated topological abelian group.

Let $A$ be a topological abelian group and $K \subset A$ be a subgroup. Then the *induced topology* on $K$ is defined by the rule that the open subgroups in $K$ are the intersections $K \cap U \subset K$, where $U \subset A$ ranges over the open subgroups of $A$.

Let $A$ be a topological abelian group and $p: A \to C$ be a surjective homomorphism of abelian groups. Then the *quotient topology* on $C$ is defined by the rule that a subgroup $W \subset C$ is open in $C$ if and only if its preimage $p^{-1}(W) \subset A$ is an open subgroup in $A$. Given a surjective homomorphism of topological abelian groups $p: A \to C$, the topology on $C$ is the quotient topology of the topology on $A$ if and only if $p$ is a continuous open map. The latter condition means that, for any open subgroup $U \subset A$, its image $p(U) \subset C$ is an open subgroup in $C$.

Let $A$ be a topological abelian group. A subgroup $K \subset A$ is closed in the topology of $A$ if and only if the set-theoretical complement $A \setminus K$ is a union of cosets with respect to open subgroups in $A$. Equivalently, a subgroup in $A$ is closed if and only if it is an intersection of open subgroups. In particular, $A$ is separated if and only if the zero subgroup is closed in $A$. The quotient group $A/K$ is separated in the quotient topology if and only if $K$ is a closed subgroup in $A$. The topological closure
$K_A \subset A$ of a subgroup $K \subset A$ is a subgroup equal to the intersection of all the open subgroups in $A$ containing $K$.

Given an injective homomorphism of topological abelian groups $i: K \to A$, the subgroup $i(K) \subset A$ is closed in $A$ and the topology on $K$ is induced by the topology of $A$ (via the embedding $i$) if and only if $i$ is a continuous closed map. The latter condition means that, for any closed subset $Z \subset K$, its image $i(Z) \subset A$ is a closed subset in $A$. It suffices to consider closed subsets of the form $Z = K \setminus W$, where $W$ ranges over the open subgroups of $K$.

The following lemma explains the connection between closures and completions. Notice that any subgroup of a separated abelian group is separated in the induced topology. As a particular case of the lemma, one obtains the assertion that any closed subgroup of a complete, separated abelian group is complete in the induced topology.

**Lemma 1.1.** Let $\mathcal{B}$ be a complete, separated topological abelian group, and let $A \subset \mathcal{B}$ be a subgroup, endowed with the induced topology. Then the morphism of completions $A^\wedge \to \mathcal{B}^\wedge = \mathcal{B}$ induced by the inclusion $A \to \mathcal{B}$ identifies the topological abelian group $A^\wedge$ with the closure $\overline{A}_{\mathcal{B}} \subset \mathcal{B}$ of the subgroup $A$ in $\mathcal{B}$, where $\overline{A}_{\mathcal{B}}$ is endowed with its induced topology as a subgroup in $\mathcal{B}$.

**Proof.** The following assertions are straightforward to check from the definitions:

(a) for any topological abelian group $B$ and a subgroup $A \subset B$ endowed with the induced topology, the map of completions $A^\wedge \to B^\wedge$ induced by the injective morphism $A \to B$ is also injective;

(b) moreover, in the context of (a), $A^\wedge$ is a closed subgroup in $B^\wedge$;

(c) moreover, in the context of (a), the topology on $A^\wedge$ as the completion of $A$ coincides with the topology induced on $A^\wedge$ as a subgroup in $B^\wedge$;

(d) for any topological abelian group $A$, the image of the completion morphism $\lambda_A: A \to A^\wedge$ is dense in $A^\wedge$, that is, the closure of $\lambda_A(A)$ in $A^\wedge$ coincides with $A^\wedge$.

In the situation at hand, according to (a), the map $A^\wedge \to \mathcal{B}$ is injective. Following (b), $A^\wedge$ is a closed subgroup in $\mathcal{B}$; hence $\overline{A}_{\mathcal{B}} \subset A^\wedge \subset \mathcal{B}$. According to (d), $A$ is dense in $A^\wedge$; so $A^\wedge \subset \overline{A}_{\mathcal{B}} \subset \mathcal{B}$. Finally, (c) tells us that the topology on $A^\wedge$ coincides with the one induced from $\mathcal{B}$. (Cf. [36, Lemma 2.1].) \(\square\)

Let $(A_i)_{i \in I}$ be a family of topological abelian groups. Then the *product topology* (also called the *Tychonoff topology*) on the direct product $\prod_{i \in I} A_i$ of the abelian groups $A_i$ is defined as follows. A base of open subgroups in $\prod_{i \in I} A_i$ is formed by the subgroups of the form $\prod_{j \in J} U_j \times \prod_{i \in I \setminus J} A_i \subset \prod_{i \in I} A_i$, where $J \subset I$ is a finite subset of indices and $U_j \subset A_j$ are open subgroups in $A_j$.

Let $(A_\gamma)_{\gamma \in \Gamma}$ be a projective system of topological abelian groups, indexed by a directed poset $\Gamma$. Then the *projective limit topology* on the projective limit $\lim_{\leftarrow \gamma \in \Gamma} A_\gamma$ of the abelian groups $A_\gamma$ is defined as follows. A base of open subgroups in $\lim_{\leftarrow \gamma \in \Gamma} A_\gamma$ is formed by the full preimages of open subgroups $U_\delta \subset A_\delta$, $\delta \in \Gamma$, under the projection maps $\lim_{\leftarrow \gamma \in \Gamma} A_\gamma \to A_\delta$.  

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One can also construct the coproduct topology on the direct sum of abelian groups $A = \bigoplus_{i \in I} A_i$ in the following way. A subgroup $U \subset \bigoplus_{i \in I} A_i$ is open if and only if, for every $j \in I$, the full preimage $U_j \subset A_j$ of the subgroup $U$ under the natural inclusion map $A_j \rightarrow \bigoplus_{i \in I} A_i$ is an open subgroup in $A_j$. Equivalently, one can say that a base of open subgroups in $\bigoplus_{i \in I} A_i$ is formed by the subgroups of the form $\bigoplus_{i \in I} U_i \subset \bigoplus_{i \in I} A_i$, where $U_i \subset A_i$ are open subgroups in $A_i$.

Denote by $\text{Top}_\mathbb{Z}$ the category of topological abelian groups and continuous additive maps. As usually, we denote by $\text{Ab}$ the category of abelian groups. Then $\text{Top}_\mathbb{Z}$ is an additive category with kernels and cokernels, as well as set-indexed products and coproducts (so $\text{Top}_\mathbb{Z}$ actually has all set-indexed limits and colimits). The forgetful functor $\text{Top}_\mathbb{Z} \rightarrow \text{Ab}$ preserves all the limits and colimits.

Specifically, for any morphism $f: A \rightarrow B$ in $\text{Top}_\mathbb{Z}$, the kernel of $f$ in $\text{Top}_\mathbb{Z}$ is the kernel of $f$ in $\text{Ab}$ endowed with the induced topology as a subgroup in $A$. The cokernel of $f$ in $\text{Top}_\mathbb{Z}$ is the cokernel of $f$ in $\text{Ab}$ endowed with the quotient topology as an epimorphic image of $B$. The products, projective limits, and coproducts in $\text{Top}_\mathbb{Z}$ are, respectively, the products, projective limits, and coproducts in $\text{Ab}$ endowed with the product, projective limit, and coproduct topologies, as described above.

Let $\text{Top}_\mathbb{Z}^\text{sc}$ denote the full subcategory of separated topological abelian groups in $\text{Top}_\mathbb{Z}$. The full subcategory $\text{Top}_\mathbb{Z}^\text{sc} \subset \text{Top}_\mathbb{Z}$ is closed under kernels and products (hence under all limits), and also under coproducts (see below). However, $\text{Top}_\mathbb{Z}^\text{sc}$ is not closed under cokernels in $\text{Top}_\mathbb{Z}$. In fact, $\text{Top}_\mathbb{Z}^\text{sc}$ is a reflective full subcategory in $\text{Top}_\mathbb{Z}$, i.e., the inclusion functor $\text{Top}_\mathbb{Z}^\text{sc} \rightarrow \text{Top}_\mathbb{Z}$ has a left adjoint functor (the reflector). The latter functor assigns to any topological abelian group $A$ its quotient group by the closure of zero subgroup in $A$, considered in the quotient topology, that is $A \mapsto A/\overline{\{0\}}_A$.

The cokernels (and all colimits) in $\text{Top}_\mathbb{Z}^\text{sc}$ can be computed by applying the reflector to the cokernel (resp., colimit) of the same morphism/diagram taken in $\text{Top}_\mathbb{Z}^\text{sc}$. In other words, the cokernel of a morphism $f: A \rightarrow B$ in $\text{Top}_\mathbb{Z}^\text{sc}$ is the quotient group $B/\overline{\{f(A)\}}_B$ of $B$ by the closure of the image of the morphism $f$, where the quotient group is endowed with the quotient topology.

Let $\text{Top}_\mathbb{Z}^\text{wc}$ denote the full subcategory of complete, separated topological abelian groups in $\text{Top}_\mathbb{Z}$. The full subcategory $\text{Top}_\mathbb{Z}^\text{wc} \subset \text{Top}_\mathbb{Z}$ is closed under kernels and products (hence under all limits). In fact, $\text{Top}_\mathbb{Z}^\text{wc}$ is a reflective full subcategory in $\text{Top}_\mathbb{Z}$. The reflector (i.e., the functor left adjoint to the inclusion $\text{Top}_\mathbb{Z}^\text{wc} \rightarrow \text{Top}_\mathbb{Z}$) assigns to any topological abelian group $A$ its completion $\hat{A} = A^\wedge$. The full subcategory $\text{Top}_\mathbb{Z}^\text{wc}$ is also closed under coproducts in $\text{Top}_\mathbb{Z}$, as the following lemma shows.

**Lemma 1.2.** (a) The direct sum of a family of separated topological abelian groups is separated in the coproduct topology.

(b) The direct sum of a family of complete topological abelian groups is complete in the coproduct topology.

**Proof.** Part (a): let $(A_i)_{i \in I}$ be a family of separated topological abelian groups. In order to show that the group $A = \bigoplus_{i \in I} A_i$ is separated it suffices to find, for any nonzero element $a \in A$, an open subgroup $U \subset A$ such that $a \notin U$. Indeed, the element $a$ can be viewed as a collection of elements $(a_i \in A_i)_{i \in I}$ such that $a_i = 0$ for all but finitely many $i \in I$. Thus, for each $i$, there is an open subgroup $U_i \subset A_i$ such that $a_i \notin U_i$. The direct sum $U = \bigoplus_{i \in I} U_i$ is an open subgroup in $A$ that contains $a$. Therefore, $a \notin U$.
for all but a finite subset of indices \( i \in I \). Since \( a \neq 0 \), there exists an index \( j \in I \) such that \( a_j \neq 0 \). Since the topological group \( A_j \) is separated, there exists an open subgroup \( U_j \subset A_j \) such that \( a_j \notin U_j \). Now \( U = U_j \oplus \bigoplus_{i \in I} A_i \subset \bigoplus_{i \in I} A_i \) is an open subgroup such that \( a \notin U \).

Part (b): we will consider the case of a family of complete, separated topological abelian groups \( A_i \); the general case can be easily reduced to that. Put \( A = \bigoplus_{i \in I} A_i \) and \( \mathfrak{A} = A^\sim \). The projection map \( A \rightarrow A_i \) is continuous, so it induces a continuous map \( q_i : \mathfrak{A} \rightarrow A^\sim_i = A_i \). Let \( b \in \mathfrak{A} \) be an element; we want to show that \( q_i(b) = 0 \) for all but a finite subset of indices \( i \in I \).

Choose an open subgroup \( U_i \subset A_i \) for every \( i \in I \) such that \( U_i = A_i \) if \( q_i(b) = 0 \) and \( q_i(b) \notin U_i \) if \( q_i(b) \neq 0 \). Then \( U = \bigoplus_{i \in I} U_i \) is an open subgroup in \( \bigoplus_{i \in I} A_i \). Let \( \mathfrak{U} = U^\sim \) be the related open subgroup in \( \mathfrak{A} \), that is, the kernel of the projection map \( \mathfrak{A} \rightarrow A/U \) (so \( A/U \cong \mathfrak{A}/\mathfrak{U} \)). Then, viewing \( A \) as a subgroup in \( \mathfrak{A} \), we have \( \mathfrak{A} = A + \mathfrak{U} \). Let \( a \in A \) be an element such that \( b - a \in \mathfrak{U} \). The subset \( J \subset I \) of all \( j \in I \) such that \( q_j(a) \neq 0 \) is finite. The square diagram of projections \( \mathfrak{A} \rightarrow A/U \rightarrow A_i/U_i, \mathfrak{A} \rightarrow A_i \rightarrow A_i/U_i \) is commutative. Hence we have \( q_i(\mathfrak{U}) \subset U_i \subset A_i \) and therefore \( q_i(b) - q_i(a) \in U_i \) for every \( i \in I \). For all \( i \notin J \), it follows that \( q_i(b) \in U_i \), thus \( q_i(b) = 0 \).

We have shown that the collection of elements \((q_i(b) \in A_i)_{i \in I}\) defines an element of \( A = \bigoplus_{i \in I} A_i \). Denote this element by \( c \in A \). Let us show that \( b = c \in \mathfrak{A} \). It suffices to check that, for every open subgroup \( U \subset A \), the image of \( b - c \) under the projection map \( \mathfrak{A} \rightarrow A/U \) vanishes. Let \( U_i \subset A_i \) denote the full preimage of \( U \) under the inclusion map \( A_i \rightarrow A \). Then \( U' = \bigoplus_{i \in I} U_i \subset \bigoplus_{i \in I} A_i \) is an open subgroup and \( U' \subset U \). So it suffices to check that the image of \( b - c \) in \( A/U' = \bigoplus_{i \in I} A_i/U_i \) vanishes. This follows from the fact that \( q_i(b) = q_i(c) \) for every \( i \in I \).  

The cokernels (and all colimits) in \( \text{Top}_{\text{sc}} \) can be computed by applying the completion functor \( A \rightarrow A^\sim \) (i.e., the reflector) to the cokernel (resp., colimit) of the same morphism/diagram computed in \( \text{Top}_{\text{sc}} \). In other words, the cokernel of a morphism \( f : A \rightarrow B \) in \( \text{Top}_{\text{sc}} \) is the completion \((B/f(A))^\sim\) of the quotient group \( B/f(A) \) in its quotient topology. The most important fact which we discuss in this paper is that the quotient group of a complete, separated topological abelian group by a closed subgroup need not be complete in the quotient topology. The construction of counterexamples, following [47] and [2], will be presented in the next section.

**Remark 1.3.** Let \( \mathfrak{A} \) be a complete, separated topological abelian group, and let \( \mathfrak{R} \subset \mathfrak{A} \) be a closed subgroup. Denote by \( Q = \mathfrak{A}/\mathfrak{R} \) the quotient group, endowed with the quotient topology. What does it mean that the topological abelian group \( Q \) is (or is not) complete?

Denote by \( \mathfrak{Q} = Q^\sim \) the completion of the topological abelian group \( Q \). By the definition, one has \( \mathfrak{Q} = \lim_{\leftarrow_{W \subset Q}} Q/W \), where \( W \) ranges over all the open subgroups in \( Q \). Let \( p : \mathfrak{A} \rightarrow Q \) denote the projection map. Then the map \( W \mapsto p^{-1}(W) = \mathfrak{U} \) is an ordered isomorphism between the poset of all open subgroups \( W \subset Q \) and the poset of all open subgroups \( \mathfrak{U} \subset \mathfrak{A} \) such that \( \mathfrak{R} \subset \mathfrak{U} \). Thus the composition
\( \mathfrak{A} \rightarrow Q \rightarrow \Omega \) can be interpreted as the natural map of projective limits
\[
\lim_{\leftarrow \mathfrak{U} \subset \mathfrak{A}} \mathfrak{A}/\mathfrak{U} \rightarrow \lim_{\leftarrow \mathfrak{K} \subset \mathfrak{U} \subset \mathfrak{A}} \mathfrak{A}/\mathfrak{U}.
\]
Here the left-hand side is the projective limit of the quotient groups \( \mathfrak{A}/\mathfrak{U} \) taken over the poset of all open subgroups \( \mathfrak{U} \subset \mathfrak{A} \). The right-hand side is the projective limit of the same quotient groups taken over the subposet of all open subgroups \( \mathfrak{U} \subset \mathfrak{A} \) containing \( \mathfrak{K} \).

Why should the map (1) be surjective? An element of the right-hand side is a compatible family of elements \( b = (\bar{a}_\mathfrak{U} \in \mathfrak{A}/\mathfrak{U})_{\mathfrak{K} \subset \mathfrak{U} \subset \mathfrak{A}} \) specified for all the open subgroups \( \mathfrak{U} \subset \mathfrak{A} \) containing \( \mathfrak{K} \). To find a preimage of \( b \) in the left-hand side means to extend the given family of elements \( \bar{a}_\mathfrak{U} \) to a similar family of elements defined for all the open subgroups \( \mathfrak{U} \subset \mathfrak{A} \) (where \( \mathfrak{U} \) no longer necessarily contains \( \mathfrak{K} \)). How to produce the missing elements \( \bar{a}_\mathfrak{U} \), simultaneously for all \( \mathfrak{U} \not\supset \mathfrak{K} \) and in a compatible way? There is no apparent way to do it, and indeed we will see in the next section that this cannot be done (generally speaking).

A positive assertion in the desired direction holds under a countability assumption.

**Proposition 1.4.** Let \( \mathfrak{A} \) be a complete, separated topological abelian group, and let \( \mathfrak{K} \subset \mathfrak{A} \) be a closed subgroup. Assume that the topological abelian group \( \mathfrak{K} \) has a countable base of neighborhoods of zero. Then the quotient group \( \mathfrak{A}/\mathfrak{K} \) is (separated and) complete in the quotient topology.

**Proof.** This is a generalization of [35, Lemma 2.2(b)], provable in a similar way with the following additional argument. Let \( \Delta \) denote the poset of all open subgroups in \( \mathfrak{A} \) with respect to the inverse inclusion, and let \( \Gamma \) be the similar poset of all open subgroups in \( \mathfrak{A} \). Let \( \psi: \Delta \rightarrow \Gamma \) be the map taking any open subgroup \( \mathfrak{U} \subset \mathfrak{A} \) to the open subgroup \( \psi(\mathfrak{U}) = \mathfrak{K} \cap \mathfrak{U} \subset \mathfrak{K} \). Then \( \psi \) is a cofinal map of posets, and \( \Gamma \) has a cofinal countable subposet. Over a countable directed poset, the derived projective limit \( \lim^1 \) of any projective system of surjective maps of abelian groups vanishes.

We need to check that \( \lim^1_{\mathfrak{U} \in \Delta} (\mathfrak{K}/(\mathfrak{K} \cap \mathfrak{U})) = 0 \). The key observation is that the inverse image with respect to a cofinal map of directed poset preserves the derived projective limits of projective systems of abelian groups, that is \( \lim^1_{\gamma \in \Gamma} K_\gamma = \lim^1_{\mathfrak{U} \in \Delta} K_\psi(\mathfrak{U}) \) for any cofinal map of directed posets \( \psi: \Delta \rightarrow \Gamma \), every projective system of abelian groups \( (K_\gamma)_{\gamma \in \Gamma} \), and all \( n \geq 0 \). This is provable using the notion of a weakly flabby (faiblement flasque) projective system, see [19, Théorème 1.8]. Moreover, the derived projective limit of a directed projective system in an abelian category with exact products is an invariant of the related pro-object [43, Corollary 7.3.7].

2. **The Construction of Counterexamples**

This section is based on [2, Theorem 4.1.48] (see [17, Proposition 11.1] for the original exposition). The exposition in [17] and [2] is very general; we specialize it to the particular case of abelian groups with linear topology. In the context of
functional analysis, for topological vector spaces over the field of real numbers with its real topology, counterexamples to completeness of quotients were found in [25], [26], §§19.5 and 31.6 and mentioned in [8, Exercise IV.4.10(b)] and [31, Proposition 11.2]. For another rather general counterexample, see [23, Problem D to Section 20].

The construction of the coproduct topology on a direct sum of topological abelian groups in Section 1 is related to the box topology on a product of topological abelian groups (cf. [2, Proposition 4.1.46]). Let \((A_i)_{i \in I}\) be a family of topological abelian groups. The box topology on the abelian group \(\prod_{i \in I} A_i\) is defined as the topology with a base of neighborhoods of zero consisting of the subgroups of the form \(\prod_{i \in I} U_i \subset \prod_{i \in I} A_i\), where \(U_i \subset A_i\) are open subgroups in the topology of \(A_i\). The coproduct topology on \(\bigoplus_{i \in I} A_i\) is induced from the box topology on \(\prod_{i \in I} A_i\) via the natural embedding \(\bigoplus_{i \in I} A_i \rightarrow \prod_{i \in I} A_i\).

**Lemma 2.1.** (a) The direct product of a family of separated topological abelian groups is separated in the box topology.

(b) The direct product of a family of complete topological abelian groups is complete in the box topology.

**Proof.** Similar to, but much simpler than, the proof of Lemma 1.2. For a more general result (which is also an “if and only if” result) applicable to both the Tychonoff and box topologies, as well as to a family of intermediate topologies between these indexed by cardinal numbers, see [2, Corollary 4.1.44 and Proposition 4.1.47]. \(\square\)

Let \((A_i)_{i \in I}\) be a family of topological abelian groups. The modified box topology on the abelian group \(\prod_{i \in I} A_i\) is defined as follows. A base of neighborhoods of zero in the modified box topology is formed by the subgroups of the form \(\prod_{j \in J} \{0\} \times \prod_{i \in I \setminus J} U_i \subset \prod_{i \in I} A_i\), where \(J \subset I\) is a finite subset of indices and \(U_i \subset A_i\) are open subgroups. For example, when the set of indices \(I\) is finite, the modified box topology on \(\prod_{i \in I} A_i\) is discrete.

**Lemma 2.2.** For any family of topological abelian groups \((A_i)_{i \in I}\), the direct product of abelian groups \(\prod_{i \in I} A_i\) is separated and complete in the modified box topology.

**Proof.** To show that the modified box topology on \(\prod_{i \in I} A_i\) is separated, it suffices to check that the intersection of all open subgroups is zero. Indeed, let \(a = (a_i)_{i \in I}\) be a nonzero element in \(\prod_{i \in I} A_i\). Then there exists \(j \in I\) such that \(a_j \neq 0\) in \(A_j\). Put \(J = \{j\}\) and \(U_l = A_l\) for all \(l \in I\), \(l \neq j\). Then the open subgroup \(\prod_{j \in J} \{0\} \times \prod_{i \in I \setminus J} U_i \subset \prod_{i \in I} A_i\) does not contain \(a\).

To prove that the group \(A = \prod_{i \in I} A_i\) is complete in the modified box topology, notice that, for every \(s \in I\), the projection map \(p_s: \prod_{i \in I} A_i \rightarrow A_s\) is continuous with respect to the modified box topology on \(\prod_{i \in I} A_i\) and the discrete topology on \(A_s\). Put \(\mathfrak{A} = A^\sim\); then we have a continuous homomorphism of the completions \(q_s: \mathfrak{A} \rightarrow A_s\) induced by the projection map \(p_s\). Given an element \(b \in \mathfrak{A}\), consider the element \(c = (q_s(b))_{i \in I} \in A\). We need to show that \(b = \lambda A(c)\) in \(\mathfrak{A}\).

For this purpose, it suffices to check that, for any open subgroup \(U \subset A\) belonging to a chosen base of open subgroups in \(A\), the images of \(b\) and \(c\) are equal in \(A/U\). We
can assume that $U = \prod_{j \in J} \{0\} \times \prod_{i \in I \setminus J} U_i$, where $J \subset I$ is a finite subset and $U_i \subset A_i$ are open subgroups. Put $U_j = 0$ for $j \notin J$. Then the square diagram of projections $\mathfrak{A} \rightarrow A/U \rightarrow A_i/U_i$, $\mathfrak{A} \rightarrow A \rightarrow A_i/U_i$ is commutative for every $i \in I$. So is the square diagram of projections $A \rightarrow A/U \rightarrow A_i/U_i$, $A \rightarrow A_i \rightarrow A_i/U_i$. The images of both $b$ and $c$ in $A_i/U_i$ are equal to the coset $q_i(b) + U_i$. Hence the images of $b$ and $c$ coincide in $A/U = \prod_{i \in I} A_i/U_i$. □

Lemma 2.3. For any family of separated topological abelian groups $(A_i)_{i \in I}$, the subgroup $\bigoplus_{i \in I} A_i \subset \prod_{i \in I} A_i$ is closed in the modified box topology on $\prod_{i \in I} A_i$.

Proof. In fact, $\bigoplus_{i \in I} A_i$ is closed already in the box topology on $\prod_{i \in I} A_i$ (hence in the modified box topology, which is finer than the box topology). Cf. the middle paragraph of the proof of Lemma 1.2(b). An even stronger and more general result can be found in [2, Proposition 4.1.46]. □

Let $(A_i)_{i \in I}$ be a family of topological abelian groups. The modified coproduct topology on the direct sum of abelian groups $\bigoplus_{i \in I} A_i$ is defined as follows. A base of neighborhoods of zero in the modified coproduct topology is formed by the subgroups of the form $\bigoplus_{j \in J} \{0\} \oplus \bigoplus_{i \in I \setminus J} U_i \subset \bigoplus_{i \in I} A_i$, where $J \subset I$ is a finite subset of indices and $U_i \subset A_i$ are open subgroups.

Corollary 2.4. For any family of separated topological abelian groups $(A_i)_{i \in I}$, the direct sum of abelian groups $\bigoplus_{i \in I} A_i$ is separated and complete in the modified coproduct topology.

Proof. By Lemma 2.2, the direct product $\prod_{i \in I} A_i$ is separated and complete in the modified box topology. According to Lemma 2.3, $\bigoplus_{i \in I} A_i$ is a closed subgroup of $\prod_{i \in I} A_i$ in the modified box topology on the product. It is clear from the definitions that the modified coproduct topology on $\bigoplus_{i \in I} A_i$ is induced from the modified box topology on $\prod_{i \in I} A_i$ via the embedding of abelian groups $\bigoplus_{i \in I} A_i \rightarrow \prod_{i \in I} A_i$. Now it remains to apply Lemma 1.1 □

Theorem 2.5 ([17, Proposition 11.1], [2, Theorem 4.1.48]). Any separated topological abelian group can be obtained as the quotient group of a complete, separated topological abelian group by a closed subgroup, endowed with the quotient topology.

Proof. Let $Q$ be a separated topological abelian group. Choose an infinite set $I$, and denote by $\mathfrak{A}_I(Q)$ the abelian group $Q^{(I)} = \bigoplus_{i \in I} Q$, endowed with the modified coproduct topology (of the direct sum of copies of $Q$). Consider the summation map $\Sigma: \mathfrak{A}_I(Q) = Q^{(I)} \rightarrow Q$, defined by the rule that, for every $i \in I$, the composition $Q \rightarrow Q^{(I)} \rightarrow Q$ of the natural inclusion map $i_i: Q \rightarrow Q^{(I)}$ with the map $\Sigma: Q^{(I)} \rightarrow Q$ is equal to the identity map, $\Sigma \circ i_i = \text{id}_Q$.

According to Corollary 2.3, the topological abelian group $\mathfrak{A}_I(Q)$ is separated and complete. In order to show that the topology of $Q$ is the quotient topology of the topology on $\mathfrak{A}_I(Q)$, it remains to check that $\Sigma$ is an open continuous map.

To show that $\Sigma$ is continuous, consider an open subgroup $W \subset Q$. Denote by $\mathfrak{U} \subset \mathfrak{A}_I(Q)$ the subgroup $\mathfrak{U} = W^{(I)} \subset Q^{(I)}$. By the definition of the modified coproduct
topology, the subgroup $U$ is open in $A$. Clearly, $\Sigma(U) \subset W$; so $U \subset \Sigma^{-1}(W)$. Hence the subgroup $\Sigma^{-1}(W) \subset \mathfrak{A}_I(Q)$ is open.

To show that $\Sigma$ is open, it suffices to check that the subgroup $f(U) \subset Q$ is open in $Q$ for any open subgroup $U \subset \mathfrak{A}_I(Q)$ belonging to a chosen base of open subgroups in $\mathfrak{A}_I(Q)$. We can assume that $U = \bigoplus_{j \in J} \{0\} \oplus \bigoplus_{I \in I, j} W_i \subset Q^{(I)}$, where $J \subset I$ is a finite subset and $W_i \subset Q_i$ are open subgroups. Then we have $\Sigma(U) = \sum_{i \in I, j} W_i \subset Q$. Since the set $I$ is infinite by assumption, and consequently the set $I \setminus J$ is nonempty, it follows that $\Sigma(U)$ is an open subgroup in $Q$. \hfill \Box

3. Vector Spaces with Linear Topology

Throughout the rest of the paper, we denote by $k$ a fixed ground field. A topological vector space $V$ is a topological abelian group with a $k$-vector space structure such that the multiplication map $k \times V \to V$ is continuous (where the topology on $k$ is discrete). In other words, this means that the multiplication with any fixed element of $k$ is a continuous endomorphism of $V$.

A topological vector space $V$ is said to have linear topology if open $k$-vector subspaces form a base of neighborhoods of zero in $V$. In this paper, all the “topological vector spaces” will be presumed to have linear topology.

All the basic theory of topological abelian groups developed in Sections 1–2 extends or specializes verbatim to topological vector spaces. In particular, the completion (of the underlying topological abelian group) of a topological vector space is a topological vector space. One can speak of complete and/or separated topological vector spaces, the induced topology on a vector subspace, the quotient topology on a quotient vector space, closed vector subspaces and closures of vector subspaces, and product and coproduct topologies for topological vector spaces in the same way as for topological abelian groups. In the same way as for topological abelian groups, one defines the box topology and the modified box topology on a direct product of topological vector spaces, and the modified coproduct topology on a direct sum.

Similarly to Section 1, we denote by $\text{Top}_k$ the category of topological vector spaces and continuous $k$-linear maps. We also denote by $\text{Vect}_k$ the category of (abstract, nontopological) $k$-vector spaces. The full subcategory of separated topological vector spaces is denoted by $\text{Top}_k^s \subset \text{Top}_k$, and the full subcategory of complete, separated topological vector spaces by $\text{Top}_k^{sc} \subset \text{Top}_k^s \subset \text{Top}_k$.

Then all the three categories $\text{Top}_k$, $\text{Top}_k^s$, and $\text{Top}_k^{sc}$ are additive categories with set-indexed limits and colimits. The full subcategories $\text{Top}_k^s$ and $\text{Top}_k^{sc}$ are reflective in $\text{Top}_k$. The forgetful functor $\text{Top}_k \to \text{Vect}_k$ preserves all limits and colimits. The forgetful functors $\text{Top}_k \to \text{Top}_Z$, $\text{Top}_k^s \to \text{Top}_Z^s$, and $\text{Top}_k^{sc} \to \text{Top}_Z^{sc}$ preserve all limits and colimits, and commute with the reflectors.

All the assertions and results of Sections 1, 2 remain valid in the topological vector space setting. Without repeating all the details, we restrict ourselves to restating Theorem 2.5 for topological vector spaces.
Theorem 3.1. Any separated topological vector space $Q$ can be obtained as the quotient vector space of a complete, separated topological vector space by a closed vector subspace (endowed with the quotient topology). Specifically, for any infinite set $I$, the open, continuous summation map $\Sigma: \mathbb{A}_I(Q) \to Q$ makes $Q$ a topological quotient vector space of the complete, separated topological vector space $\mathbb{A}_I(Q) = Q^\langle I \rangle = \bigoplus_{i \in I} Q$ with the modified coproduct topology. □

In the rest of this section we mostly discuss topological vector spaces with a countable base of neighborhoods of zero. These form a special, well-behaved class of topological vector spaces (cf. Proposition 1.4).

Lemma 3.2. A complete, separated topological vector space has a countable base of neighborhoods of zero if and only if it is isomorphic to the product of a countable family of discrete vector spaces, endowed with the product topology.

Proof. It is clear from the definition of the product topology that the product of a countable family of discrete vector spaces has a countable base of neighborhoods of zero. Such topological vector spaces are also complete and separated, since the full subcategory of complete, separated topological vector spaces is closed under infinite products in $\text{Top}_k$ and contains the discrete vector spaces.

Conversely, let $\mathfrak{Y}$ be a complete, separated topological vector space with a countable base of neighborhoods of zero. Since any countable directed poset has a cofinal subposet isomorphic to the poset of natural numbers, one can choose a descending sequence of open vector subspaces $\mathfrak{Y} = \mathcal{U}_0 \supset \mathcal{U}_1 \supset \mathcal{U}_2 \supset \cdots$ such that the set of all subspaces $\mathcal{U}_n$, $n \geq 0$, is a base of neighborhoods of zero in $\mathfrak{Y}$. Considering $\mathcal{U}_n \supset \mathcal{U}_{n+1}$ as an abstract vector space with a vector subspace, choose a complementary vector subspace $V_n \subset \mathcal{U}_n$; so $\mathcal{U}_n = V_n \oplus \mathcal{U}_{n+1}$ for every $n \geq 0$. Endow the vector spaces $V_n$ with the discrete topology. Since $\mathfrak{Y}$ is complete and separated, we have

$$\mathfrak{Y} \cong \lim_{\leftarrow n \geq 1} \mathfrak{Y}/\mathcal{U}_n \cong \lim_{\leftarrow n \geq 1} \bigoplus_{i=0}^{n} V_i \cong \prod_{i=0}^{\infty} V_i$$

as an abstract vector space; and it is clear from the definitions of a topology base and the product topology that the topologies on $\mathfrak{Y}$ and $\prod_{i=0}^{\infty} V_i$ agree. □

The next proposition shows that complete, separated topological vector spaces with a countable base of neighborhoods of zero have a rather strong injectivity property in $\text{Top}_k$.

Proposition 3.3. Let $V$ be a topological vector space and $U \subset V$ be a vector subspace, endowed with the induced topology. Let $\mathfrak{W}$ be a complete, separated topological vector space with a countable base of neighborhoods of zero. Then any continuous linear map $U \to \mathfrak{W}$ can be extended to a continuous linear map $V \to \mathfrak{W}$.

Proof. According to Lemma 3.2 the topological vector space $\mathfrak{W}$ is isomorphic to a countable product $\prod_{i=0}^{\infty} W_i$ of discrete vector spaces $W_i$, endowed with the product topology. According to the discussion above in this section and in Section 1, this means that the object $\mathfrak{W}$ is the categorical product of the objects $W_i$ in the category
Therefore, it suffices to consider the case of a discrete vector space $\mathcal{W} = W$ in order to prove the proposition.

Let $f : U \to W$ be a continuous linear map. Since $W$ is discrete, the zero subspace $\{0\} \subset W$ is open in $W$; hence the subspace $K = \ker(f) = f^{-1}(0)$ is open in $U$. By the definition of the induced topology on $U$, there exists an open vector subspace $L \subset V$ such that $U \cap L = K$. The linear map $f$ factorizes through the surjection $U \to U/K$; so we have a linear map (of discrete vector spaces) $\bar{f} : U/K \to W$. The vector space $U/K$ is a subspace in the (discrete) vector space $V/L$; so the map $\bar{f}$ can be extended to a linear map $\bar{g} : V/L \to W$. Now the composition $V \to V/L \to W$ provides a desired continuous linear extension $g : V \to W$ of the map $f$.

Corollary 3.4. Let $V$ be a topological vector space and $\mathfrak{U} \subset V$ be a vector subspace, endowed with the induced topology. Assume that the topological vector space $\mathfrak{U}$ is complete, separated, and has a countable base of neighborhoods of zero. Then there exists a topological vector space $W$ and an isomorphism of topological vector spaces $V \simeq \mathfrak{U} \oplus W$ forming a commutative triangle diagram with the inclusion $\mathfrak{U} \hookrightarrow V$ and the direct summand inclusion $\mathfrak{U} \hookrightarrow \mathfrak{U} \oplus W$.

Proof. In any additive category with kernels or cokernels, any retract is a direct summand. In the situation at hand, denote the inclusion map by $\iota : \mathfrak{U} \to V$. By Proposition 3.3 the identity map $\mathfrak{U} \to \mathfrak{U}$ can be extended to a continuous linear map $g : V \to \mathfrak{U}$ such that $g \circ \iota = \text{id}_\mathfrak{U}$. Now the topological vector space $W$ can be constructed as the kernel of $g$ or the cokernel of $\iota$ in the category $\text{Top}_k$. 

A complete, separated topological vector space $\mathfrak{V}$ (with linear topology) is called linearly compact (or pseudo-compact, or profinite-dimensional) if all its open subspaces $\mathfrak{U} \subset \mathfrak{V}$ have finite codimension in $\mathfrak{V}$, i.e., the quotient spaces $\mathfrak{V}/\mathfrak{U}$ are finite-dimensional. For any discrete vector space $V$, the dual vector space $V^* = \text{Hom}_k(V, k)$ has a natural linearly compact topology in which the open subspaces are the annihilators of finite-dimensional subspaces in $V$. The vector space $V$ can be recovered as the space of all continuous linear functions $\mathfrak{V} \to k$ on its dual vector space $\mathfrak{V} = \text{Hom}_k(V, K)$. The correspondence $V \leftrightarrow \mathfrak{V}$ is an anti-equivalence between the category of discrete vector spaces $\text{Vect}_k$ and the category of linearly compact vector spaces. Hence the category of linearly compact vector spaces is abelian.

A topological vector space is linearly compact if and only if it is isomorphic to a product $\prod_{i \in I} k$ of copies of the one-dimensional discrete vector space $k$ over some index set $I$, endowed with the product topology. Following the proofs of Proposition 3.3 and Corollary 3.4 one can see that the former holds for an arbitrary linearly compact vector space $\mathfrak{W}$, and the latter holds for any linearly compact vector space $\mathfrak{U}$ (not necessarily with a countable base of neighborhoods of zero).
4. Exact Categories

Reasonably modern expositions on exact categories (in Quillen’s sense) include Keller’s [21, Appendix A], [14, Appendix], the present author’s [33, Appendix A], and Bühler’s long paper [9]. In this section we use some material from [33].

An **exact category** \( E \) is an additive category endowed with a class of **short exact sequences** (known also in the recent literature as **conflations**) \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \). A morphism \( E' \rightarrow E \) appearing in a short exact sequence \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \) is said to be an **admissible monomorphism** (or an **inflation**), and a morphism \( E \rightarrow E'' \) appearing in such a short exact sequence is said to be an **admissible epimorphism** (or a **deflation**). The following axioms must be satisfied:

**Ex0:** The zero short sequence \( 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \) is exact. Any short sequence isomorphic to a short exact sequence is exact.

**Ex1:** In any short exact sequence \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \), the morphism \( E' \rightarrow E \) is the kernel of the morphism \( E \rightarrow E'' \), and the morphism \( E \rightarrow E'' \) is the cokernel of the morphism \( E' \rightarrow E \) in the category \( E \).

**Ex2:** Let \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \) be a short exact sequence. Then (a) for any morphism \( E' \rightarrow F' \) there exists a commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
E' \\
\downarrow \\
F' \\
\downarrow \\
F
\end{array} \quad \begin{array}{c}
E \\
\downarrow \\
E'' \\
\downarrow \\
0
\end{array}
\]

with a short exact sequence \( 0 \rightarrow F' \rightarrow F \rightarrow E'' \rightarrow 0 \); and dually, (b) for any morphism \( F'' \rightarrow E'' \) there exists a commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
E' \\
\downarrow \\
E \\
\downarrow \\
E'' \\
\downarrow \\
\downarrow \\
F' \\
\downarrow \\
F
\end{array} \quad \begin{array}{c}
E \\
\downarrow \\
E'' \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
0
\end{array}
\]

with a short exact sequence \( 0 \rightarrow E' \rightarrow F \rightarrow F'' \rightarrow 0 \).

**Ex3:** (a) The composition of any two admissible monomorphisms is an admissible monomorphism. (b) The composition of any two admissible epimorphisms is an admissible epimorphism.

The above formulation of axiom Ex2 differs from the usual one, but is actually equivalent to it under the assumption of Ex0–Ex1. The usual formulation is:

**Ex2’:** (a) For any admissible monomorphism \( E' \rightarrow E \) and any morphism \( E' \rightarrow F' \) in \( E \), there exists a pushout object \( F = F' \sqcup_{E'} E \) in \( E \), and the natural morphism \( F' \rightarrow F \) is an admissible monomorphism (in other words, the class of admissible monomorphisms is closed under pushouts).

(b) For any admissible epimorphism \( E \rightarrow E'' \) and any morphism \( F'' \rightarrow E'' \) in \( E \), there exists a pullback object \( F = F'' \cap_{E''} E \) in \( E \), and the natural morphism \( F \rightarrow F'' \) is an admissible epimorphism (in other words, the class of admissible epimorphisms is closed under pullbacks).
In fact, for any additive category $\mathcal{E}$ with a class of short exact sequences satisfying $\text{Ex}0–\text{Ex}1$ and $\text{Ex}2(a)$, and for any commutative diagram \( \mathcal{E} \) with short exact sequences $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ and $0 \rightarrow F' \rightarrow F \rightarrow E'' \rightarrow 0$, the commutative square $E' \rightarrow E \rightarrow F$, $E' \rightarrow F' \rightarrow F$ is a pushout square \[ \text{Proposition A.2}. \] So one has $F = F' \sqcup_{E'} E$, and $\text{Ex}2'(a)$ follows. Conversely, to deduce $\text{Ex}2(a)$ from $\text{Ex}2'(a)$ under $\text{Ex}0–\text{Ex}1$, it suffices to observe that $F = F' \sqcup_{E'} E$ implies $\ker(F' \rightarrow F) = \ker(E' \rightarrow E) = E''$.

Dually, for any additive category $\mathcal{E}$ with a class of short exact sequences satisfying $\text{Ex}0–\text{Ex}1$ and $\text{Ex}2(b)$, and for any commutative diagram \( \mathcal{E} \) with short exact sequences $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ and $0 \rightarrow E' \rightarrow F \rightarrow E'' \rightarrow 0$, the commutative square $F \rightarrow E \rightarrow E''$, $F \rightarrow F'' \rightarrow E''$ is a pullback square \[ \text{Section A.4}. \] So one has $F = F'' \cap_{E''} E$, and $\text{Ex}2'(b)$ follows. Conversely, to deduce $\text{Ex}2(b)$ from $\text{Ex}2'(b)$ under $\text{Ex}0–\text{Ex}1$, it suffices to notice that $F = F'' \cap_{E''} E$ implies $\ker(F \rightarrow F'') = \ker(E \rightarrow E'') = E'$.

Moreover, assuming $\text{Ex}0–\text{Ex}2$, the short sequence $0 \rightarrow E' \rightarrow F' \oplus E \rightarrow F \rightarrow 0$ is exact in the context of $\text{Ex}2(a)$ or $\text{Ex}2'(a)$, and dually, the short sequence $0 \rightarrow F \rightarrow E \oplus F'' \rightarrow E'' \rightarrow 0$ is exact in the context of $\text{Ex}2(b)$ or $\text{Ex}2'(b)$ \[ \text{21 Appendix A.} \] \[ \text{14 Appendix}. \]

The following property is known as the “obscure axiom” (see \[ \text{9, Remark 2.17 for a historical discussion}. \] It follows from $\text{Ex}0–\text{Ex}2$ (see \[ \text{21 2nd step of the proof of Proposition A.1} \] or \[ \text{9 Proposition 2.16} \].

$\text{OEx}$: (a) If a morphism $g$ has a cokernel in $\mathcal{E}$ and the composition $fg$ is an admissible monomorphism for some morphism $f$, then $g$ is an admissible monomorphism.

(b) If a morphism $f$ has a kernel in $\mathcal{E}$ and the composition $fg$ is an admissible epimorphism for some morphism $g$, then $f$ is an admissible epimorphism.

An additive category $\mathcal{A}$ is said to be \textit{idempotent-complete} if, for every object $A \in \mathcal{A}$ and an endomorphism $e: A \rightarrow A$ such that $e^2 = e$, there exists a direct sum decomposition $A \simeq B \oplus C$ in $\mathcal{A}$ such that $e$ is the composition of the direct summand projection and the direct summand inclusion $A \rightarrow B \rightarrow A$. In other words, this means that any idempotent endomorphism $e \in \mathcal{A}$ has a kernel, or equivalently, a cokernel, or equivalently, any idempotent endomorphism $e \in \mathcal{A}$ has an image. An additive category $\mathcal{A}$ is said to be \textit{weakly idempotent-complete} if, for every pair of objects $A, B \in \mathcal{A}$ and pair of morphisms $p: A \rightarrow B, \ i: B \rightarrow A$ such that $pi = \text{id}_B$, there exists a direct sum decomposition $A \simeq B \oplus C$ in $\mathcal{A}$ such that $i$ is the direct summand inclusion $B \rightarrow A$ and $p$ is the direct summand projection $A \rightarrow B$. In other words, this means that the morphism $p$ (i. e., any retraction in $\mathcal{A}$) has a kernel, or equivalently, the morphism $i$ (i. e., any section in $\mathcal{A}$) has a cokernel.

The assumption of weak idempotent completeness simplifies the theory of exact categories considerably. In particular, the following axiom for a class of short exact sequences in an additive category $\mathcal{E}$ is equivalent, under the assumption of $\text{Ex}0–\text{Ex}1$, to $\mathcal{E}$ being weakly idempotent-complete (as an additive category) \textit{and} satisfying $\text{Ex}2$ or $\text{Ex}2'$ (as an additive category with a class of short exact sequences):
Ex2′′: (a) If a composition \( fg \) is an admissible monomorphism in \( E \), then \( g \) is an admissible monomorphism.
(b) If a composition \( fg \) is an admissible epimorphism in \( E \), then \( f \) is an admissible epimorphism.
(c) If in the commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow
\end{array} \quad \begin{array}{c}
E' \\
\downarrow \quad E_1 \\
E'' \\
\downarrow \quad 0
\end{array} \quad \begin{array}{c}
E_2
\end{array}
\]

both the short sequences \( 0 \rightarrow E' \rightarrow E_1 \rightarrow E'' \rightarrow 0 \) and \( 0 \rightarrow E' \rightarrow E_2 \rightarrow E'' \rightarrow 0 \) are exact, then the morphism \( E_1 \rightarrow E_2 \) is an isomorphism.

Indeed, conditions Ex2′′(a–b) form a stronger (and simpler) version of the obscure axiom OEx. Any one of these two conditions for a class of short exact sequences satisfying (a small part of) Ex0–Ex2 easily implies that the category \( E \) is weakly idempotent-complete. Conversely, in a weakly idempotent-complete category \( E \), axioms Ex0–Ex1 and Ex2′ imply Ex2′′(a–b) [13 Appendix, part C], [21 Proposition 7.6]. Condition Ex2′′(c) is a common particular case of Ex2(a) and Ex2(b).

Finally, we should mention that, under the assumption of Ex0–Ex2, the conditions Ex3(a) and Ex3(b) are equivalent to each other, so it suffices to require only one of them [21 3rd step of the proof of Proposition A.1] (see also [33 the discussion of diagram (A.3) at the end of Section A.5]).

**Example 4.1.** Let \( E \) be an exact category and \( G \subset E \) be a full additive subcategory. We will say that the full subcategory \( G \subset E \) is closed under extensions if, for any short exact sequence \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \), the object \( E \) belongs to \( G \) whenever both the objects \( E' \) and \( E'' \) do. Similarly, we will say that \( G \) is closed under the kernels of admissible epimorphisms in \( E \) if in the same short exact sequence the object \( E' \) belongs to \( G \) whenever both the objects \( E \) and \( E'' \) do; and \( G \) is closed under the cokernels of admissible monomorphisms in \( E \) if the object \( E'' \) belongs to \( G \) whenever both the objects \( E' \) and \( E \) do.

Finally, we will say that the full subcategory \( G \subset E \) inherits the exact category structure if the class of all short sequences in \( G \) that are exact in \( E \) forms an exact category structure on \( G \). Assume that the full subcategory \( G \) is either closed under extensions, or it is closed under both the kernels of admissible epimorphisms and the cokernels of admissible monomorphisms in \( E \). Then \( G \) inherits the exact category structure from \( E \).

Indeed, if \( G \) is closed under extensions in \( E \), then the axioms Ex0–Ex2 are clearly satisfied for the class of all short sequences in \( G \) that are exact in \( E \). To see that the composition \( A \rightarrow B \rightarrow C \) of two admissible epimorphisms in \( G \) is an admissible epimorphism in \( E \), one observes that the kernel of \( A \rightarrow C \) is an extension of the kernels of \( A \rightarrow B \) and \( B \rightarrow C \) in \( E \).
If $G$ is closed under the kernels of admissible epimorphisms and the cokernels of admissible monomorphisms, then the axioms $E_0$–$E_1$ and $E_3$ are clearly satisfied in $G$. To check the axiom $E_2(a)$, consider a short exact sequence $0 \to E' \to E \to E'' \to 0$ in $G$ and a morphism $E' \to F'$ in $G$. Build a commutative diagram (2) in $E$ with a short exact sequence $0 \to F' \to F \to E'' \to 0$, and recall that the short sequence $0 \to E' \to F' \oplus E \to F \to 0$ is exact in $E$. Since the objects $E'$ and $F' \oplus E$ belong to $G$ and $G$ is closed under the cokernels of admissible monomorphisms in $E$, it follows that $F \in G$.

Given an exact category $E$, a full subcategory $G$ closed under extensions in $E$, endowed with the exact structure inherited from $E$, is called a fully exact subcategory in the surveys [22, Sections 4 and 12], [9, Sections 10.5 and 13.3] and the paper [11, Definition 2.2(ii)]. More generally, a full subcategory $G$ inheriting an exact category structure from $E$, endowed with the inherited exact category structure, is simply called an exact subcategory in [11, Definition 2.2(i)]. A characterization of exact subcategories (or in our terminology, full subcategories inheriting an exact category structure) can be found in [11, Theorem 2.6] (see also [38, Lemma 4.20]).

Let $E$ and $G$ be exact categories. An additive functor $\Psi: E \to G$ is said to be exact if it takes short exact sequences to short exact sequences.

**Example 4.2.** The following construction provides a source of examples of exact category structures that is important for the purposes of our discussion.

Let $E$ and $G$ be exact categories, and $\Psi: E \to G$ be an additive functor. We will say that $\Psi$ preserves the kernels of admissible epimorphisms if for every short exact sequence $0 \to E' \to E \to E'' \to 0$ in $E$ the morphism $\Psi(E') \to \Psi(E)$ is the kernel of the morphism $\Psi(E) \to \Psi(E'')$ in $G$. Similarly, we say that $\Psi$ preserves the cokernels of admissible monomorphisms if for every short exact sequence $0 \to E' \to E \to E'' \to 0$ in $E$ the morphism $\Psi(E) \to \Psi(E'')$ is the cokernel of the morphism $\Psi(E') \to \Psi(E)$ in $G$.

Finally, we will say that a short sequence $0 \to E' \to E \to E'' \to 0$ in $E$ is $\Psi$-exact if $0 \to E' \to E \to E'' \to 0$ is a short exact sequence in $E$ and $0 \to \Psi(E') \to \Psi(E) \to \Psi(E'') \to 0$ is a short exact sequence in $G$. Assume that the functor $\Psi$ either preserves the kernels of admissible epimorphisms, or it preserves the cokernels of admissible monomorphisms. Then we claim that the class of all $\Psi$-exact sequences in $E$ is a (new) exact category structure on $E$. We will call it the $\Psi$-exact category structure. Accordingly, we will speak of $\Psi$-admissible monomorphisms and $\Psi$-admissible epimorphisms in $E$.

Indeed, without any assumptions on $\Psi$ it is clear that the class of all $\Psi$-exact sequences satisfies $E_0$–$E_1$. Assuming that $\Psi$ preserves the kernels of admissible epimorphisms, we will check $E_2$–$E_3$.

$E_2(a)$: given a $\Psi$-exact sequence $0 \to E' \to E \to E'' \to 0$ and a morphism $E' \to F'$ in $E$, we can apply axiom $E_2(a)$ or $E_2'(a)$ in the exact category $E$ and obtain a commutative diagram (2) with $F = F' \sqcup_E E$ and a short exact sequence $0 \to F' \to F \to E'' \to 0$. Since the functor $\Psi$ preserves the kernels of admissible epimorphisms, the morphism $\Psi(F') \to \Psi(F)$ is the kernel of the morphism
Ψ(F) → Ψ(E'') in G. So the morphism Ψ(F) → Ψ(E'') has a kernel, and the composition Ψ(E) → Ψ(F) → Ψ(E'') is an admissible epimorphism in G. According to property OEEx(b) in the exact category G, it follows that Ψ(F) → Ψ(E'') is an admissible epimorphism in G. Thus 0 → Ψ(F') → Ψ(F) → Ψ(E'') → 0 is a short exact sequence in G. By the definition, this means that the short sequence 0 → F' → F → E'' → 0 is Ψ-exact in E, as desired.

Ex2(b): given a Ψ-exact sequence 0 → E' → E → E'' → 0 and a morphism F'' → E'' in E, we apply axiom Ex2(b) or Ex2'(b) in E and obtain a commutative diagram (4) with \( F = F'' \cap_{E''} E \) and a short exact sequence 0 → E' → F → F'' → 0. According to the discussion above, the short sequence 0 → F → E ⊕ F'' → E'' → 0 is exact in E. Since Ψ preserves the kernels of admissible epimorphisms, it follows that the morphism Ψ(F') → Ψ(E) ⊕ Ψ(F'') is the kernel of the morphism Ψ(E) → Ψ(E'') in G. In other words, this means that Ψ(F) = Ψ(F'') \cap_{\Psi(E'')} \Psi(E). Now axiom Ex2'(b) in the exact category G tells that the short sequence 0 → Ψ(E') → Ψ(F) → Ψ(F'') → 0 is exact in G. Thus the short sequence 0 → E' → F → F'' → 0 is Ψ-exact in E.

Ex3: following the above discussion, it suffices to check Ex3(b). Let f and g be a pair of composable Ψ-admissible epimorphisms in E. By axiom Ex3(b) in the exact category E, the composition \( fg : E \to E'' \) is an admissible epimorphism in E; so we have a short exact sequence 0 → E' → E → E'' → 0 in E. Since Ψ preserves the kernels of admissible epimorphisms, the morphism Ψ(E') → Ψ(E) is a kernel of the morphism Ψ(E) → Ψ(E''). By the definitions, the morphisms Ψ(f) and Ψ(g) are admissible epimorphisms in G; so by axiom Ex3(b) in the category G, the morphism \( \Psi(fg) : \Psi(E) \to \Psi(E'') \) is an admissible epimorphism in G, too. Hence the short sequence 0 → Ψ(E') → Ψ(E) → Ψ(E'') → 0 is exact in G, the short sequence 0 → E' → E → E'' → 0 is Ψ-exact in E, and the morphism \( fg : E \to E'' \) is a Ψ-admissible epimorphism.

5. Quasi-Abelian Categories

Quasi-abelian categories \[52, 48\] form the second most well-behaved class of additive categories after the abelian ones. We refer to \[49, 50\] for a historical and terminological discussion of quasi-abelian categories. In the language of general category theory, the quasi-abelian categories can be described as the regular and coregular additive categories (in the sense of \[3\]).

Let \( A \) be an additive category. Throughout this section, we will assume that the kernels and cokernels of all morphisms exist in \( A \) (such additive categories \( A \) are nowadays called preabelian). We will say that a morphism \( k \) in \( A \) is a kernel if \( k \) is a kernel of some morphism in \( A \) (in this case, \( k \) is a kernel of its cokernel). Similarly, a morphism \( c \) in \( A \) is said to be a cokernel if it is a cokernel of some morphism in \( A \). Obviously, any kernel is a monomorphism and any cokernel is an epimorphism, but the converse need not be true.
Given a morphism \( f: A \to B \) in \( A \) with a kernel \( K = \ker(f) \) and a cokernel \( C = \coker(f) \), one denotes the cokernel of the morphism \( K \to A \) by \( \coker(K \to A) = \text{coim}(f) \) and the kernel of the morphism \( B \to C \) by \( \ker(B \to C) = \text{im}(f) \). Then the original morphism \( f: A \to B \) decomposes naturally as \( A \to \text{coim}(f) \to \text{im}(f) \to B \), with a uniquely defined morphism \( \text{coim}(f) \to \text{im}(f) \) in between.

**Proposition 5.1.** For any additive category \( A \) with kernels and cokernels, the following four conditions are equivalent:

1. for any morphism \( f \) in \( A \), the natural morphism \( \text{coim}(f) \to \text{im}(f) \) is an epimorphism;
2. the composition of any two kernels is a kernel in \( A \);
3. if a composition \( fg \) is a kernel in \( A \), then \( g \) is a kernel;
4. for any pair of morphisms \( A' \to A \) and \( A' \to B' \) in \( A \) such that \( A' \to A \) is a kernel, there exists a commutative square

\[
\begin{array}{ccc}
A' & \longrightarrow & A \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B
\end{array}
\]

where \( B' \to B \) is a monomorphism.

**Proof.** See [33, Example (7) and “Proof of (7)” in Section A.5].

The equivalence \( (1) \iff (4) \) can be also found in [18, Proposition 1], and the implications \( (1) \implies (2), \ (1) \implies (3) \) in [18, Proposition 2]. The equivalences \( (1) \iff (2) \iff (3) \) can be found in [24, Proposition 3.1]. See [24, Section 3] for a historical discussion with references.

**Proposition 5.2.** For any additive category \( A \) with kernels and cokernels, the following four conditions are equivalent:

1. for any morphism \( f \) in \( A \), the natural morphism \( \text{coim}(f) \to \text{im}(f) \) is a monomorphism;
2. the composition of any two cokernels is a cokernel in \( A \);
3. if a composition \( fg \) is a cokernel in \( A \), then \( f \) is a cokernel;
4. for any pair of morphisms \( A \to A'' \) and \( B'' \to A'' \) in \( A \) such that \( A \to A'' \) is a cokernel, there exists a commutative square

\[
\begin{array}{ccc}
A & \longrightarrow & A'' \\
\uparrow & & \uparrow \\
B & \longrightarrow & B''
\end{array}
\]

where \( B \to B'' \) is an epimorphism.

**Proof.** Dual to Proposition 5.1. □

Let us endow the additive category \( A \) with the class of all short sequences \( 0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0 \) satisfying Ex1 (i.e., all such sequences in which \( i \) is the kernel
of $p$ and $p$ is the cokernel of $i$). An additive category $A$ with kernels and cokernels is said to be quasi-abelian if this (most obvious) class of short exact sequences is an exact category structure on $E$.

**Theorem 5.3.** An additive category $A$ with kernels and cokernels is quasi-abelian if and only if the following three conditions hold:

(a) $A$ satisfies the equivalent conditions of Proposition 5.1;
(b) $A$ satisfies the equivalent conditions of Proposition 5.2;
(c) condition Ex$^{2''}(c)$ holds for the class of all short sequences satisfying Ex1 in $A$.

Furthermore, $A$ is quasi-abelian if and only if any pushout of a kernel is a kernel and any pullback of a cokernel is a cokernel in $A$.

**Proof.** Clearly, all pushouts and pullbacks (as well as generally all finite colimits and limits) exist in an additive category with cokernels and kernels. So the condition that any pushout of a kernel is a kernel is a reformulation of Ex$^{2'}(a)$, and the condition that any pullback of a cokernel is a cokernel is a reformulation of Ex$^{2'}(b)$ (for the class of all short sequences satisfying Ex1 in $A$).

Furthermore, the condition that any pushout of a kernel is a kernel implies the condition of Proposition 5.1(4), which is equivalent to 5.1(2), which is a restatement of Ex3(a). Similarly, the condition that any pullback of a cokernel is a cokernel implies 5.2(4), which is equivalent to 5.2(2), which is a restatement of Ex3(b). Thus if the class of all short sequences satisfying Ex1 in $A$ satisfies Ex$^{2'}$, then it also satisfies Ex3. This proves the second assertion of the theorem.

Finally, the condition of Proposition 5.1(3) is a restatement of condition Ex$^{2''}(a)$, and the condition of Proposition 5.2(3) is a restatement of condition Ex$^{2''}(b)$. Since Ex$^{2''}$ is equivalent to Ex2 for a weakly idempotent-complete additive category with a class of short exact sequences satisfying Ex0–Ex1 (see the discussion in Section 4), the first assertion of the theorem follows. □

In the terminology of [48, 49], an additive category $A$ with kernels and cokernels is said to be right semi-abelian if it satisfies the equivalent conditions of Proposition 5.1, and $A$ is left semi-abelian if it satisfies the equivalent conditions Proposition 5.2. In the language of general category theory, an additive category with kernels and cokernels is left semi-abelian if and only if it has a (regular epimorphism, monomorphism)-factorization, and right semi-abelian if and only if it has an (epimorphism, regular monomorphism)-factorization.

An additive category is called semi-abelian if it is both left and right semi-abelian. In other words, an additive category $A$ with kernels and cokernels is semiabelian if and only if, for every morphism $f$ in $A$, the natural morphism $\text{coin}(f) \to \text{im}(f)$ is both an epimorphism and a monomorphism.

In the terminology of [49], an additive category $A$ with kernels and cokernels is right quasi-abelian if any pushout of a kernel is a kernel in $A$, and $A$ is left quasi-abelian if any pullback of a cokernel is a cokernel. In the language of general category theory [3], an additive category with kernels and cokernels is left quasi-abelian if and only if it is regular, and right quasi-abelian if and only if it is coregular.
It is clear from Propositions 5.1(4) and 5.2(4) that any right quasi-abelian category is right semi-abelian, and any left quasi-abelian category is left semi-abelian [48, Corollary 1]. By the second assertion of Theorem 5.3, a category is quasi-abelian if and only if it is both left and right quasi-abelian.

The following result can be found in [48, Proposition 3].

**Corollary 5.4.** An additive category is quasi-abelian if and only if it is right quasi-abelian and left semi-abelian, or equivalently, if and only if it is right semi-abelian and left quasi-abelian. In other words, a semi-abelian category is right quasi-abelian if and only if it is left quasi-abelian.

**Proof.** Following the discussion in Section 4 (or more specifically, [33, Proposition A.2 in Section A.4]), any one of the conditions Ex2'(a) or Ex2'(b) implies Ex2''(c). So the condition of Theorem 5.3(c) holds for any additive category that is either right or left quasi-abelian, and the corollary follows from the first assertion of Theorem 5.3. □

The question whether all semi-abelian categories are quasi-abelian came to be known as the *Raikov problem* or *Raikov conjecture*, with the reference to Raikov’s papers [45, 46]. The conjecture was explicitly formulated in the paper [27]. The condition of our Theorem 5.3(c) explicitly appears in the papers [45, 27, 46] as one of the axioms.

Counterexamples to Raikov’s conjecture were discovered in the papers [7, 49, 50, 53, 55]. The example in [49] has algebraic (representation-theoretic) flavour, while the examples in [7, 50, 53, 55] come from functional analysis (the theory of locally convex and bornological spaces). A historical discussion can be found in [55]. A very simple algebraic counterexample was also given in [33, Example A.5 in Section A.4].

**Example 5.5 ([33, Example A.5]).** Let A be the additive category whose objects are morphisms of (e.g., finite dimensional) k-vector spaces \( f: V'' \to V' \) endowed with the following additional datum: a vector subspace \( V \subset \text{im}(f) \) is chosen in the vector space \( \text{im}(f) \). We will denote the objects of A by the symbols \( (V'' \xrightarrow{f} V') \). Morphisms in A are defined in the obvious way.

Then the forgetful functor from A to the abelian category of morphisms of vector spaces \( f: V'' \to V' \) (forgetting the additional datum) is faithful and preserves kernels and cokernels. Hence it follows easily that the category A is semi-abelian, and moreover, any pushout of a monomorphism in A is a monomorphism and any pullback of an epimorphism is an epimorphism (so the category A is *integral* in the sense of [48, 17]). To see that A is not quasi-abelian, it suffices to consider the diagram

\[
\begin{array}{ccc}
(0 \xrightarrow{0} k) & \xrightarrow{(k \xrightarrow{id} k)} & (k \xrightarrow{0} 0) \\
\downarrow & & \downarrow \\
(k \xrightarrow{id} k)
\end{array}
\]

showing that A does not satisfy the condition of Theorem 5.3(c).
A simple algebraic example of an additive category which is right quasi-abelian but not left semi-abelian can be found in [29, Example 4.2] (while [29, Example 4.1] is a left quasi-abelian but not right semi-abelian additive category). In fact, both of these are examples of balanced additive categories with kernels and cokernels which are not abelian (where “balanced” means that any morphism which is simultaneously a monomorphism and an epimorphism is an isomorphism).

For an example of a left quasi-abelian but not right semi-abelian additive category arising in functional analysis, see [53, Example 4.2] and [24, Example 4.1].

The category of complete, separated topological vector spaces with linear topology is right quasi-abelian but not left semi-abelian, as we will see below in Section 8. For an analytic version, see [44, Propositions 4.1.6(b) and 4.1.14] or [24, Example 4.2].

6. Maximal Exact Structures

Let $A$ be a fixed additive category. Then exact category structures on $A$ are ordered by inclusion (of their classes of short exact sequences). Clearly, there is a minimal or split exact category structure, in which only the split short sequences $0 \rightarrow A' \rightarrow A' \oplus A'' \rightarrow A'' \rightarrow 0$ are considered to be exact.

It was shown in the paper [53] that any additive category with kernels and cokernels admits a maximal exact category structure. This result was generalized to weakly idempotent-complete additive categories in the paper [10]. A maximal exact category structure on an arbitrary additive category was constructed in [51].

In this section, we mostly follow [10]. So we assume $A$ to be a weakly idempotent-complete additive category (see Section 4). As in Section 5 a morphism in $A$ is said to be a (co)kernel if it is a (co)kernel of some morphism in $A$. If a morphism $i$ is a kernel and has a cokernel, then $i$ is the kernel of its cokernel. Dually, if $p$ is a cokernel and has a kernel, then $p$ is the cokernel of its kernel.

A morphism $A' \rightarrow A$ in $A$ is said to be a semi-stable kernel [10] if, for every morphism $A' \rightarrow B'$ in $A$, there exists a pushout object $B = B' \sqcup_{A'} A$, and the natural morphism $B' \rightarrow B$ is a kernel in $A$. Taking $B' = 0$, one can see that any semi-stable kernel has a cokernel.

Dually, a morphism $A \rightarrow A''$ in $A$ is said to be a semi-stable cokernel if, for every morphism $C'' \rightarrow A''$ in $A$, there exists a pullback object $C = C'' \cap_{A''} A$, and the natural morphism $C \rightarrow C''$ is a cokernel in $A$. Taking $C'' = 0$, one can see that any semi-stable cokernel has a kernel.

**Proposition 6.1.** In any weakly idempotent-complete additive category $A$, the class of all semi-stable cokernels has the following properties:

(a) any pullback of a semi-stable cokernel is a semi-stable cokernel;
(b) the direct sum of two semi-stable cokernels is a semi-stable cokernel;
(c) the composition of two semi-stable cokernels is a semi-stable cokernel;
(d) if the composition $fg$ is a semi-stable cokernel, then $f$ is a semi-stable cokernel.
Proof. Part (a) is easy (see [10, Lemma 2.2]). Part (b) is [10, Lemma 3.2]. Part (c) is [10, Proposition 3.1]. Part (d) is [10, Proposition 3.4]. □

Proposition 6.2. In any weakly idempotent-complete additive category $A$, the class of all semi-stable kernels has the following properties:

(a) any pushout of a semi-stable kernel is a semi-stable kernel;
(b) the direct sum of two semi-stable kernels is a semi-stable kernel;
(c) the composition of two semi-stable kernels is a semi-stable kernel;
(d) if the composition $fg$ is a semi-stable kernel, then $g$ is a semi-stable kernel.

Proof. Dual to Proposition 6.1. □

A short sequence $0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \rightarrow 0$ in $A$ is said to be stable exact if the morphism $i$ is a kernel of the morphism $p$, the morphism $p$ is a cokernel of the morphism $i$, the morphism $i$ is a semi-stable kernel, and the morphism $p$ is a semi-stable cokernel. In this case, $i$ is said to be a stable kernel and $p$ is said to be a stable cokernel. In other words, a stable kernel is defined as a semi-stable kernel whose cokernel is semi-stable; a stable cokernel is defined as a semi-stable cokernel whose kernel is semi-stable.

Theorem 6.3. For any weakly idempotent-complete additive category $A$, the class of all stable short exact sequences in $A$ is an exact category structure on $A$. This is the maximal exact category structure on $A$ (i.e., in any exact category structure on $A$, all short exact sequences are stable).

Proof. The second assertion follows immediately from axioms Ex1 and Ex2′. Both the assertions are the result of [10, Theorem 3.5]. □

Remark 6.4. Theorem 6.3 is surprising in the following aspect, which was overlooked in [33, Example (8) in Section A.5]. Let $0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \rightarrow 0$ be a stable short exact sequence in $A$. So, for any morphism $A' \rightarrow B'$, the pushout object $B = B' \sqcup_{A'} A$ exists, and the morphism $j: B' \rightarrow B$ is a (semi-stable) kernel; and dually, for any morphism $C'' \rightarrow B''$, the pullback object $C = C'' \cap_{A''} A$ exists, and the morphism $q: C \rightarrow C''$ is a (semi-stable) cokernel.

Why does the morphism $j$ have to be a stable kernel, and why does the morphism $q$ have to be a stable cokernel? In other words, why does a pushout object $D = B' \sqcup_{A'} C$ exist, or why does a pullback object $D = C'' \cap_{A''} B$ exist? (One can see that it must be the same object, appearing in the same stable short exact sequence $0 \rightarrow B' \rightarrow D \rightarrow C'' \rightarrow 0$, as the pushouts and the pullbacks of short exact sequences in an exact category commute with each other.)

Here is why. Consider the diagonal morphism $C'' \oplus B \rightarrow A''$, and let $E = (C'' \oplus B) \cap_{A''} A$ be the related pullback (which exists since the morphism $p$ is a semi-stable cokernel). The morphism $p$ factorizes as $A \rightarrow C'' \oplus B \rightarrow A''$, and consequently there is a morphism $A \rightarrow E$ such that the composition $A \rightarrow E \rightarrow A$ is the identity morphism. Since the category $A$ is weakly idempotent-complete by assumption, it follows that $E = A \oplus D$, where $D$ is the kernel of $E \rightarrow A$ or the
cokernel of $A 	o E$. This object $D$ is the kernel of the morphism $C'' \oplus B \to A''$, hence it is the desired pullback $D = C'' \cap_{A''} B$.

**Example 6.5.** Let $A$ be a quasi-abelian additive category (see Section 3). Then all the kernels and cokernels in $A$ are semi-stable, and consequently all of them are stable. The quasi-abelian exact category structure on $A$ (that is, the class of all short sequences satisfying Ex1) is its maximal exact category structure.

**Example 6.6.** Let $A$ be a right quasi-abelian additive category. By the definition, it means that all the morphisms in $A$ have kernels and cokernels, and all kernels in $A$ are semi-stable. It follows that all the semi-stable cokernels are stable.

If $A$ is right quasi-abelian but not left quasi-abelian (equivalently, right quasi-abelian but not left semi-abelian), then there exist cokernels in $A$ that are not semi-stable cokernels. The kernels of such cokernels are semi-stable but not stable kernels.

The maximal exact category structure on $A$ consists of all the short sequences $0 \to A' \to A \to A'' \to 0$ satisfying Ex1 such that the morphism $A \to A''$ is a semi-stable cokernel, or equivalently, the morphism $A' \to A$ is a stable kernel.

7. **Categories of Incomplete VSLTs are Quasi-Abelian**

It is claimed in [5, page 1, Section 1.1] that the category $\text{Top}^{sc}$ is quasi-abelian, and in particular it has an exact category structure in which the admissible monomorphisms are the closed embeddings and the admissible epimorphisms are the open surjections. These assertions are not true. In the next Section 8 we will explain why.

Surprisingly, the category $\text{Top}^{k}$ of arbitrary (not necessarily complete or separated) topological vector spaces with linear topology has better exactness properties; in fact, it is quasi-abelian. So is the category $\text{Top}^{s}$ of separated, but not necessarily complete topological vector spaces (with linear topology).

**Theorem 7.1.** (a) The category $\text{Top}^{Z}$ of (not necessarily complete or separated) topological abelian groups with linear topology is quasi-abelian.

A continuous homomorphism $i: K \to A$ is a kernel (i.e., an admissible monomorphism in the quasi-abelian exact structure) in $\text{Top}^{Z}$ if and only if $i$ is injective and the topology of $K$ is induced from the topology of $A$ (on $K$ viewed as a subgroup in $A$ via $i$).

A continuous homomorphism $p: A \to C$ is a cokernel (i.e., an admissible epimorphism in the quasi-abelian exact structure) in $\text{Top}^{Z}$ if and only if $p$ is surjective and the topology of $C$ is the quotient topology of the topology of $A$ (via $p$); in other words, this means that $p$ is a surjective open map.

(b) The category $\text{Top}^{k}$ of (not necessarily complete or separated) topological vector spaces with linear topology is quasi-abelian.

A continuous linear map $i: K \to V$ is a kernel (i.e., an admissible monomorphism in the quasi-abelian exact structure) in $\text{Top}^{k}$ if and only if $i$ is injective and the topology of $K$ is induced from the topology of $V$ (on $K$ viewed as a subspace in $V$ via $i$).
A continuous linear map \( p: V \to C \) is a cokernel (i.e., an admissible epimorphism in the quasi-abelian exact structure) in \( \text{Top}_k \) if and only if \( p \) is surjective and the topology of \( C \) is the quotient topology of the topology of \( V \) (via \( p \)); in other words, this means that \( p \) is a surjective open map.

**Proof.** This result is well-known; see [42, Proposition 2.6] or [48, Section 2.2]. Let us explain part (a); part (b) is similar.

Following the discussion in Section I, the forgetful functor \( \text{Top}_Z \to \text{Ab} \) preserves kernels and cokernels (as well as all limits and colimits). As this functor is also faithful, it follows immediately that, for any morphism \( f \) in \( \text{Top}_Z \), the induced morphism \( \text{coim}(f) \to \text{im}(f) \) is an epimorphism and a monomorphism (in fact, \( \text{coim}(f) \to \text{im}(f) \) is a bijective map). So the category \( \text{Top}_Z \) is semi-abelian. Furthermore, the monomorphisms in \( \text{Top}_Z \) are the injective continuous homomorphisms, and the epimorphisms are the surjective continuous homomorphisms.

Clearly, if \( i: K \to A \) is a kernel in \( \text{Top}_Z \), then the topology of \( K \) is induced from the topology of \( A \) via \( i \). Conversely, if \( i \) is an injective map and the topology of \( K \) is induced from the topology of \( A \), then \( i \) is a kernel of the continuous homomorphism \( A \to A/\text{im}(i(K)) \) (where \( A/\text{im}(i(K)) \) is endowed with the quotient topology). Similarly, if \( p: A \to C \) is a cokernel in \( \text{Top}_Z \), then the topology of \( C \) is the quotient topology of the topology of \( A \). Conversely, if \( p \) is a surjective map and the topology of \( C \) is the quotient topology of the topology of \( A \), then \( p \) is a cokernel of the continuous homomorphism \( \text{ker}(p) \to A \) (where \( \text{ker}(p) \subseteq A \) is endowed with the induced topology).

According to Corollary 5.4, in order to show that a semi-abelian category is quasi-abelian, it suffices to check that it is either right or left quasi-abelian. For the sake of clarity and completeness, let us check both the properties.

Let \( i: K \to A \) be an injective continuous homomorphism such that the topology of \( K \) is induced from the topology of \( A \) via \( i \), and let \( f: K \to L \) be an arbitrary continuous homomorphism. Then the pushout \( B = L \cup_K A \) is the abelian group \( B = \text{coker}((-f, i)) = (L \oplus A)/K \) endowed with the quotient topology. The natural map \( j: L \to B \) is injective since the map \( i \) is. We need to check that the topology of \( L \) is induced from the topology of \( B \) via \( j \).

Let \( E \subseteq L \) be an open subgroup. Then the preimage \( f^{-1}(E) \subseteq K \) is an open subgroup, too. As the topology of \( K \) is induced from \( A \), this means that there exists an open subgroup \( U \subseteq A \) such that \( f^{-1}(E) = i^{-1}(U) \). Now the image of the open subgroup \( E \oplus U \subseteq L \oplus A \) under the surjective map \( L \oplus A \to B \) is an open subgroup \( W \subseteq B \) (by the definition of the quotient topology). The preimage \( j^{-1}(W) \) is the subgroup of all elements \( l \in L \) for which there exist \( e \in E \), \( u \in U \), and \( k \in K \) such that \( l = e + f(k) \) in \( L \) and \( u - i(k) = 0 \) in \( A \). We have \( k \in i^{-1}(U) \), hence \( f(k) \in E \); so \( l \in E \), and we have shown that \( j^{-1}(W) = E \).

Let \( p: A \to C \) be an open continuous surjective homomorphism, and let \( f: D \to C \) be an arbitrary continuous homomorphism. Then the pullback \( B = D \cap_C A \) is the subgroup \( B = \text{ker}((-f, p)) \subseteq D \oplus A \) endowed with the induced topology. The projection map \( q: B \to D \) is surjective since the map \( p \) is. We need to check
that \( q \) is an open map. A base of open subgroups in \( B \) is formed by the subgroups \( W = B \cap (E \oplus U) \), where \( E \subset D \) and \( U \subset A \) are open subgroups. Now the subgroup \( q(W) \subset D \) consists of all elements \( d \in D \) such that \( d \in E \) and there exists \( u \in U \) for which \( f(d) = p(u) \). So \( q(W) = E \cap f^{-1}(p(U)) \), which is an open subgroup in \( D \) since the map \( p \) is open and the map \( f \) is continuous.

**Theorem 7.2.** (a) The category \( \text{Top}_s^2 \) of separated (but not necessarily complete) topological abelian groups with linear topology is quasi-abelian.

A continuous homomorphism \( i: K \to A \) is a kernel (i.e., an admissible monomorphism in the quasi-abelian exact structure) in \( \text{Top}_s^2 \) if and only if \( i \) is injective, the subgroup \( i(K) \) is closed in \( A \), and the topology of \( K \) is induced from the topology of \( A \) (via \( i \)); in other words, this means that \( i \) is an injective closed map.

A continuous homomorphism \( p: A \to C \) is a cokernel (i.e., an admissible epimorphism in the quasi-abelian exact structure) in \( \text{Top}_s^2 \) if and only if \( p \) is surjective and the topology of \( C \) is the quotient topology of the topology of \( A \) (via \( p \)); in other words, this means that \( p \) is a surjective open map.

(b) The category \( \text{Top}_s^2 \) of separated (but not necessarily complete) topological vector spaces with linear topology is quasi-abelian.

A continuous linear map \( i: K \to V \) is a kernel (i.e., an admissible monomorphism in the quasi-abelian exact structure) in \( \text{Top}_s^2 \) if and only if \( i \) is injective, the subspace \( i(K) \) is closed in \( V \), and the topology of \( K \) is induced from the topology of \( V \) (via \( i \)); in other words, this means that \( i \) is an injective closed map.

A continuous linear map \( p: V \to C \) is a cokernel (i.e., an admissible epimorphism in the quasi-abelian exact structure) in \( \text{Top}_s^2 \) if and only if \( p \) is surjective and the topology of \( C \) is the quotient topology of the topology of \( V \) (via \( p \)); in other words, this means that \( p \) is a surjective open map.

**Proof.** This is also well-known; see [48, Section 2.2], cf. [44, Proposition 3.1.8]. We will explain part (a); part (b) is similar.

According to the discussion in Section 1, the inclusion functor \( \text{Top}_s^2 \to \text{Top}_Z \) preserves kernels (as well as all limits). The cokernel of a morphism \( f: A \to B \) in \( \text{Top}_s^2 \) is computed as the maximal separated quotient group \( C/\{0\}_C \) of the cokernel \( C = B/f(A) \) of the morphism \( f \) in the category \( \text{Top}_Z \) (with the quotient topology on \( C/\{0\}_C \)). Equivalently, the cokernel of \( f \) in \( \text{Top}_s^2 \) is the quotient group \( B/f(A)_B \), endowed with the quotient topology.

Clearly, if \( i: K \to A \) is the kernel of a morphism \( f: A \to B \) in \( \text{Top}_s^2 \), then the topology of \( K \) is induced from the topology of \( A \) via \( i \). Furthermore, \( K = i^{-1}(0) \) is a closed subgroup in \( A \), as the zero subgroup is closed in a separated topological group \( B \). Conversely, if \( i \) is an injective map, \( i(K) \subset A \) is a closed subgroup, and the topology of \( K \) is induced from the topology of \( A \), then \( i \) is a kernel of the continuous homomorphism \( A \to A/i(K) \), where the group \( A/i(K) \) is separated in the quotient topology since the subgroup \( i(K) \) is closed in \( A \). If \( p: B \to C \) is the cokernel of a morphism \( f: A \to B \) in \( \text{Top}_s^2 \), then the map \( p \) is surjective and the topology of \( C \) is the quotient topology of the topology of \( B \). Conversely, if \( p: B \to C \) is a surjective map and the topology of \( C \) is the quotient topology of the topology of \( B \), then \( p \)
is a cokernel of the continuous homomorphism ker(p) \rightarrow B$, where the subgroup ker(p) \subset B is separated in the induced topology since the group B is separated.

So we have obtained the desired descriptions of the classes of all kernels and cokernels in \textbf{Top}_Z^*, and it is clear from these descriptions that the composition of two kernels is a kernel and the composition of two cokernels is a cokernel. Hence the category \textbf{Top}_Z^* is semi-abelian. Besides, the monomorphisms in \textbf{Top}_Z^* are the injective continuous homomorphisms, and the epimorphisms are continuous homomorphisms \gamma: C \rightarrow D with a dense image, \overline{f(C)}_D = D. For an arbitrary morphism f: A \rightarrow B, the induced morphism coim(f) \rightarrow \text{im}(f) is the natural injective map \overline{f(A)}_B \rightarrow \overline{f(A)}_B, where \overline{f(A)} is endowed with the quotient topology of the topology of A, while \overline{f(A)}_B carries the induced topology of a subgroup in B. As \overline{f(A)} is dense in \overline{f(A)}_B, the morphism coim(f) \rightarrow \text{im}(f) is both a monomorphism and an epimorphism.

In order to show that a semi-abelian category is quasi-abelian, it suffices to check that it is either right or left quasi-abelian. We will check both the properties.

Let i: K \rightarrow A be a closed continuous injective homomorphism of separated topological groups, and let f: K \rightarrow L be an arbitrary continuous homomorphism of separated topological groups. Let B = (L \oplus A)/K denote the pushout of the morphisms i and f in the category \textbf{Top}_Z^*. We claim that B is separated, so B is also the pushout B = L \cup_K A in the category \textbf{Top}_Z^*. Then we know from the proof of Theorem 7.1 that the topology of L is induced from the topology of B via the natural map j: L \rightarrow B, and it remains to check that j(L) is a closed subgroup in B.

Indeed, a subgroup C in a topological abelian group D is closed if and only if the quotient group D/C is separated in the quotient topology. Now the quotient group B/j(L) is naturally isomorphic, as a topological group, to the quotient group A/i(K). Since the subgroup i(K) is closed in A, it follows that the subgroup j(L) is closed in B. The zero subgroup is closed in L, hence it is also closed in B; so B is separated, and we are finished.

The pullbacks in the category \textbf{Top}_Z^* agree with those in \textbf{Top}_Z^*, because the kernels agree. Let p: A \rightarrow C be an open continuous surjective homomorphism of separated topological abelian groups, and let f: D \rightarrow C be an arbitrary continuous homomorphism of separated topological abelian groups. Then the subgroup B = \text{ker}((-f,p)) \subset D \oplus A endowed with the induced topology is the pullback B = D \cap_C A in both the categories \textbf{Top}_Z^* and \textbf{Top}_Z^*. It was explained already in the proof of Theorem 7.1 that q: B \rightarrow D is a surjective open map. \hfill \Box

**Corollary 7.3.** (a) The full subcategory \textbf{Top}_Z^* \subset \textbf{Top}_Z^* is closed under extensions (in the quasi-abelian exact structure of \textbf{Top}_Z^*) and subobjects. The inherited exact category structure on \textbf{Top}_Z^* from the quasi-abelian exact structure on \textbf{Top}_Z^* coincides with the quasi-abelian exact structure on \textbf{Top}_Z^*.

(b) The full subcategory \textbf{Top}_k^* \subset \textbf{Top}_k^* is closed under extensions (in the quasi-abelian exact structure of \textbf{Top}_k^*) and subobjects. The inherited exact category structure on \textbf{Top}_k^* from the quasi-abelian exact structure on \textbf{Top}_k^* coincides with the quasi-abelian exact structure on \textbf{Top}_k^*.
Proof. We will explain part (a). The monomorphisms in $\text{Top}_Z$ are the injective continuous homomorphisms, and if $f: A' \rightarrow A''$ is an injective continuous abelian group map and the topological abelian group $A''$ is separated, then the topological abelian group $A'$ is separated, too (since one always has $f(\{0\}_{A'}) \subset \{0\}_{A''}$ for a continuous map $f$). Now let $0 \rightarrow K \xrightarrow{i} A \xrightarrow{p} C \rightarrow 0$ be a short exact sequence in (the quasi-abelian exact structure on) $\text{Top}_Z$ with $A, C \in \text{Top}_Z^s$. Then $0$ is a closed subgroup in $C$, hence $p^{-1}(0) = i(K)$ is a closed subgroup in $A$. Since $0$ is a closed subgroup in $K$, the topology of $K$ is induced from the topology of $A$ via $i$, and $i(K)$ is a closed subgroup in $A$, it follows that $0$ is a closed subgroup in $A$, so $A \in \text{Top}_Z^s$.

Following Example 11.1 the inherited exact category structure on $\text{Top}_Z^s$ exists. If $0 \rightarrow K \xrightarrow{i} A \xrightarrow{p} C \rightarrow 0$ is a short sequence in $\text{Top}_Z^s$ that is exact in $\text{Top}_Z$, then $i$ is the kernel of $p$ and $p$ is the cokernel of $i$ in $\text{Top}_Z$, hence also in $\text{Top}_Z^s$. Conversely, let $0 \rightarrow K \xrightarrow{i} A \xrightarrow{p} C \rightarrow 0$ be a short exact sequence in the quasi-abelian exact structure on $\text{Top}_Z^s$. Then $i$ is the kernel of $p$ in $\text{Top}_Z^s$, hence $i$ is also the kernel of $p$ in $\text{Top}_Z$, as the kernels of morphisms in $\text{Top}_Z$ and $\text{Top}_Z^s$ agree. Moreover, the subgroup $p^{-1}(0) = i(K)$ is closed in $A$ in this case, since $C \in \text{Top}_Z^s$. Therefore, the quotient group $A/i(K)$ is separated in the quotient topology, and the cokernels of the morphism $i$ in $\text{Top}_Z$ and $\text{Top}_Z^s$ agree. So $p$ is also the cokernel of $i$ in $\text{Top}_Z^s$.

For an alternative proof, compare Propositions 10.5(a) and 10.6(a) below with Lemma 10.2(c) and Corollary 10.4(b). □

Denote by $\text{Top}_Z^c \subset \text{Top}_Z$ and $\text{Top}_Z^{c,s} \subset \text{Top}_Z^s$ the full subcategories formed by all the (unseparated or separated, respectively) topological abelian groups with a countable base of neighborhoods of zero (consisting of open subgroups). Similarly, let $\text{Top}_k^c \subset \text{Top}_k$ and $\text{Top}_k^{c,s} \subset \text{Top}_k^s$ denote the full subcategories of all topological vector spaces with a countable base of neighborhoods of zero (consisting of open subspaces).

Clearly, the full subcategories $\text{Top}_Z^c \subset \text{Top}_Z$ and $\text{Top}_Z^{c,s} \subset \text{Top}_Z^s$ are closed under kernels, cokernels, and countable products. It follows that the additive categories $\text{Top}_Z^c$ and $\text{Top}_Z^{c,s}$ are quasi-abelian. Furthermore, the full subcategories $\text{Top}_Z^c \subset \text{Top}_Z$ and $\text{Top}_Z^{c,s} \subset \text{Top}_Z^s$ inherit exact category structures from their ambient exact categories (in the sense of Example 11.1), and the inherited exact structures on $\text{Top}_Z^c$ and $\text{Top}_Z^{c,s}$ coincide with these categories’ own quasi-abelian exact structures. The full subcategories $\text{Top}_k^c \subset \text{Top}_k$ and $\text{Top}_k^{c,s} \subset \text{Top}_k^s$ have similar properties.

However, the full subcategories of topological vector spaces/abelian groups with a countable base of neighborhoods of zero are not closed under countable coproducts, as the next lemma shows.

Lemma 7.4. Let $A_0, A_1, A_2, \ldots$ be a sequence of topological abelian groups such that the separated topological abelian group $A_n/\{0\}_{A_n}$ is nondiscrete for every $n \geq 0$. Then the topological abelian group $\bigoplus_{n=0}^{\infty} A_n$ does not have a countable base of neighborhoods of zero in the coproduct topology.

Proof. For every $n \geq 0$, the natural surjective map $\bigoplus_{i=0}^{\infty} A_i \twoheadrightarrow A_n$ makes $A_n$ a quotient group of $A = \bigoplus_{i=0}^{\infty} A_i$, endowed with the quotient topology. Hence if $A$ has
a countable base of neighborhoods of zero, then so do \( A_n \) for all \( n \geq 0 \). This allows us to assume that the topological group \( A_n \) has a countable base of neighborhoods of zero for every \( n \geq 0 \). Now if the separated quotient group \( A_n / \{0\} \) is not discrete, then the topological group \( A_n \) does not have a finite base of neighborhoods of zero. Thus, for every \( n \geq 0 \), we can choose a strictly decreasing sequence of open subgroups \( A_n = U_n^0 \supseteq U_n^1 \supseteq U_n^2 \supseteq \cdots \) indexed by the nonnegative integers \( j \geq 0 \) such that the subgroups \( (U_n^j)_{j \geq 0} \) form a base of neighborhoods of zero in \( A_n \).

Then a base of neighborhoods of zero in the topological group \( A = \bigoplus_{n=0}^{\infty} A_n \) is formed by the subgroups \( U_\psi = \bigoplus_{n=0}^{\infty} U_n^{\psi(n)} \subset A \), where \( \psi \) ranges over all the maps \( \omega \to \omega \) from the set \( \omega \) of all nonnegative integers to itself. Suppose, for the sake of contradiction, that the topological group \( A \) has a countable base of neighborhoods of zero. Then the set of all the subgroups \( U_\psi \subset A \) has a countable subset forming a base of neighborhoods of zero in \( A \). This means that there is a countable subset \( F \subset \omega^\omega \) in the set of all functions \( \omega \to \omega \) such that for every \( \psi : \omega \to \omega \) there exists \( \phi \in F \) for which \( U_\phi \subset U_\psi \). In view of our choice of the open subgroups \( U_n^j \subset A_n \) and the construction of the open subgroups \( U_\psi \subset A \), the inclusion \( U_\phi \subset U_\psi \) means that \( \phi(n) \geq \psi(n) \) for all \( n \geq 0 \). A simple application of Cantor’s diagonal argument shows that such a countable set \( F \) of functions \( \omega \to \omega \) does not exist. □

8. The Maximal Exact Category Structure on VSLTs

The main result of this section is that the categories \( \text{Top}^e_{sc} \) and \( \text{Top}^c_{sc} \) of complete, separated topological abelian groups/vector spaces with linear topology are right, but not left quasi-abelian. We also describe the classes of (co)kernels, semi-stable (co)kernels, and stable (co)kernels in these additive categories, thus obtaining, in particular, a description of their maximal exact category structures.

**Proposition 8.1.** (a) A morphism \( i : \mathcal{R} \to \mathcal{A} \) in \( \text{Top}^c_{sc} \) is a monomorphism if and only if it is an injective map. A morphism \( p : \mathcal{A} \to \mathcal{C} \) in \( \text{Top}^c_{sc} \) is an epimorphism if and only if the subgroup \( p(\mathcal{A}) \) is dense in \( \mathcal{C} \).

A morphism \( i : \mathcal{R} \to \mathcal{A} \) in \( \text{Top}^c_{sc} \) is a kernel if and only if \( i \) is injective, the subgroup \( i(\mathcal{R}) \) is closed in \( \mathcal{A} \), and the topology of \( \mathcal{R} \) is induced from the topology of \( \mathcal{A} \) (via \( i \)). In other words, this means that \( i \) is an injective closed map.

A morphism \( p : \mathcal{A} \to \mathcal{C} \) in \( \text{Top}^c_{sc} \) is a cokernel if and only if \( p \) is isomorphic to the composition of natural continuous homomorphisms \( \mathcal{A} \to \mathcal{A}/\mathcal{R} \to (\mathcal{A}/\mathcal{R})^\sim \), where \( \mathcal{R} \subset \mathcal{A} \) is a closed subgroup, \( \mathcal{A} \to \mathcal{A}/\mathcal{R} \) is the quotient map, \( \mathcal{A}/\mathcal{R} \) is endowed with the quotient topology, \( (\mathcal{A}/\mathcal{R})^\sim \) is the completion of the (separated) topological abelian group \( \mathcal{A}/\mathcal{R} \), endowed with the completion topology, and \( \mathcal{A}/\mathcal{R} \to (\mathcal{A}/\mathcal{R})^\sim \) is the completion map. In other words, this means that \( p \) is the composition of a surjective continuous open homomorphism (onto an incomplete topological abelian group) followed by the completion map.
(b) A morphism $i: \mathcal{R} \rightarrow \mathcal{V}$ in $\text{Top}_{k}^{sc}$ is a monomorphism if and only if it is an injective map. A morphism $p: \mathcal{V} \rightarrow \mathcal{C}$ in $\text{Top}_{k}^{sc}$ is an epimorphism if and only if the subspace $p(\mathcal{V})$ is dense in $\mathcal{C}$.

A morphism $i: \mathcal{R} \rightarrow \mathcal{V}$ in $\text{Top}_{k}^{sc}$ is a kernel if and only if $i$ is injective, the subspace $i(\mathcal{R})$ is closed in $\mathcal{V}$, and the topology of $\mathcal{R}$ is induced from the topology of $\mathcal{V}$ (via $i$). In other words, this means that $i$ is an injective closed map.

A morphism $p: \mathcal{V} \rightarrow \mathcal{C}$ in $\text{Top}_{k}^{sc}$ is a cokernel if and only if $p$ is isomorphic to the composition of natural continuous linear maps $\mathcal{V} \rightarrow \mathcal{V}/\mathcal{R} \rightarrow (\mathcal{V}/\mathcal{R})^{\sim}$, where $\mathcal{R} \subset \mathcal{V}$ is a closed subspace, $\mathcal{V} \rightarrow \mathcal{V}/\mathcal{R}$ is the quotient map, $\mathcal{V}/\mathcal{R}$ is endowed with the quotient topology, $(\mathcal{V}/\mathcal{R})^{\sim}$ is the completion of the (separated) topological vector space $\mathcal{V}/\mathcal{R}$, endowed with the completion topology, and $\mathcal{V} \rightarrow (\mathcal{V}/\mathcal{R})^{\sim}$ is the completion map. In other words, this means that $p$ is the composition of a surjective continuous open linear map (onto an incomplete topological vector space) followed by the completion map.

Proof. We will explain part (a); part (b) is similar. According to the discussion in Section 11 the inclusion functors $\text{Top}_{k}^{sc} \rightarrow \text{Top}_{Z}^{sc} \rightarrow \text{Top}_{Z}$ preserve kernels (as well as all limits). The cokernel of a morphism $f: \mathfrak{A} \rightarrow \mathcal{B}$ in $\text{Top}_{Z}^{sc}$ is computed as the completion $C^{\sim} = (\mathcal{B}/f(\mathfrak{A}))^{\sim}$ of the cokernel $C = \mathcal{B}/f(\mathfrak{A})$ of the morphism $f$ in the category $\text{Top}_{Z}$ (with the completion topology on $C^{\sim}$).

According to Theorem 7.2(a), if $i: \mathfrak{A} \rightarrow \mathfrak{A}$ is a kernel of a morphism $f: \mathfrak{A} \rightarrow \mathcal{B}$ in $\text{Top}_{Z}^{sc}$ (hence also in $\text{Top}_{Z}^{sc}$), then $i$ is an injective closed map. Conversely, if $i$ is an injective closed map, then $i$ is the kernel of the morphism $\mathfrak{A} \rightarrow (\mathfrak{A}/i(\mathcal{R}))^{\sim}$ in $\text{Top}_{Z}^{sc}$.

Notice that the topological group $\mathfrak{A}/i(\mathcal{R})$ is separated in the quotient topology, since the subgroup $i(\mathcal{R})$ is closed in $\mathfrak{A}$; hence the completion map $\mathfrak{A}/i(\mathcal{R}) \rightarrow (\mathfrak{A}/i(\mathcal{R}))^{\sim}$ is injective and the kernel of the composition $\mathfrak{A} \rightarrow \mathfrak{A}/i(\mathcal{R}) \rightarrow (\mathfrak{A}/i(\mathcal{R}))^{\sim}$ is the subgroup $i(\mathcal{R}) \subset \mathfrak{A}$.

It is clear from the above description of the cokernels of morphisms in $\text{Top}_{Z}^{sc}$ that if $p: \mathcal{B} \rightarrow \mathcal{C}$ is the cokernel of a morphism $f: \mathfrak{A} \rightarrow \mathcal{B}$, then $p$ is the composition of a surjective continuous open homomorphism $\mathcal{B} \rightarrow \mathcal{B}/f(\mathfrak{A})$ and the completion map $\mathcal{B}/f(\mathfrak{A}) \rightarrow (\mathcal{B}/f(\mathfrak{A}))^{\sim}$. Denote by $\mathcal{K} = \overline{f(\mathfrak{A})}_{\mathcal{B}} \subset \mathcal{B}$ the closure of the subgroup $f(\mathfrak{A}) \subset \mathcal{B}$; then $p$ is also the composition of a surjective continuous open homomorphism $\mathcal{B} \rightarrow \mathcal{B}/\mathcal{K}$ onto a separated topological abelian group $\mathcal{B}/\mathcal{K}$ and an injective completion map $(\mathcal{B}/\mathcal{K}) \rightarrow (\mathcal{B}/\mathcal{K})^{\sim}$. Conversely, for any closed subgroup $\mathcal{K} \subset \mathcal{B}$, the morphism $\mathcal{B} \rightarrow (\mathcal{B}/\mathcal{K})^{\sim}$ is the cokernel of the morphism $\mathfrak{A} \rightarrow \mathcal{B}$ in $\text{Top}_{Z}^{sc}$ (where $\mathcal{K}$ is endowed with the induced topology of a subgroup in $\mathcal{B}$).

Remark 8.2. For any separated topological abelian group $B$ with the completion $\mathcal{B} = B^{\sim}$, the topology of $B$ is induced from the topology of $\mathcal{B}$ via the injective completion map $B \rightarrow \mathcal{B}$. Furthermore, let $\mathcal{L} \subset B$ be a subgroup; suppose that $\mathcal{L}$ is complete in the induced topology. Then $\mathcal{L} \subset B \subset \mathcal{B}$ can be also considered as a subgroup in $\mathcal{B}$; as the topology of $\mathcal{L}$ is induced from the topology of $B$ and the latter is induced from the topology of $\mathcal{B}$, it follows that the topology of $\mathcal{L}$ is induced from the topology of $\mathcal{B}$. According to Lemma 1.1 the closure of $\mathcal{L}$ in $\mathcal{B}$ is the completion of $\mathcal{L}$. Since $\mathcal{L}$ is complete by assumption, it follows that $\mathcal{L}$ is closed.
in \( \mathcal{B} \). Consequently, \( \mathcal{L} \) is also closed in \( B \). We have shown that a complete, separated topological abelian group is closed in any separated topological abelian group where it is embedded with the induced topology.

So the condition that \( i(\mathfrak{K}) \) is closed in \( \mathfrak{A} \) can be dropped from the characterization of kernels in \( \text{Top}_k^c \) given in Proposition 8.1(a): any morphism in \( \text{Top}_Z^c \) that is a kernel in \( \text{Top}_Z \) is also a kernel in \( \text{Top}_Z^c \). The inclusion of the full subcategory \( \text{Top}_Z^c \rightarrow \text{Top}_Z \) does not have this property (cf. Theorems 7.1(b) and 7.2(b)). Similarly, the condition that \( i(\mathfrak{K}) \) is closed in \( \mathfrak{V} \) can be dropped from the characterization of kernels in \( \text{Top}_k^c \) given in Proposition 8.1(b), and any morphism in \( \text{Top}_k^c \) that is a kernel in \( \text{Top}_k \) is also a kernel in \( \text{Top}_k^c \). The inclusion of the full subcategory \( \text{Top}_k^c \rightarrow \text{Top}_k \) does not have this property (cf. Theorem 7.1(b) and 7.2(b)).

We refer to Section 6 for the definitions of semi-stable and stable kernels and cokernels in additive categories generally, and to Example 6.6 for a discussion of these classes of morphisms in right quasi-abelian categories specifically.

**Proposition 8.3.** The additive categories \( \text{Top}_Z^c \) and \( \text{Top}_k^c \) are right quasi-abelian. In other words, all kernels in \( \text{Top}_Z^c \) and \( \text{Top}_k^c \) are semi-stable kernels, and consequently all semi-stable cokernels in these categories are stable cokernels.

**Proof.** Let us explain the assertion for topological abelian groups; the case of topological vector spaces is similar. Let \( i: \mathfrak{K} \rightarrow \mathfrak{A} \) be a kernel of some morphism in \( \text{Top}_Z^c \); by Proposition 8.1 this means that \( i \) is a closed continuous injective homomorphism of complete, separated topological abelian groups. Let \( \mathfrak{I}: \mathfrak{K} \rightarrow \mathfrak{L} \) be an arbitrary morphism in \( \text{Top}_Z^c \). Then the pushout \( B \) of the morphisms \( i \) and \( p \) in the category \( \text{Top}_Z \) is the quotient group \( B = (\mathfrak{L} \oplus \mathfrak{A})/\mathfrak{K} \), endowed with the quotient topology. Following the proof of Theorem 7.2, \( B \) is a separated topological abelian group. The pushout \( \mathfrak{B} = \mathfrak{L} \sqcup_{\mathfrak{K}} \mathfrak{A} \) in the category \( \text{Top}_Z^c \) is the completion of the topological group \( B \), i.e., \( \mathfrak{B} = B^\wedge \). We know from Theorem 7.1 that the topology of \( \mathfrak{L} \) is induced from the topology of \( B \) via the natural injective map \( \mathfrak{L} \rightarrow B \). Hence the topology of \( \mathfrak{L} \) is also induced from the topology of \( \mathfrak{B} \) via the natural morphism \( \mathfrak{L} \rightarrow \mathfrak{B} \), which is the composition of injective maps \( \mathfrak{L} \rightarrow B \rightarrow \mathfrak{B} \). According to Remark 8.2, it follows that the morphism \( \mathfrak{L} \rightarrow \mathfrak{B} \) is a kernel in \( \text{Top}_Z^c \).

For an alternative proof, compare Theorem 9.4 below with Proposition 10.9. \( \square \)

**Proposition 8.4.** (a) A morphism \( p: \mathfrak{A} \rightarrow \mathfrak{C} \) in \( \text{Top}_Z^c \) is a semi-stable cokernel if and only if \( p \) is a surjective open map.

(b) A morphism \( p: \mathfrak{B} \rightarrow \mathfrak{V} \) in \( \text{Top}_k^c \) is a semi-stable cokernel if and only if \( p \) is a surjective open map.

**Proof.** Let us explain part (a); part (b) is similar. By Proposition 8.1 a morphism \( p: \mathfrak{A} \rightarrow \mathfrak{C} \) is a cokernel in \( \text{Top}_Z^c \) if and only if it is the composition of a surjective open map \( \mathfrak{A} \rightarrow C \) onto a separated topological abelian group \( C \) and the completion map \( C \rightarrow C^\wedge = \mathfrak{C} \). So we have to show that a cokernel \( p \) is a semi-stable cokernel in \( \text{Top}_Z^c \) if and only if \( p \) is a surjective map.

Suppose that \( p \) is not surjective, and choose an element \( x \in \mathfrak{C} \) not belonging to the image of \( p \). Consider the abelian group \( \mathbb{Z} \) with the discrete topology, and the
group homomorphism $f: \mathbb{Z} \rightarrow \mathcal{C}$ taking 1 to $x$. Recall that the pullbacks in $\text{Top}_{\mathbb{Z}}^k$ agree with those in $\text{Top}_{\mathbb{Z}}$ and are preserved by the forgetful functor $\text{Top}_{\mathbb{Z}}^k \rightarrow \text{Ab}$. The map $g: \mathbb{Z} \cap_k \mathcal{A} \rightarrow \mathbb{Z}$ is not surjective, as the element $1 \in \mathbb{Z}$ does not belong to its image. As the topology on $\mathbb{Z}$ is discrete, the image of $q$ cannot be dense in $\mathbb{Z}$; so $g$ is not even an epimorphism in $\text{Top}_{\mathbb{Z}}^k$. Thus $p$ is not a semi-stable cokernel.

Suppose that $p$ is surjective; then $p$ is an open map, hence it is a cokernel in $\text{Top}_{\mathbb{Z}}^k$ and $\text{Top}_{\mathbb{Z}}^k$. Let $f: \mathcal{D} \rightarrow \mathcal{C}$ be an arbitrary morphism in $\text{Top}_{\mathbb{Z}}^k$. Once again we recall that the pullbacks in $\text{Top}_{\mathbb{Z}}^k$ agree with those in $\text{Top}_{\mathbb{Z}}$ and $\text{Top}_{\mathbb{Z}}^k$. By Theorem 7.2, the morphism $q: \mathcal{D} \cap_k \mathcal{A} \rightarrow \mathcal{A}$ is a cokernel in $\text{Top}_{\mathbb{Z}}^k$. Following the descriptions of the classes of cokernels in the categories $\text{Top}_{\mathbb{Z}}^k$ and $\text{Top}_{\mathbb{Z}}^k$ given in Theorem 7.2 and Proposition 8.1, any morphism in $\text{Top}_{\mathbb{Z}}^k$ that is a cokernel in $\text{Top}_{\mathbb{Z}}^k$ is also a cokernel in $\text{Top}_{\mathbb{Z}}^k$. (It is also true that a morphism in $\text{Top}_{\mathbb{Z}}^k$ is a cokernel if and only if it is a cokernel in $\text{Top}_{\mathbb{Z}}^k$; see Theorem 7.1.) Thus $q$ is a cokernel in $\text{Top}_{\mathbb{Z}}^k$, and $p$ is a semi-stable cokernel, as desired.

**Corollary 8.5.** (a) A morphism $i: \mathcal{R} \rightarrow \mathcal{A}$ in $\text{Top}_{\mathbb{Z}}^k$ is a stable kernel if and only if $i$ is an injective closed map and the quotient group $\mathcal{A}/i(\mathcal{R})$ is complete in the quotient topology.

(b) A morphism $i: \mathcal{R} \rightarrow \mathcal{B}$ in $\text{Top}_{\mathbb{Z}}^k$ is a stable kernel if and only if $i$ is an injective closed map and the quotient space $\mathcal{B}/i(\mathcal{R})$ is complete in the quotient topology.

**Proof.** Part (a) follows from Propositions 8.3 and 8.4(a) together with the description of the cokernels of morphisms in the category $\text{Top}_{\mathbb{Z}}^k$ (see Section 1 and/or the first paragraph of the proof of Proposition 8.1). Part (b) is similar.

**Corollary 8.6.** The categories $\text{Top}_{\mathbb{Z}}^k$ and $\text{Top}_{\mathbb{Z}}^k$ are not left quasi-abelian. In fact, they are not even left semi-abelian.

**Proof.** Let us discuss the case of topological vector spaces. Choose an incomplete separated topological $k$-vector space $C$ (with linear topology). For example, one can consider the complete, separated topological vector space $\mathcal{E} = k^{\omega} = \prod_{n=0}^{\infty} k$ (with the product topology of discrete one-dimensional vector spaces $k$), and the dense vector subspace $C = k^{(\omega)} = \bigoplus_{n=0}^{\infty} k \subset \mathcal{E}$ with the topology on $C$ induced from $\mathcal{E}$. Then $\mathcal{E}$ and consequently $C$ are even topological vector spaces with countable bases of open subspaces. Furthermore, $\mathcal{E}$ is a linearly compact (profinite-dimensional) topological vector space, while $C$ has countable dimension.

Following Theorem 8.1, there exists an open, continuous surjective linear map $\Sigma: \mathcal{A}_c(C) \rightarrow C$ onto $C$ from the complete, separated topological vector space $\mathcal{A}_c(C) = C^{(\omega)} = \bigoplus_{n=0}^{\infty} C$ with the modified coproduct topology. Put $\mathcal{U} = \mathcal{A}_c(C)$, and let $\mathcal{R} \subset \mathcal{U}$ be the kernel of $\Sigma$, endowed with the induced topology as a closed subspace in $\mathcal{U}$. Then $i: \mathcal{R} \rightarrow \mathcal{U}$ is a morphism in $\text{Top}_{\mathbb{Z}}^k$, the map $\Sigma: \mathcal{U} \rightarrow C$ is the cokernel of $i$ in $\text{Top}_{\mathbb{Z}}^k$ and in $\text{Top}_{\mathbb{Z}}^k$, and the composition $\mathcal{U} \rightarrow C \rightarrow C^\sim = \mathcal{E}$ is the cokernel of $i$ in $\text{Top}_{\mathbb{Z}}^k$. Denote this composition by $p: \mathcal{U} \rightarrow \mathcal{E}$.

Now $p$ is a cokernel in $\text{Top}_{\mathbb{Z}}^k$, which is not a surjective map. Choose a vector $x \in \mathcal{E} \setminus C$; so $x$ does not belong to the image of $p$. Consider the one-dimensional vector space $k$ with the discrete topology, and the linear map $f: k \rightarrow \mathcal{E}$ taking 1
to $x$. Then the pullback $k \cap \mathcal{V}$ of the morphisms $p$ and $f$ is $k \cap \mathcal{V} = \mathcal{R}$, and the natural morphism $q: k \cap \mathcal{V} \to k$ is the zero map. So the morphism $q: k \cap \mathcal{V} \to k$ is certainly not a cokernel (and not an epimorphism) in $\text{Top}_k^\mathcal{X}$. \hfill \Box

**Examples 8.7.** The proof of Corollary 8.6 presents an example showing that the property of Proposition 5.2(4) does not hold in the category $\text{Top}_k^\mathcal{X}$. Consequently, there should also exist counterexamples to the properties of Proposition 5.2(1–3) in $\text{Top}_k^\mathcal{X}$. Let us suggest such counterexamples here.

We keep the notation of the proof of Corollary 8.6. The induced topology on the vector subspace $kx \subset \mathcal{C}$ is separated (since $\mathcal{C}$ is separated); so it must be discrete. By Corollary 3.4, it follows that $kx$ is a direct summand in $\mathcal{C}$, so the quotient space $\mathcal{C}/kx$ is separated and complete. Therefore, we have $\text{coker}(f) = \mathcal{C}/kx$ in $\text{Top}_k^\mathcal{X}$. Denote by $g$ the split epimorphism $g: \mathcal{C} \to \mathcal{C}/kx$. Then both the morphisms $p$ and $g$ are cokernels in $\text{Top}_k^\mathcal{X}$, but the composition $gp: \mathcal{V} \to \mathcal{C}/kx$ is not a cokernel. Indeed, the kernel of $gp$ is the morphism $i: \mathcal{R} \to \mathcal{V}$, and the cokernel of $i$ is $p$ rather than $gp$. Alternatively, endow the direct sum $kx \oplus \mathcal{V}$ with the (co)product topology of the discrete topology on $kx$ and the above topology on $\mathcal{V}$. Then the morphism $\text{id}_{kx} \oplus p: kx \oplus \mathcal{V} \to kx \oplus \mathcal{C}$ is a cokernel (of the morphism $(0, i): \mathcal{R} \to kx \oplus \mathcal{V}$), and the morphism $(g, \text{id}_{\mathcal{C}}): kx \oplus \mathcal{C} \to \mathcal{C}$ is a cokernel (in fact, even a direct summand projection), but the composition $(g, \text{id}_{\mathcal{C}}) \circ (\text{id}_{kx} \oplus p) = (f, p): kx \oplus \mathcal{V} \to \mathcal{C}$ is not a cokernel. Indeed, the kernel of $(f, p)$ is the morphism $(0, i): \mathcal{R} \to kx \oplus \mathcal{C}$, and the cokernel of $(0, i)$ is $kx \oplus \mathcal{V} \to kx \oplus \mathcal{C}$ rather than $kx \oplus \mathcal{V} \to \mathcal{C}$. These are counterexamples to the property of Proposition 5.2(2) in $\text{Top}_k^\mathcal{X}$.

Furthermore, the composition $q: \mathcal{V} \to kx \oplus \mathcal{V} \to \mathcal{C}$ of the morphisms $(0, \text{id}_{\mathcal{V}}): \mathcal{V} \to kx \oplus \mathcal{V}$ and $(f, p): kx \oplus \mathcal{V} \to \mathcal{C}$ is the morphism $p: \mathcal{V} \to \mathcal{C}$, which is the cokernel of $i: \mathcal{R} \to \mathcal{C}$. But the morphism $kx \oplus \mathcal{V} \to \mathcal{C}$ is not a cokernel. This is a counterexample to the property of Proposition 5.2(3) in $\text{Top}_k^\mathcal{X}$.

Finally, the coimage of the morphism $gp: \mathcal{V} \to \mathcal{C}/k$ in the category $\text{Top}_k^\mathcal{X}$ is the object $\text{coim}(gp) = \text{coker}(i) = \mathcal{C}$, while the image is $\text{im}(gp) = \ker(\mathcal{C}/k \to 0) = \mathcal{C}/k$. The natural morphism $\text{coim}(gp) \to \text{im}(gp)$ is not a monomorphism; in fact, it is the split epimorphism $g: \mathcal{C} \to \mathcal{C}/kx$.

Alternatively, the coimage of the morphism $(f, p): kx \oplus \mathcal{V} \to \mathcal{C}$ is $\text{coim}((f, p)) = \text{coker}(0, i): \mathcal{R} \to kx \oplus \mathcal{V}) = kx \oplus \mathcal{C}$, while the image of $(f, p)$ is $\text{im}((f, p)) = \ker(\mathcal{C} \to 0) = \mathcal{C}$. The natural morphism $\text{coim}((f, p)) \to \text{im}((f, p))$ is not a monomorphism; in fact, it is the split epimorphism $(f, \text{id}_{\mathcal{C}}): kx \oplus \mathcal{C} \to \mathcal{C}$. These are counterexamples to the property of Proposition 5.2(1) in $\text{Top}_k^\mathcal{X}$.

**Corollary 8.8.** (a) The full subcategory $\text{Top}_k^\mathcal{X} \subset \text{Top}_k^\mathcal{C}$ is closed under extensions (in the quasi-abelian exact structure of $\text{Top}_k^\mathcal{C}$) and kernels. The inherited exact category structure on $\text{Top}_k^\mathcal{C}$ from the quasi-abelian exact structure on $\text{Top}_k^\mathcal{C}$ coincides with the maximal exact structure on $\text{Top}_k^\mathcal{C}$.

(b) The full subcategory $\text{Top}_k^\mathcal{X} \subset \text{Top}_k^\mathcal{C}$ is closed under extensions (in the quasi-abelian exact structure of $\text{Top}_k^\mathcal{C}$) and kernels. The inherited exact category structure on $\text{Top}_k^\mathcal{X}$ from the quasi-abelian exact structure on $\text{Top}_k^\mathcal{C}$ coincides with the maximal exact structure on $\text{Top}_k^\mathcal{X}$.

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Proof. Let us explain part (a). Let $0 \rightarrow \mathcal{K} \xrightarrow{i} A \xrightarrow{p} \mathcal{C} \rightarrow 0$ be a short exact sequence in (the quasi-abelian exact structure on) $\text{Top}_Z^\omega$ with $\mathcal{K}, \mathcal{C} \in \text{Top}_Z^\omega$. Then, for every open subgroup $U \subset A$, we have a short exact sequence of (discrete quotient) groups $0 \rightarrow \mathcal{K}/(i^{-1}(U) \cap \mathcal{K}) \rightarrow A/U \rightarrow \mathcal{C}/(p(U)) \rightarrow 0$. Consider the commutative diagram of a morphism of short sequences of abelian groups

$$
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{K} & \rightarrow & A & \rightarrow & \mathcal{C} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \lim_{\text{lim}_{U \subset A}} \mathcal{K}/(i^{-1}(U) \cap \mathcal{K}) & \rightarrow & \lim_{\text{lim}_{U \subset A}} A/U & \rightarrow & \lim_{\text{lim}_{U \subset A}} \mathcal{C}/(p(U)) & \rightarrow & 0
\end{array}
$$

The lower line is a left exact sequence, since the projective limit functor is left exact. The map $\mathcal{K} \rightarrow \lim_{\text{lim}_{U \subset A}} \mathcal{K}/(i^{-1}(U) \cap \mathcal{K})$ is an isomorphism, since the topological abelian group $\mathcal{K}$ is complete in its topology induced from the topology of $A$ via $i$. The map $\mathcal{C} \rightarrow \lim_{\text{lim}_{U \subset A}} \mathcal{C}/(p(U))$ is an isomorphism, since the topological abelian group $\mathcal{C}$ is complete in its quotient topology. It follows that the map $\lim_{\text{lim}_{U \subset A}} A/U \rightarrow \lim_{\text{lim}_{U \subset A}} \mathcal{C}/(p(U))$ is surjective and the map $A \rightarrow \lim_{\text{lim}_{U \subset A}} A/U$ is an isomorphism, so the topological abelian group $A$ is complete (cf. Lemma 10.7).

Following Example 4.1, the inherited exact structure on $\text{Top}_Z^\omega$ exists. Let $0 \rightarrow \mathcal{K} \xrightarrow{i} \mathfrak{A} \xrightarrow{p} \mathcal{C} \rightarrow 0$ be a short exact sequence in the maximal exact structure on $\text{Top}_Z^\omega$; we have to show that this short sequence is also exact in $\text{Top}_Z^\omega$. But this is clear from Proposition 8.4(a) or Corollary 8.5(a) compared with the descriptions of the kernels and cokernels of morphisms in the categories $\text{Top}_Z^\omega$ and $\text{Top}_Z^\omega$ known from the proofs of Theorem 7.2(a) and Proposition 8.1(a) (actually, even from Section 1).

For an alternative proof, compare Theorem 9.4(a) and Proposition 10.6(a) below with Corollary 10.4 and Proposition 10.9.

Denote by $\text{Top}_Z^{\omega,\text{sc}} \subset \text{Top}_Z^{\omega}$ the full subcategory formed by all the (complete, separated) topological abelian groups with a countable base of neighborhoods of zero (consisting of open subgroups). Similarly, let $\text{Top}_k^{\omega,\text{sc}} \subset \text{Top}_k^{\omega}$ denote the full subcategory of all topological vector spaces with a countable base of neighborhoods of zero. Clearly, the full subcategories $\text{Top}_Z^{\omega,\text{sc}} \subset \text{Top}_Z^{\omega}$ and $\text{Top}_k^{\omega,\text{sc}} \subset \text{Top}_k^{\omega}$ are closed under kernels, cokernels, and countable products.

Furthermore, by Proposition 1.4, the cokernel of any morphism in $\text{Top}_Z^{\omega,\text{sc}}$ (as well as the cokernel of any injective closed morphism $i: \mathcal{K} \rightarrow \mathfrak{A}$ in $\text{Top}_Z^{\omega}$ with $\mathcal{K} \in \text{Top}_Z^{\omega,\text{sc}}$) is a surjective open map. Consequently, the category $\text{Top}_Z^{\omega,\text{sc}}$ is quasi-abelian. The kernels (i. e., the admissible monomorphisms in the quasi-abelian exact structure) in $\text{Top}_Z^{\omega,\text{sc}}$ are the injective closed maps, and the cokernels (i. e., the admissible epimorphisms in the quasi-abelian exact structure) are the surjective open maps. The full subcategory $\text{Top}_Z^{\omega,\text{sc}} \subset \text{Top}_Z^{\omega}$ inherits the maximal exact category structure of the additive category $\text{Top}_Z^{\omega}$, and the inherited exact category structure on $\text{Top}_Z^{\omega,\text{sc}}$ is the quasi-abelian exact structure.

The full subcategory $\text{Top}_k^{\omega,\text{sc}} \subset \text{Top}_k^{\omega}$ has similar properties. Moreover, by Corollary 3.4, every short exact sequence in the quasi-abelian category $\text{Top}_k^{\omega,\text{sc}}$ splits.
In other words, the quasi-abelian (hence maximal) exact structure on \( \mathbf{Top}_k^{\omega, \text{sc}} \) coincides with the split (minimal) exact category structure. In the ambient categories \( \mathbf{Top}_k^{\omega, \text{sc}} \subset \mathbf{Top}_k^s \subset \mathbf{Top}_k \), the complete, separated topological vector spaces with a countable base of neighborhoods of zero have a rather strong injectivity property described in Proposition 3.3. In particular, all the objects of \( \mathbf{Top}_k^{\omega, \text{sc}} \) are injective with respect to the maximal exact category structure on \( \mathbf{Top}_k^{\omega, \text{sc}} \) (and even with respect to the quasi-abelian exact structures on \( \mathbf{Top}_k^s \) and \( \mathbf{Top}_k \)).

However, the full subcategory \( \mathbf{Top}_k^{\omega, \text{sc}} \) is not closed under countable coproducts in \( \mathbf{Top}_k^s \); see Lemma 7.4. Similarly, the full subcategory \( \mathbf{Top}_k^{\omega, \text{sc}} \) is not closed under countable coproducts in \( \mathbf{Top}_k^s \). We recall that coproducts in the category \( \mathbf{Top}_k^{\omega, \text{sc}} \) agree with those in \( \mathbf{Top}_k^s \) and in \( \mathbf{Top}_k^k \) (and similarly for topological abelian groups); see Lemma 1.2.

In order to formulate the conclusion, let us add some bits of terminology. Given a complete, separated topological abelian group (or vector space) \( \mathfrak{A} \) with linear topology and a closed injective morphism of topological abelian groups/vector spaces \( i: \mathfrak{A} \rightarrow \mathfrak{B} \), we will say that the map \( i \) is stably closed if \( i \) is a stable kernel in \( \mathbf{Top}_k^{\omega, \text{sc}} \) or in \( \mathbf{Top}_k^{\omega, \text{sc}} \) (see Corollary 8.5 for the description). In this case, the closed subgroup/subspace \( i(\mathfrak{A}) \subset \mathfrak{B} \) will be also called stably closed.

**Conclusion 8.9.** The category \( \mathbf{Top}_k^{\omega, \text{sc}} \) of complete, separated topological vector spaces with linear topology is not quasi-abelian (see Proposition 8.3 and Corollary 8.6). Moreover, contrary to [5, page 1, Section 1.1], there does not exist an exact category structure on \( \mathbf{Top}_k^{\omega, \text{sc}} \) in which all the closed embeddings would be admissible monomorphisms. In the maximal exact category structure on \( \mathbf{Top}_k^{\omega, \text{sc}} \), the admissible epimorphisms are the open surjections, and the admissible monomorphisms are the stably closed embeddings.

The problem does not arise in the categories of incomplete topological vector spaces \( \mathbf{Top}_k^{\omega, \text{sc}} \subset \mathbf{Top}_k^s \subset \mathbf{Top}_k^k \); see Theorems 7.1 and 7.2, and it also does not arise in the category of complete, separated topological vector spaces with a countable base of neighborhoods of zero. However, countable coproducts in any one of the categories \( \mathbf{Top}_k^{\omega, \text{sc}} \subset \mathbf{Top}_k^s \subset \mathbf{Top}_k^k \) do not preserve the classes of topological vector spaces with a countable base of open subspaces (by Lemmas 7.4 and 1.2).

### 9. Pro-Vector Spaces

Complete, separated topological vector spaces with linear topology form a full subcategory in the abelian category of pro-vector spaces; and incomplete topological vector spaces can be interpreted as pro-vector spaces with some additional datum. In this section we explain that the full subcategory \( \mathbf{Top}_k^{\omega, \text{sc}} \) does not inherit an exact category structure from the abelian exact category structure of \( \mathbf{Pro}(\mathbf{Vect}_k) \). In particular, \( \mathbf{Top}_k^{\omega, \text{sc}} \) is not closed under extensions in \( \mathbf{Pro}(\mathbf{Vect}_k) \).

We refer to [20, Chapter 6] for a general discussion of ind-objects; the pro-objects are dual. Given a category \( \mathbf{C} \), the category \( \mathbf{Pro}(\mathbf{C}) \) of pro-objects in \( \mathbf{C} \) is defined as the
opposite category to the full subcategory in the category of covariant functors $C \rightarrow \text{Sets}$ formed by the directed inductive limits of corepresentable functors $\text{Hom}_C(C, -)$, $C \in C$. For an additive category $C$, one can use (if one wishes, additive) functors $C \rightarrow \text{Ab}$ in lieu of the functors $C \rightarrow \text{Sets}$.

Explicitly, this means that the objects of $\text{Pro}(C)$ are the projective systems $P: \Gamma \rightarrow C$ indexed by directed posets $\Gamma$. This means that for every $\gamma \in \Gamma$ there is an object $P_\gamma \in C$ and for every $\gamma < \delta \in \Gamma$ there is a morphism $P_\delta \rightarrow P_\gamma$ such that for every $\gamma < \delta < \epsilon \in \Gamma$ the triangle diagram $P_\gamma \rightarrow P_\delta \rightarrow P_\epsilon$ is commutative. The object of $\text{Pro}(C)$ corresponding to a projective system $(P_\gamma)_{\gamma \in \Gamma}$ is denoted by

$$\lim_{\gamma \in \Gamma} P_\gamma \in \text{Pro}(C).$$

The (opposite object to the) object $\lim_{\gamma \in \Gamma} P_\gamma$ corresponds to the functor $C \rightarrow \text{Sets}$ (or, in the additive case, possibly $C \rightarrow \text{Ab}$) defined by the rule

$$X \mapsto \lim_{\gamma \in \Gamma} \text{Hom}_C(P_\gamma, X), \quad X \in C.$$

The set/group of morphisms $\lim_{\gamma \in \Gamma} P_\gamma \rightarrow \lim_{\delta \in \Delta} Q_\delta$ in $\text{Pro}(C)$ is, by the definition, the set/group of morphisms in the opposite direction between the corresponding functors $C \rightarrow \text{Sets}$ (or $C \rightarrow \text{Ab}$). This means that

$$\text{Hom}_{\text{Pro}(C)}(\lim_{\gamma \in \Gamma} P_\gamma, \lim_{\delta \in \Delta} Q_\delta) = \lim_{\delta \in \Delta} \lim_{\gamma \in \Gamma} \text{Hom}_C(P_\gamma, Q_\delta)$$

for any two projective systems $(P_\gamma)_{\gamma \in \Gamma}$ and $(Q_\delta)_{\delta \in \Delta}$ in $C$ indexed by directed posets $\Gamma$ and $\Delta$.

To any object of $C$ one can assign the object of $\text{Pro}(C)$ represented by the projective system $R_C: \{\ast\} \rightarrow C$ indexed by the singleton $\Gamma = \{\ast\}$ with $R_C(\ast) = C$. This defines a fully faithful functor $C \rightarrow \text{Pro}(C)$ such that an arbitrary object $\lim_{\gamma \in \Gamma} P_\gamma \in \text{Pro}(C)$ is the projective limit of the objects $P_\gamma \in C \subset \text{Pro}(C)$ in the category $\text{Pro}(C)$. All directed projective limits exist in the category $\text{Pro}(C)$.

The formula for the set/group of morphisms in $\text{Pro}(C)$ from an arbitrary object $\lim_{\gamma \in \Gamma} P_\gamma$ to an object $C \in C \subset \text{Pro}(C)$ is worth writing down explicitly:

$$\text{Hom}_{\text{Pro}(C)}(\lim_{\gamma \in \Gamma} P_\gamma, C) = \lim_{\gamma \in \Gamma} \text{Hom}_C(P_\gamma, C).$$

So any given morphism $\lim_{\gamma \in \Gamma} P_\gamma \rightarrow C$ in $\text{Pro}(C)$ factorizes through the canonical projection $\lim_{\gamma \in \Gamma} P_\gamma \rightarrow P_\delta$ for some $\delta \in \Gamma$.

Notice that directed projective limits in the category $C$, which may or may not exist, are in any case almost never preserved by the embedding $C \rightarrow \text{Pro}(C)$. In fact, if $(P_\gamma \in C)_{\gamma \in \Gamma}$ is a projective system indexed by a directed poset $\Gamma$ and the projective limit $\lim_{\gamma \in \Gamma} P_\gamma$ exists in $C$, then the inclusion functor $C \rightarrow \text{Pro}(C)$ preserves this projective limit if and only if the object $\lim_{\gamma \in \Gamma} P_\gamma \in \text{Pro}(C)$ belongs to the full subcategory $C \subset \text{Pro}(C)$. Then one has $\lim_{\gamma \in \Gamma} P_\gamma = \lim_{\gamma \in \Gamma} P_\gamma$. If this is the case, then there exists an index $\delta \in \Gamma$ such that the projection $\lim_{\gamma \in \Gamma} P_\gamma \rightarrow P_\delta$ is a split monomorphism in $\text{Pro}(C)$, and it follows that the projection $\lim_{\gamma \in \Gamma} P_\gamma \rightarrow P_\delta$ is a split monomorphism in $C$. So this is a kind of degenerate situation.
If the category $\mathcal{C}$ is additive, then so is $\text{Pro}(\mathcal{C})$. The following description of zero objects in $\text{Pro}(\mathcal{C})$ is helpful. Let $\mathcal{C}$ be an additive category and $(P_\gamma)_{\gamma \in \Gamma}$ be a directed projective system in $\mathcal{C}$. One can see from the above description of morphisms in $\text{Pro}(\mathcal{C})$ that the object $\limleftarrow_{\gamma \in \Gamma} P_\gamma$ vanishes in $\text{Pro}(\mathcal{C})$ if and only if the projective system $(P_\gamma)_{\gamma \in \Gamma}$ is pro-zero, in the sense that for every $\gamma \in \Gamma$ there exists $\delta \in \Delta$, $\delta \geq \gamma$ such that the transition morphism $P_\delta \to P_\gamma$ is zero.

If the category $\mathcal{C}$ is abelian, then so is $\text{Pro}(\mathcal{C})$. Let us explain this assertion in some more detail. Given a morphism $f : \limleftarrow_{\gamma \in \Gamma} P_\gamma \to \limleftarrow_{\delta \in \Delta} Q_\delta$ in $\text{Pro}(\mathcal{C})$, consider the poset $\Xi$ formed by all the triples $(\xi', \delta', g_{\gamma',\delta'})$ such that $\gamma' \in \Gamma$, $\delta' \in \Delta$, $g_{\gamma',\delta'}$ is a morphism $g_{\gamma',\delta'} : P_{\gamma'} \to Q_{\delta'}$ in $\mathcal{C}$, and the square diagram formed by $f$, $g_{\gamma',\delta'}$, and the canonical projections $\limleftarrow_{\gamma \in \Gamma} P_\gamma \to P_{\gamma'}$, $\limleftarrow_{\delta \in \Delta} Q_\delta \to Q_{\delta'}$ is commutative in $\text{Pro}(\mathcal{C})$. By the definition, $(\gamma', \delta', g_{\gamma',\delta'}) \leq (\gamma'', \delta'', g_{\gamma'',\delta''})$ in $\Xi$ if $\gamma' \leq \gamma''$ in $\Gamma$, $\delta' \leq \delta''$ in $\Delta$, and the relevant square diagram is commutative in $\mathcal{C}$. Now the morphism $f$ is naturally isomorphic to the morphism $\limleftarrow_{\xi' \in \Xi} g_{\gamma',\delta'} : \limleftarrow_{\xi' \in \Xi} P_{\xi'} \to \limleftarrow_{\xi' \in \Xi} Q_{\xi'}$. This construction shows that any morphism in $\text{Pro}(\mathcal{C})$ can be represented by a $\Xi$-indexed projective system of morphisms in $\mathcal{C}$ for a suitable directed poset $\Xi$.

Now let $(f_\xi : P_\xi \to Q_\xi)_{\xi \in \Xi}$ be a directed projective system of morphisms in $\mathcal{C}$. Then the kernel and cokernel of the morphism $\limleftarrow_{\xi \in \Xi} f_\xi : \limleftarrow_{\xi \in \Xi} P_\xi \to \limleftarrow_{\xi \in \Xi} Q_\xi$ can be simply computed termwise as

$$\ker(\limleftarrow_{\xi \in \Xi} f_\xi) = \limleftarrow_{\xi \in \Xi} \ker(f_\xi) \quad \text{and} \quad \coker(\limleftarrow_{\xi \in \Xi} f_\xi) = \limleftarrow_{\xi \in \Xi} \coker(f_\xi).$$

Hence, in particular, the embedding functor $\mathcal{C} \to \text{Pro}(\mathcal{C})$ is exact.

Let $\mathcal{C}$ be a complete abelian category (i.e., an abelian category with projective limits, or equivalently, with infinite products). Then there is a left exact functor

$$\limleftarrow : \text{Pro}(\mathcal{C}) \to \mathcal{C}$$

taking a pro-object $\limleftarrow_{\gamma \in \Gamma} P_\gamma \in \text{Pro}(\mathcal{C})$ to the projective limit of the directed projective system $(P_\gamma)_{\gamma \in \Gamma}$ in the category $\mathcal{C}$,

$$\limleftarrow(\limleftarrow_{\gamma \in \Gamma} P_\gamma) = \limleftarrow_{\gamma \in \Gamma} P_\gamma.$$

The functor $\limleftarrow$ is right adjoint to the fully faithful exact embedding functor $\mathcal{C} \to \text{Pro}(\mathcal{C})$. In particular, there is a natural transformation of endofunctors on the category $\text{Pro}(\mathcal{C})$ assigning to every pro-object $P \in \text{Pro}(\mathcal{C})$ the adjunction morphism

$$\limleftarrow_{\gamma \in \Gamma} P_\gamma \to \limleftarrow_{\gamma \in \Gamma} P_\gamma$$

in the category of pro-objects $\text{Pro}(\mathcal{C})$ (where the object $\limleftarrow_{\gamma \in \Gamma} P_\gamma \in \mathcal{C}$ is viewed as an object of $\text{Pro}(\mathcal{C})$ via the embedding functor $\mathcal{C} \to \text{Pro}(\mathcal{C})$).

The following simple categorical construction will be useful for our discussion of incomplete topological vector spaces/abelian groups with linear topology in the next Section 10. A supplemented pro-object $(C, P)$ in a category $\mathcal{C}$ is a pair consisting of an object $C \in \mathcal{C}$ and a pro-object $P \in \text{Pro}(\mathcal{C})$, endowed with a morphism $\pi : C \to P$.
in \( \text{Pro}(C) \). Morphisms of supplemented pro-objects are defined in the obvious way. We will denote the category of supplemented pro-objects in \( C \) by \( \text{Pro}^{su}(C) \).

We will say that a supplemented pro-object \((C, P)\) in \( C \) is sup-epimorphic if the morphism \( \pi : C \rightarrow P \) is an epimorphism in the abelian category \( \text{Pro}(C) \). Let \((P_\gamma)_{\gamma \in \Gamma} \) be a directed projective system in \( C \) such that \( P = \lim_{\gamma \in \Gamma} P_\gamma \); then the morphism \( \pi \) is represented by a compatible cone of morphisms \( \pi_{\gamma} : C \rightarrow P_\gamma \). Denote by \( Q_\gamma = \ker(\pi_{\gamma}) \subset P_\gamma \) the images of the morphisms \( \pi_{\gamma} \). It is clear from the above discussion of kernels and cokernels in \( \text{Pro}(C) \) that \( \pi \) is an epimorphism in \( \text{Pro}(C) \) if and only if \( \lim_{\gamma \in \Gamma} P_\gamma/Q_\gamma = 0 \), i.e., the projective system \((P_\gamma/Q_\gamma)_{\gamma \in \Gamma}\) is pro-zero.

Equivalently, this means that the morphism \( \lim_{\gamma \in \Gamma} Q_\gamma \rightarrow \lim_{\gamma \in \Gamma} P_\gamma \) induced by the projective system of monomorphisms \( Q_\gamma \rightarrow P_\gamma \) is an isomorphism in \( \text{Pro}(C) \). If this is the case, one can replace the projective system \((P_\gamma/Q_\gamma)_{\gamma \in \Gamma}\) by the projective system \((Q_\gamma)_{\gamma \in \Gamma}\). This allows one to assume that all the morphisms \( \pi_{\gamma} : C \rightarrow P_\gamma \) are epimorphisms in \( C \). Then all the transition morphisms \( P_\delta \rightarrow P_\gamma \) in the projective system \((P_\gamma)_{\gamma \in \Gamma}\) are epimorphisms, too.

Let \( C \) be a complete abelian category. Then the forgetful functor \( \text{Pro}^{su}(C) \rightarrow \text{Pro}(C) \) taking a supplemented pro-object \((C, P)\) to the pro-object \( P \) has a right adjoint functor taking a pro-object \( P \in \text{Pro}(C) \) to the supplemented pro-object \((\lim P, P) \in \text{Pro}^{su}(C)\), where \( \pi : C = \lim P \rightarrow P \) is the above adjunction morphism.

We will say that a pro-object \( P \in \text{Pro}(C) \) is limit-epimorphic if the supplemented pro-object \((\lim P, P)\) is sup-epimorphic. Simply put, a pro-object \( P \in \text{Pro}(C) \) is limit-epimorphic if and only if there exists a directed projective system \((P_\gamma)_{\gamma \in \Gamma}\) in \( C \) such that \( P \simeq \lim_{\gamma \in \Gamma} P_\gamma \) in \( \text{Pro}(C) \) and the projection morphism \( \lim_{\gamma \in \Gamma} P_\gamma \rightarrow P_\delta \) is an epimorphism in \( C \) for every \( \delta \in \Gamma \).

A pro-object \( P \) in an abelian category \( C \) is said to be strict if there exists a directed projective system \((P_\gamma)_{\gamma \in \Gamma}\) in \( C \) such that all the transition morphisms \( P_\delta \rightarrow P_\gamma \), \( \gamma < \delta \in \Gamma \) are epimorphisms in \( C \) and \( P \simeq \lim_{\gamma \in \Gamma} P_\gamma \) in \( \text{Pro}(C) \). Following the discussion above, in any sup-epimorphic supplemented pro-object \((C, P) \in \text{Pro}^{su}(C)\), the pro-object \( P \in \text{Pro}(C) \) is strict. In particular, assuming that \( C \) is complete, any limit-epimorphic pro-object in \( C \) is strict.

**Remark 9.1.** It is well-known that any pro-vector space or pro-abelian group represented by a countable directed projective system of epimorphisms in \( \text{Vect}_k \) or \( \text{Ab} \) is limit-epimorphic in the sense of the above definition. However, generally speaking, a strict pro-vector space need not be limit-epimorphic. For a counterexample of a directed projective system of surjective linear maps between nonzero, countably-dimensional vector spaces, indexed by the first uncountable ordinal \( \aleph_1 \), whose projective limit vanishes, see [IS, Section 3].

**Proposition 9.2.** For any abelian category \( C \), the full subcategory of strict pro-objects is closed under quotients and extensions in \( \text{Pro}(C) \).

**Proof.** Let \( f : P \rightarrow Q \) be a morphism in \( \text{Pro}(C) \) with a strict pro-object \( P \). Then the construction from the above discussion of kernels and cokernels in \( \text{Pro}(C) \) allows
to represent \( f \) as the morphism \( f = \lim_{\xi\in \Xi} f_\xi : \lim_{\xi\in \Xi} P_\xi \to \lim_{\xi\in \Xi} Q_\xi \) for some projective system of morphisms \( f_\xi : P_\xi \to Q_\xi \) in \( C \) indexed by a directed poset \( \Xi \). Moreover, following the construction, one can choose \((P_\xi)_{\xi\in \Xi}\) to be a projective system of epimorphisms in \( C \). Then \((\text{im}(f_\xi))_{\xi\in \Xi}\) is also a projective system of epimorphisms in \( C \). If the morphism \( P \to Q \) is an epimorphism in \( \text{Pro}(C) \), then \( Q = \lim_{\xi\in \Xi} Q_\xi \simeq \lim_{\xi\in \Xi} \text{im}(f_\xi) \) in \( \text{Pro}(C) \). This proves that the full subcategory of strict pro-objects is closed under quotients.

Let \( 0 \to P' \to P \to P'' \to 0 \) be a short exact sequence in \( \text{Pro}(C) \). According to the same argument above, one can represent this short exact sequence as the "\( \lim \)" of a projective system of short exact sequences \( 0 \to P'_\xi \to P_\xi \to P''_\xi \to 0 \) indexed by a directed poset \( \Gamma \). Let \((Q'_\delta)_{\delta\in \Delta}\) be a directed projective system in \( C \) such that \( \lim_{\delta\in \Delta} Q'_\delta \simeq \lim_{\gamma\in \Gamma} P'_\gamma \) in \( \text{Pro}(C) \). Applying the same construction to the \((\text{iso})\)morphism \( \lim_{\gamma\in \Gamma} P'_\gamma \to \lim_{\delta\in \Delta} Q'_\delta \), we obtain a directed poset \( \Xi \), a projective system of short exact sequences \( 0 \to P'_{\xi} \to P_{\xi} \to P''_{\xi} \to 0 \) indexed by \( \Xi \), whose "\( \lim \)" is the original short exact sequence \( 0 \to P' \to P \to P'' \to 0 \), and a projective system of morphisms \( P'_\xi \to Q'_\xi \) indexed by \( \Xi \), whose "\( \lim \)" is the isomorphism \( \lim_{\delta\in \Delta} P'_\gamma \to \lim_{\gamma\in \Gamma} Q'_\gamma \). Moreover, following the construction, if \((Q'_\delta)_{\delta\in \Delta}\) is a projective system of epimorphisms in \( C \), then so is \((Q'_\xi)_{\xi\in \Xi}\).

Now, for every \( \xi \in \Xi \), consider the pushout \( 0 \to Q'_\xi \to Q_\xi \to P'_\xi \to 0 \) of the short exact sequence \( 0 \to P'_\xi \to P_\xi \to P''_\xi \to 0 \) by the morphism \( P'_\xi \to Q'_\xi \). Then we have an isomorphism \( \lim_{\xi\in \Xi} Q_\xi \simeq \lim_{\xi\in \Xi} P_\xi \) in \( \text{Pro}(C) \). The original short exact sequence \( 0 \to P' \to P \to P'' \to 0 \) in \( \text{Pro}(C) \) can be obtained (up to an isomorphism of short exact sequences in \( \text{Pro}(C) \)) by applying "\( \lim \)" to the directed projective system of short exact sequences \( 0 \to Q'_\xi \to Q_\xi \to P'_\xi \to 0 \) in \( C \).

Dually, let \((S''_\lambda)_{\lambda\in \Lambda}\) be a directed projective system in \( C \) such that \( \lim_{\gamma\in \Gamma} P''_\gamma \simeq \lim_{\lambda\in \Lambda} S''_\lambda \) in \( \text{Pro}(C) \). Applying once again the same construction to the \((\text{iso})\)morphism \( \lim_{\lambda\in \Lambda} S''_\lambda \to \lim_{\xi\in \Xi} P'_\xi \), we obtain a directed poset \( \Upsilon \), a projective system of short exact sequences \( 0 \to Q''_v \to Q_v \to P''_v \to 0 \) indexed by \( \Upsilon \), whose "\( \lim \)" is the original short exact sequence \( 0 \to P'' \to P \to P'' \to 0 \), and a projective system of morphisms \( S''_\gamma \to P''_\gamma \) indexed by \( \Upsilon \), whose "\( \lim \)" is the isomorphism \( \lim_{\lambda\in \Lambda} S''_\lambda \simeq \lim_{\gamma\in \Gamma} P''_\gamma \). Moreover, if \((S''_\gamma)_{\gamma\in \Gamma}\) is a projective system of epimorphisms in \( C \), then so is \((S''_v)_{v\in \Upsilon}\). Besides, if \((Q'_\xi)_{\xi\in \Xi}\) is a projective system of epimorphisms in \( C \), then so is \((Q'_v)_{v\in \Upsilon}\).

Finally, for every \( v \in \Upsilon \), we consider the pullback \( 0 \to Q'_v \to T_v \to S''_v \to 0 \) of the short exact sequence \( 0 \to Q'_v \to Q_v \to P''_v \to 0 \) by the morphism \( S''_v \to P''_v \). Then we have an isomorphism \( \lim_{v\in \Upsilon} T_v \simeq \lim_{v\in \Upsilon} Q_v \) in \( \text{Pro}(C) \). The original short exact sequence \( 0 \to P' \to P \to P'' \to 0 \) in \( \text{Pro}(C) \) can be obtained (up to an isomorphism) by applying "\( \lim \)" to the directed projective system of short exact sequences \( 0 \to Q'_v \to T_v \to S''_v \to 0 \) in \( C \).
We have shown that any short exact sequence $0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$ in $\text{Pro}(C)$ with strict pro-objects $P'$ and $P''$ can be obtained by applying “$\lim$” to a directed projective system of short exact sequences $0 \rightarrow Q'_v \rightarrow T_v \rightarrow S''_v \rightarrow 0$ in $C$ such that both $(Q'_v)_{v \in \mathcal{T}}$ and $(S''_v)_{v \in \mathcal{T}}$ are projective systems of epimorphisms. It remains to observe that if $(Q'_v)_{v \in \mathcal{T}}$ and $(S''_v)_{v \in \mathcal{T}}$ are projective systems of epimorphisms in $C$, then so is $(T_v)_{v \in \mathcal{T}}$, because the class of all epimorphisms is closed under extensions in the category of morphisms in $C$.

\begin{lemma}
For any complete abelian category $C$, the full subcategory of limit-epimorphic pro-objects is closed under quotients in $\text{Pro}(C)$.
\end{lemma}

\begin{proof}
If $P \rightarrow Q$ is an epimorphism in $\text{Pro}(C)$ and $\lim P \rightarrow P$ is also an epimorphism in $\text{Pro}(C)$ (where $\lim P \in C \subset \text{Pro}(C)$), then it follows from commutativity of the square diagram $\lim P \rightarrow P \rightarrow Q$, $\lim P \rightarrow \lim Q \rightarrow Q$ that $\lim Q \rightarrow Q$ is an epimorphism in $\text{Pro}(C)$.
\end{proof}

We will denote the full subcategory of limit-epimorphic pro-objects by $\text{Pro}_{le}(C) \subset \text{Pro}(C)$. The following theorem explains why we are interested in limit-epimorphic pro-objects in connection with topological algebra.

\begin{theorem}
(a) There is a natural equivalence of additive categories $\text{Top}_{\text{sc}}^k \simeq \text{Pro}_{le}(\text{Ab})$ assigning to a limit-epimorphic pro-abelian group $P$ the abelian group $A = \lim P$ endowed with the topology of projective limit of discrete abelian groups.

(b) There is a natural equivalence of additive categories $\text{Top}_{\text{sc}}^k \simeq \text{Pro}_{le}(\text{Vect}_k)$ assigning to a limit-epimorphic pro-vector space $P$ the vector space $V = \lim P$ endowed with the topology of projective limit of discrete vector spaces.
\end{theorem}

\begin{proof}
Both the equivalences are almost obvious (see Propositions 10.5\cite{10.6} below for generalizations). Let us explain part (b); part (a) is similar.

Given a pro-vector space $P = \left\{ \lim_{\gamma \in \Gamma} P_\gamma \right\}$, the projective limit topology (of discrete vector spaces $P_\gamma$) on $V = \lim P = \lim_{\gamma \in \Gamma} P_\gamma$ has a base of neighborhoods of zero consisting of the kernels of the projection maps $V \rightarrow P_\gamma$. One can readily check that this topology on $\lim P$ is well-defined and functorial, i.e., for any morphism of pro-vector spaces $\left\{ \lim_{\gamma \in \Gamma} P_\gamma \right\} \rightarrow \left\{ \lim_{\delta \in \Delta} Q_\delta \right\}$ the induced map of projective limits $\lim_{\gamma \in \Gamma} P_\gamma \rightarrow \lim_{\delta \in \Delta} Q_\delta$ is continuous. Moreover, the topological vector space $\lim_{\gamma \in \Gamma} P_\gamma$ is separated and complete (in fact, it is a closed subspace in the complete, separated topological vector space $\prod_{\gamma \in \Gamma} P_\gamma$, with the product topology of discrete vector spaces on $\prod_{\gamma \in \Gamma} P_\gamma$ and the induced topology of a subspace in $\prod_{\gamma \in \Gamma} P_\gamma$ on $\lim_{\gamma \in \Gamma} P_\gamma$). This defines the desired functor $\text{Pro}_{le}(\text{Vect}_k) \subset \text{Pro}(\text{Vect}_k) \rightarrow \text{Top}_{\text{sc}}^k$.

The inverse functor $\text{Top}_{\text{sc}}^k \subset \text{Top}_k \rightarrow \text{Pro}_{le}(\text{Vect}_k)$ assigns to a topological vector space $V$ the pro-vector space $P = \left\{ \lim_{U \subset V} V/U \right\}$, where $U$ ranges over the open vector subspaces in $V$. The pro-vector space $P$ is limit-epimorphic, because the composition $V \rightarrow \lim P \rightarrow V/U$ is surjective for all open vector subspaces $U \subset V$, and consequently the map $\lim P \rightarrow V/U$ is surjective. In fact, we have constructed
a pair of adjoint functors between the categories Top_k and Pro(Vect_k), with the left adjoint functor \( V \mapsto \lim_{U \in V} V/U \) and the right adjoint functor \( P \mapsto \lim_P \). Moreover, the former functor takes values inside Pro_{le}(Vect_k) \subseteq Pro(Vect_k), while the latter one lands within Top_k \subseteq Top_k. Checking that the restrictions of these functors to Top_k and Pro_{le}(Vect_k) are mutually inverse equivalences is left to the reader. \( \Box \)

It is easy to see that the class of all limit-epimorphic pro-objects is not closed under subobjects in Pro(C). In fact, for any directed projective system \((P_\gamma)_{\gamma \in \Gamma}\) one can construct a directed projective system of split monomorphisms \(P_\gamma \to Q_\gamma\) in \(C\) such that \(\lim_{\gamma \in \Gamma} Q_\gamma \to Q_\delta\) is a split epimorphism in \(C\) for every \(\delta \in \Gamma\). It suffices to put \(Q_\delta = \prod_{\gamma \leq \delta} P_\gamma\) for every \(\delta \in \Gamma\). The following counterexamples show that the class of all limit-epimorphic pro-objects is not closed under extensions in Pro(C), either (generally speaking).

**Proposition 9.5.** (a) The full subcategory \(\text{Top}^k_{sc} \simeq \text{Pro}_{le}(\text{Ab}) \subseteq \text{Pro}(\text{Ab})\) does not inherit an exact category structure from the abelian exact category of the abelian category \(\text{Pro}(\text{Ab})\). In fact, a short sequence in \(\text{Top}^k_{sc}\) is exact in \(\text{Pro}(\text{Ab})\) if and only if it satisfies Ex1 in \(\text{Top}^k_{sc}\) (but the class of all short sequences satisfying Ex1 is not an exact category structure on \(\text{Top}^k_{sc}\)). In particular, the full subcategory \(\text{Pro}_{le}(\text{Ab})\) is not closed under extensions in \(\text{Pro}(\text{Ab})\).

(b) The full subcategory \(\text{Top}^k_{sc} \simeq \text{Pro}_{le}(\text{Vect}_k) \subseteq \text{Pro}(\text{Vect}_k)\) does not inherit an exact category structure from the abelian exact category of the abelian category \(\text{Pro}(\text{Vect}_k)\). In fact, a short sequence in \(\text{Top}^k_{sc}\) is exact in \(\text{Pro}(\text{Vect}_k)\) if and only if it satisfies Ex1 in \(\text{Top}^k_{sc}\) (but the class of all short sequences satisfying Ex1 is not an exact category structure on \(\text{Top}^k_{sc}\)). In particular, the full subcategory \(\text{Pro}_{le}(\text{Vect}_k)\) is not closed under extensions in \(\text{Pro}(\text{Vect}_k)\).

**Proof.** Let us explain part (b); part (a) is similar. Clearly, any short sequence in \(\text{Pro}_{le}(\text{Vect}_k)\) which satisfies Ex1 in \(\text{Pro}(\text{Vect}_k)\) also satisfies Ex1 in \(\text{Pro}_{le}(\text{Vect}_k)\). Conversely, the class of all short sequences satisfying Ex1 in \(\text{Top}^k_{sc}\) was described in Proposition 5.1. Such sequences have the form \(0 \to \mathbb{K} \to \mathfrak{W} \to (\mathfrak{W}/\mathbb{K})^- \to 0\), where \(\mathfrak{W}\) is a complete, separated topological vector space, \(\mathbb{K} \subseteq \mathfrak{W}\) is a closed subspace with the induced topology, \(\mathfrak{W}/\mathbb{K}\) is the quotient space with the quotient topology, and \((\mathfrak{W}/\mathbb{K})^-\) is the completion of \(\mathfrak{W}/\mathbb{K}\), endowed with the completion topology. In this context, consider the projective system of short exact sequences of vector spaces

\[
0 \longrightarrow (\mathbb{K} + \mathfrak{U})/\mathfrak{U} \longrightarrow \mathfrak{W}/\mathfrak{U} \longrightarrow \mathfrak{W}/(\mathbb{K} + \mathfrak{U}) \longrightarrow 0
\]

indexed by the directed poset of all open subspaces \(\mathfrak{U} \subseteq \mathfrak{W}\). Passing to “\(\lim\)” produces from this projective system a short exact sequence of pro-objects in \(\text{Vect}_k\) corresponding to the short sequence \(0 \longrightarrow \mathbb{K} \longrightarrow \mathfrak{W} \longrightarrow (\mathfrak{W}/\mathbb{K})^- \longrightarrow 0\) in \(\text{Top}^k_{sc}\).

The class of all short sequences satisfying Ex1 in \(\text{Top}^k_{sc}\) is not an exact category structure as the category \(\text{Top}^k_{sc}\) is not quasi-abelian (see Corollary 8.6). In view of Example 11, it follows that the full subcategory \(\text{Pro}_{le}(\text{Vect}_k)\) cannot be closed under extensions in \(\text{Pro}(\text{Vect}_k)\). \(\Box\)
Example 9.6. Let us spell out an explicit example of a short exact sequence of pro-vector spaces whose leftmost and rightmost terms are limit-epimorphic, but the middle term isn’t. A straightforward adaptation of the counterexample from Corollary 8.4 and Examples 8.7 is sufficient for this purpose.

Let $\mathcal{V}$ be a complete, separated topological vector space with a closed subspace $\mathcal{K} \subset \mathcal{V}$ such that the quotient space $C = \mathcal{V}/\mathcal{K}$ is incomplete in the quotient topology. Put $\mathcal{C} = C^\times$, and choose a vector $x \in \mathcal{C} \setminus C$. Consider the short exact sequence of pro-vector spaces

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0$$

obtained by applying “$\lim$” to (5).

For every open subspace $\mathcal{U} \subset \mathcal{V}$, denote by $W_\mathcal{U} \subset C$ the open subspace $W_\mathcal{U} = (\mathcal{K} + \mathcal{U})/\mathcal{K} \subset \mathcal{V}/\mathcal{K} = C$, and let $W'_\mathcal{U} \subset \mathcal{C}$ be the related open subspace in $\mathcal{C}$. Then we have natural isomorphisms of discrete quotient spaces $\mathcal{V}/(\mathcal{K} + \mathcal{U}) \simeq C/W_\mathcal{U} \simeq \mathcal{C}/W'_\mathcal{U}$. Hence the choice of an element $x \in \mathcal{C}$ induces a projective system of linear maps $k \longrightarrow x : \mathcal{V}/(\mathcal{K} + \mathcal{U})$ indexed by the open subspaces $\mathcal{U} \subset \mathcal{V}$. Taking the pullbacks of (5) with respect to these maps, we obtain a projective system of pullback diagrams of short exact sequences in $\text{Vect}_k$ and the related pullback diagram of short exact sequences of pro-vector spaces

$$
\begin{array}{cccc}
\text{"lim"}_{\mathcal{U} \subset \mathcal{V}} (\mathcal{K} + \mathcal{U})/\mathcal{U} & \text{"lim"}_{\mathcal{U} \subset \mathcal{V}} \mathcal{V}/\mathcal{U} & \text{"lim"}_{\mathcal{U} \subset \mathcal{V}} \mathcal{V}/(\mathcal{K} + \mathcal{U}) \\
\text{"lim"}_{\mathcal{U} \subset \mathcal{V}} (kx \cap_{\mathcal{U}/(\mathcal{K} + \mathcal{U})} \mathcal{V}/\mathcal{U}) & \text{"lim"}_{\mathcal{U} \subset \mathcal{V}} kx &
\end{array}
$$

(6)

The lower line in (6) is a short exact sequence in $\text{Pro}(\text{Vect}_k)$ (obtained by taking “$\lim$” of the projective system of short exact sequences of vector spaces $0 \longrightarrow (\mathcal{K} + \mathcal{U})/\mathcal{U} \longrightarrow kx \cap_{\mathcal{U}/(\mathcal{K} + \mathcal{U})} \mathcal{V}/\mathcal{U} \longrightarrow kx \longrightarrow 0$ indexed by the directed poset of open subspaces $\mathcal{U} \subset \mathcal{V}$). In this short exact sequence of pro-vector spaces, the leftmost term “$\lim$”$_{\mathcal{U} \subset \mathcal{V}} (\mathcal{K} + \mathcal{U})/\mathcal{U}$ is a limit-epimorphic pro-vector space (corresponding to the complete, separated topological vector space $\mathcal{K}$). The rightmost term “$\lim$”$_{\mathcal{U} \subset \mathcal{V}} kx = kx \in \text{Vect}_k \subset \text{Pro}(\text{Vect}_k)$ is also a limit-epimorphic pro-vector space (corresponding to the discrete one-dimensional topological vector space $kx$).

However, the middle term $S = \text{"lim"}_{\mathcal{U} \subset \mathcal{V}} (kx \cap_{\mathcal{U}/(\mathcal{K} + \mathcal{U})} \mathcal{V}/\mathcal{U})$ is not limit-epimorphic. In fact, one has $\text{lim} S = kx \cap \mathcal{C} \mathcal{V} = \mathcal{K}$, so the image of the morphism $\text{"lim"} S \longrightarrow S$ in $\text{Pro}(\text{Vect}_k)$ is the subobject “$\lim$”$_{\mathcal{U} \subset \mathcal{V}} (\mathcal{K} + \mathcal{U})/\mathcal{U} = \ker(S \rightarrow kx) \subset \mathcal{S}$.

10. Supplemented Pro-Vector Spaces

The definition of the category of supplemented pro-objects $\text{Pro}^s(\mathcal{C})$ in an abelian category $\mathcal{C}$ was given in the previous Section 9. The notion of a supplemented pro-object allows to formulate a category-theoretic interpretation or generalization of the theory of topological abelian groups/vector spaces with linear topology. Both
Lemma 10.1. For any abelian category $C$, the category of supplemented pro-objects $\text{Pro}^{\text{su}}(C)$ is abelian. The forgetful functors $\text{Pro}^{\text{su}}(C) \to \text{Pro}(C)$ and $\text{Pro}^{\text{su}}(C) \to C$ are exact.

Proof. The category of morphisms in an abelian category is abelian; and $\text{Pro}^{\text{su}}(C)$ is a full additive subcategory closed under kernels and cokernels in the category of morphisms in $\text{Pro}(C)$.

We refer to Section 9 for the definition of a sup-epimorphic supplemented pro-object. Furthermore, we will say that a supplemented pro-object $(C, P)$ in $C$ is jointly sup-monomorphic if for every nonzero morphism $E \to C$ in $C$ the composition $E \to C \to P$ is a nonzero morphism in $\text{Pro}(C)$. Equivalently, this means that for any nonzero morphism $E \to C$ in $C$ there exists an object $D \in C$ and a morphism $P \to D$ in $\text{Pro}(C)$ such that the composition $E \to C \to P \to D$ is a nonzero morphism in $C$. Let $(P_\gamma)_{\gamma \in \Gamma}$ be a directed projective system in $C$ such that $P = \varprojlim \gamma \in \Gamma P_\gamma$, and let $\pi_\gamma : C \to P_\gamma$, $\gamma \in \Gamma$, be a compatible cone of morphisms representing the morphism $\pi$. Then $(C, P)$ is jointly sup-monomorphic if and only if, for every nonzero morphism $E \to C$ in $C$, there exists $\gamma \in \Gamma$ such that the composition $E \to C \to P \to P_\gamma$ is nonzero.

We will denote the full additive subcategory of sup-epimorphic supplemented pro-objects by $\text{Pro}_{\text{se}}^{\text{su}}(C) \subset \text{Pro}^{\text{su}}(C)$. The full additive subcategory of sup-epimorphic, jointly sup-monomorphic supplemented pro-objects will be denoted by $\text{Pro}_{\text{se,jsm}}^{\text{su}}(C) \subset \text{Pro}_{\text{se}}^{\text{su}}(C) \subset \text{Pro}^{\text{su}}(C)$.

Lemma 10.2. Let $C$ be an abelian category. Then

(a) the full subcategory $\text{Pro}_{\text{se}}^{\text{su}}(C)$ of sup-epimorphic supplemented pro-objects is closed under quotients and extensions in the abelian category $\text{Pro}^{\text{su}}(C)$;

(b) the full subcategory of jointly sup-monomorphic supplemented pro-objects is closed under subobjects and extensions in the abelian category $\text{Pro}^{\text{su}}(C)$;

(c) the full subcategory $\text{Pro}_{\text{se,jsm}}^{\text{su}}(C)$ of sup-epimorphic, jointly sup-monomorphic supplemented pro-objects is closed under subobjects in the additive category of sup-epimorphic supplemented pro-objects in $C$.

Proof. We will only prove part (c) and closedness under extensions in part (b), which is not difficult, but all the other assertions of the lemma are even easier.

Let $0 \to (C', P') \to (C, P) \to (C'', P'') \to 0$ be a short exact sequence in the abelian category $\text{Pro}^{\text{su}}(C)$. Assume that the supplemented pro-objects $(C', P')$ and $(C'', P'')$ are jointly sup-monomorphic, and let $E \to C$ be a morphism in $C$ such that the composition $E \to C \to P$ is a zero morphism in $\text{Pro}(C)$. Then the composition $E \to C \to C'' \to P''$ is also a zero morphism in $\text{Pro}(C)$, hence the composition $E \to C \to C''$ is a zero morphism in $C$. It follows that the morphism $E \to C$ factorizes through the monomorphism $C' \to C$, leading to a morphism $E \to C'$. Now the composition $E \to C' \to P'$ is a zero morphism in $\text{Pro}(C)$,
since the morphism $P' \to P''$ is a monomorphism. Consequently, the morphism $E \to C''$ is zero, and therefore the original morphism $E \to C$ is zero as well.

Let $(f, g): (C', P') \to (C'', P'')$ be a morphism of sup-epimorphic supplemented pro-objects. Put $K = \ker(f) \in C$ and $Q = \ker(g) \in \text{Pro}(C)$. Then there is the induced morphism $K \to Q$ in $\text{Pro}(C)$, making $(K, Q)$ a supplemented pro-object. Denote by $S$ the image of the morphism $K \to Q$ in $\text{Pro}(C)$. Then $(K, S)$ is a sup-epimorphic supplemented pro-object, and one can check that the composition $(K, S) \to (K, Q) \to (C'', P'')$ is the kernel of the morphism $(f, g)$ in the category of sup-epimorphic supplemented pro-objects in $C$. In particular, if $(f, g)$ is a monomorphism in the category of sup-epimorphic supplemented pro-objects, then $K = 0$ and therefore $f: C' \to C''$ is a monomorphism in $C$.

Now assume that $(C'', P'')$ is a jointly sup-monomorphic sup-epimorphic supplemented pro-object, so the composition $E \to C'' \to P''$ is nonzero in $\text{Pro}(C)$ for any nonzero morphism $E \to C''$ in $C$. Let $E \to C'$ be a nonzero morphism in $C$; then the composition $E \to C' \to C''$ is nonzero as well. It follows that the composition $E \to C' \to C'' \to P''$ is nonzero, and consequently the composition $E \to C'' \to P''$ cannot vanish. \qed

**Lemma 10.3.** (a) Let $A$ be a left semi-abelian additive category and $B \subset A$ be a reflective full subcategory with the reflector $\Psi: A \to B$ (i.e., $\Psi$ is left adjoint to the inclusion functor $B \hookrightarrow A$). Then the full subcategory $B$ is closed under subobjects in $A$ if and only if the adjunction morphism $\psi_A: A \to \Psi(A)$ is a cokernel in $A$ for every $A \in A$.

(b) Let $A$ be a quasi-abelian additive category and $B \subset A$ be a reflective full subcategory closed under subobjects and extensions. Then the additive category $B$ is quasi-abelian, and the quasi-abelian exact category structure on $B$ coincides with the exact category structure inherited from the quasi-abelian exact category structure on $A$.

**Proof.** Part (a): assume that $\psi_A: A \to \Psi(A)$ is a cokernel in $A$ for all $A \in A$. Let $B \in B$ be an object and $A \to B$ be a monomorphism in $A$. Then the composition $A \to B \to \Psi(B)$ is a monomorphism, too, since $\psi_B: B \to \Psi(B)$ is an isomorphism. As the composition $A \to \Psi(A) \to \Psi(B)$ is the same morphism, it follows that $\psi_A: A \to \Psi(A)$ is a monomorphism. Now a morphism that is simultaneously a cokernel and a monomorphism must be an isomorphism; hence $\psi_A$ is an isomorphism, and $A \in B$ because $\Psi(A) \in B$.

Conversely, assume that $B$ is closed under subobjects in $A$. The assumption that $A$ is left semi-abelian means that any morphism $f: C \to D$ in $A$ factorizes as a cokernel $C \to \text{coim}(f)$ followed by a monomorphism $\text{coim}(f) \to D$. Given an object $A \in A$, denote by $B = \text{coim}(\psi_A)$ the coimage of the morphism $\psi_A: A \to \Psi(A)$. Since $B$ is closed under subobjects in $A$ and $\Psi(A) \in B$, we have $B \in B$. In view of the universal property of the morphism $\psi_A$, the epimorphism $A \to B$ factorizes uniquely through $\psi_A$; so we get a morphism $\Psi(A) \to B$. It follows easily that the pair of morphisms $B \to \Psi(A)$ and $\Psi(A) \to B$ are mutually inverse isomorphisms. It remains to recall that the morphism $A \to B$ is a cokernel by construction.

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Part (b): first of all, a reflective full subcategory \(B\) in a category \(A\) with kernels and cokernels also has kernels and cokernels. In fact, any reflective full subcategory \(B \subset A\) is closed under kernels; and the cokernel of a morphism \(f: B' \to B''\) in \(B\) can be constructed by applying the reflector \(\Psi\) to the cokernel of \(f\) in \(A\).

Furthermore, if \(A\) is left semi-abelian and all the adjunction morphisms \(\psi_A: A \to \Psi(A)\) are cokernels in \(A\), then a morphism \(p: B \to C\) in \(B\) is a cokernel if and only if \(p\) is a cokernel in \(A\). Indeed, let \(i: K \to B\) be a kernel of \(p\) in \(A\), then \(K \in B\) and \(i\) is a kernel of \(p\) in \(B\). If \(p\) is a cokernel of \(i\) in \(A\), then \(p\) is also a cokernel of \(i\) in \(B\). Conversely, denote by \(q: B \to A\) the cokernel of \(i\) in \(A\). If \(p\) is a cokernel of \(i\) in \(B\), then \(p\) is the composition \(B \overset{q}{\to} A \overset{\psi_A}{\to} \Psi(A)\). Since \(A\) is left semi-abelian, the composition of two cokernels is a cokernel.

Under the same assumptions, a morphism \(i: K \to B\) in \(B\) is a kernel if and only if \(i\) is a kernel in \(A\) and the cokernel \(\text{coker}_A(i)\) of the morphism \(i\) in \(A\) belongs to \(B\). Indeed, if \(i\) is a kernel in \(A\), then \(i\) is a kernel of the morphism \(B \to \text{coker}_A(i) = C\). If \(C \in B\), then \(i\) is also a kernel of the morphism \(B \to C\) in \(B\). Conversely, assume that \(i\) is a kernel in \(B\). The cokernel of \(i\) in \(B\) is computable as \(\text{coker}_B(i) = \Psi(C)\). So \(i\) is a kernel of the morphism \(B \to \Psi(C)\) in \(B\). Then \(i\) is also a kernel of the morphism \(B \to \Psi(C)\) in \(A\). By our assumptions, the composition of two cokernels \(B \to C \to \Psi(C)\) is a cokernel in \(A\). So \(B \to \Psi(C)\) is a cokernel of the morphism \(i\) in \(A\). It follows that \(C \to \Psi(C)\) is an isomorphism and \(C \in B\).

Now it is clear that a short sequence in \(B\) satisfies Ex1 if and only if it satisfies Ex1 in \(A\). Since \(B\) is closed under extensions in \(A\) by assumption, \(B\) inherits an exact category structure from the quasi-abelian exact category structure on \(A\) (by Example 4.1). So the class of all short sequences satisfying Ex1 is an exact category structure on \(B\); in other words, this means that \(B\) is quasi-abelian.

**Corollary 10.4.** (a) For any abelian category \(C\), the additive category \(\text{Pro}_{se}^{su}(C)\) of sup-epimorphic supplemented pro-objects in \(C\) is quasi-abelian. The exact category structure on \(\text{Pro}_{se}^{su}(C)\) inherited from the abelian exact structure of the ambient abelian category of supplemented pro-objects \(\text{Pro}^{su}(C)\) coincides with the quasi-abelian exact category structure on \(\text{Pro}_{se}^{su}(C)\).

(b) Assume that the intersections of families of subobjects in any given object exist in an abelian category \(C\). Then the additive category \(\text{Pro}_{se,jsm}^{su}(C)\) of sup-epimorphic, jointly sup-monomorphic supplemented pro-objects in \(C\) is quasi-abelian. The exact category structure on \(\text{Pro}_{se,jsm}^{su}(C)\) inherited from the quasi-abelian exact structure of the ambient quasi-abelian category of sup-epimorphic supplemented pro-objects \(\text{Pro}_{se}^{su}(C)\) coincides with the quasi-abelian exact category structure on \(\text{Pro}_{se,jsm}^{su}(C)\).

**Proof.** Part (a) is a particular case of the opposite version of Lemma 10.3 (with an abelian ambient category). The full subcategory \(\text{Pro}_{se}^{su}(C)\) is closed under quotients and extensions in \(\text{Pro}^{su}(C)\) by Lemma 10.2(a). Furthermore, the full subcategory \(\text{Pro}_{se}^{su}(C)\) is coreflective in \(\text{Pro}^{su}(C)\): the coreflector \(\Phi: \text{Pro}_{se}^{su}(C) \to \text{Pro}_{se}^{su}(C)\) assigns to any supplemented pro-object \((C, P) \in \text{Pro}_{se}^{su}(C)\) the sup-epimorphic supplemented pro-object \(\Phi(C, P) = (C, Q)\), where \(Q\) is the image of the morphism \(\pi: C \to P\) in
the abelian category \( \text{Pro}(\mathcal{C}) \). The adjunction morphism \( \phi_{(C,P)} : (C, Q) \rightarrow (C, P) \) is a monomorphism in the abelian category \( \text{Pro}^{\text{su}}(\mathcal{C}) \).

Part (b) is a particular case of Lemma 10.2. The full subcategory \( \text{Pro}^{\text{su}}_{\text{se,jsm}}(\mathcal{C}) \) is closed under subobjects and extensions in \( \text{Pro}^{\text{su}}(\mathcal{C}) \) by part (a) and Lemma 10.2(b–c). The full subcategory \( \text{Pro}^{\text{su}}_{\text{se,jsm}}(\mathcal{C}) \) is reflective in \( \text{Pro}^{\text{su}}(\mathcal{C}) \): the reflector \( \Psi : \text{Pro}^{\text{su}}(\mathcal{C}) \rightarrow \text{Pro}^{\text{su}}_{\text{se,jsm}}(\mathcal{C}) \) assigns to any sup-epimorphic supplemented pro-object \((C, P) \in \text{Pro}^{\text{su}}(\mathcal{C})\) the sup-epimorphic, jointly sup-monomorphic supplemented pro-object \( \Psi(C, P) = (C/K, P) \in \text{Pro}^{\text{su}}_{\text{se,jsm}}(\mathcal{C}) \). Here \( K \in \mathcal{C} \) is a subobject in \( C \) constructed as follows. Let \((P_\gamma)_{\gamma \in \Gamma} \) be a directed projective system in \( \mathcal{C} \) such that \( P = \lim_{\gamma \in \Gamma} P_\gamma \), and let \( \pi_\gamma : C \rightarrow P_\gamma, \gamma \in \Gamma, \) be the compatible cone of morphisms in \( \mathcal{C} \) representing the morphism \( \pi : C \rightarrow P \) in \( \text{Pro}(\mathcal{C}) \). Then \( K \) is the intersection of the subobjects \( \ker(\pi_\gamma) \) in the object \( C \in \mathcal{C} \). The adjunction morphism \( \psi_{(C,P)} : (C,P) \rightarrow (C/K,P) \) is a cokernel of the morphism \((K,0) \rightarrow (C,P) \) in \( \text{Pro}^{\text{su}}_{\text{se,jsm}}(\mathcal{C}) \). \( \square \)

**Proposition 10.5.** (a) There is a natural equivalence of additive categories \( \text{Top}_Z \simeq \text{Pro}^{\text{su}}_{\text{se}}(\mathcal{A}) \) assigning to a topological abelian group \( A \) with linear topology the sup-epimorphic supplemented pro-abelian group \( (C, P) \) with \( C = A \) and \( P = \lim_{U \subseteq A} A/U \), where \( U \) ranges over the directed poset of all open subgroups in \( A \).

(b) There is a natural equivalence of additive categories \( \text{Top}_k \simeq \text{Pro}^{\text{su}}_{\text{se}}(\mathcal{V}) \) assigning to a topological vector space \( V \) with linear topology the sup-epimorphic supplemented pro-vector space \( (C, P) \) with \( C = V \) and \( P = \lim_{U \subseteq V} V/U \), where \( U \) ranges over the directed poset of all open subspaces in \( V \).

**Proof.** Both the assertions are essentially obvious. Let us say a few words about (b). Given a topological vector space \( V \), the morphism of pro-vector spaces \( \pi : V \rightarrow \lim_{U \subseteq V} V/U \) is defined by the compatible cone of natural surjections \( V \rightarrow V/U \).

The morphism \( \pi \) is an epimorphism in \( \text{Pro}(\mathcal{V}) \), since all the maps \( V \rightarrow V/U \) are epimorphisms in \( \mathcal{V} \). This defines the functor \( \text{Top}_k \rightarrow \text{Pro}^{\text{su}}_{\text{se}}(\mathcal{V}) \).

The inverse functor assigns to every supplemented pro-vector space \( (C, P) \) the vector space \( C \in \mathcal{V} \) endowed with the topology in which a vector subspace \( U \subseteq C \) is open if and only if, \( U \), viewed as a subobject of the object \( C \in \mathcal{V} \), contains the subobject \( \ker(\pi) \subseteq C \). \( \ker(\pi) \in \text{Pro}(\mathcal{V}) \). Part (a) is similar. \( \square \)

**Proposition 10.6.** (a) The equivalence of additive categories of topological abelian groups and sup-epimorphic supplemented pro-abelian groups \( \text{Top}_Z \simeq \text{Pro}^{\text{su}}_{\text{se}}(\mathcal{A}) \) from Proposition 10.2(a) restricts to an equivalence between the full subcategories of separated topological abelian groups and sup-epimorphic, jointly sup-monomorphic pro-abelian groups, \( \text{Top}^Z_{\text{se}} \simeq \text{Pro}^{\text{su}}_{\text{se,jsm}}(\mathcal{A}) \).

(b) The equivalence of additive categories of topological vector spaces and sup-epimorphic supplemented pro-vector spaces \( \text{Top}_k \simeq \text{Pro}^{\text{su}}_{\text{se}}(\mathcal{V}) \) from Proposition 10.2(b) restricts to an equivalence between the full subcategories of separated topological vector spaces and sup-epimorphic, jointly sup-monomorphic pro-vector spaces, \( \text{Top}^k_{\text{se}} \simeq \text{Pro}^{\text{su}}_{\text{se,jsm}}(\mathcal{V}) \).
Proof. Part (b): given a topological vector space $V$ and the related (sup-epimorphic) supplemented pro-vector space $(V, P)$, a morphism of vector spaces $f: E \rightarrow V$ vanishes in composition with the morphism $\pi: V \rightarrow P$ if and only if the image of $f$ is contained in the closure of the zero subgroup in the topological vector space $V$. Part (a) is similar. \qed

Let $C$ be a complete abelian category. Then, following the discussion in Section 9, the exact forgetful functor $\text{Pro}^{su}(C) \rightarrow \text{Pro}(C)$ has a right adjoint functor taking a pro-object $P \in \text{Pro}(C)$ to the supplemented pro-object $(\varinjlim P, P) \in \text{Pro}^{su}(C)$. The fully faithful functor $P \mapsto (\varinjlim P, P)$ allows to consider $\text{Pro}(C)$ as a full subcategory in $\text{Pro}^{su}(C)$. Notice that all the supplemented pro-objects in $\text{Pro}(C) \subset \text{Pro}^{su}(C)$ are jointly sup-monomorphic. The following lemma implies that the full subcategory $\text{Pro}(C) \subset \text{Pro}^{su}(C)$ is closed under extensions.

Lemma 10.7. Let $A$ and $B$ be abelian categories and $\Theta: A \rightarrow B$ be an exact functor with a fully faithful right adjoint functor $\Lambda: B \rightarrow A$ (so the composition $\Theta \Lambda: B \rightarrow B$ is the identity functor). Then the essential image $\Lambda(B) \subset A$ of the functor $\Lambda$ is closed under extensions as a full subcategory in the abelian category $A$.

Proof. Let $0 \rightarrow B' \rightarrow A \rightarrow B'' \rightarrow 0$ be a short exact sequence in $A$ with $B', B'' \in \Lambda(B)$. Then $0 \rightarrow \Theta(B') \rightarrow \Theta(A) \rightarrow \Theta(B'') \rightarrow 0$ is a short exact sequence in $B$. The functor $\Lambda$ is left exact (as a right adjoint functor between abelian categories), so $0 \rightarrow \Lambda \Theta(B') \rightarrow \Lambda \Theta(A) \rightarrow \Lambda \Theta(B'')$ is a left exact sequence in $A$. Now we have a commutative diagram of adjunction morphism of short sequences

$$
\begin{array}{c}
0 & \rightarrow & B' & \rightarrow & A & \rightarrow & B'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \Lambda \Theta(B') & \rightarrow & \Lambda \Theta(A) & \rightarrow & \Lambda \Theta(B'') & \\
\end{array}
$$

where the morphisms $B' \rightarrow \Lambda \Theta(B')$ and $B'' \rightarrow \Lambda \Theta(B'')$ are isomorphisms. It follows that $\Lambda \Theta(A) \rightarrow \Lambda \Theta(B'')$ is an epimorphism and $A \rightarrow \Lambda \Theta(A)$ is an isomorphism; so $A \in \Lambda(B) \subset A$. \qed

Lemma 10.8. Let $A$ and $B$ be abelian categories and $\Theta: A \rightarrow B$ be an exact functor with a fully faithful right adjoint $\Lambda: B \rightarrow A$ (so the essential image $\Lambda(B) \subset A$ of the functor $\Lambda$ is a reflective full subcategory in $A$ with the reflector $\Lambda \Theta$). Furthermore, let $E \subset A$ be a coreflective full subcategory with the coreflector $\Phi: A \rightarrow E$; suppose that $E$ is closed under quotients and extensions in $A$. Assume that $\Phi(\Lambda(B)) \subset \Lambda(E)$ and $\Lambda \Theta(E) \subset E$ in $A$. Then

(a) the full subcategory $F = \Lambda(B) \cap E \subset A$ is right quasi-abelian;

(b) a morphism in $F$ is a kernel in $F$ if and only if it is a monomorphism in $A$;

(c) the full subcategory $F \subset A$ is closed under extensions, so it inherits an exact category structure from the abelian exact structure of the abelian category $A$.

Proof. Part (b): first of all, the full subcategory $\Lambda(B)$ is closed under kernels in $A$ (since it is reflective), while the full subcategory $E$ is closed under cokernels in $A$. 48
Furthermore, the functor $\Phi|_{\Lambda(B)} : \Lambda(B) \to F$ is the coreflector onto $F$ in $\Lambda(B)$, while the functor $\Lambda \Theta|_E : E \to F$ is the reflector onto $F$ in $E$.

Let $f : F' \to F''$ be a morphism in $F$. Then the kernel of $f$ in $F$ can be computed by applying the coreflector $\Phi$ to the kernel of $f$ in $A$ (which belongs to $\Lambda(B)$), that is $\ker_F(f) = \Phi(\ker_A(f))$; while the cokernel of $f$ in $F$ can be computed by applying the reflector $\Lambda \Theta$ to the cokernel of $f$ in $A$ (which belongs to $E$), i.e., $\coker_F(f) = \Lambda \Theta(\coker_A(f))$. Put $B = \ker_A(f) \in A$; then the composition $\Phi(B) \to F'$ of the adjunction morphism $\Phi(B) \to B$ with the natural morphism $B \to F'$ is a kernel of the morphism $f$ in $F$. The full subcategory $E$ is closed under quotients in $A$ by assumption; so the dual version of Lemma 10.3(a) tells that the morphism $\Phi(B) \to B$ is a monomorphism in $A$, and the morphism $B \to F'$ is a monomorphism in $A$ by definition. It follows that the composition $\Phi(B) \to B \to F'$ is a monomorphism in $A$, too; thus any kernel in $F$ is a monomorphism in $A$.

Conversely, let $i : K \to F$ be a morphism in $F$ such that $i$ is a monomorphism in $A$. Put $E = \coker_A(i) \in A$; then the composition $t : F \to \Lambda \Theta(E)$ of the natural morphism $F \to E$ with the adjunction morphism $E \to \Lambda \Theta(E)$ is a cokernel of the morphism $i$ in $F$. The functor $\Theta$ is exact by assumption, while the functor $\Lambda$ is left exact as a right adjoint. Applying a left exact functor $\Lambda \Theta$ to the short exact sequence $0 \to K \to F \to E \to 0$ in $A$, we obtain a left exact sequence $0 \to K \to F \to \Lambda \Theta(E) \to 0$ in $A$. Hence the adjunction morphism $E \to \Lambda \Theta(E)$ is a monomorphism in $A$. Since $i$ is a kernel of the epimorphism $F \to E$ in $A$, it is also a kernel of the composition $F \to E \to \Lambda \Theta(E)$. Now $t : F \to \Lambda \Theta(E)$ is a monomorphism in $F$; as $i$ is a kernel of $t$ in $A$, so $i$ is also a kernel of $t$ in $F$.

Part (a): let $i : K \to F$ and $g : K \to L$ be two morphisms in $F$ such that $i$ is a kernel in $F$. By part (b), this means that $i$ is a monomorphism in $A$. Denote by $E$ the pushout of the morphisms $i$ and $g$ computed in the category $A$; so $E = \coker_A((-g,i)) : K \to L \oplus F$. Then the object $\Lambda \Theta(E)$ is the pushout of $i$ and $g$ in $F$. The natural morphism $L \to \Lambda \Theta(E)$ is the composition $L \to E \to \Lambda \Theta(E)$. The morphism $L \to E$ is a monomorphism in $A$, since $A$ is an abelian category. Following the argument in the previous paragraph, the adjunction morphism $E \to \Lambda \Theta(E)$ is a monomorphism in $A$. The composition of two monomorphisms $L \to \Lambda \Theta(E)$ is again a monomorphism in $A$; being a morphism in $F$, it is a kernel in $F$ by part (b).

Part (c): the full subcategory $\Lambda(B) \subset A$ is closed under extensions by Lemma 10.7, while the full subcategory $E \subset A$ is closed under extensions by assumption. Hence the intersection $F = \Lambda(B) \cap E$ is closed under extensions in the abelian category $A$, and Example 11 is applicable.

Let $C$ be a complete additive category. Then the composition of fully faithful additive functors $\text{Pro}_{\text{le}}(C) \to \text{Pro}(C) \to \text{Pro}_{\text{su}}(C)$ allows to consider the category $\text{Pro}_{\text{le}}(C)$ of limit-epimorphic pro-objects in $C$ as a full subcategory in the category $\text{Pro}_{\text{su}}(C)$ of supplemented pro-objects. In fact, a pro-object $P \in \text{Pro}(C)$ is limit-epimorphic if and only if the corresponding supplemented pro-object $(\lim P, P) \in \text{Pro}(C)$ is sup-epimorphic; so one has $\text{Pro}_{\text{le}}(C) = \text{Pro}(C) \cap \text{Pro}_{\text{su}}(C) \subset \text{Pro}_{\text{le}}(C)$. 

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Proposition 10.9. For any complete abelian category $C$, the additive category $\text{Pro}_\text{le}(C)$ of limit-epimorphic pro-objects in $C$ is right quasi-abelian. The full subcategory $\text{Pro}_\text{le}(C)$ is closed under extensions in the abelian category $\text{Pro}^{\text{su}}(C)$, and the exact category structure on $\text{Pro}_\text{le}(C)$ inherited from the abelian exact structure of $\text{Pro}^{\text{su}}(C)$ coincides with the maximal exact structure on $\text{Pro}_\text{le}(C)$.

Proof. Consider the abelian categories $A = \text{Pro}^{\text{su}}(C)$ and $B = \text{Pro}(C)$, the forgetful exact functor $\Theta: \text{Pro}^{\text{su}}(C) \to \text{Pro}(C)$, and the fully faithful functor $\Lambda: \text{Pro}(C) \to \text{Pro}^{\text{su}}(C)$ right adjoint to $\Theta$; so $\Lambda(P) = (\lim P, P)$ for every $P \in \text{Pro}(C)$. Consider further the coreflective full subcategory $\text{Pro}^{\text{su}}(C) \subset \text{Pro}^{\text{su}}(C)$ with the coreflector $\Phi: \text{Pro}^{\text{su}}(C) \to \text{Pro}^{\text{le}}(C)$ described in the proof of Corollary 10.4(a); so $\Phi$ takes an arbitrary supplemented pro-object $(C, P) \in \text{Pro}^{\text{su}}(C)$ to the supplemented pro-object $(C, Q)$, where $Q$ is the image of the morphism $\pi: C \to P$ in $\text{Pro}(C)$.

Let $P = \varprojlim_{\gamma \in \Gamma} P_\gamma$ be a pro-object in $C$; denote by $Q_5 \subset P_3$ the image of the projection morphism $\lim_{\gamma \in \Gamma} P_\gamma \to P_5$ in the category $C$. Clearly, one has $\lim_{\gamma \in \Gamma} P_\gamma \simeq \lim_{\gamma \in \Gamma} P_\gamma$ in $C$, and it follows that $\Phi(\Lambda(B)) \subset \Lambda(B)$.

Let $(C, P)$ be a sup-epimorphic supplemented pro-object in $C$, so the morphism $\pi: C \to P$ is an epimorphism in $\text{Pro}(C)$. The morphism $C \to P$ factorizes naturally as $C \to \lim P \to P$, and it follows that $\lim P \to P$ is also an epimorphism in $\text{Pro}(C)$. Hence we have $\Lambda\Theta(E) \subset E$.

Therefore, Lemma 10.8 is applicable to our set of data, and we can conclude that the additive category $\text{Pro}_\text{le}(C) = F = \Lambda(B) \cap E$ is right quasi-abelian and closed under extensions as a full subcategory in $A = \text{Pro}^{\text{su}}(C)$. It remains to explain that the inherited exact category structure on $\text{Pro}_\text{le}(C)$ from the abelian exact structure on $\text{Pro}^{\text{su}}(C)$ is the maximal one. Indeed, let $0 \to K \xrightarrow{i} F \xrightarrow{p} G \to 0$ be a short exact sequence in the maximal exact category structure on $\text{Pro}_\text{le}(C)$. Then $i$ is a kernel in $F$, hence by Lemma 10.8(b) $i$ is a monomorphism in $A$. Furthermore, $p$ is the cokernel of $i$ in $F$; following the discussion in the proof of Lemma 10.8(b), this means that $p$ is the composition $F \to E \to G$, where $F \to E$ is the cokernel of $i$ in $A$ and $E \to \Lambda\Theta(E) = G$ is the adjunction morphism. Moreover, following the same argument, the adjunction morphism $E \to G$ is a monomorphism in $A$.

Given an object $A \in A = \text{Pro}^{\text{su}}(C)$, let us use the notation $A = (C_A, P_A)$ the related pair of objects $C_A \in C$ and $P_A \in \text{Pro}(C)$, with the structure morphism $\pi_A: C_A \to P_A$. Then the morphisms $C_K = \lim P_K \to P_K$ and $C_F = \lim P_F \to P_F$ are epimorphisms in $\text{Pro}(C)$, the short sequence $0 \to P_K \to P_F \xrightarrow{\Phi} P_E = P_G \to 0$ is exact in $\text{Pro}(C)$, the short sequence $0 \to C_K \to C_F \to C_E \to 0$ is exact in $C$, and the morphisms $C_E \to P_E$ and $C_G = \lim P_E \to P_E = P_G$ are also epimorphisms in $\text{Pro}(C)$. The morphism $C_E \to C_G$ is a monomorphism in $C$, and we have to show that it is an isomorphism.

Denote by $D$ the supplemented pro-object $D = (C_D, P_D)$ with $C_D = C_G \in C$ and $P_D = C_G \in C \subset \text{Pro}(C)$; so the morphism $\pi_D: C_D \to P_D$ is an isomorphism in $\text{Pro}(C)$. (This is our analogue of the discrete abelian group $\mathbb{Z}$ from the proof of Proposition 8.4 or the discrete vector space $kx$ from Examples 8.4.) Clearly, we have
$D \in \mathcal{F}$. Denote by $g : D \to G$ the natural morphism in $\mathcal{F}$ with the components 
$id : C_D \to C_G$ and $\pi_G : P_D \to P_G$. Then the pullback of the pair of morphisms $p$ and $g$ in the category $\mathcal{F}$ can be computed by applying the coreflector $\Phi$ to the pullback $H = (C_F, C_G \cap P_G P_F)$. In fact, the morphism $C_H = C_F \to C_E \cap P_E P_F$ is an epimorphism in $\text{Pro}(\mathcal{C})$, essentially because the morphism $F \to E$ is a cokernel of the morphism $K \to F$ in $\text{Pro}^{sa}(\mathcal{C})$ and the morphism $\pi_K : C_K \to P_K$ is an epimorphism in $\text{Pro}(\mathcal{C})$. Thus we have $\Phi(H) = (C_F, C_E \cap P_E P_F) \in \mathcal{F}$.

By axiom Ex2(b) or Ex2′(b), the class of all short exact sequences in the maximal exact category structure on $\mathcal{F}$ must be closed under pullbacks; in other words, the morphism $p : F \to G$ has to be a semi-stable cokernel in $\mathcal{F}$, which means that the natural morphism $q : \Phi(H) \to D$ must be a cokernel in $\mathcal{F}$. One has $\ker(q) = \ker_A(q) = K$ and $\coker_A(K \to \Phi(H)) = (C_E, C_E)$, hence $\coker_A(K \to \Phi(H)) = \Lambda \Theta(C_E, C_E) = (C_E, C_E)$. Hence $q$ is a cokernel in $\mathcal{F}$ if and only if $C_E \to C_G$ is a isomorphism. In the latter case, the short sequence $0 \to K \to F \to G \to 0$ is exact in $\mathcal{A}$, and we are done. □

**Conclusion 10.10.** Viewed as a full subcategory in the abelian category of pro-vector spaces $\text{Pro}(\text{Vect}_k)$, the category of complete, separated topological vector spaces with linear topology $\text{Top}^{sc}_k$ is not well-behaved. It is not closed under extensions and does not inherit an exact category structure (see Proposition 10.5(b)).

Viewed as a full subcategory in the abelian category of supplemented pro-vector spaces $\text{Pro}^{sa}(\text{Vect}_k)$, the additive category $\text{Top}^{sc}_k$ is better behaved. It is closed under extensions and inherits an exact category structure: the inherited exact category structure is the maximal exact category structure on $\text{Top}^{sc}_k$. (Compare Theorem 9.4(b) with Proposition 10.3.)

The theory developed in Sections 9–10 extends the notion of a topological abelian group/vector space with linear topology from the case of abelian groups or vector spaces to arbitrary abelian categories $\mathcal{C}$ with infinite products.

### 11. The Strong Exact Category Structure on VSLTs

The following construction plays a key role in the theory of contramodules over topological rings [32, Remark A.3], [34, Section 1.2], [35, Section 2.1], [39, Sections 1.2 and 5], [40, Section 6.2], [37, Section 2.7], [36, Sections 2.5–2.7], [41, Section 1].

Let $\mathfrak{A}$ be a complete, separated topological abelian group with linear topology, and let $X$ be a set. A family of elements $(a_x \in \mathfrak{A})_{x \in X}$ is said to *converge to zero* in the topology of $\mathfrak{A}$ if for every open subgroup $U \subset \mathfrak{A}$ the set $\{x \in X \mid a_x \notin U\}$ is finite. In the terminology of [34, Section 1], such a family of elements $(a_x)_{x \in X}$ in a complete, separated topological abelian group $\mathfrak{A}$ would be called “$S$-Cauchy” or “summable”.

We will consider infinite formal linear combinations $\sum_{x \in X} a_x x$ of elements of $X$ with families of coefficients $(a_x)_{x \in X}$ converging to zero in the topology of $\mathfrak{A}$. Such
infinite formal linear combinations form an abelian group, which we will denote by \( \mathfrak{A}[[X]] \). Equivalently, one can define \( \mathfrak{A}[[X]] \) as the projective limit
\[
\mathfrak{A}[[X]] = \lim_{\leftarrow \mathfrak{U} \in \mathfrak{A}} (\mathfrak{A}/\mathfrak{U})[X],
\]
where \( \mathfrak{U} \) ranges over the poset of all open subgroups in \( \mathfrak{A} \) and, for any abelian group \( A \), the notation \( A[X] = A^{(X)} \) stands for the direct sum of \( X \) copies of \( A \), interpreted as the set of all finite formal linear combinations of elements of \( X \) with the coefficients in \( A \). Furthermore, one can endow \( \mathfrak{A}[[X]] \) with the topology of projective limit of discrete abelian groups \( (\mathfrak{A}/\mathfrak{U})[X] \), in which the kernels of the projection maps \( \mathfrak{A}[[X]] \rightarrow (\mathfrak{A}/\mathfrak{U})[X] \) form a base of neighborhoods of zero. Obviously, this makes \( \mathfrak{A}[[X]] \) a complete, separated topological abelian group.

The rule assigning the topological abelian group \( \mathfrak{A}[[X]] \) to a topological abelian group \( \mathfrak{A} \in \text{Top}_{\text{sc}}^\mathbb{Z} \) and a set \( X \) is a covariant functor of two arguments
\[
[-[-]]: \text{Top}_{\text{sc}}^\mathbb{Z} \times \text{Sets} \longrightarrow \text{Top}_{\text{sc}}^\mathbb{Z}.
\]
Specifically, if \( g: \mathfrak{A} \rightarrow \mathfrak{B} \) is a continuous homomorphism of complete, separated topological abelian groups and \( X \) is a set, then the induced continuous homomorphism \( g[[X]]: \mathfrak{A}[[X]] \rightarrow \mathfrak{B}[[X]] \) takes a formal linear combination \( \sum_{x \in X} a_x x \in \mathfrak{A}[[X]] \) to the formal linear combination \( \sum_{x \in X} g(a_x) x \in \mathfrak{B}[[X]] \). If \( \mathfrak{A} \) is a complete, separated topological abelian group and \( f: X \rightarrow Y \) is a map of sets, then the induced continuous homomorphism \( \mathfrak{A}[[f]]: \mathfrak{A}[[X]] \rightarrow \mathfrak{A}[[Y]] \) takes a formal linear combination \( \sum_{x \in X} a_x x \in \mathfrak{A}[[X]] \) to the formal linear combination \( \sum_{y \in Y} c_y y \in \mathfrak{A}[[Y]] \), where \( c_y = \sum_{x \in X \mid f(x) = y} a_x \) is the sum of a zero-convergent family of elements in \( \mathfrak{A} \), defined as the limit of finite partial sums in the topology of \( \mathfrak{A} \).

For any complete, separated topological vector space \( \mathfrak{B} \) with linear topology and any set \( X \), the topological abelian group \( \mathfrak{B}[[X]] \) is naturally a (complete, separated) topological vector space with linear topology, too. In other words, one has a covariant functor of two arguments
\[
[-[-]]: \text{Top}_{\text{sc}}^\mathbb{K} \times \text{Sets} \longrightarrow \text{Top}_{\text{sc}}^\mathbb{K},
\]
which agrees with the functor (7) and the forgetful functor \( \text{Top}_{\text{sc}}^\mathbb{K} \rightarrow \text{Top}_{\text{sc}}^\mathbb{Z} \).

We will say that a continuous homomorphism of complete, separated topological abelian groups \( p: \mathfrak{A} \rightarrow \mathfrak{C} \) is strongly surjective if the map \( p[[X]]: \mathfrak{A}[[X]] \rightarrow \mathfrak{C}[[X]] \) is surjective for every set \( X \). A continuous homomorphism of complete, separated topological vector spaces is strongly surjective if it is strongly surjective as a map of topological abelian groups.

Furthermore, let \( \mathfrak{A} \) be a complete, separated topological abelian group (or vector space), and let \( i: \mathfrak{K} \rightarrow \mathfrak{A} \) be a closed injective morphism of topological abelian groups/vector spaces. We will say that the map \( i \) is strongly closed if the quotient group/vector space \( \mathfrak{A}/i(\mathfrak{K}) \) is complete in the quotient topology and the quotient map \( \mathfrak{A} \rightarrow \mathfrak{A}/i(\mathfrak{K}) \) is strongly surjective. In this case, the closed subgroup/subspace \( i(\mathfrak{K}) \subseteq \mathfrak{A} \) will be also called strongly closed. The notion of a strongly closed subgroup in a topological group is important for the paper [36], where it is discussed in [36, Sections 2.11–2.12], and even more important in the paper [41].
It is clear from Corollary 8.5 that any strongly closed injective map is stably closed (in the sense of the definition at the end of Section 8). The counterexample following below shows that the inverse implication fails. In other words, a surjective continuous open linear map (of complete, separated topological vector spaces) need not be strongly surjective.

**Remark 11.1.** It may be worthwhile to emphasize what the previous definitions mean. Let \( p: \mathfrak{A} \rightarrow \mathfrak{C} \) be a surjective continuous homomorphism of complete, separated topological abelian groups. Let us even assume that \( p \) is open. What does it mean that \( p \) is strongly surjective?

By the definition, the relevant question is the following one. Let \( (c_x \in \mathfrak{C})_{x \in X} \) be a family of elements in \( \mathfrak{C} \), indexed by a set \( X \) and converging to zero in the topology of \( \mathfrak{C} \). Can one lift the family \( (c_x)_{x \in X} \) to a family of elements \( (a_x \in \mathfrak{A})_{x \in X} \) such that \( p(a_x) = c_x \) for every \( x \in X \) and the family of elements \( (a_x)_{x \in X} \) converges to zero in the topology of \( \mathfrak{A} \)?

One can lift every single element \( c_x \in \mathfrak{C} \) to an element \( a_x \in \mathfrak{A} \), since the map \( p \) is surjective by assumption. But can one lift a whole zero-convergent family of elements in such a way that it remains zero-convergent? Here is an example showing that this cannot be done (generally speaking).

**Example 11.2.** Let \( Q \) be a separated topological abelian group and \( I \) be an infinite set. Consider the complete, separated topological abelian group \( \mathfrak{A}_f(Q) = Q^{(I)} \) in the modified coproduct topology, as in Theorem 2.5. We claim that no infinite family of nonzero elements in \( \mathfrak{A}_f(Q) \) converges to zero in the modified coproduct topology. In other words, this means that the natural embedding of (abstract, nontopological) abelian groups \( \mathfrak{A}[X] \rightarrow \mathfrak{A}[[X]] \), defined in the obvious way for any complete, separated topological abelian group \( \mathfrak{A} \), is a bijection for \( \mathfrak{A} = \mathfrak{A}_f(Q) \). A version of this assertion is included into the formulation of [17, Proposition 11.1].

Indeed, let \( (a_x \in \mathfrak{A})_{x \in X} \) be a family of nonzero elements in \( \mathfrak{A} \). For every \( x \in X \), the element \( a_x \in \mathfrak{A} \) can be viewed as a family of elements \( a_x = (q_{x,i} \in Q)_{i \in I} \) with \( q_{x,i} = 0 \) for all but a finite subset of indices \( i \). Given a subset \( Y \subset X \), denote by \( J_Y \subset I \) the subset of all indices \( j \in I \) for which there exists an index \( y \in Y \) such that \( q_{y,j} \neq 0 \) in \( Q \). Consider two possibilities separately: either there exists a subset \( Y \subset X \) such that the complement \( X \setminus Y \) is finite and the set \( J_Y \) is finite, or for any subset \( Y \subset X \) with a finite complement \( X \setminus Y \) the set \( J_Y \) is infinite.

In the first case, put \( J = J_Y \) and consider the open subgroup \( U = \bigoplus_{J \setminus J_Y} Q \subset \bigoplus_{J \setminus J_Y} Q = \mathfrak{A} \). For every \( y \in Y \), we have \( q_{y,i} = 0 \) for all \( i \in I \setminus J \) (by the definition of \( J = J_Y \)). Since \( a_y \neq 0 \), there should exist an index \( j \in J \) such that \( q_{y,j} \neq 0 \). Hence \( a_y \notin U \). Since the family of elements \( (a_x)_{x \in X} \) converges to zero in \( \mathfrak{A} \) by assumption, it follows that the set \( Y \) must be finite. As the complement \( X \setminus Y \) is finite, too, we can conclude that the set of indices \( X \) is finite, as desired.

In the second case, choose a pair of indices \( j_1 \in I \) and \( x_1 \in X \) such that \( q_{x_1,j_1} \neq 0 \). Since the set \( J_{X \setminus \{x_1\}} \) is infinite, we can choose a pair of indices \( j_2 \in I \) and \( x_2 \in X \) such that \( j_2 \neq j_1 \), \( x_2 \neq x_1 \), and \( q_{x_2,j_2} \neq 0 \). Since the set \( J_{X \setminus \{x_1,x_2\}} \) is infinite, we can choose a pair of indices \( j_3 \in I \) and \( x_3 \in X \) such that \( j_1 \neq j_3 \neq j_2 \), \( x_1 \neq x_3 \neq x_2 \), and
Proceeding in this way, we choose a sequence of pairwise distinct indices \( j_k \in I \) and a sequence of pairwise distinct indices \( x_s \in X, \ s = 1, 2, 3, \ldots \), such that \( q_{x_s, j_k} \neq 0 \) in \( Q \) for all \( s \geq 1 \).

Denote by \( J \) the subset \( \{ j_1, j_2, j_3, \ldots \} \subseteq I \). For every \( j = j_k \in J \), choose an open subgroup \( U_j \subseteq Q \) such that \( q_{x_s, j_k} \notin U_j \). For every \( l \in I \setminus J \), put \( U_l = Q \). Consider the open subgroup \( U = \bigoplus_{i \in I} U_i \subseteq \bigoplus_{i \in I} Q = \mathcal{A} \). For every \( s \geq 1 \), we have \( a_{x_s} \notin U \). As the subset of indices \( \{ x_1, x_2, x_3, \ldots \} \subseteq X \) is infinite, this contradicts the assumption that the family of elements \( (a_x)_{x \in X} \) converges to zero in \( \mathcal{A} \). Thus the second case is impossible, and we have proved the claim.

Now let \( \mathcal{C} \) be a complete, separated topological vector space in which an infinite family of nonzero elements \( (e_x)_{x \in X} \) converging to zero in the topology of \( \mathcal{C} \) does exist; so \( \mathcal{C}[X] \nsubseteq \mathcal{C}[[X]] \). For example, one can take \( \mathcal{C} = k^\omega = \prod_{n=0}^\infty k \) (with the product topology of discrete one-dimensional vector spaces \( k \)), as in Corollary 8.6.

Then the topological basis \( (e_n)_{n=0}^\infty \) of the linearly compact topological vector space \( \mathcal{C} \) is an infinite family of nonzero vectors converging to zero in \( \mathcal{C} \).

Let \( I \) be any infinite set; it suffices to take \( I = \omega \). According to the proof of Theorem 2.5, the summation map \( \Sigma : \mathcal{A} = \mathcal{A}_I(\mathcal{C}) \rightarrow \mathcal{C} \) is open and continuous; it is also obviously a surjective homomorphism of vector spaces. However, the image of the map \( \Sigma[[X]] : \mathcal{B}[[X]] \rightarrow \mathcal{C}[[X]] \) is contained in \( \mathcal{C}[X] \nsubseteq \mathcal{C}[[X]] \), because \( \mathcal{B}[[X]] = \mathcal{B}[X] \) according to the argument above. In other words, the infinite zero-convergent family of nonzero vectors \( (e_x)_{x \in X} \) cannot be lifted to a zero-convergent family of vectors in \( \mathcal{A} \), as there are no infinite zero-convergent families of nonzero vectors in \( \mathcal{B} = \mathcal{A}_I(\mathcal{C}) \).

Thus the map \( \Sigma \) is not strongly surjective. The kernel \( \mathfrak{K} \subset \mathcal{B} \) of the continuous homomorphism \( \Sigma \) is a stably closed, but not strongly closed vector subspace in the complete, separated topological vector space \( \mathcal{B} \).

**Proposition 11.3.** (a) For any set \( X \), the functor \( -[[X]] : \text{Top}_{sc}^\omega \rightarrow \text{Top}_{sc}^\omega \) preserves kernels and cokernels.

(b) For any set \( X \), the functor \( -[[X]] : \text{Top}_{sc}^k \rightarrow \text{Top}_{sc}^k \) preserves kernels and cokernels.

**Proof.** Let us discuss part (a). Recall that the forgetful functor \( \text{Top}_{sc}^\omega \rightarrow \text{Ab} \) preserves kernels, but not cokernels. Accordingly, in the context of the proposition, the claim is that, for any set \( X \), the functor \( -[[X]] : \text{Top}_{sc}^\omega \rightarrow \text{Ab} \) preserves the kernels (as one can easily see); but it does not preserve the cokernels.

The assertion of part (a) for the kernel and cokernel of a morphism \( f : \mathcal{A} \rightarrow \mathcal{B} \) in \( \text{Top}_{sc}^\omega \) reduces to the following properties:

- for any topological group \( \mathcal{A} \in \text{Top}_{sc}^\omega \) and a closed subgroup \( \mathcal{K} \subset \mathcal{A} \) with the induced topology, the topology of \( \mathcal{K}[[X]] \in \text{Top}_{sc}^\omega \) coincides with the induced topology on \( \mathcal{K}[[X]] \) as a subgroup in \( \mathcal{A}[[X]] \);
- for any morphism \( g : \mathcal{A} \rightarrow \mathcal{L} \) in \( \text{Top}_{sc}^\omega \) with \( g(\mathcal{A}) \) dense in \( \mathcal{L} \), the image of \( \mathcal{A}[[X]] \) is dense in \( \mathcal{L}[[X]] \); in fact, the subgroup \( g(\mathcal{A})[X] \subset \mathcal{L}[[X]] \) is already dense;
- for any for any topological group \( \mathcal{B} \in \text{Top}_{sc}^\omega \), a closed subgroup \( \mathcal{L} \subset \mathcal{B} \), and the completion \( \mathcal{C} \) of the quotient group \( \mathcal{B}/\mathcal{L} \) in its quotient topology, one has a
natural isomorphism between the topological group $\mathcal{C}[[X]]$ and the completion of the quotient group $\mathcal{B}[[X]]/\mathcal{L}[[X]]$ in its quotient topology.

The latter property holds because both the topological groups in question can be identified with the projective limit of discrete groups $\mathcal{B}[[X]]/((\mathcal{U}[X] + \mathcal{L}[X])$, where $\mathcal{U}$ ranges over the open subgroups of $\mathcal{B}$. We leave further details to the reader.

Corollary 11.4. (a) For any infinite set $X$, the functor $-[[X]]: \text{Top}^\text{sc}_{\mathbb{Z}} \longrightarrow \text{Top}^\text{sc}_{\mathbb{Z}}$ is not exact with respect to the maximal exact structure on $\text{Top}^\text{sc}_{\mathbb{Z}}$.

(b) For any infinite set $X$, the functor $-[[X]]: \text{Top}^\text{sc}_{k} \longrightarrow \text{Top}^\text{sc}_{k}$ is not exact with respect to the maximal exact structure on $\text{Top}^\text{sc}_{k}$.

Proof. This is a direct corollary of Example 11.2.

Theorem 11.5. (a) There is an exact category structure on the additive category $\text{Top}^\text{sc}_{\mathbb{Z}}$, called the strong exact structure, in which a short sequence $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \longrightarrow \mathcal{C} \longrightarrow 0$ is exact if and only if it satisfies Ex1 and the induced map $\mathcal{A}[[X]] \longrightarrow \mathcal{C}[[X]]$ is surjective for every set $X$. The admissible monomorphisms in this exact structure are the strongly closed injective maps, and the admissible epimorphisms are the open strongly surjective maps. The strong exact structure on $\text{Top}^\text{sc}_{\mathbb{Z}}$ is different from (i.e., has fewer short exact sequences than) the maximal exact structure.

(b) There is an exact category structure on the additive category $\text{Top}^\text{sc}_{k}$, called the strong exact structure, in which a short sequence $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \longrightarrow \mathcal{C} \longrightarrow 0$ is exact if and only if it satisfies Ex1 and the induced map $\mathcal{A}[[X]] \longrightarrow \mathcal{C}[[X]]$ is surjective for every set $X$. The admissible monomorphisms in this exact structure are the strongly closed injective maps, and the admissible epimorphisms are the open strongly surjective maps. The strong exact structure on $\text{Top}^\text{sc}_{k}$ is different from (i.e., has fewer short exact sequences than) the maximal exact structure.

Proof. Let us explain part (a); part (b) is similar. The argument is based on Example 11.2. Consider the exact category $\text{Top}^\text{sc}_{\mathbb{Z}}$ with its maximal exact category structure. Let us view the category of abelian groups $\text{Ab}$ as an exact category with the abelian exact structure. Fix a set $X$, and consider the functor $\Psi_X: \text{Top}^\text{sc}_{\mathbb{Z}} \longrightarrow \text{Ab}$, $\Psi_X(\mathcal{A}) = \mathcal{A}[[X]]$. Following Proposition 11.3, the functor $\Psi_X$ preserves all kernels; in particular, it preserves the kernels of admissible epimorphisms.

Alternatively, consider the functor $\Psi'_X: \text{Top}^\text{sc}_{\mathbb{Z}} \longrightarrow \text{Top}^\text{sc}_{\mathbb{Z}}$, $\Psi'_X(\mathcal{A}) = \mathcal{A}[[X]]$, viewing both the source and the target category $\text{Top}^\text{sc}_{\mathbb{Z}}$ as an exact category with the maximal exact structure. Following Proposition 11.3, the functor $\Psi'_X$ preserves all kernels and cokernels; in particular, it preserves the kernels of admissible epimorphisms and the cokernels of admissible monomorphisms.

Thus, for any one of the functors $\Psi_X$ or $\Psi'_X$, the construction of Example 11.2 is applicable, and it produces a new exact category structure on $\text{Top}^\text{sc}_{\mathbb{Z}}$. It is clear from the discussion of the maximal exact structure on $\text{Top}^\text{sc}_{\mathbb{Z}}$ in Section 8 (see Proposition 8.4) that with both approaches the same new exact category structure is produced. It can be called the $\Psi_X$-exact structure or the $\Psi'_X$-exact structure (which is the same) on $\text{Top}^\text{sc}_{\mathbb{Z}}$. In the $\Psi_X$-exact structure with a nonempty set $X$, a short
sequence $0 \rightarrow \mathcal{K} \rightarrow \mathcal{V} \rightarrow \mathcal{C} \rightarrow 0$ is exact if and only if it satisfies Ex1 and the induced map $\mathcal{V}[[X]] \rightarrow \mathcal{C}[[X]]$ is surjective.

The strong exact structure on $\text{Top}_{Z}^{sc}$ is the intersection of the $\Psi_X$-exact structures taken over all sets $X$. In other words, a short sequence is exact in the strong exact structure if and only if it exact in the $\Psi_X$-exact structure for every set $X$. Clearly, the intersection of any nonempty collection of exact structures on an additive category is an exact structure. It is obvious from the discussion of strongly surjective maps and strongly closed injective maps above in this section that the strong exact category structure on $\text{Top}_{Z}^{sc}$, defined in this way, has the properties listed in the proposition.

Finally, the strong exact structure on $\text{Top}_{Z}^{sc}$ differs from the maximal exact structure according to Corollary 11.4. In fact, following the arguments in Example 11.2, for a countable set $X$ already the $\Psi_X$-exact structure has fewer short exact sequences than the maximal exact structure.

See [36, Lemma 2.4] for a formulation and direct proof of some specific properties related to the existence of the strong exact structure.

The short sequences that are exact in the strong exact category structure on $\text{Top}_{Z}^{sc}$ or $\text{Top}_{k}^{sc}$ are called the strong short exact sequences.

The full subcategories of (complete, separated) topological abelian groups/vector spaces with a countable base of neighborhoods of zero $\text{Top}_{Z}^{w,sc}$ and $\text{Top}_{k}^{w,sc}$ inherit the strong exact category structures of the ambient additive categories $\text{Top}_{Z}^{sc}$ and $\text{Top}_{k}^{sc}$. Moreover, the inherited exact structures on the quasi-abelian categories $\text{Top}_{Z}^{w,sc}$ and $\text{Top}_{k}^{w,sc}$ coincide with their quasi-abelian exact structures (so for topological abelian groups with a countable base of neighborhoods of zero there is no difference between the maximal and strong exact structures). In fact, the following stronger version of Proposition 1.4 holds.

**Proposition 11.6.** Let $\mathfrak{A}$ be a complete, separated topological abelian group, and let $\mathfrak{K} \subset \mathfrak{A}$ be a closed subgroup. Assume that the topological abelian group $\mathfrak{K}$ has a countable base of neighborhoods of zero. Then, for any set $X$, the map $\mathfrak{A}[[X]] \rightarrow (\mathfrak{A}/\mathfrak{K})[[X]]$ induced by the open continuous homomorphism $\mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{K}$ is surjective. In other words, $\mathfrak{K}$ is a strongly closed subgroup in $\mathfrak{A}$.

**Proof.** Denote by $\mathfrak{C}$ the quotient group $\mathfrak{C} = \mathfrak{A}/\mathfrak{K}$ with the quotient topology; by Proposition 1.4, the topological abelian group $\mathfrak{C}$ is complete. Consider the projective system of short exact sequences of abelian groups

$$0 \rightarrow \mathfrak{K}/(\mathfrak{U} \cap \mathfrak{K}) \rightarrow \mathfrak{A}/\mathfrak{U} \rightarrow \mathfrak{A}/(\mathfrak{U} + \mathfrak{K}) \rightarrow 0$$

indexed by the directed poset of open subgroups $\mathfrak{U} \subset \mathfrak{A}$. Passing to the direct sum over the set $X$, we obtain a projective system of short exact sequences of abelian groups

$$0 \rightarrow (\mathfrak{K}/(\mathfrak{U} \cap \mathfrak{K}))[X] \rightarrow (\mathfrak{A}/\mathfrak{U})[X] \rightarrow (\mathfrak{A}/(\mathfrak{U} + \mathfrak{K}))[X] \rightarrow 0.$$ 

The map $\mathfrak{A}[[X]] \rightarrow \mathfrak{C}[[X]]$ which we are interested in is obtained by taking the projective limit of the projective system of surjective homomorphisms $(\mathfrak{A}/\mathfrak{U})[X] \rightarrow (\mathfrak{A}/(\mathfrak{U} + \mathfrak{K}))[X]$. Since the topological abelian group $\mathfrak{K}$ has a countable base of
neighborhoods of zero and the transition maps in the projective system of abelian
groups \((\mathcal{R}/(\mathfrak{U} \cap \mathcal{R}))[X]\) are surjective, the argument from the proof of Proposition 1.4 shows that \(\lim_{U \in \mathcal{A}}((\mathcal{R}/(\mathfrak{U} \cap \mathcal{R}))[X]) = 0\). Hence the desired surjectivity of the map of projective limits. \(\square\)

12. BEILINSON’S YOGA OF TENSOR PRODUCT OPERATIONS

In this section we discuss uncompleted versions of the topological tensor product operations defined in [5, Section 1.1]. The discussion of completed tensor products is postponed until the next Section 13.

Let \(U\) and \(V\) be topological vector spaces (with linear topology). We will define three linear topologies on the tensor product space \(U \otimes_k V\). The resulting topological vector spaces will be denoted by \(U \otimes^* V\), \(U \otimes^\leftarrow V\), and \(U \otimes^! V\).

A vector subspace \(E \subset U \otimes_k V\) is said to be open in \(U \otimes^* V\) if the following conditions hold:

1. there exist open subspaces \(P \subset U\) and \(Q \subset V\) such that \(P \otimes_k Q \subset E\);
2. for every vector \(u \in U\), there exists an open subspace \(Q_u \subset V\) such that \(u \otimes Q_u \subset E\); and
3. for every vector \(v \in V\), there exists an open subspace \(P_v \subset U\) such that \(P_v \otimes v \subset E\).

A vector subspace \(E \subset U \otimes_k V\) is said to be open in \(U \otimes^\leftarrow V\) if the following conditions hold:

1. there exists an open subspace \(P \subset U\) such that \(P \otimes_k V \subset E\); and
2. for every vector \(u \in U\), there exists an open subspace \(Q_u \subset V\) such that \(u \otimes Q_u \subset E\).

A vector subspace \(E \subset U \otimes_k V\) is said to be open in \(U \otimes^! V\) if there exist open subspaces \(P \subset U\) and \(Q \subset V\) such that \(P \otimes_k V + U \otimes_k Q \subset E\). In other words, the vector subspaces \(P \otimes_k V + U \otimes_k Q\), where \(P \subset U\) and \(Q \subset V\) are open subspaces, form a base of neighborhoods of zero in \(U \otimes^! V\).

Clearly, the identity maps \(\text{id}_{U \otimes_k V}\) are continuous homomorphisms of topological vector spaces \(U \otimes^* V \rightarrow U \otimes^\leftarrow V \rightarrow U \otimes^! V\). So one can say that the \(*\)-topology is the finest one of the three topologies, while the \(!\)-topology is the coarsest one of the three. The \(*\)-tensor product and the \(!\)-tensor product are commutative (symmetric) operations, while the \(-\)-tensor product is not. The opposite operation to the \(-\)-tensor product can be denoted by \(V \otimes^\leftarrow U = U \otimes^\leftarrow V\).

**Remark 12.1.** The reader should be warned that our notation for the three tensor product operations is actually different from the notation in [5], in that the symbols \(\otimes^*\) and \(\otimes^!\) are used to denote the **completed** tensor products in [5]. We use them to denote the uncompleted tensor products.

Thus, in our notation, the topological vector spaces \(\mathfrak{U} \otimes^* \mathfrak{V}\), \(\mathfrak{U} \otimes^\leftarrow \mathfrak{V}\), and \(\mathfrak{U} \otimes^! \mathfrak{V}\) need **not** be complete even for complete topological vector spaces \(\mathfrak{U}\) and \(\mathfrak{V}\). For example,
when \( U \) and \( V \) are linearly compact (profinite-dimensional) topological vector spaces, all the three topologies on the tensor product \( U \otimes_k V \) coincide with each other, and the vector space \( U \otimes_k V \) is incomplete in this topology (whenever both \( U \) and \( V \) are infinite-dimensional). The related completion is a linearly compact topological vector space \( \hat{U} \hat{\otimes} \hat{V} \) (the usual tensor product in the category of linearly compact topological vector spaces).

**Remark 12.2.** The following observations (essentially taken from [5, Remark (ii) in Section 1.1]) motivate the definitions of the three tensor product topologies. Let \( U, \) \( V, \) and \( W \) be three topological vector spaces, and let \( \phi: U \times V \rightarrow W \) be a bilinear map. Then the map \( \phi \) is continuous (as a function of two variables) if and only if the related linear map \( \phi^\otimes: U \otimes_k V \rightarrow W \) is continuous in the \( \ast \)-topology on \( U \otimes_k V, \) i.e., the linear map \( \phi^\otimes: U \otimes^* V \rightarrow W \) is continuous.

A **topological algebra** \( R \) (over \( k \)) is a topological vector space (with linear topology) endowed with an associative \( k \)-algebra structure such that the multiplication map \( \cdot : R \times R \rightarrow R \) is continuous. According to the previous paragraph, the multiplication in a topological algebra \( R \) can be described as a continuous linear map \( \mu: R \otimes^* R \rightarrow R. \) Now a topological algebra \( R \) has a base of neighborhoods of zero consisting of open right ideals if and only if its multiplication map is continuous in the \( \leftarrow \)-topology, i.e., the linear map \( \mu: R \otimes^\ast R \rightarrow R \) is continuous. A topological algebra \( R \) has a base of neighborhoods of zero consisting of open two-sided ideals if and only if its multiplication map is continuous in the \( ! \)-topology, i.e., the linear map \( \mu: R \otimes^! R \rightarrow R \) is continuous.

**Lemma 12.3.** All the three tensor product operations \( \otimes^*, \otimes^\ast, \) and \( \otimes^! \) are functors of two arguments \( \mathbf{Top}_k \times \mathbf{Top}_k \rightarrow \mathbf{Top}_k. \) In other words, for any continuous linear maps of topological vector spaces \( f: U' \rightarrow U'' \) and \( g: V' \rightarrow V'' \), the linear map \( f \otimes g: U' \otimes^* V' \rightarrow U'' \otimes^* V'' \) is continuous; and similarly for the \( \leftarrow \)-topology and the \( ! \)-topology on the tensor products.

**Proof.** Let us sketch a proof for the \( \ast \)-topology; the arguments for the other two topologies are similar. Let \( E'' \subset U'' \otimes^* V'' \) be an open subspace and \( E' = (f \otimes g)^{-1}(E'') \) be its preimage. We have to show that \( E' \subset U' \otimes^* V' \) is an open subspace. Let \( P'' \subset U'' \) and \( Q'' \subset V'' \) be open subspaces such that \( P'' \otimes_k Q'' \subset E'' \), and let \( P' = f^{-1}(P'') \) and \( Q' = g^{-1}(Q'') \) be their preimages. Then \( P' \subset U' \) and \( P'' \subset U'' \) are open subspaces, and \( P' \otimes_k Q' \subset E' \). Let \( u' \in U' \) be a vector; put \( u'' = f(u') \). Let \( Q_{u''} \subset V'' \) be an open subspace such that \( u'' \otimes Q_{u''} \subset E''. \) Put \( Q_{u'} = g^{-1}(Q_{u''}) \). Then \( Q_{u'} \subset V' \) is an open subspace, and \( u' \otimes Q_{u'} \subset E' \).

**Lemma 12.4.** Let \( p: U \rightarrow C \) and \( q: V \rightarrow D \) be open surjective linear maps of topological vector spaces. Then

(a) \( f \otimes g: U \otimes^* V \rightarrow C \otimes^* D \) is an open surjective linear map;
(b) \( f \otimes g: U \otimes^\ast V \rightarrow C \otimes^\ast D \) is an open surjective linear map;
(c) \( f \otimes g: U \otimes V \rightarrow C \otimes^! D \) is an open surjective linear map.

**Proof.** Let us explain part (a); parts (b) and (c) are similar. Let \( E \subset U \otimes^* V \) be an open subspace; we have to show that \( f \otimes g)(E) \subset C \otimes^* D \) is an open subspace. Let
$P \subset U$ and $Q \subset V$ be open subspaces such that $P \otimes_k Q \subset E$. Then $f(P) \subset C$ and $g(Q) \subset D$ are open subspaces, and $f(P) \otimes_k g(Q) \subset (f \otimes g)(E)$. Let $c \in C$ be a vector; choose a vector $u \in U$ such that $f(u) = c$. Let $Q_u \subset V$ be an open subspace such that $u \otimes Q_u \subset E$. Then $g(Q_u) \subset D$ is an open subspace, and $c \otimes g(Q_u) \subset (f \otimes g)(E)$. □

**Lemma 12.5.** Let $U$ and $V$ be topological vector spaces, and let $K \subset U$ and $L \subset V$ be vector subspaces endowed with the induced topologies. Then the topology of $K \otimes L$ coincides with the induced topology on $K \otimes_k L$ as a vector subspace in $U \otimes^* V$.

**Proof.** We have $K \otimes_k L \subset U \otimes_k L \subset U \otimes_k V$ and the functor $\otimes^*$ is commutative; so it suffices to show that the topology of $U \otimes^* L$ coincides with the induced topology on $U \otimes_k L$ as a vector subspace in $U \otimes^* V$. By Lemma 12.3, the inclusion $U \otimes^* L \rightarrow U \otimes^* V$ is a continuous map.

Let $F \subset U \otimes^* L$ be an open subspace. Then there exist open subspaces $P \subset U$ and $S \subset L$ such that $P \otimes_k S \subset F$. Choose an open subspace $Q \subset V$ such that $Q \cap L = S$ and $Q + L = V$. Furthermore, choose a basis $\{u_i\}_{i \in I}$ in $U$ such that a subset of $\{u_i\}$ is a basis in $P \subset U$. For every $i \in I$ such that $u_i \notin P$, there exists an open subspace $S_i \subset L$ such that $u_i \otimes S_i \subset F$. Choose an open subspace $Q_i \subset V$ such that $Q_i \cap L = S_i$. Then

$$E = F + P \otimes_k Q + \sum_{i \in I; u_i \notin P} u_i \otimes Q_i \subset U \otimes_k V$$

is an open subspace in $U \otimes^* V$ such that $E \cap (U \otimes_k L) = F$.

Indeed, let $v \in V$ be a vector. Choose vectors $q \in Q$ and $l \in L$ such that $q + l = v$. Then $P \otimes q \subset E$, and there exists an open subspace $P_l \subset U$ such that $P_l \otimes l \subset F$. Hence $P \cap P_l$ is an open subspace in $U$ for which $(P \cap P_l) \otimes v \subset E$. □

**Lemma 12.6.** Let $U$ and $V$ be topological vector spaces, and let $K \subset U$ and $L \subset V$ be vector subspaces endowed with the induced topologies. Then the topology of $K \otimes^* L$ coincides with the induced topology on $K \otimes_k L$ as a vector subspace in $U \otimes^* V$.

**Proof.** By Lemma 12.3, the inclusion $K \otimes^* L \rightarrow U \otimes^* V$ is a continuous map. Using the inclusions $K \otimes_k L \subset U \otimes_k L \subset U \otimes_k V$ (or $K \otimes_k L \subset K \otimes_k V \subset U \otimes_k V$), it suffices to consider two cases separately.

Let us show that the topology of $U \otimes^* L$ coincides with the induced topology on $U \otimes_k L$ as a vector subspace in $U \otimes^* V$. Let $F \subset U \otimes^* L$ be an open subspace. Then there exists an open subspace $P \subset U$ such that $P \otimes_k L \subset F$. Choose a basis $\{u_i\}_{i \in I}$ in $U$ such that a subset of $\{u_i\}$ is a basis in $P \subset U$. For every $i \in I$ such that $u_i \notin P$, there exists an open subspace $S_i \subset L$ such that $u_i \otimes S_i \subset F$. Choose an open subspace $Q_i \subset V$ such that $Q_i \cap L = S_i$. Then

$$E = F + P \otimes_k V + \sum_{i \in I; u_i \notin P} u_i \otimes Q_i \subset U \otimes_k V$$

is an open subspace in $U \otimes^* V$ such that $E \cap (U \otimes_k L) = F$.

Let us show that the topology of $K \otimes^* V$ coincides with the induced topology on $K \otimes_k V$ as a vector subspace in $U \otimes^* V$. Let $F \subset K \otimes^* V$ be an open subspace. Then there exists an open subspace $R \subset K$ such that $R \otimes_k V \subset F$. Choose an open
Lemma 12.7. Let $U$ and $V$ be topological vector spaces, and let $K \subset U$ and $L \subset V$ be vector subspaces endowed with the induced topologies. Then the topology of $K \otimes L$ coincides with the induced topology on $K \otimes_k L$ as a vector subspace in $U \otimes^\ast V$.

Proof. By Lemma 12.3, the inclusion $K \otimes^! L \rightarrow U \otimes^! V$ is a continuous map. Let $F \subset K \otimes^! L$ be an open subspace. Then there exist open subspaces $R \subset K$ and $S \subset L$ such that $R \otimes_k L + K \otimes_k S \subset F$. Choose open subspaces $P \subset U$ and $Q \subset V$ such that $P \cap K = R$ and $Q \cap L = S$. Put $E = F + P \otimes_k V + U \otimes_k Q \subset U \otimes_k V$. Then $E$ is an open subspace in $U \otimes^! V$ such that $E \cap (K \otimes_k L) = F$. □

Lemma 12.8. Let $U$ and $V$ be topological vector spaces, and let $K \subset U$ and $L \subset V$ be closed vector subspaces. Then the vector subspace $K \otimes_k L \subset U \otimes_k V$ is closed in the topological vector spaces $U \otimes^\ast V$, $U \otimes^\ast V$, and $U \otimes^! V$.

Proof. It suffices to consider the case of the $!$-topology, as it is the coarsest one of the three. Let $w \in U \otimes_k V$ be a vector not belonging to $K \otimes_k L$. Let $U_w \subset U$ and $V_w \subset V$ be the minimal (finite-dimensional) vector subspaces such that $w \in U_w \otimes_k V_w$. Then either $U_w \not\subset K$, or $V_w \not\subset L$ (or both). Suppose that $U_w \not\subset K$, and choose a vector $u \in U_w$ such that $u \not\in K$. Since $K$ is a closed vector subspace in $U$, there exists an open vector subspace $P \subset U$ such that the coset $u + P \subset U$ does not intersect $K$ (or equivalently, $u \not\in P + K$). Consider the open subspace $P \otimes_k V \subset U \otimes^! V$. Then we have $P \otimes_k V + K \otimes_k L \subset (P + K) \otimes_k V$ and $w \not\in (P + K) \otimes_k V$ (since $U_w \not\subset P + K$). Hence $w$ does not belong to the closure of $K \otimes_k L$ in $U \otimes_k V$. □

Lemma 12.9. Let $U$ and $V$ be topological vector spaces, and let $K \subset U$ and $L \subset V$ be dense vector subspaces. Then the vector subspace $K \otimes_k L \subset U \otimes_k V$ is dense in the topological vector spaces $U \otimes^\ast V$, $U \otimes^\ast V$, and $U \otimes^! V$.

Proof. It suffices to consider the case of the $*$-topology, as it is the finest one of the three. Let $u \in U$ and $v \in V$ be arbitrary vectors; let us show that the element $u \otimes v \in U \otimes^* V$ belongs to the closure of the subspace $K \otimes_k L \subset U \otimes^* V$. Let $E \subset U \otimes^* V$ be an open subspace. Then there exists an open subspace $Q_u \subset V$ such that $u \otimes Q_u \subset E$. Since the subspace $L$ is dense in $V$, we have $Q_u + L = V$. Let $q \in Q_u$ and $l \in L$ be vectors such that $v = q + l$. There exists an open subspace $P_l \subset U$ such that $P_l \otimes l \subset E$. Since the subspace $K$ is dense in $U$, we have $P_l + K = U$. Let $p \in P_l$ and $k' \in K$ be vectors such that $u = p + k'$. Then $u \otimes v = u \otimes q + p \otimes l + k' \otimes l \in E + k' \otimes l \subset E + K \otimes_k L$. Thus $E + K \otimes_k L = U \otimes_k V$. □

Corollary 12.10. Let $U$ and $V$ be separated topological vector spaces. Then the topological vector spaces $U \otimes^* V$, $U \otimes^! V$, and $U \otimes^! V$ are separated, too.

Proof. Follows from Lemma 12.8 (take $K = 0 = L$). □

Theorem 12.11. (a) The functor $\otimes^* : \text{Top}_k \times \text{Top}_k \rightarrow \text{Top}_k$ preserves the kernels and cokernels of morphisms (in each of its arguments). Fixing a topological vector...
space in one of the arguments, it becomes an exact endofunctor $\mathbf{Top}_k \longrightarrow \mathbf{Top}_k$ in the other argument (with respect to the quasi-abelian exact structure on $\mathbf{Top}_k$).

(b) The functor $\otimes^\ast : \mathbf{Top}_k \times \mathbf{Top}_k \longrightarrow \mathbf{Top}_k$ preserves the kernels and cokernels of morphisms (in each of its arguments). Fixing a topological vector space in any one of the arguments, it becomes an exact endofunctor $\mathbf{Top}_k \longrightarrow \mathbf{Top}_k$ in the other argument (with respect to the quasi-abelian exact structure on $\mathbf{Top}_k$).

(c) The functor $\otimes^! : \mathbf{Top}_k \times \mathbf{Top}_k \longrightarrow \mathbf{Top}_k$ preserves the kernels and cokernels of morphisms (in each of its arguments). Fixing a topological vector space in one of the arguments, it becomes an exact endofunctor $\mathbf{Top}_k \longrightarrow \mathbf{Top}_k$ in the other argument (with respect to the quasi-abelian exact structure on $\mathbf{Top}_k$).

Proof. Let us explain part (a); parts (b) and (c) are similar. The functor $\otimes^\ast$ is well-defined by Lemma 12.3. For any fixed topological vector space $U$, the functor $U \otimes^\ast - : \mathbf{Top}_k \longrightarrow \mathbf{Top}_k$ preserves the kernels of morphisms by Lemma 12.5, and it preserves the cokernels of morphisms by Lemma 12.4(a). Any additive functor between quasi-abelian categories which preserves the kernels and cokernels is exact with respect to the quasi-abelian exact structures.

Theorem 12.12. (a) The functor $\otimes^\ast : \mathbf{Top}_k^s \times \mathbf{Top}_k^s \longrightarrow \mathbf{Top}_k^s$ preserves the kernels and cokernels of morphisms (in each of its arguments). Fixing a topological vector space in one of the arguments, it becomes an exact endofunctor $\mathbf{Top}_k^s \longrightarrow \mathbf{Top}_k^s$ in the other argument (with respect to the quasi-abelian exact structure on $\mathbf{Top}_k^s$).

(b) The functor $\otimes^\ast : \mathbf{Top}_k^s \times \mathbf{Top}_k^s \longrightarrow \mathbf{Top}_k^s$ preserves the kernels and cokernels of morphisms (in each of its arguments). Fixing a topological vector space in any one of the arguments, it becomes an exact endofunctor $\mathbf{Top}_k^s \longrightarrow \mathbf{Top}_k^s$ in the other argument (with respect to the quasi-abelian exact structure on $\mathbf{Top}_k^s$).

(c) The functor $\otimes^! : \mathbf{Top}_k^s \times \mathbf{Top}_k^s \longrightarrow \mathbf{Top}_k^s$ preserves the kernels and cokernels of morphisms (in each of its arguments). Fixing a topological vector space in one of the arguments, it becomes an exact endofunctor $\mathbf{Top}_k^s \longrightarrow \mathbf{Top}_k^s$ in the other argument (with respect to the quasi-abelian exact structure on $\mathbf{Top}_k^s$).

Proof. Let us explain part (a); parts (b) and (c) are similar. The functor $\otimes^\ast$ is well-defined by Lemma 12.3 and Corollary 12.10. For any fixed separated topological vector space $V$, the functor $U \otimes^\ast - : \mathbf{Top}_k^s \longrightarrow \mathbf{Top}_k^s$ preserves the kernels by Lemma 12.5. Furthermore, let $f : V' \longrightarrow V''$ be a continuous linear map of separated topological vector spaces. Then the cokernel of the map $f$ in the category $\mathbf{Top}_k^s$ is the quotient space $V''/f(V')_{V''}$ endowed with the quotient topology. By Lemmas 12.5, 12.8, and 12.9, the closure of the vector subspace $U \otimes_k f(V')$ in the topological vector space $U \otimes V''$ is the subspace $U \otimes_k f(V')_{U \otimes V''} = V \otimes f(V')_{V''} \subset U \otimes V''$. It remains to use Lemma 12.4(a) in order to conclude that the functor $U \otimes^\ast -$ preserves the cokernels in $\mathbf{Top}_k^s$. The second assertion in part (a) follows from the first one.

The following result is an uncompleted version of [5, Corollary in Section 1.1].
Proposition 12.13. Let $U$ and $V$ be topological vector spaces. Then the short exact sequence of (abstract, nontopological) vector spaces

$$0 \longrightarrow U \otimes_k V \longrightarrow U \otimes_k V \oplus U \otimes_k V \longrightarrow U \otimes_k V \longrightarrow 0,$$

where the left arrow is the diagonal map and the right one is the difference of two projections, is exact as a short sequence

$$(9) \quad 0 \longrightarrow U^* V \longrightarrow U^* V \oplus U \rightarrow V \longrightarrow U^1 V \longrightarrow 0$$
in the quasi-abelian exact structure on the category $\text{Top}_k$.

Proof. We recall the notation $U \otimes^* V = V \otimes^* U$. The maps involved in the desired short exact sequence $(9)$ are continuous, since the $*$-topology is the finest of the three topologies on $U \otimes_k V$, while the $!$-topology is the coarsest one. On top of this observation, the proposition essentially claims two assertions:

(a) the surjective linear map $U \otimes^* V \oplus U \otimes^* V \overset{p}{\longrightarrow} U \otimes^1 V$ is open;

(b) the topology of $U \otimes^* V$ coincides with the induced topology on $U \otimes_k V$ as the diagonal subspace in $U \otimes^* V \oplus U \otimes^* V$.

Proof of (a): let $F \subset U \otimes^* V$ and $G \subset U \otimes^* V$ be open subspaces. Then there exists an open subspace $P \subset U$ such that $P \otimes_k V \subset F$. Similarly, there exists an open subspace $Q \subset V$ such that $U \otimes_k Q \subset G$. Now $P \otimes_k V + U \otimes_k Q \subset F + G = p(F \oplus G)$, hence $p(F \oplus G) \subset U \otimes_k V$ is an open subspace in $U \otimes^1 V$.

Proof of (b): let $E \subset U \otimes^* V$ be an open subspace. It suffices to find two open subspaces $F \subset U \otimes^* V$ and $G \subset U \otimes^* V$ such that $F \cap G \subset E$ in $U \otimes_k V$.

By the definition of the $*$-topology, there exist two open subspaces $P \subset U$ and $Q \subset V$ such that $P \otimes_k Q \subset E$. Choose a basis $\{u_i\}_{i \in I}$ in $U$ such that a subset of $\{u_i\}$ is a basis in $P \subset U$. Similarly, choose a basis $\{v_j\}_{j \in J}$ in $V$ such that a subset of $\{v_j\}$ is a basis in $Q \subset V$.

For every $i \in I$ such that $u_i \notin P$, there exists an open subspace $Q_i \subset V$ such that $u_i \otimes Q_i \subset E$. Similarly, for every $j \in J$ such that $v_j \notin Q$, there exists an open subspace $P_j \subset U$ such that $P_j \otimes v_j \subset E$. Put

$$F = P \otimes_k V + \sum_{i \in I; u_i \notin P} u_i \otimes (Q_i \cap Q)$$

and

$$G = U \otimes_k Q + \sum_{j \in J; v_j \notin Q} (P_j \cap P) \otimes v_j.$$

Then $F \subset U \otimes^* V$ and $G \subset U \otimes^* V$ are open subspaces, and

$$F \cap G = P \otimes_k Q + \sum_{i \in I; u_i \notin P} u_i \otimes (Q_i \cap Q) + \sum_{j \in J; v_j \notin Q} (P_j \cap P) \otimes v_j \subset E,$$
as desired. \qed
13. Refined Exact Category Structures on VSLTs

It is claimed in [5, Section 1.1, Exercise on page 2] that the completed tensor product operations are exact in the category of complete, separated topological vector spaces. As we will see, the validity of these assertions depends on what is understood by exactness; but they do not hold if one presumes exactness with respect to the maximal exact category structure on \( \text{Top}^c \).

Let \( U \) and \( V \) be two complete, separated topological vector spaces (with linear topology). Let us introduce notation for three completed topological tensor products of \( U \) and \( V \):

- \( U \hat{\otimes}^* V = (U \otimes^* V)^{\hat{\cdot}} \);
- \( U \hat{\otimes}^- V = (U \otimes^- V)^{\hat{\cdot}} \);
- \( U \hat{\otimes}^! V = (U \otimes^! V)^{\hat{\cdot}} \).

The simplest examples of complete, separated topological vector spaces for which the completed tensor products differ from the uncompleted ones were mentioned in Remark 12.1. Here are some slightly more sophisticated examples.

Examples 13.1. (1) Let \( U \) be a discrete vector space and \( V \) be a topological vector space. Then, by the definition, a vector subspace \( E \subset U \otimes_k V \) is open in \( U \hat{\otimes}^* V \) if and only if it is open in \( U \otimes^* V \), and if and only if for every vector \( u \in U \) there exists an open subspace \( Q_u \subset V \) such that \( u \otimes Q_u \subset E \).

Let \( \{u_i\}_{i \in I} \) be a basis in \( U \). Identifying the tensor product \( U \otimes_k V \) with the direct sum \( \bigoplus_{i \in I} V \), one observes that the topology of \( U \otimes^* V = U \otimes^- V \) coincides with the coproduct topology of \( \bigoplus_{i \in I} V \).

By Lemma 1.2, the direct sum of a family of complete, separated topological vector spaces is separated and complete in the coproduct topology. Thus, for any discrete vector space \( U \) and any complete, separated topological vector space \( \mathfrak{V} \), one has \( U \hat{\otimes}^* \mathfrak{V} = U \otimes^* \mathfrak{V} = \bigoplus_{i \in I} \mathfrak{V} = U \otimes^- \mathfrak{V} = U \hat{\otimes}^- \mathfrak{V} \), where \( I \) is a set indexing a basis in \( U \).

(2) Let \( U \) be a topological vector space and \( V \) be a discrete vector space. Then, by the definition, a vector subspace \( E \subset U \otimes_k V \) is open in \( U \otimes^+ V \) if and only if it is open in \( U \otimes^! V \), and if and only if there exists an open subspace \( P \subset U \) such that \( P \otimes_k V \subset E \).

Let \( \{v_x\}_{x \in X} \) be a basis in \( V \). Identifying the tensor product \( U \otimes_k V \) with the direct sum \( \bigoplus_{x \in X} U = U^{(X)} = U[X] \), one observes that a base of open subspaces in \( U \otimes^+ V = U \otimes^! V \) is formed by the subspaces \( P[X] \), where \( P \) ranges over the open vector subspaces in \( U \). This is the topology relevant in the context of the construction in the beginning of Section 11.

For a complete, separated topological vector space \( U \) and a set \( X \), the completion of the vector space \( U^{(X)} = U[X] \) in the above topology is the topological vector space denoted by \( \hat{U}[X] \) in Section 11. Thus, for any complete, separated topological vector space \( U \) and a discrete vector space \( \mathfrak{V} \), one has \( U \hat{\otimes}^+ \mathfrak{V} = U[[X]] = U \hat{\otimes}^1 \mathfrak{V} \), where \( X \) is a set indexing a basis in \( \mathfrak{V} \).
(3) Let \( \mathfrak{U} \) and \( \mathfrak{V} \) be complete, separated topological vector spaces. For any open subspace \( \mathfrak{P} \subset \mathfrak{U} \), let us view the quotient space \( \mathfrak{U}/\mathfrak{P} \) as a discrete vector space and endow the tensor product \((\mathfrak{U}/\mathfrak{P}) \otimes_k \mathfrak{V}\) with the (complete, separated) topology described in (1). Then the preimages in \( \mathfrak{U}\otimes_k \mathfrak{V} \) of open subspaces in \((\mathfrak{U}/\mathfrak{P}) \otimes_k \mathfrak{V}\) form a base of neighborhoods of zero in \( \mathfrak{U}\otimes^c \mathfrak{V} \). Hence one has \( \mathfrak{U}\otimes^c \mathfrak{V} = \lim_{\mathfrak{P} \in \mathfrak{P}_{cl}} (\mathfrak{U}/\mathfrak{P} \otimes_k \mathfrak{V}) \), where the projective limit over the poset of open subspaces \( \mathfrak{P} \subset \mathfrak{U} \) is taken in the category \( \mathrm{Top}^\mathfrak{sc}_k \), or equivalently, in any one of the categories \( \mathrm{Top}^\mathfrak{sc}_k \) or \( \mathrm{Top}_k^\mathfrak{sc} \).

As the projective limit in the categories of topological vector spaces agrees with the one in \( \mathbf{Vect}_k \), it follows that the underlying vector space of the topological vector space \( \mathfrak{U}\otimes^c \mathfrak{V} \) only depends on the underlying vector space of the topological vector space \( \mathfrak{V} \), and does not depend on a (complete, separated) topology on \( \mathfrak{V} \). So there is a well-defined functor of completed tensor product

\[
\hat{\otimes}^- : \mathrm{Top}^\mathfrak{sc}_k \times \mathbf{Vect}_k \longrightarrow \mathbf{Vect}_k,
\]

defined by the rule \( \mathfrak{U}\hat{\otimes}^- \mathfrak{V} = \lim_{\mathfrak{P} \in \mathfrak{P}_{cl}} (\mathfrak{U}/\mathfrak{P} \otimes_k \mathfrak{V}) \) for all \( \mathfrak{U} \in \mathrm{Top}^\mathfrak{sc}_k \), \( \mathfrak{V} \in \mathbf{Vect}_k \), and agreeing with the functor \( \hat{\otimes}^- : \mathrm{Top}^\mathfrak{sc}_k \times \mathrm{Top}^\mathfrak{sc}_k \longrightarrow \mathrm{Top}^\mathfrak{sc}_k \).

The following lemma holds for topological abelian groups just as well, but we will only use it for topological vector spaces.

**Lemma 13.2.** (a) The completion functors \( V \longmapsto V^- : \mathrm{Top}_k \longrightarrow \mathrm{Top}^\mathfrak{sc}_k \) and \( V \longmapsto V^- : \mathrm{Top}^\mathfrak{sc}_k \longrightarrow \mathrm{Top}^\mathfrak{sc}_k \) preserve the cokernels of morphisms.

(b) For any injective morphism of topological vector spaces \( \iota : U \longrightarrow V \) such that the subspace \( \iota(U) \) is dense in \( V \) and the topology of \( U \) is induced from the topology of \( V \) via \( \iota \), the induced morphism of the completions \( \iota^- : U^- \longrightarrow V^- \) is an isomorphism of topological vector spaces.

(c) The completion functors \( V \longmapsto V^- : \mathrm{Top}_k \longrightarrow \mathrm{Top}^\mathfrak{sc}_k \) and \( V \longmapsto V^- : \mathrm{Top}^\mathfrak{sc}_k \longrightarrow \mathrm{Top}^\mathfrak{sc}_k \) take short sequences satisfying Ex1 to short sequences satisfying Ex1.

**Proof.** Part (a): the completion functors are the reflectors, i.e., they are left adjoint to the inclusion functors \( \mathrm{Top}_k \longrightarrow \mathrm{Top}^\mathfrak{sc}_k \) and \( \mathrm{Top}^\mathfrak{sc}_k \longrightarrow \mathrm{Top}^\mathfrak{sc}_k \), respectively (cf. Section [1]). All left adjoint functors preserve all colimits, and in particular, cokernels.

Part (b): first let us assume that \( V \) is separated; then \( U \) is separated, too. For any separated topological vector space (or abelian group) \( V \) and its completion \( V^- \), the topology of \( V \) is induced from the topology of \( V^- \) via the injective completion map \( V \longrightarrow V^- \), which makes \( V \) a dense subspace/subgroup in \( V^- \). In the situation at hand, \( U \) is dense in \( V \) and \( V \) is dense in \( V^- \), hence \( U \) is dense in \( V^- \). The topology of \( U \) is induced from the topology of \( V \) and the topology of \( V \) is induced from the topology of \( V^- \), hence the topology of \( U \) is induced from the topology of \( V^- \). It remains to apply Lemma [1.1] to the embedding \( U \longrightarrow V^- \).

The general case is reduced to the separated case by the passage to the maximal separated quotient spaces \( U/\{0\}_U \) and \( V/\{0\}_V \) of the topological vector spaces \( U \) and \( V \) (see the proof of part (c) below for some additional details).

Part (c): let \( 0 \longrightarrow K \xrightarrow{i} V \xrightarrow{p} C \longrightarrow 0 \) be a short sequence satisfying Ex1 in \( \mathrm{Top}^\mathfrak{sc}_k \) (i.e., a short exact sequence in the quasi-abelian exact structure on \( \mathrm{Top}^\mathfrak{sc}_k \)). By
part (a), the morphism \( p^\wedge : V^\wedge \rightarrow C^\wedge \) is the cokernel of the morphism \( i^\wedge : K^\wedge \rightarrow V^\wedge \). Applying Lemma 1.1 to the embedding \( K \rightarrow V \rightarrow V^\wedge \), one can see that the morphism \( i^\wedge = K^\wedge \rightarrow V^\wedge \) is an injective closed map. By Proposition 5.1(b), this means that \( i^\wedge \) is a kernel in \( \text{Top}_k^\wedge \); hence \( i^\wedge \) is a kernel of its cokernel \( p^\wedge \).

Similarly, let \( 0 \rightarrow K \xrightarrow{i} V \xrightarrow{p} C \rightarrow 0 \) be a short sequence satisfying Ex1 in \( \text{Top}_k \) (i.e., a short exact sequence in the quasi-abelian exact structure on \( \text{Top}_k \)). By part (a), the morphism \( p^\wedge : V^\wedge \rightarrow C^\wedge \) is the cokernel of the morphism \( i^\wedge : K^\wedge \rightarrow V^\wedge \). Let \( K_0 = \{0\}_K \subset K \) and \( V_0 = \{0\}_V \subset V \) be the closures of the zero subgroup in \( K \) and in \( V \). Then one has \( K_0 = V_0 \cap K \), so the induced map \( i' : K/K_0 \rightarrow V/V_0 \) is injective. Moreover, the quotient topology on \( K/K_0 \) is induced from the quotient topology of \( V/V_0 \) via \( i' \). Applying Lemma 1.1 to the embedding \( K/K_0 \rightarrow V/V_0 \rightarrow (V/V_0)^\wedge = V^\wedge \), one can see that the morphism \( i^\wedge = i'^\wedge : (K/K_0)^\wedge = K^\wedge \rightarrow V^\wedge \) is an injective closed map; and the argument finishes similarly to the previous case. \( \square \)

**Example 13.3.** Very simple counterexamples show that the completion functor does **not** preserve the kernels of morphisms of topological vector spaces. In fact, it can transform an injective continuous linear map of separated topological vector spaces into a noninjective map of the completions. For example, let \( C \) be an incomplete separated topological vector space and \( C = C^\wedge \) be its completion. Choose a vector \( x \in C \setminus C \); then \( x : k \rightarrow C \) is a split monomorphism of topological vector spaces (where the one-dimensional vector space \( k \) is endowed with the discrete topology) by Corollary 3.4. Hence the quotient space \( C/kx \) is a direct summand in \( C \), so it is separated and complete. Thus the completion functor transforms the injective map \( C \rightarrow C/kx \) into the split epimorphism \( C \rightarrow C/kx \) with a nonzero kernel \( kx \).

**Proposition 13.4.** The three topological tensor products \( \hat{\otimes}^* \), \( \hat{\otimes}^\sim \), and \( \hat{\otimes}^! \) are associative: for any complete, separated topological vector spaces \( \mathfrak{U} \), \( \mathfrak{V} \), and \( \mathfrak{W} \) there are natural isomorphisms of (complete, separated) topological vector spaces

\[
\begin{align*}
(a) & \quad \( \mathfrak{U} \hat{\otimes}^* \mathfrak{V} \hat{\otimes}^* \mathfrak{W} \simeq \mathfrak{U} \hat{\otimes}^* (\mathfrak{V} \hat{\otimes}^* \mathfrak{W}) \); \\
(b) & \quad \( \mathfrak{U} \hat{\otimes}^\sim \mathfrak{V} \hat{\otimes}^\sim \mathfrak{W} \simeq \mathfrak{U} \hat{\otimes}^\sim (\mathfrak{V} \hat{\otimes}^\sim \mathfrak{W}) \); \\
(c) & \quad \( \mathfrak{U} \hat{\otimes}^! \mathfrak{V} \hat{\otimes}^! \mathfrak{W} \simeq \mathfrak{U} \hat{\otimes}^! (\mathfrak{V} \hat{\otimes}^! \mathfrak{W}) \).
\end{align*}
\]

**Proof.** First one needs to establish associativity of each one of the three uncompleted tensor products \( \otimes^* \), \( \otimes^\sim \), and \( \otimes^! \) (as functors \( \text{Top}_k \times \text{Top}_k \rightarrow \text{Top}_k \)). The best way to do it is to describe the triple (and multiple) tensor products, i.e., to formulate the definitions of \(*\text{-topology} \), the \( \longleftrightarrow\text{-topology} \), and the \( !\text{-topology} \) on the tensor product of several topological vector spaces. Such explicit definitions are given in [5, Section 1.1].

Having convinced oneself that the uncompleted tensor products are associative, the associativity of the completed tensor products becomes a formal corollary based on Lemmas 12.3, 12.7, 12.9, and 13.2(b). Let us discuss part (a). We want to show that both the iterated completed tensor products in question are naturally isomorphic to the completion of the uncompleted triple tensor product, \( (\mathfrak{U} \otimes^* \mathfrak{V} \otimes^* \mathfrak{W})^\wedge \).

Indeed, following the discussion in the proof of Lemma 13.2(b), the uncompleted tensor product \( \mathfrak{U} \otimes^* \mathfrak{V} \) is a dense subspace in the completed tensor product \( \mathfrak{U} \hat{\otimes}^* \mathfrak{V} \),
and the topology of $\mathcal{U} \otimes^* \mathcal{V}$ is induced from the topology of $\hat{\mathcal{U}} \otimes^* \mathcal{V}$. By Lemmas 12.5 and 12.9, it follows that $(\mathcal{U} \otimes^* \mathcal{V}) \otimes^* \mathcal{W}$ is a dense subspace in $(\mathcal{U} \otimes^* \mathcal{V}) \otimes^* \mathcal{W}$, and the topology of $(\mathcal{U} \otimes^* \mathcal{V}) \otimes^* \mathcal{W}$ is induced from the topology of $(\mathcal{U} \otimes^* \mathcal{V}) \otimes^* \mathcal{W}$. Applying Lemma 13.2(b), one can conclude that the embedding $(\mathcal{U} \otimes^* \mathcal{V}) \otimes^* \mathcal{W} \rightarrow (\mathcal{U} \otimes^* \mathcal{V}) \otimes^* \mathcal{W}$ becomes an isomorphism after the passage to the completions. 

**Question 13.5.** Let $\mathcal{U}$ and $\mathcal{V}$ be complete, separated topological vector spaces. By Proposition 12.13, there is a short exact sequence of uncompleted tensor products in the category $\text{Top}_k$, and consequently also $\text{Top}_{k}^{sc}$:

$$0 \longrightarrow \mathcal{U} \otimes^* \mathcal{V} \longrightarrow \mathcal{U} \otimes^* \mathcal{V} \oplus \mathcal{U} \otimes^* \mathcal{V} \longrightarrow \mathcal{U} \otimes^! \mathcal{V} \longrightarrow 0. $$

Applying the completion functor, we obtain a short sequence of complete, separated topological vector spaces

$$\text{(10)} \quad 0 \longrightarrow \hat{\mathcal{U}} \otimes^* \hat{\mathcal{V}} \longrightarrow \hat{\mathcal{U}} \otimes^* \hat{\mathcal{V}} \oplus \hat{\mathcal{U}} \otimes^* \hat{\mathcal{V}} \longrightarrow \hat{\mathcal{U}} \otimes^! \mathcal{V} \longrightarrow 0,$$

which satisfies Ex1 in $\text{Top}_{k}^{sc}$ by Lemma 13.2(c). (Here $\hat{\mathcal{U}} \otimes^* \hat{\mathcal{V}}$ is an alternative notation for $\mathcal{U} \otimes^* \mathcal{V}$.)

Is the map $\mathcal{U} \otimes^* \mathcal{V} \oplus \mathcal{U} \otimes^* \mathcal{V} \longrightarrow \hat{\mathcal{U}} \otimes^! \mathcal{V}$ surjective? In other words, is (10) a short exact sequence in the maximal exact structure on $\text{Top}_{k}^{sc}$?

**Remark 13.6.** Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be three complete, separated topological vector spaces, and let $\phi: \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{W}$ be a continuous bilinear map. Then it is clear from Remark 12.2 that the linear map $\phi^{\circ}: \mathcal{U} \otimes_k \mathcal{V} \longrightarrow \mathcal{W}$ extends uniquely to a continuous linear map $\mathcal{U} \otimes^* \mathcal{V} \longrightarrow \mathcal{W}$. So continuous bilinear pairings $\mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{W}$ correspond bijectively to continuous linear maps $\mathcal{U} \otimes^* \mathcal{V} \longrightarrow \mathcal{W}$.

Let $\mathcal{R}$ be a complete, separated topological vector space with a topological (associative) algebra structure. Following the same remark, if open two-sided ideals form a base of neighborhoods of zero in $\mathcal{R}$, then the multiplication in $\mathcal{R}$ gives rise to a continuous linear map $\mathcal{R} \otimes^! \mathcal{R} \longrightarrow \mathcal{R}$. If open right ideals form a base of neighborhoods of zero in $\mathcal{R}$, then the multiplication in $\mathcal{R}$ can be described as a continuous linear map $\mathcal{R} \otimes^* \mathcal{R} \longrightarrow \mathcal{R}$. Using Proposition 13.4, the associativity of multiplication in $\mathcal{R}$ can be formulated in terms of the topological tensor products.

Assume from now on that open right ideals form a base of neighborhoods of zero in $\mathcal{R}$. Let $N$ be a discrete $k$-vector space. Then discrete right $\mathcal{R}$-module structures on $N$ (in the sense of [51, Section VI.4], [57, Sections 2.3–2.4], [36, Section 2.4]) can be described as continuous linear maps $N \otimes^* \mathcal{R} \longrightarrow N$ (satisfying the appropriate associativity equation). See [5, Section 1.4] for a discussion.

Furthermore, assume that the algebra $\mathcal{R}$ has a unit. Let $B$ be an abstract (nontopological) vector space. Following Example 13.13, an abstract (nontopological) vector space $\mathcal{R} \hat{\otimes} B$ is well-defined. According to [31, Section 1.10] or [35, Section 2.3], left $\mathcal{R}$-contramodule structures on $B$ (in the sense of [39, 40, 37, 36, 41]) can be described as linear maps $\mathcal{R} \hat{\otimes} B \longrightarrow B$ (satisfying suitable contramultiplicativity and contramodule equations). This definition of $\mathcal{R}$-contramodules for a topological algebra $\mathcal{R}$ over a field $k$ appeared already in [32, Section D.5.2].
Proposition 13.7. (a) The functor $\hat{\otimes}^\ast : \text{Top}_k^{sc} \times \text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ preserves the cokernels of morphisms (in each of its arguments). Fixing a topological vector space in one of the arguments, it becomes an endofunctor $\text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ taking short sequences satisfying Ex1 to short sequences satisfying Ex1.

(b) The functor $\hat{\otimes}^\triangleleft : \text{Top}_k^{sc} \times \text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ preserves the cokernels of morphisms (in each of its arguments). Fixing a topological vector space in any one of the arguments, it becomes an endofunctor $\text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ taking short sequences satisfying Ex1 to short sequences satisfying Ex1.

(c) The functor $\hat{\otimes}^\downarrow : \text{Top}_k^{sc} \times \text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ preserves the cokernels of morphisms (in each of its arguments). Fixing a topological vector space in any one of the arguments, it becomes an endofunctor $\text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ taking short sequences satisfying Ex1 to short sequences satisfying Ex1.

Proof. Let us explain part (a); parts (b) and (c) are similar. Let $f : \mathcal{U} \rightarrow \mathcal{W}$ be a morphism in $\text{Top}_k^{sc}$, and $C = \mathcal{U}''/\overline{f(\mathcal{U})}$, be the cokernel of $f$ in the category $\text{Top}_k^{sc}$. Let $\mathcal{V}$ be a complete, separated topological vector space. By Theorem 12.12(a), the uncompleted tensor product $C \otimes^* \mathcal{V}$ is the cokernel of the morphism $f \otimes^* \mathcal{V} : \mathcal{U} \otimes^* \mathcal{V} \rightarrow \mathcal{U}'' \otimes^* \mathcal{V}$ in the category $\text{Top}_k^{sc}$. According to Lemma 13.2(a), it follows that the completion $(C \otimes^* \mathcal{V})^\sim$ is the cokernel of the morphism $f \otimes^* \mathcal{V} : \mathcal{U} \otimes^* \mathcal{V} \rightarrow \mathcal{U}'' \otimes^* \mathcal{V}$ in the category $\text{Top}_k^{sc}$. Finally, $C = C^\sim$ is the cokernel of the morphism $f$ in $\text{Top}_k^{sc}$; and by Lemmas 12.3 and 12.9, the topological vector space $C \otimes^* \mathcal{V}$ is a dense subspace in $C \otimes^* \mathcal{V}$ with the topology of $C \otimes^* \mathcal{V}$ induced from the topology of $C \otimes^* \mathcal{V}$. By Lemma 13.2(b), we can conclude that $(C \otimes^* \mathcal{V})^\sim \simeq (C \otimes^* \mathcal{V})^\sim = C \otimes^* \mathcal{V}$.

Let $0 \rightarrow \mathcal{R} \rightarrow \mathcal{U} \rightarrow \mathcal{C} \rightarrow 0$ be a short sequence satisfying Ex1 in $\text{Top}_k^{sc}$, and let $\mathcal{V}$ be a complete, separated topological vector space. Then $\mathcal{C} = C^\sim$, where $0 \rightarrow \mathcal{R} \rightarrow \mathcal{U} \rightarrow C \rightarrow 0$ is a short exact sequence in $\text{Top}_k^{sc}$. By Theorem 12.12(a), $0 \rightarrow \mathcal{R} \otimes^* \mathcal{V} \rightarrow \mathcal{U} \otimes^* \mathcal{V} \rightarrow C \otimes^* \mathcal{V} \rightarrow 0$ is a short exact sequence in $\text{Top}_k^{sc}$, too. Passing to the completions, by Lemma 13.2(c) we get a short sequence $0 \rightarrow \mathcal{R} \otimes^* \mathcal{V} \rightarrow \mathcal{U} \otimes^* \mathcal{V} \rightarrow (C \otimes^* \mathcal{V})^\sim \rightarrow 0$ satisfying Ex1 in $\text{Top}_k^{sc}$. Following the argument in the previous paragraph, $(C \otimes^* \mathcal{V})^\sim \simeq C \otimes^* \mathcal{V}$, and we are done. □

Question 13.8. Do the functors $\hat{\otimes}^\ast$, $\hat{\otimes}^\triangleleft$, and/or $\hat{\otimes}^\downarrow : \text{Top}_k^{sc} \times \text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ preserve the kernels of morphisms (in any one of their arguments, with the other argument fixed) in the category $\text{Top}_k^{sc}$? Notice that the completion, generally speaking, does not preserve kernels; see Example 13.3.

Corollary 13.9. (a) For any infinite-dimensional discrete vector space $\mathcal{V}$, the functor $\text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ is not exact in the maximal exact structure on $\text{Top}_k^{sc}$.

(b) For any infinite-dimensional discrete vector space $\mathcal{V}$, the functor $\text{Top}_k^{sc} \rightarrow \text{Top}_k^{sc}$ is not exact in the maximal exact structure on $\text{Top}_k^{sc}$.

Proof. In fact, the functor in parts (a) and (b) is one and the same, according to Example 13.3(2). It is not exact with respect to the maximal exact structure on $\text{Top}_k^{sc}$ by Corollary 11.4(b). □
Proposition 13.10. There exists a unique exact category structure on the category $\mathbf{Top}_{sc}$ which is maximal among all the exact structures having the property that the functors $\hat{\otimes}^*$, $\hat{\otimes}^-$, and $\hat{\otimes}^! : \mathbf{Top}_{sc} \times \mathbf{Top}_{sc} \to \mathbf{Top}_{sc}$ are exact in it (as functors of any one argument with the topological vector space in the other argument fixed).

Proof. A short sequence $0 \to K \xrightarrow{i} U \xrightarrow{p} C \to 0$ is said to be exact in the desired exact category structure (which can be called the tensor-refined exact structure on $\mathbf{Top}_{sc}$) if it is exact in the maximal exact category structure and remains exact in the maximal exact category structure after any finite number of iterated applications of the functors $- \hat{\otimes} \mathcal{V}$, $- \hat{\otimes} \mathcal{V}$, $\mathcal{V} \hat{\otimes} -$, and $- \hat{\otimes} \mathcal{V}$.

This means that the maps $i$ and $p$ are each other’s kernel and cokernel in $\mathbf{Top}_{sc}$, the map $p$ is surjective, and the map $p$ stays surjective after any repeated application of the completed tensor product functors. Any short exact sequence in the tensor-refined exact structure is also exact in the strong exact structure of Theorem 11.5 (by Example 13.1(2)), but we do not know whether the converse holds.

To convince oneself that the above definition actually produces an exact category structure on $\mathbf{Top}_{sc}$, one can use the construction of Example 4.2 (cf. the proof of Theorem 11.5). One only needs to observe that the completed tensor product functors (viewed as functors of any one argument with the other argument fixed) preserve both the cokernels of all morphisms and the kernels of admissible epimorphisms in the maximal exact structure on $\mathbf{Top}_{sc}$ by Proposition 13.7. We omit the straightforward details. □

Conclusion 13.11. The uncompleted tensor product functors $\otimes^*$, $\otimes^-$, and $\otimes^!$, acting in the categories of incomplete topological vector spaces $\mathbf{Top}_{k}$ and $\mathbf{Top}_{s}$, have nice exactness properties. They preserve the kernels and cokernels of all morphisms, and are exact in the quasi-abelian exact structures of these additive categories (by Theorems 12.11 and 12.12).

The completed tensor product functors $\hat{\otimes}^*$, $\hat{\otimes}^-$, and $\hat{\otimes}^!$, acting in the category of complete, separated topological vector spaces $\mathbf{Top}_{k}^c$, have seemingly paradoxical properties. They preserve the cokernels of all morphisms, and take short sequences satisfying Ex1 (the “kernel-cokernel pairs”) to short sequences satisfying Ex1; see Proposition 13.7. But the category $\mathbf{Top}_{k}^c$ is not quasi-abelian, and the class of all short sequences satisfying Ex1 is not well-behaved in it. More specifically, the class of all cokernels in $\mathbf{Top}_{k}^c$ is not well-behaved, because it includes nonsurjective cokernels (see Corollary 8.6 and Conclusion 8.9).

The class of all stable short exact sequences, forming the maximal exact structure on $\mathbf{Top}_{k}^c$, is better behaved. But the functors $\hat{\otimes}^-$ and $\hat{\otimes}^!$ do not preserve the semi-stability (= stability) of cokernels. Accordingly, these functors are not exact in the maximal exact structure on $\mathbf{Top}_{k}^c$ (see Corollary 13.9).

One can construct an exact structure on $\mathbf{Top}_{k}^c$ by imposing the condition of preservation of short exact sequences by the tensor product operations. But it is not clear how to describe this exact structure more explicitly.
No exactness problems arise in connection with the category \( \text{Top}^{\omega, \text{sc}} \) of complete, separated topological vector spaces with a countable base of neighborhoods of zero, as this category is quasi-abelian and all the short exact sequences in its quasi-abelian exact structure are split. But the tensor product operations \( \hat{\otimes} \) and \( \hat{\otimes}^{-} \) are not defined on the full subcategory \( \text{Top}^{\omega, \text{sc}} \subset \text{Top}^{\text{sc}} \), as they do not preserve the countable base of neighborhoods of zero (compare Example 13.1(1) with Lemma 7.3).

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