Abstract
Let $\mathcal{X}$ be a real separable Hilbert space. Let $C$ be a linear, bounded, non-negative self-adjoint operator on $\mathcal{X}$ and let $A$ be the infinitesimal generator of a strongly continuous semigroup in $\mathcal{X}$. Let $\{W(t)\}_{t \geq 0}$ be a $\mathcal{X}$-valued cylindrical Wiener process on a filtered (normal) probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Let $F : \text{Dom}(F) \subseteq \mathcal{X} \to \mathcal{X}$ be a smooth enough function. We are interested in the generalized mild solution $\{X(t, x)\}_{t \geq 0}$ of the semilinear stochastic partial differential equation

$$\begin{cases}
    dX(t, x) = (AX(t, x) + F(X(t, x)))dt + \sqrt{C}dW(t), \; t > 0; \\
    X(0, x) = x \in \mathcal{X}.
\end{cases}$$

We consider the transition semigroup defined by

$$P(t)\varphi(x) := \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in B_b(\mathcal{X}), \; t \geq 0, \; x \in \mathcal{X}.$$

If $O$ is an open set of $\mathcal{X}$, we consider the Dirichlet semigroup defined by

$$P^O(t)\varphi(x) := \mathbb{E} \left[ \varphi(X(t, x))1_{\{\omega \in \Omega : \tau_x(\omega) > t\}} \right], \quad \varphi \in B_b(O), \; x \in O, \; t > 0$$

where $\tau_x$ is the exit time defined by

$$\tau_x = \inf\{s > 0 : X(s, x) \in O^c\}.$$

We study the infinitesimal generator of $P(t)$, $P^O(t)$ in $L^2(\mathcal{X}, \nu)$, $L^2(O, \nu)$ respectively, where $\nu$ is the unique invariant measure of $P(t)$.
Keywords  Invariant measure · Generalized mild solution · Yosida approximating · Dirichlet · Reaction–diffusion equations · Dissipative systems · Semilinear stochastic partial differential equations

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1 Introduction

Let $\mathcal{X}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered (normal) probability space and let $\{W(t)\}_{t \geq 0}$ be a $\mathcal{X}$-cylindrical Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Let $A : \text{Dom}(A) \subseteq \mathcal{X} \to \mathcal{X}$ be the infinitesimal generator of strongly continuous semigroup $e^{tA}$ and let $C \in L(\mathcal{X})$ (the space of bounded and linear operators from $\mathcal{X}$ to $\mathcal{X}$) be a non-negative self-adjoint operator. Let $F : \text{Dom}(F) \subseteq \mathcal{X} \to \mathcal{X}$ (possibly non linear). We introduce the SPDE

\[\begin{align*}
&\begin{cases}
    dX(t, x) = \left(AX(t, x) + F(X(t, x))\right)dt + \sqrt{C}dW(t), \ t > 0; \\
    X(0, x) = x \in \mathcal{X}.
\end{cases} \\
&\text{(1.1)}
\end{align*}\]

This type of SPDE is widely studied in the literature, see for example [1, 5, 6, 14, 20, 22–24, 29, 33, 35, 41, 44, 47, 49]. In this paper we focus on the case of dissipative systems, where $A$ and $F$ satisfy a joint dissipativity condition (see Hypotheses 3.1(iii)).

If $\text{Dom}(F) = \mathcal{X}$, for any $x \in \mathcal{X}$ it is possible to consider the solution of the mild form of (1.1), namely

\[X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s, x))ds + W_A(t), \ \mathbb{P}\text{-a.s.} \quad (1.2)\]

where $\{W_A(t)\}_{t > 0}$ is the process defined by

\[W_A(t) := \int_0^t e^{(t-s)A}\sqrt{C}dW(s).\]

However, if $\text{Dom}(F) \subset \mathcal{X}$ is a proper subset of $\mathcal{X}$, (1.2) may not make sense for every $x \in \mathcal{X}$, since it is not guaranteed that there exists a process $\{X(t, x)\}_{t \geq 0}$ such that verifies the (1.2) and its trajectories live in $\text{Dom}(F)$. So we need a more general notion of solution. Around the nineties S. Cerrai, G. Da Prato and J. Zabczyk have considered the notion of generalized mild solution to avoid the problem of $\text{Dom}(F)$. The idea to construct a generalized mild solution is to assume that there exists a Banach space $E \subseteq \text{Dom}(F)$ densely and continuously embedded in $\mathcal{X}$ such that the operator $A$ and the function $F$ have some “good” properties on $E$. We prove that for any $x \in E$, the SPDE (1.1) has a unique mild solution $\{X(t, x)\}_{t \geq 0}$ (i.e. check (1.2)) such that its trajectories take values in $E$. After, exploiting the density of $E$, we will prove that for any $x \in \mathcal{X}$ there exists a process $\{X(t, x)\}_{t \geq 0}$, such that

[Diagram or Image]
\[
\lim_{n \to \infty} \|X(\cdot, x_n) - X(\cdot, x)\|_{C([0,T], \mathcal{X})} = 0, \quad \forall T > 0, \quad \mathbb{P}\text{-a.s.} \quad (1.3)
\]

where \( \{x_n\}_{n \in \mathbb{N}} \subseteq E \) is a sequence converging to \( x \) and \( X(t, x_n) \) is the unique mild solution of (1.1), with initial datum \( x_n \). We call the limit \( \{X(t, x)\}_{t \geq 0} \) of (1.3) generalized mild solution of (1.1). We refer to [14, Chapter 7] and [22, Chapter 4] for two examples where the generalized mild solution is constructed on an explicit space \( E \).

In Sect. 3, under suitable hypotheses (Hypotheses 3.1) we show that, for any \( x \in \mathcal{X} \), the SPDE (1.1) has a unique generalized mild solution \( \{X(t, x)\}_{t \geq 0} \).

Let \( B_b(\mathcal{X}) \) be the space of bounded and Borel measurable functions from \( \mathcal{X} \) to \( \mathbb{R} \).

We consider the semigroup \( P(t) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in B_b(\mathcal{X}), \quad x \in \mathcal{X}, \quad t > 0 \quad (1.4) \)

In Sect. 3.3, with an additional hypothesis (Hypotheses 3.17), we will prove that the semigroup \( P(t) \) has a unique invariant measure \( \nu \), such that \( \nu(E) = 1 \), and \( \nu \) has finite moments of every order. By the invariance of \( \nu \) and standard arguments, \( P(t) \) is uniquely extendable to a strongly continuous semigroup \( P_p(t) \) in \( L^p(\mathcal{X}, \nu) \), for \( p \geq 1 \). We denote by \( N_2 \) the infinitesimal generator of \( P_2(t) \). A fundamental tool to study the behavior of the transition semigroup in \( L^2(\mathcal{X}, \nu) \) will be to define a core of regular functions over which is known the action of \( N_2 \). The candidate to be the core will be the space

\[
\xi_A(\mathcal{X}) := \text{span}\{\text{real and imaginary parts of the functions } x \mapsto e^{i\langle x, h \rangle} \mid h \in \text{Dom}(A*)\}. \quad (1.5)
\]

We will prove that, on \( \xi_A(\mathcal{X}) \), \( N_2 \) acts as the following second order Kolmogorov operator defined by

\[
N_0\varphi(x) := L_0\varphi(x) + \langle F_0(x), \nabla \varphi(x) \rangle, \quad \varphi \in \xi_A(\mathcal{X}), \quad x \in \mathcal{X}, \quad (1.6)
\]

where

\[
L_0\varphi(x) := \frac{1}{2} \text{Tr}[C\nabla^2 \varphi(x)] + \langle x, A* \nabla \varphi(x) \rangle \quad (1.7)
\]

Precisely we will prove an even more significant result.

**Theorem 1.1** Assume that Hypotheses 4.5 hold true. \( N_2 \) is the closure in \( L^2(\mathcal{X}, \nu) \) of the operator \( N_0 \), defined in (1.6). In particular \( \xi_A(\mathcal{X}) \) is a core for \( N_2 \).

To prove Theorem 1.1, we should first extend the operator \( N_0 \) into \( L^2(\mathcal{X}, \nu) \), to do so is fundamental that \( \nu(E) = 1 \), since \( F \) is well defined only on \( E \). Theorem 1.1 extends those contained in [6, Sect. 3], [23], [22, Sects. 3.5 and 4.6] and [34, Sect. 11.2.2]. For a study of an analogous problem in \( L^2(E, \nu) \) in the case of a multiplicative noise we refer to [17].
Let $\mathcal{O}$ be an open set of $\mathcal{X}$ and let $B_b(\mathcal{O})$ be the space of bounded and Borel measurable functions from $\mathcal{O}$ to $\mathbb{R}$. In Sect. 5 we consider the Dirichlet semigroup

\[ P^O(t)\varphi(x) := \mathbb{E}\left[ \varphi(X(t, x)) \mathbb{1}_{\{\omega \in \Omega : \tau_x(\omega) > t\}} \right], \quad \varphi \in B_b(\mathcal{O}), \; x \in \mathcal{O}, \; t > 0 \]

(1.8)

where $\{X(t, x)\}_{t \geq 0}$ is the generalized mild solution of (1.1), and $\tau_x$ is the exit time defined by

\[ \tau_x = \inf\{s > 0 : X(s, x) \notin \mathcal{O}^c\} \]

We will prove that $\nu$ is sub-invariant for $P^O(t)$; therefore $P^O(t)$ is uniquely extendable to a strongly continuous semigroup $P^O_p(t)$ in $L^p(\mathcal{O}, \nu)$, for any $p \geq 1$. We denote by $M_2$ the infinitesimal generator of $P^O(t)$. In Sect. 5 we will restrict to the case where $F$ is a gradient perturbation, namely it has a potential. In this case the invariant measure $\nu$ is a weighted Gaussian measure and it is possible to associate a quadratic form $Q_2$ to $N_2$, see for example [2, 3, 9, 10, 21, 26–28, 31, 32, 42]. Under some additional hypotheses (Hypotheses 5.4) we will define the Sobolev space $W^{1,2}_C(\mathcal{X}, \nu)$, and we will show that there exists a quadratic form $Q_2$ on $W^{1,2}_C(\mathcal{X}, \nu)$ such that

\[ \int_{\mathcal{X}} (N_2 \varphi) \psi d\nu = Q_2(\varphi, \psi) = -\frac{1}{2} \int \left\langle C^{1/2} \nabla \varphi, C^{1/2} \nabla \psi \right\rangle d\nu, \]

\[ \forall \varphi \in \text{Dom}(N_2), \; \psi \in W^{1,2}_C(\mathcal{X}, \nu). \]

After, proceeding as in [25, Sect. 3], we will consider a suitable Sobolev space $\dot{W}^{1,2}_C(\mathcal{X}, \nu)$ of the functions $u : \mathcal{O} \to \mathbb{R}$ such that their null extension $\hat{u}$ belongs to $W^{1,2}_C(\mathcal{X}, \nu)$, and the quadratic form $Q^O_2$ on $\dot{W}^{1,2}_C(\mathcal{O}, \nu)$ defined by

\[ Q^O_2(\varphi, \psi) = Q_2(\hat{\varphi}, \hat{\psi}), \quad \forall \varphi, \psi \in \dot{W}^{1,2}_C(\mathcal{O}, \nu). \]

In Sect. 5.3 we are going to prove the last result of this paper.

**Theorem 1.2** Assume that Hypotheses 5.4 hold true. Then the infinitesimal generator $M_2$ of $P^O(t)$ is the operator $N^O_2$ associated with $Q^O_2$, namely

\[ \text{Dom}(N^O_2) := \{ \varphi \in \dot{W}^{1,2}_C(\mathcal{O}, \nu) : \exists \beta \in L^2(\mathcal{O}, \nu) \text{ s.t.} \}

\[ \int_{\mathcal{O}} \beta \psi d\nu = Q^O_2(\beta, \psi) \quad \forall \psi \in \dot{W}^{1,2}_C(\mathcal{X}, \nu) \}

\[ N^O_2 \varphi = \beta, \quad \varphi \in \text{Dom}(N^O_2). \]

This result generalizes the one contained in [25, Sect. 3] proved for $F = 0$. For a study of an analogous problem in the case where $\mathcal{X}$ is a separable Banach space and $F = 0$ we refer to [4], instead we refer to [51, 53] for other types of problems about the semigroup (1.8).
To prove the above mentioned results, we rely on the fact that \( \nu \) has finite moments of any order and \( \nu(E) = 1 \). In [24, 29, 30] the authors assume as hypothesis the existence of a measure with the properties just mentioned, in this paper we show the existence and uniqueness of such a measure for the class of dissipative systems that we are considering. In [35, Sects. 7.2 and 11.6] the authors prove existence and uniqueness of the generalized mild solution of (1.1) and of the invariant measure for \( P(t) \) in many settings that include our own. However they do not provide the estimates of the moments that we require. Instead in [22, Chapters 4] and [14, Chapters 6-7] the authors prove the estimates that we need, but in a specific context (in the same context, see [15, 16] for the case of multiplicative noise and [18] for the nonautonomous case).

In particular they assume that \( F \) is a Nemytskii operator; in Sect. 6.2 we will present an example of \( F \) that it is not of this type. Then in Sect. 3 we will prove that the SPDE (1.1) has a unique generalized mild solution \( \{X(t, x)\}_{t \geq 0} \) and the transition semigroup \( P(t) \) defined in (1.4) has a unique invariant measure \( \nu \) with finite moments of any order, and \( \nu(E) = 1 \).

We conclude this introduction comparing our assumptions and examples with the one already present in the literature. Hypotheses 3.17 include the dissipative case of the SPDE considered in [14, Sect. 6] (see [17, Hypothesis 3]). In [14, Sect. 6] the authors set \( X = L^2(\Gamma, \lambda, \mathbb{R}^n) \) and \( E = C(\Gamma, \mathbb{R}^n) \) where \( n \in \mathbb{N} \), \( \lambda \) is the Lebesgue measure and \( \Gamma \) is an open set in \( \mathbb{R}^d \) with \( d \leq 3 \). They consider as \( F \) a Nemytskii type operator

\[
F(x)(\xi) := f(\xi, x(\xi)) + cx(\xi), \quad x \in L^2(\Gamma, \lambda, \mathbb{R}^n), \ \xi \in \Gamma, \ c > 0,
\]

where \( f : \Gamma \times \mathbb{R}^n \to \mathbb{R} \) is a suitable function. This setting covers a large class of reaction-diffusion systems such as the one considered in [22, Chapter 4]. In Sect. 6.1 we are going to present a particular case of [14, Sect. 6] that verifies also the hypotheses of Theorem 1.2. In Sect. 6.2 we give an example of \( F \) that satisfies the hypotheses of Theorem 1.1, but it does not belong to the class considered in [14, Chapter 6] and [22, Chapter 4]. Finally in Sect. 6.3 we will consider a particular case of the example of Sect. 6.2, where the invariant measure \( \nu \) is concentrated on \( W^{1,2}([0, 1], \lambda) \).

## 2 Preliminaries

In this section we recall some notations, definitions and results that we will use in the rest of the paper.

### 2.1 Notations

Let \( H_1 \) and \( H_2 \) be two Banach spaces. We denote by \( \mathcal{B}(H_1) \) the family of the Borel subsets of \( H_1 \) and by \( B_b(H_1; H_2) \) the set of the \( H_2 \)-valued, bounded and Borel measurable functions. When \( H_2 = \mathbb{R} \) we simply write \( B_b(H_1) \). We denote by \( C_b(H_1; H_2) \) the set of the continuous and bounded functions from \( H_1 \) to \( H_2 \). If \( H_2 = \mathbb{R} \) we simply write \( C_b(H_1) \). We denote by \( C^k_b(H_1; H_2) \), \( k \in \mathbb{N} \cup \{\infty\} \) the set of the \( k \)-times
Fréchet differentiable functions from $H_1$ to $H_2$ with bounded derivatives up to order $k$. If $H_2 = \mathbb{R}$ we simply write $C_b^k(H_1)$. For a function $\Phi \in C_b^1(H_1; H_2)$ we denote by $\mathcal{D}\Phi(x)$ the Fréchet derivative operator of $\Phi$ at the point $x \in H_1$. Let $H_3$ be a Hilbert space, if $f \in C_b^1(H_3)$ then, for every $x \in H_3$ there exists a unique $k \in H_3$ such that for every $h \in H_3$

$$\mathcal{D} f(x)(h) = (h, k)_{H_3}.$$  

We set $\nabla f(x) := k$. If $\Phi : H_1 \to H_1$ is Gateaux differentiable we denote by $\mathcal{D}^G\Phi(x)h$ the Gateaux derivative operator of $\Phi$ at the point $x \in H_1$ along the direction $h \in H_1$.

**Definition 2.1** Let $G : \text{Dom}(G) \subseteq H_1 \to H_1$ and let $H_2 \subseteq H_1$. We call part of $G$ in $H_2$ the function $G_{H_2} : \text{Dom}(G_{H_2}) \subseteq H_2 \to H_2$ defined by

$$\text{Dom}(G_{H_2}) := \{ x \in \text{Dom}(G) \cap H_2 : G(x) \in H_2 \}, \quad G_{H_2}(x) := G(x), \ x \in \text{Dom}(G_{H_2}).$$

We denote by $\mathcal{L}(H_1)$ the set of bounded linear operators from $H_1$ to itself and by $I_{H_1} \in \mathcal{L}(H_1)$ the identity operator on $H_1$. For $\Gamma \in \mathcal{B}(H_1)$, we denote by $\mathbb{I}_\Gamma$ the characteristic function of $\Gamma$. We say that $B \in \mathcal{L}(H_1)$ is non-negative (positive) if for every $x \in H_1 \setminus \{0\}$

$$\langle Bx, x \rangle \geq 0 \ (> 0),$$

On the other hand, $B \in \mathcal{L}(H_1)$ is a non-positive (resp. negative) operator if $-B$ is non-negative (resp. positive) (see [52, Sect. IV.4]).

Let $B \in \mathcal{L}(H_1)$ be a non-negative and self-adjoint operator. We say that $B$ is a trace class operator if

$$\text{Tr}[B] := \sum_{n=1}^{+\infty} \langle Be_n, e_n \rangle < +\infty,$$

for some (and hence for all) orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $H_1$. We recall that the trace is independent of the choice of the basis (see [39, Sect. XI.6 and XI.9]).

Let $(\mathcal{O}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space we say that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is normal when

1. $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$;
2. $\mathcal{F}_t = \cap_{s > t} \mathcal{F}_s$ for any $t \geq 0$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered (normal) probability space and let $\mathcal{K}$ be a separable Banach space. Let $\xi : (\mathcal{O}, \mathcal{F}, \mathbb{P}) \to (\mathcal{K}, \mathcal{B}(\mathcal{K}))$ be a random variable, we denote by

$$\mathcal{L}(\xi) := \mathbb{P} \circ \xi^{-1}$$

the law of $\xi$ on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$, and by

$$\mathbb{E}[\xi] := \int_{\Omega} \xi(w) \mathbb{P}(d\omega) = \int_{\mathcal{K}} x \mathcal{L}(\xi)(dx)$$
the expectation of $\xi$ with respect to $\mathbb{P}$. In this paper when we refer to a $\mathcal{K}$-valued process we mean an adapted process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with values in $\mathcal{K}$. Let $\{Y_1(t)\}_{t \geq 0} \in \{Y_2(t)\}_{t \geq 0}$ be two stochastic processes; we say that $\{Y_1(t)\}_{t \geq 0}$ is a version (or a modification) of $\{Y_2(t)\}_{t \geq 0}$ if, for any $t \geq 0$ we have

$$ Y_1(t) = Y_2(t), \quad \mathbb{P}\text{-a.s.} $$

Let $\{Y(t)\}_{t \geq 0}$ be a $\mathcal{K}$-valued process we say that $\{Y(t)\}_{t \geq 0}$ is continuous if the map $Y(\cdot) : [0, +\infty) \to \mathcal{K}$ is continuous $\mathbb{P}$-a.s..

We recall the definitions of two spaces often considered in the literature (see [14, Sect. 6.2]).

**Definition 2.2**

1. Let $T > 0$ and $p \geq 1$. We denote by $\mathcal{K}^p([0, T])$ the space of progressive measurable $\mathcal{K}$-valued processes $\{Y(t)\}_{t \in [0, T]}$ endowed with the norm

$$ \|\{Y(t)\}_{t \in [0, T]}\|_{\mathcal{K}^p([0, T])}^p := \sup_{t \in [0, T]} \mathbb{E}[\|Y(t)\|_{\mathcal{K}}^p]. $$

2. Let $T > 0$ and $p \geq 1$. We denote by $C_p([0, T], \mathcal{K})$ the space of progressive measurable $\mathcal{K}$-valued processes $\{Y(t)\}_{t \in [0, T]}$ such that $Y(\cdot) : [0, T] \to \mathcal{K}$ is continuous $\mathbb{P}$-a.s., endowed with the norm

$$ \|\{Y(t)\}_{t \in [0, T]}\|_{C_p([0, T], \mathcal{K})}^p = \|\{Y(t)\}_{t \in [0, T]}\|_{C_p([0, T], \mathcal{K})}^p := \mathbb{E}[\sup_{t \in [0, T]} \|Y(t)\|_{\mathcal{K}}^p]. $$

### 2.2 Dissipative mappings

We recall some basic results about subdifferential and dissipative maps, we refer to [14, Appendix A] and [35, Appendix D] for the results of this section. Let $\mathcal{K}$ be a Banach space. For any $x \in \mathcal{K}$, we define the subdifferential $\partial \|x\|_{\mathcal{K}}$ of $\|\cdot\|_{\mathcal{K}}$ at $x \in \mathcal{K}$ as

$$ \partial \|x\| := \{x^* \in E \mid \mathcal{K} \{x, x^*\}_{\mathcal{K}^*} = \|x\|_{\mathcal{K}}, \|x^*\|_{\mathcal{K}^*} = 1\}. $$

Let $[t_0, t_1] \subset [0, +\infty)$ and let $u : [t_0, t_1] \to \mathcal{K}$ be a differentiable function. Then the function $\gamma := \|u\|_{\mathcal{K}} : [t_0, t_1] \to [0, +\infty)$ is left-differentiable in any $t_0 \in [t_0, t_1]$ and

$$ \frac{d^-\gamma}{dt}(t_0) := \lim_{h \to 0^-} \frac{\gamma(t_0 + h) - \gamma(t_0)}{h} = \min_{E \{u'(t_0), x^*\}_{\mathcal{K}^*}} : x^* \in \partial \|u(t_0)\|_{\mathcal{K}}. $$

(2.1)

Moreover, let $b \in \mathbb{R}$ and $g : [t_0, t_1] \to [0, +\infty)$ be a continuous function, if

$$ \frac{d^-\gamma}{dt}(t) \leq b\gamma(t) + g(t), \quad t \in [t_0, t_1], $$

then
then, for any \( t \in [t_0, t_1] \), we have
\[
\gamma(t) \leq e^{b(t-t_0)}\gamma(t_0) + \int_{t_0}^{t} e^{b(t-s)}g(s)ds, \quad t \in [t_0, t_1].
\] (2.2)

**Definition 2.3** A map \( f : \text{Dom}(f) \subseteq K \rightarrow K \) is said to be dissipative if, for any \( \alpha > 0 \) and \( x, y \in \text{Dom}(f) \), we have
\[
\|x - y - \alpha(f(x) - f(y))\|_K \geq \|x - y\|_K
\] (2.3)

If \( f \) is a linear operator (2.3) reads as
\[
\|(\lambda I - A)x\|_K \geq \lambda\|x\|_K, \quad \forall \lambda > 0, \ x \in \text{Dom}(A)
\]
We say that \( f \) is m-dissipative if the range of \( \lambda I - f \) is all the space \( K \) for some \( \lambda > 0 \) (and so for all \( \lambda > 0 \)).

Using the notion of subdifferential we have the following useful characterisation for the dissipative maps.

**Proposition 2.4** Let \( f : \text{Dom}(f) \subseteq K \rightarrow K \). \( f \) is dissipative if and only if, for any \( x, y \in \text{Dom}(f) \) there exists \( z^* \in \partial\|x - y\|_K \) such that
\[
K\langle f(x) - f(y), z^*\rangle \leq 0.
\] (2.4)

If \( K \) is a Hilbert space (2.4) becomes
\[
\langle f(x) - f(y), x - y\rangle_K \leq 0.
\]

**2.3 Semigroups**

In this subsection we recall some basic definitions and results of the semigroups theory. We refer to [46, Chapter 2] and [40, Chapter II].

Let \( K \) be a separable Banach space. Let \( T(t) \) be a semigroup of bounded operators on \( B_b(K) \).

(1) We say that \( T(t) \) is non-negative if for any non-negative valued \( \phi \in B_b(K) \) and for any \( t \geq 0 \), \( T(t)\phi \) has non-negative values.

(2) We say that \( T(t) \) is Feller, if for any \( t \geq 0 \) we have
\[
T(t) (C_b(K)) \subseteq C_b(K).
\]

(3) We say that \( T(t) \) is contractive, if for any \( t \geq 0 \) and \( \phi \in B_b(K) \) we have
\[
\|T(t)\phi\|_\infty \leq \|\phi\|_\infty.
\]
**Definition 2.5** Let $\mu \in \mathcal{P}(\mathcal{K})$ (the set of all Borel probability measures on $\mathcal{K}$) we say that $\mu$ is invariant for $T(t)$ if, for any $\varphi \in C_b(\mathcal{K})$ and $t \geq 0$, we have

$$\int_{\mathcal{K}} T(t)\varphi(x)\nu(dx) = \int_{\mathcal{K}} \varphi(x)\nu(dx).$$

Let $B : \text{Dom}(B) \subseteq \mathcal{K} \rightarrow \mathcal{K}$ linear. We denote by $\rho(B)$ the resolvent set of $B$ and for $\lambda \in \rho(B)$ we denote by $R(\lambda, B)$ the resolvent operator of $B$.

We consider the complexification of $\mathcal{K}$, and we still denote it by $\mathcal{K}$. Let $B : \text{Dom}(B) \subseteq \mathcal{K} \rightarrow \mathcal{K}$ be a sectorial operator, namely there exist $M > 0$, $\eta_0 \in \mathbb{R}$ and $\theta_0 \in (\pi/2, \pi]$ such that

$$S_0 := \{ \lambda \in \mathbb{C} | \lambda \neq \eta_0, \ |\arg(\lambda - \eta_0)| < \theta_0 \} \subseteq \rho(B);$$

$$\|R(\lambda, B)\|_{\mathcal{L}(\mathcal{K})} \leq \frac{M}{|\lambda - \eta_0|}, \ \forall \lambda \in S_0. \quad (2.5)$$

We denote by $e^{tB}$ the analytic semigroup generated by $B$. We recall some basic properties:

1. there exists $M_0 > 0$ such that for any $t > 0$

$$\|e^{tB}\|_{\mathcal{L}(\mathcal{K})} \leq M_0 e^{\eta_0 t}; \quad (2.6)$$

2. for any $t > 0$ and $h \in \mathbb{N}$

$$e^{tB}(\mathcal{K}) \subseteq \text{Dom}(B^h); \quad (2.7)$$

3. for any $x \in \text{Dom}(B)$

$$\lim_{n \rightarrow +\infty} nR(n, B)x = x; \quad (2.8)$$

4. let $f(t) = e^{tB}$, we have

$$f \in C^\infty((0, +\infty), \mathcal{L}(\mathcal{K})), \quad (2.9)$$

where the strong topology is considered on $\mathcal{L}(\mathcal{K})$.

**Remark 2.6** Properties analogous to (2.6) and (2.8) are also verified by strongly continuous semigroups. Moreover, for the strongly continuous semigroup, the function $f$ of (2.9) belongs to $C([0, +\infty), \mathcal{L}(\mathcal{K}))$. 
2.4 The Ornstein–Uhlenbeck case

Let $K$ be a separable Hilbert space. We recall some results about the Ornstein–Uhlenbeck semigroups that will be used in Sect. 4.3. We assume that $F = 0$ and that

$$\int_0^t \text{Tr}[e^{sA}Ce^{sA^*}]ds < +\infty, \quad \forall t \geq 0.$$

The SPDE (1.1) becomes

$$\begin{cases}
    dZ(t, x) = AZ(t, x)dt + \sqrt{C}dW(t), \quad t > 0; \\
    Z(0, x) = x \in K,
\end{cases}$$

(2.10)

where $W(t)_{t \geq 0}$ is a $K$-cylindrical Wiener process. We refer to [35, Sect. 4.1.2] and [49, Sect. 1] for a definition of cylindrical Wiener process. It is well known that $Z(t, x) = e^{tA}x + W_A(t)$ is the unique mild solution of (2.10) and, for any $t > 0$, we have

$$W_A(t) := \int_0^t e^{(t-s)A}\sqrt{C}dW(s) \sim \mathcal{N}(0, Q_t), \quad Q_t x = \int_0^t e^{sA}Ce^{sA^*}ds.$$

So, via a change of variable, for any $\varphi \in B_b(K)$ we obtain

$$T(t)\varphi(x) := \mathbb{E}[\varphi(Z(t, x))] = \int_{\Omega} \varphi(e^{tA}x + W_A(t))d\mathbb{P}$$

$$= \int_K \varphi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy).$$

(2.11)

Now we consider the Banach space

$$C_{b,2}(K) := \left\{ f : K \to \mathbb{R} \mid x \mapsto \frac{f(x)}{1 + \|x\|^2} \text{ belongs to } C_b(K) \right\}.$$ 

endowed with the norm

$$\|f\|_{b,2} := \sup_{x \in K} \left( \frac{|f(x)|}{1 + \|x\|^2} \right), \quad f \in C_{b,2}(K).$$

It is known that the semigroup $T(t)$ is not strongly continuous on $(C_{b,2}(K), \| \cdot \|_{b,2})$. For a detailed study of the semigroup $T(t)$, defined in (2.11), in spaces of continuous functions with weighted sup-norms, we refer to [12, 13], [22, Sect. 2.8.3] and [32, Sect. 2]. Instead the semigroup $T(t)$ is strongly continuous on $C_{b,2}(K)$ with respect the mixed topology. For an in-depth study of the mixed topology we refer to [44]; in the following theorems we list some properties that will be used.
Theorem 2.7  
(i) A sequence \( \{\phi_n\}_{n \in \mathbb{N}} \subseteq \text{Cb}_{1,2}(\mathcal{K}) \) converges with respect to the mixed topology to \( \phi \in \text{Cb}_{1,2}(\mathcal{K}) \) if, and only if,

\[
\sup_{n \in \mathbb{N}} \|\phi_n\|_{b,2} < +\infty,
\]

and, for any \( K \subseteq \mathcal{K} \) compact,

\[
\lim_{n \to +\infty} \sup_{x \in K} |\phi_n(x) - \phi(x)| = 0.
\]

(ii) The semigroup \( T(t) \) is strongly continuous on \( \text{Cb}_{1,2}(\mathcal{K}) \) with respect to the mixed topology.

We also state a characterisation for the uniform convergence on compacts, that will be useful next.

Proposition 2.8 A sequence \( \{\varphi_n\}_{n \in \mathbb{N}} \subseteq \text{Cb}(\mathcal{K}) \) is uniformly convergent on every compact subset of \( \mathcal{K} \) to a function \( \varphi \in \text{Cb}(\mathcal{K}) \) if, and only if, \( \{\varphi_n\}_{n \in \mathbb{N}} \) is pointwise convergent to \( \varphi \) and the family \( \{\varphi_n \mid n \in \mathbb{N}\} \) is equicontinuous, namely for any \( x_0 \in \mathcal{K} \) and \( \varepsilon > 0 \) there exists \( \delta := \delta(x_0, \varepsilon) > 0 \) such that, for any \( n \in \mathbb{N} \) and \( x \in \mathcal{K} \) with \( \|x - x_0\|_{\mathcal{K}} \leq \delta \) we have \( |\varphi_n(x) - \varphi_n(x_0)| \leq \varepsilon \).

Definition 2.9 We denote by \( (L_{b,2}, \text{Dom}(L_{b,2})) \) the infinitesimal generator, with respect to the mixed topology, of the semigroup \( T(t) \) in \( \text{Cb}_{1,2}(\mathcal{K}) \). We recall that

\[
\text{Dom}(L_{b,2}) := \left\{ \varphi \in \text{Cb}_{1,2}(\mathcal{K}) \mid \exists \lim_{t \to 0} \frac{T(t)\varphi - \varphi}{t} \text{ with respect to the mixed topology} \right\}.
\]

So by Theorem 2.7, we obtain the following characterisation of \( \text{Dom}(L_{b,2}) \).

Proposition 2.10 A function \( \varphi \in \text{Cb}_{1,2}(\mathcal{K}) \) belongs to \( \text{Dom}(L_{b,2}) \) if, and only if, there exists \( \psi \in \text{Cb}_{1,2}(\mathcal{K}) \) such that

(i) for any compact subset \( K \) of \( \mathcal{K} \),

\[
\lim_{t \to 0} \sup_{x \in K} \left( \frac{T(t)\varphi(x) - \varphi(x)}{t} - \psi(x) \right) = 0;
\]

(ii) \( \sup_{t \in (0,1]} [t^{-1} \|T(t)\varphi - \varphi\|_{b,2}] < +\infty \).

In this case \( L_{b,2}\varphi = \psi \).

Proposition 2.11 \( L_{b,2} \) is the closure in \( \text{Cb}_{1,2}(\mathcal{K}) \), endowed with the mixed topology, of the operator \( L_0 \) defined in (1.7).

We have also the following result about the resolvent of \( L_{b,2} \).
Proposition 2.12 For any $\lambda > 0$, $\varphi \in C_{b,2}(\mathcal{K})$ and $x \in \mathcal{K}$, the Riemann integral

$$J(\lambda)\varphi := \int_{0}^{+\infty} e^{-\lambda t} T(t)\varphi dt,$$

is well defined with respect to the mixed topology. Moreover, for every $\lambda > 0$, the operator

$$J(\lambda) : (C_{b,2}(\mathcal{K}), \tau_M) \to (C_{b,2}(\mathcal{K}), \tau_M)$$

is continuous (here $\tau_M$ denotes the mixed topology), and $J(\lambda) \varphi = R(\lambda, L_{b,2}) \varphi$.

We remark that, by [44, Remark 4.3], Theorem 2.7 and Proposition 2.12, the operator $L_{b,2}$ is the weak infinitesimal generator of the semigroup $T(t)$ on $C_{b,2}(\mathcal{K})$ in the sense of [12, 13]. By this fact we can use the following approximation result.

Proposition 2.13 (Propositions 2.5 and 2.6 of [32]) Let $\varphi \in \text{Dom}(L_{b,2}) \cap C_{b}^{1}(\mathcal{K})$. There exists a family $\{\varphi_{n_{1},n_{2},n_{3},n_{4}} \mid n_{1}, n_{2}, n_{3}, n_{4} \in \mathbb{N}\} \subseteq \xi_{A}(\mathcal{K})$ (see (1.5)) such that for every $x \in \mathcal{K}$

$$\lim_{n_{1} \to +\infty} \lim_{n_{2} \to +\infty} \lim_{n_{3} \to +\infty} \lim_{n_{4} \to +\infty} \varphi_{n_{1},n_{2},n_{3},n_{4}}(x) = \varphi(x);$$

$$\lim_{n_{1} \to +\infty} \lim_{n_{2} \to +\infty} \lim_{n_{3} \to +\infty} \lim_{n_{4} \to +\infty} \nabla \varphi_{n_{1},n_{2},n_{3},n_{4}}(x) = \nabla \varphi(x);$$

$$\lim_{n_{1} \to +\infty} \lim_{n_{2} \to +\infty} \lim_{n_{3} \to +\infty} \lim_{n_{4} \to +\infty} L_{b,2}\varphi_{n_{1},n_{2},n_{3},n_{4}}(x) = L_{b,2}\varphi(x).$$

Furthermore there exists a positive constant $C_{\varphi}$ such that, for any $n_{1}, n_{2}, n_{3}, n_{4} \in \mathbb{N}$ and $x \in \mathcal{K}$, it holds

$$|\varphi_{n_{1},n_{2},n_{3},n_{4}}(x)| + \nabla \varphi_{n_{1},n_{2},n_{3},n_{4}}(x)\|_{\mathcal{K}} + |L_{b,2}\varphi_{n_{1},n_{2},n_{3},n_{4}}(x)| \leq C_{\varphi} (1 + \|x\|^{2}_{\mathcal{K}}).$$

(2.12)

For a proof of the previous result we refer to [22, Sect. 2.8.3] or [32, Sect. 2]. See also [28, Sect. 8].

3 The SPDE (1.1)

In this section we will study the generalized mild solution $\{X(t, x)\}_{t \geq 0}$ of the SPDE (1.1) and the invariant measure of the transition semigroup $P(t)$ of (1.4). In Sect. 3.1 we will prove that, for any $x \in E$, the SPDE (1.1) has a unique mild solution. In Sect. 3.2 we will focus on the generalized mild solution of the SPDE (1.1) and on the transition semigroup (1.4). In Sect. 3.3 we will prove that the semigroup $P(t)$ has a unique invariant measure $\nu$ and we will investigate some properties of $\nu$.

We recall some useful inequalities that we will use frequently in this section.

$$ab \leq \frac{(q-1)(\epsilon a)^{q/(q-1)} + (b/\epsilon)^{q}}{q}, \quad \forall a, b, \epsilon > 0, \quad q > 1,$$

(3.1)
If $\mathcal{K}$ is a Banach space for every $h_1, h_2 \in \mathcal{K}$ and $r \geq 1$ it holds
\[
\|h_1 - h_2\|_{\mathcal{K}}^r \geq 2^{1-r} \|h_1\|_{\mathcal{K}}^r - \|h_2\|_{\mathcal{K}}^r.
\] (3.2)

### 3.1 The mild solution for $x$ belonging to $E$

We state the hypotheses under which we work in this subsection.

#### Hypotheses 3.1

(i) There exists a Banach space $E \subseteq \text{Dom}(F)$ densely and continuously embedded in $X$ such that $F(E) \subseteq E$.

(ii) $A$ generates a strongly continuous and analytic semigroup $e^{tA}$ on $X$ and $A_E$ (the part of $A$ in $E$) generates an analytic semigroup $e^{tA_E}$ on $E$.

(iii) There exists $\zeta \in \mathbb{R}$ such that
\begin{enumerate}[label=(a), ref=(a)]
  \item $A + F - \zeta I$ is dissipative in $X$;
  \item $A_E + F|_E - \zeta I$ is dissipative in $E$.
\end{enumerate}

(iv) $\{W_A(t)\}_{t \geq 0}$ is a $E$-valued continuous process.

(v) There exist $M > 0$ and $m \in \mathbb{N}$ such that
\[
\|F(x)\|_E \leq M (1 + \|x\|_E^m), \quad x \in E.
\]

(vi) For any $B$ bounded set of $E$ there exists a constant $L_B > 0$ such that
\[
\|F|_E(x) - F|_E(y)\|_E \leq L_B \|x - y\|_E, \quad x, y \in B.
\]

#### Remark 3.2

(1) Since $E$ is continuously embedded in $X$, by [45, Theorem 15.1] $E$ is a Borel set of $X$.

(2) Hypotheses 3.1(v) imply that $F|_E$ maps bounded sets of $E$ into bounded sets of $E$, and so, since $E$ is continuously embedded in $X$, $F$ maps bounded sets of $E$ into bounded sets of $X$.

(3) Hypotheses 3.1(vi) implies that $F|_E : E \to E$ is continuous. Moreover, since $E$ is continuously embedded in $X$, $F|_E : E \to X$ is continuous.

(4) In this abstract setting it is not meaningful to introduce the domain of $F$, since $\text{Dom}(F)$ has no topological structure. We introduce it for convenience in some definitions and in order to refer to the theory presented in [35, Sect. 7.2]. However in some specific cases $\text{Dom}(F)$ is a Banach space and it is possible to study the SPDE on it (see [22, Chapter 4]), for example if $\mathcal{X} = L^2([0, 1])$ then $\text{Dom}(F) = L^{2m}([0, 1])$.

#### Remark 3.3

Since $E$ is continuously embedded in $X$ and $\{W_A(t)\}_{t \geq 0}$ is $E$-valued stochastic process, by the definition of stochastic integral (see [35, Sects. 4.2 and 4.3]), we have
\[
\int_0^T \text{Tr}[e^{sA}Ce^{sA^*}]ds < +\infty, \quad T > 0.
\] (3.3)
For any $T > 0$, by (3.3) and [35, Theorem 5.2], the process $W_{A,T} := \{W_A(t)\}_{t \in [0,T]}$ can be seen as a $L^2([0, T], \lambda, \mathcal{X})$-valued gaussian random variable, where $\lambda$ is the Lebesgue measure. Moreover by Hypotheses 3.1(iv) and the same arguments used in [47, Remark 3.4], the process $W_{A,T}$ is a $C([0, T], E)$-valued gaussian random variable. Hence

$$\mathbb{E}\left[ \sup_{t \in [0,T]} \left\| W_A(t) \right\|_E^p \right] < +\infty, \quad \forall p \geq 1,$$

by Hypotheses 3.1(v) we have

$$\mathbb{E}\left[ \sup_{t \in [0,T]} \left\| F(W_A(t)) \right\|_E^p \right] < +\infty, \quad \forall p \geq 1,$$

and, since $E$ is continuously embedded in $\mathcal{X}$, we obtain

$$\mathbb{E}\left[ \sup_{t \in [0,T]} \left\| F(W_A(t)) \right\|_E^p + \sup_{t \in [0,T]} \left\| W_A(t) \right\|_E^p \right] < +\infty, \quad \forall p \geq 1.$$

We recall the standard definition of mild solution.

**Definition 3.4** For any $x \in E$ we call mild solution of (1.1) a $E$-valued process $\{X(t, x)\}_{t \geq 0}$ such that, for any $t \geq 0$, we have

$$X(t, x)(\omega) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s, x)(\omega))ds + W_A(t)(\omega), \quad \mathbb{P}\text{-a.s.}, \quad (3.4)$$

Moreover we say that the mild solution $\{X(t, x)\}_{t \geq 0}$ is unique if every process $\{Y(t, x)\}_{t \geq 0}$ that verifies (3.4) then $\{Y(t, x)\}_{t \geq 0}$ is a version of $\{X(t, x)\}_{t \geq 0}$.

To prove that, for any $x \in E$, the SPDE (1.1) has a unique mild solution $\{X(t, x)\}_{t \geq 0}$ we need to exploit an approximating problem. For simplicity, from here on we still denote by $A$ the part of $A$ in $E$. For any $x \in E$ and large $n \in \mathbb{N}$, we introduce the approximate problem

$$\begin{align*}
\begin{cases}
    dX_n(t, x) = (AX_n(t, x) + F(X_n(t, x)))dt + \sqrt{C}dW(t), \quad t > 0; \\
    X_n(0, x) = nR(n, A)x.
\end{cases}
\end{align*} \quad (3.5)$$

**Remark 3.5** By Hypotheses 3.1(ii), $e^{tA}$ verifies (2.5) with $\eta_0 \in \mathbb{R}$ and $M_0 > 0$. So $R(n, A)$ is defined only for $n > \eta_0$, hence if $\eta_0 > 1$, then we consider (3.5) for $n > \eta_0$.

Now we are going to prove that, for any $x \in E$ and large $n \in \mathbb{N}$ the SPDE (3.5) has unique mild solution $\{X_n(t, x)\}_{t \geq 0} \in C_p([0, T], E)$, for any $p \geq 1$ and $T > 0$ (see Definition (2.2)). To do this we consider the equation

$$\begin{align*}
\begin{cases}
    \frac{dY_n(t, x)}{dt} = AY_n(t, x) + F(Y_n(t, x) + W_A(t)), \quad t > 0; \\
    Y_n(0, x) = nR(n, A)x.
\end{cases}
\end{align*} \quad (3.6)$$
If we show that, for any \( x \in E \) and large \( n \in \mathbb{N} \), equation (3.6) has a unique mild solution \( \{Y_n(t, x)\}_{t \geq 0} \in C_p([0, T], E) \), for any \( p \geq 1 \) and \( T > 0 \) (see Definition (2.2)), then, by Remark 3.3, the process \( \{X_n(t, x)\}_{t \geq 0} \) defined by
\[
X_n(t, x) := Y_n(t, x) + W_A(t), \quad \mathbb{P}\text{-a.s.,} \quad (3.7)
\]
is the unique mild solution of (3.5) in \( C_p([0, T], E) \), for any \( p \geq 1 \) and \( T > 0 \).

**Proposition 3.6** Assume that Hypotheses 3.1 hold true. For any \( x \in E \) and large \( n \in \mathbb{N} \) problem (3.6) has a unique mild solution \( \{Y_n(t, x)\}_{t \geq 0} \in C_p([0, T], E) \), for any \( p \geq 1 \) and \( T > 0 \). Moreover there exists a sequence of processes \( \{\{Y_{n,k}(t, x)\}_{t \geq 0}\}_{k \in \mathbb{N}} \) such that
\[
t \rightarrow Y_{n,k}(t, x) \in C^1([0, T], E) \cap C([0, T], \text{Dom}(A)), \quad \forall T > 0, \forall k \in \mathbb{N}, \quad \mathbb{P}\text{-a.s.}
\]

where
\[
o_{n,k}(t, x) = \frac{dY_{n,k}}{dt}(t, x) - AY_{n,k}(t, x) - F(Y_{n,k}(t, x) + W_A(t)), \quad \mathbb{P}\text{-a.s.} \quad (3.9)
\]

In addition for any \( p \geq 1 \) there exist \( C_p := C_p(\xi) > 0 \) and \( \kappa_p := \kappa_p(\xi) \in \mathbb{R} \) such that for any \( x \in E \), large \( n \in \mathbb{N} \) and \( t > 0 \)
\[
\|Y_n(t, x)\|_p^p \leq C_p \left( e^{\kappa_p t} \|x\|_p^p + \int_0^t e^{\kappa_p (t-s)} \|F(W_A(s))\|_p^p ds \right), \quad \mathbb{P}\text{-a.s.} \quad (3.10)
\]
\[
\|Y_n(t, x)\|_E^p \leq C_p \left( e^{\kappa_p t} \|x\|_E^p + \int_0^t e^{\kappa_p (t-s)} \|F(W_A(s))\|_E^p ds \right), \quad \mathbb{P}\text{-a.s.} \quad (3.11)
\]

**Proof** We prove the statements for a fixed large \( n \in \mathbb{N} \) and \( x \in E \). By Hypotheses 3.1(iv), the trajectories of the process \( \{W_A(t)\}_{t \geq 0} \) are continuous \( \mathbb{P}\text{-a.s.} \). In this proof we work pathwise, so we will denote by \( w_A(\cdot) \) a fixed trajectory of \( \{W_A(t)\}_{t \geq 0} \). We fix \( T > 0 \) and we consider the equation
\[
\begin{cases}
\frac{d y_n}{dt}(t, x) = Ay_n(t, x) + F(y_n(t, x) + w_A(t)), \quad t \in [0, T]; \\
y_n(0, x) = nR(n, A)x,
\end{cases} \quad (3.12)
\]
and the operator
\[
V(y)(t) := e^{tA}nR(n, A)x + \int_0^t e^{(t-s)A}F(y(s) + w_A(s))ds,
\]
y \( \in C([0, T], E) \), \( t \in [0, T] \).
Let $R > M_0 \|x\|_E \sup_{t \in [0, T]} e^{\eta_0 t}$. By (2.6), (2.9), Remark 3.5 and the local Lipschitzianity of $F$, for any $y, z \in C([0, T], E)$ such that $\|y\|_{C([0, T], E)} \leq R$, we have

$$
\|V(y)\| \leq M_0 \|x\|_E \sup_{t \in [0, T]} e^{\eta_0 t} + M_0 \sup_{t \in [0, T]} \|F(y(t) + w_A(t))\|_E \sup_{t \in [0, T]} \int_0^t e^{(t-s)\eta_0} ds \\
\|V(y) - V(z)\|_{C([0, T], E)} \leq L_R M_0 \|y - z\|_{C([0, T], E)} \sup_{t \in [0, T]} \int_0^t e^{(t-s)\eta_0} ds.
$$

where $M_0$ and $\eta_0$ are the constants in Remark 3.5 and $L_R > 0$ is the Lipschitz constant of $F$ on the ball in $C([0, T], E)$ with center 0 and radius $R$. By Remark (3.2) for $T_0 \in [0, T]$ small enough $V(B(0, R)) \subseteq B(0, R)$ and $V$ is a contraction in $B(0, R)$ where $B(0, R)$ is the ball in $C([0, T_0], E)$ with center 0 and radius $R$. Hence by the contraction mapping theorem the problem (3.12) has a unique mild solution $y_{n,T_0}(\cdot, x) \in B(0, R)$. To prove that there exists a global solution $y_{n,T}$ of (3.12) in $C([0, T], E)$ it is sufficient to prove an estimate for $\|y_{n,T_0}(\cdot, x)\|_{C([0, T_0], E)}$ independent of $T_0$. By [46, Proposition 4.1.8] $y_{n,T_0}(\cdot, x)$ is the strong solution of

$$
\begin{cases}
\frac{dv_n}{dt}(t, x) = Av_n(t, x) + F(y_{n,T_0}(t, x) + w_A(t)), t \in [0, T_0]; \\
v_n(0, x) = n R(n, A)x,
\end{cases}
$$

namely there exists a sequence $\{y_{n,k,T_0}(\cdot, x)\}_{k \in \mathbb{N}} \subseteq C^1([0, T_0], E) \cap C([0, T_0], \text{Dom}(A))$ such that

$$
\begin{align*}
\lim_{k \to +\infty} \|y_{n,k,T_0}(\cdot, x) - y_{n,T_0}(\cdot, x)\|_{C([0, T_0], E)} &= 0, \\
\lim_{k \to +\infty} \left\| \frac{dy_{n,k,T_0}}{dt}(\cdot, x) - Ay_{n,k,T_0}(\cdot, x) - F(y_{n,T_0}(\cdot, x) + w_A(\cdot)) \right\|_{C([0, T_0], E)} &= 0.
\end{align*}
$$

(3.13)

For any $t \in [0, T_0]$, $x \in E$ and $n, k \in \mathbb{N}$ we set

$$
o_{n,k,T_0}(t, x) = \frac{dy_{n,k,T_0}}{dt}(t, x) - Ay_{n,k,T_0}(t, x) - F(y_{n,k,T_0}(t, x) + w_A(t)),
$$

hence we have

$$
\|o_{n,k,T_0}(t, x)\|_E \leq \left\| \frac{dy_{n,k,T_0}}{dt}(t, x) - Ay_{n,k,T_0}(t, x) - F(y_{n,T_0}(t, x) + w_A(t)) \right\|_E \\
+ \left\| F(y_{n,T_0}(t, x) + w_A(t)) - F(y_{n,k,T_0}(t, x) + w_A(t)) \right\|_E \\
\leq \left\| \frac{dy_{n,k,T_0}}{dt}(t, x) - Ay_{n,k,T_0}(t, x) - F(y_{n,T_0}(t, x) + w_A(t)) \right\|_E \\
+ L_R \|y_{n,T_0}(t, x) - y_{n,k,T_0}(t, x)\|_E,
$$

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and so, by (3.13), for any large \( n \in \mathbb{N} \) we obtain

\[
\lim_{k \to +\infty} \| o_{n,k,T_0}(x) \|_{C([0,T_0],E)} = 0, \quad \mathbb{P}\text{-a.s.}
\]

Let \( x \in E, \ p \geq 1, k, n \in \mathbb{N} \) and \( t \in [0, T_0] \). By (2.1)–(3.9) and Hypotheses 3.1(iii), there exists \( y^* \in \partial \| y_{n,k}(t, x) \|_E \), such that

\[
\frac{1}{p} \frac{d}{dt} \| y_{n,k,T_0}(t, x) \|^p_E \leq \| y_{n,k,T_0}(t, x) \|^{p-1}_E \left\{ \mathbb{E} \left[ A y_{n,k,T_0}(t, x), y^* \right] + \mathbb{E} \left[ \dot{A} y_{n,k,T_0}(t, x), \dot{y}^* \right] + \mathbb{E} \left[ \dot{A} y_{n,k,T_0}(t, x), \dot{y}^* \right] \right\} + \frac{1}{p} \| \dot{y}_{n,k,T_0}(t, x) \|^p_E.
\]

We claim that there exists \( C_1 := C_1(\xi, p) \) such that

\[
\frac{1}{p} \frac{d}{dt} \| y_{n,k,T_0}(t, x) \|^p_E \leq C_1 \| y_{n,k,T_0}(t, x) \|^p_E + \frac{1}{p} (\| F(w_A(t)) \|_E + \| o_{n,k,T_0}(t, x) \|_E)^p.
\]

Indeed for \( p = 1 \), (3.15) is verified with \( C_1 = \xi \), instead, for \( p > 1 \), applying (3.14) with \( a = \| y_{n,k,T_0}(t, x) \|^{p-1}_E, b = \left( \| F(w_A(t)) \|_E + \| o_{n,k,T_0}(t, x) \|_E \right), q = p \) and \( \epsilon = 1 \) we obtain

\[
\frac{1}{p} \frac{d}{dt} \| y_{n,k,T_0}(t, x) \|^p_E \leq (\xi + \frac{p-1}{p}) \| y_{n,k,T_0}(t, x) \|^p_E + \frac{1}{p} (\| F(w_A(t)) \|_E + \| o_{n,k,T_0}(t, x) \|_E)^p,
\]

and so (3.15) is verified with \( C_1 = \xi + \frac{p-1}{p} \). By (2.2), (2.5), Remark 3.5 and (3.15) we get

\[
\| y_{n,k,T_0}(t, x) \|^p_E \leq e^{pC_1 t} \| x \|^p_E + \int_0^t e^{pC_1 (t-s)} (\| F(w_A(t)) \|_E + \| o_{n,k,T_0}(t, x) \|_E)^p ds.
\]
and letting \( k \to +\infty \), by (3.8),

\[
\left\| y_{n,T_0}(t, x) \right\|_E^p \leq e^{pC_1 t} \| x \|_E^p + \int_0^t e^{pC_1(t-s)} \| F(w_A(t)) \|_E^p ds.
\] (3.16)

By Remark 3.3 and recalling that \( T_0 \in [0, T] \), for any \( t > 0 \) we obtain

\[
\left\| y_{n,T_0}(t, x) \right\|_E^p \leq \| x \|_E^p + \frac{1}{pC_1} (e^{pC_1 t} - 1) \sup_{t \in [0,T]} \| F(w_A(t)) \|_E^p.
\] (3.17)

and so there exists a global solution \( y_{n,T} \) of (3.12) in \( C([0, T], E) \). The uniqueness of \( y_{n,T} \) follows immediately by (3.17), the local lipschitzianity and the Gronwall inequalities of \( F \). We have proved that, for any \( T > 0 \) the equation (3.6), has unique mild solution \( y_{n,T} \in C([0, T], E) \). We consider the continuous function \( y_n(\cdot, x) : [0, +\infty) \to E \) defined by

\[
y_n(\cdot, x)[0,T] = y_{n,T}(\cdot, x), \quad \forall \, T > 0.
\]

Exploiting [46, Proposition 4.1.8] (as we have already done for \( y_{n,T_0} \)) for any \( T > 0 \), there exists a sequence \( \{y_{n,k,T_0}(\cdot, x)\}_{k \in \mathbb{N}} \subseteq C^1([0, T], E) \cap C([0, T], \text{Dom}(A)) \) such that

\[
\lim_{k \to +\infty} \left\| y_{n,k}(\cdot, x) - y_n(\cdot, x) \right\|_{C([0,T],E)} = 0,
\]

\[
\lim_{k \to +\infty} \left\| o_{n,k}(x) \right\|_{C([0,T],E)} = 0, \quad \forall \, \mathbb{P}\text{-a.s.}
\]

where

\[
o_{n,k}(t, x) = \frac{d y_{n,k}(t, x)}{dt} - Ay_{n,k}(t, x) - F(y_{n,k}(t, x) + w_A(t)), \quad \mathbb{P}\text{-a.s.}
\]

Moreover \( y_n(\cdot, x) \) verifies (3.16), for any \( p \geq 1 \) and \( t > 0 \). The process \( \{Y_n(t, x)\}_{t \geq 0} \) whose trajectories are the functions \( y(\cdot, x) \) verifies the statements of the proposition. Uniqueness follows by (3.16), local lipschitzianity of \( F \) and the Gronwall inequality. Estimates (3.10) follows in exactly the same way as (3.11) using the inner product of \( X \) instead of the duality product of \( E \) and \( E^* \). \( \square \)

**Remark 3.7** If \( e^{tA} \) is strongly continuous also on \( E \), then it is possible to replace the initial datum \( nR(n, A)x \) by \( x \), in (3.5).

By Remark 3.3, (3.2) (with \( h_1 = X_n(t, x), h_2 = W_A(t) \) and \( r = p \)) and Proposition 3.6 we obtain immediately the following result.

**Proposition 3.8** Assume that Hypotheses 3.1 hold true. For any large \( n \in \mathbb{N} \) and \( x \in E \) the process \( \{X_n(t, x)\}_{t \geq 0} \), defined in (3.7), is the unique mild solution of (3.5)
in $C_p([0, T], E)$, for any $p \geq 1$ and $T > 0$. In addition, for any $p \geq 1$, large $n \in \mathbb{N}$, $x \in E$ and $t > 0$, we have

$$
\|X_n(t, x)\|_E^p \leq C_p \left( e^{c_\text{p}^t} \|x\|_E^p + \int_0^t e^{c_\text{p}(t-s)} \|F(W_A(s))\|_E^p ds + \|W_A(t)\|_E^p \right), \ \mathbb{P}\text{-a.s.} \tag{3.18}
$$

$$
\|X_n(t, x)\|_E^p \leq C_p \left( e^{c_\text{p}^t} \|x\|_E^p + \int_0^t e^{c_\text{p}(t-s)} \|F(W_A(s))\|_E^p ds + \|W_A(t)\|_E^p \right), \ \mathbb{P}\text{-a.s.} \tag{3.19}
$$

where $C_p':=\max\left(2^{p-1}C_p, 2^{p-1}\right)$, and $C_p, \kappa_p$ are the constants of Proposition 3.6.

Now we prove a convergence result for $\{X_n(t, x)\}_{t \geq 0}$.

**Theorem 3.9** Assume that Hypotheses 3.1 hold true. For any $x \in E$, there exists $\{X(t, x)\}_{t \geq 0} \subseteq C_p((0, T], E) \cap C_p([0, T], X)$, for any $p \geq 1$ and $T > 0$, such that

$$
\lim_{n \to +\infty} \|X_n(\cdot, x) - X(\cdot, x)\|_{C([0,T],X)} = 0, \ \forall \ T > 0, \ \mathbb{P}\text{-a.s.} \tag{3.20}
$$

$$
\lim_{n \to \infty} \|X_n(\cdot, x) - X(\cdot, x)\|_{C([\varepsilon,T],E)} = 0, \ \forall \ 0 < \varepsilon \leq T, \ \mathbb{P}\text{-a.s.} \tag{3.21}
$$

For any $p \geq 1$, let $C_p'$ be the constant of Proposition 3.8 and let $\kappa_p$ be the constants of Proposition 3.6. For any $p \geq 1$, $x \in E$ and $t > 0$, we have

$$
\|X(t, x)\|_E^p \leq C_p' \left( e^{c_\text{p}^t} \|x\|_E^p + \int_0^t e^{c_\text{p}(t-s)} \|F(W_A(s))\|_E^p ds + \|W_A(t)\|_E^p \right), \ \mathbb{P}\text{-a.s.} \tag{3.22}
$$

$$
\|X(t, x)\|_E^p \leq C_p' \left( e^{c_\text{p}^t} \|x\|_E^p + \int_0^t e^{c_\text{p}(t-s)} \|F(W_A(s))\|_E^p ds + \|W_A(t)\|_E^p \right), \ \mathbb{P}\text{-a.s.} \tag{3.23}
$$

Moreover there exists a constant $\eta \in \mathbb{R}$ such that, for any $x, y \in E$ and $t > 0$, we have

$$
\|X(t, x) - X(t, y)\| \leq e^{\eta t} \|x - y\|, \ \mathbb{P}\text{-a.s.} \tag{3.24}
$$

$$
\|X(t, x) - X(t, y)\|_E \leq e^{\eta t} \|x - y\|_E, \ \mathbb{P}\text{-a.s.} \tag{3.25}
$$

**Proof** As in the proof of Proposition 3.6 we work pathwise, so we denote by $y_{n,k}(\cdot, x)$, $y_n(\cdot, x)$ and $w_A(\cdot)$ fixed trajectories of the process $\{Y_{n,k}(t, x)\}_{t \geq 0}$, $\{Y_n(t, x)\}_{t \geq 0}$ and $\{W_A(t)\}_{t \in [0,T]}$ respectively. We begin to prove (3.20) for a fixed $T > 0$. Let $x \in E$, $k, n \in \mathbb{N}$, $t \in [0, T]$. We define

$$
z_{n,k}(t, x) := y_{n,k}(t, x) + w_A(t), \ n, k \in \mathbb{N}.
$$

We stress that $z_{n,k}(t, x) - z_{m,k}(t, x) = y_{n,k}(t, x) - y_{m,k}(t, x)$, for any $n, m \in \mathbb{N}$. For any $n, m \in \mathbb{N}$, by (3.9), we have

$$
\frac{1}{2} \frac{d}{dt} \left\|z_{n,k}(t, x) - z_{m,k}(t, x)\right\|^2 \leq \left\{A(z_{n,k}(t, x) - z_{m,k}(t, x)), z_{n,k}(t, x) - z_{m,k}(t, x)\right\}
$$

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by Hypotheses 3.1(iii) we have
\[
\frac{1}{2} \frac{d}{dt} \left\| z_{n,k}(t, x) - z_{m,k}(t, x) \right\|^2 \\
\leq \zeta \left\| z_{n,k}(t, x) - z_{m,k}(t, x) \right\|^2 \\
+ \left\| o_{n,k}(t, x) - o_{m,k}(t, x) \right\| \left\| z_{n,k}(t, x) - z_{m,k}(t, x) \right\|
\]

By (3.1) (with \( \epsilon = 1 \) and \( q = 2 \)) we have
\[
\frac{1}{2} \frac{d}{dt} \left\| z_{n,k}(t, x) - z_{m,k}(t, x) \right\|^2 \\
\leq (\zeta + \frac{1}{2}) \left\| z_{n,k}(t, x) - z_{m,k}(t, x) \right\|^2 \\
+ \frac{1}{2} \left( \left\| o_{n,k}(t, x) \right\| + \left\| o_{m,k}(t, x) \right\| \right)^2.
\]

We set \( H_1 = \zeta + \frac{1}{2} \), by (2.2) we obtain
\[
\left\| z_{n,k}(t, x) - z_{m,k}(t, x) \right\|^2 \\
\leq e^{2H_1 t} \left\| (nR(n, A) - mR(m, A))x \right\| \\
+ \int_0^t e^{-2H_1 (t-s)} \left( \left\| o_{n,k}(t, x) \right\| + \left\| o_{m,k}(t, x) \right\| \right)^2 ds.
\]

Letting \( k \to +\infty \), by (3.8) and Remark 3.2 we have
\[
\left\| z_n(t, x) - z_m(t, x) \right\|^2 \\
\leq e^{2H_1 t} \left\| (nR(n, A) - mR(m, A))x \right\|.
\]

where
\[
z_n(t, x) = X_n(t, x)(w) = y_n(t, x) + w_A(t).
\]

By (2.8), we obtain that, for any \( T > 0 \) and \( x \in E \), the sequence \( \{z_n(\cdot, x)\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C([0, T], \mathcal{X}) \) and we denote by \( z_T(\cdot, x) \in C_b([0, T], \mathcal{X}) \) its limit. A continuous function \( z(\cdot, x) : [0, +\infty) \to \mathcal{X} \) such that
\[
z(\cdot, x)|_{[0,T]} := z_T(\cdot, x), \quad \forall \ T > 0,
\]
is well defined. So the process \( \{X(t, x)\}_{t \geq 0} \), whose trajectories are the functions \( z(\cdot, x) \), verifies (3.20). (3.18) and (3.20) yields (3.22) and, by Remark 3.3 and (3.22), we have \( \{X(t, x)\}_{t \geq 0} \in C_p([0, T], \mathcal{X}) \) for any \( p \geq 1 \) and \( T > 0 \).

Now we prove (3.21) for fixed \( \epsilon, T > 0 \). By (3.19), for any \( x \in E \), there exists \( R := R(x, T) > 0 \) such that for any large \( n \in \mathbb{N} \) and \( t \in [\epsilon, T] \) we have
\[
\left\| z_n(t, x) \right\|_E \leq R.
\]
Let $L := L(x, T) > 0$ be the Lipschitz constant of $F$ on $B_E(0, R)$. So, for any $x \in E$, $n, m \in \mathbb{N}$ and $t \in [\epsilon, T]$, by (2.7) Remark 3.5 and the local lipschitzianity of $F$ we have
\[
\|z_n(t, x) - z_m(t, x)\|_E \leq \left\| (nR(n, A) - mR(m, A))e^{tA}x \right\|_E + M_0L \int_0^t e^{t\eta_0} \|z_n(s, x) - z_m(s, x)\|_E ds.
\]
Hence, by the Gronwall inequality, there exists $K_2 := K_2(x, T) > 0$ such that
\[
\|z_n(t, x) - z_m(t, x)\|_E \leq K_2 \left\| (nR(n, A) - mR(m, A))e^{tA}x \right\|_E.
\] (3.27)

Letting $m, n \to +\infty$ in (3.27), by (2.8) we obtain that, for any $T > 0$, $\epsilon > 0$ and $x \in E$, the sequence $\{z_n(\cdot, x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $C([\epsilon, T], E)$ and, since $E$ is continuously embedded in $\mathcal{X}$, its limit is the same in $C([0, T], \mathcal{X})$. So the function defined in (3.26) is continuous from $(0, +\infty)$ to $E$ and the process $\{X(t, x)\}_{t \geq 0}$, which verifies (3.20), verifies also (3.21). (3.19) and (3.21) yields (3.23), and by Remark 3.3 and (3.23), we have $\{X(t, x)\}_{t \geq 0} \in C_p((0, T], E)$.

Now we prove (3.24). Let $T > 0$ and $x, y \in E$. For any, $t \in [0, T], k, n \in \mathbb{N}$, by (3.9) we have
\[
\frac{1}{2} \frac{d}{dt} \left\| z_{k,n}(t, x) - z_{k,n}(t, y) \right\|^2 \\
\leq \left\{ A(z_{k,n}(t, x) - z_{k,n}(t, y), z_{k,n}(t, x) - z_{k,n}(t, y)) \right\} \\
+ \left\{ F(z_{k,n}(t, x) + w_k(t)) - F(z_{k,n}(t, y) + w_k(t)), z_{k,n}(t, x) - y_{k,n}(t, y) \right\} \\
+ \left\{ o_{k,n}(t, x) - o_{k,n}(t, y), z_{k,n}(t, x) - z_{k,n}(t, y) \right\}
\]
and by Hypotheses 3.1(iii) we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\| z_{k,n}(t, x) - z_{k,n}(t, y) \right\|^2 \\
\leq \zeta \left\| z_{k,n}(t, x) - z_{k,n}(t, y) \right\|^2 \\
+ \left\| o_{k,n}(t, x) - o_{k,n}(t, y) \right\| \left\| z_{k,n}(t, x) - z_{k,n}(t, y) \right\|.
\]
By (3.1)(with $\epsilon = 1$ and $q = 2$) we have
\[
\frac{d}{dt} \left\| z_{k,n}(t, x) - z_{k,n}(t, y) \right\|^2 \\
\leq 2\eta \left\| z_{k,n}(t, x) - z_{k,n}(t, y) \right\|^2 \\
+ \frac{1}{2} \left\| o_{k,n}(t, x) - o_{k,n}(t, y) \right\|^2, \quad (3.28)
\]
where $\eta = \zeta + \frac{1}{2}$. By 2.2 and letting $k \to +\infty$ we obtain
\[
\left\| z_n(t, x) - z_n(t, y) \right\|^2 \leq e^{2\eta t} \left\| x - z \right\|^2.
\]
Letting \( n \to +\infty \), by (3.20) we obtain
\[
\|z(t, x) - z(t, y)\|^2 \leq e^{2\eta t} \|x - y\|^2, \quad t \in [0, T], \ x, y \in E
\]
for any \( T > 0 \) and for \( \mathbb{P}\)-a.a trajectory of \( \{X(t, x)\}_{t \geq 0} \), so (3.24) is verified. Finally (3.25) follows from (3.21) using similar arguments. \( \square \)

We make some remarks about possible variations of Theorem 3.9.

**Corollary 3.10** If the constant \( \zeta \) in Hypotheses 3.1(iii) is negative, then the constants \( \kappa_p \) and \( \eta \) are negative.

**Proof** Applying (3.1) with \( \epsilon = \zeta \) if \( \zeta \in (0, 1] \), or with \( \epsilon = 1/\zeta \) if \( \zeta > 1 \), we obtain that the constants \( C_1 \) of (3.15) and \( \eta \) of (3.28) are negative. \( \square \)

**Remark 3.11** It is possible to require that the part of \( A \) in \( E \) generates a strongly continuous semigroup instead of an analytic one, in this case we can take \( \epsilon = 0 \) in (3.21) and \( \{X(t, x)\}_{t \geq 0} \in C_p([0, T], E) \), for any \( p \geq 1 \) and \( T > 0 \).

Let \( x \in E \) and \( \{X(t, x)\}_{t \geq 0} \) be the process defined in Theorem 3.9. Now we prove that it is the unique mild solutions of (1.1).

**Theorem 3.12** Assume that Hypotheses 3.1 hold true. For any \( x \in E \), the process \( \{X(t, x)\}_{t \geq 0} \) is the unique mild solution of the SPDE (1.1) in \( C_p([0, T], X) \cap C_p((0, T], E) \), for any \( p \geq 1 \) and \( T > 0 \).

**Proof** We begin to prove uniqueness. Let \( x, y \in E \) and let
\[
\{X_1(t, x)\}_{t \geq 0}, \ \{X_2(t, x)\}_{t \geq 0} \in C_p((0, T], E), \quad \text{for any } p \geq 1 \text{ and } T > 0,
\]
be two mild solution of (1.1). For any \( 0 < t \leq T \), by Remark 3.5, we have
\[
\|X_1(t, x) - X_2(t, x)\|_E \leq M_0 \int_0^t e^{(t-s)\eta_0} \|F(X_1(t, x)) - F(X_2(t, x))\|_E ds, \quad \mathbb{P}\text{-a.s.}
\]
Since \( \{X_1(t, x)\}_{t \geq 0}, \ \{X_2(t, x)\}_{t \geq 0} \in C_p((0, T], E) \), with \( p \geq 1 \), then
\[
\sup_{t \in [0, T]} \|X_1(t, x)\|_E, \ \sup_{t \in [0, T]} \|X_2(t, x)\|_E < +\infty, \quad \mathbb{P}\text{-a.s.}
\]
so by the local lipschitzianity of \( F \), there exists \( L := L(x, T) > 0 \) such that
\[
\|X_1(t, x) - X_2(t, x)\|_E \leq M_0 L \int_0^t e^{(t-s)\eta_0} \|X_1(t, x) - X_2(t, x)\|_E ds, \quad \mathbb{P}\text{-a.s.}
\]
and by the Gronwall inequality we obtain
\[
X_1(t) = X_2(t), \quad \mathbb{P}\text{-a.s.}
\]
for any \( t \in [0, T] \) and \( T > 0 \), and so we have the uniqueness.
Now we prove that, for any \( x \in E \), the process \( \{X(t, x)\}_{t \geq 0} \) is the mild solution of (1.1). Let \( T > 0 \) and large \( n \in \mathbb{N} \). We recall that, for any \( t \in [0, T] \), we have

\[
X_n(t, x) := Y_n(t, x) + W_A(t), \quad \mathbb{P}\text{-a.s.}
\]

hence, by Proposition 3.6

\[
X_n(t, x) = e^{tA}nR(n, A)x + \int_0^t e^{(t-s)A}F(X_n(s, x))ds + W_A(t), \quad \mathbb{P}\text{-a.s. (3.29)}
\]

By (2.6), Remarks 3.2–3.3, (3.23), (3.21) and the dominated convergence theorem, we have

\[
\lim_{n \to +\infty} \left\| \int_0^t e^{(t-s)A} (F(X_n(s, x)) - F(X(s, x))) ds \right\|_E = 0, \quad \mathbb{P}\text{-a.s.,}
\]

so, letting \( n \to +\infty \) in (3.29), by (2.8) we have

\[
X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s, x))ds + W_A(t) \quad \mathbb{P}\text{-a.s.}
\]

for any \( t \in [0, T] \) and \( T > 0 \).

\[
\square
\]

### 3.2 Generalized mild solution and transition semigroup

Now we exploit the density of \( E \) in \( \mathcal{X} \) to define a process \( \{X(t, x)\}_{t \geq 0} \) for any \( x \in \mathcal{X} \).

**Proposition 3.13** Assume that Hypotheses 3.1 hold true. For any \( x \in \mathcal{X} \) there exists a unique process \( \{X(t, x)\}_{t \geq 0} \in C_p([0, T], \mathcal{X}) \), for any \( p \geq 1 \) and \( T > 0 \), such that

\[
\lim_{n \to +\infty} \left\| X(\cdot, x_n) - X(\cdot, x) \right\|_{C_p([0, T], \mathcal{X})} = 0, \quad \forall \ T > 0, \quad \mathbb{P}\text{-a.s.,}
\]

where \( \{x_n\}_{n \in \mathbb{N}} \subseteq E \) converges to \( x \) and \( \{X(t, x_n)\} \) is the unique mild solution of (1.1) with initial datum \( x_n \). In addition, for any \( p \geq 1, \ x, y \in \mathcal{X} \) and \( t > 0 \), we have

\[
\|X(t, x)\|^p \leq C_p^\prime \left( e^{\kappa_p t} \|x\|^p + \int_0^t e^{\kappa_p (t-s)} \left( \|F(W_A(s))\|^p + \|W_A(s)\|^p \right) ds + \|W_A(t)\|^p \right), \quad \mathbb{P}\text{-a.s., (3.31)}
\]

\[
\|X(t, x) - X(t, y)\| \leq e^{\eta t} \|x - y\|, \quad \mathbb{P}\text{-a.s., (3.32)}
\]

where \( \kappa_p \) is the constant of Proposition 3.6, \( C_p^\prime \) is the constant of Proposition 3.8 and \( \eta \) is the constant of Theorem 3.9. Moreover, for any \( x \in \mathcal{X}, \ p \geq 1 \) and \( T > 0 \) we have

\[
\lim_{n \to +\infty} \left\| \{X(t, x_n)\}_{t \geq 0} - \{X(t, x)\}_{t \geq 0} \right\|_{C_p([0, T], \mathcal{X})} = 0.
\]

\[\square\]
Proof Since $E$ is dense in $\mathcal{X}$, for any $x \in \mathcal{X}$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq E$ such that

$$\lim_{m \to +\infty} \|x_m - x\| = 0.$$ 

We consider the sequence of mild solutions $\{(X(t, x_m))_{t \in [0, T]}\}_{m \in \mathbb{N}} \subseteq C_p([0, T], \mathcal{X})$, for any $p \geq 1$ and $T > 0$, given by Theorem 3.12. Hence we have

$$\{(X(t, x_n))_{t \geq 0}\}_{n \in \mathbb{N}} \subseteq C([0, T], \mathcal{X}), \quad \mathbb{P}\text{-a.s.}$$

Moreover by (3.24), for any $T > 0$ and $n_1, n_2 \in \mathbb{N}$, we have

$$\lim_{n_1, n_2 \to +\infty} \|X(\cdot, x_{n_1}) - X(\cdot, x_{n_2})\|_{C([0,T],\mathcal{X})} = 0, \quad \mathbb{P}\text{-a.s.}$$

So there exists a unique $\mathcal{X}$-valued continuous process $\{X(t, x)\}_{t \geq 0}$ (see Definition 2.2) that verifies (3.30). By (3.30) the process $\{X(t, x)\}_{t \geq 0}$ verifies (3.31), (3.32) and, by Remark 3.3, $\{X(t, x)\}_{t \geq 0} \in C_p([0, T], \mathcal{X})$, for any $p \geq 1$ and $T > 0$. Finally (3.32) yields (3.33).

Definition 3.14 For any $x \in \mathcal{X}$ we call generalized mild solution of (1.1) the limit $\{X(t, x)\}_{t \geq 0}$ of Corollary 3.13. Until now we have shown that

1. for any $x \in E$ the SPDE (1.1) has a unique mild solution $\{X(t, x)\}_{t \geq 0} \in C_p([0, T], E) \cap C_p([0, T], \mathcal{X})$, for any $p \geq 1$ and for any $T > 0$, in the sense of Definition 3.4;
2. for any $x \in \mathcal{X}$ the SPDE (1.1) has a unique generalized mild solution $\{X(t, x)\}_{t \geq 0} \in C_p([0, T], \mathcal{X})$, for any $p \geq 1$ and for any $T > 0$, in the sense of Definition 3.14, in particular if $x \in E$ then the generalized mild solution of (1.1) is the mild solution of (1.1).

So we can define the following families of operators.

Definition 3.15 For every $t > 0$ we set

$$P(t)\varphi(x) := \mathbb{E}[\varphi(X(t, x))] = \int_{\Omega} \varphi(X(t, x)(\omega))\mathbb{P}(d\omega) \quad \varphi \in B_b(\mathcal{X}), \quad x \in \mathcal{X},$$

where $\{X(t, x)\}_{t \geq 0}$ is the unique generalized mild solution of (1.1). Similarly we set

$$P^E(t)\varphi(x) := \mathbb{E}[\varphi(X(t, x))] = \int_{\Omega} \varphi(X(t, x)(\omega))\mathbb{P}(d\omega) \quad \varphi \in B_b(E), \quad x \in E,$$

where $\{X(t, x)\}_{t \geq 0}$ is the unique mild solution of (1.1).

By the same arguments of [35][Proposition 9.14 and Corollary 9.15] and taking into account (3.25) and (3.32), we have the following result.
Proposition 3.16 \{P(t)\}_{t \geq 0} and \{P^E(t)\}_{t \geq 0} are two contraction, positive and Feller semigroups on $B_b(X)$ and $B_b(E)$ respectively.

3.3 Existence and uniqueness of the invariant measure

In this subsection we are going to prove that the semigroup $P(t)$ has a unique invariant measure $\nu$ verifying some useful properties. To do this we need an additional hypothesis.

Hypotheses 3.17 Assume that Hypotheses 3.1 hold true. Moreover we assume that the constant $\zeta$ in Hypotheses 3.1(iii) is negative and that

$$\sup_{t \geq 0} \mathbb{E}[\|W_A(t)\|^p] < +\infty, \quad \forall \ p \geq 1.$$  \hfill (3.34)

By Hypotheses 3.1(v) and (3.17)(3.34) we have

$$\Sigma_{p,E} := \sup_{t \geq 0} \mathbb{E}[\|F(W_A(t))\|^p + \|W_A(t)\|^p_E] < +\infty, \quad \forall \ p \geq 1,$$

and, since $E$ is continuously embedded in $\mathcal{X}$, we have

$$\Sigma_{p,X} := \sup_{t \geq 0} \mathbb{E}[\|F(W_A(t))\|^p + \|W_A(t)\|^p] < +\infty, \quad \forall \ p \geq 1.$$

For any $p \geq 1$ we set

$$\Sigma_p := \max\{\Sigma_{p,X}, \Sigma_{p,E}\}.$$  

Hence by Corollary 3.10, (3.23) and (3.31) we obtain the following result.

Proposition 3.18 Assume that Hypotheses 3.17 hold true and let $\{X(t, x)\}_{t \geq 0}$ be the generalized mild solution of (1.1). If $x \in \mathcal{X}$ then $\{X(t, x)\}_{t \geq 0} \in \mathcal{X}^p([0, \infty))$, for any $p \geq 1$, if $x \in E$ then $\{X(t, x)\}_{t \geq 0} \in E^p([0, \infty))$, for any $p \geq 1$ (see Definition 2.2). In particular, for any $p \geq 1$, there exists $K_p := K_p(\Sigma_p, C'_p)$ (where $C'_p$ is the constant of Theorem 3.8), such that

$$\mathbb{E}[\|X(t, x)\|^p] \leq K_p(1 + e^{\kappa_p t} \|x\|^p), \quad \forall \ t > 0, \forall \ x \in \mathcal{X},$$

$$\mathbb{E}[\|X(t, x)\|^p_E] \leq K_p(1 + e^{\kappa_p t} \|x\|^p_E), \quad \forall \ t > 0, \forall \ x \in E.$$  \hfill (3.35)

where $\kappa_p < 0$ is the constant of Proposition 3.6 and Corollary (3.10).

By [38, Theorem 2.1] and Proposition 3.16, it is possible to associate to the semigroups $P(t)$ and $P^E(t)$ two Markov transition functions, so we can exploit the results contained in [35, Chapter 11].

\[ Springer \]
Theorem 3.19 Assume that Hypotheses 3.17 hold true. There exists \( \nu \in \mathcal{P}(E) \) such that it is the unique invariant measure (see Definition 2.5) of both semigroups \( P^E(t) \) and \( P(t) \). Moreover \( \nu(E) = 1 \) and it verifies the following properties,

\[
\int_{\mathcal{X}} \|x\|^p \nu(dx) < +\infty, \quad \forall \ p \geq 1,
\]

\[
\int_{E} \|x\|^p \nu(dx) < +\infty, \quad \forall \ p \geq 1.
\]

Moreover we have

\[
\lim_{t \to +\infty} P(t)\varphi(x) = \int_{\mathcal{X}} \varphi(y)\nu(dy), \quad \varphi \in C_b(\mathcal{X}), \ x \in \mathcal{X}, \quad (3.38)
\]

\[
\lim_{t \to +\infty} P(t)\varphi(x) = \int_{E} \varphi(y)\nu(dy), \quad \varphi \in C_b(E), \ x \in E. \quad (3.39)
\]

**Proof** Existence and uniqueness of the invariant measures \( \nu \) and \( \nu^E \) of \( P(t) \) and \( P^E(t) \) respectively follow by the same arguments of [35, Theorems 11.33-11.34]. However we write a sketch of the proof because it is useful to know how the invariant measure is constructed to prove (3.36), (3.37), (3.38) and (3.39).

Since \( P^E(t) \) is Feller, by [35, Propositions 11.1–11.4 and Remark 11.6], if there exists \( \nu^E \in \mathcal{P}(E) \) such that, for any \( x \in E, \mathcal{L}(X(t, x)) \) narrow (or weak) converges to \( \nu^E \), then \( \nu^E \) is the unique invariant measure of \( P^E(t) \). We recall that \( \mathcal{L}(X(t, x)) \) narrow (or weak) converges to \( \nu^E \) if, for any \( \varphi \in C_b(\mathcal{X}) \), we have

\[
\lim_{n \to +\infty} \left| \int_{E} \varphi(y)\mathcal{L}(X(t, x))(dy) - \int_{E} \varphi(y)\nu^E(dy) \right| = 0. \quad (3.40)
\]

To prove (3.40) we consider the SPDE (1.1) but with an arbitrary \( s \in \mathbb{R} \) as initial time. Let \( \{W(t)\}_{t \geq 0} \) be another \( \mathcal{X} \)-cylindrical Wiener process independent of \( \{W(t, x)\}_{t \geq 0} \). For any \( t \in \mathbb{R} \) we define the process

\[
\hat{W}(t) := \begin{cases} 
W(t) & t \geq 0 \\
W(-t) & t < 0
\end{cases}
\]

and the filtration \( \{\hat{F}_t\}_{t \geq 0} \), where \( \hat{F}_t \) is the \( \sigma \)-field generated by \( \{\hat{W}(s) : s \leq t\} \). For any \( s \in \mathbb{R} \) and \( x \in \mathcal{X} \), we consider the SPDE

\[
\begin{cases}
\d X(t, s, x) = (AX(t, s, x) + F(X(t, s, x))\d t + C\d \hat{W}(t), \quad t \geq s \\
X(s, s, x) = x,
\end{cases} \quad (3.41)
\]

We remark that the method used to prove Theorem 3.12 and to define the generalized mild solution (see Corollary 3.13) also works by replacing the initial time 0 by an arbitrary \( s \in \mathbb{R} \). Hence, for any \( x \in \mathcal{X} \) and \( s \in \mathbb{R} \), the SPDE (3.41) has a unique
generalized mild solution \( \{X(t, s, x)\}_{t \geq 0} \). Moreover, as in Proposition 3.18, for any \( p \geq 1 \) we have

\[
\mathbb{E}[\|X(t, s, x)\|_E^p] \leq K_p(1 + e^{\kappa_p(t-s)}\|x\|_E^p), \quad t \geq s, \ x \in \mathcal{X}
\]

(3.42)

\[
\mathbb{E}[\|X(t, s, x)\|_E^p] \leq K_p(1 + e^{\kappa_p(t-s)}\|x\|_E^p), \quad t \geq s, \ x \in E
\]

(3.43)

where \( \kappa_p \) is the constant of Proposition 3.6, \( \eta \) is the constant of Proposition 3.9 and \( K_p \) is the constant of Proposition 3.18. By Corollary 3.10 the constants \( \eta \) and \( \kappa_p \) are negative.

Now we prove that there exists a random variable \( \xi \in L^2((\Omega, \mathbb{P}, E) \), such that, for any \( x \in E \), we have

\[
\lim_{s \to +\infty} \mathbb{E}\left[\|X(0, -s, x) - \xi\|_E^2\right] = 0,
\]

(3.44)

and after we will prove that the law of \( \xi \) is the \( \nu^E \) that verifies (3.40). We can assume that \( \{X(t, s, x)\}_{t \geq s} \) is a strict solution of (3.41), otherwise we approximate it as in Proposition 3.6. For \( \mathbb{P}\text{-a.a.} \omega \in \Omega \), for any \( x \in E \), \( s \in \mathbb{R} \), \( t \geq s \) and \( h \in [s, t] \), by Hypotheses 3.17(iii), there exist \( z^* \in \partial(\|X(t, s, x)(\omega) - X(t, h, x)(\omega)\|_E) \) such that

\[
\frac{1}{2} \frac{d\|X(t, s, x) - X(t, h, x)\|_E}{dt} = E\left[ A(X(t, s, x), z^*) \right]_{E^*} + E\left[ F(X(t, s, x)) - F(X(t, h, x)), z^* \right]_{E^*}
\]

\[
\leq \zeta \|X(t, s, x)(\omega) - X(t, h, x)(\omega)\|_E.
\]

By 2.2, taking the square root and the expectation we obtain

\[
\mathbb{E}\left[\|X(t, s, x) - X(t, h, x)\|_E^2\right] \leq e^{4\zeta(t-h)}\mathbb{E}\left[\|X(h, s, x) - x\|_E^2\right]
\]

\[
\leq 2e^{4\zeta(t-h)}(\mathbb{E}\left[\|X(h, s, x)\|_E^2\right] + \|x\|_E^2),
\]

and so

\[
\mathbb{E}\left[\|X(t, s, x) - X(t, h, x)\|_E^2\right] \leq e^{4\zeta(t-h)}C_x,
\]

(3.45)

where \( C_x := 2 \sup_{r \geq s} \left( \mathbb{E}\left[\|X(r, s, x)\|_E^2\right] \right) + 2\|x\|_E^2 \) is finite by (3.42). For any \( x \in \mathcal{X} \), by Hypotheses 3.17 and (3.45), the family \( \{X(0, -t, x)\}_{t \geq 0} \) is Cauchy in \( L^2((\Omega, \mathbb{P}, E) \), namely

\[
\lim_{s, t \to +\infty} \mathbb{E}\left[\|X(0, -t, x) - X(0, -s, x)\|_E^2\right] = 0
\]
Since $L^2((\Omega, \mathbb{P}), E)$ is complete, then $\{X(0, -t, x)\}_{t \geq 0}$ converges in $L^2((\Omega, \mathbb{P}), E)$ and by (3.43) its limit does not depend on $x$, so (3.44) is verified. Let $\nu^E = \mathcal{L}(\xi)$, where $\xi$ is the random variable that verifies (3.44). We prove that it verifies (3.40). Since $\{W'(t)\}_{t \geq 0}$ and $\{W(t)\}_{t \geq 0}$ are two independent cylindrical Wiener processes; they have the same law, and so, for any $x \in E$ and $t \geq 0$, we have

$$\mathcal{L}(X(t, x)) = \mathcal{L}(X(0, -t, x)).$$

Let $\varphi \in C_b(E)$. For any $x \in \mathcal{X}$, $t \geq 0$, we have

$$\int_{\mathcal{X}} \varphi(y) p_t(x, dy) = \int_{\mathcal{X}} \varphi(y) \mathcal{L}(X(t, x))(dy) = \int_{\mathcal{X}} \varphi(y) \mathcal{L}(X(0, -t, x))(dy)$$

$$= \int_{\Omega} \varphi(X(0, -t, x)(\omega)) \mathbb{P}(d\omega).$$

Since $\varphi \in C_b(E)$, by (3.44) and the dominated convergence theorem we have

$$\lim_{t \to +\infty} \int_{\mathcal{X}} \varphi(y) p_t(x, dy) = \lim_{t \to +\infty} \int_{\Omega} \varphi(X(0, -t, x)(\omega)) \mathbb{P}(d\omega)$$

$$= \int_{\Omega} \varphi(\xi(\omega)) \mathbb{P}(d\omega) = \int_{\mathcal{X}} \varphi(y) \nu^E(dy),$$

hence (3.40) is verified and so the measure $\nu^E$ is the unique invariant measure of the transition semigroup $\mathbb{P}^E(t)$. (3.39) follows immediately by the definition of transition semigroup $\mathbb{P}^E(t)$ and (3.46). Now we prove (3.37). For $p \geq 1$ and $b > 0$ we have

$$\int_E \frac{\|y\|_E^p}{1 + b\|y\|_E^p} p_t(x, dy) \leq \int_E \|y\|_E^p p_t(x, dy) = \mathbb{E}[\|X(t, x)\|_E^p].$$

Then, by (3.35), (3.39) and the monotone convergence theorem, we conclude

$$\int_{\mathcal{X}} \|y\|_E^p \nu^E(dy) = \lim_{b \to 0} \lim_{t \to +\infty} \int_E \frac{\|y\|_E^p}{1 + b\|y\|_E^p} p_t(x, dy) < +\infty.$$

In the same way, we can prove that the semigroup $\mathbb{P}(t)$ has a unique invariant measure $\nu$ that verifies (3.36) and (3.38). By Hypotheses 3.1(i), [8, Lemma 2.1.1] and recalling that, for any $f : \mathcal{X} \to \mathbb{R}$ and $A, B \subset \mathbb{R}$, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, it easy to prove that $\mathcal{B}(\mathcal{X})$ is continuously embedded in $\mathcal{B}(E)$. We recall that $\mathcal{B}_b(\mathcal{X})$ and $\mathcal{B}_b(\mathcal{X})$ generate $\mathcal{B}(E)$ and $\mathcal{B}(\mathcal{X})$ respectively. Hence $\mathcal{B}(\mathcal{X}) \subseteq \mathcal{B}(E)$ and the measure

$$\nu'(\Gamma) = \nu^E(\Gamma \cap E), \quad \Gamma \in \mathcal{B}(\mathcal{X}),$$

is well defined. Moreover $\nu'$ is invariant for the semigroup $\mathbb{P}(t)$, and by uniqueness, $\nu = \nu'$ and the statements of the theorem are verified. □
**Remark 3.20** Let

\[ Q_t := \int_0^t e^{tA} Ce^{tA^*} ds, \quad \Gamma(t) = Q_t^{-1/2} e^{tA}. \]

If the following conditions hold true

1. for any \( t > 0 \) \( e^{tA}(X) \subset Q_t^{1/2}(X) \);
2. there exists \( \lambda > 0 \) such that \( \int_0^{+\infty} e^{-\lambda t} \| \Gamma(t) \|_{L(X)} dt < +\infty \);

then by [33, Theorem 8.5.2] the invariant measure \( \nu \) of \( P(t) \) is absolutely continuous with respect to the Gaussian measure \( \mu \sim N(0, Q_\infty) \).

**Remark 3.21** Theorem 3.19 implies that \( \nu(H) = 1 \), for any \( H \subseteq X \) that verifies Hypotheses 3.17 (see the example of Sect. 6.3).

**Remark 3.22** In some specific settings it is possible to prove Theorem 3.19 replacing the condition \( \zeta < 0 \) with some other hypotheses on \( F \) (e.g. [14, Chapter 8]).

### 4 Behavior in \( L^2(X, \nu) \)

First of all we show that the transition semigroup \( P(t) \) is uniquely extendable to a strongly continuous and contraction semigroup in \( L^2(X, \nu) \). By the Hölder inequality and the invariance of \( \nu \), for any \( \varphi \in B_b(X) \) we have

\[ \| P(t)\varphi \|^2_{L^2(X, \nu)} = \int_X |P(t)\varphi|^2 d\nu \leq \int_X |P(t)\varphi|^2 d\nu = \int_X |\varphi|^2 d\nu = \| \varphi \|^p_{L^2(X, \nu)}. \]  

We recall that \( C_b(X) \) is dense in \( L^2(X, \nu) \). Observe that if \( \{ \varphi_n \}_{n \in \mathbb{N}} \subseteq C_b(X) \) converges to \( \varphi \) in \( L^2(X, \nu) \), then for any \( t \geq 0 \) the sequence \( \{ P(t)\varphi_n \}_{n \in \mathbb{N}} \) is Cauchy in \( L^2(X, \nu) \). Indeed, by (4.1), we have

\[ \| P(t)\varphi_n - P(t)\varphi_m \|^2_{L^2(X, \nu)} \leq \| \varphi_n - \varphi_m \|^2_{L^2(X, \nu)}. \]

Hence the transition semigroup \( P(t) \) is uniquely extendable to a strongly continuous and contraction semigroup \( P_2(t) \) in \( L^2(X, \nu) \).

**Definition 4.1** We denote by \( N_2 \) the infinitesimal generator of \( P_2(t) \).

**Remark 4.2** In a similar way, it is possible to prove that the semigroup \( P^E(t) \) is uniquely extendable to a strongly continuous semigroup \( P_2^E(t) \) in \( L^2(E, \nu) \). Moreover, by Theorem 3.19 \( \nu(E) = 1 \), and so \( L^2(X, \nu) = L^2(E, \nu) \). A result analogous to Theorem 1.1 for \( P_2^E(t) \) is verified (see [17] for the multiplicative case).

In this section we are going to prove Theorem 1.1. As a first step in the next subsection we will study the behavior of \( N_2 \) on the set

\[ \xi_A(X) := \text{span}\{\text{real and imaginary parts of the functions } x \mapsto e^{i\langle x, h \rangle} \mid h \in \text{Dom}(A^*)\}. \]
4.1 Behavior on $\xi_A(\mathcal{X})$

We recall the definition of $N_0$.

$$N_0\varphi(x) := \frac{1}{2} \text{Tr}[C \nabla^2 \varphi(x)] + \langle x, A^* \nabla \varphi(x) \rangle + \langle F_0(x), \nabla \varphi(x) \rangle, \quad \varphi \in \xi_A(\mathcal{X}), \ x \in \mathcal{X},$$

where

$$F_0(x) = \begin{cases} F(x) & x \in E, \\ 0 & x \in \mathcal{X}\setminus E. \end{cases}$$

Let $\varphi \in \xi_A(\mathcal{X})$, then there exist $m, n \in \mathbb{N}, a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{R}$ and $h_1, \ldots, h_m, k_1, \ldots, k_n \in A^*$ such that

$$\varphi(x) = \sum_{i=1}^{m} a_i \sin(\langle x, h_i \rangle) + \sum_{j=1}^{n} b_j \cos(\langle x, k_j \rangle).$$

Easy computations give for $x \in \mathcal{X}$

$$N_0\varphi(x) = \sum_{i=1}^{m} a_i \left( \langle x, Ah_i \rangle + \langle F_0(x), h_i \rangle - \frac{1}{2} \| C^{1/2} h_i \| \right) \sin(\langle x, h_i \rangle)$$

$$+ \sum_{j=1}^{n} b_j \left( \langle x, Ak_j \rangle + \langle F_0(x), k_j \rangle - \frac{1}{2} \| C^{1/2} k_j \| \right) \cos(\langle x, k_j \rangle),$$

moreover by Hypotheses 3.1(v) and Theorem 3.19 we have

$$\int_{\mathcal{X}} \| F_0(x) \|^p d\nu(x) < +\infty, \quad \forall \ p \geq 1, \quad (4.2)$$

and so $N_0\varphi$ belongs to $L^2(\mathcal{X}, \nu)$.

**Proposition 4.3** Assume that Hypotheses 3.17 hold true. $N_0$ is closable in $L^2(\mathcal{X}, \nu)$ and its closure $\overline{N}_0$ is dissipative in $L^2(\mathcal{X}, \nu)$. Moreover $N_2$ is an extension of $\overline{N}_0$, namely $\text{Dom}(\overline{N}_0) \subseteq \text{Dom}(N_2)$ and

$$\overline{N}_0\varphi = N_2\varphi, \quad \varphi \in \text{Dom}(\overline{N}_0). \quad (4.3)$$

**Proof** By Theorem 3.12, for any $x \in E$, the trajectories of $\{X(t, x)\}_{t \geq 0}$ take values in $E$. So by [22, Proof of Theorem 3.19], for any $\varphi \in \xi_A(\mathcal{X})$ and $x \in E$, we have

$$P_2(t)\varphi(x) = \mathbb{E}[\varphi(X(t, x))] = \varphi(x) + \mathbb{E} \left[ \int_{0}^{t} N_0\varphi(X(s, x)) ds \right]$$

$$= \varphi(x) + \int_{0}^{t} P(s) N_0\varphi(X(s, x)) ds, \quad (4.4)$$
and so
\[
\lim_{t \to 0} \frac{P_2(t)\varphi(x) - \varphi(x)}{t} = N_0\varphi(x). \tag{4.5}
\]

To obtain (4.3) we need to prove that
\[
\lim_{t \to 0} \int_X \left| \frac{P_2(t)\varphi(x) - \varphi(x)}{t} - N_0\varphi(x) \right|^2 \nu(dx) = 0, \quad \forall \varphi \in \xi_A(\mathcal{X}). \tag{4.6}
\]

We recall the Vitali convergence theorem (see [43, Theorem 2.24]): (4.6) is verified if and only if the following three conditions are verified.

1. \(\left\{ \frac{P(t)\varphi - \varphi}{t} \right\}_{t \geq 0}\) converges in measure to \(N_0\varphi\) with respect to the measure \(\nu\).
2. For any \(\varepsilon > 0\) there exists \(\Gamma \in \mathcal{B}(\mathcal{X})\) such that \(\nu(\Gamma) < +\infty\) and
   \[
   \frac{1}{t^2} \int_{(\mathcal{X} - \Gamma)} |P(t)\varphi(x) - \varphi(x)|^2 \nu(dx) \leq \varepsilon \quad \forall \ t > 0.
   \]
3. For any \(\varepsilon > 0\) there exists \(\delta > 0\) such that whenever \(\Gamma \in \mathcal{B}(\mathcal{X})\) with \(\nu(\Gamma) < \delta\) we have
   \[
   \frac{1}{t^2} \int_{\Gamma} |P(t)\varphi(x) - \varphi(x)|^2 \nu(dx) \leq \varepsilon \quad \forall \ t > 0.
   \]

By (4.5) and \(\nu(E) = 1\), (1) is verified. Since \(\nu\) is a probability measure then (3) implies (2). We prove (3). We fix \(\varepsilon > 0\). Since \(N_0\varphi \in L^2(\mathcal{X}, \nu)\), there exists \(\delta > 0\) such that whenever \(\Gamma \in \mathcal{B}(\mathcal{X})\) with \(\nu(\Gamma) < \delta\), then
\[
\int_{\Gamma} |N_0\varphi(x)|^2 \nu(dx) < \varepsilon.
\]

Recalling that \(\nu(E) = 1\), by the Hölder inequality, the invariance of \(P(t)\) with respect to \(\nu\) and (4.4) we have
\[
\frac{1}{t^2} \int_{\Gamma} |P(t)\varphi(x) - \varphi(x)|^2 \nu(dx) = \frac{1}{t^2} \int_{\Gamma \cap E} |P(t)\varphi(x) - \varphi(x)|^2 \nu(dx)
\]
\[
= \int_{\Gamma \cap E} \left| \int_0^t P(s)N_0\varphi(x) \frac{ds}{t} \right|^2 \nu(dx)
\]
\[
\leq \frac{1}{t} \int_0^t \left( \int_{\Gamma \cap E} |P(s)(N_0\varphi)(x)|^2 \nu(dx) \right) ds
\]
\[
\leq \frac{1}{t} \int_0^t \left( \int_{\Gamma \cap E} |P(s)(|N_0\varphi|^2)(x)| \nu(dx) \right) ds
\]
\[
= \frac{1}{t} \int_0^t \left( \int_{\Gamma \cap E} |N_0\varphi(x)|^2 \nu(dx) \right) ds = \frac{1}{t} \int_0^t \varepsilon ds = \varepsilon.
\]
Hence, by the Vitali convergence theorem, we obtain (4.6) and so (4.3). In particular, since $\nu$ is the invariant measure of $P_2(t)$, for any $\varphi \in \xi_A(\mathcal{X})$, we have

$$\int_{\mathcal{X}} N_0 \varphi d\nu = \int_E N_2 \varphi d\nu = 0. \quad (4.7)$$

Let $\varphi \in \xi_A(\mathcal{X})$, by standard calculations we obtain

$$N_0 \varphi^2(x) = 2\varphi(x)N_0 \varphi(x) + \|C^{1/2}\nabla \varphi(x)\|^2. \quad (4.7)$$

Hence integrating with respect to $\nu$ and exploiting (4.7) we have

$$\int_{\mathcal{X}} (N_0 \varphi(x))\varphi(x) d\nu(x) = -\frac{1}{2} \int_{\mathcal{X}} \|C^{1/2}\nabla \varphi(x)\|^2 \nu(x), \quad \forall \varphi \in \xi_A(\mathcal{X}),$$

so $N_0$ is dissipative and, since $\xi_A(\mathcal{X})$ is dense in $L^2(\mathcal{X}, \nu)$, it is closable in $L^2(\mathcal{X}, \nu)$ and its closure $\overline{N_0}$ is dissipative in $L^2(\mathcal{X}, \nu)$. \hfill \qed

We conclude this subsection with a useful criterium to check whether a function $\varphi : \mathcal{X} \to \mathbb{R}$ belongs to $\text{Dom}(\overline{N_0})$.

**Lemma 4.4** Assume that Hypotheses 3.17 hold true. If $\varphi \in \text{Dom}(L_{b,2}) \cap C^1_b(\mathcal{X})$, then $\varphi \in \text{Dom}(\overline{N_0})$ and

$$\overline{N_0} \varphi(x) = L_{b,2} \varphi(x) + \langle F_0(x), \nabla \varphi(x) \rangle, \quad x \in \mathcal{X};$$

where $L_{b,2}$ is the operator introduced in Theorem 2.7.

**Proof** By Proposition 2.13 a family $\{\varphi_{n_1, n_2, n_3, n_4} | n_1, n_2, n_3, n_4 \in \mathbb{N}\} \subseteq \xi_A(\mathcal{X})$ exists such that, for any $x \in \mathcal{X}$,

$$\lim_{n_1 \to +\infty} \lim_{n_2 \to +\infty} \lim_{n_3 \to +\infty} \lim_{n_4 \to +\infty} \overline{N_0} \varphi_{n_1, n_2, n_3, n_4}(x) = L_{b,2} \varphi(x) + \langle F(x), \nabla \varphi(x) \rangle.$$

whenever $\varphi \in \text{Dom}(L_{b,2}) \cap C^1_b(\mathcal{X})$. By (2.12), there exists a constant $C_\varphi$, such that for any $x \in E$

$$|\overline{N_0} \varphi_{n_1, n_2, n_3, n_4}(x)| = |N_0 \varphi_{n_1, n_2, n_3, n_4}(x)| \leq C_\varphi (1 + \|x\|^{m+2})(1 + \|F(x)\|^2),$$

so, since $\nu(E) = 1$, by (4.2), (3.36) and the Dominated Convergence theorem we obtain the statement. \hfill \qed

Before proving Theorem 1.1 we need to introduce an additional hypotheses which will allow us to use the regularizing sequence of $F$ defined in the next subsection.

**Hypotheses 4.5** Assume that Hypotheses 3.17 hold true and that there exists a constant $\zeta_F \in \mathbb{R}$ such that $F - \zeta_F I_{\mathcal{X}} : \text{Dom}(F) \subset \mathcal{X} \to \mathcal{X}$ and $F|_E - \zeta_F I_E : E \to E$ are $m$-dissipative.
4.2 A regularizing family for $F$

For any $\delta > 0$ and $x \in \mathcal{X}$, let $J_\delta(x) \in \text{Dom}(F)$ be the unique solution of

$$y - \delta(F(y) - \zeta_F y) = x.$$  \hfill (4.8)

The existence of $J_\delta(x)$, for every $x \in \mathcal{X}$ and $\delta > 0$, is guaranteed by the m-dissipativity of $F$. We define $F_\delta : \mathcal{X} \rightarrow \mathcal{X}$ as

$$F_\delta(x) := F(J_\delta(x)), \quad x \in \mathcal{X}, \delta > 0.$$  

**Lemma 4.6** Assume that Hypotheses 4.5 hold true, then the following propositions are verified.

$$\lim_{\delta \rightarrow 0} \|J_\delta(x) - x\|_{\mathcal{X}} = 0, \quad x \in \text{Dom}(F);$$  \hfill (4.9)

$$\lim_{\delta \rightarrow 0} \|J_\delta(x) - x\|_{E} = 0, \quad x \in E.$$  \hfill (4.10)

For any $0 < \delta < |\zeta_F|^{-1}$ (or $0 < \delta < +\infty$ if $\zeta_F = 0$), the function $F_\delta - \zeta_F I_{\mathcal{X}}$ is dissipative on both $X$ and $E$. Moreover for any $\delta > 0$ it hold

$$\|J_\delta(x) - x\|_{\mathcal{X}} \leq \delta \|F(x) - \zeta_F x\|_{\mathcal{X}}, \quad x \in \text{Dom}(F);$$  \hfill (4.11)

$$\|J_\delta(x) - x\|_{E} \leq \delta \|F(x) - \zeta_F x\|_{E}, \quad x \in E;$$  \hfill (4.11)

$$\|(F_\delta)(x)\|_{\mathcal{X}} \leq (3 + \delta |\zeta_F|)(\|F(x)\|_{\mathcal{X}} + \|x\|_{\mathcal{X}}), \quad x \in \text{Dom}(F);$$  \hfill (4.12)

$$\|(F_\delta)(x)\|_{E} \leq (3 + \delta |\zeta_F|)(\|F(x)\|_{E} + \|x\|_{E}), \quad x \in E;$$  \hfill (4.12)

$$\|F_\delta(x_1) - F_\delta(x_2)\|_{\mathcal{X}} \leq \left(\frac{2}{\delta} + |\zeta_F|\right)\|x_1 - x_2\|_{\mathcal{X}}, \quad x_1, x_2 \in \mathcal{X};$$  \hfill (4.13)

$$\|F_\delta(x_1) - F_\delta(x_2)\|_{E} \leq \left(\frac{2}{\delta} + |\zeta_F|\right)\|x_1 - x_2\|_{E}, \quad x_1, x_2 \in E;$$  \hfill (4.13)

**Proof** We prove the statements on $E$, the statements in $\mathcal{X}$ follow by the same arguments. We apply [33, Proposition 5.5.3] to the functions $G : E \rightarrow E$

$$G(x) := F(x) - \zeta_F x, \quad x \in E,$$  \hfill (4.14)

in this proof, in order not to burden the notation, we still denote by $F$ the restriction of $F$ to $E$. Throughout the proof we let $G_\delta(x) := G(J_\delta(x))$ for any $x \in E$ and $\delta > 0$, where $J_\delta(x)$ is defined in (4.8). We remark that (4.9) follows by [33, Proposition 5.5.3(iii)], while (4.11) follows by [14, Proposition A.2.2(4)].

Now we show that $F_\delta - \zeta_F I_E$ is dissipative in $E$. Let $\alpha > 0$, $\delta < |\zeta_F|^{-1}$ and $x_1, x_2 \in E$, by (4.8) and [33, Proposition 5.5.3(ii)] we have

$$\|x_1 - x_2 - \alpha[F_\delta(x_1) - \zeta_F x_1 - F_\delta(x_2) + \zeta_F x_2]\|_{E}$$

$$= \|x_1 - x_2 - \alpha[F_\delta(x_1) - \zeta_F [J_\delta(x_1) - \delta G_\delta(x_1)] - F_\delta(x_2) + \zeta_F [J_\delta(x_2) - \delta G_\delta(x_2)]]\|_{E}$$

$$\leq \|x_1 - x_2 - \alpha[F_\delta(x_1) - \zeta_F [J_\delta(x_1) - \delta G_\delta(x_1)] - F_\delta(x_2) + \zeta_F [J_\delta(x_2) - \delta G_\delta(x_2)]]\|_{E}$$

$$\leq \|x_1 - x_2 - \alpha[F_\delta(x_1) - \zeta_F [J_\delta(x_1) - \delta G_\delta(x_1)] - F_\delta(x_2) + \zeta_F [J_\delta(x_2) - \delta G_\delta(x_2)]])\|_{E}$$
and so $F_\delta - \xi F I_E$ is dissipative on $E$.

Now we show (4.13). By [33, Proposition 5.5.3(i)-(ii)] and (4.14), for any $x_1, x_2 \in K$ and $\delta > 0$

$$\|F_\delta(x_1) - F_\delta(x_2)\|_E \leq \|G_\delta(x_1) - G_\delta(x_2)\|_E + |\xi F|\|J_\delta(x_1) - J_\delta(x_2)\|_E$$

$$\leq \left(\frac{2}{\delta} + |\xi F|\right)\|x_1 - x_2\|_E.$$ 

This concludes the proof of (4.13).

We conclude by proving (4.12). By [33, Proposition 5.5.3(ii)], (4.11), (4.14) for any $x \in E$ and $\delta > 0$ it holds

$$\|F_\delta(x)\|_E \leq \|F(x)\|_E + \|F_\delta(x) - F(x)\|_E$$

$$\leq \|F(x)\|_E + \|G_\delta(x) - G(x)\|_E + |\xi F|\|J_\delta(x) - x\|_E$$

$$\leq \|F(x)\|_E + \|G_\delta(x)\|_E + \|G(x)\|_E + \delta|\xi F|\|G(x)\|_E$$

$$\leq \|F(x)\|_E + (2 + \delta|\xi F|)\|G(y)\|_E$$

$$\leq (3 + \delta|\xi F|)(\|F(x)\| + |\xi F|\|x\|_E).$$

So (4.12) holds true. \qed

We introduce a further regularization. For every $\delta, s > 0$ and $x \in \mathcal{X}$, we define

$$F_{\delta,s}(x) := \int_{\mathcal{X}} F_\delta(y)\mathcal{N}(e^{-(s/2)Q^{-1}}x, Q_s)(dy),$$

where $Q_s = Q(1-e^{-sQ^{-1}})$ and $Q$ is a positive and trace class operator on $\mathcal{X}$. This type of regularization is classical and it is based on the Mehler formula for the Ornstein–Uhlenbeck semigroup. By standard calculations, for any $0 < \delta < |\xi F|^{-1}, s > 0$ and $x, z \in \mathcal{X}$, we have that $F_{\delta,s}(x)$ is Lipschitz continuous and

$$\langle F_{\delta,s}(x) - F_{\delta,s}(z), x - z \rangle_{\mathcal{X}} \leq \xi F\|x - z\|^2_{\mathcal{X}}.$$ 

For any $0 < \delta < |\xi F|^{-1}$ and $x \in \mathcal{X}$ we have

$$\lim_{s \to 0} \|F_{\delta,s}(x) - F_\delta(x)\|_{\mathcal{X}} = 0. \quad (4.15)$$

We stress that, for any $s > 0$, we have

$$e^{-(s/2)Q^{-1}}(\mathcal{X}) \subseteq Q_s^{1/2}(\mathcal{X}). \quad (4.16)$$
Indeed by the analyticity of $e^{-(s/2)Q^{-1}}$ and by [46, Proposition 2.1.1(i)] the range of $e^{-(s/2)Q^{-1}}$ is contained in the domain of $Q^{-k}$ for every $k \in \mathbb{N}$. So to prove (4.16) it is sufficient to prove that $I - e^{-sQ^{-1}}$ is invertible. Since $-Q^{-1}$ is negative, we have $\|e^{-sQ^{-1}}\|_{\mathcal{L}(\mathcal{X})} < 1$, and so $I - e^{-sQ^{-1}}$ is invertible. In particular $Q_{1/2}^1(\mathcal{X}) = Q_{1/2}^1(\mathcal{X})$ and so we get (4.16). Hence by the Cameron–Martin formula, for any $0 < \delta < |\xi_F|^{-1}$ and $s > 0$, we have

$$F_{\delta,s}(x) = \int_{\mathcal{X}} F(y) e^{\left(\frac{1}{2} e^{-sQ^{-1}} x, y\right)} + \frac{1}{2} \left\|Q^{-1/2} e^{-sQ^{-1}} x\right\|_{\mathcal{X}}^2 \mathcal{N}(0, Q_s)(dy);$$

and so $F_{\delta,s}$ is Gateaux differentiable.

**Proposition 4.7** Assume that Hypotheses 4.5 hold true. Then

$$\lim_{\delta \to 0} \lim_{s \to 0} \left\| F_{\delta,s} - F \right\|_{L^2(\mathcal{X}, \nu)}.$$

**Proof** By Hypotheses 3.1(i–vi) and (4.10) we have

$$\lim_{\delta \to 0} \left\| F_{\delta}(x) - F(x) \right\|_{\mathcal{X}}, \quad x \in E. \quad (4.17)$$

We recall that for any $0 < \delta < |\xi_F|^{-1}$, $F_{\delta}$ is Lipschitz continuous and, for any $s > 0$, $\mathcal{N}(e^{-sQ^{-1}} x, Q_s)(dy)$ is a Gaussian measure. Hence for any $0 < \delta < |\xi_F|^{-1}$ we have

$$\sup_{s \geq 0} \sup_{x \in \mathcal{X}} \left\| F_{\delta,s}(x) \right\|_{\mathcal{X}} < +\infty. \quad (4.18)$$

Finally recalling that $\nu(E) = 1$, by (4.2), (4.12), (4.15), (4.17), (4.18) and applying twice the dominated convergence theorem we obtain the statement. \(\square\)

### 4.3 Proof of Theorem 1.1

To prove the main results of this chapter we need a general result about closed operators. Since we were unable to find an appropriate reference in the literature we provide its proof.

**Proposition 4.8** Let $Y$ be a Banach space and let $B_1 : \text{Dom}(B_1) \subseteq Y \to Y$ and $B_2 : \text{Dom}(B_2) \subseteq Y \to Y$ be two, possibly unbounded, linear operators. If

(i) $B_1$ is an extension of $B_2$, namely $\text{Dom}(B_2) \subseteq \text{Dom}(B_1)$ and, for any $x \in \text{Dom}(B_2)$, it holds $B_2 x = B_1 x$;

(ii) there exists a dense subset $D$ of $Y$ such that, for some $\lambda > 0$, $R(\lambda, B_1)$ and $R(\lambda, B_2)$ are well defined, and $R(\lambda, B_1)(D) \subseteq \text{Dom}(B_2)$;

then $\text{Dom}(B_1) = \text{Dom}(B_2)$ and $B_1 = B_2$. 

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Proof For any $x \in D$

$$x = (I - B_1)R(\lambda, B_1)x = \lambda R(\lambda, B_1)x - B_1 R(\lambda, B_1)x.$$ 

By the fact that $R(\lambda, B_1)(D) \subseteq \text{Dom}(B_2)$ and that $B_1$ is an extension of $B_2$, it follows

$$x = \lambda R(\lambda, B_1)x - B_2 R(\lambda, B_1)x = (I - B_2)R(\lambda, B_1)x,$$

hence, for any $x \in D$, we have $R(\lambda, B_2)x = R(\lambda, B_1)x$. So by the density of $D$ in $Y$, for any $x \in Y$, we have shown that $R(\lambda, B_2)x = R(\lambda, B_1)x$. Recalling that the domain of an operator coincides with the range of its resolvent, we get the thesis. □

Proof of Theorem 1.1 By Proposition 4.8 to prove Theorem 1.1 it is sufficient to show that there exists a dense subset $D$ of $L^2(\mathcal{X}, \nu)$ such that

$$R(\lambda, N_2)(D) \subseteq \text{Dom}(\overline{N}_0). \quad (4.19)$$

We split the proof in two steps. In the first step we assume that $F$ is Gateaux differentiable and Lipschitz continuous, and we will show that we can take $C^1_b(\mathcal{X})$ as the set $D$. In the second step we will show that, in the general case, the set $(\lambda I_\mathcal{X} - N_2)(\text{Dom}(\overline{N}_0))$ is dense in $L^2(\mathcal{X}, \nu)$ and it can be chosen as the set $D$. Throughout the proof we let $X(t, x)$ be the mild solution of (1.1).

Step 1. Assume that $F$ is Gateaux differentiable and Lipschitz continuous. For $f \in C^1_b(\mathcal{X})$ and $\lambda > 0$, we consider the function $\varphi$ defined as

$$\varphi(x) := R(\lambda, N_2)f(x) = \int_0^{+\infty} e^{-\lambda s} P(s)f(x)ds, \quad x \in \mathcal{X}.$$ 

If we prove that $\varphi \in \text{Dom}(L_{b, 2}) \cap C^1_b(\mathcal{X})$, then by Lemma 4.4, we conclude that $\varphi \in \text{Dom}(\overline{N}_0)$ and

$$\overline{N}_0 \varphi = L_{b, 2} \varphi + \langle \nabla \varphi, F \rangle.$$ 

So (4.19) is verified with $D = C^1_b(\mathcal{X})$. Hence in this first step we have to prove that $\varphi \in \text{Dom}(L_{b, 2}) \cap C^1_b(\mathcal{X})$.

We want to show that $\varphi$ is Gateaux differentiable. We begin to prove that for any $h \in \mathcal{X}$ the following limit exists

$$\lim_{\delta \to 0} \frac{\varphi(x + \delta h) - \varphi(x)}{\delta} = \lim_{\delta \to 0} \frac{\int_0^{+\infty} e^{-\lambda s} \left( \int_{\Omega} (f(X(s, x + \delta h)) - f(X(s, x))) \mathbb{P}(d\omega) \right) ds}{\delta}.$$ 

Since $f \in C^2_b(\mathcal{X})$, by [35, Theorem 9.8] and by the chain rule we have

$$\lim_{\delta \to 0} \frac{1}{\delta} [f(X(t, x + \delta h)) - f(X(t, x))] = \left\langle \nabla f(X(t, x)), \mathcal{D}^G X(t, x)h \right\rangle.$$ \hspace{1cm} (4.21)
By [35, Theorem 9.8] for any $T > 0$ the map $x \mapsto X(\cdot, x)$ is Gateaux differentiable as a function from $\mathcal{X}$ to $\mathcal{X}^2([0, T])$, and its Gateaux derivative $\{D^G X(t, x)h\}_{t \geq 0}$ is the unique mild solution of

$$
\begin{aligned}
\frac{d}{dt} S_x(t, h) &= (A + D^G F(X(t, x))) S_x(t, h), \; t > 0; \\
S_x(0, h) &= h.
\end{aligned}
$$

(4.22)

By Hypotheses 3.1(iii-a) and an easy calculations, for any $x, h \in \mathcal{X}$, we have

$$
\left\langle A + D^G F(x)h, h \right\rangle \leq \xi \|h\|^2
$$

We assume that $\{D^G X(t, x)h\}_{t \geq 0}$ is the strict solution of (4.22), otherwise we can approximate as in Proposition 3.6. We scalarly multiply both members of (4.22) by $D^G X(t, x)h$

$$
\frac{1}{2} \frac{d}{dt} \|D^G X(t, x)h\|^2 = \left\langle \frac{d}{dt} D^G X(t, x)h, D^G X(t, x)h \right\rangle
= \left\langle (A + D^G F(X(t, x)))D^G X(t, x)h, D^G X(t, x)h \right\rangle
\leq \xi \|D^G X(t, x)h\|^2.
$$

Hence we obtain $\frac{d}{dt} \|D^G X(t, x)h\|^2 \leq 2\xi \|D^G X(t, x)h\|^2$, so by the Grönwall inequality, for any $t > 0$ and $x, h \in \mathcal{X}$, it holds

$$
\|D^G X(t, x)h\| \leq e^{\xi t} \|h\|,
$$

(4.23)

Furthermore for any $\delta \neq 0$ it holds

$$
\frac{1}{|\delta|} |f(X(t, x + \delta h)) - f(X(t, x))| = \frac{1}{|\delta|} \left| \int_0^\delta \left\langle \nabla f(X(t, x + sh)), D^G X(t, x + sh)h \right\rangle ds \right|
\leq e^{-\xi t} \|\nabla f\|_\infty \|h\|,
$$

so by (4.21) and the Domained convergence theorem we obtain (4.20) For any $x, h \in \mathcal{X}$ we set

$$
Lh := \lim_{\delta \to 0} \frac{1}{|\delta|} \varphi(x + \delta h) - \varphi(x),
$$

we prove that $L$ is bounded.
\[
\lim_{\delta \to 0} \frac{1}{|\delta|} \int_{0}^{+\infty} e^{-\lambda s} \mathbb{E} \left[ \left. \int_{0}^{\delta} \left( \nabla f(X(s, x + rh)), D^G X(s, x + rh) h \right) dr \right| \right] ds
\]

\[
\leq \| \nabla f \|_\infty \| h \| \int_{0}^{+\infty} e^{-(\lambda + \zeta)s} ds = \frac{1}{\lambda + \zeta} \| \nabla f \|_\infty \| h \|,
\]

so \( L \) is a linear and bounded operator and \( \varphi \) is Gateaux differentiable. Using [50, Fact 1.13(b), p. 8] it is also possible to prove that \( \varphi \) is Fréchet differentiable, and so

\[
\| \nabla \varphi \|_\infty \leq \frac{1}{\lambda + \zeta} \| \nabla f \|_\infty.
\]

Now we prove that \( \varphi \) belongs to \( \text{Dom}(L_{b, 2}) \), to do this we are going to check the conditions of Proposition 2.10. We begin to check (i) of Proposition 2.10. Let \( \{Z(t, x)\}_{t \geq 0} \) be the unique mild solution of (2.10) with \( F = 0 \), so for any \( t \geq 0 \) we have

\[
Z(t, x) = X(t, x) - \int_{0}^{t} e^{(t-s)A} F(X(s, x)) ds, \quad \mathbb{P}\text{-a.s.}
\]

where \( \{X(t, x)\}_{t \geq 0} \) is the unique mild solution of (1.1). Then, for every \( t \geq 0 \) and \( x \in \mathcal{X} \), we have

\[
\frac{T(t) \varphi(x) - \varphi(x)}{t} = \mathbb{E} [\varphi(Z(t, x)) - \varphi(x)]
\]

\[
= \frac{1}{t} \mathbb{E} \left[ \varphi \left( X(t, x) - \int_{0}^{t} e^{(t-s)A} F(X(s, x)) ds \right) - \varphi(x) \right].
\]

By the Taylor formula we have

\[
\varphi \left( X(t, x) - \int_{0}^{t} e^{(t-s)A} F(X(s, x)) ds \right) = \varphi(X(t, x))
\]

\[
- \left< \nabla \varphi(X(t, x)), \int_{0}^{t} e^{(t-s)A} F(X(s, x)) ds \right>
\]

\[
+ o \left( \mathbb{E} \left[ \left. \left\| \int_{0}^{t} e^{(t-s)A} F(X(s, x)) ds \right\| \right| \right] \right)
\]

so in the right hand side of (4.25), we obtain

\[
\frac{T(t) \varphi(x) - \varphi(x)}{t} = \frac{P(t) \varphi(x) - \varphi(x)}{t} - \frac{1}{t} \mathbb{E} \left[ \nabla \varphi(X(t, x)), \int_{0}^{t} e^{(t-s)A} F(X(s, x)) ds \right]
\]

\[
+ \frac{1}{t} o \left( \mathbb{E} \left[ \left. \left\| \int_{0}^{t} e^{(t-s)A} F(X(s, x)) ds \right\| \right| \right] \right),
\]

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hence for any \( x \in \mathcal{X} \), we have

\[
\lim_{t \to 0} \frac{1}{t} (T(t)\varphi(x) - \varphi(x)) = N_2\varphi(x) - (\nabla \varphi(x), F(x)).
\] (4.26)

Now let \( K \) be a compact subset of \( \mathcal{X} \). Since, in this step, we have assume that \( F \) is Lipschitz continuous, by (3.35), we get

\[
\lim_{t \to 0} \sup_{x \in K} \frac{1}{t} \left( \left\| \int_0^t e^{(t-s)A} F(X(s, x))ds \right\| \right) = 0.
\] (4.27)

We set for \( t > 0 \) and \( x \in \mathcal{X} \)

\[
\Delta_t(x) := \frac{P(t)\varphi(x) - \varphi(x)}{t}, \quad R_t(x) := \frac{1}{t} \mathbb{E} \left[ \int_0^t e^{(t-s)A} F(X(s, x))ds \right].
\]

We recall that for every \( t \geq 0 \)

\[
P(t)\varphi = P(t) \int_0^{+\infty} e^{-\lambda s} P(s) f ds = e^{\lambda t} \int_t^{+\infty} e^{-\lambda s} P(s) f ds.
\]

Let \( x_0 \in K \). Since \( f \in C_b^1(\mathcal{X}) \) we know that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(x) - f(x_0)| \leq \varepsilon \), whenever \( \|x - x_0\| \leq \delta \). Now let \( \|x - x_0\| \leq \delta \)

\[
|\Delta_t(x) - \Delta_t(x_0)| = \frac{1}{t} \left| P(t)\varphi(x) - \varphi(x) - P(t)\varphi(x_0) + \varphi(x_0) \right|
\]

\[
= \frac{1}{t} \left| e^{-\lambda t} \int_t^{+\infty} e^{-\lambda s} P(s) (f(x) - f(x_0)) ds \right|
\]

\[
+ \int_0^{+\infty} e^{-\lambda s} P(s) (f(x_0) - f(x)) ds
\]

\[
= \frac{1}{t} \left| e^{-\lambda t} - 1 \right| \int_t^{+\infty} e^{-\lambda s} P(s) (f(x) - f(x_0)) ds
\]

\[
+ \int_0^{t} e^{-\lambda s} P(s) (f(x_0) - f(x)) ds
\]

\[
\leq \frac{e^{-\lambda t} - 1}{t} \int_t^{+\infty} e^{-\lambda s} P(s) \varepsilon ds + 1/t \int_0^{t} e^{-\lambda s} P(s) \varepsilon ds
\]

\[
\leq \varepsilon \left( \frac{e^{-\lambda t} - 1}{t} \int_t^{+\infty} e^{-\lambda s} ds + 1/t \int_0^{t} e^{-\lambda s} ds \right)
\]

\[
= \varepsilon \frac{e^{-\lambda t} - e^{-\lambda t}}{\lambda t} \leq \varepsilon,
\]

and so

\[
|\Delta_t(x) - \Delta_t(x_0)| \leq \varepsilon.
\] (4.28)
We observe that by the Lipschitz continuity of \( F \) there exists \( C > 0 \) such that for every \( x \in \mathcal{X} \), it holds \( \| F(x) \| \leq C(1 + \| x \|) \). Furthermore by Corollary 3.10, (3.32) and (4.24), for every \( t > 0 \), the functions \( x \mapsto \nabla \varphi(X(t, x)) \) and \( x \mapsto F(X(t, x)) \) are continuous uniformly with respect to \( t \in [0, T] \). So for every \( t \in [0, T] \), \( x_0 \in K \) and \( \varepsilon > 0 \) there exists \( \delta := \delta(\varepsilon, x_0) > 0 \) such that whenever \( \| x - x_0 \| \leq \delta \) it holds

\[
\max \{ \| \nabla \varphi(X(t, x)) - \nabla \varphi(X(t, x_0)) \|, \| F(X(t, x)) - F(X(t, x_0)) \| \} \leq \varepsilon.
\]

By the Jensen inequality and (3.35) we can write

\[
|R_t(x) - R_t(x_0)| = \frac{1}{t} \mathbb{E}\left[ \left( \nabla \varphi(X(t, x)), \int_0^t e^{(t-s)A} F(X(s, x)) ds \right) \right] 
- \mathbb{E}\left[ \left( \nabla \varphi(X(t, x_0)), \int_0^t e^{(t-s)A} F(X(s, x_0)) ds \right) \right] 
= \frac{1}{t} \mathbb{E}\left[ \left( \nabla \varphi(X(t, x)) - \nabla \varphi(X(t, x_0)), \int_0^t e^{(t-s)A} F(X(s, x)) ds \right) \right] 
+ \mathbb{E}\left[ \left( \nabla \varphi(X(t, x_0)), \int_0^t e^{(t-s)A} (F(X(s, x)) - F(X(s, x_0))) ds \right) \right] 
\leq \frac{1}{t} \mathbb{E}\left[ \| \nabla \varphi(X(t, x)) - \nabla \varphi(X(t, x_0)) \| \int_0^t e^{(t-s)A} (F(X(s, x)) - F(X(s, x_0))) ds \right] 
+ \frac{1}{t} \mathbb{E}\left[ \| \nabla \varphi(X(t, x_0)) \| \int_0^t e^{(t-s)A} (F(X(s, x)) - F(X(s, x_0))) ds \right] 
\leq \frac{C \varepsilon}{t} \int_0^t \mathbb{E}[1 + \| X(s, x) \|] ds + \frac{\| \nabla \varphi \|_{\infty}}{t} \mathbb{E}\left[ \int_0^t F(X(s, x)) - F(X(s, x_0)) ds \right],
\]

so by (3.35) there exists a constant \( C_1 > 0 \) such that

\[
|R_t(x) - R_t(x_0)| \leq \varepsilon C_1 (1 + \| x \| + \| \nabla \varphi \|_{\infty}).
\] (4.29)

Hence by Proposition 2.8, (4.26), (4.27), (4.28) and (4.29) we have

\[
\limsup_{t \to 0} \sup_{x \in K} \left| \frac{1}{t} (T(t) \varphi(x) - \varphi(x)) - N_2 \varphi(x) + \langle \nabla \varphi(x), F(x) \rangle \right| = 0,
\]

so we have checked (i) of Proposition 2.10. Using similar arguments also condition (ii) of Proposition 2.10 is verified. By Proposition 2.10 and by the one proved before \( \varphi \in \text{Dom}(L_{b,2}) \cap C^1_b(\mathcal{X}) \), so by Lemma 4.4, we conclude that \( \varphi \in \text{Dom}(\overline{N}_0) \) and

\[
\overline{N}_0 \varphi = L_{b,2} \varphi + \langle \nabla \varphi, F \rangle.
\]
Step 2. Let \( \{F_{\delta,s} \mid \delta, s > 0\} \) be the regularizing family of \( F \) defined in Sect. 4.2. Let \( f \in C^1_b(X) \), for any \( \delta, s > 0 \), we set

\[
\phi_{\delta,s}(x) := \int_0^{+\infty} e^{-\lambda t} P_{\delta,s}(t) f(x) dt, \quad x \in X,
\]

where \( P_{\delta,s}(t) \) is the transition semigroup associated to

\[
\begin{cases}
  dX(t, x) = (AX(t, x) + F_{\delta,s}(X(t, x))) dt + \sqrt{C} dW(t), \; t > 0; \\
  X(0, x) = x \in X.
\end{cases}
\]

In Sect. 4.2 we have seen that, for any \( \delta, s > 0 \), the function \( F_{\delta,s} \) is Lipschitz continuous and \( F_{\delta,s} - \xi F \) is dissipative. Hence by Step 1, for any \( \delta, s > 0 \), we have that \( \phi_{\delta,s} \in \text{Dom}(\overline{N}_0) \) and

\[
\lambda\phi_{\delta,s} - L_{b,2}\phi_{\delta,s} - \langle \nabla \phi_{\delta,s}, F_{\delta,s} \rangle = f.
\]

So

\[
\lambda\phi_{\delta,s} - \overline{N}_0\phi_{\delta,s} = f + \langle \nabla \phi_{\delta,s}, F_{\delta,s} - F \rangle,
\]

and recalling that \( \overline{N}_2 \) is an extension of \( \overline{N}_0 \) in \( L^2(X, \nu) \)

\[
\lambda\phi_{\delta,s} - \overline{N}_2\phi_{\delta,s} = f + \langle \nabla \phi_{\delta,s}, F_{\delta,s} - F \rangle,
\]

where the equality holds in \( L^2(X, \nu) \). Hence noticing that \((4.24)\) does not depend on \( \delta \) and on \( s \) and by Proposition 4.7, we have that, for any \( f \in C^1_b(X) \), there exist a family \( \{\phi_{\delta,s} \mid \delta, s > 0\} \subseteq \text{Dom}(\overline{N}_0) \), such that

\[
\lim_{\delta \to 0} \lim_{s \to 0} (\lambda I_X - \overline{N}_2)\phi_{\delta,s} = f, \quad \text{in} \; L^2(X, \nu).
\]

By the density of \( C^1_b(X) \) in \( L^2(X, \nu) \) we get the density of \( (\lambda I_X - \overline{N}_2)(\text{Dom}(\overline{N}_0)) \) in \( L^2(X, \nu) \).

5 Dirichlet semigroup associated to a dissipative gradient systems

Assume that Hypotheses 4.5 hold true. Let \( \mathcal{O} \subseteq X \) be an open set such that \( \nu(\mathcal{O}) > 0 \). We consider the family of operators \( \{P^\mathcal{O}(t)\}_{t \geq 0} \), defined by

\[
P^\mathcal{O}(t)\varphi(x) := \mathbb{E} \left[ \varphi(X(t, x)) \mathbb{I}_{\{\tau_x > t\}} \right] := \int_{\Omega} \varphi(X(t, x)) \mathbb{I}_{\{\tau_x > t\}} d\mathbb{P}, \quad \varphi \in B_b(\mathcal{O}),
\]
where \( \{X(t, x)\}_{t \geq 0} \) is the generalized mild solution of the SPDE (1.1) and \( \tau_x \) is the exit time defined by

\[
\tau_x = \inf \{ t > 0 : X(t, x) \in \mathcal{O}^c \}. \tag{5.1}
\]

We recall that the generalized mild solution \( \{X(t, x)\}_{t \geq 0} \) is a strong Markov process ([35, Sect. 9.2]) and by its uniqueness we have

\[
X(t, X(s, x)) = X(t + s, x), \quad \mathbb{P}\text{-a.s..}
\]

By the two facts mentioned above the family of operators \( \{P^O(t)\}_{t \geq 0} \) is a semigroup and it is called Dirichlet (or stopped) semigroup associated to the SPDE (1.1). We refer to [36, Chapter 2] for a general overview about this type of semigroups.

**Proposition 5.1** Assume that Hypotheses 4.5 hold true. For any \( \varphi \in B_b(\mathcal{O}) \), \( t > 0 \), and \( p \geq 1 \) we have

\[
\int_{\mathcal{O}} |(P^O(t)\varphi)(x)|^p \nu(dx) \leq \int_{\mathcal{O}} |\varphi(x)|^p \nu(dx).
\]

**Proof** Let

\[
\tilde{\varphi}(x) = \begin{cases} 
\varphi(x) & x \in \mathcal{O}, \\
0 & x \in \mathcal{O}^c.
\end{cases}
\]

By the Hölder inequality, for any \( x \in \mathcal{X} \) we have

\[
|(P^O(t)\varphi)(x)|^p \leq \mathbb{E}[|\varphi(X(t, x))|^{\mathcal{P}\mathbb{I}_{[\tau_x \geq t]}}] \\
\leq \mathbb{E}[|\tilde{\varphi}(X(t, x))|^{\mathcal{P}\mathbb{I}_{[\tau_x \geq t]}}] = (P(t)|\tilde{\varphi}^p|(x).
\]

Since \( \nu \) is invariant for \( P(t) \) and \( P(t) \) is non-negative (see definition at beginning of Sect. 2.3), then we conclude

\[
\int_{\mathcal{O}} |(P^O(t)\varphi)(x)|^p \nu(dx) \leq \int_{\mathcal{O}} (P(t)|\tilde{\varphi}^p|(x) \nu(dx) \leq \int_{\mathcal{X}} P(t)(|\tilde{\varphi}^p)d\nu \\
= \int_{\mathcal{X}} |\tilde{\varphi}(x)|^p \nu(dx) = \int_{\mathcal{O}} |\varphi(x)|^p \nu(dx).
\]

\[ \square \]

By Proposition 5.1, for any \( p \geq 1 \) the semigroup \( P^O(t) \) is uniquely extendable to a strongly continuous semigroup \( P_p^O(t) \) in \( L^p(\mathcal{O}, \nu) \).

**Definition 5.2** We denote by \( M_2 \) the infinitesimal generator of \( P_2^O(t) \).

In this section we prove Theorem 1.2. We use the technique presented in [4, 25] in the case \( F = 0 \). To do that, we have to introduce a quadratic form associated to \( N_2 \). Now we state the hypotheses and the preliminary results that will allow us to use such approach.
5.1 Sobolev spaces

**Hypotheses 5.3** Assume that Hypotheses 4.5 hold true and the following conditions hold true.

(i) $\text{Ker}(C) = \{0\}$.

(ii) $A: \text{Dom}(A) \subset X \to X$ is self-adjoint and there exist $w > 0$ and $M > 0$ such that

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq Me^{-wt}, \quad t \geq 0.$$  

(iii) $C(\text{Dom}(A)) \subseteq \text{Dom}(A)$, and $ACx = CAx$ for any $x \in \text{Dom}(A)$.

Under Hypotheses 5.3 the operator

$$Q_\infty = \int_0^\infty e^{tA}Ce^{tA^*}dt$$

is a positive and trace class operator. Let $\mu$ the gaussian measure of mean 0 and covariance operator $Q_\infty$ (we refer to [7] for a discussion of Gaussian measures in infinite dimension). Under Hypotheses 5.3, by [34, Proof of Proposition 10.1.2], the operator

$$C^{1/2}\nabla : \xi_A(X) \subseteq L^2(X, \mu) \to L^2(X, \mu, X)$$

is closable and we denote by $W^{1/2}_C(X, \mu)$ its domain.

**Hypotheses 5.4** Assume that Hypotheses 5.3 hold true and there exists a lower semi-continuous function $U: X \to \mathbb{R}$ such that

1. $\|x\|^2e^{-2U} \in L^1(X, \mu)$ and $e^{-2U} \in W^{1/2}(X, \mu)$;
2. $F = -C\nabla U$.

Under Hypotheses 5.4 the SPDE (1.1) becomes

$$\begin{cases}
  dX(t, x) = \left( AX(t, x) - C\nabla U(X(t, x)) \right)dt + \sqrt{C}dW(t), \quad t \in [0, T]; \\
  X(0, x) = x \in X,
\end{cases}$$

the operator (1.6) reads as

$$N_0\varphi(x) := \frac{1}{2}\text{Tr}[C\nabla^2\varphi(x)] + \langle x, A^*\nabla\varphi(x) \rangle - \langle C\nabla U(x), \nabla\varphi(x) \rangle, \quad \varphi \in \xi_A(X), \ x \in X,$$

Moreover the following results are verified.

**Proposition 5.5** Assume that Hypotheses 5.4 hold true.
(i) The invariant measure \( \nu \) of \( P(t) \) has the form
\[
\nu(dx) = e^{-2U(x)} B \mu(dx), \quad B := \int_X e^{-2U(x)} \mu(dx).
\]

(ii) The operator
\[
C^{1/2} \nabla : \xi_A(X) \subseteq L^2(X, \nu) \rightarrow L^2(X, \nu, X)
\]
is closable, and we denote by \( W^{1,2}_C(X, \nu) \) its domain.

(iii) For any \( \varphi \in W^{1,2}_C(X, \nu) \) and \( \psi \in \text{Dom}(N_2) \) we have
\[
\int_X (N_2 \psi) \varphi d\nu = Q_2(\varphi, \psi) := -\frac{1}{2} \int_X \langle C^{1/2} \nabla \varphi, C^{1/2} \nabla \psi \rangle d\nu.
\]

We refer to [32, Sects. 3–4] or [31] for a proof of 5.5. Similarly to [25, Sect. 2], we define the following space.

**Remark 5.6** By Hypotheses 5.4 \( Q_\infty \) is positive, so \( \mu \) is a non-degenerate Gaussian measure. Hence \( \nu \) is a non-degenerate probability measure namely \( \nu(O) > 0 \) for any open set \( O \).

**Definition 5.7** We denote by \( \hat{W}^{1,2}_C(O, \nu) \) the space of the functions \( u : O \rightarrow \mathbb{R} \) such the extension \( \hat{u} : X \rightarrow \mathbb{R} \) defined by
\[
\hat{u}(x) = \begin{cases} u(x) & x \in O \\ 0 & x \in O^c \end{cases}
\]

belongs to \( \hat{W}^{1,2}_C(X, \nu) \).

**Remark 5.8** Definition 5.7 allows to avoid regularity assumptions on \( O \) and the use of the theory of traces of Sobolev functions on \( \partial O \) (see [11] for the Gaussian measures and [28] for more general measures).

Now we define a quadratic form on \( \hat{W}^{1,2}_C(O, \nu) \).

**Definition 5.9** We denote by \( Q^O_2 \) the quadratic form defined by
\[
Q^O_2(\varphi, \psi) := -\frac{1}{2} \int_X \langle C^{1/2} \nabla \hat{\varphi}, C^{1/2} \nabla \hat{\psi} \rangle d\nu, \quad \varphi, \psi \in \hat{W}^{1,2}_C(O, \nu).
\]

Moreover, we denote by \( N^O_2 \) the self-adjoint and dissipative operator associated to \( Q^O_2 \), namely
\[
\text{Dom}(N^O_2) := \{ \varphi \in \hat{W}^{1,2}_C(O, \nu) : \exists \beta \in L^2(O, \nu) \text{ s.t.} \}
\]
\[
\int_O \beta \psi d\nu = Q^O_2(\beta, \psi) \quad \forall \psi \in \hat{W}^{1,2}_C(X, \nu),
\]
\[
N^O_2 \varphi = \beta, \quad \varphi \in \text{Dom}(N^O_2).
\]
In Sect. 5.3 we shall prove Theorem 1.2 using the following idea. For \( \lambda > 0 \) and \( f \in L^2(O, \nu) \), we consider the equation with unknown \( \varphi \in \dot{W}^{1,2}(O, \nu) \),

\[
\lambda \int_O \varphi v d\nu - Q^O_2(\varphi, v) = \int_O f v d\nu, \quad v \in \dot{W}^{1,2}(O, \nu). \tag{5.2}
\]

Since the quadratic form \(-Q^O_2\) is continuous, non-negative, coercive and symmetric, by the Lax–Milgram Theorem for every \( \lambda > 0 \) and \( f \in L^2(O, \nu) \) there exists a unique solution \( \varphi \in \dot{W}^{1,2}(O, \nu) \) of (5.2). By definition of \( N^O_2 \), for every \( \lambda > 0 \) and \( f \in L^2(O, \nu) \), we have

\[
R(\lambda, N^O_2) f = \varphi,
\]

where \( \varphi \) is the unique solution of (5.2). In Sect. 5.3, we will prove that

\[
R(\lambda, M_2) = R(\lambda, N^O_2),
\]

which yields Theorem 1.2.

**Remark 5.10** We stress that in the trivial case, where \( O = \mathcal{X} \), Theorem 1.2 follows directly from Theorem 1.1 and Proposition 5.5.

### 5.2 The approximating semigroups

In this section we define and study the Feynman–Kac approximating semigroup for the semigroup \( P^O_2 \). For \( \epsilon > 0 \) we define

1. the set

\[
O_\epsilon := \{ x \in O \mid d(x, O^c) > \epsilon \}; \tag{5.3}
\]

2. the function

\[
V_\epsilon(x) := \left( \frac{1}{\epsilon} d(x, O_\epsilon) \right) \land 1, \quad x \in \mathcal{X}. \tag{5.4}
\]

We note that \( V \in C_b(\mathcal{X}) \), \( V \equiv 0 \) on \( O^c \) and \( V \equiv 1 \) on \( O^c \);

3. the semigroup

\[
P_\epsilon(t) \varphi(x) = \mathbb{E} \left[ \varphi(X(t, x)) e^{-\frac{1}{\epsilon} \int_0^t V_\epsilon(X(s, x)) ds} \right], \quad \varphi \in B_b(\mathcal{X}), \ x \in \mathcal{X},
\]

where \( \{ X(t, x) \}_{t \geq 0} \) is the mild solution of the SPDE (1.1).

First we prove that \( P_\epsilon(t) \) is uniquely extendable to a strongly continuous semigroup in \( L^2(\mathcal{X}, \nu) \).
Lemma 5.11 For any \( \varphi \in C_b(\mathcal{X}) \) we have
\[
\| P^\epsilon(t) \varphi \|_{L^2(\mathcal{X}, \nu)} \leq \| \varphi \|_{L^2(\mathcal{X}, \nu)}.
\]

Proof By the Hölder inequality and the fact that \( V \) is nonnegative on \( \mathcal{X} \),
\[
|P^\epsilon(t)\varphi(x)|^2 \leq \mathbb{E}[\varphi^2(X(t, x))e^{-\frac{\epsilon}{2} \int_0^t V_e(X(s,x))ds}] \leq P(t)(\varphi^2)(x), \quad x \in \mathcal{X}.
\]
Hence, since \( \nu \) is invariant for \( P^\epsilon(t) \), we have
\[
\int_{\mathcal{X}} |P^\epsilon(t)\varphi(x)|^2 \nu(dx) \leq \int_{\mathcal{X}} P(t)(\varphi^2)(x)\nu(dx) \leq \int_{\mathcal{X}} \varphi^2(x)\nu(dx).
\]
By Lemma 5.11, the semigroup \( P^\epsilon(t) \) is uniquely extendable in \( L^2(\mathcal{X}, \nu) \) to a strongly continuous and contraction semigroup \( P^\epsilon_2(t) \). We denote by \( N^\epsilon_2 \) its infinitesimal generator. We recall that \( N_2 \) is both the closure of the operator \( N_0 \) in \( L^2(\mathcal{X}, \nu) \) and the infinitesimal generator of \( P_2(t) \) (see Sect. 4.3).

Proposition 5.12 Let \( \lambda > 0 \) and \( f \in L^2(\mathcal{X}, \nu) \). Then the equation
\[
\lambda \varphi_\epsilon - N_2 \varphi_\epsilon + \frac{1}{\epsilon} V_e \varphi_\epsilon = f
\]  
(5.5)
has a unique solution \( \varphi_\epsilon \in \text{Dom}(N_2) \). Moreover the following estimates are verified
\[
\| \varphi_\epsilon \|^2_{L^2(\mathcal{X}, \nu)} \leq \frac{1}{\lambda^2} \| f \|^2_{L^2(\mathcal{X}, \nu)}, \quad (5.6)
\]
\[
\| C \nabla \varphi_\epsilon \|^2_{L^2(\mathcal{X}, \nu, \mathcal{X})} \leq \frac{2}{\lambda} \| f \|^2_{L^2(\mathcal{X}, \nu)}, \quad (5.7)
\]
\[
\int_{\mathcal{O}^\epsilon} V_e \varphi_\epsilon^2 d\nu \leq \frac{\epsilon}{\lambda} \| f \|^2_{L^2(\mathcal{X}, \nu)}. \quad (5.8)
\]

Proof By Proposition 5.5, \( N_2 \) is maximal dissipative. Let \( G : L^2(\mathcal{X}, \nu) \rightarrow L^2(\mathcal{X}, \nu) \) be the operator defined by
\[
G \varphi := \frac{1}{\epsilon} V_e \cdot \varphi,
\]
then \(-G\) is dissipative. So the operator \( K : \text{Dom}(N_2) \rightarrow L^2(\mathcal{X}, \nu) \), defined by
\[
K \varphi := N_2 \varphi - G \varphi
\]
is maximal dissipative. Therefore (5.5) has a unique solution \( \varphi_\epsilon \in \text{Dom}(N_2) \) and (5.6) is verified. Multiplying both sides of (5.5) by \( \varphi_\epsilon \), integrating over \( \mathcal{X} \), and taking into account (5.5), we obtain
\[ \lambda\|\varphi_\epsilon\|^2_{L^2(\mathcal{X}, v)} + \frac{1}{2}\|C^{1/2}\nabla \varphi_\epsilon\|^2_{L^2(\mathcal{X}, v)} + \frac{1}{\epsilon} \int_{\mathcal{X}} V_\epsilon \varphi_\epsilon^2 \, dv = \int_{\mathcal{X}} f \varphi_\epsilon \, dv. \]

By the Hölder inequality \( \int_{\mathcal{X}} |f \varphi_\epsilon| \, dv \leq \|f\|_{L^2(\mathcal{X}, v)} \|\varphi_\epsilon\|_{L^2(\mathcal{X}, v)} \) and, by estimate (5.6), we obtain \( \int_{\mathcal{X}} |f \varphi_\epsilon| \, dv \leq \frac{1}{\lambda} \|f\|^2_{L^2(\mathcal{X}, v)}. \) Then

\[ \frac{1}{2}\|C^{1/2}\nabla \varphi_\epsilon\|^2_{L^2(\mathcal{X}, v)} + \frac{1}{\epsilon} \int_{\mathcal{X}} V_\epsilon \varphi_\epsilon^2 \, dv \leq \frac{1}{\lambda} \|f\|^2_{L^2(\mathcal{X}, v)}, \]

which yields (5.7) and (5.8).

Now we characterize \( N_2^\epsilon. \)

**Proposition 5.13** For any \( \epsilon > 0, \) we have \( \text{Dom}(N_2^\epsilon) = \text{Dom}(N_2) \) and

\[ N_2^\epsilon \varphi = N_2 \varphi - \frac{1}{\epsilon} V_\epsilon \varphi, \quad \forall \varphi \in \text{Dom}(N_2). \]  

**Proof** First we prove that \( \text{Dom}(N_2) \subset \text{Dom}(N_2^\epsilon). \) We begin to show that \( N_2(\mathcal{X}) \subset \text{Dom}(N_2^\epsilon). \) Let \( \varphi \in N_2(\mathcal{X}). \) For any \( x \in \mathcal{X} \) and \( h > 0, \) we have

\[ P_2^\epsilon(h) \varphi(x) - \varphi(x) = P_2(h) \varphi(x) - \varphi(x) + \mathbb{E}[\epsilon^{-\frac{1}{2}} \int_0^h V_\epsilon(X(s, x)) \, ds - 1] \varphi(X(h, x)) \]  

(5.10)

Dividing both sides of (5.10) by \( h > 0 \) we obtain

\[ \frac{P_2^\epsilon(h) \varphi(x) - \varphi(x)}{h} = \frac{P_2(h) \varphi(x) - \varphi(x) + \mathbb{E}[\epsilon^{-\frac{1}{2}} \int_0^h V_\epsilon(X(s, x)) \, ds - 1] \varphi(X(h, x))}{h}. \]

By Theorem 1.1, we know that

\[ \lim_{h \to 0} \frac{P_2(h) \varphi - \varphi}{h} = N_2 \varphi, \quad \text{in } L^2(\mathcal{X}, v) \]  

(5.11)

Since the generalized mild solution \( \{X(t, x)\}_{t \geq 0} \) is a \( \mathcal{X}-\)valued continuous process (see Theorem 3.12 and Definition 2.2), then the functions \( r \to V_\epsilon(X(r, x)) \) and \( r \to \varphi(X(r, x)) \) are paths continuous, and so, recalling that \( V_\epsilon \in C_b(\mathcal{X}), \) for any \( x \in \mathcal{X} \) we have

\[ \lim_{h \to 0} \frac{\mathbb{E}[\epsilon^{-\frac{1}{2}} \int_0^h V_\epsilon(X(s, x)) \, ds - 1] \varphi(X(h, x))}{h} = -\frac{1}{\epsilon} V_\epsilon(x) \varphi(x). \]

Hence by the Dominated Convergence theorem it follows that

\[ \lim_{h \to 0} \frac{\mathbb{E}[\epsilon^{-\frac{1}{2}} \int_0^h V_\epsilon(X(s, \cdot)) \, ds - 1] \varphi(X(h, \cdot))}{h} = -\frac{1}{\epsilon} V_\epsilon(\cdot) \varphi(\cdot), \quad \text{in } L^2(\mathcal{X}, v). \]  

(5.12)
So by (5.11) and (5.12) we obtain
\[ N_2^\varepsilon \varphi = \lim_{h \to 0} \frac{P_\varepsilon (h) \varphi - \varphi}{h} = N_2 \varphi - \frac{1}{\varepsilon} V_\varepsilon \varphi, \quad \text{in } L^2(\mathcal{X}, \nu). \]

Then, for any \( \varphi \in \xi_A(\mathcal{X}) \), we have \( \varphi \in D(N_2^\varepsilon) \) and
\[ N_2^\varepsilon \varphi = N_2 \varphi - \frac{1}{\varepsilon} V_\varepsilon \varphi. \]

Let now \( \varphi \in \text{Dom}(N_2^\varepsilon) \). By Theorem 1.1, \( \xi_A(\mathcal{X}) \) is a core for \( N_2 \), so we can take a sequence \( (\varphi_n)_{n \in \mathbb{N}} \subset \xi_A(\mathcal{X}) \) such that
\[ \lim_{n \to +\infty} \varphi_n = \varphi, \quad \lim_{n \to +\infty} N_2^\varepsilon \varphi_n = N_2 \varphi, \quad \text{in } L^2(\mathcal{X}, \nu). \]

Since \( V_\varepsilon \) is bounded, we have
\[ \lim_{n \to +\infty} \frac{1}{\varepsilon} V_\varepsilon \varphi_n = \frac{1}{\varepsilon} V_\varepsilon \varphi \quad \text{in } L^2(\mathcal{X}, \nu). \]

Hence
\[ \lim_{n \to +\infty} N_2^\varepsilon \varphi_n = \lim_{n \to +\infty} N_2 \varphi_n - \frac{1}{\varepsilon} V_\varepsilon \varphi_n = N_2 \varphi - \frac{1}{\varepsilon} V_\varepsilon \varphi, \quad \text{in } L^2(\mathcal{X}, \nu). \]

Then, for any \( \varphi \in \text{Dom}(N_2) \), we have \( \varphi \in \text{Dom}(N_2^\varepsilon) \) and (5.9) holds.

Finally we prove that \( \text{Dom}(N_2^\varepsilon) \subset \text{Dom}(N_2) \). For any \( \varphi \in \text{Dom}(N_2^\varepsilon) \), let \( \varphi_\varepsilon \) be the unique solution of (5.5) with \( f = \lambda \varphi_\varepsilon - N_2^\varepsilon \varphi_\varepsilon \). Then, by Proposition 5.12, \( \varphi_\varepsilon \in \text{Dom}(N_2) \subset \text{Dom}(N_2^\varepsilon) \). Moreover \( R(\lambda, N_2^\varepsilon) f = \varphi_\varepsilon = \varphi \) and this concludes the proof. \( \square \)

Finally we prove that the semigroups \( P_2^\varepsilon(t) \) approximate \( P_2^O(t) \) in \( L^2(O, \nu) \).

**Proposition 5.14** For any \( f \in L^2(O, \nu) \) and \( t > 0 \), we have
\[ \lim_{\epsilon \to 0} (P_2^\varepsilon(t) \widehat{f})|_O = P_2^O(t) f \quad \text{in } L^2(O, \nu), \quad \text{(5.13)} \]
and, for any \( \lambda > 0 \),
\[ \lim_{\epsilon \to 0} (R(\lambda, N_2^\varepsilon) \widehat{f})|_O = R(\lambda, M_2) f \quad \text{in } L^2(O, \nu), \quad \text{(5.14)} \]
where \( \widehat{f} \) is defined in Definition 5.7.

**Proof** We split the proof in two steps. As a first step we prove that for any \( \varphi \in C_b(\mathcal{X}) \) we have
\[ \lim_{\epsilon \to 0} \| P_2^\varepsilon(t) \varphi - P_2^O(t)(\varphi|_O) \|_{L^2(O, \nu)} = 0. \quad \text{(5.15)} \]
And as a second step we prove the statement of Proposition.

**Step 1.** Let \( \varphi \in C_b(\mathcal{X}) \). First of all we prove that

\[
\lim_{\epsilon \to 0} P^2_\epsilon (t) \varphi(x) = P^O_2 (t)(\varphi|_\mathcal{O})(x) \quad x \in \mathcal{O}, \, t > 0.
\] (5.16)

Fixed \( x \in \mathcal{O} \) we consider the exit time \( \tau_x \) defined in (5.1). Let \( t > 0 \); we define the sets

\[
\Omega_1 = \{ \tau_x > t \} = \{ w \in \Omega \mid X(s,x)(w) \in \mathcal{O}, \forall s \in [0,t) \},
\]

\[
\Omega_2 = \{ \tau_x \leq t \} = \{ w \in \Omega \mid \exists s_0 \in (0,t) \mid X(s_0,x)(w) \in \mathcal{O}^c \}.
\]

Clearly \( \Omega = \Omega_1 \cup \Omega_2 \) and \( \Omega_1, \Omega_2 \) are disjoint. Fixed \( x \in \mathcal{O} \), we have

\[
P^\epsilon_2 (t) \varphi(x) = \int_{\Omega_1} \varphi(X(t,x)) e^{-\frac{1}{\epsilon} \int_0^t V_\epsilon(X(s,x)) ds} d\mathbb{P}
\]

\[+ \int_{\Omega_2} \varphi(X(t,x)) e^{-\frac{1}{\epsilon} \int_0^t V_\epsilon(X(s,x)) ds} d\mathbb{P}. \] (5.17)

We study separately the two summands in the right hand side of (5.17). On \( \Omega_1 \), \( X(s,x) \in \mathcal{O} \), for any \( s \in [0,t) \), and then, by definition of \( V_\epsilon \) (see 5.4), there exist \( \epsilon_0 > 0 \), such that

\[V_\epsilon(X(s,x)) = 0, \quad \forall \epsilon < \epsilon_0, \, \forall s \in [0,t).\]

So for the first summand of equation (5.17), we have

\[
\lim_{\epsilon \to 0} \int_{\Omega_1} \varphi(X(t,x)) e^{-\frac{1}{\epsilon} \int_0^t V_\epsilon(X(s,x)) ds} d\mathbb{P} = \int_{\Omega_1} \varphi(X(t,x)) d\mathbb{P}.
\] (5.18)

On \( \Omega_2 \), by the fact that the generalized mild solution \( \{X(t,x)\}_{t \geq 0} \) is a \( \mathcal{X} \)-valued continuous process (see Theorem 3.12 and Definition 2.2), we know that, for \( \mathbb{P} \)-a.a. (almost all) \( w \in \Omega_2 \), there exists \( s_0(w) \in (0,t) \) such that

\[X(s_0(w),x)(w) \in \partial \mathcal{O},\]

where \( \partial \mathcal{O} \) is the boundary of \( \mathcal{O} \). Then by definition of \( V_\epsilon \), there exists \( \delta(w) > 0 \) such that

\[V_\epsilon(X(s,x))(w) \geq \frac{1}{2}, \quad \forall s \in [s_0(w) - \delta(w), s_0(w)].\]

So, for the second summand of equation (5.17), for \( \mathbb{P} \)-a.a. \( w \in \Omega_2 \), we have

\[
\lim_{\epsilon \to 0} e^{-\frac{1}{\epsilon} \int_0^t V_\epsilon(X(s,x))(w) ds} \leq \lim_{\epsilon \to 0} e^{-\frac{\delta(w)}{\epsilon}} = 0.
\] (5.19)
Therefore by (5.19) and the Dominated Convergence theorem, we have

$$\lim_{\epsilon \to 0} \int_{\Omega_2} \varphi(X(t, x)) e^{-\frac{1}{\epsilon} \int_0^t V_\epsilon(X(s, x)) ds} dP = 0. \quad (5.20)$$

Hence, (5.18) and (5.20) yield (5.16). Moreover, for each \(x \in \mathcal{O}\), we have \(|P_2^\epsilon(t)(\varphi)(x)|, |P_2^O(t)(\varphi)(x)| \leq \|\varphi\|_\infty\). Then, by (5.16) and the Dominated Convergence theorem, (5.15) is verified.

**Step 2.** Let \(f \in L^2(\mathcal{O}, \nu)\); we prove (5.13). We recall that \(C_b(\mathcal{X})\) is dense in \(L^2(\mathcal{X}, \nu)\), so there exists a sequence \((f_n) \subset C_b(\mathcal{X})\) such that, for any large \(n \in \mathbb{N}\),

$$\|\hat{f} - f_n\|_{L^2(\mathcal{X}, \nu)} \leq \frac{1}{n}.$$  

In particular

$$\|f - f_n\|_{L^2(\mathcal{O}, \nu)} = \|\hat{f} - f_n\|_{L^2(\mathcal{X}, \nu)} \leq \frac{1}{n}. \quad (5.21)$$

Therefore

$$\|P_2^\epsilon(t)\hat{f} - P_2^O(t)f\|_{L^2(\mathcal{O}, \nu)} \leq \|P_2^O(t)(f - f_n)(\mathcal{O})\|_{L^2(\mathcal{O}, \nu)} + \|P_2^\epsilon(t)(\hat{f} - f_n)\|_{L^2(\mathcal{O}, \nu)}$$

$$+ \|P_2^\epsilon(t)f_n - P_2^O(t)f_n\|_{L^2(\mathcal{O}, \nu)}.$$

By Lemma 5.11 and Proposition 5.1, we have

$$\|P_2^\epsilon(t)\hat{f} - P_2^O(t)f\|_{L^2(\mathcal{O}, \nu)} \leq \|f - f_n|\mathcal{O}\|_{L^2(\mathcal{O}, \nu)} + \|\hat{f} - f_n\|_{L^2(\mathcal{O}, \nu)}$$

$$+ \|P_2^\epsilon(t)f_n - P_2^O(t)f_n\|_{L^2(\mathcal{O}, \nu)}.$$  

Letting \(\epsilon \to 0\) and \(n \to +\infty\), the first and the second summand go to zero by (5.21), and the third summand goes to zero by Step 1. We recall the following identity in \(L^2(\mathcal{X}, \nu)\)

$$R(\lambda, N^\epsilon_2)\hat{f} = \int_0^{+\infty} e^{-\lambda t} P^\epsilon_2(t)\hat{f} dt,$$

taking the restriction to \(\mathcal{O}\) of both sides and using (5.13) we obtain (5.14). \(\square\)

### 5.3 Proof of Theorem 1.2

Finally we prove Theorem 1.2.

**Proof of Theorem 1.2** First we prove that \(\varphi \in \dot{W}^{1,2}_{C}(\mathcal{O}, \nu)\). For \(\epsilon > 0\), we set

$$\varphi_\epsilon = R(\lambda, N^\epsilon_2)\hat{f}.$$
By Proposition 5.13, $\varphi_\varepsilon$ is the unique solution of (5.5), with $f$ replaced by $\hat{f}$. Moreover, by Proposition 5.12(5.6-5.7), the $W^{1,2}_C(\mathcal{X}, \nu)$-norm of $\varphi_\varepsilon$ is bounded by a constant independent of $\varepsilon$. Therefore there exists a sub-sequence $(\varphi_{\varepsilon_k})$ weakly convergent in $W^{1,2}_C(\mathcal{X}, \nu)$ to a function $\phi$. We have to prove that $\phi = \hat{\varphi}$, namely $\phi|_{\mathcal{O}} = \varphi$ and $\phi|_{\mathcal{O}^c} = 0$. By Proposition 5.14(5.14), we know that

$$
\lim_{k \to +\infty} \|\varphi - \varphi_{\varepsilon_k}|_{\mathcal{O}}\|_{L^2(\mathcal{O}, \nu)} = 0,
$$

so that $\phi|_{\mathcal{O}} = \varphi$. Since $\varphi_{\varepsilon_k}$ weakly converges to $\phi$ in $W^{1,2}_C(\mathcal{X}, \nu)$, then it weakly converges to $\phi$ in $L^2(\mathcal{O}, \nu)$. Recalling that $V_{\varepsilon_k} \equiv 1$ in $\mathcal{O}^c$ (see (5.4)) and using (5.8), we obtain

$$
\|\phi\|^2_{L^2(\mathcal{O}^c, \nu)} = \int_{\mathcal{O}^c} \phi^2 \, d\nu \leq \limsup_{k \to +\infty} \left( \int_{\mathcal{O}^c} \varphi_{\varepsilon_k}^2 \, d\nu \right)^{1/2} \left( \int_{\mathcal{O}^c} \phi^2 \, d\nu \right)^{1/2} \leq \limsup_{k \to +\infty} \left( \frac{\varepsilon_k^2}{\lambda} \right)^{1/2} \|f\|_{L^2(H, \nu)} = 0,
$$

and so $\phi|_{\mathcal{O}^c} = 0$. Therefore, $\phi = \hat{\varphi} \in W^{1,2}_C(\mathcal{X}, \nu)$, so that $\varphi \in \hat{W}^{1,2}_C(\mathcal{O}, \nu)$.

Finally we prove that $\varphi$ is a solution of (5.2). Fixed $v \in \hat{W}^{1,2}_C(\mathcal{O}, \nu)$ and $k \in \mathbb{N}$, we multiply both members of (5.9) by $\hat{v}$ and we integrate over $\mathcal{X}\setminus(\mathcal{O}\setminus\mathcal{O}_{\varepsilon_k})$. Since $V_{\varepsilon_k} \hat{v} \equiv 0$ on $\mathcal{X}\setminus(\mathcal{O}\setminus\mathcal{O}_{\varepsilon_k})$, we have

$$
\lambda \int_{\mathcal{X}\setminus(\mathcal{O}\setminus\mathcal{O}_{\varepsilon_k})} \varphi_{\varepsilon_k} \hat{v} \, d\nu + \frac{1}{2} \int_{\mathcal{X}\setminus(\mathcal{O}\setminus\mathcal{O}_{\varepsilon_k})} \left( C^{1/2} \nabla \varphi_{\varepsilon_k}, C^{1/2} \nabla \hat{v} \right) \, d\nu = \int_{\mathcal{X}\setminus(\mathcal{O}\setminus\mathcal{O}_{\varepsilon_k})} \hat{f} \hat{v} \, d\nu.
$$

Recalling the definition of $\mathcal{O}_{\varepsilon_k}$ (see (5.3)), letting $k \to +\infty$, we obtain

$$
\lambda \int_{\mathcal{X}} \hat{\varphi} \hat{v} \, d\nu + \frac{1}{2} \int_{\mathcal{X}} \left( C^{1/2} \nabla \hat{\varphi}, C^{1/2} \nabla \hat{v} \right) \, d\nu = \int_{\mathcal{X}} \hat{f} \hat{v} \, d\nu,
$$

and so we conclude that $\varphi$ satisfies (5.2). We recall that, by the Lax-Milgram theorem, the weak solution of (5.2) is unique and so, for any $\lambda > 0$ and $f \in L^2(\mathcal{O}, \nu)$, we have

$$
R(\lambda, M_2) f = R(\lambda, N^\mathcal{O}_2) f,
$$

and so the Theorem 1.2 is proved.

\[ \square \]

### 6 Examples

We consider the SPDE

$$
\begin{align*}
\left\{ 
\begin{array}{l}
\, dX(t, x) = \left( A X(t, x) + F(X(t, x)) \right) \, dt + \sqrt{C} \, dW(t), \, t > 0; \\
\, X(0, x) = x \in \mathcal{X}.
\end{array}
\right.
\end{align*}
$$

\[ (6.1) \]
in this section we present some examples of \( A, C \) and \( F \) that verify the hypotheses of Theorems 1.1 and 1.2.

### 6.1 An example for Sect. 5

For this example we could consider a very general framework choosing as \( X, E, A \) and \( C \) as those defined in [14, Chapter 6]. However, in order not to overburden the calculations, we will consider a less general setting.

Let \( X = L^2([0, 1], \lambda) \) where \( \lambda \) is the Lebesgue measure and let \( E = C([0, 1]) \). Let \( A \) be the realization in \( L^2([0, 1]) \) of the second order derivative with Dirichlet boundary condition and \( C = I_X \). By [14, Sect. 6.1], Hypotheses 3.1(ii) and 3.1(iv) are verified, and \( A \) is dissipative in both \( L^2([0, 1], \lambda) \) and \( C([0, 1]) \). Moreover the constant \( w \) of Hypotheses 5.3 is equal to \(-\pi^2\) (see [22, Chapter 4]). By [14, Lemma 8.2.1] condition (3.34) of Hypotheses 3.17 is verified.

Now we define the function \( F \). Let \( \phi \in C^2(\mathbb{R}) \) be a function such that \( \phi' \) is increasing, and there exist \( d_1, d_2 > 0 \) and an \( m \in \mathbb{N} \) such that

\[
\begin{align*}
|\phi'(y)| &\leq d_1 (1 + |y|^m), \quad y \in \mathbb{R}; \\
|\phi''(y)| &\leq d_2 (1 + |y|^{m-1}), \quad y \in \mathbb{R};
\end{align*}
\]

Let \( \zeta_F > 0 \). We consider the function \( \phi : \mathbb{R} \to \mathbb{R} \) defined by

\[
\phi(y) = \varphi(y) + \frac{\zeta_F}{2} y^2,
\]

and the function \( U : \mathcal{X} \to \mathbb{R} \) defined by

\[
U(f) = \begin{cases}
\int_0^1 \phi(f(x)), & f \in E, \\
0, & f \notin E.
\end{cases}
\]

In this case the operator of Hypotheses 5.3 is \( Q_\infty = A^{-1} \). Let \( \mu \sim N(0, Q_\infty) \). By [26, Proposition 5.2], \( U \in W^{1,p}_C(\mathcal{X}, \mu) \), for any \( p \geq 1 \), and

\[
\nabla U(f)(x) = \phi' \circ f(x) = \varphi(f(x)) + \zeta_F f(x), \quad \forall f \in E = C([0, 1]), \quad x \in [0, 1].
\]

We set \( F = -\nabla U \), and we recall that we have taken \( C = I_X \). Hence Hypotheses 3.1(i) are verified. By (6.2) and (6.3) Hypotheses 3.1(v) and 3.1(vi) are verified. By [35, Example D.7], the fact that \( A \) is dissipative in both \( L^2([0, 1], \lambda) \) and \( C([0, 1]) \) and standard calculations Hypotheses 3.1(iii) are verified. We stress that all the hypotheses of Theorem 3.19 are verified, so \( \nu(C([0, 1])) = 1 \), where \( \nu \) is the invariant measure of the transition semigroup associated to the generalize mild solution of (6.1). Finally, by the definition of \( \phi \) and the Fernique theorem, the hypotheses of Theorem 1.2 are verified. We emphasise that in this example the semigroup \( e^{tA} \) need not be strongly continuous on \( E \).
6.2 An example where $F$ is not a Nemytskii type operator

In this subsection we consider a class of functions $F$ already presented in [6, Sect. 5.2]. We recall the notion of infinite dimensional polynomial (see [19, 37, 48]).

For every $n \in \mathbb{N}$, we say that a map $V : \mathcal{X}^n \to \mathcal{X}$ is $n$-multilinear if it is linear in each variable separately. A $n$-multilinear map $B$ is said to be symmetric if

$$V(x_1, \ldots, x_n) = V(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

for any permutation $\sigma$ of the set $\{1, \ldots, n\}$. We say that a function $P_n : \mathcal{X} \to \mathcal{X}$ is a homogeneous polynomial of degree $n \in \mathbb{N}$ if there exists a $n$-multilinear symmetric map $B$ such that for every $x \in \mathcal{X}$

$$P(x) = V(x_1, \ldots, x).$$

We consider the function $F : \mathcal{X} \to \mathcal{X}$ defined by

$$F(x) := P_n(x) + \xi_F x,$$

where $x \in \mathcal{X}$, $\xi_F \in \mathbb{R}$ and $P_n$ is a homogeneous polynomial of degree $n$ such that,

$$\langle V(h, x, \ldots, x), h \rangle \leq 0,$$

where $V$ is the $n$-multilinear map defined by (6.5). By [19, Theorem 3.4], there exists $d > 0$ such that

$$\|F(x)\| \leq d(1 + \|x\|^n), \quad x \in \mathcal{X}.$$ (6.7)

Moreover, for any $x, h \in \mathcal{X}$, we have

$$\mathcal{D}P_n(x)h = nV(h, x, \ldots, x),$$

and so, by (6.6), for any $x, y \in \mathcal{X}$, we obtain

$$\langle F(x) - F(y), x - y \rangle \leq \xi_F \|x - y\|^2.$$ (6.8)

Let now consider a particular case. Let $E = \mathcal{X} = L^2([0, 1], \lambda)$, let $A$ be the realization in $\mathcal{X}$ of the second order elliptic operator defined in [14, Sect. 6.1] and let $C \in \mathcal{L}(\mathcal{X})$ be the positive operator defined in [14, Sect. 6.1]. Let $K \in L^2([0, 1]^4)$ and assume that $K$ is symmetric ((6.4)). Let

$$[P_3(f)](\xi) := \int_0^1 \int_0^1 \int_0^1 K(\xi_1, \xi_2, \xi_3, \xi) f(\xi_1) f(\xi_2) f(\xi_3) d\xi_1 d\xi_2 d\xi_3$$

for $f \in L^2([0, 1])$. $P$ is a homogeneous polynomial of degree three on $L^2([0, 1])$ (see [37, Exercise 1.73]). (6.6) holds whenever $K$ has negative value (see [6, Sect. 5.2]). So,
by the same arguments used in the previous example, the hypotheses of Theorem 1.1 are verified. It is also possible to consider a general infinite dimensional polynomial of odd degree \( n \in \mathbb{N} \). We remark that other choices of \( A \) and \( C \) are possible, for example we could consider the ones chosen in [25].

6.3 An application of Theorem 3.19

Now we present a particular case of the previous example where \( E = W^{1,2}([0,1],\lambda) \). We assume that \( A = -\frac{1}{2}I_X \), so it verifies Hypotheses 3.1(ii) and \( A + \frac{1}{2}I \) is dissipative in both \( L^2([0,1],\lambda) \) and \( W^{1,2}([0,1],\lambda) \). Let \( B \) be the realization of the second order derivative in \( X \) with Dirichlet boundary conditions. We recall that \( B \) is a negative operator, \( \text{Dom}((−B)^{1/2}) = W^{1,2}_0([0,1],\lambda) \) and \( (−B)^{−\gamma} \) is a trace class operator, for any \( \gamma > \frac{1}{2} \) (see [22, Sect. 4.1]). Let \( \beta > 2 \) and set \( C = (−B)^{−\beta} \). Then

\[
\|W_A(t)\|_{L^2([0,1],\lambda)}^2 = \left\| (−B)^{1/2} \int_0^t e^{-s B^{−\beta/2}} dW(s) \right\|_{L^2([0,1],\lambda)}^2
= \left\| \int_0^t e^{-s B^{(1−\beta)/2}} dW(s) \right\|_{L^2([0,1],\lambda)}^2
\]

and so by [35, Theorems 4.36 and 5.11], Hypotheses 3.1(iv) and condition (3.34) of Hypotheses 3.17 are verified. Let \( F \) be as in Sect. 6.2. In addition we assume that \( K \) has weak derivative with respect to the fourth variable, such that

\[
\frac{\partial K}{\partial \xi} \in L^2([0,1]^4,\lambda).
\]

Let \( f \in W^{1,2}([0,1],\lambda) \). We have \( F(f) = P_3(f) + \xi f \in W^{1,2}([0,1],\lambda) \)(see (6.9)) and its weak derivative is

\[
(F(f))' = \int_0^1 \int_0^1 \int_0^1 \frac{\partial K}{\partial \xi} (\xi_1, \xi_2, \xi_3, \xi) f(\xi_1) f(\xi_2) f(\xi_3) d\xi_1 d\xi_2 d\xi_3 + \xi f'.
\]  

(6.10)

If we assume that \( (\partial K/\partial \xi) \in L^2([0,1]^4,\lambda) \) is symmetric (see (6.4)) and that it has negative value, then by (6.10) and the same arguments used in Sects. 6.2, (6.7) and (6.8) are verified in \( W^{1,2}([0,1],\lambda) \). Hence, by the same arguments of the previous examples, the hypotheses of Theorem 3.19 are verified and so \( \nu(W^{1,2}([0,1],\lambda)) = 1 \), where \( \nu \) is the invariant measure of the transition semigroup associated to the generalize mild solution of (6.1).

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References

1. Addona, D., Bandini, E., Masiero, F.: A nonlinear Bismut–Elworthy formula for HJB equations with quadratic Hamiltonian in Banach spaces. NODEA-Nonlinear Differ. Equ. Appl. 27 (2020)
2. Addona, D., Cappa, G., Ferrari, S. On the domain of elliptic operators defined in subsets of Wiener spaces, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 23 (2020)
3. Angiuli, L., Ferrari, S., Pallara, D.: Gradient estimates for perturbed Ornstein–Uhlenbeck semigroups on infinite-dimensional convex domains. J. Evol. Equ. 19, 677–715 (2019)
4. Assaad, J., van Neerven, J.: $L^2$-theory for non-symmetric Ornstein–Uhlenbeck semigroups on domains. J. Evol. Equ. 13, 107–134 (2013)
5. Bignamini, D.A., Ferrari, S.: Regularizing properties of (Non-Gaussian) transition semigroups in Hilbert spaces. Potential Anal. (2021). https://doi.org/10.1007/s11118-021-09931-2
6. Bignamini, D.A., Ferrari, S.: On generators of transition semigroups associated to semilinear stochastic partial differential equations. J. Math. Anal. Appl. 508(1), 125878 (2022)
7. Bogachev, V.I.: Gaussian Measures, Mathematical Surveys and Monographs, vol. 62. American Mathematical Society, Providence, RI (1998)
8. Bogachev, V.I.: Measures on topological spaces. J. Math. Sci. 91, 3033–3156 (1998)
9. Cappa, G., Ferrari, S.: Maximal Sobolev regularity for solutions of elliptic equations in infinite-dimensional Banach spaces endowed with a weighted Gaussian measure. J. Differ. Equ. 261, 7099–7131 (2016)
10. Cappa, G., Ferrari, S.: Maximal Sobolev regularity for solutions of elliptic equations in Banach spaces endowed with a weighted Gaussian measure: the convex subset case. J. Math. Anal. Appl. 458, 300–331 (2018)
11. Celada, P., Lunardi, A.: Traces of Sobolev functions on regular surfaces in infinite dimensions. J. Function. Anal. 266 (2013)
12. Cerrai, S.: A Hille–Yosida theorem for weakly continuous semigroups. Semigroup Forum 49, 349–367 (1994)
13. Cerrai, S.: Weakly continuous semigroups in the space of functions with polynomial growth. Dynam. Syst. Appl. 4, 351–371 (1995)
14. Cerrai, S.: Second order PDE’s in finite and infinite dimension. Lecture Notes in Mathematics, vol. 1762. Springer-Verlag, Berlin (2001)
15. Cerrai, S.: Stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. Probab. Theory Relat. Fields 125, 271–304 (2003)
16. Cerrai, S.: Averaging principle for systems of reaction–diffusion equations with polynomial nonlinearities perturbed by multiplicative noise. Dyn. Syst. Appl. 43, 2482–2518 (2011)
17. Cerrai, S., Da Prato, G.: A basic identity for Kolmogorov operators in the space of continuous functions related to RDEs with multiplicative noise. Ann. Probab. 42, 1297–1336 (2014)
18. Chae, S.B.: Holomorphy and Calculus in Normed Spaces, Chapman & Hall/CRC Pure and Applied Mathematics, vol. 92. Taylor & Francis Group, New York and Basel (1985)
19. Da Prato, G.: Bounded perturbations of Ornstein–Uhlenbeck semigroups. In “Evolution equations, semigroups and functional analysis (Milano, 2000)”, Progr. Nonlinear Differential Equations Appl., vol. 50, pp. 97–114 (2002), Birkhäuser, Basel
20. Da Prato, G.: Monotone gradient systems in $L^2$ spaces in “Seminar on Stochastic Analysis, Random Fields and Applications, III (Ascona, 1999)”, Progr. Probab., vol. 52, pp. 73–88 (2002), Birkhäuser, Basel
21. Da Prato, G.: Kolmogorov Equations for Stochastic PDEs, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel (2004)
22. Da Prato, G.: Transition semigroups corresponding to Lipschitz dissipative systems. Discrete Contin. Dyn. Syst. 10(1–2), 177–192 (2004)
23. Da Prato, G., Debussche, A., Goldys, B.: Invariant measures of non symmetric dissipative stochastic systems. Probab. Th. Relat. Fields 123, 355–380 (2002)
24. Da Prato, G., Lunardi, A.: On the Dirichlet semigroup for Ornstein–Uhlenbeck operators in subsets of Hilbert spaces. J. Funct. Anal. 259, 2642–2672 (2010)
25. Da Prato, G., Lunardi, A.: Sobolev regularity for a class of second order elliptic PDEs in infinite dimension. Ann. Prob. 47, 2113–2160 (2014)
27. Da Prato, G., Lunardi, A.: Maximal Sobolev regularity in Neumann problems for gradient systems in infinite dimensional domains. Ann. Inst. Henri Poincaré Probab. Stat. 51, 1102–1123 (2015)
28. Da Prato, G., Lunardi, A., Tubaro, L.: Malliavin Calculus for non Gaussian differentiable measures and surface measures in Hilbert spaces. Trans. Am. Math. Soc. 370, 2113–2160 (2016)
29. Da Prato, G., Rockner, M.: Singular dissipative stochastic equations in Hilbert spaces. Probab. Theory Relat. Fields 124, 261–303 (2002)
30. Da Prato, G., Rockner, M., Wang, F.-Y.: Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups. J. Funct. Anal. 257, 992–1017 (2009)
31. Da Prato, G., Tubaro, L.: Self-adjointness of some infinite-dimensional elliptic operators and application to stochastic quantization. Probab. Theory Relat. Fields 118, 131–145 (2000)
32. Da Prato, G., Tubaro, L.: Some results about dissipativity of Kolmogorov operators. Czechoslov. Math. J. 51, 685–699 (2001)
33. Da Prato, G., Zabczyk, J.: Ergodicity for Infinite-Dimensional Systems, London Mathematical Society Lecture Note Series, vol. 229. Cambridge University Press, Cambridge (1996)
34. Da Prato, G., Zabczyk, J.: Second Order Partial Differential Equations in Hilbert Spaces, London Mathematical Society Lecture Note Series, vol. 293. Cambridge University Press, Cambridge (2002)
35. Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, vol. 152. Cambridge University Press, Cambridge (2014)
36. Demuth, M., Van Casteren, J.: Stochastic spectral theory for selfadjoint Feller operators: a functional integration approach, Birkhauser (2000)
37. Dineen, S.: Complex Analysis on Infinite Dimensional Spaces. Springer Monographs in Mathematics, Springer-Verlag, London (1999)
38. Dynkin, E.B.: Markov Processes, vol. I. Springer-Verlag, New York (1965)
39. Dunford, N., Schwartz, J.T.: Linear Operators. Wiley, Part II, Wiley Classics Library, New York (1988)
40. Engel, K.-J., Nagel, R.: A Short Course on Operator Semigroups. Universitext, Springer, New York (2006)
41. Es-Sarhir, A., Stannat, W.: Maximal dissipativity of Kolmogorov operators with Cahn–Hilliard type drift term. J. Differ. Equ. 247, 424–446 (2009)
42. Ferrari, S.: Sobolev spaces with respect to a weighted Gaussian measure in infinite dimensions. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 22 (2019)
43. Fonseca, I., Leoni, G.: Modern methods in the calculus of variations: $L^p$ spaces. Springer monographs in Mathematics, Springer, New York (2007)
44. Goldys, B., Kocan, M.: Diffusion semigroups in spaces of continuous functions with mixed topology. J. Differ. Equ. 173, 17–39 (2001)
45. Kechris, A.S.: Classical Descriptive Set Theory, Graduate Texts in Mathematics 156. Springer, New York (2012)
46. Lunardi, A.: Analytic semigroups and optimal regularity in parabolic problems. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel (1995)
47. Masiero, F.: Stochastic optimal control problems and parabolic equations in Banach spaces. SIAM J. Control Optim. 47(1), 251–300 (2008)
48. Mujica, J.: Complex Analysis in Banach Spaces, North-Holland Mathematics Studies, vol. 120. North-Holland, Amsterdam (1985)
49. Peszat, S., Zabczyk, J.: Strong Feller property and irreducibility for diffusions on Hilbert spaces. Ann. Probab. 23, 157–172 (1995)
50. Phelps, R.R.: Convex Functions, Monotone Operators and Differentiability. Lecture Notes in Mathematics, vol. 1364. Springer-Verlag, Berlin, Heidelberg (1993)
51. Priola, E.: Dirichlet problems in a half-space of a Hilbert space. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5(2), 257–291 (2002)
52. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. I. Functional Analysis. Academic Press, New York-London (1972)
53. Talarczyk, A.: Dirichlet problem for parabolic equations on Hilbert spaces. Studia Math. 141, 109–142 (2000)

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