Interaction-free evolving states of a bipartite system

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We show that two interacting physical systems may admit entangled pure or non separable mixed states evolving in time as if the mutual interaction hamiltonian were absent. In this paper we define these states Interaction Free Evolving (IFE) states and characterize their existence for a generic binary system described by a time independent Hamiltonian. A comparison between IFE subspace and the decoherence free subspace is reported. The set of all pure IFE states is explicitly constructed for a non homogeneous spin star system model

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I. INTRODUCTION

Consider a bipartite system $S$ consisting of two quantum interacting subsystems $A$ and $B$ with free Hamiltonians $H_A$ acting on the Hilbert space $\mathcal{H}_A$ and $H_B$ acting on $\mathcal{H}_B$ respectively. The states of $A+B$ live in the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ where the Hamiltonian of the bipartite system is

$$H = H_A + H_B + H_I = H_0 + H_I,$$

$H_I$ being the operator describing the coupling between $A$ and $B$. Generally speaking, the entanglement exhibited in the initial pure or mixed state of the bipartite system, regardless of how it is measured, undergoes changes over time traceable back to the presence of $H_I$ in the Hamiltonian. Thus, for example, an initial factorized pure state or a separable mixed state evolves into an entangled state where, hence, time-dependent classical and/or quantum correlations between $A$ and $B$ generally emerge. In such a general dynamical scenario it is not surprising the increasing attention reserved to the existence in some bipartite systems of subradiant states that is selected pure factorized states which evolve keeping the system in its fully initial decorrelated condition at any time instant. Such a peculiar behavior of both fundamental [1], [2] and applicative interest [3, 4, 5, 6] , results from quantum interference effects exactly canceling in the evolved state at a generic time instant right those contributions, stemming from the superposition principle, which, otherwise, would determine the onset and possibly the persistence of correlation manifestations in between $A$ and $B$. Subradiance is a cooperative effect investigated both theoretically [1, 2, 3, 4, 5, 6, 20] and experimentally [27, 28, 29, 30] after the seminal Dicke paper [11], mainly in radiation-matter systems where it describes optically inactive states of atomic ensemble ($A$) in an electromagnetic environment ($B$). The current upsurge of interest toward these states reflects indeed the existence of many other physical contexts where this phenomenon may find promising applications [4, 5, 6, 34, 35, 36] as well as the experimental evidence that a system made up of superconducting qubits or a diatomic molecule in an optical lattice may be prepared in subradiant states. In connection with such an enlarged view we appropriately remind that the denomination subradiant states has been adopted [37] also to classify factorized states of generic bipartite systems from which the two subsystems evolve with no energy exchange between them, maintaining moreover their statistical independence. In this paper we call subradiant state a generalized state of this type, that is regardless of the specific nature of both the subsystems.

Recently, for instance, the environmental noise plaguing the unitary evolution of superconducting artificial atoms in a circuit QED setting, has been modeled coupling the dynamical variables of the circuit to the degrees of freedom of a fermionic bath. Systems of this type, where bosonic degrees of freedom are absent, might admit subradiant states under appropriate conditions [38].

In this paper we go beyond the original notion of subradiance wondering on the existence of even initially entangled pure or mixed states of the bipartite system evolving as if $A$ and $B$ were decoupled. This condition, guaranteeing the absence of energy exchanges between the two subsystems, might be of interest in any applicative protocol based on quantum processes involving storage steps. Another dynamical property of such states is that the quantum covariance of any pair of observables $O_A$ and $O_B$ acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, each one invariant with respect to the free evolution of the corresponding subsystem, keeps its initial value even if such observables do not commute with $H_I$. When states of this kind exist, we call them interaction-free evolving (IFE) states of the bipartite system. These states should not be confused with decoherence free states giving rise to celebrated decoherence free subspaces DFS (see e.g. review paper [39]). DFS are analyzed in the context of non-unitary evolution of an open quantum system living in some Hilbert space $\mathcal{H}$. One says that a linear subspace $\mathcal{H} \subset \mathcal{H}$ provides a DFS if the evolution of the system restricted to $\mathcal{H}$ is unitary. Hence if the initial state vector belongs $\mathcal{H}$ it stays there and hence does not lose quantum coherence. Here, we assume that the evolution of the bipartite system is unitary on $\mathcal{H}_A \otimes \mathcal{H}_B$. Of course the subsystem $B$ might be such to play the role of environment of $A$. We empha-
size that in this case too an IFE state is a state of the compound system A+B, unitarily evolving on $\mathcal{H}_A \otimes \mathcal{H}_B$. Generally speaking, as previously underlined, in an IFE state A and B exhibit entanglement at all times even if it might happen as well that an IFE state keeps a factorized form $|\psi_A(t)\psi_B(t)\rangle$ as time progresses. In this case the state $|\psi_A(t)\rangle$, belonging to $\mathcal{H}_A$, is indeed a decoherence-free state since $|\psi_A(t)\rangle = \exp(-iH_A t)|\psi_A(0)\rangle$, by IFE state definition. Recall that if we consider a non-unitary evolution as a reduction of the unitary one when the system S is coupled to an environment E and the interaction system-environment Hamiltonian reads $H_{SE} = \sum_a S_a \otimes E_a$, then DFS is spanned by vectors $|\psi\rangle$ satisfying $S_a|\psi\rangle = \lambda_a|\psi\rangle$ [39]. Hence, if all systems operators $S_a$ are Hermitian, then a nontrivial DFS $\tilde{\mathcal{H}}$ exists only when all $S_a$ mutually commute on $\tilde{\mathcal{H}}$. Interestingly, as we show in this paper, a similar condition governs the existence of IFE states.

The main result of this paper is the construction of the characteristic equation for both pure and mixed IFE states, that is the equation whose set of solutions singles out all and only the IFE states of a given bipartite system. In order to demonstrate the practical usefulness of such an equation, we solve it in the non trivial case of a non-homogeneous spin star system finding all its IFE pure states.

II. IFE PURE STATES

Let us consider the following

**Definition 1** A normalized vector $|\psi\rangle \in \mathcal{H}$ is an IFE pure state if it satisfies the following equation

$$e^{-iHt}|\psi\rangle \sim e^{-iH_0t}|\psi\rangle ,$$

where $\sim$ denotes an equivalence relation: $|\psi\rangle \sim |\phi\rangle$ iff $|\psi\rangle = e^{i\alpha}|\phi\rangle$ with $\alpha$ being a real number (a relative phase).

It means that $|\psi\rangle$ is an IFE state iff there exists $\alpha \in \mathbb{R}$ such that

$$e^{-iHt}|\psi\rangle = e^{-i\alpha t}e^{-iH_0t}|\psi\rangle ,$$

at any time instant $t$. In order to characterize all the IFE pure states of the system, let us begin by stating that $|\psi\rangle$ is a solution of eq. (3) if, for any nonnegative integer $n$,

$$H^n|\psi\rangle = (H_0 + \alpha I)^n|\psi\rangle$$

which implies that eq. (3) is satisfied for all $t$. For $n = 1$ one obtains

$$H_1|\psi\rangle = \alpha |\psi\rangle ,$$

that is, $|\psi\rangle$ defines an eigenvector of $H_1$ and $\alpha$ denotes the corresponding eigenvalue. It means that $|\psi\rangle$ is a zero-mode of $H_1^{(\alpha)} := H_1 - \alpha I$, i.e.

$$|\psi\rangle \in \text{Ker} H_1^{(\alpha)} .$$

Moreover, starting from eq. (4) and exploiting eq. (6) we also obtain

$$H_1^{(\alpha)}H_0|\psi\rangle = 0$$

and, by induction

$$H_1^{(\alpha)k}H_0^n|\psi\rangle = 0 ,$$

for all $n$. Now, for any eigenvalue $\alpha$ of $H_1$ let us define

$$N_\alpha := \bigcap_n \text{Ker} (H_1^{(\alpha)k}H_0^n) .$$

It is clear that $N_\alpha$ defines a linear subspace of $\mathcal{H}$. Of course it may happen that $N_\alpha = \{0\}$. It is easy to show that if $|\psi\rangle \in N_\alpha \neq \{0\}$, then equation (3) holds. In this way we have proved

**Theorem 1** A vector $|\psi\rangle \in \mathcal{H}$ is an IFE state iff $|\psi\rangle \in N_\alpha \neq \{0\}$ for some eigenvalue $\alpha$ of the interaction part $H_I$.

It is clear that the space $\mathcal{N}$ of IFE states is stratified into mutually orthogonal sectors

$$\mathcal{N} = \bigcup_\alpha N_\alpha ,$$

with $N_\alpha \perp N_\beta$ for $\alpha \neq \beta$. In particular if $|\psi\rangle \in N_0$ then

$$e^{-iHt}|\psi\rangle = e^{-iH_0t}|\psi\rangle ,$$

at any time instant $t$.

Now, we show that the formula (10) defining $N_\alpha$ may be considerably simplified. Note that

$$[H_0,H_I]_{|N_0} = 0 .$$

Indeed, for any $|\psi\rangle \in N_0$ one finds $H_0H_I|\psi\rangle - H_IH_0|\psi\rangle = 0$. Conversely, if $|\psi\rangle \in \text{Ker}H_I$ and $[H_0,H_I]|\psi\rangle = 0$, then $H_IH_0^n|\psi\rangle = 0$ for $n = 1,2,\ldots$. To prove this let $\mathcal{M} = \text{Ker}[H_0,H_I]$ and let $\{|e_1\rangle,\ldots,|e_r\rangle\}$ be an orthonormal basis in $\mathcal{M}$ such that

$$H_0|e_k\rangle = \sum_{k=1}^r a_k |e_k\rangle \langle e_k| ,$$

and

$$H_I|e_k\rangle = \sum_{k=1}^r b_k |e_k\rangle \langle e_k| ,$$

provide spectral decompositions of $H_0$ and $H_I$ restricted to $\mathcal{M}$. Now, let $|\psi\rangle \in \text{Ker}H_I$ and $|\psi\rangle \in \text{Ker}[H_0,H_I]$, that is, we assume that $\text{Ker}H_I \cap \mathcal{M} \neq \{0\}$. Suppose that $\text{Ker}H_I \cap \mathcal{M}$ is spanned by $\{|e_1\rangle,\ldots,|e_l\rangle\}$ with $l \leq r$, that is, $H_I|e_k\rangle = \sum_{k=i+1}^r b_k |e_k\rangle \langle e_k|$ due to $H_I|e_k\rangle = 0$ for $k = 1,\ldots,l$. One immediately finds

$$H_IH_0^n|\psi\rangle = \sum_{k=l+1}^r a_k^n b_k |e_k\rangle \langle e_k| |\psi\rangle = 0 ,$$

for $n = 1,2,\ldots,l$.
due to the fact that \(|\psi\rangle = \sum_{k=1}^{l} x_k |e_k\rangle \in \text{Ker} H_I \cap \mathcal{M}\). Hence, \(H_I H_I^\dagger |\psi\rangle = 0\) whenever \(H_I |\psi\rangle = 0\) and \([H_0, H_I] |\psi\rangle = 0\). In a similar way one shows that \(H_I^{(\alpha)} H_I^\dagger |\psi\rangle = 0\) whenever \(H_I^{(\alpha)} |\psi\rangle = 0\) and \([H_0, H_I^{(\alpha)}] |\psi\rangle = 0\).

**Corollary 1** The subspace \(\mathcal{N}_0\) may be represented as follows:

\[
\mathcal{N}_0 = \text{Ker} H_I \cap \text{Ker} [H_0, H_I],
\]

and similarly

\[
\mathcal{N}_\alpha = \text{Ker} H_I^{(\alpha)} \cap \text{Ker} [H_0, H_I^{(\alpha)}],
\]

for any eigenvalue \(\alpha\) of the interaction part \(H_I\).

It is clear that to define \(\mathcal{N}_\alpha\) one has to solve eigenvalues of \(H_I\) which might be highly nontrivial. One may ask a simpler question, namely, how to check whether IFE states do exist. Combining (16) and (17) one arrives at the following existence condition:

**Corollary 2** A Hamiltonian \(H = H_0 + H_I\) allows for IFE states if and only if \(\text{Ker} [H_0, H_I]\) is nontrivial.

Indeed if \(|\psi\rangle\) is an IFE state then there exists \(\alpha \in \mathbb{R}\), eigenstate of \(H_I\), such that \(\mathcal{N}_\alpha\) is not trivial. This existence in turn implies that \(|\psi\rangle \in \text{Ker} [H_0, H_I]\) = \(\text{Ker} [H_0, H_I]\). Viceversa if \(\mathcal{M} = \text{Ker} [H_0, H_I]\) is not trivial, \(H_0\) and \(H_I\) may be simultaneously diagonalized in \(\mathcal{M}\) and each common eigenstate is an IFE state since it belongs to \(\mathcal{N}_\alpha\) for some \(\alpha\). We emphasize that had we put \(\alpha = 0\) in eq. (3), the existence of IFE states belonging to the restricted set accordingly defined, would not be guaranteed by the condition expressed by corollary 2. The reason is that we cannot be sure to find zero among the eigenvalues of \(H_I\) restricted to \(\mathcal{M}\).

Suppose now that one deals with a bipartite system in \(\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B\) described by

\[
H_0 = H_A + H_B,
\]

and the interaction term \(H_I\) (to simplify notation we identify \(H_A\) with \(\mathcal{H}_A \otimes \mathbb{I}_B\) and similarly for \(H_B\)). Note that the corresponding bipartite IFE states do exhibit absence of energy exchanges between subsystems \(A\) and \(B\). Indeed, for any \(t\) one finds

\[
\mathcal{E}_A(t) := \langle \psi | e^{iH_I t} H_A e^{-iH_I t} |\psi\rangle = \langle \psi | e^{iH_0 t} H_A e^{-iH_0 t} |\psi\rangle = \langle \psi | H_A |\psi\rangle,
\]

and

\[
\mathcal{E}_B(t) := \langle \psi | e^{iH_I t} H_B e^{-iH_I t} |\psi\rangle = \langle \psi | e^{iH_0 t} H_B e^{-iH_0 t} |\psi\rangle = \langle \psi | H_B |\psi\rangle,
\]

which shows that energies \(\mathcal{E}_A(t)\) and \(\mathcal{E}_B(t)\) of two subsystems are conserved. Of course the converse is generally not true. Let us consider for example the time evolution obtained starting from a stationary state of \(H\). Under this condition the mean values of both \(H_A\) and \(H_B\), as well as of any time-independent observable of the system, are obviously stationary but the eigenstates of \(H\) do not in general satisfy eq. (3).

### III. IFE MIXED STATES

In this section we generalize the notion of IFE for mixed states. Denote by \(S(\mathcal{H})\) the space of density operators living in \(\mathcal{H}\) and consider the Hamiltonian dynamics generated by \(H\). One has the following generalization of Definition 1.

**Definition 2** A density operator \(\rho \in S(\mathcal{H})\) is an IFE mixed state if it satisfies the following equation

\[
e^{-iH_I t} \rho e^{iH_I t} = e^{-iH_0 t} \rho e^{iH_0 t},
\]

at any time instant \(t \in \mathbb{R}\).

It is clear that if \(|\psi\rangle \langle \psi|\), then the above definition reproduces Definition 1.

Let \(|\psi_\alpha^i\rangle\) denotes an orthonormal basis in \(\mathcal{N}_\alpha\), that is,

\[
H_I |\psi_\alpha^i\rangle = \alpha |\psi_\alpha^i\rangle,
\]

for \(i = 1, \ldots, n_\alpha = \text{dim} \mathcal{N}_\alpha\). One immediately has

**Corollary 3** A density operators \(\rho\) defines an IFE mixed state iff

\[
\rho = \sum_\alpha \sum_{i,j=1}^{n_\alpha} p_{\alpha}^{(i,j)} |\psi_\alpha^i\rangle \langle \psi_\alpha^j|,
\]

where \(p_{\alpha}^{(i,j)} \geq 0\) and \(\sum_\alpha \sum_{i,j=1}^{n_\alpha} p_{\alpha}^{(i,j)} = 1\).

Let us observe that any IFE mixed state define a direct sum of positive operators

\[
\rho = \bigoplus_\alpha \rho_\alpha,
\]

where

\[
\rho_\alpha = \sum_{i,j=1}^{n_\alpha} p_{\alpha}^{(i,j)} |\psi_\alpha^i\rangle \langle \psi_\alpha^j|,
\]

is supported on \(\mathcal{N}_\alpha\). Hence, any IFE pure state belongs to single sector \(\mathcal{N}_\alpha\) whereas a genuine IFE mixed state defines a mixture of positive operators supported on all sectors \(\mathcal{N}_\alpha\).

Again, it is clear that if one deals with a bi-partite system and if \(\rho_{AB}\) is IFE state then

\[
\mathcal{E}_A(t) = \text{Tr}(e^{-iH_I t} \rho_{AB} e^{iH_I t} H_A) = \text{Tr}(\rho_{AB} H_A),
\]

and the same for \(\mathcal{E}_B(t)\). Hence, there is no energy exchange between subsystems \(A\) and \(B\) for any IFE mixed state.
Consider a non-homogeneous spin star system consisting of a central spin coupled to $N$ mutually not interacting spins around it. The Hamiltonian describing such a system has the form \[ H_0 = \omega_0 \sigma_z + \omega \sum_{i=1}^{N} \sigma_z^{(i)} \] (25) and
\[ \sigma_z \equiv \frac{1}{2}(\sigma_x + i\sigma_y) \] whereas the Pauli operators $\sigma_z$ describing the $i$-th spin are denoted by $\sigma_z^{(i)}$, $\sigma_\sigma^{(i)} \equiv \frac{1}{2}(\sigma_x^{(i)} + i\sigma_y^{(i)})$.

The dynamical variables of the central spin are represented by the Pauli operators $\sigma_z$, $\sigma_\pm \equiv \frac{1}{2} \left( \sigma_x \pm i\sigma_y \right)$ whereas the Pauli operators describing the $i$-th $(i = 1, \ldots, N)$ spin are denoted by by $\sigma_\sigma^\pm \equiv \frac{1}{2} \left( \sigma_x^\pm \pm i\sigma_y^\pm \right)$.

Considering this physical system as bipartite and the central spin as one of the two subsystems, the main aim of this section is the construction of the set of all IFE pure states associated to the spin star system under scrutiny.

In order to do this let us begin by observing that a normalized state of our bipartite system can be always written in the form $|\Psi\rangle = (-)|\psi_-\rangle + (+)|\psi_+\rangle$ where $|\pm\rangle$ are the eigenstates of $\sigma_z$ with eigenvalues +1 and -1 respectively whereas $|\psi_\pm\rangle$ belong to the Hilbert space of the system constituted by the spins $1, \ldots, N$ and satisfying the condition $|\psi_+|^2 + |\psi_-|^2 = 1$.

In view of corollary (1) and corollary 2, we must diagonalize $H_0$ and $H_I$ within the vectorial space $\text{Ker}(H_0, H_I)$ provided $\text{dim}(\text{Ker}(H_0, H_I)) > 0$. It is easy to demonstrate that the equation $[H_0, H_I] |\psi\rangle = 0$ may be rewritten as follows
\[ [H_0, H_I] |\psi\rangle = 2(\omega - \omega) \left[ \sum_{i=1}^{N} \gamma_i \sigma_\sigma^{(i)} |\psi_-\rangle - \sum_{i=1}^{N} \gamma_i \sigma_\sigma^{(i)} |\psi_+\rangle \right] = 0 \] (27)
which in turn requires the existence of solutions for the two equations
\[ \sum_{i=1}^{N} \gamma_i \sigma_\sigma^{(i)} |\psi_\pm\rangle = 0 \] (28)

We solve eq. \[ \text{(28)}, \] exploiting the method reported in Ref. \[ \text{[1]} \]: let us introduce the operators $A_\pm$ given by
\[ A_\pm = \exp\left( \sum_{i=1}^{N} g_{\pm i}^{(i)} \sigma^{(i)}_z \right) \] (29)
where the complex parameters $g_{\pm i}^{(i)}$ will be chosen later.

The two operators $A_+$ and $A_-$ thus defined are in general neither unitary nor Hermitian. However they are not singular and thus $A_\pm^{-1}$ there exist. Accordingly eq. \[ \text{(28)} \] may be transformed as follows
\[ A_\pm^{-1} \sum_{i=1}^{N} \gamma_i \sigma_\sigma^{(i)} A_\pm A_\pm^{-1} |\psi\rangle = 0 \] (30)

On the other hand, it is easy to demonstrate that
\[ A_\pm^{-1} \sigma_\sigma^{(i)} A_\pm = \sigma_\sigma^{(i)} e^{\pm 2g_{\pm i}^{(i)}} \] (31)
and then, choosing the parameters $g_{\pm i}^{(i)}$ $(i = 1, \ldots, N)$ in such a way that $\gamma_i = \gamma e^{\pm 2g_{\pm i}^{(i)}}$ with $\gamma = \sqrt{\sum_{i=1}^{N} \gamma_i^2}$, the condition under which the state $|\Psi\rangle = (-)|\psi_-\rangle + (+)|\psi_+\rangle$ belongs to the kernel of $[H_0, H_I]$ becomes
\[ \sum_{i=1}^{N} \gamma_i \sigma_\sigma^{(i)} (A_\pm^{-1} |\psi_\pm\rangle) = 0 \] and
\[ \sum_{i=1}^{N} \gamma_i \sigma_\sigma^{(i)} (A_\pm^{-1} |\psi_-\rangle) = 0 \] (32)

These equations show that due to the operators $A_\pm$ we get rid of the non homogeneous character of Eq. \[ \text{(30)} \] which it appears through the $i$-dependence of the coupling constants ($\gamma_i$).

Let us note that the choice of the parameters $g_{\pm i}^{(i)}$ guarantees that the two operators $A_+$ and $A_-$ satisfy $A_+ A_- = A_- A_+ = I$. Let’s moreover observe that the states $|\tilde{\psi}_\pm\rangle = A_\pm^{-1} |\psi_\pm\rangle$ satisfying eq. \[ \text{(32)} \] are well known in terms of the simultaneous eigenstates $|r, m, \nu\rangle$ of the square and of the $z$-component of the total angular momentum of the $N$ uncoupled spins
\[ S^2 |r, m, \nu\rangle = \frac{1}{2} \sum_{i=1}^{N} \sigma^{(i)}_z^2 |r, m, \nu\rangle = (r + 1) |r, m, \nu\rangle \] (33)
where $r = 0, 1, \ldots, \frac{N}{2}$ if $N$ is even and $r = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N - 1}{2}$ if $N$ is odd. Moreover
\[ S_z |r, m, \nu\rangle = \frac{1}{2} \sum_{i=1}^{N} \sigma^{(i)}_z |r, m, \nu\rangle = m |r, m, \nu\rangle \] (34)

with $m = r - r + 1, \ldots, r$. The quantum number $\nu = 1, 2, \ldots, \nu(r)$ with
\[ \nu(r) = \left( \frac{N}{2} - r \right) + \left( \frac{N}{2} - r - 1 \right) \] (35)
and $\left( \frac{N}{2} - r - 1 \right) = 0$ allows to distinguish between different states of the coupled angular momentum basis characterized by the same $r$ and $m$. It is possible to convince oneself that $|\tilde{\psi}_+\rangle \equiv \sum_{r, \nu} C_{r, \nu}^+ |r, r, \nu\rangle$ and $|\tilde{\psi}_-\rangle \equiv \sum_{r, \nu} C_{r, \nu}^- |r, -r, \nu\rangle$ with $C_{r, \nu}^\pm \in \mathbb{C}$. We may thus claim that a generic state $|\psi\rangle$ satisfying eq. \[ \text{(27)} \] may be written as follows
\[ |\psi\rangle = (+) \sum_{r, \nu} C_{r, \nu}^+ A_+ |r, r, \nu\rangle + (-) \sum_{r, \nu} C_{r, \nu}^- A_- |r, -r, \nu\rangle \] (36)
It is remarkable that $Ker[H_0, H_I]$ for the Hamiltonian model under scrutiny coincides with $Ker H_I$ which means that $H_I |\psi\rangle = 0$ iff $|\psi\rangle$ is given by eq. (38). This result is a direct consequence of the fact that the resolution of the equation $H_I |\psi\rangle = 0$ leads exactly to eqs. (32). In view of corollary 2 we may thus claim that $N_{\alpha}$ is empty for each eigenvalue $\alpha \neq 0$ of $H_I$. We thus may conclude that the space $\mathcal{N}$ of IFE pure states for our Hamiltonian model coincides with $N_0$. It is interesting to investigate the diagonalization problem of $H_0$ within $\mathcal{N} \equiv N_0$. To this end let’s observe that both the operators $A_+$ and $A_-$ commute with the $z$ component of the total angular momentum operator $S_z$ of the $N$ spins. This property directly implies that the states $A_+ |r, r, \nu\rangle$ as well as the states $A_- |r, -r, \nu\rangle$ are eigenstates of $S_z$ with eigenvalues $r$ and $-r$ respectively. We have indeed

$$S_z A_\pm |r, \pm r, \nu\rangle = A_\pm A_\mp S_z A_\pm |r, \pm r, \nu\rangle$$

$$= A_\pm S_z |r, \pm r, \nu\rangle = \pm r A_\pm |r, \pm r, \nu\rangle \quad (37)$$

On the other hand, it is immediate to convince oneself that they are also eigenstates of $H_0$ correspondent to the eigenvalues $(\omega + 2r\omega)$ and $-(\omega + 2r\omega)$ respectively. This circumstance in turn means that these states are also eigenstates of the total Hamiltonian given by eq. 11 being simultaneous eigenstates of $H_0$ and $H_I$. In other words the IFE states space may be represented as a direct sum of appropriate vectorial subspaces invariant under the action of the total Hamiltonian $H$. As a consequence we might envision initial conditions starting from which the system effectively evolves conserving the value of its initial entanglement no matter the measure used. Our results on the structure of $N_0$ play an important role in the context of the problem of the diagonalization of non-homogeneous spin star system Hamiltonian model under scrutiny in this section. In the near past, indeed, many efforts have been made in order to find the spectrum of such Hamiltonian but, until now only a particular set of eigensolutions are known [10].

V. CONCLUSIVE REMARKS

In this paper we have introduced a new class of states of a bipartite system christened IFE states. This set of states encompasses all those initial conditions of the compound system from where each subsystem evolves with no energy exchange with the other one and leaving unmodified the level of mutual entanglement whatever measure is adopted. These properties stem from cooperative effects leading through quantum interference processes, to the cancellation of any dynamical consequence of the coupling term $H_I$. We stress that since the constructions of the IFE states space requires the resolution of their characteristic equations in the Hilbert state of the given bipartite system, it may happen that it is empty. It is however worth noticing that when subradiant states exist then they are IFE states too, allowing us to claim that our definition of IFE states generalizes indeed that of subradiant state. Our main result is constituted by the two characteristic equations of the states (Theorem 1 and Corollary 3) as well as construction of the set of all the IFE states of a nontrivial Hamiltonian model of evergreen interest. A remarkable merit of such a result is its universality with respect to time-independent Hamiltonian models which means that the characteristic equations here reported are applicable to any bipartite system evolving unitarily. The more intriguing situation corresponding to the evolution of a bipartite system in presence of an environment is currently under investigation and will be presented elsewhere.

VI. ACKNOWLEDGEMENTS

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