The 8k - 3 Instanton

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Abstract
We give an explicit parameterization of the general 8k -3 instanton.
It seems that gauge field theories have a good chance for describing nature. At macroscopic and microscopic levels, Abelian and non-Abelian classical and quantum field theories have an amazing degrees of success in physics or mathematics. The simplicity and the beauty, manifest and hidden symmetry mix together in one frame. Non–Abelian gauge theory proved to be a very rich field of research. Quark confinement stands as a great obstacle to achieve the first and the oldest dream of high energy physics: What bind nucleons together? A question that is still rigorously unanswered for decades. For sure, non–perturbative effects must be taken into account[1]. Constructing all ADHM instantonic solutions explicitly is important task[2][3]. In this letter, we would like to announce an explicit full parametrization of instantons solutions in classical gauge field theory over $\mathbb{R}^2$ and $\mathbb{R}^4$.

Initially, we solved the ADHM conditions by iteration

$$S^0 \rightarrow S^1 \rightarrow S^3 \rightarrow S^7 \rightarrow \cdots$$

We used the real ADHM over $S^1$ (the conformal compactification of $\mathbb{R}^1$) to find explicitly the complex ADHM then extending it to quaternions was not difficult. Here, we will not go into details of the method but we derive the corresponding 8$k$ – 3 gauge field by means of a simple systematic way that can be verified easily and compared to the 5$k$ multi instanton solutions and at the same time avoiding many technicalities.

We work over $\mathbb{R}^4$ which can be endowed with a natural quaternionic structure

$$x = x_\mu e_\mu = x_0 e_0 + x_i e_i ,$$

for $e_0 = 1$ and $e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k$, where $i,j$ and $k$ run from 1 to 3 and $\epsilon_{ijk}$ is the three dimensional skew symmetric Levi–Civita tensor. We can also represent $e_0$ by the 4 × 4 real identity matrix and $e_i$ by the canonical left $E_i$ or right $\bar{E}_i$ quaternionic (for differential geometric minded readers, we mean hyperKahler) structure[4](page 159). We have the following three 4 × 4 real antisymmetric matrices

$$(E_i)_{\mu\nu} = \left(-\delta_{0\mu}\delta_{i\nu} + \delta_{0\nu}\delta_{i\mu} - \epsilon_{i\mu\nu}\right) ,

\left(\bar{E}_i\right)_{\mu\nu} = \left(-\delta_{0\mu}\delta_{i\nu} + \delta_{0\nu}\delta_{i\mu} + \epsilon_{i\mu\nu}\right) ,$$

explicitly

$$(E_i)_{jk} = -\epsilon_{ijk} , \quad (E_i)_{0\nu} = -\delta_{i\nu} , \quad (E_i)_{00} = 0 .$$

For example

$$E_1 \quad = \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ,$$
and so on for the other $E_i$ and $\overline{E}_i$. These quaternionic structures are widely known as the $\eta$ symbols \cite{5}. We can check that

$$E_i E_j = -\delta_{ij} + \epsilon_{ijk} E_k,$$

$$\overline{E}_i \overline{E}_j = -\delta_{ij} - \epsilon_{ijk} \overline{E}_k.$$  

One may look at \cite{6} and references therein for properties of these $E$'s.

We would like to find the classical $su(2)$ gauge field $A_\mu$ that satisfies the antiself duality equation

$$F_{\mu\nu} = -\frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}.$$  

First we introduce the antiself dual $so(4)$ basis

$$\vartheta_{\mu\nu} = (E_{\mu} E_{\nu} - E_{\nu} E_{\mu}),$$  

by $E$, we mean conjugation i.e. $(E_0 = E_0, E_i = -E_i)$, we find

$$\vartheta_{\mu\nu} = -\frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \vartheta_{\alpha\beta}.$$  

Consider the ansatz

$$A_\mu = -\frac{1}{4} \vartheta_{\mu\nu} \partial_{\nu} \ln (\phi).$$  

After simple calculation

$$A_0 = \frac{-(E_1 \partial_1 \phi + E_2 \partial_2 \phi + E_3 \partial_3 \phi)}{2\phi},$$

$$A_1 = \frac{-(E_2 \partial_3 \phi - E_3 \partial_2 \phi - E_1 \partial_0 \phi)}{2\phi},$$

$$A_2 = \frac{-(E_3 \partial_1 \phi - E_1 \partial_3 \phi - E_2 \partial_0 \phi)}{2\phi},$$

$$A_3 = \frac{-(E_1 \partial_2 \phi - E_2 \partial_1 \phi - E_3 \partial_0 \phi)}{2\phi},$$  

notice that $A_\mu$ is purely imaginary quaternions since it lies in the $su(2)$ algebra. By straightforward substitution in

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$  

"miraculous" cancellations due to the magic duality leads to

$$F_{01} + F_{23} = a_1 E_1 + \frac{\partial_1 \partial_2 \phi - \partial_2 \partial_1 \phi}{\phi} E_2 + \frac{\partial_2 \partial_3 \phi + \partial_3 \partial_2 \phi}{\phi} E_3$$

$$= \begin{pmatrix}
0 & -a_1 & -\partial_1 \partial_3 \phi + \partial_3 \partial_1 \phi & -\partial_1 \partial_2 \phi + \partial_2 \partial_1 \phi \\
-a_1 & 0 & -\partial_2 \partial_3 \phi + \partial_3 \partial_2 \phi & \partial_1 \partial_3 \phi - \partial_3 \partial_1 \phi \\
\partial_1 \partial_2 \phi - \partial_2 \partial_1 \phi & -\partial_2 \partial_3 \phi + \partial_3 \partial_2 \phi & 0 & 0 \\
\partial_1 \partial_3 \phi + \partial_3 \partial_1 \phi & -\partial_1 \partial_2 \phi + \partial_2 \partial_1 \phi & 0 & a_1
\end{pmatrix},$$  

(7)
\[ F_{02} + F_{31} = \frac{\partial_1 \partial_2 \phi + \partial_1 \partial_3 \phi}{\phi} E_1 + a_2 E_2 + \frac{\partial_2 \partial_3 \phi - \partial_3 \partial_1 \phi}{\phi} E_3 \]
\[
= \begin{pmatrix}
0 & -\frac{\partial_1 \partial_3 + \partial_3 \partial_1 \phi}{\phi} & -a_2 \\
\frac{\partial_1 \partial_2 + \partial_2 \partial_1 \phi}{\phi} & 0 & \frac{\partial_2 \partial_3 + \partial_3 \partial_2 \phi}{\phi} \\
\frac{\partial_2 \partial_3 - \partial_3 \partial_2 \phi}{\phi} & -a_2 & 0
\end{pmatrix},
\]
\[ (8) \]

\[ F_{03} + F_{12} = \frac{\partial_1 \partial_3 \phi - \partial_3 \partial_2 \phi}{\phi} E_1 + \frac{\partial_2 \partial_3 \phi + \partial_3 \partial_1 \phi}{\phi} E_2 + a_3 E_3 \]
\[
= \begin{pmatrix}
0 & -\frac{\partial_1 \partial_3 + \partial_3 \partial_1 \phi}{\phi} & -a_3 \\
\frac{\partial_1 \partial_2 + \partial_2 \partial_1 \phi}{\phi} & 0 & \frac{\partial_2 \partial_3 + \partial_3 \partial_2 \phi}{\phi} \\
\frac{\partial_2 \partial_3 - \partial_3 \partial_2 \phi}{\phi} & -a_3 & 0
\end{pmatrix},
\]
\[ (9) \]

where
\[ a_1 = \frac{\partial_0 \partial_0 \phi + \partial_1 \partial_1 \phi - \partial_2 \partial_2 \phi - \partial_3 \partial_3 \phi}{2\phi}, \]
\[ a_2 = \frac{\partial_0 \partial_0 \phi - \partial_1 \partial_1 \phi + \partial_2 \partial_2 \phi - \partial_3 \partial_3 \phi}{2\phi}, \]
\[ a_3 = \frac{\partial_0 \partial_0 \phi - \partial_1 \partial_1 \phi - \partial_2 \partial_2 \phi + \partial_3 \partial_3 \phi}{2\phi}. \]
\[ (10, 11, 12) \]

Putting together all the conditions arisen from (7)–(12), we conclude that the antiself duality condition holds if and only if
\[ \partial_\mu \partial_\nu \phi = 0, \]
\[ \partial_0 \partial_0 \phi = \partial_1 \partial_1 \phi = \partial_2 \partial_2 \phi = \partial_3 \partial_3 \phi. \]
\[ (13) \]

\[ F_{\mu\nu} \] simplifies tremendously
\[ F_{\mu\nu} = \frac{-\partial_{\mu\nu} \left( -2 (\partial_0 \partial_0 \phi) \phi + (\partial_0 \phi)^2 + (\partial_1 \phi)^2 + (\partial_2 \phi)^2 + (\partial_3 \phi)^2 \right)}{4\phi^2}. \]
\[ (14) \]

Now, we want to solve (13). Taking into account that \( F_{\mu\nu} \) decays at infinity with rate \( \frac{1}{r^2} \). Actually, there are two different kinds of solutions that we can find: Singular and non–singular. We start by the non–singular instanton, the easy solution is what we call the free part (all the \( a \)'s are quaternions and \( x \) as given in (9))
\[ \phi_{free} = \lambda_1 \overline{\lambda_1} + (x - a_1) (\overline{x - a_1}), \]
\[ (15) \]
after substitution in (4) and (14), we find directly the $k = 1$ BPST instanton [1].

Since (4) and (13) are additive, we can construct solutions with higher topological index $k$

$$\phi^k_{\text{free}} = \lambda_1 \bar{\lambda}_1 + (x - a_1) \bar{(x - a_1)} + \cdots + \lambda_k \bar{\lambda}_k + (x - a_k) \bar{(x - a_k)} ,$$

we recover the $5k$ multi–instanton solution. It is not hard to find the interaction term. From some simple quaternionic reasoning,

$$\phi^2_{\text{int}} = \lambda_1 \bar{\lambda}_2 + \lambda_2 \bar{\lambda}_1 + (x - a_1) \bar{(x - a_2)} + (x - a_2) \bar{(x - a_1)} ,$$

the important thing to notice is: $\phi_{\text{int}}$ is real exactly like $\phi_{\text{free}}$. Unambiguously, we can assemble any $k$ instanton by allowing all the interacting terms. Let us count the number of parameters: Any quaternionic $\lambda$ and $a$ contribute $4 + 4 = 8$ real parameters, so for generic $k$ solution, we have $\lambda_1 \cdots \lambda_k$ and $a_1 \cdots a_k$ quaternions or equivalently $8k$ real parameters. From our familiarity with the ADHM solution [3], we know that not all of $\lambda_1 \cdots \lambda_k$ are independent. The general solution composed of the free part and the interaction term is invariant under $\lambda \rightarrow \lambda T$, where $T$ is any generic quaternion of unit norm. Then the true number of parameters, after substracting the dimension of the automorphism group, will be $8k - 3$ as it should be. Without loss of generality, we can fix any of the $\lambda$ to be real and the others remain quaternions. for example, the $k = 5$

$$\phi^5_{\text{int}} = \lambda_1 \bar{\lambda}_2 + \lambda_2 \bar{\lambda}_1 + (x - a_1) \bar{(x - a_2)} + (x - a_2) \bar{(x - a_1)} ,$$

we know that the Abelian Seiberg–Witten monopole equations over $R^4$ admits only singular solutions. So, we also look for singular solutions of the self–dual Yang–Mills instanton. We know from the index theorem that the most general instanton, whether singular or non–singular has $8k - 3$ real parameters. We can try to plug some minus signs into (15) and (17) but we take a different route. We start by

$$\phi_{\text{free}} = \lambda_1 (x - a_1) \bar{(x - a_1)} \bar{\lambda}_1 ,$$

it satisfies (13) and any generic free singular solution can be written as

$$\phi^k_{\text{free}} = \lambda_1 (x - a_1) \bar{(x - a_1)} \bar{\lambda}_1 + \cdots + \lambda_k (x - a_k) \bar{(x - a_k)} \bar{\lambda}_k ,$$

finding the interacting term is only possible by allowing mixing between different indices and at the same time maintaining the reality of $\phi$, then

$$\phi^{12}_{\text{int}} = \lambda_1 (x - a_1) \bar{(x - a_2)} \lambda_2 + \lambda_2 (x - a_2) \bar{(x - a_1)} \bar{\lambda}_1 .$$

In fact singular solutions may be related to the Abelian monopole equation over $R^4$ but the details may take us too far. It is crucial to observe that the
numerator of $F_{\mu\nu}$ (in (14)) is a constant i.e. it has no $x$ dependence for both of the singular and non–singular solutions.

Returning to the self-duality equation $F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$, we have to use the right instead of the left quaternionic structure

$$E \rightarrow \overline{E}.$$ 

The corresponding $so(4)$ self–dual basis are

$$\bar{\vartheta}_{\mu\nu} = \left( \overline{E}_\mu \overline{E}_\nu - \overline{E}_\nu \overline{E}_\mu \right) \implies \bar{\vartheta}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \bar{\vartheta}^{\alpha\beta}.$$ 

and proceeding as above, one rederives (13).

For the sake of illustration, we conclude with a similar 2 dimensional problem. We work with u(1) gauge fields (Quantum Electrodynamics QED) over $\mathbb{R}^2$ and look for solutions that lead to an integral first Chern class

$$\frac{i}{2\pi} \int F_{\mu\nu} dx_\mu dx_\nu \in \mathbb{Z}.$$ 

This model in its own is very interesting, it shares many properties of the 4 dimensional problem \[7\]. Reducing from 4 dimensions to two dimensions, we must set to zero all the $e_2$ and $e_3$ components which amounts to change quaternions by complex. For example, $x = x_0 + e_1 x_1$ i.e. only $E_1$ survives

$$E_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ 

and so on. To make clear the discrepancy between the $5k$ and the $8k–3$ solution, we consider explicitly the $k = 2$ case, For

$$a_1 = 1 + 2i,$$
$$a_2 = 5 + 6i,$$
$$\lambda_1 = 4 + 0i,$$
$$\lambda_2 = 7 + 8i.$$ 

For the non–singular case, we have

$$\phi^2_{full} = 285 - 24x_0 + 4x_0^2 - 32x_1 + 4x_1^2$$ \hspace{1cm} (21)$$

which leads to

$$F_{01} = \frac{740 e_1}{(285 - 24x_0 + 4x_0^2 - 32x_1 + 4x_1^2)^2}.$$ \hspace{1cm} (22)$$

Whereas the singular case, we find

$$\phi^2_{full} = 16 \left( 151 - 10x_0 + 7x_0^2 - 88x_1 + 7x_1^2 \right)$$ \hspace{1cm} (23)$$
and

\[ F_{01} = \frac{-904 \epsilon_1}{(151 - 10x_0 + 7x_0^2 - 88x_1 + 7x_1^2)^2}. \] (24)

To make the difference clear, we plot them in Fig. 1 and 2 respectively.

The aim of this article is to provide the general form of instantons in gauge field theories. But this topological construction is governed by certain conditions, not every gauge group \( G \) over any manifold will lead to instantons. Over \( \mathbb{R}^2 \), it is \( \pi_1(\mathbb{G}) = \mathbb{Z} \), and in \( \mathbb{R}^4 \), we have \( \pi_3(\mathbb{su}(2) \sim S^3) = \mathbb{Z} \) that made possible the construction of instantons. Hence \( \mathbb{u}(1) \) in 4 dimensions does not admit instantonic solutions and the same for \( \mathbb{su}(2) \) in 6 dimensions \( (\pi_5(\mathbb{su}(2) = 0)) \). Our \( \mathbb{su}(3) \) at ten dimensions is quite different. The homotopical condition is

\[ \text{Over } \mathbb{R}^n, \quad \pi_{n-1}(\mathbb{G}) = \text{non-trivial}. \]

If instantons have to do anything with confinement then one expects that the Abelian projection idea should be supplied by some topological considerations. It will be also interesting to investigate higher dimensional cases.

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Figure 1: The full non–singular $k = 2$ instanton.
Figure 2: The full singular $k = 2$ instanton.