A Secure Self-synchronized Stream Cipher

Amir Daneshgar*
Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11155–9415, Tehran, Iran.

Fahimeh Mohebbipoor
Faculty of Mathematics and Computer Science
Kharazmi University
P.O. Box 15719-14911, Tehran, Iran.

Abstract
We follow two main objectives in this article. On the one hand, we introduce a security model called LORBACPA* for self-synchronized stream ciphers which is stronger than the blockwise LOR-IND-CPA, where we show that standard constructions as delayed CBC or similar existing self-synchronized modes of operation are not secure in this stronger model. Then, on the other hand, following contributions of G. Millérioux et.al., we introduce a new self-synchronized stream cipher and prove its security in LORBACPA* model.

1 Introduction
Analysis and design of stream ciphers is a classic and among the oldest subjects in cryptography, however, amazingly, there are still some challenging problems to be addressed by the experts in the field (e.g. see [6,15]).

Among these interesting and challenging problems one may recall the analysis and design of self-synchronized stream ciphers where despite the efforts made so far, there does not exist a deep understanding of design methods, analysis and security models for this kind of stream ciphers yet.

1.1 Main results
Although, it is hopeless to think of CCA secure self-synchronized stream ciphers because of their error-correction properties, we show that it is possible to design such systems which are secure in quite stronger models than the classical CPA setting. Our main contribution in this article is to propose such a self-synchronized stream cipher along with the security model.

Before we proceed, it is instructive to note that our results are based on three fundamental contributions in cryptography.

On the one hand, not only our security model are built on the basic contribution of Bellare, Desai, Jokipi and Rogaway in [2] who introduced the concept of left-or-right indistinguishability (LOR – IND) but also we adopt their fundamental approach for the method of security proof in Section [4].

*Correspondence should be addressed to daneshgar@sharif.ir.
On the other hand, we will basically adopt the notion of blockwise security introduced independently by Jouix, Martinet and Valette [12] and Bellare, Kohno and Namprempre [4] (also see [5, 9, 10, 13] for the background) since our self-synchronized stream cipher can be considered as an extension of modes of operation for block-ciphers.

Although, the adaption of these ideas are crucial in our contribution, but our system would not be secure without applying basic ideas from what G. Millérioux et.al. has done on connecting the design principles of self-synchronized stream cipher to basic concepts of synchronized control systems (see [16], [17] and references therein).

The canonical form of a Self-Synchronizing Stream Cipher (an SSSC in short) is made of a combination of a shift register, which acts as a state register with the ciphertext as input, together with a filtering function that provides the running key stream and an output function which combines the running key stream with plaintext to produce the cipher text. In particular, it is known that such canonical SSSC’s are IND–CPA secure under some conditions on the filtering function [3,8].

As a matter of fact, using the control-theoretic approach of G. Millérioux et.al., we have been able to make sure about the existence of free random initial states in our self-synchronized stream cipher that will guarantee the security of the system in a stronger setting than the CPA model traditionally used for modes of operation. These basic ideas will give rise to our proposed system introduced in Section 3. Also, we will discuss some more control-theoretic aspects of our designs in Section 5.

However, to begin, after covering the necessary background in the rest of this section, we will concentrate on the details of the proposed security model (i.e. the LORBACPA+ setup) in Section 2. Also, in this section we will elaborate on mentioning the basic ideas and problems using the classical CBC and CFB modes of operation and will mention two premature modified versions of these modes that will both illustrate the basic design principles on the one hand, and shows that the basic CBC and CFB modes are not flexible enough to lead to a fully secure system in our security model. This in conjunction with the control theoretic approach of G. Millérioux et.al. shows the importance of handling the initial vectors to gain maximal security and operational efficiency in the design of stream ciphers. The main setup of our proposed self-synchronized stream cipher will be introduced in Section 3 where we also present the full security proof in our security model in Section 4.

As for the notations, the symbol $\mathbb{F}_q = GF(q)$ for $q$ a prime power, stands for the finite field on $q$ elements, $\mathbb{R}$ is the field of real numbers, and $[n]$ stand for the set $\{1, 2, \ldots, n\}$, respectively. Also, if $b \in \{0, 1\}$ then $\bar{b} \in \{0, 1\}$ is the other complement bit, i.e, $\bar{b} \overset{\text{def}}{=} b - 1$ modulo 2.

A vector of $n$ elements in $\mathbb{F}_q^n$ is denoted by $s = (s_1, s_2, \ldots, s_n)^T$. If the elements of a vector $s$ are functions of a variable $t$, then we write $s(t)$ to refer to the vector $s$ at time $t$. The set of $n \times n$ matrices with entries in $\mathbb{F}_q$ is denoted by $M^{n \times n}$. The symbol $I$ stands for the identity matrix, and $0$ stands for the zero matrix (when the dimension is clear from the context). The symbol $s \overset{\text{def}}{\triangleleft} \mathbb{F}_q^n$ demonstrates the action of picking $s$ uniformly at random from the set $\mathbb{F}_q^n$.

### 1.2 Self-synchronized stream-ciphers

From a data-transmission point of view a self-synchronized stream cipher is a master-slave communication system consisting of a transmitter and a receiver as

\[
\Sigma_\varphi: \begin{cases} 
\mathbf{z}(t) = f_\varphi(\mathbf{c}(t - l), \ldots, \mathbf{c}(t - l')), \\
\mathbf{c}(t + d) = enc_\varphi(\mathbf{z}(t), \mathbf{p}(t)) \end{cases}, \\
\Sigma'_\varphi: \begin{cases} 
\tilde{\mathbf{z}}(t + d) = f_{\varphi'}(\mathbf{c}(t - l), \ldots, \mathbf{c}(t - l')), \\
\tilde{\mathbf{p}}(t + d) = dec_{\varphi'}(\tilde{\mathbf{z}}(t + d), \mathbf{c}(t + d)) \end{cases},
\]

(1)
where \( f_s \) is the function that generate the key-streams \( \{z(t)\} \) and \( \hat{z} \). The ciphertext \( c(t + d) \) is worked out through an encryption function \( enc_s \) and the decryption is performed through a function \( dec_s \) depending on the ciphertext \( c(t + d) \), in which \( l' \geq l \geq 1 \) and \( d \geq 0 \) is the system delay. Note that if \( dec_s = enc_s^{-1} \) and for some finite time synchronization delay \( t_s > d \), we have

\[
\forall t \geq t_s, \quad \hat{z}(t + d) = z(t),
\]

then

\[
\forall t \geq t_s \geq 1 \quad \hat{p}(t + d) = dec_s(\hat{z}(t + d), c(t + d)) = p(t).
\]

In this setting, \( t_s \) is transmitted on the channel with a delay \( d \) of \( \Sigma \), and hence in this setting, for instance, equations \( 1 \) imply that \( c(t + d) \) is generated at time \( t \) using the data \( z(t) \) and \( p(t) \), but is transmitted on the channel \( d \) clocks later, at time \( t + d \). On the other hand, the equations of \( \Sigma' \) show that \( \hat{p}(t + d) \) is computed at time \( t + d \) using \( \hat{z}(t + d) \) and \( c(t + d) \) that is received through the communication channel with a delay \( d \). These facts imply that, generically, in a self-synchronized setting, and when both systems are synchronized, we have \( \hat{z}(t + d) = z(t) \) and \( \hat{p}(t + d) = p(t) \) for \( t \geq 1 \). In what follows we always assume that the delayed signals \( z(t) \) and \( p(t) \) are set to \( \perp \) when they are not defined.

A study of stream ciphers through a control theoretic approach has been pioneered by G. Millérioux et.al. (see [17] and references therein) from which we know that the setup introduced in Equations \( 1 \) is equivalent to the following setup under flatness condition (see Lemma \( 1 \)).

\[
\Gamma_s : \begin{cases} 
    s(t + 1) = \phi_s(s(t), p(t)), \\
    c(t + d) = \varepsilon_s(s(t), p(t)),
\end{cases}
\]

\[
\Gamma'_s : \begin{cases} 
    \hat{s}(t + 1) = \beta_s(\hat{s}(t), c(t)), \\
    \hat{p}(t) = \delta_s(\hat{s}(t), c(t)),
\end{cases}
\]

where \( \phi_s \) and \( \beta_s \) are the functions that generate \( s(t) \) and \( \hat{s}(t) \) and also \( \varepsilon_s, \delta_s \) are the encryption and the decryption functions. Such a system is said to be finite-time self-synchronized if there exists some finite time \( t_s \) such that for any initial condition

\[
\forall t \geq t_s, \quad \hat{s}(t + d) = s(t).
\]

Note that after synchronization, i.e. for all \( t \geq t_s \), we have

\[
\hat{p}(t + d) = p(t).
\]

Although systems \( 1 \) and \( 2 \) are essentially equivalent under flatness condition (see Lemma \( 1 \)), but as our first crucial observation, we will see that there is a subtlety about the mapping between initial conditions. In other words, the definition of initialization vector, IV, as it is usually referred to in a cryptographic context is based on initializing System \( 1 \) while for System \( 2 \) one may choose parts of the initial conditions randomly and still recover the plaintext correctly using self-synchronization. This important technique of mapping self-synchronization to a random initial condition is fundamental in our security analysis and
shows the importance of the approach proposed by G. Millérioux et al. for the design of self-synchronized stream ciphers which is based on the design of flat systems of type 2.

Since we are going to analyze our proposed systems in a provable security model, we elaborate on emphasizing on the computational details of our model as follows.

A self-synchronized dynamical cryptosystem scheme

\[ \Gamma_{d,t_s} = (Gen_{Enc}, Init_{Enc}, Enc_\theta, Dec_\theta) \]

is defined to be a set of efficient algorithms \( Gen_{Enc}, \) \( Init_{Enc}, \) \( Enc_\theta \) and \( Dec_\theta \) described as follows.

- \( k \): is the security parameter,
- \( \theta \) is a vector containing system parameters determined by \( \kappa \) and the public information.
- \( Init_{Enc} \): is a randomized algorithm that on input \( \kappa \) outputs an initial condition vector \( ICV \) and the state initialization vector \( s(0) \).
- \( Init_{Dec} \): is a randomized algorithm that on inputs \( \kappa \) and the vector IV (received through the communication channel) outputs an initial condition vector \( ICV \) and the state initialization vector \( \hat{s}(d) \).
- \( Enc_\theta \): is a randomized algorithm that using initializations, a message block \( p(t) \) and a state vector \( s(t) \), outputs a ciphertext block \( c(t) \) and an updated state vector \( s(t+1) \) as follows,

\[
Enc_\theta(t) : \begin{cases} 
(s(0), ICV) \leftarrow Init_{Enc}, \\
If \ t = 0 \ then \\
\ s(1) = \phi'_\theta(s(0), ICV), \\
\ c(d) = \epsilon'_\theta(s(0), ICV), \\
Output : \bot \\
Else \\
\ s(t+1) = \phi_\theta(s(t), p(t)), \\
\ c(t+d) = \epsilon_\theta(s(t), p(t)), \\
Output : c(t), 
\end{cases}
\]  

(3)

where \( \phi'_\theta \) and \( \phi_\theta \) are the functions that generate \( s(1) \) and \( s(t+1) \) and also \( \epsilon'_\theta \) and \( \epsilon_\theta \) are encryption functions.

- \( Dec_\theta \): is a randomized algorithm that using initializations, a ciphertext block \( c(t) \) and a state vector \( \hat{s}(t) \), outputs a (recovered plaintext) block \( \hat{p}(t) \) and an updated state vector \( \hat{s}(t+1) \) as follows,

\[
Dec_\theta(t) : \begin{cases} 
(\hat{s}(d), ICV) \leftarrow Init_{Dec}, \\
If \ t < d \ then \\
Output : Ack \\
If \ t = d \ then \\
\hat{s}(t+1) = \beta'_\theta(\hat{s}(t), c(d), ICV), \\
Output : Ack \\
Else \\
\hat{s}(t+1) = \beta_\theta(\hat{s}(t), c(t)), \\
\hat{p}(t) = \delta_\theta(\hat{s}(t), c(t)), \\
Output : \hat{p}(t). 
\end{cases}
\]  

(4)

Where \( \beta'_\theta \) and \( \beta_\theta \) are the functions that generate \( \hat{s}(1) \) and \( \hat{s}(t+1) \) and also \( \delta_\theta \) is decryption functions.
Here again $d$ is the system delay and $t_s > d$ is the synchronization delay, respectively. We usually assume that $\delta_t = \varepsilon_s^{-1}$ and that the first $(t_s - 1)$ plaintexts are selected randomly to ensure safe decryption of the plaintext. As a natural condition we always assume that $\varepsilon$ resists collisions in the sense that for any pair of states $(s, s')$, the probability of not having a collision like

$$\varepsilon(s, p) = \varepsilon(s', p')$$

for some $p \neq p'$ is greater than a noticeable function of the security parameter.

Also, note that the initial condition vector is generated using the secret key $\kappa$, however the whole vector ICV is not necessarily transmitted to the receiver on the transmission channel along with the ciphertext. For this, in what follows, IV always stands for part of ICV that is transmitted (plainly) along with the ciphertext to the receiver. Note that in this setting the rest of ICV is either determined by the secret key or is chosen at random (of course the IV itself can also depend on the $\kappa$ or be chosen at random but the difference falls in the fact that IV is transmitted through the channel to the receiver).

The parametric function $\phi_\omega$ is called the update of the encryption process $\text{Enc}$, and the parametric function $\beta_\omega$ is called the update of the decryption process $\text{Dec}$.

The following lemma essentially reflects the main contribution of G. Millérioux et.al. (see [14] and references therein) expressed in our setting.

**Lemma 1.** Consider the communication system $\Gamma_{d,t_s}$. Assuming that for all $t \geq t_s$ we have $\hat{s}(t + d) = s(t)$, then,

1. If $\delta_t = \varepsilon_s^{-1}$ then for all $t \geq t_s$ we have $\hat{p}(t + d) = p(t)$.
2. If $\text{Enc}_\omega$ is flat, i.e. if there exist constants $l, l'$ and functions $F_\omega$ and $G_\omega$ such that

$$\Gamma_{d,t_s} : \begin{cases} s(t) = F_\omega(c(t - l), \ldots, c(t - l')), \\
p(t) = G_\omega(c(t - l), \ldots, c(t - l')) \end{cases}$$

then $\Gamma_{d,t_s}$ is equivalent to the following system which is in the form of System [1] (note that the converse is trivial),

$$\Sigma_\omega(t) : \begin{cases} \text{Initialize} \ (c(t_s - l), \ldots, c(t_s - l')) \\
\text{If} \ t \geq t_s \text{ then} \\
z(t) = H_\omega(c(t - l), \ldots, c(t - l')), \\
c(t + d) = \varepsilon_\omega(z(t), p(t)). \end{cases}$$

$$\Sigma'_\omega(t) : \begin{cases} \text{Receive} \ (c(t_s - l), \ldots, c(t_s - l')) \\
\text{If} \ t \geq t_s \text{ then} \\
\hat{z}(t + d) = H_\omega(c(t - l), \ldots, c(t - l')), \\
\hat{p}(t + d) = \delta_\omega(\hat{z}(t + d), c(t + d)). \end{cases}$$

Nowadays, self-synchronized stream ciphers (at least the classic ones) are much more related to block ciphers than to synchronous stream ciphers. In this regard, it is well-known that several modes of operation for block ciphers have been proposed among which a couple of them (as CBC and CFB modes) are self-synchronized (e.g. see [18]).

**Example 1. CBC and CFB modes**

Given a block cipher with encryption $E_\kappa$ and decryption $D_\kappa$, we may describe the CBC and CFB modes as follows. Note that in both cases $\text{Init}_{\text{Enc}}$ and $\text{Init}_{\text{Dec}}$ are randomized algorithms that on input $1^k$, output a random initial value IV (here $t_s = t_c = 1$ and $d = 0$).
The CBC mode:

\[
\begin{align*}
\text{Enc}_\theta(t) : & \quad \begin{cases} 
\text{Initialize IV}, \\
\text{s}(0) = \text{IV}, \\
\text{If } t = 0 \text{ then} \\
\text{s}(1) = \text{s}(0), \\
\text{Output : } \bot \\
\text{Else} \\
\text{e}(t) = E_\kappa(\text{p}(t) \oplus \text{s}(t)), \\
\text{s}(t+1) = \text{e}(t), \\
\text{Output : } \text{e}(t).
\end{cases} \\
\text{Dec}_\theta(t) : & \quad \begin{cases} 
\text{Receive IV}, \\
\hat{s}(0) = \text{IV}, \\
\text{If } t = 0 \text{ then} \\
\hat{s}(1) = \hat{s}(0), \\
\text{Output : } \text{Ack} \\
\text{Else} \\
\hat{s}(t+1) = \text{e}(t), \\
\hat{\text{p}}(t) = D_\kappa(\text{e}(t)) \oplus \hat{s}(t), \\
\text{Output : } \hat{\text{p}}(t).
\end{cases}
\end{align*}
\]

The CFB mode:

\[
\begin{align*}
\text{Enc}_\theta(t) : & \quad \begin{cases} 
\text{Initialize IV}, \\
\text{s}(0) = \text{IV}, \\
\text{If } t = 0 \text{ then} \\
\text{s}(1) = E_\kappa(\text{IV}), \\
\text{Output : } \bot \\
\text{Else} \\
\text{e}(t) = E_\kappa(\text{e}(t)), \\
\text{c}(t) = \text{p}(t) \oplus \text{s}(t), \\
\text{Output : } \text{c}(t).
\end{cases} \\
\text{Dec}_\theta(t) : & \quad \begin{cases} 
\text{Receive IV}, \\
\hat{s}(0) = \text{IV}, \\
\text{If } t = 0 \text{ then} \\
\hat{s}(1) = E_\kappa(\text{IV}), \\
\text{Output : } \text{Ack} \\
\text{Else} \\
\hat{s}(t+1) = E_\kappa(\text{e}(t)), \\
\hat{\text{p}}(t) = \text{c}(t) \oplus \hat{s}(t), \\
\text{Output : } \hat{\text{p}}(t).
\end{cases}
\end{align*}
\]

2 Security models

In [12] Joux et al. introduced a new attack to some modes of operation including the CBC mode which was the initiating fundamental contribution leading to the blockwise models of security for stream ciphers (also see [4] for other motivations). In [11] the authors proposed a simple delay procedure to prevent the proposed attack as follows.

Example 2. DCBC: the delayed CBC mode [11]

- Here Init_{Enc} and Init_{Dec} are randomized algorithms that on input $1^k$, output a random initial value IV (note that here we have $t_s = t_c = d = 1$).

- Enc_{θ}(t) takes as inputs a plaintext block or the special symbol “stop” for any time.

\[
\begin{align*}
\text{Enc}_{\theta}(t) : & \quad \begin{cases} 
\text{Initialize IV}, \\
\text{s}(0) = \text{IV}, \\
\text{If } t = 0 \text{ then} \\
\text{e}(1) = \text{s}(0), \\
\text{s}(1) = \text{s}(0), \\
\text{Output : } \bot \\
\text{If } \text{p}(t) = \text{stop} \text{ then} \\
\text{Output : } \text{e}(t) \\
\text{Else} \\
\text{e}(t+1) = E_\kappa(\text{s}(t) \oplus \text{p}(t)), \\
\text{s}(t+1) = E_\kappa(\text{s}(t) \oplus \text{p}(t)), \\
\text{Output : } \text{e}(t).
\end{cases} \\
\text{Dec}_{\theta}(t) : & \quad \begin{cases} 
\text{Receive IV}, \\
\hat{s}(0) = \text{IV}, \\
\text{If } t = 1 \text{ then} \\
\hat{s}(2) = \text{IV}, \\
\hat{\text{p}}(1) = \text{Ack}, \\
\text{Output : } \hat{\text{p}}(1) \\
\text{Else} \\
\hat{s}(t+1) = \text{e}(t), \\
\hat{\text{p}}(t) = D_\kappa(\text{e}(t)) \oplus \hat{s}(t), \\
\text{Output : } \hat{\text{p}}(t).
\end{cases}
\end{align*}
\]

Although the proposed DCBC mode is secure in the blockwise security model (the details will follow) let us consider the following setup showing that the delay procedure also has some drawbacks.

Assume that the adversary has oracle access to the decryption update (not necessarily to the whole decryption procedure) and uses the decryption update to do the encryption in
Figure 1: The interaction between the adversary and the oracle Enc′θ.

the following setting.

\[
\text{Enc′}_θ(t) : \begin{cases} 
\text{Initialize IV,} \\
\hat{s}(1) = IV, \\
\text{If } t = 1 \text{ then} \\
\hat{c}(1) = IV, \\
\hat{s}(2) = IV, \\
\text{Output : } \hat{c}(1), \\
\text{If } p(t-1) = \text{stop then} \\
\hat{s}(t+1) = \hat{c}(t), \\
\text{Else} \\
\hat{c}(t) = E_κ(\hat{s}(t) ⊕ p(t-1)), \\
\text{Output : } \perp \\
\end{cases}
\] (11)

Note that in a synchronized regime the response of Enc′θ is the same as that of Encθ.

The following attack using the encryption oracle Enc′θ succeeds in a distinguishability test as it is described below. This kind of attack is our basic motivation for our propose security model in Section 2.1.

- The adversary uniformly chooses two messages \(\{p^0(t)\}, \{p^1(t)\}\), each consisting of two blocks, such that \(p^0(2) ≠ p^1(2)\), and queries its oracle on this pair.

- The encryption of \(\{p^b(t)\}\) as \((c^b(1), c^b(2), c^b(3), IV)\) is given to the adversary. (The goal of the adversary will be to guess the value of \(b\).)

- The adversary queries a pair \((\tilde{p}^0(1), \tilde{p}^1(1))\) for plaintexts \(\{\tilde{p}^0(t)\}, \{\tilde{p}^1(t)\}\) whose first blocks, \(\tilde{p}^0(1)\) and \(\tilde{p}^1(1)\) are chosen uniformly at random.

- The adversary receives the encryption of \(\tilde{p}^b(1)\) as \((\tilde{c}^b(1), \tilde{c}^b(2))\) and then sends the second query as \((\tilde{p}^0(2), \tilde{p}^1(2))\) for which \(\tilde{p}^0(2) = \tilde{p}^1(2) = p^0(2) ⊕ c^b(2) ⊕ \tilde{c}^b(2)\).

- The adversary receives \(\tilde{c}^b(3)\).

- If \(\tilde{c}^b(3) = c^b(3)\) the adversary guesses \(b = 0\), otherwise it sets \(b = 1\).

Note that if \(b = 0\) then,
\[
\tilde{c}^b(3) = E_κ(\tilde{s}^b(3) ⊕ \tilde{p}^b(2)) = E_κ(\tilde{c}^b(2) ⊕ p^0(2) ⊕ c^b(2) ⊕ \tilde{c}^b(2)) = c^b(3),
\]
while if \(b = 1\) with a chance of at least \(\frac{1}{2}\) we have,
\[
\tilde{c}^b(3) = E_κ(\tilde{s}^b(3) ⊕ \tilde{p}^b(2)) = E_κ(\tilde{c}^b(2) ⊕ p^0(2) ⊕ c^b(2) ⊕ \tilde{c}^b(2)) ≠ c^b(3).
\]
It is instructive to note that a similar attack is not applicable using the \( \text{Enc}_\theta \) oracle. Also, this example shows that although applying a delay procedure may prevent attacks in the blockwise security model but at the same time it may still make the whole system vulnerable to the attacks using the decryption update (which is quite feasible in practice).

### 2.1 The LORBACPA\(^+\) security model

Our main objective in this section is to formalize a security model that can be used for both random and nonrandom (nonce) initialization scenarios within the blockwise model using decryption update. It will become clear that this security model, hereafter called LORBACPA\(^+\), is stronger than the blockwise CPA model, where our ultimate aim in the next sections will be to propose a LORBACPA\(^+\) secure self-synchronized stream cipher.

To begin, we have to formalize the oracles we are going to use in our security model.

**Definition 1. LORBA encryption oracles**

A left-or-right blockwise adaptive encryption oracle \( E(-,b) \) is an oracle applying the function \( \text{Enc}_\theta \) in the following way (depending on the specified parameters).

- Each communication with the oracle is either an initialization request of a new session or a request for encryption.
- For each initialization request we have the following cases:
  - If the oracle is a randomly initialized oracle, denoted by \( E(\text{IV}, b) \), then the oracle starts a new session using a random initializing data and sends the number of the session along with the (randomly fixed) IV to the main program.
  - If the oracle is of chosen IV type, denoted by \( E(\text{IV}, b) \), then the initialization request contains an IV chosen by the program (i.e. the adversary) and the oracle uses this IV along with the key and possibly other randomly chosen initializing parameters (if any) and returns the session number to the main program.
  - The oracle uses \( \text{Init}_{\text{enc}} \) to determine \( s(0) \).
- Each encryption request is of the form \( (p^0, p^1, i) \) containing two plaintext-blocks \( p^0 \) and \( p^1 \) with same length and session number \( i \). The oracle is capable of saving the history of each session (in particular the state vectors \( s(t) \)), therefore, upon receiving such a request the oracle returns the encryption of the block \( p^b \) using the history of the session \( i \) along with the IV by applying

\[
\text{Enc}_\theta(t) : \begin{cases}
\text{Use history of } (s(0), \text{IV}, \text{ICV}), \\
\text{If } t = 0 \text{ then} \\
\quad s(1) = \phi_s'(s(0), \text{ICV}), \\
\quad c(d) = \varepsilon_s'(s(0), \text{ICV}), \\
\quad \text{Output: } c(0) \\
\text{Else} \\
\quad s(t + 1) = \phi_s(s(t), p^b(t)), \\
\quad c(t + d) = \varepsilon_s(s(t), p^b(t)), \\
\quad \text{Output: } c(t). 
\end{cases}
\]

Note that the oracle is capable of answering different queries related to different sessions intermittently.
Definition 2. Synchronized LORBA encryption oracles

A left-or-right blockwise adaptive synchronized encryption oracle \( SE(−, b) \) is an oracle applying the update of the decryption process in the following way (depending on the specified parameters).

- The function of the oracle is the same as that of a LORBA encryption oracle as far as the initialization of the sessions are concerned but uses Init\_Dec to determine \( ˆs(d) \).
- Each encryption request is of the form \((p^0, p^1, i)\) containing two plaintext-blocks \( p^0 \) and \( p^1 \) along with a session number \( i \).
- The oracle is capable of saving the history of each session, therefore, upon receiving such a request the oracle returns the encryption of the block \( p^b \) using the history of the session \( s \) along with the IV by applying the update of the decryption procedure as

\[
Enc'_s(t) : \begin{cases} 
    \text{Use history of } (\hat{s}(d), \text{IV}, \text{ICV}), \\
    \text{If } t = d \text{ then} \\
    c(t) = \varepsilon'_s(\hat{s}(t), \text{ICV}), \\
    \hat{s}(t + 1) = \beta'_s(\hat{s}(t), \text{ICV}), \\
    \text{Output : } c(d) \\
    \text{Else} \\
    c(t) = \varepsilon_s(\hat{s}(t), p^b(t - d)), \\
    \hat{s}(t + 1) = \beta_s(\hat{s}(t), c(t)), \\
    \text{Output : } c(t).
\end{cases}
\]

Note that the oracle is capable of answering different queries related to different sessions intermittently.

Now we focus on the adversary model.

Definition 3. The LORBACPA\(^+\) model

We refer to our security model as LORBACPA\((−, −)\) where in what follows we describe the model and its dependence on the two undetermined parameters.

- A LORBACPA\((−, −)\) adversary \( A \) is a nonuniform probabilistic polynomial time oracle Turing machine.
- The adversary uses either a LORBA encryption oracle or a synchronized LORBA encryption oracle or both of them. The second parameter denotes the oracle type as in LORBACPA\((−, E)\) or LORBACPA\((−, SE)\) which are referred to as simple oracle models or LORBACPA\((−, (E\&SE))\) which is referred to as a mixed oracle model.
- The first parameter denotes whether the adversary can choose the initial vector IV or not. In particular, for an LORBACPA\((\text{IV, } −)\) oracle the initial vector IV can be chosen by the adversary for each encryption session and be sent to the oracle, while for a LORBACPA\((\$IV, −)\) oracle the initial vector of each session is chosen at random by the oracle and will be sent to the adversary in response to each one of the queries. It is instructive to recall that IV is the part of the initialization information that is transmitted on the communication channel. In what follows the security model LORBACPA\((\text{IV, } (E\&SE))\) is referred to as LORBACPA\(^+\).
• The adversary may query its oracle(s) about the encryption of different blocks of different messages during its computation. For each one of these messages a session must be initialized by the adversary and after that queries about consecutive blocks may be sent to the oracle. The oracle has the capability to save the history of each session so that the adversary may submit queries concerning the encryption of different blocks of different messages (sessions) intermittently.

The adversary simulates the following experiment $eval(A)$.

• A key $\kappa$ is generated by $Gen(1^k)$.

• A bit $b$ is chosen uniformly at random (unknown to the adversary) and the adversary is given access to its LORBA oracle(s) ($E(\cdot, b)$ and/or $SE(\cdot, b)$) of type $b$.

• The adversary may send queries to its oracle concurrently.

• The adversary outputs a bit $b'$ as the result.

• The output of the experiment is defined to be 1 (it succeeds) if $b = b'$ and 0 otherwise. If it returns 1, we say that $A$ succeeds and otherwise it fails.

We define the advantage of an adversary $A$ attacking a system $S$ in the LORBACPA($-, -$) model as

$$Adv_{A,S}^{\text{LORBACPA}(-,-)}(k) = 2 \left| Pr(\text{eval}(A) = 1) - \frac{1}{2} \right|$$

$$= \left[ Pr(\text{eval}(A) = 1|b = 1) - Pr(\text{eval}(A) = 1|b = 0) \right]$$

$$= \left[ Pr(\text{eval}(A) = 1) - Pr(\text{eval}(A) = 1) \right].$$

Also, the insecurity of such a system $S$ in the LORBACPA($-, -$) model is defined as

$$\text{Insec}^{\text{LORBACPA}(-,-)}_S(k) = \max_A Adv_{A,S}^{\text{LORBACPA}(-,-)}(k).$$

Clearly, such a system $S$ is said to be secure in the corresponding LORBACPA model if $\text{Insec}^{\text{LORBACPA}(-,-)}_S(k)$ is negligible compared to $k$ e.t.

$$\text{Insec}^{\text{LORBACPA}(-,-)}_S(k) < \text{negl}(k).$$

Where $\text{negl}(k)$ is negligible function compared to $k$.

On the other hand, for sufficiently large $k$, we have,

$$\text{Insec}^{\text{LORBACPA}(\text{IV},E)}_S(k) \leq \text{Insec}^{\text{LORBACPA}(\text{IV},E)}_S(k)$$

and

$$\text{Insec}^{\text{LORBACPA}(\text{IV},SE)}_S(k) \leq \text{Insec}^{\text{LORBACPA}(\text{IV},SE)}_S(k),$$

indicating that LORBACPA$^+$ is the strongest security model in this setting.

Next, we are going to consider a fundamental type of attack based on detecting collisions, which will be our basic guideline in Section 3 for the security analysis. But, before introducing the attack let us first fix some notations and concepts to be used in what follows.

Note that an adversary $A$ in a LORBACPA($-, -$) security model, is essentially a randomized algorithm having interactions with LORBA encryption oracles $E$ or $SE$. In this setting we use the following notations:

1 Sometimes this is referred to as concurrent blockwise encryption ability.
• $t$: is the global counter for the clock of the algorithm referring to the global real time.

• $o$: refers to an oracle of type $E$ or $SE$. Note that the algorithm may initiate different sessions and ask queries intermittently and without any loss in generality we may assume that there are at most two oracles, one of type $E$ and the other of type $SE$, where they are capable of initiating and answering queries for different sessions (appropriately keeping the history of each session separately).

• $i$: refers to the session number.

• $q(o, i, \tau, \nu)$: refers to the query asked from oracle $o$, in the $i$th session, while this query is the $\tau$th query of the $i$th session, and it is the $\nu$th query asked by the algorithm (counting all queries from the beginning). In this setting we may write $(p^0, p^1, i) = q(o, i, \tau, \nu)$ to indicate that $(p^0, p^1, i)$ is the corresponding query in the $i$th session.

Now, let us define a collision attack as follows.

**Definition 4.** Within a LORBCPA$(-,-)$ security model for a self-synchronized system $S$ with delay $d$ and synchronization time $t_s$ consider an adversary interacting with its oracle(s) $o$ and $o'$ each of which can be of type $E$ or $SE$, and assume that the adversary has initiated a number of sessions of queries. In this setting, a collision for sessions $i$ and $i'$ (not necessarily distinct) is the event $Coll((o, i, \tau), (o', i', \tau'), b)$ for which the adversary have the following collision for the ciphertexts,

$$
\varepsilon_o(s(\tau + d\delta_o), p_{i,o}^b(\tau)) = \varepsilon_o(s'(\tau'), \hat{p}_{i',o'}^b(\tau'))
$$

and

$$
\varepsilon_o(s(\tau + d\delta_o), p_{i,o}^b(\tau)) \neq \varepsilon_o(s'(\tau'), \hat{p}_{i',o'}^b(\tau'))
$$

where $\delta_{SE} = 1, \delta_E = 0$ and we have

$$(p^0_{i,o}, p^1_{i,o}, i) = q(o, i, \tau, \nu) \quad \text{and} \quad (\hat{p}^0_{i',o'}, \hat{p}^1_{i',o'}, i) = q(o', i', \tau', \nu').$$

A LORBCPA$(-,-)$ collision attack is a LORBCPA$(-,-)$ adversary for which the probability of detecting a collision is non-negligible.

Clearly, the main objective of a collision attack is to provide an algorithm (whose output depends on the queries) that can maximize the probability of a collision. For this, a straightforward approach is to provide a sequence of queries in different sessions such that it leads to a collision. To see this, note that the attack provided in Section 2 for the DCBC mode is essentially a collision attack using only one oracle of type $SE$.

We should emphasize that the straightforward scenario mentioned above is not the only possible setup for a collision attack where one may think of many different approaches to detect collisions. To see one more scenario, consider an adversary who uses one oracle of type $E$ and one other oracle of type $SE$, while the adversary tries to detect $t_s$ consecutive ciphertexts such that

$$\forall 0 \leq j \leq t_s - 1 \quad c^b_{E}((\tau - t_s) + j) = c^b_{SE}((\tau' + d - t_s) + j),$$

in which the flag $b$ refers to the oracle action type (i.e. left or right). Note that as a consequence of synchronization we have $s(\tau) = \bar{s}(\tau' + d)$.

Then choosing $p^0_{i,E}(\tau) = p^0_{i,SE}(\tau')$ for $b = 0$ we have,

$$c^b_{E}((\tau + d) - c^b_{SE}(\tau' + d) = \varepsilon_o(s(\tau), p^0_{i,E}(\tau)) - \varepsilon_o(\bar{s}(\tau' + d), p^0_{i,SE}(\tau')) = 0. \quad (14)$$

On the other hand, choosing a random $p^1_{i,E}(\tau) = p_1 \neq 0$ for $b = 1$ and setting $p^1_{i,SE}(\tau') = 0$ we have,

$$c^b_{E}((\tau + d) - c^b_{SE}(\tau' + d) = \varepsilon_o(s(\tau), p^1_{i,E}(\tau)) - \varepsilon_o(\bar{s}(\tau' + d), p^1_{i,SE}(\tau')) = \varepsilon_o(s(\tau), p_1) - \varepsilon_o(s(\tau), 0). \quad (15)$$

Then effectiveness of the algorithm follows from the fact that $\varepsilon_o$ is collision resistant.
Corollary 1. A self-synchronized stream cipher $S$ for which there exists $\tau_0$ and a function $f$ such that $s(\tau_0) = f(IV)$ (resp. $\tilde{s}(\tau_0) = f(IV)$), is not LORBACPA(IV, $E$) (resp. LORBACPA(IV, $SE$)) secure.

Proof. For a collision attack, the adversary initializes two sessions $i$ and $i'$ with the same initial values IV, respectively. Then it chooses $p_{i,o}^0(\tau_0) = p_{i',o}^1(\tau_0) = 0$ uniformly at random and set $p_{i',o}^0(\tau_0) = 0$. Since $s(\tau_0) = f(IV)$ (resp. $\tilde{s}(\tau_0) = f(IV)$), this gives rise to a collision for the queries $(p, p, i) = q(o, i, \tau_0, \nu)$ and $(p, 0, i') = q(o, i', \tau_0, \nu')$.

An important consequence of this corollary is the fact that the state vectors of a secure LORBACPA(IV, $-\cdot$) self-synchronized stream cipher must possess a pseudorandom nature and should not be a deterministic function of the initial vector IV.

Before we proceed any further, let us consider some well-known self-synchronized block-cipher modes and their security in our proposed models.

In [12] A. Joux et al. have shown that the CBC encryption mode cannot be IND secure in the blockwise adversarial model, which implies that, in our language, the CBC mode is not LORBACPA($IV, E$) secure. Subsequently, P. Fouque et al. [11] using an output delay procedure, introduced the delayed CBC mode, DCBC (see Example 2), and proved that it is secure in the blockwise model (assuming the security of the underlying block cipher). This implies that the DCBC mode is LORBACPA($IV, E$) secure under the same assumptions (note that our LORBACPA($IV, E$) security is equivalent to the LORC − BCPA security in [11]). On the other hand, the DCBC mode is not LORBACPA(IV, $-\cdot$) secure by Corollary 1. Clearly, the DCBC mode is also not secure in the LORBACPA($IV, SE$) setting by the collision attack presented at the beginning of this section.

For the CFB mode, by the results of [11], we know that the scheme is provably secure in the blockwise model, assuming the security of the underlying block cipher (or function) which is equivalent to our LORBACPA($IV, E$) security. It is instructive to note that the update and output functions of an SE($IV, b$) oracle for the CFB mode is identical to those of the oracle $E($IV, $b$). This, in particular, proves that the CFB mode is also LORBACPA($IV, SE$) secure. On the other hand, the CFB mode is not LORBACPA(IV, $-\cdot$) secure by Corollary 1.

2.2 A modification

This section is going to serve as an appetizer before we focus on our final proposal in the next section. Based on our results of the previous section we understand that we have to choose a pseudorandom state update procedure in order to prevent insecurity in LORBACPA+$\cdot$ model. Our major objective in this section is to analyze the performance of a modification on well-known encryption modes based on choosing this approach and adding iterations of a nilpotent linear function for finite-time self-synchronization to the scheme. We will see that although this modification will not give rise to better encryption modes but the analysis will pave the way to introduce our proposed scheme in the next section in which we will have to prevent linearity and we also force the iteration function to depend on the secret key.

2.2.1 A modified DCBC mode

Let us introduce the modified DCBC mode as follows.

Definition 5. The MDCBC mode.

\begin{definition}
\end{definition}
Note that for the MDCBC mode we have 

\[ f^{n_o}(x) = f(f(...(f(x)))) = 0. \]

Lemma 2. The MDCBC mode is finite-time self-synchronized with \( t_s = n_o \), i.e.

\[ \forall t \geq t_s = n_o, \quad \mathbf{s}(t + 1) = \mathbf{s}(t). \]

Proof.

\[
\begin{align*}
\mathbf{e}(t + 1) & \overset{\text{def}}{=} \mathbf{s}(t + 2) - \mathbf{s}(t + 1) \\
& = \mathbf{c}(t + 1) \oplus f(\mathbf{s}(t + 1)) - (E_n(\mathbf{s}(t) \oplus \mathbf{p}(t)) \oplus f(\mathbf{s}(t))) \\
& = f(\mathbf{s}(t + 1)) - f(\mathbf{s}(t)) = f(\mathbf{s}(t + 1) - \mathbf{s}(t)) = f(\mathbf{e}(t)).
\end{align*}
\]

Hence, according to the definition of the MDCBC system, the error is equal to zero for \( t \geq t_s = n_o \).

The following lemma shows that our modification is not effective in the presence of an \( SE \) oracle.

Lemma 3. The MDCBC mode is not \( \text{LORBACPA}(\&IV, SE) \) secure.

Proof. Recall that an \( SE(\&IV, b) \) oracle operates as follows,

\[
\begin{align*}
\text{Enc}_\theta(t) : & \quad \begin{cases}
\text{Initialize IV}, \\
\hat{\mathbf{s}}(1) \overset{\text{def}}{=} \mathbf{F}_n^\theta, \\
\text{If } t = 0 \text{ then} \\
\hat{\mathbf{s}}(2) = f(\hat{\mathbf{s}}(1)) \oplus \mathbf{IV}, \\
\mathbf{c}(1) = \mathbf{IV}, \\
\text{Output : } \mathbf{c}(1) \\
\text{Stop} \text{ then} \\
\text{Output : } \bot \\
\text{Else} \\
\mathbf{c}(t) = E_n(\hat{\mathbf{s}}(t) \oplus \mathbf{p}(t - 1)), \\
\hat{\mathbf{s}}(t + 1) = \mathbf{c}(t) \oplus f(\hat{\mathbf{s}}(t)), \\
\text{Output : } \mathbf{c}(t).
\end{cases}
\end{align*}
\]

We introduce a collision attack whose success probability is equal to one. For this, the adversary initializes two sessions \( i \) and \( i' \) with two random initial values IV and IV, respectively. Then it chooses at random a sequence \( \{\mathbf{p}_{i,SE}^0(\tau), \mathbf{p}_{i',SE}^1(\tau)\}, 0 < \tau < t, \)
with \( p_{c,SE}(t_e) = p_{c,SE}(t_e) = p_c \neq 0 \) uniformly at random and receives \( \{c^b(1), \ldots, c^b(t_e)\} \).

Then it chooses two random sequences \( \{p_{c,SE}(\tau), p^\prime_{c,SE}(\tau)\}, 0 < \tau < t_e \) and receives \( \{c^b(1), \ldots, c^b(t_e)\} \). Then it sets \( p^\prime_{c,SE}(t_e) = 0 \) and \( p^0_{c,SE}(t_e) = p_c \), with

\[
p_c = p_c \oplus c^b(t_e) \oplus \cdots \oplus f^{t_e-1}(IV) \oplus c^b(t_e) \oplus \cdots \oplus f^{t_e-1}(IV)
\]

for session \( t' \). Since we have

\[
\tilde{s}(t_e + 1) = c^b(t_e) \oplus f(c^b(t_e - 1)) \oplus \cdots \oplus f^{t_e-1}(IV)
\]

and

\[
\tilde{s}'(t_e + 1) = c^b(t_e) \oplus f(c^b(t_e - 1)) \oplus \cdots \oplus f^{t_e-1}(IV),
\]

this gives rise to a collision for the queries

\[
(p_c, \tilde{p}_c, i) = q(SE, i, t_e + 1, \nu) \quad \text{and} \quad (p_c, 0, i') = q(SE, i', t_e + 1, \nu').
\]

\[\Box\]

### 2.2.2 A modified CFB mode

Now let us concentrate on a modified version of the CFB mode as follows,

**Definition 6.** The MCFB mode.

- \( f \): is a known linear function such that there exists a natural number \( n_0 > 1 \) such that

\[
\forall x \quad f^{n_0}(x) \equiv f(f(...(f(x)))) = 0.
\]

**Encryption**

\[
\text{Enc}_a(t) : \begin{cases}
\text{Initialize IV,} \\
\tilde{s}(0) \overset{\$}{\leftarrow} \mathbb{F}_2^\nu, \\
\text{If } t = 0 \text{ then } \tilde{s}(1) = E_c(IV) \oplus f(s(0)), \\
\text{Else } s(t + 1) = E_c(c(t)) \oplus f(s(t)), \\
\hat{e}(t) = p(t) \oplus \tilde{s}(t), \\
\text{Output: } \hat{e}(t).
\end{cases}
\]

**Decryption**

\[
\text{Dec}_a(t) : \begin{cases}
\text{Receive IV,} \\
\tilde{s}(0) \overset{\$}{\leftarrow} \mathbb{F}_2^\nu, \\
\text{If } t = 0 \text{ then } \tilde{s}(1) = E_c(IV) \oplus f(s(0)), \\
\text{Else } \hat{e}(t + 1) = E_c(c(t)) \oplus f(s(t)), \\
\hat{e}(t) = \hat{s}(t) \oplus c(t), \\
\text{Output: } \hat{e}(t).
\end{cases}
\]

Note that for the MCFB mode we have \( t_e = 1 \) and \( d = 0 \). In what follows we show that also \( t_e = n_0 \).

\[\Box\]

**Lemma 4.** The MCFB mode is finite-time self-synchronized with \( t_e = n_0 \), i.e.

\[
\forall t \geq t_e, \; \tilde{s}(t) = s(t).
\]

**Proof.**

\[
e(t + 1) \overset{\text{def}}{=} \tilde{s}(t + 1) - s(t + 1) = E_c(c(t)) \oplus f(s(t)) - E_c(c(t)) - f(s(t)) = f(s(t)) - f(s(t)) = f(e(t)).
\]

Hence, according to the definition of the MCFB system, the error is equal to zero for \( t \geq t_e = n_0 \).

\[\Box\]

**Lemma 5.** The MCFB mode is not LORBACPA(IV,E) secure.
Proof. The MCFB mode is CFB mode with a linear transformation MCFB mode is a simple reduction of MCFB mode. For any \( t > 0 \), with Linear transformation \( f^t_s(.) \) over encryption update and output functions MCFB mode, we have

\[
\text{Enc''}_\sigma(t) : \begin{cases} 
\text{Initialize IV,} \\
\text{s(0) } \leftarrow \mathbb{F}_2^n, \\
\text{If } t = 0 \text{ then} \\
\text{s(1) } = \text{E}_\kappa(\text{IV}) \oplus \text{f(s(0))}, \\
\text{Output} : \bot \\
\text{Else} \\
\text{s(t+1) } = \text{E}_\kappa(\text{c(t)}) \oplus \text{f(s(t))}, \\
\text{c'(t) } = \text{f}^t_s^{-1}(\text{c(t)}) = \text{f}^t_s^{-1}(\text{p(t)}) \oplus \text{f}^t_s^{-1}(\text{s(t)}), \\
\text{Output} : \text{c(t)}. 
\end{cases}
\] (21)

The following collision attack finds a collision with probability 1. For this, the adversary initializes two sessions \( i \) and \( i' \) with the same initial values IV. Then it chooses \( p_0^i, E(1) = p_0^{i'}, E(1) = p_0 \neq 0 \) uniformly at random. On the other hand, it sets \( \tilde{p}_0^i, E(1) = p_0, \tilde{p}_0^{i'}, E(1) = 0 \).

Since \( c'(t) = f^t_s^{-1}(s(t) \oplus p(t)), s(1) = E_\kappa(\text{IV}) \oplus f(s(0)) \) and \( \tilde{s}(1) = E_\kappa(\text{IV}) \oplus f(\tilde{s}(0)) \), this gives rise to a collision for the queries \( (p_0, p, i) = q(E, i, 1, \nu) \) and \( (p_0, 0, i') = q(E, i', 1, \nu') \).

\[\blacksquare\]

This shows that MCFB mode is not LORBACPA(IV, −) secure. The following table summarizes our results so far.

| System | LORBACPA(IV, E) | LORBACPA(IV, SE) | LORBACPA(SE, IV) | LORBACPA(SE, E) | Ref. |
|--------|----------------|-----------------|-----------------|-----------------|------|
| CBC    | ×              | ×               | ×               | ×               | [11] |
| DCBC   | √              | ×               | ×               | ×               | [11], Example 2, Cor. 1 |
| CFB    | √              | ×               | √               | ×               | [11], Cor. 1 |
| \( S^4_\sigma \) | √              | √               | √               | √               | Sec. 4 |

3 The \( S^4_\sigma(\mathbb{F}) \) cryptosystem

In this section we define our proposed cryptosystem \( S^4_\sigma(\text{Gen}, \text{Enc}_\kappa, \text{Dec}_\kappa, \mathbb{F}) \) which is based on basic ideas coming from the design of cryptographic modes of operations and the contributions of G. Millérioux et.al. [17] in design and analysis of switching cryptosystems.

3.1 System description

Here we go through the details of our system’s description.

- **Definition of parameters**
  - The integer \( n \) is system’s dimension which also determines the length of each block of plaintext or ciphertext.

2The acronym stands for *Switching Self-Synchronized Stream-cipher*
– The integer \( q \) stands for the size of the finite field \( \mathbb{F}_q \).
– The security parameter is \( k = n \log q \).
– The key generator \( \text{Gen} \): is a probabilistic algorithm that on input \( 1^k \) outputs the secret key \( \kappa \).
– The family \( \mathcal{P} \): is a family of pseudorandom permutations on the elements of \( GF(q) \) as \( \pi_\kappa: \mathbb{F}_q \to \mathbb{F}_q \) indexed by the (secret) key string \( \kappa \).

In this setting the one-to-one function \( \varphi_\kappa: \mathbb{F}_q^\ast \to \mathbb{F}_q^\ast \) (resp. \( \varphi_\kappa^{-1}: \mathbb{F}_q^\ast \to \mathbb{F}_q^\ast \)) is defined by the action of the (secret) random permutation \( \pi_\kappa \) (resp. \( \pi_\kappa^{-1} \)) on each entry of an \( n \)-vector in \( \mathbb{F}_q^n \).

– The switching function \( \sigma(t): \mathbb{N} \to [\ell] \) (see [17]): The switching function must depend on the output of the system, while the motivation of such a dependence lies in that the switching rule must also be self-synchronizing. Thus, it must depend on shared variables and so on the output or a finite sequence of delayed outputs.

– Matrices: The matrix \( \mathbf{W} \) and for \( j \in [\ell] \), the matrices \( \mathbf{L}_j, \mathbf{F}_j \) are secret invertible matrices in \( \mathcal{M}_{m \times n} \). The set \( \{ \mathbf{Q}_j | j \in [\ell] \} \) is a public nilpotent semigroup of matrices of index \( n_0 \) (see e.g. [17]).

On the other hand, for any \( j \in [\ell] \), the matrices \( \mathbf{E}_j, \mathbf{B}_j \) in \( \mathcal{M}_{m \times n} \) are public invertible matrices, where, for any \( j \in [\ell] \), we define \( \mathbf{A}_j \defeq \mathbf{E}_j \mathbf{F}_j^{-1} \mathbf{B}_j \), \( \mathbf{R}_j \defeq \mathbf{E}_j \mathbf{F}_j^{-1} \) and \( \mathbf{D}_j \defeq \mathbf{E}_j \mathbf{F}_j^{-1} \mathbf{L}_j - \mathbf{Q}_j \).

– The matrix \( \mathbf{M} \) is an upper triangular public matrix with zeros on the diagonal in \( \mathcal{M}_{m \times m_0} \).

– At any time \( t \in \mathbb{N} \), \( p(t) \in \mathbb{F}_q \) and \( c(t) \in \mathbb{F}_q \) are the plaintext and the ciphertext at time \( t \).

\[
p(t) = (p((t - 1)n + 1), p((t - 1)n + 2), \ldots, p(tn))^T
\]

and

\[
c(t) = (c((t - 1)n + 1), c((t - 1)n + 2), \ldots, c(tn))^T
\]

are the \( t \geq 1 \)th blocks of plaintext and ciphertext, respectively. We assume that each block of data is of length \( n \) whose symbols are numbered from 1 to \( n \), and that the end of encryption is indicated by sending a predefined block \( p(t) = \text{stop} \).

Also, we assume that if the decryption algorithm does not have to output a block, it sends, as an acknowledgment, a predefined block \( \text{Ack} \).

**Encryption procedure (\text{Enc}_\kappa)**

– The transmitter chooses a vector \( \mathbf{s}(0) = (s(0)_1, \ldots, s(0)_n)^T \) at random in \( \mathbb{F}_q^n \).
– The transmitter chooses two random vectors \( \mathbf{m}(0), \mathbf{c}(0) \in \mathbb{F}_q^n \) and forms the initial-value vector \( \mathbf{IV} \defeq (\mathbf{m}(0), \mathbf{c}(0)) \) with

\[
\mathbf{c}(0) = (c(1 - n), \ldots, c(0))^T,
\]

\[
\mathbf{m}(0) = (m(0)_1, \ldots, m(0)_{m_0})^T
\]

and transmits this vector over the public channel.

– The general description of the encryption procedure is as follows (see Figures 2 and 3).
Decryption procedure \((\text{Dec})\)

The input to the receiver is the ciphertext stream and the initial-vector \(\{c(t)\}, IV\).
In \(S^4_n(\mathcal{Q})\) we have \(t_c = 1\) as the cipher delay and the receiver operates as an unknown input observer as follows.

- The receiver selects a vector \(\hat{s}(1) = (\hat{s}(1)_1, \cdots, \hat{s}(1)_n)^T\) at random in \(\mathbb{F}_q^n\).
- The general setup of the receiver procedure is as follows,

\[
\text{Dec}_\kappa : \begin{cases}
\text{Input : } \{\hat{c}(t)\}_t, IV \\
\text{Initial : } \pi_\kappa, \sigma(t), \hat{s}(1), IV, \{B_j\}, \{E_j\}, \{F_j\}, \{L_j\}, \mathcal{W}, \mathcal{M} \\
\text{Output : } \{\hat{p}(t)\}_t, t \geq 2.
\end{cases}
\]
Figure 3: The general scheme of $S_4^\sigma(P)$ cryptosystem steps.

KernelDec$(c(t)) :$

\[
\begin{cases}
\text{IV} = (m(0), c(0)), \quad \hat{s}(1) \overset{\text{R}}\leftarrow F_q \\
\text{if } t = 0 \text{ then} \quad \text{Output} : \text{Ack} \\
\text{if } t = 1 \text{ then} \quad \hat{s}(2) = \hat{s}(1), \\
\hat{m}(2) = m(0), \quad \text{Output} : \text{Ack} \\
\text{Else} \quad \text{update Dec,} \\
\hat{p}(t) = \delta_{s}(\hat{s}(t), c(t), \{F_j^{-1}\}), \\
\text{Output} : \hat{p}(t).
\end{cases}
\]

where the details of the functions in update procedure are as follows,

update Dec :

\[
\begin{cases}
\text{Initial} : j = \sigma(t - 1), \\
\hat{s}(t) = L_j \hat{s}(t) + B_j \hat{m}(t), \\
\hat{m}(t + 1) = M \hat{m}(t) + c(t - 1),
\end{cases}
\]

Output Dec : $\hat{p}(t) = \varphi_{\sigma}^{(t)}(F_j^{-1}(c(t) - \hat{s}(t)))$.

3.2 Verification of system properties

The following lemma describes the main properties of $S_4^\sigma(P)$.

**Lemma 6.** The $S_4^\sigma(P)$ cryptosystem is finite-time self-synchronized with $t_s = n_0$, delay 1 and dummy ciphertext symbols $t_c \leq t_s + m_0$. Also, $\text{Enc}_s$ is a flat dynamical systems.

**Proof.** Clearly the lemma follows from the following four claims.

i) The algorithm Dec$_s$ operates as an unknown input observer for Enc$_s$ in a self-synchronized setup, i.e.

\[
\forall t \geq t_s, \quad \hat{s}(t + 1) = s(t).
\]

ii) For any $t \geq t_s$ we have $\hat{p}(t + 1) = p(t)$ and system’s delay is $d = 1$.

iii) For any $t \geq t_s$, we have

\[
s(t + 1) = (\prod_{i=1}^{t} Q_{s(i)} s(1)) + \sum_{h=1}^{t} \left[ (\prod_{i=h+1}^{t} Q_{s(i)} W m(h) + (\prod_{i=h+1}^{t} Q_{s(h)} E s(h)) F_{s(h)}^{-1} c(h)) \right].
\]
iv) For all \( t \geq t_s \), the vector \( s(t+1) \) and \( \hat{s}(t+2) \) depends on a finite number of previous ciphertexts, i.e.

\[
\hat{s}(t+2) = s(t+1) = \begin{cases} 
F_n(c(t-t_s-m_0), \ldots, c(t)) & t > m_0 \\
F_n(m(0), \ldots, m(0), c(1), \ldots, c(t)) & t \leq m_0.
\end{cases}
\]  

(27)

Now, we prove each one of these claims as follows.

For Part (i) note that,

\[
s(t + 1) = \hat{s}(t + 1) + \sum_{j \in [\ell]} \sum_{\sigma \in \mathbb{S}} \mathcal{D}_\sigma(s(t) + \mathcal{A}_\sigma \phi_{\sigma}(\hat{s}(t + 1)) + \mathcal{R}_j(c(t + 1) - \hat{s}(t + 1)))
\]

(28)

Hence, since \( \{Q_j \mid j \in [\ell]\} \) is a nilpotent semigroup of matrices of index \( n_0 \),

\[
\forall t \geq t_s = n_0, \quad \hat{s}(t + 1) = s(t).
\]

Part (ii) is a direct consequence of Part (i), i.e.,

\[
\hat{s}(t + 1) = \rho^{-1}(F^{t+1}(c(t+1) - \hat{s}(t+1)))
\]

(29)

On the other hand, since \( c(t+1) = \varepsilon_\sigma(z(t), p(t)) \) we deduce that the delay is \( d = 1 \).

For Part (iii) we use induction on time \( t \geq 2 \).

- For \( t = 2 \):

  Considering \( c(t+1) = z(t) + F_j \phi_\sigma(p(t)) \), we have,

\[
s(2) = Wm(1) + D_{\sigma(1)} s(1) + A_{\sigma(1)} \phi_\sigma(s(1)) + E_{\sigma(1)} \phi_\sigma(p(1))
\]

(30)

- The induction step for time \( t \):

  Assuming \([26]\); that is,

\[
s(t) = \sum_{h=1}^{t-1} \left[ \left( \prod_{h=1}^{t-1} Q_{\sigma(h)} \right) Wm(h) + \left( \prod_{h=t+1}^{t-1} Q_{\sigma(h)} \right) E_{\sigma(h)} F_{\sigma(h)} c(h+1) \right].
\]

(31)

then using equations \([22]\) and \([14]\) for \( s(t+1) \) we may conclude that,

\[
s(t+1) = Wm(t) + A_{\sigma(t)} \phi_\sigma(s(t)) + E_{\sigma(t)} \phi_\sigma(p(t)) + D_{\sigma(t)} s(t)
\]

(32)

\[
= \sum_{h=1}^{t-1} \left[ \left( \prod_{h=1}^{t-1} Q_{\sigma(h)} \right) Wm(h) + \left( \prod_{h=t+1}^{t-1} Q_{\sigma(h)} \right) E_{\sigma(h)} F_{\sigma(h)} c(h+1) \right] + \sum_{h=t+1}^{t-1} \left[ \left( \prod_{h=1}^{t-1} Q_{\sigma(h)} \right) Wm(h) + \left( \prod_{h=t+1}^{t-1} Q_{\sigma(h)} \right) E_{\sigma(h)} F_{\sigma(h)} c(h+1) \right].
\]

Thus, \([26]\) holds for time \( t + 1 \), and the proof of the induction step is complete.

For Part (iv) note that \( \{Q_j \mid j \in [\ell]\} \) is a nilpotent semigroup of matrices of index \( t_s = n_0 \), and consequently, for \( t \geq t_s \) we have \( (\prod_{h=1}^{t-1} Q_{\sigma(h)} s(1) = 0 \). Hence, using \([32]\), for any \( t \geq t_s \) we have

\[
s(t+1) = \sum_{h=1}^{t+1} \left[ \left( \prod_{h=1}^{t+1} Q_{\sigma(h)} \right) Wm(h) + \left( \prod_{h=t+2}^{t+1} Q_{\sigma(h)} \right) E_{\sigma(h)} F_{\sigma(h)} c(h+1) \right].
\]

(33)

Also, since for \( h > n_s \), \( m(h) \) is a linear function of \( c(h-1), \ldots, c(h-1) \) and for \( h \leq m_0 \), we know that \( m(h) \) is a linear function of \( c(1), \ldots, c(h-1) \) and \( m(0)_h, \ldots, m(0)_m \), we may conclude that for all \( t \geq t_s \) and .
sequently, for time \( t \) and \( t = 23 \) with \( s \) we wanted to show.

Example 3. An illustrative example

We use induction on time \( t \geq 3 \).

- For \( t = 3 \):
  
  We have,
  \[
  \hat{s}(3) = W\hat{m}(2) + D_{\sigma(1)}\hat{s}(2) + A_{\sigma(1)}\hat{s}(2) + R_{\sigma(1)}(c(t) - \hat{s}(2))
  = (D_{\sigma(1)} - B_{\sigma(1)}F_{\sigma(1)}L_{\sigma(1)})\hat{s}(2) + W\hat{m}(2) + R_{\sigma(1)}(c(t)).
  \]

- The induction step for time \( t \):
  
  Assuming \([26]\); that is,
  \[
  \hat{s}(t+1) = (\prod_{i=1}^{t-1} Q_{\sigma(i)})\hat{s}(2) + \sum_{h=1}^{t-1} \left( [\prod_{i=1}^{h-1} Q_{\sigma(i)}]W\hat{m}(h+1) + [\prod_{i=h+1}^{t-1} Q_{\sigma(i)}]R_{\sigma(h)}c(h+1) \right).
  \]
  
  then using equations \([22]\) and \([14]\) for \( s(t+1) \) we may conclude that,
  \[
  \hat{s}(t+2) = W\hat{m}(t+1) + D_{\sigma(t)}\hat{s}(t+1) + A_{\sigma(t)}\hat{s}(t+1) + R_{\sigma(t)}(c(t+1) - \hat{s}(t+1))
  = (\prod_{i=1}^{t} Q_{\sigma(i)})\hat{s}(2) + \sum_{h=1}^{t} \left( [\prod_{i=1}^{h} Q_{\sigma(i)}]W\hat{m}(h+1) + [\prod_{i=h+1}^{t} Q_{\sigma(i)}]R_{\sigma(h)}c(h+1) \right).
  \]
  
  Since \( \{Q_j \mid j \in [\ell]\} \) is a nilpotent semigroup of matrices of index \( t_s = n_0 \), and consequently, for \( t \geq t_s \) we have \( \prod_{i=1}^{t-1} Q_{\sigma(i)}\hat{s}(2) = 0 \). Hence, using \([32]\), for any \( t \geq t_s \) we have
  \[
  \hat{s}(t+2) = \sum_{h=t-t_s}^{t} \left( [\prod_{i=1}^{h} Q_{\sigma(i)}]W\hat{m}(h+1) + [\prod_{i=h+1}^{t} Q_{\sigma(i)}]E_{\sigma(h)}F_{\sigma(h)}^{-1}c(h) \right).
  \]
  
  Also, since \( \hat{m}(h+1) = m(h) \) then
  \[
  \hat{s}(t+2) = \left\{ \begin{array}{ll}
  F_{\sigma(t)}(c(t) - m_0), \ldots, c(t) & t > m_0, \\
  F_{\sigma(t)}(m(0), \ldots, m(0), c(1), \ldots, c(t)) & t \leq m_0,
  \end{array} \right.
  \]
  as we wanted to show.

Example 3. An illustrative example

We consider a 3-dimensional \( S_3^3 \) cryptosystem with following matrices (see Equations \([22]\) and \([23]\) with \( s(t) \in F_3^3 \), \( p(t) \in F_3^3 \) and \( c(t) \in F_3^3 \) and applying a simple switching function \( \sigma(t) = t \mod 2 \):

\[
\begin{align*}
Q_1 &= \begin{pmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 6 & 1 & 0 \end{pmatrix} & D_1 &= \begin{pmatrix} 6 & 0 & 2 \\ 0 & 6 & 3 \\ 6 & 3 & 2 \end{pmatrix} & A_1 &= \begin{pmatrix} 0 & 2 & 5 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} & E_1 &= \begin{pmatrix} 0 & 6 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} & L_1 &= \begin{pmatrix} 1 & 1 & 5 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
B_1 &= \begin{pmatrix} 0 & 5 & 0 \\ 0 & 6 & 0 \\ 3 & 0 & 1 \end{pmatrix} & F_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} & Q_2 &= \begin{pmatrix} 2 & 5 & 0 \\ 2 & 5 & 0 \\ 2 & 5 & 0 \end{pmatrix} & D_2 &= \begin{pmatrix} 0 & 3 & 5 \\ 0 & 2 & 4 \\ 5 & 4 & 6 \end{pmatrix} & A_2 &= \begin{pmatrix} 2 & 6 & 0 \\ 5 & 0 & 4 \\ 3 & 1 & 1 \end{pmatrix}
\end{align*}
\]
Figure 4 depicts the error vector between the \( p(t) \) (regular line) and \( \hat{p}(t) \) (dotted line) with \( s(0) = (2, 4, 1)^T, \hat{s}(0) = (0, 2, 4)^T, c = (1, 4, 4)^T \) and \( m = (0, 0, 0)^T \). Note that when \( t \geq t_s = 2 \) we have \( \hat{s}(t + 1) = s(t) \) and after two clocks both sequences are synchronized.

\[
E_2 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 6 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 6 & 1 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 3 & 0 \\ 5 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & 0 & 4 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[ s^E(t) = (2, 4, 1)^T, \hat{s}(0) = (0, 2, 4)^T, \ c = (1, 4, 4)^T \] and \( m = (0, 0, 0)^T \). Note that when \( t \geq t_s = 2 \) we have \( \hat{s}(t + 1) = s(t) \) and after two clocks both sequences are synchronized.

\[
\text{Figure 4: System synchronization (see Example 3)}
\]

4 Security analysis

In this section we analyze the security of \( S^4_{\sigma}(\mathbb{P}) \) in the LORBACPA\(^+\) model. In what follows we always assume that the secret key, \( \kappa \), is a \( k \)-bit string. Also, \( \mathcal{U} \) refers to the ensemble of truly (i.e. uniformly) random permutations of \( F_q \), while \( \mathbb{P} \) is a pseudorandom ensemble of permutations used in the cryptosystem \( S^4_{\sigma}(\mathbb{P}) \).

Recall that \( S^4_{\sigma}(\mathbb{P}) \) is said to be LORBACPA\(^+\) secure, if for all probabilistic polynomial-time adversaries \( A \) (as in Definition 3),

\[
\text{Insec}^\text{LORBACPA\(^+\)}_{S^4_{\sigma}(\mathbb{P})}(k) = \max_A \text{Adv}^\text{LORBACPA\(^+\)}_{A,S^4_{\sigma}(\mathbb{P})} \leq \text{negl}(k).
\]

Before we proceed, let us fix the setup. Assuming the existence of an adversary \( A \) in the security model LORBACPA\(^+\), then \( 1 \leq i \leq s \) is the indicator of query sessions where it is assumed that the whole number of sessions is equal to \( s \). In this setting \( s_E \) is the number of query sessions initiated by the oracle \( E \) and \( s_{SE} \) is the corresponding number for the oracle \( SE \). Similarly, \( \nu_E \) (resp. \( \nu_{SE} \)) is the whole number of queries asked from the oracle \( E \) within \( s_E \) sessions (resp. \( SE \) within \( s_{SE} \) sessions).

Note that each one of \( \nu_E \) queries from \( E \) made by the adversary \( A \) consists of a pair of equal length messages \( p^0_{i,E}(t) \) and \( p^1_{i,E}(t) \) as vectors in \( F_q^n \) along with the vector \( IV_i \) related to initialization of session \( i \). For each \( 1 \leq i \leq \nu_E \) the vector \( (c^b_{i,E}(\tau), IV_i) \) stands for the answer of the oracle \( E \) to the query \( (p^0_{i,E}(\tau), p^1_{i,E}(\tau), IV_i) = q(E, i, \tau, \nu) \) for which \( s_i(\tau) \) is the internal state of the oracle \( E \) simulating the encryption scheme in session \( i \) with the memory vector \( m_i(\tau) \). Similarly, for each \( 1 \leq i \leq \nu_{SE} \) the vector \( (c^b_{i,SE}(\tau), IV_i) \) stands for the answer to the query \( (p^0_{i,SE}(\tau), p^1_{i,SE}(\tau), IV_i) \) from the oracle \( SE \) with the internal
state $\hat{s}_i(\tau)$ and the memory vector $\hat{m}_i(\tau)$. Recall that in this setting $\tau$ is the local counter dedicated to each session while $\nu$ is the global query counter of the algorithm.

Also, note that for $b \in \{0, 1\}$, within the oracle $E(IV, b)$ we have
\[
c^b_{i,E}(\tau + 1) = Ls_i(\tau) + B\varphi_s(s_i(\tau)) + F\varphi_s(p^b_{i,E}(\tau)),
\]
and within $SE(IV, b)$ we have
\[
c^b_{i,SE}(\tau + 1) = L\hat{s}_i(\tau + 1) + B\varphi_s(\hat{s}_i(\tau + 1)) + F\varphi_s(p^b_{i,SE}(\tau)).
\]

As an standard stage of the security proof, we first reduce the security of $S^4(\mathcal{P})$ to that of $S^4(\mathcal{U})$ in the following proposition.

**Proposition 1.** Considering two cryptosystems $S^4(\mathcal{P})$ and $S^4(\mathcal{U})$ with the same set of parameters, then
\[
|\text{Adv}^{\text{LORBACPA}^+}_{A,S^4(\mathcal{P})}(k) - \text{Adv}^{\text{LORBACPA}^+}_{A,S^4(\mathcal{U})}(k)| < \text{negl}(k).
\]

**Proof.** Consider an adversary $A$ against $S^4(\mathcal{P})$ within the setting of LORBACPA$^+$ security model. Using this adversary, we construct a distinguisher $D$, for $\mathcal{P}$.

More precisely, the distinguisher $D$ interacts with a permutation oracle $O_\pi$, that in the beginning of the game, flips a random bit $b$ and if $b = 0$, chooses a random permutation $\pi \leftarrow \mathcal{U}$, while otherwise, if $b = 1$, the oracle chooses a permutation $\pi \leftarrow \mathcal{P}$.

To this end, since $D$ uses the adversary $A$ as a subroutine, it has to simulate the environment of the adversary $A$. For this, let us describe how $D$ answers the queries made by $A$. The distinguisher $D$ runs $A$ and has to concurrently answer the block queries used to be answered by the oracles $SE(IV, b)$ or $E(IV, b)$. In detail, $D$ works as follows (see Algorithm $\Pi$):

**Distinguisher $D$:**

The input of $D$ is $1^k$ and has access to a permutation oracle $O_\pi$.

1. **Initialization:** $D$ picks at random a bit $b \in \{0, 1\}$ chosen uniformly at random and sets $O_\pi$.

2. **Running $A$ and answering queries:** $D$ runs $A$, answering queries via the subroutine $LR$ described below.

   - **$A$ feeds:** $A$ can submit queries $(p^b_{i,E}(\tau), p^1_{i,E}(\tau), IV_i)$ or $(p^b_{i,SE}(\tau), p^1_{i,SE}(\tau), IV_i')$ where $i$ and $i'$ are session indicators.

   - **$D$ answers:** If $D$ receives a query $(p^b_{i,E}(\tau), p^1_{i,E}(\tau), IV_i)$ it returns $(c^b_{i,E}(\tau), IV_i)$ to $A$, and if it receives a query $(p^b_{i,SE}(\tau), p^1_{i,SE}(\tau), IV_i')$ it returns $(c^b_{i,SE}(\tau), IV_i')$ to $A$. The key points are as follows:

     - If $b = 1$, $D$’s oracle uses $\pi \leftarrow \mathcal{P}$ (a pseudorandom permutation). In this case if $D$ receives query $(p^b_{i,E}(\tau), p^1_{i,E}(\tau), IV_i)$ it always simulates the oracle $E(IV, 1)$ for $S^4(\mathcal{P})$ and returns $(c^b_{i,E}(\tau), IV_i)$ to $A$. Also, if a query $(p^b_{i,SE}(\tau), p^1_{i,SE}(\tau), IV_i)$ is received, then $SE(IV, 1)$ is simulated for $S^4(\mathcal{P})$ and $(c^b_{i,SE}(\tau), IV_i')$ is returned to $A$.

     - If $b = 0$, $D$’s oracle uses $\pi \leftarrow \mathcal{U}$ (a truly random permutation). In this case if $D$ receives query $(p^b_{i,E}(\tau), p^1_{i,E}(\tau), IV_i)$ it always simulates the oracle $E(IV, 0)$ for $S^4(\mathcal{U})$ and returns $(c^b_{i,E}(\tau), IV_i)$ to $A$. Also, if a query $(p^b_{i,SE}(\tau), p^1_{i,SE}(\tau), IV_i)$ is received, then $SE(IV, 0)$ is simulated for $S^4(\mathcal{U})$ and $(c^b_{i,SE}(\tau), IV_i)$ is returned to $A$.

22
Algorithm 1 Distinguisher $D$ (with access to $O_\pi$)

\begin{verbatim}
procedure INITIALIZE
  $b \xleftarrow{} \{0, 1\}$
end procedure

procedure RUN $\mathcal{A}$
  $b' \xleftarrow{} \mathcal{A}^{LR}$
end procedure

procedure FINALIZATION
  Output 1 if $b' = b$,
  Output 0 otherwise.
end procedure
\end{verbatim}

2- Final stage: Continue answering any oracle queries of $\mathcal{A}$ as described above, and at
the end of the game, let $b'$ be the output of $\mathcal{A}$. Then, $D$ outputs 1 if $b = b'$ and outputs
0 otherwise.

We have

$$\text{Adv}^{\text{LORBACPA}^+}_{\text{AS}_2}(\Psi) = 2|Pr(output(\mathcal{A}) = 1) - \frac{1}{2}|$$

and

$$\text{Adv}^{\text{LORBACPA}^+}_{\mathcal{A},\mathcal{S}_2}(\Psi) = 2|Pr(b' = b | b = 1) - \frac{1}{2}|$$

Since we know that $\text{Adv}_{\Psi,D}(k)$ is a negligible function of the security parameter $k$, we have

$$\text{Adv}_{\Psi,D}(k) = 2|Pr(output(D) = 1) - \frac{1}{2}|$$

$$= |Pr(b' = b | b = 1) - Pr(b' = b | b = 0)|$$

$$= \frac{1}{2}|\text{Adv}^{\text{LORBACPA}^+}_{\mathcal{A},\mathcal{S}_2}(\Psi) - \text{Adv}^{\text{LORBACPA}^+}_{\mathcal{A},\mathcal{S}_2}(\Psi)| < \text{negl}(k).$$

Let $\mathcal{A}$ be an adversary in the LORBACPA$^+$ security model as a randomized algorithm
having interactions with LORBA encryption oracles $E$ and $SE$. In this setting, let

$$(p^0_1, p^1_1, i) = q_1(o_1, i, 1, \nu_1), (p^0_2, p^1_2, i) = q_2(o_2, i, 2, \nu_2), \ldots, (p^0_l, p^1_l, i) = q_l(o_l, i, l, \nu_l),$$

be the consecutive $r$ queries that the adversary asks from its oracles, and also let

$$c^1(1) = \epsilon_\pi(s(\tau_{\nu_1} + \delta_{\nu_1}), p^0_1(\tau_{\nu_1})), c^1(2) = \epsilon_\pi(s(\tau_{\nu_2} + \delta_{\nu_2}), p^0_2(\tau_{\nu_2})), \ldots, c^1(r) = \epsilon_\pi(s(\tau_{\nu_l} + \delta_{\nu_l}), p^0_l(\tau_{\nu_l})),$$

be the corresponding consecutive answers, with $\delta_{SE} = 1$ and $\delta_k = 0$. Note that in this
setting and for a fixed session $i$, $l$ consecutive queries of session $i$ can be described as

$$(p^0_1, p^1_1, i) = q_1(o_1, i, 1, \nu_1), (p^0_2, p^1_2, i) = q_2(o_2, i, 2, \nu_2), \ldots, (p^0_l, p^1_l, i) = q_l(o_l, i, l, \nu_l),$$

where the answer to the $\tau$th query $q_{\nu_\tau}(o_{\nu_\tau}, i, \nu_\tau) = (p^0_{\nu_\tau}, p^1_{\nu_\tau}, i)$ is

$$c^b(\nu_{\tau+1}) = \Psi^b(p^b(\tau)) = \Psi^b(c^b(\nu_{\tau-1}), \ldots, c^b(\nu_{\tau-1})).$$ (43)
where $\Psi_b^b$ is defined using Equations 41 and 42. Moreover, the probability space is generated by the random bits used in the experiment for choosing the bit $b$, the key $k$ (containing the information necessary to reconstruct the secret matrices and the secret random permutation) and the initial states of the oracles

$$s(\delta_{0_1}), s(\delta_{0_2}), \ldots, s(\delta_{0_n}),$$

as well as the random bits used by the adversary (i.e. the randomized algorithm). Also, note that $r$ is bounded by a polynomial of the security parameter $k$ since the adversary is a polynomial-time algorithm.

We define the following partial order on the sequence of queries (and consequently on answers),

$$q_v(o_v, i_v, \tau_v, \nu_v) \leq q_v(o_v, i_v, \tau_v, \nu_v) \iff (i_v = i_v \& \tau_v \leq \tau_v).$$

Note that the inequality also implies that $\nu_v \leq \nu_v$ where, clearly, the partial order turns the set of queries (and consequently on answers) into a well-founded (i.e. Noetherian) set.

Now, if $\nu > 0$, let $\text{Col}_\nu$ be the event that for some $j_1 \leq \nu$ and $j_2 \leq \nu$ we have $c^b(j_1) = c^b(j_2)$ for some $b \in \{0, 1\}$. Also, define $\text{Col} \overset{\text{def}}{=} \text{Col}_{j_1}$.

Let $\xi \overset{\text{def}}{=} (c_0, \ldots, c_n)$ be fixed sequence of blocks, let $q_v(o_v, i, \tau, \nu)$ be the $v$th query (i.e. $\nu \overset{\text{def}}{=} \nu_v$), define $\mu_v \overset{\text{def}}{=} m_0 + n_0 = m_0 + t_0$ and also let

$$q_{\nu_{r-1}-\mu_0}(o_{\nu_{r-1}-\mu_0}, i, \tau - 1 - \mu_0, \nu_{\tau-1}), \ldots, q_{\nu_{\nu_{\tau-1}}-1}(o_{\nu_{\tau-1}}, i, \tau - 1),$$

be $\mu_v + 1$ of its consecutive predecessors (in the $i$th session) with the following vector of answers,

$$\varrho^b = (c^b(\nu_{\nu-1}), \ldots, c^b(\nu_{\tau-1})).$$

Define $H_\nu(\xi) \overset{\text{def}}{=} A^0(\xi) \cup A^1(\xi)$ in which

$$A^0(\xi) \overset{\text{def}}{=} \{ (x_{n_0}, \ldots, x_n) \mid \exists j \leq \nu - 1, \ c_j - F_\eta(p^b_j) = \Psi^b_o(x_{n_0}, \ldots, x_{n_0}) \}.$$

Our main objective in what follows is to prove that not only the probability of the event $\text{Col}$ is negligible but also the probability of success for the adversary conditioned to having no collision is also a negligible function of the security parameter. Formally, we have to prove the following statements.

**Proposition 2.** Let $A$ be an adversary for $S^b_o(\Omega)$ within the setting of LORBACPA$^+$ security model. Then for any $b \in \{0, 1\}$ there exist a negligible function $\text{negl}_k$ such that

a) $Pr_0(\text{output}(A) = 1|\text{Col}) = Pr_1(\text{output}(A) = 1|\overline{\text{Col})}.$

b) $Pr_0(\text{Col}) = Pr_1(\text{Col}).$

c) $Pr_b(\text{Col}) \leq \text{negl}_k(k).$

**Proof.** (a) $\Rightarrow$ For this part, let $\xi \overset{\text{def}}{=} (c_0, \ldots, c_n)$ be fixed sequence of answers, let $q_v(o_v, i, \tau, \nu)$ be the $v$th query (i.e. $\nu \overset{\text{def}}{=} \nu_v$) and also let

$$q_{\nu_{r-1}-\mu_0}(o_{\nu_{r-1}-\mu_0}, i, \tau - 1 - \mu_0, \nu_{\tau-1}), \ldots, q_{\nu_{\nu_{\tau-1}}-1}(o_{\nu_{\tau-1}}, i, \tau - 1),$$

be $\mu_v + 1$ of its consecutive predecessors (in the $i$th session) with the following vector of answers,

$$\varrho^b = (c^b(\nu_{\nu-1}), \ldots, c^b(\nu_{\tau-1})).$$
Let \( \xi' \overset{\text{def}}{=} (c_{\nu-1-\nu_0}, \cdots, c_{\nu-1}) \). We use well-founded induction to prove that for any \( \nu \) we have

\[
Pr_0(\xi^0 | Col_\nu) = Pr_1(\xi^1 | Col_\nu).
\]

Since the distribution of the initial states, and consequently, the answers to the first queries are uniformly distributed the base of the well-founded induction holds.

For induction step, note that,

\[
Pr_{b,\overline{\text{Col}_\nu}}(\xi^b = \xi') = Pr_{b,\overline{\text{Col}_\nu}}(\xi^b = \xi'| H_\nu(\xi))Pr_{b,\overline{\text{Col}_\nu}}(\xi' \in H_\nu(\xi)) + \Pr_{b,\overline{\text{Col}_\nu}}(\xi^b = \xi'| \xi' \notin H_\nu(\xi))\Pr_{b,\overline{\text{Col}_\nu}}(\xi' \notin H_\nu(\xi)).
\]

First, note that the event \( \xi' \in H_\nu(\xi) \) does not depend on \( b \), since the size of the set \( H_\nu(\xi) \) does not depend on \( b \). On the other hand, \( \xi' \in H_\nu(\xi) \) implies the event \( \text{Col}_\nu \), indicating that the first term in the sum is equal to zero. Also, the union of \( \text{Col}_\nu \) and \( \xi' \notin H_\nu(\xi) \) does not depend on \( b \) by induction hypothesis.

(b) \( \Rightarrow \) For this case, we use well-founded induction to prove that for any \( \nu \) we have \( Pr_0(\text{Col}_\nu) = Pr_1(\text{Col}_\nu) \). Note that the equality is trivially true for the minimal elements (corresponding to the initial states).

Within the same setting as in Part (a), for any \( b \in \{0,1\} \) we have,

\[
Pr_b(\text{Col}_\nu) = Pr_b(\text{Col}_\nu|\text{Col}_{\nu-1})Pr_b(\text{Col}_{\nu-1}) + Pr_b(\text{Col}_\nu|\overline{\text{Col}_{\nu-1}})Pr_b(\overline{\text{Col}_{\nu-1}}).
\]

Since \( Pr_b(\text{Col}_\nu|\text{Col}_{\nu-1}) = 1 \), by induction the first term of the sum is independent of \( b \). Hence, by induction, it suffices to prove that \( Pr_b(\text{Col}_\nu|\overline{\text{Col}_{\nu-1}}) \) is independent of \( b \). But,

\[
Pr_b(\text{Col}_\nu|\overline{\text{Col}_{\nu-1}}) = \sum_{\xi'} Pr_b(\text{Col}_\nu|\overline{\text{Col}_{\nu-1}} \& \xi^b = \xi') Pr_b(\xi^b = \xi'|\overline{\text{Col}_{\nu-1}}).
\]

In each term, the second component is independent of the bit \( b \) by the proof of part (a). For the first term, note that the probability only depends on the vectors \( c^{(h+1)} \) satisfying Equation [43] But, since the initial states are uniformly chosen random vectors, the matrix \( \mathbf{W} \) and for any \( j \in [\ell] \), the matrices \( \mathbf{L}_j, \mathbf{F}_j \) are uniformly chosen random invertible matrices, and \( \varphi_n \) is a uniformly chosen random permutation, the distributions of vectors \( \mathbf{F}_{\sigma^{-1}(h)}^{-1}(c(h)), \mathbf{F}_{\sigma^{-1}(h)}^{-1}(c(h)), \mathbf{W}_m(h), \mathbf{W}_m(h+1), \mathbf{L}_n(\tau), \mathbf{L}_n(\tau+1), \mathbf{F}_{\varphi_n}(p^b(\tau)) \) are the same, and consequently, by Equations [41], [42] and [43] the probability \( Pr_b(\text{Col}_\nu|\overline{\text{Col}_{\nu-1}} \& \xi^b = \xi') \) does not depend on \( b \) (see Section [3] for a discussion on this part).

(c) \( \Rightarrow \) For this part note that,

\[
Pr_b(\text{Col}) \leq \sum_{\nu=2}^{r} Pr_b(\text{Col}_\nu|\overline{\text{Col}_{\nu-1}}).
\]

But considering Part (b) we know that \( c_\nu \) is uniformly distributed,

\[
Pr_b(\text{Col}) \leq \sum_{\nu=2}^{r} Pr_b(\text{Col}_\nu|\overline{\text{Col}_{\nu-1}}) \leq \sum_{\nu=2}^{r} \frac{2(\nu-1)}{q^{\nu}} \leq 2r^{2}\cdot 2^{-k},
\]

which is a negligible function of \( k \) since \( r \) is bounded by a polynomial function of \( k \).\[\blacksquare\]
Clearly, using Proposition 2, one may prove the main security result as follows.

**Theorem 1.** The self-synchronized stream cipher \[ S^4(\mathcal{P}) \] is LORBACPA+ secure.

**Proof.** By Proposition 1, it suffices to prove the claim for \[ S^4(U) \]. For this we have

\[
\text{Adv}_{A,S^4(\mathcal{U})}^{\text{LORBACPA}+}(k) = Pr_1(\text{output}(A) = 1) - Pr_0(\text{output}(A) = 1) \\
= Pr_1(\text{output}(A) = 1 | \text{Col}^1) Pr_1(\text{Col}^1) + Pr_1(\text{output}(A) = 1 | \text{Col}^2) Pr_1(\text{Col}^2) \\
- Pr_0(\text{output}(A) = 1 | \text{Col}^2) Pr_0(\text{Col}^2) - Pr_0(\text{output}(A) = 1 | \text{Col}^1) Pr_0(\text{Col}^1) \\
= (Pr_1(\text{output}(A) = 1 | \text{Col}^1) - Pr_0(\text{output}(A) = 1 | \text{Col}^1)) \cdot \text{negl}(k) \\
\leq \text{negl}(k).
\]

Consequently, we obtain

\[
\text{Adv}_{A,S^4(\mathcal{P})}^{\text{LORBACPA}+}(k) \leq \text{Adv}_{A,S^4(\mathcal{U})}^{\text{LORBACPA}+}(k) + \text{negl}(k) \\
\leq \text{negl}(k).
\]

\[\blacksquare\]

## 5 Concluding remarks

In this article, we introduced the security model LORBACPA+ for self-synchronized stream ciphers which is stronger than the standard blockwise LOR-IND-CPA, and based on contributions of G. Milléroux et al., we introduced a new self-synchronized stream cipher \[ S^4(\mathcal{P}) \] which is secure in this stronger model. It is instructive to note that the main idea giving rise to this stronger security property is the fact that in the new setup which is based on control-theoretic unknown input observer design for the receiver, one is able to use totally random initial state vectors for encryption.

It is also interesting to have a control-theoretic view to our security proof as an uncontrollability result. From this point of view, an adversary is a stochastic discrete dynamical system gaining information from the answers it receives to its queries as control-inputs. Hence, the aim of the adversary is to make collisions for the answers since having no collision gives rise to a no-information state because of the uniform distribution of the answers (as a consequence of the uniform distribution of the initial states of the oracles and system secret parameters). This can be thought of as a game in which the adversary tries to make collisions while in the state of a self-avoiding walk (i.e., no collision) of this dynamical system, the adversary gains no information about \( b \). Therefore, the whole scenario is to control the system for the objective of maximizing the probability of a collision (hopefully to become noticeable), where from this point of view, security can be interpreted as an uncontrollability property. In this setting, a simple intuition supporting our security proof is based on the facts that being able to choose i.i.d random vectors for the initial states of the oracles guarantees that different oracles have independent trajectories, while the fact that the length of runs (i.e., walks) of each oracle is bounded by a polynomial function of the security parameter (since the adversary is a polynomial-time algorithm) and the fact that each step of the run as an \( n \)-dimensional vector has an exponential number of possibilities, makes sure that the probability of a collision is bounded by a negligible function of the security parameter.

Let us also add a couple of comments on practical issues. First, note that Part (b) of Proposition 2 is still valid even if only the matrix \( L \) and the permutation \( \varphi \), are secretly chosen uniformly at random, however, since in practice, and in particular for small parameters used in a lightweight setting, real simulations fail to completely satisfy theoretical assumptions, we have also added the matrices \( F \) and \( W \) to the secret parameters of the cipher system to ensure uniform randomization mixing (see Equations 41 and 42). On the other hand, we
would like to mention that from a practical point of view, our ciphersystem is as close as to a CCA secure streamcipher while it still is error-resistant and self-synchronized, which in our opinion, makes it more applicable in comparison to a CCA secure stream cipher along with a synchronizer, at least in noisy environments.

Naturally, more practical issues and applications concerning the implementation of this new streamcipher should be the subject of further investigations.

References

[1] F. Anstett, G. Millerioux and G. Bloch, Message-Embedded Cryptosystems: Cryptanalysis and Identifiability, Proc. of the 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC05), Sevilla, Spain, December 12-15.

[2] M. Bellare, A. Desai, E. Jokipii, and P. Rogaway, A concrete security treatment of symmetric encryption, In FOCS 1997, IEEE Computer Society, (1997), 394–403.

[3] P. Guillot, B. Dravie and G. Millerioux, Security proof of the canonical form of self-synchronizing stream ciphers, In Proceedings of Workshop on Cryptography and Coding, WCC 2015, Paris, France, (2015).

[4] M. Bellare, T. Kohno, and C. Namprempre, Authenticated encryption in SSH: provably fixing the ssh binary packet protocol, ACM transactions on information and system security, 7(2), May (2004), 206-241.

[5] M. Bellare and P. Rogaway, Introduction to modern cryptography, InUCSD CSE 207 Course Notes, (2005), 207.

[6] J. Daemen and P. Kitsos, The self-synchronizing stream cipher moustique, In New Stream Cipher Designs - The eSTREAM Finalists, (2008), 210223.

[7] N. Courtois, W. Meier, Algebraic attacks on stream ciphers with linear feedback, In: Biham E. (ed.) EUROCRYPT, Volume, 2656 of Lecture Notes in Computer Science, Springer, Berlin(2003), 345-359.

[8] B. Dravie, P. Guillot and G. Millérioux, Security Proof of the Canonical Form of Self-Synchronizing Stream Ciphers, Designs, Codes and Cryptography, Designs Codes and Cryptography, February (2016), 1-12.

[9] P. Fouque, A. Joux, and G. Poupard, Blockwise Adversarial Model for On-line Ciphers and Symmetric Encryption Schemes, Selected Areas of Cryptography, (2004).

[10] P. Fouque, A. Joux, G. Martinet, and F. Valette, Authenticated On-Line Encryption, Proceedings of the Selected Areas of Cryptography Conference, (2003).

[11] P. Fouque, G. Martinet and G. Poupard, Practical Symmetric On-line Encryption, FSE 2003, LNCS, vol. 2887, 362-375.

[12] A. Joux, G. Martinet, and F. Valette, Blockwise-Adaptive Attacks. Revisiting the (In)Security of Some Provably Secure Encryption Modes: CBC, GEM, IACBC, Proceedings of Advances in Cryptology, (2002).

[13] A. Joux, On the security of blockwise secure modes of operation beyond the birthday bound, IEEE Transactions on Information Theory, Volume 56 Issue 3, March (2010), 1239-1246.
[14] G. Millerioux, J. Daafouz, Flatness of switched linear discrete-time systems, IEEE Transactions on Automatic Control. 54, (2009).

[15] B. Preneel, Cryptanalysis and Design of Stream Ciphers, PHD thesis, Katholieke Universiteit Leuven (Belgium), (2008).

[16] J. Parriaux, P. Guillot, G. Millerioux, Towards a spectral approach for the design of self-synchronizing stream ciphers, in: Cryptography and Communications, Springer, New York, (2011).

[17] J. Parriaux and G. Millerioux, Designing self-synchronizing switched linear systems: An application to communications, Nonlinear Analysis: Hybrid Systems 7, (2013), 68-79.

[18] National Bureau of Standards, DES mode of operations, Technical report, Institute for Computer Sciences and Technology, National Bureau of Standards, Springfield, VA, Decembre (1980).