Spherical simplices
generating discrete reflection groups

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Abstract. Let \( P \) be a simplex in \( S^n \) and \( G_P \) be a group generated by the reflections with respect to the facets of \( P \). We are interested in the case when the group \( G_P \) is discrete. In this case we say that \( G \) generates the discrete reflection group \( G_P \). We develop the criteria for a simplex generating a discrete reflection group. We also describe all indecomposable spherical simplices generating discrete reflection groups.

Introduction

1. Let \( P \) be a convex polyhedron in the spherical space \( S^n \), Euclidean space \( \mathbb{E}^n \) or hyperbolic space \( \mathbb{H}^n \). Consider a group \( G_P \) generated by the reflections with respect to the facets of \( P \). We call \( G_P \) a reflection group generated by \( P \). The problem is to list the polyhedra generating discrete reflection groups. Although the problem is old and easy to state, the answer is known only for some very simple combinatorial types of polyhedra. Already in 1873, Schwarz [8] listed the spherical triangles generating discrete groups. In 1998, E. Klimenko and M. Sakuma [7] solved the problem for hyperbolic triangles. In [3], [4], [5], [6] the problem was solved for hyperbolic quadrilaterals, bounded hyperbolic pyramids and triangular prisms, hyperbolic simplices, and Lambert cubes in \( S^3, \mathbb{E}^3, \mathbb{H}^3 \).

In this paper, we solve this problem for spherical simplices.

2. In [3], this problem is solved for hyperbolic simplices. The methods of [3] are valid for the spherical case, and these methods are sufficient to solve the problem for any partial case. Namely, if \( G \) is a reflection group in \( S^n \), then it is possible to find all the simplices, generating the group \( G \). It is shown in [3] that for any \( G \) the procedure takes finite (but not uniformly bounded!) time.

A spherical discrete reflection group \( G \) is a group generated by one of the spherical Coxeter simplices. Any indecomposable spherical Coxeter simplex is one of \( A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4 \) and \( G_2^{(m)} \) (we use the standard notation, see Table [1]). The rest spherical Coxeter simplices are the direct products of some indecomposable Coxeter simplices. It is sufficient to classify simplices generating indecomposable discrete reflection groups (other simplices generating discrete groups are the direct products of these ones).

Since there exist indecomposable Coxeter simplices in any dimension, it is not possible to solve the general problem case by case. One could expect that

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the answer become stable, while \( n \) goes to infinity. But the list of answers grows exponentially.

E. B. Vinberg suggested a way to reduce the number of items in the answer. Any simplex in the spherical \( n \)-space is bounded by \( n + 1 \) hyperplanes. Observe, that these hyperplanes decompose the \( n \)-dimensional sphere into \( 2^{n+1} \) simplices. We call this set of simplices a family. Evidently, the simplices in a family generate one and the same group. We say that a family generates a reflection group. In this paper we classify all the families of simplices, generating discrete groups.

The classification up to a family is shorter than one for the ordinary simplices is. The number of families for \( A_n, B_n \) and \( D_n \) still goes to infinity with growth of the dimension. But in this situation it is possible to classify the families in terms of graphs. In Section 2, we prove that

- The families generating \( A_n \) are in one-to-one correspondence with the trees with \( n + 1 \) vertices.
- The families generating \( B_n \) are in one-to-one correspondence with the trees with \( n \) vertices exactly one of which is marked.
- The families generating \( D_n \) are nearby in one-to-one correspondence with the connected graphs with \( n \) vertices containing exactly one cycle (possibly the cycle contains only two vertices, but the cycle containing exactly one vertex is permitted).

Once the classification for \( A_n, B_n \) and \( D_n \) is done, we can solve the problem for the rest groups case by case. In fact, the answers for some groups are huge even in terms of the families. For example, there are 78 families generating \( H_4 \), 223 families generating \( E_7 \), and 1242 families generating \( E_8 \). In Section 3, we derive a criterium in terms of the dihedral angles for a family generating a discrete group. The complete answer for each of the groups \( E_6, E_7, E_8, F_4, H_3 \) and \( H_4 \) is contained in the Appendix. (The answer for the group \( G_2^{(m)} \) is quite evident: two reflections generate \( G_2^{(m)} \) if and only if the angle between the mirrors equals \( \frac{1}{m} \pi \), where \( k \) is prime to \( m \).)

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1 Preliminaries

Spherical reflection groups. Let \( S^n \) be the \( n \)-dimensional sphere, and \( G \) be a group acting on \( S^n \) and generated by reflections. Suppose that \( G \) acts discretely,
that is \( G \) is a finite group for the spherical case. Then \( G \) is called a Coxeter group and \( G \) is generated by the reflections with respect to the facets of a spherical Coxeter polyhedron (a polyhedron is called Coxeter polyhedron, if its dihedral angles are the integer parts of \( \pi \)). The classification of the spherical Coxeter polyhedra is due to Coxeter [1]. Any spherical Coxeter polyhedron containing no pair of antipodal points of \( S^n \) is a simplex. Any spherical indecomposable Coxeter simplex is one of the simplices \( A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4 \) and \( G_2^{(m)} \) (a simplex is called indecomposable if it is not a direct product of some other simplices).

To describe the Coxeter simplices, one can use the Coxeter diagrams. The Coxeter diagram of a Coxeter simplex \( P \) is a graph whose vertices \( v_i \) correspond to the faces \( f_i \) of \( P \), the vertices \( v_i \) and \( v_j \) are joined by a \((k-2)\)-fold edge if the dihedral angle formed up by \( f_i \) and \( f_j \) equals \( \frac{\pi}{k} \) (if \( f_i \) is orthogonal to \( f_j \), \( v_i \) and \( v_j \) are disjoint). See Table 1 for the Coxeter diagrams of the indecomposable spherical Coxeter simplices. For more information about the reflection groups see [2].

We use one and the same notation for the Coxeter simplex and the group it generates.

**Simplices generating discrete reflection groups.** Let \( \mathbb{E}^{n+1} \) be a \((n+1)\)-dimensional Euclidean space and \( S^n \) be a unit \( n \)-sphere centered at the origin. Any hyperplane in \( S^n \) corresponds to a hyperplane in \( \mathbb{E}^{n+1} \) containing the origin. An \( n \)-dimensional simplex \( P \) in \( S^n \) corresponds to some \((n+1)\)-faced cone \( C \) in \( \mathbb{E}^{n+1} \) with tip at the origin. A simplex can be represented by a system of \( n+1 \) unit vectors \( f_1, ..., f_{n+1} \) orthogonal to the faces of the cone \( C \) and faced outside of \( C \).

\[
\begin{align*}
A_n (n \geq 1) & \quad \bullet \quad \cdots \quad \bullet \\
B_n (n \geq 2) & \quad \bullet \quad \cdots \quad \bullet \\
D_n (n \geq 4) & \quad \bullet \quad \cdots \quad \bullet \\
E_6 & \quad \bullet \quad \bullet \quad \bullet \\
E_7 & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
E_8 & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
F_4 & \quad \bullet \quad \bullet \quad \bullet \\
G_2^{(m)} & \quad \bullet \quad \bullet \quad m \\
H_3 & \quad \bullet \quad \bullet \\
H_4 & \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{align*}
\]

Table 1: Coxeter diagrams of indecomposable spherical simplices.
Let \(\Pi_1, \ldots, \Pi_{n+1}\) be the faces bounding \(P\). The hyperplanes containing \(\Pi_1, \ldots, \Pi_{n+1}\) decompose \(S^n\) into \(2^{n+1}\) simplices \(P_1, \ldots, P_{2^{n+1}}\) encoded by the vectors \(\{\pm f_1, \ldots, \pm f_{n+1}\}\). Throughout this paper the set of simplices \(P_1, \ldots, P_{2^{n+1}}\) is called a family. Obviously, any of \(P_1, \ldots, P_{2^{n+1}}\) generates the same reflection group as \(P\) does. Thus, we can study families instead of ordinary simplices. (In fact, any family contains at most \(2^n\) simplices up to an isometry: the simplex \(\{f_1, \ldots, f_{n+1}\}\) is always congruent to \(\{-f_1, \ldots, -f_{n+1}\}\).

Let \(P\) be a simplex generating a discrete reflection group \(G\). Clearly, the dihedral angles of \(P\) are some rational numbers multiplied by \(\pi\). Moreover, if \(G\) is one of \(A_n\), \(D_n\), \(E_6\), \(E_7\) and \(E_8\), any dihedral angle of \(P\) is either right angle or \(\pi\). If \(G\) is either \(B_n\) or \(F_4\), the dihedral angle can be also \(\frac{\pi}{4}\) or \(\frac{3\pi}{4}\).

Observe, that in these cases any dihedral angle is either \(\pi\) or \(\frac{\pi}{k}\) with \(k = 3, 4, 5\). If \(f_1\) is orthogonal to \(f_j\), \(f_i\) and \(f_j\) are either \(\pi\) or \(\frac{\pi}{k}\) (if \(f_i\) is disjoint).

Clearly, any two simplices in one family have one and the same family diagram. To specify the simplex in the family, one can use a labeled diagram: we assign an edge with \("+\) if the correspondent dihedral angle is acute, and with \("-\) otherwise. Since the spherical simplices are determined (up to an isometry) by their dihedral angles, different simplices have different labeled diagrams. Note, that some diagrams and some labeled diagrams correspond to no spherical simplex.

We are aimed now to prove that any family diagram corresponds to at most one family of spherical simplices up to an isometry. In general, any dihedral angle of \(P\) is \(\frac{\pi}{k}\) or \(\pi\), where \(k = 3, 4, 5\). This means that if \(\Sigma\) is a family diagram for a family \(\Phi\) then any labeling of \(\Sigma\) either determines no spherical simplex or determines a simplex in \(\Phi\) (all simplices are considered up to an isometry).

**Lemma 1.** Let \(\Gamma\) be a cycle, possibly with some 2-fold edges. Then \(\Gamma\) is a family diagram for at most one family of spherical simplices.

**Proof.** Let \(\pm f_1, \ldots, \pm f_{n+1}\) be a spherical simplex whose family diagram is \(\Gamma\). We can assume that \(f_i\) is orthogonal to \(f_j\) if \(|i - j| > 1\) and \(\{i, j\} \neq \{1, n + 1\}\). Fix \(f_1\) and choose the sign for \(f_2\) to make the angle \(\angle f_1f_2\) non-acute. Then choose the sign for \(f_3\) to make the angle \(\angle f_2f_3\) non-acute, and so on. When the sign for \(f_{n+1}\) is chosen, we have two possibilities: either the angle \(\angle f_{n+1}f_1\) is acute, or non-acute. Observe, that if \(\angle f_{n+1}f_1\) is non-acute, then the simplex \(P\) determined by \(f_1, \ldots, f_{n+1}\) is acute-angled. Moreover, any dihedral angle of \(P\) is either right angle or \(\frac{\pi}{k}\), or \(\frac{\pi}{k}\). Thus, \(P\) is a Coxeter simplex, and the Coxeter diagram of \(P\) cannot be cyclic. The Coxeter diagram of \(P\) coincides with the family diagram of \(P\) which is a cycle. The contradiction shows that \(\angle f_{n+1}f_1\) is acute.
Suppose that $\Gamma$ corresponds to two different families. Choosing the signs as above for both families, we arrive with one and the same simplex (since the dihedral angles determine spherical simplex up to an isometry). Thus, the families coincide.

\[ \square \]

**Lemma 2.** Let $\Gamma$ be a graph, possibly with some 2-fold edges. Then $\Gamma$ is a family diagram for at most one family of spherical simplices.

**Proof.** The proof is by induction on the number of vertices of $\Gamma$. If $\Gamma$ consists of a unique vertex, the lemma is evident. Assume, inductively, that the lemma is true for any graph with at most $n$ vertices.

Let $\Gamma$ be a graph with $n+1$ vertices $v_1, \ldots, v_{n+1}$. Suppose that $\Gamma$ corresponds to two different families, and $f_1, \ldots, f_{n+1}$ and $g_1, \ldots, g_{n+1}$ are the representatives of these families. Let $\Gamma^f$ and $\Gamma^g$ be the labeled diagrams for these simplices. It is sufficient to prove that after some changes of signs of vectors $f_1, \ldots, f_{n+1}$ the labeled diagram $\Gamma^f$ coincides with $\Gamma^g$.

Remove from $\Gamma$ the vertex $v_1$ and all the edges ended in $v_1$. By the assumption, $\Gamma \setminus v_1$ corresponds to at most one family. Thus, up to an isometry we have $f_i = \pm g_i$, $i = 2, \ldots, n+1$ (we choose the sign for each $i$ independently). Without loss of generality we can assume that $f_i = g_i$, $i = 2, \ldots, n+1$, and that the labeled diagram $\Gamma^f \setminus v_1$ coincides with $\Gamma^g \setminus v_1$.

Denote the connected components of $\Gamma \setminus v_1$ by $\Gamma_1, \ldots, \Gamma_k$. Suppose that $v_1$ is joined with $\Gamma_i$ by the edges $e_1^i, \ldots, e_n^i$. We can assume that the edge $e_1^i$ of $\Gamma^f$ is labeled by “+”. If it is not, we change the sign of every vector $f_i$ correspondent to some vertex of $\Gamma_1$. Note that these changes preserve the labels assigned to the edges of $\Gamma_i$, $i = 1, \ldots, k$. Analogously, we assume that $e_1^i$ is labeled by “+” for both diagrams $\Gamma^f$ and $\Gamma^g$ and for any $i = 1, \ldots, k$. Note, that the labeled subdiagrams $\Gamma_1 \setminus v_1$ and $\Gamma_i \setminus v_1$ still coincide.

Consider the edge $e_1^i$ joining $v_1$ with $\Gamma_1$. Since $\Gamma_1$ is connected, $e_1^i$ belongs to some cycle in $\Gamma_1 \cup v_1$. By Lemma 2, any cycle corresponds to at most one family of simplices. Thus, the label of $e_1^i$ is determined by the labeled diagram $\Gamma^f \setminus v_1 = \Gamma^g \setminus v_1$. Therefore, the edge $e_1^i$ has one and the same label in $\Gamma^f$ and $\Gamma^g$. Analogously, the same is true for any edge ended in $v_1$. Thus, we have $\Gamma^f = \Gamma^g$, and the lemma is proved.

\[ \square \]

The above lemma is useful for studying of families generating discrete reflection groups distinct from $H_3$ and $H_4$. To deal with $H_3$ and $H_4$, we extend the definition of the family diagram. The dihedral angles of any simplex generating $H_3$ or $H_4$ belong to the set \( \{ \frac{\pi}{2}, \frac{k\pi}{3}, \frac{l\pi}{3} \} \), where $k = 1, 2$; $l = 1, 2, 3, 4$. In the **family diagram**, the vertices $v_i$ correspond to the faces of the simplex; $v_i$ and $v_j$ are joined by a $(k-2)$-fold edge, if $\angle f_i f_j$ is either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$, $v_i$ and $v_j$ are joined by a 3-fold edge decomposed into 2 parts, if $\angle f_i f_j$ is either $\frac{\pi}{3}$ or $\frac{2\pi}{3}$. See Table 3 for some examples.

Clearly, all the simplices in one family have one and the same family diagram. Unfortunately, the uniqueness of the family correspondent to a diagram does
not hold now. For example, the triangles \((\frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{3})\) and \((\frac{4\pi}{5}, \frac{\pi}{3}, \frac{\pi}{3})\) have the same diagram, but these triangles belong to different families. Nevertheless, usually a family diagram corresponds to at most one family generating \(H_3\) or \(H_4\):\

**Lemma 3.** Let \(\Gamma\) be a family diagram for a family \(\Phi\) generating \(H_3\) or \(H_4\). If \(\Gamma\) is not a cycle with 4 vertices, then \(\Phi\) is the only family with family diagram \(\Gamma\). If \(\Gamma\) is a cycle with 4 vertices, then \(\Gamma\) corresponds to at most two families.\

**Proof.** First, suppose that \(\Phi\) generates \(H_3\). The list of the triangles generating \(H_3\) is contained in [8]. We list the families of the triangles in Table 5. It is easy to see that different families have different family diagrams in this case.

Now, suppose that \(\Phi\) generates \(H_4\) and \(\Gamma\) is not a cycle with 4 vertices. In this case, the lemma follows from the argument of the proof of Lemma 2 together with the result of the previous paragraph.

Finally, if \(\Gamma\) is a cycle with 4 vertices, we can assume that three edges correspond to the acute angles of the simplex \(P \in \Phi\). The rest edge corresponds to either acute or non-acute dihedral angle. Thus, there are at most two ways to specify the dihedral angles of \(P\), and \(\Gamma\) corresponds to at most two families.\

**Remark.** In Lemma 3, we do not specify the types of edges in the cycle. Thus, there could be up to 20 cyclic diagrams corresponding to two families, generating \(H_4\), each. In fact, there are only two such diagrams.

## 2 Simplices generating \(A_n\), \(B_n\) and \(D_n\)

Let \(G\) be one of the reflection groups \(A_n\), \(B_n\) and \(D_n\). We will say that a reflection subgroup \(R\) of \(G\) is of **maximal rank**, if only the origin is fixed by \(R\). Equivalently, \(R\) is generated by \(n\) reflections, or \(R\) is generated by some \((n-1)\)-dimensional simplex.

Let \(\Phi\) be a family generating \(G\) and \(f_1, \ldots, f_n\) be the vectors orthogonal to the faces of \(\Phi\). Denote by \(\Delta\) the correspondent root system \(A_n\), \(B_n\) or \(D_n\). Then suitably normalized vectors \(f_1, \ldots, f_n\) belong to \(\Delta\). Moreover, any linearly independent \(n\)-tuple of vectors in \(\Delta\) determines some family generating a maximal rank subgroup of \(G\).

To classify all the families generating \(G\) it suffices to classify all the linearly independent \(n\)-tuples of vectors in \(\Delta\) and then eliminate families generating proper subgroups of \(G\). Clearly, we are not interested in the ordering of the vectors in the \(n\)-tuple. The Weyl group acts on \(E^n\) by reflections, and thus, it acts on the linearly independent \(n\)-tuples of vectors in \(\Delta\). We will not distinguish \(n\)-tuples equivalent under this action.

**Definition.** Let \(W\) be a Weyl group of a root system \(\Delta\). We say that a family \(\Phi\) is **embedded** into \(W\), if \(\Phi\) is represented by some vectors \(\pm f_1, \ldots, \pm f_n\) chosen from \(\Delta\).
2.1 Simplices generating \( A_n \)

Let \( P \) be a simplex generating a group \( A_n \). Then we can assume that the vectors \( f_1, ..., f_n \) belong to \( \Delta(A_n) = \{ \pm (h_i - h_j) \mid 0 \leq i < j \leq n \} \) (where \( h_0, ..., h_n \) is a standard basis of \( E^{n+1} \)). From the other side, any linear independent system of vectors in \( \Delta(A_n) \) corresponds to a simplex generating \( A_n \) (according to [2], \( A_n \) has no maximal rank subgroups).

For any simplex \( P = \{f_1, ..., f_n\} \) generating \( A_n \) we construct the following graph \( \Sigma \): the vertices \( v_0, ..., v_n \) of \( \Sigma \) correspond to the vectors \( h_0, ..., h_n \), the vertices \( v_i \) and \( v_j \) are joined by the edge \( e_{i,j} \) if one of the vectors \( (h_i - h_j) \) and \( -(h_i - h_j) \) belong to the set \( \{f_1, ..., f_n\} \). Clearly, two simplices in one family have one and the same graph.

To show that the families are in one-to-one correspondence with the graphs, we need a notion of a dual graph:

**Definition.** A complete subgraph \( C \) of a graph \( \Sigma \) is called maximal, if \( \Sigma \) contains no complete subgraph \( C_1 \) such that \( C \) is a subgraph of \( C_1 \).

A graph \( \Sigma^* \) is dual to a graph \( \Sigma \) if the vertices of \( \Sigma^* \) correspond to the maximal complete subgraphs of \( \Sigma \), and two vertices of \( \Sigma^* \) are joined by a line if and only if they correspond to subgraphs in \( \Sigma \) having a common vertex.

If \( \Sigma \) is a tree, the vertices of \( \Sigma^* \) correspond to the edges of \( \Sigma \), and two vertices are joint if the edges are adjacent.

**Lemma 4.** Let \( \Sigma_1 \) and \( \Sigma_2 \) be the trees with \( n + 1 \) vertices. If \( \Sigma_1^* = \Sigma_2^* \), then \( \Sigma_1 = \Sigma_2 \).

**Proof.** A graph \( \Sigma^* \) dual to a tree looks like a “cactus”, that is

1. any vertex belongs to at most two maximal complete subgraphs of \( \Sigma^* \);
2. any pair of maximal complete subgraphs of \( \Sigma^* \) has at most one common vertex;
3. any cycle in \( \Sigma^* \) belongs to some complete subgraph.

Any cactus is a graph dual to some tree. The lemma follows from the fact that the tree can be reconstructed from the cactus by the following algorithm:

1. replace each maximal complete subgraph \( S \) in \( \Sigma^* \) by a vertex \( v_S \);
2. join the vertices \( v_{S_1} \) and \( v_{S_2} \), if \( S_1 \) and \( S_2 \) have a common vertex;
3. for each vertex \( v_S \) attach some additional edges to make the valence of \( v_S \) equal to the number of vertices in \( S \).

See Fig. 1 for an example.

**Remark.** The proof of Lemma 4 shows that \( (\Sigma^*)^* \) is \( \Sigma \setminus L \), where \( L \) is the set of all leaves of the tree \( \Sigma \) (a vertex is called a leaf if its valency equals 1).

Lemma 4 actually shows that the trees with \( n + 1 \) vertices are in one-to-one correspondence with the cacti with \( n \) vertices.


Theorem 1. The families, generating $A_n$, are in one-to-one correspondence with the trees with $n + 1$ vertices.

Proof. Let $P = \{f_1, \ldots, f_n\}$ be a simplex generating $A_n$, and $\Sigma$ be its graph. Since $f_0, \ldots, f_n$ are linearly independent, $\Sigma$ has no cycles. Thus, any family generating $A_n$ corresponds to some tree with $n + 1$ vertices.

Let $\Sigma$ be a tree with vertices $v_0, \ldots, v_n$. Then $\Sigma$ contains exactly $n$ edges $e_{i,j}$. Let $f_1, \ldots, f_n$ be the vectors $h_i - h_j$, $0 \leq i < j \leq n$, such that $e_{i,j}$ is an edge of $\Sigma$. Since $\Sigma$ is a tree, the vectors $f_1, \ldots, f_n$ are linearly independent. These vectors define a simplex generating $A_n$. Thus, the map from the families to the trees is surjective. It is sufficient to prove that this map is injective and that this map does not depend on the embedding of the family in $A_n$.

Consider a graph $\Sigma^*$ dual to $\Sigma$. Note, that $\Sigma^*$ coincides with the family diagram discussed in the previous section (the vertices correspond to the vectors $f_1, \ldots, f_n$, two vertices are joined if the vectors are not mutually orthogonal). Since the graphs dual to the trees are in one-to-one correspondence with the trees, it is sufficient to prove that the family diagrams for the families generating $A_n$ are in one-to-one correspondence with the families generating $A_n$. The last statement follows from Lemma 2.

Corollary 1. Let $\Phi$ be a family of simplices generating $A_n$. There exists a unique embedding of $\Phi$ into $A_n$ (up to the action of the Weyl group of $A_n$).

Proof. Let $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ be two embeddings of $\Phi$, and let $\Sigma_f$ and $\Sigma_g$ be the graphs for these embeddings. Recall, that the vertices of the graphs $\Sigma_f$ and $\Sigma_g$ correspond to the basis vectors $h_0, \ldots, h_n$. Theorem 1 shows that $\Sigma_f$ coincides with $\Sigma_g$ up to a permutation of vertices. The Corollary follows from the fact that the group of permutations of the vectors $h_0, \ldots, h_n$ coincides with the Weyl group of $A_n$. 

8
2.2 Simplices generating $B_n$

Let $P$ be a simplex generating a group $B_n$. Then we can assume that the vectors $f_1, \ldots, f_n$ belong to $\Delta(B_n) = \{\pm h_i, \pm h_i \pm h_j\}, 1 \leq i < j \leq n$ (where $h_1, \ldots, h_n$ is a standard basis of $E^n$). Any linear independent system of vectors in $\Delta(B_n)$ corresponds to a simplex generating either $B_n$ or some reflection subgroup of $B_n$.

According to [2], the only maximal rank indecomposable subgroup of $B_n$ is $D_n$. To generate $B_n$, the system of vectors $f_1, \ldots, f_n$ should be indecomposable and it should contain at least one of the vectors $\pm h_i$.

For any simplex $P = \{f_1, \ldots, f_n\}$ generating $B_n$ we construct the following **graph** $\Sigma$: the vertices $v_1, \ldots, v_n$ of $\Sigma$ correspond to the vectors $h_1, \ldots, h_n$, the vertices $v_i$ and $v_j$ are joined if $\{f_1, \ldots, f_n\}$ contains either $\pm (h_i + h_j)$ or $\pm (h_i - h_j)$ the vertex $h_i$ is **marked** if $\{f_1, \ldots, f_n\}$ contains $\pm h_i$. If $\{f_1, \ldots, f_n\}$ contains both $\pm (h_i + h_j)$ and $\pm (h_i - h_j)$ then $v_i$ and $v_j$ are joined by two edges.

Since $f_1, \ldots, f_n$ is indecomposable, the graph $\Sigma$ is connected. Clearly, $\Sigma$ contains at least one marked vertex. It is easy to see, that if $\Sigma$ contains two marked vertices, then the vectors $f_1, \ldots, f_n$ are linearly dependent. Thus, $\Sigma$ contains a unique marked vertex, and the number of edges in $\Sigma$ is $n - 1$. Since $\Sigma$ has $n$ vertices, $\Sigma$ is a tree with one marked vertex (in particular, no pair of vertices is joined by two edges). Clearly, any tree with one marked vertex corresponds to a family generating $B_n$.

To show that the graphs of this type are in one-to-one correspondence with the families generating $B_n$, we define the **dual** graphs as the graphs obtained by the following algorithm:

1. Construct the graph $\Sigma^*$ dual to $\Sigma$ as usually: the vertices of $\Sigma^*$ correspond to the edges of $\Sigma$, two vertices are joined if the edges are adjacent.
2. Put an additional vertex $v$ in $\Sigma^*$ for the marked vertex $h_{marked}$ of $\Sigma$.
3. Join $v$ by 2-fold line with the vertices standing for $\pm h_{marked} \pm h_i$.

**Lemma 5.** Let $\Sigma_1$ and $\Sigma_2$ be the trees with $n$ vertices containing exactly one marked vertex each. If $\Sigma_1^* = \Sigma_2^*$ then $\Sigma_1 = \Sigma_2$.

**Proof.** A graph $\Sigma^*$ dual to a tree with one marked vertex looks like a “bouquet of cacti”, see Fig. [2]. Any bouquet of cacti is dual to some tree, and the tree can be easily reconstructed from the bouquet.

**Theorem 2.** The families generating $B_n$ are in one-to-one correspondence with the trees with $n$ vertices containing exactly one marked vertex.

**Proof.** The proof follows the proof of Theorem [1]. It is already shown that any family generating $B_n$ corresponds to a graph described in the theorem, and it is easy to see that any graph under consideration corresponds to some family generating $B_n$. Thus, we have the map from the trees to the families and back. To see that this map constitutes the one-to-one correspondence, note, that the
Corollary 2. Let $\Phi$ be a family of simplices generating $B_n$. There exists a unique embedding of $\Phi$ into $B_n$ (up to the action of the Weyl group of $B_n$).

Proof. The proof follows from the fact that the group of permutations of the vectors $h_1, ..., h_n$ is a subgroup of the Weyl group of $B_n$.

2.3 Simplices generating $D_n$

Let $P$ be a simplex generating a group $D_n$. Then we can assume that the vectors $f_1, ..., f_n$ belong to $\Delta(D_n) = \{\pm h_i \pm h_j\}, 1 \leq i < j \leq n$ (where $h_1, ..., h_n$ is a standard basis of $\mathbb{E}^n$). Any linear independent system of vectors in $\Delta(D_n)$ corresponds to a simplex generating either $D_n$ or some reflection subgroup of $D_n$. It follows from [2], that any maximal rank proper reflection subgroup of $D_n$ is decomposable. Thus, $P$ generates $D_n$ if and only if the system of vectors $f_1, ..., f_n$ is indecomposable (the system of vectors is said to be decomposable if its Gram matrix is decomposable).

For any simplex $P = \{f_1, ..., f_n\}$ generating $D_n$ we construct the following colored graph $\Sigma$: the vertices $v_1, ..., v_n$ of $\Sigma$ correspond to the vectors $h_1, ..., h_n$, the vertices $v_i$ and $v_j$ are joined by a black edge if $\{f_1, ..., f_n\}$ contains $\pm (h_i + h_j)$, $v_i$ and $v_j$ are joined by a red edge if $\{f_1, ..., f_n\}$ contains $\pm (h_i - h_j)$. If $\{f_1, ..., f_n\}$ contains both $\pm (h_i + h_j)$ and $\pm (h_i - h_j)$, $v_i$ and $v_j$ are joined by two edges.

We assume the system $f_1, ..., f_n$ to be indecomposable. Hence, the graph $\Sigma$ is connected. Since $\Sigma$ consists of $n$ vertices and $n$ edges, $\Sigma$ contains exactly one cycle. The cycle has at least one red edge, otherwise the vectors $f_1, ..., f_n$ are linearly dependent. After some changes of signs of $h_1, ..., h_n$ one can make all but one edges of $\Sigma$ black. Moreover, one can choose any edge of the cycle, to make this edge the only red edge in $\Sigma$ (the proof follows the proof of Lemma 1).
Hence, the graph itself, without colors, prescribes the family of simplices. The only condition for the uncolored connected graph is that $\Sigma$ contains a unique cycle (possibly, the cycle contains two vertices only).

Let $\Sigma$ be a connected graph with exactly one cycle. Define the dual graph $\Sigma^*$ as follows: the vertices of $\Sigma^*$ correspond to the edges of $\Sigma$, two vertices are joined if the edges have a common vertex; if two edges make up a cycle, the correspondent vertices of $\Sigma^*$ are joined by a dotted edge.

**Lemma 6.** Let $\Sigma_1$ and $\Sigma_2$ be the graphs with $n$ vertices containing exactly one cycle each. If $\Sigma_1^* = \Sigma_2^*$, then $\Sigma_1 = \Sigma_2$.

**Proof.** Let $\Sigma$ be a graph with $n$ vertices containing exactly one cycle. Then $\Sigma^*$ looks like a “cactus necklace”, see Fig. 3. If the cycle in $\Sigma$ contains two edges only, then the string of the necklace is just the dotted edge. To prove the lemma, we check that the graph $\Sigma$ can be recovered from the necklace.

![Cactus necklace](image)

Figure 3: Cactus necklace.

Suppose that the string of $\Sigma^*$ contains at least four edges. Then, to find the graph $\Sigma$ correspondent to the necklace $\Sigma^*$, one can use the same procedure as for $A_n$.

If the string of $\Sigma^*$ consists of two edges, one can use the usual procedure, but the vertices correspondent to two maximal complete subgraphs having the dotted edge in common should be joined by a 2-fold edge.

From now on in this proof suppose that the string of $\Sigma^*$ consists of three edges. Then $\Sigma$ could be reconstructed with the usual procedure, but one should put no vertex for the string of the necklace (the string is a complete subgraph with 3 vertices). Unfortunately, sometimes it is impossible to recognize the string: see Table 2, second row, column $\Sigma_1^*$. We denote this graph by $T$.

To recognize the string in the other cases, note, that 1) any edge contained in two different maximal complete subgraphs belongs to the string; 2) if the string contains at least three vertices and the maximal complete subgraph $C$ intersects the string, then $C$ contains exactly two vertices of the string. Combining these properties, it is easy to see that any necklace with at least four vertices contains $T$ as a subgraph, and that if the necklace contains at least five vertices, the string could be recognized. Thus, the graph $\Sigma$ could be reconstructed, unless $\Sigma^*$ is $T$. The last case does not cause a problem, since $T$ is symmetrical, and
there is no difference which of the two triangles is chosen for a string. Therefore, Σ could be recovered from Σ∗, and the lemma is proved. □

If Σ∗ has a dotted edge, denote by Σ∗ with the dotted edge removed. If Σ∗ contains no dotted edge, define Σ∗ = Σ∗.

**Lemma 7.** Let Σ1 and Σ2 be the graphs with n vertices containing exactly one cycle each. If Σ∗ = Σ∗, then either Σ1 = Σ2 or (Σ1, Σ2) is one of two pairs described in Table 2.

**Proof.** Suppose that neither Σ∗1 nor Σ∗2 contain the dotted line. Then the lemma follows from Lemma 6.

Suppose that both Σ∗1 and Σ∗2 contain the dotted line, and Σ∗1 = Σ∗2. The proof of the preceding lemma shows that the dotted lines in Σ∗1 and Σ∗2 could be reconstructed. Thus, if Σ∗1 = Σ∗2, then Σ∗1 = Σ∗2, and the lemma follows from Lemma 6.

We are left with the case when Σ∗1 contains a dotted line, Σ∗2 contains no dotted line, and Σ∗1 = Σ∗2. Suppose that the 2-fold edge in Σ1 is incident to at least three other edges. Then Σ∗1 has at least two cycles whose vertices are not the vertices of a complete subgraph. This is impossible, since Σ∗2 contains at most one such a cycle (this cycle is a string in Σ∗2, unless Σ2 = T, where T is a graph formed up by two triangles). Suppose that the 2-fold edge in Σ1 is incident to exactly two other edges. Then Σ∗1 has a cycle C whose vertices are not the vertices of a complete subgraph. No edge of C belongs to a complete subgraph having at least one vertex not in C. This is impossible for Σ∗2, unless any vertex of Σ∗2 is contained in C. Thus, Σ1 and Σ2 contain four vertices each, and (Σ1, Σ2) is one of two pairs described in Table 2. Finally, suppose that the 2-fold edge in Σ1 is incident to a unique edge. Then Σ∗1 has two leaves with a common vertex, that is impossible for Σ∗2. (If the 2-fold edge is incident to no other edges, the lemma is obvious).

Denote by Γ1 and Γ2 the graphs shown in the column Σ2 of Table 2.

| Σ1 | Σ2 | Σ∗1 | Σ∗2 | Σ∗1 = Σ∗2 |
|----|----|-----|-----|-----------|
| 1  |  |   |   |   |
| 2  |  |   |   |   |

Table 2: Two exclusions.

Denote by Γ1 and Γ2 the graphs shown in the column Σ2 of Table 2.
Theorem 3. Let $M$ be the set of the connected graphs with $n$ vertices containing exactly one cycle (possibly the cycle contains only two edges, but the cycle containing exactly one vertex is permitted). Then the families, generating $D_n$ are in one-to-one correspondence with the graphs contained in $M \setminus \{\Gamma_1, \Gamma_2\}$.

Proof. The proof follows the proof of Theorem 1. It is already shown that any family generating $D_n$ corresponds to some connected graph with a unique cycle. Any connected graph with $n$ vertices with a unique cycle corresponds to some linearly independent indecomposable system of vectors $f_1, \ldots, f_n$ (to construct these vectors take any edge in the cycle, make it red, and put $h_i - h_j$ for black edges and $h_i + h_j$ for the red one). Thus, we have the map from the graphs with $n$ vertices and exactly one cycle to the families generating $D_n$ and back. To see that this map constitutes the one-to-one correspondence, note, that the graph $\Sigma^*$ coincides with the family diagram for $f_1, \ldots, f_n$, and use Lemma 7 and Lemma 2.

Denote by $\Theta_1$ and $\Theta_2$ the graphs shown in the last column of Table 2.

Corollary 3. Let $\Phi$ be a family of simplices generating $D_n$. If the family diagram for $\Phi$ differs from $\Theta_1$ and $\Theta_2$, then there exists a unique embedding of $\Phi$ into $D_n$ (up to the action of the Weyl group of $D_n$). If the family diagram for $\Phi$ is either $\Theta_1$ or $\Theta_2$, then there are exactly two embeddings.

Proof. If the family diagram for $\Phi$ differs from $\Theta_1$ and $\Theta_2$, the proof follows from the fact that the group of permutations of the vectors $h_1, \ldots, h_n$ is a subgroup of the Weyl group of $D_n$.

Suppose that the family diagram of $\Phi$ is either $\Theta_1$ or $\Theta_2$. Theorem 3 shows that $\Phi$ has at most two embeddings (these embeddings are described by the graphs in the left column of Table 2). To show that these embeddings are different, note that the Weyl group of $D_4$ never takes the pair $\{h_1 - h_2, h_1 + h_2\}$ to the pair $\{h_1 - h_2, h_3 - h_4\}$.

3 Simplices generating other groups

When all the families generating $A_n$, $B_n$ and $D_n$ are listed, we are left with finitely many indecomposable spherical reflection groups. These groups are $E_6$, $E_7$, $E_8$, $F_4$, $H_3$ and $H_4$ (we omit $G_2^{(m)}$ since the question is trivial for these groups). To treat these groups, one can use case-by-case check organized as follows:

1. Let $\Gamma$ be a finite indecomposable reflection group in $S^{n-1}$; list the unit vectors orthogonal to the planes of the reflections of the group $\Gamma$ under the consideration (this vectors are collinear to the roots in the correspondent root system, unless $\Gamma_n = H_3$ or $\Gamma_n = H_4$);
2. for each \( n \)-tuple of vectors from this list check that the system is linearly independent and indecomposable;

3. each linearly independent \( n \)-tuple corresponds to some family generating \( \Gamma \) or some subgroup of \( \Gamma \) (but we get every family a lots of times);

4. list the different families obtained at the previous steps and eliminate the families generating the proper subgroups of \( \Gamma \).

Unfortunately, the lists one obtains after all are not too short (see Appendix for the lists). To get more compact criteria, we specify the necessary condition for a family generating a discrete spherical reflection group.

**Notation.** Let \( P = f_1, \ldots, f_{n+1} \) be a spherical simplex, and \( \alpha \) be a face of \( P \) orthogonal to \( f_{n+1} \). An \( (n-1) \)-dimensional spherical simplex \( f_1, \ldots, f_n \) will be denoted by \( P \setminus \alpha \).

**Definition.** Let \( S \) be a Coxeter simplex generating a reflection group \( G \). A simplex \( P \) satisfies the subgroup property for a group \( G \), if for any facet \( \alpha \) of \( P \) there exists a facet \( \beta \) of \( S \) such that the group generated by \( P \setminus \alpha \) coincides with the group generated by \( S \setminus \beta \).

**Lemma 8.** If a spherical simplex \( P \) generates a discrete reflection group \( G \), then \( P \) satisfies the subgroup property for \( G \).

**Proof.** Let \( \alpha \) be a face of \( P \) and \( v \) be a vertex of \( P \) opposite to a face \( \alpha \). Then the group generated by \( P \setminus \alpha \) is a stabilizer \( \text{Fix}(v) \) of \( v \) in \( G \). Let \( S \) be a fundamental chamber for \( G \) containing \( v \), and \( \beta \) be a face of \( S \) opposite to \( v \). Then \( \text{Fix}(v) \) is a group generated by \( S \setminus \beta \).

A case-by-case treating with the computer shows, that the subgroup property is almost sufficient criteria for a discreteness of the group generated by \( P \):

**Theorem 4.** Let \( P \) be a simplex in \( S^n \), \( n \geq 3 \). Suppose that the dihedral angles of \( P \) are in the set \( \{ \pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4 \} \). Let \( G_P \) be a reflection group generated by \( P \). Let \( G \) be an indecomposable spherical Coxeter group.

The group \( G_P \) is a subgroup of \( G \) if and only if \( P \) satisfies the subgroup property for the group \( G \).

Given a spherical simplex \( P \) without the dihedral angles \( k\pi/5 \), one can determine whether the group \( G_P \) generated by \( P \) is discrete or not. Really, if the dihedral angles of \( P \) satisfy the conjecture of Theorem 4 one can use the theorem. Otherwise, \( G_P \) is not discrete. Note, that if an indecomposable simplex \( P \) has a dihedral angle \( \pi/5 \) and \( P \) generates a discrete group \( G_P \), then \( G_P = H_3 \) or \( G_P = H_4 \). The analog of Theorem 4 for tetrahedra with a dihedral angle \( k\pi/5 \) has some exclusions:

**Proposition 1.** Let \( P \) be a spherical tetrahedron having a dihedral angle \( k\pi/5 \). Then \( P \) generates a discrete reflection group if and only if \( P \) satisfies the subgroup property for a group \( H_4 \) and the family diagram for \( P \) differs from the seventeen diagrams listed in Table 3.
depends on the ordering of vectors to the binary number. This is a doubled unsigned Gram matrix of the system $f \pm E$.

Proposition 1 occurs, and how can one find these exclusions a priori? But one can state that any non-Euclidean simplex whose dihedral angles $\pm \pi$, $\pi/2$, $\pi/3$, $\pi/5$, $\pi/6$, $\pi/7$, $\pi/8$ depend only on the family of simplices. Lemma 2 shows that different families have different matrices. Thus, one can easily recover the family from the number $p$.

Remark. The proof of this proposition as well as of Theorem 4 is just a coincidence of the results of two long case-by-case computer checks. It is not clear if it is possible to prove Theorem 4 a priori. Why do the exclusions in Proposition 1 occur, and how can one find these exclusions a priori?

The Appendix contains the list of families generating $E_6$, $E_7$, $E_8$, $F_4$, $H_3$ and $H_4$. Families generating $E_6$, $E_7$ and $E_8$ are encoded in the following way. For a family $f_1, \ldots, f_n$ construct a symmetric matrix $G^+ = \{g_{i,j}\}$, where $g_{i,j} = 2(|f_i, f_j|)$. This is a doubled unsigned Gram matrix of the system $f_1, \ldots, f_n$. The upper triangle of $G^+$ is filled up by 0 and 1. Let $p$ be a decimal number which is equal to the binary number $g_1.2g_1g_2.3g_2g_3.4g_3g_4.5g_4g_5.6g_5g_6.7g_6g_7.8g_7g_8.9g_8$. The number $p$ depends on the ordering of vectors $f_1, \ldots, f_n$. We choose $p$ the smallest possible. Then $p$ depends only on the family of simplices. Lemma 3 shows that different families have different matrices. Thus, one can easily recover the family from the number $p$.

### Appendix

The Appendix contains the list of families generating $E_6$, $E_7$, $E_8$, $F_4$, $H_3$ and $H_4$. Families generating $E_6$, $E_7$ and $E_8$ are encoded in the following way. For a family $f_1, \ldots, f_n$ construct a symmetric matrix $G^+ = \{g_{i,j}\}$, where $g_{i,j} = 2(|f_i, f_j|)$. This is a doubled unsigned Gram matrix of the system $f_1, \ldots, f_n$. The upper triangle of $G^+$ is filled up by 0 and 1. Let $p$ be a decimal number which is equal to the binary number $g_1.2g_1g_2.3g_2g_3.4g_3g_4.5g_4g_5.6g_5g_6.7g_6g_7.8g_7g_8$. The number $p$ depends on the ordering of vectors $f_1, \ldots, f_n$. We choose $p$ the smallest possible. Then $p$ depends only on the family of simplices. Lemma 3 shows that different families have different matrices. Thus, one can easily recover the family from the number $p$.

### Families generating $E_6$

1187 1195 1197 1199 1214 1259 1260 1261 1263 1270 1271 1278 1279 1983 3451
3452 3453 3455 3687 3949 3967 4831 5717 5791 7903 7915 7917 9719
17598 20287

### Families generating $E_7$

35050 35051 35052 35053 35054 35055 35070 35071 35098 35099 35102 35103
35195 35196 35197 35199 35431 35693 35711 36208 36209 36211 36212 36213

Table 3: Family diagrams for spherical tetrahedra, that satisfy the subgroup property for $H_4$ and generate non-discrete reflection groups. To represent dihedral angles $\pm \pi/5$ and $\pm 2\pi/5$ we use 3-fold edge decomposed into 2 parts.
Families generating $E_8$
Families generating $F_4$

Table 4: Families generating $F_4$ are represented by their family diagrams (in most cases these diagrams are not Coxeter diagrams).
Families generating $H_3$

\[
\begin{array}{cccc}
(\frac{1}{5}, \frac{1}{5}, \frac{2}{5}) & (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}) & (\frac{1}{5}, \frac{3}{5}, \frac{4}{5}) & (\frac{2}{5}, \frac{3}{5}, \frac{4}{5}) \\
(\frac{1}{5}, \frac{1}{5}, \frac{2}{5}) & (\frac{2}{5}, \frac{1}{5}, \frac{3}{5}) & (\frac{2}{5}, \frac{2}{5}, \frac{3}{5}) & (\frac{3}{5}, \frac{1}{5}, \frac{2}{5}) \\
(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}) & (\frac{2}{5}, \frac{2}{5}, \frac{3}{5}) & (\frac{3}{5}, \frac{1}{5}, \frac{3}{5}) & (\frac{3}{5}, \frac{2}{5}, \frac{4}{5}) \\
\end{array}
\]

Table 5: The families of triangles are represented by triangles with the smallest angle sum (and thus, smallest area). The triangle with angles $\frac{k_1 \pi}{l_1}, \frac{k_2 \pi}{l_2}, \frac{k_3 \pi}{l_3}$ is denoted by $(\frac{k_1 \pi}{l_1}, \frac{k_2 \pi}{l_2}, \frac{k_3 \pi}{l_3})$.

Families generating $H_4$

Families generating $H_4$ are encoded in the following way. For a family determined by $\pm f_1, \ldots, \pm f_4$ construct a symmetric matrix $G^+ = \{g_{i,j}\}$, where $g_{i,j} = 0$ if $f_i$ is orthogonal to $f_j$, $g_{i,j} = 1$ if $\angle f_i f_j = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, $g_{i,j} = 2$ if $\angle f_i f_j = \frac{\pi}{5}$ or $\frac{4\pi}{5}$, and $g_{i,j} = 3$ if $\angle f_i f_j = \frac{2\pi}{5}$ or $\frac{3\pi}{5}$. Let $p$ be a decimal number which is equal to the base four number $g_1, g_2, g_3, \ldots g_1, g_2, g_3, \ldots g_i, g_{i+1}, \ldots g_{i,n}$, $g_i, g_{i+1}, \ldots g_{n-1}, n$. The number $p$ depends on the numbering of vectors $f_1, \ldots, f_n$. We choose $p$ the smallest possible. Then $p$ depends only on the family of simplices. Lemma 3 shows that different families have different matrices, unless the family diagram is a cycle with 4 vertices. Thus, in most cases one can recover the family from the number $p$.

If the family diagram is a cycle with 4 vertices and the family generates $H_4$, then $p$ is one of the numbers 344, 348, 364, 420, 500 and 760. It is easy to check that 348 and 500 are the only numbers correspondent to two different families each.

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