Inelastic impact of a sphere on a massive plane: Nonmonotonic velocity-dependence of the restitution coefficient

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Abstract – We have studied the coefficient of restitution, \( \eta \), in normal collisions of a non-rotating sphere on a massive plate for a range of materials, impact velocity and sphere size. The measured coefficient of restitution does not monotonically vary with velocity. This effect is attributed to dynamics that occur during the finite duration of impact, since the timescales of the vibrational modes of the plate are comparable to the contact time. The measured effect is robust and is expected to be ubiquitous in fluidized granular media. We also find that \( \eta \) is a decreasing function of sphere radius, with a dependence that is not captured by existing models of impact.

Collisions of macroscopic objects —such as a ball with the floor— are typically inelastic: some fraction of their total translational kinetic energy is siphoned off into viscoelastic work, plastic deformations, vibrations of the objects, and into producing sound. After careful experiments on normal collisions of spheres, Newton [1] suggested that the degree of inelasticity could be characterized by the ratio \( \eta = -v' / v \), where \( v \) and \( v' \) are the relative velocities before and after impact. The ratio \( \eta \), called the coefficient of restitution, was at first thought to be a constant whose value was determined solely by the geometry and the material properties of the colliding objects. It is now well known that \( \eta \) also depends on the relative velocity of impact: experiments as well as theoretical models [2,3] indicate that \( \eta \to 1 \) as \( v \to 0 \), i.e. the gentler the impact, the closer it is to an elastic collision. In this article we study a particularly simple inelastic collision, that of a sphere colliding normally with a massive wall, and present data that show that \( \eta \) is nonmonotonic in \( v \) and is a decreasing function of the radius, \( R \), of the sphere. The data suggest that an understanding of the velocity dependence of \( \eta(v) \) requires a fuller consideration of the elastodynamics of the colliding objects in the finite duration of the impact.

The starting point for most models of normal inelastic collisions is the Hertz solution to the static problem of a sphere that is being pushed into a wall [4]. This solution—which specifies the stress field in terms of compression of the sphere, its radius \( R \), and the elastic moduli of sphere and wall—is also assumed to obtain at any instant during a collision under the condition that the contact time, \( t_c \), is much longer than \( T \), the slowest elastic response time scale of the sphere [5,6]. This “Love criterion” leads to a restriction of the incoming velocity:

\[
T/t_c < 1 \implies (v/C_0)^{1/5} < 1,
\]

where \( C_0 \) is the speed of sound in the sphere and \( T \approx 2R/C_0 \).

To compute the coefficient of restitution, a model for what is judged to be the dominant dissipation mechanism supplements the Hertzian specification of the elastic force. In recent calculations [7–11], the dissipation mechanism has been modeled by a viscous damping term that is linear in the local strain rate. This yields the prediction \( \eta_{\text{visc}}(v) \sim 1 - CR^{-1}v^{4/5} \), where \( C \) is a material-dependent constant. A different calculation that attributes the dissipation to plastic deformation [3,10] predicts \( \eta_{\text{plastic}}(v) \approx 1.18(v/v_y)^{-1/4} \) for \( v \gg v_y \), the velocity at which the yield stress is first exceeded. These and other, more phenomenological models that incorporate hysteretic loading are reviewed and evaluated in refs. [12,13].

Experiments on ball-ball and ball-plane collisions [2] generally show that \( \eta \) decreases with increasing \( v \), in qualitative agreement with theoretical expectations. At high impact velocities the data are limited in range but moder-
ately good agreement has been claimed [3] with a $v^{-1/4}$ dependence. At lower impact velocities the situation is less clear: the data of Labous et al. [14] for collisions between nylon beads are not fit very well by either $\eta_{\text{plastic}}$ or $\eta_{\text{visc}}$. The data of Hatzes et al. [15] on collisions of smooth ice spheres with ice bricks have been fit by $C \exp(-\gamma v)$. Falcon et al. [16] find $\eta$ almost independent of $v$ for collisions of a carbide sphere with a steel surface. They point out that better agreement with their data is obtained with a dissipation model that is sublinear in strain rate. Stevens et al. find some experimental agreement in restitution coefficient and contact duration with viscoelastic models [7] and a hysteretic force model [17].

Non-monotonicity in the velocity dependence of the restitution coefficient of colliding spheres was previously observed by Sorace et al. [18] and by Grasselli et al. [19]. Their data show that $\eta$ decreases at low impact velocity, as we discuss later in this article. Sorace et al. attribute this behavior to van der Waals adhesion between the spheres, while pointing out that the measured surface energy of the materials is one or two orders of magnitude lower than necessary to quantitatively explain their observations.

Furthermore, the models of $\eta(v)$ yield different dependencies on size of the impinging sphere with $1 - \eta_{\text{visc}} \propto 1/R$, whereas $\eta_{\text{plastic}} \propto R^\beta$. A review of simulational schemes [20] for granular materials catalogues simulation models in which $\eta$ increases, decreases, or is independent of $R$. Experiments that winnow down this wide range of choices are currently lacking. Thus, it is our view that in spite of some high-quality experiments, the available data pool does not allow a decisive experimental test of models for the size or velocity dependence of $\eta$.

We have studied normal collisions of a non-rotating sphere on a massive wall while trying to explore a range of material parameters, impact velocities and sphere size. Most of our results are for collisions on two surfaces: the first is a surface-ground steel plate 2.5 cm in thickness and 22 cm in diameter resting on a 1.25 cm thick sorbothane pad to suppress reflections from the bottom of the plate. The second is a granite optical table 120 × 80 × 30 cm$^3$ in size. The lateral extent of these surfaces is large enough to eliminate end-effects in the impact [21]. We have used steel, brass, aluminium, copper and plastic (delrin) spheres of radius $R = 0.47$ cm. The brass spheres were varied in size from $R = 0.312$ to 0.938 cm.

The ball is held by a vacuum and released without spin from a height of about 1 cm onto a massive plane surface [22] by using a solenoid valve to break the vacuum (see fig. 1). The ball bounces repeatedly on the plane and finally comes to rest. The voltage pulse that releases the vacuum also triggers acquisition of the times of successive impacts. An accelerometer mounted to the plate about 5 cm away from the location of the impact detects elastic waves excited by the impact. The instant of the beginning of the $i$-th collision, $t_i$, is obtained from the leading edge of the accelerometer pulse (to within a fixed time offset of $\approx 10 \mu s$ corresponding to the time of propagation).

When both sphere and plate are metallic, then a second determination of $t_i$ is obtained by applying a small dc voltage between ball and plate and finding the instant when the circuit closes. The electrical method is more sensitive and allows us to measure gentler impacts; it also yields the contact duration of the $i$-th impact, $t_{c,i}$. We have directly verified that electrostatic forces are negligible in the collision since our results are unchanged when the applied voltage is varied by a factor of 120, or when an ac voltage is used. Where both measurements are possible, they yield consistent results for initial time of contact, as seen in fig. 1. Given a set of collision times and contact durations, $t_i$ and $t_{c,i}$, $\eta$ at the $i$-th bounce can be determined as $\eta(v_i) = -v_{i+1}/v_i = \frac{t_{c,i+1} - (t_i + t_{c,i})}{t_{c,i} - (t_i + t_{c,i-1})}$. When either ball or plate was not metallic, the accelerometer data were used to calculate $\eta$ from $t_i$, ignoring $t_{c,i}$ in the expression. This approximation is only valid when the time of flight is much longer than the contact time, however, the sensitivity of the accelerometer limited the accessible velocity range more strongly than the validity of this approximation.

In fig. 2 we show the variation of $\eta$ with impact velocity for a brass sphere on steel. The cloud of small points represents raw data while the solid squares are averages of these data taken in logarithmically spaced bins. The novel and striking implication of these data is that the velocity dependence of $\eta$ is non-monotonic; there
Velocity dependent restitution coefficient

![Graph showing restitution coefficient vs. impact velocity](image)

Fig. 2: $\eta$ vs. impact velocity, $v$ (m/s), plotted on a log-scale, for a brass sphere with $R = 0.47$ cm bouncing against a steel plate. The small dots represent 1130 individual collisions, taken over 100 launches of a sphere. The solid circles are averages of these data taken in logarithmically spaced bins. The vertical bars are the standard deviation about these values and represent the width of the distribution of measurements. The capped gray bars at the top of the figure are error bars showing the precision of individual measurements.

is a range of velocities in which the collisions become more elastic as the impact becomes harder. This is at variance with the theories described above which prescribe a monotonic increase towards the limit of $\eta(v = 0) = 1$. It might appear surprising that this non-monotonic dependence is not routinely observed, since our collision geometry is fairly typical and the precision of some previous measurements (e.g., ref. [16]) is comparable to ours. Our understanding of this apparent inconsistency is that resolving the broad and shallow features seen in fig. 2 requires a much larger number of data points than were taken in previous measurements, and that these data be gathered over a large range in log $v$. The scatter in the raw data of fig. 2 is much larger than the experimental error in a single determination of $\eta$ (indicated by the error bars in the figure). We ascribe the bounce-to-bounce variability due to slight imperfections or asperities falling within the area of contact of these macroscopically smooth objects: only averaging over repeated bounces reveals the underlying behaviour. This interpretation is consistent with the observation that the scatter grows at small velocities: as $v$ is varied from 0.001 m/s to 1 m/s, the Hertzian radius of contact grows from 20 $\mu$m to 350 $\mu$m, conceivably averaging over the rough topography of the ball, as illustrated in fig. 3.

In fig. 4 we show that the non-monotonic behaviour is extremely robust and can be seen for impacts between several pairs of materials. Figure 4A shows $\eta$ as a function of velocity for several metals on a steel plate. While the magnitude of the inelasticity and the position of the minimum and peak in the data vary from one material to another, the overall trends in $\eta(v)$ are maintained. In fig. 4B we show $\eta$ for collisions of a plastic (delrin) sphere on steel and granite surfaces. Once again, $\eta(v)$ clearly shows a peak, though due to the fact that we are not able to use our electrical method of detection, we are unable to go to very small $v$. The fact that the data are displaced from each other demonstrates the important role of the plate, even when there is a considerable mismatch in the elastic modulus of the materials (delrin being much softer than either steel or granite). Finally, in fig. 4C, we

![Graph showing restitution coefficient vs. log impact velocity](image)

Fig. 3: (Colour on-line) Roughness of the surface topography, indicated by deviations of the surface profile from a circle. The three curves show line scans by a profilometer at three location on a 0.47 cm brass ball. A and B show, on the same vertical and horizontal scales, the Hertzian impact region computed for a smooth sphere at impact velocities of 0.002 m/s and 0.05 m/s. Harder impacts (B) average over more surface asperities than softer collisions (A), which are, therefore, more sensitive to local topography.

![Graph showing restitution coefficient vs. impact velocity for different materials](image)

Fig. 4: (Colour on-line) $\eta$ vs. log $v$ (m/s) for (A) brass (●), aluminium (○), and copper (▲) spheres bouncing on a steel plate; (B) delrin on granite (♦) and steel (○); and (C) brass (●) and delrin (♦) bouncing on granite. $R = 0.47$ cm in all these data.
compare the impact of brass and plastic spheres on the granite surface. We emphasize that both plates are much thicker than typical containing walls used in experiments on granular media.

In all the cases above, the lowest impact velocity is determined either by the precision of our technique, or by the ultimate contact of the ball with the plate. (Reference [16] argues that the last stages of this process are an elastic oscillation of the ball and plate under gravity.) The highest velocity we use is determined by an elastic oscillation of the ball and plate under gravity [16] as arguments in refs. [3,23].

Vander Waals interactions can modify collision dynamics by providing greater attractive forces during the outbound phase of the impact than in the inbound phase, when the contact is forming [24,25]. Below a critical impact velocity there is not enough kinetic energy to break the adhesion in the contact. While these attractive forces may be operative in our experiment, they cannot be the source of the observed nonmonotonicity, as the restitution coefficient first decreases with decreasing $v$, but then increases again, at the lowest velocities. This is indicated in fig. 5, where we show alongside an example of our data, the calculated upper limit [18] on critical sticking velocities predicted. We also show for comparison the data of refs. [18,19]. As noted in [18], the calculated critical velocities are substantially below their lowest impact velocities.

We have tried to eliminate other sources of adhesion between sphere and plate by repeatedly cleaning both surfaces between launches of the ball. Our results are unchanged when the experiments are done in a dry $N_2$ atmosphere and when the sphere and plate are both held at high temperature to expel adsorbed water and volatiles. The viscous drag of the air is also negligible: in the extreme case of a plastic sphere at $v = 100$ cm/s, weight/stokes drag $\approx 2 \times 10^4$. Thus we believe we are close to an experimental idealization of the impact problem in which the important forces operative are gravity and the elastic stresses of the media. Why then is there such a discrepancy between theory and our observations?

We believe that the answer lies in dynamical effects that occur during the collision. In fig. 6A we show measurements of the contact time, $t_c$, of the sphere with the plate which varies with the impact velocity as $t_c \sim v^{-1/3}$, in approximate agreement with the contact time predicted from an elastic, Hertzian collision [4]. In fig. 6B, we show the velocity of the plate as a function of time with $t_c$ marked on the time-axis for various impact velocities. It is evident that as $t_c(v)$ changes, the phase of motion of the plate at the instant the ball leaves the plate can change substantially. Thus the peak observed in $\eta(v)$ could be viewed as an elastic mode of the plate slinging the ball upward. The minimum, likewise, could correspond to the plate receding downward at the instant the ball leaves the plate. Since we did not measure the acceleration at the location of the impact, or the vibrations of the ball, we do not have a direct verification of this, but the data of fig. 6 make this explanation quite compelling.
show that the dependence on radius is consistent with a scaling of $1/R^{1/2}$ [10] over the narrow range of $R$ that we explore. The measured dependence is inconsistent with the size dependence of $\eta_{\text{visc}}$, which displays an increase in $\eta$ with increase in $R$, as well as with $\eta_{\text{plastic}}$, which is independent of $R$. Labous et al. [14] state that their data for collisions of nylon spheres shows an increase in $\eta$ with size consistent with a scaling of $1 - \eta \propto R^{-1/2}$, a trend opposite to that shown in fig. 7. The variations in $\eta$ for their different sizes, however, are close to the scatter in the data so it difficult to ascertain whether our results are in contradiction.

We have thus presented data that reveal an unexpectedly complex, nonmonotonic, functional dependence of coefficient of restitution on impact velocity. We have made measurements over a broad range of impact velocity, materials, and particle sizes and find this behaviour to be quite robust. Our experiments suggest that the origin of this nonmonotonic behaviour lies in the fact that the characteristic modes of vibration in the objects participating in a collision are comparable to the contact time in an impact. We should thus not be surprised by apparent disagreement with work that relies on quasistatic approximations [2–4,7–9,11,23]. More generally, we point out that the quasistatic regime is rather difficult to attain and we should expect elastodynamic effects to be typical rather than unusual for collisions in granular media. It has long been known [27] that elastic vibrations can contribute to $\eta$ even without any further dissipation mechanism. These ideas have been elaborated in recent continuum [26] and lattice [28] simulations of normal impacts of discs against rigid walls. However, we do not have direct evidence in favour of any specific dissipation mechanism in our experiments, though our results for the $R$-dependence of $\eta$ may provide a useful benchmark in developing quantitative theories.

Recent work [29,30] has predicted macroscopic consequences of a velocity-dependent $\eta$; it remains to be seen whether there are new consequences that arise from the specific behaviour of $\eta(v)$ that we report. It seems likely that interesting resonant phenomena might occur in sound propagation in granular solids that stem from the velocity dependence we find.

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