16-vertex graphs with automorphism groups $A_4$ and $A_5$ from the icosahedron

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Abstract

The article deals with the problem of finding vertex-minimal graphs with a given automorphism group. We exhibit two undirected 16-vertex graphs having automorphism groups $A_4$ and $A_5$. It improves Babai’s bound for $A_4$ and the graphical regular representation bound for $A_5$. The graphs are constructed using projectivisation of the vertex-face graph of the icosahedron.

Keywords: graph, icosahedron, hemi-icosahedron, automorphism group, alternating group

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This article addresses a problem in graph representation theory of finite groups - finding undirected graphs with a given automorphism group and minimal number of vertices. Denote by $\mu(G)$ the minimal number of vertices of undirected graphs having automorphism group isomorphic to $G$, $\mu(G) = \min_{\Gamma: \text{Aut}(\Gamma) \cong G} |V(\Gamma)|$. It is known [1] that $\mu(G) \leq 2|G|$, for any finite group $G$ which is not cyclic of order 3, 4 or 5. See Babai [2] for an exposition of this area. There are groups which admit a graphical regular representation, for such groups $\mu(G) \leq |G|$. For some recent work see [4].

For alternating groups $A_n$, $\mu(A_n)$ is known for $n \geq 13$, see Liebeck [6]. If $n \equiv 0$ or 1$(mod$ 4), then $\mu(A_n) = 2^n - n - 2$. Additionally, for $n \geq 5$ $A_n$ admits a graphical regular representation, see [8]. Thus for $A_5$ the best published estimate until now seemed to be $\mu(A_5) \leq 60$.

In this paper we exhibit graphs $\Gamma_i = (V, E_i), i \in \{4, 5\}$, such that $|V| = 16$ and $\text{Aut}(\Gamma_i) \cong A_i$. 

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\(\Gamma_4\) (also denoted \(\Xi_I\)) improves Babai’s bound for \(A_4\). \(\Gamma_5\) (also denoted \(\Pi_I\)) has fewer vertices than the graphical regular representation of \(A_5\). \(\Gamma_5\) is listed in [3] together with the order of its automorphism group. The new graphs are based on projectivisation of the vertex-face incidence relation of the regular icosahedron.

We use standard notation for undirected graphs, see Diestel [5]. A bipartite graph \(\Gamma\) with vertex partition sets \(V_1\) and \(V_2\) is denoted as \(\Gamma = (V_1, V_2, E)\). Given a polyhedron \(P\), we denote its vertex, edge and face sets as \(V = V(P), E = E(P)\) and \(F = F(P)\), respectively. We can think of \(P\) as the triple \((V, E, F)\). If \(S\) is a subset of \(\mathbb{R}^3\) not containing the origin, then its image under the projectivisation map to \(P(\mathbb{R}^3)\) is denoted by \(\pi(S)\) or \([S]\), \([S]\) = \(\bigcup_{x \in S}[x]\).

1. Main results

In this section we define objects used for our construction - projective vertex-face graphs. We prove that the automorphism group of the projective vertex-face graph of the regular icosahedron is \(A_5\). We further show that after adding three extra edges we get a graph with the automorphism group \(A_4\).

1.1. Vertex-face graphs of polyhedra

**Definition 1.1.** Let \(P = (V, E, F)\) be a polyhedron. An undirected bipartite graph \(\Gamma_P = (V, F, I)\) is the **vertex-face graph of** \(P\) if \(v \sim f\) iff \(v \in V, f \in F\) and \(v \in f\). In other words, \(\Gamma_P\) corresponds to the vertex-face incidence relation in \(V \times F\).

**Definition 1.2.** Let \(S = (V, E, F)\) be a centrally symmetric polyhedron. Let \(S\) be positioned in \(\mathbb{R}^3\) so that its center is at \((0, 0, 0)\). We call the undirected bipartite graph \(\Pi_S = ([V], [F], I_p)\) **projective vertex-face graph** if for any \(v_p \in [V], f_p \in [F]\) we have \(v_p \sim f_p\) iff \(v \in f\) for some \(v \in \pi^{-1}(v_p)\) and \(f \in \pi^{-1}(f_p)\).

1.2. Projective vertex-face graph of the icosahedron and \(A_5\)

Let \(I = (V, E, F)\) be the regular icosahedron. Define \(\Gamma_5 = \Pi_I\), it is shown in Fig.1, an adjacency matrix of \(\Pi_I\) is given in Appendix A. \(\Pi_I\) can be interpreted in terms of the hemi-icosahedron, see [7].

![Fig.1. - \(\Pi_I\).](image-url)
**Proposition 1.1.** Let \( I \) be the regular icosahedron. Then \( \text{Aut}(\Pi_I) \simeq A_5 \).

*Proof.* We prove that \( \text{Rot}(I) \simeq \text{Aut}(\Pi_I) \) in two steps. First we show that there is a subgroup in \( \text{Aut}(\Pi_I) \) isomorphic to \( \text{Rot}(I) \) - the group of rotational symmetries of \( I \), rotations of \( \mathbb{R}^3 \) preserving \( V \) and \( E \). It is known that \( \text{Rot}(I) \simeq A_5 \). There is an injective group morphism \( f : \text{Rot}(I) \xrightarrow{\text{bij}} \text{Aut}(\Pi_I) \). \( f_1 : \text{Rot}(I) \to \text{Aut}(\Gamma_I) \) maps every \( \rho \in \text{Rot}(I) \) to \( f_1(\rho) \in \text{Aut}(\Gamma_I) \) which is the permutation of \( V \cup F \) induced by \( \rho \): \( f_1(\rho)(x) = \rho(x) \) for any \( x \in V \cup F \). Rotations of \( I \) preserve the vertex-face incidence relation and \( f_1 \) is a group morphism. \( f_2 : \text{Aut}(\Gamma_I) \to \text{Aut}(\Pi_I) \) maps every \( \varphi \in \text{Aut}(\Gamma_I) \) to \( \varphi \circ \rho \in \text{Aut}(\Pi_I) \) defined by the rule \( \varphi([x]) = [\varphi(x)] \) for any \( x \in V(\Gamma_I) \). Projectivization and composition commute therefore \( f_2 \) is a group morphism. \( f \) is injective since there is no nontrivial rotation of \( I \) sending each vertex to another vertex in the same projective class.

In the second step we prove that \( |\text{Aut}(\Pi_I)| \leq 60 \) by a counting argument. Every vertex \( v \in [V] \) is contained in a subgraph \( \sigma(v) \) shown in Fig.2.

![Fig.2. - \( \sigma(v) \).](image)

All \( \Pi_I \)-vertices in \([V]\) have degree 5, all \( \Pi_I \)-vertices in \([F]\) have degree 3. It follows that \([V]\) and \([F]\) both are unions of \( \text{Aut}(\Pi_I) \)-orbits. \( v \) can be mapped by a \( \Pi_I \)-automorphism in at most 6 possible ways. After fixing the image of \( v \) it follows by \( \text{Aut}(\Pi_I) \)-invariance of \([V]\) that the subgraph \( \sigma(v) \) can be mapped in at most 10 ways. Any permutation of \([V]\) by an automorphism determines a unique permutation of \([F]\). Thus \( |\text{Aut}(\Pi_I)| \leq 60 \). We have proved that \( \text{Aut}(\Pi_I) = f(\text{Rot}(I)) \simeq A_5 \). \( \square \)

**Remark 1.1.** A graph isomorphic to \( \Pi_I \) is listed without discussion of its construction and automorphism group in [3] as ET16.5.

1.3. A modification of the projective vertex-face graph of the icosahedron and \( A_4 \)

Since \( A_5 \) has subgroups isomorphic to \( A_4 \), we can try to modify \( \Pi_I \) so that the automorphism group of the modified graph is isomorphic to \( A_4 \). We find generators for a subgroup \( H \leq \text{Rot}(I) \), such that \( H \simeq A_4 \), and add three extra edges to \( \Pi_I \) which are permuted only by elements of \( H \).

Denote by \( I_1 \) the polyhedral (1-skeleton) graph of \( I \), \( \text{Aut}(I_1) \simeq \text{Sym}(I) \simeq A_5 \times \mathbb{Z}_2 \).

**Proposition 1.2.** Choose a 6-subset of vertices \( W = \{O, A, B, C, D, E\} \subseteq V(I) \) such that \( I_1[W] \) is isomorphic to the 5-wheel, see Fig.3.
Define an undirected graph \( \Gamma_4 = \Xi_I = ([V] \cup [F], I_p \cup J) \) by adding three edges to \( \Pi_I \):
\[
J = \{ [A] \sim [C], [B] \sim [O], [D] \sim [E] \},
\]
see Fig.4, Fig.5 and Appendix B. Then \( \text{Aut}(\Xi_I) \cong A_4 \).

**Proof.** Consider the subgroup \( H = \langle a, b \rangle \leq \text{Rot}(I) \) generated by two rotations: \( a \) - a rotation of order 2 around the line passing through the center of the edge \( OB \) and the center of \( I \), \( b \) - a rotation of order 3 around the line passing through the center of the face \( OCD \) and the center of \( I \). We prove that \( H \cong A_4 \) and \( f(H) = \text{Aut}(\Xi_I) \) where \( f \) is as in Proposition 1.1.

To prove that \( H \cong A_4 \) we investigate subgroups of \( A_5 \) generated by two elements of order 2 and 3. If \( H' = \langle a', b' \rangle \leq A_5, \text{ord}(a') = 2, \text{ord}(b') = 3 \), then there are 3 possibilities for the isomorphism type of the functional graph ("cycle type") of the pair \( (a', b') \): \( (a_1, b_1) = ((12)(34), (345)) \), \( (a_2, b_2) = ((12)(34), (134)) \) or \( (a_3, b_3) = ((12)(34), (135)) \). It can be checked that \( \langle a_1, b_1 \rangle \cong \Sigma_3 \), \( \langle a_2, b_2 \rangle \cong A_4 \), \( \langle a_3, b_3 \rangle \cong A_5 \). Additionally, \( \text{ord}(a_1b_1) = 2, \text{ord}(a_2b_2) = 3, \text{ord}(a_3b_3) = 5 \). Now, in our case \( \text{ord}(ab) = 3 \), thus \( H = \langle a, b \rangle \cong \langle a_2, b_2 \rangle \cong A_4 \).

Next we prove that \( \text{Aut}(\Xi_I) = f(H) \). Note that \( O, A, B, C, D, E \) in Fig.3 and Fig.4 represent \([V]\).

First we prove that \( f(H) \leq \text{Aut}(\Xi_I) \). \( \Xi_I \) differs from \( \Pi_I \) by three extra edges. Elements of \( f(H) \) permute \( \Pi_I \)-edges so we only need to check that they permute the new edges. The restrictions
of \( f(a) \) and \( f(b) \) to \([V]\) are, respectively, \(([O][B])\) and \((([O][C][D])([A][E][B]))\) (in cycle notation). It follows that \( f(b) \) cyclically permutes the three extra edges and \( f(a) \) fixes them.

To prove that \( \text{Aut}(\Xi_I) \leq f(H) \) we observe that only \([F]\)-type vertices have degree 3 in both \( \Pi_I \) and \( \Xi_I \), only \( V\)-type vertices have degree 5 in \( \Pi_I \). Thus any \( \text{Aut}(\Xi_I)\)-element as a permutation of \([V] \cup [F]\) belongs to \( \text{Aut}(\Pi_I) \) and thus is the \( f\)-image of a \( \text{Rot}(I)\)-element. We show that for any rotation \( r' \in \text{Rot}(I) \right\} H \), \( f(r') \) does not permute the three extra edges and thus \( f(r') \notin \text{Aut}(\Xi_I) \).

We have that \( \text{Rot}(I) = \langle a, b, c \rangle \) where \( c \) is any rotation of order 5. Since \( |\text{Rot}(I) : H| = 5 \) it follows that any element of \( \text{Rot}(I) \) is in form \( c^nh \) where \( h \in \langle a, b \rangle = H \). Let \( c \) be the rotation around the line passing through the center of \( I \) and \( O \) corresponding to the vertex permutation \((ABCDE)\). The edge \([O] \sim [B]\) is the only extra edge having \([O]\) as a vertex, all edges from \([O]\) are rotationally permuted by \( f(c^n) \), see Fig.4. It follows that nontrivial elements \( f(c^n) \) do not permute the three extra edges in \( \Xi_I \).

\[\text{Remark 1.2.} \] If \( D \) is the dodecahedron then \( \Pi_D \cong \Pi_I \cong A_5 \).

\section{Appendices}

\subsection{A - An adjacency matrix of \( \Pi_I \)}

\textbf{Remark 2.1.} In the standard ordering vertices \( \{1, \ldots, 10\} \) correspond to \([F]\) and vertices \( \{11, \ldots, 16\} \) correspond to \([V]\).
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$B$ - An adjacency matrix of $\Xi_I$

\[
\begin{array}{cccccccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

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