Multipolar corrections for Lense-Thirring precession

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For stationary axially symmetric spacetimes we find a simple expression for the Lense-Thirring precession in terms of the Ernst potential. This expression is used to compute, in the weak field approximation, the major non-spherical contributions to the precession of a gyroscope orbiting the Earth. We reproduce previously known results and give a new estimation for non-spherical contributions.

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I. INTRODUCTION

The Gravity Probe B satellite launched in 2004 is a test of the general theory of relativity sensitive to off-diagonal components of the metric tensor resulting from the Earth rotation. The satellite contains a set of gyroscopes[1] that, according to Einstein theory of gravity, over the course of a year will precess about 6.606 arcsec/year (0.0018 degrees/year) in the orbital plane, an effect known as geodetic precession and 39 milliarcsec/year (0.00011 degrees/year) in the plane of the Earth equator, an effect known as Lense-Thirring precession[2][3]. The experiment is now in its final phase of data analysis[4] and the results are intended to be published in the present year[1]. The Lense-Thirring part of the precession is the one related directly to the off diagonal components of the metric tensor and so its experimental verification will test the Einstein theory for gravitation.

The aim of this paper is to relate the expression for the Lense-Thirring precession with the mathematically sound theory of multipolar moments for axially symmetric spacetimes due to Geroch[5] and Hansen[6] and to discuss multipolar corrections. In Section IV we summarize and discuss our results.

II. THE VECTOR \( \Omega_{LT} \)

Along the paper we use units such that \( c = G = 1 \) and metric signature \((+,+,+,-)\). The symbol \(<>\) denotes the symmetric and trace free part of a tensor. \( A_l \) is a shorthand notation for \( a_{a_1 \ldots a_l} \). Greek letters run from 1 to 4 and Latin from 1 to 3.

From the weak field approximation of General Relativity and the fact that the spin of an orbiting gyroscope is Fermi-Walker transported along its worldline[10] we can show[2][8] that the gyroscope spin will precess with an angular velocity \( \Omega \) that can be separated in three parts according to their physical origin,

\[
\Omega = \Omega_T + \Omega_{DS} + \Omega_{LT}.
\]

where

\[
\Omega_T = -\frac{1}{2} \nu \times a
\]

(2)

\[
\Omega_{DS} = \frac{3}{2} \nu \times \nabla U
\]

(3)

\[
\Omega_{LT} = -\frac{1}{2} \nu \times h.
\]

(4)

The expression (1) is general and it is valid for any gyroscope describing a timelike world line. The first term is known as the Thomas precession, it depends on the gyroscope three-acceleration, \( a \), as well as, on the three-velocity, \( \nu \). It is null when the gyroscope moves along a geodesic. The second term is known as the geodetic or de Sitter precession and is related to the Newtonian potential \( U \). The third term is the angular velocity associated to the Lense-Thirring effect. It is related to non-diagonal part of the metric, \( h = g_{ij} e_j \), where \( e_j \) denotes the spatial part of a tetrad base. An important characteristic of this term is that it does not depend on the gyroscope
velocity along the wordline. This fact suggests to study the gyroscope precession when it is at rest relative to a far observer. Since $\mathbf{v}$ is null in such a frame the Thomas and the geodetic precession will vanish and we are left only with the Lense-Thirring precession.

The spacetime to be considered in this work has the form,

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{34}dx^3 dx^4 + g_{44}(dx^4)^2.$$  \hspace{1cm} (5)

We will choose the frame associated to the tetrads

$$e_1 = \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial x^1}, \quad e_2 = \frac{1}{\sqrt{g_{22}}} \frac{\partial}{\partial x^2},$$  \hspace{1cm} (6)

$$e_3 = \frac{1}{\sqrt{F}} \frac{\partial}{\partial x^3} - \frac{g_{34}}{g_{44}} \frac{1}{\sqrt{F}} \frac{\partial}{\partial x^4},$$  \hspace{1cm} (7)

$$e_4 = \frac{1}{\sqrt{-g_{44}}} \frac{\partial}{\partial x^4},$$  \hspace{1cm} (8)

where $F = g_{33} - g_{34}/g_{44}$ (see also 11). The four velocity $u^\mu$ in this frame is given by $u^\mu = (0, 0, 0, (-g_{44})^{-1/2})$.

To calculate the angular velocity of precession relative to the chosen tetrads we use the Fermi-Walker transport law,

$$\nabla_u \mathbf{S} = u(a \mathbf{S}), \quad a = \nabla_u u.$$  \hspace{1cm} (9)

Therefore

$$\frac{dS_j}{d\tau} = \nabla_u (S \cdot e_j),$$  \hspace{1cm} (10)

$$= S_{\alpha}(\nabla_u e_j)^{\alpha},$$  \hspace{1cm} (11)

where $\nabla_u$ is the total derivative operator, $u$ the four-velocity and $\mathbf{S}$ the spin vector. Using the fact that $e_\alpha \cdot e_\nu = \eta_{\mu\nu}$ and considering that the spinning particle is at rest relative to the tetrads we have $u = e_4$. Hence the condition $u \cdot S = 0$ gives us $S_4 = 0$. From $\nabla_u e_\alpha = e_\alpha \cdot u$, we find that the Fermi-Walker transport law can be cast as,

$$\frac{dS_j}{d\tau} = (S \cdot e_\nu)(m^\nu_j \cdot u),$$  \hspace{1cm} (12)

$$= S_\alpha (m^\alpha_j \cdot e_4).$$  \hspace{1cm} (13)

Now using the connection coefficients given in Appendix A and omitting the subscript $LT$ in $\Omega_{LT}$ we find,

$$\frac{d\mathbf{S}}{d\tau} = \Omega \times \mathbf{S},$$  \hspace{1cm} (14)

where

$$\Omega = \Gamma_{234}^1 \mathbf{m}^1 + \Gamma_{314}^3 \mathbf{m}^2.$$  \hspace{1cm} (15)

The symbols $\Gamma_{314}$ and $\Gamma_{324}$ are given by

$$\Gamma_{314} = \frac{g_{34}}{2\sqrt{-g_{44}g_{11}}} F \left[ \ln \left( \frac{g_{34}}{g_{44}} \right) \right]_1,$$  \hspace{1cm} (16)

$$\Gamma_{324} = \frac{g_{34}}{2\sqrt{-g_{44}g_{22}}} F \left[ \ln \left( \frac{g_{34}}{g_{44}} \right) \right]_2.$$  \hspace{1cm} (17)

We can write (16) and (17) in a simpler way introducing the norm $\lambda$ and the twist $\omega_\mu$ of the time-like Killing vector field $\xi^\mu = (0, 0, 0, 1)$,

$$\lambda = \xi_\mu \xi^\mu,$$  \hspace{1cm} (18)

$$\omega_\mu = \epsilon_{\mu\alpha\beta} \xi^\nu \nabla_\nu \xi^\alpha.$$  \hspace{1cm} (19)

From (12), (19), (19) and (18) we find that $\Gamma_{234}$ and $\Gamma_{314}$ can be written as,

$$\Gamma_{234} = \frac{1}{2\lambda} \omega_1, \quad \Gamma_{314} = \frac{1}{2\lambda} \omega_2.$$  \hspace{1cm} (20)

From the vacuum Einstein equations we know that $\omega_\mu = \nabla_\mu \omega$ 12. Hence the angular velocity $\Omega_{\mu}$ can be cast as,

$$\Omega_{\mu} = \frac{1}{2\lambda} \nabla_\mu \omega.$$  \hspace{1cm} (21)

The scalars $\lambda$ and $\omega$ are related to the Ernst potential, $\Gamma$, by the relation $\Gamma = -\lambda + i \omega$ 13. The expression (21) is particularly interesting because it relates, in an exact manner, the angular velocity of precession to the two fundamental scalars of a stationary spacetime. We were unable to find this expression in the literature.

To express the scalar $\omega$ in terms of Thorne moments using an harmonic coordinate system we use the definition (19) and the Thorne metric for a weak gravitational field 37. This metric is given by,

$$g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}, \quad \gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta_{\alpha\beta} \Gamma^{\alpha\beta},$$  \hspace{1cm} (22)

$$\gamma_{44} = \frac{4M}{r} + \sum_{l=2}^{\infty} (-1)^{l+1} \frac{4M A_l}{l!} [r^{-l}], A_l,$$  \hspace{1cm} (23)

$$\gamma_{4j} = -\frac{2\epsilon_{jpq} S_p n_q}{r^2} - \sum_{l=2}^{\infty} (-1)^{l+1} \frac{4l \epsilon_{jpq} S_p A_{l-1}}{(l+1)!} [r^{-l}], A_{l-1},$$  \hspace{1cm} (24)

$$\gamma_{jk} = 0,$$  \hspace{1cm} (25)

where $\eta_{\mu\nu}$ is the Minkowski metric. $M_{A_l}$ and $S_{A_l}$ are the mass and angular momentum multipole moments respectively. A useful expression is,

$$[r^{-l}], A_l = (-1)^l (2l - 1)! \frac{N_{A_l}}{r^{l+1}},$$  \hspace{1cm} (26)

where $N_{A_l} = n_{a_1} \cdots n_{a_l}$ and $n_a = x^a / r$.

Now from the previous equations we find that $\omega_\alpha$ can be written as

$$\omega_\alpha = \epsilon_{abc} \gamma_{4c,b}.$$  \hspace{1cm} (27)

From Eqs. (21) and (26) get

$$\omega = \sum_{l=1}^{\infty} 4l(2l - 1)! \frac{S_{A_l} N_{A_l}}{(l+1)!} \frac{N_{A_l}}{r^{l+1}}.$$  \hspace{1cm} (28)
Now we shall write this solution in spherical coordinates. Since we are working in an axially symmetric spacetime our multipole moments \( M_{A_l} \) and \( S_{A_l} \) are multiples of \( \hat{z} \langle A_i \rangle \), the symmetric, trace-free outer product of the axis-vector with itself. The moments are completely determined by the numbers \( M_l \) and \( S_l \) defined by

\[
M_l = M_{A_l} \hat{z} \langle A_i \rangle, \quad S_l = S_{A_l} \hat{z} \langle A_i \rangle.
\]  

(29)

We can show that,

\[
M_{A_l} = \frac{(2l - 1)!!}{l!} M_l \hat{z} \langle A_i \rangle, \quad S_{A_l} = \frac{(2l - 1)!!}{l!} S_l \hat{z} \langle A_i \rangle.
\]  

(30)

To compute the quantity \( S_{A_l} N_{A_l} \) we use \( S_{A_l} N_{A_l} = S_l P_l (\cos \theta) \), where \( P_l (\cos \theta) \) are the usual Legendre polynomials. Hence,

\[
\omega = \sum_{l=1}^{\infty} \frac{4l(2l-1)!!}{(l+1)!} S_l P_l (\cos \theta) \hat{r}
\]

(31)

To be consistent in our approximation we use \( g_{44} = -1 \) in (21). Then

\[
\Omega_{\mu} = -\frac{1}{2} \nabla_\mu \omega.
\]  

(32)

Therefore in spherical coordinates,

\[
\Omega = \sum_{l=1}^{\infty} \frac{2l(2l-1)!!}{(l+1)!} \frac{S_l}{r^{l+2}} \left[ (l+1) P_l (\cos \theta) \hat{r} + \sin \theta P_l'(\cos \theta) \hat{\theta} \right],
\]  

(33)

where \( (') \) denotes derivative with respect to \( \cos \theta \). Equation (33) shows that, in this approximation, the contributions to \( \Omega \) are only from the angular momentum moments. In Appendix B we show that the above formula is equivalent to formula (23) of [14]. Relating \( \Omega \) to the Cartesian unit vectors \( \hat{x}, \hat{y} \) and \( \hat{z} \), we get

\[
\Omega = \sum_{l=1}^{\infty} \frac{2l(2l-1)!!}{(l+1)!} \frac{S_l}{r^{l+2}} f(l, \theta),
\]  

(34)

where the components of \( f(l, \theta) \) are

\[
\begin{align*}
{f^x} &= (l+1) P_l (\cos \theta) \sin \theta + P_l' (\cos \theta) \cos \theta \sin \theta, \\
{f^y} &= 0, \\
{f^z} &= (l+1) P_l (\cos \theta) \cos \theta - P_l'(\cos \theta) \sin^2 \theta.
\end{align*}
\]  

(35 - 37)

In deriving the previous expressions, for convenience, it was assumed that the orbit is on the plane \( y = 0 \).

III. \( \Omega \) FOR THE GYROSCOPE EXPERIMENT

The ideal orbit for the Gravity Probe B satellite is a circular one with altitude 642km. Since the Earth has non-vanishing multipole moments this circle is slightly distorted. The radius of such orbit in the lowest order correction of the quadrupole moment is given in [15]. In terms of Thorne quadrupole moment \( Q \) the orbit can be written as,

\[
r = r_0 \left( 1 - \frac{3Q \cos 2\theta}{8Mr_0^2} \right),
\]  

(38)

where \( M \) and \( Q \) are the mass and Thorne quadrupole moment of the source, respectively. From Eq. (33), expanding in powers of \( 1/r_0 \), we obtain

\[
\Omega = \frac{S}{2r_0^3} \left[ g_1(\theta) - \frac{27Q}{16Mr_0^2} (g_2(\theta) - \frac{5MS_3}{2QS} g_3(\theta)) \right],
\]  

(39)

where the vectors \( g_j(\theta) \), \( j = 1, 2, 3 \) are,

\[
\begin{align*}
g_1 &= 2f(1, \theta), \\
g_2 &= \frac{4}{3} f(1, \theta) \cos \theta, \\
g_3 &= \frac{16}{9} f(3, \theta).
\end{align*}
\]  

(40)

The vectors \( g_j \) have the property \( \langle g_j^2 \rangle = \langle g_j^4 \rangle = 0 \) and \( \langle g_j^5 \rangle = 1 \), where \( \langle g_j^k \rangle \) means to take the average over \( g_j^{k}(\theta) \), which in this case is a simple integration over \( \theta \) and in view of the orbit symmetry, it is enough to consider half orbit from one pole to the other \([0, \pi]\). The average of \( \Omega \) reads,

\[
\langle \Omega \rangle = A \left[ 1 - B \left( 1 - \frac{5C}{2} \right) \right] \hat{z},
\]  

(41)

where

\[
A = \frac{S}{2r_0^3}, \quad B = \frac{27Q}{16Mr_0^2}, \quad C = \frac{MS_3}{QS}.
\]  

The case \( C = 2/5 \) is special, since we have no correction although the spacetime is deformed.

The Thorne mass moments can be accurately computed using de data published in [16], but the angular momentum moments are independent of the mass moments. To determine the constant \( C \) we need a model for an oblate Earth. Some authors [17] [14] have already calculated the non-spherical contributions to \( \Omega \). We shall use our approach to compare the precession for these models. Also, based on the Earth model of reference [18], we give a new estimative for \( \Omega \).

A. Teyssandier model

Comparing the general form of Thorne metric with the one in [14], we get

\[
Q = -\frac{2MR^2 J_2}{3}, \quad S_3 = -\frac{4SR^2 K_2}{5}, \quad C = 0.97.
\]  

(42)
where we have used the values $J_2 = (1082.64 \pm 0.01) \times 10^{-6}$ e $K_2 = 0.874 \times 10^{-3}$ given by the author. Therefore we will have

$$\langle - \Omega \rangle = A(1 + 1.42B) \hat{\varepsilon}. \quad (43)$$

This value is different from the one in [14] because we use the angle $\theta$ in [34] instead of $\psi = \pi/2 - \theta$ as in the quoted reference. Therefore on averaging over [39] we get a different, but equivalent result.

**B. Adler-Silbergeit model**

For Adler-Silbergeit model B [17], we obtain

$$Q = -\frac{2MR^2J_2}{3}, \quad S_3 = -\frac{8\omega MR^4J_2}{35}, \quad S = I\omega. \quad (44)$$

From this expression we calculate the constant $C = MS_3/QS$, and Eqs (41) gives us,

$$\langle - \Omega \rangle = A \left[ 1 - B \left( 1 - \frac{6M R^2}{7I} \right) \right] \hat{\varepsilon}. \quad (45)$$

Confronting this result with Eq. (61) of [17] we see that our third term differs by a factor two (the numerical factor in the quoted reference is $3/7$). Using $MR^2/I = 3.024$ given by the authors we obtain

$$\langle - \Omega \rangle = A(1 + 1.59B) \hat{\varepsilon}. \quad (46)$$

They get $\langle - \Omega \rangle = A(1 + 0.30B) \hat{\varepsilon}$.

**C. Adler model**

For this model [15] we have

$$Q = -\frac{2Ma^2}{9}, \quad S_3 = -\frac{4Sa^2}{25}, \quad C = 0.72. \quad (47)$$

Hence the averaged angular velocity is

$$\langle - \Omega \rangle = A(1 + 0.80B) \hat{\varepsilon}. \quad (48)$$

**D. Exact solution**

It is instructive to calculate the constant $C$ for the case our geometry is described by an exact solution of the Einstein vacuum equations with arbitrary quadrupole moment. We choose the version of this solution presented in [19]. Equivalent solutions has been obtained by several authors using different methods [12]. The first four nonzero Geroch-Hansen moments are,

$$M_0 = k(1 + \alpha^2)/(1 - \alpha^2)$$

$$J_1 = -2a k^2(1 + \alpha^2)/(1 - \alpha^2)^2$$

$$M_2 = -k^3 \beta + 4 \alpha^2(1 + \alpha^2)(1 - \alpha^2)^{-3}$$

$$J_3 = 4ak^3 \beta + 2 \alpha^2(1 + \alpha^2)(1 - \alpha^2)^{-3}/(1 - \alpha^2)$$

where $M_n, J_n$ are the mass and current moments, respectively, and $\alpha, \beta$ and $k$ are parameters. We find the following relation,

$$J_3 = \frac{J}{M}(2M_2 - M_2^{Kerr}) \quad (50)$$

where $M_2^{Kerr} = -J^2/M$ is the Kerr quadrupole moment. Since $M_2^{Kerr}$ has vanishing Newtonian limit, in our approximation, we are only left with

$$J_3 = \frac{2M_2J}{M}. \quad (51)$$

Now using the correspondence between Geroch-Hansen and Thorne moments [9] we get $C = 4/15$. Hence equation (41) reads

$$\langle - \Omega \rangle = A(1 - 0.33B) \hat{\varepsilon}. \quad (52)$$

that differs considerably from the previous models.

**IV. SUMMARY AND CONCLUSIONS**

By using an adequate tetrad basis we derive an exact expression for the Lense-Thirring precession in terms of the norm and the twist of the spacetime timelike killing vector. We calculated the imaginary part of Ernst potential using the weak field approximation and the Lense-Thirring precession to any desired multipolar correction. We consider the case of the Gravity Probe B experiment expanding the precession angular velocity up to the order of $1/r_0^5$ and averaging over a half trajectory. The final expression is given in terms of Thorne multipole moments. In our opinion, this general expression, may be interesting for the interpretation of exact solutions of Einstein field Equations. We have reproduced known results and given new estimatives for the non-spherical contributions to precession angular velocity. We conclude that the earth model plays a significant role in the multipolar contribution. The results produced by different Earth models are accounted for by our constant $C$, fact that makes our relation (41) particularly useful.

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**APPENDIX A: CONNECTION COEFFICIENTS**

For easy reference we give in this appendix the connection coefficients relative to the tetrads [13], [15]. The dual base is:

$$m^1 = \sqrt{g_{11}}dx^1, \quad m^2 = \sqrt{g_{22}}dx^2, \quad (A1)$$

$$m^3 = \sqrt{F}dx^3, \quad m^4 = \frac{g_{34}}{\sqrt{-g_{44}}}dx^4. \quad (A2)$$
From the first Cartan structure equations [12],
\[ \text{d}m^\mu + m_\nu^\sigma \times m^\nu = 0, \]  
we find the connection coefficients,
\[ m_{12} = \Gamma_{121} m^1 + \Gamma_{122} m^2, \quad (A3) \]
\[ m_{13} = \Gamma_{131} m^1 + \Gamma_{132} m^2, \quad (A4) \]
\[ m_{23} = \Gamma_{231} m^1 + \Gamma_{232} m^2, \quad (A5) \]
\[ m_{24} = \Gamma_{241} m^1 + \Gamma_{242} m^2, \quad (A6) \]
\[ m_{42} = \Gamma_{421} m^1 + \Gamma_{422} m^2, \quad (A7) \]
\[ m_{43} = \Gamma_{431} m^1 + \Gamma_{432} m^2, \quad (A8) \]
\[ m_{44} = \Gamma_{441} m^1 + \Gamma_{442} m^2, \quad (A9) \]

where
\[ \Gamma_{i21} = \frac{g_{11,2}}{2g_{11} \sqrt{g_{22}}} \quad \Gamma_{212} = \frac{g_{22,1}}{2g_{22} \sqrt{g_{11}}} \quad (A10) \]
\[ \Gamma_{3i3} = \frac{F_i}{2F \sqrt{g_{11}}} \quad \Gamma_{333} = \frac{g_{11}}{2F \sqrt{g_{11}}} \quad (A11) \]
\[ \Gamma_{4i4} = \frac{g_{44,1}}{2g_{44} \sqrt{g_{11}}} \quad \Gamma_{442} = \frac{g_{44,2}}{2g_{44} \sqrt{g_{22}}} \quad (A12) \]
\[ \Gamma_{3i4} = \frac{g_{44}}{2 \sqrt{-g_{44} g_{11} F}} \left[ \ln \left( \frac{g_{44}}{g_{44}} \right) \right] \quad (A13) \]
\[ \Gamma_{324} = \frac{g_{44}}{2 \sqrt{-g_{44} g_{22} F}} \left[ \ln \left( \frac{g_{44}}{g_{44}} \right) \right] \quad (A14) \]

and
\[ \Gamma_{i23} = \Gamma_{32i}, \quad \Gamma_{i31} = \Gamma_{13i}, \quad (A15) \]
\[ \Gamma_{i43} = \Gamma_{34i}, \quad \Gamma_{i42} = \Gamma_{24i}. \quad (A16) \]

**APPENDIX B: EQUIVALENT EXPRESSIONS**

In this appendix we show that our Eq. (33) is equivalent to Eq. (23) of Ref. [14]. Using the relation,
\[ \hat{\theta} = \frac{1}{\sin \theta} (\cos \vartheta \hat{r} - \hat{z}), \quad (B1) \]
we can write equation (33) in terms of the vectors \( \hat{r} \) and \( \hat{z} \),
\[ \Omega = \sum_{l=1}^{\infty} \frac{2(2l-1)!}{l+1} \frac{S_l}{r^{l+2}} \left[ P_{l+1}(\cos \vartheta) \hat{r} - P_l(\cos \vartheta) \hat{z} \right]. \quad (B2) \]

From the relation,
\[ P_{l+1}(\cos \vartheta) = (l+1)P_l(\cos \vartheta) + \cos \vartheta P_l'(\cos \vartheta), \quad (B3) \]
we can cast (B2) in the form,
\[ \Omega = \sum_{l=1}^{\infty} \frac{2(2l-1)!}{(l+1)!} \frac{S_l}{r^{l+2}} \left[ P_{l+1}(\cos \vartheta) \hat{r} - P_l(\cos \vartheta) \hat{z} \right]. \quad (B4) \]

or as,
\[ \Omega = \frac{S}{r^3} \left[ 3 \cos \vartheta \hat{r} - \hat{z} \right] + \sum_{l=1}^{\infty} \frac{2(2l+1)! (S_l/S)_{l+2}}{(l+2)!} \left[ P_{l+1}(\cos \vartheta) \hat{r} - P_l(\cos \vartheta) \hat{z} \right]. \quad (B5) \]

Identifying,
\[ S_{l+1} = -\frac{(l+2)!SKl}{2(2l+1)!}, \quad (B6) \]
we get Eq. (23) of Teyssandier paper.
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