Existence Proofs of Some EXIT Like Functions

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Abstract—The Extended BP (EBP) Generalized EXIT (GEXIT) function introduced in [4] plays a fundamental role in the asymptotic analysis of sparse graph codes. For transmission over the binary erasure channel (BEC) the analytic properties of the EBP GEXIT function are relatively simple and well understood. The general case is much harder and even the existence of the curve is not known in general. We introduce some tools from non-linear analysis which can be useful to prove the existence of EXIT like curves in some cases. The main tool is the Krasnoselskii-Rabinowitz (KR) bifurcation theorem.

I. INTRODUCTION

The Extended BP (EBP) GEXIT function introduced in [4] plays an important role in the analysis of iterative coding systems. For transmission over the binary erasure channel (BEC) this function encodes both the behavior of the BP as well as the MAP decoder in the asymptotic limit of infinite blocklengths. Further, in this case the EBP GEXIT function has a very simple analytic expression in terms of the degree distribution of the ensemble.

It is conjectured that the fundamental characteristic of EBP GEXIT functions remains valid also for general (binary memoryless symmetric) channels. Figure 1 shows the EBP GEXIT function for the degree distribution pair \((\lambda(x) = 0.25x + 0.75x^2, \rho(x) = x^2)\), assuming that transmission takes place over the binary symmetric channel (BSC). Note that this curve smoothly connects the point \((1,1)\), corresponding to the channel BSC(\(\frac{1}{2}\)), with the point \((h_{\text{stab}}^{0}, 0)\), where \(h_{\text{stab}}^{0}\) corresponds to that channel parameter at which the coding system changes its stability behavior. The curve was computed using a procedure suggested in [4].

This procedure guarantees in general the existence of a fixed point density for every point on the vertical axis. Unfortunately, it does not guarantee that the set of fixed points so computed forms a smooth one-dimensional manifold. Such a property however, is required in order to complete the theory of EBP GEXIT functions. E.g., it is known that if the curve is smooth then the area it encloses is equal to the code rate. Combined with the General Area Theorem (first proved for the BEC in [5] and then extended to the general case in [4]) this statement on the area gives rise to bounds on the MAP performance for sparse graph codes. For the BEC it has been shown that in many cases the bound is tight and it is conjectured to be tight not only for the BEC but also in the general case.

The existence of the EBP GEXIT function is therefore a fundamental question at the heart of the asymptotic theory of sparse graph codes. We introduce some tools from non-linear analysis which can be useful to prove the existence of EXIT like curves in some cases. The main tool is the Krasnoselskii-Rabinowitz (KR) bifurcation theorem.

II. DEFINITIONS AND THEOREM RELATED TO THE EXISTENCE OF FIXED POINTS

As discussed in the last section, it is a difficult task to prove the existence of the EBP GEXIT curve for general channels. I.e., it is difficult to prove that the set of fixed point densities of density evolution forms a differentiable one-dimensional manifold.

Although we currently do not know how to prove the existence for the general case, a fundamental theorem of non-linear analysis, called the Krasnoselskii-Rabinowitz (KR) theorem ([1], [2]), can be helpful in some instances to establish the existence of an unbounded connected component of fixed points. To be more precise: density evolution represents a non-linear map in the space of densities. If we are given a degree distribution pair with a non-zero fraction of degree-two variable nodes and a family of BMS channels, then this map has a bifurcation point for that channel parameter which corresponds to the stability condition. In other words, consider the channel parameter for which the linearization of the density evolution map around the density corresponding to perfect decoding has its largest eigenvalue equal to one. Then this channel parameter is a bifurcation point. Under some technical conditions the KR theorem then guarantees that there is a connected set of fixed points which starts at this bifurcation point and which either extends to infinity or which connects back to another bifurcation point. This is not quite as strong a statement as we would wish: we are not guaranteed that this connected set forms a smooth manifold, nor do we know...
that the curve connects to the fixed point corresponding to the worst density and worst channel. Nevertheless, if the theorem applies, we at least know the existence of the EBP GEXIT curve locally around the stability point. Before we can show some cases where the KR theorem can be applied let us quickly review the main notation and the main statement.

We denote a generic Banach space by $X$ (e.g. $X = \mathbb{R}^N$). We denote elements of $X$ in boldface letters, i.e., $\mathbf{x} \in X$. We denote the space of bounded linear operators from $X$ to $X$ by $L(X)$. We are interested in maps of the form $G : \mathbb{R} \times X \to X$. The argument $\gamma$ of $G(\gamma, \mathbf{x})$ is called the parameter. In our setting the parameter will be the channel parameter (e.g., the erasure probability of the BEC or the cross-over probability for the BSC). Recall the following definitions:

- Completely Continuous (CC) Map: A map $G : \mathbb{R} \times X \to X$ is CC if it maps every bounded set $A$ of $\mathbb{R} \times X$ to a relatively compact set in $X$.
- Frechet differentiable: Let $G : \mathbb{R} \times X \to X$ be a map such that $G(\gamma, \mathbf{0}) = \mathbf{0}$. $G$ is Frechet differentiable at $\mathbf{x} = \mathbf{0}$ if there exists $T \in L(X)$ such that, given $\varepsilon > 0$ and an interval $[\gamma_0, \gamma_1]$ of $\mathbb{R}$, there exists $\delta > 0$ with the property that $||y|| < \delta$ implies $\frac{||G(\gamma, y) - \gamma Ty||}{||y||} < \varepsilon$

for all $\gamma \in [\gamma_0, \gamma_1]$. Note that $\delta$ depends on both the choice of interval and the value of $\varepsilon$. We say that $\gamma T$ is the Frechet derivative of $G$ at $\mathbf{0}$.

We denote the set of non trivial fixed points of $G$ by $S = \{ (\gamma, \mathbf{x}) : G(\gamma, \mathbf{x}) = \mathbf{x}, \mathbf{x} \neq \mathbf{0} \}$ and the closure of $S$ by $\overline{S}$. If a point $(\mu, \mathbf{0}) \in \overline{S}$, then the number $\mu$ is called a bifurcation point for the solutions to $G(\gamma, \mathbf{x}) = \mathbf{x}$.

**Theorem 1 (KR Theorem):** [1, Theorem 17.8] Let $X$ be a Banach space and let $G : \mathbb{R} \times X \to X$ be a map. Let $S = \{ (\gamma, \mathbf{x}) : G(\gamma, \mathbf{x}) = \mathbf{x}, \mathbf{x} \neq \mathbf{0} \}$ be the set of non trivial fixed points of $G$ and let $\overline{S}$ denote the closure of $S$. Assume that the following hypothesis holds.

1. $G(\gamma, \mathbf{x})$ is a continuously continuous map.
2. $G(\gamma, \mathbf{x})$ is Frechet differentiable at $\mathbf{0}$, with Frechet derivative $\gamma T$.
3. Let $\frac{1}{\mu}$ be an eigenvalue of $T$ which is of odd algebraic multiplicity.

Then there exists a maximal closed connected subset $C_\mu$ of $\overline{S}$ which contains $(\mu, 0)$ and one of the following is true.

1. $C_\mu$ is unbounded in $\mathbb{R} \times X$.
2. $C_\mu$ contains $(\mu^*, \mathbf{0})$ for some other bifurcation point $\mu^* \neq \mu$.

A graphical representation of the KR theorem is shown in Figure 2.

Our basic plan of attack is the following. In our setting $\mathbf{x}$ will denote a density, and $G$ will be the density evolution map. We want to parametrize the space in such a way that $\mathbf{0}$ denotes the desired fixed point corresponding to perfect decoding. The parameter $\mu$ will parametrize the channel. If we can show that the linearization of the density evolution map around $\mathbf{0}$ has eigenvalue $1/\mu$, where $\mu$ denotes the channel parameter which corresponds to the stability condition, and if the linearization fulfills the desired technical conditions, then there is a connected component of fixed-points which either extends to infinity or is connected to another bifurcation point. At least locally, we will therefore have proved the existence of a connected component of fixed points.

In the following it is also good to know the following fact.

**Theorem 2:** [1, Theorem 17.4] Let $G : \mathbb{R} \times X \to X$ be a continuously continuous and Frechet differentiable at $\mathbf{0}$, with derivative $\gamma T$. If $\frac{1}{\mu}$ is not an eigenvalue of the compact linear operator $T$, then there exist $\varepsilon, \eta > 0$ such that $G(\gamma, \mathbf{x}) \neq \mathbf{x}$ for all $(\gamma, \mathbf{x})$ for which $|\gamma - \mu| < \varepsilon$ and $0 < ||\mathbf{x}|| < \eta$. In particular, $\mu$ is not a bifurcation point for the solutions to $G(\gamma, \mathbf{x}) = \mathbf{x}$.

We also use the following terminology in the rest of the paper. Let $G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a map of the form $G = \{ G_i \}_{i=1}^N$, where $G_i : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a multivariate polynomial in the components of $\mathbf{x}$ and $G_i(\mu, \mathbf{x}) = G_i^1(\mathbf{x}) + \mu G_i^2(\mathbf{x})$. Then we say that $G$ is a vector polynomial map.

### III. EXAMPLES

In principle, we would like to apply the KR theorem directly to the BP or min-sum decoder. But there are some technical conditions that make the direct application difficult. For example, the bifurcation point for the BP decoder appears when the Bhattacharyya parameter is equal to $\frac{1}{\lambda(0)p(1)}$. This suggest that the Bhattacharyya parameter should play the role of the parameter in the setting of the KR theorem. The theorem requires that the parameter $\gamma$ takes on values in $\mathbb{R}$ and not only on $[0, 1]$. Therefore, we can not just work in the space of symmetric densities (for which the Bhattacharyya parameter is in the range $[0, 1]$) but we are required to extend the space. How this is best done is currently an open question. Because of these technical difficulties, we consider quantized decoders. First we show the application of the KR theorem to the simplest possible case.

**Example 1 (BP Decoder for Binary Erasure Channel):** It is instructive (and easy) to analyze the fixed points of the density evolution map for the BEC(\varepsilon). Consider a degree distribution pair $C(\lambda, \rho)$ with $\lambda^2(0)p^2(1) > 0$.

The density evolution recursion reads

$$x_l = \varepsilon \lambda (1 - \rho (1 - x_{l-1})).$$
We take the space \( X = \mathbb{R} \) and set \( G(\varepsilon, x) = \varepsilon \lambda(1 - p(1 - x)) \). Here the erasure probability \( \varepsilon \) plays the role of the parameter. As \( G(\varepsilon, x) \) is a polynomial map, it is completely continuous by Lemma 1 and Frechet differentiable by Lemma 2. From Lemma 2, the Frechet derivative of \( G(\varepsilon, x) \) is given by \( \varepsilon T x = \varepsilon \lambda'(0)p'(1)x \). Thus the parameter \( \varepsilon \) appears multiplicatively, as required by the KR theorem.

Trivially, \( \lambda'(0)p'(1) \) is the eigenvalue of the operator \( T \) and this eigenvalue has multiplicity one (the space is one-dimensional), which is odd. Since by assumption \( \lambda'(0)p'(1) > 0 \), this eigenvalue is strictly positive. Thus \( 1 / (\lambda'(0)p'(1)) \) is a bifurcation point. As there can be only one eigenvalue of \( T \), there can be at most one bifurcation point (Theorem 2). Thus the first conclusion of Theorem 1 holds true: the connected component of fixed points containing the bifurcation point \( (1 / (\lambda'(0)p'(1)), 0) \) is unbounded.

Of course, for this simple example we even have an explicit characterization of this connected set of fixed points and an application of the powerful KR theorem is not needed. But for only slightly more elaborate examples an explicit characterization is typically no longer available.

Consider now transmission over the Binary Symmetric Channel (BSC) with transition probability \( p \) and min-sum (MS) decoding. For iteration \( i \), let \( M_m^{(i)} \) be the message sent from check node \( m \) to variable node \( n \) and \( M_{n'}^{(i)} \) be the message sent from variable node \( n \) to check node \( m \). We denote the set of neighbors of a node \( m \) by \( \mathcal{N}(m) \). If we assume that we represent messages as log-likelihood ratios then the processing rules in each iteration are as follows:

1. Processing rule at check nodes—for each \( m \) and each \( n \in \mathcal{N}(m) \),
   \[
   M_m^{(i)} = \prod_{n' \in \mathcal{N}(m) \setminus m} \text{sgn}(M_{n'}^{(i)}) \min_{n' \in \mathcal{N}(m)} |M_{n'}^{(i)}|
   \]
   \[
   \text{(1)}
   \]
2. Processing rule at variable nodes—for each \( n \) and each \( m \in \mathcal{N}(n) \),
   \[
   M_n^{(i)} = L_n + \sum_{m' \in \mathcal{N}(n) \setminus m} M_{m'}^{(i-1)}
   \]
   \[
   \text{(2)}
   \]
   where \( L_n \) denotes the initial log-likelihood ratio received by node \( n \).

We claim that there exists a one-to-one mapping between the messages of the min-sum decoder and the set of integers \( \mathbb{Z} \). More precisely, the messages of the min-sum decoder are of the form \( i \ln(1/p) \), \( i \in \mathbb{Z} \). This can be easily seen by induction. The initial messages from the variable nodes to the check nodes are \( \pm \ln(1/p) \). At the check nodes if all the incoming messages are of the form \( i \ln(1/p) \), then by inspecting the check node processing rule given in Equation 1 we see that the outgoing message is again of this form. At the variable nodes, all the messages are added up which clearly preserve this property. We can therefore equivalently formulate message-passing under min-sum on the lattice \( \mathbb{Z} \) by assuming that the initial messages are from the set \( \{ \pm 1 \} \) and have probabilities \((1 - p) \) and \( p \), respectively.

In order to be able to apply the KR theorem, below we consider bounded versions of min-sum, i.e., we bound the absolute value of the messages to \( M \), where \( M \) is a fixed integer. More precisely, we assume that message alphabet is \( \mathcal{M} = \{-M, -M + 1, \ldots, -1, 0, 1, \ldots, M - 1, M\} \). As mentioned before, \( \mathcal{M}_p = \{-1, 1\} \). The message passing rule for the check node side is the same as given by Equation 1. On the other hand, to enforce the boundedness constraint, we need to slightly modify the message-passing rule for variable nodes. For a node of degree \( d \), the rule is defined by:

\[
\Psi_v(m_0, m_1, \ldots, m_{d-1}) = \begin{cases} 
\exists i \text{ s.t. } m_i = M & M \\
\exists j \text{ s.t. } m_j \neq -M & -M \\
\exists i \text{ s.t. } m_i = -M & M \\
0 & M \\
0 & -M \\
Q \left( \sum_{i=0}^{d-1} m_i \right) & \text{otherwise,} \end{cases}
\]

where the quantization function \( Q(x) = M \) if \( x \geq M \), \( Q(x) = -M \) if \( x \leq -M \) and equal to \( x \) otherwise. Note that the exact rule for the case when both \( M \) and \( -M \) are incoming to the variable node is not really important since this should hardly ever happen if \( M \) is large enough. This is because if \( M \) is large, the quantized decoder will mimic more and more the min-sum decoder.

For future reference, consider the ensemble \( \mathcal{A}(x) = 0.4x^2 + 0.6x^3, \Gamma(x) = x^4 \). It has design rate \( r = 0.05 \). The Shannon threshold for this rate is \( p_{Sh} = 0.369 \). Table 1 shows the threshold values of this ensemble for increasing values of \( M \) as well as the threshold under true min-sum decoding. We see that the thresholds for finite \( M \) quickly converge to the unbounded case. Note that this quantizer and

\[
M = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \infty
\]

\[
\approx p^* \quad 0.0319 \quad 0.0962 \quad 0.0974 \quad 0.1219 \quad 0.1318 \quad 0.148
\]

TABLE 1: Thresholds of \( \mathcal{A}(x) = 0.4x^2 + 0.6x^3, \Gamma(x) = x^4 \) under quantized min-sum decoding.

the message passing rules satisfy the symmetry conditions of [7]. Thus we can perform the density evolution under the all-one codeword assumption. Recall that the alphabet has \( 2M + 1 \) elements. But since the probability of the individual elements sums up to one, the density evolution recursion \( G \) can be written as a function of \( 2M \) variables. Thus, the underlying space is \( X = \mathbb{R}^{2M} \). As can be easily seen, the density evolution map is again a vector polynomial map. Thus such a map is completely continuous by Lemma 1 and Hypothesis 1 of Theorem 1 is satisfied. The first condition for the second hypothesis to hold true is that \( G(p, 0) = 0 \). Note that \( x = 0 \) implies that with probability one, the message is equal to \( M \). Now at the check node side if all the incoming messages are equal to \( M \), then the outgoing is also equal
to M. The same holds true for the variable node side by the definition of $\Psi_v$ given in Equation (3). Also the channel transition probability $p$ appears only as $p$ and $1 - p$. Thus the Frechet derivative of the map is of the form $pT + T'$, where both $T, T' \in \mathbb{R}^{2M \times 2M}$. In order to satisfy Hypothesis 2 of Theorem 1 we need to modify the density evolution map. We use Lemma 3 and consider the derived map with Frechet derivative $p(I_{2M} - T')^{-1}T$.

Example 2 (Min-Sum Decoder with $M = 2$): For our running example consider $M = 2$. The Frechet derivative is of the form $pT + T'$, where $T'$ is not identically zero. Fortunately $(I_4 - T')^{-1}$ exists. As mentioned before, by Lemma 3 we need to study the eigenvalues of the matrix $(I_4 - T')^{-1}T$. The matrix $(I_4 - T')^{-1}T$ has eigenvalues $\frac{1}{\mu_1} = 3.50027, \frac{1}{\mu_2} = -2.70249$ and the other two eigenvalues are zero. Both $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ have multiplicity one (i.e., the multiplicities are odd). This implies that the KR theorem is applicable to both the eigenvalues and at least one of the conclusion of the KR theorem must hold true for both of them. In particular $(\mu_1, 0)$ and $(\mu_2, 0)$ are bifurcation points. Let $C_{\mu_1}$ and $C_{\mu_2}$ be the fixed point component containing $\mu_1$ and $\mu_2$ respectively. Now by the KR theorem either the fixed point connected component $C_{\mu_1}$ and $C_{\mu_2}$ are unbounded or $C_{\mu_1} = C_{\mu_2}$.

We can compute the fixed points explicitly in this case. The result is shown in Figure 3. Since the fixed points are elements of $\mathbb{R}^4$ we need to project them into $\mathbb{R}$ in order to be able to plot them. We choose to apply the error probability operator. As the density evolution is done assuming that the all-one codeword has been transmitted, so the error probability operator sums up the component corresponding to negative indices and adds to this sum half the weight of index zero as it is like an erasure.

$$P_e(x) = \sum_{i=-M}^{-1} x_i + \frac{x_0}{2}.$$  \hspace{1cm} (4)

As we can see, the second conclusion of Theorem 1 holds i.e. $C_{\mu_1} = C_{\mu_2} = C'$. The fixed point connected component $C'$ containing the point $a = (\mu_1, 0) = (0.28569, 0)$ also contains the point $d = (\mu_2, 0) = (-0.37003, 0)$. In the component $C'$, the branch from $a$ to $b$ is stable, $b$ to $c$ is unstable and $c$ to $d$ is stable. The component $C'$ is stable. The threshold is $p^* = 0.0962$. The fixed point of iterative decoder at the threshold is represented by point $c$ of the fixed point component $C'$. Above the threshold, the fixed points of iterative decoder moves upward along $C'$ as the channel transition probability $p$ increases.

Example 3 (Min-sum decoder with $M = 3$): For our running example we consider $M = 3$. The Frechet derivative is again of the form $pT + T'$. In this case also the inverse $(I_6 - T')^{-1}$ exists. By Lemma 3 we need to study the eigenvalues of $(I_6 - T')^{-1}T$. The matrix $(I_6 - T')^{-1}T$ has the only non-zero real eigenvalue as $\frac{1}{\mu} = 2.09804$ and its multiplicity is one. So the KR theorem is applicable in this case. Note that as there is only one non-zero eigenvalue, there can be at most one bifurcation point by Theorem 2. Thus the second conclusion of KR theorem can not be true. This implies that the first conclusion holds: there is an unbounded component $C_{\mu}$ of fixed point containing the bifurcation point $\mu$. In this case we can compute this component explicitly. As the fixed points are element of $\mathbb{R}^6$, in order to plot them we project them to one dimension by the error probability operator given in Equation 4. The plot is shown in Figure 4. The bifurcation point is $a = (\mu_0, 0) = (0.476636, 0, 0)$. As far as the stability of the fixed point in $C_{\mu}$ is concerned, the branch $a$ to $b$ is stable. The fixed points in branch $b$ to $c$ is unstable and from point $c$ onwards the fixed points are stable. The point $e$ represents the fixed point at which the iterative decoder get stuck at threshold $p^* \approx 0.0974$. Above the threshold, the fixed points of iterative decoder moves upward along $C_{\mu}$ as the channel transition probability $p$ increases.

Discussion: We presented the examples $M = 2$ and $M = 3$. It is tempting to increase $M$ and see how the fixed point structure changes. By taking $M$ to infinity, one would hope to recover the structure of the fixed point components of the un-quantized min-sum decoder.

Example 4 (Decoder with Erasure): The decoder with erasure was introduced in [7]. The underlying channel is $BSC(p)$. On the variable node side the message-passing rule for a node of degree 1 reads

$$\Psi_v(m_0, m_1, \ldots, m_{l-1}) = \text{sgn} \left( m_0 + \sum_{i=1}^{l-1} m_i \right).$$

The rule for a check node of degree $r$ is

$$\Psi_v(m_1, \ldots, m_{l-1}) = \prod_{i=1}^{r-1} m_i.$$  \hspace{1cm} (5)

Note that for this decoder if there are degree two variable nodes then the threshold is 0 i.e. 0 can not be a fixed point. To see this, suppose that all the incoming messages to variables nodes are equal to one. Then with probability $p$, the
outgoing message from a variable node is equal to 0. Thus the probability of 0 is equal to $\lambda_2 p$. Hence we assume that $\lambda_2 = 0$. For this example $M = 1$, hence the underlying space is $X = \mathbb{R}^2$. The density evolution equation can be found in [7]. The Frechet derivative of the density evolution map can again be computed and it turns out that its only eigenvalue is $2\lambda_3 \rho'(1)$. But now this eigenvalue has even multiplicity. So we cannot apply the KR theorem to this case. In [6], it was investigated if the conclusions of the KR theorem is still applicable to an eigenvalue of even multiplicity. We are currently investigating whether the result of [6] is applicable to the decoder with erasure. However numerical computation of fixed point structure reveals that $2\lambda_3 \rho'(1)$ is 10, so supposedly $p = 0.1$ is a bifurcation point. We can see from Figure 5 that point a which corresponds to $p = 0.1$ is indeed a bifurcation point. The threshold for this ensemble is $p^* = 0.0708$. The point b represents the fixed point at which decoder gets stuck at the threshold. The branch a to b is unstable. From b onwards the fixed points are stable.

**Fig. 5: Fixed point component for the decoder with erasure for (3,6) LDPC ensemble.**

IV. OUTLOOK

We have shown how the tools of non-linear analysis can be used in proving the existence of fixed points. Our ultimate goal is to understand the fixed point structure of the BP and the min-sum decoder. For the min-sum decoder we hope to accomplish our goal by considering a sequence of quantized decoders where the number of quantization points tends to infinity. Whether a similar strategy can be devised for the BP decoder is still an open question.

APPENDIX

**Lemma 1:** Every vector polynomial map $G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a completely continuous map.

**Proof:** Consider any bounded set $S$ in $\mathbb{R}^N$. As $S$ is bounded, so will be all the components $G_i(S)$. Hence the set $G(S)$ is also bounded. Clearly this would imply that the closure $\overline{G(S)}$ is also bounded. In a finite dimensional vector space a closed and bounded set is a compact. Hence $G(S)$ is relatively compact. Thus the map $G$ is Completely continuous.

**Lemma 2:** Let $G : \mathbb{R}^N \to \mathbb{R}^N$ be a vector polynomial map such that $G(\mathbf{0}) = \mathbf{0}$. Then $G$ is Frechet differentiable. The Frechet derivative $T$ of $G$ is a matrix whose entries are given by $t_{ij}$ where $1 \leq i, j \leq N$ and

\[ t_{ij} = \frac{\partial G_i}{\partial x_j}(\mathbf{0}). \]

**Proof.** Consider $||G(x) - Tx||$. As $|x| \leq ||x||$, there are no linear term in $G(x) - Tx$ and $G(\mathbf{0}) = \mathbf{0}$ implies that $||G(x) - Tx|| = o(||x||^2)$. Hence

\[ \frac{||G(x) - Tx||}{||x||} = o(||x||). \]

This proves the lemma. \[ \square \]

Note that Hypothesis 2 of Theorem 1 implies that the parameter $\gamma$ must appear multiplicatively in the Frechet derivative. But in many cases we see that the Frechet derivative is of the form $\gamma T + T'$. The following lemma says that in this case also the KR theorem can be applied provided the linear operator $I - T'$ is invertible.

**Lemma 3:** Let $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector polynomial map and Frechet differentiable with Frechet derivative $\gamma T + T'$. Let us assume that $(I_n - T')^{-1}$ exists. Let $F(\gamma, x) = (I_n - T')^{-1}(G(\gamma, x) - T'x)$. Then $F$ is a vector polynomial map and Frechet differentiable with Frechet derivative $\gamma(I_n - T')^{-1}T$. Also the set of fixed points of $F$ is same as set of fixed points of $G$.

**Proof.** The fact that $F$ is a vector polynomial map is obvious. For the Frechet differentiability of $F$ we need that $F(\gamma, \mathbf{0}) = \mathbf{0}$. Now, $F(\gamma, \mathbf{0}) = (I_n - T')^{-1}(G(\gamma, \mathbf{0}) - T'\mathbf{0}) = \mathbf{0}$, as $G(\gamma, \mathbf{0}) = \mathbf{0}$. Now the Frechet derivative of $(G(\gamma, x) - T'x)$ is given by $\gamma T'x$. This implies that the Frechet derivative of $F(\gamma, x)$ is equal to $\gamma(I_n - T')^{-1}T$. To see that $F$ and $G$ have the same set of fixed points, let $x$ be a fixed point of $G$. Then $G(\gamma, x) - T'x = x - T'x$ which implies $(I_n - T')^{-1}(G(\gamma, x) - T'x) = x$ i.e. $F(\gamma, x) = x$. \[ \square \]

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