Open induction in a bounded arithmetic for $\text{TC}^0$

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Abstract

The elementary arithmetic operations $+, \cdot, \leq$ on integers are well-known to be computable in the weak complexity class $\text{TC}^0$, and it is a basic question what properties of these operations can be proved using only $\text{TC}^0$-computable objects, i.e., in a theory of bounded arithmetic corresponding to $\text{TC}^0$. We will show that the theory $\text{VTC}^0$ extended with an axiom postulating the totality of iterated multiplication (which is computable in $\text{TC}^0$) proves induction for quantifier-free formulas in the language $\langle +, \cdot, \leq \rangle$ ($\text{IOpen}$), and more generally, minimization for $\Sigma^b_0$ formulas in the language of Buss’s $S_2$.

1 Introduction

Proof complexity is sometimes presented as the investigation of a three-way correspondence between propositional proof systems, theories of bounded arithmetic, and computational complexity classes. In particular, we can associate to a complexity class $C$ satisfying suitable regularity conditions a theory $T$ such that on the one hand, the provably total computable functions of $T$ of certain logical form define exactly the $C$-functions in the standard model of arithmetic, and on the other hand, $T$ proves fundamental deductive principles such as induction and comprehension for formulas that correspond to $C$-predicates. In this sense $T$ provides a formalization of $C$-feasible reasoning: we can interpret provability in $T$ as capturing the idea of what can be demonstrated when our reasoning capabilities are restricted to manipulation of objects and concepts of complexity $C$. The complexity class corresponding to a “minimal” theory that proves a given logical or combinatorial statement can be seen as a gauge of its proof complexity. Then a particularly natural question is, given a function or predicate $X$, which properties of $X$ can be proved by reasoning whose complexity does not exceed that of $X$, that is, in a theory corresponding to the complexity class for which $X$ is complete.

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The main theme of this paper is what we can feasibly prove about the basic integer arithmetic operations \(+, \cdot, \leq\). The matching complexity class is \(\text{TC}^0\): \(+\) and \(\leq\) are computable in \(\text{AC}^0 \subseteq \text{TC}^0\), while \(\cdot\) is in \(\text{TC}^0\), and it is in fact \(\text{TC}^0\)-complete under \(\text{AC}^0\) (Turing) reductions. (In this paper, all circuit classes like \(\text{TC}^0\) are assumed DLOGTIME-uniform unless stated otherwise.) \(\text{TC}^0\) also includes many other functions related to arithmetic. First, \(+\) and \(\cdot\) are also \(\text{TC}^0\)-computable on rationals or Gaussian rationals. An important result of Hesse, Allender, and Barrington [11] based on earlier work by Beame et al. [3] and Chiu et al. [7] states that integer division and iterated multiplication are \(\text{TC}^0\)-computable. As a consequence, one can compute in \(\text{TC}^0\) approximations of functions presented by sufficiently nice power series, such as \(\log\), \(\sin\), or \(x^{1/k}\), see e.g. Reif [26], Reif and Tate [27], Maciel and Thérien [22], and Hesse et al. [11].

The more-or-less canonical arithmetical theory corresponding to \(\text{TC}^0\) is \(\text{VTC}^0\) (see Cook and Nguyen [8]). This is a two-sorted theory in the setup of Zambella [33], extending the base \(\text{AC}^0\)-theory \(\text{V}^0\) by an axiom stating the existence of suitable counting functions, which gives it the power of \(\text{TC}^0\). \(\text{VTC}^0\) is equivalent (\(\text{RSUV}\)-isomorphic) to the one-sorted theory \(\Sigma^b_1\text{-CR}\) by Johannsen and Pollett [19], which is in turn \(\forall \exists \Sigma^b_1\)-conservative under the theory \(\text{C}^0\) [18].

\(\text{VTC}^0\) can define addition and multiplication on binary integers, and it proves basic identities governing these operations, specifically the axioms of discretely ordered rings (\(\text{DOR}\)). We are interested in what other properties of integers expressible in the language \(\text{L}_{\text{OR}} = \langle 0, 1, +, -, \cdot, \leq \rangle\) of ordered rings are provable in \(\text{VTC}^0\), and in particular, whether the theory can prove induction for a nontrivial class of formulas. Note that we should not expect the theory to prove induction for bounded existential formulas, or even its weak algebraic consequences such as the Bézout property: this would imply that integer \(\gcd\) is computable in \(\text{TC}^0\), while it is not even known to be in \(\text{NC}\). However, this leaves the possibility that \(\text{VTC}^0\) could prove induction for \(\text{open}\) (quantifier-free) formulas of \(\text{L}_{\text{OR}}\), i.e., that it includes the theory \(\text{IOpen}\) introduced by Shepherdson [29].

Using an algebraic characterization of open induction and a witnessing theorem for \(\text{VTC}^0\), the provability of \(\text{IOpen}\) in this theory is equivalent to the existence of \(\text{TC}^0\) algorithms for approximation of real or complex roots of constant-degree univariate polynomials whose soundness can be proved in \(\text{VTC}^0\). The existence of such algorithms in the “real world” is established in [15], but the argument extensively relies on tools from complex analysis (Cauchy integral formula, …) that are not available in bounded arithmetic, hence it is unsuitable for formalization in \(\text{VTC}^0\) or a similar theory.

The purpose of this paper is to demonstrate that \(\text{IOpen}\) is in fact provable in a mild extension of \(\text{VTC}^0\). The argument naturally splits into two parts. We first formalize by a direct inductive proof a suitable version of the Lagrange inversion formula (LIF), which was also the core ingredient in the algorithm in [15]. This allows us to compute approximations of a root of a polynomial \(f\) by means of partial sums of a power series expressing the inverse function of \(f\), but only for polynomials obeying certain restrictions on coefficients. The second part of the argument is model-theoretic, using basic results from the theory of valued fields. The question whether a given \(\text{DOR}\) is a model of \(\text{IOpen}\) can be reduced to the question whether the completion of its fraction field under a valuation induced by its ordering is real-
closed, and there is a simple criterion for recognizing real-closed valued fields. In our situation, LIF ensures the relevant field is henselian, which implies that the criterion is satisfied.

We do not work with $VTC^0$ itself, but with its extension $VTC^0 + IMUL$ including an axiom ensuring the totality of iterated multiplication. This theory corresponds to $TC^0$ just like $VTC^0$ does, as iterated multiplication is $TC^0$-computable. We need the extra axiom because it is not known whether $VTC^0$ can formalize the $TC^0$ algorithms for division and iterated multiplication of Hesse et al. [11], and this subtle problem is rather tangential to the question of open induction and root approximation. As explained in more detail in Section 3, the $IMUL$ axiom is closely related to the integer division axiom $DIV$ which is implied by $IOpen$, hence its use is unavoidable in one way or another. In terms of the original theory $VTC^0$, our results show that $VTC^0 \vdash IOpen$ if and only if $VTC^0 \vdash DIV$.

We can strengthen the main result if we switch from $L_{OR}$ to the language of Buss’s one-sorted theories of bounded arithmetic. By formalizing the description of bounded Σb0-definable sets due to Mantzivis [23], $VTC^0 + IMUL$ can prove the $RSUV$-translation of Buss’s theory $T_2^0$, and in fact, of the Σb0-minimization schema. In other words, $T_2^0$ and $Σ^b_0-MIN$ are included in the theory $Δ^b_1-CR + IMUL$.

## 2 Preliminaries

A structure $⟨D, 0, 1, +, −, ⋅, ≤⟩$ is an ordered ring if $⟨D, 0, 1, +, −, ⋅⟩$ is a commutative (associative unital) ring, ≤ is a linear order on $D$, and $x ≤ y$ implies $x + z ≤ y + z$ and $xz ≤ yz$ for all $x, y, z ∈ D$ such that $z ≥ 0$. If $D$ is an ordered ring, $D^+$ denotes $\{a ∈ D : a > 0\}$. A discretely ordered ring (DOR) is an ordered ring $D$ such that 1 is the least element of $D^+$. Every DOR is an integral domain. An ordered field is an ordered ring which is a field. A real-closed field (RCF) is an ordered field $R$ satisfying any of the following equivalent conditions:

- Every $a ∈ R^+$ has a square root in $R$, and every $f ∈ R[x]$ of odd degree has a root in $R$.
- $R$ has no proper algebraic ordered field extension.
- The field $R(\sqrt{-1})$ is algebraically closed.
- $R$ is elementarily equivalent to $\mathbb{R}$.

(In a RCF, ≤ is definable in terms of the ring structure, thus we can also call a field $⟨R, +, ⋅⟩$ real-closed if it is the reduct of a RCF.) The real closure of an ordered field $F$ is a RCF $\tilde{F}_{\text{real}} ⊇ F$ which is an algebraic extension of $F$. Every ordered field has a unique real closure up to a unique $F$-isomorphism.

The theory $IOpen$ consists of the axioms of ordered rings and the induction schema

$$\varphi(0) \land \forall x (\varphi(x) → \varphi(x + 1)) \rightarrow \forall x ≥ 0 \varphi(x)$$

for open formulas $\varphi$ (possibly with parameters). An integer part of an ordered field $F$ is a discretely ordered subring $D ⊆ F$ such that every element of $F$ is within distance 1 from an element of $D$. The following well-known characterization is due to Shepherdson [29].
Theorem 2.1 Models of $I\text{Open}$ are exactly the integer parts of real-closed fields.

The criterion is often stated with the real closure of the fraction field of the model instead of a general real-closed field, but these two formulations are clearly equivalent, as an integer part $D$ of a field $R$ is also an integer part of any subfield $D \subseteq R' \subseteq R$.

In particular, models of $I\text{Open}$ are integer parts of their fraction fields. This amounts to provability of the division axiom

$$\forall x > 0 \forall y \exists q, r \ (y = qx + r \land 0 \leq r < x)$$

in $I\text{Open}$. (The uniqueness of $q$ and $r$ holds in any DOR.)

We define $AC^0$ as the class of languages recognizable by a DLOGTIME-uniform family of polynomial-size constant-depth circuits using $\neg$ and unbounded fan-in $\land$ and $\lor$ gates, or equivalently, languages computable by an $O(\log n)$-time alternating Turing machine with $O(1)$ alternations, or by a constant-time CRAM with polynomially many processors [12]. If we represent an $n$-bit binary string $w$ by the finite structure $\langle \{0, \ldots, n-1\}, <, +, \cdot, P_w \rangle$, where $P_w(i)$ iff the $i$th bit of $w$ is 1, then $AC^0$ coincides with FO (languages definable by first-order sentences). A language $B$ is $AC^0$-reducible to a language $A$ if $B$ is computable by a DLOGTIME-uniform family of polynomial-size constant-depth circuits using unbounded fan-in $\land$, $\lor$, $\neg$, and $A$-gates. The class of languages $AC^0$-reducible to $A$ is its $AC^0$-closure.

$TC^0$, originally introduced as a nonuniform class by Hajnal et al. [10], is defined for our purposes as the $AC^0$-closure of $\text{Majority}$. (Several problems $TC^0$-complete under $AC^0$ reductions are noted in Chandra et al. [6], any of these could be used in place of $\text{Majority}$.) Equivalently, $TC^0$ coincides with languages computable by $O(\log n)$-time threshold Turing machines with $O(1)$ thresholds, or by constant-time TRAM with polynomially many processors [25]. In terms of descriptive complexity, a language is in $TC^0$ iff the corresponding class of finite structures is definable in FOM, i.e., first-order logic with majority quantifiers [1].

In connection with bounded arithmetic, it is convenient to consider not just the complexity of languages, but of predicates $P(x_1, \ldots, x_n, X_1, \ldots, X_m)$ with several inputs, where $X_i$ are binary strings as usual, and $x_i$ are natural numbers written in unary. It is straightforward to generalize $AC^0$, $TC^0$, and similar classes to this context, see [8, §IV.3] for details. Likewise, we can consider computability of functions: if $C$ is a complexity class, a unary number function $f(\vec{x}, \vec{X})$ is in $FC$ if it is bounded by a polynomial in $\vec{x}$ and the lengths of $\vec{X}$, and its graph $f(\vec{x}, \vec{X}) = y$ is in $C$; a string function $F(\vec{x}, \vec{X})$ is in $FC$ if the length of the output is polynomially bounded as above, and the bitgraph $G_F(\vec{x}, \vec{X}, y) \Leftrightarrow (F(\vec{x}, \vec{X}))_y = 1$ is in $C$. For simplicity, functions from $FC$ will also be called just $C$-functions.

We will work with two-sorted (second-order) theories of bounded arithmetic in the form introduced by Zambella [33] as a simplification of Buss [5]. We refer the reader to Cook and Nguyen [8] for a general background on these theories as well as a detailed treatment of $VTC^0$, however, we include the main definitions here in order to fix our notation.

The language $L_2 = \langle 0, S, +, \cdot, \leq, \in, \| - \| \rangle$ of second-order bounded arithmetic is a first-order language with equality with two sorts of variables, one for unary natural numbers, and one for finite sets thereof, which can also be interpreted as binary strings, or binary integers. The standard convention is that variables of the first sort are written with lowercase
letters $x, y, z, \ldots$, and variables of the second sort with uppercase letters $X, Y, Z, \ldots$. While we adhere to this convention in the introductory material on the theories and their basic properties, we will not follow it in the less formal main part of the paper (we will mostly work with binary integers or rationals, and it looks awkward to write them all in uppercase). The symbols $0, S, +, \cdot, \leq$ of $L_2$ denote the usual arithmetic operations and relation on the unary sort; $x \in X$ is the elementhood predicate, and the intended meaning of the $\|X\|$ function is the least unary number strictly greater than all elements of $X$. This function is usually denoted as $|X|$, however (apart from the section on Buss’s theories) we reserve the latter symbol for the absolute value on binary integers and rationals, which we will use more often.

We write $x < y$ as an abbreviation for $x \leq y \land x \neq y$.

Bounded quantifiers are introduced by

\[
\exists x \leq t \varphi \Leftrightarrow \exists x (x \leq t \land \varphi), \\
\exists X \leq t \varphi \Leftrightarrow \exists X (\|X\| \leq t \land \varphi),
\]

where $t$ is a term of unary sort not containing $x$ or $X$ (resp.). Universal bounded quantifiers, as well as variants of bounded quantifiers with strict inequalities, are defined in a similar way. A formula is $\Sigma^B_0$ if it contains no second-order quantifiers, and all its first-order quantifiers are bounded. The $\Sigma^B_0$-definable predicates in the standard model of arithmetic are exactly the $\mathrm{AC}^0$ predicates. A formula is $\Sigma^B_i$ if it consists of $i$ alternating (possibly empty) blocks of bounded quantifiers, the first of which is existential, followed by a $\Sigma^B_0$ formula. We define $\Pi^B_i$ formulas dually. Similarly, a formula is $\Sigma^1_i$ ($\Pi^1_i$) if it consists of $i$ alternating blocks of (possibly unbounded) quantifiers, the first of which is existential (universal, resp.), followed by a $\Sigma^B_0$ formula\(^1\).

The theory $V^0$ in $L_2$ can be axiomatized by the basic axioms

\[
\begin{align*}
x + 0 &= x \\
x \cdot 0 &= 0 \\
x + Sy &= S(x + y) \\
x \cdot Sy &= x \cdot y + x \\
Sy &\leq x \rightarrow y < x \\
x \in X \rightarrow x < \|X\| \\
x \in X \rightarrow x \in Y \rightarrow X = Y
\end{align*}
\]

and the comprehension schema

\[
(\varphi\text{-COMP}) \quad \exists X \leq x \forall u < x (u \in X \leftrightarrow \varphi(u))
\]

for $\Sigma^B_0$ formulas $\varphi$, possibly with parameters not shown (but with no occurrence of $X$). We denote the set $X$ whose existence is postulated by $\varphi\text{-COMP}$ as $\{u < x : \varphi(u)\}$. Using $\text{COMP}$, $V^0$ proves the induction and minimization schemata

\[
\begin{align*}
(\varphi\text{-IND}) & \quad \varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow \forall x \varphi(x), \\
(\varphi\text{-MIN}) & \quad \varphi(x) \rightarrow \exists y (\varphi(y) \land \forall z < y \neg \varphi(z))
\end{align*}
\]

\(^1\)Notice that bounded second-order quantifiers still count towards $i$, so these formula classes do not correspond in the one-sorted setting to the usual arithmetical hierarchy $\Sigma^0_i$, but to its restricted version where the formula after the main quantifier prefix is sharply bounded. We follow \[8\] in this usage; they only appear to define $\Sigma^1_1$, but we find it convenient to extend this notation to higher levels as well.
for $\Sigma^B_0$ formulas $\varphi$. In particular, $V^0$ includes $I\Delta_0$ on the unary number sort.

Let $\langle x, y \rangle$ be a $V^0$-definable pairing function on unary numbers, e.g., $\langle x, y \rangle = (x + y)(x + y + 1)/2 + y$. We define $X^{[u]} = \{x : \langle u, x \rangle \in X\}$; this provides an encoding of sequences of sets by sets. We can encode sequences of unary numbers by putting $X^{(u)} = ||X^{[u]}||$ (this is easily seen to be a $\Sigma^B_0$-definable function). For convenience, we also extend the pairing function to (standard-length) $k$-tuples by $\langle x_1, \ldots, x_{k+1} \rangle = \langle \langle x_1, \ldots, x_k \rangle, x_{k+1} \rangle$, and we write $X^{[u_1, \ldots, u_k]} = X^{\langle \langle u_1, \ldots, u_k \rangle \rangle}$, $X^{(u_1, \ldots, u_k)} = X^{\langle \langle u_1, \ldots, u_k \rangle \rangle}$.

$VTC^0$ is the extension of $V^0$ by the axiom

$$\forall n, X \exists Y (Y^{(0)} = 0 \land \forall i < n ((i \notin X \to Y^{(i+1)} = Y^{(i)}) \land (i \in X \to Y^{(i+1)} = Y^{(i)} + 1)))$$

whose meaning is that for every set $X$ there is a sequence $Y$ supplying the counting function $Y^{(i)} = \text{card}(X \cap \{0, \ldots, i - 1\})$.

Let $T$ be a class of formulas, and $\Gamma$ an extension of $V^0$. A string function $F(\vec{x}, \vec{X})$ is a provably total $\Gamma$-definable function of $T$ if its graph is definable in $\mathbb{N}$ by a formula $\varphi(\vec{x}, \vec{X}, Y) \in \Gamma$ such that $\Gamma \vdash \forall \vec{x}, \vec{X} \exists Y \varphi(\vec{x}, \vec{X}, Y)$; similarly for number functions. If $\Gamma = \Sigma^1_1$, such functions are also called provably total recursive functions of $T$. Note that one function may have many different definitions that are not $T$-provably equivalent; some of them may be provably total, while other are not.

The provably total recursive functions of $V^0$ and $VTC^0$ are FAC$^0$ and FTC$^0$, respectively. Moreover, we can use these functions freely in the sense that if we expand the languages of the theories with the corresponding function symbols, the resulting conservative extensions of $V^0$ and $VTC^0$ (respectively) prove the comprehension and induction schemata for $\Sigma^B_0$ formulas of the expanded language; we will see more details in the next section.

Being AC$^0$, the ordering on binary integers is definable by a $\Sigma^B_0$ formula, and addition is provably total in $V^0$. Likewise, multiplication and iterated addition are provably total $\Sigma^B_1$-definable functions of $VTC^0$. In fact, as shown in [8], the natural $\Sigma^B_0$ definitions of $X < Y$ and $X + Y$ provably satisfy basic properties like commutativity and associativity in $V^0$, and similarly, there are natural definitions of $X \cdot Y$ and $\sum_{i<n} X^{[i]}$ provably total in $VTC^0$ such that $VTC^0$ proves their basic properties, including the inductive clauses:

$$\sum_{i<0} X^{[i]} = 0,$$
$$\sum_{i<n+1} X^{[i]} = \sum_{i<n} X^{[i]} + X^{[n]}.$$

While Cook and Nguyen [8] normally use second-sort objects to denote nonnegative integers, it will be more convenient for us to make them represent all integers, which is easily accomplished by using one bit for sign. The definitions of $<, +, \cdot$, and $\sum_{i<n} X^{[i]}$ can be adapted in a straightforward way to this setting so that $VTC^0$ still proves their relevant properties, that is, the axioms of discretely ordered rings.
3 Iterated multiplication and division

As we already mentioned, it is not known whether \( VTC^0 \) can formalize the \( TC^0 \) algorithms of Hesse, Allender, and Barrington [11] for integer division and iterated multiplication. In particular, it is not known whether \( VTC^0 \) proves the sentence \( DIV \) (formulated for binary integers), which is a consequence of \( IOpen \). This problem is rather tangential to the formalization of root finding, whence we bypass it by strengthening our theory appropriately.

It might seem natural just to work in the theory \( VTC^0 + DIV \), however we will instead consider an axiom stating the totality of iterated multiplication in the following form:

\[
(IMUL) \quad \forall X, n \exists Y \forall i \leq j < n \left( Y^{[i,j]} = 1 \land Y^{[i,j+1]} = Y^{[i,j]} \cdot X^{[j]} \right).
\]

(The meaning is that for any sequence \( X \) of \( n \) binary integers, there is a triangular matrix \( Y \) with entries \( Y^{[i,j]} = \prod_{k=i}^{j-1} X^{[k]} \).) One reason is simply that we need to use iterated multiplication at various places in the argument (in particular, to compute partial sums of power series), and we do not know whether \( VTC^0 + DIV \vdash IMUL \). The more subtle reason is that we need the theory to be well-behaved in a certain technical sense that we will describe in more detail below, and it turns out that \( VTC^0 + IMUL \) is the smallest well-behaved extension of \( VTC^0 + DIV \).

Consider an extension \( T \supseteq V^0 \) proving that a particular polynomially bounded recursive (i.e., \( \Sigma^1_1 \)-definable) function \( F \) is total, e.g. \( DIV \) or \( IMUL \). While the most simplistic arguments employing \( F \) can get away with the mere fact that the value computed by \( F \) exists for a particular input, usually we need more than that. For example, we may want to use induction on a formula \( \varphi(x) \) which involves \( F \) applied to an argument depending on \( x \); since induction is obtained over \( V^0 \) by considering the least element of the set \( \{ x < a : \neg \varphi(x) \} \), we effectively need comprehension for (simple enough) formulas containing \( F \), say, \( \Sigma^B_0(F)\)-\( COMP \).

From a computational viewpoint, it is desirable that we can combine provably total recursive functions in various ways. For example, one of the basic \( TC^0 \) functions is iterated addition, and a natural way how we would like to apply it is to compute \( \sum_{x<a} F(x) \) for a given provably total function \( F \). More generally, we want the class of provably total recursive functions to be closed under \( AC^0 \) (or even \( TC^0 \) in our case) reductions, and as a simple special case, under parallel repetition: if we can compute a function \( F(X) \), we want to be able to compute its aggregate function \( F^* : \langle X_0, \ldots, X_{n-1} \rangle \mapsto \langle F(X_0), \ldots, F(X_{n-1}) \rangle \) (where \( n \) is a part of the input). In more logical terms, it is desirable that \( T \) is closed under the choice rule \( \Sigma^B_0 - AC^R \): if \( T \vdash \forall X \exists Y \varphi(X,Y) \), where \( \varphi \in \Sigma^B_0 \), then also \( T \vdash \forall n \forall W \exists Z \forall i < n \varphi(W^{[i]}, Z^{[i]}) \).

This is a derived rule corresponding to the axiom of choice, also called replacement or bounded collection:

\[
(\Sigma^B_0 - AC) \quad \forall i < n \exists Y \leq m \varphi(i,Y,P) \rightarrow \exists Z \forall i < n \varphi(i,Z^{[i]},P).
\]

Unfortunately, none of the desiderata mentioned in the last two paragraphs hold automatically, even for theories of the simple form \( V^0 + \forall X \exists Y F(X) = Y \) (note that \( VTC^0 + DIV \) is of such form): this axiom implies the totality of functions making a constant number of calls to \( F \), but we cannot a priori construct functions involving an unbounded number of
applications of \( F \), such as the aggregate function \( F^* \). However, Cook and Nguyen [8] show that the simple expedient of using \( F^* \) in the axiomatization instead of \( F \) leads to theories satisfying all the properties above.

\[ \text{Theorem 3.2} \]

Let \( F \) be a partial function, then \( V \) for some term \( t(X) \). The Cook–Nguyen (CN) theory\(^2\) associated with \( \delta \) is

\[
V(\delta) = V^0 + \forall W, n \exists Z \forall i < n \delta(W[i], Z[i]).
\]

(That is, if \( F \) is a polynomially bounded function with an AC\(^0\) graph defined by \( \delta \), which \( V^0 \) proves to be a partial function, then \( V(\delta) \) is axiomatized by the statement that the aggregate function \( F^* \) is total.)

For example, \( VTC^0 \) can be formulated as a CN theory, as shown in [8, §IX.3].

\[ \text{Theorem 3.2} \]

Let \( V(\delta) \) be a CN theory, and \( F \) the function whose graph is defined by \( \delta \).

(i) The provably total \( \Sigma_1^1 \)-definable (or \( \Sigma_1^B \)-definable) functions of \( V(\delta) \) are exactly the functions in the \( AC^0 \)-closure of \( F \).

(ii) \( V(\delta) \) has a universal definitional (and therefore conservative) extension \( \overline{V(\delta)} \) in a language \( L_{\overline{V(\delta)}} \) consisting of \( \Sigma_1^B \)-definable functions of \( V(\delta) \). The theory \( \overline{V(\delta)} \) has quantifier elimination for \( \Sigma_0^B(L_{\overline{V(\delta)}}) \)-formulas, and it proves \( \Sigma_0^B(L_{\overline{V(\delta)}})^{-\text{COMP}}, \Sigma_0^B(L_{\overline{V(\delta)}})^{-\text{IND}}, \) and \( \Sigma_0^B(L_{\overline{V(\delta)}})^{-\text{MIN}} \).

(iii) \( V(\delta) \) is closed under \( \Sigma_0^B \)-AC\(^R \), and \( V(\delta) + \Sigma_0^B \)-AC is \( \Pi_2 \)-conservative over \( V(\delta) \).

\textbf{Proof:} (i) and (ii) are Theorems IX.2.3, IX.2.14, and IX.2.16 in Cook and Nguyen [8].

(iii): If \( V(\delta) \models \forall X \exists Y \varphi(X, Y) \) with \( \varphi \in \Sigma_0^B \), there is an \( L_{\overline{V(\delta)}} \)-term \( G(X) \) such that \( \overline{V(\delta)} \models \varphi(X, G(X)) \) by Herbrand’s theorem, as \( \overline{V(\delta)} \) is a universal theory, and \( \varphi \) is equivalent to an open formula. Then \( \overline{V(\delta)} \), hence \( V(\delta) \), proves

\[
\forall W, n \exists Z \exists \{i, y : i < n, y \in G(W[i])\}
\]

using \( \Sigma_0^B(L_{\overline{V(\delta)}})^{-\text{COMP}} \).

The \( \Pi_2 \)-conservativity of \( \Sigma_0^B \)-AC over \( V(\delta) \) follows from the closure under \( \Sigma_0^B \)-AC\(^R \) by cut elimination. Alternatively, see [14, Thm. 4.19] for a model-theoretic proof generalizing the result of Zambella [33] for \( V^0 \). \( \square \)

\(^2\)In [8], \( V(\delta) \) is denoted \( VC \), where the complexity class \( C \) is the \( AC^0 \)-closure of \( F \), and it is called the minimal theory associated with \( C \). We refrain from this terminology as the theory is not uniquely determined by the complexity class: it depends on the choice of the \( C \)-complete function \( F \), and of a particular \( \Sigma_0^B \)-formula defining the graph of \( F \) in \( \mathbb{N} \). In particular, both \( VTC^0 \) and \( VTC^0 + \text{IMUL} \) are “minimal” theories for the same class (TC\(^0\)), and it would be rather confusing to call them as such.
Lemma 3.3

(i) \( VTC^0 + IMUL \) is a CN theory.

(ii) \( VTC^0 + IMUL \vdash DIV \).

Proof: (i): The main observation is that \( VTC^0 + IMUL \) proves the totality of the aggregate function of iterated multiplication, that is,

\[ IMUL^* \quad \forall W, m, n \exists Z \forall k < m \forall i < j < n \left( Z^{[k,i,j]} = 1 \land Z^{[k,i,j+1]} = Z^{[k,i,j]} \cdot W^{[k,j]} \right). \]

Thus, \( VTC^0 + IMUL = VTC^0 + IMUL^* \). The latter looks almost like a CN theory, except that the graph of the function specified in the axiom is not \( \Sigma^B_0 \), as it involves multiplication. (The official definition also does not allow an extra unary input, but this is benign as we could easily code \( X, n \) into a single set.) There are several ways how to get around this problem. For one, the whole machinery from [8, §IX.2] works fine if we take \( VTC^0 \) instead of \( V^0 \) as a base theory, and allow the use of \( \Sigma^B_0(L_{VTC^0}) \) formulas. Alternatively, we can rewrite \( IMUL \) to incorporate the definition of multiplication, say

\[ \forall X, n \exists Y, Z \forall i < j < n \forall x < \|X\| \left( (Y^{[i,]}) = 1 \land Z^{[i,j,0]} = 0 \land Z^{[i,j,\|X\|]} = Y^{[i,j+1]} \right) \]

\( IMUL' \)

\[ \land (x \notin X^{[j]} \rightarrow Z^{[i,j,x+1]} = Z^{[i,j,x]}) \]

\[ \land (x \in X^{[j]} \rightarrow Z^{[i,j,x+1]} = Z^{[i,j,x]} + 2^x Y^{[i,j]}) \), \]

where + and multiplication by \( 2^x \) can be given easy \( \Sigma^B_0 \) definitions. Since the entries of \( Z \) can be expressed as products of suitable \( \Sigma^B_0 \)-definable sequences of integers, one can show in the same way as above that \( IMUL' \), as well as the axiom \( IMUL^* \) stating the totality of the corresponding aggregate function, is provable in \( VTC^0 + IMUL \). Conversely, the CN theory \( V^0 + IMUL^* \) proves \( VTC^0 \) (as it implies the totality of usual multiplication), hence it is equivalent to \( VTC^0 + IMUL \).

(ii) can be shown by formalizing the reduction from [3]. Assume that we want to find \([Y/X]\), where \( X \geq 1 \). Choose \( n, m > 0 \) such that \( 2^{n-1} \leq X \leq 2^n \) and \( Y \leq 2^m \), and put

\[ Z = \sum_{i < m} (2^n - X)^i 2^{n(m-1-i)}. \]

An easy manipulation of the sum shows that \( XZ = 2^{nm} - (2^n - X)^m \), hence

\[ 2^{nm} - 2^{(n-1)m} \leq XZ \leq 2^{nm}. \]

Put \( Q = \lfloor YZ/2^{nm} \rfloor \). Then

\[ 2^{nm}Y \geq XYZ \geq 2^{nm}QX > XYZ - 2^{nm}X \geq 2^{nm}(Y - X - 1), \]

hence \( QX \leq Y \leq (Q + 1)X \). \( \square \)
The more complicated converse reduction of iterated multiplication to division was formalized in bounded arithmetic by Johannsen [17] (building on Johannsen and Pollett [18]), but in a different setting, so let us see what his result gives us here. Johannsen works with a one-sorted theory $C_2^0[\text{div}]$, whose language consists of the usual Buss’s language for $S_2$ expanded with $\sim$, $\text{MSP}$, and most importantly $\lfloor x/y \rfloor$. It is axiomatized by a suitable version of $\text{BASIC}$, the defining axiom for division, the quantifier-free $\text{LIND}$ schema, and the axiom of choice $\text{BB}$ for $\Sigma^b_0$ formulas in the expanded language.

We claim that $C_2^0[\text{div}]$ is $\text{RSUV}$-isomorphic to the theory $VTC^0 + \text{DIV} + \Sigma^B_0$-$\text{AC}$. We leave the interpretation of the latter theory in $C_2^0[\text{div}]$ to the reader as we will not need it, and focus on the other direction. It is straightforward to translate the symbols of the language save division to the corresponding operations on binary integers, and prove the translation of $\text{BASIC}$ in $VTC^0$. Of course, $\text{DIV}$ allows us to translate the division function and prove its defining axiom, hence the only remaining problem is with the $\text{LIND}$ and $\text{BB}$ schemata. Here we have to be a bit careful, as $\Sigma^b_0$ (or even quantifier-free) formulas in the language of $C_2^0[\text{div}]$ do not translate to $\Sigma^b_0$ formulas in the language of $V^0$.

Let $\text{DIV}^*$ denote the axiom stating the totality of the aggregate function of division, or rather, of its expanded version with witnesses for multiplication as in the proof of Lemma 3.3, so that $T = VTC^0 + \text{DIV}^*$ is a CN theory. By an application of choice, $VTC^0 + \text{DIV} + \Sigma^B_0$-$\text{AC}$ proves $\text{DIV}^*$. Let $\overline{T}$ be the universal conservative extension of $T$ from Theorem 3.2, which includes function symbols for division and for $\text{TC}^0$ functions like multiplication. Since $\Sigma^b_0$ formulas in the language of $C_2^0[\text{div}]$ translate to $\Sigma^B_0(L_{\overline{T}})$ formulas, Theorem 3.2 implies that $\overline{T}$, and therefore $T \subseteq VTC^0 + \text{DIV} + \Sigma^B_0$-$\text{AC}$, proves the translation of open (or even $\Sigma^b_0$) $\text{LIND}$. As for the axiom of choice, every $\Sigma^B_0(L_{\overline{T}})$ formula is equivalent to a $\Sigma^B_1$ formula in the language of $V^0$, and $\Sigma^B_0$-$\text{AC}$ implies $\Sigma^B_1$-$\text{AC}$, hence the translation of $\text{BB}\Sigma^b_0$ is provable in $\overline{T} + \Sigma^B_0$-$\text{AC}$, and thus in $VTC^0 + \text{DIV} + \Sigma^B_0$-$\text{AC}$ by the conservativity of $\overline{T}$ over $T$.

This, together with provability of iterated multiplication in $C_2^0[\text{div}]$, implies the following:

**Theorem 3.4 (Johannsen [17])** $VTC^0 + \text{DIV} + \Sigma^B_0$-$\text{AC}$ proves $\text{IMUL}$.  

**Corollary 3.5** $VTC^0 + \text{IMUL} = VTC^0 + \text{DIV}^*$ is the smallest CN theory including $VTC^0 + \text{DIV}$.  

**Proof**: Since $VTC^0 + \text{DIV}^*$ is a CN theory, Theorem 3.2 implies that $VTC^0 + \text{DIV} + \Sigma^B_0$-$\text{AC}$ is $\Pi^b_1$-conservative over $VTC^0 + \text{DIV}^*$, hence $VTC^0 + \text{DIV}^* + \text{IMUL}$ by Theorem 3.4. Conversely, every CN theory (such as $VTC^0 + \text{IMUL}$, by Lemma 3.3) that proves $\text{DIV}$ also proves $\text{DIV}^*$, using its closure under $\Sigma^B_0$-$\text{AC}^R$.  

**Corollary 3.6** $VTC^0 + \text{DIV}$ if and only if $VTC^0 + \text{IMUL}$.  

**Proof**: $VTC^0$ is a CN theory.
does not seem to be any particular reason we should expect to get a CN theory if we formulate
the axiom more economically, using only a one-dimensional array consisting of the products
\[ \prod_{j < i} X[j]. \]
In view of this, the decision to axiomatize the theory using \textsc{imul} rather than \textsc{div} is mostly a matter of esthetic preference and convenience. Even in its triangular form, the
\textsc{imul} axiom is a fairly natural rendering of the idea of computing iterated products, whereas
the usage of an aggregate function in \textsc{div} is overtly a technical crutch. Moreover, we will be
using iterated products more often than division, and while \textsc{div} has a straightforward proof
in \textsc{vtc} \textsuperscript{0} + \textsc{imul} as indicated above, we would have to rely on the complicated argument
from [17] to derive \textsc{imul} if we based the theory on \textsc{div}, making the main result of the
paper less self-contained.

We mention another possibility for axiomatization of our theory, using the powering axiom

\[(\text{POW}) \quad \forall X, n \exists Y \forall i < n \left( Y^{[0]} = 1 \land Y^{[i+1]} = Y^{[i]} \cdot X \right)\]

(here it makes no difference whether we use a linear or triangular array of witnesses) and its
aggregate function version \textsc{pow}*. Over \textsc{vtc} \textsuperscript{0}, we clearly have \textsc{imul} \vdash \textsc{pow}* \vdash \textsc{pow}.
The argument in Lemma 3.3 (ii) only needed the sequence of powers \( (2^n - X)^i, i \leq m \) apart
from \textsc{vtc} \textsuperscript{0}, hence it actually shows \textsc{pow} \vdash \textsc{div}. Since \textsc{vtc} \textsuperscript{0} + \textsc{pow}* is a CN theory, this
implies \textsc{vtc} \textsuperscript{0} + \textsc{pow}* = \textsc{vtc} \textsuperscript{0} + \textsc{imul}. In fact, one can also show that \textsc{vtc} \textsuperscript{0} + \textsc{pow} = \textsc{vtc} \textsuperscript{0} + \textsc{div}
by formalizing the reduction of powering to division from [3]. The key point
is that the result of a single division is enough to reconstruct the whole sequence of powers
\( X^0, \ldots, X^n \), hence we do not need any aggregate functions. If \( X < 2^k \) and \( m = k(n + 1) + 1 \),
let \( 2^{nm} = (2^m - X)Q + R \) with \( R < 2^m - X \) using \textsc{div}, write \( Q = \sum_{i < n} Y^{[i]} 2^{(n-1-i)m} \) with
\( Y^{[i]} < 2^m \), and put \( Y^{[n]} = R \). Then one can show \( Y^{[0]} = 1 \) and

\[ Y^{[j]} \leq 2^{kj} \land \forall i < j Y^{[i+1]} = XY^{[i]} \]

by induction on \( j \leq n \). We leave the details to the interested reader.

Let us also mention that while it is unclear whether the soundness of the Hesse–Allender–
Barrington algorithms for division and iterated multiplication is provable in \textsc{vtc} \textsuperscript{0}, it seems
very likely that it is provable in \textsc{vtc} \textsuperscript{0} + \textsc{imul}. If true, this would imply that \textsc{vtc} \textsuperscript{0} + \textsc{imul}
is \Pi_1\textsuperscript{1}-axiomatizable over \textsc{vtc} \textsuperscript{0} by the sentence asserting the soundness of the algorithm, and
it can be formulated as a purely universal theory in the language of \textsc{vtc} \textsuperscript{0}. A priori, the
\textsc{imul} axiom is only \forall \Sigma_1^\text{B}.

Even though we do not know whether \textsc{imul} is provable in \textsc{vtc} \textsuperscript{0} itself, we can place it
reasonably low in the usual hierarchy of theories for small complexity classes: it is straight-
forward to show that \textsc{vtc} \textsuperscript{0} + \textsc{imul} is included in the theory \textsc{vnc} \textsuperscript{2} (and even \textsc{vtc} \textsuperscript{1}, if
anyone bothered to define such a theory) by formalizing the computation of iterated products
by a balanced tree of binary products.

As stated in the Introduction, the provability of \textsc{iopen} in \textsc{vtc} \textsuperscript{0} or \textsc{vtc} \textsuperscript{0} + \textsc{imul} can be
phrased in terms of \textsc{tc} \textsuperscript{0} root-finding algorithms. There are several ways of expressing this
connection precisely; one version reads as follows.
Proposition 3.7 \( \text{VTC}^0 + \text{IMUL} \) proves IOpen if and only if for every constant \( d > 0 \) there exist \( L_{\text{VTC}^0+\text{IMUL}} \)-terms \( R_-(A_0, \ldots, A_d, X, Y, E) \) and \( R_+(A_0, \ldots, A_d, X, Y, E) \) such that the theory proves

\[
\begin{align*}
(1) \quad & X < Y \land F(X) < 0 < F(Y) \land E > 0 \land Z_\pm = R_{\pm}(A_0, \ldots, A_d, X, Y, E) \\
& \quad \rightarrow X < Z_- < Z_+ < Y \land Z_- - Z_+ < E \land F(Z_-) < 0 < F(Z_+),
\end{align*}
\]

where all second-sort variables are interpreted as binary rational numbers (fractions), and \( F(X) \) denotes \( A_dX^d + A_{d-1}X^{d-1} + \cdots + A_0 \).

Proof: Left-to-right: the statement that for every \( A_0, \ldots, A_d, X, Y, E \) there exist \( Z_-, Z_+ \) satisfying (1) is provable in IOpen (in the real closure of the model, there is a root of \( F \) between \( X \) and \( Y \) where \( F \) changes sign, and this root can be arbitrarily closely approximated from either side in the fraction field of the model using Theorem 2.1). By assumption, the same statement is also provable in \( \text{VTC}^0 + \text{IMUL} \). Since the latter is a universal theory whose terms are closed under definitions by cases, Herbrand’s theorem implies that there are terms \( R_-, R_+ \) witnessing \( Z_-, Z_+ \).

Right-to-left: Let \( D \) be a DOR induced by a model of \( \text{VTC}^0 + \text{IMUL} \), \( K \) its fraction field, and \( F \) a polynomial with coefficients in \( D \). Since \( F \) can change sign only \( \deg(F) \) times, a repeated use of (1) gives us elements \( Z_0 < Z_1 < \cdots < Z_k \) of \( K \) such that \( F \) has (in \( K \)) a constant sign on each interval \(( -\infty, Z_0 ), (Z_k, \infty) \), and \((Z_i, Z_{i+1}) \), except when \( Z_{i+1} - Z_i < 1 \).

We have \( D \vdash \text{DIV} \), hence we can approximate each \( Z_i \) in \( D \) within distance 1; it follows that in \( D \), \( F \) is positive on a finite union of (possibly degenerate) intervals. Every \( L_{\text{OR}} \) open formula \( \varphi \) is equivalent to a Boolean combination of formulas of the form \( F(X) > 0 \), hence \( \{ X \in D : X \geq 0 \land \neg \varphi(X) \} \) is also a finite union of intervals, and as such it has a least element if nonempty. Thus, \( D \) satisfies induction for \( \varphi \).

Note that \( L_{\text{VTC}^0+\text{IMUL}} \)-terms denote \( \text{TC}^0 \) algorithms (employing iterated multiplication), hence the gist of the conclusion of Proposition 3.7 is that \( \text{VTC}^0 + \text{IMUL} \) proves the soundness of a \( \text{TC}^0 \) degree-\( d \) polynomial root-approximation algorithm for each \( d \). The details can be varied; for example, we could drop \( X \) and \( Y \), and make the algorithm output approximations to all real roots of the polynomial, or even complex roots. However, such modifications make it more difficult to state what exactly the “soundness” of the algorithm means.

4 Working in \( \text{VTC}^0 + \text{IMUL} \)

As we already warned the reader, the objects we work with most often in this paper are binary numbers (integer or rational), and we will employ common mathematical notation rather than the formal conventions used in [8]: in particular, we will typically denote numbers by lowercase letters (conversely, we will occasionally denote unary numbers by capital letters), and we will write \( x_i \) for the \( i \)th member of a sequence \( x \) (which may be a constant-length tuple, a variable-length finite sequence encoded by a set as in Section 2, or an infinite sequence given by a \( \text{TC}^0 \) function with unary input \( i \)). We do not distinguish binary and unary numbers in notation; we will either explicitly mention which numbers are unary, or it will be assumed.
from the context: unary natural numbers appear as indices and lengths of sequences, as powering exponents, and as bound variables in iterated sums $\sum_{i=0}^{n} x_i$ and products $\prod_{i=0}^{n} x_i$.

By Theorem 3.2, we can use $L_{\text{VC}^0 + \text{IMUL}}^+$ function symbols (i.e., $\text{TC}^0$ algorithms) freely in the arguments. In particular, we can use basic arithmetic operations on integers, including iterated sums and products. Iterated sums satisfy the recursive identities

$$
\sum_{i<0} x_0 = 0,
$$

$$
\sum_{i<n+1} x_i = \sum_{i<n} x_i + x_n,
$$

and other basic properties can be easily proved by induction, for example

$$
\sum_{i<n} (x_i + y_i) = \sum_{i<n} x_i + \sum_{i<n} y_i,
$$

(2)

$$
\sum_{i<n} y x_i = y \sum_{i<n} x_i,
$$

$$
\sum_{i<n+m} x_i = \sum_{i<n} x_i + \sum_{i<m} x_{n+i}.
$$

In particular, $\text{VC}^0 + \text{IMUL}$ proves that if $\pi$ is a permutation of $\{0, \ldots, n-1\}$, then

$$
\sum_{i<n} x_i = \sum_{i<n} x_{\pi(i)}.
$$

(3)

(In order to see this, show $\sum_{i<n} x_i = \sum_{i<n} x_{\pi(i)}[\pi(i) < m]$ by induction on $m \leq n$ using (2), where $[\cdot]$ denotes the Iverson bracket.) This allows us to make sense of more general sums $\sum_{i \in I} x_i$ where the indices run over a $\text{TC}^0$-definable collection of objects (e.g., tuples of unary numbers) that can be enumerated by a subset of some $\{0, \ldots, n-1\}$; the identity (3) shows that the value of such a sum is independent of the enumeration. For example, we can write

$$
f(n) = \sum_{i+j=n} x_{i,j},
$$

meaning a sum over all pairs of numbers $(i, j)$ such that $i + j = n$. We can also prove the double counting identity

$$
\sum_{i<n} \sum_{j<m} x_{i,j} = \sum_{i<n} \sum_{j<m} x_{i,j} = \sum_{j<m} \sum_{i<n} x_{i,j}
$$

(4)

by first showing $\sum_{i<n} \sum_{j<m} x_{i,j} = \sum_{k<nm} x_{\lfloor k/m \rfloor, k \mod m}$ by induction on $n$ using (2), and then (3) implies that other enumerations of the same set of pairs give the same result. Likewise, we can show

$$
(\sum_{i<n} x_i)(\sum_{i<n} y_i) = \sum_{i<n} \sum_{j<m} x_i y_j.
$$

(5)

Iterated products can be treated the same way as sums, mutatis mutandis.
Rational numbers can be represented in $VTC^0 + IMUL$ as pairs of integers standing for fractions $a/b$, where $b > 0$. We will not assume fractions to be reduced, as we cannot compute integer $gcd$. Arithmetic operations can be extended to rational numbers in $VTC^0 + IMUL$ in the obvious way, for example

$$\sum_{i<n} a_i \cdot b_i := \frac{\sum_{i<n} a_i \prod_{j\neq i} b_j}{\prod_{i<n} b_i}.$$ 

$VTC^0 + IMUL$ knows the rationals form an ordered field, being the fraction field of a DOR. The properties of iterated sums and products we established above for integers also hold for rationals.

Using iterated products, we can define factorials and binomial coefficients

$$n! = \prod_{i=1}^{n} i, \quad \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

for unary natural numbers $n \geq m$. A priori, $n!$ is a binary integer, and $\binom{n}{m}$ a binary rational; however, the definition easily implies the identities

$$\binom{n}{0} = \binom{n}{n} = 0, \quad \binom{n+1}{m+1} = \binom{n}{m} + \binom{n}{m+1},$$

from which one can show by induction on $n$ that $\binom{n}{m}$ is an integer for all $m \leq n$. We can also prove by induction on $n$ the binomial formula

$$(x + y)^n = \sum_{i \leq n} \binom{n}{i} x^i y^{n-i}$$

for rational $x, y$. More generally, we can define the multinomial coefficients

$$\binom{n}{n_1, \ldots, n_d} = \frac{n!}{n_1! \cdots n_d!} = \binom{n}{n_1} \binom{n-n_1}{n_2} \cdots \binom{n-n_1-\cdots-n_{d-1}}{n_d}$$

for a standard constant $d$ and unary $n = n_1 + \cdots + n_d$, and we can prove the multinomial formula

$$(x_1 + \cdots + x_d)^n = \sum_{n_1 + \cdots + n_d = n} \binom{n}{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

by metainduction on $d$.

## 5 Lagrange inversion formula

The Lagrange inversion formula (LIF) is an expression for the coefficients of the (compositional) inverse $g = f^{-1}$ of a power series $f$. In this section, we will formalize in $VTC^0 + IMUL$ variants of LIF for the special case where $f$ is a constant-degree polynomial; we first show that $g$ inverts $f$ as a formal power series, and then with the help of a suitable bound on the
coefficients of $g$, we show that the series $g(w)$ is convergent for small enough $w$; this means that under some restrictions, partial sums of $g(-a_0)$ approximate a root of the polynomial $f(x) + a_0$.

LIF, specifically the equivalent identity (9), has a simple combinatorial interpretation in terms of trees which allows for a straightforward bijective proof. However, this proof relies on exact counting of exponentially many objects, and as such it cannot be formalized in $VTC^0 + IMUL$. In contrast, the inductive proof we give below proceeds by low-level manipulations of sums and products; while it lacks conceptual clarity, it is elementary enough to go through in our weak theory.

We introduce some notation for convenience. Let us fix a standard constant $d \geq 1$. We are going to work extensively with sequences $m = (m_2, \ldots, m_d)$ of length $d - 1$ of unary nonnegative integers. We will use subscripts $i = 2, \ldots, d$ to extract elements of the sequence as indicated, and we will employ superscripts (and primes) to label various sequences used at the same time; these do not denote exponentiation. If $m_1$ and $m_2$ are two such sequences, we define $m_1 + m_2$ and $m_1 - m_2$ coordinatewise (i.e., $(m_1 + m_2)_i = m_1^i + m_2^i$), we write $m_1 \leq m_2$ if $m_1^i \leq m_2^i$ for all $i = 2, \ldots, d$, and $m_1 \preceq m_2$ if $m_1 \leq m_2$ and $m_1 \neq m_2$. We define the generalized Catalan numbers

$$C_m = \frac{(\sum_{i=2}^d im_i)!}{(\sum_{i=2}^d (i-1)m_i + 1)! \prod_{i=2}^d m_i!}.$$  

**Theorem 5.1** $VTC^0 + IMUL$ proves the following for every constant $d \geq 1$: let

$$f(x) = x + \sum_{k=2}^d a_k x^k$$

be a rational polynomial, and let

$$g(w) = \sum_{n=1}^{\infty} b_n w^n$$

be the formal power series (with unary indices) defined by

(7) $$b_n = \sum_{\sum_{i=2}^d (i-1)m_i = n-1} C_m \prod_{i=2}^d (-a_i)^{m_i}.$$  

Then $f(g(w)) = w$ as formal power series.

**Remark 5.2** The sum in (7) runs over sequences $m = (m_2, \ldots, m_d)$ satisfying the constraint $\sum_{i=2}^d (i-1)m_i = n - 1$; since this implies $m_2, \ldots, m_d < n$, there are at most $n^{d-1}$ such sequences, hence the sum makes sense in $VTC^0 + IMUL$.

The power series identity $f(g(w)) = w$ in the conclusion of the theorem amounts to $b_1 = 1$, and the recurrence

(8) $$b_n = \sum_{k=2}^d (-a_k) \sum_{n_1 + \cdots + n_k = n} b_{n_1} \cdots b_{n_k} \quad (n > 1).$$
Rather than developing a general theory of formal power series in \( VTC^0 + IMUL \), we take this as a definition of \( f(g(w)) = w \).

**Proof:** After plugging in the definition of \( b_n \), both sides of (8) can be written as polynomials in \(-a_2, \ldots, -a_d\) with rational (actually, integer) coefficients by several applications of (5). Moreover, \( b_{n,j} \) contains only monomials \( \prod_i (-a_i)^{m_i} \) with \( \sum_i (i - 1)m_i^j = n_j - 1 \). Thus, the right-hand side contains monomials \( \prod_i (-a_i)^{m_i} \) with \( m_i = \sum_j m_i^j + \delta_i^k \), where \( \delta_i^k \) is Kronecker’s delta. We have \( \sum_i (i - 1)m_i = \sum_i, (i - 1)m_i^j + k - 1 = \sum_j (n_j - 1) + k - 1 = n - 1 \), which is the same constraint as on the left-hand side. In order to prove (8), it thus suffice to show that the coefficients of the monomials \( \prod(-a_i)^{m_i} \) satisfying \( \sum_i (i - 1)m_i = n - 1 \) are the same on both sides of (8). This is easily seen to be equivalent to the following identity for every sequence \( m \):

\[
C_m = \sum_{k=2}^{d} \sum_{m_1^2 + \cdots + m_k^k = m - \delta^k} C_m \cdots C_m \quad (m \neq 0).
\]

(Here, we treat Kronecker’s delta as the sequence \( \delta^k = (\delta^k_2, \ldots, \delta^k_d) \).) We will prove (9) by induction on \( \sum_i m_i \), simultaneously with the identities

\[
\sum_{m' + m'' = m} (\sum_i (i - 1)m_i^j + 1)C_{m'}C_{m''} = (\sum_i im_i + 1)C_m, \tag{10}
\]

\[
\sum_{m_1 + \cdots + m_k = m} C_{m_1} \cdots C_{m_k} = \frac{(\sum_i im_i + k - 1)!k}{(\sum_i (i - 1)m_i + k)! \prod_i m_i!} \quad (k = 1, \ldots, d). \tag{11}
\]

The reader may find it helpful to consider the following combinatorial explanation of the identities, even though it cannot be expressed in \( VTC^0 + IMUL \). First, \( C_m \) counts the number of ordered rooted trees with \( m_2, \ldots, m_d \) nodes of out-degree 2, \ldots, \( d \), respectively, and the appropriate number (i.e., \( \sum_i (i - 1)m_i + 1 \)) of leaves. Indeed, such a tree can be uniquely described by the sequence of out-degrees of its nodes in preorder. One checks easily that every string with \( m_2, \ldots, m_d \) occurrences of 2, \ldots, \( d \), resp., and \( \sum_i (i - 1)m_i + 1 \) occurrences of 0, has a unique cyclic shift that is a valid representation of a tree, so there are

\[
\frac{1}{\sum_i im_i + 1} \left( \sum_i im_i + 1 \right) = C_m
\]

such trees. The left-hand side of (11) thus counts \( k \)-tuples of trees with a prescribed total number of nodes of out-degree 2, \ldots, \( d \); a similar argument as above shows their number equals the right-hand side (every string with the appropriate number of symbols of each kind has exactly \( k \) cyclic shifts that are concatenations of representations of \( k \) trees). The main identity (9) expresses that a tree with more than one node can be uniquely decomposed as a root of out-degree \( k = 2, \ldots, d \) followed by a \( k \)-tuple of trees. Finally, (10) expresses that a pair of trees \( t', t'' \) together with a distinguished leaf \( x \) of \( t' \) uniquely represent a tree \( t \) with a distinguished node \( x \), namely the tree obtained by identifying the root of \( t'' \) with \( x \).

Let us proceed with the formal proof by induction. Assume that (9), (10), and (11) hold for all \( m' \) such that \( m' \leq m \), we will prove them for \( m \).
(9): If \( m \neq \vec{0} \), we have
\[
\sum_{k=2}^{d} \sum_{m^1+\ldots+m^k = m-\delta^k} C_{m^1} \cdots C_{m^k} = \sum_{k=2}^{d} \frac{(\sum_i im_i - 1)!}{(\sum_i (i-1) m_i + 1)! \prod_{i \neq k} m_i! (m_k - 1)!} \cdot \delta^k i \cdot \sum_{k=2}^{d} \sum_{m^1+\ldots+m^k = m-\delta^k} C_{m^1} \cdots C_{m^k}
\]
\[
= \frac{(\sum_i im_i - 1)!}{(\sum_i (i-1) m_i + 1)! \prod_{i \neq k} m_i!} \cdot \delta^k i \cdot \sum_{k=2}^{d} \sum_{m^1+\ldots+m^k = m-\delta^k} C_{m^1} \cdots C_{m^k}
\]
\[
= \frac{(\sum_i im_i)!}{(\sum_i (i-1) m_i + 1)! \prod_{i \neq k} m_i!} = C_m,
\]
using (11) for \( m - \delta^k \leq m \).

(10): If \( m = \vec{0} \), the statement holds. Otherwise, we have
\[
(\sum_i im_i + 1)C_m = C_m + \sum_{k=2}^{d} \sum_{m^1+\ldots+m^k = m-\delta^k} C_{m^1} \cdots C_{m^k}
\]
\[
= C_m + \sum_{k=2}^{d} \sum_{m^1+\ldots+m^k = m-\delta^k} \sum_{k=1}^{k} (\sum_i im_i^j + 1)C_{m^1} \cdots C_{m^k}
\]
\[
= C_m + \sum_{k=2}^{d} \sum_{m^1+\ldots+m^k+\ldots+\delta^k} (\sum_i (i-1) m_i^k + 1)C_{m^1} \cdots C_{m^k} C_m
\]
\[
= C_m + \sum_{m' + m'' = m} C_{m'} \sum_{k=2}^{d} \sum_{m^1+\ldots+m^k = m' - \delta^k} \sum_{j=1}^{k} (\sum_i (i-1) m_i^j + 1)C_{m^1} \cdots C_{m^k}
\]
\[
= C_m + \sum_{m' + m'' = m} C_{m'} C_{m''} \sum_{k=2}^{d} \sum_{m^1+\ldots+m^k = m' - \delta^k} \sum_{j=1}^{k} (\sum_i (i-1) m_i^j + 1)C_{m^1} \cdots C_{m^k}
\]
\[
= C_m + \sum_{m' + m'' = m} \sum_{m' + m'' = m} (\sum_i (i-1) m_i^j + 1)C_{m'} C_{m''}
\]
\[
= \sum_{m' + m'' = m} (\sum_i (i-1) m_i^j + 1)C_{m'} C_{m''},
\]
using (9) for \( m \) and \( m' \leq m \), and (10) for \( m^k \leq m \). We derive line (12) by observing that the
We can estimate

\[
\sum_{m^1 + \ldots + m^k = m - \delta^k} \left( \sum_i im_i^j + 1 \right) C_{m^1} \cdots C_{m^k} \quad (j = 1, \ldots, k)
\]

have the same value due to symmetry (i.e., by an application of (3)). Line (13) is similar.

(11): By metainduction on \(k = 1, \ldots, d\). The case \(k = 1\) is the definition of \(C_m\). Assuming the statement holds for \(k\), we prove it for \(k + 1\) from the identity

\[
k \left( \sum_i (i - 1)m_i + k + 1 \right) \sum_{m^1 + \ldots + m^{k+1} = m} C_{m^1} \cdots C_{m^{k+1}}
\]

\[
= k \sum_{m^1 + \ldots + m^{k+1} = m} \left( \sum_j (i - 1)m_i^j + 1 \right) C_{m^1} \cdots C_{m^{k+1}}
\]

\[
= k \sum_{m^1 + \ldots + m^{k+1} = m} \left( \sum_i (i - 1)m_i^{k+1} + 1 \right) C_{m^1} \cdots C_{m^{k+1}}
\]

\[
= k \sum_{m^1 + \ldots + m^{k+1} = m} \left( \sum_i (i - 1)m_i^{k+1} + 1 \right) C_{m^1} \cdots C_{m^{k+1}}
\]

\[
= k \left( \sum_i im_i + k \right) \sum_{m^1 + \ldots + m^k = m} C_{m^1} \cdots C_{m^k}
\]

using (10) for \(m^k \leq m\). \(\square\)

**Lemma 5.3** \(VTC^0 + IMUL\) proves: let \(f, g\) be as in Theorem 5.1, and \(a = \max \{1, \sum_i |a_i|\}\). Then \(|b_n| \leq (4a)^{n-1}\) for every \(n\).

**Proof:** We can estimate

\[
|b_n| \leq a^{n-1} \sum_{\sum_i (i - 1)m_i = n - 1} C_m \prod_{i=2}^d \left( a^{1-i} |a_i| \right)^{m_i}
\]

\[
= \frac{a^{n-1}}{n} \sum_{\sum_i (i - 1)m_i = n - 1} \left( n - 1 + \sum_i m_i \right) \prod_{i=2}^d \left( a^{1-i} |a_i| \right)^{m_i}
\]

\[
\leq \frac{a^{n-1}}{n} \sum_{t=n-1} 2^{(n-1)} \sum_{s+\sum_i m_i = t} \left( s, m_2, \ldots, m_d \right) \prod_{i=2}^d \left( a^{1-i} |a_i| \right)^{m_i}
\]

\[
= \frac{a^{n-1}}{n} \sum_{t=n-1} 2^{(n-1)} \left( 1 + a^{-1} \sum_{i=2}^d |a_i| \right)^t
\]

\[
\leq a^{n-1} \left( 1 + a^{-1} \sum_{i=2}^d |a_i| \right)^{2(n-1)} \leq a^{n-1} 2^{2(n-1)}
\]

using the multinomial formula (6). \(\square\)
**Example 5.4** The bound in Lemma 5.3 is reasonably tight even in the “real world”. Let $a > 0$ be a real number, and put $f(x) = x - ax^2$. Then $g$ is its inverse function $g(w) = (1 - \sqrt{1 - 4aw})/2a$, whose radius of convergence is the modulus of the nearest singularity, namely $1/4a$. Thus, for every $\varepsilon > 0$, $|b_n| \geq (4a - \varepsilon)^n$ for infinitely many $n$. In fact, the Stirling approximation for Catalan numbers gives $b_n = \Theta((4a)^n n^{-3/2})$.

**Theorem 5.5** $VTC^0 + IMUL$ proves the following for every constant $d \geq 1$. Let $h(x) = \sum_{i=0}^{d} a_i x^i$ be a rational polynomial with linear coefficient $a_1 = 1$. Put $f = h - a_0$, let $g$ and $b_n$ be as in Theorem 5.1, $a = \max\{1, \sum_{i=2}^{d} |a_i|\}$, $\alpha = 4a|a_0|$, and let

$$x_N = \sum_{n=1}^{N} b_n (-a_0)^n$$

denote the $N$th partial sum of $g(-a_0)$ for every unary natural number $N$. If

$$|a_0| < \frac{1}{4a},$$

then

$$|x_N| \leq \frac{|a_0|}{1 - \alpha},
$$

(14)

$$|x_N - x_M| \leq \frac{|a_0|\alpha^{N-1}}{1 - \alpha},
$$

(15)

$$|h(x_N)| \leq N^d |a_0|\alpha^N
$$

(16)

for every unary $M \geq N \geq 1$.

**Proof:** Lemma 5.3 gives

$$|x_N| \leq \sum_{n=1}^{N} |a_0|^n (4a)^{n-1} = |a_0| \sum_{n=0}^{N-1} \alpha^n \leq \frac{|a_0|}{1 - \alpha}.$$

The proof of (15) is similar. As for (16), we have

$$h(x_N) = a_0 + \sum_{k=1}^{d} a_k \sum_{n_1, \ldots, n_k = 1}^{N} b_{n_1} \cdots b_{n_k} (-a_0)^{n_1 + \cdots + n_k}
$$

(17)

$$= \sum_{k=1}^{d} a_k \sum_{n_1, \ldots, n_k = 1}^{N} b_{n_1} \cdots b_{n_k} (-a_0)^{n_1 + \cdots + n_k},$$

as

$$\sum_{k=1}^{d} a_k \sum_{n_1 + \cdots + n_k = n} b_{n_1} \cdots b_{n_k} = \delta_n^1$$

for every unary natural number $N$. If $|a_0| < 1/4a$, then

$$|x_N| \leq \frac{|a_0|}{1 - \alpha},$$

(14)

$$|x_N - x_M| \leq \frac{|a_0|\alpha^{N-1}}{1 - \alpha},
$$

(15)

$$|h(x_N)| \leq N^d |a_0|\alpha^N
$$

(16)
for all $n \leq N$ by Theorem 5.1. Note that the inner sum in (17) is empty for $k = 1$, thus

$$
|h(x_N)| \leq \sum_{k=2}^{d} |a_k| \sum_{n_1, \ldots, n_k=1}^{N} (4a)^{-k} \left(4a|a_0|\right)^{n_1+\cdots+n_k}
$$

$$
\leq \sum_{k=2}^{d} |a_k| \left(\frac{N}{4a}\right)^k \alpha^{N+1}
$$

$$
\leq a \max \left\{ \frac{N^2}{(4a)^2}, \frac{N^d}{(4a)^d} \right\} \alpha^{N+1}
$$

$$
\leq \max \left\{ \frac{N^2}{4}, \frac{N^d}{4d-1} \right\} |a_0| \alpha^N \leq N^d |a_0| \alpha^N,
$$

using Lemma 5.3 and $a \geq 1$. \qed

Intuitively, the conclusion of Theorem 5.5 says that $x_N$ is a Cauchy sequence with an explicit modulus of convergence whose limit is a root of $h$ of bounded modulus.

## 6 Valued fields

Theorem 5.5 shows that $VTC^0 + IMUL$ can compute roots of polynomials of a special form, however it would still be rather difficult to extend it to a full-blown root-finding algorithm. We will instead give a model-theoretic argument using well-known properties of valued fields to bridge the gap between Theorem 5.5 and approximation of roots of general polynomials.

In order to prove $VTC^0 + IMUL \vdash IOpen$, it suffices to show that every model of $VTC^0 + IMUL$ is a model of $IOpen$. First, since $VTC^0 + IMUL \vdash DIV$, we can reformulate Theorem 2.1 in terms of fields.

**Lemma 6.1** Let $D$ be a DOR, and $F$ its fraction field. The following are equivalent.

(i) $D \vDash IOpen$.

(ii) $D \vDash DIV$, and $F$ is a dense subfield of a RCF $R$. \qed

The condition that $F$ is dense in $R$ means that elements of $R$ can be well approximated in $F$, i.e., $R$ cannot be too large, while the condition that $R$ is real-closed (or at least contains the real closure $\hat{F}_{\text{real}}$) means that $R$ cannot be too small, so these two conditions work against each other. One canonical choice of $R$ is the smallest RCF extending $F$, i.e., $\hat{F}_{\text{real}}$. We obtain that a DOR $D \vDash DIV$ is a model of $IOpen$ iff $F$ is dense in $\hat{F}_{\text{real}}$. However, it will be useful for us to consider another choice: it turns out that there exists the largest ordered field extension $\hat{F} \supset F$ in which $F$ is dense, and a DOR $D \vDash DIV$ is a model of $IOpen$ iff $\hat{F}$ is real-closed.

The existence of $\hat{F}$ was shown by Scott [28]. One way to prove it is by generalization of the construction of $\mathbb{R}$ using Dedekind cuts. Consider pairs $\langle A, B \rangle$, where $F = A \cup B$, $B$ has no smallest element, and

$$
\inf \{b - a : a \in A, b \in B\} = 0.
$$

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One can show that the collection of all such cuts can be given the structure of an ordered field in a natural way, and it has the property needed of \( \hat{\mathbb{F}} \). However, we will use a different construction of \( \hat{\mathbb{F}} \) which may look more complicated on first sight, but has the advantage of allowing us to employ tools from the theory of valuations to explore its properties (such as being real-closed). It can be thought of as generalizing the construction of \( \mathbb{R} \) by means of Cauchy sequences.

We refer the reader to [9] for the theory of valued fields, however we will review our notation and some basic facts below to make sure we are on the same page.

A valuation on a field \( K \) is a surjective mapping \( v : K \to \Gamma \cup \{\infty\} \), where \( (\Gamma, +, \leq) \) is a totally ordered abelian group (called the value group), and \( v \) satisfies

(i) \( v(a) = \infty \) only if \( a = 0 \),

(ii) \( v(ab) = v(a) + v(b) \),

(iii) \( v(a + b) = \min\{v(a), v(b)\} \),

where we put \( \infty + \gamma = \gamma + \infty = \infty \) and \( \gamma \leq \infty \) for every \( \gamma \in \Gamma \). (Elements with large valuation should be thought of as being small; the order is upside down for historical reasons.)

Valuations \( v : K \to \Gamma \cup \{\infty\} \), \( v' : K \to \Gamma' \cup \{\infty\} \) are equivalent if there is an ordered group isomorphism \( f : \Gamma \to \Gamma' \) such that \( v' = f \circ v \).

The valuation ring of \( v \) is \( O = \{a \in K : v(a) \geq 0\} \),

with its unique maximal ideal being

\[ I = \{a \in K : v(a) > 0\} \].

The quotient field \( k = O/I \) is called the residue field. If \( a \in O \), we will denote its image under the natural projection \( O \to k \) as \( \overline{a} \).

More abstractly, a valuation ring for a field \( K \) is a subring \( O \subseteq K \) such that \( a \in O \) or \( a^{-1} \in O \) for every \( a \in K^\times \). Any such ring corresponds to a valuation: we take \( \Gamma = K^\times/O^\times \) ordered by \( aO^\times \leq bO^\times \) iff \( b \in aO \), and define \( v \) as the natural projection \( v(a) = aO^\times \). A valuation is determined uniquely up to equivalence by its valuation ring; thus, either of the structures \( \langle K, v \rangle \) and \( \langle K, O \rangle \) can be called a valued field. A valued field \( \langle K', v' \rangle \) is an extension of \( \langle K, v \rangle \) if \( K \) is a subfield of \( K' \), and \( v \subseteq v' \). (In terms of valuation rings, the latter means \( O = O' \cap K \).) A valuation (or valuation ring or valued field) is nontrivial if \( \Gamma \neq \{0\} \), or equivalently, if \( O \neq K \).

A valuation \( v : K \to \Gamma \cup \{\infty\} \) induces a topology on \( K \) with basic open sets

\[ B(a, \gamma) = \{b \in K : v(b - a) > \gamma\}, \quad a \in K, \gamma \in \Gamma \].

(Note that \( B(a, \gamma) = B(a', \gamma) \) for any \( a' \in B(a, \gamma) \).) This makes \( K \) a topological field, and as with any topological group, it also makes \( K \) a uniform space (with a fundamental system of entourages of the form \( \{(a, b) \in K^2 : v(a - b) > \gamma\} \) for \( \gamma \in \Gamma \)). Consequently, we have the notions of Cauchy nets, completeness, and completion; for the particular case of valued
fields, they can be stated as follows. A Cauchy sequence in $K$ is $\{a_\gamma : \gamma \in \Gamma\} \subseteq K$ such that $v(a_\gamma - a_\delta) > \min\{\gamma, \delta\}$ for every $\gamma, \delta \in \Gamma$. (Alternatively, it would be enough if Cauchy sequences were indexed over a cofinal subset of $\Gamma$.) Such a sequence converges to $a \in K$ if $v(a - a_\gamma) > \gamma$ for every $\gamma \in \Gamma$. The valued field $\langle K, v \rangle$ is complete if every Cauchy sequence in $K$ converges. A completion of $\langle K, v \rangle$ is an extension $\langle \hat{K}, \hat{v} \rangle$ of $\langle K, v \rangle$ which is a complete valued field such that $K$ is (topologically) dense in $\hat{K}$. (The last condition implies that $\hat{K}$ is an immediate extension of $K$, i.e., the natural embeddings $\Gamma \subseteq \hat{\Gamma}$ and $k \subseteq \hat{k}$ are isomorphisms.)

**Theorem 6.2 ([9, Thm.2.4.3])** Every valued field $\langle K, v \rangle$ has a completion, which is unique up to a unique valued field isomorphism identical on $K$. \(\square\)

Now we turn to the interaction of valuation and order [9, §2.2.2]. Let $\langle K, O \rangle$ be a valued field. If $\leq$ is an order on $K$ (i.e., $\langle K, \leq \rangle$ is an ordered field) such that $O$ is convex (i.e., $a \leq b \leq c$ and $a, c \in O$ implies $b \in O$), then an order is induced on the residue field $k$ by $a \leq b \iff a \leq b$. Conversely, any order on $k$ is induced from an order $\leq$ on $K$ making $O$ convex in this way. If $\Gamma$ is 2-divisible, such $a \leq$ is unique, and can be defined explicitly by

$$a > 0 \iff \exists b \in K (ab^2 \in O^* \land \overline{ab^2} > 0).$$

In general, the structure of all such orders $\leq$ is described by the Baer–Krull theorem [9, Thm. 2.2.5]. Notice also that every convex subring of an ordered field is a valuation ring.

**Lemma 6.3** If $\langle K, \leq \rangle$ is an ordered field, and $O$ a nontrivial convex subring of $K$, then the valuation topology on $K$ coincides with the interval topology. In particular, a subset $X \subseteq K$ is topologically dense iff it is order-theoretically dense.

**Proof:** The convexity of $O$ implies that every $B(a, \gamma)$ is also convex. If $c \in (a, b)$, and $\gamma \geq v(c - a), v(c - b)$, then $c \in B(c, \gamma) \subseteq (a, b)$. On the other hand, if $c \in B(a, \gamma)$, pick $e > 0$ with $v(e) > \gamma$ (which exists as the valuation is nontrivial). Then $c \in (c - e, c + e) \subseteq B(a, \gamma)$. \(\square\)

For any ordered field $\langle K, \leq \rangle$, the set of its bounded elements

$$O = \{a \in K : \exists q \in \mathbb{Q}^+ (-q \leq a \leq q)\}$$

is a convex valuation ring for $K$ with the set of infinitesimal elements

$$I = \{a \in K : \forall q \in \mathbb{Q}^+ (-q \leq a \leq q)\}$$

being its maximal ideal. The corresponding valuation is the *natural valuation* induced by $\leq$. The residue field is an archimedean ordered field, and as such it can be uniquely identified with a subfield $k \subseteq \mathbb{R}$. Here is the promised construction of the largest dense extension of an ordered field.

**Lemma 6.4** Let $\langle K, \leq \rangle$ be a nonarchimedean ordered field, $v$ its natural valuation, and $\langle \hat{K}, \hat{v} \rangle$ its completion. There is a unique order on $\hat{K}$ extending $\leq$ that makes $\hat{O}$ convex. Its natural valuation is $\hat{v}$, and it satisfies:
(i) $\hat{K}$ is an ordered field extension of $K$ such that $K$ is dense in $\hat{K}$.

(ii) If $K'$ is any ordered field extension of $K$ in which $K$ is dense, there is a unique ordered field embedding of $K'$ in $\hat{K}$ identical on $K$.

Proof: Since $\hat{K}$ is an immediate extension of $K$, for every $a \in \hat{K}^\times$ there exists an $a_0 \in K^\times$ such that $aa_0^{-1} \in 1 + I$, or equivalently, $\hat{v}(a - a_0) > 0 = \hat{v}(a)$. Any order $\preceq$ on $\hat{K}$ extending $\leq$ such that $\hat{O}$ is convex (which implies $1 + I \subseteq \hat{K}^+$) must satisfy

$$a > 0 \iff a_0 > 0,$$

which specifies it uniquely. On the other hand, we claim that (18) defines an order on $\hat{K}$. First, the definition is independent of the choice of $a_0$: if $a_1 \in K^\times$ is such that $aa_1^{-1} \in 1 + I$, then $a_0a_1^{-1} \in 1 + I$ is positive, whence $a_0$ and $a_1$ have the same sign. Clearly, exactly one of $a$ and $-a$ is positive for any $a \in \hat{K}^\times$. Let $a, b \in \hat{K}^\times$, $a, b > 0$. Since $(ab)(a_0b_0)^{-1} \in 1 + I$, we have $ab > 0$. Also, $\min\{\hat{v}(a_0), \hat{v}(b_0)\}$ as they have the same sign, thus

$$\hat{v}((a + b) - (a_0 + b_0)) = \min\{\hat{v}(a - a_0), \hat{v}(b - b_0)\} > \min\{\hat{v}(a_0), \hat{v}(b_0)\} = \hat{v}(a_0 + b_0).$$

This means we can take $a_0 + b_0$ for $(a + b)_0$, showing that $a + b > 0$.

If $a < b < c$, $a, c \in O$, we may assume $(c - a)_0 = (c - b)_0 + (b - a)_0$ by the argument above, hence $(c - b)_0 + (b - a)_0 \in O$. Since $(c - b)_0, (b - a)_0 > 0$, this implies $(b - a)_0 \in O$, hence $b - a \in \hat{O}$, and $b \in \hat{O}$. Thus, $\hat{O}$ is convex under $\preceq$.

Since $\langle K, \preceq \rangle$ is nonarchimedean, the valuations $v$ and $\hat{v}$ are nontrivial. Thus, $K$ is an order-theoretically dense subfield of $\hat{K}$ by Lemma 6.3, which shows (i). Also, in view of the convexity of $\hat{O}$, this implies that $O$ is dense in $\hat{O}$, hence

$$\hat{O} = \{a \in \hat{K} : \exists q \in \mathbb{Q}^+ (-q \preceq a \preceq q)\},$$

i.e., $\hat{v}$ is the natural valuation of $\langle \hat{K}, \preceq \rangle$.

(ii): Let $\hat{v}'$ be the natural valuation on $K'$, and $\langle \hat{K}', \hat{v}' \rangle$ its completion. By Lemma 6.3, $\langle K, v \rangle$ is topologically dense in its complete extension $\langle K', v' \rangle$, hence there is an isomorphism of $\langle \hat{K}', \hat{v}' \rangle$ and $\langle \hat{K}, \hat{v} \rangle$ identical on $K$ by Theorem 6.2. It restricts to an embedding $f : \langle K', v' \rangle \rightarrow \langle \hat{K}, \hat{v} \rangle$. For any $a \in K'$, we can see from (18) that $f(a) > 0$ implies $a_0 > 0$ for some $a_0 \in K^\times$ such that $aa_0^{-1} \in 1 + I'$, whence $a >' 0$. Thus, $f$ is order-preserving. The uniqueness of $f$ follows from the density of $K$ in $\hat{K}$. □

(If $K$ is archimedean, its natural valuation is trivial, hence the induced topology is discrete, and $\hat{K} = K$. However, the largest ordered field extension of $K$ where $K$ is dense is $\mathbb{R}$.)

We will rely on the following important characterization of real-closed fields in terms of valuations [9, Thm. 4.3.7].

**Theorem 6.5** Let $\langle K, \preceq \rangle$ be an ordered field, and $O$ a convex valuation ring of $K$. The following are equivalent.

(i) $K$ is real-closed.
(ii) $\Gamma$ is divisible, $k$ is real-closed, and $O$ is henselian. \hfill \Box

There are many equivalent definitions of henselian valuation rings or valued fields (cf. [9, Thm. 4.1.3]). It will be most convenient for our purposes to adopt the following one: a valuation ring $O$ or a valued field $\langle K, O \rangle$ is henselian iff every polynomial $h(x) = \sum_{i=0}^{d} a_i x^i \in O[x]$ such that $a_0 \in I$ and $a_1 = 1$ has a root in $I$.

The basic intuition behind Theorem 6.5 is that in order to find a root $a$ of a polynomial in $K$, we use the divisibility of $\Gamma$ to get a ballpark estimate of $a$, we refine it to an approximation up to an infinitesimal relative error using the real-closedness of $k$, and then use the henselian property to compute $a$. Complications arise from interference with other roots of the polynomial.

It is well known that the completion of a henselian valued field is henselian. In fact, we have the following simple criterion, where we define a valued field $\langle K, O \rangle$ to be almost henselian if for every polynomial $h$ as above, and every $\gamma \in \Gamma$, there is $a \in I$ such that $v(h(a)) > \gamma$. (Equivalently, $\langle K, O \rangle$ is almost henselian iff the quotient ring $O/P$ is henselian for every nonzero prime ideal $P \subseteq O$ [31].)

**Lemma 6.6** The completion $\langle \hat{K}, \hat{v} \rangle$ is henselian iff $\langle K, v \rangle$ is almost henselian.

**Proof:** First, we observe that if $h = \sum_{i=0}^{d} a_i x^i \in O[x]$ has $a_1 = 1$, then

$$v(h(b) - h(c)) = v(b - c)$$

for any $b, c \in I$. Indeed, if $b \neq c$, we have

$$\frac{h(b) - h(c)}{b - c} = a_1 + \sum_{i=2}^{d} a_i (b^{i-1} + b^{i-2}c + \cdots + c^{i-1}) \in 1 + I \subseteq O^\times.$$

Left to right: assume that $h = \sum_{i=0}^{d} a_i x^i \in O[x]$, $a_1 = 1$, $a_0 \in I$, and $\gamma \in \Gamma$. Without loss of generality, $\gamma \geq 0$. Since $\hat{K}$ is henselian, there is $\hat{a} \in \hat{I}$ such that $h(\hat{a}) = 0$. By the density of $K$ in $\hat{K}$, we can find $a \in K$ such that $\hat{v}(a - \hat{a}) > \gamma$. Then $a \in I$, and $v(h(a)) > \gamma$ by (19).

Right to left: let $h = \sum_{i=0}^{d} a_i x^i \in \hat{O}[x]$ with $a_1 = 1$ and $a_0 \in \hat{I}$. For any $\gamma \in \Gamma$, $\gamma \geq 0$, we choose $a_{i, \gamma} \in K$ such that $\hat{v}(a_i - a_{i, \gamma}) > \gamma$, and put $h_{\gamma} = \sum_{i} a_{i, \gamma} x^i$. Then $h_{\gamma} \in \hat{O}[x]$, $a_{0, \gamma} \in I$, and we could have picked $a_{1, \gamma} = 1$, hence by assumption, there is $b_{\gamma} \in I$ such that $v(h_{\gamma}(b_{\gamma})) > \gamma$. By the choice of $h_{\gamma}$, this implies $\hat{v}(h(b_{\gamma})) > \gamma$. Moreover, $v(b_{\gamma} - b_{\delta}) = \hat{v}(h(b_{\gamma}) - h(b_{\delta})) > \min\{\gamma, \delta\}$ by (19), hence $\{b_{\gamma} : \gamma \geq 0\}$ is a Cauchy sequence. Since $\hat{K}$ is complete, there is $b \in \hat{K}$ such that $\hat{v}(b - b_{\gamma}) > \gamma$ for every $\gamma$. Then $b \in I$. Since $\hat{v}(h(b) - h(b_{\gamma})) > \gamma$ by (19), we have $\hat{v}(h(b)) > \gamma$ for every $\gamma \in \Gamma$, i.e., $h(b) = 0$. \hfill \Box

Putting all the things together, we obtain the following characterization of open induction. We note that the fact that the completion of a real-closed field is real-closed was shown by Scott [28].

**Lemma 6.7** Let $D$ be a nonstandard DOR such that $D \models \text{DIV}$, $F$ its fraction field endowed with its natural valuation, and $\hat{F}$ its completion. The following are equivalent.


(i) $D \models IOpen$.

(ii) $\hat{F}$ is real-closed.

(iii) $F$ is almost henselian, its value group is divisible, and its residue field is real-closed.

Proof: (ii) and (iii) are equivalent by Theorem 6.5 and Lemma 6.6, using the fact that $\hat{F}$ is an immediate extension of $F$.

(ii) $\rightarrow$ (i) follows from Lemma 6.1 as $F$ is dense in $\hat{F}$. Conversely, assume that $F$ is a dense subfield of a RCF $R$. By Theorem 6.5, $R$ is henselian, its value group is divisible, and its residue field is a RCF. The completion $\hat{R}$ is also henselian by Lemma 6.6, and it has the same $\Gamma$ and $k$ as $R$, hence it is a RCF by Theorem 6.5. However, the density of $F$ in $\hat{R}$ implies $\hat{F} \simeq \hat{R}$ by Lemma 6.4, hence $\hat{F}$ is a RCF. □

We remark that we could have used any nontrivial convex subring in place of the natural valuation in Lemma 6.4 (any two such valuations determine the same uniform structure by Lemma 6.3, which means that their completions are the same qua topological fields, and one checks easily that they also carry the same order). Likewise, Lemma 6.7 continues to hold when $F$ is endowed with any nontrivial valuation with a convex valuation ring; this may make a difference for verification of condition (iii). Notice that such valuation rings correspond to proper cuts (in the models-of-arithmetic sense) on $D$ closed under multiplication.

We can now prove the main result of this paper.

**Theorem 6.8** $VTC^0 + IMUL$ proves $IOpen$ on binary integers.

Proof: Let $M \models VTC^0 + IMUL$, and $D$ be its ring of binary integers, we need to show that $D \models IOpen$. We may assume without loss of generality that $M$ is $\omega$-saturated. Since $VTC^0 + IMUL \vdash DIV$, it suffices to check the conditions of Lemma 6.7 (iii).

As we have mentioned above, the residue field $k$ of any ordered field under its natural valuation is a subfield of $\mathbb{R}$. The $\omega$-saturation of $D$ implies that every Dedekind cut on $\mathbb{Q}$ is realized by an element of $F$, hence in fact $k = \mathbb{R}$, which is a real-closed field.

Every element of the value group $\Gamma$ is the difference of valuations of two (positive) elements of $D$. Let thus $a \in D^+$, and $k \in \mathbb{Z}^+$. Put $n = \|a\| - 1$, which is a unary integer of $M$ such that $2^n \leq a < 2^{n+1}$. Put $m = \lfloor n/k \rfloor$ and $b = 2^m$. Then $b^k \leq a < 2^kb^k$, hence $kv(b) = v(a)$. This shows that $\Gamma$ is divisible.

Let $\gamma \in \Gamma$, and $h(x) = \sum_{i \leq d} a_i x^i \in F[x]$ be such that $v(a_i) \geq 0$, $v(a_0) > 0$, and $a_1 = 1$. Then $a = \max\{1, \sum_{i=2}^d |a_i|\}$ is bounded by a standard integer, whereas $a_0$ is infinitesimal, thus $a = 4a|a_0|$ is also infinitesimal. Let $N$ be a nonstandard unary integer of $M$ such that $v(2^{-N}) > \gamma$, and let $x_N$ be as in Theorem 5.5. Then using a crude estimate,

$$|h(x_N)| \leq N^d |a_0| \alpha^N \leq 2^N 4^{-N} = 2^{-N},$$

which means that $v(h(x_N)) > \gamma$. Moreover, $|x_N| \leq |a_0|/(1 - \alpha)$ is infinitesimal. Thus, $F$ is almost henselian. □
As explained in Section 3, Theorem 6.8 implies that for any constant $d$, $VTC^0 + IMUL$ can formalize a $TC^0$ algorithm for approximation of roots of degree $d$ rational polynomials. The reader might find it disappointing that we have shown its existence nonconstructively using the abstract nonsense from this section, so let us give at least a rough idea how this algorithm may actually look like; it is somewhat different from the one in [15].

Clearly, one ingredient is Theorem 5.5, which gives an explicit description of a $TC^0$ algorithm for approximation of roots of polynomials of a special form (small constant coefficient and large linear coefficient). The remaining part is a reduction of general root approximation to this special case, and this happens essentially in Theorem 6.5. This theorem has a proof with a fairly algorithmic flavour using Newton polygons (cf. [2, §2.6], where a similar argument is given in the special case of real Puiseux series). The Newton polygon of a polynomial $f(x) = \sum_{i=0}^{d} a_i x^i \in K[x]$ is the lower convex hull of the set of points \[ \{ e_i = (i, v(a_i)) : i = 0, \ldots, d \} \subseteq \mathbb{Q} \times \Gamma. \]

The basic idea is as follows. Take an edge of the Newton polygon with endpoints $e_{i_0}, e_{i_1}$. The slope of the edge is in $\Gamma$ due to its divisibility, hence we can replace $f(x)$ by a suitable polynomial of the form $af(bx)$ to ensure $v(a_{i_0}) = v(a_{i_1}) = 0$. Then $f \in O[x]$, its image $\overline{f} \in k[x]$ has degree $i_1$, and the least exponent of its nonzero coefficient is $i_0$. If we find a nonzero root $\overline{v} \in k^\times$ of $\overline{f}$ of multiplicity $m$ using the real-closedness of $k$, the Newton polygon of the shifted polynomial $f(x + a)$ will have an edge whose endpoints satisfy $i'_0 < i'_1 \leq m \leq i_1 - i_0$, since $m$ is the least exponent with a nonzero coefficient in $\overline{f}(x + \overline{v})$. This is strictly shorter than the original edge unless $\overline{f}$ is a constant multiple of $x^{i_0}(x-\overline{v})^{i_1-i_0}$, which case has to be handled separately. If we set up the argument properly, we can reduce $f$ by such linear substitutions in at most $d$ steps into a polynomial whose Newton polygon has $e_0, e_1$ for vertices, and then we can apply the henselian property to find its root in $K$.

One can imagine that a proper $TC^0$ algorithm working over $\mathbb{Q}$ instead of a nonarchimedean field can be obtained along similar lines by replacing “infinitesimal” with a suitable notion of “small enough” (e.g., employing an approximation of $- \log |a|$ as a measure of magnitude in place of $v(a)$). However, the details are bound to be quite unsightly due to complications arising from the loss of the ultrametric inequality of $v$.

### 7 Application to Buss’s theories

While $VTC^0 + IMUL$ does not stand much chance of proving induction for interesting classes of formulas with quantifiers in the language of ordered rings, we will show in this section that we can do better in the richer language $L_B = \langle 0, 1, +, \cdot, \leq, \#, |x|, |x/2| \rangle$ of Buss’s one-sorted theories of bounded arithmetic—$VTC^0 + IMUL$ proves the $RSUV$-translation of $T_2^0$, and even minimization for sharply bounded formulas ($\Sigma^b_0$-$MIN$). The main tool is a description of $\Sigma^b_0$-definable sets discovered by Mantzivis [23], whose variants were also given in [4, 20]: in essence, a $\Sigma^b_0$-definable subset of $[0, 2^n)$ can be written as a union of $n^{O(1)}$ intervals on each residue class modulo $2^n$, where $c$ is a standard constant. As we will see, this property can be formalized in $VTC^0 + IMUL$ using the provability of $I\textit{Open}$ for the base case of polynomial inequalities, and as a consequence, our theory proves minimization and induction for $\Sigma^b_0$.
formulas. (We stress that as in the case of IOpen, these are minimization and induction over binary numbers. Despite the same name, the schemata denoted as IND and MIN in the two-sorted framework only correspond to LIND and minimization over lengths in Buss’s language, respectively.) We will present the messier part of the argument as a normal form for Σ₀^b formulas over a weak base theory, in the hope that this will make the result more reusable.

We will assume the reader is familiar with definitions of Buss’s theories (see e.g. [5, 21]), in particular, with BASIC. Recall that a formula is sharply bounded if all its quantifiers are of the form ∃x ≤ |t| or ∀x ≤ |t|. We reserve Σ₀^b for the class of sharply bounded formulas of L_B, whereas sharply bounded formulas in an extended language L_B ∪ L’ will be denoted Σ₀^b(L’). Let BASIC⁺ denote the extension of Buss’s BASIC by the axioms

\[
\begin{align*}
(20) & \quad x(yz) = (xy)z, \\
(21) & \quad y \leq x \rightarrow \exists z \ (y + z = x), \\
(22) & \quad u \leq |x| \rightarrow \exists y \ ([y] = u), \\
(23) & \quad z < x \# y \rightarrow |z| \leq |x||y|, \\
(24) & \quad |x| \leq x.
\end{align*}
\]

(The quantifiers in (21), (22) could be bounded by x, if desired.) On top of BASIC, axioms (20) and (21) imply the theory of nonnegative parts of discretely ordered rings, hence we can imagine the universe is extended with negative numbers in the usual fashion. In particular, we can work with integer polynomials. We introduce two extra functions by

\[
x - y = z \iff y + z = x \lor (x < y \land z = 0),
\]

\[
2^\min\{u,|x|\} = z \iff z \# 1 = 2z \land (u \leq |x| \land |z| = u + 1) \lor (u > |x| \land |z| = |x| + 1)).
\]

BASIC⁺ proves that − and 2^\min\{u,|x|\} are well-defined total functions. Notice that BASIC⁺ is universally axiomatizable in a language with − and 2^\min\{u,|x|\}. We will write 2^u for 2^\min\{u,|x|\} when a self-evident value of x such that u ≤ |x| can be inferred from the context (e.g., when u is a sharply bounded quantified variable).

If p is a polynomial with nonnegative integer coefficients, one can construct easily a term t such that BASIC⁺ ⊢ p(|x_1|, \ldots, |x_k|) ≤ |t(\vec{x})|. Conversely, one can check that BASIC⁺ proves |xy| ≤ |x| + |y|; together with other axioms, this implies that for every term t (even using − and 2^\min\{u,|x|\}) there is a polynomial p such that BASIC⁺ ⊢ |t(\vec{x})| ≤ p(|\vec{x}|).

**Lemma 7.1** Let φ(x₁, \ldots, x_k) be a Σ₀^b(−,2^\min\{u,|x|\}) formula. Then BASIC⁺ proves φ(\vec{x}) equivalent to a formula of the form

\[
\bigvee_{\sigma_1, \ldots, \sigma_k < 2^c} \left( \bigwedge_{i=1}^k (x_i \equiv \sigma_i \pmod{2^c}) \land Q_1 u_1 \leq p(|x|) \cdots Q_l u_l \leq p(|x|) f_\vec{\sigma}(x_1, \ldots, x_k, u_1, \ldots, u_l, 2^{u_1}, \ldots, 2^{u_l}) \geq 0 \right),
\]

where c is a constant, Q₁, \ldots, Qₗ ∈ \{∃, ∀\}, p is a nonnegative integer polynomial, xᵢ ≡ σᵢ (mod 2^c) stands for xᵢ = σᵢ + 2^c \lfloor \frac{|xᵢ/2|}{2} \rfloor \cdots \frac{|xᵢ/2|}{2}, and f_\vec{σ} is an integer polynomial.
Proof: Using the remark before the lemma, we can find a nonnegative integer polynomial \( p \) such that \( p(\lfloor t \rfloor) \) bounds the values of \( |t| \) for every subterm \( t(x, u) \) occurring in \( \varphi \) and all possible values of the quantified variables \( u \). Then we can rewrite \( \varphi \) in the form

\[
Q_1 u_1 \leq p(|x|) \cdots Q_l u_l \leq p(|x|) \psi(x, u),
\]

where \( \psi \) is open. The next step is elimination of unwanted function symbols. Let \( |t| \) be a subterm of \( \psi \), and write \( \psi(x, u) = \psi'(x, u, |t|) \). Then \( \psi(x, u) \) is equivalent to

\[
\exists u \leq p(|x|) (|t| = u \land \psi'(x, u, u)).
\]

Using the axioms of BASIC\(^+\) and the definition of \( 2^u \), this is equivalent to

\[
\exists u \leq p(|x|) (\lfloor 2^u/2 \rfloor \leq t < 2^u \land \psi'(x, u, u)).
\]

Likewise,

\[
\psi(x, u, t \neq s) \leftrightarrow \exists u, v, w \leq p(|x|) (u = |t| \land v = |s| \land w = uv \land \psi(x, u, 2^w)),
\]

\[
\psi(x, u, 2^{\min\{t, s\}}) \leftrightarrow \exists u, v \leq p(|x|) (u = |s| \land v = \min\{t, u\} \land \psi(x, u, 2^v)),
\]

where we further eliminate \( |t| \) and \( |s| \) as above, and \( \min\{t, u\} \) in an obvious way. Applying successively these reductions, we can eventually write \( \varphi \) as

\[
Q_1 u_1 \leq p(|x|) \cdots Q_l u_l \leq p(|x|) \psi(x, u, 2^u),
\]

where \( \psi \) is an open formula in the language \( \{0, 1, +, \cdot, \div, \lfloor x/2 \rfloor, \leq\} \).

Claim 1 Let \( t(x) \) be a \( \langle \{0, 1, +, \cdot, \div, \lfloor x/2 \rfloor\} \rangle \)-term such that the nesting depth of \( \lfloor x/2 \rfloor \) in \( t \) is \( c \), and the number of occurrences of \( \div \) is \( r \). For every \( \bar{s} < 2^c \), there are integer polynomials \( g_1, \ldots, g_r \) and \( \{ f_{\bar{\alpha}} : \alpha_1, \ldots, \alpha_r \in \{0, 1\} \} \) such that BASIC\(^+\) proves

\[
\bigwedge_{i=1}^r (g_i(x) \geq 0)^{\alpha_i} \rightarrow t(2^c \bar{x} + \bar{s}) = f_{\bar{\alpha}}(x),
\]

where \( \varphi^1 = \varphi, \varphi^0 = \lnot \varphi. \)

Proof: By induction on the complexity of \( t \). For example, assume (26) holds for \( t \), and consider the term \( \lfloor t/2 \rfloor \). Let \( \bar{r} < 2 \), and assume that \( f_{\bar{\alpha}}(\bar{r}) \equiv \rho \pmod{2}, \rho \in \{0, 1\} \). Notice that all coefficients of \( f_{\bar{\alpha}}(2\bar{x} + \bar{r}) - \rho \) are even, so \( h_{\bar{\alpha}}(\bar{x}) = \frac{1}{2}(f_{\bar{\alpha}}(2\bar{x}) - \rho) \) is again an integer polynomial, and BASIC\(^+\) proves

\[
\bigwedge_{i=1}^r (g_i(2\bar{x} + \bar{r}) \geq 0)^{\alpha_i} \rightarrow t(2^{c+1}\bar{x} + (2^c\bar{r} + \bar{s})) = \left[ \frac{f_{\bar{\alpha}}(2\bar{x})}{2} \right] = \left[ \frac{2h_{\bar{\alpha}}(\bar{x}) + \rho}{2} \right] = h_{\bar{\alpha}}(\bar{x}).
\]

\( \square \) (Claim 1)
Claim 2  Every open formula $\psi(\vec{x})$ in the language $\langle 0, 1, +, -, |x/2|, \leq \rangle$ is equivalent to a formula of the form

$$\bigvee_{\vec{\sigma} < 2^c} \left( \bigwedge_{i = 1}^k (x_i \equiv \sigma_i \pmod{2^c}) \land \psi_{\vec{\sigma}}(\vec{x}) \right)$$

over BASIC$^+$, where each $\psi_{\vec{\sigma}}$ is a Boolean combination of integer polynomial inequalities.

Proof: Using Claim 1 and BASIC$^+$-provable uniqueness of the representation $x = 2^c y + \sigma$, $\sigma < 2^c$, we obtain an equivalent of $\psi$ in almost the right form except that $\psi_{\vec{\sigma}}$ is a Boolean combination of inequalities of the form

$$f(2^{-c}(\vec{x} - \vec{\sigma})) \geq 0,$$

where $f$ is an integer polynomial. If $d = \deg(f)$, $g(\vec{x}) = 2^{cd} f(2^{-c}(\vec{x} - \vec{\sigma}))$ is an integer polynomial, and the inequality above is equivalent to $g(\vec{x}) \geq 0$. \qed (Claim 2)

Let us apply Claim 2 to the formula $\psi(\vec{x}, \vec{u}, \vec{\sigma})$ in (25). Since BASIC$^+$ knows that $2^0 = 1$ and $2^{u+1} = 2 \cdot 2^u$, we can replace $2^u \equiv \sigma \pmod{2^c}$ with $2^u = \sigma \lor (u_i \geq c \land \sigma = 0)$. Moreover, $u_i \equiv \sigma \pmod{2^c}$ can be written as $\exists v \leq p(|\vec{x}|) (u_i = 2^c v + \sigma)$, and $x_i \equiv \sigma \pmod{2^c}$ can be moved outside the quantifier prefix. Thus, $\varphi(\vec{x})$ is equivalent to

$$\bigvee_{\vec{\sigma} < 2^c} \left( \bigwedge_{i = 1}^k (x_i \equiv \sigma_i \pmod{2^c}) \land Q_1 u_1 \leq p(|\vec{x}|) \cdots Q_t u_t \leq p(|\vec{x}|) \psi_{\vec{\sigma}}(\vec{x}, \vec{u}, \vec{\sigma}) \right),$$

where $\psi_{\vec{\sigma}}$ is a Boolean combination of integer polynomial inequalities. We can reduce $\psi_{\vec{\sigma}}$ to a single inequality using

$$\neg (f \geq 0) \leftrightarrow -f - 1 \geq 0,$$

$$f \geq 0 \land g \geq 0 \leftrightarrow \forall v \leq p(|\vec{x}|) (vf + (1-v)^2 g \geq 0),$$

assuming $p(|\vec{x}|) \geq 1$. \qed

Lemma 7.2 VTC$^0$ + IMUL proves the following for every constant $d$: if $\{f_u : u < n\}$ is a sequence of integer polynomials of degree $d$ (each given by a $(d+1)$-tuple of binary integer coefficients), and $a > d$ is a binary integer, there exists a double sequence $w = \{w_u : u < n, i \leq d + 1\}$ such that $0 = w_{u,0} < w_{u,1} < \cdots < w_{u,d+1} = a$ and $f_u(x)$ has a constant sign on each interval $[w_{u,i}, w_{u,i+1})$, that is,

$$\forall u < n \forall x \bigwedge_{i \leq d} (w_{u,i} \leq x < w_{u,i+1} \rightarrow (f_u(x) \geq 0 \leftrightarrow f_u(w_{u,i}) \geq 0)).$$

(27)

Proof: Using IOpen, $\{x < a : f(x) \geq 0\}$ is a union of at most $d$ intervals for every polynomial $f$ of degree at most $d$, i.e., VTC$^0$ + IMUL proves

$$\forall f \forall a > d \exists 0 = x_0 < \cdots < x_{d+1} = a \forall x \bigwedge_{i \leq d} (x_i \leq x < x_{i+1} \rightarrow (f(x) \geq 0 \leftrightarrow f(x_i) \geq 0)).$$

Now we would like to invoke $\Sigma^B_1 AC^R$ to find a sequence $w$ satisfying (27), but we cannot directly do that as the conclusion is only $\Pi^B_1$. 29
Let \( M \models \text{VTC}^0 + \text{IMUL} \), and \( R \) be its real closure. Quantifier elimination for RCF furnishes an open formula \( \vartheta \) in \( L_{\text{OR}} \) such that \( M \models \vartheta(x, y, a_0, \ldots, a_d) \) iff \( f(x) = \sum_{i \leq d} a_i x^i \) has no roots in the interval \( (x, y)_R \). By replacing \( f \) with \( 2f + 1 \) if necessary, we may assume \( f \) has no integral roots. Let \( \alpha_1 < \cdots < \alpha_c \), \( c \leq d \), be the list of all roots of \( f \) in \( (0, a]_R \), and let \( x_0, \ldots, x_{d+1} \in M \) be the sequence of integers 0, \( \lceil \alpha_1 \rceil \), \( \ldots \), \( \lceil \alpha_c \rceil \), \( a \) (which exist due to IOpen) with duplicates removed, and dummy elements added if necessary to make it the proper length. Then \( f \) has no roots in the intervals \( (x_i, x_{i+1} - 1)_R \). This means we can prove in \( \text{VTC}^0 + \text{IMUL} \) the statement

\[
\forall f \forall a > d \exists 0 = x_0 < x_1 < \cdots < x_{d+1} = a \bigwedge_{i \leq d} \vartheta(x_i, x_{i+1} - 1, f),
\]

which has the right complexity, hence we can use \( \Sigma_1^B - \text{AC}^R \) to derive the existence of a sequence \( w \) such that

\[
\forall u < n \left( w_{u,0} = 0 \land w_{u,d+1} = a \land \bigwedge_{i \leq d} (w_{u,i} < w_{u,i+1} \land \vartheta(w_{u,i}, w_{u,i+1} - 1, f_u)) \right).
\]

This implies (27).

\[ \square \]

**Theorem 7.3** The RSUV-translation of \( \text{BASIC}^+ + \Sigma_0^b(-, 2^{\min\{u,|x|\}})-\text{MIN} \), and a fortiori of \( T_2^0 \), is provable in \( \text{VTC}^0 + \text{IMUL} \).

**Proof:** Work in \( \text{VTC}^0 + \text{IMUL} \). It is straightforward but tedious to verify the axioms of \( \text{BASIC}^+ \). Let \( \varphi(x) \) be (the translation of) a \( \Sigma_0^b(-, 2^{\min\{u,|x|\}}) \) formula (possibly with other parameters), and \( a \) a binary number such that \( \varphi(a) \), we have to find the least such number. Since it is enough to do this separately on each residue class modulo \( 2^c \), we can assume using Theorem 7.3 that \( \varphi(x) \) is equivalent to

\[
Q_1 w_1 \leq n \cdots Q_l w_l \leq n \ f(x, \bar{u}, 2^m \bar{u}) \geq 0
\]

for \( x < a \), where \( n \) is a unary number, and \( f \) is a polynomial with binary integer coefficients. By Lemma 7.2, there is a sequence \( w \) such that

\[
w_{\bar{u},i} \leq x < w_{\bar{u},i+1} \rightarrow (f(x, \bar{u}, 2^m \bar{u}) \geq 0 \leftrightarrow f(w_{\bar{u},i}, \bar{u}, 2^m \bar{u}) \geq 0)
\]

for all \( x < a, \bar{u} \leq n \), and \( i \leq d \). As \( \text{VTC}^0 \) proves that every sequence of integers can be sorted, there is an increasing sequence \( \{w_j' : j < m\} \) whose elements include every \( w_{\bar{u},i} \). Consequently, the truth value of \( \varphi(x) \) is constant on each interval \( [w_j', w_{j+1}') \), and the minimal \( x < a \) satisfying \( \varphi(x) \), if any, is \( w_{j_0}' \), where

\[
j_0 = \min\{j < m : \varphi(w_j')\}.
\]

The latter exists by \( \Sigma_0^b(L_{\text{VTC}^0 + \text{IMUL}})-\text{COMP} \).  

\[ \square \]
We remark that the proof used nothing particularly special about division by 2, except that $BASIC$ conveniently includes the $\lceil x/2 \rceil$ function symbol and the relevant axioms. We could allow more general instances of division as long as the values of all denominators encountered when evaluating a $\Sigma^b_0$ formula on $[0,a]$ have a common multiple which is a length (unary number); in particular, Theorem 7.3 (along with an appropriate version of Lemma 7.1) holds for $\Sigma^b_0$ formulas in a language further expanded by function symbols for $\lceil x/2 \rceil\|y\|$ and $\lfloor x/\max\{1,\|y\|\}\rfloor$.

We formulated Theorem 7.3 for $VTC^0 + IMUL$ as we have been working with this two-sorted theory throughout the main part of the paper, however here it is perhaps more natural to state the result directly in terms of one-sorted arithmetic to avoid needless $RSUV$ translation. A theory $\Delta^b_1-CR$ corresponding to $TC^0$ was defined by Johannsen and Pollett [19], and shown $RSUV$-isomorphic to $VTC^0$ by Nguyen and Cook [24]. Recall also Johannsen’s theory $C^0_2[div]$ from Section 3.

**Corollary 7.4** The theories $\Delta^b_1-CR + IMUL$ and $C^0_2[div]$ prove $\Sigma^b_0(\lceil -2^{\min\{u,|x|\}} \rceil)-MIN$ and therefore $T^0_2$.

To put Theorem 7.3 in context, there has been a series of results to the effect that various subsystems of bounded arithmetic axiomatized by sharply bounded schemata are pathologically weak. Takeuti [30] has shown that $S^0_2 = \Sigma^b_0-PIND$ does not prove the totality of the predecessor function, and Johannsen [16] extended his method to show that $S^0_2$ in a language including $\lceil -, \lfloor x/2 \rfloor \rceil$, and bit counting does not prove the totality of division by three (or even of the $AC^0$ function $\lceil 2^{\lfloor x \rfloor}/3 \rceil$). Boughattas and Kołodziejczyk [4] have shown that $T^0_2 = \Sigma^b_0-IND$ does not prove that nontrivial divisors of powers of two are even, and by Kołodziejczyk [20], it does not even prove $3 \div 2^{\lfloor x \rfloor}$. These results also apply to certain mild extensions of $T^0_2$, nevertheless no unconditional independence result is known for $\Sigma^b_0-MIN$, or its subtheory $T^0_2 + S^0_2$.

What makes such separations possible is a lack of computational power. It is no coincidence that there are no result of this kind for two-sorted Zambella-style theories, where already the base theory $V^0$ proves the totality of all $AC^0$-functions: we can show $V^0(p) \not\subseteq V^0(q)$ for primes $p \neq q$ using the known lower bounds for $AC^0[p]$, but we have no independence results for stronger theories without complexity assumptions such as $AC^0[6] \neq PH$. This is directly related to the expressive power of sharply bounded formulas: while $\Sigma^B_0$ formulas can define all $AC^0$ predicates, the ostensibly quite similar $\Sigma^b_0$ formulas (that even involve the $TC^0$-complete multiplication function) have structural properties that preclude this, as witnessed by Mantzívis’s result. Indeed, the pathological behaviour of $T^0_2$ disappears if we slightly extend its language: as proved in [13], $T^0_2(\lfloor x/2 \rfloor) = PV_1$, and this can be easily extended to show $\Sigma^b_0(\lfloor x/2 \rfloor 2)-MIN = T^1_2$.

Theorem 7.3 formally implies only conditional separations: in particular, $PV_1 \not\subseteq \Sigma^b_0-MIN$ unless $P = TC^0$, and $\Sigma^b_0-MIN \not\subseteq T^1_2$ unless $PH = BH \subseteq TC^0/poly$ and $PLS = FTC^0$ (probably in $\Sigma^b_0-MIN$). However, heuristically it gives us more. If $\Sigma^b_0-MIN$ were a “computationally reasonable” theory, we would expect it to coincide with $T^1_2$ due to its shape, or at the very least to correspond to a class closer to $PLS$ than $TC^0$. Thus, Theorem 7.3 indicates that it
might be a pathologically weak theory in some way, and therefore amenable to unconditional independence results by means of a direct combinatorial construction of models in the spirit of [4, 20].

8 Conclusion

The weakest theory of bounded arithmetic in the setup of [33, 8] that can talk about elementary arithmetic operations on binary integers is $VTC^0$. We have shown that its strengthening $VTC^0 + IMUL$ proves that these operations are fairly well behaved in that they satisfy open induction. Despite that the theory $VTC^0 + IMUL$ corresponds to the complexity class $TC^0$ similarly to $VTC^0$, it is still an interesting problem what properties of integer arithmetic operations are provable in plain $VTC^0$. In view of Theorem 6.8 and Corollary 3.6, we have:

**Corollary 8.1** $VTC^0$ proves $I\text{Open}$ if and only if it proves $DIV$. □

**Question 8.2** Does $VTC^0$ prove $DIV$? In particular, does it prove the soundness of the division algorithm by Hesse et al. [11]?

While the analysis of the algorithm in [11] generally relies on quite elementary tools, its formalization in $VTC^0$ suffers from “chicken-and-egg” problems. For instance, the proof of Lemma 6.1, whose goal is to devise an algorithm for finding small powers in groups, assumes there is a well-behaved powering function, and uses its various properties to establish that its value is correctly computed by the algorithm. This is no good if we need the very algorithm to construct the powering function in the first place. Similarly, integer division is employed throughout Section 4. It is not clear whether one can circumvent these circular dependencies in $VTC^0$. On the other hand, the requisite operations such as division are available in $VTC^0 + IMUL$, which makes it plausible that $VTC^0 + IMUL$ can formalize the arguments.

We remark that it is not difficult to do division by standard integers in $VTC^0$. This means $VTC^0$ knows that binary integers form a $\mathbb{Z}$-ring, and in particular, they satisfy all universal consequences of $I\text{Open}$ by a result of Wilkie [32]. ($I\text{Open}$ itself is a $\forall\exists$-axiomatized theory, and likewise, $DIV$ is a $\forall\exists$ sentence.)

As explained in Section 7, our main result implies that $VTC^0 + IMUL$ (or better, the corresponding one-sorted theory $\Delta^b_1\text{-CR} + IMUL$) proves minimization for $\Sigma^b_0$ formulas in Buss’s language, which suggests that the theory axiomatized by $\Sigma^b_0\text{-MIN}$ is rather weak. Consequently, it might be feasible to unconditionally separate this theory from stronger fragments of $S_2$, nevertheless our argument gives no clue how to do that.

**Problem 8.3** Prove that $\Sigma^b_0\text{-MIN}$ is strictly weaker than $T^1_2$ without complexity-theoretic assumptions.

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