Classical Sphaleron Rate on Fine Lattices

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Abstract

We measure the sphaleron rate for hot, classical Yang-Mills theory on the lattice, in order to study its dependence on lattice spacing. By using a topological definition of Chern-Simons number and going to extremely fine lattices (up to $\beta = 32$, or lattice spacing $a = 1/(8g^2T)$) we demonstrate nontrivial scaling. The topological susceptibility, converted to physical units, falls with lattice spacing on fine lattices in a way which is consistent with linear dependence on $a$ (the Arnold-Son-Yaffe scaling relation) and strongly disfavors a nonzero continuum limit. We also explain some unusual behavior of the rate in small volumes, reported by Ambjørn and Krasnitz.
Baryon number is not a conserved quantity in the standard model. Rather, because of the anomaly, its violation is related to the electromagnetic field strength of the SU(2) weak group \[ N_G \frac{g^2}{32\pi^2} \epsilon_{\mu
u\alpha} \text{Tr} F^\mu\nu F^{\alpha\beta} = N_G \frac{g^2}{8\pi^2} F^a_i B^a_i, \] (1)

where \( N_G = 3 \) is the number of generations. The right-hand side of this equation is not surprisingly a total derivative, with an associated charge called the Chern-Simons number, \[ \frac{N_B}{N_G} = \frac{1}{N_G} \int d^3x J_B^0 = N_{CS} \equiv \text{(integer)} + \frac{g^2}{32\pi^2} \int d^3x \epsilon_{ijk} \left( F^a_{ij} A^a_k - \frac{g}{3} f_{abc} A^a_i A^b_j A^c_k \right), \] (2)

\[ N_{CS}(t_1) - N_{CS}(t_2) \equiv \int_{t_2}^{t_1} dt \int d^3x \frac{g^2}{8\pi^2} E^a_i B^a_i. \] (3)

Chern-Simons number \( N_{CS} \) has topological meaning; its change in a vacuum to vacuum process is the (integer) second Chern class of the gauge connection. Note that, according to the first equation, the total baryon number \( N_B \) need only be an integer in vacuum, when \( N_{CS} \) is an integer. The baryon number in vacuum fixes the constant of integration in the definition of \( N_{CS} \).

In vacuum, baryon number can be violated by a vacuum fluctuation large enough to have a nonzero integer Chern-Simons number. The efficiency of baryon number violation by this mechanism is totally negligible; but at a sufficiently high temperature baryon number violation can proceed by thermal excitations which change \( N_{CS} \), and the rate for such a process is not necessarily very small. This can have very interesting cosmological significance, since it complicates GUT baryogenesis mechanisms and opens the possibility of baryogenesis from electroweak physics alone. This motivates a more careful investigation of baryon number violation in the standard model at high temperatures.

The baryon number violation rate relevant in cosmological settings can be related by a fluctuation dissipation relation to the diffusion constant for Chern-Simons number, \[ \Gamma \equiv \lim_{V \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{\langle (N_{CS}(t) - N_{CS}(0))^2 \rangle}{V t}, \] (4)

where the angular brackets \( \langle \rangle \) represent a trace over the thermal density matrix. This quantity is called the sphaleron rate for historical reasons. There has been some controversy not only to its size in the symmetric electroweak phase (which closely resembles pure Yang-Mills theory, an approximation we will make from here on) but even to its parametric dependence. On purely dimensional grounds, at high temperature it must scale as \( T^4 \), but the dependence on the coupling constant has been more controversial. Since the natural non-perturbative length scale of hot Yang-Mills theory at weak coupling is \( \sim 1/(g^2T) \) it was long believed that \( \Gamma \propto \alpha^{-4}_w T^4 \). In this case, \( \Gamma \) could equal its value in classical Yang-Mills theory, as originally suggested by Grigoriev and Rubakov.

\[^1\text{There is also a contribution from the hypercharge fields, but it will not be relevant here because the topological structure of the abelian vacuum does not permit a permanent baryon number change.}\]
More recently Arnold, Son, and Yaffe (ASY) have argued that while the natural length scale for hot, weakly coupled Yang-Mills theory is $\sim 1/(g^2 T)$, the natural time scale is different; because of interactions between the nonperturbative infrared excitations and essentially perturbative but very numerous UV excitations, the time evolution of IR Yang-Mills fields should be overdamped and the natural time scale for their evolution should be $\sim 1/(g^4 T)$, or, restoring $\hbar$, $\sim 1/(\hbar g^4 T)$. The appearance of $\hbar$ in this expression precludes any simple correspondence between the classical theory and the quantum theory. They then argue that, since only the nonperturbative IR fields can contribute to the diffusion of $N_{CS}$, the correct parametric behavior for $\Gamma$ is $\Gamma \propto \alpha_5^5 T^4$. The analogous expression in the classical theory has some UV regulator serving the role of $\hbar$ in the quantum theory; for instance, for Yang-Mills theory on a lattice under the standard lattice action, the UV regulator scale is the inverse of the lattice spacing $a$, so the natural time scale should be of form $1/(g^4 a T^2)$, leading to $\Gamma \propto a a_5^5 T^5$. Subsequently, Bödeker showed that the coefficient of the $\alpha_5^5$ law should contain a further logarithmic dependence on $g^2$, or, on the lattice, on $g^2 a T$.

The arguments of ASY are still considered somewhat controversial. While numerical simulations of classical Yang-Mills theory, supplemented with added degrees of freedom intended to serve the role of the “hard” quantum UV degrees of freedom, clearly support their arguments, results for $\Gamma$ in pure classical Yang-Mills theory on the lattice have never convincingly displayed linear scaling in lattice spacing. Furthermore, the more recent work of Ambjørn and Krasnitz finds two other results which are problematic to interpret if Arnold, Son, and Yaffe are correct; they find overly strong lattice spacing dependence for $\Gamma$ in small volumes, and they find unexpectedly rapid falloff for unequal time, Coulomb gauge fixed correlators.

We will not address the question of unequal time, Coulomb gauge correlators here, except to note that we believe such correlators should show a strong volume dependence even at fixed $k$. Since the physical volume was varying along with the lattice spacing in Ambjørn and Krasnitz’ results, it is difficult to disentangle these two dependencies. We leave settling this problem to future work, but it is our general belief that such correlators will not prove very useful probes of infrared dynamics.

This work is intended to answer the other two questions about the applicability of the ASY picture to classical Yang-Mills theory. First, we present results for $\Gamma$ on a much wider range of lattice spacings, including much smaller spacings $a$, than has previously been done. Although the results at fairly large $a$ show weak lattice spacing dependence, as $a$ becomes smaller a strong $a$ dependence sets in, which is incompatible with $\alpha_4^4$ scaling, but is in accordance with expectations if Arnold, Son, and Yaffe are correct. We also present a re-analysis of the behavior of $\Gamma$ in a small, fixed volume. When care is taken to ensure that the physical volume really remains fixed as the lattice spacing is varied, we find $\Gamma$ to depend on $a$ slightly more weakly than in large volumes. This is also expected in the ASY picture.

2 Expected scaling behaviors

To explain the different proposed scaling behaviors for $\Gamma$, we will briefly review the thermodynamics of classical Yang-Mills theory. The partition function for classical Yang-Mills theory is equivalent to the path integral for three dimensional, quantum Yang-Mills the-
ory with an added adjoint scalar, as originally shown by Ambjørn and Krasnitz [11]. The classical Yang-Mills partition function, absorbing $g$ into the definition of the connection so covariant derivatives are $D_i = \partial_i + iA_i$, is

$$Z = \int D A_i^a D E_i^a \delta((D_i E_i)^a) \exp(-H/T),$$

$$H = \int d^3 x \left( \frac{1}{4g^2} F_{ij}^a F_{ij}^a + \frac{1}{2} E_i^a E_i^a \right).$$

The delta function enforces Gauss’ law. It is convenient to rewrite it by introducing a Lagrange multiplier,

$$\delta((D_i E_i)^a) = \int D A_0^a \exp \left( \frac{i}{g} \int A_0^a (D_i E_i)^a \right),$$

where $A_0$ has the same normalization as $A_i$. The integral over $E$ is then Gaussian. Performing it yields the partition function

$$Z = \int D A_i D A_0 \exp(-H/T),$$

$$H/T = \int d^3 x \left( \frac{1}{4g^2T} F_{ij}^a F_{ij}^a + \frac{1}{2g^2T} (D_i A_0)^a (D_i A_0)^a \right).$$

Physically the $A_0$ field corresponds to the time component of the connection, but if we choose to interpret the result as a path integral for a 3-D quantum field theory then the $A_0$ field just corresponds to some massless adjoint scalar.

The form for the partition function coincides, except for the absence of a mass term for the $A_0$ field, with the path integral for the full quantum theory in the dimensional reduction approximation [14, 15, 16], and so it quite accurately reproduces the thermodynamics of infrared Yang-Mills fields. This motivates the belief that the dynamics of the full Yang-Mills theory will also coincide with the dynamics of the classical theory. Looking carefully at Eq. (9), we observe that $g^2$ and $T$ only enter in the combination $g^2T$. Now $\Gamma$ has engineering dimension 4, so since $T$ is the only dimensionful quantity in hot Yang-Mills theory at weak coupling, we must have $\Gamma \propto T^4$. The “naive” scaling argument for $\Gamma$ corresponds to requiring that each $T$ carry a factor of $g^2$, as is motivated by the form of the partition function.

The problem with this argument, and with the argument that the dynamics of the classical and quantum theories should correspond, is that it neglects the effects of UV divergences. For thermodynamic quantities the UV divergences of the classical theory arise from a finite number of graphs and can be absorbed by counterterms. However, from the way that the $A_0$ field arose above, the classical theory is obliged to have a zero bare mass squared for this field. Hence the physical mass squared will approximately equal the linear one loop divergent contribution,$ m_0^2 \propto g^2 T \Lambda$, with $\Lambda$ some UV cutoff. In particular, on the lattice and using the conventional (Wilson) action, $1/a$ serves the role of $\Lambda$, and [14]

$$m_0^2 = \frac{\Sigma g^2 T}{\pi a}, \quad \Sigma = 3.175911536 \ldots$$

[1] The result in the reference differs by a factor of 5/4 because it includes a contribution from a Higgs field, which is absent here since we treat pure Yang-Mills theory.
The thermodynamics of the gauge fields $A_i$ do not care about this divergence, since their thermodynamics have a finite limit as $m^2_D \to \infty$. Therefore the argument that the only natural length scale for the $A_i$ fields is $1/g^2 T$, remains valid in the presence of UV divergences. However, since the dynamics of the $A_i$ and $A_0$ fields are intertwined, it is not at all clear that unequal time phenomena involving the $A_i$ fields should be UV regulation insensitive.

Arnold, Son, and Yaffe [7] examined the propagator of the full quantum theory at soft momentum, including the UV influences at leading order in the coupling by including the hard thermal loop (HTL) self-energy contribution [17]. For spatial momenta $p$ in the parametric regime $p \ll gT$ they conclude that the evolution of the transverse (magnetic) degrees of freedom responsible for baryon number violation is overdamped, of form

$$\frac{dA(p)}{dt} = -\frac{4p^3}{\pi m_D^2} A(p) + \text{(noise)}.$$ (11)

The perturbative treatment which gives this result holds provided that $p \gg g^2 T$, but it breaks down when $p$ is of order $g^2 T$, the scale which we are interested in. Ignoring this difficulty and applying it for $p \sim g^2 T$, we find that the time scale for significant change in $A$ is $\sim m_D^3/g^6 T^3$, which in the continuum theory is $\sim 1/(g^4 T)$ and on the lattice is $\sim 1/(g^4 a T^2)$.

Of course it is problematic to apply Eq. (11) beyond its range of applicability. However, for $g$ sufficiently small, we may apply it at the scale $p \sim g^2 T$ to find that the natural time scale here is $(1/t) \sim g^{4-3\epsilon} T$. Since the time scale for field evolution at $p \sim g^2 T$ cannot be faster than that at a larger value of $p$, Eq. (11) does place a bound on the natural time scale for fields with $p \sim g^2 T$; the time scale cannot differ from the estimate $\sim 1/(g^4 T)$ by a nonzero power of $g$. Thus, Eq. (11) is enough to ensure the ASY result up to corrections weaker than any power of $g$. A much more careful study of the HTL effective theory by Bödeker [8, 18] finds that log$(1/g)$ corrections do occur; however, they prove to be numerically small [19].

Note however that the ASY argument is a parametric treatment which relies on a large separation between the $gT$ and $g^2 T$ scales. Numerical results, for instance for the subleading contributions to the Debye screening length [20], suggest that $g^2$ (or, on the lattice, $g^2 a T$), may need to be fairly small before such a separation of scales really exists. Therefore, we might expect that the ASY scaling behavior only sets in at reasonably small lattice spacings $a$. This motivates the study of the sphaleron rate on very fine lattices, which we take up in the next section.

3 Results: large lattices, fine spacings

To address the question: “How does $\Gamma$ depend on lattice spacing $a$ when $a$ is small?” we track $N_{CS}$ on the lattice. We use the conventional Kogut-Susskind lattice action [21], which is the Minkowski time analog of the Wilson action for Euclidean lattice Yang-Mills theory. We use the same discrete implementation for the time evolution as in Ambjørn et. al. [22] and almost all subsequent work. Besides the size of the lattice there is one variable, $\beta_L$, which is a reciprocal temperature in lattice units. At tree level it is $\beta_L = 4/(g^2 a T)$, but this relation receives radiative corrections because the UV lattice and continuum fields behave differently. These are treated, for the case $m_D^2 \ll 1/a^2$, in [23] and extended to larger $m_D$.


in Appendix B of [10]. We use the expression from there for the one loop improved relation between $\beta_L$ and lattice spacing $a$,

$$
\beta_L = \frac{4}{g^2aT} + \left( \frac{1}{3} + \frac{37\xi}{6\pi} \right) - \left[ \left( \frac{4}{3} + \frac{2a^2m_D^2}{3} + \frac{a^4m_D^4}{18} \right) \frac{\xi(am_D)}{4\pi} - \left( \frac{1}{3} + \frac{a^2m_D^2}{18} \right) \frac{\Sigma(am_D)}{4\pi} \right]
$$

\[
\beta_L \approx \frac{4}{g^2aT} + 0.6 \quad \text{or} \quad a \approx \frac{4}{(\beta_L - 0.6)g^2T},
\]

where $\xi = 0.152859 \ldots$ and the functions $\xi(x)$ and $\Sigma(x)$ are defined in [10]; the approximation that the correction term is 0.6 is good to 10% for all $a$ considered here but we will use the full expression. In this section the difference between the naive and improved match will not be important, but in the next section it is essential. Henceforth, when we refer to $\beta_L$ we will mean the “unimproved” quantity appearing in Eq. (12); while the quantity $\beta$ will mean $\beta \equiv 4/(g^2aT)$ using the improved relation for $a$.

The measurement of Chern-Simons number deserves some comment. Early work on Chern-Simons number diffusion [22, 11, 24, 25] used “naive” definitions of $N_{CS}$ in which the right hand side of Eq. (3) is implemented as the integral of a local operator over the unmodified lattice fields. This approach is not topological because such a local operator on the lattice is never a total derivative, as it should be for $N_{CS}$ to have topological meaning. Because of this, such a definition of $N_{CS}$ shows diffusive behavior even when there is no true diffusion of baryon number occurring [24, 7]. The response to true topology change can also get renormalized by UV fluctuations. The latter problem gets less severe as $a$ is reduced, but we find that the “spurious” diffusion per physical 4-volume grows as $a^{-1}$ and is therefore disastrous. Therefore we should use a topological definition of $N_{CS}$.

Technically topology is not well defined for lattice fields, but it can be well defined on a restricted class of lattice fields which are sufficiently “smooth” [27, 28, 29]. In our context topology is well defined for suitably small lattice spacing; in practice there is no problem if $\beta \geq 6 \ (a < 2/3g^2T)$, which will be the case for almost all lattices we consider. For the current application, two topological means have been developed; the “slave field” method [12], similar to the method of Woit [28]; and “calibrated cooling” [30], an improvement on the field smearing proposal of Ambjørn and Krasnitz [13]. The “slave field” method is numerically efficient but noisy, which means in practice that a longer numerical evolution is needed to get good statistics. Therefore we will use “calibrated cooling”. The philosophy of the method is that the topological content of the connection cannot be modified by small local changes; therefore we may “smear” the connection, removing UV noise which is responsible for the poor performance of the “naive” measurement method. After the smearing we measure $N_{CS}$ by integrating an $O(a^2)$ improved local operator for $E_i B_i$, and we cure the slight residual error in the algorithm by “calibrating” it with occasional coolings all the way to vacuum; the full details can be found in [30].

The problem with the above approach is its numerical cost; measuring $N_{CS}$ as a function of time is much more expensive than generating the Hamiltonian trajectory. This is only really a problem on very fine lattices, as numerical cost rises as $a^{-4}$. However, in this case the topology changing configurations we are after are many, many lattice spacings across and are highly over-resolved; no topological information is lost by “blocking” the lattice and then measuring $N_{CS}$. That is, we construct a “blocked” lattice with $B$ times the original lattice
We do this first, before applying any smearing. This introduces some white noise into $N_{\text{CS}}$ but does not change the diffusion at all, which means that results for $\Gamma$ will be unchanged. Figure 1 shows a test of blocking in which we track $N_{\text{CS}}$ for the same Hamiltonian trajectory, with and without blocking. The difference is small and purely spectrally white. On the other hand, the difference in numerical effort is enormous; without blocking, measuring $N_{\text{CS}}$ took 5 times as much CPU time as updating the fields, while with blocking it took around 10% as much. We will block for all data with $a \leq 1/3g^2T$ in this work, which means that the CPU time taken to measure $N_{\text{CS}}$ is negligible for all the most numerically intensive cases.

We should also be sure to use a large enough volume to eliminate finite volume systematics (effectively, to perform the $V \to \infty$ limit in Eq. 4). To do so we measure $\Gamma$ as a function of $L = V^{1/3}$ at a fixed lattice spacing, $a = 1/2g^2T$. The result is shown in Figure 2. The dependence on $L$ is in general accordance with the results of [11], except at small volumes where ours go to zero and theirs do not because of the spurious UV contributions in their definition of $N_{\text{CS}}$. We will use $L = 10/g^2T$ ($N = 2.5\beta$) for all large volume results.
Figure 2: Volume dependence of the sphaleron rate in cubic volumes $L$ on a side with periodic boundary conditions and $a = 1/2g^2T$ ($\beta = 8$). Large volume behavior is obtained by $L = 8/g^2T$. Around $L = 5/g^2T$ the sphaleron rate falls off abruptly and it is virtually zero already at $L = 3/g^2T$.

| $N$ | $L \times g^2T$ | $\Gamma/\alpha^4T^4$ |
|-----|-----------------|---------------------|
| 6   | 3.0             | 0.0023 ± 0.0016     |
| 7   | 3.5             | 0.029 ± 0.008       |
| 8   | 4.0             | 0.112 ± 0.012       |
| 9   | 4.5             | 0.276 ± 0.015       |
| 10  | 5.0             | 0.545 ± 0.035       |
| 12  | 6.0             | 1.18 ± 0.04         |
| 14  | 7.0             | 1.40 ± 0.05         |
| 16  | 8.0             | 1.61 ± 0.07         |
| 20  | 10.0            | 1.68 ± 0.03         |

Table 1: Volume dependence of the sphaleron rate for an $a = 1/2g^2T$ lattice. The volume dependence is very strong at $L = 4/g^2T$ but is consistent with zero above $L = 8/g^2T$. 
Figure 3: Chern-Simons number diffusion constant plotted against lattice spacing $a$, and a linear fit for the last 5 points. Around $a \sim 0.5/g^2T$ the dependence is somewhat weak, but at larger $a$ a rapid falloff is evident.

Table 2: Sphaleron rate $\Gamma$ as a function of lattice spacing, $\beta = 4/g^2aT$. On the finer lattices an approximately linear dependence on $a$ (or $1/\beta$) becomes apparent. For those who prefer it we also include $\Gamma$ converting to physical units using the naive relation between $\beta_L$ and $a$. 

| $a \times g^2T$ | $\beta$ | $N$ | time/$a$ | $\Gamma/(\alpha T)^4$ | same, converting using $\beta_L$ |
|-----------------|--------|-----|----------|------------------------|---------------------------------|
| 1.00            | 4      | 10  | 40000    | 2.123 ± 0.075          | 3.78 ± 0.12                     |
| 0.67            | 6      | 16  | 58000    | 1.800 ± 0.055          | 2.66 ± 0.08                     |
| 0.50            | 8      | 20  | 157000   | 1.683 ± 0.032          | 2.25 ± 0.04                     |
| 0.33            | 12     | 32  | 170000   | 1.336 ± 0.036          | 1.62 ± 0.04                     |
| 0.31            | 13     | 32  | 188000   | 1.264 ± 0.035          | 1.51 ± 0.04                     |
| 0.25            | 16     | 40  | 105000   | 1.123 ± 0.046          | 1.30 ± 0.05                     |
| 0.167           | 24     | 60  | 234400   | 0.853 ± 0.031          | 0.94 ± 0.03                     |
| 0.125           | 32     | 80  | 202000   | 0.626 ± 0.030          | 0.67 ± 0.03                     |
Our results at large volumes are presented in Table 2 and plotted in Figure 3. These constitute the main result of this paper. While $\Gamma$ is not a strong function of lattice spacing around $a \simeq 0.5/g^2T$ ($\beta \simeq 8$), it then turns over and falls roughly linearly in $a$ at finer lattices. This indicates that the lattice spacing needed before the ASY scaling behavior sets in is around $\beta = 12$ ($a \simeq 1/3g^2T$).

If we insist on believing that the small $a$ scaling behavior is of form $c_1 + c_2g^2aT$ with $c_2$ representing a correction to scaling, then a fit to the points with $a \leq 1/3g^2T$ ($\beta \geq 12$) gives $c_1 = 0.257 \pm 0.044$ and $c_2 = 3.29 \pm 0.18$ ($\xi^2/\nu = 5.4/3$). The “correction” to scaling only becomes subdominant below $a = 1/12.8g^2T$ ($\beta > 51$). The original motivation for believing in a finite small $a$ limit for $\Gamma$ is the belief that the UV lattice behavior is not important to the infrared dynamics. Such an enormous correction to scaling contradicts this belief. This makes it very difficult to reconcile our data with a finite small $a$ limit for $\Gamma$.

On the other hand, if ASY are correct, it makes more sense to plot our results as $\Gamma/(\alpha^5aT^5)$, as we do in Figure 4. The extrapolation to small $a$ looks much better behaved here. If we fit $\Gamma$ to the form $\Gamma = \alpha^5aT^5(c_1 + c_2g^2aT)$ we get $c_1 = 74.5 \pm 3.4$ and $c_2 = -73 \pm 12$ ($\xi^2/\nu = 1.6/3$), which means the correction to scaling comes of order 1 at $a = g^2T$ ($\beta = 4$).

In fact, as Bödeker has shown, we should not expect a finite intercept in this figure, there
should be a weak logarithmic divergence as $a \to 0$. In the continuum its amplitude is

$$\log \text{part of } \Gamma = (10.7 \pm .7) \frac{g^2 T^2}{m_D^2} \log \left( \frac{m_D}{g^2 T} \right) \alpha^5 T^4. \tag{13}$$

The appearance of $1/m_D^2$ is changed somewhat on the lattice; $m_D^2$ is reduced by a weighted average over $k$ of the group velocity under lattice dispersion relations, which is about 0.68. Combining this with the expression for $m_D^2$ on the lattice, Eq. (10), gives

$$\log \text{part of } \Gamma = [7.8 \pm .5] \log \left( \frac{1}{g^2 a T} \right) \alpha^5 a T^5. \tag{14}$$

Strictly speaking such logarithmic behavior only pertains in the regime where $\log(1/g^2 a T) \gg 1$, which is probably not satisfied at any conceivable lattice spacing. Nevertheless we perform the fit to see what happens. The fit in Fig. [4] which curves assumes $\Gamma = \log \text{part of } \Gamma + \alpha^5 a T^5(c_1 + c_2 g^2 a T)$, which still has only two free fitting parameters. The best fit value is $c_1 = 54.6 \pm 3.4$, $c_2 = -39 \pm 12$ ($\xi^2/\nu = 2.1/3$). In this case the leading $c_1$ behavior dominates over the scaling correction out to $a = 1.4 g^2 T$ ($\beta = 3$). Note that the data do not yet justify either believing or disbelieving in the log behavior. This is because the coefficient of the log is numerically small.

4 Small volumes

Ambjørn and Krasnitz report another puzzling problem with the ASY picture. They analyzed the dependence of $\Gamma$ on lattice spacing at a fixed, small volume, small enough that $\Gamma$ is much smaller than its large volume limit. The idea was that, in such a small volume, $N_{CS}$ fluctuates about an integer value with occasional, abrupt changes from integer to integer, which can be identified even with the naive definition of $N_{CS}$. Therefore, $N_{CS}$ can be tracked topologically without needing a true topological definition. Their results, replotted here in Figure 5, are puzzling. Not only is there a strong lattice spacing dependence in evidence; it is too strong. The rate appears to fall off faster than $\Gamma \propto a$, and certainly faster than the rate falls off in large volumes, where the corrections to scaling at finite $a$ make the $a$ dependence somewhat weaker than linear.

To see why this result does not jive with the ASY picture, we will review again the basic ASY argument. Examining the propagator for the gauge field at momentum $p$ gives an equation of motion for $A(p)$ which, viewed on suitably long time scales, is approximately

$$\frac{d^2 A(p)}{dt^2} + \frac{\pi m_D^2}{4p} \frac{dA(p)}{dt} = -p^2 A(p). \tag{15}$$

The damping term, with the single derivative, is more and more important as $p$ gets smaller. We expect that, in a constrained volume, the field configurations responsible for changing $N_{CS}$ are spatially smaller than in a large volume. Therefore they are composed of excitations of larger $p$, and should be more weakly damped. In particular it should take a larger value of $m_D^2$ (smaller $a$) for the overdamped regime to apply, and the corrections to linear in $a$ scaling should be larger.
Figure 5: Dependence of $\Gamma$ on $a$ in a small volume, $L \simeq 4/g^2T$. Left, the results Ambjørn and Krasnitz reported in [13]. Right, results when more care is taken to keep the lattice volume fixed in physical units. The overly strong $a$ dependence in the Ambjørn-Krasnitz data is due to small changes in the physical volume, which $\Gamma$ depends on very strongly.

The problem lies in the data. Ambjørn and Krasnitz worked in a lattice with $N = \beta_L$, which at the unimproved level means $L = 4/g^2aT$. However, they used the tree relation between the lattice spacing and the reciprocal temperature $\beta_L$. As can be seen from Eq. 12, the true lattice spacing is larger than the tree relation indicates, and the error is worse as the lattice is made coarser. Therefore, their larger $a$ lattices possessed more physical volume than their smaller $a$ lattices. The effect is fairly small; the difference in linear dimension between their $\beta_L = 10$ and $\beta_L = 20$ lattices is only about 3%. However, as Fig. 2 shows, $\Gamma$ is a very strong function of volume in the regime where they were working. Around $N = \beta (L = 4/g^2T)$, the data in the figure give roughly $d(\log \Gamma)/d(\log L) \simeq 10$, so a 3% change in length could lead to a 30% change in $\Gamma$, which is significant. In fact this effect could be as large as or larger than the $a$ dependence due to hard thermal loop dynamics.

To fix this problem we recompute $\Gamma$ in a fixed, small volume, but using the improved relation between the lattice spacing and the reciprocal temperature. We choose to use a volume equivalent (at the improved level) to the volume of Ambjørn and Krasnitz’ finest volume, $\beta_L = 20$ and $N = 20$, which, using the improved relation, gives $L = 4.114/g^2T$. The results are presented in Figure 3 and include a re-calculation of the finest lattice point, using a topological definition of $N_{CS}$ (rather than counting integer winding changes by eye from the old “unimproved” $N_{CS}$ definition). The new data show a lattice spacing dependence similar to the large volume case, although to make a good comparison we would need data at smaller $a$ (larger $\beta$). The behavior of $\Gamma$ in small volumes is in accord with what we expect if the ASY argument is correct.
5 Conclusion

Chern-Simons number diffusion in pure Yang-Mills theory follows the Arnold-Son-Yaffe scaling behavior, $\Gamma \propto a$; however it takes a fairly fine lattice to demonstrate this in a convincing way. The same behavior, with similar corrections to scaling, is observed in small volumes, but only after taking care to keep the physical volume constant beyond leading (tree) level while the lattice spacing is varied.

The original goal of measuring $\Gamma$ on the lattice was to determine the rate at which a baryon number excess, present before the electroweak phase transition, would be erased. We can do this with the results of this paper by using Arnold’s study of the matching between the lattice and continuum theories \[^8\]. He shows that $\Gamma$ on the lattice matches the continuum value when

\[(.68 \pm .20)m_D^2(\text{latt}) = m_D^2(\text{continuum}),\]

where the error is all systematic, but the error estimate is considered generous \[^8\]. In the minimal standard model (MSM), at leading order $m_D^2 = (11/6)g^2 T^2$, which for $g^2 = 0.42$ and using Eq. (14), means the MSM value is obtained on a lattice with $a = .157/g^2 T$, or $\beta = 25.4$. Our data at $\beta = 24$ give $\Gamma = 0.85a_4T^4$, with an error insignificant compared to the 30% error estimate in the lattice to continuum matching procedure. Since $\Gamma \propto a$ is well satisfied in this regime, our estimate for the Standard Model value of $\Gamma$ is

\[\Gamma(\text{MSM}) = (0.82 \pm .24)a_4T^4 \quad \text{or} \quad \Gamma = (45 \pm 13)\left(\frac{g^2T^2}{m_D^2}\right) a_5T^4,\]

where the latter form shows the correct parametric dependence on the Debye mass. This result is in good agreement with results obtained when classical Yang-Mills theory is supplemented with particle degrees of freedom \[^10\] to induce the hard thermal loop effects. Therefore the diffusion constant for $N_{CS}$ in classical Yang-Mills theory is consistent with both analytic expectations and numerical results obtained by explicit inclusion of hard thermal loops.

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