Abstract. It has been proved by various authors that a normalized, 1-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$ is a nullcorrelation bundle, while a normalized, 2-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$ is an instanton bundle of charge 2. We find that the same is not true for 3-Buchsbaum rank 2 vector bundles on $\mathbb{P}^3$, and propose a conjecture regarding the classification of such objects.

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Introduction

A coherent sheaf $E$ on $\mathbb{P}^3$ is said to be $p$-Buchsbaum if $p$ is the minimal power of the irrelevant ideal which annihilates $H^1_*(E)$. The complete list of $p$-Buchsbaum rank 2 bundles on $\mathbb{P}^3$ for $p \leq 2$ has been established by several authors, see for example [7, 9, 14, 15, 16]. More precisely, we have the following.

Theorem 1. Let $E$ be a normalized $p$-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$. Then

- $p = 0$ if and only if $E$ is direct sum of line bundles;
- $p = 1$ if and only if $E$ is a null correlation bundle, i.e. an instanton bundle of charge 1;
- $p = 2$ if and only if $E$ is an instanton bundle of charge 2.

After examining this list, two questions naturally arise. First, is every rank 2 instanton bundle of charge $k$ on $\mathbb{P}^3$ $k$-Buchsbaum? Second, since every bundle is $p$-Buchsbaum for some sufficiently high $p$, for which values of $p$ can we find a $p$-Buchsbaum rank 2 bundle which is not instanton?

The goal of this paper is to provide partial answers to these questions. In particular, we show that every rank 2 instanton bundle of charge 3 is 3-Buchsbaum. However, this is false for instantons of higher charge. On the other
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hand, we show that the generic instanton of charge 4 or 5 is also 3-Buchsbaum. In addition, we provide an explicit example of a 3-Buchsbaum bundle of rank 2 which is not an instanton, and conjecture that every 3-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is one of these.

1. Preliminaries

In this section we will fix the notation and recall the basic definitions used throughout this paper.

1.1. Buchsbaum sheaves

Let $K$ be an algebraically closed field of characteristic zero. Let us denote by $S = K[x_0, x_1, x_2, x_3]$ the ring of polynomials in four variables, so that $\mathbb{P}^3 := \text{Proj}(S)$, and let $m = (x_0, x_1, x_2, x_3)$ denote the irrelevant ideal.

Let $V$ be a $K$-vector space of dimension $m + 1$, with $V^*$ denoting its dual. The projective space $\mathbb{P}(V) = \mathbb{P}^m$ is understood as the set of equivalence classes of $m$-dimensional subspaces of $V$, or, equivalently, the equivalence classes of the lines of $V^*$.

Given a coherent sheaf $E$ on $\mathbb{P}^3$, consider the following graded $S$-module:

$$H^1_\ast(E) = \bigoplus_{n \in \mathbb{Z}} H^1(E(n)).$$

**Definition 1.1.** A coherent sheaf $E$ on $\mathbb{P}^3$ is said to be $p$-Buchsbaum if and only if $p$ is the minimal power of the irrelevant ideal which annihilates the $S$-module $H^1_\ast(E)$, i.e.

$$p = \min \{ t \mid m^t H^1_\ast(E) = 0 \}.$$ 

In this work, we will only consider locally free sheaves on $\mathbb{P}^3$.

1.2. Monads and regularity

Recall that a monad on a projective variety $X$ of dimension $n$ is a complex of locally free sheaves on $X$ of the form

$$M_\bullet : A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

such that the map $\alpha$ is injective and the map $\beta$ is surjective. It follows that $E := \ker \beta / \text{Im} \alpha$ is the only nontrivial cohomology of the complex $M_\bullet$. The coherent sheaf $E$ is called the cohomology of $M_\bullet$; it is locally free if and only if the map $\alpha$ is injective in every fiber.

The monad $M_\bullet$ is called a Horrocks monad if, in addition:
i) $A$ and $C$ are direct sum of invertible sheaves,

ii) $H^1(B) = H^{n-1}(B) = 0$.

Furthermore, the monad is also called minimal if it satisfies

iii) no direct sum of $A$ is isomorphic to a direct sum of $B$,

iv) no direct sum of $C$ is the image of a line subbundle of $B$.

Let us recall the following result on minimal Horrocks monads, cf. [12, Theorem 2.3].

**Theorem 1.2.** Let $X$ be an arithmetically Cohen–Macaulay variety of dimension $n \geq 3$, and let $E$ be a locally free sheaf on $X$. Then there is a 1-1 correspondence between collections

$$\{n_1, \ldots, n_r, m_1, \ldots, m_s\}$$

with $n_i \in H^1(E^\vee \otimes \omega_X(k_i))$ and $m_j \in H^1(E(-l_j))$

for integers $k_i$'s and $l_j$'s, and equivalence classes of Horrocks monads of the form

$$M_\bullet: r \bigoplus_{i=1}^r \omega_X(k_i) \xrightarrow{\alpha} F \xrightarrow{\beta} s \bigoplus_{j=1}^s \mathcal{O}_X(l_j),$$

whose cohomology is isomorphic to $E$.

Moreover, the correspondence is such that $M_\bullet$ is minimal if and only if the elements $m_j$ generate $H^1_*(E)$ and the elements $n_i$ generate $H^1_*(E^\vee \otimes \omega_X)$ as modules.

Recall that a coherent sheaf $E$ on $\mathbb{P}^n$ is said to be $m$-regular in the sense of Castelnuovo–Mumford if $H^i(\mathbb{P}^n, E(m-i)) = 0$ for $i > 0$. Costa and Miró-Roig studied in [3] the Castelnuovo–Mumford regularity of the cohomology of a certain class of monads which include monads of the following form:

$$\mathcal{O}_{\mathbb{P}^3}(-l)^{\oplus k} \xrightarrow{\alpha} \bigoplus_{j=1}^{2+2k} \mathcal{O}_{\mathbb{P}^3}(b_j) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(d)^{\oplus k},$$

where $l, k, c \geq 1$ and $-l < b_1 \leq \cdots \leq b_{2+2k} < d$. Specializing [3, Theorem 3.2] to monads of the form (1), one obtains the following result.

**Proposition 1.3.** If $E$ is the cohomology of a monad of the form (1), then $E$ is $m$-regular for any integer $m$ such that

$$m \geq \max\{(k+2)d - (b_1 + \cdots + b_{k+3}) - 2, l\}.$$
1.3. Cohomology of generic instanton bundles

Recall that a bundle $E$ of rank 2 on $\mathbb{P}^3$ is called an instanton bundle if it is isomorphic to the cohomology of a monad of the following form:

$$O_{\mathbb{P}^3}(-1)^{\oplus k} \xrightarrow{\alpha} O_{\mathbb{P}^3}^{\oplus 2 + 2k} \xrightarrow{\beta} O_{\mathbb{P}^3}(1)^{\oplus k}$$

(2)

The integer $k$ is called the charge of $E$; notice that $c_1(E) = 0$ and $c_2(E) = k$. Note also that nullcorrelation bundles are precisely instanton bundles of charge 1.

Alternatively, an instanton bundle can also be defined as a bundle $E$ on $\mathbb{P}^3$ with $c_1(E) = 0$ and satisfying the following cohomological conditions:

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

The Hilbert polynomial of an instanton bundle is given by

$$P_E(t) = 2(k + 1)\chi(O_{\mathbb{P}^3}(t)) - k\chi(O_{\mathbb{P}^3}(t - 1)) - k\chi(O_{\mathbb{P}^3}(t + 1))$$

(3)

$$= \frac{1}{3}(t + 2)((t + 3)(t + 1) - 3k)$$

$$= \frac{1}{3}(t + 2)(t + 2 + \sqrt{3k} + 1)(t + 2 - \sqrt{3k} + 1).$$

Note also that $P_E(t) = h^0(E(t)) - h^1(E(t))$ for $t \geq -2$.

On another direction, recall that a coherent sheaf $F$ on $\mathbb{P}^3$ is said to have natural cohomology if for each $t \in \mathbb{Z}$, at most one of the cohomology groups $H^p(F(t))$, where $p = 0, \ldots, 3$, is nonzero; every torsion free coherent sheaf with natural cohomology is in fact locally free [10, Lemma 1.1]. In addition, every rank 2 locally free sheaf with $c_1 = 0$, $c_2 > 0$ and natural cohomology is an instanton bundle [10, p. 365].

Hartshorne and Hirschowitz have shown in [10] that the generic instanton bundle has natural cohomology. More precisely, let $\mathcal{I}(k)$ denote the moduli space of rank 2 locally free instanton sheaves of charge $k$; this is known to be an affine [4], irreducible [18, 19], nonsingular variety of dimension $8k - 3$ [13]. Let $\mathcal{N}(k)$ denote the subset of $\mathcal{I}(k)$ consisting of instanton bundles with natural cohomology; it is easy to see that $\mathcal{N}(k)$ is open within $\mathcal{I}(k)$, and [10, Theorem 0.1 (a)] tells us that it is nonempty.

More recently, Eisenbud and Schreyer have introduced the notion of supernatural bundles, see [6, p. 862]: a locally free sheaf on $\mathbb{P}^3$ is called supernatural if it has natural cohomology and its Hilbert polynomial has distinct integral roots. Therefore we see that there exists a rank 2 supernatural bundle with $c_1 = 0$ and $c_2 = k > 0$ if and only if $3k + 1$ is a perfect square; the first three possible values for $k$ are $k = 1, 5, 8$. 

2. Instanton vs Buchsbaum

We start by introducing the following function on the positive integers

\[ m(k) = \left\lfloor \sqrt{3k + 1} - 2 \right\rfloor, \]

where \( \lfloor \cdot \rfloor \) denotes the largest positive integer which is smaller than or equal to the argument.

**Proposition 2.1.** A rank 2 instanton bundle \( E \) is \( p \)-Buchsbaum if and only if \( h^1(E(p-2)) \neq 0 \) and \( h^1(E(p-1)) = 0 \). In addition, every rank 2 instanton bundle of charge \( k \) is \( p \)-Buchsbaum for some \( m(k) + 2 \leq p \leq k \).

**Proof.** By Theorem 1.2, we get that \( H^1_\ast(E) \) is generated in \( H^1(E(-1)) \). Thus if \( h^1(E(p-2)) \neq 0 \) and \( h^1(E(p-1)) = 0 \) (and hence \( h^1(E(t)) = 0 \) for every \( t \geq p-1 \)), then \( H^1_\ast(E) \) must be \( p \)-Buchsbaum. Conversely, if \( E \) is \( p \)-Buchsbaum, then \( h^1(E(p-2)) \neq 0 \) (otherwise, \( H^1_\ast(E) \) would be annihilated by the \((p-1)\)-th power of the irrelevant ideal) and \( h^1(E(p-1)) = 0 \).

By Proposition 1.3, we have that \( E \) is \( k \)-regular (cf. also [3, Corollary 3.3]). Hence \( H^1(E(k-1)) = 0 \), and it follows that every rank 2 instanton bundle is at most \( k \)-Buchsbaum.

Finally, note from (3) that for \(-1 \leq t \leq m(k)\) we have \( \chi(E(t)) < 0 \). Since \( h^3(E(t)) = 0 \) in this range, it follows that \( h^1(E(t)) \neq 0 \) for \( t = m(k) \). Thus every rank 2 instanton bundle is at least \((m(k) + 2)\)-Buchsbaum.

Since \( m(3) + 2 = 3 \), the first immediate consequence of the previous Proposition is given by the following Corollary.

**Corollary 2.2.** Every rank 2 instanton bundle of charge 3 is 3-Buchsbaum.

However, it is not true that every rank 2 instanton bundle of charge 3 has natural cohomology, as observed in [10, Example 1.6.1]. Indeed, recall that an instanton bundle \( E \) is called a \( \text{'t Hooft instanton} \) if \( h^0(E(1)) \neq 0 \), cf. [1]; more formally, consider the set

\[ \mathcal{H}(k) := \{ E \in \mathcal{I}(k) \mid h^0(E(1)) \neq 0 \}, \]

which is known to be a locally closed subvariety of \( \mathcal{I}(k) \) of dimension \( 5k + 4 \), irreducible and rational [1, Theorem 2.5]. On the other hand, let \( \mathcal{U}(k) := \mathcal{I}(k) \setminus \mathcal{N}(k) \), the subvariety of “unnatural” instanton bundles.

**Lemma 2.3.** For every \( k \geq 3 \), we have \( \mathcal{H}(k) \subset \mathcal{U}(k) \), while \( \mathcal{H}(3) = \mathcal{U}(3) \).
Proof. If $E$ is a rank 2 instanton bundle of charge $k \geq 3$, then $h^1(E(1)) \neq 0$ (because $\chi(E(-1)) < 0$). Hence if $E$ is a ’t Hooft instanton, then it does not have natural cohomology, showing that $\mathcal{H}(k) \subset \mathcal{U}(k)$.

Conversely, let now $E$ be a rank 2 instanton bundle of charge 3 which does not have natural cohomology. We then know that

(i) $h^0(E(t)) = 0$ for $t \leq 0$;
(ii) $h^1(E(t)) = 0$ for $t \neq -1, 0, 1$;
(iii) $h^2(E(t)) = 0$ for $t \neq -5, -4, -3$;
(iv) $h^3(E(t)) = 0$ for $t \geq -4$.

The last two claims are obtained by Serre duality and the fact $E \simeq E^\ast$. Therefore the only way in which $E$ may fail to have natural cohomology is if $h^0(E(1)) = h^3(E(-5)) \neq 0$. It follows that $\mathcal{U}(3) \subset \mathcal{H}(3)$. 

It would be interesting to determine properties of the $\mathcal{U}(k)$ for $k \geq 4$, particularly its dimension and number of irreducible components. The previous lemma tells us that $\dim \mathcal{U}(k) \geq 5k + 4$.

Another immediate consequence of Proposition 2.1 is the following.

Corollary 2.4. The generic rank 2 instanton bundle of charge $k$ is $(m(k)+2)$-Buchsbaum.

In particular, since $m(4) + 2 = m(5) + 2 = 3$, the generic rank 2 instanton bundle of charges 4 and 5 are 3-Buchsbaum, while instanton bundles of charge $k \geq 6$ are at least 4-Buchsbaum.

3. A 3-Buchsbaum rank 2 bundle with $c_1 = -1$

Theorem 1 tells us, in particular, that the first Chern class of every 1- and 2-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is zero. In this section, we show that the same is not true for $p$-Buchsbaum bundles with $p \geq 3$, providing an example of a 3-Buchsbaum rank 2 bundle with $c_1 = -1$.

Indeed, consider the monad

$$
\begin{align*}
O_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} O_{\mathbb{P}^3}^\oplus \oplus O_{\mathbb{P}^3}(-1)^\oplus \xrightarrow{\beta} O_{\mathbb{P}^3}(1),
\end{align*}
$$

which is the simplest example of a class of monads originally introduced by Ein in [5, eq. 3.1.4]. The existence of such monads can be easily established; consider for instance the following explicit maps

$$
\alpha = \begin{pmatrix} -z^2 \\ -w^2 \\ x \\ y \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x \\ y \\ z^2 \\ w^2 \end{pmatrix}
$$
where $[x:y:z:w]$ are homogeneous coordinates on $\mathbb{P}^3$.

Let $F$ denote the locally free cohomology of a monad of the form (4); it is a rank 2 bundle with $c_1(F) = -1$ and $c_2(F) = 2$. Ein claims in [5, p. 21], without proof, that $F$ is $\mu$-stable. For the sake of completeness, we include a proof below.

**Lemma 3.1.** Every locally free sheaf $F$ obtained as the cohomology of a monad of the form (4) is $\mu$-stable.

**Proof.** First consider the kernel bundle $K := \ker \beta$ defined by the sequence

$$0 \rightarrow K \rightarrow \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(1).$$

It follows from [2, Theorem 2.7] that $K$ is $\mu$-semistable (but not $\mu$-stable). Therefore, since $\mu(K) = -1$, we must have $h^0(K) = 0$. Now, from the sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} K \rightarrow F \rightarrow 0$$

we have that $h^0(F) = 0$, which implies that $F$ is $\mu$-stable. \qed

We now show that the bundles considered in this Section are 3-Buchsbaum.

**Proposition 3.2.** Every locally free sheaf $F$ obtained as cohomology of a monad of the form (4) is 3-Buchsbaum.

**Proof.** By Theorem 1.2, we get that $H^1_*(F)$ is generated in $H^1(F(-1))$. On the other hand, Proposition 1.3 tells us that $F$ is 3-regular, thus $h^1(F(2)) = 0$.

If we also had $h^1(F(1)) = 0$, $F$ would be 2-Buchsbaum, which, by Theorem 1 cannot happen. Therefore $F$ must be 3-Buchsbaum. \qed

Note also that, since $h^0(F(1)) = h^1(F(1)) = 1$ [11, 2.2], such bundles do not have natural cohomology.

Based on the evidence here presented and also motivated by results due to Roggero and Valabrega in [17], specially Propositions 5 and 6 and Theorem 2 there, we propose the following classification of 3-Buchsbaum rank 2 bundles on $\mathbb{P}^3$.

**Conjecture 3.3.** Every normalized, 3-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is either an instanton bundle of charge 3, 4 or 5, if $c_1 = 0$, or the cohomology of a monad of the form (4), if $c_1 = -1$.

Finally, let us comment on $p$-Buchsbaum rank 2 bundles on $\mathbb{P}^3$ for $p \geq 4$. An interesting, possible source of examples of such bundles is provided by Ein’s *generalized nullcorrelation bundles*, described in [5]. These are bundles obtained as cohomologies of monads of the following two types:
\[ O_{\mathbb{P}^3}(-d) \longrightarrow O_{\mathbb{P}^3}(-b) \oplus O_{\mathbb{P}^3}(-a) \oplus O_{\mathbb{P}^3}(a) \oplus O_{\mathbb{P}^3}(b) \longrightarrow O_{\mathbb{P}^3}(d) , \]  
and
\[ O_{\mathbb{P}^3}(-(d-1)) \longrightarrow O_{\mathbb{P}^3}(-(b-1)) \oplus O_{\mathbb{P}^3}(-(a-1)) \oplus O_{\mathbb{P}^3}(a) \oplus O_{\mathbb{P}^3}(b) \longrightarrow O_{\mathbb{P}^3}(d) , \]
where \( d > b \geq a \geq 0 \). Let us denote the cohomology of such monads by \( E_{a,b,d} \) and \( F_{a,b,d} \), respectively.

Note that, by Theorem 1.2 and Proposition 1.3, \( H^1(\mathbb{P}^2) \) is generated in degree \(-d\), and that \( E_{a,b,d} \) is \((3d - 2)\)-regular when \( d \geq 1 \). Therefore, such bundles are at most \((4d - 3)\)-Buchsbaum, being precisely \((4d - 3)\)-Buchsbaum provided \( h^1(E_{a,b,d}(3d - 4)) \neq 0 \).

Similarly, note that \( H^1(F_{a,b,d}) \) is generated in degree \(-d\), and that \( F_{a,b,d} \) is \(3d\)-regular. Therefore, such bundles are at most \((4d - 1)\)-Buchsbaum, being precisely \((4d - 1)\)-Buchsbaum provided \( h^1(F_{a,b,d}(3d - 2)) \neq 0 \).

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Authors’ addresses:
Marcos Jardim
Departamento de Matemática
IMECC-UNICAMP
Rua Sérgio Buarque de Holanda 651
13083-859 Campinas, SP, Brazil
E-mail: jardim@ime.unicamp.br

Simone Marchesi
Departamento de Matemática
IMECC-UNICAMP
Rua Sérgio Buarque de Holanda 651
13083-859 Campinas, SP, Brazil
E-mail: marchesi@ime.unicamp.br

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