New Converse Bounds on the Mismatched Reliability Function and the Mismatch Capacity Using an Auxiliary Genie Receiver

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Abstract
We develop a novel framework for proving converse theorems for channel coding, which is based on the analysis technique of multicast transmission with an additional auxiliary receiver, which serves as a “genie-aided-genie” to the original receiver. The genie provides the original receiver a certain narrowed list of codewords to choose from that includes the transmitted one. This technique is used to derive upper bounds on the mismatch capacity as well as the reliability function with a mismatched decoding metric of discrete memoryless channels. Unlike previous works, our bounding technique exploits also the inherent symmetric requirement from the codewords, leading to these new upper bounds which are tighter. Since the computations of most of the known bounds on the mismatch capacity are rather complicated, we further present a method to obtain relaxed bounds that are easier to compute. We conclude by presenting simpler bounds on the reliability function, and provide sufficient conditions for their tightness in certain ranges of rates.

I. INTRODUCTION
This paper addresses the problem of determining the fundamental limits of reliable communication over a discrete memoryless channel $W$ with a given decoding metric $q$. This setup is usually referred to as mismatched decoding, since the decoding metric differs from the optimal maximum likelihood (ML) metric that is matched to the communication channel. While the ML decoder minimizes the error probability, in certain cases it is not applicable for various reasons such as channel estimation errors or for practical decoder implementation considerations. The highest achievable rate with decoding metric $q$ is referred to as the mismatch capacity. The problem of determining the fundamental bounds on channels with mismatched decoding is also related to other information-theoretic setups such as zero-error transmission over communication channels.

There have been quite a few works on achievable rates for channels with mismatched decoding from the information theoretic viewpoint. A partial list of works is [1]–[15], and for a survey on the subject see [16], and references therein.

Early results with a converse flavor were derived in [17], where a necessary and sufficient condition for the positivity of the mismatch capacity was determined, as well as a single-letter expression for the mismatch capacity in the case of the binary input binary output channel. Later works presented multi-letter expressions and upper bounds on the mismatch capacity [18] [19], including, among other results, a multi-letter max-min upper bound based on the approach of upper bounding the mismatch capacity by an achievable rate of a different channel, and establishing a soft converse bound [19].

In [20], a single-letter upper bound was derived for the mismatch capacity of the DMC, as the minimum of achievable rates of a set of channels which are formed by a transformation of the channel into a translated channel.

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Tighter single-letter upper bounds on the mismatch capacity were derived in [21] (see also [22]), which are based on a proof technique via multicast transmission of only one common message over a broadcast channel. The multicast transmission technique is based on extending the single-user channel $W$ with output $Y$ to a channel having an additional output $Z$, with the property that the intersection event of correct $q$-decoding of the $Y$-receiver and erroneous decoding of the auxiliary $Z$-receiver has zero probability for any codebook of a certain composition $P$. This approach led to a strictly tighter bound with a significantly simpler proof, which holds also for continuous alphabet channels. A tighter bound of a more involved form was presented in [22], which relied on considering a $Z$-receiver which is genie-aided and is informed of the actual joint empirical distribution of the transmitted codeword and the $Z$-output sequence. Equivalence classes of isomorphic channel-metric pairs $(W, q)$ were further introduced in [21], that enabled to derive a sufficient condition for the tightness of the bound.

In a later work, [23], the class of two-receivers (broadcast) channels was enlarged to include channels satisfying that if the $Z$-receiver makes an error, then with high probability (approaching 1) so does the $Y$-receiver. An improved bound was derived in [24], [25], which further enlarged the class of channels. The above mentioned bounds are described in detail in this paper, but most of them are quite complicated to compute in the sense that it is required to solve an optimization problem in order to determine whether a certain channel belongs to the set or not. The tightest bound known to date which is easily computable in this sense is the basic bound of [21].

The study of the reliability function (error exponents) of channels with ML decoding has been quite extensive (see, e.g., [26]–[29]). Clearly, the known upper bounds are applicable also to mismatched decoding. Lower bounds on the exponents with mismatched decoding were derived in several works such as [3], [5], [14] (see also [16], and references therein). Recently, an upper bound for zero-rate codes was derived in [30]. For a wide class of channel-metric pairs, this bound was shown to be tight at $R = 0^+$. In [24], a new upper bound on the reliability function with mismatched decoding was derived for all rates up to the aforementioned bound of [24] on the mismatch capacity.

In this paper, we refine our multicast approach to allow the genie-aided auxiliary $Z$-receiver of the channel to serve as a genie for the original $Y$-receiver. We call this approach “transmission with a genie-aided-genie”. The idea is very simple: the genie-aided auxiliary $Z$-receiver informs the original $Y$-receiver of the list of all the codewords that share the same empirical statistics (joint type-class) with the channel $Z$-output as that of the actual transmitted codeword. Doing so, it narrows down the list of competing hypothesized codewords that the original mismatched decoder needs to choose from, and yields a lower bound on the probability of error. This leads to a basic upper bound on the mismatch capacity and the reliability function. We further present possibly looser bounds which are easily computable.

This paper is organized as follows: In Section II we present notation conventions. A formal statement of the mismatched decoding setup appears in Section III. In Section IV-A, we present the transmission with a Genie-Aided-Genie proof technique. Section IV-B summarizes our main results. Sections V and VI are dedicated to the proofs of the main theorems regarding reliability function and mismatch capacity theorems, respectively. In Section VII we establish some simpler bounds on the reliability function and sufficient conditions for tightness. In Section VIII we study the case of binary-input DMCs. In Section IX we compare our results to former results. Section X discusses some concluding remarks. Proofs of additional results and lemmas appear in the appendix.

II. Notation

Throughout this paper, scalar random variables are denoted by capital letters, their sample values are denoted by their respective lower case letters, and their alphabets are denoted by their respective calligraphic letters; e.g. $X$, $x$, and $\mathcal{X}$, respectively. A similar convention applies to random vectors of dimension $n$ and their sample values, which are denoted in boldface; e.g., $x$. The set of all $n$-vectors with components taking values in a certain finite alphabet are denoted by the same alphabet superscripted by $n$, e.g., $\mathcal{X}^n$. Logarithms are taken to the natural base $e$, unless stated otherwise.
For a given sequence \( x \in \mathcal{X}^n \), where \( \mathcal{X} \) is a finite alphabet, \( \hat{P}_x \) denotes the empirical distribution on \( \mathcal{X} \) extracted from \( x \); in other words, \( \hat{P}_x \) is the vector \( \{ \hat{P}_x(x), x \in \mathcal{X} \} \), where \( \hat{P}_x(x) \) is the relative frequency of the symbol \( x \) in the vector \( x \). The type-class of \( x \) is the set of \( x' \in \mathcal{X}^n \) such that \( \hat{P}_{x'} = \hat{P}_x \), which is denoted \( T_n(\hat{P}_x) \). Similarly, the joint empirical distribution of two sequences \( x, y \), denoted \( \hat{P}_{xy} \), is the vector \( \{ \hat{P}_{xy}(x, y), (x, y) \in \mathcal{X} \times \mathcal{Y} \} \), where \( \hat{P}_{xy}(x, y) \) is the relative frequency of the pair of symbols \( (x, y) \) in the vector \( (x, y) \); i.e., the number of indices \( i \) such that \( (x_i, y_i) = (x, y) \) normalized by \( n \). The conditional type-class of \( y \) given \( x \) is the set of \( \tilde{y} \)'s such that \( \hat{P}_{x,\tilde{y}} = \hat{P}_{xy} \), which is denoted \( T_n(\hat{P}_{y|x} | x) \).

The set of all probability distributions on \( \mathcal{X} \) is denoted following the usual conventions in the information theory literature, e.g., \( \mathcal{M} \) of messages is denoted by \( \mathcal{P} \). Similarly, the set of all conditional distributions from \( \mathcal{X} \) to \( \mathcal{Y} \) is denoted \( \mathcal{P}(\mathcal{Y}|\mathcal{X}) \), and the set of empirical distributions of order \( n \) on alphabet \( \mathcal{X} \) is denoted \( \mathcal{P}_n(\mathcal{X}) \).

Information theoretic quantities, such as entropy, conditional entropy, and mutual information are denoted following the usual conventions in the information theory literature, e.g., \( H(X), H(X|Y), I(X; Y) \) and so on. To emphasize the dependence of a quantity on a certain underlying probability distribution, say \( \mu \), we at times use notations such as \( H(\mu), H(\mu_{X|Y}), I(\mu_{X,Y}) \), or \( H_{\mu}(X), H_{\mu}(Y|X) \), etc. For \( P \in \mathcal{P}(\mathcal{X}) \), and \( V, Q \in \mathcal{P}(\mathcal{Y}|\mathcal{X}) \) we denote the conditional divergence as \( D(V||Q) = \sum_{x,y} P(x)V(y|x) \log \frac{V(y|x)}{Q(y|x)} \).

The expectation operator is denoted by \( \mathbb{E}(\cdot) \), and to make the dependence on the underlying distribution clear, it is denoted by \( \mathbb{E}_{\mu}(\cdot) \). The cardinality of a finite set \( \mathcal{A} \) is denoted by \( |\mathcal{A}| \). The indicator function of an event \( \mathcal{E} \) is denoted by \( 1(\mathcal{E}) \).

For two measures \( P, Q \) defined on the same measurable space \( (\Omega, \mathcal{F}) \) the measure \( P \) is said to be absolutely continuous w.r.t. \( Q \) if for every \( \mathcal{E} \in \mathcal{F} \) such that \( Q(\mathcal{E}) = 0 \) it also holds that \( P(\mathcal{E}) = 0 \); this is denoted \( P \ll Q \).

### III. A Formal Statement of the Problem

Consider transmission over a memoryless channel described by a conditional probability rule \( W(y|x) \), with input \( x \in \mathcal{X} \) and output \( y \in \mathcal{Y} \) finite alphabets \( \mathcal{X} \) and \( \mathcal{Y} \); in particular, \( W(y|x) \) is a conditional probability mass function. We define \( W^n(y|x) = \prod_{k=1}^{n} W(y_k|x_k) \) for input/output sequences \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n \) and \( y = (y_1, \ldots, y_n) \in \mathcal{Y}^n \). The corresponding random variables are denoted by \( X \) and \( Y \).

An encoder maps a message \( m \in \{1, \ldots, M_n\} \) to a channel input sequence \( x_m \in \mathcal{X} \), where the number of messages is denoted by \( M_n \). The message, represented by the random variable \( M \), is assumed to take values in \( \{1, \ldots, M_n\} \) equi-probably. This mapping induces an \( (n, M_n) \)-codebook \( \mathcal{C}_n = \{ x_1, \ldots, x_{M_n} \} \) with rate \( R_n = \frac{1}{n} \log M_n \).

Upon observing the channel output \( y \), the decoder produces an estimate of the transmitted message \( \hat{m} \). We consider the decoding rule

\[
\hat{m} = \arg \max_{i \in \{1, \ldots, M_n\}} q(x_i, y), \tag{1}
\]

where \( q(x_i, y) \) is a certain additive decoding metric \cite{27} Ch. 2] defined by a single-letter mapping \( q : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) such that

\[
q(x, y) = \frac{1}{n} \sum_{i=1}^{n} q(x_i, y_i) = \mathbb{E}_{\hat{P}_{xy}}[q(X, Y)] \triangleq q(\hat{P}_{xy}), \tag{2}
\]

where for convenience we slightly abuse notation using \( q \) for both the per-letter metric \( q(x, y) \) and the \( n \)-letter metric \( q(\hat{P}_{xy}) \). Throughout the paper it is assumed that ties are broken uniformly between the maximizers.

Denoting the random variable corresponding to the decoded message by \( \hat{M}_q(Y) \), we denote the average probability of error as \( P_e(W, \mathcal{C}_n, q) = \Pr[\hat{M}_q(Y) \neq M] \).

A rate \( R \) is said to be achievable with decoding metric \( q \) if there exists a sequence of codebooks \( \mathcal{C}_n, \ n = 1, 2, \ldots \) such that \( \frac{1}{n} \log |\mathcal{C}_n| \geq R \) and \( \lim_{n \to \infty} P_e(W, \mathcal{C}_n, q) = 0 \). The channel capacity w.r.t. metric
q, denoted $C_q(W)$, is defined as the supremum of achievable rates, and is referred to as the mismatch capacity.

Since the optimal decoding rule w.r.t. average probability of error (of equiprobable messages) is ML, which is additive for DMCs, Shannon’s channel capacity $C(W)$ can be viewed in fact as the channel capacity w.r.t. the metric $q(x, y) = \log W(y|x)$; that is,

$$C(W) = C_q(W)|_{q(x,y)=\log W(y|x)}.$$ (3)

A rate-exponent pair $(R, E)$ is said to be achievable for channel $W$ with decoding metric $q$ if there exists a sequence of codebooks $C_n$, $n = 1, 2, ...$ such that for all $n$, $\frac{1}{n} \log |C_n| \geq R$ and

$$\liminf_{n \to \infty} \frac{1}{n} \log P_e(W, C_n, q) \geq E.$$ (4)

Equivalently, we say that $E$ is an achievable error exponent at rate $R$ if $(R, E)$ is an achievable rate-exponent pair.

The reliability function of the channel with decoding metric $q$ is the supremum of achievable error exponents as a function of the code rate, and is denoted by $E^q(R, W)$. The reliability function with ML decoding metric is denoted $E(R, W)$.

Define the highest achievable exponent with $P$ constant composition codebooks of block length $n$ as

$$e^q_n(R, P, W) \triangleq \max_{C_n \subseteq \mathcal{T}_n(P): |C_n| \geq e^{nR}} \frac{1}{n} \log P_e(q, W, C_n).$$ (5)

Using standard arguments that follow from the fact that the number of type-classes grows polynomially with $n$, it can be shown that

$$E^q(R, W) = \liminf_{n \to \infty} \max_{P_n \in \mathcal{P}_n(\mathcal{X})} e^q_n(R, P_n, W),$$ (6)

and for this reason, the main focus of this paper is on analyzing constant composition codes.

**IV. MAIN RESULTS**

In this section we present new upper bounds on the mismatch capacity and the reliability function of the DMC when the decoder uses a mismatched decoding metric $q$. Before we present our new bounds, we describe the main idea behind our proof technique.

**A. The Multicast Transmission with a Genie-Aided-Genie Proof Technique**

We refine our multicast transmission setup which was introduced in [21], that extends the single-user channel $W_{Y|X}$ to a two-receivers channel $W_{Y,Z|X}$ having an additional output $Z$ over some finite alphabet $Z$. An encoder uses a codebook $C_n = \{\mathbf{x}_i\}_{i=1}^M$ of size $M_n = e^{nR}$ to transmit a message over the channel.

As in [21], we add a type-genie that informs the $Z$-receiver of the actual joint empirical distribution $\hat{P}_{xz}$ of the input and output signals $(x, z)$. The refinement of the proof technique in this paper, is in that the $Z$-receiver, which observes $z$, serves as a genie to the $Y$-receiver by providing it with the list

$$\mathcal{L}(z, \hat{P}_{xz}) \triangleq C_n \cap \mathcal{T}_n(\hat{P}_{xz}|z) \triangleq \{x_1(z, \hat{P}_{xz}), \ldots, x_{|\mathcal{L}(z, \hat{P}_{xz})|}(z, \hat{P}_{xz})\}.$$ (7)

of all the codewords, which lie in $\mathcal{T}_n(\hat{P}_{xz}|z)$ (the conditional type-class given the received signal $z$). The $Y$-receiver compares the metrics of all the codewords in the list and outputs

$$\hat{m} = \arg\max_{i: \mathbf{x}_i \in \mathcal{L}(z, \hat{P}_{xz})} q(\mathbf{x}_i, y),$$ (8)

where ties are broken uniformly between the maximizers. This setup is depicted in Fig. [1].
Since by definition the true codeword belongs to this narrowed down list, the error probability in mismatched decoding of the $Y$ receiver cannot exceed that of the original single-user setup.

Further, we lower bound the average probability of error of this genie-aided setup. We show that for rates which exceed our upper bound on the mismatch capacity, $C_q(W)$, the average probability of error is bounded away from zero. As for rates below $C_q(W)$, we upper bound the exponent of the average error probability.

Note that, as opposed to [21], where the terminology of multicast transmission over a broadcast channel was used, in this work we modify the terminology to a two-receivers channel, or a two-outputs channel, which better reflects the fact that the message is not decoded by the $Z$-receiver.

B. An Overview of the Main Results

In this section we present the new upper bounds on the achievable error exponent using decoding metric $q$, and the mismatch capacity $C_q(W)$. Throughout this paper, we adopt the shorthand notation that $W$ without subscript signifies the original single-user channel $W_{Y|X}$. Whenever we refer to other marginal distributions; i.e., $W_{Y|XZ}$, $W_{Z|XY}$ or $W_{Y|Z}$, the subscript is mentioned explicitly.

1) The Main Upper Bound on $C_q(W)$: Consider the set of two-outputs channels:

$$W_q(P_X) = \left\{ P_{YZ|X} : \min_{V_{UXXZ} = V_{UXXZ}} \mathbb{E}_{V_{UXXZ}P_{Y|XZ}} q(\tilde{X}, Y) \geq \mathbb{E} q(X, Y) \right\},$$

(9)

where $U$ is an auxiliary random variable with alphabet size $|U| \leq |X|^2 |Z|$, and the condition $V_{UXXZ} = V_{UX\tilde{X}Z}$ signifies that for all $(u, x_1, x_2, z), V_{UXXZ}(u, x_1, x_2, z) = V_{UX\tilde{X}Z}(u, x_2, x_1, z)$. Let

$$\overline{C_q}(W) \triangleq \max_{P_X} \min_{P_{YZ|X} \in W_q(P_X), P_{Y|X} = W} I(X; Z),$$

(10)

By inspecting (9) it is evident that without loss of generality one can take $U$ such that $Z$ is a deterministic function of $U$ (perhaps with larger alphabet for $U$), because one can replace any $U'$ by $U' = (U, Z)$. Therefore, one can add the constraint $H(Z|U) = 0$ to the set of the minimization in (9) without changing the resulting bound (10).
where throughout this paper, we adopt the convention that the minimum and maximum over an empty set equal ∞ and −∞, respectively. Our basic bound on \( C_q(W) \) of the following theorem is proved in Section [V].

**Theorem 1.** For any \( W \), additive metric \( q \in \mathbb{R} \cup \{-\infty\} \), and finite alphabet \( \mathcal{Z} \),

\[
C_q(W) \leq \overline{C}_q(W).
\]

The bound \( \overline{C}_q(W) \) is tighter compared to previously known bounds, see Section [IX] for a comparison to previous works. Note that \( \overline{C}_q(W) \) is quite difficult to compute, since in order to determine whether a two-outputs channel \( \tilde{P}_{YZ|X} \) belongs to the set \( \mathcal{W}_q(P_X) \) or not, one needs to solve the minimization problem in (12). A similar problem arises with the computation of many of the previous bounds (e.g., those of [23], [24]). For this reason, in the next section we present a few looser bounds that are easier to compute.

It is worth mentioning though, that denoting

\[
\Delta^q(P_{XZU}, P_{Y|XZ}) \triangleq \sum_{x,z,u,x,y} P_{UZ}(u, z)P_{X|UZ}(x|u, z)P_{X|Y|Z}(y|x, z)[q(\bar{x}, y) - q(x, y)],
\]

the bound \( \overline{C}_q(W) \) in (10) can also be expressed as

\[
\overline{C}_q(W) = \max_{P_X} \min_{P_{Y|X}: \min_{P_{U|XZ}} \Delta^q(P_{XZU}, P_{Y|XZ}) \geq 0} I(X; Z) \quad (13)
\]

\[
= \max_{(P_X, P_{U|XZ})} \min_{P_{Y|X}: \min_{P_{U|XZ}} \Delta^q(P_{XZU}, P_{Y|XZ}) \geq 0} I(X; Z). \quad (14)
\]

In this form of the bound, for every given \( (P_X, P_{U|XZ}) \), it is easy to determine whether the channel \( P_{Y|X} \) satisfies \( \Delta^q(P_{XZU}, P_{Y|XZ}) \geq 0 \) or not, nevertheless, one still needs to optimize over the pair \( (P_X, P_{U|XZ}) \).

2) Possibly Looser Easier to Compute Bounds on \( C_q(W) \): We next present several sets\(^2\) of two-outputs channels, \( \mathcal{W}^\text{sym}_q(P_X) \), and \( \mathcal{W}^\text{sym}_q(P_X), \mathcal{W}^\text{sym}_q(P_X), \mathcal{W}^\text{psd}_q(P_X) \), which are subsets of \( \mathcal{W}_q(P_X) \).

Consider the following set of symmetric distributions:

\[
\mathcal{P}^\text{sym}(\mathcal{X}^2 \times \mathcal{Z}) \triangleq \{ P_{\bar{X}XZ} \in \mathcal{P}(\mathcal{X}^2 \times \mathcal{Z}) : \forall (x, \bar{x}, z), \; P_{\bar{X}XZ}(\bar{x}, x, z) = P_{\bar{X}XZ}(x, \bar{x}, z) \}, \quad (15)
\]

and define

\[
\mathcal{W}^\text{sym}_q(P_X) \triangleq \left\{ P_{YZ|X} : \min_{\bar{X}XZ \in \mathcal{P}^\text{sym}(\mathcal{X}^2 \times \mathcal{Z}) : V_{XZ} = P_{XZ}} E_{V_{\bar{X}XZ}P_{Y|XZ}} q(\bar{X}, Y) \geq E_q(X, Y) \right\}, \quad (16)
\]

\[
\mathcal{W}^\text{sym}_q(P_X) \triangleq \left\{ P_{YZ|X} : \min_{\bar{X}XZ \in \mathcal{P}^\text{sym}(\mathcal{X}^2 \times \mathcal{Z}) : V_{XZ} = P_{XZ}, \forall (x, z), \; V_{X_{x,z}(x,z)} \geq P_{X|Z}(x|z)} E_{V_{\bar{X}XZ}P_{Y|XZ}} q(\bar{X}, Y) \geq E_q(X, Y) \right\}. \quad (17)
\]

Further, define the third set of two-outputs channels

\[
\widetilde{\mathcal{W}}_q(P_X) \triangleq \left\{ P_{YZ|X} : \sum_{x,\bar{x},y} V(x|z)V(\bar{x}|z)P_{Y|XZ}(y|x, z)[q(\bar{x}, y) - q(x, y)] \geq 0 \right\} \quad (18)
\]

\[
= \left\{ P_{YZ|X} : \sum_{x,\bar{x},y} V(x|z)V(\bar{x}|z)P_{Y|XZ}(y|x, z)[q(\bar{x}, y) - q(x, y)] \geq 0 \right\} \quad (19)
\]

\(^2\)In fact, the sets, as well as \( \mathcal{W}_q(P) \) are also functions of the alphabet cardinality \( |\mathcal{Z}| \), but for the sake of simplicity we omit this dependence from our notation.
where as mentioned in Section II \( \ll \) denotes absolute continuity.

Now, denote w.l.o.g. \( \mathcal{X} = \{1, \ldots, |\mathcal{X}|\} \), and consider the collection of symmetric \( |\mathcal{X}| \times |\mathcal{X}| \) matrices \( \{\mathcal{D}^q(P_{Y|X,Z=z})\} \), indexed by \( z \in \mathcal{Z} \), whose \((i, j)\)-th entries are given by:

\[
\{\mathcal{D}^q(P_{Y|X,Z=z})\}_{i,j} = \sum_y \left( P_{Y|XZ}(y|i, z)[q(j, y) - q(i, y)] + P_{Y|XZ}(y|j, z)[q(i, y) - q(j, y)] \right) 
\]

\[
= \sum_y [P_{Y|XZ}(y|i, z) - P_{Y|XZ}(y|j, z)] \cdot [q(j, y) - q(i, y)] .
\]

Denoting that a matrix \( \mathcal{D} \) is positive semi-definite (p.s.d.) by \( \mathcal{D} \succeq 0 \), we define the last set of two-outputs channels:

\[
\mathcal{W}^{psd}_q(P_X) \triangleq \{ P_{Y|X} : \forall z \in \mathcal{Z}, \mathcal{D}^q(P_{Y|X,Z=z}) \succeq 0 \} .
\]

Let

\[
C^\text{sym}_q(W) \triangleq \max_{P_X} \min_{P_{Y,Z|X} \in \mathcal{W}^\text{sym}_q(P_X), P_{Y|X}=W} I(X; Z) \quad (23)
\]

\[
\bar{C}^\text{sym}_q(W) \triangleq \max_{P_X} \min_{P_{Y,Z|X} \in \bar{\mathcal{W}}^\text{sym}_q(P_X), P_{Y|X}=W} I(X; Z) \quad (24)
\]

\[
\tilde{C}_q(W) \triangleq \max_{P_X} \min_{P_{Y,Z|X} \in \tilde{\mathcal{W}}_q(P_X), P_{Y|X}=W} I(X; Z) . \quad (25)
\]

\[
C^{psd}_q(W) \triangleq \max_{P_X} \min_{P_{Y,Z|X} \in \mathcal{W}^{psd}_q(P_X): P_{Y|X}=W} I(X; Z) . \quad (26)
\]

The possibly looser bounds compared to \( \bar{C}_q(W) \) are presented in the following proposition, which is proved in Appendix A.

**Proposition 1.** For any \( P \in \mathcal{P}(\mathcal{X}) \),

\[
\mathcal{W}^{psd}_q(P) \subseteq \tilde{\mathcal{W}}_q(P) \subset \mathcal{W}_q(P), \quad (27)
\]

\[
\tilde{\mathcal{W}}^\text{sym}_q(P_X) \subseteq \bar{\mathcal{W}}^\text{sym}_q(P) \subseteq \mathcal{W}_q(P), \quad (28)
\]

and therefore

\[
\bar{C}_q(W) \leq \tilde{C}_q(W) \leq C^{psd}_q(W), \quad (29)
\]

\[
\bar{C}_q(W) \leq C^\text{sym}_q(W) \leq \tilde{C}^\text{sym}_q(W). \quad (30)
\]

As mentioned before, the bound \( C^{psd}_q(W) \) has a significant advantage over \( \bar{C}_q(W) \), since it is easier to compute in the sense that determining whether a two-outputs channel \( P_{Y|X,Z} \) belongs to the set \( \mathcal{W}^{psd}_q(P_X) \) requires a simple calculation. In particular, one needs to verify for all \( z \in \mathcal{Z}, \mathcal{D}^q(P_{Y|X,Z=z}) \succeq 0 \) by checking that the determinants of the \( 2|\mathcal{X}| - 1 \) minors of each of these \(|\mathcal{Z}|\) matrices are all non-negative [31]. This is in contrast to the calculation of \( \bar{C}_q(W) \) and similarly several other previous bounds (e.g., those of [23], [24]), which require to solve a certain minimization problem, such as the minimization in [9], in order to determine whether a two-outputs channel belongs to the set \( \mathcal{W}_q(P_X) \) (see Section VIII). Furthermore, the bound \( C^\text{sym}_q(W) \) is also easier to compute compared to \( \bar{C}_q(W) \) and previous bounds, since the constraints in the definition of the set \( \mathcal{W}^\text{sym}_q(P) \) limit the range and the number of degrees of freedom of the solution of the optimization problem. In Section VIII, we analyze the various bounds for the binary-input channel case (\( |\mathcal{X}| = 2, |\mathcal{Y}| < \infty \)).
3) Upper Bounds on the Reliability Function with Decoding Metric \( q \): In this section we present our main bound \( E_{sp}^{q}(R, W) \) on the reliability function with mismatched decoding, \( E^{q}(R, W) \), and similar to the mismatch capacity, we present looser bounds that are easier to compute.

For \( P \in \mathcal{P}(\mathcal{X}) \) define
\[
E_{sp}^{q}(R, P, W) \triangleq \min_{P_{Y|X} \in \mathcal{W}_{q}(P), I(X;Z) \leq R} D(P_{Y|X} \parallel W|P) \tag{31}
\]

Due to (6), our main result concerning the reliability function is presented in terms of upper bounds on \( e_{n}^{q}(R, P, W) \).

**Theorem 2.** Let \( |Z| < \infty \), then for all \( n \), and any \( P \in \mathcal{P}_{n}(\mathcal{X}) \)
\[
e_{n}^{q}(R, P, W) \leq E_{sp}^{q}(R - \epsilon_{n,a}, P, W) + \epsilon_{n,b}, \tag{32}
\]
where \( \epsilon_{n,a} = O(\frac{\log n}{n}) \), and \( \epsilon_{n,b} = O(\frac{\log n}{n}) \).

The main idea of the proof of Theorem 2 was presented in Section IV-A, the full proof can be found in Section V as well as the exact quantities \( \epsilon_{n,a} \), and \( \epsilon_{n,b} \).

Now, define further
\[
\tilde{E}_{sp}^{q}(R, P, W) \triangleq \min_{P_{Y|X} \in \tilde{\mathcal{W}}_{q}(P), I(X;Z) \leq R} D(P_{Y|X} \parallel W|P) \tag{33}
\]
\[
E_{sp}^{q,psd}(R, P, W) \triangleq \min_{P_{Y|X} \in \mathcal{W}_{q}^{psd}(P), I(X;Z) \leq R} D(P_{Y|X} \parallel W|P) \tag{34}
\]
\[
E_{sp}^{q,sym}(R, P, W) \triangleq \min_{P_{Y|X} \in \mathcal{W}_{q}^{sym}(P), I(X;Z) \leq R} D(P_{Y|X} \parallel W|P) \tag{35}
\]
\[
\tilde{E}_{sp}^{q,sym}(R, P, W) \triangleq \min_{P_{Y|X} \in \tilde{\mathcal{W}}_{q}^{sym}(P), I(X;Z) \leq R} D(P_{Y|X} \parallel W|P). \tag{36}
\]

Further, let \( E_{sp}^{q}(R, W), \tilde{E}_{sp}^{q}(R, W), E_{sp}^{q,psd}(R, W), E_{sp}^{q,sym}(R, W), \tilde{E}_{sp}^{q,sym}(R, W) \) denote the maximum over \( P \in \mathcal{P}(\mathcal{X}) \) of the above quantities (31)-(36), respectively. The following theorem states the additional bounds.

**Theorem 3.** The following inequalities hold for all \( (R, P, W) \)
\[
E_{sp}^{q}(R, P, W) \leq \tilde{E}_{sp}^{q}(R, P, W) \leq E_{sp}^{q,psd}(R, P, W) \tag{37}
\]
\[
E_{sp}^{q}(R, P, W) \leq E_{sp}^{q,sym}(R, P, W) \leq \tilde{E}_{sp}^{q,sym}(R, P, W). \tag{38}
\]

The inequalities of (37)-(38) follow from (27)-(28).

The following corollary follows from (6), and Theorems 2 and 3.

**Corollary 1.** For any \( |Z| < \infty \),
\[
E^{q}(R, W) \leq E_{sp}^{q}(R, W) \leq \tilde{E}_{sp}^{q}(R, W) \leq E_{sp}^{q,psd}(R, W) \tag{39}
\]
\[
E^{q}(R, W) \leq E_{sp}^{q}(R, W) \leq E_{sp}^{q,sym}(R, W) \leq \tilde{E}_{sp}^{q,sym}(R, W). \tag{40}
\]

The following corollary, which is easily verified, states that \( \bar{C}_{q}(W) \) is a rate above which the average probability of error in \( q \) decoding cannot vanish exponentially fast.

**Corollary 2.** For all \( R > \bar{C}_{q}(W) \), \( E_{sp}^{q}(R, P, W) = 0 \).
4) Upper Bound on the Reliability Function for Type-Dependent Metrics: Our next corollary extends the results of Theorems 2 and 3 to the case of type-dependent metrics, similar to the extension in [21]. The class of type-dependent metrics generalizes additive metrics in the following manner. It is assumed that the decoding metric $q(x, y)$ depends on $x, y$ solely via their joint empirical distribution; i.e., $q(x, y) = q(\hat{P}_{X,Y})$, so $q$ can be viewed as a mapping from the empirical distributions to the reals $q: \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}$. More generally, in order not to restrict attention to a specific block-length $n$, we assume that it maps the simplex to a real number; i.e.,

$$q : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}.$$  

We refer to this class of metrics as type-dependent (formerly referred to as $\alpha$-decoders by Csiszár and Körner [3]). In the case of type-dependent metrics, (1) becomes:

$$\hat{m} = \arg \max_{i \in \{1, \ldots, M_n\}} q(\hat{P}_{i,Y}).$$  

The equivalent of the set of two-outputs channels $\tilde{W}_q^{sym}(P_X)$ in (16) for type-dependent metrics is given by

$$\tilde{W}_q^{sym}(P_X) \triangleq \left\{ P_{Y|X} : \min_{V_{\hat{X}XZ}: V_{\hat{X}XZ} \in \mathcal{P}_{sym}(\mathcal{X}^2 \times \mathcal{Z})} q(V_{\hat{X}Y}) \geq q(P_{XY}) \right\},$$

yielding $\tilde{E}_q^{sym}(R, P, W)$ as defined in (36).

The result pertaining to type-dependent metrics is the following.

**Corollary 3.** Let $|\mathcal{Z}| < \infty$, and let $q(P_{XY})$ be convex in $P_{Y|X}$ for fixed $P_X$, then for all $n$, and any $P \in \mathcal{P}_n(\mathcal{X})$

$$e_n^q(R, P, W) \leq \tilde{E}_q^{sym}(R - \epsilon_{n,c}, P, W) + \epsilon_{n,d},$$

where $\epsilon_{n,c} = O\left(\frac{\log n}{n}\right)$ and $\epsilon_{n,d} = O\left(\frac{\log n}{n}\right)$, and consequently

$$E_n^q(R, W) \leq \tilde{E}_q^{sym}(R, W).$$

The corollary is proved in Appendix C.

V. PROOF OF THEOREM 2

In this section we prove Theorem 2 based on the outline described in Section IV-A.

Let a DMC $W = W_{Y|X}$ be given. Fix $P_n \in \mathcal{P}_n(\mathcal{X})$, and let $C_n = \{\overline{X}_i\}_{i=1}^{M_n}$ be a $P_n$-constant composition codebook of rate $R$ for the channel $W_{Y|X}$. Consider another channel from $\mathcal{X} \times \mathcal{Y}$ to a finite set $\mathcal{Z}$ denoted by $W_{Z|XY}$, which along with $W_{Y|X}$ constitutes a two-outputs channel $W_{YZ|X}$. For technical reasons, we assume that $W_{Z|XY}$ takes the following form:

$$\forall (x, y, z), \ W_{Z|XY}(z|x, y) = (1 - \tau_n) \cdot W_{Z|XY}^*(z|x, y) + \tau_n \cdot \frac{1}{|\mathcal{Z}|},$$

where $\tau_n = \frac{1}{n}$, and $W_{Z|XY}^*(z|x, y)$ is some conditional distribution from $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{Z}$. Note that this implies that

$$\forall (x, y, z), \ W_{Y|XZ}(y|x, z) = \frac{W(y|x)W(z|x, y)}{W(z|x)} \geq W_{Y|X}(y|x) \cdot \frac{\tau_n}{|\mathcal{Z}|}.$$  

Denote

$$w_{\min} \triangleq \min_{(x,y) : W(y|x) > 0} W_{Y|X}(y|x),$$

(48)
Given the channel input $X \in \mathcal{T}_n(P_n)$, and the joint type-class of $(X, Z)$, $\hat{P}_{XZ} = \hat{P}_{XZ}$, we clearly have that $Z$ is distributed uniformly over $\mathcal{T}_n(P_{Z|X}|X)$, i.e.,

$$\Pr(Z = z|X = x, \hat{P}_{XZ} = \hat{P}_{XZ}) = \frac{1}{|\mathcal{T}_n(P_{Z|X}|X)|}. \quad (50)$$

Recall the definition of $\mathcal{L}(z, \hat{P}_{xz})$, and $x_i(z, \hat{P}_{xz})$ in (7), and note that by definition, $|\mathcal{L}(z, \hat{P}_{XZ})| \geq 1$ for any $\hat{P}_{XZ}$ which is a possible joint empirical distribution of a channel input-output sequences pair $x, z$. Further recall the assumption that the Y-decoder is informed of the list $\mathcal{L}(Z, \hat{P}_{XZ}) = \{x_i(Z, \hat{P}_{XZ})\}$ and employs the decoding rule (8).

It is easily verified that for any possible channel output $z$ such that $\hat{P}_z = \hat{P}_z$, it holds that $\{x_i(z, \hat{P}_{XZ})\}$ are equiprobable given $\{Z = z, \hat{P}_{XZ} = \hat{P}_{XZ}\}$; that is,

$$P(X = x_i(z, \hat{P}_{XZ})|Z = z, \hat{P}_{XZ} = \hat{P}_{XZ}) = \frac{1}{|\mathcal{L}(z, \hat{P}_{XZ})|}. \quad (51)$$

To see this, note that by applying Bayes’ law twice we have

$$\Pr(X = x_i(z, \hat{P}_{XZ}), Z = z|\hat{P}_{XZ} = \hat{P}_{XZ}) = \Pr(Z = z|\hat{P}_{XZ} = \hat{P}_{XZ}) \cdot \Pr(X = x_i(z, \hat{P}_{XZ})|Z = z, \hat{P}_{XZ} = \hat{P}_{XZ})$$

$$= \Pr(X = x_i(z, \hat{P}_{XZ})|\hat{P}_{XZ} = \hat{P}_{XZ}) \cdot \Pr(Z = z|X = x_i(z, \hat{P}_{XZ}), \hat{P}_{XZ} = \hat{P}_{XZ}). \quad (52)$$

Now, since the code is constant composition, the actual joint type-class $\hat{P}_{XZ}$ does not depend on the codeword $X$, and hence $\Pr(X = x_i(z, \hat{P}_{XZ})|\hat{P}_{XZ} = \hat{P}_{XZ}) = \frac{1}{M_n}$, and further we have $\Pr(Z = z|X = x_i(z, \hat{P}_{XZ}), \hat{P}_{XZ} = \hat{P}_{XZ}) = \frac{1}{|\mathcal{T}_n(P_{Z|X}|X)|}$. This yields

$$\Pr(X = x_i(z, \hat{P}_{XZ})|Z = z, \hat{P}_{XZ} = \hat{P}_{XZ}) = \frac{1}{|\mathcal{T}_n(P_{Z|X}|X)| \cdot M_n \cdot \Pr(Z = z|\hat{P}_{XZ} = \hat{P}_{XZ})}. \quad (54)$$

Since the r.h.s. does not depend on $i$ we obtain the desired result (51).

Next, let

$$E_{ij} \triangleq E_{ij}(z, \hat{P}_{XZ}) \triangleq \{y : q(x_j(z, \hat{P}_{XZ}), y) \geq q(x_i(z, \hat{P}_{XZ}), y)\}, \quad (55)$$

and adopt the shorthand notation

$$x_i \triangleq x_i(z, \hat{P}_{XZ}), \mathcal{L} \triangleq \mathcal{L}(z, \hat{P}_{XZ}). \quad (56)$$

Since $\Pr(error|x_i, z) = \Pr(\cup_{j \neq i} E_{ij}|x_i, z)$, we have the following lower bound on the average error probability in $q$-mismatched decoding at the Y-receiver given that $Z = z$ and $\hat{P}_{XZ} = \hat{P}_{XZ}$,

$$\Pr(error|z, \hat{P}_{XZ} = \hat{P}_{XZ}) \geq \left\{ \frac{1}{|\mathcal{L}||\mathcal{L}|-1} \sum_{i,j \in \mathcal{L}, j \neq i} \Pr(E_{ij}|x_i, z) \right\} \left\{ \frac{1}{|\mathcal{L}|} \geq 2 \right\} \left\{ |\mathcal{L}| = 1 \right\}. \quad (57)$$

Evidently, we shall focus on the case where $(z, \hat{P}_{XZ})$ are such that $|\mathcal{L}| \geq 2$ to obtain a meaningful lower bound on the error probability. Note that

$$\Pr(E_{ij}|x_i, z) = \sum_{y : q(x_j, y) \geq q(x_i, y)} W^n_{Y|XZ}(y|x_i, z) \quad (58)$$
where for an empirical distribution $P_{XZX} \in \mathcal{P}_n(\mathcal{X}^2 \times \mathcal{Z})$ satisfying $P_{XZ} = P_{XZ}$ we define

$$\Omega_n(P_{XZX}, W_{Y|XZ}) \triangleq \min_{V_{XXYZ} \in \mathcal{P}_n(\mathcal{X}^2 \times \mathcal{Z} \times \mathcal{Y}) \cap S(P_{XZX})} D(V_{Y|XZX} \| W_{Y|XZ} | P_{XZX}).$$

Consider the function $\Omega(P_{XZX}, W_{Y|XZ})$ which extends $\Omega_n(P_{XZX}, W_{Y|XZ})$ in a twofold manner: (a) it is defined for $P_{XZX} \in \mathcal{P}(\mathcal{X}^2 \times \mathcal{Z})$ which need not necessarily be an empirical distribution of order $n$, and (b) the minimization is over the simplex $\mathcal{P}(\mathcal{X}^2 \times \mathcal{Z} \times \mathcal{Y})$ rather than empirical distributions; that is,

$$\Omega(P_{XZX}, W_{Y|XZ}) \triangleq \min_{V_{XXY} \in \mathcal{P}(\mathcal{X}^2 \times \mathcal{Z} \times \mathcal{Y})} D(V_{Y|XZX} \| W_{Y|XZ} | P_{XZX}).$$

We present the following approximation lemma whose proof appears in Appendix B

**Lemma 1.** For all $P_{XZX} \in \mathcal{P}_n(\mathcal{X}^2 \times \mathcal{Z}),$

$$\Omega_n(P_{XZX}, W_{Y|XZ}) \leq \Omega(P_{XZX}, W_{Y|XZ}) + \delta_n,$$

where $\delta_n = 2 \cdot \frac{|\mathcal{X}|^2|\mathcal{Z}||\mathcal{Y}|}{n} \log \frac{|\mathcal{X}|^2}{w_{\min}}$, with $w_{\min}$ defined in (47).

Thus,

$$\Pr(\mathcal{E}_{ij} | x_i, z) \geq \frac{1}{(n + 1)|\mathcal{X}|^2|\mathcal{Z}||\mathcal{Y}|} e^{-n\Omega_n(\hat{P}_{x_i, x_j, z}, W_{Y|XZ}) + \delta_n}. \quad (65)$$

Further,

$$\sum_{i \in L, j \neq i} e^{-n\Omega_n(\hat{P}_{x_i, x_j, z}, W_{Y|XZ})}$$

$$= \sum_{i \in L, j \neq i} \frac{1}{2} \left[ e^{-n\Omega_n(\hat{P}_{x_i, x_j, z}, W_{Y|XZ})} + e^{-n\Omega_n(\hat{P}_{x_j, x_i, z}, W_{Y|XZ})} \right]$$

$$\geq \sum_{i \in L, j \neq i} \frac{1}{2} e^{-n\min \{\Omega_n(\hat{P}_{x_i, x_j, z}, W_{Y|XZ}), \Omega_n(\hat{P}_{x_j, x_i, z}, W_{Y|XZ})\}}. \quad (66)$$

where (66) follows by switching the roles of the summation indices and multiplying and dividing by 2, (67) follows since for positive $A, B$, we have, $A + B \geq \max\{A, B\}$, and since $f(t) = e^{-t}$ is a monotonically decreasing function.

Now, in analogy to the definition of $S(P_{XZX})$ in (62), define the following set of conditional distributions rather than joint distributions

$$S_{cond}(P_{XZX}) \triangleq \left\{ V_{Y|XXY} : P_{XZX} \times V_{Y|XXY} \in S(P_{XZX}) \right\}. \quad (68)$$

Next, observe that since $\Omega_n(\hat{P}_{XZX}, W_{Y|XZ}) = \min_{V_{Y|XXY} \in S_{cond}(\hat{P}_{XZX})} D(V_{Y|XZX} \| W_{Y|XZ} | \hat{P}_{XZX}),$

$$\min \left\{ \Omega(\hat{P}_{x_i, x_j, z}, W_{Y|XZ}), \Omega(\hat{P}_{x_j, x_i, z}, W_{Y|XZ}) \right\}$$

$$\geq \frac{1}{(n + 1)|\mathcal{X}|^2|\mathcal{Z}||\mathcal{Y}|} e^{-n\Omega_n(\hat{P}_{x_i, x_j, z}, W_{Y|XZ}) + \delta_n}. \quad (65)$$

Further,
To this end we present a few definitions and two lemmas. 

Next, we show that for any $x \in L$ 

\[\min_{V|XZ} D(V|XZ) \leq \min \left\{ \min_{V|XY \in S_{\text{cond}}(\hat{P}_{x|z})} D(V|XZ), \min_{V|XZ \in S_{\text{cond}}(\hat{P}_{x|z})} D(V|XZ) \right\} \]

\[= \min_{V|XZ \in S_{\text{cond}}(\hat{P}_{x|z}) \cup S_{\text{cond}}(\hat{P}_{x|z})} D(V|XZ) \]

\[\leq \min_{V|XZ \in S_{\text{cond}}(\hat{P}_{x|z}) \cup S_{\text{cond}}(\hat{P}_{x|z})} D(V|XZ), \]

where (70) holds since $D(V|XZ|\hat{P}_{x|z}) = D(V|XZ|\hat{P}_{x|z}) + I(V; \hat{X}|X, Z)$, and since $\forall i, \hat{P}_{x|z} = \hat{P}_{x|z}, (70)$ holds since $\min_{\hat{P}_{x|z}} \{ f(t), \min_{\hat{P}_{x|z}} f(t) \} = \min_{\hat{P}_{x|z}} f(t), (71)$ follows since $S_{\text{cond}}(1/2) \subseteq S_{\text{cond}}(\hat{P}_{x|z}) \cup S_{\text{cond}}(\hat{P}_{x|z})$ which follows from the linearity of the expectation; that is, 

\[S_{\text{cond}}(\hat{P}_{x|z}) \cup S_{\text{cond}}(\hat{P}_{x|z}) \]

\[= \left\{ V|XZ \right\} \left\{ \hat{P}_{x|z} \times V|XZ \Rightarrow [q(\hat{X}, Y) - q(X, Y)] \geq 0 \text{ or } \hat{P}_{x|z} \times V|XZ \Rightarrow [q(\hat{X}, Y) - q(X, Y)] \geq 0 \right\} \]

\[\geq \left\{ V|XZ \right\} \left\{ \hat{P}_{x|z} \times V|XZ \Rightarrow [q(\hat{X}, Y) - q(X, Y)] \geq 0 \right\}. \]

Gathering (57), (65), (67), (71), and denoting $k_n = \delta_n + \frac{|X||Z||Y|}{n} \log(n + 1) + \frac{\ln(2)}{n}$ we obtain that whenever $|L(z, \hat{P}_{XZ})| \geq 2$, 

\[\text{Pr}(\text{error}|z, \hat{P}_{XZ} = \hat{P}_{XZ}) \geq \min_{x_i \in L, x_j \in L, j \neq i} \exp \left\{ -n \left[ \min_{V_{XXY} \in S_{\text{cond}}(1/2)} D(V|XZ) \right] \right\}. \]

Next, we show that for any $x_i, x_j \in L$ 

\[\min_{V_{XYZ} \in S_{\text{cond}}(1/2)} D(V|XZ) \leq \min_{V_{XYZ} \in W_q(\hat{P}_{X})} D(V|XZ) \]

by establishing that if $V_{XYZ}$ is such that $V_{YZ|X} \in W_q(\hat{P}_{X})$ and $V_{XZ} = \hat{P}_{XZ}$, then for any $z$ and $x_i, x_j \in L(z, \hat{P}_{XZ})$, we have 

\[\mathbb{E}_{\hat{P}_{x|z}} [q(\hat{X}, Y) - q(X, Y)] \geq 0. \]

To this end we present a few definitions and two lemmas. 

For any collection of codewords $L' \subseteq L$ such that $|L'| \geq 2$, let 

\[P_{XZ}(x, \bar{x}, z) \triangleq \frac{1}{|L'|(|L'|-1)} \sum_{i,j \in L', j \neq i} \hat{P}_{x|z}(x, \bar{x}, z), \]

We next define an additional distribution $P_{TZX \bar{X}}^{L'}$. Let $T$ be a random variable uniformly distributed over $\{1, \ldots, n\}$ and define 

\[P_{TZ}(t, z) \triangleq \frac{1}{n} \cdot \mathbb{1}_{\{z = z(t)\}}. \]
The following lemma detects a few relations between $P_{X|T}^{C'}$ and $\overline{P}_{X|T}^{C'}$.

**Lemma 2.** For any collection of codewords $C' \subseteq C$ such that $|C'| \geq 2$ and any $z$, $x \neq \tilde{x}$,

$$P_{XZ}^{avg,C'}(x, \tilde{x}, z) = \frac{|C'|}{|C'| - 1} \cdot P_{XZ}^{C'}(x, \tilde{x}, z). \tag{81}$$

Further, for any given $P_{Y|XZ}$ letting $P_{XZY}^{C'} = P_{XZ}^{C'} \times P_{Y|XZ}$, and $P_{XZ}^{avg,C'} = P_{XZ}^{avg,C'} \times P_{Y|XZ}$, one has

$$q(P_{XY}^{avg,C'}) - q(P_{XY}^{avg,C'}) \geq 0 \iff q(P_{XY}^{C'}) - q(P_{XY}^{C'}) \geq 0 \tag{82}$$

**Proof.** Recall that all members of $C'$ lie in $\mathcal{T}_n(P_X|Z)$. Next, based on Plotkin’s counting idea (similar to [28]), we obtain that for all $(x, \tilde{x}, z) \in \mathcal{X}^2 \times \mathcal{Z}$ such that $x \neq \tilde{x}$,

$$|C'|(|C'|-1) \cdot P_{XZ}^{avg,C'}(x, \tilde{x}, z) \tag{83}$$

$$= \sum_{i, j \notin \in C'} \hat{P}_{x_j|z}(x, \tilde{x}, z) \tag{84}$$

$$= \sum_{j \in C', i \notin C'} \hat{P}_{x_j|z}(x, \tilde{x}, z) - \sum_{j \in C'} \hat{P}_{x_j|z}(x, \tilde{x}, z) \tag{85}$$

$$= \frac{1}{n} \sum_{t=1}^{n} \left[ \sum_{j \in C', i \notin C'} \hat{1}_{(x_j(t)=x, z(t)=z, x_i(t)\neq \tilde{x})} \right] \tag{86}$$

$$= \frac{1}{n} \sum_{t=1}^{n} \hat{1}_{(z(t)=z)} \left[ \sum_{j \in C', i \notin C'} \hat{1}_{(x_j(t)=x)} \cdot \hat{1}_{(x_i(t)\neq \tilde{x})} \right] \tag{87}$$

$$= \frac{1}{n} \sum_{t=1}^{n} \hat{1}_{(z(t)=z)} \left[ \sum_{j \in C'} \hat{1}_{(x_j(t)=x)} \cdot \sum_{i \in C'} \hat{1}_{(x_i(t)\neq \tilde{x})} \right] \tag{88}$$

$$\triangleq |C'|^2 \sum_{t=1}^{n} P_{TZ}^{C'}(t, z) \cdot P_{X|T}^{C'}(x|t) \cdot P_{X|T}^{C'}(x|t),$$

where (85) follows since $\sum_{j \in C'} \hat{P}_{x_j|z}(x, \tilde{x}, z) = 0$ whenever $x \neq \tilde{x}$, and (87) follows since the indicator $\hat{1}_{(x_j(t)=x)}$ does not depend on $i$. This yields (81). To prove (82), for given $P_{Y|XZ}$ denote $P_{XZY}^{C'} = P_{XZ}^{C'} \times P_{Y|XZ}$, and $P_{XZ}^{avg,C'} = P_{XZ}^{avg,C'} \times P_{Y|XZ}$, and note that from (81) it follows that

$$q(P_{XY}^{avg,C'}) - q(P_{XY}^{avg,C'}) = \frac{|C'|}{|C'| - 1} \cdot \left( q(P_{XY}^{C'}) - q(P_{XY}^{C'}) \right), \tag{89}$$

and thus (82) follows.

The following lemma shows that the random variable $T$ (uniformly distributed over $\{1, ..., n\}$) can be replaced by another random variable $U$ of finite alphabet cardinality that does not increase with $n$.

**Lemma 3.** There exists a random variable $U$ whose alphabet size is $|U| \leq |\mathcal{X}|^2 |\mathcal{Z}|$, and a joint distribution $P_{UXZ} \in \mathcal{P}(\mathcal{U} \times \mathcal{X} \times \mathcal{Z})$ such that for any $(x, \tilde{x}, z)$,

$$\sum_{u} P_{UZ}(u, z) P_{X|UZ}(x|u, z) P_{X|UZ}(\tilde{x}|u, z) = P_{XZ}(x, \tilde{x}, z). \tag{90}$$
There also exists a joint distribution $P_{UXZ} \in \mathcal{P}(U \times X \times Z)$ such that for any $(x, \bar{x}, z)$ (90) holds, $Z$ is a deterministic function of $U$, and $|U| \leq |X|^2|Z| + 1$.

Lemma 3 is proved in Appendix D. It remains to show that (76) holds for any $V_{XYZ}$ such that $V_{YZ|X} \in \mathcal{W}_q(\hat{P}_X)$ and $V_{XZ} = \hat{P}_{XZ}$. Let $\hat{\mathcal{L}} = \{x_i, \bar{x}_j\}$. Obviously, $\hat{\mathcal{L}} \subseteq \mathcal{L}$, and thus we can invoke (82) to obtain

$$
\mathbb{E}_{\hat{\mathcal{L}}}[\hat{P}_{u,x,z} + \hat{P}_{\bar{x},z}] \cdot \frac{(91)}{|\mathcal{L}| - 1} \sum_{u,x,z,y} P_{UXZ}(x, \bar{x}, z) \cdot V(y|x, z) [q(\bar{x}, y) - q(x, y)]
$$

(92)

$$
\geq \min_{P_{UXZ}} \frac{|\mathcal{L}|}{|\mathcal{L}| - 1} \sum_{u,x,z,y} P_{UXZ}(u, z) P_{X|Z}(x|u, z) P_{X|Z}(\bar{x}|u, z) \cdot V(y|x, z) [q(\bar{x}, y) - q(x, y)]
$$

(93)

$$
\geq 0,
$$

(94)

where (91) follows by definition of $P_{UXZ}^\hat{\mathcal{L}}$ (92) follows from (81), (93) follows by definition of $P_{UXZ}^\hat{\mathcal{L}}$ and from Lemma 3 and (94) follows since $V_{YZ|X} \in \mathcal{W}_q(\hat{P}_X)$ and $V_{XZ} = \hat{P}_{XZ}$. Thus, we have shown that (76) holds and consequently also (75).

Hence, from (74) and (75), we obtain that whenever $|\mathcal{L}(z, \hat{P}_{XZ})| \geq 2$,

$$
\Pr(\text{error} | z, \hat{P}_{XZ} = \hat{P}_{XZ}) \geq e^{-n \left[ \min_{V_{YZ|X} \in \mathcal{W}_q(\hat{P}_X)} V_{XYZ} = \hat{V}_{XZ} \right] D(V_{YZ|X} || V_{YZ|X}) + h_n}\]

(95)

We next present a lemma which enables to assess the size of the list $\mathcal{L}(Z, \hat{P}_{XZ})$ defined in (7).

Lemma 4. Let a codebook $C_n = \{x_i \}_{i=1}^{M_n}$ be given, let $X$ denote the random codeword (distributed uniformly over $C_n$), and let $Z$ denote the output of the channel $W_{Z|X}$ when fed by $X$. For any $\tau \geq 0$,

$$
\Pr \left( |\mathcal{L}(Z, \hat{P}_{XZ})| \geq e^{n\tau} | \hat{P}_{XZ} = \hat{P}_{XZ} \right) \geq 1 - (n + 1)^{|X||Z|-1} \cdot e^{-n[4(\hat{P}_{XZ}) - R - \tau]}
$$

(96)

Proof. From the law of total probability

$$
\Pr \left( |\mathcal{L}(Z, \hat{P}_{XZ})| < e^{n\tau} | \hat{P}_{XZ} = \hat{P}_{XZ} \right)
$$

(97)

$$
= \frac{1}{M_n} \sum_{i=1}^{M_n} \Pr \left( |\mathcal{L}(Z, \hat{P}_{XZ})| < e^{n\tau} | X = x_i, \hat{P}_{XZ} = \hat{P}_{XZ} \right)
$$

(98)

$$
= \frac{1}{M_n} \sum_{i=1}^{M_n} \left| \{z \in T_n(\hat{P}_{XZ}|x_i) : |\mathcal{L}| < e^{n\tau} \} \right| \cdot \frac{1}{|T_n(\hat{P}_{XZ}|x_i)|}
$$

(99)

$$
\leq \frac{1}{M_n} \cdot e^{n\tau} \cdot \frac{|T_n(\hat{P}_{XZ})|}{|T_n(\hat{P}_{XZ}|x_i)|}
$$

(100)

$$
\leq \frac{1}{M_n} \cdot e^{n\tau} \cdot \frac{|T_n(\hat{P}_{XZ})|}{|T_n(\hat{P}_{XZ}|x_i)|}
$$

(101)

$$
\leq (n + 1)^{|X||Z|-1} \cdot e^{-n[4(\hat{P}_{XZ}) - R - \tau]}
$$

(102)

$$
eq e^{-n[4(\hat{P}_{XZ}) - R - \tau] - \frac{|X||Z|-1}{n}| \log(n+1)|}
$$

(103)

where (99) follows since $Z$ is uniform over $T_n(\hat{P}_{XZ}|x_i)$ given $x_i$, (101) holds by replacing the count over codewords by a count over sequences $z$, and (102) follows by a standard bound on the size of a type-class.
Note that Lemma 4 implies that for any \( \hat{P}_{XZ}, \tilde{e}_n > 0 \), and \( \epsilon_n > 0 \), such that
\[
R \geq I(\hat{P}_{XZ}) + \epsilon_n + \frac{\|X\|Z| - 1}{n} \log(n + 1) + \tilde{e}_n,
\] (104)
it holds that
\[
\Pr \left( |\mathcal{L}(Z, \hat{P}_{XZ})| < e^{n\epsilon_n} | \hat{P}_{XZ} = \hat{P}_{XZ} \right) \leq e^{-n\tilde{e}_n}.
\] (105)

Consequently, if \( \hat{P}_{XZ} \) is a possible joint empirical distribution of a codeword and a channel output \( Z \) such that (104) holds, we have for \( \epsilon_n > 1/n \) that
\[
\Pr(\text{error} | \hat{P}_{XZ}) \geq \Pr(\text{error}, \mathcal{L}(Z, \hat{P}_{XZ}) \geq e^{n\epsilon_n} | \hat{P}_{XZ} = \hat{P}_{XZ})
\]
\[
\geq (1 - e^{-n\tilde{e}_n}) \cdot \Pr(\text{error} | \hat{P}_{XZ} = \hat{P}_{XZ}, \mathcal{L}(Z, \hat{P}_{XZ}) \geq e^{n\epsilon_n})
\]
\[
\geq (1 - e^{-n\tilde{e}_n}) \cdot \min_{z \in \mathcal{T}_n(\hat{P}_{XZ}) : |z| \geq e^{n\epsilon_n}} \Pr(\text{error} | \hat{P}_{XZ} = \hat{P}_{XZ}, Z = z)
\]
\[
\geq (1 - e^{-n\tilde{e}_n}) \cdot e^{-n\epsilon_n} (15) \cdot \frac{1}{|X||Z|} \log(1 - e^{-n\tilde{e}_n}) + \frac{\|X\|Z|}{n} | \log(n + 1) + \tilde{e}_n, (107)
\]
where the last step follows from (95).

Now, let
\[
\Psi \left( P_{XZ}, W_{Y|XZ} \right) \triangleq \min_{V_{XYZ} \in \mathcal{W}_{q}(\hat{P}_{X}) : V_{XZ} = P_{XZ}} D(V_{Y|XZ} \| W_{Y|XZ} | P_{XZ}).
\] (110)

Since (109) holds for every \( \hat{P}_{XZ} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Z}) \) such that \( \hat{P}_{X} = P_{X} \), which is a possible empirical distribution of \( X, Z \), and which satisfies (104), and since
\[
\Pr(\text{error}) \geq \sum_{\hat{P}_{XZ} : \hat{P}_{X} = P_{X}} \frac{1}{(n + 1)^{|X||Z|}} e^{-nD(\hat{P}_{Z|X} \| P_{XZ} \| P_{X})}. \Pr(\text{error} | \hat{P}_{XZ})
\] (111)
denoting \( \delta''_n = k_n - \frac{1}{n} \log(1 - e^{-n\tilde{e}_n}) + \frac{|X||Z|}{n} | \log(n + 1) + \tilde{e}_n \), we get
\[
- \frac{1}{n} \log \Pr(\text{error}) \leq \min_{\hat{P}_{XZ} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Z}) : \hat{P}_{X} = P_{X}} D(\hat{P}_{Z|X} \| W_{Z|X} | \hat{P}_{X}) + \Psi \left( \hat{P}_{XZ}, W_{Y|XZ} \right) + \delta''_n,
\] (112)
where \( \epsilon''_n \triangleq \epsilon_n + \frac{|X||Z|}{n} | \log(n + 1) + \tilde{e}_n \). The following lemma shows that the minimization over empirical conditional distributions \( \mathcal{P}_n(\mathcal{X} \times \mathcal{Z}) \) can be approximated by minimization over the simplex \( \mathcal{P}(\mathcal{X} \times \mathcal{Z}) \).

**Lemma 5.** For any \( P_{X} \in \mathcal{P}_n(\mathcal{X}) \), and any \( P_{YZ|X} \in \mathcal{W}_{q}(P_{X}) \), there exists \( \hat{P}_{XZ} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Z}) \) such that \( \hat{P}_{X} = P_{X} \), \( \hat{P}_{Z|X} \times P_{Y|XZ} \in \mathcal{W}_{q}(P_{X}) \),
\[
\|P_{X} \times P_{Z|X} - \hat{P}_{XZ}\| \leq \frac{|X||Z|}{n}, \quad \text{and} \quad P_{Z|X}(z|x) = 0 \Rightarrow \hat{P}_{Z|X}(z|x) = 0.
\] (113)

**Lemma 5** is proved in Appendix 1.3.

Next, let \( P_{XZ}^* \) be the minimizer of (110). Note that \( \|P_{XZ}^* - \hat{P}_{XZ}\| \leq \frac{|X||Z|}{n} \) implies \( D(P_{Z|X}^* \| W_{YZ|X} | P_{X}) - D(\hat{P}_{Z|X} \times P_{Y|XZ}^* \| W_{YZ|X} | P_{X}) \) \( \leq 2 \frac{|X||Z|}{n} | \log n + \frac{|X||Z|}{n} | \log \frac{n}{w_{\min}} \), where \( w_{\min} \) is defined in (49) and further, \( I(P_{XZ}^*) - I(P_{X} \times \hat{P}_{Z|X}) \leq 2 \frac{|X||Z|}{n} | \log n \), \( \delta_{1,n} \).

This implies that for any \( P_{X} \in \mathcal{P}_n(\mathcal{X}) \),
\[
\min_{\hat{P}_{XZ} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Z}) : \hat{P}_{X} = P_{X}, \ I(\hat{P}_{XZ}) \leq R} D(\hat{P}_{Z|X} \| W_{Z|X} | \hat{P}_{X}) + \Psi \left( \hat{P}_{XZ}, W_{Y|XZ} \right)
\]
where $P$.

Lemma 6. If $P_{XZ} \in A_q(W_{Y|XZ}, P_n) \cap P_n(\mathcal{X} \times \mathcal{Z})$, then for any subset $\mathcal{L}' \subseteq \mathcal{L}(z, \widehat{P}_{XZ})$ such that $|\mathcal{L}'| \geq 2$, $x, \tilde{x}, z$ in $\mathcal{L}$, either

$$
\sum_{x, \tilde{x}, z} P_{XZ}^\text{avg,}\mathcal{L}'(x, \tilde{x}, z) \cdot W(y|x, z) [q(\tilde{x}, y) - q(x, y)] \geq 0,
$$

where $P_{XZ}^\text{avg,}\mathcal{L}'$ is defined in (77), and consequently, for any pair of codewords $x, z$ in $\mathcal{L}$, either

$$
\sum_{x, \tilde{x}, z} \widehat{P}_{x, z}(x, \tilde{x}, z) \cdot W(y|x, z) [q(\tilde{x}, y) - q(x, y)] \geq 0.
$$
Proof. Recall the definition of $\overline{P}_{X\tilde{X}Z}$ in (80), hence similar to (94),

$$
\frac{1}{|\mathcal{L}'|(|\mathcal{L}'| - 1)} \sum_{i,j \in \mathcal{L}'} \sum_{x,\bar{x},z,y} \hat{P}_{x_{i \neq j}}(x, \bar{x}, z) \cdot W(y|x, z) \left[ q(\bar{x}, y) - q(x, y) \right]
$$

(124)

$$
= \sum_{x,\bar{x},z,y} P_{\text{avg},L'}(x, \bar{x}, z) \cdot W(y|x, z) \left[ q(\bar{x}, y) - q(x, y) \right]
$$

(125)

$$
= \frac{|\mathcal{L}'|}{|\mathcal{L}'| - 1} \sum_{x,\bar{x},z,y} \mathcal{P}_{X\tilde{X}Z}(x, \bar{x}, z) \cdot W(y|x, z) \left[ q(\bar{x}, y) - q(x, y) \right]
$$

(126)

$$
\geq \min_{P_{U|XZ} : P_{XZ} = \overline{P}_{XZ}} \frac{|\mathcal{L}'|}{|\mathcal{L}'| - 1} \sum_{u,x,\bar{x},z,y} P_{UZ}(u, z) P_{X|UZ}(x|u, z) P_{X|UZ}(\bar{x}|u, z) \cdot W(y|x, z) \left[ q(\bar{x}, y) - q(x, y) \right]
$$

(127)

$$
\geq 0,
$$

(128)

where we abbreviate $W = W_{Y|XZ}$, (126) follows (81), (127) follows by definition of $\overline{P}_{X\tilde{X}Z}$ (see (80)), and the last step is by definition of $\mathcal{A}_q(P_n, W_{Y|XZ})$.

The next lemma shows that for at least half of the pairs of distinct codewords, the pairwise error probability is bounded away from zero. It is based on [17] Lemma 3], which was used by Csiszár and Narayan to establish a necessary and sufficient condition for the positivity of the mismatch capacity.

Lemma 7. Assume (118) holds. There exists $\nu > 0$, such that for all $n$ sufficiently large, for any $P_n \in \mathcal{P}_n(X)$, any $\overline{P}_{XZ} \in \mathcal{A}_q(W_{Y|XZ}, P_n) \cap \mathcal{T}_n(\overline{P}_{XZ})$ and any $z \in \mathcal{T}_n(\overline{P}_{XZ})$ which satisfies $|\mathcal{L}(z, \overline{P}_{XZ})| \geq 2$, any pair of codewords $x_{\ell}, x_k$ in $\mathcal{L}(z, \overline{P}_{XZ})$ such that $z$ is a possible output of $W^n_{Z|X}$ when fed by $x_{\ell}$ or $x_k$ satisfies either

$$
W^n_{Y|XZ}(q(x_k, Y) \geq q(x_{\ell}, Y) | X = x_{\ell}, Z = z) > \nu,
$$

(129)

or

$$
W^n_{Y|XZ}(q(x_{\ell}, Y) \geq q(x_k, Y) | X = x_k, Z = z) > \nu.
$$

(130)

Proof: From Lemma 6 we know that for any pair of codewords $x_{\ell}, x_k$ in $\mathcal{L}(z, \overline{P}_{XZ})$ we have either (122) or (123). Assume w.l.o.g. that (122) holds, then clearly,

$$
W^n_{Y|XZ}(q(x_k, Y) \geq q(x_{\ell}, Y) | X = x_{\ell}, Z = z)
$$

$$
\geq W^n_{Y|XZ} \left( q(x_k, Y) \geq q(x_{\ell}, Y) + \left[ \sum_{x,\bar{x},z,y} \hat{P}_{x_{i \neq j}}(x, \bar{x}, z) W(y|x, z) \left[ q(\bar{x}, y) - q(x, y) \right] \right] | X = x_{\ell}, Z = z \right)
$$

(131)

$$
\Delta = W^n_{Y|XZ} \left( \sum_{i=1}^n S_i \geq 0 | X = x_{\ell}, Z = z \right),
$$

(132)

where $\{S_i\}$ are the following random variables satisfying $\mathbb{E}(S_i | x_{\ell}, z) = 0$,

$$
S_{i,\ell,k} = [q(x_{\ell,i}, Y_i) - q(x_{k,i}, Y_i)] - \mathbb{E}[q(x_{\ell,i}, Y_i) - q(x_{k,i}, Y_i)|Z_i = z_i, X_i = x_{\ell,i}].
$$

(133)
Note also that the distribution of $S_i$ given $(x_\ell, z)$ depends only on $(x_{\ell,i}, x_{k,i}, z_i)$, and therefore, there is a finite set, denoted $\hat{\mathcal{P}}$ (of size not exceeding $|Z|\|X\|^2$) of conditional distributions that $S_i$ can have given $(x_\ell, z)$. Hence, it follows from [17 Lemma 3] that if the $q(\cdot)$ values are bounded, there exists a constant $\nu > 0$ such that for all $n$,

$$W^n_{XZ} \left( \sum_i S_i \geq 0 \middle| x_\ell, z \right) > \nu. \tag{134}$$

It remains to treat infinite values of $\mathbb{E}(q(x_k, Y)|x_\ell, z)$ and $\mathbb{E}(q(x_\ell, Y)|x_\ell, z)$: Since we assume metric values $q \in \mathbb{R} \cup \{-\infty\}$, the cases of interest for their values are limited to $(c, -\infty), (-\infty, c), (-\infty, -\infty)$, where $c$ represents a finite constant, respectively. The case $(-\infty, c)$ yields the inequality (134) trivially.

Since we assume that $z$ is a possible output of $W^n_{Z|X}$ when fed by $x_\ell$ (or $x_k$), the cases $(c, -\infty)$ and $(-\infty, -\infty)$; i.e., $\mathbb{E}(q(x_\ell, Y)|x_\ell, z) = -\infty$, imply that there exists a symbol $x_0$ in $x_\ell$ and a pair $(x_0, y_0)$ such that $W(y_0|x_0) > 0$ and $q(x_0, y_0) = -\infty$, in contradiction to assumption (118).

Now, since the members of $\mathcal{L}(z, \hat{P}_{XZ})$ are equiprobable given $(z, \hat{P}_{XZ})$ (see (51)), we obtain from Lemma [7] that if $|\mathcal{L}(z, \hat{P}_{XZ})| \geq 2$, there exists $\nu > 0$ such that

$$P(error|\hat{P}_{XZ} = \hat{P}_{XZ}, Z = z) \geq \frac{1}{2\nu} \cdot \mathbb{1}\{\hat{P}_{XZ} \in \mathcal{A}_q(W_{Y|X,Z}, P_n)\}. \tag{135}$$

Next recall (104)-(108), stating that for any $\hat{P}_{XZ}$, $\epsilon_n > 0$, and $\tilde{\epsilon}_n > 0$ such that

$$R_n \geq I(\hat{P}_{XZ}) + \epsilon_n + \frac{|\mathcal{X}| |Z| - 1}{n} \log(n + 1) + \tilde{\epsilon}_n, \tag{136}$$

it holds that $\Pr\left(\mathcal{L}(z, \hat{P}_{XZ}) < e^{n\epsilon_n}|\hat{P}_{XZ} = \hat{P}_{XZ}\right) \leq e^{-n\tilde{\epsilon}_n}$, and also

$$\Pr(error|\hat{P}_{XZ} = \hat{P}_{XZ}) \geq (1 - e^{-n\tilde{\epsilon}_n}) \cdot \min_{z \in T_n(\hat{P}_{XZ}) \cap \mathcal{L}(z, \hat{P}_{XZ}) \geq e^{n\epsilon_n}} \Pr(error|\hat{P}_{XZ} = \hat{P}_{XZ}, Z = z). \tag{137}$$

and thus, we get from (135) and (137) that if (136) holds,

$$\Pr(error|\hat{P}_{XZ} = \hat{P}_{XZ}) \geq (1 - e^{-n\tilde{\epsilon}_n}) \cdot \frac{1}{2\nu} \cdot \mathbb{1}\{\hat{P}_{XZ} \in \mathcal{A}_q(W_{Y|X,Z}, P_n)\}. \tag{138}$$

Next, take a vanishing sequence $c_n$ that satisfies $\lim_{n \to \infty} n[c_n - \frac{1}{n}|\mathcal{X}| |Z| \log(n + 1)] = \infty$; we have

$$\Pr(D(\hat{P}_{Z|X}||W_{Z|X}|P_n) > c_n|X = x) = \sum_{z: D(\hat{P}_{Z|X}||W_{Z|X}|P_n) > c_n} W^n_{Z|X}(z|x) \tag{139}$$

$$\leq (n + 1)^{|\mathcal{X}| |Z| - 1} \max_{\hat{P}_{Z|X}: D(\hat{P}_{Z|X}||W_{Z|X}|P_n) > c_n} e^{-nD(\hat{P}_{Z|X}||W_{Z|X}|P_n)} \tag{140}$$

$$\leq e^{-n[c_n - \frac{1}{n}|\mathcal{X}| |Z| \log(n + 1)]}. \tag{141}$$

denote $f_n = c_n - \frac{1}{n}|\mathcal{X}| |Z| \log(n + 1)$, and $d_n = \epsilon_n + \frac{|\mathcal{X}| |Z| - 1}{n} \log(n + 1) + \tilde{\epsilon}_n$, we have

$$\Pr(error) = \sum_{\hat{P}_{XZ} \in \mathcal{A}_q(W_{Y|X,Z}, P_n) \cap \mathcal{T}_n(\hat{P}_{XZ}) \cap \mathcal{L}(\hat{P}_{XZ})} \Pr(\hat{P}_{XZ} = \hat{P}_{XZ}) \cdot \Pr(error|\hat{P}_{XZ} = \hat{P}_{XZ}) \geq \sum_{\hat{P}_{XZ} \in \mathcal{A}_q(W_{Y|X,Z}, P_n) \cap \mathcal{T}_n(\hat{P}_{XZ}) \cap \mathcal{L}(\hat{P}_{XZ})} \Pr(\hat{P}_{XZ} = \hat{P}_{XZ}) \cdot \Pr(error|\hat{P}_{XZ} = \hat{P}_{XZ}) \tag{142}$$
\[ \geq (1 - e^{-nf_n}) \cdot (1 - e^{-n\epsilon_n}) \cdot \frac{1}{2} \nu \cdot 1 \left\{ \exists \hat{P}_{XZ} \in \mathcal{K}_q(R, P_n, W_{YZ|X}) \right\}, \] (143)

where
\[ \mathcal{K}_q(R, P_n, W_{YZ|X}) \triangleq \left\{ \hat{P}_{XZ} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Z}) : \begin{array}{l}
D(\hat{P}_{Z|X}||W_{Z|X}|P_n) \leq f_n, \\
R_n \geq I(\hat{P}_{XZ}) + d_n, \\
\hat{P}_{XZ} \in A_q(W_{YZ|X}, P_n) \end{array} \right\}. \] (144)

The following lemma concludes the proof of Theorem 1.

**Lemma 8.** For any \( P_n \in \mathcal{P}_n(\mathcal{X}) \) and \( W_{YZ|X} \in \mathcal{W}_q(P_n) \) if \( R > I(P_n \times W_{Z|X}) + \epsilon \), then the set \( \mathcal{K}_q(R, P_n, W_{YZ|X}) \) is non-empty, and consequently there exists \( \nu > 0 \) (which does not depend on \( P_n \)), such that for any \( C_n \subseteq T_n(P_n) \), such that \( |C_n| > e^{nR} \), we have \( P_e(W, C_n, q) > (1 - e^{-nf_n}) \cdot (1 - e^{-n\epsilon_n}) \cdot \frac{1}{2} \nu \).

Lemma 8 is proved in Appendix E, and this concludes the proof of Theorem 1.

**VII. Simpler Bounds on the Reliability Function and Sufficient Conditions for Tightness**

We next show how looser yet simpler bounds on the reliability function using the method of [21] can be derived, and these bounds provide sufficient conditions for tightness for certain ranges of rates.

In [21], the multicast transmission proof technique was proposed, where the same message is transmitted simultaneously to two decoders over a two-outputs (broadcast) channel \( W_{YZ|X} \) with two outputs \( Y \) and \( Z \). The \( Z \)-decoder employs an additive decoding metric \( \rho \) that can be optimized, and the two-outputs channel belongs to the set
\[ \Gamma(q, \rho) \triangleq \left\{ P_{YZ|X} : \forall (x, y, z) : \rho(x, z) - q(x, y) < \max_{x' \in \mathcal{X}} [\rho(x', z) - q(x', y)] \right\}. \] (145)

It is easily verified (see details in [21]), that any two-outputs channel in this class has the property that an error occurs at the \( Z \) decoder, only if the \( Y \)-receiver makes an error. Thus, for any codebook \( C_n \), we have \( P_e(W_{Y|X}, C_n, q) \geq P_e(W_{Z|X}, C_n, \rho) \) and the following bound holds for any stationary memoryless channel.
\[ C_q(W) \leq \max_{P_{X}} \min_{P_{Y|X} \in \Gamma(q, \rho) : P_{Y|X} = W} I(X; Z), \] (146)

For exactly the same reason, the following bound on the reliability function can be deduced.

**Theorem 4.** For all \( Z \), additive metrics \( q, \rho \), and a stationary memoryless channel \( W \)
\[ E^q(R, W) \leq \min_{P_{Y|X} \in \Gamma(q, \rho) : P_{Y|X} = W} E(R, P_{Y|X}). \] (147)

Additionally, as mentioned above, in [21] equivalence classes of isomorphic channel-metric pairs \( (W, q) \) that share the same mismatch capacity for additive metrics were introduced. The following definition of [21] is repeated here for completeness.

**Definition 1.** A channel-metric pair \( (P_{Z|X}, \rho) \) is superior to the channel-metric pair \( (P_{Y|X}, q) \) if there exists a joint conditional distribution \( P_{YZ|X} \in \Gamma(q, \rho) \), whose marginal conditional distributions are \( P_{Y|X} \) and \( P_{Z|X} \). The superiority relation is denoted by \( (P_{Y|X}, q) \rightarrow (P_{Z|X}, \rho) \). If both \( (P_{Y|X}, q) \rightarrow (P_{Z|X}, \rho) \) and \( (P_{Z|X}, \rho) \rightarrow (P_{Y|X}, q) \), denote \( (P_{Y|X}, q) \leftrightarrow (P_{Z|X}, \rho) \), and the pairs are called isomorphic.

It was proved that if one of the pairs in the class is matched, then the mismatch capacity of the entire class is fully characterized and equal to that of the matched pair. The following theorem follows straightforwardly for exactly the same reason of superiority/equivalence.
Theorem 5. If \((W, q) \Rightarrow (P_{Z|X}, \rho)\) then
\[ E^q(R, W) \leq E^\rho(R, P_{Z|X}), \]  
(148)
and consequently, if \((W, q) \Leftrightarrow (P_{Z|X}, \rho)\) then
\[ E^q(R, W) = E^\rho(R, P_{Z|X}). \]  
(149)
If there exists a matched channel-metric pair \((\tilde{P}_{Z|X}, \tilde{q}_{ML})\) where \(\tilde{q}_{ML} = \log \tilde{P}_{Z|X}\) is the maximum likelihood metric w.r.t. \(\tilde{P}_{Z|X}\) such that \((W, q) \Leftrightarrow (\tilde{P}_{Z|X}, \tilde{q}_{ML})\) then
\[ E^q(R, W) = E(R, \tilde{P}_{Z|X}). \]  
(150)
Note that the theorem implies that for the range of rates such that \(E(R, \tilde{P}_{Z|X})\) is known; e.g., at \(R = 0^+\) or above the critical rate where the tangential straight line bound meets the sphere packing bound, the \(E^q(R, W)\) is known as well.

Theorem \([4]\) can be extended to yield tighter bounds for larger classes of channels that depend also on the codebook composition \(P\) using a similar approach. For example, using the set (see \([21]\))
\[ \Gamma(q, \rho, P) = \left\{ P_{YZ|X} : \min_{V_{XYZ}: V_{X}=V_{Y}=V_{Z}} \left[ q(V_{XY}) - q(V_{YY}) \right] \geq 0 \right\}. \]  
(151)

VIII. Binary-Input Channels

As mentioned above, in \([17]\), a single-letter expression for the mismatch capacity in the case of the binary input binary output channel. The single-letter converse result reported in \([32]\) for binary-input DMCs (with \(2 < |Y| < \infty\)) was disproved in \([15]\). Specifically, a rate based on superposition coding was shown to exceed the claimed mismatch capacity of \([32]\).

The following lemma specifies explicit expressions for \(C_q^{sym}(W), \tilde{C}_q^{sym}(W), \) and \(C_q^{psd}(W)\) for binary input DMCs.

Lemma 9. Let \(W_{Y|X}\) be a DMC with a binary input alphabet \(X = \{0, 1\}\), and let \(|Z| < \infty\), then
\[ C_q^{sym}(W) = \max_{P_X} \min_{P_{YZ|X}: \forall z, d_q(P_{YZ|X};z) < 0 = P_{Z}(z)P_X(z)P_Z(1|z)P_X(1|z) = 0} \quad I(X; Z) \]  
(152)
\[ \tilde{C}_q^{sym}(W) = \max_{P_X} \min_{P_{YZ|X}: \forall z, d_q(P_{YZ|X};z) \geq 0} \quad I(X; Z) \]  
(153)
\[ C_q^{psd}(W) = \max_{P_X} \min_{P_{YZ|X}: \forall z, d_q(P_{YZ|X};z) = 0} \quad I(X; Z), \]  
(154)
where
\[ d_q(P_{YZ|X}, z) = \sum_y [P(y|0, z) - P(y|1, z)][q(1, y) - q(0, y)]. \]  
(155)

For given \(P_X\), it is easy to check whether the condition in the minimization in \((152)\) is satisfied for a suggested channel \(P_{YZ|X}\). Further, it is evident that in addition to the obvious inequality, \(C_q^{sym}(W) \leq \tilde{C}_q^{sym}(W)\), in the binary-input case \(\tilde{C}_q^{sym}(W) \leq C_q^{psd}(W)\).

Proof. Consider a binary input DMC, whose transition probability distribution is given by
\[ W(y|x)_{x \in X, y \in Y}, \quad X = \{0, 1\}, \quad |Y| < \infty. \]  
(156)
Let $\mathcal{Z}$ be a finite set, and consider the bound $C_q^{sym}(W)$ in (23), where $W_q^{sym}(P_X)$ is defined in (17), and can be expressed as

$$W_q^{sym}(P_X) \triangleq \left\{ P_{YZ|X} : \min_{V_{\tilde{X}XZ} \in \mathcal{P}^{sym}(X^2 \times Z), \forall (x,z), V_{\tilde{X}XZ}(x,x|z) \geq P_{X|Z}(x|z)^2} \mathbb{E}_{V_{\tilde{X}XZ}P_Y|XZ}[q(\tilde{X}, Y) - q(X, Y)] \geq 0 \right\}. \quad (157)$$

We have from symmetry of $V$

$$\mathbb{E}_{V_{\tilde{X}XZ}P_Y|XZ}[q(\tilde{X}, Y) - q(X, Y)] = \sum_{z,x,\bar{x}} P_Z(z) V_{X|Z}(x, \bar{x}|z) P(y|x, z)[q(\bar{x}, y) - q(x, y)] + \sum_{z,x \in \{0,1\}, \bar{x} \neq x} P_Z(z) V_{X|Z}(x, \bar{x}|z) \sum_y P(y|x, z)[q(\bar{x}, y) - q(x, y)] \quad (158)$$

$$= \sum_{z,x \in \{0,1\}, \bar{x} \neq x} \frac{1}{2} P_Z(z) V_{X|Z}(x, \bar{x}|z) d_q(P_{YZ|X}, z)$$

$$= \sum_{z} P_Z(z) a(z) d_q(P_{YZ|X}, z), \quad (161)$$

where $d_q(P_{YZ|X}, z) = \sum_y [P(y|0, z) - P(y|1, z)] [q(1, y) - q(0, y)]$. Note that the constraint $V_{\tilde{X},X|Z}(0,0|z) \geq P_{X|Z}(0|z)$ combined with $V_{\tilde{X},X|Z}(0,0|z) + V_{\tilde{X},X|Z}(0,1|z) = P_{X|Z}(0|z)$ becomes $a(z) \leq P_{X|Z}(0|z) P_{X|Z}(1|z)$. Therefore, the minimizing $a(z)$ equals

$$a_{opt}(z) = \begin{cases} 0 & d_q(P_{YZ|X}, z) \geq 0 \\ d_q(P_{YZ|X}, z) & d_q(P_{YZ|X}, z) < 0 \end{cases}. \quad (162)$$

This yields

$$\min_{V_{\tilde{X}XZ} \in \mathcal{P}^{sym}(X^2 \times Z), \forall (x,z), V_{\tilde{X}XZ}(x,x|z) \geq P_{X|Z}(x|z)^2} \mathbb{E}_{V_{\tilde{X}XZ}P_Y|XZ}[q(\tilde{X}, Y) - q(X, Y)]$$

$$= \sum_{z} P_Z(z) a_{opt}(z) d_q(P_{YZ|X}, z)$$

$$= \sum_{z : d_q(P_{YZ|X}, z) < 0} P_Z(z) P_{X|Z}(0|z) P_{X|Z}(1|z) d_q(P_{YZ|X}, z). \quad (164)$$

Now, this quantity is non-negative if and only if for all $z$

$$d_q(P_{YZ|X}, z) < 0 \Rightarrow P_Z(z) P_{X|Z}(0|z) P_{X|Z}(1|z) = 0,$$

which yields (152), and concludes the proof of Lemma 9.

The bound (153) follows similarly by removing the constraint $V_{\tilde{X},X|Z}(x,x|z) \geq P_{X|Z}(x|z)$. As for (154), note that

$$D^q(P_{Y|X,z=0}) = \begin{pmatrix} 0 & d_q(P_{YZ|X}, z) \\ d_q(P_{YZ|X}, z) & 0 \end{pmatrix}, \quad (166)$$

hence in this case, $D^q(P_{Y|X,z=0}) \succeq 0$ is equivalent to $d_q(P_{YZ|X}, z) = 0$, which yields (154) so in this case we have $C_q^{sym}(W) \leq C_q^{psd}(W)$, which may imply that the inequality can be strict.

IX. A COMPARISON TO PREVIOUS RESULTS

We next discuss the relationship between our new results and some relevant previous converse results.
1) Mismatch Capacity: In [20], a single-letter bound was derived by forming a transformation of the channel into another translated channel from $\mathcal{X}$ to $\mathcal{Y}$ such that $q$-decoding error at the latter implies $q$-decoding error at the original channel. The idea was to connect the two channels by means of a graph in the output space $\mathcal{Y}^n$.

In [21], the multicast transmission proof technique (discussed in Section VII) was proposed, which yielded a few tighter bounds, the first of which is (146). The bound of In [20] is obtained as a special case for the suboptimal choice of $\rho = q$.

Another bound that was derived in [21] and is applicable to DMCs, enlarges the set of channels to one that depends also on the input distribution $P_X$. This bound was obtained by including a genie that helps the receiver $Z$ by providing it with the actual value of the joint empirical distribution of the transmitted codeword and the output sequence $Z$. In [23] the bound was improved to include a minimization over the larger class of channels

$$\Theta^*(q, P_X) \triangleq \left\{ P_{YZ|X} : \min_{V_{XYZ} : V_X = P_X, V_{XZ} = P_X} \mathbb{E}_V q(\tilde{X}, Y) \geq \mathbb{E}q(X,Y) \right\}.$$ 

These channels satisfy the condition that if the $Y$-decoder successfully decodes the message, then with high probability (approaching 1 as the block length tends to infinity), also does the genie-aided $Z$-decoder.

A further improved bound was derived in [24] where the set of channels is given by

$$\mathcal{M}_{max}(q, P_X) \triangleq \left\{ P_{YZ|X} : \min_{V_{XYZ} : V_X = P_X, V_{XZ} = P_X} \mathbb{E}_V q(\tilde{X}, Y) \geq \mathbb{E}q(X,Y) \right\},$$

which includes the additional constraint $\tilde{X} \prec (X, Z) - Y$. The resulting bound is denoted $\tilde{R}(W, q)$.

Note that

$$\left\{ P_{X|Z} : P_{XZ} \in \mathcal{P}_{sym}(\mathcal{X}^2 \times \mathcal{Z}) \right\} \subseteq \left\{ P_{X|Z} : P_{XZ} = P_{XZ} \right\},$$

and that there are $|\mathcal{Z}| \cdot |\mathcal{X}| + 1$ symmetry constraints imposed in the l.h.s. set of (167) and $|\mathcal{Z}| \cdot |\mathcal{X}| + 1$ marginal distribution constraints in the r.h.s. set of (167). Thus, for $|\mathcal{X}| > 3$, the inclusion is strict.

Therefore, from (11) we obtain

$$\tilde{C}_q^{sym}(W) \leq \tilde{R}(W, q),$$

and it is likely that the strict inclusion cases result in a strict inequality. Furthermore, since $\overline{C}_q^{sym}(W) \leq \tilde{C}_q^{sym}(W) \leq \overline{C}_q^{sym}(W)$, and since the additional constraint which appears in $C_q^{sym}(W)$; i.e., $P_{X|Z}(x, z) \neq [P_X(z) | Z]^2$ may be active, it is conjectured that there could be cases of strict inequality even for $|\mathcal{X}| \leq 3$.

But, even more importantly, the above mentioned previous bounds except (147) are rather complicated to compute, compared to the bounds $C_q^{psd}(W)$ and $C_q^{sym}(W)$.

2) Reliability Function: The general upper bounds which hold for ML decoding are applicable for mismatched decoding as well, and in particular, the classical sphere-packing bound [27], which is given by:

$$E_{sp}(R, P, W) = \min_{P_Y|X : I(P \times P_Y|X) \leq R} D(P_{Y|X} \| W | P),$$

as it holds for all metrics $q$ including the ML metric.

Consider (37)-(38), yielding that $E_{sp}^q(R, P, W)|q = \log W$ essentially upper bounds $E_{sp}^q(R, P, W)|q = \log W$. Taking $Z = Y$ (instead of minimizing) in our bound (31), $E_{sp}^q(R, P, W)|q = \log W$, and noting that $P_{Y|X}$
\begin{equation}
\mathbb{1}_{\{Y = Z\}} \in \mathcal{W}_q(P_X) \quad \text{(since } P_{XZ} = P_{X\hat{Z}} \text{ implies that } P_{XY} = P_{\hat{X}Y} \text{ and thus } \mathbb{E}q(\hat{X}, Y) - \mathbb{E}q(X, Y) = 0 \text{ yields } E_{sp}^q(R, P, W)|_{q = \log W} \leq E_{sp}(R, P, W).)
\end{equation}

In \cite{24}, an upper bound on the mismatched reliability function was derived, which we denote (to avoid confusion) as

\begin{equation}
f_{sp}^q(R, P, W) \triangleq \min_{P_{XY}: I_p(X; Y) \leq R, \; \min P_{\hat{X}|X} = P_{\hat{X}Y}, \; \mathbb{E}q(\hat{X}, Y) \geq \mathbb{E}q(X, Y)} D(P_{Y|X||W_{Y|X}|P_X}).
\end{equation}

Note that we have (treating \(\hat{Y}\) in the role of \(Z\))

\begin{equation}
\min_{P_{\hat{X}|X: P_{XZ} = P_{\hat{X}Z}, \; \mathbb{E}q(\hat{X}, Y) \leq \mathbb{E}P_{XZ\hat{X}}P_{Y|XZ} \mathbb{E}q(\hat{X}, Y)} \min_{P_{XZ} \in \mathcal{P}(X^2 \times Z)} \mathbb{E}P_{XZ\hat{X}}P_{Y|XZ} q(\hat{X}, Y)
\end{equation}

where the r.h.s. is the minimization which appears in the definition of \(\mathcal{W}_{sym}^q(P_X)\) (see (16)) and thus \(E_{sp}^{q, sym}(R, P, W) \leq f_{sp}^q(R, P, W).\)

In addition to being tighter, the bound \(E_{sp}^{q, sym}(R, P, W)\) has the advantage that due to the symmetry, solving the associated minimization problem on the r.h.s. of (169) involves fewer degrees of freedom to optimize over and parameters range compared to the l.h.s. of (169). A similar comment holds even more so when \(E_{sp}^{q, sym}(R, P, W)\) is concerned, due to the constraint \(\forall (x, z), \; P_{\hat{X}|XZ} (x|x, z) \geq P_{XZ}(x|z)\) in (17). Furthermore, our bounding technique using a two-outputs channel, in which the \(Z\)-receiver serves as a genie to the \(Y\)-receiver is significantly simpler compared to the derivation of \(\hat{R}(W, q)\), since it does not involve the construction of a graph and graph-theoretic tools used in \cite{24}.

X. CONCLUDING REMARKS

The new technique presented in the paper yields the tightest bounds known to date on the mismatch capacity and the reliability function with mismatched decoding, except for \(R = 0^+\) which was calculated in \cite{30} for certain cases. One of the main contributions of our work is the derivation of bounds that are easier to compute compared to previous bounds, either by reducing the number of degrees of freedom of the parameters that are optimized in the calculation of the bounds (such as in \(C_{sym}^q(W)\)), or by providing a looser bound that is considerably easier to compute, \(\hat{C}_{q, psd}(W)\) (yet still tighter compared to previous work) in addition to our previous bound (146) of \cite{21}.

It would be interesting to see whether there are cases for which \(\mathcal{C}_q(W)\) can be strictly tighter than \(C_{sym}^q(W)\) or not; another interesting question is what are the exact relations between the bounds \(\mathcal{C}_q(W)\), \(\hat{C}_{q, sym}(W)\), \(C_{sym}^q(W)\) and \(C_{q, psd}(W)\). A partial answer to this question was given by analyzing the binary input channels.

Similar questions apply also to the bounds on the reliability function.

APPENDIX

A. Proof of Proposition 1

Proof of \(\mathcal{C}_q(W) \leq C_{sym}^q(W)\) :

Clearly, the marginal \(P_{X\bar{X}Z}\) distribution of \(P_{X\bar{X}ZU}\) where \(\bar{X} - (U, Z) - X\) and \(P_{XZU} = P_{X\bar{Z}U}\) is symmetric, and thus satisfies \(P_{\bar{X}XZ} \in \mathcal{P}_{sym}(X^2 \times \bar{Z})\).

Secondly, note that any distribution \(P_{\bar{X}XZU}\) where \(\bar{X} - (U, Z) - X\) and \(P_{XZU} = P_{X\bar{Z}U}\) satisfies for any \((x, z) \in X \times \bar{Z}\):

\[P_{\bar{X}XZ}(x, x|z) = \sum_u P_{U|Z}(u|z) \left[P_{X|UZ}(x|u, z)\right]^2\]
\[
\begin{align*}
\sum_{u} P_{U|Z}(u|z) P_{X|UZ}(x|u, z) & \geq \left[ \sum_{u} P_{U|Z}(u|z) P_{X|UZ}(x|u, z) \right]^2 \\
= [P_{X|Z}(x|z)]^2.
\end{align*}
\] 

(170)

(171)

where (170) follows from Jensen’s inequality. Hence, \( P_{X|Z}(x|x, z) \geq P_{X|Z}(x|z) \).

Thus the claim \( \overline{C}_q(W) \leq C_{q^{sym}}(W) \) follows since

\[
\min_{P_{X|Z}: P_{X|Z} \in P_{q^{sym}}(X^2 \times Z)} \sum_{x, \tilde{x}, z} P_{XZ}(x, z) P_{X|Z}(\tilde{x}|x, z) P_{Y|XZ}(y|x, z) \left[ q(\tilde{x}, y) - q(x, y) \right] \\
\leq \min_{P_{U|XZ}} \sum_{x, \tilde{x}, z} P_{XZ}(x, z, u) P_{X|UZ}(\tilde{x}|u, z) P_{Y|XZ}(y|x, z) \left[ q(\tilde{x}, y) - q(x, y) \right].
\]

(172)

Proof of \( \overline{C}_q(W) \leq \tilde{C}_q(W) \):

We show that \( \tilde{W}_q(P_X) \subseteq \tilde{W}_q(P_X) \). Fix \( P_X \), and let \( P_{Y|X} \in \tilde{W}_q(P_X) \) and \( P_{U|XZ} \) be given. This also induces \( P_{XUZ} = P_{XZ} \times P_{U|XZ} \).

Since \( P_{Y|X} \in \tilde{W}_q(P_X) \), we have for all \((z, V)\) such that \( z \in Z \), and \( V_{X|Z} \) such that \( P_{Z|V_{X|Z}} \ll P_{Z|X} \),

\[
\sum_{x, \tilde{x}, y} V(x|z) V(\tilde{x}|z) P_{Y|XZ}(y|x, z) \left[ q(\tilde{x}, y) - q(x, y) \right] \geq 0.
\]

(173)

Therefore, for all \((z, u)\in Z \times U \) such that either \( P_{X|U=u, Z=z} \ll P_{X|Z=z} \) or \( P_{UZ}(u, z) = 0 \), it must hold that

\[
P_{UZ}(u, z) \cdot \sum_{x, \tilde{x}} P_{X|UZ}(x|u, z) P_{X|UZ}(\tilde{x}|u, z) P_{Y|XZ}(y|x, z) \left[ q(\tilde{x}, y) - q(x, y) \right] \geq 0.
\]

(174)

Summing over \((z, u)\) we obtain

\[
\sum_{x, \tilde{x}, u, z} P_{UZ}(u, z) P_{X|UZ}(x|u, z) P_{X|UZ}(\tilde{x}|u, z) P_{Y|XZ}(y|x, z) \left[ q(\tilde{x}, y) - q(x, y) \right] \geq 0.
\]

(175)

Since for any \( P_{U|XZ} \) it holds that for all \((z, u)\in Z \times U \) either \( P_{X|U=u, Z=z} \ll P_{X|Z=z} \) or \( P_{UZ}(u, z) = 0 \), this yields that \( P_{Y|X} \in \tilde{W}_q(P_X) \).

Proof of \( \overline{C}_q(W) \leq C_{q^{psd}}(W) \):

Observe that

\[
\begin{align*}
\sum_{x, \tilde{x}, y} V(x|z) V(\tilde{x}|z) P_{Y|XZ}(y|x, z) [q(\tilde{x}, y) - q(x, y)] \\
= \frac{1}{2} \sum_{x, \tilde{x}, y} V(x|z) V(\tilde{x}|z) \left( P_{Y|XZ}(y|x, z) [q(\tilde{x}, y) - q(x, y)] + P_{Y|XZ}(y|\tilde{x}, z) [q(x, y) - q(\tilde{x}, y)] \right) \\
= \frac{1}{2} \sum_{x, \tilde{x}} V(x|z) V(\tilde{x}|z) \mathcal{D}^q(P_{Y|X,Z=z})_{x, \tilde{x}}.
\end{align*}
\]

(176)

(177)

Hence, we get for any \( z \in Z \),

\[
\mathcal{D}^q(P_{Y|X,Z=z}) \geq 0 \Rightarrow \\
\forall V_{X|Z}: \ P_{Z|V_{X|Z}} \ll P_{XZ}, \sum_{x, \tilde{x}, y} V(x|z) V(\tilde{x}|z) P_{Y|XZ}(y|x, z) [q(\tilde{x}, y) - q(x, y)] \geq 0,
\]

(178)

because the requirement \( \sum_{z} \sum_{\tilde{x}} V(x|z) V(\tilde{x}|z) \mathcal{D}^q(P_{Y|X,Z=z})_{x, \tilde{x}} \geq 0 \) for any \((V(1|z), ..., V(|X||z))\) which is a probability vector (that has non-negative entries) is looser than the same requirement for any real vector \( V \in \mathbb{R} \), which is nothing but the definition of a p.s.d. matrix.
B. Proof of Lemma [7]

Consider the argument of the minimization in (63); i.e.,
\[
D(V_{Y|X\tilde{X}} \| W_{Y|XZ} | P_{\tilde{X}XZ}) = \sum_{x,\tilde{x},z} P_{\tilde{X}XZ}(\tilde{x}, x, z) \sum_y V_{Y|X\tilde{X}}(y|x, \tilde{x}, z) \log \frac{V_{Y|X\tilde{X}}(y|x, \tilde{x}, z)}{W_{Y|XZ}(y|x, z)},
\]
and recall the definition of \(S^{cond}(P_{\tilde{X}XZ})\) in (68).

Clearly, if \(P_{\tilde{X}XZ}\) is such that \(\Omega(P_{\tilde{X}XZ}, W_{Y|XZ}) = \infty\), the inequality (64) holds trivially, thus assume \(\Omega(P_{\tilde{X}XZ}, W_{Y|XZ}) < \infty\), which implies that \(S^{cond}(P_{\tilde{X}XZ})\) is non-empty, and that a minimizer satisfies \(V_{Y|X\tilde{X}}(y|x, \tilde{x}, z) = 0\) whenever \(W(y|x, z) = 0\). Let \(V^{\ast}_{Y|X\tilde{X}} \in S^{cond}(P_{\tilde{X}XZ})\) be such a distribution.

We next show that \(V^{\ast}_{Y|X\tilde{X}}\) can be approximated by an empirical distribution \(V^{(n)}_{Y|X\tilde{X}}(y|x, \tilde{x}, z) \in S^{cond}(P_{\tilde{X}XZ})\).

To this end, we introduce two additional technical lemmas. The first lemma is obtained as a special case of Krein-Milman Theorem. It asserts that any distribution in \(\mathcal{P}(A)\) where \(A\) is finite, can be expressed as a convex combination of no more than \(2^{|A|}\) empirical distributions in \(\mathcal{P}_\ell(A)\), which are all \(1/\ell\) close to it in the \(L_\infty\)-sense:

**Lemma 10.** Let \(\xi = (\xi_1, \xi_2, \ldots, \xi_{|A|}) \in \mathcal{P}(A)\) be a given distribution, let \(\ell \geq 1\) be an integer, and consider the following convex subset of \(\mathcal{P}(A)\)

\[
\Pi(\xi, \ell) \triangleq \left\{ V \in \mathcal{P}(A) : \forall j \in A, \frac{|\xi_j|}{\ell} \leq V(j) \leq \frac{\lfloor \ell \xi_j \rfloor}{\ell} \right\}.
\]

There exist \(K \leq 2^{|A|}\) empirical distributions \(\{P^{(i)}\}_{i=1}^K\) in \(\Pi(\xi, \ell) \cap \mathcal{P}_\ell(A)\) such that any \(P \in \Pi(\xi, \ell)\) (and in particular, \(\xi\)) can be expressed as:

\[
P = \sum_{i=1}^K \alpha_i \cdot P^{(i)}
\]

for some \(\{\alpha_i\}\), such that \(\alpha_i \in [0, 1]\) and \(\sum_{i=1}^K \alpha_i = 1\).

**Proof.** By Krein-Milman’s Theorem, \(\Pi(\xi, \ell)\) is the closed convex hull of its extreme points. The extreme points of \(\Pi(\xi, \ell)\) are empirical distributions of order \(\ell\). There are no more than \(2^{|A|}\) empirical distributions in \(\Pi(\xi, \ell) \cap \mathcal{P}_\ell(A)\), because each entry \(V(j)\) can only take 2 values. \(\square\)

The next lemma is a straightforward consequence of Lemma [10]

**Lemma 11.** Let \(A, B\) be finite sets, and \(\ell\) an integer. For any \(Q_A \in \mathcal{P}_\ell(A)\) and \(V_{B|A} \in \mathcal{P}(B|A)\), there exist empirical distributions \(P_{AB}^{(j)} \in \mathcal{P}_\ell(A \times B)\), \(i = 1, 2, \ldots, K\), with \(K \leq |A| \cdot 2^{|B|}\), such that

\[
Q_A \times V_{B|A} = \sum_{j=1}^K \beta_j P_{AB}^{(j)}
\]

and

\[
\forall j, \quad P_A^{(j)} = Q_A, \quad \|Q_A \times V_{B|A} - P_{AB}^{(j)}\| \leq \frac{|A||B|}{\ell}, \quad \text{and} \quad V_{B|A}(b|a) = 0 \Rightarrow P_{B|A}^{(j)}(b|a) = 0.
\]

**Proof.** For each \(a \in A\), we use Lemma [10] to express \(V_{B|A=a}\) as a convex combination of empirical distributions,

\[
V_{B|A=a}(b|a) = \sum_{i_a=1}^{K_a} \alpha_{i_a} V_{B|A}^{(i_a)}(b|a),
\]

\(25\)
where $K_a \leq 2^{|B|}$, and naturally, if $V_{B|A=a} = 0$, we take $V_{B|A}^{(i_a)}(b | a) = 0$ for all $i_a$, and by definition we have $V_{B|A}^{(i_a)} \in \mathcal{P}_{\ell_a}(A)$, where $\ell_a = \ell \cdot Q(a)$, and for all $i_a$, $\|V_{B|A=a} - V_{B|A=a}^{(i_a)}\| \leq \frac{|B|}{\ell_a}$.

Now, let $j = (i_1, \ldots, i_{|A|})$ denote the index that takes $K = \sum_{x=1}^{|A|} K_a$ values, denote further

$$P_{AB}^{(j)}(a, b) = Q(a) \cdot V_{B|A}^{(i_a)}(b | a),$$

(186)

thus, we obtain (183), where for $j = (i_1, \ldots, i_{|A|})$ we have $\beta_j = \alpha_{i_j}$, and clearly (184) holds.

Next, we invoke Lemma [11] with $\left( n, \mathcal{X}^2 \times \mathcal{Z}, \mathcal{P}_{X|XZ}, \mathcal{S}, V_{Y|XXZ}^* \right)$ in the roles of $\left( \ell, \mathcal{A}, Q_A, B, P_{B|A} \right)$, respectively, and we let $V_{Y|XXZ}^{(j,n)}$ denote $P_{Y|XXZ}^{(j)}$. By this construction we have for all $j$, $\|P_{X|XZ} \times V_{Y|XXZ}^* - P_{X|XZ} \times V_{Y|XXZ}^{(j,n)}\| \leq \frac{|X|^2|Z|}{n}$, and by affinity of $q(\cdot)$, we have that there must exist at least one index $j_0$ such that $q(V_{XY}^*) - q(V_{XY}) \geq 0$, and hence $V_{XY}^{(j_0,n)} \in \mathcal{S}(P_{XY|XZ})$.

Recall again that the argument of the minimization is $D(V_{Y|XZ} || W_{Y|XZ}|P_{X|XZ})$, which is equal to $-H(V_{Y|XZ}) - \mathbb{E}P_{XZ|Y} \log W_{Y|XZ}$. Hence, denoting $A = |\mathcal{X}|^2|\mathcal{Z}||\mathcal{Y}|$, and $c_n = \frac{|X|^2|Z|}{n}$ from [27], Lemma 2.7], if follows that

$$|D(V_{Y|XZ} || W_{Y|XZ}|P_{X|XZ}) - D(V_{Y|XZ} || W_{Y|XZ}|P_{X|XZ})| \leq -2 \cdot c_n \log \frac{c_n}{A} + c_n \log \frac{1}{t_{n,\text{min}}},$$

(187)

where $t_{n,\text{min}}$ is defined (49) and this concludes the proof of (64).

C. Proof of Corollary [2]

The proof is similar to that of Theorem [2] so it is advisable to read the latter up to (74), before reading this proof. For convenience, we introduce the following notation for a joint distribution $V_{X|XXZ}$; Let

$$r_q(V_{X|XXZ}) \triangleq q(V_{XY}).$$

(188)

Denote also

$$\mathcal{A}^* \triangleq \{ V_{Y|XZ} : I_V(\tilde{X}; Y|XZ) = 0 \}.$$

(189)

We follow the steps of the proof of Theorem [2] up to (74) where step (71) follows since

$$\mathcal{S}^{\text{cond}}\left( \frac{1}{2}[\hat{P}_{x,xxz} + \hat{P}_{x,xiz}] \right) \cap \mathcal{A}^*$$

$$\subseteq \left( \mathcal{S}^{\text{cond}}(\hat{P}_{x,xxz}) \cap \mathcal{A}^* \right) \cup \left( \mathcal{S}^{\text{cond}}(\hat{P}_{x,xiz}) \cap \mathcal{A}^* \right)$$

(190)

is also valid for convex type-dependent metrics. To realize this, note that since $\hat{P}_{x,zz} = \hat{P}_{x,z}$,

$$(\mathcal{S}^{\text{cond}}(\hat{P}_{x,xxz}) \cap \mathcal{A}^*) \cup (\mathcal{S}^{\text{cond}}(\hat{P}_{x,xiz}) \cap \mathcal{A}^*)$$

$$= \left\{ V_{Y|XZ} : r_q \left( \hat{P}_{x,xxz} \times V_{Y|XZ} \right) \geq q(V_{XY}) \right\} \text{ or } r_q \left( \hat{P}_{x,xiz} \times V_{Y|XZ} \right) \geq q(V_{XY}) \right\}$$

(191)

$$= \left\{ V_{Y|XZ} : \frac{1}{2} \left( r_q \left( \hat{P}_{x,xxz} \times V_{Y|XZ} \right) + r_q \left( \hat{P}_{x,xiz} \times V_{Y|XZ} \right) \right) \geq q(V_{XY}) \right\}$$

(192)

$$\supseteq \left\{ V_{Y|XZ} : r_q \left( \frac{1}{2}[\hat{P}_{x,xxz} + \hat{P}_{x,xiz}] \times V_{Y|XZ} \right) \geq q(V_{XY}) \right\},$$

(193)

where the last step follows from convexity of $q(P_{XY})$ in $P_{Y|X}$ for fixed $P_X$. 


Thus, we have (74), i.e.,
\[
\Pr(\text{error}|z, \hat{P}_{XZ} = \hat{P}_{XZ}) \\
\geq \exp \left\{ -n \max_{x_i \in \mathcal{L}, x_j \in \mathcal{L}, i \neq j} \left( \min_{V_{XYYZ} \in \mathcal{S}(\frac{1}{n}[\hat{P}_{XZ} + \hat{P}_{XZ}])} D(V_{X}|XZ|\hat{P}_{XZ} + k_n) \right) \right\}
\] (194)
\[
\geq \exp \left\{ -n \max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(P_{X\tilde{X}Z}, W_{Y|XZ}) + k_n \right\},
\] (195)

where
\[
F(P_{X\tilde{X}Z}, W_{Y|XZ}) \triangleq \min_{V_{XYYZ} \in \mathcal{S}(\frac{1}{n}[P_{X\tilde{X}Z} + \tilde{P}_{XZ}])} D(V_{X}|XZ|\tilde{P}_{XZ}).
\] (196)

Next, we use (106)-(108) and this gives for \( \hat{P}_{XZ} \) which is a possible joint empirical distribution of a codeword and a channel output \( Z \) such that (104) holds, for \( \tilde{v}_n > 1/n \)
\[
\Pr(\text{error}|\hat{P}_{XZ}) \geq (1 - e^{-n\tilde{v}_n}) \cdot e^{-n\min_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} \max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(P_{X\tilde{X}Z}, W_{Y|XZ}) + k_n}
\geq (1 - e^{-n\tilde{v}_n}) \cdot e^{-n\max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(P_{X\tilde{X}Z}, W_{Y|XZ}) + k_n},
\] (197)

And, similar to (111)-(112) this gives
\[
- \frac{1}{n} \log \Pr(\text{error}) \leq \min_{\hat{P}_{XZ} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Z}): \hat{P}_{XZ} = P_n, \ I(\hat{P}_{XZ}) \leq R - \epsilon_n''} D(\hat{P}_{Z|X}||W_{Z|X}|\hat{P}_{X}) + \max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(P_{X\tilde{X}Z}, W_{Y|XZ}) + \delta_n''
\] (199)

where \( \epsilon_n'' \triangleq \epsilon_n + \frac{1}{n} \log(n+1) + \tilde{v}_n \). It remains to show that the minimization over empirical conditional distributions \( \mathcal{P}_n(\mathcal{X} \times \mathcal{Z}) \) can be approximated by a minimization over the simplex \( \mathcal{P}(\mathcal{X} \times \mathcal{Z}) \), as asserted in the following lemma whose proof appears in Appendix [G].

**Lemma 12.**
\[
\min_{\hat{P}_{XZ} \in \mathcal{P}(\mathcal{X} \times \mathcal{Z}): \hat{P}_{XZ} = P_n, \ I(\hat{P}_{XZ}) \leq R - \epsilon_{1,n}} D(\hat{P}_{Z|X}||W_{Z|X}|\hat{P}_{X}) + \max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(\mu_{X\tilde{X}Z}, W_{Y|XZ})
\leq \min_{P_{X\tilde{X}Z} \in \mathcal{P}(\mathcal{X} \times \mathcal{Z}): P_{XZ} = P_n, \ I(P_{XZ}) \leq R - \epsilon_{1,n}} D(P_{Z|X}||W_{Z|X}|P_X) + \max_{P_{X\tilde{X}Z}: P_{XZ} = P_{XZ}} F(\mu_{X\tilde{X}Z}, W_{Y|XZ}) + \tilde{\gamma}_n + \epsilon_{1,n},
\leq \min_{P_{X\tilde{X}Z} \in \mathcal{P}(\mathcal{X} \times \mathcal{Z}): P_{XZ} = P_n, \ I(P_{XZ}) \leq R - \epsilon_{1,n}} D(P_{Z|X}||W_{Z|X}|P_X) + \max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(\mu_{X\tilde{X}Z}, W_{Y|XZ}) + \min_{q_{V_{XY}}: q_{V_{XY}} \geq q(V_{XY}), \ I(\hat{P}_{XZ}) = 0} D(\hat{V}_{XY}||V_{XY}|\hat{P}_{XZ})
\leq \min_{P_{X\tilde{X}Z} \in \mathcal{P}(\mathcal{X} \times \mathcal{Z}): P_{XZ} = P_n, \ I(P_{XZ}) \leq R - \epsilon_{1,n}} D(P_{Z|X}||W_{Z|X}|P_X) + \max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(\mu_{X\tilde{X}Z}, W_{Y|XZ}) + \min_{q_{V_{XY}}: q_{V_{XY}} \geq q(V_{XY}), \ I(\hat{P}_{XZ}) = 0} D(\hat{V}_{XY}||V_{XY}|\hat{P}_{XZ}) + \epsilon_{1,n} + \delta_{1,n}
\leq \min_{P_{X\tilde{X}Z} \in \mathcal{P}(\mathcal{X} \times \mathcal{Z}): P_{XZ} = P_n, \ I(P_{XZ}) \leq R - \epsilon_{1,n}} D(P_{Z|X}||W_{Z|X}|P_X) + \max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(\mu_{X\tilde{X}Z}, W_{Y|XZ}) + \min_{q_{V_{XY}}: q_{V_{XY}} \geq q(V_{XY})} D(\hat{V}_{XY}||V_{XY}|\hat{P}_{XZ}) + \epsilon_{1,n} + \delta_{1,n}
\]
(200)

where \( \epsilon_{1,n} = 2\frac{|X||Y|}{n} \log n \) and \( \tilde{\gamma}_n = 2\frac{|X||Y|}{n} \log n + \frac{|X||Y|}{n} \log \frac{n}{w_{\min}} \), where \( w_{\min} \) is defined in (49).

Thus, denoting \( a_n = \tilde{\gamma}_n + \epsilon_{1,n} + \delta_{1,n} \) we proved that
\[
- \frac{1}{n} \log \Pr(\text{error}) \leq \min_{P_{X\tilde{X}Z} \in \mathcal{P}(\mathcal{X} \times \mathcal{Z}): P_{XZ} = P_n, \ I(P_{XZ}) \leq R - \epsilon_{1,n}} D(P_{Z|X}||W_{Z|X}|P_X) + \max_{P_{X\tilde{X}Z}: P_{XZ} = \tilde{P}_{XZ}} F(\mu_{X\tilde{X}Z}, W_{Y|XZ}) + \min_{q_{V_{XY}}: q_{V_{XY}} \geq q(V_{XY})} D(\hat{V}_{XY}||V_{XY}|\hat{P}_{XZ}) + \epsilon_{1,n} + \delta_{1,n}
\]
(201)

and since \( D(\hat{V}_{XY}||V_{XY}|\hat{P}_{XZ}) = D(P_{Z|X}||W_{Z}|P_X) + D(V_{Y|XZ}||W_{Y}|P_{XZ}) \) and \( W_{Z|XY} \) can be optimized, we can choose in (46) \( W_{Z|XY} = P_{Z|XY}, \) yielding \( ||W_{Z|XY} - P_{Z|XY}|| \leq \tilde{\gamma}_n \). Since we also
have $P_{Z|XY} \ll W_{Z|XY}$, from Pinsker’s inverse inequality, we get $D(P_{Z|XY} \| W_{Z|XY} | P_n \times P_{Z|X}) \leq 2 \frac{\epsilon_n}{|Z|}$, yielding
\begin{equation}
- \frac{1}{n} \log \Pr(error) \leq \tilde{E}_{sp}(R - \epsilon''_n - \epsilon_{1,n}, P_n, W) + a_n + 2 \frac{\epsilon_n}{|Z|}.
\end{equation}
which concludes the proof.

D. Proof of Lemma 3

We prove that while $T$ is a random variable uniformly distributed over $\{1, ..., n\}$, indeed the cardinality of the alphabet of the random variable $U$ can be limited without loss of generality by $|\mathcal{X}|^2 \cdot |Z|$. This is done by an application of the support Lemma (Caratheodory’s Theorem). Note that by Bayes’ Law
\begin{equation}
\Pr(V) = \sum_P \Pr(V \| P) \Pr(P),
\end{equation}
with $(P)$. We use Lemma 11 to express $P_{ZXU}$:
\begin{equation}
\left\{ \frac{V_{ZX|U}(z, x|u)V_{Z|U}(z|u)}{V_{Z|U}(z|u)} \right\}_{(z, x, \bar{x}) \in \mathcal{X}^2},
\end{equation}
preserve those of $P_{T|ZX}$; i.e.,
\begin{equation}
\forall (z, x, \bar{x}), \sum_u V_U(u) \frac{V_{ZX|U}(z, x|u)V_{Z|U}(z|u)}{V_{Z|U}(z|u)} = \sum_t P_T(t) \frac{P_{ZX|T}(z, x|t)P_{Z|T}(z|t)}{P_{Z|T}(z|t)}
\end{equation}
because there are in fact $|Z| \cdot |\mathcal{X}|^2 - 1$ degrees of freedom in $P_{ZX\bar{x}}(z, x, \bar{x})$, it suffices to preserve only $|\mathcal{X}|^2 \cdot |Z| - 1$ of the functionals.

As for the last assertion of the lemma, note that preserving the expectation of one additional functional $H_V(Z|U = u) = - \sum_{x, z} V_{Z|X}(z, x|u) \log \sum_{x'} V_{Z|X}(z, x'|u)$ yields
\begin{equation}
\sum_u V_U(u) H_{P}(Z|U = u) = \sum_t P_T(t) H_{P_{Z|T}}(Z|T = t),
\end{equation}
since $H_{P}(Z|T) = 0$, $Z$ is also deterministic function of $U$, where the alphabet cardinality increase is 1.

E. Proof of Lemma 5

Let $P_{XZ} = P_n \times P_{Z|X}$, recall the definition of $\Delta^q(P_{XZU}, P_{Y|XZ})$ in (12), and let
\begin{equation}
\mathcal{Q}_{diff}(P_{XZ}, P_{Y|XZ}) \triangleq \min_{P_{U|XZ}} \Delta^q(P_{XZU}, P_{Y|XZ}).
\end{equation}
We use Lemma 11 to express $P_{XZ} = P_n \times P_{Z|X}$ as a convex combination of empirical distributions, with $(\mathcal{X}, Z, P_{XZ}, n)$ in the roles of $(\mathcal{A}, B, P_{AB}, \ell)$, respectively, to obtain
\begin{equation}
P_{XZ} = \sum_i \beta_i \cdot P_{XZ}^{(i)}
\end{equation}
where
\begin{equation}
\forall i, P_{X}^{(i)} = P_X = P_n, \|P_{XZ} - P_{XZ}^{(i)}\| \leq \frac{|\mathcal{X}| |Z|}{n}, \text{ and } P_{Z|X}(z|x) = 0 \Rightarrow P_{Z|X}^{(i)}(z|x) = 0.
\end{equation}
Denote by $\{P_{U|XZ}^{(i)}\}$ the minimizers corresponding to $\{P_{XZ}^{(i)}\}$ in (206), respectively; that is, those $\{P_{U|XZ}^{(i)}\}$ which satisfy
\begin{equation}
\mathcal{Q}_{diff}(P_{XZ}^{(i)}, P_{Y|XZ}) \triangleq \Delta^q(P_{XZ}^{(i)} \times P_{U|XZ}^{(i)}, P_{Y|XZ}).
\end{equation}
Let \( \{ P_{XZU}^{(i)} \} \) be the corresponding induced probabilities (by Bayes’ Law); i.e.,

\[
P_{XZU}^{(i)}(x, z, u) = P_{XZ}^{(i)}(x, z) P_{U|XZ}^{(i)}(u|x, z) \cdot \frac{P_{U|XZ}^{(i)}(u|x, z) \cdot P_{XZ}^{(i)}(x, z)}{P_{U|XZ}^{(i)}(u, z)}. \tag{210}
\]

Let \( \overline{T} \) be the random variable whose distribution is \( [\beta_1, \ldots, \beta_K] \), and signifies the value of “\( i \)” in \( \overline{T} \), we have \( P_{XZU}^{(i)}(\bar{x}, x, z, u) = \beta_i P_{XZU}^{(i)}(\bar{x}, x, z) \), and therefore

\[
\begin{align*}
&\max_i Q_{\text{diff}}(P_{XZ}^{(i)}, P_{Y|XZ}) \\
&\geq \sum_i \beta_i Q_{\text{diff}}(P_{XZ}^{(i)}, P_{Y|XZ}) \\
&= \sum_i \beta_i \mathbb{E}_{P_{XZ}^{(i)}}\left[ q(\bar{X}, Y) - q(X, Y) \right] \\
&\triangleq \Delta^q(P_n \times P_{Z|X} \times P_{U|XZ}, P_{Y|XZ}) \\
&\triangleq \Delta^q(P_n \times P_{Z|X} \times P_{U|XZ}, P_{Y|XZ}) \\
&\geq Q_{\text{diff}}(P_n \times P_{Z|X}, P_{Y|XZ}) \geq 0, \tag{211}
\end{align*}
\]

where \( \Delta^q \) follows since the average is upper bounded by the maximal value, in \( \Delta^q \) we define \( \bar{U} = (U, \overline{T}) \), and \( \Delta^q \) follows by definition of \( Q_{\text{diff}} \) as the minimum over \( P_{U|XZ} \) of \( \Delta^q(P_n \times P_{Z|X} \times P_{U|XZ}, P_{Y|XZ}) \). The fact that the alphabet of \( \bar{U} \) is larger than that of \( U \) does not impair the derivation as by Caratheodory’s Theorem, any \( \bar{U} \) can be substituted with \( U \) of alphabet size not larger than \( |X|^2 |Z| \) as asserted in Lemma 5. The last step \( \Delta^q \) follows since \( P_{Y|XZ} \in \mathcal{W}_q(P_X) \).

Finally, let \( i^* \) be the maximizer of \( \Delta^q \), thus \( P_{XZ}^{(i^*)} \) serves as the empirical distribution whose existence is claimed in Lemma 5.

**F. Proof of Lemma 8**

We continue the proof of Lemma 5 (in Appendix E) applied to \( W_{YZ|X} \) in the role of \( P_{Y|XZ} \).

To conclude the proof of Lemma 8 we verify that \( P_{XZ}^{(i^*)} \in \mathcal{K}_q(R, P_n, W_{YZ|X}) \) by checking that the 3 requirements in its definition \( \mathfrak{A} \) are met.

1. From \( \mathfrak{A} \) we have that \( P_{XZ}^{(i^*)} \in \mathcal{A}_q(W_{Y|XZ}, P_n) \).

2. From \( \mathfrak{B} \) we have \( D(P_{Z|X}^{(i^*)}||W_{Z|X}|P_n) \leq 2n^{2-\min|\mathcal{X}| \log(n+1)} |\mathcal{Z}| \log(n+1) \), where \( c_n \gg \frac{1}{n} \) (for example \( c_n = n^{-1/2} \)), for \( n \) sufficiently large, we clearly have \( D(P_{Z|X}^{(i^*)}||W_{Z|X}|P_n) \leq f_n \).

3. By continuity of the mutual information, inequality \( \mathfrak{B} \) also implies that \( |I(P_n \times W_{Z|X}) - I(P_{XZ}^{(i^*)})| \leq |\mathcal{X}| |\mathcal{Z}| \log(n+1) \) (see [27] Lemma 2.7). Since \( I(P_n \times W_{Z|X}) \leq R - \epsilon \), this gives \( I(P_{XZ}^{(i^*)}) \leq R - \epsilon + |\mathcal{X}| |\mathcal{Z}| \log(n+1) \). Since \( d_n = \epsilon_n + |\mathcal{X}| |\mathcal{Z}| \log(n+1) + \epsilon_n \), where both \( \epsilon_n, \epsilon_n \) are vanishing sequences, for \( n \) sufficiently large we thus have \( I(P_{XZ}^{(i^*)}) \leq R - d_n \).

Hence, \( P_{XZ}^{(i^*)} \in \mathcal{K}_q(R, P_n, W_{YZ|X}) \), and this concludes the proof of Lemma 8.

**G. Proof of Lemma 12**

Let \( P_{\hat{XZ}} \) denote the minimizer of the following function

\[
\min_{P_{XZ} \in \mathcal{P}(X \times Z), P_X = P_n, \mu_{X\hat{X}Z} = \mu_{XZ}} D(P_{Z|X}||W_{Z|X}|P_X) + \max_{\mu_{X\hat{X}Z} = \mu_{XZ}} F(\mu_{X\hat{X}Z}, W_{Y|XZ})
\]
\[ \Delta \triangleq D(P^*_{Z|X}) + \max_{\mu_{X\bar{X}Z}: \mu_{X\bar{X}Z}=\mu_{\bar{X}XZ}, \mu_{XZ}=P^*_{XZ}} F(\mu_{X\bar{X}Z}, W_Y|XZ). \]  

(218)

Next, we use Lemma 11 to express \( P^*_{Z|X} \) as a convex combination of empirical distributions \( P^*_{XZ} = \sum_i \beta_i P^{(i)}_{XZ} \), with \( \left( n, X, P_n, Z, P^*_{Z|X} \right) \) in the roles of \( (\ell, A, Q_A, B, P_B|A) \), respectively, and we let \( P^{(i)}_{Z|X} \) denote \( P^{(i)}_{B|A} \).

By this construction we have for all \( i \), \( \| P_n \times P^*_{Z|X} - P_n \times P^{(i)}_{Z|X} \| \leq \frac{\|X\|Z}{n} \). Now, for each \( i \), denote by \( P^{(i)}_{\bar{X}|XZ} \) the maximizing \( \mu^{(i)}_{\bar{X}|XZ} \) of:

\[ \max_{\mu_{X\bar{X}Z}: \mu_{X\bar{X}Z}=\mu_{\bar{X}XZ}, \mu_{XZ}=P^*_{XZ}} F(\mu_{X\bar{X}Z}, W_Y|XZ) \]

\[ \geq F(\sum_i \beta_i P^{(i)}_{XZ}, W_Y|XZ) \]

\[ = \min_{V_{Y|XZ} \in S^{cond}(\frac{1}{2} \sum_i \beta_i [P^{(i)}_{XZ} + P^{(i)}_{\bar{X}XZ}])} D(V_{Y|XZ} || W_Y|XZ) \sum_i \beta_i P^{(i)}_{XZ}). \]  

(219)

(220)

where (219) follows since by definition, the marginal \((X, Z)\) distribution of \( \sum_i \beta_i P^{(i)}_{XZ} \) is equal to \( \sum_i \beta_i P^{(i)}_{XZ} \), which is equal to \( P^*_{XZ} \), and thus \( \sum_i \beta_i P^{(i)}_{XZ} \) yields an \( F(\cdot) \) value that cannot exceed the maximal possible value. Step (220) is by definition of \( F(\cdot) \). Now, by affinity of the expectation, there must exist \( i_0 \) such that the minimizing \( V_{Y|XZ} \) belongs to \( S^{cond}(\frac{1}{2} [P^{(i_0)}_{XZ} + P^{(i_0)}_{\bar{X}XZ}]) \), and hence

\[ \min_{V_{Y|XZ} \in S^{cond}(\frac{1}{2} \sum_i \beta_i [P^{(i)}_{XZ} + P^{(i)}_{\bar{X}XZ}])} D(V_{Y|XZ} || W_Y|XZ) \sum_i \beta_i P^{(i)}_{XZ}) \]

\[ \geq \min_{V_{Y|XZ} \in S^{cond}(\frac{1}{2} \sum_i \beta_i [P^{(i)}_{XZ} + P^{(i)}_{\bar{X}XZ}])} D(V_{Y|XZ} || W_Y|XZ) \sum_i \beta_i P^{(i)}_{XZ}) \]

\[ \geq \min_{V_{Y|XZ} \in S^{cond}(\frac{1}{2} \sum_i \beta_i [P^{(i)}_{XZ} + P^{(i)}_{\bar{X}XZ}])} D(V_{Y|XZ} || W_Y|XZ) \sum_i \beta_i P^{(i)}_{XZ}) - \delta_n \]

\[ = \max_{\mu_{X\bar{X}Z}: \mu_{X\bar{X}Z}=\mu_{\bar{X}XZ}, \mu_{XZ}=P^*_{XZ}} F(\mu_{X\bar{X}Z}, W_Y|XZ) - \delta_n, \]  

(221)

(222)

(223)

where (222) follows since by definition \( \sum_i \beta_i P^{(i)}_{XZ} = P^*_{XZ} \), and since \( \| P^*_{XZ} - P^{(i)}_{XZ} \| \leq \frac{\|X\|Z}{n} \), which implies that for any \( P_{Y|XZ} \), \( |D(P_{Y|XZ})||W_{Y|XZ}|P^{(i)}_{XZ}) - D(P_{Y|XZ})||W_{Y|XZ}|P^*_{XZ})| \leq 2 \frac{\|X\|Z}{n} \log n + \frac{|X||Y|Z}{n} \log t_{n,\text{min}} \leq \delta_n \), where \( t_{n,\text{min}} \) is defined in (49). The last step follows by definition of \( P^{(i)}_{\bar{X}|XZ} \) as the maximizing \( \mu^{(i)}_{\bar{X}|XZ} \) of:

\[ \max_{\mu_{X\bar{X}Z}: \mu_{X\bar{X}Z}=\mu_{\bar{X}XZ}, \mu_{XZ}=P^{(i)}_{XZ}} F(\mu_{X\bar{X}Z}, W_Y|XZ). \]

Now,

\[ D(P^*_{Z|X} || W_{Z|X} | P_X) + \max_{\mu_{X\bar{X}Z}: \mu_{X\bar{X}Z}=\mu_{\bar{X}XZ}, \mu_{XZ}=P^*_{XZ}} F(\mu_{X\bar{X}Z}, W_Y|XZ) \]

\[ \geq D(P^{(i)}_{Z|X} || W_{Z|X} | P_X) + \max_{\mu_{X\bar{X}Z}: \mu_{X\bar{X}Z}=\mu_{\bar{X}XZ}, \mu_{XZ}=P^{(i)}_{XZ}} F(\mu_{X\bar{X}Z}, W_Y|XZ) - \epsilon_{1,n} \]  

(224)

(225)

where both (224) and (225) hold since \( \| P^*_{XZ} - P^{(i)}_{XZ} \| \leq \frac{\|X\|Z}{n} \), which implies \( |I(P^*_{XZ}) - I(P_n \times P^{(i)}_{XZ})| \leq 2 \frac{\|X\|Z}{n} \log n \leq \epsilon_{1,n} \). The last step follows since \( P^{(i)}_{Z|X} \) is an empirical distribution of order \( n \), and since \( I(P^*_{XZ}) \leq \epsilon_{1,n} \).
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