Is $S = 1$ for $c = 1$?  

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ABSTRACT

The $c = 1$ string in the Liouville field theory approach is shown to possess a nontrivial tree-level $S$-matrix which satisfies factorization property implied by unitary, if all the extra massive physical states are included.

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String amplitudes and their factorization properties in the Liouville field theory \cite{1,2} are not yet well understood at present. In this note we show that the spherical $c = 1$ string amplitudes computed from the Liouville theory according to conventional vertex operator normalization are nontrivial and satisfy correct factorization. The renormalized amplitudes computed by Gross and Klebanov \cite{3} imply that $S = 1$. This result agrees with the so-called “bulk” $S$-matrix in the collective field theory of $c = 1$ quantum gravity. We, on the other hand, follow Polyakov \cite{4} who considered unrenormalized amplitudes and find a unitary $S$-matrix not equal to 1. The $S$-matrix we are considering is also different from the ones computed in matrix models \cite{5} (see also \cite{6,7}), which are “wall” amplitudes. As a necessary byproduct of our analysis, a host of extra states appears in the spectrum of the theory. These states are not present in the naive light-cone analysis of the spectrum (according to this there is only one degree of freedom, the center of mass of the string, so-called “tachyon”), but appear naturally both in factorization of amplitudes as intermediate states and in the pole structure of the corresponding world-sheet operator product expansion ($OPE$).

We hope that our results will shed some light on difficult issues such as factorization and nature of spectrum in the Liouville theory. Due to the apparent difference of our results and conclusions on \cite{3}, a subtle relation is to be expected between the continuous collective field theory à la Das-Jevicki and the related Liouville theory of $c = 1$ quantum gravity.

The organizaton of this note is as follows. First, we review some well-known facts about $c = 1$ string theory. Then we discuss factorization of amplitudes at tree level. Lastly, we concentrate on the $OPE$'s of the underlying world-sheet theory in attempt to understand the pole structure of the factorization formulae.

In the path integral approach to string theory, the $S$-matrix is computed from sum over surfaces with insertion of local vertex operators of correct dimensions. One checks
for unitarity to verify the consistency of the result. At tree level, the unitarity reduces to
factorization of amplitudes \[8\]. The underlying reason is of course the OPE of the world-
sheet conformal field theory. A singularity in the OPE \[9\] is directly related to the pole
structure in the factorization formula. Indeed, in their original computation of the 2 point
function in c=1 string theory, Gross, Klebanov and Newman \[10\] have derived some OPEs
and observed their pole structure. In light of their analysis we consider later on the OPEs
of the world-sheet theory and find signs of a nontrivial spectrum and S-matrix.

Let us start from the general expression for the N-point tachyon amplitude of the c = 1
string with the zero cosmological constant in the path integral approach \[11, 12\]:

\[
A_N = \int D t D \phi \prod_{i=1}^N \int d^2 z_i \sqrt{\hat{g}(z_i)} V_{\beta_i, p_i}(z_i) 
\exp\left\{ -\frac{1}{8\pi} \int d^2 \sigma \sqrt{\hat{g}} \left[ \partial_\alpha t \partial^\alpha t + \partial_\alpha \phi \partial^\alpha \phi - 2\sqrt{2} \hat{R} \phi \right] \right\}. \tag{1}
\]

Here the vertex operator \( V_{\beta_i, p_i} \) creates a massless “tachyon” at momentum \( p_i \) with chirality
\( \epsilon(i) = sgn(p_i) \), \( t \) represents Euclidean time, and \( \phi \) is understood as a spatial coordinate.
Conformal invariance dictates the form of the vertex operator

\[
V_{\beta_i, p_i}(z_i) = \exp(ip_i t(z_i) - \sqrt{2}\phi(z_i) + |p_i|\phi(z_i)) 
\equiv \exp(ip_i t(z_i) + \beta_i \phi(z_i)). \tag{2}
\]

Apart from some non-covariant factors, (1) defines transition amplitudes in string theory.
Evaluation of the corresponding path integral in \( D = 26 \) results in the well-known Virasoro-
Shapiro amplitude \[13\].

In what follows kinematics will play a major role. There exists a standard procedure
that leads to the peculiar kinematical constraints in the Liouville theory with the Euclidean
signature. One integrates over the zero modes of \( t \) and \( \phi \) in (1) (\( t_0 \) and \( \phi_0 \) respectively). Integration over \( t_0 \) gives the momentum conservation law. Integration over \( \phi_0 \) is not well-
defined, so one rotates \( \phi_0 \) to \( i\phi_0 \) and deduces an “energy” conservation law. All in all one
is left with \[3,4\],
\[
\sum_{i=1}^{N} p_i = 0, \quad \sum_{i=1}^{N} |p_i| = (N - 2) \sqrt{2}.
\] (3)

We prefer to think about physical scattering in 2D Minkowski space-time. Consider, for example, a three-point tachyon amplitude. For that purpose, prepare two suitably normalized wave packets of tachyons and let them collide. This is of course a well-defined physical process: insertion of wave packets should render previously divergent integrals analytically well behaved. Indeed, if the initial state is represented as
\[
|i\rangle \sim \int dp_1 dp_2 f_1(p_1) f_2(p_2)|p_1, p_2, in\rangle,
\] (4)
where \(p_1\) and \(p_2\) are incoming momenta and \(f_1(p_1), f_2(p_2)\) are peaked around the actual momenta of the incident particles with a finite and small width, and the final state is approximated by a plane wave, integration over \(t_0\) implies momentum conservation. On the other hand, integration over \(\phi_0\) is now well-defined with the corresponding integral
\[
\sim \int d\phi_0 dp_1 dp_2 f_1(p_1) f_2(p_2) \exp[i(p_1 + p_2 - p_3 + i\sqrt{2})\phi_0],
\] (5)
leading to the “energy sum rule” \(p_1 + p_2 - p_3 = -i\sqrt{2}\) in Minkowski space-time. Therefore, the formal procedure based on the Wick rotation of \(\phi_0\) is supported by a reasonable physical picture, and leads to the same result.

In what follows we will use \(A_{m,N-m}\) to denote an \(N\)-point amplitude (1) of \(m\) points with + chirality and \((N-m)\) points with − chirality. The evaluation of (1) has been done in \[7, 3, 4\], where the integrals calculated in \[14\] were used. The integral representation of (1) is
\[
A(1, 2, \cdots, N) = \int \prod_{i=1}^{N} d^2 z_i \mu(z_i) \langle \prod_{i=1}^{N} V_{\beta, p_i} \rangle
= \int \prod_{i=1}^{N-3} d^2 z_i |z_i|^{-2s_i,N-2}|1 - z_i|^{-2s_i,N-1} \prod_{i<j} |z_i - z_j|^{-2s_{ij}},
\] (6)

1The following picture was suggested to us by J. Polchinski.
where $s_{ij} = (-\sqrt{2}+|p_i|)(-\sqrt{2}+|p_j|) - p_ip_j$ and \(\langle\rangle\) means free field contraction. The function $\mu(z_i)$ is an appropriate Faddeev-Popov determinant from the $SL(2,C)$ gauge fixing. In particular [14, 3, 6],

$$
A_{2,1} = A_{1,2} = 1, \quad A_{3,1} = \pi \prod_{i=1}^{3} \frac{\Gamma(1-\sqrt{2}p_i)}{\Gamma(\sqrt{2}p_i)}, \quad A_{N-1,1} = \frac{\pi^{N-3}}{(N-3)!} \prod_{i=1}^{N-1} \frac{\Gamma(1-\sqrt{2}p_i)}{\Gamma(\sqrt{2}p_i)}.
$$

The last formula can be established if one uses symmetries of the integrand in (6) in conjunction with convenient kinematical constraints and makes an ansatz such that successive reduction from the \((N-1,1)\) case leads to the 4-point function, for which the analytic expression can be explicitly written.

We illustrate the outlined procedure for $A_{4,1}$ amplitude. In this case, (6) is invariant under the following substitutions:

$$
\begin{align*}
  z_1 &\rightarrow 1 - u_1, \quad s_{13} \rightarrow s_{14} \\
  z_2 &\rightarrow 1 - u_2, \quad s_{23} \rightarrow s_{23} - s_{13} + s_{14},
\end{align*}
$$

where $s_{ij} = 2 - \sqrt{2}(p_i + p_j)$, $p_i \geq 0$ for $i = 1, \ldots, 4$, and $p_5 = -3/\sqrt{2}$. Also $s_{24} = s_{23} - s_{13} + s_{14}$ as a consistency requirement. Then by writing $A_{4,1}$ as

$$
A_{4,1} = T(\frac{2 - s_{12} - s_{13} + s_{23}}{2})T(\frac{2 - s_{21} - s_{23} + s_{13}}{2})
T(\frac{2 - s_{31} - s_{23} + s_{12}}{2})T(\frac{2 + s_{12} + s_{13} - s_{23} - 2s_{14}}{2})f(s_{13}, s_{14}, s_{23}, s_{12}),
$$

where $T(x) = \Gamma(1-x)/\Gamma(x)$, one can see that $f(s_{13}, s_{14}, s_{23}, s_{12})$ has to satisfy the above symmetry relations. Following Dotsenko and Fateev [14], one proves that $f$ is a bounded and analytic function of $s_{ij}$ in the entire complex plane. Then $f$ is a constant that can be deduced by truncating $A_{4,1}$ to $A_{3,1}$. Apparently the same procedure persists for higher
order correlation functions of the form \((N-1,1)\) \[3\]. Observe that from the expression for \(A_{N-1,1}\) one can get \(A_{N-2,2}\) by picking an appropriate value for one of the \((N-1)\) momenta, and so on for \(A_{M-m,m}\). That means that generically \(A_{N-m,m} = 0\) if \(m = 2, \cdots, N-2\). We illustrate this by displaying \(A_{2,2}\)

\[
A_{2,2} = \pi \frac{\Gamma(1 - s_{14}) \Gamma(1 - s_{24}) \Gamma(1 - s_{34})}{\Gamma(s_{14}) \Gamma(s_{24}) \Gamma(s_{34})},
\]

(10)

where \(p_{1,2} \geq 0\) and \(p_{3,4} \leq 0\). The conservation laws imply (3) that \(p_1 + p_2 = -(p_3 + p_4) = \sqrt{2}\), thus \(s_{34} = 0\) and \(A_{2,2} = 0\). It is clear that \(A_{N-1,1}\) has poles for exceptional values of the momenta \(p_i = (M+1)/\sqrt{2}, M = 0, 1, \cdots\). If we Wick-rotate back to the Minkowski signature \((p \rightarrow ip)\), the poles are at imaginary momenta. It has been suggested \[3\] that one should simply absorb them through a wave function normalization. After the renormalization all amplitudes vanish, giving \(S = 1\) for \(c = 1\). Such an S-matrix is of course trivially unitary. Here we propose an alternative interpretation of (1), which leads to a tree level unitary, yet non-trivial S-matrix. First we recall what is meant by tree-level factorization (see, for example, \[8, 9\] where the correct vertex operator normalization is derived from tree-level factorization). When the total momentum \(p_\mu\) of some set of external legs, say \(1, 2, \cdots, L\), approaches the mass shell of a physical particle of mass \(m\) and type \(j\), the S-matrix must have a pole with

\[
A(1, 2, \cdots, N) = -i(p^2 + m^2)^{-1} \sum_j A(1, 2, \cdots, L, j) A(j, L + 1, L + 2, \cdots, N),
\]

(11)

where \(p \equiv p_1 + \cdots + p_L\). There are only minor changes in the above equation when applied to our situation. Firstly, since the tachyon is really massless in \(c = 1\) string theory we have \(m = 0\). More importantly, in string theory, the propagator is in general defined as \((L_0 - 1)^{-1}\). For an off-shell tachyon vertex operator \(\int d^2 \sigma \sqrt{g} \exp (ipt - \sqrt{2} \phi \pm q\phi), L_0\) is \(1 + p^2 - q^2\). So the tachyon propagator is just \((p^2 - q^2)^{-1}\). Finally, as we will see, a satisfactory factorization must include the extra states discovered in \[4\] in the physical
Hilbert space.

Before investigating the general case, let’s work out the factorization of (1) for the 2 particle scattering amplitude $A_{3,1}$. We assume that $p_3 + p_4 + p = 0$, $p_3 - p_4 - q = \sqrt{2}$, i.e., the off-shell intermediate state in the $s$ channel (figure 1) carries 2-momentum $(p, -\sqrt{2} - q)$. For the $(3, 1)$ kinematic region, $p_4$ is easily worked out as $p_4 = -\sqrt{2}$. So we can express $p_3$ as $p_3 = -p + \sqrt{2}$. The on-shell condition of the intermediate state is then $p = q = 1/\sqrt{2}$, which also implies that $p_2 = 1/\sqrt{2} - p_1$. Now we can rewrite $A_{3,1}$ as

$$A_{3,1} = \pi \frac{\Gamma(1 - \sqrt{2}p_1) \Gamma(1 - \sqrt{2}p_2) \Gamma(\sqrt{2}p - 1)}{\Gamma(\sqrt{2}p_1) \Gamma(\sqrt{2}p_2) \Gamma(2 - \sqrt{2}p)}, \quad (12)$$

As $p \to 1/\sqrt{2}, \Gamma(\sqrt{2}p - 1) \to (\sqrt{2}p - 1)^{-1}$, and the last $\Gamma$ function in the denominator approaches $1$. (8) then leads to

$$A_{3,1} \to \pi \frac{\Gamma(1 - \sqrt{2}p_1) \Gamma(1 - \sqrt{2}(1/\sqrt{2} - p_1))}{\Gamma(\sqrt{2}(1/\sqrt{2} - p_1)) \sqrt{2}p - 1} \frac{1}{\sqrt{2}p - 1} \quad (13)$$

$$\to \frac{\pi}{p^2 - q^2}.$$ 

Since $A_{2,1} = A_{1,2} = 1$, (9) satisfies the correct factorization.

It appears paradoxical that $A_{2,2} = 0$, since the unitarity requires that $A_{2,2} \neq 0$. For example, in the situation indicated in figure 2, when the intermediate state is close to the tachyon mass-shell $p + q \sim 0$,

$$A_{2,2} \to \frac{1}{p^2 - q^2}. \quad (14)$$

The resolution is also simple. If the intermediate state is on the mass-shell, the kinematical relation tells us that $p_1 = p_2 = 1/\sqrt{2}$. One can easily work out that $s_{1,4} = s_{2,4} = 1$, so

$$A_{2,2} = \pi \frac{\Gamma(1) \Gamma(0) \Gamma(0)}{\Gamma(0) \Gamma(1) \Gamma(1)}. \quad (15)$$

To be sure, (15) really means a Dirac $\delta$-function at $p_1 = 1/\sqrt{2}$. Indeed, after Wick rotation with suitable $i\epsilon$ prescription, one can show that $A_{2,2}$ is $i\delta(p_1 - 1/\sqrt{2})$. Thus unitarity
holds in a strange way. Clearly, many analytical properties familiar in the case of a four
dimensional S-matrix are lost here.

Generalization to $A(N - 1, 1)$ is easy (figure 3). The factorization we are interested in
is $A(N - 1, 1) \rightarrow A(N - 2, 1)A(2, 1)$. Again we look for an on-shell pole at total momenta
$p_{N-1} + p_N + p = 0$, $p_{N-1} - p_N - q = \sqrt{2}$. Since $p_N$ can be worked out to be $(2 - N)/\sqrt{2}$,
the on-shell condition is $p = q = (N - 3)/\sqrt{2}$. Using the kinematic relation we can express
$p_{N-1}$ as $(N - 2)/\sqrt{2} - p$. Substituting this into (4) we find, near the mass shell

$$A_{N-1,1} \rightarrow \frac{\pi^{N-3}}{(N-3)!} \prod_{i=1}^{N-2} \frac{\Gamma(1 - \sqrt{2}p_i)}{\Gamma(\sqrt{2}p_i)} \frac{1}{\sqrt{2}p - (N - 3)}$$
$$\rightarrow \frac{\pi^{N-3}}{(N-4)!} \prod_{i=1}^{N-2} \frac{\Gamma(1 - \sqrt{2}p_i)}{\Gamma(\sqrt{2}p_i)} \frac{1}{p^2 - q^2}$$
$$\rightarrow \frac{\pi}{p^2 - q^2} A_{N-2,1} A_{1,2},$$

which is again what one would expect from factorizability. Note that although the pole
in (8) corresponds to a resonance of high momentum $p = (N - 3)/\sqrt{2}$ of the intermediate
state, it is the lowest pole in external legs. The resonant state is the ordinary massless
tachyon with peculiar values of its momenta. We will discuss higher order poles in the
external legs later.

The tachyon poles at discrete momenta can also be observed in the structure of the
OPE in the underlying “gravitationally dressed” conformal field theory. This should be
expected on general grounds [9]. Consider the same $N$ tachyon amplitude as in the last
paragraph. From (4), if we fuse the last two vertex operator using the free field OPE, we
obtain for the first short distance singularity ($z = z_N - z_{N-1}$),

$$A_{N-1,1} \sim \int d^2 z |z|^{-4 + 2\sqrt{2}(p_{N-1} - p_N) + 4p_{N-1}p_N} \prod_{i=1}^{N-1} d^2z_i \mu(z_i)$$
$$\langle V_{\beta_1, p_1}(z_1) \cdots V_{\beta_{N-1} + \beta, p_{N-1} + p_N}(z_{N-1}) \rangle$$
$$\sim \frac{1}{\sqrt{2}p - (N - 3)} \prod_{i=1}^{N-1} d^2z_i \mu(z_i) \langle V_{\beta_1, p_1}(z_1) \cdots V_{\beta_{N-1} + \beta, p_{N-1} + p_N}(z_{N-1}) \rangle, (17)$$

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where we have used kinematic relation $p_N = (2 - N)/\sqrt{2}$ and $p_N + p_N + p = 0$. Indeed, the lowest tachyon pole which appears in the factorization of the $N$-point amplitude in the Liouville theory (eq.(9)) can be clearly seen.

Now we offer a similar explanation for higher poles in external leg as exchange of extra states \( \mathcal{A} \). Take, for example, a 4-point amplitude in (6) and work out the free field OPE of $V_{\beta_3,p_3}$ and $V_{\beta_4,p_4}$,

$$V_{\beta_3,p_3}(z_3, \bar{z}_3)V_{\beta_4,p_4}(z_3 + z, \bar{z}_3 + \bar{z}) = \sum_{M,M'} |z|^{-4 + 2\sqrt{2}(p_3 - p_4) + 4p_3p_4} z^M \bar{z}^{M'} O_M^4(z_3) O_{M'}^4(z_3), \quad (18)$$

where

$$O_M^4(z_3) = \frac{1}{M!} : \exp[\beta_3 \phi(z_3) + ip_3 t(z_3)] \partial_z^M \exp[\beta_4 \phi(z_3) + ip_4 t(z_3)] : \quad (19)$$

and similarly for the antiholomorphic operator $O_M^4(\bar{z}_3)$. By inserting the OPE into (6) and using the kinematical relations $\beta_4 = 0$ and $p_4 = -\sqrt{2}$, we obtain the singular contribution

$$A_{3,1} \sim \sum_{M,M'} \int d^2z |z|^{-4 + 2\sqrt{2}(p_3 - p_4) + 4p_3p_4} z^M \bar{z}^{M'} \int \prod_{i=1}^3 d^2z_i \mu(z_i) \langle V_{\beta_1,p_1}(z_1) V_{\beta_2,p_2}(z_2) O_M^4(z_3) O_{M'}^4(\bar{z}_3) \rangle \quad (20)$$

Clearly the external leg poles in (7) correspond to $A_{3,1}$ factorizing into $O_M^4(z_3) O_{M'}^4(\bar{z}_3)$ at special momenta $p_3 = (M + 1)/\sqrt{2}$, and $O_M^4(z)$ takes the form

$$O_M^4(z_3) = \frac{1}{M!} : \exp[(M - 1) \phi(z_3)/\sqrt{2} + i(M + 1) t(z_3)/\sqrt{2}] \partial_z^M \exp[-i\sqrt{2} t(z_3)] : \quad (21)$$

These are precisely the special operators considered by Danielsson and Gross [13] in the $c = 1$ conformal field theory. Generalization to $A_{N-1,1}$ presents no difficulty. We will find that even more special operators appear. Now let us fuse $V_{\beta_{N-1},p_{N-1}}$ and $V_{\beta_{N+1}}$ in $A_{N-1,1}$ and integrate over $z$ and $\bar{z}$. We obtain

$$A_{N-1,1} \sim \sum_M \frac{1}{\sqrt{2}p_{N-1} - (M + 1)} \int \prod_{i=1}^{N-1} d^2z_i \mu(z_i) \langle V_{\beta_{i,p_i}} \cdots V_{\beta_{N-1,p_{N-1}}} O_M^N(z_{N-1}) O_M^N(\bar{z}_{N-1}) \rangle, \quad (22)$$
where

\[ O_M^N(z_{N-1}) = \frac{1}{[M(N-3)]!} : \exp[\beta_{N-1} \phi + ip_{N-1} t] \partial_z^{M(N-3)} \exp[\beta_N \phi + ip_N t] : \]  \hspace{1cm} (23)

and \( \beta_N = (N-4)/\sqrt{2}, p_N = (2-N)/\sqrt{2} \). In deriving (23) we have taken into account the fact that some poles in the OPE do not show up in the final answer (7). When \( p_{N-1} = (M+1)/\sqrt{2} \), we get the following special operators

\[ O_M^N = \frac{1}{[M(N-3)]!} : \exp[(M-1) \phi/\sqrt{2} + i(M+1) t/\sqrt{2}] \partial_z^{M(N-3)} \exp[(N-4) \phi/\sqrt{2} + i(2-N) t/\sqrt{2}] : . \]  \hspace{1cm} (24)

Unlike \( O_M^4 \), \( O_M^N (N > 4) \) mixes \( \phi \) and \( t \) in a nontrivial way. However, simple arguments show that \( O_M^N \) can be reduced to a product of a nontrivial \( t \) primary field with a pure exponential operator of \( \phi \), plus spurious operators. By the way, looking at the propagator corresponding to the special operators, one finds that they describe massive intermediate states.

Since extra massive states exist in the intermediate channel, they should also appear as external states for the reason of unitarity. One then has to consider factorization of amplitudes involving them. These states appear only at discrete momenta, thus the physical picture of their scattering is not very clear.

Recently a number of authors have considered extra degrees of freedom in \( c = 1 \) string theory from different points of view. Comparing our results with we find an agreement. Their analysis was done from the point of view of matrix models which indicates a more precise relation between the Liouville theory and matrix models. Closely related is the work of where the issue of extra states was discussed directly in the Liouville theory. We would like to point out that our \( S \)-matrix obviously does not agree with the so-called “bulk” \( S \)-matrix calculated in the collective string field theory by Gross and Klebanov. Therefore it is not clear how the collective string field theory exhibits stringy degrees of freedom found in the Liouville theory.
In conclusion, we have shown that the naive definition of string amplitudes gives rise to a non-trivial $S$-matrix with the desired property of factorization, if we take into account the extra physical states. The nature of extra states is obscure though. It is not obvious to us that they can be related to topological degrees of freedom of $D < 1$ string theory \cite{4}.

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References

[1] N. Seiberg, Rutgers preprint RU-90-29, to appear in the proceedings of the 1990 Yukawa International Seminar.

[2] J. Polchinski, Texas preprint UTTG-19-90, to appear in the Proceedings of String ’90.

[3] D.J. Gross and I.R. Klebanov, Nucl. Phys. B359 (1991) 3.

[4] A.M. Polyakov, Mod. Phys. Lett. A6 (1991) 635.

[5] K. Demeterfi, A. Jevicki and J.P. Rodrigues, Brown/Witwatersrand preprint BROWN-HET-795, CLN-91-03 (1991);
G. Mandal, A.M. Sengupta and S.R. Wadia, IAS preprint IASSNS-HEP/91/8 (1991);
G. Moore, Rutgers/Yale preprint RU-91-12, YCTP-P8-91 (1991);
J. Polchinski, Texas preprint UTTG-06-91 (1991).
[6] P. Di Francesco and D. Kutasov, Phys. Lett. 261B (1991) 385.

[7] A. Gupta, S.P. Trivedi and M.B. Wise, Nucl. Phys. B340 (1990) 475.

[8] S. Weinberg, Phys. Lett. 156B (1985) 309.

[9] A.M. Polyakov, Gauge Fields and Strings (Harwood, Chur, 1987); J. Polchinski, Nucl. Phys. B307 (1988) 61.

[10] D.J. Gross, I.R. Klebanov and M.J. Newman, Nucl. Phys. B350 (1991) 621.

[11] A. Polyakov, Phys. Lett. 103B (1981) 207.

[12] T.L. Curtright and C.B. Thorn, Phys. Rev. Lett. 48 (1982) 1309; F. David, Mod. Phys. Lett. A3 (1988) 1651; J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.

[13] M.A. Virasoro, Phys. Rev. 177 (1969) 2309; J. Shapiro, Phys. Lett. 33B (1970) 361.

[14] V.S. Dotsenko and V.A. Fateev, Nucl. Phys. B251 (1985) 691.

[15] U.H. Danielsson and D.J. Gross, Princeton preprint PUPT-1258 (1991).

[16] C. Bachas and S. Hwang, Phys. Lett. 247B (1990) 265.

[17] G. Moore and N. Seiberg, Rutgers/Yale preprint RU-91-29, YCTP-P19-91 (1991).

[18] S. Mukherji, S. Mukhi and A. Sen, Tata preprint TIFR/TH/91-25 (1991).

[19] A. Jevicki and B. Sakita, Nucl. Phys. B165 (1980) 511; S.R. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639.