MONOPOLE CONDENSATION AND ANTISYMMETRIC TENSOR FIELDS: COMPACT QED AND THE WILSONIAN RG FLOW IN YANG-MILLS THEORIES

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Abstract

A field theoretic description of monopole condensation in strongly coupled gauge theories is given by actions involving antisymmetric tensors $B_{\mu\nu}$ of rank 2. We rederive the corresponding action for 4d compact QED, summing explicitly over all possible monopole configurations. Its gauge symmetries and Ward identities are discussed. Then we consider the Wilsonian RGs for Yang-Mills theories in the presence of collective fields (again tensors $B_{\mu\nu}$) for the field strengths $F_{a\mu\nu}$ associated to the U(1) subgroups. We show that a “vector-like” Ward identity for the Wilsonian action involving $B_{\mu\nu}$, whose validity corresponds to monopole condensation, constitutes a fixed point of the Wilsonian RG flow.

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1 Introduction

Recently progress has been made in the field theoretic formulation of monopole condensation in strongly coupled gauge theories, which has been proposed by t’Hooft and Mandelstam [1] as the underlying mechanism of confinement in QCD. Quevedo and Trugenberger [2] have proposed the “kinetic parts” (quadratic in the fields) of actions in $d$ space-time dimensions describing the condensation of $(d - r - 1)$ dimensional topological defects by the means of antisymmetric tensors of rank $r$. Applied to monopole condensation in gauge theories in $d = 4$ the corresponding action requires the introduction of a Kalb-Ramond field $B_{\mu\nu}$ of rank 2.

Starting with the dual formulation of compact QED in $d = 3$ [3] (where the topological defects have dimension 0), Polyakov [4] derived a partition function involving $B_{\mu\nu}$, which couples to the surface of the Wilson loop. (This coupling has also been considered in [2].) One of the aims of Polyakov was to show how a second quantized string theory emerges due to the multivaluedness of the action of $B_{\mu\nu}$ in Minkowski space. The massive Kalb-Ramond field $B_{\mu\nu}$ in Polyakov’s action plays exactly the role assigned to it in [2]. Employing a duality transformation beyond the semiclassical approximation (which was used by Polyakov), and working in $d = 4$, Diamantini, Quevedo and Trugenberger [5] rederived the result of Polyakov, again with the dual formulation of compact QED (involving a massive vector in $d = 4$) as a starting point.

In [6] (see also [7]) we had introduced rank 2 tensor fields in Yang-Mills theories as “collective fields” for the field strength tensor $F^a_{\mu\nu}$, in some analogy with the field strength formulation of Yang-Mills theories [8,9]. Using the Wilsonian exact renormalization group equations (ERGEs) in the presence of the collective fields we have argued in [6] that the quadratic part of the Wilsonian effective action in the infrared limit assumes a particular form, which is equivalent to the Quevedo-Trugenberger and Polyakov action [2,4]. A non-trivial “phenomenological” test of this approach consists in the computation of the field strength two-point function, which is now given in terms of the two-point function of $B_{\mu\nu}$ [10,11], and which agrees well with lattice results.

The aim of the present paper is twofold: First, in section 2, we reconsider four-dimensional compact QED on the lattice. Without passing by the dual formulation
involving the massive vector field, we will show directly, how monopole condensation lets the Kalb-Ramond field $B_{\mu\nu}$ appear, and we will rederive its action by explicit summation over monopole configurations. Our approach also allows to discuss explicitly, how vector-like gauge symmetries (under which $B_{\mu\nu}$ transforms) together with gauge fixing terms appear in a formulation, where both $B_{\mu\nu}$ and the original gauge field $A_\mu$ are present in the action. We emphasize the role of an associated Ward identity in this formulation. We also discuss how $A_\mu$ can be “gauged away” without modifying the number of degrees of freedom, whereupon one recovers Polyakov’s formulation involving just a massive $B_{\mu\nu}$ field without manifest gauge invariance.

Second, in section 3, we consider the Wilsonian ERGEs for Yang-Mills theories in the maximal Abelian gauge, and in the presence of collective fields $B_{\mu\nu}^a$ for the “diagonal” components of the field strength tensor $F_{\mu\nu}^a$ (the index $a$ being associated with the generators of the $N - 1$ U(1) subgroups of SU(N)). We present a modified vector-like Ward identity (depending explicitly on the Wilsonian infrared cutoff $k$) which a) is invariant under the ERG flow, and describes thus fixed “points” (actually still an infinite dimensional stable subclass) of Wilsonian effective actions, and b) which turns into the Ward identity satisfied by the Quevedo-Trugenberger and Polyakov actions for $k \to 0$. The role of this Ward identity for Wilsonian Yang-Mills effective action in the infrared limit is discussed in section 4.

2 Compact QED

Let us start with the partition function of compact QED on the lattice, following closely the presentation of Polyakov [3]. We will work in $d = 4$ Euclidean dimensions; our results can, however, straightforwardly be carried over to arbitrary dimensions, and sometimes we will let $d$ to be arbitrary.

On a lattice with lattice spacing $\ell = 1$ the action of compact QED is given by

$$S = \frac{1}{2e^2} \sum_x (1 - \cos F_{\mu\nu}(x)) \ , \quad (2.1)$$

where the sum over $\mu, \nu$ at each lattice site $x$ is understood, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.2)$$
with \( \partial_\mu \) being the lattice (forward) derivative. The fields \( A_\mu \) are restricted to the domain \(-\pi \leq A_\mu \leq \pi\). Neglecting higher powers than \( F_{\mu\nu}^2 \) in the action, but respecting the periodicity of the cosine, the partition function can be written as

\[
Z = \sum_{n_{\mu\nu}(x)} \int_{-\pi}^{+\pi} dA_\mu \delta(\partial_\mu A_\mu) e^{-\frac{1}{4e^2} \sum_x (F_{\mu\nu}(x) - 2\pi n_{\mu\nu}(x))^2}. \tag{2.3}
\]

Here the antisymmetric tensor \( n_{\mu\nu}(x) \) represents a set of 6 independent integers (in \( d = 4 \)) at each lattice site. For later convenience we have added a gauge fixing \( \delta \)-function \( \delta(\partial_\mu A_\mu) \). Due to the restricted domain of integration over \( A_\mu \) such a gauge fixing is actually not mandatory; it just amounts to the multiplication of \( Z \) by a finite factor per lattice site.

Next we introduce a Hodge decomposition of the 6 independent integers \( n_{\mu\nu} \) per lattice site:

\[
n_{\mu\nu} = \partial_\mu m_\nu + B_{\mu\nu}. \tag{2.4}
\]

The vector \( m_\mu \) satisfies \( \partial_\mu m_\mu = 0 \) and represents thus \( d - 1 = 3 \) independent degrees of freedom. The antisymmetric tensor \( B_{\mu\nu} \) satisfies \( \partial_\mu B_{\mu\nu} = 0 \), which constitute \( d - 1 = 3 \) constraints in \( d = 4 \). The remaining 3 degrees of freedom in \( B_{\mu\nu} \) can be represented in terms of a conserved monopole current density \( \tilde{q}_\mu \) (with \( \partial_\mu \tilde{q}_\mu = 0 \)) in the form

\[
\frac{1}{2} \epsilon_{\sigma\mu\nu\rho} \partial_\mu B_{\nu\rho} = \tilde{q}_\sigma. \tag{2.5}
\]

Integrating (2.3) over a lattice cube with surface \( \sum_i \) one obtains

\[
\frac{1}{2} \oint \sum_i B_{\mu\nu} d\sigma_{\mu\nu\rho} = q_\rho(z_i). \tag{2.6}
\]

The integer monopole currents \( q_\rho \), situated at centres \( z_i \) of the lattice cubes, are related to the density \( \tilde{q}_\sigma \) by

\[
\tilde{q}_\rho(z) = \sum_i q_\rho(z_i) \delta(z - z_i) \tag{2.7}
\]

where \( \delta(z - z_i) \) denotes the Kronecker symbol, \( \delta(z - z_i) = 1 \) for \( z = z_i \), \( \delta(z - z_i) = 0 \) otherwise. Introducing a dual field strength \( H_\sigma \) for \( B_{\mu\nu} \),

\[
H_\sigma = \frac{1}{2} \epsilon_{\sigma\mu\nu\rho} \partial_\mu B_{\nu\rho}. \tag{2.8}
\]
the sum over \( n_{\mu\nu} \) in the partition function (2.3) can be replaced by sums over \( m_\mu \) and \( B_{\mu\nu} \), together with the corresponding constraints:

\[
Z = \sum_{m_\mu(x)} \sum_{B_{\mu\nu}(x)} \int_{-\pi}^{+\pi} \prod_x [dA_\mu \delta (\partial_\mu A_\mu) \tilde{\delta} (\partial_\mu m_\mu) \tilde{\delta}^{d-1} (\partial_\mu B_{\mu\nu})] \\
\times \left( \sum_{\tilde{q}(z)} \prod_z \tilde{\delta}^{d-1} (H_\sigma(z) - \tilde{\tilde{q}}_\sigma(z)) \right) e^{-\frac{1}{4\pi} \sum_z (F_{\mu\nu} - 2\pi \partial_\mu m_\nu - 2\pi B_{\mu\nu})^2}. \quad (2.9)
\]

Here \( \tilde{\delta} \) denote again Kronecker symbols, now in field space. The \( d-1 \) dimensional Kronecker symbol of a conserved vector \( v_\mu \), \( \tilde{\delta}^{d-1}(v_\mu) \) with \( \partial_\mu v_\mu = 0 \), can be represented as

\[
\tilde{\delta}^{d-1}(v_\mu) = \text{const.} \int_{-\pi}^{+\pi} \prod_x [dC_\mu \delta (\partial_\mu C_\mu)] e^{i \sum_x C_\mu v_\mu}. \quad (2.10)
\]

Next we can combine the gauge field \( A_\mu \) with the integers \( m_\mu \) into a single field \( A'_\mu \), which varies from \(-\infty\) to \(+\infty\) at each lattice site:

\[
A'_\mu = A_\mu + 2\pi m_\mu \quad . \quad (2.11)
\]

At this point the introduction of the gauge fixing \( \delta \) function in (2.3) proves to be convenient. Omitting the primes of \( A'_\mu \), the partition function becomes

\[
Z = \sum_{B_{\mu\nu}(x)} \int_{-\infty}^{+\infty} \prod_x [dA_\mu \delta (\partial_\mu A_\mu) \tilde{\delta}^{d-1} (\partial_\mu B_{\mu\nu})] \left( \sum_{\tilde{q}_\sigma(z)} \prod_z \tilde{\delta}^{d-1} (H_\sigma(z) - \tilde{\tilde{q}}_\sigma(z)) \right) \\
\times e^{-\frac{1}{4\pi} \sum_x (F_{\mu\nu} - 2\pi B_{\mu\nu})^2}. \quad (2.12)
\]

Our next aim is the explicit evaluation of the sum over monopole current configurations \( \tilde{q}_\sigma(z) \). First, we decompose all possible configurations \( \tilde{q}_\sigma(z) \) into configurations, which are nonvanishing at \( N \) centres of the lattice cubes:

\[
\sum_{\tilde{q}_\sigma(z)} = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{z_1 \ldots z_N} \sum_{q_\sigma(z_i) \neq 0} \left| \tilde{q}_\sigma(z) = \sum_{i=1}^N q_\sigma(z_i) \tilde{\delta}(z - z_i) \right|. \quad (2.13)
\]

Second, we observe that contributions to the partition function with \( |q_\sigma(z_i)| \geq 2 \) are suppressed relatively to configurations with \( q_\sigma(z_i) = \pm 1 \); therefore we will restrict the sum over \( q_\sigma(z_i) \) to these values subsequently. Third, the corresponding term \( \sim B_{\mu\nu}^2 \) in the action in (2.12) is problematic even for these restricted configurations: Near a monopole current situated at \( z' \) the field \( B_{\mu\nu}(x) \) behaves like \( |x - z'|^{-3} \) (in
$d = 4$), consequently the continuum integral $\int d^4 x B^2_{\mu\nu}$ would diverge quadratically, whereas on the lattice we are left with an ambiguity in the form of a factor

$$\xi = e^{-\frac{\text{const}}{x^2}}$$  \hspace{1cm} (2.14)

per centre with nonvanishing monopole current. Taking the $N$ powers of $\xi$ into account we thus rewrite the sum over $\tilde{q}_\sigma(z)$ as

$$\sum_{\tilde{q}_\sigma(z)} \prod \tilde{\delta}^{d-1}(H_\sigma(z) - \tilde{q}_\sigma(z))$$

$$\to \prod_{z} \sum_{N=0}^\infty \frac{1}{N!} \xi^N \sum_{z_1 \cdots z_N} \sum_{q_\sigma(z_i) = \pm 1} \tilde{\delta}^{d-1}(H_\sigma(z) - \sum_{i=1}^N q_\sigma(z_i) \delta(z - z_i)) . \hspace{1cm} (2.15)$$

Let us now fix $z$, $N$ and the Lorentz index $\sigma$ and investigate, in which cases the sums over $z_i$ and $q_\sigma(z_i)$ give a nonvanishing contribution to the monopole current density $\tilde{q}_\sigma(z)$ in the argument of the Kronecker symbol $\tilde{\delta}^{d-1}(H_\sigma(z) - \tilde{q}_\sigma(z))$. Let us assume that the lattice has $V$ sites. Then, a sum over a variable $z_i$ and the associated monopole current $q_\sigma(z_i) = \pm 1$ gives $2V$ terms. In $(2V - 2)$ cases the contribution to $\tilde{q}_\sigma(z)$ vanishes, since $z_i$ differs from $z$ and $\delta(z - z_i)$ is zero. The two remaining cases give contributions $\Delta \tilde{q}_\sigma = \pm 1$ to $\tilde{q}_\sigma$. Turning to the sums over $N$ variables $z_i$ and associated monopole currents $q_\sigma(z_i) = \pm 1$ we can decompose the result into powers of $(2V - 2)$:

We have $(2V - 2)^N$ cases, where no $z_i$ coincides with $z$, and where $\tilde{q}_\sigma$ vanishes. We have $2N(2V - 2)^{N-1}$ cases, where one $z_i$ coincides with $z$; these $2N(2V - 2)^{N-1}$ cases can be decomposed into $N(2V - 2)^{N-1}$ cases with $\tilde{q}_\sigma = +1$ and $N(2V - 2)^{N-1}$ cases with $\tilde{q}_\sigma = -1$. Next we have $\frac{1}{2} \cdot (2N) \cdot (2N - 2) \cdot (2V - 2)^{N-2}$ cases where two $z_i$ coincide with $z$; a quarter of them corresponds to $\tilde{q}_\sigma = +2$, half of them to $\tilde{q}_\sigma = 0$ (since $\tilde{q}_\sigma(z_i) = -\tilde{q}_\sigma(z_j)$ with $z_i = z_j = z$), and the remaining quarter to $\tilde{q}_\sigma = -2$. In general we have \left(\begin{array}{c} N \\ m \end{array}\right)(2V - 2)^{N-m} 2^m$ cases where $m$ variables $z_i$ coincide with $z$.

The corresponding contributions to $\tilde{q}_\sigma$ are generally different; if we distinguish these contributions to $\tilde{q}_\sigma$ we can write the different cases as

$$\left(\begin{array}{c} N \\ m \end{array}\right)(2V - 2)^{N-m} \left[ \sum_{\nu=0}^m \left(\begin{array}{c} m \\ \nu \end{array}\right) \right]_{\tilde{q}_\sigma = 2\nu - m} . \hspace{1cm} (2.17)$$


where the expression in the squared brackets gives \( 2^m \) terms. Summing over \( m \) we can thus rewrite the expression (2.13) as

\[
\prod_z \sum_{N=\nu}^{\infty} \frac{1}{N!} \xi^N \sum_{m=0}^{N} \binom{N}{m} (2V - 2)^{N-m} \sum_{\nu=0}^{m} \binom{m}{\nu} \delta^{d-1} (H_\sigma(z) - 2\nu + m). \tag{2.18}
\]

The sums can be rearranged and partially evaluated; as an intermediate result one obtains

\[
\prod_z e^{\xi(2V - 2)} \sum_{\nu=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!\nu!} \xi^{n+\nu} \sum_{\nu=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!\nu!} \delta^{d-1} (H_\sigma(z) - \nu + n). \tag{2.19}
\]

Employing the Kronecker symbol in order to perform either of the sums over \( \nu \) or \( n \) one ends up with a standard series representation for a modified Bessel function [12] and, omitting the field independent prefactor \( \exp \xi(2V - 2) \), the result becomes simply

\[
\prod_{z,\sigma} I_{H_\sigma(z)} (2\xi) = e^{\sum_{z,\sigma} \log I_{H_\sigma(z)}(2\xi)} \tag{2.20}
\]

where we have restored the summation over the Lorentz index \( \sigma \).

The same result has been obtained previously by Diamantini, Quevedo and Trugenberger [5] with the help of an exact duality transformation of the partition function of the massive dual gauge field. As noted in [4], in the semiclassical approximation \( H_\sigma \rightarrow \infty, \xi \rightarrow \infty, H_\sigma/\xi \) fixed the exponent in (2.20) becomes [13]

\[
\sum_{z} \left( -H_\sigma \arcsinh \left( \frac{H_\sigma}{2\xi} \right) + \sqrt{H_\sigma^2 + 4\xi^2} \right) \equiv -S_P(H) \tag{2.21}
\]

whereupon one recovers the Euclidean four-dimensional version of Polyakov’s action \( S_P(H) \) in [4].

With the result for the summation over monopole configurations at hand we are in a position to rewrite the partition function (2.12). For convenience we switch to a continuum notation, and rescale the fields such that the kinetic terms are properly normalized: \( A_\mu \rightarrow e\Lambda_0^{-1} A_\mu, \ B_{\mu\nu} \rightarrow \sqrt{2\xi} \Lambda_0^{-1} B_{\mu\nu} \) such that \( \Lambda_0 S_P(H) = \frac{1}{2} H_\mu^2 + O(H^4) \), where \( \Lambda_0 \) is the inverse lattice spacing. We obtain

\[
Z = \int \mathcal{D}A_\mu \mathcal{D}B_{\mu\nu} \delta(\partial_\mu A_\mu) \delta^{d-1}(\partial_\mu B_{\mu\nu}) \ e^{-\int d^4x \left\{ \frac{1}{4}(F_{\mu\nu} - mB_{\mu\nu})^2 + S_P(H) \right\}} \tag{2.22}
\]

where the mass \( m \) of the Kalb-Ramond field \( B_{\mu\nu} \) is given by

\[
m^2 = \frac{8\pi^2}{e^2} \xi \Lambda_0^2. \tag{2.23}
\]
In the weak field limit of $S_P(H)$, the action in (2.22) coincides with the 4d version of the action proposed by Quevedo and Trugenberger [2] to describe the condensation of topological defects.

Now there are two options concerning a subsequent treatment of the partition function (2.22): The first option consists in a field redefinition

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \frac{1}{m} F_{\mu\nu} .$$

(2.24)

Since $H_{\mu}$ is invariant under this redefinition of $B_{\mu\nu}$ by a Bianci identity, the gauge field $A_{\mu}$ disappears completely from the action and appears only in the $\delta$ functions

$$\delta(\partial_{\mu}A_{\mu}) \delta^{d-1}(\partial_{\mu}B_{\mu\nu} + \frac{1}{m} \partial_{\mu}F_{\mu\nu}).$$

Since the number of $\delta$ functions matches precisely the number of degrees of freedom of $A_{\mu}$, the $A_{\mu}$ path integral can be performed giving just trivial (field independent) determinants. The resulting partition function reads

$$Z^{(1)} = \int D B_{\mu\nu} e^{-\int d^4 x \left\{ \frac{1}{4} m^2 B_{\mu\nu}^2 + S_P(H) \right\}} ,$$

(2.25)

which is the version derived by Polyakov [4].

The second way to treat the partition function (2.22) consists in the standard procedure to promote the $\delta$ functions to gauge fixing terms in the action: First, we replace the $\delta$ functions by

$$\delta (\partial_{\mu}A_{\mu} - C_{\mu}) \delta^{d-1} (\partial_{\mu}B_{\mu\nu} - C_{\nu})$$

(2.26)

where $C_{\mu}$ are arbitrary functions with $\partial_{\mu}C_{\mu} = 0$. Next, we integrate over these functions with a Gaussian measure involving arbitrary gauge fixing parameters $\alpha$ and $\beta$; the resulting partition functions reads (again up to field independent Fadeev-Popov determinants)

$$Z^{(2)} = \int D A_{\mu} D B_{\mu\nu} e^{-S_{inv} - S_{gf}}$$

(2.27)

with

$$S_{inv} = \int d^4 x \left\{ \frac{1}{4} (F_{\mu\nu} - m B_{\mu\nu})^2 + S_P(H) \right\} ,$$

$$S_{gf} = \int d^4 x \left\{ \frac{1}{2\alpha} (\partial_{\mu}A_{\mu}) + \frac{1}{2\beta} (\partial_{\mu}B_{\mu\nu})^2 \right\} .$$

(2.28)

Here $S_{gf}$ serves to "gauge fix" the gauge symmetries of $S_{inv}$:

a) $\delta A_{\mu} = \partial_{\mu} \Lambda$, $\delta B_{\mu\nu} = 0$

b) $\delta A_{\mu} = m \Lambda_{\mu}$, $\delta B_{\mu\nu} = \partial_{\mu} \Lambda_{\nu} - \partial_{\nu} \Lambda_{\mu}$ .

(2.29)
These gauge symmetries are actually not independent; the transformation \( a \) is obtained from \( b \) after identifying \( \Lambda_{\mu} = m^{-1} \partial_{\mu}A \). A remnant of the gauge invariance \( b \) is the following Ward identity, which is satisfied by the total action \( S_T = S_{\text{inv}} + S_{gf} \):

\[
\frac{\delta S_T}{\delta A_{\mu}} + \frac{2}{m} \partial_{\nu} \frac{\delta S_T}{\delta B_{\mu\nu}} + \frac{1}{\alpha} \partial_{\mu} \partial_{\nu} A_{\nu} - \frac{\Box}{\beta m} \partial_{\nu} B_{\nu\mu} = 0 .
\]

(2.30)

The standard Ward identity related to the gauge invariance \( a \) is obtained by contracting (2.30) with \( \partial_{\mu} \).

Herewith we conclude this section and turn now to a possible relation between \( S_T \) and the low energy effective action of pure Yang-Mills theories.

### 3 Wilsonian RG flow for Yang-Mills theories with antisymmetric tensor fields

Let us start this section with the definition of the partition function of an Euclidean Yang-Mills theory. Subsequently we will employ the maximal abelian gauge [7,9,14]. Abelian gauges were originally introduced by t’Hooft [1], who showed that they lead to the appearance of magnetic monopoles in Yang-Mills theories. Below it will be useful that abelian Ward identities related to U(1) subgroups of SU(N) remain valid in the maximal abelian gauge [14]. Including gauge fixing terms and ghosts, the Yang-Mills partition function reads:

\[
\exp \{-G(J, \chi, \bar{\chi})\} = \int D_{\text{reg}}(A, c, \bar{c}) \exp \{-S_{YM} - S_{gf} - S_{gh} + J \cdot A + \bar{\chi} \cdot c + \chi \cdot \bar{c}\} \quad (3.1)
\]

where we used the short-hand notation

\[
J \cdot A = \int d^4 x \ J^\alpha_{\mu}(x) \ A^\alpha_{\mu}(x) \quad \text{etc.} \quad (3.2)
\]

The index “reg” attached to the path integral measure indicates an ultraviolet regularisation. \( S_{YM} \) denotes the standard Yang-Mills action, \( S_{gf} \) the gauge fixing terms, and \( S_{gh} \) the terms depending on the ghost fields. In the maximal abelian gauge it is convenient to adopt following conventions: We decompose the \( N^2 - 1 \) generators of SU(N) indexed by \( \alpha, \beta = 1 \ldots N^2 - 1 \) into the \( N - 1 \) generators of the \( N - 1 \)
U(1) subgroups with \(N - 1\) indices \(a, b\) and the \(N(N - 1)\) non-diagonal “charged” generators indexed by \(i, j = 1 \ldots N(N - 1)\). It is helpful to introduce a U(1)-covariant derivative \(D_\mu\), which acts on the charged fields \(\phi^i = \{A^i_\mu, c^i, \bar{c}^i\}\) as

\[
D_\mu \phi^i = \partial_\mu \phi^i + gf^i_{\alpha j}A^\alpha_\mu \phi^j .
\]  

(3.3)

The Yang-Mills action thus decomposes as

\[
S_{YM} = \int d^4x \frac{1}{4} F^a_\mu \cdot F^a_\mu = \int d^4x \left\{ \frac{1}{4} F^i_\mu \cdot F^i_\mu + \frac{1}{4} F^a_\mu \cdot F^a_\mu \right\} .
\]  

(3.4)

The maximal abelian gauge corresponds to gauge fixing terms of the form

\[
S_{gf} = \int d^4x \left\{ \frac{1}{2\alpha} (\partial_\mu A^a_\mu)^2 + \frac{1}{2\alpha(c)} (D_\mu A^i_\mu)^2 \right\} ,
\]  

(3.5)

i.e. the gauge fixing of the charged gauge fields \(A^a_\mu\) is U(1) gauge invariant. The form of \(S_{gh}\) is not relevant subsequently.

Now we add collective fields \(B^a_\mu\) for the U(1) field strengths \(F^a_\mu\) to the partition function (3.1). The addition of collective fields corresponds to a multiplication of the integrand of the path integral with

\[
1 = \frac{1}{N} \int DB \exp\{ -\hat{S} \} , \quad \hat{S} = \int d^4x \left\{ \frac{1}{4} \left( F^a_\mu - B^a_\mu \right)^2 \right\}
\]  

(3.6)

Moreover we add the following source term to the exponent under the path integral:

\[
\hat{J}^a_\mu \cdot B^a_\mu
\]  

(3.7)

If one performs the Gaussian path integral over \(B\) in the presence of the source terms (3.7), one finds that the sources \(\hat{J}^a_\mu\) couples to the operator \(F^a_\mu\). In addition one obtains terms quadratic in the sources. The expression for the partition function finally becomes

\[
\exp \left\{ -G \left( J, \chi, \bar{\chi}, \hat{J} \right) \right\} = \int DA \cdot c \cdot \bar{c} \cdot B \exp \left\{ -S(A, B) - S_{gf} - S_{gh} + Sources \right\}
\]  

(3.8)

with
\[ S(A, B) = \int d^4x \left\{ \frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i + \frac{1}{2} F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2} F_{\mu\nu}^a B_{\mu\nu}^a + \frac{1}{4} B_{\mu\nu}^a B_{\mu\nu}^a \right\} \]  

(3.9)

The expression \( \text{Sources} \) reads

\[ \text{Sources} = J \cdot A + \bar{\chi} \cdot c + \chi \cdot \bar{c} + \hat{J} \cdot B \]  

(3.10)

where we employ the convention (3.2). For later use we introduce the effective action \( \Gamma(A, c, \bar{c}, B) \), the Legendre transform of \( G(J, \chi, \bar{\chi}, \hat{J}) \):

\[ \Gamma(A, c, \bar{c}, B) = G(J, \chi, \bar{\chi}, \hat{J}) + J \cdot A + \bar{\chi} \cdot c + \chi \cdot \bar{c} + \hat{J} \cdot B \]  

(3.11)

The Wilsonian ERGEs [15–18] are obtained by adding an “artificial” infrared cutoff \( k \) to the partition function (3.1) or (3.8). One exploits the facts that the corresponding \( k \) dependent effective action \( \Gamma_k \) a) is equal to the full quantum effective action \( \Gamma \) for \( k = 0 \), b) corresponds to the classical action \( S_{cl} \) in the limit \( k \to \infty \) (up to additional terms determined by modified Slavnov-Taylor identities [16,17]), and c) that an exact functional differential equation with respect to \( k \) (the ERGE) can be derived. The integration of the ERGEs from some large value \( k = \Lambda \) down to \( k = 0 \) provides us with a non-perturbative method for calculating \( \Gamma_{k=0} \) in terms of some “high energy” effective action \( \Gamma_\Lambda \sim S_{cl} \). The formalism can straightforwardly be extended towards partition functions involving sources for collective fields [18,6,11], provided \( S_{cl} \) is replaced by \( S_{YM} + \hat{S} = S(A, B) \).

Let us consider directly the partition function (3.8) with the collective fields included. In the presence of an infrared cutoff it becomes

\[ \exp \left\{ -G_k \left( J, \chi, \bar{\chi}, \hat{J} \right) \right\} \]

\[ = \int D(A, c, \bar{c}, B) \exp \left\{ -S(A, B) - S_{gf} - S_{gh} - \Delta S_k + \text{Sources} \right\} \]  

(3.12)

where \( \Delta S_k \) is quadratic in the gluon and ghost fields:

\[ \Delta S_k = \int \frac{d^4p}{(2\pi)^4} \left[ \frac{1}{2} A_{\mu}^a(-p) R^k_{\mu\nu}(p^2) A_{\nu}^a(p) + \bar{c}^a(-p) R^k_{\nu\alpha}(p^2) c^a(p) \right] \]  

(3.13)
The functions $R^k(p^2)$ modify the propagators such that modes with $p^2 \ll k^2$ are suppressed. Possible choices are

$$R^k_{\mu\nu}(p^2) = \left( p^2 \delta_{\mu\nu} - \left( \frac{1}{\alpha} - 1 \right) p_\mu p_\nu \right) \tilde{R}^k(p^2),$$

$$R^k_g(p^2) = p^2 \tilde{R}^k(p^2),$$

$$\tilde{R}^k(p^2) = \frac{e^{-p^2/k^2}}{1 - e^{-p^2/k^2}}. \quad (3.14)$$

Infrared cutoffs for the collective fields could also be introduced \[18\] but are not mandatory. The ERGEs for the functional $G_k$ follow after differentiation of both sides of (3.12) with respect to $k$, and expressing the expectation value $< \partial_k \Delta S_k >$ through variations with respect to the sources. One obtains

$$\partial_k G_k = \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2} \partial_k R^k_{\mu\nu} \left( \frac{\delta G_k}{\delta J^\alpha_\mu(p)} \frac{\delta G_k}{\delta J^\alpha_\nu(-p)} - \frac{\delta^2 G_k}{\delta J^\alpha_\mu(p) \delta J^\alpha_\nu(-p)} \right) \right. \right.$$

$$\left. + \partial_k R^k_g \left( \frac{\delta G_k}{\delta \chi^\alpha(p)} \frac{\delta G_k}{\delta \bar{\chi}^\alpha(-p)} - \frac{\delta^2 G_k}{\delta \chi^\alpha(p) \delta \bar{\chi}^\alpha(-p)} \right) \right\}. \quad (3.15)$$

The ERGEs for $\Gamma_k$, the Legendre transform of $G_k$, can easily be obtained from (3.15), but they are not needed in the following.

Now we wish to show that a slight variant of the Ward identity (2.30), which implies the (gauge fixed) vector like gauge symmetry b) in eq. (2.29), constitutes a (quasi) fixed point of the ERGEs. To this end we study the following functional $\Omega^{k,a}_\mu$, which is constructed in terms of sources, $G_k$ (satisfying the ERGEs (3.15)), and the IR cutoff function $R^k$:

$$\Omega^{k,a}_\mu = J^a_\mu(p) + 2i \frac{1}{m} p_\nu \bar{J}^a_{\mu\nu}(p) + \frac{1}{\alpha} p_\mu p_\nu \frac{\delta G_k}{\delta J^\alpha_\mu(p)} - \frac{ip^2}{\beta m} p_\nu \frac{\delta G_k}{\delta \bar{J}^a_{\nu\mu}(p)} + R^k_{\mu\nu} \frac{\delta G_k}{\delta \bar{J}^a_{\nu\mu}(p)}. \quad (3.16)$$

First we note that, for $k \to 0$, $R^k$ and hence the last term in $\Omega^{k,a}_\mu$ vanishes. The remaining 4 terms in $\Omega^{0,a}_\mu$ can be expressed in terms of $\Gamma_0$, the effective action at vanishing IR cutoff $k$, through the Legendre transform (3.11):

$$\Omega^{0,a}_\mu = \frac{\delta \Gamma_0}{\delta A^a_\mu(-p)} + 2i \frac{1}{m} p_\nu \frac{\delta \Gamma_0}{\delta B^a_{\mu\nu}(-p)} - \frac{1}{\alpha} p_\mu p_\nu A^a_\nu(p) + \frac{ip^2}{\beta m} p_\nu B^a_{\nu\mu}(p). \quad (3.17)$$
Thus the functional equation
\[ \Omega^{0,a}_{\mu} = 0 \] (3.18)
is equivalent to the statement that \( \Gamma_0 \) satisfies the Ward identity (2.30).

Next we consider the variation of \( \Omega^{k,a}_{\mu} \) with respect to the infrared cutoff \( k \). When one evaluates \( \partial_k \Omega^{k,a}_{\mu} \) with \( \Omega^{k,a}_{\mu} \) as in eq. (3.16), the derivative \( \partial_k \) hits \( G_k \) and the IR cutoff function \( R_k \). Using the ERGE (3.15) for \( \partial_k G_k \) one obtains the following important result:
\[ \partial_k \Omega^{k,a}_{\mu} = 0 \quad \text{if} \quad \Omega^{k,a}_{\mu} = 0 . \] (3.19)
Hence the functional equation \( \Omega^{k,a}_{\mu} = 0 \) — either in terms of \( G_k \) or in terms of \( \Gamma_k \) — constitutes a quasi-fixed point of the ERGEs. If it is satisfied by \( G_k \) or \( \Gamma_k \) for some \( k \), it will also be satisfied by \( G_0 \) and \( \Gamma_0 \), i.e. \( \Gamma_0 \) will satisfy the Ward identity (2.30).

Let us define an “Abelian projection” \( \tilde{\Gamma}_0 \) of \( \Gamma_0 \) by
\[ \tilde{\Gamma}_0 = \Gamma_0 \mid_{\varphi^i = 0} , \] (3.20)
where \( \varphi^i \) denote all “charged” fields \( A_{\mu}^i, c^i, \bar{c}^i \) with respect to the U(1) subgroups, cf. our convention for the indices below eq. (3.2). Trivially, once \( \Gamma_0 \) satisfies the Ward identity (2.30), it is also satisfied by \( \tilde{\Gamma}_0(A_{\mu}^a, B_{\mu\nu}^a) \). Up to terms quadratic in the fields and derivatives, the general solution for \( \tilde{\Gamma}_0 \) is then necessarily of the form of the weak field limit of \( S_{inv} + S_{gf} \) in eqs. (2.28), i.e. of the form of the Quevedo-Trugenberger action describing the condensation of magnetic monopoles. The terms involving higher powers of the dual field strength \( H_{\mu} \) of the Kalb-Ramond field \( B_{\mu\nu} \), which appear in Polyakov’s action \( S_P(H) \) in (2.21) and (2.28), also satisfy the Ward identity, but cannot be derived by the Ward identity alone.

Remember that, in the process of computing \( \Gamma_{k=0} \) by integrating the ERGEs with respect to \( k \), a “boundary condition” \( \Gamma_{\Lambda} \) at some large scale \( k = \Lambda \) has to be specified. In our case \( \Gamma_{\Lambda} \) is given by the action \( S(A, B) \) of eq. (3.9), up to gauge fixing terms for the gluons, and up to additional terms of \( O(g^2 R^A) \) in order to satisfy the modified Slavnov-Taylor identities [16,17]. It is easily checked that the Abelian projection \( \Gamma_{\Lambda} \) does not satisfy the Ward identity (2.30) (for \( \beta \to \infty \)), since the three terms \( \frac{1}{2} F^a F^a - \frac{1}{2} F^a B^a + \frac{1}{4} B^a B^a \) are not of the form of a square. Trivially, the classical Yang-Mills action does not describe confinement by monopole condensation, even if collective fields \( B_{\mu\nu}^a \) are introduced.
During the ERGE flow all parameters in $\Gamma_k$ will vary with $k$; the ERGE flow can be represented as a motion in the infinite dimensional space of couplings (parameters) of $\Gamma_k$. Couplings, which are absent in $\Gamma_\Lambda$, but not protected by the modified Slavnov-Taylor identities, will become non-zero, as powers of $H_\mu$ or terms of the form $(\partial_\mu B_{\mu\nu})^2$. A priori it is an open question whether, for some value of $k$, the terms involving $F^a$ combine with terms involving $B^a$ into combinations of the form $(F^a - mB^a)$, where a $k$-independent parameter $m$ is generated dynamically by dimensional transmutation. A necessary condition for this to happen is that the “fixed point” $\Omega^k_{\mu a} = 0$, with some arbitrary ($k$ independent dynamically generated value) of $\beta$, is infrared attractive.

Indications for such an “infrared attractiveness” can be obtained from the results in [6], where the RG flow of the $A_\mu/B_{\mu\nu}$ system has been studied in a simple approximation (However, in [6] antisymmetric tensor fields were introduced for all $N^2 - 1$ components of $F_{\mu\nu}$, and the Landau gauge was employed): Within a parametrization of $\Gamma_k(A, B)$ of the form

$$\Gamma_k(A, B) = \frac{Z}{4}(F_{\mu\nu})^2 - \frac{n}{2}F_{\mu\nu}B_{\mu\nu} + \frac{m^2}{4}(B_{\mu\nu})^2 + \text{gauge fixing terms} \quad (3.21)$$

it was shown that

$$Z_{\text{eff}} = Z - \frac{n^2}{m^2} = 0 \quad , \quad (3.22)$$

whereupon $F$ and $B$ combine into a perfect square, constitutes an infrared fixed point. At the starting point, where $\Gamma_\Lambda(A, B)$ is given by $S(A, B)$ of eq. (3.9), we have $Z = 2$, $n = m$ and hence $Z_{\text{eff}}(\Lambda) = 1$. However, already perturbatively, $Z_{\text{eff}}(k)$ decreases with decreasing $k$ and approaches thus the fixed point (3.22).

Within the simple approximation in [6], on the other hand, a Landau singularity in the running gauge coupling prevented a detailed analysis of the regime $k \to 0$. This problem disappeared within a less trivial truncation of $\Gamma_k$ in [11], where the gauge coupling became even vanishingly small for $k \to 0$.

Whereas the dependence of the results of [6,11] on the truncation of $\Gamma_k$ is an open problem, we emphasize that the fixed point nature of the Ward identity, eq. (3.19), is completely general.

A final remark concerns the relevance of the maximal abelian gauge for our results:
Since the abelian Ward identities (several in the case of several U(1)'s) follow from the vector Ward identity $\Omega_{\mu}^{k,a} = 0$ after contraction with $p_\mu$, their validity is a necessary condition on $\Gamma_k$, if $\Gamma_k$ is assumed to satisfy $\Omega_{\mu}^{k,a} = 0$. This necessary condition is guaranteed to be satisfied precisely in the maximal abelian gauge.

4 Summary and Conclusions

The aim of the present paper is to emphasize the role of antisymmetric tensor fields for the description of monopole condensation in strongly coupled gauge theories. In the first part of the paper we have studied four-dimensional compact QED on the lattice, and we have rederived two equivalent versions of the partition function: The first version involves just a massive Kalb-Ramond field $B_{\mu\nu}$, and the original Abelian gauge field $A_\mu$, has disappeared completely (it has been “eaten” by $B_{\mu\nu}$ in order to become massive, in the same way as Goldstone bosons are eaten by massive vector fields in the case of spontaneous gauge symmetry breaking). The corresponding action had been obtained by Polyakov [4] and Diamantini, Quevedo and Trugenberger [5] before, starting with the dual action involving a massive vector field. Here we have shown how to obtain this action directly from compact QED.

The second version of the partition function involves both the gauge field $A_\mu$ and the Kalb-Ramond field $B_{\mu\nu}$, but additional gauge fixing terms which fix, in particular, the vector-like gauge symmetry under which $B_{\mu\nu}$ transforms. The quadratic part of the action is a special ($4d$) case of actions proposed by Quevedo and Trugenberger [2] in order to describe the condensation of topological defects. Here we have emphasized a Ward identity related to the vector gauge symmetry, whose validity is a sufficient condition on the action in order to be of the Quevedo-Trugenberger form.

In the second part of the paper we have studied the Wilsonian exact renormalization group flow of pure Yang-Mills theories in the maximal Abelian gauge, and in the presence of auxiliary fields $B^a_{\mu\nu}$ for the “diagonal” components $F^a_{\mu\nu}$ of the field strength. We have introduced a modified ($k$-dependent) vector Ward identity $\Omega_{\mu}^{k,a}(\Gamma_k) = 0$ and shown that its validity is stable under the ERG flow. At vanishing IR cutoff $k$ it coincides with the Ward identity above. Its validity does not fix $\Gamma_k$ completely, but constrains the infinitely many couplings in $\Gamma_k$ to lay inside a
fixed “hyperplane” in the infinite dimensional space of couplings. This picture can be represented schematically as in Fig. 1: The plane in Fig. 1 represents the infinite dimensional space of couplings of actions depending on Abelian gauge fields $A_\mu$ and Kalb-Ramond fields $B_{\mu\nu}$. The curve $W$ represents the “hyperplane” on which the vector Ward identity is satisfied.

Wilsonian actions of Yang-Mills theories can be represented on this plane, once auxiliary fields $B^a_{\mu\nu}$ are introduced, and once they are projected onto the Abelian subsector. Their ERG flow is represented by the curve $YM$ in Fig. 1. The starting point of the ERG flow is denoted by the point $P$ (“perturbation theory”), which is certainly not on the curve $W$. At the point $Q$ the Wilsonian Yang-Mills action would satisfy the vector Ward identity, and the interesting question is whether it is assumed in the limit of vanishing IR cutoff $k$. We have shown that it is a fixed point of the ERG flow, and that it is IR stable in a particular direction in the space of couplings; the general IR stability remains to be shown. Furthermore, perturbatively the ERG flow is from $P$ towards $Q$, which is indicated by the arrow – this is related to the decrease of the wave function renormalization of the diagonal gluons, which gives the increase of the gauge coupling in the maximal Abelian gauge due to the Abelian Ward identity [14]. Clearly further investigations of properties of Wilsonian Yang-Mills actions near the point $Q$ are highly desirable.
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**Figure Caption**

**Fig. 1**: Schematic representation of the space of couplings of effective actions
depending on abelian gauge fields $A_\mu$ and antisymmetric tensor fields $B_{\mu\nu}$. The
meaning of the curves $W$, $YM$ and the points $P$, $Q$ is explained in section 4.
Fig. 1