Uniformly bounded Lebesgue constants for scaled cardinal interpolation with Matérn kernels

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Abstract

For \( h > 0 \) and positive integers \( m, d \), such that \( m > d/2 \), we study non-stationary interpolation at the points of the scaled grid \( h\mathbb{Z}^d \) via the Matérn kernel \( \Phi_{m,d} \)—the fundamental solution of \( (1 - \Delta)^m \) in \( \mathbb{R}^d \). We prove that the Lebesgue constants of the corresponding interpolation operators are uniformly bounded as \( h \to 0 \) and deduce the convergence rate \( O(h^{2m}) \) for the scaled interpolation scheme. We also provide convergence results for approximation with Matérn and related compactly supported polyharmonic kernels.

Keywords: approximation order; cardinal interpolation; compactly supported RBF; Lebesgue constant; Matérn kernel; non-stationary ladder.

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1 Introduction

Cardinal interpolation at the points of the lattice \( \mathbb{Z}^d \) provides an ideal model for studying multivariable kernel interpolation, which extends Schoenberg’s theory of univariate cardinal spline interpolation [21]. For a kernel \( \phi : \mathbb{R}^d \to \mathbb{R} \) decaying sufficiently fast, let

\[
S(\phi) := \left\{ \sum_{j \in \mathbb{Z}^d} c_j \phi(\cdot - j) : c \in \ell_\infty \right\}.
\]

The problem of cardinal interpolation with \( \phi \) is to find, for any bounded sequence of data values \( \{y_j\}_{j \in \mathbb{Z}^d} \), a function \( s \in S(\phi) \), such that \( s(j) = y_j \), for all \( j \in \mathbb{Z}^d \). If such a function exists and it is unique, cardinal interpolation with \( \phi \) is deemed to be ‘correct’, or ‘solvable’.

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One way to analyze the approximation properties of such an interpolation method is to consider the associated stationary scheme for interpolation on the scaled lattice $h\mathbb{Z}^d$, for a scaling parameter $h > 0$. For this, consider the space of dilations

$$S^h(\phi) = \{ s(\cdot/h) : s \in S(\phi) \},$$

and, for a bounded data function $f : \mathbb{R}^d \to \mathbb{R}$, let $s_{f,h} \in S^h(\phi)$ be the unique bounded interpolant to the values of $f$ on $h\mathbb{Z}^d$. For $k \geq 0$, this interpolation method is said to achieve the $L_\infty$-approximation order $k$ if, for any sufficiently smooth $f \in L_\infty := L_\infty(\mathbb{R}^d)$, we have $\| f - s_{f,h} \|_\infty = O(h^k)$, as $h \to 0$, with the $L_\infty$-norm $\| \cdot \|_\infty$. The last condition implies, in particular, that the $L_\infty$-distance from any such $f$ to the space $S^h(\phi)$ also decays at the rate $O(h^k)$, hence, by a well-known result (de Boor and Ron [8, Theorem 3.1]), $\phi$ must satisfy the Strang-Fix conditions of order $k$, i.e. its Fourier transform $\hat{\phi}$ must have a zero of order $k$ at every $j \in 2\pi \mathbb{Z}^d \setminus 0$ (for $k = 1$, cf. Buhmann [9, Theorem 23]).

In case $\phi$ is a box-spline kernel, the approximation order of stationary cardinal interpolation can be elegantly expressed in terms of the direction matrix defining $\phi$ (see the monograph by de Boor et al. [7]). But, in general, decaying kernels (e.g. Gaussian, generalized multiquadric, or Matérn kernels) may not satisfy the Strang-Fix conditions. In such cases, one may consider a non-stationary scheme, where dilations are selected from a space $S(\phi_h)$ based on a $h$-dependent kernel $\phi_h$, usually itself a dilation of $\phi$. A natural choice is $\phi_h := \phi(h\cdot)$, in which case the collection $\{ S^h(\phi_h) \}_h$—the flat ladder generated by $\phi$—is defined as:

$$S^h(\phi_h) := \{ \sum_{j \in \mathbb{Z}^d} c_j \phi_h(h^{-1} \cdot - j) : c \in \ell_\infty \} = \{ \sum_{j \in \mathbb{Z}^d} c_j \phi(\cdot - hj) : c \in \ell_\infty \}. \quad (1.1)$$

Recently, the approximation order of flat ladder interpolation on $h\mathbb{Z}^d$ has been studied, for the Gaussian kernel, by Hangelbroek et al. [15], and, for the generalized multiquadric kernel, by Hamm and Ledford [14]. The analysis employed in these two works makes use of an intermediate band-limited interpolant and ultimately relies on the fact that the Fourier transform $\hat{\phi}$ of the Gaussian or the generalized multiquadric kernel decays exponentially at infinity.

Here, we propose a different method, based on bounding uniformly the associated Lebesgue constants, in order to obtain the rate of approximation of the flat ladder interpolation scheme with the Matérn kernel, whose Fourier transform decays only algebraically at infinity. For a positive integer $m > d/2$, the Matérn kernel $\phi = \Phi_{m,d}$ is defined, up to a constant factor, as the fundamental solution of the elliptic operator $(1 - \Delta)^m$ in $\mathbb{R}^d$, where $\Delta$ is the Laplace operator. This kernel is commonly used as a covariance function in statistical modeling (e.g. Gneiting et al. [13], Chen et al. [10]).

The basic properties of cardinal interpolation on $\mathbb{Z}^d$ with the Matérn kernel $\Phi_{m,d}$ are provided in our recent work [5, Example 5.2], which also covers the case of non-integer $m$, as well as the related model of ‘semi-cardinal’ interpolation on half-space lattices. In particular, the corresponding Lagrange function $\chi \in$
$S(\Phi_{m,d})$, satisfying $\chi(j) = \delta_{j0}$ for all $j \in \mathbb{Z}^d$, is shown to decay exponentially at infinity.

In the present paper, we establish a much stronger version of this result (Theorem 3.1), by proving that, for $\phi = \Phi_{m,d}$, the exponential decay of the Lagrange function for cardinal interpolation on $\mathbb{Z}^d$ from the space $S(\phi_h)$ holds with constants that are independent of the scale parameter $h$. As a direct consequence (Corollary 4.1), we derive a scale independent bound on the Lebesgue constant for interpolation on $h\mathbb{Z}^d$ from the space $S^h(\phi_h)$ (no similar result in the non-stationary setting seems to have been obtained previously in the literature). This, in turn, allows us to deduce the convergence rate $O(h^{2m})$ for the Matérn flat ladder interpolation scheme on $h\mathbb{Z}^d$ from the corresponding rate for approximation in $S^h(\phi_h)$ implied by the work of de Boor and Ron [8].

Section 2 contains preliminary material on cardinal interpolation with Matérn kernels, including the Fourier representation of the corresponding Lagrange functions. Section 3 proves the main result of the paper, Theorem 3.1, while the convergence results are obtained in section 4. In section 5, the rate of approximation $O(h^{2m})$ by finite linear combinations of shifted Matérn kernels is also transferred to the class of compactly supported ‘perturbation’ kernels defined by Ward and Unser [23], and two families of polyharmonic radial kernels constructed by Johnson [19] are shown to belong to this class.

2 Cardinal interpolation with Matérn kernels

Matérn kernels and their basic properties. For an integer $m$, such that $m > \frac{d}{2}$, the Matérn kernel $\Phi := \Phi_{m,d}$ is expressed as

$$\Phi(x) = \|x\|^{m-\frac{d}{2}}K_{m-\frac{d}{2}}(\|x\|), \quad x \in \mathbb{R}^d,$$

(2.1)

where $K_\nu$ denotes the modified Bessel function of the third kind of order $\nu$. It is known that $\Phi$ is continuous on $\mathbb{R}^d$ and $\Phi(x) = O(\|x\|^{m-\frac{d}{2}}e^{-\|x\|})$, as $\|x\| \to \infty$, hence, there exists $\alpha \in (0,1]$ and $C_0 := C_0(m,d) > 0$, such that

$$|\Phi(x)| \leq C_0 e^{-\alpha\|x\|}, \quad x \in \mathbb{R}^d.$$  

(2.2)

It follows from [24, Theorem 6.13] that the Fourier transform of $\Phi$ is given by

$$\hat{\Phi}(t) = \rho_{m,d}(1 + \|t\|^2)^{-m}, \quad t \in \mathbb{R}^d,$$

(2.3)

for a constant $\rho_{m,d} > 0$.

Non-stationary interpolation scheme. For each parameter $h > 0$, we let

$$\Phi_h(x) := \Phi(hx), \quad x \in \mathbb{R}^d,$$

(2.4)

and define the associated scaled shift-invariant space $S_h(\Phi_h)$ via (1.1). Since the exponential decay property (2.2) ensures that

$$\sum_{j \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |\Phi_h(y - j)| < \infty,$$

(2.5)
each element of $S^h(\Phi_h)$ is a continuous function on $\mathbb{R}^d$, being defined by a series which converges uniformly on compact sets. Also, such an element is a bounded function on $\mathbb{R}^d$. The collection $\{S^h(\Phi_h)\}_h$ is the non-stationary flat ladder generated by $\Phi$.

The main problem addressed in this paper is to interpolate a data function at the points of the scaled grid $h\mathbb{Z}^d$ from the space $S^h(\Phi_h)$, i.e. using series representations of $h\mathbb{Z}^d$-translates of $\Phi$ with bounded coefficients. Due to translation invariance, this problem amounts to the construction of a Lagrange function $\chi_h \in S^h(\Phi_h)$, which satisfies $\chi_h(hj) = \delta_{j0}$, $j \in \mathbb{Z}^d$. Note that $\chi_h$ indicates a generic dependence on $h$, while the specific notation of (2.4) applies only to $\Phi_h$.

In the sequel, it is convenient to employ the change of variables $y = h^{-1}x$, by which the above problem is equivalently formulated as cardinal interpolation at the points of the lattice $\mathbb{Z}^d$ from the corresponding shift-invariant space $S(\Phi_h) = \{ \sum_{j \in \mathbb{Z}^d} c_j \Phi_h(\cdot - j) : c \in \ell_\infty \}$.

The existence and uniqueness of a solution to the latter problem (e.g. Chui et al. [11, Lemma 1.1]) depend on the non-vanishing of the symbol function defined by the absolutely convergent Fourier series:

$$\sigma_m(t, h) := \sum_{k \in \mathbb{Z}^d} \Phi_h(k)e^{ikt}, \ t \in \mathbb{R}^d. \quad (2.6)$$

Note that, by (2.3) and usual transform laws, we have

$$\widehat{\Phi}_h(t) = h^{-d}\widehat{\Phi}(h^{-1}t) = \frac{\rho_m,dh^{2m-d}}{(h^2 + \|t\|^2)^m}, \ t \in \mathbb{R}^d. \quad (2.7)$$

Hence, since $2m > d$, an application of the Poisson Summation Formula provides

$$\sigma_m(t, h) = \sum_{k \in \mathbb{Z}^d} \widehat{\Phi}_h(t + 2\pi k) = \rho_m,dh^{2m-d} \sum_{k \in \mathbb{Z}^d} (h^2 + \|2\pi k\|^2)^{-m} > 0, \quad (2.8)$$

for all $t \in \mathbb{R}^d$ and $h > 0$, the series being uniformly convergent on compact sets.

It follows that cardinal interpolation on $\mathbb{Z}^d$ with $\Phi_h$ is ‘correct’ and the corresponding Lagrange function $\widetilde{\chi}_h$ satisfying $\widetilde{\chi}_h(j) = \delta_{j0}$, $j \in \mathbb{Z}^d$, is given by

$$\widetilde{\chi}_h(y) = \sum_{k \in \mathbb{Z}^d} a_k^h \Phi_h(y - k), \ y \in \mathbb{R}^d, \quad (2.9)$$

where, by Wiener’s lemma, $\{a_k^h\}_{k \in \mathbb{Z}^d} \in \ell_1$ is the sequence of Fourier coefficients of $1/\sigma_m(\cdot, h)$. Hence, reverting to $x = hy$, the above Lagrange function $\chi_h$ for interpolation with $\Phi$ on $h\mathbb{Z}^d$ can be identified as

$$\chi_h(x) = \chi_h(h^{-1}x) = \sum_{k \in \mathbb{Z}^d} a_k^h \Phi(x - hk), \ x \in \mathbb{R}^d.$$
The following lemma shows that the Fourier transform of $\tilde{\chi}_h$ is the function $\omega_m(\cdot, h)(h^2 + \|\cdot\|^2)^{-m}$, where, by (2.3),

$$
\omega_m(t, h) := \frac{1}{\sum_{k \in \mathbb{Z}^d} (h^2 + \|t + 2\pi k\|^2)^{-m}} = \frac{\rho_m, dh^{2m-d}}{\sigma_m(t, h)}, \quad t \in \mathbb{R}^d. \quad (2.10)
$$

**Lemma 2.1** For each $h > 0$, $\omega_m(\cdot, h)$ is a continuous positive-valued function, $2\pi$-periodic in each of its $d$ variables, with the inverse Fourier representation:

$$
\tilde{\chi}_h(y) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} a_k^h \int_{\mathbb{R}^d} e^{iyt} \frac{\omega_m(t, h)}{(h^2 + \|t\|^2)^m} dt, \quad y \in \mathbb{R}^d. \quad (2.11)
$$

**Proof.** The continuity of $\omega_m(\cdot, h)$ is a consequence of the uniform convergence of the series (2.8) on compact sets. By (2.7), we have $\hat{\Phi}_h \in L_1(\mathbb{R}^d)$. Hence, using the Fourier inversion formula and the fact that $(a_k^h)_{k \in \mathbb{Z}^d}$ is absolutely summable, (2.9) implies

$$
\tilde{\chi}_h(y) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} a_k^h \int_{\mathbb{R}^d} e^{iyt} \hat{\Phi}_h(t) dt
$$

$$
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iyt} \hat{\Phi}_h(t) \sum_{k \in \mathbb{Z}^d} a_k^h e^{-itk} dt, \quad y \in \mathbb{R}^d.
$$

Since $\sum_{k \in \mathbb{Z}^d} a_k^h e^{-itk} = [\sigma_m(t, h)]^{-1}$, the required representation follows via (2.7) and (2.10). ■

**Remark.** The inverse symbol $\omega_m$ is well-defined by the middle fraction of (2.10) even for $h = 0$. In this case, $\omega_m$ acquires a zero at $t = 0$ of the same order as the denominator of the integrand in (2.11). This integral representation, for $h = 0$, was used by Madych and Nelson [20] as definition of the Lagrange function for cardinal interpolation with the $m$-harmonic kernel

$$
\Phi_0(x) = \left\{ \begin{array}{ll}
\|x\|^{2m-d} \ln \|x\|, & \text{if } d \text{ is even},
\|x\|^{2m-d}, & \text{if } d \text{ is odd},
\end{array} \right. \quad x \in \mathbb{R}^d,
$$

for which the cardinal symbol (2.6) cannot be defined classically.

### 3 Scale independent exponential decay

For each $h > 0$, estimate (2.2) implies that the kernel $\Phi_h$ decays exponentially: $|\Phi_h(y)| \leq C_0 e^{-\alpha h \|y\|}$, $y \in \mathbb{R}^d$. Hence, by [3] Theorem 2.7, this decay is transferred to the Lagrange function $\tilde{\chi}_h$ as $|\tilde{\chi}_h(y)| \leq A_h e^{-B_h \|y\|}$, $y \in \mathbb{R}^d$, for some positive constants $A_h, B_h$ that, *a priori*, may depend on $h$.

The main result of this paper, stated next, asserts that the exponential decay of $\tilde{\chi}_h$ actually holds with constants independent of $h \in (0, 1]$.

**Theorem 3.1** There exist $A, B > 0$, depending only on $d$ and $m$, such that

$$
|\tilde{\chi}_h(y)| \leq A e^{-B \|y\|}, \quad y \in \mathbb{R}^d, \quad h \in (0, 1],
$$

where $|y| = |y_1| + \cdots + |y_d|$.  

\[5\]
For \( d = 1 \), this result was proved by Bejancu et al. \cite[Theorem 4.1]{Bejancu} in the quite different setting of multivariate ‘polyspline’ interpolation of continuous data prescribed on equally spaced parallel hyperplanes. In fact, in that context, \( h \) does not play the role of a scaling parameter, denoting instead the norm of a certain frequency variable \( \xi \).

As in \cite{Bejancu}, our proof of Theorem 3.1 uses the Fourier transform representation of \( \tilde{\chi}_h \) given in Lemma 2.1. The main technical ingredient is Lemma 3.2 below (which extends \cite[Lemma 3.1]{Bejancu}, for \( d = 1 \)), based on ideas of Madych and Nelson \cite[Lemma 1]{Madych}. To state it, we introduce the notation:

\[
q_h(z) := z_1^2 + \cdots + z_d^2 + h^2, \quad z = (z_1, \ldots, z_d) \in \mathbb{C}^d, \ h \geq 0.
\] (3.2)

Also, for a given set \( \Omega \subset \mathbb{R} \) and a positive number \( \alpha \), we let

\[
\Omega_\alpha = \{ \zeta \in \mathbb{C} : \text{Re}\zeta \in \Omega \text{ and Im}\zeta \in (-\alpha, \alpha) \}.
\]

The Cartesian product of \( d \) copies of \( \Omega_\alpha \) is denoted by \( \Omega_\alpha \subset \mathbb{C}^d \).

The next lemma extends the symbol function \( \omega_m(\cdot, h) \) defined by (2.10) as an analytic function of \( z \) on a certain tube \( \mathbb{R}_\alpha \subset \mathbb{C}^d \):

\[
\omega_m(z, h) := \left( \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|q_h(z + 2\pi k)|^m} \right)^{-1}.
\] (3.3)

**Lemma 3.2** Let \( Q := [-\pi, \pi] \). There exists \( \alpha := \alpha(d, m) > 0 \) such that, for all \( h \in [0, 1], \omega_m(z, h) \) and \( \omega_m(z, h)[q_h(z)]^{-m} \) are analytic functions of \( z \) in the common tube \( \mathbb{R}_\alpha \subset \mathbb{C}^d \).

**Proof.** For the sake of exposition, we split the proof in three parts.

1. Let \( \alpha \in (0, \sqrt{(\pi^2 - 1)/d}] \). Then we claim that, for all \( h \in [0, 1], z \in Q_\alpha^d, \) and \( k \in \mathbb{Z}^d \setminus \{0\} \), we have \( q_h(z + 2\pi k) \neq 0 \), and the series

\[
G_{m,h}(z) := \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|q_h(z + 2\pi k)|^m},
\]
is absolutely and uniformly convergent, so analytic, for \( z \in Q_\alpha^d, Q := (-\pi, \pi) \).

Indeed, for all \( k \in \mathbb{Z}^d \setminus \{0\}, h \geq 0, \) and \( z = u + iv \in Q_\alpha^d (\text{i.e. } |u_p| \leq \pi, |v_p| \leq \alpha, \text{ for all } p = 1, \ldots, d) \), we have

\[
|q_h(z + 2\pi k)| \geq \text{Re}[q_h(z + 2\pi k)] = |u + 2\pi k|^2 - |v|^2 + h^2 \geq \pi^2 - do^2 \geq 1. \tag{3.4}
\]

Also, if \( \|k\| \geq \sqrt{d} \), then \( \|u + 2\pi k\| \geq 2\pi\|k\| - \|u\| \geq \pi(2\|k\| - \sqrt{d}) > 0 \) and \( \|v\|^2 < do^2 < \pi^2 \) imply

\[
|q_h(z + 2\pi k)| \geq |u + 2\pi k|^2 - |v|^2 + h^2 > \pi^2(2\|k\| - \sqrt{d})^2 - \pi^2. \tag{3.5}
\]
Thus, using (3.4) for the terms of index \( k \in \mathbb{Z}^d \setminus \{0\} \) satisfying \( \|k\| < \sqrt{d} \), and (3.5) for the terms corresponding to \( \|k\| \geq \sqrt{d} \), we obtain the estimate

\[
|G_{m,h}(z)| \leq \sum_{0 < \|k\| < \sqrt{d}} |q_h(z + 2\pi k)|^{-m} + \sum_{\|k\| \geq \sqrt{d}} |q_h(z + 2\pi k)|^{-m} \leq N + \pi^{-m} \sum_{\|k\| \geq \sqrt{d}} [(2\|k\| - \sqrt{d})^2 - 1]^{-m} =: M, \tag{3.6}
\]

where \( N := N(d) \) is the number of indices \( k \in \mathbb{Z}^d \setminus \{0\} \) such that \( \|k\| < \sqrt{d} \), and \( M := M(d, m) < \infty \), since \( 2m > d \). This estimate implies that the above claim is true.

2. Next, we note that, by its definition (3.2), \( q_h(z) \) is continuous as a function of \((z, h) \in \mathbb{C}^d \times [0, \infty)\), while \( G_{m,h}(z) \) is continuous as a function of \((z, h) \in Q_d^\alpha \times [0, \infty)\), since the estimate (3.6) is valid for all \( h \geq 0 \). Hence, the product \([q_h(z)]^m G_{m,h}(z)\) is continuous for \((z, h) \in Q_d^\alpha \times [0, \infty)\). Further, since this product is nonnegative for \( z := t \in Q^d \) and \( h \geq 0 \), we have

\[
1 + [q_h(t)]^m G_{m,h}(t) \geq 1, \quad t \in Q^d, \quad h \geq 0.
\]

Using the fact that \( Q^d \times [0, 1] \) is compact, it follows, by reducing \( \alpha \) if necessary, that the function \( 1 + [q_h(z)]^m G_{m,h}(z) \), which is continuous in variables \((z, h)\), remains bounded away from zero in modulus on the set \( Q_d^\alpha \times [0, 1] \), hence

\[
1 + [q_h(z)]^m G_{m,h}(z) \neq 0, \quad (z, h) \in Q_d^\alpha \times [0, 1]. \tag{3.7}
\]

3. From parts 1 and 2 above, we deduce that, for \( z \in Q_d^\alpha \) and \( h \in [0, 1] \), the definition (3.3) provides \( \omega_m(z, h) \in \mathbb{C} \), if \( q_h(z) \neq 0 \). We also let \( \omega_m(z, h) := 0 \), if \( q_h(z) = 0 \). Therefore, the following identity holds:

\[
\frac{\omega_m(z, h)}{[q_h(z)]^m} = \frac{1}{1 + [q_h(z)]^m G_{m,h}(z)}, \quad z \in Q_d^\alpha, \quad h \in [0, 1], \tag{3.8}
\]

where the left-hand side is assigned the value 1, if \( q_h(z) = 0 \). This shows that \( \omega_m(z, h)[q_h(z)]^{-m} \), hence \( \omega_m(z, h) \) as well, are analytic functions of \( z \) on \( Q_d^\alpha \), for all \( h \in [0, 1] \). By its periodicity and continuity on \( Q_d^\alpha \), \( \omega_m(z, h) \) extends analytically to \( \mathbb{R}_d^\alpha \) as a function of \( z \), for all \( h \in [0, 1] \). Now, (3.4) implies that \( q_h(z) \neq 0 \) for \( z \in \mathbb{R}_d^\alpha \setminus Q_d^\alpha \), hence \( \omega_m(z, h)[q_h(z)]^{-m} \) is also analytic on \( \mathbb{R}_d^\alpha \) as a function of \( z \), for all \( h \in [0, 1] \). \( \blacksquare \)

Remark. It is possible to establish Lemma 3.2 for all \( h \geq 0 \), by replacing the above compactness argument for (3.7) with the extension to arbitrary \( d \) of the inequality [9 (14)].

Proof of Theorem 3.1. Let \( \alpha \) be the value provided by Lemma 3.2 which implies that the integrand of the FT representation (2.11) can be extended as an analytic function in the common tube \( \mathbb{R}_d^\alpha \), for all \( h \in [0, 1] \). We pick \( B \in (0, \alpha) \) and proceed to prove (3.1), with some \( A > 0 \), for all \( h \in (0, 1] \).
To this aim, for \( y \in \mathbb{R}^d \), we intend to estimate the integral (2.11), after changing its contour of integration from \( \mathbb{R}^d \) to \( \mathbb{R}^d + i\gamma \), where \( \gamma = (\gamma_1, \ldots, \gamma_d) \), \( \gamma_p = \pm B \), and \( \gamma_p \) has the same sign as \( y_p \), for \( p \in \{1, \ldots, d\} \) (this sign choice being irrelevant if \( y_p = 0 \)). We will also employ the notation

\[
\Psi_{m,h}(z) := \omega_m(z,h)[q_h(z)]^{-m}, \quad z \in \mathbb{R}^d, \; h \in [0,1].
\]

The change of contour is obtained in \( d \) steps, via successive applications of Cauchy’s Theorem. In the first step, this theorem implies

\[
\int_{\Gamma_1} e^{iy_1 z_1} \left( \int_{\mathbb{R}^{d-1}} e^{i(y_2 t_2 + \cdots + y_d t_d)} \Psi_{m,h}(z_1, t_2, \ldots, t_d) \ dt_2 \ldots dt_d \right) dz_1 = 0, \quad (3.9)
\]

where \( \Gamma_1 \subset \mathbb{C} \) is the rectangle of horizontal (long) sides \([ -R, R ]\) and \([ -R, R ] + i\gamma_1 \), and corresponding vertical sides \( \pm R + i\gamma[0,1] \). The integral inside brackets is analytic in \( z_1 \), since \( \Psi_{m,h}(z_1, t_2, \ldots, t_d) \) has this property for each \((t_2, \ldots, t_d)\).

Next, note that the outside integral along the two vertical sides tends to zero as \( R \to \infty \), due to the boundedness of \( \omega_m \) and the sufficient power growth in the denominator of \( \Psi_{m,h} \). Thus, (2.11) and (3.9) imply

\[
\tilde{\chi_h}(y) = \frac{-1}{(2\pi)^d} \int_{\mathbb{R}^{d} + i\gamma} e^{iy_1 z_1} \int_{\mathbb{R}^{d-1}} e^{i(y_2 t_2 + \cdots + y_d t_d)} \Psi_{m,h}(z_1, t_2, \ldots, t_d) \ dt_2 \ldots dt_2 \ dz_1.
\]

Repeating this argument for each of the remaining \( d - 1 \) variables, we obtain, via Fubini’s Theorem,

\[
\tilde{\chi_h}(y) = \frac{(-1)^d}{(2\pi)^d} \int_{\mathbb{R}^{d} + i\gamma} e^{i(y_1 z_1 + \cdots + y_d z_d)} \Psi_{m,h}(z_1, \ldots, z_d) \ dz.
\]

On this integration contour, we use \( z_p = t_p + i\gamma_p \) for \( p = 1, \ldots, d \), hence

\[
\sum_{p=1}^{d} y_p z_p = y t + i \sum_{p=1}^{d} y_p \gamma_p = y t + i B \sum_{p=1}^{d} |y_p|,
\]

which implies the estimate:

\[
|\tilde{\chi_h}(y)| \leq \frac{e^{-\|y\|}}{(2\pi)^d} \int_{\mathbb{R}^{d}} |\Psi_{m,h}(t + i\gamma)| \ dt.
\]

Therefore, to obtain (3.10), it is sufficient to prove the existence of a constant \( C > 0 \), independent of \( h \), such that

\[
\int_{\mathbb{R}^{d}} \frac{|\omega_m(t + i\gamma, h)|}{|q_h(t + i\gamma)|^m} \ dt \leq C, \quad h \in [0,1]. \quad (3.10)
\]

To achieve this, we estimate the above integral by splitting it over two regions: \( \|t\| \leq \alpha \sqrt{d} \) and \( \|t\| > \alpha \sqrt{d} \).

In the first region, the restriction on \( \alpha \) from the first line of the proof of Lemma 3.2 implies \( |t_p| \leq \|t\| \leq \alpha \sqrt{d} < \pi \) for all \( p = 1, \ldots, d \), hence \( t + i\gamma \in Q_\alpha^d \).
Since the identity (3.8) shows that \( \omega_m(t + i\gamma, h)[q_h(t + i\gamma)]^{-m} \) is a continuous (hence, bounded) function of \((t, h)\) on \(Q^d \times [0, 1]\), it follows that there exists \( C_1 := C_1(d, m, \alpha) > 0 \), such that
\[
\int_{\|t\| \leq \alpha \sqrt{d}} \frac{|\omega_m(t + i\gamma, h)|}{|q_h(t + i\gamma)|^m} dt \leq C_1, \quad h \in [0, 1]. \tag{3.11}
\]

For \( \|t\| > \alpha \sqrt{d} \), we use the fact that (3.8) also implies the continuity of \( \omega_m(z, h) \) as a function of \((z, h)\) on \(Q^d \alpha \times [0, 1]\). By the 2\(\pi\)-periodicity of \(\omega_m\) in each component of \(z\), we deduce the existence of a constant \( C_2 := C_2(d, m, B) > 0 \), such that
\[
|\omega_m(t + i\gamma, h)| \leq C_2, \quad t \in \mathbb{R}^d, \quad h \in [0, 1].
\]
Using this bound, coupled with the estimate, for \( \|t\| > \alpha \sqrt{d} > B \sqrt{d} \):
\[
\int_{\|t\| > \alpha \sqrt{d}} \frac{|\omega_m(t + i\gamma, h)|}{|q_h(t + i\gamma)|^m} dt \leq \int_{\|t\| > \alpha \sqrt{d}} \frac{C_2 dt}{(\|t\|^2 - dB^2)^m} \leq C_3, \quad h \in [0, 1],
\tag{3.12}
\]
for some constant \( C_3 := C_3(d, m, \alpha, B) < \infty \).

Thus, (3.11) and (3.12) imply (3.10), which completes the proof. \(\blacksquare\)

4 Convergence rates

This section uses the main result to obtain a uniform bound on the Lebesgue constant for non-stationary cardinal interpolation with the Matérn kernel, which eventually enables the transfer of the approximation order of the flat ladder \(\{S^h(\Phi_h)\}_h\) over to the scaled cardinal interpolation scheme.

For each \( h \in (0, 1] \), let \( I_h \) denote the interpolation operator taking any bounded function \( f \) on \( \mathbb{R}^d \) to its interpolant \( I_h f \) generated with the kernel \( \Phi \) on the scaled grid \( h\mathbb{Z}^d \), i.e.
\[
I_h f(x) := \sum_{j \in \mathbb{Z}^d} f(hj)\chi_h(x - hj), \quad x \in \mathbb{R}^d.
\tag{4.1}
\]
Since the change of variables \( y = h^{-1}x \) equivalently expresses this operator as interpolation to any function \( f(h\cdot) \) on the standard cardinal grid \( \mathbb{Z}^d \) by means of the kernel \( \Phi_h \), it follows that all basic properties of cardinal interpolation on \( \mathbb{Z}^d \) (e.g. [5, §2.1]) also apply to \( I_h \). In particular, the norm (or ‘Lebesgue constant’) of \( I_h \) as a linear bounded operator on \( L^\infty(\mathbb{R}^d) \) can be expressed as:
\[
\|I_h\|_\infty = \sup_{x \in \mathbb{R}^d} \sum_{j \in \mathbb{Z}^d} |\chi_h(x - hj)|.
\tag{4.2}
\]
Thus, the scale-independent decay of \( \chi_h = \chi_h(h\cdot) \) established in the previous section leads to the following immediate consequence for Lebesgue constants.
Corollary 4.1 The Lebesgue constant $\| I_h \|_\infty$ for non-stationary interpolation with the Matérn kernel $\Phi$ on the scaled grid $h \mathbb{Z}^d$ admits a uniform bound for all $h \in (0, 1]$, i.e. there exists $C := C(d, m, \alpha, B) > 0$, such that

$$\| I_h \|_\infty \leq C, \quad h \in (0, 1]. \quad (4.3)$$

Proof. Since $\chi_h(x) = \tilde{\chi}_h(h^{-1}x)$, from (4.2) and Theorem 3.1 we obtain:

$$\| I_h \|_\infty \leq \sup_{x \in \mathbb{R}^d} \sum_{j \in \mathbb{Z}^d} A e^{-B \| h^{-1}x - j \|} = A \sup_{y \in \mathbb{R}^d} \sum_{j \in \mathbb{Z}^d} e^{-B \| y - j \|} =: C < \infty,$$

as required. ■

The next theorem specializes to the Matérn kernel $\Phi$ a result of de Boor and Ron [8] on the approximation order of the non-stationary ladder $\{ S_h(\Phi_h) \}_h$. We employ the notation

$$\text{dist}(f, A; X) := \inf_{s \in A} \| f - s \|_X,$$

for the distance from a function $f$ to a set of functions $A$, measured in the norm of a space $X$. Also, we denote by $L^{2m,1}$ the Bessel potential space consisting of all functions $f \in L_1(\mathbb{R}^d)$, such that $(1 + \| \cdot \|^2)^m \hat{f} \in L_1(\mathbb{R}^d)$. Note that the Schwartz class of rapidly decaying smooth functions on $\mathbb{R}^d$ is a subspace of $L^{2m,1}$. Further, we let

$$S^h_0(\Phi_h) := \text{span}\{ \Phi_h(h^{-1} \cdot - j) : j \in \mathbb{Z}^d \}, \quad (4.4)$$

the space of finite linear combinations of the translates $\{ \Phi(h \cdot - j) : j \in \mathbb{Z}^d \}$.

Theorem 4.2 If $f \in L^{2m,1}$, there exists $C_f := C(f, d, m) > 0$, such that

$$\text{dist}(f, S^h_0(\Phi_h); L_\infty) = \text{dist}(f, S^h(\Phi_h); L_\infty) \leq C_f h^{2m}, \quad h \in (0, 1]. \quad (4.5)$$

Proof. The right-side inequality expresses the fact that $\{ S^h(\Phi_h) \}_h$ provides $L_\infty$-approximation of order $2m$. This result follows from [8] Theorem 2.37], which holds under three general assumptions. Firstly, the family $\{ \Phi_h \}_h$ should satisfy the so-called synthesis condition of order $2m$. In the case of the Matérn kernel, this condition is equivalent to the existence of $\delta > 0$, such that

$$\sup_{\| t \| \leq \delta} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \left| \frac{\hat{\Phi}_h(ht + 2\pi j)}{\hat{\Phi}_h(ht)} \right| = O(h^{2m}). \quad (4.6)$$

Since $2m > d$, (4.6) is satisfied, for a sufficiently small $\delta$, due to the estimate:

$$\left| \frac{\hat{\Phi}_h(ht + 2\pi j)}{\hat{\Phi}_h(ht)} \right| \leq \frac{h^{2m}(1 + \| t \|^2)^m}{(h^2 + \| ht + 2\pi j \|^2)^m} \leq \frac{h^{2m}(1 + \delta^2)^m}{\| ht + 2\pi j \|^2m} \leq \frac{h^{2m}(1 + \delta^2)^m}{(2\pi \| j \| - \delta)^{2m}}.$$
A second assumption that needs to be verified is the boundedness of the semi-discrete convolution operator generated by $\Phi_h$, for each $h$. Using [8, Proposition 2.3], this is ensured by condition (2.5), due to the exponential decay of $\Phi_h$.

Thirdly, the function $f$ needs to be $2m$-admissible, i.e. $(1 + \| \cdot \|_2^m)\hat{f} \in L_1(\mathbb{R}^d)$, which is easily verified for any $f \in L_2m.1$.

As for the left-side equality of (4.5), which transfers the rate of convergence to approximation with finite, rather than infinite, linear combinations of the translates $\{\Phi(\cdot - hj) : j \in \mathbb{Z}^d\}$, this follows from a result of Johnson [16, Proposition 2.2], due to condition (2.5) satisfied by $\Phi_h$, and to the fact that any function $f \in L_2m.1$ is necessarily bounded and has the limit 0 at infinity.

Remarks. (i) The convergence rate $2m$ for approximation from a non-stationary ladder generated by the Matérn kernel $\Phi = \Phi_{m,d}$ defined in (2.1) also follows from the more general result of Johnson [18, Theorem 3.7]. That result replaces the $\mathbb{Z}^d$-shifts by a set of translations $\Xi$ which is a sufficiently small perturbation of $\mathbb{Z}^d$, it considers errors in $L_p$-norms, for all $1 \leq p \leq \infty$, and it applies to sufficiently smooth functions $f$ from a Besov space.

(ii) Note that, for approximation of functions in Bessel potential spaces by finite linear combinations of quasi-uniform translates of $\Phi_{m,d}$, Ward [22, §3.1] and Ward and Unser [23, §3.2] obtain the convergence rate $O(h^{2m})$, if $d$ is odd, and the slower rate $O(h^{2m-1})$, if $d$ is even. Hence, Theorem 4.2 improves this rate for even $d$ and a uniform grid. In [22] and [23], $L_p$-error bounds, as well as kernels of non-integer order, are also considered.

The uniform bound on the Lebesgue constant given in Corollary 4.1 can now be used to transfer the convergence order of Theorem 4.2 to non-stationary interpolation on the grid $h\mathbb{Z}^d$ with the Matérn kernel $\Phi$.

**Corollary 4.3** If $f \in L_2m.1$, there exists a constant $C_f := C(f, d, m, \alpha, B)$, such that the interpolant (4.1) satisfies

$$\|f - I_hf\|_\infty \leq C_f h^{2m}, \quad h \in (0, 1].$$

**Proof.** We employ a classical estimate based on the Lebesgue constant, which relates the error at interpolating $f$ by $I_hf$ on $h\mathbb{Z}^d$ to the error for approximating $f$ by any $s_h \in S^h(\Phi_h)$:

$$\|f - I_hf\|_\infty \leq \|f - s_h\|_\infty + \|s_h - I_hf\|_\infty \leq (1 + \|I_h\|_\infty)\|f - s_h\|_\infty,$$

where we have used the projection property $I_h s_h = s_h$ (see [5, Corollary 2.2]).

Now (1.5) and (4.6) imply (4.7).

## 5 Compactly supported perturbation kernels

In this section, we transfer the approximation result of Theorem 4.2 to the compactly supported radial kernels introduced and studied independently by Johnson [19] and Ward and Unser [23], and we discuss examples of such kernels.
We start by recalling (e.g. [24, Theorem 5.26]) that, if \( \Psi = \psi(\|\cdot\|) \) is an integrable radial function on \( \mathbb{R}^d \), with profile \( \psi \) defined on \([0, \infty)\), then its Fourier transform \( \hat{\Psi} \) is also radial, namely \( \hat{\Psi} = (F_d \psi)(\|\cdot\|) \), with profile given by

\[
(F_d \psi)(r) = r^{1-d/2} \int_0^\infty \psi(t) t^{d/2} J_{d-1}(rt) \, dt, \quad r > 0,
\]

where \( J_\nu \) is the Bessel function of the first kind of order \( \nu \).

For \( m > d/2 \), Ward and Unser [23] defined the class of continuous profiles \( \psi : [0, \infty) \to \mathbb{R} \), such that the \( d \)-dimensional Fourier transform of the radial function \( \Psi := \psi(\|\cdot\|) \) has a profile of the form

\[
(F_d \psi)(r) = Cr^{-2m}\lambda(r), \quad r > 0,
\]

where \( C > 0 \) and \( \lambda : [0, \infty) \to \mathbb{R} \) satisfies three conditions:

1. There exist a positive integer \( K \), a set of nodes \( 0 < r_1 < r_2 < \cdots < r_K \), and a sequence of real coefficients \( \{a_j\}_{j=1}^K \), such that:

\[
\lambda(r) = 1 + \sum_{j=1}^K a_j (r_j r)^{1-d/2} J_{d-1}(r_j r), \quad r \geq 0.
\]

2. \( \lambda(r) > 0 \) for \( r > 0 \).

3. \( \lambda(r) \geq \beta > 0 \).

As detailed in the last part of this section, examples of profiles \( \psi \) satisfying the above conditions have first been constructed by Johnson [19].

Building on the Paley-Wiener approach of Baxter [2, 3] for kernel engineering, Ward and Unser [23, Proposition 2.2] proved that conditions 1 and 2 above imply that \( \psi \) must be compactly supported. Further, in [23, Lemma 2.3], they showed that a radial function \( \Psi \) whose Fourier transform profile has the form (5.1) is a ‘perturbation’ of \( \Phi \), the \( d \)-dimensional Matérn kernel (2.1) of corresponding parameter \( m \), in the sense that it can be expressed as a convolution

\[
\Psi = \mu \ast \Phi,
\]

for an invertible finite Borel measure \( \mu \). Using this fact, [23, Theorem 3.11] proved that such a kernel \( \Psi \) satisfies similar approximation properties as \( \Phi \). Consequently, the convergence rate of Theorem 3.12 can also be transferred to approximation by finite linear combinations of translates of \( \Psi_h \).

**Corollary 5.1** Let \( \Psi \) be a compactly supported kernel, with a Fourier transform profile of the form (5.1). If \( f \in L^{2m,1} \), there exists \( \hat{C}_f := \hat{C}(f, d, m) > 0 \), such that

\[
\text{dist}(f, S_0^h(\Psi_h))_{L_\infty} \leq \hat{C}_f h^{2m}, \quad h \in (0, 1],
\]

where \( S_0^h(\Psi_h) \) is defined analogously to (4.14).
Proof. Let \( \nu := \mu^{-1} \), where \( \mu \) is the invertible finite Borel measure from (5.3). The proof of (5.4) follows from the estimates given in the proof of [23, Theorem 3.11], which relate the error of approximating \( f \) from \( S_h^0(\Psi_h) \) to the error of approximating \( \nu * f \) from \( S_h^0(\Phi_h) \). Thus, one only needs to ensure that \( f \in L^{2m,1} \) implies \( \nu * f \in L^{2m,1} \), which is seen to hold, due to the boundedness of the Fourier transform of \( \nu \). \qed

Remark. For \( d \geq 3 \), the approximation order \( 2m \) from the non-stationary ladder generated by the perturbed shifts of the kernel \( \Psi \) can also be obtained directly from the general result of Johnson [18, Theorem 3.7] described after Theorem 4.2. Indeed, for this kernel, the hypotheses of Johnson’s theorem are seen to be verified due to the form of the Fourier profile (5.1) and the asymptotic properties of the Bessel functions which appear in (5.2).

In the remaining part of this section, we show that two of the families of profiles constructed by Johnson [19], for \( d = 2 \) and \( d = 3 \), belong to the above class defined by Ward and Unser. Note that, in place of the Paley-Wiener approach, these constructions use an L-spline approach based on the finite dimensional representations of piecewise polyharmonic radial functions developed in [17].

Johnson’s compactly supported profiles \( \{ \eta_m \} \) for \( d = 2 \). This family of profiles is described in [19, §3]. For each integer \( m \geq 1 \), \( \eta_m \) is a piecewise defined function with nodes at 0, 1, 2, \ldots, \( m \), such that \( \eta_m = 0 \) on \( (m, \infty) \), \( \eta_m \in C^{2m-2}(0, \infty) \), and each non-trivial piece of \( \eta_m \) belongs to the \( 2m \)-dimensional null-space of \( L^m \), where \( L = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \) is the radial Laplacean in \( \mathbb{R}^2 \). Up to a suitable normalization, \( \eta_m \) is uniquely determined by imposing certain \( m - 1 \) boundary conditions on its restriction to the first interval \( (0, 1) \). Therefore \( \eta_m(\| \cdot \|) \) is a compactly supported, piecewise \( m \)-harmonic, radially symmetric function on \( \mathbb{R}^2 \). It follows from [19 (3.4)] that (5.2) holds with \( K = m \) for \( \lambda := (\cdot)^{2m}F_2\eta_m \), since

\[
(F_2\eta_m)(r) = \frac{4^{m-1}(m-1)!^2}{\mu^{2m}} \left( 1 + \sum_{j=1}^{m} a_j J_0(jr) \right), \quad r > 0, \tag{5.5}
\]

where \( \{ a_j \}_{j=1}^{m} \) are uniquely determined by the fact that \( \lim_{r \to 0^+} (F_2\eta_m)(r) =: \beta \in \mathbb{R} \) exists (as \( \eta_m \) has compact support). Next, using an integral representation of \( J_0 \), Johnson [19 (3.5)] proves the remarkable property that \( (F_2\eta_m)(r) > 0 \) for \( r > 0 \), i.e. condition 3 holds for \( \lambda \). Since \( K = m \) and \( F_2\eta_m \) is an entire function, it also follows that \( \beta > 0 \) is satisfied automatically, hence \( \lambda \) satisfies all three conditions listed after (5.1).

For \( m = 2 \), the two nontrivial pieces of \( \eta_2 \) are displayed explicitly in [19] as:

\[
\eta_2(t) = \begin{cases} 
4 \ln 2 + (\ln 2 - 3)t^2 + 3t^2 \ln t, & t \in (0, 1], \\
(4 \ln 2 - 4) - 4 \ln t + (\ln 2 + 1)t^2 - t^2 \ln t, & t \in (1, 2].
\end{cases} \tag{5.6}
\]
The scaled profile \( \eta_2(t) \) supported on \([0,1]\), was also obtained by Ward and Unser in [23, Example 2.4]. Further, \( \eta_2(\|\|) \) was identified in [4] as a radially symmetric thin plate spline.

**Johnson’s compactly supported profiles \( \{\psi_{3,m}\} \) for \( d = 3 \).** For each integer \( m \geq 1 \), let \( \psi_m \) be the restriction to \([0,\infty)\) of the polynomial B-spline which has a double knot at 0 and simple knots at \( \pm 1, \ldots, \pm m \). This family of B-splines was studied by Al-Rashdan and Johnson [1], who proved:

\[
(F_1 \psi_m)(r) = \frac{\delta_m}{r^{2m}} \left( 1 + \sum_{j=1}^{m} b_j \frac{\sin(jr)}{r} \right) > 0, \quad r > 0,
\]

where \( \delta_m > 0 \) and the coefficients \( \{b_j\}_{j=1}^{m} \) are uniquely determined by the fact that \( \lim_{r \to 0^+} (F_1 \psi_m)(r) =: \beta \in \mathbb{R} \) exists.

As part of a larger class of profiles, the family \( \{\psi_{3,m}\} \) was defined by Johnson [19, §8] via

\[
\psi_{3,m} := D \psi_m, \quad m \geq 1,
\]

where \( D = -\frac{d}{dr} \). Hence, the ‘dimension walk’ formula \( F_3(D \psi_m) = F_1 \psi_m \), the identity \( r^{-1/2}J_{1/2}(r) = \sqrt{\frac{\pi}{2}} \frac{\sin r}{r} \), and the last two displays imply

\[
(F_3 \psi_{3,m})(r) = \frac{\delta_m}{r^{2m}} \left( 1 + \sqrt{\frac{\pi}{2}} \sum_{j=1}^{m} b_j (jr)^{-1/2} J_{1/2}(jr) \right) > 0, \quad r > 0.
\]

It follows that, for \( d = 3 \), the profile \( F_3 \psi_{3,m} \) is of the form (5.1), with \( K = m \). Further, by [19, Theorem 6.1], \( \psi_{3,m}(\|\|) \) is a compactly supported, piecewise \( m \)-harmonic, radially symmetric function on \( \mathbb{R}^3 \).

In the special case \( m = 2 \), [1] gives the explicit expression

\[
\psi_{2}(t) = \begin{cases} 
8 - 24t^2 + 24t^3 - 7t^4, & t \in [0,1], \\
(2 - t)^4, & t \in (1,2],
\end{cases}
\]

from which the expression of \( \psi_{3,2} \) can be calculated via (5.8). The scaled version \( \frac{1}{\sqrt{3}} \psi_{3,2}(2r) \) is also provided as Example 2.5 by Ward and Unser [23], while their Example 2.6 is seen to coincide with \( \frac{1}{\sqrt{5}} \psi_{3,3}(3r) \).

**Remarks.** (i) It follows from the proofs of [1, Theorem 2.8] and [19, Theorem 3.11], that the three conditions imposed on the function \( \lambda \) of (5.1) imply that the corresponding profile \( \psi \) has ‘Sobolev regularity’ \( (d,m) \), i.e.

\[
A(1 + r^2)^{-m} \leq (F_d \psi)(r) \leq B(1 + r^2)^{-m}, \quad r > 0,
\]

for some constants \( B \geq A > 0 \). Hence, a natural application of such profiles \( \psi \) may occur in the field of multilevel interpolation algorithms; cf. Farrell et al. [12]. Another potential application, to tomographic image reconstruction via
X-ray transform, is discussed by Ward and Unser \cite{23}, where the plots of the profiles $\eta_2$ and $\psi_{3,2}$, rescaled on the support interval $[0, 1]$, are also presented.

(ii) Since a function $\lambda$ of the form \((5.2)\) is even and entire, the condition that $\lim_{r \to 0^+} (F_\lambda\psi(r))$ exists is seen to imply $K \geq m$. For $K > m$, currently there do not seem to be any constructions of such functions $\lambda$ satisfying the three conditions described at the beginning of the section. For $K = m$, as shown by Johnson \cite{19} \((3.2)\), the coefficients $\{a_j\}$ of \((5.2)\) are uniquely determined by the requirement that $\lim_{r \to 0^+} (F_\lambda\psi(r))$ exists. Hence, it is remarkable that the above two families of profiles $\{\eta_m\}$ and $\{\psi_{3,m}\}$, for $d = 2$ and $d = 3$, also satisfy the extra condition that $\lambda(r) > 0$ for $r > 0$. As already noted above, this automatically ensures $\lim_{r \to 0^+} (F_\lambda\psi(r)) > 0$.

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