Uniqueness in Law for a Class of Degenerate Diffusions with Continuous Covariance

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Abstract

We study the martingale problem associated with the operator

$$Lu = \partial_s u(s, x) + \frac{1}{2} \sum_{i,j=1}^{d_0} a^{ij} \partial_{ij} u(s, x) + \sum_{i=1}^{d} B^i x^j \partial_i u(s, x),$$

where $d_0 \leq d$. We show that the martingale problem is well-posed when the function $a$ is continuous and strictly positive-definite on $\mathbb{R}^{d_0}$ and the matrix $B$ takes a particular lower-diagonal, block form. We then localize this result to show that the martingale problem remains well-posed when $B$ is replaced by a sufficiently smooth vector field whose Jacobian matrix satisfies a nondegeneracy condition.

Keywords: Martingale problem, Stochastic differential equations, Degenerate parabolic operators, Homogeneous groups

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1 Introduction

In this paper we consider stochastic differential equations (SDEs) of the form

$$dX_t = b(t, X_t) \, dt + \sigma(t, X_t) \, dW_t,$$

where the process $X$ takes values in $\mathbb{R}^d$ and $W$ is a Brownian motion of dimension $d_0 \leq d$. We provide conditions which are sufficient to ensure that the solution to this SDE is unique in law when the covariance function $a = \sigma \sigma^T$ is degenerate and continuous.

When $d_0 = d$, the drift is bounded, and the covariance function is bounded and uniformly positive definite on compact sets, then a number of sufficient conditions are available which ensure uniqueness in law for the SDE (1.1). Stroock

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and Varadhan [24] have shown that weak uniqueness holds when the covariance is continuous in the spacial variables, and this continuity is uniform on compact time sets. More recently, Bramanti and Cerutti [2] have provided an estimate which implies that the solution is unique when the covariance function is VMO continuous in space and time, and Krylov [14] has relaxed this condition to VMO-continuity in the spacial variables only.

If we retain the assumption that the covariance is uniformly positive definite on compact sets and we further assume that the covariance is a function of the spacial variables only, then more results are available. Krylov [10] has shown that uniqueness holds for all measurable covariance functions when \( d \leq 2 \). Bass and Pardoux [1] show that uniqueness holds when \( \mathbb{R}^d \) can divided into a finite number of polyhedrons such that the covariance function is constant on each polyhedron. Cerutti et al. [5] show that uniqueness holds when the covariance is continuous outside of a countable set that has a single cluster point. Gao [7] shows that uniqueness holds when the covariance function is continuous on the sets \( \{ x \in \mathbb{R}^d : x_1 > 0 \} \) and \( \{ x \in \mathbb{R}^d : x_1 \leq 0 \} \). Safonov [19] shows that weak uniqueness holds when the set of discontinuities of the covariance function has zero \( \alpha \)-Hausdorff measure for sufficiently small \( \alpha \). Krylov [13] gives a number of results which may be combined to produce weak uniqueness results for a variety of settings. Finally, Nadirashvili [18] provides a counterexample which shows that uniqueness may not hold if the covariance function is only assumed to be measurable and \( d \geq 3 \).

In the case of multidimensional diffusions with degenerate covariance, fewer results are available. It is a classical result that pathwise uniqueness holds if the coefficients \( \sigma \) and \( b \) are Lipschitz continuous. Figalli [6] has shown that uniqueness holds for the associated Stochastic Lagrangian Flow when the covariance is a bounded, deterministic function of time and the drift is a BV vector field whose divergence is controlled. Le Bris and Lions [17] provide estimates for the forward equation associated with the SDE (1.1), and they sketch how these results may be used to produce weak uniqueness results for SDEs whose coefficients possess sufficient Sobolev regularity.

In contrast to the results just mentioned, we consider a setting where the null space of the covariance may be nontrivial everywhere and the covariance is only assumed to be a continuous function time and space. We are able to obtain weak uniqueness results in this setting by imposing conditions on the drift which ensure that the process is, in some sense, locally hypoelliptic.

We will delay the precise statement of our results to Section 5 and instead give two examples that illustrate the kinds of SDEs that can be handled. To present the first example, suppose that \( d = nd_0 \) for some \( n \geq 2 \) and let \( X_t = \)}
(\(X_t^1, \ldots, X_t^d\)). We then define the SDE:
\[
\begin{align*}
X_t^i &= \hat{b}^i(t, X_t) \, dt + \sum_{j=1}^{d_0} \hat{\sigma}^{ij}(t, X_t) \, dW_t^j, \quad 1 \leq i \leq d_0, \\
X_t^i &= X_t^{i-d_0} \, dt, \quad d_0 < i \leq d,
\end{align*}
\]
where \(W\) is a \(d_0\)-dimensional Brownian motion and the functions \(\hat{b}\) and \(\hat{\sigma}\) may depend upon all of the components of the process \(X\). Notice that if we rewrite the equation (1.2) in the form (1.1), then \(\sigma \sigma^T\) is of rank \(d_0 < d\) everywhere.

Theorem 5.10 asserts that existence and uniqueness in law hold for the SDE (1.2) when \(\hat{b}\) and \(\hat{\sigma}\) satisfy a linear growth condition and \(\hat{\sigma} \hat{\sigma}^T\) is continuous and strictly positive definite on \(\mathbb{R}^{d_0}\).

We can also handle a situation where the drift of the finite variation components of \(X\) is given by a sufficiently smooth function that satisfies a local nondegeneracy condition. To state this example, fix \(d_0 \geq d/2\), and write \(x \in \mathbb{R}^d\) and \(X\) in the form \(x = (x', x'')\) and \(X = (X', X'')\), where the first coordinate denotes the first \(d_0\) components, and the second component denotes the remaining \(d - d_0\) components. Now consider the following SDE written in vector form:
\[
\begin{align*}
X_t' &= b'(t, X_t) \, dt + \tilde{\sigma}(t, X_t) \, dW_t, \\
X_t'' &= b''(t, X_t) \, dt,
\end{align*}
\]
where \(W\) is a \(d_0\)-dimensional Brownian motion, \(b'\) takes values in \(\mathbb{R}^{d_0}\), \(\tilde{\sigma}\) takes values in the space of \(d_0 \times d_0\)-matrices, and \(b''\) takes values in \(\mathbb{R}^{d-d_0}\). We now assume that all of the coefficients satisfy a linear growth condition, \(\tilde{\sigma}\) is continuous, and \(b'' \in C^2\). We also need to impose nondegeneracy conditions on both \(\tilde{\sigma}\) and \(b''\). We assume that \(\tilde{\sigma} \tilde{\sigma}^T\) is strictly positive definite, and we assume that the Jacobian matrix of \(b''\) with respect to the variables \(x'\) is of rank \(d - d_0\) at each point. Under these conditions, it follows from Theorem 5.14 that existence and uniqueness in law hold for the SDE (1.3).

To obtain these results, we follow the approach developed by Stroock and Varadhan [22, 23, 24]. We first produce a Calderón-Zygmund-type estimate for the solutions of Kolmogorov’s backward equation
\[
\partial_t u + L^{a,B} u = f,
\]
where
\[
L^{a,B} u(s, x) = \frac{1}{2} \sum_{i, j=1}^{d_0} a^{ij}(s, x) \partial_{ij} u(s, x) + \sum_{i, j=1}^{d} B^{ij} x^i \partial_j u(s, x),
\]
with \(d_0 \leq d\) and \(B\) is a fixed matrix that satisfies a structural condition given in Section [2]. We then make a perturbation argument to produce a local uniqueness result, followed by a localization argument to produce a global result.

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Before we close the introduction, we should mention that equation (1.4) has been studied rather extensively, and we will not attempt to give a comprehensive account of the literature. Instead, we refer the reader to the survey article [15] and we mention only two references. Lanconelli and Polidoro [16] identify a homogeneous group with respect to which the operator \( \partial_s + L^{a,B} \) is left-translation invariant when \( a \) is constant. We make extensive use of this group structure in everything that follows. Bramanti, Cerutti, and Manfredini [4] give estimates for solutions of (1.4) when the \( a \) is a VMO-continuous function of space and time. These results are obtained by combining estimates from the constant coefficient case with deep results about the commutators of singular integrals on homogeneous spaces from [3]. While these result are in many ways more sophisticated than the approach that we take in Section 4, they do not imply the estimates that we obtain. In particular, we study the case where \( a \) is a measurable function of time only, and coefficients in this class need not be VMO-continuous. Moreover, there are some technical challenges which must be overcome before the estimates obtained in [4] may be used to obtain weak uniqueness results for SDEs with discontinuous and degenerate covariance. We refer the reader to Remark 5.5 for a more detailed discussion of the issue that arises.

The outline of the paper is as follows. In Section 2 we introduce notation. In Section 3 we study the transition function which will play the role of a fundamental solution for the equation (1.4). In Section 4 we derive the \( L^p \)-estimate upon which our local uniqueness result depends, and in Section 5 we provide the announced uniqueness results.

## 2 Notation and Geometric Structure

We let \( | \cdot | \) denote the Euclidean norm with associated inner product \( \langle \cdot, \cdot \rangle \). We use superscripts to access the components of a vector and we start numbering our components at zero when the first coordinate corresponds to time. We let \( M^{d_0 \times d_1} \) denote the set of \( d_0 \times d_1 \) matrices and \( \| \cdot \| \) denotes the operator norm on matrices which is compatible with the Euclidean norm. We abbreviate \( M^{d \times d} \) to \( \mathbb{M}^d \) and let \( I^d \subset \mathbb{M}^d \) denote the identity matrix. We let \( B^d_r(x) \subset \mathbb{R}^d \) denote the open ball of radius \( r \) centered at \( x \) and \( \overline{B}^d_r(x) \) denotes the closed ball. We let \( S^d_+ \subset \mathbb{M}^d \) denote the symmetric, nonnegative-definite matrices, we write \( A \geq B \) if \( A - B \in S^d_+ \), and we let \( S^d_{\mu} \subset S^d_+ \) denote the matrices whose eigenvalues are contained in the interval \([1/\mu, \mu]\) when \( \mu \geq 1 \).

We let \( C_K(\mathbb{R}^d) \) denote the continuous functions with compact support, we set \( \mathbb{R}_+ = [0, \infty) \), and we let \( C^n(\mathbb{R}_+ \times \mathbb{R}^d) \) denotes the class of functions that possess \( n \) continuous derivatives in \((0, \infty) \times \mathbb{R}^d\), each of which admits a continuous extension to \( \mathbb{R}_+ \times \mathbb{R}^d \). If \( \alpha \) is a multiindex, then \( D^\alpha f \) denote the partial derivative corresponding to \( \alpha \). If no multiindex is given, then \( Df \) denote the gradient of a scalar function and the Jacobian matrix of a vector-valued
function, and \( D^2 f \) denotes the Hessian of a scalar function. If the components of \( \mathbb{R}^d \) have been partitioned as \( x = (y, z) \), then \( D_y \) denotes the gradient or Jacobian matrix restricted to the components in \( y \). We will also use \( \frac{\partial^2 f}{\partial x_i \partial x_j} \), \( \partial_s \), and \( \partial_{ij} \) to denote partial derivatives, but we will never use subscripts.

We let \( X \) denote the canonical process on \( C(\mathbb{R}^+; \mathbb{R}^d) \), and we equip this space with the locally uniform topology. We let \( \mathcal{C}^d \) denote the Borel \( \sigma \)-field on \( C(\mathbb{R}^+; \mathbb{R}^d) \), we set \( \mathcal{C}^d_t = \sigma(X_s : s \leq t) \), and we set \( \mathcal{C}^d = (\mathcal{C}^d_t)_{t \geq 0} \). The filtration \( \mathcal{C}^d \) does not satisfy the usual conditions of right-continuity and completeness, but this will not cause any problems in what follows. If \( Y \) is an \( \mathbb{R}^d \)-valued process, \( f : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^r \) is measurable, and \( B \in \mathcal{M}^{r \times d} \), then \( f(Y) \) denotes the process \( t \mapsto f(t, Y_t) \) and \( BY \) denotes the process \( t \mapsto BY_t \). The following definition is convenient when dealing with processes whose covariance is degenerate.

**Definition 2.1.** Let \( d_0 \leq d \), let \((\Omega, \mathcal{F}, \mathbb{P} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space, let \( a : \mathbb{R}^+ \times \Omega \rightarrow S^d_{+} \) and \( b : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^d \) be \( \mathcal{F} \)-progressive processes, and let \( \hat{L}^{a,b} \) denote the stochastic operator

\[
\hat{L}^{a,b} f(s, x) = \partial_s f(s, x) + \frac{1}{2} \sum_{i,j=1}^{d_0} a_{ij} \partial_{ij} f(s, x) + \sum_{i=1}^{d} b_i \partial_i f(s, x).
\]

We say that a continuous, \( \mathcal{F} \)-adapted, \( \mathbb{R}^d \)-valued process \( Y \) is a solution to the \((a, b)\)-martingale problem if the canonical process is a solution to the martingale problem under the measure \( \mathbb{P} \), and we say that the \((a, b)\)-martingale problem is well-posed if there exists a unique measure which solves the \((a, b)\)-martingale problem for each initial condition.

We now give a brief description of the geometric setting in which we shall be working. The reader may consult [10] or [4] for a more thorough discussion. Fix some \( d \geq 1 \) and let \( B \in \mathcal{M}^d \) denote a matrix which takes the following lower-triangular, block form

\[
B = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_n
\end{bmatrix},
\]

where \((d_0, \ldots, d_n)\) is a nonincreasing sequence with \( \sum_i d_i = d \), \( B_i \subset \mathbb{M}^{d_i \times d_{i-1}} \) is a matrix of rank \( d_i \), and 0 denotes a matrix of zeros whose dimensions may
vary with each appearance. It follows from this block structure that $B^i = 0$ when $i > n$.

Once a matrix $B$ has been fixed, we define the following binary operation on $\mathbb{R} \times \mathbb{R}^d$:

$$(s, x) \circ (t, y) = (s + t, e^{tB}x + y).$$

It is easy to check that $(\mathbb{R}^{1+d}, \circ )$ is a group with identity element $(0, 0)$ whose inverse operation is given by

$$(s, x)^{-1} = (-s, -e^{-tB}x).$$

The reader can also check that the operator $\partial_s + L^{a,B}$ is left-translation invariant with respect to this group when $L^{a,B}$ is defined as in (1.4) and $a$ is constant.

Now let $\bar{\delta}_\lambda \in M^{1+d}$ denote the diagonal matrix

$$
\begin{bmatrix}
\lambda^2 & 0 & 0 & \ldots & 0 \\
0 & \lambda I^{d_0} & 0 & \ldots & 0 \\
0 & 0 & \lambda^3 I^{d_1} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda^{2n+1} I^{d_n}
\end{bmatrix},
$$

and let $\delta_\lambda \in M^d$ denote the matrix obtained by removing the first row and column from $\bar{\delta}_\lambda$. It follows easily from the block structure of $B$ that $\delta_\lambda B^i \delta_\lambda^{-1} = \lambda^{2i} B^i$. As $B$ is nilpotent, we have

$$\delta_\lambda e^{tB} \delta_\lambda^{-1} = e^{\lambda^2 tB}.$$ 

One can then check that each dilation $\delta_\lambda$ is an automorphism of the group $(\mathbb{R}^{1+d}, \circ )$, so the collection $(\mathbb{R}^{1+d}, \circ, \delta)$ forms a homogeneous group in the sense of Definition XIII.5.2 in [21]. We then set $\bar{d} = 2 + d_0 + 3d_1 + \cdots + (2n+1)d_n$, so $\det \delta_\lambda = \lambda^\bar{d}$ and $\bar{d}$ gives the homogeneous dimension of the group $(\mathbb{R}^{1+d}, \circ, \bar{\delta})$.

Finally, let

$$\rho(\bar{x}) = \inf \{ \lambda > 0 : |\delta_\lambda^{-1} \bar{x}| \leq 1 \}, \quad \bar{x} \in \mathbb{R}^{1+d},$$

denote the homogeneous norm associated with these dilations and observe that $\rho(\delta_\lambda \bar{x}) = \lambda \rho(\bar{x})$ for all $\bar{x} \in \mathbb{R}^{1+d}$. Moreover, $\bar{\delta}_\lambda \leq \lambda I^{1+d}$ when $\lambda \leq 1$ and $\lambda I^{1+d} \leq \delta_\lambda$ when $\lambda \geq 1$, so it follows immediately that $|\bar{x}| \leq \rho(\bar{x})$ when $|\bar{x}| \leq 1$ and $\rho(\bar{x}) \leq |\bar{x}|$ when $|\bar{x}| \geq 1$.

In the following sections, we will always assume that $B$ is a matrix which satisfies the structural conditions given above. We also observe that the matrix $B$ fully determines the constants $d$, $\bar{d}$, $n$, and $(d_0, \ldots, d_n)$, the binary operation $\circ$, and the matrix $\delta_\lambda$. 
3 Initial Estimates for a Transition Function

We now begin to study the \((c, BX)\)-martingale problem on \(C(\mathbb{R}_+; \mathbb{R}^d)\) when \(c : \mathbb{R}_+ \to S^d_{\mu}\) for some \(\mu \geq 1\) and \(B\) satisfies the structural conditions given in Section 2. Let \(\sigma(t) = \sqrt{c(t)}\) denote the symmetric, positive-definite square root of \(c(t)\), and consider the vector-valued SDE

\[
\begin{align*}
\frac{dY_t(s, x)}{dt} &= BY_t dt + \begin{bmatrix} \sigma(t) \\ 0 \end{bmatrix} dW_t, & t > s, \\
Y_t(s, x) &= x, & t \leq s,
\end{align*}
\]

where \(W\) is a \(d_0\)-dimensional Brownian motion and \(0 \in M^{(d-d_0) \times d_0}\). A solution to this equation is given by

\[
Y_t(s, x) = e^{(t-s)+B}x + \int_s^t e^{(t-u)B} \begin{bmatrix} \sigma(u) \\ 0 \end{bmatrix} dW_u.
\]

The coefficients in equation (3.1) are Lipschitz continuous in space, so this solution is both pathwise unique and unique in law. In particular, if we let \(\mathbb{P}_{s,x}\) denote the law of the process \(Y(s, x)\), then we see that \(\mathbb{P}_{s,x}\) is the unique solution to the \((c, BX)\)-martingale problem starting at \((s, x)\), and the collection of measures \(\{\mathbb{P}_{s,x}\} (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) form a strong Markov family on \((C(\mathbb{R}_+; \mathbb{R}^d), \mathbb{C}^d)\). The transition function associated with this strong Markov family is given by

\[
p_{c, B}(s, t; y) = 1_{\{t > s\}} (2\pi)^{-d/2} \left( \det C_{c, B}(s, t) \right)^{-1/2} \times \exp \left\{ -\frac{1}{2} \langle C_{c, B}^{-1}(s, t)(y - e^{(t-s)B})x, y - e^{(t-s)B}x \rangle \right\},
\]

where

\[
C_{c, B}(s, t) = \int_s^t e^{(t-u)B} \begin{bmatrix} c(u) \\ 0 \end{bmatrix} e^{(t-u)B^T} du.
\]

We will now study this transition function using analytic tools. This will be more pleasant if the transition function is defined on the entire time line, so we will now assume that \(c : \mathbb{R} \to S^d_{\mu}\) and we will define \(p_{c, B}(s, x; t, y)\) for all \((s, x; t, y) \in \mathbb{R}^{2(1+d)}\). In Lemma 3.2 we will see that \(C_{c, B}(s, t)\) is invertible when \(t > s\), so \(p_{c, B}\) is well-defined. The fact that \(C_{c, B}(s, t)\) is invertible for all \(s > t\) reflects the fact that the backward equation associated with the process \(Y(s, x)\) is hypoelliptic when \(c\) is constant. We now define the Green’s operator associated with this transition function:

\[
G_{c, B}f(x) = \int_{\mathbb{R}^{1+d}} p_{c, B}(x; y) f(y) dy, \quad x \in \mathbb{R}^{1+d}, f \in C_K(\mathbb{R}^{1+d}).
\]

We will also need the operator

\[
L^{c, B}u(s, x) = \frac{1}{2} \sum_{i,j=1}^{d_0} c^{ij}(s) \partial_{ij} u(s, x) + \langle Bx, D_x u(s, x) \rangle.
\]
If we apply the operator $L_{c,B}$ to the transition function $p_{c,B}(x; y)$, then we will add a subscript to indicate which set of coordinates the operator should act upon.

We first give a number of scaling properties possessed by the transition function. These properties follow easily from the fact that $e^{MtB} = \delta_t^{1/2} e^{tB} \delta_t^{-1/2}$ and $\det \delta_t = \lambda^{d-2}$, so we leave their verification to the reader.

**Lemma 3.1.** Let $c : \mathbb{R} \to S_{\mu}^{d_0}$ be a measurable function, let $\bar{x} = (s, x) \in \mathbb{R}^{1+d}$, $\bar{y} = (t, y) \in \mathbb{R}^{1+d}$, $\bar{z} = (u, z) \in \mathbb{R}^{1+d}$, let $\lambda \geq 0$, and set $c_1(\tau) = c(u + \tau)$ and $c_2(\tau) = c(\lambda^2 \tau)$. Then

\[(3.4) \quad p_{c,B}(\bar{z} \circ \bar{x}; \bar{z} \circ \bar{y}) = p_{c_1}(\bar{x}; \bar{y}),\]

\[(3.5) \quad C_{c,B}(\lambda^2 s, \lambda^2 t) = \delta_{\lambda} C_{c_2}(s, t) \delta_{\lambda},\]

\[(3.6) \quad p_{c,B}(\tilde{\delta}_\lambda \bar{x}; \tilde{\delta}_\lambda \bar{y}) = \lambda^{2-d} p_{c_2}(\bar{x}; \bar{y}).\]

We will soon need bounds on the transition function which only depend upon $c$ through $\mu$, so we will now develop some lemmas for uniformly dominating the transition function.

**Lemma 3.2.** Let $c : \mathbb{R} \to S_{\mu}^{d_0}$ be measurable. Then $C_{c,B}(s, t)$ is invertible when $t > s$ and there exist polynomials $P$ and $Q$ with positive coefficients such that $\|C_{c,B}(s, t)\| \leq P(t - s)$ and $\|C_{c,B}^{-1}(s, t)\| \leq Q(1/(t - s))$ when $t > s$. Moreover, the coefficients of the polynomials $P$ and $Q$ only depend upon $B$ and $\mu$.

**Proof.** Set $A = \begin{bmatrix} I_{d_0} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}^d$ and define

\[\tilde{C}(\tau) = \int_0^{\tau} e^{(\tau - u)B} A e^{(\tau - u)B^T} du.\]

Then $\tilde{C}(\tau)$ is strictly positive definite when $\tau > 0$. This follows from the fact that $A(B^T)^i x = 0$ for all $i \geq 0$ if and only if $x = 0$. The reader may consult Proposition A.1 of [16] for the details of this argument. It follows from the definition of $C_{c,B}(s, t)$ that $\tilde{C}(t - s)/\mu \leq C_{c,B}(s, t) \leq \mu \tilde{C}(t - s)$ with respect to the natural partial ordering on symmetric matrices. As a result, we see that $C_{c,B}(s, t)$ is invertible and $C_{c,B}^{-1}(s, t) \leq \mu \tilde{C}^{-1}(t - s)$ when $t > s$. Using the fact that $\delta_{\lambda} e^{tB} \delta_{\lambda}^{-1} = e^{\lambda tB}$, one can easily check that $\tilde{C}(t) = \delta_t^{1/2} \tilde{C}(1) \delta_t^{1/2}$. Finally, we recall that the Euclidean operator norm of a symmetric, positive-definite matrix is equal to the largest eigenvalue of the matrix. Combining these observations, we see that

\[
\|C_{c,B}(s, t)\| \leq \mu \|\tilde{C}(t - s)\| \leq \mu \|\tilde{C}(1)\| \|\delta_{(t-s)}^{1/2}\|^2
\leq \mu \|\tilde{C}(1)\| \{(t - s) + (t - s)^{2n+1}\},
\]

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where $n$ denote the constant that appears in Section 2. As $\|\hat{C}(1)\|$ and $n$ are fully determined by $B$, we have produced the polynomial $P$. Arguing in the same way, we see that

$$\|C_{c,B}^{-1}(s,t)\| \leq \mu \|\hat{C}(1)\| \|\delta^{1/2}_{(t-s)}\|^2$$

$$\leq \mu \|\hat{C}(1)\| \{(t-s)^{-1} + (t-s)^{-(2n+1)}\}.$$ 

This gives the polynomial $Q$ and completes the proof.

**Lemma 3.3.** Let $0 < a < b$ and let $c : \mathbb{R} \to S^{d_0}_\mu$ be measurable. Then there exist constants $N = N(B,\mu)$ and $\varepsilon = \varepsilon(B,\mu) > 0$ such that $p_{c,B}(s,x;t,y) \leq N \exp \{N|x||y| - \varepsilon(|x|^2 + |y|^2)\}$ when $t - s \in [a,b]$.

**Proof.** Let $\hat{C}$ be defined as in Lemma 3.2 let $\lambda_1 > 0$ denote the smallest eigenvalue of $\hat{C}^{-1}(b)$, and set $\delta = \inf \{|e^{sB}x| : s \in [a,b], x \in \mathbb{R}^d, |x| = 1\}$. As this infimum is achieved and $e^{sB}$ is invertible, we have $\delta > 0$. When $t - s \in [a,b]$, we have $\hat{C}^{-1}(b)/\mu \leq C_{c,B}^{-1}(s,t) \leq \mu \hat{C}^{-1}(a)$ and

$$\langle C_{c,B}^{-1}(s,t)(y - e^{(t-s)B}x), y - e^{(t-s)B}x \rangle$$

$$\geq \lambda_1 |y|^2/\mu + \lambda_1 \delta^2 |x|^2/\mu - 2\mu \|\hat{C}^{-1}(a)\| |y|e^{(b-a)}\|B\| |x|.$$ 

In particular, if we set $N_1 = \mu \|\hat{C}^{-1}(a)\|e^{(b-a)}\|B\|$ and $\varepsilon = \lambda_1 (1 \land \delta^2)/(2\mu)$, then we have

$$p_{c,B}(s,x;t,y) \leq (2\mu\pi)^{d/2} \{\det(\hat{C}(a))\}^{-1/2} \exp \{N_1|x||y| - \varepsilon(|x|^2 + |y|^2)\},$$

when $t - s \in [a,b]$.

**Lemma 3.4.** Let $c : \mathbb{R} \to S^{d_0}_\mu$ be measurable and let $\alpha, \beta \in (\{0\} \cup \mathbb{N})^d$ be multindices. Then there exists a polynomial $P$ in four variables with positive coefficients such that

$$|D^\alpha_x D^\beta_y p_{c,B}(s,x;t,y)| \leq P(t-s,1/(t-s),|x|,|y|) \ p_{c,B}(s,x;t,y),$$

when $t > s$. If we further assume that $c \in C^\infty(\mathbb{R};S^{d_0}_\mu)$, then $\partial_s p_{c,B}(s,x;t,y) + L^c_x p_{c,B}(s,x;t,y) = 0$ when $s < t$, and we may find polynomials $Q$ and $R$ of the same form as $P$ such that

$$|\partial_s D^\alpha_x D^\beta_y p_{c,B}(s,x;t,y)| \leq Q(t-s,1/(t-s),|x|,|y|) \ p_{c,B}(s,x;t,y),$$

$$|\partial_s D^\alpha_x D^\beta_y p_{c,B}(s,x;t,y)| \leq R(t-s,1/(t-s),|x|,|y|) \ p_{c,B}(s,x;t,y),$$

when $t > s$. Moreover, the coefficients of $P$, $Q$, and $R$ may be chosen so that they only depend upon $B$, $\mu$, $\alpha$ and $\beta$. 

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Proof. Define the vector-valued functions
\[ f_m(s, x; t, y) = e^{m(t-s)B^T}C_c^{-1}(s, t)(y - e^{(t-s)B}x), \quad m \in \{0, 1\}, \]
the matrix-valued functions
\[ g_{mn}(s, t) = e^{m(t-s)B^T}C_c^{-1}(s, t)e^n(t-s)B, \quad m, n \in \{0, 1\}, \]
and the scalar functions
\[
\begin{align*}
h_0(s, x; t, y) &= \frac{1}{2} \sum_{i,j=1}^{d_0} c^{ij}(t)(f_{0}^{i} f_{0}^{j} - g_{00}^{ij}) + (By, f_0), \\
h_1(s, x; t, y) &= \frac{1}{2} \sum_{i,j=1}^{d_0} c^{ij}(t)(g_{11}^{ij} - f_{1}^{i} f_{1}^{j}) - (Bx, f_1),
\end{align*}
\]
where the arguments of \( f_m \) and \( g_{mn} \) have been suppressed. We then see that \( Dg_{0}p_{c,B} = -f_{0}p_{c,B}, Dg_{1}p_{c,B} = f_{1}p_{c,B}, Dg_{m}f_{m} = g_{m0}, \) and \( Dg_{m}f_{m} = -g_{m1} \) when \( t > s \). If \( c \) is smooth, then we also have \( \partial_s p_{c,B} = h_{0}p_{c,B} \) and \( \partial_s p_{c,B} = h_{1}p_{c,B} \). Using these expressions, one can check directly that \( \partial_s p_{c,B}(s, x; t, y) + L^B_x p_{c,B}(s, x; t, y) = 0 \) when \( s < t \).

An inductive argument using the product rule for differentiation shows that is enough to check that \( f_n, g_{nm}, \) and \( h_n \) are dominated by polynomials of the required form for \( n, m \in \{0, 1\} \). Examining the expressions above, we then see that it is actually enough to bound \( \|c(\tau)\|, \|e^{(t-s)B}\| \) and \( \|C_c^{-1}(s, t)\| \) by polynomials of the desired form. We have \( \|c(\tau)\| \leq \mu \) for all \( \tau \). As the matrix \( B \) is nilpotent, the expression \( \|e^{(t-s)B}\| \) is bounded by a polynomial in the single variable \( (t-s) \) whose coefficients are determined by \( B \). Finally, Lemma 3.2 asserts that \( \|C_c^{-1}(s, t)\| \) is bounded by a polynomial in the single variable \( (t-s)^{-1} \) whose coefficients are determined by \( B \) and \( \mu \), so the proof is complete.

We will need the following cancellation properties in the next section.

Lemma 3.5. Let \( c : \mathbb{R} \to S_{d_0}^d \) be measurable, let \( 1 \leq i, j \leq d \), and let \( s < t \). Then \( \int_{\mathbb{R}^d} \frac{\partial^2}{\partial x_i \partial x_j} p_{c,B}(s, x; t, y) \, dx = 0 \) and \( \int_{\mathbb{R}^d} \frac{\partial^2}{\partial x_i \partial x_j} p_{c,B}(s, x; t, y) \, dy = 0 \).

Proof. Let \( f_1 \) and \( g_{11} \) be defined as in the last lemma, so \( D^2_x p_{c,B} = (f_1 f_1^T - g_{11}) p_{c,B} \). It follows from Lemma 3.3 and Lemma 3.4 that
\[
\lim_{x' \to \pm \infty} \frac{\partial p_{c,B}}{\partial x_i'}(s, x^1, \ldots, x^i, \ldots, x^d, t; y) = 0,
\]
so \( \int_{\mathbb{R}^d} \frac{\partial^2}{\partial x_i \partial x_j} p_{c,B}(s, x; t, y) \, dx = 0 \) by the Fundamental Theorem of Calculus and Fubini’s Theorem. To handle the second integral, we let \( Y \) denote a \( d \)-dimensional,
normally distributed random variable on some probability space with mean zero and covariance $C_{c,B}(s,t)$. With this notation, we have

$$
\int_{\mathbb{R}^d} D_s^2 p_{c,B}(s, x; t, y) \, dy
= \mathbb{E}\left[ e^{(t-s)B^T C^{-1}_c(s,t) Y Y^T C^{-1}_c(s,t) e^{(t-s)B}} \right] - g_{11}(s,t)
= 0,
$$

which completes the proof.

We now show that $G_{c,B}$ plays the role of a fundamental solution for the operator $L^{c,b}$. We let $C^{0,\infty}(\mathbb{R} \times \mathbb{R}^d)$ denotes the class of continuous functions that are infinitely continuously differentiable with respect to the spacial variables.

Lemma 3.6. Let $c : \mathbb{R} \rightarrow S^d_0$ be measurable and fix some $f \in C^{0,\infty}(\mathbb{R} \times \mathbb{R}^d)$. Then $G_{c,B}f \in C^{0,\infty}(\mathbb{R} \times \mathbb{R}^d)$ and for each multiindex $\alpha \in (\{0\} \cup \mathbb{N})^d$ and fixed time $s \in \mathbb{R}$ there exists a constant $N = N(B, f, \alpha, s)$ such that

$$
\| D^\alpha_x G_{c,B} f \|_{L^\infty([s,\infty) \times \mathbb{R}^d)} \leq N.
$$

Moreover, $D^\alpha_x G_{c,B} f$ admits the representation

$$
D^\alpha_x G_{c,B} f(s, x) = \lim_{\varepsilon \to 0^+} \int_{s+\varepsilon}^{s+1/\varepsilon} \int_{\mathbb{R}^d} D^\alpha_x p_{c,B}(s, x; t, y) f(t, y) \, dy \, dt.
$$

If we further assume that $c \in C^{\infty}(\mathbb{R}_+; S^d_0)$, then $u = G_{c,B}f$ is a classical solution to the backwards equation $\partial_s u + L^{c,b}u = -f$ on $\mathbb{R}^{1+d}$.

Proof. Set $g(s, x; t, y) = f(t, e^{(t-s)B}x + y)$ and define

$$
\begin{align*}
\varepsilon u^\varepsilon(s, x) &= \int_{s+\varepsilon}^{s+1/\varepsilon} \int_{\mathbb{R}^d} p_{c,B}(s, x; t, y) f(t, y) \, dy \, dt, \\
\alpha u^\alpha(s, x) &= \int_{s}^{\infty} \int_{\mathbb{R}^d} p_{c,B}(s, 0; t, y) D^\alpha_x g(s, x; t, y) \, dy \, dt.
\end{align*}
$$

As $p_{c,B}(s, x; t, y) = p_{c,B}(s, 0; t, y - e^{(t-s)B}x)$, we have

$$
\begin{align*}
\varepsilon u^\varepsilon(s, x) &= \int_{s+\varepsilon}^{s+1/\varepsilon} \int_{\mathbb{R}^d} p_{c,B}(s, 0; t, y) g(s, x; t, y) \, dy \, dt.
\end{align*}
$$

We then choose $T$ so that the support of $f$ is contained in the set $(-\infty, T] \times \mathbb{R}^d$, and we observe that

$$
\| D^\alpha_x g(s, x; t, y) \|_{L^\infty(\mathbb{R}^{1+d})} \leq \| D^\alpha_x f \|_{L^\infty(\mathbb{R}^{1+d})},
$$

which completes the proof.
for any multiindex $\beta$. In particular, we may differentiate $3.11$ repeatedly to obtain

$$D_x^\alpha w^\varepsilon(s, x) = \int_{s+1/\varepsilon}^{s+1} \int_{\mathbb{R}^d} p_{c,B}(s,0;t,y)D_x^\alpha g(s,x;t,y)\,dy\,dt.$$  

It follows from Lemma $3.3$ and dominated convergence that $D_x^\alpha w^\varepsilon$ is continuous. It then follows from (3.10), (3.12), and (3.13) that

$$|v^\alpha(s,x) - D_x^\alpha w^\varepsilon(s,x)| \leq \varepsilon e^{|\alpha|B} \|D_x^\alpha f\|_{L^\infty(\mathbb{R}^{1+d})},$$

when $s + 1/\varepsilon > T$, so $D_x^\alpha w^\varepsilon$ converges to $v^\alpha$ uniformly on each set of the form $[u, \infty) \times \mathbb{R}^d$ as $\varepsilon \to 0$. As a result, we may conclude that $v^\alpha$ is continuous, $D_x^\alpha G_{c,B}f$ exists, and $D_x^\alpha G_{c,B}f = v^\alpha$ for each multiindex $\alpha$. In particular, (3.11) follows immediately from (3.12).

It follows from Lemma $3.3$, Lemma $3.4$, and dominated convergence that we may differentiate $3.9$ to obtain

$$D_x^\alpha w^\varepsilon(s, x) = \int_{s+1/\varepsilon}^{s+1} \int_{\mathbb{R}^d} D_x^\alpha p_{c,B}(s,x,t,y) f(t,y)\,dy\,dt.$$  

In particular, we see that $D_x^\alpha G_{c,B}f$ admits the representation $3.8$.

When $c \in C^\infty(\mathbb{R}^+;S^d_{\mu})$, another domination argument shows that we have $\partial_s w^\varepsilon(s,x) = -w_0^\varepsilon(s,x) + w_1^\varepsilon(s,x) + w_2^\varepsilon(s,x)$, where

$$w_0^\varepsilon(s,x) = \int_{\mathbb{R}^d} p_{c,B}(s,x,s+\varepsilon,y)f(s+\varepsilon,y)\,dy,$$

$$w_1^\varepsilon(s,x) = \int_{s+1/\varepsilon}^{s+1} \int_{\mathbb{R}^d} \partial_s p_{c,B}(s,x,t,y)f(t,y)\,dt\,dy,$$

$$w_2^\varepsilon(s,x) = \int_{\mathbb{R}^d} p_{c,B}(s,x,s+1/\varepsilon,y)f(s+1/\varepsilon,y)\,dy.$$  

As $f$ is uniformly continuous, $w_0^\varepsilon \to f$ uniformly on $\mathbb{R}^{1+d}$. As $f$ has compact support, $w_2^\varepsilon \to 0$ uniformly on sets of the form $[u, \infty) \times \mathbb{R}^d$. Lemma $3.4$ asserts that $\partial_s p_{c,B}(s,x,t,y) + L^{c,B}_x p_{c,B}(s,x,t,y) = 0$ when $s < t$, so

$$w_1^\varepsilon(s,x) = -\int_{s+1/\varepsilon}^{s+1} \int_{\mathbb{R}^d} L^{c,B}_x p_{c,B}(s,x,t,y)f(t,y)\,dt\,dy.$$  

We have already shown that $D_x^\alpha w^\varepsilon$ converges uniformly to $D_x^\alpha G_{c,B}f$ on sets of the form $[u, \infty) \times \mathbb{R}^d$ for each multiindex $\alpha$. The coefficients of the operator $L^{c,B}$ are locally bounded, so $w_1^\varepsilon$ converges to $L^{c,B}G_{c,B}f$ uniformly on compact sets. Putting this all together, we see that $\frac{d}{dt} u^\varepsilon$ converges to $-f - L^{c,B}G_{c,B}f$ and this convergence is uniform on compact sets, so $\partial_s G_{c,B}f$ exists and equals $-f - L^{c,B}G_{c,B}f$. 

\[\square\]
4 An $L^p$-estimate

The following theorem is main result of this section.

**Theorem 4.1.** Let $c: \mathbb{R} \to S^d_{\mu}$ be measurable, $i, j \in \{1, \ldots, d_0\}$, $p \in (1, \infty)$, and $f \in C^\infty_c(\mathbb{R}^{1+d})$. Then $\|\partial_{ij} G_{c, B} f\|_{L^p(\mathbb{R}^{1+d})} \leq N(B, \mu, p) \|f\|_{L^p(\mathbb{R}^{1+d})}$.

We will obtain this estimate by studying the singular integral representation of $\partial_{ij} G_{c, B}$. The approach that we follow is essentially a mixture of techniques from Section XIII.5 of [21] and Section A.2 of [24]. To reduce the notational burden in this section, we will collect all of the information that we need to specify a kernel in a single tuple. Let $B(\mathbb{R}; S^d_{\mu})$ denote the set of bounded, measurable functions from $\mathbb{R}$ to $S^d_{\mu}$ and set $\mathcal{A}(\mu) = B(\mathbb{R}; S^d_{\mu}) \times \{1, \ldots, d_0\} \times \{0, 1\}$. Given a tuple $\alpha = (c, k, \ell, m) \in \mathcal{A}(\mu)$, we define the singular kernel

$$h_\alpha(s, x; t, y) = \mathbb{1}_{\{m=0\}} \frac{\partial^2 p_{c, B}}{\partial x^i \partial x^j} (s, x; t, y) + \mathbb{1}_{\{m=1\}} \frac{\partial^2 p_{c, B}}{\partial x^i \partial x^j} (t, y; s, x).$$

We will also need the truncated kernels

$$h^i_\alpha(s, x; t, y) = \mathbb{1}_{(1,4]}(\|t-s\|/4^i) h_\alpha(s, x; t, y), \quad i \in \mathbb{Z},$$

and the operators

$$H^i_\alpha f(\bar{x}) = \int_{\mathbb{R}^{1+d}} h^i_\alpha(\bar{x}; \bar{y}) f(\bar{y}) d\bar{y}, \quad i \in \mathbb{Z},$$

$$K^j_\alpha f(\bar{x}) = \sum_{i=-j}^{j} H^i_\alpha f(\bar{x}), \quad j \in \mathbb{N}.$$  

Each kernel $h^i_\alpha$ is bounded, so these operators are defined in a pointwise sense for all $f \in C_K(\mathbb{R}^{1+d})$. The main task in this section is to show that the collection of operators $\mathcal{K} = \{K^j_\alpha : \alpha \in \mathcal{A}(\mu), j \in \mathbb{N}\}$ is uniformly bounded with respect to the $L^p$-operator norm for each $p \in (1, \infty)$. Once this is done, Theorem 4.1 follows easily from Fatou’s Lemma.

We will first obtain a uniform bound with respect to the $L^2$-operator norm using the Cotlar-Stein Almost Orthogonality Lemma, which we now recall for the reader’s convenience. One proof of this lemma may be found in Section VII.2 of [21].

**Lemma 4.2** (Cotlar-Stein Lemma). Let $\{T_i\}$ be a sequence of bounded operators on some $L^2$-space and let $\{c_i\}_{i=-\infty}^\infty$ be a sequence of positive constants with $N = \sum_{i=-\infty}^\infty c_i < \infty$. If $\|T_i^* T_j\| \leq c^2_{i-j}$ and $\|T_i T_j^*\| \leq c^2_{i-j}$, then $\|\sum_{i=-n}^n T_i\| \leq N$ for all $n \geq 1$.

Once we have a uniform bound in the $L^2$-operator norm, we will check that the kernels associated with the operators $K^j_\alpha$ satisfy an integrable Hörmander
condition which is adapted to our geometric setting. This will allow us to obtain a uniform bound with respect to the $L^p$-operator norm for $p \in (1,2)$ using the following theorem.

**Theorem 4.3.** Let $p \in (1,2)$, let $k$ be a bounded, measurable function, and set

$$Kf(x) = \int_{\mathbb{R}^{1+d}} k(x; y) f(y) \, dy, \quad f \in C^\infty_K(\mathbb{R}^{1+d}).$$

Suppose that there exists a constant $N_1$ such that

$$\left| k(x; y) - k(x; z) \right| \, dx \leq N_1,$$

and a constant $N_2$ such that $\|Kf\|_{L^2} \leq N_2\|f\|_{L^2}$ when $f \in C^\infty_K(\mathbb{R}^{1+d})$. Then there exists a constant $N = N(N_1, N_2, p)$ such that $\|Hf\|_{L^p} \leq N\|f\|_{L^p}$ when $f \in C^\infty_K(\mathbb{R}^{1+d})$.

The reader may consult Theorem 3 in Section I.5 of [21] for a proof of Theorem 4.3 under weaker hypotheses. As the collection of operators $\mathcal{K}$ is closed with respect to taking the (formal) adjoint, uniform bounds with respect to the $L^p$-operator norm for $p \in (2, \infty)$ will then follow from duality. It is this last case that we will need in Section 5.

We will begin the process by listing the translation and scaling properties of the kernels $h_\alpha$. These properties follow easily from the properties of $p_{r,\beta}$ given in Lemma 3.1, the explicit formula for $\frac{\partial^2}{\partial x^\alpha \partial \xi^\beta} p_{t,\beta}$ given in the proof of Lemma 3.2, and the fact that $B(\mathbb{R}^d)$ is closed with respect to translation and dilation, so we leave the verification of this lemma to the reader. We remind the reader that $d \geq d + 2$ denotes the homogeneous dimension of the group $p_{r,\beta}$ as defined in Section 2. We also point out that the exponent in the dilation law (4.4) becomes less favorable if we attempt to differentiate $p_{r,\beta}$ with respect to $x^i$ with $i > d$. This explains to a large extent why we must wait until we get to the probabilistic level to make any changes to the drift.

**Lemma 4.4.** Let $\alpha \in \mathcal{A}(\mu)$ and fix some $\bar{z} = (u, z) \in \mathbb{R}^{1+d}$ and $\lambda > 0$. Then we may find $\beta, \gamma \in \mathcal{A}(\mu)$ such that

$$h_\alpha(\bar{z} \circ \bar{x}; \bar{z} \circ \bar{y}) = h_\beta(\bar{x}; \bar{y}),$$

$$h_\alpha(\bar{z} \circ \bar{x}; \bar{z} \circ \bar{y}) = \lambda^{-d} h_\gamma(\bar{x}; \bar{y}).$$

**Remark 4.5.** For each $\alpha \in \mathcal{A}(\mu)$, we may find $\beta \in \mathcal{A}(\mu)$ such that

$$h^i_\alpha(\bar{x}; \bar{y}) = 2^{-i\delta} h^0_\beta(\bar{\delta}_{2^{-i}} \bar{x}; \bar{\delta}_{2^{-i}} \bar{y}).$$

But the Jacobian determinant of the map $\bar{x} \mapsto \bar{\delta}_{\lambda} \bar{x}$ is $\lambda^d$, so

$$\int_{\mathbb{R}^{1+d}} h^i_\alpha(\bar{x}; \bar{y}) f(\bar{x}) \, d\bar{x} = \int_{\mathbb{R}^{1+d}} h^0_\beta(\bar{x}; \bar{\delta}_{2^{-i}} \bar{y}) f(\bar{\delta}_{2^{-i}} \bar{x}) \, d\bar{x},$$

when either integral is well-defined. In the remainder of this section, when we say “by dilation”, we are making use of (minor variations on) this observation.
Lemma 4.6. There exists an function $\hat{h}_\mu : \mathbb{R}^d \to \mathbb{R}_+$ such that

\begin{align*}
(4.5) & \quad |h_\alpha(s; x; \bar{y})| \leq \tilde{\hat{h}}_\mu(x), \\
(4.6) & \quad |h_\alpha(s; x; \bar{y}) - h_\alpha(s; x; 0)| \leq |\bar{y}| \hat{\tilde{h}}_\mu(x),
\end{align*}

for all $|s| \in [1/2, 5]$, $|\bar{y}| \leq 1/4$, and $\alpha \in \mathcal{A}(\mu)$. Moreover, $\hat{h}_\mu$ may be chosen such that $\int_{\mathbb{R}^d} |f(x)| \hat{h}_\mu(x) dx < \infty$ for every function $f$ of polynomial growth.

Proof. Let $\tilde{\bar{y}} = (s, x)$, $\bar{y} = (t, y)$, and set $k^{ij}_c(s, x; t, y) = \frac{\partial^2}{\partial x^i \partial x^j} p_{c, B}(s; x; t, y)$ for $0 \leq i, j \leq d_0$ and $c \in C^\infty(\mathbb{R}; S^d_{\mu})$. Using Lemma 3.3, we may find a polynomial $P$ in four variables with positive coefficients such that

\begin{align*}
|k^{ij}_c(s, x; t, y)| + |D_x k^{ij}_c(s, x; t, y)| + |D_y k^{ij}_c(s, x; t, y)| & \leq P(t - s, 1/(t - s), |x|, |y|) p_{c, B}(s; x, t, y),
\end{align*}

when $t > s$. Next we use Lemma 3.3 to produce constants $N = N(B, \mu)$ and $\varepsilon = \varepsilon(B, \mu) > 0$ such that $p_{c, B}(s; x, t, y) \leq N \exp\{N|x||y| - \varepsilon|x|^2 + |y|^2\}$ when $t - s \in [1/4, 6]$, and we set

\begin{align*}
\hat{h}_\mu(x) &= N\{P(6, 4, 1/4) + P(6, 4, 1/4, x)\} \exp\{N|x|/4 - \varepsilon|x|^2\}.
\end{align*}

As $P$ is a polynomial, $f \hat{h}_\mu$ is integrable when $f$ has polynomial growth.

We now check that (4.5) and (4.6) holds. Let $\alpha = (c, i, j, 0) \in \mathcal{A}(\mu)$ with $c \in C^\infty(\mathbb{R}; S^d_{\mu})$ and $1 \leq i, j \leq d_0$. If $s \in [-5, -1/2]$ and $|\bar{y}| \leq 1/4$, then $t - s \in [1/4, 6]$,

\begin{align*}
|h_\alpha(s; x; \bar{y})| &= |k^{ij}_c(s; x; \bar{y})| \\
&\leq P(6, 4, |x|, 1/4) \sup_{|\bar{z}| \leq 1/4} p_{c, B}(s; x; \bar{z}) \\
&\leq P(6, 4, |x|, 1/4) N \exp\{N|x|/4 - \varepsilon|x|^2\} \leq \hat{h}_\mu(x),
\end{align*}

and

\begin{align*}
|h_\alpha(s; x; \bar{y}) - h_\alpha(s; x; 0)| &= |k^{ij}_c(s; x; \bar{y}) - k^{ij}_c(s; x; 0)| \\
&\leq |\bar{y}| \sup_{|\bar{z}| \leq 1/4} |D_y k^{ij}_c(s; x; \bar{z})| \leq |\bar{y}| \hat{\tilde{h}}_\mu(x).
\end{align*}

Of course, if $s \in [1/2, 5]$ and $|\bar{y}| \leq 1/4$, then $h_\alpha(s; x; \bar{y}) - h_\alpha(s; x; 0) = 0$, and the inequalities holds trivially.

If we choose any $\beta = (c, i, j, 1) \in \mathcal{A}(\mu)$ with $c \in C^\infty(\mathbb{R}; S^d_{\mu})$ and $1 \leq i, j \leq d_0$, then $h_\alpha(s; x; \bar{y}) - h_\alpha(s; x; 0) = k^{ij}_c(\bar{y}; s, x) - k^{ij}_c(0; s, x)$ and the estimate follows in the same way as the previous case. Finally, to handle the case where $\gamma = (c, i, j, k) \in \mathcal{A}(\mu)$, but $c$ is only measurable, we choose $c_n \in C^\infty(\mathbb{R}; S^d_{\mu})$ with $\int_I \|c(s) - c_n(s)\| ds \to 0$ for each compact interval $I$. Then $h_{(c_n, i, j, k)} \to h_{\gamma}$ pointwise, and (4.5) and (4.6) hold for each $h_{(c_n, i, j, k)}$, so they also hold for $h_{\gamma}$. \qed
We will make use the following easy corollary a couple of times.

**Corollary 4.7.** Let $\alpha \in \mathcal{A}(\mu)$ and $\bar{y} \in \mathbb{R}^{1+d}$. Then there exists a constant $N = N(B, \mu)$ such that $\int_{\mathbb{R}^{1+d}} |\mathcal{H}_\alpha^0(\bar{x}; \bar{y})| \, d\bar{x} \leq N$.

**Proof.** Let $\hat{h}_\mu$ denote function defined in the previous lemma. Making use of left-translation and change of variable, we may find $\beta \in \mathcal{A}(\mu)$ such that

$$\int_{\mathbb{R}^{1+d}} |\mathcal{H}_\alpha^0(\bar{x}; \bar{y})| \, d\bar{x} = \int_{\mathbb{R}^{1+d}} |\mathcal{H}_\beta^0(\bar{x}; 0)| \, d\bar{x} \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} 1_{|\alpha|}(|s|) \hat{h}_\mu(x) \, dx \, ds,$$

and this last integral is finite. \hfill $\square$

**Lemma 4.8.** Let $\alpha = (c, k, \ell, m) \in \mathcal{A}(\mu)$ and let $p \in (1, \infty)$. Then there exists a unique bounded linear operators $T : L^p(\mathbb{R}^{1+d}) \to L^p(\mathbb{R}^{1+d})$ such that $T$ agrees with $H_{\mathcal{A}}$ on $C^\infty(\mathbb{R}^{1+d})$. Moreover, if we set $q = p/(p-1)$ and $\alpha^* = (c, k, \ell, 1 - m) \in \mathcal{A}(\mu)$, then $T^* : L^q(\mathbb{R}^{1+d}) \to L^q(\mathbb{R}^{1+d})$ is the unique bounded operator that agrees with $H_{\mathcal{A}}^*$ on $C^\infty(\mathbb{R}^{1+d})$.

**Proof.** By dilation, it is enough to show that the lemma holds for $H_{\mathcal{A}}^0$. Let $f, g \in C^\infty(\mathbb{R}^{1+d})$, set $E = \mathbb{R}^{1+d}$, and choose $N = N(B, \mu)$ as in Corollary 4.7 so that $\int_{\mathbb{R}^{1+d}} |\mathcal{H}_\alpha^0(\bar{x}; \bar{y})| \, d\bar{x} \leq N$ for all $\alpha \in \mathcal{A}$ and $\bar{y} \in \mathbb{R}^{1+d}$. It then follows from Tonelli’s Theorem and Young’s Inequality that

$$\int_E |H_{\mathcal{A}}^0 f(\bar{x})g(\bar{x})| \, d\bar{x} \leq \int_E \int_E |\mathcal{H}_\alpha^0(\bar{x}; \bar{y})| \frac{|f(\bar{y})|^p}{p} \, d\bar{y} \, d\bar{x} + \int_E \int_E |\mathcal{H}_\alpha^0(\bar{x}; \bar{y})| \frac{|g(\bar{y})|^q}{q} \, d\bar{x} \, d\bar{y} \leq N \left( \|f\|_{L^p} + \|g\|_{L^q} \right).$$

Taking the supremum over $f$ and $g$ with $\|f\|_{L^p} \leq 1$ and $\|g\|_{L^q} \leq 1$, we see that $\|H_{\mathcal{A}}^0 f\|_{L^p} \leq N \|f\|_{L^p}$, so $H_{\mathcal{A}}^0$ extends uniquely to a bounded operator $T$ on $L^p(\mathbb{R}^{1+d})$. Moreover, if $f, g \in C^\infty(\mathbb{R}^{1+d})$, then

$$\int_E f(\bar{x}) T^* g(\bar{x}) \, d\bar{x} = \int_E T f(\bar{x}) g(\bar{x}) \, d\bar{x} = \int_E H_{\mathcal{A}}^i f(\bar{x}) g(\bar{x}) \, d\bar{x} = \int_E f(\bar{x}) H_{\mathcal{A}}^i g(\bar{x}) \, d\bar{x},$$

where the use of Fubini’s Theorem in the last equality is justified by the previous inequality. By varying $f$, we may conclude that $H_{\mathcal{A}}^i g$ is a version of $T^* g$. \hfill $\square$

**Lemma 4.9.** There exists a constant $N = N(B, \mu)$ such that

\begin{equation}
\int_{\mathbb{R}^{1+d}} |\mathcal{H}_\alpha^i(\bar{x}; \bar{y}) - \mathcal{H}_\alpha(\bar{x}; \bar{y})| \, d\bar{x} \leq N 2^{-i} \rho (\bar{z}^{-1} \circ \bar{y}),
\end{equation}

for all $\alpha \in \mathcal{A}(\mu)$ and $i \in \mathbb{Z}$. 

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Proof. By left-translation and dilation, it is enough to produce a constant \( N = N(\mu) \) such that

\[
(4.8) \quad \int_{\mathbb{R}^{1+d}} |h^0_\alpha(\bar{x}; \bar{y}) - h^0_\alpha(\bar{x}; 0)| \, d\bar{x} \leq N \rho(\bar{y}).
\]

Corollary 4.7 asserts that we may choose a constant \( N_1 = N_1(B, \mu) \) such that

\[
(4.9) \quad \int_{\mathbb{R}^{1+d}} |h^0_\alpha(\bar{x}; \bar{y}) - h^0_\alpha(\bar{x}; 0)| \, d\bar{x} \leq N_1.
\]

This is a useful bound when \( \rho(\bar{y}) \) is large.

We now set \((t, y) = \bar{y}\) and consider the case where \( |\bar{y}| \leq 1/4\). Using Lemma 2.6 we may choose an integrable function \( \widehat{h}_\mu : \mathbb{R}^d \to \mathbb{R}_+ \) such that \( |h_\beta(s, x; \bar{y})| \leq \widehat{h}_\mu(x) \) and \( |h_\beta(s, x; \bar{y}) - h_\beta(s, x; 0)| \leq |\bar{y}| \widehat{h}_\mu(x) \) when \( \beta \in \mathcal{A}(\mu) \), \( |\bar{y}| \leq 1/4 \), and \( |s| \in [1/2, 5] \). Set \( N_2 = N_2(B, u) = \int_{\mathbb{R}^d} \widehat{h}_\mu(x) \, dx \). When \( |\bar{y}| \leq 1/4 \), we have \( |\mathbf{1}_{[1,4]}(t - s) - \mathbf{1}_{[1,4]}(|s|)| \leq 2|\mathbf{1}_{[1/2,5]}(|s|)| \) and

\[
\int_{\mathbb{R}^{1+d}} |h^0_\alpha(\bar{x}; \bar{y}) - h^0_\alpha(\bar{x}; 0)| \, d\bar{x} \\
\leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\mathbf{1}_{[1,4]}(|t - s|) - \mathbf{1}_{[1,4]}(|s|)| \cdot |h_\alpha(s, x; \bar{y})| \, dx \, ds \\
+ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_{[1,4]}(|s|) \cdot |h_\alpha(x; \bar{y}) - h_\alpha(x; 0)| \, dx \, ds \\
\leq 4|t|N_1 + 3|\bar{y}|N_1.
\]

As \( \rho(\bar{y}) \geq |\bar{y}| \) when \( |\bar{y}| \leq 1 \), we may conclude that

\[
(4.10) \quad \int_{\mathbb{R}^{1+d}} |h^0_\alpha(\bar{x}; \bar{y}) - h^0_\alpha(\bar{x}; 0)| \, d\bar{x} \leq 7N_1 \rho(\bar{y}), \quad \text{when } \rho(\bar{y}) \leq 1/4.
\]

We may then produce a constant such that (4.8) holds by using (4.10) when \( \rho(\bar{y}) \) is small and using (4.9) when \( \rho(\bar{y}) \) is large. \( \square \)

We now produce the desired bound with respect to the \( L^2 \)-operator norm.

**Lemma 4.10.** There exists a constant \( N = N(B, \mu) \) with \( \|K_\alpha^j f\|_{L^2} \leq N \|f\|_{L^2} \) for all \( \alpha \in \mathcal{A}(\mu) \), \( j \in \mathbb{N} \) and \( f \in C^\infty_b(\mathbb{R}^{1+d}) \).

Proof. Set \( E = \mathbb{R}^{1+d} \), and let \( T^i_\alpha : L^2(E) \to L^2(E) \) denote the unique, bounded operator that agrees with \( H^0_\alpha \) on \( C^\infty_b(E) \). We will show that the collection of operators \( \mathcal{T} = \{ \sum_{i=-n}^n T^i_\alpha : \alpha \in \mathcal{A}(\mu), n \in \mathbb{N} \} \) is uniformly bounded.
By the Cotlar-Stein Lemma, it is enough to produce a constant $N$ such that
$$
\|[T_\alpha^i]^* T_\alpha^j\| \leq N2^{-|i-j|} \quad \text{and} \quad \|[T_\alpha^i T_\alpha^j]^*\| \leq N2^{-|i-j|}
$$
for all $\alpha \in \mathcal{A}(\mu)$ and $i, j \in \mathbb{Z}$. But Lemma 4.8 asserts that the class $\mathcal{T}$ is closed with respect to taking adjoints, so it sufficient to show that the first of these inequalities holds.

We will, in fact, produce a constant $N = N(B, \mu)$ such that

$$
(4.11) \quad \int_{E^4} |h_\alpha^i(\bar{x}; \bar{y})h_\alpha^j(\bar{x}; \bar{z})| \, d\bar{x} \, d\bar{z} \leq N2^{-|i-j|}, \quad \text{for all } \bar{y} \in E.
$$

To see that this is sufficient to prove the theorem, assume that (4.11) holds and choose any $f, g \in C_\infty^\infty(\mathbb{R}_{1+d})$. We then have

$$
\langle (T_\alpha^i)^* T_\alpha^j f, g \rangle = \langle T_\alpha^i f, T_\alpha^j g \rangle
$$

$$
= \int_{E^2} h_\alpha^i(\bar{x}; \bar{y})h_\alpha^j(\bar{x}; \bar{z}) f(\bar{y})g(\bar{z}) \, d\bar{x} \, d\bar{y} \, d\bar{z}
$$

$$
\leq \frac{1}{2} \int_{E^2} \left( \int_{E^2} |h_\alpha^i(\bar{x}; \bar{y})h_\alpha^j(\bar{x}; \bar{z})| \, d\bar{x} \, d\bar{z} \right) f^2(\bar{y}) \, d\bar{y}
$$

$$
+ \frac{1}{2} \int_{E^2} \left( \int_{E^2} |h_\alpha^i(\bar{x}; \bar{y})h_\alpha^j(\bar{x}; \bar{z})| \, d\bar{x} \, d\bar{y} \right) g^2(\bar{z}) \, d\bar{z}
$$

$$
\leq \frac{1}{4} N2^{-|i-j|} (\|f\|_{L^2} + \|g\|_{L^2}),
$$

where we have used Fubini’s Theorem and Young’s inequality. If we then take the supremum over $f$ and $g$ with $\|f\|_{L^2} \leq 1$ and $\|g\|_{L^2} \leq 1$, then we see that $\|[T_\alpha^i]^* T_\alpha^j\| \leq N2^{-|i-j|}$ and the theorem follows.

We now show that (4.11) holds. Using Corollary 4.7 we find a constant $N_2 = N_2(B, \mu)$ such that $\int_{\mathbb{R}_{1+d}} |h_\alpha^0(\bar{x}; 0)| \rho(\bar{x}) \, d\bar{x} \leq N_2$ for all $\alpha \in \mathcal{A}(\mu)$. By left-translation and dilation, we may choose $\beta, \gamma \in \mathcal{A}(\mu)$ such that

$$
\int_{E} h_\alpha^i(\bar{x}; \bar{y}) h_\alpha^j(\bar{x}; \bar{z}) \, d\bar{x} = \int_{E} h_\beta^0(\bar{x}; 0) h_\gamma^j(\bar{\delta}_2 \bar{x}; \bar{\gamma}^{-1} \circ \bar{z}) \, d\bar{x}.
$$

After a change of variable and an application of the cancellation property given in Lemma 4.8, we have

$$
\int_{E} \int_{E} h_\beta^0(\bar{x}; 0) h_\gamma^j(\bar{\delta}_2 \bar{x}; \bar{\gamma}^{-1} \circ \bar{z}) \, d\bar{x} \, d\bar{z}
$$

$$
= \int_{E} h_\beta^0(\bar{x}; 0) \left\{ \int_{E} h_\gamma^j(\bar{\delta}_2 \bar{x}; \bar{z}) - h_\gamma^j(0; \bar{z}) \, d\bar{z} \right\} \, d\bar{x}.
$$
Letting $N_3 = N_3(B, \mu)$ denote the constant obtained in Lemma 4.3, we see that
\[
\int_{E^2} |h^i_\alpha(x; \bar{y})h^j_\alpha(x; \bar{z})| \, dx \, dz \\
\leq \int_E |h^0_\beta(x; 0)| \left\{ \int_E |h^i_\beta(\delta_{2, x}; \bar{y}) - h^0_\beta(0; \bar{z})| \, dz \right\} \, dx \\
\leq 2^{-j} N_3 \int_E |h^0_\beta(x; 0)| \rho(\delta_{2, x}) \, dx \\
= 2^{i-j} N_3 \int_E |h^0_\beta(x; 0)| \rho(x) \, dx \\
\leq N_2 N_3 2^{i-j}.
\]

To handle the case where $i > j$, we choose new $\beta, \gamma \in \mathcal{A}(\mu)$ such that
\[
\int_E h^i_\alpha(x; \bar{y}) h^j_\alpha(x; \bar{z}) \, dx \, dz = \int_E h^i_\beta(\delta_{2, x}; \bar{z}^{-1} \circ \bar{y}) h^0_\gamma(x; 0) \, dx.
\]
The matrix $B$ is strictly lower triangular, so $\det e^{-sB} = \pm 1$. In particular, if we fix some $\bar{y} = (t, y) \in E$, then we see that the absolute value of the Jacobian determinant of the map $\bar{z} = (u, \bar{z}) \mapsto \bar{z}^{-1} \circ \bar{y} = (t-u, y-e(t-u)Bz)$ is one. This means that
\[
\int_E h^i_\beta(\delta_{2, x}; \bar{z}^{-1} \circ \bar{y}) \, d\bar{z} = \int_E h^i_\beta(\delta_{2, x}; \bar{z}) \, d\bar{z},
\]
for each fixed $x$. Arguing as in the previous case, we see that
\[
\int_{E^2} |h^i_\alpha(x; \bar{y})h^j_\alpha(x; \bar{z})| \, dx \, dz \leq \int_E \left\{ \int_E |h^i_\beta(\delta_{2, x}; \bar{z}) - h^0_\beta(0; \bar{z})| \, dz \right\} h^0_\gamma(x; 0) \, dx \\
\leq N_2 N_3 2^{j-i}.
\]
We have now shown that (4.11) holds, so the proof is complete.

\[\square\]

**Lemma 4.11.** Let $p \in (1, 2)$. Then there exists a constant $N = N(B, \mu, p)$ such that $\|K^\alpha_f\|_{L^p} \leq N\|f\|_{L^p}$ for all $\alpha \in \mathcal{A}(\mu)$, $j \in \mathbb{N}$, and $f \in C^\infty_E(\mathbb{R}^{1+d})$.

**Proof.** First we observe that it is enough to produce a constant $N_1 = N_1(B, \mu)$ such that
\[
\sum_{i=-\infty}^{\infty} \int_{\rho(\bar{z} \circ \delta \bar{y}) \geq N_1 \rho(\bar{z} \circ \bar{y})} |h^i_\alpha(x; \bar{y}) - h^i_\alpha(x; \bar{z})| \, dx \leq N_1.
\]

To see this, suppose that (4.12) holds and set $k^i_\alpha(x; \bar{y}) = \sum_{i=-j}^{i} h^i_\alpha(x; \bar{y})$. Then $k^i_\alpha$ is bounded, $K^\alpha_f(x) = \int_{\mathbb{R}^{1+d}} k^0_\alpha(x; \bar{y}) f(\bar{y}) \, d\bar{y}$ when $f \in C^\infty_E$, and
\[
\int_{\rho(\bar{z} \circ \delta \bar{y}) \geq N_1 \rho(\bar{z} \circ \bar{y})} |k^i_\alpha(x; \bar{y}) - k^i_\alpha(x; \bar{z})| \, dx \leq N_1.
\]

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Lemma 4.10 asserts that we may choose a constant \( N_2 = N_2(B, \mu) \) such that \( \| K_n f \|_{L^2} \leq N \| f \|_{L^2} \) for all \( f \in C_0^\infty(\mathbb{R}^{1+d}) \). We may then invoke Theorem 3.1 to produce a constant \( N = N(N_1, N_2, p) \) such that \( \| K_n f \|_{L^p} \leq N \| f \|_{L^p} \) for all \( f \in C_0^\infty(\mathbb{R}^{1+d}) \). The constants \( N_1 \) and \( N_2 \) only depend upon \( B \) and \( \mu \), so the constant \( N \) only depends upon \( B, \mu, \) and \( p \).

Rather than prove (4.12) directly, we will instead produce a constant \( N_1 = N_1(B, \mu) \), such that

\[
(4.13) \quad \sum_{i=-\infty}^{\infty} \int_{\rho(\bar{x}) \geq N_{1/2}} |h^i_\alpha(x; \bar{y}) - h^i_\alpha(x; 0)| \, d\bar{x} \leq N_1, \quad \text{when } \rho(\bar{y}) \leq 1.
\]

It is easy to check that (4.12) follows from (4.13) after a left-translation that moves \( \bar{z} \) to zero and a dilation that puts \( \rho(\bar{y}) \in (1/2, 1] \).

We will show that (4.13) holds by handling the terms where \( i \geq 0 \) and \( i < 0 \) separately. To handle the terms where \( i \geq 0 \), we invoke Lemma 4.6 to produce a function \( \hat{h} \) such that

\[
\int_E |h^i_\alpha(x; \bar{y}) - h^i_\alpha(x; 0)| \, d\bar{x} \leq 2^{-i}N_2 \rho(\bar{y}).
\]

In particular, we have

\[
(4.14) \quad \sum_{i=0}^{\infty} \int_E |h^i_\alpha(x; \bar{y}) - h^i_\alpha(x; 0)| \, d\bar{x} \leq 2N_2, \quad \text{when } \rho(\bar{y}) \leq 1.
\]

We now handle the terms where \( i < 0 \). The map \( (\bar{x}, \bar{z}) \mapsto \rho(\bar{z} \circ \bar{x}) \) is continuous and bounded on the compact set \( \{ (\bar{x}, \bar{z}) \in \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} : \rho(\bar{x}) \leq 1, \rho(\bar{z}) \leq 1 \} \), so we may choose \( N_3 = N_3(B) \) so large that \( \rho(\bar{z} \circ \bar{x}) \geq 1 \) when \( \rho(\bar{z} \circ \bar{x}) \geq N_3 \) and \( \rho(\bar{z}) \leq 1 \). We then choose \( \beta, \gamma \in \mathcal{A}(\mu) \) such that

\[
\int_{\rho(\bar{x}) \geq N_3} |h^i_\alpha(\bar{x}; \bar{z})| \, d\bar{z} = \int_{\rho(\bar{z} \circ \bar{x}) \geq N_3} |h^i_\beta(\bar{z}; 0)| \, d\bar{z}
\]

\[
\leq \int_{\rho(\bar{x}) \geq 1} |h^i_\beta(\bar{z}; 0)| \, d\bar{z} = \int_{\rho(\bar{z}) \geq 4^{-i}} |h^0_\gamma(\bar{z}; 0)| \, d\bar{z},
\]

when \( \rho(\bar{z}) \leq 1 \).

Now define the function \( \phi(x) = \sum_{i=1}^{\infty} \mathds{1}_{\{ \rho(\bar{x}) \geq 4^i \}} \) and observe that \( \phi \) has sublinear growth. We may then use Lemma 4.6 to produce a function \( \hat{h}_\mu : \mathbb{R}^d \to \mathbb{R}_+ \) such that \( N_4 = N_4(B, \mu) = \int_{\mathbb{R}^d} \phi(x) \hat{h}_\mu(x) \, dx < \infty \) and \( h_\alpha(s, x; 0, 0) \leq \hat{h}_\mu(x) \) for all \( \alpha \in \mathcal{A}(\mu) \) and \( |s| \in [1, 4] \). We then see that

\[
\sum_{i=-\infty}^{-1} \int_{\rho(\bar{x}) \geq N_3} |h^i_\alpha(\bar{x}; \bar{z})| \, d\bar{z} \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathds{1}_{|s| \in [1, 4]}(\hat{\phi}(s, x)) \hat{h}(x) \, dx \, ds \leq 6 N_4.
\]
when $\rho(\bar{z}) \leq 1$. In particular, we have

$$\sum_{i=-\infty}^{1} \int_{\rho(\bar{z}) \geq N} |h_{\alpha}^{i}(\bar{x}, \bar{y}) - h_{\alpha}^{i}(\bar{x}, 0)| \, d\bar{x} \leq 12 N_4, \quad \text{when } \rho(\bar{y}) \leq 1.$$ 

We have now shown that (4.13) holds, so the proof is complete. \( \square \)

The remaining case then follows easily by duality.

**Corollary 4.12.** Let $p \in (2, \infty)$. Then there exists a constant $N = N(B, \mu, p)$ such that $\|K_{\alpha} f\|_{L^p} \leq N \|f\|_{L^p}$ for all $\alpha \in \mathcal{A}(\mu)$, $j \in \mathbb{N}$, and $f \in C_{\mathcal{K}}(\mathbb{R}^{1+d})$.

**Proof.** Let $\alpha = (c, k, l, m) \in \mathcal{A}(\mu)$, set $\alpha^{*} = (c, k, l, 1 - m)$ and $q = p/(p - 1) \in (1, 2)$, and choose $N(B, \mu, q)$ as in Lemma 4.11 such that $\|K_{\alpha} g\|_{L^q} \leq N \|g\|_{L^q}$ for all $g \in C_{\mathcal{K}}(\mathbb{R}^{1+d})$. We then have

$$\int_{\mathbb{R}^{1+d}} K_{\alpha} f(\bar{x}) g(\bar{x}) \, d\bar{x} = \int_{\mathbb{R}^{1+d}} f(\bar{x}) K_{\alpha} g(\bar{x}) \, d\bar{x} \leq \|f\|_{L^p} \|K_{\alpha} g\|_{L^q} \leq N \|f\|_{L^p} \|g\|_{L^q}.$$ 

Taking the supremum over $g$ with $\|g\|_{L^q} \leq 1$, we see that $\|K_{\alpha} f\|_{L^p} \leq N \|f\|_{L^p}$. \( \square \)

The proof of Theorem 4.1 now follows in a few lines.

**Proof of Theorem 4.1.** Choose any $p \in (1, \infty)$, $1 \leq i, j \leq d_0$, and measurable $c : \mathbb{R} \to S_{d_0}^{d_0}$, and set $\alpha = (c, i, j, 0)$. Using Lemma 4.10, Lemma 4.11 or Corollary 4.12, we may find a constant $N = N(B, \mu, p)$ such that $\|K_{\alpha} f\|_{L^p} \leq \|f\|_{L^p}$ for all $\ell \in \mathbb{N}$ and $f \in C_{\mathcal{K}}(\mathbb{R}^{1+d})$. Theorem 3.8 asserts that the functions $K_{\alpha} f$ converge to $\partial_{ij} G_{c, B} f$ pointwise on $\mathbb{R}^{1+d}$ as $\ell \to \infty$, so we may invoke Fatou’s Lemma to conclude that $\|\partial_{ij} G_{c, B} f\|_{L^p} \leq \|f\|_{L^p}$ for all $f \in C_{\mathcal{K}}(\mathbb{R}^{1+d})$.

## 5 Uniqueness for the Martingale Problem

We now use the estimate obtained in the previous section to obtain uniqueness results for a class of degenerate martingale problems. Given a law $\mathbb{P}$ on $C(\mathbb{R}^{1+d})$, we will refer to the functionals $f \mapsto \mathbb{E}^{\mathbb{P}}[\int_{0}^{T} f(t, X_t) \, dt]$ informally as Green’s functionals. We will obtain an *a priori* estimate for the Green’s functionals associated with the solutions to martingale problems in a particular class. More specifically, we will show that they are bounded functionals on $L^p([0, T] \times \mathbb{R}^{d})$ for each $T$. Once this is done, we may use the estimates obtained in the previous section to obtain a local uniqueness result. We will then extend this uniqueness result using a localization procedure. Finally, we will relax the drift conditions by employing a second localization step.
In the previous section, it was convenient to work with the operator $G_{c,B}$ which operated on functions in $C_K([0,1]^d)$. We would now prefer to work with a restricted version of $G_{c,B}$ which operates on functions in $C_K([0,T] \times \mathbb{R}^d)$. Given a function $c : \mathbb{R}_+ \to \mathbb{R}^d$, we define

$$G_T^{c,B}f(s,x) = \int_0^T \int_{\mathbb{R}^d} p_{c,B}(s,t,y) f(t,y) \, dy \, dt, \quad (s,x) \in [0,T] \times \mathbb{R}^d.$$ 

We start by giving some estimates for the operator $G_T^{c,B}$. Recall that $d$ denotes the homogeneous dimension of the group associated with the matrix $B$.

**Lemma 5.1.** Let $c : \mathbb{R}_+ \to S^d_{\mu}$, and let $i,j \in \{1, \ldots, d_0\}$, $p \in (1, \infty)$, and $f \in C_K([0,T] \times \mathbb{R}^d)$. Then $\|\partial_{ij} G_T^{c,B}f\|_{L^p([0,T] \times \mathbb{R}^d)} \leq N(B, \mu, p) \|f\|_{L^p([0,T] \times \mathbb{R}^d)}$.

**Proof.** Let $\tilde{f} \in C_K^\infty(\mathbb{R}^{1+d})$ with $\tilde{f}(s,x) = f(s,x)$ when $s \in [0,T]$ and $\tilde{f}(s,x) = 0$ when $s \geq T$. The existence of such an extension is shown in [20]. Now let $\phi \in C^\infty(\mathbb{R}; [0,1])$ with $1_{[0,\infty)} \leq \phi \leq 1_{[-1,\infty)}$, let $N = N(B, \mu, p)$ denote the constant obtained in Theorem 4.11, and set $\tilde{f}_n(s,x) = \phi(ns) \tilde{f}(s,x)$. Then $G_T^{c,B}f(s,x) = G_T^{c,B}\tilde{f}_n(s,x)$ for all $s \in [0,T]$ and $n \in \mathbb{N}$, and

$$\|\partial_{ij} G_T^{c,B}f\|_{L^p([0,T] \times \mathbb{R}^d)} \leq \liminf_{n \to \infty} \|\partial_{ij} \tilde{G}_T^{c,B}\tilde{f}_n\|_{L^p([0,T] \times \mathbb{R}^d)} \leq \liminf_{n \to \infty} N \|\tilde{f}_n\|_{L^p([0,T] \times \mathbb{R}^d)} = N \|f\|_{L^p([0,T] \times \mathbb{R}^d)},$$

so the result follows. \qed

**Lemma 5.2.** Let $f \in C_K([0,T] \times \mathbb{R}^d)$, let $c : \mathbb{R}_+ \to S^d_{\mu}$ be measurable, and let $p \in (d/2, \infty)$. Then

$$\|G_T^{c,B}f\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leq N(B, \mu, p) T^{1-d/2p} \|f\|_{L^p([0,T] \times \mathbb{R}^d)}.$$ 

**Proof.** Let $\underline{c}(t) = I_{d_0}^0/\mu$ and $\underline{c}(t) = \mu I_{d_0}$, so $C_{\underline{c}}(s,t) \leq C_c(s,t) \leq C_{\overline{c}}(s,t)$ for all $t > s \geq 0$. The functions $\underline{c}$ and $\overline{c}$ are translation and dilation invariant, so we have (see 3.5)

$$\frac{\det C_c(s,t)}{\det C_{\overline{c}}(s,t)} = \frac{\det \{ \delta_{(t-s)}^{1/2} G_{\underline{c}}(0,1) \delta_{(t-s)}^{1/2} \}}{\det \{ \delta_{(t-s)}^{1/2} C_{\overline{c}}(0,1) \delta_{(t-s)}^{1/2} \}} = \frac{\det C_{\underline{c}}(0,1)}{\det C_{\overline{c}}(0,1)}.$$ 

In particular, if we set $N_1 = \det C_{\underline{c}}(0,1)/\det C_{\overline{c}}(0,1) > 0$, then we have $p_{c,B}(s,x;t,y) \leq N_1^{-1/2} p_{\overline{c}}(s,x;t,y)$.

Now, if $q \in (1, 1 + 2/(d-2))$ and $p = q/(q-1) > d/2$, then

$$\int_{\mathbb{R}^d} \{p_{c,B}(s,x;t,y)\}^q \, dy \leq N_1^{-q/2} \int_{\mathbb{R}^d} \{p_{\overline{c}}(0,0; t-s, z)\}^q \, dz 
= N_1^{-q/2}(t-s)^{-(q-1)(d-2)/2} \int_{\mathbb{R}^d} \{p_{\overline{c}}(0,0; 1, z)\}^q \, dz,$$
where the first inequality follows by left-translation (see (3.4)), the second equality follows by dilation (see (3.6)), and the last integral is finite. We now observe that \( q(1 - d/2p) = 1 - (q - 1)(d - 2)/2 \), so

\[
\int_s^T \int_{\mathbb{R}^d} \{p_{c,B}(s, x; u, z)\}^q dz \, du \leq \frac{N^{q/2}}{q(1 - d/2p)} \int_{\mathbb{R}^d} \{p_{c,0}(0,0, z)\}^q dz,
\]

and (5.1) follows by duality. \( \square \)

The next step is to show that the Green’s functionals can be expressed in terms of \( G_{c,B}^T \) and a stochastic correction term.

**Lemma 5.3.** Let \( a : \mathbb{R}_+ \times C(\mathbb{R}; \mathbb{R}^d) \rightarrow S^d_{\mu} \) be \( C^d \)-progressive, let \( c : \mathbb{R} \rightarrow S^d_{\mu} \) be measurable, let \( f \in C^\infty([0, T] \times \mathbb{R}^d) \). Also let \( \mathbb{P} \) denote a solution to the \((a(X), BX)\)-martingale problem starting at \((s, x) \in [0, T] \times \mathbb{R}^d\), and define the process \( \phi_t(a,c,B,f) = \frac{1}{2} \sum_{i,j=1}^{d_0} \{a^{ij} - c^iJ(t)\} \partial_{ij} G_{c,B}^T f(t, X_t) \) for \( t \in [s, T] \). Then

\[
\mathbb{E}^T \left[ \int_s^T f(t, X_t) \, dt \right] = G_{c,B}^T f(s, x) + \mathbb{E}^\phi \left[ \int_s^T \phi_t(a,c,B,f) \, dt \right].
\]

**Proof.** As \( G_{c,B}^T f(t, x) = 0 \) for \( t = T \), it is enough to show that

\[
M_t = G_{c,B}^T f(t, X_t) + \int_s^t f(u, X_u) - \phi_u(a,c,B,f) \, du, \quad t \in [s, T].
\]

is a martingale. Using Lemma 5.6 we may find a constant \( N = N(B, f) \) such that \( G_{c,B}^T f \leq N \) and \( \partial_{ij} G_{c,B}^T f \leq N \) on \([0, T] \times \mathbb{R}^d\) when \( 1 \leq i, j \leq d_0 \). Although Lemma 5.6 is stated in terms of \( G_{c,B} \), one may repeat the argument given in Lemma 5.1 to see that the conclusions also holds for \( G_{c,B}^T \). As a result, \( M \) is bounded and the bound does not depend upon \( c \). If \( c \in C^\infty(\mathbb{R}_+; S^d_{\mu}) \), then we also have \( \partial_{ij} G_{c,B}^T f + L^c G_{c,B}^T f = -f \) by the same lemma, and the result follows from Ito’s Lemma. To handle the general case, we choose a sequence \( c_n \in C^\infty(\mathbb{R}_+; S^d_{\mu}) \) with \( c_n \rightarrow c \) in \( L^p([0, T] \times \mathbb{R}^d) \), and we let \( M^n \) denote the process obtained by replacing \( c \) with \( c_n \) in (5.3). Then \( M^n \) is a uniformly bounded sequence of martingales that converges pointwise to \( M \), so we may conclude that \( M \) is a martingale. \( \square \)

We now produce the desired estimate for the Green’s functionals. We do this by imposing conditions which ensure that the stochastic correction term in the previous lemma is sufficiently small.

**Lemma 5.4.** Let \( a : \mathbb{R}_+ \times C(\mathbb{R}_+; \mathbb{R}^d) \rightarrow S^d_{\mu} \) be \( C^d \)-progressive, let \( c : \mathbb{R}_+ \rightarrow S^d_{\mu} \) be measurable, let \( p \in (d/2, \infty) \), and suppose that \( \mathbb{P} \) is a solution to the \((a(X), BX)\)-martingale problem starting at \((s, x) \). Then there exists constants \( N = N(d, \mu, B, p) \) and \( \varepsilon = \varepsilon(d, \mu, B, p) > 0 \) such that

\[
\mathbb{E}^\mathbb{P} \left[ \int_s^T |f(t, X_t)| \, dt \right] \leq NT^{1-d/2p} \|f\|_{L^p([0, T] \times \mathbb{R}^d)}.
\]

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for all $T \geq 0$ and $f \in C^\infty_{\text{per}}([0,T] \times \mathbb{R}^d)$ when

$$\|a(s,x) - c(s)\| \leq \varepsilon, \quad \text{for all } (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{5.5}$$

**Proof.** The proof of this lemma is somewhat involved and follows in the same way as Lemmas 7.1.2, 7.1.3, and 7.1.4 of [24], so we only recall the main ideas for the convenience of the reader. First we consider the case where $a$ is a simple process with respect to a deterministic time partition and (5.5) may not hold. In this case, one can show that (5.4) holds when $N$ is replaced by a constant $N_1$ which depends upon the number of points in the time partition. This is done by conditioning on the information available at the start of each time interval in the partition and applying the estimate (5.1) to the conditioned process.

We continue to consider the case where $a$ is a simple process, but we now produce a constant which does not depend upon the number of steps in the partition. Set $\varepsilon = \sup_{t,s,x} |a(t,s,x) - c(t)|$ and let $N_2 = N_2(\mathbb{P})$ denote the smallest constant such that (5.4) holds when $N$ is replaced by $N_2$. The previous step ensures that $N_2$ is finite. We will now show that $N_2$ is bounded by a constant which does not depend upon $\mathbb{P}$ when $\varepsilon$ is sufficiently small. First we use Lemma 5.2 and Lemma 5.3 to produce a constant $N_3 = N_3(B, \mu, p)$ such that

$$\mathbb{E}_\mathbb{P}\left[\int_0^T |f(t,X_t)| \, dt\right] \leq N_3 T^{1-d/2p} \|f\|_{L^p([0,T] \times \mathbb{R}^d)} + \frac{1}{2} \varepsilon T^{1-d/2p} \sum_{i,j=1}^d |\partial_{ij} G^T_{e,B} f\|_{L^p([0,T] \times \mathbb{R}^d)}.$$

We next apply Theorem 4.1 to produce a constant $N_4 = N_4(d, B, \mu, p)$, such that

$$\mathbb{E}_\mathbb{P}\left[\int_0^T |f(t,X_t)| \, dt\right] \leq (N_3 + \frac{1}{2} \varepsilon d^2 N_2 N_4) T^{1-d/2p} \|f\|_{L^p([0,T] \times \mathbb{R}^d)}$$

We then take the supremum over the set of functions $f \in C^\infty_{\text{per}}([0,T] \times \mathbb{R}^d)$ with $\|f\|_{L^p([0,T] \times \mathbb{R}^d)} \leq 1$ to see that $N_2 \leq 2N_3$ when $\varepsilon \leq 1/(d^2 N_4)$. This gives a bound which depends upon $B, \mu$, and $p$ but does not depend upon the number of time steps in the partition. The general case may then be handled by approximation. \hfill \square

**Remark 5.5.** When $d = d_0$, $a$ is of rank $d$ everywhere and much stronger results are available. Let $a : \mathbb{R}_+ \times \mathbb{R}^d \to S^d_0$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be bounded, measurable functions. Then Krylov [11] has shown that there exists a constant $N$ which depends upon $d, \mu, \|b\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)}$, and $T$ such that

$$\mathbb{E}_\mathbb{P}\left[\int_0^T |f(t,X_t)| \, dt\right] \leq N \|f\|_{L^{1+d}([0,T] \times \mathbb{R}^d)} \tag{5.6}$$
for any solution $P$ to the $(a, b)$-martingale problem. In particular, this result does not require $a$ to be well-approximated by a deterministic function of time. One consequence of this estimate is the existence of weak solutions to SDEs with measurable, uniformly positive-definite covariance. The reader may consult Theorem 2.3.4 and Theorem 2.6.1 of [12] for the proof of these results. Another consequence of the estimate (5.6) is that weak uniqueness holds for SDEs in which the covariance function is VMO-continuous in the spatial variables. The reader may consult Remark 2.2 of [14] for a brief discussion of this fact.

The estimate (5.6) is closely related the Aleksandrov-Bakelman-Pucci estimate of PDE theory and the parabolic extension due to Krylov and Tsu. These results depend in an essential way upon the geometry of convex functions in $\mathbb{R}^d$, and analogous results are not yet available for our geometric setting. One can consult Section 9 of [8] for a further discussion of these issues in the context of the Heisenberg group. The lack of such an estimate is the main impediment to obtaining a weak uniqueness result using the estimates obtained by Bramanti, Cerutti, and Manfredini in [4].

We are dealing with martingale problems where the drift is unbounded, but linear, so the next two lemmas will prove useful.

**Lemma 5.6.** Let $a^\alpha : \mathbb{R}_+ \times \Omega^\alpha \to S_+^d$ and $b^\alpha : \mathbb{R}_+ \times \Omega^\alpha \to \mathbb{R}^d$ be progressive processes, possibly defined on different spaces, and suppose that $X^\alpha$ is a continuous solution to the $(a^\alpha, b^\alpha)$-martingale problem for each $\alpha \in A$. Further suppose that the random variables $\{X^\alpha_0\}_{\alpha \in A}$ are uniformly bounded, and that there exists a constant $N$ such that

\[
\|a^\alpha_t\| + |b^\alpha_t|^2 \leq N \left(1 + \sup_{s \leq t} (X^\alpha_s)^2\right), \quad t \in \mathbb{R}_+, \alpha \in A.
\]

Then the collection of processes $\{X^\alpha\}_{\alpha \in A}$ is tight.

**Proof.** Let $T^\alpha_\infty$ denote the stopping time $T^\alpha_\infty = \inf\{t \in \mathbb{R}_+ : |X^\alpha_t| \geq t\}$, and set $X^{\alpha, n} = (X^\alpha)^{T^\alpha_\infty}$. The process $X^{\alpha, n}$ is a solution to the $(\mathbb{1}_{[0, T^\alpha_\infty]} a^\alpha, \mathbb{1}_{[0, T^\alpha_\infty]} b^\alpha)$-martingale problem, and the processes $\mathbb{1}_{[0, T^\alpha_\infty]} a^\alpha$ and $\mathbb{1}_{[0, T^\alpha_\infty]} b^\alpha$ are uniformly bounded with respect to $\alpha$, so we may conclude that the collection of processes $\{X^{\alpha, n}\}_{\alpha \in A}$ is tight for each fixed $n$.

It then follows from (5.7), the uniform boundedness of the random variables $\{X^\alpha_0\}_{\alpha \in A}$, the Burkholder-Davis-Gundy inequalities, Gronwall’s Lemma, and Chebyshev’s inequality that

\[
\lim_{n \to \infty} \sup_{\alpha \in A} P^\alpha[T^\alpha_n \leq t] = 0, \quad \text{for each } t \geq 0.
\]

The tightness of the collection $\{X^\alpha\}_{\alpha \in A}$ then follows from (5.8) and the tightness of $\{X^{\alpha, n}\}_{\alpha \in A}$ for each fixed $n$. □

**Lemma 5.7.** Let $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, let $a : \mathbb{R}_+ \times C(\mathbb{R}_+ \times \mathbb{R}^d) \to S_+^d$ and $b : \mathbb{R}_+ \times C(\mathbb{R}_+ \times \mathbb{R}^d) \to \mathbb{R}^d$ be $C^d$-progressive, and suppose that the maps $\omega \to a(t, \omega)$
and \( \omega \rightarrow b(t, \omega) \) are continuous for each fixed \( t \geq 0 \). Further suppose that there exists a constant \( N \) such that
\[
\|a_t\| + |b_t|^2 \leq N \left(1 + (X_t^*)^2\right), \quad t \in \mathbb{R}_+.
\]
Then there exists a solution to the \((a, b)\)-martingale problem starting at \((s, x)\).

**Proof.** When \( a \) and \( b \) are bounded, the result follows by approximating the processes \( a \) and \( b \) using an Euler-type scheme and checking that any weak limit point of the approximations is a solution to the desired martingale problem. The reader may consult Theorem 6.1.6 of \cite{24} for the details. The general case follows by truncating the coefficients and using the previous tightness result to find a limit point that solves the desired martingale problem. \( \square \)

We now use this existence result to show that the localized martingale problem is well-posed.

**Lemma 5.8.** Let \( a : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow S_{\mu}^{d_0} \), \( b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \), and \( c : \mathbb{R}_+ \rightarrow S_{\mu}^{d_0} \) be bounded, measurable functions, and suppose that \( b^i(t, x) = 0 \) for \( i > d_0 \). Then there exists a constant \( \varepsilon = \varepsilon(B, \mu) > 0 \) such that the \((a(X), b(X) + BX)\)-martingale problem is well-posed when
\[
(a(s, x) - c(s)) \leq \varepsilon, \quad \text{for all } (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d.
\]

**Proof.** As \( a \) is uniformly positive definite and \( b \) is bounded, we may use Girsanov’s Theorem to place solutions to the \((a(X), b(X) + BX)\)-martingale problem in one-to-one correspondence with solutions to the \((a(X), BX)\)-martingale problem. As a result, we may assume without loss of generality that \( b = 0 \). Now fix some \( p > d/2 \), and let \( N_1 = N(B, \mu, p) \) and \( \varepsilon_1 = \varepsilon_1(B, \mu, p) \) denote the constants obtained in Lemma 5.4.

We now show the existence of a solution for each initial condition when \( \varepsilon \leq \varepsilon_1 \). By mollification in the spacial directions, we may find a sequence of functions \( a_n : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow S_{\mu}^{d_0} \) such that \( \lim_{n \to \infty} \|a - a_n\|_{L^p([0,T] \times \mathbb{R}^d)} = 0 \) for each fixed \( T \); sup\((s,x)\) \( \mathbb{E}_{\mu} \|a_n(s, x) - c(s)\| \leq \varepsilon \); and the functions \( x \mapsto a(s, x) \) are continuous for each fixed \( s \geq 0 \). The existence of a solution \( \mathbb{P}_n \) to the \((a_n(X), BX)\)-martingale problem starting at \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \) then follows from Lemma 5.7. The collection \( \{\mathbb{P}_n\}_{n < \infty} \) is tight by Lemma 5.6 so may find a weak limit point \( \mathbb{P}_\infty \). The inequality (5.4) holds with \( \mathbb{P} \) replaced by \( \mathbb{P}_n \) for any \( n \), so it also holds with \( \mathbb{P} \) replaced by \( \mathbb{P}_\infty \). As a result, we have
\[
\mathbb{E}_{\mathbb{P}_n} \left[ \int_s^T \|a_n(u, X_u) - a_t(u, X_u)\| \, du \right] \leq NT^{1-d/2p} \|a_n - a_t\|_{L^p([0,T] \times \mathbb{R}^d)},
\]
for any \( \ell, m, n \in \mathbb{N} \cup \{\infty\} \) if we set \( a_\infty = a \). It then follows easily that \( \mathbb{P}_\infty \) is a solution to the \((a(X), BX)\)-martingale problem.
We now show that the solution to the \((a(X), BX)\)-martingale problem with initial condition \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) is unique. Choose and \(T > s\) and let \(K^T\) denote the operator

\[
K^T f(s, x) = \frac{1}{2} \sum_{i,j=1}^d \{a^{ij}(s, x) - c^{ij}(s)\} \partial_{ij} G^T_{c,B} f,
\]

for \(f \in C^\infty_K([0, T] \times \mathbb{R}^d)\). Now let \(U^T : L^p([0, T] \times \mathbb{R}^d) \to L^p([0, T] \times \mathbb{R}^d)\) denote the unique, bounded operator which agrees with \(K^T\) on \(C^\infty_K([0, T] \times \mathbb{R}^d)\), and let \(V^T_{s,x} : L^p([0, T] \times \mathbb{R}^d) \to \mathbb{R}\) denote the unique, bounded functional which agrees with the map \(f \mapsto G^T_{c,B} f(s, x)\) on \(C^\infty_K([0, T] \times \mathbb{R}^d)\). The existence and uniqueness of these extensions follows from Lemma 5.1 and Lemma 5.2. Moreover, it follows from Lemma 5.2, that we may choose \(\varepsilon_2 = \varepsilon_2(B, \mu, p) > 0\) so small that the operator \(I - U^T\) is invertible.

Set \(\varepsilon = \varepsilon_1 \wedge \varepsilon_2\) and assume \((5.9)\) holds. Now fix any \(f_\infty \in C^\infty_K([0, T] \times \mathbb{R}^d)\) and choose \(f_n \in C^\infty_K([0, T] \times \mathbb{R}^d)\) such that \(f_n\) converges to \((I - U^T)^{-1} f_\infty\) in \(L^p([0, T] \times \mathbb{R}^d)\). It follows from Lemma 5.3 that

\[
\mathbb{E}^P \left[ \int_s^T (I - K^T) f_n(t, X_t) \, dt \right] = G^T_{c,B} f_n(s, x),
\]

for each \(n\). We may then use Lemma 5.4 to pass to the limit in this expression and obtain

\[
(5.10) \quad \mathbb{E}^P \left[ \int_s^T f_\infty(t, X_t) \, dt \right] = V^T_{s,x}(I - U^T)^{-1} f_\infty.
\]

As \((5.10)\) holds for any solution to the \((a(X), BX)\)-martingale problem with initial condition \((s, x)\) and the right-hand side does not depend upon \(P\), we may conclude that the solution to the \((a(X), BX)\)-martingale problem starting at \((s, x)\) is unique (e.g. Cor. 6.2.5 of [24]).

To extend the local result to a global result, we use a localization procedure due to Stroock and Varadhan. The main technical difficulty that we encounter here is that the drift in our local solutions must be unbounded; we cannot truncate the map \(x \mapsto Bx\) without disturbing the geometrical structure. As a result, we need to extend the localization results provided by Stroock and Varadhan to allow for unbounded coefficients. We should also point out that when we make use this result later in Theorem 5.14 we will not have any uniform control on the growth of the local coefficients \((a_\alpha, b_\alpha)_{\alpha \in \mathcal{A}}\), so it is important that the following result only imposes conditions on the functions \(a\) and \(b\).

**Theorem 5.9.** Let \(a : \mathbb{R}_+ \times \mathbb{R}^d \to S^d_+\) and \(b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d\) be locally bounded, measurable functions, and suppose that there exists a constant \(N\) such that

\[
\|a(s, x)\| + |b(s, x)|^2 \leq N(1 + |x|^2), \quad \text{for all } (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d.
\]

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Let \( \{G_\alpha\}_{\alpha \in A} \) be an open cover of \( \mathbb{R}_+ \times \mathbb{R}^d \), and suppose that each open set in \( \{G_\alpha\}_{\alpha \in A} \) is associated with a pair of locally bounded, measurable functions \( a_\alpha : \mathbb{R}_+ \times \mathbb{R}^d \to S^d_+ \) and \( b_\alpha : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \) such that \( a_\alpha = a \) and \( b_\alpha = b \) on the set \( G_\alpha \), and the \((a, b)\)-martingale problem is well-posed. Then the \((a, b)\)-martingale problem is well-posed.

Proof. As \( \mathbb{R}_+ \times \mathbb{R}^d \) is \( \sigma \)-compact, we may always find a countable subcover, so we may assume that \( A = \mathbb{N} \). For notational convenience, we will show the existence and uniqueness of a solution to \((a, b)\)-martingale problem starting at \((0, 0) \in \mathbb{R}_+ \times \mathbb{R}^d \); however, it will be clear that the same argument works for any initial condition.

For each compact set \( K \subset \mathbb{R}_+ \times \mathbb{R}^d \), let \( n(K) \) denote the smallest natural number such that \( K \subset \bigcup_{i=1}^{n(K)} G_i \), and define

\[
\rho_0(K) = \inf_{\bar{x} \in K} \max_{i \leq n(K)} \text{dist}(\bar{x}, G_i^c),
\]

where \( G_i^c \) denotes the complement of the set \( G_i \). Observe that \( \rho_0(K) > 0 \) for each compact set \( K \). Now define

\[
(5.11) \quad \rho(t, r) = 1 \wedge \sup_{u \geq t, q \geq r} \rho_0([0, u] \times \mathbb{T}_q^t(0))/2, \quad t, r \geq 0.
\]

We introduce the supremum to ensure that the function \( \rho \) is nonincreasing in each coordinate. Finally, let \( \phi(s, x) \) denote the smallest natural number such that \( \text{dist}((s, x), G_{\phi(s, x)}^c) > \rho(s, |x|) \), so \( B_{\phi(s, x)}^{1+d}(s, x) \subset G_{\phi(s, x)} \) for all \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \). It not hard to check that \( \phi \) is finitely-valued and measurable.

We will proceed by patching together local solutions, so we will need notion for the concatenation of measures. The notation that we adopt here is a slight modification of the notation adopted in Chapter 6 of [24]. If \( \mathbb{P} \) is a probability measure on \( C(\mathbb{R}_+; \mathbb{R}^d) \), \( T \) is a \( \mathcal{C}^d \)-stopping time, \( Y \) is an \( \mathbb{N} \)-valued, \( \mathcal{C}^d \)-measurable random variable, and \( Q : \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{C}^d \to [0, 1] \) is a measurable probability kernel with

\[
(5.12) \quad Q_{s, x}^n[X_t = x, \ t \leq s] = 1, \quad n \in \mathbb{N}, \ (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d,
\]

then we let \( \mathbb{P} \otimes_T Y Q \) denote the probability measure on \( C(\mathbb{R}_+; \mathbb{R}^d) \) which is uniquely characterized by the following properties:

1. If \( f : C(\mathbb{R}_+; \mathbb{R}^d) \to \mathbb{R} \) is a bounded, measurable function, then we have
\[
\mathbb{E}^{\mathbb{P} \otimes_T Y Q}[f(X^T)] = \mathbb{E}^\mathbb{P}[f(X^T)].
\]

2. If \( f : C(\mathbb{R}_+; \mathbb{R}^d) \to \mathbb{R} \) is a bounded, measurable function, then the random variable
\[
(5.13) \quad \omega \to \mathbb{E}^{Q_{T(\omega), X_T(\omega)}}[f(X(\omega)1_{[0, T(\omega))] + X1_{[T(\omega), \infty))}]
\]

is a version of \( \mathbb{E}^{\mathbb{P} \otimes_T Y Q}[f(X) \mid \mathcal{C}_T] \).
Intuitively, $P \otimes_{T,Y} Q$ corresponds to the law of a process that begins evolving according to $P$ prior to time $T$. At time $T$, a new law is selected from the collection $\{Q^n\}$ using the random variables $T, X_T$, and $Y$, and the process then begins evolving in accordance with this new law. Observe that, as a consequence of (5.12), the process $X(\omega)1_{[0,T(\omega))} + X 1_{T(\omega), \infty})$ is $Q^Y(\omega)$ a.s. continuous, and the right-hand side of (5.13) is well-defined.

We now begin patching together measures to produce a solution to the $(a(X), b(X))$-martingale problem starting at $(0, 0)$. To do this, we inductively define the sequence of stopping times: $T_0 = 0$ and

$$T_n = \inf \{ t \geq T_{n-1} : t - T_{n-1} + |X_t - X_{T_{n-1}}| \geq \rho(T_{n-1}, |X_{T_{n-1}}|) \},$$

for $n \geq 1$. This is the hitting time of a closed set, so it is a $\mathcal{C}^d$-stopping time, even though $\mathcal{C}^d$ is not right continuous. We then define the $\mathcal{C}^d$-measurable random variables $Y_0 = \phi(0, 0)$ and $Y_n = \phi(T_n, X_{T_n})$ for $n \geq 1$. For each $n$, we may find a measurable kernel $Q^n$ such that $Q^n_{s,x}$ is the unique solution to the $(a_n(X), b_n(X))$-martingale problem starting at $(s, x)$. The measurability of the map $(s, x) \mapsto Q^n_{s,x}$ follows immediately from the fact that the $(a_n(X), b_n(X))$-martingale problem is well-posed. The reader may consult Exercise 6.7.4 of [24] for a proof of this fact. We now inductively define a sequence of probability measures: $P^0 = Q^\phi(0, 0)$, and $P^n = P^{n-1} \otimes_{T_n,Y_n} Q^n$.

We may characterize the measures $P^n$ as solutions to martingale problems. Define the processes

$$A^n_t = 1_{[0,T_{n+1})}a(t, X_t) + 1_{(T_{n+1}, \infty)}a^n(t, X),$$

$$B^n_t = 1_{[0,T_{n+1})}b(t, X_t) + 1_{(T_{n+1}, \infty)}b^n(t, X).$$

The pairs of functions $(a, b)$ and $(a^n, b^n)$ agree on the set $G^n(\omega)$ and $X_t(\omega) \in G^n(\omega)$ for all $t \in [T^n(\omega), T^{n+1}(\omega))$. It then follows as in the proof of Lemma 6.6.4 of [24] that $P^n$ is a solution to the $(A^n, B^n)$-martingale problem starting at $(0, 0)$.

We now show that

$$\lim_{n \to \infty} P^n[T_{n+1} \leq t] = 0, \quad \text{for each } t > 0. \tag{5.14}$$

Fix $t > 0$ and $\varepsilon > 0$ and set $X^n = X^{T^n}$. It follows from the previous characterization of $P^n$, that $X^{n+1}$ is a solution to the $(1_{[0,T_{n+1})}a(X^n), 1_{[0,T_{n+1})}b(X^n))$-martingale problem starting at $(0, 0)$ under $P^n$. It then follows from Lemma 5.6 that the sequence $\mathcal{L}(X^{n+1} \mid P^n)$ is tight. In particular, we may choose $M$ and then $\delta > 0$ such that

$$E^n[X^{n+1}]_\delta \geq M \leq \varepsilon/2, \quad \text{for all } n \in \mathbb{N}, \tag{5.15}$$

$$E^n[m^\delta(X^{n+1}) \geq \rho(t, M)/2] \leq \varepsilon/2, \quad \text{for all } n \in \mathbb{N}, \tag{5.16}$$

where $m^\delta_t$ denotes the modulus of continuity

$$m^\delta_t(\omega) = \sup_{0 \leq r \leq r(\delta) \wedge t} |\omega(s) - \omega(r)|, \quad \omega \in C(\mathbb{R}_+; \mathbb{R}^d). \tag{5.17}$$
We now show that
\[
\{T_n \leq t\} \cap \{X_{T_n}^* < M\} \cap \{m_i^n(X^n) < \rho(t, M)/2\} \subseteq \{T_n \geq n(\delta \wedge \rho(t, M)/2)\}.
\]

If we suppose that \(X_{T_n}^* < M\), then the process \(X\) remains in the ball \(B_d^0(0)\) during \([0, T_n]\). If we further suppose that \(T_n \leq t\), then we must have \(T_m - T_{m-1} + |X_{T_m} - X_{T_{m-1}}| \geq \rho(t, M)\) for each \(m \leq n\) because \(\rho\) is nonincreasing. In particular, we must have \(T_m - T_{m-1} \geq \rho(t, M)/2\) or \(|X_{T_m} - X_{T_{m-1}}| \geq \rho(t, M)/2\). But if \(m_i^n(X^n) < \rho(t, M)/2\), then it takes the process \(X^n\) at least \(\delta\) units of time to move by the amount \(\rho(t, M)/2\), so we must have \(T_m - T_{m-1} > \delta\) when \(|X_{T_m} - X_{T_{m-1}}| \geq \rho(t, M)/2\). Either way, we have \(T_m - T_{m-1} > \delta \wedge \rho(t, M)/2\) for all \(m \leq n\) and summing over \(1 \leq m \leq n\) gives (5.15).

It then follows immediately from (5.18), that
\[
\{(X^n)^*_t < M\} \cap \{m_i^n(X^n) < \rho(M)\} \subseteq \{T_n \geq n(\delta \wedge \rho(t, M)/2)\} \cup \{T_n > t\}.
\]
If we choose \(n\) so large that \((n+1)(\delta \wedge \rho(t, M)/2) > t\), then we may apply (5.15) and (5.16) to conclude that \(\mathbb{P}^n[T_{n+1} \leq t] \leq \varepsilon\). We have now shown that (5.14) holds.

We are now essentially done. It follows from (5.14) that there exists a unique probability measure \(\mathbb{P}\) which agrees with \(\mathbb{P}_n\) on \(\mathcal{G}_{T_{n+1}}\) for all \(n\) (e.g. [24, Theorem 1.3.5]). Moreover, we have shown that \(X^{n+1}\) is a solution to the \((1_{[0,T_{n+1}]}a(X^n), 1_{[0,T_{n+1}]}b(X^n))\)-martingale problem starting at \((0, 0)\) under \(\mathbb{P}\) for all \(n\), and \(T_n \rightarrow \infty\), \(\mathbb{P}\)-a.s., so we may conclude that \(X\) is a solution to the \((a(X), b(X))\)-martingale problem starting at \((0, 0)\) under \(\mathbb{P}\). Finally, if \(\hat{\mathbb{P}}\) is another solution to the \((a(X), b(X))\)-martingale problem starting at \((0, 0)\) then it follows as in Lemma 6.6.4 in [24] that \(\hat{\mathbb{P}}\) must agree with \(\mathbb{P}\) on \(\mathcal{G}_{T_{n+1}}\) for each \(n\). As \(\mathbb{P}\) is the unique measure with this property, we must have \(\hat{\mathbb{P}} = \mathbb{P}\). \(\square\)

We now use this localization result to produce a global uniqueness result.

**Theorem 5.10.** Let \(a : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow S^d_+\) and \(b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) be measurable functions and suppose that
\[
\inf_{s \in [0,T]} \inf_{\theta \in \mathbb{R}^d \setminus \{0\}} \|a(s, x)\theta - a(s, \theta)\|^2 > 0, \quad \text{for all } T > 0, x \in \mathbb{R}^d,
\]
\[
\lim_{y \to x} \sup_{s \in [0,T]} \|a(s, y) - a(s, x)\| = 0, \quad \text{for all } T > 0, x \in \mathbb{R}^d,
\]
\[
b^i(s, x) = 0, \quad d_0 < i \leq d, (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d.
\]

Further suppose that there exists a constant \(N\) such that
\[
\|a(s, x)\| + |b(s, x)|^2 \leq N(1 + |x|^2), \quad \text{for all } (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d.
\]

Then the \((a(X), b(X) + BX)\)-martingale problem is well-posed.
Proof. Let $\varepsilon(\mu) = \varepsilon(B, \mu) > 0$ denote the constant obtained in Lemma 5.8 as a function of $\mu$. For each point $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we may choose $\mu(s, x) > 0$ such that $a(t, x) \in S_{\mu(s, x)}^{d_0}$ when $|t - s| \leq 1$. This follows from (5.19) and (5.22).

We may then use (5.20) to choose $r(s, x) \in (0, 1)$ such that $a(t, y) \in S_{\mu(s, x)}^{d_0}$ and $|a(t, y) - a(t, x)| \leq \varepsilon(\mu(s, x))$ when $(t, y) = (s - r(s, x), s + r(s, x)) \times B_{r(s, x)}^d(x)$.

We now let $\pi_{s, x}$ denote the unique projection onto the closed, convex set $[s - r(s, x)/2, s + r(s, x)/2] \times B_{r(s, x)/2}^d(x)$, and we define the functions $a_{s, x}(t, y) = a \circ \pi_{s, x}(t, y)$, $c_{s, x}(t) = a\{s - r(s, x)/2\} \vee\{s + r(s, x)/2\}$, and $b_{s, x}(t, y) = b(t, y) \circ \pi_{s, x}(t, y)$. We then see that $a_{s, x} : \mathbb{R}_+ \times \mathbb{R}^d \to S_{\mu(s, x)}^{d_0}$, $c_{s, x} : \mathbb{R}_+ \to S_{\mu(s, x)}^{d_0}$, and $b_{s, x}$ are bounded; $b_{s, x}^i = 0$ for $d_0 < i \leq d$; and $|a_{s, x}(t, y) - c_{s, x}(t)| \leq \varepsilon(\mu(s, x))$ for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^d$, so we may invoke Theorem 5.8 to conclude that the $(a_{s, x}(X), b_{s, x}(X) + BX)$-martingale problem is well-posed. We then define the open cover $\{G_{s, x} \}_{(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d}$ where $G_{s, x} = (s - r(s, x)/2, s + r(s, x)/2) \times B_{r(s, x)/2}^d(x)$, and apply the previous localization result to conclude that the $(a, b)$-martingale problem is well-posed. \hfill \Box

The results provided up to this point provide very little flexibility regarding the drift of the last $d - d_0$ components of the process: The drift must take the form $b''(t, x) = B''x$ where $B''$ denotes the last $d - d_0$ rows of a matrix $B$ which satisfies the structure conditions given in Section 2. As a result, we cannot apply the previous theorem to an SDE of the form:

\begin{equation}
\begin{aligned}
&dx_t = \sigma(t, X_t, Y_t) \, dW_t, \\
&dy_t = (X_t + Y_t) \, dt.
\end{aligned}
\end{equation}

However, if we define the process $Z = X + Y$, then we see that the process $(Z, Y)$ solves the SDE:

\begin{equation}
\begin{aligned}
&dz_t = Z_t \, dt + \tilde{\sigma}(t, Z_t, Y_t) \, dW_t, \\
&dy_t = Z_t \, dt,
\end{aligned}
\end{equation}

where $\tilde{\sigma}(t, z, y) = \sigma(t, z - y, y)$. Written in this form, the drift now satisfies the required structure condition with $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Moreover, the set of weak solutions to the SDE (5.22) are in one-to-one correspondence with the set of weak solutions to the SDE (5.24). In particular, if $\tilde{\sigma}$ satisfies conditions which ensure existence and uniqueness in law for the SDE (5.24), then existence and uniqueness in law also hold for the SDE (5.22). We now generalize this observation slightly to obtain a second local uniqueness result. Recall that $Db''$ denotes the $d_1 \times (1 + d)$ Jacobian matrix of the function $b''$ when $b''$ takes values in $\mathbb{R}^{d_1}$.

**Lemma 5.11.** Suppose that $d = d_0 + d_1$ with $d_1 \leq d_0$. Let $a : \mathbb{R}_+ \times \mathbb{R}^d \to S_{\mu}^{d_0}$, $b' : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d_0}$, and $c : \mathbb{R}_+ \to S_{\mu}^{d_0}$ be bounded, measurable functions, let $b'' \in C^{1,2,1}(\mathbb{R}_+ \times \mathbb{R}^{d_0} \times \mathbb{R}^{d_1} ; \mathbb{R}^{d_1})$ denote a function with bounded derivatives,
and let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ denote the function $b(s, x) = (b'(s, x), b''(s, x))$. Let $A_0 \in \mathbb{M}^{d_1 \times 1}, A_1 \in \mathbb{M}^{d_1 \times d_0}$ and $A_2 \in \mathbb{M}^{d_1 \times d_1}$ be matrices, and suppose that $A_1$ is of rank $d_1$. Finally, let $A = [A_0 \ A_1 \ A_2] \in \mathbb{M}^{d_1 \times (1+d)}$ denote the matrix obtained by appending the columns of $A_1$ and $A_2$ to $A_0$. Then there exists a constant $\varepsilon = \varepsilon(A, \mu) > 0$ such that there is at most one solution to the $(a(X), b(X))$-martingale problem starting at $(s, x)$ when

$$(5.25) \quad \|a(t, y) - c(t)\| + \|Db''(t, y) - A\| \leq \varepsilon, \quad \text{for all } (t, y) \in \mathbb{R}_+ \times \mathbb{R}^d.$$ \hspace{1cm} \text{Proof.} As in the proof of Lemma 5.8 we may assume that $b' = 0$. We first show that the result holds when $d_1 = d_0$. Until further notice, we will write a generic point $x \in \mathbb{R}^d$ in the form $x = (x', x'') \in \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}$, and we will let $\pi''$ denote the projection $\pi''(x', x'') = x''$. Define

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ A_0 & A_1 & A_2 \end{bmatrix} \in M^{(1+d)},$$

$$\hat{B} = \begin{bmatrix} 0 & 0 \\ I_{d_0} & 0 \end{bmatrix} \in M^d,$$

where 0 denotes a matrix of zeros whose dimensions vary at each occurrence, and set $\hat{\mu} = 4\mu$. The assumption that $A_1$ is of full rank ensures that $\hat{A}$ is invertible and the matrix $\hat{B}$ clearly satisfies the structure conditions given in Section 2. We then let $\varepsilon_1 = \varepsilon_1(\hat{B}, \hat{\mu})$ denote the constant obtained in Lemma 5.8 and set

$$\varepsilon = \min \left\{ \|A_1\|, \frac{1}{2}\|A^{-1}\|, \frac{\varepsilon_1}{6(1+\mu)\|A_1\|(1+\|A_1\|)} \right\}.$$ \hspace{1cm} \text{Let } \hat{\alpha} : \mathbb{R}_+ \times \mathbb{R}^d \to S_{\hat{A}^{-1}}^{d_0} \text{ and } \hat{\beta} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \text{ denote the functions}

$$\hat{\alpha}(s, x) = D_x b''(s, x) a(s, x) D_x^T b''(s, x),$$

$$\hat{\beta} = \left\{ \begin{array}{cl} \{ \partial_s b'' + \frac{1}{2} \text{tr}(D_x D_x^T b'' a) \} & \text{if } 1 \leq i \leq d_1 \\
\{ D_{x^{(i)}} b'' + (D_{x^{(i)}} b'', b'') \} & \text{if } 1 \leq i \leq d_1 \\
\{ 0 \} & \text{otherwise} \end{array} \right.$$ \hspace{1cm} \text{where the arguments of } \hat{\beta} \text{ have been suppressed. The derivatives of } b'' \text{ are bounded, so } \hat{\beta} \text{ satisfies a linear growth condition. Finally, let } \hat{c} : \mathbb{R}_+ \to S_{\hat{A}^{-1}}^{d_0} \text{ denote the function } \hat{c}(t) = A_1 c(t) A_1^T.$$ \hspace{1cm} \text{We now assume that } (5.25) \text{ holds and show that } \hat{\alpha} \text{ and } \hat{\beta} \text{ take values in } S_{\hat{A}^{-1}}^{d_0}. \text{ If } z \in \mathbb{R}^{d_0} \text{ and } (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \text{ then}

$$\frac{|z|}{\|A_1^{-1}\|} \leq |A_1^T z| \leq |D_x^T b''(s, x) z| + \|D_x^T b''(s, x) - A_1^T\| |z|. \hspace{1cm} \text{It then follows from our choice of } \varepsilon \text{ that}

$$|z|/(2\|A_1^{-1}\|) \leq |D_x^T b''(t, x) z| \leq 2\|A_1\| |z|. \hspace{1cm} \text{32}$$
Moreover, as \( \langle a(s, x)z, z \rangle \in [\mu^{-1}|z|^2, \mu|z|^2] \) and \( \langle c(s)z, z \rangle \in [\mu^{-1}|z|^2, \mu|z|^2] \) by assumption, we see that \( \hat{a} \) and \( \hat{c} \) take values in \( S_{\hat{\mu}}^{\hat{a}} \). We also have

\[
\|\hat{a}(s, x) - \hat{c}(s)\| \leq \|Dv' b''(s, x)\| \|a(s, x)\| \|Dv' b''(s, x) - A_1\|
+ \|Dv' b''(s, x)\| \|a(s, x) - c(s)\| \|A_1\|
+ \|Dv' b''(s, x) - A_1\| \|c(s)\| \|A_1\|
\leq 6(1 + \mu) \|A_1\| (1 + \|A_1\|) \varepsilon,
\]

so \( \|\hat{a}(s, x) - \hat{c}(s)\| \leq \varepsilon_1 \) for all \( (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \).

Now let \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) denote two solutions to the \((a(X), b(X))\)-martingale problem starting at \((s, x) = (s, x', x'')\), let \( f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+ \times \mathbb{R}^d \) denote the function \( f(s, x', x'') = (s, b''(s, x', x''), x'') \), and let \( Y : \mathbb{R}_+ \times C(\mathbb{R}_+; \mathbb{R}^d) \to \mathbb{R}^d \) denote the process

\[
Y_t = 1_{\{t < s\}} (b''(s, x), x'') + 1_{\{t \geq s\}} (b''(t, X_t), \pi''(X_t)).
\]

We now show that \( f \) maps \( \mathbb{R}_+ \times \mathbb{R}^d \) onto itself and admits a continuous inverse. To see this, fix any \((t_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}^d\) and define the function \( \phi_{t_0, y_0} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+ \times \mathbb{R}^d \) that removes these extra coordinates. Let \( \tilde{A} \) denote the matrix such that \( \tilde{A}_{ij} = A_{ij} \) if \( 1 \leq i, j \leq d \) and \( \tilde{A}_{ij} = 0 \) otherwise, and set \( \hat{A} = [\hat{A}_0 \hat{A}_1 \hat{A}_2] \). Finally, let \( \varepsilon = \varepsilon(\hat{A}, \hat{\mu}) \) denote the constant that was obtained in the previous case.

For the remainder of the lemma, we will write a generic point in \( x \in \mathbb{R}^{2d_0} \) in the form \( x = (x', x'') \in \mathbb{R}^d \times \mathbb{R}^{d_0 - d_1} \), so \( x'' \) denotes the extra coordinates that we are adding to place ourselves in the previous case. We will let \( \pi \) denote the projection \( \pi(s, x', x'') = (s, x') \) that removes these extra coordinates. Let \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) denote solutions to the \((a(X), b(X))\)-martingale problem starting at \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \) and assume that \( \text{[5.25]} \) holds. Set \( \tilde{a} = a \circ \pi \), set \( \tilde{b}' = b' \circ \pi \), let
The function \( \tilde{b}'' : \mathbb{R}_+ \times \mathbb{R}^{2d_0} \to \mathbb{R}^{d_0} \) denotes the function

\[
\tilde{b}''(s, x) = \mathbb{I}_{\{1 \leq i \leq d_0\}} b''(s, \pi(x)) + \mathbb{I}_{\{d_1 < i \leq d_0\}} \sum_{j=1}^{d_0} \tilde{A}^{i}_{j} x^j, \quad 1 \leq i \leq d_0,
\]

and let \( \tilde{b} : \mathbb{R}_+ \times \mathbb{R}^{2d_0} \to \mathbb{R}^{2d_0} \) denote the function \( \tilde{b}(s, x) = (\tilde{b}'(s, x), \tilde{b}''(s, x)) \).

Now let \( Z : \mathbb{R}_+ \times C(\mathbb{R}_+ ; \mathbb{R}^d) \to \mathbb{R}^{2d_0} \) denote the process

\[
Z^i = \mathbb{I}_{\{1 \leq i \leq d_0\}} X^i_t + \mathbb{I}_{\{d_1 < i \leq 2d_0\}} \sum_{j=1}^{d_0} \int_s^{\sqrt{t}} \tilde{A}^{(i-d_0), j} X^j_u du, \quad 1 \leq i \leq 2d_0.
\]

It's not hard to check that \( Z \) is a solution to \((\tilde{a}(Z), \tilde{b}(Z))\)-martingale problem starting from \((s, x, 0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d_0-d_1} \) under both \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \). It is also easy to see that \((5.25)\) implies

\[
\|\tilde{a}(t, x) - \tilde{c}(t)\| + \|D\tilde{b}''(t, x) - \tilde{A}\| \leq \varepsilon, \quad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^{2d_0}.
\]

As a result, we may apply the previous case to conclude that \( Z \) has the same law under \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \), but then \( \mathbb{P}_1 = \mathbb{P}_2 \).

We now give a simple approximation lemma that we will need to extend the previous result.

**Lemma 5.12.** Let \( f \in C^1(\mathbb{R}^d) \) and fix any \( x \in \mathbb{R}^d \) and \( \varepsilon > 0 \). Then there exists a constant \( r > 0 \) and a function \( g \in C^1(\mathbb{R}^d) \) such that \( g = f \) on \( B_r(x) \) and \( \|Dg(y) - DF(x)\| \leq \varepsilon \) for all \( y \in \mathbb{R}^d \).

**Proof.** Let \( \eta \in C^\infty_K(\mathbb{R} ; [0, 1]) \) denote a function with \( \eta(z) = 1 \) when \( |z| \leq 1 \), \( \eta(z) = 0 \) when \( |z| \geq 3 \), and \( \left| \eta'(z) \right| \leq 1 \) everywhere. By translation and the addition of an affine function, we may assume that \( x = DF(0) = 0 \) and \( f(0) = 0 \). We now choose \( r \) such that \( \|f(x)/|x|\| \leq \varepsilon / 64\sqrt{d} \) and \( \|DF(x)\| \leq \varepsilon / 2\sqrt{d} \) when \( |x| \leq 4r \), and set \( g(x) = \eta(|x|^2/r^2)f(x) \). It is clear that \( g(x) = f(x) \) on \( B_r(x) \) and \( Dg(x) = 0 \) when \( |x| \geq 4r \). But when \( |x| \leq 4r \), we have

\[
|\partial_i g(x)| \leq 2|x||f(x)/r^2 + |\partial_i f(x)| < \varepsilon / \sqrt{d},
\]

so \( g \) possesses the desired properties.

**Corollary 5.13.** Let \( f \in C^{1,2,1}(\mathbb{R}_+ \times \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}) \), \( (s, x) \in \mathbb{R}_+ \times \mathbb{R}^{d_0+d_1} \), and let \( \varepsilon > 0 \). Then there exists a function \( g \in C^{1,2,1}(\mathbb{R}_+ \times \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}) \) with bounded derivatives and a constant \( r > 0 \) such that

\[
g(t, y) = f(t, y), \quad \text{for all } (t, y) \in (s - r, s + r) \times B^{d_0+d_1}_r(x),
\]

\[
|Dg(t, y) - DF(s, x)| \leq \varepsilon, \quad \text{for all } (t, y) \in \mathbb{R}_+ \times \mathbb{R}^{d_0+d_1}.
\]
Proof. We may construct a function \( \hat{f} \in C^{1,2,1}(\mathbb{R} \times \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}) \) with \( \hat{f}(t,y) = f(t,y) \) when \( t \geq 0 \) (e.g. [9] 6.37), so the corollary follows immediately from proof of the previous lemma. \( \square \)

We now have all the tools that we need for the final result of the paper.

**Theorem 5.14.** Let \( d = d_0 + d_1 \) with \( d_1 \leq d_0 \) and let \( x = (x', x'') \in \mathbb{R}^{d_0} \times \mathbb{R}^{d_1} \) denote the generic point in \( \mathbb{R}^d \). Let \( a: \mathbb{R}_+ \times \mathbb{R}^d \to S_{d_0}^d \) and \( b': \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d_0} \) be measurable functions, let \( b'' \in C^{1,2,1}(\mathbb{R}_+ \times \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}; \mathbb{R}^{d_1}) \), and let \( b: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \) denote the function \( b(s,x) = (b'(s,x), b''(s,x)) \). Suppose that

\[
\inf_{s \in [0,T]} \inf_{\theta \in \mathbb{R}^{d_1}} |\langle a(s,x)\theta, \theta \rangle|/|\theta|^2 > 0, \quad \text{for all } T > 0, x \in \mathbb{R}^d, \tag{5.26}
\]

\[
\lim_{y \to x} \sup_{s \in [0,T]} \| a(s,y) - a(s,x) \| = 0, \quad \text{for all } T > 0, x \in \mathbb{R}^d. \tag{5.27}
\]

Further suppose that there exists a constant \( N \) such that

\[
\|a(s,x)\| + |b(s,x)|^2 \leq N(1 + |x|^2), \quad \text{for all } (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{5.28}
\]

and that

\[
D_x b''(s,x) \text{ is of rank } d_1 \text{ for all } (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{5.29}
\]

Then the \( (a(X), b(X)) \)-martingale problem is well-posed.

Proof. For each point \( (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d \), we may again choose \( \mu(s,x) > 0 \) such that \( a(t,x) \in S_{2\mu(s,x)}^d \) when \( |t-s| \leq 1 \). Now let \( \varepsilon(A, \mu) > 0 \) denote the constant obtained in Lemma 5.11 and set \( \delta(s,x) = \varepsilon(Db''(s,x), \mu(s,x))/2 \). It follows from Corollary 5.13 and condition 5.27 that we may find a function \( b''_{s,x} \in C^{1,2,1}(\mathbb{R}_+ \times \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}; \mathbb{R}^{d_1}) \) with bounded derivatives and a radius \( r(s,x) > 0 \) such that:

- \( a(t,y) \in S_{\mu(s,x)}^d, \|a(t,y) - a(t,x)\| \leq \delta(s,x), \text{ and } b''_{s,x}(t,y) = b''(t,y) \) when \((t,y) \in (s-r, s+r) \times B^d_{r/2}(x), \) and
- \( \|Db''_{s,x}(t,y) - Db''(s,x)\| \leq \delta(s,x) \) for all \( (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d \).

We now set \( G_{s,x} = (s-r(s,x)/2, s+r(s,x)/2) \times B_{r(s,x)/2}(x) \), let \( \pi_{s,x} \) denote the unique projection onto the closure of \( G_{s,x} \), and define the functions

\[
a_{s,x}(t,y) = a \circ \pi_{s,x}(t,y),
\]

\[
c_{s,x}(t) = a\{(s-r(s,x)/2) \vee t \land (s+r(s,x)/2), x\},
\]

\[
b'_{s,x}(t,y) = b'(t,y) \circ \pi_{s,x}(t,y),
\]

\[
b_{s,x}(t,y) = (b'_{s,x}(t,y), b''_{s,x}(t,y)).
\]

The functions \( a_{s,x} \) and \( b'_{s,x} \) are continuous in the spatial variable and satisfy a linear growth condition, the function \( a_{s,x} \) is uniformly positive-definite, and
the function $b'_{s,x}$ is bounded. As a result, the existence of a solution to the $(a_{s,x}, b_{s,x})$-martingale problem for each initial condition follows from Lemma 5.6 and Girsanov’s Theorem.

The functions $a_{s,x}$ and $c_{s,x}$ take values in $S^d_{\mu(s,x)}$ and

$$\|a_{s,x}(t, y) - c_{s,x}(t)\| + \|Db''_{s,x}(t, y) - Db''(s, x)\| \leq \varepsilon(Db''(s, x), \mu(s, x)),$$

for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^d$. It then follows from Lemma 5.11 that the $(a_{s,x}, b_{s,x})$-martingale problem is well-posed. Finally, we have $a_{s,x} = a$ and $b_{s,x} = b$ on $G_{s,x}$ and $\{G_{s,x}\}_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ is an open cover of $\mathbb{R}_+ \times \mathbb{R}^d$, so we may invoke Theorem 5.9 to conclude that that the $(a, b)$-martingale problem is well-posed.

\[\square\]

References

[1] R. F. Bass and É. Pardoux. Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields*, 76(4):557–572, 1987.

[2] M. Bramanti and M. C. Cerutti. $W^{1,2}_p$ solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients. *Comm. Partial Differential Equations*, 18(9-10):1735–1763, 1993.

[3] M. Bramanti and M. C. Cerutti. Commutators of singular integrals on homogeneous spaces. *Boll. Un. Mat. Ital. B (7)*, 10(4):843–883, 1996.

[4] M. Bramanti, M. C. Cerutti, and M. Manfredini. $L^p$ estimates for some ultrasymmetric operators with discontinuous coefficients. *J. Math. Anal. Appl.*, 200(2):332–354, 1996.

[5] M. C. Cerutti, L. Escauriaza, and E. B. Fabes. Uniqueness for some diffusions with discontinuous coefficients. *Ann. Probab.*, 19(2):525–537, 1991.

[6] A. Figalli. Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal.*, 254(1):109–153, 2008.

[7] P. Gao. The martingale problem for a differential operator with piecewise continuous coefficients. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 135–141. Birkhäuser Boston, Boston, MA, 1993.

[8] N. Garofalo and F. Tournier. New properties of convex functions in the Heisenberg group. *Trans. Amer. Math. Soc.*, 358(5):2011–2055 (electronic), 2006.

[9] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second orders*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
[10] N. V. Krylov. A diffusion on the plane with reflection. I. Construction of the process. Sibirsk. Mat. Z., 10:343–354, 1969.

[11] N. V. Krylov. A certain estimate from the theory of stochastic integrals. Teor. Veroyatnost. i Primenen., 16:446–457, 1971.

[12] N. V. Krylov. Controlled diffusion processes, volume 14 of Applications of Mathematics. Springer-Verlag, New York, 1980. Translated from the Russian by A. B. Aries.

[13] N. V. Krylov. On weak uniqueness for some diffusions with discontinuous coefficients. Stochastic Process. Appl., 113(1):37–64, 2004.

[14] N. V. Krylov. Parabolic and elliptic equations with VMO coefficients. Comm. Partial Differential Equations, 32(1-3):453–475, 2007.

[15] E. Lanconelli, A. Pascucci, and S. Polidoro. Linear and nonlinear ultra-parabolic equations of Kolmogorov type arising in diffusion theory and in finance. In Nonlinear problems in mathematical physics and related topics, II, volume 2 of Int. Math. Ser. (N. Y.), pages 243–265. Kluwer/Plenum, New York, 2002.

[16] E. Lanconelli and S. Polidoro. On a class of hypoelliptic evolution operators. Rend. Sem. Mat. Univ. Politec. Torino, 52(1):29–63, 1994. Partial differential equations, II (Turin, 1993).

[17] C. Le Bris and P.-L. Lions. Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients. Comm. Partial Differential Equations, 33(7-9):1272–1317, 2008.

[18] N. Nadirashvili. Nonuniqueness in the martingale problem and the Dirichlet problem for uniformly elliptic operators. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24(3):537–549, 1997.

[19] M. V. Safonov. On a weak uniqueness for some elliptic equations. Comm. Partial Differential Equations, 19(5-6):943–957, 1994.

[20] R. T. Seeley. Extension of $C^\infty$ functions defined in a half space. Proc. Amer. Math. Soc., 15:625–626, 1964.

[21] E. M. Stein. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, volume 43 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[22] D. W. Stroock and S. R. S. Varadhan. Diffusion processes with continuous coefficients. I. Comm. Pure Appl. Math., 22:345–400, 1969.

[23] D. W. Stroock and S. R. S. Varadhan. Diffusion processes with continuous coefficients. II. Comm. Pure Appl. Math., 22:479–530, 1969.
[24] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.