ON HECKE EIGENVALUES OF CUSP FORMS IN ALMOST ALL SHORT INTERVALS

JISEONG KIM

Abstract. Let $\psi$ be a function such that $\psi(x) \to \infty$ as $x \to \infty$. Let $\lambda_f(n)$ be the $n$-th Hecke eigenvalue of a fixed holomorphic cusp form $f$ for $SL(2, \mathbb{Z})$. We show that for any real valued function $h(x)$ such that $(\log X)^{2-2\alpha} \ll h(X) = o(X)$,

$$\sum_{x+h(X)}^{x+h(X)} |\lambda_f(n)| \ll h(X)\psi(X)(\log X)^{\alpha-1}$$

for all but $O_f(X\psi(X)^{-2})$ many integers $x \in [X, 2X - h(X)]$, in which $\alpha$ is the average value of $|\lambda_f(p)|$ over primes. We generalize this for $|\lambda_f(n)|^{2k}$ for $k \in \mathbb{Z}^+$.

1. Introduction

Let $f(z)$ be a holomorphic Hecke cusp form of even integral weight $k$ for the full modular group $SL(2, \mathbb{Z})$. Let $e(z) = e^{2\pi iz}$. It is well known that $f(z)$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} c_n n^{k-1/2} e(nz)$$

for some real numbers $c_n$. For each $n \in \mathbb{N}$,

$$T_n f(z) := \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right) = \lambda_f(n) f(z),$$

in which $T_n$ is the $n$-th Hecke operator, $\lambda_f(n)$ is the $n$-th Hecke eigenvalue. The Hecke eigenvalues $\{\lambda_f(n)\}_{n \in \mathbb{N}}$ satisfy the following properties.

$$c_1 \lambda_f(n) = c_n,$$

$$\lambda_f(m)\lambda_f(n) = \sum_{d|\langle n,m \rangle} \lambda_f\left(\frac{nm}{d^2}\right),$$

$$|\lambda_f(n)| \leq d(n),$$

in which $d(n) := \sum_{m|n} 1$ (the inequality (4) is called the Deligne bound). For details, see Chapter 14, [2].
We say that $\alpha$ is the average value of $|\lambda_f(p)|$ when
\begin{equation}
\sum_{p<x} \frac{|\lambda_f(p)|}{p} = \sum_{p<x} \frac{\alpha}{p} + O_f(1)
\end{equation}
for big enough $X$. Sato-Tate conjecture implies that $\alpha = \frac{8}{15} (= 0.848826...).$ In [1] P. D. T. A Elliott, C. J. Moreno and F. Shahidi proved that $\alpha \leq \frac{17}{18}$ without assuming Sato-Tate conjecture.

When $h = X^\delta$ for some $\delta \in (0, 1]$, by Shiu’s theorem (see Lemma 2.2),
\begin{equation}
\sum_{n=X}^{X+h} |\lambda_f(n)| \ll f \delta h \prod_{p=1}^{X} \left(1 + \frac{\alpha - 1}{p}\right) \ll h (\log X)^{\alpha - 1}
\end{equation}
for big enough $X$, but when $h(X) = o_\delta(X^\delta)$ for any $\delta > 0$, we can not use Shiu’s theorem because the interval is too short. In Section 2, we prove some lemmas by using some arguments of the papers [4], [5] to overcome this obstacle.

Although the results in this paper are stated for holomorphic cusp forms, the same arguments in this paper apply equally well to Maass cusp forms on $SL(2, \mathbb{Z})$, if we assume (1.4) (The Ramanujan-Petersson conjecture).

We give some notations that will be used throughout in this paper. We use $\varphi$ to denote the Euler totient function. We use $\psi$ to denote a function from $\mathbb{R}$ to $\mathbb{R}$ such that $\psi(x) \to \infty$ as $x \to \infty$. For any two functions $k(x)$ and $l(x)$, we use $k(x) \ll l(x)$ (and $k(x) = O(l(x))$) to denote that there exists a constant $C$ such that $|k(x)| \leq C l(x)$ for all $x$. We use $k(x) = o(l(x))$ to denote $\frac{k(x)}{l(x)} \to 0$ as $x \to \infty$ and $n \sim x$ to denote $n \in [X, 2X]$. Summing over the index $p$ denotes summing over primes. For the convenience, we denote $h := h(X)$.

1.1. Main results.

**Theorem 1.1.** Let $X > 0$ be big enough, let $q$ be a natural number smaller than $X$. Let $h$ be a real valued function such that $\varphi(q)(\log X)^{2 - 2\alpha} \ll h = o(X)$. Then there exists a Dirichlet character $\chi$ modulo $q$ such that
\begin{equation}
\sum_{n=X}^{X+h} |\lambda_f(n)| \ll f \psi(X) \varphi(q)^{-0.5} (\log X)^{\alpha - 1}
\end{equation}
for all but $O_f(X \psi(X)^{-2})$ many integers $x \in [X, 2X - h]$.

When $q = 1$, $\chi$ in Theorem 1.1 should be the trivial character. Therefore, we obtain the following corollary.

**Corollary 1.2.** Let $X > 0$ be big enough. Let $h$ be a real valued function such that $(\log X)^{2 - 2\alpha} \ll h = o(X)$. Then
\begin{equation}
\sum_{n=X}^{X+h} |\lambda_f(n)| \ll f \psi(X)(\log X)^{\alpha - 1}
\end{equation}
for all but at most $O_f(X \psi(X)^{-2})$ integers $x \in [X, 2X - h]$. 
It is well known that for big enough $X$,

\[
\sum_{n=1}^{X} |\lambda_f(n)|^2 = c_1 X + O_f(X^{\frac{4}{5}}),
\]

(1.8)

\[
\sum_{n=1}^{X} |\lambda_f(n)|^4 = c_2 X \log X + c_3 X + O_f(X^{\frac{7}{8} + \epsilon})
\]

for some $c_1, c_2, c_3$ (see [3]). In our method, the upper bound of the short sum (1.7) and the sizes of $h$ in Corollary 1.2 are only depend on the long sums (1.8), first equation) and the average of $|\lambda_f(p)|$ over primes (for the detail, see (2.8)). Therefore, we generalize Corollary 1.2 to arbitrary $2^k$ power of $|\lambda_f(n)|$ for $k \in \mathbb{Z}^+$.

**Theorem 1.3.** Let $X > 0$ be big enough. Let $k$ be a fixed non-negative integer. Assume that there exist positive constants $\beta$ and $\gamma$ such that both inequalities

\[
\sum_{n=X}^{2X} |\lambda_f(n)|^{2k+1} \ll_f X(\log X)^{\beta},
\]

(1.9)

\[
\sum_{p=1}^{X} \frac{|\lambda_f(p)|^{2k}}{p} - \sum_{p=1}^{X} \frac{\gamma}{p} = O_f(1)
\]

(1.10)

hold. Then for any real valued function $h$ such that $(\log X)^{\beta - 2\gamma + 2} \ll_f h = o(X)$,

\[
\sum_{n=x}^{x+h} |\lambda_f(n)|^{2k} \ll_f h(\log X)^{\gamma-1}\psi(X)
\]

for all but $O_f(X\psi(X)^{-2})$ many integers $x \in [X, 2X - h]$.

In Lemma 2.5, we prove that the average of $\lambda_f(p)^2$ over primes is 1. Therefore, the upper bound of $\sum_{n=X}^{2X} |\lambda_f(n)|^2$ from Shiu’s theorem is also $O(X)$. From the above facts, we obtain the following corollary.

**Corollary 1.4.** Let $X > 0$ be big enough. Let $h$ be a real valued function such that $\log X \ll_f h = o(X)$. Then

\[
\sum_{n=x}^{x+h} |\lambda_f(n)|^2 \ll_f h\psi(X)
\]

(1.11)

for all but $O_f(X\psi(X)^{-2})$ many integers $x \in [X, 2X - h]$.

**Remark 1.5.** We apply Shiu’s theorem to get some trivial bounds. Let

\[
R_1(x) := \sum_{n=x}^{x+h} |\lambda_f(n)|\chi(n),
\]

\[
K_1(X) := \{x \in [X, 2X - h] : h\psi(X)\varphi(q)^{-0.5}(\log X)^{\alpha-1} \ll R_1(x)\}.
\]
Then
\[ |K_1(X)| h \psi(X) \varphi(q)^{-0.5} (\log X)^{\alpha - 1} \leq \sum_{X \leq x \leq 2X} |R_1(x)| \]

\[ \leq \sum_{X \leq x \leq 2X} |R_1(x)| \]

\[ \leq \sum_{X \leq x \leq 2X} \sum_{n=x}^{x+h} |\lambda_f(n)| \]

\[ \ll f h X (\log X)^{\alpha - 1}. \]

(1.12)

Therefore, \( X(\psi(X) \varphi(q)^{-0.5})^{-1} \) is a trivial bound for \( |K_1(X)| \). Thus the upper bound of \( |K_1(X)| \) from Corollary 1.2 saves \( \psi(X) \varphi(q)^{0.5} \) from the trivial one.

Let
\[
R_2(x) := \sum_{n=x, (n,q)=1}^{x+h} |\lambda_f(n)|^2,
\]

\[
K_2(X) := \{ x \in [X, 2X - h] : h \psi(X) \ll R_2(x) \}.
\]

Then
\[ |K_2(X)| h \psi(X) \leq \sum_{X \leq x \leq 2X, h \psi(X) \ll R_2(x)} |R_2(x)| \]

\[ \leq \sum_{X \leq x \leq 2X} |R_2(x)| \]

\[ \leq \sum_{X \leq x \leq 2X} \sum_{n=x}^{x+h} |\lambda_f(n)|^2 \]

\[ \ll_f h X \]

(1.13)

Therefore, \( X(\psi(X))^{-1} \) is a trivial bound for \( |K_2(X)| \). Thus the upper bound of \( |K_2(X)| \) from Corollary 1.4 saves \( \psi(X) \) from the trivial one.

2. Lemmas

The following lemma shows that one can get some information about the average of \( |\lambda_f(n)| \chi(n) \) in almost all short intervals from the upper bounds of the second moment of the Dirichlet polynomial
\[
F(s) := \sum_{n \sim X} \frac{|\lambda_f(n)| \chi(n)}{n^s}.
\]

Lemma 2.1. Let \( X > 0 \) be big enough, let \( q \) be a natural number smaller than \( X \), and let \( h = o(X) \). Then
By the mean value theorem, the right hand side of (2.2) is bounded by
\[
\ll \int_0^{Xh^{-1}} \left| \sum_{n \sim X} \frac{|\lambda_f(n)| \chi(n)}{n^{1+it}} \right|^2 dt + \max_{T>Xh^{-1}} \frac{Xh^{-1}}{T} \int_T^{2T} \left| \sum_{n \sim X} \frac{|\lambda_f(n)| \chi(n)}{n^{1+it}} \right|^2 dt.
\]

**Proof.** The proof of this basically follows from [4, Lemma 14]. Since we choose the Dirichlet polynomial \( F(s) \) instead of \( \sum_{n=1}^\infty \frac{|\lambda_f(n)| \chi(n)}{n^{it}} \), there is no issue on absolute convergence of \( F(s) \). By Perron’s formula,
\[
\sum_{x \leq n \leq x+h} \frac{|\lambda_f(n)| \chi(n)}{n} = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} F(s) \frac{(x+h)^s - x^s}{s} ds.
\]
Let
\[
V = \frac{1}{h^2X} \int_X^{2X} \left| \int_1^{1+i\infty} F(s) \frac{(x+h)^s - x^s}{s} ds \right|^2 dx.
\]
Since
\[
\frac{(x+h)^s - x^s}{s} = \frac{1}{2h} \left[ \int_h^{2h} \frac{(x+w)^s - x^s}{s} dw - \int_h^{3h} \frac{(x+w)^s - (x+h)^s}{s} dw \right],
\]
\[
V \ll Xh^{-4} \int_X^{2X} \left| \int_0^{2h} \int_1^{1+i\infty} F(s) x^s \frac{(1+w)^s - 1}{s} ds dw \right|^2 dx
\]
\[
+ Xh^{-4} \int_X^{2X} \left| \int_0^{2h} \int_1^{1+i\infty} F(s)(x+h)^s \frac{(1+w)^s - 1}{s} ds dw \right|^2 dx.
\]
By the mean value theorem, the right hand side of (2.2) is bounded by
\[
\ll \frac{1}{h^2X} \int_X^{2X} \left| \int_1^{1+i\infty} F(s) x^s \frac{(1+u)^s - 1}{s} ds \right|^2 dx
\]
\[
+ \frac{1}{h^2X} \int_{X+h}^{2X+2h} \left| \int_1^{1+i\infty} F(s)x^s \frac{(1+u)^s - 1}{s} ds \right|^2 dx
\]
for some \( u \ll \frac{1}{X} \). Let \( V_1 \) be the first summand, \( V_2 \) be the second summand of (2.3). Let \( g_1 \) be a smooth function supported on \( [\frac{X}{2}, 4X] \), \( g_1(x) = 1 \) for \( x \in [X, 2X] \), and \( g'_1(x) \ll \frac{1}{X} \). Let \( s_1 = 1 + it_1, s_2 = 1 + it_2 \). Then
\[
V_1 \ll \frac{1}{h^2X} \int g_1(x) \left| \int_1^{1+i\infty} F(s) x^s \frac{(1+u)^s - 1}{s} ds \right|^2 dx
\]
\[
\ll \frac{1}{h^2X} \int_1^{1+i\infty} \left| F(s_1)F(s_2) \min\left\{ \frac{h}{X}, \frac{1}{|t_1|} \right\} \min\left\{ \frac{h}{X}, \frac{1}{|t_2|} \right\} \right| \left| \int g_1(x)x^{s_1+s_2} dx \right| ds_1ds_2.
\]
Since

\[
\int g_1(x)x^{s_1+s_2}dx \ll \frac{1}{X} \int_\frac{-X}{2}^{4X} \left| \frac{x^{s_1+s_2+1}}{s_1+s_2+1} \right| dx,
\]

\[
V_1 \ll \frac{1}{h^2X} \int_1^{1+i\infty} \int_1^{1+i\infty} |F(s_1)F(s_2)\min\{\frac{h}{X}, \frac{1}{|t_1|}\} \min\{\frac{h}{X}, \frac{1}{|t_2|}\} \sqrt{1+\frac{X^3}{|t_1-t_2|^2+1}}|ds_1ds_2|
\ll \frac{X^2}{h^2} \int_1^{1+i\infty} \int_1^{1+i\infty} |F(s_1)|^2 \min\{(\frac{h}{X})^2, |t_1|^{-2}\} + |F(s_2)|^2 \min\{(\frac{h}{X})^2, |t_2|^{-2}\}|ds_1ds_2|
\ll \int_1^{1+i\frac{X}{h}} |F(s)|^2 |ds| + \frac{X^2}{h^2} \int_1^{1+i\frac{X}{h}} \frac{|F(s)|^2}{|t|^2} |ds|.
\]

Since \(|t|^{-2} \ll \int_{it}^{2it} |T|^{-3}dT\),

\[
V_1 \ll \int_1^{1+i\frac{X}{h}} |F(s)|^2 |ds| + \frac{X^2}{h^2} \int_\frac{-X}{2}^{X} \frac{1}{T^3} \int_1^{1+2iT} |F(s)|^2 |ds|dT|\]
\ll \int_1^{1+i\frac{X}{h}} |F(s)|^2 |ds| + \frac{X^2}{h^2} \frac{1}{X} \max_{T>\frac{X}{h}} \int_1^{1+2iT} |F(s)|^2 |ds|
\ll \int_0^{Xh^{-1}} \left| \sum_{n=X}^{2X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} \right|^2 dt + \max_{T>\frac{Xh^{-1}}{T}} Xh^{-1} \int_T^{2T} \left| \sum_{n=X}^{2X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} \right|^2 dt.
\]

Let \(g_2\) be a smooth function supported on \([\frac{X+h}{2}, 4X+4h]\), \(g_2(x) = 1\) for \(x \in [X+h, 2X+2h]\), and \(g_2'(x) \ll \frac{1}{X}\). By the similar arguments of the bounding \(V_1\) (replacing \(g_1\) with \(g_2\)),

\[
V_2 \ll \int_0^{Xh^{-1}} \left| \sum_{n=X}^{2X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} \right|^2 dt + \max_{T>\frac{Xh^{-1}}{T}} Xh^{-1} \int_T^{2T} \left| \sum_{n=X}^{2X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} \right|^2 dt.
\]

In the proof of Lemma 2.3, we bound some type of the integral

\[
(2.4) \quad \int_{-T}^{T} |F(1+it)|^2 dt
\]

by some terms in which are related to the average of \(|\lambda_f(n)|^2\) over \([X, 2X]\) and the average of the shifted sums \(\sum_{X \leq m \leq 2X} |\lambda_f(m)\lambda_f(m+hq)|\) over \(h \in [1, \frac{T}{Xq}]\). The following lemma allows us to compute them.

**Lemma 2.2.** (Shiu’s theorem \[5\] Lemma 2.3)
Let $0 < \delta \leq 1$. Let $1 \leq q \leq X^\delta$, $1 \leq H$. Let $r(n)$ be a non-negative multiplicative function such that $r(n) \ll d(n)^k$ for some $k \in \mathbb{N}$. For $2 \leq X^\delta \leq Y$,

\begin{equation}
\sum_{n=X}^{X+Y} r(n) \ll \delta \ Y \prod_{p<X} \left(1 + \frac{r(p) - 1}{p}\right),
\end{equation}

\begin{equation}
\sum_{\lvert h \rvert \leq H} \sum_{\frac{X}{\tau} \leq n \leq X + Y} r(n) r(n + hq) \ll \delta \ Y \prod_{\substack{p \leq X, \ p \nmid q}} \left(1 + \frac{r(p) - 1}{p}\right)^2 \prod_{\substack{p | q}} \left(1 - \frac{1}{p}\right).
\end{equation}

**Proof.** See [5, Lemma 2.3].

Let

\begin{equation}
A(s) := \sum_{n \sim X} a_n n^{-s}
\end{equation}

for some $\{a_n\} \in \mathbb{C}$. The standard method for bounding the second moment of $A(s)$ is the mean value theorem

\[
\int_{-T}^T |A(it)|^2 \, dt = O((T + X) \sum_{n \sim X} |a_n|^2)
\]

(see [2, Theorem 9.2]). By factoring $A(s)$ to reduce the size of the length of the Dirichlet polynomials, one can obtain some nontrivial bounds of the second moment of $A(s)$ from the above mean value theorem (see Section 5, [4]. In [4], K. Matomäki, M. Radziwiłł applied the Ramare identity, an analogue of the Buchstab identity). But in our case, we are unable to reduce the size of $X$. Therefore, we apply the following lemma.

**Lemma 2.3.** Let $X > 0$ be big enough, let $q$ be a natural number smaller than $X$. Then

\begin{equation}
\sum_{\chi \pmod{q}} \int_{-T}^T \left| \sum_{n \sim X} \frac{\lambda_f(n) \chi(n)}{n^{1+it}} \right|^2 \, dt \ll_f \frac{T \varphi(q)}{X^2} \sum_{n \sim X} |\lambda_f(n)|^2 + (\log X)^{2\alpha-2}.
\end{equation}

**Proof.** The proof of this basically follows from [5, Lemma 5.2]. Let

\[
I := \sum_{\chi \pmod{q}} \int_{-T}^T \left| \sum_{n \sim X} \frac{\lambda_f(n) \chi(n)}{n^{1+it}} \right|^2 \, dt.
\]

Let $\phi$ be a non-negative smooth function such that $\phi \geq 1$ for $|x| \leq 1$, $\hat{\phi}(x) = 0$ for $1 < |x|$, in which

\[
\hat{\phi}(x) := \int_{-\infty}^\infty \phi(t) e(-xt) \, dt.
\]

Then

\[
I \leq \sum_{\chi \pmod{q}} \int_{-\infty}^\infty \left| \sum_{n \sim X} \frac{\lambda_f(n) \chi(n)}{n^{1+it}} \right|^2 \phi\left(\frac{t}{T}\right) \, dt
\]

\[
= \sum_{\chi \pmod{q}} \sum_{m,n \sim X} \frac{|\lambda_f(m)\lambda_f(n)|}{(mn)} \chi(m) \chi(n) T \hat{\phi}(T \log \frac{m}{n})
\]
For each fixed $n$, the range of $m$ is decided by the compact support of $\hat{\phi} (m = n + h, |h| \leq \frac{2X}{T})$, and by averaging over characters $\chi \pmod{q}$,

$$I \ll \varphi(q) \frac{T}{X^2} \sum_{n \sim X \atop (n,q)=1} \lambda_f(n) + \varphi(q) \frac{T}{X^2} \sum_{0 < |h| < \frac{2X}{Tq}} \sum_{n \sim X \atop (n,q)=1} |\lambda_f(n)\lambda_f(n + h)|.$$

By Lemma 2.2, (2.6),

$$\sum_{0 < |h| < \frac{2X}{Tq}} \sum_{n \sim X \atop (n,q)=1} |\lambda_f(n)\lambda_f(n + h)| \ll_f 2X \prod_{p \leq X \atop p \nmid q} (1 + \frac{|\lambda_f(p)| - 1}{p})^2 \prod_{p \mid q} (1 - \frac{1}{p}).$$

By (1.4), $||\lambda_f(p)| - 1| \leq 1$ for all prime $p$. By Taylor expansion and (1.5),

$$\log \left( \prod_{p \leq X} \left(1 + \frac{|\lambda_f(p)| - 1}{p}\right) \right) = \sum_{p \leq X} \log \left(1 + \frac{|\lambda_f(p)| - 1}{p}\right) = \sum_{p \leq X} \frac{|\lambda_f(p)| - 1}{p} + O(1) = \sum_{p \leq X} \frac{\alpha - 1}{p} + O_f(1).$$

By (1.4),

$$\log \left( \frac{\varphi(q) \prod_{p \mid q} (1 + \frac{|\lambda_f(p)| - 1}{p})^2 (1 - \frac{1}{p})}{\varphi(q) \prod_{p \mid q} (1 + \frac{|\lambda_f(p)| - 1}{p})^2 (1 - \frac{1}{p})} \right) = \sum_{p \mid q} \frac{-2|\lambda_f(p)|}{p} + O(1) \ll 1.$$

Therefore, the 2nd term of the right-hand side of (2.8) is bounded by

$$(\log X)^{2\alpha - 2}.$$

Notice that the absolute constant of the inequality (2.8) does not depend on $q$, but in (2.9), one can produce a saving factor from $\sum_{p \mid q} \frac{-2|\lambda_f(p)|}{p}$ for some $q$. This saving factor can be crucial when we treat $|\lambda_f(n)|^{2k}$ for some big $k$. Let $k \in \mathbb{N}$, $|\lambda_f(2)| = 2$, $q = 2$. Then

$$\log \left( \frac{\varphi(q) \prod_{p \mid q} (1 + \frac{|\lambda_f(p)|^{2k} - 1}{p})^2 (1 - \frac{1}{p})}{\varphi(q) \prod_{p \mid q} (1 + \frac{|\lambda_f(p)|^{2k} - 1}{p})^2 (1 - \frac{1}{p})} \right) = \log \left(1 - \frac{1}{2}\right)^2 \left(1 + \frac{2^{2k} - 1}{2}\right)^{-2} = -2 \log(2^{2k} + 1).$$
Therefore, the last term of (2.10) is heavily depend on \( k \). So we only generalize Lemma 2.3 for \( q = 1 \).

**Lemma 2.4.** Let \( X > 0 \) be big enough. Let \( k \) be a fixed non-negative integer. Assume that there exist positive constants \( \beta \) and \( \gamma \) such that both inequalities

\[
\sum_{n = X}^{2X} |\lambda_f(n)|^{2k+1} \ll_f X (\log X)^\beta,
\]

\[
\sum_{p = 1}^{X} \frac{|\lambda_f(p)|^{2k}}{p} - \sum_{p = 1}^{X} \frac{\gamma}{p} = O_f(1)
\]

hold. Then

\[
\int_{-T}^{T} \left| \sum_{n \sim X} \frac{|\lambda_f(n)|^{2k}}{n^{1+it}} \right|^2 dt \ll_f \frac{T}{X} (\log X)^\beta + (\log X)^{2\gamma - 2}.
\]

**Proof.** Let

\[
I_k := \int_{-T}^{T} \left| \sum_{n \sim X} \frac{|\lambda_f(n)|^{2k}}{n^{1+it}} \right|^2 dt.
\]

By the similar argument for \( I \) in Lemma 2.3 \((q = 1)\),

\[
I_k \ll \frac{T}{X^2} \sum_{n \sim X} |\lambda_f(n)|^{2k+1} + \frac{T}{X^2} \sum_{0 < |h| < \frac{2X}{T}} \sum_{n \sim X} |\lambda_f(n)|^{2k} |\lambda_f(n + h)|^{2k}.
\]

By (2.11),

\[
\frac{T}{X^2} \sum_{n \sim X} |\lambda_f(n)|^{2k+1} \ll_f \frac{T}{X} (\log X)^\beta.
\]

By Lemma 2.2 (2.6),

\[
\sum_{0 < |h| < \frac{2X}{T}} \sum_{n \sim X} |\lambda_f(n)|^{2k} |\lambda_f(n + h)|^{2k} \ll \frac{2X}{T} X \prod_{p \leq X} \left( 1 + \frac{|\lambda_f(p)|^{2k} - 1}{p} \right)^2
\]

\[
\ll \frac{T}{X} (\log X)^{2\gamma - 2}.
\]

Therefore,

\[
I_k \ll_f \frac{T}{X} (\log X)^\beta + (\log X)^{2\gamma - 2}.
\]

\( \square \)

The following lemma shows that the average of \( \lambda_f(p)^2 \) over primes is 1.
Lemma 2.5. Let $X > 0$ be big enough. Then

$$
\sum_{p < X} \frac{\lambda_f(p)^2}{p} = \sum_{p < X} \frac{1}{p} + O_f(1)
$$

Proof. Let

$$
L(g, s) := \sum_{n = 1}^{\infty} \lambda_f(n)n^{-s}.
$$

Let $\Lambda$ be the von Mangoldt function. $L(g \otimes \overline{g}, s)$ has a zero free region by [2, Theorem 5.44]. By [2, Theorem 5.13],

$$
\sum_{p \leq x} \lambda_f(p)^2 \Lambda(p) = x + O(x(\log x)e^{-C\log \frac{1}{2} x})
$$

for some absolute constant $C > 0$ depending only on $g$. Partial summation over $p$ gives

$$
\sum_{1 < p \leq x} \frac{\lambda_f(p)^2}{p} = \int_2^x \frac{1}{t \log t} d\left(\sum_{p \leq t} \lambda_f(p)^2 \Lambda(p)\right) + O_f(1) = \log \log x + O_f(1) = \sum_{p \leq x} \frac{1}{p} + O_f(1).
$$

□

3. Propositions

In this section, we prove Proposition 3.1, Proposition 3.2. We need Proposition 3.1, Proposition 3.2 for Theorem 1.1, Theorem 1.3 respectively.

Proposition 3.1. Let $X > 0$ be big enough, let $q$ be a natural number smaller than $X$. Then there exists a Dirichlet character $\chi$ modulo $q$ such that when $\varphi(q)(\log X)^{2-2\alpha} \ll_f h = o(X)$,

$$
\frac{1}{X} \int_X^{2X} |\frac{1}{h} \sum_{n = x}^{x+h} |\lambda_f(n)|\chi(n)|^2 dx \ll_f \varphi(q)^{-1}(\log X)^{2\alpha-2}.
$$

Proof. Dropping all but one term, there exists a character $\chi$ modulo $q$ such that for all $T > 0$,

$$
\int_{-T}^{T} \left| \sum_{n \sim X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} \right|^2 dt \leq \frac{1}{\varphi(q)} \sum_{\chi(\text{mod } q)} \int_{-T}^{T} \left| \sum_{n \sim X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} \right|^2 dt.
$$

By Lemma 2.3, (2.8),

$$
\frac{1}{\varphi(q)} \sum_{\chi(\text{mod } q)} \int_{0}^{Xh^{-1}} \left| \sum_{n \sim X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} \right|^2 dt \ll_f \frac{1}{Xh} \sum_{n \sim X} |\lambda_f(n)|^2 + \varphi(q)^{-1}(\log X)^{2\alpha-2}
$$

$$
\ll \frac{1}{h} + \varphi(q)^{-1}(\log X)^{2\alpha-2} \ll \varphi(q)^{-1}(\log X)^{2\alpha-2}.
$$
By the similar argument of (3.2),

\[
\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \max_{T > Xh^{-1}} \frac{Xh^{-1}}{T} \int_{T}^{2T} | \sum_{n \sim X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} |^2 dt \\
\ll_f \max_{T > Xh^{-1}} Xh^{-1}T^{-1} \left( \frac{T}{X} + \varphi(q)^{-1}(\log X)^{2\alpha-2} \right) \\
\ll \frac{1}{h} + \varphi(q)^{-1}(\log X)^{2\alpha-2} \\
\ll \varphi(q)^{-1}(\log X)^{2\alpha-2}.
\]

By Lemma 2.1,

\[
\frac{1}{X} \int_{X}^{2X} \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^2 dx \\
\ll \int_{0}^{Xh^{-1}} | \sum_{n \sim X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} |^2 dt \\
+ \max_{T > Xh^{-1}} Xh^{-1} \frac{1}{T} \int_{T}^{2T} | \sum_{n \sim X} \frac{\lambda_f(n)\chi(n)}{n^{1+it}} |^2 dt.
\]

Therefore,

\[
\frac{1}{X} \int_{X}^{2X} \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^2 dx \\
\ll_f \varphi(q)^{-1}(\log X)^{2\alpha-2}.
\]

By the similar arguments of the proof of Proposition 3.1, we generalize Proposition 3.1 to arbitrary \(2^k\) power of \(|\lambda_f(n)|\).

**Proposition 3.2.** Let \(X > 0\) be big enough. Let \(k\) be a fixed non-negative integer. Assume that there exist positive constants \(\beta\) and \(\gamma\) such that both inequalities

\[
(3.4) \sum_{n=X}^{2X} |\lambda_f(n)|^{2^{k+1}} \ll_f X(\log X)^{\beta},
\]

\[
(3.5) \sum_{p=1}^{X} \frac{|\lambda_f(p)|^{2^k}}{p} - \sum_{p=1}^{X} \frac{\gamma}{p} = O_f(1)
\]

hold. Then for any real valued function \(h\) such that \((\log X)^{\beta-2\gamma+2} \ll_f h = o(X),

\[
(3.6) \frac{1}{X} \int_{X}^{2X} \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^{2^k} dx \ll_f (\log X)^{2\gamma-2}.
\]
Proof. By the similar argument of the proof of Lemma 2.1 (one just need to replace $|\lambda_f(n)|^{2k}$, $q = 1$),

\begin{equation}
\frac{1}{X} \int_X^{2X} \left| \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^2 \right|^2 \, dx \ll \int_0^{Xh^{-1}} |\sum_{n \sim X} \frac{|\lambda_f(n)|^{2k}}{n^{1+it}}|^2 \, dt + \max_{T > Xh^{-1}} \frac{Xh^{-1}}{T} \int_T^{2T} |\sum_{n \sim X} \frac{|\lambda_f(n)|^{2k}}{n^{1+it}}|^2 \, dt.
\end{equation}

By Lemma 2.4,

\begin{equation}
\int_0^{Xh^{-1}} |\sum_{n \sim X} \frac{|\lambda_f(n)|^{2k}}{n^{1+it}}|^2 \, dt \ll h^{-1}(\log X)^\beta + (\log X)^{2\gamma - 2},
\end{equation}

\begin{equation}
\max_{T > Xh^{-1}} \frac{Xh^{-1}}{T} \int_T^{2T} |\sum_{n \sim X} \frac{|\lambda_f(n)|^{2k}}{n^{1+it}}|^2 \, dt \ll \max_{T > Xh^{-1}} \frac{Xh^{-1}T^{-1}}{T} (\frac{T}{X}(\log X)^\beta + (\log X)^{2\gamma - 2}) \ll h^{-1}(\log X)^\beta + (\log X)^{2\gamma - 2}.
\end{equation}

Since $h^{-1} \ll f(\log X)^{-\beta + 2\gamma - 2}$, (3.8),(3.9) are bounded by

\begin{equation}
(\log X)^{2\gamma - 2}.
\end{equation}

\[\Box\]

4. Proof of Theorem 1.1, Theorem 1.3, Corollary 1.4

4.1. Proof of Theorem 1.1. By Proposition 3.1, there exists a \( \chi \) modulo \( q \) such that

\[ \int_X^{2X} \left| \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|\chi(n)|^2 \right|^2 \, dx \ll f \varphi(q)^{-1}(\log X)^{2\alpha - 2}. \]

Let \( B(X) = \psi(X)^2 \varphi(q)^{-1}(\log X)^{2\alpha - 2} \). By the Chebyshev inequality,

\[ \left| \{ x \in [X, 2X - h] : \left| \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|\chi(n)| \right| \gg f B(X)^{\frac{1}{2}} \} \right| \ll f B(X)^{-1} \int_X^{2X} \left| \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^2 \right|^2 \, dx \]

\[ = O_f(X\psi(X)^{-2}). \]
4.2. **Proof of Theorem 1.3, Corollary 1.4.** By Proposition 3.2,

\[
\frac{1}{X} \int_X^{2X} \left( \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^{2k} \right)^2 dx \ll_f (\log X)^{2\gamma - 2}.
\]

Let

\[ B_k(X) = \psi(X)^2 (\log X)^{2\gamma - 2}. \]

By the Chebyshev inequality,

\[
\left| \left\{ x \in [X, 2X - h] : \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^{2k} \gg_f B_k(X)^{1/2} \right\} \right| \ll_f B_k(X)^{-1} \int_X^{2X} \left( \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^{2k} \right)^2 dx
\]

\[
= O_f(X^{\psi(X)^{-2}}).
\]

When \( k = 1 \), by (1.8), \( \beta = 1 \). And by Lemma 2.5, \( \gamma = 1 \). Therefore,

\[
\left| \left\{ x \in [X, 2X - h] : \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^2 \gg_f \psi(X) \right\} \right| \ll_f (\psi(X))^{-2} \int_X^{2X} \left( \frac{1}{h} \sum_{n=x}^{x+h} |\lambda_f(n)|^2 \right)^2 dx
\]

\[
= O_f(X^{\psi(X)^{-2}}).
\]

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Email address: Jiseongk@buffalo.edu

University at Buffalo, Department of Mathematics 244 Mathematics Building Buffalo, NY 14260-2900