On the resolution of quantum paradoxes by weak measurements

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In this presentation, I argue that weak measurements empirically support the notion of quantum superpositions as statistical alternatives. In short, weak measurements show that Schrödinger’s cat is already dead or alive before the measurement. The collapse of the wavefunction in a strong measurement should therefore be separated into the statistical selection of one of the available alternatives and a physical interaction that causes decoherence. The application to entanglement reveals that measurements in A have no physical effect in B, resolving the paradox of Bell’s inequality violation in favor of locality and against (non-empirical) realism.

Keywords: wavefunction collapse, entanglement, non-locality

I. INTRODUCTION

As Schrödinger famously illustrated in his paradox of a cat in a superposition of life and death [1], it is not at all obvious how one can reconcile the intuitive notion of an external reality with the concept of quantum coherence between distinguishable alternatives. Although decoherence may seem to solve the problem, Bell correctly pointed out that the theory would have to clarify at what point the quantum “and” turns into a classical “or” [2]. Bohr would probably have objected to this criticism by pointing out that only the results of experiments are real, and that it is the purpose of the theory to describe this reality, not to define it. In fact, our only hope for a scientific resolution of the quantum measurement problem is the design of experiments that provide empirical answers to our questions.

In 1988, Aharonov, Albert, and Vaidman proposed such an empirical approach in the form of weak measurements [3]. Essentially, weak measurements can access the quantum statistics of a system between its initial preparation and a final measurement outcome with negligible measurement interaction. Recently, this method has been applied to experimental realizations of quantum paradoxes, suggesting that an empirical resolution of the paradoxes can be achieved by interpreting the weak values observed in the measurements as negative probabilities [4, 5, 6, 7]. This seems to be an exciting possibility, but it assumes that the negative probabilities observed in weak measurements are well understood and not at all paradoxical. Otherwise, we have merely replaced one paradox with another.

In the following, I will argue that the real resolution of quantum paradoxes by weak measurement should be based on the complete quantum statistics of the states between preparation and measurement that can be experimentally determined by weak measurement tomography [8]. Specifically, weak measurement tomography shows that the only empirically correct answer to Bell’s question is that quantum superpositions are actually statistical alternatives. The quantum mechanical “and” is a misinterpretation, it is always an “or”. Once this misunderstanding is cleared up, it is possible to resolve quantum paradoxes by identifying the fallacies that cause the contradictions with reality.

II. WEAK MEASUREMENT TOMOGRAPHY

The key to understanding weak measurements is quantum statistics. An individual weak measurement results in a completely random outcome, with no discernible relation to the properties of the quantum state. However, the average of a large number of weak measurements reveals small differences in the probability distribution that can be identified with the quantum statistics of the input state. In terms of measurement theory, the operator defining the probability of obtaining an outcome \( m \) in any given measurement can be written as

\[
\hat{E}_m = w_m (\hat{1} + \epsilon \hat{S}_m),
\]

where \( \| \epsilon \hat{S}_m \| \leq 1 \). Thus, the small differences between the experimentally observed probabilities \( p(m) = \text{Tr}\{\hat{\rho}_i \hat{E}_m\} \) and \( w_m \) can be used to determine the expectation values of \( \hat{S}_m \). The complete density matrix \( \hat{\rho}_i \) can be reconstructed from the expectation values of at least \( d^2 \) linearly independent operators \( \hat{S}_m \). However, weak measurements have negligible back-action, so the quantum state is still available for further measurements. It is therefore possible to determine the quantum statistics of a system between the initial preparation \( i \) given by \( \hat{\rho}_i \) and a final measurement outcome \( f \) given by a measurement operator \( \hat{\Pi}_f \). As explained in more detail in [8], the joint probabilities of the measurement outcomes \( m \) and \( f \) can be given as

\[
p(m, f|i) = \text{Tr}\{\hat{E}_m \frac{1}{2} (\hat{\rho}_i \hat{\Pi}_f + \hat{\Pi}_f \hat{\rho}_i)\}.
\]
The measurement \( f \) therefore defines a decomposition of the initial density matrix \( \hat{\rho}_i \) into a mixture of subensemble density matrices \( \hat{R}_{if} \) associated with the individual outcomes \( f \),

\[
\hat{\rho}_i = \sum_f p(f|i) \hat{R}_{if},
\]

where \( p(f|i) = \text{Tr}\{\hat{\rho}_i \hat{\Pi}_f\} \) and

\[
\hat{R}_{if} = \frac{1}{2\text{Tr}\{\hat{\rho}_i \hat{\Pi}_f\}} \left( \hat{\rho}_i \hat{\Pi}_f + \hat{\Pi}_f \hat{\rho}_i \right).
\]

This decomposition is valid even if the density matrix \( \hat{\rho}_i \) describes a pure state that is a coherent superposition of different outcomes \( f \). In other words, eq. (3) shows that there is no contradiction between a statistical interpretation of the alternatives \( f_1 \) and \( f_2 \) and quantum coherence between the eigenstates associated with those outcomes. Specifically, the problem is resolved by attributing half of the coherence to \( f_1 \) and the other half to \( f_2 \).

III. SCHRÖDINGER’S CAT AND DOUBLE SLIT INTERFERENCE

The significance of the result derived above emerges when it is applied to the examples that are used to illustrate the strangeness of quantum mechanics. Firstly, eq. (3) suggests that the death or survival of Schrödinger’s cat is not decided by the measurement - the measurement merely uncovers which of the alternatives has already happened.

It may be surprising that a simple argument about quantum measurements should lead to new insights into the nature of quantum coherence. However, it seems that much of the perplexity caused by Schrödinger’s cat was due to the analogy of quantum states and classical waves, where interference patterns appear to require the presence of two sources. It is therefore useful to consider the statistical explanation of coherence in terms of the double slit interference of a single particle. Here, it is commonly argued that the interference pattern could not exist if the particle either went through one slit or through the other slit. However, weak measurement tomography shows that the states defined by a two slit superposition \(|1\rangle+|2\rangle\) and a subsequent which-path measurement are

\[
\hat{R}_{+1} = |1\rangle(1|+\frac{1}{2}(1\langle 1|2\rangle+|2\rangle\langle 1|)
\]

\[
\hat{R}_{+2} = |2\rangle(2|+\frac{1}{2}(1\langle 1|2\rangle+|2\rangle\langle 1|).
\]

Thus, weak measurement shows that the assumption that which-slit information prevents interference is unfounded - the interference pattern associated with the coherences between path 1 and path 2 shows up equally in the weak measurements performed on photons found in path 1 and on photons found in path 2.

By analogy, the same should be true for Schrödinger’s cat. Theoretically, there exists a microscopic interference pattern that arises from the superposition of death and survival. If we were to perform weak measurement tomography on the cat before confirming its death or survival, we would find that (a) the dead cats were already dead when the tomography was performed and (b) both dead and living cats showed the microscopic signature of a “+” superposition before the final measurement confirmed their death or survival.

IV. ENTANGLEMENT AND NON-LOCALITY

Entanglement is widely considered to be the most counter-intuitive feature of quantum mechanics. In many introductions of entanglement, it is assumed that the non-local change of the wavefunction caused by a local measurement is an effect that has no classical explanation. However, such claims are rather misleading, since the same non-local change is caused by a classical (Bayesian) update of probabilities if the initial probability distribution (or prior) includes correlations between the two systems. In the following, I will show that this is indeed the correct analogy.

Consider a maximally entangled state \(|E\rangle\) of two d-level systems, \( A \) and \( B \). The notion of non-locality arises because any measurement represented by a projection on a pure state \(|f\rangle\) in \( A \) results in a corresponding pure state \(|f^*\rangle\) in \( B \),

\[
A\langle f | E \rangle_{AB} = \frac{1}{\sqrt{d}} |f^*\rangle_B.
\]

However, the density matrix of the maximally entangled state can already be written as a mixture of \( d \) alternatives \( f \) given by

\[
\hat{R}_{E f} = \frac{d}{2} (|E\rangle\langle E|)_{AB} (|f\rangle\langle f|)_A
\]

\[
+ (|f\rangle\langle f|)_A (|E\rangle\langle E|)_{AB}
\]

\[
= \frac{\sqrt{d}}{2} (|E\rangle\langle f; f^*| + |f; f^*\rangle\langle E|).
\]

Here, the coherences between \(|f; f^*\rangle\) and orthogonal components of \(|E\rangle\) represent the non-local correlations between properties of \( A \) and \( B \) other than \( f \) and \( f^* \). Weak measurement tomography therefore shows that (a) the property \( f^* \) was already present in \( B \) before the measurement in \( A \), and (b) the two systems are strongly correlated in the properties of the systems other than those determined by \( f \) and \( f^* \) until the correlations are lost in the back-action related randomization of the local properties in system \( A \).

One significant consequence of this analysis is that the physical back-action of the measurement is completely local. The changes of the quantum state in \( B \) are completely explained by the selection of an alternative already present in the initial density matrix. To confirm
that system $B$ is already in the pure state $| f^* \rangle$ even before the measurement in $A$ is performed, we can trace out system $A$ to find the local statistics described by $R_{Ef}$,

\[
\text{Tr}_A\{R_{Ef}\} = d(f | E)\langle E | f \rangle = | f^* \rangle\langle f^* | .
\]

(8)

Initially, system $B$ is in an incoherent mixture of the states $| f^* \rangle$, and the measurement in $A$ merely identifies the alternative that is actually realized in a given individual case. Thus, the non-locality of entanglement does not involve any physical non-locality, it may be of particular interest to explain Bell’s inequality violation in terms of negative joint probabilities. In weak measurements, the possibility of negative joint probabilities shows up in the form of negative eigenvalues of the density matrix $\hat{R}_{ij}$.

If $\hat{\Phi}_g$ is the operator for the probability of obtaining an outcome $g$ in a given measurement, then the probability of $g$ obtained in weak measurements is given by

\[
p(g|i, f) = \text{Tr}\{\hat{R}_{ij}\hat{\Phi}_g\},
\]

(9)
as in conventional measurement theory. This formalism accurately describes the negative probabilities observed in experimental resolutions of quantum paradoxes by weak measurements such as the one reported in [8]. Alternatively, it is possible to define the joint probability of $f$ and $g$ for the initial state $\hat{\rho}_i$, as shown in [9]. The result is a joint probability given by

\[
p(f, g|i) = \text{Tr}\{\hat{\mu}^\dagger \frac{1}{2} (\hat{I}_f\hat{\Phi}_g + \hat{\Phi}_g\hat{I}_f)\}.
\]

(10)

Since this definition of joint probabilities is the only one consistent with weak measurement tomography, it is independent of the specific experimental realization which is used to confirm it. In particular, it does not matter whether $g$ or $f$ is obtained in the weak measurement. Moreover, the joint probability of $f$ and $g$ is consistent with all strong measurements of $g$ and $f$. It is therefore possible to explain quantum paradoxes constructed from the measurement results of strong measurements in terms of the negative joint probabilities observed in weak measurements.

Since it was shown in the previous section that entanglement does not involve any physical non-locality, it may be of particular interest to explain Bell’s inequality violation in terms of negative joint probabilities. In the formulation of Bell’s inequalities, values of $\pm 1$ are assigned to orthogonal components $X$ and $Y$ of a two level system. The operators $\hat{\Pi}(X, Y)$ describing the joint probabilities can be obtained from the projectors $(1 \pm \hat{X})/2$ and $(1 \pm \hat{Y})/2$. They read

\[
\hat{\Pi}(+1; +1) = \frac{1}{4}(1 + \hat{X} + \hat{Y})
\]

\[
\hat{\Pi}(+1; -1) = \frac{1}{4}(1 + \hat{X} - \hat{Y})
\]

\[
\hat{\Pi}(-1; +1) = \frac{1}{4}(1 - \hat{X} + \hat{Y})
\]

\[
\hat{\Pi}(-1; -1) = \frac{1}{4}(1 - \hat{X} - \hat{Y}).
\]

(11)

The eigenvalues of each of these operators are $(1 \pm \sqrt{2})/4$, corresponding to probabilities of $60\%$ and $-10\%$. The eigenstates are those of the spin operator $\hat{S}_z$ along the diagonal between $\hat{X}$ and $\hat{Y}$ for $\hat{\Pi}(+1; +1)$ and $\hat{\Pi}(-1; -1)$, and those of the orthogonal spin component $\hat{S}_x$ for $\hat{\Pi}(+1; -1)$ and $\hat{\Pi}(-1; +1)$. If the initial state is maximally entangled, the values of $S_+ = \pm 1$ and $S_- = \pm 1$ in system $B$ can be determined by measurements in system $A$. The joint probabilities for $X$, $Y$, $S_+$ and $S_-$ then result in the following total probabilities,

- $60\%$ for $(X + Y)S_+ = +2$
- $60\%$ for $(X + Y)S_+ = 0$
- $-10\%$ for $(X + Y)S_+ = -2$
- $-10\%$ for $(X + Y)S_+ = 0$

Clearly, the value of $(X + Y)S_+ + (X - Y)S_-$ is either $+2$ or $-2$. However, local negative probabilities result in an average of $2\sqrt{2}$.

This result can be confirmed experimentally by weak measurements of the joint probabilities. Significantly, the negative probabilities originate from some mysterious action at a distance, but from the local correlations between the spin components of the same system. The paradox of Bell’s inequality violation is therefore resolved in favor of locality and against realism.

VI. THE PROBLEM WITH REALITY

Weak measurement tomography shows that a measurement result $f$ can be considered real even before the measurement is performed. Specifically, the quantum statistics of a pure state $\hat{\rho}_i = | \psi \rangle\langle \psi |$ can be decomposed into a mixture of non-positive statistical operators $R_{ij}$ in anticipation of the measurement $f$. However, the decomposition is only justified if the measurement of $f$ is actually performed. The negative joint probabilities of measurements that cannot be performed jointly indicate that reality cannot be simultaneously attributed to the outcomes $f$ and $g$ of both measurements.
Weak measurements therefore indicate that quantum paradoxes arise from the assumption of a measurement independent reality. In everyday life, we naturally assume that such a reality exists. Obviously, it would be crazy to assume that a room ceases to exist when we close the door from the outside. However, it may be foolish to extrapolate this experience without reflecting on its foundations. In fact, almost every philosopher who thought deeply about the origin of our sense of reality noticed that it is somehow rooted in experience and perception. The list includes famous names such as Descartes (“I think therefore I am”), Berkeley (“to exist is to be perceived”), Kant (no experience of the “thing in itself”), and Schopenhauer (no object without subject). In practice, objective reality is established through the consistency of observations. As Kant explained in perhaps excessive detail, we identify realities by extending our experience using chains of causality relations. Normally, this results in a tightly knit web with very little wiggle space—especially in the physical sciences. Nevertheless, the primary condition for the reality of an object is always the possibility of experiencing its effects.

The uncertainty principle of quantum mechanics defines an absolute limit for the effects of an individual quantum object on the web of empirical reality. However, quantum mechanics can provide a complete description of the measurement statistics obtained from a certain class of objects. Intuitively, we try to identify the individual representative with the whole class, and are therefore confused by the appearance of irreconcilable differences between representatives observed in different ways. However, there is really no reason why a particle with the observable property \( f \) should also have a hidden property \( g \), just because this property has been observed on a different member of the same class of objects.

Ultimately, our imagination is pushed to its limits when it comes to quantum mechanics. Nevertheless, we might achieve some insights by identifying the correct analogies and rejecting false and misleading ones. As far as reality is concerned, it may be useful to discard the notion of fundamental material objects in favor of a model that emphasizes the role of perspective. Quantum objects seem to show us only their front, whereas the backside is hidden by uncertainty. When we look at different objects of the same class, we can construct something that looks like a complete image, but there seem to be inconsistencies when we try to identify the backside of an individual object. It is almost as if the individual quantum objects were stage props showing different sides of the same building in different scenes of a play. In our minds, we construct the complete building, and for the purpose of understanding the action on the stage, this is quite useful. But a look backstage would show that the people who built the set did not really bother to construct the whole building, but achieved their purpose much more economically with just a bit of wood and canvas. Now, should we be disappointed by this absence of “realism”? If the world is a stage, this might actually depend on whether the kind of realism we have in mind has any impact on our appreciation of the plot.

VII. CONCLUSIONS

Empirically, quantum mechanics is a statistical theory that predicts only the probabilities of events. In the case of weak measurements, it is particularly important to keep this in mind since the results of weak measurements are averages over a large number of individual measurements. Nevertheless weak measurements can provide a complete description of a quantum state defined by both initial conditions \( i \) and final conditions \( f \). The non-positive density matrix \( R_{if} \) of this state can be determined experimentally by weak measurement tomography [8].

Weak measurement tomography shows that the collapse of the wavefunction caused by the final measurement \( f \) is very similar to the collapse of a classical probability distribution when additional information becomes available. In fact, the initial state \( \hat{\rho}_i \) can always be written as a mixture of the non-positive density matrices \( R_{if} \) associated with the different possible outcomes \( f \). Coherences between different \( f \) are divided equally between the alternative outcomes.

In the case of entangled states, the non-locality of the collapse of the wavefunction can be explained entirely in terms of the statistical correlations between the systems. The non-locality is therefore statistical and corresponds to the non-locality of an update of probability in classically correlated systems. Quantum paradoxes arise only from the negative eigenvalues of \( R_{if} \) that define negative joint probabilities for the outcome \( f \) and the outcome \( g \) of an alternative measurement.

While it is possible to attribute reality to the measurement result \( f \) even before the measurement happens, it is not possible to attribute reality to a measurement result that is not actually obtained. The negativity of joint probabilities shows that any assignment of hidden variables would be inconsistent with the empirical results obtained in weak measurements. Weak measurements therefore indicate that non-empirical realism (that is, the dogmatic insistence on a measurement independent reality) is inconsistent with the statistical predictions of quantum mechanics. Thus, weak measurements resolve quantum paradoxes by identifying non-empirical realism as the crucial fallacy in the formulation of the paradox.

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