Inverse problems for one dimensional conformable fractional Dirac type integro differential system

Baki Keskin

Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, Turkey
E-mail: bkeskin@cumhuriyet.edu.tr

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Abstract
In this paper, one dimensional conformable fractional Dirac-type integro differential system is considered. The asymptotic formulae for the solutions, eigenvalues and nodal points are obtained. We investigate the inverse nodal problem and give an effective procedure for solving the inverse nodal problem with respect to given a dense subset of nodal points.

Keywords: conformable fractional Dirac system, intego-differential operators, inverse nodal problem

1. Introduction

The Dirac operator is the relativistic Schrödinger operator in quantum physics. The basic and comprehensive results about Dirac operators were given in [34]. Inverse problems for the Dirac operators have been extensively well studied in various publications (see [14, 18, 20, 24–26, 42] and the references therein). The subject of fractional calculus has acquired significant popularity and major attention from several authors in various science due mainly to its direct involvement in the problems of differential equations in mathematics, physics (classic and quantum mechanics, thermodynamics, etc), engineering, signal and image processing, control theory and others. Fractional calculus is also a powerful and effective tool for modelling nonlinear systems. This topic is initiated by [1, 29]. In the past few years, fractional calculus has been investigated by several author [2, 3] and references therein. In recent years, scholars have focussed on a fractional generalization of the well known Sturm–Liouville and Dirac problems [4–8, 16, 19, 30, 31, 40, 53].

Inverse nodal problem was started for the Sturm–Liouville operator by McLaughlin [36] in 1988. In 1989, Hald and McLaughlin showed that it is sufficient to know just the nodal points to determine the potential function of the regular Sturm–Liouville problem with more general boundary conditions and gave some numerical schemes for the reconstruction of the potential...
from nodal points [23]. Yang proposed an algorithm to solve an inverse nodal problem for the Sturm–Liouville operator in 1997 [47]. Such problems have been considered by several researchers in [11, 15, 21, 37, 39, 41, 43, 44, 46, 48, 50, 51] and other works. The inverse nodal problems for the Dirac operators with various boundary conditions have been studied and shown that a dense subset of the zeros of the first component of the eigenfunctions alone can determine the coefficients of discussed problem [22, 49, 52]. In [38], the authors have developed the spectral theory for a conformable fractional Sturm–Liouville problem and have proved uniqueness theorem with respect to the nodal points.

Nowadays, the studies concerning the perturbation of a differential operator by a Volterra type integral operator, namely the integro-differential operator has acquired significant popularity and major attention from several authors and take significant place in the literature [9, 10, 12, 13, 17, 32, 33, 45]. Integro-differential operators are nonlocal, and therefore they are more difficult for investigation, than local ones. New methods for solution of these problems are being developed. The inverse nodal problem for Dirac type integro-differential operators was first studied by Keskin and Ozkan in [27]. In their study, it is shown that the coefficients of the differential part of the operator can be determined by using nodal points and nodal points also gives the partial information about integral part. In [28], the authors considered Dirac type integro-differential operators with boundary conditions depend on the spectral parameter linearly.

2. Conformable fractional preliminaries

Firstly, we want to recall some basic definitions and properties of conformable fractional calculus which can be found in [1, 29].

**Definition 1.** Let $f : [0, \infty) \to \mathbb{R}$ be a given function. Then the conformable fractional derivative of $f$ of order $\alpha$ is defined by:

$$
D^\alpha_{t} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad D^\alpha_{t} f(0) = \lim_{t \to 0^+} D^\alpha f(t),
$$

for all $t > 0$, $\alpha \in (0, 1]$. If this limit exist and finite at $t_0$, we say $f$ is $\alpha -$differentiable at $t_0$. Note that if $f$ is differentiable, then $D^\alpha f(t) = t^{1-\alpha} f'(t)$.

**Definition 2.** The conformable fractional integral starting from 0 of order $\alpha$ is defined by

$$
I^\alpha_{t} f(t) = \int_{0}^{t} f(x) \, d_{C} x = \int_{0}^{x} x^{\alpha-1} f(x) \, d x, \quad \text{for all } t > 0.
$$

**Lemma 1.** Let $f : [a, \infty) \to \mathbb{R}$ be any continuous function. Then, for all $t > a$, we have $D^\alpha_{t} I^\alpha_{t} f(t) = f(t)$.

**Lemma 2.** Let $f : (a, b) \to \mathbb{R}$ be any differentiable function. Then, for all $t > a$, we have $I^\alpha_{t} D^\alpha_{t} f(t) = f(t) - f(a)$.

**Theorem 1** ($\alpha$-integration by parts). Let $f, g : [a, b] \to \mathbb{R}$ be two conformable fractional differentiable functions. Then,

$$
\int_{a}^{b} f(t)D^\alpha_{t} g(t) \, d_{C} t = f(b)g(b) - f(a)g(a) - \int_{a}^{b} g(t)D^\alpha_{t} f(t) \, d_{C} t
$$
**Theorem 2** (α-Leibnitz rule). Let \( f^\alpha(x,t) \) and \( f^\alpha_a(x,t) \) be continuous in \( x \) on some regions of the \((x,t)\)-plane, including \( a(t) \leq t \leq b(t) \), \( x_0 \leq x \leq x_1 \). If \( a(x) \) and \( b(x) \) are both \( \alpha \)-differentiable for \( x_0 \leq x \leq x_1 \), then

\[
D_x^\alpha \left[ \int_{a(x)}^{b(x)} f(x,t) \, dt \right] = f(x,b(x))D_x^\alpha b(x) - f(x,a(x))a^{\alpha-1}(x)D_x^\alpha a(x) + \int_{a(x)}^{b(x)} D_x^\alpha f(x,t) \, dt.
\]

**Definition 3.** The space \( C_\alpha^n[a,b] \) consists of all functions defined on the interval \([a, b]\) which are continuously \( \alpha \)-differentiable up to order \( n \).

### 3. Conformable fractional Dirac systems

In this work, we consider the following one-dimensional conformable fractional Dirac type integro-differential system

\[
BY + \Omega(x)Y + \int_0^x M(x,t)Y_d\,dt = \lambda Y, \quad x \in (0,\pi),
\]

with the boundary conditions

\[
y_1(0) \sin \theta + y_2(0) \cos \theta = 0 \tag{2}
\]

\[
y_1(\pi) \sin \beta + y_2(\pi) \cos \beta = 0 \tag{3}
\]

where, \( 0 \leq \theta, \beta < \pi \) are real numbers, \( \lambda \) is the spectral parameter, \( B = \begin{pmatrix} 0 & D_x^\alpha \\ -D_x^\alpha & 0 \end{pmatrix} \), \( \Omega(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}, \ M(x,t) = \begin{pmatrix} M_{11}(x,t) & M_{12}(x,t) \\ M_{21}(x,t) & M_{22}(x,t) \end{pmatrix}, \ Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ p(x), \ r(x), \ M(x,t) \) are real-valued conformable fractional differentiable functions and \( x^{\alpha-1}p(x) \) and \( x^{\alpha-1}r(x) \) are continuous on \((0,\pi)\).

Let \( \varphi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda))^T \) be the solution of (1) satisfying the initial condition \( \varphi(0,\lambda) = (\cos \theta, -\sin \theta)^T \). It is clear that \( \varphi(x,\lambda) \) is an entire function of \( \lambda \) satisfies the following conformable fractional Volterra integral equations:

\[
\varphi_1(x,\lambda) = \cos \theta \cos \left( \lambda x^{\alpha-1} \right) + \sin \theta \sin \left( \lambda x^{\alpha-1} \right) + \int_0^x \sin \left( \lambda x^{\alpha-1} - \lambda x^{\alpha-1} \right) p(t)\varphi_1(t,\lambda)\,dt + \int_0^x \cos \left( \lambda x^{\alpha-1} - \lambda x^{\alpha-1} \right) r(t)\varphi_2(t,\lambda)\,dt
\]

\[
+ \int_0^x \int_0^t \sin \left( \lambda x^{\alpha-1} - \lambda x^{\alpha-1} \right) \left[ M_{11}(t,\xi)\varphi_1(\xi,\lambda) + M_{12}(t,\xi)\varphi_2(\xi,\lambda) \right] \, d_\xi d_\xi + \int_0^x \cos \left( \lambda x^{\alpha-1} - \lambda x^{\alpha-1} \right) \left[ M_{21}(t,\xi)\varphi_1(\xi,\lambda) + M_{22}(t,\xi)\varphi_2(\xi,\lambda) \right] \, d_\xi d_\xi \tag{4}
\]
\[
\varphi_2(x, \lambda) = \cos \theta \sin \left( \frac{\lambda x^o}{\alpha} \right) - \sin \theta \cos \left( \frac{\lambda x^o}{\alpha} \right) \\
- \int_0^x \cos \left( \frac{\lambda x^o - \epsilon'}{\alpha} \right) p(t) \varphi_1(t, \lambda) d_u t + \int_0^x \sin \left( \frac{\lambda x^o - \epsilon'}{\alpha} \right) r(t) \varphi_2(t, \lambda) d_u t \\
- \int_0^x \int_0^\epsilon' \cos \left( \frac{\lambda x^o - \epsilon'}{\alpha} \right) \left[ M_{11}(t, \xi) \varphi_1(\lambda, \xi) + M_{12}(t, \xi) \varphi_2(\lambda, \xi) \right] d_u \xi d_u t \\
+ \int_0^x \int_0^\epsilon' \sin \left( \frac{\lambda x^o - \epsilon'}{\alpha} \right) \left[ M_{21}(t, \xi) \varphi_1(\lambda, \xi) + M_{22}(t, \xi) \varphi_2(\lambda, \xi) \right] d_u \xi d_u t
\]

(5)

The proof of the following lemma is clear from [35] (lemma 1.3.1).

**Lemma 3.** Let \( f(x) \) be a function in \( C^4_0[0, \pi] \), \( \alpha \in (0, 1) \), then

\[
\lim_{|\lambda| \to \infty} \exp \left( - |\text{Im} \frac{\pi^0}{\alpha}| \right) \int_0^\pi f(x) \cos \frac{\lambda x^o}{\alpha} d_u x = 0
\]

and

\[
\lim_{|\lambda| \to \infty} \exp \left( - |\text{Im} \frac{\pi^0}{\alpha}| \right) \int_0^\pi f(x) \sin \frac{\lambda x^o}{\alpha} d_u x = 0
\]

**Theorem 3.** For \( |\lambda| \to \infty \), the following asymptotic formulae are valid:

\[
\varphi_2(x, \lambda) = \cos \theta \sin \left( \frac{\lambda x^o}{\alpha} - \mu(x) - \theta \right) - \frac{1}{2\lambda} \nu(0) \cos \left( \frac{\lambda x^o}{\alpha} - \mu(x) + \theta \right) + \frac{1}{2\lambda} \sin \left( \frac{\lambda x^o}{\alpha} - \mu(x) - \theta \right) \int_0^x \nu^2(t) d_u t \\
- \frac{1}{2\lambda} K(x) \cos \left( \frac{\lambda x^o}{\alpha} - \mu(x) - \theta \right) - \frac{1}{2\lambda} L(x) \sin \left( \frac{\lambda x^o}{\alpha} - \mu(x) - \theta \right) \\
+ o \left( \frac{1}{\lambda} \exp(|\tau| \frac{x^o}{\alpha}) \right).
\]

(6)

\[
\varphi_2(x, \lambda) = \sin \left( \frac{\lambda x^o}{\alpha} - \mu(x) - \theta \right) - \frac{1}{2\lambda} \nu(0) \sin \left( \frac{\lambda x^o}{\alpha} - \mu(x) + \theta \right) + \frac{1}{2\lambda} \cos \left( \frac{\lambda x^o}{\alpha} - \mu(x) - \theta \right) \int_0^x \nu^2(t) d_u t \\
- \frac{1}{2\lambda} K(x) \sin \left( \frac{\lambda x^o}{\alpha} - \mu(x) - \theta \right) + \frac{1}{2\lambda} L(x) \cos \left( \frac{\lambda x^o}{\alpha} - \mu(x) - \theta \right) \\
+ o \left( \frac{1}{\lambda} \exp(|\tau| \frac{x^o}{\alpha}) \right).
\]

uniformly in \( x \in [0, \pi] \), where \( \mu(x) = \frac{1}{2} \int_0^x (p(t) + r(t)) d_u t, \ \nu(x) = \frac{1}{2} (p(x) - r(x)), \ K(x) = \int_0^x (M_{11}(t, \xi) - M_{22}(t, \xi)) d_u \xi d_u t, \ L(x) = \int_0^x (M_{12}(t, \xi) - M_{21}(t, \xi)) d_u \xi d_u t \) and \( \tau = \text{Im} \lambda. \)
Proof. We denote

\[ \varphi_{1,0}(x, \lambda) = \cos \left( \frac{\lambda x^0}{\alpha} - \theta \right), \]

\[ \varphi_{1,n+1}(x, \lambda) = \int_0^x \sin \left( \frac{\lambda x^0 - t^0}{\alpha} \right) p(t) \varphi_{1,n}(t, \lambda) dt + \int_0^x \cos \left( \frac{\lambda x^0 - t^0}{\alpha} \right) r(t) \varphi_{2,n}(t, \lambda) dt + \int_0^x \int_0^x \sin \left( \frac{\lambda x^0 - t^0}{\alpha} \right) \{ M_{11}(t, \xi) \varphi_{1,n}(\lambda, \xi) + M_{12}(t, \xi) \varphi_{2,n}(\lambda, \xi) \} \, d\xi \, d\eta, \]

\[ \varphi_{2,0}(x, \lambda) = \sin \left( \frac{\lambda x^0}{\alpha} - \theta \right), \]

\[ \varphi_{2,n+1}(x, \lambda) = -\int_0^x \cos \left( \frac{\lambda x^0 - t^0}{\alpha} \right) p(t) \varphi_{1,n}(t, \lambda) dt - \int_0^x \sin \left( \frac{\lambda x^0 - t^0}{\alpha} \right) r(t) \varphi_{2,n}(t, \lambda) dt - \int_0^x \int_0^x \cos \left( \frac{\lambda x^0 - t^0}{\alpha} \right) \{ M_{11}(t, \xi) \varphi_{1,n}(\lambda, \xi) + M_{12}(t, \xi) \varphi_{2,n}(\lambda, \xi) \} \, d\xi \, d\eta, \]

applying successive approximations method to the equations (4) and (5) and using lemma 3, we get the estimates (6) and (7).

The characteristic function \( \Delta(\lambda) \) of the problem (1)–(3) is defined by the relation

\[ \Delta(\lambda) = \varphi_1(\pi, \lambda) \sin \beta + \varphi_2(\pi, \lambda) \cos \beta, \quad (8) \]

It is obvious that \( \Delta(\lambda) \) is an entire function and its zeros, namely \( \{ \lambda_n \}_{n \in \mathbb{Z}} \), coincide with the eigenvalues of the problem (1)–(3). Using the asymptotic formulae (6) and (7), one can easily obtain

\[ \Delta(\lambda) = \sin \left( \frac{\lambda x^0}{\alpha} - \mu(x) - \theta + \beta \right) - \frac{1}{2\lambda} v(x) \sin \left( \frac{\lambda x^0}{\alpha} - \mu(x) - \theta - \beta \right) - \frac{1}{2\lambda} v(0) \sin \left( \frac{\lambda x^0}{\alpha} - \mu(x) + \theta + \beta \right) - \frac{1}{2\lambda} \cos \left( \frac{\lambda x^0}{\alpha} - \mu(x) - \theta + \beta \right) \int_0^x v(t) \, dt - \frac{1}{2\lambda} K(x) \sin \left( \frac{\lambda x^0}{\alpha} - \mu(x) - \theta + \beta \right) + \frac{1}{2\lambda} L(x) \cos \left( \frac{\lambda x^0}{\alpha} - \mu(x) - \theta + \beta \right) + o \left( \frac{1}{\lambda} \exp(\tau |\lambda x^0|) \right), \quad (9) \]

for sufficiently large \( |\lambda| \). Since the eigenvalues of the problem (1)–(3) are the roots of \( \Delta(\lambda_n) = 0 \), we can write the following equation for them:
\[
\left( 1 - \frac{1}{2\lambda_n} v(\pi) \cos 2\beta - \frac{1}{2\lambda_n} v(0) \cos 2\theta - \frac{1}{2\lambda_n} K(\pi) \right) \tan(\lambda_n^{\frac{\pi^\alpha}{\alpha}} - \mu(\pi) - \theta + \beta) \\
= - \frac{1}{2\lambda_n} v(\pi) \sin 2\beta - \frac{1}{2\lambda_n} v(0) \sin 2\theta + \frac{1}{2\lambda_n} \int_0^\pi v^2(t) dt - \frac{1}{2\lambda_n} L(\pi) + o \left( \frac{1}{\lambda_n} \right)
\]

which implies that

\[
\tan(\lambda_n^{\frac{\pi^\alpha}{\alpha}} - \mu(\pi) - \theta + \beta) = \left( 1 - \frac{1}{2\lambda_n} v(\pi) \cos 2\beta - \frac{1}{2\lambda_n} v(0) \cos 2\theta - \frac{1}{2\lambda_n} K(\pi) \right)^{-1} \\
\times \left( - \frac{1}{2\lambda_n} v(\pi) \sin 2\beta - \frac{1}{2\lambda_n} v(0) \sin 2\theta + \frac{1}{2\lambda_n} \int_0^\pi v^2(t) dt - \frac{1}{2\lambda_n} L(\pi) + o \left( \frac{1}{\lambda_n} \right) \right)
\]

for sufficiently large \( n \).

We obtain from the last equation,

\[
\lambda_n = \frac{\alpha}{\pi^{\alpha - n}} + \theta + \frac{\mu(\pi) - \beta}{\pi^n} \\
+ \frac{\alpha}{2\pi \pi^n} \left( v(\pi) \sin 2\beta - v(0) \sin 2\theta + \int_0^\pi v^2(t) dt - L(\pi) \right) + o \left( \frac{1}{n} \right) \quad (10)
\]

for \( |n| \to \infty \).

4. Main results

In this section, we obtain the asymptotic formula for the nodal points of considered problem and prove an inverse nodal problem for the one-dimensional conformable fractional Dirac-type integro differential system.

**Lemma 4.** For sufficiently large \( n \), the first component \( \varphi_1(x; \lambda_n) \) of the eigenfunction \( \varphi(x; \lambda_n) \) has exactly \( n \) nodes \( \{ x_j^n : j = 0, 1, \ldots, n - 1 \} \) in the interval \( (0, \pi) \): \( 0 < x_0^n < x_1^n < \cdots < x_{n-1}^n < \pi \). The numbers \( \{ x_j^n \} \) satisfy the following asymptotic formula:

\[
\left( x_j^n \right)^\alpha = \frac{(j + 1/2)^{\pi^\alpha}}{n} + \frac{\mu(x_0^n) + \theta}{n \pi^{1-\alpha}} - \frac{(j + 1/2)^{\pi^\alpha}}{n \pi} \left( \theta + \mu(\pi) - \beta \right) - \frac{\theta + \mu(\pi) - \beta}{\pi^{2-\alpha} n^2} \\
\times \left( \mu(x_0^n) + \frac{\alpha}{2\pi n^2} \left( v(0) \sin 2\theta + \int_0^{x_0^n} v^2(t) dt - L(x_0^n) \right) + o \left( \frac{1}{n^2} \right) \right) \quad (11)
\]

**Proof.** From (6), the following asymptotic formula can be written for sufficiently large \( n \)

\[
\varphi_1(x; \lambda_n) = \cos \left( \lambda_n^{\frac{x^\alpha}{\alpha}} - \mu(x) - \theta \right) + \frac{1}{2\lambda_n} v(x) \cos \left( \lambda_n^{\frac{x^\alpha}{\alpha}} - \mu(x) - \theta \right) \\
- \frac{1}{2\lambda_n} v(0) \cos \left( \lambda_n^{\frac{x^\alpha}{\alpha}} - \mu(x) + \theta \right) + \frac{1}{2\lambda_n} \sin \left( \lambda_n^{\frac{x^\alpha}{\alpha}} - \mu(x) - \theta \right) \int_0^x v^2(t) dt \\
- \frac{1}{2\lambda_n} K(x) \cos \left( \lambda_n^{\frac{x^\alpha}{\alpha}} - \mu(x) - \theta \right) - \frac{1}{2\lambda_n} L(x) \sin \left( \lambda_n^{\frac{x^\alpha}{\alpha}} - \mu(x) - \theta \right) \\
+ o \left( \frac{1}{\lambda_n} \exp|\tau| \frac{x^\alpha}{\alpha} \right),
\]
from \( \varphi_1((x^j)^\alpha, \lambda_n) = 0 \), we get

\[
\cos \left( \lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta \right) = -\frac{1}{2\lambda_n} v(x^j_\alpha) \cos \left( \lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta \right)
\]

\[
+ \frac{1}{2\lambda_n} v(0) \cos \left( \lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta \right) \cos 2\theta
\]

\[
- \frac{1}{2\lambda_n} v(0) \sin \left( \lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta \right) \sin 2\theta
\]

\[
- \frac{1}{2\lambda_n} \sin \left( \lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta \right) \int_0^\alpha v^2(t)dt t
\]

\[
+ \frac{1}{2\lambda_n} K(x^j_\alpha) \cos \left( \lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta \right)
\]

\[
+ \frac{1}{2\lambda_n} L(x^j_\alpha) \sin \left( \lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta \right) + o \left( \frac{1}{\lambda_n} \right),
\]

\[
\left( 1 + \frac{1}{2\lambda_n} v(x^j_\alpha) - \frac{1}{2\lambda_n} v(0) \cos 2\theta - \frac{1}{2\lambda_n} K(x^j_\alpha) \right) \tan \left( \lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta - \frac{\pi}{2} \right)
\]

\[
= \frac{1}{2\lambda_n} v(0) \sin 2\theta + \frac{1}{2\lambda_n} \int_0^\alpha v^2(t)dt t - \frac{1}{2\lambda_n} L(x^j_\alpha) + o \left( \frac{1}{\lambda_n} \right),
\]

Taking into account Taylor’s expansion formula for the arctangent, we get

\[
\lambda_n \frac{(x^j)^\alpha}{\alpha} - \mu(x^j_\alpha) - \theta - \frac{\pi}{2} = j\pi + \frac{1}{2\lambda_n} \left( v(0) \sin 2\theta + \int_0^\alpha v^2(t)dt t - L(x^j_\alpha) \right) + o \left( \frac{1}{\lambda_n} \right).
\]

It follows from the last equality

\[
\frac{(x^j)^\alpha}{\alpha} = \frac{(j + \frac{1}{2}) \pi + \mu(x^j_\alpha) + \theta}{\lambda_n} + \frac{1}{2\lambda_n^2} \left( v(0) \sin 2\theta + \int_0^\alpha v^2(t)dt t - L(x^j_\alpha) \right) + o \left( \frac{1}{\lambda_n} \right).
\]

The relation (11) is proven by using the asymptotic formula

\[
\lambda_n^{-1} = \frac{\pi^{n-1}}{2n\alpha} \left( 1 - \frac{\mu(\pi) + \theta - \beta}{\pi} - \frac{v(\pi) \sin 2\beta - v(0) \sin 2\theta + \int_0^\alpha v^2(t)dt t - L(\pi)}{2\pi^2} \right)
\]

\[
+ o \left( \frac{1}{n^2} \right)
\]

\[
\square
\]

Let \( X \) be the set of nodal points. For each fixed \( x \in (0, \pi) \) and \( \alpha \in (0, 1] \) we can choose a sequence \( \{x^j_\alpha\} \subset X \) so that \( x^j_\alpha \) converges to \( x \). Then the following limits are exist and finite:

\[
\lim_{|n| \to \infty} n \left( (x^j_\alpha)^\alpha - \frac{(j + 1/2) \pi^n}{n} \right) = f(x).
\]
where

$$f(x) = \frac{\mu(x) + \theta}{\pi^{1-\alpha}} = \frac{x^\alpha}{\pi} (\theta + \mu(\pi) - \beta)$$  \hspace{1cm} (12)$$

and

$$\lim_{|n| \to \infty} 2n^2 \left( (x_n^0)^\alpha - \frac{(j + 1/2) \pi^\alpha}{n} - \frac{\mu(x_j^0) + \theta}{n \pi^{1-\alpha}} + \frac{(j + 1/2) \pi^\alpha}{n} \left( \frac{\theta + \mu(\pi) - \beta}{n \pi} \right) \right) = g(x),$$

where

$$g(x) = \alpha \left( \nu(0) \sin 2\theta + \int_0^x \nu^2(t) \, dt - L(x) \right)$$  \hspace{1cm} (13)$$

Therefore, proof of the following theorem is clear.

**Theorem 4.** Let $\mu(\pi) = 0$. The given dense subset of nodal points $X$ uniquely determines the coefficients $\theta$ and $\beta$ of the boundary conditions and if $L(x)$ is known, $X$ also uniquely determines the potential $\Omega(x)$ a.e. on $(0, \pi)$. Moreover, $\Omega(x)$, $L(x)$, $\theta$ and $\beta$ can be reconstructed by the following formulas:

**Step 1:** for each fixed $x \in (0, \pi)$ and $\alpha \in (0, 1]$, choose a sequence $(x_n^0) \subset X$ such that $\lim_{|n| \to \infty} x_n^0 = x$;

**Step 2:** find the function $f(x)$ from (12) and calculate

$$\theta = f(0) \pi^{1-\alpha}$$

$$\beta = f(\pi) \pi^{1-\alpha}$$

$$D^\alpha_x \mu(x) = \pi^{1-\alpha} D^\alpha_x f(x) + \frac{\alpha}{\pi^{2\alpha - 1}} (f(0) - f(\pi))$$  \hspace{1cm} (14)$$

**Step 3:** find the function $g(x)$ from (13) and calculate

$$\nu(x) = \frac{1}{\sqrt{\alpha}} \sqrt{D^\alpha_x (g(x) + \alpha L(x))}$$  \hspace{1cm} (15)$$

**Step 4:** if $L(x)$ is known then from (14) and (15) calculate

$$p(x) = \nu(x) + D^\alpha_x \mu(x)$$

$$r(x) = D^\alpha_x \mu(x) - \nu(x)$$

If $p(x)$ and $r(x)$ are known then from (13) calculate

$$L(x) = \nu(0) \sin 2\theta + \int_0^x \nu^2(t) \, dt - \frac{g(x)}{\alpha}$$

**ORCID iDs**

Baki Keskin \(https://orcid.org/0000-0003-1689-8954\)
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