ON THE FIRST-ORDER DIFFERENTIAL SUBORDINATION AND SUPERORDINATION RESULTS FOR $p$-VALENT FUNCTIONS

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Abstract. In this paper, we obtain some applications of first-order differential subordination, superordination and sandwich-type results involving operator for certain normalized $p$-valent analytic functions. Further, properties of $p$-valent functions such as; $\lambda$-spirallike and $\lambda$-Robertson of complex order are considered.

1. Introduction

Let $\mathcal{H}(U)$ denote the class of holomorphic functions in the open unit disc $U := \{ z \in \mathbb{C} : |z| < 1 \}$ on the complex plane $\mathbb{C}$, and let $\mathcal{H}[a, n]$ denote the subclass of the functions $p \in \mathcal{H}(U)$ of the form:

$$ p(z) = a + a_n z^n + \cdots ; \quad (a \in \mathbb{C}, \ n \in \mathbb{N} := \{1, 2, \ldots \}) . $$

Let $\mathcal{A}_p$ denote the class of all $p$-valent functions $f \in \mathcal{H}$ of the following form:

$$ f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k , \quad (1.1) $$

which are analytic in the open unit disk $U$. The class $\mathcal{A}_1$ denoted by $\mathcal{A}$.

Let $g$ and $h$ be analytic in $U$. We say that the function $g$ is subordinate to $h$, or the function $h$ is superordinate to $g$, and express it by $g \prec h$ or conventionally by $g(z) \prec h(z)$ if $g = h \circ \omega$ for some analytic map $\omega : U \to U$ with $\omega(0) = 0$. When $h$ is univalent, the condition $g \prec h$ is equivalent to $g(U) \subset h(U)$ and $g(0) = h(0)$.

For some non-zero complex numbers $b$ and real $\lambda; \ (|\lambda| < \frac{\pi}{2})$, we define classes $\mathcal{S}_p^\lambda(\alpha, b)$ and $\mathcal{K}_p^\lambda(\alpha, b)$ as follows:

$$ \mathcal{S}_p^\lambda(\alpha, b) := \left\{ f \in \mathcal{A}_p : \text{Re} \left( \frac{1}{b \cos \lambda} \left[ e^{i\lambda} \frac{zf'(z)}{pf(z)} - (1 - b) \cos \lambda - i \sin \lambda \right] \right) > \alpha \right\} , $$

and

$$ \mathcal{K}_p^\lambda(\alpha, b) := \left\{ f \in \mathcal{A}_p : \text{Re} \left( \frac{1}{b \cos \lambda} \left[ \frac{e^{i\lambda} z f''(z)}{pf'(z)} - (1 - b) \cos \lambda - i \sin \lambda \right] \right) > \alpha \right\} . $$

For a function $f$ belonging to the class $\mathcal{S}_p^\lambda(\alpha, b)$, we say that $f$ is multivalent $\lambda$-spirallike of complex order $b$ and type $\alpha; \ (0 \leq \alpha < 1)$ in $U$. Also for a function $f$ belonging to...
the class \( \mathcal{K}_a^\lambda(x, b) \), we say that \( f \) is multivalent \( \lambda \)-Robertson of complex order \( b \) and type \( \alpha; (0 \leq \alpha < 1) \) in \( \mathbb{U} \). This classes for \( \alpha = 0 \) were introduced and studied by Al-Oboudi and Haidan [2].

In particular for \( p = b = 1 \), we denote
\[
\mathcal{S}^\lambda(\alpha) := \mathcal{S}_1^\lambda(\alpha, 1),
\]
is the class of \( \lambda \)-spirallike functions of order \( \alpha \) with \( 0 \leq \alpha < 1 \) and
\[
\mathcal{K}^\lambda(\alpha) := \mathcal{K}_1^\lambda(\alpha, 1),
\]
is the class of \( \lambda \)-Robertson functions of order \( \alpha \) with \( 0 \leq \alpha < 1 \).

Let \( \eta \) and \( \mu \) be complex numbers not both equal to zero and \( f \in \mathcal{A}_p \) given by (1.1). Define the differential operator \( \mathcal{F}^{\eta, \mu}_{p} : \mathcal{A}_p \rightarrow \mathcal{H}[1, 1] \) as follows:
\[
\mathcal{F}^{\eta, \mu}_{p}[f](z) := \left[ \frac{f'(z)}{p z^{p-1}} \right]^\eta \left[ \frac{z^p}{f(z)} \right]^\mu = 1 + \left( \eta - \mu + \frac{\eta}{\mu} \right) a_{p+1} z + \cdots; \quad (z \in \mathbb{U}), \quad (1.2)
\]
with \( \mathcal{F}^{\eta, \mu}_{p}[f](z) \big|_{z=0} = 1 \). Here, all powers are mean as principal values (see [8]).

2. Definitions and Preliminaries

In order to achieve our aim in this section, we recall some definitions and preliminary results from the theory of differential subordination and superordination.

**Definition 1** ([11][12]). Let \( \psi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C} \) and the function \( h(z) \) be univalent in \( \mathbb{U} \). If the function \( p(z) \) is analytic in \( \mathbb{U} \) and satisfies the following first-order differential subordination
\[
\psi(p(z), z p'(z); z) \prec h(z); \quad (z \in \mathbb{U}), \quad (2.1)
\]
then \( p(z) \) is called a solution of the differential subordination.

A function \( q \in \mathcal{H} \) is said to be a dominant of the differential subordination (2.1) if \( p \prec q \) for all \( p \) satisfying (2.1). An univalent dominant that satisfies \( \tilde{q} \prec q \) for all dominants \( q \) of (2.1), is said to be best dominant of the differential subordination.

**Definition 2** ([13]). Let \( \varphi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C} \) and the function \( h(z) \) be univalent in \( \mathbb{U} \). If the function \( p(z) \) and \( \varphi(p(z), z p'(z); z) \) are univalent in \( \mathbb{U} \) and satisfies the following first-order differential superordination
\[
h(z) \prec \varphi(p(z), z p'(z); z); \quad (z \in \mathbb{U}), \quad (2.2)
\]
then \( h(z) \) is called a solution of the differential superordination.

An analytic function \( q \in \mathcal{H} \) is called a subordinant of the solution of the differential superordination (2.2), or more simply a subordinant if \( q \prec p \) for all the functions \( p \) satisfying (2.2). An univalent subordinant that satisfies \( q \prec \tilde{q} \) for all of the subordinants \( q \) of (2.2), is said to be the best subordinant.

Miller and Mocanu [13] obtained sufficient condition on the functions \( p \) and \( q \) for which the following implication holds:
\[
h(z) \prec \varphi(p(z), z p'(z); z) \Rightarrow q(z) \prec p(z).
\]
Using these results, in [5] were obtained sufficient conditions for certain normalized analytic function $f$ to satisfy

$$q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent normalized function in $U$.

**Definition 3** (cf. Miller and Mocanu[10, Definition 2.2b, p.21]). Denote by $Q$, the set of all functions $f(z)$ that are analytic and injective on $\mathbb{C} \setminus E(f)$, where

$$E(f) := \left\{ \zeta \in \partial U \text{ and } \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $\min |f'(\zeta)| = \rho > 0 \text{ for } \zeta \in \partial U \setminus E(f)$.

**Lemma 2.1** (cf. Miller and Mocanu[10, Theorem 3.4h, p.132]). Let $q$ be univalent in $U$, and let $\varphi$ and $\theta$ be analytic in a domain $\Omega$ containing $q(U)$, with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := z\varphi'(z)\varphi(q(z)); h(z) := \theta(q(z)) + Q(z)$ and suppose that

(i) $Q(z)$ is starlike function in $U$,

(ii) $\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \frac{\varphi'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0 \text{ for } z \in U$.

If $p(z)$ is analytic in $U$, with $p(0) = q(0)$, and $p(U) \subset \Omega$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) = h(z); \quad z \in U, \quad (2.3)$$

then $p(z) \prec q(z)$ and $q$ is the best dominant of Eq. (2.3).

**Lemma 2.2** ([17]). Let $q(z)$ be convex function in $U$ and $\gamma \in \mathbb{C}$ with $\Re \{ \gamma \} > 0$. If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + \gamma zp'(z)$ is univalent in $U$, then

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z) \quad (2.4)$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant of Eq. (2.4).

**Lemma 2.3** ([14]). The function

$$q_\lambda(z) := (1 - z)^\lambda \equiv e^{\lambda \log(1 - z)} = 1 - \lambda z + \frac{\lambda(\lambda - 1)}{2} z^2 - \frac{\lambda(\lambda - 1)(\lambda - 2)}{6} z^3 + \ldots$$

for some $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}, z \in U$ is univalent in $U$ if and only if $\lambda$ is either in the closed disk $|\lambda + 1| \leq 1$ or $|\lambda - 1| \leq 1$.

**Lemma 2.4.** For the univalent functions

$$(UF.1) \quad q(z) = (1 + Bz)^\lambda \text{ with } -1 \leq B \leq 1; B \neq 0 \quad \text{ and } \quad \lambda \in \mathbb{C}^* \text{ with } |\lambda + 1| \leq 1 \text{ or } |\lambda - 1| \leq 1,$$

$$(UF.2) \quad \text{and}$$

$$q(z) = \frac{1 + Az}{1 + Bz}; \quad (-1 \leq B < A \leq 1, z \in U),$$

we have

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0; \quad (z \in U). \quad (2.5)$$
Lemma 2.5. Let \( q(z) = (1 + A z) / (1 + B z) \); \((-1 \leq B < A \leq 1, z \in \mathbb{U}) \), then we have
\[
\Re \left\{ 1 + \frac{2q''(z)}{q'(z)} - \frac{2q'(z)}{q(z)} \right\} = \Re \left\{ \frac{1}{1 + B z} \right\} > \frac{1}{1 + |B|} > 0.
\]

UF.2. Let \( q(z) = (1 + A z) / (1 + B z) \); \((-1 \leq B < A \leq 1, z \in \mathbb{U}) \), then we have
\[
\Re \left\{ 1 + \frac{2q''(z)}{q'(z)} - \frac{2q'(z)}{q(z)} \right\} = \Re \left\{ \frac{1 - AB z^2}{(1 + A z)(1 + B z)} \right\}.
\]

The function
\[
p_{A,B}(z) = \frac{1 - AB z^2}{(1 + A z)(1 + B z)}; \quad (-1 \leq B < A \leq 1),
\]
does not have any poles in \( \mathbb{U} \) and is analytic in \( \mathbb{U} \). Then
\[
\min \{ \Re \{ p_{A,B}(z) \}; \ |z| < 1 \}
\]
attains its minimum value on the boundary \( \{ z \in \mathbb{C} : |z| = 1 \} \). If take \( z = e^{i\theta} \) with \( \theta \in (-\pi, \pi] \), then
\[
\Re \left\{ \frac{1 - AB e^{2i\theta}}{(1 + A e^{i\theta})(1 + B e^{i\theta})} \right\} = \frac{(1 - AB)[1 + AB + (A + B) \cos \theta]}{|1 + A e^{i\theta}|^2 |1 + B e^{i\theta}|^2}. \tag{2.6}
\]
If \( A + B \geq 0 \), it follows that \( 1 + AB + (A + B) \cos \theta \geq (1 - A)(1 - B) \geq 0 \), and if \( A + B \leq 0 \), it follows that \( 1 + AB + (A + B) \cos \theta \geq (1 + A)(1 + B) \geq 0 \). Therefore, the minimum value of expression (2.6) is equal to 0. \( \square \)

Lemma 2.5 \((7)\). Let \( q \) be function in \( \mathbb{U} \) with \( q(0) \neq 0 \). If \( q \) satisfy the condition (2.5), then for all \( z \in \mathbb{U} \), \( q(z) \neq 0 \).

Lemma 2.6. For the function \( q(z) = (1 + A z) / (1 + B z); -1 \leq B < A \leq 1, z \in \mathbb{U} \) the condition
\[
\Re \left\{ 1 + \frac{2q''(z)}{q'(z)} \right\} > \max \{ 0, -\Re(\zeta) \}; \quad (z \in \mathbb{U}, \zeta \in \mathbb{C}), \tag{2.7}
\]
equivalent to \( \Re \{ \zeta \} \geq \frac{|B|-1}{|B|+1} \).

Proof. The function \( \omega(z) = 1 + \frac{2q''(z)}{q'(z)} = 1 + B z^2 / (1 + B z); (-1 \leq B < A \leq 1, B \neq 0) \), maps unit disk \( \mathbb{U} \) onto the disk
\[
\left| \omega(z) - \frac{1 + B^2}{1 - B^2} \right| < \frac{2|B|}{1 - B^2}; \quad (z \in \mathbb{U}),
\]
which implies that
\[
\Re \{ \omega(z) \} > \frac{1 - |B|}{1 + |B|}; \quad (z \in \mathbb{U}).
\]
From (2.4) we have
\[
\frac{1 - |B|}{1 + |B|} \geq \max \{ 0, -\Re(\zeta) \}
\]
and this is equivalent to \( \Re \{ \zeta \} \geq ((|B| - 1) /(|B| + 1)) \). \( \square \)
Lemma 2.7. Let
\[ \omega(z) = \frac{u + vz}{1 + Bz}; \quad (u, v \in \mathbb{C}; \text{ with } (u, v) \neq (0, 0), \quad -1 < B < 1, \quad z \in \mathbb{U}). \]

Suppose that \( \text{Re}\{u - vB\} \geq |v - uB| \), then \( \text{Re}\{\omega(z)\} > 0; \quad (z \in \mathbb{U}). \)

Proof. The function \( \omega(z) = \frac{u + vz}{1 + Bz} \) maps \( \mathbb{U} \) onto the disk
\[ \left| \omega(z) - \frac{u - vB}{1 - B^2} \right| < \frac{|v - uB|}{1 - B^2}; \quad (z \in \mathbb{U}), \]
which implies that
\[ \text{Re}\{\omega(z)\} > \text{Re}\{u - vB\} - \frac{|v - uB|}{1 - B^2} \geq 0; \quad (z \in \mathbb{U}). \] \( \square \)

Some interesting results of differential subordination and superordination were obtained recently (for example) Bulboacă [4, 5, 6], Shammugam et al. [16], Zayed et al. [18], Ebadian and Sokál [9] and Aouf et al. [3].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator \( \mathcal{F}_p^{n, \mu}. \)

3. Subordination Results

For convenience, let
\[ \mathcal{A}_0 := \left\{ f \in \mathcal{A}_p : \mathcal{F}_p^{n, \mu}[f](z) \bigg|_{z=0} = 1, \quad \eta, \mu \in \mathbb{C}; \quad (\eta, \mu) \neq (0, 0) \right\}. \]
\[ \mathcal{B} := \{ z \in \mathbb{C} : |z + 1| \leq 1 \quad \text{or} \quad |z - 1| \leq 1 \}. \]

We assume in the remainder of this paper that \( \sigma \) be complex number, \( \gamma \in \mathbb{C}^* \), \( \alpha, \lambda \) are real numbers with \( 0 \leq \alpha < 1, \quad -\frac{\pi}{2} < \lambda < \frac{\pi}{2} \), respectively, and all the powers are principal ones.

Theorem 3.1. Let \( q \) be univalent in \( \mathbb{U} \) with \( q(0) = 1 \), and \( q \) satisfy the condition (2.3). If the function \( f \in \mathcal{A}_0 \) with \( \mathcal{F}_p^{n, \mu}[f](z) \neq 0; \quad (z \in \mathbb{U}) \) satisfies the following subordination condition:
\[ 1 + \gamma \left[ \frac{1}{n} \left( 1 - p + \frac{zf''(z)}{f'(z)} \right) + \mu \left( p - \frac{zf'(z)}{f(z)} \right) \right] < 1 + \gamma \frac{zq'(z)}{q(z)}; \quad (z \in \mathbb{U}), \] (3.1)
then
\[ \mathcal{F}_p^{n, \mu}[f](z) \prec q(z); \quad (z \in \mathbb{U}), \]
and \( q \) is the best dominant of Eq. (3.1).

Proof. If we choose \( \theta(w) = 1 \) and \( \varphi(w) = \frac{w}{w} \), then \( \theta, \varphi \in \mathcal{H}(\Omega); \quad (\Omega := \mathbb{C}^*). \) The condition \( q(\mathbb{U}) \subset \Omega \) from Lemma 2.1 is equivalent to \( q(z) \neq 0 \) for all \( z \in \mathbb{U} \). For \( w \in q(\mathbb{U}) \), we have \( \varphi(w) \neq 0 \). Define
\[ Q(z) := zq'(z)q(z) = \gamma \frac{zq'(z)}{q(z)}; \quad (z \in \mathbb{U}). \]
From Lemma 2.5, \( q(z) \neq 0 \) for all \( z \in U \), then \( Q \in \mathcal{H}(U) \). Further, \( q \) is an univalent function, implies \( q'(z) \neq 0 \) for all \( z \in U \), \( Q(0) = 0 \) and \( Q'(0) = \gamma \frac{q''(0)}{q'(0)} \neq 0 \), and

\[
\Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} = \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{2q'(z)}{q(z)} \right\} > 0; \quad (z \in U),
\]

hence \( Q \) is a starlike function in \( U \). Moreover, if \( h(z) := \theta(q(z)) + Q(z) = 1 + \gamma \frac{zq'(z)}{q(z)} \),

we also have

\[
\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} > 0; \quad (z \in U).
\]

For \( f \in A_0 \), the function \( F_{\eta, \mu} \left[ f \right] (z) \) given by (1.2), we have \( F_{\eta, \mu} \left[ f \right] (U) \subset \Omega \) and the subordinations (2.3) and (3.1) are equivalent, then all the conditions of Lemma 2.1 are satisfied and the function \( q \) is the best dominant of (3.1). \( \square \)

Taking \( \eta = 0 \), \( \gamma = 1 \) and \( q(z) = (1 + Az)/(1 + Bz); \ (-1 \leq A < B \leq 1, z \in U) \) in Theorem 3.1 and applying item (UF.2), we get the following result:

**Corollary 3.1.1.** Let \(-1 \leq A < B \leq 1, \mu \neq 0 \) and \( f \in A_p \) satisfy the conditions

\[
\left[ \frac{z^p}{f(z)} \right]^\mu = 1 \quad \text{and} \quad \frac{z^p}{f(z)} \neq 0; \quad (z \in U).
\]

If the function \( f \) satisfies the following subordination condition:

\[
1 + \mu \left( p - \frac{zf'(z)}{f(z)} \right) < \frac{(A - B) z}{(1 + Az)(1 + Bz)}; \quad (z \in U), \tag{3.2}
\]

then

\[
\left( \frac{z^p}{f(z)} \right)^\mu \approx \frac{1 + Az}{1 + Bz}; \quad (z \in U),
\]

and \((1 + Az)/(1 + Bz)\) is the best dominant of Eq. (3.2).

Taking \( \mu = 0 \), \( \gamma = 1 \) and \( q(z) = (1 + Az)/(1 + Bz); \ (-1 \leq A < B \leq 1, z \in U) \) in Theorem 3.1 and applying item (UF.2), we get the following result:

**Corollary 3.1.2.** Let \(-1 \leq A < B \leq 1, \eta \neq 0 \) and \( f \in A_p \) satisfy the conditions

\[
\left[ \frac{f'(z)}{(p + p^{-1})} \right]^\eta = 1 \quad \text{and} \quad \frac{f'(z)}{(p + p^{-1})} \neq 0; \quad (z \in U).
\]

If the function \( f \) satisfies the following subordination condition:

\[
1 + \eta \left[ 1 - p + \frac{zf''(z)}{f'(z)} \right] < \frac{(A - B) z}{(1 + Az)(1 + Bz)}; \quad (z \in U), \tag{3.3}
\]
then
\[
\left[ \frac{f'(z)}{pz^{p-1}} \right]^n < \frac{1 + Az}{1 + Bz}; \quad (z \in \mathbb{U}),
\]
and \((1 + Az)/(1 + Bz)\) is the best dominant of [3.3]

Taking \(\gamma = \frac{e^{i\lambda}}{pab \cos \lambda}, \mu = -a, \eta = 0\) and \(q(z) = (1 - z)^{-2pab(1-\alpha)e^{-i\lambda} \cos \lambda}\) in Theorem 3.1 and combining this together with item [(UF.1)] we obtain the following result:

**Corollary 3.1.3.** Let \(f \in S^\lambda_p(\alpha, b).\) Then
\[
\left[ \frac{f(z)}{z^p} \right]^a < \frac{1}{(1 - z)^{2pab(1-\alpha)e^{-i\lambda} \cos \lambda}}; \quad (a \in \mathbb{C}^*, \ z \in \mathbb{U}).
\]
or, equivalently
\[
1 + \frac{e^{i\lambda} f(z)'}{b \cos \lambda \left( 1 + \frac{zf(z')}{pf(z')} - 1 \right)} < \frac{1 + (1 - 2\alpha)z}{1 - z} \implies \left[ \frac{f(z)}{z^p} \right]^a < \frac{1}{(1 - z)^{2pab(1-\alpha)e^{-i\lambda} \cos \lambda}},
\]
where \(2pab(1-\alpha)e^{-i\lambda} \cos \lambda \in \mathbb{B} \) and \(q(z) = (1 - z)^{-2pab(1-\alpha)e^{-i\lambda} \cos \lambda}\) is the best dominant.

For example, for \(a = \frac{1}{2}\) and \(p = b = 1\) we get
\[
f \in S^\lambda_p(\alpha) \implies \sqrt{\frac{f(z)}{z}} < \frac{1}{(1 - z)^{(1-\alpha)e^{-i\lambda} \cos \lambda}}; \quad (z \in \mathbb{U}).
\]

**Remark 1.** A special case of Corollary 3.1.3 when \(p = 1, \alpha = 0\) and \(f \in \mathcal{A}\) was given by Aouf et al. [1 Theorem 1].

Taking \(\gamma = \frac{e^{i\lambda}}{pab \cos \lambda}, \mu = 0, \eta = a\) and \(q(z) = (1 - z)^{-2pab(1-\alpha)e^{-i\lambda} \cos \lambda}\) in Theorem 3.3 and combining this together with item [(UF.1)] we obtain the following result:

**Corollary 3.1.4.** Let \(f \in K^\lambda_p(\alpha, b).\) Then
\[
\left[ \frac{f'(z)}{pz^{p-1}} \right]^a < \frac{1}{(1 - z)^{2pab(1-\alpha)e^{-i\lambda} \cos \lambda}}; \quad (z \in \mathbb{U}),
\]
or, equivalently
\[
1 + \frac{e^{i\lambda} f(z)''}{b \cos \lambda \left( 1 + \frac{zf''(z)}{f'(z)} - 1 \right)} < \frac{1 + (1 - 2\alpha)z}{1 - z} \implies \left[ \frac{f'(z)}{pz^{p-1}} \right]^a < \frac{1}{(1 - z)^{2pab(1-\alpha)e^{-i\lambda} \cos \lambda}},
\]
where \(2pab(1-\alpha)e^{-i\lambda} \cos \lambda \in \mathbb{B} \) and \(q(z) = (1 - z)^{-2pab(1-\alpha)e^{-i\lambda} \cos \lambda}\) is the best dominant.

For example, for \(a = \frac{1}{2}\) and \(p = b = 1\) we get
\[
f \in K^\lambda_p(\alpha) \implies \sqrt{f'(z)} < \frac{1}{(1 - z)^{(1-\alpha)e^{-i\lambda} \cos \lambda}}; \quad (z \in \mathbb{U}).
\]
Remark 2. A special case of Corollary 3.1.4 when \( p = 1, \alpha = 0 \) and \( f \in \mathcal{A} \) was given by Aouf et al. [1, Corollary 1].

**Theorem 3.2.** Let \( q \) be univalent in \( U \) with \( q(0) = 1 \). Further, assume that \( f \in \mathcal{A}_0 \) and \( q \) satisfy the condition

\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{\sigma}{\gamma} \right) \right\}; \quad (z \in U). \tag{3.4}
\]

If the function \( \Psi \) defined by

\[
\Psi(z) := \left[ \frac{f'(z)}{p z^{p-1}} \right]^{n} \left[ \frac{z^{p}}{f(z)} \right]^{\mu} \left\{ \sigma + \gamma \left[ \eta \left( 1 - p + \frac{zf''(z)}{f'(z)} \right) + \mu \left( p - \frac{zf'(z)}{f(z)} \right) \right] \right\}, \tag{3.5}
\]

satisfies the following subordination condition:

\[
\Psi(z) < \sigma q(z) + \gamma z q'(z); \quad (z \in U). \tag{3.6}
\]

Then

\[
\mathcal{F}^n_{p, \mu} f(z) < q(z); \quad (z \in U).
\]

and \( q \) is the best dominant of Eq. (3.6).

**Proof.** If we choose \( \theta(w) = \sigma w \) and \( \varphi(w) = \gamma \), then \( \theta, \varphi \in \mathcal{H}(\Omega); (\Omega := \mathbb{C}) \). Also, for all \( w \in q(U) \), \( \varphi(w) \neq 0 \). Define

\[
Q(z) := z q'(z) \varphi(q(z)) = \gamma z q'(z),
\]

The function \( q \) is an univalent, then \( q'(z) \neq 0 \) for all \( z \in U \), \( Q(0) = 0 \) and \( Q'(0) = \gamma q'(0) \neq 0 \), and from condition (3.4)

\[
\Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} = \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > 0; \quad (z \in U).
\]

Thus \( Q \) is a starlike function in \( U \). Moreover, if

\[
h(z) := \theta(q(z)) + Q(z) = \sigma q(z) + \gamma z q'(z),
\]

then from condition (3.4), we deduce

\[
\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} + \frac{\sigma}{\gamma} \right\} > 0; \quad (z \in U).
\]

For \( f \in \mathcal{A}_0 \), the function \( \mathcal{F}^n_{p, \mu} f(z) \) given by (1.2), we have \( \mathcal{F}^n_{p, \mu} f(U) \subset \Omega \) and the subordinations (2.3) and (3.4) are equivalent, then all the conditions of Lemma 2.1 are satisfied and the function \( q \) is the best dominant of (3.1). \( \square \)

Taking \( q(z) = (1 + Az)/(1 + Bz); \) \((-1 \leq B < A \leq 1, z \in U) \) in Theorem 3.2, and then applying Lemma 2.6, we obtain the following result:

**Corollary 3.2.1.** Let \(-1 \leq B < A \leq 1 \) and

\[
\Re \left( \frac{\sigma}{\gamma} \right) \geq \frac{|B| - 1}{|B| + 1}.
\]
If \( f \in A_0 \) and the function \( \Psi \) given by (3.5) satisfies the subordination
\[
\Psi(z) \prec \sigma \left( \frac{1 + Az}{1 + Bz} \right) + \frac{\gamma(A - B)z}{(1 + Bz)^2}; \quad (z \in \mathbb{U}),
\]
(3.7)
then
\[
\mathcal{F}_p^{\eta, \mu}[f](z) \prec \frac{1 + Az}{1 + Bz}; \quad (z \in \mathbb{U}).
\]
and \((1 + Az)/(1 + Bz)\) is the best dominant of Eq. (3.7).

For \( q(z) = e^{Cz}; \quad (|C| < \pi) \) in Theorem 3.2, we obtain the following corollary.

**Corollary 3.2.2.** Let
\[
\text{Re} \left\{ \frac{\sigma}{\gamma} \right\} \geq |C| - 1; \quad (|C| < \pi).
\]
If \( f \in A_0 \) and the function \( \Psi \) given by (3.5) satisfies the subordination
\[
\Psi(z) \prec (\sigma + \gamma Cz)e^{Cz}; \quad (z \in \mathbb{U}),
\]
(3.8)
then
\[
\mathcal{F}_p^{\eta, \mu}[f](z) \prec e^{Cz}; \quad (z \in \mathbb{U}),
\]
and \( e^{Cz} \) is the best dominant of Eq. (3.8).

Taking \( q(z) = (1 + Az)/(1 + Bz); \quad (-1 < B < A \leq 1, \ z \in \mathbb{U}) \) in Theorem 3.2, we obtain the following result:

**Corollary 3.2.3.** Let \(-1 < B < A \leq 1\) and \( \text{Re} \{u - vB\} \geq |v - uB| \) where \( u = 1 + \frac{\sigma}{\gamma} \) and \( v = \frac{B(\sigma - \gamma)}{\gamma} \). If \( f \in A_0 \) and the function \( \Psi \) given by (3.5) satisfies the subordination
\[
\Psi(z) \prec \sigma \left( \frac{1 + Az}{1 + Bz} \right) + \frac{(A - B)\gamma z}{(1 + Bz)^2}; \quad (z \in \mathbb{U}),
\]
(3.9)
then
\[
\mathcal{F}_p^{\eta, \mu}[f](z) \prec \frac{1 + Az}{1 + Bz}; \quad (z \in \mathbb{U}),
\]
and \((1 + Az)/(1 + Bz)\) is the best dominant of Eq. (3.9).

**Proof.** Let \( q(z) = (1 + Az)/(1 + Bz) \), then we have
\[
zq'(z) = \frac{(A - B)z}{(1 + Bz)^2} \quad \text{and} \quad 1 + \frac{zq''(z)}{q'(z)} = \frac{1 - Bz}{1 + Bz}.
\]
Thus
\[
\frac{\sigma}{\gamma} + 1 + \frac{zq''(z)}{q'(z)} = \frac{u + vz}{1 + Bz},
\]
where \( u = 1 + \frac{\sigma}{\gamma} \) and \( v = \frac{B(\sigma - \gamma)}{\gamma} \). According to Lemma 2.7, it follows that
\[
\text{Re} \left\{ \frac{\sigma}{\gamma} + 1 + \frac{zq''(z)}{q'(z)} \right\} > \frac{\text{Re} \{u - vB\} - |v - uB|}{1 - B^2} \geq 0.
\]
By using Theorem 3.2 we obtain the required result. \qed

4. Superordination Results

**Theorem 4.1.** Let $q$ be a convex function in $U$ with $q(0) = 1$. Further, assume that $\text{Re}\left\{\frac{a}{g}\right\} > 0$ and the functions $f \in A_0$ and $q$ satisfy the conditions

$$F_{\eta, \mu}^p[f](z) \in H[q(0), 1] \cap Q; \quad (z \in U).$$

If the function $\Psi$ given by (3.5) is univalent in $U$, and satisfies the following subordination condition:

$$\sigma q(z) + \gamma z q'(z) < \Psi(z); \quad (z \in U). \quad (4.1)$$

Then

$$q(z) < F_{\eta, \mu}^p[f](z); \quad (z \in U),$$

and $q$ is the best subordinant of Eq. (4.1).

**Proof.** Let $f \in A_0$. Define the function $g$ by

$$g(z) := F_{\eta, \mu}^p[f](z) = \left[\frac{f'(z)}{pz^p - 1}\right]^{\eta} \left[\frac{z^p}{f(z)}\right]^\mu; \quad (z \in U).$$

Differentiating $g(z)$ logarithmically with respect to $z$, we get

$$\frac{z g'(z)}{g(z)} = \eta \left(1 - \frac{zf''(z)}{f'(z)}\right) + \mu \left(p - \frac{zf'(z)}{f(z)}\right); \quad (z \in U),$$

hence the subordination (4.1) is equivalent to

$$\sigma q(z) + \gamma z q'(z) < \sigma g(z) + \gamma z g'(z).$$

By using Lemma 2.2 we obtain the required result. \qed

Taking $\eta = 1$ and $\mu = 0$ in Theorem 4.1, we obtain the following result:

**Corollary 4.1.** Let $q$ be a convex function in $U$ with $q(0) = 1$. Further, assume that the functions $f \in A_p$ and $q$ satisfy the conditions

$$\frac{f''(z)}{pz^p - 1} \in H[q(0), 1] \cap Q; \quad (z \in U).$$

If the function

$$\phi(z) := \frac{f'(z)}{pz^p - 1} \left[2 - p + \frac{zf''(z)}{f'(z)}\right] = \left[\frac{zf'(z)}{pz^p - 1}\right]',$$

is univalent in $U$, and satisfies the following subordination condition:

$$[z q(z)]' < \left[\frac{zf'(z)}{pz^p - 1}\right]'; \quad (z \in U). \quad (4.2)$$

Then

$$q(z) < \frac{f'(z)}{pz^p - 1}; \quad (z \in U),$$

and $q$ is the best subordinant of (4.2).
Taking $\mu = \eta = 1$ in Theorem 4.1, we obtain the following result:

**Corollary 4.1.2.** Let $q$ be convex function in $U$ with $q(0) = 1$. Further, assume that the functions $f \in A_p$ and $q$ satisfy the conditions

$$\frac{1}{p} \frac{zf''(z)}{f''(z)} \in H[q(0), 1] \cap Q; \quad (z \in \mathbb{U}).$$

If the function

$$\Psi(z) := \frac{1}{p} [2 + \frac{zf''(z)}{f''(z)} - \frac{zf'(z)}{f(z)}] \frac{zf'(z)}{f(z)} = \left[ \frac{1}{p} \frac{zf'(z)}{f(z)} \right]'$$

is univalent in $U$, and satisfies the following subordination condition:

$$[zq(z)]' \preceq \left[ \frac{1}{p} \frac{zf'(z)}{f(z)} \right]'; \quad (z \in \mathbb{U}). \quad (4.3)$$

Then

$$q(z) \preceq \frac{1}{p} \frac{zf'(z)}{f(z)}; \quad (z \in \mathbb{U}),$$

and $q$ is the best subordinant of Eq. (4.3).

Combining Theorem 3.2 with Theorem 4.1, we obtain the following “sandwich result”.

**Theorem 4.2.** Let $q_1$ and $q_2$ be convex and convex (univalent) functions in $U$ with $q_1(0) = q_2(0) = 1$ respectively. Further, assume that $\text{Re} \left\{ \frac{\sigma}{\gamma} \right\} > 0$ and function $f \in A_0$ satisfy the condition

$$F_{\eta, \mu}^\eta f(z) \in H[1, 1] \cap Q; \quad (z \in \mathbb{U}).$$

If the function $\Psi$ given by (3.5) is univalent in $U$, and satisfies the following subordination condition:

$$\sigma q_1(z) + \gamma zq_1'(z) \prec \Psi(z) \prec \sigma q_2(z) + \gamma zq_2'(z); \quad (z \in \mathbb{U}). \quad (4.4)$$

Then

$$q_1(z) \prec F_{\eta, \mu}^\eta f(z) \prec q_2(z); \quad (z \in \mathbb{U}),$$

and $q_1$ and $q_2$ are respectively the best subordinant and best dominant of Eq. (4.4).

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