Graphs defined on groups

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### An apology

The phrase “Graphs defined on groups” will inevitably suggest “Cayley graphs” to many of you. But this is not what I will be talking about. I hope to show you a part of algebraic graph theory which is full of interesting unsolved problems, and give you a taste of some of these.

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### Graphs and groups

Graphs and groups represent very contrasting parts of the mathematical universe.

Groups measure symmetry; they are highly structured, elegant objects.

Graphs, on the other hand, are “wild”: we can put in edges however we please. Some graphs are beautiful, but most are scruffy.

Algebraic graph theory is the area where these two very different subjects can meet and have a productive relationship.

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### Brauer and Fowler

The story begins some time later than Cayley, with a paper in 1955 by Brauer and Fowler. There are several remarkable things about this paper.

- As in all of Brauer’s early papers, a group is denoted by $G$, its order by $g$, and a typical element by $x$.
- The main theorem of the paper is that, given a finite group $H$, there are only finitely many finite simple groups containing an involution whose centralizer is $H$. This could be regarded as the beginning of the Classification of Finite Simple Groups, in which characterizations of simple groups by the centralizer of an involution plays such a big part. But this result is not formally stated as a theorem in the paper.

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### Some other graphs

Before plunging in, I will define a few more graphs on the vertex set $G$. In each case, I give the rule for joining $x$ to $y$.

- The **power graph**: one of $x$ and $y$ is a power of the other.
- The **enhanced power graph**: $x$ and $y$ are both powers of an element $z$ (equivalently, $(x,y)$ is cyclic).
- The **generating graph**: $(x,y) = G$.
- The **non-generating graph**, the complement of the generating graph.

This is not the complete *dramatis personae*, just the big stars. Some others will come in later. Indeed you can imagine some for yourself. Noting that $x$ and $y$ are joined in the commuting graph if and only if $(x,y)$ is abelian, we could define a graph where the joining rule is $(x,y)$ is nilpotent, or solvable, or …

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### Some history

I wrote a paper on these graphs earlier this year and posted it on the arXiv. Two things happened:

- Alireza Abdollahi from Isfahan saw it and invited me to submit it to the *International Journal of Group Theory*, where it has now appeared, as 11 (2022), 43–124; doi: 10.22108/ijgt.2021.127679.1681
- Ambat Vijaykumar from Kochi saw it and decided to set up a research discussion on graphs and groups, which ran for five months and stimulated a lot of new research, including some of the results reported here.

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Some philosophy

Each of the types of graph I mentioned earlier has a huge literature. For example, a recent survey of the power graph includes 82 references, mostly published since an earlier survey in 2013. My intention is to show that we gain something by considering these graphs together rather than individually. So I will mostly not present detailed results about a particular family. In order to get started, we observe that these graphs form a hierarchy; each is contained in the next as a spanning subgraph. This is the main reason for taking the vertex set in each case to be the whole group.

Equality?

Once we have a hierarchy, it is natural to ask when adjacent terms can be equal. Some are trivial.

- The power graph is null if and only if \( G \) is the trivial group (for the identity is joined to all other vertices).
- The non-generating graph is complete if and only if \( G \) is not 2-generated.
- The commuting graph is equal to the non-generating graph if and only if \( G \) is a minimal non-abelian group. Such groups were determined by Miller and Moreno in 1904.

The hierarchy

Here is the hierarchy, with notation and a brief reminder of the definition.

- The null graph.
- The power graph \( \text{Pow}(G) \): \( x = y^n \) or \( y = x^m \).
- The enhanced power graph \( \text{EPow}(G) \): \( (x, y) \) is cyclic.
- The commuting graph \( \text{Com}(G) \): \( xy = yx \).
- The non-generating graph \( \text{NGen}(G) \): \( (x, y) \notin G \).
- The complete graph.

Each is contained in the next, except that the commuting graph is contained in the non-generating graph if and only if \( G \) is either non-abelian or has more than two generators.

The other cases

Theorem

Let \( G \) be a finite group.

- \( \text{Pow}(G) = \text{EPow}(G) \) if and only if \( G \) has no subgroup isomorphic to \( C_p \times C_q \) where \( p \) and \( q \) are distinct primes.
- \( \text{EPow}(G) = \text{Com}(G) \) if and only if \( G \) has no subgroup isomorphic to \( C_p \times C_q \) where \( p \) is a prime.

The groups in each of these classes have been determined. Before explaining this, let me mention another graph associated with a finite group.

The Gruenberg–Kegel graph

The Gruenberg–Kegel graph, sometimes called the prime graph of \( G \), has vertices the prime divisors of \( |G| \), with an edge joining \( p \) and \( q \) if \( G \) contains an element of order \( pq \). Gruenberg and Kegel showed that the augmentation ideal of the integral group ring of \( G \) is decomposable if and only if this graph is disconnected. They gave a structural description of such groups in an unpublished manuscript; the result was later published by Gruenberg’s student Williams.

Theorem

Let \( G \) be a finite group whose Gruenberg–Kegel graph is disconnected. Then either

- \( G \) is a Frobenius or 2-Frobenius group; or
- \( G \) is an extension of a nilpotent \( \pi \)-group by a simple group by a \( \pi \)-group, where \( \pi \) is the set of primes in the connected component containing 2.

EPPO groups

The group \( G \) is an EPPO group (“Elements of Prime Power Order”) if every element of \( G \) has prime power order. These groups were studied by Higman in the 1950s; he determined the solvable ones. Following the discovery of his infinite family of simple groups, Suzuki was able to determine the simple EPPO groups. Now we have a complete classification.

Theorem

For a finite group \( G \), the following conditions are equivalent:

- \( \text{Pow}(G) = \text{EPow}(G) \);
- the Gruenberg–Kegel graph of \( G \) is a null graph;
- \( G \) is an EPPO group.
### Universal Property

**Definition**
A class of finite graphs is **universal** if every finite graph can be embedded as induced subgraph in a graph in that class. The power graphs of finite groups do not form a universal class. For these graphs are comparability graphs of partial orders, and hence are perfect; in particular, they do not contain odd cycles of length greater than 3 or their complements. But this is the only restriction:

**Theorem**
If \( \Gamma \) is the comparability graph of a finite partial order, then there is a finite group \( G \) such that \( \Gamma \) is isomorphic to an induced subgraph of \( \text{Pow}(G) \).

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### Approximate Equality?

We can ask for generalizations of these results along the following lines. Let \( \rho \) be a monotone graph parameter (that is, adding edges to a graph cannot decrease the value of the parameter). Now for each consecutive pair of graphs in the hierarchy, we can ask: for which groups, do the values of \( \rho \) on the corresponding graphs coincide? There are plenty of open questions here; the only case to have been looked at (as far as I know) is the power graph and enhanced power graph. Again not many results are known. Recall that these graphs are equal for a group \( G \) if and only if every element of \( G \) has prime power order.

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### An Example

We do not expect to be able to classify groups in which the largest order of an element is a prime power. Nevertheless, it is an interesting question.

**Theorem**
Let \( q \) be a prime power, and \( G = \text{PGL}(2,q) \). Then \( \omega(\text{Pow}(G)) = \omega(\text{EPow}(G)) \) if and only if one of

- \( q \) is a Mersenne prime;
- \( q + 1 \) is a Fermat prime;
- \( q = 8 \).

For the largest order of an element of \( G \) is \( q + 1 \). If \( q \) and \( q + 1 \) are both proper powers, then \( q = 8 \), by the Catalan conjecture (proved fairly recently by Mihăilescu). Otherwise either \( q \) or \( q + 1 \) is prime, giving the remaining cases. So our problem includes the determination of all Fermat and Mersenne primes!

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### Differences

For each consecutive pair of graphs in the hierarchy, we can ask: If they are not equal, what can be said about their difference? For example, is it connected? This has been very little studied, apart from the difference between the non-generating graph and the commuting graph, where Saul Freedman has detailed results, specifically about its connectedness and diameter.

I will look at one further property to illustrate the benefit of treating the graphs as a hierarchy.

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### Theorem
- Let \( \omega \) denote clique number, the size of the maximal complete subgraph. Then \( \omega(\text{Pow}(G)) = \omega(\text{EPow}(G)) \) if and only if the largest order of an element of \( G \) is a prime power.
- Let \( \mu \) denote matching number, the maximum number of pairwise disjoint edges. Then every finite group \( G \) satisfies \( \mu(\text{Pow}(G)) = \mu(\text{EPow}(G)) \).

One slightly surprising thing about the second result is that we do not have a formula for the matching number of \( \text{Pow}(G) \) for an arbitrary group \( G \). The theorem is proved by showing that, given any matching in \( \text{EPow}(G) \), we can find another matching of the same size which has fewer edges which don’t belong to \( \text{Pow}(G) \).

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### Example

We saw that a group whose enhanced power graph is equal to its commuting graph has no subgroup \( C_p \times C_p \). These groups can be determined as well. By a theorem of Burnside, the Sylow subgroups of \( G \) are all cyclic or generalized quaternion. In the cyclic case, using Burnside’s transfer theorem, we see that \( G \) is metacyclic. In the other case, let \( O(G) \) be the maximal normal subgroup of \( G \) of odd order. By the Brauer–Suzuki theorem, \( H = G/O(G) \) contains a unique subgroup \( Z \) of order 2, and \( H/Z \) has dihedral Sylow 2-subgroups, so is determined by the Gorenstein–Walter theorem. Then a cohomological argument due to Glauberman shows that if a group \( K \) has dihedral Sylow 2-subgroups, then there is a unique \( H \) (up to isomorphism) containing a unique subgroup \( Z \) of order 2 such that \( H/Z \cong K \).

Then we just have to impose the extra condition that the other Sylow subgroups are cyclic. This implies in particular that \( O(G) \) is metacyclic.

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### EPow\((G) = \text{Com}(G)\)

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I will look at one further property to illustrate the benefit of treating the graphs as a hierarchy.
By contrast we have:

**Theorem**
The classes of enhanced power graphs, commuting graphs, or non-generating graphs of finite groups are universal.

But using our hierarchy, we can strengthen the last result.

### More on the generating graph

Let \( d(G) \) (the rank of \( G \)) denote the minimal number of generators of \( G \).

If \( d(G) > 2 \), then the generating graph is null and gives no information. In recent work, Andrea Lucchini has proposed a way around this. We define two new graphs, the independence graph and the rank independence graph. In each case, the vertex set is \( G \).

The independence graph has an edge \( \{x, y\} \) whenever \( \{x, y\} \) is a subset of a minimal (with respect to inclusion) generating set of \( G \); the rank independence graph has an edge \( \{x, y\} \) whenever \( \{x, y\} \) is a subset of a generating set of minimum cardinality \( d(G) \).

Note that
- if \( x \) and \( y \) are joined in the power graph, they are not joined in the independence graph (for if, say, \( y \) is a power of \( x \), then \( y \) could be dropped from the generating set);
- if \( x \) and \( y \) are joined in the enhanced power graph, they are not joined in the rank independence graph (since if \( x \) and \( y \) are both powers of \( z \), we get a smaller generating set by deleting \( x \) and \( y \) and including \( z \)).

When do these implications reverse?

### Clique number of the power graph

As a final topic, there is a sense in which the enhanced power graph is not much larger than the power graph. For example, while \( \omega(\text{Pow}(G)) \leq \omega(\text{EPow}(G)) \) it is true the \( \omega(\text{EPow}(G)) \) is bounded above by a function of \( \omega(\text{Pow}(G)) \). This can be seen by looking more closely at the clique number of \( \text{Pow}(G) \).

Any edge of \( \text{Pow}(G) \) is contained in a cyclic subgroup; and if every pair of vertices of a set \( S \) in a group are contained in a cyclic subgroup, then \( S \) is contained in a cyclic subgroup. So \( \omega(G) \) is equal to the maximum of \( \omega(C) \) over all cyclic subgroups \( C \) of \( G \).

Similarly, \( \omega(\text{EPow}(G)) \) is equal to the order of the largest cyclic subgroup of \( G \).

So it suffices to look at cyclic groups.
In a cyclic group

Let \( f(n) \) be the clique number of \( \text{Pow}(C_n) \), where \( C_n \) is the cyclic group of order \( n \).
Then \( f(n) \) is given by the recurrence

\[
\begin{align*}
\text{for } n > 1, \quad f(n) &= \phi(n) + f(n/p), \text{ where } \phi \text{ is Euler’s totient function and } p \text{ is the smallest divisor of } n.
\end{align*}
\]

From this it follows easily that \( f(n) \leq 3\phi(n) \). Hence \( n \) is bounded above by \( cm \log \log m \), where \( m = f(n) \); and the same bound holds for the clique numbers \( m \) and \( n \) of the power graph and enhanced power graph of an arbitrary group.

In fact,

\[
\limsup f(n)/\phi(n) = 2.6481017597 \ldots,
\]

where the constant on the right is

\[
\sum_{k \geq 0} \prod_{i=1}^{k} \frac{1}{p_i - 1},
\]

where \( p_1, p_2, \ldots \) are the primes in order.
This suggests several questions, such as

\[
\begin{align*}
\text{is this constant rational, algebraic or transcendental?} \\
\text{what other numbers are limit points of the set } \{f(n)/\phi(n) : n \in \mathbb{N}\}?
\end{align*}
\]

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... for your attention.