Two-body correlations in pionic systems*

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Abstract

In close analogy to fermionic many-body theory the truncation of the bosonic BBGKY density matrix hierarchy on the two-body level leads to a coupled set of nonlinear equations of motion for the one-body density matrix and the two-body correlation function. These equations provide a nonperturbative description of the nonequilibrium time evolution of particle number conserving bosonic many-body systems including the dynamical resummation of parquet-like diagrams. Within this framework we study the properties of a pionic system as a function of temperature and density with focus on two-body quantities. For each temperature we find a related pion density for which the relative strength of the two-body correlation function assumes a maximum and the pionic system is far from the mean-field limit. Since these correlated phases up to $T=200$ MeV only appear at rather low pion density, the hot and dense pion gas as generated in ultrarelativistic nucleus-nucleus collisions should be well described within mean-field theory; i.e. the HBT analysis of pion sources from $\pi-\pi$ correlations should remain valid even in the case of strongly interacting pions.

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1 Introduction

The theoretical description of strongly interacting hadronic many-body systems in general requires nonperturbative methods. A systematic way for constructing a wide class of such nonperturbative approaches is given by the truncation of Green function hierarchies as obtained by inserting the cluster expansion in terms of the corresponding connected Green functions. In the lowest order truncation scheme one then ends up with the mean-field level, i.e. TDHF\(^1\). The next order truncation scheme, i.e. neglecting the connected six-point function (the three-body correlation function) as well as all higher order correlation functions leads to the equations of two-body correlation dynamics, which in the fermionic case have been denoted by NQCD\(^2\). The NQCD-method has already been successfully applied to the most important fermionic many-body problem in hadronic physics, the nuclear many-body problem \([1]-[9]\). It has been shown to guarantee a simultaneous resummation of ring- and ladder-diagrams in vertical and horizontal direction (a parquet resummation) and thus to adequately take into account long- as well as short-range nucleon-nucleon correlations \([5]\).

It is the purpose of the present paper, to develop a related nonperturbative theory for bosonic systems. The most important bosonic many-body problem on the hadron level is that of a pion gas, which can be generated e.g. in an ultrarelativistic heavy-ion collision \([10]-[12]\). Here especially the Hanbury Brown-Twiss (HBT) analysis of pion sources from \(\pi^-\pi^+\) correlations \([13]-[22]\) might become questionable due to the strong interaction between the pions. In this work we will apply our method to study the properties of two-body quantities in such systems.

The paper is organized as follows: In Sect. \(\S\) we will derive the equations of motion of two-body boson correlation dynamics starting from the Heisenberg equations of motion for an arbitrary, particle number conserving many-body Hamiltonian with a two-body interaction. In Sect. \(\S\) we will investigate the topological structure of the equations and analyze the various interaction terms diagrammatically.

Sect. \(\S\) to Sect. \(\S\) are devoted to a presentation of numerical results for pionic systems, i.e. the comparison of different limiting cases for the resummation of diagrams (Sect. \(\S\)), the computation of correlated near-equilibrium states (Sect. \(\S\)), the effect of correlations on two-body quantities (Sect. \(\S\)) and signatures for a phase transition of the system as a function of density and temperature (Sect. \(\S\)). We will shift the specification of the Hamiltonian density for interacting pions as well as the final explicit equations of motion to the Appendices \(\S\) and \(\S\), respectively.

2 The equations of motion

In this Sect. we develop the mathematical apparatus of correlation dynamics for bosonic many-particle systems described by an arbitrary, particle number conserving Hamiltonian with a two-body interaction.

In this respect, we first introduce the formalism of density matrices: let \(a_\alpha\) and \(a_\alpha^\dagger\) be the creation and annihilation operators for a system of bosons, where \(\alpha\) stands for a complete set of quantum numbers characterizing an element of an arbitrary, orthonormal

\(^1\)Time Dependent Hartree Fock

\(^2\)Nuclear Quantum Correlation Dynamics
basis of the corresponding one-body Hilbert space, with canonical equal-time commutation relations

\[ [a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \quad [a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0. \] (2.1)

The matrix elements of the n-body density matrix are then defined by

\[ (\rho_n)_{\alpha_1...\alpha_n\alpha_1'...\alpha_n'} = \langle a_{\alpha_1'}^\dagger...a_{\alpha_n'}^\dagger a_{\alpha_n}...a_{\alpha_1} \rangle, \] (2.2)

i.e., by the expectation value\(^3\) of the normal-ordered operator product of the corresponding creation and annihilation operators, all of these considered at equal-time \( t \) in the Heisenberg picture. The matrix elements of the density matrices can thus be considered as equal-time Green functions.

From the canonical commutation relations (2.1) we immediately obtain the symmetry relations

\[ (\rho_1)_{\alpha\alpha'} = (\rho_1)^*_{\alpha'\alpha}, \quad (\rho_2)_{\alpha\beta\alpha'\beta'} = (\rho_2)^*_{\alpha'\beta'\alpha\beta}, \]

\[ (\rho_2)_{\alpha\beta\alpha'\beta'} = (\rho_2)_{\beta\alpha\alpha'\beta'} = (\rho_2)_{\alpha\beta'\alpha'\beta}. \] (2.3)

Analogous relations hold for all higher density matrices.

We now consider a Hamiltonian of the general form

\[ H = \sum_{\alpha\alpha'} t_{\alpha\alpha'} a_{\alpha'}^\dagger a_{\alpha}, \quad \frac{1}{2} \sum_{\alpha\beta\alpha'\beta'} \langle \alpha|v|\alpha'\beta'\rangle a_{\alpha'}^\dagger a_{\alpha}^\dagger a_{\alpha'} a_{\beta'}, \] (2.4)

with a pure two-body interaction \( v \). From the hermeticity of the Hamiltonian we get

\[ t_{\alpha\alpha'} = t^*_{\alpha'\alpha}, \quad \langle \alpha|v|\alpha'\beta'\rangle = \langle \alpha'\beta'|v|\alpha\beta \rangle^*, \] (2.5)

The time evolution of an arbitrary operator \( O \) with no explicit time dependence is given by the Heisenberg equation

\[ i\partial_t O = [O, H], \] (2.6)

where \( \partial_t \) denotes the total time derivative. Then the time derivative of the operator product

\[ a_{\alpha_1'}^\dagger...a_{\alpha_n'}^\dagger a_{\alpha_n}...a_{\alpha_1} = a_{\alpha_1'}^\dagger...a_{\alpha_n'}^\dagger a_{\alpha_1}...a_{\alpha_n} = \prod_{i=1}^{n} a_{\alpha_i'}^\dagger \prod_{k=1}^{n} a_{\alpha_k}, \] (2.7)

which defines the n-body density matrix (2.2) via the expectation value, reads:

\[ i\partial_t \left[ \left( \prod_{i=1}^{n} a_{\alpha_i'} \right) \left( \prod_{k=1}^{n} a_{\alpha_k} \right) \right] \]

\[ = \sum_{j=1}^{n} \sum_{\lambda} t_{\alpha_j\lambda} \left( \prod_{i=1}^{n} a_{\alpha_i'} \right) \left( \prod_{k\neq j=1}^{n} a_{\alpha_k} \right) a_{\lambda} - t_{\lambda\alpha_j} \left( \prod_{i\neq j=1}^{n} a_{\alpha_i'} \right) a_{\lambda}^\dagger \left( \prod_{k=1}^{n} a_{\alpha_k} \right) \]

\[ + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{j-1} \sum_{\lambda_1 \lambda_2} \langle \alpha_j|\alpha_k|v|\lambda_1\lambda_2 \rangle \left( \prod_{i=1}^{n} a_{\alpha_i'} \right) \left( \prod_{l\neq j, k=1}^{n} a_{\alpha_l} \right) a_{\lambda_1} a_{\lambda_2}, \]

\(^3\)In general an ensemble average with an arbitrary nonequilibrium quantum statistical density operator
\[-\frac{1}{2} \sum_{j=1}^{n} \sum_{k=j+1}^{n} \langle \lambda_1 \lambda_2 | v | \alpha'_j \alpha'_k \rangle_S a_{\lambda_1}^\dagger a_{\lambda_2}^\dagger \left( \prod_{i \neq j, k} a_{\alpha_i}^\dagger \right) \left( \prod_{l=1}^{n} a_{\alpha_l} \right) \]
\[+ \frac{1}{2} \sum_{j=1}^{n} \sum_{\lambda_1 \lambda_2 \lambda_3} \left\{ \langle \alpha_j \lambda_1 | v | \lambda_2 \lambda_3 \rangle_S \left( \prod_{i=1}^{n} a_{\alpha_i}^\dagger \right) a_{\lambda_1}^\dagger \left( \prod_{k \neq j}^{n} a_{\alpha_k} \right) a_{\lambda_2} a_{\lambda_3} \right\} \]
\[-\langle \lambda_1 \lambda_2 | v | \lambda_3 \alpha'_j \rangle_S \left( \prod_{i \neq j}^{n} a_{\alpha_i}^\dagger \right) a_{\lambda_1}^\dagger a_{\lambda_2}^\dagger \left( \prod_{k=1}^{n} a_{\alpha_k} \right) a_{\lambda_3} \right\} \right\}
(2.8)

with the symmetrized two-body matrix elements
\[\langle \alpha_j \lambda_1 | v | \lambda_2 \lambda_3 \rangle_S = \langle \alpha_j \lambda_1 | v | \lambda_2 \lambda_3 \rangle + \langle \alpha_j \lambda_1 | v | \lambda_3 \lambda_2 \rangle\]  
(2.9)

In the middle two terms one can use either \(\sum_{k=1}^{j-1}\) or \(\sum_{k=j+1}^{n}\), because the expressions in the sums are symmetric in \(i\) and \(j\). The equations of the BBGKY density matrix hierarchy now follow by taking the expectation value on both sides of (2.8)\(^4\). With (2.2) we obtain:

\[i \partial_t (\rho_n)_{\alpha_1...\alpha_n \alpha'_1...\alpha'_n} = \sum_{j=1}^{n} \sum_{\lambda} \left\{ t_{\alpha_j \lambda} (\rho_n)_{\alpha_1...\alpha_{j-1} \alpha_{j+1}...\alpha_n \alpha'_1...\alpha'_n} \right\} + \frac{1}{2} \sum_{j=1}^{n} \sum_{\lambda_1 \lambda_2} \left\{ \langle \alpha_j \alpha_k | v | \lambda_1 \lambda_2 \rangle_S (\rho_n)_{\alpha_1...\alpha_{k-1} \alpha_{k+1}...\alpha_{j-1} \alpha_{j+1}...\alpha_n \alpha'_1...\alpha'_n} \right\} - \langle \lambda_1 \lambda_2 | v | \lambda_3 \alpha'_j \rangle_S (\rho_n+1)_{\alpha_1...\alpha_{j-1} \alpha_{j+1}...\alpha_n \alpha'_1...\alpha'_n \alpha'_j} \}
(2.10)

The time evolution of the density matrices consequently is given by a coupled system of equations of first order in time, where the equation of motion for \(\rho_n\) couples to \(\rho_{n+1}\). For practical purposes the hierarchy (2.10) has to be truncated.

For the special cases of \(\rho_1\) and \(\rho_2\) – which we will examine furtheron – the equations read explicitly:

\[i \partial_t (\rho_1)_{\alpha \alpha'} = \sum_{\lambda} \left\{ t_{\alpha \lambda} (\rho_1)_{\lambda \alpha'} - t_{\lambda \alpha'} (\rho_1)_{\alpha \lambda} \right\} \]
(2.11)

\[i \partial_t (\rho_2)_{\alpha \beta \alpha' \beta'} = \sum_{\lambda} \left\{ t_{\alpha \beta \lambda} (\rho_2)_{\lambda \alpha' \beta'} + t_{\beta \lambda} (\rho_2)_{\alpha \alpha' \beta'} - t_{\lambda \alpha'} (\rho_2)_{\alpha \beta \lambda} - t_{\lambda \beta'} (\rho_2)_{\alpha \beta \alpha'} \right\} \]
(2.12)

\(^4\)One can therefore regard the BBGKY hierarchy as an operator identity.
A suitable truncation scheme for (2.10) is based on the cluster decomposition of Green functions, i.e. on the fact that any n-point Green function can be decomposed into a sum of products of connected Green functions of equal or lower order. The truncation then is performed by neglecting all connected Green functions higher than a certain order, i.e. in our case all connected Green functions of higher order than the four-point function (or two-body correlation function) \[23\]. This strategy is based on the assumption that the connected (correlated) parts of the corresponding Green functions become increasingly unimportant at higher order \[3, 23\].

A reduction to the two-body level requires the equations of motion for \(\rho_1\) \((2.11)\) and \(\rho_2\) \((2.12)\), which contain all density matrices up to \(\rho_3\). Thus we need the cluster expansions of \(\rho_1\), \(\rho_2\) and \(\rho_3\) up to the two-body correlation function. With \((\rho_1)_{\alpha\alpha'} = \rho_{\alpha\alpha'}\) and \(c_n\) denoting the n-body correlation function we obtain:

\[
(\rho_1)_{\alpha\alpha'} = \rho_{\alpha\alpha'} ,
\]

\[
(\rho_2)_{\alpha\beta\alpha'\beta'} = \rho_{\alpha\alpha'}\rho_{\beta\beta'} + \rho_{\alpha\beta'}\rho_{\beta\alpha'} + (c_2)_{\alpha\beta\alpha'\beta'}
= (\rho_{20})_{\alpha\beta\alpha'\beta'} + (c_2)_{\alpha\beta\alpha'\beta'} ,
\]

\[
(\rho_3)_{\alpha\beta\gamma\alpha'\beta'\gamma'} = \rho_{\alpha\alpha'}\rho_{\beta\beta'}\rho_{\gamma\gamma'} + \rho_{\alpha\beta'}\rho_{\beta\gamma'}\rho_{\gamma\alpha'} + \rho_{\alpha\gamma'}\rho_{\beta\alpha'}\rho_{\gamma\beta'}
+ \rho_{\alpha\alpha'}(c_2)_{\beta\beta'\gamma\gamma'} + \rho_{\alpha\beta'}(c_2)_{\beta\gamma\alpha'\gamma'} + \rho_{\alpha\gamma'}(c_2)_{\beta\gamma\beta'\alpha'}
+ \rho_{\beta\beta'}(c_2)_{\alpha\gamma\alpha'\gamma'} + \rho_{\beta\gamma'}(c_2)_{\alpha\gamma\beta'\alpha'} + \rho_{\beta\gamma'}(c_2)_{\alpha\beta\gamma'\beta'}
+ \rho_{\gamma\gamma'}(c_2)_{\alpha\beta\alpha'\gamma'} + \rho_{\gamma\alpha'}(c_2)_{\alpha\beta\beta'\gamma'} + \rho_{\gamma\beta'}(c_2)_{\alpha\beta\gamma'\alpha'}
+ (c_3)_{\alpha\beta\gamma\alpha'\beta'\gamma'}
= (\rho_{30})_{\alpha\beta\gamma\alpha'\beta'\gamma'}
+ (1 + P_{\alpha\beta} + P_{\alpha\gamma})(1 + P_{\alpha'\beta'} + P_{\alpha'\gamma'})\rho_{\alpha\alpha'}(c_2)_{\beta\gamma\beta'\alpha'}
+ (c_3)_{\alpha\beta\gamma\alpha'\beta'\gamma'} ,
\]

where \(P_{\alpha\beta}\) is the permutation operator interchanging the indices \(\alpha\) and \(\beta\);

\[
(\rho_{20})_{\alpha\beta\alpha'\beta'} = (1 + P_{\alpha\beta})\rho_{\alpha\alpha'}\rho_{\beta\beta'}
\]

and

\[
(\rho_{30})_{\alpha\beta\gamma\alpha'\beta'\gamma'} = (1 + P_{\alpha\beta} + P_{\alpha\gamma})(1 + P_{\beta\gamma})\rho_{\alpha\alpha'}\rho_{\beta\beta'}\rho_{\gamma\gamma'}
\]

are the uncorrelated parts of the two- and three-body density matrices.

The explicit expressions for the cluster expansions can be derived from the generating functionals of full and connected Green functions \[24\].

In our present case we make the additional assumption, that operator products with different numbers of creation and annihilation operators, i.e. operators not conserving particle number, have vanishing expectation values, e.g.

\[
\langle a_\alpha^{\dagger} a_\beta^{\dagger} \rangle = \langle a_\alpha a_\beta \rangle = 0 .
\]
Neglecting $c_3$ in (2.15) and inserting the truncated cluster expansions into the equations of motion (2.11) and (2.12), we arrive at a closed system of coupled nonlinear differential equations for $\rho$ and $c_2$. In order to compactify these lengthy equations (in close analogy to fermionic correlation dynamics [3, 4]) we introduce the following abbreviations:

$$U_{\alpha\alpha'} = \sum_{\lambda_1\lambda_2} \rho_{\lambda_1\lambda_2} \langle \alpha \lambda_2 | v | \alpha' \lambda_1 \rangle_S \quad \text{(mean – field)} \quad (2.19)$$

$$h_{\alpha\alpha'} = t_{\alpha\alpha'} + U_{\alpha\alpha'} \quad \text{(mean – field – Hamiltonian)} \quad (2.20)$$

$$Q_{\alpha\beta\alpha'}^{\pi} = \delta_{\alpha\alpha'} \delta_{\beta\beta'} + \rho_{\alpha\alpha'} \delta_{\beta\beta'} + \rho_{\beta\beta'} \delta_{\alpha\alpha'} \quad \text{(vertical Bose – factor)} \quad (2.21)$$

$$Q_{\alpha\beta\alpha'}^{\perp} = \rho_{\alpha\alpha'} \delta_{\beta\beta'} - \rho_{\alpha\beta} \delta_{\alpha\alpha'} \quad \text{(horizontal Bose – factor)} \quad (2.22)$$

Since no higher correlation functions are included, $c_2$ is simply denoted by $c$ furtheron. The compactified equations of motion for $\rho$ and $c$ then read:

$$i\partial_t \rho_{\alpha\alpha'} = \sum_{\lambda} (h_{\alpha\lambda} \rho_{\lambda\alpha'} - h_{\lambda\alpha'} \rho_{\alpha\lambda}) + \sum_{\lambda_1\lambda_2\lambda_3} \left( \langle \alpha \lambda_1 | v | \lambda_2 \lambda_3 \rangle c_{\lambda_2\lambda_3\alpha'\lambda_1} - \langle \lambda_1 \lambda_2 | v | \alpha' \lambda_3 \rangle c_{\alpha\lambda_3\lambda_1\lambda_2} \right) \quad (2.23)$$

and

$$i\partial_t c_{\alpha\beta\alpha'} = \sum_{\lambda} (h_{\alpha\lambda} c_{\lambda\beta\alpha'} + h_{\beta\lambda} c_{\alpha\lambda\beta'} - h_{\lambda\beta} c_{\alpha\beta\lambda'} - h_{\lambda\beta'} c_{\alpha\beta\lambda'\lambda}) + \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \langle \lambda_3 \lambda_4 | v | \lambda_1 \lambda_2 \rangle \left\{ Q_{\alpha\beta\lambda_3\lambda_4}^{\pi} (\rho_{20})_{\lambda_1\lambda_2\alpha'\beta'} - Q_{\lambda_1\lambda_2\alpha'\beta'}^{\pi} (\rho_{20})_{\alpha\beta\lambda_3\lambda_4} \right\} + \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \langle \lambda_3 \lambda_4 | v | \lambda_1 \lambda_2 \rangle \left\{ Q_{\alpha\beta\lambda_3\lambda_4}^{\pi} c_{\alpha\beta\lambda_2\lambda_4} - Q_{\lambda_1\lambda_2\alpha'\beta'}^{\pi} c_{\alpha\beta\lambda_3\lambda_4} \right\} + (1 + P_{\alpha\beta}) \left( 1 + P_{\alpha'\beta'} \right) \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \langle \lambda_3 \lambda_4 | v | \lambda_1 \lambda_2 \rangle S_{\alpha\lambda_1\lambda_2\lambda_3} Q_{\lambda_1\alpha'\lambda_3}^{\pi} c_{\lambda_2\lambda_3\lambda_4\beta'} \quad (2.24)$$

The approximation of the bosonic BBGKY density matrix hierarchy given by (2.23) and (2.24) will be denoted as NCBCD (Number-Conserving Boson Correlation Dynamics) furtheron. The equations conserve e.g. particle number $\langle N \rangle = \sum_{\alpha} \rho_{\alpha\alpha}$ and total energy $\langle H \rangle$ for a Hamiltonian with no explicit time dependence.

We note in passing that the compactified bosonic NCBCD equations are formally very similar to the compactified NQCD equations for the nonrelativistic description of fermionic nuclear systems [4] with Bose-factors instead of Pauli-blocking-factors and symmetrized instead of antisymmetrized matrix elements of the two-body interaction and the two-body density matrix.

The NCBCD equations guarantee a dynamical, nonperturbative resummation of ring- and ladder-diagrams in both vertical and horizontal direction and of the corresponding mixed diagrams without double-counting. Topologically this corresponds to considering parquet-like diagrams.
3 The diagrammatical structure of the equations of motion

In this Sect. we investigate the topological structure of the NCBCD equations of motion for the pionic model specified in Appendix A and give a graphical representation of the terms appearing in the equations in order to illustrate the classes of diagrams that are resummed.

The interaction \( v \) (see Appendix A) is split up in the t-, u- and s-channel part:

\[
\langle \alpha |v|\alpha' \rangle = \langle \alpha |v_{tu}|\alpha' \rangle + \langle \alpha |v_s|\alpha' \rangle .
\]

(3.25)

In order to allow for a unique identification, we now label the individual terms in the equations of motion:

\[
i\partial_t \rho_{\alpha\alpha'} = \sum_\lambda (t_{\alpha\lambda} \rho_{\lambda\alpha} - t_{\lambda\alpha'} \rho_{\alpha\lambda})
\]

(3.26a)

\[
+ \sum_\lambda (U_{\alpha\lambda} \rho_{\lambda\alpha'} - U_{\lambda\alpha'} \rho_{\alpha\lambda})
\]

(3.26b)

\[
+ \sum_{\lambda_1 \lambda_2 \lambda_3} (\langle \alpha_1 |v| \lambda_2 \lambda_3 \rangle c_{\lambda_2 \lambda_3 \alpha' \lambda_1} - \langle \lambda_1 \lambda_2 |v| \alpha' \lambda_3 \rangle c_{\alpha \lambda_3 \lambda_1 \lambda_2}) ,
\]

(3.26c)

\[
i\partial_t c_{\alpha' \beta'} = \sum_\lambda (t_{\alpha\lambda} c_{\lambda\beta' \beta'} + t_{\beta\lambda} c_{\alpha\lambda' \beta'} - t_{\lambda\alpha'} c_{\alpha\beta \lambda' \beta'} - t_{\lambda\beta'} c_{\alpha\beta \alpha' \lambda})
\]

(3.27a)

\[
+ \sum_\lambda (U_{\alpha\lambda} c_{\lambda\beta' \beta'} + U_{\beta\lambda} c_{\alpha\lambda' \beta'} - U_{\lambda\alpha'} c_{\alpha\beta \lambda' \beta'} - U_{\lambda\beta'} c_{\alpha\beta \alpha' \lambda})
\]

(3.27b)

\[
+ \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (\langle \lambda_3 \lambda_4 |v_{tu}| \lambda_1 \lambda_2 \rangle (Q^\alpha_{\alpha\lambda_3 \lambda_4} (\rho_{20})_{\lambda_1 \lambda_2 \alpha' \beta'} - Q^\alpha_{\lambda_1 \lambda_2 \alpha' \beta'} (\rho_{20})_{\alpha \beta \lambda_3 \lambda_4}) - Q^\alpha_{\lambda_1 \lambda_2 \alpha' \beta'} (\rho_{20})_{\alpha \beta \lambda_3 \lambda_4})
\]

(3.27c)

\[
+ \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (\langle \lambda_3 \lambda_4 |v_s| \lambda_1 \lambda_2 \rangle (Q^\alpha_{\alpha\lambda_3 \lambda_4} (\rho_{20})_{\lambda_1 \lambda_2 \alpha' \beta'} - Q^\alpha_{\lambda_1 \lambda_2 \alpha' \beta'} (\rho_{20})_{\alpha \beta \lambda_3 \lambda_4})
\]

(3.27d)

\[
+ \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (\langle \lambda_3 \lambda_4 |v_s| \lambda_1 \lambda_2 \rangle (Q^\alpha_{\alpha\lambda_3 \lambda_4} c_{\lambda_1 \lambda_2 \alpha' \beta'} - Q^\alpha_{\lambda_1 \lambda_2 \alpha' \beta'} c_{\alpha \beta \lambda_3 \lambda_4})
\]

(3.27e)

\[
+ \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (\langle \lambda_3 \lambda_4 |v_s| \lambda_1 \lambda_2 \rangle (Q^\alpha_{\alpha\lambda_3 \lambda_4} c_{\lambda_1 \lambda_2 \alpha' \beta'} - Q^\alpha_{\lambda_1 \lambda_2 \alpha' \beta'} c_{\alpha \beta \lambda_3 \lambda_4})
\]

(3.27f)

\[
+ (1 + P_{\alpha \beta})(1 + P_{\alpha' \beta'}) \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (\langle \lambda_3 \lambda_4 |v_{tu}| \lambda_1 \lambda_2 \rangle S Q^\alpha_{\alpha \lambda_1 \alpha' \lambda_3} c_{\lambda_2 \lambda_4 \beta'}
\]

(3.27g)

\[
+ (1 + P_{\alpha \beta})(1 + P_{\alpha' \beta'}) \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (\langle \lambda_3 \lambda_4 |v_s| \lambda_1 \lambda_2 \rangle S Q^\alpha_{\alpha \lambda_1 \alpha' \lambda_3} c_{\lambda_2 \lambda_4 \beta'} .
\]

(3.27h)

3.1 The one-body equation

The term (3.26a) accounts for the free propagation of the one-body density matrix. In general all terms are of hermitean structure, i.e. for each process there is the conjugate process with the incoming and outgoing indices interchanged.
The term (3.26b) accounts for the influence of the unrenormalized mean-field on the propagation of the one-body density matrix, (3.26a) and (3.26b) together then are equivalent to the TDHF approximation.

In order to investigate the structure of (3.26b), we first examine the part with positive overall sign:

\[
\sum_{\lambda} U_{\alpha\lambda} \rho_{\lambda\alpha'} = \\
\sum_{\lambda_1\lambda_2\lambda_3} \rho_{\lambda_1\lambda_2}\langle \alpha \lambda_2 | v_{t,u} | \lambda_3 \lambda_1 \rangle \rho_{\lambda_3\alpha'} \tag{3.28a}
\]

\[+ \sum_{\lambda_1\lambda_2\lambda_3} \rho_{\lambda_1\lambda_2}\langle \alpha \lambda_2 | v_{t,u} | \lambda_1 \lambda_3 \rangle \rho_{\lambda_3\alpha'} \tag{3.28b}
\]

\[+ \sum_{\lambda_1\lambda_2\lambda_3} \rho_{\lambda_1\lambda_2}\langle \alpha \lambda_2 | v_s | \lambda_3 \lambda_1 \rangle \rho_{\lambda_3\alpha'} \tag{3.28c}
\]

\[+ \sum_{\lambda_1\lambda_2\lambda_3} \rho_{\lambda_1\lambda_2}\langle \alpha \lambda_2 | v_s | \lambda_1 \lambda_3 \rangle \rho_{\lambda_3\alpha'} \tag{3.28d}
\]

We now use the following convention: In all diagrams of this Sect. solid lines stand for the link between two equal indices or for external indices. They describe the propagation of the pions participating in the process. In the graphical representation of terms containing a \(\sigma\)-propagator we use a horizontal dashed line for a spacelike and a vertical dashed line for a timelike \(\sigma\). At this point it is useful to note, that in momentum space \(v_{t,u}\) is proportional to a \(\sigma\)-propagator depending on the difference of two on-shell four momenta while \(v_s\) is proportional to a \(\sigma\)-propagator containing the sum of two on-shell four momenta, where for \(v_s\) (up to the formfactor) the factor multiplying the propagator is only half of that for \(v_{t,u}\) (see Appendix A).

The contributions (3.28a) and (3.28d) are depicted in fig. 1 and correspond to the Hartree approximation. The terms (3.28b) and (3.28c) are depicted in fig. 2 and correspond to the exchange terms in the Hartree-Fock approximation. The graphical representation of the conjugate terms is obtained by interchanging the incoming and outgoing external indices.

The coupling of the equation of motion for the one-body density matrix to the two-body correlation function is given by (3.26c). In fig. 3 we show only one of the two conjugate processes. Both conjugate processes together have a gain-loss structure as it is known e.g. from collision terms in transport equations [1]. However, (3.26c) cannot directly be identified with a collision term, since it also renormalizes the mean-field.

In order to present the last point more explicitly, it is useful to introduce a G-matrix and a selfenergy within the formalism of equal-time density matrices. The G-matrix \(G_{\alpha\beta\alpha'\beta'}\) is defined by

\[
\sum_{\lambda_1\lambda_2} G_{\alpha\beta\lambda_1\lambda_2}(\rho_{20})\langle \lambda_1\lambda_2 | \lambda_2\lambda_1 \rangle = \sum_{\lambda_1\lambda_2} \langle \alpha\beta | v | \lambda_1\lambda_2 \rangle \langle \rho_2 | \lambda_1\lambda_2 \rangle. \tag{3.29}
\]

Inserting (3.29) into the equation of motion for \(\rho\) leads to

\[
i\partial_t \rho_{\alpha\alpha'} = \sum_{\lambda} t_{\lambda\alpha} \rho_{\lambda\alpha'} - t_{\lambda\alpha'} \rho_{\lambda\alpha} + \sum_{\lambda_1\lambda_2\lambda_3} \left( G_{\alpha\lambda_1\lambda_2}(\rho_{10})\rho_{2\lambda_2\lambda_3} - G_{\lambda_1\lambda_2\alpha\lambda_3}(\rho_{\alpha\lambda_1})\rho_{\lambda_2\lambda_3} \right) \tag{3.30}
\]

\[\text{The difference of two on-shell four momenta of particles of equal rest mass is always spacelike, the sum is always timelike.}\]
with \( G_{\alpha\beta\alpha'\beta'} = G_{\alpha\beta\alpha'\beta'} + G_{\alpha\beta\beta'\alpha'} \). The G-matrix terms in (3.30) contain all corrections to the free propagation, i.e. also the renormalized mean-field.

A further compactification of the one-body equation can be achieved by introducing the selfenergy according to

\[
\Sigma_{\alpha\alpha'} = \sum_{\lambda_1\lambda_2} \lambda_1 \lambda_2 \rho_{\lambda_1\lambda_2} G_{\alpha\lambda_2\alpha'\lambda_1}^S .
\]

Inserting (3.31) into (3.30) gives

\[
i\partial_t \rho_{\alpha\alpha'} = \sum_{\lambda} \left\{ (t_{\alpha\lambda} + \Sigma_{\alpha\lambda}) \rho_{\lambda\alpha'} - (t_{\lambda\alpha'} + \Sigma_{\lambda\alpha'}) \rho_{\alpha\lambda} \right\} .
\]

The renormalized mean-field can now be obtained as the hermitean part of \( \Sigma_{\alpha\alpha'} \), i.e.

\[
U_{\alpha\alpha'}^{ren} = (Re \Sigma)_{\alpha\alpha'} = \frac{1}{2} \left( \Sigma + \Sigma^\dagger \right)_{\alpha\alpha'} = \frac{1}{2} \Sigma_{\alpha\alpha'} + \frac{1}{2} \Sigma_{\alpha'\alpha}^* .
\]

i.e. as the selfenergy obtained from the hermitean part of the symmetrized G-matrix.

The renormalized mean-field Hamiltonian can then be defined by

\[
h_{\alpha\alpha'}^{ren} = t_{\alpha\alpha'} + U_{\alpha\alpha'}^{ren} .
\]

With the above definitions the equation of motion for the one-body density matrix reads

\[
i\partial_t \rho_{\alpha\alpha'} - \sum_{\lambda} \left\{ h_{\alpha\lambda}^{ren} \rho_{\lambda\alpha'} - h_{\lambda\alpha'}^{ren} \rho_{\alpha\lambda} \right\} = \sum_{\lambda} \left\{ (Im \Sigma)_{\alpha\lambda} \rho_{\lambda\alpha'} - \rho_{\alpha\lambda} (Im \Sigma)_{\lambda\alpha'} \right\} ,
\]

where the term on the right hand side of (3.33) corresponds to the collision term of a transport theory. The explicit derivation of a transport equation with a collision term containing the correct in-medium factors and transport coefficients from the NCBCD equations can be carried out in complete analogy to the fermionic case \[1, 25\].

### 3.2 The mean-field terms in the two-body equation

The term (3.27a) in analogy to the corresponding one in the one-body equation accounts for the free propagation of the two-body correlation function.

The mean-field part (3.27b) of the two-body equation has the same structure as in (3.26b); for each of the external indices there is a diagram with a mean-field loop at the corresponding line.

### 3.3 The Born-term

The expressions (3.27c) and (3.27d) only contain matrix elements of the two-body interaction coupled to a Bose-factor \( Q^\equiv \) and two one-body density matrices, but not the two-body correlation function itself; they describe exactly one elementary interaction process between two pions and thus correspond to the first Born approximation of scattering theory, i.e. (3.27c) in the t- and u- channel and (3.27d) in the s-channel. The two terms together are hence referred to as Born-term.
The scattering process is subject to an in-medium modification by the Bose-factor in the incoming or (for the conjugate process) in the outgoing channel. Using the single particle basis that diagonalizes $\rho$, i.e.

$$\rho_{\alpha\alpha'} = \delta_{\alpha\alpha'} n_\alpha,$$  
(3.36)

the summations over the two intermediate indices appearing in $Q^\pm$ in (3.27d) and (3.27c) break down and we obtain factors of $1 + n_\alpha + n_\beta$ in the incoming channel and $1 + n'_{\alpha} + n'_{\beta}$ in the outgoing channel; the Bose-factors thus cause an enhancement of the contribution of the corresponding process to the total amplitude proportional to the occupation numbers of the respective states. As an example, two of the diagrams generated by (3.27c) and (3.27d) are represented in fig. 4.

### 3.4 Dynamical iteration in the vertical direction

The terms (3.27e) and (3.27f) each contain a matrix element of the two-body correlation function itself coupled to a matrix element of the two-body potential, leading to a dynamical iteration of the interaction in "vertical" direction (i.e. along the time direction of the diagrams). The same Bose-factor $Q^\pm$ as for the Born-term appears in the incoming or the outgoing channel. The two terms with $Q^\pm$ in the incoming channel are graphically represented in fig. 4.

The left diagram in fig. 5 shows, that the term (3.27e) leads to a dynamical resummation of ladder-diagrams in vertical direction, where in the intermediate states as well as in one of the external channels there is a Bose-factor $Q^\pm$. The right diagram in fig. 5 shows, that the term (3.27f) leads to a dynamical resummation of ring-diagrams in vertical direction with the same factors of $Q^\pm$ appearing as for the ladder diagrams. The first step of the resummation in each case is given by the Born-term; (3.27e) and (3.27f) together lead to a mutual dynamical iteration of the ring- and ladder-diagrams as shown in fig. 5.

In analogy to the fermionic NQCD case one obtains a Bethe-Goldstone equation for the $G$-matrix by neglecting (3.27g) and (3.27h) and considering the equations in the stationary limit. The terms (3.27e) and (3.27f) are therefore in the following denoted as $G$-matrix-terms.

In order to simplify the notation, the single particle basis is now chosen to diagonalize the unrenormalized mean-field Hamiltonian, i.e.

$$h_{\alpha\alpha'} = \delta_{\alpha\alpha'} \epsilon_\alpha.$$

One can then rewrite the equation of motion for $c$ in the limit described above as

$$\begin{aligned}
(\omega - \epsilon_\alpha - \epsilon_\beta) c_{\alpha\beta\alpha'\beta'} - (\omega - \epsilon_\alpha' - \epsilon_\beta') c_{\alpha\beta\alpha'\beta'} &= \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \langle \lambda_3 \lambda_4 \mid v \mid \lambda_1 \lambda_2 \rangle \left\{ Q_{\alpha\beta\lambda_3 \lambda_4}^\pm (\rho_2)_{\lambda_1 \lambda_2 \alpha' \beta'} - Q_{\lambda_1 \lambda_2 \alpha' \beta'}^\pm (\rho_2)_{\alpha\beta\lambda_3 \lambda_4} \right\}.
\end{aligned}$$

(3.38)

This equation is fulfilled for $(\eta \to 0^+)$, if

$$c_{\alpha\beta\alpha'\beta'} = \frac{1}{\omega - \epsilon_\alpha - \epsilon_\beta + i\eta} \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} Q_{\alpha\beta\lambda_3 \lambda_4}^\pm G_{\lambda_3 \lambda_4 \lambda_1 \lambda_2} (\rho_2)_{\lambda_1 \lambda_2 \alpha' \beta'}$$

(3.39)
and the conjugate equation hold, where at the right hand side the definition of the G-matrix has been inserted. Matrix-multiplication of (3.39) with the matrix element of the two-body potential and eliminating \( c \) on the left hand side via \( c = \rho_2 - \rho_{20} \) leads to

\[
\sum_{\gamma_1 \gamma_2} G_{\alpha \beta \gamma_1 \gamma_2} (\rho_{20}) \gamma_1 \gamma_2 \alpha' \beta' = \sum_{\gamma_1 \gamma_2} \langle \alpha \beta | v | \gamma_1 \gamma_2 \rangle (\rho_{20}) \gamma_1 \gamma_2 \alpha' \beta' + \sum_{\gamma_1 \gamma_2 \lambda_1 \lambda_2 \lambda_3 \lambda_4} \langle \alpha \beta | v | \gamma_1 \gamma_2 \rangle \frac{Q_{\gamma_1 \gamma_2 \lambda_3 \lambda_4}}{\omega - \epsilon_{\gamma_1} - \epsilon_{\gamma_2} + i\eta} G_{\lambda_3 \lambda_4 \lambda_1 \lambda_2} (\rho_{20}) \lambda_1 \lambda_2 \alpha' \beta' \]

(3.40)

for an arbitrary \( \rho_{20} \), i.e. we have

\[
G_{\alpha \beta \alpha' \beta'} = \langle \alpha \beta | v | \alpha' \beta' \rangle + \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \langle \alpha \beta | v | \lambda_1 \lambda_2 \rangle \frac{Q_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}}{\omega - \epsilon_{\alpha} - \epsilon_{\beta} + i\eta} G_{\lambda_3 \lambda_4 \alpha' \beta'} .
\]

(3.41)

Equation (3.41) is a Bethe-Goldstone-equation for the G-matrix which guarantees the resummation of ring- and ladder-diagrams in vertical direction.

### 3.5 Dynamical iteration in the horizontal direction

The terms (3.27g) and (3.27h) each also contain a two-body correlation function coupled to a matrix element of the two-body potential, which leads to a dynamical resummation of diagrams. The topological structure of the terms, graphically represented in figs. 6 and 7, in contrast to (3.27e) and (3.27f) leads to a resummation in “horizontal” direction, where the factor \( Q_{\perp} \) appears in the intermediate states and at a pair of an incoming and an outgoing index.

In the single particle basis with \( \rho_{\alpha \alpha'} = \delta_{\alpha \alpha'} n_\alpha \), as for the terms accounting for the resummation in vertical direction, the summation over the intermediate indices breaks down and one remains with a factor of \( n'_{\alpha'} - n_\alpha \). In our notation each term containing one horizontal factor \( Q_{\perp} \) comprises a process together with its conjugate process.

The direct term in (3.27g), with the indices in the matrix element of the two-body potential not interchanged, is shown in fig. 6 at the top. It leads to a dynamical resummation of ring-diagrams in horizontal direction. The exchange term in (3.27g) accounts for the dynamical resummation of ladder-diagrams in horizontal direction.

In (3.27h) the direct term and the exchange term are equal; together they account for the resummation of ladder-diagrams in horizontal direction.

All terms of the NCBCD approximation together lead to a dynamical resummation and mutual iteration of ring- and ladder-diagrams in vertical and horizontal direction on the two-body level, i.e. a parquet-like resummation [26, 27] which also renormalizes the one-body equation of motion.

### 4 Numerical studies for pionic systems

#### 4.1 Comparison of partial resummations

In the following we consider a pion gas as described in Appendix A confined to a box of sidelongths \( L_x = L_y = L_z = 10 \text{fm} \). In order to reduce the numerical effort we assume the pion gas to be in an isospin-symmetric configuration and thereby eliminate the internal
isospin degrees of freedom as described in Appendix B. As a single particle basis for our numerical simulations we choose standing waves, which vanish at the box boundaries. The calculations are performed using the 11 (in the noninteracting case) energetically lowest basis elements.

Since the NCBCD equations only describe the propagation of the density matrices in time, we first have to solve the problem of finding proper initial conditions for the system at some given starting time. In the bosonic case there is no direct access to the manifold of stationary solutions of the NCBCD equations and, furthermore, it is e.g. not clear, what further constraints – besides the above mentioned symmetry properties – these solutions have to fulfill in order to remain physical. In [28] it was shown for the case of an analytically solvable model (Lipkin-model), that only requiring the density matrices to be positively definite and to satisfy the trace theorems for fermionic systems with good particle number leads to a sensible solution within the framework of the NQCD formalism. However, in our case the trace theorems do not hold.

The most obvious starting point for the initial condition is given by a stationary self-consistent Hartree-Fock solution. In order to generate this solution for a given temperature and particle number, the one-body density matrix is occupied with a Bose-Einstein distribution with respect to the eigenstates of the mean-field Hamiltonian, where the chemical potential is adjusted in order to give the right particle number.

In fig. 8 the Hartree-Fock energy levels and occupation numbers are shown for the example of systems with 60 and 90 pions for various temperatures, where the occupation numbers are given for only one isospin quantum number; the total occupation numbers are obtained by multiplication with a factor of 3. Due to the spatial symmetries of the system, some of the higher lying levels are degenerate. As one expects, the error due to the finite number of basis elements reduces in going to higher densities and lower thermal excitations.

After providing a first approximation for the initial condition of the pion gas, we now investigate how the system evolves in time using the complete NCBCD equations or certain limiting cases for the partial resummation of diagrams on the two-body level. We aim at deciding, if for an approximately correct description of two-body quantities in the pionic many-body system one really needs to use the complete, numerically very involved NCBCD equations.

The limiting case, in which only the resummation of ring- and ladder-diagrams in vertical direction is taken into account, is obtained from the complete equations (2.23) and (2.24) by neglecting (2.24d) and corresponds to time-dependent G-Matrix theory (TDGMT). It will be denoted by vertical approximation in the following.

The limiting case, which only takes into account the resummation of ring- and ladder-diagrams in horizontal direction, is obtained from (2.23) and (2.24) by neglecting (2.24c) and will be denoted by horizontal approximation furtheron. In the fermionic case the approximation generated from the NQCD equations in an analogous way can be identified with a two-body RPA theory [2, 3].

Using the isospin channel quantities defined in Appendix B, we get the expression for the total energy of the system:

\[ E = \langle H \rangle = 3 \sum_{aa'} t_{a'a} \rho_{aa'} \]  

(4.42a)
The kinetic energy is given by (4.42a), the mean-field energy by (4.42b) and the correlation energy by (4.42c); the total energy, which is conserved in time, is the sum of these three contributions.

Even though in the investigation for fermionic systems in [5] the vertical as well as the horizontal approximation showed distinct deviations from the full NQCD approximation, the correlation energies still had about the same order of magnitude as in the complete correlation dynamics approach. For the pionic systems under investigation here this is no longer true. While at lower densities the 3 approximations do not differ significantly, at higher densities the horizontal approximation yields absolute correlation energies which are drastically higher than those extracted from the NCBCD calculations.

In order to illustrate this behaviour, we show in the upper part of fig. 9 the correlation energies (averaged over time) for the 3 approximations as a function of the pion number at a temperature of 100 MeV. The system in each case was propagated up to the time $t = 20$ fm/c. One clearly sees the dramatic overestimation of correlation energy by the horizontal approximation with increasing density. In the lower part of fig. 9 we have left out the horizontal approximation in order to allow for a better comparison between NCBCD and the vertical approximation. The vertical approximation always generates a correlation energy, which is about a factor of 2 too small as compared to the NCBCD correlation energy.

This demonstrates the importance of mixed diagrams consisting of contributions from the vertical and the horizontal approximation and of the interference of the different classes of diagrams. We are thus lead to the conclusion, that only the full NCBCD approach is able to yield adequate results for two-body quantities in the pionic system under consideration. Especially we infer, that a description of two-body dynamics by a Bethe-Goldstone approach or a time-dependent G-Matrix theory is not appropriate. For all further investigations we, therefore, use the complete NCBCD approach.

4.2 Dynamical generation of a correlated state

In this Sect. we will demonstrate, how one can dynamically generate an approximately stationary physical state for the NCBCD equations, which is fully correlated on the two-body level.

A first attempt for the solution of this problem consists in initializing the Hartree-Fock solution and propagating the system in time until all correlations have built up and the system has reached a quasi-stationary state. However, in this state all density matrix elements will show more or less stable oscillations about an equilibrium value and
not necessarily approach a stationary state asymptotically, as we have checked numerically. Furthermore, this approach has the additional problem, that the proper energy of the correlated state cannot be reached, since the NCBCD approximation conserves total energy.

For the reasons mentioned above we introduce a different method by modifying the NCBCD equations of motion in the following way: The terms (2.23a) and (2.24a), which describe the free and mean-field propagation of $\rho$ and $c$, remain unchanged. In all other terms, i.e. in (2.23b), (2.24b), (2.24c) and (2.24d), the matrix elements of the two-body potential $\langle \alpha\beta | v | \alpha'\beta' \rangle$ are multiplied by a time-dependent dimensionless factor $g(t)$. This alteration of the equations of motion, although at first sight appearing to be arbitrary, does not lead to an unphysical propagation of the system, since it is equivalent to an alteration of the Hamiltonian according to $H \rightarrow H'(t)$, where

$$H'(t) = \sum_{\alpha\alpha'} \left\{ t_{\alpha\alpha'} + (1 - g(t)) \sum_{\lambda_1\lambda_2} \rho_{\lambda_1\lambda_2} \langle \alpha\lambda_2 | v | \alpha'\lambda_1 \rangle \right\} a_{\alpha}^\dagger a_{\alpha'} + g(t) \sum_{\alpha\beta\alpha'\beta'} \langle \alpha\beta | v | \alpha'\beta' \rangle a_{\alpha}^\dagger a_{\beta}^\dagger a_{\alpha'} a_{\beta'}.$$  \hfill (4.43)

If $g(t) = 0$, one recovers the TDHF approximation; if $g(t) = 1$ one recovers the full NCBCD approximation with respect to $H$.

By initializing a stationary Hartree-Fock solution at time $t = 0$ and propagating the system using the NCBCD equations induced by (4.43) up to the time $t = t_f$, where $g(0) = 0$ and $g(t_f) = 1$, we continuously switch on the residual interaction on the two-body level. The total energy is not conserved during this process, thus enabling the system to energetically approach the correlated equilibrium configuration.

Numerical experiments using different forms of $g(t)$ have shown that a linear function requires the smallest amount of computation time in order to guarantee the convergence of the method. We, therefore, choose $g(t) = \lambda t$ and hence $t_f = 1/\lambda$.

In fig. 10, the change $\Delta E$ of the total energy reached at the end of the process is plotted versus $t_f$ for 30 pions in the volume of $10 \text{ fm}^3$. The picture clearly shows, that there is a convergence as $t_f \rightarrow \infty$ (adiabatic limit); i.e. if the parameter $g(t)$ changes slowly, the response of the system is fast enough to generate an equilibrium configuration with respect to $H'(t)$ at any given time $t$. The last statement implies, that for $t_f \rightarrow \infty$ a convergence of the trajectory of the system in the overall configuration space has to occur. This is indeed the case, as can e.g. be seen from fig. 11, where the correlation energy with respect to $H$ during the process is depicted as a function of $t/t_f$ for various values of $t_f$. Furthermore, fig. 11 illustrates, that the oscillations caused by switching on the residual interaction too fast progressively vanish as one goes to higher values of $t_f$, which clearly indicates that indeed in the limit $t_f \rightarrow \infty$ the system equilibrates at each time during the process.

The change in total energy during the adiabatic process is not mainly caused by a buildup of correlation energy, but rather by a change in kinetic and mean-field energy by about the same order of magnitude, since the generation of two-body correlations causes a redistribution at the one-body level as well. This can be seen from fig. 12, where the changes in the single contributions to the total energy and the change of the total energy with respect to the Hartree-Fock solution are plotted as a function of $t_f$. Furthermore, we note that for all densities and temperatures considered in this work, the total energy is lowered when the residual interaction is adiabatically switched on.
4.3 Two-body observables in coordinate space

We now turn to the investigation of the two-body density matrix resulting from the adiabatic method described above. For this purpose the residual interaction in each case is switched on with $t_f = 160 \text{ fm}/c$, and the values for $\rho$ and $c$ reached at $t = t_f$ are used for the evaluation of the observables of interest.

The two-body density matrix $\rho_2$ in coordinate space is given by

$$\rho_2(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2) = \sum_{\alpha\beta\alpha'\beta'} (\rho_2)_{\alpha\beta\alpha'\beta'} \psi_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_2)\psi^*_\alpha(\vec{r}'_1)\psi^*_\beta(\vec{r}'_2)$$

(4.44)

and the uncorrelated part of $\rho_2$, i.e. $\rho_{20}$, and the two-body correlation function $c$ by

$$\rho_{20}(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2) = \sum_{\alpha\beta\alpha'\beta'} (\rho_{20})_{\alpha\beta\alpha'\beta'} \psi_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_2)\psi^*_\alpha(\vec{r}'_1)\psi^*_\beta(\vec{r}'_2)$$

(4.45)

and

$$c(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2) = \sum_{\alpha\beta\alpha'\beta'} c_{\alpha\beta\alpha'\beta'} \psi_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_2)\psi^*_\alpha(\vec{r}'_1)\psi^*_\beta(\vec{r}'_2)$$

(4.46)

with

$$\rho_2(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2) = \rho_{20}(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2) + c(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2)$$

(4.47)

and

$$\rho_{20}(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2) = \rho(\vec{r}_1; \vec{r}'_1)\rho(\vec{r}_2; \vec{r}'_2) + \rho(\vec{r}_1; \vec{r}'_2)\rho(\vec{r}_2; \vec{r}'_1)$$

(4.48)

The probability distribution for a simultaneous measurement of a particle at position $\vec{r}_1$ and a particle at $\vec{r}_2$ is given by $\rho_2(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2)$. In fig. $\rho(x, x; x, x) = \int dydz \rho_2(\vec{r}, \vec{r}; \vec{r}_1, \vec{r}_2)$ with $\vec{r} = (x, y, z)$ is plotted as a function of $x$ for the temperatures 50 and 150 MeV at various pion densities (solid line). In addition, the same quantity is shown without the correlated part of $\rho_2$, i.e. only for $\rho_{20}$ (dashed line), and for the classical two-body distribution

$$\rho_2^{\text{class}}(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2) = \rho(\vec{r}_1; \vec{r}'_1)\rho(\vec{r}_2; \vec{r}'_2)$$

(4.49)

i.e. neglecting Bose statistics (dotted line). At the boundaries of the box all distributions vanish due to the boundary conditions imposed on the single particle basis elements. The inclusion of Bose statistics leads to an enhancement of the distribution obtained with $\rho_{20}$ by a factor of 2 with respect to the distribution obtained with $\rho_2^{\text{class}}$. Including the two-body correlation function leads to a further increase in the probability of finding two particles at the same place, which is due to the fact that the $\sigma$-interaction used is purely attractive.

We now investigate the strength of the enhancement of $\rho_2(\vec{r}, \vec{r}; \vec{r}_1, \vec{r}_2)$ relative to $\rho(\vec{r}; \vec{r})\rho(\vec{r}; \vec{r})$ as a function of temperature and density. Naively one might expect, that with increasing density the system will be more correlated and therefore show a stronger enhancement. However, this is not true. In fig. one can already observe, that at equal temperature an increase in density does not necessarily imply an increase in the relative enhancement of the two-body probability. While at $T=50$ MeV the enhancement decreases with increasing density over the whole density range considered, at $T=150$ MeV there is a maximum in enhancement between 45 and 60 pions in (10 fm)$^3$.
In order to illustrate this point more clearly the quantity

\[
\frac{\rho_2}{\rho\rho} = \frac{\int d^3r \rho_2(\vec{r}, \vec{r}; \vec{r}, \vec{r})}{\int d^3r \rho(\vec{r}; \vec{r}) \rho(\vec{r}; \vec{r})}
\]  

(4.50)

is plotted in fig. 14 as a function of the pion density for various temperatures. For higher temperatures there are distinct maxima of the relative enhancement at certain densities. For all temperatures the relative enhancement then decreases again for higher densities. This behaviour is an indication of the nonperturbative nature of n-point correlation dynamics (the lowest order of which in our case is given by the Hartree-Fock approximation), since a denser and therefore more strongly interacting system does not imply a worse convergence of the cluster expansion.

The next point to consider is the probability distribution for simultaneously finding two particles at different places in coordinate space. The question is, how this two-body density is altered by the inclusion of the two-body correlation function as a function of the relative distance of the two particles.

In fig. 15 a two-dimensional cut through the six-dimensional quantities

\[
\frac{\rho_{20}^3(\vec{r}_1, \vec{r}_2; \vec{r}_1, \vec{r}_2)}{\rho(\vec{r}_1; \vec{r}_1) \rho(\vec{r}_2; \vec{r}_2)} \quad \frac{\rho_2^3(\vec{r}_1, \vec{r}_2; \vec{r}_1, \vec{r}_2)}{\rho(\vec{r}_1; \vec{r}_1) \rho(\vec{r}_2; \vec{r}_2)}
\]  

(4.51)

at \(y_1 = y_2 = z_1 = z_2 = 5\) fm (i.e. in the center of the box of sidelength 10 fm) is depicted as a contour-plot as a function of \(x_1\) and \(x_2\) for two parameter pairs of density and temperature; in the left part of the figure only Bose-statistics are taken into account. The inclusion of the two-body correlation function (right part) leads to an increased gradient with respect to the relative distance, i.e. at smaller distances the two-body density is increased and at larger distances decreased.

In fig. 16 we show the same quantity as a function of \(x\) with \(x_1 = x\) and \(x_2 = L - x\), i.e. along a cut from the center to the lower right corner in the previous figure, for 50 and 150 MeV temperature and various densities, where \(L = 10\) fm is the sidelength of the box; the two-body densities evaluated with \(\rho_2\) (solid lines) and with \(\rho_{20}\) (dashed lines) are directly compared.

Due to Bose-statistics, the curve for \(\rho_{20}\) always approaches 2 as \(x \to 0\). Again the increased gradient with respect to the relative distance caused by the inclusion of the two-body correlation function can be seen, which is most pronounced for those parameters, that also show the strongest enhancement of probability at relative distance 0, i.e. at the maxima in fig. 14.

### 4.4 Relative strength of two-body correlations

In the last Sect. we discovered, that for fixed temperature the relative enhancement of the probability for simultaneously detecting two particles at the same place in coordinate space exhibits a maximum at a certain density. This behaviour of the pionic many-body system will now be analyzed more closely. In this respect, we introduce the quantity

\[
\frac{c}{\rho_2} = \frac{\sum_{\alpha \beta \alpha' \beta'} |c_{\alpha \beta \alpha' \beta'}|^2}{\sum_{\alpha \beta \alpha' \beta'} |(\rho_2)_{\alpha \beta \alpha' \beta'}|^2},
\]  

(4.52)

which will be denoted as "correlation strength" in the following. While in the two-body density in coordinate space only the isospin channel correlation functions \(c^{stu}\) contribute
(since we are looking at pairs of identical pions), all possible isospin configurations enter into the evaluation of the correlation strength (4.52). Also all off-diagonal elements of the two-body density matrix are considered without having the possibility of mutually cancelling out each other. Although (4.52) is not invariant under unitary transformations of the single particle basis, it is still a useful measure of the relative importance of two-body correlations in the system.

The curves in fig. 17 were generated by adiabatically switching on the residual interaction as described in Sect. 4.2 using \( t_f = 160 \text{ fm}/c \) and plotting the correlation strength reached at \( t = t_f \) versus density for various temperatures. Again there are distinct maxima as in the case of the relative enhancement of the two-body density in coordinate space. If we compare fig. 17 to fig. 14 we observe a very good agreement of the shape of the curves, implying that the observed result is not an artifact of the way we defined our measure for the relative importance of two-body correlations.

As one approaches the maxima, the stationary Hartree-Fock solution used for the initialization at \( t = 0 \) becomes increasingly useless. It is therefore tempting to assume, that the convergence of the cluster expansion in general might also become worse; an assumption which would have to be checked by explicitly taking into account higher order than the two-body correlation functions. However, within the framework of the NCBCD approximation it is not possible to verify this assumption.

A possible interpretation of the observed maximum is that the system runs through a second order phase transition at a certain critical density (for fixed temperature), which corresponds to a critical chemical potential in the parameter space of the theory. In such a scenario one might expect the cluster expansion to break down as one approaches the critical region. Moving away from the critical region the mean-field (Hartree-Fock) solution progressively becomes better and the convergence of the cluster expansion improves [24].

A problem we want to point out in this context is that in a finite system, as considered in this work, one cannot draw any definite conclusions about the real order of a phase transition, since in such systems usually also first order phase transitions are smeared out and appear as second order phase transitions [29].

Finally, we explicitly demonstrate that not only a higher density, but also a stronger coupling does not automatically imply stronger correlations in the pionic system. For that purpose we show in fig. 18 the correlation strength obtained by adiabatically switching on the residual interaction plotted versus density for 100 MeV temperature for the coupling given by (A.77) (solid line) and for a coupling twice as large (dashed line). The correlation strength for the higher value of the coupling constant is distinctly smaller than that for the lower one. This again clearly proves the nonperturbative nature of the cluster expansion.

5 Summary

In this work we have derived the NCBCD approximation for the nonperturbative, dynamical description of particle number conserving bosonic many-body systems including two-body correlations. This approach leads to a dynamical resummation and mutual iteration of ring- and ladder-diagrams in vertical and horizontal direction on the two-body level, i.e. a parquet-like resummation [26, 27], which also renormalizes the one-body equation of motion.
For our numerical simulations we assumed a pion gas confined to a box and a single particle basis of standing waves. The $\pi\pi$ interaction used was derived from a covariant Lagrangian in a three-dimensional reduction scheme and fitted to $\pi\pi$ phase shifts. Within this model we showed, that a restriction of the equations to a resummation in horizontal direction leads to a dramatic overestimation of correlation energy and a restriction to a resummation in vertical direction leads to an underestimation of correlation energy by about a factor of 2. Thus an adequate description of dynamical two-body correlations in pionic systems requires the use of the complete NCBCD equations; e.g. a description within the TDGTM approximation is inappropriate.

We, furthermore, showed a way to obtain a correlated, stationary and physical configuration by adiabatically switching on the residual interactions on the two-body level starting from a stationary Hartree-Fock solution. For this configuration we investigated two-body probability distributions in coordinate space and obtained an increased probability for the simultaneous detection of two pions at the same place and a decreased probability for the simultaneous detection of two pions at large distances caused by the inclusion of the two-body correlation function. The two-body correlations in coordinate space as well as the correlation strength (defined in Sect. 4.4) showed a distinct maximum in the relative strength of two-body correlations as compared to the disconnected parts of the two-body density matrix at a certain critical density at fixed temperature. This might be interpreted as a signature for a phase transition at a certain critical density. However, in the investigated temperature region of $T \leq 200$ MeV this maximum in each case was located at a density below $0.09$ fm$^{-3}$, which is much smaller than those estimated in ultrarelativistic nucleus-nucleus collisions \[10\].

Furthermore we found, that at higher densities the two-body correlations progressively decrease in relative importance, which implies the possibility of an adequate description of the system on the one-body level in such cases as ultrarelativistic heavy-ion collisions. We thus conclude that the conventional Hanbury Brown-Twiss (HBT) analysis of pion sources from $\pi - \pi$ correlations in such reactions appears justified.

Appendix

A Specification of the model Hamiltonian

In order to describe an interacting pion system we consider the Lagrangian

$$L = \frac{1}{2} \left( \partial_{\mu} \Phi \right) \left( \partial^{\mu} \Phi \right) - \frac{1}{2} m^2 \Phi^2 + \frac{1}{2} (\partial_{\mu} \sigma) (\partial^{\mu} \sigma) - \frac{1}{2} M^2 \sigma^2 - g \sigma \Phi^2 ,$$

(A.53)

where $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ is the pion field of mass $m$ and $\sigma$ is the scalar, isoscalar field of mass $M$ mediating the interaction between the pions.

From the Lagrangian \[(A.53)\] we get the coupled equations of motion for the classical fields or equivalently for the quantized field operators of the system:

$$\partial^{\mu} \partial_{\mu} \Phi_i + m^2 \Phi_i = -2g\sigma \Phi_i ,$$

(A.54)

and

$$\partial^{\mu} \partial_{\mu} \sigma + M^2 \sigma = -g \Phi^2 .$$

(A.55)
In order to obtain a two-body interaction for the pions, we integrate out the \( \sigma \)-field. Formally inverting (A.55) we get

\[
\tilde{\sigma}(k) = \frac{1}{k^\mu k_\mu - M^2 + i\epsilon} \int d^4 x \ e^{ik_\mu x^\mu} \tilde{\Phi}^2(x). \tag{A.56}
\]

With \( \tilde{G}_\sigma(k) = \frac{1}{k^\mu k_\mu - M^2 + i\epsilon} \) and \( G_\sigma(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik_\mu x^\mu} \tilde{G}_\sigma(k) \) we have

\[
\tilde{\sigma}(k) = g \tilde{G}_\sigma(k) \int d^4 x' e^{ik_\mu x'^\mu} \tilde{\Phi}^2(x') \Rightarrow \sigma(x) = g \int d^4 x' G_\sigma(x - x') \tilde{\Phi}^2(x'). \tag{A.57}
\]

Insertion of (A.57) into (A.54) leads to

\[
\left( \partial^\mu \partial_\mu + m^2 \right) \Phi_i(x) = -2g^2 \Phi_i(x) \int d^4 x' G_\sigma(x - x') \tilde{\Phi}^2(x'), \tag{A.58}
\]

i.e. the \( \sigma \)-field has been eliminated from the equations of motion.

We aim at a Hamiltonian, which yields (A.58) via the Heisenberg equations for \( \Phi_i \) and its canonically conjugate field momentum \( \Pi_i \).

This implies, that we have to carry out a three-dimensional reduction of the four-dimensional integration in (A.58). As a first step we neglect the zeroth component of the four momentum \( k^\mu \) in the denominator of \( \tilde{G}_\sigma(k) \), i.e.

\[
\tilde{G}_\sigma(k) \approx \tilde{G}_\sigma(\vec{k}) = \frac{1}{-\vec{k}^2 - M^2}. \tag{A.59}
\]

and

\[
G_\sigma(\vec{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \frac{1}{-\vec{k}^2 - M^2}. \tag{A.60}
\]

The intrinsic flaws of this naive instantaneous approximation will in part be cured later on. The easiest way to uncover these defects is to use a mode-expansion of the pion fields and to establish, which terms are responsible for s-, t- or u-channel scattering.

With (A.59) and (A.60) we get

\[
\int d^4 x' G_\sigma(x - x') \tilde{\Phi}^2(x') \approx \int d^4 x' \int d^4 k \frac{1}{(2\pi)^4} e^{-i\vec{k} \cdot (x' - x)} \frac{1}{-\vec{k}^2 - M^2} \tilde{\Phi}^2(x')
\]

\[
= \int d^4 x' \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot (x' - x)} \tilde{\Phi}^2(x') \frac{1}{-\vec{k}^2 - M^2} \delta(x^0 - x'^0)
\]

\[
= \int d^3 x' G_\sigma(\vec{x} - \vec{x'}) \tilde{\Phi}^2(\vec{x'}, x^0). \tag{A.61}
\]

The equation of motion from Hamiltonian dynamics in the instantaneous approximation (A.61) thus reads:

\[
\left( \partial^\mu \partial_\mu + m^2 \right) \Phi_i(\vec{x}) = -2g^2 \Phi_i(\vec{x}) \int d^3 x' G_\sigma(\vec{x} - \vec{x'}) \tilde{\Phi}^2(\vec{x'}), \tag{A.62}
\]

where all field operators are taken at the same time \( t \). This is fulfilled for

\[
H = \frac{1}{2} \int d^3 x \left\{ \Pi^2(\vec{x}) + \left( \nabla \Phi(\vec{x}) \right)^2 + m^2 \tilde{\Phi}^2(\vec{x}) \right\}
\]

\[
+ \frac{1}{2} g^2 \int d^3 x d^3 x' \tilde{\Phi}^2(\vec{x}) G_\sigma(\vec{x} - \vec{x'}) \tilde{\Phi}^2(\vec{x'}), \tag{A.63}
\]

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since (A.63) together with (2.6) and the canonical equal-time commutation relations

\[ [\Phi_i(\vec{x}, t), \Phi_j(\vec{x}', t)] = [\Pi_i(\vec{x}, t), \Pi_j(\vec{x}', t)] = 0 , \]

\[ [\Phi_i(\vec{x}, t), \Pi_j(\vec{x}', t)] = i\delta_{ij}\delta^{(3)}(\vec{x} - \vec{x}') \] (A.64)

generate the equations of motion

\[ \partial_t \Phi_i(\vec{x}) = \Pi_i(\vec{x}) \] (A.65)

and

\[ \partial_t \Pi_i(\vec{x}) = \nabla^2 \Phi_i(\vec{x}) - m^2 \Phi_i(\vec{x}) - 2g^2 \Phi_i(\vec{x}) \int d^4x' G_\sigma(\vec{x} - \vec{x}') \Phi^2(\vec{x}') , \] (A.66)

equivalent to (A.62).

The mode expansions for the pion field operators now read:

\[ \Phi_i(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{2\omega(\vec{k})}} [a(\vec{k}, i) + a^\dagger(-\vec{k}, i)] \] (A.67)

and

\[ \Pi_i(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{\omega(\vec{k})^2 + m^2}} [a(\vec{k}, i) - a^\dagger(-\vec{k}, i)] \] (A.68)

with

\[ \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2} \] (A.69)

and

\[ [a(\vec{k}, i), a^\dagger(\vec{k}', j)] = \delta_{ij}\delta^{(3)}(\vec{k} - \vec{k}') , \] (A.70)

where \( i \) denotes the isospin of the corresponding operator with regard to the cartesian representation of SU(2). Inserting the mode expansions (A.67) and (A.68) into (A.63) leads to

\[ H = \sum \int d^3k \omega(\vec{k}) a^\dagger(\vec{k}, \tau)a(\vec{k}, \tau) \]

\[ + \sum_{\tau',\tau''} \int d^3k_1 d^3k_2 d^3k_3 d^3k_4 \frac{g^2}{8(2\pi)^3} \left( \omega(\vec{k}_1)\omega(\vec{k}_2)\omega(\vec{k}_3)\omega(\vec{k}_4) \right)^{-\frac{1}{2}} \times \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)G_\sigma(\vec{k}_1 + \vec{k}_2) \times \]

\[ \times \left\{ a(\vec{k}_1, \tau)a(\vec{k}_2, \tau)a(\vec{k}_3, \tau')a(\vec{k}_4, \tau') + a(\vec{k}_1, \tau)a(\vec{k}_2, \tau)a(\vec{k}_3, \tau')a^\dagger(-\vec{k}_4, \tau') \right. \]

\[ + a(\vec{k}_1, \tau)a(\vec{k}_2, \tau)a^\dagger(-\vec{k}_3, \tau')a(\vec{k}_4, \tau') + a(\vec{k}_1, \tau)a(\vec{k}_2, \tau)a^\dagger(-\vec{k}_3, \tau')a^\dagger(-\vec{k}_4, \tau') \]

\[ + a(\vec{k}_1, \tau)a^\dagger(-\vec{k}_2, \tau)a(\vec{k}_3, \tau')a(\vec{k}_4, \tau') + a(\vec{k}_1, \tau)a^\dagger(-\vec{k}_2, \tau)a(\vec{k}_3, \tau')a^\dagger(-\vec{k}_4, \tau') \]

\[ + a(\vec{k}_1, \tau)a^\dagger(-\vec{k}_2, \tau)a^\dagger(-\vec{k}_3, \tau')a(\vec{k}_4, \tau') + a(\vec{k}_1, \tau)a^\dagger(-\vec{k}_2, \tau)a^\dagger(-\vec{k}_3, \tau')a^\dagger(-\vec{k}_4, \tau') \]

\[ + a^\dagger(-\vec{k}_1, \tau)a(\vec{k}_2, \tau)a(\vec{k}_3, \tau')a(\vec{k}_4, \tau') + a^\dagger(-\vec{k}_1, \tau)a(\vec{k}_2, \tau)a^\dagger(-\vec{k}_3, \tau')a^\dagger(-\vec{k}_4, \tau') \]

\[ + a^\dagger(-\vec{k}_1, \tau)a^\dagger(-\vec{k}_2, \tau)a(\vec{k}_3, \tau')a(\vec{k}_4, \tau') + a^\dagger(-\vec{k}_1, \tau)a^\dagger(-\vec{k}_2, \tau)a^\dagger(-\vec{k}_3, \tau')a^\dagger(-\vec{k}_4, \tau') \}

\]
where now $\tau$ denotes the isospin quantum number.

The Hamiltonian (A.71) still contains terms (underlined) not commuting with the particle number operator. In view of Sect. 2, we neglect such terms in our approach, i.e. we only consider elastic scattering of pions (also including particle-number conserving $s$-channel scattering). This is equivalent to the assumption, that the $-\tau$ particle number violating operator products generated by the underlined terms in (A.71) (after normal ordering) in the equations of motion for particle number conserving operator products have vanishing or at least negligible expectation values, in agreement with assumption (2.18) of Sect. 2.

Normal ordering the remaining terms in (A.71) gives:

\[
H = \sum_\tau \int d^3k \, \omega(\vec{k}) a^\dagger(\vec{k}, \tau) a(\vec{k}, \tau) \\
+ \sum_{\tau, \tau'} \int d^3k_1 d^3k_2 d^3k_3 d^3k_4 \frac{g^2}{8(2\pi)^3} \left( \omega(\vec{k}_1) \omega(\vec{k}_2) \omega(\vec{k}_3) \omega(\vec{k}_4) \right)^{-\frac{1}{2}} \\
\times \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \tilde{G}_{\sigma}(\vec{k}_1 + \vec{k}_2) \\
\times \left\{ a^\dagger(-\vec{k}_3, \tau') a^\dagger(-\vec{k}_4, \tau') a(\vec{k}_1, \tau) a(\vec{k}_2, \tau) + a^\dagger(-\vec{k}_2, \tau) a^\dagger(-\vec{k}_4, \tau') a(\vec{k}_1, \tau) a(\vec{k}_3, \tau') \\
+ a^\dagger(-\vec{k}_2, \tau) a^\dagger(-\vec{k}_3, \tau') a(\vec{k}_1, \tau) a(\vec{k}_4, \tau') + a^\dagger(-\vec{k}_4, \tau') a^\dagger(-\vec{k}_3, \tau') a(\vec{k}_1, \tau) a(\vec{k}_2, \tau) \\
+ a^\dagger(-\vec{k}_1, \tau) a^\dagger(-\vec{k}_3, \tau') a(\vec{k}_2, \tau) a(\vec{k}_4, \tau') + a^\dagger(-\vec{k}_1, \tau) a^\dagger(-\vec{k}_2, \tau) a(\vec{k}_3, \tau') a(\vec{k}_4, \tau') \right\} \\
= \sum_\tau \int d^3k \, \omega(\vec{k}) a^\dagger(\vec{k}, \tau) a(\vec{k}, \tau) \\
+ \sum_{\tau, \tau'} \int d^3k_1 d^3k_2 d^3k_3 d^3k_4 \frac{g^2}{8(2\pi)^3} \left( \omega(\vec{k}_1) \omega(\vec{k}_2) \omega(\vec{k}_3) \omega(\vec{k}_4) \right)^{-\frac{1}{2}} \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 - \vec{k}_4) \\
\times \left\{ 2\tilde{G}_{\sigma}(\vec{k}_1 + \vec{k}_2) a^\dagger(\vec{k}_1, \tau) a^\dagger(\vec{k}_2, \tau) a(\vec{k}_3, \tau') a(\vec{k}_4, \tau') \\
+ 4\tilde{G}_{\sigma}(\vec{k}_1 - \vec{k}_3) a^\dagger(\vec{k}_1, \tau) a^\dagger(\vec{k}_2, \tau') a(\vec{k}_3, \tau) a(\vec{k}_4, \tau') \right\} . \quad (A.72a)
\]

As advertised before, one can now extract the terms corresponding to the different scattering channels; i.e. (A.72a) is responsible for the $s$-channel scattering of pions, since the intermediate $\sigma$-propagator depends on the sum of momenta in the in- or outgoing channel and the (cartesian) isospin quantum numbers of the two pions in each of these channels have to be equal.

Term (A.72b) is responsible for $t$- and $u$-channel scattering of pions, since the intermediate $\sigma$-propagator depends on the momentum transfer from one of the incoming to one of the outgoing particles, while these two particles have to carry the same isospin quantum number as well as the remaining two pions.

We are now in the position to improve the three-dimensional reduction scheme. In the center of momentum-frame of two colliding pions the zeroth component of the four momentum in the $\sigma$-propagator vanishes for $t$- or $u$-channel scattering, while for $s$-channel scattering the three momentum in the $\sigma$-propagator vanishes. This implies that the naive instantaneous approximation (A.59) is inappropriate for the $s$-channel term of the two-body potential.
The situation is now cured by simply placing the zeroth components of the pion four-momenta on-shell. The ambiguity, which of the up to now equivalent three-momentum combinations in the $\sigma$-propagator have to be used, we resolve by averaging over both possibilities. This method is in close analogy to the well-known three-dimensional reduction schemes of Gross, Blankenbecler-Sugar and Thompson [30], except for the difference, that with these methods one carries out the three-dimensional reduction of $T$- or $G$-matrix equations, where in addition to the external three momenta the invariant mass $\sqrt{s}$ of the colliding system is fixed. In the latter case one has different possibilities of placing the particles off-shell, such that their four momenta give the correct $\sqrt{s}$. In a dynamical many-body calculation as in this work, the $\sqrt{s}$ for an elementary two-body scattering process can only be reconstructed from the three-momenta involved, so that the schemes cited above are not directly applicable.

Since point-like interaction concepts are inappropriate for hadron scattering we introduce formfactors for the $\pi\pi\pi\sigma$-vertices containing momentum cutoffs to regularize the theory. We use a formfactor of the form

$$F(k^2) = \frac{2(\Lambda^2 - M^2)\Lambda^2 + M^4}{2(\Lambda^2 - k^2)\Lambda^2 + k^4},$$  \hfill (A.73)  

where $\Lambda$ is the cutoff parameter and $k$ is the four momentum of the $\sigma$-particle coupling to the vertex. The formfactor (A.73) has the distinct advantage of leading to a uniform parameterization for spacelike and timelike four momenta without running into a pole in one of the two domains.

The Hamiltonian we use in our explicit computations then reads:

$$H = \sum_\tau \int d^3k \omega(\vec{k})a^\dagger(\vec{k}, \tau)a(\vec{k}, \tau)$$

$$+ \frac{1}{2} \sum_{\tau, \tau'} \int d^3k_1d^3k_2d^3k_3d^3k_4 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)$$

$$\times \left\{ \langle k_1k_2|v_s\bar{k}_3\bar{k}_4\rangle a^\dagger(k_1, \tau)a^\dagger(k_2, \tau')a(\bar{k}_3, \tau)a(\bar{k}_4, \tau') + \langle k_1k_2|v_s\bar{k}_4\bar{k}_3\rangle a^\dagger(k_1, \tau)a^\dagger(k_2, \tau')a(\bar{k}_3, \tau)a(\bar{k}_4, \tau') \right\}$$ \hfill (A.74)

with

$$\langle k_1k_2|v_s\bar{k}_3\bar{k}_4\rangle = \frac{g^2}{4(2\pi)^3} \left( \frac{\omega(k_1)\omega(k_2)\omega(k_3)\omega(k_4)}{\left(\omega(k_1) + \omega(k_2)\right)^2 - (\vec{k}_1 + \vec{k}_2)^2} \right)^{1/2}$$

$$\times \left\{ \frac{\left(\omega(k_1) + \omega(k_2)\right)^2 - (\vec{k}_1 + \vec{k}_2)^2 - M^2}{\left(\omega(k_3) + \omega(k_4)\right)^2 - (\vec{k}_3 + \vec{k}_4)^2 - M^2} \right\}$$ \hfill (A.75)

and

$$\langle k_1k_2|v_s\bar{k}_4\bar{k}_3\rangle = \frac{g^2}{2(2\pi)^3} \left( \frac{\omega(k_1)\omega(k_2)\omega(k_3)\omega(k_4)}{\left(\omega(k_1) - \omega(k_3)\right)^2 - (\vec{k}_1 - \vec{k}_3)^2} \right)^{1/2}$$

$$\times \left\{ \frac{\left(\omega(k_1) - \omega(k_3)\right)^2 - (\vec{k}_1 - \vec{k}_3)^2 - M^2}{\left(\omega(k_1) - \omega(k_3)\right)^2 - (\vec{k}_1 - \vec{k}_3)^2 - M^2} \right\}$$
The $\sigma$ can be associated with the resonances in the $(J^{PC} = 0^{++}, I^G = 0^+)$-channel ($\sigma$-channel) in $\pi\pi$-scattering, where the theory described can easily be generalized to a coupling of the pion field to more than a single scalar and isoscalar field; we simply obtain the matrix elements of the two-body potential as a sum over those of the form (A.75), (A.76) for the individual $\sigma$-particles $\sigma_i$ of masses $M_i$ with couplings $g_i$ and cutoffs $\Lambda_i$. The parameters used in this work are taken from [31], where additional particles besides the scalar and isoscalar ones are used to fit scattering data, but the couplings for the latter are directly deduced from the widths of the resonances in the $\sigma$-channel by means of a Breit-Wigner formula. Using only $\sigma$-mesons – in order to keep the numerical task within a reasonable range – should at least guarantee a realistic description of the short-range attractive part of the $\pi\pi$-interaction. The parameters taken from [31] are:

\begin{align*}
M_1 &= 980 \text{ MeV}, \quad \Lambda_1 = 1200 \text{ MeV}, \quad g_1 = 595 \text{ MeV} \\
M_2 &= 1300 \text{ MeV}, \quad \Lambda_2 = 1200 \text{ MeV}, \quad g_2 = 1854 \text{ MeV} \\
m &= m_\pi = 140 \text{ MeV}.
\end{align*}

In view of the complexity of the NCBCD equations we do not claim to have a fully realistic model of a pion gas, which would require the inclusion of $\rho$-mesons and kaons in a coupled channel-calculation [32]-[38] in order to fit the experimentally measured phase shifts for the higher partial waves [39, 40]. The interaction should work at moderate relative momenta.

## B Equations of motion for the isosymmetric pion system

Since for a numerical solution of the NCBCD equations the required computation time increases like $N^6$ and the required memory space increases like $N^4$, where $N$ denotes the number of single particle basis states, it is furthermore desirable to eliminate all internal degrees of freedom from the equations and thereby reduce $N$ by a factor of 3. We therefore assume, that the system is in an isospin-symmetric state and that all density matrices are diagonal in isospin space, which in the spherical representation of isospin SU(2) implies, that the sum of the isospin-z-components in the incoming and outgoing channels have to be equal. As one easily verifies, isosymmetry and isodiagonality are conserved dynamically; therefore these assumptions do not lead to an inconsistent theory.

Now we use $\tilde{\alpha} = (\alpha, \tau)$, where $\alpha$ denotes the quantum numbers of the space-time degrees of freedom in an arbitrary orthonormal single particle basis and $\tau$ denotes the isospin quantum number in the cartesian representation. The Hamiltonian of the pionic system can then be written as

\begin{equation}
H = \sum_{\tilde{\alpha} \tilde{\alpha}'} t_{\tilde{\alpha} \tilde{\alpha}'} a_{\tilde{\alpha}}^\dagger a_{\tilde{\alpha}'} + \frac{1}{2} \sum_{\tilde{\alpha} \tilde{\beta} \tilde{\beta}'} \langle \tilde{\alpha} \tilde{\beta} | v | \tilde{\alpha} \tilde{\beta} \rangle a_{\tilde{\alpha}}^\dagger a_{\tilde{\beta}} a_{\tilde{\alpha}} a_{\tilde{\beta}}',
\end{equation}

We use a slightly different form of the formfactor, which however agrees reasonably well with their form in the kinematical regime under investigation.

\begin{equation}
+ \left\{ F \left( (\omega(\vec{k}_2) - \omega(\vec{k}_4))^2 - (\vec{k}_2 - \vec{k}_4)^2 \right) \right\}^2 \left\{ \omega(\vec{k}_2) - \omega(\vec{k}_4) \right\}^2 - (\vec{k}_2 - \vec{k}_4)^2 - M^2 \right\}.
\end{equation}
with
\[ t_{\alpha \alpha'} = t_{\alpha \alpha'} \delta_{\tau_a \tau_{\alpha'}} \]  
(B.79)

and
\[ \langle \tilde{\alpha} \tilde{\beta} | t | \tilde{\alpha'} \tilde{\beta'} \rangle = (\alpha \beta | v_{\tau_a \tau_{\alpha'}} | \alpha' \beta') \delta_{\tau_a \tau_{\beta'}} \delta_{\tau_{\alpha'} \tau_{\beta'}} + (\alpha \beta | v_{\tau_u \tau_{\alpha'}} | \alpha' \beta') \delta_{\tau_a \tau_{\alpha'}} \delta_{\tau_{\beta} \tau_{\beta'}} . \]  
(B.80)

Isosymmetry implies, that in the cartesian representation of SU(2) all density matrices have to be invariant under an overall permutation of isospin quantum numbers.

For the one-body density matrices we assume
\[ \rho_{\alpha \alpha'} = \rho_{\alpha \alpha'} \delta_{\tau_a \tau_{\alpha'}} . \]  
(B.81)

For the two-body correlation function \( c_{\tilde{\alpha} \tilde{\beta} \tilde{\alpha}' \tilde{\beta}'} \) there are 4 possible distinct isospin configurations:

1. \( \tau_a = \tau_{\beta} = \tau_{\alpha'} = \tau_{\beta'} \)
2. \( \tau_a = \tau_{\beta}, \tau_{\alpha'} = \tau_{\beta'}, \tau_{\alpha} \neq \tau_{\alpha'} \)
3. \( \tau_a = \tau_{\alpha'}, \tau_{\beta} = \tau_{\beta'}, \tau_{\alpha} \neq \tau_{\beta} \)
4. \( \tau_a = \tau_{\beta'}, \tau_{\beta} = \tau_{\alpha'}, \tau_{\alpha} \neq \tau_{\beta} . \)  
(B.82)

All other configurations, which do not contain two pairs of equal isospin indices, violate the diagonality of the two-body density matrix in isospin space; the corresponding correlation functions are thus assumed to vanish.

Configuration 1. corresponds to the situation, where s-, t- and u-channel scattering contribute to the Born amplitude for the elementary process; configuration 2. corresponds to the situation, where only s-channel scattering contributes and configurations 3. and 4. reflect the situation, where only t-channel scattering contributes.

We can therefore decompose the two-body correlation function in the following way:
\[ c_{\tilde{\alpha} \tilde{\beta} \tilde{\alpha}' \tilde{\beta}'} = \delta_{\tau_a \tau_{\beta}} \delta_{\tau_{\alpha'} \tau_{\beta'}} \delta_{\tau_{\alpha} \tau_{\tau_{\alpha'}}} c_{\alpha \beta \alpha' \beta'}^{stu} + \delta_{\tau_{\tau_a} \tau_{\beta}} \delta_{\tau_{\alpha} \tau_{\tau_{\alpha'}}} (1 - \delta_{\tau_{\tau_a} \tau_{\alpha'}}) c_{\alpha \beta \alpha' \beta'}^{s1} + \delta_{\tau_{\alpha} \tau_{\alpha'}} \delta_{\tau_{\tau_{\beta}} \tau_{\beta'}} (1 - \delta_{\tau_{\tau_{\beta}} \tau_{\beta'}}) c_{\alpha \beta \alpha' \beta'}^{r2} . \]  
(B.83)

with the isospin channel correlation functions \( c_{stu}, c_{s}, c_{r1}, c_{r2} \).

The one-body density matrices fulfill
\[ \rho_{\alpha \alpha'} = (\rho_{\alpha' \alpha})^* , \]  
(B.84)

while the isospin channel correlation functions fulfill
\[ c_{\alpha \beta \alpha' \beta'}^{stu} = c_{\beta \alpha' \beta \alpha}^{stu}, c_{\alpha \beta \alpha' \beta'}^{stu} = (c_{\alpha' \beta \alpha \beta}^{stu})^* \]  
(B.85)

and
\[ c_{\alpha \beta \alpha' \beta'}^{r1} = c_{\beta \alpha' \beta \alpha}^{r2}, c_{\alpha \beta \alpha' \beta'}^{r2} = (c_{\alpha' \beta \alpha \beta}^{r1})^*, c_{\alpha \beta \alpha' \beta'}^{r2} = (c_{\alpha' \beta' \alpha \beta}^{r2})^* . \]  
(B.86)

We note in passing, that for our purpose the cartesian representation of SU(2) is technically superior to the spherical one, because in the spherical representation we have to consider 7 instead of 4 different isospin channel correlation functions.

We can now explicitly carry out all summations over isospin indices appearing in the NCBCD equations, which leads to a coupled system of a single one-body equation of
motion and four two-body equations of motion, where all considered quantities now only depend on space-time degrees of freedom.

With the abbreviations

\[
U_{aa'} = \sum_{\lambda_1, \lambda_2} \rho_{\lambda_1 \lambda_2} \left( 3 \langle \alpha \lambda_2 | v_{t,u} | \alpha' \lambda_1 \rangle + \langle \alpha \lambda_2 | v_{t,u} | \lambda_1 \alpha' \rangle + \langle \alpha \lambda_2 | v_s | \alpha' \lambda_1 \rangle \right),
\]

\[
h_{aa'} = t_{aa'} + U_{aa'},
\]

\[
Q^-_{\alpha \beta \alpha' \beta'} = \delta_{\alpha \alpha'} \delta_{\beta \beta'} + \rho_{aa'} \delta_{\beta \beta'} + \rho_{\beta \beta'} \delta_{aa'},
\]

\[
Q^+_{\alpha \beta \alpha' \beta'} = \rho_{\beta \beta'} \delta_{\alpha \alpha'} - \rho_{\alpha \alpha'} \delta_{\beta \beta'},
\]

we obtain the NCBCD equations for the isosymmetric, isodiagonal pionic system:

\[
i \partial_t \rho_{aa'} = \sum_{\lambda} \left( h_{\lambda \alpha} \rho_{\lambda \alpha'} - h_{\lambda \alpha'} \rho_{\lambda \alpha} \right)
+ \sum_{\lambda_1, \lambda_2, \lambda_3} \left\{ \langle \alpha \lambda_1 | v_{t,u} | \lambda_2 \lambda_3 \rangle \left( c^{stu}_{\lambda_2, \lambda_3} \delta_{\alpha \alpha'} + 2c^{t1}_{\lambda_2, \lambda_3} \delta_{\alpha \alpha'} \right) + \langle \alpha \lambda_1 | v_s | \lambda_2 \lambda_3 \rangle \left( c^{stu}_{\lambda_2, \lambda_3} \delta_{\alpha \alpha'} + 2c^{t1}_{\lambda_2, \lambda_3} \delta_{\alpha \alpha'} \right) \right\}
- \langle \alpha \lambda_1 | v_{t,u} | \alpha' \lambda_3 \rangle \left( c^{stu}_{\alpha \lambda_3 \lambda_1 \lambda_2} \delta_{\alpha \alpha'} + 2c^{t1}_{\alpha \lambda_3 \lambda_1 \lambda_2} \delta_{\alpha \alpha'} \right) - \langle \alpha \lambda_1 | v_s | \alpha' \lambda_3 \rangle \left( c^{stu}_{\alpha \lambda_3 \lambda_1 \lambda_2} \delta_{\alpha \alpha'} + 2c^{t1}_{\alpha \lambda_3 \lambda_1 \lambda_2} \delta_{\alpha \alpha'} \right) \}
\]

\[
and
\]

\[
i \partial_t \rho_{aa'} = \sum_{\lambda} \left( h_{\lambda \alpha} \rho_{\lambda \alpha'} - h_{\lambda \alpha'} \rho_{\lambda \alpha} \right)
+ \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \left\{ \langle \lambda_3 \lambda_4 | v_{t,u} | \lambda_1 \lambda_2 \rangle Q^-_{\alpha \beta \lambda_3 \lambda_4} \right\}
+ \langle \lambda_3 \lambda_4 | v_s | \lambda_1 \lambda_2 \rangle Q^-_{\alpha \beta \lambda_3 \lambda_4}
- \langle \lambda_3 \lambda_4 | v_{t,u} | \lambda_1 \lambda_2 \rangle Q^+_{\lambda_1 \lambda_2 \alpha' \beta'}
- \langle \lambda_3 \lambda_4 | v_s | \lambda_1 \lambda_2 \rangle Q^+_{\lambda_1 \lambda_2 \alpha' \beta'}
\]
\[ -\langle \lambda_3 \lambda_4 | v_{t,u} | \lambda_1 \lambda_2 \rangle Q_{\lambda_1 \lambda_2 \alpha' \beta'}^{\perp} \begin{bmatrix} c_{\alpha \beta \lambda_3 \lambda_4}^s + 2 c_{\alpha \beta \lambda_3 \lambda_4}^t \\ c_{\alpha \beta \lambda_3 \lambda_4}^s \\ c_{\alpha \beta \lambda_3 \lambda_4}^t \\ c_{\alpha \beta \lambda_3 \lambda_4}^t \end{bmatrix} - \langle \lambda_3 \lambda_4 | v_s | \lambda_1 \lambda_2 \rangle Q_{\lambda_1 \lambda_2 \alpha' \beta'}^{\perp} \begin{bmatrix} c_{\alpha \beta \lambda_3 \lambda_4}^s + 2 c_{\alpha \beta \lambda_3 \lambda_4}^t \\ c_{\alpha \beta \lambda_3 \lambda_4}^s \\ c_{\alpha \beta \lambda_3 \lambda_4}^t \\ c_{\alpha \beta \lambda_3 \lambda_4}^t \end{bmatrix} \]

\[ + \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \begin{cases} Q_{\alpha \lambda_{1,a'} \lambda_3}^s \langle \lambda_3 \lambda_4 | v_{t,u} | \lambda_1 \lambda_2 \rangle & \begin{bmatrix} c_{\lambda_2 \lambda_4 \alpha}^{s t u} + 2 c_{\lambda_2 \lambda_4 \alpha}^{t t} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \end{bmatrix} + Q_{\alpha \lambda_{1,a'} \lambda_3}^s \langle \lambda_3 \lambda_4 | v_s | \lambda_1 \lambda_2 \rangle_s & \begin{bmatrix} c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \end{bmatrix} \\ + Q_{\beta \lambda_{1,b'} \lambda_3}^s \langle \lambda_3 \lambda_4 | v_{t,u} | \lambda_1 \lambda_2 \rangle & \begin{bmatrix} c_{\lambda_2 \lambda_4 \alpha}^{s t u} + 2 c_{\lambda_2 \lambda_4 \alpha}^{t t} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \end{bmatrix} + Q_{\beta \lambda_{1,b'} \lambda_3}^s \langle \lambda_3 \lambda_4 | v_s | \lambda_1 \lambda_2 \rangle_s & \begin{bmatrix} c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \end{bmatrix} \\ + Q_{\alpha \lambda_{1,b'} \lambda_3}^s \langle \lambda_3 \lambda_4 | v_{t,u} | \lambda_1 \lambda_2 \rangle & \begin{bmatrix} c_{\lambda_2 \lambda_4 \alpha}^{s t u} + 2 c_{\lambda_2 \lambda_4 \alpha}^{t t} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \end{bmatrix} + Q_{\alpha \lambda_{1,b'} \lambda_3}^s \langle \lambda_3 \lambda_4 | v_s | \lambda_1 \lambda_2 \rangle_s & \begin{bmatrix} c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \end{bmatrix} \\ + Q_{\beta \lambda_{1,a'} \lambda_3}^s \langle \lambda_3 \lambda_4 | v_{t,u} | \lambda_1 \lambda_2 \rangle & \begin{bmatrix} c_{\lambda_2 \lambda_4 \alpha}^{s t u} + 2 c_{\lambda_2 \lambda_4 \alpha}^{t t} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \end{bmatrix} + Q_{\beta \lambda_{1,a'} \lambda_3}^s \langle \lambda_3 \lambda_4 | v_s | \lambda_1 \lambda_2 \rangle_s & \begin{bmatrix} c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \\ c_{\lambda_2 \lambda_4 \alpha}^{s t u} \end{bmatrix} \end{cases} \]
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3 Figure Captions

**fig.1:** The Hartree-terms in the t-, u- (l.h.s.) and s-channel (r.h.s.); dashed horizontal lines are spacelike, dashed vertical lines are timelike σ-fields, solid lines represent pions.

**fig.2:** The Fock-terms in the t-, u- (l.h.s.) and s-channel (r.h.s.).

**fig.3:** The term coupling to the two-body correlation function; t-, u-channel (l.h.s.), s-channel (r.h.s.).

**fig.4:** Contributions to the Born-term.

**fig.5:** Resummation terms in vertical direction with $Q^x$ in the incoming channel; t-, u-channel term (l.h.s.), s-channel term (r.h.s.).

**fig.6:** Resummation terms in horizontal direction in the t-, u-channel; direct term (upper part), exchange term (lower part). Note that $Q^\perp$ connects $\alpha$ and $\lambda_3$, $\alpha'$ and $\lambda_1$.

**fig.7:** Resummation terms in horizontal direction in the s-channel; direct and exchange term are identical.

**fig.8:** Single particle energy levels and occupation numbers of the Hartree-Fock initialization for 60 (upper part) and 90 (lower part) pions in a volume of $(10\text{ fm})^3$ at various temperatures. The symbols denote the position of the discrete energy levels.

**fig.9:** Time-averaged correlation energy as a function of the pion number at 100 MeV temperature for NCBCD, vertical and horizontal approximation (upper part); magnification (lower part).

**fig.10:** Change in total energy after switching on the residual interaction as a function of $t_f$.

**fig.11:** Correlation energy with respect to $H$ as function of $t/t_f$ for different values of $t_f$.

**fig.12:** Change of the single contributions to the total energy as function of $t_f$; A kinetic energy, B mean-field-energy, C total energy, D correlation energy.

**fig.13:** $\int d\vec{y}d\vec{z}\rho_2(\vec{r},\vec{r};\vec{r},\vec{r})$ (solid line), $\int d\vec{y}d\vec{z}\rho_{20}(\vec{r},\vec{r};\vec{r},\vec{r})$ (dashed line) and $\int d\vec{y}d\vec{z}\rho(\vec{r};\vec{r})\rho(\vec{r};\vec{r})$ (dotted line) as a function of $x$ at 50 MeV (upper part) and 150 MeV (lower part) temperature for various densities.

**fig.14:** $\int d^3r\rho_2(\vec{r},\vec{r};\vec{r},\vec{r})/\int d^3r\rho(\vec{r};\vec{r})\rho(\vec{r};\vec{r})$ for various temperatures as a function of density.

**fig.15:** Cut through the two-body density divided by the product of one-body densities at $y_1 = y_2 = z_1 = z_2 = 5\text{ fm}$ at 50 MeV temperature and 30 pions in $(10\text{fm})^3$ (upper part) and at 200 MeV and 90 pions (lower part); $\rho_{20}/\rho\rho$ (l.h.s.), $\rho_2/\rho\rho$ (r.h.s.).
fig.16: One-dimensional cut through the two-body density divided by the product of one-body densities at 50 MeV (upper part) and 150 MeV (lower part) temperature and for various densities; the relative distance is parameterized by $x$. Solid line $\rho_2/\rho \rho$, dashed line $\rho_{20}/\rho \rho$.

fig.17: Relative correlation strength after switching on the residual interaction with $t_f = 160\text{fm}/c$.

fig.18: Correlation strength at 100 MeV temperature; solid line with normal coupling, dashed line with doubled coupling.
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