Strong Influence of a Small Fiber on Shear Stress in Fiber-Reinforced Composites

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Abstract
In stiff fiber-reinforced material, the high shear stress concentration occurs in the narrow region between fibers. With the addition of a small geometric change in cross-section, such as a thin fiber or an overhanging part of fiber, the concentration is significantly increased. This paper presents mathematical analysis to explain the rapidly increased growth of the stress by a small particle in cross-section. To do so, we consider two crucial cases where a thin fiber exists between a pair of fibers, and where one of two fibers has a protruding small lump in cross-section. For each case, the optimal lower and upper bounds on the stress associated with the geometrical factors of fibers is established to explain the strongly increased growth of the stress by a small particle.

MSC-class: 35J25, 73C40

1 Introduction
In this paper, we concern ourselves with the high stress concentration occurring in the stiff fiber-reinforced composites when fibers are located closely. The primary investigation focuses on the case when a smaller fiber is located in-between area of two fibers, see Figure 1 and Figure 2. This paper reveals that, with the addition of a smaller fiber, the growth of stress is significantly increased: if the diameter \(d\) of the fiber in the middle is sufficiently small and the distance between adjoining fibers is \(\epsilon\), then the stress blows up at the rate of \(\frac{1}{\sqrt{d\epsilon}}\) in the narrow region, even though the blow-up rate has been known as \(\frac{1}{\sqrt{\epsilon}}\) as in the case of a pair of fibers. This means that the defect of fiber as a protrusion causes much lower strengths in composites than had been thought. To derive it, we estimate the optimal lower and upper bounds of the stress concentration in terms of the diameters of fibers and the distances between them. These bounds explain the dramatic change of the growth of stress when the diameter of the fiber placed in the middle is relatively smaller than other two fibers.

In the anti-plane shear model, the stress tensor represents the electric field in the two dimensional space, where the out-of-plane elastic displacement satisfies a conductivity equation, and the cross-section of stiff fibers corresponds to the embedded conductors. In this

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respect, we consider the gradient of the solution to a conductivity problem to estimate the stress. Adjacent stiff fiber-reinforcement induces the high stress concentration in the narrow region between the fibers. This implies the blow-up of the gradient of the solution between adjoining conductors, see \[2, 3, 4, 6, 7, 10, 14, 15\].

Meanwhile, the extreme conductivities are indispensable to the blow-up phenomena: when the inclusion’s conductivity is away from zero and infinity, the boundedness of the stress function has been derived by Li and Vogelius \[12\], see also \[9\], and it was generalized to elliptic systems by Li and Nirenberg \[11\]. In \[3, 4\], for conductivities including both bounded and extreme cases, Ammari et al. have established the optimal bounds of the gradient of solutions to the conductivity equation, when conductors are of circular shape in two dimensions, and the optimal bounds provides $\epsilon^{-1/2}$ blow-up rate, where $\epsilon$ is the distance between two conductors. Yun \[14, 15\] has extended this blow-up result for the case of two adjacent perfect conductors of a sufficiently general shape in two dimensions. In Bao, Li and Yin’s paper \[7\], it has been also investigated as the blow-up phenomena in higher dimensional spaces, also see \[2, 13\]. They have also done a natural follow-up in \[5, 8\] that the blow-up rate known only for a pair of fibers is still valid for the multiple inclusions in any dimensions.

In contrast, our paper witnesses an unexpected fact on multiple inclusions that the growth of stress can be significantly increased by a little geometric change of an inclusion, even though the blow-up rate is still $\epsilon^{-1/2}$.

For $l = 1, \ldots, L$, let $D_l$ be conducting inclusions in $\mathbb{R}^2$, that is cross-sections of stiff fibers. Then, under the action of the applied field $H$, the electric potential $u$ satisfies the following conductivity equation:

\[
\begin{aligned}
\Delta u &= 0, \\
& \quad \text{in } \mathbb{R}^2 \setminus \bigcup_{l=1}^{L} D_l, \\
& \quad \text{as } |x| \to \infty, \\
& \quad \text{as } |x| \to \infty, \\
& \quad \text{as } |x| \to \infty, \\
& \quad \text{as } |x| \to \infty, \\
& \quad \text{as } |x| \to \infty, \\
\end{aligned}
\]

\begin{aligned}
u(n) u\bigg|_{\partial D_l} &= C_l \text{ (constant)} \\
\int_{\partial D_l} \partial_n u \ dS &= 0, \\
& \quad \text{for } l = 1, \ldots, L, \\
& \quad \text{for } l = 1, \ldots, L.
\end{aligned}

\tag{1}

Figure 1: Case (A) and case (B)

Figure 2: Case (C) and case (D)
where \( H \) is an entire harmonic function \( H \) in \( \mathbb{R}^2 \) and \( x = (x_1, x_2) \). In this paper, we only consider the case of \( L = 2, 3 \).

As we mentioned previously, for the two fibers with the circular cross-sectional shape, Ammari et al. \([3, 4]\) obtained the optimal blow-up rate \( \epsilon^{-1/2} \) for \( \nabla u \), and this result is extended by Yun \([14, 15]\) to general shaped fibers. Building on the prior results, we extend these into the following interesting direction: we first consider the circular inclusions in Case (A) and Case (B), and second extend the result into general shaped ones in Case (C) and Case (D). In Case (A) and Case (B), we add a small circular inclusion between two others so that three disk centers are lined up in one straight line. The additional disk can be embedded disjointly from other disks, or it partially overlap one of two disks, and we formulate these two cases as follows.

(A) One disk and a pair of partially overlapping disks, Figure 1, there is a portion of disk protruding from one of circular inclusions, i.e., \( L = 2 \), and \( D_1 \) and \( D_2 \) are \( \epsilon \)-distanced domains defined as

\[
D_1 = B_{r_1}(c_1) \text{ and } D_2 = B_{r_2}(c_2) \cup B_{r_3}(c_3),
\]

where \( B_{r_1}(c_1) \) is the disk with the radius \( r_1 \) and centered at \( c_1 \), and

\[
c_1 = (-r_1 - \frac{\epsilon}{2}, 0), \quad c_2 = (r_2 + \frac{\epsilon}{2}, 0), \quad \text{and} \quad c_3 = (r_3 + a + \frac{\epsilon}{2}, 0).
\]

Here, \( B_{r_2}(c_2) \) is a small disk protruding from \( B_{r_3}(c_3) \), and we assume

\[
B_{r_2}(c_2) \cap B_{r_3}(c_3) \neq \emptyset, \text{ i.e., } 0 < a < 2r_2,
\]

\[
\text{dist}(B_{r_1}(c_1), B_{r_3}(c_3)) \approx r_2
\]

and

\[
0 < \epsilon \ll r_2 < r_1 \approx r_3.
\]

(B) Three disjoint disks, Figure 1, a small disk is disjointly embedded into the in-between area of two disks, i.e., \( L = 3 \), and

\[
D_l = B_{r_l}(c_l), \quad l = 1, 2, 3,
\]

where \( c_1 = (-r_1 - \frac{\epsilon}{2}, 0), c_2 = (r_2 + \frac{\epsilon}{2}, 0) \) and \( c_3 = (r_3 + r_2 + \frac{\epsilon}{2} + \epsilon_2, 0) \). Hence, the distance between \( D_1 \) and \( D_2 \) is \( \epsilon_1 \), and the distance between \( D_2 \) and \( D_3 \) is \( \epsilon_2 \). Here, \( D_2 \) is regarded as the cross-section of the thin fiber between a pair of fibers with the cross-section \( D_1 \) and \( D_3 \). Thus, we assume that

\[
0 < \epsilon_1 \ll r_2 \ll r_1 \approx r_3 \quad \text{for } i = 1, 2.
\]

In both cases, the blow-up rate is remarkably increased due to the existence of \( B_{r_2}(c_2) \) as follows:

**Theorem 1.1 (Case A: Protruding small disk)** Let \( D_1 \) and \( D_2 \) be defined as (2). Then there is a positive constant \( C \) independent of \( \epsilon, r_1, r_2 \) and \( r_3 \) such that

\[
|u|_{\partial D_2} - |u|_{\partial D_1} \geq C \frac{r_1r_3}{r_1 + r_3 \sqrt{r_2}} \sqrt{\epsilon},
\]
where \( u \) is the solution to \((1)\) with \( H(x_1, x_2) = x_1 \). As a result, by the Mean Value Theorem, there is a point \( x_0 \) in the narrow region between \( D_1 \) and \( D_2 \) such that

\[
|\nabla u(x_0)| \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2} \sqrt{\epsilon}}.
\]

For any entire harmonic function \( H \), let \( u \) be the solution to \((1)\) with \( H \). Then, there is a positive constant \( C \) independent of \( \epsilon, r_1, r_2 \) and \( r_3 \) such that

\[
|\nabla u(x)| \leq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2} \sqrt{\epsilon}}
\]

in the narrow region between \( D_1 \) and \( D_2 \).

**Theorem 1.2 (Case B: Disjointly embedded small disk)** Let \( D_i, i = 1, 2, 3 \), be balls defined as \((4)\). Then there is a positive constant \( C \) independent of \( \epsilon_1, \epsilon_2, r_1, r_2 \) and \( r_3 \) such that

\[
|u|_{\partial D_2} - |u|_{\partial D_1} \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2} \sqrt{\epsilon_1}},
\]

and

\[
|u|_{\partial D_3} - |u|_{\partial D_2} \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2} \sqrt{\epsilon_2}},
\]

where \( u \) is the solution to \((1)\) with \( H(x_1, x_2) = x_1 \). As a result, by the Mean Value Theorem, there exists points; \( x_1 \) in the narrow region between \( D_1 \) and \( D_2 \); \( x_2 \) in the narrow region between \( D_2 \) and \( D_3 \), which satisfy that

\[
|\nabla u(x_i)| \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2} \sqrt{\epsilon_i}} \text{ for } i = 1, 2.
\]

For any entire harmonic function \( H \), let \( u \) be the solution to \((1)\) with \( H \). Then, there is a positive constant \( C \) independent of \( \epsilon_1, \epsilon_2, r_1, r_2 \) and \( r_3 \) such that

\[
|\nabla u(x)| \leq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2} \sqrt{\epsilon_i}} \text{ for } i = 1, 2.
\]

in the narrow regions between \( D_1 \) and \( D_2 \), and between \( D_2 \) and \( D_3 \), respectively.

In this paper, we first estimate the lower-bounds in terms of the radii of inclusions. Based on this estimates we derive the remarkable blow-up rate increasing phenomena when a small conducting inclusion is located in-between region of two inclusions. This paper is organized as follows: In section 2, we explain the method to calculate the potential difference in the case of two disks. We then derive the lower bound of Case (A) in section 3; Case (B) in section 4. In the case of the upper bounds, the major part of derivation overlaps in Case (A) and Case (B). Thus, the derivation is presented in Subsection 4.5. Based on the similar derivation, we can also obtain the analogues of Theorem 1.1 and 1.2 for the inclusions associated by a sufficiently general class of shapes.
Analogues of Theorem 1.1 and 1.2 for a sufficiently general class of shapes

The proofs of Theorem 1.1 and 1.2 are flexible enough even though the results are restricted to circular inclusions. The estimates presented in Theorem 1.1 and 1.2 can be extended to the inclusions associated by a sufficiently general class of shapes. To consider a large class of shapes, we make the geometric assumptions more precise. To define \( D_1 \), \( D_2 \) and \( D_3 \), we consider three domains \( D_{\text{right}} \), \( D_{\text{center}} \) and \( D_{\text{left}} \) in \( \mathbb{R}^2 \). In addition, we assume that \( \varphi_{\text{right}} : \mathbb{C} \setminus B_1(0) \to \mathbb{R}^2 \setminus D_{\text{right}} \), \( \varphi_{\text{center}} : \mathbb{C} \setminus B_1(0) \to \mathbb{R}^2 \setminus D_{\text{center}} \) and \( \varphi_{\text{left}} : \mathbb{C} \setminus B_1(0) \to \mathbb{R}^2 \setminus D_{\text{left}} \) are conformal mappings in \( C^2(\mathbb{C} \setminus B_1(0)) \) such that \( \varphi'_{\text{right}}(z) \neq 0 \) and \( \varphi'_{\text{left}}(z) \neq 0 \) for \( z \in \partial B_1(0) \). Here, we do not distinguish \( \mathbb{R}^2 \) from \( \mathbb{C} \). The \( C^2 \) regularity condition of these conformal mappings does not allow non-smooth inclusions such as polygons, but Riemann mapping theorem yields a sufficiently general class of shapes: refer to Ahlfors [1]. Now, we consider the analogues of Theorem 1.1 and 1.2 for two cases as follows:

(C) One domain and a pair of partially overlapping domains, similarly to Figure 2: there is a small portion of another domain protruding from an inclusion, i.e., \( L = 2 \), and \( D_1 \) and \( D_2 \) are \( \epsilon \)-distanced domains defined as

\[
D_1 = D_{\text{left}} \quad \text{and} \quad D_2 = (r_2 D_{\text{center}}) \cup D_{\text{right}},
\]

where \( r_2 D_{\text{center}} \) is the \( r_2 \) times diminished domain of \( D_{\text{center}} \). We suppose that \( D_2 \) is a connected domain, \( \text{dist}(D_1, D_2) = \text{dist}(D_1, r_2 D_{\text{center}}) \),

\[
\text{dist}(D_1, D_{\text{right}}) \preceq r_2,
\]

\[
D_1 \subset \mathbb{R}^- \times \mathbb{R} \quad \text{and} \quad D_2 \subset \mathbb{R}^+ \times \mathbb{R}.
\]

In addition, we also assume that \( r_2 \) is small enough and

\[
0 < \epsilon \ll r_2,
\]

and that the boundaries \( \partial D_1 \), \( \partial D_2 \) and \( \partial D_{\text{right}} \) are strictly convex in the narrow region between \( D_1 \) and \( D_2 \).

(D) Three disjoint domains \( D_1 \), \( D_2 \) and \( D_3 \), Figure 2: a small inclusion \( D_2 \) is disjointly embedded into the in-between area of two other domains, i.e., \( L = 3 \), and

\[
D_1 = D_{\text{left}}, \quad D_2 = r_2 D_{\text{center}} \quad \text{and} \quad D_3 = D_{\text{right}}
\]

where \( r_2 D_{\text{center}} \) is the \( r_2 \) times diminished domain of \( D_{\text{center}} \). We assume that \( D_1 \) and \( D_2 \) are \( \epsilon_1 \) apart, \( D_2 \) and \( D_3 \) are \( \epsilon_2 \) apart, and \( D_1 \) distances enough from \( D_3 \) that \( r_2 \) is sufficiently small and

\[
0 < \epsilon_i \ll r_2 \quad \text{for} \quad i = 1, 2,
\]

since \( D_2 \) is regarded as the cross-section of the thin fiber, that the boundaries \( \partial D_1 \), \( \partial D_2 \) and \( \partial D_{\text{right}} \) are strictly convex in the narrow region between \( D_1 \) and \( D_2 \),

\[
D_1 \subset \mathbb{R}^- \times \mathbb{R} \quad \text{and} \quad D_2 \cup D_3 \subset \mathbb{R}^+ \times \mathbb{R}.
\]
Theorem 1.3 (Case C: Protruding small lump) Let \( D_1 \) and \( D_2 \) be defined as (5). Then there is a positive constant \( C \) independent of \( \epsilon \) and \( r_2 \) such that

\[
|u|_{\partial D_2} - u|_{\partial D_1} \geq C \frac{1}{\sqrt{r_2}} \sqrt{\epsilon},
\]

where \( u \) is the solution to (1) with \( H(x_1, x_2) = x_1 \). As a result, by the Mean Value Theorem, there is a point \( x_0 \) in the narrow region between \( D_1 \) and \( D_2 \) such that

\[
|\nabla u(x_0)| \geq C \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon}}.
\]

For any entire harmonic function \( H \), let \( u \) be the solution to (1) with \( H \). Then, there is a positive constant \( C \) independent of \( r_2 \) and \( \epsilon \) such that

\[
|\nabla u(x)| \leq C \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon}}
\]
in the narrow region between \( D_1 \) and \( D_2 \).

Theorem 1.4 (Case D: Disjointly embedded small inclusion) Let \( D_i \), \( i = 1, 2, 3 \), be balls defined as (6). Then there is a positive constant \( C \) independent of \( r_2, \epsilon_1 \) and \( \epsilon_2 \) such that

\[
|u|_{\partial D_2} - u|_{\partial D_1} \geq C \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_1},
\]

and

\[
|u|_{\partial D_3} - u|_{\partial D_2} \geq C \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_2},
\]

where \( u \) is the solution to (1) with \( H(x_1, x_2) = x_1 \). As a result, by the Mean Value Theorem, there exists points; \( x_1 \) in the narrow region between \( D_1 \) and \( D_2 \); \( x_2 \) in the narrow region between \( D_2 \) and \( D_3 \), which satisfy that

\[
|\nabla u(x_i)| \geq C \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon_i}} \text{ for } i = 1, 2.
\]

For any entire harmonic function \( H \), let \( u \) be the solution to (1) with \( H \). Then, there is a positive constant \( C \) independent of \( r_2, \epsilon_1 \) and \( \epsilon_2 \) such that

\[
|\nabla u(x_i)| \leq C \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon_i}} \text{ for } i = 1, 2.
\]
in the narrow regions between \( D_1 \) and \( D_2 \), and \( D_2 \) and \( D_3 \), respectively.

We derive the lower bound of Case (C) in section 3; Case (D) in section 4. In the case of the upper bounds, the major part of derivation overlaps in Case (A), Case (B), Case (C) and Case (D). Thus, the main idea is presented in Subsection 4.5.
2 Preliminary

2.1 Calculation of the potential difference

We explain the main idea to calculate the difference of potential between two adjacent, possibly disconnected, conductors.

In this section, differently from (1), \(D_i, i = 1, 2\), could be also the union of two disjoint domains. Define \(u\) as the solution to (1), where it is assigned one constant value throughout \(D_i\) even when \(D_i\) is disconnected. Now, define \(h\) as the solution to

\[
\begin{align*}
\Delta h &= 0, & \text{in } \mathbb{R}^2 \setminus (D_1 \cup D_2), \\
h &= O(|x|^{-1}), & \text{as } |x| \to \infty, \\
h_{|\partial D_i} &= k_i \text{ (constant)}, & \text{for } i = 1, 2, \\
\int_{\partial D_i} \partial_n h \, dS &= (-1)^i, & \text{for } i = 1, 2,
\end{align*}
\]

where \(\nu\) is the outward unit normal vector of \(\mathbb{R}^n \setminus (D_1 \cup D_2)\), i.e., directed inward of \(D_i\). To indicate the dependence of \(u\) and \(h\) on \(D_1\) and \(D_2\), we denote them as

\[
\begin{align*}
u & \Phi[D_1, D_2], \\
h & \Psi[D_1, D_2].
\end{align*}
\]

The potential difference of \(u\) in \(D_1\) and \(D_2\) is represented in terms of \(h\) as follows.

Lemma 2.1 (14)

\[
|u|_{\partial D_2} - |u|_{\partial D_1} = \int_{\partial D_1} H \partial_n h \, dS + \int_{\partial D_2} H \partial_n h \, dS.
\]

The lemma above can be derived by the Divergence Theorem, see [14].

2.2 Two disks in \(\mathbb{R}^2\)

Using Lemma 2.1 we can easily calculate the potential difference \(|u|_{D_2} - |u|_{D_1}\) of the solution \(u\) to (1) when

\[
D_1 = B_{r_1}(c_1) \text{ and } D_2 = B_{r_2}(c_2),
\]

where \(c_1 = (-r_1 - \frac{\epsilon}{2}, 0)\) and \(c_2 = (r_2 + \frac{\epsilon}{2}, 0)\).

Let \(R_i\) be the reflection with respect to \(D_i\), in other words,

\[
R_i(x) = \frac{r_i^2(x - c_i)}{|x - c_i|^2} + c_i, \text{ for } i = 1, 2,
\]

and \(p_1 \in D_1\) be the fixed point of \(R_1 \circ R_2\), then \(R_2(p_1) (=: p_2)\) is the fixed point of \(R_2 \circ R_1\), and

\[
p_1 = \left(-\sqrt{2} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon + O(\epsilon)}, 0\right) \text{ and } p_2 = \left(\sqrt{2} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon + O(\epsilon)}, 0\right).
\]

Moreover, we can easily show that

\[
\Phi[D_1, D_2] = \frac{1}{2\pi} \left(\log |x - p_1| - \log |x - p_2|\right).
\]
By an elementary calculation, it can be shown that the middle point \( \frac{p_1 + p_2}{2} \) exists between two approaching points \((-\frac{\epsilon}{2}, 0)\) and \((\frac{\epsilon}{2}, 0)\). Applying the middle point property to estimate for \( \Psi[\Omega] \) we can get the following lemma.

**Lemma 2.2** There is a constant \( C > 0 \) independent of \( \epsilon, r_1 \) and \( r_2 \) such that
\[
\frac{1}{C} \sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon} \leq \Psi[D_1, D_2]|_{\partial D_2} - \Psi[D_1, D_2]|_{\partial D_1} \leq C \sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon}
\]
for small \( \epsilon > 0 \).

From Lemma 2.1 we calculate the potential difference of \( u \).

**Lemma 2.3** Let \( H(x_1, x_2) \) be an entire harmonic function. The solution \( u \) to \( u \) where \( L = 2 \) and \( D_i, i = 1, 2 \), are given as \( D_i \) satisfies
\[
u|_{\partial D_2} - u|_{\partial D_1} = H(p_2) - H(-p_1) = 2\sqrt{2} |\partial x_i, H(0, 0)| \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon} + O(\epsilon).
\]

**Remark 2.4** Referring to the mean value theorem, there exists a point \( x_2 \) between \( \partial D_1 \) and \( \partial D_2 \) such that
\[
|\nabla u(x_2)| \geq 2\sqrt{2} |\partial x_i, H(0, 0)| \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon}
\]
for any sufficiently small \( \epsilon > 0 \). Moreover, as a result in \( [4] \), there is a constant \( C \) independent of \( \epsilon, r_1 \) and \( r_2 \) such that
\[
\|\nabla u\|_{L^\infty(\Omega \setminus (D_1 \cup D_2))} \leq \|\nabla H\|_{L^\infty(\Omega)} \frac{2\sqrt{2} |\partial x_i, H(0, 0)| \sqrt{r_1 r_2}}{r_1 + r_2} \sqrt{\epsilon}
\]
where \( \Omega = B_4(r_1 + r_2)(0, 0) \).

### 3 One disk and a pair of partially overlapping disks

In this section, we consider two \( \epsilon \)-distanced domains \( D_1 \) and \( D_2 \), see Case (A) at Figure I where
\[
D_1 = B_{r_1}(c_1) \quad \text{and} \quad D_2 = B_{r_2}(c_2) \cup B_{r_3}(c_3),
\]
where
\[
c_1 = (-r_1 - \frac{\epsilon}{2}, 0), \quad c_2 = (r_2 + \frac{\epsilon}{2}, 0), \quad \text{and} \quad c_3 = (r_3 + a + \frac{\epsilon}{2}, 0).
\]
Here, \( B_{r_2}(c_2) \) is a small lump of \( B_{r_3}(c_3) \), and we assume
\[
B_{r_2}(c_2) \cap B_{r_3}(c_3) \neq \emptyset, \quad \text{i.e.}, \quad 0 < a < 2r_2,
\]
and
\[
0 < \epsilon \ll r_2 \ll \min(r_1, r_3).
\]

Define
\[
h = \Psi[\Omega], \quad \text{and} \quad u = \Phi[\Omega],
\]
and
\[
h_j = \Psi[D_1, B_{r_j}(c_j)], \quad \text{and} \quad u_j = \Phi[D_1, B_{r_j}(c_j)], \quad j = 2, 3,
\]
where \( \Psi \) and \( \Phi \) defined in section 2.1.
3.1 Properties of $h$ and $h_j$, $j = 2, 3$

Lemma 3.1 Let $h = \Psi[D_1, D_2]$, then we have

$$\partial_\nu h(x) = O(\sqrt{\epsilon}), \quad x \in \partial D_2 \setminus B_{r_2}(c_2).$$  \hspace{1cm} (17)

Proof. Set

$$B_i = B_{r_i}(c_i), \quad \text{for } i = 1, 2, 3.$$  

We choose a smooth domain $\tilde{\Omega}$ as follows:

$$\tilde{\Omega} \subset (B_2 \cup B_3), \quad B_3 \subset \tilde{\Omega}$$

$$\partial \tilde{\Omega} \setminus B_2 = \partial B_3 \setminus B_2$$

$$(\partial \tilde{\Omega} \cap \partial B_2) \setminus \partial B_3 = (\frac{1}{2}\epsilon, 0)$$

Let $\tilde{h} = \Psi[B_3, \tilde{\Omega}]$. Then, we consider $V$ defined in $\mathbb{R}^2 \setminus (B_2 \cup B_3)$ as follows:

$$V = h - \frac{h|_{\partial B_1} - h|_{\partial (B_2 \cup B_3)}}{\tilde{h}|_{\partial B_1} - \tilde{h}|_{\partial \tilde{\Omega}}} \tilde{h}.$$  

Then, it follows that

$$V|_{\partial B_1} = V|_{\partial \tilde{\Omega} \setminus B_2} = \text{a constant}.$$  

Since $h|_{\partial (B_2 \cup B_3)} > h|_{\partial B_1}$ and $\tilde{h}|_{\partial \tilde{\Omega}} > \tilde{h}|_{\partial B_1}$, the minimum of $V$ attains on $\partial \tilde{\Omega} \setminus B_2$ and $\partial B_1$. Thus, we have

$$\partial_\nu h - \frac{h|_{\partial B_1} - h|_{\partial (B_2 \cup B_3)}}{h|_{\partial B_1} - \tilde{h}|_{\partial \tilde{\Omega}}} \partial_\nu \tilde{h} \leq 0 \text{ on } \partial B_1 \cup (\partial \tilde{\Omega} \setminus B_2).$$  \hspace{1cm} (18)

By the integration on $\partial B_1$, we have

$$0 < \frac{h|_{\partial B_1} - h|_{\partial (B_2 \cup B_3)}}{h|_{\partial B_1} - \tilde{h}|_{\partial \tilde{\Omega}}} \leq 1.$$  

Using the bound (18) once more, we have

$$\partial_\nu h \leq \partial_\nu \tilde{h}|_{\partial \tilde{\Omega}} \text{ on } \partial \tilde{\Omega} \setminus B_2 = \partial B_3 \setminus B_2.$$  

The domain $\tilde{\Omega}$ is smooth so that we can use the method presented by Yun [14, 15]. Then, up to a conformal mapping to a circle, $\partial_\nu h$ is bounded by constant times the Poisson Kernel with respect to a interior point $\sqrt{\epsilon}$ distanced from the boundary (refer to the inequality (9) in [15]). Note that $\partial B_3 \setminus B_2$ distances enough from $(\epsilon, 0)$. Thus, we have

$$\partial_\nu h \leq \partial_\nu \tilde{h}|_{\partial \tilde{\Omega}} \leq C \sqrt{\epsilon} \text{ on } \partial B_3 \setminus B_2.$$  

Therefore, we have completed the proof of the lemma. \hfill \blacksquare
Lemma 3.2

$$\partial_v h(x) \leq M \partial_v h_3(x), \quad x \in \partial D_1,$$

where

$$M = \frac{h|_{\partial D_2} - h|_{\partial D_1}}{h_3|_{\partial B_{r_3}(c_3)} - h_3|_{\partial D_1}}.$$  \hfill (20)

**Proof.** Define

$$W = h - Mh_3, \quad \text{in } \mathbb{R}^2 \setminus (D_1 \cup D_2).$$

Since $h$ is constant on $\partial D_2$, $M > 0$, and $h_3$ takes it's maximum on $\partial B_{r_3}(c_3)$,

$$W|_{\partial D_2 \setminus B_{r_3}(c_2)} - W|_{\partial D_2 \setminus B_{r_3}(c_3)} = -M(h_3|_{\partial B_{r_3}(c_3)} - h_3|_{\partial D_2 \setminus B_{r_3}(c_3)}) < 0,$$

and

$$W|_{\partial D_2 \setminus B_{r_3}(c_2)} - W|_{\partial D_1} = h|_{\partial D_2} - h|_{\partial D_1} - M(h_3|_{\partial B_{r_3}(c_3)} - h_3|_{\partial D_1}) = 0.$$

Therefore, $W$ takes its minimum on $\partial D_1$, and

$$\partial_v W \leq 0, \quad \text{on } \partial D_1.$$  \hfill \Box

Lemma 3.3

$$h|_{\partial D_2} - h|_{\partial D_1} = h_2|_{\partial B_{r_3}(c_2)} - h_2|_{\partial D_1} + O(\epsilon).$$  \hfill (21)

**Proof.** Note that

$$\int_{\partial D_1} \partial_v (h - h_2) \, dS = 0,$$

and

$$\int_{\partial D_2} \partial_v (h - h_2) \, dS = \int_{\partial D_2} \partial_v h \, dS - \int_{\partial B_{r_3}(c_2)} \partial_v h_2 \, dS - \int_{\partial (D_2 \setminus B_{r_3}(c_2))} \partial_v h_2 \, dS = 1 - 1 - 0 = 0.$$

With the fact that $h|_{\partial D_1}$ and $h|_{\partial D_2}$ are constants and the (exterior) Divergence Theorem, we have that

$$0 = \int_{\partial D_1} \partial_v (h - h_2)h \, dS + \int_{\partial D_2} \partial_v (h - h_2)h \, dS$$

$$= \int_{\partial D_1} (h - h_2) \partial_v h \, dS + \int_{\partial D_2} (h - h_2) \partial_v h \, dS.$$

Hence,

$$h|_{\partial D_1} - h|_{\partial D_2} = \int_{\partial D_1} h \partial_v h \, dS + \int_{\partial D_2} h \partial_v h \, dS$$

$$= \int_{\partial D_1} h_2 \partial_v h \, dS + \int_{\partial D_2} h_2 \partial_v h \, dS$$

$$= h_2|_{\partial D_1} - h_2|_{\partial B_{r_3}(c_2)} + \int_{\partial D_2} (h_2 - h_2|_{\partial B_{r_3}(c_2)}) \partial_v h \, dS.$$
From (12), there is a constant $C$ dependent of $a$, see (3), such that
\[
\left| (h_2 - h_2)_{\partial B_{r_2}(c_2)}(x) \right| \leq C \sqrt{\epsilon}, \quad \text{for all } x \in \partial D_2 \setminus B_{r_2}(c_2).
\]
Therefore, with (17) as well, we obtain (21). \hfill \square

3.2 Proof Theorem 1.1

Let $H(x_1, x_2) = x_1$ and $\nu$ be the unit normal vector of $\mathbb{R}^2 \setminus (D_1 \cup D_2)$, i.e., directed inward to $D_i$, $i = 1, 2$. Remind that we defined
\[
h = \Psi[D_1, D_2], \quad u = \Phi[D_1, D_2],
\]
and
\[
h_j = \Psi[D_1, B_{r_j}(c_j)], \quad u_j = \Phi[D_1, B_{r_j}(c_j)], \quad j = 2, 3,
\]
where $\Psi$ and $\Phi$ defined in section 2.1.

Note that $\partial \nu h |_{\partial D_2} < 0$, $H < 0$ on $\partial D_1$ and $H > 0$ on $\partial D_2$, and, as a result, from Lemma 2.1, we have
\[
\left| u \bigg|_{\partial D_2} - u \bigg|_{\partial D_1} \right| \geq \int_{\partial D_1} H \partial \nu h \, dS.
\]

Applying the lemma 3.2, Lemma 3.3, (24) becomes
\[
\left| u \bigg|_{\partial D_2} - u \bigg|_{\partial D_1} \right| \geq \frac{h_2 |_{\partial B_{r_2}(c_2)} - h_2 |_{\partial D_1} + O(\epsilon)}{h_3 |_{\partial B_{r_3}(c_3)} - h_3 |_{\partial D_1}} \sqrt{2} \frac{r_1 r_3}{r_2}.
\]

It follows from Lemma 2.2 that
\[
h_2 |_{\partial B_{r_2}(c_2)} - h_2 |_{\partial D_1} \geq C \sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon} + O(\epsilon)
\]
and
\[
h_3 |_{\partial B_{r_3}(c_3)} - h_3 |_{\partial D_1} \leq C \sqrt{\frac{r_1 + r_3}{r_1 r_3}} \sqrt{r_2} + O(r_2).
\]

Therefore,
\[
\left| u \bigg|_{\partial D_2} - u \bigg|_{\partial D_1} \right| \geq \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon}.
\]

This proves Theorem 1.1. \hfill \square
3.3 Proof Theorem 1.3

We consider the general shaped domain in Theorem 1.3. But, we take an advantage of the properties of circular inclusions. To make a connection between circular domains and general shaped domains, we need to establish the monotonic property of $\Psi$ as follows:

**Lemma 3.4** [Monotonic property of $\Psi$] Let $D_A, D_B, D\tilde{A}$ and $D\tilde{B}$ be domains. Assume that $D_A \subseteq D\tilde{A}$ and $D_B \subseteq D\tilde{B}$.

Then, we have

\[
0 \leq \Psi[D\tilde{A}, D\tilde{B}] - \Psi[D_A, D_B] \leq \Psi[D, D_B] - \Psi[D_A, D_B].
\]

**Proof.** Without any loss of generality, we consider only the case of $D_A = D\tilde{A}$. Let

\[
G = \Psi[D_A, D_B] - M\Psi[D_A, D_B]
\]

where

\[
M = \frac{\Psi[D_A, D_B] - \Psi[D_A, D\tilde{B}]}{\Psi[D, D_B] - \Psi[D_A, D_B]}.
\]

The minimum of $G$ attains on $\partial D_A$. By the Hopf’s Lemma, we have

\[
\partial\nu G \leq 0 \text{ on } \partial D_A.
\]

Integrating $\partial\nu G$ on $\partial D_A$, we have

\[
0 \leq \Psi[D_A, D_B] - M\Psi[D_A, D_B].
\]

Repeating the same argument again, we can obtain the disable inequality.

Applying $D_{left}, r_2D_{center}$ and $D_{right}$ instead of $B_{r_i}(c_i), i = 1, 2, 3$, to the argument presented in the proof of Theorem 1.1, we can obtain

\[
u|_{\partial r_2 D_{center}} - \nu|_{\partial D_{left}} \geq C h_2|_{\partial (r_2 D_{center})} - h_2|_{\partial D_{left}} + O(\epsilon) \sqrt{r_2} - h_3|_{\partial D_{right}} - h_3|_{\partial D_{left}}\]

when $H(x_1, x_2) = x_1$. Here, $h_3 = \Psi[D_{left}, r_2 D_{center}]$ and $h_2 = \Psi[D_{left}, D_{right}]$.

It follows that from Yun [14, 15] that

\[
h_3|_{\partial D_{right}} - h_3|_{\partial D_{left}} \simeq \sqrt{r_2}.
\]

To estimate $h_2|_{\partial (r_2 D_{center})} - h_2|_{\partial D_{left}}$, we choose two disks $B_{left}$ and $B_{center}$ containing $D_{left}$ and $D_{center}$ such that the distance between $B_{left}$ and $r_2 B_{center}$ is $\epsilon$. Using Lemma 3.4 and 2.2 we have

\[
h_2|_{\partial (r_2 D_{center})} - h_2|_{\partial D_{left}} > \frac{\epsilon}{r_2}.
\]
Note that $D_1 = D_{\text{left}}$, $D_2 = r_2D_{\text{center}}$ and $D_3 = D_{\text{right}}$ in this theorem. Therefore,

$$u \bigg|_{\partial D_2} - u \bigg|_{\partial D_1} \geq C \frac{1}{\sqrt{r_2}} \sqrt{\epsilon}.$$ 

This proves Theorem 1.3. □

4 Three disjoint smooth domains

We consider three disjoint inclusion case, see Figure 1 and 2, a small one is disjointly embedded into the in-between area of two others, and prove Theorem 1.2 and 1.4. We assume that $D_1$ and $D_2$ are closely spaced with the distance $\epsilon_1$, and $D_2$ and $D_3$ are closely spaced with $\epsilon_2$, but $D_1$ and $D_3$ are not close, and that $D_1$, $D_2$ and $D_3$ have the boundary regularity given in Theorem 1.4.

4.1 Solution representation of $u$

Let $H^c$ be a harmonic function outside of $\bigcup_{i=1}^3 D_i$ and have the same constant value in $\bigcup_{i=1}^3 D_i$ satisfying that

$$\begin{cases} 
\Delta H^c = 0, & \text{in } \mathbb{R}^2 \setminus \bigcup_{i=1}^3 D_i, \\
H^c(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & \text{as } |\mathbf{x}| \to \infty, \\
H^c|_{\partial D_i} = C_H \text{ (constant)}. 
\end{cases}$$

(25)

Since $H^c - H$ is harmonic at infinity, $H^c - H$ attains maximum only at the boundary points of $D_i$, $i = 1, 2, 3$. To make $H^c - H$ attains zero at infinity, $C_H$ should satisfy

$$-\|H\|_{L^\infty(\bigcup_{i=1}^3 D_i)} \leq C_H \leq \|H\|_{L^\infty(\bigcup_{i=1}^3 D_i)}.$$ 

(26)

Moreover, $H^c$ satisfies

$$\sum_{i=1}^3 \int_{\partial D_i} \partial_\nu H^c \, dS = 0.$$ 

The solution $u$ to (1) is represented as

$$u(\mathbf{x}) = H^c(\mathbf{x}) + c_1 h_1(\mathbf{x}) + c_2 h_2(\mathbf{x}),$$

(27)

where

$$h_1 = \Psi[D_1, (D_2 \cup D_3)], \quad h_2 = \Psi[(D_1 \cup D_2), D_3],$$

and

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = - \begin{pmatrix} -1 & \int_{\partial D_1} \partial_\nu h_2 \, dS \\ \int_{\partial D_2} \partial_\nu h_1 \, dS & \int_{\partial D_2} \partial_\nu h_2 \, dS \end{pmatrix}^{-1} \begin{pmatrix} \int_{\partial D_1} \partial_\nu H^c \, dS \\ \int_{\partial D_2} \partial_\nu H^c \, dS \end{pmatrix},$$

(28)

where $\Psi$ is defined as (7) and (9). The equality (28) is from the integration of $\partial_\nu u$ on $\partial D_1$ and $\partial D_2$.

Applying the upper bound on the gradient of solution without the potential difference among the boundaries to conductivity equation derived in Bao et al. [7], we can show that
$\nabla H^c$ does not blow-up (also refer to [14]). Using Lemma 4.3 in the following section, we have
\[ \int_{\partial D} \frac{\partial_{\nu} h_1}{\partial S} = 1 + O(\sqrt{\epsilon_1}). \]
This implies that
\[ \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right) \approx - \left( \begin{array}{cc} -1 & 0 \\ 1 & -1 \end{array} \right)^{-1} \left( \begin{array}{c} \int_{\partial D_1} \frac{\partial_{\nu} H^c}{\partial S} \\ \int_{\partial D_2} \frac{\partial_{\nu} H^c}{\partial S} \end{array} \right). \]
Thus, the coefficient $c_i$, $i = 1, 2$, is bounded independently of $\epsilon_1$ and $\epsilon_2$. Therefore, the blow-up rate of $\nabla u$ essentially relies on $\nabla h_i$. In this respect, we consider the properties of $h_i$ in the following section.

### 4.2 Properties of $h_1$ and $h_2$

We build the optimal bounds of $u$ based on (27); it is essential to drive properties of $h_1$ and $h_2$ in the narrow regions between inclusions. Let $h_1$ and $h_2$ be as follows:
\[ h_1 = \Psi[D_1, (D_2 \cup D_3)] \quad \text{and} \quad h_2 = \Psi[(D_1 \cup D_2), D_3]. \]

**Proposition 4.1** There are the following estimates for $h_1$ and $h_2$:

(i) In the narrow region between $D_1$ and $D_2$, we have
\[ \nabla h_1 = O\left(\frac{1}{\sqrt{\epsilon_1}}\right) \quad \text{and} \quad \nabla h_2 = O(\sqrt{\epsilon_2}). \]

(ii) In the narrow region between $D_2$ and $D_3$, we have
\[ \nabla h_1 = O(\sqrt{\epsilon_1}) \quad \text{and} \quad \nabla h_2 = O\left(\frac{1}{\sqrt{\epsilon_2}}\right). \]

(iii)
\[ h_1|_{\partial D_2 \cup \partial D_3} - h_1|_{\partial D_1} \simeq \sqrt{\epsilon_1} \]

and
\[ h_2|_{\partial D_3} - h_2|_{\partial D_1 \cup \partial D_2} \simeq \sqrt{\epsilon_2}. \]

**Proof.** We consider $\nabla h_1$. By Lemma 4.2 and 4.4, we have
\[ 0 > \partial_{\nu} h_1 \geq C \partial_{\nu} \Psi[D_1, D_4] \quad \text{on} \quad \partial D_1 \]
and
\[ 0 < \partial_{\nu} h_1 \leq \partial_{\nu} \Psi[D_1, D_2] \quad \text{on} \quad \partial D_2, \]
and by Lemma 4.3
\[ 0 < |\partial_{\nu} h_1| \leq C \sqrt{\epsilon_1} \quad \text{on} \quad \partial D_3, \]
where $D_4$ is defined in Lemma 4.4. Without any loss of generality, we assume that
\[ \left( -\frac{\epsilon_1}{2}, 0 \right) \in \partial D_1, \quad \left( \frac{\epsilon_1}{2}, 0 \right) \in \partial D_2 \quad \text{and} \quad \text{dist}(D_1, D_2) = \epsilon_1. \]
Let

\[ p(x) = \log |x - (\sqrt{\epsilon_1}, 0)| - \log |x + (\sqrt{\epsilon_1}, 0)|. \]

Referring to the inequality (9) in [15], there is a constant \( C_1 \) such that

\[ 0 < |\nabla h_1| \leq C_1 |\nabla p| \text{ on } \partial(D_1 \cup D_2 \cup D_3). \]

Regarding \((x_1, x_2)\) as a complex number \( z = x_1 + x_2i \), we consider

\[ \rho(z) = \frac{\partial_1 h_1(z) - \partial_2 h_1(z)i}{C_1(\partial_1 p(z) - \partial_2 p(z)i)}. \]

Then, \( \rho(z) \) can be extended to \( \infty \) as an analytic function. From definition, \( |\rho(z)| < 1 \) on \( \partial(D_1 \cup D_2 \cup D_3) \). By the maximum principle,

\[ |\rho(z)| < 1 \text{ in } \mathbb{C} \setminus (D_1 \cup D_2 \cup D_3). \]

Thus, we have

\[ |\nabla h_1| \leq C_1 |\nabla p| \text{ in } \mathbb{R}^2 \setminus (D_1 \cup D_2 \cup D_3). \]

Therefore, \( \nabla h_1 = O\left(\frac{1}{\sqrt{\epsilon_1}}\right) \) in the narrow region between \( D_1 \) and \( D_2 \), and \( \nabla h_1 = O(\sqrt{\epsilon_1}) \) in the narrow region between \( D_2 \) and \( D_3 \). Similarly, we have \( \nabla h_2 = O\left(\frac{1}{\sqrt{\epsilon_2}}\right) \) in the narrow region between \( D_2 \) and \( D_3 \), and \( \nabla h_2 = O(\sqrt{\epsilon_2}) \) in the narrow region between \( D_1 \) and \( D_2 \).

We have proven (i) and (ii).

The estimate (iii) is presented by Lemma 4.4. \( \square \)

**Lemma 4.2** We have the following properties:

(i) \[ 0 < h_1|_{\partial D_2} - h_1|_{\partial D_1} \leq \Psi[D_1, D_2]|_{\partial D_2} - \Psi[D_1, D_2]|_{\partial D_1}. \]

(ii) \[ 0 < \partial \nu h_1 \leq \partial \nu \Psi[D_1, D_2] \text{ on } \partial D_2. \]

**Proof.** Let

\[ M = \frac{\Psi[D_1, (D_2 \cup D_3)]|_{\partial D_1} - \Psi[D_1, (D_2 \cup D_3)]|_{\partial D_2 \cup \partial D_3}}{\Psi[D_1, D_2]|_{\partial D_1} - \Psi[D_1, D_2]|_{\partial D_2}}, \]

and

\[ G(x) = \Psi[D_1, (D_2 \cup D_3)](x) - \Psi[D_1, (D_2 \cup D_3)]|_{\partial D_2 \cup \partial D_3} \]

\[ - M \left( \Psi[D_1, D_2](x) - \Psi[D_1, D_2]|_{\partial D_2} \right). \]

Then, \( G = 0 \) on \( \partial D_1 \cup \partial D_2 \), and \( G > 0 \) on \( \partial D_3 \). By Hopf’s lemma,

\[ \partial \nu G < 0 \text{ on } \partial D_1. \]
This means that  
\[ \partial_{\nu} h_1 \leq M \partial_{\nu} \Psi[D_1, D_2] \text{ on } \partial D_1. \] (29)

Note that \( h_1 = \Psi[D_1, (D_2 \cup D_3)] \). By integrating \( G \) on \( \partial D_1 \), we have the inequality (i).

On the other hand, by Hopf’s lemma,
\[ \partial_{\nu} G < 0 \text{ on } \partial D_2. \]

This means that
\[ \partial_{\nu} h_1 \leq M \partial_{\nu} \Psi[D_1, D_2] \text{ on } \partial D_2. \]

From the inequality (i), \( M < 1 \). Therefore, we have (ii).

\[ \square \]

**Lemma 4.3** There is a constant \( C \) such that
\[ 0 \leq \partial_{\nu} h_1 \leq C \sqrt{\epsilon_1} \text{ on } \partial D_3. \]

**Proof.** We use the method similar to Lemma 4.2. Let
\[ M = \frac{\Psi[D_1, (D_2 \cup D_3)]|_{\partial D_3} - \Psi[D_1, (D_2 \cup D_3)]|_{\partial D_2 \cup \partial D_3}}{\Psi[D_1, D_3]|_{\partial D_1} - \Psi[D_1, D_3]|_{\partial D_3}}, \]

and
\[ G(x) = \Psi[D_1, (D_2 \cup D_3)](x) - \Psi[D_1, (D_2 \cup D_3)]|_{\partial D_2 \cup \partial D_3} - M \left( \Psi[D_1, D_3](x) - \Psi[D_1, D_3]|_{\partial D_3} \right). \]

Then, \( G = 0 \) on \( \partial D_1 \cup \partial D_3 \), and \( G > 0 \) on \( \partial D_2 \). By Hopf’s lemma,
\[ \partial_{\nu} G < 0 \text{ on } \partial D_3. \]

Since \( h_1 = \Psi[D_1, (D_2 \cup D_3)] \), this inequality means that
\[ 0 \leq \partial_{\nu} h_1 \leq M \partial_{\nu} \Psi[D_1, D_3] \text{ on } \partial D_3. \]

Now, we estimate the gradient of \( M \Psi[D_1, D_3] \). To do so, we consider the potential difference between \( \partial D_1 \) and \( \partial D_3 \) as follows:
\[ M \Psi[D_1, D_3]|_{\partial D_3} - M \Psi[D_1, D_3]|_{\partial D_1} = h|_{\partial D_3} - h|_{\partial D_1} = h|_{\partial D_2} - h|_{\partial D_1} \leq \Psi[D_1, D_2]|_{\partial D_2} - \Psi[D_1, D_2]|_{\partial D_1} \leq C \sqrt{\epsilon_1} \]

The last inequality above was proven by Yun in his paper [14, 15], since \( \Psi[D_1, D_2] \) is only for two domains. Note that \( D_3 \) is not close to \( D_2 \). Owing to the method in Bao et al. [2], we have
\[ \| \partial_{\nu} M \Psi[D_1, D_3] \|_{L^\infty(\partial D_3)} \leq C \sqrt{\epsilon_1}. \]

Therefore, we can obtain the result. \[ \square \]
Lemma 4.4 Let $D_4$ be a disk containing $D_2$ and $D_3$ with
\[\text{dist}(D_1, D_4) = \text{dist}(D_1, D_2).\]

(i) There is a positive constant $C$ such that
\[0 > \partial_\nu h_1 \geq C \partial_\nu \Psi[D_1, D_4] \text{ on } \partial D_1.
\]

(ii) \[h_1|_{\partial D_2} - h_1|_{\partial D_1} \geq \Psi[D_1, D_4]|_{\partial D_4} - \Psi[D_1, D_4]|_{\partial D_1}.
\]

(iii) \[h_1|_{\partial D_2 \cup \partial D_3} - h_1|_{\partial D_1} \simeq \sqrt{\epsilon_1}.
\]

Proof. To prove (i) and (ii), we use the same derivation to Lemma 4.2. So, we set
\[M = \Psi[D_1, (D_2 \cup D_3)]|_{\partial D_1} - \Psi[D_1, (D_2 \cup D_3)]|_{\partial D_2 \cup \partial D_3},
\]
and
\[G(x) = \Psi[D_1, (D_2 \cup D_3)](x) - \Psi[D_1, (D_2 \cup D_3)]|_{\partial D_2 \cup \partial D_3}
- M \left( \Psi[D_1, D_4](x) - \Psi[D_1, D_4]|_{\partial D_4} \right).
\]

Then $G|_{\partial D_1} = 0$ and $G \leq 0$ on $\partial D_4$. By Hopf’s lemma, we have
\[\partial_\nu G > 0 \text{ on } \partial D_1.
\]

By the integration on $\partial D_1$, we have (ii) and $M < 1$. Therefore, the inequality $\partial_\nu G > 0$ can also yield (i).

From (i) of Lemma 4.2 and (ii) in this lemma, we have
\[h_1|_{\partial D_2} - h_1|_{\partial D_1} \geq \Psi[D_1, D_4]|_{\partial D_4} - \Psi[D_1, D_4]|_{\partial D_1}
\]
and
\[h_1|_{\partial D_2} - h_1|_{\partial D_1} \leq \Psi[D_1, D_2]|_{\partial D_2} - \Psi[D_1, D_2]|_{\partial D_1}.
\]

The potential $\Psi[D_1, D_i] (i = 1, 4)$ is only for two domains and thus, its difference between $D_1$ and $D_i (i = 1, 2)$ was already estimated in Yun [14, 15] as follows: for $i = 1, 2$,
\[\Psi[D_1, D_i]|_{\partial D_i} - \Psi[D_1, D_i]|_{\partial D_i} \simeq \sqrt{\epsilon_1}.
\]

Therefore, we have (iii). \qed
Lemma 4.5 We have
\[ \left| \int_{\bigcup_{i=1}^{3} \partial D_i} H \partial_{\nu} h_1 \, dS \right| \leq C \sqrt{\epsilon_1}. \]

Proof. Without loss of generality, we assume that
\[ \left( -\frac{\epsilon_1}{2}, 0 \right) \in \partial D_1, \quad \left( \frac{\epsilon_1}{2}, 0 \right) \in \partial D_2, \quad \text{dist}(D_1, D_2) = \epsilon_1 \quad \text{and} \quad (-1, 0) \in D_2. \]

We consider \( \tilde{H} \) as follows:
\[ \tilde{H} = H - \partial_2 H(0, 0) \frac{x_2}{|x - (1, 0)|^2}. \]

It follows from the Divergence Theorem that
\[ \int_{\partial D_1 \cup \partial D_2 \cup \partial D_3} \frac{x_2}{|x - (1, 0)|^2} \partial_{\nu} h \, dS = \int_{\partial D_1 \cup \partial D_2 \cup \partial D_3} \partial_{\nu} \left( \frac{x_2}{|x - (1, 0)|^2} \right) h \, ds = 0, \]
since \( \frac{x_2}{|x - (1, 0)|^2} = O(|x|^{-1}) \) as \( |x| \to \infty \). Hence, we have
\[ \int_{\bigcup_{i=1}^{3} \partial D_i} H \partial_{\nu} h_1 \, dS = \int_{\partial D_1 \cup \partial D_2} \tilde{H} \partial_{\nu} h_1 \, dS + \int_{\partial D_3} \tilde{H} \partial_{\nu} h_1 \, dS. \]

We first consider \( \int_{\partial D_1 \cup \partial D_2} \tilde{H} \partial_{\nu} h_1 \, dS \). By Lemma 4.2 and 4.4 we have
\[ 0 > \partial_{\nu} h_1 \geq C \partial_{\nu} \Psi[D_1, D_4] \quad \text{on} \quad \partial D_1 \]
and
\[ 0 < \partial_{\nu} h_1 \leq \partial_{\nu} \Psi[D_1, D_2] \quad \text{on} \quad \partial D_2. \]

From definition, \( \partial_2 \tilde{H} = 0 \). Hence, we can use Lemma 3.2 in [15] so that
\[ \left| \int_{\partial D_1} \tilde{H} \partial_{\nu} h_1 \, dS \right| \leq C \int_{\partial D_1} |\tilde{H} \Psi[D_1, D_4]| \, dS \leq C \sqrt{\epsilon_1} \]
and
\[ \left| \int_{\partial D_2} \tilde{H} \partial_{\nu} h_1 \, dS \right| \leq \int_{\partial D_2} |\tilde{H} \Psi[D_1, D_2]| \, dS \leq C \sqrt{\epsilon_1}. \]

We second consider \( \int_{\partial D_3} \tilde{H} \partial_{\nu} h_1 \, dS \). By Lemma 4.3 we can have
\[ \left| \int_{\partial D_3} \tilde{H} \partial_{\nu} h_1 \, dS \right| \leq C \sqrt{\epsilon_1}. \]

Therefore, we have done it.

\[ \square \]

Remark 4.6 We draw attention of readers to the independent work of Bao, Li and Yin in [5] and [8]. Bao et al. have shown that the blow-up rate known only for a pair of inclusion is still valid to the multiple inclusions cases. As a byproduct of our work, the blow-up rate of the gradient for three inclusions is established in Theorem 4.7.
Theorem 4.7 Let $D_1, D_2$ and $D_3$ be as assumed in the beginning of Section 4. Note that $D_2$ is not assumed to be smaller than the others.

(i) Optimal upper bounds: For any entire harmonic function $H(x_1, x_2)$, we have the following: in the narrow region between $D_1 \cup D_2$,

$$|\nabla u| \leq C \frac{1}{\sqrt{\epsilon_1}},$$

and, in the narrow region between $D_2 \cup D_3$,

$$|\nabla u| \leq C \frac{1}{\sqrt{\epsilon_2}}.$$

(ii) Existence of blow-up: Without loss of generality, we assume that

$$\left(-\frac{\epsilon_1}{2}, 0\right) \in \partial D_1, \left(\frac{\epsilon_1}{2}, 0\right) \in \partial D_2 \text{ and } \text{dist}(D_1, D_2) = \epsilon_1.$$

For $H(x_1, x_2) = x_1$, there exist $x_0$ in the narrow region between $D_1$ and $D_2$ such that

$$|\nabla u(x_0)| \geq C \frac{1}{\sqrt{\epsilon_1}},$$

and, similarly, there is a linear function $H(x_1, x_2)$ with $y_0$ between $D_2$ and $D_3$ such that

$$|\nabla u(y_0)| \geq C \frac{1}{\sqrt{\epsilon_2}}.$$

Proof. From Subsection 4.1, we have a representation (28) for $u$ and the coefficient $c_i$, $i = 1, 2$, is bounded independently of $\epsilon_1$ and $\epsilon_2$. Proposition 4.1 yields the upper bound of Theorem 4.7.

Now, we consider the existence of the blow-up. Using the result of Subsection 4.1 again, we have a constant $C$ independent of $\epsilon$ such that

$$\|u\|_{L^{\infty}(\cup_{i=1}^3 \partial D_i)} \leq C\|H\|_{L^{\infty}(\cup_{i=1}^3 D_i)}.$$ 

Applying the Green’s identity to $\int_{\cup_{i=1}^3 \partial D_i} u\partial_\nu h_1 \ dS$, we have

$$\int_{\cup_{i=1}^3 \partial D_i} H\partial_\nu h_1 \ dS = \int_{\cup_{i=1}^3 \partial D_i} u\partial_\nu h_1 \ dS$$

$$= -u|_{\partial D_1} + u|_{\partial D_2} (\int_{\partial D_2} \partial_\nu h_1 \ dS) + u|_{\partial D_3} (\int_{\partial D_3} \partial_\nu h_1 \ dS)$$

$$= -u|_{\partial D_1} + u|_{\partial D_2} (1 - \int_{\partial D_3} \partial_\nu h_1 \ dS) + u|_{\partial D_3} (\int_{\partial D_3} \partial_\nu h_1 \ dS). \quad (30)$$

By Lemma 4.3 we have

$$u|_{\partial D_2} - u|_{\partial D_1} \leq C\sqrt{\epsilon_1}, \quad (31)$$

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where the constant $C$ above depends on $\|H\|_{L^{\infty}(\cup_{i=1}^{3} \partial D_i)}$. Similarly, we have

$$u|_{\partial D_3} - u|_{\partial D_2} \leq C \sqrt{\epsilon_2}. \quad (32)$$

Using (30) again, we have

$$u|_{\partial D_3} - u|_{\partial D_2} = O(\sqrt{\epsilon_1 \epsilon_2}) = \int_{\cup_{i=1}^{3} \partial D_i} H \partial_{\nu} h_1 \, dS \geq \int_{\partial D_1} H \partial_{\nu} h_1 \, dS. \quad (33)$$

The last inequality can be derived from the fact that $H > 0$ on $\partial D_2 \cup \partial D_3$.

To get the last inequality above, we took an advantage of $H = x_1$. By (29),

$$\partial_{\nu} h \leq \left( \frac{h_1|_{\partial D_1} - h_1|_{\partial D_2 \cup \partial D_3}}{\Psi[D_1, D_2]|_{\partial D_1} - \Psi[D_1, D_2]|_{\partial D_2}} \right) \partial_{\nu} \Psi[D_1, D_2] < 0 \text{ on } \partial D_1.$$ 

By (iii) in Proposition 4.1, we have

$$\frac{\partial_{\nu} h}{w_1|_{\partial D_2} - w_1|_{\partial D_1}} \simeq \sqrt{\epsilon_1}.$$ 

By the Mean Value Theorem, we have the desirable lower bound in the narrow region between $D_1$ and $D_2$. Similarly, we can also obtain the other lower bound. □

### 4.3 Proof of Theorem 1.2

We derive the optimal bounds of the gradient of the solution to (1), when there are adjacent three disks:

$$D_l = B_{r_l}(c_l), \ l = 1, 2, 3,$$

where $c_1 = (-r_1 - \frac{\epsilon_1}{2}, 0)$, $c_2 = (r_2 + \frac{\epsilon_1}{2}, 0)$ and $c_3 = (r_3 + r_2 + \frac{\epsilon_2}{2} + \epsilon_2, 0)$. As defined before, $h_1 = \Psi[D_1,(D_2 \cup D_3)]$. Let $w_1 = \Psi[D_1, D_2]$.

We begin the proof by showing that

$$w_1|_{\partial D_2} - w_1|_{\partial D_1} \simeq h_1|_{\partial D_2} - h_1|_{\partial D_1}.$$ 

By the monotonic property of Lemma 3.4 we have

$$h_1|_{\partial D_2} - h_1|_{\partial D_1} \leq w_1|_{\partial D_2} - w_1|_{\partial D_1}.$$ 

Considering

$$h_1 \leq \left( \frac{h_1|_{\partial D_2} - h_1|_{\partial D_1}}{w_1|_{\partial D_2} - w_1|_{\partial D_1}} \right) w_1,$$
we can obtain, from the Hopf’s Lemma,
\[ \int_{\partial D_2} \partial_{\nu} h_1 dS \leq \left( \frac{h_1|_{\partial D_2} - h_1|_{\partial D_1}}{w_1|_{\partial D_2} - w_1|_{\partial D_1}} \right) \int_{\partial D_2} \partial_{\nu} w_1 dS. \]

By Lemma 4.3, we have
\[ \int_{\partial D_3} \partial_{\nu} h_1 dS = O(\sqrt{\epsilon_1}). \]

Since \( \int_{\partial D_2 \cup \partial D_3} \partial_{\nu} h_1 dS = 1 \), we have
\[ \left( w_1|_{\partial D_2} - w_1|_{\partial D_1} \right) \left( 1 + O(\sqrt{\epsilon_1}) \right) \leq h_1|_{\partial D_2} - h_1|_{\partial D_1}. \]

Therefore, we can obtain \( \text{(38)} \). Owing to the estimate for \( w_1|_{\partial D_2} - w_1|_{\partial D_1} \) in Lemma 2.2, we have
\[ h_1|_{\partial D_2} - h_1|_{\partial D_1} \simeq \sqrt{r_1 + r_2} \sqrt{\epsilon_1}. \]  

(36)

Let \( w_2 = \Psi[D_1, D_3] \). Considering
\[ h_1 - \left( \frac{h_1|_{\partial D_3} - h_1|_{\partial D_1}}{w_2|_{\partial D_3} - w_2|_{\partial D_1}} \right) w_2, \]

from the Hopf’s Lemma, we obtain
\[ \partial_{\nu} h_1 \leq \left( \frac{h_1|_{\partial D_3} - h_1|_{\partial D_1}}{w_2|_{\partial D_3} - w_2|_{\partial D_1}} \right) \partial_{\nu} w_2 \leq 0 \text{ on } \partial D_1. \]

Here, we estimate the coefficient in the right hand side. Note that \( h_1|_{\partial D_2} = h_1|_{\partial D_3} \). Thus, we have
\[ h_1|_{\partial D_3} - h_1|_{\partial D_1} \simeq \sqrt{r_1 + r_2} \sqrt{\epsilon_1}. \]

Since \( r_2 \ll r_1 \) and \( r_2 \ll r_3 \), we also have
\[ w_2|_{\partial D_3} - w_2|_{\partial D_1} \simeq \sqrt{r_1 + r_3} \sqrt{r_2}. \]

This implies that
\[ \partial_{\nu} h_1 \leq \frac{\sqrt{r_1 + r_2} \sqrt{r_3}}{\sqrt{r_1 + r_3} \sqrt{r_2}} \partial_{\nu} w_2 \leq 0 \text{ on } \partial D_1. \]

Therefore, we have
\[
\begin{align*}
\int_{\partial D_1} H \partial_{\nu} h_1 dS & \geq \sqrt{\frac{r_1 + r_2}{r_1 + r_3}} \sqrt{r_2} \sqrt{\epsilon_1} \int_{\partial D_1} H \partial_{\nu} w_2 dS \\
& \geq \sqrt{\frac{r_1 + r_2}{r_1 + r_3}} \sqrt{r_2} \sqrt{\epsilon_1} \sqrt{\frac{r_1 r_3}{r_1 + r_3}} \sqrt{r_2} \\
& \geq \frac{r_1 r_3}{r_1 + r_3} \sqrt{\epsilon_1} \geq 0.
\end{align*}
\]

(37)
Owing to (30), (31) and (32), we have
\[
\int_{\cup_{i=1}^{3} \partial D_i} u \partial_{\nu} h_1 \, dS = \left(1 - \int_{\partial D_3} \partial_{\nu} h_1 \, dS\right) \left(u \big|_{\partial D_2} - u \big|_{\partial D_1}\right) + \left(\int_{\partial D_3} \partial_{\nu} h_1 \, dS\right) \left(u \big|_{\partial D_3} - u \big|_{\partial D_1}\right) = (1 - O(\sqrt{\epsilon_1})) \left(u \big|_{\partial D_2} - u \big|_{\partial D_1}\right) + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})).
\]
Therefore, we have
\[
u_{1D_2} - \nu_{1D_1} \geq \frac{1}{2} \int_{\cup_{i=1}^{3} \partial D_i} u \partial_{\nu} h_1 \, dS + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})) \]
\[
= \frac{1}{2} \int_{\partial D_1} H \partial_{\nu} h_1 \, dS + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})) \]
\[
\geq \frac{1}{2} \int_{\partial D_1} H \partial_{\nu} h_1 \, dS + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})) \]
\[
\geq C \frac{r_1 r_3}{r_1 + r_3 \sqrt{\epsilon_1}} \geq C r_1 r_3 \sqrt{\epsilon_1}.\]
Therefore, we have completed the proof. \(\square\)

### 4.4 Proof of Theorem 1.4

We pursue the proof of Theorem 1.2, taking an advantage of the monotonic property of Lemma 3.4. The domains \(D_1, D_2\) and \(D_3\) are as assumed in Theorem 1.4. As assumed before, \(h_1 = \Psi[D_1, (D_2 \cup D_3)]\). Let \(w_1 = \Psi[D_1, D_2]\). By the same way as Theorem 1.2, we have
\[
w_1 \big|_{\partial D_2} - w_1 \big|_{\partial D_1} \simeq h_1 \big|_{\partial D_2} - h_1 \big|_{\partial D_1}.
\]
Here, we use the monotonic property of Lemma 3.4 to estimate the difference between domains. Choosing two pairs of proper disks containing \(D_1\) and \(D_2\), and contained \(D_1\) and \(D_2\), respectively, we can obtain
\[
h_1 \big|_{\partial D_2} - h_1 \big|_{\partial D_1} \simeq \sqrt{\epsilon_1} \sqrt{r_2}
\]
under the assumption that \(r_2\) is small.

Let \(w_2 = \Psi[D_1, D_3]\). Choosing two pairs of proper disks containing \(D_1\) and \(D_3\), and contained \(D_1\) and \(D_3\), respectively, then, we have
\[
w_2 \big|_{\partial D_3} - w_2 \big|_{\partial D_1} \simeq \sqrt{r_2}.
\]
By the same argument as Theorem 1.2, we have
\[
\int_{\partial D_1} H \partial_{\nu} h_1 \, dS \geq \sqrt{\epsilon_1} \sqrt{r_1} \geq 0.
\]
Note that $D_1 \subset \mathbb{R}^- \times \mathbb{R}$ and $D_2 \cup D_3 \subset \mathbb{R}^+ \times \mathbb{R}$. Continuing to follow the proof of Theorem 1.2 we can obtain
\[ u_{|\partial D_2} - u_{|\partial D_1} \geq C \sqrt{\frac{\epsilon_1}{r_1}} + O(\sqrt{\epsilon_1}) \left( O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2}) \right). \]
Therefore, we have done the proof.

4.5 Derivation for the optimal upper bounds

We consider the optimal upper bounds presented in Theorem 1.1, 1.2, 1.3 and 1.4. These proofs have essential thing in common. In this respect, we prove only the optimal upper bound presented in Theorem 1.2. As have assumed them before, we set
\[ h_1 = \Psi[D_1, D_2 \cup D_3], \]
\[ h_2 = \Psi[D_1, D_2], \]
\[ h_3 = \Psi[D_1, D_3], \]
\[ h_4 = \Psi[D_1, D_4]. \]

Here, the domain $D_4$ is given in Lemma 4.4, which is a disk containing $D_2$ and $D_3$ with $\text{dist}(D_1, D_4) = \text{dist}(D_1, D_2)$, and the diameter of $D_4$ is in proportion as $r_3$, because $r_2$ is sufficiently small. Then, we compare $h_1$ with $h_2$, $h_3$ and $h_4$. The proof of Lemma 4.2 contains
\[ 0 \leq \partial_{x_2}h_1 \leq \left( \frac{h_1_{|\partial (D_2 \cup D_3)} - h_1_{|\partial D_1}}{h_2_{|\partial D_2} - h_2_{|\partial D_1}} \right) \partial_{x_2}h_2 \text{ on } \partial D_2, \]
the proof of Lemma 4.3 yields
\[ 0 \leq \partial_{x_2}h_1 \leq \left( \frac{h_1_{|\partial (D_2 \cup D_3)} - h_1_{|\partial D_1}}{h_3_{|\partial D_3} - h_3_{|\partial D_1}} \right) \partial_{x_2}h_3 \text{ on } \partial D_3, \]
and the proof of Lemma 4.4 implies
\[ 0 \leq -\partial_{x_2}h_1 \leq - \left( \frac{h_1_{|\partial (D_2 \cup D_3)} - h_1_{|\partial D_1}}{h_4_{|\partial D_4} - h_4_{|\partial D_1}} \right) \partial_{x_2}h_4 \text{ on } \partial D_4. \]
In the same way as Lemma 4.3, we can consider $\tilde{H}$ by choosing the point in $D_3$. In this respect, without any loss of generality, we can assume that
\[ \partial_{x_2}H(0, 0) = 0. \]
The reason why we assumed above is because the integration representation for the potential difference is not good enough, refer to [15]. The geometrical assumption of Case (B) implies that $D_1$ and $D_2 \cup D_3$ are separated by $x_1 = 0$ and they are approaching to $(0, 0)$. 23
Therefore, by the proof of Theorem 1.2 and Lemma 2.3, we have

\[
|u|_{\partial D_2} - u|_{\partial D_1} + O(\sqrt{c_1}) (O(\sqrt{c_1}) + O(\sqrt{c_2}))
\]

\[
= \left| \int_{\partial(D_1 \cup D_2)} H \partial_n h_1 \text{d}S \right|
\]

\[
= \lambda \left( h_1 \left|_{\partial(D_2 \cup D_3)} - h_1 \left|_{\partial D_1} \right. \right) \right. \right. \right. \right.
\]

\[
+ \lambda \left( h_2 \left|_{\partial D_2} - h_2 \left|_{\partial D_1} \right. \right) \right. \right. \right. \right.
\]

\[
+ \lambda \left( h_3 \left|_{\partial D_3} - h_3 \left|_{\partial D_1} \right. \right) \right. \right. \right. \right.
\]

\[
+ \lambda \left( h_4 \left|_{\partial D_4} - h_4 \left|_{\partial D_1} \right. \right) \right. \right. \right. \right.
\]

and

\[
h_1 \left|_{\partial(D_2 \cup D_3)} - h_1 \left|_{\partial D_1} \right. \right) \approx h_2 \left|_{\partial D_2} - h_2 \left|_{\partial D_1} \right. \right)
\]

Here, note that the radius of \( D_4 \) can be choosen between \( 2r_3 \) and \( 2r_3 \). Lemma 2.2 implies that

\[
\left| u \right|_{\partial D_2} - u|_{\partial D_1} \leq \frac{r_1 r_3}{r_1 + r_3} \sqrt{c_1}.
\]

Therefore, we establish the optimal upper bound for \( u|_{\partial D_2} - u|_{\partial D_1} \).

Based on this, the optimal upper bound on the gradient of \( u \) in the narrow region be obtained. Here, the main idea to get the gradient estimate from the potential difference has already been presented by Bao et al. (Theorem 1.3, Lemma 2.2 and 2.3 in [1]), and has been modified to fit our problem by Lim and Yun in [13]. Thus, we give a brief description on the method. We choose a large domain \( D_0 \) containing \( D_1, D_2 \) and \( D_3 \), where \( \partial D_0 \) is at a sufficient distance from \( D_1, D_2 \) and \( D_3 \). Then, \( u \) can be decomposed as follows:

\[
u = C_0 + v_0 + C_1 v_1 + C_3 v_3
\]

where for \( i = 0, 1, 3 \), \( v_i \) is a harmonic function in \( D_0 \setminus (D_1 \cup D_2 \cup D_3) \) with the boundary data

\[
v_i = \delta_{0j} \text{ on } \partial D_j \text{ for } i = 1, 3
\]

and

\[
v_0 = \delta_{ij} u \text{ on } \partial D_j
\]

for any \( j = 0, 1, 2, 3 \). Thus, the constants \( C_1 \) and \( C_3 \) keep

\[
|C_1| \leq \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{c_1}}
\]

and

\[
|C_3| \leq \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{c_1}}.
\]
To estimate $\nabla v_0$, we consider a harmonic function $\rho$ in $D_0 \setminus (D_1 \cup D_2 \cup D_3)$ with the boundary data 
\[ \rho = \delta_{0j} \text{ on } \partial D_j \]
for any $j = 0, 1, 2, 3$. By comparing with the harmonic function $\rho_i$ in $D_0 \setminus D_i$ with $\rho_i = 0$ on $\partial D_i$ and $\rho_i = 1$ on $\partial D_0$, the Hopf’s Lemma yields
\[
\| \nabla \rho \|_{L^\infty(D_0 \setminus (D_1 \cup D_2 \cup D_3))} \leq \max\{ \| \nabla \rho_1 \|_{L^\infty(\partial D_1)}, \| \nabla \rho_2 \|_{L^\infty(\partial D_2)}, \| \nabla \rho_3 \|_{L^\infty(\partial D_3)}, \| \nabla \rho \|_{L^\infty(\partial D_0)} \} < C.
\]
Applying the Hopf’s Lemma again, we can have that the gradient of $v_0$ is bounded independent of $\epsilon_1$, refer to Lemma 2.2 in [7].

We estimate $C_1 \nabla v_1$ in the narrow region between $D_1$ and $D_2$. Since $v_1$ is constat on the boundaries and the boundaries is smooth enough in the narrow region, the proof of Lemma 4.3 implies that $v_1$ can be extend into the interior areas of $D_1$ and $D_2$ by the distance almost $\epsilon$ from the boundaries in the narrow region, independently of $r_1$ and $r_2$. By the gradient estimate for harmonic functions allows
\[
|C_1 \nabla v_1| \lesssim \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon_1}}
\]
in the narrow region between $D_1$ and $D_2$. Note that the inequality above is a local property independent of choosing $D_0$.

Now, we consider $C_3 \nabla v_3$ in the narrow region between $D_1$ and $D_2$. Let $\tilde{\rho}$ be a harmonic function in $D_0 \setminus (D_2 \cup D_3)$ with the boundary data
\[ \tilde{\rho} = 0 \text{ on } \partial(D_0 \cup D_2) \text{ and } \tilde{\rho} = 1 \text{ on } \partial D_3. \]
By the maximum principle, we have
\[ 0 \leq v_3 \leq \tilde{\rho} \text{ in } D_0 \setminus (D_1 \cup D_2 \cup D_3) \]
Considering the standard estimate for $\Psi[D_2, D_3]$, we can obtain
\[
|C_3 v_3| \leq C \sqrt{\epsilon_2}.
\]
Similarly to the estimate for $C_1 \nabla v_1$, the gradient estimate for harmonic functions yields
\[
|C_3 \nabla v_3| \lesssim \frac{\sqrt{\epsilon_2}}{\epsilon_1}
\]
in the narrow region between $D_1$ and $D_2$.

Therefore, we can obtain the desirable upper bound. Here, it is noteworthy that the upper bound is dominated only by the estimate for $C_2 \nabla v_1$, which is independent of choosing $D_0$. In this respect, the constant $C$ of the upper bound in Theorem 1.2 is independent of $r_1$, $r_2$, $r_3$, $\epsilon_1$ and $\epsilon_2$. \]
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