I. INTRODUCTION

Recent experiments on transport phenomena in low dimensional spin systems have attracted much interest. There the behavior of conductivity is sensitively affected by the magnetic phases. Therefore the thermal conductivity can be a strong experimental instrument to investigate the magnetic properties in materials. In experiments, magnetic energy transport is much larger than the phononic transport at most temperatures. This fact is considered to be attributed to large exchange coupling constants and nondiffusive behavior of transport. Sr$_2$CuO$_3$ described by the isotropic Heisenberg chain has large exchange coupling constant $J \sim 2000$K and the magnon can transmit energy even at the order of 100K. The mean free path is experimentally estimated in the similar way as the Peierls-Boltzmann approach for phononic transport and nondiffusive behavior is found. In another isotropic Heisenberg chain CuGeO$_3$, where the mean free path is calculated as 500 times lattice size. It is noteworthy that even when the alternation between bonds occurs, the magnetic transport may be still dominant and nontrivial enhancement of conductivity is observed. Spin-ladder system (Sr, La, Ca)$_{14}$Cu$_{24}$O$_{41}$ and La$_{24}$Cu$_{16}$O$_{41}$ also show extremely large mean free path.

As studied in classical systems, an integrable system shows ballistic energy transport. Even if the energy current is not conserved, the low-dimensional systems with conserved quantities tend to show the divergence of conductivity. These tendencies are also valid in quantum systems. For the Heisenberg chain by Meisner et al., the isotropic Heisenberg chain CuGeO$_3$, and Sr$_2$CuO$_3$ described by the isotropic Heisenberg chain has large exchange coupling constant $J \sim 2000$K and the magnon can transmit energy even at the order of 100K. The mean free path is experimentally estimated in the similar way as the Peierls-Boltzmann approach for phononic transport and nondiffusive behavior is found. In another isotropic Heisenberg chain CuGeO$_3$, the mean free path is calculated as 500 times lattice size. It is noteworthy that even when the alternation between bonds occurs, the magnetic transport may be still dominant and nontrivial enhancement of conductivity is observed.

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II. EFFECTIVE ENERGY CURRENT IN LOW TEMPERATURE REGIME

We first consider an alternate spin chain whose Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1,$$

$$\mathcal{H}_0 = J \sum_\ell \mathbf{S}_\ell \cdot \mathbf{S}_{\ell+1}, \quad \mathcal{H}_1 = J \delta \sum_\ell (-1)^{\ell+1} \mathbf{S}_\ell \cdot \mathbf{S}_{\ell+1}. \quad (2)$$

The total Hamiltonian is divided into the isotropic part $\mathcal{H}_0$ and the dimerized part $\mathcal{H}_1$. Thermal conductivity $\kappa(T)$ at a temperature $T(=1/\beta)$ is calculated by the Green-Kubo formula which reads as

$$\kappa(T) = \lim_{\omega \to 0} \kappa(\omega) = \frac{\beta}{2L} \int_{-\infty}^{\infty} dt e^{-i\omega t} \int_0^\beta d\tau \langle j(t-i\tau) j \rangle. \quad (3)$$
where \( j(...) \) is the total current operator in the Heisenberg picture. From the continuity equation of energy, the total current operator \( j \) is written as \( j = -i \sum \left[ h(t), h(t+1) \right] \) with the local Hamiltonian \( h(t) = J(1 + (-1)^{t+1} \delta) S_t \cdot S_{t+1} \).

In the absence of the alternation, \( j \) is the conserved quantity [24], whereas the existence of alternation does not allow the conservation, i.e.,

\[
\left[ j, h(0) \right] = 0, \text{ but } \left[ j, h \right] \neq 0.
\]

At low energy regime, the Hamiltonian can be effectively represented by the boson field [24, 25],

\[
\mathcal{H}_B = \int dx \left[ \pi v p^2 + \frac{v}{4\pi} (\partial_x \Theta_+)^2 - B \cos \Theta_+(x) \right],
\]

where \( v = \frac{L}{T} \) and \( B = J \delta \). The Boson field is defined using the fermion field \( \Psi_R \) and \( \Psi_L \) and satisfies the commutation relation as,

\[
\frac{1}{2\pi} \partial \Theta_+ = : \Psi_R^\dagger(x) \Psi_R(x) : = : \Psi_L^\dagger(x) \Psi_L(x) : ,
\]

\[
p(x) = -\frac{1}{2\pi} \partial_x \Theta_-(x),
\]

\[
[\Theta_+(x), \Theta_-(x')] = 2\pi i \text{sgn}(x-x').
\]

We define the local Hamiltonian density at \( x \) as, \( h(x) = \pi v p^2 + \frac{v}{4\pi} (\partial_x \Theta_+)^2 - B \cos \Theta_+(x) \). Using the relations,

\[
\frac{\partial \Theta_+}{\partial t} = -2\pi v p, \quad \frac{\partial p}{\partial t} = -\frac{v}{2\pi} \partial_x \Theta_+(x) + B \sin \Theta_+,
\]

we find the Heisenberg equation of motion given by,

\[
\frac{\partial h(x, t)}{\partial t} = -\frac{v^2}{2} \partial_x \left( p \partial_x \Theta_+ + (\partial_x \Theta_+) p \right).
\]

This leads to the effective total energy current written as,

\[
j_B = v^2 \int dx p(x) \partial_x \Theta_+(x).
\]

Interestingly, this effective current operator \( j_B \) does not include the perturbation term (Umklapp term) \( B \cos \Theta_+(x) \). Furthermore we find that even if the energy current is not rigorously conserved [24], it behaves as the conserved quantity in the low temperature regime, i.e.,

\[
\left[ j, h \right] \neq 0, \text{ but } \left[ j_B, h_B \right] = 0.
\]

The perturbation term \( B \cos \Theta_+(x) \) causes the Umlkapp process in the scattering between fermions [24]. Nevertheless at the level of the energy transport, this process is irrelevant to decrease the mean free path at finite low temperatures.

This situation is also the case in the other realistic perturbations. We consider the two following examples [24].

\[
\mathcal{H}_2 = J_0 \sum \ell \mathbf{S}_\ell \cdot \mathbf{S}_{\ell+2}, \quad \mathcal{H}_3 = J_3 \sum \ell (-1)^\ell \mathbf{S}_\ell^z,
\]

The perturbation \( \mathcal{H}_2 \) causes the frustration and the non-trivial phases appears as a function of \( \Theta^z \). Recent numerical studies treat this perturbation and the effect of frustration is investigated [24, 25]. \( \mathcal{H}_3 \) is the term of staggered magnetic field. The effective expressions at the low energy regime for the perturbations (13) are written as,

\[
\mathcal{H}_2 \sim \int dx \left[ c_2 (\partial_x \Theta_+)^2 + c_2^' \cos(2\Theta_+) \right],
\]

\[
\mathcal{H}_3 \sim c_3 \int dx \cos(2\Theta_+),
\]

with constants, \( c_2, c_2', c_3 \). These cases have the same form as the Hamiltonian (11), and the effective total current is easily proved to be conserved. This feature is generally common for the Hamiltonian with the form,

\[
\int dx \left[ a p^2 + b (\partial_x \Theta_+)^2 + \sum \ell c_\ell \cos(d_\ell \Theta_+(x) + \phi_\ell) \right].
\]

Here \( a, b, c_\ell, d_\ell, \) and \( \phi_\ell \) are constants. Thus we conclude that the effective energy current is conserved in most realistic spin chains at least as far as we consider in the low temperature regime.

### III. DRUDE WEIGHT

Now let us calculate the prefactor of thermal conductivity, i.e., the Drude weight \( D_{th} \) in the low energy regime and confirm that finite value of \( D_{th} \) really exists. Since the energy current does not depend on a time, we write the Green-Kubo formula as,

\[
\kappa = D_{th} \delta(\omega)
\]

\[
D_{th} = \beta \pi J \int_0^\beta d\tau \int_0^L dx \int_0^L dx' \langle j_B(x, \tau) j_B(x', 0) \rangle,
\]

where \( j(x, \tau) \) is the local energy current at \( x \) and the imaginary time \( \tau \) in the Heisenberg picture. The current is expressed as \( j(x, \tau) = -\frac{i \pi}{2} \partial_x \Theta_+(x, \tau) \partial_x \Theta_+(x, \tau) \). We confine ourselves in the alternate spin chain effectively described by Hamiltonian (11). At very low temperatures, the Hamiltonian (11) is approximated by the mean field theory [25],

\[
\cos(\Theta_+) \sim e^{-(\Theta^z_+)^2/2} \left( 1 - \frac{\Theta^2_+ - (\Theta^2_+)^2}{2} \right).
\]

The Hamiltonian is reduced to,

\[
\mathcal{H}_B \sim \int dx \left[ \frac{v}{4\pi} (\partial_x \Theta_+)^2 + C \Theta^2_+ + \pi v p^2 + \text{const.} \right],
\]

where \( C = B e^{-<\Theta^2_+>/2} \) and \( <\Theta^2_+>/2 \) is the average value at the ground state which is self-consistently determined.
This approximation yields the accurate ground state energy within 10 percent error in comparison with the exact numerical data. Thus we expect that this approximation well describes properties in very low temperatures. Using the Hamiltonian, we calculate the Drude weight $D_{th}$ using the path integral (e.g.). The Drude weight is given by explicit calculation of

$$D_{th} = \frac{-v^2}{4\pi L} \int \int d\tau_1 d\tau_2 \partial_{\tau_1} \partial_{\tau_2} \tau_1 \tau_2 \sum_{n_1, n_4 k_1, k_4} \sum_{k_2, k_3} e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)} e^{i(\omega_n x_1 + \omega_n x_2 + \omega_n x_3 + \omega_n x_4)}$$

where we take $x_1 = x_2 = x, x_3 = x_4 = x', \tau_1 = \tau_2 = \tau$, and $\omega_n = 2\pi n$ respectively. The Fourier transformation is calculated by the Wick’s theorem using the two point correlation

$$\langle \Theta_+(k_1, n_1) \Theta_+(k_2, n_2) \Theta_+(k_3, n_3) \Theta_+(k_4, n_4) \rangle,$$

The existence of finite alternation $\delta$ assures the anomalous energy transport in one-dimensional gapped systems. This aspect is consistent with the numerical studies.

IV. SUMMARY

We find the possible mechanism of anomalous transport in low energy region of realistic spin chains. In the isotropic Heisenberg chain, the energy current is exactly conserved so that the Green-Kubo formula trivially diverges. On the other hand, the presence of perturbations, the Umklapp terms appear. However this term is irrelevant to the energy current of long wave length at low temperatures. We considered the quantitative estimation of the Drude weight for the alternate chain, and find exponential decrease of Drude weight at low temperature. This aspect is consistent with the numerical studies.

Here we should make clear the difference between classical nonintegrable Fermi-Pasta-Ulam (FPU) chains, where the mechanism of anomalous transport is still argued. In this classical system, anomalous heat transport is attributed to a slow relaxation due to the total momentum conservation, whereas the the auto-correlation function of energy current vanishes in the long time limit, i.e., the mixing property is satisfied. However, in perturbed quantum Heisenberg spin chains we discuss here, Numerical evidence of the finite Drude weight means the violation of the mixing property. Thus the mechanism in quantum spin chains is different from the one in classical FPU system.

The alternate spin chain directly related to the spin-Peiers system. In the realistic mechanism, the alternation is formed by the fluctuation of the phonon. In this sense, in more realistic treatment, $\delta$ must be exchanged by the time-dependent bond length between atoms. In the case where the bond fluctuation is not negligible, how the magnetic energy transport is affected is also interesting.

Note added—After submitting this paper, we find the preprint where the Drude weight in the dimerized xy spin chain and two-leg ladder system are extensively studied. The paper includes the similar formula as (22) in these systems.

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A.V. Sologubenko, K. Gioannò, H. R. Ott, A. Vietkine, and A. Revcolevschi, Phys. Rev. B 62 R6108 (2000), Phys. Rev. B 64 054412 (2001).

Y. Ando, J. Takeya, D. L. Sisson, S. G. Doettinger, I. Tanaka, R. S. Feigelman, and A. Kapitulnik, Phys. Rev. B 64 054412-1 (2001).

M. Hofmann, T. Lorenz, A. Freimuth, G. S. Uhrig, H. Kageyama, Y. Ueda, G. Dhalenne and A. Revcolevschi, Physica B 312-313 597 (2002).

K. Kudo, S. Ishikawa, T. Noji, T. Adachi, Y. Koike, K. Maki, S. Tsuji, and K. Kumagai, J. Phys. Soc. Jpn 70 437 (2001).

A.V. Sologubenko, K. Gioannò, H. R. Ott, U. Ammerahl, A. Revcolevschi, D. F. Brewer, and A. L. Thomson, Physica B 284-288 1595 (2000).

A.V. Sologubenko, K. Gioannò, H. R. Ott, U. Ammerahl, and A. Revcolevschi: Phys. Rev. Lett. 84 2714 (2000).

C. Hess, C. Baumann, U. Ammeral, B. Büchner, F. Heidrich-Meissner, W. Brenig, and A. Revcolevschi, Phys. Rev. B 64 184305 (2001).

K. Kudo, T. Noji, Y. Koike, T. Nishizaki, and N. Kobayashi, J. Phys. Soc. Jpn 70 1448 (2001).

M. Hofmann, T. Lorenz, G. S. Uhrig, H. Kierspel, O. Zabara, A. Freimuth, H. Kageyama, and Y. Ueda, Phys. Rev. Lett. 87 047202 (2001).

P. G. Klemens, Solid State Physics 7 1 (1958), R. P. Tye ed., Thermal conductivity, Academic Press 1969, R. Berman, Thermal Conduction in Solids, Clarendon Press, Oxford 1976

J. Callaway, Phys. Rev. 113 1046 (1959).

Z. Rieder, J. L. Lebowitz, and E. Lieb, J. Math. Phys. 8, 1073 (1967).

S. Lepri, R. Livi, and A. Politi, Phys. Rev. Lett. 78, 1896 (1997), Europhys. Lett. 43, 271 (1998), O. Narayan, and S. Ramaswamy, Phys. Rev. Lett. 89 200601 (2002).

X. Zotos, J. Low Tem. Phys. 126 1185 (2002).

K. Saito and S. Miyashita, J. Phys. Soc. Jpn, 71 2485 (2002).

J.V. Alvarez and C. Gros, cond-mat/0201300.

J.V. Alvarez and C. Gros, Phys. Rev. B 66, 094403 (2002).Phys. Rev. Lett., 88, 077203 (2002).

F. Heidrich-Meissner, A. Honecker, D. C. Cabra, and W. Brenig, Phys. Rev. B 66 140406(R) (2002).

K. Saito, Europhys Lett. to be published.

D. L. Huber, Prog. Theo. Phys. 39, 1170 (1968), D. L. Huber and J. S. Semura, Phys. Rev. 182, 602 (1969).

X. Zotos, F. Naef and P. Prelovsek, Phys. Rev. B 55, 11029 (1997).

R. Kubo, M. Toda, and N. Hashitsume, Statistical Physics II, Springer-Verlag, New York, 1985.

A. Klümper and K. Sakai, J. Phys. A 35 2173 (2002).

M. C. Cross and D. S. Fisher, Phys. Rev. B, 19 402 (1979).

T. Nakano and H. Fukuyama, J. Phys. Soc.Jpn, 49 1679 (1980).

R. Kubo, M. Toda, and N. Hashitsume, Statistical Physics II, Springer-Verlag, New York, 1985.